LENGTHS AND MULTIPLICITIES OF INTEGRALLY CLOSED MODULES OVER A TWO-DIMENSIONAL REGULAR LOCAL RING

VIJAY KODIYALAM AND RADHA MOHAN

Dedicated to Bill Heinzer on his seventy fifth birthday

Abstract. Let \((R, \mathfrak{m})\) be a two-dimensional regular local ring with infinite residue field. We prove an analogue of the Hoskin-Deligne length formula for a finitely generated, torsion-free, integrally closed \(R\)-module \(M\). As a consequence, we get a formula for the Buchsbaum-Rim multiplicity of \(F/M\), where \(F = M^*\).

1. Introduction

The theory of integrally closed or complete ideals in a two-dimensional regular local ring was founded by Zariski in \([Zrs1938]\). Since, then this theory has received a good deal of attention and has been refined and generalized. The first named author generalized this theory to finitely generated, torsion-free, integrally closed modules in \([Kdy1993, Kdy1995]\). While the structural cornerstones of this theory are Zariski’s product and unique factorization theorems, the basic numerical result here is the Hoskin-Deligne length formula.

This formula has several proofs beginning with the one by Hoskin in \([Hsk1956]\), through proofs by Deligne in \([Dlg1973]\), by Rees in \([Res1981]\), by Lipman in \([Lpm1987]\), by Johnston and Verma in \([JhnVrm1992]\), and by the first named author in \([Kdy1993]\) (which is based on techniques of \([Lpm1987]\) and \([Hnk1989]\)), to the one in \([DbrLrd2002]\).

In this paper we obtain an analogue of the Hoskin-Deligne formula for finitely generated, torsion-free, integrally closed modules over a two-dimensional regular local ring. A consequence of this is a formula for the Buchsbaum-Rim multiplicity of a certain finite length module associated to an integrally closed module.

We now summarise the rest of the paper. In \(\S 2\), we collect various facts and results about integrally closed modules and reductions from \([Res1987]\, about integrally closed modules and their transforms over two-dimensional regular local rings from \([Kdy1995]\) and about Buchsbaum-Rim multiplicities from \([BchRim1964]\). In \(\S 3\), we prove the analogue of the Hoskin-Deligne formula for integrally closed modules which expresses their colength in terms of those of modules contracted from the order valuation rings of various quadratic transforms of the base ring. In the final \(\S 4\) we apply our analogue of the Hoskin-Deligne formula to prove a Buchsbaum-Rim multiplicity formula for such modules.

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2. Preliminaries

2.1. Integral Closures and Reductions of Modules. We review the notions of integral closures and reductions for torsion-free modules over arbitrary Noetherian domains as developed by Rees [Res1987].

Throughout this subsection, $R$ will be a Noetherian domain with field of fractions $K$ and $M$ will be a finitely generated, torsion-free $R$-module. We denote its rank by $r r_{R}(M)$. By $M_{K}$ we denote the $r r_{R}(M)$-dimensional $K$-vector space $M \otimes_{R} K$. If $N$ is a submodule of $M$ then $N_{K}$ is naturally identified with a subspace of $M_{K}$.

Any ring between $R$ and $K$ is said to be a birational overring of $R$. For any such birational overring $S$ of $R$, we let $MS$ denote the $S$-submodule of $M_{K}$ generated by $M$. There is a canonical $R$-module homomorphism from $M \otimes_{R} S$ onto $MS$ with kernel being the submodule of $S$-torsion (equivalently $R$-torsion) elements. Hence, $M \otimes_{R} S$ modulo $S$-torsion and $MS$ are isomorphic as $S$-modules.

Let $S(M)$ denote the image of the symmetric algebra $\text{Sym}^{R}(M)$ in the algebra $\text{Sym}^{K}(M_{K})$ under the canonical map. As an $R$-algebra $S(M)$ is $\text{Sym}^{R}(M)$ modulo its ideal of $R$-torsion elements. Let $S_{n}(M)$ denote the $n$-th graded component of the positively graded $R$-algebra $S(M)$.

Similarly, let $E(M)$ denote the image of the exterior algebra $\wedge^{R}(M)$ in the algebra $\wedge^{K}(M_{K})$ under the canonical map. As an $R$-algebra $E(M)$ is $\wedge^{R}(M)$ modulo its ideal of $R$-torsion elements. Let $E_{n}(M)$ denote the $n$-th graded component of the positively graded $R$-algebra $E(M)$. Observe that if $rk(M) = r$ then $\wedge_{r}(M)$ is an $R$-module of rank 1 contained in $K$ and hence is isomorphic to an ideal of $R$. If $N$ is a submodule of $M$ of the same rank, then $E_{r}(N)$ is contained in $E_{r}(M)$ and fixing an isomorphism of $E_{r}(M)$ with an ideal of $R$, we can identify $E_{r}(N)$ as a subideal.

The following fundamental result of Rees - see Theorems 1.2 and 1.5 of [Res1987] - is the basic theorem in integral closures and reductions of modules.

**Theorem 2.1.** Let $R$ be a Noetherian domain with field of fractions $K$ and let $M$ be a finitely generated, torsion-free $R$-module of rank $r$. For an element $v \in M_{K}$, the following conditions are equivalent:

- **Valuative criterion:** $v \in MV$ for every discrete valuation ring $V$ of $K$ containing $R$.
- **Equational criterion:** The element $v \in M_{K} = \text{Sym}^{K}_{1}(M_{K})$ is integral over $S(M)$.
- **Determinantal criterion:** Under some (any) isomorphism of $E_{r}(M + Rv)$ with an ideal $I$ of $R$, the subideal $J$ corresponding to $E_{r}(M)$ is a reduction of $I$.

**Definition 2.2.** With notation as above, the element $v \in M_{K}$ is said to be integral over $M$ if the equivalent conditions of Theorem 2.1 hold. The integral closure of $M$, denoted $\overline{M}$, is the set of all elements of $M_{K}$ that are integral over $M$. The module $M$ is integrally closed if $\overline{M} = M$. A submodule $N$ is a reduction of $M$ if $M \subseteq N$. Over a local domain, a minimal reduction is one for which the minimal number of generators is minimal among all reductions.

It is clear that $M \subseteq \overline{M} \subseteq M_{K}$. In addition, Rees shows in [Res1987] that if $R$ is a Noetherian normal domain then there are natural inclusions $\overline{M} \subseteq M^{**} \subseteq M_{K}$ where $(\cdot)^{*}$ denotes the functor $\text{Hom}_{R}(\cdot, R)$. 

The following result of Rees - see Lemma 2.1 of [Res1987] - generalizes to modules the theorem that any $m$-primary ideal of a $d$-dimensional, Noetherian local ring $(R, m)$ with infinite residue field has a $d$-generated reduction where $d > 0$.

