QUASIDERIVATIVE METHOD FOR DERIVATIVE
ESTIMATES OF SOLUTIONS TO DEGENERATE ELLIPTIC
EQUATIONS

WEI ZHOU

Abstract. We introduce the concept of quasiderivative and give an example constructed by random time change, Girsanov’s Theorem and Levy’s Theorem. As an application, we investigate the smoothness and estimate the derivatives up to second order for probabilistic solutions to Dirichlet problems for linear degenerate elliptic partial differential equations of second order, under the assumption of non-degeneracy with respect to the normal to the boundary and an interior condition to control the moments of quasiderivatives, which is weaker than non-degeneracy. The probabilistic solution is the unique solution to the associated Dirichlet problem.

1. Introduction and Background

We consider the Dirichlet problem for the linear degenerate elliptic partial differential equation of second order

\begin{aligned}
\left\{ \begin{array}{ll}
Lu(x) - c(x)u(x) + f(x) = 0 & \text{in } D \\
u = g & \text{on } \partial D,
\end{array} \right.
\end{aligned}

(1.1)

where \( Lu(x) := a^{ij}(x)u_{x^i}u_{x^j}(x) + b^i(x)u_{x^i}(x) \), with \( a = (1/2)\sigma\sigma^* \), and summation convention is understood. The probabilistic solution of (1.1) is known as

\begin{equation}
C^2(D) \cap \mathbb{C} \quad (1.2)
\end{equation}

\begin{equation}
E \left[ g(x_\tau(x))e^{-\phi_\tau} + \int_0^\tau f(x_t(x))e^{-\phi_t} \, dt \right],
\end{equation}

with \( \phi_t = \int_0^t c(x_s(x)) \, ds \).

where \( x_t(x) \) is the solution to the Itô equation

\begin{equation}
x_t = x + \int_0^t \sigma(x_s) \, dw_s + \int_0^t b(x_s) \, ds
\end{equation}

(1.3)

and \( \tau = \tau_D(x) \) is the first exit time of \( x_t(x) \) from \( D \).

If we know a priori that \( u \in C^2(D) \cap \mathbb{C} \) and \( u \) solves (1.1), then \( u \) satisfies (1.2) via Itô’s formula, which implies the uniqueness of the solution of (1.1) provided the uniqueness of the solution of (1.3). However, in general, \( u \) defined by (1.2) doesn’t necessarily have first and second derivatives.
in $L$, especially when $a$ is degenerate, and the differential equation is understood in a generalized sense. We are interested in understanding under what conditions $u$ defined by (1.2) is twice differentiable and does satisfy (1.1).

The accumulation of the research on existence, uniqueness and regularity of degenerate elliptic or parabolic partial differential equations has become vast. See, for example, Hörmander [2], Kohn-Nirenberg [3] and Oleinik-Radkevich [13], in which analysis techniques for PDEs are used. For probabilistic approaches, we refer to Freidlin [1] and Stroock-Varadhan [14], to name a few.

Our approach, quasiderivative method, is also probabilistic. The concept of quasiderivative was first introduced by N. V. Krylov in [6] (1988), in which this probabilistic technique is applied to find weaker and more flexible conditions on $\sigma$, $b$ and $c$ such that $u$ in (1.2) is twice continuously differentiable in manifolds without boundary. Since then, this technique has been applied to investigate the interior smoothness of solutions of various elliptic and parabolic partial differential equations. The first derivatives of various linear elliptic and parabolic PDEs have been estimated under various conditions in Krylov [8] (1992), [9] (1993) and [11] (2004), where each case was treated by its particular choice of quasiderivatives. In Krylov [12] (2004), a unified method is presented, also based on quasiderivatives, while $\sigma$ and $b$ are assumed to be constant. As far as the applications to nonlinear equations, for example, in Krylov [7] (1989), derivative estimates are obtained when controlled diffusion processes and consequently fully nonlinear elliptic equations are considered.

Compared to the operators considered in [6, 8, 9, 11, 12], the differential equation in this article is more general. $L$ in (1.1) is the general linear elliptic differential operator, and $c$ and $f$ are non-trivial. Also, we estimate the derivatives up to the second order, not just the first order. More precisely, our main target is investigating first derivatives of $u$ if we only assume $f, g \in C^{0,1}(\bar{D})$, as well as the second derivatives therein when assuming $f, g \in C^{1,1}(\bar{D})$. Note that, in these cases, one cannot assert that the first and second derivatives of $u$ are bounded up to the boundary (for example, see Remark 1.0.2 and Example 4.2.1 in [11]). One can only expect to prove that inside $D$ the derivatives of $u$ exist. We show that under our assumptions, the first and second derivatives of $u$ in (1.2) exist almost everywhere in $D$, which implies the existence and uniqueness for degenerate boundary value problem (1.1) in our setting. We also obtain first and second derivative estimates.

This article is organized as follows: In Section 2, we review the concept of quasiderivative and give an example of it. In Section 3, we take this approach to show the existence of, and then estimate, the first and second derivatives of $u$ in (1.2), under the assumption of the non-degeneracy of $a$ with respect to the normal to the boundary and an interior condition to control the moments of quasiderivatives, which is weaker than the nondegeneracy of $a$. 
To conclude this section, we introduce the notation: Above we have already defined $C^k(\bar{D})$, $k = 1$ or 2, as the space of bounded continuous and $k$-times continuously differentiable functions in $\bar{D}$ with finite norm given by

$$|g|_{k,D} = |g|_{0,D} + |g_x|_{0,D},$$

respectively, where

$$|g|_{0,D} = \sup_{x \in D} |g(x)|,$$

$g_x$ is the gradient vector of $g$, and $g_{xx}$ is the Hessian matrix of $g$. For $\alpha \in (0, 1]$, the Hölder spaces $C^{k,\alpha}(\bar{D})$ are defined as the subspaces of $C^k(\bar{D})$ consisting of functions with finite norm

$$|g|_{k,\alpha,D} = |g|_{k,D} + |g|_{\alpha,D},$$

where $|g|_{\alpha,D} = \sup_{x,y \in D} \frac{|g(x) - g(y)|}{|x - y|^{\alpha}}$.

$\mathbb{R}^d$ is the $d$-dimensional Euclidean space with $x = (x^1, x^2, \ldots, x^d)$ representing a typical point in $\mathbb{R}^d$, and $(x, y) = \sum_{i=1}^d x^i y^i$ is the inner product for $x, y \in \mathbb{R}^d$. For $x, y, z \in \mathbb{R}^d$, set

$$u(y) = \sum_{i=1}^d u_i y^i, \quad u(y)(z) = \sum_{i,j=1}^d u_{ij} z^i y^j, \quad u_i^2(y) = (u_i(y))^2.$$

For any matrix $\sigma = (\sigma_{ij})$,

$$||\sigma||^2 := \text{tr} \sigma \sigma^* = \sum_{i,j} (\sigma_{ij})^2.$$

For any $s, t \in \mathbb{R}$, we define

$$s \wedge t = \min(s, t), \quad s \vee t = \max(s, t).$$

Constants $K, N$ and $\lambda$ appearing in inequalities are usually not indexed. They may differ even in the same chain of inequalities.

2. Definition and Examples of Quasiderivative

In what follows, we consider the Itô stochastic equation

$$(2.1) \quad x_t = x + \int_0^t \sigma^i(x_s)dw^i_s + \int_0^t b(x_s)ds$$

on a given complete probability space $(\Omega, \mathcal{F}, P)$, where $x \in \mathbb{R}^d$, $\sigma^i$ and $b$ are (nonrandom) $\mathbb{R}^d$-valued functions with bounded domain $D$ in $\mathbb{R}^d$, defined for $i = 1, \ldots, d_1$ with $d_1$ possibly different from $d$, and $w_t := (w_1^t, \ldots, w_{d_1}^t)$ is a $d_1$-dimensional Wiener process with respect to a given increasing filtration $\{\mathcal{F}_t, t \geq 0\}$ of $\sigma$-algebras $\mathcal{F}_t \subset \mathcal{F}$, such that $\mathcal{F}_t$ contain all $P$-null sets. We denote by $\sigma$ the $d \times d_1$ matrix composed of the column-vectors $\sigma^i$, $i = 1, \ldots, d_1$. We also assume that $\sigma$ and $b$ are twice continuously differentiable in $\mathbb{R}^d$. Based on the assumptions above, for any $x \in D$, it is known that equation (2.1) has a unique solution $x_t(x)$ on $[0, \tau(x))$, where

$$\tau(x) = \inf\{t \geq 0 : x_t(x) \notin D\} \quad (\inf\emptyset := \infty).$$
Definition 2.1. We write
\[ u \in \mathcal{M}^k(D, \sigma, b) \]
if \( u \) is a real-valued \( k \) times continuously differentiable function given on \( D \) such that the process \( u(x_t(x)) \) is a local \( \{ \mathcal{F}_t \} \)-martingale on \([0, \tau(x))\) for any \( x \in D \).

We abbreviate \( \mathcal{M}^k(D, \sigma, b) \) by \( \mathcal{M}^k(D) \), or simply \( \mathcal{M}^k \) when this will cause no confusion.

Definition 2.2. Let \( x \in D \), and let \( \tau \) be a stopping time, \( \tau \leq \tau(x) \). Assume that \( \xi \in \mathbb{R}^d \), \( \xi_t \) and \( \xi^0_t \) are adapted continuous processes defined on \([0, \tau] \cap [0, \infty)\) with values in \( \mathbb{R}^d \) and \( \mathbb{R} \), respectively, such that \( \xi_0 = \xi \).

We say that \( \xi_t \) is a first quasiderivative of \( x_t(y) \) in the direction of \( \xi \) at point \( x \) on \([0, \tau)\) if for any \( u \in \mathcal{M}^1(D, \sigma, b) \) the following process
\[ u((\xi_t)_{\xi_t}')(x_t(x)) + \xi^0_t u(x_t(x)) \]
is a local martingale on \([0, \tau)\). In this case the process \( \xi^0_t \) is called a first adjoint process for \( \xi_t \). If \( \tau = \tau(x) \) we simply say that \( \xi_t \) is a first quasiderivative of \( x_t(y) \) in \( D \) in the direction of \( \xi \) at \( x \).

Definition 2.3. Under the assumptions of Definition 2.2 additionally assume that \( \eta \in \mathbb{R}^d \), \( \eta_t \) and \( \eta^0_t \) are adapted continuous processes defined on \([0, \tau] \cap [0, \infty)\) with values in \( \mathbb{R}^d \) and \( \mathbb{R} \), respectively, such that \( \eta_0 = \eta \).

We say that \( \eta_t \) is a second quasiderivative of \( x_t(y) \) associated with \( \xi_t \) in the direction of \( \eta \) at point \( x \) on \([0, \tau)\) if for any \( u \in \mathcal{M}^2(D, \sigma, b) \) the following process
\[ u((\xi_t)_{\eta_t}')(x_t(x)) + u((\eta_t)'(x_t(x)) + 2 \xi^0_t u(x_t(x)) + \eta^0_t u(x_t(x)), \]
where \( \xi_t \) and \( \xi^0_t \) are first quasiderivative and first adjoint process.
is a local martingale on \([0, \tau)\). In this case the process \( \eta^0_t \) is called a second adjoint process for \( \eta_t \). If \( \tau = \tau(x) \) we simply say that \( \eta_t \) is a second quasiderivative of \( x_t(y) \) associated with \( \xi_t \) in \( D \) in the direction of \( \eta \) at \( x \).

Let us consider
\[ u(x) = E g(x_\tau(x)), \]
that is, we temporarily let \( f = c = 0 \) in \([1, 2]\). Based on the definitions above, if \( u \in C^2(D) \), then the strong Markov property of \( x_t(x) \) implies that \( u \in \mathcal{M}^2(D) \), and the usual first and second “derivatives” with respect to \( x \) of \( x_t(x) \), which are defined as the solutions of the Itô equations
\[ \xi_t = \xi + \int_0^t \sigma^k_{(\xi_s)}(x_s)dw^k_s + \int_0^t b_{(\xi_s)}(x_s)ds \]
\[ \eta_t = \eta + \int_0^t \left[ \sigma^k_{(\xi_s)}(x_s) + \sigma^k_{(\eta_s)}(x_s) \right]dw^k_s + \int_0^t \left[ b_{(\xi_s)}(x_s) + b_{(\eta_s)}(x_s) \right]ds \]
are first and second quasiderivatives with zero adjoint processes. This means, the concept “quasiderivative” is a generalization of the usual “derivative”.

Now we additionally assume that the domain $D$ is of class $C^2$ with $\partial D$ bounded, $\tau(x) < \infty$ (a.s.), and $g$ is twice continuously differentiable on $\partial D$.

If the process $[2.2]$, $\xi_{\tau(x)}$ is tangent to $\partial D$ at $x_{\tau(x)}(x)$ (a.s.), then we have

$$u_{(\xi)}(x) = E[u_{(\xi)}(x_{\tau}) + \xi^0 \cdot u(x_{\tau})] = E[g_{(\xi)}(x_{\tau}) + \xi^0 g(x_{\tau})].$$

This shows how we can apply first quasiderivatives to get interior estimates of $u_{(\xi)}$ through $|g|_{1,D}$ or $|g|_{1,\partial D}$.

As far as second derivatives are concerned, first notice that

$4u_{(\xi)(\eta)}(x) = u_{(\xi+\eta)(\xi+\eta)}(x) - u_{(\xi-\eta)(\xi-\eta)}(x).$

So to estimate $u_{(\xi)(\eta)}(x), \forall \xi, \eta \in \mathbb{R}^d$, it suffices to estimate $u_{(\xi)(\xi)}(x), \forall \xi \in \mathbb{R}^d$.

Again, if the process $[2.3]$ is a uniformly integrable martingale on $[0, \tau(x))$, $\xi_{\tau(x)}$ and $\eta_{\tau(x)}$ are tangent to $\partial D$ at $x_{\tau(x)}(x)$ (a.s.), then by letting $\eta = 0$, we have

$$u_{(\xi)(\eta)}(x) = u_{(\xi)(\xi)}(x) + u_{(\eta)}(x)$$

$$= E[u_{(\xi)}(x_{\tau}) + u_{(\eta)}(x_{\tau}) + 2\xi^0 u_{(\xi)}(x_{\tau}) + \eta^0 u(x_{\tau})]$$

$$= E[g_{(\xi)}(x_{\tau}) + u_{(\eta)}(x_{\tau}) + \xi^0 \cdot D^2 h_{x_{\tau}}(0)\xi_{\tau} + g_{(\eta)}(x_{\tau}) + 2\xi^0 g_{(\xi)}(x_{\tau}) + \eta^0 g(x_{\tau})],$$

where $n(x)$ is the unit inward normal at $x \in \partial D$ and $h_p : T_p(\partial D) \rightarrow \mathbb{R}$ is a local representation of $\partial D$ as a graph over tangent space of $\partial D$ at $p$. (Notice that it is different from the first order case that generally $u_{(\xi)(\eta)}(x_{\tau}) \neq g_{(\xi)(\eta)}(x_{\tau})$.) Since $D$ is of class $C^2$ and $\partial D$ is bounded,

$$\xi^T D^2 h_{x_{\tau}}(0)\xi_{\tau} \leq N|\xi_{\tau}|^2,$$

where $N$ is constant. This shows how we can apply second quasiderivatives to get interior estimates of $u_{(\xi)(\xi)}$ through $|g|_{2,D}$, or even $|g|_{2,\partial D}$, provided that $u_{(\eta)}(y)$ can be estimated on $\partial D$ in terms of $|g|_{2,D}$ or $|g|_{2,\partial D}$.

It is also worth mentioning that $\eta_{\tau(x)}$ need not be tangent to $\partial D$ at $x_{\tau(x)}(x)$, provided that we can control the moments of $\eta_{\tau(x)}$ and estimate the normal derivative of $u$, because we can write $\eta_{\tau(x)}$ as the sum of the tangential component and the normal component.

