Lyapunov exponents for families of rotated linear cocycles

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Abstract

Consider a compact metric space $X$, a homeomorphism $T : X \to X$ and an ergodic $T$-invariant measure $\mu$. In this work, we are interested in the study of the upper Lyapunov exponent $\lambda^*(\theta)$ associated to the periodic family of cocycles defined by $A_\theta(x) := A(x)R_\theta$, $x \in X$, where $A : X \to GL^+(2, \mathbb{R})$ is a linear cocycle orientation–preserving and $R_\theta$ is a rotation of angle $\theta \in \mathbb{R}$. We show that if the cocycle $A$ has dominated splitting, then there exists a non empty open set $U$ of parameters $\theta$ such that the cocycle $A_\theta$ has dominated splitting and the function $U \ni \theta \mapsto \lambda^*(\theta)$ is real analytic and strictly concave. As a consequence, we obtain that the set of parameters $\theta$ where the cocycle $A_\theta$ does not have dominated splitting is non empty.

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1. Introduction

Consider a compact metric space $X$, let $T : X \to X$ be a homeomorphism and let $\mu$ be an ergodic invariant measure for $T$. Let $A : X \to GL(2, \mathbb{R})$ be continuous and consider the \textit{linear cocycle generated by $A$ over $T$}, $A_T : X \times \mathbb{R}^2 \to X \times \mathbb{R}^2$, defined by $A_T(x, v) = (Tx, A(x)v)$.

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For $n \in \mathbb{Z}$, the iterates of $A_T$ are given by $A_T^n(x, v) = (T^n(x), A^n(x))$ where $A^0(x) = \text{Id}$, and for any integer $n > 0$,

$$A^n(x) = A(T^{n-1}x) \cdot \ldots \cdot A(Tx) \cdot A(x),$$

$$A^{-n}(x) = A(T^{-n}x)^{-1} \cdot A(T^{-(n-1)}x)^{-1} \cdot \ldots \cdot A(T^{-1}x)^{-1}.$$ 

The upper Lyapunov exponent of the cocycle $A_T$ at $x \in X$ is defined by

$$\lambda^+(A_T, x) = \lim_{n \to +\infty} \frac{1}{n} \log \|A^n(x)\|.$$ 

In the same way, we define the lower Lyapunov exponent of the cocycle $A_T$ at $x \in X$ by

$$\lambda^-(A_T, x) = \lim_{n \to +\infty} \frac{1}{n} \log \|A^{-n}(x)\|^{-1}.$$ 

The classical theory of linear cocycles was initiated by Furstenberg and Kesten [5, 6]. They proved that the limits above exist for almost every point $x \in X$ and they do not depend on the point when $\log^+ \|A^{\pm 1}\| \in L^1(\mu)$. More information is provided in Oseledet’s Theorem [9, 10] which establishes the existence of a measurable invariant splitting $\mathbb{R}^2 = E^-(x) \oplus E^+(x)$ such that for $\mu$-almost all $x \in X$ and every $v \in E^+(x) \setminus \{0\}$

$$\lim_{n \to \pm \infty} \frac{1}{n} \log \|A^n(x)v\| = \lambda^+(A_T, x).$$ 

Similarly, for every $u \in E^-(x) \setminus \{0\}$, we have

$$\lim_{n \to \pm \infty} \frac{1}{n} \log \|A^n(x)u\| = \lambda^-(A_T, x).$$ 

We assume that the linear cocycle $A_T$ has dominated splitting, that is, there exists a continuous decomposition $\mathbb{R}^2 = E(x) \oplus F(x)$ such that for every $x \in X$:

(i) $\dim E(x) = \dim F(x) = 1$,

(ii) $A(x)E(x) = E(Tx), A(x)F(x) = F(Tx),$ and

(iii) there exists $l \geq 1$ such that

$$\|A^l(x)|E(x)\| \cdot \|A^{-l}(T^l x)|F(T^l x)\| < \frac{1}{2}.$$ 

We recall that in our case dominated splitting is a continuous extension of Oseledec splitting. Moreover, the angle of the splitting varies continuously with the point and it is bounded away from zero. That means, there exists an $\alpha > 0$ such that

$$\inf \{\angle(E(x), F(x)) : x \in X\} \geq \alpha. \tag{1.1}$$

In particular we can assume $l \geq 1$ is large enough satisfying

$$\|A^l(x)|E(x)\| \cdot \|A^{-l}(T^l x)|F(T^l x)\| \sin \alpha < 1. \tag{1.2}$$

We denote by $\mathbb{GL}^+(2, \mathbb{R})$ the set of non singular matrices $A$ satisfying $\det A > 0$. We are interested in the periodic family of cocycles defined by

$$A_\theta(x) := A(x)R_\theta, \quad x \in X,$$

where $A : X \to \mathbb{GL}^+(2, \mathbb{R})$ is a linear cocycle orientation–preserving and $R_\theta$ is a rotation of angle $\theta \in \mathbb{R}$. For every $\theta \in \mathbb{R}$, the system underlying on the base $X$ is the original one $(T, X, \mu)$. For this reason, we will simplify the notation omitting $T$ if it is not necessary to explicit it.

When the matrices $A$ belong to $\mathbb{SL}(2, \mathbb{R})$ the previous family acquires interesting properties. For instance, Herman proved that the average of the upper Lyapunov exponent
of $A_{\theta}$ is bounded above by a certain value involved with the norm of the original linear cocycle [7, section 6.2]. Later, Avila and Bochi showed that the previous relation is in fact an equality [1]. The relation above was used by Knill to prove that there exists a dense set of bounded $\mathbb{G}L(2, \mathbb{R})$ cocycles that have non-zero Lyapunov exponents (see [8, proposition 2.4, theorem 3.1]). We refer the reader to [2] for an alternative proof of the Herman–Avila–Bochi equality and [4, section 12.5.2], [14, lemma 2] and [3, lemma 6.6] for another remarkable properties related with the family $A_{\theta}$.

We are interested how the Lyapunov exponent $\lambda^+(A_{\theta})$ varies with respect to $\theta$, when $A = A_0$ has dominated splitting. Earlier Ruelle [11] proved that the upper Lyapunov exponents varies analytically with respect to the cocycles with dominated splitting. On the other hand, concavity of the Lyapunov exponents was studied by Shub and Wilkinson [12] in the setting of perturbations of Anosov times identity on the three-torus. Roughly speaking, they studied how Lyapunov exponents change when the derivative $Df$ restricted to the center unstable subbundle $E^{cu}$ is composed by a matrix of type

$$
\begin{pmatrix}
1 & t \\
0 & 1
\end{pmatrix}, \quad t \neq 0.
$$

Our main Theorem is the following.

**Theorem A.** Assume that the cocycle $A : X \to \mathbb{G}L^+(2, \mathbb{R})$ is continuous and has dominated splitting. Then, there exists an open set $U \subseteq \mathbb{R}$ such that for any $\theta \in U$ the cocycle $A_{\theta}$ has dominated splitting and the function $U \ni \theta \mapsto \lambda^+(A_{\theta}) \in \mathbb{R}$ is real analytic and strictly concave.

In theorem A the set $U$ is formed by the parameters $\theta \in \mathbb{R}$ such that $A_{\theta}$ has dominated splitting. However, it is not possible to have $U = \mathbb{R}$ due the concavity. In fact, if we denote

$$
D = \{ \theta \in [0, 2\pi] : \text{the cocycle } A_{\theta} \text{ has not dominated splitting}\}
$$

and we assume that $D = \emptyset$, it follows from theorem A that the function $\theta \mapsto \lambda^+(A_{\theta})$ is real analytic, concave and periodic in the interval $[0, 2\pi]$. In particular, the function $\theta \mapsto \lambda^+(A_{\theta})$ must have a minimum in a point $\theta_0 \in [0, 2\pi]$. If the cocycle $B = A_{\theta_0}$ has dominated splitting, by a change in the parameter, we can apply theorem A to the cocycle $B$ having a contradiction with the concavity. Summarizing,

**Corollary B.** The set $\mathbb{R} \setminus U$ is not empty and for any $\theta \in \mathbb{R} \setminus U$, the cocycle $A_{\theta}$ does not have dominated splitting.

