Confinings of QCD at purely classic levels

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It is well known that confinings and asymptotic freedom are properties of quantum chromodynamics (QCD). But they can also be observed at purely classic levels. For this purpose we need to find solutions to the colorly-sourceful Yang-Mills equations with both confining and asymptotic freedom features. We provide such a solution in this paper which at the near-source region is of serial form, while at the far-away region is expressed through simple fundamental functions approximately. From the solution, we derive out a non-perturbative beta function describing the running of effective coupling constant, which is linear in the couplings both in the infrared and ultraviolet region.

As is well know, exact solutions to the Einstein equation shape our knowledge structure of general relativities remarkably. So finding exact solutions to the classic Yang-Mills equation like the one of Schwarzschild or Riessner-Nordström to Einstein equations must also be meaningful. But to our surprise, when we search on the internet, we almost find no such solutions. The literature we obtained most closely related to this topic is some on the construction of colorless-monopoles [1,2] or colorful black-holes [3,4], or non-gravity but horizon-carrying Yang-Mills-Scalar black holes [5]. The purpose of this paper is to provide solutions to the purely classic Yang-Mills theories with both confining and asymptotic freedom features.

Another motivation of this work is reference [6] (see also [7] for earlier ideas in a totally different environment), in which G. Dvali et al observed that by studying the classic field stimulated by fixed external sources, many features of the underlying quantum theories, such as the renormalization group structure, the phenomenon of dimensional transmutation, running coupling constant, asymptotic freedom et al can all be exhibited out at the purely classic level. G. Dvali et al take the $\lambda \phi^4(\lambda < 0)$ theory as examples to illustrate these ideals. Obviously, it will be more instructive if for Yang-Mills theories we can also see asymptotical freedom and confinement et al at purely classic levels.

Let us begin our stories from the basic action describing classic Yang-Mills fields coupled with external quarks

$$\mathcal{L} = -\frac{1}{2g_{YM}^2} \text{tr} F_{\mu\nu} F^{\mu\nu} + \frac{2}{g_{YM}} \text{tr} A_{\mu} J^{\mu}$$

(1)

where $A_{\mu} \equiv A^a_{\mu} t^a$, $J_{\mu} = J^a_{\mu} t^a$, $J^{\mu a} = \bar{\psi} i t^a \gamma^\mu \psi$, with $t^a$s denoting the generator of $SU(N)$ group while

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$$

(2)

Using these notations, the covariant Yang-Mills equation can be written as

$$D_\mu F^{\mu\nu} \equiv \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} F^{\mu\nu}) + i[A_\mu, F^{\mu\nu}] = g_{YM} J^\nu$$

(3)

where $\sqrt{-g}$ is the root of metric determinant in some specific coordinating system.

For our purpose, we need to find solutions to the above Yang-Mills equation with point like static quark sources. The corresponding 4-component quark-current vector can be written as

$$J^\mu = (Q^a t^a 4\pi \delta(\vec{r}), 0, 0, 0)$$

(4)

By symmetry reasoning, we know that gauge fields stimulated by this quark should be spherically symmetric. According to reference [3] and [4], the most simple color-fields with this symmetry could be written as

$$A^a_{\mu} dx^\mu = A_t dt + A_r dr + A_\theta d\theta + A_\phi d\phi$$

(5)

$$A_t = D \cdot \frac{J(r)}{2r}, \quad A_r \equiv 0, \quad A_\theta = -\frac{i}{2} (C - C^\dagger) K(r)$$

(6)

$$A_\phi = -\frac{1}{2} [(C + C^\dagger) K(r) \sin \theta + D \cos \theta]$$

(7)