The discussion above motivates us on attempting to construct as many quasiderivatives as possible.

**Theorem 2.1.** Let $\tau_1, \tilde{\tau}_1, \pi_1, \tilde{\pi}_1, P_1, \tilde{P}_1$ be jointly measurable adapted processes with values in $\mathbb{R}$, $\mathbb{R}$, $\mathbb{R}^{d_1}$, $\mathbb{R}^{d_1}$, Skew$(d_1, \mathbb{R})$, Skew$(d_1, \mathbb{R})$, respectively,
Then \( \text{Skew}(d_1, \mathbb{R}) \) denotes the set of all \( d_1 \times d_1 \) skew-symmetric real matrices. Assume that 
\[
\int_{0}^{T} (|r_t|^4 + |\pi_t|^4 + |\hat{\pi}_t|^4 + |P_t|^4 + |\dot{P}_t|^2) dt < \infty
\]
for any \( T \in [0, \infty) \). For \( x \in D, \xi \in \mathbb{R}^d \) and \( \eta \in \mathbb{R}^d \), on the time interval \([0, \infty)\), define the process \( \xi_t \) and \( \eta_t \) as solutions of the following (linear) equations:

\[
(2.4) \quad \xi_t = \xi + \int_{0}^{t} \left[ \sigma(\xi_s) + r_s \sigma + \sigma P_s \right] dw_s + \int_{0}^{t} \left[ b(\xi_s) + 2r_s b - \sigma \pi_s \right] ds,
\]
\[
\eta_t = \eta + \int_{0}^{t} \left[ \sigma(\eta_s) + \hat{\pi}_s \sigma + \sigma \hat{P}_s + \sigma(\xi_s)(\xi_s) + 2r_s \sigma(\xi_s) \right] dw_s + \int_{0}^{t} \left[ b(\eta_s) + 2\hat{\pi}_s b - \sigma \hat{\pi}_s + b(\xi_s)(\xi_s) + 4r_s b(\xi_s) \right] ds,
\]

where in \( \sigma, b \) and their derivatives we dropped the argument \( x_s(x) \). Also define:

\[
(2.6) \quad \xi_t^0 = \int_{0}^{t} \pi_s dw_s,
\]
\[
(2.7) \quad \eta_t^0 = (\xi_t^0)^2 - \langle \xi_t^0 \rangle_t + \int_{0}^{t} \hat{\pi}_s dw_s.
\]

Then \( \xi_t \) is a first quasiderivative of \( x_t(y) \) in \( D \) in the direction of \( \xi \) at \( x \) and \( \xi_t^0 \) is its first adjoint process, and \( \eta_t \) is a second quasiderivative of \( x_t(y) \) associated with \( \xi_t \) in \( D \) in the direction of \( \eta \) at \( x \) and \( \eta_t^0 \) is its second adjoint process.

**Remark 2.1.** Equations \((2.4)\) and \((2.5)\) give the most general forms of the first and second quasiderivatives known so far. On one hand, they contain various auxiliary processes, \( r_t, \pi_t, P_t, \hat{\pi}_t, \hat{P}_t \), which supply us fruitful quasiderivatives for our applications. On the other hand, in specific applications, many of the auxiliary processes are defined to be zero (processes), which make the equations \((2.4)\) and \((2.5)\) shorter.

**Proof.** Mimic the proof of Theorem 3.2.1 in [11] by replacing \( y_t(\varepsilon, x) \) as the solution to the Itô equation

\[
dy_t = \sqrt{1 + 2\varepsilon r_t + \varepsilon^2 \pi_t(\varepsilon) e^{\varepsilon P_t} e^{\frac{1}{2} \varepsilon^2 \hat{P}_t} dw_t + \left[ (1 + 2\varepsilon r_t + \varepsilon^2 \hat{r}_t) b(y_t) \right.}
- \sqrt{1 + 2\varepsilon r_t + \varepsilon^2 \pi_t(\varepsilon) e^{\varepsilon P_t} e^{\frac{1}{2} \varepsilon^2 \hat{P}_t}(\varepsilon \pi_t + \frac{1}{2} \varepsilon^2 \hat{\pi}_t)} dt
\]
with initial condition \( y = x + \varepsilon \xi + \frac{1}{2} \varepsilon^2 \eta \), and then differentiating the local martingale
\[
u(y_t(\varepsilon, x)) \exp \left( \int_0^t (\varepsilon \pi_s + \frac{1}{2} \varepsilon^2 \tilde{\pi}_s) dw_s - \frac{1}{2} \int_0^t [\varepsilon \pi_s + \frac{1}{2} \varepsilon^2 \tilde{\pi}_s]^2 ds \right)
\]
twice which turns out to be a local martingale also.

Before ending this section, we introduce two local martingales to be used in applications.

**Theorem 2.2.** Let \( c, f, g \) and \( u \) be real-valued twice continuously differentiable functions in \( D \). Suppose that \( u \) satisfies (1.1). Take the processes \( r_t, \hat{r}_t, \pi_t, \tilde{\pi}_t, P_t, \hat{P}_t, \xi_t, \eta_t, \xi_0, \eta_0 \) from Theorem 2.1 Then for any \( x \in D \), the processes
\[
X_t := e^{-\phi_t} \left[ u(\xi_t)(x_t) + \tilde{\xi}_0 u(x_t) \right] + \int_0^t e^{-\phi_s} \left[ f(\xi_s)(x_s) + (2r_s + \tilde{\xi}_0) f(x_s) \right] ds,
\]
\[
Y_t := e^{-\phi_t} \left[ u(\xi_t)(x_t) + u(\eta_t)(x_t) + 2\tilde{\xi}_0 u(\xi_t)(x_t) + \tilde{\eta}_0 u(x_t) \right]
+ \int_0^t e^{-\phi_s} \left[ f(\xi_s)(\xi_s) + f(\eta_s)(x_s) + (4r_s + 2\tilde{\xi}_0) f(\xi_s)(x_s) \right.
+ \left. (2\hat{r}_s + 4\tilde{\xi}_0 r_s + \tilde{\eta}_0) f(x_s) \right] ds,
\]
with
\[
\phi_t := \int_0^t c(x_s) ds,
\]
\[
\xi_0^{d+1} := -\int_0^t c(\xi_s)(x_s) + 2r_s c(x_s) ds,
\]
\[
\tilde{\xi}_0 := \xi_0^0 + \xi_0^{d+1},
\]
\[
\eta_0^{d+1} := -\int_0^t c(\xi_s)(\xi_s) + c(\eta_s)(x_s) + 4r_s c(\xi_s)(x_s) + 2\hat{r}_s c(x_s) ds,
\]
\[
\tilde{\eta}_0 := \eta_0^0 + 2\tilde{\xi}_0^{d+1} + (\xi_0^{d+1})^2 + \eta_0^{d+1},
\]
are local martingales on \([0, \tau_D(x)]\). (We keep writing \( x_t \) in place of \( x_t(x) \) and drop this argument in many places.)

**Proof.** Introduce two additional equations
\[
x_t^{d+1} = -\int_0^t c(x_s) ds, \quad x_t^{d+2} = \int_0^t \exp(x_s^{d+1}) f(x_s) dt.
\]
For \( \bar{x} = (x, x^{d+1}, x^{d+2}) \in D \times \mathbb{R} \times \mathbb{R} \), define
\[
\bar{u}(\bar{x}) = \exp(x^{d+1}) u(x) + x^{d+2}.
\]
Itô’s formula and the assumption that \( a^{ij}(x)u_{x^i x^j} + b^i(x)u_{x^i} - c(x)u + f(x) = 0 \) in \( D \) imply that \( \bar{u}(\bar{x}_t(x,0)) \) is a local martingale on \([0, \tau_D(x))\). That means, \( \bar{u}(\bar{x}_t) \in \mathcal{M}^2 \).

According to definitions 2.2 and 2.3

\[
\bar{u}(\bar{\xi}_t)(\bar{x}_t) + \xi_t^0 \bar{u}(\bar{x}_t) \quad \text{and} \quad \bar{u}(\bar{\eta}_t)(\bar{x}_t) + \xi_t^0 \bar{u}(\bar{x}_t) + 2 \xi_t^0 \bar{u}(\bar{\xi}_t)(\bar{x}_t) + \eta_t^0 \bar{u}(\bar{x}_t)
\]

are local martingales on \([0, \tau_D(x))\), where \( \bar{\xi}_t = (\xi_t, \xi_t^{d+1}, \xi_t^{d+2}) \) and \( \bar{\eta}_t = (\eta_t, \eta_t^{d+1}, \eta_t^{d+2}) \) are first and second quasiderivatives of \( \bar{x}_t((x,0)) \) in the directions of \( \xi = (\xi, 0, 0) \) and \( \eta = (\eta, 0, 0) \), respectively.

Direct computation leads to

\[
\bar{u}(\bar{\xi}_t)(\bar{x}_t) = \exp(x_t^{d+1} [u(\bar{\xi}_t)(x_t) + \xi_t^{d+1} u(x_t)]) + \xi_t^{d+2},
\]

\[
\bar{u}(\bar{\eta}_t)(\bar{x}_t) = \exp(x_t^{d+1} [u(\bar{\xi}_t)(x_t) + 2 \xi_t^{d+1} u(\bar{\xi}_t)(x_t) + (\xi_t^{d+1})^2 u(x_t)]),
\]

\[
\bar{u}(\bar{\eta}_t)(\bar{x}_t) = \exp(x_t^{d+1} [u(\bar{\eta}_t)(x_t) + \eta_t^{d+1} u(x_t)]) + \eta_t^{d+2},
\]

with

\[
\xi_t^{d+1} = - \int_0^t c(\bar{\xi}_s)(x_s) + 2r_s c(x_s) ds,
\]

\[
\xi_t^{d+2} = \int_0^t \exp(x_s^{d+1} [f(\xi_s)(x_s) + (\xi_s^{d+1} + 2r_s) f(x_s)]) ds,
\]

\[
\eta_t^{d+1} = - \int_0^t c(\bar{\xi}_s)(\bar{\xi}_s)(x_s) + 2r_s c(x_s) ds,
\]

\[
\eta_t^{d+2} = \int_0^t \exp(x_s^{d+1} [f(\xi_s)(x_s) + f(\eta_s)(x_s) + (2 \xi_s^{d+1} + 4r_s) f(x_s)] + (\xi_s^{d+1})^2 + \eta_s^{d+1} + 4r_s \xi_s^{d+1} + 2r_s^2) f(x_s)] ds.
\]

It remains to notice that \( \xi_t^0 \) and \( \eta_t^0 \) are local martingales, so by Lemma II.8.5(c) in [10]

\[
\xi_t^{d+2} = \int_0^t \int_0^t c(x_s) ds, \quad \xi_t^{d+2} = \int_0^t \int_0^t \xi_t^{d+2} ds + \int_0^t \int_0^t \xi_t^{d+2} ds
\]

are local martingales.

\[\square\]

3. APPLICATION OF QUASIDERIVATIVES TO DERIVATIVE ESTIMATES OF NON-HOMOGENEOUS LINEAR DEGENERATE ELLIPTIC EQUATIONS

In this section, we use quasiderivative to derive first and second derivative estimates of (1.2), which is the probabilistic solution of (1.1).

To be precise, let \( \sigma, b \) and \( c \) in (1.2) and (1.3) be twice continuously differentiable in \( \mathbb{R}^d \), and \( c \) be non-negative. Let \( D \subset C^4 \) be a bounded domain in \( \mathbb{R}^d \), then there exists a function \( \psi \in C^4 \) satisfying

\[
\psi > 0 \text{ in } D, \quad \psi = 0 \text{ and } |\psi| \geq 1 \text{ on } \partial D.
\]
We also assume that
\begin{equation}
L\psi := a^{ij}(x)\psi_{x^ix^j} + b^i(x)\psi_{x^i} \leq -1 \text{ in } D.
\end{equation}
\begin{equation}
|\sigma^{ij}|_{2,D} + |b^i|_{2,D} + |c|_{2,D} + |\psi|_{4,D} \leq K_0,
\end{equation}
with constant $K_0 \in [1, \infty)$.

Let $\mathcal{B}$ be the set of all skew-symmetric $d_1 \times d_1$ matrices. For any positive constant $\lambda$, define
$$D_\lambda = \{ x \in D : \psi(x) > \lambda \}.$$

**Assumption 3.1.** *(non-degeneracy along the normal to the boundary)\(\)\(\)

\( (an,n) > 0 \text{ on } \partial D, \)

where $n$ is the unit normal vector.

**Assumption 3.2.** *(interior condition to control the moments of quasiderivatives, weaker than the non-degeneracy)\(\)\(\)

There exist functions $\rho(x), Q(x,y) : D \times \mathbb{R} \to \mathbb{R}$, bounded with respect to $x$ in $D\_\lambda$ for all $\lambda > 0$, $y \in \mathbb{R}^d$ and linear in $y$;

such that for any $x \in D$ and $|y| = 1$,
$$\|\sigma(y)(x) + (\rho(x),y)\sigma(x) + \sigma(x)Q(x,y)\|^2 + 2(y,b(y)(x) + 2(\rho(x),y)b(x)) \leq c(x) + M(x)(a(x)y,y).$$

Our main result is the following:

**Theorem 3.1.** Define $u$ by \(\text{(1.2)}\), in which $x_1(x)$ is the solution of \(\text{(1.3)}\).

Suppose that Assumption 3.1 and Assumption 3.2 are satisfied.

(1) If $f, g \in C^{0,1}(\bar{D})$, then $u \in C^{0,1}_\text{loc}(D)$, and for any $\xi \in \mathbb{R}^d$,
\begin{equation}
|u(\xi)| \leq N\left(\|\xi\| + \frac{|\psi(\xi)|}{\psi^\frac{1}{2}}\right)(|f|_{0,1,D} + |g|_{0,1,D}) \text{ a.e. in } D,
\end{equation}
where $N = N(K_0, d, d_1, D)$.

(2) If $f, g \in C^{1,1}(\bar{D})$, then $u \in C^{1,1}_\text{loc}(D)$, and for any $\xi \in \mathbb{R}^d$,
\begin{equation}
|u(\xi)| \leq N\left(\|\xi\|^2 + \frac{|\psi(\xi)|^2}{\psi}\right)(|f|_{1,1,D} + |g|_{1,1,D}) \text{ a.e. in } D,
\end{equation}
where $N = N(K_0, d, d_1, D)$. Furthermore, $u$ is the unique solution in $C^{1,1}_\text{loc}(D) \cap C^{0,1}(\bar{D})$ of
\begin{equation}
\left\{ \begin{array}{l}
Lu(x) - c(x)u(x) + f(x) = 0 \text{ a.e. in } D \\
u = g \text{ on } \partial D.
\end{array} \right.
\end{equation}

**Remark 3.1.** The author doesn't know whether the estimates \(\text{(3.3)}\) and \(\text{(3.4)}\) are sharp.

The following two remarks are reductions of Theorem 3.1.
Remark 3.2. Without loss of generality, we may assume that \( c \geq 1 \) and replace condition (3.2) by

\[
\frac{1}{\psi + 1} \left\| \sigma(y)(x) + (\rho(x),y)\sigma(x) + \sigma(x)Q(x,y) \right\|^2 + 2\left( y, b(y)(x) + 2(\rho(x),y)b(x) \right) \leq c(x) - 1 + \tilde{M}(x)(\tilde{a}(x)y, y).
\]

Indeed, letting \( \tilde{u} = \frac{u}{\psi + 1} \) in \( D \), we have

\[
\tilde{u}_{x_i} = (\psi + 1)\tilde{u}_{x_i} + \psi \tilde{u}, \quad \tilde{u}_{x_i x_j} = (\psi + 1)\tilde{u}_{x_i x_j} + \psi \tilde{u}_{x_i} + \psi \tilde{u}_{x_j} + \psi \tilde{u}_{x_i x_j} \tilde{u}
\]

Hence (3.3) turns into

\[
\begin{cases}
\tilde{a}^{ij}(x)\tilde{u}_{x_i x_j} + \tilde{b}^{i}(x)\tilde{u}_{x_i} - \tilde{c}(x)\tilde{u} + f(x) = 0, \\
\tilde{u} = \tilde{g} := g/(1 + \psi), \quad \text{on } \partial D
\end{cases}
\]

with

\[
\tilde{a}^{ij} = (\psi + 1)a^{ij}, \quad \tilde{b}^{i} = 2a^{ij}\psi_{x_j} + (\psi + 1)b^{i}, \quad \tilde{c} = -L\psi + (1 + \psi)c.
\]

Notice that \( \tilde{\sigma}^{ij} = \sqrt{\psi + 1} \sigma^{ij} \). So a direct computation implies that

\[
|\tilde{\sigma}^{ij}|_{2,D} + |\tilde{b}^{i}|_{2,D} + |\tilde{c}|_{2,D} + |\psi|_{4,D} \leq (d^2 + 2d + 2)K_0^3,
\]

which plays the same role as (3.3).

