Inviscid limit for SQG in bounded domains

Peter Constantin, Mihaela Ignatova, and Huy Q. Nguyen

ABSTRACT. We prove that the limit of any weakly convergent sequence of Leray-Hopf solutions of dissipative SQG equations is a weak solution of the inviscid SQG equation in bounded domains.

1. Introduction

The behavior of high Reynolds number fluids is a broad, important and mostly open problem of nonlinear physics and of PDE. Here we consider a model problem, the surface quasi-geostrophic equation, and the limit of its viscous regularizations of certain types. We prove that the inviscid limit is rigid, and no anomalies arise in the limit.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Denote

$$\Lambda = \sqrt{-\Delta}$$

where $-\Delta$ is the Laplacian operator with Dirichlet boundary conditions. The dissipative surface quasi-geostrophic (SQG) equation in $\Omega$ is the equation

$$\partial_t \theta^\nu + u^\nu \cdot \nabla \theta^\nu + \nu \Lambda^s \theta^\nu = 0, \quad \nu > 0, \ s \in (0, 2],$$

(1.1)

where $\theta^\nu = \theta^\nu(x, t)$, $u^\nu = u^\nu(x, t)$ with $(x, t) \in \Omega \times [0, \infty)$ and with the velocity $u^\nu$ given by

$$u^\nu = R^\perp \nabla \theta^\nu : = \nabla^\perp \Lambda^{-1} \theta^\nu,$$

(1.2)

We refer to the parameter $\nu$ as “viscosity”. Fractional powers of the Laplacian $-\Delta$ are based on eigenfunction expansions. The inviscid SQG equation has zero viscosity

$$\partial_t \theta + u \cdot \nabla \theta = 0,$$

(1.3)

The dissipative SQG (1.1) has global weak solutions for any $L^2$ initial data:

THEOREM 1.1. For any initial data $\theta_0 \in L^2(\Omega)$ there exists a global weak solution $\theta$ $\in C([0, \infty); L^2(\Omega) \cap L^2(0, \infty; D(\Lambda^\frac{1}{2})))$ to the dissipative SQG equation (1.1). More precisely, $\theta$ satisfies the weak formulation

$$\int_0^\infty \int_\Omega \theta \varphi(x) dx \partial_t \varphi(t) dt + \int_0^\infty \int_\Omega u \cdot \nabla \varphi(x) dx \varphi(t) dt - \nu \int_0^\infty \int_\Omega \Lambda^s \theta \Lambda^s \varphi(x) dx \varphi(t) dt = 0$$

(1.4)

for any $\varphi \in C_c^\infty((0, \infty))$ and $\varphi \in D(\Lambda^2)$. Moreover, $\theta$ obeys the energy inequality

$$\frac{1}{2} \|\theta(\cdot, t)\|^2_{L^2(\Omega)} + \nu \int_0^t \int_\Omega |\Lambda^{\frac{1}{2}} \theta|^2 dx dr \leq \frac{1}{2} \|\theta_0\|^2_{L^2(\Omega)}$$

(1.5)

and the balance

$$\frac{1}{2} \|\theta(\cdot, t)\|^2_{D(\Lambda^{-\frac{1}{2}})} + \nu \int_0^t \int_\Omega |\Lambda^{-\frac{1}{2}} \theta|^2 dx dr \leq \frac{1}{2} \|\theta_0\|^2_{D(\Lambda^{-\frac{1}{2}})}$$

(1.6)

for a.e. $t > 0$. In addition, $\theta \in C([0, \infty); D(\Lambda^{-\varepsilon}))$ for any $\varepsilon > 0$ and the initial data $\theta_0$ is attained in $D(\Lambda^{-\varepsilon})$. 



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We refer to any weak solutions of (1.1) satisfying the properties (1.4), (1.5), (1.6) as a “Leray-Hopf weak solution”.

REMARK 1.2. Theorem 1.1 for critical dissipative SQG $s = 1$ was obtained in [6].

REMARK 1.3. Note that $C_c^\infty(\Omega)$ is not dense in $D(\Lambda^2)$ since the $D(\Lambda^2)$ norm is equivalent to the $H^2(\Omega)$ norm and $C_c^\infty(\Omega)$ is dense in $H^2_0(\Omega)$ which is strictly contained in $D(\Lambda^2)$.

The existence of $L^2$ global weak solutions for inviscid SQG (1.3) was proved in [8]. More precisely, (see Theorem 1.1, [8]) for any initial data $\theta_0 \in L^2(\Omega)$ there exists a global weak solution $\theta \in C_w(0, \infty; L^2(\Omega))$ satisfying

$$\int_0^\infty \int_\Omega \theta \partial_t \varphi dt dx + \int_0^\infty \int_\Omega u \theta \cdot \nabla \varphi dt dx = 0 \quad \forall \varphi \in C_c^\infty(\Omega \times (0, \infty)), \tag{1.7}$$

and such that the Hamiltonian

$$H(t) := \|\theta(t)\|^2_{D(\Lambda^{-\frac{1}{2}})} \tag{1.8}$$

is constant in time. Moreover, the initial data is attained in $D(\Lambda^{-\varepsilon})$ for any $\varepsilon > 0$.

Our main result in this note establishes the convergence of weak solutions of the dissipative SQG to weak solutions of the inviscid SQG in the inviscid limit $\nu \to 0$.

**THEOREM 1.4.** Let $\{\nu_n\}$ be a sequence of viscosities converging to 0 and let $\{\theta_{0\nu_n}\}$ be a bounded sequence in $L^2(\Omega)$. Any weak limit $\theta$ in $L^2(0, T; L^2(\Omega))$, $T > 0$, of any subsequence of $\{\theta_{0\nu_n}\}$ of Leray-Hopf weak solutions of the dissipative SQG equation (1.1) with viscosity $\nu_n$ and initial data $\theta_{0\nu_n}$ is a weak solution of the inviscid SQG equation (1.3) on $[0, T]$. Moreover, $\theta \in C(0, T; D(\Lambda^{-\varepsilon}))$ for any $\varepsilon > 0$, and the Hamiltonian of $\theta$ is constant on $[0, T]$.

