Interval min-plus algebraic structure and matrices over interval min-plus algebra

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Abstract. Max-plus algebra is the set \( \mathbb{R}_{\max} \) or \( \mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\} \) where \( \mathbb{R} \) is the set of all real number and \( \varepsilon = -\infty \) which is equipped with maximum (\( \oplus \)) and plus (\( \odot \)) operations. The structure of max-plus algebra is semifield. Another semifield that can be learned is min-plus algebra. Min-plus algebra is the set \( \mathbb{R}_{\min} \) or \( \mathbb{R}_{\varepsilon'} = \mathbb{R} \cup \{\varepsilon'\} \) where \( \varepsilon' = \infty \) which is equipped with minimum (\( \ominus \)) and plus (\( \odot \)) operations. Max-plus algebra has been generalized into interval max-plus algebra, so that min-plus algebra can be developed into an interval min-plus algebra. Interval min-plus algebra is defined as a set \( I(\mathbb{R})_\varepsilon = \{x = [\underline{x}, \overline{x}] | \underline{x}, \overline{x} \in \mathbb{R}, \underline{x} \leq x < \overline{x} < \varepsilon\} \) which have minimum (\( \ominus \)) and addition (\( \odot \)) operations. A matrix in which its components are the element of \( \mathbb{R}_\varepsilon \) is called matrix over max-plus algebra. Matrices over max-plus algebra has been generalized into interval matrices in which its components are the element of \( I(\mathbb{R})_\varepsilon \). This research will discusses the interval min-plus algebraic structure and matrices over interval min-plus algebra.

1. Introduction

Let \( \mathbb{R} \) is a set of all real numbers. Max-plus algebra is the set \( \mathbb{R}_{\max} = \mathbb{R} \cup \{\varepsilon\} \) with \( \varepsilon = -\infty \) which is equipped with maximum (\( \oplus \)) and plus (\( \odot \)) operations (Tam[7]). Then max-plus algebra will be denoted by \( \mathbb{R}_{\max} \). The structure of max-plus algebra is an idempotent commutative semiring (Subiono[6]). Furthermore, \( (\mathbb{R}_{\max}, \oplus, \odot) \) is an idempotent semifield (Baccelli et al.[1]). Let \( A \) be a matrix with its components being all real numbers \( \mathbb{R} \) with \( m \) row and \( n \) column. A matrix in which its components are the element of \( \mathbb{R}_\varepsilon \) is called matrix over max-plus algebra. Matrices over max-plus algebra has been generalized into interval matrices in which its components are the element of \( I(\mathbb{R})_\varepsilon \).

In 2008, Rudhito et al.[4] has discussed about interval max-plus algebra. Interval max-plus algebra denoted by \( I(\mathbb{R})_{\max} \) is the set of interval over \( \mathbb{R}_{\max} \) which can be defined as the set \( I(\mathbb{R})_{\max} = \{x = [\underline{x}, \overline{x}] | \underline{x}, \overline{x} \in \mathbb{R}, \varepsilon < x < \overline{x} \} \) which is equipped with two operations i.e. maximum (\( \oplus \)) and plus (\( \odot \)). Then Rudhito et al.[5] has expanded the operators \( \oplus \) and \( \odot \) into the set of matrices. Matrices over interval max-plus algebra has components which are elements of \( I(\mathbb{R})_\varepsilon \), denoted by \( I(\mathbb{R})_{\max}^{\max} \). In 2014, Nowak[3] has discussed about min-plus algebra. Min-plus algebra is the set or \( \mathbb{R}_{\varepsilon'} = \mathbb{R} \cup \{\varepsilon'\} \) with \( \varepsilon' = \infty \) which is equipped with minimum (\( \ominus \)) and plus (\( \odot \)) operations. Then min-plus algebra...
will be denoted by \( \mathbb{R}_{\text{min}} \). If the components of a matrix are the elements of \( \mathbb{R}_{\text{min}} \) is called matrix over min-plus algebra. According to the research by Rudhito et al. about interval max-plus algebra and min-plus algebra concept, this research will discussed about interval min-plus algebraic structure and matrices over interval min-plus algebra.