Also as consequence of concavity, the Lyapunov exponents do not remain constant with respect to the parameter in the domain of domination. More precisely:

**Corollary C.** There exists $\varepsilon > 0$, an open interval $I \subseteq \mathbb{R}$, with $|I| = \varepsilon$ and $0 \in \partial I$ such that for any $\theta \in I$, we have

$$
\lambda^+(A_{\theta}) < \lambda^+(A)
$$

and

$$
\lambda^-(A_{\theta}) > \lambda^-(A).
$$

The remaining of the paper is organized as follows. We will divide the proof of theorem A in two parts.

In section 2, we introduce the notion of quasi–conjugation of cocycles. We prove that for quasi–conjugated cocycles, their upper Lyapunov exponents are equals. We establish the
existence of a linear cocycle $H$ quasi–conjugated to $A$. The new cocycle $H$ is formed by upper triangular matrices and it exhibits dominated splitting when $A$ has dominated splitting.

The core of the proof of theorem A is in section 3. There, we define a new metric that allows an easy calculation of the Lyapunov exponent. Then, inspired in the work of Shub and Wilkinson [12], we give an explicit expression for the Lyapunov exponent $\lambda^+(A_0)$ and we show that the expressions involved are real analytic functions and thus, we can study the concavity of $\lambda^+(A_0)$.

In section 4, we detailed an application of theorem A. Let $X = N$ be the compact nilmanifold obtained from the quotient of the Heisenberg group $H$ with the lattice $\Gamma = \{(x, y) : x \in \mathbb{Z}^2, y \in \frac{1}{2}\mathbb{Z}\}$. Let $\Phi : \mathcal{H} \to \mathcal{H}$ be an automorphism and $f : N \to N$ the diffeomorphism induced by $\Phi$. Then $f$ is partially hyperbolic with splitting $TN = E_s \oplus E_c \oplus E_u$, and the Lebesgue measure in $N$ is $f$-invariant and ergodic.

We consider the natural linear cocycle induced from $f$ and its derivative $F : TN \to TN$ defined by

$$F((x, y), v) = (f(x, y), Df(x, y)v).$$

In such a case, the cocycle defined by the restriction to the center-unstable direction $(F|E^{cu}) : E^{cu} \to E^{cu}$ is orientation–preserving and it has dominated splitting. Moreover, the Lyapunov exponents are

$$\lambda^u(F) = \lambda^s(F) < \lambda^c(F) = 0 < \lambda^u(F).$$

In particular, for the cocycle restricted to the center unstable direction we have

$$\lambda^+(F|E^{cu}) = \lambda(Df|E^u) = \lambda^u(F) > 0,$$

and

$$\lambda^-(F|E^{cu}) = \lambda(Df|E^c) = \lambda^c(F) = 0.$$

Then, we consider the one–parameter family of continuous cocycles $F_\theta : TN \to TN$ defined by

$$F_\theta((x, y), v) = \begin{cases} Df(x, y)R_\theta v, & v \in E^{cu}; \\ Df(x, y)v, & v \in E^s. \end{cases}$$

Then we can apply corollary C to $F_\theta|E^{cu}$ and conclude the following.

**Corollary D.** There is an open set $I \subseteq [0, 2\pi]$ such that for every $\theta \in I$, the cocycle $F_\theta$ is partially hyperbolic with splitting $TN = E^s_\theta \oplus E^c_\theta \oplus E^u_\theta$ and $F_\theta$ is non uniformly hyperbolic.

2. Quasi–conjugation

Let $X$ be a compact metric space, let $T : X \to X$ be an homeomorphism and let $\mu$ be an ergodic invariant measure for $T$. Consider the continuous maps $A : X \to \text{GL}(2, \mathbb{R})$ and $H : X \to \text{GL}(2, \mathbb{R})$ and consider the linear cocycle generated by $A$ over $T$, $A_T : X \times \mathbb{R}^2 \to X \times \mathbb{R}^2$, and the linear cocycle generated by $H$ over $T$, $H_T : X \times \mathbb{R}^2 \to X \times \mathbb{R}^2$.

The linear cocycles $A_T$, $H_T$ are quasi–conjugated if there exist two families of matrices $\mathfrak{B} = \{B(x) \in \text{SO}(2, \mathbb{R}) : x \in X\}$ and $\mathfrak{D} = \{D(x) \in \text{SO}(2, \mathbb{R}) : x \in X\}$ satisfying:

(i) $H(x) = (D(x))^{-1} \cdot A(x) \cdot B(x)$, for every $x \in X$; and

(ii) $B(Tx) = s(x)D(x)$ where $s(x) \in \{1, -1\}$, for every $x \in X$. 

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We remark that it is not required some continuity on the variable $x$ for the families of matrices. Recall that the element of $SO(2, \mathbb{R})$ are isometries of $\mathbb{R}^2$ and they form a commutative group.

We have the following.

**Lemma 1.** Let $A_T$ and $H_T$ be quasi–conjugated linear cocycles. Then

(i) The cocycles $A_0$ and $H_0$ are quasi–conjugated,

(ii) $\lambda^*(A) = \lambda^*(H)$.

**Proof.** Since $R_\theta \in SO(2, \mathbb{R})$ for every $\theta \in [0, 2\pi]$ and $H(x) = (D(x))^{-1} \cdot A(x) \cdot B(x)$ for every $x \in X$, then we have

\[
A_\theta (x) = A(x) \cdot R_\theta = A(x) \cdot B(x) \cdot R_\theta \cdot (B(x))^{-1} = D(x)[(D(x))^{-1} \cdot A(x) \cdot B(x)] \cdot R_\theta \cdot (B(x))^{-1} = D(x) \cdot H(x) \cdot R_\theta \cdot (B(x))^{-1} = D(x) \cdot H_\theta (x) \cdot (B(x))^{-1}.
\]

On the other hand, for each $n \in \mathbb{N}$ and $x \in X$ we have

\[
\|A^n(x)\| = \|H^n(x)\|. \tag{2.1}
\]

In fact, arguing by induction it is not difficult to see that

\[
H^n(x) = \prod_{i=0}^{n-1} s(T^i x) \cdot (D(T^{n-1} x))^{-1} \cdot A^n(x) \cdot B(x) \tag{2.2}
\]

and therefore

\[
\|H^n(x)\| = \|(D(T^{n-1} x))^{-1} \cdot A^n(x) \cdot B(x)\| = \|A^n(x)\|.
\]

To conclude, it follows from Furstenberg–Kesten Theorem and (2.1) that

\[
\lambda^*(A) = \lim_{n \to \infty} \frac{1}{n} \int \log \|A^n(y)\| \, d\mu(y) = \lim_{n \to \infty} \frac{1}{n} \int \log \|H^n(y)\| \, d\mu(y) = \lambda^*(H).
\]

As an immediate consequence of lemma 1, if $A$ and $H$ are quasi–conjugated, then for every $\theta \in \mathbb{R}$ the cocycles $A_\theta$ and $H_\theta$ are quasi–conjugated. Applying item (ii) of lemma 1 to the quasi–conjugated cocycles $A_\theta$ and $H_\theta$ we obtain that

\[
\lambda^*(A_\theta) = \lambda^*(H_\theta), \quad \text{for all } \theta \in \mathbb{R}. \tag{2.3}
\]

**Proposition 2.** For every $A : X \to GL(2, \mathbb{R})$ continuous and having dominated splitting, there exists $H : X \to GL(2, \mathbb{R})$, depending on $A$, satisfying:

(i) $H$ is continuous and it has dominated splitting,

(ii) for each $x \in X$, $H(x)$ is a lower triangular matrix, and

(iii) $H_T$ is quasi–conjugated to $A_T$. 

