CLOSED IDEALS IN THE ALGEBRA OF COMPACT-BY-APPROXIMABLE OPERATORS

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Abstract. We construct various examples of non-trivial closed ideals of the compact-by-approximable algebra \( A_X := \mathcal{K}(X)/\mathcal{A}(X) \) on Banach spaces \( X \) failing the approximation property. The examples include the following: (i) if \( X \) has cotype 2, \( Y \) has type 2, \( \mathcal{A}_X \neq \{0\} \) and \( \mathcal{A}_Y \neq \{0\} \), then \( \mathcal{A}_{X \oplus Y} \) has at least 2 closed ideals, (ii) there are closed subspaces \( X \subset \ell^p \) for \( 4 < p < \infty \) and \( X \subset c_0 \) such that \( \mathcal{A}_X \) contains a non-trivial closed ideal, (iii) there is a Banach space \( Z \) such that \( \mathcal{A}_Z \) contains an uncountable lattice of closed ideal having the reverse order structure of the power set of the natural numbers. Some of our examples involve non-classical approximation properties associated to various Banach operator ideals. We also discuss the existence of compact non-approximable operators \( X \to Y \), where \( X \subset \ell^p \) and \( Y \subset \ell^q \) are closed subspaces for \( p \neq q \).

1. Introduction

Given Banach spaces \( X \) and \( Y \), let \( \mathcal{K}(X,Y) \) be the class of compact operators \( X \to Y \). The uniform norm closure \( \mathcal{A}(X,Y) = \overline{\mathcal{F}(X,Y)} \) defines the class of approximable operators, where \( \mathcal{F}(X,Y) \) is the linear subspace consisting of the bounded finite-rank operators \( X \to Y \). We abbreviate \( \mathcal{K}(X) = \mathcal{K}(X,X) \) and \( \mathcal{A}(X) = \mathcal{A}(X,X) \) for \( X = Y \). One obtains the quotient algebra \( \mathfrak{A}_X = : \mathcal{K}(X)/\mathcal{A}(X) \) of the compact-by-approximable operators on \( X \), since \( \mathcal{A}(X) \) is a closed two-sided ideal of \( \mathcal{K}(X) \). The quotient \( \mathfrak{A}_X \) is a non-unital Banach algebra equipped with the quotient norm

\[
\|S + \mathcal{A}(X)\| = \text{dist}(S, \mathcal{A}(X)), \quad S \in \mathcal{K}(X).
\]

If \( X \) has the approximation property, then it is well known that \( \mathfrak{A}_X = \{0\} \). Recall that Banach spaces \( X \) failing the approximation property are complicated objects to construct or recognise. As a consequence the class of compact-by-approximable algebras \( \mathfrak{A}_X \) is quite intractable, and it has largely been neglected in the literature. Nevertheless, these quotient algebras are natural examples of (typically) non-commutative radical Banach algebras, that is, the quotient elements \( S + \mathcal{A}(X) \) are quasi-nilpotent for all \( S \in \mathcal{K}(X) \). Recently various facts and problems about such algebras were highlighted by Dales [10], and his questions motivated the results and examples in [53] about the size of the quotient algebras \( \mathfrak{A}_X \) for classes of Banach spaces \( X \). In this paper we complement and expand our earlier study by looking more carefully at the algebraic structure of \( \mathfrak{A}_X \), in particular at examples of non-trivial closed two-sided ideals.

If \( X \) fails to have the approximation property and \( \mathfrak{A}_X \neq \{0\} \), then it is a serious challenge to construct non-trivial closed ideals of \( \mathfrak{A}_X \). In Section 2 we provide the first examples of this kind. In Theorem 2.6 we exhibit a class of direct sums \( X \oplus Y \), where \( \mathfrak{A}_{X \oplus Y} \) has at least

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two non-trivial, incomparable closed ideals. As a by-product of our discussion we also point out (Corollary 2.4) that the class of non-approximable operators does not contain a universal operator. In Theorem 2.8 we find closed subspaces \( X \subset \ell^p \) for \( p \in [1, \infty) \) and \( p \neq 2 \), as well as \( X \subset c_0 \), for which the quotient algebra \( \mathfrak{A}_X \) is non-nilpotent and infinite-dimensional. This result improves on [53, section 2], and it will also be crucial in some of our later examples of closed ideals of the compact-by-approximable algebra.

In Section 3 we systematically discuss the following natural problem: for which parameters \( p \neq q \) is it possible to find closed subspaces \( X \subset \ell^p \) and \( Y \subset \ell^q \), such that \( \mathcal{A}(X,Y) \not\subset \mathcal{K}(X,Y) \)? This is not always the case (see Theorem 2.2), and our discussion is motivated by Theorem 2.6. The cases \( p,q \in [1,2) \) turn out to be related to earlier factorisation results of Figiel [19], Alexander [2] and Bachelis [3] for compact operators. For \( p,q \in (2,\infty) \) we will use non-classical approximation properties with respect to certain Banach operator ideals \( \mathcal{I} \) which are contained in the ideal \( \mathcal{K} \) of the compact operators. The result (Theorem 3.9) re-examines a sophisticated example of Reinov [46] about the failure of duality for \( p \)-nuclear operators. The concepts and results in Section 3 enable us to revisit the setting of Theorem 2.6, and to exhibit in Example 3.14 such direct sums \( X \oplus Y \), for which \( \mathfrak{A}_X \) contains a non-trivial closed ideal. Moreover, in Examples 3.12 and 3.13 we uncover closed subspaces \( X \subset \ell^p \) for \( p \in (4,\infty) \) and \( X \subset c_0 \), where \( \mathfrak{A}_X \) contains a non-trivial closed ideal.

In Section 4 we find spaces \( X \) for which \( \mathfrak{C}_A(X) \cap \mathcal{K}(X) \) determines a non-trivial closed ideal of \( \mathfrak{A}_X \), where \( \mathfrak{C}_A \) is the class of compactly approximable operators. Finally, given any Banach space \( X \) such that \( X \) has the approximation property but \( X^* \) fails this property, we construct in Example 4.5 an associated space \( Z \) for which \( \mathfrak{A}_Z \) carries an uncountable family of non-trivial closed ideals having an explicit order structure. We also draw attention to some problems raised by our results.

Preliminaries. We briefly recall some standard concepts that will freely be used later. The Banach space \( X \) has the approximation property (A.P.) if for all compact subsets \( K \subset X \) and \( \varepsilon > 0 \) there is a bounded finite-rank operator \( U \in \mathcal{F}(X) \) such that

(1.1) \[ \sup_{x \in K} \| x - Ux \| < \varepsilon. \]

If there is a uniform bound \( C < \infty \) such that the approximating operator \( U \in \mathcal{F}(X) \) in (1.1) can be chosen to satisfy \( \| U \| \leq C \), then \( X \) has the bounded approximation property (B.A.P.). If compact operators \( U \in \mathcal{K}(X) \) are allowed in condition (1.1), then one obtains analogously the compact approximation property (C.A.P.) and its bounded version B.C.A.P. For a comprehensive discussion of the classical approximation properties we refer to [36, 1.e and 2.d], [37, 1.g], and the survey [7]. In general, [1], [16] and [36] are references for unexplained concepts and results related to Banach spaces.

Let \( \mathcal{L}(X,Y) \) be the space of bounded linear operators \( X \to Y \) for Banach spaces \( X \) and \( Y \). We say here that \( (\mathcal{I}, \| \cdot \|_\mathcal{I}) \) is a Banach operator ideal if \( (\mathcal{I}, \| \cdot \|_\mathcal{I}) \) is a complete normed operator ideal in the sense of Pietsch [11]. More precisely, this entails that the following conditions hold for all Banach spaces \( X \) and \( Y \):

(BOI1) the ideal component \( \mathcal{I}(X,Y) \) is a linear subspace of \( \mathcal{L}(X,Y) \), and \( \| \cdot \|_\mathcal{I} \) is a complete norm in \( \mathcal{I}(X,Y) \) such that \( \| S \| \leq \| S \|_\mathcal{I} \) for all \( S \in \mathcal{I}(X,Y) \),
(BOI2) the bounded finite-rank operators \( F(X,Y) \subset \mathcal{I}(X,Y) \) and \( |x^* \otimes y|_I = \|x^*\| \cdot \|y\| \) for all \( x^* \in X^* \) and \( y \in Y \),

(BOI3) for all Banach spaces \( Z \) and \( W \) the product \( BSA \in \mathcal{I}(Z,W) \) for all \( S \in \mathcal{I}(X,Y) \) and all bounded operators \( A \in \mathcal{L}(Z,X) \) and \( B \in \mathcal{L}(Y,W) \), and in addition

\[
|BSA|_Z \leq \|B\| \cdot \|A\| \cdot |S|_I.
\]

Above \( x^* \otimes y \in F(X,Y) \) denotes the operator \( x \mapsto x^*(x)y \). Banach operator ideals of particular interest in Section 3 will be the ideal \( K\mathcal{S}_r \) of the operators that factor compactly through a closed subspace of \( \ell^r \), the ideal \( \mathcal{S}K_r \) of the (Sinha-Karn) \( r \)-compact operators, and the ideal \( \mathcal{Q}N_r \) of the quasi \( r \)-nuclear operators. We refer to [41], [12] and [16] as general sources for classical examples of operator ideals and for related unexplained concepts and constructions.

Recall that a complex Banach algebra \( \mathcal{A} \) is a radical algebra if the spectrum \( \sigma(x) = \{0\} \) for all \( x \in \mathcal{A} \), where the spectrum is computed in the unitisation \( \mathcal{A}^\mathbb{F} \) of \( \mathcal{A} \). It is known [9, Theorem 2.5.8(iv)] that for complex Banach spaces \( X \) the quotient algebra \( \mathfrak{A}_X \) is radical. For real Banach spaces \( X \) our interpretation is that the real quotient algebra \( \mathfrak{A}_X \) is radical in the sense that the real spectrum

\[
\sigma_{\mathbb{R}}(S + \mathcal{A}(X)) =: \{ \lambda \in \mathbb{R} : \lambda 1 - (S + \mathcal{A}(X)) \text{ is invertible in } (\mathfrak{A}_X)^\# \} = \{0\}
\]

for all \( S \in \mathcal{K}(X) \). The fact that this holds in the real case is a consequence of classical Riesz-Fredholm theory, which does not depend of the scalar field (see Proposition 2.7 for the precise details). With this understanding our results and examples are independent of the scalar field \( \mathbb{R} \) or \( \mathbb{C} \) of the underlying Banach space \( X \). Recall also from [9, Question 2.2.A, page 182] that it is a longstanding open problem whether there are topologically simple radical Banach algebras \( \mathcal{A} \), that is, \( \mathcal{A} \) has a non-trivial product and no non-trivial closed ideals.

2. Non-trivial closed ideals of \( \mathfrak{A}_Z \) and algebraic properties

In this section we construct the first examples of non-trivial closed two-sided ideals in the compact-by-approximable quotient algebra \( \mathfrak{A}_Z \) for certain classes of Banach spaces \( Z \). We also exhibit closed subspaces \( X \subset \ell^p \), where \( 1 \leq p < \infty \) and \( p \neq 2 \), such that the algebra \( \mathfrak{A}_X \) is non-nilpotent. This result improves [52, section 2], and it will be essential for some of our later examples of closed ideals.

Let \( Z \) be a Banach space. In view of Proposition 2.1 below we will equivalently exhibit operator norm closed two-sided (algebraic) ideals \( \mathcal{J} \) of \( \mathcal{K}(Z) \), such that \( \mathcal{A}(Z) \subsetneq \mathcal{J} \subsetneq \mathcal{K}(Z) \). Some of the ideals will be defined by internal conditions for particular Banach spaces \( Z \), in which case the task is to verify that the class really is a non-trivial closed ideal of \( \mathcal{K}(Z) \). Banach operator ideals \( (\mathcal{I}, \|\cdot\|_I) \) provide another important source of examples. Here \( \mathcal{I}(Z) =: \mathcal{I}(Z,Z) \) is a two-sided (algebraic) ideal of \( \mathcal{L}(Z) \), but \( \mathcal{I}(Z) \) is typically not closed in the uniform operator norm. We will reserve the notation \( \overline{\mathcal{I}(X,Y)} \) for the operator norm closure of \( \mathcal{I}(X,Y) \) in \( \mathcal{L}(X,Y) \) for Banach spaces \( X \) and \( Y \). Evidently \( \overline{\mathcal{I}(Z)} \) is a closed (algebraic) ideal of \( \mathcal{L}(Z) \) for every space \( Z \). We emphasize that \( \mathcal{A}(Z) \subset \mathcal{I} \) whenever \( \{0\} \neq \mathcal{I} \subseteq \mathcal{K}(Z) \) is a closed (two-sided algebraic) ideal of \( \mathcal{K}(Z) \). In fact, let \( U \in \mathcal{I} \) be a non-zero operator, and pick \( x \in Z \) and \( x^* \in Z^* \) such that \( \langle Ux, x^* \rangle = 1 \). If \( y \in Z \) and \( y^* \in Z^* \) are arbitrary, then

\[
(x^* \otimes y) \circ (y^* \otimes Ux) = (Ux, x^*) (y^* \otimes y) = y^* \otimes y \in \mathcal{I},
\]

since \( y^* \otimes Ux = U \circ (y^* \otimes x) \in \mathcal{I} \). This implies that \( \mathcal{A}(Z) \subset \mathcal{I} \).
Proof. (i) Note that \(q(I)\) is an ideal of \(\mathfrak{A}_Z\), since \(q\) is an algebra homomorphism. To verify that \(q(I)\) is closed in \(\mathfrak{A}_Z\), suppose that \(S \in K(Z)\) with \(q(S) \in q(I)\). Hence there is a bounded sequence \((S_n) \subseteq I\) such that

\[
\|q(S) - q(S_n)\| = \text{dist}(S - S_n, \mathcal{A}(Z)) \to 0, \quad n \to \infty.
\]

Pick \(V_n \in \mathcal{A}(Z)\) for \(n \in \mathbb{N}\) so that \(\|S - S_n - V_n\| \to 0\) as \(n \to \infty\). Here \(S_n + V_n \in I\) for each \(n\), so that \(S \in I = I\). Clearly \(q(I) \subseteq \mathfrak{A}_Z\), since otherwise \(I = q^{-1}(q(I)) = K(Z)\). Moreover, a simple verification shows that if \(I_1 \neq I_2\), then \(q(I_1) \neq q(I_2)\).

(ii) If \(U \in q^{-1}(J)\) and \(S \in K(Z)\), then \(q(SU) = q(S)q(U) \in J\), so that \(SU \in q^{-1}(J)\). Similarly \(US \in q^{-1}(J)\). Moreover, \(\{0\} \neq J \subseteq \mathfrak{A}_Z\) implies that \(\mathcal{A}(Z) \subseteq q^{-1}(J) \subseteq K(Z)\). \(\square\)

If the closed ideal \(J\) of \(K(Z)\) can be written as \(J = I(Z)\) for some Banach operator ideal \(I\), then \(J\) is an ideal of \(L(Z)\). However, we will also encounter closed ideals \(J\) of \(K(Z)\) that fail to be an ideal of \(L(Z)\), see Remark 4.8.

Our first class of examples contains the result that for \(p < 2 < q\) there are closed subspaces \(X \subseteq \ell^p\) and \(Y \subseteq \ell^q\), such that \(\mathfrak{A}_{X \oplus Y}\) contains at least two non-trivial incomparable closed ideals. The construction is based on the following theorem, which combines classical factorisation results of Kwapien and Maurey (see [44, Theorem 3.4 and Corollary 3.6]) with an observation of John [28, Lemma 2]. It will be crucial that the equality (2.1) below is independent of any approximation properties on \(X\) or \(Y\). For other results of this type, see [24, Theorem 2.2] and the subsequent comments for the case of separable reflexive spaces, as well as the remark on [16, p. 248]. We refer e.g. to [1, section 6.2] for the notions of type and cotype for Banach spaces. We will require the facts, see [1, Theorem 6.2.14] that \(L^p(\mu)\) (as well as all its closed subspaces) has type 2 for \(2 \leq p < \infty\), and cotype 2 for \(1 \leq p \leq 2\).

