Global Existence for Compressible Euler Equations with Damping in Partial Space-Period Domains

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The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract
In this paper, we are concerned with the global existence of solutions to isentropic compressible Euler equations with damping in partial space-period domains. Based on the uniform energy estimates, we obtain the global existence for any spatial dimension if the initial data is sufficiently close to an equilibrium. Simultaneously, we show that the vorticity and its derivatives decay exponentially to zero in two and three dimensions.

Keywords: Compressible Euler equations; damping; partial space-period domains; energy estimates; global existence.

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1 Introduction

In this paper, we consider the isentropic compressible Euler equations with damping for \((t, x) \in [0, +\infty) \times \Omega:\)
\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u + p I_d) &= -\rho a, \\
\end{align*}
\] where \(\Omega = \mathbb{T}^n \times \mathbb{R}^m, \mathbb{T}^n = [0, 2\pi]^n\) is the \(n\)-dimensional torus, \(m + n = d, n \geq 1, d \geq 2\), \(\rho\) is the density, \(u = (u_1, u_2, \ldots, u_d)\) is the velocity, \(p = p(\rho)\) is the pressure with state equation \(p(\rho) = \gamma \rho^n\) with constants \(A > 0\) and \(\gamma > 1\). From now on, for convenience, we assume that \(A = \frac{\gamma}{\gamma - 1}\).

The initial data of the system (1.1) are given as follows
\[
(\rho(0, x), u(0, x)) = (1 + \varepsilon \rho_0(x), \varepsilon u_0(x)), \quad x \in \Omega,
\] where \(1 + \varepsilon \rho_0(x) > 0\) for all \(x \in \Omega, \varepsilon > 0\) is sufficiently small. There are many literatures on the global existence of the Cauchy problem (1.1)-(1.2). In one-dimensional case, the global existence was proved by T. Nishida in [1] and the long-time behavior of \((\rho, u)\) was established in [2]-[4]. For more results about the \(L^\infty\) entropy weak solution and the \(BV\) solution, one can see [5], [6] and the references therein. When it comes to the general multi-dimensional case, the global existence and long-time behavior of the solution were established in [7]-[11]. In particular, Lu used the semigroup method and proved the exponential stability of constant steady state on torus in [12]. Fang and Xu obtained the global \(C^1\) solution in the framework of Besov space in [13]. It should be pointed out that the smallness of the parameter \(\varepsilon\) plays a key role in the above results. For instance, the authors in [8] proved that the \(C^1\) solutions will blow up for large amplitude initial data in three dimension. For the non-isentropic compressible Euler equations with constant damping, the global solutions were also obtained, we refer it to [14], [15] and the references therein. Moreover, the work on the general hyperbolic systems including (1.1) was studied in [16], [17], [18] and so on.

In the present paper, we will consider the problem in partial space-period domains. Motivated by the energy method in [8], we establish the uniform energy estimates and obtain the global existence for any spatial dimension. In addition, we can also obtain that the vorticity and its derivatives decay exponentially to zero for \(d = 2, 3\).

For convenience, we shall use the following convention throughout this paper:

- \(f \lesssim g\) means there exists a generic constant \(C\) such that \(f \leq CG\);
- The differential operator \(\partial\) denotes the time-space derivatives, i.e. \(\partial = (\partial_t, \partial_{x_1}, \ldots, \partial_{x_d})\);
- The Latin letters \(a, b, c\) denote multiple indices, for example, \(a = (a_0, a_1, \ldots, a_d), |a| = a_0 + a_1 + \ldots + a_d\) in its length;
- For two multiple indices \(a, b\), \(b \leq a\) means \(b_i \leq a_i\) for all \(i = 0, 1, \ldots, d\);
- Define the operator \(\partial^a = \partial_1^{a_1} \partial_2^{a_2} \cdots \partial_d^{a_d}\) and \(\partial^{\leq a} = \sum_{0 \leq b \leq a} \partial^b\), where \(a_i\) is some non-negative integer for all \(i = 0, 1, \ldots, d\);
- \(\|f(t, \cdot)\|_2\) stands for \(\|f(t, x, y)\|_{L^2(\mathbb{T}^n) L^2_2(\mathbb{R}^m)}\) and \(\|f(t, \cdot)\|_\infty\) stands for \(\|f(t, x, y)\|_{L^\infty(\mathbb{T}^n) L^\infty_2(\mathbb{R}^m)}\).