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First we will explicitly define the cocycle $H : X \rightarrow \mathbb{GL}(2, \mathbb{R})$. Given a unitary vector $u = (u_1, u_2) \in \mathbb{R}^2$, we denote by $u^\perp$ the orthogonal vector $(u_2, -u_1)$. The column matrix

$$B_u := [u^\perp u] = \begin{pmatrix} u_2 & u_1 \\ -u_1 & u_2 \end{pmatrix}$$

is a rotation and therefore it belongs to $SO(2, \mathbb{R})$. Given $A \in GL(2, \mathbb{R})$ with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

denote $v = Au/\|Au\|$ and $D_u = [v^\perp v]$. It is easy to see that

$$H_u = D_u^{-1} \cdot A \cdot B_u = \begin{pmatrix} \lambda_u & 0 \\ \sigma_u & \eta_u \end{pmatrix}$$

and $\eta_u > 0$.

On the other hand, $A : X \rightarrow \mathbb{GL}(2, \mathbb{R})$ has dominated splitting, so for every $x \in X$ let $\mathbb{R}^2 = E_A(x) \oplus F_A(x)$ be the dominated splitting of $A$. Let $\{V_1, \ldots, V_n\}$ be an open cover of $X$ such that there exist continuous functions $u_i : V_i \rightarrow \mathbb{R}^2$ such that $u_i(x) \in F_A(x)$ and $\|u_i(x)\| = 1$ for every $x \in V_i$, $i = 1, \ldots, n$. For every $x \in V_i$, $i = 1, \ldots, n$, we write

$$v_i(x) := A(x)u_i(x)/\|A(x)u_i(x)\|,$$
$$B_i(x) := [u_i^\perp(x) u_i(x)],$$
$$D_i(x) := [v_i^\perp(x) v_i(x)].$$

Define

$$H_i(x) = (D_i(x))^{-1} \cdot A(x) \cdot B_i(x) = \begin{pmatrix} \lambda_i(x) & 0 \\ \sigma_i(x) & \eta_i(x) \end{pmatrix}.$$

Finally, define the function $H : X \rightarrow \mathbb{GL}(2, \mathbb{R})$ by $H(x) = H_i(x)$ when $x \in V_i$.

**Lemma 3.** The function $H : X \rightarrow \mathbb{GL}(2, \mathbb{R})$ is well defined and therefore it is continuous.

**Proof.** Given $x \in V_i \cap V_j$ we have that

$$s_{ij}(x) := (u_i(x), u_j(x)) = \pm 1$$

and this is constant in each connected component of $V_i \cap V_j$. Therefore, it is easy to see that $B_i(x) = s_{ij}(x)B_j(x)$, $D_i(x) = s_{ij}(x)D_j(x)$ and $D_j(x)^{-1} = s_{ij}(x)D_i(x)^{-1}$ for every $x \in V_i \cap V_j$. Then we have

$$H_j(x) = (D_j(x))^{-1} \cdot A(x) \cdot B_j(x)$$
$$= s_{ij}(x)(D_j(x))^{-1} \cdot A(x) \cdot s_{ij}(x)B_i(x)$$
$$= (s_{ij}(x))^2(D_j(x))^{-1} \cdot A(x) \cdot B_i(x)$$
$$= H_i(x).$$

We can write the function $H : X \rightarrow \mathbb{GL}(2, \mathbb{R})$ by

$$H(x) = \begin{pmatrix} \lambda(x) & 0 \\ \sigma(x) & \eta(x) \end{pmatrix}$$

with $\eta(x) > 0$. We also denote

$$H^n(x) = \begin{pmatrix} \lambda_n(x) & 0 \\ \sigma_n(x) & \eta_n(x) \end{pmatrix}.$$
where
\[ \lambda_n(x) = \prod_{j=0}^{n-1} \lambda(T^j x) \quad \text{and} \quad \eta_n(x) = \prod_{j=0}^{n-1} \eta(T^j x). \] (2.4)

**Lemma 4.** The cocycle \( H_T \) is quasi–conjugated to \( A_T \).

**Proof.** Let \( x \in V_i \) with \( Tx \in V_j \). Note that by construction there exists \( s_{ij}(x) \in \{1, -1\} \) such that \( v_i(x) = s_{ij}(x) \cdot u_j(Tx) \) and therefore \( D_j(x) = s_{ij}(x)B_j(Tx) \). We define the families \( \mathfrak{B} \) and \( \mathfrak{D} \) inductively as follows:

(i) for each \( x \in V_1 \), we let \( B(x) := B_1(x) \) and \( D(x) := D_1(x) \),

(ii) for each \( x \in V_i \setminus \bigcup_{j<i} V_j \), we let \( B(x) := B_i(x) \) and \( D(x) := D_i(x) \)

and our lemma it follows from the construction above. \( \square \)

**Lemma 5.** There is an continuous splitting \( \mathbb{R}^2 = E(x) \oplus F(x) \) invariant by \( H_T \).

**Proof.** Define \( F(x) = \text{span}(e_2) \) and \( E(x) = (B(x))^{-1}E_A(x) \) where \( e_2 = (0, 1) \). Clearly \( H(x)(F(x)) = F(Tx) \). Note that
\[
E(Tx) = (B(x))^{-1}E_A(Tx) = \text{span} \left( (B(Tx))^{-1}E_A(Tx) \right)
= \text{span} \left( s(x)(D(x))^{-1}E_A(Tx) \right)
= (D(x))^{-1}E_A(Tx).
\]
If \( v \in E_A(x) \), then \( (B(x))^{-1}v \in E(x) \). The equality
\[
w = A(x)v = D(x) \cdot H(x) \cdot (B(x))^{-1}v \in E_A(Tx)
\]
implies that \( (D(x))^{-1}w = H(x)(B(x))^{-1}v \in E(Tx) \) as required. \( \square \)

Recall that if \( u, v \in \mathbb{R}^2 \setminus \{(0, 0)\} \) are linearly independent and \( A \in \mathbb{GL}(2, \mathbb{R}) \), then
\[
| \det A | = \frac{\phi(Au, Av)}{\phi(u, v)},
\]
where
\[
\phi(u, v) = \|u\| \cdot \|v\| \sin(\angle(u, v)).
\]

**Lemma 6.** Consider \( l \geq 1 \) satisfying (1.2). Then for each \( x \in X, \eta_l(x) > |\lambda_l(x)| \).

**Proof.** Fix \( x \in X \) and let \( u \in F(x) \) and \( v \in E(x) \) be unitary vectors such that \( \angle(v, u) = \angle(E(x), F(x)) \). Then \( \|A^l(x)u\| = \|A^l(x)|F(x)|\| \), \( \|A^l(x)v\| = \|A^l(x)|E(x)|\| \) and
\[
| \det A^l(x) | = \frac{\phi(A^l(x)u, A^l(x)v)}{\phi(u, v)}
= \frac{\|A^l(x)v\| \cdot \|A^l(x)u\| \sin(\angle(A^l(x)v, A^l(x)u))}{\sin(\angle(v, u))}
\leq \frac{\|A^l(x)|E(x)|\| \cdot \|A^l(x)|F(x)|\| \sin(\angle(E(x), F(x)))}{\sin(\angle(v, u))}
< \|A^l(x)|F(x)|\|^2 \frac{\sin \alpha}{\sin(\angle(E(x), F(x)))}
\leq \|A^l(x)|F(x)|\|^2.
\]
Let \( u \) be such that \( (B(x))^{-1}u = e_2 \). Since \( D(T^{n-1}x) \) is an isometry and \( \prod_{i=0}^{n-1} s(T^i x) = \pm 1 \) it follows from (2.2) that
\[
\|A^+(x)u\| = \|H^+(x)e_2\| = \eta(x),
\]
and therefore
\[
|\det(A^+(x))| = |\det(H^+(x))| = |\lambda_1(x)| \cdot \eta(x) < (\eta(x))^2
\]
as desired.

