INSCRIBABLE FANS II: INSCRIBED ZONOTOPES, SIMPLICIAL ARRANGEMENTS, AND REFLECTION GROUPS

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Abstract. An arrangement of hyperplanes is strongly inscribable if it has an inscribed (or ideal hyperbolic) zonotope. We characterize inscribed zonotopes and prove that the family of strongly inscribable arrangements is closed under restriction and localization. Moreover, we show that (strongly) inscribable arrangements are simplicial. We conjecture that only reflection arrangements and their restrictions are strongly inscribable and we verify our conjecture in rank-3 using the conjecturally complete list of irreducible simplicial rank-3 arrangements.

1. Introduction

A convex polytope $P \subset \mathbb{R}^d$ is inscribed if its vertices lie on a common sphere. A polytope $P$ is inscribable if there is an inscribed polytope $P'$ that is combinatorially equivalent to $P$. The question which combinatorial types of 3-polytopes are inscribable was raised by Steiner [23] and settled by Steinitz [24] and Rivin [21]. In stark contrast, our understanding of the inscribability problem in dimensions four and up is rather exiguous [20, 10]. In [18], we replaced combinatorial equivalence with the discrete-geometric condition of normal equivalence. A polytope $P$ is normally inscribable (or strongly isomorphic) if there is an inscribed polytope $P'$ normally equivalent to $P$. We showed that the collection $I_+(P)$ of (translation-classes of) inscribed polytopes normally equivalent to $P$ has the structure of an open polyhedral cone with respect to Minkowski addition. This brought to light a number of remarkable structural and algorithmic properties; cf. Section 2. Polytopes $P$ and $P'$ are normally equivalent if and only if they have the same normal fan $\mathcal{N}$. A fan $\mathcal{N}$ is inscribable if there is an inscribed polytope $P$ with normal fan $\mathcal{N}(P) = \mathcal{N}$. The simplest normal fans are induced by arrangements of linear hyperplanes. Due to strong connections to convex geometry and algebra, the geometry and combinatorics of arrangements of hyperplanes is a very active area of research. In this paper we study inscribable arrangements, that is, hyperplane arrangements whose associated fan is inscribable.

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement of linear hyperplanes in $\mathbb{R}^d$. The decomposition $\mathbb{R}^d \setminus \bigcup_i H_i$ into open polyhedral cones, called regions, induces a fan that we will also denote
Bolker [1] calls a polytope $P$ with normal fan $\mathcal{A}$ a belt polytope. The most prominent belt polytopes are zonotopes: If $H_i = \{x \in \mathbb{R}^d : \langle z_i, x \rangle = 0\}$ for $i = 1, \ldots, n$, then for any $\lambda \in \mathbb{R}_{\geq 0}^n$ the Minkowski sum of segments

$$Z_\lambda := \lambda_1[-z_1, z_1] + \lambda_2[-z_2, z_2] + \cdots + \lambda_n[-z_n, z_n]$$

is a polytope with $N(Z) = \mathcal{A}$. Zonotopes are distinguished among belt polytopes by many favorable geometric and combinatorial properties. We show that inscribed zonotopes also stand out among inscribed polytopes. For an inscribed polytope $P$, we write $c(P) \in \text{aff}(P)$ for the center of the inscribing sphere relative to its affine hull. For a segment $e \subset \mathbb{R}^d$, we let $M_e$ be the hyperplane of points equidistant to the endpoints of $e$ and we write $\pi_e : \mathbb{R}^d \to M_e$ for the orthogonal projection onto $M_e$.

**Theorem 1.1.** For an inscribed polytope $P \subset \mathbb{R}^d$ the following are equivalent:

(i) $P$ is an inscribed zonotope;

(ii) The projection $\pi_e(P)$ is inscribed with center $c(P)$ for every edge $e$ of $P$;

(iii) The section $P \cap M_e$ is inscribed with center $c(P)$ for every edge $e$ of $P$.

It is quite unusual that projections of inscribed polytopes are inscribed.

We call an arrangement $\mathcal{A}$ strongly inscribable if $\mathcal{A}$ has an inscribed zonotope (as opposed to an inscribed belt polytope). Theorem 1.1 implies that strongly inscribable arrangements constitute a structurally interesting class of hyperplane arrangements.

**Theorem 1.2.** Let $\mathcal{A}$ be a strongly inscribable arrangement and $L$ a flat of $\mathcal{A}$. Then the restriction $\mathcal{A}^L$ and the localization $\mathcal{A}_L$ are strongly inscribable arrangements.

A finite group $W \subset \text{GL}(\mathbb{R}^d)$ is a (finite) reflection group if it is generated by reflections in linear hyperplanes. The collection of reflecting hyperplanes is the reflection arrangement $\mathcal{A}(W)$ of $W$. We show that reflection arrangements are paragons of strongly inscribable arrangements.

**Proposition 1.3.** Reflection arrangements and their restrictions are strongly inscribable.

Note that while localizations of reflection arrangements are reflection arrangements, this does not hold for restrictions; cf. [19, Example 6.83].

In [18], we showed that verifying whether a fan $\mathcal{N}$ is inscribable can be reduced to a linear programming feasibility problem. However, the feasibility problem requires complete knowledge of the fan. Quite remarkably, we give a simple procedure to test if a hyperplane arrangement is strongly inscribable: the linear programming feasibility problem depends on the 2-dimensional flats spanned by the vector configuration $z_1, \ldots, z_n$; see Theorem 4.12 and Remark 6.6.

A hyperplane arrangement $\mathcal{A}$ is simplicial if every region of $\mathcal{A}$ is a simplicial cone. The class of simplicial arrangements is closed with respect to restrictions and localizations. In particular reflection arrangements yield prime examples of simplicial arrangements.
Theorem 1.4. If $\mathcal{A}$ is inscribable, then $\mathcal{A}$ is simplicial. Equivalently, inscribed belt polytopes are simple.

Simplicial arrangements are fascinating but rare. A collection of two infinite families and 90 sporadic examples of simplicial arrangements of rank 3 up to projective transformation was described by Grünbaum [14]. Cuntz [4, 5] contributed five further examples. It is conjectured that the Grünbaum–Cuntz catalog is complete up to combinatorial isomorphism; see also Section 5.2. Using techniques from computational algebra, we show the following.

Theorem 1.5. The only strongly inscribable arrangement in the Grünbaum–Cuntz catalog are restrictions of reflection arrangements.

Assuming that the Grünbaum–Cuntz catalog contains all essential simplicial rank-3 arrangements up to combinatorial isomorphism, we show the following.

Theorem 1.6. If the Grünbaum–Cuntz catalog is complete, then for each $d \geq 3$, there exist only finitely many irreducible strongly inscribable arrangements of rank $d$ up to combinatorial isomorphism.

To show these claims, we computationally re-establish that all simplicial arrangements in the Grünbaum–Cuntz catalog are projectively unique, that is, every combinatorial isomorphism stems from a linearly isomorphism. We also show that under the assumption that the catalog is complete every simplicial arrangement of rank $d \geq 3$ is projectively unique. This observation might be of independent interest.

Further computations in higher rank fuel our main conjecture.

Conjecture 1.7. Every strongly inscribable arrangement of rank $d \geq 3$ is combinatorially isomorphic to the restriction of a reflection arrangement.

Inscribed polytopes correspond to ideal hyperbolic polytopes, that is, polytopes in hyperbolic space with all vertices at infinity. Conjecture 1.7 gives a conjectural classification of ideal hyperbolic zonotopes.

The paper is organized as follows: In Section 2, we recap results on inscribable fans and hyperplane arrangements. In particular, we recall that the collection of translation-classes of inscribed polytopes with normal fan $\mathcal{N}$ is a relatively open polyhedral subcone of the type cone of $\mathcal{N}$ and we discuss inscribed virtual polytopes. We also show that reflection arrangements are strongly inscribable. In Section 3, we give geometric and algebraic characterizations of (strongly) inscribable line arrangements. As in the case of reflection groups, the rank-2 situation serves as building blocks for higher ranks. In Section 4, we investigate operations on (strongly) inscribed arrangements. We show that products, localizations, and restrictions retain (strong) inscribability. In Section 4.3 we derive a polyhedral representation of the cone of inscribed zonotopes of a fixed arrangement $\mathcal{A}$. In Section 5 we show that inscribable arrangements are simplicial and we discuss projective uniqueness of simplicial arrangements. We extend a local characterization of zonotopes due to Bolker to inscribed...
zonotopes (Theorem 4.13). In Section 6, we adopt a broader perspective and investigate arrangements inscribed in a general quadric. This allows us to treat all arrangements from the Grünbaum–Cuntz catalog. We first show that there are only finitely many strongly inscribed arrangements in the two infinite families (Section 6.1). The remaining finitely many arrangements are treated using techniques from computational algebra in Section 6.2. In Section 6.3, we complete the classification of all quadrics for which restrictions of reflection arrangements are inscribable. We close in Section 7 with two remarks and pretty pictures.

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2. INSCRIBABLE FANS OF HYPERPLANE ARRANGEMENTS

In this section, we recall background and central results on inscribable fans (following [18]) and hyperplane arrangements (following [19]). Throughout our ambient space is \( \mathbb{R}^d \) equipped with a fixed inner product \( \langle \cdot, \cdot \rangle \).

A complete fan is a collection \( \mathcal{N} \) of polyhedral cones in \( \mathbb{R}^d \) such that

(F1) if \( C \in \mathcal{N} \) and \( F \subseteq C \) is face, then \( F \in \mathcal{N} \);
(F2) if \( C, C' \in \mathcal{N} \), then \( C \cap C' \) is a face of both;
(F3) \( \bigcup_{C \in \mathcal{N}} C = \mathbb{R}^d \).

The inclusion-maximal cones of \( \mathcal{N} \) are all of dimension \( d \) and are called the regions of \( \mathcal{N} \) and we will usually identify \( \mathcal{N} \) with its set of regions.

Let \( P \subset \mathbb{R}^d \) be a convex polytope with vertex set \( V(P) \). For \( c \in \mathbb{R}^d \), we write

\[ P^c := \{ x \in P : \langle c, x \rangle \geq \langle c, y \rangle \text{ for all } y \in P \} \]

for the non-empty face that maximizes the linear function \( x \mapsto \langle c, x \rangle \). The set of all non-empty faces of \( P \) is denoted by \( \mathcal{F}(P) \). This is a graded lattice with respect by inclusion.

The normal cone of a face \( F \in \mathcal{F}(P) \) is

\[ N_F P := \{ c \in \mathbb{R}^d : F \subseteq P^c \} \]

and the normal fan of \( P \) is \( \mathcal{N}(P) := \{ N_F P : F \in \mathcal{F}(P) \} \). The regions of \( \mathcal{N}(P) \) correspond to normal cones of vertices

\[ N_v P = \{ c \in \mathbb{R}^d : \langle c, v \rangle \geq \langle c, u \rangle \text{ for all } u \in V(P) \} . \]

We call a fan polytopal, if it is the normal fan of a polytope. Two polytopes \( P, P' \) are called normally equivalent if \( \mathcal{N}(P) = \mathcal{N}(P') \). In [18] we studied the question when for a polytope \( P \) there is some inscribed \( P' \) normally equivalent to \( P \). Since this depends only on the normal fan, we call a polytopal fan \( \mathcal{N} \) inscribable if there is an inscribed polytope \( P \) with \( \mathcal{N}(P) = \mathcal{N} \).

Let \( T_d \cong \mathbb{R}^d \) be the group of translations. We defined the inscribed cone of \( \mathcal{N} \) as

\[ \mathcal{I}_+(\mathcal{N}) := \{ P \subset \mathbb{R}^d \text{ inscribed polytope} : \mathcal{N}(P) = \mathcal{N} \} / T_d . \]

The name is justified by the following fundamental result.

**Theorem 2.1** ([18, Theorem 1.1]). Let \( \mathcal{N} \) be a polytopal fan in \( \mathbb{R}^d \). Then \( \mathcal{I}_+(\mathcal{N}) \) is closed with respect to Minkowski addition and has the structure of an open polyhedral cone of dimension \( \leq d \).

It turns out that there is a simple embedding of \( \mathcal{I}_+(\mathcal{N}) \) into \( \mathbb{R}^d \). For an inscribed polytope \( P \), let \( c(P) \in \text{aff}(P) \) be the center of the inscribing sphere relative to the affine hull of \( P \).

**Theorem 2.2** ([18, Corollary 2.8]). Let \( \mathcal{N} \) be an inscribable fan and let \( R_0 \in \mathcal{N} \) be a region. The map \( \nu_{R_0} : \mathcal{I}_+(\mathcal{N}) \to \text{int} R_0 \) with \( \nu_{R_0}(P) := v \), where

\[ \{ v \} = V(P - c(P)) \cap \text{int} R_0 , \]
is linear and injective. The image of \( v_{R_0} \) is called the based inscribed cone \( \mathcal{L}_+ (\mathcal{N}, R_0) \) of \( \mathcal{N} \).

In this paper, we focus on the class of fans induced by linear hyperplane arrangements. A linear hyperplane is of the form \( H = z^\perp = \{ x : \langle z, x \rangle = 0 \} \) for some \( z \in \mathbb{R}^d \setminus \{ 0 \} \). An arrangement of linear hyperplanes is a collection \( \mathcal{A} = \{ H_1, \ldots, H_n \} \) where \( H_i = z_i^\perp \) for \( i = 1, \ldots, n \). For a generic \( c \in \mathbb{R}^d \), let \( \sigma = (\sigma_1, \ldots, \sigma_n) \in \{-1, +1\}^n \) with \( \sigma_i = \text{sgn}(\langle z_i, c \rangle) \). Then \( c \) is contained in the interior of the cone
\[
R_{\sigma} := \{ x : \sigma_i \langle z_i, x \rangle \geq 0 \text{ for all } i = 1, \ldots, n \}.
\]
The collection of such cones \( R_{\sigma} \) defines a fan \( \mathcal{N}(\mathcal{A}) \) induced by \( \mathcal{A} \). It is the closure of connected components of \( \mathbb{R}^d \setminus \bigcup \mathcal{A} \). Since \( \mathcal{A} \) is uniquely determined by \( \mathcal{N}(\mathcal{A}) \), we do not distinguish between \( \mathcal{A} \) and its fan. The lineality space of an arrangement \( \mathcal{A} \) is \( \text{lineal}(\mathcal{A}) := \bigcap_{H \in \mathcal{A}} H \) and we call \( \mathcal{A} \) essential, if \( \text{lineal}(\mathcal{A}) = \{ 0 \} \). The rank of \( \mathcal{A} \) is \( \text{rk}(\mathcal{A}) := d - \dim(\text{lineal}(\mathcal{A})) \). The essentialization \( \text{ess}(\mathcal{A}) := \{ H \cap \text{lineal}(\mathcal{A})^\perp : H \in \mathcal{A} \} \) is a hyperplane arrangement in \( \text{lineal}(\mathcal{A})^\perp \) and, by passing to the essentialization, we often will assume that \( \mathcal{A} \) is essential.

Every fan induced by a hyperplane arrangement is polytopal. The zonotope associated to \( z_1, \ldots, z_n \) is the polytope
\[
Z = [-z_1, z_1] + [-z_2, z_2] + \cdots + [-z_n, z_n]
\]
and it is straightforward to verify that \( R_{\sigma} \) is the normal cone of the vertex \( \sum_i \sigma_i z_i \) of \( Z \).

