ON CONSECUTIVE HAPPY NUMBERS

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Abstract. Let $e \geq 1$ and $b \geq 2$ be integers. For a positive integer $n = \sum_{j=0}^{k} a_j \times b^j$ with $0 \leq a_j < b$, define

$$T_{e,b}(n) = \sum_{j=0}^{k} a_j^e.$$

$n$ is called $(e,b)$-happy if $T_{e,b}(n) = 1$ for some $r \geq 0$, where $T_{e,b}$ is the $r$-th iteration of $T_{e,b}$. In this paper, we prove that there exist arbitrarily long sequences of consecutive $(e,b)$-happy numbers provided that $e - 1$ is not divisible by $p - 1$ for any prime divisor $p$ of $b - 1$.

1. Introduction

For an arbitrary positive integer $n$, let $T(n)$ be the sum of the squares of 10-adic digits of $n$. That is, if we write $n = \sum_{j=0}^{k} a_j \times 10^j$ with $0 \leq a_1, a_2, \ldots, a_k < 10$, then $T(n) = \sum_{j=0}^{k} a_j^2$. Also, let $T^r$ denote the $r$-th iteration of $T$, i.e.,

$$T^r(n) = T(T(\ldots T(n) \ldots)).$$

In particular, we set $T^0(n) = n$. If $T$ is iteratively applied to $n$, it is easy to see (cf. [4]) that we either get 1 or fall into a cycle

$$4 \rightarrow 16 \rightarrow 37 \rightarrow 58 \rightarrow 89 \rightarrow 145 \rightarrow 42 \rightarrow 20 \rightarrow 4.$$

We say that $n$ is a happy number if we get 1 by applying $T$ to $n$ iteratively, i.e., $T^r(n) = 1$ for some positive integer $r$.

Obviously here squares and 10-adic digits are very specialized. In general, we may replace the square by an arbitrary positive $e$-th power, and replace the base 10 by an integer $b \geq 2$. Let $T_{e,b}(n)$ be the sum of the $e$-th powers of $b$-adic digits of $n$, i.e.,

$$T_{e,b} \left( \sum_{j=0}^{k} a_j \times b^j \right) = \sum_{j=0}^{k} a_j^e.$$
where $0 \leq a_j < b$. And $n$ is called $(e,b)$-happy provided that there exists $r \geq 0$ such that $T_{e,b}^r(n) = 1$. Observe that

$$T_{e,b}(n) < (b - 1)^e(\log_b n + 1).$$

So if we iteratively apply $T_{e,b}$ to $n$, the process must reach some fixed points or cycles. And the fixed points and cycles of $T_{e,b}$ and $T_{3,b}$ ($2 \leq b \leq 10$) have been listed in [2].

In the second edition of his famous book [3, Problem E34], Guy asked whether there exists arbitrarily long sequences of consecutive $(2,10)$-happy numbers? For example, the least five consecutive $(2,10)$-happy numbers are

$$44488, 44489, 44490, 44491, 44492.$$ 

In [1], El-Sedy and Siksek gave an affirmative answer to Guy’s question. As we will see soon, a key of El-Sedy and Siksek’s proof is to find $h > 0$ such that $h + x$ is $(2,10)$-happy for each $x \in \{1, 4, 16, 20, 37, 42, 58, 89, 145\}$.

However, El-Sedy and Siksek’s result cannot be extended to every power $e$ and base $b$. In fact, assume that $p$ is a prime divisor of $b - 1$ and $e \equiv 1 \pmod{p - 1}$. Then by Fermat’s little theorem

$$T_{e,b}\left(\sum_{j=0}^{k} a_j \times b^j\right) = \sum_{j=0}^{k} a_j^e \equiv \sum_{j=0}^{k} a_j \equiv \sum_{j=0}^{k} a_j \times b^j \pmod{p},$$

that is, $T_{e,b}(n) \equiv n \pmod{p}$ for every $n$. Hence now $n$ is $(e,b)$-happy only if $n \equiv 1 \pmod{p}$. In this paper, we shall show that the above examples are the only exceptions.