\( \square \)

Lemma 7. The splitting \( \mathbb{R}^2 = E(x) \oplus F(x) \) is dominated.

Proof. We remark that by construction, there exists \( \alpha > 0 \) such that for every \( x \in X \),
\[
\zeta(E(x), F(x)) > \alpha.
\]
Therefore, there exists \( C > 0 \) such that
\[
\frac{1}{\sin(\zeta(E(x), F(x)))} < C.
\]
Taking \( v \in E(x) \) unitary and \( e_2 \in F(x) \), it follows that
\[
\frac{\|H^n(x)|E(x)\|}{\|H^n(x)|F(x)\|} = \frac{\|H^n(x)v\|}{\|H^n(x)e_2\|} = \frac{\|H^n(x)v\| \cdot \|H^n(x)e_2\|}{\|H^n(x)e_2\|^2} = \frac{\|H^n(x)e_2\|}{|\det(H^n(x))|} \cdot \sin(\zeta(E(x), F(x))) < C \cdot \frac{\|H^n(x)e_2\|^2}{|\det(H^n(x))|} = C \cdot \frac{|\lambda_n(x)|}{\eta_n(x)}.
\]
From lemma 6 and equation (2.4), \( \frac{|\lambda_n(x)|}{\eta_n(x)} \to 0 \) when \( n \to \infty \). This implies that
\[
\frac{\|H^{nl}(x)|E(x)\|}{\|H^{nl}(x)|F(x)\|} < C \cdot \frac{|\lambda_{nl}(x)|}{\eta_{nl}(x)} < 1
\]
for \( n \) large enough, and therefore \( H \) has dominated splitting as desired.

\( \square \)

3. Proof of theorem A.

This section is devoted to prove theorem A. We remark that theorem A follows from the next result.

Theorem 8. Assume that the cocycle \( A : X \to \mathbb{GL}^+(2, \mathbb{R}) \) is continuous and has dominated splitting. Then, there exists \( \varepsilon > 0 \) such that:

(i) There exist one–parametric continuous functions
\[
a_0, c_0, u_0 : X \to \mathbb{R}, \quad \theta \in (-\varepsilon, \varepsilon),
\]
such that for all \( \theta \in (-\varepsilon, \varepsilon) \), we have
\[
\lambda^+(A_\theta) = \lambda^+(A) + \lambda^-(A) - \int_X \log(a_\theta(x) - c_\theta(x)u_\theta(T^i x)) \, d\mu(x).
\]
(ii) For each \( x \in X \), the functions \((-\varepsilon, \varepsilon) \ni \theta \mapsto a_\theta(x), c_\theta(x), u_\theta(x)\) are real analytic.

(iii) The function \( \lambda^+ : (-\varepsilon, \varepsilon) \ni \theta \mapsto \lambda^+(A_\theta)\) is real analytic and
\[
\frac{d^2 \lambda^+(0)}{d\theta^2} < 0.
\]

From proposition 2, it is enough to give the proof for the cocycle \( H_\theta \).

### 3.1. Implicit expression for the Lyapunov exponent

Fix \( \varepsilon > 0 \) such that \( H_\theta \) has dominated splitting. We denote the invariant splitting of \( H_\theta \) by \( E_\theta(x) \oplus F_\theta(x) = \mathbb{R}^2 \). Since \( F_0(x) = \text{span}(e_2) \), there exist a continuous function \((-\varepsilon, \varepsilon) \ni (\theta, x) \mapsto u_\theta(x)\) such that \( F_\theta(x) = \text{span}((u_\theta(x), 1))\). Therefore by construction \( u_0(x) = 0 \) for each \( x \in X \).

Consider \( X \times \mathbb{R}^2 = \bigcup_{x \in X} T_x \), where \( T_x = \{x\} \times \mathbb{R}^2 \). In all the preceding, we consider the standard inner product over \( T_x \cong \mathbb{R}^2 \). For \((x, \theta)\), we consider \( \langle \cdot, \cdot \rangle_{x, \theta} \) the inner product over \( T_x \) such that the set \( \{(1, -u_\theta(x)), (u_\theta(x), 1)\} \) is an orthonormal base. Then, for each \( \theta \in [0, 2\pi] \) and \( u, v \in \mathbb{R}^2 \) the map \( x \mapsto \langle u, v \rangle_{x, \theta} \) is continuous. Since \( X \) is compact and all metric in \( \mathbb{R}^2 \) are equivalent, we can choose a constant \( C > 0 \) such that \( C^{-1} \|\cdot\|_{x, \theta} \leq \|\cdot\| \leq C \|\cdot\|_{x, \theta} \) for all \( x \in X \). Moreover, for every \( \theta \in [0, 2\pi] \), we have
\[
\lambda^+(\theta) = \lambda^+(H_\theta) = \int_X \log \|H_\theta(y)|F_\theta(y)\|_{T_y, \theta} d\mu(y).
\]

In fact, it follows from Birkhoff’s Theorem that for \( \mu \)-a.e. \( x \in X \) we have
\[
\lambda^+(\theta) = \lim_{n \to \infty} \frac{1}{n} \log \|H^n(x)(u_\theta(x), 1)\|
= \lim_{n \to \infty} \frac{1}{n} \log \|H^n(x)(u_\theta(x), 1)\|_{T^n x, \theta}
= \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|H_\theta(T^j x)|F_\theta(T^j x)\|_{T^{j+1} x, \theta}
= \int_X \log \|H_\theta(y)|F_\theta(y)\|_{T_y, \theta} d\mu(y).
\]

Item (i) of theorem 8 follows directly from the next result.

**Proposition 9.** For every \( \theta \in [0, 2\pi] \) define
\[
H_\theta(x) = \begin{pmatrix} a_\theta(x) & b_\theta(x) \\ c_\theta(x) & d_\theta(x) \end{pmatrix}
\]
for all \( x \in X \).

Then, there exists \( \varepsilon > 0 \) such that for \( |\theta| < \varepsilon \), we have
\[
\lambda^+(\theta) = \lambda^+(0) + \lambda^-(0) - \int_X \log(a_\theta(x) - c_\theta(x) u_\theta(T x)) d\mu(x).
\]

**Proof.** For every \( \theta \in [0, 2\pi] \) and every \( x \in X \), define
\[
r_\theta(x) = \|H_\theta(x)|F_\theta(x)\|_{T_x, \theta}.
\]

