LikeNs a point of view on natural numbers, II

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Abstract

In this paper we continue our research on the concept of liken. This notion has been defined as a sequence of non-negative real numbers, tending to infinity and closed with respect to addition in \( \mathbb{R} \). The most important examples of likens are clearly the set of natural numbers \( \mathbb{N} \) with addition and the set of positive natural numbers \( \mathbb{N}^* \) with multiplication, represented by a sequence \( (\ln(n+1))_{n=0}^{\infty} \). The set of all likens can be parameterized by the points of some infinite dimensional, complete metric space. In this "space of likens" we consider elements up to isomorphism and define "properties of likens" as such, that are isomorphism invariant. The main result of this work is a theorem characterizing the liken \( \mathbb{N}^* \) of natural numbers with multiplication in the space of all likens.

1 Introduction

We will begin by recalling the content of the paper \[1\], which is necessary to formulate and prove the main result of this paper, i.e. Theorem \[1.1\]. As it was mentioned in \[1\], the notion of a liken may be considered as some way of talking about of the so-called Beurling numbers \[2\]. The family of all likens - say \( \mathcal{H} \) - (we will say also the space of likens) described in \[1\], constitutes a kind of a natural environment where "live" the two fundamental mathematical structures: \( (\mathbb{N},+) \) (the natural numbers with addition) and \( (\mathbb{N}^*,\cdot) \), (the natural numbers with multiplication) which - as mathematical structures - are ordered semigroups. Let us pay attention here, that by a liken we mean a sub-semigroup of the additive semigroup \( \mathbb{R}^+ \), so we replace \( \mathbb{N}^* \) by the sequence \( (\ln(n+1))_{n=0}^{\infty} \), however without changing the notation, and we call elements of this last sequence also "the natural numbers". Although \( \mathcal{H} \) is a rather big space (infinite dimensional complete metric space), the most of likens seem to be of little interest and if they were brought to life in \[1\], it was only to look at the liken \( \mathbb{N}^* \) from a slightly different point of view.
The exact definition of a liken (in different versions) will be recalled below, and at the beginning it is enough to know, that a liken $L$ is a strictly increasing sequence $L = (x_n)_{n=0}^{\infty}$ of real numbers, which is a sub-semigroup of the semigroup $\mathbb{R}^+$. Hence in each liken $L$ we have two types of mathematical structures inherited from $\mathbb{R}^+$, i.e. the algebraic structure of the sub-semigroup with addition and the structure of the ordered space with respect to the inequality in $\mathbb{R}$. This make possible to define the isomorphism of likens as a bijection which preserves both structures - algebraic and ordinal.

Different details concerning the relation of the isomorphism of likens will be discussed in the next section. It appears - and this is in a sense a typical situation - that all ”interesting” likens (infinitely generated and with uniqueness) are algebraically isomorphic to each other, and at the same time, they are always isomorphic as ordered spaces to each other. On the other hand, as it was proved in [1], they are isomorphic as likens if and only if their sets of generators are homothetic. As it was mentioned above, this situation is ”typical”. To understand better the meaning of the term ”typical”, let us consider the example of the family of all infinite dimensional, separable Banach spaces. Each two such spaces are isomorphic as vector spaces since they have the (vector) bases of the same cardinality, and each two such spaces are homeomorphic as topological spaces by the theorem of Kadec-Anderson, but two such spaces are isomorphic as Banach spaces only when there exists a linear isomorphism which is also the topological homeomorphism. It seems that the basic advantage of using the abstract language of likens lies in the fact that we can formulate different properties of likens in this language and consequently distinguish between them. Roughly speaking, a property of liken is each property, which is preserved by isomorphisms of likens. Perhaps the most important of such properties of likens is that they are generated by their irreducible elements (just like natural numbers by prime numbers in the semigroup $\mathbb{N}^*$), but this property is common for all likens. We will provide non-trivial examples of a few such properties later in the paper, but for now let us note, for (a trivial) example, that property: the element $x_2$ is undecomposable is fulfilled in $\mathbb{N}^*$ but is not true in $\mathbb{N}$.

The main result of this paper is Theorem 4.1 which gives a characterization of the liken $\mathbb{N}^*$ among all likens. For this, first we will formulate two, among others, properties of likens called convexity, denoted by (C), and Ockham’s razor property denoted by (OR). Then the main theorem states: if a liken $L$ has the properties (C) and (OR) then it is isomorphic to $\mathbb{N}^*$.

The paper is organized as follows. In Section 2 we recall some definitions, notations and theorems proved in [1], which will be used in this paper. In fact the contents of Section 2 is to be found in [1], but because of some small differences in notations it will be better to collect in Section 2 all we will need about likens in this paper. In Section 3 we formulate a number of general properties of likens, in particular the mentioned properties (C) and (OR). In Section 4 we present the proof of Theorem 4.1. In the last section we formulate a number of remarks.
2 Definitions, notations and the main results about likens

In this paper, as in [1], we will use the following notations:

\[ \mathbb{R}^+ = [0, \infty), \]  
\[ \mathbb{Q}^+ = [0, \infty) \cap \mathbb{Q}, \]  
\[ \mathbb{R}^N = \{ \overrightarrow{a} = (a_i)^\infty_{i=1} : a_i \in \mathbb{R} \}. \]  
\[ (\mathbb{R}^+)^N = \{ \overrightarrow{a} \in \mathbb{R}^N : a_i \geq 0 \}. \]  
\[ \mathbb{R}_0^N = \{ \overrightarrow{a} \in \mathbb{R}^N : \exists j : i > j \Rightarrow a_i = 0 \}. \]  
\[ \mathbb{Q}^N = \{ \overrightarrow{a} = (a_i)^\infty_{i=1} : a_i \in \mathbb{Q} \}. \]  
\[ (\mathbb{Q}^+)^N = \{ \overrightarrow{a} \in \mathbb{Q}^N : a_i \geq 0 \}. \]  
\[ \mathbb{Q}_0^N = \{ \overrightarrow{m} \in \mathbb{Q}^N : \exists j : i > j \Rightarrow a_i = 0 \}. \]  
\[ \mathbb{N}_0^N = \{ \overrightarrow{a} \in \mathbb{N}^N : \exists j : i > j \Rightarrow a_i = 0 \}. \]  

Moreover, for \( \overrightarrow{a} \in \mathbb{R}^N \), and for \( \overrightarrow{m} \in \mathbb{N}_0^N \) we set:

\[ \langle \overrightarrow{a}, \overrightarrow{m} \rangle = m_1a_1 + m_2a_2 + \ldots \]  

Let us note, that although \( \overrightarrow{a} \) may tend to infinity, the righthand side sum is always finite, since the sequence \( \overrightarrow{m} \) in fact is finite.

The definition of the liken given in [1] is following

**Definition 2.1.** A liken \( \mathbb{L} \) is a sequence \( (x_n)^\infty_0 \) of real numbers such that:

a) For all \( n \in \mathbb{N} \) we have \( 0 = x_0 \leq x_n < x_{n+1} \).

b) For all \( n \in \mathbb{N} \ni m \) there is \( k \in \mathbb{N} \) such that \( x_n + x_m = x_k \).

As it was observed in [1], a liken \( \mathbb{L} \) is an increasing sequence of nonnegative real numbers, which is closed with respect to the addition and tends to infinity.

Now we recall the notion of the isomorphism of likens.