By properly choosing the direction of source quarks’ charge $Q^a$ in the color-space, we can always write the matrices $C$ and $D$ involved in the above expressions as follows (noting we use $i$ denoting $N - i$)

$$D \equiv \text{diag.}\{N - 1, N - 3, \cdots, 3 - N, 1 - N\}$$

(8)

$$C \equiv \begin{bmatrix} 0 & \sqrt{1 - 1} \\ 0 & \sqrt{2 - 2} \\ & \ddots \ddots \\ 0 & \sqrt{1 - 1} & \cdots & \cdots & 0 \end{bmatrix}_{N \times N}$$

(9)
sponding field strength, we can easily derive out equations satisfied by $J(r)$ and $K(r)$
\[ r^2 J'' - 2JK^2 = g_{YM} Qr \delta(r) \] \[ r^2K'' - K(K^2 - 1 - J^2) = 0 \]
where $Q$ is now only a number quantifying the amount of color-charge, its direction in the color-space has been specified when we choose the form of matrices $C$ and $D$. Although this is a source equation, we will neglect the source term when solving differential equations. This is very similar to the case of solving Poinsson equations $\nabla^2 \phi = \rho \delta(r)$ to get Column potentials in electrostatics or solving Einstein equations $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$ to get Schwarzschild metric in general relativity. The difference is, in Column potentials and Schwarzschild metric, we have definite scheme to relate the solution parameter with the source charges (masses is looked as charges of gravity). While in the Yang-Mills case, due to the feature of asymptotical freedom and confinings, we still have no appropriate schemes to relate the solution parameter with both constant color-magnetic field and Column-like increasing $r$. This is very similar to the case of solving Poinsson equations $\nabla^2 \phi = \rho \delta(r)$ to get Column potentials in electrostatics or solving Einstein equations $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$ to get Schwarzschild metric in general relativity. The difference is, in Column potentials and Schwarzschild metric, we have definite scheme to relate the solution parameter with the source charges (masses is looked as charges of gravity). While in the Yang-Mills case, due to the feature of asymptotical freedom and confinings, we still have no appropriate schemes to relate the solution parameter $J$, $K$ in the following to the source charge $Q$.

Now let us give out the first two simple but non-trivial solutions

\[ (i) \quad J \equiv 0, \quad K \equiv 0, \]
\[ (ii) \quad J = ar + b, \quad K = 0 \quad (b \neq 0) \]

Their non-triviality can be looked out from their corresponding field strength

\[ (i) F_{0r}^a t^a \equiv 0, \quad F_{\phi r}^a t^a = \frac{1}{2}D \sin \theta \]
\[ (ii) F_{0r}^a t^a = \frac{D}{2r^2}, \quad F_{\phi r}^a t^a = \frac{1}{2}D \sin \theta \]

The first solution describes objects with constant color-magnetic field, while the second one describes object with both constant color-magnetic field and Column-like color-electric field. Obviously, the solution parameter $b$ is equal to the source charge $Q$. If we calculate energies stored in this two field configurations (see expression (20) in the following), we will see that both of them contain singularity at $r \to 0$, i.e. $E \propto \int_0^\infty dr/r^2$. This is a singularity similar to that of point-like electric charges in classic electrodynamics.

The previous two solutions are both non-confining. Maybe the more attractive solutions should be the following two-parametors solution family with both asymptotical freedom and confining features

\[ (iii) J(r) \xrightarrow{r \to 0} \sum_{n=2,4}^6 g_{YM} J_n r^n, \quad K(r) \xrightarrow{r \to 0} \sum_{n=0,2}^6 K_n r^n \]

With the corresponding field strength given by

\[ (iii) F_{0r} = -\frac{rJ' - J}{2r^2} D, \quad F_{\phi r} = \frac{JK}{2r} (C + C^\dagger) \]
\[ F_{0\phi} = -i\frac{JK}{2r} (C - C^\dagger) \sin \theta, \quad F_{r\theta} = -\frac{K'}{2} (C - C^\dagger) \]
\[ F_{r\phi} = -i\frac{K'}{2} (C + C^\dagger) \sin \theta, \quad F_{\theta \phi} = -\frac{(K^2 - 1)}{2} D \sin \theta \]