REMARK 1.5. The same result holds true on the torus $\mathbb{T}^2$. The case of the whole space $\mathbb{R}^2$ was treated in [1].

REMARK 1.6. With more singular constitutive laws $u = \nabla^\perp \Lambda^{-\alpha} \theta$, $\alpha \in [0, 1)$, $L^2$ global weak solutions of the inviscid equations were obtained in [3, 15]. Theorem 1.4 could be extended to this case. It is also possible to consider $L^p$ initial data in light of the work [12].

It is worth noting that in order for a given weak solution $\theta$ of the inviscid SQG to conserve the Hamiltonian, the Onsager-type critical condition requires $\theta \in L^3_{t,x}$ (see [14] for $\Omega = \mathbb{T}^2$). On the other hand, the vanishing viscosity solutions obtained in Theorem 1.4 conserve the Hamiltonian even though they are only in $L^\infty_t L^2_x$.

As a corollary of the proof of Theorem 1.4 we have the following weak rigidity of inviscid SQG in bounded domains:

**COROLLARY 1.7.** Any weak limit in $L^2(0, T; L^2(\Omega))$, $T > 0$, of any sequence of weak solutions of the inviscid SQG equation (1.3) is a weak solution of (1.3). Here, weak solutions of (1.3) are interpreted in the sense of (1.7).

REMARK 1.8. On tori, this result was proved in [14]. If the weak limit occurs in $L^\infty(0, T; L^2(\Omega))$ and the sequence of weak solutions conserves the Hamiltonian then so is the limiting weak solution.

The paper is organized as follows. Section 2 is devoted to basic facts about the spectral fractional Laplacian and results on commutator estimate. The proofs of Theorems 1.1 and 1.4 are given respectively in sections 3 and 4. Finally, an auxiliary lemma is given in Appendix A.
2. Fractional Laplacian and commutators

Let \( \Omega \subset \mathbb{R}^d, d \geq 2 \), be a bounded domain with smooth boundary. The Laplacian \(-\Delta\) is defined on \( D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega) \). Let \( \{ w_j \}_{j=1}^{\infty} \) be an orthonormal basis of \( L^2(\Omega) \) comprised of \( L^2 \)–normalized eigenfunctions \( w_j \) of \(-\Delta\), i.e.

\[
-\Delta w_j = \lambda_j w_j, \quad \int_\Omega w_j^2 dx = 1,
\]

with \( 0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_j \to \infty \).

The fractional Laplacian is defined using eigenfunction expansions,

\[
\Lambda^s f \equiv (-\Delta)^{\frac{s}{2}} f := \sum_{j=1}^{\infty} \lambda_j^{\frac{s}{2}} f_j w_j \quad \text{with} \quad f = \sum_{j=1}^{\infty} f_j w_j, \quad f_j = \int_\Omega f w_j dx
\]

for \( s \geq 0 \) and \( f \in \mathcal{D}(\Lambda^s) \) where

\[
\mathcal{D}(\Lambda^s) := \{ f \in L^2(\Omega) : (\lambda_j^{\frac{s}{2}} f_j) \in \ell^2(\mathbb{N}) \}.
\]

The norm of \( f \) in \( \mathcal{D}(\Lambda^s) \) is defined by

\[
\| f \|_{\mathcal{D}(\Lambda^s)} := \| (\lambda_j^{\frac{s}{2}} f_j) \|_{\ell^2(\mathbb{N})}.
\]

It is also well-known that \( \mathcal{D}(\Lambda) \) and \( H_0^1(\Omega) \) are isometric. In the language of interpolation theory,

\[
\mathcal{D}(\Lambda^\alpha) = [L^2(\Omega), D(-\Delta)]_{\frac{\alpha}{2}} \quad \forall \alpha \in [0, 2].
\]

As mentioned above,

\[
H_0^1(\Omega) = \mathcal{D}(\Lambda) = [L^2(\Omega), D(-\Delta)]_{\frac{1}{2}},
\]

hence

\[
\mathcal{D}(\Lambda^\alpha) = [L^2(\Omega), H_0^1(\Omega)]_{\alpha} \quad \forall \alpha \in [0, 1].
\]

Consequently, we can identify \( \mathcal{D}(\Lambda^\alpha) \) with usual Sobolev spaces (see Chapter 1, [17]):

\[
\mathcal{D}(\Lambda^\alpha) = \begin{cases} 
H_0^1(\Omega) \cap H^\alpha(\Omega) & \text{if } \alpha \in (1, 2], \\
H_0^\alpha(\Omega) & \text{if } \alpha \in (\frac{1}{2}, 1], \\
H^\frac{1}{3} H_0^1(\Omega) := \{ u \in H^\frac{1}{3} H_0^1(\Omega) : u/\sqrt{d(x)} \in L^2(\Omega) \} & \text{if } \alpha = \frac{1}{2}, \\
H^\alpha(\Omega) & \text{if } \alpha \in [0, \frac{1}{2}).
\end{cases}
\]

Here and below \( d(x) \) denote the distance from \( x \) to the boundary \( \partial \Omega \).

Next, for \( s > 0 \) we define

\[
\Lambda^{-s} f = \sum_{j=1}^{\infty} \lambda_j^{-\frac{s}{2}} f_j w_j
\]

if \( f = \sum_{j=1}^{\infty} f_j w_j \in \mathcal{D}(\Lambda^{-s}) \) where

\[
\mathcal{D}(\Lambda^{-s}) := \left\{ \sum_{j=1}^{\infty} f_j w_j \in \mathcal{D}'(\Omega) : f_j \in \mathbb{R}, \sum_{j=1}^{\infty} \lambda_j^{-\frac{s}{2}} f_j w_j \in L^2(\Omega) \right\}.
\]

The norm of \( f \) is then defined by

\[
\| f \|_{\mathcal{D}(\Lambda^{-s})} := \| \Lambda^{-s} f \|_{L^2(\Omega)} = \left( \sum_{j=1}^{\infty} \lambda_j^{-s} f_j^2 \right)^{\frac{1}{2}}.
\]

It is easy to check that \( \mathcal{D}(\Lambda^{-s}) \) is the dual of \( \mathcal{D}(\Lambda^s) \) with respect to the pivot space \( L^2(\Omega) \).
LEMA 2.1 (Lemma 2.1, [15]). The embedding
\[ D(\Lambda^s) \subset H^s(\Omega) \] (2.2)
is continuous for all \( s \geq 0 \).

