Quantum Cosmology of Kantowski–Sachs like Models

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Abstract

The Wheeler–DeWitt equation for a class of Kantowski–Sachs like models is completely solved. The generalized models include the Kantowski–Sachs model with cosmological constant and pressureless dust. Likewise contained is a joined model which consists of a Kantowski–Sachs cylinder inserted between two FRW half–spheres. The (second order) WKB approximation is exact for the wave functions of the complete set and this facilitates the product structure of the wave function for the joined model. In spite of the product structure the wave function can not be interpreted as admitting no correlations between the different regions. This problem is due to the joining procedure and may therefore be present for all joined models. Finally, the symmetric initial condition (SIC) for the wave function is analyzed and compared with the “no boundary” condition. The consequences of the different boundary conditions for the arrow of time are briefly mentioned.
1 Introduction

Ever since the advent of canonical quantum gravity with DeWitt’s paper [1], minisuperspace models have played an important part in the discussion. The popularity of this finite dimensional models is based both on the otherwise tremendous technical problems and the difficulty to find even classical solutions for the general case. Although the relevance of minisuperspace models is not clear, it is hoped that they shed some light on the general theory and perhaps on some conceptual issues, such as the problems of how to pose boundary conditions and the arrow of time.

While far the most papers on minisuperspace quantization are concerned with the isotropic FRW model, the Kantowski–Sachs spacetime is the most popular anisotropic model. The more realistic Bianchi–I X spacetime, unfortunately, has turned out to be rather complicated; the classical model was a long time believed to be chaotic and only a few exact solutions are known. For the Kantowski–Sachs model in comparison all solutions are known analytically even if some particular types of matter are coupled to gravity. It is thus a good toy model for studying the influence of anisotropy in quantum cosmology or more generally for models with additional gravitational degrees of freedom to the one FRW degree. The Kantowski–Sachs model has for example played a crucial role in Hawking’s argument why the arrow of time is asymmetric. In addition, since the Kantowski–Sachs metric may describe the interior of black holes, it has been used to discuss the entropy and the quantum states of black holes. The last point has recently gained renewed interest in the investigations of two–dimensional black holes following the paper of Callan et. al. [2]. This last point mentions already one of the drawbacks of the quantization of the Kantowski–Sachs spacetime: namely that the Kantowski–Sachs spacetime is not a spacetime itself but only part of it. What does it mean to quantize only a part of the system? This restriction holds for the vacuum model but disappears if a perfect fluid is coupled to gravity. Unfortunately, quantum cosmology with a perfect fluid is somewhat artificial; however it is not sufficient to introduce a scalar field in order to render the Kantowski–Sachs metric complete.

In the next part of this paper some aspects of the classical Kantowski–Sachs model are reviewed, thereby emphasizing certain exceptional aspects. Moreover a joined model is introduced which is compact, inhomogeneous and anisotropic. It consists of a Kantowski–Sachs cylinder which is inserted between two FRW half–spheres. In Sec. 3 a whole class of Kantowski–Sachs like models is quantized and a complete set of exact solutions of the corresponding Wheeler–DeWitt equation is derived. In Sec. 4 a Hamiltonian formalism for the joined model is derived which makes obvious that it can be quantized along the lines of Sec. 3. Special emphasize is given to the interpretation of the resulting wave function. It turns out that there is a hidden inconsistency in the quantization the junction.

In order to render the Wheeler–DeWitt equation physically meaningful it is necessary to pose some sort of boundary conditions at the wave functions. In section 5 the symmetric initial condition (SIC) is investigated for the Kantowski–Sachs model. This initial condition is then compared with the “no boundary” condition of Hartle and Hawking and the consequences of the different boundary conditions on the arrow of time are outlined.

Natural units $\hbar = c = G = 1$ are used throughout.
2 The classical Kantowski–Sachs model

This section is intended to give a short review of the Kantowski–Sachs model by highlighting some of its features. The Kantowski–Sachs spacetime which was first investigated by Kompaneets and Chernov \[3\] and independently by Kantowski and Sachs \[4\] combines spherical symmetry with a translational symmetry in the “radial” direction. The spacelike hypersurfaces of constant times are therefore cylinders

\[ds^2 = z^2 \, dr^2 + b^2 \, d\Omega^2.\]  

(1)

Here \(b = b(t)\) is the surface measure of the two–spheres \(S_2\) with metric \(d\Omega^2\); \(z = z(t)\) measures the spacelike distance between the two–spheres and \(r\) is the radial coordinate, \(r \in \mathbb{R}\), or after compactification \(r \in I\), with \(I\) an arbitrary interval of \(\mathbb{R}\). Neither \(b\) nor \(z\) depend on the angle variables of \(S_2\) or on the radial coordinate \(r\).

It is important that the variables \(b\) and \(z\) are playing different rôles since a cylinder is defined only by the homogeneity of \(b\) and would not be deformed by an inhomogeneous \(z\). Actually an inhomogeneous \(z = z(t, r)\) is even dynamically consistent, e. g. for matter in form of pressureless dust \[5\], but this case is not considered in the remainder of the paper.

There are two types of singularities in this geometry: cigarlike singularities as \(b \to 0\) and disklike singularities as \(z \to 0\). However, the curvature of the hypersurface \(3\mathcal{R} = 2/b^2\) is divergent for the cigarlike singularity only. An investigation of the spacetime structure is thus unavoidable in order to analyze the disklike singularity. It turns out that without matter it is a coordinate singularity where the right–handed coordinate system is changed into a left–handed one. Moreover it is an indication for the incompleteness of the vacuum Kantowski–Sachs spacetime; in the case of a vanishing cosmological constant it describes the interior parts of the Kruskal spacetime\[1\]. The model can be rendered complete by adding matter e. g. in form of pressureless dust, since then the 4–curvature \(4\mathcal{R}\) diverges as the matter density becomes infinite.

For this reason the case of pressureless dust is considered here, although the quantization of this model is somewhat artificial. A scalar field as a more realistic matter field is not considered. In the first place because the differential equations with a massive scalar field are much more difficult and neither the classical nor the quantized ones have yet been analytically solved. Second, the model *seems to remain* incomplete because as \(z \to 0\) the scalar field behaves like an effective cosmological constant; that means, for \(z \to 0\) the model with scalar field *would be* as incomplete as the vacuum one. Furthermore the construction of the joined model at the end of this section is impossible with a scalar field. However, the scalar field can, at least for minisuperspace models, be conveniently used in order to describe a dust field by considering the limiting case when \(8\pi \rho = \ddot{\phi}^2/N^2 + m^2 \phi^2 \to \text{const}\) and simultaneously \(p = \dot{\phi}^2/N^2 - m^2 \phi^2 \to 0\). The equation of motion for the scalar field

\[\frac{1}{N} \left( \frac{\dot{\phi}}{N} \right)^2 + \frac{1}{N^2} \ddot{\nu} + m^2 \phi^2 = 0,\]  

(2)

\[\text{1} \quad \text{Although a compactification of the radial axis does change the coordinate singularity into a topological singularity, it does not improve the incompleteness of the model.}\]
where \( v = zb^2 \) is the volume measure of the hypersurface, proves this possibility if the second term can be neglected (which is fulfilled for great volumes) for \( \phi = \sqrt{8\pi \rho/m \sin m\tau} \), with \( d\tau = Ndt \). The dust is then described by a parameter instead of an variable in the Lagrangian, c. f. Eq. (3). This approximation breaks down, of course, for vanishing volumes. However, for small volumes the scalar field behaves as an effective cosmological constant. It may therefore be more realistically to consider the regions for small and for great volumes separately. It should be noted that this approach which treats dust as a mere parameter is essentially equivalent to a more sophisticated approach starting point of which is a Lagrangian for the dust degrees of freedom, see e. g. [3] and the literature cited therein. This is, because the latter approach leads in the homogeneous models to a Lagrangian (Hamiltonian) which consists of only one term containing a cyclic momentum as the only remaining variable.