Although converges only at a small region around $r = 0$, the serial expressions (15a) exactly satisfy equations (10a) - (10b) as long as

\[ J_4 = \frac{2}{5} J_2 K_2, \quad J_6 = \frac{6J_2 K_2^2 - J_4}{35} - \frac{J_6}{70} \]
\[ J_8 = \frac{104J_2 K_2^3}{1575} - \frac{4J_4 K_2^2}{225}, \ldots \]

\[ K_0 = \pm 1, \quad K_4 = \frac{3K_2^2 - J_2^2}{10}, \quad K_6 = \frac{K_2^3}{10} - \frac{3J_2 K_2}{35} \]
\[ K_8 = \frac{37J_4^2}{12600} - \frac{606J_2 K_2^2}{12600} + 413K_2^4, \ldots \]

with $J_2$ and $K_2$ being two parameters characterizing the color-electric and color-magnetic charges of the source. The analytical expressions (15a) exactly satisfies eq (10a) but satisfies (10b) only up to an order $r \cos[r \ln(r)]$ oscillating errors. It can be easily proved that $J(r)$ in this expression approaches $r \ln r$ asymptotically. FIG. displays one of our numerical solutions explicitly.

From the field strength expression eq (15a) and the serial solution (15a), we easily see that both the color-electric and color-magnetic field have behavior $F_{0r}$, $F_{\phi r} \xrightarrow{r \to 0}$ constant instead of the Column-like increasing $r^{-2}$. This is a feature of asymptotical freedoms. Another way to see this is using the fact that $J(r) \xrightarrow{r \to 0} r^2$ and that $Jr' - J \xrightarrow{r \to 0} 0$ has the meaning of effective color-electric charges. The color-magnetic charge has similar properties. The confinement of the solution can be observed from the corresponding field-configuration energies, which are defined and can be calculated as follows

\[ E \equiv \int d^3X \sum_{\nu} T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L)}{\delta g_{\mu\nu}} \]
Because, due to the running of effective couplings, the right panel is that of color-electric or color-magnetic couplings. Reversing eqs (23) we can write $J_2r_0^2 = \sum c_{j,n}^n Q^n$, $K_2r_0^2 = \sum c_{k,n}^n Q^n$. Substituting them into eq (23) we will get

$$g_c(r)Q = \sum_n Q^n c_n^z(z, \frac{r}{r_0})$$

$$g_m(r)zQ = \sum_n Q^n c_m^z(z, \frac{r}{r_0})$$

If the running of coupling constants (23) has RGS, then the coefficient $c_{n}^{z,m}[z, r/r_0]$’s should satisfy the following structure equation (26),

$$\frac{dc_n^{z,m}[z, x]}{dx} = \sum_{k=1}^{n} k c_k^{e,m}[x] c_{n+1-k}^{e,m}[0], x = \ln \frac{r}{r_0}$$

This is a highly non-trivial constraints on the definition of effective couplings (22)-(23). But regretfully, we find except the lowest two level of coefficients $c_1^{z,m}[z, x]$, $c_2^{z,m}[z, x]$, all the other coefficients do not satisfy this RGS equation. Maybe, $J_2$, $K_2$ and $Q$ are not connected through so simple a relation like (22). For example, they maybe connected by the following relation

$$[(rJ' - Jr')^2 + 2J^2K^2 + 2r^2K'^2 + (K^2 - 1)^2] = \alpha(r)Q^4$$

$$\alpha(r = r_0) = 1, F[J_2r_0^2, K_2r_0^2] = Q$$

with $F(x, y)$ a properly chosen function different from $\sqrt{\alpha(r_0)}$ so that the running of $\alpha(r)$ has exact RGS.

