1. Introduction

Let $P = -h^2 \Delta_g + V(x)$ be a self-adjoint Schrödinger operator on a compact Riemannian $n$-manifold, $(X, g)$, $V \in C^\infty(X; \mathbb{R})$. The spectral asymptotics as $h \to 0$ are given by the celebrated Weyl law – see [10] and [16] for recent advances and numerous references. If we assume that the zero energy surface is nondegenerate,

$$p \overset{\text{def}}{=} |\xi|^2_g + V(x) = 0 \implies dp \neq 0,$$

then

$$(1.1) \quad |\text{Spec}(P) \cap D(0, Ch)| = O(h^{-n+1}),$$

where $D(0, r) = \{z \in \mathbb{C} : |z| < r\}$, though of course in this case the eigenvalues are all real – see §6.1 for yet another proof of this well known result.

Let $H_p$ be the Hamilton vector field of $p$ on $T^*X$, locally given by

$$H_p = \sum_{j=1}^n \frac{\partial p}{\partial \xi_j} \partial x_j - \frac{\partial p}{\partial x_j} \partial \xi_j, \quad (x, \xi) \in T^*\mathbb{R}^n.$$

When the flow, $\exp tH_p : p^{-1}(E) \to p^{-1}(E)$, has the property that the set of its closed orbits has Liouville measure zero on $p = 0$, then we have the infinitesimal version of the Weyl law:

$$(1.2) \quad |\text{Spec}(P) \cap D(0, Ch)| = \frac{2Ch}{(2\pi h)^n} \int_{p(x,\xi)=0} d\mathcal{L}(x, \xi) + o(h^{-n+1}),$$

where $d\mathcal{L}$ is the Liouville measure on $p = 0$, that is $d\mathcal{L}dp = dx d\xi$. This result is the mathematical starting point of many recent investigations, mostly in physical literature, of the finer structure of the spectrum and its relation to classical dynamics – see [11] and references given there.

When the manifold is non-compact the situation is dramatically different. The simplest case is that of a manifold which is Euclidean outside of a compact set and $V+1 \in C^\infty_c(X; \mathbb{R})$. The discrete eigenvalues of $P$ are replaced by quantum resonances which are defined as the poles of the meromorphic continuation of

$$(P - z)^{-1} : C^\infty_c(X) \xrightarrow{1} C^\infty(X), \quad \text{Im } z > 0,$$
and we denote the set of resonances by Res($P(h)$). The basic physical interpretation is that a resonance at $z = E_0 - i\Gamma/2$ corresponds to a state with time evolution given by $\exp(-itE_0/h - \Gamma t/2h)$, and to a Breit-Wigner peak in energy density, $\Gamma/((E - E_0)^2 + \Gamma^2/4)$.

This intuitive picture becomes however complicated when many resonances are present which is natural for $h$ is small – see [24] and [3],[5],[27] for recent results and references.

Here we provide upper bounds for the number of resonances of $P$ in $D(0,Ch)$. The main result (Theorem 3) states that for classical Hamiltonians $p$ with hyperbolic flow on $p = 0$,

$$|\text{Res} P(h) \cap D(0,Ch)| = \mathcal{O}(h^{-\nu}),$$

where $2\nu + 1$ is essentially the dimension of the trapped (non-wandering) set in $p^{-1}(0)$,

$$K \overset{\text{def}}{=} \{(x,\xi) \in T^*X : p(x,\xi) = 0, \exp(tH_p)(x,\xi) \not\to \infty, \ t \to \pm \infty\}.$$

In the case of a compact manifold $\nu = n - 1$ so that (1.3) reduces to (1.1). By dimension we always mean the Minkowski dimension

$$m_0 = 2n - 1 - \sup\{d : \limsup_{\epsilon \to 0} \epsilon^{-d} \text{vol}(\{\rho \in p^{-1}(0) : d(\rho,K) < \epsilon\}) < \infty\}.$$

A simple example is provided by a three bump potential shown in Fig.1.

**Figure 1.** A three bump potential exhibiting hyperbolic dynamics at an interval of energies.
The first estimate of this form was proved by the first author in [29, Theorems 4.6, 5.5, and 5.7]: there exists constants $C_0, C_1 > 0$, such that for $\delta_0 > 0$ fixed and small enough

$$|\text{Res}(P(h)) \cap \{z : |z| < \delta, \quad \text{Im } z > -\mu\}| \leq C_1 \delta \left(\frac{h}{\mu}\right)^{-n} \mu^{-\frac{3}{2} \tilde{m}},$$

where now $\tilde{m}$ is any number greater than the dimension of the trapped set in $p^{-1}([-2\delta_0, 2\delta_0])$.

In homogeneous situations, such as for instance obstacle scattering, $\tilde{m} = m + 1$. When $\mu = C_0 h$, the improvement in Theorem 3 lies in allowing $\delta \approx h$, which is the natural limit for this type of spectral estimates.

Earlier, non-geometric, bounds on the number of resonances (scattering poles) were obtained by Melrose [21], [22] and the second author [38], [39]. In the case of convex co-compact Schottky quotients (and any convex co-compact quotients in dimension two) the analogue of (1.4) was proved in [12] using zeta function techniques, improving earlier estimates of [40]; the proof of which was largely based on [29]. These technique gave similar results for the zeros of zeta functions of rational maps [8], [35], in which case the dimension of the trapped set becomes essentially the dimension of the Julia set.

**Figure 2.** A sample of results of [17]: the plot on the left shows resonances for $h = 0.015$, and the plot on the right is the log-log plot of the number of resonances vs. $\hbar$ with $\circ$ denoting numerical data, $\ast$ the density predicted by the fractal Weyl law, and $\bullet$, the least square interpolants.
Numerical investigations in different settings of semiclassical three bump potentials \[17,18\] (see Fig.2), Schottky quotients \[12\], three disc scattering \[19\], and Cantor-like Julia sets for \(z \mapsto z^2 + c, c < -2\) \[35\], suggest that for \(\mu \approx Ch\) and \(\delta \approx 1\) the estimate (1.4) is optimal. A different model was recently considered in \[25\]: quantum resonances were defined using an open quantum map with a classical “trapped set” corresponding to \(K\) intersected with a hypersurface transversal to the flow – see Fig.5. The numerical results and a simple linear algebraic toy model suggest that the fine estimate (1.3) is optimal – see Fig.3. A similar model was also used in \[28\] where the fractal Weyl law gave corrections to the applications of random matrix theory to open quantum systems.

We now state the general assumptions on the operator \(P\). We reiterate that the simplest case to keep in mind is

\[
P = -h^2\Delta + V(x) - 1, \quad V \in C^\infty_c(\mathbb{R}^n).
\]

In general we consider

\[
P(h) \in \Psi(X), \quad P(h) = P(h)^*,
\]

where the calculus of semiclassical pseudodifferential operators is reviewed in \[31\] (1.5)

\[
P(h) = p^u(x, hD) + hp_1^u(x, hD; h), \quad p_1 \in s^{0,2}(T^*X),
\]

\[
|\xi| \geq C \implies p(x, \xi) \geq |\xi|^2/C, \quad p = 0 \implies dp \neq 0,
\]

\[
\exists R, \quad u \in C^\infty(X \setminus B(0, R)) \implies P(h)u(x) = Q(h)u(x), \quad Q(h)u = \sum_{|\alpha| \leq 2} a_\alpha(x; h)(hD_x)^\alpha u,
\]

where \(a_\alpha(x; h) = a_\alpha(x)\) is independent of \(h\) for \(|\alpha| = 2\), \(a_\alpha(x; h) \in C^\infty_b(\mathbb{R}^n)\) are uniformly bounded with respect to \(h\), here \(C^\infty_b(\mathbb{R}^n)\) denotes the space of \(C^\infty\) functions on \(\mathbb{R}^n\) with bounded derivatives of all orders,

\[
\sum_{|\alpha| = 2} a_\alpha(x)\xi^\alpha \geq (1/c)|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \text{ for some constant } c > 0,
\]

(1.6)

\[
\sum_{|\alpha| \leq 2} a_\alpha(x; h)\xi^\alpha \to \xi^2 - 1 \text{ uniformly with respect to } h \text{ as } |x| \to \infty.
\]

We also need the following analyticity assumption in a neighbourhood of infinity: there exist \(\theta \in [0, \pi)\), \(\epsilon > 0\) and \(R \geq R_0\) such that the coefficients \(a_\alpha(x; h)\) of \(Q(h)\) extend holomorphically in \(x\) to

\[
\{r\omega : \omega \in \mathbb{C}^n, \quad \text{dist}(\omega, S^n) < \epsilon, \quad r \in \mathbb{C}, \quad |r| > R, \quad \arg r \in [-\epsilon, \theta_0 + \epsilon]\},
\]

with (1.6) valid also in this larger set of \(x\)’s. We remark that in \[13,29\] the operators were required to be globally real analytic but the conditions at infinity were much more general. A particularly nice feature of the theory developed in \[13\] is allowing arbitrary homogeneous polynomials as potentials (see \[29\] (c.31)-(c.33)).
Figure 3. An example of numerical results of [25]. The eigenvalues of a $3N \times 3N$ matrix $A_{3N}$ model the resonances in a disc of size $h \sim 1/N$. Dashed lines corresponding to counting the resonances (eigenvalues of $A_{3N}$) with $\exp(\text{Im } z/h) \sim r$. Full lines give the counting function rescaled using the dimension of the trapped set in this problem. The coalescence of the graphs confirms the power law based on the dimension of the trapped set. A similar correction based on [18],[19],[29] was given for the random matrix models of open system in [28].

The first theorem we present fits naturally in the methodology of this paper. It is a slight generalization of a result of Martinez [20] which in turn was a $C^\infty$ version of a similar result in [29], already implicit in [13]. As discussed at the end of §4 it is essentially optimal.

**Theorem 1.** Suppose that $X$ is non compact and euclidean outside of a compact set, and that the operator $P \in \Psi^{0.2}(X)$ satisfies the assumptions below. Suppose in addition that no orbit of $H_p$ on $p^{-1}(0)$ is trapped:

\begin{equation}
\forall K \subseteq p^{-1}(0), \exists T_K, \ (x, \xi) \in K \implies \exp(tH_p(x, \xi)) \notin K, \ t > T_K.
\end{equation}
Then, for any \( M > 0 \) there exists \( h(M) \) such that for \( 0 < h < h(M) \),

\[
| \text{Res}(P) \cap D(0, Mh \log(1/h)) | = 0.
\]

Here \( \text{Res}(P) \) denotes the set of resonances of \( P \) defined in some \( h \)-independent neighbourhood of 0.

Before stating the main result which requires hyperbolicity of the flow on the energy surface we first give the general upper bound which generalizes slightly\(^1\) the results of Bony \[2\] which in turn generalized earlier results of Petkov and the second author \[26\], \[27\].

**Theorem 2.** Suppose that \( X \) is Euclidean outside a compact set, and that the operator \( P \in \Psi^{2,0}(X) \) satisfies the general assumptions below. Then, as \( h \to 0 \),

\[
| \text{Res}(P) \cap D(0, Ch) | = \mathcal{O}(h^{-n+1}).
\]

The estimate (1.9) is also optimal in the same way that the analogous estimate for eigenvalues of a self-adjoint operator with a compact and smooth energy surface. That follows for instance from applying \[24\, \text{Corollary, §5}].

The basic hyperbolicity assumption at an energy \( E \) can be stated as follows: for \( \rho \in p^{-1}(E) \) lying in a neighbourhood of the trapped set \( K_E \) we have,

\[
T_\rho(p^{-1}(E)) = \mathbb{R} H_\rho(\rho) \oplus E_+(\rho) \oplus E_- (\rho) , \quad \dim E_\pm(\rho) = n - 1, \\
p^{-1}(E) \ni \rho \mapsto E_\pm(\rho) \subset T_\rho(p^{-1}(E)) \text{ is continuous}, \\
d(\exp tH_\rho)_{\rho}(E_\pm(\rho)) = E_\pm(\exp tH_\rho(\rho)), \\
\exists \lambda > 0 \quad \| d(\exp tH_\rho)_{\rho}(X) \| \leq C e^{\pm \lambda t} \| X \| , \text{ for all } X \in E_\pm(\rho), \quad \forall t \geq 0.
\]

An example of a potential satisfying this assumption at a range of non-zero energies is given in Fig.1 – see \[23\] and \[29\, \text{Appendix c}]]. Following \[29\] we will formulate a weaker dynamical hypothesis in \[7\].

The main result of this paper is

**Theorem 3.** Suppose that \( P(h) \) satisfies our general assumptions (1.5), the flow of \( H_\rho \) near zero energy is hyperbolic in the sense of (1.10), or the weaker sense given in \[7\] and that the trapped set at zero energy has Minkowski dimension \( m_0 = 2\nu_0 + 1 \). Then for any \( \nu > \nu_0 \), and \( C_0 > 0 \) there exists \( C_1 \) such that

\[
| \text{Res}(P(h)) \cap D(0, C_0h) | \leq C_1 h^{-\nu}.
\]

When the trapped is set is of pure dimension, \( \nu \) can be replaced by \( \nu_0 \).

A sharp rigorous lower bound is known when \( K \) is an isolated hyperbolic trajectory. A very precise asymptotic description of resonances in that case is given in \[11\] and it implies (1.11) with \( \nu = 0 \). In spite of the convincing numerical evidence cited above no rigorous

\(^1\) Here we consider an arbitrary manifold in the compact part. It is clear that the methods of \[2\] easily allow this type of generalization and the point is that our method is different and more robust.
examples with non-integral values \( \nu_0 \) are known. A recent indication of the delicate nature of lower bounds for resonances was given in [7] where a class of complex compactly supported potentials in \( \mathbb{R}^3 \) with no resonances at all. We have no reasonable hope of obtaining any analogue of (1.2) at the present moment.

The methods of this paper apply also to a simpler problem of operators with complex absorbing barriers. Let \( V \in C_0^\infty(B(0,R_0);\mathbb{R}) \) be a potential for which \( H_p, p = \xi^2 + V(x) - 1, \) has hyperbolic flow on \( p = 0, \) for instance a “three bump” potential [23]. Now let \( W \in C^\infty(\mathbb{R}^n) \) satisfy

\[
W(x) \geq 0, \quad \text{supp} W \cap B(0,R_0) = \emptyset, \quad W(x) \geq 2C_0 > 0 \quad \text{for } |x| > 2R_0.
\]

Consider then (see [34] and references given there)

\[
\tilde{P}(h) = -h^2 \Delta + V(x) - iW(x).
\]

The spectrum of this non-selfadjoint operator lies in \( \overline{\mathbb{C}}_\pm \) and we have the exact analogue of (1.3):

\[
|\text{Spec} \tilde{P}(h) \cap D(0,Ch)| = \mathcal{O}(h^{-\nu}).
\]

Acknowledgements. The second author would like to thank Jean-Yves Chemin and Jean-Michel Bony for useful discussions, the National Science Foundation for partial support under the grant DMS-0200732, and École Polytechnique for its generous hospitality in Fall 2004.

2. Outline of the proof

To prove the main result on fractal upper bounds (Theorem 3) we first develop methods for proving the natural results on the absence of resonances (Theorem 1) and on general upper bounds at non-degenerate energies (Theorem 2). In this section we present the general ideas. All of them have origins in other works and pointers to the literature will be given in corresponding sections.

The absence of resonances for operators with \( C^\infty \) coefficients in domains of size \( h \log(1/h) \) around an energy level hold under a nontrapping condition:

\[
\exists \epsilon_0 > 0, \quad \forall K \in p^{-1}(0), \quad \exists T_K, \quad (x, \xi) \in K \quad \Longrightarrow \quad \exp(tH_p(x,\xi)) \notin K, \quad t > T_K.
\]

This implies the existence of an escape function in a neighbourhood of \( p^{-1}(0) \):

\[
\exists G_1 \in C^\infty(T^*X), \quad H_pG_1(x,\xi) \geq c_0 > 0, \quad \text{for } |p(x,\xi)| < \epsilon_0.
\]

The resonances of \( P \) are given by the eigenvalues of the deformed operator \( P_\theta \). In the case of \( P = -h^2 \Delta + V(x) \) with \( V \) analytic in a conic neighbourhood of \( \mathbb{R}^n \),

\[
V(x) + 1 \longrightarrow 0, \quad |x| \longrightarrow \infty,
\]

the scaled operator is simply

\[
P_\theta = -h^2 e^{-2i\theta} \Delta + V(e^{i\theta}x),
\]
and it behaves as $-\hbar^2 e^{-2i\theta} \Delta - 1$ near infinity. For $\theta > 0$ that last operator is clearly invertible.

We can introduce a modified $G = \chi G_1, \chi \in C_c^\infty(X)$, so that for $\theta \sim \epsilon \sim M\hbar \log(1/\hbar)$, we have

$$|\text{Re} p_\theta| < \delta \implies -\text{Im} p_\theta + \epsilon H_p G \geq c_0 \epsilon, \quad p_\theta = \sigma(P_\theta).$$

The operators $\exp(\pm \epsilon G_w(x, hD)/\hbar)$ are now pseudodifferential operators in a mildly exotic class and we consider

$$P_{\theta, \epsilon} = e^{-\epsilon G_w/\hbar} P_{\theta} e^{\epsilon G_w/\hbar}.$$

The spectrum of $P_{\theta}$ in $D(0, M\hbar \log(1/\hbar))$ is the same as that of $P_{\theta, \epsilon}$ but the properties of $G$ imply that

$$\|P_{\theta, \epsilon}^{-1}\| \leq C/\epsilon$$

showing that in fact there is no spectrum in $D(0, M'\hbar \log(1/\hbar))$. This approach allows us to obtain the absence of resonances very directly.

In Theorem 2 we show that if 0 is a non-critical energy level then

$$|\text{Res} P \cap D(0, Ch)| = O(h^{-n+1}).$$

The proof follows from a “robust” proof of the same estimate for an operator with a compact resolvent (for instance, an elliptic operator on a compact manifold). Let $P$ be such an operator, say, $P = -\hbar^2 \Delta_g - 1$, on a compact Riemannian manifold. We would like to consider a modified operator

$$\tilde{P}(h) \overset{\text{def}}{=} P(h) - iM\hbar \psi(MP(h)/\hbar), \quad \psi \in C_c^\infty(\mathbb{R}),$$

whose “symbol”, $p - iM\hbar \psi(Mp/h)$, has the absolute value bounded from below by $M\hbar/2$ everywhere. That does not make sense at first since

$$\psi(MP(h)/h)$$

is not an $h$-pseudodifferential operator. To remedy this we construct a second microlocal calculus with a new Planck constant $\hbar \sim 1/M$. The new operator $\tilde{P}(h)$ becomes elliptic in this calculus and for $\hbar$ small enough, independent of $h$, it is invertible. We then have

$$(P(h) - z)^{-1} = (I + K(z))^{-1}(\tilde{P}(h) - z)^{-1}, \quad K(z) \overset{\text{def}}{=} i(\tilde{P}(h) - z)^{-1} M\hbar \psi(MP/h)$$

and the eigenvalues of $P(h)$ near 0 coincide with the zeros of $\det(I + K(z))$. The zeros of this determinant are the same as the zeros of a determinant $\det(I + R(z))$ where $R(z)$ is a finite rank operator with the rank proportional to $h^{-n}$ times the volume of the support of $\psi(Mp/h)$. That gives estimates on the determinant which imply (1.1). A slight modification of this argument is needed to obtain Theorem 2.

We now assume that the flow of $H_p$ is hyperbolic and introduce the sets

$$\Gamma_{\pm} \overset{\text{def}}{=} \{(x, \xi) \in T^*X : p(x, \xi) = 0, \exp(tH_p)(x, \xi) \not\to \infty, \quad t \to \mp \infty\}.$$
depicted in a simple case in Fig. 4. The trapped set at zero energy is

\[ K = \Gamma_+ \cap \Gamma_- \]  

If we assume that \( K \subset \overline{\Gamma_+ \setminus \Gamma_-} \), that is \( K \) has no component isolated from infinity, then \( K \) is a set of Liouville measure 0.

To prove an upper bound involving the dimension of \( K \) we combine the methods used to prove Theorems 1 and 2. There exist functions \( \varphi_\pm \in C^{1,1}(T^*X) \) such that, uniformly on compact sets,

\[
H_p \varphi_\pm \sim \mp \varphi_\pm, \quad \varphi_\pm \sim d(\Gamma_\pm, \bullet)^2, \quad \varphi_+ + \varphi_- \sim d(K, \bullet)^2.
\]

A local model for the simplest case of one trajectory is given by \( p = \xi_1 + x_2 \xi_2, \ (x, \xi) \in T^*\mathbb{R}^2 \), so that

\[ H_p = \partial_{x_1} + x_2 \partial_{x_2} - \xi_2 \partial_{\xi_2}, \quad \varphi_+ = \xi_2^2, \quad \varphi_- = x_2^2, \quad K = \{(t, 0, 0, 0) : t \in \mathbb{R}\}. \]
A new escape function is given by
\begin{equation}
G \overset{\text{def}}{=} (\log(C\epsilon + \hat{\varphi}_-) - \log(C\epsilon + \hat{\varphi}_+)) , \quad \epsilon \sim Mh , \quad M \gg 1 ,
\end{equation}
where \(\hat{\varphi}_\pm\) are suitable \(h\)-dependent regularizations of \(\varphi_\pm\).

The logarithmic flattening of the more straightforward escape function \(\varphi_- - \varphi_+\) is forced by the requirement that \(G = O(\log(1/h))\) so that the conjugation used in the proof of Theorem 1 can be applied. However, even for uniformly smooth \(\hat{\varphi}_\pm\) the regularization of \(G\) is essentially in the symbolic class \(S_{1/2}\) and the situation becomes more complicated in general. Nevertheless we obtain the following estimates:
\[\partial_{(x,\xi)} a(x,\xi) H^k p G = O(\epsilon^{-|\alpha|/2}) , \quad \text{for } |\alpha| + k \geq 1, \text{ uniformly on compact sets},\]
and
\[d((x,\xi), K)^2 \geq C\epsilon \implies H^p G \geq 1/C .\]
As in the proof of Theorem 1 (but with very different parameters and escape functions) we introduce a conjugated operator,
\[P_{\theta,t}(h) \overset{\text{def}}{=} e^{-t\hat{G}w} P_{\theta}(h) e^{t\hat{G}w} ,\]
which now is in an exotic \(1/2\)-class, with the second Planck constant \(\hat{h} \sim 1/M\) playing the rôle of the asymptotic parameter. The escape function used here, \(G\), has compact support.

We now build a second microlocal calculus which combines this exotic class with the one used in the proof of Theorem 2. The first allows us the use of the irregular escape function and the second allows a localization to an \(h\)-neighbourhood of the energy surface. In the new calculus the operator
\[\widetilde{P}_{\theta,t} = P_{\theta,t} - iMh\widetilde{O}_h^w(a) , \quad a(x,\xi) \overset{\text{def}}{=} \chi \left( \frac{p(x,\xi; K'h)}{K'h} \right) \chi \left( \frac{C_0 H^p G(x,\xi)}{K'h} \right) ,\]
is globally elliptic (here \(\widetilde{O}_h^w\) describes a second microlocal quantization operator). As in the proof of Theorem 2 the number of eigenvalues of \(P_{\theta,t}\), and hence \(P_{\theta}\), near 0, is estimated by \(h^{-n}\) times the volume of the support of \(a\). A cross-section of that support with a hypersurface transversal to the flow is illustrated in Fig. 5. That volume is bounded by \(h^{1+(2n-2-2\nu)/2}\), where \(\nu > \nu_0\), \(2\nu_0 + 1\) is the dimension of \(K\). That gives (1.3).

3. Preliminaries

3.1. Review of semiclassical pseudodifferential calculus. Let \(X\) be a \(C^\infty\) manifold which is agrees with \(\mathbb{R}^n\) outside a compact set, or more generally
\[X = X_0 \sqcup (\mathbb{R}^n \setminus B(0, R_0)) \sqcup \cdots \sqcup (\mathbb{R}^n \setminus B(0, R_0)), \quad X_0 \subset X .\]

We introduce the usual class of semiclassical symbols on \(X\):
\[S^{m,k}(T^*X) = \{ a \in C^\infty(T^*X \times (0,1]) : |\partial_x^\alpha \partial_{\xi}^\beta a(x,\xi; h)| \leq C_{\alpha,\beta} h^{-m}|\xi|^{k-|\beta|} \} ,\]
Figure 5. The trapped set $K \subset p^{-1}(0)$ intersected with a hypersurface $M \subset T^*X$ transversal to the flow and surrounded by an $h$-dependent neighbourhood admissible in the second microlocal calculus. For $K$ of dimension less than $2\nu + 1$, the volume of this neighbourhood is bounded by $h^{n-\nu}$, $n = \dim X$.

where outside $X_0$ we take the usual $\mathbb{R}^n$ coordinates in this definition. The corresponding class of pseudodifferential operators is denoted by $\Psi_h^{m,k}(X)$, and we have the quantization and symbol maps:

$$\begin{align*}
\text{Op}_h^w : & \quad S^{m,k}(T^*X) \longrightarrow \Psi_h^{m,k}(X) \\
\sigma_h : & \quad \Psi_h^{m,k}(X) \longrightarrow S^{m,k}(T^*X)/S^{m-1,k-1}(T^*X) ,
\end{align*}$$

with both maps surjective, and the usual properties

$$\begin{align*}
\sigma_h(A \circ B) = \sigma_h(A)\sigma_h(B) , \\
0 \to \Psi^{m-1,k-1}(X) \hookrightarrow \Psi^{m,k}(X) \xrightarrow{\sigma_h} S^{m,k}(T^*X)/S^{m-1,k-1}(T^*X) \to 0 ,
\end{align*}$$

a short exact sequence, and

$$\begin{align*}
\sigma_h \circ \text{Op}_h^w : & \quad S^{m,k}(T^*X) \longrightarrow S^{m,k}(T^*X)/S^{m-1,k-1}(T^*X) ,
\end{align*}$$
the natural projection map. The class of operators and the quantization map are defined locally using the definition on $\mathbb{R}^n$:

\begin{equation}
\text{Op}_h^w(a)(x) = \frac{1}{(2\pi h)^n} \int \int a\left(\frac{x + y}{2}, \xi\right) e^{i(x-y, \xi)/h} u(y) dy d\xi,
\end{equation}

and we refer to [10] for a detailed discussion. We remark only that when we consider the operators acting on half-densities we can define the symbol map, $\sigma_h$, onto $S^{m,k}(T^*X)/S^{m-2,k-2}(T^*X)$, see [33, Appendix]. We keep this in mind but for notational simplicity we suppress the half-density notation.