The Lagrangian of the Kantowski–Sachs model is

\[
L(z, b) = \frac{I}{2} \left( -\frac{2b\dot{z}^2}{N} - \frac{\dot{z}b^2}{N} + N \left( z - z_m - \Lambda zb^2 \right) \right)
\]

\[(3)\]

where \( I \) is the compactification interval, \( \Lambda \) is a cosmological constant and \( z_m = \rho zb^2 = \text{const} \) is the dust potential (analogous to \( a_m = \rho a^3 \) in the FRW model). Considering the above comment on dust in quantum cosmology one may thus approximate the scalar field for small volumes by \( \Lambda \neq 0 \) and \( z_m = 0 \) and for large volumes by \( \Lambda = 0 \) and \( z_m \neq 0 \). If a two dimensional pseudosphere or plane is considered instead of the two–sphere in (1) a different sign of the first term of the potential will result: \( -z \) respectively 0 instead of \( +z \). These models are special cases of the generalized Kantowski–Sachs model which is introduced and quantized in Sec. 3.

The Legendre transformation with the canonical momenta

\[
\pi_b = -\frac{I(zb)}{N} \quad \text{and} \quad \pi_z = -\frac{I\dot{b}b}{N}
\]

\[(4)\]

leads to the Hamiltonian constraint

\[
z\dot{b}^2 + 2\dot{z}\dot{b}b + N^2 \left( z - z_m - \Lambda zb^2 \right) = 0
\]

\[(5)\]

written in the configuration variables and their velocities. The equations of motion are obtained by taking the Poisson brackets with the Hamiltonian,

\[
\frac{\dot{b}^2}{N^2} + 1 - \frac{b_m}{b} - \frac{1}{3}\Lambda b^2 = 0
\]

\[(6)\]

\[
b\ddot{z} + \dot{b}\dot{z} + \ddot{b}z - (zb) \frac{\dot{N}}{N} - N^2zb\Lambda = 0
\]

\[(7)\]

where the equation of motion for \( b(t) \) has already been integrated. Eq. (6) therefore contains \( b_m \) as an arbitrary constant of motion; for a vanishing cosmological constant \( b_m \) is the maximum of \( b(t) \). Although this equation is identical to the equation of motion for the scale factor \( a(t) \) of the closed FRW model, in contrast to that model the matter content of the Kantowski–Sachs model does not fix the constant of integration \( b_m \). Instead, the matter content fixes the constant of
integration for the $z$ dynamics as already indicated by the notation: $\rho z b^2 := z_m = z(t_m)$ where $t_m$ is defined by $b(t_m) = b_m$. It is worth noting that Eq. (6) provides one with an expression for the classical range of the variable $b$: $b_m > b(1 - \frac{1}{3}\Lambda b^2)$. Since the equation of motion for $b(t)$ is independent of $z(t)$, these equations confirm that an inhomogeneous $z(t, r)$ might be dynamically consistent. A rigorous proof must, however, use the equations of motion for the general spherically symmetric model. There it follows directly since, in Kuchař’s notation \[\text{(7)}\] (there are two fields $\Lambda(t, r)$ and $R(t, r)$ with $\Lambda(t, r) \to z(t, r)$ and $R(t, r) \to b(t)$ for the Kantowski–Sachs model), terms which contain $\Lambda'$ do always contain $R'$ too and do thus not appear in the Kantowski–Sachs model.

In order to solve the equations of motion the lapse function has to be fixed. The gauge $N = b$ has the advantage of covering the whole manifold while, for example, the gauge $b = t$ covers only half of it. For a vanishing cosmological constant, the explicit solutions for the gauge $N(t) = b(t)$ are

$$
\begin{align*}
  b(t) &= b_m \sin^2 \left(\frac{t}{2}\right), \\
  z(t) &= K \cot \left(\frac{t}{2}\right) + z_m \left(1 - \frac{t}{2} \cot \left(\frac{t}{2}\right)\right),
\end{align*}
$$

where the initial time has been set to zero and where $K$ is an arbitrary constant of integration. For an illustration see the left diagram in Fig. 1 where the model is depicted in Kruskal–type coordinates. It is worthwhile to note that in the vacuum model the constant of integration $K$ is meaningless due to a possible redefinition of the radial coordinate $r \to Kr$. Solutions with different values of $K$ are thus identical. The inhomogeneity of $z(r, t)$ is due to either inhomogeneous dust $z_m'(r)$ or to an inhomogeneous constant of integration $K(r)$. Unlike the FRW model, the Kantowski–Sachs model is not time–symmetric with respect to $b_m$ since $z(t)$ is not.

For the case of $z_m = 0$ but $\Lambda \neq 0$ one obtains a similar set of solutions which admits the exceptional solution of a constant $b(t)$: $b = \sqrt{1/\Lambda}$ and $z = e^t$ (again in the gauge $N = b$).

Since both the Kantowski–Sachs and the FRW hypersurface are spherically symmetric, it is possible to construct a new spherically symmetric 3–geometry by joining the respective hypersurfaces. In Fig. 2 such a hypersurface is visualized. In order to be dynamically consistent, however, the dynamics of the surface measure of the two–sphere, the metric at fixed $r$, has to be identical on both sides of the junction. This is trivially fulfilled in this case, provided the constants of integration are equal: $b_m = a_m$. That means, the matter content of the FRW part fixes the constant of integration of the Kantowski–Sachs part of the spacetime. In addition the metric of the hypersurface has to be smooth (more precisely: in the notation by Kuchař one has to require $\dot{R}$ and $R'/\Lambda$ to be smooth in order to obtain a well defined Hamiltonian). This requires the junction to take place at the equator of the FRW 3–sphere which is thus cut into halves. It is therefore even possible to split the FRW 3–sphere by inserting a Kantowski–Sachs cylinder of arbitrary length; this has no influence on the FRW dynamics. For an illustration see the right diagram in Fig. 1 where the model is represented in Kruskal–type coordinates. A Hamiltonian formulation and quantization of this compact, anisotropic and inhomogeneous model is given in Sec. 4. The joining procedure is impossible with a scalar field as matter.
source. Technically, this is because then the $b$–dynamics is not decoupled from the $z$–dynamics. This means physically that the scalar field can not be contained in one region: it leaks into the other regions, thereby displaying the inhomogeneity of the model.

