Higher derivative scalar-tensor monomials and their classification

Xian Gao*

School of Physics and Astronomy, Sun Yat-sen University, Guangzhou 510275, China

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We make a full classification of scalar monomials built of the Riemann curvature tensor up to the quadratic order and of the covariant derivatives of the scalar field up to the third order. From the point of view of the effective field theory, the third or even higher order covariant derivatives of the scalar field are of the same importance as the higher curvature terms, and thus should be taken into account. Moreover, the higher curvature terms and the higher order derivatives of the scalar field are complementary to each other, of which novel ghostfree combinations may exist. We make a systematic classification of all the possible monomials, according to the numbers of the Riemann tensor and the higher derivatives of the scalar field in each monomial. A complete basis of monomials at each order is derived, of which the linear combinations may yield novel ghostfree Lagrangians. We also develop diagrammatic representations for the monomials, which may help to simplify the analysis.

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1 Introduction

Scalar-tensor theory has been extensively studied in the past few decades as one of the main theories of modification of gravity. In particular, much effort has been made to introduce the higher derivatives of the scalar field without the Ostrogradsky ghost(s) [1]. The representative achievements are the Horndeski theory [2-5] as well as the degenerate higher-order theory [6-9] (see refs. [10, 11] for reviews).

However, the previous studies mostly stopped at the second order in the derivative of the scalar field1). This is partly because the second derivatives of the scalar field provide a “playground” for the higher derivative scalar-tensor theories that are sufficiently nontrivial but not too exhausting to be studied. Although focusing on the second derivative is consistent by itself, there are at least two motivations to go beyond, as we shall explain below.

First, from the point of view of the effective field theory, operators of the same order in derivatives are of the same importance and should be treated in the same footing. As an example, let us recall that \( \mathcal{L}_4^{\text{H}} \) of the Horndeski theory with \( G_4 = X^2 \) takes the form

\[
\mathcal{L}_4^{\text{H}} = X^2 R + 2X \left( \Box \phi \right)^2 - 2X \left( \nabla_a \nabla_b \phi \right)^2 ,
\]

with \( X = -\frac{1}{2} \nabla_a \phi \nabla^a \phi \). Since the Riemann tensor arises from the commutator of two covariant derivatives, we may think that \( R \sim O(\nabla^2) \). As a result, all the three terms in the above are of \( O(\nabla^6) \). This partly explains why these three terms arise together in the Horndeski theory. On the other hand, there are other terms that are of \( O(\nabla^6) \) as well. Schematically, one example is

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*Corresponding author (email: gaoxian@mail.sysu.edu.cn)

1) Non-polynomial derivative terms that are infinite order in derivatives have also been studied, see (e.g.,) ref. [12] and references therein.
\[ \sim X \nabla_a \phi \nabla^n \phi, \]

in which the third order derivative of the scalar field arises. Of course, for this particular case (i.e., \( \mathcal{L}_X^H \)), this term is trivial since it can be reduced to the Horndeski form by integrations by parts. Nevertheless, things become less trivial if we consider \( \mathcal{L}_X^H \) with

\[
\mathcal{L}_X^H = X^2 G^{ab} \nabla_a \phi \nabla_b \phi - \frac{1}{3} X (\Box \phi)^3 + X \Box \phi (\nabla_a \nabla_b \phi)^2 - \frac{2}{3} X (\nabla_a \nabla_b \phi)^3,
\]

where \( G_{ab} \) is the Einstein tensor. All the terms in the above are of \( O(\nabla^3) \). There is another type of term, schematically

\[ \sim \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi, \]

which is also of \( O(\nabla^3) \) and thus should be considered in the same footing. In particular, this type of term cannot be fully reduced by integrations by parts.

The second motivation comes from the study of the higher derivative scalar-tensor theories, and in particular, the higher curvature gravity theories that suffer from the ghosts generally, but are ghostfree when the gradient of the scalar field is assumed to be timelike. With this assumption, sometimes it is convenient to work in the so-called unitary gauge, in which the Lagrangian is significantly simplified since the scalar field is chosen to be spatially uniform\(^2\). Perhaps in this sense the simplest example is the Chern-Simons gravity\(^{[13,14]} \), which suffers from ghosts generally but the ghosts are eliminated when the scalar field is timelike. Another interesting example was introduced in ref. \([15]\), which is of the quadratic order in the Weyl tensor and is also ghostfree when the scalar field is timelike. Some exotic parity-violating scalar-tensor theories that appear to be ghostfree when assuming a timelike scalar field were identified in ref. \([16]\), in which terms of the quadratic order in the curvature tensor also arise. Despite the behavior of such kind of theory in a general background still under debate\(^7\), such theories provide us an even broader framework of the higher derivative scalar-tensor theories that have many applications. Refs. \([15,16]\) focus on the direct couplings between the curvature and derivatives of the scalar field up to the second order. For the same reason in the above, the terms built of derivatives of the scalar field higher than the second order should also be considered. For example, the terms of the quadratic order in the Riemann tensor were introduced in refs. \([15,16]\), which take the schematic form

\[ \sim \nabla^a \phi \nabla^b \phi \nabla^c \phi R^d, \]

where \( R \) is a shorthand for the Riemann tensor, and there are \( n \) first order derivatives of the scalar field. These terms are of \( O(\nabla^{n+4}) \). While terms in the form

\[ \sim \nabla^a \phi \cdots \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi \]

are of the same order and thus should be considered as well.

From another point of view, the terms with derivatives of the scalar field higher than the second order can be treated as being “complimentary” to the higher curvature terms. This is similar to the Horndeski theory, in which the Lagrangian can be split into the “curvature sector” and the “scalar field sector”, and one is complimentary to the other\(^3\). For example, in \( \mathcal{L}_X^H \), the “curvature sector” is \( X^2 R \) and the “scalar field sector” is \( 2X(\Box \phi)^2 - 2X(\nabla_a \nabla_b \phi)^2 \). Neither the curvature nor the scalar field sector can be tuned to be ghostfree individually. Only their linear combination can yield a ghostfree covariant Lagrangian. Then it is natural to ask what the complementary terms of (e.g.,) the quadratic curvature terms are, and whether their combinations can yield new ghostfree theories.

In order to answer the above questions, a systematic investigation of more general higher derivatives of the scalar field and their couplings with the curvature is required. As the first step, this paper is devoted to the classification of monomials built of the derivatives of the scalar field up to the third order as well as their couplings with the curvature tensor.

### 2 The formalism

A general Lagrangian that is built of a single scalar field and the Riemann tensor as well as their covariant derivatives takes the form

\[
\mathcal{L} \left( g^{ab}, \varepsilon_{abcd}, R_{abcd}, \phi, \nabla_a \right),
\]

where the completely antisymmetric Levi-Civita tensor is

\[
\varepsilon_{abcd} = \sqrt{-g} \varepsilon_{abcd},
\]

with \( \varepsilon_{0123} = 1 \). The form of the Lagrangian can be quite arbitrary in general. In this paper we concentrate on the case that the Lagrangian is a polynomial, in which the monomials are scalar invariants that are built of the Riemann curvature tensor, the scalar field and their covariant derivatives, with possible coefficients in the form \( X^n \), where

\[ X = -\frac{1}{2} \nabla_a \phi \nabla^a \phi \]

---

2) The price is that only part of the phase space of the scalar field can be explored, since the part in which the gradient of scalar field is spacelike or null is suppressed in the unitary gauge. As a result, conclusions made in the unitary gauge are valid only when the scalar field possesses a timelike gradient.

3) This is similar to the “covariantization” procedure in refs. \([4,18]\).
is the canonical kinetic term of the scalar field.