**Definition 2.2.** Let \( (\mathbb{G},+) \) be a semigroup and let \( \mathbb{L} \) be a liken. We will say that a map \( \varphi : \mathbb{G} \rightarrow \mathbb{L} \) is

a) an algebraic homomorphism, when \( \varphi(x + y) = \varphi(x) + \varphi(y) \),

b) an algebraic monomorphism, when it is an injective homomorphism,

c) an algebraic isomorphism, when it is a surjective monomorphism.
In particular we know now, what it means that two likens $L$ and $K$ are algebraically isomorphic. It is also clear, that each two likens are isomorphic as ordered spaces, since they are similar to the ordered space $(\mathbb{N}, \leq)$. Let us mention, that the map $K \ni x_n \to y_n \in L$ is not (in general) a homomorphism of likens, and let us mention also, that if $\varphi : K \to L$ is an ordinal isomorphism, then it is unique. Finally we set

**Definition 2.3.** Two likens $L$ and $K$ are isomorphic if the (unique) ordinal isomorphism is also an algebraic homomorphism.

A very important consequence of the axioms of liken is the existence of *undecomposable elements* (called also *irreducible elements* or *prime elements*).

**Definition 2.4.** Let $L$ be a liken and let $u \in L$. We will say, that $u$ is undecomposable if

$$u = v + w, v \in L \ni w \Rightarrow v = 0 \lor w = 0.$$ 

As it was observed in [1]

**Proposition 2.5.** Each liken $L = (x_n)^{\infty}_{0}$ has at least one undecomposable element.

Also (see [1])

**Proposition 2.6.** Let $L$ be a liken, and let $P_L$ be the set of all undecomposable elements of $L$. Then each element of $x \in L$ can be written in the form

$$x = m_1 \cdot a_1 + m_2 \cdot a_2 + \ldots + m_k \cdot a_k,$$

(11)

where $m_1, m_2, \ldots, m_k \in \mathbb{N}$, $a_1, a_2, \ldots, a_k \in P_L$, and $k \in \mathbb{N}$.

One may ask now about the uniqueness of the representation from Proposition 2.6. In general, as it was discussed in [1], such representations are not unique. So the above Definition 2.1 of a liken admits likens without uniqueness. This is for example the case of the so-called *numerical semigroups* [3] with the associated *Apéry sets* (see also Remark 1 in Section 5). However in this paper we will be interested only in likens with uniqueness, so further, in this paper, "liken" means "liken with uniqueness". This implies, as it will be discussed later, that all likens are isomorphic algebraically. We recall below shortly the description of this situation presented widely in [1].

Let $\mathcal{E} = \mathbb{N}_{0}^{\mathbb{N}}$ denote, as in (9), the set of all sequences of natural numbers, with almost all terms vanishing, ie.:

$$\mathbb{N}_{0}^{\mathbb{N}} := \{ \overrightarrow{n} = (n_1, n_2, ...) : (n_j \in \mathbb{N}) \land (\exists i \in \mathbb{N} : k > i \Rightarrow n_k = 0) \}. $$

(12)
In the set \( E = \mathbb{N}_0^\mathbb{N} \) we may consider the operations: “+” - addition and “·” - multiplication by natural numbers - defined as usually in a cartesian product. With these operations \( \mathbb{N}_0^\mathbb{N} \) is an algebraic structure, which may be called semimodule or a cone over \( \mathbb{N} \).

We set: \( e_k = (0, 0, \ldots, 0, 1, 0, \ldots) \), i.e. \( e_k \) is an element of \( \mathbb{N}_0^\mathbb{N} \), with all terms equal 0 except the \( k \)-th, which is 1. So we have for \( n \in E \):

\[
\overrightarrow{n} = (n_1, n_2, \ldots) = n_1 \cdot e_1 + n_2 \cdot e_2 + \ldots
\]  

(13)

Using the terminology from the linear algebra we may say, that \( (e_k)_{k=1}^\infty \) is a basis of the cone \( \mathbb{N}_0^\mathbb{N} \). This means precisely, that each element from \( \mathbb{N}_0^\mathbb{N} \) can be, in a unique way, written as a linear combination of \( (e_k)_{k=1}^\infty \) with the coefficient from \( \mathbb{N} \). Clearly \( E = \mathbb{N}_0^\mathbb{N} \) is a semigroup.

Clearly \( \mathbb{R}^+ \) is a cone over \( \mathbb{N} \). A map \( \varphi : E \to \mathbb{R}^+ \) will be called a homomorphism of semigroups (or of cones), when

\[
\varphi(n_1 \cdot e_1 + n_2 \cdot e_2 + \ldots) = n_1 \cdot \varphi(e_1) + n_2 \cdot \varphi(e_2) + \ldots
\]  

(14)

It is evident, that a homomorphism \( \varphi : \mathbb{N}_0^\mathbb{N} \to \mathbb{R}^+ \) cannot be an ephimorphism, since \( \mathbb{N}_0^\mathbb{N} \) is countable, but \( \mathbb{R}^+ \) is uncountable. However there exist monomorphisms \( \varphi : \mathbb{N}_0^\mathbb{N} \to \mathbb{R}^+ \), and the mentioned above the space of likens can be considered as the space of all such monomorphisms. We will now give description of this situation (for details see [1]).

**Proposition 2.7.** Each function \( a : \mathbb{N} \to \mathbb{R}^+ \) can be "extended" in a unique way to a homomorphism \( \tilde{a} : \mathbb{N}_0^\mathbb{N} \to \mathbb{R}^+ \) "by linearity" (i.e. \( \tilde{a}(\overrightarrow{n}) = \langle \overrightarrow{a}, \overrightarrow{n} \rangle \)).

Now we will recall the description of the variety of infinitely generated likens. Let \( L \) be a liken and let the set \( \mathcal{P}_L = \{a_1, a_2, \ldots\} \) be infinite. We will assume, that \( \mathcal{P}_L \) is linearly independent in the vector space \((\mathbb{R}, \mathbb{Q})\) of real numbers over rational numbers. This assumption is sufficient to have the uniqueness of the representation (11). We have observed in [1] that in the case when \( \mathcal{P}_L \) is infinite we must have \( \lim_{k \to \infty} a_k = +\infty \). The converse is also true. Namely

**Proposition 2.8.** Let \( \overrightarrow{a} = (a_i)_1^\infty \) be a sequence from \((\mathbb{R}^+)^\mathbb{N}\), which is linearly independent in the vector space \((\mathbb{R}, \mathbb{Q})\) and tends to infinity. Then \( \tilde{a}(\mathbb{N}_0^\mathbb{N}) \) is a liken.

The essence of the concept of "the space of likens", denoted above by \( \mathcal{H} \), lies in latter Propositions. Namely, as we see, there exists one to one correspondence between infinite dimensional likens with uniqueness and the sequences of positive numbers tending to infinity and linearly independent in the vector space \((\mathbb{R}, \mathbb{Q})\). If we abandon the assumption of uniqueness, the space \( \mathcal{H} \) looks better from the topological point of view. Some further details are to be found in [1].

At the end of this section we recall one more theorem from [1].