![Diagram of Kantowski–Sachs model](image1)

Fig. 1: The left diagram shows the dust filled Kantowski–Sachs model in Kruskal–like coordinates. In the limit of vanishing dust the limiting hyperbolas degenerate into the usual lightlike event horizon of the Schwarzschild black hole. In that case it is possible to continue the Kantowski–Sachs model into the outer regions of the Kruskal diagram. This is impossible in the dust case since there the hyperbolas build a spacelike borderline; the complete spacetime is represented by the shaded regions only. It is possible to continue paths (as e. g. the $r = \text{const}$ geodesics which are represented in this diagram by straight lines through the origin) from the lower into the upper Kantowski–Sachs region. The right diagram shows as an example those geodesics which constitute the junction to the FRW regions of the joined model. In order to make graphical explicit that there is no crossing of the geodesics, right and left have been interchanged in the upper (lower) region.

![Diagram of FRW-S1-S2](image2)

Fig. 2: The diagram shows a hypersurface of constant time for the joined model. Every $S_1$–circle represents a $S_2$–sphere. The arrows indicate the joining procedure which enforces $a_1 = b = a_2$. 
3 Solving the Wheeler–DeWitt equation

As usual, the Kantowski–Sachs model is quantized following Dirac’s quantization scheme by turning the variables $b$, $z$, $\pi_b$ and $\pi_z$ into operators which fulfill the standard commutation relations and the Hamiltonian constraint (5) becomes an operator which annihilates the physical states: $H = 0 \rightarrow \hat{H}\Psi = 0$. In the field representation this equation is called the Wheeler–DeWitt equation. For the Kantowski–Sachs model it takes the form

$$-2Iz b^2 \hat{H}\Psi = \left[ z^2 \partial_z^2 + k_1 z \partial_z - 2 zb \partial_b \partial_z + I^2 z b^2 (zf_1(b) + f_2(b)) \right]\Psi(b, z) = 0 ,$$

with

$$f_1 = 1 - \Lambda b^2 \quad \text{and} \quad f_2 = -z_m ,$$

where $\partial_z$ and $\partial_b$ are the partial derivatives with respect to $z$ and $b$, respectively. Factor ordering is partially left open as indicated by the arbitrary parameter $k_1$. This one–parameter family of orderings includes the important Laplace–Beltrami ordering with $k_1 = 1/2$.

The particular form (10) of the Wheeler–DeWitt operator with the two functions $f_i(b)$ has been chosen because it is possible to solve the equation exactly for arbitrary functions $f_i(b)$. This general form of the Hamiltonian does not only include the ordinary Kantowski–Sachs model with dust and cosmological constant but also the joined model (see Sec. 4) and the rotational symmetric Bianchi–I and Bianchi–III models which result when the two–spheres in the metric (1) are replaced by two–dimensional planes or pseudospheres, respectively. In the literature one finds several sets of exact solutions for the vacuum model, see e. g. [8, 9, 10]. One of these is the general solution (15) specialized for the vacuum case [8]. The other two sets of solutions utilize transformations which separate the kinetic part of the Wheeler–DeWitt equation and at the same time keep the potential part simple.

In order to solve the Wheeler–DeWitt (9) equation it is convenient to introduce the operator

$$2I^2 \hat{M} = -\frac{1}{b} \partial_z^2 + I^2 \int^b f_1(b')db'$$

(for the special case (10) this simplifies to $2I^2 \hat{M} = -(1/b) \partial_z^2 + I^2 (b - b^3)$). This operator is known from the general spherical symmetric model where $\hat{M}(r)$ is interpreted as the gravitating mass inside the radius $r$. (For an explicit account of the general spherical model see [11, 12]).

One can use this operator in the Kantowski–Sachs model since $\hat{M}$ is the first time–integral of the equation of motion for $b(t)$ (that means, Eq. (11) is identical to the operator version of Eq. (3) with $\hat{M} = 2\hat{b}_m$), although neither the interpretation as gravitational mass nor its usual derivation (which makes use of the supermomentum density) is valid. This is true also for the general spherically symmetric models (assuming vanishing shift–function). It is essential that the new operator $\hat{M}$ has a vanishing commutator with the Wheeler–DeWitt operator

$$[\hat{M}, \hat{H}] = 0 .$$

One can thus find simultaneous eigenfunctions for both operators and $\hat{M}$ can be interpreted as formal observable. The eigenvalue equation $\hat{M}\psi_M = M\psi_M$ can, in contrast to the Wheeler–
DeWitt equation (9), easily be solved

$$\psi_M(z,b) = h(b) \exp \{ \pm i I z W_M(b) \} ,$$  \hspace{1cm} (13)

with

$$W_M^2(b) = 2Mb - b \int_b^b f_1(b')db' .$$

The arbitrary function $h(b)$ is then fixed by requiring $\psi_M$ to solve the Wheeler–DeWitt equation. The resulting differential equation for $h(b)$ is again simple,

$$h'(b) h(b) = - \partial_b W_M(b) W_M(b) + k_1^2 b - iI^2 G b f_2(b) W_M(b) .$$  \hspace{1cm} (14)

This gives (up to an arbitrary constant)

$$\Psi_M(z,b) = \sqrt{b} k_1 W_M(b) \exp \left\{ \pm i I \left( z W_M(b) - \frac{1}{2} \int_b^b \frac{b' f_2(b')}{W_M(b')} db' \right) \right\} ,$$  \hspace{1cm} (15)

as a one–parameter set of exact solutions for the Wheeler–DeWitt equation. In the particular model with $f_1(b) = 1 - \Lambda b^2$ and $f_2(b) = -z_m$, the wave function is thus

$$\Psi_M(z,b) = \sqrt{b} k_1 \frac{b^{k_1-2}}{\sqrt{\Lambda b^2/3 + 2M/b - 1}} \exp \left\{ i I \left( z b \sqrt{\frac{\Lambda b^2}{3} + \frac{2M}{b} - 1 + g(b)} \right) \right\} ,$$  \hspace{1cm} (16)

with

$$g(b) = \frac{z_m}{2} \int_b^b \frac{b'}{\sqrt{\Lambda b'^3/3 + 2M - b'}} db' .$$  \hspace{1cm} (17)

The integral $g(b)$ can be solved analytically in special cases. For $M = 0$, that is for large values of $b$, one gets

$$g_M(b) = \frac{z_m}{2} \sqrt{\frac{3}{\Lambda}} \ln \left( b + \sqrt{b^2 - \frac{3}{\Lambda}} \right) ,$$  \hspace{1cm} (18)

while for vanishing $\Lambda$, that is for small values of $b$, one gets

$$g_M(b) = -\frac{z_m}{2} \left( b \sqrt{\frac{2M}{b}} - 1 + M \arcsin \left( \frac{M-b}{M} \right) \right) .$$  \hspace{1cm} (19)

The solution for the model which approximates the scalar field at vanishing $z_m$ for small volumes and at vanishing $\Lambda$ for great volumes is therefore given by Eq. (15) with vanishing $g(b)$ and with $g(b)$ given by Eq. (18), respectively. These solutions have, of course, to be matched at that volume where the character of the equation changes.