For $a \in S^{m,k}(T^*X)$ we define

\[ \text{ess-supp}_h a \subset T^*X \cup S^*X, \quad S^*X \overset{\text{def}}{=} (T^*X \setminus 0)/\mathbb{R}_+, \]

where the usual $\mathbb{R}_+$ action is given by multiplication on the fibers: $(x, \xi) \mapsto (x, t\xi)$, as

\[
\text{ess-supp}_h a = \mathcal{C}\{ (x, \xi) \in T^*X : \exists \epsilon > 0, \partial_x^\alpha \partial_{\xi}^\beta a(x', \xi') = O(h^\epsilon), \quad d(x, x') + |\xi - \xi'| < \epsilon \}
\]

\[
\cup \mathcal{C}\{ (x, \xi) \in T^*X \setminus 0 : \exists \epsilon > 0, \partial_x^\alpha \partial_{\xi}^\beta a(x', \xi') = O(h^\epsilon \langle \xi' \rangle^{-\epsilon}) ,
\quad d(x, x') + 1/|\xi'| + |\xi'/|\xi| - \xi'/|\xi'| < \epsilon \}/\mathbb{R}_+ ,
\]

where the second complement is in $S^*X$. For $A \in \Psi^m(X)$, then define

\[ \text{WF}_h(A) = \text{ess-supp}_h a, \quad A = \text{Op}_h^w(a), \]

noting that, as usual, the definition does not depend on the choice of $\text{Op}_h^w$, and

\[ \text{Char}(A) = \bigcup \left\{ \text{WF}_h(B) : B \in \Psi^m(X), \quad \sigma_h(B) = \sigma_h(A) \right\} , \]

where $\sigma_h$ is the principal symbol map in (3.1). For

\[ u \in C^\infty((0, 1]_h; C^\infty(X)), \quad \forall K \subset X, N \in \mathbb{N} \exists P, h_0, \quad \|u\|_{C^N(K)} \leq h^{-P}, \quad h < h_0. \]

we define

\[ \text{WF}_h(u) = \left( \bigcup \left\{ \text{Char}(A) : A \in \Psi^{0,0}(X) : \quad Au \in h^\infty C^\infty((0, 1]_h; C^\infty(X)) \right\} \right)^c , \]

where the complement is taken in $T^*X \cup S^*X$. Here we will be concerned with a purely semiclassical theory and deal only with compact subsets of $T^*X$.

To illustrate the $h$-pseudodifferential calculus at work we prove two simple lemmas which will be used later. We say that $A \in \Psi^m(X)$ is elliptic on $K \subset T^*X$ if $|\sigma(A)|_K > h^{-m}/C$. This is equivalent to saying

**Lemma 3.1.** Suppose $Q \in \Psi^{0,m}(X)$ is elliptic at $(x_0, \xi_0)$, $\|u\|_{L^2} = 1$, and $\text{WF}_h(u)$ is contained in a sufficiently small neighbourhood of $(x_0, \xi_0)$. Then for $h$ small enough,

\[ \|Qu\|_{L^2} \geq 1/C. \]
Lemma 3.2. Suppose that $\psi_j \in C_0^\infty(T^*X)$, $\psi_1^2 + \psi_2^2 = 1$, supp $\psi \subset \{(x, \xi) : |\xi| \leq C\}$. Then, there exist $\Psi_1 \in \Psi^{0,-\infty}(X)$ and $\Psi_2 \in \Psi^{0,0}(X)$, with principal symbols $\psi_1$ and $\psi_2$ respectively, such that

$$\Psi_1^2 + \Psi_1^2 = I + R, \quad R \in \Psi^{-\infty,-\infty}(X), \quad \Psi_j^* = \Psi_j.$$

Proof. Functional calculus gives

$$\psi^w(\psi_1^w)^2 + (\psi_2^w)^2 = I + r_1^w, \quad r_1 \in S^{-1,-\infty}(T^*X),$$

in particular $r = \mathcal{O}(h) : H^{-M}(X) \to H^M(X)$. If $h$ is small enough we put

$$\Psi_j^1 = (1 + r_1^w)^{-\frac{1}{2}}\psi_j^w(1 + r_1^w)^{-\frac{1}{2}},$$

so that

$$(\Psi_1^1)^2 + (\Psi_2^1)^2 = I + r_2^w, \quad r_2 \in S^{-2,-\infty}(T^*X), \quad (\Psi_1^1)^* = \Psi_1^1.$$

and we can then proceed by iteration. \hfill \Box

The semiclassical Sobolev spaces, $H^s_h(X)$ are defined by choosing a globally elliptic, self-adjoint operator, $A \in \Psi^{1,0}(X)$ (that is an operator satisfying $\sigma(A) \geq \langle \xi \rangle/C$ everywhere) and putting

$$\|u\|_{H^s_h} = \|A^s u\|_{L^2(X)}.$$

When $X = \mathbb{R}^n$,

$$\|u\|^2_{H^s_h} \sim \int_{\mathbb{R}^n} \langle h\xi \rangle^{2s} |F u(\xi)|^2 \, d\xi, \quad F u(\xi) \overset{\text{def}}{=} \int_{\mathbb{R}^n} u(x) e^{-i(x,\xi)} \, dx.$$

The following lemma will also be useful:

Lemma 3.3. Suppose that $P_t$, $t \in (0, \infty)$, is a family of operators such that

$$P_t : H^s_h(X) \to H^{s-m}_h(X),$$

$$\forall A \in \Psi^{0,-\infty}(X), \quad \text{ad}_{P_t} A = \mathcal{O}(h) : L^2(X) \to L^2(X), \quad 0 < h < h_0(t).$$

Let $\Psi_j$ be as in Lemma 3.2 and suppose that

$$\|P_t \Psi_j u\| \geq th \|\Psi_j u\| - \mathcal{O}(h/t) \|u\|, \quad j = 1, 2, \quad u \in C_c^\infty(X).$$

Here the constants in $\mathcal{O}$ are independent of $h$ and $t$. Then for $t > t_0 \gg 1$ and $0 < h < h_0(t)$,

$$\|P_t u\| \geq th \|u\|/2.$$

Proof. We recall from Lemma 3.2 that

$$(3.3) \quad \|\Psi_1 v\|^2 + \|\Psi_2 v\|^2 = \|v\|^2 + \langle Rv, v \rangle = \|v\|^2 + O(h^\infty) \|v\|_{H^{-\infty}_h},$$

and hence with $v = P_t u,$
\[ \|P_t u\|^2 = \|\Psi_1 P_t u\|^2 + \|\Psi_2 P_t u\|^2 - \mathcal{O}(h^\infty)\|u\|^2 \]
\[ \geq \|P_t \Psi_1 u\|^2 + \|P_t \Psi_2 u\|^2 - \|\Psi_2, P_t]\|u\| - \|\Psi_2, P_t u\|^2 \]
\[ - 2 (\|\Psi_1 P_t u\|\|\Psi_1, P_t\|u\|^2 + \|\Psi_2 P_t u\|\|\Psi_2, P_t\|u\|^2) \frac{1}{2} - \mathcal{O}(h^\infty)\|u\|^2 \]
\[ \geq \|P_t \Psi_1 u\|^2 + \|P_t \Psi_2 u\|^2 - 2C(\|\Psi_1, P_t\|u\|^2 + \|\Psi_2, P_t\|u\| - \|P_t\|u\|/C - \mathcal{O}(h^\infty)\|u\|^2 \]
\[ \geq \|P_t \Psi_1 u\|^2 + \|P_t \Psi_2 u\|^2 - C'h^2\|u\|^2 - \|P_t\|u\|^2/C. \]

We now use the hypothesis of the lemma and (3.3) with \( v = u \) to obtain
\[ \|P_t u\|^2 \geq t^2 h^2 (\|\Psi_1 u\|^2 + \|\Psi_2 u\|^2 - C'h^2\|u\|^2 - \|P_t u\|/C \]
\[ \geq t^2 h^2\|u\|^2 - C'h^2\|u\|^2 - \|P_t u\|^2/C \]
and the lemma follows. \( \square \)

3.2. Semiclassical Fourier integral operators. We now follow [33] and review some aspects of the theory of semiclassical Fourier Integral Operators. We take a point of view which will be used in showing invariance of the second microlocal calculus developed below in [15].

Thus let \( A(t) \) be a smooth family of pseudodifferential operators,
\[ A(t) = \text{Op}_h^w(a(t)), \quad a(t) \in C^\infty([-1, 1]; S^{0,-\infty}(T^*X)), \]
such that for all \( t \), \( \text{WF}_h(A(t)) \subseteq T^*X \). We then define a family of operators
\[ U(t) : L^2(X) \to L^2(X), \]
\[ hD_t U(t) + U(t)A(t) = 0, \quad U(0) = U_0 \in \Psi_{h,0,0}(X). \]

This is an example of a family of \( h \)-Fourier Integral Operators, \( U(t) \), associated to canonical transformations \( \kappa(t) \), generated by the Hamilton vector fields \( H_{a_0(t)} \), where the real valued \( a_0(t) \) is the \( h \)-principal symbol of \( A(t) \),
\[ \frac{d}{dt}\kappa(t)(x, \xi) = (\kappa(t))\ast(H_{a_0(t)}(x, \xi)), \quad \kappa(0)(x, \xi) = (x, \xi), \quad (x, \xi) \in T^*X. \]

We will often need the Egorov theorem which can be proved directly from this definition: when \( U_0 \) in (3.4) is elliptic (that is \( |\sigma(U_0)| > c > 0 \)) on \( T^*X \), then for \( B \in \Psi_{h}^{m,k}(X) \)
\[ \sigma(V(t)BU(t)) = (\kappa(t))\ast\sigma(B), \]
where \( V(t) \) is an approximate inverse to \( U(t) \),
\[ V(t)U(t) - I, U(t)V(t) - I \in \Psi_{h, -\infty,-\infty}(T^*X). \]

The approximate inverse is constructed by taking
\[ hD_t V(t) - A(t)V(t) = 0, \quad V(0) = V_0, \quad V_0 U_0 - I, U_0 V_0 - I \in \Psi_{h, -\infty,-\infty}(T^*X), \]
the existence of $V_0$ being guaranteed by the ellipticity of $U_0$. The proof of (3.5) follows from writing $B(t) = V(t)B(t)$, so that, in view of the properties of $V(t)$,

$$hD_tB(t) \equiv [A(t), B(t)] \mod \Psi^{-\infty, -\infty}_h, \quad B(0) = B_0.$$ 

Since the symbol of the commutator is given by $(h/i)H_{a_0(t)}\sigma(B(t))$, (3.5) follows directly from the definition of $\kappa(t)$.

If $U = U(1)$, say, and the graph of $\kappa(1)$ is denoted by $C$, we conform to the usual notation and write

$$U \in I^0_h(X \times X; C'), \quad C' = \{(x, \xi; y, -\eta) : (x, \xi) = \kappa(y, \eta)\},$$

which means that $U$ is an $h$-Fourier Integral Operator associated to the canonical graph $C$. Locally all $h$-Fourier Integral Operators associated to canonical graphs are of the form $U(1)$ since each local canonical transformation with a fixed point can be deformed to identity, see [33, Lemma 3.2] and the proof of Lemma 5.8 below.

Our definitions of pseudo-differential operators and of (the special class of) $h$-Fourier Integral Operators were global. It is useful and natural to consider the operators and their properties microlocally. We consider classes of tempered operators:

$$T : C^\infty(X) \to C^\infty(X),$$

and for any semi-norm $\| \cdot \|_1$ on $C^\infty(X)$ there exist a seminorm $\| \cdot \|_2$ on $C^\infty(X)$ and a constant $M_0$ such that

$$\|Tu\|_1 = \mathcal{O}(h^{-M_0})\|u\|_2.$$ 

We remark that since we deal with compact subsets of $T^*X$ here, we could consider operators $T : \mathcal{D}'(X) \to C^\infty(X)$ in which case we can ask for existence of $M_0$ for any two seminorms $\| \cdot \|_j, \ j = 1, 2$.

For open sets, $V \subset T^*X, \ U \subset T^*X$, the operators defined microlocally near $V \times U$ are given by equivalence classes of tempered operators given by the relation

$$T \sim T' \iff A(T - T')B = \mathcal{O}(h^\infty) : \mathcal{D}'(X) \to C^\infty(X),$$

for any $A, B \in \Psi^{0, 0}_h(X)$ such that

$$WF_h(A) \subset \tilde{V}, \quad WF_h(B) \subset \tilde{U},$$

$$\triangledown \in \tilde{V} \subset T^*X, \quad \triangledown \in \tilde{U} \subset T^*X, \quad \tilde{U}, \tilde{V} \text{ open}.$$ 

The equivalence class $T$, $h$-Fourier Integral Operator associated to a local canonical graph $C$ if, again for any $A$ and $B$ above

$$ATB \in I^0(h \times h; \tilde{C'}),$$

where $C$ needs to be defined only near $U \times V$.

We say that $P = Q$ microlocally near $U \times V$ if $APB - AQB = \mathcal{O}_{L^2 \to L^2}(h^\infty)$, where because of the assumed pre-compactness of $U$ and $V$ the $L^2$ norms can be replaced by any other norms. For operator identities this will be the meaning of equality of operators in
this paper, with \( U, V \) specified (or clear from the context). Similarly, we say that \( B = T^{-1} \)
 microlocally near \( U \times V \), if \( BT = I \) microlocally near \( U \times U \), and \( TB = I \) microlocally
 near \( V \times V \). More generally, we could say that \( P = Q \) microlocally on \( W \subset T^*X \times T^*X \)
 (or, say, \( P \) is microlocally defined there), if for any \( U, V, U \times V \subset W \), \( P = Q \) microlocally
 in \( U \times V \).

If the open sets \( U \) or \( V \) in (3.6) are small enough, so that they can be identified with
 neighbourhoods of points in \( T^*\mathbb{R}^n \), we can use that identification to state that \( T \) is microlo-
 cally defined near, say, \((m, (0, 0)), m \in T^*X, (0, 0) \in T^*\mathbb{R}^n \).

To give a useful example of this formalism we state a semiclassical version of Egorov’s theorem.

**Proposition 3.4.** Suppose that \( F \) is an \( h \)-Fourier integral operator, microlocally defined
 near \( U \times V \subset T^*X \times T^*X \), and associated to a locally defined canonical transformation \( \kappa : V \to U \),
 and elliptic near \((\kappa(\rho), \rho) \in U \times V \). Let \( F^{-1} \) be the microlocal inverse of \( F \)
 near \((\kappa(\rho), \rho) \). Then for any \( A \in \Psi^{m,k}(X) \),

\[
F^{-1} \circ A \circ F = B \in \Psi^{m,k}(X), \quad \sigma_\hbar(B) = \sigma_\hbar(A) \circ \kappa,
\]

microlocally near \( \rho \in T^*X \).

The proof is a localized adaptation of the argument giving (3.5) and the observation
 recalled above that any canonical transformation with a fixed point (as we can assume that
 \( \kappa(m) = m \)) can be deformed to the identity.

### 3.3. \( S_{1/2} \) spaces with two parameters.

We define the following symbol class:

\[
a \in S^{m,\tilde{m},k}_{1/2}(T^*\mathbb{R}^n) \iff |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha \beta} h^{-\tilde{m}} \tilde{h}^{-m} \left( \frac{\hbar}{\hbar} \right)^{\frac{1}{2}(|\alpha| + |\beta|)} |\xi|^{k-|\beta|},
\]

where in the notation we suppress the dependence of \( a \) on \( h \) and \( \hbar \). We define the Weyl
 quantization of \( a \) in the usual way

\[
a^w(x, hD_x)u = \frac{1}{(2\pi h)^n} \int a \left( \frac{x + y}{2}, \xi \right) e^{\frac{i}{h}(x-y,\xi)} u(y) dy d\xi,
\]

and the standard results (see [10]) show that if \( a \in S^{m,\tilde{m},k}_{1/2}(T^*\mathbb{R}^n) \) and \( b \in S^{m',\tilde{m}',k'}_{1/2}(T^*\mathbb{R}^n) \)
 then

\[a(x, hD_x) \circ b(x, hD_x) = c(x, hD_x) \quad \text{with} \quad c \in S^{m+m',\tilde{m}+\tilde{m}',k+k'}_{1/2}(T^*\mathbb{R}^n).\]

The presence of the additional parameter \( \hbar \) allows us to conclude that

\[c \equiv \sum_{|\alpha| < M} \frac{1}{\alpha!} \partial^\alpha_x a D_x^\alpha b \mod S^{m+m',\tilde{m}+\tilde{m}-M,k+k'-M}_{1/2}(T^*\mathbb{R}^n),\]
that is, we have a symbolic expansion in powers of \( \tilde{h} \). We could also consider an expansion in the Weyl quantization – see (3.11).

We denote our class of operators by \( \Psi^m_{\tilde{m},k}(T^*\mathbb{R}^n) \). For simplicity we will only state the characterization à la Beals for a simpler class of symbols:

**Lemma 3.5.** Suppose that \( A : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n) \). Then \( A = \text{Op}_h^w(a) \) with

\[
(3.9) \quad \partial_x^\alpha \partial_\xi^\beta a = \mathcal{O}(h^{-m}\tilde{h}^{-\tilde{m}}) \left( \frac{\tilde{h}}{h} \right)^{\frac{1}{4}(|\alpha| + |\beta|)},
\]

if and only if for any sequence \( \{\ell_j\}_{j=1}^N \) of linear functions on \( T^*\mathbb{R}^n \) we have

\[
\| \text{ad}_{\text{Op}_h^w(\ell_1)} \circ \cdots \circ \text{ad}_{\text{Op}_h^w(\ell_N)} A u \|_{L^2(\mathbb{R}^n)} \leq C h^{-m-N/2} \tilde{h}^{-\tilde{m}+N/2} \| u \|_{L^2(\mathbb{R}^n)},
\]

for any \( u \in \mathcal{S}(\mathbb{R}^n) \).

**Proof.** We can assume that \( m = \tilde{m} = 0 \). The statement follows from the proof in [10, Chapter 8] and a rescaling:

\[
(\tilde{x}, \tilde{\xi}) = \left( \frac{h}{\tilde{h}} \right)^{\frac{1}{2}} (x, \xi).
\]

In fact, we define the following unitary operator on \( L^2(\mathbb{R}^n) \):

\[
U_{h, \tilde{h}} u(\tilde{x}) = \left( \frac{\tilde{h}}{h} \right)^{\frac{1}{2}} u((h/\tilde{h})^{\frac{1}{2}} \tilde{x}),
\]

for which we can check that

\[
a(x, hD_x) = U_{h, \tilde{h}}^{-1} a_{h, \tilde{h}}(\tilde{x}, \tilde{h}D_{\tilde{x}}) U_{h, \tilde{h}}, \quad a_{h, \tilde{h}}(\tilde{x}, \tilde{\xi}) = a((h/\tilde{h})^{\frac{1}{2}} (\tilde{x}, \tilde{\xi})).
\]

Clearly \( a \) satisfies (3.9) if and only if \( a_{h, \tilde{h}} \in C^\infty_b(T^*\mathbb{R}^n) \). The Beals condition for \( \tilde{h} \)-pseudodifferential operators is

\[
\| \text{ad}_{\ell_j}(\tilde{x}, \tilde{h}D_{\tilde{x}}) \circ \cdots \circ \text{ad}_{\ell_N}(\tilde{x}, \tilde{h}D_{\tilde{x}}) a_{h, \tilde{h}}(\tilde{x}, \tilde{h}D_{\tilde{x}}) u \|_{L^2} \leq C \tilde{h}^N \| u \|_{L^2}.
\]

But this is the condition in the lemma since we should take

\[
\ell_j = (\ell_j)_{h, \tilde{h}} = (h/\tilde{h})^{\frac{1}{2}} \ell_j,
\]

and this completes the proof. \( \square \)

We remark that the proof given in [13, 10, Chapter 8] is recalled, in a more complicated setting, in the proof of Proposition 5.2 below. It is based on showing that

\[
\forall \alpha \in \mathbb{N}^{2n}, \quad (\| (\partial^\alpha a)(x, D) u \|_{L^2} \leq C_\alpha \| u \|_{L^2} \iff \forall \beta \in \mathbb{N}^{2n}, \quad \sup \| \partial^\beta a \| \leq C'_\beta,
\]

which in the setting presented in Lemma 3.5 becomes

\[
\forall \alpha \in \mathbb{N}^{2n}, \quad (\| (\partial^\alpha a)(x, hD) u \|_{L^2} \leq C_\alpha (\tilde{h}/h)^{|\alpha|/2} \| u \|_{L^2} \iff \forall \beta \in \mathbb{N}^{2n}, \quad \sup \| \partial^\beta a \| \leq C'_\beta (\tilde{h}/h)^{|\alpha|/2}.
\]

(3.10)
We will also need the following application of the semi-classical calculus:

**Lemma 3.6.** Suppose that $\partial^\alpha a$, $\partial^\alpha b = O_a((\hbar/h)|^a/2)$, and that $c^w(x, hD) = a^w(x, hD) \circ b^w(x, hD)$. Then

$$
(3.11) \quad c(x, \xi) = \sum_{k=0}^{N} \frac{1}{k!} \left( \frac{ih}{2} \sigma(D_x, D_\xi; D_y, D_\eta) \right)^k a(x, \xi)b(y, \eta)|_{x=y, \xi=\eta} + O_N(x, \xi),
$$

where for some $M$

$$
(3.12) \quad |\partial^\alpha e_N| \leq C_N h^{N+1} \times \sup_{\alpha_1 + \alpha_2 = \alpha} \sup_{(x, \xi) \in T^*\mathbb{R}^n} \sup_{\beta \in \mathbb{N}^{2n}} \left| (h^{\frac{1}{2}} \partial_{(x, \xi, \eta, \eta)} \beta) (i\sigma(D)/2)^{N+1} \partial^{\alpha_1} a(x, \xi) \partial^{\alpha_2} b(y, \eta) \right|,
$$

where $\sigma(D) = \sigma(D_x, D_\xi; D_y, D_\eta)$.

**Proof.** This follows from from the standard estimates of symbolic calculus (see [10, Proposition 7.6]): suppose that $A(D)$ is a non-degenerate real quadratic form. Then there exists $M$ such that

$$
|\partial^{\alpha} \exp(iA(D)) a(x, \xi)| \leq C \sum_{|\beta| \leq M} \sup_{(x, \xi) \in T^*\mathbb{R}^n} |\partial^{\alpha+\beta} a(x, \xi)|.
$$

We observe that a rescaling $\tilde{x} = x/\sqrt{s}$, $s > 0$, shows that

$$
|\partial^{\alpha} \exp(isA(D)) a(x, \xi)| \leq C \sum_{|\beta| \leq M} \sup_{(x, \xi) \in T^*\mathbb{R}^n} |\partial^{\alpha} (\sqrt{s} \partial)^{\beta} a(x, \xi)|.
$$

To obtain an expansion we use the Taylor expansion:

$$
\exp(i\hbar A(D)) = \sum_{k=0}^{N} \frac{(ihA(D))^k}{k!} + \frac{1}{N!} \int_0^1 (1-t)^N \exp(i\hbar A(D))(ihA(D))^{N+1} dt.
$$

In the notation of the lemma and with $A(D) = \sigma(D_x, D_\xi; D_y, D_\eta)/2$,

$$
\exp(iA(D)) a(x, \xi) b(y, \eta)|_{x=y, \eta=\xi},
$$

and the lemma follows. \hfill \square

As a particular consequence we notice that if $a \in S^0,0,-\infty(T^*\mathbb{R}^n)$ and $b \in S^0,-\infty(T^*\mathbb{R}^n)$ then

$$
a^w(x, hD) \circ b^w(x, hD) = c^w(x, hD) \circ b^w(x, hD) = c^w(x, hD),
$$

and the usual pseudodifferential calculus allows a remainder improvement to

$$
O(h^{N+1-\frac{N+1}{2}})(\xi^{-\infty}).
$$
3.4. **One parameter groups of elliptic operators.** We recall a special case of a result of Bony and Chemin \[4, \text{Théorème 6.4}\]. Let $m(x, \xi)$ be an order function in the sense of \[10\]:

\[
m(x, \xi) \leq Cm(y, \eta)((x - y, \xi - \eta))^N.
\]

The class of symbols, $S(m)$, corresponding to $m$ is defined as

\[
a \in S(m) \iff |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta}m(x, \xi).
\]

If $m_1$ and $m_2$ are order functions in the sense of (3.13), and $a_j \in S(m_j)$ then (we put $h = 1$ here),

\[
a_1^w(x, D)a_2^w(x, D) = b^w(x, D), \quad b \in S(m_1m_2),
\]

with $b$ given by the usual formula,

\[
b(x, \xi) = a_1 \# a_2(x, \xi)
\]

\[
\text{def} \quad \exp(i\sigma(D_{x^1}, D_{\xi^1}; D_{x^2}, D_{\xi^2})/2)a_1(x^1, \xi^1)a_2(x^2, \xi^2)|_{x^1 - x^2 = x, \xi^1 = \xi^2}.
\]

A special case of \[4, \text{Théorème 6.4}\] gives

**Proposition 3.7.** Let $m$ be an order function in the sense of (3.13) and suppose that $G \in C_\infty(T^*\mathbb{R}^n; \mathbb{R})$ satisfies

\[
G(x, \xi) - \log m(x, \xi) = O(1), \quad \partial_x^\alpha \partial_\xi^\beta G(x, \xi) = O(1), \quad |\alpha| + |\beta| \geq 1.
\]

Then

\[
\exp(tG^w(x, D)) = B_t^w(x, D), \quad B_t \in S(m^t).
\]

Here $\exp(tG^w(x, D))$ is constructed using spectral theory of bounded self-adjoint operators. The estimates on $B_t \in S(m^t)$ depend only on the constants in (3.15) and in (3.13). In particular they are independent of the support of $G$.

In Appendix at the end of the paper we give a simple direct proof of this proposition. We should stress that the main difficulties in \[4\] came from considering general Weyl calculi of pseudodifferential operators. Here we need only the case of the simplest metric $g = dx^2 + d\xi^2$.

3.5. **Review of complex scaling.** We very briefly recall the procedure described in \[30\]. It follows the long tradition of the complex scaling method – see \[31\] for the presentation for compactly supported perturbations and references to earlier work.

Let $\Gamma_{\theta} \subset \mathbb{C}^n$ be a totally real contour with the following properties:

\[
\Gamma_{\theta} \cap B_{\mathbb{C}^n}(0, R_0) = B_{\mathbb{R}^n}(0, R_0),
\]

\[
\Gamma_{\theta} \cap \mathbb{C}^n \setminus B_{\mathbb{C}^n}(0, 2R_0) = e^{i\theta}\mathbb{R}^n \cap \mathbb{C}^n \setminus B_{\mathbb{C}^n}(0, 2R_0),
\]

\[
\Gamma_{\theta} = \{x + if_{\theta}(x) : x \in \mathbb{R}^n\}, \quad \partial_x f_{\theta}(x) = O(\theta).
\]
The contour can be considered as a deformation of the manifold $X$ as nothing is being done in the compact region. The operator $P$ defines a dilated operator:

$$P_\theta \overset{\text{def}}{=} P|_{\Gamma_\theta}, \quad P_\theta u = \widetilde{P}(\tilde{u})|_{\Gamma_\theta},$$

where $\widetilde{P}$ is the holomorphic continuation of the operator $P$, and $\tilde{u}$ is an almost analytic extension of $u \in \mathcal{C}_c^\infty(\Gamma_\theta)$ (here we are only concerned with $\Gamma_\theta \cap B_{\mathbb{C}^n}(0, R_0)$).

For $\theta$ fixed, the scaled operator, $P_\theta$, is uniformly elliptic in $\Psi^0(X)$ outside a compact set (see (4.9) below) and hence the resolvent, $(P_\theta - z)^{-1}$, is meromorphic for $z \in D(0, 1/C)$. We can also take $\theta$ to be $h$ dependent and the same statement holds for $z \in D(0, \theta/C)$. The spectrum of $P_\theta$ in $z \in D(0, \theta/C)$ is independent of $\theta$ and consists of quantum resonances of $P$ which are defined as the poles of the meromorphic continuation of

$$(P - z)^{-1} : \mathcal{C}_c^\infty(X) \rightarrow \mathcal{C}^\infty(X).$$

In fact, that is one of the ways of defining resonances, and in this paper we will be estimating the number of eigenvalues of $P_\theta$.

4. Resonance free regions under the non-trapping assumption

4.1. Estimates using weight functions. We follow the presentation given in [9, §4.2] and inspired by many previous works, including [20].

