Derivation of the Verlinde Formula
from Chern-Simons Theory
and the G/G Model

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Abstract

We give a derivation of the Verlinde formula for the $G_k$ WZW model from Chern-Simons theory, without taking recourse to CFT, by calculating explicitly the partition function $Z_{\Sigma \times S^1}$ of $\Sigma \times S^1$ with an arbitrary number of labelled punctures. By a suitable gauge choice, $Z_{\Sigma \times S^1}$ is reduced to the partition function of an Abelian topological field theory on $\Sigma$ (a deformation of non-Abelian BF and Yang-Mills theory) whose evaluation is straightforward. This relates the Verlinde formula to the Ray-Singer torsion of $\Sigma \times S^1$.

We derive the $G_k/G_k$ model from Chern-Simons theory, proving their equivalence, and give an alternative derivation of the Verlinde formula by calculating the $G_k/G_k$ path integral via a functional version of the Weyl integral formula. From this point of view the Verlinde formula arises from the corresponding Jacobian, the Weyl determinant. Also, a novel derivation of the shift $k \to k + h$ is given, based on the index of the twisted Dolbeault complex.

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The Verlinde formula \cite{1, 2, 3} is one of the most surprising and interesting results to have emerged from mathematical physics in recent years. From the conformal field theory point of view it is a formula for the number of conformal blocks of a rational conformal field theory on a punctured Riemann surface. In this context it is a consequence of the well-established, but nevertheless still somewhat enigmatic, fact that the modular matrix $S$ implementing the modular transformation $\tau \rightarrow -1/\tau$ on the space of genus one conformal blocks ‘diagonalizes’ the fusion rules.

On the other hand, from the mathematical point of view, the Verlinde formula is an expression for the dimension of the space of holomorphic sections of some line bundle on a moduli space of vector bundles. As such, it should yield to a standard mathematically rigorous derivation but has withstood these attempts so far in all but the simplest of cases, the $SU(2)$ Wess-Zumino-Witten (WZW) model. For some work in this direction see e.g. \cite{4}. In that case, denoting the vector space of conformal blocks of the WZW model at level $k$ on a genus $g$ surface

\footnote{We understand that a rigorous general proof has recently been obtained by Faltings as well as by Narasimhan and Ramadas.}
Σₙ by Vₙ,k, the Verlinde formula reads

$$\dim V_{g,k} = \left(\frac{k+2}{2}\right)^{g-1} \sum_{j=0}^{k} \left[ \sin \left(\frac{(j+1)\pi}{k+2}\right) \right]^{2-2g}.$$  \hfill (1.1)

The expression on the right hand side of (1.1) has several notable (and non-obvious) features, not the least of which are that it is indeed an integer and a finite polynomial in k. Its generalization to arbitrary compact Lie groups G is

$$\dim V_{g,k} = (C(k+h)^r)^{g-1} \sum_{\lambda \in \Lambda_k} \prod_{\alpha \in \Delta} \left(1 - e^{i \frac{\alpha(\lambda + \rho)}{k+h}}\right)^{1-g},$$  \hfill (1.2)

where C, h, r are the order of the center, the dual Coxeter number, and the rank of G, Λ_k denotes the space of integrable highest weights at level k, Δ the set of roots of G, and ρ the Weyl vector (half the sum of the positive roots). We want to draw attention to the fact that in both these formulae the square of the Weyl denominator makes a prominent appearance.

The purpose of this paper is to shed some light on these formulae (and their counterparts for surfaces with labelled punctures) by deriving them directly from Chern-Simons theory using only gauge theory techniques and without recourse to either conformal field theory or mathematics more advanced than elementary group theory. As the basic chain of arguments leading to the derivation of (1.1,1.2) from Chern-Simons theory is quite simple, we present it at the end of the Introduction. The length of this paper is due to the fact that we have attempted to make it reasonably self-contained and not to appeal to calculations which have been performed elsewhere in superficially similar but different contexts (here we have in mind in particular the derivation of the ‘shift’ k → k + h of section 6 or the explicit evaluation of the Ray-Singer torsion of S¹ and Σ × S¹ in section 3).

Our derivation, which is based on an ‘Abelianization’ of the Chern-Simons path integral via an appropriate gauge choice, explains and highlights certain features of the Verlinde formula and its relation to various topological field theories in two and three dimensions. For instance, it explains the appearance of the square of the Weyl denominator in the Verlinde formula by relating it to either the Ray-Singer torsion of Σ × S¹ (section 3) or the Weyl integral formula (section 5). Moreover, in the course of the derivation we also prove the equivalence of Chern-Simons theory on Σ × S¹ with the G/G gauged WZW model on Σ, thus establishing conjectures of Spiegelglas [5] anticipated by Verlinde and Verlinde [6] and partially verified by Witten [7]. We also establish the equivalence of the G/G model with a compact (‘q-deformed’) version of BF theory which, like Yang-Mills theory, is related to classical representation theory and has been extensively...
studied in e.g. \[8, 9, 10\]. Via the methods of \[11\] this brings one closer to a fixed point theorem interpretation of the Verlinde formula (either for path integrals, as we will explain in \[12\], or for some finite dimensional integral). Finally, our derivation of the fusion rules in section 7.6 shows that they arise naturally from Chern-Simons theory, and already in diagonalized form, the traces of Wilson loops only ever receiving contributions from those gauge field configurations where the classical characters satisfy the quantum fusion rules.

The methods we use or develop in this paper can also be applied to other problems arising e.g. in coset models or Yang-Mills theory. For instance, they lead to a very simple derivation of the 2d Yang-Mills partition function (see \[12\]). They also provide us with Lagrangian realizations of \((G/H)/(G/H)\) models (e.g. \(S^2/S^2\)-models) whose partition functions would calculate the number of conformal blocks of the \(G/H\) coset model. These methods can also be applied to non-compact groups like \(SL(2, \mathbb{R})\) (where the conjugation into standard form needs to be performed seperately for the elliptic, parabolic, and hyperbolic elements). In principle, they also allow us calculate Witten three-manifold invariants of mapping tori \(\Sigma_g \times_f S^1\) directly from gauge theory. It remains to be seen, however, if this can be turned into an effective calculational tool for \(g > 1\).

### 1.1 Some Background

To explain why a derivation of the Verlinde formula from Chern-Simons theory is possible in principle, and to indicate how we will proceed in practice, we will have to briefly recall some basic facts of Chern-Simons theory. For all the other things that should be said about Chern-Simons theory, see e.g. \[2, 13, 14, 15\].

Choosing a closed oriented three-manifold \(M\) and a compact gauge group \(G\) (which we will assume to be simply-connected so that any principal \(G\)-bundle over \(M\) is trivial), the Chern-Simons action for \(G\) gauge fields \(A\) on \(M\) (we reserve the notation \(A\) for spatial gauge fields) is defined by

\[
kS_{CS}(A) = \frac{k}{4\pi} \int_M \text{tr}(dA + \frac{2}{3} A^3) .
\]

The trace (invariant form on the Lie algebra \(g\) of \(G\)) is normalized in such a way that invariance of \(\exp ikS_{CS}\) under large gauge transformations requires \(k \in \mathbb{Z}\).

Traditionally, two approaches have been used to evaluate the partition function of this action. One of them is perturbative in nature and is conveniently performed in a background field expansion. In this theory this amounts to expanding about flat connections. Then, to one loop order, one finds that the effective theory is

\[
Z_M(S_{CS}, k)^{(1)}(A_c) = T_M(A_c)^{1/2} \exp \left( i(k + h)S_{CS}(A_c) \right),
\]
(modulo contributions coming from the ghost and one-form zero modes) with the integration over the moduli space of flat connections $\mathcal{A}$, still to be done. Here $T_M$ is the Ray-Singer torsion \[16\], a particular metric independent ratio of determinants of twisted Laplacians on $M$.

This is a Gaussian approximation and on a general three manifold represents the first term in a $1/k$ expansion of the path integral. Such a perturbative approach is particularly useful for exploring the relationship of Chern-Simons theory with knot-invariants. At higher loops, however, it manages to hide quite effectively the basic simplicity and elegance of Chern-Simons theory.

On the other hand, the relation of Chern-Simons theory to conformal field theory can be used to evaluate the partition function non-perturbatively using either surgery or Heegard splitting techniques, see e.g. [2, 17]. In certain cases the large $k$ limit of these results has been shown to compare favourably with the evaluation of \[14\]. What we will show in this paper is that for three-manifolds of the form $\Sigma \times S^1$ the partition function can be evaluated exactly by ordinary gauge theory techniques (essentially because on these manifolds there is a gauge choice available which makes the theory one-loop exact).

An alternative approach to the quantization of Chern-Simons theory is canonical quantization. On three-manifolds of the form $\Sigma \times \mathbb{R}$, Chern-Simons theory can be subjected to a canonical analysis. Upon choosing the gauge $A_0 = 0$, one determines the classical reduced phase space to be the moduli space $\mathcal{M}$ of flat connections on $\Sigma$. This is a symplectic space (as it should be) and becomes Kähler once one chooses a complex structure on $\Sigma$. Hence one can use the recipe of geometric quantization to quantize this system, the Hilbert space being the space of holomorphic sections of a line bundle over $\mathcal{M}$ whose curvature is $(i$ times) the symplectic form. It follows from Quillen’s calculation \[18\] and the fact that the symplectic form for ‘level $k$’ Chern-Simons theory is $k$ times the fundamental symplectic form $\frac{1}{2\pi} \int_{\Sigma} \delta A \delta A$, that the line bundle in question is (for $SU(n)$) the $k$-th power of the determinant line bundle associated to the family of operators \{\bar{\partial}_A\}. In [2] and [3] it is shown that the space of holomorphic sections of this line bundle (the space of states of Chern-Simons theory) coincides with the space $V_{g,k}$ of holomorphic blocks of the $G_k$ WZW model.

What is important for us is the fact that the dimension of this vector space is given by a path integral of Chern-Simons theory. In fact, since the Hamiltonian of Chern-Simons theory is zero (like that of any generally covariant or topological theory), the statistical mechanics formula

$$Z_{\Sigma \times S^1} = \text{Tr} \ e^{-\beta H} \quad \quad (1.5)$$
for a circle of radius (imaginary time) \( \beta \) reduces to

\[
Z_{\Sigma_g \times S^1}(S_{CS}, k) = \dim V_{g,k} .
\]  

(1.6)

This is the key equation which allows us to derive the Verlinde formula \([1.1][1.2]\) by evaluating the Chern-Simons partition function of \( \Sigma_g \times S^1 \). If one does not want to make use of the identification of Chern-Simons states with conformal blocks, one can nevertheless regard this derivation as a calculation of the dimension of the Chern-Simons Hilbert space on \( \Sigma_g \times \mathbb{R} \).

### 1.2 Outline of the Derivation

The derivation of the Verlinde formula from Chern-Simons theory is more or less straightforward but somewhat lengthy once one pays due attention to certain technicalities. In order not to distract from the basic simplicity of the argument, in the following we sketch the main ideas and give a general outline of the derivation. The presentation in this section will be informal.

The strategy will be to exploit the large gauge symmetry present in Chern-Simons theory in a way which a) is compatible with the geometry of the problem, and b) simplifies the action to the extent that the path integral can indeed be evaluated explicitly.

The first of these desiderata we fulfill by choosing the gauge \( \partial_0 A_0 = 0 \) (the more obvious choice \( A_0 = 0 \) not being available as \( A_0 \) may have a non-trivial holonomy along the circle). This still leaves us with the time-independent gauge transformations (and certain ‘large’ time-dependent gauge transformations) to play with. The former can be used to conjugate \( A_0 \) into the Lie algebra \( t \) of a maximal torus \( T \) of \( G \). By integrating over the time-dependent modes of the \( t \)-components of \( A \) and all the modes of the remaining components of the gauge fields and the ghosts (all these integrals are Gaussian) one is left with an effective two-dimensional Abelian topological field theory (sections 2 and 3), which can then be easily evaluated.

Alternatively, one may wish to trade the constant mode of \( A_0 \) for the holonomy \( g = \exp A_0 \) (section 4). If one does that and eliminates the time-dependent modes of the gauge fields, e.g. by thinking of the path integral on \( \Sigma \times S^1 \) as the trace of an amplitude on the cylinder \( \Sigma \times I \) with (time-independent) boundary conditions, one arrives at the action of the \( G/G \) coset model. The fact that all the determinants work out to produce precisely the Haar measure on the group from the linear measure on the Lie algebra is one of the numerous miracles we witness in this theory. As the notation ‘\( /G \)’ indicates, this model has a local \( G \) gauge invariance.
which can be used to conjugate $g$ into $T$ (section 5). Again, the integration over the non-Abelian components of $A$ is easily performed, leaving one with the same effective two-dimensional Abelian theory.

In whichever way one proceeds one finds that the partition function has now been reduced to the manageable form

$$Z_{\Sigma \times S^1}(S_{CS}) = \int D[\phi, A] \exp(i(k + h) S_{\phi F}(\phi, A)),$$

where $A$ is a $t$-valued gauge field and $\phi$ is a compact scalar field taking values in $T$. The action (a compact Abelian BF action) and the measure are

$$S_{\phi F}(\phi, A) = \frac{1}{2\pi} \int_{\Sigma} \text{tr} \phi F_A$$

and

$$D[\phi, A] = D\phi DA \det(1 - \text{Ad}(e^{i\phi}))^{1 - g}$$

respectively. The action is rather obviously a remnant of the Chern-Simons action. What deserves attention, though, are the occurrence of the shift $k \rightarrow k + h$ and the non-trivial measure. The former could have been more or less anticipated from other investigations of Chern-Simons theory. Here it arises from a gauge invariant regularization of the ghost and gauge field determinants (via the index of the twisted Dolbeault complex, section 6).

These determinants, which also almost cancel each other (a typical feature of topological field theories), leave behind the finite dimensional determinant appearing in the measure. From the point of view of sections 2 and 3 it arises as the square-root of the Ray-Singer torsion of $\Sigma_g \times S^1$. In fact, by a theorem of Ray and Singer one has $T_{\Sigma \times S^1} = T_{S^1}^{2 - 2g}$, while the torsion of a gauge field with holonomy $t \in T$ on the circle is just $\det(1 - \text{Ad}(t))$ (section 3.1), so that the usual one-loop approximation ($14$) would suggest the appearance of such a term. We want to emphasize, however, that the above result is exact.