A polytope \( P \) with \( \mathcal{N}(P) = \mathcal{A} \) is called a belt polytope [1]. The name derives from the following fact. For a set \( S \subset \mathbb{R}^d \) denote by \( \text{aff}(S) \) its affine hull and by \( \text{aff}_0(S) := \text{aff}(S) - S \) the linear space parallel to it. Two faces \( F, F' \) of a belt polytope \( P \) are normally equivalent if and only if \( \text{aff}_0(F) = \text{aff}_0(F') \). The collections of faces that are normally equivalent are the belts of \( P \). Moreover, if \( L = \text{aff}_0(F)^\perp \), then the faces normally equivalent to \( F \) are in bijection to the regions of the restriction
\[
\mathcal{A}_L := \{ H \cap L : H \in \mathcal{A}, L \not\subset H \},
\]
which is a hyperplane arrangement in \( L \). If we denote the localization of \( \mathcal{A} \) at \( L \) by
\[
\mathcal{A}^L := \{ H \in \mathcal{A} : L \subset H \}
\]
then one checks that \( F \) is a belt polytope with respect to \( \mathcal{A}^L \). Such a subspace \( L \), which is an intersection of hyperplanes in \( \mathcal{A} \), is called a flat. The lattice of flats \( \mathcal{L}(\mathcal{A}) \) is the collection of flats of \( \mathcal{A} \) partially ordered by reverse inclusion. This is a (graded) lattice with minimal element \( \mathbb{R}^d \) and maximal element \( \text{lineal}(\mathcal{A}) \). We denote by \( \mathcal{L}_k(\mathcal{A}) \) the subset of \( k \)-dimensional flats (or \( k \)-flats, for short) and by \( \mathcal{F}_L(P) \subset \mathcal{F}(P) \) the collection of faces in the belt determined to \( L \).

The upcoming local characterization of belt polytopes and zonotopes is essentially due to Bolker [1, Thm. 3.3].
Theorem 2.3. A polytope $P$ is a belt polytope if and only if for every 2-dimensional face $F \subseteq P$ the following holds: $F$ has an even number of vertices and combinatorially antipodal edges are parallel. A polytope $P$ is a zonotope if and only if all 2-dimensional faces are centrally-symmetric.

We give a local characterization of inscribed belt polytopes and zonotopes akin to the above in Theorem 4.13.

An arrangement $\mathcal{A}$ is inscribable if $\mathcal{I}_+(\mathcal{A}) := \mathcal{I}_+(N(\mathcal{A})) \neq \emptyset$, that is, if there is an inscribed belt polytope $P$ with $N(P) = \mathcal{A}$. If there is an inscribed zonotope $Z$ with $N(Z) = \mathcal{A}$, then $\mathcal{A}$ is strongly inscribable. Since Minkowski sums of zonotopes are zonotopes, this prompts the definition of the strongly inscribed cone of $\mathcal{A}$ as the subcone of the inscribed cone

$$Z_+(\mathcal{A}) := \{Z \in \mathcal{I}_+(\mathcal{A}) : Z \text{ zonotope} \} \subseteq \mathcal{I}_+(\mathcal{A}).$$

A polytopal fan $\mathcal{N} = N(P)$ is even if every 2-face of $P$ has an even number of vertices or, equivalently, if the link of every codimension-2 cone of $\mathcal{N}$ is an even cycle. Theorem 2.3 yields that every arrangement is even and together with Theorem 4.13 in [18] we get:

Corollary 2.4. Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{R}^d$. If $\mathcal{A}$ is inscribable, then $\dim \mathcal{I}_+(\mathcal{A}) = d$.

2.1. Reflection arrangements. An important class of inscribable arrangements comes from reflection arrangements. For $\alpha \in \mathbb{R}^d \setminus \{0\}$, let $s_\alpha : \mathbb{R}^d \to \mathbb{R}^d$ be the reflection in $\alpha^\perp$, that is, the orthogonal transformation that fixes $\alpha^\perp$ pointwise and that satisfies $s_\alpha(\alpha) = -\alpha$.

A finite reflection group $W$ is a finite subgroup of $O(\mathbb{R}^d)$ that is generated by reflections. The associated reflection arrangement is $\mathcal{A}_W = \{\alpha^\perp : s_\alpha \in W\}$.

For $q \in \mathbb{R}^d$, the $W$-permutahedron is the polytope $P_W(q) := \text{conv}(W \cdot q)$. It follows for example from Theorem 1.12 in [17] that whenever $q$ is not contained in a reflection hyperplane, then $N(P_W(q)) = \mathcal{A}_W$. Since $W$ acts by orthogonal transformations, $P_W(q)$ is inscribed which proves the following:

Proposition 2.5. Every reflection arrangement $\mathcal{A}_W$ is inscribed and all $P \in \mathcal{I}_+ (\mathcal{A}_W)$ are $W$-permutahedra.

In particular, by Corollary 2.4, $\mathcal{I}_+ (\mathcal{A}_W)$ is full-dimensional. The fans induced by reflection arrangements play a distinguished role. A fan $\mathcal{N}$ is called full if the linear map $v_{R_0} : \mathcal{I}_+(\mathcal{N}) \to \text{int} \ R_0$ given in Theorem 2.2 is an isomorphism for all $R_0$.

Theorem 2.6. Let $\mathcal{N}$ be a fan. Then $\mathcal{N}$ is full if and only if $\mathcal{N} = N(\mathcal{A}_W)$ for some reflection arrangement $\mathcal{A}_W$.

Proof. Let $\mathcal{N}$ be a full and inscribable fan. By Theorem 4.13 in [18] $\mathcal{N}$ is even. Let $C \in \mathcal{N}$ be a cone of codimension-2. It follows from Corollary 4.16 of [18] that $\mathcal{N}_C$ is also full. In
fact, \( v_{R_0} : \mathcal{I}_+(\mathcal{N}_C) \to \text{int} \ R_0 \) is an isomorphism for every cone \( R_0 \in \mathcal{N}_C \). But this means that \( \mathcal{N}_C \) is a 2-dimensional complete fan with all regions isometric to each other. Since the number of regions is even, this means that \( \mathcal{N}_C = \mathcal{N}(\mathcal{A}') \) for some line arrangement \( \mathcal{A}' \). Via Theorem 2.3 this implies that \( \mathcal{N} = \mathcal{N}(\mathcal{A}) \) for some hyperplane arrangement. Lemma 2.5 in [11] states that if the restriction of \( \mathcal{A} \) to any flat of codimension 2 is a reflection arrangement, then \( \mathcal{A} \) is a reflection arrangement. This finishes the proof. \( \Box \)

To see that reflection arrangements are strongly inscribed, recall that a root system is a non-empty, finite collection \( \Phi \subset \mathbb{R}^d \setminus \{0\} \) such that for all \( \alpha \in \Phi \)

(R1) \( \Phi \cap \mathbb{R} \alpha = \{-\alpha, \alpha\} \) and

(R2) \( s_\alpha(\Phi) = \Phi \).

The group generated by the reflections in the hyperplanes \( \{\alpha^\perp : \alpha \in \Phi\} \) is a finite reflection group and, conversely, every reflection group has a root system. The Coxeter zonotope associated to \( \Phi \)

\[ Z_\Phi := \sum_{\alpha \in \Phi} [-\alpha, \alpha] \]

has normal fan \( \mathcal{A}_W \) and \( w \cdot Z_\Phi = Z_\Phi \) every \( w \in W \). In fact \( W \) acts transitively on the regions of \( \mathcal{A}_W \) [17, Sect. 1.15] and thus \( W \) acts transitively on the vertices of \( Z_\Phi \). This implies that all vertices lie on a common sphere and proves

**Proposition 2.7.** \( \mathcal{A}_W \) is strongly inscribed for all reflection groups \( W \).

The converse is also true:

**Proposition 2.8.** If \( Z \) is an inscribed zonotope with \( \mathcal{N}(Z) = \mathcal{A}_W \), then \( Z \) is a translate of \( Z_\Phi \) for some root system of \( W \). In particular \( \dim Z_+(\mathcal{A}_W) \) is equal to the number of orbits of \( \Phi \) under \( W \).

**Proof.** We may assume that \( Z = \sum_{i=1}^n [-z_i, z_i] \). If we require the vectors \( z_i \) to be pairwise linearly independent, then the Minkowski sum decomposition is unique up to relabelling. Since \( Z \in \mathcal{I}_+(\mathcal{A}) \), \( Z \) is an \( \mathcal{W} \)-permutahedra by Proposition 2.5 and hence \( wZ = \sum_i [-wz_i, wz_i] = Z \) for all \( w \in W \). This shows that \( \Phi = \{\pm z_1, \ldots, \pm z_n\} \) is a root system for \( W \). For the second claim, we note that root systems of \( W \) are unique up to scaling, where (R2) requires that roots in the same orbit have the same length. \( \Box \)

Finite reflection groups have been classified [17, Section 2.7]: there are four infinite families of irreducible reflection groups \( A_n, B_n, D_n, I_2(k) \), where the subscript denotes rank. In addition there are six sporadic examples \( F_4, E_6, E_7, E_8, H_3 \) and \( H_4 \). The following table gives the dimensions of the strong inscribed cones:

| \( A_n \) | \( B_n \) | \( D_n \) | \( F_4 \) | \( E_6 \) | \( E_7 \) | \( E_8 \) | \( H_3 \) | \( H_4 \) | \( I_2(k) \) |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 1 | 2 | 1 | 1 | 1 | 1 | \( k \mod 2 \) |

We come back to these examples in Sections 6.2 to 6.3.
2.2. Inscribed virtual Zonotopes. In [18], we observed that questions of inscribability naturally extend beyond polytopes to virtual polytopes. In this paper we will explore inscribed virtual zonotopes and belt polytopes, so we give an executive summary of the theory tailored to our needs.

The collection $\tilde{T}_+$ of polytopes $P$ with fixed normal fan $\mathcal{N}$ forms a convex cone with respect to Minkowski sums. The support function of $P$ is the function $h_P: \mathbb{R}^d \to \mathbb{R}$ with

$$h_P(c) = \max\{\langle c, v \rangle : v \in V(P)\}.$$  

This is a strictly-convex piecewise-linear function supported on $\mathcal{N}$. Since $h_P$ uniquely determines $P$ and $h_{P+Q} = h_P + h_Q$, we get a faithful representation of $\tilde{T}_+$ as the (open) polyhedral cone of strictly-convex piecewise-linear functions supported on $\mathcal{N}$. The formal Minkowski difference of $P$ and $Q$ is the PL function $h_P - h_Q$. Conversely, every PL function is the difference of two support functions and hence $\text{PL}(\mathcal{N}) = \tilde{T}_+ + (-\tilde{T}_+)$. Convex functions in $\text{PL}(\mathcal{N})$ correspond to weak Minkowski summands of $P$. Non-convex functions $\ell \in \text{PL}(\mathcal{N})$ are called virtual polytopes and denoted by $P$, whenever $\ell = h_P - h_Q$. The lineality space of $\tilde{T}_+$ is given by translations. The type cone of $\mathcal{N}$ is $\mathcal{T}_+(\mathcal{N}) = \tilde{T}_+(\mathcal{N})/T_d$, where $T_d$ is the group of translations. Consequently, the type space $\mathcal{T}(\mathcal{N}) = \mathcal{T}_+(\mathcal{N}) + (-\mathcal{T}_+(\mathcal{N}))$ is isomorphic to $\text{PL}(\mathcal{N})/(\mathbb{R}^d)^*$. If $P - Q \in \mathcal{T}(\mathcal{N})$ and $c \in \mathbb{R}^d \setminus \{0\}$, then we say that $P^c - Q^c$ is a face of $P - Q$.

If $\mathcal{A} = \{H_i = z_i^+ \cdot z^+ : i = 1, \ldots, n\}$ is an arrangement of hyperplanes, then up to translation every zonotope of $\mathcal{A}$ is of the form $Z_\lambda = \lambda_1 [-z_1, z_1] + \cdots + \lambda_n [-z_n, z_n]$ for a unique $\lambda \in \mathbb{R}^n$. Thus, the subcone of $\mathcal{T}_+(\mathcal{A})$ spanned by zonotopes is isomorphic to $\mathbb{R}^n_{>0}$. If $Z_{\lambda^+}, Z_{\lambda^-}$ are two zonotopes with $\lambda^+, \lambda^- \in \mathbb{R}^n_{>0}$, then

$$h_{Z_{\lambda^+}}(c) - h_{Z_{\lambda^-}}(c) = \sum_{i=1}^n \lambda_i |\langle z_i, c \rangle|,$$

where $\lambda = \lambda^+ - \lambda^- \in \mathbb{R}^n$. Every virtual zonotope $Z_\lambda = Z_{\lambda^+} - Z_{\lambda^-}$ is, up to translation, uniquely represented by $\lambda \in \mathbb{R}^n$.

For a PL function $\ell \in \text{PL}(\mathcal{N})$ and a region $R \in \mathcal{N}$, there is $v_R \in \mathbb{R}^d$ such that $\ell(x) = \langle v_R, x \rangle$ for all $x \in R$. The collection $V(\ell) := \{v_R : R \in \mathcal{N} \text{ region}\}$ is the vertex set of $\ell$. We call $\ell$ inscribed if $V(\ell)$ is contained in a sphere and we define $c(\ell)$ to be the center of this sphere (if $V(\ell)$ is contained in an affine subspace of $\mathbb{R}^d$ the center is contained in this subspace). If $\ell = h_P$, then $V(\ell) = V(P)$ and $P$ is inscribed precisely when $h_P$ is. The inscribed cone $\mathcal{I}_+(\mathcal{N})$ is naturally a relatively open subcone of $\mathcal{T}_+(\mathcal{N})$.

If a polytopal fan $\mathcal{N}$ is inscribable, we call $\mathcal{I}(\mathcal{N}) := \mathcal{I}_+(\mathcal{N}) + (-\mathcal{I}_+(\mathcal{N})) \subseteq \mathcal{T}(\mathcal{N})$ the inscribed space of $\mathcal{N}$. The precise definition of $\mathcal{I}(\mathcal{N})$ for $\mathcal{N}$ not inscribable is a bit subtle, we refer to [18, Section 5] for details. Indeed, not every inscribed PL function $\ell \in \mathcal{T}(\mathcal{N})$ is contained in $\mathcal{I}(\mathcal{N})$. For example, if $\mathcal{N} = \mathcal{N}(\mathcal{A})$ with $z_i^+, z_i^- \in \mathcal{A}$, then $S_i = [-z_i, z_i] \in \mathcal{T}(\mathcal{N})$ is inscribed for $i = 1, 2$ but $S_i \notin \mathcal{I}(\mathcal{N})$ if and only if $z_1 \perp z_2$. The following Lemma follows easily from the definition of the inscribed space in [18, Section 5], but we might also take
it as a definition for our purposes here. For any two regions \( R, S \in \mathcal{N} \) such that \( R \cap S \) is of codimension 1, let \( \alpha_{RS} \) be a normal vector to the hyperplane \( \text{lin}(R \cap S) \). We have:

**Lemma 2.9.** Let \( P - Q \in \mathcal{T}(\mathcal{N}) \) and let \( V(P - Q) = \{v_R : R \in \mathcal{N} \text{ region}\} \). Then \( P - Q \in \mathcal{I}(\mathcal{N}) \) if and only if \( P - Q \) is inscribed and

\[
\langle \alpha_{RS}, c([v_R, v_S]) - c(P - Q) \rangle = 0
\]

for all adjacent regions \( R, S \in \mathcal{N} \).

If \( \mathcal{N} = \mathcal{N}(\mathcal{A}) \), then \( \alpha_{RS} = z_i \) for some \( i \) for any two adjacent regions \( R, S \). We will exploit Lemma 2.9 in Section 4.3 to give a simple representation of \( \mathcal{I}_+(\mathcal{A}) \) and \( \mathcal{I}(\mathcal{A}) \).

---

**Figure 1**

**Example 2.10.** Figure 1 shows an inscribed virtual zonotope \( Z_\lambda \) presented as the Minkowski difference of two (inscribed) non-virtual zonotopes. The vertex of \( Z_\lambda \) corresponding to the green region \( R \) is \( v_R = (\frac{5}{2}) \). Thus, the corresponding PL function \( \ell(c) := h_{Z_{\lambda^+}}(c) - h_{Z_{\lambda^-}}(c) \) satisfies \( \ell_R(c) = \langle v_R, c \rangle \) for all \( c \in R \).