These exact solutions (15) of the Wheeler–DeWitt equation have the peculiar structure that

$$\Psi = D e^{\pm i S_M} ,$$  \hspace{1cm} (20)

where $S_M$ is a solution of the Hamilton–Jacobi equation and $D$ is the (first order) WKB prefactor. One may interprete these solutions as a complete, one parameter family of exact
solutions; complete in the sense that $S_M$ is a complete Integral of the Hamilton–Jacobi equation. The prefactor $D$ does not depend on the $f_2$–part of the potential; that means it does not depend on the dust content. The considered factor ordering ambiguity has completely been taken care of by the single factor $b^{\frac{\alpha_i}{2}}$. This specific structure of the solutions has important consequences in the next sections. The solutions are exact in spite of the divergency at the borderline of the classically allowed region. It is possible to get rid of this apparent divergency. Consider first the second term in the exponent. This is an integral with a lower limit of integration which has not yet been fixed. The integral can therefore be forced to vanish at the classical borderline by requiring the integration to start just there (provided there is only one). Since thus only the first part of the Hamilton–Jacobi function is concerned one can simply consider the particular superposition of the solution which is proportional to the sine. At the borderline the wave function behave then as $(1/x)\sin x$ and is regular.

An exact WKB solution has recently been found for the much more complicated Bianchi–IX model [13, 14]. That solution even has a constant prefactor but it requires a rather artificial factor ordering and it is a solitary solution instead of a complete set. It can be obtained by transforming the trivial solution of the Wheeler–DeWitt equation in Ashtekars variables into geometrodynamical variables. In this sense it is a “ground state” solution.

4 The joined model

In this section a Hamiltonian formulation for the joined model which consists of two FRW half–spheres and a connecting Kantowski–Sachs cylinder is derived and then the model is quantized along the usual lines. Without a Hamiltonian formalism for the whole model it is only possible to join the WKB solutions of different sections as, e. g., in [15], but not to go beyond this approximation. However, it becomes apparent at the end that it is not possible to really circumvent the difficulties of the junction procedure with this approach.

The Lagrangian for the whole model is

$$S(a, z, b) = \int \int \pi \mathcal{L}_{FRW_1} d\chi dt + \int \int_{r_1}^{r_2} \mathcal{L}_{KS} dr dt + \int \int_{\pi}^{\pi} \mathcal{L}_{FRW_2} d\chi dt,$$  \hspace{1cm} \text{(21)}$$

where $\chi$ is the third angle variable of the FRW three–sphere and it has already been implied that the Kantowski–Sachs model must join the FRW model at the equator of the FRW spheres. The length of the Kantowski–Sachs cylinder is not restricted; that is, $r_1$ and $r_2$ are arbitrary. All together, the following junction conditions have to be fulfilled for the joined model

$$b = a_i \sin \chi_i ,$$
$$\sin \chi_i = 1, \hspace{1cm} i = 1, 2$$
$$b_m = a_{m_i} = a^3 \rho_{FRW_i} ,$$

in order to built a dynamically consistent smooth hypersurface. The $a_{m_i}$ are fixed by the dust contents of the respective FRW regions. The last condition thus means in particular that the dust density of both FRW regions has to be identical; but there is no requirement concerning
the dust content of the Kantowski–Sachs region. Inserting these conditions into the Lagrangian results in a Lagrangian which describes the whole model

\[ L_{\text{Ges}}(z, b) = \frac{3I_1}{2} \left( -\frac{b b^2}{N} + N \left( b - a_m - \frac{1}{3} \Lambda b^3 \right) \right) + \frac{I_2}{2} \left( -\frac{2b b \dot{z}}{N} - \frac{z b^2}{N} + N \left( z - z_m - \Lambda z b^2 \right) \right) \].

(23)

Here \( b \) is both a Kantowski–Sachs variable and (two times) a FRW variable while \( z \) is a genuine Kantowski–Sachs variable. \( I_1 = \int_0^\pi \sin^2 \chi d\chi \) and \( I_2 = \int_{r_1}^{r_2} dr \) are the volume elements for the FRW model and for the Kantowski–Sachs model, respectively. This Lagrangian can now be used to construct the Hamiltonian for the joined model. It is important to notice that it is not possible to incorporate the condition \( a_m = b_m \) into the Lagrangian since \( b_m \) does not occur explicitly. Therefore still one junction condition has to be imposed.

The Legendre transformation with the canonical momenta

\[ \pi_z = -\frac{I_2}{N} \dot{b}, \quad \pi_b = -\frac{3I_1}{N} \dot{b} - \frac{I_2}{N} (zb) \]

then leads to the Hamiltonian

\[ H = \frac{1}{2I_2 z b^2} \left\{ \left( z^2 \pi_z^2 - 2zb \pi_z \pi_b - I_2^2 z b^2 \left( z - z_m - \Lambda z b^2 \right) \right) + 3 \frac{I_1}{I_2} z b \left( \pi_z^2 - I_2^2 \left( b^2 - ba_m - \frac{1}{3} \Lambda b^4 \right) \right) \right\} = 0 \]

(25)

which is constrained to vanish. However, due to the neglected junction condition this Hamiltonian alone does not reproduce the correct equations of motion. It has to be accompanied by another constraint which enforces \( a_m = b_m \). It is here chosen to be

\[ I_2^2 \left( M - \frac{a_m}{2} \right) := \frac{1}{2b} \pi_z^2 + \frac{I_2^2}{2b} \left( 1 - \frac{\Lambda}{3} b^2 - \frac{a_m}{b} \right) = 0 \].

(26)

The field \( M \) which is defined as in the last section is fixed by this constraint to the value \( a_m/2 \). The full Hamiltonian \( H_{\text{full}} \) leading to the correct equations of motion is therefore given by

\[ H_{\text{full}} := N_H H + N_M M \]

(27)

with the two Lagrange multipliers \( N_H \) and \( N_M \). This procedure is consistent due to the vanishing of the Poisson bracket of the two constraints

\[ \{ H, M \} = 0 \],

(28)

that is \( H = 0 \) and \( M = 0 \) are first class constraints. This relation holds for the corresponding operators too, if for example the simplest ordering for the operator \( \hat{M} \) is chosen and the Laplace–Beltrami factor ordering is chosen for the Wheeler–DeWitt equation which then reads

\[ \hat{H} \Psi = -\frac{1}{2I_2 z b^2} \left[ \left( \frac{z^2 + 3 \frac{I_1}{I_2}}{I_2} \right) \partial_z^2 + z \partial_z - 2zb \partial_z \partial_b + \right. \]

\[ + \left. I_2^2 z b^2 \left( z - z_m - \Lambda z b^2 + 3 \frac{I_1}{I_2} \left( b - a_m - \frac{1}{3} \Lambda b^3 \right) \right) \right] \Psi = 0 \].