From the $G/G$ model point of view, on the other hand, the measure can be understood as the Jacobian arising in the Weyl integral formula (section 5.1) relating the integral over $G$ of a conjugation invariant function to an integral over $T$ (and almost cancelled by the determinant arising from the gauge field integration).

The above Abelian $\phi F$ theory is closely related to ordinary non-Abelian BF theory with action $\int tr BF_A$, $B$ a non-compact Lie algebra valued scalar field, and Yang-Mills theory (whose action differs from that of BF theory by a term $\sim B^2$). In both these theories the partition function can be expressed as a sum over all irreducible representations of $G$ $[4, 11]$. Here the compactness of $\phi$ imposes a
cutoff on the representations contributing and reduces the integral to a sum over integrable representations. To see this directly, one can make a change of variables from $A$ to its curvature $F_A$ (plus longitudinal components). The $F_A$-integral (constrained by the condition that $F_A$ be an integral two-form) will then impose a delta function constraint on $\phi$ allowing only certain discrete values of $\phi$ (corresponding to the highest weights of the integrable representations) to contribute. The details of this calculation will be explained in section 7, where we also discuss the inclusion of punctures ('vertical' Wilson loops in Chern-Simons theory). The result is then precisely the Verlinde formula, up to an overall normalization which we determine in section 7.5 (again taking care not to make use of results from conformal field theory or the representation theory of loop groups).

2 Chern-Simons Theory on $\Sigma \times S^1$

In the Introduction we mentioned the usual approach to the evaluation of the Chern-Simons partition function via a background field expansion. On a three manifold of the form $\Sigma \times S^1$ one may follow a different approach. This will give us an exact evaluation of the path integrals involved but, nevertheless, has some features in common with the background field method. At an intermediate stage of the calculation we will obtain a formula quite like (1.4), except that only gauge fields with values in the Cartan subalgebra $t \subset g$ of the Lie algebra enter and, indeed, they are not flat (for what is meant by the Ray-Singer torsion in this case see section 6.2). The final step in the calculation is then easy to perform. The reason that one may go so far in this instance is that for a manifold of the type $\Sigma \times S^1$ there is a gauge choice available which ensures that all of the path integrals that we will encounter are Gaussian.

2.1 Gauge Fixing

It makes sense, on $\Sigma \times S^1$, to split the gauge field into components

$$A = A + A_0 dt ,$$

in an obvious notation. In order to perform the path integral we will need to fix on a gauge. On the line, with suitable boundary conditions, it would be possible to set $A_0 = 0$. On the circle, however, this gauge choice is not possible. There is a simple reason for this. In terms of Fourier modes, the gauge transform of the gauge field acquires the shift $n\Lambda_n + \ldots$, so that all the non-constant modes may be eliminated by an appropriate choice of the $\Lambda_n$. This shift is absent in the case
of the constant mode. This discussion shows us that the natural choice here is
the gauge
\[ \partial_0 A_0 = 0 \]  
(2.2)
This eliminates almost all of the time dependent gauge transformations, see (3.38).

But time independent transformations can still be used to impose a stronger
restriction on \( A_0 \), and it is a particular such choice which will simplify matters to
the extent that the Chern-Simons partition function becomes explicitly calculable.
In order to motivate this choice of gauge condition it is useful to pass to a more
group theoretic description of the above discussion.

We denote by \( g \) the holonomy of \( A_0 \) around the circle \( S^1 \), i.e. \( g \) is the path
ordered exponential
\[ g = P \exp \left( \int A_0 \right) . \]  
(2.3)
Under a gauge transformation by \( h \), the holonomy \( g \) is conjugated,
\[ A_0 \rightarrow A^h_0 \equiv h^{-1} A_0 h + h^{-1} \partial_0 h \Rightarrow g \rightarrow h^{-1} g h . \]  
(2.4)
This also makes it clear that one may not impose the gauge \( A_0 = 0 \), for that
would mean that we could conjugate \( g \) to the identity, which is not possible unless
\( g \) already happens to be the identity. What we may do, however, is conjugate any
\( g \in G \) into a maximal torus \( T \) of the group \( G \), i.e. we can ‘diagonalize’ \( g \), and we
learn that the only gauge invariant degree of freedom of a gauge field on the circle
is the conjugacy class of its holonomy. Thus, if we eliminated \( A_0 \) in favour of \( g \), it
would be possible to impose the gauge condition \( g \in T \). This is the procedure we
will adopt in section 5 to abelianize the \( G/G \) action. Here, however, we are still
working at the Lie algebra level, and we will instead make use of the possibility
to conjugate an element of the Lie algebra \( g \) into the Lie algebra \( t \) of \( T \). We may
thus augment (2.2) with the condition
\[ A^k_0 = 0 , \]  
(2.5)
where we have split the gauge field into the part \( A^t \) taking values in \( t \) and into
\( A^k \) taking values in the complement \( k \) of \( t \) in \( g = t \oplus k \).

In order to impose (2.2) and (2.3) simultaneously we add the following gauge
fixing terms to the action,
\[ \int_{\Sigma \times S^1} \text{tr} \left[ B A_0 + \bar{c} D_0 c \right] , \]  
(2.6)
where
\[ D_0 = \partial_0 + A_0 , \]  
(2.7)
and we furthermore impose the conditions

\[ \oint B^t = \oint c^t = \oint \bar{c}^t = 0. \]  

(2.8)

This may need some explanation. Firstly, (2.6) is BRST exact and so does not generate any unwanted metric dependence in the path integral. The conditions (2.8) may also be imposed in a BRST invariant manner, and so also do not give rise to spurious metric dependence (the way to do this is explained e.g. in [8, 15]). For this reason we have refrained from introducing the metric explicitly in the above expressions. The condition on the multiplier field \( B \) is clearly needed so as to impose precisely the conditions (2.2) and (2.5). The condition on the anti-ghost \( \bar{c} \) follows from the requirement that the anti-ghost modes be in one to one correspondence with those of the multiplier field.

Perhaps it is surprising that there is also a constraint on the ghost \( c \). But that this requirement is correct is easily seen if we note that the combined conditions (2.2, 2.5) are invariant under constant torus valued gauge transformations. These transformations are therefore not used to arrive at the gauge fixing conditions that we have chosen and hence those components of the ghost should not appear.

### 2.2 One Loop Exactness

We wish to perform the path integral over all the modes of \( A^k \) and the ghosts and the non-constant (in time) modes of \( A^t \). This will leave us with an effective two-dimensional Abelian theory in \( A^k_0 \) and (the constant mode of) \( A^t \). The modes to be integrated over enter quadratically in the action so the integrals to be performed are simply Gaussian. Hence, as far as all these modes are concerned, the one-loop approximation is exact. The Chern-Simons path integral then becomes a two dimensional path integral. The reason for approaching the problem in this way is that at this point we may employ techniques developed for two-dimensional gauge theory to complete the evaluation of the three-dimensional path integral.

In the gauge chosen the (quantum) Chern-Simons action takes on a particularly simple form,

\[ S_{CS}(A) = \int_{\Sigma \times S^1} dt \, \text{tr} \left( A_0^t dA^t + A^t \partial_0 A^t + A^k D_0 A^k + \bar{c}^k D_0 \bar{c}^k + \bar{c}^t \partial_0 \bar{c}^t \right), \]  

(2.9)

where it is understood that \( A_0, c \) and \( \bar{c} \) satisfy the constraints (2.2), (2.3) and (2.8).

Performing the integration over the modes specified above generates the fol-
lowing ratios of determinants:

\[
\frac{\text{Det}'_t[\partial_0]\Omega^p(\Sigma) \otimes \Omega^p(S^1)}}{\text{Det}^{1/2}_t[\partial_0]\Omega^1(\Sigma) \otimes \Omega^p(S^1)} \cdot \frac{\text{Det}_k[D_0]\Omega^p(\Sigma) \otimes \Omega^p(S^1)}}{\text{Det}^{1/2}_k[D_0]\Omega^1(\Sigma) \otimes \Omega^p(S^1)}
\]

(2.10)

The subscript labels refer to the spaces on which the operators act, and the prime indicates that one should not include the \(S^1\) zero mode. We will perform a precise analysis of this ratio of determinants in section 6. Some insight into what the final result will look like, and why, can however also be gained by proceeding in a less rigorous fashion and this is what we will do here.

The first thing to note is that the determinants almost cancel, as a one-form in two dimensions is ‘like’ a pair of scalars. To understand the qualification ‘almost’ in the above, recall that by the Hodge decomposition theorem any \(p\)-form \(\omega\) on a compact \(n\)-dimensional Riemannian manifold \(M\) may be uniquely written as the sum of an exact, a coexact, and a harmonic form,

\[
\omega = d\alpha + d^*\beta + \gamma
\]

(2.11)

Here \(d^*\) is the adjoint of \(d\) with respect to the scalar product

\[
(\omega_1, \omega_2) = \int_M \omega_1 \ast \omega_2
\]

(2.12)

It follows from \(\ast^2 = (-1)^{p(n-p)}\) that, acting on \(p\)-forms,

\[
d^* = (-1)^{np+n+1}d^*\ast
\]

(2.13)

Thus, in two dimensions, (2.11) refines the above statement in the sense that it expresses a one-form in terms of two scalars \(\alpha\) and \(\ast\beta\) (modulo constants which are the harmonic zero forms) and a harmonic form representing an element of \(H^1(\Sigma, \mathbb{R})\). This allows us to figuratively decompose the space of one-forms as

\[
\Omega^1(\Sigma) = \Omega^0(\Sigma) \oplus \Omega^0(\Sigma) \oplus H^1(\Sigma) \ominus 2H^0(\Sigma)
\]

(2.14)

Recalling that the harmonic modes are orthogonal to the others we may then deduce that (2.10) is more or less equal to

\[
\frac{\text{Det}'_t[\partial_0]H^p(\Sigma) \otimes \Omega^p(S^1)}}{\text{Det}^{1/2}_t[\partial_0]H^1(\Sigma) \otimes \Omega^p(S^1)} \cdot \frac{\text{Det}_k[D_0]H^p(\Sigma) \otimes \Omega^p(S^1)}}{\text{Det}^{1/2}_k[D_0]H^1(\Sigma) \otimes \Omega^p(S^1)}
\]

(2.15)

This mechanism for the cancellation of modes is similar to that used by Witten in [11]. However, (2.13) is not quite right on two counts. Firstly, as \(A_0\) has both harmonic and non-harmonic modes it mixes the two in the determinants. Secondly, to give meaning to these functional determinants one should regularise.
Our expectation is that a gauge invariant regularization scheme will reproduce (2.13) plus the shift $k \to k + h$. As the latter is not our most immediate concern, we postpone a discussion of this issue until section 6. There we will also discuss the case of non-constant $A_0$. For our present purposes, however, it is enough to take $A_0$ to be a constant (and hence flat) as the integral over $A^t$ will eventually impose a delta function constraint to this effect. With $A_0$ understood to be constant the determinants become

$$T_{S^1}(A_0)\chi(\Sigma)/2 \equiv \left[ \text{Det}'(\partial_0)_{\Omega^0(S^1)}, \text{Det}_k(D_0)_{\Omega^0(S^1)} \right]^{b_0 - b_1/2}$$

(2.16)

where the Betti numbers $b^i = \dim H^i(\Sigma)$ are $b^0 = 1, b^1 = 2g$. In the next section we show that (2.16) equals the square root of the Ray-Singer analytic torsion of $\Sigma \times S^1$ (and that $T_{S^1}(A_0)$ is the torsion of the circle, explaining the notation adopted in (2.16)).

As all the non-constant time modes have been integrated out, the Chern-Simons path integral on $\Sigma \times S^1$ becomes a path integral on $\Sigma$. This path integral is

$$\int DA_0 DA^t T_{S^1}(A_0)\chi(\Sigma)/2 \exp \left( \frac{ik}{2\pi} \int_\Sigma A_0 F_A \right).$$

(2.17)

As this bears a formal resemblance to the result (1.4) of a one-loop calculation in a background field, it is useful to point out the differences as well. First of all, we have not chosen a background field about which to expand. Secondly, we have therefore also not made any approximation (i.e. dropped higher orders in the quantum field). And finally, to arrive at (2.17) we have only integrated over parts of the modes, whereas the one-loop background field expansion entails an integral over all quantum fields.

### 2.3 A Supersymmetry

The cancellation of contributions between the one-forms and zero-forms may be formalised by introducing a supersymmetry relating the two. Indeed (2.9) has a number of BRST like symmetries, the one of interest to us here being

$$\delta A^k = \epsilon c^k, \quad \delta c^k = 2\epsilon A^k,$$

(2.18)

where $\epsilon$ is a one form on $\Sigma$. Now (2.18) allows us to conclude that the non-harmonic modes of $A^k$ are precisely paired against the non-harmonic modes of $c^k$ and of $\bar{c}^k$. 

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For the harmonic modes the situation is somewhat different. In this case we need to take $\epsilon$ to be harmonic. The first of (2.18) shows us that, as there is only one $c^k$ harmonic mode, only one of the $g A^k$ zero-modes is paired against the zero-mode of $c^k$. The other $g - 1$ modes are not transformed in (2.18). The second of the equations (2.18) can be satisfied for some $\bar{c}^k$, when $\epsilon$ is harmonic and the gauge field is taken to be the harmonic mode that makes an appearance in the first transformation.

There is a similar supersymmetry relating the non-constant modes of $A^t$ to those of $c^t$ and $\bar{c}^t$. These considerations lead us once more to the conclusion that the ratio of determinants that appears in (2.15) is given correctly by (2.16).

We will also find a similar supersymmetry in the (appropriately gauge fixed) $G/G$ model, see section 6. There we will also give a more rigorous derivation of the above result. In particular, using a heat kernel regularization, the remaining finite dimensional determinant will be seen to arise as a consequence of the gravitational contribution to the chiral anomaly.