---

### 3. Strongly inscribable fans in dimension 2

Throughout the section, let \( \mathcal{A} \) be an arrangement of \( n \geq 2 \) lines in \( \mathbb{R}^2 \). Belt polytopes and zonotopes are easily described in the 2-dimensional setting: a **belt polygon** is a polygon with an even number of edges with opposite edges parallel and a **zonogon** is a centrally symmetric polygon, i.e. a belt polygon, whose opposites edges are not only parallel but equal in length. In the following two subsections, we give geometric and algebraic characterizations of (strongly) inscribed line arrangements. These characterizations are indispensable for subsequent results in higher dimensions. In particular, we show that the notions of **inscribed** and **strongly inscribed** coincide in the plane.

**Theorem 3.1.** A rank-2 arrangement \( \mathcal{A} \) is (virtually) strongly inscribed if and only it is (virtually) inscribed.
If $\mathcal{A}$ has an inscribed belt polytope $P$, then $-P$ is also inscribed and $\mathcal{N}(-P) = \mathcal{N}(P) = \mathcal{A}$. We simply observe that $P + (-P)$ is a centrally-symmetric polygon and hence inscribed by Theorem 2.1. This shows that inscribed arrangements are also strongly inscribed. The same argument does not work for inscribed virtual polytopes: For example, if $\mathcal{A}$ is the reflection arrangement of type $A_2$, then $P = P_W(1, 2, 4)$ is an inscribed hexagon, which is not centrally symmetric. The virtual polytope $P - (-P)$ given by $\ell = h_P - h_{-P}$ is inscribed by Lemma 2.9 but its symmetrization satisfies

$$\ell(c) + \ell(-c) = h_P(c) - h_{-P}(c) + h_P(-c) - h_{-P}(-c) = 0$$

and thus is the zero function. We proof Theorem 3.1 in full generality in Section 3.1.

### 3.1 A geometric perspective

We order the $2n$ regions of $\mathcal{A}$ counterclockwise and denote them by $R_0, \ldots, R_{2n-1}$ as in Figure 2. For $i = 0, \ldots, 2n - 1$, let $\beta_i \in (0, \pi)$ be the angle of $R_i$. Note that by central symmetry, $\beta_i = \beta_{n+i}$ for $i = 0, \ldots, n - 1$ and thus $\beta_0 + \beta_1 + \cdots + \beta_{n-1} = \pi$. We call $\beta(\mathcal{A}) = (\beta_0, \ldots, \beta_{n-1}, \beta_0, \ldots, \beta_{n-1})$ the profile of $\mathcal{A}$ and $\bar{\beta}(\mathcal{A}) = (\beta_0, \ldots, \beta_{n-1})$ the reduced profile. The reduced profile $\bar{\beta}(\mathcal{A})$ determines $\mathcal{A}$ up to rotation. More generally, the profile $\beta(\mathcal{N})$ of a 2-dimensional fan $\mathcal{N}$ is given by the angles of its regions ordered counterclockwise [18, Section 3]. Clearly $\beta(\mathcal{A}) = \beta(\mathcal{N}(\mathcal{A}))$.

![Figure 2. Left: A 2-dimensional arrangement $\mathcal{A}$ with regions and its reduced profile $\bar{\beta} = (\beta_0, \beta_1, \beta_2)$. Right: An inscribed zonogon with normal fan $\mathcal{A}$, face angles $\alpha_0, \alpha_1, \ldots, \alpha_5$, and interior angles $\gamma_0, \gamma_1, \ldots, \gamma_5$.](image)

To give a complete geometric picture, let us introduce two additional sets of angles. Let $P$ be a virtual belt polygon with normal fan $\mathcal{A}$ and assume that $P$ is inscribed into a circle with center at the origin $0$. We label the vertices $v_0, v_1, \ldots, v_{2n-1}$ such that $v_i$ has normal cone $R_i$ for $i = 0, \ldots, 2n - 1$. For convenience, set $v_{-1} := v_{2n-1}$ and $v_{2n} := v_0$. The face angle $\alpha_i, i = 0, \ldots, 2n - 1$, is the oriented angle $\angle(v_{i-1}, 0, v_i) \in (-\pi, \pi]$. Furthermore, we define the interior angle $\gamma_i, i = 0, \ldots, 2n - 1$, to be the angle $\angle(v_{i+1}, v_i, v_{i-1})$ (cf. Figure 2). These angles are related to $\beta(\mathcal{A})$ via the following equations for all $i = 0, \ldots, n - 1$:

$$\frac{1}{2}(\alpha_i + \alpha_{i+1}) \equiv \beta_i \equiv \pi - \gamma_i.$$
Moreover, we have $\gamma_i = \gamma_{i+n}$. If $P$ is a zonotope, then $\alpha_i = \alpha_{i+n}$ by central symmetry.

**Proof of Theorem 3.1.** We only need to show that if $\mathcal{A}$ has an inscribed virtual belt polygon $P$, then is also has an inscribed virtual zonogon.

Let $\alpha_1, \ldots, \alpha_{2n}$ be the face angles of $P$. Let $Z$ be the inscribed virtual zonogon with face angles $\alpha'_i = \alpha'_{n+i} = \frac{1}{2}(\alpha_i + \alpha_{n+i})$, for $i = 1, \ldots, n$. We observe that

$$\frac{1}{2}(\alpha'_i + \alpha'_{i+1}) = \frac{1}{2}(\alpha_i + \alpha_{n+i} + \alpha_{i+1} + \alpha_{n+i+1}) = \frac{1}{2}(\beta_i + \beta_{n+i}) = \beta_i,$$

and hence $\tilde{\beta}(\mathcal{N}(Z)) = \tilde{\beta}(\mathcal{A})$. Thus, up to rotation, $Z$ is an inscribed zonogon with normal fan $\mathcal{A}$. □

**Theorem 3.2.** $\beta$ is the reduced profile of an inscribed arrangement if and only if $2 \beta$ is the profile of an inscribed fan.

**Proof.** Let $\beta = \tilde{\beta}(\mathcal{A})$ be the reduced profile of an inscribed arrangement $\mathcal{A}$. By Theorem 3.1, there exist an inscribed zonotope $Z \in Z_+(\mathcal{A})$. Let $\alpha_1, \ldots, \alpha_{2n}$ be the face angles of $Z$. These are positive real numbers with $\alpha_i = \alpha_{n+i}$, and therefore $\alpha_1 + \cdots + \alpha_n = \pi$.

Let $w_1, \ldots, w_n$ be $n$ points on the unit sphere such that $\angle(w_{i-1}, w_i) = 2 \alpha_i$ for $i = 2, \ldots, n$. Then $Q := \text{conv}(w_1, \ldots, w_n)$ is an inscribed polygon with face angles $\alpha'_i := 2 \alpha_i$, $1 \leq i \leq n$. It is not hard to see that for $\beta' := \beta(\mathcal{N}(Q))$:

$$\beta'_i = \frac{1}{2}(\alpha'_i + \alpha'_{i+1}) = \alpha_i + \alpha_{i+1} = 2 \beta_i.$$

Starting with $Q$ and reversing the process completes the proof. □

We conclude the following from [18, Theorem 3.5]:

**Proposition 3.3.** Let $\mathcal{A}$ be a 2-dimensional line arrangement with $n \geq 2$ lines, and reduced profile $(\beta_0, \beta_1, \ldots, \beta_{n-1})$. If $n$ is odd, then $\mathcal{A}$ is (strongly) inscribable, if and only if for all $0 \leq j < n$:

$$\beta_{j+0} - \beta_{j+1} + \cdots - \beta_{j+n-2} + \beta_{j+n-1} > 0.$$

If $n = 2m$ is even, then $\mathcal{A}$ is (strongly) inscribable, if and only if for all $0 \leq h < m$ and $0 \leq j < n$

$$\beta_0 + \beta_2 + \cdots + \beta_{n-2} = \frac{\pi}{2}, \quad \sum_{i=1}^{h} \beta_{2i+j} + \sum_{i=h+1}^{m-1} \beta_{2i+1+j} < \frac{\pi}{2}.$$

**Example 3.4.** Let us explicitly state these equations and inequalities for $n = 2, 3$ and 4.

For $n = 2$, we have $\beta_1 = \beta_2$ and $\beta_1 + \beta_2 = \pi$, so

$$\beta_1 = \beta_2 = \frac{\pi}{2}.$$

Thus, an inscribed parallelogram is a rectangle.

For $n = 3$, we get $\beta_1 + \beta_2 + \beta_3 = \pi$, so $\pi - \beta_3 = \beta_1 + \beta_2 > \beta_3 > 0$, so by symmetry:

$$\beta_1 + \beta_2 + \beta_3 = \pi, \quad 0 < \beta_1, \beta_2, \beta_3 < \frac{\pi}{2}.$$
Finally, for \( n = 4 \), the inequalities can be reduced to
\[
\beta_1 + \beta_3 = \beta_2 + \beta_4 = \frac{\pi}{2}, \quad 0 < \beta_1, \beta_2, \beta_3, \beta_4.
\]
For \( n \geq 5 \), the inequalities are more involved.

We close this section with a important corollary, which will help us to show in Theorem 1.4 that inscribed belt polytopes are simple:

**Corollary 3.5.** Let \( A \) be a 2-dimensional inscribed line arrangement with \( n \geq 2 \) lines and reduced profile \((\beta_0, \ldots, \beta_{n-1})\). Then \( \beta_i \leq \frac{\pi}{2} \leq \gamma_i \) for \( i = 0, \ldots, n - 1 \).

**Proof.** If \( n = 2 \), then by Example 3.4 \( \beta_0 = \beta_1 = \frac{\pi}{2} = \gamma_0 = \gamma_1 \), so any inscribed realization is a rectangle. Otherwise:
\[
\pi = \beta_0 + \beta_1 + \cdots + \beta_n > \beta_{i-1} + \beta_i + \beta_{i+1}
\]
\[
> \frac{1}{2}\alpha_{i-1} + \alpha_i + \alpha_{i+1} + \frac{1}{2}\alpha_{i+2} > \alpha_i + \alpha_{i+1} = 2\beta_i.
\]
\( \square \)

### 3.2. An algebraic perspective

While the description of inscribable rank-2 arrangements in the previous subsection in terms of the reduced profile is very simple, it does not give a description of the strongly inscribed cone.

Let \( A \) be an arrangement of \( n \geq 2 \) lines in \( \mathbb{R}^2 \). The lines are given by \( z_1^+, \ldots, z_n^+ \) for some non-zero vectors \( z_1, \ldots, z_n \). We assume \( z_1, \ldots, z_n, -z_1, \ldots, -z_n \) to be cyclically ordered. The **skew-Gram matrix** of an ordered collection of vectors \( z_1, \ldots, z_n \) is the skew-symmetric matrix \( R = R(z_1, \ldots, z_n) \in \mathbb{R}^{n \times n} \) with \( R_{ij} = -R_{ji} = \langle z_i, z_j \rangle \) for \( i < j \) and \( R_{ii} = 0 \).

**Theorem 3.6.** Let \( A = \{z_1^+, \ldots, z_n^+\} \) as above and let \( R = R(z_1, \ldots, z_n) \). As vector spaces, \( Z(A) \cong \ker R \). As cones, \( Z_+(A) \cong \ker R \cap \mathbb{R}^n_\geq 0 \).

These isomorphisms can be described as follows: For \( \lambda \in \ker R \cap \mathbb{R}^n_\geq 0 \) the corresponding zonogon is
\[
Z_\lambda := \sum_{i=1}^n \lambda_i [-z_i, z_i] \in Z_+(A).
\]

For \( \lambda \in \ker R \), let \( \lambda^+, \lambda^- \in \mathbb{R}^n_\geq 0 \) such that \( \lambda = \lambda^+ - \lambda^- \). The virtual zonogon associated to \( \lambda \) is then \( Z_\lambda := Z_{\lambda^+} - Z_{\lambda^-} \in Z(A) \).

**Proof of Theorem 3.6.** Let \( Z \in Z(A) \) with \( c(Z) = 0 \), i.e., that \( Z = Z_\lambda \) for some \( \lambda \in \mathbb{R}^n \). The vertices of the (virtual) zonogon \( Z_\lambda \) are
\[
v_i = \lambda_1 z_1 + \cdots + \lambda_{i-1} z_{i-1} - \lambda_i z_i - \cdots - \lambda_n z_n
\]
for \( i = 1, \ldots, n \) together with their reflections \(-v_1, \ldots, -v_n\). Thus, the midpoint of the edge \( e_i := [v_i, v_{i+1}] = v_i + [0, 2\lambda_i z_i] \) is
\[
c(e_i) = v_i + \lambda_i z_i = \lambda_1 z_1 + \cdots + \lambda_{i-1} z_{i-1} - \lambda_{i+1} z_{i+1} - \cdots - \lambda_n z_n.
\]
Lemma 2.9 gives \( \langle z_i, c(e_i) \rangle = 0 \) for all \( i = 1, \ldots, n \) which is equivalent to \( R\lambda = 0 \). \( \square \)
Recall that a skew-symmetric matrix of odd order is singular. If $R$ is a skew-symmetric matrix of order $n = 2m$, then $\det R = (\text{pf } R)^2$, where $\text{pf } R$ denotes the Pfaffian of $R$.

**Corollary 3.7.** Let $A$ be an arrangement of $n$ lines in $\mathbb{R}^2$. If $n$ is odd, then $A$ is virtually inscribable. Otherwise, $A$ is virtually inscribable if and only if $\text{pf } R(A) = 0$.

As a refinement of Corollary 3.7, we have the following:

**Corollary 3.8.** Let $A$ be an arrangement of $n$ lines in $\mathbb{R}^2$. If $n$ is even, $\mathcal{I}(A) = \mathcal{Z}(A)$. If $n$ is odd, then $\dim \mathcal{Z}(A) = 1$.

*Proof.* Let $n$ be even. Since $\mathcal{Z}(A) \subseteq \mathcal{I}(A)$, we are left to show that $\dim \mathcal{Z}(A) = \dim \mathcal{I}(A)$. The statement is trivial if $\dim \mathcal{I}(A) = 0$, so assume $\dim \mathcal{I}(A) > 0$. The rank of every skew-symmetric matrix is even and since $\text{rk } R(A) \neq 0$, we get $\dim \ker R = n - \text{rk } R \geq 2$. By [18, Proposition 3.1], we see that $2 = \dim \mathcal{I}(A) \geq \dim \mathcal{Z}(A) = \dim \ker R$ and therefore $\dim \mathcal{Z}(A) = \dim \mathcal{I}(A)$.

If $n$ is odd, then $\dim \ker R = n - \text{rk } R \geq 1$ is odd, but since $\dim \mathcal{Z}(A) \leq \dim \mathcal{I}(A) = 2$ and $\ker R \cong \mathcal{Z}(A)$, we have in fact $\dim \mathcal{Z}(A) = 1$. $\square$

**Example 3.9.** We will list the immediate consequences of the last results for $n = 2, 3, 4$ to compare with Example 3.4. For $n = 2, 3, 4$, let $A_n = \{z_1^1, \ldots, z_n^1\}$ be an arrangement of $n$ lines $z_1^1, \ldots, z_n^1$ in $\mathbb{R}^2$. Assume that $z_1, \ldots, z_n, -z_1, \ldots, -z_n$ is cyclically ordered and write $r_{ij} = \langle z_i, z_j \rangle$ for $i, j = 1, \ldots, n$. The skew-Gram matrices $R_n = R(A_n)$ are then:

$$R_2 = \begin{pmatrix} 0 & r_{12} \\ -r_{12} & 0 \end{pmatrix} \quad R_3 = \begin{pmatrix} 0 & r_{12} & r_{13} \\ -r_{12} & 0 & r_{23} \\ -r_{13} & -r_{23} & 0 \end{pmatrix} \quad R_4 = \begin{pmatrix} 0 & r_{12} & r_{13} & r_{14} \\ -r_{12} & 0 & r_{23} & r_{24} \\ -r_{13} & -r_{23} & 0 & r_{34} \\ -r_{14} & -r_{24} & -r_{34} & 0 \end{pmatrix}.$$

For $n = 2$, we have $\text{pf } R_2 = r_{12}$, so $A_2$ is inscribed if and only if $r_{12} = 0$. The kernel of $R_2$ is simply $\mathbb{R}^2$. Geometrically, this is clear: If $r_{12} = 0$, then $A_2$ consists of two orthogonal lines. All (virtual) polygons with this normal fan are rectangles and therefore inscribed. The space of such rectangles is parameterized by the two side lengths.

