ANALYSIS OF D-BMAP/G/1 QUEUEING SYSTEM UNDER N-POLICY AND ITS COST OPTIMIZATION

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Abstract. This article studies an infinite buffer single server queueing system under N-policy in which customers arrive according to a discrete-time batch Markovian arrival process. The service times of customers are independent and obey a common general discrete distribution. The idle server begins to serve the customers as soon as the queue size becomes at least N and serves the customers until the system becomes empty. We determine the queue length distribution at post-departure epoch with the help of roots of the associated characteristic equation of the vector probability generating function. Using the supplementary variable technique, we develop the system of vector difference equations to derive the queue length distribution at random epoch. An analytically simple and computationally efficient approach is also presented to compute the waiting time distribution in the queue of a randomly selected customer of an arrival batch. We also construct an expected linear cost function to determine the optimal value of N at minimum cost. Some numerical results are demonstrated for different service time distributions through the optimal control parameter to show the key performance measures.

1. Introduction. The different kinds of threshold policies have received great significant attention over the last few decades due to their utility in many practical real life queueing systems. The threshold policies are used to reduce total setup cost and get economic benefit. The N-policy is the most general threshold policy among many well-known threshold policies. According to this policy, the server stops service and remains idle whenever a customer leaves the empty system. The server stays idle until the number of waiting customers reaches a predefined threshold value N. As soon as the number of waiting customers reaches N, the server turns to busy state and serves the customers exhaustively.

Over the last few decades many authors have studied on the variants of N-policy

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queueing model due to its wide applications in the areas of production management in manufacturing industries, service systems, computer and telecommunication systems, wireless sensor network (WSN), and optimal design operating scheme of queues. More details on this scheme of queues, readers are referred to the works of Aksoy and Gupta [1], Buzacott and Shanthikumar [4], Kavusturucu and Gupta [13], Jiang et al. [11] and references therein.

The concept of $N$-policy was initiated with the pioneering work of Yadin and Naor [30], who studied the $M/G/1$ queueing system to reduce the number of switchovers within idle and busy states. Lee et al. [19] concerned $M^{X}/G/1$ queue with $N$-policy and single vacation. From an application perspective, they considered that the raw materials in production system arrive in batches and the machine goes for some additional works (machine repair, maintenance, supplementary jobs, etc.) when there are no raw materials in the production system to process. Lee and Seo [20] investigated the $M/G/1$ queueing system under the $\text{Min}(N,D)$-policy, where the idle server begins to serve the waiting customers if either $N$ customers are in the system or the total service times of the waiting customers exceeds $D$, whichever occurs earlier. Kasahara et al. [12] analyzed the stationary queue length and the actual waiting time distributions of $\text{MAP}/G/1$ queue under $N$-policy with and without vacations, where customers arrive according to the Markovian arrival process (MAP). From an application point of view, they considered that one of the machines in manufacturing plant may be converted as needed from production mode to a repair mode in order to rework all of the defective items exhaustively. Lee et al. [17] investigated $\text{MAP}/G/1$ queueing model under $N$-policy with single and multiple vacations. They derived an explicit expression of matrix decompositions for the queue length distributions at random and departure epochs based on the supplementary variable technique. More works on the variants of $N$-policy queueing model can be found in Reddy et al. [14], Krishnamoorthy and Deepak [15], Sreenivasan et al. [27], and to name a few.

To the best of our knowledge, there are very less works on discrete-time queueing system under $N$-policy compare to its continuous-time counterpart. Discrete-time queues have gained special importance due to their vast applications in modeling slotted digital communication systems and computer networks. The sources of information like audio recording and playback, graphic art, animation, audio and video broadcast are progressively produced, manipulated, preserved, and transmitted in digital form to make it easier and faster to convey information. Lee and Yang [21] carried out discrete-time $\text{Geo}/G/1$ queue with disasters and shown the applications of $N$-policy to the power consumption control in a wireless sensor networks. Wang et al. [29] performed the sensitivity analysis of a cost model for a random $N$-policy of $\text{Geo}/G/1$ queue with repairable breakdown server, where the threshold value of $N$ is redetermined at every regenerative cycle using the probabilistic global search Lausanne algorithm. A few more works on discrete-time queue with different $N$-policies can be found in the reference list of Lee and Yang [21], Wang et al. [29] and Lan and Tang [16].

Modern communication system operates in a packet switching mode and usually there exists a strong considerable correlation between arrival of packets which violates the independence assumption associated with Poisson or Bernoulli process. Thus a smooth traditional Poisson or Bernoulli traffic model is not a boundless compatible assumption for non-renewal arrival process of network traffic in digital
communication systems. The discrete-time batch Markovian arrival process (D-BMAP) proposed by Blondia [3] is a good representation of bursty and correlated traffics arising in digital communication systems and computer networks. Lee et al. [18] demonstrated the factorization property to derive the queue length distribution for the D-BMAP/G/1 queueing system under double threshold policy and setup time. For a detailed study on applications of D-BMAP, readers are referred to Hofkens et al. [9], Zhao et al. [32], Turck et al. [28] and others.

In the analysis of various queues, the probability generating function (p.g.f.) of the queue length is represented as a rational function with a number of unknown constant coefficients. The classical way of finding the unknown constant coefficients is based on the analyticity of the p.g.f. inside the unit disk, see e.g., Choi and Yoon [6], Samanta [26], Oblakova et al. [25], and references therein. Also, Zhao and Campbell [31] pointed out that both the inside and outside roots of the unit disk are playing a significant role in the calculation of system state characteristics. In this paper, we investigate the D-BMAP/G/1 queueing system under N-policy and first derive the vector generating function (v.g.f.) of the steady state probabilities at post-departure epoch. The analyticity implies that each zero inside and on the unit disk of the denominator of the rational function is also a zero of the numerator which yields a system of linear equations for the unknown probability vectors. The remaining probability vectors are obtained by inverting the v.g.f. using the zeros outside the unit circle. The main advantage of roots method in computation of the steady state probability distribution due to their existence of closed-form expression. For this reason, the roots method is often faster in time rather than other methods to obtain the steady state probability distribution, even though size of the matrices gets large. For a comprehensive and systematic treatment of roots method, readers are referred to Zhao and Campbell [31], Gail et al. [8], Janssen and Leeuwaarden [10], and related references therein.

Further, we derive the queue length distribution at random epoch using the supplementary variable technique with remaining service time as the supplementary variable. The queue length distributions at arrival epoch of a batch and intermediate epoch are also obtained by making relations with the queue length distribution at random epoch. Later, we provide an analytically simple approach to obtain the waiting time distribution in the queue measured in slots of a randomly selected customer of a batch. To do this, we first obtain the v.g.f. of the waiting time distribution using the matrix generating function of number of customers arrive during idle period of the system and the service time distribution with remaining service time as the supplementary variable. We then obtain the partial fraction expansion through the calculation of non-zero simple and multiple zeros of the denominator of the v.g.f. of the waiting time distribution. The simple and multiple zeros can be evaluated accurately using one of the several commercially software packages such as MAPLE, MATHEMATICA and MATLAB. Moreover, we demonstrate several numerical outcomes through the optimal control parameter for a variety of service time distributions.

Our proposed N-policy queueing model seems to be applicable in a flexible manufacturing system to improve machine reliability and optimize maintenance operations which transforms raw materials to industrial materials controlled by an automated computer system. In such system industrial material production does not start until at least some specified number of raw materials, say N, coming from different machineries are accumulated during an idle period. As soon as all raw
materials are used for production of industrial materials and no new raw materials are available for production of industrial materials, then the machine becomes idle and a decision is taken for maintenance operations of the machine, and waits for at least \( N \) new raw materials. The process of material processing which consists of few steps such as crushing, roasting, magnetic separation, flotation, and leaching can be represented as the D-BMAP. To get better efficiency in manufacturing industry, the provision of \( N \)-policy in queueing environment provides the proper utilisation of valuable system and economic benefit instead of functioning the automatic production system in specified time intervals. Such a control mechanism is usually efficient where the machine holding cost and setup costs are comparably effective. Buzacott and Shanthikumar [4], and Curry and Feldman [7] provided several applications of flexible manufacturing system in a wide variety of production and manufacturing industries using queueing models.

The remainder of this paper is structured as follows. In Section 2, we give the description of the model and introduce the notations to describe the model parameters. The steady state queue length distributions at various time epochs are obtained in Section 3. Section 4 analyzes the waiting time distribution of a randomly selected customer of an arrival batch. The optimal \( N \)-policy is addressed in Section 5. Computational experiences with a variety of numerical results are discussed in Section 6. Section 7 concludes the paper.

2. Model description. We consider the \( D-BMAP/G/1 \) queueing system under \( N \)-policy, where customers arrive according to the discrete-time batch Markovian arrival process (D-BMAP) which is a tractable class of Markov renewal processes. The D-BMAP is a discrete-time version of the versatile Markovian point process introduced by Neuts [23] and further studied using a transparent notation batch Markovian arrival process (BMAP) by Lucantoni [22]. Many well-known discrete-time arrival processes (Bernoulli arrival process, batch Bernoulli process with correlated batch arrivals, switched batch Bernoulli process (SBBP), Markov modulated Bernoulli process (MMBP), discrete-time PH-renewal process, discrete-time Markovian arrival process (D-MAP), and superposition of D-MAP’s) can be obtained as the special cases of the D-BMAP. It is a powerful arrival process which captures dependent and non-geometrically distributed interarrival times, and correlated batch sizes. The arrival process D-BMAP is characterized by a sequence of \( m \times m \) substochastic matrices \( D_k = [(D_k)_{ij}], k = 0, 1, 2, \ldots \), where the matrix \( D_0 \) corresponds to state transitions with no arrivals during a time slot, and the matrix \( D_k, k \geq 1 \), corresponds to state transitions with a batch arrivals of size \( k \) during a time slot. The \( m \)-state of the D-BMAP is usually referred to as the phase (state) of the underlying Markov chain (UMC) corresponding to the D-BMAP. In most of the practical situations, the number of customers of an arrival batch is finite size. Therefore, we consider that the maximum batch size of the D-BMAP is \( b \). This gives \( D_k = 0 \), for \( k \geq b + 1 \). We define the matrix generating function \( D(z) = \sum_{k=0}^{b} D_k z^k \), for \( |z| \leq 1 \), and an \( m \)-state discrete-time Markov chain with transition probability matrix \( D = \sum_{k=0}^{b} D_k \). Let \( \pi \) be the \( 1 \times m \) stationary probability vector of the UMC, i.e., \( \pi D = \pi \), and \( \pi e = 1 \), where \( e \) denotes a column vector with an appropriate order whose all elements are 1. The average number of customers arrive per slot under D-BMAP is given by \( \lambda^* = \pi \sum_{k=1}^{b} k D_k e \). We can get the above well-known special cases of the D-BMAP by choosing the matrices \( D_k \) appropriately. For a detailed discussion on these special cases, readers are referred to Blondia [3] and
Chakravarthy [5].