Since \( L\psi \leq -1 \) and \( c \geq 0 \), \( \tilde{c} \geq 1 \).

We also have \( (\tilde{a}, n) > 0 \) on \( \partial D \). Under the substitutions on \( \sigma, b \) and \( c \), by inequality (3.3), we have

\[
\begin{align*}
&\frac{1}{\psi + 1} \left\| \tilde{\sigma}(y)(x) - \frac{1}{2\psi + 1} \sigma(x) + (\rho(x),y)\tilde{\sigma}(x) + \tilde{\sigma}(x)Q(x,y) \right\|^2 \\
&+ \frac{2}{\psi + 1} \left( y, \tilde{b}(y)(x) - \frac{\psi(y)}{\psi + 1} \tilde{b}(x) + 2(\rho(x),y)\tilde{b}(x) \right) \\
&\leq \tilde{c}(x) + L\psi + M(x)(a(x)y, y) + 2\left( y, \frac{2a\psi_{x}}{\psi + 1} (y) + 2(\rho(x),y)\frac{2a\psi_{x}}{\psi + 1} \right).
\end{align*}
\]

Collecting similar terms and noticing that \( L\psi \leq -1 \), we get

\[
\left\| \tilde{\sigma}(y)(x) + (\tilde{\rho}(x),y)\tilde{\sigma}(x) + \tilde{\sigma}(x)Q(x,y) \right\|^2 + 2\left( y, \tilde{b}(y)(x) + 2(\tilde{\rho}(x),y)\tilde{b}(x) \right) \\
\leq \tilde{c}(x) - 1 + \tilde{M}(x)(\tilde{a}(y)x, y) + 4(\tilde{a}(y)(x)\psi_{x}, y),
\]

with

\[
\tilde{\rho}(x) := \rho(x) - \frac{\psi_{x}}{2(\psi + 1)},
\]

and \( \tilde{M}(x) \) is in terms of \( M(x), K_0 \) and \( |\rho(x)| \).

The term \( 4(\tilde{a}(y)\psi_{x}, y) \) can not be bounded by \( \tilde{M}(x)(\tilde{a}(x)y, y) \). However, notice that

\[
\tilde{a}(y)(x) = \tilde{\sigma}(x)\tilde{\sigma}^*(y)(x).
\]

So \( \tilde{M}(x)(\tilde{a}(x)y, y) + 4(\tilde{a}(y)\psi_{x}, y) \) can be rewritten in the form of

\[
\left( \tilde{\sigma}(x)\frac{\tilde{M}(x)}{2}\tilde{\sigma}^*(x)y + 4\tilde{\sigma}^*(y)(x)\psi_{x}, y \right),
\]
which can play the same role as that of \( M(x)(a(x)y, y) \), which, in the proof, will be rewritten in the form of

\[
\left( \sigma(x) \cdot \frac{M(x)}{2}\sigma(x)y, y \right).
\]

A direct computation shows that if \( \tilde{u} \) satisfies estimates (3.3) and (3.4), we have the same estimates for \( u \).

**Remark 3.3.** Without loss of generality, we may assume that \( u \in C^1(D) \) and \( f, g \in C^1(D) \) when investigating first derivatives of \( u \), and \( u \in C^2(D) \) and \( f, g \in C^2(D) \) when investigating second derivatives of \( u \).

Let us take the first situation for example, in which \( u, f, g \) can be assumed to be of class \( C^1 \). The second situation can be discussed by almost the same argument.

We define the process \( x_t^\varepsilon(x) \) to be the solution to the equation

\[
x_t^\varepsilon(x) = x_0 + \int_0^t \sigma(x_s)dw_s + \int_0^t \varepsilon Id\tilde{w}_s + \int_0^t b(x_s)ds
\]

where \( \tilde{w}_t \) is a \( d \)-dimensional Wiener process independent of \( w_t \) and \( I \) is the identity matrix of size \( d \times d \), and we define \( \tau^\varepsilon(x) \) to be the first exit time of \( x_t^\varepsilon(x) \) from \( D \), then for the function

\[
u^\varepsilon(x) := E\left[ g\left(x_{\tau^\varepsilon(x)}(x)\right)e^{-\phi^\varepsilon(x)} + \int_0^{\tau^\varepsilon(x)} f\left(x_t^\varepsilon(x)\right)e^{-\phi^\varepsilon(x)} dt \right],
\]

with \( \phi^\varepsilon_t := \int_0^t c(x_t^\varepsilon(x))dt \),

the relation \( u^\varepsilon \to u \) holds as \( \varepsilon \to 0 \). Indeed, notice that

\[
E|g(x_{\tau^\varepsilon(x)}(x)) - g(x_\tau(x))| \leq KE(|x_{\tau^\varepsilon(x) \land \tau}(x) - x_{\tau^\varepsilon \land \tau}(x)| + (\tau^\varepsilon \lor \tau - \tau^\varepsilon \land \tau) + (\tau^\varepsilon \lor \tau - \tau^\varepsilon \land \tau)^{1/2}),
\]

\[
E|e^{-\phi^\varepsilon(x)} - e^{-\phi(x)}| \leq Ee^{-\tau^\varepsilon \land \tau}|\phi^\varepsilon(x) - \phi(x)| \leq KEe^{-\tau^\varepsilon \land \tau}(\tau^\varepsilon \lor \tau \cdot \sup_{t \leq \tau^\varepsilon \land \tau} |x_t^\varepsilon(x) - x_t(x)| + (\tau^\varepsilon \lor \tau - \tau^\varepsilon \land \tau) + (\tau^\varepsilon \lor \tau - \tau^\varepsilon \land \tau)^{1/2})
\]

\[
\leq KE\left( \sup_{t \leq \tau^\varepsilon \land \tau} |x_t^\varepsilon(x) - x_t(x)| + (\tau^\varepsilon \lor \tau - \tau^\varepsilon \land \tau) + (\tau^\varepsilon \lor \tau - \tau^\varepsilon \land \tau)^{1/2},
\]

\[
E\left| \int_0^{\tau^\varepsilon(x)} f\left(x_t^\varepsilon(x)\right)e^{-\phi^\varepsilon(x)} dt - \int_0^\tau f\left(x_t(x)\right)e^{-\phi(x)} dt \right|
\]

\[
\leq E\int_0^{\tau^\varepsilon \land \tau} |f\left(x_t^\varepsilon(x)\right)e^{\phi^\varepsilon(x)} - f\left(x_t(x)\right)e^{-\phi(x)| dt
\]

\[
+ KE(\tau^\varepsilon \lor \tau - \tau^\varepsilon \land \tau) + (\tau^\varepsilon \lor \tau - \tau^\varepsilon \land \tau)^{1/2})
\]

\[
\leq E\int_0^{\tau^\varepsilon \land \tau} K(|x_t^\varepsilon(x) - x_t(x)| + t \cdot \sup_{s \leq \tau} |x_s^\varepsilon(x) - x_s(x)|)e^{-t} dt
\]
\[ + KE(\tau^\varepsilon \lor \tau - \tau^\varepsilon \land \tau + (\tau^\varepsilon \lor \tau - \tau^\varepsilon \land \tau)^{1/2}) \]
\[ \leq KE(\sup_{t \leq \tau^\varepsilon \land \tau} |x_t^\varepsilon(x) - x_t(x)| + (\tau^\varepsilon \lor \tau - \tau^\varepsilon \land \tau) + (\tau^\varepsilon \lor \tau - \tau^\varepsilon \land \tau)^{1/2}) , \]

where \( K \) is a constant depending on \(|g|_{0,1,D}, |f|_{0,1,D} \) and \( K_0 \). It follows that
\[ |u^\varepsilon(x) - u(x)| \leq KE(\sup_{t \leq \tau^\varepsilon \land \tau} |x_t^\varepsilon(x) - x_t(x)| + (\tau^\varepsilon \lor \tau - \tau^\varepsilon \land \tau) + (\tau^\varepsilon \lor \tau - \tau^\varepsilon \land \tau)^{1/2}) \]
\[ \leq K \left( E(\sup_{t \leq \tau^\varepsilon \land \tau \land T} |x_t^\varepsilon(x) - x_t(x)| + K P(\tau > T) + E I_1 + E I_2 + \sqrt{E I_1} + \sqrt{E I_2} \right) , \]

where
\[ I_1 = (\tau^\varepsilon \lor \tau - \tau^\varepsilon \land \tau) I_{\tau^\varepsilon < \tau^\varepsilon} = (\tau - \tau^\varepsilon) I_{\tau^\varepsilon < \tau^\varepsilon} , \]
\[ I_2 = (\tau^\varepsilon \lor \tau - \tau^\varepsilon \land \tau) I_{\tau < \tau^\varepsilon} = (\tau^\varepsilon - \tau) I_{\tau < \tau^\varepsilon} . \]

It remains to notice that
\[ E \sup_{t \leq \tau^\varepsilon \land \tau \land T} |x_t(x) - x_t^\varepsilon(x)| \leq e^{KT^\varepsilon} \varepsilon \rightarrow 0, \text{ as } \varepsilon \rightarrow 0, \]
\[ P(\tau > T) \leq \frac{E T}{T} \leq \frac{1}{T} E \int_0^T \left( - L \psi(x_t(x)) \right) dt = \frac{\psi(x) - \psi(x_\tau(x))}{T} \leq \frac{K_0}{T} , \]

and
\[ E(\tau - \tau^\varepsilon) I_{\tau^\varepsilon < \tau^\varepsilon} = E \int_{\tau^\varepsilon < \tau^\varepsilon} 1 dt \]
\[ \leq - E \int_{\tau^\varepsilon < \tau^\varepsilon} L \psi(x_t(x)) dt \]
\[ = - E \left( \psi(x_\tau(x)) - \psi(x_\tau^\varepsilon(x)) \right) I_{\tau^\varepsilon < \tau^\varepsilon} \]
\[ = E \psi(x_\tau^\varepsilon(x)) I_{\tau^\varepsilon < \tau^\varepsilon} \]
\[ = E \left( \psi(x_\tau^\varepsilon(x)) - \psi(x_\tau^\varepsilon(x)) \right) I_{\tau^\varepsilon < \tau^\varepsilon} \]
\[ \leq E \left( \psi(x_\tau^\varepsilon(x)) - \psi(x_\tau^\varepsilon(x)) \right) I_{\tau^\varepsilon < \tau^\varepsilon \leq \tau} + 2 K_0 P(\tau > T) \]
\[ \leq K_0 E \sup_{t \leq \tau^\varepsilon \land \tau \land T} |x_t(x) - x_t^\varepsilon(x)| + \frac{2 K^2}{T} \]
\[ E(\tau^\varepsilon - \tau) I_{\tau^\varepsilon < \tau^\varepsilon} \leq - 2 E \int_{\tau^\varepsilon < \tau^\varepsilon} L^\varepsilon \psi(x_t^\varepsilon(x)) dt \]
\[ \leq \cdots \leq K_0 E \sup_{t \leq \tau^\varepsilon \land \tau \land T} |x_t(x) - x_t^\varepsilon(x)| + \frac{2 K^2}{T} . \]

Hence by first letting \( \varepsilon \downarrow 0 \) and then \( T \uparrow \infty \), we conclude that
\[ |u^\varepsilon(x) - u(x)| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \]

Moreover, for small \( \varepsilon \) the condition \((3.1)\) holds for \( 2 \psi \), taken instead of \( \psi \) and \( L^\varepsilon \) associated to the process \( x_\tau^\varepsilon(x) \). The matrix \( \sigma^\varepsilon \) corresponding to the process \( x_\tau^\varepsilon(x) \) is obtained by attaching the identity matrix, multiplied by \( \varepsilon \), to the right of the original matrix \( \sigma \). In this connection we modify \( P(x,y) \) by
adding zero entries on the right and below to form a \((d_1 + d) \times (d_1 + d)\) matrix. Then the condition (3.6) corresponding to the process \(x_t^\varepsilon(x)\) will differ from the original condition by the fact that the term \(\varepsilon^2(\rho(x), y)^2 d\) appears on the left, and \(\frac{1}{2} M(x)\varepsilon^2\) on the right. From this it is clear that the condition (3.6) for the process \(x_t^\varepsilon(x)\) (for all \(\varepsilon\)) also holds when \(M(x)\) is replaced by \(M(x) + 2|\rho(x)|^2 d\).

Finally, from analysis of PDE, we know that for \(\varepsilon \neq 0\) the nondegenerate elliptic equation \(L^\varepsilon w = 0\) in \(D\) with the boundary condition \(w = g\) on \(\partial D\) has a solution that is continuous in \(\bar{D}\) and twice continuously differentiable in \(D\), and \(w^\varepsilon = w\) in \(D\) by Itô's formula. From this it follows that it suffices to prove the theorem for small \(\varepsilon \neq 0\), the process \(x_t^\varepsilon(x)\), and a function \(u^\varepsilon\) that is continuously differentiable in \(D\). Of course, we must be sure that the constants \(N\) in (3.3) is chosen to be independent of \(\varepsilon\), which is true as we can see in the proof of the theorem. Observing further that for each fixed \(\varepsilon \neq 0\) the functions \(f\) and \(g\) can be uniformly approximated in \(D\) by infinitely differentiable functions, in such a way that the last factor in (3.3) increases by at most a factor of two when \(f\) and \(g\) are replaced by the approximating functions, while for the latter the function \(w\) (i.e., \(u^\varepsilon\)) has continuous and bounded first derivatives in \(D\), we conclude that we may assume \(u\) has continuous first derivatives in \(D\) and \(f, g \in C^1(D)\) when investigating first derivatives of \(u\).

Before proving the theorem, let us prove four lemmas. In Lemma 3.1 we estimate the first exit time. It is a well-known result, but we still prove for the sake of completeness. Lemma 3.2 concerns the estimate of the first derivative along the normal to the boundary, to be used when estimating the second derivatives. In Lemma 3.3 and Lemma 3.4 we construct two supermartingales, which will play the roles of barriers near the boundary and in the interior of the domain, respectively.