For $n = 3$, $\text{pf } R_3 = 0$, since $n$ is odd, so $A_3$ is always virtually inscribed. The kernel of $R_3$ is $(r_{23}, -r_{13}, r_{12})$ and therefore 1-dimensional. Finally, $A_3$ is strongly inscribed, if and only if $r_{12}, -r_{13}, r_{23} > 0$. In relation to reduced profiles, $r_{13} < 0$ is equivalent to $\beta_3 < \frac{\pi}{2}$, while $r_{23} > 0$ if and only if $\beta_1 < \frac{\pi}{2}$ and $\beta_3 > 0$ if and only if $\beta_2 < \frac{\pi}{2}$. These are precisely the inequalities in Example 3.4 (that $\beta_i > 0$ follows from the cyclic ordering).

For $n = 4$, $\text{pf } R_4 = r_{12}r_{34} - r_{13}r_{24} + r_{14}r_{23}$. Plugging

$$\cos \beta_1 = r_{12}, \quad \cos \beta_2 = r_{23}, \quad \cos \beta_3 = r_{34}, \quad \cos(\beta_1 + \beta_3) = r_{13}, \quad \cos(\beta_2 + \beta_3) = r_{24}, \quad \cos(\beta_1 + \beta_2 + \beta_3) = r_{14}$$

into $\text{pf } R_4$, a quick struggle with trigonometric identities reveals $\text{pf } R_4 = \cos(\beta_1 + \beta_3)$. This is in line with the angle conditions of Example 3.4.
4. Operations and strongly inscribed cones

We show that the class of inscribed arrangements is closed under the basic operations of taking products and localizations. While this is clear from the geometric point of view, we prove that strong inscribability is retained under restrictions as well. Said differently, the orthogonal projection of an inscribed zonotope onto any of its flats is inscribed. Up to an additional assumption, this characterizes inscribed zonotopes.

In Section 4.3 we derive a simple representation of the strongly inscribed cone of an arrangement and give a first extension of Bolker’s characterization to inscribed zonotopes.

4.1. Products and reducible arrangements. Let \( P_1 \subset \mathbb{R}^d, P_2 \subset \mathbb{R}^e \) be inscribed polytopes. Then \( P_1 \times P_2 \subset \mathbb{R}^{d+e} \) is also inscribed. If \( P_1, P_2 \) are belt polytopes for \( \mathcal{A}_1, \mathcal{A}_2 \), respectively, then \( P_1 \times P_2 \) is a belt polytope for the arrangement \( \mathcal{A}_1 \times \mathcal{A}_2 := \{ H \times \mathbb{R}^e : H \in \mathcal{A}_1 \} \cup \{ \mathbb{R}^d \times H : H \in \mathcal{A}_2 \} \).

This shows:

**Proposition 4.1.** If \( \mathcal{A}_1, \mathcal{A}_2 \) are (strongly) inscribable, then so is \( \mathcal{A}_1 \times \mathcal{A}_2 \).

An essential hyperplane arrangement \( \mathcal{A} \) in \( \mathbb{R}^d \) is called **reducible**, if there exists a partition \( \mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2 \) into non-empty subarrangements, such that \( \text{lineal}(\mathcal{A}_1) \oplus \text{lineal}(\mathcal{A}_2) = \mathbb{R}^d \). Equivalently, \( \text{ess} \mathcal{A} \) is linearly isomorphic to \( \text{ess} \mathcal{A}_1 \times \text{ess} \mathcal{A}_2 \). If \( \mathcal{A} \) is not reducible, it is called **irreducible**.

Towards a classification of inscribed arrangements, the following proposition yields that we focus on irreducible arrangements:

**Proposition 4.2.** Let \( \mathcal{A} \) be an inscribable arrangement. If \( \mathcal{A} \) is reducible with \( \mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2 \) as above, then \( \text{lineal} \mathcal{A}_1 \perp \text{lineal} \mathcal{A}_2 \).

**Proof.** Choose \( H_1 = z_1^+ \in \mathcal{A}_1 \) and \( H_2 = z_2^+ \in \mathcal{A}_2 \). Then \( L := H_1 \cap H_2 \) is a flat of codimension 2 and no other hyperplane in \( \mathcal{A} \) contains \( L \). The localization \( \mathcal{A}_L \) is inscribed, and thus its essentialization is a 2-dimensional inscribed arrangement with 2 lines. By Example 3.4, we see that \( \beta(\text{ess}(\mathcal{A}_L)) = (\frac{\pi}{2}, \frac{\pi}{2}) \), or in other words, \( z_1 \perp z_2 \). Varying over \( H_1, H_2 \) shows the claim. \( \square \)

**Remark 4.3.** The same argument shows that for two polytopes \( P, Q \subset \mathbb{R}^d \) if \( P + Q \) is inscribed and \( P + Q \) is combinatorially isomorphic to \( P \times Q \), then \( \text{aff}_0(P) \perp \text{aff}_0(Q) \) and \( P \) and \( Q \) are both inscribed. Note that it is not enough to assume that \( P + Q \) is combinatorially isomorphic to an inscribed polytope, as seen for example by irregular inscribed quadrilaterals or more generally irregular inscribed cubes.

4.2. Restrictions and localizations. Let \( P \) be an inscribed polytope. If \( F \subseteq P \) is a face, then \( F \) is clearly inscribed as well. Recall from Section 2 that if \( P \) is a belt polytope with arrangement \( \mathcal{A} \), then \( F \) is a belt polytope with respect to the localization \( \mathcal{A}_L \),
where $L = \text{aff}_0(F)$. This shows that (strongly) inscribed arrangements are closed under localizations:

**Lemma 4.4.** Let $L$ be a flat of a (strongly) inscribed arrangement $A$. Then $A_L$ is (strongly) inscribed.

For a flat $L$, let $\pi_L : \mathbb{R}^d \to L$ be the orthogonal projection onto $L$. The image of a belt polytope $P$ with $A(P) = A$ under $\pi_L$ is a belt polytope with respect to the restriction $A^F$. The vertices of $\pi_L(P)$ are precisely the images $\pi_L(F)$, where $F$ ranges over the faces in the belt corresponding to $L^\perp$.

Projections of inscribed polytopes are rarely inscribed. Inscribed zonotopes however have the fascinating property, that they remain inscribed under orthogonal projection onto their flats:

**Proposition 4.5.** Let $Z$ be an inscribed zonotope and $H \in A = A(Z)$. Then $\pi_H(Z)$ is inscribed with center $c(Z)$. Equivalently, $A^H$ is strongly inscribed.

Theorem 1.2 follows from Proposition 4.5 by repeated application.

**Proof.** We may assume that $c(Z) = 0$. Let $E$ be the belt of edges orthogonal to $H$. If $e = [u, v] \in E$ is an edge, then all points of $H$ have the same Euclidean distance to $u$ and to $v$. In particular, $H$ bisects every edge in $E$. Since $Z$ is a zonotope, any two edges in $E$ are translates of each other and thus have the same length. Therefore, all the triangles $T_e := \text{conv}(e \cup \{0\})$ with $e \in E$ are congruent. The segments $[0, c(e)]$ are the altitudes of $T_e$ and therefore $c(e) = \pi_H(e)$ have the same distance to the origin for all $e \in E$. The claim now follows from $\pi_H(Z) = \text{conv}(c(e) : e \in E)$. □

**Remark 4.6.** Note that Proposition 4.5 is not true for belt polytopes. If $W$ is the reflection group of type $A_3$ acting on $\mathbb{R}^4$ by permuting coordinates, one checks that projections of the $W$-permutahedron $P_W(1, 2, 3, 5)$ along edges are not inscribed.

In addition to zonotopes, products of polygons and simplices also have the property that inscribability is preserved under projection along edges. Thus, we propose the following question:

**Question 4.7.** Which inscribed polytopes remain inscribed when projecting along any of their edge directions?

Unlike products of polygons and simplices, inscribed zonotopes have the property that the center of the inscribing sphere is preserved under projection. It turns out that this characterizes inscribed zonotopes.

**Proof of Theorem 1.1.** We may assume that $c(P) = 0$ and hence $M_e = e^\perp$.

(i) $\implies$ (ii) follows from Proposition 4.5. Since $P$ is a belt polytope, it follows that the vertices of the projection of $P$ along $e$ are precisely the images of edges $e'$ parallel to $e$. The
proof of Proposition 4.5 now shows that the projection and intersection coincide and thus yields (i) \(\implies\) (iii).

(ii) \(\implies\) (i) and (iii) \(\implies\) (i): We will show that every 2-face of \(P\) is centrally symmetric. The result then follows from Theorem 2.3. If \(P\) satisfies (ii) or (iii) respectively, then so does every 2-face of \(P\). Therefore, we can assume that \(P\) is 2-dimensional and the result follows from the next two Lemmas.

\[\square\]

**Lemma 4.8.** Let \(P \subseteq \mathbb{R}^2\) be an inscribed polygon with \(c(P) = 0\). If for every edge \(e\) the orthogonal projection \(\pi_e(P)\) of \(P\) onto \(e^\perp\) satisfies \(c(\pi_e(P)) = 0\), then \(P\) is a zonogon.

**Proof.** If \(P\) is not a zonogon, then there exists a vertex \(v\), such that \(-v \notin P\). Let \(u, u'\) be the neighbors of \(-v\) in \(\text{conv}(P \cup \{-v\})\). Since \(P\) is inscribed, the segment \(e = uu'\) is an edge of \(P\) and the affine line spanned by \(e\) separates \(-v\) from \(P\). Let \(\eta\) be the outer unit normal vector to \(\text{aff}(e)\). The image \(\pi_e(P)\) is a segment \(aa'\), with

\[||a|| = \max\{\langle x, \eta \rangle : x \in P\} = \langle u, \eta \rangle, \quad ||a'|| = \max\{\langle x, -\eta \rangle : x \in P\},\]

and \(||a|| = ||a'||\) by assumption. But then

\[||a'|| \geq \langle v, -\eta \rangle = \langle -v, \eta \rangle > \langle u, \eta \rangle = ||a||,\]

a contradiction. \(\square\)

**Lemma 4.9.** Let \(P \subseteq \mathbb{R}^2\) be an inscribed polygon with \(c(P) = 0\). If \(c(P \cap e^\perp) = 0\) for every edge \(e\), then \(P\) is a zonogon.

**Proof.** Let \(e = [x, y]\) be an edge of \(P\). We first establish the following claims:

(i) *No vertex \(v\) of \(P\) is contained in \(e^\perp\).* Otherwise, \(P \cap e^\perp = [c(e), v]\). Since \(P\) is inscribed, \(||v|| = ||x|| = ||y||\), but as \(c(P \cap e^\perp) = 0\), we have \(||v|| = \|c(e)|| < ||x||\), contradiction.

(ii) *If \(-e\) is an edge of \(P\), then \(e \cap f^\perp\) is empty for all other edges \(f\) of \(P\).* This follows from the observation that if \(e \cap f^\perp \neq \emptyset\), then also \(-e \cap f^\perp \neq \emptyset\), which contradicts \(f \cap f^\perp = \{c(f)\} \neq \emptyset\), since at most two edges of \(P\) intersect any line.

(iii) *Let \(q := c(-e)\). If \(\{q\} = e^\perp \cap f\) for an edge \(f = [u, v]\), we have \(||x - y|| \leq ||u - v||\) with equality if and only if \(f = -e\).* By the Chord Theorem from plane geometry, we have

\[||u - q|| \cdot ||q - v|| = ||(-x) - q|| \cdot ||q - (-y)|| = \frac{1}{2}||x - y||^2.\]

The inequality of arithmetic and geometric means now yields

\[||u - q|| \cdot ||q - v|| \leq \left(\frac{1}{2}(||u - q|| + ||q - v||)\right)^2 = \frac{1}{4}(||u - v||)^2,\]

with equality if and only if \(||u - q|| = ||q - v||\), i.e. if \(f = -e\).

Now, assume that \(e_1\) is an edge of \(P\) such that \(-e_1\) is not an edge of \(P\). Because of (i), \(e_1^\perp\) intersects another edge \(e_2 \neq e_1\) of \(P\). Iterating this, there exists an edge \(e_{n+1} \neq e_n\) intersecting \(e_n^\perp\) for all \(n \geq 1\). By (ii), we will never have \(e_{n+1} = -e_n\), so by (iii) the lengths of the \(e_i\)'s will strictly increase, a contradiction. \(\square\)
Figure 3. An inscribed zonotope for each of the combinatorial type of rank 3 which can be strongly inscribed.

Theorem 1.2 furnishes further examples of inscribed arrangements by restrictions of reflection arrangements. These restrictions are well understood, see [19, Section 6.5], forming two infinite families and several sporadic examples (all of rank $\leq 8$). A Hasse diagram of the restrictions is given in Figure 9. For example, there are 17 such restrictions of rank 3 and Figure 3 shows inscribed zonotopes for each one of them (we refer to Section 5.1 for the naming scheme).

4.3. Representation of the strongly inscribed cone. In this section we give a simple representation of the strongly inscribed cone and we prove a characterization of inscribed zonotopes that generalizes that of Bolker.

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an essential arrangement of $n$ distinct hyperplanes with $H_i = z_i^\perp$ for $i = 1, \ldots, n$. We assume that $z_1, \ldots, z_n$ are contained in an open halfspace. For every region $R$ of $\mathcal{N}(\mathcal{A})$ there is unique $\sigma = (\sigma_1, \ldots, \sigma_n) \in \{-1, +1\}^n$ such that

$$R = \{x \in \mathbb{R}^d : \langle \sigma_i z_i, x \rangle \geq 0 \text{ for } i = 1, \ldots, n\}.$$

Such sign vectors are called topes. The tope graph $\mathcal{T}(\mathcal{A})$ is the undirected simple graph with nodes given by topes and edges $\sigma \sigma' \in E(\mathcal{T})$ if and only if the corresponding regions
are adjacent. Two topes $\sigma, \sigma'$ represent adjacent regions if and only if the separation set $S(\sigma, \sigma') := \{ j \in [n] : \sigma_j \neq \sigma'_j \}$ of the corresponding topes contains a single index $j$. We can identify edges by sign vectors $\tau \in \{-1, 0, +1\}^n$ with a unique zero entry at $j = j(\tau)$ and such that $\tau \pm e_j$ are topes.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n$ and let $Z_\lambda$ be the corresponding (virtual) zonotope, where

$$Z_\lambda := \lambda_1 [-z_1, z_1] + \lambda_2 [-z_2, z_2] + \cdots + \lambda_n [-z_n, z_n].$$

Every vertex of $Z_\lambda$ is of the form $v_\sigma := \sigma_1 \lambda_1 z_1 + \cdots + \sigma_n \lambda_n z_n$ for a unique tope $\sigma$, and every edge $e = [v_\sigma, v_{\sigma'}]$, represented by $\tau = \frac{1}{2}(\sigma + \sigma')$, has center $c(e) = \sum_i \tau_i \lambda_i z_i$. Let $T(\mathcal{A}) \in \mathbb{R}^{E(\mathcal{T}) \times n}$ be the matrix with entries $T_{\tau,i} = \tau_i (z_j(\tau), z_i)$ for $\tau \in E(\mathcal{T})$ and $i = 1, \ldots, n$. Applying Lemma 2.9 to all edges of $Z_\lambda$ yields:

**Proposition 4.10.** Let $\mathcal{A} = \{z^1_1, \ldots, z^n_1\}$ be a hyperplane arrangement in $\mathbb{R}^d$ and let $T = T(\mathcal{A})$ as above. Then

$$Z(\mathcal{A}) \cong \{ \lambda \in \mathbb{R}^n : T\lambda = 0 \} \quad \text{and} \quad Z_+(\mathcal{A}) \cong \{ \lambda \in \mathbb{R}^n_{>0} : T\lambda = 0 \}.$$

Note that if $\dim \mathcal{A} = 2$, then up to a permutation of rows $T(\mathcal{A})$ and $R$ as defined in Section 3.2 are identical.