Precisely, each monomial takes the general structure\(^4\)
\[
\sum_{c_0} \cdots R^{c_0} \cdots \nabla \cdots \nabla \phi \cdots \nabla \phi \cdots \\
\sum_{d_1} \cdots \nabla \nabla \phi \cdots \nabla \nabla \phi \cdots 
\]
where “⋯” denotes the multiple Riemann curvature tensor (we schematically denote it as \(R\)), the scalar field and their covariant derivatives. All the indices are contracted by the metric \(g^{ab}\) and/or the completely antisymmetric tensor \(\epsilon^{abcd}\). Thus, we may assign each monomial a set of integers
\[
(c_0, c_1, c_2, \cdots ; d_1, d_2, d_3, d_4, \cdots),
\]
where
- \(c_0, c_1, c_2, \cdots\) are numbers of the Riemann curvature tensor and of the first, the second order derivatives of it, etc.,
- \(d_1, d_2, d_3, d_4, \cdots\) are numbers of the first, the second, the third and the fourth order covariant derivatives of \(\phi\), etc.

We assume all the \(c_n\)’s and \(d_n\)’s are non-negative except \(d_1\). This is simply because we allow the monomials to be divided by some powers of \(X\), which will be convenient in the classification of the monomials. As an example, the term
\[
\sim X R^2 \nabla R \nabla \phi \nabla \phi \nabla \phi
\]
corresponds to
\[
(2, 1, 0, \cdots ; 2, 2, 1, 0, \cdots).
\]

We shall classify various monomials according to the order and the partition of the derivatives. First, the total number of the derivatives of a given monomial with \((c_0, c_1, c_2, \cdots ; d_1, d_2, d_3, d_4, \cdots)\) is
\[
D \equiv 2c_0 + 3c_1 + 4c_2 + \cdots + d_1 + 2d_2 + 3d_3 + 4d_4 + \cdots,
\]
that is
\[
D \equiv \sum_{n=0} \left[ (n+2)c_n + n d_n \right].
\]
which is justified by the fact that the Riemann tensor arises from the commutator of two covariant derivatives. That is, we will treat each Riemann tensor as of order 2 in the covariant derivatives, or schematically, \(R \sim O(\nabla^2)\). Thus a monomial corresponding to \((c_0, c_1, c_2, \cdots ; d_1, d_2, d_3, d_4, \cdots)\) is of \(O(\nabla^2)\) with \(D\) given in eq. \((7)\).

Second, one of the main points of view taken in this work is that different monomials with the same \(D\) should be treated as “of the same order”. This is also important when one tries to build ghostfree combinations (polynomials) of various scalar-tensor monomials. In fact, only the polynomials that are linear combinations of the monomials with the same \(D\) can be possibly tuned to be degenerate, and therefore to be ghostfree.

As a simple and illustrative example, let us recall that \(L^H_4\) of the Horndeski theory with \(G_4 = X^2\) is given in eq. \((1)\). The first term of \(L^H_4\) corresponds to \((c_0, c_1, \cdots ; d_1, d_2, \cdots) = (1, 0, \cdots ; 4, 0, \cdots)\) and the last two terms correspond to \((c_0, c_1, \cdots ; d_1, d_2, \cdots) = (0, 0, \cdots ; 2, 2, \cdots)\). They all correspond to \(D = 6\) and thus arise together with finely tuned coefficients to make a ghostfree polynomial.

This counting, although applicable by itself, does not fully capture the crucial structure of the monomials as well as of their combinations, which is encoded in the higher derivatives of the scalar field. In fact, the number of the first order derivative \(d_1\) is not crucial as the first order derivative will not affect the degeneracy structure of the theory. Let us take \(L^H_4\) in eq. \((1)\) as an example again. Alternatively, we may think of this specific Lagrangian in the form
\[
L^H_4 = X^2 \times \left[ R + \frac{2}{X} \left[ (\Box \phi)^2 - (\nabla \phi \nabla \phi)^2 \right] \right],
\]
where in the curly bracket, all the monomials are recast in the form such that the dimensions of the scalar field are completely cancelled\(^5\). The trick is simply to divide each derivative term of the scalar field by the factor \(\sim \nabla \phi\), i.e., schematically \(\nabla \nabla \phi \rightarrow \frac{1}{\nabla \phi} \nabla \phi \nabla \phi, \nabla \nabla \phi \rightarrow \frac{1}{\nabla \phi} \nabla \nabla \phi\), etc. As a result, instead of using \(D\) directly, it is convenient to define another number
\[
d \equiv D - (d_1 + d_2 + d_3 + d_4 + \cdots),
\]
that is
\[
d \equiv \sum_{n=0} [(n + 2)c_n + (n + 1) d_{n+2}] .
\]
Since \(d_1\) completely drops out in \(d\), from now on we may suppress \(d_1\) and write \((c_0, c_1, \cdots ; d_2, d_3, \cdots)\).

As being mentioned in the Introduction, there exist higher derivative scalar-tensor theories where the ghost can be made invisible when the scalar field is timelike. With a timelike scalar field, it is possible to fix the so-called unitary gauge such that the generally covariant scalar-tensor theories can be written in terms of the spatially covariant gravity theories [19, 20], in which the basic building blocks are the spatially covariant tensors, such as the spatial metric \(h_{ab}\), the extrinsic curvature \(K_{ab}\) and the intrinsic curvature \(\bar{R}_{ab}\) as well as

\(^4\) We use the word “monomial” since it is a scalar built of the products of several tensors with all the indices being contracted. Of course, in the sense of tensor components, each monomial is actually a summation of many terms.

\(^5\) That is, each monomial is invariant under \(\phi \rightarrow \lambda \phi\) with constant \(\lambda\).
their spatial and temporal derivatives. The spatially covariant theories of gravity can be traced back to the study of the effective field theory (EFT) of inflation [21, 22] and of the Hořava gravity [23, 24] (see also Appendix), which were also extended in the study of the EFT of dark energy [25-32], of non-singular cosmology [33, 34] and of the teleparallel gravity [35]. The spatially covariant gravity theories with at most three degrees of freedom have been generalized by including the kinetic term of the lapse function [36, 37], and by including a non-dynamical scalar field [38]. Inversely, one can derive the generally covariant scalar-tensor theories from the spatially covariant geometric quantities by the so-called “Stückelberg trick” [39-42] (see also ref. [43]). The classification of the monomials in this work makes the connection between the generally covariant scalar-tensor theories and the spatially covariant gravity theories transparent [44].

In this work, we consider monomials up to \( d = 4 \). It is thus eligible to consider \( c_n \)'s up to \( c_2 \) and \( d_n \)'s up to \( d_4 \). We classify all the possible categories of a given \( d \) with different \( c_n \)'s and \( d_n \)'s. The results are summarized in Table 1.

Comments are in order.

1. We split all the categories into two cases, which we dub the irreducible and reducible cases, respectively. The monomials belonging to the reducible case can be reduced to (the linear combinations of) the monomials belonging to the irreducible case. In the right column of Table 1, there are 6 reducible categories. For example, the category \((0, 0, 0; 0, 1, 0)\), which corresponds to the monomials of the schematic form

\[
\nabla \phi \cdots \nabla \phi \nabla \nabla \nabla \phi,
\]

can always be reduced by

\[
\nabla \phi \cdots \nabla \phi \nabla \nabla \nabla \phi \equiv \nabla \phi \cdots \nabla \phi \nabla \nabla \nabla \nabla \phi,
\]

up to the total derivatives. In this sense we refer to the category \((0, 0, 0; 0, 1, 0)\) as being reducible and schematically write

\[
(0, 0, 0; 0, 1, 0) \equiv (0, 0, 0; 2, 0, 0).
\]

It is easy to verify that all the 6 categories in the right column in Table 1 can be reduced by integrations by parts.

2. We notice that all the irreducible categories in Table 1 have \( c_1 = c_2 = d_1 = 0 \). For the sake of brevity, we therefore suppress \( c_1, c_2, d_1 \) in their notation and use 3 integers to denote

\[
(c_0; d_2, d_3) \equiv (c_0, 0, 0; d_2, d_3, 0)
\]

in the middle column of Table 1 and in the rest part of this paper. Thanks to this fact, up to \( d = 4 \) we do not need to consider the fourth order derivatives of the scalar field, nor the derivatives of the Riemann tensor.