Suppose, that we have two sequences \( \overrightarrow{a} = (a_k)_1^\infty \) and \( \overrightarrow{b} = (b_k)_1^\infty \), which generates two likens with uniqueness denoted by \( L_a \) and \( L_b \) respectively. We have the following
Theorem 2.9. In the notations as above the likens $L_a$ and $L_b$ are isomorphic, if and only if there exists a positive number $\lambda$ such that $\overrightarrow{a} = \lambda \cdot \overrightarrow{b}$.

Remark 2.10. As we have observed above (Proposition 2.6), given a set of generators (finite or infinite) $\overrightarrow{a} = (a_k)_1^{\infty}$, the likeness $L_a$ does not depend on the sequence $(a_k)_1^{\infty}$ but depends only on the set of its elements. The only property we need from $(a_k)_1^{\infty}$ is to be locally finite. Clearly each finite set is locally finite, and for infinite sequences $\overrightarrow{a} = (a_k)_1^{\infty}$ of generators, it is evident, that such a sequence is locally finite if and only if $\lim_{k \to +\infty} a_k = +\infty$. In other words for a likeness $L_a$ we can always assume (and we do it in particular in this paper) that its sequence of generators is strictly increasing.

3 The different properties of likens

Let $L = (x_n)_0^{\infty}$ be a likeness. As it was mentioned above, by a property of likenes we will mean - roughly speaking - all the conditions concerning likenes and formulated only using the language (and properties) of the addition and order in $\mathbb{R}$ and the addition and order in $\mathbb{N}$. Clearly, the properties of likenes are preserved by isomorphisms of likenes. We present below a few examples of such properties. The method of construction of these properties is as follows. We take into account a particular likeness (for example $\mathbb{N}^*$), we take into account a particular property of this likeness (for example the twin primes conjecture), we formulate this property in the language of likenes, and this way we obtain a "property of likeness".

Property 3.1. We will say that the dimension of $L$ equals $k \in \mathbb{N}$ if $L$ has exactly $k$ irreducible elements. In other words $\dim(L) = k \iff \text{card}(\mathcal{P}_L) = k$. Such a likeness will be said finitely generated.

Let us mention here, that $\mathbb{N}$ is a one dimensional likeness. The so-called numerical semigroups (for definition see [3]) are finite dimensional likenes. A numerical semi-group is a semi-group generated in $(\mathbb{N}, +)$ by the complements of finite sets. For example the set $\mathbb{N} \setminus \{1, 2\}$ is a numerical semi-group, which is - a three dimensional likeness - (its generators are $\{3, 4, 5\}$). This likeness is a likeness without uniqueness, since, for example $8 = 3 + 5 = 4 + 4$.

This way we have

Property 3.2. Suppose that $\overrightarrow{a} = (a_k)_1^{\infty}$ is a set of generators (finite or infinite) of the likeness $L$. As we have mentioned above, the likeness $L$ has the uniqueness property if for each $x \in L$ there exists exactly one $\overrightarrow{n} \in \mathbb{N}_0^\mathbb{N}$ such that $x = \langle \overrightarrow{a}, \overrightarrow{n} \rangle$.

Property 3.3. Suppose, that $\mathbb{P} = \{1 < p_1 < p_2 < \ldots\}$ is a subset of the set of natural numbers (finite, or infinite). We will say that $L$ has its generators exactly in $\mathbb{P}$ when for each $n \in \mathbb{N}$ we have: $x_n \in L$ is irreducible if and only if $n \in \mathbb{P}$.
It is not hard to see, that if $\mathbb{P}$ is finite then one always can find a liken which has the generators exactly in $\mathbb{P}$. When the set $\mathbb{P}$ is infinite then the problem of the existence of a liken which has its generators "exactly" in $\mathbb{P}$ is more complicated. There is an obvious necessary condition for such a property, namely the set $\mathbb{N} \setminus \mathbb{P}$ must be infinite, but we do not know any reasonable characterization of those $\mathbb{P}$ for which there exists a liken which has the generators in $\mathbb{P}$.

Let us recall here, that Definition 2.1 does not assure the uniqueness, and that in this paper, for simplicity, we mean liken as liken with uniqueness. If $L$ is a liken with uniqueness then each element of this liken can be identified with a sequence of its coefficients in the representation

$$x = \langle \overrightarrow{a}, \overrightarrow{n} \rangle = n_1a_1 + n_2a_2 + ... = n_1(x)a_1 + n_2(x)a_2 + ....$$

We set $\text{supp}(x) = \{i \in \mathbb{N} : n_i(x) \neq 0\}$ and we call this set the support of $x$.

**Property 3.4.** We will say that a liken $L$ has a "disjoint support property" if for each $n \in \mathbb{N}$ we have $\text{supp}(x_n) \cap \text{supp}(x_{n+1}) = \emptyset$.

Clearly, if a liken $L$ has the disjoint support property, then it is infinitely dimensional. For example the liken $\mathbb{N}^*$ has this property. If $i \in \text{supp}(x)$ than we will say that $a_i$ divides $x$ (in symbol $a_i | x$).

**Property 3.5.** We will say that $L$ has the "parity property" if for each $n \in \mathbb{N}$ we have $(x_1 | x_n) \Rightarrow \neg (x_1 | x_{n+1})$.

In [1] we studied the sequence of gaps in likens, i.e. the sequence of differences $\delta_L(k) = \delta_k = \delta(x_k) = x_{k+1} - x_k$. By the definition of liken the sequence $\delta_k$ is strictly positive and as it was observed in [1], if $\dim(L) \geq 2$ then $\lim_{k \to \infty} \delta_L(k) = 0$. However in general, in particular in the case of finite dimensional likens, the sequence $\delta_k$ is not strictly decreasing. On the other hand there are the likens, such that $\delta_L(k)$ is strictly decreasing. Since the property $\delta_k$ is strictly decreasing is equivalent to: for each $k \in \mathbb{N}$ we have $\delta_k > \delta_{k+1}$ or equivalently, $2x_{k+1} > x_k + x_{k+2}$, then we formulate the property of convexity as follows.

**Property 3.6.** A liken $L$ is said to be convex if and only if for each $k \in \mathbb{N}$ the following inequality holds

$$2x_{k+1} > x_k + x_{k+2}.$$
It is clear, that the convexity of a liken is preserved by isomorphisms, but one may give example of two convex likens $K$ and $L$ which are not isomorphic. Indeed, let $L = N^* = (x_n)_0^\infty$ and let $N^{**} = (y_n)_0^\infty = K = (\ln(2n+1))_0^\infty$ be a liken of all odd natural numbers (see Introduction). This likens are both convex, but are not isomorphic. Indeed, let $x_3$ is composed and $y_3$ is irreducible. The same is true if one considers the liken $K_p = (\ln(pn+1))_0^\infty$ for $p = 1, 2, ...$

Let us look at the liken $N^*$ and notice that except for one case, of two consecutive elements of this liken, at most one is irreducible. The situation is different in the liken $N^{**}$, where every twin primes (in $N^*$) are consecutive elements of the liken $N^{**}$. We can therefore formulate the following property of likens:

**Property 3.7.** We will say that in liken $L$ almost all irreducible elements are separated when the number of such pairs $(x_n, x_{n+1})$ in which both elements are irreducible, is finite.

It not difficult to show, that there exist likens without the separation property. On the other hand, the example of $N^{**}$ shows that the problem of proving that a particular liken has the separation property, may be very difficult.

The name of the next property refers to an old philosophical principle. The so-called Ockham’s razor principle states that entities should not be multiplied beyond necessity.