**Theorem 2.3.** Let $R$ be a $d$-dimensional, Noetherian local domain with infinite residue field and $M$ be a non-free, finitely generated, torsion-free $R$-module. Then $M$ has a minimal reduction which is generated by at most $\text{rk}(M) + d - 1$ elements. Further, a minimal generating set of a minimal reduction of $M$ forms part of a minimal generating set for $M$. In particular, when $d = 2$, $M$ has a $\text{rk}(M) + 1$ generated minimal reduction.

### 2.2. Contracted Modules and Module Transforms.

In this subsection, we summarise the results of [Kdy1999] that we will use in the sequel. Throughout this subsection, $R$ will be a two-dimensional regular local ring with maximal ideal $m$, infinite residue field $k$ and field of fractions $K$. The discrete valuation determined by the powers of $m$ is denoted $\text{ord}_R(\cdot)$ and the associated valuation ring is denoted by $V_R$ or simply by $V$.

Throughout, $M$ be a finitely generated, torsion-free $R$-module and the notations $\lambda_R(\cdot)$ and $\nu_R(\cdot)$ will denote respectively the length and minimal number of generators functions on $R$-modules. We will reserve $F$ to stand for the double dual $M^{**}$. It is a fact that $F$ is free (of rank $\text{rk}_R(M)$) and canonically contains $M$ with quotient of finite length. Further, these properties characterise $M^{**}$ up to unique isomorphism (restricting to the identity on $M$).

Let $G$ be any free module containing $M$ and of the same rank as $M$. Choose a basis for $G$ and a minimal generating set for $M$ and consider the matrix expressing this set of generators in terms of the chosen basis of $G$. Considering the elements of $G$ as column vectors we get a $\text{rk}_R(M) \times \nu_R(M)$ representing matrix for $M$. The ideal of maximal minors, i.e., $\text{rk}(M)$-sized minors, of this matrix is denoted $I_G(M)$.

It is easy to see that $I_G(M)$ is independent of the choices made and is an invariant of the imbedding $M \subseteq G$. If $G = F = M^{**}$, then we will write $I(M)$ for $I_G(M)$. If $M$ is a free module, $I(M) = R$ and if $M$ is non-free then it follows from the fact that $F/M$ is of finite length that $I(M)$ is $m$-primary. We define the order of $M$ denoted by $\text{ord}_R(I(M))$ to be $\text{ord}_R(I(M))$.

We will also have occasion to use the following simple lemma.

**Lemma 2.4.** Let $R$ be a Noetherian local domain, $G$ be a free module of finite rank and $M \subseteq G$ a submodule of rank equal to $\text{rk}_R(G)$. Then $M$ is free iff $I_G(M)$ is principal.

**Proof.** One implication being trivial, we prove the other. Suppose that $I_G(M)$ is principal. Since $R$ is local, some maximal minor of a representing matrix of $M$ generates $I_G(M)$. Say it corresponds to some $\text{rk}_R(G)$ columns of such a matrix. These generate some free submodule of $M$ of rank $\text{rk}_R(G)$. We claim that this submodule is $M$ itself. For take any other column of $M$. Write it as a linear combination of the chosen $\text{rk}_R(G)$ columns with coefficients in $K$ - the field of fractions of $R$. Consider the minor of $M$ obtained by replacing one of the $\text{rk}_R(G)$ columns by this column. On the one hand this gives the corresponding coefficient times the generating maximal minor of $I_G(M)$. On the other hand this lies in $I_G(M)$. It follows that the coefficient is in $R$. So this column lies in the $R$-submodule generated by the chosen $\text{rk}_R(G)$ columns.

$\square$
Definition 2.5. Let $M$ be a finitely generated, torsion-free $R$-module. Let $S$ be a birational overring of $R$ of the form $R[\frac{m}{F}]$ where $x$ is a minimal generator of $m$. We call $MS$ the transform of $M$ in $S$. The module $M$ is said to be contracted from $S$ if $M = MS \cap F$ regarded as submodules of $FS$.

The following proposition - see Proposition 2.5 of Kdy1995 - is a useful characterisation of contracted modules.

Proposition 2.6. With notation as above $\nu_R(M) \leq \text{ord}_R(M) + rk_R(M)$. Further, the following conditions are equivalent:

1. There exists $x \in m \setminus m^2$ such that $M$ is contracted from $S = R[\frac{m}{x}]$.
2. There exists $x \in m \setminus m^2$ such that $(M :_F x) = (M :_F m)$.
3. There exists $x \in m \setminus m^2$ such that $\lambda_R(F/(xF + M)) = \nu_R(M) - rk_R(M)$.
4. $\text{ord}_R(M) = \nu_R(M) - rk_R(M)$. 
5. For any $x \in m \setminus m^2$ such that $\text{ord}_R(M) = \lambda_R(R/(x, I(M)))$, $M$ is contracted from $S = R[\frac{m}{x}]$.

A first quadratic transform of $R$ is a ring obtained by localizing a ring of the form $S = R[\frac{m}{x}]$ (as above) at a maximal ideal containing $mS$. Such a ring is itself a two-dimensional regular local ring and we define an $n$-th quadratic transform of $R$ as a first quadratic transform of an $(n - 1)$st quadratic transform of $R$. In general, a quadratic transform of $R$ is a $n$-th quadratic transform of $R$ for some $n$. By convention, we regard $R$ itself as a quadratic transform of $R$ with $n = 0$.

By well-known results on quadratic transforms - see p392 of ZrsSmI1960 - if $T$ is a quadratic transform of $R$, there is a unique sequence of quadratic transforms, $R = T_0 \subset T_1 \subset \ldots \subset T_n = T$, where each $T_{i+1}$ is a first quadratic transform of $T_i$ for $i = 0, \ldots, n - 1$. Further, if $m = (x, y)R$, then any first quadratic transform of $R$ is either a localisation of $R[\frac{m}{x}]$ or the localisation of $R[\frac{m}{y}]$ at the maximal ideal $(y, \frac{x}{y})R[\frac{m}{x}]$.

For a finitely generated, torsion-free $R$-module $M$ and a quadratic transform $T$ of $R$, the transform of $M$ in $T$ is defined to be the module $MT$. For an ideal $I$, we consider also the related notion of proper transform, denoted $IT$, which is defined to be the $m_T$-primary ideal $x^{-\text{ord}_R(IT)}IT$, where $x \in m_T$ is a generator.

We will use the following result - see Proposition 4.3, Proposition 4.6, Theorem 5.2, Theorem 5.3, Theorem 5.4 of Kdy1995.