(29)
It is not surprising that these two operators commute because $\hat{H}$ can be transformed into the generalized Kantowski–Sachs model \([3]\) by

$$\bar{b} := b \quad \text{and} \quad \bar{z} := z + \frac{I_1}{I_2} \bar{b}.$$ \(30\)

In the notation of the last section $k_1 = \frac{1}{2}$ and

$$f_1(b) = 1 - \Lambda b^2 \quad \text{and} \quad f_2(b) = - \left( z_m + \frac{I_1}{I_2} 3b_m \right) + 2 \frac{I_1}{I_2} \bar{b}.$$ \(31\)

Only $f_2$ thus differs from the Wheeler–DeWitt equation of the pure Kantowski–Sachs model. $\hat{M}$ is not altered by the transformation.

Instead of one equation as in Sec. 3, two differential equations have now to be satisfied

$$\hat{H}\Psi = 0 \quad \text{and} \quad \left( 2\hat{M} - a_m \right) \Psi = 0 \quad (32)$$

which can be viewed as a consequence of the inhomogeneity of the model. This does not pose any problems since in Sec. 3 the solutions of the Wheeler–DeWitt equation $\hat{H}\Psi = 0$ were actually found by using $\hat{M}\Psi = M\Psi$. The sole consequence of the second equation is the uniqueness of the solution (up to the sign in the exponent)

$$\Psi_{\text{join}}(\bar{b}, \bar{z}) = \frac{1}{\sqrt{\Lambda b^2 + a_m - \bar{b}}} \exp \left\{ \pm i\mathcal{I}_2 \left( \sqrt{\frac{\Lambda b^2}{3} + a_m \frac{b}{\bar{b}} - 1 + g(\bar{b})} \right) \right\}$$ \(33\)

with

$$g(\bar{b}) = \frac{1}{2} \int_{\bar{b}}^{b} \frac{z_m + 3 \frac{I_1}{I_2} a_m - 2 \frac{I_1}{I_2} \bar{b}}{\sqrt{\Lambda \bar{b}^2 / 3 + a_m / \bar{b} - 1}} d\bar{b}'$$ \(34\)

since the eigenvalue of $\hat{M}$ is now fixed by the second constraint. It is of course possible to make the same distinction between small volumes with vanishing dust and large volumes with vanishing cosmological constant as in the last section.

The WKB structure ($\Psi = De^{\pm iS}$) of this exact solution allows one to write the wave function in another form. This is most directly seen in the case of a vanishing cosmological constant, but holds in the general case, too. The Hamilton–Jacobi function $S$ is the sum of the Hamilton–Jacobi functions of the separate regions $S = S_{KS} + S_{FRW}$ written in the original variables $b$ and $z$

$$S = \mathcal{I}_2 \left( b \left( z - \frac{z_m}{2} \right) \sqrt{\frac{a_m}{b} - 1 - \frac{z_m}{4} a_m} \arcsin \left( \frac{a_m - 2b}{a_m} \right) \right) + 3 \mathcal{I}_1 \left( \frac{b^2}{4} \left( 2 - \frac{a_m}{b} \right) \sqrt{\frac{a_m}{b} - 1 - \frac{a_m}{8} \arcsin \left( \frac{a_m - 2b}{a_m} \right) } \right).$$ \(35\)

Furthermore, the WKB prefactor (for the whole model) is here simply the prefactor of the Kantowski–Sachs region, because only $f_1$ contributes to the prefactor which contains no FRW term and $\bar{b} = b$. Thus the wave function \([33]\) has a product structure

$$\Psi_{\text{join}} = \Psi_{KS} \Psi_{FRW},$$ \(36\)
where the Kantowski–Sachs part is an exact solution of the corresponding Wheeler–DeWitt equation, whereas the FRW wave function is a WKB solution only. It is only the WKB structure of the wave function (33) plus the form of the prefactor which facilitates the product structure. This product structure seems to implicate that there are no correlations between the different regions in the quantized model. One would expect this behaviour at least in the classical limit. This conclusion however is not valid; the separation of the different regions in Eq. (36) is thus an artificial one. First, it is invalidated by every superposition and in particular for the regular sine solution. Second, and more important, it is not even true for the above wave function, simply because the variable $b$ is a variable for both the Kantowski–Sachs and the FRW region. That means that expectation values for this wave function would neither give the FRW nor the Kantowski–Sachs expectation value nor a product as one would expect if there were no correlations between the regions. The explanation for this behaviour seems to be an insufficient treatment of the junction conditions. While technically nothing seems to be wrong the very act of the junction itself seems to be improper in a quantum field theory. Joining the regions by demanding $a = b$ and $a_m = b_m$ means that one is simultaneously fixing a variable and its conjugate momentum, which is obviously not in accordance with a quantum treatment. In other words, while quantizing the spacetime in each region one demands the junction to be undisturbed and smooth; there are no “quantum fluctuations” from one region to the other. The same comment holds of course for every joined model as for example the Oppenheimer–Snyder model or the bubble spacetimes. The only way to circumvent this problem at least formally may be to transform classically to the true degrees of freedom. For the example of a bubble spacetime this would mean that only the bubble radius remains which is to be quantized.

5 The symmetric initial condition for the Kantowski–Sachs model

One of the most controversial issues in quantum cosmology is the question of how to pose a boundary condition at the wave function. The first proposal is due to DeWitt who proposed in his 1967 paper on canonical quantum gravity a vanishing wave function for vanishing volume. In the 80th then the “no boundary” condition of Hartle and Hawking caused some new interest in quantum cosmology which was followed by several other suggestions of how the boundary conditions should look like. Here I restrict myself mostly to the symmetric initial condition (SIC) but compare the SIC with the “no boundary” condition.