Let us suppose that $P_\theta \in \Psi^{0,2}(X)$ (we identify $X$ and $X_\theta$ here) is a complex scaled operator with $\theta = M_1 h \log(1/h)$. We choose $\epsilon$

$$\epsilon \leq M_2 h \log \frac{1}{h},$$

where $M_2 > M_1$ is a large constant to be fixed later.

Let $G \in \mathcal{C}_c^\infty(T^*X)$ and define

$$P_{\epsilon, \theta} \overset{\text{def}}{=} e^{-\epsilon G/h} P_\theta e^{\epsilon G/h} = e^{-\frac{1}{h} \text{ad}_G} P_\theta \sim \sum_{k=0}^\infty \frac{\epsilon^k}{k!} (-\frac{1}{h} \text{ad}_G)^k(P_\theta), \quad G = G^w(x, hD).$$

We note that the assumption on $\epsilon$ and the boundedness of $\text{ad}_G/h$ show that the expansion makes sense. The operators $\exp(\epsilon G/h)$ are pseudo-differential in an exotic class $\mathcal{S}_\delta^{C_2}$ for any $\delta > 0$ (see [33]) but that is not relevant here.

Using the same letters for operators and and the corresponding symbols, we see that

$$P_{\epsilon, \theta} = P_\theta - i\epsilon \{P_\theta, G\} + \mathcal{O}(\epsilon^2) = P_\theta - i\epsilon \{p_\theta, G\} + \mathcal{O}(h + \epsilon^2),$$

so that

$$\text{Re} P_{\epsilon, \theta} = \text{Re} p_\theta + \epsilon \{\text{Im} p_\theta, G\} + \mathcal{O}(h + \epsilon^2) = \text{Re} p_\theta + \mathcal{O}(h + \theta \epsilon + \epsilon^2),$$

$$\text{Im} P_{\epsilon, \theta} = \text{Im} p_\theta - \epsilon \{\text{Re} p_\theta, G\} + \mathcal{O}(h + \epsilon^2).$$
We now make the following assumption: for a fixed $\delta > 0$
\begin{equation}
|\text{Re} p_\theta| < \delta \implies -\text{Im} p_\theta + \epsilon H_\rho G \geq c_0 \epsilon .
\end{equation}

Now let $\psi_1, \psi_2 \in C^\infty_b(T^* X)$ be two functions satisfying
\begin{equation}
\psi_1 + \psi_2^2 = 1, \quad \psi_1|_{|\text{Re} p_\theta| < \delta/2} = 1, \quad \text{supp } \psi \subset \{|\text{Re} p_\theta| < \delta\} .
\end{equation}

Lemma 3.2 gives two selfadjoint operators $\Psi_1$ and $\Psi_2$ with principal symbols $\psi_1$ and $\psi_2$ respectively, such that
\begin{equation}
\Psi_2 + \Psi_1 = I + R, \quad R = O(h^{\infty}) : H^{-M}(X) \to H^M .
\end{equation}

We then write $P_{\epsilon, \theta} = A_{\epsilon, \theta} + iB_{\epsilon, \theta}$, where
\begin{equation}
A_{\epsilon, \theta} = \frac{1}{2}(P_{\epsilon, \theta} + P^*_{\epsilon, \theta}), \quad B_{\epsilon, \theta} = \frac{1}{2i}(P_{\epsilon, \theta} - P^*_{\epsilon, \theta}) .
\end{equation}

The principal symbol of $B_{\epsilon, \theta}$ is given by $\text{Im} p_\theta - \epsilon H_\rho G$ and on the essential support of $\Psi_1$ it is bounded below by $c_0 \epsilon \gg h$. Hence the sharp Gårding inequality (see [10, Theorem 7.12]) implies that for $h$ small enough
\begin{equation}
\|P_{\epsilon, \theta} \Psi_1 u\| \geq \frac{\epsilon}{C} \|\Psi_1 u\| ,
\end{equation}

and hence
\begin{equation}
\|P_{\epsilon, \theta} \Psi_1 u\| \geq \frac{\epsilon}{C} \|\Psi_1 u\| .
\end{equation}

On the support of $\psi_2$ the operator $A_{\epsilon, \theta}$ is elliptic and by Lemma 3.1
\begin{equation}
\|P_{\epsilon, \theta} \Psi_2 u\| \geq \frac{1}{C} \|\Psi_2 u\| - O(h^{\infty})\|u\| .
\end{equation}

We conclude from Lemma 3.3 that
\begin{equation}
\|P_{\epsilon, \theta} u\| \geq \frac{\epsilon}{C} \|u\| .
\end{equation}

This shows that the conjugated operator has no spectrum in $D(0, \epsilon/(2C))$.

4.2. Construction of an escape function. Using the results of §4.1 all we need to do is to construct $G$ so that (4.2) holds. For that we modify a standard argument with the presentation borrowed in part from [36, Sect.4].

We recall that (4.1) implies the same condition with $p^{-1}([-\epsilon_0, \epsilon_0])$ for some $\epsilon_0 > 0$. That follows from the compactness of the trapped set in $p^{-1}([-\delta, \delta])$ – see [11, Appendix] for a detailed discussion.

Let us now fix $R$ a large parameter. We will define $G_\rho \in C^\infty_c(T^* X)$, a local escape function supported in a neighbourhood of the bicharacteristic segment
\begin{equation}
I_\rho = \{\exp(tH_\rho)(\rho) : t \in [-T, T]\} ,
\end{equation}
and which satisfies $H_p G_\rho \geq 1$ on the part of $I_\rho$ lying over

\[(4.3)\]

$K' = \{ \rho' \in T^*X : |x(\rho')| \leq R \}$

For that, let $\Gamma$ be a hypersurface through $\rho$ which is transversal to $H_p$. Then there is a

neighbourhood $U_\rho$ of $\rho$, such that

\[V_\rho = \{ \exp(t(U_\rho \cap \Gamma)) : t \in (-T - 1, T + 1) \} \subset p^{-1}([-\epsilon_0/2, \epsilon_0/2]) , \]

is a neighbourhood of $I_\rho$. That neighbourhood can be identified with a product,

\[V_\rho \simeq (-T - 1, T + 1) \times (U_\rho \cap \Gamma) , \]

and, in this identification, we will choose $T$ and $0 < \alpha < 1$ so that

\[(((-T - 1, -\alpha T) \cup (\alpha T, T + 1)) \times (U_\rho \cap \Gamma)) \cap K' = \emptyset . \]

We now need the following elementary

Lemma 4.1. For any $0 < \alpha < 1/2$ and $T > 0$ there exist as function $\chi = \chi_{T, \alpha} \in C^\infty(\mathbb{R}; \mathbb{R})$ such that

\[\chi(t) = \begin{cases} 0 & |t| > T \\ t & |t| < \alpha T \end{cases}, \quad \chi'(t) \geq -2\alpha . \]

Proof. The piecewise linear function

\[\chi_\#(t) = \begin{cases} 0 & |t| > T \\ t & |t| < \alpha T \\ \pm\alpha(T - t)/(1 - \alpha) & \alpha T \leq \pm t \leq T \end{cases} \]

satisfies $\chi_\#' \geq -\alpha/(1 - \alpha) > -2\alpha$ wherever the derivative is defined. A regularization of

this function gives $\chi_{T, \alpha}$. \hfill \Box \]

Now let $\phi_\rho \in C^\infty_c(U_\rho \cap \Gamma)$ be identically 1 near $\rho$, and let $\chi_T$ be given by the lemma.

Using the product coordinates, we can think of $\phi_\rho$, $t$, and hence $\chi(t)$, as functions on $T^*X$. The functions $\phi_\rho$ and $\chi_T(t)$ have compact support in $V_\rho$. Let

\[\psi \in C^\infty_c((-\epsilon_0, \epsilon_0)), \quad \psi|_{[-\epsilon_0/2, \epsilon_0/2]} \equiv 1 , \]

and put

\[(4.4)\]

$G_\rho = \chi_T(t)\phi_\rho \psi(p) , \quad G_\rho \in C^\infty_c(V_\rho) . \]

so that

\[(4.5)\]

$H_p G_\rho = \chi_T^\prime \phi_\rho \psi(p) , \]

satisfies

\[H_p G_\rho = 1 \text{ on } V_\rho \cap \{|x| < R\} \text{ and } H_p G_\rho \geq -2\alpha \text{ everywhere.} \]

Now let $K \subset T^*X$ be the compact set

\[(4.6)\]

$K = \{ \rho \in p^{-1}([-\epsilon_0/3, \epsilon_0/3]) : |x(\rho)| \leq R/2 \}.$
Since $K$ is compact, applying the previous argument for every $\rho \in K$ gives a $U_{\rho}$, and a $U'_\rho \subset U_\rho$ on which $\phi_\rho = 1$. Since $\{U'_\rho : \rho \in K\}$ covers $K$, the compactness of $K$ shows that we can pass to a finite subcover, $\{U'_\rho : j = 1, \ldots, N\}$. We let

$$G = \sum_{j=1}^N G_{\rho_j}. \tag{4.7}$$

The construction of $G_{\rho_j}$'s now shows that by choosing $\alpha$ small enough (depending on the maximal number of support overlaps we obtain

$$H_p G(\rho) \geq 1, \rho \in p^{-\epsilon_0/2} \cap \{|x(\rho)| < R\} \text{ and } H_p G(\rho) \geq -\delta, \rho \in T^*X. \tag{4.8}$$

4.3. **Resonance free region.** We now want to choose the scaling so that (4.2) holds with $G$ satisfying (4.8). Once that is done the results of §4.1 will give Theorem 3.

For that we choose the complex scaling so that

$$-\text{Im} p_\theta(x, \xi) \geq \theta \text{ when } |p(x, \xi)| \leq \epsilon_0 \text{ and } |x| \geq R,$$

$$\text{Im} p_\theta \leq C_1 \theta \text{ when } |p(x, \xi)| \leq \epsilon_0,$$

where $R$ is independent of $\theta$. With $\epsilon = M_2h \log(1/h)$ we now choose $\theta = M_1h \log(1/h)$ such that

$$M_1 < M_2/C_1, \quad \delta M_2 < M_1,$$

where $C_1$ comes from (4.9) and $\delta$ comes from (4.8). Since we can choose $\delta$ as small as we want this can certainly be arranged leading to (4.2).

For completeness we include a quantitative corollary of Theorem 1 from [24].

**Theorem 1.** Suppose that the assumptions of Theorem 1 are satisfied and that $(P_\theta - z)^{-1}$ is the scaled resolvent defined for $0 < \theta < 2M_2h \log h$, $M \gg 1$. Then for $0 < h < h_0(M)$ we have

$$\|(P_\theta - z)^{-1}\|_{L^2(\Gamma_\phi) \to L^2(\Gamma_\phi)} = C \exp(C|\text{Im} z|/h), \quad z \in D(0, Mh \log(1/h)).$$

Theorem 1 is essentially optimal as shown by the well known one dimensional result going back to Regge (see [58] for a proof and references): if $V \in C^N([a, b])$ is extended by 0 to a potential on $\mathbb{R}$, and

$$V(x) \sim \begin{cases} 
(x - a)^p & x \approx a+ \\
(x - b)^q & x \approx b-. \end{cases} \quad p, q < N,$$

then the scattering poles for $-\Delta + V(x)$ are given at high energies by the sequence

$$\lambda_k = \frac{\pi k}{b - a} - i\alpha\log |k| + \mathcal{O}(1), \quad k \in \mathbb{Z}, \quad \alpha = \frac{p + q + 4}{2(b - a)}.$$
The semiclassical resonances, \( z_k(h) \), of \(-h^2 \Delta + h^2 V(x)\) are related to these scattering poles by the formula \( z_k(h) = h^2 \lambda_k^2 \). Hence

\[
\text{Re } z_k(h) \sim 1 \implies \text{Im } z_k(h) \sim h \log(1/h).
\]

5. Second microlocal calculus associated to a hypersurface

To obtain Theorem 2 we need to localize to an \( h \)-size neighbourhood of the energy surface

\[
\Sigma \overset{\text{def}}{=} \left\{ (x, \xi) \in T^* X : p(x, \xi) = 0 \right\}.
\]

That means that we have to work with functions of the form

\[
a(x, \xi; h) = \psi(p(x, \xi)/h).
\]

The usual quantization procedure (the passage from symbols to pseudodifferential operators) is prohibited as

\[
\partial_{x, \xi}^\alpha a \sim h^{-|\alpha|}.
\]

The troublesome symbols have a special form and we can construct a calculus which includes them by straightening \( \Sigma \) locally by means of canonical transformations. That means moving \( \Sigma \) to

\[
\Sigma_0 = \{ \xi_1 = 0 \}.
\]

We may then localize to rectangles in the \((x_1, \xi_1)\)-space of length \( \sim 1 \) in \( x_1 \) and of length \( \sim h \) in \( \xi_1 \). This amounts to a form of semiclassical second microlocalization. The presentation here is essentially self-contained and we refer to \([32]\) for pointers to the literature.

For Theorem 2 we need even more singular calculus related to the \( \Psi^1_+ \) calculus described in \([33]\).

We assume that \( \Sigma \) is a compact \( C^\infty \) hypersurface in \( T^* X \). Since the delicate constructions will only be used in a compact set and since we are working in the \( C^\infty \) category this creates no restrictions.

5.1. Basic properties. To construct the calculus, let \( \Sigma \in T^* X \) be a \( C^\infty \) compact hypersurface. We consider a class of symbols associated to \( \Sigma \), a multiindex

\[
\mathbf{p} \overset{\text{def}}{=} (m, \bar{m}, k_1, k_2),
\]

and depending on \textit{two} small parameters,

\[
0 < h < \tilde{h}.
\]
\[ \partial \]

Following \[ \Psi \]

Lemma 5.1. Lemma 5.1. Definition of \((5.1)\) is independent of the choice of \(q\) and the vector fields applied to \(a\) can be taken in any order.

Proof. The independence of the order of vector fields follows from the fact that \([V_q, H_u]\) is a vector field tangent to \(\Sigma\). To see independence of \(q\) we note that \(H_{wq} = uH_q + qH_u\). The vector field \(H_u\) can be considered as an arbitrary vector field and \(qH_u\) is tangent to \(\Sigma\). Hence the application of the second term is estimated by

\[
|qH_u a| \leq C \min(|q|h^{-1}h^{-\delta}), h^{-\delta}) \leq C^\prime \langle h^{-1}q \rangle^\delta,
\]

and this estimate can be iterated. Hence we have the same estimates for \(q\) replaced by \(uq\), \(u \neq 0\) near \(\Sigma\). \(\square\)

The symbol classes in \((5.1)\) are best understood in the simple case when \(q = \xi_1\), say. The condition for \(|\xi| \leq C\), means that, with \(\xi = (\xi_1, \xi')\),

\[
(5.2) \quad \partial_{x_i}^\alpha \partial_{x^j}^\beta (\xi_1 \xi_i) \partial_{x_j} \partial_{\xi_i} \partial_{\xi_j} a = \mathcal{O} \left( h^{-m - \tilde{m}} \langle \tilde{h}/h \rangle^{\delta(|\alpha| + |\beta|)} \right) \left( (\tilde{h}/h) \xi_1 \right)^{k_2 + \tilde{k} - (1 - \delta)f_2},
\]

or, if we eliminate the vector fields vanishing at \(\xi_1 = 0\),

\[
(5.3) \quad \partial_{x_1}^\alpha \partial_{x'}^\beta \partial_{\xi_1}^\delta ((h/\tilde{h}) \partial_{\xi_1}) \partial_{\xi_1} a = \mathcal{O} \left( h^{-m - \tilde{m}} \langle \tilde{h}/h \rangle^{\delta(|\alpha| + |\beta|)} \right) \left( (\tilde{h}/h) \xi_1 \right)^{k_2 - (1 - \delta)p + \tilde{k}}.
\]

We used the fact that if \(|\xi_1| \leq C\) then

\[
(\tilde{h}/h) \xi_1^{\delta-1} \leq (\tilde{h}/h) \xi_1^{-1} (\tilde{h}/h) \delta,
\]

to eliminate the need for the \(\xi_1 \partial_{\xi_1}\). The advantage of the formulation in \((5.1)\) and \((5.2)\) is the geometric invariance. In explicit coordinates \((5.3)\) is however more transparent.

Since it is sufficient for our purposes, and for simplicity of presentation we will consider the case of \(\delta = 1/2\) only.

The second microlocalization associates to the space of symbols \(\Psi^p_{\Sigma, T^* X}\), a space of operators, \(\Psi^p_{\Sigma, T^* X}\), defined in \((5.4)\) below. The basic properties of these spaces are described in the following...
Theorem 4. Let us define two multiindices,\[ \mathbf{p} = (m, \tilde{m}, k_1, k_2), \quad \mathbf{p}' = (m, \tilde{m} - 1, k_1 - 1, k_2). \]
With the definitions of \( S^\mathbf{p}_{\Sigma, \delta} \) above and \( \Psi^\mathbf{p}_{\Sigma, \delta} \) in \([5, 4]\) below, there exist maps
\[ \text{Op}_{\Sigma, h, \tilde{h}} : S^\mathbf{p}_{\Sigma, \frac{1}{2}}(T^*X) \rightarrow \Psi^\mathbf{p}_{\Sigma, \frac{1}{2}}(X) \]
\[ \sigma_{\Sigma, h, \tilde{h}} : \Psi^\mathbf{p}_{\Sigma, \frac{1}{2}}(X) \rightarrow S^\mathbf{p}_{\Sigma, \frac{1}{2}}(T^*X)/S^\mathbf{p}'_{\Sigma, \frac{1}{2}}(T^*X) \]
such that
\[ (5.4) \quad \sigma_{\Sigma, h, \tilde{h}}(A \circ B) = \sigma_{\Sigma, h, \tilde{h}}(A) \sigma_{\Sigma, h, \tilde{h}}(B) \]
\[ 0 \rightarrow \Psi^\mathbf{p}_{\Sigma, \frac{1}{2}}(X) \rightarrow \Psi^\mathbf{p}_{\Sigma, \frac{1}{2}}(X) \xrightarrow{\sigma_{\Sigma, h, \tilde{h}}} S^\mathbf{p}_{\Sigma, \frac{1}{2}}(T^*X)/S^\mathbf{p}'_{\Sigma, \frac{1}{2}}(T^*X) \rightarrow 0 \]
is a short exact sequence and
\[ \sigma_{\Sigma, h, \tilde{h}} \circ \text{Op}_{\Sigma, h, \tilde{h}} : S^\mathbf{p}_{\Sigma, \frac{1}{2}}(T^*X) \rightarrow S^\mathbf{p}_{\Sigma, \frac{1}{2}}(T^*X)/S^\mathbf{p}'_{\Sigma, \frac{1}{2}}(T^*X) \]
is the natural projection map. If \[ a \in S^\mathbf{p}_{\Sigma, \frac{1}{2}}(T^*X), \quad d(\text{supp} a, \Sigma) \geq 1/C \]
then
\[ a \in S_{\frac{1}{2}}^{m+k_2, \tilde{m}-k_1, k_1}(T^*X), \quad \text{Op}_{\Sigma, h, \tilde{h}}(a) = \text{Op}_{\tilde{h}}^\mathbf{p}(a) \in \Psi_{\tilde{h}}^{m+k_2, \tilde{m}-k_1, k_1}(X), \]
where \( \Psi_{\tilde{h}}^\mathbf{p}(X) \) is the class of pseudodifferential operators defined in \([8, 3]\).

5.2. Calculus in the model case. To define \( \Psi^\mathbf{p}_{\Sigma, \frac{1}{2}}(X) \) we proceed locally and put \( \Sigma \) into a normal form \( \Sigma_0 = \{ \xi_1 = 0 \} \) (locally).

The model case is obtained by taking symbols satisfying \([5, 3]\) globally and defining
\[ a = a(x, \xi, \lambda, h, \tilde{h}), \quad \lambda = \tilde{h} \xi_1/h, \]
satisfying
\[ (5.5) \quad \partial_x^k \partial_{\xi}^\alpha \partial_{\tilde{h}}^\beta \partial_\lambda^\rho a(x, \xi, \lambda; h) = O(h^{-m} \tilde{h}^{-\tilde{m}})(\tilde{h}/h)^{(|\alpha|+|\beta|)/2} \lambda^{k_2+k/2-n/2}, \]
which is the same as \([5, 1]\). We will write \([5, 3]\) as
\[ a \in \tilde{O}_{\frac{1}{2}} \left( h^{-m} \tilde{h}^{-\tilde{m}} \lambda^{k_2} \right). \]
and define an exact quantization in the usual way,
\[ \tilde{\text{Op}}_{h, \tilde{h}}(a) u(x) = \]
\[ (5.6) \quad \frac{1}{2\pi h (2\pi h)^{n-1}} \int a(x, \xi', (h/\tilde{h})\lambda, \lambda; h) e^{i(x'-y, \xi')/(h+i(x_1-y_1, \lambda)/\tilde{h})} u(y) dy dyd\lambda, \]
n = \text{dim } X, \quad \text{and where} \]
\[ \lambda = (\tilde{h}/h) \xi_1. \]
For \( a \in \mathcal{O}(\langle \lambda \rangle^{k_2}) \) and \( b \in \mathcal{O}(\langle \lambda \rangle^{k_2}) \) we have
\[
\tilde{\text{Op}}_{h,\hat{h}}(a) \circ \tilde{\text{Op}}_{h,\hat{h}}(b) = \tilde{\text{Op}}_{h,\hat{h}}(a\hat{h}^{-1}b), \quad a \hat{h}^{-1}b = \mathcal{O}
\]
(5.7)
where the asymptotic sum is defined up to terms in
\[
\mathcal{O}^2(\langle \lambda \rangle^{k_2+k_2}).
\]
To see this we write \( \tilde{\text{Op}}_{h,\hat{h}}(a) \) as a quantization of an operator valued symbol, \( \text{Op}_h(a)(x_1, \lambda) \),
\[
\text{Op}_h(a)(x_1, \lambda) = \frac{1}{(2\pi h)^{n-1}} \int a(x, \xi', (h/\hat{h})\lambda, \lambda; \hat{h}) e^{i(x'-y',\xi')/h} d\xi',
\]
so that
\[
(5.8) \quad \tilde{\text{Op}}_{h,\hat{h}}(a) = \text{Op}_h(a)(x_1, \hat{h}D_{x_1}).
\]
In view of (5.7) this symbol map is a homomorphism onto the quotient of symbol spaces:
\[
(5.9) \quad \tilde{\text{Op}}_{h,\hat{h}}(a) \mapsto [a] \in \mathcal{O}^{2}(\langle \lambda \rangle^{k_2})/\mathcal{O}^{2}(\langle \hat{h}\lambda \rangle^{k_2}).
\]
We note that in the local model we are not concerned here with the behaviour as \( |\xi| \to \infty \).

It will be useful to have the analogue of Beals’s characterization of pseudodifferential operators by stability under taking commutators. It follows from the proof of the semiclassical analogue of Beals’s result [14] and its adaptations in [10, Chapter 8] and [32, Lemma 4.2].
Proposition 5.2. Let $A = A_h : S(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n)$ and put $x' = (x_2, \ldots, x_n)$. For $p \in \mathbb{R}$, we define the norms

$$\|u\|_p = \|\tilde{h}Dx_1\|^p u\|_{L^2}.$$ 

Then

$$A = \tilde{O}_{p_h, \tilde{h}}(a)$$

for $a = \tilde{O}_\frac{1}{2}((\lambda)^k)$, $a = a(x, \xi, \lambda, h, \tilde{h})$, if and only if for all $N, p, q \geq 0$ and every sequence $\ell_1(x', \xi'), \ldots, \ell_N(x', \xi')$ of linear forms on $\mathbb{R}^{2(n-1)}$ there exists $C > 0$ for which

$$\|\text{ad}_{\ell_1(x', hD_x')} \circ \cdots \circ \text{ad}_{\ell_N(x', hD_x')} \circ \tilde{h}Dx_1\circ (\text{ad}_{x_1})^p \circ (\text{ad}_{x_1})^q Au\|_{(q/2-\min(k,0))} \leq C h^{N/2} h^{N/2+p+q} \|u\|_{p/2+(\text{max}(k,0))}.$$ 

Proof. We first observe that for $A = A = \tilde{O}_{p_h, \tilde{h}}(a)$ for $a = \tilde{O}_\frac{1}{2}((\lambda)^k)$, (5.10) follows from the calculus.