The customers are served by single server according to the first-come first-served (FCFS) queueing discipline. The service can start only at slot boundaries and service duration is integral multiple of a slot duration. The service times $S$ are assumed to be independent and identically distributed (i.i.d.) discrete random variables (r.v.’s) with common probability mass function (p.m.f.) $s_k = Pr(S = k)$, $k \geq 1$, and the probability generating function (p.g.f.) $S(z) = \sum_{k=1}^{\infty} s_k z^k$, $|z| \leq 1$.

The mean service time is $E[S] = 1/\mu = \sum_{k=1}^{\infty} k s_k$. The traffic intensity is then given by $\rho = \lambda^* / \mu < 1$. In this $N$-policy queueing system, the server stops service and remains idle whenever a customer leaves the empty system. The server stays idle until the number of waiting customers reaches or exceeds a predefined threshold value $N$ ($N \geq 1$). As soon as the number of waiting customers reaches or exceeds $N$, the server turns to busy state and serves the customers exhaustively. This paper discusses with the late arrival system with delayed access (LAS-DA), that is, batch arrival occurs just before the end of a slot boundary and a departure occurs just after the beginning of a slot boundary. For mathematical clarity, let us assume that the time axis is partitioned into a sequence of equal intervals of unit duration and is marked as $0, 1, 2, \ldots$. To make it more clear, the various time epochs are depicted in Figure 1.

![Figure 1. Various time epochs in LAS-DA.](image)

3. Analysis of the model. In this section, we determine the queue length distributions at different time epochs, viz., post-departure, random, arrival and intermediate epochs. In order to determine these distributions, we define some probability matrices which will be needed in the next.

Let $N(t)$ denote the number of customers arrived in the first $t$ time slots and $J(t)$ the state of the underlying Markov chain (called arrival phase) after the same amount of time. Let $P(n,t)$, $n \geq 0, t \geq 0$, be the matrix of order $m \times m$ whose $(i,j)$th element $P_{ij}(n,t)$ is defined by

$$P_{ij}(n,t) = Pr\{N(t) = n, J(t) = j|N(0) = 0, J(0) = i\}, 1 \leq i, j \leq m, n \geq 0,$$

where $P_{ij}(n,t)$ is the conditional probability that the Markov chain $D$ is in phase $j$ at time $t$ and $n$ customers are arrived in $(0,t]$, given that the Markov chain starts in phase $i$ at time $t = 0$. Therefore, the matrices $P(n,t)$, $n \geq 0, t \geq 0$, satisfy the
forward Chapman-Kolmogorov matrix difference equations

\[
P(n,t) = \begin{cases} 
I_m, & n = t = 0, \\
\sum_{r=0}^{n} D_r P(n-r,t-1), & n \geq 0, \quad t \geq 1,
\end{cases}
\]  

(1)

where \(P(n,0) = 0, n \geq 1\), and \(I_m\) is the \(m\)-dimensional identity matrix.

Let \(A_n, n \geq 0\), denote the square matrix of order \(m\) whose \((i,j)\)th element represents the conditional probability that given a departure which leaves at least one customer in the queue with the batch arrival process in phase \(i\), there are \(n\) customers arrive during the service time of a customer with the batch arrival process in phase \(j\) at the next departure. Then

\[
A_n = \sum_{k=1}^{\infty} s_k P(n,k), \quad n \geq 0.
\]

Further, let \(B_n, n \geq N - 1\), denote the square matrix of order \(m\) whose \((i,j)\)th element represents the conditional probability that given a departure which leaves the system empty with the batch arrival process in phase \(i\), there are \((n+1)\) customers arrive so that the queue length becomes at least \((N - 1)\) with the batch arrival process in phase \(j\) at just after the next departure. Then, \(B_n\) can be written as

\[
B_n = \sum_{r=0}^{N-1} \sum_{v=0}^{n-N+1} D_{n-r-v+1} s_k P(v,k-u),
\]

\[
= \sum_{r=0}^{N-1} \sum_{u=1}^{\infty} P(r,u-1) \sum_{v=0}^{n-N+1} D_{n-r-v+1} s_k P(v,k),
\]

\[
= \sum_{r=0}^{N-1} \Xi(r) \sum_{v=0}^{n-N+1} D_{n-r-v+1} A_v, \quad n \geq N - 1,
\]

where

\[
\Xi(r) = \sum_{u=1}^{\infty} P(r,u-1), \quad r \geq 0.
\]

The simplified expression for \(\Xi(r), r \geq 0\), can be easily obtained as follows. Using (1) in (2), for \(r = 0\), we have

\[
\Xi(0) = \sum_{k=1}^{\infty} D_0^{k-1} = (I_m - D_0)^{-1}.
\]

Similarly, using (1) in (2), for \(r \geq 1\), we have

\[
\Xi(r) = \sum_{k=2}^{\infty} \sum_{i=0}^{r} D_i P(r-i,k-2),
\]

\[
= \sum_{i=0}^{\infty} D_i \sum_{k=2}^{\infty} P(r-i,k-2) + D_r \sum_{k=2}^{\infty} P(0,k-2),
\]

\[
= \sum_{i=0}^{\infty} D_i \Xi(r-i) + D_r (I_m - D_0)^{-1}.
\]
which gives

$$
\Xi(r) = (I_m - D_0)^{-1} \sum_{i=1}^{r} D_i \Xi(r-i), \quad r \geq 1.
$$

### 3.1. Queue length distribution at post-departure epoch.

In this subsection, we carry out the queue length distribution at departure epoch. Let $\tau_k$, $k = 0, 1, \ldots$, be the time instant until the $k$th departure with $\tau_0 = 0$, and also let $\tau_k^+$ denote the departure epoch of $k$th customer, i.e., the time epoch just after a departure instant $\tau_k$. Then the state of the system at $\tau_k^+$ defined by $\Lambda_k = \{N_{\tau_k^+}, J_{\tau_k^+}\}$ is a Markov chain, where $N_{\tau_k^+}$ and $J_{\tau_k^+}$ denote, respectively, the number of customers in the queue and the phase of the UMC immediately after the departure. Now viewing the state of the system at two consecutive departure epochs, we have an irreducible and aperiodic discrete-time Markov chain $\{\Lambda_k, k \geq 0\}$ on the state space $\{(n,j) : n \geq 0, 1 \leq j \leq m\}$. Then the one-step transition probability matrix (TPM) $\mathcal{P}$ of the $M/G/1$ type Markov chain can be written in the block matrix form as

$$
\mathcal{P} = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & B_{N-1} & B_N & \cdots \\
A_0 & A_1 & A_2 & \cdots & A_{N-2} & A_{N-1} & A_N & \cdots \\
0 & A_0 & A_1 & \cdots & A_{N-3} & A_{N-2} & A_{N-1} & \cdots \\
0 & 0 & A_0 & \cdots & A_{N-4} & A_{N-3} & A_{N-2} & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & \cdots & A_0 & A_1 & A_2 & \cdots \\
0 & 0 & 0 & \cdots & 0 & A_0 & A_1 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
$$

Let $\Pi^+ = [\pi^+(0), \pi^+(1), \pi^+(2), \ldots]$ denote the stationary probability vector of the TPM $\mathcal{P}$, where $\pi^+(n) = [\pi_1^+(n), \ldots, \pi_m^+(n)]$, $n \geq 0$. The $j$th component $\pi_j^+(n)$ of $\pi^+(n)$ represents the steady state probability that $n$ customers are in the queue immediately after departure epoch and batch arrival process in phase $j$. In order to obtain $\Pi^+$, we write $\Pi^+ = \Pi^+ \mathcal{P}$ in the form of the system of vector difference equations as

$$
\pi^+(n) = \sum_{i=0}^{n} \pi^+(i+1) A_{n-i}, \quad 0 \leq n \leq N - 2,
$$

$$
\pi^+(n) = \pi^+(0) B_n + \sum_{i=0}^{n} \pi^+(i+1) A_{n-i}, \quad n \geq N - 1.
$$

Multiplying (4) and (5) by an appropriate power of $z$, using $\Pi^+(z) = \sum_{n=0}^{\infty} \pi^+(n) z^n$, $A(z) = \sum_{n=0}^{\infty} A_n z^n$ and after simplification, we obtain

$$
\Pi^+(z) = \pi^+(0) \left[ \sum_{r=0}^{N-1} \Xi(r) \sum_{k=N-r}^{b} D_k z^{k+r} - I_m \right] \left[ A(z) \det[I_m - A(z)] \right]^{-1}. \tag{6}
$$

We rewrite the right hand side of $\Pi^+(z)$ in (6) as

$$
\Pi^+(z) = \begin{bmatrix}
G_1(z) & G_2(z) & \cdots & G_m(z)
\end{bmatrix},
$$

\begin{align*}
G_1(z) &= \sum_{n=0}^{\infty} \pi^+(n) G(n), \\
G_2(z) &= \sum_{n=0}^{\infty} \pi^+(n) G(n), \\
&\vdots \\
G_m(z) &= \sum_{n=0}^{\infty} \pi^+(n) G(n),
\end{align*}

where $G(n)$ is the generating function of the batch arrival process. In the next section, we shall consider the analysis of $D$-BMAP/G/1 queueing system under $N$-policy.
where \( G(z) = \det[zI_m - A(z)] \) and \( G_j(z) \) is the \( j \)th component of the vector
\[
\pi^+(0) \left[ \sum_{r=0}^{N-1} \Xi(r) \sum_{k=N-r}^{b} D_k z^{k+r} - I_m \right] A(z) \text{adj}[zI_m - A(z)].
\]

To evaluate the vectors \( \pi^+(n), n \geq 0 \), from (7), we apply the method of roots which involves the determination of roots of the so-called characteristic equation \( G(z) = 0 \). For this purpose, we have to determine the roots of \( G(z) = 0 \) whose absolute value is less than or equal to one. Using the same interpretation given in Gail et al. [8], it can be shown that if \( D \) is irreducible and \( \rho < 1 \), then \( G(z) = 0 \) has exactly \( (m-1) \) roots in \(|z| < 1 \) and one root at \( z = 1 \). Further, we assume that \( G(z) = 0 \) has \( \psi \) roots (which depends on the arrival batch size and the service time distribution) whose absolute value is greater than one. Let \( \gamma_1, \gamma_2, \ldots, \gamma_{m-1} \) are the roots of \( G(z) = 0 \) whose absolute value is less than one, and \( \vartheta_1, \vartheta_2, \ldots, \vartheta_\psi \) are the roots of \( G(z) = 0 \) whose absolute value is greater than one. Our first objective is to compute the unknown vector \( \pi^+(0) \) from (7) accurately. Now, as each component \( \pi_j^+(z) \) of \( \Pi^*(z) \) is convergent for \(|z| \leq 1 \) and \( \gamma_1, \gamma_2, \ldots, \gamma_{m-1} \) are the roots of \( G(z) = 0 \) whose absolute value is less than one, they must be the zeros of numerator of each component of (7) and therefore we have
\[
G_j(\gamma_k) = 0, \quad k = 1, 2, \ldots, m-1,
\]
and using the normalization condition \( \Pi^*(1)e = 1 \), we have
\[
\lim_{z \to 1} \sum_{j=1}^{m} G_j(z) = \sum_{j=1}^{m} G_j^{(1)}(1) = 1,
\]
where \( V_j^{(k)}(\sigma) \) and \( V_j^{(k)}(\sigma) \) represent the \( k \)th order derivatives of functions \( V_j(z) \) and \( V_j(z) \), respectively, at \( z = \sigma \).