Assuming that such a function exist, then from the basic equation of motion (10) and the solution (15), we will be able to derive out a classic beta function,

$$\beta \equiv r \frac{d\alpha}{dr} \rightarrow \left\{ \begin{array}{l} 4\alpha, \alpha \rightarrow 0 \\ \alpha \rightarrow \frac{1}{e} \text{ProductLog}[e\alpha], \alpha \rightarrow \infty \end{array} \right.$$
As can be easily proved that, the quantum set trivalent cut off. Maybe a natural question is, can we not match the quantum expression, \( \beta \) not involve the theory with \( \Lambda \) in other words, Yang-Mills in this case is taken as effective theory with \( \Lambda \). It is derived from exact solutions of the theory.

The beta function (28) involves no averaging the small scale degrees of freedom, i.e. in this case Yang-Mills is just taken as an effective description of physics. Obvi-ously, when \( \Lambda_0 = \infty \) (correspondingly \( \alpha = 0 \)), we do not know how \( \beta(\alpha) \) is. From the viewpoint of effective theories, we also need not to know how it is. While the classic beta function (28) involves no averaging the small scale physics, it takes the theory as an ultimate description of physics. It is derived from exact solutions of the theory.

In other words, Yang-Mills in this case is taken as effective theory with \( \Lambda_0 = \infty \). It is completely possible that \( \beta(\alpha)_{\Lambda_0=\infty} \alpha \) while \( \beta(\alpha)_{\Lambda_0=\text{finite}} \propto \alpha^2 \). In this sense, the beta function we defined in this paper is different from that defined by G. Dvali et al in [3]. As conclusions, we construct three solutions to the classic Yang-Mills equation. The first two are exact but non-confining. While the third one manifests both asymptotical freedom and confinings. It is of serial form and exact at the near-source region, but approximate and analytical at the far-away region. From this solution we derive out non-perturbative beta functions describing the running of effective coupling constant, which are linear in the couplings both in the infrared and ultraviolet region. Our result is sensible to understand confinings of QCD at purely classical levels and non-perturbatively. It may provide new ingredients for studies of QCD using gauge/gravity dualities [12]. More directly, it will be helpful for looking for fully analytical solutions to the Yang-Mills equation [13]. On this point, the value of our work is similar to that of Julia and Zee [1], which makes preparations for Prasad and Sommerfield’s discoveries [8].

As a discussion, we must point out that, as a non-linear differential equation array, (10) also allows other solutions which are different from what we provide in this paper. For example, it can be easily verified that \( J_{r \to 0} \to r^{-1} \), \( K_{r \to 0} \to 1 \) also leads to reasonable solutions. More solutions are possible if \( K_{r \to 0} \neq 1 \). We numerically find that all such solutions are confining but not necessarily asymptotic free. This point to a conclusion that, confinings have no direct relation with asymptotical freedom, which is just the viewpoint of [14]. The real deep reason of confinement is the topological structure of the gauge group.

Appendix

In numeric practices, we find that no matter how we set the \( r \to 0 \) boundary condition, as long as the equation (10) can be smoothly integrated from \( r = 0 \) to \( r \to \infty \), we always get confining solutions at \( r \to \infty \). So it maybe very interesting if we can prove that \( J(r) \propto r \ln(r) \) analytically, since it means that we prove the confinings of classic Yang-Mills theory somehow. Let us try the following idea. Consider possibilities,

\[
a) \quad K^2 - 1 - J^2 \to \infty -r^{p} \\
b) \quad K^2 - 1 - J^2 \to \infty +r^{p} \\
c) \quad K^2 - 1 - J^2 \to \infty -r^2 \ln^2(r) \\
d) \quad K^2 - 1 - J^2 \to \infty +r^2 \ln^2(r)
\]

where \( p \) is allowed to take any real values. Of course, enumeration of \( K^2 - 1 - J^2 \)'s asymptotical behavior cannot be completely listed. But through careful examinations in the following, we will see that for self-consistencies, further listing of other possibilities such as \( e^{pr} \) is of no use. Substituting these possibilities into equation (10b), we will get