Theorem 2.7. If $M$ is an integrally closed module over a two-dimensional regular local ring $R$, then, for most $x \in m \setminus m^2$, $M$ is contracted from $S = R[\frac{m}{x}]$. The ideal $I(M)$ is integrally closed. All the symmetric powers of $M$ modulo $R$-torsion, $S_n(M)$, are integrally closed. The transform $MT$ of $M$ in a quadratic transform $T$ of $R$ is integrally closed. If $M$ and $N$ are both integrally closed, so is $MN$.

2.3. Buchsbaum-Rim Multiplicity. Let $R$ be a Noetherian local ring of dimension $d$. Let $P$ be an $R$-module of finite length with a free presentation

$$G \to F \to P \to 0.$$ 

Buchsbaum and Rim - see Theorem 3.1 of BchRim1964 - showed that if $R$ is a Noetherian local ring of dimension $d$ and $P$ is a finite length, non-zero $R$-module and $S(G)$ is the image of $\text{Sym}^R(G)$ in $\text{Sym}^R(F)$, then $\lambda_R(\text{Sym}^R(F)/S(G))$ is asymptotically given by a polynomial function, $p(n)$, of $n$ of degree $rk(F) + d - 1$ and that the normalized leading coefficient is independent of the presentation chosen.
Definition 2.8. With notation as above, the normalized leading coefficient of \( p(n) \), is an invariant of \( P \) and is called the Buchsbaum-Rim multiplicity of \( P \). The Buchsbaum-Rim multiplicity of the zero module is defined to be zero.

In a two-dimensional regular local ring \((R, \mathfrak{m}, k)\) with infinite residue field and a finitely generated, torsion-free \( R \)-module \( M \), we let \( e_R(M) \) denote the Buchsbaum-Rim multiplicity of \( F/M \) where \( F = M^{**} \). We will need the following result that is a consequence of Corollary 4.5 of [BchRim1964] and Proposition 3.8 of [Kdy1995].

Proposition 2.9. Let \((R, \mathfrak{m}, k)\) be a two-dimensional regular local ring with infinite residue field and \( M \subseteq F = M^{**} \) a finitely generated, torsion-free \( R \)-module with minimal reduction \( N \). Then

\[
e_R(M) = e_R(N) = \lambda_R(F/N)
\]

3. Analogue of the Hoskin-Deligne length formula

All notation in this section will be as in §2.2. In particular, \( R \) will be a two-dimensional regular local ring with maximal ideal \( \mathfrak{m} \), infinite residue field \( k \) and field of fractions \( K \) and \( M \) will be a finitely generated, torsion-free \( R \)-module with double dual \( F \). The order valuation ring of \( R \) will be denoted by \( V \). These notations will be in force in the statements of all results of this section.

3.1. On modules contracted from the order valuation ring. The goal of this subsection is to study some properties of modules contracted from the order valuation ring \( V \) of \( R \). These will form the basic building blocks in the analogue of the Hoskin-Deligne formula.

We begin with the following lemma which will be used to compute the contraction of a module extended to \( V \).

Lemma 3.1. There exists a free submodule \( C \subseteq M \) of rank equal to \( \text{rk}_R(M) \) such that \( CV \cap F = MV \cap F \). Further, \( \text{ord}_R(\text{det}(C)) = \text{ord}_R(M) \).

Proof. Consider a matrix representation of \( M \subseteq F \) as a \( k \text{rk}_R(M) \times \nu_R(M) \) matrix over \( R \) so that \( M \) is generated by the columns of this matrix. Hence so is \( MV \subseteq FV \), as a module over \( V \). Since \( V \) is a principal ideal domain and \( MV \) and \( FV \) are of equal rank, some \( k \text{rk}_R(M) \) columns of this matrix generate \( MV \). Let \( C \) be the \( R \)-submodule of \( M \) generated by these columns. Then \( C \) is free of rank \( k \text{rk}_R(M) \) and \( CV = MV \) so that \( CV \cap F = MV \cap F \). Appeal to Lemma 3.9 to see that \( \text{ord}_R(\text{det}(C)) = \text{ord}_R(M) \). \( \square \)

Lemma 3.2. Suppose that \( C \subseteq F \) are free modules of equal rank and let \( n = \text{ord}_R(\text{det}(C)) \). Then \( CV \cap F = (\mathfrak{m}^n C :_F \text{det}(C)) \).

Proof. Let \( w \in (\mathfrak{m}^n C :_F \text{det}(C)) \). Then \( w \text{det}(C) \in \mathfrak{m}^n C \subseteq \mathfrak{m}^n CV \). Since \( \text{det}(C) \) generates \( \mathfrak{m}^n V \), it follows that \( w \in CV \cap F \) showing that \( (\mathfrak{m}^n C :_F \text{det}(C)) \subseteq CV \cap F \).

To see the opposite inclusion, observe that \( \text{det}(C)(CV \cap F) = \text{det}(C)CV \cap \text{det}(C)F \). Since, \( \text{det}(C) = C \text{adj}(C) \) we have that \( \text{det}(C)CV \cap \text{det}(C)F \subseteq \mathfrak{m}^n CV \cap C = \mathfrak{m}^n C \). This proves that \( CV \cap F \subseteq (\mathfrak{m}^n C :_F \text{det}(C)) \). \( \square \)

Proposition 3.3. Suppose that \( M = MV \cap F \), \( \text{ord}_R(M) = n \) and that the residue field \( k \) of \( R \) is algebraically closed. Then, \( I(M) = \mathfrak{m}^n \).
Proof. Choose \( C \subseteq F \) as in Lemma 3.1 so that \( \text{ord}_R(\det(C)) = n \) and set \( I = I(M) \) so that \( \det(C) \in I \subseteq m^n \). Since \( I \) is \( m \)-primary, there is a smallest \( t \) so that \( m^t \subseteq I \). We will show that \( t > n \) leads to a contradiction.

Observe that since \( M = MV \cap F \) and all \( S = R[\frac{m}{x}] \) are contained in \( V \), we also have \( M = MS \cap F \) for all \( x \in \mathfrak{m}_V \mathfrak{m}^2 \). It then follows from Proposition 2.6 that \( I \) is also contracted from all \( S = R[\frac{m}{x}] \). Thus, again by Proposition 2.6, \( I : x = I : \mathfrak{m} \) for every minimal generator \( x \) of \( \mathfrak{m} \).