Equations (8) and (9) give \( m \) linearly independent simultaneous equations in \( m \) unknowns, \( \pi_j^+(0), 1 \leq j \leq m \). Solving these \( m \) equations, we get the \( m \) unknowns \( \pi_j^+(0), 1 \leq j \leq m \). This determines the unknown vector \( \pi^+(0) \).

Now, after substituting the value of \( \pi^+(0) \) in (7) and let \( \overline{\pi}^+ = \Pi^*(1) \), we have
\[
\pi^+ = \begin{bmatrix} G^{(1)}_1(1) & G^{(1)}_2(1) & \cdots & G^{(1)}_m(1) \\ G^{(1)}_1(1) & G^{(1)}_2(1) & \cdots & G^{(1)}_m(1) \end{bmatrix}.
\]

Having found \( \pi^+(0) \), we now concentrate our focus on calculating the remaining state probability vectors \( \pi^+(n), n \geq 1 \). After substituting the value of \( \pi^+(0) \) in (7), we have completely known polynomials both in the numerator and the denominator of each component \( \pi_j^+(z) \) of \( \Pi^*(z) \). Let the degrees of the polynomials \( G_j(z) \) and \( G(z) \) are \( \eta_j \) \((j = 1, 2, \ldots, m)\) and \( \eta_d \), respectively. Now, applying the partial fraction method on the \( j \)th component \( \pi_j^+(z) \) of \( \Pi^*(z) \), we have
\[
\pi_j^+(z) = \sum_{k=0}^{\eta_j-\eta_d} T_{k,j} z^k + \sum_{k=1}^{\psi} \frac{R_{k,j}}{\vartheta_k - z}, \quad 1 \leq j \leq m,
\]
where constant coefficients of the above expression can be written explicitly as

\[ R_{k,j} = \frac{-G_j(\vartheta_k)}{G^{(3)}(\vartheta_k)}, \quad k = 1, 2, \ldots, \psi, \]

\[ T_{k,j} = \frac{1}{k!} \Gamma^{(k)}(0) - \sum_{i=1}^{\psi} \frac{R_{i,j}}{\vartheta_{k+1}}, \quad 0 \leq k \leq \eta_j - \eta_d, \]

with the \( k \)th order derivative \( \Gamma^{(k)}_{j,H}(0) \) of function \( \Gamma_{j,H}(z) = \frac{H_j(z)}{H(0)} \), at \( z = 0 \), is given by

\[ \Gamma^{(0)}_{j,H}(0) = \frac{H_j(0)}{H(0)}, \]

\[ \Gamma^{(k)}_{j,H}(0) = \frac{H_j^{(k)}(0) - \sum_{r=0}^{k-1} \binom{k}{r} \Gamma^{(r)}_{j,H}(0) H^{(k-r)}(0)}{H(0)}, \quad k = 1, 2, \ldots, \eta_j - \eta_d. \]

Now, collecting the coefficients of \( z^n \) from both the sides of (11), we have

\[ \pi_j^+(n) = T_{n,j} + \sum_{k=1}^{\psi} \frac{R_{k,j}}{\vartheta_{k+1}}, \quad 1 \leq j \leq m, \quad 0 \leq n \leq \eta_j - \eta_d, \]

\[ \pi_j^+(n) = \sum_{k=1}^{\psi} \frac{R_{k,j}}{\vartheta_{k+1}}, \quad 1 \leq j \leq m, \quad n \geq \eta_j - \eta_d + 1. \]

Thus, in vector form, we obtain

\[ \pi^+(n) = \begin{bmatrix} T_{1,n} + \sum_{k=1}^{\psi} \frac{R_{k,1}}{\vartheta_{k+1}}, & \ldots, & T_{n,j} + \sum_{k=1}^{\psi} \frac{R_{k,j}}{\vartheta_{k+1}}, & \ldots, & T_{n,m} + \sum_{k=1}^{\psi} \frac{R_{k,m}}{\vartheta_{k+1}} \end{bmatrix}, \quad 0 \leq n \leq \eta_j - \eta_d, \]

\[ \pi^+(n) = \begin{bmatrix} \sum_{k=1}^{\psi} \frac{R_{k,1}}{\vartheta_{k+1}}, & \ldots, & \sum_{k=1}^{\psi} \frac{R_{k,j}}{\vartheta_{k+1}}, & \ldots, & \sum_{k=1}^{\psi} \frac{R_{k,m}}{\vartheta_{k+1}} \end{bmatrix}, \quad n \geq \eta_j - \eta_d + 1. \]

This completes the evaluation procedure of steady state probability distribution at post-departure epoch.

3.2. Queue length distribution at random epoch. To get the queue length distribution at random epoch, we first obtain the basic difference equations using the remaining service time as the supplementary variable. Let \( \mathcal{N}_r \) denote the number of customers in the queue excluding the one in service (if any) at time \( t^* \), and \( \mathcal{J}_r \) be the state of the underlying Markov chain of the D-BMAP at time \( t^* \) (see Figure 1). Let \( \mathcal{U}_r \) be the remaining service time of a customer in service (if any) at time \( t^* \). Let \( \Delta_r \) be the state of the server at time \( t^* \), i.e., \( \Delta_r = 1 \) or 0 corresponding to whether the server being busy, or idle, respectively.

In the steady state, let us define their joint probabilities, for \( 1 \leq i \leq m \), as

\[ \omega_i(n) = \lim_{t^* \to \infty} Pr[\mathcal{N}_r = n, \mathcal{J}_r = i, \Delta_r = 0], \quad 0 \leq n \leq N - 1, \]

\[ \pi_i(n, u) = \lim_{t^* \to \infty} Pr[\mathcal{N}_r = n, \mathcal{J}_r = i, \mathcal{U}_r = u, \Delta_r = 1], \quad n \geq 0, u \geq 0. \]

Further, let \( \omega(n) \) and \( \pi(n, u) \) be the row vectors of order \( m \) whose \( i \)th components are \( \omega_i(n) \) and \( \pi_i(n, u) \), respectively. Now, relating the state of the system at two
consecutive random epochs $t^*$ and $t^* + 1$, (using the supplementary variable technique and considering various possible phase transitions), we set up the steady state vector difference equations, for $u \geq 1$, as:

\[
\omega(0) = \omega(0)D_0 + \pi(0, 0)D_0, \quad (14)
\]

\[
\omega(n) = \sum_{k=0}^{n} \omega(k)D_{n-k}, \quad 1 \leq n \leq N - 1, \quad (15)
\]

\[
\pi(n, u - 1) = \sum_{i=0}^{n} \pi(i, u)D_{n-i} + \sum_{i=0}^{n+1} \pi(i, 0)D_{n+1-i}S_u, \quad 0 \leq n \leq N - 2, \quad (16)
\]

\[
\pi(n, u - 1) = \sum_{i=0}^{n} \pi(i, u)D_{n-i} + \sum_{i=0}^{n+1} \pi(i, 0)D_{n+1-i}S_u + \sum_{i=0}^{N-1} \omega(i)D_{n+1-i}S_u, \quad n \geq N - 1. \quad (17)
\]

Define the following vector generating function as

\[
\pi^*(n, z) = \sum_{u=0}^{\infty} \pi(n, u)z^u, \quad |z| \leq 1, \quad n \geq 0,
\]

with $\pi(n) = \pi^*(n, 1)$, where $\pi(n)$ is a row vector of order $m$ whose $i$th component $\pi_i(n)$ denotes the probability of $n$ customers in the queue excluding the one in service and the batch arrival process in phase $i$, when the server being busy at random epoch. 

Multiplying both the sides of (16) and (17) by $z^u$ and summing over $u$ from 1 to $\infty$, we have

\[
z\pi^*(n, z) = \sum_{i=0}^{n} \pi^*(i, z)D_{n-i} + \sum_{i=0}^{n+1} \pi(i, 0)D_{n+1-i}S(z) - \sum_{i=0}^{n} \pi(i, 0)D_{n-i}, \quad 0 \leq n \leq N - 2, \quad (18)
\]

\[
z\pi^*(n, z) = \sum_{i=0}^{n} \pi^*(i, z)D_{n-i} + \sum_{i=0}^{n+1} \pi(i, 0)D_{n+1-i}S(z) - \sum_{i=0}^{n} \pi(i, 0)D_{n-i}
+ \sum_{i=0}^{N-1} \omega(i)D_{n+1-i}S(z), \quad n \geq N - 1. \quad (19)
\]

In order to determine the steady state queue length distribution at random epoch, we make relationship between distributions of the number of customers in the queue immediately after departure and random epochs. Thus, the queue length distribution at random epoch can be obtained as follows.

Post-multiplying (14) and (15) by $e$, and adding them, we obtain

\[
\pi(0, 0)D_0e = \sum_{n=0}^{N-1} \omega(n)e - \sum_{n=0}^{N-1} \omega(n) \sum_{k=0}^{N-1-n} D_k e. \quad (20)
\]

Post-multiplying (18) and (19) by $e$, then adding them and using (20), we obtain

\[
\sum_{n=0}^{\infty} \pi^*(n, z)e = \left( \frac{S(z) - 1}{z - 1} \right) \sum_{n=0}^{\infty} \pi(n, 0)e. \quad (21)
\]
Taking the limit as \( z \to 1 \) in (21) and using the normalization condition 
\[
\sum_{n=0}^{N-1} \omega(n)e + \sum_{n=0}^{\infty} \pi(n)e = 1, \]
we obtain 
\[
1 - \sum_{n=0}^{N-1} \omega(n)e \quad E[S] = \sum_{n=0}^{\infty} \pi(n, 0)e. \quad (22)
\]

Now, we develop relationship between \( \pi^+(n) \) and \( \pi(k, 0) \), \( 0 \leq k \leq n, n \geq 0 \), as follows. Applying the conditional probability argument, we obtain 
\[
\pi^+_j(n) = \lim_{t^* \to \infty} \frac{\sum_{k=0}^{n} \sum_{i=1}^{m} \Pr[N_{i,t^*} = k, J_{i,t^*} = i, \xi_{i,t^*} = 1 | U_{i,t^*} = 0](D_{n-k})_{ij}}{\sum_{k=0}^{n} \pi(k, 0)(D_{n-k})_{ij}}, \quad 1 \leq j \leq m, n \geq 0.
\]

In vector and matrix notations, we write the above as 
\[
\pi^+(n) = E^* \sum_{k=0}^{n} \pi(k, 0)D_{n-k}, \quad n \geq 0, \quad (23)
\]
where \( E^* = \frac{1}{\sum_{n=0}^{\infty} \pi(n, 0)e} \) represents the mean inter-departure time, and hence \( \frac{1}{E^*} \) represents the average number of departures per slot.