Before we formulate this property for likens, let us establish some notations. Suppose that $L = (x_m)_0^\infty$ is a liken. For $n \in \mathbb{N}$ we set $L^{(n)} = L(x_1, x_2, ..., x_n)$ i.e. $L^{(n)}$ is a liken generated by all elements not greater than $x_n$, which is clearly a sub-liken of $L$. Let us observe that $L^{(n)} = L(a_1, a_2, ..., a_k)$ where $(a_1, a_2, ..., a_k)$ are all irreducible elements such that $a_i \leq x_n$.

Let

$$z_n = \min \{x : x \in L^{(n)}, x > x_n\}.$$  (15)

**Property 3.8.** We will say that a liken $L$ has the Ockham’s razor property if

$$\text{supp}(x_n) \cap \text{supp}(z_n) = \emptyset \implies x_{n+1} = z_n.$$

For the sake of explaining the name of Ockham’s razor let us consider the following: Suppose we want to construct a liken $L$ with the disjoint support property and the construction runs recursively. Suppose we construct $x_n$ and want to construct $x_{n+1}$. We do this: we determine the smallest element of the liken generated by the already constructed among bigger than $x_n$ and denote it with $z_n$. If the support of $z_n$ is disjoint with the support of $x_n$ then we take $z_n$ as $x_{n+1}$. This is just the considered property. And what happens, when $\text{supp}(x_n) \cap \text{supp}(z_n) \neq \emptyset$? Because the ”necessity” (for us ) is the disjoint support property, then we must ”multiply the entities” and set $x_{n+1} = a_{k+1}$. Let us notice here, that if $x_{n+1} = a_{k+1}$ then necessarily $x_{n+2} = z_n$ since undoubtedly $z_n \in L^{(n+1)}$ and $z_{n+1} = z_n$. Indeed, if $a_{k+1} \not\in \text{supp}(z_{n+1})$ then $z_{n+1} \in L^{(n)}$ and by definition $z_n = z_{n+1}$. On the other hand $a_{k+1} \not\in \text{supp}(x_{n+2})$ since in such case the disjoint support property would be violated.
Property 3.9. We will say, that a liken $\mathbb{L}((a_k)_{0}^{\infty})$ has \textit{the Bertrand property} when for each $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $x_n \leq a_k \leq x_n + a_{1}$.

Property 3.10. We will say, that a liken $\mathbb{L}((a_k)_{0}^{\infty})$ has \textit{the Legendre property} when

$$\lim_{n \to \infty} \frac{\text{card}\{k : a_k \leq x_n\}}{n} = 0.$$

All properties, $(3.2, 3.10)$, are true in the liken $\mathbb{N}^{*}$, so they are consistent. On the other hand it is obvious that the conjunctions of some of the properties on the list above imply other or even all of the others.

In this situation it is natural to ask if there are other likens besides $\mathbb{N}^{*}$ that have all of properties listed above, or which of these properties characterize the liken of natural numbers with multiplication.

Note that both properties (C) and (OR) are fulfilled in $\mathbb{N}^{*}$ while $\mathbb{N}^{**}$ has property (C) and no property (OR). Indeed, in this case we have (in multiplicative model): $x_1 = 3$, $x_2 = 5$, $x_2 = 9$ and $x_3 = 7$. Hence convexity do not imply the Ockham’s razor property. On the other hand, the property (C) implies the disjoint support property. Indeed suppose that $x_{k+1} = x_p + a_{1}$ and $x_k = x_q + a_{1}$. Hence $\delta(x_k) = x_{k+1} - x_k = x_p - x_q \geq x_{q+1} - x_q = \delta(x_q)$. But this is impossible, since in convex likens $q < k$ implies $\delta(x_q) > \delta(x_k)$.

4 The main theorem

In this section we are going to prove the main theorem, which gives a characterization of the liken $\mathbb{N}^{*}$ in the space of all likens. Suppose, that $\overrightarrow{a} = (a_k)_{0}^{\infty}$ is a sequence of positive real numbers generating a liken $\mathbb{L}(\overrightarrow{a})$ denoted shortly by $\mathbb{L}_{a}$. (Let us recall, that in this paper, ”liken” means ”liken with uniqueness”). In this notations we formulate the main result of this paper as follows:

\begin{quote}
\textbf{Theorem 4.1.} If the liken $\mathbb{L}_{a}$ is convex and has the Ockham’s razor property, then it is isomorphic to the liken $\mathbb{N}^{*}$.
\end{quote}

First we will make a number of observations, that we will use in the proof.

4.1 Multiplicative notation

Our Definition 2.1 of a liken determines, that a liken $\mathbb{L} = (x)_{0}^{\infty}$ is an increasing sequence of non-negative real numbers closed under addition in $\mathbb{R}$. Consider a new sequence defined by the formula

$$\hat{x}_n = \exp(x_{n-1}),$$

(16)
(for \( n = 1, 2, \ldots \)). This sequence \( \hat{\mathbb{L}} = (\hat{x}_n)_{1}^{\infty} \) is a strictly increasing sequence of positive real numbers closed with respect to the multiplication in \( \mathbb{R} \) and obviously

\[
\hat{x}_p \cdot \hat{x}_q = \hat{x}_{p+1} \cdot \hat{x}_{q+1} \tag{17}
\]

We may say, that \( \hat{\mathbb{L}} \) is the same liken as \( \mathbb{L} \), but we write “ \( \cdot \) ” instead of “ + ”. The number 0 is replaced by 1 and indices go from 1 to +\( \infty \). Conversely, if we have a liken \( \hat{\mathbb{L}} = (\hat{x}_n)_{0}^{\infty} \) with the multiplicative notation, than the sequence \( (x_n)_{0}^{\infty} \) defined by the formula \( x_n = \ln(\hat{x}_{n+1}) \) for \( n = 0, 1, \ldots \) is a liken with additive notation. In consequence, if in an additive liken \( \mathbb{L} \) we consider the gaps \( \delta_k = x_{k+1} - x_k \) then in the multiplicative version we use the fraction

\[
\hat{\delta}_k = \frac{\hat{x}_{k+1}}{\hat{x}_{k}},
\]

and conversely the quotients are replaced by the differences. Let us agree, that if there is a "hat" above the symbols referring to the liken \( \mathbb{L} \) then the formulas refer to the multiplicative model of \( \mathbb{L} \).

### 4.2 The isomorphism "exponent"

Let us take into account the set \( \mathcal{E} = \mathbb{N}_0^\mathbb{N} \), called in the sequel the space of exponents and let \( \mathbb{L}_a = (x_n)_{0}^{\infty} \) be a liken. Hence, as we have observed above, the map

\[
\Omega_{\mathbb{L}} : \mathbb{N}_0^\mathbb{N} \ni \overrightarrow{m} \mapsto \langle \overrightarrow{a}, \overrightarrow{m} \rangle \in \mathbb{L}_a \tag{18}
\]

is a bijection and it is an isomorphism of semigroups.

The inverse map

\[
\Omega_{\mathbb{L}}^{-1} : \mathbb{L}_a \ni x_n \mapsto \Omega_{\mathbb{L}}^{-1}(x_n) \in \mathbb{N}_0^\mathbb{N} \tag{19}
\]

is also a bijection and is an isomorphism of semigroups.