**Lemma 3.1.** Let \(\tau_{D_0}(x)\) be the first exit time of \(x_t(x)\) from \(D_0\), which is a sub-domain of \(D\) containing \(x\). Then we have

\[
E\tau_{D_0}(x) \leq E\tau_D(x) \leq |\psi(x)|_{0,D},
\]

\[
E\tau_{D_0}^2(x) \leq E\tau_D^2(x) \leq 2|\psi|_{0,D}^2 \leq 2|\psi|_{0,D}^2.
\]

**Proof.** The fact that \(D_0 \subset D\) implies \(E\tau_{D_0}(x) \leq E\tau_D(x)\) and \(E\tau_{D_0}^2(x) \leq E\tau_D^2(x)\). Now we abbreviate \(\tau_D(x)\) by \(\tau(x)\), or simply \(\tau\) when this will cause no confusion. By (3.1) and Itô's formula, we have

\[
E\tau = E\int_0^\tau 1\,dt \leq -E\int_0^\tau L\psi\,dt = \psi(x) - E\psi(x_\tau) = \psi(x),
\]

\[
E\tau^2 = 2E\int_0^\tau (\tau - t)I_{\tau > t}\,dt = 2E\int_0^\infty I_{\tau > t}E\psi(x_\tau)\,dt
\]

\[
\leq 2\sup_{y \in D} E\tau(y) \cdot E\int_0^\infty I_{\tau > t}\,dt = 2\sup_{y \in D} E\tau(y) \cdot E\tau \leq 2|\psi|_{0,D}^2 \psi(x).
\]
Lemma 3.2. If $f, g \in C^2(\bar{D})$, and $u \in C^1(\bar{D})$, then for any $y \in \partial D$ we have
\begin{equation}
|u(y)| \leq K(|g|_{2,D} + |f|_{0,D}),
\end{equation}
where $n$ is the unit inward normal on $\partial D$ and the constant $K$ depends only on $K_0$.

Proof. Fix a $y \in \partial D$, and choose $\epsilon_0 > 0$ so that $y + \epsilon_0 n \in D$ as long as $0 < \epsilon \leq \epsilon_0$. Also, fix an $\epsilon \in (0, \epsilon_0]$ and let $x := y + \epsilon n$. By Itô's formula,
\begin{align*}
d(g(x_t) e^{-\phi_t}) &= e^{-\phi_t} g(x_t) du_t^D + e^{-\phi_t}(L g(x_t) - c(x_t) g(x_t))dt, \\
d(u(x_t) e^{-\phi_t}) &= e^{-\phi_t} u(x_t) du_t^D + e^{-\phi_t} f(x_t)dt.
\end{align*}
Notice that
\begin{align*}
E \int_0^\infty \left( e^{-\phi_t} g(x_t) \right)^2 dt &\leq N |g|_{1,D}^2 E \tau < \infty, \\
E \int_0^\infty \left( e^{-\phi_t} u(x_t) \right)^2 dt &\leq N |u|_{1,D}^2 E \tau < \infty.
\end{align*}
The Wald identities hold:
\begin{align*}
E \int_0^\tau e^{-\phi_t} g(x_t) dt = 0, & \quad E \int_0^\tau e^{-\phi_t} u(x_t) dt = 0.
\end{align*}
Thus
\begin{align*}
u(x) &= g(x) + E \int_0^\tau e^{-\phi_t} (L g(x_t(x)) - c(x_t(x)) g(x_t(x))) dt \\
& \quad + E \int_0^\tau f(x_t(x)) e^{-\phi_t} dt \\
& \leq g(x) + (|L g|_{0,D} + |c|_{0,D} |g|_{0,D} + |f|_{0,D}) E \tau \\
& \leq g(x) + K (|g|_{2,D} + |f|_{0,D}) \psi(x).
\end{align*}
Notice that $u(y) = g(y)$ and $\psi(y) = 0$. So we have
\begin{equation}
\frac{u(y + \epsilon_n) - u(y)}{\epsilon} \leq \frac{g(y + \epsilon_n) - g(y)}{\epsilon} + K (|g|_{2,D} + |f|_{0,D}) \frac{\psi(y + \epsilon_n) - \psi(y)}{\epsilon}.
\end{equation}
Letting $\epsilon \downarrow 0$, we get
\begin{equation}
|u(y)| \leq K (|g|_{2,D} + |f|_{0,D}).
\end{equation}
Replacing $u$ with $-u$ yields the same estimate of $(-u)_{(n)}$ from above, which is an estimate of $u_{(n)}$ from below. Combining the estimates from above and from below leads to (3.7) and proves the lemma. \qed
For constants $\delta$ and $\lambda$, such that $0 < \delta < \lambda^2 < \lambda < 1$, define
\[
D_\delta = \{ x \in D : \delta < \psi \}, \quad D^\lambda = \{ x \in D : \psi < \lambda \}, \quad D_\delta^\lambda = \{ x \in D : \delta < \psi < \lambda \}, \quad D_\lambda^2 = \{ x \in D : \lambda^2 < \psi \}.
\]

**Lemma 3.3.** Introduce
\[
\varphi(x) = \lambda^2 + \psi(1 - \frac{1}{4\lambda^2}), \quad B_1(x, \xi) = \left[ \lambda + \sqrt{\psi(1 + \psi^2)}\right] |\xi|^2 + K_1 \varphi^2 \frac{\psi^2(\xi)}{\psi},
\]
where $K_1 \in [0, \infty)$ is a constant depending only on $K_0$.

In $D_\lambda^2$, if we construct first and second quasiderivatives by (2.4) and (2.5), in which
\[
r(x, \xi) := \rho(x, \xi) + \frac{\psi(\xi)}{\psi}, \quad r_t := r(x_t, \xi_t),
\]
where $\rho(x, \xi) := -\frac{1}{A} \sum_{k=1}^{d} \psi^{(\sigma_k)}(\psi^{(\sigma_k)})(\xi)$, with $A := \sum_{k=1}^{d} \psi^{(\sigma_k)}$;
\[
\hat{r}(x, \xi) := \frac{\psi^2(\xi)}{\psi^2}, \quad \hat{r}_t := \hat{r}(x_t, \xi_t);
\]
\[
\pi^k(x, \xi) := \frac{2\psi^{(\sigma_k)}(\psi(\xi))}{\varphi}, \quad k = 1, \ldots, d_1, \quad \pi_t := \pi(x_t, \xi_t);
\]
\[
P^{ik}(x, \xi) := \frac{1}{A} \left[ \psi^{(\sigma_k)}(\xi) - \psi^{(\sigma_i)}(\psi^{(\sigma_k)})(\xi) \right], \quad i, k = 1, \ldots, d_1, \quad P_t := P(x_t, \xi_t);
\]
\[
\hat{\pi}^k_t = \hat{P}^{ik} = 0, \quad \forall i, k = 1, \ldots, d_1, \forall t \in [0, \infty).
\]

Then for sufficiently small $\lambda$, when $x_0 \in D_\delta^\lambda$, $\xi_0 \in \mathbb{R}^d$ and $\eta_0 = 0$, we have
\begin{enumerate}
\item $B_1(x_t, \xi_t)$ and $\sqrt{B_1(x_t, \xi_t)}$ are local supermartingales on $[0, \tau^1_1]$, where $\tau^1_1 = \tau_{D_\delta^\lambda}(x_0)$;
\item $E \int_0^{\tau^1_1} |\xi_t|^2 + \frac{\psi^2(\xi_t)}{\psi^2} \, dt \leq NB_1(x_0, \xi_0)$;
\item $E \sup_{t \leq \tau^1_1} |\xi_t|^2 \leq NB_1(x_0, \xi_0)$;
\item $E|\eta_t| \leq E \sup_{t \leq \tau^1_1} |\eta_t| \leq NB_1(x_0, \xi_0)$;
\item $E \left( \int_0^{\tau^1_1} |\eta_t|^2 \, dt \right)^{\frac{1}{2}} \leq NB_1(x_0, \xi_0)$;
\end{enumerate}
where $N$ is a constant depending on $K_0$ and $\lambda$.

**Proof.** Throughout the proof, keep in mind that the constant $K$ depend only on $K_0$, while the constants $N \in [1, \infty)$ and $\lambda_0 \in (0, 1)$ depend on $K_0$ and $\lambda$. 
First, notice that, on $\partial D$, we have
\[ A = \sum_{k=1}^{d_1} \psi_x^2 = 2(a\psi_x, \psi_x) = 2|\psi_x|(an, n) \geq 2\delta, \]
where the constant $\delta > 0$, because of the compactness of $\partial D$. Replacing $\psi$ by $\psi/2\delta$ if needed, we may, therefore, assume that $A \geq 1$.

By Itô’s formula, for $t < \tau_1$, we have
\[ d\psi(\xi_t) = [(\psi(\sigma^i))(\xi_t) + r_t\psi(\sigma^i) + \psi(\sigma^k)P_{ki}^t]dw_t^i + [(L\psi)(\xi_t) + 2r_tL\psi - \psi(\sigma^i)\pi_t^k]dt. \]
A crucial fact about this equation is that owing to our choice of $\lambda$ and $P$
\[ (\psi(\sigma^i))(\xi_t) + r_t\psi(\sigma^i) + \psi(\sigma^k)P_{ki}^t = \frac{\psi(\xi_t)}{\psi} \psi(\sigma^i). \]
Thus
\[ d\psi(\xi_t) = \frac{\psi(\xi_t)}{\psi} \psi(\sigma^i)dw_t^i + [(L\psi)(\xi_t) + 2r_tL\psi - \psi(\sigma^i)\pi_t^k]dt. \]
Let
\[ \bar{a} := a(\xi) + r\sigma + \sigma P, \quad \bar{b} := b(\xi) + 2rb. \]
We have
\[ \|\bar{a}\| \leq K(\|\xi\| + \frac{|\psi(\xi)|}{\psi}), \]
(3.10)
\[ |\bar{b}| \leq K(\|\xi\| + \frac{|\psi(\xi)|}{\psi}). \]

By Itô’s formula,
\[ dB_1(x, \xi_t) = \Gamma_1(x, \xi_t)dt + \Lambda_1^k(x, \xi_t)dw_t^k \]
with
\[ \Gamma_1(x, \xi) = I_1 + I_2 + \ldots + I_{13} \]
where
\[ I_1 = \lambda\|2(\xi, \bar{b}) + \|\bar{a}\|^2 \leq \lambda K(\|\xi\|^2 + \frac{\psi^2(\xi)}{\psi^2}) \leq K\lambda^2 \frac{|\xi|^2}{\psi^2} + K\frac{\varphi^2}{\psi^2} \frac{|\psi|^2(\xi)}{\psi^2}, \]
here we apply (3.9), (3.10) and $\lambda \leq \varphi^2$,\n\[ I_2 = -\lambda^2(\xi, \sigma^k)\pi^k \leq \frac{K\lambda\|\xi\|\psi(\xi)}{\varphi^2} \leq \frac{\lambda^2\|\xi\|^2}{32\cdot 2\varphi^2} + \varphi^2 \frac{\psi^2}{\psi^2} \]
\[ \leq \frac{|\xi|^2}{32\cdot 2\varphi^2} + \varphi^2 \frac{\psi^2(\xi)}{\psi^2} \leq \frac{|\xi|^2}{32\varphi^2} + \varphi^2 \frac{\psi^2(\xi)}{\psi^2}, \]
here we apply $\lambda^2 \leq \varphi$, and then observe that $\psi \leq 2\varphi$,\n\[ I_3 = \sqrt{\psi}(1 + \sqrt{\psi})2(\xi, \bar{b}) \leq \sqrt{\psi}K\|\xi\|(|\xi| + \frac{|\psi(\xi)|}{\psi}) \leq K\lambda\|\xi|^2}{\psi^2}, \]
here we apply (3.11),

\[ I_4 = -\sqrt{\psi}(1 + \sqrt{\psi})2(\xi, \sigma^k)\pi^k \leq \frac{K \sqrt{\psi} |\psi| |\psi(\xi)|}{\varphi \psi} \leq \frac{\psi |\xi|^2}{32 \cdot 2^\frac{3}{2} \varphi^\frac{1}{2}} + \frac{K \varphi^\frac{1}{2} \psi^2 (\xi)}{\psi^2}, \]

here we observe that \( \psi \leq 2\varphi, \)

\[ I_5 = \sqrt{\psi}(1 + \sqrt{\psi})|\sigma|^2 \leq K \sqrt{\psi} |\xi|^2 + \frac{\psi^2 (\xi)}{\psi^2} \leq K \lambda^2 \frac{\psi^2 |\xi|}{\psi^2} + K \varphi^\frac{1}{2} \frac{\psi^2 (\xi)}{\psi^2}, \]

here we apply (3.9),

\[ I_6 = (1 + 2\sqrt{\psi})|\xi|^2 \frac{L\psi}{2\sqrt{\psi}} - \frac{A}{8\psi^\frac{1}{2}} \leq -\frac{|\xi|^2}{8\psi^\frac{1}{2}}, \]

\[ I_7 = \frac{A}{4\psi} |\xi|^2 \leq K \frac{|\xi|^2}{\psi} \leq K \frac{\lambda}{\psi^\frac{1}{2}}, \]

\[ I_8 = (1 + 2\sqrt{\psi}) \frac{\psi_{\varphi}}{\varphi} (\xi, \sigma^k) \leq K \frac{|\xi|}{\sqrt{\psi}} |\xi| \leq K \lambda \frac{|\xi|^2}{\psi^\frac{1}{2}} + \frac{|\xi|^2}{32\psi^\frac{1}{2}} + K \varphi^\frac{1}{2} \frac{\psi^2 (\xi)}{\psi^2}, \]

here we apply (3.9) and \( \psi \leq 2\varphi, \)

\[ I_9 = K_1 \frac{3}{2} \varphi^\frac{1}{2} \left[ (1 - \frac{\psi}{2\lambda})L\psi - \frac{A}{4\lambda} \right] \frac{\psi^2 (\xi)}{\psi} + K_1 \frac{3}{2} \frac{\psi^2 (\xi)}{\psi^2} L\psi \leq 0, \]

\[ I_{10} = K_1 \frac{3}{8} \varphi^{-\frac{1}{2}} (1 - \frac{\psi}{2\lambda})^2 \frac{\psi^2 (\xi)}{\psi} \leq K_1 \frac{3}{8} \frac{\psi}{\varphi^\frac{1}{2}} A \frac{\psi^2 (\xi)}{\psi} \leq K_1 \frac{3}{4} \varphi^\frac{1}{2} A \frac{\psi^2 (\xi)}{\psi^2}, \]

here we use \( \psi \leq 2\varphi, \)

\[ I_{11} = K_1 \varphi^\frac{3}{2} \frac{\psi^2 (\xi)}{\psi} \left[ (L\psi)(\xi) + 2\rho L\psi \right] \leq K_1 K \varphi^\frac{1}{2} \frac{|\psi(\xi)|}{\psi} |\xi| \leq K_1 K \lambda \varphi^\frac{1}{2} \frac{|\psi(\xi)|}{\psi} |\xi| \]

\[ \leq K_1 \lambda \varphi^\frac{1}{2} \frac{|\psi^2 (\xi)}{\psi^2} + K_1 K \lambda^3 \frac{|\xi|^2}{\psi^\frac{1}{2}}, \]

here we first notice that \( \varphi \leq 2\lambda, \) and then apply \( \psi \leq 2\varphi, \)

\[ I_{12} = -K_1 \varphi^\frac{3}{2} \frac{A}{\varphi} \frac{\psi^2 (\xi)}{\psi^2} = -K_1 4\varphi^\frac{1}{2} A \frac{\psi^2 (\xi)}{\psi^2}, \]

\[ I_{13} = K_1 \frac{3}{2} \varphi^\frac{1}{2} (1 - \frac{\psi}{2\lambda}) A \frac{\psi^2 (\xi)}{\psi^2} \leq K_1 \frac{3}{2} \varphi^\frac{1}{2} A \frac{\psi^2 (\xi)}{\psi^2}. \]

Collecting our estimates above we see that, when \( x \in D^\lambda, \)

\[ \Gamma_1(x, \xi) \leq \left( K(\lambda^\frac{3}{2} + \sqrt{\lambda}) + K_1 K \lambda^3 + \left( \frac{3}{32} - \frac{1}{8} \right) \right) \frac{|\xi|^2}{\psi^\frac{1}{2}} \]
Recall that \( K \) and \( K_1 \) depend only on \( K_0 \). By first choosing \( K_1 \) such that \( K_1 \geq K \), then letting \( \lambda \) be sufficiently small, we get

\[
\text{(3.12)} \quad \Gamma_1(x, \xi) \leq -\frac{1}{64} \frac{1}{\psi^2} \frac{\psi^2(\xi)}{\psi^2} \leq \frac{1}{2} \frac{\lambda \psi^2(\xi)}{\psi^2} \leq 0.
\]

It follows that \( B_1(x_t, \xi_t) \) is a local supermartingale on \([0, \tau^\delta]\).