We further analyze the kernel of $T$. For fixed $j = 1, \ldots, n$, let $T^j$ the matrix with rows given by those $\tau \in E(\mathcal{T})$ such that $j(\tau) = j$. Define the diagonal matrix $D^j \in \mathbb{R}^{n \times n}$ with diagonal $\langle z_j, z_i \rangle$ for $i = 1, \ldots, n$. Then $T\lambda = 0$ if and only if $T^j D^j \lambda = 0$ for all $j = 1, \ldots, n$. The rows of $T^j$ are strongly related to the topes of the restriction $\mathcal{A}_{H_j}$. Every hyperplane in $\mathcal{A}_{H_j}$ is of the form $L = H_s \cap H_j$ for some $s \neq j$. If $H_s \cap H_j = H_t \cap H_j$, then the corresponding columns $s$ and $t$ of $T^j$ are either identical or differ by $-1$, which is the case when $z_s$ and $z_t$ induce different orientations on $L$. Choose an orientation for every hyperplane of $\mathcal{A}_{H_j}$ and for $s \neq j$ set $\rho^j_s = 1$ if $H_s \cap H_j$ induces the chosen orientation and $\rho^j_s = -1$ otherwise.

We will need the following technical fact.

**Lemma 4.11.** The collection of topes of an arrangement of $n$ distinct hyperplanes has full rank $n$.

**Proof.** By replacing some $z_i$ by $-z_i$ if necessary, we may assume that $\sigma = (+1, \ldots, +1)$ is a tope of $\mathcal{A}$. Pick a shortest path in the tope graph of $\mathcal{A}_{H_j}$ from $\sigma$ to $-\sigma$. Up to relabelling hyperplanes if necessary, the sequence of topes visited along the path is $(+1, \ldots, +1), (+1, \ldots, +1, -1), (+1, \ldots, +1, -1, -1), \ldots, (-1, \ldots, -1)$, which is a collection of vectors of full rank $n$. \qed

Lemma 4.11 implies $T^j D^j \lambda = 0$ if and only if for every $L \in \mathcal{A}_{H_j}$

$$\sum_{s \neq j, L \subseteq H_s} \rho^j_s \langle z_j, z_s \rangle \lambda_s = 0. \quad (1)$$
Now \( L \in A_{H_j} \) is a codimension-2-flat of \( A \). We identify \( L \) with the ordered sequence \( L = (z_{i_1}, \ldots, z_{i_k}) \) such that \( L \subset H_s \) if and only if \( s \in \{i_1, \ldots, i_k\} \) and such that the vector configuration \( z_{i_1}, z_{i_2}, \ldots, z_{i_k}, -z_{i_1}, -z_{i_2}, \ldots, -z_{i_k} \) is cyclically ordered in the 2-dimensional subspace \( L^\perp \). We refer to this tuple as an ordered codimension-2-flat. We may now choose \( \rho_{i_s}^{i_j} = -1 \) if \( i_s < i_j \) and \( = +1 \) otherwise. Lastly, let \( R_L := R(z_{i_1}, z_{i_2}, \ldots, z_{i_k}) \) be the skew-Gram matrix and define \( \lambda_L = (\lambda_{i_1}, \ldots, \lambda_{i_k}) \). Then combining conditions (1) for all \( j = 1, \ldots, k \) then yields the following representation.

**Theorem 4.12.** Let \( A = \{H_i = z_i^\perp : i = 1, \ldots, n\} \) be an arrangement in \( \mathbb{R}^d \). Then

\[
\mathcal{Z}(A) \cong \{ \lambda \in \mathbb{R}^n : R_L \lambda_L = 0, L \in \mathcal{L}_{d-2}(A) \} \quad \text{and} \quad \mathcal{Z}_+(A) \cong \{ \lambda \in \mathbb{R}^n_{>0} : R_L \lambda_L = 0, L \in \mathcal{L}_{d-2}(A) \}.
\]

There is nice interpretation of the conditions of Theorem 4.12: If \( F \subset Z_\lambda \) is a 2-face with \( \text{aff}_0(F)^\perp = L \), then \( F \) is a translate of \( \lambda_i[z_{i_1}^-, z_{i_1}] + \lambda_{i_2}[z_{i_2}^-, z_{i_2}] + \cdots + \lambda_{i_k}[-z_{i_k}, z_{i_k}] \). Theorem 3.6 yields that \( Z_\lambda \) is inscribed if and only if all 2-faces of \( Z_\lambda \) are inscribed. Combining this with Theorem 2.3, this gives an extension of Bolker’s characterization of zonotopes.

**Theorem 4.13.** A (virtual) polytope \( P \) is an inscribed (virtual) zonotope if and only if all 2-dimensional faces are inscribed and centrally-symmetric.

Using description of inscribed cones for general fans in [18, Theorem 5.20] with the additional constraint that edges in the same belt have the same length gives a different path to Theorem 4.12. Since \( \mathcal{N}(A) \) has typically many edges, this approach is much more involved. A byproduct to our approach is a simple way to decide if an arrangement is virtually inscribable.

**Corollary 4.14.** Let \( A \) be an arrangement in \( \mathbb{R}^d \). We have \( \mathcal{I}(A) \cong \mathbb{R}^d \) if and only if \( \text{pf} R_L = 0 \) for all \( L \in \mathcal{L}_{d-2}(A) \).

**Proof.** It follows from Corollary 4.14 in [18] that \( A \) is virtually inscribable if and only if the localizations \( A_L \) are inscribable for all \( L \in \mathcal{L}_{d-2}(A) \). Thus, the result follows from Theorem 3.1 together with Theorem 3.6. \( \square \)

## 5. Simplicial arrangements

A hyperplane arrangement is simplicial, if all of its regions are simplicial cones, that is, spanned by linearly independent vectors. It is well-known [17, Sect. 1.15] that reflection arrangements are simplicial. Restrictions and localizations preserve simpliciality. Therefore, all examples of inscribed arrangements given so far are simplicial. This is true for general inscribed arrangements. We use this to prove the extension of Bolker’s characterization to inscribed belt polytopes and zonotopes (Theorem 4.13).

Simplicial arrangements are fascinating but rare. They arise, for example, in the study of reflection arrangements (see the comments in Section 4) as well as in the classification
of simply connected Cartan schemes \[8, 15\] by way of *crystallographic* arrangements \[2\]. Via localization and restriction, this is a rich source of simplicial arrangements of any rank. Grünbaum \[13\] studied simplicial arrangements of rank 3 (as line arrangements in the projective plane) and gave a list of two infinite families and 90 sporadic examples. Five further arrangements were discovered by Cuntz \[4, 5\] by computer search. It is believed that the two infinite families together with the 95 sporadic examples give a complete list of the combinatorial types of simplicial arrangements in rank 3. Apart from restrictions of reflection arrangements we are aware of only two further simplicial arrangements \[12\], both in rank 4.

We adapt the naming scheme of Grünbaum \[14\] and denote simplicial arrangements by \(A_r(n,k)\), where \(k\) is the position in the list of simplicial arrangements in rank \(r\) with \(n\) hyperplanes. For example, the two infinite families in rank 3 will be denoted by \(A_3(2n,1)\) and \(A_3(4m+1,1)\).

We confirm that all known simplicial arrangements in rank 3 are projectively unique. Assuming that the Grünbaum–Cuntz catalog is complete, we show in Section 5.2 that simplicial arrangements of all ranks \(\geq 3\) are projectively unique and that there are only finitely many irreducible strongly inscribed arrangements of every rank \(\geq 3\).

5.1. **Inscribable arrangements are simplicial.** In this section we prove Theorem 1.4. Recall that a \(d\)-polytope \(P\) is simple if every vertex is incident to \(d\) edges. We will need the following result on polyhedral cones.

**Lemma 5.1.** Let \(a_1, \ldots, a_d \in \mathbb{R}^d\) be linearly independent vectors such that \(\langle a_i, a_j \rangle \leq 0\) for all \(1 \leq i < j \leq d\). Let \(b_1, \ldots, b_d\) be such that

\[
\text{cone}(b_1, \ldots, b_d) = \{ x \in \mathbb{R}^d : \langle a_i, x \rangle \leq 0 \text{ for } 1 \leq i \leq d \}.
\]

Then \(\langle b_i, b_j \rangle \geq 0\) for all \(1 \leq i < j \leq d\).

**Proof.** We prove the claim by induction on \(d\). The case \(d \leq 2\) is clear. We may assume that \(\langle a_i, a_j \rangle = 1\) for all \(i\). Let \(C\) be the stated simplicial cone. Consider the facet \(C' := \{ x \in C : \langle a_d, x \rangle = 0 \} = \text{cone}(b_1, \ldots, b_{d-1})\). This is a \((d-1)\)-dimensional cone in the hyperplane \(H = a_d^\perp\) given by

\[
C' = \{ x \in H : \langle \bar{a}_i, x \rangle \leq 0 \text{ for } i = 1, \ldots, d-1 \},
\]

where \(\bar{a}_i = a_i - \langle a_d, a_i \rangle a_d\) is the orthogonal projection of \(a_i\) onto \(H\). One checks that \(\langle \bar{a}_i, \bar{a}_j \rangle \leq 0\) for all \(1 \leq i < j < d\) and thus, by induction, \(\langle b_i, b_j \rangle \geq 0\) for all \(1 \leq i < j < d\).

Let \(b_d = b_d - \langle a_d, b_d \rangle a_d\) be the orthogonal projection of \(b_d\) onto \(H\). Note that for all \(i < d\) we have \(\langle a_i, b_d \rangle = \langle a_i, b_d \rangle - \langle a_d, b_d \rangle \langle a_i, a_d \rangle \leq 0\) and hence \(\bar{b} \in C'\). Thus, there are \(\mu_1, \ldots, \mu_{d-1} \geq 0\) such that \(b_d = \mu_1 b_1 + \cdots + \mu_{d-1} b_{d-1} + \langle a_d, b_d \rangle a_d\). For \(j < d\), it follows that \(\langle b_j, b_d \rangle = \sum_{i=1}^{d-1} \mu_i \langle b_j, b_i \rangle \geq 0\), which finishes the proof. \(\square\)

A ridge \(F\) of \(P\) is a face of codimension 2. There are precisely two facets \(G, G'\) of \(P\) containing \(F\). The **dihedral angle** of \(F\) is the angle between \(G\) and \(G'\). If \(b\) and \(b'\) are
outer facet normals of \( G \) and \( G' \) respectively, then the dihedral angle is \( \pi - \angle(b,0,b') \). In particular the dihedral angle is non-acute, if \( \langle b,b' \rangle \geq 0 \). We call a polytope non-acute, if the dihedral angles at all ridges are non-acute.

Proof of Theorem 1.4. We proceed by induction on the dimension \( d \). The induction step is divided into two parts. First, assuming that every facet is non-acute, we show that \( P \) is simple. In the second step we then show, that if the inscribed belt polytope \( P \) is simple, then it is non-acute. The dihedral angles of a polygon coincide with its interior angles, so Corollary 3.5 proves the base case of the induction.

For the first step, let \( P \) be a \( d \)-dimensional inscribed belt polytope and let \( F \) be a \((d-3)\)-face of \( P \). The link of \( F \) in the face lattice is a cycle. Let \( G \) be a facet of \( P \) containing \( F \). Then \( F \) is a ridge of \( G \) and the dihedral angle of \( F \) in \( G \) is non-acute by the induction hypothesis. Since the sum of all dihedral angles has to be strictly smaller than \( 2\pi \), there are precisely three facets in \( P \) containing \( F \), which is equivalent to \( P \) being simple.

Thus far, we can assume that \( P \) is a simple and inscribed belt polytope. To show that the dihedral angles of \( P \) are non-acute, let \( v \in V(P) \) be a vertex, and let \( a_1, \ldots, a_d \in \mathbb{R}^d \) be the \( d \) edge directions emanating from \( v \). By Corollary 3.5, we have \( \langle a_i, a_j \rangle \leq 0 \) for all \( 1 \leq i < j \leq d \). Let \( G_i, 1 \leq i \leq d \), be the facets of \( P \) containing \( F \) and let \( b_i \) be their outer normal vectors. Then the dihedral angle of \( G_i \cap G_j \) is non-acute if and only if \( \langle b_i, b_j \rangle \leq 0 \), which is the conclusion of Lemma 5.1.

Remark 5.2. The proof of Theorem 1.4 is considerably simpler for strongly inscribed arrangements. Let \( Z \) be an inscribed zonotope and \( F \) a ridge of \( Z \). By Theorem 1.2, the orthogonal projection of \( Z \) along \( F \) is an (non-acute) inscribed zonogon, showing that the dihedral angle of \( F \) is non-acute. This removes the need for Lemma 5.1.

The proof does not go through for virtually inscribed zonotopes.

Question 5.3. Is there a non-simplicial arrangement \( A \) that is (strongly) virtually inscribable?

The simplicity of inscribed zonotopes is the key to the following refinement of Theorem 2.3, which extends Theorem 4.13 to belt polytopes.

Theorem 5.4. A polytope \( P \) is an inscribed belt polytope if and only if all 2-dimensional faces are inscribed belt polygons. A polytope \( P \) is an inscribed zonotope if and only if all 2-dimensional faces are inscribed and centrally-symmetric.

We recall that for simple polytopes inscribability is governed by faces of fixed dimension.

Corollary 5.5 ([18, Corollary 4.10]). Let \( P \subseteq \mathbb{R}^d \) be a simple \( d \)-polytope and \( 2 \leq k \leq d \). Then \( P \) is inscribed if and only if all its \( k \)-faces are inscribed.

Proof of Theorem 5.4. If all 2-faces are belt polygons (centrally-symmetric), then \( P \) is a belt polytope (zonotope) by Theorem 2.3. Furthermore, by Corollary 3.5, the interior angles of
every 2-face are $\geq \frac{\pi}{2}$. Using the same argument as in the proof of Theorem 1.4 we see that $P$ is simple, thus by Corollary 5.5 it is inscribed.

5.2. Projective uniqueness of simplicial arrangements. We call arrangements $\mathcal{A}$ and $\mathcal{A}'$ (combinatorially) isomorphic ($\mathcal{A} \cong \mathcal{A}'$) if $\mathcal{N}(\mathcal{A}) \cong \mathcal{N}(\mathcal{A}')$. In that sense the following has been conjectured by Grünbaum and Cuntz.

**Conjecture 5.6** (Grünbaum [14], Cuntz [5]). Every simplicial arrangement of rank 3 is isomorphic to an arrangement in the Grünbaum–Cuntz catalog.

For $g \in \text{GL}(\mathbb{R}^d)$, let $g\mathcal{A} := \{gH : H \in \mathcal{A}\}$. Clearly $g\mathcal{A} \cong \mathcal{A}$ for all $g \in \text{GL}(\mathbb{R}^d)$. We define

$$\mathfrak{R}(\mathcal{A}) := \{\mathcal{A}' \text{ arrangement in } \mathbb{R}^d : \mathcal{A} \cong \mathcal{A}'\} / \text{GL}(\mathbb{R}^d).$$

An arrangement $\mathcal{A}$ is **projectively unique**, if $|\mathfrak{R}(\mathcal{A})| = 1$, that is, if for any arrangement $\mathcal{A}' \cong \mathcal{A}$ there is $g \in \text{GL}(\mathbb{R}^d)$ such that $\mathcal{A}' = g\mathcal{A}$.