When we have another liken \( \mathbb{K}_b = (y_n)_{0}^{\infty} \) then we can consider an analogous isomorphisms

\[
\Omega_{\mathbb{K}} : \mathbb{N}_0^\mathbb{N} \ni \overrightarrow{m} \mapsto \langle \overrightarrow{b}, \overrightarrow{m} \rangle \in \mathbb{K}_b \tag{20}
\]

as well as

\[
\Omega_{\mathbb{K}}^{-1} : \mathbb{K}_b \ni y_n \mapsto \Omega_{\mathbb{K}}^{-1}(y_n) \in \mathbb{N}_0^\mathbb{N}. \tag{21}
\]

The superposition

\[
\Psi_{\mathbb{K},\mathbb{L}} : \mathbb{K}_b \ni y_n \mapsto \Omega_{\mathbb{L}}(\Omega_{\mathbb{K}}^{-1}(y_n)) \in \mathbb{L}_a \tag{22}
\]

is an algebraic isomorphism of the likens \( \mathbb{K}_b \) and \( \mathbb{L}_a \), which allows us to say, that each two infinitely generated likens are algebraically isomorphic. Let us notice, that in the case, when
the sequences of generators are strictly increasing, then the described isomorphism $\Psi_{K,L}$ is unique.

Now we take as $K_b$ the particular liken $N^* = (\ln(n+1))^{\infty}$, denoted as $(y_n)_0^{\infty}$ and we consider the analogous isomorphisms $\Omega_n^*$ and $\Omega_n^{−1}$. We will write simply $\Omega$, the when lower index is implied by the context.

The composed isomorphism $\Psi_{K,L}$ in this special case will be denoted simply by $\Psi$. We have

$$\Psi : N^* \ni y_n \rightarrow \Omega_L(\Omega_N^1(y_n)) \in \mathbb{L}_a = \Omega^{-1}(\gamma_n)) \in \mathbb{L}_a.$$  \hfill (23)

### 4.3 The beginning of the inductive proof

As we see, the map $\Psi$ is an algebraic isomorphism of $\mathbb{N}^*$ and $\mathbb{L}_a$. It remains to show, that $\Psi$ is also ordinal. This last assertion will be proved by induction. In fact we want to prove, that for each $n \in \mathbb{N}$ we have $\Psi(y_n) = x_n$. Since clearly $\Psi(y_0) = x_0$ then the induction step is: if $\Psi(y_k) = x_k$ for $k \leq n$ then $\Psi(y_{n+1}) = x_{n+1}$. Or, in other words we must prove the implication:

**Theorem 4.2.** If for each $0 \leq i < j \leq n$ the inequality $x_i < x_j$ is equivalent to the inequality $y_i < y_j$ then $\Psi(y_{n+1}) = x_{n+1}$.

First we shall verify, that for ”small” $n$ the function $\Psi$ has the claimed property. Clearly, for $n = 0$ we have $x_0 = 0$ (i.e. $\Psi(y_0) = x_0$) , as in each liken. Although, from the formal point of view, this is not necessary, we will check in details that $\Psi(y_k) = x_k$ for a few initial $k \in \mathbb{N}$ in order to see how the properties (C) and (OR) ”work”.

**Case n = 1.** It must be $x_1 = a_1$, since $x_1$ must be indecomposable. Indeed suppose that $x_1 = u + v$, where $u \in \mathbb{L} \ni u > 0$ and $v > 0$. Hence $0 < u < x_1$, but this is impossible, since $x_1$ is next after $x_0$. In other words $x_1 = a_1$. Hence $\Psi(y_1) = x_1$. Let us observe, that the equality $\Psi(y_1) = x_1$ does not require any additional assumption (i.e. it is true in all likens).

**Case n = 2.** It must be $x_2 = a_2$. Indeed $z(x_1) = 2a_1$ (the definition of $z(x)$ is in $[13]$ and we see, that the supports of $z(x_1)$ and $x_1 = a_1$ are not disjoint. Hence, by (OR) $x_2 = a_2$. Then, clearly $x_1 = a_1 < a_2 = x_2 < 2a_1$.

**Case n = 3.** We have clearly $x_2 = a_2 < 2a_1 < a_1 + a_2$. Hence $z(x_2) = 2a_2$ is disjoint with $x_2 = a_2$. In consequence $x_3 = 2a_1$. Let us remark, that here we use (OR).

**Case n = 4.** We see, that $x_3 = 2a_1$ is still in $\mathbb{L}^{(2)}$. It is also easy to check, that $z(x_3) = a_1 + a_2$. Since the support of $z(x_3)$ is not disjoint with the support of $x_3$, then $x_4 = a_3$. Here we use once more the (OR) property.

**Case n = 5.** Clearly $x_4 \in \mathbb{L}^{(3)}$ and $z(x_4) = a_1 + a_2$ (this follows from $a_1 + a_2 < 2a_3$). We see that $z(x_4)$ has the support disjoint with the support of $x_4 = a_3$. Hence (by (OR) $x_5 = a_1 + a_2$.
Case \( n = 6 \). Since \( 2a_2 < 2a_3 \) then \( z(x_5) \in L^2 \). It is clear, that \( 3a_1 < 2a_1 + a_2 < a_1 + 2a_2 \). Using the property (C) for \( n = 2 \) we obtain \( 3a_1 = a_1 + 2a_1 = x_1 + x_3 < 2x_2 = 2a_2 \). In consequence \( z(x_5) = 3a_1 \). Since \( z(x_5) \) is not disjoint with \( x_5 \), then \( x_6 = a_4 \).

Case \( n = 7 \). Here, as before, and as we will do later, we may apply a general remark: if \( x_n < z(x_n) \) and \( x_n \) and \( z(x_n) \) are not disjoint, then from (OR) we have: \( x_{n+1} = a_{k+1} \) and \( x_{n+2} = z(x_n) \). This follows from the inequality \( z(x_n) < 2a_{k+1} \). Thus \( x_7 = 3a_1 \).

Case \( n = 8 \). It follows from the considerations for \( n = 6 \) and \( n = 7 \) that \( z(x_7) = 2a_2 \), hence \( x_8 = 2a_2 \).

Case \( n = 9 \). We are now in \( \mathbb{L}^{(4)} \), and we calculate \( z(x_8) \), which belongs "a priori" to \( \mathbb{L}^{(4)} \). But, we have

\[
a_1 + a_3 - x_8 = a_1 + a_3 - 2a_2 = a_1 + 2a_3 - 2a_2 - a_3 = a_1 + 2x_4 - 2a_2 - a_3 >
\]

\[
a_1 + x_3 + x_5 - 2a_2 - a_3 = a_1 + 2a_1 + a_1 + a_2 - 2a_2 - a_3 = 4a_1 - (a_2 + a_3) = 2x_3 - (x_2 + x_4) > 0.
\]

Since \( a_1 + a_3 < 2a_1 + a_2 \) (because \( x_4 < x_5 \)) and clearly \( a_1 + a_3 < a_1 + a_4 \), then \( z(x_8) = a_1 + a_3 \). Since \( z(x_8) \) and \( x_8 = 2a_2 \) are disjoint, then \( x_9 = a_1 + a_3 \).

Case \( n = 10 \). Since \( x_4 < x_5 \) then \( x_9 = a_1 + a_3 < 2a_1 + a_2 \). Clearly \( 2a_1 + a_2 < a_1 + a_4 \) and \( 2a_1 + a_2 < a_2 + a_3 \). Then \( z(x_9) = 2a_1 + a_2 \) and hence, \( x_{10} = a_5 \).