LEMA 2.2. For \( s, r \in \mathbb{R} \) with \( s > r \), the embedding \( D(\Lambda^s) \subset D(\Lambda^r) \) is compact.

PROOF. Let \( \{ u_n \} \) be a bounded sequence in \( D(\Lambda^s) \). Then \( \{ \Lambda^r u_n \} \) is bounded in \( D(\Lambda^{s-r}) \). Choosing \( \delta > 0 \) smaller than \( \min(s-r, \frac{1}{2}) \) we have \( D(\Lambda^{s-r}) \subset D(\Lambda^\delta) = H^\delta(\Omega) \subset L^2(\Omega) \) where the first embedding is continuous and the second is compact. Consequently the embedding \( D(\Lambda^{s-r}) \subset L^2(\Omega) \) is compact and thus there exist a subsequence \( u_{n_j} \) and a function \( f \in L^2(\Omega) \) such that \( \Lambda^r u_{n_j} \) converge to \( f \) strongly in \( L^2(\Omega) \). Then \( u_{n_j} \) converge to \( u := \Lambda^{-r} f \) strongly in \( D(\Lambda^r) \) and the proof is complete. \( \Box \)

A bound for the commutator between \( \Lambda \) and multiplication by a smooth function was proved in [6] using the method of harmonic extension:

THEOREM 2.3 (Theorem 2, [6]). Let \( \chi \in B(\Omega) \) with \( B(\Omega) = W^{2,d}(\Omega) \cap W^{1,\infty}(\Omega) \) if \( d \geq 3 \), and \( B(\Omega) = W^{2,p}(\Omega) \) with \( p > 2 \) if \( d = 2 \). There exists a constant \( C(d, p, \Omega) \) such that
\[ \| [\Lambda, \chi] \psi \|_{D(\Lambda^{\frac{3}{2}})} \leq C(d, p, \Omega) \| \chi \|_{B(\Omega)} \| \psi \|_{D(\Lambda^{\frac{3}{2}})} \]
Pointwise estimates for the commutator between fractional Laplacian and differentiation were established in [8]:

THEOREM 2.4 (Theorem 2.2, [8]). For any \( p \in [1, \infty] \) and \( s \in (0, 2) \) there exists a positive constant \( C(d, s, p, \Omega) \) such that for all \( \psi \in C_c^\infty(\Omega) \) we have
\[ \| \Lambda^s, \nabla \| \psi(x) \| \leq C(d, s, p, \Omega) d(x)^{-s-1-\frac{d}{p}} \| \psi \|_{L^p(\Omega)} \]
holds for all \( x \in \Omega \).

This pointwise bound implies the following commutator estimate in Lebesgue spaces:

THEOREM 2.5. Let \( p, q \in [1, \infty] \), \( s \in (0, 2) \) and \( \varphi \) satisfy
\[ \varphi(\cdot) d(\cdot)^{-s-1-\frac{d}{p}} \in L^q(\Omega). \]
Then the operator \( \varphi[\Lambda^s, \nabla] \) can be uniquely extended from \( C_c^\infty(\Omega) \) to \( L^p(\Omega) \) such that there exists a positive constant \( C = C(d, s, p, \Omega) \) such that
\[ \| \varphi[\Lambda^s, \nabla] \psi \|_{L^q(\Omega)} \leq C \| \varphi(\cdot) d(\cdot)^{-s-1-\frac{d}{p}} \|_{L^q(\Omega)} \| \psi \|_{L^p(\Omega)} \] (2.3)
holds for all \( \psi \in L^p(\Omega) \).

(2.3) is remarkable in that the commutator between an operator of order \( s \in (0, 2) \) and an operator of order 1 is an operator of order 0.

3. Proof of Theorem 1.1

We use Galarkin approximations. Denote by \( P_m \) the projection in \( L^2(\Omega) \) onto the linear span \( L^2_{\omega_m} \) of eigenfunctions \( \{ w_1, \ldots, w_m \} \), i.e.
\[ P_m f = \sum_{j=1}^m f_j w_j \quad \text{for} \quad f = \sum_{j=1}^\infty f_j w_j. \] (3.1)
The \( m \)th Galerkin approximation of (1.1) is the following ODE system in the finite dimensional space \( L_m^2 \):

\[
\begin{cases}
\dot{\theta}_m + \mathbb{P}_m(u_m \cdot \nabla \theta_m) + \nu \Lambda^s \theta_m = 0 & t > 0, \\
\theta_m = P_m \theta_0 & t = 0,
\end{cases}
\]

with \( \theta_m(x, t) = \sum_{j=1}^m \theta_j^{(m)}(t)w_j(x) \) and \( u_m = R_D^{-1} \theta_m \) satisfying \( \text{div} \ u_m = 0 \). Note that (3.2) is equivalent to

\[
\frac{d\theta_j^{(m)}}{dt} + \sum_{j,k=1}^m \gamma_{jkl}^{(m)} \theta_j^{(m)} \theta_k^{(m)} + \nu \lambda_j^{(m)} \theta_l^{(m)} = 0, \quad l = 1, 2, \ldots, m,
\]

with

\[
\gamma_{jkl}^{(m)} = \lambda_j^{(m)} \int_\Omega \left( \nabla w_j \cdot \nabla w_k \right) w_l \, dx.
\]