It was shown in [3] that $|\mathfrak{R}(\mathcal{A})|$ is finite for all examples of simplicial arrangements of rank 3. In fact, using methods in [3], one computationally confirms the following for the sporadic examples. For the two infinite families this is explicitly stated in [3].

**Proposition 5.7.** All arrangements in the Grünbaum–Cuntz catalog are projectively unique.

Conditional on the Grünbaum–Cuntz conjecture, this implies projective uniqueness in all dimensions $\geq 3$.

**Corollary 5.8.** If Conjecture 5.6 holds, then every simplicial arrangement of rank $\geq 3$ is projectively unique.

The proof of Corollary 5.8 depends on the following observation that we did not find in the literature and that might be of independent interest.

**Lemma 5.9.** Let $\mathcal{A}$ be an arrangement of rank $d \geq 3$. Then $|\mathfrak{R}(\mathcal{A})| \leq \prod_{H \in \mathcal{A}} |\mathfrak{R}(\mathcal{A}^H)|$.

**Proof.** We may assume that $\mathcal{A}$ is an essential arrangement in $\mathbb{R}^d$ and $|\mathfrak{R}(\mathcal{A}^H)| < \infty$ for all $H \in \mathcal{A}$. Let $H_1, \ldots, H_d \in \mathcal{A}$ be hyperplanes with $H_1 \cap \cdots \cap H_d = \{0\}$. Up to linear transformation, we may assume that $H_i = \{x \in \mathbb{R}^d : x_i = 0\}$ for $i = 1, \ldots, d$. This gives a canonical realization of $\mathcal{A}$ within $\mathfrak{R}(\mathcal{A})$ and, in fact, canonical realizations of $\mathcal{A}^{H_i} \in \mathfrak{R}(\mathcal{A}^{H_i})$ for $i = 1, \ldots, d$. Since $d \geq 3$, we have

$$H = (H_1 \cap H) + (H_1 \cap H) + \cdots + (H_d \cap H),$$

for every $H \in \mathcal{A}$ and hence $\mathcal{A}$ can be reconstructed from $\mathcal{A}^{H_i}$, $i = 1, \ldots, d$. This gives an embedding $\mathfrak{R}(\mathcal{A}) \hookrightarrow \prod_{i=1}^d \mathfrak{R}(\mathcal{A}^{H_i})$ and yields the claim.

**Proposition 6.2** in Section 6.1 implies that only finitely many arrangements from the Grünbaum–Cuntz catalog are strongly inscribable.
Theorem 5.10. If Conjecture 5.6 holds, then there are only finitely many combinatorial types of irreducible and strongly inscribable arrangements in every dimension $\geq 3$.

Let $m$ be the maximal number of hyperplanes that meet a codimension-2 flat of an irreducible and strongly inscribable arrangement of rank 3. Equivalently, $2m$ is the maximal number of edges of a 2-face of an irreducible and inscribed 3-dimensional zonotopes. (If Conjecture 5.6 is true, then the results of Section 6 show that $m = 12$.) We call a codimension-2 flat large if it is contained in more than $m$ hyperplanes.

Lemma 5.11. Assume that $m$ is finite. Let $A$ be a strongly inscribable arrangement of rank $d$. If $A$ has a large codimension-2 flat, then $A$ is reducible.

Proof. Let $A = \{H_1, \ldots, H_n\}$ and w.l.o.g. let $L = H_1 \cap \cdots \cap H_k$ be a large codimension-2 flat with $k > m$. For any $i > k$, consider the codimension-3 flat $M = L \cap H_i$. The localization $A_M$ is a strongly inscribed arrangement of rank 3 with the large codimension-2 flat $L$ and therefore reducible. Hence $H_i \perp L$ for $i = k + 1, \ldots, n$ and thus

$$A = \text{ess}\{H_1, \ldots, H_k\} \times \text{ess}\{H_{k+1}, \ldots, H_n\}. \quad \square$$

The last ingredient for the proof of Theorem 5.10 is the following result in [9]:

Lemma 5.12 ([9, Lemma 3.11]). Let $A$ be a simplicial and irreducible hyperplane arrangement. Then $A_L$ is irreducible for all $L \in \mathcal{L}(A)$.

Proof of Theorem 5.10. Let $N_d$ be the maximal number of hyperplanes of an irreducible inscribable arrangement of rank $d$. It suffices to show that $N_d < \infty$ for all $d \geq 3$. Assuming the validity of Conjecture 5.6, it follows from Proposition 6.2 that there are only finitely many combinatorial types of irreducible and inscribable arrangements of rank 3 and so $N_3$ is finite.

Let $A$ be an irreducible inscribable arrangement of rank $d > 3$ and let $H \in A$. The restriction $A^H$ is irreducible and inscribable by Lemma 5.12 and Theorem 1.2. Any $H' \in A \setminus \{H\}$ meets $H$ in a hyperplane of $A^H$. Conversely, any $J \in A^H$ is a codimension-2 flat of $A$ and hence is contained in at most $m$ hyperplanes by Lemma 5.11. We compute

$$|A| \leq (m - 1) \cdot |A^H| + 1 \leq (m - 1)N_{d-1} + 1$$

and hence $N_d$ is finite. \quad \square

6. Inner Products and Algebraic Computations

Let $A$ be a (simplicial) arrangement of $n$ hyperplanes in $\mathbb{R}^d$. In this section we consider the question when $A$ is inscribable up to a change of coordinates, that is, if there is a $g \in \text{GL}(\mathbb{R}^d)$ such that $gA$ is inscribable. In light of Corollary 5.8, this allows us to treat the combinatorial types of all known simplicial arrangements of rank 3. For this, we propose a change in perspective.
Throughout $Q$ denotes a real symmetric $d \times d$-matrix and we write $\langle x, y \rangle_Q := x^t Q y$ for the associated bilinear form. A fan $\mathcal{N}$ is (virtually) $Q$-inscribable if there is a (virtual) polytope $P$ with $\mathcal{N}(P) = \mathcal{N}$ such that $V(P) \subset E_Q := \{ x \in \mathbb{R}^d : \langle x, x \rangle_Q = 1 \}$. If $Q$ is positive definite, then $E_Q$ is an ellipsoid. The following lemma is apparent.

**Lemma 6.1.** Let $\mathcal{N}$ be a complete fan in $\mathbb{R}^d$ and $g \in \text{GL}(\mathbb{R}^d)$. Then $g\mathcal{N}$ is (strongly or virtually) inscribable if and only if $\mathcal{N}$ is (strongly or virtually) $Q$-inscribable with respect to the positive definite matrix $Q := g^t g$.

Testing if $\mathcal{A}$ is strongly inscribable up to linear transformation leads to a finite set of bilinear equations in $\lambda$ and $Q$, for which we seek a solution over $\mathbb{R}^n_{>0} \times \text{PSD}_d$. We treat the two infinite families in Section 6.1. For the sporadic simplicial arrangement from the Grübaün–Cuntz catalog we employ techniques from Gröbner basis theory to prove the existence or non-existence of a strongly inscribed realization. Assuming Conjecture 5.6, this shows that the only strongly inscribable rank-$3$ arrangements are restrictions of reflection arrangements.

Our approach also yields information about the space of $Q$’s for which $\mathcal{A}$ is strongly $Q$-inscribable. We determine these for all known inscribable arrangements.

### 6.1 Infinite families of rank-3

Recall from Section 5 that the two infinite families in the Grübaün–Cuntz catalog are denoted $\mathcal{A}_3(2n, 1)$ for $n \geq 3$ and $\mathcal{A}_3(4m + 1, 1)$ and $m \geq 2$. To construct $\mathcal{A}_3(2n, 1)$ as an arrangement of lines in the projective plane, one starts with the $n$ lines spanned by the sides of a regular polygon $\mathfrak{P}_n$ with $n$ sides and adds the $n$ lines of mirror symmetry of $\mathfrak{P}_n$. To obtain $\mathcal{A}_3(4m + 1, 1)$ one adds the line at infinity to $\mathcal{A}_3(4m, 1)$.

We can realize $\mathcal{A}_3(2n, 1)$ as $\mathcal{R}_n := \{ z_0^+, \ldots, z_{n-1}^+, z_0^-, \ldots, z_{n-1}^- \}$, where

$$z_j := (\text{Re}(\zeta^j), \text{Im}(\zeta^j), 0), \quad z'_j := (-\text{Im}(\zeta^j), \text{Re}(\zeta^j), 1),$$

for $j = 0, \ldots, n - 1$ and $\zeta := \exp(2\pi i/n)$. We will say that hyperplanes of the form $z_j^+$ are of type $M$, corresponding to mirrors of $\mathfrak{P}_n$, and that those of the form $z_j^\perp$ are of type $E$, corresponding to edges of $\mathfrak{P}_n$. If $n$ is even, then we distinguish those mirrors that pass through vertices of $\mathfrak{P}_n$ and those that connect edge midpoints and denote them by $M_v$ and $M_e$, respectively. For $m \geq 2$, we realize $\mathcal{A}_3(4m + 1, 1)$ as $\mathcal{R}'_m := \mathcal{R}_{2n} \cup \{(0, 0, 1)^\perp\}$ and denote $(0, 0, 1)^\perp$ by $\infty$. Examples are shown in Figure 4.

The codimension-2 flats (i.e., the 1-dimensional flats) of $\mathcal{R}_n$ and $\mathcal{R}'_m$ correspond to intersection points (possibly at infinity) of lines in the projective picture. The possible cases are illustrated in Figure 4. We list them according to the types of hyperplanes in which they are contained. The subscript in $M_v$ and $M_e$ is to be ignored, if $n$ is odd.

- Intersection of all $n$ lines of type $M$: $M_vM_vM_v\ldots M_vM_v$;
- 1-flats of type $M_vE$;
- 1-flats of type $M_vEE$ and $M_vEE$;
- For $\mathcal{A}(4m + 1, 1)$ there are $m$ 1-flats of type $M_v\infty$;
- For $\mathcal{A}(4m + 1, 1)$ there are $m$ 1-flats of type $M_vE\infty E$. 
The main result of this section is the following.

**Proposition 6.2.** The only strongly inscribable simplicial arrangements in the infinite families $A_3(2n, 1)$ and $A_3(4m + 1, 1)$ are $A_3(6, 1)$, $A_3(8, 1)$, $A_3(9, 1)$ and $A_3(13, 1)$.

We first derive conditions on $Q$ for which $R_n$ and $R'_m$ are virtually $Q$-inscribable.

**Lemma 6.3.** Let $A = R_n$, $n \geq 5$ or $A = R'_m$, $m \geq 3$. Then $A$ is virtually $Q$-inscribable if and only if $Q = \text{diag}(a, a, t)$, where $a, t \in \mathbb{R}^\times$.

**Proof.** We only consider the arrangement $A = R_n$ of type $A_3(2n, 1)$ for $n \geq 5$. The argument for $A_3(4m + 1, 1)$ is analogous. Let $Q$ be a symmetric $3 \times 3$-matrix.

The codimension-2 flats of type $M_{E}$ are precisely $L_j := z_j^+ \cap z_j'^-$ for $j = 0, \ldots, n - 1$. Using Example 3.9, we deduce that $\langle z_j^+, z_j'^- \rangle_Q = 0$ and thus

$$0 = -\text{Re}(\zeta^j)\text{Im}(\zeta^j)(Q_{11} - Q_{22}) + (\text{Re}(\zeta^j))^2 - \text{Im}(\zeta^j)^2)Q_{12} + \text{Re}(\zeta^j)Q_{13} + \text{Im}(\zeta^j)Q_{23}.$$ 

Using $\text{Re}(\zeta^j) = \frac{1}{2}(\zeta^j + \zeta^{-j})$ and $\text{Im}(\zeta^j) = \frac{i}{2}(\zeta^j - \zeta^{-j})$ we get a linear system of equations for $j = 0, 1, 2, 3$:

$$\begin{pmatrix}
0 & 1 & 1 & 0 \\
\frac{1}{2}(\zeta^2 + \zeta^{-2}) & \frac{1}{2}(\zeta^2 + \zeta^{-2}) & \frac{1}{2}(\zeta + \zeta^{-1}) & -\frac{1}{2}i(\zeta + \zeta^{-1}) \\
\frac{1}{2}(\zeta^4 + \zeta^{-4}) & \frac{1}{2}(\zeta^4 + \zeta^{-4}) & \frac{1}{2}(\zeta^2 + \zeta^{-2}) & -\frac{1}{2}i(\zeta^2 + \zeta^{-2}) \\
\frac{1}{2}(\zeta^6 + \zeta^{-6}) & \frac{1}{2}(\zeta^6 + \zeta^{-6}) & \frac{1}{2}(\zeta^3 + \zeta^{-3}) & -\frac{1}{2}i(\zeta^3 + \zeta^{-3})
\end{pmatrix}
\begin{pmatrix}
Q_{11} - Q_{22} \\
Q_{12} \\
Q_{13} \\
Q_{23}
\end{pmatrix} = 0,$$

where $X := X$.

The determinant of $X$ is

$$\det(X) = -\frac{1}{8\zeta^7}(\zeta - 1)^2(\zeta^2 - 1)(\zeta^3 - 1)^2(\zeta^4 - 1).$$

Thus, for $n \geq 5$, $\det X \neq 0$ and we conclude that $Q_{12} = Q_{13} = Q_{23} = 0$ and $Q_{11} = Q_{22}$. 

**Figure 4.** Two members of the infinite families.
Conversely, to show that $\mathcal{A}$ is virtually $Q$-inscribable for $Q = \text{diag}(a, a, t)$, $a, t \neq 0$, we show for each codimension-2 flat $L$ that $\mathcal{A}_L$ is virtually inscribable. For $L$ of type $M_v E$ we have $\text{pf} R_L = 0$ by construction. If $L$ is of type $M_v M_v M_v \ldots M_v M_v$, then $\mathcal{A}_L$ is the arrangement of a regular polygon and is therefore (virtually) inscribable. For the types $M_v EE$ and $M_v EE$, the localization $\mathcal{A}_L$ has an odd number of hyperplanes, thus $\det R_L = 0$. \hfill \Box$

Let us denote by $I_2(n)$ the symmetry group of the regular polygon $\mathcal{P}_n$ acting on $\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}$ by fixing the last coordinate. By construction, $\mathcal{R}_n$ and $\mathcal{R}'_{n/2}$ are $I_2(n)$-symmetric.

**Proposition 6.4.** If $\mathcal{A}_3(2n, 1)$ or $\mathcal{A}_3(4m + 1, 1)$ (for $n = 2m$) is inscribable, then there exists an inscribable realization that is $I_2(n)$-symmetric.

**Proof.** We again only consider the case when $\mathcal{A}$ is an inscribable arrangement of type $\mathcal{A}_3(2n, 1)$. Since $\mathcal{R}_n$ is projectively unique by Proposition 5.7, there is some $g \in \text{GL}(\mathbb{R}^3)$ with $\mathcal{A} = g \mathcal{R}_n$. Thus $\mathcal{R}_n$ is $Q$-inscribable with respect to $Q = g'g$. From Lemma 6.3 we conclude that up to an orthogonal transformation, we can assume that $g = \text{diag}(\sqrt{a}, \sqrt{a}, \sqrt{t})$ for some $a, t > 0$. Since $g$ commutes with the action of $I_2(n)$ on $\mathbb{R}^3$, $\mathcal{A}$ is $I_2(n)$-symmetric up to orthogonal transformation. \hfill \Box

**Proof of Proposition 6.2.** All of the arrangements $\mathcal{A}_3(6, 1)$, $\mathcal{A}_3(8, 1)$, $\mathcal{A}_3(9, 1)$ and $\mathcal{A}_3(13, 1)$ are restrictions of reflection arrangements, see Figure 9, and therefore strongly inscribable by Theorem 1.2. Therefore, we assume that $\mathcal{A}$ is a strongly inscribable arrangement of type $\mathcal{A}_3(2n, 1)$ for $n \geq 5$ or of type $\mathcal{A}_3(4m + 1, 1)$ for $m \geq 4$.

