LEAPFROG IN POSETS

Abstract. We consider the following solitaire game whose rules are reminiscent of the children’s game of leapfrog. The player is handed an arbitrary ordering \( \pi = (x_1, x_2, \ldots, x_n) \) of the elements of a finite poset \((P, \prec)\). At each round an element may “skip over” the element in front of it, i.e. swap positions with it. For example, if \( x_i \prec x_{i+1} \), then it is allowed to move from \( \pi \) to the ordering \((x_1, x_2, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n)\). The player is to carry out such steps as long as such swaps are possible. When there are several consecutive pairs of elements that satisfy this condition, the player can choose which pair to swap next. Does the order of swaps matter for the final ordering or is it uniquely determined by the initial ordering? The reader may guess correctly that the latter proposition is correct. What may be more surprising, perhaps, is that this question is not trivial. The proof works by constructing an appropriate system of invariants.

1. Introduction

Let \((P, \prec)\) be a finite poset. We say that the elements \(x, y \in P\) are comparable if either \(x \prec y\) or \(y \prec x\). Otherwise we say they are incomparable and write \(x \parallel y\). Let \(\pi = (x_1, x_2, \ldots, x_n)\) be an ordering of \(P\)’s elements. A swap changes this permutation to \((x_1, x_2, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n)\) for some index \(i\). This swap is permissible if \(x_i \prec x_{i+1}\). We say that a permutation \(\sigma\) of \(P\)’s elements is reachable from \(\pi\) if it is possible to move from \(\pi\) to \(\sigma\) through a sequence of permissible swaps. A permutation of \(P\)’s elements is called terminal if no swap is permissible. Clearly, every series of permissible swaps is finite, since no two elements can be swapped more than once. The main result in this note is the following:

**Theorem 1.** For every permutation \(\pi\) of the elements of a finite poset \((P, \prec)\) there is exactly one terminal permutation reachable from \(\pi\).

2. The Proof

*Proof.* Let \(\tau\) be a terminal permutation that is reachable from \(\pi\). The uniqueness of \(\tau\) is proved by providing a criterion, depending only on \(\pi\), as to which pairs of elements appear in the same order in \(\pi\) and \(\tau\) and which are reversed.

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An \((x, y)\)-fence in \(\pi\) is a sequence \(x = z_1, z_2, \ldots, z_k = y\) that appear in this order (not necessarily consecutively) in \(\pi\) such that \(z_\alpha \parallel z_{\alpha+1}\) for every \(\alpha \in [k-1]\).

Clearly, if \(x \parallel y\) no permissible swap can change the relative order of \(x\) and \(y\). Consequently:

- No sequence of permissible swaps can change the relative order of \(x\) and \(y\) if an \((x, y)\)-fence exists.
- No sequence of permissible swaps can create or eliminate an \((x, y)\)-fence.

We say that \((x, y)\) is a critical pair in \(\pi\) if (i) \(x \prec y\), (ii) \(x\) precedes \(y\) in \(\pi\) and, (iii) there is no \((x, y)\)-fence in \(\pi\).

We now assert and prove the criterion for whether any two elements \(x\) and \(y\) in \(\pi\), with \(x\) preceding \(y\), preserve or reverse their relative order in a terminal permutation:

1. If \(y \prec x\), the order is preserved.
2. If there exists an \((x, y)\)-fence, the order is preserved.
3. Otherwise, i.e. if \((x, y)\) is a critical pair, the order is reversed.

The first element of the criterion is trivial and the second has already been dealt with. It remains to show the third and last element: Since an \((x, y)\)-fence cannot be created or eliminated by permissible swaps, an equivalent statement to this claim is that a permutation \(\tau\) with a critical pair cannot be terminal. We prove this by induction on the number of elements in \(\tau\) separating \(x\) and \(y\):

At the base of induction, if \(x\) and \(y\) are neighbor elements, the assertion is true since as \(x \prec y\) the permutation is not terminal. Now let \(k\) be the number of elements separating \(x\) and \(y\), and the induction hypothesis is that if the number of elements separating a pair is less than \(k\) it cannot be critical.

Let \(z\) be an element between \(x\) in \(y\) in \(\tau\). Assume \(x \prec z\). Then by the induction hypothesis there exists an \((x, z)\)-fence. Now consider the relation between \(z\) and \(y\): \(z \parallel y\) is impossible, because then \(y\) could be concatenated to the \((x, z)\)-fence to form a \((x, y)\)-fence, contrary to the assumption that \((x, y)\) is a critical pair. Similarly, \(z \prec y\) would by the induction hypothesis prove the existence of a \((z, y)\)-fence, but this is impossible as it could be concatenated to the \((x, z)\)-fence to form an \((x, y)\)-fence. This leaves \(y \prec z\) as the only possibility.

In summary \(x \prec z \Rightarrow y \prec z\).

By similar reasoning \(z \prec y \Rightarrow z \prec x\).

Furthermore, the possibility \(x \parallel z\) together with \(z \parallel y\) can be dismissed as constituting an \((x, y)\)-fence, leaving just two possible scenarios satisfied by each \(z\) between \(x\) and \(y\):

- \(z\) is “small”, i.e. \(z \prec x\) and \(z \prec y\).
- \(z\) is “large”, i.e. \(x \prec z\) and \(y \prec z\).
Since $\tau$ is terminal, $x$’s immediate neighbor must be “small”, and $y$’s immediate neighbor must be “large”. Between these two, there must exist two consecutive elements $z_1, z_2$ such that $z_1$ is “small” and $z_2$ is “large”. But this leads to a contradiction as $z_1 < x < z_2 \Rightarrow z_1 < z_2$, which implies that $\tau$ is not terminal. Therefore $(x, y)$ cannot be critical, completing the proof by induction that a terminal permutation cannot have a critical pair.

This completes the proof for the uniqueness of the terminal permutation. □

3. Remarks

As the proof shows, the relative final order of every pair of elements $x$ and $y$ is determined by the existence of a fence that connects them. If a fence exists, then $x, y$ maintain their initial relative order in the final ordering. If no such fence exists, their final relative order agrees with their mutual order relation.

We note that the proof not only shows the uniqueness of the final ordering. It also implies that the number of swaps carried by the player until termination is uniquely defined. This also yields an efficient algorithm that determines the final ordering given an initial ordering: Perform an arbitrary swap until no more swaps are possible.