Using (23) in (14), and then substitute the values in (15) repeatedly, we obtain 
\[
\omega(n) = \frac{1}{E^*} \pi^+(0) \Xi(n), \quad 0 \leq n \leq N - 1, \quad (24)
\]
where the expression for \( E^* \) can be obtained as follows.

Post-multiplying (24) by \( e \), and adding them, we have 
\[
\sum_{n=0}^{N-1} \omega(n)e = \frac{1}{E^*} \pi^+(0) \sum_{n=0}^{N-1} \Xi(n)e. \quad (25)
\]

Using (22) in (25), after simplification, we obtain 
\[
E^* = E(S) + \pi^+(0) \sum_{n=0}^{N-1} \Xi(n)e.
\]

Now, setting \( z = 1 \) in (18) and (19), using (23), we obtain 
\[
\pi(0) = \frac{1}{E^*} \left[ \pi^+(1) - \pi^+(0) \right] (I_m - D_0)^{-1},
\]
\[
\pi(n) = \sum_{k=0}^{n-1} \pi(k)D_{n-k} + \frac{1}{E^*} \left[ \pi^+(n + 1) - \pi^+(n) \right] (I_m - D_0)^{-1}, \quad 1 \leq n \leq N - 2,
\]
\[
\pi(n) = \sum_{k=0}^{n-1} \pi(k)D_{n-k} + \frac{1}{E^*} \left[ \pi^+(n + 1) - \pi^+(n) \right] + \sum_{k=0}^{n-1} \omega(k)D_{n+1-k} \quad (I_m - D_0)^{-1}
\]
\[n \geq N - 1.
\]
The average number of customers in the queue excluding the one in service (if any) at random epoch is given by \( L_q = \sum_{n=1}^{N-1} n\omega(n)e + \sum_{n=1}^{\infty} n\pi(n)e \). From the Little’s law, we also deduce the average waiting time in the queue as \( W_q = \frac{L_q}{\lambda^*} \).

3.3. Queue length distribution at arrival epoch. Here, we deduce the queue length distribution just before an arrival epoch of a batch. Let us define the row vectors \( \pi^-(n), n \geq 0, \) and \( \omega^-(n), 0 \leq n \leq N - 1, \) each of order \( n \), and \( \omega^-(n), 0 \leq n \leq N - 1, \) each of order \( m \), represent the arrival epoch probability vectors that batch arrival finds \( n \) customers in the queue with the server being busy and idle, respectively. Thus, we have

\[
\omega^-(n) = \frac{1}{\lambda_g} \omega(n) \sum_{k=1}^{b} D_k, \quad 0 \leq n \leq N - 1,
\]

\[
\pi^-(n) = \frac{1}{\lambda_g} \pi(n) \sum_{k=1}^{b} D_k, \quad n \geq 0,
\]

where \( \lambda_g = \pi \sum_{k=1}^{b} D_k e \) is the average number of batch arrivals per slot.

3.4. Queue length distribution at intermediate epoch. We now derive explicit expression for the steady state queue length distribution at intermediate epoch. Let us define the row vectors \( \omega^*(n), 0 \leq n \leq N - 1, \) and \( \pi^*(n), n \geq 0, \) each of order \( m \) represent the intermediate epoch probability vectors that there are \( n \) customers in the queue with the server being idle and busy, respectively. Since an intermediate epoch \((t)\) takes place after a batch arrival and before a departure epoch (see Figure 1), the probability vectors \( \omega^*(n) \) and \( \pi^*(n) \) can be effortlessly derived by making a relationship between a random epoch \((t^*)\) and an intermediate epoch \((t)\) with a possibility of a batch arrival or no arrivals. Thus, we have

\[
\omega^*(n) = \sum_{r=0}^{n} \omega(r)D_{n-r}, \quad 0 \leq n \leq N - 1,
\]

\[
\pi^*(n) = \sum_{r=0}^{n} \pi(r)D_{n-r}, \quad 0 \leq n \leq N - 2,
\]

\[
\pi^*(n) = \sum_{r=0}^{n} \pi(r)D_{n-r} + \sum_{r=0}^{N-1} \omega(r)D_{n+1-r}, \quad n \geq N - 1.
\]

4. Waiting time distribution. In this section, we obtain the waiting time distribution in the queue measured in slots of a randomly selected customer of an arrival batch under the FCFS queueing discipline. Let \( w(u) = [w_1(u), . . . , w_j(u), . . . , w_m(u)], u \geq 0, \) denote the row vector of order \( m \), where \( w_j(u) \) represents the stationary joint probability that the waiting time in the queue of a randomly selected customer in his batch is \( u \) time slots and the batch arrival process being in phase \( j \). Further, let \( W_q = \sum_{k=1}^{\infty} kw(k)e \) denote the average waiting time in the queue of a randomly selected customer of an arrival batch.

Let \( W_1(z) \) be the v.g.f. of the waiting time distribution of a randomly selected customer of an arrival batch of size at most \( (N - 1 - n) \) which arrives during an idle period of the system with \( n \) customers are already present in the queue. In this case, idle period is not terminated. The random customer of an arrival batch must wait until the number of customers in the queue becomes at least \( N \) and the
service times of all waiting customers in the queue including the customers in the same batch waiting in front of him. Therefore, $W_1(z)$ is given by

$$W_1(z) = \sum_{n=0}^{N-2} \omega(n) \sum_{k=1}^{N-1-n} \sum_{i=0}^{N-1-k-n} \frac{D_k}{\lambda^*} \sum_{j=N-n-k-i}^b \mathbf{P}^*(i, z) \sum_{r=1}^k D_r z^r [S(z)]^{n+r-1},$$

where $\mathbf{P}^*(n, z) = \sum_{k=0}^\infty \mathbf{P}(n, k) z^k$, $|z| \leq 1$, $n \geq 0$.

Now, multiplying (1) by $z^t$, for fixed $n$, $n \geq 0$, and summing over $t$ from 0 to $\infty$, we obtain

$$\mathbf{P}^*(0, z) = (\mathbf{I}_m - D_0 z)^{-1},$$
$$\mathbf{P}^*(n, z) = (\mathbf{I}_m - D_0 z)^{-1} \sum_{r=1}^n D_r z^r \mathbf{P}^*(n-r, z), \quad n \geq 1.$$

Let $W_2(z)$ be the v.g.f. of the waiting time distribution of a randomly selected customer of an arrival batch of size at least $(N - n)$ which arrives during an idle period of the system with $n$ customers are already present in the queue. In this case, idle period ends as at least $N$ customers are in the queue. Therefore, $W_2(z)$ is given by

$$W_2(z) = \sum_{n=0}^{N-1} \omega(n) \sum_{k=N-n}^b \frac{D_k}{\lambda^*} \sum_{r=1}^k [S(z)]^{n+r-1}.$$

Let $W_3(z)$ be the v.g.f. of the waiting time distribution of a randomly selected customer of an arrival batch which arrives during a busy period of the system. For this purpose, we need to take into account the remaining service time of the customer in service during busy period.

Define the following vector generating functions as

$$\Phi^*(\nu, z) = \sum_{k=0}^\infty \pi^*(k, z) \nu^k, \quad \text{and} \quad \Omega^*(\nu, 0) = \sum_{k=0}^\infty \pi(k, 0) \nu^k, \quad \text{for} \quad |\nu| \leq 1.$$

Multiplying $\nu^n$ on both the sides of (18) and (19), and then summing them over $n$ from 0 to $\infty$, we obtain

$$\Phi^*(\nu, z) \left[ z\mathbf{I}_m - \mathbf{D}(\nu) \right] = \Omega^*(\nu, 0) \mathbf{D}(\nu) \left[ \frac{1}{\nu} S(z) - 1 \right] + \sum_{k=0}^{N-1} \omega(k) \sum_{n=N-1-k}^{b+k-1} \mathbf{D}_{n+1-k} [S(z)]^{n+1} - \frac{1}{\nu} \pi(0, 0) \mathbf{D}_0 \right] S(z). (26)$$

Substitute $\nu = S(z)$ in (26) and using (23), we have

$$\Phi^*(S(z), z) = \sum_{k=0}^{N-1} \omega(k) \sum_{n=N-1-k}^{b+k-1} \mathbf{D}_{n+1-k} [S(z)]^{n+1} - \frac{1}{E^*} \pi^+(0) \right] [z\mathbf{I}_m - \mathbf{D}(S(z))]^{-1}. (27)$$

Therefore, $W_3(z)$ is given by

$$W_3(z) = \sum_{n=0}^{\infty} \pi^*(n, z) \sum_{k=1}^b \sum_{r=1}^k [S(z)]^{n+r-1},$$
$$= \frac{\Phi^*(S(z), z) \mathbf{D} - \mathbf{D}(S(z))}{\lambda^*(1 - S(z))}. (28)$$
Using (27) in (28), we obtain
\[
W_3(z) = \sum_{k=0}^{N-1} \omega(k) \sum_{n=0}^{N-1} D_{n+1-k}[S(z)]^{n+1} - \frac{1}{E^*} \pi^+(0) \left[ zI_m - D(S(z)) \right]^{-1} \left[ D - D(S(z)) \right].
\]

Let \( W(z) = [W_1(z), \ldots, W_j(z), \ldots, W_m(z)] \) denote the v.g.f. of the waiting time distribution of a randomly selected customer of an arrival batch. Then \( W(z) = W_1(z) + W_2(z) + W_3(z). \) Therefore, after little algebraic manipulation, we obtain
\[
W(z) = \sum_{n=0}^{N-2} \omega(n) \sum_{k=1}^{N-1-n} D_k \sum_{i=0}^{N-1-k-n} P^*(i, z) \sum_{j=0}^{b} D_j z \left( \frac{(S(z))^n - (S(z))^{n+k}}{\lambda^*(1 - S(z))} \right)
\]
\[
\times \left[ zI_m - D(S(z)) \right]^{-1} \left[ D - D(S(z)) \right].
\]