At first, we reformulate the Cauchy problem (1.1)-(1.2). Let
\[
\sigma(\rho) = \frac{2}{\gamma - 1} (c(\rho) - 1),
\]
where \( c(\rho) = \sqrt{p'(\rho) + \frac{\rho}{\rho + 1}} \) is the sound speed. Then the problem (1.1)-(1.2) can be rewritten as the following symmetric form

\[
\begin{align*}
\partial_t \sigma + u \cdot \nabla \sigma + \text{div} u + \frac{2}{\rho+1} \sigma \text{div} u &= 0, \\
\partial_t u + u \cdot \nabla u + u + \nabla \sigma + \frac{2}{\rho+1} \sigma \nabla \sigma &= 0, \\
\sigma(0, x) &= \frac{2}{\rho+1} \left[ (1 + \varepsilon \rho_0(x)) \frac{2}{\rho+1} - 1 \right], \\
u(0, x) &= \varepsilon u_0(x).
\end{align*}
\]

By the standard method in [19], the problem (1.4) has a local solution \((\sigma, u) \in C^4([0, t] \times \mathbb{R}^d)\) for some \( t > 0 \). To obtain the global existence, the main task is to derive the uniform prior estimates by making full use of the damping term. To this end, for any \( k \in \mathbb{N}^+ \), we define the energy functionals

\[
E[f](t) = \|f(t, \cdot)\|_2, \quad E_k[f](t) = \sum_{1 \leq |m| \leq k} E[\partial^m f](t)
\]

and the dissipation functionals

\[
D[f](t) = \left( \int_0^t \int_{\mathbb{T}^n} f^2(\tau, x, y) dxdy \right)^{\frac{1}{2}}, \quad D_k[f](t) = \sum_{1 \leq |m| \leq k} D[\partial^m f](t).
\]

Denote

\[
E[f_1, f_2](t) = E[f_1](t) + E[f_2](t), \quad E_k[f_1, f_2](t) = E_k[f_1](t) + E_k[f_2](t)
\]

and the similar definitions for \( D[f_1, f_2](t), D_k[f_1, f_2](t) \).

Our main results are

**Theorem 1.1.** Assume that \((\rho_0(x), u_0(x)) \in H^{2N+1}(\mathbb{R}^d)\), where \( N = \left[ \frac{d}{2} \right] + \left[ \frac{d-2}{2} \right] + 2, 1 + \varepsilon \rho_0(x) > 0 \) for all \( x \in \Omega \), \( \varepsilon > 0 \) is sufficiently small. Then there exists a global solution \((\rho, u) \in C([0, +\infty), H^{2N+1}(\mathbb{T}^n \times \mathbb{R}^m))\) for the problem (1.1)-(1.2) and the solution satisfies

\[
\|(\rho - 1)(t, \cdot)\|_{H^{2N+1}(\mathbb{R}^d)} + \|u(t, \cdot)\|_{H^{2N+1}(\mathbb{R}^d)} \lesssim \varepsilon, \quad \forall \ t > 0.
\]

**Theorem 1.2.** Under the conditions in theorem 1.1, for any \( C^1 \) solution \((\rho, u)\), we have that the vorticity \( \omega = \nabla \times u \) and its derivates decay exponentially to zero when \( d = 2, 3 \).

**Remark 1.1.** The condition that \( 1 + \varepsilon \rho_0(x) > 0 \) for all \( x \in \Omega \) is very important to exclude the appearance of vacuum for the classical solutions to (1.1)-(1.2). In fact, for any solution \((\rho, u) \in C^1([0, t] \times \Omega)\) to (1.1)-(1.2), if we set

\[
\frac{dx}{dt} = u(t, x), \quad x(0) = x_0,
\]

then by the method of characteristic and the first equation of (1.1), we can get

\[
\rho(t, x(t)) = (1 + \varepsilon \rho_0(x_0)) \exp \left( - \int_0^t \text{div} u(s, x(s)) ds \right) > 0.
\]

**Remark 1.2.** The unknown function transformation (1.3) is valid without vacuum. In this sense, for any \( t > 0 \), \((\rho, u) \in C^1([0, t] \times \Omega)\) solves the problem (1.1)-(1.2) with \( \rho > 0 \), then \((\sigma, u) \in C^1([0, t] \times \Omega)\) solves the problem (1.4) with \( \frac{2}{\rho+1} \sigma + 1 > 0 \); Conversely, if \((\sigma, u) \in C^1([0, t] \times \Omega)\) solves the problem (1.4) with \( \frac{2}{\rho+1} \sigma + 1 > 0 \), let \( \rho = c^{-1}(\frac{2}{\rho+1} \sigma + 1) \), then \((\rho, u) \in C^1([0, t] \times \Omega)\) solves the problem (1.1)-(1.2) with \( \rho > 0 \).