Let

$$V_h u(x_1, x') \overset{\text{def}}{=} h^{-\frac{n}{2} - \frac{1}{2}} u(x_1, h\frac{1}{2}, x')$$

so that

$$A = V_h^{-1} \tilde{A} V_h,$$

where, as in the proof of Proposition 3.5, we have

$$\|\text{ad}_{\ell_1(x', D_x')} \circ \cdots \circ \text{ad}_{\ell_N(x', D_x')} \circ (\text{ad}_{D_x})^p \circ (\text{ad}_{x_1})^q \tilde{A} u\|_{(q/2-\min(k,0))} \leq C h^{N/2} h^{N/2+p+q} \|u\|_{p/2+(\text{max}(k,0))}.$$ 

We can write $\tilde{A} = a_h(x, D_x)$, so that, $A = \tilde{O}_{p_h, \tilde{h}}(a)$, where

$$a(x, \lambda, \xi; h) = a_{h, \tilde{h}}(x_1, h^{-1/2} x', \tilde{h}^{1/2} \lambda, h^{-1/2} \xi').$$

The required estimate on $a$ becomes

$$\partial_{x_1}^p \partial_{\xi_1}^p \partial_{x'}^p \partial_{\xi'}^p a_{h, \nu} = \mathcal{O}(1) h^{(\text{max}(k,0)-1)k - \frac{1}{2} + q} u(x_1, \tilde{h}^{1/2} \lambda, h^{-1/2} \xi').$$ 

We have

$$\langle \tilde{A} \psi, \phi \rangle = \frac{1}{(2\pi)^n} \int e^{i(x, \xi)} a_{h, \nu}(x, \xi, \tilde{\psi}(\xi)) \phi(x) dx d\xi,$$

with

$$\tilde{\psi}(\xi) = (\mathcal{F} \psi)(\xi) = \int e^{-i(x, \xi)} \psi(x) dx,$$

and for $\phi, \psi \in S(\mathbb{R}^n)$. Let us fix $(x_0, \xi_0), (y_0, \eta_0) \in T^* \mathbb{R}^n$, and $\lambda \gg 1$. With $\phi, \psi \in S(\mathbb{R}^n)$ we put

$$\psi_{x_0, \xi_0}(x) = \lambda^{\frac{1}{4}} \psi \left( \lambda^{\frac{1}{2}} (x_1 - x_0), x', x'_0 \right) e^{i(x, \xi_0)},$$

$$\phi_{y_0, \eta_0}(x) = \lambda^{\frac{1}{2}} \phi \left( \lambda^{\frac{1}{2}} (x_1 - y_0), x', y'_0 \right) e^{i(x, y_0, \eta_0)}.$$
We see that
\[
\hat{\psi}_{x_0,\xi_0}(\xi) = \lambda^{-\frac{1}{2}} \psi \left( \lambda^{-\frac{1}{2}} (\xi_1 - \xi_{0,1}), \xi' - \xi'_{0} \right) e^{-i(\xi - \xi_{0,0})},
\]
\[
\hat{\phi}_{\eta_0,\eta_0}(\xi) = \lambda^{-\frac{1}{2}} \phi \left( \lambda^{-\frac{1}{2}} (\xi_1 - \eta_{0,1}), \xi' - \eta'_{0} \right) e^{-i(\xi - \eta_0)}.
\]
We have
\[
B \overset{\text{def}}{=} \text{ad}_{x'}^{\alpha'} \text{ad}_{p}^{\beta'} \text{ad}_{x_1}^{\gamma} \text{ad}_{p_1}^{\delta} A = (-i)^{\alpha' + |\beta'| + q + p} b_{h,\tilde{h}}(x, D),
\]
\[
b_{h,\tilde{h}}(x, \xi) = (-\partial_{\xi})^{\alpha'} \partial_{x'}^{\gamma} (-\partial_{\xi_1})^{q} \partial_{x_1}^{\delta} a_{h,\tilde{h}}(x, \xi).
\]
Let us now assume we have the commutator estimate in the lemma with \( k \geq 0 \). Since
\[
\|u\|_{(-q/2)} = \|\tilde{H} D_{x_1}\|_{q/2} u\|_{L^2}
\]
is the norm dual to \( \bullet \|_{(q/2)} \), we get from (5.14)
\[
|\langle B \psi_{x_0,\xi_0}, \phi_{\eta_0,\eta_0} \rangle| \leq \tilde{I}^{(\alpha' + |\beta'|)/2 + q} \|\psi_{x_0,\xi_0}\|_{p/2 + k} \|\phi_{\eta_0,\eta_0}\|_{(-q/2)}.
\]
Let \( \bullet_{0,1} \) denote the first component of \( \bullet \in \mathbb{R}^n \). For fixed \( \psi, \phi \in \mathcal{S}(\mathbb{R}^n) \), we have
\[
\|\psi_{x_0,\xi_0}\|^2_{p/2 + k} \leq C_N I^N_{p/2 + k}(\lambda, \xi_{0,1}), \quad \|\phi_{\eta_0,\eta_0}\|^2_{(-q/2)} \leq C_N I^N_{-q/2}(\lambda, \eta_{0,1})
\]
(5.15)
\[
I^N_{\gamma}(\lambda, \tau) \overset{\text{def}}{=} \lambda^{-\frac{1}{2}} \left( \int_{\mathbb{R}} (\tilde{h} \rho)^{2\gamma} (\lambda^{-\frac{1}{2}} (\rho - \tau)) - N d\rho \right)
\]
Using (5.13) we rewrite the left hand side of (5.14) as
\[
\frac{1}{(2\pi)^n} \left| \int \int e^{i(x, \xi)} b_{h,\tilde{h}}(x, \xi) \hat{\psi}_{x_0,\xi_0}(\xi) \hat{\phi}_{\eta_0,\eta_0}(x) dx d\xi \right|.
\]
Decomposing the first exponent in the integral as
\[
\langle x, \xi \rangle = \langle y_0, \xi_0 \rangle + \langle x - y_0, \xi_0 \rangle + \langle \xi - \xi_0, y_0 \rangle + \langle x - y_0, \xi - \xi_0 \rangle
\]
and using the formulæ for \( \hat{\psi}_{x_0,\xi_0}, \hat{\phi}_{\eta_0,\eta_0} \), we rewrite it further as
\[
\frac{1}{(2\pi)^n} \left| \int \int b_{h,\tilde{h}}(x, \xi) \exp(i(x - y_0, \xi - \xi_0)) \hat{\psi}(\lambda^{-\frac{1}{2}} (\xi_1 - \xi_{0,1}), \xi' - \xi'_{0}) \hat{\phi}(\lambda^{-\frac{1}{2}} (x_1 - y_{0,1}), x' - y'_{0}) \right| dx d\xi.
\]
Summing up, we get
\[
\mathcal{F}(\chi b_{h,\tilde{h}})(\eta_0 - \xi_0, x_0 - y_0) = \mathcal{O}(1) \tilde{I}^{(\alpha' + |\beta'|)/2 + q} \|\psi_{x_0,\xi_0}\|_{p/2 + k} \|\phi_{\eta_0,\eta_0}\|_{(-q/2)},
\]
(5.16)
\[
\chi(x, \xi) = e^{i(x - y_0, \xi - \xi_0)} \hat{\psi}(\lambda^{-\frac{1}{2}} (\xi_1 - \xi_{0,1}), \xi' - \xi'_{0}) \hat{\phi}(\lambda^{-\frac{1}{2}} (x_1 - y_{0,1}), x' - y'_{0}).
\]
Writing
\[
\zeta_1 \overset{\text{def}}{=} \frac{\eta_{0,1} - \xi_{0,1}}{\lambda^{\frac{1}{2}}}, \quad z_1 \overset{\text{def}}{=} \lambda^{\frac{1}{2}} (x_0,1 - y_{0,1}), \quad \zeta' = \eta'_{0} - \xi'_{0}, \quad z' = x'_{0} - y'_{0},
\]
we therefore get

\[ \mathcal{F} \left( \widetilde{b}_{h,\tilde{h}} \right)(\zeta, z) = O(1) \tilde{h}^{(\alpha' + |\beta'|)/2 + q} \|\psi_{x_0,\xi_0}\|_{(p/2+k)} \|\phi_{y_0,\eta_0}\| \left( -q/2 \right), \]

where

\[ \widetilde{b}_{h,\tilde{h}}(X, \Xi) \overset{\text{def}}{=} b_{h,\tilde{h}}(y_0 + \lambda^{-1/2}X, \xi_0 + \lambda^{1/2} \Xi), \]

and

\[ \widetilde{\chi}(X, \Xi) \overset{\text{def}}{=} e^{i\langle X, \Xi \rangle} \psi(\Xi) \bar{\phi}(X). \]

We then have

\[ \mathcal{F} \left( \partial_{\zeta}^{\tilde{q}} \partial_{\xi}^{\tilde{p}} \partial_{X}^{\tilde{\alpha}} \partial_{\Xi}^{\tilde{\beta}} \widetilde{b}_{h,\tilde{h}} \right)(\zeta, z) = \]

\[ O(1) \tilde{h}^{(\alpha' + |\beta'|)/2 + q} I_N^{(p/2+k-\tilde{p}/2)}(\lambda, \xi_0, 1) I_N^{(-q/2-\tilde{q}/2)}(\lambda, \eta_0, 1) \lambda^{-\tilde{p}/2+\tilde{q}/2}, \]

which putting

\[ \tilde{\alpha} = (\tilde{q}, \tilde{\alpha}', \tilde{\beta}) = (\tilde{p}, \tilde{\beta}') \],

we rewrite as

\[ z^\tilde{\alpha} \zeta^{\tilde{\beta}} \mathcal{F} \left( \widetilde{b}_{h,\tilde{h}} \right)(\zeta, z) = \]

\[ O(1) \tilde{h}^{(\alpha' + |\beta'|)/2 + q} I_N^{(p/2+k-\tilde{p}/2)}(\lambda, \xi_0, 1) I_N^{(-q/2-\tilde{q}/2)}(\lambda, \eta_0, 1) \lambda^{-\tilde{p}/2+\tilde{q}/2}. \]

We now go back to (5.15) and choose \( \lambda = \langle \tilde{h} \xi_0, 1 \rangle \). Then

\[ I_N^{(p/2+\tilde{p}/2+k)}(\langle \tilde{h} \xi_0, 1 \rangle, \xi_0, 1) = \left( \int_\mathbb{R} \langle \tilde{h} \xi_0, 1 + \langle \tilde{h} \xi_0, 1 \rangle \frac{1}{2} r \rangle^{p+\tilde{p}+2k} \langle r \rangle^{-N} dr \right)^{1/2} \]

\[ = C_N \left( \int_\mathbb{R} \langle R + \langle R \rangle \frac{1}{2} \tilde{h} r \rangle \rangle^{p+\tilde{p}+2k} \langle r \rangle^{-N} dr \right)^{1/2} \]

\[ \sim \langle R \rangle^{(p+\tilde{p}+2k)/2} \bigg|_{R=\tilde{h} \xi_0} = \langle \tilde{h} \xi_0, 1 \rangle^{p/2+\tilde{p}/2+k}. \]

It follows that for any \( \tilde{\alpha} \) and \( N \) we have

\[ (5.17) \ |z^\tilde{\alpha} \mathcal{F} \left( \widetilde{b}_{h,\tilde{h}} \right)(\zeta, z)| = O_N(1) \tilde{h}^{(\alpha' + |\beta'|)/2 + q} I_N^{(p/2+\tilde{p}/2+k)}(\zeta, N, \lambda = \langle \tilde{h} \xi_0, 1 \rangle). \]
We now integrate the left hand side in $\zeta_1 = (\eta_{0,1} - \xi_{0,1})/\lambda$: 

$$\int_{\mathbb{R}} I_{-q/2-\bar{q}/2} \left( \frac{\eta_{0,1} - \xi_{0,1}}{\langle \hat{h}\xi_{0,1} \rangle} \right)^{-N} \langle \hat{h}\xi_{0,1} \rangle^{-\frac{1}{2}} d\eta_{0,1}$$

$$= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \langle \hat{h}(\eta_{0,1} + \langle \hat{h}\xi_{0,1} \rangle^\frac{1}{2} r) \rangle^{-q-\bar{q}} \langle r \rangle^{-N} d\eta_{0,1} \right)^{\frac{1}{2}} \left( \frac{\eta_{0,1} - \xi_{0,1}}{\langle \hat{h}\xi_{0,1} \rangle} \right)^{-\frac{1}{2}} \langle \hat{h}\xi_{0,1} \rangle^{-\frac{1}{2}} d\eta_{0,1}$$

$$\leq C_N \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \langle \hat{h}(\xi_{0,1} + \langle \hat{h}\xi_{0,1} \rangle^\frac{1}{2} (r + \zeta_1)) \rangle^{-q/2-\bar{q}/2} \langle r \rangle^{-N} d\eta_{0,1} \right)^{\frac{1}{2}} \langle \zeta_1 \rangle^{-N} d\zeta_1$$

$$\leq \langle \hat{h}\xi_{0,1} \rangle^{-q/2-\bar{q}/2}.$$

Returning to (5.17) we see that 

$$\int_{\mathbb{R}^n} |F \left( \widetilde{b}_{h,\bar{h}} \right) (\zeta, z)| d\zeta = O(1)\langle \hat{h}(\xi_{0,1}) \rangle^{k+p/2}$$

and consequently 

$$\widetilde{b}_{h,\bar{h}}(0,0) = O(1)\langle \hat{h}(\xi_{0,1}) \rangle^{k+p/2-2q/2}.$$

Combining this with the definition of $b_{h,\bar{h}}$ and $\widetilde{b}_{h,\bar{h}}$ gives 

$$\partial_{\xi_1}^p \partial_{\xi_1}^q a_{h,\bar{h}}(y_0, \xi_0) = O(1)\langle \hat{h}(\xi_{0,1}) \rangle^{k+p/2-2q/2},$$

which is (5.12) for $k \geq 0$. When $k < 0$ we check that the assumption is satisfied for $\langle \hat{h}D_{x_1} \rangle^{-k} A$ with $k$ replaced by 0. The composition formula then gives the result. 

5.3. Modified Sobolev spaces. The norm used in Proposition 5.2 can be defined globally. We generalize it to include standard Sobolev spaces by adding additional information at a smooth compact hypersurface $\Sigma \subset T^*X$.

Let $Q \in \Psi_{h,0}^0$ be an operator, $Q = Q^*$, with the principal symbol $q$ satisfying

$$(5.18) \quad \Sigma = \{ (x, \xi) : q(x, \xi) = 0 \}, \quad dq|_{\Sigma} \neq 0, \quad |q(x, \xi)| \geq 1 \text{ for } |\xi| \geq C. \quad \text{ }$$

For $s, m \in \mathbb{R}$ we define

$$(5.19) \quad H_{\Sigma}^{s,m}(X) = \{ u \in \mathcal{S}'(X) : \langle (\hat{h}/h)Q \rangle^m u \in H_{h}^s(X) \}, \quad \| u \|_{H_{\Sigma}^{s,m}(X)} \overset{\text{def}}{=} \| \langle (\hat{h}/h)Q \rangle^m u \|_{H_{h}^s(X)}.$$

Here $H_{h}^s(X)$ denotes the usual semiclassical Sobolev spaces defined in (3.1). The spaces are (complex) interpolation spaces, in $m$, and in $s$. 

When $m \in \mathbb{Z}$ the definition is equivalent to 

$$H_{\Sigma}^{s,m}(X) = \begin{cases} 
\{ u : (\hat{h}/h)^k Q^k u \in H_{h}^s(X), \ 0 \leq k \leq m \} & m \geq 0 \\
\{ u : u = \sum_{k=0}^{m}[\hat{h}/h]^k Q^k u_k, \ u_k \in H_{h}^s(X) \} & m \leq 0 
\end{cases}$$
with the norms equivalent to the same natural way as for Sobolev spaces:

\[
\|u\|_{H^s,m,\Sigma}(X) \approx \begin{cases} 
\sum_{k=0}^m \| (\tilde{h}/h)^k Q u \|_{H^s(X)}, & m \geq 0 \\
\inf \left\{ \sum_{k=0}^{|m|} \| u_k \|_{H^s(X)} : u = \sum_{k=0}^{|m|} (\tilde{h}/h)^k Q u_k \right\}, & m \leq 0
\end{cases}
\]

This can be seen using a spectral decomposition of \(Q\) which is assumed to be bounded and self-adjoint. We use the following simplified notation

\[
H^s_{\Sigma}(X) \overset{\text{def}}{=} H^0_{\Sigma}(X), \ m \in \mathbb{Z}.
\]

The spaces \(H^{s,m}_{\Sigma}(X)\) have the following basic invariance property:

**Lemma 5.3.** The definition \((5.19)\) does not depend on the choice of \(Q \in \Psi^{-0,0}(X)\) satisfying \((5.18)\). If \(A \in \Psi^{-0,0}(X)\) has the property that \(d(\text{WF}_h(A), \Sigma) > 1/C\) then for \(M \geq 0\)

\[
\| Au \|_{H^{-s,M}_{\Sigma}(X)} \leq (h/\tilde{h})^M \| u \|_{L^2(X)},
\]

\[
\| Au \|_{L^2(X)} \leq (h/\tilde{h})^M \| u \|_{H^s_{\Sigma}(X)},
\]

for \(u \in C^\infty_c(X)\).

Also, suppose that \(F\) is a 0th order \(\hbar\)-Fourier Integral operator associated to a canonical transformation which maps \(\Sigma\) to another hypersurface \(\Sigma'\) satisfying our hypothesis. Then

\[
F : H^{s,m}_{\Sigma}(X) \longrightarrow H^{s,m}_{\Sigma'}(X).
\]

**Proof.** Let \(Q'\) be another operator satisfying \((5.18)\). Then

\[
Q' = AQ + E = QA + E',
\]

where \(A \in \Psi^{0,0}(X)\) is uniformly elliptic, and \(E, E' \in \Psi^{-1,-1}(X)\). Because of the interpolation property of \(H^{s,m}_{\Sigma}\) we only need to check the independence for \(k \in \mathbb{Z}\). Then, by induction on \(k\),

\[
\|(\tilde{h}/h)^k (Q')^k u\|_{H^s(X)} \leq C_k \sum_{\ell=0}^k \tilde{h}^{k-\ell} \|(\tilde{h}/h)^\ell u\|_{H^s(X)},
\]

where \(C_k\) depends only on \(A\) and \(E\). This shows that the definition for \(m \geq 0\) is independent of the choice of \(Q\). The case of \(m < 0\) is similar. To see the first inequality in \((5.20)\) we recall that

\[
\| Au \|_{H^{-s,M}_{\Sigma}(X)} = \inf \left\{ \sum_{k=0}^M \| u_k \|_{L^2(X)} : Au = \sum_{k=0}^M (\tilde{h}/h)^k Q^k u_k \right\}.
\]

Because of the ellipticity of \(Q\) on \(\text{WF}_h(A)\) we can find \(B\) such that

\[
Au = Q^k B Au + v, \quad \| v \|_{L^2} = O(h^\infty) \| u \|_{L^2}.
\]
Hence we can take $u_0 = v$ and $u_M = (h/\tilde{h})^M BAu$, $u_k = 0$ for all other $k$’s, so that
\[
\|Au\|_{H^{-M}_\Sigma(X)} \leq C(h/\tilde{h})^M \|u\|_{L^2(X)} + O(h^\infty) \|u\|_{L^2(X)}
\]
proving the first estimate in (5.20) ($h < \tilde{h}$ everywhere here). Since $H^{-M}_\Sigma(X)$ is the dual of $H^M_\Sigma(X)$ this is equivalent to
\[
\forall v \in C^\infty_c(X) \langle Au, v \rangle_{L^2(X)} \leq C(h/\tilde{h})^M \|u\|_{L^2(X)} \|v\|_{H^M_\Sigma(X)} ,
\]
which in turn proves the second estimate with $A$ replaced by $A^*$.

The Egorov theorem (Proposition 3.4 above) shows that $QF$ is equal to $FQ' + E$ where $Q'$ satisfies (5.18) with $\Sigma$ replaced by $\Sigma'$. The mapping property follows from the boundedness of $F$ on $H^s(X)$ and an argument similar to that above. \qed

5.4. Globally defined class of operators. For the global definition we cannot use the classes
\[
\widetilde{O}_P h, \tilde{h} \left( \tilde{O}_{\frac{1}{2}}((\lambda)^m) \right)
\]
due to the presence of propagation in the $h$-sense. To see the problem let us consider the operator $P$ introduced in §1 and define
\[
A = h^{-m} \chi \left( \frac{hP}{\tilde{h}} \right), \quad \chi \in C^\infty_c(\mathbb{R}).
\]
In the model theory of §5.2 this operator is obtained from the symbol $h^{-m} \chi(\lambda)$. However, when $\tilde{h}$ is fixed, the operator $A$ is an $h$-Fourier integral operator:
\[
A = \frac{h^{-m}}{2\pi \tilde{h}} \int_\mathbb{R} \tilde{\chi}(t/\tilde{h}) \exp(itP/\tilde{h}) dt ,
\]
associated to the relation
\[
\{ (\rho, \rho') : \exists t, \exp(tH_\rho)(\rho) = \rho', \ p(\rho) = p(\rho') = 0 \}.
\]
Hence the presence of almost closed orbits of the flow prevents a definition which would be purely local in the $O(h^\infty)$ sense. We observe however that the non-local contributions in $A$ are of the order $O(h^\infty)$.

These global concerns suggest the following definition of the residual class. First we introduce a useful cut-off operator. Let $V_1, V_2$ be two open neighbourhoods of $\Sigma$, satisfying
\[
\nabla_1 \subset V_2.
\]
Then we choose
\[
\gamma_\Sigma \in \Psi^{0,0}(X), \quad WF_h(\gamma_\Sigma) \subset V_2, \quad WF_h(I - \gamma_\Sigma) \subset \mathcal{C}V_1.
\]
We now define the spaces of operators. Let
\[
p = (m, \bar{m}, k_1, k_2), \quad p_\infty = (m, -\infty, -\infty, k_2).
\]
As before we start with the definition of a residual class:

**Definition 1.** We say that $A \in \Psi_{\Sigma, \frac{1}{2}}^{p, \infty}(X)$ if

$$A : \mathcal{C}^{\infty}(X) \to \mathcal{C}^{\infty}(X), \quad (I - \gamma_{\Sigma}) A, \quad A(I - \gamma_{\Sigma}) \in \Psi_{\Sigma, \frac{1}{2}}^{m+k, \infty, -\infty}(X),$$

and for any $u \in \mathcal{C}^{\infty}(X)$, any sequence $\{b_j\}_{j=1}^{N} \subset S^{0}(T^*X)$, and any $k, p,$ and $M,$ we have

$$\| \text{ad}_{\text{Op}_h^w(b_1)} \circ \cdots \circ \text{ad}_{\text{Op}_h^w(b_N)} \circ \text{ad}_Q^k pu \|_{H^{p+k/2}_{\Sigma}} \leq Ch^{-m+k} \tilde{h}^M \|u\|_{H^{p+k/2}_{\Sigma}},$$

where $Q$ is the operator in (5.19).

**Definition 2.** We say that $A \in \Psi_{\Sigma, \frac{1}{2}}^{p, \Sigma}(X)$ if

$$A : \mathcal{C}^{\infty}(X) \to \mathcal{C}^{\infty}(X), \quad (I - \gamma_{\Sigma}) A, \quad A(I - \gamma_{\Sigma}) \in \Psi_{\Sigma, \frac{1}{2}}^{m+k, \Sigma, \infty, \infty}(X),$$

for some cut-off operator $\gamma_{\Sigma}$ satisfying (5.21).

- For any $\chi \in \mathcal{C}_c^{\infty}(T^*X)$,

$$A = A^{\chi} + A^{\flat}_{\chi}, \quad A^{\chi}_{\chi} \in \Psi_{\Sigma, \frac{1}{2}}^{p, \infty}(X),$$

so that for any $p,$ and any sequences

$$\{b_j\}_{j=1}^{N}, \quad \{a_j\}_{j=1}^{M} \subset S^{0}(T^*X), \quad H_p a_j \big|_{\text{supp}\ \chi} \equiv 0,$$

$$\| \text{ad}_{\text{Op}_h^w(b_1)} \circ \cdots \circ \text{ad}_{\text{Op}_h^w(b_N)} \circ \text{ad}_{\text{Op}_h^w(a_M)} \circ \text{ad}_Q^k \chi^w \gamma_{\Sigma} A^{\chi}_{\chi} \chi^w p u \|_{H^{p+k/2}_{\Sigma}} \leq Ch^{M/2+k-m} \tilde{h}^M \|u\|_{H^{p+k/2}_{\Sigma}}, \quad \chi^w = \text{Op}_h^w(\chi),$$

for any $u \in \mathcal{C}_c^{\infty}(X)$.

- For any $\psi_1, \psi_2 \in \mathcal{C}_c^{\infty}(T^*X)$, $\text{supp}\ \psi_1 \cap \text{supp}\ \psi_2 = \emptyset,$

$$\text{Op}_h^w(\psi_1) A \text{Op}_h^w(\psi_2) \in \Psi_{\Sigma, \frac{1}{2}}^{p, \infty}(X).$$

It is important to record

**Lemma 5.4.** Definitions 1 and 2 are independent of the choice of the cut-off operator satisfying (5.21).

**Proof.** Suppose that $\gamma'_{\Sigma}$ is another cut-off operator satisfying (5.21). We need to show that

$$\gamma_{\Sigma} - \gamma'_{\Sigma} \quad A, \quad A(\gamma_{\Sigma} - \gamma'_{\Sigma}) \in \Psi_{\Sigma, \frac{1}{2}}^{m+k, \Sigma, \infty, \infty}(X).$$
and for that we will use only the commutators involving $b_j$'s in Definitions 1 and 2. The first inclusion is, in view of Lemma 3.3 and the fact that $\gamma_\Sigma - \gamma_\Sigma'$ is smoothing, equivalent to

$$
\| \text{ad}_{\text{Op}_h^w(b_1)} \circ \cdots \circ \text{ad}_{\text{Op}_h^w(b_N)} (\gamma_\Sigma - \gamma_\Sigma') Au \|_{L^2(X)} \leq Ch^{-m-k_2+N/2}h^{-\tilde{m}+k_2+N/2}\|u\|_{L^2(X)}.
$$

To see this we first use (5.20) to see that if $E \in \Psi_{\Sigma}^{0,0}(X)$ and $d(\text{WF}_h(E), \Sigma) > 1/C$ then for any subsequence $\{i_j\}_{j=1}^J$ of $1, \ldots, N$,

$$
\| E \text{ad}_{\text{Op}_h^w(b_1)} \circ \cdots \circ \text{ad}_{\text{Op}_h^w(b_{i_j})} Au \|_{L^2(X)} \leq C(h/\tilde{h})^{J/2-k_2} \| \text{ad}_{\text{Op}_h^w(b_1)} \circ \cdots \circ \text{ad}_{\text{Op}_h^w(b_{i_j})} Au \|_{H_{\Sigma}^{J/2-k_2}}
\leq Ch^{-m}(h/\tilde{h})^{J/2-k_2}h^{-\tilde{m}}\|u\|_{L^2(X)} = Ch^{-m-k_2+J/2}h^{-\tilde{m}+k_2+J/2}\|u\|_{L^2(X)}.
$$

Using the derivation property of $\text{ad}_{\text{Op}_h^w(b_j)}$ and the fact that

$$
\text{ad}_{\text{Op}_h^w(b_j)} (\gamma_\Sigma - \gamma_\Sigma') = hE_j, \ E_j \in \Psi_{\Sigma}^{0,0}(X), \ d(\text{WF}_h(E_j), \Sigma) > 1/C,
$$

we can estimate the left hand side of (5.23) by a linear combination of terms of the form

$$
h^{-J} \| E \text{ad}_{\text{Op}_h^w(b_1)} \circ \cdots \circ \text{ad}_{\text{Op}_h^w(b_{i_j})} Au \|_{L^2(X)}, \ E \in \Psi_{\Sigma}^{0,0}(X), \ d(\text{WF}_h(E), \Sigma) > 1/C.
$$

Consequently (5.23) follows from (5.24).

Since, by the calculus of (5.3)

$$
B \in \Psi_{\Sigma, 1/2}^{m+k_2,-\infty,-\infty}(X) \iff B^* \in \Psi_{\Sigma, 1/2}^{m+k_2,-\infty,-\infty}(X),
$$

and since

$$
\text{ad}_{\text{Op}_h^w(b_1)} \circ \cdots \circ \text{ad}_{\text{Op}_h^w(b_N)} B^* = (-1)^N \left( \text{ad}_{\text{Op}_h^w(b_1)} \circ \cdots \circ \text{ad}_{\text{Op}_h^w(b_N)} B \right)^*;
$$

the second inclusion in (5.22) follows from the first one. \hfill \Box

We now have a natural mapping and invariance properties:

**Proposition 5.5.** The operators $\Psi_{\Sigma, 1/2}^{0,0,0}(X)$ form an algebra and $\Psi_{\Sigma, 1/2}^{0,-\infty,-\infty,0}(X)$ is an ideal in that algebra. If $A \in \Psi_{\Sigma, 1/2}^{p,q}(X)$ then

$$
A = \mathcal{O}(h^{-m}\tilde{h}^M) : H_{\Sigma}^{M,r}(X) \to H_{\Sigma}^{M,r-k_2}(X),
$$

for any $m, M$. If $A \in \Psi_{\Sigma, 1/2}^{p,q}(X)$ then

$$
A = \mathcal{O}(h^{-m}\tilde{h}^{-\tilde{m}}) : H_{\Sigma}^{s+k_1,p+k_2}(X) \to H_{\Sigma}^{s,p}(X),
$$

for any $s \in \mathbb{R}$ and $p \in \mathbb{Z}$. 
Also, suppose that \( F, G \) are 0th order \( h \)-Fourier Integral operators, \( F \) associated to a canonical transformation which maps \( \Sigma \) to another hypersurface \( \Sigma' \) satisfying our hypothesis, and \( G \) to its inverse (both transformation need to be defined only locally). Then

\[
A \in \Psi^{p, q, k, l}_\Sigma(X) \implies F \circ A \circ G \in \Psi^{p, q, k, l}_\Sigma(X).
\]

**Proof.** We first show that \( \Psi^{0, -\infty, -\infty, 0}_\Sigma(X) \) is an algebra. In fact, in the notation of Definition 1,

\[
\| \text{ad}_{\operatorname{Op}}(b_1) \circ \cdots \circ \text{ad}_{\operatorname{Op}}(b_N) \circ \text{ad}_Q ABu \|_{H^{p+N/2}(X)}, \quad A, B \in \Psi^{0, -\infty, -\infty, 0}_\Sigma(X),
\]

is bounded by a linear combination of

\[
\| \text{ad}_{\operatorname{Op}}(b_1) \circ \cdots \circ \text{ad}_{\operatorname{Op}}(b_j) \circ \text{ad}_Q A \circ \cdots \circ \text{ad}_{\operatorname{Op}}(b_{N-j}) \circ \text{ad}_Q Bu \|_{H^{p+N/2}(X)},
\]

\[
\{i_j\}_{j=1}^J \cup \{k_j\}_{j=1}^{N-J} = \{1, \ldots, N\}.
\]

These terms are estimated by

\[
Ch^{l-1}M_1 \| \text{ad}_{\operatorname{Op}}(b_1) \circ \cdots \circ \text{ad}_{\operatorname{Op}}(b_{N-j}) \circ \text{ad}_Q Bu \|_{H^{p+N/2}(X)} \leq C'h^{l}M_1 \| u \|_{p+N/2+k},
\]

where we used Definition 1 with \( A \) and with \( B \). Here \( M_1 \) and \( M_2 \) are any large integers. Checking that

\[
(I - \gamma_\Sigma)AB, \ AB(I - \gamma_\Sigma) \in \Psi^{0, -\infty, -\infty, 0}_\Sigma(X),
\]

is the same as in the proof of Lemma 5.4.

