Magnets in an electric field: hidden forces and momentum conservation

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Abstract. In 1967 Shockley and James addressed the situation of a magnet in an electric field. The magnet is at rest and contains electromagnetic momentum, but there was no obvious mechanical momentum to balance this for momentum conservation. They concluded that some sort of mechanical momentum, which they called “hidden momentum”, was contained in the magnet and ascribed this momentum to relativistic effects, a contention that was apparently confirmed by Coleman and Van Vleck. Since then, a magnetic dipole in an electric field has been considered to have this new form of momentum, but this view ignores the electromagnetic forces that arise when an electric field is applied to a magnet or a magnet is formed in an electric field. The electromagnetic forces result in the magnet gaining electromagnetic momentum and an equal and opposite amount of mechanical momentum so that it is moving in its original rest frame. This moving reference frame is erroneously taken to be the rest frame in studies that purport to show hidden momentum. Here I examine the analysis of Shockley and James and of Coleman and Van Vleck and consider a model of a magnetic dipole formed in a uniform electric field. These calculations show no hidden momentum.

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1 Introduction

In 1891 J.J. Thompson pointed out an apparent paradox [1]. It appeared as if electromagnetic systems at rest could contain non-zero electromagnetic momentum. Later [2], he calculated the (Lorentz) electromagnetic momentum of an at-rest system consisting of a point charge in the vicinity of an Amperian dipole (for example, a loop of current) to be, in SI units, \( \epsilon_0 E \times B V \), where \( E \) is the electric field at the magnetic dipole, \( B \) is the uniform magnetic field inside the solenoid he used to approximate the Amperian dipole, and \( V \) is the solenoid’s volume. However, it does not appear he ever postulated a mechanical momentum, equal and opposite that of the electromagnetic momentum, presumably necessary to preserve momentum conservation [3].

In 1967 paper, Shockley and James [4] introduced the term “hidden momentum” to describe this “hitherto disregarded momentum” they thought necessary to conserve linear momentum in their charge and magnet model. In this model there are two small spheres of opposite charge in the vicinity of a magnet consisting of counter-rotating oppositely charged disks contained within a “pill box”. They were puzzled by the lack of mechanical momentum to balance the electromagnetic momentum in this at-rest system. Then when the magnet is demagnetized by bringing the rotating disks slowly to rest, the force they calculated acting on the charged spheres was not accompanied by an obvious equal and opposite force on the magnet necessary to conserve linear momentum. They proposed the existence of mechanical hidden momentum in the rotating dipoles that balanced the electromagnetic momentum before they were brought to rest. The release of this momentum as the disks slowed, they conjectured, would result in a force acting on the magnet. They speculated that relativistic effects were involved in this momentum.

A year later Coleman and Van Vleck [5] published a detailed analysis on the question of a point charge in the vicinity of a magnet to see if the conjecture of Shockley and James was correct. They based their analysis on a Lagrangian for electromagnetic systems derived by Darwin [6], and concluded that the demagnetizing magnet would experience an equal and opposite impulse to that experienced by the point charge and equal in magnitude to that found by Shockley and James. Hence, they conclude momentum is conserved in this system and the hidden momentum is due to relativistic effects as proposed by Shockley and James. Unlike the magnet of Shockley and James,
that of Coleman and Van Vleck was not model dependent, though they alluded to a specific model in footnote 9 of their paper.

In this paper, rather than beginning with a point charge-magnet system already intact, I apply the electric field to the magnet of Shockley and James by bringing in the charges originally very distant from the pill box magnet. There can be no question that there is zero momentum, electromagnetic and mechanical, as well as zero interaction energy, for this starting point. I show that upon assembly of the system (hereafter called the SJ model) Lorentz forces arise such that, if you want to keep this system at rest, you need an external source of mechanical force to counteract the electromagnetic momentum in the system.

Hence, to the charges and magnet in this reference frame you must add an external agent – let us say initially containing zero momentum – that can exert forces on the charges and magnet as the SJ model is assembled. You end up with the external agent having a mechanical momentum equal to the mechanical momentum which the SJ model would acquire were it not constrained to remain at rest, equal and opposite to the electromagnetic momentum in the system.

Thus if the SJ model contains hidden momentum equal and opposite to its electromagnetic momentum after being held stationary, you get the following paradox. The momenta of all components of the model in the original rest frame were zero, as was the momentum of the external agent, but now, in the same rest frame, the sum of the mechanical momentum of the external agent, the electromagnetic momentum of the SJ model, and the hidden mechanical momentum is no longer zero.

Next I revisit the paper of Coleman and Van Vleck and explicitly calculate the sums in the Darwin Lagrangian of the point charge-magnet system for a model where the current consists of non-interacting charges that flow along a circular frictionless track. (This is the model they alluded to in footnote 9.) I will show that the hidden momentum of the model only appears to be present because the calculations are, in effect, carried out in the rest frame of the track rather than in the center of momentum frame of the system. The effect of this is that the model turns out to be very contrived and does not correspond to the application of an electric field to a uniform current loop of non-interacting particles.

In my final calculation I consider the formation of a magnetic dipole inside a spherical shell with a dipolar electric charge distribution. This distribution produces a uniform electric field inside the shell and a dipolar electric field on the outside. The mechanical momentum acquired by the shell when the magnetic dipole is formed is equal and opposite to the momentum in the resulting electromagnetic field. No hidden momentum is present. (All accelerations in the calculations of this paper are assumed to be small enough that radiation effects may be ignored.) After this calculation, due to suggestions by a referee, I have added comments on a number of papers, some very recent.

An argument often made for the presence of hidden momentum is that the center-of-energy theorem [7] of relativity requires its presence. Some physicists with whom I have had private communications have claimed that it is not necessary to consider the assembly of the charge-magnet system; that is, applying an electric field to a magnet or creating a magnet in an electric field. They feel you can treat the system with this theorem without worrying about how the system came to be.

This is undoubtedly true for many systems. In the case of this sort of system, however, its formation imparts mechanical momentum to the system (equal and opposite to the electromagnetic momentum it receives), which will change its inertial reference frame unless held stationary. If held stationary, energy and momentum are exchanged with the system’s environment due to the necessity of countering the Lorentz forces when either an electric field is applied to a magnet or a magnet is formed in an electric field. To treat the restrained system as if were isolated in the universe, so to speak, is invalid in my view. Somewhere in this “universe” is the momentum that was exchanged, so to say the system must have zero momentum to satisfy the center-of-energy theorem ignores this exchange.

2 The assembly of the model of Shockley and James and the resulting momentum

To calculate the electromagnetic field momentum in the Lorentz formulation, you evaluate the volume integral of the Lorentz electromagnetic field momentum density. This integral can be written in SI units as follows

$$P_{em} = \varepsilon_0 \int_V (E \times B) \, dV,$$

where $E$ is the electric field and $B$ the magnetic field. When this momentum is nonzero and the system containing this momentum is at rest, it is generally thought [8] the electromagnetic momentum must be balanced by an equal amount of hidden momentum somehow mechanically present in the system. The problem with the models in which hidden momentum is inferred is the lack of appreciation of the effect of the Lorentz forces involved in assembling the models (namely, applying an electric field to a magnet or forming a magnet in an electric field). When the momentum imparted to the model in its assembly is taken into account, the total momentum, electromagnetic plus mechanical, is conserved without the need to postulate a hidden form.

To illustrate this, I will assemble the model of Shockley and James [9]. They considered a model (Fig. 1) consisting of two “plastic” coaxial disks at the origin of a coordinate system, their areas perpendicular to the $z$ direction with opposite charge distributions on their rims and rotating in opposite directions such that they produce a current $I$. This establishes a magnetic moment $\mathcal{M} = IA$ at the origin pointing in the positive $z$ direction where $A$ is the area of a disk.
The rotating disks are housed in a pill box with arms extending outward a distance $r$ along the positive and negative $x$ axis. At the end of these arms are two small stationary spheres containing equal and opposite charges $\pm q$, the positive charge on the sphere in the negative $x$ direction and the negative charge in the positive direction. They imagine there is a brake applying torque to the disks, as an example of a mechanical force. The external agent will experience an impulse equal and opposite to that in Eq. (2) as well as be shown below.

Each charge in the SJ model interacting with the magnetic moment will, as is well known [9], contribute an amount of momentum $(\epsilon_o \mu_o E \times \mathcal{M})$ stored in the electromagnetic field. The charges approach the disks along the $x$ axis, in the “equator” of the magnetic moment, so the magnetic field at their positions a distance $r$ from the disks is

$$B = \frac{-\mu_o q I A}{4\pi r^3} \hat{k}, \quad (3)$$

The Lorentz force experienced by each charge is of the same magnitude and direction. The total Lorentz force on the charges is

$$F = 2q \vec{v} \times \left(-\frac{\mu_o q I A}{4\pi r^3} \hat{k}\right) = \frac{\mu_o q I A v}{2\pi r^3} \hat{j}, \quad (4)$$

where $v$ is the speed of the charges. When you integrate the force over time you get the impulse delivered to the charges,

$$\Delta P = \frac{\mu_o q I A}{4\pi r^3} \hat{j} = -\mu_o \epsilon_o E \times \mathcal{M}, \quad (5)$$

a result also obtained by Boyer [10] for different model.