**Remark 1.3.** Our result has no restriction on the spatial dimensions. In addition, based on the method in this paper, the conclusion can be extended to the compressible Euler equations with constant damping in the half space \( \mathbb{R}^d_+ \).
2 Proof of Theorem 1.1

Throughout this section, we will always assume that

\[ \sup_{0 \leq \tau \leq t} E[\sigma, u](\tau) + \sup_{0 \leq \tau \leq t} E_{2N}[\sigma, u](\tau) \lesssim \varepsilon, \quad (2.1) \]

where \( N = \lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor + 2 \). First, we introduce a Sobolev embedding theorem, one can see [20] for details.

**Lemma 2.1.** Assume that \( f(t, x, y) \in H^{\tilde{N}}(\Omega), (x, y) \in \mathbb{T}^n \times \mathbb{R}^m \), then we have

\[ \|f(t, \cdot)\|_{\infty} \lesssim \|\partial^\delta_x \sigma(\cdot)\|_{\infty} + \sup_{0 \leq \tau \leq t} \|\partial^\delta u(t, \cdot)\|_{\infty} \lesssim \varepsilon, \quad 0 \leq |a| \leq N. \]

(2.2)

Thus it follows from lemma 2.1 and the assumption (2.1) that

\[ \sup_{0 \leq \tau \leq t} \|\partial^\alpha \sigma(\tau, \cdot)\|_{\infty} + \sup_{0 \leq \tau \leq t} \|\partial^\alpha u(\tau, \cdot)\|_{\infty} \lesssim \varepsilon, \quad 0 \leq |a| \leq N. \]

(2.3)

In the following, to prove theorem 1.1, we split the process into several lemmas.

**Lemma 2.2.** Under the assumption (2.1), for all \( t > 0 \), we have the following connection between the dissipation functionals \( D_k[\sigma](t) \) and \( D_k[u](t) \) that

\[ D_k[\sigma](t) \lesssim D_k[u](t) + D[u](t), \quad 1 \leq k \leq 2N + 1. \]

(2.4)

**Proof.** Rewrite the equations in (1.4) as follows

\[ \begin{cases}
\partial_t \sigma = -\text{div} u - u \cdot \nabla \sigma - \frac{a+1}{2} \sigma \text{div} u, \\
\nabla \sigma = -\partial_t u - u \cdot \nabla u - \frac{a+1}{2} \sigma \nabla \sigma.
\end{cases} \]

(2.5)

By taking the \( L^2 \) norm of (2.5), squaring, integrating them over \([0, t]\) and adding up the two resulting equalities, we arrive at

\[ D_k^2[\sigma](t) \lesssim D_k^2[u](t) + D^2[u](t) \]

\[ + \int_0^t \int_{\mathbb{T}^n \times \mathbb{R}^m} \left( |u \cdot \nabla \sigma|^2 + |u \cdot \nabla u|^2 + |\sigma \text{div} u|^2 + |\sigma \nabla \sigma|^2 \right) dx dy dt. \]

(2.6)

This together with (2.3) yields

\[ D_k^2[\sigma](t) \lesssim D_k^2[u](t) + D^2[u](t) \]

\[ + \|u\|_{\infty}^2 \int_0^t \int_{\mathbb{T}^n \times \mathbb{R}^m} (|\nabla \sigma|^2 + |\nabla u|^2) dx dy dt + \|\sigma\|_{\infty} \int_0^t \int_{\mathbb{T}^n \times \mathbb{R}^m} (|\text{div} u|^2 + |\nabla \sigma|^2) dx dy dt \]

\[ \lesssim D_k^2[u](t) + D^2[u](t) + \varepsilon^2 D_k^2[\sigma, u](t). \]

(2.7)

By the smallness of \( \varepsilon \), we get

\[ D_k^2[\sigma](t) \lesssim D_k^2[u](t) + D^2[u](t). \]