We now check that for \( B \in \Psi^{0, -\infty, -\infty, 0}_\Sigma(X) \) and \( A \in \Psi^{0, 0, 0, 0}_\Sigma(X) \) we have

\[
BA \in \Psi^{0, -\infty, -\infty, 0}_\Sigma(X).
\]

In the notation of Definition 2, we can write

\[
B = \gamma_\Sigma B_1 \gamma_\Sigma + (1 - \gamma_\Sigma)B_1 + \gamma_\Sigma B_1 (1 - \gamma_\Sigma) + B_2,
\]

and it suffices to check that

\[
A \gamma_\Sigma B_1 \gamma_\Sigma \in \Psi^{0, -\infty, -\infty, 0}_\Sigma(X).
\]

Since \( \Sigma \) is compact (and hence we can take \( \chi^w = \gamma_\Sigma \)) and Definition 2 is independent of the choice of \( \gamma_\Sigma \) we have, as its special case,

\[
\| \text{ad}_{\operatorname{Op}}(b_1) \circ \cdots \circ \text{ad}_{\operatorname{Op}}(b_N) \circ \text{ad}_Q \gamma_\Sigma B_1 \gamma_\Sigma u \|_{H^{p+N/2}(X)} \leq Ch^{k}h^N \| u \|_{H^{p+k/2}(X)},
\]

and the verification of (5.26) follows the proof of the algebra property of \( \Psi^{0, -\infty, -\infty, 0}_\Sigma(X) \).

To conclude the algebraic part of the proof we show that \( \Psi^{0, 0, 0, 0}_\Sigma(X) \) is closed under composition of operators. Let \( B_1, B_2 \in \Psi^{0, 0, 0, 0}_\Sigma(X) \) and let \( \chi \in C_c^\infty(T^*X) \). Since we already
established that composition with elements of $\Psi^{0,-\infty,-\infty,0}_{\Sigma,\frac{1}{2}}(X)$ produces an operator in that space we only have to show that

$$B_1 B_2 = A_1 + A_2 , \ A_2 \in \Psi^{0,-\infty,-\infty,0}_{\Sigma,\frac{1}{2}}(X),$$

so that for any $p$, and any sequences

$$\{b_j\}_{j=1}^N, \ \{a_j\}_{j=1}^M \subset S^0(T^*X), \ H_p a_j|_{\text{supp } \chi} \equiv 0,$$

and

$$\| \text{ad}_{\text{Op}^p(a_1)} \cdots \text{ad}_{\text{Op}^p(a_M)} \circ \text{ad}_{\text{Op}^p(b_1)} \cdots \text{ad}_{\text{Op}^p(b_N)} \circ \text{ad}_{\text{Op}^k} \chi^w \gamma_\Sigma A \chi^w u \|_{H^{p+N/2-k_2}_{\Sigma}} \leq C h^{M/2-k-m} h^{M/2-N-\bar{m}} \| u \|_{H^{p+k/2}_{\Sigma}},$$

To find the decomposition of $B_1 B_2$ we introduce $\chi_j \in C^\infty_c(T^*X)$ such that $\chi_1 \equiv 1$ on $\text{supp } \chi_0$, and $\chi_0 \equiv 1$ on $\text{supp } \chi$. We choose $\text{supp } \chi_j$ sufficiently close to the support of $\chi$ so that the functions $a_j$ satisfy $H_p a_j|_{\text{supp } \chi_1} \equiv 0$. We then put

$$A_2 = \chi_0^w(B_1)^{\sharp}_{\chi_1}(1-\chi_1^w)(B_2)^{\sharp}_{\chi_1} + B_1(B_2)^{\sharp}_{\chi_1} + (B_1)^{\sharp}_{\chi_1}B_2,$$

which is in $\Psi^{0,-\infty,-\infty,0}_{\Sigma,\frac{1}{2}}(X)$ since $\text{supp } \chi_0 \cap \text{supp } (1-\chi_1^w) = \emptyset$, and we can use the second part of Definition 2.

We have

$$A_1 = (1-\chi_0^w)(B_1)^{\sharp}_{\chi_1}(B_2)^{\sharp}_{\chi_1} + \chi_0^w(B_1)^{\sharp}_{\chi_1} \chi_1(1)(B_1)^{\sharp}_{\chi_1}.$$ 

Up to negligible, $O(h^\infty)$, errors

$$\chi^w(1-\chi_0^w) \equiv 0, \ \chi^w \chi_0^w \equiv \chi^w.$$

Hence we need to check that

$$\text{ad}_{\text{Op}^p(b_1)} \cdots \text{ad}_{\text{Op}^p(b_N)} \circ \text{ad}_{\text{Op}^p(a_1)} \cdots \text{ad}_{\text{Op}^p(a_M)} \circ \text{ad}_{\text{Op}^k} \chi^w_B \gamma_\Sigma A \chi^w = O(h^{M/2-k-m} h^{M/2-N-\bar{m}}) : H^{p+k/2}_{\Sigma}(X) \rightarrow H^{p+k/2}_{\Sigma}(X),$$

and this follows from the Leibnitz rule for ad_{b_j} and assumptions on $(B_j)^{\sharp}_{\chi_1}$.

The mapping properties are immediate from the definitions: we apply them with no commutators, and in the case of Definition 2, with $\chi^w = \gamma_\Sigma$.

Lemma 5.3 shows that the spaces $H^p_{\Sigma}$ transform correctly under $F$ and hence Proposition 3.4 (Egorov’s Theorem) shows that for

$$A \in \Psi_{\Sigma,\frac{1}{2}}^{p,N}(X) \implies F \circ A \circ G \in \Psi_{\Sigma,\frac{1}{2}}^{p,N}(X),$$

that is $\Psi_{\Sigma,\frac{1}{2}}^{p,N}$ holds for the residual class. Since the conditions on $a_j, b_j$’s in Definition 2 are symplectically invariant, Egorov’s theorem (Proposition 3.4) again shows that $F \circ A \circ G \in \Psi_{\Sigma,\frac{1}{2}}^{p,N}(X)$. □
5.5. **The symbol map.** To define the symbol map we will use the invariance given by Proposition 5.5 and the symbol map in the model case. We start with

**Lemma 5.6.** In the notation of Definition 2, suppose that

\[ \Sigma \cap V = \{ \xi_1 = 0 \} \cap V, \quad V \subset T^* \mathbb{R}^n, \quad \text{is open, and supp}\ \chi \in \mathcal{V}. \]

Then

\[ \mathrm{Op}_h^w(\chi) A_1 \mathrm{Op}_h^w(\chi) = \tilde{\mathrm{Op}}^w_h(\tilde{a}_\chi), \quad \tilde{a}_\chi = \tilde{O}_{\frac{1}{2}}(h^{-m} \tilde{h}^{-\tilde{m}} \langle \lambda \rangle^k). \]

**Proof.** This is a consequence of Proposition 5.2 and the properties of the term \( A_1 \) in Definition 2. \( \square \)

To construct a symbol map, that is a homomorphism

\[ \sigma_{\Sigma, h, \tilde{h}} : \Psi_{\Sigma, \frac{1}{2}}^p(X) \longrightarrow S_{\Sigma, \frac{1}{2}}^p(T^*X)/S_{\Sigma, \frac{1}{2}}^p(T^*X), \]

\[ (m, \tilde{m}, k_1, k_2), \quad (m, \tilde{m} - 1, k_1 - 1, k_2), \]

such that the sequence

\[ 0 \longrightarrow \Psi_{\Sigma, \frac{1}{2}}^p(X) \longrightarrow \Psi_{\Sigma, \frac{1}{2}}^p(X) \xrightarrow{\sigma_{\Sigma, h, \tilde{h}}} S_{\Sigma, \frac{1}{2}}^p(T^*X)/S_{\Sigma, \frac{1}{2}}^p(T^*X) \longrightarrow 0 \]

is exact, for an arbitrary \( \Sigma \) we will use Lemma 5.6. That requires putting \( \Sigma \) locally to the model hypersurface \( \xi_1 = 0 \) which on the quantum level is done using \( h \)-Fourier integral operators. Hence we need a local invariance statement given in the next

**Proposition 5.7.** Let \( U \) be an \( h \)-Fourier Integral Operator, elliptic in \( V \times V \), where \( V \) is a neighbourhood of \((0,0) \in T^* \mathbb{R}^n \), the compact hypersurface \( \Sigma \) satisfies,

\[ \Sigma \cap V = \{ \xi_1 = 0 \} \cap V. \]

Assume also that the canonical transformation associated to \( U, \kappa, \) satisfies:

\[ \kappa(0,0) = (0,0), \quad \kappa(\{ \xi_1 = 0 \} \cap V) \subset \{ \xi_1 = 0 \}. \]

Let \( A = \widetilde{\mathrm{Op}}_{h, \tilde{h}}(a) \) where \( a = \tilde{O}_{\frac{1}{2}}(h^{-m} \tilde{h}^{-\tilde{m}} \langle \lambda \rangle^k), \quad a(x, \xi, \lambda) \equiv 0 \quad \text{for } (x, \xi) \notin V. \)

If \( U^{-1} \) is the microlocal inverse of \( U \) near \( V \times V \), then

\[ U^{-1} \circ A \circ U = \widetilde{\mathrm{Op}}_{h, \tilde{h}}(b) + E, \quad b = a \circ K, \]

\[ E \in \Psi_{\Sigma}^p(\mathbb{R}^n), \quad (m, \tilde{m} - 1, -\infty, k - 1), \]

where \( K \) is the natural lift of \( \kappa \) to the \((x, \xi, \lambda)\) variables:

\[ K(y, \eta, \mu) = (x, \xi, \lambda) \iff (x, \xi) = \kappa(y, \eta), \quad \lambda = (\xi_1/\eta_1)\mu. \]
Proof. We start by observing that the proposition holds in the special cases $\kappa(x, \xi) = (x, \xi)$ and $\kappa(x, \xi) = (-x, -\xi)$. The first special case concerns conjugation with elliptic classical $h$-pseudodifferential operators and it follows from the discussion after (5.7). The second special case follows from the first one and the fact that the proposition is easily checked for $U u(x) = u(-x)$. As a consequence we can assume that $\kappa$ preserves the sign of $\xi_1$:

$$\kappa(y, \eta) = (x, \xi) \implies \xi_1 \eta_1 \geq 0.$$ 

We will prove the proposition by a deformation method inspired by the “Heisenberg picture of quantum mechanics” and for that we need the following geometric lemma.

**Lemma 5.8.** Let $\kappa$ be a smooth canonical transformation satisfying (5.27) and (5.29). Then we can find a piecewise smooth family of canonical transformations $[0, 1] \ni t \mapsto \kappa_t$ satisfying (5.27), (5.29) and such that $\kappa_0 = id$ and $\kappa_1 = \kappa$.

**Proof.** Let us denote by $\Sigma$ the hypersurface given by $\xi_1 = 0$. We first observe that if $\kappa$ is a linear symplectic transformation preserving $\Sigma$ then we can find a family of linear symplectic transformations, $\kappa_t$, satisfying the conclusions of the lemma: the subgroup of elements of $Sp(n, \mathbb{R})$ preserving a half space bounded by $\Sigma$ is connected.

Hence we can assume that $d\kappa(0, 0) = Id$. Now introduce

$$\kappa_t(x, \xi) = t^{-1} \kappa(tx, t\xi),$$

a smooth family of symplectic transformations, preserving a half space bounded by $\Sigma$, with $\kappa_1 = \kappa$ and $\kappa_0 = Id$. \hfill $\Box$

We now return to the proof of Proposition 5.7. For simplicity we can assume that $m = \tilde{m} = 0$. Let $\kappa_t$ be a piecewise smooth family with $\kappa_1 = \kappa$, $\kappa_0 = id$. Using (3.4) we construct a piecewise smooth family of classical elliptic $h$-Fourier integral operators, $U_t$, defined microlocally near $(0, 0)$ and associated to $\kappa_t$. If we demand that $U_1 = U$ then $U_0$ is a pseudodifferential operator elliptic at $(0, 0)$. For notational convenience we assume that our deformation is smooth in $t$ – the piecewise smooth case follows from the same argument applied in several steps and that $U_0 = Id$ (the last condition can be arranged by composing $U$ with an elliptic pseudodifferential operator). Thus we have

$$hD_t U_t + U_t Q_t = 0,$$

where $Q_t$ is a smooth family of classical $h$-pseudodifferential operators of order 0 with the leading symbol $q_t$ satisfying

$$\frac{d}{dt} \kappa_t(x, \xi) = (\kappa_t)_* (H_{q_t}(x, \xi)),$$

in a neighbourhood of $V$. It follows from (5.31) that $H_{q_t}$ is tangent to $\Sigma$ and hence

$$\partial_{x_1} q_t(x, \xi) = \xi_1 r_t(x, \xi).$$

We extend $q_t$ to a globally defined function in $S(T^*\mathbb{R}^n, \langle x, \xi \rangle)$, keeping the property (5.32). This defines a family of global canonical transformation which coincide with $\kappa_t$ near $V$. 

**Proof.**
Let \( V_t \) satisfy
\[
(5.33) \quad hD_t V_t = Q_t V_t, \quad V_0 = I_d.
\]
It follows that \( V_t = U_t^{-1} \), and if we take \( Q_t \) to be self-adjoint, \( V_t^* = U_t \).

We will now use (5.30) and (5.31) to prove Egorov’s theorem (3.5) for the new class. Thus we consider \( A_t = U_t^{-1} A U_t \), so that \( A_1 \) is the operator we want to study and \( A_0 = A \) is the given operator. From Proposition 5.5 we already know that \( A_t \in \Psi^{0,0,k}(\mathbb{R}^n) \). Using Lemma 5.6 we can write
\[
A_t \equiv \widetilde{O}_{p,h_\lambda}(a_t) \mod \Psi_{\Sigma,\frac{1}{2}}^{0,-\infty,-\infty,k}(\mathbb{R}^n),
\]
microlocally near \( V \). From (5.30) and (5.33), we get
\[
(5.34) \quad \begin{cases}
  hD_t A_t = [Q_t, A_t], \\
  A_0 = A.
\end{cases}
\]

To compute the commutator on the symbolic level we need:

**Lemma 5.9.** Suppose that \( a \in \widetilde{O}_{1,\frac{1}{2}}((\lambda) ) \) and that \( b \in S^{0,-\infty}(T^*\mathbb{R}^n) \) (that is, \( b \) is a symbol in the sense of \( (3.7) \), does not depend on \( \lambda \). In addition let us assume that
\[
\partial_{x_1} b = \xi_1 r(x, \xi).
\]
Then
\[
(5.35) \quad i \hbar \frac{\partial}{\partial \lambda} [\widetilde{O}_{p,h_\lambda}(b), \widetilde{O}_{p,h_\lambda}(a)] = \widetilde{O}_{p,h_\lambda}(c), \quad c = (H_b - r \lambda \partial_\lambda) a + \widetilde{O}_{\frac{1}{2}}(\hbar \langle \lambda \rangle^k).
\]

**Proof.** We will use (5.8) to compute \( a^{\sharp}_{p,h_\lambda} b \) (and \( b^{\sharp}_{p,h_\lambda} a \)), noting that
\[
\text{Op}_p a(x_1, \hbar D_{x_1}) \circ \text{Op}_p b(x_1, \hbar D_{x_1}) = (\text{Op}_p a)^{\sharp}_{p,h_\lambda} \text{Op}_p(b) (x_1, \hbar D_{x_1}).
\]

Since \( b \in S^{0,-\infty} \), Lemma 3.6 shows that the composition formula in the \( (x', \xi') \) is given by an asymptotic series in \( (\hbar \hbar)^{\frac{1}{2}} \), and a \( O(h^\infty) \) error. The only subtlety lies in the dependence on \( (x_1, \lambda) \) and to explain it we suppress the other variables. We have
\[
\partial_{x_1}^\ell \partial_\lambda^\ell a = O(\langle \lambda \rangle^{k-\ell/2+p/2}). \quad \text{Consequently the terms in the expansions of } a^{\sharp}_{p,h_\lambda} b \text{ and } b^{\sharp}_{p,h_\lambda} a \text{ are bounded by}
\]
\[
C_p h \hbar^{p-1}(\lambda)^{k-p/2+1} \langle (\hbar / \hbar) \lambda \rangle^{-\infty}, \quad C_p(h/\hbar)^{p} \hbar^{p} \langle (\hbar / \hbar) \lambda \rangle^{-\infty}, \quad \text{for } p > 0,
\]
respectively. Hence we have expansions such that for \( p > 1 \) the terms are bounded by \( h \hbar^{p-1}(\lambda)^{k} \), and the errors are \( O(h \hbar^\infty(\lambda)^k) \). This argument shows that
\[
a^{\sharp}_{p,h} b = ab + h \left( \sum_{i=1}^n \partial_{x_i} a D_{x_i} b + r \lambda \partial_\lambda a \right) + \widetilde{O}_{\frac{1}{2}}(\hbar \hbar \langle \lambda \rangle^k),
\]
and

\[ b \star_h a = ab + h \sum_{i=1}^{n} \partial_{\xi_i} b D_{\xi_i} a + \tilde{O}_{1/2} \left( h\tilde{h}\langle \lambda \rangle^k \right), \]

from which the lemma follows. \(\square\)

Since \(q_t\) satisfies (5.32), Lemma 5.9 gives

\[ i \hbar \left[ Q_{t,\tilde{\gamma}_{\tilde{h}},\tilde{\gamma}_h}(a_t) \right] = \tilde{O}_{1/2}(\tilde{h}(\langle \lambda \rangle^k) \]

If we write \(a_t = a_0^0 + O_{1/2}(\tilde{h}(\langle \lambda \rangle^k))\) then

\[ \partial_t a_0^0 = (H_{q_t} - \kappa_t(x,\xi)\lambda \partial_{\lambda}) a_0^0, \quad a_0^0 \equiv a \pmod{O_{1/2}(\tilde{h}(\langle \lambda \rangle^k))}. \]

We now note that

\[ (H_{q_t} - \kappa_t(x,\xi)\lambda \partial_{\lambda}) a_0^0 = H_{q_t}(a_0^0|_{\lambda = (\tilde{h}/h)\xi_1}). \]

Hence, if \(K_t\) is the transformation in \((x,\xi,\lambda)\)-space corresponding to \(\kappa_t\) as in the statement of the proposition, it follows that

\[ a_t^0 = a \circ K_t, \]

that the principal symbol of \(A_t\) is \(a \circ K_t\) and the proposition follows. \(\square\)

We can now define the symbol map,

\[ \Psi_{\Sigma,\frac{1}{2}}(X) \ni A \mapsto \sigma_{\Sigma,\tilde{h},\tilde{h}}(A) \in S^p_{\Sigma,\frac{1}{2}}(T^*X)/S^p_{\Sigma,\frac{1}{2}}(T^*X). \]

We recall from §3.1 and §3.3 that we already have the symbol map for the \(S^\frac{1}{2}\) calculus:

\[ \Psi_{\frac{1}{2}}(X) \ni B \mapsto \sigma_h(B) \in S^m_{\frac{1}{2}}(T^*X)/S^m_{\frac{1}{2}}(T^*X). \]

The definition of the symbol classes (5.1) shows that

\[ (S^p_{\Sigma,\frac{1}{2}}(T^*X) \cap C^\infty(T^*X \setminus U_{\Sigma})) / S^p_{\Sigma,\frac{1}{2}}(T^*X) = \]

\[ (S^{m+k_2,\tilde{m}-k_2,1}(T^*X) \cap C^\infty(T^*X \setminus U_{\Sigma})) / S^{m+k_2,\tilde{m}-k_2-1,1}(T^*X), \]

for any open \((h\text{-independent})\) neighbourhood of \(\Sigma, U_{\Sigma}\). Hence we define

\[ \sigma_{\Sigma,\tilde{h},\tilde{h}}((I - \gamma_{\Sigma})A) \triangleq \sigma_h((I - \gamma_{\Sigma})A), \]

and as in the proof of Lemma 5.4 we see that this definition is independent of the choice of \(\gamma_{\Sigma}\).

To define \(\sigma_{\Sigma,\tilde{h}}(\gamma_{\Sigma}A)\) we use Lemma 5.6. We choose a partition of the cut-off operator, \(\gamma_{\Sigma}\):

\[ \sum_{j=1}^{J} \gamma_j^2 = \sigma_h(\gamma_{\sigma}), \quad \gamma_j \in C^\infty_c(T^*X), \]

and as in the proof of Lemma 5.4 we see that this definition is independent of the choice of \(\gamma_{\Sigma}\).
such that for each \( j \), \( \text{supp} \gamma_j \cap \Sigma \) can be put into the normal form \( \Omega \cap \{ \xi_1 = 0 \} \) by a local canonical transformation,

\[
\kappa_j : \Omega \longrightarrow \Omega_j, \quad (0, 0) \in \Omega \subset T^*\mathbb{R}^n,
\]

\[
\kappa_j(\{ \xi_1 = 0 \} \cap \Omega) = \Sigma \cap \Omega_j.
\]

(5.37)

We then choose elliptic \( h \)-Fourier Integral Operators, \( U_j \), microlocally defined in neighbourhoods of \( \Omega \times \Omega_j \) and associated to \( \kappa_j \)'s. By Proposition 5.5

\[
U_j^{-1} \text{Op}_h^w(\chi_j) \text{Op}_h^w(\bar{\chi}_j) U_j = \text{Op}_h^w(\bar{\chi}_j) \text{Op}_h^w(\chi_j) \in \Psi^{p'}_{\Sigma', \frac{1}{2}}(\mathbb{R}^n), \quad \Sigma' \cap \Omega = \{ \xi_1 = 0 \}.
\]

In the notation of Lemma 5.6 we then define

\[
(5.38) \sigma_{\Sigma, h, \tilde{h}}(\gamma \Sigma A) = \sum_{j=1}^{J} (\kappa_j^{-1})^* a_{\tilde{x}_j} \gamma_j^2.
\]

Proposition 5.7 shows that this definition is independent of the choice \( s \) made here.

5.6. Global quantization map. From the local quantization given in \( \S 5.2 \) we can define a global map \( \text{Op}_{\Sigma, h, \tilde{h}} \). Thus let \( a \in S^m_T(T^*X) \) be a symbol in the class defined in \( 5.1 \).

Let \( \gamma \Sigma \) (where we will use the same letter for the symbol and the operator) and \( V_j \)'s be as in \( 5.21 \). By shrinking \( V_2 \) if necessary we can find a finite open cover

\[
V_2 \subset \bigcup_{j=1}^{J} \Omega_j
\]

such that for each \( j \) there exists a canonical transformation \( \kappa_j \) satisfying \( 5.37 \).

Let \( \phi_j \) be a partition of unity on \( V_2 \) subordinate to the cover by \( \Omega_j \)'s. Let \( a_j \) be the unique symbol of the form

\[
a_j = a_j(x, \xi_2, \cdots, \xi_n, \lambda; h)
\]

such that

\[
(a_j)_{\lambda=(\bar{h}/h)\xi_1} = (\gamma \Sigma \phi_j a) \circ \kappa_j.
\]

Using the \( h \)-Fourier integral operators defined after \( 5.37 \) we put

\[
(5.39) \text{Op}_{\Sigma, h, \tilde{h}}(a) \overset{\text{def}}{=} \text{Op}_h^w((1 - \gamma \Sigma)a) + \sum_{j} U_j \tilde{\text{Op}}_{h, \tilde{h}}(a_j) U_j^{-1}.
\]

In view of Proposition 5.5 we have \( \text{Op}_{\Sigma, h, \tilde{h}}(a) \in \Psi^{p'}_{\Sigma', \frac{1}{2}}(X) \).

The construction of the symbol map in \( 5.5 \) shows that

\[
\sigma_{\Sigma, h, \tilde{h}}(\text{Op}_{\Sigma, h, \tilde{h}}(a)) \equiv a \mod S^p_{\Sigma}(X), \quad p' = (m, \bar{m} - 1, k_1 - 1, k_2).
\]

We recall from \( 5.1 \) that away from \( \Sigma \),

\[
S^p_{\Sigma, \frac{1}{2}} \text{ becomes } S^{m+k_2, \bar{m}-k_2, k_1}_{\frac{1}{2}},
\]
and

$$S_{\Sigma, \frac{1}{2}}^\prime \text{ becomes } S_{\frac{1}{2}}^{m+k_2, \bar{m}-k_2-1, k_1-1}.$$  

If $\delta < 1/2$ in (5.1) then we have the usual filtration of the $h$-pseudodifferential calculus: near $\Sigma$ we only gain in $\tilde{h}$ and away from $\Sigma$, in $h$. In the case of $\delta = 1/2$ considered here in detail we only gain in $\tilde{h}$, near and away $\Sigma$.

This completes the proof of Theorem 4 and provides an explicit quantization $\text{Op}_{\Sigma, h, \tilde{h}}$.

5.7. Approximation by finite rank operators. To estimate the number of resonances we will need to use approximation by finite rank operators.

For $a \in S_{\Sigma, \frac{1}{2}}^{m, \bar{m}, -\infty, k_2}(T^*X)$ we need a notion of essential support. Unlike the essential support defined in §3.1 it now has to depend on $h, \tilde{h}$. As in [32], rather than introduce an equivalence class of families of sets, we will say that for an $(h, \tilde{h})$-dependent family of sets $W_{h, \tilde{h}} \subset T^*X$

$$\text{ess supp } a \subset W_{h, \tilde{h}} \iff \exists a' \in S_{\Sigma, \frac{1}{2}}^{m, \bar{m}, -\infty, k_2}(T^*X), \supp a' \subset W_{h, \tilde{h}}, a - a' \in S_{\Sigma, \frac{1}{2}}^{m, -\infty, -\infty, k_2}(T^*X).$$

We notice that

$$\text{ess supp } a \subset V_{h, \tilde{h}}^j, j = 1, \cdots, N \implies \text{ess supp } a \subset V_{h, \tilde{h}}^1 \cap \cdots \cap V_{h, \tilde{h}}^N,$$

so that this formal notion of essential support behaves correctly under finite products and sums. We can now state

**Proposition 5.10.** Suppose that $a \in S_{\Sigma, \frac{1}{2}}^{0, 0, -\infty, -\infty}(T^*X)$ and

$$\text{ess supp } a \subset W_{h, \tilde{h}},$$

where $W_{h, \tilde{h}}$ satisfies

$$W_{h, \tilde{h}} \subset \{(x, \xi) : d((x, \xi), \Sigma \cap W_{h, \tilde{h}}) \leq C_1 h/\tilde{h}\},$$

$$W_{h, \tilde{h}} \subset \bigcup_{k=1}^{K(h)} \exp([-1, 1]H_q)(B_k), \quad \text{diam } B_k \leq C_1(h/\tilde{h})^{\frac{1}{2}}, .$$

Then for $0 < h < h_0$, there exists a finite rank operator $R(h)$ such that for $\text{Op}_{\Sigma, h, \tilde{h}}(a) - R(h) \in \Psi_{\Sigma, \frac{1}{2}}^{0, -\infty, -\infty, -\infty}(X)$, $\text{rank } R(h) = C_2 h^{-n} K(h)$.