The next task is to compute the impulse delivered to the disks through the action of the displacement current. (Refer to Fig. 2.) Assuming the disks are sufficiently small, the displacement at their position due to both charges is

$$D = \frac{1}{2\pi} \frac{q}{\gamma^2 r^4} \approx \frac{1}{2}\frac{q}{\gamma^2 r^2},$$

for the slow-motion case where $\gamma \approx 1$. The displacement current responsible for the magnetic field at the edges of the disks at a given point along the $x$ axis will depend on the cross-sectional area defined by a circle with a diameter equal to the distance between the edges of the disks parallel to $y$. (See Fig. 2.) This cross-sectional area is

$$A = \pi a^2 \sin^2 \phi, \quad (7)$$

where $a$ is the radius of the disk and $\phi$ is the usual azimuth angle of a spherical coordinate system. The displacement current as a function of $\phi$ is therefore

$$I_D = \frac{dD}{dt} = \frac{2}{\gamma^2 r^5} q \alpha a^2 \sin^2 \phi = \frac{qv a^2}{r^5} \sin^2 \phi, \quad (7)$$

where $v = (-dr/dt) \hat{i}$ when you take the time derivative of $D$. From the integral form of Ampere’s law, you find
the magnetic field at the rims of the disks as a function of \( \omega \), the angular speed of the disks.

\[
\mathbf{B}_D = \frac{\mu_0 J_D}{2\pi a \sin \phi} \mathbf{k} = \frac{\mu_0 q a v}{2\pi r^2} \sin \phi \mathbf{k}. \tag{8}
\]

To find the Lorentz force on the disks, you perform the following integration.

\[
F_D = I \oint dl \times \mathbf{B}_D = I \oint d\phi \times \mathbf{B}_D = \frac{\mu_0 q I a v}{2\pi r^3} \mathbf{j}, \tag{9}
\]

where \( \mathbf{j} = -\sin \phi \mathbf{i} + \cos \phi \mathbf{j}. \) When you integrate this over time to get the impulse, you find

\[
\Delta P_D = \frac{\mu_0 q I a v}{4\pi r^2} \mathbf{j} = -\mu_0 e_o E \times \mathbf{M}. \tag{10}
\]

Note that this is equal to the impulse applied to the charges, Eq. (5), both in magnitude and direction. (This result was also previously obtained by Boyer using a different approach \[10\].)

To get the total impulse applied to the system by its assembly, you add Eq. (5) to Eq. (10). The result is equal and opposite that possessed by the external agent. The total momentum, linear and angular, of the SJ system plus external agent remains zero throughout this process, and no hidden momentum due to stresses in the disks as supposed by Shockley and James is necessary to conserve linear momentum. Rather the “hidden momentum” is contained in the external agent.

3 A model-dependent examination of the Coleman-Van Vleck calculation

The Darwin Lagrangian used by Coleman and Van Vleck \[3\] needs to be slightly reformulated to be usable in calculations of the model I examine. Referring to the system of Fig. 3, where there is a single charge in the vicinity of a magnet, the Lagrangian becomes, in SI units (symbols defined below),

\[
L = \frac{1}{2} m v^2 + \frac{1}{8\epsilon_0 c^2} m v^4 + \frac{1}{2} M V^2 + \frac{1}{8\epsilon_0 c^2} M V^4 \\
+ \frac{1}{2} \sum_i m_i v_i^2 + \frac{1}{8\epsilon_0 c^2} \sum_i m_i v_i^4 \\
- \frac{q}{4\pi \epsilon_0} \sum_i \frac{q_i}{|\mathbf{r} - \mathbf{a}_i|} - \frac{1}{8\pi \epsilon_0} \sum_{ij} \frac{q_i q_j}{|\mathbf{a}_i - \mathbf{a}_j|} \\
+ \frac{q}{8\pi \epsilon_0 c^2} \sum_i \frac{q_i}{|\mathbf{r} - \mathbf{a}_i|} \left[ \mathbf{v} \cdot \mathbf{v}_i + \frac{(\mathbf{v} \cdot (\mathbf{r} - \mathbf{a}_i))(\mathbf{v}_i \cdot (\mathbf{r} - \mathbf{a}_i))}{|\mathbf{r} - \mathbf{a}_i|^2} \right] \\
+ \frac{1}{8\pi \epsilon_0 c^2} \sum_{ij} \frac{q_i q_j}{|\mathbf{a}_i - \mathbf{a}_j|} \left[ \mathbf{v}_i \cdot \mathbf{v}_j + \frac{(\mathbf{v}_i \cdot (\mathbf{a}_i - \mathbf{a}_j))(\mathbf{v}_j \cdot (\mathbf{a}_i - \mathbf{a}_j))}{|\mathbf{a}_i - \mathbf{a}_j|^2} \right].
\]

The model addressed in this section assumes the current producing the magnetism consists of a frictionless circular tube with the electrical properties of the vacuum, a radius equal to that of the magnet, and containing non-interacting positive charges moving in the positive sense in the \( x-y \) plane. This positive charge is neutralized by a coaxial ring of negative charge of the same radius that is
stationary. The origin of the coordinate system is the position of the center of the magnet. The third and fourth summations in the above Lagrangian can be considered zero if, as is assumed by both Shockley and James and by Coleman and Van Vleck, the magnet is neutral and electrically unpolarized. In this equation (see also Fig. 3) the quantities and variables are defined as follows.

- \( \rho \) = the speed of light,
- \( m \) = the mass of the point charge \( q \),
- \( \mathbf{v} \) = the velocity of the point charge,
- \( q \) = the positive point charge,
- \( M \) = the mass of the magnet,
- \( V \) = the velocity of the center of mass of the magnet,
- \( r \) = the position of the point charge (in the \(-x\) direction) from the origin,
- \( q_i \) = a (tiny) charge of a magnet particle,
- \( m_i \) = the mass of \( q_i \),
- \( \mathbf{a}_i \) = the position of \( q_i \) from the center of mass of the magnet,
- \( a \) = the radius of magnet,
- \( \rho \) = the position of the center of mass of the magnet from the origin,
- \( a' = a + \rho \),
- \( v_i = \mathbf{v}_i + \mathbf{V} \),
- \( v_i = \mathbf{v}_i \) = the velocity of \( q_i \) with respect to the center of mass of the magnet.

If the (neutral and unpolarized) magnet is massive enough, \( a' \) can be replaced with \( v_i \) in the Lagrangian. Also the current in the magnet is assumed here to be positive, so a \( q_i \) associated with its \( v_i \) is positive, and negative charges do not appear in the Lagrangian when \( V \) is absent.

When converting the sums to integrals, the following relationships are used.

- \( q_i \rightarrow \lambda d\rho \), where
- \( \lambda \) = the linear charge density of the positive current,
- \( a' = a + \rho \),
- \( \mathbf{v}_i = \omega \times a = \omega d\rho \), where,
- \( \phi = -\sin\phi + \cos\phi \).

### 3.1 The general momentum of the point charge

This result will come out the same as that of Coleman and Van Vleck [5], their equation (22). I call the momentum of the point charge \( P \). It is found by taking the gradient of the Darwin Lagrangian, \( L \), with respect to the velocity, \( \mathbf{v} \), of the point charge. The equation for the general momentum for the point charge is

\[
P = \frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v} \left( 1 + \frac{v^2}{2c^2} \right)
\]

\[+ \frac{q}{8\pi\epsilon_0 c^2} \sum_{i} \frac{q_i}{|\mathbf{r} + \rho - \mathbf{a}_i|} \left[ \mathbf{v}_i + \frac{(v_i \cdot (\mathbf{r} + \rho - \mathbf{a}_i))}{|\mathbf{r} + \rho - \mathbf{a}_i|^2} (\mathbf{r} + \rho - \mathbf{a}_i) \right].
\]

The first term on the right-hand side is the relativistic mechanical momentum to order \( c^{-2} \), and the second term is the electromagnetic momentum to the same order of approximation.

In converting the summation above into an integral, it is convenient to minimize the "busyness" of the math displayed by dropping \( \rho \) from the equations. This can be done because the length \( \rho \) is negligible compared to the length \( r \) and can be negligible to \( a \) if \( M \), the mass of the magnet, is sufficiently large. These assumptions appear to be compatible with those of Coleman and Van Vleck. Converting the sum in the above equation to an integral using the relationships above, you obtain

\[
P = m\mathbf{v} \left( 1 + \frac{v^2}{2c^2} \right)
\]

\[+ \frac{q\omega a^2}{8\pi\epsilon_0 c^2} \left[ \int_{0}^{2\pi} \frac{-\sin\phi + \cos\phi}{(r^2 + a^2 + 2racos\phi)^{1/2}} d\phi \right]
\]

\[+ \int_{0}^{2\pi} \frac{r \cdot (-\sin\phi + \cos\phi)}{(r^2 + a^2 + 2racos\phi)^{1/2}} d\phi \] .