**Theorem 2.2.** (Kwapien-Maurey-John) Suppose that \(X\) has type 2 and \(Y\) has cotype 2. Then

\[
K(X, Y) = A(X, Y).
\]

**Proof.** Since the argument combines two results of different nature we review the relevant ideas for completeness.

Suppose that \(T \in K(X, Y)\) is arbitrary. It follows from the classical factorisation results of Kwapien and Maurey, see [44, Corollary 3.6], that there is a Hilbert space \(H\) and bounded operators \(A \in L(X, H), B \in L(H, Y)\) such that \(T = BA\).
Suppose first that $H$ is separable and let $(P_n)$ be the sequence of orthogonal projections from $H$ onto the linear span $[e_k : 1 \leq k \leq n]$, where $(e_n)$ is a fixed orthonormal basis of $H$. Observe that

$$(BP_nA)^*x^* = \langle A^*x^*, P_n^*B^*y^* \rangle \to \langle A^*x^*, B^*y^* \rangle = \langle T^*x^*, y^* \rangle$$

as $n \to \infty$ for any $x^* \in X^*$ and $y^* \in Y^*$, because $\|P_n^*B^*y^* - B^*y^*\| \to 0$ as $n \to \infty$. Since $T$ is a compact operator this means that $BP_nA \to T$ weakly in $K(X,Y)$ as $n \to \infty$, see e.g. Corollary 3. By Mazur’s theorem, for any given $\varepsilon > 0$ there is a finite convex combination

$$\sum_{n=p}^q \lambda_n BP_n A \in F(X,Y)$$

such that

$$\|T - \sum_{n=p}^q \lambda_n BP_n A\| < \varepsilon.$$ 

Hence $T$ is an approximate operator.

The general case reduces to the separable case by the following elementary observation. (Alternatively, one may apply Remark 3 from [28, page 512], but that result depends on more sophisticated facts.) Namely, if the compact operator $T = BA \in K(X,Y)$ factors through a Hilbert space $H$, then we may actually factor $T$ as $T = B_0A_0$ through a closed separable subspace $H_1$ of $H$. Indeed, it suffices to show that $T$ is approximable considered as an operator $X \to \overline{T(X)}$, where $\overline{T(X)} \subset Y$ is a separable subspace, since $\overline{TBX}$ is separable by the compactness of $T$. Hence we may also suppose that $Y$ is separable. We first factor $B = \hat{B}P$ through $H_1 = \text{Ker}(B)^\perp$, where $P$ is the orthogonal projection of $H$ onto $H_1$ and $\hat{B} = B_{\text{Ker}(B)^\perp}$. Thus $T = B_0A_0$, where $B_0 = \hat{B}$ is an injective operator $H_1 \to Y$ and $A_0 = PA$. Finally, it follows that $H_1$ is separable from the general fact stated separately in Lemma 2.3 below.

Above we applied the following general observation, which we include for completeness.

**Lemma 2.3.** Let $Z$ and $W$ be Banach spaces such that $Z$ is reflexive and $W$ is separable. If there is a bounded linear injection $S : Z \to W$, then $Z$ is a separable space.

**Proof.** Fix an isometric embedding $J : W \to \ell_\infty$ and let $D : \ell_\infty \to \ell^2$ be the injective diagonal operator $(a_n) \mapsto (a_nx_n)$, where $(a_n) \in \ell^2$ and $a_n \neq 0$ for all $n$. Then $U = DSJ$ is a bounded linear injection $Z \to \ell^2$. Since $Z$ is reflexive, the range $U^*(\ell^2)$ is norm-dense in $Z^*$ by the Hahn-Banach and the Mazur theorems, so that $Z^*$ is separable. □

As a brief digression we note that the result of John [28] used in Theorem 2.2 also yields that there are no $\mathcal{A}^c$-universal operators for the class $\mathcal{A}^c$ of the non-approximable operators, which was mentioned as a problem in [4, 2.1]. Let $I$ be a Banach operator ideal. Recall from [12, 1.12] that the operator $U \in \mathcal{L}(X,Y)$, where $U \notin \mathcal{I}(X,Y)$, is $\mathcal{I}^c$-universal if for every Banach space $Z$, $W$ and for every bounded operator $V \in \mathcal{L}(Z,W) \setminus \mathcal{I}(Z,W)$ there are bounded operators $A \in \mathcal{L}(X,Z)$ and $B \in \mathcal{L}(W,Y)$ such that $U = BVA$. For instance, Johnson [30] showed that the canonical inclusion $J : \ell^1 \to \ell^\infty$ is a $\mathcal{K}^c$-universal operator. We refer to the recent paper by Beanland and Causey [4] for a systematic study of universal factoring operators for various classes of operators.

**Corollary 2.4.** There are no $\mathcal{A}^c$-universal operators.

**Proof.** Suppose to the contrary that $U \in \mathcal{L}(X,Y)$ is a $\mathcal{A}^c$-universal operator. Fix a compact non-approximable operator $V \in K(Z,W) \setminus \mathcal{A}(Z,W)$ for suitable Banach spaces $Z$ and $W$. By
assumption there are bounded operators $A$ and $B$ for which $U = BVA$, so that $U \in \mathcal{K}(X,Y)$. Moreover, by assumption $U$ also factors through the identity $I_{q^2}$, so that $U$ is a compact operator that factors through a Hilbert space. It follows from [28, Lemma 2] that $U \in \mathcal{A}(X,Y)$, which is not possible.

We will in the sequel several times require various classical examples related to the existence of closed subspaces of $\ell^p$-spaces that fail the A.P. To avoid repetition we collect these results here, together with their sources, for convenient reference.

**Facts 2.5.** Let $1 \leq p < \infty$ and $p \neq 2$. Then

(i) there are closed subspaces $X \subset \ell^p$ and $X \subset c_0$ that fail the A.P., and

(ii) there are closed subspaces $Z \subset \ell^p$ and $Z \subset c_0$ such that $\mathcal{A}(Z) \subsetneq \mathcal{K}(Z)$.

For part (i) recall that the first examples of closed subspaces $X$ failing the A.P. were constructed by Enflo [18] for $c_0$, by Davie [11] for $\ell^p$ and $2 < p < \infty$, and by Szankowski [49] for $\ell^p$ and $1 \leq p < 2$. We also refer to [36, Section 2.d], [37, Section 1.g] and [41, Section 10.4] for systematic expositions.

Concerning (ii) Alexander [2] found closed subspaces $Z \subset \ell^p$ for $2 < p < \infty$ such that $\mathcal{A}(Z) \subsetneq \mathcal{K}(Z)$, and she observed that similar examples can be deduced from the compact factorisation result [19, Theorem 7.4] of Figiel. Moreover, a general result of Bachelis [3, Theorem 2] immediately implies (ii), once the examples in (i) are known.

We point out that in part (ii) the case $Z \subset c_0$ can also be deduced from earlier factorisation results. In fact, if the closed subspace $X \subset c_0$ fails to have the A.P., then by [36, Theorem 1.e.4] there is a Banach space $Y$ and an operator $T \in \mathcal{K}(Y,X) \setminus \mathcal{A}(Y,X)$. By a compact factorisation theorem of Terzioğlu [52] (see also Randtke [45]) we may factor $T$ compactly through a closed subspace of $c_0$, that is, there is a closed subspace $Z_0 \subset c_0$ and $A \in \mathcal{K}(Y,Z_0)$, $B \in \mathcal{K}(Z_0,X)$ such that $T = BA$. Consider $Z = Z_0 \oplus X \subset c_0$ and define $U \in \mathcal{L}(Z)$ by

$$U(x,y) = (0, Bx), \quad (x,y) \in Z.$$ 

It follows that $U \in \mathcal{K}(Z) \setminus \mathcal{A}(Z)$, since $B$ cannot be an approximable operator.

We will often use the operator matrix notation

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$$

for bounded operators $U \in \mathcal{L}(X \oplus Y)$ on direct sums $X \oplus Y$. The alternative notation $(U)_{i,j} = U_{ij}$ for $U \in \mathcal{L}(X \oplus Y)$ and $i,j = 1,2$ will also be used where more appropriate. The operator ideal property of $\mathcal{K}$ implies that $U \in \mathcal{K}(X \oplus Y)$ if and only if each component operator $U_{ij} \in \mathcal{K}$, and a similar fact holds for the components of $U \in \mathcal{I}(X \oplus Y)$ for any Banach operator ideal $\mathcal{I}$. Let $\mathcal{I}_{i1} \subset \mathcal{K}(X)$, $\mathcal{I}_{i2} \subset \mathcal{K}(Y,X)$, $\mathcal{I}_{21} \subset \mathcal{K}(X,Y)$ and $\mathcal{I}_{22} \subset \mathcal{K}(Y)$ be given classes of operators. We introduce the convenient notation

$$\left( \mathcal{I}_{i1} \quad \mathcal{I}_{i2} \\ \mathcal{I}_{21} \quad \mathcal{I}_{22} \right) =: \{U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \in \mathcal{K}(X \oplus Y) : U_{ij} \in \mathcal{I}_{ij} \text{ for } i,j = 1,2\},$$

for the resulting class of compact operators on $X \oplus Y$.

We proceed to construct a class of direct sums $Z = X \oplus Y$, for which $\mathfrak{A}_Z$ admits non-trivial closed ideals. The fact that the quotient algebra $\mathfrak{A}_{X \oplus Y}$ has a lower triangular form in the operator matrix representation will play a crucial role. There is an analogy with the classical result that the Banach algebra $\mathcal{L}(\ell^p \oplus \ell^q)$ contains two incomparable maximal closed ideals for $p < q$, see e.g. [41, 5.3.2].
Theorem 2.6. Suppose that $X$ and $Y$ are Banach spaces such that $X$ has cotype 2, $Y$ has type 2, as well as $\mathcal{A}(X) \subseteq K(X)$ and $\mathcal{A}(Y) \subseteq K(Y)$. Let

$$\mathcal{I} = \begin{pmatrix} K(X) & \mathcal{A}(Y, X) \\ K(X, Y) & \mathcal{A}(Y) \end{pmatrix}$$

and

$$\mathcal{J} = \begin{pmatrix} \mathcal{A}(X) & \mathcal{A}(Y, X) \\ K(X, Y) & K(Y) \end{pmatrix},$$

where $K(Y, X) = \mathcal{A}(Y, X)$ in view of Theorem 2.2.

Then $\mathcal{I}$ and $\mathcal{J}$ are non-trivial incomparable closed ideals of $K(X \oplus Y)$, and $q(\mathcal{I})$ and $q(\mathcal{J})$ are non-trivial incomparable closed ideals of $\mathfrak{A}_{X \oplus Y}$. In particular, for $1 \leq p < 2 < q < \infty$ there are closed subspaces $X \subset \ell^p$ and $Y \subset \ell^q$ for which $\mathfrak{A}_{X \oplus Y}$ contains (at least) two non-trivial incomparable closed ideals.

Proof. We show that $\mathcal{I}$ and $\mathcal{J}$ are actually closed ideals of $\mathcal{L}(X \oplus Y)$. Clearly $\mathcal{I}$ and $\mathcal{J}$ are closed subspaces of $\mathcal{K}(X \oplus Y)$.

Let $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$, $V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \in \mathcal{L}(X \oplus Y)$, so that

$$UV = \begin{pmatrix} U_{11}V_{11} + U_{12}V_{21} & U_{11}V_{12} + U_{12}V_{22} \\ U_{21}V_{11} + U_{22}V_{21} & U_{21}V_{12} + U_{22}V_{22} \end{pmatrix}.$$  \hfill (2.2)

We claim that $UV \in \mathcal{I}$ and $VU \in \mathcal{I}$ for arbitrary $U \in \mathcal{I}$ and $V \in \mathcal{L}(X \oplus Y)$. Note first that both $UV$ and $VU$ belong to $\mathcal{K}(X \oplus Y)$, so that by Theorem 2.2 the component

$$(UV)_{1,2} \in \mathcal{K}(Y, X) = \mathcal{A}(Y, X),$$

and similarly $(VU)_{1,2} \in \mathcal{A}(Y, X)$. It remains to verify that the components $(UV)_{2,2} \in \mathcal{A}(Y)$ and $(VU)_{2,2} \in \mathcal{A}(Y)$. By (2.2) we know that

$$(UV)_{2,2} = U_{21}V_{12} + U_{22}V_{22}.$$ 

In view of the factorisation theorem of Kwapien and Maurey (see [44, Theorem 3.4 and Corollary 3.6]) we may factor $V_{12} = BA$, where $B \in \mathcal{L}(H, X)$ and $H$ is a Hilbert space. Since $U_{21} \in \mathcal{K}(X, Y)$ it follows that $U_{21}B \in \mathcal{K}(H, Y) = \mathcal{A}(H, Y)$ and $U_{21}V_{12} = U_{12}BA \in \mathcal{A}(Y)$. Consequently $(UV)_{2,2} \in \mathcal{A}(Y)$, since $U_{22} \in \mathcal{A}(Y)$ by assumption. Moreover, the corresponding component $(VU)_{2,2} = V_{21}U_{12} + V_{22}U_{22} \in \mathcal{A}(Y)$, since $U_{12}$ and $U_{22}$ are approximable operators.

In the case of $\mathcal{J}$ it remains to show that the $(1,1)$-components of $UV$ and $VU$ are approximable for $U \in \mathcal{J}$ and $V \in \mathcal{L}(X \oplus Y)$. From (2.2) we have

$$(VU)_{1,1} = V_{11}U_{11} + V_{12}U_{21},$$

where $U_{11} \in \mathcal{A}(X)$ by assumption. As above one uses the Kwapien-Maurey factorisation theorem to deduce that $V_{12}U_{21} \in \mathcal{A}(X)$. In addition, $(UV)_{1,1} = U_{11}V_{11} + U_{12}V_{21} \in \mathcal{A}(X)$ since the operators $U_{11}$ and $U_{12}$ are approximable.

From our assumptions there are operators $U_{11} \in \mathcal{K}(X) \setminus \mathcal{A}(X)$ and $V_{22} \in \mathcal{K}(Y) \setminus \mathcal{A}(Y)$, so that

$$U = \begin{pmatrix} U_{11} & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{I} \setminus \mathcal{J}$$

and

$$V = \begin{pmatrix} 0 & 0 \\ 0 & V_{22} \end{pmatrix} \in \mathcal{J} \setminus \mathcal{I}.$$ 

Hence $\mathcal{I}$ and $\mathcal{J}$ are incomparable ideals, and it follows that

$$\mathcal{A}(X \oplus Y) \subseteq \mathcal{I} \subseteq \mathcal{K}(X \oplus Y), \quad \mathcal{A}(X \oplus Y) \subseteq \mathcal{J} \subseteq \mathcal{K}(X \oplus Y).$$

Proposition 2.7 yields that $q(\mathcal{I})$ and $q(\mathcal{J})$ are non-trivial incomparable closed ideals of $\mathfrak{A}_{X \oplus Y}$. 
For the final claim recall from Facts 2.3.11(ii) that for \( 1 \leq p \leq q < \infty \) there are closed subspaces \( X \subset \ell^p \) and \( Y \subset \ell^q \) such that \( \mathfrak{A}_X \neq \{0\} \) and \( \mathfrak{A}_Y \neq \{0\} \). Moreover, recall that \( X \subset \ell^p \) has cotype 2 for \( 1 \leq p < 2 \) and \( Y \subset \ell^q \) has type 2 for \( 2 < q < \infty \). \( \square \)

It is relevant to ask how many non-trivial closed ideals of \( \mathcal{K}(X \oplus Y) \) we may construct for direct sums \( X \oplus Y \) belonging to the class of spaces in Theorem 2.5. For instance, the closed ideal \( \mathcal{I} \cap \mathcal{J} \) of \( \mathcal{K}(X \oplus Y) \) is non-trivial if and only if \( \mathcal{A}(X, Y) \subset \mathcal{K}(X, Y) \). In Example 3.13 we will construct a direct sum \( X \oplus Y \) from this class of spaces for which \( \mathcal{K}(X \oplus Y) \), and consequently also \( \mathfrak{A}_{X \oplus Y} \), contains at least 8 non-trivial closed ideals. Such examples require more preparation, including a study of the strict inclusion \( \mathcal{A}(X, Y) \subset \mathcal{K}(X, Y) \) among closed subspaces \( X \subset \ell^p \) and \( Y \subset \ell^q \) for \( p \neq q \), as well as non-classical approximation properties associated to Banach operator ideals.