Also, notice that \( f(x) = \sqrt{x} \) is concave, so \( \sqrt{B_1(x_t, \xi_t)} \) is a local supermartingale on \([0, \tau^\delta]\). Thus (1) is proved.

From (3.12), there exists a sufficiently small positive \( \lambda_0 \), such that

\[
\Gamma_1(x, \xi) + \lambda_0 (|\xi|^2 + \frac{\psi^2(\xi)}{\psi^2}) \leq 0, \forall x \in D^\lambda_\delta.
\]

Therefore,

\[
\lambda_0 E \int_0^{\tau^\delta} |\xi|^2 + \frac{\psi^2(\xi)}{\psi^2} \, dt \leq - E \int_0^{\tau^\delta} \Gamma_1(x_t, \xi_t)
\]

\[
= B_1(x_0, \xi_0) - E B_1(x_{\tau^\delta}, \xi_{\tau^\delta}) \leq B_1(x_0, \xi_0),
\]

which proves (2).

Since

\[
|\xi|^2 = |\xi_0|^2 + \int_0^{\tau^\delta} 2(\xi_s, \bar{b}) + \|\bar{\sigma}\|^2 \, ds + \int_0^{\tau^\delta} 2(\xi_s, \bar{\sigma}) \, dw_s,
\]

by Burkholder-Davis-Gundy inequality, for \( \tau_n = \tau^\delta \wedge \inf\{ t \geq 0 : |\xi_t| \geq n \} \), we have,

\[
E \sup_{t \leq \tau_n} |\xi_t|^2 \leq |\xi_0|^2 + \int_0^{\tau_n} \left( 2|\xi_t| : |\bar{b}| + \|\bar{\sigma}\|^2 \right) \, dt + 6E \left( \int_0^{\tau_n} |(\xi_t, \bar{\sigma})|^2 \right)^{\frac{1}{2}}
\]

\[
\leq |\xi_0|^2 + NE \int_0^{\tau_n} |\xi_t|^2 + \frac{\psi^2(\xi_t)}{\psi^2} \, dt + E \left( \int_0^{\tau_n} N|\xi_t|^2 (|\xi_t|^2 + \frac{\psi^2(\xi_t)}{\psi^2}) \, dt \right)^{\frac{1}{2}}
\]

\[
\leq NB_1(x_0, \xi_0) + E \left[ \sup_{t \leq \tau_n} |\xi_t| \cdot \left( \int_0^{\tau_n} N(|\xi_t|^2 + \frac{\psi^2(\xi_t)}{\psi^2}) \, dt \right)^{\frac{1}{2}} \right]
\]

\[
\leq NB_1(x_0, \xi_0) + \frac{1}{2} E \sup_{t \leq \tau_n} |\xi_t|^2 + \frac{1}{2} E \left( \int_0^{\tau_n} N(|\xi_t|^2 + \frac{\psi^2(\xi_t)}{\psi^2}) \, dt \right)
\]

\[
\leq NB_1(x_0, \xi_0) + \frac{1}{2} E \sup_{t \leq \tau_n} |\xi_t|^2,
\]

which implies that

\[
E \sup_{t \leq \tau_n} |\xi_t|^2 \leq NB_1(x_0, \xi_0).
\]

Now (3) is obtained by letting \( n \to \infty \).
Now we estimate the moments of second quasiderivative $\eta_t$. Based on our definition, we have
\[
d\eta_t = [\sigma(\eta_t) + G(x_t, \xi_t)]dw_t + [b(\eta_t) + H(x_t, \xi_t)]dt,
\]
with
\[
G(x, \xi) = \sigma(\xi)(x) + 2r\sigma(\xi) + (2\sigma(\xi) + 2r\sigma + \sigma P + (\hat{r} - r^2)\sigma),
\]
\[
H(x, \xi) = b(\xi)(x) + 4rb(\xi) + 2\frac{\psi^2(\xi)}{\psi^2}b.
\]
Therefore, we have the estimates
\[
\|G\| \leq N|\xi|(|\xi| + \frac{\|\psi(\xi)\|}{\psi}), \quad |H| \leq N(|\xi|^2 + \frac{\|\psi(\xi)\|^2}{\psi^2}).
\]
Itô’s formula implies
\[
d(|\eta_t|^2 e^{2\varphi}) = \theta(x_t, \xi_t, \eta_t)dt + \mu^k(x_t, \xi_t, \eta_t)dw^k_t,
\]
where
\[
\theta(x, \xi, \eta) = e^{2\varphi} \left[ 2|\eta|^2 \left[ \frac{1}{4\lambda}L\psi - \frac{A}{4\lambda} + (1 - \frac{\psi}{2\lambda})^2A \right] + \|\sigma(\eta) + G(x, \xi)\|^2 + 2(\eta, b(\eta) + H(x, \xi)) + 2(\eta, \sigma(\eta) + G(x, \xi))[(1 - \frac{\psi}{2\lambda})\psi(\sigma(\eta))] \right].
\]
It is not hard to see that, for any $x \in D^\lambda_0$,
\[
\theta(x, \xi, \eta) \leq e^{2\varphi} \left[ 2\left(1 - \frac{1}{4\lambda}\right)A|\eta|^2 + N|\eta|^2 + |\xi|^2(\|\xi\|^2 + \frac{\|\psi(\xi)\|^2}{\psi^2}) + |\eta|(|\xi|^2 + \frac{\|\psi(\xi)\|^2}{\psi^2}) \right].
\]
So for sufficiently small $\lambda$, we have
\[
\theta(x, \xi, \eta) + \lambda_0|\eta|^2 \leq Ne^{2\varphi}(|\xi|^2 + |\eta|)(|\xi|^2 + \frac{\|\psi(\xi)\|^2}{\psi^2}).
\]
Then for any bounded stopping time $\gamma$ with respect to $\{\mathcal{F}_t\}$, we have
\[
E(e^{2\varphi}|\eta_{\gamma}|^2) + \lambda_0E \int_0^\gamma |\eta_t|^2 dt \leq E \int_0^\gamma Ne^{2\varphi}(|\xi_t|^2 + |\eta_t|)(|\xi_t|^2 + \frac{\|\psi(\xi_t)\|^2}{\psi^2}) dt.
\]
Let $\tau_n = \tau^\lambda_t \wedge \inf\{t \geq 0 : e^{\varphi}|\eta_t| \geq n\}$. Recall that $\eta_0 = 0$. By Theorem III.6.8 in [10], we have
\[
E \sup_{t \leq \tau_n} (e^{\varphi}|\eta_t|) \leq 3E \left( \int_0^{\tau_n} Ne^{2\varphi}(|\xi_t|^2 + |\eta_t|)(|\xi_t|^2 + \frac{\|\psi(\xi_t)\|^2}{\psi^2}) dt \right)^{\frac{1}{2}}
\]
\[
\leq E \left[ \left( \int_0^{\tau_n} 9Ne^{2\varphi}|\xi_t|^2(|\xi_t|^2 + \frac{\|\psi(\xi_t)\|^2}{\psi^2}) dt \right)^{\frac{1}{2}} + \left( \int_0^{\tau_n} 9Ne^{2\varphi}|\eta_t|(|\xi_t|^2 + \frac{\|\psi(\xi_t)\|^2}{\psi^2}) dt \right)^{\frac{1}{2}} \right]
\]
\[
\leq E \left[ N \sup_{t \leq \tau_n} |\xi_t| \cdot \left( \int_0^{\tau_n} |\xi_t|^2 + \frac{\|\psi(\xi_t)\|^2}{\psi^2} dt \right)^{\frac{1}{2}} + \sup_{t \leq \tau_n} \sqrt{e^{\varphi}|\eta_t|} \cdot \left( \int_0^{\tau_n} N(|\xi_t|^2 + \frac{\|\psi(\xi_t)\|^2}{\psi^2}) dt \right)^{\frac{1}{2}} \right].
\]
Lemma 3.4. Introduce

\[ B_2(x, \xi) = \lambda^\frac{4}{7} |\xi|^2. \]

If we construct first and second quasiderivatives by (2.4) and (2.5), in which
\[
\begin{align*}
  r(x, y) &= (\rho(x), y), \quad r_t := r(x_t, \xi_t), \quad \tilde{r}_t := r(x_t, \eta_t), \\
  \pi(x, y) &= \frac{M(x)}{2} \sigma^*(x)y, \quad \pi_t := \pi(x_t, \xi_t), \quad \tilde{\pi}_t := \pi(x_t, \eta_t), \\
  P(x, y) &= Q(x, y), \quad P_t := P(x_t, \xi_t), \quad \tilde{P}_t := P(x_t, \eta_t).
\end{align*}
\]
Then for sufficiently small \( \lambda \), when \( x_0 \in D_{\lambda^2}, \xi_0 \in \mathbb{R}^d \) and \( \eta_0 = 0 \), we have

1. \( e^{-\phi_t} B_2(x_t, \xi_t) \) and \( \sqrt{e^{-\phi_t} B_2(x_t, \xi_t)} \) are local supermartingales on \([0, \tau_2]\), where \( \tau_2 = \tau_{D_{\lambda^2}}(x) \);
2. \( E \int_0^{\tau_2} e^{-\phi_t} |\xi_t|^2 dt \leq NB_2(x_0, \xi_0) \);
3. \( E \sup_{t \leq \tau_2} e^{-\phi_t} |\xi_t|^2 \leq NB_2(x_0, \xi_0) \);
4. \( E e^{-\phi_{\tau_2}} |\eta_\tau| \leq E \sup_{t \leq \tau_2} e^{-\phi_t} |\eta_t| \leq NB_2(x_0, \xi_0) \);
5. \( E \left( \int_0^{\tau_2} e^{-2\phi_t} |\eta_t|^2 dt \right)^{\frac{1}{2}} \leq NB_2(x_0, \xi_0) \);
6. The above inequalities are still all true if we replace \( \phi_t \) by \( \phi_t - \frac{1}{2} t \). More precisely, we have
\[
\begin{align*}
  E \int_0^{\tau_2} e^{-\phi_t + \frac{1}{2} t} |\xi_t|^2 dt &\leq NB_2(x_0, \xi_0), \quad \sup_{t \leq \tau_2} e^{-\phi_t + \frac{1}{2} t} |\xi_t|^2 \leq NB_2(x_0, \xi_0), \\
  E \left( \int_0^{\tau_2} e^{-2\phi_t + t} |\eta_t|^2 dt \right)^{\frac{1}{2}} &\leq NB_2(x_0, \xi_0), \quad \sup_{t \leq \tau_2} e^{-\phi_t + \frac{1}{2} t} |\eta_t| \leq NB_2(x_0, \xi_0),
\end{align*}
\]
where \( N \) is constant depending on \( K_0 \) and \( \lambda \).

Proof. First of all, replacing \( K_0 \) by
\[
\max \left\{ K_0, \sup_{x \in D_{\lambda^2}} |\rho(x)|, \sup_{x \in D_{\lambda^2}, y \in \mathbb{R}^d} \frac{\|Q(x, y)\|}{|y|}, \sup_{x \in D_{\lambda^2}} M(x) \right\},
\]
we may assume that
\[ \sup_{x \in D_{\omega}} |\rho(x)| \leq K_0, \quad \sup_{x \in D_{\omega}} \|Q(x, y)\| \leq K_0|y|, \quad \forall y \in \mathbb{R}^d, \quad \sup_{x \in D_{\omega}} M(x) \leq K_0. \]

By Itô’s formula, for \( t < \tau_2 \), we have
\[
d\|\xi_t\|^2 = \Lambda_2^k(x_t, \xi_t)dw_t^k + \Gamma_2(x_t, \xi_t)dt,
\]
where
\[
\Lambda_2(x, \xi) = 2(\xi_t \sigma(\xi_t) + r_t \sigma + \sigma P_t),
\]
\[
\Gamma_2(x, \xi) = \left[2(\xi, b(\xi) + 2rb - \sigma \pi) + \|\sigma(\xi) + r \sigma + \sigma P\|^2\right] \leq (c - 1)|\xi|^2.
\]
So
\[
d(e^{-\phi_t}|\xi_t|^2) = e^{-\phi_t} \left[\Gamma_2(x_t, \xi_t) - c(x_t)|\xi_t|^2\right] dt + dm_t
\]
\[
\leq -e^{-\phi_t}|\xi_t|^2 dt + dm_t,
\]
where \( m_t \) is a local martingale.

Thus \( e^{-\phi_t}B_2(x_t, \xi_t) \) is a local supermartingale on \([0, \tau_2)\).

Also, notice that \( f(x) = \sqrt{x} \) is concave, so \( \sqrt{e^{-\phi_t}B_2(x_t, \xi_t)} \) is a local supermartingale on \([0, \tau_2)\). (1) is proved.

From (3.13), we also have
\[
E \int_{0}^{\tau_2} e^{-\phi_t}|\xi_t|^2 dt = B_2(x_0, \xi_0) - Ee^{-\phi_{\tau_2}}B_2(x_{\tau_2}, \xi_{\tau_2}) \leq B_2(x_0, \xi_0),
\]
which proves (2).

Since
\[
e^{-\phi_t}|\xi_t|^2 = |\xi_0|^2 + \int_{0}^{t} e^{-\phi_s} \left[2(\xi, \bar{b}) + \|\sigma\|^2 - c|\xi_t|^2\right] ds + \int_{0}^{t} e^{-\phi_s} 2(\xi, \bar{\sigma})dw_s,
\]
by Burkholder-Davis-Gundy inequality, for \( \tau_n = \tau_2 \wedge \inf\{t \geq 0 : |\xi_t| \geq n\} \), we have,
\[
E \sup_{t \leq \tau_n} e^{-\phi_t}|\xi_t|^2 \leq |\xi_0|^2 + \int_{\tau_n}^{\tau_n} e^{-\phi_t} \left[2|\xi_t| \cdot |\bar{b}| + \|\bar{\sigma}\|^2 + c|\xi_t|^2\right] dt
\]
\[
+ 12E\left( \int_{\tau_n}^{\tau_n} e^{-2\phi_t}|(\xi_t, \bar{\sigma})|^2 dt \right)^{\frac{1}{2}}
\]
\[
\leq |\xi_0|^2 + NE \int_{\tau_n}^{\tau_n} e^{-\phi_t}|\xi_t|^2 dt + E\left( \int_{\tau_n}^{\tau_n} Ne^{-2\phi_t}|\xi_t|^4 dt \right)^{\frac{1}{2}}
\]
\[
\leq NB_2(x_0, \xi_0) + E\left[ \sup_{t \leq \tau_n} e^{-\frac{1}{2}\phi_t} |\xi_t| \cdot \left( \int_{\tau_n}^{\tau_n} Ne^{-\phi_t}|\xi_t|^2 dt \right)^{\frac{1}{2}} \right]
\]
\[
\leq NB_2(x_0, \xi_0) + \frac{1}{2}E \sup_{t \leq \tau_n} e^{-\phi_t}|\xi_t|^2 + \frac{1}{2}E\left( \int_{\tau_n}^{\tau_n} Ne^{-\phi_t}|\xi_t|^2 dt \right)
\]
\[
\leq NB_2(x_0, \xi_0) + \frac{1}{2}E \sup_{t \leq \tau_n} e^{-\phi_t}|\xi_t|^2,
\]
which implies that
\[ E \sup_{t \leq \tau_n} e^{-\phi t} |\xi_t|^2 \leq NB_2(x_0, \xi_0). \]
So (3) is true by letting \( n \to \infty \).