\[
a_1) \quad K \propto C_1 r^{\frac{1}{p}} J - \frac{2}{p} \left[ \ln r \right]^2 + C_2 r^{\frac{1}{p}} J \left[ \frac{2}{p} r^{\frac{1}{p}} \right] \to r \to \infty, \quad r^{\frac{1}{p}} \frac{1}{p} - \frac{1}{p}
\]
\[ b_1 \quad K \propto C_1 r^2 \frac{I_{-\frac{1}{2}} \left( \frac{2 r}{p} \right)}{p} + C_2 r^2 \frac{I_{\frac{1}{2}} \left( \frac{2 r}{p} \right)}{p} \]
\[ \rightarrow \infty \quad r^2 \propto \frac{2 r}{p} r^{p/2} \tag{34} \]
\[ c_1 \quad K \rightarrow \infty \quad C_1 \frac{\cos[r \ln(r)]}{\sqrt{1 + \ln(r)}} + C_2 \frac{\sin[r \ln(r)]}{\sqrt{1 + \ln(r)}} \tag{35} \]
\[ d_1 \quad K \rightarrow \infty \quad C_1 \frac{e^{-r \ln(r)}}{\sqrt{1 + \ln(r)}} + C_2 \frac{e^{r \ln(r)}}{\sqrt{1 + \ln(r)}} \tag{36} \]
where \( I_n(x) \) and \( J_n(x) \) denote the usual Bessel functions.

In case a), if \( p = 2 \) then both the two equations (10a)-(10b) can be solved exactly with results \( J \propto C_1 r^2 + C_2 r^{-1}, \quad K \propto D_1 \cos[r] + D_2 \sin[r] \). So \( K^2 - 1 - J^2 \propto -r^4 + O(r) \), which is obviously inconsistent with the assumption (29). So we need only to explore the fact that

\[
\begin{align*}
\text{if } p > 2 \quad & \text{eq}(29) \Rightarrow J \xrightarrow{r \rightarrow \infty} r^{p/2}, \\
& \text{eq}(10a) \Rightarrow J \xrightarrow{r \rightarrow \infty} r^2 e^{-r^{(2-p)/4}} \tag{37}
\end{align*}
\]
\[
\begin{align*}
\text{if } p < 2 \quad & \text{eq}(29) \Rightarrow J \xrightarrow{r \rightarrow \infty} \sqrt{r^p + r^{p/2}}, \\
& \text{eq}(10a) \Rightarrow J \xrightarrow{r \rightarrow \infty} r^2 e^{-r^{(2-p)/4}} \tag{38}
\end{align*}
\]

In either condition, the final conclusions from equations (29) and (10a) are inconsistent. So, the a) case can be excluded. Similarly, the case b) can also be excluded. But case c) cannot be excluded by self-consistency. While in case d)

\[
\begin{align*}
& \text{if } C_1 = 0 \quad \text{eq}(32) \Rightarrow J \xrightarrow{r \rightarrow \infty} -r^2 \ln^2(r), \\
& \text{eq}(10a) \Rightarrow J \xrightarrow{r \rightarrow \infty} r \ln r \\
& \text{if } C_2 = 0 \quad \text{eq}(32) \Rightarrow J \xrightarrow{r \rightarrow \infty} e^{r \ln(r)} < K, \\
& \text{eq}(10a) \Rightarrow J \xrightarrow{r \rightarrow \infty} \sqrt{1 + \ln(r)} \tag{39}
\end{align*}
\]

So this is also excluded. The only possibility which cannot be excluded is c). While through observing equations (29), (37) and (38), we know that for self-consistency, the asymptotic of \( K^2 - 1 - J^2 \) should be something between \(-r^2\) and \(-r^{2+e}\), but cannot be of the \( r^p \) form. So, up to corrections of order \( r^2 \ln[r] \), \(-r^2 \ln^2[r]\) is almost the unique possibility admitted by self-consistency. This finishes our proof of conings of QCD at the purely classic levels.

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