3. The necessity of including the third order derivatives of the scalar field \( \nabla \nabla \nabla \phi \) becomes transparent according to Table 1. On one hand, the monomials with \( \nabla \nabla \nabla \phi \) can be of the same order as the monomials with derivatives only up to the second order. For example, \((0; 1, 1)-(0; 3, 0)\) and \((0; 2, 1)-(0; 0, 2)-(0; 0, 2)-(0; 0, 4, 0)\). On the other hand, as being discussed in the Introduction, the monomials built of the higher derivatives of the scalar field can be viewed as the complementary terms of the monomials built of the Riemann tensor together with its direct couplings with the scalar field. For example, \( \mathcal{L}^H_4 \) shows \((0; 2, 0)-(1; 0, 0)\) and \( \mathcal{L}^H_5 \) shows \((0; 3, 0)-(1; 2, 0)\). While the fact that \((0; 1, 1)-(1; 1, 0)\) and \((0; 2, 1)-(0; 0, 2)-(1; 2, 0)-(2; 0, 0)-(1; 1, 0)\) implies that there might be more general combinations of the higher derivative terms of the scalar field and the higher curvature terms that can be ghostfree. In particular, the monomials that are of the quadratic order in the curvature tensor were considered in refs. [15, 16]. The classification in this work indicates that the terms built of the higher derivatives of the scalar field may act as the complementary terms such that novel ghostfree combinations may arise.

In the rest part of this paper, we construct all the monomials for each category in the irreducible case according to Table 1.

### 3 \( d = 1 \)

The case of \( d = 1 \) is simple, which we will use to illustrate our strategy and formalism. For \( d = 1 \), according to Table 1, there is only one irreducible category \((0; 1, 0)\), corresponding to the monomials in

| \( d \) | Irreducible \((c_0; d_2, d_3)\) | Reducible \((c_0, c_1, c_2; d_2, d_3, d_4)\) |
| --- | --- | --- |
| 1 | \((0; 1, 0)\) | - |
| 2 | \((0; 0, 2)(1; 0, 0)\) | \((0, 0, 0; 0, 1, 0)\) |
| 3 | \((0; 3, 0)(0; 1, 1)(1; 1, 0)\) | \((0, 0, 0; 0, 0, 1)(0, 1, 0; 0, 0, 0)\) |
| 4 | \((0; 4, 0)(0; 2, 1)(0; 0, 2)(1; 2, 0)(2; 0, 0)(1; 0, 1)\) | \((0, 0, 0; 1, 0, 1)(0, 1, 0; 1, 0, 0)(0, 0, 1; 0, 0, 0)\) |
which the second order derivatives enter linearly. There are two monomials, which are

\[ E_1^{(0;1,0)} = \frac{1}{\sigma} \Box \phi, \]

\[ E_2^{(0;1,0)} = \frac{1}{\sigma^3} \nabla_a \phi \nabla_b \phi \nabla^a \nabla^b \phi, \]

where and in what follows we define

\[ \sigma = \sqrt{2X} \]

for short.

There are several comments we would like to make.

(1) Here and throughout this paper, we use the notation \( E_{(c_0;d_2,d_3)}^{(n)} \) to denote the monomials built of \( c_0 \) Riemann curvature tensors, \( d_2 \) second order derivatives and \( d_3 \) third order derivatives of the scalar field. Again, all the indices are contracted by the metric and/or the Levi-Civita tensor.

(2) We deliberately divide the monomials by powers of \( \sigma \) such that the resulting monomials are dimensionless with respect to \( \phi \). This is especially convenient when one tries to construct ghostfree combinations of several monomials, in which the coefficients are purely numerical constants.

(3) Both monomials \( E_1^{(0;1,0)} \) and \( E_2^{(0;1,0)} \) cannot be further factorized. That is, they cannot be reduced in terms of the products of more than one monomials. In this sense we dub them “unfactorizable” (or “prime”) monomials. Throughout this paper we shall concentrate on these unfactorizable monomials.

Although it is simple for \( d = 1 \), when \( d \) becomes large the number of the corresponding monomials (even the unfactorizable ones) becomes huge. It is helpful to derive diagrammatic representations for these monomials, which are similar to those employed in the study of the tensor networks (see (e.g.,) refs. [45, 46] for reviews). For the covariant derivatives of the scalar field, we denote

\[ \nabla_a \phi = \bullet_a, \]

\[ \nabla_b \nabla_a \phi = \bullet_a \bullet_b, \]

\[ \nabla_c \nabla_b \nabla_a \phi = \bullet_a \bullet_b \bullet_c, \]

where a black dot stands for the scalar field \( \phi \) and each leg stands for one derivative \( \nabla \). For the Riemann tensor and the Levi-Civita tensor, we denote

\[ R_{abcd} = \bullet_c \bullet_d \bullet_a \bullet_b, \]

\[ \varepsilon_{abcd} = \bullet_c \bullet_d \bullet_a \bullet_b, \]

where each leg stands for one spacetime index. In this work, we do not need to consider the covariant derivatives of the Riemann tensor, which may be represented by drawing more legs in the above diagram. The contraction between two indices is thus represented by connecting two legs.

According to these simple rules, the diagrammatic representations of the two monomials \( E_1^{(0;1,0)} \) and \( E_2^{(0;1,0)} \) are shown in Figure 1.

Clearly, we may dub the first and the second diagrams the “loop” and “tree” diagrams, respectively.

4 \( d = 2 \)

In this section, we derive the monomials of \( d = 2 \). We first derive the unfactorizable monomials for each category according to Table 1.

4.1 Monomials

4.1.1 \((0;2,0)\)

For the category \((0; 2, 0)\), there are 2 unfactorizable monomials, which are

\[ E_1^{(0;2,0)} = \frac{1}{\sigma^2} \nabla_a \nabla_b \phi \nabla^a \nabla^b \phi, \]

\[ E_2^{(0;2,0)} = \frac{1}{\sigma^4} \nabla^a \phi \nabla^b \phi \nabla_a \phi \nabla_b \phi. \]

In fact, for all the categories \((0; n, 0)\) with \( n = 1, 2, 3, 4 \), there are only 2 monomials that cannot be factorized. On the other hand, the products of the two monomials of \( d = 1 \) also yield the monomials of \( d = 2 \). In our notation, this can be explained as the decomposition

\[ (0; 2, 0) = (0; 1, 0) + (0; 1, 0), \]

where each leg stands for one spacetime index. In this work, we do not need to consider the covariant derivatives of the Riemann tensor, which may be represented by drawing more legs in the above diagram. The contraction between two indices is thus represented by connecting two legs.

Figure 1  Diagrammatic representations of the monomials of \( d = 1 \).
which implies that the products of two monomials $E_n^{(0,1,0)} E_n^{(0,1,0)}$ are also of the category $(0; 2, 0)$. Since there are 2 unfactorizable monomials of $(0; 1, 0)$, there are $\frac{2(2+1)}{2} = 3$ monomials that are factorizable, i.e., can be expressed in terms of products of multiple unfactorizable monomials. In the case of $(0; 2, 0)$, these 3 factorizable monomials are

$$E_1^{(0,1,0)} E_1^{(0,1,0)}, \quad (E_2^{(0,1,0)})^2,$$

where $E_1^{(0,1,0)}$ and $E_2^{(0,1,0)}$ are defined in eqs. (10) and (11), respectively. For the sake of briefness, here and throughout this paper, we do not write the explicit expressions for these factorizable monomials, which are not important for our purpose.

In total, there are 5 monomials of $(0; 2, 0)$, which are consistent with the result in ref. [47] (see eq. (2.7)).

4.1.2 $(0; 0,1)$

There are 3 monomials that are not factorizable, which are

$$E_1^{(0,0,1)} = \frac{1}{\sqrt{2}} \nabla^a \phi \nabla_a \phi, \quad (17)$$

$$E_2^{(0,0,1)} = \frac{1}{\sqrt{2}} \nabla^a \phi \nabla_a \phi, \quad (18)$$

$$E_3^{(0,0,1)} = \frac{1}{\sqrt{2}} \nabla^a \phi \nabla^b \phi \nabla_a \nabla_b \phi. \quad (19)$$

At the order of $d = 2$, i.e., if the above monomials enter the Lagrangian linearly with the coefficients being functions of $X$, they can be reduced by the integrations by parts, as shown in Table 1. Here we derive their expressions, which will be used to construct factorizable monomials of $d = 3$ and $d = 4$.