### 3 Ray-Singer Torsion

We will show presently that the ratio of determinants that we have to evaluate in (2.16) is known as the Ray-Singer Torsion for the circle. Formally, given a flat vector bundle with flat connection $A$ over a Riemannian manifold $M$ of dimension $n$ and denoting the corresponding twisted Laplacian on $k$-forms by

$$\Delta_k = d_A d_A^* + d_A^* d_A$$

(3.1)

the Ray-Singer Torsion \cite{16} is defined to be\footnote{There is some latitude in the definition of the Torsion. Here we have adopted the definition of Ray and Singer, though the reader should be warned that some authors find it more convenient to call the inverse of this the Torsion.}

$$T_M(A) = \prod_{k=0}^{n} [\text{Det} \Delta_{k}]^{(-1)^{k+1}k/2},$$

(3.2)

provided that the $\Delta_k$ have no harmonic modes (i.e. the twisted de Rham complex is acyclic). Even in that case, however, the product, as it stands, is still not well defined. To give meaning to the infinite product of eigenvalues, Ray and Singer introduced the $\zeta$-function regularization of determinants. Each determinant that appears in (3.2) is defined by

$$\text{Det} \Delta = \exp (-\zeta'(0)),$$

(3.3)
with
\[ \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{tr} \exp(-t\Delta) dt. \quad (3.4) \]
The remarkable fact established by Ray and Singer is that, with these definitions, the Ray-Singer Torsion does not depend on the Riemannian metric that went into its definition.

If there are non-trivial cohomology groups \( H^k_A \), there are two things that need to be changed in the above. First of all, the \( \zeta \)-function will have to include a projector onto the non-zero eigenvalues of \( \Delta_k \). Technically this is achieved by defining
\[ \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{tr} \left( \exp(-t\Delta) - P \right) dt, \quad (3.5) \]
where \( P = \lim_{t \to \infty} \exp(-t\Delta) \) is the projector onto the harmonic modes. Secondly, the Ray-Singer torsion should then properly be thought of as an object assigning a number to a choice of volume element on the cohomology groups, i.e. as an element of the one-dimensional vector space
\[ T_M(A) \in \bigotimes_{k=0}^n (\det H^k_A)^{(-1)^k}. \quad (3.6) \]
Here \( \det V \) denotes the highest exterior power of \( V \),
\[ \det V = \bigwedge^{\dim V} V. \]

With the appropriate definitions the torsion can then again be shown to be metric independent. This minor ambiguity (a scale factor) in the definition of the Torsion \( T_M(A) \) as a number will not concern us in the following as we will use a different kind of argument to fix the overall normalization of the path integral.

In a further development Schwarz \[19\] gave simple field theoretic representations of the Ray-Singer Torsion. This form of the Torsion allows for standard path integral manipulations and so makes its determination, in good circumstances, possible. We use such a representation to make contact with Chern-Simons theory. From the path integral point of view, the above ambiguity is also easy to understand. It corresponds to the ambiguity one encounters when attempting to gauge away or soak up the zero modes in the path integral. For more on the relation between the Ray-Singer torsion and path integrals, see \[8\].
3.1 Ray-Singer Torsion on $S^1$

All gauge fields on a circle are flat so that it makes sense to define the Torsion for any connection $A_0$ on $S^1$. From (3.2) we see that, in this case,

$$T_{S^1}(A_0) = [\text{Det } \Delta_1]^{1/2},$$

(3.7)

where the positive square root is to be taken. It is possible to get rid of the troublesome square root since

$$\Delta_1 = d_{A_0} * d_{A_0}^* = (d_{A_0}^*)^2.$$  

(3.8)

The last line makes sense as one-forms and zero-forms are in one to one correspondence via the Hodge operator. We therefore want to calculate $\text{det } d_{A_0}^*$. The field theoretic form for this determinant is

$$T_{S^1}(A_0) = N \int_{g} D\eta D\bar{\eta} \exp \left( i \int_{0}^{1} dt \bar{\eta}(t)(\partial_0 + A_0)\eta(t) \right),$$

(3.9)

with periodic boundary conditions on the anti-commuting fields $\eta$ and $\bar{\eta}$. These fields take values in the adjoint representation of $g$ and the subscript on the integral ($g$ or $t$ or $k$) will be used to indicate which of those fields we are integrating over. By making use of the gauge invariance of $T_{S^1}(A_0)$, it is possible to give a simple evaluation of the path integral in (3.9). From (3.9) we have that

$$T_{S^1}(A_0^g) = T_{S^1}(A_0).$$

(3.10)

On the circle, as we have seen, any gauge field is gauge equivalent to a constant gauge field (on the line it would be gauge equivalent to the zero connection). By making use of time independent gauge transformations the constant gauge field may be conjugated into a given torus of the Lie algebra of the group under consideration. By (3.10) in order to evaluate the path integral we need only consider the gauge field to be constant and to lie in a torus. At this point we encounter a technical difficulty. The $\eta$ and $\bar{\eta}$ lying in the Cartan subalgebra have zero modes that do not appear in the action so that at this point the path integral vanishes. These are precisely the (twisted) harmonic modes that we have been instructed to drop. We may simply define the right hand side of (3.9) to be an integral over the fields with values in $k$.

In path integral language this amounts to gauge fixing the zero modes to zero (there is clearly enough symmetry to do this and is the analogue of projecting them out) and to soaking up the contribution of the rest of the modes lying in $t$. 

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into the normalisation of $T_{S^1}(A_0)$. This factor is

$$
\int_t D\eta D\bar{\eta} \exp \left( \oint \bar{\eta} \partial_0 \eta(t) \right) = \prod_{n>0} n^{2\dim t} = \exp \left( 2 \dim t \sum_{n>0} \ln n \right) = 1/(2\pi)^{\dim t}
$$

(3.11)

We have just evaluated one of the determinants that appear in (2.16), namely

$$
\text{Det}(\partial_0)_{\Omega^0(S^1)} = 1/(2\pi)^{\dim t}.
$$

(3.12)

The part of the path integral that is left to evaluate runs over the fields with values in $k$. This is well defined, up to regularization, and corresponds precisely to the other determinant that we encountered in Chern-Simons theory (2.16). This functional integral is a standard representation of

$$
\text{tr}_{k} \left[ (-1)^F \exp(\text{ad}(A_0)) \right],
$$

(3.13)

with the trace restricted to $k$. By linear algebra this is simply

$$
T(A_0) \sim \text{tr}_{k} \left[ (-1)^F \exp(\text{ad}(A_0)) \right] = \det [1 - \text{Ad}_{k}(\exp A_0)].
$$

(3.14)

In order to fix the constants and at the same time answer some questions about regularization we compute the path integral. One could do this by expanding $\eta$ and $\bar{\eta}$ in Fourier modes but we follow a different path here, which will prove useful when we wish to exhibit the equivalence of the $G/G$ models and Chern-Simons theory on $\Sigma \times S^1$.

First define the path integral on the interval with boundary conditions $\eta(0) = \eta$ and $\bar{\eta}(1) = \bar{\eta}$,

$$
Z[A_0; \eta, \bar{\eta}] = \int_k D\eta D\bar{\eta} \exp \left( i \int_0^1 dt \bar{\eta}(t)(\partial_0 + A_0)\eta(t) - i\bar{\eta}(1)\eta(1) \right).
$$

(3.15)

The boundary term is needed to ensure that the $\eta(t)$ variation of the action is well defined. The path integral on the circle is now given by

$$
Z[A_0] = \frac{1}{(2\pi)^{\dim t}} \int_k d\eta d\bar{\eta} e^{i\bar{\eta}\eta} Z[A_0; \eta, \bar{\eta}].
$$

(3.16)

In order to evaluate (3.15) we perform a gauge transformation

$$
\eta(t) \to \text{Ad}(g_t)\eta(t), \quad \bar{\eta}(t) \to \text{Ad}(g_t)\bar{\eta}(t),
$$

(3.17)
with the gauge parameter defined by
\[ g_t = P \exp \left( \int_0^t A_0 \right) = \exp (tA_0). \] (3.18)

Clearly \( g_0 = 1 \) and we set \( g_1 = \exp A_0 \equiv g \). With respect to the new fields (3.13) becomes
\[ \int_k D\eta D\bar{\eta} \exp \left( i \int_0^1 dt \bar{\eta}(t) \partial_0 \eta(t) - i \text{Ad}(g^{-1})\bar{\eta}(1) \right), \] (3.19)
the Jacobian of the transformation being unity. The only point to note is that in terms of the new fields the boundary data does not change for \( \eta(t), \eta(0) = \eta \), while for \( \bar{\eta}(t) \) one has \( \bar{\eta}(1) = \text{Ad}(g^{-1})\bar{\eta} \). This means that
\[ Z[A_0; \eta, \text{Ad}(g)\bar{\eta}] = \int_k D\eta D\bar{\eta} \exp \left( i \int_0^1 dt \bar{\eta}(t) \partial_0 \eta(t) - i \eta(t) \right), \] (3.20)
with \( \eta(0) = \eta \) and \( \bar{\eta}(1) = \bar{\eta} \). We may now evaluate (3.20) by shifting fields
\[ \eta(t) = \eta + \eta_q(t), \]
\[ \bar{\eta}(t) = \bar{\eta} + \bar{\eta}_q(t), \] (3.21)
so that the boundary conditions on the “quantum” fields are \( \eta_q(0) = 0 \) and \( \bar{\eta}_q(1) = 0 \). This leads to
\[ Z[A_0; \eta, \text{Ad}(g)\bar{\eta}] = \exp (-i\bar{\eta}) \int_k D\eta_q D\bar{\eta}_q \exp \left( i \int_0^1 dt \bar{\eta}_q(t) \partial_0 \eta_q(t) \right). \] (3.22)

This path integral may now be thought of as a path integral on the circle, with the zero mode neglected. It can be evaluated just as in (3.11) to be
\[ \int_k D\eta_q D\bar{\eta}_q \exp \left( i \int_0^1 dt \bar{\eta}_q(t) \partial_0 \eta_q(t) \right) = 1/(2\pi)^{\dim k}. \] (3.23)

Putting all the pieces together we find, as expected from (3.14),
\[ Z[A_0] = \frac{1}{(2\pi)^{\dim k}} \int d\eta d\bar{\eta} \exp \left( i\bar{\eta}(1 - \text{Ad}(g^{-1}))\eta \right) = \frac{1}{(2\pi)^{\dim k}} \det (1 - \text{Ad}_k(g)). \] (3.24)

In going to the second line of this equation we have made use of the fact that \( \det \text{Ad}(g) = 1 \) (for the groups under consideration) and that \( \dim(k) \) is even.

### 3.2 Normalisation of the Ray-Singer Torsion

We would like to fix the constant \( N \) that appears in (3.9). Recall that for the trivial connection, when one ignores harmonic modes, the torsion should be unity.
As this amounts to projecting out the zero modes of η and \( \tilde{\eta} \) in this instance, we have
\[
T_{S^1}(A_0 = 0) = N \text{Det}' \partial_0 = N/(2\pi)^{\dim g} \frac{1}{2} = 1 ,
\]
so that
\[
T_{S^1}(A_0) = NZ[A_0] = \det(1 - \text{Ad}_k(g)) .
\]
This agrees with the results obtained by Freed \[20\] and Witten \[10\] using spectral sequences and a Meyer-Vietoris argument.

### 3.3 Ray-Singer Torsion on \( \Sigma \times S^1 \)

The discussion above shows us that the determinants encountered in the Chern-Simons theory, for constant \( A_0 \), are representations of the Ray-Singer Torsion on the circle and for that Torsion one has the concrete form
\[
T_{S^1}(A_0) = \det (1 - \text{Ad}_k(g)) .
\]
Comparing with the semi-classical approximation (1.4) we would expect to obtain the square root of the Ray-Singer Torsion \( T_{\Sigma \times S^1} \) of \( \Sigma \times S^1 \) rather than just some power of the Torsion on the circle. That these are indeed the same follows from (a slight generalization of) a theorem of Ray and Singer. This theorem \[16\] states that if \( M \) is a closed simply connected even-dimensional manifold, then the torsion of the product manifold \( M \times N \) (no restrictions on the dimension or fundamental group of \( N \)) is
\[
T_{M \times N} = T_N^{\chi(M)} .
\]
The theorem then applies to our case if \( M = S^2 \) and \( N = S^1 \) and
\[
T_{S^2 \times S^1}(A_0) = T_{S^1}(A_0)^2
\]
agrees with the result we have obtained. So what about higher genus surfaces? Looking at the proof of the theorem one sees that the restriction to simply connected \( M \) is there to ensure that a flat connection on \( M \times N \) is completely specified by its holonomies around the non-trivial 1-cycles of \( N \). This will be the case for arbitrary flat vector bundles over \( M \times N \) provided that \( M \) is simply connected.

Let us turn our attention to those flat vector bundles over \( M \times N \), where \( M \) is not necessarily simply connected, with connections of the local form \( A \in \Omega^1(N) \otimes \Omega^0(M) \), up to gauge equivalence. For such bundles the connection is once more completely specified by its holonomies around the 1-cycles of \( N \). In this setting (3.28) is once more correct and is then a slight extension of the theorem.
of Ray and Singer. As these conditions are met in the case at hand (after all, we are calculating the Ray-Singer torsion of $A_0 \, dt$ on $\Sigma \times S^1$), we find that

$$T_{\Sigma \times S^1}(A_0) = T_{S^1}(A_0)^{\chi(\Sigma)},$$

so that the determinant appearing in (2.17) is indeed precisely the square root of the torsion of $\Sigma \times S^1$.

One may give a standard mathematical proof of this generalization following almost line for line that of Ray and Singer. Turning the argument on its head, we see that path integral manipulations can be used to provide an alternative proof of this theorem, solving a problem raised in [8].

### 3.4 Relationship with Yang-Mills Theory on $S^1$

We could have expected the result we have derived for the Ray-Singer Torsion on the circle on general grounds. Suppose we wished to construct a Yang-Mills theory on the circle. Then, as $F_A$ is automatically zero, the only terms that would appear in the action are the gauge fixing and ghost terms. But the path integral obtained in this way is simply (3.9) integrated over all gauge fixed connections, with $\eta$ and $\bar{\eta}$ playing the role of the ghost $c$ and the antighost $\bar{c}$ respectively. This would be the integral of the Ray-Singer Torsion over the maximal Torus of the group $G$ (or more precisely over $T/W$, where $W$ is the Weyl group; see section 5.1 for the group theory used in the following).