Now we estimate the moments of the second quasiderivative \( \eta_t \). Based on our definition, we have
\[ d\eta_t = [\tilde{\sigma} + G]dw_t + [\tilde{b} + H]dt, \]
where
\[
\begin{align*}
\tilde{\sigma} &= \tilde{\sigma}(x, \eta) = \sigma(\gamma) + \tilde{r}\sigma + \sigma\tilde{P}, \\
\tilde{b} &= \tilde{b}(x, \eta) = b(\gamma) + 2\tilde{r}b - \sigma\tilde{n}, \\
G &= G(x, \xi) = \sigma(\xi)(\xi) + 2r\sigma(\xi) - r^2\sigma + (2\sigma(\xi) + 2r\sigma + \sigma P)P, \\
H &= H(x, \xi) = b(\xi)(\xi) + 4rb(\xi) - 2(\sigma(\xi) + r\sigma - \sigma P)\pi.
\end{align*}
\]
From the expressions above, we have the estimates
\[ \|G\| \leq N|\xi|^2; \quad |H| \leq N|\xi|^2. \]
Hence Itô’s formula implies
\[ d(e^{-2\phi t}|\eta_t|^2) = e^{-2\phi t} [2(\eta_t, \tilde{b} + H) + \|\tilde{\sigma} + G\|^2 - 2c|\eta_t|^2] dt + 2e^{-2\phi t}(\eta_t, \tilde{\sigma} + G)dw_t. \]
Notice that
\[
\begin{align*}
2(\eta_t, \tilde{b} + H) + \|\tilde{\sigma} + G\|^2 - 2c|\eta_t|^2 \\
= 2(\eta_t, \tilde{b}) + \|\tilde{\sigma}\|^2 - 2c|\eta|^2 + 2(\eta_t, H) + |H|^2 + 2(\tilde{\sigma}, G) \\
\leq (c - 1)|\eta|^2 - 2c|\eta|^2 + |\eta|^2 + N|\xi|^4 \\
\leq -|\eta|^2 + N|\xi|^4.
\end{align*}
\]
So for any bounded stopping time \( \gamma \) with respect to \( \{F_t\} \), we have
\[ Ee^{-2\phi \tau} |\eta_\tau|^2 \leq E \int_0^\gamma e^{-2\phi t}|\eta_t|^2 dt \leq E \int_0^\gamma Ne^{-2\phi t}|\xi_t|^4 dt. \]
Recall that \( \eta_0 = 0 \). By Theorem III.6.8 in [10], we have
\[
E \sup_{t \leq \tau_2} e^{-\phi t}|\eta_t| \leq 3E \left( \int_0^{\tau_2} Ne^{-2\phi t}|\xi_t|^4 dt \right)^{\frac{1}{2}} \leq E \left[ \sup_{t \leq \tau_2} e^{-\frac{1}{2}\phi t}|\xi_t| : \left( \int_0^{\tau_2} 9Ne^{-\phi t}|\xi_t|^2 dt \right)^{\frac{1}{2}} \right] \leq \frac{1}{2} E \sup_{t \leq \tau_2} e^{-\phi t}|\xi_t|^2 + \frac{1}{2} E \int_0^{\tau_2} 9Ne^{-\phi t}|\xi_t|^2 dt \leq NB_2(x_0, \xi_0),
\]
which implies that (4) and (5) are true.
Finally, rewriting $c - 1$ by $(c - \frac{1}{2}) - \frac{1}{2}$ and repeating the argument above, we conclude that (6) is true.

\[\square\]

Now we are ready to prove the theorem.

**Proof of (3.3).** Denote $\tau_{D_1}(x)$ and $\tau_{D,2}(x)$ by $\tau_1^d$ and $\tau_2$, respectively. From (1.2) we immediately have

\begin{equation}
(3.14) \quad |u|_{0,D} \leq |g|_{0,D} + |f|_{0,D} E \int_0^T e^{-t} dt \leq |g|_{0,D} + |f|_{0,D}.
\end{equation}

When $x_0 \in D_0^\delta$, by Theorem 2.2 we have

\[u(x_0) = X_0 = EX_{\tau_1^d}^\delta.\]

So from (2.8) and (3.14),

\[
|u(x_0)| \leq E\left|u(x_{\tau_1^d}) + (\xi_{\tau_1^d} + \xi_{\tau_1^d}^{d+1})u(x_{\tau_1^d})\right|
\]

\[+ |f|_{1,D} E \int_{0}^{\tau_1^d} e^{-s}\left(|\xi| + 2r + |\xi|^{d+1}\right) ds
\]

\[\leq E\left|u(x_{\tau_1^d})\right| + \left(|g|_{0,D} + |f|_{0,D}\right)\left(E\xi_{\tau_1^d}^0 + E\xi_{\tau_1^d}^{d+1}\right)
\]

\[+ |f|_{1,D} E \int_{0}^{\tau_1^d} |\xi| + 2r ds + E \sup_{t \leq \tau_1^d} |\xi| + E \sup_{t \leq \tau_1^d} |\xi|^{d+1}.
\]

By Lemma 3.3 Davis inequality and Hölder inequality,

\[E|u(x_{\tau_1^d})| \leq \sup_{x \in \partial D_0^\delta} \frac{|u(x)|}{\sqrt{B_1(x, \xi)}} \cdot E \sqrt{B_1(x_{\tau_1^d}, \xi_{\tau_1^d})}
\]

\[\leq \sup_{x \in \partial D_0^\delta} \frac{|u(x)|}{\sqrt{B_1(x, \xi)}} \cdot \sqrt{B_1(x_{0}, \xi_0)},
\]

\[E|\xi_{\tau_1^d}^0| \leq E \sup_{t \leq \tau_1^d} |\xi|^0 \leq 3 E \xi_{\tau_1^d}^0 \tau_1^d \leq 3 \left(E \xi_{\tau_1^d}^0 \right)^\frac{1}{2}
\]

\[\leq N \left(E \int_{0}^{\tau_1^d} \frac{\psi(\xi)}{\psi^2} ds\right)^\frac{1}{2} \leq N \sqrt{B_1(x_{0}, \xi_0)},
\]

\[E|\xi_{\tau_1^d}^{d+1}| \leq E \sup_{t \leq \tau_1^d} |\xi|^{d+1} \leq NE \int_{0}^{\tau_1^d} |\xi| + \frac{|\psi(\xi)|}{\psi} ds
\]

\[\leq NE \int_{0}^{\tau_1^d} I_{s \leq \tau_1^d} \cdot I_{s \leq \tau_1^d} \left(|\xi| + \frac{|\psi(\xi)|}{\psi}\right) ds
\]

\[\leq N \left(E \tau_1^d \right)^\frac{1}{2} \left(E \int_{0}^{\tau_1^d} |\xi|^2 + \frac{\psi(\xi)}{\psi^2} ds\right)^\frac{1}{2}
\]

\[\leq N \sqrt{B_1(x_{0}, \xi_0)},
\]
Similarly, when

\[ E \int_{0}^{\tau_{2}} |\xi_{s}| + 2r_{s} ds \leq NE \int_{0}^{\tau_{2}} |\xi_{s}| + \frac{|\psi(\xi_{s})|}{\psi} ds \leq N \sqrt{B_{1}(x_{0}, \xi_{0})}. \]

Collecting all estimates above, we conclude that

\[ |u(\xi_{0})(x_{0})| \leq \sup_{x \in \partial D_{0}^{\lambda}} \frac{|u(\xi)(x)|}{\sqrt{B_{1}(x, \xi)}} \cdot \sqrt{B_{1}(x_{0}, \xi_{0})} + N(|g|_{0,D} + |f|_{1,D}) \sqrt{B_{1}(x_{0}, \xi_{0})}. \]

So for any \( x_{0} \in D_{0}^{\lambda}, \xi_{0} \in \mathbb{R}^{d} \setminus \{0\} \), we have

\[ |u(\xi_{0})(x_{0})| \leq \sup_{x \in \partial D_{0}^{\lambda}} \frac{|u(\xi)(x)|}{\sqrt{B_{1}(x, \xi)}} + N_{1}, \]

with

\[ N_{1} = N(|g|_{1,D} + |f|_{1,D}). \]

Similarly, when \( x_{0} \in D_{\lambda,2} \), by Theorem 2.2 we have

\[ u(\xi_{0})(x_{0}) = X_{0} = EX_{\tau_{2}}. \]

Again, from (2.8) and (3.14),

\[ |u(\xi_{0})(x_{0})| \leq E e^{-\frac{1}{2} \phi_{s}} |u(\xi_{2})|_{x_{\tau_{2}}} + (\xi_{0}^{0} + \xi_{d+1}^{0}) u(x_{\tau_{2}}) \]

\[ + |f|_{1,D} E \int_{0}^{\tau_{2}} e^{-\phi_{s}} \left(|\xi_{s}| + 2r_{s} + |\xi_{s}^{0}| + |\xi_{s}^{d+1}|\right) ds \]

\[ \leq E e^{-\frac{1}{2} \phi_{s}} |u(\xi_{2})|_{x_{\tau_{2}}} + \left(E e^{-\frac{1}{2} \phi_{s}} |\xi_{0}^{0}| + E e^{-\frac{1}{2} \phi_{s}} |\xi_{d+1}^{0}|\right) \]

\[ + |f|_{1,D} \left(E \int_{0}^{\tau_{2}} e^{-\phi_{s}} (|\xi_{s}| + 2r_{s}) ds + 4E \sup_{t \leq \tau_{2}} e^{-\frac{1}{2} \phi_{s}} |\xi_{s}^{d+1}| + 2E \sup_{s \leq \tau_{2}} e^{-\frac{1}{2} \phi_{s}} |\xi_{s}^{0}|\right). \]

By Lemma 3.4 Davis inequality and Hölder inequality,

\[ E e^{-\frac{1}{2} \phi_{s}} |u(\xi_{2})|_{x_{\tau_{2}}} \leq \sup_{x \in \partial D_{\lambda,2}} \frac{|u(\xi)(x)|}{\sqrt{B_{2}(x, \xi)}} \cdot E \sqrt{e^{-\phi_{s}} B_{2}(x_{\tau_{2}}, \xi_{\tau_{2}})} \]

\[ \leq \sup_{x \in \partial D_{\lambda,2}} \frac{|u(\xi)(x)|}{\sqrt{B_{2}(x, \xi)}} \cdot \sqrt{B_{2}(x_{0}, \xi_{0})}, \]

\[ E e^{-\frac{1}{2} \phi_{s}} |\xi_{d+1}^{0}| \leq E \sup_{s \leq \tau_{2}} e^{-\frac{1}{2} \phi_{s}} |\xi_{s}^{0}| = E \sup_{s \leq \tau_{2}} \left| \int_{0}^{s} e^{-\frac{1}{2} \phi_{s}} \pi_{r} dw_{r} \right| \]

\[ \leq 3E \left( \int_{0}^{\tau_{2}} e^{-\phi_{s}} |\pi_{r}|^{2} dr \right)^{\frac{1}{2}} \]

\[ \leq NE \left( \int_{0}^{\tau_{2}} e^{-\phi_{s}} |\pi_{r}|^{2} dr \right)^{\frac{1}{2}} \]

\[ \leq N \sqrt{B_{2}(x_{0}, \xi_{0})}, \]

\[ E e^{-\phi_{s}} |\xi_{s}^{d+1}| \leq E \sup_{t \leq \tau_{2}} e^{-\frac{1}{2} \phi_{t}} |\xi_{s}^{d+1}| \leq NE \int_{0}^{\tau_{2}} e^{-\frac{1}{2} \phi_{s}} |\xi_{s}| ds \]
\[
\leq N E \left( \int_0^{T_2} e^{-\frac{1}{2} \phi_s} ds \right)^{\frac{1}{2}} \left( \int_0^{T_2} e^{-\phi_s} |\xi_s|^2 ds \right)^{\frac{1}{2}}
\]
\[
\leq N \left( E \int_0^{T_2} e^{-\phi_s} |\xi_s|^2 ds \right)^{\frac{1}{2}}
\]
\[
\leq N \sqrt{B_2(x_0, \xi_0)},
\]
\[
E \int_0^{T_2} e^{-\phi_s} (|\xi_s| + 2r_s) ds \leq NE \int_0^{T_2} e^{-\phi_s} |\xi_s| ds \leq N \sqrt{B_2(x_0, \xi_0)}.
\]
Collecting all estimates above, we conclude that
\[
|u(\xi_0)(x_0)| \leq \sup_{x \in \partial D_\lambda} \frac{|u(\xi)(x)|}{\sqrt{B_2(x, \xi)}} \cdot \sqrt{B_2(x_0, \xi_0)} + N(|g|_{0,D} + |f|_{1,D}) \sqrt{B_2(x_0, \xi_0)}.
\]
So for any \( x_0 \in D_{2\lambda}, \xi_0 \in \mathbb{R}^d \setminus \{0\} \), we have
\[
(3.17) \quad \frac{|u(\xi_0)(x_0)|}{\sqrt{B_2(x_0, \xi_0)}} \leq \sup_{x \in \partial D_\lambda} \frac{|u(\xi)(x)|}{\sqrt{B_2(x, \xi)}} + N_1,
\]
with \( N_1 \) defined by (3.16).

Notice that
\[
B_1(x, \xi) \begin{cases} 
\geq \sqrt{\psi}(1 + \sqrt{\psi})|\xi|^2 \geq \lambda^\frac{1}{2}|\xi|^2 & \text{on } \{ \psi = \lambda \} \\
\leq \lambda(2 + \lambda)|\xi|^2 + K_1(2\lambda)^\frac{1}{2}\psi(\xi) \leq K\lambda|\xi|^2 & \text{on } \{ \psi = \lambda^2 \}.
\end{cases}
\]
Recall that \( K \) doesn’t depend on \( \lambda \). So for sufficiently small \( \lambda \), we have \( B_1(x, \xi) \geq 4B_2(x, \xi) \) when \( \psi = \lambda \), \( 4B_1(x, \xi) \leq B_2(x, \xi) \) when \( \psi = \lambda^2 \).