If now, \( t > n \), there exists \( z \in I \) of order \( t - 1 \). For instance \( z \) could be chosen to be an appropriate multiple of \( \det(C) \). Since \( k \) is assumed to be algebraically closed the image of \( z \) in the graded ring \( gr_m(R) \cong k[X,Y] \) is a product of \( t - 1 \) linear factors. Lifting back to \( R \) shows that \( z - z_1z_2\cdots z_{t-1} \in \mathfrak{m}^t \) where each \( z_k \) is a minimal generator of \( \mathfrak{m} \). Thus \( z_1\cdots z_{t-1} \in I \). Since each \( I : z_k = I : \mathfrak{m} \), it follows that \( \mathfrak{m}^{t-1} \subseteq I \), contradicting choice of \( t \).

Thus \( t \leq n \) and so \( I = \mathfrak{m}^n \). \( \square \)

Recall - see Proposition 6.8.2 of Chapter 0 of [GruDco1971] - that a local ring \((R, \mathfrak{m})\) admits a faithfully flat local overring \((\tilde{R}, \tilde{\mathfrak{m}})\) such that \( \tilde{R} = \mathfrak{m} \tilde{R} = \mathfrak{m} \) and such that \( \tilde{R} \) has algebraically closed residue field. If \( R \) is a regular local ring, then the dimension formula - see Theorem 15.1 of [Mts1986] - implies that so is \( \tilde{R} \). We will use this to drop that requirement that \( k \) be algebraically closed from Proposition 3.3.

**Proposition 3.4.** If \( M = MV \cap F \) and \( \text{ord}_R(M) = n \), then \( I(M) = \mathfrak{m}^n \).

**Proof.** Let \((\tilde{R}, \tilde{\mathfrak{m}})\) be a two-dimensionel regular local ring with algebraically closed residue field that is a faithfully flat overring of \( R \). We claim that \((\tilde{R}, \tilde{\mathfrak{m}})\) admits a faithfully flat local overring \((\tilde{\tilde{R}}, \tilde{\tilde{\mathfrak{m}}})\) such that \( \tilde{\tilde{R}} \) has algebraically closed residue field. If \( \tilde{R} \) is a regular local ring, then the dimension formula implies that so is \( \tilde{\tilde{R}} \). We will use this to drop that requirement that \( k \) be algebraically closed from Proposition 3.3. The only thing that needs verification is that \( \tilde{M} \) is contracted from \( \tilde{V} \). Choose \( C \subseteq M \) as in Lemma 3.1. By Lemma 3.2, we have \( (\mathfrak{m}^nC : \mathcal{F} \det(C)) = M \). Let \( \tilde{C} \) denote the \( \tilde{R} \)-submodule of \( \tilde{F} \) generated by \( C \). We claim that \((\tilde{\mathfrak{m}}^n\tilde{C} : \tilde{\mathcal{F}} \det(\tilde{C})) = \tilde{M} \). Given this, it follows from Lemma 3.2 that \( \tilde{M} \) is contracted from \( \tilde{V} \), as desired.

Note that \( \det(C) = \det(\tilde{C}) \) (they are both represented by the same matrix with entries regarded in \( \tilde{R} \) and \( R \) respectively). Also note that if \( \mathfrak{m}^nC \) is generated by \( f_1, \cdots, f_k \in F \), then the \( \tilde{R} \)-module generated by these (regarded as elements of \( \tilde{F} \)) is exactly \( \tilde{\mathfrak{m}}^n\tilde{C} \).

To show that \( \tilde{M} \subseteq (\tilde{\mathfrak{m}}^n\tilde{C} : \tilde{\mathcal{F}} \det(\tilde{C})) \), note that \( \det(C) \mathfrak{m} \subseteq \mathfrak{m}^nC \). Since the \( \tilde{R} \)-modules generated by \( M \) and \( \mathfrak{m}^nC \) are exactly \( \tilde{M} \) and \( \tilde{\mathfrak{m}}^n\tilde{C} \) (and \( \det(C) = \det(\tilde{C}) \)), the desired containment is clear.

To show the opposite containment, consider the sets
\[
S = \{(f, z_1, \cdots, z_k) \in F \oplus R^k : \det(C)f = z_1f_1 + \cdots + z_kf_k \}, \quad \text{and}
\]
\[
\tilde{S} = \{ (\tilde{f}, \tilde{z}_1, \cdots, \tilde{z}_k) \in \tilde{F} \oplus \tilde{R}^k : \det(\tilde{C})\tilde{f} = \tilde{z}_1f_1 + \cdots + \tilde{z}_kf_k \}.
\]
Choosing a basis of \( F \) identifies elements of \( F \) with elements of \( R^r \) (where \( r = \text{rk}_R(F) \)). Suppose that \( f \) is identified with the column vector \([x_1, x_2, \ldots, x_r]^T\) and the \( f_j \) with the column vectors \([f_{j1}, f_{j2}, \ldots, f_{jk}]^T\). Thus \( S \) (respectively \( \tilde{S} \)) is the solution set in \( R^{r+k} \) (respectively \( \tilde{R}^{r+k} \)) of the set of homogeneous linear equations

\[
\text{det}(C) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1k} \\ f_{21} & f_{22} & \cdots & f_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ f_{r1} & f_{r2} & \cdots & f_{rk} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_k \end{bmatrix}
\]

in the variables \( x_1, \cdots, x_r \) and \( z_1, \cdots, z_k \) and coefficients given by \( \text{det}(C) \) and the \( f_{ij} \). The equational criterion for flatness - see Theorem 7.6 of [Mts1986] - now implies that any element of \( \tilde{S} \) is an \( \tilde{R} \)-linear combination of elements of \( S \). In particular, projecting an element of \( \tilde{S} \) onto its \( \tilde{F} \) part gives an \( \tilde{R} \)-linear combination of projections of elements of \( S \) onto their \( F \) parts. Thus any element of \( (\tilde{m}^nC : \tilde{f} \cap \text{det}(\tilde{C})) \) is an \( \tilde{R} \)-linear combination of elements of \( (m^nC : f \cap \text{det}(C)) = M \). Since \( M \) is exactly the set of \( \tilde{R} \)-combinations of elements of \( M \), we have established the other containment and finished the proof.

\[\square\]

**Remark 3.5.** A natural question that arises from the proof of Proposition 3.3 is whether an ideal that is contracted from all \( S = R[\frac{m}{2}] \) is necessarily a power of \( m \) (even without the residue field being algebraically closed). We give an example to show that this need not be the case. Consider, for instance, the ideal \( I = (x^2, x^2y, x^2 + y^2) \) in the ring \( R = \mathbb{R}[x, y] \). It is easy to check that \( (I : x) = m^2 = (I : m) \) and so \( I \) is contracted from \( S = R[\frac{m}{2}] \). Further \( IS = x^2(x, 1 + \frac{y^2}{2})S \) which is the product of a principal ideal and the maximal ideal \( (x, 1 + \frac{y^2}{2})S \) and therefore integrally closed. It follows that \( I \) itself is integrally closed.