Then, equation (3.1) implies that
\[
\lambda^+(\theta) = \int_X \log r_\theta(x) d\mu(x).
\]
Note that there exists $t_\theta(x) \in \mathbb{R}$ such that
\[(HR_\theta)(x)(u_\theta(x), 1) = t_\theta(x)(u_\theta(Tx), 1)\]
and hence
\[\|HR_\theta(x)(u_\theta(x), 1)\|_{T_x, \theta} = |t_\theta(x)| = |t_\theta(x)| = r_\theta(x).
\]
Writing
\[(HR_\theta)(x) = \begin{pmatrix} \lambda(x) \cos \theta & -\lambda(x) \sin \theta \\
\sigma(x) \cos \theta + \eta(x) \sin \theta & \eta(x) \cos \theta - \sigma(x) \sin \theta \end{pmatrix} = \begin{pmatrix} a_\theta(x) & b_\theta(x) \\
c_\theta(x) & d_\theta(x) \end{pmatrix}, \tag{3.3}\]
we have the equations
\[
\begin{cases}
a_\theta(x)u_\theta(x) + b_\theta(x) = t_\theta(x)u_\theta(Tx) \\
c_\theta(x)u_\theta(x) + d_\theta(x) = t_\theta(x) \\
\end{cases}
\tag{3.4}
\]
and multiplying the first equation by $-c_\theta(x)$ and the second one by $a_\theta(x)$ we deduce that
\[t_\theta(x) = \frac{\eta(x)\lambda(x)}{a_\theta(x) - c_\theta(x)u_\theta(Tx)}.
\]
Since $\lambda > 0$ and $u_\theta = 0$, it follows that $a_\theta(x) - c_\theta(x)u_\theta(Tx) > 0$ for $\theta$ close to 0 and for each $x \in X$. Then, there exists $\epsilon > 0$ such that if $\theta \in (-\epsilon, \epsilon)$ then $t_\theta(x) > 0$. We conclude
\[\lambda^+(\theta) = \int \log t_\theta(x) d\mu(x) \]
\[= \int \log \eta(x) d\mu(x) + \int \log \lambda(x) d\mu(x) - \int_X \log (a_\theta(x) - c_\theta(x)u_\theta(Tx)) d\mu(x).
\]
Recalling that
\[\lambda^+(0) = \int \log \|H(x)\|_{F(x)} \|d\mu(x) = \int \log \eta(x) d\mu(x)
\]
and
\[\lambda^+(0) + \lambda^-(0) = \lim_{n \to \infty} \frac{1}{n} \log |\det(H^n(x))| = \int \log(\lambda(x)\eta(x)) d\mu(x)
\]
for $\mu$–a.e. $x \in X$, we obtain
\[\lambda^+(\theta) = \lambda^+(0) + \lambda^-(0) - \int_X \log (a_\theta(x) - c_\theta(x)u_\theta(Tx)) d\mu(x).
\]
□

3.2. Analyticity of the Lyapunov exponent

This subsection is devoted to explain the main tools to prove that the map $\theta \mapsto u_\theta(x)$ is real analytic.

For the real matrix
\[A = \begin{pmatrix} a & b \\
c & d \end{pmatrix},\]
we denote the action of $A$ in the Riemann $\overline{C}$ by a Möbius transformation by
\[A \cdot z = \frac{az + b}{cz + d}, \quad z \in \overline{C}.
\]
Lemma 10. A linear cocycle $A$ has dominated splitting if and only if there exist a family of open disks $D = \{D(x)\}_{x \in X}$ with $D(x) \subset \mathbb{C}$ and $N \geq 0$ such that
\[
A^n(x) \cdot D(x) \subset D(T^n x)
\] (3.5)
for each $x \in X$ and $n \geq N$.

**Proof.** The reader can find the complete proof of this Lemma in [13]. Nevertheless, we explain the main steps by completeness.

For the sufficient direction, a classical argument of contraction for family of cones, gives us the existence of the family of disks.

For the reciprocal, first we observe that the linear cocycle $A$ over $X \times \mathbb{C}^2$ has dominated splitting if and only if the projective action $A(x) \cdot [v] = [A(x)v]$ is hyperbolic on $X \times \mathbb{C}$. More precisely, there exist sections $\tau, \sigma: X \to \mathbb{C}$ and constants $C > 0$ and $0 < \lambda < 1$ such that

(i) $A(x) \cdot \tau(x) = \tau(Tx)$ and $A(x) \cdot \sigma(x) = \sigma(Tx)$,
(ii) $\|A^n(x) \cdot \tau(x)\| \leq C\lambda^n$ for each $n \geq 0$, and
(iii) $\|A^{-n}(x) \cdot \sigma(x)\| \leq C\lambda^n$ for each $n \geq 0$,

where the norm is provided by the spherical metric. Moreover, we can show that in order to obtain dominated splitting it is enough to exhibit a contractive or an expansive section (the $\tau$ or $\sigma$ section respectively).

Actually, it is sufficient the existence of $\tau_x \in \mathbb{C}$ such that $\|A^n(x) \cdot \tau_x\| \leq C\lambda^n$. (3.6)

This follows from the fact that a contractive direction must be unique, and therefore the correspondence $x \mapsto \tau_x$ is continuous. Finally, we observe that the condition of contraction of disks and Schwartz Lemma give us the existence of $\tau_x \in D(x)$ for each $x \in X$ such that $\tau_x$ satisfies (3.6).

Let $A$ be a linear cocycle with dominated splitting $E \oplus F = \mathbb{R}^2$. For $F(x) = \text{span}(v)$ with $v = (v_1, v_2) \in \mathbb{R}^2$ and $v_2 \neq 2$, we define $\xi(x) = v_1/v_2$ and for $F(x) = \text{span}((1, 0))$ define $\xi(x) = \infty$. Therefore
\[
\xi(x) = \bigcap_{n \geq 0} A^n(T^{-n}x) \cdot \mathbb{D}(T^{-n}x).
\]
Moreover, the function $\xi: X \to \mathbb{C}$ is continuous and $\xi(x) \in D(x)$.

Let $\Lambda \subset \mathbb{C}$ be an open connected set. We say that the family of cocycles $\{A_\lambda\}_{\lambda \in \Lambda}$ is holomorphic if for every $x \in X$ and $\lambda \in \Lambda$
\[
A_\lambda(x) = \begin{pmatrix} a_{11}(x, \lambda) & a_{12}(x, \lambda) \\ a_{21}(x, \lambda) & a_{22}(x, \lambda) \end{pmatrix},
\]
and the functions $\lambda \mapsto a_{ij}(x, \lambda)$ for $i, j = 1, 2$ are holomorphic for every $x \in X$.

The family $\{A_\lambda\}_{\lambda \in \Lambda}$ has dominated splitting if for each $\lambda \in \Lambda$, $A_\lambda$ has dominated splitting. In this setting, we consider the correspondence $\lambda \mapsto \xi_\lambda(x)$ as before.

**Proposition 11.** If the family $\{A_\lambda\}_{\lambda \in \Lambda}$ has dominated splitting, then for each $x \in X$ the map $\lambda \mapsto \xi_\lambda(x)$ is holomorphic.

**Proof.** The proof follows from Montel’s theorem. We recall the basic fact.

A family $\mathcal{F}$ of holomorphic functions defined over a fixed domain $\Lambda \subset \mathbb{C}$ is said to be normal if every sequence of members of $\mathcal{F}$ has a subsequence that converges uniformly on compact subsets of $\Lambda$. We recall that if a sequence of holomorphic functions converge
uniformly on compact sets, then the limit function is also holomorphic. Finally, Montel theorem assert that a family \( F = \{ f : \Lambda \to \mathbb{D} \} \) of holomorphic functions is normal.

Fix \( \lambda \in \Lambda \). We recall that if \( D \) is the family of disks given by Lemma 10 for the cocycle \( A_\zeta \), then the equation (3.5) hold for \( A_\zeta \) for \( \zeta \in D(\lambda) \) in an small disk around \( \lambda \), with the same family of disks.

Fix \( x \in X \). Note that the function \( f_n(\zeta) = A_n(T^{-n}x) \cdot \xi_n(T^{-n}x) \) for \( n \in \mathbb{N} \) is holomorphic. From equation (3.5) we have that
\[
A_n(T^{-n}x) \cdot \mathbb{D}(T^{-n}x) \subset \mathbb{D}(x)
\]
and since that \( \xi_n(T^{-n}x) \in \mathbb{D}(T^{-n}x) \) we conclude that \( f_n(\zeta) \) uniformly on compact sets, as required.

\( \square \)

The next proposition summarize the proof of item (ii) and the first statement of item (iii) in theorem 8.

**Proposition 12.** For each \( x \in X \), the functions \( (-\varepsilon, \varepsilon) \ni \theta \mapsto a_\theta(x), c_\theta(x), u_\theta(x) \) and the function \( \lambda^+: (-\varepsilon, \varepsilon) \to \mathbb{R} \) given by \( \lambda^+(\theta) := \lambda^+(A\mathbb{R}_0) \) are real analytic.

**Proof.** Since \( a_\theta(x) = \lambda(x) \cos \theta \) and \( c_\theta(x) = \sigma(x) \cos \theta + \eta(x) \sin \theta \), it only remains to prove this fact for the function \( u_\theta(x) \).