The local existence of \( \theta_m \) on some time interval \([0, T_m]\) follows from the Cauchy-Lipschitz theorem. On the other hand, the antisymmetry property \( \gamma_{jkl}^{(m)} = -\gamma_{lkj}^{(m)} \) yields

\[
\frac{1}{2} \| \theta_m(\cdot, t) \|^2_{L_2(\Omega)} + \nu \int_0^t \int_\Omega | \Lambda^s \theta_m |^2 \, dx \, dr = \frac{1}{2} \| \mathbb{P}_m \theta_0 \|^2_{L_2(\Omega)} \leq \frac{1}{2} \| \theta_0 \|^2_{L_2(\Omega)}
\]

for all \( t \in [0, T_m] \). This implies that \( \theta_m \) is global and (3.4) holds for all positive times. The sequence \( \theta_m \) is thus uniformly bounded in \( L^\infty(0, \infty; L^2(\Omega)) \cap L^2(0, \infty; D(\Lambda^s)) \). Upon extracting a subsequence, we have \( \theta_m \) converge to some \( \theta \) weakly-* in \( L^\infty(0, \infty; L^2(\Omega)) \) and weakly in \( L^2(0, \infty; D(\Lambda^s)) \). In particular, \( \theta \) obeys the same energy inequality as in (3.4). On the other hand, if one multiplies (3.3) by \( \lambda_l^{(m)} \theta_l^{(m)} \) and uses the fact that \( \gamma_{jkl}^{(m)} \lambda_l^{(m)} = -\gamma_{lkj}^{(m)} \lambda_l^{(m)} \), one obtains

\[
\frac{1}{2} \| \theta_m(\cdot, t) \|^2_{D(\Lambda^{s/2})} + \nu \int_0^t \int_\Omega | \Lambda^{s/2} \theta_m |^2 \, dx \, dr = \frac{1}{2} \| \mathbb{P}_m \theta_0 \|^2_{D(\Lambda^{s/2})}.
\]

We derive next a uniform bound for \( \partial_t \theta_m \). Let \( N > 0 \) be an integer to be determined. For any \( \varphi \in D(\Lambda^{2N}) \) we integrate by parts to get

\[
\int_\Omega \partial_t \theta_m \varphi \, dx = -\int \mathbb{P}_m \text{div}(u_m \theta_m) \varphi \, dx - \int_\Omega \nu \Lambda^s \theta_m \varphi \, dx
\]

\[
= \int_\Omega (u_m \theta_m) \cdot \nabla (\mathbb{P}_m \varphi) \, dx - \int \nu \theta_m \Lambda^s \varphi \, dx.
\]

The first term is controlled by

\[
\left| \int_\Omega (u_m \theta_m) \cdot \nabla (\mathbb{P}_m \varphi) \, dx \right| \leq \| u_m \theta_m \|_{L_1(\Omega)} \| \nabla \mathbb{P}_m \varphi \|_{L_\infty(\Omega)} \leq C \| \mathbb{P}_m \varphi \|_{H^1(\Omega)}.
\]

According to Lemma A.1 for \( N \) and \( k \) satisfying \( N > \frac{k}{2} + 1 \) there exists a positive constant \( C_{N,k} \) such that

\[
\| \mathbb{P}_m \varphi \|_{H^k(\Omega)} \leq C_{N,k} \| \varphi \|_{D(\Lambda^{2N})} \quad \forall m \geq 1, \quad \forall \varphi \in D(\Lambda^{2N}).
\]

With \( k = 3 \) and \( N = 3 \) we have

\[
\left| \int_\Omega (u_m \theta_m) \cdot \nabla (\mathbb{P}_m \varphi) \, dx \right| \leq C \| \varphi \|_{D(\Lambda^6)}.
\]

On the other hand,

\[
\left| \int_\Omega \nu \theta_m \Lambda^s \varphi \, dx \right| \leq C \| \theta_m \|_{L^2(\Omega)} \| \varphi \|_{D(\Lambda^2)}.
\]

We have proved that

\[
\left| \int_\Omega \partial_t \theta_m \varphi \, dx \right| \leq C \| \varphi \|_{D(\Lambda^6)} \quad \forall \varphi \in D(\Lambda^6).
\]
Because $L^2(\Omega) \times D(\Lambda^6) \ni (f, g) \mapsto \int_\Omega f g dx$ extends uniquely to a bilinear from on $D(\Lambda^{-2}) \times D(\Lambda^6)$, we deduce that $\partial_t \theta_m$ are uniformly bounded in $L^\infty(0, \infty; D(\Lambda^{-6}))$. Note that we have used only the uniform regularity $L^\infty(0, \infty; L^2(\Omega))$ of $\theta_m$. We have the embeddings $D(\Lambda^2) \subset D(\Lambda^{s-1/2}) \subset D(\Lambda^{-6})$ where the first one is compact by virtue of Lemma 2.2 and the second is continuous. Fix $T > 0$. Aubin-Lions’ lemma (see [16]) ensures that for some function $f$ and along some subsequence $\theta_m$ converge to $f$ weakly in $L^2(0, T; D(\Lambda^2))$ and strongly in $L^2(0, T; D(\Lambda^{s-1/2}))$. Apriori, both $f$ and the subsequence depend on both $T$. However, we already know that $\theta_m \to \theta$ weakly in $L^2(0, \infty; D(\Lambda^2))$. Therefore, $f = \theta$ and the convergences to $\theta$ hold for the whole sequence. Similarly, applying Aubin-Lions’ lemma with the embeddings $L^2(\Omega) \subset D(\Lambda^{-6}) \subset D(\Lambda^{-6})$ for sufficiently small $\varepsilon > 0$ we obtain that $\theta_m \to \theta$ strongly in $C([0, T]; D(\Lambda^{-6}))$. Integrating (3.2) against an arbitrary test function of the form $\phi(t) \varphi(x)$ with $\varphi \in C^\infty_c((0, T))$, $\varphi \in D(\Lambda^6)$ yields

$$
\int_0^T \int_\Omega \theta_m \varphi(x) dx \partial_t \phi(t) dt + \int_0^T \int_\Omega \nabla \varphi \theta_m dx \phi(t) dt - \nu \int_0^T \int_\Omega \Lambda^2 \theta_m \Lambda^2 \varphi dx \phi(t) dt = 0.
$$