Instead of working at the level of the connection, we could define Yang-Mills theory directly in terms of the structure group. That is we could simply associate to the connection its holonomy (this is like the first step of gauge fixing, namely $\partial_0 A_0 = 0$). The Yang-Mills path integral would then devolve to

$$\int_G dg = 1,$$

with a particular choice of Haar measure. However, in our previous considerations, we had moved down to the maximal Torus of $G$ (this corresponds to the next stage of gauge fixing, $A_0^k = 0$). We would like, therefore, to express the integral over $G$ as an integral over $T$. This may be done using the Weyl integral formula (section 5.1) to give

$$\int_G dg = \frac{1}{|W|} \int_T \det (1 - \text{Ad}_k(t)) dt,$$

in which we recognise the Ray-Singer Torsion.

As an aside we would like to point out that this Yang-Mills theory calculates
a topological invariant in its own right. There is a theorem that states that

\[ \chi(G/T) = \int T \det(1 - \text{Ad}_k(t)) dt, \]  

(3.33)

and this suggests that the properly normalised Yang-Mills path integral on the circle is the Euler character of the homogeneous space \( G/T \). It is not too difficult to see that the gauge fixed Yang-Mills action is equivalent to \( N = 2 \) supersymmetric quantum mechanics on \( G/T \), thus guaranteeing that this path integral yields the Euler character. This statement may be made more palatable by noting that the one-dimensional Yang-Mills action is \( Q \)-exact (\( Q \) is the BRST operator),

\[ S = Q \int \text{tr} \, c^k A^k_0. \]  

(3.34)

This is one of the characteristic features of cohomological field theories. Furthermore there is a second supersymmetry obtained by exchanging \( c \) and \( \bar{c} \), suggesting that this model is related to de Rham cohomology. And finally these BRST symmetries are seen to be the typical topological ‘shift’ symmetries at the group level. Namely, for

\[ g(t) = P \exp(\int_0^t A_0)g(0) \]  

(3.35)

one finds

\[ Qg(t) = c(t)g(t), \]

as one also transforms \( g(0) \) according to \( Qg(0) = c(0)g(0) \). This shows that we have a Witten type supersymmetric quantum mechanics model, so it is bound to calculate the Euler number of some space and the analysis may proceed from here.

### 3.5 The Path Integral to be Evaluated

We have now reduced the three dimensional path integral to a well specified two dimensional theory, namely

\[ \int DA_0 DA^t \det \left( 1 - \text{Ad}(e^{A_0}) \right)^{\chi(\Sigma)/2} \exp \left( \frac{ik}{2\pi} \int_\Sigma A_0 F_A \right). \]  

(3.37)

However, this is not quite the end of the story yet. It should be kept in mind that the gauge conditions

\[ \partial_0 A_0 = 0, \quad A^k_0 = 0, \]

still do not fix the time dependent gauge transformations completely. There are still ‘large’ periodic gauge transformations wrapping around the maximal torus \( T \) which shift \( A^k_0 \) by elements of the integer lattice \( I \) of \( t \). Explicitly (denoting
the time parameter by \( s \) to avoid confusion) these gauge transformations can be written as

\[
t(s) = t(0) \exp s\gamma \quad , \quad t(0) = t(1) \leftrightarrow \gamma \in I \quad ,
\]

\[
t^{-1}A_0 t(s) + t^{-1}\partial_0 t(s) = A_0 + \gamma \quad .
\]

(3.38)

As \( t/I = T \), eliminating these shifts is tantamount to regarding \( A_0 \) as a compact scalar field \( \phi^k \) taking values in \( T \). We have thus found that Chern-Simons theory on \( \Sigma \times S^1 \) is equivalent to an Abelian topological field theory,

\[
Z_{\Sigma \times S^1}(S_{CS}) = \int D[\phi, A] \exp (i(k + h) S_{\phi_F}(\phi, A)) \quad ,
\]

(3.39)

with action and measure given by

\[
S_{\phi_F}(\phi, A) = \frac{1}{2\pi} \int_{\Sigma} \text{tr} \phi F_A \quad ,
\]

(3.40)

\[
D[\phi, A] = D\phi \, DA \, \det(1 - \text{Ad}(e^{i\phi}))^{1-g} \quad .
\]

(3.41)

respectively. Here we have already included the shift \( k \rightarrow k + h \), whose occurrence we will establish in section 6.

### 3.6 Comparison with BF Theory

The action (3.40) is a ‘compact’ counterpart of the so-called BF theories, studied extensively in e.g. [8, 10, 11, 15, 21, 22], and defined in any dimension \( n \) by

\[
S_{BF} = \frac{1}{2\pi} \int_M \text{tr} BF_A \quad ,
\]

(3.42)

where \( B \) is an ordinary (i.e. non-compact) \((n - 2)\)-form. It will be useful to keep in mind the following differences between the compact and non-compact models in two dimensions:

1. The non-compact Abelian BF action (or rather \( \exp i k S_{BF} \)) also enjoys the invariance \( B \rightarrow B + \gamma, \gamma \in I \), since

\[
\int_{\Sigma} F_A \in 2\pi \mathbb{Z} \quad .
\]

(3.43)

The (crucial) difference however is that here this symmetry is not a consequence of some underlying remnant gauge invariance but just some global symmetry of the action. As such it need not and should not be eliminated, and all one can expect is to find it unitarily represented on the Hilbert space of the theory (something that is rather trivially true in this case).
2. In two dimensions (and, with some caveat also in general, see [22]) the integral over $B$ simply imposes the delta function constraint $F_A = 0$, so that the partition function calculates the volume of the moduli space of flat connections, with measure given by the Ray-Singer torsion. In $n = 2$ this measure coincides with the symplectic measure [10] and hence, with proper normalization, the partition function is the symplectic volume of $\mathcal{M}$. The compactness of $\phi$ in (3.40) on the other hand implies that the partition function is no longer a simple delta function but some deformation thereof. In fact, in terms of a suitably chosen mode expansion (spectral representation of the delta function) one finds that sufficiently high modes of the delta function are cut off due to the compactness of $\phi$.

3. Note also that, in the case of BF theories, any prefactor (coupling constant) like $k$ in the path integral can be absorbed by a rescaling of $B$, so that the result is essentially independent of $k$. This is something that, due to the compactness of $\phi$, cannot be done in the action $S_{\phi F}$, as a rescaling of $\phi$ would change its radius. We thus expect the partition function (and hence that of Chern-Simons theory) to depend in a much more subtle manner on $k$, something that is indeed borne out by the result, the Verlinde formula. One would, however, expect the large $k$ limit of this result to agree with the partition function of BF theory since, by rescaling, the large $k$ limit corresponds to a larger and larger radius of $\phi$. This can indeed be verified and is in agreement with the expectation that in the semi-classical limit of Chern-Simons theory the dimension of the Hilbert space is equal to the volume (number of cells) of phase space.

4. The $G/G$ model from Chern-Simons theory on $\Sigma \times S^1$

The purpose of this and the following sections is to give an alternative two-dimensional derivation of the results of the previous section. In particular, this will allow us to understand the determinant in (3.41) as arising not from the Ray-Singer torsion of $\Sigma \times S^1$ but this time from an infinite dimensional version of the Weyl integral formula applied to the partition function of the $G/G$ model. As a preliminary result we establish the equivalence of Chern-Simons theory on $\Sigma \times S^1$ and the $G/G$ model on $\Sigma$ directly at the level of the action and the path integral. This implies that the partition function of the $G/G$ model is the dimension of the space of conformal blocks of the $G$ WZW model and that the fusion rules are reproduced by the correlation functions of the traces of the group valued fields.
The strategy will be to write the partition function $Z_{\Sigma \times S^1}$ of Chern-Simons theory as the trace of an amplitude on $\Sigma \times I$ and to trade $A_0$ for the time independent group valued field $g = P \exp \int A_0 dt$, the holonomy of $A_0$, which captures the entire gauge invariant information carried by $A_0$. This is just the three-dimensional counterpart of the procedure employed in section 3.1 to determine the torsion of $S^1$. In that way we will end up with a two-dimensional theory which is expressed solely in terms of $g$ and the (time independent) boundary values of the spatial components of the connection. In order to recognize this as the $G/G$ model, we begin by recalling the action of the $G$ WZW model and the $G/H$ coset models.

### 4.1 The Gauged WZW Model

In complex coordinates on $\Sigma$, the action of the WZW model is (we follow the conventions of [7])

$$S_G(g) = S_0(g) - i \Gamma(g),$$  \hspace{1cm} (4.1)

$$S_0(g) = -\frac{1}{4\pi} \int_{\Sigma} d^2z g^{-1} \partial_z g^{-1} \partial_{\bar{z}} g,$$  \hspace{1cm} (4.2)

$$\Gamma(g) = \frac{1}{12\pi} \int_{N} d^3x \epsilon^{ijk} g^{-1} \partial_i g^{-1} \partial_j g^{-1} \partial_k g,$$  \hspace{1cm} (4.3)

where $\partial N = \Sigma$, i.e. $N$ is what is known as a handlebody. $S_G$ is invariant under a $G_L \times G_R$ symmetry, $g \rightarrow agb^{-1}$, which (for the particular choice of coefficient in (4.3)), is extended to a $G_L(\bar{z}) \times G_R(z)$ Kac-Moody symmetry. Only ‘anomaly-free’ subgroups of $G_L \times G_R$ can be gauged; these include, in particular, all subgroups $H$ of the adjoint group $G_{adj}$ ($g \rightarrow aga^{-1}$). Introducing an $H$ gauge field $A$, the action $S_{G/H}$ of the gauged WZW model is

$$S_{G/H}(g, A) = S_G(g) + S_{/H}(g, A),$$  \hspace{1cm} (4.4)

$$S_{/H}(g, A) = -\frac{1}{2\pi} \int_{\Sigma} d^2z (A_z \partial_z g^{-1} - A_{\bar{z}} g^{-1} \partial_{\bar{z}} g + A_{\bar{z}} A_z - g^{-1} A_z g A_{\bar{z}}).$$

This action gives a field-theoretic realization of the GKO coset models [23]. Taking $H = G$ in (4.4) one obtains the action of the topological $G/G$ model discussed by Verlinde and Verlinde [6] and more recently in [5, 7]. For later purposes it will be convenient to have this action written in terms of differential forms,

$$S_{G/G}(g, A) = -\frac{1}{8\pi} \int_{\Sigma} \text{tr} g^{-1} d_A g \ast g^{-1} d_A g - i \Gamma(g, A),$$  \hspace{1cm} (4.5)

$$\Gamma(g, A) = \frac{1}{12\pi} \int_{N} \text{tr} (g^{-1} dg)^3 - \frac{1}{4\pi} \int_{\Sigma} \text{tr} (A(dg g^{-1} + g^{-1} dg) + Ag^{-1}Ag).$$
4.2 Gauge Fixing

To streamline the derivation of $S_{G/G}$ from $S_{CS}$, we make some preliminary observations. First of all, as we have already seen in section 2.1, when discussing Chern-Simons theory on a three-manifold of the form $\Sigma \times S^1$, it is permissible and convenient to choose the gauge $\partial_0 A_0 = 0$. In this gauge

$$g_t \equiv P \exp \int_0^t A_0 = \exp t A_0$$

(4.6)

with $g_0 = 1$ and $g_1 = g$. Note also that $g^t A_0 = 0$ where we have introduced the notation $g^A = A g^{-1}$.

We will enforce this gauge condition algebraically as in section 2.1 by imposing the conditions (2.8) on the ghost and multiplier fields. The Faddeev-Popov determinant we obtain in this way will be $\det' D_0$ instead of the $\det D_0$ of section 2.2, as the conditions (2.8) are now imposed on all the Lie algebra components of the ghosts. As on constant modes $D_0$ reduces to $\text{ad}(A_0)$, these two determinants are related by

$$\det D_0 = \det' D_0 \times \det \text{ad}(A_0)$$

(4.7)

The difference between these two determinants will turn out to be crucial below.

4.3 Boundary Conditions and Boundary Terms

For the amplitude on $\Sigma \times I$, with $A_0$ regarded as an external source, we choose the holomorphic representation of the path integral. Calling the quantum field $B$, we wish to fix the boundary conditions

$$B_z|_{\Sigma \times \{0\}} = A_z, \quad B_{\bar{z}}|_{\Sigma \times \{1\}} = A_{\bar{z}}.$$

(4.8)

In order to do that, we have to add boundary terms to the action,

$$S_{CS}(A_0, B) \rightarrow S_{CS}(A_0, B) - \frac{1}{4\pi} \int_{\Sigma \times \{0\}} B_z B_{\bar{z}} - \frac{1}{4\pi} \int_{\Sigma \times \{1\}} B_z B_{\bar{z}}.$$

(4.9)

The path integral amplitude with these boundary conditions we will denote by $Z_{\Sigma \times I}[A_0; A_z, A_{\bar{z}}]$. In terms of this, the partition function of Chern-Simons theory on $\Sigma \times S^1$ can be written as

$$Z_{\Sigma \times S^1} = \int DA_0 DA \det' D_0 Z_{\Sigma \times I}[A_0; A_z, A_{\bar{z}}] \exp \left( \frac{ik}{2\pi} \int_{\Sigma} A_z A_{\bar{z}} \right),$$

(4.10)

where the last factor is the Kähler potential measure required in the holomorphic representation and compensating for the boundary terms in (4.9).
Finally, we need to keep track of the fact that on a manifold with boundary \( \exp ikS_{CS} \) is not invariant under arbitrary gauge transformations. Rather, one has
\[
\exp ikS_{CS}(A^h) = \Theta(A, h)^k \exp ikS_{CS}(A),
\]
where
\[
\Theta(A, h) = \exp(-\frac{i}{12\pi} \int_M (h^{-1} dh)^3 + \frac{i}{4\pi} \int_{\partial M} Adh h^{-1})
\]
(4.12)

As a consequence, the amplitude transforms non-trivially under gauge transformations and for \( \Sigma \times I \) and \( g_t = \exp tA_0 \) one finds
\[
Z_{\Sigma \times I}[A_0; A_z, A^g_z] = Z_{\Sigma \times I}[g_t A_0 = 0; A_z, A^z_\bar{z}] C[A^\bar{z}, g],
\]
(4.13)
where the Polyakov-Wiegmann cocycle \( C[A^\bar{z}, g] \) is
\[
C[A^\bar{z}, g] = \exp ikS_{G/G}(g^{-1}, A^\bar{z} = 0, A_z).
\]
(4.14)
This is nothing but the familiar transformation behaviour of Chern-Simons wave functionals under gauge transformations, see e.g. \[13\].