First, we introduce some preliminaries about min-plus algebra structure, matrices over min-plus algebra, and general explanation of intervals in a semiring.

2. Preliminaries

**Definition 2.1.** ([7]) Let \( \mathbb{R} \) be a set of all real number and \( \epsilon' = \infty \), min-plus algebra is a set of \( \mathbb{R}_{\epsilon'} \) i.e. \( \mathbb{R}_{\epsilon'} = \mathbb{R} \cup \{ \epsilon' \} \) which is equipped with minimum (\( \oplus' \)) and plus (\( \otimes \)) operations. Both operations can be defined that for all \( a, b, c \in \mathbb{R}_{\epsilon'} \) then \( a \oplus' b = \min(a, b) \) and \( a \otimes c = a + c \).

**Theorem 2.2.** ([3]) \( (\mathbb{R}_{\epsilon'}, \oplus', \otimes) \) is an idempotent commutative semiring.

Furthermore, idempotent commutative semiring \( (\mathbb{R}_{\epsilon'}, \oplus', \otimes) \) has inverse element to the \( \otimes \) operation, because for every \( x, y \in \mathbb{R}_{\epsilon'} \) there is \( -x \) as inverse element so that \( x \otimes (-x) = 0 = (-x) \otimes x \) and 0 is identity element of \( \mathbb{R}_{\epsilon'} \). Therefore, \( (\mathbb{R}_{\epsilon'}, \oplus', \otimes) \) is a semifield. In order to be more concise, writing idempotent commutative semiring \( (\mathbb{R}_{\epsilon'}, \oplus', \otimes) \) is written as \( \mathbb{R}_{\text{min}} \).

**Definition 2.3.** ([4]) Matrices over min-plus algebra is denoted by \( \mathbb{R}_{\text{min}}^{m \times n} \).

\[
\mathbb{R}_{\text{min}}^{m \times n} = \{ M = [M_{ij}] | M_{ij} \in \mathbb{R}_{\text{min}} \}
\]

The entry of matrix \( M \) in the \( i^{th} \) row and \( j^{th} \) column is denoted by \( M_{ij} \). Using the definition of the operators in min-plus algebra then for all \( A, B \in \mathbb{R}_{\text{min}}^{m \times n} \), \( C \in \mathbb{R}_{\text{min}}^{n \times l} \), \( D \in \mathbb{R}_{\text{min}}^{l \times n} \) and \( \kappa \in \mathbb{R}_{\text{min}} \) can be define

\[
[A \oplus' B]_{ij} = A_{ij} \oplus' B_{ij} \\
[C \otimes D]_{ij} = \bigoplus_{k=1}^{m} (C_{ik} \otimes D_{kj}) \\
[\kappa \otimes A]_{ij} = \kappa \otimes A_{ij}
\]

It has been known before that the zero element in \( \mathbb{R}_{\text{min}} \) is \( \epsilon' = \infty \) and its identity element is 0. The identity matrix for \( \oplus' \) is \( E \) with \( (\forall i, j) E_{ij} = \epsilon' \) and for \( \otimes \) is \( E = \begin{cases} 0, & i = j \\ \epsilon', & i \neq j \end{cases} \) such that based on the neutrality, if \( A \in \mathbb{R}_{\text{min}}^{m \times n} \) then \( A \oplus' E = A = E \oplus' A \) and \( A \otimes E = E = E \otimes A \) According to Rudhito, et al.[5] \( (\mathbb{R}_{\text{min}}^{m \times n}, \oplus', \otimes) \) is an idempotent commutative semigroup and \( (\mathbb{R}_{\text{min}}^{m \times n}, \oplus', \otimes) \) is an idempotent semiring.