4.1.3 $(1; 0,0)$

This category involves the Riemann curvature tensor, which implies the direct coupling between the curvature and the derivatives of the scalar field. There are 2 unfactorizable monomials, which are

$$E_1^{(1,0,0)} = R, \quad (20)$$

$$E_2^{(1,0,0)} = \frac{1}{\sqrt{2}} R_{ab} \nabla^a \phi \nabla_b \phi. \quad (21)$$

For later convenience, noting by $[\nabla, \nabla_a] \phi = R_{ab} \nabla^b \phi$, we have

$$E_2^{(0,0,1)} = E_1^{(0,0,1)} + E_2^{(1,0,0)}.$$

We emphasize that this is an equality, instead of the integration by parts. The above fact indicates explicitly that direct couplings between the curvature and the derivatives of the scalar field are of the same importance as the derivatives of the scalar field higher than the second order.

4.2 Complete basis

We are now in the position to derive a set of monomials such that any polynomial of $d = 2$ can be expressed as a linear combination of the monomials in this set. For this reason, we refer to this set of monomials as the “complete basis”. Since the categories belonging to the reducible case in Table 1 can be suppressed from the beginning, we consider the complete basis for the irreducible case only. We emphasize that the “completeness” is in the sense of linear combination, by taking into account the (anti)symmetries including the Bianchi identities of the Riemann tensor. At the level of the Lagrangian, there might be further reduction after performing the integrations by parts.

Since all the 3 monomials of $(0; 0, 1)$ are reducible, the complete basis of $d = 2$ consists of 4 unfactorizable monomials

$$E_1^{(0,2,0)}, \quad E_2^{(0,2,0)}, \quad E_1^{(1,0,0)}, \quad E_2^{(1,0,0)}, \quad (23)$$

Together with 3 factorizable monomials in eq. (16). As an immediate application, the Horndeski Lagrangian $\mathcal{L}_1^H$ in eq. (1) now can be written briefly as:

$$\mathcal{L}_4^H = X^2 \left( E_1^{(1,0,0)} + 4 \left( E_1^{(0,1,0)} \right)^2 - 4 E_1^{(0,2,0)} \right). \quad (24)$$

The diagrammatic representations of these 7 monomials are shown in Figure 2.

The upper 4 diagrams are the unfactorizable monomials in eq. (23), and the lower 3 diagrams are the factorizable monomials in eq. (16). Clearly, the unfactorizable monomials correspond to the “connected” diagrams, while the factorizable monomials correspond to the “disconnected” diagrams.

Note all the monomials discussed in the above are parity preserving. It is not possible to construct parity violating monomials of $d = 2$.
5 \ d = 3

In this section, we investigate the monomials of \( d = 3 \). Again, we first derive all the monomials according to Table 1, then discuss the possible linear dependence among different monomials in order to derive the complete basis of the monomials.

5.1 Monomials

5.1.1 \((0;3,0)\)

There are 2 unfactorizable monomials, which are
\[
E_1^{(0,3,0)} = \frac{1}{\sigma^3} \nabla_a \nabla_b \phi \nabla_c \phi \nabla_d \phi, \\
E_2^{(0,3,0)} = \frac{1}{\sigma^3} \phi \nabla_b \phi \nabla_c \phi \nabla_d \phi. 
\]

Since
\[
(0; 3, 0) = (0; 1, 0) + (0; 1, 0) + (0; 1, 0) \\
= (0; 1, 0) + (0; 2, 0),
\]

there must be another \( 4 + 2 \times 2 = 8 \) monomials that can be factorized, which are
\[
\begin{align*}
&\left(E_1^{(0,1,0)}\right)^3, \quad \left(E_1^{(1,0)}\right)^2 E_2^{(0,1,0)}, \\
&E_1^{(0,1,0)} \left(E_2^{(0,1,0)}\right)^2, \quad \left(E_2^{(0,1,0)}\right)^3, \\
&E_1^{(0,1,0)} E_1^{(0,2,0)}, \quad E_1^{(1,0)} E_2^{(0,2,0)}, \\
&E_2^{(0,1,0)} E_1^{(0,2,0)}, \quad E_2^{(1,0)} E_2^{(0,2,0)}. 
\end{align*}
\]

In total, these 10 terms are consistent with the result in ref. [47] (see eq. (2.8)).

5.1.2 \((0;1,1)\)

There are 5 monomials that are not factorizable
\[
\begin{align*}
&E_1^{(0,1,1)} = \frac{1}{\sigma^3} \nabla_a \phi \nabla_b \phi \nabla_c \phi \\
&E_2^{(0,1,1)} = \frac{1}{\sigma^3} \nabla_a \phi \nabla_b \phi \nabla_c \phi \nabla_d \phi, \\
&E_3^{(0,1,1)} = \frac{1}{\sigma^3} \nabla_a \phi \nabla_b \phi \nabla_c \phi \nabla_d \phi, \\
&E_4^{(0,1,1)} = \frac{1}{\sigma^3} \nabla_a \phi \nabla_b \phi \nabla_c \phi \nabla_d \phi, \\
&E_5^{(0,1,1)} = \frac{1}{\sigma^3} \nabla_a \phi \nabla_b \phi \nabla_c \phi \nabla_d \phi \nabla_e \phi. 
\end{align*}
\]

Since
\[
(0; 1, 1) = (0; 1, 0) + (0; 0, 1),
\]

there are also another \( 2 \times 3 = 6 \) monomials that can be factorized.

There is a single parity violating term
\[
F_1^{(0,1,1)} = \frac{1}{\sigma^3} \epsilon_{abcd} \nabla_a \phi \nabla_b \phi \nabla_c \phi \nabla_d \phi, 
\]
which cannot be factorized. Clearly, \( F_1^{(0,1,1)} \) can be recast in terms of the Levi-Civita tensor due to the antisymmetry of the curvature tensor and to the commutator of two covariant derivatives, and thus is not a linearly independent term. Here we show its expression for notational completeness.

5.1.3 \((1;1,0)\)

There are 3 unfactorizable monomials
\[
\begin{align*}
&E_1^{(1,1,0)} = \frac{1}{\sigma^3} \nabla_a \phi \nabla_b \phi, \\
&E_2^{(1,1,0)} = \frac{1}{\sigma^3} \nabla_a \phi \nabla_c \phi \nabla_d \phi, \\
&E_3^{(1,1,0)} = \frac{1}{\sigma^3} \nabla_a \phi \nabla_c \phi \nabla_d \phi. 
\end{align*}
\]

Since
\[
(1; 1, 0) = (1; 0, 0) + (0; 1, 0),
\]

there are another \( 2 \times 2 = 4 \) factorizable monomials.

In the case of parity violation, there is a single term
\[
F_1^{(1,1,0)} = \frac{1}{\sigma^3} \epsilon_{abcd} R_{ef} \nabla_a \phi \nabla_b \phi \nabla_c \phi \nabla_d \phi, 
\]
which has also been considered in ref. [16] (see eq. (3.12)).

5.2 Complete basis

By using the antisymmetry of the Levi-Civita tensor and the fact that the Riemann tensor arises from the commutator of two covariant derivatives, we get the following linear dependence among different monomials:
\[
\begin{align*}
&E_2^{(0,1,1)} = E_1^{(0,1,1)} + E_3^{(1,1,0)}, \\
&E_4^{(0,1,1)} = E_3^{(0,1,1)} + E_2^{(1,1,0)}. 
\end{align*}
\]

As a result, the complete basis of \( d = 3 \) consists of \( 2 + 3 + 3 = 8 \) unfactorizable monomials
\[
\begin{align*}
&E_1^{(0,3,0)}, \quad E_2^{(0,3,0)}, \\
&E_1^{(0,1,1)}, \quad E_3^{(0,1,1)}, \quad E_5^{(0,1,1)}, \\
&E_1^{(1,1,0)}, \quad E_2^{(1,1,0)}, \quad E_3^{(1,1,0)}, 
\end{align*}
\]

which together with 84+4=16 factorizable monomials\(^6\). The diagrammatic representations of the 8 unfactorizable monomials are shown in Figure 3.