Then by Young’s inequality, we conclude that

\[ D_k[\sigma](t) \lesssim D_k[u](t) + D[u](t), \]

(2.8)

which implies that (2.4) holds for \( k = 1 \). Next we prove (2.4) for \( k > 1 \). Applying the operator \( \partial^a(1 \leq |a| \leq 2N) \) to (2.5) together with the Leibniz’s rule derives

\[ \begin{cases}
\partial_t (\partial^a \sigma) = -\text{div}(\partial^a u) - Q_1^a, \\
\nabla(\partial^a \sigma) = -\partial_t (\partial^a u) - \partial^a u - Q_2^a.
\end{cases} \]

(2.9)
where
\[ Q_i^{bc} = \partial^i u \cdot \nabla (\partial^i \sigma) + \frac{\gamma - 1}{2} \partial^i \sigma \mathrm{div} (\partial^i u), \quad Q_2^{bc} = \partial^i u \cdot \nabla (\partial^i \sigma) + \frac{\gamma - 1}{2} \partial^i \sigma \nabla (\partial^i \sigma), \tag{2.10} \]
and \( C_0^c \) are some constants. Employing the similar process as (2.6), we can get
\[ D_Z^2[\sigma](t) \lesssim D_Z^2[u](t) + \sum_{|b| + |c| \leq 1} D^2[Q_i^{bc}, Q_2^{bc}](t). \tag{2.11} \]
By Hölder inequality, the estimate (2.3) and (2.8), one has
\[ \sum_{|b| + |c| \leq 1} D^2[Q_i^{bc}, Q_2^{bc}](t) \lesssim \varepsilon^2 D_Z^2[\sigma, u](t) + \varepsilon^2 D^2[\sigma](t) + \varepsilon^2 D_Z^2[u](t) \tag{2.12} \]
\[ \lesssim \varepsilon^2 D_Z^2[\sigma, u](t) + \varepsilon^2 D^2[u](t). \tag{2.13} \]
This together with (2.11) and the smallness of \( \varepsilon \) yields
\[ D_Z^2[\sigma](t) \lesssim D_Z^2[u](t) + \varepsilon^2 D^2[u](t) + \sum_{1 \leq k \leq 2N + 1} D^2[Q_i^{bc}, Q_2^{bc}](t), \tag{2.14} \]
which shows (2.4) holds for \( k = 2 \). After an iterative process, we have
\[ D_Z^2[\sigma](t) \lesssim D_Z^2[u](t) + \varepsilon^2 D_Z^2[u](t) + \sum_{|b| + |c| \leq k - 1} D^2[Q_i^{bc}, Q_2^{bc}](t), \tag{2.15} \]
by Hölder inequality and taking the \( L^\infty \) norm in \( Q_i^{bc}, Q_2^{bc} \) for those terms with \( |b| \leq N \) or \( |c| \leq N - 1 \) along with (2.3), one has
\[ \sum_{|b| + |c| \leq k - 1} D^2[Q_i^{bc}, Q_2^{bc}](t) \lesssim \varepsilon^2 D^2[\sigma, u](t), \quad 1 \leq k \leq 2N + 1. \tag{2.15} \]
Substituting (2.15) into (2.14) and using the smallness of \( \varepsilon \) derive
\[ D_k[\sigma](t) \lesssim D_k[u](t) + D[u](t), \quad 1 \leq k \leq 2N + 1, \]
where we have used the Young’s inequality. Thus the proof of lemma 2.2 is finished. \( \square \)

Now, we begin to estimate the \( L^2 \) norm of the solution \( (\sigma, u) \).

**Lemma 2.3.** Under the assumption (2.1), for all \( t > 0 \), it holds that
\[ E^2[\sigma, u](t) + D^2[u](t) \lesssim E^2[\sigma, u](0) + \varepsilon D^2[u](t). \tag{2.16} \]