**Definition 2.4.** ([3]) Let \( (\mathcal{S}, +, \times) \) is an idempotent semiring and does not contain a zero divisor, with a zero element 0, then interval in a semiring is defined as

\[
I(\mathcal{S}) = \{ x = [\underline{x}, \overline{x}] | \underline{x}, \overline{x} \in \mathcal{S}, 0 < \underline{x} \leq \overline{x} \} \cup \{[0, 0]\}.
\]

Interval \( x \) is called pure interval.

\( (I(\mathcal{S}), \overline{+}, \overline{\times}) \) is an idempotent semiring which has binary operations \( \overline{+} \) and \( \overline{\times} \), such that for every \( x, y \in I(\mathcal{S}) \) then

(1) \( x \overline{+} y = [\underline{x} + \underline{y}, \overline{x} + \overline{y}] \) and

(2) \( x \overline{\times} y = [\underline{x} \times \underline{y}, \overline{x} \times \overline{y}] \).

**Theorem 2.5.** ([4]) \( (I(\mathcal{S}), \overline{+}, \overline{\times}) \) is an idempotent semiring with identity element.
Definition 2.6. ([4]) Let $(\mathcal{S}, +, \times)$ is an idempotent semiring and has no zero divisor, with neutral element 0. Defined that

$$ I(\mathcal{S})^* = I(\mathcal{S}) \cup \{ x = [\overline{x}, \overline{\sigma}] | x, \overline{x} \in \mathcal{S}, 0 < x < \overline{x} \} \text{ Interval } x^* \in I(\mathcal{S})^* \text{ is called non-pure interval.} $$

Theorem 2.7. ([4]) If $(I(\mathcal{S}), +, \overline{\cdot})$ is a semifield, then $(I(\mathcal{S})^*, +, \overline{\cdot})$ is a semifield which has the binary operations that similar to the binary operations that applied in $I(\mathcal{S})$.

3. Main Results

This section will discuss the results of this research about interval min-plus algebraic structure and matrices over interval min-plus algebra.

3.1 Interval Min-Plus Algebraic Structure

It is known that min-plus algebra $\mathbb{R}_{\min}$ is an idempotent commutative semiring and does not contain a zero divisor, with a zero element $\varepsilon' = \infty$, therefore interval min-plus algebra defined in the Definition 3.1.

Definition 3.1. ([7]) Interval min-plus algebra is a set of $I(\mathbb{R})_{\min}$ or $I(\mathbb{R})_{\min}$, and defined as

$$ I(\mathbb{R})_{+} = \{ x = [\overline{x}, \overline{\sigma}] | x, \overline{x} \in \mathbb{R}, \overline{x} \leq x < \overline{\sigma} \} \cup \{ [\varepsilon', \varepsilon'] \} \text{ Interval min-plus algebra.} $$

which is equipped with $\overline{\cdot}$ and $\otimes$ operations, that for all $x, y \in I(\mathbb{R})_{\min}$ then

(i) $x \oplus y = [\overline{x} \bigoplus \overline{y}, \overline{x} \bigoplus \overline{y}]$

(ii) $x \otimes y = [\overline{x} \bigotimes \overline{y}, \overline{x} \bigotimes \overline{y}]$.

For example, given $[2,3], [-2,5] \in I(\mathbb{R})_{\min}$ then $[2,3] \bigoplus [-2,5] = [-2,3]$ and $[2,3] \bigotimes [-2,5] = [0,8]$. Based on operations that apply to $I(\mathbb{R})_{\min}$ and $I(\mathcal{S})$ then Theorem 2.2 is obtained.

Theorem 2.2. $(I(\mathbb{R})_{\min}, \bigoplus, \bigotimes)$ is a semiring with identity element.

Proof. Let $x, y, z \in I(\mathbb{R})_{\min}$ with $x = [\overline{x}, \overline{\sigma}], y = [\overline{y}, \overline{v}]$ and $z = [\overline{z}, \overline{w}]$. According to Definition 3.1, it will be proven first that $(I(\mathbb{R})_{\min}, \bigoplus)$ is a semigroup with zero element $\varepsilon' = [\varepsilon', \varepsilon']$.