**Theorem 1.1.** Let $e \geq 1$ and $b \geq 2$ be integers. Suppose that $e \not\equiv 1 \pmod{p - 1}$ for any prime divisor $p$ of $b - 1$. Then for arbitrary positive integer $m$, there exists $l > 0$ such that $l + 1, l + 2, \ldots, l + m$ are all $(e,b)$-happy.

In view of Theorem 1.1, we know that there exist arbitrarily long sequences of consecutive $(2, b)$-happy numbers if $b$ is even. For example, the least nine consecutive $(2,16)$-happy numbers are

$$65988605 + i, \quad i = 0, 1, \ldots, 8.$$ 

However, there exists no pair of consecutive $(2,12)$-happy numbers less than $2^{32} - 1$ (the maximal value of unsigned long integers).

The proof of Theorem 1.1 will be given in the next section.

2. Proof of Theorem 1.1

Let $\mathbb{Z}^+$ denote the set of all positive integers. Since $2 - 1 = 1 \mid e - 1$, below we always assume that $b$ is even. And for convenience, we abbreviate
‘(e, b)-happy’ to ‘happy’ since e and b are always fixed. The following lemma is motivated by El-Sedy and Siksek’s proof in [1].

Lemma 2.1. Let x and m be an arbitrary non-negative integers. Then for any \( r \geq 1 \), there exists a positive integer \( l \) such that

\[
T_{e,b}^r(l + y) = T_{e,b}^r(l) + T_{e,b}^r(y) = x + T_{e,b}^r(y)
\]

for each \( 0 \leq y \leq m \).

Proof. We make an induction on \( r \). When \( r = 1 \), choose a positive integer \( s \) such that \( b^s > m \) and let

\[
l_1 = \sum_{j=0}^{x-1} b^{s+j}.
\]

Clearly

\[
T_{e,b}(l_1 + y) = T_{e,b}(l_1) + T_{e,b}(y) = x + T_{e,b}(y)
\]

for any \( 0 \leq y \leq m \).

Now assume \( r > 1 \) and the assertion of Lemma 2.1 holds for the smaller values of \( r \). Since \( T_{e,b}(n) \leq (b-1)^e (\log_b n + 1) \), there exists an \( m' \) satisfying that \( T_{e,b}(y) \leq m' \) for all \( 0 \leq y \leq m \). Thus by the induction hypothesis, there exists an \( l_{r-1} \) such that

\[
T_{e,b}^{r-1}(l_{r-1} + T(y)) = T_{e,b}^{r-1}(l_{r-1}) + T_{e,b}^{r-1}(T(y)) = x + T_{e,b}^r(y).
\]

whenever \( 0 \leq y \leq m \).

Let

\[
l_r = \sum_{j=0}^{l_{r-1}-1} b^{s+j}.
\]

where \( s \) satisfies that \( b^s > m \). Then

\[
T_{e,b}^r(l_r) = T_{e,b}^{r-1}(T(l_r)) = T_{e,b}^{r-1}(l_{r-1}) = x,
\]

and for each \( 0 \leq y \leq m \)

\[
T_{e,b}^r(l_r + y) = T_{e,b}^{r-1}(T_{e,b}(l_r + y)) = T_{e,b}^{r-1}(T_{e,b}(l_r) + T(y)) = T_{e,b}^{r-1}(l_{r-1} + T(y)) = T_{e,b}^{r-1}(l_{r-1}) + T_{e,b}^r(y) = T_{e,b}^r(l_r) + T_{e,b}^r(y).
\]

Suppose that a subset \( D_{e,b} \) of positive integers satisfies that:

1. For any \( n \in \mathbb{Z}^+ \), there exists \( r \geq 0 \) such that \( T_{e,b}^r(n) \in D_{e,b} \).
2. For any \( x \in D_{e,b} \), \( T(x) \in D_{e,b} \).
3. For any \( x \in D_{e,b} \), there exists \( r \geq 1 \) such that \( T_{e,b}^r(x) = x \).
Then we say that \( D_{e,b} \) is a cycle set for \( T_{e,b} \). It is not difficult to see that \( D_{e,b} \) is finite and uniquely determined by \( e \) and \( b \). For example, \( D_{2,10} = \{1, 4, 16, 2037, 42, 58, 89, 145\} \).