We see, that for \( 0 \leq n \leq 10 \) the map \( \Psi \) satisfies the claimed properties on isomorphism of likens.

### 4.4 The induction step

As we are used to the multiplicative structure of the \( \mathbb{N}^* \) semigroup, we will write the proof of the main Theorem 4.1 in the multiplicative convention of both likens \( \mathbb{L}_a \) and \( \mathbb{N}^* \). Moreover, the role played by even numbers in \( \mathbb{N}^* \) incline to the some reformulation of the inductive step.

Let us say also, that \( \widehat{x}_k \) is even when \( \widehat{x}_1 | \widehat{x}_k \). Let us recall, that

\[
\widehat{\Psi} : \mathbb{N} \ni n \rightarrow \widehat{\Psi}(n) \in \mathbb{L}_a
\]

is the (unique) algebraic isomorphism of the considered likens, i.e. for each \( i, j \in \mathbb{N} \) we have

\[
\widehat{\Psi}(i \cdot j) = \widehat{\Psi}(i) \cdot \widehat{\Psi}(j).
\]

Thus to prove, that \( \widehat{\Psi} \) is an isomorphism of likens we must prove that \( \widehat{\Psi} \) is an order isomorphism, which means, as usually for likens, that for each \( i \in \mathbb{N} \) we have: \( \widehat{\Psi}(i) = \widehat{x}_i \). So to prove Theorem 4.1 it is sufficient to prove the following theorem ("even" version of the induction step):

**Theorem 4.3.** Suppose, that \( n \in \mathbb{N} \) and that for each \( 1 \leq i \leq 2n \) we have \( \widehat{\Psi}(i) = \widehat{x}_i \). Then \( \widehat{\Psi}(2n + 1) = \widehat{x}_{2n+1} \) and \( \widehat{\Psi}(2n + 2) = \widehat{x}_{2n+2} \).
We will start by formulating a number of observations.

i). Let consider the elements \( \hat{\Psi}(2j) \) for \( 1 \leq j \leq 2n \). Since \( \hat{\Psi} \) is an algebraic isomorphism, for each \( j \leq 2n \) we have \( \hat{\Psi}(2j) = \hat{\Psi}(2) \cdot \hat{\Psi}(j) = \hat{x}_2 \cdot \hat{x}_j \). If \( j \leq n \) then we can write (by induction hypothesis) \( \hat{x}_2 \cdot \hat{x}_j = \hat{x}_2 \cdot \hat{x}_2 \cdot \hat{x}_j = \hat{x}_2 \cdot \hat{x}_j \). If \( j \leq n \) then we can write (by induction hypothesis) \( \hat{x}_2 \cdot \hat{x}_j = \hat{x}_2 \cdot \hat{x}_2 \cdot \hat{x}_j = \hat{x}_2 \cdot \hat{x}_j \). If \( j \leq n \) then we can write (by induction hypothesis) \( \hat{x}_2 \cdot \hat{x}_j = \hat{x}_2 \cdot \hat{x}_2 \cdot \hat{x}_j = \hat{x}_2 \cdot \hat{x}_j \).

ii). Since \( \hat{x}_n < \hat{x}_{n+1} \) then \( \hat{x}_{2n} = \hat{x}_2 \cdot \hat{x}_n < \hat{x}_2 \cdot \hat{x}_{n+1} \). But in \( L_a \) we have the disjoint support property, so we must have

\[
\hat{x}_{2n} < \hat{x}_{2n+1} < \hat{x}_2 \cdot \hat{x}_{n+1} \quad (26)
\]

In other words this means, that between \( \hat{x}_{2n} \) and \( \hat{x}_2 \cdot \hat{x}_{n+1} \) there are some elements of the liken \( L_a \) but we do not now how many of these elements are there, and what they are.

iii). Let us consider the set

\[
D = (\hat{x}_2 \cdot \hat{x}_n, \hat{x}_2 \cdot \hat{x}_{n+1}) \cap L^{(2n)} \quad (27)
\]

and let us call it "a box".

First we will prove that

**Lemma 4.4.** If \( 2n + 1 \) is composed, then \( \hat{\Psi}(2n + 1) \in (\hat{x}_{2n}, \hat{x}_2 \cdot \hat{x}_{n+1}) \).

**Proof.** Let us assume then that \( 2n + 1 = p \cdot q \). Then clearly \( p \geq 2 \) and \( q \geq 2 \) and we have to prove the following inequalities:

\[
\hat{x}_{2n} < \hat{\Psi}(2n + 1) \quad (28)
\]

and

\[
\hat{\Psi}(2n + 1) < \hat{x}_2 \cdot \hat{x}_{n+1} \quad (29)
\]

The first inequality follows directly from the inductive assumption. Indeed, the inductive assumption says, in particular, that

\[
\hat{\Psi} : [1, 2, \ldots, 2n] \rightarrow [1, \hat{x}_2, \ldots, \hat{x}_{2n}]
\]

is a bijection. But \( 2n + 1 \notin [1, 2, \ldots, 2n] \) then \( \hat{\Psi}(2n + 1) \notin [1, \hat{x}_2, \ldots, \hat{x}_{2n}] \) and in consequence \( \hat{x}_{2n} < \hat{\Psi}(2n + 1) \).

The proof of the second inequality is more complicated. Clearly we may assume that \( p \leq q \) and since \( p \cdot q \) is odd then \( p \) and \( q \) are both odd, and we have the inequality

\[
3 \leq p \leq q < n.
\]

Indeed, suppose \( q \geq n \). Then we have \( 2n + 1 = p \cdot q \geq 3 \cdot n \) which is possible only in \( n = 1 \) but in our case \( n \geq q \geq 3 \).
Let us denote

$$A = \frac{\hat{\Psi}(2n + 2)}{\Psi(2n + 1)}.$$

Our aim is to show that $A > 1$. We have (recall that $\hat{\Psi}$ is an algebraic isomorphism on the whole $\mathbb{N}^*$ and recall that the quotients corresponds to differences in the additive models).

$$A = \frac{\hat{\Psi}(2n + 2)}{\Psi(2n + 1)} = \frac{\hat{\Psi}(2(n + 1)))}{\Psi(p \cdot q)} = \frac{\hat{x}_2 \cdot \hat{x}_{n+1}}{\hat{x}_p \cdot \hat{x}_q}.$$

Let us notice here, that in this moment we cannot write $\hat{x}_p \cdot \hat{x}_q = \hat{x}_{pq}$ since $pq > 2n$. But we know, that $p$ is odd, and then $p + 1$ is even and $p + 1 \leq n$. Thus $p + 1 = 2s \leq n$ and then we may write $\hat{x}_{p+1} = \hat{x}_2 \cdot \hat{x}_s$. So we may also write

$$A = \frac{\hat{\Psi}(2n + 2)}{\Psi(2n + 1)} = \frac{\hat{\Psi}(2(n + 1)))}{\Psi(p \cdot q)} = \frac{\hat{x}_2 \cdot \hat{x}_{n+1}}{\hat{x}_p \cdot \hat{x}_q} = \frac{x_{p+1}}{x_p} \cdot \frac{\hat{x}_2 \cdot \hat{x}_{n+1}}{\hat{x}_s \cdot \hat{x}_q} = \frac{x_{p+1}}{x_p} \cdot \frac{\hat{x}_{n+1}}{\hat{x}_s \cdot \hat{x}_q}.$$