There are four nonzero integrals over \( \phi \) to evaluate. These integrals can be approximated by noting that, consistent with the model of Shockley and James and that of Coleman and Van Vleck, \( r \) can be considered to be much larger than \( a \). This allows you to expand the denominators in powers of \( a/r \) before performing the integrations. With these approximations, the general momentum of the point charge becomes

\[
P \approx m\mathbf{v} \left( 1 + \frac{v^2}{2c^2} \right)
\]

\[+ \frac{q\omega a^2}{8\pi\epsilon_0 c^2} \left[ -\frac{\pi a_1}{r^2} - 3\frac{\pi a_1 \cdot \hat{r}}{r^2} \hat{r} - \frac{\pi a_1 \cdot \hat{r}}{r^2} \hat{r} + \frac{\pi a_1 \cdot \hat{i}}{r^2} \hat{i} \right],
\]

where \( \hat{r} \) is the unit vector in the \( r \) direction. Now \( r \) is a long position vector in the \(-x\) direction before the disks are brought to rest, so you might think you could replace \( \hat{r} \) to a good approximation by \(-\hat{i}\). Also, replace \( c^{-2} \) with \( \mu_0\epsilon_0 a_1 \). When this is done, the above equation becomes

\[
P \approx m\mathbf{v} \left( 1 + \frac{v^2}{2c^2} \right) - \frac{\mu_0\epsilon_0\omega a^3}{4\pi r^3} \hat{j}.
\]

Next note the following.

\[
(\omega a)(a^2 \hat{j}) = IA \hat{j} = IA(\hat{i} \times \hat{k}) = \hat{r} \times IA \hat{k}
\]

\[
= \frac{r \times \mathcal{M}},
\]

where \( \mathcal{M} \) is the magnetic moment of the magnet and, as before, \( \hat{r} \) has been identified with \(-\hat{i}\). This lets you express Eq. (14) as

\[
P \approx m\mathbf{v} - \frac{\mu_0\epsilon_0 a^3 r \times \mathcal{M}}{4\pi r^3},
\]

which is Coleman and Van Vleck’s equation (22) to the same degree of approximation as they used. This result can also be written as

\[
P = m\mathbf{v} + \mu_0\epsilon_0 E \times \mathcal{M},
\]

where \( E \) is the electric field at the position of the magnetic moment.
3.2 Motion of the center of energy and equation of motion of the magnet

The center of energy, following Coleman and Van Vleck [5], is,

\[
\frac{EX}{c^2} = m \left( 1 + \frac{v^2}{2c^2} \right) r + M \left( 1 + \frac{V^2}{2c^2} \right) \rho \\
+ \sum_i m_i \left( 1 + \frac{\omega^2}{2c^2} \right) a_i + \frac{q}{4\pi\epsilon_0c^2} \sum_i \frac{q_i}{|a_i - r|} \rho \\
+ \frac{1}{8\pi\epsilon_0c^2} \sum_{i \neq j} q_i q_j \left( \frac{1}{|a_i - a_j|} \right) \lambda, 
\]

where \( E \) here is the energy and \( V \) is the velocity of the magnet. (Note that this equation has the units of mass times position rather than just position due to the inclusion of electromagnetic energy.) Again following Coleman and Van Vleck, I write the time derivative of the center of energy as follows.

\[
\frac{1}{c^2} \frac{d(EX)}{dt} = \left[ \frac{1}{c^2} \frac{dx}{dt} \right] \rho + \sum_i \left( 1 + \frac{\omega^2}{2c^2} \right) v_i \\
+ \frac{q}{8\pi\epsilon_0c^2} \sum_i \frac{q_i}{|r - a_i|} \left[ v_i + \frac{(v_i \cdot (r - a_i))}{|r - a_i|^2} (r - a_i) \right] \\
+ \frac{1}{8\pi\epsilon_0c^2} \sum_{i \neq j} \frac{q_i q_j}{|a_i - a_j|} \left[ v_j + \frac{(v_j \cdot (a_i - a_j))}{|a_i - a_j|^2} (a_i - a_j) \right],
\]

which has the units of momentum.

In order to evaluate Eq. (20), the speeds of the particles must be determined by the applied electric field. The non-interacting masses \( m' \), confined to a frictionless circular track, experience an angle-dependent, radially-directed normal force, \( N \), and an electric force in the positive \( x \) direction of magnitude \( Eq' \) due to the external charge, where \( q' \) is the charge on the mass \( m' \). (Here, primed quantities are associated with the magnet.) Let \( a \) be the radius of the track, \( \phi \) the azimuth angle that locates the particle \( m' \) at time \( t \) with speed \( v \), and \( v_0 \) the initial azimuth angle where the particle has the initial speed \( v_0 \) at the time, \( t = 0 \), when the electric field is applied. Taking \( x = \alpha \cos \phi \) as the zero point of the electrical potential energy, the energy conservation equation for a particle is

\[
m'\gamma c^2 = m'\gamma_0 c^2 + q' Ea(\cos \phi - \cos \phi_0) 
\]

where \( \gamma \) and \( \gamma_0 \) are the Lorentz factors for \( v \) and \( v_0 \). Solving for \( v \) by expanding its Lorentz factor, you get

\[
v^2 = c^2 \left[ 1 + \frac{\gamma_0^2}{m'\gamma_0 c^2} (\cos \phi - \cos \phi_0) \right].
\]

Expanding \( \gamma_0 \) in terms of \( \gamma_0^2/c^2 \) to first order, you find

\[
v = v_0 \left[ 1 + \frac{2q'Ea}{m'\gamma_0 c^2} (\cos \phi - \cos \phi_0) \right]^\frac{1}{2},
\]

where \( 2q'Ea/(m'\gamma_0 c^2) \) is assumed to be very small compared to 1. This will be true for a weak electric field where particle accelerations are small enough that radiation can be neglected. Since \( v = \omega a \) and \( v_0 = \omega_0 a \) due to the circular motion, this equation becomes

\[
\omega = \omega_0 \left( 1 + \frac{2q'E}{m'\gamma_0 \omega_0^2 a} (\cos \phi - \cos \phi_0) \right)^\frac{1}{2},
\]

where the revolution is in the positive direction in the \( x-y \) plane. (A similar calculation was done by Boyer [10].)

The above is the angular motion of a single particle, beginning at \( \phi_0 \) with speed \( v_0 \). To get the non-relativistic motion, all you have to do is set \( \gamma_0 = 1 \). Since the non-relativistic motion is sufficient to obtain the result of Coleman and Van Vleck, this will be done in subsequent work in this section.

To get the motion of a collection of particles moving around the track and creating a uniform current, imagine inserting particles at \( \phi_0 = \pi/2 \) one at a time in equal intervals with speed \( v_0 \) until the current is complete. This process will add momentum to the system in the negative \( x \) direction but not in the \( y \) direction. Also, ignore the Lorentz forces that arise to imitate the disregard by Coleman and Van Vleck of how the system came to be. Eq. (24) with \( \cos \phi_0 = 0 \) will now describe the motion of each particle at a given angle \( \phi \). The non-uniform angular motion will also result in a non-uniform linear charge density, \( \lambda \), the motion of which constitutes the current. A short calculation shows, as expected, \( \lambda \) to vary inversely as \( \omega \). That is,

\[
\lambda = \lambda_0 \left( 1 + \frac{2q'E}{m'\omega_0^2 a} (\cos \phi - \cos \phi_0) \right)^\frac{-1}{2},
\]

where \( \lambda_0 \) is the linear charge density at \( \phi = \pi/2 \). This means \( I = \lambda_0 a = \lambda a \omega_0 \) and the evaluation of Eq. (15) is the same as before. (The current remains uniform as required.) The general momentum is once again Eq. (18).

The mass distribution in the magnet is also not uniform due to the non-uniform angular speed. Like the linear charge distribution, mass becomes more concentrated when the speed slows and less concentrated when the speed increases. The linear mass density of the current particles is

\[
d = d_0 \left( 1 + \frac{2q'E}{m'\omega_0^2 a} (\cos \phi - \cos \phi_0) \right)^\frac{-1}{2},
\]

where \( d_0 \) is the mass density at \( \phi = \pi/2 \).

The time derivative of the general momentum of Eq. (18) is taken to be zero, as in the analysis of Coleman and Van Vleck (to their degree of approximation), since the magnet is assumed to be neutral and unpolarized and terms on the order of \( v^2/c^2 \) are neglected. (See their equation (7)). This means, as Coleman and Van Vleck found in the equation below their equation (22), the force on the charge is

\[
m \frac{dv}{dt} = -\mu_0 c E \times \frac{dM}{dt},
\]

where

\[
M = \frac{m_0 v_0^2}{\gamma_0}.
\]
Turning next to the time derivative of the velocity of the center of energy, Eq. (20). The second sum has already been evaluated in the calculation of Eq. (12), resulting in Eq. (18). The double summation is zero due to the assumption on non-interaction, but the first sum, the kinetic energy, does not vanish. The result is, to the same approximation as Coleman and Van Vleck,

\[
\frac{1}{c^2} \frac{d(EX)}{dt} = mv + \sum_i m_i \frac{v^2}{2c^2} v_i + M\mathbf{v} + \mu_0 e_o \mathbf{E} \times \mathbf{M}.
\]

(28)

Using Eq. (29) and the relationships previously listed to turn summations into integrals, the summation in the above equation becomes

\[
\sum_i m_i \frac{v^2}{2c^2} v_i \rightarrow \frac{a^2 d\omega_0^2}{2c^2} \int_0^{2\pi} \left(1 + \frac{2q_i E}{m\omega_0^2} \cos \phi \right) \phi d\phi
\]

\[
= \frac{a^2 \lambda_0 E \omega_0^2}{c^2} \int_0^{2\pi} \cos^2 \phi d\phi = \frac{\mu_0 q \lambda_0 \lambda m^3}{4r^2}
\]

(29)

Then, using Eqs. (27) and (28), the derivative of the velocity of the center of energy becomes

\[
\frac{1}{c^2} \frac{d^2(EX)}{dt^2} = 0 = -2\mu_0 e_o \mathbf{E} \times \frac{d\mathbf{M}}{dt}
\]

(30)

\[
\frac{d}{dt}(EX) = m(1 + \frac{v^2}{2c^2}) \mathbf{v} + \sum_a m_a (1 + \frac{v^2}{2c^2}) \mathbf{v}_a
\]

(32)

which is the same result obtained by Coleman and Van Vleck. (For those possibly confused by the sign difference between the above equation and their equation (27), note that \( E = -1/(4\pi \varepsilon_0)|/r^2 \) since \( r = -r_i \).)