Further we claim that for any \( z \in m, m^2 \), we have that \( (I : z) = (I : m) \) ( = \( m^2 \)), so that \( I \) is contracted from \( S = R[\frac{m}{2}] \). To prove this claim, write \( z = \sum_{n \geq 1} p_n(x, y) \) where \( p_n \) is a homogeneous polynomial of degree \( n \) and \( p_1(x, y) \neq 0 \). Suppose that \( u \in (I : z) \). Note that \( u \in m \) and hence we may write \( u = \sum_{n \geq 1} q_n(x, y) \) where \( q_n \) is a homogeneous polynomial of degree \( n \). It suffices to see that \( u \in m^2 \), or equivalently, that \( q_1(x, y) = 0 \).

Observe that the degree 2 component of any element of \( I \) is a scalar multiple of \( x^2 + y^2 \) while the degree 2 component of \( uz \) is \( q_1(x, y)p_1(x, y) \). Since \( x^2 + y^2 \) does not factor into linear polynomials over \( \mathbb{R} \), it follows that \( q_1(x, y) = 0 \), as desired.

**Corollary 3.6.** If \( M = MV \cap F \), then for any quadratic transform \( T \) of \( R \) other than \( R \) itself, \( MT \) is a free \( T \)-module.

**Proof.** By Proposition 3.4 it follows that \( I(M) = m^n \) with \( n = \text{ord}_R(M) \). Thus any quadratic transform \( T \) of \( R \) other than \( R \) itself, \( I_{FT}(MT) = m^nT \) is principal and so by Lemma 2.4 \( MT \) is a free \( T \)-module. \[\square\]

We now have a useful characterisation of modules contracted from the order valuation ring.

**Theorem 3.7.** The following conditions are equivalent for \( M \):

1. \( M = MV \cap F \).

Proof. By Proposition 3.4 it follows that \( I(M) = m^n \) with \( n = \text{ord}_R(M) \). Thus any quadratic transform \( T \) of \( R \) other than \( R \) itself, \( I_{FT}(MT) = m^nT \) is principal and so by Lemma 2.4 \( MT \) is a free \( T \)-module. \[\square\]
(2) $M$ is contracted from every $S = R[\frac{m}{n}]$ and $I(M)$ is a power of $m$.

(3) $M$ is contracted from some $S = R[\frac{m}{n}]$ and $I(M)$ is a power of $m$.

(4) $M$ is integrally closed and $I(M)$ is a power of $m$.

Proof. (1) $\Rightarrow$ (2) holds since each $S = R[\frac{m}{n}]$ $\subseteq V$, and by an application of Proposition 3.4 while (2) $\Rightarrow$ (3) is obvious. To show that (3) $\Rightarrow$ (1), observe first that for all first quadratic transforms $T$ of $R$, the $T$-module $MT$ is free since $I_{FT}(MT) = I(M)T$ is principal. Now apply Proposition 3.4 to conclude that $M = MV \cap F$. To see that (1) $\Rightarrow$ (4) observe that the valuative criterion of Theorem 2.1 implies that $M \subseteq \overline{M} = F \cap (\cap V MV)$ over all valuation rings $V$ of $K$ containing $R$. It follows that if $M = MV \cap F$ for any one valuation ring $V \supseteq R$, then $M$ is integrally closed. The other part of (4) follows from Proposition 3.4. Finally (4) $\Rightarrow$ (3) is clear from Theorem 2.7. □

We will have occasion in the sequel to use the following corollary of Theorem 3.7.

**Corollary 3.8.** For any $M$ and any $n \geq 1$, $S_n(MV \cap F) = S_n(M)V \cap S_n(F)$.

*Proof.* Begin by noticing that the characterisation of the double dual implies that $S_n(F)$ is the double dual of $S_n(M)$. First suppose that $M$ is contracted from $V$, so that $M = MV \cap F$. In this case, we need to see that so are all $S_n(M)$, for $n \geq 1$. By Theorem 3.7, $M$ is integrally closed (with $I(M)$ a power of $m$) and then by Theorem 2.7, all $S_n(M)$ are integrally closed. Further it is easy to see that $I(S_n(M))$ is, in general, a power of $I(M)$ (with exponent given by an appropriate binomial coefficient) and so, in this case, is a power of $m$. By Theorem 3.7 again, $S_n(M)$ is contracted from $V$.

For a general $M$, it is clear that $S_n(MV \cap F) \subseteq S_n(M)V \cap S_n(F)$. For the opposite inclusion, use that $S_n(M \cap F)$ is contracted from $V$ (by the previous paragraph) and contains $S_n(M)$ (obviously) to conclude that it contains $S_n(MV \cap F)$. □

### 3.2. The analogue of the Hoskin-Deligne formula.

We recall that the Hoskin-Deligne formula for integrally closed ideals in $R$ asserts that if $I$ is an integrally closed $m$-primary ideal of $R$, then

$$\lambda_R(R(T)) = \sum_{T \supseteq R} \binom{ord_T(I(T)) + 1}{2} [T : R],$$

where the sum is over all quadratic transforms $T$ of $R$, $ord_T(\cdot)$ denotes the order in the discrete valuation ring $V_T$ associated to the maximal ideal of $T$, and $[T : R]$ is the residue field extension degree.

We begin the proof of the analogue of the Hoskin-Deligne length formula for integrally closed modules by proving a preparatory lemma for the basic inductive step.

**Lemma 3.9.** $ord_R(M) = \lambda_V(FV/MV)$.

*Proof.* By definition, $ord_R(M) = ord_R(I(M))$. To compute this, we may extend $I(M)$ to $V$ and take its colength. Note that $MV \subseteq FV$ are free modules of equal rank (= $rk_R(M)$) over the PID $V$ and so the extension of $I(M)$ to $V$ may also be described as the ideal generated by the determinant of the matrix representing a basis of $MV$ written in terms of a basis of $FV$. Now an appeal to the structure of modules over a principal ideal domain shows that the colength of the ideal generated by the determinant is the length of $FV/MV$, as desired. □
Next we prove the inductive step which holds more generally for contracted modules. One of the observations that we will use in its proof is that if $S = R[\frac{m}{x}]$ for $x \in m \setminus m^2$, then,

$$MS = \bigcup_{n \geq 0} \frac{m^n}{x^n} M,$$

regarded as subsets of $M_K$.

**Proposition 3.10.** If $M$ is contracted from $S = R[\frac{m}{x}]$ for $x \in m \setminus m^2$, then the natural map

$$\frac{MV \cap F}{M} \rightarrow \frac{(MV \cap F)S}{MS}$$

is an isomorphism (of $R$-modules).