---

6) We also have the linear dependence \( E_2^{(0,0,1)} = E_1^{(0,0,1)} + E_2^{(1,0,0)} \). As a result, there are actually \( 2 \times 2 = 4 \) (instead of 6) factorizable monomials of \((0;1,1)\) according to eq. (35).
In the case of parity violation, we have

\[ F_{1}^{(0,1,1)} = -\frac{1}{2} F_{1}^{(1,1,0)} , \]

thus \( F_{1}^{(1,1,0)} \) is the only linearly independent parity violating monomial of \( d = 3 \). The diagrammatic representation of \( F_{1}^{(1,1,0)} \) is shown in Figure 4.

6 \( d = 4 \)

In this section we investigate the monomials of \( d = 4 \), which are the most involved ones in this work.

6.1 Monomials

6.1.1 \((0;4,0)\)

There are 2 unfactorizable monomials, which are

\[
E_{1}^{(0,4,0)} = \frac{1}{\sigma^4} \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi \nabla^e \phi \nabla^f \phi , \tag{46}
\]

\[
E_{2}^{(0,4,0)} = \frac{1}{\sigma^6} \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi \nabla^e \phi \nabla^f \phi \nabla^g \phi \nabla^h \phi . \tag{47}
\]

Since

\[
(0;4,0) = (0;1,0) + (0;1,0) + (0;1,0) + (0;1,0) = (0;1,0) + (0;1,0) = (0;2,0) + (0;2,0) = (0;1,0) + (0;3,0) , \tag{51}
\]

there are \( 5 + 3 \times 2 + 3 + 2 \times 2 = 18 \) monomials that can be factorized, which are

\[
\begin{align*}
E_{1}^{(0,4,0)} = & E_{2}^{(0,4,0)} = & E_{3}^{(0,4,0)} = & E_{4}^{(0,4,0)} = & E_{5}^{(0,4,0)} = & E_{6}^{(0,4,0)} = & E_{7}^{(0,4,0)} = & E_{8}^{(0,4,0)} = & E_{9}^{(0,4,0)} = & E_{10}^{(0,4,0)} = & E_{11}^{(0,4,0)} = & E_{12}^{(0,4,0)} = & E_{13}^{(0,4,0)} = & E_{14}^{(0,4,0)} = & E_{15}^{(0,4,0)} = & E_{16}^{(0,4,0)} = & E_{17}^{(0,4,0)} = & E_{18}^{(0,4,0)} = \\
\end{align*}
\]
there are another $3 \times 3 + 2 \times 3 = 11$ monomials due to their length, which can be read straightforwardly.

In the case of parity violation, there are 6 unfactorizable monomials

$$F_{(0;2,1)}^{(0,2)} = \frac{1}{\sigma^2} E_{abcd} \phi \phi^\dagger \phi \phi^\dagger f \phi \phi^\dagger f \phi \phi^\dagger f \phi, \quad (65)$$

$$F_{(0;2,1)}^{(0,2)} = \frac{1}{\sigma^2} E_{abcd} \phi \phi^\dagger \phi \phi^\dagger f \phi \phi^\dagger f \phi \phi^\dagger f \phi, \quad (66)$$

$$F_{(0;2,1)}^{(0,2)} = \frac{1}{\sigma^2} E_{abcd} \phi \phi^\dagger \phi \phi^\dagger f \phi \phi^\dagger f \phi \phi^\dagger f \phi, \quad (67)$$

$$F_{(0;2,1)}^{(0,2)} = \frac{1}{\sigma^2} E_{abcd} \phi \phi^\dagger \phi \phi^\dagger f \phi \phi^\dagger f \phi \phi^\dagger f \phi, \quad (68)$$

$$F_{(0;2,1)}^{(0,2)} = \frac{1}{\sigma^2} E_{abcd} \phi \phi^\dagger \phi \phi^\dagger f \phi \phi^\dagger f \phi \phi^\dagger f \phi, \quad (69)$$

$$F_{(0;2,1)}^{(0,2)} = \frac{1}{\sigma^2} E_{abcd} \phi \phi^\dagger \phi \phi^\dagger f \phi \phi^\dagger f \phi \phi^\dagger f \phi, \quad (70)$$

In the case of parity violation, since $(0; 2, 1) = (0; 1, 0) + (0; 1, 1), there are another 2 monomials that are factorizable, i.e.,

$$E_{(0,1,0)}^{(0,1,0)} F_{(0,1,1)}^{(0,1,1)}, \quad E_{(0,1,0)}^{(0,1,0)} F_{(0,1,1)}^{(0,1,1)}, \quad (71)$$

or equivalently (recall eq. (45)),

$$E_{(0,1,0)}^{(0,1,0)} F_{(1,1,0)}^{(1,1,0)}, \quad E_{(0,1,0)}^{(0,1,0)} F_{(0,1,1)}^{(0,1,1)}, \quad (72)$$

6.1.3 $(0; 0, 2)$

There are 11 unfactorizable monomials:

$$E_{(0,0,2)}^{(0,0,2)} = \frac{1}{\sigma^2} \nabla_a C_a \phi \nabla^a C^a \phi, \quad (73)$$

$$E_{(0,0,2)}^{(0,0,2)} = \frac{1}{\sigma^2} \nabla_a C_a \phi \nabla^a C^a \phi, \quad (74)$$

$$E_{(0,0,2)}^{(0,0,2)} = \frac{1}{\sigma^2} \nabla_a C_a \phi \nabla^a C^a \phi, \quad (75)$$

$$E_{(0,0,2)}^{(0,0,2)} = \frac{1}{\sigma^2} \nabla_a C_a \phi \nabla^a C^a \phi, \quad (76)$$

$$E_{(0,0,2)}^{(0,0,2)} = \frac{1}{\sigma^2} \nabla_a C_a \phi \nabla^a C^a \phi, \quad (77)$$

$$E_{(0,0,2)}^{(0,0,2)} = \frac{1}{\sigma^2} \nabla_a C_a \phi \nabla^a C^a \phi, \quad (78)$$

$$E_{(0,0,2)}^{(0,0,2)} = \frac{1}{\sigma^2} \nabla_a C_a \phi \nabla^a C^a \phi, \quad (79)$$

$$E_{(0,0,2)}^{(0,0,2)} = \frac{1}{\sigma^2} \nabla_a C_a \phi \nabla^a C^a \phi, \quad (80)$$

$$E_{(0,0,2)}^{(0,0,2)} = \frac{1}{\sigma^2} \nabla_a C_a \phi \nabla^a C^a \phi, \quad (81)$$

$$E_{(0,0,2)}^{(0,0,2)} = \frac{1}{\sigma^2} \nabla_a C_a \phi \nabla^a C^a \phi, \quad (82)$$

$$E_{(0,0,2)}^{(0,0,2)} = \frac{1}{\sigma^2} \nabla_a C_a \phi \nabla^a C^a \phi, \quad (83)$$

Since

$$(0; 0, 2) = (0; 0, 1) + (0; 0, 1), \quad (84)$$

there are another $\frac{1}{2} (3 + 1) = 6$ monomials that are factorizable.

In the case of parity violation, there are 3 monomials:

$$F_{(0,0,2)}^{(0,0,2)} = \frac{1}{\sigma^2} E_{abcd} \phi \phi^\dagger \phi \phi^\dagger f \phi \phi^\dagger f \phi \phi^\dagger f \phi, \quad (85)$$

$$F_{(0,0,2)}^{(0,0,2)} = \frac{1}{\sigma^2} E_{abcd} \phi \phi^\dagger \phi \phi^\dagger f \phi \phi^\dagger f \phi \phi^\dagger f \phi, \quad (86)$$

$$F_{(0,0,2)}^{(0,0,2)} = \frac{1}{\sigma^2} E_{abcd} \phi \phi^\dagger \phi \phi^\dagger f \phi \phi^\dagger f \phi \phi^\dagger f \phi, \quad (87)$$

which are not factorizable. There is no factorizable monomial with parity violation, since although $(0; 0, 2) = (0; 0, 1) + (0; 0, 1)$, there is no parity violating monomial of $(0; 0, 1)$.