**Proof.** We take an open covering of $W_{h, \tilde{h}}$,

$$W_{h, \tilde{h}} \subset \bigcup_{k=1}^{K'(h)} U_k, \quad K'(h) \leq C' K(h), \quad U_k = \exp([-1/C, 1/C]H_q)V_k$$

$$\text{diam } (V_k) \leq C(h/\tilde{h})^{\frac{1}{4}}, \quad \text{sup}_{(x, \xi) \in U_k} d((x, \xi), U_k \cap \Sigma) \leq C h/\tilde{h},$$
with a partition of unity on $W_{\tilde{h},\tilde{h}}$, 
\[
\sum_{k=1}^{K'(h)} \chi_k = 1 \text{ on } W_{\tilde{h},\tilde{h}}, \quad \text{supp} \chi_k \subset U_k, \quad \chi_k \in \mathcal{S}_{\Sigma,\frac{1}{2}}^{0,0,-\infty,-\infty}(T^*X).
\]

If $\psi = 1 - \sum_k \chi_k \in \mathcal{S}_{\Sigma,\frac{1}{2}}^{0,0,-\infty,-\infty}$ then the condition on the support of $a$ shows that
\[
\forall \alpha, \beta \in \mathbb{N}^n, \quad \partial^\alpha a \partial^\beta \psi \equiv 0.
\]

Consequently the calculus of Theorem 4 gives
\[
\text{Op}_{\Sigma,\tilde{h}}(\tilde{h})A \in \Psi_{\Sigma,\frac{1}{2}}^{0,-\infty,-\infty,-\infty}(X).
\]

Hence it suffices to show that for each $k$ there exists an operator $R_k$ such that
\[
\text{Op}_{\Sigma,\tilde{h}}(\chi_k)A - R_k \in \Psi_{\Sigma,\frac{1}{2}}^{0,-\infty,-\infty,-\infty}(X), \quad \text{rank}(R_k) \leq C\tilde{h}^{-n},
\]
with $C$ independent of $k$. By taking a finer partition (with a number of elements $K''(h) \leq C'' \tilde{h}$) we can assume that
\[
\text{Op}_{\Sigma,\tilde{h}}(\chi_k)A = U_k A_k V_k
\]
where $U_k, V_k$ are $h$-semiclassical Fourier Integral Operators of the form used in the construction of $\text{Op}_{\Sigma,\tilde{h}}$, and
\[
A_k = \text{Op}_{\Sigma,\tilde{h}}(a_k)
\]
\[
\text{supp } a_k \subset \{(x, \xi', \lambda) : |\lambda| \leq C, \quad |x'| + |\xi'| \leq C(h/\tilde{h})^{\frac{1}{2}}, |x_1| < 1/C\}
\]

Consider commuting operators
\[
Q = \left(\tilde{h} D_{x_1}\right)^2 + x_1^2 + (h D_{x_2})^2 + x_2^2 + \cdots (h D_{x_n})^2 + x_n^2,
\]
\[
Q = \text{Op}_{\Sigma,\tilde{h}}(q), \quad q = \lambda^2 + x_1^2 + \cdots \xi_n^2 + x_n^2,
\]
\[
Q' = (h D_{x_2})^2 + x_2^2 + \cdots (h D_{x_n})^2 + x_n^2.
\]

If $\chi \in C_c^\infty(\mathbb{R})$, $\chi(t) = 1$ for $t \leq \tilde{C}$, $\chi(t) = 0$ for $t > 2\tilde{C}$, then
\[
\chi(\tilde{h}Q'/h) \in \Psi_{\Sigma,\frac{1}{2}}^{0,0,0,0}(\mathbb{R}^n),
\]
\[
\chi(\tilde{h}Q'/h)\chi(Q)A_k - A_k \in \Psi_{\Sigma,\frac{1}{2}}^{0,-\infty,-\infty,-\infty}.
\]

The standard analysis of the spectrum of harmonic oscillators shows that $\chi(\tilde{h}Q'/h)\chi(Q)$ is a finite rank operator and its rank is bounded by $C'\tilde{h}^{-n}$. Hence we can take $R_k = \chi(Q)\chi(\tilde{h}Q'/h)A_k$. □
6. General upper bounds in regions of size $h$

6.1. Bound for the number of eigenvalues of a self-adjoint operator. With the calculus developed in §5 we can follow the standard procedure of modifying the operator near the energy surface, now at the limiting scale. For that we introduce the second small parameter $\tilde{h}$ which eventually will be fixed, as $h \to 0$.

Let $\chi \in C_c^\infty(\mathbb{R}; [0, 1])$ be equal to one near 0. We then define

$$a(x, \xi, h) \overset{\text{def}}{=} \chi \left( \frac{\tilde{h}p(x, \xi)}{h} \right).$$

Then, in terms of Definition 5.1, $a \in S^{0,0}_{\tilde{h}}(X, \sigma; \Sigma)$, $\Sigma = p^{-1}(0)$. Although this class of symbols corresponds to a class $\Psi_{\Sigma,0}^{0}$ we will use the larger class $\Psi_{\Sigma}^{1,2}$ since it was presented in detail in §5. We stress that this is done for convenience only and an examination of §5 shows how the simpler calculus is constructed without the $S^{1/2}$ complications.

The operator

$$\tilde{P} \overset{\text{def}}{=} P + i(h/\tilde{h})A - z \ , \ A \overset{\text{def}}{=} \tilde{O}_p h, \tilde{h}(a),$$

is elliptic in $\Psi_{\Sigma}^{0,0,1}(X)$, in the sense that

$$|\sigma_{\Sigma, h}(\tilde{P})| > C \left( d(\bullet, \Sigma) + h/\tilde{h} \right).$$

This is most clearly seen locally when $p = \xi_1$ and $\lambda = (h/\tilde{h})^{-1} \xi_1$:

$$\tilde{p} = \frac{h}{\tilde{h}} (\lambda + i\chi(\lambda)), \quad |\tilde{p}| > C \frac{h}{\tilde{h}}(|\lambda|).$$

Using Theorem 4 we can construct a parametrix for $\tilde{P} - z$, $|z| \leq Ch$, uniformly in $\tilde{h}$, which for $\tilde{h}$ small enough (keeping $\tilde{h}M$ large and constant) gives an exact inverse:

$$(\tilde{P} - z)^{-1} = \mathcal{O}\left( \frac{h}{\tilde{h}} \right) : L^2(X) \to L^2(X), \quad |z| \leq Ch.$$

Proposition 5.10 gives a finite rank operator $R$ such that

$$A = R + E, \quad \text{rank}(R) \leq C\tilde{h}^{-n}h^{-n+1}, \quad E \in \Psi_{\Sigma}^{0,\infty,\infty,\infty},$$

so that in particular, by Proposition 5.3

$$\|E\|_{L^2 \to L^2} = \mathcal{O}(\tilde{h}^{\infty}).$$

Now we write

$$P - z = (\tilde{P} - z)(I - i(h/\tilde{h})(\tilde{P} - z)^{-1}(R + E))$$

$$= (\tilde{P} - z)(I - i(h/\tilde{h})(\tilde{P} - z)^{-1}E)(I + K(z)),$$
where
\[ K(z) = -i(h/\hat{h})(I + i(h/\hat{h})(\hat{P} - z)^{-1}E)^{-1}(\hat{P} - z)^{-1}R, \]
\[ \text{rank}(K(z)) \leq Mh^{-n+1}, \quad K(z) = O(1) : L^2(X) \rightarrow L^2(X). \]
This implies that
\[ h(z) \overset{\text{def}}{=} \det(I + K(z)) = O(\exp(CMh^{-n+1})), \quad |z| \leq C_0 h, \]
and that the zeros of \( h(z) \) are the eigenvalues of \( P \) in \( |z| \leq Ch \).

The bound on the number of eigenvalues will follow standard estimates \(^\dagger\) once we show that
\[ |h(z_0)| > \exp(-CMh^{-n+1}) \text{ at some } z_0, \quad |z_0| \leq C_1 h, \quad C_1 < C_0. \]
For that we take \( z_0 \) with \( \text{Im } z_0 > C_1 h \) so that, by self-adjointness,
\[ (P - z_0)^{-1} = O((C_1 h)^{-1}) : L^2(X) \rightarrow L^2(X). \]
We then see that
\[ (I + K(z_0))^{-1} = I + L(z_0), \]
where
\[ I + L(z) = (I + i(h/\hat{h})(P - z)^{-1}(R + E))(I + i(h/\hat{h})(\hat{P} - z)^{-1}E), \quad |\text{Im } z| > C_1 h. \]
The operator \( L(z_0) \) is of trace class and using the rank of \( R \) we have the estimate
\[ h(z_0)^{-1} = \det(I + L(z_0)) = O(\exp(CMh^{-n+1})), \]
which shows that
\[ |\text{Spec } P \cap D(0, h)| = O(h^{-n+1}). \]

6.2. Proof of Theorem 2. To establish the estimate (1.9) we proceed as in the case of eigenvalues but using the scaled operator \( P_\theta \) instead – see §3.5. In this section we take
\[ \theta = C_0 h \]
for \( C_0 \), a large and fixed constant. We recall from §3.1 that
\[ X = X_0 \sqcup (\mathbb{R}^n \setminus B(0, R_0)), \]
We can also have more neighbourhoods of infinity (see §3.1) but for simplicity of notation we restrict ourselves to the case above.

\(^\dagger\)The estimate we need is this: if \( \log|h(z)| \leq K \) in \( R_1 \), where \( R_1 \) is a rectangle, and \( R_2 \) is another rectangle strictly inside \( R_2 \), \( \log|h(z_0)| > -K \), for some \( z_0 \in R_2 \), then the number of zeros of \( h(z) \) in \( R_2 \) is bounded by \( C(R_1, R_2)K \), with a dilation invariant constant, \( C(tR_1, tR_2) = C(R_1, R_2) \).
The operator \( P_\theta \) can be written as \( P_\theta = P_1 + iP_2 \), where \( P_j \) are formally self-adjoint on \( L^2(X) \). We consider the Weyl symbols, \( p_1 \) and \( p_2 \) (defined, we recall, modulo \( \mathcal{O}(h^2) \)) of these two operators. In view of (3.17) we have

\[
\begin{align*}
|p_1| & \leq c_1 h \langle \xi \rangle^2, \\
|p_2| & \leq c_1 h \langle \xi \rangle^2.
\end{align*}
\]

We also note that our assumptions give

\[
|p_1| \leq \delta \implies \langle \xi \rangle \sim |\xi|,
\]

with some small fixed \( \delta \). Now let \( \chi \in C^\infty_c(\mathbb{R},[0,1]) \), satisfy

\[
\chi(t) = \begin{cases} 1 & |t| \leq 1 \\ 0 & |t| \geq 2 \end{cases}
\]

With this \( \chi \) we define

\[
a(x,\xi,h) \overset{\text{def}}{=} \chi \left( \frac{\tilde{h} p(x,\xi)}{h} \right) \chi \left( \frac{|x|}{3R_0} \right),
\]

where \( \tilde{h} \) is small,

\[
\tilde{h}^{-1} \sim C_1 \sim c_1 \sim C_0,
\]

in (6.3) but eventually fixed.

We now choose a compact \( C^\infty \) hypersurface \( \Sigma \) so that

\[
\Sigma \cap T^*_X = \{ (x,\xi) : p(x,\xi) = 0 \}.
\]

It follows that

\[
a(x,\xi,h) \in S^0_{\Sigma,0,-\infty,0}(T^*X),
\]

and continuing in the same spirit as in §6.1, we put

\[
A \overset{\text{def}}{=} \tilde{\Theta}_{h\tilde{h}}(a).
\]

On the level of symbols we have

\[
|p_1| \leq (h/\tilde{h}) \implies -p_2(x,\xi) + (h/\tilde{h})a(x,\xi,h) \geq (h/\tilde{h})/C.
\]

Hence if we put

\[
\tilde{P} = P_\theta - i(h/\tilde{h})A,
\]

then for \( |\text{Re} z| \leq Ch, \text{Im} z \geq -Ch \),

\[
(h/\tilde{h})(\tilde{P} - z) \in \Psi^0_{\Sigma,0,2,1}(X).
\]

We claim that \( \tilde{P} - z \) is invertible for \( z \in D(0,Ch) \) and

\[
\|(\tilde{P} - z)^{-1}\| = \mathcal{O}(\tilde{h}/h).
\]

In fact, let \( W \subset T^*X \) be set in which \( (h/\tilde{h})(\tilde{P} - z) \) is elliptic in \( S^0_{\Sigma,0,2,1}(T^*X) \). The estimate (6.3) shows that \( \text{Im}(\tilde{P} - z) \leq -C(h/\tilde{h}) \) on a neighbourhood of \( \mathcal{C}W \cap \{|\text{Re}(\tilde{P} - z)| < \delta\} \).
Now, let $\Psi_1$ and $\Psi_2$ be as in Lemma 3.2 with $\text{ess sup} \Psi_1 \subset W$. Then proceeding as in §4.1 we obtain
\[
\| (\tilde{P} - z) \Psi_2 u \| \geq (\tilde{h}/\tilde{h}) \| \Psi_2 u \| / C - O(\tilde{h}) \| u \| , \quad u \in C^\infty_c (X).
\]
Ellipticity of $(\tilde{h}/h)(\tilde{P} - z)$ in $W$ shows that
\[
\| (\tilde{h}/h)(\tilde{P} - z) \Psi_1 u \| \geq \| \Psi_1 u \| / C - O(\tilde{h}) \| u \| ,
\]
that is
\[
\| (\tilde{P} - z) \Psi_1 u \| \geq (\tilde{h}/h) \| \Psi_1 u \| / C - O(\tilde{h}) \| u \| .
\]
Lemma 3.3 then gives (6.5).

We can now proceed as in §6.1 and obtain the bound on the number of resonances (1.9). The complex analytic argument outlined in the footnote is used in a rectangle
\[
[\!\! -Ch, Ch \!\!] + i[\!\! C_2 h, -Ch \!\!]
\]
where $C_2$ is large enough to guarantee a lower bound for $-\text{Im}(P_\theta - z)$ when $\text{Im} z \sim C_2 h$. We can still take $C_2$ proportional to the other large constants.

7. The escape function for hyperbolic flows and its $h$ dependent regularizations

In this section we modify [29, Sect.5] and construct a regularized escape function depending on a small parameter, essentially $h/\tilde{h}$. We recall that we assume that $p \in C^\infty (T^*X; \mathbb{R})$ satisfies
\[
(7.1) \quad \begin{align*}
|p(x,\xi)| < 2\delta & \implies \exp tH_p(x,\xi) \to \infty \quad \text{for either } t \to \infty \text{ or } t \to -\infty.
\end{align*}
\]
We also recall the result of [11, Appendix]:

Proposition 7.1. Suppose that (7.1) holds and that $\tilde{K}$ is the trapped set,
\[
(7.2) \quad \tilde{K} \overset{\text{def}}{=} \{ \rho \in T^*X : \exp(tH_p)(\rho) \not\to \infty , \ t \to \pm \infty , |p(\rho)| \leq \delta \} \subseteq T^*X.
\]
Then for any two neighbourhoods, $U, V$, of $\tilde{K}$, $U \subset V$ there exists $G_0 \in C^\infty (T^*X)$ such that
\[
(7.3) \quad \begin{align*}
\text{supp } G_0 & \subset T^*X \setminus U , \quad H_p G_0 \geq 0 , \quad H_p G_0 \big|_{p^{-1}([2\delta,2\delta])} \leq C , \\
H_p G_0 \big|_{p^{-1}([-\delta,\delta]) \setminus V} & \geq 1 .
\end{align*}
\]
7.1. Dynamical assumptions. We start with the hyperbolicity assumptions [22, §5] weaker than the more standard assumptions in §7. Let $\widehat{K}$ be the compact trapped set near zero energy given by (7.2). The trapped set at zero energy is given by $K = \widehat{K} \cap p^{-1}(0)$. We also have $\widehat{K} = \widehat{\Gamma}_+ \cap \widehat{\Gamma}_-$, where

$$\widehat{\Gamma}_\pm \overset{\text{def}}{=} \{(x, \xi) \in T^*X : |p(x, \xi)| \leq \delta, \exp(tH_p)(x, \xi) \not\rightarrow \infty, t \rightarrow \mp \infty\},$$

and the sets $\widehat{K}, \widehat{\Gamma}_\pm$ are clearly invariant under the flow,

$$\exp(tH_p)(\widehat{K}) \subset \widehat{K}, \exp(tH_p)(\widehat{\Gamma}_\pm) \subset \widehat{\Gamma}_\pm.$$

We can now state the dynamical hypothesis.

- In a neighbourhood, $\Omega_{\rho_0}$ of any $\rho_0 \in K$,

$$\widehat{\Gamma}_\pm = \bigcup_{\rho \in \Omega_{\rho_0} \cap \widehat{\Gamma}_\pm} \widehat{\Gamma}_{\pm, \rho}, \quad \rho \in \widehat{\Gamma}_{\pm, \rho},$$

$$\widehat{\Gamma}_{\pm, \rho} \cap \widehat{\Gamma}_{\pm, \rho'} = \emptyset, \text{ or } \widehat{\Gamma}_{\pm, \rho} = \widehat{\Gamma}_{\pm, \rho'}.$$

- Each $\widehat{\Gamma}_{\pm, \rho}$ is a closed $C^1$ manifold of dimension $n + d$, with $d \geq 0$ fixed, and the dependence

$$\Omega_{\rho_0} \cap \widehat{\Gamma}_\pm \ni \rho \mapsto T_\rho \widehat{\Gamma}_{\pm, \rho}$$

is continuous.

- If $E^\pm_\rho \overset{\text{def}}{=} T_\rho \widehat{\Gamma}_{\pm, \rho}$, then $E^+_\rho + E^-_\rho = T_\rho p^{-1}(p(\rho)) \subset T_\rho (T^*X)$, $\mathbb{R} H_p(\rho) \in E^\pm_\rho$, and

$$\|d(\exp tH_p)(X)\| \leq Ce^{\pm \lambda t}\|X\|, \quad \rho \in K, \quad \text{for all } X \in T_\rho (T^*X)/E^\pm_\rho, \quad t \geq 0.$$  

The above definition makes sense since by (7.5) $d(\exp tH_p)_\rho (E^+_\rho) = E_{\exp tH_p(\rho)}, \rho \in \widehat{\Gamma}_\pm$, we have

$$d(\exp tH_p)_\rho T_\rho (T^*X)/E^+_\rho \rightarrow T_{\exp tH_p(\rho)} (T^*X)/E^+_\rho_{\exp tH_p(\rho)}, \quad \rho \in K,$$

and we choose continuously dependent norms in the last estimate in (7.6). We also note that $X \in T_\rho (T^*X)/E^+_\rho$ implies that $X$ can be identified with a vector tangent to $p^{-1}(p(\rho))$.

In [22, §5] it is shown that there exist two functions, $\varphi_\pm \in C^{1,1}(T^*X)$, $\varphi_\pm \geq 0$, $H^k_p \varphi_\pm \in C^{1,1}(T^*X), \ k \in \mathbb{N}$, such that for $\rho$ in a small neighbourhood of $K$,

$$H^k_p \varphi_\pm(\rho) \sim \varphi_\pm(\rho), \quad H^k_p \varphi_\pm(\rho) = \mathcal{O}(\varphi_\pm(\rho)), \quad k \in \mathbb{N},$$

$$\varphi_\pm(\rho) \sim d(\rho, \widehat{\Gamma}_\pm), \quad \varphi_+(\rho) + \varphi_-(\rho) \sim d(\rho, \widehat{K})^2,$$

and where $d(\bullet, \Gamma)$ is the distance to a closed set $\Gamma$. The notation $f \sim g$, means that there exists a constant $C > 0$ such

$$0 \leq g/C \leq f \leq C g.$$

The simple model (2.3) is given in [22] Here we modify the construction to obtain suitably regularized functions $\widehat{\varphi}_\pm$. 

[raw_text]
7.2. Regularization of \( \varphi_\pm \). We start with two general lemmas:

**Lemma 7.2.** Suppose \( \Gamma \subset \mathbb{R}^m \) is a closed set. For any \( \epsilon > 0 \) there exists \( \varphi_\epsilon \in C^\infty(\mathbb{R}^m) \) such that

\[
\varphi_\epsilon \geq \epsilon, \quad \varphi_\epsilon \sim d(\bullet, \Gamma)^2 + \epsilon, \quad \partial^\alpha \varphi_\epsilon = O(\varphi_\epsilon^{1-|\alpha|/2}),
\]

uniformly on compact sets.

**Proof.** We can find a sequence \( x_j \in \mathbb{R}^m \) such that

\[
\bigcup_j B(x_j, d(x_j, \Gamma)/8) = \mathbb{R}^m \setminus \Gamma,
\]

every \( x \in Q \setminus \Gamma, Q \in \mathbb{R}^m \), is in at most \( N_0 = N_0(Q) \) balls \( B(x_j, d(x_j, \Gamma)/2) \).

Let \( \chi \in C^\infty_c(\mathbb{R}^m; [0,1]) \) be supported in \( B(0,1/4) \), and be identically one in \( B(0,1/8) \). We define

\[
\varphi_\epsilon(x) \overset{\text{def}}{=} \epsilon + \sum_{d(x_j, \Gamma) > \sqrt{\epsilon}} d(x_j, \Gamma)^2 \chi\left(\frac{x - x_j}{d(x_j, \Gamma) + \sqrt{\epsilon}}\right).
\]

We first note that the number non-zero terms in the sum is uniformly bounded by \( N_0 \). In fact, \( d(x_j, \Gamma) + \sqrt{\epsilon} < 2d(x_j, \Gamma) \), and hence if \( \chi((x - x_j)/(d(x_j, \Gamma) + \sqrt{\epsilon})) \neq 0 \) then

\[
1/4 \geq |x - x_j|/(d(x_j, \Gamma) + \sqrt{\epsilon}) \geq (1/2)|x - x_j|/d(x_j, \Gamma),
\]
and \( x \in B(x_j, d(x_j, d(x_j, \Gamma))/2) \). This shows that \( \varphi_\epsilon(x) \leq 2N_0(\epsilon + d(x, \Gamma)^2) \), and

\[
\partial^\alpha \varphi_\epsilon(x) = O((d(x, \Gamma)^2 + \epsilon)^{1-|\alpha|/2}),
\]
uniformly on compact sets.

To see the lower bound on \( \varphi_\epsilon \) we first consider the case when \( d(x, \Gamma) \leq C\sqrt{\epsilon} \).

\[
\varphi_\epsilon(x) \geq \epsilon \geq (\epsilon + d(x, \Gamma)^2)/C'.
\]

If \( d(x, \Gamma) > C\sqrt{\epsilon} \) then for at least one \( j \), \( \chi((x - x_j)/(d(x_j, \Gamma) + \sqrt{\epsilon})) = 1 \) (since the balls \( B(x_j, d(x_j, \Gamma)/8) \) cover the complement of \( \Gamma \), and \( \chi(t) = 1 \) if \( |t| \leq 1/8 \)). Thus

\[
\varphi_\epsilon(x) \geq \epsilon + d(x_j, \Gamma)^2 \geq (\epsilon + d(x, \Gamma)^2)/C,
\]
which concludes the proof. \( \square \)

For future use we also record the following

**Lemma 7.3.** Suppose \( \varphi \in C^{1,1}(\mathbb{R}^m), \varphi \geq 0 \), and for a vectorfield \( V \in C^\infty(\mathbb{R}^m; \mathbb{R}^m), V^k \varphi = O(\varphi) \), \( V^k \varphi \in C^{1,1}(\mathbb{R}^m) \), \( k \in \mathbb{N} \). Then, uniformly on compact sets,

\[
dV^k \varphi = O(\varphi^{1/2}), \quad k \in \mathbb{N}.
\]
Proof. For some $C > 0$ the $C^1$ function $C \varphi - V^k \varphi$ is non-negative. Hence using the standard estimate based on Taylor’s formula,
\[ |d \varphi|^2 = O(\varphi), \quad |d(C \varphi - V^k \varphi)|^2 = O(C \varphi - V^k \varphi) = O(\varphi). \]
The lemma follows. \hfill \Box

We now have

**Proposition 7.4.** Let $\hat{\Gamma}_\pm$ be given by (1.3). For any small $\epsilon > 0$ there exist functions $\hat{\varphi}_\pm \in C^\infty(T^*X; [0, \infty))$ such that in a neighbourhood of $\hat{K}$,
\[
\hat{\varphi}_\pm(\rho) \sim d(\rho, \hat{\Gamma}_\pm)^2 + C\epsilon, \\
\mp H_p \hat{\varphi}_\pm(\rho) + C\epsilon \sim \hat{\varphi}_\pm(\rho), \\
\partial^\alpha H_p^k \hat{\varphi}_\pm(\rho) = O(\hat{\varphi}_\pm(\rho)^{1-|\alpha|/2}), \quad k \in \mathbb{N}, \\
\hat{\varphi}_+(\rho) + \hat{\varphi}_-(\rho) \sim d(\rho, \hat{K})^2 + C\epsilon.
\]

**Proof.** We modify the arguments of [29, §5], roughly speaking, adding an $O(\epsilon)$ error to all the estimates. Let $\varphi_\pm$ be the functions obtained using Lemma 7.2 with $\Gamma = \Gamma_\pm$. We now put
\[
\hat{\varphi}_\pm(\rho) \overset{\text{def}}{=} \int g_T(t) \varphi_\pm(\exp t H_p(\rho))dt,
\]
where $g_T \in C^\infty((-1, T + 1))$, $\text{supp } g_T' \subset [-1, 1] \cup [T - 1, T + 1]$, $g_T'[-1, 1] \geq 0$, $g_T'[T-1, T+1] \leq 0$, $g_T'(0) = 1$, $g_T'(T) = -1$.

To check (7.4) we note that, by definition, $\varphi_\pm(\rho) \sim d(\rho, \hat{\Gamma}_\pm)^2 + C\epsilon$. The assumptions (7.6) imply (see [29, Lemma 5.2]) that
\[
\exists C, \forall T \geq 0, \exists \Omega_T \supset K, \text{an open set}, \quad d(\exp(\pm T H_p)(\rho), \hat{\Gamma}_\pm) \leq C e^{-T/C} d(\rho, \hat{\Gamma}_\pm).
\]
Hence, with constants depending on $T$,
\[
\hat{\varphi}_+(\rho) \sim \varphi_+(\exp(T H_p(\rho))) \sim \varphi_+(\rho) \sim d(\rho, \Gamma_\pm)^2 + C\epsilon, \\
\hat{\varphi}_-(\rho) \sim \varphi_-(\rho) \sim d(\rho, \Gamma_-)^2 + C\epsilon.
\]
This shows the first statement in (7.4).