**Proof.** Begin by observing that $M \subseteq MV \cap F \subseteq F$ and so the characterisation of the double dual from §2.2 implies that $F$ is also the double dual of $MV \cap F$. Next, observe by an application of Lemma 3.9 that $ord_R(M) = ord_R(MV \cap F)$ since both of these extend to the same submodule, namely $MV$, of $FV$.

Now, since $MV \cap F$ is contracted from $V$, it is certainly also contracted from its subring $S$ (it is easily seen that $V$ is the localisation of $S$ at its height 1 prime $mS$). We now apply Proposition 2.6 (to both $M$ and $MV \cap F$) and the above equality of orders to conclude that

$$\lambda_R(F/MV \cap F + xF) = ord_R(MV \cap F) = ord_R(M) = \lambda_R(F/M + xF),$$

or equivalently that $MV \cap F + xF = M + xF$. In particular, $MV \cap F \subseteq M + xF$, and therefore

$$(3.2)\quad MV \cap F = M + x(MV \cap F :_F x) = M + x(MV \cap F :_F m),$$

where the last equality follows from Proposition 2.6 applied to $MV \cap F$.

We will now prove by induction that

$$(3.3)\quad m^n(MV \cap F) = m^nM + x^n(MV \cap F),$$

for all $n \geq 0$. The case $n = 0$ is trivial and the basis case $n = 1$ follows easily from Equation (3.2). As for the inductive step, we have that for $n \geq 1$,

$$m^{n+1}(MV \cap F) = m^{n+1}M + x^n(m(MV \cap F)) = m^{n+1}M + x^n(mM + x(MV \cap F)) = m^{n+1}M + x^{n+1}(MV \cap F),$$

where the first equality follows from the inductive assumption and the second from the basis case.

Equation (3.3) implies that

$$\bigcup_{n \geq 0} \frac{m^n}{x^n}(MV \cap F) = \bigcup_{n \geq 0} \frac{m^n}{x^n}M + (MV \cap F),$$

and so in view of the observation preceding the statement of this proposition,

$$(MV \cap F)S = MS + (MV \cap F).$$

Hence the natural map of Equation (3.1) is surjective and its injectivity follows easily since $M$ is contracted. \qed

The next proposition restates Proposition 3.10 in terms of quadratic transforms.
Proposition 3.11. If $M$ is contracted from $S = R[\frac{m}{x}]$ for $x \in m\setminus m^2$, then,

$$\frac{MV \cap F}{M} \cong \bigoplus_T (MT)^{**}$$

where the direct sum extends over all first quadratic transforms $T$ of $R$ (and the corresponding summand vanishes except for finitely many $T$).

Proof. In view of Proposition 3.10 it is to be seen that

$$\frac{(MV \cap F)S}{MS} \cong \bigoplus_T (MT)^{**}$$

where the direct sum extends over all first quadratic transforms $T$ of $R$.

Observe that the module $N = \frac{(MV \cap F)S}{MS}$ is an $S$-module that is of finite length as an $R$-module and hence also an $S$-module. Thus its support is a set of finitely many maximal ideals, say $Q_1, \cdots, Q_k$, of $S$. Each of these maximal ideals necessarily contains $mS$, since any other maximal ideal of $S$ contracts to a height 1 prime of $R$, at which the localisation of $N$ vanishes. Hence $N$ is isomorphic to the direct sum of its localisations at these maximal ideals.

Let $Q$ be one such maximal ideal of $S$ in the support of $N$. By definition, $T = S_Q$ is a first quadratic transform of $R$. Then,

$$N_Q \cong \frac{(MV \cap F)T}{MT},$$

which is a $T$-module of finite length (being a summand of $N$). Since $(MV \cap F)T$ is a free $T$-module by Corollary 3.6 the characterisation of the double dual shows that $(MV \cap F)T = (MT)^{**}$.

We see therefore that $N$ is isomorphic to a direct sum of all $(MT)^{**}/MT$ where the direct sum ranges over all (first) quadratic transforms $T$ of $R$ that are localisations of $S$. It only remains to see that if $T$ is the first quadratic transform of $R$ that is not a localisation of $S$, then $MT$ is a free $T$-module. Thus $T = R[\frac{m}{y}]_{(y,1)}$ where $m = (x,y)R$. Since $M$ is contracted from $S$, it follows from Proposition 2.6 that so is $I(M)$ and hence from Equation (3.2) (applied to $I(M)$) that

$$m^n = I(M) + x(I(M) : m) = I(M) + xm^{n-1} \Rightarrow y^n \in I(M) + xm^{n-1},$$

where $n = ord_R(I(M)) (= ord_R(M))$. Divide by $y^n$ and read the equation in $T$ to conclude that $1 \in I(M)^T + m_T$ (where $m_T$ is the maximal ideal of $T$) and therefore that $I(M)^T = T$. Thus $I_{PT}(MT) = I(M)T$ is principal (generated by $y^n$) and so $MT$ is free by Lemma 2.4.

We now prove the following analogue of the Hoskin-Deligne length formula for integrally closed modules.

Theorem 3.12. Let $R$ be a two-dimensional regular local ring with maximal ideal $m$ and infinite residue field. Let $M$ be a finitely generated, torsion-free, integrally closed $R$-module. Then,

$$(3.4) \quad \lambda_R\left(\frac{F}{M}\right) = \sum_T \lambda_T\left(\frac{(MT)^{**}}{MV_T \cap (MT)^{**}}\right)[T : R]$$

where the sum extends over all quadratic transforms $T$ of $R$. 

Proof. The proof proceeds by induction on the length of $F/M$. If $\lambda_R(F/M) = 0$, then $M = F$ is a free $R$-module. Hence, for each quadratic transform $T$ of $R$, the module $MT$ is a free $T$-module and therefore all terms on the right in equation (3.4) also vanish.

Next, we suppose that $\lambda_R(F/M) \geq 1$. We may write

$$\lambda_R(F/M) = \lambda_R(F/M) + \lambda_R(MV \cap F/M).$$

Note that $MV \cap F \neq F$ since otherwise $MV = FV$ and so by Lemma 3.9 $ord_R(M) = 0$. This implies that $I(M) = R$ and so $M$ would be free (and therefore equal to $F$).

Since $M$ is integrally closed, it is contracted from $S = R[x]$ for some $x \in m \setminus m^2$ by Proposition 2.6. Now applying Proposition 3.11 to the last term in Equation (3.5) gives:

$$\lambda_R(F/M) = \lambda_R(F/M) + \sum_T \lambda_T((MT)^*/MT)[T : R]$$

where the sum extends over all first quadratic transforms $T$ of $R$. The multiplicative factor $[T : R]$ arises since any finite length $T$-module is also a finite length $R$-module and its lengths as $R$- and $T$-modules differ by exactly this factor.