4.4 The Measure

We would also like to convert the linear measure \( DA_0 \) on the Lie algebra to the Haar measure \( Dg \) on the group. It follows from Duhamel’s formula
\[
\exp(-X) \delta \exp(X) = \int_0^1 ds \exp(-sX) \delta X \exp(sX),
\]
(4.15)
that these are related non-trivially by \[24\]
\[
Dg = \text{Det} \left( \frac{(1 - \text{Ad}(e^{A_0}))}{\text{ad}(A_0)} \right) DA_0.
\]
(4.16)
It is quite remarkable that, as a consequence of (4.7) and the calculations of section 3, \( \text{Det}'D_0 \) provides precisely this conversion factor (the missing \( \text{Det}'\partial_0 \) being kindly supplied by the \( B \) integration below).

4.5 Synthesis

Putting all this together and using the gauge invariance of the gauge field measure one finds that
\[
Z_{\Sigma \times S^1} = \int DgDAZ_{\Sigma \times I}[A_0 = 0; A_z, A^\bar{z}] C[A^\bar{z}, g] \exp(\frac{ik}{2\pi} \int_{\Sigma} A_z A^\bar{z}_g).
\]
(4.17)
Thus, finally, the remaining path integral we need is $Z_{\Sigma \times I}[A_0 = 0; A_z, A_{\bar{z}}]$ which is not a functional of $g$ and may be straightforwardly evaluated, providing due care of the boundary data is taken. One can introduce these boundary conditions into the path integral via Lagrange multipliers and then solve the resulting equations of motion for $B$. Alternatively, one uses the fact, that the boundary terms in $Z_{\Sigma \times I}[A_0 = 0; A_z, A_{\bar{z}}]$ were designed to cancel those arising from the variation of the Chern-Simons action, so that one can read off directly the equations of motion $\partial_0 B = 0$ which (with the boundary conditions (4.8)) imply $B = A$.

Either way one obtains

$$Z_{\Sigma \times I}[A_0 = 0; A_z, A_{\bar{z}}] = N \exp(-\frac{ik}{2\pi} \int_{\Sigma \times \{0\}} A_{\bar{z}} A_z) . \quad (4.18)$$

Here $N^{-1}$ is the factor Det $\partial_0$ which we already absorbed into the proper normalization of the measure (4.16). Putting all the pieces together from equations (4.13-4.18) one obtains from the Chern-Simons path integral on $\Sigma \times S^1$ precisely the $G_k/G_k$ gauged WZW action on $\Sigma$ (4.4) (with $g \to g^{-1}$, which is due to our choice of orientation for $\Sigma \times I$),

$$S_{G/G}(g^{-1}, A) = S_G(g^{-1}) - \frac{1}{2\pi} \int_{\Sigma} d^2z (A_{\bar{z}} \partial_z g^{-1} + A_z A_{\bar{z}} - A_{\bar{z}} A_\bar{z}) \quad (4.19)$$

with the correct Haar measure $Dg$. It is also evident from this derivation, that correlators of ‘vertical Wilson loops’ in Chern-Simons theory (corresponding to the fusion rules) at level $k$ are equal to the correlators of $\text{tr} g(z, \bar{z})$ in the $G_k/G_k$ coset model.

5 Abelian Reduction of the $G/G$ Theory

The next task is to evaluate the partition function of the $G/G$ model obtained in the previous section. We will do this by making use of the Weyl integral formula. This formula, whose precise form and derivation we will recall below, relates the integral of a conjugation invariant function on a compact group $G$ to an integral over the maximal torus $T$, an Abelian group. Applied to the path integral of the $G/G$ model it thus permits one to effectively reduce the path integral to that of an Abelian theory which can be exactly calculated.

Before proceeding we want to point out that the range of applicability of the Weyl integral formula and its relatives (valid e.g. for Lie algebras or non-compact semi-simple Lie groups) to path integrals is not limited to the $G/G$ models considered here but can also be used to significantly simplify the evaluation
of path integrals in other theories with local symmetries. We just mention that
this provides possibly the shortest available derivation of the partition function
of Yang-Mills theory on an arbitrary closed surface [12], obtained previously in
e.g. [10, 9] by various other methods.

5.1 The Weyl Integral Formula

To write down and explain the Weyl integral formula we will have to introduce
some notation. Thus let \( G \) be a compact Lie group (which we will also assume
to be semi-simple and simply connected later on) and \( T \) a maximal torus of \( G \).
The rank \( r = \text{rk}(G) \) of \( G \) is the dimension of \( T \). Any element of \( G \) can be
conjugated into \( T \) (‘diagonalized’) and the residual conjugation action of \( G \) on
\( T \) (permutation of the diagonal entries) is that of a finite group, the Weyl group
\( W = N(T)/T \) (\( N(T) \) denotes the normalizer of \( T \) in \( G \)). Furthermore any two
maximal tori are conjugate to each other and the set \( G_r \) of regular elements of \( G \n\) (i.e. those whose centralizer is conjugate to \( T \)) is open and dense in \( G \). It follows
that the conjugation map

\[
G/T \times T_r \rightarrow G_r \\
(g, t) \mapsto g^{-1}tg
\]  

(5.1)
is a \(|W|\)-fold covering onto \( G_r \). Corresponding to a choice of \( T \) we have a
direct sum decomposition of the Lie algebra \( g \) of \( G \), \( g = t \oplus k \), orthogonal with
respect to the Killing-Cartan metric on \( g \). \( G \) acts on \( g \) via the adjoint represen-
tation \( \text{Ad} \). This induces an action of \( T \) which acts trivially on \( t \) and leaves \( k \)
invariant (the isotropy representation \( \text{Ad}_k \) of \( T \) on \( k \)). Thus the complexified Lie
algebra \( g_C \) splits into \( t_C \) and the one-dimensional eigenspaces \( g_\alpha \) of the isotropy
representation, labeled by the roots \( \alpha \) (\( g_\alpha = g_{-\alpha} \)) and one obtains the Cartan
decomposition

\[
g_C = t_C \oplus \sum_\alpha g_\alpha \) .  
\]  

(5.2)

On \( G \) and \( T \) there exist natural invariant Haar measures \( dg \) and \( dt \) normalized
to \( \int_G dg = \int_T dt = 1 \). For the purpose of integration over \( G \) we may restrict
ourselves to \( G_r \) and we can thus use (5.1) to pull back the measure \( dg \) to \( G/T \times T \).
Calculating the corresponding Jacobian one finds the Weyl integral formula

\[
\int_G dg \ f(g) = \frac{1}{|W|} \int_{T} dt \ \det \Delta_W(t) \int_{G/T} dg \ f(g^{-1}tg) ,  
\]

(5.3)

where

\[
\Delta_W(t) = (1_k - \text{Ad}_k(t)) \) .  
\]  

(5.4)
In particular, if \( f \) is conjugation invariant (a class function), it is determined entirely by its restriction to \( T \) (where it is \( W \)-invariant) and (5.3) reduces to
\[
\int_G dg \ f(g) = \frac{1}{|W|} \int_T dt \ \det \Delta_W(t) f(t) ,
\]
which is the version of the Weyl integral formula which we will make use of later on. Note that the determinant of \( \Delta_W(t) \) is precisely equal to the Ray-Singer torsion on the circle we derived in section 3, where we also sketched a physical explanation for this coincidence.

It follows from (5.2) that
\[
\det(1 - \text{Ad}_k(t)) = \prod_{\alpha}(1 - e^{\alpha}(t)) .
\]
Decomposing the set of roots into positive \((\alpha > 0)\) and negative roots and introducing the Weyl vector \( \rho = \frac{1}{2} \sum_{\alpha > 0} \alpha \), this can also be written in terms of the denominator \( Q(t) \) of the Weyl character formula,
\[
Q(t) = \sum_{w \in W} \det(w) e^{w(\rho)(t)} ,
\]
as
\[
\det(1 - \text{Ad}_k(t)) = Q(t)Q(t) .
\]

5.2 Example: \( SU(2) \)

Let us illustrate the above in the case when \( G = SU(2) \). We parametrize elements of \( SU(2) \) and \( T = U(1) \) as
\[
g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} , \quad t = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} .
\]
The Weyl group \( W = \mathbb{Z}_2 \) acts on \( T \) as \( t \mapsto t^{-1} \). We use the trace to identify the Lie algebra \( \mathfrak{t} \) of \( T \) with its dual and introduce the positive root \( \alpha \) and the fundamental weight \( \lambda \),
\[
\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \lambda = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,
\]
satisfying the relations
\[
\text{tr} \ \alpha^2 = 2 , \quad \text{tr} \ \alpha \lambda = 1 , \quad \rho = \frac{1}{2} \alpha = \lambda .
\]
Later on we will find it convenient to parametrize elements of \( T \) in terms of weights. Thus, we write \( t = \exp i\lambda \phi \), where \( \phi \) is related to \( \varphi \) by \( \phi = 2\varphi \). Then
the expression \( \exp(\alpha)(t) \) entering (5.6) becomes \( \exp(\alpha)(t) = \exp i\phi \), and the Weyl denominator (5.7) and the determinant (5.4) are

\[
Q(t) = 2i \sin \frac{\phi}{2} , \\
\det \Delta_W(t) = 4 \sin^2 \frac{\phi}{2} .
\] (5.12)

Hence the Weyl integral formula for class functions is (with \( f(\phi) \equiv f(\exp i\lambda\phi) \))

\[
\int_G dg f(g) = \frac{1}{2} \int_0^{4\pi} \frac{d\phi}{4\pi} 4 \sin^2(\phi/2) f(\phi) \\
= \frac{1}{2\pi} \int_0^{4\pi} d\phi \sin^2(\phi/2) f(\phi) \\
= \frac{1}{\pi} \int_0^{2\pi} d\phi \sin^2(\phi/2) f(\phi) .
\] (5.13)

Here the last line follows e.g. from writing \( 2 \sin^2(\phi/2) = 1 - \cos \phi \) and is a useful reformulation because it effectively incorporates the action of the Weyl group.

### 5.3 Faddeev-Popov Derivation

We mention in passing that these formulae can be obtained à la Faddeev-Popov by ‘gauge fixing’ the non-torus part of \( g \) to zero (i.e. by imposing \( g \in T \) as a gauge condition). This amounts to inserting 1 in the form

\[
1 = \frac{1}{|W|} \int_{G/T} dh \int_T dt \delta(h^{-1}ght^{-1}) \det \Delta_W(t)
\] (5.14)

into the integral on the lhs of (5.3) and performing the integral over \( g \),

\[
\int_G dg f(g) = \frac{1}{|W|} \int_G dg \int_{G/T} dh \int_T dt \delta(h^{-1}ght^{-1}) \det \Delta_W(t) \\
= \frac{1}{|W|} \int_T dt \det \Delta_W(t) \int_{G/T} dh f(h^{-1}th) .
\] (5.15)

The Faddeev-Popov determinant \( \det \Delta_W(t)/|W| \) can then be obtained directly from the (BRST) variation of the condition \( g \in T \). In the \( SU(2) \) case this amounts to fixing the gauge \( g_{12} = g_{21} = 0 \). Since infinitesimally \( g_{12} \) transforms under conjugation as \( (a \in g) \)

\[
\delta g = [g, a] \Rightarrow \delta g_{12} = 2i a_{12} \sin \varphi ,
\] (5.16)

the resulting Faddeev-Popov determinant is just (5.12), while the additional factor of \( 1/2 \) accounts for the residual gauge freedom (conjugations leaving \( T \) invariant).
5.4 Abelianization of the WZW model

It is now straightforward to apply the Weyl integral formula to the partition function of the $G/G$ model. As the functional of $g$ that one obtains after having performed the path integral over the gauge fields is locally and pointwise conjugation invariant,

$$F(g) \equiv \int DA \exp(ikS_{G/G}(g, A)) = F(h^{-1}gh) \; , \tag{5.17}$$

one can formally use the Weyl integral formula pointwise to reduce the remaining path integral over the group valued fields to one over fields taking values in the torus $T$ of $G$. Or, in other words, one can use the gauge invariance of the $G/G$ action to impose the gauge condition $g \in T$. Either way one will generate a functional version of the determinant $\det \Delta_W(t)$ we encountered in section 5.1,

$$\int DgF(g) = \int Dt DA \det (1 - \text{Ad}_k(t)) \exp(ikS_{G/G}(t, A)) \; . \tag{5.18}$$

Let us now in turn take a look at the two parts $S_G(t)$ and $S_{G/G}(t, A)$ of the action $S_{G/G}(t, A)$ defined in (4.4). For notational simplicity only we will assume in the following that $G = SU(n)$. It should be apparent how to extend this to arbitrary compact $G$. We will also simplify things as in section 5.2 by identifying $t$ with its dual $t^*$. In particular, we will regard the roots of $G$ as elements of $t$ and a set $\{\alpha_l, l = 1, \ldots, n - 1 = r\}$ of simple roots as a basis of $t$. We thus expand the gauge field as $A^t = i\alpha_l A^l$. It is then convenient to parametrize the torus valued field $t$ in terms of the dual basis $\{\lambda^l\}$ of fundamental weights,

$$A^t = i\alpha_l A^l \; , \quad t = \exp i\phi^t \; , \quad \phi^t = \phi_l \lambda^l \; . \tag{5.19}$$

It is clear from this description that the $\phi_l$ are compact scalar fields. We will determine their range (radii) in section 7.4 below when we take into account the effect of the Weyl group.

In the WZW action $S_G(t)$ the kinetic term $S_0(t)$ reduces to the standard kinetic term

$$S_0(t) = \frac{1}{4\pi} \int_{\Sigma} \lambda^{kl} \partial_k \phi_l \partial_k \phi_l \; \tag{5.20}$$
for compact bosons. Here $\lambda^{kl} = \text{tr}(\lambda^k \lambda^l)$ is the inverse of the Cartan matrix.