By Lemma A.1,

$$
\| (\mathbb{I} - \mathbb{P}_m) \varphi \|_{L^\infty(\Omega)} \leq C \| (\mathbb{I} - \mathbb{P}_m) \varphi \|_{H^3(\Omega)} \to 0 \quad \text{as} \quad m \to \infty.
$$

The weak convergence of $\theta_m$ in $L^2(0, T; D(\Lambda^2))$ allows one to pass to the limit in the two linear terms. The strong convergence of $\theta_m$ in $L^2(0, T; L^2(\Omega))$ together with the weak convergence of $\theta_m$ in the same space allows one to pass to the limit in the nonlinear term and conclude that $\theta$ satisfies the weak formulation (1.4) with $\varphi \in D(\Lambda^6)$. In fact, $\theta \in L^2(0, \infty; D(\Lambda^2)) \subset L^2(0, \infty; L^p(\Omega))$ for some $p > 2$, hence $u \theta \in L^2(0, \infty; L^q(\Omega))$ for some $q > 1$. In addition, if $\varphi \in D(\Lambda^2)$ then $\nabla \varphi \in L^r$ for all $r < \infty$, and thus the nonlinearity $\int_\Omega u \theta \nabla \varphi dx$ makes sense. Then because $D(\Lambda^2)$ is dense in $D(\Lambda^6)$, (1.4) holds for $\varphi \in D(\Lambda^2)$.

We now pass to the limit in (3.5). The strong convergence $\theta_m \to \theta$ in $C([0, T]; D(\Lambda^{-6}))$ gives the convergence of the first term. On the other hand, the strong convergence $\theta_m \to \theta$ in $L^2(0, T; D(\Lambda^{s-1/2}))$ yields the convergence of the second term. The right hand side converges to $\frac{1}{2} \| \theta_0 \|^2_{D(\Lambda^{-\frac{1}{2}})}$ since $\mathbb{P}_m \theta_0$ converge to $\theta_0$ in $L^2(\Omega)$. We thus obtain (1.6).

Since $\theta_m \to \theta$ in $C([0, T]; D(\Lambda^{-6}))$ we deduce that

$$
\theta_0 = \lim_{m \to \infty} \mathbb{P}_m \theta_0 = \lim_{m \to \infty} \theta_m |_{t=0} = \theta |_{t=0} \quad \text{in} \quad D(\Lambda^{-6}).
$$

For a.e. $t \in [0, T]$, $\theta_m(t)$ are uniformly bounded in $L^2(\Omega)$, and thus along some subsequence $\theta_{m_j}$, a priori depending on $t$, we have $\theta_{m_j}(t)$ converge weakly to some $f(t)$ in $L^2(\Omega)$. But we know $\theta_m(t) \to \theta(t)$ in $D(\Lambda^{-6})$. Thus, $f(t) = \theta(t)$ and $\theta_{m_j}(t) \to \theta(t)$ in $L^2(\Omega)$ as a whole sequence for a.e. $t \in [0, T]$. Recall that $\frac{d}{dt} \theta_m$ are uniformly bounded in $L^\infty(0, T; D(\Lambda^{-6}))$. For all $\varphi \in D(\Lambda^6)$ and $t \in [0, T]$ we write

$$
\langle \theta_m(t), \varphi \rangle_{L^2(\Omega), L^2(\Omega)} = \langle \theta_m(0), \varphi \rangle_{L^2(\Omega), L^2(\Omega)} + \int_0^t \langle \frac{d}{dt} \theta_m(r), \varphi \rangle_{D(\Lambda^{-6}), D(\Lambda^6)} dr.
$$

Because $\frac{d}{dt} \theta_m$ converge to $\frac{d}{dt} \theta$ weakly-* in $L^\infty(0, T; D(\Lambda^{-6}))$, letting $m \to \infty$ yields

$$
\langle \theta(t), \varphi \rangle_{L^2(\Omega), L^2(\Omega)} = \langle \theta_0, \varphi \rangle_{L^2(\Omega), L^2(\Omega)} + \int_0^t \langle \frac{d}{dt} \theta(r), \varphi \rangle_{D(\Lambda^{-6}), D(\Lambda^6)} dr
$$

for a.e. $t \in [0, T]$. Taking the limit $t \to 0$ gives

$$
\lim_{t \to 0} \langle \theta(t), \varphi \rangle_{L^2(\Omega), L^2(\Omega)} = \langle \theta_0, \varphi \rangle_{L^2(\Omega), L^2(\Omega)}
$$

for all $\varphi \in D(\Lambda^6)$. Finally, since $D(\Lambda^6)$ is dense in $L^2(\Omega)$ and $\theta \in L^\infty(0, T; L^2(\Omega))$ we conclude that $\theta \in C^w_w(0, T; L^2(\Omega))$ for all $T > 0$. 
4. Proof of Theorem 1.4

First, using approximations and commutator estimates we justify the commutator structure of the SQG nonlinearity derived in [8].

**Lemma 4.1.** For all $\psi \in H^1_0(\Omega)$ and $\varphi \in C_c^\infty(\Omega)$ we have
\[
\int_\Omega \Lambda \psi \nabla \psi \cdot \nabla \varphi dx = -\frac{1}{2} \int_\Omega |\Lambda| \nabla \psi \cdot \nabla \varphi dx - \frac{1}{2} \int_\Omega \nabla \psi \cdot [\Lambda, \nabla \varphi] dx.
\]
Here, the commutator $[\Lambda, \nabla \psi] \cdot \nabla \varphi$ is understood in the sense of the extended operator defined in Theorem 2.5.