Let \( \varepsilon > 0 \) as in Proposition 9 and let \( U(1, r) = \{ z \in \mathbb{C} : |1 - z| < r < 1 \} \) such that \( S^1 \cap U(1, r) \subset [e^{i\theta} : |\theta| < \varepsilon] \). For each \( z \in U(1, r) \), define
\[
S_z = \left( \begin{array}{c} \frac{z + z^{-1}}{2} \\ \frac{z - z^{-1}}{2} \end{array} \right).
\]
We have that \( S_z^{\varepsilon} = R_\theta \). Define \( H_z = (H_z)_{\mathbb{R}} \) where \( H_z(x) = H(x)S_z \). Then for \( 0 < r < 1 \) the family \( \mathcal{F} = \{ H_z : z \in U(1, r) \} \) has dominated splitting and therefore, for each \( x \in X \) the map \( z \mapsto \xi_\zeta(x) \) is holomorphic. Our assertion it follows from the fact \( \xi_{\varepsilon z}(x) = u_\theta(x) \). The assertion of analyticity corresponding to the function \( \lambda^+(\theta) := \lambda^+(A\mathbb{R}_0) \) is immediate from (3.2).

\( \square \)

### 3.3. Calculating the derivatives of \( \lambda^+(\theta) \)

From proposition 12 we know that the function \( \lambda^+: (-\varepsilon, \varepsilon) \to \mathbb{R} \) defined by \( \lambda^+(\theta) := \lambda^+(A\mathbb{R}_0) \) is real analytic. Now we proceed to calculate the derivatives of the Lyapunov exponent \( \lambda^+(\theta) \) in \( \theta = 0 \) and to study the concavity finishing the proof of item (iii) in theorem 8.

**Lemma 13.** For every \( x \in X \) we have
\[
\frac{d\lambda^+(0)}{d\theta} = \int \frac{\sigma(x)}{\lambda(x)} \tilde{u}_0(Tx) \, d\mu(x)
\] (3.7)
and
\[
\frac{d^2\lambda^+(0)}{d\theta^2} = \int \left[ \frac{2\sigma(x)\tilde{u}_0(Tx)}{\lambda(x)} + 1 + \frac{\sigma(x)\tilde{u}_0(Tx)}{\lambda(x)} \right] \, d\mu(x).
\] (3.8)
Proof. From equation (3.2) it follows that \( \lambda^* \) is differentiable and
\[
\frac{d\lambda^*(\theta)}{d\theta} = -\int \frac{d}{d\theta} \log [a_0(x) - c_0(x)u_0(Tx)] d\mu(x) = -\int \frac{\dot{a}_0(x) - \dot{c}_0(x)u_0(Tx) - c_0(x)\dot{u}_0(Tx)}{a_0(x) - c_0(x)u_0(Tx)} \frac{d\mu(x)}{a_0(x) - c_0(x)u_0(Tx)}.
\]
Since \( u_0(x) = 0 \) for every \( x \in X \), we conclude
\[
\frac{d\lambda^*(0)}{d\theta} = \int \frac{c_0(x)\dot{u}_0(Tx) - \dot{a}_0(x)}{a_0(x)} d\mu(x).
\]
From (3.3) we have \( a_0(x) = \lambda(x) \cos \theta \) and \( c_0(x) = \sigma(x) \cos \theta + \eta(x) \sin \theta \), for every \( x \in X \). A simple calculation allow us to conclude that
\[
\frac{d\lambda^*(0)}{d\theta} = \int \frac{\sigma(x)}{\lambda(x)} \dot{u}_0(Tx) d\mu(x).
\]
Similarly, we can show that
\[
\frac{d^2\lambda^*(\theta)}{d\theta^2} = \int \left[ \frac{\ddot{c}_0(x)u_0(Tx) + 2\dot{c}_0(x)\dot{u}_0(Tx) + c_0(x)\ddot{u}_0(Tx) - \ddot{a}_0(x)}{a_0(x) - c_0(x)u_0(Tx)} + \left( \frac{\dot{c}_0(x)u_0(Tx) + c_0(x)\dot{u}_0(Tx) - \dot{a}_0(x)}{a_0(x) - c_0(x)u_0(Tx)} \right)^2 \right] d\mu(x)
\]
and therefore
\[
\frac{d^2\lambda^*(0)}{d\theta^2} = \int \left[ \frac{2\ddot{c}_0(x)\dot{u}_0(Tx) + c_0(x)\ddot{u}_0(Tx) - \ddot{a}_0(x)}{a_0(x)} + \left( \frac{c_0(x)\dot{u}_0(Tx) - \dot{a}_0(x)}{a_0(x)} \right)^2 \right] d\mu(x)
\]
\[
= \int \left[ \frac{2\eta(x)\dot{u}_0(Tx)}{\lambda(x)} + 1 + \frac{\sigma(x)\ddot{u}_0(Tx)}{\lambda(x)} + \left( \frac{\sigma(x)\dot{u}_0(Tx)}{\lambda(x)} \right)^2 \right] d\mu(x).
\]
\[\square\]

Lemma 14. For each \( x \in X \), we have
\[
\dot{u}_0(x) = -\sum_{k=1}^{\infty} \frac{\lambda_k(T^{-k}x)}{\eta_k(T^{-k}x)} \tag{3.9}
\]
and
\[
\ddot{u}_0(x) = -\sum_{k=1}^{\infty} \frac{\lambda_k(T^{-k}x)}{\eta_k(T^{-k}x)} \cdot \alpha(T^{-k}x) \cdot \sigma(T^{-k}x) \tag{3.10}
\]
where
\[
\alpha(x) = \frac{2(\dot{u}_0(x) - 1)^2}{\eta(x)}
\]
and \( \lambda_k, \eta_k \) are as in equation (2.4).
The relations in (3.4) establish that, for every \( \theta \in [0, 2\pi] \) and every \( x \in X \),
\[
 u_\theta(x) = \frac{a_\theta(T^{-1}x)u_\theta(T^{-1}x) + b_\theta(T^{-1}x)}{c_\theta(T^{-1}x)u_\theta(T^{-1}x) + d_\theta(T^{-1}x)}.
\]
(3.11)

Regarding that from (3.3) we have \( a_\theta(x) = \lambda(x) \cos \theta \), \( b_\theta(x) = -\lambda(x) \sin \theta \), \( c_\theta(x) = \sigma(x) \cos \theta + \eta(x) \sin \theta \) and \( d_\theta(x) = \eta(x) \cos \theta - \sigma(x) \sin \theta \), define
\[
 f_\theta(x) = a_\theta(T^{-1}x)u_\theta(T^{-1}x) + b_\theta(T^{-1}x)
\]
\[
 = \lambda(T^{-1}x) \cos \theta \cdot u_\theta(T^{-1}x) - \lambda(T^{-1}x) \sin \theta \tag{3.12}
\]
and
\[
 g_\theta(x) = c_\theta(T^{-1}x)u_\theta(T^{-1}x) + d_\theta(T^{-1}x)
\]
\[
 = (\sigma(T^{-1}x) \cos \theta + \eta(T^{-1}x) \sin \theta) u_\theta(T^{-1}x) \tag{3.13} 
\]
\[
 + \eta(T^{-1}x) \cos \theta - \sigma(T^{-1}x) \sin \theta
\]

So, the derivatives of both function with respect to \( \theta \) are
\[
f_\theta'(x) = -\lambda(T^{-1}x)u_\theta(T^{-1}x) \sin \theta
\]
\[
 + \lambda(T^{-1}x)u_\theta(T^{-1}x) \cos \theta - \lambda(T^{-1}x) \cos \theta,
\]
(3.14)
and
\[
g_\theta'(x) = -\sigma(T^{-1}x)u_\theta(T^{-1}x) \sin \theta + \eta(T^{-1}x)u_\theta(T^{-1}x) \cos \theta
\]
\[
 + (\sigma(T^{-1}x) \cos \theta + \eta(T^{-1}x) \sin \theta) u_\theta(T^{-1}x) \sin \theta
\]
\[
 - \eta(T^{-1}x) \sin \theta - \sigma(T^{-1}x) \cos \theta.
\]
(3.15)