Recall that the Banach algebra \( A \) is nilpotent if there is \( m \in \mathbb{N} \) such that \( x_1 \cdots x_m = 0 \) for all \( x_1, \ldots, x_m \in A \). It follows from a general result, see [3], Proposition 1.5.6.(iv), that a non-nilpotent complex radical Banach algebra is infinite-dimensional. In the sequel we also wish to apply this fact to the real quotient algebra \( \mathfrak{A}_X = \mathcal{K}(X)/\mathcal{A}(X) \) in the case of real Banach spaces \( X \). One reason is that the problem of the closed ideals of \( \mathfrak{A}_X \), that is, the closed ideals \( \mathcal{A}(X) \subset \mathcal{J} \subset \mathcal{K}(X) \), is also relevant for real spaces \( X \). To justify this application we must verify that for real Banach spaces \( X \) the real Banach algebra \( \mathfrak{A}_X \) is radical in the interpretation [1.3] from Section 1 which is one of the equivalent conditions of radicality for complex Banach algebras. Actually, we will need a general version for quotient algebras which are formed from nested closed ideals contained in the class of inessential operators.

Recall that the operator \( S \in \mathcal{L}(X, Y) \) is inessential, denoted \( S \in \mathcal{R}(X, Y) \) if \( I_X - TS \) is a Fredholm operator for all \( T \in \mathcal{L}(Y, X) \), that is, the kernel \( \text{Ker}(I_X - TS) \) is finite-dimensional and the range \( \text{Im}(I_X - TS) \) has finite codimension in \( X \). The closed Banach operator ideal \( \mathcal{R} \) was introduced by Kleinecke [34]. Thus \( \mathcal{K}(X) \subset \mathcal{R}(X) \), and it is known that \( \mathcal{R}(X) \) is the largest closed ideal of \( \mathcal{L}(X) \) for which Atkinson’s characterisation of Fredholm operators holds for any Banach space \( X \) (see below). Recall that the Banach operator ideal \( \mathcal{I} \) is closed if the components \( \mathcal{I}(X, Y) \) are closed in the operator norm for all spaces \( X \) and \( Y \).

**Proposition 2.7.** Let \( X \) be an infinite-dimensional real Banach space. If \( \mathcal{I} \) and \( \mathcal{J} \) are closed Banach operator ideals such that \( \mathcal{A}(X) \subset \mathcal{I}(X) \subset \mathcal{J}(X) \subset \mathcal{R}(X) \), then the real quotient algebra

\[
\mathfrak{A}_X =: \mathcal{J}(X)/\mathcal{I}(X)
\]

is radical in the sense of [1.3]. In particular, \( \mathfrak{A}_X \) is a real radical Banach algebra.

**Proof.** Put \( \mathbb{K} = \mathbb{R} \). Recall that the abstract unitisation \( (\mathfrak{A}_X)^\# \) of \( \mathfrak{A}_X \) consists of \( \mathbb{K} \oplus \mathfrak{A}_X \) with the product

\[
(\alpha, S + \mathcal{I}(X)) \cdot (\beta, T + \mathcal{I}(X)) = (\alpha \beta, \alpha T + \beta S + ST + \mathcal{I}(X)), \quad \alpha, \beta \in \mathbb{K}, \ S, T \in \mathcal{J}(X),
\]

and the algebra norm \( \|(\alpha, S + \mathcal{I}(X))\| = |\alpha| + \|S + \mathcal{I}(X)\| \). Observe next that the unitisation \( (\mathfrak{A}_X)^\# \) can also be concretely identified with the closed subalgebra

\[
B =: (\mathbb{K}I_X \oplus \mathcal{J}(X))/\mathcal{I}(X)
\]

of \( \mathcal{L}(X)/\mathcal{I}(X) \). (Since \( \mathcal{J}(X) \subset \mathcal{R}(X) \) this observation will connect the unitisation to classical Fredholm theory, which is independent of the scalar field.) To see this identification note that the map

\[
(\alpha, S + \mathcal{I}(X)) \mapsto \alpha I_X + S + \mathcal{I}(X)
\]
defines an algebra isomorphism $\theta : \mathbb{K} \oplus \mathbb{J}_X^\ell \to B$, since

$$(1/3)(|\alpha| + \|S + I(X)\|) \leq \|\alpha I_X + S + I(X)\| \leq |\alpha| + \|S + I(X)\|$$

for $\alpha \in \mathbb{K}$ and $S \in \mathcal{J}(X)$. In fact, for $S \in \mathcal{J}(X)$ we have

$$|\alpha| = \|\alpha I_X + \mathcal{J}(X)\| = \|\alpha I_X + S + \mathcal{J}(X)\| \leq \|\alpha I_X + S + I(X)\|,$$

so that $|\alpha| + \|S + I(X)\| \leq 3\|\alpha I_X + S + I(X)\|$ by the triangle inequality. Above we used that $\|I_X + \mathcal{J}(X)\| = 1$. Namely, if $\|I_X - U\| < 1$ for some $U \in \mathcal{J}(X)$, then $U = I_X - (I_X - U)$ is invertible. This is impossible since $X$ is infinite-dimensional and $U \in \mathcal{J}(X) \subset \mathcal{R}(X)$.

Next let $\alpha \neq 0$ and $S \in \mathcal{J}(X)$ be arbitrary. To identify the inverse $(\alpha I_X + S + I(X))^{-1}$ in $B$, let $\pi : \mathcal{L}(X) \to \mathcal{L}(X)/\mathcal{J}(X)$ be the quotient map. Recall that an operator $U \in \mathcal{L}(X)$ is invertible modulo $\mathcal{A}(X)$ (that is, a Fredholm operator) if and only if it is invertible modulo $\mathcal{R}(X)$, see e.g. [34, Theorem 2] or [41, section 26.3 and 26.7.2]. Note that this result holds equally well for real Banach spaces. We know that $\pi(\alpha I_X + S) = \alpha I_X + \mathcal{J}(X)$ has the inverse $\alpha^{-1}I_X + \mathcal{J}(X)$ in $\mathcal{L}(X)/\mathcal{J}(X)$. Hence the above invertibility fact implies that there is $V \in \mathcal{J}(X)$ and $R_1, R_2 \in \mathcal{I}(X)$ for which

$$(\alpha^{-1}I_X + V)(\alpha I_X + S) = I_X + R_1 \text{ and } (\alpha I_X + S)(\alpha^{-1}I_X + V) = I_X + R_2.$$ 

In other words, the inverse $(\alpha I_X + S + I(X))^{-1} = \alpha^{-1}I_X + V + I(X)$ belongs to $B$, so that $\sigma_R(S + I(X)) = \{0\}$ for all $S \in \mathcal{J}(X)$ (that is, condition [13] holds).

Finally, $\mathfrak{A}_X$ is obtained for $\mathcal{I}(X) = \mathcal{A}(X)$ and $\mathcal{J}(X) = \mathcal{K}(X)$. \hfill $\square$

The above argument is also valid for the complex scalars $\mathbb{K} = \mathbb{C}$, so that $\mathbb{J}_X^\ell = \mathcal{J}(X)/\mathcal{I}(X)$ is a radical Banach algebra in the classical sense. We also point out that the non-zero real quotient algebras $\mathfrak{A}_X \neq \{0\}$ (as well as $\mathbb{J}_X^\ell \neq \{0\}$) cannot have a unit element in view of Proposition [27]. In fact, if $S + \mathcal{A}(X)$ were the unit in $\mathfrak{A}_X$ for some $S \in \mathcal{K}(X)$, then $I_X - S + \mathcal{A}(X)$ is invertible in the unitisation $B$, because $\sigma_R(-S + \mathcal{A}(X)) = \{0\}$. Let $\alpha I_X + T + \mathcal{A}(X)$ be the inverse in $B$, so that

$$I_X + \mathcal{A}(X) = (I_X - S + \mathcal{A}(X))(\alpha I_X + T + \mathcal{A}(X)) = \alpha I_X - \alpha S + T - ST + \mathcal{A}(X),$$

$$= \alpha I_X - \alpha S + \mathcal{A}(X),$$

since $ST - T \in \mathcal{A}(X)$ by assumption. This implies that $\alpha = 1$ and $S \in \mathcal{A}(X)$, which contradicts the assumption.

We showed in [53, Proposition 3.1.(i)] that if the Banach space $X$ has the B.C.A.P., but fails the A.P., then there is a compact operator $U \in \mathcal{K}(X)$ such that $U^m \notin \mathcal{A}(X)$ for any $m \in \mathbb{N}$. In particular, the quotient algebra $\mathfrak{A}_X$ is non-nilpotent. In [53, Proposition 2.2 and Corollary 2.4] we constructed linear isomorphic embeddings $\psi : c_0 \to \mathfrak{A}_X$ for certain closed subspaces $X \subset \ell^p$, where $1 \leq p < \infty$ and $p \neq 2$, and $X \subset c_0$. However, by [53, Proposition 2.5] the embedding $\psi$ cannot preserve any of the multiplicative structure of $c_0$, and the construction does not ensure that $\mathfrak{A}_X$ is non-nilpotent. We next improve and complement these results from [53], and show there are also closed subspaces $X \subset \ell^p$, where $1 \leq p < \infty$ and $p \neq 2$, and $X \subset c_0$, such that the quotient algebra $\mathfrak{A}_X$ is non-nilpotent and infinite-dimensional. These subspaces will also be used in later examples. By contrast with [53, section 2] the subsequent construction in Theorem [28] will be based on a general compact factorisation result of Bachelis [3], which allows for greater generality and additional features.
Let $E$ be a real Banach space with a 1-unconditional basis $(e_n)$, and suppose that $(E_n)$ is a sequence of Banach spaces. The corresponding $E$-direct sum is defined by
\[
(\oplus_{n\in\mathbb{N}} E_n)_E = \{ (x_n) : x_n \in E_n, \sum_{n=1}^{\infty} \|x_n\|e_n \text{ converges in } E \}.
\]
It is not difficult to check by using [36, Proposition 1.c.7] that
\[
\| (x_n) \|_E = \left\| \sum_{n=1}^{\infty} \|x_n\|e_n \right\|_E,
\]
defines a complete norm in $(\oplus_{n\in\mathbb{N}} E_n)_E$. We use $Y \approx Z$ for linearly isomorphic Banach spaces $Y$ and $Z$.

**Theorem 2.8.** Suppose that the Banach space $E$ has the B.A.P., and that
\[
(\oplus_{n\in\mathbb{N}} E)_E \approx E
\]
for some real Banach space $E$ which has a 1-unconditional basis. If $E$ has a closed subspace $X_0$ that fails the A.P., then there is a closed subspace $X \subset E$ such that $\mathfrak{A}_X$ is infinite-dimensional, and for which there is an operator $U \in \mathcal{K}(X)$ that satisfies $U^m \notin \mathcal{A}(X)$ for all $m \in \mathbb{N}$. In particular, $\mathfrak{A}_X$ is a non-nilpotent quotient algebra.

The assumptions apply e.g. to $E = t^p$ for $1 \leq p < \infty$ and $p \neq 2$, or $E = c_0$.

**Proof.** Since $X_0$ fails the A.P. there is a Banach space $W$ and a compact non-approximable operator $T_0 : W \to X_0$, see e.g. [36, Theorem 1.e.4]. By Bachelis' factorisation theorem [3, Theorem 2'] there is a closed subspace $Z_1$ of $E$ and a compact factorisation $T_0 = B_1 A_1$, that is, $A_1 \in \mathcal{K}(W,Z_1)$ and $B_1 \in \mathcal{K}(Z_1,X_0)$. Note that $B_1 \notin \mathcal{A}(Z_1,X_0)$, since $T_0$ is not an approximable operator.

By successive applications of [3, Theorem 2'] we obtain a sequence $(Z_n)$ of closed subspaces of $E$ as well as sequences $(A_n)$ and $(B_n)$ of compact operators, such that
\[
B_n = B_{n+1} A_{n+1}, \quad \text{where } A_{n+1} \in \mathcal{K}(Z_n,Z_{n+1}) \quad \text{and } B_{n+1} \in \mathcal{K}(Z_{n+1},X_0) \setminus \mathcal{A}(Z_{n+1},X_0)
\]
for all $n \in \mathbb{N}$.

Let $X = (X_0 \oplus (\oplus_{n\in\mathbb{N}} Z_n))_E$. By assumption $X$ is, up to a linear isomorphism, a closed linear subspace of $E$. Let $J_n$ denote the natural inclusion maps and $P_n$ the natural coordinate projections for $n \geq 0$, such that $P_0 : X \to X_0$ and $J_0 : X_0 \to X$, whereas $P_n : X \to Z_n$ and $J_n : Z_n \to X$ for $n \geq 1$. (Here we canonically identify $X_0$ and $Z_n$ with closed subspaces of $X$.) Define operators $\widehat{B}_n \in \mathcal{L}(X)$ for $n \in \mathbb{N}$ and $\widehat{A}_n \in \mathcal{L}(X)$ for $n \geq 2$ as follows:
\[
\widehat{B}_n(x,z_1,z_2,\ldots) = (B_n z_n, 0, 0, \ldots) \quad \text{and} \quad \widehat{A}_n(x,z_1,\ldots) = (0, 0, \ldots, A_n z_{n-1}, 0, \ldots),
\]
where $A_n z_{n-1} \in Z_n$ (in the $n$:th position in $\oplus_{k\in\mathbb{N}} Z_k$). This means that $\widehat{B}_n = J_0 B_n P_n \in \mathcal{K}(X)$ for all $n \in \mathbb{N}$ and $\widehat{A}_n = J_n A_n P_{n-1} \in \mathcal{K}(X)$ for all $n \geq 2$. Moreover,

\[
\widehat{B}_1 = \widehat{B}_2 \widehat{A}_2 = \widehat{B}_3 \widehat{A}_3 \widehat{A}_2 = \ldots = \widehat{B}_n \circ \left( \prod_{k=1}^{n-1} \widehat{A}_{n+1-k} \right)
\]
for $n \in \mathbb{N}$. Namely, by successive evaluations one gets that
\[
\widehat{B}_n \widehat{A}_n \cdots \widehat{A}_2(x_1,z_1,z_2,\ldots) = (B_n A_n \ldots A_2 z_1, 0, \ldots)
\]
\[
= (B_1 z_1, 0, \ldots) = \widehat{B}_1 (x_1,z_1,z_2,\ldots)
\]
for all \((x_1, z_1, \ldots) \in X\). It follows from (2.3) that the quotient algebra \(A_X\) is not nilpotent, since \(\hat{B}_1 \notin A(X)\).

We are now in position to repeat the argument of [53, Proposition 3.1]. In fact, \(A_X\) is a non-nilpotent radical Banach algebra, which is infinite-dimensional by [53, Proposition 1.5.6.(iv)]. Note that for real Banach spaces \(X\) we need Proposition 2.7 to ensure that \(A_X\) is a real radical Banach algebra in the sense of (1.3). Moreover, by the Baire category argument from 2.7 there is an operator \(U \in K(X)\) such that \(U^n \notin A(X)\) for any \(m \in \mathbb{N}\).

In the case of complex scalars the operator \(U \in K(X)\) from Theorem 2.8 has the stronger property that \(U^n - U^m \notin A(X)\) for all \(n \neq m\). Namely, if \(U^{n+k} - U^n \in A(X)\) for some \(n, k \in \mathbb{N}\), then by iteration \(U^{n+k} - U^n \in A(X)\) for all \(s \in \mathbb{N}\). Conclude that

\[
\lim_{s \to \infty} \|U^{n+k} - A(X)\|^{1/(n+k)} = \lim_{s \to \infty} \|U^n + A(X)\|^{1/(n+k)} = 1,
\]

since \(U^n \notin A(X)\). This contradicts the fact that \(A_X\) is a radical Banach algebra, because \(\lim_{m \to \infty} \|V^m + A(X)\|^{1/m} = 0\) for all \(V \in K(X)\) by the spectral radius formula.