6.1.4 $(1; 2, 0)$

There are 7 unfactorizable monomials:

$$E_{(1,2,0)}^{(1,2,0)} = \frac{1}{\sigma^2} \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi, \quad (88)$$

$$E_{(1,2,0)}^{(1,2,0)} = \frac{1}{\sigma^2} \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi, \quad (89)$$

$$E_{(1,2,0)}^{(1,2,0)} = \frac{1}{\sigma^2} \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi, \quad (90)$$

$$E_{(1,2,0)}^{(1,2,0)} = \frac{1}{\sigma^2} \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi, \quad (91)$$

$$E_{(1,2,0)}^{(1,2,0)} = \frac{1}{\sigma^2} \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi, \quad (92)$$

$$E_{(1,2,0)}^{(1,2,0)} = \frac{1}{\sigma^2} \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi, \quad (93)$$

$$E_{(1,2,0)}^{(1,2,0)} = \frac{1}{\sigma^2} \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi, \quad (94)$$

Since

$$(1; 2, 0) = (1; 0, 0) + (0; 2, 0), \quad (95)$$

$$= (1; 0, 0) + (0; 1, 0) + (0; 1, 0), \quad (96)$$

$$= (1; 1, 0) + (0; 1, 0), \quad (97)$$

there are another $2 \times 2 + 2 \times 3 + 3 \times 2 = 16$ factorizable monomials, of which the expressions can be read straightforwardly.

In the case of parity violation, there are 8 unfactorizable monomials:

$$F_{(1,2,0)}^{(1,2,0)} = \frac{1}{\sigma^2} E_{abcd} \phi \phi^\dagger \phi \phi^\dagger f \phi \phi^\dagger f \phi \phi^\dagger f \phi, \quad (98)$$

$$F_{(1,2,0)}^{(1,2,0)} = \frac{1}{\sigma^2} E_{abcd} \phi \phi^\dagger \phi \phi^\dagger f \phi \phi^\dagger f \phi \phi^\dagger f \phi, \quad (99)$$

$$F_{(1,2,0)}^{(1,2,0)} = \frac{1}{\sigma^2} E_{abcd} \phi \phi^\dagger \phi \phi^\dagger f \phi \phi^\dagger f \phi \phi^\dagger f \phi, \quad (100)$$

$$F_{(1,2,0)}^{(1,2,0)} = \frac{1}{\sigma^2} E_{abcd} \phi \phi^\dagger \phi \phi^\dagger f \phi \phi^\dagger f \phi \phi^\dagger f \phi, \quad (101)$$
\[ F^{(1,2,0)}_5 = \frac{1}{\sigma^4} e_{abcd} R_{ef}^2 \phi \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi, \] (102)

\[ F^{(1,2,0)}_6 = \frac{1}{\sigma^4} e_{abcd} R_{ef}^2 \phi \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi, \] (103)

\[ F^{(1,2,0)}_7 = \frac{1}{\sigma^4} e_{abcd} R_{ef}^2 \phi \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi, \] (104)

\[ F^{(1,2,0)}_8 = \frac{1}{\sigma^6} e_{abcd} \phi \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi \nabla^e \phi. \] (105)

Some of these monomials have been considered in ref. [16] (see eq. (3.13)). Since \((1; 2, 0) = (1; 1, 0) + (0; 1, 0)\), there are another two monomials that are factorizable.

6.1.5 \((2; 0, 0)\)

There are 6 unfactorizable monomials:

\[ E^{(2,0,0)}_1 = R_{abcd} \phi^{abcd}, \] (106)

\[ E^{(2,0,0)}_2 = R_{ab} \phi^{ab}, \] (107)

\[ E^{(2,0,0)}_3 = \frac{1}{\sigma^2} R_{\alpha}^{\phi \beta} \phi^{\beta \gamma \delta} \nabla^\alpha \phi \nabla^\beta \phi, \] (108)

\[ E^{(2,0,0)}_4 = \frac{1}{\sigma^2} R_{\alpha \beta \gamma}^{\phi \delta} \phi^{\phi \gamma \delta} \nabla^\alpha \phi \nabla^\beta \phi, \] (109)

\[ E^{(2,0,0)}_5 = \frac{1}{\sigma^2} R_{\alpha \beta}^{\phi \gamma} \phi^{\phi \gamma \delta} \nabla^\alpha \phi \nabla^\beta \phi, \] (110)

\[ E^{(2,0,0)}_6 = \frac{1}{\sigma^2} R_{\alpha \beta}^{\phi \gamma} \phi^{\phi \gamma \delta} \nabla^\alpha \phi \nabla^\beta \phi. \] (111)

Since \((2; 0, 0) = (1; 0, 0) + (1; 0, 0),\) there are another \(\frac{1}{2} (2 + 1) = 3\) factorizable monomials.

In the case of parity violation, there are 5 unfactorizable monomials:

\[ F^{(2,0,0)}_1 = e_{abcd} R_{ef}^2 \phi \nabla^a \phi \nabla^b \phi, \] (112)

\[ F^{(2,0,0)}_2 = e_{abcd} R_{ef}^2 \phi \nabla^a \phi \nabla^b \phi, \] (113)

\[ F^{(2,0,0)}_3 = e_{abcd} R_{ef}^2 \phi \nabla^a \phi \nabla^b \phi, \] (114)

\[ F^{(2,0,0)}_4 = e_{abcd} R_{ef}^2 \phi \nabla^a \phi \nabla^b \phi, \] (115)

\[ F^{(2,0,0)}_5 = e_{abcd} R_{ef}^2 \phi \nabla^a \phi \nabla^b \phi, \] (116)

\[ F^{(2,0,0)}_6 = e_{abcd} R_{ef}^2 \phi \nabla^a \phi \nabla^b \phi. \] (117)

Some of these monomials have been considered in ref. [16] (see eq. (3.1)). There is no factorizable monomial since although \((2; 0, 0) = (1; 0, 0) + (1; 0, 0),\) there is no parity violating monomial of \((1; 0, 0).\)

6.1.6 \((1; 0, 1)\)

There are 8 unfactorizable monomials:

\[ E^{(1,0,1)}_1 = \frac{1}{\sigma^2} R_{abcd} \phi \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi, \] (118)

\[ E^{(1,0,1)}_2 = \frac{1}{\sigma^2} R_{abcd} \phi \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi, \] (119)

\[ E^{(1,0,1)}_3 = \frac{1}{\sigma^2} R_{abcd} \phi \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi, \] (120)

\[ E^{(1,0,1)}_4 = \frac{1}{\sigma^2} R_{abcd} \phi \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi, \] (121)

\[ E^{(1,0,1)}_5 = \frac{1}{\sigma^2} R_{abcd} \phi \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi, \] (122)

\[ E^{(1,0,1)}_6 = \frac{1}{\sigma^2} R_{abcd} \phi \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi, \] (123)

\[ E^{(1,0,1)}_7 = \frac{1}{\sigma^2} R_{abcd} \phi \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi, \] (124)

\[ E^{(1,0,1)}_8 = \frac{1}{\sigma^2} R_{abcd} \phi \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi. \] (125)

Since \((1; 0, 1) = (1; 0, 0) + (0; 0, 1),\) there are another \(2 \times 3 = 6\) factorizable monomials.