Now our objective is to extract the p.m.f. \( w(u), u \geq 0, \) from the v.g.f. \( W(z) \) given in (29) using the partial fraction expansion. Each component \( W_j(z) \) of \( W(z) \) is a rational function with completely known polynomials both in the numerator and the denominator. As \( W_j(z) \) is convergent for \( |z| \leq 1, \) the zeros of the denominator of \( W_j(z) \) whose absolute value is less than or equal to one must be the zeros of the numerator of \( W_j(z) \) and they are automatically cancelled out from it. Therefore in making partial fraction expansion, these zeros do not play any further role. Thus we have concentrated only those zeros which are lie outside the unit circle. The number of zeros of the denominator of \( W_j(z) \) outside the unit circle depends on the arrival batch size of D-BMAP and the service time distribution function. Let us assume that the p.g.f. \( S(z) \) can be expressed in rational form as \( S(z) = \frac{P(z)}{Q(z)} \), where both numerator and denominator are polynomials in \( z \) of finite degrees, say \( Y_1 \) and \( Y_2 \), respectively. Now, we need to have the knowledge of the roots of \( S(z) = 1, \) \( Q(z) = 0, \) \( \det[I_m - D_0z] = 0, \) and \( \det[zI_m - D(S(z))] = 0 \) whose absolute values are greater than one. Since \( S(z) \) is a probability generating function, therefore \( 1 - S(z) = 0 \) has exactly one root at \( z = 1 \) and no other roots inside and on the unit circle. This implies that \( Q(z) - P(z) = 0 \) has \( \max(Y_1, Y_2) - 1 \) outside roots and we call these roots as \( \delta_k, k = 1, 2, \ldots, \max(Y_1, Y_2) - 1. \) By the same argument, \( Q(z) = 0 \) has \( Y_2 \) outside roots and therefore, \( |Q(z)|^{b+N-2} = 0 \) has \( Y_2 \) outside roots, say \( \xi_k, k = 1, 2, \ldots, Y_2 \) with each of multiplicity \( b+N-2 \). Using Rouche’s theorem, it can be shown that \( \det[I_m - D_0z] = 0 \) has \( m \) roots in \( |z| > 1 \) and no roots in \( |z| \leq 1. \) We call these \( m \) outside roots of \( \det[I_m - D_0z]^{N-1} = 0 \) as \( \beta_i, i = 1, 2, \ldots, m \) with each of multiplicity \( N - 1. \) It can be shown with the argument given in Gail et al. [8] that when \( D \) is irreducible and \( \rho < 1, \) then \( \det[zI_m - D(S(z))] = 0 \) has exactly \( (m - 1) \) roots in \( |z| < 1 \) and one zero at \( z = 1. \) Let \( \det[zI_m - D(S(z))] = 0 \) has \( Y_3 \) roots with each of absolute value is greater than one, viz., \( \alpha_i, i = 1, 2, \ldots, Y_3. \) In order to efficiently derive \( w(u), u \geq 0, \) from (29), we rewrite \( W_j(z) \) in its rational form as
\[
W_j(z) = \frac{F_j(z)}{F(z)}, \quad j = 1, 2, \ldots, m,
\]
where

\[ F(z) = \prod_{r=1}^{\max(Y_1, Y_2)-1} (z - \delta_r) \prod_{r=1}^{Y_2} (z - \xi_r)^{b+N-2} \prod_{r=1}^{m} (z - \beta_r)^{N-1} Y_2 \prod_{r=1}^{Y_2} (z - \alpha_r), \]

and \( F_j(z) \) is the \( j \)th component of the vector \( W(z)F(z) \). Let the degrees of \( F_j(z) \) and \( F(z) \) are \( M_j \) \((1 \leq j \leq m)\) and \( \Upsilon, \) respectively. On employing the partial fraction expansion, one can uniquely write \( W_j(z), 1 \leq j \leq m, \) as

\[
W_j(z) = \begin{cases}
M_j - \Upsilon \\
\sum_{i=0}^{M_j - \Upsilon} K_{ij} z^i + \sum_{k=1}^{\max(Y_1, Y_2)-1} \frac{C_{kj}}{z - \alpha_k} + \sum_{k=1}^{\max(Y_1, Y_2)-1} \frac{Z_{kj}}{z - \delta_k} \\
+ \sum_{i=1}^{N-1} \sum_{k=1}^{m} U_{ikj} \frac{(z - \beta_k)^i}{z - \xi_k} + \sum_{i=1}^{N-1} \sum_{k=1}^{m} U_{ikj} \frac{(z - \xi_k)^i}{z - \alpha_k}, & \text{for } M_j \geq \Upsilon,
\end{cases}
\]

\( \text{(30)} \)

The constant coefficients of the above expression can be obtained as

\[
C_{kj} = \frac{F_j(\alpha_k)}{\prod_{r=1}^{\max(Y_1, Y_2)-1} (\alpha_k - \delta_r) \prod_{r=1}^{Y_2} (\alpha_k - \xi_r)^{b+N-2} \prod_{r=1}^{m} (\alpha_k - \beta_r)^{N-1} Y_2 \prod_{r=1; r \neq k}^{Y_2} (\alpha_k - \alpha_r)},
\]

\( k = 1, 2, \ldots, Y_2, \)

\[
Z_{kj} = \frac{F_j(\delta_k)}{\prod_{r=1; r \neq k}^{\max(Y_1, Y_2)-1} (\delta_k - \delta_r) \prod_{r=1}^{Y_2} (\delta_k - \xi_r)^{b+N-2} \prod_{r=1}^{m} (\delta_k - \beta_r)^{N-1} Y_2 \prod_{r=1}^{\max(Y_1, Y_2)-1} (\delta_k - \alpha_r)},
\]

\( k = 1, 2, \ldots, \max(Y_1, Y_2) - 1, \)

\[
U_{ikj} = \frac{\phi^{(N-1-i)}_j(L)(\xi_k)}{N-1-i)!}, \quad j = 1, 2, \ldots, N-1; \quad k = 1, 2, \ldots, m,
\]

\[
L_{ikj} = \frac{\phi^{(b+N-2-i)}_j(U)(\xi_k)}{(b+N-2-i)!}, \quad i = 1, 2, \ldots, b+N-2; \quad k = 1, 2, \ldots, Y_2,
\]

\[
K_{ij} = \frac{1}{i!} \left[ \Gamma_{j,F}(0) + \sum_{k=1}^{\max(Y_1, Y_2)} \frac{i! C_{kj}}{\alpha_k^{i+1}} + \sum_{k=1}^{\max(Y_1, Y_2)-1} \frac{i! Z_{kj}}{\delta_k^{i+1}} \right. \\
- \sum_{r=1}^{N-1} \sum_{k=1}^{m} \frac{(r+i-1)! U_{rkj}}{(r+i-1)! \beta_k^{i+1}} - \sum_{r=1}^{b+N-2} \sum_{k=1}^{m} \frac{(r+i-1)! L_{rkj}}{(r+i-1)! \xi_k^{i+1}}, \quad i = 0, 1, \ldots, M_j - \Upsilon.
\]

where \( \Gamma_{j,F}(0) \) can be obtained using the iterative scheme given in (12) and (13), and the \( i \)th order derivatives \( \phi^{(i)}_{j,l}(\xi_k) \) of \( \phi_{j,l}(z, \xi_k) = \frac{F_j(z)}{X_l(z, \xi_k)}, \) at \( z = \xi_k, \) (where \( l = U \) when \( \xi_k = \beta_k, \) and \( l = L \) when \( \xi_k = \xi_k, \)) are given by

\[
\phi^{(0)}_{j,l}(\xi_k) = \frac{F_j(\xi_k)}{X_l(\xi_k, \xi_k)},
\]

\[
\phi^{(i)}_{j,l}(\xi_k) = \frac{F_j^{(i)}(\xi_k) - \sum_{r=0}^{i-1} \binom{i}{r} \phi^{(r)}_{j,l}(\xi_k) X_l^{(i-r)}(\xi_k, \xi_k)}{X_l(\xi_k, \xi_k)}, \quad i \geq 1,
\]
with
\[ X_U(z, \beta_k) = \prod_{r=1}^{m} (z - \delta_r)^{Y_2} \prod_{r=1; r \neq k}^{m} (z - \beta_r)^{Y_1} \prod_{r=1}^{r=1; r \neq k}^{N-1} (z - \alpha_r), \]
\[ X_L(z, \xi_k) = \prod_{r=1}^{m} (z - \delta_r)^{Y_2} \prod_{r=1; r \neq k}^{m} (z - \xi_r)^{b+N-2} \prod_{r=1}^{r=1; r \neq k}^{N-1} (z - \alpha_r). \]

Now, collecting the coefficient of \( z^n, n \geq 0 \), from both the sides of (30), we obtain the p.m.f. of waiting time distribution as

\[ w_j(n) = \begin{cases} 
K_{nj} \sum_{k=1}^{m} C_{kj} + \sum_{k=1}^{m} \sum_{i=1}^{N-1} \sum_{i=1}^{N-1} U_{ikj} f(n, i, k) \\
+ \sum_{k=1}^{m} \sum_{i=1}^{N-1} L_{ikj} h(n, i, k), & 0 \leq n \leq M_j - \Upsilon, \\
b + N - 2 \sum_{k=1}^{m} Y_2 \\
- \sum_{k=1}^{m} \sum_{i=1}^{N-1} Z_{kj} \frac{1}{\delta_k^{n+1}} + \sum_{i=1}^{N-1} \sum_{i=1}^{N-1} U_{ikj} f(n, i, k) \\
+ \sum_{k=1}^{m} \sum_{i=1}^{N-1} L_{ikj} h(n, i, k), & n > M_j - \Upsilon, \\
b + N - 2 \sum_{k=1}^{m} Y_2 \\
- \sum_{k=1}^{m} \sum_{i=1}^{N-1} Z_{kj} \frac{1}{\delta_k^{n+1}} + \sum_{i=1}^{N-1} \sum_{i=1}^{N-1} U_{ikj} f(n, i, k) \\
+ \sum_{k=1}^{m} \sum_{i=1}^{N-1} L_{ikj} h(n, i, k), & n \geq 0, \quad \text{for } M_j < \Upsilon, 
\end{cases} \]

where \( \sum_{n=0}^{\infty} f(n, i, k) z^n = \frac{1}{(z - \beta_k)^{i}} \) and \( \sum_{n=0}^{\infty} h(n, i, k) z^n = \frac{1}{(z - \xi_k)^{i}} \).