**Corollary 2.1.** Let \( D_{e,b} \) is the cycle set for \( T_{e,b} \). Assume that there exists \( h \in \mathbb{Z}^+ \) such that \( h + x \) is happy for any \( x \in D_{e,b} \). Then for arbitrary \( m \in \mathbb{Z}^+ \), there exists \( l \in \mathbb{Z}^+ \) such that \( l + 1, l + 2, \ldots, l + m \) are all happy.

**Proof.** By the definition of cycle sets, there exists \( r \in \mathbb{Z}^+ \) such that \( T_{e,b}^r(y) \in D \) for all \( 1 \leq y \leq m \). Applying Lemma 2.1, we can find an \( l \in \mathbb{Z}^+ \) such that \( T_{e,b}^r(l + y) = h + T_{e,b}^r(y) \) whenever \( 1 \leq y \leq m \). Thus by noting that \( x \) is happy if and only if \( T_{e,b}^r(x) \) is happy, we are done. \( \square \)

However, in general, it is not easy to search such \( h \) for \( D_{e,b} \). With help of computers, when \( e = 2 \) and \( b = 10 \), El-Sedy and Siksek found such

\[
h = \sum_{r=1}^{233192} 9 \times 10^{r+4} + 20958
\]

by noting that \( 233192 \times 9^2 + 2^2 + T_{2,10}(958 + x) \) is happy for any \( x \in D_{2,10} \). Fortunately, the following lemma will reduce the requirement of \( h \).

**Lemma 2.2.** Let \( D_{e,b} \) is the cycle set for \( T_{e,b} \). Assume that for any \( x \in D_{e,b} \), there exists \( h_x \in \mathbb{Z}^+ \) such that both \( h_x + 1 \) and \( h_x + x \) are happy. Then there exists \( h \in \mathbb{Z}^+ \) such that \( h + x \) is happy for each \( x \in D_{e,b} \).

**Proof.** We shall prove that under the assumptions of Lemma 2.2, for any subset \( S \) of \( D_{e,b} \) with \( 1 \in S \) and \( |S| \geq 2 \), there exists \( h_S \in \mathbb{Z}^+ \) such that \( h_S + x \) is happy for any \( x \in S \).

The cases \( |S| = 2 \) are trivial. Assume that \( |S| > 2 \) and the assertion holds for any smaller value of \( |S| \). For any \( 1 \neq x \in S \), since \( h_x + 1 \) and \( h_x + x \) are happy, there exists \( r \in \mathbb{Z}^+ \) such that

\[
T_{e,b}^r(h_x + 1) = T_{e,b}^r(h_x + x) = 1,
\]

and

\[
T_{e,b}^r(h_x + y) \in D_{e,b} \quad \text{for all } y \in S
\]

by the definition of the cycle set. Let

\[
S^* := \{T_{e,b}^r(h_x + y) : y \in S\}.
\]

Then \( 1 \in S^* \subseteq D_{e,b} \) and \( |S^*| < |S| \). Thus by the induction hypothesis, we can find \( h_{S^*} \in \mathbb{Z}^+ \) such that \( h_{S^*} + T_{e,b}^r(h_x + y) \) is happy for any \( y \in S \). Also in view of Lemma 2.1, there exists \( l \in \mathbb{Z}^+ \) satisfying that

\[
T_{e,b}^r(l + h_x + y) = h_{S^*} + T_{e,b}^r(h_x + y)
\]
provided that \( y \in S \). It follows that \((l + h_x) + y\) is happy for any \( y \in S \).

All are done. \(\Box\)

**Lemma 2.3.** Suppose that for any integer \( a \), there exists a happy number \( h \) such that

\[ h \equiv a \pmod{(b-1)^e}. \]

Then for any \( x \in \mathbb{Z}^+ \), there exists an arbitrarily large happy number \( l \) such that \( l + x \) is also happy.