The WZ term $\Gamma(t)$ does not vanish, as one might naively expect in the case of Abelian groups, but as a ‘topological’ term it only depends on the winding numbers of the field $\phi$. The reason for the appearance of this contribution is, that
maps from $\Sigma$ to $T$ with non-trivial windings cannot necessarily be extended to the interior $N$ of $\Sigma$ within $T$, as some (half) of the non-contractable cycles of $\Sigma$ become contractible in the handlebody $N$. The general form of this term is 

$$\Gamma(t) = \int_\Sigma \mu^{kl} d\phi_k d\phi_l ,$$  

(5.21)

where $\mu^{kl}$ is some antisymmetric matrix. As we will show below (cf. the discussion after (5.25)) that the non-trivial winding sectors do not contribute to the partition function, we will not have to be more precise about this term here.

5.5 The Gauge Field Contribution

In the action $S\!/_\!G(t, A)$ the contributions from the $t$ and $k$ components $A^t$ and $A^k$ of the gauge field $A$ are neatly separated so that it is easy to perform the path integral over the $A^k$, leaving behind an effective Abelian theory. In fact, because $t$ and $k$ are orthogonal to each other with respect to the invariant scalar product (trace), only $A^t$ will contribute to the terms of the form $A_z t^{-1} \partial_z t$ and $A_z \partial_z t t^{-1}$ (cf. equation (4.4)),

$$-\frac{1}{2\pi} \int_\Sigma (A_z \partial_z t t^{-1} - A_z t^{-1} \partial_z t) = \frac{1}{2\pi} \int_\Sigma (A^t \partial_z \phi - A^t \partial_z \phi) .$$  

(5.22)

We now observe that we can eliminate (5.20) altogether by a shift of the gauge field,

$$A^t \rightarrow A^t + \frac{1}{4} * d\phi^t ,$$  

(5.23)

(note that this is not a gauge transformation) leaving us with the simple action

$$\frac{1}{2\pi} \int_\Sigma \text{tr} A d\phi .$$  

(5.24)

As we will see in section 7.2 that only the constant modes of $\phi$ contribute to the path integral, we could have just as well carried the term (5.20) along until the end. And while this would have avoided the seeming nuisance of a metric dependent field redefinition, it is nicer to work with the action (5.24) because of its resemblance to other topological gauge theories in two dimensions.

We would now like to integrate by parts in (5.24) to put it into the form of the action of a BF theory (3.42), whose action in 2d, we recall, is

$$S_{BF} = \frac{1}{2\pi} \int_\Sigma \text{tr} B F_A ,$$  

(5.25)

where $B$ is an ordinary (non-compact) scalar field.
At first, the compactness of $\phi$ may cast some doubt on this procedure since, with $\phi$ being an ‘angular variable’, $d\phi$ is not necessarily exact. One would therefore expect to pick up ‘boundary’ terms from the monodromy of $\phi$. The following argument shows that in the $G/G$ model (and hence in Chern-Simons theory on $\Sigma \times S^1$) the non-trivial winding sectors of these fields do not contribute to the partition function: as it is only the harmonic modes of $A$ that couple to the non-exact (winding) parts of $d\phi$, integration over these modes will set the non-zero winding modes of $\phi$ to zero. As there is no Jacobian involved in going from $A$ to ‘harmonic modes plus rest’, this shows that we can indeed integrate by parts in (5.24) (with the understanding that the harmonic modes of $A$ no longer appear) and we thus arrive at the BF like action

$$S_{\phi F} = \frac{1}{2\pi} \int_{\Sigma} \phi_l F^l.$$  \hspace{1cm} (5.26) 

Here $F^l = dA^l$ is the curvature of the Abelian gauge field $A^l$. This argument also takes care of the WZ term and the result is in pleasant agreement with that obtained by quite different means in (3.40). One of the important differences between this theory and the ordinary BF models is of course, as already pointed out in section 3.6, that here the scalar fields $\phi_l$ are compact which implies that the integral over them will not simply produce a delta function onto flat connections as is the case in the non-compact BF theories.

While $A^k$ has made no appearance in the above, it is $A^t$ that will drop out of the remaining term $A_z \bar{A}_z - g^{-1} A_z g A_z$ which becomes simply

$$A_z \bar{A}_z - g^{-1} A_z g A_z \rightarrow A^k_z (1 - \text{Ad}_k(t)) A^k_{\bar{z}}.$$  \hspace{1cm} (5.27) 

Thus the $A^k$ integral will give rise to a determinant that formally (cf. the considerations in section 2.2) cancels against the Faddeev-Popov determinant in (5.18). Of course this is not correct, as certainly the zero modes will leave behind a finite dimensional determinant. Furthermore, the determinants should be properly regularized, and we will perform this in the following section. Suffice it to say here that this gives rise to the shift $k \rightarrow k + h$. In fact, the residual finite dimensional determinant $\det^{1-g} \Delta_W(t)$ and the shift will arise simultaneously as the gravitational and gauge field contributions to the chiral anomaly.

Anticipating this result we have thus deduced that the $G/G$ model on $\Sigma$ (and hence Chern-Simons theory on $\Sigma \times S^1$) is equivalent to the two-dimensional Abelian $\phi F$ theory (5.26) with measure $\det^{1-g} \Delta_W$. In particular this is in perfect agreement with the results of section 3 (cf. equations (3.32-3.41)).
6 The shift \( k \to k + h \)

Thus far we have cancelled the determinants arising from the gauge field integration against those of the ghosts, up to harmonic modes, with gay abandon. This was the case e.g. in sections 2 and 3, where we claimed that the ratio (2.10) of determinants would give rise to the Ray-Singer torsion of \( \Sigma \times S^1 \) and where we also promised that a gauge invariant regularization would produce the shift \( k \to k + h \). This was also the case in the previous section, where we argued that the ratio of the determinants \( \text{Det} \Delta_W(t) \) leads to the same result. It is thus high time to give a precise meaning to the ratios of determinants involved and to declare the regularisation that is used to do this. We start by considering the determinants that arise in the \( G/G \) model. As we have shown the equivalence of Chern-Simons theory with the \( G/G \) model, this also takes care of the former. However, a small additional argument allows us to reduce the calculation of the Chern-Simons determinants directly to the \( G/G \) result and the calculation of the Ray-Singer torsion on \( S^1 \) and we indicate in section 6.4 how this is done.

6.1 The Dolbeault Complex in the \( G/G \) Model

The first thing to note is that our gauge fixing so far has only been partial as we have been careful to preserve the Abelian \( T \) invariance. We should thus regularize in a manner which respects this residual gauge invariance and we will accomplish this by using a heat kernel (or \( \zeta \)-function) regularization based on the \( t \) covariant Laplacian \( \Delta_A = -(d_A^* d_A + d_A d_A^*) \) where \( A \) is the \( T \) gauge field. For an operator \( \mathcal{O} \) we set

\[
\log \text{Det} \mathcal{O} = \text{Tr} e^{-\epsilon \Delta_A} \log \mathcal{O} ,
\]

where we use \( \text{Tr} \) to denote a functional trace (e.g. including an integration).

We begin with the determinant that arises on integrating out \( A^k \), see (5.27). Up to an overall factor, the relevant part of the action is (using the differential form version (4.5))

\[
\text{tr}(A^k * A^k - A^{k t^{-1}}(i + *)A^k t) .
\]

To put this into a more explicit form we recall that, on the root space \( g_\alpha \subset \mathfrak{k}_C \), \( \text{Ad}(t) \) acts by multiplication by \( \exp i\alpha(\phi) \). Furthermore, with respect to the Killing-Cartan metric (trace), \( g_\alpha \) and \( g_\beta \) are orthogonal unless \( \beta = -\alpha \). Thus, expanding \( A^k \) in terms of basis vectors \( E_\alpha \) of \( g_\alpha \) such that

\[
A^k = \sum_\alpha E_\alpha A^\alpha , \quad \text{tr}(E_\alpha E_{-\alpha}) = 1 ,
\]

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we can break up (6.2) into a sum of terms depending only on the pair \( \pm \alpha \). We obtain

\[
(6.2) = \sum_{\alpha} A^\alpha \ast A^{-\alpha} - A^\alpha e^{-i\alpha(\phi)}(i + \ast)A^{-\alpha} \\
= \sum_{\alpha > 0} \left[ A^\alpha (i + \ast) M_{-\alpha} A^{-\alpha} - A^\alpha(i - \ast)M_{\alpha} A^{-\alpha} \right] ,
\]

where \( M_{\alpha} \) is the number

\[
M_{\alpha} = \left(1 - e^{i\alpha(\phi)}right),
\]

\[
\prod_{\alpha} M_{\alpha} = \det (1 - Ad_k(e^{i\phi})) . \tag{6.5}
\]

Writing this in terms of the scalar product (2.12) on 1-forms, one sees that the path integral over \( A^k \) yields

\[
\prod_{\alpha > 0} \text{Det} \left[(1 + i\ast)M_{\alpha} + (1 - i\ast)M_{-\alpha}\right]^{-1} . \tag{6.6}
\]

Here we recognize the projectors

\[
P_\pm = \frac{1}{2}(1 \pm i\ast) \tag{6.7}
\]

onto the spaces of \((1, 0)\)-forms \((\sim dz)\) and \((0, 1)\)-forms \((\sim d\bar{z})\) respectively and thus (6.6) exhibits quite clearly the chiral nature of the (gauged) WZW model. As a consequence of the presence of the projectors \( P_\pm \) in (6.6), the two summands act on different spaces. For each \( \alpha \) we may thus write the determinant as a product of the \((1, 0)\) and \((0, 1)\) pieces,

\[
\prod_{\alpha > 0} \text{Det} \left[(1 + i\ast)M_{\alpha} + (1 - i\ast)M_{-\alpha}\right]^{-1} = \left[\text{Det}_{(1,0)}M_{\alpha}\right]^{-1} \times \left[\text{Det}_{(0,1)}M_{-\alpha}\right]^{-1} . \tag{6.8}
\]

Before evaluating this, we will combine it with the contributions from the ghosts (equivalently, the Weyl integral formula). The ghost action has the form

\[
\sum_{\alpha > 0} \left[ c^\alpha \ast M_{-\alpha} c^{-\alpha} + \bar{c}^{-\alpha} \ast M_{\alpha} c^\alpha \right] , \tag{6.9}
\]

and therefore the ghost determinant is

\[
\prod_{\alpha > 0} \text{Det}_0 M_{\alpha} \text{Det}_0 M_{-\alpha} . \tag{6.10}
\]

Combining this with (6.8), we see that we need to determine and make sense of

\[
\prod_{\alpha > 0} \left[\text{Det}_0 M_{\alpha} \text{Det}_{(1,0)}^{-1}M_{\alpha}\right] \left[\text{Det}_0 M_{-\alpha} \text{Det}_{(0,1)}^{-1}M_{-\alpha}\right] . \tag{6.11}
\]
This we will accomplish by relating the products of these determinants to the Witten index of the Dolbeault complex. Indeed, suppose that $M_\alpha$ is a constant. Then

$$\log \det_0 M_\alpha \det^{-1}_{(1,0)} M_\alpha = \left[ \text{Tr}_0 e^{-\epsilon \Delta_A} - \text{Tr}_{(1,0)} e^{-\epsilon \Delta_A} \right] \log M_\alpha ,$$  \hfill (6.12)

where we need to remember that the Laplacian $\Delta_A$ acts to the right on one-forms taking values in $g_{(-\alpha)}$, the root space of $(-\alpha)$. There we have

$$d_A|(-\alpha) = d - i\alpha(A) \equiv d + \text{tr}(\alpha \omega_l) A^l ,$$  \hfill (6.13)

so that the ‘charge’ is $\text{tr}(\alpha \omega_l)$. The term in brackets is nothing but the index of the Dolbeault complex,

$$\left[ \text{Tr}_0 e^{-\epsilon \Delta_A} - \text{Tr}_{(1,0)} e^{-\epsilon \Delta_A} \right] = \sum_{p=0}^1 (-1)^p b^{p,0} = \text{Index } \bar{\partial}_A .$$  \hfill (6.14)

This index can of course be calculated directly from the heat kernel expansion, but if one does not want to reinvent the wheel one may call upon the known result that for the Dolbeault operator coupled to a vector bundle $V$ with connection $A$ one has (see e.g. [26])

$$\text{Index } \bar{\partial}_A = \int_M \text{Td}(T^{(1,0)}(M)) \text{ch}(V) .$$  \hfill (6.15)

In two dimensions this reduces to

$$\text{Index } \bar{\partial}_A = \frac{i}{2} \chi(\Sigma) + c_1(V) .$$  \hfill (6.16)

Therefore, in the case at hand, one finds that (6.12) equals

$$\text{Index } \bar{\partial}_{(-\alpha)} \log M_\alpha = \left[ \frac{i}{2} \chi(\Sigma) + c_1(V_{(-\alpha)}) \right] \log M_\alpha$$

$$= \left[ \frac{1}{8\pi} \int_\Sigma R + \frac{1}{2\pi} \int_\Sigma \text{tr}(\alpha \omega_l) F^l \right] \log M_\alpha .$$  \hfill (6.17)

When $M_\alpha$ is not a constant, one simply has to move $\log M_\alpha$ into the integral, so that one obtains

$$\log \det_0 M_\alpha \det^{-1}_{(1,0)} M_\alpha = \frac{1}{8\pi} \int_\Sigma R \log M_\alpha + \frac{1}{2\pi} \int_\Sigma \text{tr}(\alpha \omega_l) F^l \log M_\alpha .$$  \hfill (6.18)

To see that this is correct, we write

$$\text{Tr} \log M_\alpha e^{-\epsilon \Delta_A} \equiv \int dx \langle x| \log M_\alpha e^{-\epsilon \Delta_A} |x \rangle$$

$$= \int dx \log M_\alpha \langle x| e^{-\epsilon \Delta_A} |x \rangle ,$$  \hfill (6.19)
and note that $R$ and $F$ arise as the first Seeley coefficients in the expansion of $\langle x | e^{-\epsilon \Delta A} | x \rangle$. It is worthwhile remarking that the result (6.18) is finite, the $1/\epsilon$ poles cancelling between the scalar and one-form contributions. This is another manifestation of the supersymmetry discussed in section 2. Looking at the relevant gauge field and ghost terms in the action, (6.2) and (6.9), we see that in the $G/G$ model it appears in the (chiral) form

$$\delta A^{-\alpha} = \epsilon e^{-\alpha}, \quad \delta \bar{c}^\alpha = *(\epsilon(-i + *)A^\alpha)$$

(with a similar expression for the other components). Again, the presence of this supersymmetry implies (formally) that only the finite dimensional spaces of zero modes contribute to the ratio of determinants.