Theorem 2.8 enables us to revisit the setting of [53, Proposition 2.2 and Corollary 2.4], and to obtain linear isomorphic embeddings \(c_0 \to A_X\) that display quite extreme properties for particular closed subspaces \(X \subset \ell^p\), where \(1 \leq p < \infty\) and \(p \neq 2\), or \(X \subset c_0\) for which

(i) there is a linear isomorphic embedding \(\theta : c_0 \to A_X\), such that \(\theta(a)^n \neq 0\) for all \(a \neq 0\) and \(n \in \mathbb{N}\), or

(ii) there is a linear isomorphic embedding \(\psi : c_0 \to A_X\), such that \(\psi(a)\psi(b) = 0\) for all \(a, b \in c_0\). In particular, the closed subalgebra \(A(\psi(c_0))\) of \(A_X\) generated by \(\psi(c_0)\) satisfies

\[A(\psi(c_0)) = \psi(c_0).\]

**Proof.** For unity of notation we only construct the desired closed subspaces \(X \subset \ell^p\). The case where \(X \subset c_0\) is similar.

(i) By Theorem 2.8 there is a closed linear subspace \(Y \subset \ell^p\) for \(p \neq 2\) as well as \(U \in K(Y)\) such that \(U^m \notin A(Y)\) for all \(m \in \mathbb{N}\). Consider \(X = (\oplus_{n \in \mathbb{N}} Y)_{\ell^p}\), which can be identified with a closed subspace of \(\ell^p\). Define \(\beta : c_0 \to K(X)\) by

\[
\beta(a) = \sum_{k=1}^{\infty} a_k U_k, \quad a = (a_k) \in c_0,
\]

where \(U_k\) denotes the operator \(U\) defined on the \(k\):th copy of \(Y\) in \(X\). Let \(a = (a_k) \in c_0\) be arbitrary. Since \(U_k \circ U_r = 0\) for all \(k \neq r\) by definition, we get that

\[
\left( \sum_{k=1}^{m} a_k U_k \right) \circ \left( \sum_{r=1}^{m} a_k U_k \right) = \sum_{k=1}^{m} a_k^2 U_k^2
\]

for all \(m \in \mathbb{N}\). Since \((a_k) \in c_0\) we may pass to the limit in \(K(X)\) as \(m \to \infty\), and obtain that \(\beta(a)^2 = \beta(a) \cdot \beta(a) = \sum_{k=1}^{\infty} a_k^2 U_k^2\). By induction we get that

\[
\beta(a)^n = \sum_{k=1}^{\infty} a_k^n U_k^n, \quad a = (a_k) \in c_0.
\]
Suppose next that \(a = (a_k) \neq 0\) and pick \(k \in \mathbb{N}\) such that \(a_k \neq 0\). Clearly \(\beta(a)^n \notin \mathcal{A}(X)\), since the \(k\):th term \(a_k^n U_k^n \notin \mathcal{A}(X)\).

Finally, one verifies as in the proof of [53, Proposition 2.2] that \(\theta = q \circ \beta\) is a linear isomorphic embedding \(c_0 \to \mathcal{A}_X\), where \(q : \mathcal{K}(X) \to \mathcal{A}_X\) is the quotient map. From the above computation it follows that \(\theta(a)^n \neq 0\) whenever \(a \neq 0\) and \(n \in \mathbb{N}\).

(ii) The argument modifies the original construction in [53, section 2]. According to Facts 2.5(ii) we may pick a closed linear subspace \(Z \subset \ell^p\) and an operator \(U \in \mathcal{K}(Z) \setminus \mathcal{A}(Z)\).

Consider \(Y = Z \oplus Z\) and define \(V \in \mathcal{K}(Y)\) by \(V(x, y) = (0, Ux)\) for \((x, y) \in Z \oplus Z\). Then \(V \notin \mathcal{A}(Y)\) and \(V^2 = 0\).

Consider \(X = (\oplus_{n \in \mathbb{N}} Y)_{\ell^p}\), and define \(\beta : c_0 \to \mathcal{K}(X)\) by

\[\beta(a) = \sum_{k=1}^{\infty} a_k V_k, \quad a = (a_k) \in c_0,\]

where \(V_k\) denotes the operator \(V\) on the \(k\):th copy of \(Y = Z \oplus Z\) in \(X\). As above \(\psi = q \circ \beta\) is a linear isomorphic embedding \(c_0 \to \mathcal{A}_X\). Let \(a = (a_k), b = (b_k) \in c_0\) be arbitrary. Since \(V_k \circ V_r = 0\) for all \(k, r \in \mathbb{N}\) by definition, we get that

\[\left(\sum_{k=1}^{m} a_k V_k\right) \circ \left(\sum_{r=1}^{m} b_k V_k\right) = \sum_{k=1}^{m} a_k b_k V_k^2 = 0\]

for all \(m \in \mathbb{N}\). Deduce that in the limit \(\beta(a)\beta(b) = 0\), so that \(\psi(a)\psi(b) = 0\).

Finally, finite linear combinations of the products \(\psi(u_1) \cdots \psi(u_s)\), where \(u_1, \ldots, u_s \in c_0\) and \(s \in \mathbb{N}\), are dense in \(\mathcal{A}(\psi(c_0))\). This implies that \(\mathcal{A}(\psi(c_0)) = \psi(c_0)\). \(\square\)

3. Compact non-approximable operators between subspaces of \(\ell^p\) and \(\ell^q\)

In this section we first discuss the existence of compact non-approximable operators between closed subspaces \(X \subset \ell^p\) and \(Y \subset \ell^q\) for \(p \neq q\), that is, whether there are such subspaces for which

\[(3.1) \quad \mathcal{A}(X, Y) \subsetneq \mathcal{K}(X, Y).\]

Recall from Facts 2.5(ii) that for any \(p \neq 2\) there are closed subspaces \(Z \subset \ell^p\) for which \(\mathcal{A}(Z) \subsetneq \mathcal{K}(Z)\). However, Theorem 2.2 implies that (3.1) cannot hold for \(q < 2 < p\), so the situation becomes subtler once \(X\) and \(Y\) are specified from different classes of spaces. The cases \(p = 2\) or \(q = 2\) are excluded from our discussion since (3.1) is impossible in this event.

Our motivation is the quest for additional examples in the setting of Theorem 2.6 but the question in (3.1) has fundamental interest. It turns out to involve results and concepts which were devised for other purposes. In particular, we will use the Banach operator ideal of the operators that factor compactly through a subspace of \(\ell^p\) as well as non-classical approximation properties. We apply these results to exhibit several examples related to Theorem 2.6 including a direct sum \(X \oplus Y\) in Example 3.14 where \(\mathcal{K}(X \oplus Y)\) contains (at least) 8 non-trivial closed ideals.

The following Banach factorisation ideals will be essential for our purposes. Let \(r \in [1, \infty)\) be fixed. For Banach spaces \(X\) and \(Y\) we define

\[(3.2) \quad \mathcal{KS}_r(X, Y) = \{T \in \mathcal{K}(X, Y) : T = BA, A \in \mathcal{K}(X, Z), B \in \mathcal{K}(Z, Y), Z \subset \ell^r\ \text{a closed subspace}\}.\]
The associated factorisation norm of $T \in \mathcal{KS}_r(X,Y)$ is

$$|T|_{\mathcal{KS}} = \inf \{ \|B\| : \|A\| : T = BA \text{ factors as in (3.2)} \}$$

Recall further that the class of classical $r$-compact operators $\mathcal{K}_r$ is defined by

$$\mathcal{K}_r(X,Y) = \{ T \in \mathcal{K}(X,Y) : T = BA, \text{ where } A \in \mathcal{K}(X,\ell^r), \ B \in \mathcal{K}(\ell^r,Y) \},$$

and the related factorisation norm is

$$|T|_{\mathcal{K}_r} = \inf \{ \|B\| : \|A\| : T = BA, \text{ where } A \in \mathcal{K}(X,\ell^r), \ B \in \mathcal{K}(\ell^r,Y) \}.$$

It is known that $(\mathcal{K}_r, |\cdot|_{\mathcal{K}_r})$ is a Banach operator ideal, see e.g. [29, Proposition 1] and [22, Theorem 2.1], or [41, 18.3]. The class $(\mathcal{KS}_r, |\cdot|_{\mathcal{KS}_r})$ is also a Banach operator ideal, but this fact is less well-documented in the literature. For $r = 2$ one has $\mathcal{KS}_2(X,Y) = \mathcal{K}_2(X,Y) \subset \mathcal{A}(X,Y)$ for all spaces $X$ and $Y$, so this case will not be of interest for us.

Let $(I, |\cdot|_I)$ be a Banach operator ideal. Recall that the injective hull $I^{\text{inj}}$ of $I$ is the Banach operator ideal

$$I^{\text{inj}}(X,Y) = \{ T \in \mathcal{L}(X,Y) : J_T \in I(X,\ell^\infty(B\ell^r)) \},$$

which is normed by $|T|_{I^{\text{inj}}} = |J_T|_I$ for $T \in I^{\text{inj}}(X,Y)$. Here $J_T$ is the natural isometric embedding from $Y$ into $\ell^\infty(B\ell^r)$. The following result was proved by Fourie [20, Theorem 2.1] (it is also stated without proof on lines 4-6 of [42, p. 529]).

**Proposition 3.1.** If $1 \leq r < \infty$, then

$$\mathcal{KS}_r = \mathcal{K}_r^{\text{inj}},$$

with equality of the respective operator ideal norms. In particular, $(\mathcal{KS}_r, |\cdot|_{\mathcal{KS}_r})$ is a Banach operator ideal.

**Remarks 3.2.** (i) The argument in [20] uses the additional fact [22, Theorem 2.3] that in the factorisation $T = BA$ from (3.2) it suffices to assume that $B$ is a bounded operator. We note that it is possible to avoid this step by using the compact extension property of $\ell^\infty(I)$. There are also direct approaches: $(\mathcal{KS}_r, |\cdot|_{\mathcal{KS}_r})$ can be shown a Banach operator ideal either by modifying the argument [40, Hilfssatz 1] for the class of operators that factors boundedly through $\ell^r$, or the argument from [29, Proposition 1] for the case $\mathcal{K}_r(X,Y)$. We leave the details for the reader.

(ii) There is an explicit characterisation of $\mathcal{K}_r^{\text{inj}}$ in [21, Theorem 3.6].

Our primary interest lies in the closed ideals $\overline{\mathcal{KS}_r(X)}$, for which $\mathcal{A}(X) \subset \overline{\mathcal{KS}_r(X)} \subset \mathcal{K}(X)$ for all Banach spaces $X$. To make our notation less cumbersome we have chosen to introduce the abbreviation $\overline{\mathcal{KS}_r(X)} := \mathcal{K}_r^{\text{inj}}$. We note that Proposition 3.3(ii) below implies that for any $r \neq 2$ there is a Banach space $X$ such that

$$\overline{\mathcal{K}_r(X)} \subset \mathcal{KS}_r(X) = \mathcal{K}_r^{\text{inj}}(X).$$

Moreover, the class $\mathcal{KS}_r$ is not monotone for $r \in [1, \infty)$ by Proposition 3.6.

We first observe that certain ideal components $\overline{\mathcal{KS}_r(X,Y)}$ and $\overline{\mathcal{KS}_r(X,Y)}$ can be identified among the closed subspaces of $\ell^2$-spaces thanks to Theorem 2.2 and the compact factorisation theorems of Figiel [19] and Bachelis [3].
Proposition 3.3. Suppose that \( p, q \in [1, \infty) \), where \( p \neq 2 \) and \( q \neq 2 \), and let \( X \subset \ell^p \) and \( Y \subset \ell^q \) be arbitrary closed subspaces. Then the following holds:

(i) \( \mathcal{K}(Z, Y) = \mathcal{K}_q(Z, Y) = \overline{\mathcal{K}_q(Z, Y)} \) for any Banach space \( Z \), so that
\[
\mathcal{K}(Y) = \mathcal{K}_q(Y) = \overline{\mathcal{K}_q(Y)}.
\]

(ii) There is a closed subspace \( Z \subset \ell^p \) such that \( \mathcal{K}_p(Z) \not\subset \mathcal{K}_q(Z) = \mathcal{K}^{int}(Z) \).

(iii) If \( p < 2 < q \), then
\[
\mathcal{K}_q(Z, X) = \mathcal{A}(Z, X) \text{ and } \overline{\mathcal{K}_q(Y, Z)} = \mathcal{A}(Y, Z)
\]
for any Banach spaces \( Z \).

Proof. (i) Suppose that \( T \in \mathcal{K}(Z, Y) \) is arbitrary. The compact factorisation results in [19, Theorem 7.4] or [3, Theorem 2] imply that there is a closed subspace \( W \subset \ell^q \) together with compact operators \( A \in \mathcal{K}(W, Y) \), \( B \in \mathcal{K}(W, Y) \) such that \( T = BA \). In other words, \( \mathcal{K}(Z, Y) = \mathcal{K}_q(Z, Y) \).

(ii) Recall from Facts 2.2(ii) that for all \( p \neq 2 \) there is a closed subspace \( Z \subset \ell^p \) such that \( \mathcal{A}(Z) \not\subset \mathcal{K}(Z) \). Clearly \( \mathcal{K}_p(Z) = \mathcal{A}(Z) \), while \( \mathcal{K}_q(Z) = \mathcal{K}(Z) \) by part (i).

(iii) If \( T \in \mathcal{K}_q(Z, X) \), then there is a closed subspace \( W \subset \ell^p \) and compact operators \( A \in \mathcal{K}(Z, W) \), \( B \in \mathcal{K}(W, X) \) such that \( T = BA \). Theorem 2.2 implies that here \( B \in \mathcal{A}(W, X) \), since \( p < 2 < q \). The first equality follows after passing to the uniform closure. The argument for the second equality is similar.

□

In Proposition 3.3(i) the equality \( \mathcal{K}(Z, Y) = \mathcal{K}_q(Z, Y) \) implies that \( | \cdot |_{\mathcal{K}_q} \) and \( \| \cdot \| \) are equivalent norms on \( \mathcal{K}(Z, Y) \) by the open mapping theorem.

We will split our discussion of the examples of strict inclusion (3.1) into the cases \( p, q \in [1, 2] \) and \( p, q \in (2, \infty) \), since they require quite different tools. We first record for comparison the following (essentially known) version of Pitt’s theorem for closed subspaces of \( \ell^r \)-spaces. The argument is a straightforward modification of the classical perturbation argument for that result, see e.g. [1, Theorem 2.1.4] or [36, Proposition 2.c.3], and it will not be reproduced here.

Proposition 3.4. Suppose that \( 1 \leq s < r < \infty \), and let \( X \subset \ell^r \) and \( Y \subset \ell^s \) be arbitrary closed subspaces. Then
\[
\mathcal{L}(X, Y) = \mathcal{K}(X, Y).
\]
This identity is also valid for closed subspaces \( X \subset c_0 \).

The cases \( p, q \in [1, 2] \) related to (3.1) revisit a factorisation theorem of Figiel [19] for compact operators mapping into \( L^p(\mu) \)-spaces. Let \( X \subset \ell^p \) and \( Y \subset \ell^q \) be closed subspaces. The following result demonstrates that Theorem 2.2 fails to hold in the range \( 1 \leq p < q < 2 \), and that Proposition 3.3 does not include the class \( \mathcal{A}(X, Y) \) for \( 1 \leq q < p < 2 \). (Analogous remarks also apply to the cases \( 2 < q < p < \infty \) in view of Theorem 3.5 below.)

Theorem 3.5. (i) If \( 1 \leq q \leq p < 2 \), then
\[
\mathcal{K}_q(X) = \overline{\mathcal{K}_q(X)} = \mathcal{K}(X)
\]
for any closed subspace \( X \subset \ell^p \).