The “no boundary” condition of Hartle and Hawking is an initial condition for paths although it may be that it makes sense for classical trajectories only. This point of view was for example expressed by Hawking in his comment to Zeh’s contribution at the Huelva conference on the arrow of time. At least for practical purposes it is used mostly as a means for defining initial conditions for the Hamilton–Jacobi equation or alternatively for the classical trajectories. In contrast to the “no boundary” condition the SIC is an initial condition for the Wheeler–DeWitt equation. The main underlying idea of the SIC is to demand the wave function to be initially as “simple” as possible in configuration space. That this boundary condition may be sensibly imposed is confirmed by the structure of the potential in the Wheeler–DeWitt
equation which goes to zero for $\alpha \to -\infty$. For the FRW model with arbitrary matter the potential even has the form

$$V(\alpha, \{\beta_i\}) \to V(\alpha) \to 0 \quad \text{for} \quad \alpha \to -\infty.$$ (37)

Here and in the following $\alpha$ is the timelike variable in configuration space. $\alpha$ can always be chosen to be a function of the volume $v$ of the 3–geometry and is here and in the following defined by $\alpha = \frac{1}{3} \ln v$. The $\beta_i$ characterize all the spacelike variables like matter fields or gravitational variables for more complicated models than the FRW model. In view of the behavior of the potential (37), a first formulation of the SIC may therefore be

$$\Psi(\alpha, \{\beta_i\}) \to \Psi(\alpha) \to \text{const} \quad \text{for} \quad \alpha \to -\infty.$$ (38)

But what does it mean that the wave function behaves like a constant? The obvious choice

$$\partial_{\beta_i} \Psi(\alpha, \{\beta_i\}) \to 0 \quad \text{for} \quad \alpha \to -\infty$$ (39)

is ambiguous since it does not specify how (39) has to be fulfilled. For the example of oscillator eigenfunctions with $\omega \propto e^{3\alpha}$, as in the FRW model with scalar field, all superpositions of these eigenfunctions would satisfy this criterion. Furthermore, it cannot be generalized for non–FRW potentials. In order to decide whether a wave function is constant or not one needs a basis which is here canonically given by the “spacelike Hamiltonian” $H_{sp}$ defined by

$$\partial_\alpha^2 \Psi = H_{sp} \Psi.$$ (40)

$H_{sp}$ is called spacelike Hamiltonian since it contains only spacelike momenta and is thus a Schrödinger like operator. Its eigenfunctions

$$H_{sp} h_n = E_n h_n$$ (41)

then define canonically what a “simple” wave function is supposed to mean:

$$\int \Psi h_n \prod_i d\beta_i \to \int \text{const} \cdot h_n \prod_i d\beta_i \quad \text{for} \quad \alpha \to -\infty.$$ (42)

This initial condition for the wave function is called symmetric initial condition (SIC) in order to emphasize the character of the wave function as a particular superposition of all eigenfunctions. One can simplify this condition by using a more intuitive approach: A wave function is constant if it is broader than all the eigenfunctions

$$\frac{b_\Psi}{b_{h_n}} \to +\infty \quad \text{for} \quad \alpha \to -\infty,$$ (43)

where $b_\Psi$ and $b_{h_n}$ are the widths of the wave function and the eigenfunctions, respectively.

The definitions (42) and (43) of the SIC suppose that the eigenfunctions $h_n$ are normalized in the spacelike degrees of freedom. This seems to be a reasonable condition which is indeed fulfilled by some important cosmological models, as e. g. the FRW model with arbitrary matter,
Bianchi–IX or the perturbed FRW model. The Kantowski–Sachs model, however, admits an unbounded potential and does not possess normalizable eigenfunctions (see Eq. (50) below).

Before addressing the problem of the Kantowski–Sachs model, I will first review models which are better tractable by considering only those potentials which behave as in (37) [18]. One can then show that with respect to these degrees of freedom, the condition (39) is a consequence of the SIC as defined in (42). Consider, for example, the Wheeler–DeWitt equation for the FRW model with a scalar field
\[
H\Psi = \left[ \partial_\alpha^2 - 3 \partial_\phi^2 - 9 \mathcal{I}_1^2 \left( e^{4\alpha} - \frac{1}{3} m^2 \phi^2 e^{6\alpha} \right) \right] \Psi = 0 ,
\]
where \( a = e^\alpha \) is the FRW scale factor. The uncommon prefactors of the terms containing the scalar field result from a reparametrization of the scalar field suited for the spherical symmetric model but different from the usual FRW reparametrization \( \phi = \frac{1}{\sqrt{3}} \phi_{FRW} \). The partial differential equation (44) then reduces to a one dimensional differential equation
\[
H\Psi \approx \left[ \partial_\alpha^2 - 9 \mathcal{I}_1^2 \left( e^{4\alpha} - \frac{1}{3} m^2 \phi^2 e^{6\alpha} \right) \right] \Psi = 0 \quad \text{for} \quad \alpha \to -\infty
\]
which can be analyzed with the common methods of one–dimensional quantum mechanics. Since the spacelike Hamiltonian \( H_{sp} \) has the form of a harmonic oscillator, the spacelike eigenfunctions are known and the discussion of the condition (42) or (43) can be carried through. It turns out that it is convenient for several reasons to introduce a repulsive potential in the Planck era, as e. g.
\[
V_p = -C^2 e^{-2\alpha}
\]
with the arbitrary constant \( C^2 \). The WKB approximation can only then be used for \( \alpha \to -\infty \) and the solutions behave exponentially. This in turn facilitates the possibility of selecting one solution by a normalization condition. Furthermore the SIC in its strong formulation (42) can be fulfilled. The Planck potential seems to be artificial, however, since the physics at the Planck scale is entirely unknown and quantum gravity is often regarded as the low energy limit of some GUT, a Planck potential may arise as an effective potential of such a theory (the effective axion potential, e. g., almost leads to a Planck potential). Furthermore, one can get completely rid of the Planck potential by regarding it solely as a regulator which gives one a unique wave function.

With the Planck potential the FRW model with massive scalar field can easily be solved (in the considered regions) [17, 18]. For this model and for the perturbed FRW model, the SIC approximately leads to the same wave functions as the “no boundary” condition [16, 20]. This is to be expected since in both examples the condition (39) for matter fields has been used by either boundary condition. However, there is an important difference: While the SIC leads to a solution of the Wheeler–DeWitt equation, this is not so for the “no boundary” condition: It has often been emphasized after the 1985 debate between Page and Hawking [21, 22] that the condition \( \phi = const \) is valid only for the beginning of the classical trajectories but not for the end and that therefore the wave function possesses an additional oscillating term which reflects this behaviour. This means that the condition \( \Psi \to 1 \) is regarded not as an initial condition for the Wheeler–DeWitt equation but as an initial condition for the Hamilton–Jacobi
equation. As more explicitly argued in the conclusion it is exactly this difference which leads to the disagreement for the arrow of time derived with both boundary conditions, for a more detailed discussion see e. g. [17, 22, 23, 24, 25] and the literature cited therein. For a critique of using semiclassical methods only, see in particular [26].

In the context of this paper I am only interested in the dust case where the Wheeler–DeWitt equation is given by

\[ H \Psi = \left[ \partial_{\alpha}^2 - C^2 e^{-2\alpha} - 9I_1^2 \left( e^{4\alpha} - a_m e^{3\alpha} \right) \right] \Psi = 0. \]  (47)

By generalizing the SIC wave function from the scalar field model one finds

\[ \Psi = \exp \left\{ 3iI_1 \left( \frac{a^2}{4} \left( 2 - \frac{a_m}{a} \right) \sqrt{\frac{a_m}{a} - 1} - \frac{a_m^2}{8} \arcsin \left( \frac{a_m}{a} - 2a \right) \right) \right\} \]  (48)

in the region, where the Planck potential can be neglected. Strictly speaking, however, it is not possible to determine a SIC wave function for the FRW–dust model since the spacelike Hamiltonian vanishes.