Since $MV \cap F \neq F$, Equation (3.6) shows that $\lambda_T((MT)^*/MT) < \lambda_R(F/M)$ for any $T$ occurring in the sum. Also, by Theorem 2.7 $MT$ is an integrally closed module and so by induction we may assume that for any such $T$ we have

$$\lambda_T((MT)^*/MT) = \sum_T \lambda_T((MT)^*/MT)[T' : T].$$

where the sum extends over all quadratic transforms $T'$ of $T$.

Finally, substituting the expression from Equation (3.7) into Equation (3.6) and using the facts - see [2.2] - that (i) any quadratic transform of $R$ is either $R$ itself or a quadratic transform of a unique first quadratic transform of $R$ and (ii) multiplicativity of the residue field extension degree, we have the desired result.

Remark 3.13. We note the sense in which Theorem 3.12 is an analogue of the Hoskin-Deligne formula for ideals. The terms on the right in Equation (3.4) depend only on the modules $MV_T \cap (MT)^*$. These are all modules that are contracted from the order valuation rings of various quadratic transforms of $R$. Thus the colength (in its double dual) of an integrally closed module is expressed in terms of the colengths of modules contracted from the order valuation rings of quadratic transforms of $R$.

When $M = I$ - an $m$-primary ideal of $R$, $(MT)^*$ is just $m_{ord_T(IT)}^R T$ and so we recover the ideal form of the result.

Remark 3.14. Observe that our proof of Theorem 3.12 shows that it is more than just a numerical result. What is actually seen is that the finite length $R$-module $F/M$ has a filtration where the successive quotients are exactly those of the form $(MT)^*/(MV_T \cap (MT)^*)$ where $T$ is a quadratic transform of $R$.

An immediate corollary of the analogue of the Hoskin-Deligne formula is an expression for the Buchsbaum-Rim multiplicity in terms of those of modules contracted from the order valuation rings of various quadratic transforms of $R$. 

...
Corollary 3.15. If $M$ is an integrally closed $R$-module, then,

$$e_R(M) = \sum e_T(MV_T \cap (MT)^**) [T : R]$$

where the sum extends over all quadratic transforms $T$ of $R$.

Proof. By Theorem 2.7, for each $n \geq 1$, $S_n(M)$ is integrally closed and by applying Theorem 3.12 to $S_n(M)$ we get:

$$\lambda_R \left( \frac{S_n(F)}{S_n(M)} \right) = \sum \lambda_T \left( \frac{(S_n(M)T)^**}{S_n(M)VT \cap (S_n(M)T)^**} \right) [T : R],$$

where the sum extends over all quadratic transforms $T$ of $R$ and the second equality follows from Corollary 3.8 applied to $MT$.

Only finitely many $T$ contribute to the sum on the right and these are those for which $MV_T \cap (MT)^** \neq (MT)^**$. For any such $T$, the term corresponding to $T$ is given by a polynomial in $n$ of degree $1 + \text{rk}_RMT = 1 + \text{rk}_R(M)$ and leading coefficient $e_T(MV_T \cap (MT)^**)$, if $n$ is sufficiently large. The desired equality follows. □

4. A Multiplicity Formula for Integrally Closed Modules

The main result in this section - Corollary 4.3 - establishes a relation between the colengths and multiplicities of an integrally closed module and its ideal of minors. While the result itself refers only to a single two-dimensional regular local ring, the proof seems to need the machinery of quadratic transforms, exactly as in the proof of, say, Zariski’s product theorem. Also, note that this result is genuinely a module statement - the ideal case of this being a triviality.

Proposition 4.1. If $M$ is integrally closed with $I(M) = m^n$ and $N$ is a minimal reduction of $M$, then $\lambda_R(\frac{M}{N}) = \binom{n}{2}$.

Proof. We first reduce to the case that $M \subseteq mF$. Write $M = G \oplus \tilde{M}$ where $G$ is free and $\tilde{M}$ has no free direct summand. Taking double duals gives $F = G \oplus \tilde{F}$, with the obvious notation. Now $\tilde{M} \subseteq m\tilde{F}$ and is integrally closed. Further $I(M) = I(\tilde{M}) = m^n$ and if $\tilde{N}$ is a minimal reduction of $\tilde{M}$, then $G \oplus \tilde{N}$ is a minimal reduction of $M$. Thus it would follow that

$$\lambda_R(\frac{M}{\tilde{N}}) = \lambda_R(\frac{\tilde{M}}{\tilde{N}}) = \binom{n}{2},$$

by the reduction. Hence we assume that $M \subseteq mF$. Thus $n \geq rk_R(M) = r$, say. Note that $\nu_R(M) = r + n$.

The proof proceeds by induction on $n$. If $n = 0$, then $r = 0$, so $M = 0$ and the result is clear. If $n = 1$ and $M \neq 0$, then $r = 1$ and so $M = N = m$ and here too the result is easily verified.

Now suppose that $n > 1$. Extend a minimal generating set of $N$ (with $r + 1$ elements) to one of $M$ (with $r + n$ elements). Use these generating sets to construct minimal resolutions of $N$ and $M$ and extend the inclusion map $N \hookrightarrow M$ to the
following map of complexes:

\[
\begin{array}{cccccc}
0 & \longrightarrow & R & \xrightarrow{\Delta} & R^{r+1} & \longrightarrow & N & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R^n & \xrightarrow{A} & R^{n+r} & \longrightarrow & M & \longrightarrow & 0.
\end{array}
\]

Here \( A \) is a \((n + r) \times n\) presenting matrix of \( M \), \( I \) is the identity matrix of size \( r + 1 \), \( 0 \) is the zero matrix of size \((n - 1) \times (r + 1)\) and \( \Delta \) presents \( N \) (so that the entries of \( \Delta \) generate \( I(N) \)).

Standard homological algebra implies that the mapping cone of the map of complexes above resolves \( M/N \). Explicitly, the following complex is exact:

\[
\begin{array}{cccccc}
0 & \longrightarrow & R & \xrightarrow{(\begin{array}{c} X \\ -\Delta \end{array})} & R^n \oplus R^{r+1} & \longrightarrow & \left( \begin{array}{c} B \\ C \end{array} \right) & \longrightarrow & \frac{M}{N} & \longrightarrow & 0,
\end{array}
\]

where \( B \) is the matrix formed by the first \( r + 1 \) rows of \( A \) and \( C \) is matrix formed by the remaining rows of \( A \).