However, there is a serious oversight in the derivation of the above equation. For one thing notice how you can make a uniform current corresponding to Eq. (24) by adding particles one at a time at equal intervals at an arbitrary angle \( \phi \). Instead of an initial momentum of zero in the \( y \) direction and \(-N m v_s \), in the \( x \) direction, there would be a non-zero momentum of \( N m v_s \cos \phi \) in the \( y \) direction and \(-N m v_s \sin \phi \) in the \( x \) direction added to the system, where \( N \) is the number of inserted particles. The consequences of this observation is discussed in the next section.

### 4 A critical examination of the Coleman-Van Vleck result

The Coleman-Van Vleck result [5] would seem to confirm the suspicion of Shockley and James [4] that hidden momentum can result from relativistic kinetic energy effects. Thus it would seem that the force applied to the external charge, Eq. (27), is balanced by the opposite force applied to the magnet, Eq. (30), and momentum is conserved.

The claim by Coleman and Van Vleck that the magnet in their model experiences an equal and opposite force to that of the point charge is expressed in their equation (15). There is, in my view, an error in the interpretation of this equation. This equation is, in their notation,

\[
m\frac{d^2 \mathbf{r}}{dt^2} = -M \frac{d^2 \mathbf{X}_m}{dt^2},
\]

(31)

where \( m \) and \( \mathbf{r} \) are the mass and position of the point charge, \( M \) is the mass of the magnet, and \( \mathbf{X}_m \) is the magnet’s center of energy. The terms that go into \( M \) and \( \mathbf{X}_m \) are terms that could apply to the particles in the magnet and/or the electromagnetic field. Therefore, I argue that the term on the right hand side of the above equation could just as well correspond to the loss of electromagnetic and mechanical momentum. When the magnetic field decays, as in the SJ model, the momentum in the field is converted to mechanical momentum of the charge, and the system, which was moving in its original rest frame, now has zero mechanical momentum.

Coleman and Van Vleck calculate the force presumably applied to the magnet in their equation (26). This result comes from the time rate of change of the center of energy, their equation (11). I rewrite this equation below in their notation, separating the terms involving the point charge.

\[
\frac{d}{dt}(EX) = m(1 + \frac{v^2}{2c^2}) \mathbf{v} + \sum_a m_a (1 + \frac{v^2}{2c^2}) \mathbf{v}_a
\]

(32)

\[
+ \frac{e}{c^2} \sum_a \frac{e_a}{|r - r_a|} \left[ \mathbf{v} + \frac{(r - r_a)(\mathbf{v} \cdot (r - r_a))}{|r - r_a|^2} \right]
\]

\[
+ \frac{e}{c^2} \sum_a \frac{e_a}{|r - r_a|} \left[ \mathbf{v}_a + \frac{(r - r_a)(\mathbf{v}_a \cdot (r - r_a))}{|r - r_a|^2} \right]
\]

\[
+ \frac{1}{2c^2} \sum_{a \neq b} \frac{e_a e_b}{|r_a - r_b|} \left[ \mathbf{v}_b + \frac{(r_a - r_b)(\mathbf{v}_b \cdot (r_a - r_b))}{|r_a - r_b|^2} \right] - \mu_0 e_o \mathbf{E} \times \mathbf{M}
\]

where \( m \), \( \mathbf{v} \), \( \mathbf{r} \), and \( e \) refer to the mass, velocity, position, and charge, respectively, of the point charge. The subscripted quantities refer to the particles comprising the magnet. The last two terms on the right hand side of the above equation are gotten from their equation (14). Their equation (25) is

\[
M \frac{d\mathbf{X}_m}{dt} = \mathbf{P}_m - \frac{e}{c^2} \sum_a e_a \frac{(r_a \cdot \mathbf{v}_a) r_a}{r^3},
\]

(33)

where \( \mathbf{P}_m \) is the momentum of the isolated magnet and the second term on the right is what remains of the magnetic
portion of Eq. 8 after approximations have been made and the velocity of the external charge has been taken as zero \((v = 0)\). This means they have applied the center of energy theorem without taking into account the original momentum involved in the assembly of the system. At the beginning there were components destined to be parts of this system which had zero linear momentum. Assuming for the present case that no external agent kept the system stationary when it was put together, it would contain the mechanical momentum due to the Lorentz forces that arose during assembly. It also has the equal and opposite electromagnetic momentum in the fields.

If the system of Coleman and Van Vleck has hidden momentum equal and opposite to the electromagnetic momentum, when the magnetism dies out (for example, by the brake mechanism of Shockley and James) the electromagnetic momentum is converted to the mechanical momentum of the charge and the hidden momentum to the mechanical momentum of the magnet. They are equal and opposite, but the system is still moving with respect to its original rest frame due to the mechanical momentum imparted by the Lorentz forces. There is nothing to balance this momentum such that momentum is not conserved: This reference frame originally had zero momentum and now it has a net momentum. The paradox of Shockley and James is therefore not a paradox at all; hidden momentum, if it exists, is.

A side issue is the presence of electromagnetic angular momentum in the single-charge Coleman-Van Vleck model. A charge-magnet system has both linear and angular momentum in its fields [11]. For the model of Shockley and James, the presence of two equal and opposite charges at equal distances from the magnet results in zero angular field momentum, such that when the magnetic field decays, the equal impulses applied to the charges result in zero mechanical angular momentum. In the Coleman-Van Vleck case, however, there is a net angular momentum in the fields. When the magnetic field decays this results in mechanical angular momentum due to the impulse received by the single charge.

The circular frictionless track model based on the work of Coleman and Van Vleck and their footnote 9, appears in a more accessible version in Babson et al [8] in their “cleanest example of hidden momentum”. There is an error common to both these models. Other than ignoring the Lorentz forces that arise during the assembly of the systems, also ignored is the momentum transfer between the particles and the track they follow. The result is the system starts out already containing momentum in the frame of reference in which it is viewed. As pointed out in the above section, you can start out with the magnet consisting of charged particles traveling freely on a circular track having any momentum whatsoever and still get the results of Coleman and Van Vleck. This is because you are either viewing the magnet in its new rest frame, or the magnet is held stationary and momentum is added to its environment. The hidden momentum calculated in the above section is only seen in the frame of reference at rest with the track.

You might think that this artificial situation can be resolved by starting out with a magnet consisting of non-interacting charge carriers that form a uniform current before the electric field is applied. However, as will be shown below, the Coulomb forces cannot transfer net mechanical momentum to either the circular track model or the Babson et al model in a direction perpendicular to the field. In all the analyses that follow, it will be assumed that an external agent provides forces to counter the (magnetic) Lorentz forces so that the effects of the Coulomb forces can be isolated. This mimics the disregard of the Lorentz forces in the assembly of the systems.

It is perhaps clearest to see this oversight by looking at the example of Babson et al ("cleanest example of hidden momentum"). In one scenario, their model is a rectangular frictionless track containing non-interacting positive particles that nevertheless respond to a uniform electric field applied across the track. Imagine the bottom of the track defining the \(x\) axis and the left side the \(y\) axis with the origin of the axes at the lower left corner of the track. The corners of the rectangular track are curved so that the particles can move around the track without making a normal collision with a side (see Fig. 4).

The electric field is in the positive \(y\) direction such that particles accelerate in that direction (“up”) on the left side of the track, turn the corner and move with constant speed in the positive \(x\) direction (to the right) at the top of the track, decelerate while moving in the negative \(y\) direction (“down”) on the right side of the track, then finally move to the left along the bottom of the track at a speed less than what they had at the top of the track. The number of particles on the bottom of the track in their model is taken to be larger than that on the top by an amount necessary to maintain a uniform current.

They examine two versions of their particle scenario: a Newtonian version where the mass of the particles does not depend on their speed and a relativistic version that includes the relativistic increase of mass with speed. In the latter version the total momentum of the particles at the top of the track to the right is purported to be greater than that of the larger number of particles at the bottom of the track to the left due to this relativistic effect that increases the mass to first order by a factor of \(1 + v^2/c^2\). This is the source of the hidden momentum the track is supposed to hold. There is supposedly a net (hidden) momentum in the positive \(x\) direction equal and opposite to the electromagnetic field momentum that results from the presence of the electric and magnetic fields, presumably conserving momentum.

The error arises when Babson et al do not consider how the system evolves under the action of Coulomb forces after the electric field is applied. For simplicity and clarity, imagine their track is fixed so it can’t move parallel to the \(y\) direction but can slide frictionlessly parallel to the \(x\) direction. To illustrate the problem involving the momentum exchange at the corners of the track, follow a single particle after the electric field has been applied. Have it start at the origin moving with a small initial speed in the positive \(y\) direction (up the left side of the track). It
accelerates until it reaches the upper left corner at which point it is deflected to the right. The total momentum in the \( x \) direction was zero to begin with, and now it is still zero since the track must recoil to the left. (There is, of course, momentum transfer to the track in the positive \( y \) direction. If the track is held stationary in that direction, this momentum will be absorbed by the environment.)

Momentum transfer between the particle and track also occurs at the other three corners while the total momentum in the \( x \) direction remains zero. Finally the initial situation is regained with the particle moving up the left side of the track. Note that it doesn’t matter whether or not the mass of the particle is increased by relativistic effects at the top of the track and that the electric field does not change the momentum of the system in a direction perpendicular to itself (parallel to the \( x \) direction). Also note that, just as in the circular track model, you can start the particle anywhere and begin with a system containing arbitrary momentum. However, at any given time the track and the particle can each have a non-zero momentum, but these will be equal and opposite to maintain momentum conservation.