Since \( u_\theta(x) = f_\theta(x)/g_\theta(x) \), we can take the derivative with respect to \( \theta \) using the expressions (3.12)–(3.15) above and, evolving in \( \theta = 0 \), we obtain
\[
 \dot{u}_\theta(x) = \frac{\lambda(T^{-1}x)}{\eta(T^{-1}x)} (\dot{u}_\theta(T^{-1}x) - 1). \tag{3.16}
\]

Using the recurrence given by the expression (3.16) and taking (2.4) into account, we obtain that for every integer \( n \geq 1 \)
\[
 \dot{u}_\theta(x) = \left( \prod_{j=1}^{n} \frac{\lambda(T^{-j}x)}{\eta(T^{-j}x)} \right) \dot{u}_\theta(T^{-n}x) - \sum_{k=1}^{n} \frac{\lambda_k(T^{-k}x)}{\eta_k(T^{-k}x)} \cdot \dot{u}_\theta(T^{-n}x) 
\]
\[
 = \frac{\lambda_n(T^{-n}x)}{\eta_n(T^{-n}x)} \cdot \dot{u}_\theta(T^{-n}x) - \sum_{k=1}^{n} \frac{\lambda_k(T^{-k}x)}{\eta_k(T^{-k}x)} \cdot \dot{u}_\theta(T^{-n}x).
\]

Let \( 0 < \tau < 1 \) such that \( \lambda(x)/\eta(x) < \tau \) for all \( x \in X \). Since \( u_\theta \) is real analytic, then \( \dot{u}_\theta(x) \) is bounded and we conclude
\[
 \left| \frac{\lambda_n(T^{-n}x)}{\eta_n(T^{-n}x)} \cdot \dot{u}_\theta(T^{-n}x) \right| \leq \tau^n |\dot{u}_\theta(T^{-n}x)| \to 0
\]
when \( n \to \infty \). Therefore,
\[
 \dot{u}_\theta(x) = -\sum_{k=1}^{\infty} \frac{\lambda_k(T^{-k}x)}{\eta_k(T^{-k}x)}.
\]

Arguing in a similar fashion, we obtain that the second derivative of \( u_\theta(x) \) with respect to \( \theta \) evaluated in \( \theta = 0 \) is given by
\[
 \ddot{u}_\theta(x) = \frac{\lambda(T^{-1}x)}{\eta(T^{-1}x)} \left( \ddot{u}_\theta(T^{-1}x) - \frac{2(\dot{u}_\theta(T^{-1}x) - 1)^2}{\eta(T^{-1}x)} \cdot \alpha(T^{-1}x) \right)
\]
\[
 = \frac{\lambda(T^{-1}x)}{\eta(T^{-1}x)} \left( \ddot{u}_\theta(T^{-1}x) - \alpha(T^{-1}x) \cdot \sigma(T^{-1}x) \right). \tag{3.17}
\]
Again, the recurrence in (3.17) allows us to conclude that
\[
\ddot{u}_0(x) = \lambda_n(T^{-n}x) \cdot \eta_n(T^{-n}x) \cdot \ddot{u}_0(T^{-n}x) - \sum_{k=1}^{n} \lambda_k(T^{-k}x) \cdot \eta_k(T^{-k}x) \cdot \alpha(T^{-k}x) \cdot \sigma(T^{-k}x)
\]
\[
= - \sum_{k=1}^{\infty} \lambda_k(T^{-k}x) \cdot \eta_k(T^{-k}x) \cdot \alpha(T^{-k}x) \cdot \sigma(T^{-k}x)
\]
\[\Box\]

Now, we can study the growth and concavity of \(\lambda^+(\theta)\) in a neighbourhood of \(\theta = 0\).

**Lemma 15.** If \(\sigma(x) = 0\) for \(\mu-a.e.\) \(x \in X\) then \(\lambda^+(\theta)\) has a local maximum in \(\theta = 0\).

**Proof.** From (3.7), we have that
\[
\frac{d\lambda^+(0)}{d \theta} = 0
\]
and from (3.8) we have
\[
\frac{d^2\lambda^+(0)}{d \theta^2} = \int \left[ \frac{2\eta(x)\dot{u}_0(Tx)}{\lambda(x)} + 1 \right] d\mu(x).
\]
Note that from (3.9)
\[
-\dot{u}_0(Tx) = \sum_{k=0}^{\infty} \lambda_k(T^{-k}x) = \sum_{k=1}^{\infty} \lambda_k(T^{-k}x) + \frac{\lambda(x)}{\eta(x)},
\]
and that implies that
\[
\frac{2\eta(x)\dot{u}_0(Tx)}{\lambda(x)} > -2.
\]
Then,
\[
\frac{2\eta(x)\dot{u}_0(Tx)}{\lambda(x)} + 1 < 0
\]
and so,
\[
\frac{d^2\lambda^+(0)}{d \theta^2} < 0.
\]
Therefore \(\lambda^+(\theta)\) has a maximum in \(\theta = 0\). \[\Box\]

**Proposition 16.**
\[
\frac{d^2\lambda^+(0)}{d \theta^2} < 0.
\]

**Proof.** Let \((C(X), \| \cdot \|_\infty)\) be the Banach space of all continuous functions \(f : X \to \mathbb{R}\) provided with the supremum norm. Define the linear continuous operator
\[
L(f) = -\sum_{k=1}^{\infty} \left[ \frac{\lambda_k}{\eta_k} \cdot \alpha \cdot f \right] \circ T^{-k+1}.
\]
Let
\[
F_0 = \int \left[ \frac{2\eta(x)\dot{u}_0(Tx)}{\lambda(x)} + 1 \right] d\mu(x).
\]
and $F_1 : C(X) \to \mathbb{R}$ defined by

$$F_1(f) = \int \left[ \frac{f(x)L(f)(x)}{\lambda(x)} + \left( \frac{f(x)\sigma_0(Tx)}{\lambda(x)} \right)^2 \right] d\mu(x).$$

Then, we consider the functional $F : C(X) \to \mathbb{R}$ defined by $F = F_0 + F_1$. It is not difficult to see that $F$ is continuous and that

$$d\lambda(0) + (0) = F(\sigma).$$

In particular, for $\sigma = 0$ we have that $F_1(0) = 0$ and from the proof of Lemma 15 we obtain that $F(0) = F_0 < 0$. Then there exists $r > 0$ such that $F_1(f) < 0$ for all $\|f\|_\infty < r$. Let $t_0 > 0$ and $\sigma_0 \in C(X)$ such that $\|\sigma_0\| < r$ and $t_0\sigma_0 = \sigma$. Then, we have

$$\frac{d\lambda^*(0)}{dt^2} = F(\sigma) = F_0 + F_1(t_0\sigma_0) = F_0 + t_0^2 F_1(\sigma_0) < 0$$

as desired. □

4. Heisenberg nilmanifold

Let $H \equiv \mathbb{R}^3$ be the Heisenberg group of upper triangular $3 \times 3$ matrices with ones in the diagonal and consider $\Phi : H \to H$ be the automorphism defined by $\Phi(x, y) = (Bx, l(x, y))$, where $x = (x_1, x_2)$ and

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } l(x, y) = y + x_1^2 + \frac{1}{2}x_2^2 + x_1x_2. \quad (4.1)$$

Let $N$ be the compact nilmanifold obtained from the quotient of the Heisenberg group $H$ by the lattice $\Gamma = \{(x, y) : x \in \mathbb{Z}^2, y \in \frac{1}{2}\mathbb{Z}\}$. Let $\pi : H \to N$ be the projection and $f : N \to N$ be the induced diffeomorphism from $\Phi$ by

$$f \circ \pi = \pi \circ \Phi.$$

Then $f$ is a strong partially hyperbolic diffeomorphism. We remark that $f$ is ergodic with respect to Lebesgue measure with zero central Lyapunov exponent.