(i) Associative

$$ x \bigoplus (y \bigoplus z) = [x \bigoplus (y \bigoplus z), x \bigoplus (y \bigoplus z)] $$

$$ = \left( (x \bigoplus y) \bigoplus (\varepsilon', \varepsilon') \right) \bigoplus \left( (\varepsilon', \varepsilon') \bigoplus (\varepsilon', \varepsilon') \right) $$

$$ = (x \bigoplus y) \bigoplus z $$

(ii) There is zero element $\varepsilon'$

$$ x \bigoplus \varepsilon' = [x \bigoplus \varepsilon', x \bigoplus \varepsilon'] = [x, x] \text{ and } $$

$$ \varepsilon' \bigoplus \overline{\sigma} x = [\varepsilon' \bigoplus \overline{\sigma} x, \varepsilon' \bigoplus \overline{\sigma} x] = [x, x] . $$

$(I(\mathbb{R})_{\min}, \bigotimes)$ is also proved that it is a semigroup with unit element $\overline{0} = [0,0]$.

(i) Associative

$$ x \bigotimes (y \bigotimes z) = [y \bigotimes z, y \bigotimes z] $$

$$ = \left( x \bigotimes y \bigotimes z, x \bigotimes y \bigotimes z \right) $$

$$ = \left( x \bigotimes y \bigotimes z \right) \bigotimes \left( x \bigotimes y \bigotimes z \right) $$

$$ = (x \bigotimes y) \bigotimes z $$

(ii) There is zero element $\overline{0}$

$$ x \bigotimes \overline{0} = [x \bigotimes 0, x \bigotimes 0] = [x, x] $$

and
\[ \overline{0} \otimes x = [0 \otimes x, 0 \otimes x] = [x, x]. \]

Semiring \((I(\mathbb{R})_{\min}, \otimes, \odot)\) has distributive properties.

(i) \[ x \odot (y \oplus z) = [x \otimes (y \oplus z), x \otimes (y \oplus z)] = \left( [x \otimes y] \oplus (x \otimes z), (x \otimes y) \oplus (x \otimes z) \right) = (x \otimes y) \oplus (x \otimes z) \text{ and} \]

(ii) \[ (x \oplus y) \odot z = \left( [x \oplus y] \otimes z, (x \oplus y) \otimes z \right) = \left( [x \otimes z] \oplus (y \otimes z), (x \otimes z) \oplus (y \otimes z) \right) = (x \otimes z) \oplus (y \otimes z). \]

Furthermore, to be more concise, semiring \((I(\mathbb{R})_{\min}, \oplus, \otimes)\) is written as \(I(\mathbb{R})_{\min}\).

**Theorem 3.3.** \(I(\mathbb{R})_{\min}\) is an idempotent commutative semiring.

**Proof.** Let \(x, y, z \in I(\mathbb{R})_{\min}\) with \(x = [x, x]\) and \(y = [y, y]\), then

(i) commutative in \(\oplus\)
\[ x \oplus y = [x \oplus y, x \oplus y] = [y \oplus y, y \oplus y] = y \oplus x \]

(ii) commutative in \(\otimes\)
\[ x \otimes y = [x \otimes y, x \otimes y] = [y \otimes y, y \otimes y] = y \otimes x \]

(iii) idempotent
\[ x \odot x = [x \otimes x, x \otimes x] = [x, x] = x \]