(ii) Suppose that \( 1 \leq p, q < 2 \) and \( p \neq q \). Then there are closed subspaces \( X \subset \ell^p \) and \( Y \subset \ell^q \) for which the strict inclusion (3.1) holds.
(iii) For $1 \leq p < q < 2$ there is a closed subspace $X \subset \ell^p$ such that the quotient algebra
\[
\overline{\mathcal{KS}_q(X)} / \mathcal{A}(X)
\]
is non-nilpotent and infinite-dimensional.

**Proof.** (i) Let $1 \leq q \leq p < 2$ and $X \subset \ell^p$ be a closed subspace. Suppose that $T \in \mathcal{K}(X)$ is arbitrary. It is a classic fact that $\ell^p$ embeds isometrically into $L^q(0,1)$ for $1 \leq q \leq p < 2$, see e.g. [1], Theorem 6.4.18, so that $X$ embeds isometrically into $L^q(0,1)$. Hence it follows from [19, Theorem 7.4] that $T \in \mathcal{K}(X)$ factors compactly through a closed subspace of $\ell^q$, that is, there is a closed subspace $Z \subset \ell^q$ as well as compact operators $A \in \mathcal{K}(X,Z)$ and $B \in \mathcal{K}(Z,X)$ so that $T = BA$. In other words, $\mathcal{K}(X) = \mathcal{KS}_q(X) = \overline{\mathcal{KS}_q(X)}$.

(ii) Suppose that $1 \leq q < p < 2$. According to Facts [2,5](ii) there is a closed subspace $X \subset \ell^p$ that carries a compact non-approximable operator $T \in \mathcal{K}(X) \setminus \mathcal{A}(X)$. By part (i) we know that $T \in \mathcal{KS}_q(X) = \mathcal{K}(X)$, so there is a closed subspace $Z \subset \ell^q$ together with $A \in \mathcal{K}(X,Z)$, $B \in \mathcal{K}(Z,X)$, so that $T = BA$. Here neither $A$ nor $B$ can be approximable operators, so that
\[
A(X,Z) \subsetneq \mathcal{K}(X,Z) \text{ and } A(Z,X) \subsetneq \mathcal{K}(Z,X).
\]
The first strict inclusion gives the claim for $1 \leq q < p < 2$, while the claim for $1 \leq p < q < 2$ follows from the second strict inclusion in (3.3) after exchanging the roles of $p$ and $q$. (In fact, part (iii) contains a much stronger result for $1 \leq p < q < 2$.)

(iii) Let $1 \leq p < q < 2$. In view of Theorem [2,8] there is a closed subspace $Z \subset \ell^p$ and a compact operator $T \in \mathcal{K}(Z)$ for which $T^n \notin \mathcal{A}(Z)$ for any $n \in \mathbb{N}$. Since $1 \leq p < q < 2$, we know that $T \in \mathcal{KS}_p(Z)$ by part (i), so there is a closed subspace $X \subset \ell^p$ and a factorisation $T = BA$, where $A \in \mathcal{K}(Z,X)$ and $B \in \mathcal{K}(X,Z)$. Here $AB \in \mathcal{KS}_q(X)$, because $AB$ factors compactly through $Z \subset \ell^q$. Moreover, $(AB)^n \notin \mathcal{A}(X)$ for any $n \in \mathbb{N}$, since
\[
T^{n+1} = (BA)^{n+1} = (BA)^n A \notin \mathcal{A}(X).
\]
For complex scalars the conclusion that $\overline{\mathcal{KS}_q(X)} / \mathcal{A}(X)$ is infinite-dimensional follows from general Banach algebra theory. Namely, the quotient $\overline{\mathcal{KS}_q(X)} / \mathcal{A}(X)$ is a closed ideal of the radical Banach algebra $\mathcal{K}(X) / \mathcal{A}(X)$, so that $\overline{\mathcal{KS}_q(X)} / \mathcal{A}(X)$ is also a radical algebra. Moreover, since $(AB)^n \notin \mathcal{A}(X)$ for any $n \in \mathbb{N}$, it follows that $\overline{\mathcal{KS}_q(X)} / \mathcal{A}(X)$ is non-nilpotent. We conclude that the radical quotient algebra $\overline{\mathcal{KS}_q(X)} / \mathcal{A}(X)$ is infinite-dimensional by [2, Proposition 1.5.6(iv)].

For real scalars we apply Proposition [2,7] to the closed ideals $\mathcal{A}(X) \subset \overline{\mathcal{KS}_q(X)}$ of $\mathcal{K}(X)$ and get that the real quotient algebra $\overline{\mathcal{KS}_q(X)} / \mathcal{A}(X)$ is radical in the sense of (1.3). Since $\overline{\mathcal{KS}_q(X)} / \mathcal{A}(X)$ is non-nilpotent by construction, we may also apply [2, Proposition 1.5.6(iv)] in the real case to deduce that $\overline{\mathcal{KS}_q(X)} / \mathcal{A}(X)$ is infinite-dimensional. $\square$

The uniform closures $\overline{\mathcal{KS}_r(Z)}$ provide examples of non-trivial closed ideals of $\mathcal{K}(Z)$ for certain Banach spaces $Z$. This also gives an alternative approach to particular instances of the examples contained in Theorem [2,9].

**Proposition 3.6.** Suppose that $1 \leq p < 2 < q < \infty$, and use Facts [2,5](ii) to pick closed subspaces $X \subset \ell^p$ and $Y \subset \ell^q$ for which $\mathcal{A}(X) \subsetneq \mathcal{K}(X)$ and $\mathcal{A}(Y) \subsetneq \mathcal{K}(Y)$. Then
\[
\mathcal{A}(X \oplus Y) \subsetneq \overline{\mathcal{KS}_q(X \oplus Y)} \subsetneq \mathcal{K}(X \oplus Y)
\]
for \( r = p \) and \( r = q \), where \( \mathcal{KS}_p(\mathbb{X} \oplus \mathbb{Y}) \) and \( \mathcal{KS}_q(\mathbb{X} \oplus \mathbb{Y}) \) are incomparable ideals.

In addition, \( \mathcal{KS}_p(\mathbb{X} \oplus \mathbb{Y}) \) and \( \mathcal{KS}_q(\mathbb{X} \oplus \mathbb{Y}) \) are also incomparable, so the classes \( \mathcal{KS}_r \) are not monotone for \( r \in [1, \infty) \).

**Proof.** Since \( \mathcal{KS}_p \) is a Banach operator ideal it follows from parts (i) and (iii) of Proposition 3.3 that the components of the closure \( \overline{\mathcal{KS}_p(\mathbb{X} \oplus \mathbb{Y})} \) satisfy
\[
\overline{\mathcal{KS}_p(\mathbb{X} \oplus \mathbb{Y})} = \left( \begin{array}{c} \mathcal{K}(X) \\ \mathcal{K}_p(X,Y) \\ \mathcal{A}(Y) \end{array} \right).
\]
The assumptions on the diagonal components imply that (3.5) holds.

For \( r = q \) one similarly get from parts (i) and (iii) of Proposition 3.3 that
\[
\overline{\mathcal{KS}_q(\mathbb{X} \oplus \mathbb{Y})} = \left( \begin{array}{c} \mathcal{A}(X) \\ \mathcal{K}(X,Y) \\ \mathcal{K}(Y) \end{array} \right).
\]

Our assumptions on \( X \) and \( Y \) again yield that \( \mathcal{K}_p(X,Y) \) is a non-trivial closed ideal of \( \mathcal{K}(X \oplus Y) \), which cannot be compared to \( \mathcal{K}_q(X \oplus Y) \).

For \( \mathcal{KS}_p(X \oplus Y) \) the respective components satisfy \( \mathcal{KS}_p(X) = \mathcal{K}(X) \) and \( \mathcal{KS}_p(Y) \subset \mathcal{A}(Y) \) by parts (i) and (iii) of Proposition 3.3. Similarly, \( \mathcal{KS}_q(X) \subset \mathcal{A}(X) \) and \( \mathcal{KS}_q(Y) = \mathcal{K}(Y) \), so the classes \( \mathcal{KS}_p(X \oplus Y) \) and \( \mathcal{KS}_q(X \oplus Y) \) cannot be compared. \( \square \)

The strict inclusion (3.1) for the cases \( p, q \in (2, \infty) \) involves non-classical approximation properties associated to certain Banach operator ideals. Let \( 1 \leq p < \infty \) and let \( X \) be a Banach space. We denote by \( \ell^p(X) \) the vector space of the strongly \( p \)-summable sequences in \( X \), and by \( \ell^p_w(X) \) that of the weakly \( p \)-summable sequences in \( X \). Recall that \( \ell^p(X) \) and \( \ell^p_w(X) \) are Banach spaces equipped with their respective natural norms,
\[
\| (x_k) \|_p = \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}, \quad (x_k) \in \ell^p(X),
\]
\[
\| (x_k) \|_{p,w} = \left( \sup_{x* \in B_{X*}} \sum_{k=1}^{\infty} |x^*(x_k)|^p \right)^{1/p}, \quad (x_k) \in \ell^p_w(X).
\]

We will require the following Banach operator ideals. Let \( 1 \leq p < \infty \), and \( X, Y \) be Banach spaces. Recall from [41, 18.1 and 18.2] that \( U \in \mathcal{L}(X,Y) \) is \( p \)-nuclear, denoted \( U \in \mathcal{N}_p(X,Y) \), if there is a scalar sequence \((\sigma_j) \in \ell^p\), a bounded sequence \((x^*_j) \subset X^*\), and a weakly \( p' \)-summable sequence \((y_j) \in \ell^p_w(Y) \) such that
\[
(3.5) \quad U x = \sum_{j=1}^{\infty} \sigma_j x^*_j(x) y_j, \quad x \in X.
\]

Here \( p' \) is the dual exponent of \( p \). The \( p \)-nuclear norm is
\[
|U|_{\mathcal{N}_p} = \inf \{ \| (\sigma_j) \|_{\ell^p} \cdot \| (x^*_j) \|_{\ell^p} \cdot \| (y_j) \|_{\ell^p_w} : (3.5) \text{ holds} \}.
\]
It is known that \( (\mathcal{N}_p, \cdot, |\cdot|_{\mathcal{N}_p}) \) is a Banach operator ideal, see [41, 18.1 and 18.2].

Following Persson and Pietsch [39, section 4], the operator \( U \in \mathcal{L}(X,Y) \) is quasi \( p \)-nuclear, denoted \( U \in \mathcal{Q}\mathcal{N}_p(X,Y) \), if there is a strongly \( p \)-summable sequence \((x^*_j) \in \ell^p(X^*) \) such that
\[
(3.6) \quad \| U x \| \leq \left( \sum_{j=1}^{\infty} |x^*_j(x)|^p \right)^{1/p}, \quad x \in X.
\]
Then $(\mathcal{Q}N_p, | \cdot |_{\mathcal{Q}N_p})$ is also a Banach operator ideal, where 
\[
|U|_{\mathcal{Q}N_p} = \inf \{ \| (x^*_j) \|_p : (3.7) \text{ holds for } (x^*_j) \}.
\]
We will require the following facts: $\mathcal{Q}N_p = \mathcal{N}^n_p$, see [39, Satz 39], and $\mathcal{Q}N_p \subset \mathcal{Q}N_q$ for $p < q$, see [39, Satz 24]. Moreover, 
\[
(3.7) \quad \mathcal{Q}N_p(X,Y) \subset \mathcal{K}S_p(X,Y)
\]
for any $1 \leq p < \infty$ and any $X$ and $Y$, see the proof of [39, Lemma 5]. Another proof is obtained by combining Proposition 3.1 with [42, Theorem 6] and the inclusion displayed on line 9 of [42, page 529].

We will also require the following duality relation (see [14, Proposition 3.8] as well as [42, Theorem 7] or [23, Corollary 2.7]).

**Fact 3.7.** Let $1 \leq p < \infty$, and $X,Y$ be Banach spaces. Then 
\[
T \in \mathcal{K}S_p(X,Y) \text{ if and only if } T^* \in \mathcal{Q}N_p(Y^*,X^*).
\]

Let $(I, | \cdot |_I)$ be a Banach operator ideal such that $I \subset \mathcal{K}$, that is, $I(X,Y) \subset \mathcal{K}(X,Y)$ for all spaces $X$ and $Y$. We will say that the Banach space $X$ has the *uniform $I$-approximation property* (abbreviated uniform $I$-A.P.), if 
\[
(3.8) \quad I(Y,X) \subset \mathcal{F}(Y,X) = \mathcal{A}(Y,X)
\]
holds for all Banach spaces $Y$.

Special instances of the uniform $I$-A.P. have recently been studied (under varying terminology) for various Banach operator ideals $I$, see e.g. [48], [13], [35] and [33]. For example, Sinha and Karn [48] introduced (an equivalent version of) the uniform $\mathcal{K}S_p$-A.P. as the $p$-approximation property, see [14, Theorem 2.1] for the equivalence with (3.8). We have found it convenient to slightly modify the terminology suggested by Lassalle and Turco [35, page 2460] in order to distinguish the uniform $I$-A.P. from the following property considered e.g. by Oja [38]: the Banach space $X$ is said to have the *$I$-approximation property* ($I$-A.P.) if 
\[
I(Y,X) = \mathcal{F}(Y,X)^{|I|}
\]
holds for all Banach spaces $Y$. Note that if $X$ has the $I$-A.P., then $X$ also has the uniform $I$-A.P., since $\| \cdot \| \leq | \cdot |_I$. The classical A.P. coincides with the uniform $\mathcal{K}$-A.P. as well as the $\mathcal{K}$-A.P. We will here (mostly) be concerned with the uniform $I$-A.P., since we are interested in closed ideals of $\mathcal{K}(X)$.

We will require the following auxiliary results, which connect the failure of the uniform $I$-A.P. for certain Banach operator ideals $I$ to the existence of non-approximable operators
in $\mathcal{KS}_p$. In parts (ii) and (iii) the uniform $\mathcal{SK}_p\text{-}A.P.$ is only relevant for $2 < p < \infty$, since any Banach space $X$ has this property for $1 \leq p \leq 2$, see [48, Theorem 6.4] or [13, Corollary 2.5]. Recall for Banach operator ideals $\mathcal{I}$ and $\mathcal{J}$ that $\mathcal{I} \subset \mathcal{J}$ means that $\mathcal{I}(X,Y) \subset \mathcal{J}(X,Y)$ for all $X, Y$, and $|\cdot|_{\mathcal{J}} \leq |\cdot|_{\mathcal{I}}$.

**Lemma 3.8.** (i) Let $1 \leq p < \infty$ and suppose that the Banach space $X$ fails to have the uniform $\mathcal{I}\text{-}A.P.$ for some Banach operator ideal $\mathcal{I} \subset \mathcal{KS}_p$. Then there is a closed subspace $Z \subset \ell^p$ such that

$$\mathcal{A}(Z, X) \not\subset \mathcal{K}(Z, X).$$

(ii) Let $2 < p < \infty$ and suppose that $X$ fails to have the uniform $\mathcal{SK}_p\text{-}A.P.$ Then there is a closed subspace $Z \subset \ell^p$ such that

$$\mathcal{A}(X^*, Z) \not\subset \mathcal{K}(X^*, Z).$$

(iii) Let $2 < p < \infty$ and suppose that $X$ fails to have the uniform $\mathcal{SK}_p\text{-}A.P.$ Then there is a closed subspace $Z \subset \ell^p$ such that

$$\mathcal{A}(X^{**}, Z) \not\subset \mathcal{K}(X^{**}, Z).$$

If $X$ is reflexive, then $\mathcal{A}(X, Z) \not\subset \mathcal{K}(X, Z)$.

**Proof.** (i) From the failure of (3.8) there is a Banach space $Y$ and an operator $U \in \mathcal{I}(Y, X)$ such that $U \not\in \mathcal{A}(Y, X)$. Since $U \in \mathcal{I}(Y, X) \subset \mathcal{KS}_p(Y, X)$ by assumption, there is a closed subspace $Z \subset \ell^p$ and a compact factorisation $U = BA$, where $A \in \mathcal{K}(Y, Z)$ and $B \in \mathcal{K}(Z, X)$. Here $B \not\in \mathcal{A}(Z, X)$.