We recall that the group $H$ is a Lie group with the usual product of matrices, then the automorphism $\Phi$ is conjugated via the exponential map (of the Lie algebra) with $D\Phi(\mathbf{0}, \mathbf{0})$.

Therefore, the stable, unstable and central leaves of $\Phi$ at the point $(x, y) \in H$ are explicitly

$$W^s((x, y), \Phi) = \left\{ \left( tv^s_B, \frac{tx_1p_2 + tx_2p_1 + t^2p_1p_2}{2} \right) : t \in \mathbb{R} \right\},$$

$$W^u((x, y), \Phi) = \left\{ \left( tv^u_B, \frac{tx_1q_2 + tx_2q_1 + t^2q_1q_2}{2} \right) : t \in \mathbb{R} \right\}$$

and

$$W^c((x, y), \Phi) = \{(x, y + t) : t \in \mathbb{R} \},$$

where $v^s_B = (p_1, p_2)$ and $v^u_B = (q_1, q_2)$ are the eigenvector associated to the eigenvalues $\lambda^s_B$ and $\lambda^u_B$ respectively. Thus, the invariant spaces for $\Phi$ at $(x, y)$ are given by

$$E^s_\Phi((x, y)) = \left\{ \left( v^s, \frac{tx_1p_2 + tx_2p_1}{2} \right) \right\},$$

$$E^u_\Phi((x, y)) = \left\{ \left( v^u, \frac{tx_1q_2 + tx_2q_1}{2} \right) \right\}.$$
Thus \( N \) base

The sets described above are in fact leaves of invariant foliations and they are projected onto the invariant foliations in \( N \). Observe that \( W^c((x, y), \Phi) \) is projected onto a circle in \( N \) and hence the projection of the central foliation is a foliation by circles. Moreover, these circles are collapsed by the projection \( p_N \) and hence the central foliation is a nontrivial fibration with base \( T^2 \) and fiber \( S^1 \).

Recall that \( E^c_\Phi((x, y)) \) is the space generated by the vector \((v^a, \frac{x_1(y_1 + y_2)}{2})\) and \( E^c_\Phi((x, y)) \) is the space generated by the vector \((0, 1)\). Then, \( E^c_\Phi((x, y)) \) is generated by the vectors \((v^a, 0)\) and \((0, 1)\), so \( E^c_\Phi \) does not depend of the point \((x, y)\). The derivative of \( \Phi \) is given by

\[
D\Phi(x, y) = \begin{pmatrix}
2 & 1 & 0 \\
1 & 1 & 0 \\
2x_1 + x_2 & x_1 + x_2 & 1
\end{pmatrix}.
\]

Evaluating \( D\Phi(x, y) \) in the vectors \((v^a, 0)\) and \((0, 1)\) we obtain

\[
D\Phi(x, y)(v^a, 0) = (\lambda^u_A v^a, A x \cdot v^a) = (\lambda^u_A v^a, x \cdot A v^a) = \lambda^u_A (v^a, 0) + \lambda^u_A x \cdot v^a(0, 1)
\]

and

\[
D\Phi(x, y)(0, 1) = (0, 1).
\]

Then, we obtain \( D\Phi(x, y)|E^c_\Phi \) in terms of the base \([v^a, 0], (0, 1)\),

\[
D\Phi(x, y)|E^c_\Phi = \begin{pmatrix}
\lambda^u_A & 0 \\
\lambda^u_A x \cdot v^a & 1
\end{pmatrix}.
\]  

(4.2)

Clearly \( E^c_\Phi(\pi(x, y)) = D\pi(x, y) E^c_\Phi \). We assert that the sub–bundle \( E^c_\Phi \subset TN \) is trivial. In fact, observe that a fundamental domain for \( N = \mathcal{H}/\Gamma \) is the cube

\[
\mathcal{C} = \{(x, y) \in \mathcal{H} : 0 \leq x_1, x_2 \leq 1, 0 \leq y \leq 1/2\}.
\]

Since the product in \( \mathcal{H} \) can be written by \((x, y) \cdot (a, b) = (x + a, y + b + x_1 a_2)\), then in the box \( \mathcal{C} \), we identify the top face with the bottom face because

\[
(x, 0) \cdot (0, 1/2) = (0, 1/2) \Rightarrow (x, 0) \sim (x, 1/2),
\]

and the front face with the back face because

\[
((0, x_2), y) \cdot (e_2, 0) = ((1, x_2), y) \Rightarrow ((0, x_2), y) \sim ((1, x_2), y).
\]

The group multiplication

\[
(x, y) \cdot (e_2, 0) = ((x_1, x_2 + 1), y + x_1)
\]

tells us to join left and right faces of the cube by the identification

\[
((x_1, 0), y) \sim \left( (x_1, 1), y + x_1 \mod \frac{1}{2} \right).
\]

(4.3)

Thus \( N = \mathcal{C}/\sim \), that is, the nilmanifold is the cube \( \mathcal{C} \) with the previous relations. To see a representation of \( E^c_\Phi \) it is necessary to give relations in the tangents of \( E^c_\Phi \) restricted to the border of the cube.

For top (resp. bottom) faces and front (resp. back) faces, the relation in the vectors is the identity. On the other hand, let \( h \) be the right multiplication of \((x, y)\) by \((e_2, 0)\). Since

\[
Dh(x, y) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix}
\]
then $Dh((x_1, 0), y)(0, 1) = (0, 1)$ and $Dh((x_1, 0), y)(v^w_B, 0) = (v^w_B, q_1)$. It follows from the relation (4.3) and the last equalities the following vector relations:

$$\begin{align*}
(0, 1) &\in T_{((x_1, 0), y)}H \sim (0, 1) \in T_{((x_1, 1), y + x_1 \mod 1/2)}H
\intertext{and}
(v^w_B, 0) &\in T_{((x_1, 0), y)}H \sim (v^w_B, q_1) \in T_{((x_1, 1), y + x_1 \mod 1/2)}H.
\end{align*}$$

Thus $E^c_{f} = \left( \bigsqcup_{v \in \mathbb{P}} E^c_{f, v} \right) / \sim$. To see that $E^c_{f}$ is trivial, observe that the sections $v_1$ and $v_2$ over $E^c_{f}$ defined by

$$v_j((x_1, x_2), y) = (0, 1)$$

and

$$v_2((x_1, x_2), y) = (v^w_B, x_2q_1)$$

are global zero–zero sections that define a base in each fiber of $E^c_{f}$.

Finally, we define the cocycle $F : TN \to TN$ as the cocycle induced by $f$ and its derivative, that is,

$$F((x_1, x_2), y) = (f(x_1, x_2), Df((x_1, x_2))y).$$

and we define the one–parameter family of continuous cocycles $F_\theta : TN \to TN$ defined by

$$F_\theta((x_1, x_2), y) = \begin{cases} Df(x_1, x_2)R_\theta y, & v \in E^{cu}, \\ Df(x_1, x_2)y, & v \in E^{s}. \end{cases}$$

From item (i) of theorem A, there exists an open set $I \subset \mathbb{R}$ such that for every $\theta \in I$, the cocycle $(F_\theta|E^{cu})$ has dominated splitting. Moreover $(F_\theta|E^{cu})$ is partially hyperbolic, and so is the cocycle $F_\theta$ whose splitting is given by $TN = E^{s} \oplus E^{u} \oplus E^{cu}$. From item (ii) of theorem A, reducing the open set $I$ if necessary, the central Lyapunov exponent of $F_\theta$ is positive and then we prove the corollary D.

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