The assumptions on $g_T$ also show that
\[
H_p \hat{\varphi}_\pm(\rho) \sim \varphi_\pm(\exp T H_p(\rho)) - \varphi_\pm(\rho) \sim d(\exp T H_p(\rho), \hat{\Gamma}_\pm)^2 - d(\rho, \hat{\Gamma}_\pm)^2 + O(\epsilon).
\]
so that for $T$ large enough and for $\rho$ in a small neighbourhood of $K$, (again with $T$ dependent constants)
\[
\mp H_p \hat{\varphi}_\pm(\rho) + C\epsilon \sim d(\rho, \hat{\Gamma}_\pm)^2 + C'\epsilon \sim \hat{\varphi}_\pm(\rho).
\]
This proves the second part of (7.7). The third part is proved using Lemma 7.3 for \(|\alpha| = 1\) and the estimates on \(\varphi_\pm\) in general.

To prove the last statement in (7.7) we first see that the transversality, \(E^\pm_{\rho_0} + E^-_{\rho_0} = T_{\rho_0}(T^*X)\), and the continuity, \(\rho \mapsto E^\pm_{\rho}\), assumed in (7.6) imply that for \(\rho, \rho_1, \rho_2\), near a point \(\rho_0 \in K\),

\[
d(\rho, \hat{\Gamma}_{+,.\rho_1} \cap \hat{\Gamma}_{-.\rho_2}) \sim d(\rho, \hat{\Gamma}_{+.\rho_1} + d(\rho, \hat{\Gamma}_{-.\rho_2}).
\]

Hence

\[
\hat{\varphi}_+(\rho) + \hat{\varphi}_-(\rho) + O(\epsilon) \sim d(\rho, \hat{\Gamma}_+)^2 + d(\rho, \hat{\Gamma}_-)^2 + C\epsilon
\]

\[
\leq d(\rho, \hat{\Gamma}_{+.\rho'})^2 + d(\rho, \hat{\Gamma}_{-.\rho'})^2 + C\epsilon
\]

\[
\sim d(\rho, \hat{\Gamma}_{+.\rho'} \cap \hat{\Gamma}_{-.\rho'})^2 + C\epsilon.
\]

If we choose \(\rho' \in K\) so that \(d(\rho, \hat{K}) = d(\rho, \rho')\) then

\[
d(\rho, \hat{\Gamma}_{+.\rho'} \cap \hat{\Gamma}_{-.\rho'})^2 \leq d(\rho, \rho')^2 = d(\rho, \hat{K})^2,
\]

proving that

\[
\hat{\varphi}_+(\rho) + \hat{\varphi}_-(\rho) \leq d(\rho, \hat{K})^2 + O(\epsilon).
\]

The opposite inequality is obtained by choosing \(\rho_\pm \in \hat{\Gamma}_\pm\) such that \(d(\rho, \rho_\pm) = d(\rho, \hat{\Gamma}_\pm)\). Then using the transversality of \(\hat{\Gamma}_+, \hat{\Gamma}_-\)

\[
d(\rho, \hat{K})^2 \leq d(\rho, \hat{\Gamma}_{+.\rho_+} \cap \hat{\Gamma}_{-.\rho_-})^2 \sim d(\rho, \hat{\Gamma}_{+.\rho_+})^2 + d(\rho, \hat{\Gamma}_{-.\rho_-})^2
\]

\[
\leq d(\rho, \rho_+)^2 + d(\rho, \rho_-)^2 = d(\rho, \hat{\Gamma}_+)^2 + d(\rho, \hat{\Gamma}_-)^2
\]

\[
\leq \hat{\varphi}_+(\rho) + \hat{\varphi}_-(\rho) + O(\epsilon).
\]

\[\square\]

### 7.3. Regularized escape function.

We now use the functions constructed in Proposition 7.4 to obtain an escape function near \(K\). We first need the following

**Lemma 7.5.** Then for \(|\alpha| + k \geq 1\) we have

\[
\partial^\alpha_p H^k_p \log(\hat{\varphi}_\pm) = O(\hat{\varphi}_\pm^{-\frac{|\alpha|}{2}})
\]

**Proof.** Let \(f(t) = \log(t)\). Then

\[
f^{(k)}(\hat{\varphi}_\pm) = O\left(\frac{1}{\hat{\varphi}_\pm^k}\right), \quad k \geq 1,
\]

and for \(|\alpha| + k \geq 1\), \(\partial^\alpha_p H^k_p f(\hat{\varphi}_\pm)\) is a finite linear combination of terms

\[
f^{(l)}(\hat{\varphi}_\pm) \left(\partial^\alpha_p H^k_p \hat{\varphi}_\pm\right) \cdots \left(\partial^\alpha_p H^k_p \hat{\varphi}_\pm\right) = O(1) \prod_{j=1}^\ell \frac{\partial^\alpha_p H^k_p \hat{\varphi}_\pm}{\hat{\varphi}_\pm},
\]
with
\[ |\alpha_j| + k_j \geq 1, \quad \alpha_1 + \cdots + \alpha_\ell = \alpha, \quad k_1 + \cdots + k_\ell = k. \]
The estimates in (7.7) show that \( \partial^\alpha H_p^k \hat{\varphi}_\pm / \varphi_\pm = \mathcal{O}(\varphi_\pm^{-|\alpha|/2}) \), and hence
\[ \partial^\alpha H_p^k (\hat{\varphi}_\pm) = \mathcal{O}(\varphi_\pm^{-|\alpha|}), \]
proving the lemma. □

We are now ready for the main results of this section.

**Lemma 7.6.** Let \( \hat{\varphi}_\pm \) be given in Proposition 7.4 and
\begin{equation}
(7.8) \quad \hat{G} \overset{\text{def}}{=} (\log(M\epsilon + \hat{\varphi}_-) - \log(M\epsilon + \hat{\varphi}_+)).
\end{equation}
Then in a neighbourhood of \( K \) we have
\begin{equation}
(7.9) \quad \partial^\alpha H_p^k \hat{G} = \mathcal{O}(\min(\hat{\varphi}_+, \hat{\varphi}_-) - |\alpha|) = \mathcal{O}(\epsilon^{-|\alpha|/2}), \quad |\alpha| + k \geq 1,
\end{equation}
due to the second estimate, \( M \) has to be chosen large enough, independently of \( \epsilon \), and \( C \) is a large constant.

**Proof.** We observe that, with constants depending on \( M, \hat{\varphi}_\pm + M\epsilon \) has the same properties as \( \hat{\varphi}_\pm \). Hence the estimates on \( \partial^\alpha H_p^k \hat{G} \) follow directly from the definition (7.8) and from Lemma 7.5. To check the second part of (7.9), we compute, using Proposition 7.4,
\[ H_p \hat{G} = \left( \frac{H_p \hat{\varphi}_-}{\hat{\varphi}_- + M\epsilon} - \frac{H_p \hat{\varphi}_+}{\hat{\varphi}_+ + M\epsilon} \right) \geq \frac{1}{C_1} \left( \frac{\hat{\varphi}_- - C_2 \epsilon}{\hat{\varphi}_- + M\epsilon} + \frac{\hat{\varphi}_+ - C_2 \epsilon}{\hat{\varphi}_+ + M\epsilon} \right). \]
From (7.7) we also have
\[ d(\rho, \hat{K})^2 \geq C\epsilon \quad \Rightarrow \quad \max(\hat{\varphi}_+, \hat{\varphi}_-) \geq (C/2 - O(1))\epsilon > C_3 \epsilon, \]
where \( C_3 \) can be as large as we like depending on the choice of \( C \). Hence, since \( x \mapsto (x - C_2)/(x + M) \) is increasing,
\[ H_p \hat{G} \geq \frac{1}{C_1} \left( \frac{C_3 - C_2}{C_3 + M} - \frac{C_2}{M} \right) \geq \frac{1}{C}, \]
if we choose \( C_3 \gg M \gg C_2 \). □

We now modify \( \hat{G} \) using \( G_0 \) given in Proposition 7.4.

**Proposition 7.7.** Let us fix \( \delta_0 > 0 \). Then there exist \( \hat{\chi}, \chi_0 \in C^\infty_c(T^*X) \), \( C_0 > 0 \), and a neighbourhood \( V \) of \( K \), such that
\[ G \overset{\text{def}}{=} \hat{\chi} \hat{G} + C_0 \left( \log \frac{1}{\epsilon} \right) \chi_0 G_0, \]
satisfies
\[
\partial^\alpha H^k_p G = \begin{cases} O(\log(1/\epsilon)) & \alpha = 0 \\ O(\epsilon^{-|\alpha|/2}) & \text{otherwise} \end{cases},
\]
\quad (7.10)
d \in (\rho, \delta)^2 \geq C\epsilon, \rho \in V \implies H_p G(\rho) \geq 1/C, \\
\rho \in p^{-1}([-\delta, \delta]) \setminus V, |x(\rho)| \leq 3R_0 \implies H_p G(\rho) \geq \log(1/\epsilon), \\
H_p G(\rho) \geq -\delta_0 \log(1/\epsilon), \rho \in T^* X.
\]
In addition we have
\[
\exp G(\rho) \leq C_0 \left( \frac{\rho - \mu}{\sqrt{\epsilon}} \right)^{N_0},
\]
for some constants $C_0$ and $N_0$.

Proof. We obtain $G_0$ from Proposition 7.1 taking for $V$ a neighbourhood of $\hat{K}$ in which the estimates of Lemma 7.6 hold. We have $\partial^\alpha H^k_p G_0 = O_{k,|\alpha|}(1)$, and consequently for any $\chi_0 \in \mathcal{C}_c^\infty(T^* X)$,
\[
\partial^\alpha H^k_p (\log(1/\epsilon)\chi_0 G_0) = O_{k,|\alpha|}(\log(1/\epsilon)) = \begin{cases} O(\log(1/\epsilon)) & \alpha = 0 \\ O(\epsilon^{-|\alpha|/2}) & \text{otherwise} \end{cases}.
\]
From Lemma 7.6 we obtain, again for any $\hat{\chi} \in \mathcal{C}_c^\infty(T^* X)$,
\[
\partial^\alpha H^k_p (\hat{\chi} \hat{G}) = \begin{cases} O(\log(1/\epsilon)) & \alpha = 0 \\ O(\epsilon^{-|\alpha|/2}) & \text{otherwise} \end{cases}.
\]
The loss compared to $(7.9)$ is due to the presence of the cut-off function.

We take $\chi_0 \in \mathcal{C}_c^\infty(T^* X; [0, 1])$ to be identically equal to 1 in
\[
p^{-1}([-\delta, \delta]) \cap \{(x, \xi) : |x| \leq 3R_0\}.
\]
For $\hat{\chi} \in \mathcal{C}_c^\infty(T^* X)$ we take a function which is supported in a neighbourhood of $\hat{K}$ where $(7.9)$ holds, and identically 1 in $V$. Hence for $\rho \in p^{-1}([-\delta, \delta]) \setminus V, |x(\rho)| \leq 3R_0$, $H_p G(\rho) = C_0 \log(1/\epsilon) H_p G_0(\rho) + H_p (\hat{\chi} \hat{G})(\rho) \geq C_0 \log(1/\epsilon) - O(1) \log(1/\epsilon) \geq \log(1/\epsilon)$, if $C_0$ is taken large enough. For $\rho \in V, \hat{\chi}(\rho) = 1$, and
\[
H_p G(\rho) = C_0 \log(1/\epsilon) H_p G_0(\rho) + H_p \hat{G}(\rho) \geq H_p \hat{G}(\rho),
\]
and if $d(\rho, \hat{K}) \geq C\epsilon, H_p G(\rho) \geq 1/C$. To complete the proof of $(7.10)$ we need to define $\chi_0$ for $|x| \geq R_0$. That is essentially done as in $(7.2)$ where it was based on Lemma 4.1. Let $T$ and $R$ be large positive constants to be fixed later, $\chi(t)$ be given by Lemma 4.1 and let $\psi \in \mathcal{C}_c^\infty(\mathbb{R}; [0, 1])$ be equal to 1 for $|t| \leq 1$, and to 0 for $|t| \geq 2$. We define
\[
\chi_0(\rho) \overset{\text{def}}{=} \frac{\chi(G_0(\rho))}{G_0(\rho)} \psi \left( \frac{p(\rho)}{\delta} \right) \psi \left( \frac{|x(\rho)|}{R} \right).
\]
Then
\[ H_p(\chi_0 G_0)(\rho) = \chi'(G_0(\rho)) H_p G_0(\rho) \psi \left( \frac{p(\rho)}{\delta} \right) \psi \left( \frac{|x(\rho)|}{R} \right) \]
\[ + \frac{1}{R} \chi(G_0(\rho)) \psi \left( \frac{p(\rho)}{\delta} \right) \psi' \left( \frac{|x(\rho)|}{R} \right) H_p(|x|)(\rho), \]
and
\[ H_p(\chi_0 G_0)(\rho) \geq -C_1 \left( \alpha + \frac{T}{R} \right), \]
where \( C_1 \) is independent of \( T \) and \( R \): we note that \( \text{Lemma 7.6} \) guarantees the boundedness of \( H_p G_0 \), and the assumptions on \( p \) imply that \( H_p(|x|) \) is uniformly bounded for \( |p| \leq 2 \delta \). For any \( \alpha > 0 \) we can choose \( T = T(\alpha) \) such that \( |G_0(\rho)| \leq \alpha T \) for \( |x(\rho)| \leq 3R_0, |p(\rho)| \leq 2 \delta \).

We then choose \( \alpha \) and \( R \) so that
\[ C_0 C_1 (\alpha + T(\alpha)/R) < \delta_0. \]
Hence for \( |x(\rho)| \geq R_0 \)
\[ H_p G = C_0 \log(1/\epsilon) H_p(\chi_0 G_0) \geq -\delta_0 \log(1/\epsilon), \]
which is the last statement in \( \text{Lemma 7.6} \).

It remains to show \( \text{Lemma 7.10} \) and for simplicity of presentation we replace \( T^* X \) with \( \mathbb{R}^{2n} \). We first prove that
\[ (7.12) \quad \hat{\varphi}_\pm(\rho) + M\epsilon \leq \frac{C_1}{\varphi_\pm(\mu) + M\epsilon}, \quad M \geq 0, \]
with constants depending on \( M \). We can replace \( \hat{\varphi}_\pm + M\epsilon \) with \( \hat{\varphi}_\pm \), as \( \hat{\varphi}_\pm + M\epsilon \sim_M \hat{\varphi}_\pm \).

Thus we claim that,
\[ \frac{\hat{\varphi}_\pm(\rho)}{\varphi_\pm(\mu)} \leq C_1 \left( \frac{\rho - \mu}{\sqrt{\epsilon}} \right)^2. \]

Since \( \hat{\varphi}_\pm \sim d(\bullet, \Gamma_\pm)^2 + \epsilon, \varphi_\pm \geq \epsilon, \) we have
\[ \hat{\varphi}_\pm(\rho) \leq C(d(\rho, \Gamma_\pm)^2 + \epsilon) \leq C(d(\mu, \Gamma_\pm)^2 + |\mu - \rho|^2 + \epsilon) \leq C'(\hat{\varphi}_\pm(\mu) + |\mu - \rho|^2) = C'((\hat{\varphi}_\pm(\mu) + \epsilon(|\mu - \rho|)/\sqrt{\epsilon})^2) \leq 2C'\hat{\varphi}_\pm(\mu)((\rho - \mu)/\sqrt{\epsilon})^2. \]

In the notation of \( \text{Lemma 7.6} \) \( \text{Lemma 7.12} \) gives
\[ |\hat{\chi}(\rho) - \hat{G}(\mu)| \leq C + 2 \log(|\rho - \mu|/\sqrt{\epsilon}), \]
and with \( \hat{\chi} \in C_\infty^\infty \),
\[ |\hat{\chi}(\rho)\hat{G}(\rho) - \hat{\chi}(\mu)\hat{G}(\mu)| \leq C|\rho - \mu| \log(1/\epsilon) + C\log(|\rho - \mu|/\sqrt{\epsilon}). \]

Clearly,
\[ |\chi_0(\rho)G_0(\rho) - \chi_0(\mu)G_0(\mu)| \leq C|\rho - \mu| \log(1/\epsilon), \]
and hence to obtain (7.11) we need
\[ |\rho - \mu| \log(1/\varepsilon) \leq C \log((\rho - \mu)/\sqrt{\varepsilon}) + C, \quad \rho, \mu \in Q \subset \mathbb{R}^{2n}. \]
If we put \( \delta = \sqrt{\varepsilon}, t = |\rho - \mu|/(C\delta) \) this becomes
\[
\delta \log \frac{1}{\delta} \leq \frac{\log(t) + 1}{t}, \quad 0 \leq t \leq \frac{1}{\delta},
\]
and that is clear as \( t \mapsto (\log(t) + 1)/t \) is decreasing. \( \square \)

8. Proof of the main result

Let \( G \) be the escape function given in Proposition 7.7, \( \varepsilon = \hbar/\tilde{\hbar} \). and let \( G^w \) be its Weyl quantization,
\[ G^w = \mathcal{O}(\log(\tilde{\hbar}/\hbar)) : L^2(X) \rightarrow L^2(X). \]
We define a family of conjugated operators:
\[
P_{\theta,t} \overset{\text{def}}{=} e^{-tG^w}Pe^{tG^w}, \quad \theta = C_0 \hbar \log(1/\hbar).
\]
It is easy to see that, in the notation of §3.3,
\[
\exp(-tG^w)Q \exp(tG^w) \in \Psi^0_{1/2}(\mathbb{R}^n),
\]
that is \( \exp(tG^w) = B^w_t, \partial^\alpha B_t = \mathcal{O}(\hbar^{-|\alpha|/2}\tilde{\hbar}^{|\alpha|/2}). \) Finer estimates are however possible thanks to the results of Bony-Chemin [4]. The first of these is given in

Lemma 8.1. Suppose that \( Q \in \Psi^{0,0,0}_{1/2}(\mathbb{R}^n) \). Then
\[ \exp(-tG^w)Q \exp(tG^w) \in \Psi^{0,0,0}_{1/2}(\mathbb{R}^n). \]

Proof. We follow §3.3 and change to the variables
\[
(\tilde{x}, \tilde{\xi}) = (\tilde{\hbar}/\hbar)^{1/2} (x, \xi),
\]
\[ \tilde{G}(\tilde{x}, \tilde{\xi}) = G(x, \xi), \quad \tilde{Q}_t(\tilde{x}, \tilde{\xi}) = Q_t(x, \xi), \]
\[ U^{-1}G^w(x, hD)U = \tilde{G}^w(\tilde{x}, \tilde{\hbar}D_{\tilde{x}}), \quad U^{-1}Q^w_t(x, hD)U = \tilde{Q}^w_t(\tilde{x}, \tilde{\hbar}D_{\tilde{x}}), \]
\[ Uv(\tilde{x}) = (\tilde{\hbar}/\hbar)^{1/2} v((\hbar/\tilde{\hbar})^{1/2} \tilde{x}). \]
We also note that
\[ R \in \Psi^{0,0,0}_{1/2}(\mathbb{R}^n) \iff U^{-1}RU \in \Psi^{0,0}(\mathbb{R}^n), \]
where on the right, \( \tilde{\hbar} \) is the small parameter – see the proof of Lemma 3.3. The estimate (7.11) shows that, in \((\tilde{x}, \tilde{\xi})\) coordinates, \( \tilde{G} \) satisfies the hypothesis of Proposition 3.7 and that proves (8.3). \( \square \)

The basic properties of \( P_{t,\theta} \) are given in
Proposition 8.2. Let $P_{\theta,t}$ be given by (8.1) and let $\Sigma \Subset T^*X$ be a compact surface coinciding with $p^{-1}(0)$ in a neighbourhood of the support of $G$. Then for $|t| \leq C$,

$$P_{\theta,t} = P_{\theta} - ithOp_h^w(H_pG) + E_t, \quad P_{\theta} - ithOp_h^w(H_pG) \in \Psi_{\Sigma, \frac{1}{2}}^{0,0,2}(X) \cap \Psi_{\frac{1}{2}}^{0,0,2}(X),$$

(8.4)

$$E_t \in \Psi_{\frac{1}{2}}^{-1,-1,0}(X), \quad E_t = O(h\tilde{h}) : L^2(X) \to L^2(X),$$

uniformly in $h$ and $\tilde{h}$.

Proof. Let $V_1, V_2$ be open neighbourhoods of $\text{supp} G$,

$$\text{supp} G \subset V_1 \Subset V_2 \Subset T^*X.$$ 

We first observe that if $\Psi \in \Psi_{\Sigma, \frac{1}{2}}^{0,0,2}(X)$ satisfies

$$\text{WF}_h(\Psi) \subset V_2, \quad \text{WF}_h(I - \Psi) \subset \overline{V}_1,$$

then

$$\exp(tG^w), \Psi \in \Psi_{\Sigma, \frac{1}{2}}^{0,0,2}(X), \quad (I - \Psi)(\exp(tG^w) - I) \in \Psi_{\Sigma, \frac{1}{2}}^{0,0,2}(X), \quad |t| \leq 1.$$ 

In fact, using the calculus in §3.3 we see that

$$\frac{d}{dt}[\exp(tG^w), \Psi] = G^w[\exp(tG^w), \Psi] + [G^w, \Psi] \exp(tG^w)$$

$$= G^w[\exp(tG^w), \Psi] + A_t, \quad A_t \in \Psi_{\Sigma, \frac{1}{2}}^{0,0,2}(X).$$

Thus

$$[\exp(tG^w), \Psi] = \int_0^t \exp((t - s)G^w)A_s ds \in \Psi_{\Sigma, \frac{1}{2}}^{0,0,2}(X),$$

which is the first statement in (8.5). We also compute

$$\frac{d}{dt}(I - \Psi)(\exp(tG^w) - I) = (I - \Psi)G^w \exp(tG^w) \in \Psi_{\Sigma, \frac{1}{2}}^{0,0,2}(X),$$

and the second statement in (8.5) follows. Treating the equivalence of $(I - \Psi)P_{\theta}e^{tG^w}$ and $(I - \Psi)P_{\theta}$ similarly we conclude that

$$P_{\theta,t} - e^{-tG^w}P_{\theta}e^{tG^w} - (I - \Psi)P_{\theta} \in \Psi_{\Sigma, \frac{1}{2}}^{0,0,2}(X).$$

We now put

$$Q_{\theta} \defeq \Psi P_{\theta} \in \Psi^{0,0,0}(X), \quad Q_{\theta,t} \defeq e^{-tG^w}Q_{\theta}e^{tG^w},$$

and we only need to prove (8.4) with $P_{\theta}$ replaced by $Q_{\theta}$. By a localization argument similar to the one used to construct $Q_{\theta}$, we can assume that $X = \mathbb{R}^n$ when applying Lemma 8.1 and that shows that

$$Q_{\theta,t} \in \Psi_{\frac{1}{2}}^{0,0,0}(X).$$

We now establish the expansion in (8.4). Lemma 8.1 implies that

$$[Q_{\theta}, G^w] = (h/i)Op_h^w(H_pG) + R,$$
where $R \in \Psi^{-3/2,-3/2,0}(X) \subset \Psi^{-1,-1,0}(X)$. It also shows that

$$[[Q_\theta, G^u], G^u] = (h/i)[\text{Op}_h(H_{p_0}G), G^u] + [R, G^u] \in \Psi^{-1,-1,0}(X).$$

Here we used the special structure of $G$,

$$G = \tilde{\chi} \hat{G} + C_0 \log(1/h) \chi_0 G_0,$$

where $\tilde{\chi}, \chi_0$ and $G_0$ are uniformly smooth. When derivatives fall on these terms in error estimates (3.12) the gain in $h$ compensates for the logarithmic growth, while for $|\alpha| > 0$, $\partial^\alpha \hat{G} \in S^{(\alpha/2, -|\alpha|/2)}$.

This gives,

$$\frac{d}{dt} E_t = [Q_{\theta,t}, G^u] - (h/i)\text{Op}_h^u(H_{p_0}G) + (h/i)\text{Op}_h^u(H_{p_0-p}G) = [Q_{\theta,t} - Q_\theta, G^u] + R_t,$$

with 

$$E_0 = (h/i)\text{Op}_h^u(H_{p_0-p}G) \in (h \log(1/h))^2 \Psi^{0,0,0}_1(X) \subset \Psi^{-1,-1,0}(X),$$

and $R_t \in \Psi^{-1,-1,0}(X)$. We also have

$$\frac{d}{dt} [(Q_{\theta,t} - Q_\theta), G^u] = e^{-tG^u}[[Q_\theta, G^u], G^u]e^{tG^u} \in \Psi^{-1,-1,0}(X), \quad Q_{\theta,0} - Q_\theta = 0.$$

Hence $[Q_{\theta,t} - Q_\theta, G^u] \in \Psi^{-1,-1,0}(X)$, and consequently $E_t \in \Psi^{-1,-1,0}$.

To show that

$$Q^0_{\theta,t} \overset{\text{def}}{=} Q_\theta - thi\text{Op}_h^u(H_{p}G) \in \Psi^{-0,0,0}_1(X),$$

it suffices to show, in view of Definition 2, that for any sequence 

$$\{a_j\}_{j=1}^M \subset S^0(T^*X),$$

we have

$$\|\text{ad}_{\text{Op}_h^u(a_1)} \cdots \text{ad}_{\text{Op}_h^u(a_M)} \circ \text{ad}_{P}^k Q^0_{\theta,t} u\|_{L^2(X)} \leq C h^{M/2+k} \tilde{h}^{M/2} \|u\|_{L^2(X)}.$$

This will follow if we show that

$$\text{ad}_{P}^k Q^0_{\theta,t} \in \Psi^{-k,0,0}_1(X),$$

and since that is clear for $Q_\theta$ we only need to check that

$$h \text{ad}_{P}^k \text{Op}_h^u(H_p G) \in \Psi^{-k,0,0}_1(X).$$

This follows from the following stronger result

**Lemma 8.3.** Let $G$ and $P$ be as above. Then for any $\epsilon > 0$

$$\text{ad}_{P}^k (H_p G)^w \in \Psi^{-k-\epsilon,0,0}_1(X), \quad \ell \in \mathbb{N}, \quad \ell \geq 1.$$
Proof. We start by proving that
\begin{equation}
\text{ad}_P(H_p G_1)^w \in \Psi^{-\ell+e,0,0}_{\frac{1}{2}}(X), \quad \ell \geq 1,
\end{equation}
where $G_1 = \hat{G} \chi$ is given in Proposition 7.7. We claim that $\text{ad}_P(H_p G_1)^w = E_\ell^w$, where
\[
E_\ell \sim \left(\frac{\hbar}{\ell}\right)^\ell \left( H_p^{\ell+1} G_1 + \sum_{s=0}^{\ell-1} \sum_{r=2(\ell-s)}^\infty h^r V^r_{rs}(H_p^{s+1} G_1) \right),
\]
with $V^r_{rs}$ a differential operator of order less than or equal to $r + \ell - s$, and where in view of good symbolic properties of $P$ the error is $O(h^\infty)$. In fact, Lemma 3.6 gives for any $a \in S_{0,0,0}^1$, $\text{ad}_P a^w = a_1^w$,
\[
a_1 \sim (h/i) \left( H_p a + \sum_{r=2}^\infty h^r V_r(a) \right),
\]
where $V_r$ is a differential operator of order $r+1$. The expansion for $E_\ell$ comes from iterating this and observing that for any differential operator $B$ of order $q$
\begin{equation}
H^m_p B H^k_p G_1 = \sum_{s=0}^m B_s H^{k+s}_p G_1,
\end{equation}
where each $B_s$ is a differential operator of order $q$. Using (7.9) we now see that for $r \geq 2(\ell - s)$,
\[
h^r h^r V^r_{rs}(H_p^{s+1} G_1) \in h^{r+\ell-2s+1} S^{(r+\ell-2s)/2+\epsilon,(r-\ell-s)/2,0}_{\frac{1}{2}} \subset S^{-(\ell-1)/2+\epsilon,0,0}_{\frac{1}{2}},
\]
where the $\epsilon > 0$ correction in the order comes from the term $(H_p^{s+1} \hat{G}) \in S^{r,0,0}_{\frac{1}{2}}$. Hence
\[
E_\ell = (h/i)^\ell (H_p^{\ell+1} G_1)^w + O(h^{\ell+1}) \Psi^{-\ell+e,0,0}_{\frac{1}{2}}(X) \in \Psi^{-\ell+e,0,0}_{\frac{1}{2}}(X),
\]
which gives (8.7). On the other hand, again in the notation of Proposition 7.7,
\[
\text{ad}_P(H_p G_0)^w \in \Psi^{-\ell,0}_{\frac{1}{2}}(X),
\]
and consequently
\[
\text{ad}_P(\log(1/\hbar) H_p G_0)^w \in \log(1/\hbar) \Psi^{-\ell,0,0}_{\frac{1}{2}}(X) \subset \Psi^{-\ell+e,0,0}_{\frac{1}{2}}(X).
\]
Since $G = G_1 + \log(\hbar/\hbar) \chi \chi_0 G_0$ the lemma follows. □

The lemma immediately gives (8.6) completing the proof of Proposition 8.2. □

The next lemma follows from Proposition 7.7.
Lemma 8.4. Let $\widehat{G}$ be given in Lemma 7.3, $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$, and let $\psi \in \mathcal{C}_c^\infty(T^*X)$ be one in a fixed small neighbourhood of $K$ and zero outside of another sufficiently small neighbourhood of $K$. Then

$$\psi(x, \xi) \chi(\mathcal{H}_p \widehat{G}(x, \xi)) \in S^{0,0,-\infty,0}(T^*X).$$

Proof. Lemma 7.6 gives a stronger condition

$$H_p^{\ell} \partial_p^m \mathcal{H}_p^k(\psi(\rho) \chi(\mathcal{H}_p G(\rho))) = \mathcal{O}((\tilde{h}/h)^{|a|/2}),$$

as can be verified using (8.8).