**Proof.** Multiplying the first equation of (1.4) by \( \sigma \) and the second one by \( u \), and adding the resulting equations together derive
\[ \frac{1}{2} \partial_t (\sigma^2 + |u|^2) + |u|^2 + \mathrm{div}(\sigma u) + \gamma - 1 \frac{1}{2} \mathrm{div}(\sigma^2 u) = -\sigma u \cdot \nabla \sigma - \langle u \cdot \nabla u, u \rangle + \gamma - 1 \frac{1}{2} \sigma \nabla \sigma \cdot u, \tag{2.17} \]
where \( \langle \cdot, \cdot \rangle \) represents the standard inner product in \( \mathbb{R}^d \). Integrating (2.17) by parts over \([0, t] \times \Omega \) gives
\[ E^2[\sigma, u](t) + D^2[u](t) \lesssim E^2[\sigma, u](0) + \int_0^t \int_{\mathbb{T}^n \times \mathbb{R}^m} |\sigma||u||\nabla \sigma| dx dy d\tau \]
\[ + \int_0^t \int_{\mathbb{T}^n \times \mathbb{R}^m} |\nabla u||u|^2 dx dy d\tau. \tag{2.18} \]
Applying Cauchy-Schwarz inequality to (2.18) and using (2.3), one can obtain
\[
E^2_1[\sigma, u](t) + D^2_2[u](t) \\
\leq E^2_2[\sigma, u](0) + \|\sigma\|_\infty \int_0^t \int_{\mathbb{T}^n \times \mathbb{R}^m} (|u|^2 + |\nabla \sigma|^2) dx dy dt + \|\nabla u\|_\infty \int_0^t \int_{\mathbb{T}^n \times \mathbb{R}^m} |u|^2 dx dy dt
\]
\[
\leq E^2_2[\sigma, u](0) + \varepsilon D^2_2[u](t) + \varepsilon D^2_1[\sigma](t),
\]
which together with lemma 2.2 yields
\[
E^2_2[\sigma, u](t) + D^2_2[u](t) \lesssim E^2_2[\sigma, u](0) + \varepsilon D^2_2[u](t) + \varepsilon D^2_1[\sigma](t).
\] Combining (2.20) and the smallness of \(\varepsilon\) derives the desired result (2.16).

Next, we establish the estimate of the derivatives for the solution \((\rho, u)\).

**Lemma 2.4.** Under the assumption (2.1), for all \(t > 0, 1 \leq k \leq 2N + 1\), it holds that
\[
E^2_2[\sigma, u](t) + D^2_2[u](t) \lesssim E^2_2[\sigma, u](0) + \varepsilon D^2_2[u](t).
\]  

**Proof.** Multiplying the first equation of (2.9) by \(\partial^a \sigma\) and the second one by \(\partial^a u\), and adding the resulting equations together give
\[
\frac{1}{2} \partial_t (|\partial^a \sigma|^2 + |\partial^a u|^2) + |\partial^a u|^2
\]
\[
+ \operatorname{div} \left[ \partial^a \sigma \partial^a u + \frac{1}{2} u (|\partial^a \sigma|^2 + |\partial^a u|^2) + \frac{1}{2} \frac{1}{2} \partial^a \sigma \partial^a u \nabla \sigma \right]
\]
\[
= \frac{1}{2} \operatorname{div} u \left( |\partial^a \sigma|^2 + |\partial^a u|^2 \right) + \frac{1}{2} \frac{1}{2} \partial^a \sigma \partial^a u \cdot \nabla \sigma
\]
\[
- \sum_{\{\alpha, \beta\} \in \mathbb{N}^m |\alpha| \leq |\beta|} \frac{1}{2} \partial^a \sigma Q^{\alpha\beta}_{1,2} + \partial^a u \cdot Q^{\alpha}_{1,2},
\]
where \(Q^{\alpha}_{1,2}, i = 1, 2\) are defined in (2.10) and the multi-index \(a\) satisfies \(1 \leq |a| \leq k \leq 2N + 1\). Integrating (2.22) by parts over \([0, t] \times \Omega\) and adding the resulting equalities with \(|a|\) from 1 to \(k\), we can obtain
\[
E^2_2[\sigma, u](t) + D^2_2[u](t) \lesssim E^2_2[\sigma, u](0)
\]
\[
+ \sum_{1 \leq |a| \leq k} \left\{ \int_0^t \int_{\mathbb{T}^n \times \mathbb{R}^m} \left( |\operatorname{div} u (|\partial^a \sigma|^2 + |\partial^a u|^2) \right) dx dy dt + \int_0^t \int_{\mathbb{T}^n \times \mathbb{R}^m} |\partial^a \sigma||\partial^a u||\nabla \sigma| dx dy dt
\]
\[
+ \sum_{\{\alpha, \beta\} \in \mathbb{N}^m |\alpha| \leq |\beta|} \int_0^t \int_{\mathbb{T}^n \times \mathbb{R}^m} \left( |\partial^a \sigma||Q^{\alpha\beta}_{1,2}| + |\partial^a u||Q^{\alpha}_{1,2}| \right) dx dy dt \right\}.
\]  