Then on \( \{ x \in D : \psi(x) = \lambda \} \), we have
\[
\frac{|u(\xi)(x)|}{\sqrt{B_1(x, \xi)}} \begin{array}{l}
\leq \frac{1}{2} \frac{|u(\xi)(x)|}{\sqrt{B_2(x, \xi)}} \\
\leq \frac{1}{2} \left( \sup_{\{ \psi = \lambda^2 \}} \frac{|u(\xi)(x)|}{\sqrt{B_2(x, \xi)}} + N_1 \right) \\
\leq \frac{1}{4} \sup_{\{ \psi = \lambda^2 \}} \frac{|u(\xi)(x)|}{\sqrt{B_1(x, \xi)}} + N_1 + \frac{N_1}{4} \\
\leq \frac{1}{4} \left( \sup_{\{ \psi = \lambda \}} \frac{|u(\xi)(x)|}{\sqrt{B_1(x, \xi)}} + \sup_{\{ \psi = \delta \}} \frac{|u(\xi)(x)|}{\sqrt{B_1(x, \xi)}} + N_1 \right) + \frac{N_1}{4} \\
= \frac{1}{4} \sup_{\{ \psi = \lambda \}} \frac{|u(\xi)(x)|}{\sqrt{B_1(x, \xi)}} + \frac{1}{4} \sup_{\{ \psi = \delta \}} \frac{|u(\xi)(x)|}{\sqrt{B_1(x, \xi)}} + \frac{3N_1}{4},
\end{array}
\]
which implies that
\[
(3.18) \quad \sup_{\{ \psi = \lambda \}} \frac{|u(\xi)(x)|}{\sqrt{B_1(x, \xi)}} \leq \frac{1}{3} \sup_{\{ \psi = \delta \}} \frac{|u(\xi)(x)|}{\sqrt{B_1(x, \xi)}} + N_1.
\]
Meanwhile, on \( \{ x \in D : \psi(x) = \lambda^2 \} \), we have
\[
\frac{|u(\xi)(x)|}{\sqrt{B_2(x, \xi)}} \leq \frac{1}{2} \frac{|u(\xi)(x)|}{\sqrt{B_1(x, \xi)}} \\
\leq \frac{1}{2} \left( \sup_{\psi = \lambda} \frac{|u(\xi)(x)|}{\sqrt{B_1(x, \xi)}} + \sup_{\psi = \delta} \frac{|u(\xi)(x)|}{\sqrt{B_1(x, \xi)}} + N_1 \right) \\
\leq \frac{1}{2} \left( \frac{1}{3} \sup_{\psi = \delta} \frac{|u(\xi)(x)|}{\sqrt{B_1(x, \xi)}} + N_1 \right) + \frac{1}{2} \sup_{\psi = \delta} \frac{|u(\xi)(x)|}{\sqrt{B_1(x, \xi)}} + \frac{N_1}{2} \\
= \frac{2}{3} \sup_{\psi = \delta} \frac{|u(\xi)(x)|}{\sqrt{B_1(x, \xi)}} + N_1.
\]

Therefore,
\[
(3.19) \quad \sup_{\psi = \lambda} \frac{|u(\xi)(x)|}{\sqrt{B_2(x, \xi)}} \leq \frac{2}{3} \sup_{\psi = \delta} \frac{|u(\xi)(x)|}{\sqrt{B_1(x, \xi)}} + N_1.
\]

Combining (3.15) and (3.18), we get, for any \( x \in D^\lambda_0, \xi \in \mathbb{R}^d \setminus \{0\}, \)
\[
(3.20) \quad \frac{|u(\xi)(x)|}{\sqrt{B_1(x, \xi)}} \leq \frac{4}{3} \sup_{\psi = \delta} \frac{|u(\xi)(x)|}{\sqrt{B_1(x, \xi)}} + 2N_1.
\]

Combining (3.17) and (3.19), we get, for any \( x \in D^\lambda_2, \xi \in \mathbb{R}^d \setminus \{0\}, \)
\[
(3.21) \quad \frac{|u(\xi)(x)|}{\sqrt{B_2(x, \xi)}} \leq \frac{2}{3} \sup_{\psi = \delta} \frac{|u(\xi)(x)|}{\sqrt{B_1(x, \xi)}} + 2N_1.
\]

Thus it remains to estimate
\[
\lim_{\delta \downarrow 0} \left( \sup_{\psi = \delta} \frac{|u(\xi)(x)|}{\sqrt{B_1(x, \xi)}} \right).
\]

Notice that for each \( \delta, \) there exist \( x(\delta) \in \{\psi = \delta\} \) and \( \xi(\delta) \in \{\xi : |\xi| = 1\}, \) such that
\[
\sup_{\psi = \delta} \frac{|u(\xi)(x)|}{\sqrt{B_1(x, \xi)}} = \frac{|u(\xi(\delta))(x(\delta))|}{\sqrt{B_1(x(\delta), \xi(\delta))}}.
\]

A subsequence of \((x(\delta), \xi(\delta))\) converges to some \((y, \zeta), \) such that \( y \in \partial D \) and \( |\zeta| = 1. \)

If \( \psi(\zeta)(y) \neq 0, \) then \( B_1(x(\delta), \xi(\delta)) \rightarrow \infty \) as \( \delta \downarrow 0. \) In this case,
\[
\lim_{\delta \downarrow 0} \left( \sup_{\psi = \delta} \frac{|u(\xi)(x)|}{\sqrt{B_1(x, \xi)}} \right) = \lim_{\delta \downarrow 0} \frac{|u(\xi(\delta))(x(\delta))|}{\sqrt{B_1(x(\delta), \xi(\delta))}} = 0.
\]

If \( \psi(\zeta)(y) = 0, \) then \( \zeta \) is tangential to \( \partial D \) at \( y. \) In this case,
\[
(3.22) \quad \lim_{\delta \downarrow 0} \left( \sup_{\psi = \delta} \frac{|u(\xi)(x)|}{\sqrt{B_1(x, \xi)}} \right) = \lim_{\delta \downarrow 0} \frac{|u(\xi(\delta))(x(\delta))|}{\sqrt{B_1(x(\delta), \xi(\delta))}} = \frac{|g(\zeta)(y)|}{\sqrt{\lambda}} \leq N \sup_{\partial D} |g_x|.
From (3.20), (3.21) and (3.22), we have
\[
\frac{|u(\xi)(x)|}{\sqrt{B_1(x,\xi)}} \leq N(|f|_{1,D} + |g|_{1,D}), \quad \text{when } x \in D^\lambda;
\]
\[
\frac{|u(\xi)(x)|}{\sqrt{B_2(x,\xi)}} \leq N(|f|_{1,D} + |g|_{1,D}), \quad \text{when } x \in D_{\lambda^2}.
\]
Notice that \(D^\lambda \cup D_{\lambda^2} = D\), and
\[
\sqrt{B_1(x,\xi)} \leq N(|\xi| + \frac{|\psi(\xi)|}{\psi_1^2}), \quad \text{when } x \in D^\lambda;
\]
\[
\sqrt{B_2(x,\xi)} \leq N(|\xi| + \frac{|\psi(\xi)|}{\psi_1^2}), \quad \text{when } x \in D_{\lambda^2}.
\]
We conclude that, for any \(x \in D\) and \(\xi \in \mathbb{R}^d\),
\[
|u(\xi)(x)| \leq N(|\xi| + \frac{|\psi(\xi)|}{\psi_1^2})(|f|_{1,D} + |g|_{1,D}).
\]
The inequality (3.3) is proved.

The proof of (3.4) is similar.

Proof of (3.4). When \(x_0 \in D_\delta^\lambda\), by Theorem 2.2, we have
\[
u(\xi_0)(x_0) = u(\xi_0)(x_0) + u(y_0)(x_0) = Y_0 = EY_{\tau_1^\delta}.
\]
From (2.9) and (3.14),
\[
|u(\xi_0)(x_0)| \leq E|u(\xi_1)(\xi_1^d)(x_{\tau_1^d})| + \sup_{x \in \partial D_{\delta^\lambda}, |\xi| = 1} |u(\xi)(x)| \cdot Ee^{-\tau_1^d}\left(\eta_{\tau_1^d} + 2\tilde{\eta}_{\tau_1^d}^0\right)
\]
\[
+ \left(|g|_{0,D} + |f|_{0,D}\right)Ee^{-\tau_1^d}\left|\tilde{\eta}_{\tau_1^d}^1\right| + |f|_{2,D}E \int_{\tau_1^d}^{T^\delta_s} e^{-s}\left[|\xi_s|^2 + |\eta_s| + (4r_s + 2|\tilde{\xi}_s^0|)|\xi_s| + 2\tilde{\xi}_s + 4|\tilde{\xi}_s^0|r_s + |\tilde{\eta}_s^0|\right] ds.
\]
Recall that in this case,
\[
\tilde{\xi}_s^0 = \xi_s^0 + \xi_s^{d+1}, \quad \tilde{\eta}_s^0 = 2\xi_s^0\xi_s^{d+1} + (\xi_s^{d+1})^2 + \eta_s^{d+1}.
\]
It follows that
\[
|u_{(\xi_0)}(\xi_0)(x_0)| \leq E|u(\xi_t^\delta)(\xi_t^\delta)(x_t^\delta)| + N\left(|g|_0D + |f|_0D + \sup_{x \in \partial D_\delta^1,\xi = 1} |u(\xi)|(x)\right)
\]
\[
\cdot e^{-\frac{\delta t}{\tau}} \left(|\eta_0^\delta| + |\xi_t^\delta|^2 + |\xi_t^0|^2 + |\xi_t^d|^2 + |\eta_0|^2 + |\eta_t^d|^2 + |\eta_t|^2 + \tau_\delta + \hat{r}_s\right) ds
\]
\[
+N|f|_{2,D} E\int_0^{\tau_\delta} e^{-s}\left(|\xi_s|^2 + |\eta_s| + |\xi_s^0|^2 + |\xi_s^d|^2 + |\eta_s|^2 + \tau_s^2 + \hat{r}_s\right) ds
\]
\[
\leq E|u(\xi_t^\delta)(\xi_t^\delta)(x_t^\delta)| + N\left(|g|_0D + |f|_{2,D} + \sup_{x \in \partial D_\delta^1,\xi = 1} |u(\xi)|(x)\right)
\]
\[
\cdot \left(E \sup_{t \leq \tau_\delta} |\eta_t| + E \sup_{t \leq \tau_\delta} |\xi_t|^2 + E \sup_{t \leq \tau_\delta} |\xi_t^0|^2 + E \sup_{t \leq \tau_\delta} e^{-\frac{1}{2}t} |\xi_t^d|^2 + E \sup_{t \leq \tau_\delta} e^{-\frac{1}{2}t} |\eta_t|^2 + E \int_0^{\tau_\delta} \tau_s^2 + \hat{r}_s ds\right).
\]

By Lemma 3.3, Davis inequality and Hölder inequality,
\[
E|u(\xi_t^\delta)(\xi_t^\delta)(x_t^\delta)| \leq \sup_{x \in \partial D_\delta^1} \frac{|u(\xi)|_1(x)}{B_1(x, \xi)} \cdot E\{1(x_t^\delta, \xi_t^\delta)
\]
\[
\leq \sup_{x \in \partial D_\delta^1} \frac{|u(\xi)|_1(x)}{B_1(x, \xi)} \cdot B_1(x_0, \xi_0),
\]

\[
E \sup_{t \leq \tau_\delta} \eta_t \leq NB_1(x_0, \xi_0),
\]

\[
E \sup_{t \leq \tau_\delta} |\xi_t|^2 \leq NB_1(x_0, \xi_0),
\]

\[
E \sup_{t \leq \tau_\delta} |\xi_t^0|^2 \leq 4E(\xi_t^0, \tau_\delta) \leq NE \int_0^{\tau_\delta} \frac{\psi_2(\xi_t))}{\psi^2} dt \leq NB_1(x_0, \xi_0),
\]

\[
E \sup_{t \leq \tau_\delta} e^{-\frac{1}{2}t} |\xi_t^d|^2 \leq NE \sup_{t \leq \tau_\delta} e^{-\frac{1}{2}t} \left(\int_0^t |\xi_s|^2 + \psi_2(\xi_s) ds \right)^2
\]
\[
\leq NE \sup_{t \leq \tau_\delta} e^{-\frac{1}{2}t} \int_0^t |\xi_s|^2 + \psi_2(\xi_s) ds
\]
\[
\leq NE \int_0^{\tau_\delta} |\xi_t|^2 + \psi_2(\xi_t) dt
\]
\[
\leq NB_1(x_0, \xi_0),
\]

\[
E \sup_{t \leq \tau_\delta} e^{-\frac{1}{2}t} |\eta_t|^2 \leq NE \sup_{t \leq \tau_\delta} e^{-\frac{1}{2}t} \int_0^t |\eta_t|^2 + \psi_2(\eta_t) + \eta_t ds
\]
\[
\leq NE \sup_{t \leq \tau_\delta} e^{-\frac{1}{2}t} \left(\int_0^t |\eta_t|^2 + \psi_2(\eta_t) ds + \sqrt{b} \left(\int_0^t |\eta_t|^2 ds\right)^{\frac{1}{2}}\right)
\]
Recall that in this case, \( x \in (3.23) \)

\[ \leq NB_1(x_0, \xi_0), \]

\[ E \int_0^{\tau_1^d} r_s^2 + \hat{r}_s ds \leq NE \int_0^{\tau_1^d} |\xi_s|^2 + \frac{\psi^2(\xi_s)}{\psi^2} dt \leq NB_1(x_0, \xi_0). \]

Collecting all estimates above, we conclude that

\[ |u(x_0(x_0))| \leq \sup_{x \in \partial D^\lambda_1} \frac{|u(x)(x)|}{B_1(x, \xi_0)} \cdot B_1(x_0, \xi_0) \]

\[ + N \left( |g|_{2, D} + |f|_{2, D} + \sup_{x \in \partial D^\lambda_1, |\xi| = 1} |u(x)| \right) B_1(x_0, \xi_0). \]

So for any \( x_0 \in D^\lambda_\delta, \xi_0 \in \mathbb{R}^d \setminus \{0\} \), we have

\[ |u(x_0(x_0))| \leq \sup_{x \in \partial D^\lambda_1} \frac{|u(x)|}{B_1(x, \xi_0)} \cdot B_1(x_0, \xi_0) + N, \]

with

\[ N_2 = N \left( |g|_{2, D} + |f|_{2, D} + \sup_{x \in \partial D^\lambda_1, |\xi| = 1} |u(x)| \right). \]

When \( x_0 \in D^\lambda_\delta \), by Theorem \( 2.2 \) we have

\[ u(x_0(x_0)) = u(x_0(x_0)) + u(x_0(x_0)) = Y_0 = EY_{\tau_2}. \]

Again, from \( (2.9) \) and \( (3.14) \),

\[ |u(x_0(x_0))| \leq Ee^{-\phi_{\tau_2}} |u(x)_{\tau_2}(x_{\tau_2})| + \sup_{x \in \partial D^\lambda_\delta, |\xi| = 1} |u(x)| \cdot E e^{-\phi_{\tau_2}} \left( |\eta_{\tau_2}| + 2|z_{\tau_2}| \right) \]

\[ + \left( |g|_{1, D} + |f|_{1, D} \right) E e^{-\phi_{\tau_2}} \left( |\eta_{\tau_2}| + 2|z_{\tau_2}| + |\eta_{\tau_2}| \right) \]

\[ \cdot E e^{-\phi_{\tau_2}} \left( |\eta_{\tau_2}| + 2|z_{\tau_2}| + |\eta_{\tau_2}| \right) \]

\[ \cdot E e^{-\phi_{\tau_2}} \left( |\eta_{\tau_2}| + 2|z_{\tau_2}| + |\eta_{\tau_2}| \right) \]

Recall that in this case,

\[ \tilde{\xi}_t^0 = \xi_t^0 + \xi_t^{d+1}, \quad \tilde{\eta}_t^0 = \eta_t^0 + 2\xi_t^0 \xi_t^{d+1} + \eta_t^{d+1}. \]

Also, notice that by \( (3.3) \)

\[ \sup_{x \in \partial D^\lambda_\delta, |\xi| = 1} |u(x)| \leq N \left( 1 + \frac{|g|_{1, D}}{\lambda^2} \right) \left( |f|_{1, D} + |g|_{1, D} \right) \leq N \left( |f|_{1, D} + |g|_{1, D} \right). \]