Idempotent commutative semiring \(I(\mathbb{R})_{\min}\) is not a semifield because not every nonzero element has an inverse. Let \(x \in I(\mathbb{R})_{\min}\) with \(x = [x, x]\) then \(-x = [-x, -x]\). If \(x\) and \(-x\) is operated using \(\otimes\) then it does not produce unit elements \(\overline{0} = [0, 0]\) because \(\overline{0} \otimes (-x) = [\overline{x} \otimes (-x), \overline{x} \otimes (-x)]\), but produces an interval of length \(2x - 2x\). In the introduction it is known that \(\mathbb{R}_{\min}\) is a semifield. \(I(\mathbb{R})_{\min}\) can be semifield if the inverse element set is the opposite of the element on \(I(\mathbb{R})_{\min}\). Suppose \(x = [x, x]\) and \(x^{-1} = opp(x) = [-x, -x]\) assuming that \(0 < x < \overline{x}\) then it produces the zero element, \(\overline{0} \otimes \overline{x} = [\overline{x} \otimes \overline{x}, \overline{x} \otimes \overline{x}] = [0, 0]\). Therefore according to Definition 1.6 then \(I(\mathbb{R})_{\min}\) can be defined in Definition 3.4.

**Definition 3.4.** Interval min-plus algebra \(I(\mathbb{R})_{\min}\) can be generalized to \(I(\mathbb{R})_{\min}^1\).
\[ I(\mathbb{R})_{\min}^1 = I(\mathbb{R})_{\min} \cup \{ x = [x, x] | x, x \in \mathbb{R}, x < \overline{x} < \varepsilon \} \]

Let \(p, q \in I(\mathbb{R})_{\min}^1\) if there is an interval equation \(p \otimes x = q\) then its solution is \(x = q \otimes opp(p)\), because \(p \otimes opp(p) = [0, 0]\). For example, if there is an interval equation
\[ [3, 4] \otimes [x, x] = [4, 6] \]

Equation (1) has interval solution \(x = [4, 6] \otimes opp[3, 4] = [4, 6] \otimes [-3, -4] = [1, 2]\). Can be seen that the solution \([1, 2]\) satisfies interval in equation (1), that \([3, 4] \otimes [1, 2] = [4, 6]\). Because \(x \in I(\mathbb{R})_{\min}\) is non-pure interval and based on Theorem 2.7 then Theorem 3.5 is obtained.
**Theorem 2.5.** \((I(\mathbb{R})^{\min}_{m,n}, \oplus, \otimes)\) is a semifield.

**Proof.** \((I(\mathbb{R})^{\min}_{m,n}, \oplus, \otimes)\) is known to be a semifield, then according to Theorem 1.7 \((I(\mathbb{R})^{\min}_{m,n}, \oplus^T, \otimes)\) is a semifield. Let \(x \in I(\mathbb{R})^{\min}_{m,n}\) and \(x = [x, \overline{x}] \neq [0,0]\), there is \(\in I(\mathbb{R})^{\min}_{m,n}\) with \(x^{-1} = [-\overline{x}, -x]\) and \(x \otimes (-x) = [0,0]\). ■

3.2 Matrices over Interval Min-Plus Algebra

Based on the definition of matrices over min-plus algebra, the interval min-plus algebra can be expanded into a matrix set. We can define the matrix over interval min-plus algebra in the Definition 3.6.

**Definition 3.6.** Matrices over interval min-plus algebra is defined as the set

\[ I(\mathbb{R})^{m \times n}_{\min} = \{ A = [A_{ij}] | A_{ij} \in I(\mathbb{R})^{\min}, i = 1,2,...,n, j = 1,2,...,n \} \]

A matrix in which its component is the element of \(I(\mathbb{R})^{m \times n}_{\min}\) is called an interval min-plus matrix or we can call only with an interval matrix. Moreover, the operator \(\oplus^T\) and \(\otimes\) also applies to the set of matrices over interval min-plus algebra that we can define in the Definition 3.7 and Definition 3.8.