To determine \( f(n, i, k) \), taking the natural logarithm on both the sides of \( \sum_{n=0}^{\infty} f(n, i, k) z^n = \frac{1}{(z - \beta_k)^{i}} \), we obtain

\[ \log \left( \sum_{n=0}^{\infty} f(n, i, k) z^n \right) = -i \log(z - \beta_k). \]  

(31)

Differentiating both the sides of (31) w.r.t. \( z \), we have

\[ (z - \beta_k) \sum_{n=1}^{\infty} n f(n, i, k) z^{n-1} = -i \sum_{n=0}^{\infty} f(n, i, k) z^n. \]  

(32)

Now collecting the coefficient of \( z^n, n \geq 0 \), from both the sides of (32), we obtain

\[ f(0, i, k) = \frac{1}{(-\beta_k)^{i}}, \]
\[ f(n, i, k) = \left( \frac{n-1+i}{n\beta_k} \right) f(n-1, i, k), \quad n \geq 1, \quad i = 1, 2, \ldots, N - 1; k = 1, 2, \ldots, m. \]

Similarly, the unknown coefficient \( h(n, i, k) \) of \( z^n \) in \( \sum_{n=0}^{\infty} h(n, i, k) z^n = \frac{1}{(z - \xi_k)^{i}} \) can be determined using the following iterative scheme:

\[ h(0, i, k) = \frac{1}{(-\xi_k)^{i}}, \]
\[ h(n, i, k) = \left( \frac{n-1+i}{n\xi_k} \right) h(n-1, i, k), \quad n \geq 1, \quad i = 1, 2, \ldots, b + N - 2; k = 1, 2, \ldots, Y_2. \]
5. **Optimal N-policy.** In this section, we consider an expected operating cost function \( \Theta(N) \) for this queueing model in which \( N \) is the queue based parameter. To formulate the expression of \( \Theta(N) \), it is necessary to construct period related functions such as idle period, busy period and busy cycle. A cycle holds a busy period with consecutive idle period. The idle period, busy period and busy cycle of the system are defined as follows:

1. **Idle period** \((\mathcal{I}_N)\): An idle period per cycle is the length of time from the instant of the system becomes empty to the time instant when at least \( N \) customers accumulate in the queue.

2. **Busy period** \((\mathcal{B}_N)\): A busy period per cycle is the length of time when the number of customers arrive during idle period touches \( N \) to the instant the system becomes empty.

3. **Busy cycle** \((\mathcal{C}_N)\): Busy cycle is the length of time from the beginning of the busy period to the end of the next idle period.

Let \( E[\mathcal{I}_N] \), \( E[\mathcal{B}_N] \), and \( E[\mathcal{C}_N] \) denote the expected length of idle period, busy period and busy cycle, respectively. Hence, we have \( E[\mathcal{C}_N] = E[\mathcal{I}_N] + E[\mathcal{B}_N] \). Since the length of times between two consecutive batch arrivals in the D-BMAP are independent identically distributed random variables with mean \( \frac{1}{\lambda^*} \), therefore the expected length of idle period for the D-BMAP/G/1 queueing system under \( N \)-policy is given by

\[
E[\mathcal{I}_N] = \frac{N}{\lambda^*}.
\]

The long-run proportion of times that the server being idle and busy are given by

\[
\Pr\{\text{server idle}\} = \frac{E[\mathcal{I}_N]}{E[\mathcal{C}_N]} = 1 - \rho,
\]

\[
\Pr\{\text{server busy}\} = \frac{E[\mathcal{B}_N]}{E[\mathcal{C}_N]} = \rho.
\]

Thus, we have

\[
E[\mathcal{C}_N] = \frac{N}{\lambda^*(1 - \rho)},
\]

\[
E[\mathcal{B}_N] = \frac{N}{\mu(1 - \rho)}.
\]

Now, we build up an expected cost function per unit time for the D-BMAP/G/1 queueing system under \( N \)-policy in which \( N \) is the key decision variable. Our objective is to determine optimum \( N \), say \( N^* \), which minimizes the cost function. Since \( \frac{E[\mathcal{I}_N]}{E[\mathcal{C}_N]} \) and \( \frac{E[\mathcal{B}_N]}{E[\mathcal{C}_N]} \) both are not functions of the decision variable \( N \), we have neglected them in construction of the expected cost function. Therefore, we consider the following cost parameters:

\( C_h \equiv \) the holding cost per unit time for each customer present in the queue;

\( C_s \equiv \) the setup cost per busy cycle.
The long-run expected cost function per unit time under the linear cost structure is given by

$$\Theta(N) = C_h L_q + C_s \frac{1}{E[C_N]}$$

$$= C_h \left[ \sum_{n=0}^{N-1} n\omega(n)e + \sum_{n=0}^{\infty} n\pi(n)e \right] + C_s \frac{1}{E[C_N]} \tag{33}$$

Now, we examine the characteristics of the expected cost function $\Theta(N)$. A sequence is said to be monotonically increasing if each consecutive term is greater than or equal to its previous term. Considering a sequence $\{L^N_q\}_{N=1}^{\infty}$ such that

$$L^N_q = \sum_{n=0}^{N-1} n\omega(n)e + \sum_{n=0}^{\infty} n\pi(n)e,$$

which is non-negative. Therefore, we conclude that $L^N_q \geq L^{N-1}_q$, for $N \geq 1$. Thus, $\{L^N_q\}_{N=1}^{\infty}$ is a monotonically increasing sequence. Similarly, for the sequence $\{E[C_N]\}_{N=1}^{\infty}$, we have

$$E[C_N] - E[C_{N-1}] = \frac{1}{\lambda^*(1-\rho)},$$

which is strictly positive. Therefore, we conclude that $\frac{1}{E[C_N]} < \frac{1}{E[C_{N-1}]}$, for $N \geq 1$. Thus, $\frac{1}{E[C_N]}$ is a strictly monotonically decreasing sequence. Since $L_q$ is an increasing function of $N$, therefore smaller value of $N$ is effective to minimize the expected cost function. On the other hand, since $\frac{1}{E[C_N]}$ is a decreasing function of $N$, therefore larger value of $N$ is effective to reduce the expected cost function. Thus, the convexity of the expected cost function $\Theta(N)$ is basically dependent on choice of $C_h$ and $C_s$, but it is independent on choice of arrival- and service time distributions. Note that since the increasing rate of $L_q$ is much larger than the decreasing rate of $\frac{1}{E[C_N]}$, it is prospective that the expected cost function $\Theta(N)$ is a convex function of $N$ whenever $C_h \ll C_s$. Since the expected cost function given in (33) is extremely complex, we cannot obtain the explicit expression for $N$ from $\Theta(N)$ and therefore, its a difficult task to carry out analytic result for the optimum value of $N^*$. In order to optimize the unconstrained expected cost function $\Theta(N)$, we use direct search method to obtain the optimal value of $N$.

6. **Numerical results.** The main objective of this section is to provide a few numerical results based on the proposed analytical procedure discussed in previous sections. Some numerical results are presented in self explanatory tables and graphs so that one can easily get a good idea about the behavior of system parameters. All the digital operations are performed on a PC having configuration Intel® Core™ i5-4200U CPU Processor @ 2.30 GHz with 6.00 GB of RAM in Windows 10 environment. The queue length distributions at various time epochs and the waiting time distribution of a randomly selected customer of an arrival batch for $D-BMAP/D-PH/1$ queue with $N = 7$ are given in Tables 1-5. All the numerical results are reported here rounded to eight decimal places. It is desired that numerical results presented in this paper facilitate engineers, researchers and practitioners to identify...
how the change of model parameters effects the various system performances. To show
the impact of model parameters on the system performance, we choose the
input matrices $D_n$, $n \geq 0$, of the arrival process (D-BMAP) as

$$D_0 = \begin{bmatrix} 0.451 & 0.046 & 0.020 & 0.000 & 0.011 \\ 0.031 & 0.612 & 0.042 & 0.012 & 0.001 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.030 & 0.198 & 0.042 & 0.014 & 0.046 \\ 0.041 & 0.033 & 0.061 & 0.020 & 0.013 \end{bmatrix}$$

$$D_2 = \begin{bmatrix} 0.101 & 0.087 & 0.557 & 0.031 & 0.012 \\ 0.050 & 0.060 & 0.010 & 0.421 & 0.012 \end{bmatrix}, \quad D_3 = \begin{bmatrix} 0.021 & 0.004 & 0.012 & 0.012 & 0.011 \\ 0.042 & 0.054 & 0.012 & 0.191 & 0.015 \end{bmatrix}$$

$$D_4 = \begin{bmatrix} 0.011 & 0.002 & 0.021 & 0.010 & 0.006 \\ 0.010 & 0.000 & 0.005 & 0.041 & 0.012 \end{bmatrix}, \quad D_5 = \begin{bmatrix} 0.000 & 0.011 & 0.000 & 0.012 & 0.013 \\ 0.000 & 0.011 & 0.002 & 0.021 & 0.001 \end{bmatrix}$$

with $D_k = 0$, $k \geq 6$, having lag-1 coefficient of correlation ($C_{corr}$) 0.00877827. The coefficient of correlation between arrivals is calculated using the formula given in Blondia [3]. Hence, we have

$$\bar{\pi} = \begin{bmatrix} 0.17895473 & 0.33796971 & 0.19708476 & 0.20065459 & 0.08533621 \end{bmatrix},$$

and with this, we get $\lambda_y = 0.35205720$ and $\lambda^* = 0.64802638$. Since various discrete probability distributions can be either represented or approximated by discrete phase-type distribution (D-PH), we take the service time distribution to be of discrete phase-type having the representation $(\theta, T)$, where $\theta$ and $T$ are of order 3. The p.g.f. of the service time distribution is $S(z) = z\theta(I_3 - zT)^{-1}T^0$, where $T^0 = (I_3 - T)e$. For D-PH service time distribution, the explicit form of $A_n$, $n \geq 0$, is given by

$$A_n = (I_m \otimes \theta)\chi_n(I_m \otimes T^0), \quad n \geq 0,$$

where

$$\chi_0 = (I_{3m} - D_0 \otimes T)^{-1}(D_0 \otimes I_3),$$

$$\chi_1 = (I_{3m} - D_0 \otimes T)^{-1}(D_1 \otimes I_3)(I_{3m} - D_0 \otimes T)^{-1},$$

$$\chi_n = \left[(I_{3m} - D_0 \otimes T)^{-1}(D_n \otimes I_3) + \sum_{r=1}^{n-1} \chi_r(D_{n-r} \otimes T) \right](I_{3m} - D_0 \otimes T)^{-1}, n \geq 2,$$

with $T^0 = (I_3 - T)e$, and the symbol $\otimes$ denotes the Kronecker product of two matrices. The proof is straightforward extension of Neuts [24, pp. 67-70] for PH-type service and so we have skipped here the proof. In our numerical computation,
we consider $\theta$ and $T$ as
\[
\theta = \begin{bmatrix} 0.30 & 0.20 & 0.50 \end{bmatrix}, \quad T = \begin{bmatrix} 0.02 & 0.10 & 0.20 \\ 0.01 & 0.25 & 0.10 \\ 0.15 & 0.05 & 0.10 \end{bmatrix}.
\]

This leads to $\mu = 0.67935893$, and hence the traffic intensity is $\rho = 0.95387924$.