One can proceed analogously for the second factor in (6.11). In this case it is the index of $\partial A$ that makes an appearance and which differs by the sign of the second summand from (6.16),

$$\text{Index } \partial A = \frac{1}{2} \chi(\Sigma) - c_1(V).$$

(6.21)

As the Laplacian still acts on $g_{(-\alpha)}$, one obtains a $\log M_{\alpha} - \log M_{-\alpha}$ contribution to the gauge field part of the index, while it is the sum of the two terms that contributes to the gravitational part. Hence one finds that the regularized determinant (6.11) is

$$\text{(6.11)} = \prod_{\alpha > 0} \exp \left( \frac{1}{8\pi} \int_\Sigma R \log M_\alpha M_{-\alpha} + \frac{1}{2\pi} \int_\Sigma \text{tr}(\alpha \alpha \ell) F^\ell \log \frac{M_\alpha}{M_{-\alpha}} \right).$$

(6.22)

We will now consider separately the two contributions to this expression.

### 6.2 Generalized Ray-Singer Torsion

Rewriting the term in (6.22) that depends on the curvature as

$$\exp \left[ \frac{1}{8\pi} \int_\Sigma R \sum_{\alpha > 0} \log M_\alpha M_{-\alpha} \right],$$

(6.23)

one recognizes it as a dilaton like coupling to the metric, the role of the dilaton being played by

$$\Phi = \sum_{\alpha > 0} \log M_\alpha M_{-\alpha} = \log \det(1 - \text{Ad}(e^{i\phi})).$$

(6.24)

For our purposes, however, the most useful way of looking at (6.23) is to regard it as a (metric dependent) generalization of the (metric independent) Ray-Singer
torsion on $\Sigma \times S^1$ to non-flat connections. Indeed, $A_0 dt$ is flat iff $A_0$ is constant iff $\phi$ is constant iff $\Phi$ is constant, and in that case (6.23) reduces to

$$
\exp \chi(\Sigma) \Phi/2 = \left[ \det (1 - \text{Ad}(e^{i\phi})) \right]^{\chi(\Sigma)/2}
= T_{\Sigma \times S^1}(A_0) .
$$

(6.25)

On the other hand, when $A_0$ is not flat on $\Sigma \times S^1$, we can think of each point $x \in \Sigma$ as indexing a flat connection on $S^1$. From this point of view, (6.23) is an averaging over the different $S^1$ connections weighted by the curvature. This is, of course, not metric independent in general but turns out to be so for flat connections on $\Sigma \times S^1$.

We will show in section 7.2 that only constant $\phi$ configurations contribute to the path integral. Hence the expression (6.23) will then indeed collapse to the Ray-Singer torsion, as anticipated by our more naive considerations in sections 3 and 5.

6.3 The Shift $k \to k + h$

We now come to the crux of the matter. The second term in (6.22) is responsible for the shift in $k$. To see this note that

$$
\frac{M_\alpha}{M_{-\alpha}} = \frac{1 - e^{i\alpha \phi}}{1 - e^{-i\alpha \phi}} = -e^{i\alpha \phi} ,
$$

(6.26)

so that

$$
\prod_{\alpha > 0} \exp \left( \frac{1}{2\pi} \int_{\Sigma} \text{tr}(\alpha \alpha l) F^l \log \frac{M_\alpha}{M_{-\alpha}} \right) = \exp \left[ \frac{i}{2\pi} \sum_{\alpha > 0} \int_{\Sigma} \text{tr}(\alpha \alpha l) \alpha(\phi) F^l \right] .
$$

(6.27)

Here we have suppressed the imaginary contribution to the log, as it will make no appearance for simply connected groups (where $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$ is integral).

We now put the exponent in more manageable form by noting that the (negative of the) Killing-Cartan metric $b$ of $g$, restricted to $t$,

$$
b(X, Y) = - \text{tr} \text{ad}(X) \text{ad}(Y) ,
$$

(6.28)

can be written in terms of the roots as

$$
b(X, Y) = 2 \sum_{\alpha > 0} \alpha(X) \alpha(Y) .
$$

(6.29)

Moreover, with our convention that $\text{ad}(X) \big|_\alpha = i\alpha(X)$, $b(X, Y)$ is related to the Coxeter number (or quadratic Casimir of the adjoint representation) $h$ via

$$
b(X, Y) = 2h \text{tr}(XY)
$$

(6.30)

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\((h = n \text{ for } SU(n))\). Hence the exponent becomes
\[
\frac{i}{2\pi} \sum_{\alpha > 0} \int_{\Sigma} \text{tr}(\alpha \alpha_l) \alpha(\phi) F^l = \frac{i}{4\pi} \int_{\Sigma} b(\phi, \alpha_l) F^l = \frac{i\hbar}{2\pi} \int_{\Sigma} \phi_l F^l ,
\]
which produces precisely the long awaited shift \(k \to k + \hbar\) in the action \(S_{\phi F}\),
\[
\frac{i}{2\pi} \int_{\Sigma} \phi_l F^l \longrightarrow \frac{i(k + \hbar)}{2\pi} \int_{\Sigma} \phi_l F^l .
\]

### 6.4 The Calculation for Chern-Simons Theory

We now indicate briefly how the calculation of the determinants arising in Chern-Simons theory on \(\Sigma \times S^1\) can be reduced to those performed for the torsion of the circle \(S^1\) in section 3.1 and for the \(G/G\) model above. The ratio of determinants that needs to be regularized is given in (2.10). To make our calculation as easy as possible and to make contact with the \(\zeta\)-function regularization of section 3 and the heat-kernel regularization of of the \(G/G\) model, we use a hybrid regularization.

As only differential forms of the type \(\Omega^\ast(\Sigma) \otimes \Omega^0(S^1)\) and time derivatives enter into (2.10), it is convenient to expand the forms as Fourier series. Then the higher modes can be dealt with essentially as in the case of the torsion of the circle so that one finds e.g.
\[
\frac{\text{Det}' \partial_0 |_{\Omega^1(\Sigma) \otimes \Omega^0(S^1)}}{\text{Det} t |_{\Omega^1(\Sigma) \otimes \Omega^0(S^1)}} = (2\pi)^{-\chi(\Sigma) \dim t/2} .
\]
This then reduces these determinants to those of the purely algebraic operators (no derivatives) of the kind we encountered in the \(G/G\) model. The chiral nature of these determinants, which is not obvious from the Chern-Simons point of view, arises because once one has chosen a metric on \(\Sigma\) to implement the heat kernel regularization, one has in particular chosen a complex structure on \(\Sigma\). It is then natural to decompose the differential forms into their \((1,0)\) and \((0,1)\) parts. Note that this is the way the (projective) dependence on the complex structure arises quite generally in the canonical quantization of Chern-Simons theory.

### 7 Evaluation of the Abelian theory and the Verlinde formula

In this section we will evaluate the partition function
\[
Z_\Sigma(S_{\phi F}, k) = \int D\phi DA \det(1 - \text{Ad}(e^{i\phi}))^{\chi(\Sigma)/2} \exp \left( \frac{i(k + \hbar)}{2\pi} \int_{\Sigma} \text{tr} \phi F_A \right) ,
\]
\[(7.1)\]
which, as we have seen in the preceding sections, is equal both to the partition function of Chern-Simons theory on $\Sigma \times S^1$ and that of the $G/G$ model on $\Sigma$.

### 7.1 A Trivializing Map

This task can be simplified significantly by making use of a suitable change of variables from $A$ to $F_A$, introduced in [22], which trivializes the path integral and is in some sense an analogue of the Nicolai map of supersymmetric field theories. In order to implement this we will have to choose some gauge fixing condition $G(A) = 0$ for the Abelian gauge symmetry of the path integral (7.1). Here $G(A)$ is some scalar $t$-valued function like e.g. $G(A) = \partial \cdot A$ or $G(A) = n \cdot A$. This will give rise to a Faddeev-Popov determinant

$$\text{Det} \Delta_{FP} = \text{Det} \left( \frac{\delta G(A)}{\delta A} d_A \right). \quad (7.2)$$

The next step is to perform the change of variables

$$A \rightarrow (F_A, G(A)) . \quad (7.3)$$

This maps the one-form $A$ to a pair of scalars and is well defined because we have already eliminated the harmonic modes of $A$ (cf. the discussions on the Hodge decomposition in section 2 and the elimination of the winding modes in section 5.5). It is easy to see that, for any choice of $G(A)$, the Jacobian of this change of variables cancels precisely against the Faddeev-Popov determinant (7.2),

$$\text{Det} \left( \frac{\delta (F_A, G(A))}{\delta A} \right) = \text{Det} \Delta_{FP} . \quad (7.4)$$

This is a manifestation of the fact that the Ray-Singer torsion is trivial in even (and hence in particular in two) dimensions. After performing the (trivial) integrals over $G(A)$ and its multiplier field one is then left with a path integral from which all derivatives have disappeared,

$$Z_\Sigma(S_{\phi F}, k) = \int D\phi \int D\phi F_A \det(\Delta_W(e^{i\phi}))^{\chi(\Sigma)}/2 \exp \left( \frac{i(k+h)}{2\pi} \int_{\Sigma} \text{tr} \phi F_A \right) , \quad (7.5)$$

and which can be straightforwardly evaluated.

### 7.2 Non-trivial T Bundles and Integrality Conditions

The only point that needs some care is that the integral over $F_A$ should not extend over all two-forms but only over those that arise as the curvature $F_A$ of
some connection $A$. As two-forms in two dimensions are automatically closed, all that we need to require is the integrality condition

$$\int_{\Sigma} F^l_A \in 2\pi \mathbb{Z} \quad \forall l = 1, \ldots, r = \dim \mathfrak{t} .$$

(7.6)

We do this by inserting the periodic delta function

$$\delta^P(\int_{\Sigma} F^l_A) = \sum_{n=-\infty}^{\infty} \exp\left(in \int_{\Sigma} F^l_A\right)$$

(7.7)

into the path integral for each $l$ and can henceforth drop the (now irrelevant) label $A$ on $F_A$. Recalling that the label $l$ on $F^l$ refers to an expansion of $A$ in terms of simple roots $\alpha_i$, and that the fundamental weights $\lambda^k$ are dual to these (5.19), we see that we can write the sum over $r$-tuples of integers arising from (7.7) as a sum over elements $\lambda = \sum_{k} n_k \lambda^k$ of the weight lattice $\Lambda = \mathbb{Z}[\lambda^1, \ldots, \lambda^r]$ (7.8) of $G = SU(n)$. Thus (7.5) becomes

$$(\ref{7.5}) = \sum_{\lambda \in \Lambda} \int D\phi \det(\Delta_W(e^{i\phi}))^{\chi(\Sigma)/2} \exp\left(\frac{i(k+h)}{2\pi} \int_{\Sigma} \phi F - i \int_{\Sigma} \lambda F\right) .$$

(7.9)

Let us briefly pause to explain the necessity of including non-trivial $T$ bundles in this sum although the $G$ bundle we started off with was (necessarily for simply connected $G$) trivial. The reason for this is that any $T$ connection on $\Sigma$, be it a connection on a trivial bundle or not, can also be regarded as a particular $G$ connection (technically speaking, the structure group can always be extended from $T$ to $G$). Thus an integral over connections on a trivial $G$ bundle will necessarily have to include contributions from all the non-trivial $T$ sectors. We encountered a similar phenomenon in section 5.4, where we saw that an integral over maps from $\Sigma$ to $G$, which are all homotopic to the identity, devolved to an integral over maps to $T$ including all non-trivial winding sectors (that these turned out not to contribute to the path integral is tangential to the present discussion). Regarded as maps from $\Sigma$ to $G$ which just happen to take their values in $T \subset G$, these maps of course become contractible.

After this excursion we return to the path integral (7.9). The integration over $F$ can now be performed, giving rise to a delta function constraint on $\phi$, so this step in our evaluation of (7.9) leads to

$$Z_{\Sigma}(S_{\phi F}, k) = \frac{1}{\prod_{k=1}^{r} \sum_{n_k} \int D\phi \det(\Delta_W(e^{i\phi}))^{\chi(\Sigma)/2} \delta(\frac{k+h}{2\pi} \phi - \lambda)}$$

$$= \frac{1}{\prod_{k=1}^{r} \sum_{n_k} \int D\phi \det(\Delta_W(e^{i\phi}))^{\chi(\Sigma)/2} \delta(\frac{k+h}{2\pi} \phi_k - n_k)} .$$

(7.10)
The first thing to note is that this equation implies in particular that only the constant modes of \( \phi \) contribute to the partition function. This has the two consequences mentioned above, namely a) that the dilaton term of section 6.2 turns into the metric independent Ray-Singer torsion of \( \Sigma \times S^1 \), and b) that had we carried around the kinetic term (5.20) for the compact scalars until now (instead of eliminating it by the shift (5.23)), it would disappear now.

One can gradually see the structure of the Verlinde formula emerging. The correct integrand (which becomes a summand via the delta function) has been around for some time. We have also been forced to include a sum over the weight lattice, whose summation range will be restricted by the compactness of \( \phi \). In order to recognize the result as a sum over the highest weights of integrable representations at level \( k \), it is convenient to restrict the integration range for \( \phi \) to a fundamental domain of the action of the Weyl group \( W \) on \( T \) (which is the only piece of gauge freedom we have not yet fixed). As \( W \) is a finite group and the integral (7.10) is manifestly \( W \)-invariant, the result will be the same as dividing the integral by \( |W| \), but not necessarily manifestly so, as the sum will then extend over all the weights in the \( W \)-orbits of highest weights of integrable representations. We postpone a discussion of the overall normalization of the path integral, which is subject to a standard renormalization ambiguity anyway, until after we have convinced ourselves that up to normalization the partition function is equal to the Verlinde formula.