By Proposition 6.4 we can assume that $\mathcal{A}$ is symmetric with respect to $I_2(n)$. Let $Z$ be an inscribed zonotope for $\mathcal{A}$. Using Corollary 2.9 of [18] we can assume that $Z$ is also symmetric with respect to $I_2(n)$.

The type of a hyperplanes of $\mathcal{A}$ is invariant under $I_2(n)$ and hence the edges of $Z$ of the same type $M_v$, $M_v$, $E$, or $\infty$ have the same length. As inscribed polygons are uniquely determined by their edge lengths, we see that all facets corresponding to codimension-2 flats of the same type are congruent. Hexagons of type $M_v EE$ have two different angles $\alpha_1$, $\beta_1$ (between the edges of type $M_v$ and $E$ and between edges of type $E$ and $E$), likewise for $M_v EE$. For $\mathcal{A}_3(2n, 1)$ and $n \geq 5$ or $\mathcal{A}_3(4m + 1, 1)$ and $m \geq 4$, $Z$ contains the configuration of four hexagons depicted in Figure 5. Considering the interior angle sum of each of the hexagons gives $4\alpha_1 + 2\beta_1 = 4\pi$ and $4\alpha_2 + 2\beta_2 = 4\pi$, but considering the angles at the two vertices in the center gives $\beta_1 + 2\alpha_2 < 2\pi$ and $\beta_2 + 2\alpha_1 < 2\pi$, a contradiction. \hfill \Box

**Remark 6.5.** It remains an open question which of the arrangements $\mathcal{A}$ in the two infinite families are (non-strongly and non-virtually) $Q$-inscribable. Based on experiments, we conjecture, that $\mathcal{A}$ can always be $Q$-inscribed for $Q = \text{diag}(a, a, t)$ with $0 < t \ll a$.

### 6.2. Algebraic computations and sporadic simplicial arrangements

For an arrangement $\mathcal{A} = \{z_i^\perp : i = 1, \ldots, n\}$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{R}^\times)^n$, and $Q \in \mathbb{R}^{d \times d}$, it is a simple matter of linear algebra to check if $Z_\lambda = \sum_i \lambda_i [-z_i, z_i]$ is $Q$-inscribable: We may use
Lemma 2.9 to conclude that $Z_\lambda$ is $Q$-inscribed if and only if

$$0 = \langle z_{j(\tau)}, c(e_\tau) \rangle_Q = \sum_{i=1}^{n} \tau_i \lambda_i \langle z_{j(\tau)}, z_i \rangle_Q$$

for all edges $\tau \in E(T)$. This gives a linear subspace \{\(Q : Z_\lambda \text{ is } Q\text{-inscribed}\)\} $\subset \mathbb{R}^{d \times d}$. Now to test if $Z_\lambda$ is inscribed into an ellipsoid, it suffices to test the linear subspace contains a positive definite matrix, which can be done by semidefinite programming.

Conversely, for a fixed $Q$, Theorem 4.12 may be adapted to test if there is $\lambda \in \mathbb{R}^n$ such that $Z_\lambda$ is (virtually) $Q$-inscribed. Let $L = (z_{i_1}, \ldots, z_{i_k})$ be an ordered codimension-2 flat of $A$ and write $R_L(Q) \in \mathbb{R}^{k \times k}$ for the skew-Gram matrix with respect to $Q$:

$$R_L(Q)_{ij} = -R_L(Q)_{ji} = \langle z_{i}, z_{j} \rangle_Q$$

for $1 \leq i < j \leq k$ and $R_L(Q)_{ii} = 0$. Then $A$ is strongly $Q$-inscribable if and only if there is $\lambda \in \mathbb{R}_{>0}^n$ with $R_L(Q)\lambda_L = 0$ for all ordered codimension-2 flats $L$ of $A$.

**Remark 6.6.** Note that it is quite simple to enumerate all ordered codimension-2 flats: Given $z_1, \ldots, z_n \in \mathbb{R}^d$, we can assume that there are $c, w \in \mathbb{R}^d$ such that $\langle c, z_i \rangle = 1$ for $i = 1, \ldots, n$ and $\langle w, z_i \rangle \neq \langle w, z_j \rangle$ for $i \neq j$. The codimension-2 flat of $A$ correspond to 2-dimensional subspaces spanned by subsets of $z_1, \ldots, z_n$. The total order induced by $w$ induces a cyclic order on each 2-dimensional subspace. For given $Q$, this give a simple algorithm to set up the linear programming feasibility problem of finding $\lambda \in \mathbb{R}^n$ with $\lambda > 0$ and $R_L(Q)\lambda_L = 0$ for all ordered codimension-2 flats $L$.
Let \( S := \mathbb{C}[Q_{ij}, \lambda_k : 1 \leq i \leq j \leq n, 1 \leq k \leq n] \). The ideal
\[
\mathcal{J}_A := \langle R_L(Q)\lambda_L : L \in \mathcal{L}_{d-2}(A) \rangle : (\lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n \cdot \det Q)^\infty \subseteq S
\]
contains the Zariski closure of the collection of pairs \((\lambda, Q) \in (\mathbb{R}^\times)^n \times \text{GL}(\mathbb{R}^d)\) such that \(Z_\lambda\) is \(Q\)-inscribed. The saturation with respect to \(\lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n \cdot \det Q\) ensures that the closure takes place over \((\mathbb{R}^\times)^n \times \text{GL}(\mathbb{R}^d)\). Standard facts from computational algebra now give the following simple criterion for non-inscribability.

**Proposition 6.7.** Let \( A \) be a simplicial arrangement. If \( \mathcal{J}_A = \langle 1 \rangle \), then there is no non-singular \( Q \) such that \( A \) is strongly virtually \( Q \)-inscribable.

We applied this criterion to the 95 sporadic simplicial arrangements of the Grünbaum–Cuntz catalog. Except for the arrangements stemming from restrictions of reflection arrangements only the arrangements \( A_3(10, 1) \), \( A_3(12, 1) \), \( A_3(15, 4) \), \( A_3(16, 4) \), \( A_3(17, 1) \), and \( A_3(21, 2) \) were not ruled out. In these 6 cases the dimension of the ideal \( \mathcal{J}_A \) is 2. Since \( \mathcal{J}_A \) is homogeneous in \( Q \) and \( \lambda \) it follows that there is a unique \( \lambda \) and \( Q \) up to scaling such that \( Z_\lambda \) is \( Q \)-inscribed. In the above cases it turns out that \( \lambda \) and \( Q \) are independent of each other and determined by linear equations; see Example 6.10 below. For \( A_3(10, 1) \), \( A_3(12, 1) \), \( A_3(15, 4) \), \( A_3(17, 1) \) the matrix \( Q \) is indefinite. For the two arrangements \( A_3(16, 4) \), \( A_3(21, 2) \) the unique \( \lambda \) had negative entries. Thus, while these arrangements posses a virtual inscribed zonotope, there exists no non-virtual inscribed zonotope with these hyperplane arrangements as normal fan.

Using the projective uniqueness of the arrangements in the Grünbaum–Cuntz catalog, the approach outlined above gives a computational proof of the following result.

**Proposition 6.8.** The only combinatorial types of strongly inscribable arrangements in the Grünbaum–Cuntz catalog are restrictions of reflection arrangements.

Moreover, assuming Conjecture 5.6, an irreducible rank-3 arrangement is strongly inscribable if and only if it is combinatorially isomorphic to a restriction of a reflection arrangement.

In addition to restrictions of reflection arrangements, there are 2 further known simplicial arrangements of rank 4, denoted \( A_4(27, 1) \) and \( A_4(28, 1) \), that arise as subarrangements of \( H_4 \); see [12]. For \( A_4(27, 1) \), the ideal \( \mathcal{J}_A \) is trivial, for \( A_4(28, 1) \) the ideal \( \mathcal{J}_A \) has dimension 2 but the matrix \( Q \) is indefinite. Alternatively, restrictions of both arrangements are simplicial of rank 3 and contained in the Grünbaum–Cuntz catalog, see Figure 6. Both arrangements have non-strongly inscribable restrictions and Theorem 1.2 together with Proposition 6.8 refute strong inscribability.

**Corollary 6.9.** No arrangement combinatorially isomorphic to \( A_4(27, 1) \) or \( A_4(28, 1) \) is strongly inscribable.

We implemented this algorithm in SAGE [22] and our code is part of the arXiv submission. Normal vectors (roots) to all known simplicial arrangements of rank 3 are found in [6] and
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Figure 6. Hasse diagram of the restrictions of \( \mathcal{A}_4(27,1) \) and the non-inscribable arrangements for which \( \mathcal{J}_A \neq \langle 1 \rangle \). For the labeling, we refer to Figure 9.

as a SAGE database [7] by the same authors. Roots for the restrictions can be found in [8], while [12] contains roots for the rank-4 examples \( \mathcal{A}_4(27,1) \) and \( \mathcal{A}_4(28,1) \). Example 6.10 shows the computation for \( \mathcal{A}_3(10,1) \).

Example 6.10. The arrangement \( \mathcal{A}_3(10,1) \) is an arrangement in \( \mathbb{R}^3 \) with 10 hyperplanes given by normals

\[
(z_1, \ldots, z_{10}) = \begin{pmatrix}
2\tau + 1 & 2\tau + 2 & 1 & \tau + 1 & 2\tau & \tau + 1 & 1 & 0 & 1 & \tau + 1 \\
2\tau & 2\tau + 1 & 1 & \tau + 1 & 2\tau & \tau + 1 & 1 & 1 & 0 & \tau \\
\tau & \tau + 1 & 1 & \tau & \tau & 1 & 0 & 0 & 0 & \tau \\
\end{pmatrix}
\]

where \( \tau = \frac{\sqrt{5}+1}{2} \) is the golden ratio. The 16 ordered codimension-2 flats are

\[
(z_8, z_3), (z_9, z_6), (z_{10}, z_5), (z_1, z_1), (z_7, z_2), (z_5, z_1, z_9), (z_8, z_6, z_1), (z_7, z_1, z_{10}), (z_9, z_{10}, z_3), \\
(z_8, z_5, z_2), (z_4, z_2, z_9), (z_8, z_7, z_9), (z_8, z_4, z_{10}), (z_6, z_2, z_{10}), (z_3, z_2, z_1), (z_7, z_6, z_5, z_4, z_3)
\]

The ideal \( \mathcal{J}_A \) is generated by the polynomials

\[
\lambda_9 - \lambda_{10}, \lambda_8 - \lambda_{10}, \lambda_7 + (\tau + 1)\lambda_{10}, \lambda_6 + (\tau + 1)\lambda_{10}, \lambda_5 + (\tau + 1)\lambda_{10}, \\
\lambda_4 + (\tau + 1)\lambda_{10}, \lambda_3 + (\tau + 1)\lambda_{10}, \lambda_2 - \lambda_{10}, \lambda_1 - \lambda_{10}, \\
10Q_{23} - (\tau - 3)Q_{33}, 10Q_{13} + (\tau + 2)Q_{33}, 10Q_{22} + (2\tau - 1)Q_{33}, \\
10Q_{12} + (-\tau - 2)Q_{33}, 10Q_{11} + (2\tau - 1)Q_{33}
\]

Setting \( \lambda_{10} = s \) and \( Q_{33} = t \), this yields

\[
\lambda = s(1,1,-(\tau + 1),-(\tau + 1),-(\tau + 1),-(\tau + 1),1,1,1)
\]
and
\[
Q = \frac{t}{10} \begin{pmatrix}
-2\tau + 1 & \tau + 2 & -\tau - 2 \\
\tau + 2 & -2\tau + 1 & \tau - 3 \\
-\tau - 2 & \tau - 3 & 10
\end{pmatrix}.
\]

It is straightforward to verify that \(Q\) is never positive-definite and \(\lambda \notin \mathbb{R}_{>0}^{10}\).

As an implementation detail, we compute \(J_A\) by adding the equations \(R_L\lambda_L = 0\) step by step, where we order the flats \(L\) by the number of hyperplanes in which they are contained, and reduce to a Gröbner basis. Thus, we start with flats contained in precisely two hyperplanes, which are especially restrictive, as they do not involve any of the \(\lambda_i\)'s (compare Example 3.9). It helps tremendously to saturate with \(\lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n\) after each step to cut down the computational cost. This allowed us to inspect the ideals not only for all the sporadic examples in dimension 3 but for all restrictions of sporadic reflection arrangements of all ranks up to combinatorial symmetry. The computation of a Gröbner basis for \(J_A\) for the largest arrangement \(E_8\), took about 20 minutes on a contemporary laptop. With a modest level of parallelization, the remaining examples can be checked. The computation finishes long before that for \(E_8\). This allowed us to compute the dimension of \(J_A\) of all sporadic reflection arrangements and their restrictions, again up to combinatorial isomorphism. The dimensions are collected in Figure 9.

The same ideas yield a criterion for the non-existence of a \(Q\)-inscribed belt polytope for a given arrangement \(A\). We appeal to Corollary 4.14: \(A\) has a (virtual) belt polytope \(P\) that is \(Q\)-inscribed if and only if \(R_L(Q)\) is singular for all ordered codimension-2 flats \(L \in \mathcal{L}_{d-2}(A)\). This gives rise to the ideal
\[
J'_A := \langle \text{pf } R_L(Q) : L \in \mathcal{L}_{d-2}(A) \rangle : (\det Q)^\infty \subseteq \mathbb{C}[Q_{ij} : 1 \leq i \leq j \leq d].
\]

Similarly to Proposition 6.7, we get

**Proposition 6.11.** Let \(A\) be a simplicial arrangement. If \(J'_A = \{1\}\), then there is no non-singular \(Q\) such that \(A\) is virtually \(Q\)-inscribable.

Note that the existence of \(\lambda \in \mathbb{R}^n\) with \(R_L(Q)\lambda_L = 0\) for all codimension-2 flats \(L\) is a much stronger condition than \(\text{pf } R_L(Q) = 0\). Moreover, the dimension of \(J'_A\) is hard to interpret.

**Remark 6.12.** We computed \(J'_A\) for the known simplicial arrangements which are not part of an infinite family. The computations show that in 77 examples we have \(J'_A \neq \{1\}\). In all of these cases, \(J'_A\) describes a linear subspace. With the help of semidefinite programming we could certify that in all but two cases, \(A_3(14, 3)\) and \(A_3(15, 4)\), there is a positive definite matrix \(Q\), such that \(A\) is virtually \(Q\)-inscribable. We did not check, whether one can find a \(Q\) such that \(A\) is (non-virtually) \(Q\)-inscribable. The computations show that there are inscribable arrangements which are not strongly inscribable. Moreover, there exist inscribable arrangements \(A\) such that \(A^2\) is not inscribable, contrary to Theorem 1.2.

**Remark 6.13.** If \(A\) is a strongly inscribed arrangement and \(H \in A\), then Theorem 1.2 implies that there is a linear projection \(\mathbb{Z}_+(A) \to \mathbb{Z}_+(A^H)\). This projection is in general
not onto, as the following example shows: Let \( A = A_4(22,1) \) and \( H \) in \( A \) such that \( A^H \cong A_3(13,2) \) (compare Figure 9). Our computations show that \( \dim Z^+_+(A) = 1 \), but \( \dim Z^+_+(A^H) = 2 \).

6.3. Restrictions of arrangements of type \( A, B, D \). The algorithm of the last section also yields the possible bilinear forms \( Q \) for which the known sporadic simplicial arrangements of ranks \( \leq 8 \) are strongly inscribable. This included the reflection arrangements \( F_4, E_6, E_7, E_8, H_3 \) and \( H_4 \) as well as their restrictions. In this section, we explicitly determine the bilinear forms \( Q \) for which the reflection arrangements of type \( A_n, B_n, D_n \) as well as their restrictions are inscribable. Restrictions of reflection arrangements of type \( A \) are combinatorially, and hence projectively, of type \( A \). For the arrangements of type \( B \) and \( D \), the restrictions are well understood and captured by a family of simplicial arrangements \( \mathcal{D}_{n,s} \) for \( 0 \leq s \leq n \); see Figure 9.