(ii) If $X$ fails to have the uniform $\mathcal{SK}_p\text{-}A.P.$, then from (3.8) there is a Banach space $Y$ and an operator $T \in \mathcal{SK}_p(Y, X) \setminus \mathcal{A}(Y, X)$. From Fact 3.7 and the inclusion 3.7 we have

$$T^* \in \mathcal{QN}_p(X^*, Y^*) \subset \mathcal{KS}_p(X^*, Y^*).$$

Hence there is a closed subspace $Z \subset \ell^p$ and a compact factorisation $T^* = BA$ through $Z$. Note that $T^*$ cannot be approximable. Namely, if $T^*$ were approximable $X^* \rightarrow Y^*$, then the principle of local reflexivity implies that $T$ is also approximable, see e.g. [6, Propositions 2.5.1-2.5.2] or [11, Theorem 11.7.4]. We get that $\mathcal{A}(X^*, Z) \not\subset \mathcal{K}(X^*, Z)$, since $A \in \mathcal{K}(X^*, Z)$ cannot be an approximable operator.

(iii) If $X$ fails to have the uniform $\mathcal{SK}_p\text{-}A.P.$, then $X^*$ also fails this property by the duality result in [8, Theorem 2.7]. Part (ii) implies that there is a closed subspace $Z \subset \ell^p$ such that

$$\mathcal{A}(X^{**}, Z) \not\subset \mathcal{K}(X^{**}, Z).$$

The desired examples of a strict inclusion for $p, q \in (2, \infty)$ revisits an intricate construction of Reinov [46] concerning the failure of duality for $p$-nuclear operators, which we reinterpret in terms of the uniform $\mathcal{QN}_p\text{-}A.P.$ Reinov’s argument in part (i) of [46, Lemma 1.1 and Corollary 1.1] covers the case $p = q \in (2, \infty)$, which extends by monotonicity to $2 < q < p < \infty$ (see the argument of Theorem 3.9). For $2 < p < q < \infty$ we somewhat modify Reinov’s construction, and in the interest of readability we have added details to the condensed explanation in [46]. Recall that $\mathcal{QN}_p \subset \mathcal{A}$ for $1 \leq p \leq 2$ by monotonicity and [41, 18.1.8 and 18.1.4], so any Banach space $X$ has the uniform $\mathcal{QN}_p\text{-}A.P.$ for $p \in [1, 2]$.

**Theorem 3.9.** Let $2 < p, q < \infty$ and $p \neq q$. Then there exists a closed subspace $Y \subset \ell^p$ such that $Y$ fails to have the uniform $\mathcal{QN}_p\text{-}A.P.$, and a closed subspace $X \subset \ell^p$ for which

$$\mathcal{A}(X, Y) \not\subset \mathcal{K}(X, Y).$$
Proof. We first consider the details for $2 < p < q < \infty$, where we will exhibit (by closely following Reinov's outline) a closed subspace $Y \subset \ell^q$ together with an operator

$$U \in \mathcal{QN}_p(Z, Y) \setminus \mathcal{A}(Z, Y)$$

for some Banach space $Z$. The second claim follows immediately from Lemma 3.8 (i), since $\mathcal{QN}_p \subset \mathcal{K}S_p$ by (3.7). Hence the compact factorisation $U = BA$ through some closed subspace $X \subset \ell^p$ provides an operator $B \in \mathcal{K}(X, Y) \setminus \mathcal{A}(X, Y)$.

The starting point is the construction of Davie [11] (see also [36, Section 2.d]): There is a matrix $A = (a_{ij})_{i,j=1}^{\infty}$ with the following properties:

(i) $A^2 = 0$,
(ii) $tr A = \sum_{k=1}^{\infty} a_{kk} = 1$ and
(iii) $\sum_{n=1}^{\infty} \lambda_n^\alpha < \infty$ for all $\alpha > 2/3$, where $\lambda_n = \sup_{k \in \mathbb{N}} |a_{nk}| > 0$ for all $n \in \mathbb{N}$.

The matrix $A$ defines a 1-nuclear operator $\ell^1 \to \ell^1$ through

$$Ax = \left( \sum_{n=1}^{\infty} a_{kn}x_n \right)^{\infty}_{k=1}, \quad x = (x_n) \in \ell^1.$$ 

Define bounded operators $V : \ell^1 \to \ell^\infty$ and $\Delta : \ell^\infty \to \ell^1$ as follows:

$$Vx = \left( \lambda_k^{-1} \sum_{n=1}^{\infty} a_{kn}x_n \right)^{\infty}_{k=1}, \quad x \in \ell^1, \quad \Delta x = \left( \lambda_k x_k \right), \quad x \in \ell^\infty.$$ 

Clearly $A = \Delta V$, where $\Delta = \sum_{n=1}^{\infty} \lambda_n e_n \otimes e_n$ is a 1-nuclear diagonal operator. Here $(e_n)$ denotes the unit vector basis in $\ell^1$ (and subsequently also in $\ell^r$ for appropriate $r$). It follows from (i) and (ii) that

(iv) $(\Delta V)^2 = 0$ and
(v) $tr(\Delta V) = \sum_{k=1}^{\infty} \langle \Delta V e_k, e_k \rangle = \sum_{k=1}^{\infty} a_{kk} = 1$.

Let $\frac{2}{3} < \alpha < 1 - \frac{2}{3p}$ (this is possible because $p > 2$) and put $\lambda_n^{(1)} = \lambda_n^{1-\alpha}$ and $\lambda_n^{(2)} = \lambda_n^{\alpha}$ for all $n \in \mathbb{N}$. It follows from (iii) that $(\lambda_n^{(1)}) \in \ell^p$ and $(\lambda_n^{(2)}) \in \ell^1$. Consider the following commuting diagram:

Here $Z = \ell^1/\ker V$ and $Y_0 = V\ell^1 \subset \ell^\infty$; the operators $V_1$ and $\tilde{V}$ are induced by $V$ and $j_1$ is the isometric inclusion map;

$$\Delta_1 = \sum_{k=1}^{\infty} \lambda_k^{(1)} e_k \otimes e_k \text{ and } \Delta_2 = \sum_{k=1}^{\infty} \lambda_k^{(2)} e_k \otimes e_k,$$

and $i$ is the canonical inclusion; $Y = i\Delta_1j_1Y_0 \subset \ell^q$ and $\tilde{\Delta}_1 = j_2^{-1}i\Delta_1j_1$, where $j_2$ is the isometric inclusion map and $U = \tilde{\Delta}_1\tilde{V}$. 
Clearly $\Delta_1 \in \mathcal{N}_p(\ell^\infty, \ell^p) \subset \mathcal{Q}N_\rho(\ell^\infty, \ell^p)$, and it is easy to check by using (3.10) that the restriction $\Delta_1 \in \mathcal{Q}N_\rho(Y_0, Y)$. Deduce that $U = \Delta_1 V \in \mathcal{Q}N_\rho(Z, Y)$. Next we verify that $U$ is not an approximable operator. Towards this define the bounded functional $\Phi \in \mathcal{L}(Z, Y)^*$ by

$$\Phi(S) = \sum_{k=1}^{\infty} \lambda_k^{(2)} \langle SV_1 e_k, j_2^* e_k \rangle$$

for $S \in \mathcal{L}(Z, Y)$.

**Claim:** $\Phi(T) = 0$ for all $T \in \mathcal{A}(Z, Y)$.

By linearity and continuity it will be enough to show that $\Phi(z^* \otimes y) = 0$ for all $z^* \in Z^*$ and $y \in Y$. Moreover, by the definition of $Y$ we may also (by continuity) assume that $y \in j_2^{-1} i_\Delta_1 V a$. We get that

$$\Phi(z^* \otimes y) = \sum_{k=1}^{\infty} \lambda_k^{(2)} z^* (V_1 e_k) \langle y, j_2^* e_k \rangle = \sum_{k=1}^{\infty} \lambda_k^{(2)} z^* (V_1 e_k) \langle i_\Delta_1 V a, e_k \rangle$$

$$= z^* \left( V_1 \left( \sum_{k=1}^{\infty} (\lambda_k^{(2)} i_\Delta_1 V a, e_k) e_k \right) \right) = z^* (V_1 \Delta_2 i_\Delta_1 V a) = z^* (V_1 \Delta V a) = 0.$$

The last equality follows from the fact that $V_1 \Delta V = 0$. Namely, by (iv) we have $0 = \Delta V \Delta V = \Delta_1 V_1 V \Delta V$, where $\Delta_1 V$ is injective. Thus $\Phi(T) = 0$ for all $T \in \mathcal{A}(Z, Y)$. On the other hand, condition (v) implies that

$$\Phi(U) = \sum_{k=1}^{\infty} \lambda_k^{(2)} \langle UV_1 e_k, j_2^* e_k \rangle = \sum_{k=1}^{\infty} \langle \lambda_k^{(2)} j_2 UV_1 e_k, e_k \rangle = \sum_{k=1}^{\infty} \langle \Delta V e_k, e_k \rangle = 1.$$

Hence $U \notin \mathcal{A}(E, F)$, while $U \in \mathcal{Q}N_\rho(Z, Y)$.

Suppose next that $2 < q < p < \infty$. According to part (i) of [46, Lemma 1.1 and Corollary 1.1] there is a closed subspace $Y \subset \ell^q$ together with an operator $U \in \mathcal{Q}N_q(Z, Y) \setminus \mathcal{A}(Z, Y)$, where $Z$ is a suitable Banach space. Since $q < p$ we get from the monotonicity of the classes $\mathcal{Q}N_r$ and $\mathcal{K}S_p$ that

$$U \in \mathcal{Q}N_p(Z, Y) \subset \mathcal{Q}N_\rho(Z, Y) \subset \mathcal{K}S_p(Z, Y).$$

Thus $Y$ fails the uniform $\mathcal{Q}N_p$-A.P. Moreover, there is a closed subspace $X \subset \ell^p$ and a compact factorisation $U = BA$ through $X$. In particular, $B \in \mathcal{K}(X, Y) \setminus \mathcal{A}(X, Y)$ is the desired operator.

**Remarks 3.10.** (i) The diagram of Reinov [46, page 127] for the cases $p = q \in (2, \infty)$ is similar to the one displayed above. Our diagram adds the inclusion map $i : \ell^p \to \ell^q$, but removes the DFJP-factorisation of $V$ through a reflexive space, which is not relevant for our purposes. The argument of Reinov produces a closed subspace $Y \subset \ell^p$ such that $Y$ fails the uniform $\mathcal{Q}N_\rho$-A.P., which strengthens Facts 2.30(i) for $p > 2$ (see also Example 3.16 below).

(ii) It is possible to recover part of Theorem 3.9 for certain combinations of $p, q \in (2, \infty)$ from an example of Choi and Kim [8] concerning the uniform $\mathcal{SK}_p$-A.P. (Their example is also based on Davie’s construction [11].) In fact, for $2 < p < \infty$ and $q > \frac{2p}{p-2}$ there is in view of Corollary 2.9(a) a closed subspace $X \subset \ell^q$ that fails the uniform $\mathcal{SK}_p$-A.P. Hence Lemma 3.8(iii) provides a closed subspace $Y \subset \ell^p$ for which $\mathcal{A}(X, Y) \subset \mathcal{K}(X, Y)$.

After these preparations we return to the setting of Theorem 2.10 and the question of how many non-trivial closed ideals we may find in $\mathfrak{A}_Z$ for spaces $Z$ belonging to that class of
Consequently, by Proposition 2.1 we get the non-trivial closed ideals.

**Example 3.11.** Suppose that \( p < q < 2p/(p-2) \). Then there are closed subspaces \( X \subset \ell^p \) and \( Y \subset \ell^q \) such that the quotient algebra \( A_{X^* \oplus Y} \) contains (at least) three non-trivial closed ideals.

**Proof.** From \([8, Corollary 2.9]\) we find a closed subspace \( X_0 \subset \ell^q \) that fails the uniform \( \mathcal{SK}_{p^*} \)-A.P. According to part (ii) of Lemma \([3, Section 5.2]\) there is a closed subspace \( Y_0 \subset \ell^p \) together with an operator

\[
S \in \mathcal{K}(X_0^*, Y_0) \setminus A(X_0^*, Y_0).
\]

Next we use Facts \([2, (ii)]\) to pick closed subspaces \( M \subset \ell^p \) and \( N \subset \ell^q \) such that \( A_M \neq \{0\} \) and \( A_N \neq \{0\} \). Put

\[
X = X_0 \oplus M \quad \text{and} \quad Y = Y_0 \oplus N.
\]

Thus \( X \) is a closed subspace of \( \ell^q \), where \( q > 2 \), and \( Y \) a closed subspace of \( \ell^p \), so that both \( X \) and \( Y \) have type 2. It follows from known facts about type and cotype that \( X^* = X_0^* \oplus M^* \) has type 2, see e.g. \([10, Proposition 11.10]\). Since \( A_M \neq \{0\} \) and \( A_N \neq \{0\} \), the quotient algebras \( A_X \) and \( A_Y \) are both non-trivial by \([53, Proposition 4.2]\). Finally, since \( A_X \) embeds isometrically into \( A_Y \)-by duality \([53, Proposition 4.1]\), we also know that \( A_Y \neq \{0\} \).

Altogether \( X^* \) and \( Y \) satisfy the conditions of Theorem 2.6.

Let \( Z = X^* \oplus Y \), and let \( I \) and \( J \) be the closed ideals of \( \mathcal{K}(Z) \) obtained in Theorem 2.6 for which \( q(I) \) and \( q(J) \) define two non-trivial incomparable closed ideals of \( A_Z \).

**Claim.** \( I \cap J = \begin{pmatrix} \mathcal{A}(X^*) & \mathcal{A}(Y, X^*) \\ \mathcal{K}(X^*, Y) & \mathcal{A}(Y) \end{pmatrix} \) is a third non-trivial ideal in \( \mathcal{K}(X^* \oplus Y) \).

In fact, \( I \cap J \subset I \) and \( I \cap J \subset J \), since \( \mathcal{A}(X^*) \subset \mathcal{K}(X^*) \) and \( \mathcal{A}(Y) \subset \mathcal{K}(Y) \) by construction. Pick \( S \in \mathcal{K}(X_0^*, Y_0) \setminus A(X_0^*, Y_0) \) by (3.9), and let \( \tilde{S} := JSI : X^* \to Y \), where \( P : X^* \to X_0^* \) is the natural projection and \( J : Y_0 \to Y \) is the inclusion map. Hence \( \tilde{S} \in \mathcal{K}(X^*, Y) \setminus \mathcal{A}(X^*, Y) \), and

\[
\begin{pmatrix} 0 & 0 \\ \tilde{S} & 0 \end{pmatrix} \in (I \cap J) \setminus \mathcal{A}(X^* \oplus Y).
\]

Consequently, by Proposition 2.1 we get the non-trivial closed ideal \( q(I) \cap q(J) \) of \( A_Z \), which differs from both \( q(I) \) and \( q(J) \).

We note in passing that \( q(I) \cap q(J) \) is a nilpotent ideal of \( A_Z \), that is, if \( U, V \in I \cap J \), then \( UV \in \mathcal{A}(X \oplus Y) \). In fact, by (2.2) the component

\[
(\mathcal{U} \mathcal{V})_{2,1} = U_{21}V_{11} + U_{22}V_{21} \in \mathcal{A}(X, Y),
\]

since \( V_{11} \) and \( U_{22} \) are approximable operators.