**Proof.** Let $\psi_n \in C_c^\infty(\Omega)$ converging to $\psi$ in $H^1_0(\Omega)$. Integrating by parts and using the fact that $\nabla \psi = 0$ gives
\[
\int_\Omega \Lambda \psi_n \nabla \psi_n \cdot \nabla \varphi dx = -\int_\Omega \psi_n \nabla \Lambda \psi_n \cdot \nabla \varphi dx,
\]
Because $\psi_n$ is smooth and has compact support inside $\Omega$, $\nabla \psi_n \in D(\Lambda)$, and thus we can commute $\nabla$ with $\Lambda$ to obtain
\[
\int_\Omega \Lambda \psi_n \nabla \psi_n \cdot \nabla \varphi dx
= -\int_\Omega \psi_n [\nabla, \Lambda] \psi_n \cdot \nabla \varphi dx - \int_\Omega \psi_n \Lambda \nabla \psi_n \cdot \nabla \varphi dx
= -\int_\Omega \psi_n [\nabla, \Lambda] \psi_n \cdot \nabla \varphi dx - \int_\Omega \nabla \psi_n \cdot \Lambda (\psi \nabla \varphi) dx
= -\int_\Omega [\nabla, \Lambda] \psi_n \cdot \nabla \psi_n dx - \int_\Omega \nabla \psi_n \cdot [\Lambda, \nabla \varphi] \psi dx - \int_\Omega \nabla \psi_n \cdot \nabla \varphi \Lambda \psi_n dx.
\]
Noticing that the last term on the right-hand side is exactly the negative of the left-hand side, we deduce that
\[
\int_\Omega \Lambda \psi_n \nabla \psi_n \cdot \nabla \varphi dx = \frac{1}{2} \int_\Omega [\Lambda, \nabla] \psi_n \cdot \nabla \varphi dx - \frac{1}{2} \int_\Omega \nabla \psi_n \cdot [\Lambda, \nabla \varphi] \psi dx.
\]
The commutator estimates in Theorems 2.3 and 2.5 then allow us to pass to the limit in the preceding representation and conclude that (4.1) holds.

Now let $\nu_n \to 0^+$ and let $\theta_n$ be a bounded sequence in $L^2(\Omega)$. For each $n$ let $\theta_n \equiv \theta_n^\infty$ be a Leray-Hopf weak solution of (1.1) with viscosity $\nu_n$ and initial data $\theta_n^\infty$. In view of the energy inequality (1.5), $\theta_n$ are uniformly bounded in $L^\infty(0, \infty; L^2(\Omega))$ and satisfies
\[
\int_0^\infty \int_\Omega \theta_n \varphi(x) dx \partial_t \phi(t) dt + \int_0^\infty \int_\Omega u_n \theta_n \cdot \nabla \varphi(x) dx \phi(t) dt - \nu_n \int_0^\infty \int_\Omega \Lambda \varphi(x) dx \phi(t) dt = 0
\]
for all $\varphi \in C_c^\infty((0, \infty))$ and $\varphi \in D(\Lambda^2)$. Fix $T > 0$. Assume that along a subsequence, still labeled by $n$, $\theta_n$ converge to $\theta$ weakly in $L^2(0, T; L^2(\Omega))$. We prove that $\theta$ is a weak solution of the inviscid SQG equation. We first prove a uniform bound for $\partial_t \theta_n$ provided only the uniform regularity $L^\infty(0, T; L^2(\Omega))$ of $\theta_n$. To this end, let us define for a.e. $t \in [0, T]$ the function $f_n(t) \in H^{-\delta}(\Omega)$ by
\[
\langle f_n(t), \varphi \rangle_{H^{-\delta}(\Omega), H^\delta(\Omega)} := \int_\Omega (u_n(x, t) \theta_n(x, t) \cdot \nabla \varphi(x) - \nu, \theta_n(x, t) \Lambda \varphi(x)) dx
\]
for all $\varphi \in H^\delta(\Omega) \subset D(\Lambda^2)$, where $H^\mu(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $H^\mu(\Omega)$ for any $\mu > 0$. Indeed, we have
\[
\left| \int_\Omega (u_n(x, t) \theta_n(x, t) \cdot \nabla \varphi(x) - \nu \theta_n(x, t) \Lambda \varphi(x)) dx \right| \leq C(\|\theta_n(t)\|_{L^2(\Omega)}^2 + 1) \|\varphi\|_{H^\delta(\Omega)}.
\]
This shows that \( f_n \) are uniformly bounded in \( L^\infty(0, T; H^{-3}(\Omega)) \). Then for any \( \phi \in C_c^\infty((0, T)) \), it follows from (4.2) that
\[
\int_0^T \theta_n \partial_t \phi dt = -\int_0^T f_n \phi dt
\]
in \( H^{-3}(\Omega) \). In other words, \( \partial_t \theta_n = f_n \) and the desired uniform bound for \( \partial_t \theta_n \) follows. Fix \( \varepsilon \in (0, \frac{1}{2}) \). Aubin-Lions’ lemma applied with the embeddings \( L^2(\Omega) \subset D(\Lambda^{-\varepsilon}) \subset H^{-3}(\Omega) \) then ensures that \( \theta_n \) converge to \( \theta \) strongly in \( C(0, T; D(\Lambda^{-\varepsilon})) \). Consequently \( \psi_n \) converge to \( \psi := \Lambda^{-1}\theta \) strongly in \( C(0, T; D(\Lambda^{1-\varepsilon})) \). Now we take \( \phi \in C_c^\infty((0, \infty)) \) and \( \varphi \in C_c^\infty(\Omega) \). By virtue of Lemma 4.1 the weak formulation (1.4) gives
\[
\int_0^T \int_\Omega \theta_n \varphi(x) dx \partial_t \phi(t) dt + \frac{1}{2} \int_0^T \int_\Omega [\Lambda, \nabla^\perp] \psi_n \cdot \nabla \varphi(x) \psi_n dx \phi(t) dt
\]
\[
- \frac{1}{2} \int_0^T \int_\Omega \nabla^\perp \psi_n \cdot [\Lambda, \nabla \varphi(x)] \psi_n dx \phi(t) dt - \nu_n \int_0^T \int_\Omega \partial_t \Lambda^s \varphi(x) dx \phi(t) dt = 0,
\]
where \( \psi_n := \Lambda^{-1}\theta \) are uniformly bounded in \( L^\infty(0, T; H^1_0(\Omega)) \). The weak convergence \( \theta_n \rightarrow \theta \) in \( L^2(0, T; L^2(\Omega)) \) readily yields
\[
\lim_{n \to \infty} \int_0^T \int_\Omega \theta_n \varphi(x) dx \partial_t \phi(t) dt = \int_0^T \int_\Omega \theta \varphi(x) dx \partial_t \phi(t) dt
\]
and
\[
\lim_{n \to \infty} \nu_n \int_0^T \int_\Omega \theta_n \Lambda^s \varphi(x) dx \phi(t) dt = 0.
\]
Next we pass to the limit in the two nonlinear terms. Applying the commutator estimate in Theorem 2.3 we have
\[
\left| \int_0^T \int_\Omega \nabla^\perp \psi_n \cdot [\Lambda, \nabla \varphi] \psi_n dx \phi(t) dt - \int_0^T \int_\Omega \nabla^\perp \psi \cdot [\Lambda, \nabla \varphi] \psi dx \phi dt \right|
\]
\[
\leq \int_0^T \int_\Omega \nabla^\perp (\psi_n - \psi) \cdot [\Lambda, \nabla \varphi] \psi dx \phi dt + \| \phi \nabla^\perp \psi_n \|_{L^2(0,T;L^2(\Omega))} \| [\Lambda, \nabla \varphi] (\psi_n - \psi) \|_{L^2(0,T;L^2(\Omega))}
\]
\[
\leq \int_0^T \int_\Omega \nabla^\perp (\psi_n - \psi) \cdot [\Lambda, \nabla \varphi] \psi dx \phi dt + C \| \psi_n - \psi \|_{L^2(0,T;D(\Lambda^\frac{1}{2}))}.
\]
The first term converges to 0 due to the weak convergence of \( \psi_n \) to \( \psi \) in \( L^2(0, T; H^1_0(\Omega)) \) and the fact that \( [\Lambda, \nabla \varphi] \psi \in D(\Lambda^\frac{1}{2}) \subset L^2(\Omega) \) in view of Theorem 2.3. The second term also converges to 0 due to the strong convergence of \( \psi_n \) to \( \psi \) in \( C(0,T;D(\Lambda^{1-\varepsilon})) \) with \( \varepsilon \in (0, \frac{1}{2}) \). Finally, we apply the commutator estimate in Theorem 2.3 to obtain
\[
\left| \int_0^T \int_\Omega [\Lambda, \nabla^\perp] \psi_n \cdot \nabla \varphi \psi_n dx \phi(t) dt - \int_0^T \int_\Omega [\Lambda, \nabla^\perp] \psi \cdot \nabla \varphi \psi dx \phi dt \right|
\]
\[
\leq \| \nabla \varphi [\Lambda, \nabla^\perp](\psi_n - \psi) \|_{L^2(0,T;L^2(\Omega))} \| \nabla \psi_n \|_{L^2(0,T;L^2(\Omega))}
\]
\[
+ \| [\Lambda, \nabla^\perp] \psi \cdot \nabla \varphi \|_{L^2(0,T;L^2(\Omega))} \| \phi(\psi_n - \psi) \|_{L^2(0,T;L^2(\Omega))}
\]
\[
\leq C \| \psi_n - \psi \|_{L^2(0,T;L^2(\Omega))}
\]
which converges to 0. Putting together the above considerations leads to
\[
\int_0^T \int_\Omega \theta \varphi(x) dx \partial_t \phi(t) dt + \int_0^T \int_\Omega u \theta \cdot \nabla \varphi(x) dx \phi(t) dt = 0, \quad \forall \phi \in C_c^\infty((0, T)), \ \varphi \in C_c^\infty(\Omega).
\]
Therefore, \( \theta \) is a weak solution of the inviscid SQG equation on \( [0, T] \).
Finally, let us show the Hamiltonian conservation of $\theta$. We have the energy balance (1.6) for each $\theta_n$. If $s \leq 1$, then the uniform boundedness of $\theta_n$ in $L^\infty(0, T; L^2(\Omega))$ implies
\[
\lim_{n \to \infty} \nu_n \int_0^t \int_\Omega |\Lambda^{s-1} \theta_n|^2 \, dx \, dr = 0, \quad t \in [0, T].
\] (4.3)