**Proof.** Choose \( s \in \mathbb{Z}^+ \) satisfying that \( b^s > x \) and let \( x^* = b^s - x \). Suppose that \( h \) is the happy number such that

\[ h \equiv T_{e,b}(x^*) \pmod{(b-1)^e}. \]

Note that \( hb^{\phi((b-1)^e)} \) is also happy and

\[ hb^{\phi((b-1)^e)} \equiv h \pmod{(b-1)^e} \]

where \( \phi \) is the Euler totient function. We may assume that \( h > T_{e,b}(x^*) \).

Write \( h = (b-1)^ek + T_{e,b}(x^*) \). Let

\[ l = x^* + \sum_{j=0}^{k-1} (b-1) b^{s+j}. \]

Then

\[ T_{e,b}(l) = k(b-1)^e + T_{e,b}(x^*) = h \]

and

\[ T_{e,b}(l + x) = T_{e,b}(b^s + \sum_{j=0}^{k-1} (b-1) b^{s+j}) = T_{e,b}(b^{s+k}) = 1. \]

It follows that both \( l \) and \( l + x \) are happy. \(\Box\)

**Lemma 2.4.** Let \( n \) be a positive odd integer. Then for any \( a \) with \( a \equiv 1 \pmod{n} \) and positive integer \( k \), there exists \( r \in \mathbb{Z}^+ \) such that

\[ (n+1)^r \equiv a \pmod{n^k}. \]

**Proof.** Assume that \( n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} \) where \( p_1, p_2, \ldots, p_s \) are distinct odd primes and \( \alpha_1, \alpha_1, \ldots, \alpha_s \geq 1 \). For \( 1 \leq i \leq s \), let \( g_i \) be a prime root of \( p_i^{\alpha_i k} \).

Assume that

\[ n + 1 \equiv g_i^{\beta_i} \pmod{p_i^{\alpha_i k}} \]

and \( a \equiv g_i^{\gamma_i} \pmod{p_i^{\alpha_i}} \)

for each \( 1 \leq i \leq s \). Clearly both \( \beta_i \) and \( \gamma_i \) are divisible by \( \phi(p_i^{\alpha_i}) \) since \( n + 1 \equiv a \equiv \pmod{p_i^{\alpha_i}} \). So we only need to find \( r \) satisfying that

\[ \beta_i r \equiv \gamma_i \pmod{\phi(p_i^{\alpha_i})} \]

for all \( i \), or equivalently,

\[ (\beta_i/\phi(p_i^{\alpha_i})) r \equiv \gamma_i/\phi(p_i^{\alpha_i}) \pmod{p_i^{\alpha_i(k-1)}}. \]
Note that \( p_i \nmid \beta_i / \phi(p_i^{\alpha_i}) \) since \( n + 1 \not\equiv 1 \pmod{p_i^{\alpha_i+1}} \). Thus such \( r \) always exists in view of the Chinese remainder theorem. \( \square \)

**Corollary 2.2.** Assume that for any integer \( a \), there exists a happy number \( h \) such that

\[ h \equiv a \pmod{b - 1}. \]

Then we can find a happy number \( h' \) such that

\[ h' \equiv a \pmod{(b - 1)^e}. \]

**Proof.** Suppose that

\[ \sum_{j=1}^{h-1} b^j \equiv k_1(b - 1) + a - 1 \pmod{(b - 1)^e}. \]

And suppose that

\[ b^{-h}(-k_1(b - 1) + 1) \equiv k_2(b - 1) + 1 \pmod{(b - 1)^e}. \]

In light of Lemma 2.4, there exists \( r \in \mathbb{Z}^+ \) such that

\[ b^r \equiv k_2(b - 1) + 1 \pmod{(b - 1)^e}. \]

Therefore

\[ \sum_{j=1}^{h-1} b^j + b^{h+r} \equiv \sum_{j=1}^{h-1} b^j + b^h(k_2(b - 1) + 1) \equiv a \pmod{(b - 1)^e}, \]

which is apparently happy. \( \square \)