We split off free direct summands to get the following minimal resolution:

\[
0 \longrightarrow R \xrightarrow{X} R^n \xrightarrow{C} R^{n-1} \longrightarrow \frac{M}{N} \longrightarrow 0.
\]

Let \( P \) be the image of \( C \), which is a submodule of \( mR^{n-1} \) (since all entries of \( C \) are in \( m \)) with finite length quotient isomorphic to \( \frac{M}{N} \), and let \( \overline{P} \) be its integral closure (which is also a submodule of \( mR^{n-1} \), since \( mR^{n-1} \) is integrally closed). It follows from the above resolution that \( I(P) \) is generated by the entries of \( X \).

We claim that \( P \) is a minimal reduction of \( \overline{P} \) and that \( I(\overline{P}) = m^{n-1} \). To see this, note that \( rk_R(P) = n - 1 \) while \( \nu_R(P) = n = rk_R(P) + 1 \) and so \( P \) is indeed a minimal reduction of \( \overline{P} \). Next we have by Theorems 2.1 and 2.7 that \( I(\overline{P}) = I(P) \). Thus we need to see that \( I(P) \) is a reduction of \( m^{n-1} \). Clearly, \( I(P) \subseteq m^{n-1} \) since it is the ideal of maximal minors of \( C \) which has all entries in \( m \). Also the map of complexes above shows that \( AX = \left[ \begin{array}{c} \Delta \\ 0 \end{array} \right] \) and therefore that \( I(N) \subseteq mI(P) \) (since the entries of \( \Delta \) generate \( I(N) \) and those of \( X \) generate \( I(P) \)). Taking integral closures (and using Zariski’s product theorem) we get \( m^n = I(M) = I(N) \subseteq mI(\overline{P}) \). By the determinant trick, \( m^{n-1} \subseteq I(\overline{P}) = I(P) \). Thus, \( I(P) \) is a reduction of \( m^{n-1} \) establishing the claim. It now follows by induction on \( n \) that \( \lambda_R(\overline{P}) = \binom{n-1}{2} \).

Next, we claim that \( \overline{P} = mR^{n-1} \). We already know that \( \overline{P} \subseteq mR^{n-1} \). To see the opposite inclusion, first apply Theorem 3.7 to see that \( \overline{P} \) is contracted from \( V \) and then by Lemmas 3.1 and 3.2 find \( C \subseteq \overline{P} \) so that \( (m^{n-1}C) \cap R^{n-1} = \overline{P} \). Now note that \( \det(C)R^{n-1} \subseteq m^{n-2}C \) since \( \det(C) = C \det(\text{adj}(C)) \) and all entries of \( \text{adj}(C) \) are in \( m^{n-2} \), and therefore \( \det(C)mR^{n-1} \subseteq m^{n-1}C \). So \( mR^{n-1} \subseteq \overline{P} \), as needed. Thus \( \lambda_R(mR^{n-1}) = \lambda_R(\frac{mR^{n-1}}{m}) = n - 1 \).

Finally, we have

\[
\lambda_R(\frac{M}{N}) = \lambda_R(\frac{R^{n-1}}{P}) = \lambda_R(\frac{R^{n-1}}{P}) + \lambda_R(\frac{\overline{P}}{P}) = (n - 1) + \binom{n - 1}{2} = \binom{n}{2}.
\]

\( \square \)
An immediate corollary of Proposition 4.1 is the following ‘local’ version of the multiplicity formula. Here, we use ‘local’ in the sense of ‘valid for modules contracted from the order valuation ring’.

**Corollary 4.2.** If \( M = MV \cap F \) and \( n = \text{ord}(M) \), then,

\[
e_R(M) = \lambda_R \left( \frac{F}{M} \right) + \left( \frac{n}{2} \right).
\]

**Proof.** If \( N \subseteq M \) is a minimal reduction of \( M \), then, by Proposition 2.9, \( e_R(M) = \lambda_R(F/N) \). Now appeal to Proposition 4.1. \( \square \)

As an application of the analogue of the Hoskin-Deligne length formula for integrally closed modules, we now derive an interesting numerical relationship between the multiplicities and colengths of an integrally closed module \( M \) and its ideal of minors \( I(M) \).

**Corollary 4.3.** If \( M \) is integrally closed with ideal of minors \( I(M) \), then,

\[
e_R(M) = e_R(I(M)) - \lambda_R \left( \frac{R}{I(M)} \right) + \lambda_R \left( \frac{F}{M} \right).
\]

**Proof.** Note that by Theorem 2.7, \( I(M) \) is integrally closed. We have expressions for each term above in terms of modules contracted from order valuations of various quadratic transforms of \( R \). Explicitly, we have by Corollary 3.15 applied to \( M \) that,

\[
e_R(M) = \sum_T e_T(MV_T \cap (MT)^{**})[T : R],
\]

and by the same corollary applied to \( I(M) \) that,

\[
e_R(I(M)) = \sum_T e_T(I(M)V_T \cap (I(M)T)^{**})[T : R]
\]

\[
= \sum_T e_T(m_T^{\text{ord}_T(I(M)^T)})[T : R]
\]

\[
= \sum_T \text{ord}_T(I(M)^T)^2[T : R].
\]

Similarly, by Theorem 3.12 applied to \( M \) we have

\[
\lambda_R \left( \frac{F}{M} \right) = \sum_T \lambda_T \left( \frac{(MT)^{**}}{MV_T \cap (MT)^{**}} \right)[T : R],
\]

and by the same theorem applied to \( I(M) \) that

\[
\lambda_R \left( \frac{R}{I(M)} \right) = \sum_T \lambda_T \left( \frac{(I(M)T)^{**}}{I(M)V_T \cap (I(M)T)^{**}} \right)[T : R]
\]

\[
= \sum_T \left( \frac{T}{m_T^{\text{ord}_T(I(M)^T)}} \right)[T : R]
\]

\[
= \sum_T \left( \text{ord}_T(I(M)^T) + 1 \right)[T : R].
\]
To finish the proof it therefore suffices to see that
\[
e_T(MV_T \cap (MT)^{**}) = \text{ord}_T(I(M)^T)^2 - \left(\frac{\text{ord}_T(I(M)^T) + 1}{2}\right) + \lambda_T\left(\frac{(MT)^{**}}{MV_T \cap (MT)^{**}}\right)
\]
\[
= \left(\frac{\text{ord}_T(I(M)^T)}{2}\right) + \lambda_T\left(\frac{(MT)^{**}}{MV_T \cap (MT)^{**}}\right)
\]
A little thought now shows that this equality is exactly the content of Corollary 4.2 applied to the \(T\)-module \(MV_T \cap (MT)^{**}\). \(\square\)

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