Let \( Z = \ell^p \), where \( 1 \leq p < \infty \) and \( p \neq 2 \), or \( Z = c_0 \). A classical result of Gohberg, Markus and Feldman says that \( \mathcal{K}(Z) \) is the unique non-trivial closed ideal in \( \mathcal{L}(Z) \), see e.g. \([41, Section 5.2]\). It is relevant to ask whether there are non-trivial closed ideals

\[
\mathcal{A}(X) \not\subset J \not\subset \mathcal{K}(X)
\]

among the closed subspaces \( X \subset Z \). Our next two results demonstrate that this is indeed the case (at least) for \( p > 4 \) and for \( c_0 \). For subspaces of \( \ell^p \) this is based on \([8, Corollary 2.9]\) and Theorem 2.8 as well as properties of quasi-\( p \)-nuclear operators. We stress that the resulting direct sums do not belong to the class of spaces in Theorem 2.6.
Example 3.12. Let \( p > 2 \) and \( q > 2p/(p - 2) \). By combining \cite[Corollary 2.9]{28} and \cite[Theorem 2.7]{28}, there is a closed subspace \( X \subset \ell^q \) such that \( X^* \) fails the uniform \( SK_p \)-A.P. By definition there is an operator \( T \in SK_p(X_0, X^*) \setminus A(X_0, X^*) \) for some Banach space \( X_0 \). Consequently \( T^* \in QN_p(X, X_0^*) \setminus A(X, X_0^*) \) by Fact 3.7 and the reflexivity of \( X \). According to the proof of \cite[Lemma 5]{39} there is a factorisation \( T^* = BA \) through a closed subspace \( Y \subset \ell^p \) such that \( A \in QN_p(X, Y) \). Moreover, by Theorem 2.8 there is a closed subspace \( Z \subset \ell^p \) together with a compact operator \( S \in K(Z) \) such that
\[
(3.10) \quad S^n \notin A(Z) \text{ for all } n \in \mathbb{N}.
\]
We consider the closed subspace \( W = X \oplus Y \oplus Z \) of \( \ell^q \oplus \ell^p \) and claim that
\[
(3.11) \quad A(W) \subseteq \overline{QN}_p(W) \subseteq K(W),
\]
\[
(3.12) \quad A(W^*) \subseteq \overline{SK}_p(W^*) \subseteq K(W^*).
\]
In particular, for \( p = q > 4 \) there is a closed subspace \( W \subset \ell^p \) such that \( (3.11) \) holds.

Proof. First note that \( A \) is not approximable, since otherwise \( T^* \) would also be approximable. Thus \( A \in QN_p(X, Y) \setminus A(X, Y) \), which implies for \( W = X \oplus Y \oplus Z \)
\[
A(W) \subseteq \overline{QN}_p(W).
\]
We claim that \( S \in K(Z) \setminus \overline{QN}_p(Z) \) (which immediately yields that \( \overline{QN}_p(W) \subseteq K(W) \)). Suppose to the contrary that \( S \in \overline{QN}_p(Z) \) and let \( (R_n) \subset \overline{QN}_p(Z) \) be a sequence such that
\[
(3.13) \quad ||R_n - S|| \to 0 \text{ as } n \to \infty.
\]
Recall from \cite[section 7]{39} that \( UV \in QN_r \) whenever \( U \in QN_s \) and \( V \in QN_t \) are compatible operators for which \( 1/r = 1/s + 1/t \leq 1 \). Fix an integer \( m \) such that \( p/2 \leq m < p \). By iterating the above product formula we get that \( R_n^m \in QN_{p/m}(Z) \) for all \( n \in \mathbb{N} \). Since
\[
(3.14) \quad QN_{p/m} \subset QN_2 = N_2 \subset A
\]
by monotonicity and \cite[18.1.8 and 18.1.4]{41}, the operators \( R_n^m \in A(Z) \) for all \( n \in \mathbb{N} \). Conclude from \( (3.13) \) that \( S^m \in A(Z) \), which contradicts \( (3.10) \). Thus \( S \notin \overline{QN}_p(Z) \).

The strict inclusions in \( (3.12) \) follow by duality. In fact, since \( W \) is reflexive, Fact 3.7 implies that \( T \in \overline{QN}_p(W) \) if and only if \( T^* \in \overline{SK}_p(W^*) \). Finally, note that it is possible to choose \( p = q \) for \( p > 4 \) and \( q > 2p/(p - 2) \), in which case \( W \) is a closed subspace of \( \ell^p \). \( \square \)

In the analogous example of a closed subspace \( X \subset c_0 \) we may simultaneously involve the closures \( \overline{QN}_p(X) \) and \( \overline{SK}_p(X) \), and the details are somewhat different.

Example 3.13. Let \( 2 < p < \infty \). Then there is a closed subspace \( X \subset c_0 \) such that
\[
A(X) \subseteq \overline{SK}_p(X) \subseteq K(X) \quad \text{and} \quad A(X) \subseteq \overline{QN}_p(X) \subseteq K(X),
\]
where \( \overline{SK}_p(X) \) and \( \overline{QN}_p(X) \) are incomparable closed ideals.

Proof. We establish the claim in two parts.

Claim 1. There are closed subspaces \( M_0 \) and \( M_1 \) of \( c_0 \) such that
\[
(3.15) \quad \overline{SK}_p(M_0, M_1) \not\subseteq \overline{QN}_p(M_0, M_1).
\]

Claim 2. There are closed subspaces \( M_2 \) and \( M_3 \) of \( c_0 \) such that
\[
(3.16) \quad \overline{QN}_p(M_2, M_3) \not\subseteq \overline{SK}_p(M_2, M_3).
\]
We may then take
\[ X = M_0 \oplus M_1 \oplus M_2 \oplus M_3 \subset c_0. \]
Denote \( I = \overline{SK_p(X)} \) and \( J = \overline{QN_p(X)} \). Then (3.15) implies that \( I \not\subset J \) and (3.16) implies that \( J \not\subset I \). Hence \( I \) and \( J \) are incomparable closed ideals. Since \( I \) and \( J \) lie between \( A(X) \) and \( K(X) \), we get the strict inclusions \( A(X) \not\subset I \not\subset K(X) \) and \( A(X) \not\subset J \not\subset K(X) \).

**Proof of Claim 1.** Let \( F \) be a reflexive Banach space with type 2 that fails the uniform \( SK_p \)-A.P. \( [3, \text{Corollary 2.9}] \). In view of \( [3, \text{Theorem 2.7}] \) the dual \( F^* \) also fails the uniform \( SK_p \)-A.P. By definition there is an operator \( T \in SK_p(E,F^*) \setminus A(E,F^*) \) for a suitable Banach space \( E \).

According to \( [23, \text{Proposition 2.9}] \) we can factor \( T = VRV_0 \) where \( V_0 \) and \( V \) are compact operators and \( R \in SK_p \). Moreover, \( V_0 \) factors compactly through a reflexive space \( W \) by the Figiel-Johnson factorisation result \( [19, \text{Proposition 3.1}] \) or the DFJP-factorisation (see e.g. \( [37, \text{Theorem 2.g.11}] \) or \( [26, \text{Theorem 3.2.1}] \)). Let \( V_0 = BA \) be the corresponding factorisation, so that \( T = VRBA \). We claim that
\[
(3.17) \quad S =: VRB \in SK_p(W,F^*) \setminus QN_p(W,F^*).
\]
Firstly, \( S \in SK_p(W,F^*) \) because \( R \in SK_p \). Secondly, in view of inclusion (3.6) and Example \( 3.16 \) (ii) below for the cotype 2 space \( F^* \), we obtain that
\[
QN_p(W,F^*) \subset KS_p(W,F^*) \subset A(W,F^*).
\]
Thus if \( S \in QN_p(W,F^*) \), then \( S \) must be approximable. But this would mean that \( T = SA \) is approximable, which is a contradiction. Hence \( S \notin QN_p(W,F^*) \).

Next we use Terzioğlu’s factorisation result \( [52] \) to obtain a compact factorisation \( B = B_2B_1 \) through a closed subspace \( M_0 \subset c_0 \). Similarly, \( V \) has a compact factorisation \( V = V_2V_1 \) through a closed subspace \( M_1 \subset c_0 \). Consider \( U = : V_1RB_2 \in SK_p(M_0,M_1) \), so that
\[
S = VRB = V_2V_1RB_2B_1 = V_2UB_1.
\]
From (3.17) we get that \( U \in SK_p(M_0,M_1) \setminus QN_p(M_0,M_1) \), which proves Claim 1.

**Proof of Claim 2.** By using the reflexivity of \( W \) and \( F^* \) and the duality in Fact 3.7 together with (3.17) we obtain that
\[
(3.18) \quad S^* \in QN_p(F,W^*) \setminus SK_p(F,W^*).
\]
According to \( [39, \text{Lemma 5}] \) the operator \( S^* \) factors as \( S^* = QT_0P \), where \( P \) and \( Q \) are compact operators and \( T_0 \) is also quasi \( p \)-nuclear. As above, by Terzioğlu’s result we can further factor \( P \) and \( Q \) compactly through closed subspaces \( M_2 \subset c_0 \) and \( M_3 \subset c_0 \), respectively. Let \( P = P_1P_2 \) and \( Q = Q_2Q_1 \) be the corresponding factorisations, so that \( S^* = Q_2Q_1T_0P_2P_1 \). Consider \( R =: Q_1T_0P_2 \in QN_p(M_2,M_3) \). From (3.18) we obtain that
\[
R \in QN_p(M_2,M_3) \setminus SK_p(M_2,M_3),
\]
which establishes Claim 2. \( \square \)

We are now in position to combine the spaces appearing in Examples 3.11 and 3.12, and to obtain a particular direct sum \( X \oplus Y \) in the setting of Theorem 2.6 for which the quotient algebra \( A_{X \oplus Y} \) contains (at least) 8 non-trivial closed ideals.
Example 3.14. Suppose that $p > 2$ and $q > 2p/(p - 2)$. Let $W \subset \ell^p + \ell^q$ be the closed subspace constructed in Example 3.12 for which the strict inclusions (3.11) and (3.12) hold. By arguing as at the beginning of Example 3.11 let $X_0 \subset \ell^p$ and $Y_0 \subset \ell^p$ be closed subspaces such that $\mathcal{A}(X_0, Y_0) \subset \mathcal{K}(X_0, Y_0)$. Consider the direct sum

$$X \oplus Y =: (W^* \oplus X_0^*) \oplus (W \oplus Y_0).$$

It is easy to check that the direct sums $W \oplus X_0$ and $Y = W \oplus Y_0$ have type 2, so that $X = W^* \oplus X_0^*$ has cotype 2 by [16, Proposition 11.10]. Hence $X \oplus Y$ belongs to the class of spaces from Theorem 2.6, for which $\mathcal{K}(Y, X) = \mathcal{A}(Y, X)$ by Theorem 2.2. By using suitable component operators as before one verifies the following strict inclusions by using (3.11), (3.12) and the fact that $\mathcal{A}(X_0, Y_0) \subset \mathcal{K}(X_0, Y_0)$:

(i) $\mathcal{A}(X) \subset \mathcal{I}_0 =: SK_p(X) \subset \mathcal{K}(X)$,
(ii) $\mathcal{A}(Y) \subset \mathcal{J}_0 =: QN_p(Y) \subset \mathcal{K}(Y)$,
(iii) $\mathcal{A}(X, Y) \subset \mathcal{K}(X, Y)$.

We claim that $\mathcal{K}(X \oplus Y)$ contains (at least) the following 8 non-trivial closed ideals, where $\mathcal{I}$ and $\mathcal{J}$ are the ideals constructed in Theorem 2.6

$$\mathcal{I} = \left( \begin{array}{c}
\mathcal{K}(X) \\
\mathcal{K}(X, Y) \\
\mathcal{A}(Y) \\
\mathcal{A}(Y, X) \\
\mathcal{A}(Y, X)
\end{array} \right), \quad \mathcal{J} = \left( \begin{array}{c}
\mathcal{A}(X) \\
\mathcal{K}(X, Y) \\
\mathcal{A}(Y) \\
\mathcal{K}(X, Y) \\
\mathcal{K}(X, Y)
\end{array} \right), \quad \mathcal{I} \cap \mathcal{J},$$

$$\mathcal{J}_1 = \left( \begin{array}{c}
\mathcal{I}_0 \\
\mathcal{K}(X, Y) \\
\mathcal{A}(Y, X) \\
\mathcal{A}(Y, X) \\
\mathcal{A}(Y, X)
\end{array} \right), \quad \mathcal{I}_1 = \left( \begin{array}{c}
\mathcal{K}(X) \\
\mathcal{A}(Y) \\
\mathcal{K}(X, Y) \\
\mathcal{J}_0 \\
\mathcal{J}_0
\end{array} \right), \quad \mathcal{I}_1 \cap \mathcal{J}_1,$$

$$\mathcal{I}_2 = \left( \begin{array}{c}
\mathcal{I}_0 \\
\mathcal{K}(X, Y) \\
\mathcal{A}(Y, X) \\
\mathcal{A}(Y, X) \\
\mathcal{A}(Y, X)
\end{array} \right), \quad \mathcal{J}_2 = \left( \begin{array}{c}
\mathcal{A}(X) \\
\mathcal{K}(X, Y) \\
\mathcal{A}(Y, X) \\
\mathcal{K}(X, Y) \\
\mathcal{K}(X, Y)
\end{array} \right).$$

It is a straightforward exercise to verify from (2.2) that the new classes above are closed ideals. We consider $\mathcal{I}_2$ as an example, and leave the cases $\mathcal{I}_1$, $\mathcal{J}_1$ and $\mathcal{J}_2$ for the reader.

Suppose first that $U \in \mathcal{I}_2$ and $V \in \mathcal{K}(X \oplus Y)$. It follows that the component

$$(UV)_{1,1} = U_{11}V_{11} + U_{12}V_{21} \in \mathcal{I}_0$$

since $U_{11} \in \mathcal{I}_0$, while $(UV)_{2,2} = U_{21}V_{12} + U_{22}V_{22} \in \mathcal{A}(Y)$ since $V_{12}, U_{22} \in \mathcal{A}$. Suppose next that $U \in \mathcal{K}(X \oplus Y)$ and $V \in \mathcal{I}_2$. In this case

$$(UV)_{1,1} = U_{11}V_{11} + U_{12}V_{21} \in \mathcal{I}_0$$

since $V_{11} \in \mathcal{I}_0$ and $U_{12} \in \mathcal{A}$, while $(UV)_{2,2} = U_{21}V_{12} + U_{22}V_{22} \in \mathcal{A}(Y)$ since $V_{12}, V_{22} \in \mathcal{A}$. The strict inclusions (i) - (iii) for the component ideals imply that all the ideals in the above list are non-trivial, as well as pairwise different. In particular, $\mathcal{A}(X \oplus Y) \subset \mathcal{I} \cap \mathcal{J}$ by (iii).

A closer inspection reveals the order structure among the ideals in the above list is the following, where a line indicates a strict inclusion (from left to right):

$$\mathcal{I}_2 \rightarrow \mathcal{I} \rightarrow \mathcal{I}_1 \rightarrow \mathcal{J} \rightarrow \mathcal{J}_1 \rightarrow \mathcal{K}(X \oplus Y).$$

Thus $\mathcal{K}(X \oplus Y)$ contains several chains of closed ideals, as well as incomparable pairs. □
Let $1 \leq p < q < \infty$. Recall that the closed ideals
\[
\begin{pmatrix}
\mathcal{L}(\ell^p) & \mathcal{K}(\ell^q, \ell^p) \\
\mathcal{L}(\ell^p, \ell^q) & \mathcal{K}(\ell^q)
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\mathcal{K}(\ell^p) & \mathcal{K}(\ell^q, \ell^p) \\
\mathcal{L}(\ell^p, \ell^q) & \mathcal{L}(\ell^q)
\end{pmatrix}
\]
are maximal in $\mathcal{L}(\ell^p \oplus \ell^q)$, see e.g. [41, 5.3.2]. Suppose that $X \subset \ell^p$ and $Y \subset \ell^q$ are closed subspaces, where $p < 2 < q$. It is a natural question whether the corresponding closed ideals
\[
\mathcal{I} = \begin{pmatrix}
\mathcal{K}(X) & \mathcal{A}(Y, X) \\
\mathcal{K}(X, Y) & \mathcal{A}(Y)
\end{pmatrix}
\quad \text{and} \quad
\mathcal{J} = \begin{pmatrix}
\mathcal{A}(X) & \mathcal{A}(X, Y) \\
\mathcal{K}(X, Y) & \mathcal{K}(Y)
\end{pmatrix}
\]
given by Theorem 2.6 are maximal in $\mathcal{K}(X \oplus Y)$. The following variant of the preceding examples demonstrate that neither $\mathcal{I}$ nor $\mathcal{J}$ are in general maximal ideals in the setting of Theorem 2.6.