In the scenario involving the frictionless track version of the Darwin Lagrangian calculation, the particles were introduced at \( \phi = \pi/2 \) with an initial speed in the negative \( x \) direction at equal time intervals to create a uniform current. No net momentum in the \( y \) direction is introduced to the magnet by this procedure. The Lagrangian calculation for that situation resulted in a net momentum in a direction perpendicular to the electric field (parallel to \( y \)). Why is that? To find out you can once again look at the analogous Babson et al. model.

Issue four particles from the lower left corner at equal time intervals with +1 unit of momentum in the \( y \) direction and have the track held immobile so that any momentum transferred to the track goes into its environment. Let the time interval be such that at any given time each particle is on a different side of the track. Due to the action of a constant Coulomb force in the positive \( y \) direction, say the particles gain +1 unit of momentum in the \( y \) direction on the left (speeding up) and right (slowing down) sides of the track. The particles will collide elastically with the corners at different times but will always be separated from each other in time by the chosen time interval.

After \( N \) (where \( N \geq 3 \)) collisions have taken place at the top left corner of the track, \( N - 1 \) have taken place at the upper right corner, \( N - 2 \) at the lower right corner, and \( N - 3 \) at the lower left corner. Looking only at momentum parallel to the \( x \) direction, a collision at the upper left will impart a momentum of -2 units to the track, one at the upper right will impart +2 units, one at the lower right +1 unit, and one at the lower left -1 unit. So the net momentum imparted to the track (and subsequently transferred to its environment) after \( N \) collisions at the upper left is \(-2N + 2(N - 1) + (N - 2) - (N - 3) = -1\) unit in the \( x \) direction.

The track’s environment gained -1 unit of momentum in the \( x \) direction after the first two sets of collisions, and its momentum does not change afterwards. However, there is always a particle at the top of the track with a momentum of +2 and one at the bottom with a momentum of -1, so the total momentum of the system, particles plus track plus environment, is always zero. Adding more particles in a way to keep the current uniform will not change this picture qualitatively: The environment will contain momentum in the negative \( x \) direction while the particles will contain and equal and opposite momentum in the positive \( x \) direction.

The momentum of the system is zero, but if you view it from the track’s rest frame, it appears that there is a net (hidden) particle momentum of \( n/4 \) units in the positive \( x \) direction, where \( n \) is the number of particles. No relativistic effects are needed. The calculation performed earlier for the Coleman and Van Vleck model with the frictionless track does just that. The calculation is in effect performed in the rest frame of the track.

The above scenario is, of course, quite contrived. A more realistic scenario is one where the particles are initially moving around the track with the same speed equally spaced. The total momentum of the particles is zero. Once the electric field is applied, particles on the left side of the track moving up will gain momentum and particles on the right moving down will lose momentum. As a result of the particles having different speeds, the faster ones will pass the slower ones, assuming the particles are non-interacting as was done by both Coleman and Van Vleck and by Babson et al.

So, although you start out with a uniform current, this will change due to the effect of the electric field. As the system evolves, the total momentum of the particles moving right at the top of the track will be equal and opposite to that of the particles moving to the left at the bottom due to equal and opposite collisions at the corners. No hidden momentum results.

This situation is also true for a circular track if you start out with a uniform current instead of artificially creating one by introducing particles one by one. From Eq. \[21\] you find, approximately,

\[
dt = \frac{d\phi}{\omega} = \frac{1}{\omega_o} \left[ 1 - \frac{q'E}{m'\omega_o^2} (\cos \phi - \cos \phi_o) \right] d\phi. \tag{34}
\]
When you integrate this from $\phi_o$ to $\phi_o + 2\pi$, you find that the period of a particle starting out at $\phi_o$ with angular speed $\omega_0$ when the electric field is applied is about
\[ T = T_o \left(1 + \frac{qE}{m\omega_o^2} \cos \phi_o \right), \tag{35} \]
where $T_o = 2\pi/\omega_o$ is the original period. In general each particle will have a different average angular speed, $\bar{\omega} = 2\pi/T$. In fact, after a time
\[ t = \frac{2\pi - (\phi_2 - \phi_1)}{\omega_2 - \omega_1} = \frac{m'E_o}{q'E} \left[ \frac{2\pi - (\phi_2 - \phi_1)}{\cos \phi_2 - \cos \phi_2} \right] \tag{36} \]
goes by, a faster particle originally at $\phi_2$ will catch up to a slower one originally at $\phi_1$.

A way to get around this unphysical result is to assume the particles are interacting such that the speed of the particles remains the same (or, equivalently, the non-relativistic linear mass density remains uniform). Due to the uniform speed, the relativistic mass effect will not work to produce hidden momentum. However there will still be relativistic momentum due to relativistic effects involving pressure. Babson et al address this in their scenario of an incompressible and frictionless fluid flowing through a tube shaped like their track.

Babson et al use the relativistic expression for the momentum density in an incompressible frictionless electric fluid due to pressure ($= \gamma P v/c^2$ where $P$ is the pressure and $v$ is the fluid velocity) to show that there is a momentum difference, $\gamma/c^2 \left( P_T - P_B \right) \epsilon / A$, between the top and bottom of the tube due to the pressure difference, $P_T - P_B$, produced by the electric field, where $v$ is the fluid speed, $l$ is the length of the track segments, and $A$ is the cross-sectional area. This is certainly the case, but to say this is the net (hidden) momentum in the system runs into difficulties.

There is a simple thought experiment that is apropos to this situation. Instead of having a uniform electric field exert a force on a charged incompressible circulating fluid, say you apply a uniform gravitational field to an uncharged incompressible fluid. There is, of course, no electromagnetic momentum in the system, yet the fluid will still acquire the relativistic momentum that appears in the Babson et al example due to the pressure created by the gravitational force. Although it is unclear to me how a gravitational field might contain hidden momentum (apparently, there has to be a Maxwell’s theory of gravity of some sort), such a hidden momentum has been proposed [12]. Covering this is beyond the scope of this paper, but I argue below there is no need for hidden momentum in the gravitational field.

I look upon this situation as a case of momentum flow due to energy flow [13]. As the fluid flows in the direction of the gravitational field, it gains energy due to the increasing pressure. This energy then flows horizontally after it encounters the corner toward which it is moving. After encountering the next corner it flows counter to the gravitational field, losing energy. In the Newtonian view the energy lost or gained by the fluid is due to the gain or loss, respectively, of energy stored in the gravitational field.

Looking at the momentum, there is the same situation here regarding interaction between track and particles as in the non-interacting particle scenarios. The momentum transfer between the fluid and tube at the top left corner is equal and opposite to that at the top right corner (similarly for the bottom corners). The corner collisions produce a stress in the tube – a tension equal and opposite to the pressure in the fluid. Due to time differences between when a particular slug of fluid collides with the upper left corner and the upper right corner, there is a flow of stress in the tube, resulting in a steady state flow after the electric field reaches full strength. This stress counters the pressure in the fluid so that there is no net momentum flow. Planck [13] conjectured that in relativity theory when there is energy flow there is also momentum. If this is true, there can be no net momentum in the system since the net energy flow is zero. Momentum flow – that is momentum traveling through non-moving media – is important in other paradoxes, for example, the paradox of Tronton and Noble [14,15,16].

Summing up, the problem with both the model of Babson et al and the application of the circular frictionless track to the model of Coleman and Van Vleck is the same. Both ignore the momentum interactions within their models, and both ignore the Lorentz forces that arise when the models were formed or assembled. As has been shown, the Coulomb forces due to the application of an electric field cannot add net mechanical momentum to the models (plus environment if appropriate) in a direction perpendicular to the electric field.

When the electric field is turned off in the case of the incompressible fluid, the pressure in the fluid and stress in the tube will disappear as will the relativistic momentum. Both the internal energy flow and the consequent momentum flow will also disappear, and there would be no contribution to the motion of the track from this change, as the thought experiment with the gravitational field shows. (Of course, in all models you will get motion from Lorentz forces when the electric field is removed.) You might indeed use Boyer’s term “internal momentum” for this relativistic momentum [17], but it has no effect on the motion of the magnet as the net momentum flow in the direction perpendicular to the electric field is zero before, during, and after the field application. Hence this is different from the concept of hidden momentum: The net hidden momentum in the particle scenarios (if it existed) can affect the motion of the track, but this form of momentum cannot.

A complicating factor is an erroneous assumption made by Coleman and Van Vleck (and implicitly by Babson et al) with regard to their non-interacting particle scenarios. They assumed that the polarization of the magnet is negligible due to the distance between the magnet and the external charge when they implemented the Darwin Lagrangian, but this fails to appreciate the polarization of the charge in the magnet due to the non-uniform charge pattern (for example, more charge on the bottom of the Babson et al track than on the top).
If you examine the special case of a single particle orbiting a circular track and calculate the effective electric dipole by integrating \( \text{adq'} = a(a\lambda d\phi) \) over the loop while making the assumption that \( 2q'E/m\omega_0^2a \) is small compared to 1, you get \( -(q'E)/(m\omega_0^2a)(aq'=2) \). The magnitude of the effective dipole is equal to the ratio of the electric force on the charge element to the mechanical centripetal force on that element times a dipole moment of \( aq'/2 \). The effective dipole field in concert with the magnetic dipole field will contribute a small amount to the electromagnetic field momentum — about one-sixth of that due to the interaction of the magnetic dipole field with the electric field of the external charge if you approximate the dipoles with spheres of radius \( a \) containing uniform fields inside and dipolar fields outside. Of course, the incompressible fluid scenario does not exhibit this defect.