### 7.3 The Verlinde Formula for \( SU(2) \)

In this case, \( \phi \) is a single compact scalar, \( r = 1 \) and \( \Lambda \sim \mathbb{Z} \) in (7.10) and the Weyl determinant (torsion) is \( 4 \sin^2(\phi/2) \). By the action of the Weyl group the range of \( \phi \) is cut down from \([0, 4\pi)\) to \([0, 2\pi]\) and it is convenient to use the form of the Weyl integral formula given in the last line of (5.13). Thus, for \( SU(2) \) we have

\[
Z_\Sigma(S_{\phi F}, k) = \sum_{n=-\infty}^{+\infty} \int_0^{2\pi} d\phi \, \sin^2\left(\frac{\phi}{2}\right) \, \delta\left(\frac{k+2}{2\pi}\phi - n\right).
\] (7.11)

In particular, only certain discrete values of \( \phi \) contribute to the path integral and due to the compactness of \( \phi \) only a finite number of \( n \)'s give a non-vanishing contribution. Ignoring the boundary values \( n = 0 \) and \( n = k + 2 \) for a moment (we will come back to them below) we see that the allowed values of \( \phi \) are

\[
\phi = \frac{2n\pi}{k+2}, \quad n = 1, \ldots, k+1.
\] (7.12)

These points are in one-to-one correspondence with the \( k+1 \) integrable representations of the \( SU(2) \) WZW model at level \( k \) and indeed we see that, up to
normalization, the partition function

\[ Z_{\Sigma}(S_{\phi F}, k) = \sum_{n=1}^{k+1} \sin^{2-2g}\left(\frac{n\pi}{k+2}\right) \quad (7.13) \]

is precisely equal to the Verlinde formula

\[ \dim V_{g,k} = \left(\frac{k+2}{2}\right)^{g-1} \sum_{j=0}^{k} \left(\sin^2\left(\frac{j+1}{k+2}\right)\pi\right)^{1-g}. \quad (7.14) \]

It remains to come to terms with the values \( \phi = 0 \) and \( \phi = 2\pi \), arising from \( n = 0 \) and \( n = k + 2 \) and arising as the boundary points of the reduced \( \phi \)-range \([0, 2\pi]\). The first thing to note is, that these values correspond to the connections on the circle with holonomy group \{1\} (the trivial connection) and \{1, -1\} respectively. As such they are the most reducible connections on the circle and require a special treatment in the path integral. This can also be seen from our use of the Weyl integral formula which, strictly speaking, only covers the regular elements of \( G \) or \( T \), i.e. excludes precisely the two special values of \( \phi \) (for which the Weyl determinant vanishes).

The usual procedure would be to either declare their contributions to be zero because of ghost zero modes or to ignore these singular points. Technically, this can be achieved by choosing the integration range for \( \phi \) to be \([\epsilon, 2\pi - \epsilon]\) and taking the limit \( \epsilon \to 0 \). This also takes care of the problem that these boundary values give rise to infinities in the partition function in genus \( g > 1 \) (as may be seen from (7.13)) and any other method of regulating these infinities would also amount to ignoring these contributions. We can take the attitude that the WZW models are defined by integrating over fields with values in \( G_r \). Such configurations are dense in the space of fields and the path integral is naturally regularized by the restriction to \( G_r \).

As this procedure may nevertheless seem somewhat ad hoc, we want to point out that there is also another reason for ‘dropping’ the boundary values and (more generally) the points on the boundary of the Weyl alcove. Namely, as is well known there is a quantization ambiguity in Chern-Simons theory (see e.g. \([14, 27]\)), corresponding to the option to work with \( W \)-even or \( W \)-odd wave functions. While both of these appear to lead to perfectly unitary quantizations of Chern-Simons theory, it is only the latter which turns out to be related to the current blocks of \( G_k \) WZW models. In particular, for \( SU(2) \) this forces the wave functions to vanish at \( \phi = 0 \) and \( \phi = 2\pi \). And, while the differences between the two alternatives are quite far-reaching and subtle in general, for the purposes of calculating the partition function they indeed only amount to including or dropping the boundary values.
7.4 The Verlinde Formula for $SU(n)$

The only complication that arises for $SU(n)$, $n > 2$, is that we have to prescribe a fundamental domain for the action of the Weyl group on the torus $T = U(1)^{n-1}$. Alternatively, we are looking for a fundamental domain of the action on $t$ of the semi-direct product of the integral lattice (acting via translations) with the Weyl group (acting via reflections). The advantage of this reformulation is that one now recognizes this as a fundamental domain for the affine Weyl group (for simply connected groups the integral lattice and the coroot lattice coincide), which is known as a Weyl alcove or Stiefel chamber. In particular, given such an alcove $P$, we obtain a refinement of the covering conjugation map (5.1) to a universal covering \[ G/T \times P \to G_r, \]

\[(g, X) \to g^{-1} \exp(X) g.\]  

(7.15)

Hence this is an isomorphism if $G$ is simply connected and therefore $P$ is precisely the integration domain we require in the Weyl integral formula instead of $T$ if we want to mod out by the Weyl group explicitly.

For $SU(n)$ such a Weyl alcove is determined by $\alpha_l > 0$ (fixing a Weyl chamber) and the one additional condition $\sum \alpha_l < 2\pi$. As the fundamental weights are dual to the simple roots, this amounts to the following conditions on the integration range of $\phi$:

\[ P = \{ \phi_l : \phi_l > 0, \sum_{i=1}^{r-n-1} \phi_l < 2\pi \}. \]

(7.16)

Introducing this constraint into the path integral (7.10), one finds that only those weights ($r$-tuples of integers) contribute to the partition function which satisfy $n_l > 0$ and $\sum n_l < k + n$, i.e. the allowed values of $\phi$ are

\[ \phi_l = \frac{2\pi n_l}{k + n}, \quad n_l > 0, \quad \sum n_l < k + n. \]

(7.17)

Again these are in one-to-one correspondence with the integrable representations of the $SU(n)$ WZW model at level $k$ and, up to an overall normalization, the partition function agrees with the Verlinde formula given in the Introduction (with $\phi = \lambda + \rho$).

7.5 The Normalization of the Path Integral

In the course of the derivation of the partition function we have so far paid little attention to the various factors contributing to the overall normalization. And
while it is certainly possible in principle to keep track of these, it is also quite
cumbersome. One might hence be tempted to leave it at that, in particular when
one keeps in mind that the normalization is subject to a standard renormalization
ambiguity $Z \to v^{2-g} Z$ arising from the possibility to add terms $\sim \int R$ to the
action without violating any of the symmetries of the theory. However, the very
fact that we are calculating a dimension, which should at the very least be an
integer, forces us (and permits us) to go further than that. We will now sketch
how one could proceed.

The relation between the (naive) partition function $Z_{g,k}$ of Chern-Simons the-
ory at level $k$, (7.13), and the dimension $\dim V_{g,k}$, consistent with standard renor-
malization, is

$$\dim V_{g,k} = ab^{g-1} Z(g,k) ,$$

(7.18)

where we allow both $a$ and $b$ to depend on $k$. First of all, it follows from the fact
that the moduli space of flat connections on the two-sphere is a single point, that
canonical quantization of Chern-Simons theory on $S^2 \times \mathbb{R}$ will give rise to a one-
dimensional Hilbert space. Demanding $\dim V_{0,k} = 1$ and using $Z_{0,k} = (k+2)/2$
(as follows easily from (7.13)), one finds that $b$ is determined in terms of $a$ and
(7.18) reduces to

$$\dim V_{g,k} = a^g \frac{(k+2)^{g-1}}{2} Z_{g,k} .$$

(7.19)

If we permitted ourselves to use the fact that in genus one the dimension of the
Hilbert space is equal to the number of integrable representations, this would fix
$a = 1$ and lead to the correct result. However, this fact relies not only on the
knowledge that the Hilbert space of Chern-Simons theory is the space of conformal
blocks of the WZW model, but also on the knowledge of that space’s dimension.
And, in the spirit of this paper, we will continue without making recourse to the
connection between Chern-Simons theory and conformal field theory.

By demanding that $\dim V_{1,k}$ be an integer one learns that $a(k+1)$ has to be
an integer. And one might suspect that (7.19) cannot possibly be an integer for
all $k$ and $g$ unless $a$ itself is an integer. This is in fact correct and the remainder
of this section serves to establish just this (and to fix $a = 1$).

First of all, one can e.g. demand that in the large $k$ limit $\dim V_{1,k}$ approach
the volume of the moduli space of flat connections. In genus one this moduli
space is two-dimensional and hence the determination of its symplectic volume is
elementary. As the symplectic form (or first Chern class of the prequantum line
bundle) is assumed to be $k$ times the generator of the second cohomology, the
volume (divided by $k$) is just $\text{Vol}(\mathcal{M}_{g=1}) = 1$. This fixes $a(k+1)$ to be linear in
and leads to the somewhat improved expression

\[ \dim V_{g,k} = (\frac{k+c}{k+1})^g(\frac{k+2}{2})^{g-1}Z_{g,k}. \] (7.21)

This is as much as \( g = 0,1 \) tell us. Furthermore, \( k = 1 \) does not provide any additional information in any genus, as the dimension comes out to be \( (c + 1)^g \), which is an integer. However, by considering e.g. genus 2 and successively calculating \( \dim V_{2,k} \) from (7.21) for \( k = 2, 4, \ldots \), one finds that \( c \) has to be of the form \( c = 1 + 3m \), that \( m \) has to be of the form \( m = 5n, \ldots \), leaving finally as the only possibility \( c = 1 \). This leads to the correctly normalized Verlinde formula.

While this line of argument was not very elegant (and other ways of fixing the normalization are certainly also possible), its purpose was to illustrate that, in principle, the correct normalization can be determined by elementary methods. It remains a challenge to determine the regularization which automatically leads to the correct normalization.

### 7.6 Punctured Surfaces and the Fusion Rules

On the basis of what we have achieved so far it turns out to be surprisingly easy to derive the fusion rules, and more generally the dimension \( \dim V_{g,s,k} \) of the space of conformal blocks for a punctured Riemann surface, from Chern-Simons theory or the \( G/G \) model by explicit calculation. These formulae arise as the expressions for correlation functions of vertical Wilson loops (in Chern-Simons theory) or traces of \( g \) (the corresponding observables of the \( G/G \) model) directly in terms of the modular matrix \( S \), as in \[^1,3\] . We will again be content with illustrating this in the case of \( SU(2) \).

Let \( \chi_l(h) \) be the character (trace) of \( h \in SU(2) \) in the \((l + 1)\)-dimensional representation of \( SU(2) \). We will only consider integrable representations, \( l \leq k \). What we wish to prove is that the correlator of \( s \) such operators is given by the Verlinde formula for \( \dim V_{g,s,k} \),

\[ \dim V_{g,s,k}(l_1, \ldots, l_s) = \langle \chi_{l_1}(h) \cdots \chi_{l_s}(h) \rangle_g. \] (7.22)

As the characters are conjugation invariant, we need to know them only on the maximal torus, where they can be expressed as

\[ \chi_l(\phi) \equiv \chi_l(e^{i\phi}) = \frac{\sin(l+1)\phi/2}{\sin\phi/2}. \] (7.23)
Repeating the steps of sections 5 and 7.1-7.2 for this correlator instead of the partition function one arrives at (7.23) with an insertion of $s$ characters in the form (7.24). Then, just as in the case without insertions, evaluation of the delta function will lead to a sum over the discrete allowed values of $\phi$ and one finds

$$
\langle \prod_{i=1}^s \chi_{l_i}(\phi) \rangle_g = (\frac{k+2}{2})^{g-1} \sum_{j=0}^k \left( \sin \left( \frac{(j+1)\pi}{k+2} \right) \right)^{2g-s-2} \prod_{i=1}^s \sin \left( \frac{(j+1)(l_i+1)\pi}{k+2} \right). 
$$

(7.24)

It is now convenient to introduce the $(k+1) \times (k+1)$ matrix $S$,

$$
S_{ij} = (\frac{2}{k+2})^{1/2} \sin \left( \frac{(i+1)j\pi}{k+2} \right).
$$

(7.25)

This symmetric and orthogonal matrix is usually introduced as the modular matrix which implements the modular transformation $\tau \rightarrow -1/\tau$ on the genus 1 conformal blocks (Weyl-Kac characters),

$$
\chi_i^{wk}(-1/\tau) = \sum_j S_{ij} \chi_j^{wk}(\tau).
$$

(7.26)

We see that within our formalism it arises quite naturally as well. In terms of $S$ the above correlator is

$$
\langle \prod_{i=1}^s \chi_{l_i}(\phi) \rangle_g = \sum_{j=0}^k (S_{j0})^{2g-s} \prod_{i=1}^s S_{jl_i}.
$$

(7.27)

This is precisely the formula obtained in [1, 3] for the dimension of the space of conformal blocks on a surface of genus $g$ with punctures labelled by the representations $\{l_i\}$. In particular, for the two- and three-point functions on the sphere we obtain

$$
\langle \chi_l(\phi)\chi_m(\phi) \rangle_{g=0} = \delta_{lm},
$$

$$
\langle \chi_l(\phi)\chi_m(\phi)\chi_n(\phi) \rangle_{g=0} = \sum_{j=0}^k \frac{S_{jl}S_{jm}S_{jn}}{S_{j0}} \equiv N_{lmn}.
$$

(7.28)

It is interesting to note that in this derivation the fusion rules appear naturally in already diagonalized form. In particular, once the delta function constraint on $\phi$ has been imposed, the characters appear only in the form

$$
\chi_l^{(j)} = \frac{S_{lj}}{S_{0j}}.
$$

(7.29)

These ‘discrete characters’ are the eigenvalues of the fusion matrix $(N_l)_{mn} = N_{lmn}$ and are known to satisfy the fusion rules all on their own,

$$
\chi_l^{(j)}\chi_m^{(j)} = N_{lmn}\chi_n^{(j)}.
$$

(7.30)
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