**Inscribed arrangements of type \( A \).** The reflection arrangement of type \( A_n \) is canonically realized in \( \mathbb{R}^{n+1} \) by the \( \binom{n+1}{2} \) hyperplanes

\[
H_{ij} = \{ x \in \mathbb{R}^{n+1} : x_i = x_j \}
\]

for \( 1 \leq i < j \leq n+1 \). In [19, Proposition 6.73], it is shown that the restriction of \( A_n \) to any hyperplane is combinatorially isomorphic to \( A_{n-1} \). Since \( A_3 \) is projectively unique, it follows from Lemma 5.9 that every \( A_n \) is projectively unique. Hence it suffices to determine the possible inner products for a fixed realization of \( A_n \) for each \( n \). The above realization is not essential. Restricting to the hyperplane \( \{ x : x_{n+1} = 0 \} \cong \mathbb{R}^n \) gives the realization

\[
A_n := \{ f_{ij} : 1 \leq i < j \leq n+1 \},
\]

where we set \( f_{ij} := e_i - e_j \) for \( 1 \leq i < j \leq n \) and \( f_{in+1} := e_i \). Let \( J_n \in \mathbb{R}^{n \times n} \) be the matrix of all ones.

**Theorem 6.14.** For \( n \geq 3 \), \( A_n \) is virtually \( Q \)-inscribable if and only if

\[
Q = \text{diag}(a_1, \ldots, a_n) + a_{n+1}J_n
\]

for some \( a_1, \ldots, a_n, a_{n+1} \in \mathbb{R} \). Moreover, if \( Z_\lambda = \sum_{i<j} \lambda_{ij}[f_{ij}, f_{ij}] \) is a \( Q \)-inscribed zonotope, then \( a_1, \ldots, a_n, a_{n+1} > 0 \) and \( \lambda_{ij} = \frac{1}{a_i a_j} \) for \( 1 \leq i < j \leq n+1 \) and a unique \( t \in \mathbb{R}_{>0} \).

**Proof.** The arrangement \( A_n \) is virtually \( Q \)-inscribable if and only if \( \text{pf} R_L(Q) = 0 \) for every ordered codimension-2 flat of \( A_n \). It follows from general theory ([17]) or simply by inspection that every ordered codimension-2 flat is the intersection of 2 or 3 hyperplanes and of the form

1. \( L = (f_{ij}, f_{kl}) \) for \( \{i,j\} \cap \{k,l\} = \emptyset \), or
2. \( L = (f_{ij}, f_{ik}, f_{jk}) \) for \( i < j < k \).
Theorem 6.16. Let \( i \) is virtually \( Q \) for \( \lambda \). Represent a zonotope for \( D \). Inscribed arrangements from types \( V \) and \( D \). For the type \( D \) arrangement of type \( n, s \). Based on these computations, we conclude: 

\[
\begin{pmatrix}
0 & a_i & -a_j \\
-a_i & 0 & a_{n+1} \\
a_j & -a_{n+1} & 0
\end{pmatrix}
\]

For \( j = n + 1 \), we thus get \( 0 = pf R_L(Q) = Q_{ik} - Q_{kl} \) and hence all off-diagonal entries of \( Q \) are equal to some \( a_{n+1} \in \mathbb{R} \). This implies \( Q = a_{n+1}J_n = \text{diag}(a_1, \ldots, a_n) \). Conversely, \( f_{ij}^t Q f_{kl} = 0 \) for all flats of type (1) whenever \( Q \) is of this form. This shows the first claim.

Assume that \( Z_\lambda = \sum_{i<j} \lambda_{ij} [-f_{ij}, f_{ij}] \) is a \( Q \)-inscribed zonotope. In particular \( \lambda_{ij} > 0 \) for all \( i < j \). Consider the ordered flat \( L = (f_{ij}, f_{jk}, f_{jk}) \) with \( k = n + 1 \). Then

\[
R_L(Q) = \begin{pmatrix}
0 & a_i & -a_j \\
-a_i & 0 & a_{n+1} \\
a_j & -a_{n+1} & 0
\end{pmatrix}
\]

and \( R_L(Q)\lambda_L = 0 \) if and only if \( \lambda_L = (\lambda_{ij}, \lambda_{in+1}, \lambda_{jn+1}) = s_L(a_{n+1}, a_j, a_i) \) for some \( s_L \in \mathbb{R} \). Since \( \lambda > 0 \), we conclude that all \( a_i \) have the same sign. By inspecting all flats of this type, we infer that there is a \( t \in \mathbb{R} \) with \( \lambda_{ij} = \frac{t}{a_i a_j} \) for all \( 1 \leq i < j \leq n + 1 \). Since \( V(Z_\lambda) \subseteq E_Q = \{ x : \langle x, x \rangle Q = 1 \} \), the scaling parameter \( t \) is positive and unique. \( \square \)

Based on these computations, we conclude:

**Corollary 6.15.** Let \( A = A_n, n \geq 3 \). Then \( \dim J_A = n + 2 \) and \( \dim J'_A = n + 1 \).

Inscribed arrangements from types \( B \) and \( D \). For \( 0 \leq s \leq n \), define the arrangement in \( \mathbb{R}^n \)

\[
\mathcal{D}_{n,s} := \{ e_1^+ \ldots, e_s^+ \} \cup \{ (e_i \pm e_j)^\perp : 1 \leq i < j \leq n \}.
\]

For \( s = 0 \), \( \mathcal{D}_{n,0} \) is the reflection arrangement of type \( D_n \). For \( s = n \), \( \mathcal{D}_{n,n} \) is the reflection arrangement of type \( B_n \). In particular \( \mathcal{D}_{n,s} \) is an \( s \)-fold restriction of the arrangement of type \( D_{n+s} \). It is shown in Orlik-Terao [19, Chapter 6.4] that the restrictions of \( \mathcal{D}_{n,0} \) are combinatorially isomorphic to \( \mathcal{D}_{n-1,1} \), while \( B_n \) uniquely restricts to \( B_{n-1} \). Moreover, for \( 0 < s < n \), the restrictions of \( \mathcal{D}_{n,s} \) is combinatorially of type \( B_{n-1} \), \( \mathcal{D}_{n-1,s-1} \) (if \( s > 1 \)), \( \mathcal{D}_{n-1,s} \), or \( \mathcal{D}_{n-1,s+1} \) (if \( s < n - 1 \)). Figure 9 shows the Hasse diagram of the restrictions of reflection arrangements. Since the arrangements \( \mathcal{D}_{3,1}, \mathcal{D}_{3,2} \) and \( \mathcal{D}_{3,3} \) are projectively unique, we conclude by Lemma 5.9 that the same is true for all \( \mathcal{D}_{n,s} \) with \( n \geq 3 \). We represent a zonotope for \( \mathcal{D}_{n,s} \) by

\[
Z_\lambda = \sum_{1 \leq i < j \leq n} \lambda_{ij}^- [e_i - e_j, e_j - e_i] + \sum_{1 \leq i < j \leq n} \lambda_{ij}^+ [-e_j - e_i, e_i + e_j] + \sum_{k=1}^s \lambda_k [-e_k, e_k]
\]

for \( \lambda_{ij}^+, \lambda_{ij}^-, \lambda_k \in \mathbb{R} \) for \( 1 \leq i < j \leq n \) and \( 1 \leq k \leq s \).

**Theorem 6.16.** Let \( n \geq 4 \) and \( 0 \leq s \leq n \) and \( Q \in \mathbb{R}^{n \times n} \) non-singular. Then \( \mathcal{D}_{n,s} \) is virtually \( Q \)-inscribable if and only if there are \( a_1, \ldots, a_s, a \in \mathbb{R}^s \) with \( a_i \neq a \) for \( i = 1, \ldots, s \) such that \( Q = \text{diag}(a_1, \ldots, a_s, a, \ldots, a) \). Moreover, \( Z_\lambda \) is \( Q \)-inscribed if and
Proof. For \(i < j\) and \(k < l\) with \(\{i, j\} \cap \{k, l\} = \emptyset\), \(L = (e_i + \sigma e_j, e_k + \tau e_l)\) is an ordered codimension-2 flat for all \(\sigma, \tau \in \{-1, +1\}\). If \(D_{n,s}\) is virtually \(Q\)-inscribable, then for all \(\sigma, \tau \in \{-1, +1\}\)

\[
0 = \text{pf} R_L = (e_i + \sigma e_j)^t Q(e_k + \tau e_l).
\]

This implies \(Q_{ij} = 0\) for all \(i \neq j\) and hence \(Q = \text{diag}(a_1, \ldots, a_n)\) for some \(a_1, \ldots, a_n \in \mathbb{R}\). For \(i < j\) with \(i > s\), \(L = (e_i - e_j, e_i + e_j)\) is an ordered codimension-2 flat and

\[
0 = \text{pf} R_L = (e_i - e_j)^t Q(e_i + e_j) = Q_{ii} + Q_{ij} - Q_{jj} = a_i - a_j
\]

shows that \(a_{s+1} = \cdots = a_n = a\). It is a routine calculation to check that \(\text{pf} R_L(Q) = 0\) for all codimension-2 flats if and only if \(Q\) is of the stated form.

For the second claim, observe that for the codimension-2 flat \(L = (e_i - e_j, e_i - e_k, e_j - e_k)\) the condition \(R_L(Q)\lambda_L = 0\) translates into

\[
0 = (e_i - e_j)^t Q(\lambda^-_{ik} e_i - e_k) + \lambda^-_{jk}(e_j - e_k)) = \lambda^-_{ik} a_i - \lambda^-_{jk} a_j.
\]

\[
0 = (e_j - e_k)^t Q(\lambda^-_{ij} e_i - e_k) + \lambda^-_{ik}(e_i - e_k)) = \lambda^-_{ik} a_k - \lambda^-_{ij} a_j.
\]

Set \(t := a_1 a_2 \lambda^-_{12}\). The second equation now implies \(\lambda^-_{jk} = -\frac{t}{a_1 a_k}\) for all \(1 < k \leq n\) and the first in turn implies \(\lambda^-_{ik} = -\frac{t}{a_i a_k}\) for all \(1 \leq j < k \leq n\).

Likewise, the ordered codimension-2 flats \(L = (e_i - e_j, e_i + e_k, e_j + e_k)\) give the condition \(0 = \lambda^+_{ik} a_k - \lambda^+_{ij} a_j\) and thus \(\lambda^+_{ik} = -\frac{t}{a_ia_k}\) for all \(1 \leq i < j < k \leq n\). A similar calculation for \(L = (e_i + e_j, e_i - e_k, e_j + e_k)\) shows that this also holds if \(k = i + 1\).

If \(s < n\), then \(L = (e_i + e_n, e_i, e_i - e_n)\) implies \(0 = a_i \lambda_i - (a_i - a_i) \lambda^+_i\). Thus, \(\lambda_i = \frac{t(a_i - a)}{a_i a}\), which is nonzero if and only if \(a_i \neq a\).

Finally, if \(s = n\), then the ordered codimension-2-flat \(L = (e_j, e_i + e_j, e_i, e_i - e_j)\) yields \(0 = a_i \lambda_i - a_j \lambda_j + (a_i - a_j) \lambda^+_j\). Setting \(t' := a_1 \lambda_1 - \frac{t(a_i - a)}{a_1 a}\) then gives for \(i = 1\)

\[
a_j \lambda_j = a_1 \left(\frac{t(a_i - a)}{a_1 a} + \frac{t'}{a_1}\right) + (a_1 - a_j) \frac{t}{a_1 a_j} = \frac{t a_j a - ta_1 a_j}{a_1 a_j} + t' + \frac{ta_1 a - ta_j a}{a_1 a_j} = a_j^{t} = \frac{t(a_i - a_j)}{a_j a} + t',
\]

thus \(\lambda_j = \frac{t(a_i - a_j)}{a_j a} + \frac{t'}{a_j}\), as required. It is now easy (but tedious) to verify that \(Z_\lambda\) is \(Q\)-inscribed for the stated choice of \(\lambda\). \(\Box\)

**Remark 6.17.** The proof does not cover the case \(n = 3\), since no codimension-2 flats of the form \((e_i + \sigma e_j, e_k + \tau e_l)\) exist. Instead, we determined \(J_\mathcal{A}\) and \(J'_\mathcal{A}\) with the computer as explained in the previous section.
Again, we directly conclude:

**Corollary 6.18.** Let $\mathcal{A} = D_{n,s}$, $n \geq 4$, $0 \leq s \leq n$. Then $\dim J_\mathcal{A} = s + 2$ and $\dim J'_\mathcal{A} = \min(s + 1, n)$.

## 7. General quadrics and infinite arrangements

We close with two brief remarks and some appealing pictures.

### 7.1. General quadrics.** Our algebraic approach for checking $Q$-inscribability in Section 6.2 showed that for the arrangements $\mathcal{A}_3(10, 1)$, $\mathcal{A}_3(12, 1)$, $\mathcal{A}_3(15, 4)$, $\mathcal{A}_3(16, 4)$, $\mathcal{A}_3(17, 1)$, and $\mathcal{A}_3(21, 2)$, the corresponding ideal $J_\mathcal{A}$ had dimension 2. Since $J_\mathcal{A}$ is homogeneous with respect to $Q$ as well as $\lambda$, this implies that the bilinear form $Q$ and lengths $\lambda$ are unique if we require that $\langle v, v \rangle_Q = 1$ for all vertices $v \in V(Z_\lambda)$. In all all but two cases, the bilinear form $Q$ was indefinite and the zonotope $Z_\lambda$ virtual. The two arrangements $\mathcal{A}_3(16, 4)$, $\mathcal{A}_3(21, 2)$ only have virtual zonotopes that are inscribed into an ellipsoid. Figure 7 shows all six examples.

![Virtual zonotopes for the six combinatorial types not excluded from Grünbaums list by our program. They are inscribed into some quadric, not necessarily a sphere.](image)

**Figure 7.** Virtual zonotopes for the six combinatorial types not excluded from Grünbaums list by our program. They are inscribed into some quadric, not necessarily a sphere.

It is noteworthy that among all finite reflection arrangements, only the arrangements of type $A_n$ and $B_n$ have inscribable realizations into quadrics of different type. Figure 7 fuels the question for a better geometric understanding of inscribable (virtual) zonotopes. Also, we do not know if arrangements which are virtually inscribable into other quadrics need to be simplicial.

### 7.2. Infinite arrangements.** An affine reflection arrangement $\mathcal{A}$ is the infinite collection of affine hyperplanes

$$H_{a,k} = \{x \in \mathbb{R}^d : \langle \alpha, x \rangle = k\},$$

where $k \in \mathbb{Z}$ and $\alpha$ ranges over all roots in a crystallographic root system $\Phi$. The associated affine Weyl group $W_a$ is the group of affine transformations generated by the affine reflections $s_{a,k}$ in $H_{a,k}$; see [17, Chapter 4]. The localization of $\mathcal{A}$ at any 0-flat is the translate of
a finite reflection arrangement. Hence the orbit of a point $p$ under $W_a$ yields a tessellation of $\mathbb{R}^d$ into inscribed belt polytopes.

For a suitable choice of $p$, the belt polytopes of the tessellation are zonotopes. Theorem 1.2 now yields that the restriction of $\mathcal{A}$ to a flat $L$ is a strongly inscribed infinite arrangement and the tessellation into inscribed zonotopes is not induced by an affine reflection arrangement. Figure 8 shows a 2-dimensional restriction of $\tilde{E}_8$.

![Figure 8. A tiling of $\mathbb{R}^2$ by inscribed zonogons.](image)

It would be very interesting to investigate geometric and combinatorial properties of tilings of space into inscribed zonotopes.
Figure 9. Hasse diagram of the restrictions of reflection arrangements and \( \dim \mathcal{J}_A \) and \( \dim \mathcal{J}'_A \) for all entries of rank greater or equal to 3. The intensity of the blue shading increases with \( \dim \mathcal{J}_A \).
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