One can observe in Table 2 that $\pi = \sum_{n=0}^{N-1} \omega(n) + \sum_{n=0}^{\infty} \pi(n)$, as desired. It may note here that $\sum_{n=0}^{N-1} \omega(n)e = 1 - \rho$, which represents the probability that the server is idle. This also acts as an internal check of our results. It is pointed out from the bottom of Tables 2 and 4 that $\sum_{n=0}^{N-1} \omega(n) + \sum_{n=0}^{\infty} \pi(n) = \sum_{n=0}^{N-1} \omega^*(n) + \sum_{n=0}^{\infty} \pi^*(n)$. One must also note that, at the bottom of the Tables 2 and 5, the average waiting time in the queue ($W_q$) of a random customer of an arrival batch obtained from the Little’s law matches with the one obtained from the waiting time distribution. These facts confirm the correctness of our analytical and numerical results. To understand the behavior of system characteristics straightforwardly, the computational results of the queue length distribution at random epoch given in Table 2 for idle and busy modes are also plotted in Figure 2 and Figure 3, respectively. Figure 4 graphically displays the results of the waiting time distribution given in Table 5.

To show the effect of correlation between arrivals on the mean queue length ($L_q$) for the $D$-BMAP$G/1$ queueing model under $N$-policy, we consider three different D-BMAPs having the same fundamental arrival rate ($\lambda^*$) and different coefficients of correlation with lag-1 in the D-BMAP. The input parameter matrices $D_k$, $k \geq 1$, for three D-BMAPs are defined as
\[
D_k = \frac{(1-c)e^{k-1}}{1-c} D, \quad \text{for } k = 1, 2, \ldots, b.
\]

where $c = 0.785$ and $D = D - D_0$ is a fixed matrix.

Here, we have taken $b = 10$ and pair of matrices $D_0, \overline{D}$ are given by

- The first D-BMAP labeled as D-BMAP$_1$ having $C_{corr} = 0.19231967$ is characterized by the matrices
  \[
  D_0 = \begin{bmatrix} 0.089 & 0.419 \\ 0.400 & 0.578 \end{bmatrix}, \quad \overline{D} = \begin{bmatrix} 0.491 & 0.001 \\ 0.020 & 0.002 \end{bmatrix}.
  \]

- The second D-BMAP labeled as D-BMAP$_2$ having $C_{corr} = 0.17641922$ is characterized by the matrices
  \[
  D_0 = \begin{bmatrix} 0.091 & 0.417 \\ 0.419 & 0.559 \end{bmatrix}, \quad \overline{D} = \begin{bmatrix} 0.489 & 0.003 \\ 0.001 & 0.021 \end{bmatrix}.
  \]

- The third D-BMAP labeled as D-BMAP$_3$ having $C_{corr} = 0.16581892$ is characterized by the matrices
  \[
  D_0 = \begin{bmatrix} 0.113 & 0.395 \\ 0.411 & 0.567 \end{bmatrix}, \quad \overline{D} = \begin{bmatrix} 0.467 & 0.025 \\ 0.009 & 0.013 \end{bmatrix}.
  \]

We have defined D-BMAP$_1$, D-BMAP$_2$ and D-BMAP$_3$ in such a way that the corresponding fundamental rates are equal to $\lambda^* = 0.94471274$. In this experiment, we consider the service time distribution to be arbitrarily distributed with p.m.f. $a_1 = 0.9930$, $a_2 = 0.0032$, $a_4 = 0.0019$, $a_5 = 0.0017$, and $a_6 = 0.0002$. This
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leads to $\mu = 0.98357431$. Hence $\rho = 0.96048944$. Figure 5 shows the effect of coefficients of correlation ($C_{\text{corr}}$) in the D-BMAP on the mean queue length ($L_q$) for different threshold value ($N$). It is observed in Figure 5 that $L_q$ increases as the threshold value $N$ increases. Further, one more interesting observation is that the value of $L_q$ also significantly depends on the coefficient of correlation in D-BMAP. With the same fundamental arrival rate $\lambda^*$, the mean number of customers in the queue increases as the coefficient of correlation increases. This study reveals that not only service time and threshold value play an important role in N-policy queueing processes, but also the correlation between arrivals plays a major role. This means that the correlation in D-BMAP can significantly influence the performance measures of the system and should be taken into account when evaluating the system performance measures.

We consider a practical problem concerning the flexible manufacturing system mentioned in introduction (Section 1). It is assumed that the raw materials arrive at the system according to a D-BMAP with the input matrices $D_n, n \geq 0$, taken for Tables 1-5. In order to show the applicability of the proposed model in the flexible manufacturing system, we consider three numerical examples for three different service time distributions (i.e., production time of each industrial material) such as D-PH distribution, general discrete distribution and mixed geometric distribution. They are taken as follows:

- For D-PH, we consider the above (34) representation of $(\theta, T)$.
- For general discrete-time distribution, we take $s_1 = 0.04, s_5 = 0.01, s_7 = 0.02, s_{10} = 0.03$. This implies that $\mu = 0.71428571$, and so $\rho = 0.90723694$.
- For mixed geometric distribution with representation $(\theta, T)$ (see, Alfa [2, pp. 41]), we choose

$$
\theta = \begin{bmatrix} 0.230 & 0.320 & 0.450 \end{bmatrix}, \quad T = \begin{bmatrix} 0.312 & 0.000 & 0.000 \\ 0.000 & 0.217 & 0.000 \\ 0.000 & 0.000 & 0.197 \end{bmatrix}.
$$

This leads to $\mu = 0.76723279$, and hence $\rho = 0.84462811$.

To demonstrate the optimum value of $N$, we perform the numerical experiment by considering the following cost elements: $C_s = 0.225$, $C_h = 0.001$, and vary the threshold value $N$ from 1 to 15. The computational results for the long-run expected cost per unit time and threshold value $N$ under three different service time distributions mentioned above are plotted in Figure 6. This confirms that the expected cost function is convex, and so the solution gives unique optimum threshold value for each service time distribution. It is seen in Figure 6 that $\Theta(4) = 0.02754288$ is the minimum value of the long-run expected operating cost per unit time, i.e., the optimal threshold value of $N$ is $N^* = 4$ for D-PH service time distribution. We also observed that the minimum long-run expected operating costs per unit time for general discrete distribution and mixed geometric distribution are $\Theta(5) = 0.02104816$ and $\Theta(7) = 0.01175413$ corresponding to their optimal threshold value $N^* = 5$ and $N^* = 7$, respectively. It is concluded from the above observation that the optimal threshold value $N^*$ of flexible manufacturing system significantly depends on the traffic intensity of the system.
Table 1. Queue length distribution at post-departure epoch

| n  | $\pi_1(n)$   | $\pi_2(n)$   | $\pi_3(n)$   | $\pi_4(n)$   | $\pi_5(n)$   | $\pi_6(n)$   |
|----|--------------|--------------|--------------|--------------|--------------|--------------|
| 0  | 0.00114837   | 0.00255731   | 0.00122101   | 0.00057458   | 0.00030733   | 0.00580861   |
| 1  | 0.00196732   | 0.00413389   | 0.00211277   | 0.00132765   | 0.00067006   | 0.01021170   |
| 2  | 0.00268813   | 0.00542042   | 0.00287623   | 0.0014745   | 0.00100214   | 0.01413437   |
| 3  | 0.00336532   | 0.00676685   | 0.00362099   | 0.00302599   | 0.00135476   | 0.01813911   |
| 4  | 0.00407581   | 0.00806720   | 0.0042181    | 0.00388853   | 0.00172228   | 0.02217563   |
| 5  | 0.00480560   | 0.00931208   | 0.00519387   | 0.00478110   | 0.00209565   | 0.02619131   |
| 6  | 0.00545231   | 0.01038279   | 0.00590375   | 0.00577743   | 0.00247998   | 0.02999627   |
| 7  | 0.00550115   | 0.01037107   | 0.00599819   | 0.00616987   | 0.00258839   | 0.03062867   |
| 10 | 0.00518961   | 0.00969826   | 0.00573383   | 0.00601163   | 0.00253004   | 0.02916336   |
| 20 | 0.00355502   | 0.00663085   | 0.00392615   | 0.00413473   | 0.00174204   | 0.01999780   |
| 50 | 0.00113109   | 0.00214262   | 0.00125016   | 0.00131658   | 0.00055470   | 0.00636770   |
| 100| 0.00016808   | 0.00031392   | 0.00018562   | 0.00019549   | 0.00008236   | 0.00094547   |
| 200| 0.00000371   | 0.00000692   | 0.00000499   | 0.00000431   | 0.00000182   | 0.00002084   |
| 300| 0.00000008   | 0.00000015   | 0.00000009   | 0.00000010   | 0.00000004   | 0.00000046   |
| 500| 0.00000000   | 0.00000000   | 0.00000000   | 0.00000000   | 0.00000000   | 0.00000000   |

sum 0.17884285 0.33699076 0.19680812 0.20180982 0.08554844 1.00000000

Figure 2. Queue length distribution at random epoch in idle mode
### Table 2. Queue length distribution at random epoch

| $n$ | $\omega_1(n)$ | $\omega_2(n)$ | $\omega_3(n)$ | $\omega_4(n)$ | $\omega_5(n)$ | $\omega(n)e$ |
|-----|----------------|----------------|---------------|---------------|---------------|--------------|
| 0   | 0.00222695     | 0.00529393     | 0.00242011    | 0.00090775    | 0.00048977    | 0.01133850   |
| 1   | 0.00122073     | 0.00248229     | 0.00141283    | 0.00079891    | 0.00049111    | 0.00634588   |
| 2   | 0.00105468     | 0.00179357     | 0.00112890    | 0.00103029    | 0.00047010    | 0.00547755   |
| 3   | 0.00089757     | 0.00190533     | 0.00096004    | 0.00118332    | 0.00048134    | 0.00542760   |
| 4   | 0.00096295     | 0.00185514     | 0.00111938    | 0.00116282    | 0.00051063    | 0.00561091   |
| 5   | 0.00111788     | 0.00198211     | 0.00120152    | 0.00116430    | 0.00051719    | 0.00598300   |
| 6   | 0.00106199     | 0.00200132     | 0.00116720    | 0.00115114    | 0.00049567    | 0.00587732   |