Applying (2.3) to the terms \(\operatorname{div} u\) and \(\nabla \sigma\) in the second and third terms of (2.23) on the right-hand side, we arrive at
\[
\sum_{1 \leq |a| \leq k} \int_0^t \int_{\mathbb{T}^n \times \mathbb{R}^m} \left( |\operatorname{div} u (|\partial^a \sigma|^2 + |\partial^a u|^2) + |\partial^a \sigma||\partial^a u||\nabla \sigma| \right) dx dy dt
\]
\[
\lesssim \sum_{1 \leq |a| \leq k} \int_0^t \int_{\mathbb{T}^n \times \mathbb{R}^m} \left( \|\operatorname{div} u\|_\infty + \|\nabla \sigma\|_\infty \right) \left( |\partial^a \sigma|^2 + |\partial^a u|^2 \right) dx dy dt
\]
\[
\lesssim \varepsilon D^2_1[\sigma, u](t),
\]
where the Cauchy-Schwarz inequality has been used. Taking the $L^\infty$ norm of in $Q^{bc}_1$, $Q^{bc}_2$ for $|b| \leq N$ or $|c| \leq N-1$ and using (2.3) again, one has

$$
\sum_{1 \leq |b| \leq k} \sum_{|c| < |b|} \int_0^1 \int_{\mathbb{T}^n \times \mathbb{R}^m} \left( |\partial^a \sigma||Q^{bc}_1| + |\partial^a u||Q^{bc}_2| \right) dx dy d\tau \leq \varepsilon D_2^n[\sigma,u](t). \tag{2.25}
$$

Substituting (2.24), (2.25) into (2.23), we get

$$
E_2^2[\sigma,u](t) + D_2^2[u](t) \leq E_2^2[\sigma,u](0) + \varepsilon D_2^2[\sigma,u](t).
$$

This together with lemma 2.2 yields

$$
E_2^2[\sigma,u](t) + D_2^2[u](t) \leq E_2^2[\sigma,u](0) + \varepsilon (D_2^2[u](t) + D^2[u](t)) \tag{2.26}
$$

which implies (2.21) by the smallness of $\varepsilon$. Thus we complete the proof of lemma 2.4.

Based on lemma 2.3-2.4, we are ready to prove theorem 1.1.

**Proof of Theorem 1.1** By lemma 2.3-2.4 and the smallness of $\varepsilon$, we get

$$
E_2^2[\sigma,u](t) + E_{2N+1}^2[\sigma,u](t) + D_{2N}^2[u](t) + D_{2N+1}^2[u](t) \leq E_2^2[\sigma,u](0) + E_{2N+1}^2[\sigma,u](0).
$$

Since $(\rho_0(x), u_0(x)) \in H^{2N+1}(\mathbb{R}^d)$, then we have

$$
E_2^2[\sigma,u](t) + E_{2N+1}^2[\sigma,u](t) \leq \varepsilon^2. \tag{2.27}
$$

Combining (2.27) with the local existence result and the continuation argument, we obtain the global $C^1$ solution $(\rho, u)$ to (1.1).}

**3 Proof of Theorem 1.2**

When $d = 3$, by the second equation of (1.1), we get the equation of the vorticity that

$$
\partial_t \omega + \omega \cdot \nabla u + \omega \text{ div } u = \omega \cdot \nabla u, \tag{3.1}
$$

where $\omega = \text{curl } u = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1)^T$. Applying the operator $\partial^a(0 \leq |a| \leq 3)$ to (3.1) gives

$$
\partial_t (\partial^a \omega) + \partial^a \omega + u \cdot \nabla \partial^a \omega = - \sum_{0 \leq b \leq a} K_{a,b} \partial^b u \cdot \nabla \partial^{a-b} \omega - \sum_{0 \leq c \leq a} K_{a,c} (\partial^c \omega \cdot \partial^{a-c} \text{div } u - \partial^c \omega \cdot \nabla \partial^{a-c} u), \tag{3.2}
$$

where $K_{a,b}, K_{a,c}$ are some constants. Multiplying (3.2) by the factor $\partial^a \omega$ and integrating the resulting equality over $\Omega$ deriv