5 A magnetic dipole in a uniform electric field

In this section I will show that there is no hidden momentum when a magnet is formed in a uniform electric field, using as the model a spherical shell containing the uniform field with a magnet at its center. First I will calculate the mechanical momentum imparted to the shell as the magnetic dipole is formed and then show that the electromagnetic field momentum is equal and opposite to this. No hidden momentum is necessary to conserve linear momentum.

5.1 Mechanical momentum in the sphere-magnetic dipole system

A uniform Coulomb electric field directed in the positive \( y \) direction, given by

\[
E_o = \frac{1}{4\pi\epsilon_o a^3} \frac{p}{\hat{z}},
\]

will exist inside a spherical shell of radius \( a \) with an angular surface charge distribution given by

\[
\sigma = -\frac{3psin\theta sin\phi}{4\pi a^3},
\]

where \( p \) is the dipole moment of the shell. In this equation \( \theta \) and \( \phi \) are the polar and azimuth angles, respectively, of a system of spherical coordinates with the origin at the center of the sphere. Outside the sphere there is a dipole electric field with dipole moment \( p \).

Imagine that such a shell has a magnetic dipole formed at its center pointing in the positive \( z \) (\( \hat{k} \)) direction, increasing uniformly from zero to \( \mathcal{M}\hat{k} \). This could be done, for example, by having two small nested spherical shells at the origin containing equal and opposite surface charge (varying as \( sin\theta \)) with virtually no space between them. Torques provided by an external agent could set the spheres rotating about the \( z \) axis, gradually picking up speed. The torques could be equal and opposite, as would those arising from Lenz’ law, so that no net angular momentum is imparted.

The vector electromagnetic potential of the magnetic dipole at the location of the shell is

\[
A = \frac{\mu_o \mathcal{M}\hat{k} \times \hat{r}}{4\pi a^2} = \frac{\mu_o \mathcal{M}}{4\pi a^2} \sin \theta (\sin \phi \hat{i} + \cos \phi \hat{j}).
\] (39)

The increasing magnetic flux in the shell will induce a Faraday electric field given by

\[
E = \frac{\partial A}{\partial t} = -\frac{\mu_o \dot{\mathcal{M}}}{4\pi a^2} \sin \theta (\sin \phi \hat{i} + \cos \phi \hat{j}),
\] (40)

where \( \dot{\mathcal{M}} \) is the time derivative of the magnetic moment. This electric field will exert a force on each charge element on the shell. The charge on an element of area \( dA \) is given by

\[
dq = \sigma dA = -\frac{3psin\theta sin\phi}{4\pi a^3} \sin \theta d\phi d\theta d\phi = -\frac{3psin^2\theta sin\phi}{4\pi a} d\phi d\theta d\phi.
\] (41)

An element of force is then

\[
dF = Edq = \frac{3\mu_o p \mathcal{M}}{16\pi a^3} \sin \theta (\sin^2 \phi \hat{i} + \sin \phi \cos \phi \hat{j}) d\phi d\theta d\phi.
\] (42)

Integrating the above equation over \( \phi \) from 0 to \( 2\pi \) gets rid of the \( \hat{j} \) term, whereas the integral of \( sin^2 \phi \) results in \( \pi \). The integral over \( \theta \) of \( sin^2 \theta \) from 0 to \( \pi \) is \( 4/3 \). The force exerted on the shell due to the formation of the magnetic dipole is therefore

\[
F = -\frac{\mu_o p \mathcal{M}}{4\pi a^3} \hat{i}.
\] (43)

(There is no Lorentz force on the magnetic dipole.) Since the time rate of change of linear momentum equals the force, the mechanical momentum acquired by the shell is

\[
P_{mech} = -\frac{\mu_o p \mathcal{M}}{4\pi a^3} \hat{i} = -\epsilon_o \mu_o E_o \times \mathcal{M},
\] (44)

using Eq. (37) to get the right-hand result.

5.2 The electromagnetic momentum of the shell-magnetic dipole system

Although point magnetic dipoles don’t exist in nature, the magnetic field of such an object is given by

\[
B = \frac{\mu_o}{4\pi r^3} [3(\mathcal{M} \cdot \hat{r}) \hat{r} - \mathcal{M}] + \frac{2}{3} \epsilon_o \mathcal{M} \delta^3(r),
\] (45)

where \( \delta^3(r) \) is the Dirac delta function. This magnetic field should be applicable for a very small magnet inside the spherical shell. The creation of this dipole imparts a momentum to the shell given by Eq. (44). The magnetic
dipole rests at the center of the sphere in a uniform electric field given by Eq. (37). The electromagnetic momentum is calculated by Eq. (11) which inside the sphere is given by

\[
P_{em} = \frac{\mu_0 E_o}{4\pi} \int_0^\pi \int_0^{2\pi} \left[ \frac{3cos^2 \theta - 1}{r^3} \hat{i} 
- \frac{3sin\theta cos\theta \hat{o}}{r^3} \right] r^2 sin\theta dr d\theta d\phi + \frac{2}{3} \epsilon_0 \mu_0 E_o \hat{a}.
\] (46)

The integrals over the angles in Eq. (46) are zero, leaving only contribution to the momentum from the Dirac delta term. The electromagnetic momentum inside the shell is therefore

\[
P_{em-in} = \frac{2}{3} \epsilon_0 \mu_0 E_o \times \mathbf{M}.
\] (47)

Outside the sphere the electric field is that of a dipole, given by

\[
E = \frac{1}{4\pi \epsilon_0 r^2} [3(\hat{p} \cdot \hat{r}) \hat{r} - \hat{p}] .
\] (48)

The electromagnetic momentum outside the shell is

\[
P_{em-out} = \frac{\mu_0 M_p}{16\pi^2} \int_V \frac{1}{r^6} \left[ 3cos\theta \hat{r} \times \hat{j}
+ 3sin\theta sin^2 \hat{o} \hat{k} \times \hat{r} + \hat{i} \right] dV
= \frac{\mu_0 M_p}{16\pi^2} \int_0^a \int_0^\pi \int_0^{2\pi} \frac{1}{r^6} \left[ -3cos^2 \theta \hat{i} 
+ 3cos\theta sin\theta \hat{o} \hat{k} - 3sin^2 \theta sin^2 \hat{i} 
+ 3sin^2 \theta sin\theta \hat{o} \hat{k} + \hat{i} \right] r^2 sin\theta dr d\theta d\phi .
\] (49)

The integrals over \( \phi \) involving \( sin\theta cos\theta \) from 0 to 2\( \pi \) are zero and that over \( sin^2 \theta \) is \( \pi \). The other integrals over \( \phi \) are 2\( \pi \). Hence the electromagnetic momentum outside the shell becomes

\[
P_{em-out} = \frac{\mu_0 M_p \hat{i}}{16\pi^2} \int_a^\infty \int_0^\pi \int_0^{2\pi} \left[ -6cos^2 \theta sin\theta + 2sin\theta 
- 3sin^3 \theta \right] d\theta = \frac{1}{3} \epsilon_0 \mu_0 M_p \hat{i}.
\] (50)

Therefore

\[
P_{em-out} = \frac{1}{3} \epsilon_0 \mu_0 E_o \times \mathbf{M} .
\] (51)

Clearly the sum of Eqs. (41), (17), and (51) is zero. There is no hidden momentum necessary to achieve momentum conservation.

6 A review of other work on hidden momentum

One of the referees suggested that I include reviews of rather recent work – some of which predated the submission of this manuscript. This was a good suggestion, and it led me to other physics literature that was referenced in these papers that I felt also needed to be commented on. The following is the result of that effort.

Boyer [21] argued that an Amperian current loop (as a model of a neutron) experiences a force while passing a charged wire; that is, traveling through an electrostatic field. In response Aharonov et al [22] claimed that Boyer overlooked the role of hidden momentum. This question is central to the Aharonov-Casher effect [23], where there is a phase difference between a neutron passing on one side of the wire compared to one passing on the other side. Aharonov et al argue that the time rate of change of the hidden momentum of the neutron counters the force \((p \cdot \nabla)E\) where \(p\) is the induced electric dipole moment and \(E\) is the electric field) acting on the neutron such that the total force on the neutron is zero. Here I intend to show there is no hidden momentum in the neutron and it does experience a force in general while moving in an electric field.

First, Aharonov et al make the same error as others who examine charge-magnetic dipole interactions. There is momentum, electromagnetic momentum, in such a system when it is at rest. As argued here, the application of an electric field to a magnet results in both mechanical and electromagnetic momentum. If the neutron can be modeled as a tiny current loop, you can envision how it comes to rest in an electric field as follows.

The neutron enters the region of the field with mechanical momentum. As it progresses through the field, Lorentz forces impart additional mechanical momentum as an equal and opposite amount of momentum is stored in the electromagnetic field. If the additional mechanical momentum is just the right amount to bring the neutron to rest in the lab frame, there will be an amount of electromagnetic momentum associated with the neutron, but the mechanical momentum will be zero; the missing momentum has been transferred to the environment (for example, to the wire) by field interaction.

Whether or not the neutron is brought to rest, the system involved consists of the source of the electric field and the neutron – not just the neutron alone. This is what Aharonov et al fail to take into account in their argument that a neutron at rest in an electric field must contain hidden mechanical momentum. With no hidden momentum in a neutron modeled as a current loop, you can calculate the force acting on the neutron as Boyer did.