As mentioned above, the SIC (42) is not directly applicable to the Kantowski–Sachs model, whose Wheeler–DeWitt equation reads

\[ \left[ z^2 \partial_z^2 + \frac{1}{2} z \partial_z - 2zb \partial_b \partial_z + \partial_{\sigma}^2 + I^2 z b^2 \left( z - \Lambda b \right) - z b^2 m^2 \phi^2 \right] \Psi = 0 \]  (49)

\[ \Leftrightarrow \left[ \partial_{\alpha}^2 - 3 \partial_{\sigma}^2 + 3I^2 \left( -e^{4\alpha+2\sigma} + e^{6\alpha}(\Lambda + m^2 \phi^2) \right) \right] \Psi = 0. \]  (50)

The Wheeler–DeWitt equation is here displayed with a scalar field, and the Laplace–Beltrami factor ordering is chosen (for the factor ordering the scalar field is neglected). Again \( \alpha := \frac{1}{3} \ln zb^2 \) is the timelike variable and \( \sigma := \frac{1}{3} \ln(z/b) \) and \( \phi \) are spacelike. While the scalar field poses no problem (its potential is bounded; and since (39) can be used for \( \alpha \to -\infty \), it is equivalent to a cosmological constant and therefore not considered in the following), the potential for \( \sigma \) is unbounded for \( \sigma \to \infty \). It is therefore impossible to find normalizable eigenfunctions for the spacelike Hamiltonian. Condition (39), too, cannot be used because the \( \sigma \)–dependence of the potential does not vanish for \( \alpha \to -\infty \).

As a first attempt to apply the SIC in spite of this difficulty one can investigate the effects of a “Planck potential” for the Kantowski–Sachs model. The Planck potential is given by exploiting the fact that the dynamics for \( b \) in the Kantowski–Sachs model should be identical to the dynamics of \( a \) in the FRW model, regardless of a Planck potential. Using the ansatz (46) for the FRW Planck potential one gets

\[ V_P = -10C^2 \frac{z}{b^6} + \tilde{V}(b), \]  (51)

as a Planck potential for the Kantowski–Sachs model. Here \( \tilde{V}(b) \) is an arbitrary function which for simplicity is set to \( \tilde{V} = D^2/b^6 \) in the following. Another ansatz for the FRW Planck potential or a general \( \tilde{V} \) does not alter the conclusions. The additional Planck potential does not render the potential bounded in \( \sigma \), since in the \( \{\alpha, \sigma\} \) representation it has the form

\[ \frac{1}{3I^2} V_P = 10C^2 e^{-2\alpha+8\sigma} - D^2 e^{-3\alpha+6\sigma}. \]  (52)
Obviously, the potential is still unbounded.

However, one can use the FRW model with its SIC wave function in another way by considering the joined model. The FRW part of the wave function fixes the Kantowski–Sachs part of the wave function by the product structure of the solution. One therefore has to analyze the joined model with a Planck potential included for each region. It turns out that again this model has a potential of the form

$$V = \mathcal{I}_2^2 z b^2 (z f_1(b) + f_2(b))$$  \hspace{1cm} (53)

with

$$f_1(b) = 1 - \Lambda b^2 - 10 C^2 b^{-6}$$
$$f_2(b) = - \left( z_m + 3 \frac{\mathcal{I}_1}{\mathcal{I}_2} b_m \right) + 2 \frac{\mathcal{I}_2}{\mathcal{I}_1} b + D^2 b^{-6} - 13 \frac{\mathcal{I}_2}{\mathcal{I}_1} C^2 b^{-5}.$$  \hspace{1cm} (54)

The joined model with a Planck potential is thus exactly solvable and its solution is given by (33). The SIC wave function for the FRW model in the region where the Planck potential can be neglected is given by (48). This is identical with the FRW part of the joined model as given by (35). The Kantowski–Sachs part of the wave function can thus be interpreted as fulfilling the SIC. For the vacuum FRW and Kantowski–Sachs model and in the region where the Planck potential can be neglected, the wave function simplifies to

$$\Psi = \frac{1}{\sqrt{\frac{1}{3} \lambda b^3 - b}} \exp \left\{ \pm i \mathcal{I}_2 \left( z b \sqrt{\frac{1}{3} \lambda b^2 - 1} \right) \right\}.$$  \hspace{1cm} (55)

Except for the prefactor this is the wave function derived by Laflamme and Shellard in order to fulfill the “no boundary” condition [27]. As emphasized in particular by Halliwell and Louko [28] a more sophisticated treatment of the “no boundary” condition shows that due to a freedom in choosing a “contour of integration” one could get either one of the signs in the exponent or the sine or the cosine as superpositions of these solutions. Only an additional choice restricts one to the exponential solution again. But while Halliwell and Louko discuss the vacuum model only, Laflamme and Shellard are primarily interested in the model with scalar field. In this case the solution is derived by assuming $\phi = \text{const}$ initially; that is the scalar field behaves as a cosmological constant. It is important in their approach that a classical scalar field behaves like a constant only for one end of the evolution. Since the “no boundary” wave function is given by the exponent of the action evaluated at the classical trajectory the above solution is argued to be valid “initially” only. It is thus interpreted as a solution of the Hamilton–Jacobi equation while the corresponding solution of the Wheeler–DeWitt equation would admit an additional oscillating term.

6 Conclusion

By reviewing the classical analysis of the Kantowski–Sachs model, I have emphasized the pathological character of this model: The incompleteness of the model, the possibility of an inhomogeneous $z$ and the unboundedness of its spacelike potential. It is, nevertheless, useful as
a toy model since it is the most simple anisotropic model and the joined model is even inhomogeneous. The more realistic Bianchi–IX model is far more complicated and an analytical treatment has so far not been given.

In Sec. 3 the Wheeler–DeWitt equation was solved exactly for a whole class of Kantowski–Sachs like models and a complete set of wave functions was given. The treatment of the general spherically symmetric model served here as a guide. The class of solved models includes the Kantowski–Sachs model with a cosmological constant and dust as well as the joined model. The derived exact solutions are of WKB form

\[ \Psi_M = De^{\pm iS_M}, \]  

(56)

where \( S_M \) are solutions of the Hamilton–Jacobi solution, and \( D \) is the WKB prefactor. This is somewhat surprising and one therefore has to be careful when basing conclusions on the breakdown of the WKB approximation as in \([29]\). Compared to the other sets of solutions given below, the \( \Psi_M \) have the advantage of great applicability. A cosmological constant and dust and even the joined model are contained in the class of solvable models.