**Lemma 2.5.** Let \( a \) be a positive integer. Assume that there exists a happy number \( h \) such that

\[ l \equiv T_{e,b}(h') \pmod{b - 1} \]

for some \( h' \equiv a \pmod{b - 1} \). Then we can find a happy number \( h \) such that

\[ h \equiv a \pmod{b - 1}. \]

**Proof.** In view of the proof of Lemma 2.2, we may assume that \( l > T_{e,b}(h') \).

And let

\[ h = \sum_{j=0}^{l-T_{e,b}(h')} b^{s+j} + h', \]

where we choose \( s \) such that \( b^s > h' \). Clearly

\[ h \equiv l - T_{e,b}(h') + h' \equiv a \pmod{b - 1}. \]

And

\[ T_{e,b}(h) = l - T_{e,b}(h') + T_{e,b}(h') = l. \]

Thus \( h \) is the desired happy number. \( \square \)


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Note that the property of $e$ is not used until now.

Proof of Theorem 1.1. Write $b - 1 = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ where $p_1, p_2, \ldots, p_s$ are distinct odd primes and $\alpha_1, \alpha_2, \ldots, \alpha_s$ are positive integers. Let $1 \leq g_i \leq p_i^{\alpha_i}$ be a primitive root of $p_i^{\alpha_i}$ for $1 \leq i \leq s$. For every positive integer $0 \leq a \leq b - 1$, let $L(a) \in \{0, 1, \ldots, b - 1\}$ be the integer such that

$$L(a) \equiv \begin{cases} a - g_i + g_i^e \pmod{p_i^{\alpha_i}} & \text{if } a \not\equiv 1 \pmod{p_i^{\alpha_i}}, \\ 1 \pmod{p_i^{\alpha_i}} & \text{if } a \equiv 1 \pmod{p_i^{\alpha_i}} \end{cases},$$

$i = 1, 2, \ldots, s.$

And let $r_a \geq 0$ be the minimal integer such that $L^r(a) \equiv 1 \pmod{b - 1}$, where $L^r$ denotes $r$-th iteration of $L$. Since $e \not\equiv 1 \pmod{p_i - 1}$, we have $g_i^e - g_i$ is prime to $p_i$ for every $i$. Hence $r_a$ always exists.

Combining Corollary 2.1, Lemma 2.2, Lemma 2.3 and Corollary 2.2, now it suffices to show that for every integer $0 \leq a \leq b - 1$ there exists a happy number $h$ such that

$$h \equiv a \pmod{b - 1}.$$ 

We use an induction on $r_a$. If $r_a = 0$, then $a \equiv 1 \pmod{b - 1}$. There is noting to do. Now assume that $r_a \geq 1$ and the assertion holds for any $a'$ with $r_a < r_a$. Clearly $r_{L(a)} = r_a - 1$. Hence by the induction hypothesis, there exists a happy number $l$ such that

$$l \equiv L(a) \pmod{b - 1}.$$ 

Let $0 \leq g \leq b - 1$ be the integer such that

$$g \equiv \begin{cases} g_i \pmod{p_i^{\alpha_i}} & \text{if } a \not\equiv 1 \pmod{p_i^{\alpha_i}}, \\ 1 \pmod{p_i^{\alpha_i}} & \text{if } a \equiv 1 \pmod{p_i^{\alpha_i}} \end{cases},$$

$i = 1, 2, \ldots, s.$

And let

$$h' = \sum_{j=1}^{a+b-1-g} b^j + g.$$ 

Then

$$h' \equiv a + b - 1 - g + g \equiv a \pmod{b - 1}.$$ 

And

$$T_{e,b}(h') \equiv a + b - 1 - g + g^e \equiv \begin{cases} a - g_i + g_i^e \pmod{p_i^{\alpha_i}} & \text{if } a \not\equiv 1 \pmod{p_i^{\alpha_i}}, \\ 1 \pmod{p_i^{\alpha_i}} & \text{if } a \equiv 1 \pmod{p_i^{\alpha_i}} \end{cases},$$

for every $1 \leq i \leq s$. Hence

$$T_{e,b}(h') \equiv L(a) \equiv l \pmod{b - 1}.$$ 

Thus in light of Lemma 2.3, we are done. \qed
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