$$
\frac{d}{dt} \|\partial^a \omega(t,\cdot)\|^2_2 + \|\partial^a \omega(t,\cdot)\|^2_2
\leq \int_{\mathbb{T}^n \times \mathbb{R}^m} |\text{div } u||\partial^a \omega|^2 dx dy
+ \sum_{0 \leq b \leq a} \int_{\mathbb{T}^n \times \mathbb{R}^m} |\partial^b u||\nabla \partial^{a-b} \omega||\partial^a \omega| dx dy
+ \sum_{0 \leq c \leq a} \int_{\mathbb{T}^n \times \mathbb{R}^m} (|\partial^c \omega||\partial^{a-c} \text{div } u||\partial^a \omega| + |\partial^c \omega||\partial^{a-c} \nabla u||\partial^a \omega|) dx dy. \tag{3.3}
$$

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Then summing up (3.3) with $|a|$ from 0 to 3 yields

$$
\frac{d}{dt} \|\partial^{|a|}\omega(t, \cdot)\|_2^2 + \|\partial^{|a|}\omega(t, \cdot)\|_2^2 \\
\lesssim \|\text{div}(t, \cdot)\|_\infty \sum_{|a|\leq 3} \int_{T^d \times \mathbb{R}^d} |\partial^a\omega|^2 \, dx \, dy \\
+ \|\partial^a u(t, \cdot)\|_\infty \sum_{|a|\leq 3} \sum_{0 \leq b \leq |a|} \int_{T^d \times \mathbb{R}^d} (|\nabla \partial^b \omega|^2 + |\partial^a \omega|^2) \, dx \, dy \\
+ \|\omega(t, \cdot)\|_\infty \sum_{|a|\leq 3} \sum_{0 \leq c \leq |a|} \sum_{0 \leq b \leq c} \int_{T^d \times \mathbb{R}^d} (|\partial^c \omega|^2 + |\partial^a \omega|^2) \, dx \, dy \\
\lesssim \|\partial^{|a|}\omega(t, \cdot)\|_2^2 + \|\omega(t, \cdot)\|_\infty \|\partial^{|a|}\omega(t, \cdot)\|_2 \|\partial^{|a|}\omega(t, \cdot)\|_2.
$$

By lemma 2.1, we have $\|\omega(t, \cdot)\|_\infty \lesssim \|\partial^{|a|}\omega(t, \cdot)\|_2$. Since $\|\partial^{|a|}\omega(t, \cdot)\|_\infty \lesssim \varepsilon$, $\|u(t, \cdot)\|_H^4 \lesssim \varepsilon$, then (3.4) becomes

$$
\frac{d}{dt} \|\partial^{|a|}\omega(t, \cdot)\|_2^2 + \|\partial^{|a|}\omega(t, \cdot)\|_2^2 \lesssim \varepsilon \|\partial^{|a|}\omega(t, \cdot)\|_2^2.
$$

This together with the smallness of $\varepsilon$ yields that there exists a constant $C > 0$ such that

$$
\frac{d}{dt} \|\partial^{|a|}\omega(t, \cdot)\|_2^2 + C \|\partial^{|a|}\omega(t, \cdot)\|_2^2 \leq 0.
$$

Thus it follow from (3.6) that

$$
\|\partial^{|a|}\omega(t, \cdot)\|_2 \leq C \varepsilon e^{-\frac{t}{2}}
$$

and we obtain the exponential decay of the vorticity and its derivatives when $d = 3$.

**Remark 3.1.** For $d = 2$, the equation of the vorticity becomes

$$
\partial_t \omega + \omega \cdot \nabla u + \omega \cdot \text{div} u + \omega = 0,
$$

where $\omega = \text{curl} u = \partial_1 u_2 - \partial_2 u_1$. Compared with the equation of the vorticity for $d = 3$, (3.8) is much simpler. One only needs to remove the term deriving from $\omega \cdot \nabla u$ in the proof for $d = 3$ above and can also obtain the exponential decay of the vorticity and its derivatives when $d = 2$. Thus we finish the proof of Theorem 1.2.

### 4 Conclusions

In this paper, if the initial data is small in appropriate Sobolev norm, then the global solution exists for the Cauchy problem of the compressible Euler equations in partial space-period domains. In addition, we show the vorticity and its derivatives decay exponentially to zero in two and three dimensions.
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Competing Interests

Author has declared that no competing interests exist.

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