As in §6.2 we modify our operator to obtain global invertibility. Thus we define $a \in S^{0,0,-\infty,-\infty}(T^*X)$ as follows

$$a(x, \xi) \overset{\text{def}}{=} \chi \left( \frac{\tilde{h}}{h} \rho(x, \xi) \right) \chi(\mathcal{H}_p G(x, \xi)) \psi(x, \xi),$$

$$\chi \in \mathcal{C}_c^\infty(\mathbb{R}; [0, 1]), \quad \chi(t) \equiv 1, \quad |t| \leq 1,$$

and $\psi$ is as in Lemma 8.3. In particular by taking its support close to $K$ we can replace $G$ by $\widehat{G}$ in the definition of $a$.

We then put

$$(8.9) \quad \tilde{P}_{\theta, t} = P_{\theta, t} - i(h/\tilde{h})\mathcal{O}_{\tilde{H}, t}(a) \in \Psi^{0,0,2,1}_c(X),$$

and first treat the region away from the trapped set:

Lemma 8.5. Suppose that $P_{\theta, t}$ is given by (8.9) and $\Psi_0 \in \Psi^{0,0}(T^*X)$ satisfies

$$\text{WF}_h(\Psi_0) \cap K = \emptyset.$$ 

Then for $u \in \mathcal{C}_c^\infty(X), z \in D(0, Ch)$ we have

$$\|(\tilde{P}_{\theta, t} - z)\Psi_0 u\|_{L^2} \geq th\|\Psi_0 u\|_{L^2(X)}/C - \mathcal{O}(h^\infty)\|u\|_{L^2(X)}.$$

$$0 < h \leq h_0(\tilde{h}), \quad 0 < \tilde{h} \leq \tilde{h}_0(t).$$

Proof. Let us assume that $\|u\| = 1$. Once $h$ is small enough, $a \equiv 0$ in a neighbourhood of $\text{WF}_h(\Psi_0)$ and Theorem 7 gives

$$\|(\tilde{P}_{\theta, t} - z)\Psi_0 u\|_{L^2} = \|(P_{\theta, t} - z)\Psi_0 u\|_{L^2} + \mathcal{O}(h\tilde{h}^\infty).$$

We then observe that $P_{\theta, t} \in \Psi^{0,0,2}_c(X)$, and consequently we can use the simpler calculus of §8.3. Microlocally near $\text{WF}_h(\Psi_0)$, for $z \in D(0, Ch)$, and for $t$ sufficiently large, Proposition 7.7 and the choice of the angle of scaling give,

$$P_{\theta, t} - z = \mathcal{O}_p(u)(\text{Re} p_\theta - \text{Re} z) + i\mathcal{O}_p(u)(\text{Im} p_\theta - iht\mathcal{H}_p G - \text{Im} z) + \mathcal{O}_t(h\tilde{h} + h^2 \log(1/h)),$$

$$|\text{Re} p_\theta - \text{Re} z| < \delta \implies -\text{Im} p_\delta + h\tilde{t}\mathcal{H}_p G + \text{Im} z \geq th/C.$$
(This is the analogue of (1.2) in the non-trapping case of [1]) Lemma 3.3 applied with \( \Psi_j \)'s such that \(| \Re p_\theta - \Re z | > \delta \) on \( \WF_h(\Psi_1) \) (with \( \Psi_j \)'s constructed using Lemma 3.2) completes the proof. \( \square \)

Near the trapped set we use the second microlocal calculus to obtain

**Lemma 8.6.** Suppose that \( P_{\theta,t} \) is given by (8.9) and let \( z \in D(0, Ch) \). For \( u \in C_c^\infty(X) \), \( \| u \| = 1 \), with \( \WF_h(u) \) in a fixed small neighbourhood of \( K \) we have

\[
\| (P_{\theta,t} - z) u \|_{L^2(X)} \geq t h \| u \|_{L^2(X)}/C, \quad 0 < h \leq h_0(t), 0 < \tilde{h} \leq h_0(t).
\]

provided that \( t \) is large enough.

**Proof.** In a small neighbourhood of \( K \) the operator is microlocally equal to

\[
P_{\theta,t} \defeq P - i t h \tilde{\Omega}_{h,h}(H_p G) - i(h/\tilde{h}) \tilde{\Omega}_{h,h}(a) + \mathcal{O}_{L^2 \to L^2}(h\tilde{h}),
\]

that is,

\[
\| (P_{\theta,t} - z) u \|_{L^2(X)} = \| (P_{\theta,t} - z) u \|_{L^2(X)} + \mathcal{O}(h\tilde{h}) \text{, } \| u \|_{L^2(X)} = 1,
\]

for \( u \) with \( \WF_h(u) \) near \( K \). For \( z \in D(0, Ch) \),

\[
P_{\theta,t} - Z \defeq (h/\tilde{h})(P_{\theta,t} - z) \in \Psi^{0,0.2,1}_\Sigma, \quad Z \defeq (h/\tilde{h}) Z,
\]

has the symbol given by

\[
p^z - Z = \lambda - Z - i h H_p G - i \chi(\lambda) \chi(H_p G) + \mathcal{O}(h^2), \quad Z \in D(0, C\tilde{h}), \quad \lambda = (h/\tilde{h}) p.
\]

Now let \( \psi_0, \psi_1 \in C_c^\infty(\mathbb{R}) \) satisfy

\[
\psi_0(t)^2 + \psi_1^2(t) = 1, \quad \text{supp } \psi_0 \subset \{ t : \chi(t) = 1 \}, \quad \psi_1(t) \equiv 0, \quad |t| \leq 1/2.
\]

As in Lemma 3.2 we can now find two operators \( \Psi_j \in \Psi^{0,0,0}_\Sigma, T^*(X) \) such that

\[
\sigma_{h, \tilde{h}}(\Psi_j) = \psi_j(H_p G), \quad (\Psi_j)^2 + (\Psi_j^*)^2 = Id + \mathcal{O}_{L^2 \to L^2}(h\tilde{h}).
\]

In a neighbourhood of the support of \( \psi_1(H_p G) \) the operator \( P_{\theta,t} - Z \) is elliptic in \( \Psi^{0,0,2,1}_\Sigma, T^*(X) \):

\[
|\lambda - Z - i h H_p G - i \chi(\lambda) \chi(H_p G) | \geq (|\lambda - Z| + \chi(\lambda)\chi(H_p G))/2 \geq (\lambda)/C,
\]

when \( Z \in D(0, C\tilde{h}) \) and \( \chi(H_p G) > 1/2 \), say. This implies (see Lemma 3.1) that for \( u \) with \( \WF_h(u) \) near \( K \), \( \| u \|_{L^2(X)} = 1 \),

\[
\| (P_{\theta,t} - Z) \Psi_0 u \|_{L^2(X)} \geq \| \Psi_0 u \|_{L^2(X)}/C - \mathcal{O}(h\tilde{h}), \quad 0 < \tilde{h} \leq h_0.
\]

To estimate \( \| (P_{\theta,t} - Z) \Psi_0 u \| \) from below we proceed as in (4.4). Let \( \hat{B}^z_1 = \frac{1}{2i} \left( P_{\theta,t} - (P_{\theta,t}^*)^2 \right) \), so that the Weyl symbol (in the sense of \( \Psi^{0,0,2,1}_\Sigma, T^*(X) \)) of \( \hat{B}^z_1 \) is equal to

\[
-\chi(\lambda) \chi(H_p G) + \mathcal{O}(h) - t h H_p G + \mathcal{O}_t(h^2),
\]
where we indicated the dependence on \( t \) in the second bound. Since \( H_p G \geq 1/C \) in a
neighbourhood of the support of \( \psi_1(H_p) \) we see that for \( u \) with \( \text{WF}_h(u) \) near \( K \), \( \|u\|_{L^2(X)} \),
\[-\langle B^2_t \Psi^2_1 u, \Psi^2_1 u \rangle \geq (t h - O(h) - \mathcal{O}(h^2)) \|\Psi^2_1 u\|^2 - \mathcal{O}(h^\infty) .\]

We now first take \( t \) large enough to dominate the first error term and then \( \tilde{h} \) small enough
to dominate the second one. Hence,
\[ \|(P^2_t - Z)\Psi^2_1 u\| \geq \|(P^2_t - Z)\Psi^2_1 u, \Psi^2_1 u\| \geq |\text{Im}( (P^2_t - Z)\Psi^2_1 u, \Psi^2_1 u) | \]
\[ = - \langle (B^2_t - \text{Im} Z)\Psi^2_1 u, \Psi^2_1 u \rangle \geq t \tilde{h} \|\Psi^2_1 u\|^2/2 - \mathcal{O}(h^\infty) , \]
provided that \( t \) was large enough, and then \( \tilde{h} \) small enough. Lemma 3.3 (or rather its proof) gives
\[ \|(P^2_t - Z)u\| \geq \tilde{h}\|u\|/C , t \geq t_0 \gg 1 , \ 0 < h < h_0(t) , \]
for \( u \) with \( \text{WF}_h(u) \) near \( K \).

We complete the proof by writing
\[ \|(\tilde{P}_{\theta,t} - z)u\|_{L^2(X)} = \|(P^2_t - z)u\|_{L^2(X)} + \mathcal{O}(h^\infty) \|u\|_{L^2(X)} \]
\[ = (h/\tilde{h}) \|(P^2_t - Z)u\|_{L^2(X)} + \mathcal{O}(h^\infty) \|u\|_{L^2(X)} \]
\[ \geq t h \|u\|_{L^2(X)}/C . \]

\[ \square \]

The two lemmas are now combined using Lemma 3.3 which gives for large \( t \), \( 0 < \tilde{h} \leq \tilde{h}_0(t) \), and \( 0 < h < h_0(t, \tilde{h}) \), the invertibility of \( \tilde{P}_{\theta,t} - z \), \( z \in D(0, C\tilde{h}) \):
\[ (\tilde{P}_{\theta,t} - z)^{-1} = \mathcal{O}(1/\tilde{h}) : L^2(X) \rightarrow L^2(X) . \]

As in 6.2 Theorem 3 is a consequence of writing
\[ (8.11) \quad \tilde{\Omega}_{h,\tilde{h}}(a) = R + E , \ \text{rank}(R) = \mathcal{O}(h^{-\nu}) , \ E = \mathcal{O}(\tilde{h}^\infty) : L^2(X) \rightarrow L^2(X) , \]
\( \nu > \nu(E) \), where \( m(E) = 2\nu(E) + 1 \) is the dimension of the trapped set at energy \( E \),
allowing \( \nu = \nu(E) \) if the trapped set is of pure dimension.

The decomposition (8.11) follows from Proposition 5.10 and the definition of the Minkowski dimension:
\[ m_0 = 2n - 1 - \sup \{d : \limsup_{\epsilon \rightarrow 0} \epsilon^{-d} \text{vol}(\{\rho \in p^{-1}(0) : d(\rho, K) < \epsilon\}) < \infty\} , \]
with the set being of pure dimension if
\[ \limsup_{\epsilon \rightarrow 0} \epsilon^{-2n+1+m_0} \text{vol}(\{\rho \in p^{-1}(0) : d(\rho, K) < \epsilon\}) < \infty . \]
In other words, for \( \epsilon \) small
\[ \text{vol}(\{\rho \in p^{-1}(0) : d(\rho, K) < \epsilon\}) \leq C \epsilon^{2n-1-m} , \ m > m_0 , \]
and $m$ replaceable by $m_0$ when $K$ is of pure dimension. In particular,
\[
\text{vol}(\text{supp} \, a \cap p^{-1}(0)) \leq C_{\tilde{h}} h^{(2n-1-m)/2} = C_{\tilde{h}} h^{n-\nu-1}, \quad m = 2\nu + 1 > m_0,
\]
with equality if $K$ is of pure dimension. Since
\[
\text{supp} \, a \cap p^{-1}(0) \subset \bigcup_{\rho \in K} B_{\Sigma}(\rho, M(h/\tilde{h})^{1/2}),
\]
where $B_{\Sigma}$ are balls in $\Sigma$ with respect to some fixed smooth metric, and since $K$ is invariant under the flow, the standard covering arguments (see [29, Lemma 3.3]) show that the hypothesis of Proposition 5.10 are satisfied with
\[
K(h) \leq C_{\tilde{h}} h^{-\nu},
\]
which completes the proof of Theorem 3.

Appendix

We present a direct proof of Proposition 3.7. The hypotheses on $G$ in (3.15) are equivalent to the statement that $\exp(tG) \in S(m^t)$, for all $t \in \mathbb{R}$. We start with

**Lemma A.1.** Let $U(t) \overset{\text{def}}{=} (\exp tG)^w(x,D) : S(\mathbb{R}^n) \rightarrow S(\mathbb{R}^n)$. For $|t| < \epsilon_0(G)$, the operator $U(t)$ is invertible, and
\[
U(t)^{-1} = B_t^w(x,D), \quad B_t \in S(m^{-t}).
\]

**Proof.** We apply the composition formula (3.14) to obtain
\[
U(-t)U(t) = \text{Id} + E_t^w(x,D), \quad E_t \in S(1).
\]
More explicitly we write (see [10, Proposition 7.7] and Lemma 3.6 here)
\[
E_t(x_1,\xi) = \int_0^s e^{sA(D)}A(D)(e^{-tG(x_1,\xi_1)+tG(x_2,\xi_2)}|_{x_2=x_1=x,\xi_2=\xi_1=\xi} \, ds
\]
\[
= \int_0^s (it/2)e^{sA(D)}(D_{\xi_1}GD_{x_2}G - D_{x_1}GD_{\xi_2}G)e^{-tG(x_1,\xi_1)+tG(x_2,\xi_2)}|_{x_2=x_1=x,\xi_2=\xi_1=\xi} \, ds,
\]
where $A(D) = i\sigma(D_{x_1},D_{\xi_1};D_{x_2},D_{\xi_2})/2$.

Hence $E_t = t\tilde{E}_t$ where $\tilde{E}_t \in S(1)$ uniformly, and thus
\[
E_t^w(x,D) = O(t) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n).
\]
This shows that for $|t|$ small enough $\text{Id} + E_t^w(x,D)$ is invertible, and Beals’s lemma (see for instance [10, Proposition 8.3]) gives
\[
(\text{Id} + E_t^w(x,D))^{-1} = C_t^w(x,D), \quad C_t \in S(1).
\]
Hence $B_t = C_t\# \exp(-tG(x,\xi)) \in S(m^{-t})$. \qed
We now observe that
\begin{equation}
\frac{d}{dt} (U(-t) \exp(tG^w(x,D))) = V(t) \exp(tG^w(x,D)) ,
\end{equation}
where
\begin{equation}
V(t) = A^w_t(x,D) , \ A_t \in S(m^{-t}) .
\end{equation}
In fact, we see that
\begin{equation}
\frac{d}{dt} U(-t) = -(G \exp(-tG))^w(x,D) , \ U(-t)G^w(x,D) = (\exp(tG) \# G)^w(x,D).
\end{equation}
As before, the composition formula (3.14) gives
\begin{equation}
\exp(-tG)\# G - G \exp(-tG) = \int_0^1 \exp(sa(D))A(D) \exp(-tG(x^1,\xi^1)G(x^2,\xi^2)_{x^1=x^2=x,\xi^1=\xi^2} ,
\end{equation}
where
\begin{equation}
A(D) = i\sigma(D_{x^1},D_{\xi^1};D_{x^2},D_{\xi^2})/2 .
\end{equation}
The hypothesis on \( G \) shows that \( A(D) \exp(tG(x^1,\xi^1))G(x^2,\xi^2) \) is a sum of terms of the form \( a(x^1,\xi^1)b(x^2,\xi^2) \) where \( a \in S(m^{-t}) \) and \( b \in S(1) \). The continuity of \( \exp(A(D)) \) on the spaces of symbols (see [10] Proposition 7.6) gives (A.1).

If we put
\begin{equation}
C(t) \overset{\text{def}}{=} -V(t)U(-t)^{-1} ,
\end{equation}
then by Lemma A.1 \( C(t) = c_t^w \) where \( c_t \in S(1) \). Symbolic calculus shows that \( c_t \) depends smoothly on \( t \) and
\begin{equation}
(\partial_t + C(t))(U(-t) \exp(tG^w(x,D))) = 0 .
\end{equation}
The proof of Proposition 3.7 is now reduced to showing
\begin{lemma}
Suppose that \( C(t) = c_t^w(x,D) \), where \( c_t \in S(1) \), depends continuously on \( t \in (-\epsilon_0,\epsilon_0) \). Then the solution of
\begin{equation}
(\partial_t + C(t))Q(t) = 0 , \ Q(0) = q^w(x,D) , \ q \in S(1) ,
\end{equation}
is given by \( Q(t) = q_t(x,D) \), where \( q_t \in S(1) \) depends continuously on \( t \in (-\epsilon_0,\epsilon_0) \).
\end{lemma}
\begin{proof}
The Picard existence theorem for ODEs shows that \( Q(t) \) is bounded on \( L^2 \). If \( \ell_j(x,\xi) \) are linear functions on \( T^*\mathbb{R}^n \) then
\begin{equation}
\frac{d}{dt} \text{ad}_{\ell_1(x,D)} \circ \cdots \circ \text{ad}_{\ell_N(x,D)} Q(t) + \text{ad}_{\ell_1(x,D)} \circ \cdots \circ \text{ad}_{\ell_N(x,D)}(C(t)Q(t)) = 0 ,
\end{equation}
where
\begin{equation}
\text{ad}_{\ell_1(x,D)} \circ \cdots \circ \text{ad}_{\ell_N(x,D)} Q(0) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n) .
\end{equation}
If we show that for any choice of \( \ell_j's \) and any \( N \)
\begin{equation}
\text{ad}_{\ell_1(x,D)} \circ \cdots \circ \text{ad}_{\ell_N(x,D)} Q(t) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n) ,
\end{equation}

\end{proof}
then Beals’s lemma (see [10, Chapter 8]) concludes the proof. We proceed by induction on $N$:

$$\text{ad}_{\ell_1(x,D)} \circ \cdots \circ \text{ad}_{\ell_N(x,D)}(C(t)Q(t)) = C(t) \text{ad}_{\ell_1(x,D)} \circ \cdots \circ \text{ad}_{\ell_N(x,D)} Q(t) + R(t),$$

where $R(t)$ is the sum of terms of the form

$$A_k(t) \text{ad}_{\ell_1(x,D)} \circ \cdots \circ \text{ad}_{\ell_k(x,D)} Q(t), \quad k < N,$$

where $a_k(t) \in S(1)$ depend continuously on $t$ (this statement can also be proved by induction using the derivation property of $\text{ad}_\ell$: $\text{ad}_\ell(CD) = (\text{ad}_\ell C)D + C(\text{ad}_\ell D)$). Hence by the induction hypothesis $R(t)$ is bounded on $L^2$, and depends continuously on $t$. Thus

$$\left(\frac{d}{dt} + C(t)\right) \text{ad}_{\ell_1(x,D)} \circ \cdots \circ \text{ad}_{\ell_N(x,D)} Q(t) = R(t) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n).$$

Since (A.3) is valid at $t = 0$ we obtain it for all $t \in (-\epsilon_0, \epsilon_0)$.

\[\square\]

References

[1] E. Bogomolny, Spectral statistics, in Proc. Int. Congress of Mathematicians (Doc. Math. Extra vol. 3) 99–108, Springer Verlag, Berlin, 1998.

[2] J.-F. Bony, Résonances dans des domaines de taille $h$. Int. Math. Res. Notices, 16(2001), 817–847.

[3] J.-F. Bony and J. Sjöstrand, Traceformula for resonances in small domains, J. Funct. Anal. 184(2001), 402–418.

[4] J.-M. Bony and J.-Y. Chemin, Espaces fonctionnels associés au calcul de Weyl-Hörmander, Bull. Soc. math. France, 122(1994), 77-118.

[5] V. Bruneau and V. Petkov, Meromorphic continuation of the spectral shift function, Duke Math. J. 116(2003), 389–430.

[6] N. Burq, Semiclassical estimates for the resolvent in non trapping geometries. Int. Math. Res. Not. 5(2002), 221–241.

[7] T. Christiansen, Schrödinger operators with complex-valued potentials and no resonances, Duke Math. J., to appear, math-phys/0505065.

[8] H. Christianson, Growth and zeros of the zeta function for hyperbolic rational maps, Can. J. Math., to appear, math.DS/0404543

[9] N. Dencker, J. Sjöstrand, and M. Zworski, Pseudospectra of semiclassical (pseudo)differential operators, Comm. Pure Appl. Math., 57(2004), 384–415.

[10] M. Dimassi and J. Sjöstrand, Spectral Asymptotics in the semiclassical limit, Cambridge University Press, 1999.

[11] C. Gérard and J. Sjöstrand, Semiclassical resonances generated by a closed trajectory of hyperbolic type, Comm. Math. Phys. 108(1987), 391–421.

[12] L. Guillopé, K. Lin, and M. Zworski, The Selberg zeta function for convex co-compact Schottky groups, Comm. Math. Phys. 245(2004), 149 - 176.

[13] B. Helffer and J. Sjöstrand, Résonances en limite semiclassique. Mémoires de la S.M.F. 114(3)(1986).

[14] B. Helffer and J. Sjöstrand, Équation de Schrödinger avec champ magnétique et équation de Harper. Springer Lecture Notes in Physics 345, 118–197, Springer Verlag, Berlin, 1989.

[15] L. Hörmander, The Analysis of Linear Partial Differential Operators, vol.III,IV, Springer Verlag, 1985.

[16] V. Ivrii, Microlocal Analysis and Precise Spectral Asymptotics, Springer Verlag, 1998.
[17] K. Lin, *Numerical study of quantum resonances in chaotic scattering*, J. Comp. Phys. **176**(2002), 295-329.

[18] K. Lin and M. Zworski, *Quantum resonances in chaotic scattering*, Chem. Phys. Lett. **355**(2002), 201-205.

[19] W. Lu, S. Sridhar, and M. Zworski, *Fractal Weyl laws for chaotic open systems*, Phys. Rev. Lett. **91**(2003), 154101.

[20] A. Martinez, *Resonance free domains for non globally analytic potentials*, Ann. Henri Poincaré, **4**(2002), 739–756.

[21] R.B. Melrose, *Polynomial bounds on the number of scattering poles*, J. Funct. Anal. **53**(1983), 287-303.

[22] R.B. Melrose, *Polynomial bounds on the distribution of poles in scattering by an obstacle*, Journées "Équations aux dérivées Partielles", Saint-Jean-des-Monts, 1984.

[23] T. Morita, *Periodic orbits of a dynamical system in a compound central field and a perturbed billiards system*, Ergodic Theory Dynam. Systems **14**(1994), 599–619.

[24] S. Nakamura, P. Stefanov, and M. Zworski, *Resonance expansions of propagators in the presence of potential barriers*, J. Funct. Anal. **205**(2003), 180-205.

[25] S. Nonnenmacher and M. Zworski, *Distribution of resonances for open quantum maps*, preprint 2005, math-ph/0505034.

[26] V. Petkov and M. Zworski, *Breit-Wigner approximation and distribution of resonances*, Comm. Math. Phys., **204**(1999), 329-351. Correction, **214**(2000), 733-735.

[27] V. Petkov and M. Zworski, *Semiclassical estimates on the scattering determinant*, Annales H. Poincaré, **2**(2001), 675-711.

[28] H. Schomerus and J. Tworzydło, *Quantum-to-classical crossover of quasi-bound states in open quantum systems*, Phys. Rev. Lett. Phys. Rev. Lett. **93**(2004), 154102.

[29] J. Sjöstrand, *Geometric bounds on the density of resonances for semiclassical problems*, Duke Math. J., **60**(1990), 1–57

[30] J. Sjöstrand, *A trace formula and review of some estimates for resonances*, in *Microlocal analysis and spectral theory* (Lucca, 1996), 377–437, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 490, Kluwer Acad. Publ., Dordrecht, 1997.

[31] J. Sjöstrand and M. Zworski, *Complex scaling and the distribution of scattering poles*, J. Amer. Math. Soc. **4**(1991), 729–769.

[32] J. Sjöstrand and M. Zworski, *Asymptotic distribution of resonances for convex obstacles*, Acta Math. **183**(1999), 191-253.

[33] J. Sjöstrand and M. Zworski, *Quantum monodromy and semiclassical trace formulae*, J. Math. Pure Appl. **81**(2002), 1–33.

[34] P. Stefanov, *Approximating resonances with the complex absorbing potential method*, preprint 2004, math-ph/0409020

[35] J. Strain and M. Zworski, *Growth of the zeta function for a quadratic map and the dimension of the Julia set*, Nonlinearity, **17**(2004), 1607-1622.

[36] A. Vasy and M. Zworski, *Semiclassical estimates in asymptotically euclidean scattering*, Comm. Math. Phys. **212**(2000), 205–217.

[37] J. Wunsch and M. Zworski, *Distribution of resonances for asymptotically Euclidean manifolds*, J. of Diff. Geom. **55**(2000), 43-82.

[38] M. Zworski, *Distribution of poles for scattering on the real line*, J. Funct. Anal. **73**(1987), 277-296.

[39] M. Zworski, *Sharp polynomial bounds on the distribution of scattering poles*, Duke Math. J. **59**(1989), 311-323.

[40] M. Zworski, *Dimension of the limit set and the density of resonances for convex co-compact Riemann surfaces*, Inv. Math. **136**(1999), 353-409.
Centre de Mathématiques, École Polytechnique, UMR 7460, CNRS, F-91128 Palaiseau
E-mail address: johannes@math.polytechnique.fr

Mathematics Department, University of California, Evans Hall, Berkeley, CA 94720
E-mail address: zworski@math.berkeley.edu