In the case of parity violation, there are 6 unfactorizable monomials:

\[ F^{(1,0,1)}_1 = \frac{1}{\sigma^2} e_{abcd} R_{ef}^2 \phi \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi, \] (126)

\[ F^{(1,0,1)}_2 = \frac{1}{\sigma^2} e_{abcd} R_{ef}^2 \phi \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi, \] (127)

\[ F^{(1,0,1)}_3 = \frac{1}{\sigma^2} e_{abcd} R_{ef}^2 \phi \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi, \] (128)

\[ F^{(1,0,1)}_4 = \frac{1}{\sigma^2} e_{abcd} R_{ef}^2 \phi \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi, \] (129)

\[ F^{(1,0,1)}_5 = \frac{1}{\sigma^2} e_{abcd} R_{ef}^2 \phi \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi, \] (130)

\[ F^{(1,0,1)}_6 = \frac{1}{\sigma^2} e_{abcd} R_{ef}^2 \phi \nabla^a \phi \nabla^b \phi \nabla^c \phi \nabla^d \phi. \] (131)

There is no factorizable monomial of \((1; 0, 1)\) since although \((1; 0, 1) = (1; 0, 0) + (0; 0, 1),\) there is no parity violating monomial of neither \((1; 0, 0)\) nor \((0; 0, 1).\)

6.2 Complete basis

We are now ready to derive the complete basis for the monomials of \(d = 4\). First we have to suppress those monomials that are not linearly independent after taking into account the antisymmetry of Levi-Civita tensor as well as the fact that the Riemann tensor arises from the commutator of two covariant derivatives. After some manipulations, we find the following linear dependence among various monomials:

\[ E^{(0,2,1)}_2 = E^{(0,2,1)}_1 + E^{(1,2,0)}_6, \] (132)

\[ E^{(0,2,1)}_4 = E^{(0,2,1)}_3 + E^{(1,2,0)}_6, \] (133)

\[ E^{(0,2,1)}_6 = E^{(0,2,1)}_5 + E^{(1,2,0)}_3, \] (134)

\[ E^{(0,2,1)}_8 = E^{(0,2,1)}_7 + E^{(1,2,0)}_4, \] (135)

\[ E^{(0,2,2)}_2 = E^{(0,2,2)}_1 + 2 E^{(1,0,1)}_2 + E^{(2,0,0)}_5, \] (136)

\[ E^{(0,2,2)}_3 = E^{(0,2,2)}_1 + E^{(1,0,1)}_2, \] (137)

\[ E^{(0,2,2)}_4 = E^{(0,2,2)}_1 + E^{(1,0,1)}_2, \] (138)
unfactorizable monomials, which are shown in Figure 1. Monomials are expressed as a function of the derivatives of the Riemann tensor and the scalar field. The diagrammatic representations of the 29 unfactorizable monomials are shown in Figure 5.

As the result, the complete basis for the parity preserving monomials of \( d = 4 \) consists of \( 2 + 5 + 5 + 7 + 6 + 4 = 29 \) unfactorizable monomials, which are

\[
E_{4}^{(0,0,2)} \equiv E_{4}^{(0,0,2)} - \frac{1}{2} F_{3}^{(2,0,0)},
\]
\[
E_{7}^{(0,0,2)} \equiv E_{6}^{(0,0,2)} + E_{8}^{(1,0,1)},
\]
\[
E_{9}^{(0,0,2)} \equiv E_{8}^{(0,0,2)} + E_{6}^{(2,0,0)} - 2 E_{7}^{(1,0,1)},
\]
\[
E_{10}^{(0,0,2)} \equiv E_{8}^{(0,0,2)} - E_{7}^{(1,0,1)},
\]

and
\[
E_{1}^{(1,0,1)} \equiv - \frac{1}{2} F_{3}^{(2,0,0)},
\]
\[
E_{2}^{(1,0,1)} \equiv E_{2}^{(1,0,1)} + E_{5}^{(2,0,0)},
\]
\[
E_{3}^{(1,0,1)} \equiv E_{5}^{(1,0,1)} - E_{6}^{(2,0,0)},
\]
\[
E_{6}^{(1,0,1)} \equiv E_{7}^{(1,0,1)} - E_{6}^{(2,0,0)}.
\]

As the result, the complete basis for the parity violating monomials of \( d = 4 \) consists of \( 1 + 8 + 5 + 1 = 15 \) unfactorizable monomials, which are

\[
F_{2}^{(1,0,1)} \equiv - \frac{1}{2} F_{3}^{(2,0,0)},
\]
\[
F_{3}^{(1,0,1)} \equiv \frac{1}{2} F_{4}^{(2,0,0)},
\]
\[
F_{5}^{(1,0,1)} \equiv F_{4}^{(1,0,1)} - F_{5}^{(2,0,0)},
\]
\[
F_{6}^{(1,0,1)} \equiv \frac{1}{2} F_{5}^{(2,0,0)}.
\]

Thus the complete basis for the parity violating monomials of \( d = 4 \) consists of \( 2 + 3 + 6 + 6 + 3 + 4 = 60 \) factorizable monomials, of which the expressions can be read straightforwardly. The diagrammatic representations of the 15 unfactorizable monomials are shown in Figure 6.

7 Conclusions

In this work, we investigated the necessity and possibility of extending the scalar-tensor theories by including the third or even higher order derivatives of the scalar field as well as more general couplings between the curvature and the higher derivatives of the scalar field. The ghostfree higher curvature terms have been studied in the literature [15, 16]. From the point of view of the effective field theory, the third or even higher order covariant derivatives of the scalar field are of the same importance as the higher curvature terms. Thus a full investigation of all the possible monomials built of both the higher curvature terms and the higher derivatives of the scalar field is necessary.

As described in sect. 2 in detail, we assigned each monomial with a set of integers \((c_0, c_1, \cdots ; d_2, d_3, \cdots)\), which are the numbers of different orders of the derivatives of the Riemann tensor and of the scalar fields, respectively. The hierarchy of the monomials was made according to the integer \( d \) defined in eq. (8). For each \( d \), we classified the monomials into different categories according to the integers \((c_0, c_1, \cdots ; d_2, d_3, \cdots)\). This classification is summarized in Table 1, which is one of the main results in this work. We argued that all the monomials of the same value of \( d \) are of the same order and thus should be treated in the same footing. This not only explains the natural arising of the derivatives of the scalar field beyond the second order, but also indicates
that novel ghostfree Lagrangians may appear by combining the higher curvature terms and the higher derivatives of the scalar field.

In sects. 3-6, we made a systematic and complete investigation of all the monomials for \( d = 1, 2, 3, 4 \). We concentrated on the unfactorizable monomials in the irreducible case, and derived their explicit expressions for each category \((c_0; d_2, d_3)\). Both the parity preserving and parity violating cases were discussed. The main results in this work are the complete basis of the monomials for \( d = 2, 3, 4 \) present at the end of each section. Considering the complexity and the large amount of the monomials, we also developed diagrammatic representations for the monomials, which may help us construct and classify the monomials in a transparent manner. The diagrammatic representations of the unfactorizable monomials in the complete basis were presented at the end of

Figure 5  (Color online) Diagrammatic representations of the 29 unfactorizable parity preserving monomials of \( d = 4 \).
The results derived in this work may serve as the starting point of exploring more general viable higher derivative scalar-tensor theories. Further investigations are under way.