| $\pi_1(n)$ | $\pi_2(n)$ | $\pi_3(n)$ | $\pi_4(n)$ | $\pi_5(n)$ | $\pi(n)$ |
|------------|------------|------------|------------|------------|----------|
| 0          | 0.00162452 | 0.00345778 | 0.00174092 | 0.00103506 | 0.00052768 | 0.00838598 |
| 1          | 0.00234181 | 0.00477333 | 0.00250811 | 0.00179593 | 0.00085359 | 0.01227277 |
| 2          | 0.00300104 | 0.00603923 | 0.00322414 | 0.00261562 | 0.00118356 | 0.01606358 |
| 3          | 0.00366851 | 0.00729361 | 0.00397075 | 0.00344304 | 0.00152948 | 0.01990539 |
| 4          | 0.00435868 | 0.00849806 | 0.00471593 | 0.00428708 | 0.00188379 | 0.02374353 |
| 5          | 0.00500101 | 0.00957287 | 0.00541216 | 0.00520452 | 0.00226049 | 0.02747355 |
| 10         | 0.00486647 | 0.00905994 | 0.00535865 | 0.00563201 | 0.00238020 | 0.02729727 |
| 20         | 0.00330525 | 0.00617334 | 0.00365030 | 0.00384423 | 0.00161965 | 0.01859277 |
| 50         | 0.00105246 | 0.00196571 | 0.00116233 | 0.00122408 | 0.00051573 | 0.00592030 |
| 100        | 0.00015627 | 0.00029187 | 0.00017258 | 0.00018175 | 0.00007658 | 0.00087904 |
| 200        | 0.00000345 | 0.00000643 | 0.00000301 | 0.00000196 | 0.00000000 | 0.00000000 |
| 300        | 0.00000008 | 0.00000014 | 0.00000008 | 0.00000009 | 0.00000003 | 0.00000002 |
| 500        | 0.00000000 | 0.00000000 | 0.00000000 | 0.00000000 | 0.00000000 | 0.00000000 |

sum 0.17895473, 0.33796971, 0.19708476, 0.20065459, 0.08533621, 1.00000000

$L_q = 27.32669405, W_q \equiv L_q/\lambda^* = 42.16910717$
Figure 3. Queue length distribution at random epoch in busy mode

Table 3. Queue length distribution at prearrival epoch

| n  | $\omega_0^-(n)$ | $\omega_1^-(n)$ | $\omega_2^-(n)$ | $\omega_3^-(n)$ | $\omega_4^-(n)$ | $\omega_5^-(n)$ | $\omega_-^-(n)$ e |
|----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0  | 0.00174343      | 0.00277470      | 0.00221558      | 0.00262957      | 0.00132854      | 0.01069181      | 0.01069181      |
| 1  | 0.00097452      | 0.00160317      | 0.00122966      | 0.00163056      | 0.00074979      | 0.00618770      | 0.00618770      |
| 2  | 0.00083109      | 0.00141140      | 0.00103647      | 0.00156583      | 0.00062849      | 0.00547327      | 0.00547327      |
| 3  | 0.00547327      | 0.00133665      | 0.00103247      | 0.00165829      | 0.00060400      | 0.00546386      | 0.00546386      |
| 4  | 0.00085171      | 0.00139904      | 0.00105950      | 0.00167026      | 0.00063105      | 0.00561157      | 0.00561157      |
| 5  | 0.00090939      | 0.00152759      | 0.00113284      | 0.00173509      | 0.00068207      | 0.00598698      | 0.00598698      |
| 6  | 0.00089418      | 0.00148113      | 0.00111396      | 0.00171177      | 0.00066572      | 0.00586676      | 0.00586676      |
| 10 | 0.00128465      | 0.00209323      | 0.00162002      | 0.00213136      | 0.00097544      | 0.00810471      | 0.00810471      |
| 1  | 0.00187638      | 0.00307986      | 0.00235698      | 0.00326222      | 0.00141681      | 0.01199226      | 0.01199226      |
| 2  | 0.00245421      | 0.00402943      | 0.00307416      | 0.00440903      | 0.00184105      | 0.01580788      | 0.01580788      |
| 3  | 0.00308200      | 0.00499321      | 0.00379932      | 0.00556828      | 0.00227196      | 0.01970977      | 0.01970977      |
| 4  | 0.00362127      | 0.00597046      | 0.00452247      | 0.00673302      | 0.00270420      | 0.02355141      | 0.02355141      |
| 5  | 0.00418155      | 0.00691160      | 0.00521393      | 0.00790838      | 0.00311484      | 0.02730300      | 0.02730300      |
| 10 | 0.00415110      | 0.00687971      | 0.00516200      | 0.00810034      | 0.00307831      | 0.02737146      | 0.02737146      |
| 20 | 0.00282716      | 0.00468171      | 0.00351552      | 0.00552180      | 0.00209539      | 0.01864157      | 0.01864157      |
| 50 | 0.00090022      | 0.00194075      | 0.00111941      | 0.00175825      | 0.00066721      | 0.00593585      | 0.00593585      |
| 100| 0.00013366      | 0.00022135      | 0.00016621      | 0.00026106      | 0.00009907      | 0.00088135      | 0.00088135      |
| 200| 0.00000295      | 0.00000488      | 0.00000366      | 0.00000576      | 0.00000218      | 0.00001943      | 0.00001943      |
| 300| 0.00000006      | 0.00000010      | 0.00000008      | 0.00000012      | 0.00000005      | 0.00000042      | 0.00000042      |
| 500| 0.00000000      | 0.00000000      | 0.00000000      | 0.00000000      | 0.00000000      | 0.00000000      | 0.00000000      |

sum 0.15214569 0.25164805 0.18938631 0.29385469 0.11296526 1.00000000
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Figure 4. Waiting time distribution

Table 4. Queue length distribution at intermediate epoch

| $n$ | $\omega_1^*(n)$ | $\omega_2^*(n)$ | $\omega_3^*(n)$ | $\omega_4^*(n)$ | $\omega_5^*(n)$ | $\omega^*(n)e$ |
|-----|----------------|----------------|----------------|----------------|----------------|----------------|
| 0   | 0.00000000     | 0.00000000     | 0.00000000     | 0.00000000     | 0.00000000     | 0.00000000     |
| 1   | 0.00109382     | 0.00243612     | 0.00116304     | 0.00054705     | 0.00029262     | 0.00553266     |
| 2   | 0.00187471     | 0.00393963     | 0.00201337     | 0.00126468     | 0.00063839     | 0.00973078     |
| 3   | 0.00256220     | 0.00516670     | 0.00274151     | 0.00194630     | 0.00095506     | 0.01347178     |
| 4   | 0.00328050     | 0.00645099     | 0.00345180     | 0.00228409     | 0.00129132     | 0.01727824     |
| 5   | 0.00382930     | 0.00769135     | 0.00421561     | 0.00288409     | 0.00164183     | 0.02111427     |
| 6   | 0.00545120     | 0.00887902     | 0.00495214     | 0.00370673     | 0.00199793     | 0.02497183     |
| 10  | 0.00498630     | 0.00928331     | 0.00549189     | 0.00357555     | 0.00243227     | 0.02795104     |
| 20  | 0.00339147     | 0.00633383     | 0.00374553     | 0.00164183     | 0.00166190     | 0.01907779     |
| 50  | 0.000170991    | 0.00016304     | 0.00054705     | 0.00029262     | 0.00001904     | 0.000973078    |
| 100 | 0.00016034     | 0.000121699    | 0.00119265     | 0.00256011     | 0.00259218     | 0.00607475     |
| 200 | 0.000000533    | 0.000000000    | 0.00017708     | 0.00018649     | 0.0007857     | 0.00090197     |
| 300 | 0.000000008    | 0.000000015    | 0.00000009     | 0.00000009     | 0.00000003     | 0.00000004     |
| 500 | 0.000000000    | 0.000000000    | 0.00000000     | 0.00000000     | 0.00000000     | 0.00000000     |

sum 0.17895473 0.33796971 0.19708476 0.20065459 0.08533621 1.00000000
Figure 5. Mean queue length versus threshold value $N$ for different coefficients of correlation in the D-BMAPs

Table 5. Waiting time distribution

| $k$ | $w_1(k)$ | $w_2(k)$ | $w_3(k)$ | $w_4(k)$ | $w_5(k)$ | $w(k)e$ |
|-----|-----------|-----------|-----------|-----------|-----------|---------|
| 0   | 0.00047483 | 0.00077372 | 0.00059879 | 0.00078790 | 0.00036054 | 0.00299578 |
| 1   | 0.00076314 | 0.00119978 | 0.00097247 | 0.00140311 | 0.00061681 | 0.00495532 |
| 2   | 0.00110386 | 0.00172634 | 0.00140867 | 0.00210776 | 0.00091559 | 0.00726221 |
| 3   | 0.00147097 | 0.00227929 | 0.00190462 | 0.00286252 | 0.00123500 | 0.00975240 |
| 4   | 0.00185631 | 0.00287564 | 0.00242983 | 0.00368458 | 0.00157512 | 0.01241248 |
| 5   | 0.00226328 | 0.00349641 | 0.00297861 | 0.00455765 | 0.00193035 | 0.01526260 |
| 10  | 0.00342239 | 0.00526449 | 0.00490328 | 0.00694631 | 0.00290268 | 0.02302618 |
| 20  | 0.0053361 | 0.00388041 | 0.00331762 | 0.00511929 | 0.00214814 | 0.01699908 |
| 50  | 0.0015231 | 0.00176460 | 0.00150943 | 0.00233341 | 0.00097725 | 0.00773727 |
| 100 | 0.00031795 | 0.00048690 | 0.00041649 | 0.00064384 | 0.00026972 | 0.00213489 |
| 200 | 0.0002421 | 0.0003707 | 0.00030171 | 0.0004902 | 0.0002053 | 0.0016254 |
| 300 | 0.0000184 | 0.0000282 | 0.0000241 | 0.0000373 | 0.0000156 | 0.0001237 |
| 500 | 0.0000001 | 0.0000002 | 0.0000001 | 0.0000002 | 0.0000001 | 0.0000007 |
| 1000| 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |

sum 0.14901642 0.22858763 0.19513341 0.30097685 0.12628569 1.000000

$W_q \equiv \sum_{k=1}^{\infty} kw(k)e = 42.16910716$
ANALYSIS OF $D-BMAP/G/1$ QUEUEING SYSTEM UNDER $N$-POLICY

7. **Conclusion.** In this paper, we have carried out an analysis of the $D-BMAP/G/1$ queue under $N$-policy. We have provided a simple procedure to obtain the steady state probability distributions of number of customers in the queue at post-departure, random, arrival, intermediate epochs and some key performance measures. We then have investigated the waiting time distribution in the queue of a randomly selected customer of an arrival batch. An expected linear cost function per unit time is constructed to determine the optimal threshold value of $N$ which minimizes the expected cost function. We have also demonstrated extensive numerical results in the form of tables and graphs to show the performance measures. It is hoped that the analysis of this paper would be useful to make more accurate capacity planning decisions in telecommunication, computer network, transportation, manufacturing, inventory systems and so on. The procedure presented in this paper may be applied to the analysis of $D-BMAP/G/1$ queue with single and multiple vacations under $N$-policy.

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