**Definition 3.7.** Matrices over interval min-plus algebra have binary operations \(\oplus^T\) and \(\otimes\) is defined that for all \(A,B \in I(\mathbb{R})^{m \times n}_{\min}\) and \(\kappa \in I(\mathbb{R})^{\min}\) then

\[ [A \oplus^T B]_{ij} = A_{ij} \oplus^T B_{ij} \text{ and} \]
\[ [\kappa \otimes A]_{ij} = \kappa \otimes A_{ij}, \]

for \(i = 1,2,...,n \) and \(j = 1,2,...,n\).

**Definition 3.8.** For all \(A \in I(\mathbb{R})^{m \times l}_{\min}\) and \(B \in I(\mathbb{R})^{l \times n}_{\min}\) defined that

\[ [A \otimes B]_{ij} = \oplus^T_{k=1} A_{ik} \otimes B_{kj}, \]

for \(i = 1,2,...,n \) and \(j = 1,2,...,n\).

The interval matrix \(A,B \in I(\mathbb{R})^{m \times n}_{\min}\) is called the same matrix if \(\overline{A}_{ij} = B_{ij}\) and \(\overline{A}_{ij} = \overline{B}_{ij}\) for all \(i \) and \(j\). From the definition of binary operations of \(I(\mathbb{R})^{m \times n}_{\min}\) in Definition 2.7 and 2.8, so we obtained Theorem 3.9 and Theorem 3.10.

**Theorem 3.9.** \((I(\mathbb{R})^{m \times n}_{\min}, \oplus^T)\) is an idempotent commutative semigroup.

**Proof.** Based on the definition of the operator \(\oplus^T\) on the interval matrix, then it is easy to prove that the operator \(\oplus^T\) for each element in \(I(\mathbb{R})^{m \times n}_{\min}\) is closed, associative, commutative and idempotent. Then there is the zero element \(\overline{E} \in I(\mathbb{R})^{m \times n}_{\min}\) so for each \(A \in I(\mathbb{R})^{m \times n}_{\min}\)

\[ [A \oplus^T \overline{E}]_{ij} = A = [\overline{E} \oplus^T A]_{ij} \]

where \(\overline{E}_{ij} = [\overline{e}', \overline{e}']\) for all \(i\) and \(j\). Therefore it can be deduced that \((I(\mathbb{R})^{m \times n}_{\min}, \oplus^T)\) is proven to be an idempotent commutative semigroup. ■

**Theorem 3.10.** \((I(\mathbb{R})^{m \times n}_{\min}, \oplus^{T}, \otimes)\) is an idempotent semiring with identity element.

**Proof.** It has been proven in Theorem 2.9 that \((I(\mathbb{R})^{m \times n}_{\min}, \oplus^T)\) is an idempotent commutative semigroup with \(\oplus^T\) as the binary operation. Therefore, for the square matrix of interval matrix also applies. By understanding the definition of the operator \(\otimes\) on the interval matrix, it is easy to prove that the \(\otimes\)
operator is closed, associative and contains an identity element $\bar{E} \in I(\mathbb{R})_{\min}^{n \times n}$ where $E_{ij} = \{0, 0\}$, $i = j$, $i \neq j$, so for each $A \in I(\mathbb{R})_{\min}^{n \times n}$ then

$$[A \overline{\otimes} E]_{ij} = A = [E \overline{\otimes} A]_{ij}$$

So it can be deduced that $(I(\mathbb{R})_{\min}^{n \times n})^{\otimes}$ is proven to be an idempotent semiring with identity element.

An interval matrix can be formed into matrix interval, which the definition of matrix interval we can see in the Definition 3.11.

**Definition 3.11.** Matrix interval of $A \in I(\mathbb{R})_{\min}^{m \times n}$ is defined as

$$[A, \bar{A}] = \{A, \bar{A} \in \mathbb{R}_{\min}^{m \times n} | A \leq \bar{A} \}$$

such that the interval matrix $A$ can be represented with the interval matrix $[A, \bar{A}]$, which is symbolized $A \approx [A, \bar{A}]$. As well as defined the set of interval matrix

$$I(\mathbb{R})_{\min}^{m \times n}_b = \{A = [A, \bar{A}] | A \in I(\mathbb{R})_{\min}^{m \times n} \}$$

Based on the Definition 3.11 that interval matrix $A \approx [A, \bar{A}]$, so the operators $\overline{\oplus}$ and $\overline{\otimes}$ can be defined in Definition 3.12 as the binary operation of matrix interval over min-plus algebra.