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References

1. R. Woodard, Scholarpedia 10, 32243 (2015).
2. G. W. Horndeski, Int. J. Theor. Phys. 10, 363 (1974).
3. A. Nicolis, R. Rattazzi, and E. Trincherini, Phys. Rev. D 79, 064036 (2009), arXiv: 0811.2197.
4. C. Deffayet, X. Gao, D. A. Steer, and G. Zahariade, Phys. Rev. D 84, 064039 (2011), arXiv: 1103.3260.
5. T. Kobayashi, M. Yamaguchi, and J. Yokoyama, Prog. Theor. Phys. 126, 511 (2011), arXiv: 1105.5723.
6. J. Gleyzes, D. Langlois, F. Piazza, and F. Vernizzi, Phys. Rev. Lett. 114, 211101 (2015), arXiv: 1404.6495.
7. J. Gleyzes, D. Langlois, F. Piazza, and F. Vernizzi, J. Cosmol. Astropart. Phys. 2015, 18 (2015), arXiv: 1408.1952.
8. D. Langlois, and K. Noui, J. Cosmol. Astropart. Phys. 2016, 34 (2016), arXiv: 1510.06930.
9. H. Motohashi, K. Noui, T. Suyama, M. Yamaguchi, and D. Langlois, J. Cosmol. Astropart. Phys. 2016, 33 (2016), arXiv: 1603.09555.
10. D. Langlois, Int. J. Mod. Phys. D 28, 1942006 (2019), arXiv: 1811.06271.
11. T. Kobayashi, Rep. Prog. Phys. 82, 086901 (2019), arXiv: 1901.07183.
12. L. Buoninfante, G. Lambiase, and M. Yamaguchi, Phys. Rev. D 100, 026019 (2019), arXiv: 1812.10105.
13. A. Lue, L. Wang, and M. Kamionkowski, Phys. Rev. Lett. 83, 1506 (1999), arXiv: astro-ph/9812088.
14. R. Jackiw, and S. Y. Pi, Phys. Rev. D 68, 104012 (2003), arXiv: gr-qc/0308071.
15. N. Deruelle, M. Sasaki, Y. Sendouda, and A. Youssef, J. High Energy Phys. 9, 9 (2012).
16. M. Crisostomi, K. Noui, C. Charmousis, and D. Langlois, Phys. Rev. D 97, 044034 (2018), arXiv: 1710.04531.
17. A. De Felice, D. Langlois, S. Mukohyama, K. Noui, and A. Wang, Phys. Rev. D 98, 084024 (2018).
18. C. Deffayet, G. Esposito-Farèse, and A. Vikman, Phys. Rev. D 79, 084003 (2009), arXiv: 0901.1314.
19. X. Gao, Phys. Rev. D 90, 081501 (2014), arXiv: 1406.0822.
20. X. Gao, Phys. Rev. D 90, 104033 (2014), arXiv: 1409.6708.
21. P. Creminelli, M. A. Luty, A. Nicolis, and L. Senatore, J. High Energy Phys. 2006, 80 (2006), arXiv: hep-th/0606090.
22. C. Cheung, A. L. Fitzpatrick, J. Kaplan, L. Senatore, and P. Creminelli, J. High Energy Phys. 2008, 14 (2008), arXiv: 0709.0293.
23. P. Hofta, Phys. Rev. D 79, 084008 (2009), arXiv: 0901.3775.
24. D. Blas, O. Pujolás, and S. Sibiryakov, Phys. Rev. Lett. 104, 181302 (2010), arXiv: 0909.3525.
25. P. Creminelli, G. D’Amico, J. Noreña, and F. Vernizzi, J. Cosmol. Astropart. Phys. 2009, 18 (2009), arXiv: 0811.0827.
26. G. Gabitosi, F. Piazza, and F. Vernizzi, J. Cosmol. Astropart. Phys. 2013, 32 (2013), arXiv: 1210.0201.
27. J. Bloomfield, E. E. Flanagan, M. Park, and S. Watson, J. Cosmol. Astropart. Phys. 8, 10 (2013), arXiv: 1211.7054.
28. J. Gleyzes, D. Langlois, F. Piazza, and F. Vernizzi, J. Cosmol. Astropart. Phys. 8, 25 (2013), arXiv: 1304.4840.
29. J. Bloomfield, J. Cosmol. Astropart. Phys. 12, 44 (2013), arXiv: 1304.6712.
30. J. Gleyzes, D. Langlois, and F. Vernizzi, Int. J. Mod. Phys. D 23,
Appendix A brief review of the Hořava gravity and the Einstein-Aether theory

There exist higher derivative scalar-tensor theories that are ghostfree when the scalar field is assumed to be timelike. These theories are highly related to the Lorentz breaking gravity theories such as the Hořava gravity [23] (see also ref. [48] for a recent review) and the Einstein-Aether theory [49].

The Hořava gravity was originally proposed to improve the renormalizability of the general relativity by introducing the higher order spatial derivatives only while keeping the temporal derivatives up to the second order. The basic operators are the spatial curvature $\mathcal{R}_{ij}$, the extrinsic curvature $K_{ij}$, the acceleration $a_i$ as well as their spatial derivatives $\nabla_i$. Hořava assumed that the full spacetime diffeomorphism is broken to $t \to t^\phi(t)$ and $x^i \to x^\phi(t, x^i)$ so that the spatial diffeomorphism remains. It is easy to show that the basic operators have the dimensions

$$[\mathcal{R}_{ij}] = 2, \quad [K_{ij}] = 3, \quad [a_i] = 1, \quad [\nabla_i] = 1,$$

under such coordinate transformations. With these basic building blocks, the total Lagrangian of the Hořava gravity is a summation of the following scalar operators order by order [50]:

$$\dim 6: \quad K_{ij} K^{ij}, K^2, 3^2 \mathcal{R}_{ij} 3^2 \mathcal{R}^{ij}, (\mathcal{R}^3)^3, 3^2 \mathcal{R}_i \mathcal{R}_k^3 \mathcal{R}_i^{3k}, (\nabla_i 3^2 \mathcal{R}_{ij})^2 \left( a_i, a_d \right), 3^2 \mathcal{R}_i \left( a_i a_d^3 \right) a_d a_i \mathcal{R}^{ij}, a_i a_d, \cdots,$$

$$\dim 5: \quad K_{ij} 3^2 \mathcal{R}^{ij}, \epsilon^{ijk} 3^2 \mathcal{R}_i \nabla_j \mathcal{R}_k, \epsilon^{ijk} a_i a_j \nabla^3 \mathcal{R}_k a_i a_j K^{ij}, \epsilon^{ijk} a_i a_j \nabla^3 \mathcal{R}_k a_i a_j K^{ij}, \cdots,$$

$$\dim 4: \quad (\mathcal{R}^2)^3, 3^2 \mathcal{R}_i 3^2 \mathcal{R}^{ij}, (a_i a_d)^2, (a_i a_d)^3 \mathcal{R}_i a_i a_j 3^2 \mathcal{R}^{ij}, \cdots,$$

$$\dim 3: \quad \epsilon^{ijk} \left( \Gamma_{il}^{\mu} \Gamma_{km}^l + \frac{2}{3} \Gamma_{il}^l \Gamma_{jm}^m \Gamma_{lm}^n \right),$$

$$\dim 2: \quad 3^2 \mathcal{R}, a_i a_d,$$

$$\dim 1: \quad \text{None}.$$

Several extensions to the original version of the Hořava gravity have been proposed. In particular, the so-called “healthy extension” [24], which abandons the projectability condition and includes all the possible operators listed above, attracted much attention. The healthy extension of the Hořava gravity can be viewed as the (unitary) gauge-fixed version of the higher derivative scalar-tensor theories with a timelike scalar field [40, 43], in which the Lorentz invariance is spontaneously broken by the existence of the timelike scalar field. On the other hand, since the higher order spatial derivatives are allowed in the Hořava gravity, the covariant formulation of the healthy extension of the Hořava gravity will also contain higher order covariant derivatives.

This is also similar to the Einstein-Aether theory, in which the Lorentz symmetry is broken by the existence of a preferred frame defined by a timelike unit vector field $u_\mu$. The general action of the Einstein-Aether theory takes the form [51]:

$$\mathcal{L} = R + M^{\mu\nu}_{\mu\nu} \nabla_\alpha u^\alpha \nabla_\beta u^\beta + \lambda \left( u_\mu u^\mu + 1 \right),$$

with

$$M^{\mu\nu}_{\mu\nu} \equiv a_1 g_{\mu\nu}^\phi + a_2 g_{\mu\nu}^{\phi^2} + a_3 g_{\mu\nu}^\phi + a_4 u^\mu u^\nu g_{\mu\nu}. $$

The Lagrange multiplier $\lambda$ ensures that the vector field $u_\mu$ is timelike. When further assuming that $u_\mu$ is hypersurface orthogonal, i.e., $u_\mu \propto \nabla_\mu \phi$ with $\phi$ a scalar field, the resulting theory can be regarded as the projectable version of the infrared limit of the Hořava gravity when the potential for the scalar field is constant [52]. In this sense, the scalar version of the Einstein-Aether theory also provides a generally covariant formulation for the projectable Hořava gravity.