Here is the time to replace $\hat{x}_s \cdot \hat{x}_q$ by $\hat{x}_{sq}$ but for this we must evaluate $sq$ from above. We have $pq = 2n + 1$ and $sq < pq = 2n + 1$, hence $sq$ is a natural number satisfying $sq \leq 2n$. This is sufficient for our purposes (for the use the induction hypothesis) although a more detailed analysis allows us to prove that $sq \leq \frac{3p}{2}$. So, by induction hypothesis, we may write $\hat{x}_s \cdot \hat{x}_q = \hat{x}_{sq}$ and in consequence we obtain

$$A = \frac{\hat{x}_{p+1}}{\hat{x}_p} \cdot \frac{\hat{x}_{n+1}}{\hat{x}_s \cdot \hat{x}_q} = \frac{\hat{x}_{p+1}}{\hat{x}_p} \cdot \frac{\hat{x}_{n+1}}{\hat{x}_{sq}}.$$

iv). Here we will need a simple lemma resulting from the convexity property. Suppose, that $\mathbb{L} = (x_n)_{0}^{\infty}$ is a liken (in additive convention). For fixed $j$ we can consider the sequence $\delta^j(n) = x_{n+j} - x_n$. It appears, that in convex likens (for each $j$) such a sequence is also strictly decreasing. First we have

**Lemma 4.5.** Let $\mathbb{L} = (x_n)_{0}^{\infty}$ be a liken satisfying the convexity property and let $p$ and $q$ be arbitrary positive integers such that $1 \leq p < q$. Then $x_{q-1} - x_{p-1} > x_q - x_p$.

Let us recall the notation $\delta(k) = x_{k+1} - x_k$ and recall that in a convex liken we have $\delta(k+1) < \delta(k)$. Hence

$$x_q - x_p = x_q - x_{q-1} + x_{q-1} - x_{q-2} + ... + x_{p+1} - x_p =$$

$$= \delta(q - 1) + \delta(q - 2) + ... + \delta(p) \leq \delta(q - 2) + \delta(q - 3) + ... + \delta(p - 1) =$$

$$= x_{q-1} - x_{q-2} + x_{q-2} - x_{q-3} + ... + x_{p} - x_{p-1} = x_{q-1} - x_{p-1}.$$

From this lemma, by induction, we obtain the following inequality: if $1 \leq p < q$ and $k \leq p$ then $x_{q-k} - x_{p-k} > x_q - x_p$.

The same, but in multiplicative notation, may be formulated as follows.
Lemma 4.6. Let us suppose, that \( x_p, x_q \) are two elements of a convex liken \( L = (x_n)_{1}^{\infty} \) (in multiplicative convention), and \( 1 \leq k < p < q \). Then
\[
\frac{x_p}{x_q} > \frac{x_{p-k}}{x_{q-k}}.
\]

v). Now we return to the evaluation from below of the quantity \( A \). Our aim is to prove that \( A > 1 \). We have proved that
\[
A = \frac{x_{p+1}}{x_p} \cdot \frac{x_{n+1}}{x_{s}} > \frac{x_{p+1}}{x_p} \cdot \frac{x_{n+1}}{x_{s}}.
\]

The inequality \( A > 1 \) is evident when \( n+1 \leq sq \), then assume that \( n+1 < sq \). By Lemma 4.6 we have
\[
A = \frac{x_{p+1}}{x_p} \cdot \frac{x_{n+1}}{x_{sq}} > \frac{x_{p+1}}{x_p} \cdot \frac{x_{n+1-s}}{x_{sq-s}}.
\]

As we have observed above, we have \( n + 1 - s < 2n \) and \( sq - s < 2n \) and additionally we have
\[
\frac{n + 1 - s}{sq - s} = \frac{p}{p + 1}.
\]

Indeed we have the sequence of equivalent equalities
\[
\frac{n + 1 - s}{sq - s} = \frac{p}{p + 1} \iff (p + 1)(n + 1 - s) = p(sq - s) \iff 2s(n + 1 - s) = ps(q - 1) \iff 2(n + 1 - s) = pq - p \iff 2n + 2 - p - 1 = 2n + 1 - p.
\]
The last equality is true since we assumed that \( pq = 2n + 1 \) and \( p + 1 = 2s \).

Hence there exists a number \( t \in \mathbb{N} \) such that \( n + 1 - s = tp \) and \( sq - s = t(p + 1) \). From the induction assumption we have
\[
A = \frac{x_{p+1}}{x_p} \cdot \frac{x_{n+1}}{x_{sq}} > \frac{x_{p+1}}{x_p} \cdot \frac{x_{n+1-s}}{x_{sq-s}} = \frac{x_{p+1}}{x_p} \cdot \frac{x_{n+1-s}}{x_{t(p+1)}} = \frac{x_{p+1}}{x_p} \cdot \frac{x_{t}}{x_{p}} \cdot \frac{x_{p+1}}{x_{p}} = 1.
\]

This ends the proof of the inequality (29) and at the same time of Lemma 4.4.

vi). Consider now the situation, when between \( \hat{x}_{2n} \) and \( \hat{x}_2 \cdot \hat{x}_{n+1} \) there are no elements of the liken \( L(2n) \), i.e. the box is empty. In this case \( z(2n) = \hat{x}_2 \cdot \hat{x}_{n+1} \). The razor property implies then, that \( \hat{x}_{2n+1} = a_{k+1} \). But in this case \( 2n + 1 \) cannot be composed, since, when \( 2n + 1 \) is composed, then \( \hat{\Psi}(2n + 1) \) is in \( L(2n) \), and, as we have proved above
\[
\hat{\Psi}(2n + 1) \in (\hat{x}_{2n}, \hat{x}_2 \cdot \hat{x}_{n+1}),
\]
contrary to our assumption. Hence \(2n + 1 = p_{k+1}\) \((p_{k+1}\) is the \((k+1)\)-th prime in \(\mathbb{N}^*\)) and 
\(a_{k+1} = \hat{x}_{2n+1}\), and we see that in this case \(\Psi(2n + 1) = x_{2n+1}\).

vii). Summarizing, we have proved, that the element \(\hat{\Psi}(2n + 1)\) is always in the interval 
\((\hat{x}_{2n}, \hat{x}_{2} \cdot \hat{x}_{n+1})\). To end the proof of the main Theorem it is enough to show, that in the interval 
\((\hat{x}_{2n}, \hat{x}_{2} \cdot \hat{x}_{n+1})\) there are no other elements of \(\hat{\mathbb{L}}_{(2n)}\) besides, possibly, \(\hat{\Psi}(2n + 1)\).