Vaidman [24] discussed the torque and force on a magnetic dipole. The situation addressed was that of a current loop with its magnetic moment in the positive \(x\) direction lying in the \(y-z\) plane. There is a uniform electric field in the positive \(z\) direction. The loop is at rest in the \(S'\) frame and moving in the positive \(x\) direction in the \(S\) frame. In the \(S\) frame there is no magnetic field and no torque observed on the loop. However, in the \(S'\) frame traveling with the loop there is a magnetic field, Lorentz forces acting on the current in the loop, and a subsequent torque given by

\[
\tau = \mathbf{M} \times \mathbf{B}' = -\mathbf{M} \mathbf{E} \hat{k} .
\] (52)
where \( \mathbf{B}' = \gamma(v/c^2)\mathbf{E}'\mathbf{j} = B_y \mathbf{j} \) is the magnetic field in the \( S' \) frame. This poses a paradox much like that of Mansuripur \[25\], Vaidman considers the work of Namias \[26\] and Bedford and Krumm \[27\] as pertaining to the resolution of the paradox. The model used by Namias for the loop involves charges shielded by conducting material. The model of Bedford and Krumm is like that of Colemen and Van Vleck \[5\], where the current charges are non-interacting but exposed to the electric field. Vaidman then proposes his own resolution based on a model like that of Babson et al, where there is a charged, incompressible fluid in a circular tube. Here I will address the models where the charges are exposed to the electric field, since they have hidden momentum-like mechanisms.

My resolution is much like that I presented concerning the paradox of Mansuripur \[28\]. When the dipole is at rest in the \( S \) frame the external electric field and magnetic field of the loop contain linear electromagnetic momentum given by \[11\]

\[
P = \mathbf{E} \times \mathbf{M}/c^2 = [(EM)/c^2]\mathbf{j}.
\]

(53)

There is no angular electromagnetic momentum in this case due to the uniformity of the electric field, but when you Lorentz-transform the angular momentum four-tensor, \( L^{\mu,\nu} \), containing the linear momentum times \( c^2 \) in the \( L^{2,4} \) slot to the \( S' \) frame, you get the following angular momentum in the positive \( z \) direction,

\[
L_z' = \gamma vtEM/c^2 = MB't.
\]

(54)

Note that this angular momentum is increasing in the positive \( z \) direction. Meanwhile the angular momentum due to the Lorentz forces is increasing in the negative \( z \) direction by the same amount. That is,

\[
\tau_z = \frac{dL_z'}{dt} - MB' = 0,
\]

(55)

where Eqs. \( 53 \) and \( 52 \) have been used. There is no torque on the loop in either frame.

An interesting example of hidden momentum discussed by Vaidman is a point charge at the center of a current carrying toroidal coil. The coil lies in the \( x-y \) plane with its center at the origin. Also at the origin is a point charge. Vaidman calculates the electromagnetic momentum of this combination as approximately \(-\epsilon_0\mu_0N\mathbf{M} \times \mathbf{E} \), where \( N \) is the number of turns in the coil, \( \mathbf{M} \) is the magnetic moment of a turn, and \( \mathbf{E} \) is the electric field at the coil due to the point charge. For a current that creates a magnetic field in the positive \( \phi \) direction around the coil and a positive point charge, this momentum is in the positive \( z \) direction.

You can assemble the charge-coil system loop-by-loop using the results given above for the Shockley-James model. As each loop is brought toward the charge in turn, an impulse of \((1/2)\epsilon_0\mu_0\mathbf{M} \times \mathbf{E} \) is imparted to the charge and to the loop. For \( N \) loops this is just the opposite of the electromagnetic momentum of the coil.

I would be remiss if I didn’t acknowledge that there are other ways to put the system together that don’t appear to lead to a balance of mechanical and electromagnetic momentum. For example, you can grow a charge in the center of the coil from zero to \( q \) (like charging one side of a capacitor). The opposing mechanical momentum received by the coil is half that of the electromagnetic momentum. Or, you can bring the charge in from a far distance along the axis of the coil towards its center. The combined mechanical momentum supplied to the coil from the Lorentz forces and lost to the charge due to the time rate of change of the vector potential at the position of the moving charge yields the same result \[29\]. It is clear that in the loop-by-loop assembly the changing magnetic field of each loop in turn exerts a force on the charge, a force that does not appear in my calculations of the other two methods of assembly. Possibly this can be resolved, but none of these methods of assembly is consistent with the presence of hidden momentum.

Recently, Boyer \[17\] has performed calculations for a model of a charge-magnet system in which he claims to have found a new form of hidden momentum. (He prefers the term “internal momentum” to “hidden momentum”). In Boyer’s model the magnet consists of a negative charge of \(-Ne \), where \( N \) is a positive integer, at the center of a circular frictionless track around which \( N \) charges of \(+e\) may orbit. In the vicinity of this magnet is a positive charge distant enough (and/or the magnet small enough) that its electric field at the magnet is essentially uniform. He includes calculations for both non-interacting and interacting magnet charges.

In the non-interacting version he finds the mechanical hidden momentum of Colemen and Van Vleck. In the interacting version he has two interacting electric charges circling the track and finds a hidden momentum consisting of both mechanical and electromagnetic contributions. However, the central argument regarding the ignoring of Lorentz forces when the magnet is placed in an electric field applies whether or not the charges in the magnet are interacting.

Mansuripur \[25\] proposed a paradox in which there is a charge in the vicinity of a current loop. An observer stationary with respect to these objects sees no interaction between them (with the possible exception of an induced electrical polarization of the loop). An observer moving with respect to the charge and loop will, in general, see an electric dipole resulting from the Lorentz transformation of the magnetic dipole of the loop (but not a dipole due to a separation of charge, as I discuss below). Hence, this observer should see a torque acting on the loop due to the electric field of the charge acting on the electric dipole. One observer sees a torque on the loop and another does not. Mansuripur argues that this invalidates the Lorentz force law and proposes the force law of Einstein and Laub \[30\] is the correct one instead. To save the Lorentz force, it was proposed that the torque due to the charge-dipole interaction was countered, in the moving observer’s frame, by a torque due to hidden momentum \[31\].

However, I have shown \[28\] that there is no need for hidden momentum to solve this paradox. There is both linear and angular momentum in the electromagnetic fields of a charge-magnet system \[11\]. When seen in the moving frame, these fields contain a torque that counters the
one resulting from charge-dipole interaction. Both these torques are actually time rate of change of angular momentum in the electromagnetic fields themselves. (In my reference [28] I referred to this torque as “fictitious”. This was a poor choice of words. The point is it is electromagnetic torque, not mechanical torque.)

Saldanha and Filho [36] recently placed a paper on the arXiv preprint server regarding the role of hidden momentum in physical media. Their starting point was the non-relativistic Lagrangian for a particle with both an electric dipole and magnetic dipole moving through a magnetic and electric field. The Lagrangian is

$$\mathcal{L} = \frac{1}{2} m v^2 + E \cdot p + B \cdot \mathbf{M}. \quad (56)$$

Here, $m$ is the mass of the particle with speed $v$, $E$ and $B$ are the electric and magnetic fields, and $p$ and $\mathbf{M}$ are the electric and magnetic dipole moments, respectively, as seen in the lab frame. If $p_0$ and $\mathbf{M}_0$ are the dipole moments in the particle’s rest frame, then $p = p_0 + \epsilon_0 \mu_0 v \times \mathbf{M}$ and $\mathbf{M} = \mathbf{M}_0 - v \times p_0$, where $v$ is the particle’s velocity. (These relationships are for $v << c$, such that the Lorentz factor $\gamma$ is taken to be one.)

To get the canonical (total) momentum of the particle they take the velocity gradient of the Lagrangian.

$$\nabla_v \mathcal{L} = m v - p_0 \times B + \epsilon_0 \mu_0 \mathbf{M}_0 \times E. \quad (57)$$

They attribute the last term on the right-hand side of the above equation to hidden momentum. However, this interpretation is erroneous, once again, due to ignoring the Lorentz forces that arise upon applying electromagnetic fields to magnets. Their argument for hidden momentum parallels that of Coleman and Van Vleck [5], where work is done on orbiting non-interacting charged particles by the fields, resulting in greater speed on one side of the orbit than the other. The argument is that even though linear charge density remains the same around the orbit, the linear momentum density does not due to the relativistic increase in mass. I have shown above that this view is not viable.

Saldanha and Filho examine four different types of media for the presence of hidden momentum, assuming there could be magnetic charges. No hidden momentum is found in media where the dipoles consist of electric and magnetic charges separated by a small distance as there are no orbiting charges on which work can be done. The media where they find hidden momentum are the three others where there are orbiting charges forming the dipoles – one with all dipoles due to orbiting charges and two others missing either orbiting electric charges or orbiting magnetic charges. In particular, they claim orbiting magnetic charges form electric dipoles that contain hidden momentum. They apply the supposed hidden momentum in these media to the question of Minkowski and Abraham momenta.

Consider magnetic charges forming a circuit and creating the Amperean version of an electric dipole at the origin of a Cartesian coordinate system. To assemble a model analogous to that of Shockley and James [4], bring magnetic charges from a large distance toward the magnetic circuit in the plane of the circuit. As before, have the positive (north pole) charge move along the $x$ axis in the positive direction and the negative (south pole) charge move along the $x$ axis in the negative direction.

Due to the changes of sign in Maxwell’s equations when interchanging the roles of the electric and magnetic quantities, the counterclockwise magnetic circuit produces an electric dipole in the negative $z$ direction. Also the Lorentz force on the magnetic charges moving with velocity $v$ will be $-(q_m/c^2)v \times \mathbf{E}$, where $\mathbf{E}$ is the electric field (in the positive $z$ direction at the location of the magnetic charges) and $q_m$ is the magnetic charge of a north pole. The result of these changes means the force on both magnetic charges will be in the positive $y$ direction and will equal to $vBp/r$ in magnitude, where $p$ is the magnitude of the electric dipole moment. Integrating over time to find the impulse, as was done for the model of Shockley and James, yields a total impulse of $B \times p$ acting on the charges in the positive $y$ direction.