In addition, $\theta_n \to \theta$ strongly in $C(0, T; D(\Lambda^{-\varepsilon})) \subset C(0, T; D(\Lambda^{-\frac{1}{2}}))$. Letting $\nu = \nu_n \to 0$ in the balance (1.6) we conclude that the Hamiltonian of $\theta$ is constant on $[0, T]$. Consider next the case $s \in (1, 2]$. Then since $\Lambda^{s-1} \in (0, \frac{s}{2})$ it follows by interpolation that
\[
\|\Lambda^{s-1} \theta_n\|_{L^2(\Omega)}^2 \leq \|\theta_n\|_{L^2(\Omega)}^{2(1-\lambda)} \|\Lambda^s \theta_n\|_{L^2(\Omega)}^{2\lambda} \leq C\|\Lambda^s \theta_n\|_{L^2(\Omega)}^{2\lambda}
\]
for some $\lambda \in (0, 1)$ depending only on $s$. Thus, for any $\delta > 0$,
\[
\nu_n \int_0^t \|\Lambda^{s-1} \theta_n\|_{L^2(\Omega)}^2 \, dt \leq C \nu_n \delta^{-\frac{\lambda}{1-\lambda}} + C \delta \nu_n \int_0^T \|\Lambda^s \theta_n\|_{L^2(\Omega)}^2 \, dr, \quad t \in [0, T].
\]

By virtue of (1.5), the energy dissipations $\nu_n \int_0^t \int_\Omega |\Lambda^s \theta_n|^2 \, dx \, dt, \ t \in [0, T]$, are uniformly bounded. Sending $\nu_n \to 0$ and then $\delta \to 0$ yields (4.3) for this case. This completes the proof.

**Appendix A. A bound on $P_m$**

Recall the definition (3.1) of $P_m$. The following lemma is essentially taken from [8]. We include the proof for the sake of completeness.