viii). Suppose that there exists an element \(\hat{x}\) such that 
\(\hat{x} \in \hat{\mathbb{L}}_{(2n)} \cap (\hat{x}_{2n}, \hat{x}_{2} \cdot \hat{x}_{n+1})\) and 
\(\hat{x} \neq \hat{\Psi}(2n + 1)\). Since \(\hat{x} \in \hat{\mathbb{L}}_{(2n)}\) then there exist two natural numbers \(r\) and \(s\), such that 
\(r \leq 2n, s \leq 2n\), \(\hat{x} = \hat{x}_r \cdot \hat{x}_s\) and 
\(\hat{x}_{2n} < \hat{x}_r \cdot \hat{x}_s < \hat{x}_2 \cdot \hat{x}_{n+1}\). Since \(\hat{x} \neq \hat{\Psi}(2n + 1)\) than 
\(r \cdot s > 2n + 1\) and since \(\hat{x}_{2n} < \hat{x}_r \cdot \hat{x}_s < \hat{x}_2 \cdot \hat{x}_{n+1}\) then both \(r\) and \(s\) are odd. Clearly, we can assume that 
\(r \leq s\) and observe, that in fact we have the inequalities

\[
3 \leq r \leq s < n
\]  
(30)

Since \(r > 1\) and \(r\) is odd, we have \(r \geq 3\), so we must show that \(s < n\). Suppose that \(s \geq n\). But \(s \leq 2n\) and \(\hat{\Psi}\) is increasing in the interval \([1, 2n]\) (induction), thus \(\hat{x}_s \geq \hat{x}_n\) and in consequence

\[
\hat{x}_2 \cdot \hat{x}_{n+1} > \hat{x}_r \cdot \hat{x}_s > \hat{x}_3 \cdot \hat{x}_n
\]  
(31)

This gives the inequality (in the multiplicative convention)

\[
\hat{x}_2 \cdot \hat{x}_{n+1} > \hat{x}_3 \cdot \hat{x}_n
\]

and this gives (in additive convention) the inequality

\[
x_{n+1} - x_n > x_2 - x_1
\]

which impossible in convex likens. Since \(n > r \geq 3\) and \(r\) is odd, then \(r - 1\) is even and we can put \(r - 1 = 2t\). In consequence we have

\[
\hat{x}_2 \cdot \hat{x}_t \cdot \hat{x}_s = \hat{x}_{r-1} \cdot \hat{x}_s < \hat{x}_r \cdot \hat{x}_s < \hat{x}_2 \cdot \hat{x}_{n+1}
\]  
(32)

which gives the inequality

\[
\hat{x}_t \cdot \hat{x}_s < \hat{x}_{n+1}.
\]  
(33)

Since we are in the interval \([1, \hat{x}_2, \hat{x}_3, ..., \hat{x}_{2n}]\), and \(\hat{\Psi}^{-1}\) is increasing then \(t \cdot s < n + 1\).

ix). The end of our reasoning is similar as above. We know, that we may set \(rs = 2m + 1\) where \(m > n + 1\). Let us denote

\[
B = \frac{\hat{\Psi}(2m + 1)}{\Psi(2m)}
\]  
(34)

Our aim is to prove, that \(B > 1\). We have
\[
B = \frac{\hat{\Psi}(2m + 1)}{\hat{\Psi}(2m)} = \frac{\hat{\Psi}(r + s)}{\hat{\Psi}(2m)} = \frac{\hat{x}_r \cdot \hat{x}_s}{\hat{x}_2} \cdot \frac{\hat{x}_{s-1}}{\hat{x}_m} = \frac{\hat{x}_r}{\hat{x}_2} \cdot \frac{\hat{x}_{s-1}}{x_m} = B_{s-1} \cdot \hat{x}_s
\]

By a similar argument as before, we check that \(ts - t = w(r - 1)\) and \(m - t = wr\). Since \(\hat{\Psi}\) is an algebraic isomorphism in the whole \(\mathbb{N}^*\), we have \(\hat{x}_{ts-t} = \hat{x}_w \cdot \hat{x}_{r-1}\) and \(\hat{x}_{m-t} = \hat{x}_w \cdot \hat{x}_r\).

Let us observe some inequalities. Since we have proved that \(st < n + 1\) then \(ts - t \leq n\) and hence \(w \cdot (r - 1) \leq n\) and by induction hypothesis, we have

\[
\hat{x}_{ts-t} = \hat{x}_w(r-1) = \hat{x}_w \cdot \hat{x}_r
\]

We must also bound \(m - t\) from above. We have \(rs > 2m\). Thus \((r - 1 + 1)s > 2m\) and \((r - 1)s + s > 2m\). But \((r - 1) = 2t\) then \(2ts + s > 2m\). We have proved that \(s < n\) and \(ts < n + 1\). In consequence \(2m < 2ts + s < 2n + 2 + n \leq 3n + 1\). Hence \(m - t < 2n\) and we can use the induction hypothesis for \(m - t = wr\). Hence

\[
B = \frac{\hat{\Psi}(2m + 1)}{\hat{\Psi}(2m)} > \frac{\hat{x}_r}{\hat{x}_r-1} \cdot \frac{\hat{x}_{ts-t}}{\hat{x}_{m-t}} = \frac{\hat{x}_r}{\hat{x}_r-1} \cdot \frac{\hat{x}_{w(r-1)}}{\hat{x}_m} = \frac{\hat{x}_r}{\hat{x}_r-1} \cdot \frac{\hat{x}_w \cdot \hat{x}_{r-1}}{\hat{x}_m} = 1
\]

Finally we obtain

\[
\hat{x}_2 \cdot \hat{x}_{n+1} > \hat{x}_r \cdot \hat{x}_t = \hat{\Psi}(2m + 1) > \hat{\Psi}(2m) = \hat{x}_2 \cdot \hat{x}_m.
\]

In consequence \(\hat{x}_{n+1} > \hat{x}_m\). This means that \(\hat{x}_m \in [1, \hat{x}_2, \ldots, \hat{x}_{2n}]\) so we may use the induction hypothesis and we obtain \(n + 1 > m\). But we know, that \(m > n + 1\) and this contradiction ends the proof of the inductive step, and at the same time, the proof of the main theorem.

### 5 Some additional remarks

**Remark 5.1.** As we have observed, the space of likens is big, but there are only a few examples of likens which could be described as "suitable for counting". A natural method of obtaining such kind of examples is to choose a subset \(K \subset \mathbb{N}^*\) and consider the sub-semigroup \(\mathbb{L}(K)\) generated by \(K\) (i.e. the smallest semigroup containing the set \(K\)) which is ordered by the order inherited from \(\mathbb{N}^*\). Hence we obtain a liken, a sub- liken of \(\mathbb{N}^*\). In particular we may consider only the likens generated by the subsets of the set of prime numbers. Even this family, "small" compared to the family of all likens, is nevertheless "rich", since it contains a continuum of non-isomorphic likens. Theorem 4.1 shows, that only one of these likens is convex and has the razor property.
Remark 5.2. It is commonly known, that the Cauchy functional equation of the type $f(x \cdot y) = f(x) + f(y)$ has many "bad" solutions and only one (up to a constant factor) "good" solution if we claim $f$ to be continuous (or monotone, or locally bounded etc.) and this solution is the logarithmic function. It follows from Theorem 4.1 that the condition of convexity for likens together with the razor property may be considered as a kind of condition guaranteeing the uniqueness of the logarithmic function.

References

[1] Tutaj, E., LikeN's-a point of view on natural numbers. Annales Universitatis Paedagogicae Cracoviensis, Studia Mathematica XVI (2017) 95-115

[2] Beurling, A., Analyse de la loi asymptotique de la distribution de nombre premiers generalises. I. Acta Math.68.no.1.(1937) 255-291

[3] Rosales, J.C., Garcia - Sanchez, P.A., Numerical semigroups, Springer (2009)