Performing an analogous calculation for the impulse on the disks as was done in the SJ model, the surface integral of the “displacement current” is $-\pi \mu_0 q_m v^2 \sin^2 \phi/(2\pi r^3)$, resulting in an electric field on the rims of the disks of $-\mu_0 q_m \sin \phi k/4\pi r^3$. The electric field acting on the magnetic current results in an impulse on the disk equal to that on the charges and in the same direction. (Here, $\phi$ is the same azimuthal angle as in the SJ model and $a$ is the radius of the disks.)

So, the total impulse due to applying a magnetic field to the magnetic charge–“Amperean” electric dipole system is $2B \times p$. The negative of this is the electromagnetic momentum. This is easily shown by starting with Eq. (1) for the magnetic charge-Amperean electric dipole system and manipulating it algebraically until it is identical to that of the SJ system. There is therefore no hidden momentum necessary for momentum conservation for an Amperean electric dipole in a magnetic field.

Filho and Saldanha [37] recently performed a quantum calculation to find the hidden momentum in a hydrogen atom using degenerate perturbation theory. As in classical calculations, they consider the atom to be at rest in the electric field, ignoring what happens when the electric field rises from zero to its final value. It would appear that time-dependent perturbation theory should be used to give a complete picture. There is no reason a quantum calculation should show hidden momentum when correct classical calculations do not as hidden momentum is a non-quantum conjecture.

Finally, Spavieri [38] found a role for hidden momentum in the spin-orbit effect in a hydrogen atom. Much of his discussion involves the interaction between the induced electric dipole $p = v \times \mathbf{M}$ on the electron due to its magnetic dipole being in motion around the nucleus with speed $v$ and the electric field of the nucleus. Spavieri finds the electromagnetic interaction energy $U$ between the induced electric dipole and the electric field $\mathbf{E}$ of the nucleus.
by performing a volume integral as follows.

\[ U = \epsilon_0 \int (E_p \cdot E) dV = -p \cdot E, \quad (58) \]

where \( E_p \) is the electric field due to the induced dipole. There is a problem with this result.

The magnetic field of a magnetic dipole with magnetic moment \( M \) at the origin of the coordinate system in its rest (primed) frame is

\[ B' = \frac{\mu_0}{4\pi} \left[ \frac{3(M' \cdot r')r'}{r'^5} - \frac{M'}{r'^3} \right] - \frac{2\mu_0}{3} \delta^3(r'), \quad (59) \]

where \( \delta^3(r') \) is the Dirac delta function. Transforming this to the moving (unprimed) frame as seen in the coordinate system at rest with the nucleus (see [39] for the technique) you find

\[ B = \frac{\mu_0}{4\pi v^2} \left[ \frac{3(M \cdot r)r}{\gamma^2 v^2[1 - (v^2/c^2)\sin^2\alpha]^{3/2}} \right] - \frac{\mu_0}{3} \delta^3(r), \quad (60) \]

where \( \alpha \) is the (instantaneous) angle between \( r \) and \( v \). The corresponding electric field is

\[ E_p = -v \times B. \quad (61) \]

Now the volume integral of the field of an actual dipole due to a separation of charge will indeed equal the dipole itself due to the delta function; however, it is clear that the electric field \( E_p \) is not that of an actual dipole. For one thing there is no electric field parallel to the direction of motion of the magnetic dipole. In the slow motion case where \( v \ll c \), the field can be written as

\[ E_p = E_{dp} - \frac{3p}{4\pi \epsilon_0 r^3} \sin^2\theta - \frac{3m \times p}{4\pi \epsilon_0 r^3} \sin \theta \cos \theta \sin \phi. \quad (62) \]

Here \( E_{dp} \) is the electric field of a dipole \( p \) that consists of an actual separation of charge. The angles are those of a spherical system of coordinates, where the velocity is in the positive \( z \) direction. (Thus \( \theta \) is the same as \( \alpha \) in the earlier equations.) The third term on the right hand side of the above equation cancels the field in the \( z \) direction of the first term such that there is no field parallel to \( z \). The problem in taking the volume integral of \( E_p \) is due to the second term on the right hand side. It results in a term proportional to \( ln(1/r) \), which is undefined at the limits of the integral. (The third term integrates to zero due to \( \sin \phi \).

Of course, there can’t be an actual charge separation on an electron, and the Lorentz transformation of the fields is consistent with that. (I have performed a calculation that indicates the apparent dipole on a current loop is due to a relativity-of-simultaneneity effect rather than an actual charge separation [40].) Since the rest of the argument of Spavieri depends on this calculation, his results are brought into question.

7 Discussion and conclusion

This paper examines models where a magnet is embedded in an electric field. The model due to Shockley and James [4], with its electrically charged rotating disks in the vicinity of two equal and opposite external charges, acquires mechanical momentum when the charges are moved in from a distance, unless the charges and magnet are held stationary by an external agent. This mechanical momentum is equal and opposite to the electromagnetic momentum it acquires at the same time.

No hidden momentum appears in this model and no force is applied to the magnet when it is demagnetized, which is the paradox Shockley and James thought they had discovered. Rather, the so-called paradox is simply due to analyzing this model in a reference frame that is moving with respect to the frame in which it was assembled (by either applying an electric field to the magnet or forming the magnet in an electric field). If the model is held stationary, then the “hidden momentum” actually resides in the external agent that held the model at rest.

Coleman and Van Vleck examined the hidden momentum proposal of Shockley and James with a calculation involving a point charge and a model-free Amperian magnet using the Lagrangian of Darwin [5]. Their work appeared to confirm a force on the magnet when it is demagnetized. A calculation based on a specific model alluded to by Coleman and Van Vleck shows their result only obtains for a very contrived model that already has momentum. The force they claim is experienced by the magnet when it is demagnetized is actually the loss of momentum by the electromagnetic field.

The calculation of Coleman and Van Vleck assumed the electric charges that produced the magnetism were non-interacting, as did Babson et al in a model they examined [5]. If formed by applying an electric field to non-interacting particles constituting a current, the resulting particle motion is not the same as that of their models and no hidden momentum results.

The incompressible, frictionless fluid magnet model of Babson et al contains a relativistic momentum in the fluid due to pressure, which is balanced by the relativistic momentum in the tube containing the fluid due to it being subject to stress. This momentum might be viewed as the “internal momentum” proposed by Boyer [17]. It disappears with no effect on the motion of the magnet when the electric field is removed. Thus, this is not the type of hidden momentum envisioned by Shockley and James and by Coleman and Van Vleck, since their momentum would create an impulse on the magnet when its magnetism goes to zero.

The total momentum of a magnetic dipole created in a uniform electric field is shown to be zero without the need for hidden momentum. You must consider the effect of the creation of the dipole on the charges responsible for the uniform Coulomb field, whether that of the finite system used here or of infinite parallel plates [11]. Taking the moving frame of this model as its rest frame for analysis is misleading, as it is for the other models. There is an electromagnetic momentum given by the sum of Eqs. (47).
and \( \text{[42]} \), but due to observation in the new rest frame there is no observed mechanical momentum. This is not the case in the original rest frame where the magnet was formed in an electric field. Hence whether an electric field is applied to a magnet or a magnet is created in an electric field, no hidden momentum appears.

Finally, some previous results in the literature were examined. In the dispute over whether or not a neutron experiences a force in an electric field \([21,22,23]\), I found that if the neutron is modeled as a current loop, it does not contain hidden momentum, since a current loop contains no hidden momentum. The paradox explored by Vaidman \([24]\) where a current loop is moving in an electric field, like that of Mansuripur \([25,26]\) is resolved by transforming linear electromagnetic momentum of a current loop in an electric field in a frame at rest with the loop to a frame in which the loop is moving. The transformation produces an electromagnetic torque that cancels the anomalous torque in the moving frame. Also shown is that an Amperian electric dipole consisting of circulating magnetic charges, like its magnetic counterpart, holds no hidden momentum. Lastly, a calculation by Spavieri \([27]\) showing a connection between hidden momentum and spin-orbit splitting in hydrogen-like atoms fails due to the electric field of a moving magnetic dipole not being identical to that of an electric dipole with a charge separation.

The center-of-energy theorem is often invoked to support the idea of hidden momentum \([7]\). In this view hidden momentum is necessary to satisfy the theorem. However, it should be clear from the discussion in this paper that center-of-energy arguments erroneously assume the charge-magnet systems are at rest in their original frames of reference. This ignores the Lorentz forces necessarily involved when either an electric field is applied to an Amperian magnet or a magnet is formed in an electric field. A charge-magnet system can be assembled and gain both electromagnetic field and mechanical momentum (equal and opposite). So, it is possible for an observer to adjust her velocity so that the system is stationary in her reference frame. In her frame the system contains field momentum but no mechanical momentum.

Of course, the presence of a net force on a charge-magnetic dipole system during its assembly with no equal and opposite force violates the usual view of Newton’s third law of motion. However, it is clear from the calculations given here that the four-force acting on the system is zero if you include the electromagnetic momentum in the space component along with the mechanical momentum in the momentum four-vector as the system is assembled. This renders the spatial component zero. The time derivative of this is the relativistic three-force, which is also zero.

I have calculated other charge-magnet systems that are claimed to contain hidden momentum \([12]\) and have found no hidden momentum when the assembly of the systems is considered. Although these calculations concern specific systems, if hidden momentum exists for a magnet in an electric field, it should be evident in these calculations. I conclude that hidden momentum does not exist in these systems and that such systems at rest with respect to an observer can contain electromagnetic momentum.

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