Therefore,

\[ |u(x_0(x_0))| \leq E e^{-\phi_{\tau_2}} |u(x)_{\tau_2}(x_{\tau_2})| + N \left( |g|_{1, D} + |f|_{1, D} \right) \]

\[ \cdot E e^{-\phi_{\tau_2}} \left( |\eta_{\tau_2}| + |\xi_{\tau_2}|^2 + |\xi_{\tau_2}^0|^2 + |\xi_{\tau_2}^{d+1}|^2 + |\eta_{\tau_2}^0| + |\eta_{\tau_2}^0| \right) \]

\[ + N |f|_{2, D} E \int_0^{\tau_2} e^{-\phi_s} \left( |\xi_s|^2 + |\eta_s| + |\xi_s^0|^2 + |\xi_s^{d+1}|^2 + |\eta_s^0| + |\eta_s^0| + r_s^2 + \hat{r}_s \right) ds. \]
By Lemma 3.4, Davis inequality and Hölder inequality,

\[
E e^{-\phi t} |u(\xi_2, \xi_3)(x_{32})| \leq N \left( |g|_1 + |f|_2 \right) \cdot \left( E \sup_{t \leq \tau_2} e^{-\phi t + \frac{1}{2} t} \eta_t + E \sup_{t \leq \tau_2} e^{-\phi t + \frac{1}{4} t} |\xi_t|^2 + E \sup_{t \leq \tau_2} e^{-\phi t + \frac{1}{2} t} \eta_t^0 \right)
\]

\[
+ E \sup_{t \leq \tau_2} e^{-\phi t + \frac{1}{4} t} |\xi_t| + E \int_{0}^{\tau_2} e^{-\phi s} (\xi_s^2 + \eta_s) ds.
\]

\[
\leq 4 E \int_{0}^{\tau_2} e^{-\phi t + \frac{1}{4} t} |\xi_t|^2 dt
\]

\[
\leq N E \int_{0}^{\tau_2} e^{-\phi t + \frac{1}{4} t} |\xi_t|^2 dt
\]

\[
\leq N E \sup_{t \leq \tau_2} e^{-\frac{1}{4} t} \left( \int_{0}^{t} e^{-\frac{1}{2} \phi_s + \frac{1}{4} s} |\xi_s| ds \right)^2
\]

\[
\leq N E \sup_{t \leq \tau_2} e^{-\frac{1}{2} t} \left( \int_{0}^{t} e^{-\frac{1}{2} \phi_s + \frac{1}{4} s} |\xi_s| ds \right)^2
\]

\[
\leq N E \sup_{t \leq \tau_2} e^{-\frac{1}{2} t} \cdot t \int_{0}^{t} e^{-\phi s + \frac{1}{4} s} |\xi_s|^2 ds
\]

\[
\leq N E \int_{0}^{\tau_2} e^{-\phi s + \frac{1}{4} s} |\xi_s|^2 ds
\]

\[
\leq N E \int_{0}^{\tau_2} e^{-\phi s + \frac{1}{4} s} |\xi_s|^2 ds + N E \sup_{t \leq \tau_2} e^{-\frac{1}{4} t} \int_{0}^{t} e^{-\phi_s + \frac{1}{4} s} |\eta_s| ds
\]
Collecting all estimates above, we conclude that

\[
\leq \text{NB}_2(x_0, \xi_0) + NE \sup_{t \leq T_2} e^{-\frac{1}{4}t} \cdot \sqrt{t} \left( \int_0^t e^{-2\beta_s + \tau_s} |\eta_s|^2 \, ds \right)^{\frac{1}{2}}
\]

\[
\leq \text{NB}_2(x_0, \xi_0) + NE \left( \int_0^{T_2} e^{-2\beta_s + \tau_s} |\eta_s|^2 \, ds \right)^{\frac{1}{2}}
\]

\[
\leq \text{NB}_2(x_0, \xi_0),
\]

\[
E \sup_{t \leq T_2} e^{-\phi_t + \frac{1}{2}t} |\eta_t'| \leq E \sup_{t \leq T_2} \left( \left| \int_0^t e^{-\frac{1}{4}\phi_t + \frac{1}{4}t} \pi_s \, ds \right|^2 + \int_0^t e^{-\phi_t + \frac{1}{2}t} |\pi_s|^2 \, ds \right)
\]

\[
+ \left| \int_0^t e^{-\phi_t + \frac{1}{2}t} \hat{\pi}_s \, ds \right|
\]

\[
\leq 5E \int_0^{T_2} e^{-\phi_t + \frac{1}{2}t} |\pi_s|^2 \, ds + 3E \left( \int_0^{T_2} e^{-\phi_t + \frac{1}{2}t} |\hat{\pi}_s|^2 \right)^{\frac{1}{2}} \, ds
\]

\[
\leq \text{NB}_2(x_0, \xi_0),
\]

\[
E \int_0^{T_2} e^{-\phi_s} (r_s^2 + \hat{r}_s) \, ds \leq NE \int_0^{T_2} e^{-\phi_s} (|\xi_s|^2 + |\eta_s|) \, ds
\]

\[
\leq \text{NB}_2(x_0, \xi_0) + N \left( \int_0^{T_2} e^{-s} \, ds \right)^{\frac{1}{2}} \left( \int_0^{T_2} e^{-2\beta_s + \tau_s} |\eta_s|^2 \, ds \right)^{\frac{1}{2}}
\]

\[
\leq \text{NB}_2(x_0, \xi_0).
\]

Collecting all estimates above, we conclude that

\[
|u_{(\xi_0)(\xi_0)}(x_0)| \leq \sup_{x \in \partial \mathcal{D}_2} \frac{|u_{(\xi)(\xi)}(x)|}{B_2(x, \xi)} \cdot B_2(x_0, \xi_0) + N(|g|_{1, D} + |f|_{2, D})B_2(x_0, \xi_0).
\]

So for any \( x_0 \in \mathcal{D}_\lambda \), \( \xi_0 \in \mathbb{R}^d \setminus \{0\} \), we have

\[
\left(3.25\right) \quad \frac{|u_{(\xi)(\xi)}(x_0)|}{B_2(x_0, \xi_0)} \leq \sup_{x \in \partial \mathcal{D}_\lambda} \frac{|u_{(\xi)(\xi)}(x)|}{B_2(x, \xi)} + N_2,
\]

with \( N_2 \) defined by \( (3.24) \).

Then on \( \{ x \in D : \psi(x) = \lambda \} \), we have

\[
\frac{|u_{(\xi)(\xi)}(x)|}{B_1(x, \xi)} \leq \frac{1}{4} \frac{|u_{(\xi)(\xi)}(x)|}{B_2(x, \xi)}
\]

\[
\leq \frac{1}{4} \left( \sup_{\{\psi = \lambda\}} \frac{|u_{(\xi)(\xi)}(x)|}{B_2(x, \xi)} + N_2 \right)
\]

\[
\leq \frac{1}{16} \left( \sup_{\{\psi = \lambda\}} \frac{|u_{(\xi)(\xi)}(x)|}{B_1(x, \xi)} + N_2 \right) + \frac{N_2}{4}
\]

\[
\leq \frac{1}{16} \left( \sup_{\{\psi = \lambda\}} \frac{|u_{(\xi)(\xi)}(x)|}{B_1(x, \xi)} + \sup_{\{\psi = \delta\}} \frac{|u_{(\xi)(\xi)}(x)|}{B_1(x, \xi)} + N_2 \right) + \frac{N_2}{4}
\]

\[
= \frac{1}{16} \sup_{\{\psi = \lambda\}} \frac{|u_{(\xi)(\xi)}(x)|}{B_1(x, \xi)} + \frac{1}{16} \sup_{\{\psi = \delta\}} \frac{|u_{(\xi)(\xi)}(x)|}{B_1(x, \xi)} + \frac{5N_2}{16}.
\]
which implies that

$$
(3.26) \quad \sup_{\{\psi=\lambda\}} \frac{|u(\xi)(x)|}{B_1(x, \xi)} \leq \frac{1}{15} \sup_{\{\psi=\delta\}} \frac{|u(\xi)(x)|}{B_1(x, \xi)} + \frac{N_2}{3}.
$$

Meanwhile, on \( \{x \in D : \psi(x) = \lambda^2\} \), we have

$$
\frac{|u(\xi)(x)|}{B_2(x, \xi)} \leq \frac{1}{4} \frac{|u(\xi)(x)|}{B_1(x, \xi)} \leq \frac{1}{15} \left( \sup_{\{\psi=\lambda\}} \frac{|u(\xi)(x)|}{B_1(x, \xi)} + \sup_{\{\psi=\delta\}} \frac{|u(\xi)(x)|}{B_1(x, \xi)} + N_2 \right)
$$

$$
\leq \frac{1}{15} \left( \frac{1}{15} \sup_{\{\psi=\delta\}} \frac{|u(\xi)(x)|}{B_1(x, \xi)} + 4 \right) + \frac{1}{4} \sup_{\{\psi=\delta\}} \frac{|u(\xi)(x)|}{B_1(x, \xi)} + \frac{N_2}{4}.
$$

Therefore,

$$
(3.27) \quad \sup_{\{\psi=\lambda^2\}} \frac{|u(\xi)(x)|}{B_2(x, \xi)} \leq \frac{4}{15} \sup_{\{\psi=\delta\}} \frac{|u(\xi)(x)|}{B_1(x, \xi)} + \frac{N_2}{3}.
$$

Combining (3.23) and (3.26), we get, for any \( x \in D_\delta^\lambda, \xi \in \mathbb{R}^d \setminus \{0\} \),

$$
(3.28) \quad \frac{|u(\xi)(x)|}{B_1(x, \xi)} \leq \frac{16}{15} \sup_{\{\psi=\delta\}} \frac{|u(\xi)(x)|}{B_1(x, \xi)} + \frac{4N_2}{3}.
$$

Combining (3.25) and (3.27), we get, for any \( x \in D_\lambda^2, \xi \in \mathbb{R}^d \setminus \{0\} \),

$$
(3.29) \quad \frac{|u(\xi)(x)|}{B_2(x, \xi)} \leq \frac{4}{15} \sup_{\{\psi=\delta\}} \frac{|u(\xi)(x)|}{B_1(x, \xi)} + \frac{4N_2}{3}.
$$

Thus it remains to estimate

$$
\lim_{\delta \downarrow 0} \left( \sup_{\{\psi=\delta\}} \frac{|u(\xi)(x)|}{B_1(x, \xi)} \right) \text{ and } \lim_{\delta \downarrow 0} \sup_{x \in \partial D_\delta^\lambda, |\xi|=1} |u(\xi)(x)|.
$$

First, notice that

$$
\lim_{\delta \downarrow 0} \sup_{x \in \partial D_\delta^\lambda, |\xi|=1} |u(\xi)(x)| \leq \sup_{x \in \partial D_\delta, |\xi|=1} |u(\xi)(x)| + \sup_{x \in \{\psi=\lambda\}, |\xi|=1} |u(\xi)(x)|
$$

$$
\leq \sup_{x \in \partial D, |l|=1, l \parallel \partial D} |u(l)(x)| + \sup_{x \in \partial D, |n|=1, n \perp \partial D} |u(n)(x)|
$$

$$
+ \sup_{x \in \{\psi=\lambda\}, |\xi|=1} |u(\xi)(x)|.
$$

Apply Lemma 3.2 and first derivative estimate (3.3), we get

$$
\lim_{\delta \downarrow 0} \sup_{x \in \partial D_\delta^\lambda, |\xi|=1} |u(\xi)(x)| \leq \sup_{x \in \partial D, |l|=1, l \parallel \partial D} |g(l)(x)| + N(|g|_{2, D} + |f|_{0, D})
$$
\[ + N\left(1 + \frac{|\psi|_{1,D}}{\lambda}\right)(|g|_{1,D} + |f|_{1,D}) \]
\[ \leq N(|g|_{2,D} + |f|_{1,D}). \]

Second, notice that for each \( \delta \), there exist \( x(\delta) \in \{ \psi = \delta \} \) and \( \xi(\delta) \in \{ \xi : |\xi| = 1 \} \), such that
\[ \sup_{\{\psi = \delta\}} \frac{|u(\xi(\delta))(x)|}{B_1(x, \xi)} = \frac{|u(\xi(\delta))(x(\delta))|}{B_1(x(\delta), \xi(\delta))}. \]

A subsequence of \((x(\delta), \xi(\delta))\) converges to some \((y, \zeta)\), such that \( y \in \partial D \) and \(|\zeta| = 1\).

If \( \psi(\zeta)(y) \neq 0 \), then \( B_1(x(\delta), \xi(\delta)) \rightarrow \infty \) as \( \delta \downarrow 0 \). In this case,
\[ \lim_{\delta \downarrow 0} \left( \sup_{\{\psi = \delta\}} \frac{|u(\xi(\delta))(x)|}{B_1(x, \xi)} \right) = \lim_{\delta \downarrow 0} \frac{|u(\xi(\delta))(\xi(\delta))|}{B_1(\xi(\delta), \xi(\delta))} = 0. \]

If \( \psi(\zeta)(y) = 0 \), then \( \zeta \) is tangential to \( \partial D \) at \( y \). In this case,
\[ \lim_{\delta \downarrow 0} \left( \sup_{\{\psi = \delta\}} \frac{|u(\xi(\delta))(x)|}{B_1(x, \xi)} \right) = \lim_{\delta \downarrow 0} \frac{|u(\xi(\delta))(\xi(\delta))|}{B_1(\xi(\delta), \xi(\delta))} = \frac{|g(\zeta(\zeta))(y)| + K|u(\alpha)(y)|}{\lambda}. \]

By Lemma (3.2), we have
\[ \frac{|g(\zeta(\zeta))(y)| + K|u(\alpha)(y)|}{\lambda} \leq N(|g|_{2,D} + |f|_{0,D}). \]

Therefore, we have
\[ \frac{|u(\xi(\delta))(x)|}{B_1(x, \xi)} \leq N(|f|_{2,D} + |g|_{2,D}), \text{ when } x \in D^\lambda; \]
\[ \frac{|u(\xi(\delta))(x)|}{B_2(x, \xi)} \leq N(|f|_{2,D} + |g|_{2,D}), \text{ when } x \in D_{\lambda^2}. \]

It follows that, for any \( x \in D \) and \( \xi \in \mathbb{R}^d \),
\[ |u(\xi)(x)| \leq N(|\xi|^2 + \frac{\psi^2}{\psi})(|f|_{2,D} + |g|_{2,D}). \]

The inequality (3.4) is proved.

\[ \square \]

**Proof of the existence and uniqueness of (3.3).** The fact that \( u \) given by (1.2) satisfies (3.5) follows from Theorem 1.3 in [5].

To proof the uniqueness, assume that \( u_1, u_2 \in C^{1,1}_{\text{loc}}(D) \cap C^{0,1}(\bar{D}) \) are solutions of (3.3). Let \( \Lambda = \{ u_1_{|0,D} \vee u_2_{|0,D} \} \). For constants \( \delta \) and \( \varepsilon \) satisfying \( 0 < \delta < \varepsilon < 1 \), define
\[ \Psi(x, t) = \varepsilon(1 + \psi(x))\Lambda e^{-\delta t}, \quad U(x, t) = u(x)e^{-\varepsilon t} \text{ in } \bar{D} \times (0, \infty), \]
\[ F[U] = U_t + LU - cU + f \text{ in } D \times (0, \infty). \]

Notice that a.e. in \( D \), we have
\[ F[U_1 - \Psi] = -\varepsilon e^{-\varepsilon t}u_1 + \delta \Psi - \varepsilon \Lambda e^{-\delta t}L\psi + e\Psi \geq \varepsilon \Lambda(e^{-\delta t} - e^{-\varepsilon t}) \geq 0, \]
\[ F[U_2 + \Psi] = \varepsilon e^{-\varepsilon t}u_2 - \delta \Psi + \varepsilon \Lambda e^{-\delta t} L\psi - c \Psi \leq \varepsilon \Lambda (e^{-\varepsilon t} - e^{-\delta t}) \leq 0. \]

On \( \partial D \times (0, \infty) \), we have
\[ U_1 - U_2 - 2\Psi = -2\Psi \leq 0. \]

On \( \bar{D} \times T \), where \( T = T(\varepsilon, \delta) \) is a sufficiently large constant, we have
\[ U_1 - U_2 - 2\Psi = (u_1 - u_2)e^{-\varepsilon T} - 2\varepsilon (1 + \psi) \Lambda e^{-\delta T} \leq 2\Lambda (e^{-\varepsilon T} - e^{-\delta T}) \leq 0. \]

Applying Theorem 1.1 in [4], we get
\[ U_1 - U_2 - 2\Psi \leq 0 \text{ a.e. in } \bar{D} \times (0, T). \]

It follows that
\[ u_1 - u_2 \leq 2\varepsilon (1 + \psi) \Lambda e \to 0, \text{ as } \varepsilon \to 0, \text{ a.e. in } D. \]

Similarly, \( u_2 - u_1 \leq 0 \text{ a.e. in } D. \) The uniqueness is proved.

\[ \square \]

**Remark 3.4.** Based on our proof, if we replace \( \sigma(x), b(x), c(x), f(x) \) and \( g(x) \) in (1.3) and (1.2) by \( \sigma(\omega, t, x), b(\omega, t, x), c(\omega, t, x), f(\omega, t, x) \) and \( g(\omega, t, x) \), defined on \( \Omega \times [0, \infty) \times D \), under appropriate measurable assumptions, the first and second derivative estimates (3.3) and (3.4) are still true.

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