**Lemma A.1.** Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with smooth boundary. For every $N$ and $k \in \mathbb{N}$ satisfying $N > \frac{k}{2} + \frac{d}{2}$ there exists a positive constant $C_{N,k}$ such that
\[
\|P_m \varphi\|_{H^k(\Omega)} \leq C_{N,k} \|\varphi\|_{D(\Lambda^{2N})}
\] (A.1)

for all $m \geq 1$ and $\varphi \in D(\Lambda^{2N})$; moreover, we have
\[
\lim_{m \to \infty} \|(I - P_m)\varphi\|_{H^k(\Omega)} = 0.
\] (A.2)

**Proof.** As $\varphi \in D(\Lambda^{2N})$, we have $\Delta^\ell \varphi \in H^0_0(\Omega)$ for all $\ell = 0, 1, ..., N - 1$. This allows repeated integration by parts with $w_j$ using the relation $-\Delta w_j = \lambda_j w_j$. Using Hölder’s inequality and the fact that $w_j$ is normalized in $L^2$, we obtain
\[
|\varphi_j| \leq \lambda_j^{-N} \|\Delta^N \varphi\|_{L^2}, \quad \varphi_j = \int_\Omega \varphi w_j \, dx.
\]

By elliptic regularity estimates and induction, we have for all $k \in \mathbb{N}$ that
\[
\|w_j\|_{H^k(\Omega)} \leq C_k \lambda_j^k.
\]

We know from the easy part of Weyl’s asymptotic law that $\lambda_j \geq C j^{\frac{d}{2}}$. Consequently, with $N > \frac{k}{2} + \frac{d}{2}$ we deduce that
\[
\sum_{j=1}^{\infty} \|\varphi_j\|^2_{H^k(\Omega)} \leq C_k \|\Delta^N \varphi\|_{L^2} \sum_{j=1}^{\infty} \lambda_j^{-N+\frac{k}{2}} \leq C_k \|\varphi\|_{D(\Lambda^{2N})} \sum_{j=1}^{\infty} j^{(-N+\frac{k}{2})\frac{2}{d}} = C_{N,k} \|\varphi\|_{D(\Lambda^{2N})}
\]
where $C_{N,k} < \infty$ depends only on $N$ and $k$. Because

$$(I - P_m)\varphi = \sum_{j=m+1}^{\infty} \varphi_j w_j,$$

this proves both (A.1) and (A.2). The proof is complete. □

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References

[1] L. C. Berselli. Vanishing Viscosity Limit and Long-time Behavior for 2D Quasi-geostrophic Equations. Indiana Univ. Math. J. 51(4) (2002), 905–930.
[2] T. Buckmaster, S. Shkoller, V. Vicol. Nonuniqueness of weak solutions to the SQG equation. arXiv:1610.00676, to appear in Communications on Pure and Applied Mathematics.
[3] D. Chae, P. Constantin, D. Córdoba, F. Gancedo, J. Wu. Generalized surface quasi-geostrophic equations with singular velocities. Comm. Pure Appl. Math., 65 (2012) No. 8, 1037-1066.
[4] A. Cheskidov, M. C. Lopes Filho; H. J. Nussenzveig Lopes; R. Shvydkoy. Energy conservation in two-dimensional incompressible ideal fluids. Comm. Math. Phys. 348 (2016), no. 1, 129–143.
[5] P. Constantin, D. Cordoba, J. Wu. On the critical dissipative quasi-geostrophic equation. Indiana Univ. Math. J., 50 (Special Issue): 97–107, 2001. Dedicated to Professors Ciprian Foias and Roger Temam (Bloomington, IN, 2000).
[6] P. Constantin, M. Ignatova. Remarks on the fractional Laplacian with Dirichlet boundary conditions and applications. Internat. Math. Res. Notices, (2016), 1-21.
[7] P. Constantin, M. Ignatova. Critical SQG in bounded domains. Ann. PDE (2016) 2:8.
[8] P. Constantin, H.Q. Nguyen. Global weak solutions for SQG in bounded domains. Comm. Pure Appl. Math, 71 (2018), no. 11, 2323-2333.
[9] P. Constantin, H. Q. Nguyen. Local and global strong solutions for SQG in bounded domains. Phys. D Vol. 376-378 (2018), Special Issue in Honor of Edriss Titi, 195-203.
[10] P. Constantin, A.J. Majda, and E. Tabak. Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar. Nonlinearity, 7(6) (1994), 1495–1533.
[11] P. Constantin, A. Tarfulea, V. Vicol. Absence of anomalous dissipation of energy in forced two dimensional fluid equations. Arch. Ration. Mech. Anal. 212 (2014), 875-903.
[12] F. Marchand. Existence and Regularity of Weak Solutions to the Quasi-Geostrophic Equations in the Spaces $L^p$ or $\dot{H}^{-1/2}$. Comm. Math. Phys. (2008) 277(1): 45–67.
[13] I.M. Held, R.T. Pierrehumbert, S.T. Garner, and K.L. Swanson. Surface quasi-geostrophic dynamics. J. Fluid Mech., 282 (1995),1–20.
[14] P. Isset and V. Vicol. Hölder continuous solutions of active scalar equations. Ann. PDE 1 (2015), no. 1, 1–77.
[15] H. Q. Nguyen. Global weak solutions for generalized SQG in bounded domains. Anal. PDE, Vol. 11 (2018), No. 4, 1029–1047.
[16] J.L. Lions, Quelque methodes de r´esolution des problemes aux limites non lin´eaires. Paris: Dunod-Gauth, 1969.
[17] J. L. Lions, E. Magenes, Non-homogeneous boundary value problems and applications. Vol. I, Translated from the French by P. Kenneth. Die Grundlehren der mathematischen Wissenschaften, Band 181. Springer-Verlag, New York-Heidelberg, 1972.
[18] S. Resnick, Dynamical problems in nonlinear advective partial differential equations. ProQuest LLC, Ann Arbor, MI, 1995, Thesis (Ph.D.)–The University of Chicago.