For the vacuum model an alternative set of exact WKB solutions \( \Psi_K = e^{iS_K} = e^{iS_K} \) (with constant prefactor) is known which is given for the vacuum model by

\[ \Psi_K = \exp \left\{ \pm \frac{i\mathcal{I}}{2} \left( -Kz^2b - \frac{b}{K} \right) \right\} = e^{\pm iS_K}, \]  

(57)

where \( K \) is an arbitrary (complex) constant \([8, 9]\). This solution is found by transforming the original Wheeler–DeWitt equation \([31]\) into a Klein–Gordon equation by \( u = (\mathcal{I}/4)z^2b \) and \( v = Th \). The transformation can be generalized to an arbitrary \( f_1(b) \), but a generalization to a nonvanishing \( f_2(b) \) has not yet been found. This set of solutions is complementary to the one discussed in this paper in the following sense: \( S_M \) and \( S_K \) both solve the Hamilton–Jacobi equation but while \( S_M \) represents the classical solutions with a fixed constant of integration \( 2M = b_n \) for \( b(t) \), \( S_K \) represents the classical solutions with a fixed constant of integration \( K \) for \( z(t) \), c. f. Eq. (8). It is worth noting, however, that all solutions corresponding to \( S_K \) are identical in the vacuum model.

Another set of solutions for the vacuum model was first given by Ryan \([10]\) by separating the Wheeler–DeWitt equation by the transformation \( x = z \) and \( y = zb \). For a certain factor ordering one then gets the solutions

\[ \Psi_k = z^{\pm ik}K_{ik}(zb), \]  

(58)

where \( k \) is an arbitrary constant of separation and \( K_{ik}(zb) \) is the MacDonald function (the MacDonald function as solution of a modified Bessel equation has been chosen since it vanishes exponentially for great values of \( y = zb \)). Unlike the other sets of solutions these functions do not have a WKB structure. These solutions can be generalized to a rather arbitrary factor ordering and even for some arbitrary functions in the potentials. Unfortunately however, neither a cosmological constant nor the dust case fits into these class of models, so that is usage seems to be restricted to the vacuum model and the coupling of a massless scalar field.
The detailed analysis of how the different set of solutions are related requires a well–defined scalar product for this model. This is an open problem at the moment.

The joined model which consist of a Kantowski–Sachs cylinder inserted between two FRW half–spheres was investigated in detail in Sec. 4. It is mainly the WKB structure of the wave function which allows the unique solution of this model to be written as a product of solutions of the respective spacetime regions. In spite of the product structure the solution can not be interpreted as admitting no correlations between the different spacetime regions since the variable $b$ is defined for the whole spacetime. This reflects the fact that although one has quantized the spacetime in the different regions one does not allow fluctuations between them — the junction is treated classically.

One can nevertheless use the wave function for the joined model with its product structure in order to determine a wave function for the pure Kantowski–Sachs model which satisfies the symmetric initial condition (SIC). This initial condition cannot be directly applied because of the structure of the potential in the Wheeler–DeWitt equation. Neither does it simplify to a pure function of the volume for small volumes nor does it admit normalizable spacelike eigenfunction due to the unboundedness of the spacelike potential like other reasonable models as the Bianchi–IX model. The SIC wave function for the Kantowski–Sachs model is then determined by the Kantowski–Sachs part of the product wave function since its FRW part is the respective SIC wave function. It turns out that the wave function which satisfies the SIC is approximately the same as the one which was given by Laflamme and Shellard for the “no boundary” condition [27].

The interpretation of the solution is, however, different for the two boundary conditions. In case of the “no boundary” condition the wave function is regarded as a solution of the Hamilton–Jacobi equation. This is due to the fact that only classical trajectories were considered and that the condition of $\phi = const$ can only be satisfied at one end of the classical solution while at the other one $\phi$ behaves in a more complicated way. (This is a special case of the general observation that a spacetime which “starts” regular will develop inhomogeneities and anisotropies and thus end up irregular. The beginning respective the end is thereby defined by the homogeneity of the state.) Since the “no boundary” wave function is given by the exponent of the classical action Laflamme and Shellard concluded that the approximation and thus the wave function is good only “initially”. This means that their wave function can only be regarded as a solution of the Hamilton–Jacobi equation while in this interpretation a solution of the Wheeler–DeWitt equation would admit an additional (oscillating) factor which corresponds to the complicated “end” of the classical trajectory. Since they argue that their wave function admits no initial anisotropies its interpretation as a solution of the Hamilton–Jacobi equation means that the wave function predicts a universe which ends up anisotropic.

The SIC on the other hand is interpreted as an initial condition for the solutions of the Wheeler–DeWitt equation. This marks a decisive difference because of the dynamical structure of this equation. The Wheeler–DeWitt equation is a wave equation (the signature of its kinetic term admits one minus sign, the rest are plus signs) which thus admits one internal timelike variable. This timelike variable, the so called intrinsic time, has nothing to do a priori with the physical time but is that variable with respect to which the dynamics of the wave is defined.
The timelike variable can always be chosen to be the volume of the three–geometry. Since the dynamics is a dynamics with respect to the volume it follows directly that a prediction of a homogeneous and isotropic state for small volumes is irrespective of an expanding or a contracting universe. Moreover, even the very notion of expanding and contracting does not make sense in quantum cosmology due to the absence of an external time parameter; one can only conclude how a variable changes with varying volume. Since the SIC arrives at approximately the “no boundary” wave function which is thought to predict an isotropic initial state (or more generally a homogeneous and isotropic initial state) it can be concluded that the wave function predicts an isotropic state for small volumes. The universe in this interpretation is thus predicted to be isotropic at its “beginning” and at its “end”.

The different interpretations of the wave function lie at the heart of the ongoing controversy on the symmetry or asymmetry of the arrow of time. Since the “no boundary” wave function is interpreted as Hamilton–Jacobi function it is thus consequent that with this assumptions an asymmetric arrow of time is derived (for a critique of the consistency of using semiclassical methods only in quantum cosmology see \[26\]) while the SIC as an initial condition for the Wheeler–DeWitt equation arrives at an symmetric arrow of time. A symmetric arrow of time does not lead to paradoxes in the recollapsing part of the universe since it is not possible to build a wave function which represents a recollapsing universe: Quantum interference effects are dominant at the turning point, see \[24\] for a more detailed argument. This is a direct consequence of the above considerations but can be explicitly verified for some models too.

The exact solutions \[50\] were in this paper mainly used in order to find a wave function which satisfies the SIC and then to confirm that the SIC leads to an isotropic beginning of the classical universe. There are, of course, other open questions in this class of models for which the exact solutions may be helpful. I will mention but two of them. First, what is the right scalar product? The given wave function for example have either a apparent singularity at the classical borderline or the wave function diverges exponentially for great \(b\). In neither case it seems to be appropriate simply to interpret the wave function as the probability for a some configuration. Second, one should include a more realistic matter source like a massive scalar field. This is in particular important in order to analyze the behavior of the wave function at the classical turning point: Does there exist a wave function which resembles a recollapsing universe? For the FRW model Kiefer was able to show that this is impossible \[30\]. Does the additional degree of freedom change this behaviour by avoiding the interference of the “expanding” and “recollapsing” part of the wave packet due to the extra dimension of the configuration space? In order to answer this question, one already needs a well defined scalar product.

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