**Definition 3.12.** If $A, B \in I(\mathbb{R})_{\min}^{m \times n}_b$ and $\kappa \in I(\mathbb{R})_{\min}$ then

$$\kappa \overline{\otimes} A = \kappa \overline{\otimes} A, \kappa \overline{\otimes} \bar{A}$$

and $A \overline{\oplus} B = [A \oplus B, \bar{A} \oplus \bar{B}]$.

If $C \in I(\mathbb{R})_{\min}^{m \times n}_b$ and $D \in I(\mathbb{R})_{\min}^{m \times n}_b$ then

$$C \overline{\otimes} D = \{C \otimes D, C \otimes \bar{D} \}$$

The matrix interval $[A, \bar{A}], [B, \bar{B}] \in I(\mathbb{R})_{\min}^{m \times n}_b$ is called the same matrix if $A = B$ and $\bar{A} = \bar{B}$.

**Theorem 3.13.** $(I(\mathbb{R})_{\min}^{m \times n}_b, \overline{\oplus})$ is an idempotent commutative semigroup.

**Proof.** Based on the definition of the operator $\overline{\oplus}$ on the matrix interval, then it is easy to prove that the operator $\overline{\oplus}$ for each element in $I(\mathbb{R})_{\min}^{m \times n}_b$ is closed, associative, commutative and idempotent. Then there is the zero element $\bar{E} \in I(\mathbb{R})_{\min}^{m \times n}_b$ with $\bar{E} = [E, \bar{E}]$ so for each $A \in I(\mathbb{R})_{\min}^{m \times n}_b$ then

$$[A \overline{\oplus} E]_{ij} = [A, \bar{A}] \overline{\oplus} \bar{E}$$

$$= A$$

$$= [E, \bar{E}] \overline{\oplus} [A, \bar{A}]$$

$$= [E \overline{\oplus} \bar{A}]_{ij}$$

where $E_{ij} = e'$ for all $i$ and $j$. Therefore it can be deduced that $(I(\mathbb{R})_{\min}^{m \times n}_b, \overline{\oplus})$ is proven to be an idempotent commutative semigroup.

**Theorem 3.14.** $(I(\mathbb{R})_{\min}^{m \times n}_b, \overline{\oplus}, \overline{\otimes})$ is an idempotent semiring with identity element.

**Proof.** It has been proven in Theorem 2.13 that $I(\mathbb{R})_{\min}^{m \times n}_b$ is an idempotent commutative semigroup with $\overline{\oplus}$ as the binary operation. Therefore, for the square matrix of interval matrix also applies. By understanding the definition of the operator $\overline{\otimes}$ on the interval matrix, it is easy to prove that the $\overline{\otimes}$ operator is closed, associative and contains an identity element $\bar{E} \in I(\mathbb{R})_{\min}^{m \times n}_b$ with $\bar{E} = [E, \bar{E}]$ where $E_{ij} = \begin{cases} 0, & i = j \\ e', & i \neq j \end{cases}$. So for each $A \in I(\mathbb{R})_{\min}^{m \times n}_b$ we have
\[ [A \otimes E]_{ij} = [A, A] \otimes [E, E] = A = [E, E] \otimes [A, A] = [E \otimes A]_{ij} \]

So, it can be deduced that \((I(\mathbb{R}^{m \times n}_{\min})_B, \oplus, \otimes)\) is proven to be an idempotent semiring with identity element.

4. Conclusions

The structure of interval min-plus algebra to the min and plus operations is semifield and the structure of matrices over interval min-plus algebra to the min and plus operation is idempotent semiring with identity element.

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