**Example 3.15.** Suppose that $1 \leq p < 2$ and $q > 4$. Let $W \subset \ell^q$ be the closed subspace from Example 3.12 such that (3.11) and (3.12) hold. In view of Facts 2.5(ii) let $X \subset \ell^p$ be a closed subspace such that $\mathcal{A}(X) \subseteq \mathcal{K}(X)$. The direct sum $X \oplus W$ satisfies the assumptions of Theorem 2.6 so that $\mathcal{K}(W, X) = \mathcal{A}(W, X)$. Let $\mathcal{J}_0 = \mathcal{QN}_q(W)$. By arguing as in Example 3.14 we get that
\[
\mathcal{I} = \begin{pmatrix}
\mathcal{K}(X) & \mathcal{A}(W, X) \\
\mathcal{K}(X, W) & \mathcal{A}(W)
\end{pmatrix}
\quad \text{and} \quad
\mathcal{J} = \begin{pmatrix}
\mathcal{A}(X) & \mathcal{A}(X, W) \\
\mathcal{K}(X, W) & \mathcal{J}_0
\end{pmatrix}
\]
are both non-trivial closed ideals in $\mathcal{K}(X \oplus W)$. Thus $\mathcal{I}$ is not a maximal ideal in $\mathcal{K}(X \oplus W)$, where $X \oplus W$ is a closed subspace of $\ell^p \oplus \ell^q$.

Actually, it is not difficult to check that the Banach algebra $\mathcal{K}(X \oplus W)$ contains (at least) the following four non-trivial closed ideals: $\mathcal{I}$ and $\mathcal{J}$ from Theorem 2.6 together with
\[
\begin{pmatrix}
\mathcal{A}(X) & \mathcal{A}(W, X) \\
\mathcal{K}(X, W) & \mathcal{J}_0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\mathcal{K}(X) & \mathcal{A}(X, W) \\
\mathcal{K}(X, W) & \mathcal{J}_0
\end{pmatrix}
\]

Next, let $r > 2$ and pick a closed subspace $Y \subset \ell^r$ such that $\mathcal{A}(Y) \subseteq \mathcal{K}(Y)$. Then $W^* \oplus Y$ satisfies the assumptions of Theorem 2.6 As in Example 3.14 we get with $\mathcal{I}_0 = \mathcal{SK}_p(W^*)$ from (3.12) that
\[
\mathcal{J} = \begin{pmatrix}
\mathcal{A}(W^*) & \mathcal{A}(Y, W^*) \\
\mathcal{K}(W^*, Y) & \mathcal{K}(Y)
\end{pmatrix}
\quad \text{and} \quad
\mathcal{J} = \begin{pmatrix}
\mathcal{I}_0 & \mathcal{A}(Y, W^*) \\
\mathcal{K}(W^*, Y) & \mathcal{K}(Y)
\end{pmatrix}
\]
are again non-trivial closed ideals of $\mathcal{K}(W^* \oplus Y)$. In particular, $\mathcal{J}$ is not maximal in $\mathcal{K}(W^* \oplus Y)$. In this case $W^*$ is a quotient space of $\ell^p$. \qed

Recently Kim [33] studied the $K^{inj}_{p,q}$-A.P. and the uniform $K^{inj}_{p,q}$-A.P. for a scale $K_{p,q}$ of Banach operator ideals that include the classical $p$-compact operators as $K_p := K_{p,p'}$, where $p'$ is the dual exponent of $p$. In particular, he obtained the surprising result [33, Proposition 4.3] that $X$ has the $K^{inj}_{p,q}$-A.P. if and only if $X$ has the uniform $K^{inj}_{p,q}$-A.P. However, [33] does not include explicit examples concerning this approximation property. Recall that $\mathcal{K}S_p = K^{inj}_p$ by Proposition 3.1 so that the following examples about the (uniform) $K^{inj}_{p,q}$-A.P. are contained in Proposition 3.9 Theorem 3.3 and Theorem 3.9.

**Example 3.16.** Let $1 \leq p < \infty$ and $p \neq 2$.

(i) If $X \subset \ell^p$ is a closed subspace, then $X$ has the (uniform) $\mathcal{K}S_p$-A.P. if and only if $X$ has the A.P.
(ii) Let $2 < q < \infty$. If $X$ has cotype 2, then $X$ has the (uniform) $\mathcal{KS}_q$-A.P. In particular, for $1 \leq p < 2$ there is a closed subspace $X \subset \ell^p$ that has the (uniform) $\mathcal{KS}_q$-A.P., but fails the A.P.

(iii) If $p,q \in [1,2)$ or $p,q \in (2,\infty)$, then there is a closed subspace $X \subset \ell^p$ such that $X$ fails the (uniform) $\mathcal{KS}_q$-A.P.

**Proof.** (i) If $X \subset \ell^p$ is a closed subspace, then $\mathcal{K}(Z,X) = \mathcal{KS}_p(Z,X)$ for any space $Z$ by Proposition 3.3(i). Hence, if $X$ has the (uniform) $\mathcal{KS}_p$-A.P., then

$$\mathcal{K}(Z,X) = \mathcal{KS}_p(Z,X) = \mathcal{A}(Z,X)$$

for all $Z$, so that $X$ has the A.P. Conversely, if $X$ has the A.P., then

$$\mathcal{KS}_p(Z,X) \subset \mathcal{K}(Z,X) = \mathcal{A}(Z,X)$$

for any $Z$, so that $X$ has the uniform $\mathcal{KS}_p$-A.P. (The converse implication is also noted in [32, Corollary 4.5].)

(ii) Let $Z$ be an arbitrary Banach space, and suppose that $T \in \mathcal{KS}_q(Z,X)$. Hence there is a closed subspace $M \subset \ell^q$ and a compact factorisation $T = BA$, where $B \in \mathcal{A}(M,X)$. Theorem 2.2 implies that $B \in \mathcal{A}(M,X)$, since $X$ has cotype 2 and $M$ has type 2. This means that $\mathcal{KS}_q(Z,X) \subset \mathcal{A}(Z,X)$, so that $X$ has the (uniform) $\mathcal{KS}_q$-A.P. Finally, recall from Facts 2.5(i) that for $1 \leq p < 2$ there are closed subspaces $X \subset \ell^p$ that fail the A.P.

(iii) We may assume that $p \neq q$, since the case $p = q$ follows from part (i) together with Facts 2.5(i). If $1 \leq q < p < 2$, then recall from Theorem 3.5(i) that $\mathcal{KS}_q(X) = \mathcal{K}(X)$ for any closed subspace $X \subset \ell^p$. Hence $X$ fails the uniform $\mathcal{KS}_q$-A.P. whenever $X \subset \ell^p$ is a closed subspace such that $\mathcal{A}(X) \nsubseteq \mathcal{K}(X)$, where such subspaces again exists by Facts 2.5(ii).

Let $1 \leq p < q < 2$. According to the proof of Theorem 3.5(iii) there is a closed subspace $X \subset \ell^p$ together with an operator

$$U = AB \in \mathcal{KS}_q(X) \setminus \mathcal{A}(X).$$

This means that $X$ does not have the uniform $\mathcal{KS}_q$-A.P.

Let $p,q \in (2,\infty)$. By Theorem 3.5 there is a closed subspace $X \subset \ell^p$ such that $X$ fails the uniform $\mathcal{QN}_q$-A.P. Consequently $X$ also fails the (uniform) $\mathcal{KS}_q$-A.P. in view of (3.7).

\[ \square \]

4. Concluding examples and problems

In this section we find a natural Banach operator ideal, which is equipped with the operator norm, that induces a non-trivial closed ideal of $\mathfrak{M}_X$ for a class of Banach spaces $X$. Moreover, we exhibit a class of spaces $X$ such that $\mathfrak{M}_X$ has uncountably many closed ideals with an explicit order structure. Both examples are related to the failure of duality for the A.P.

Let $X$ and $Y$ be Banach spaces. The operator $T \in \mathcal{L}(X,Y)$ is *compactly approximable*, denoted $T \in \mathcal{CA}(X,Y)$, if for any compact subset $K \subset X$ and $\varepsilon > 0$ there is a bounded finite rank operator $V \in \mathcal{F}(X,Y)$ so that

$$\sup_{x \in K} \|Tx - Vx\| < \varepsilon.$$  

In other words, $\mathcal{CA}(X,Y) = \overline{\mathcal{F}(X,Y)}^\tau$, where the closure is taken in $\mathcal{L}(X,Y)$ with respect to the topology $\tau$ of uniform convergence on compact sets in $\mathcal{L}(X,Y)$. By definition $X$ has the A.P. if and only if the identity operator $I_X \in \mathcal{CA}(X)$.  

The class $\mathcal{CA}$ was used by Pisier [43] (see also [14, 0.2] and [12, 31.5]). It defines a Banach operator ideal, which has not been much studied, though the related class $\mathcal{K}(X,Y)^\tau$ appears in [23] and [24]. We first list the relevant basic properties of $\mathcal{CA}$.

**Proposition 4.1.** (i) $\mathcal{CA}$ is a closed Banach operator ideal.

(ii) If $X$ or $Y$ has the A.P., then $\mathcal{CA}(X,Y) = \mathcal{L}(X,Y)$.

(iii) $\mathcal{CA}(X,Y) = \mathcal{A}(X,Y)^\tau$.

(iv) If $V \in \mathcal{K}(Z,X)$ and $U \in \mathcal{CA}(X,Y)$, then $UV \in \mathcal{A}(Z,Y)$.

**Proof.** (i) It is straightforward to check the operator ideal properties of $\mathcal{CA}$, and we leave this to the reader. Suppose that $U \in \mathcal{CA}(X,Y)$. Let $K \subset X$ be a compact subset, $\varepsilon > 0$ and put $M = \sup_{x \in K} \|x\|$. First pick $T \in \mathcal{CA}(X,Y)$ such that $\|U - T\| < \varepsilon/(M + 1)$, and then $V \in \mathcal{F}(X,Y)$ such that $\|Tx - Vx\| < \varepsilon$ for all $x \in K$. Hence, for any $x \in K$ we have

$$\|Ux - Vx\| \leq M\|U - T\| + \|Tx - Vx\| < 2\varepsilon.$$ 

It follows that $U \in \mathcal{CA}(X,Y)$.

(ii) By assumption $I_X$ or $I_Y$ is compactly approximable, so that $\mathcal{CA}(X,Y) = \mathcal{L}(X,Y)$ by the operator ideal property of $\mathcal{CA}$.

(iii) This is an simple variant of the argument in part (i).

(iv) Let $\varepsilon > 0$ be arbitrary. Since $\overline{VB_Z}$ is a compact subset of $X$, there is $T \in \mathcal{F}(X,Y)$ so that

$$\|UV - TV\| = \sup_{z \in \overline{VB_Z}} \|Uz - Tz\| < \varepsilon.$$ 

We obtain that $UV \in \mathcal{A}(Z,Y)$ since $TV \in \mathcal{F}(Z,Y)$.

We next use the failure of duality for the approximation property to show that the closed Banach operator ideal $\mathcal{CA} \cap \mathcal{K}$ gives a non-trivial closed ideal inside the compact operators for certain Banach spaces. Recall that there are Banach spaces $X$ such that $X$ has the A.P., but $X^*$ fails to have the A.P. In fact, for every separable Banach space $Y$ that fails the A.P. there is by [36, Theorems 1.d.3 and 1.e.7.(b)] a Banach space $Z$ such that $Z^*$ has a Schauder basis and $Z^{**}/Z$ is isomorphic to $Y$, so that $Z^{***} \cong Z^* \oplus Y^*$ fails the A.P. (since $Y^*$ also fails the A.P.) There are also concrete spaces of this kind: the space $X$ first constructed by Willis [55], or

$$\{z \in L^2 \mid z \text{ is a compact operator on } L^2\}$$

fails the A.P. By a celebrated result of Szankowski [50]. Recall further that spaces $X$ having the above property cannot be reflexive, cf. [30, Theorem 1.e.7.(a)].

**Example 4.2.** Let $X$ be a Banach space such that $X$ has the A.P., but $X^*$ fails the A.P. By [36, Theorem 1.e.5] there is a Banach space $Y$ such that $\mathcal{A}(X,Y) \not\subseteq \mathcal{K}(X,Y)$. Suppose that $W = X \oplus Y \oplus Z$, where either

(i) the Banach space $Z$ has the B.C.A.P., but fails to have the A.P. (such spaces were first constructed by Willis [55]), or

(ii) $Z \subset \ell^p$ is the closed subspace constructed in Theorem [28] for $1 \leq p < \infty$ and $p \neq 2$.

**Claim.** For $\mathcal{I} =: \mathcal{CA} \cap \mathcal{K}$ we have

$$\mathcal{A}(W) \not\subseteq \mathcal{I}(W) \subseteq \mathcal{K}(W),$$

where the induced quotient ideal $q(\mathcal{I}(W))$ is nilpotent in $\mathfrak{A}_W$. Moreover, there is $V \in \mathcal{K}(W)$ such that $V^n \not\in \mathcal{I}(W)$ for any $n \in \mathbb{N}$, so that the radical quotient algebra $\mathcal{K}(W)/\mathcal{I}(W)$ is non-nilpotent and infinite-dimensional.
Proof. We will again systematically use the fact that Banach operator ideals are uniquely determined on finite direct sums by their respective ideal components. Since $X$ has the A.P., it follows from Proposition 1.1(ii) that $\mathcal{A}(X,Y) = \mathcal{L}(X,Y)$. Hence $\mathcal{A}(X,Y) \subseteq \mathcal{C}(X,Y) \cap \mathcal{K}(X,Y) = \mathcal{K}(X,Y)$, as $\mathcal{A}(X,Y) \not\subseteq \mathcal{K}(X,Y)$ by our choice of $X$ and $Y$. This implies that $\mathcal{A}(W) \not\subseteq \mathcal{I}(W)$. We need the fact that there is a compact operator $U \in \mathcal{K}(Z)$ such that $U^n \notin \mathcal{A}(Z)$ for all $n \in \mathbb{N}$. This follows from [23, Proposition 3.1] in the case (i) and from Theorem 2.3 in the case (ii). Hence $U \notin \mathcal{A}(Z)$, since otherwise $U^2 \in \mathcal{A}(Z)$ by Proposition 4.1(iv). It follows that $U \in \mathcal{K}(Z) \setminus (\mathcal{A}(Z) \cap \mathcal{K}(Z))$, so that also $\mathcal{I}(W) \not\subseteq \mathcal{K}(W)$.

Define $V \in \mathcal{K}(W)$ by $V(x,y,z) = (0,0,Uz)$ for $(x,y,z) \in W$. The above properties yield that $V^n \notin \mathcal{A}(W)$ for $n \in \mathbb{N}$, so that the quotient algebra $\mathcal{K}(W)/\mathcal{I}(W)$ is non-nilpotent. By Proposition 2.7 the quotient $\mathcal{K}(W)/\mathcal{I}(W)$ is a radical algebra both in the real and complex cases. It follows once more from [1, Proposition 1.5.6.(iv)] that the algebra $\mathcal{K}(W)/\mathcal{I}(W)$ is infinite-dimensional.

Finally, Proposition 4.1(iv) implies that $ST \in \mathcal{A}(W)$ for any $S,T \in \mathcal{C}(A(W) \cap \mathcal{K}(W)$, so that $q(\mathcal{I}(W))$ is a nilpotent closed ideal of $\mathcal{A}(W)$.

The preceding example also demonstrates that in Proposition 4.1(iv) the order of composition matters. Namely, if we pick $U \in \mathcal{K}(X,Y) \setminus \mathcal{A}(X,Y)$ according to Example 4.2 then $I_X \in \mathcal{C}(A(X)$ as $X$ has the A.P., but $U \circ I_X \notin \mathcal{A}(X,Y)$.

In Example 4.2 the space $W = X \oplus Y \oplus Z$ is not reflexive, since $X$ cannot be reflexive as we noted above. We observe next that actually $\mathcal{A} \cap \mathcal{K}$ coincides with $\mathcal{A}$ within the class of reflexive Banach spaces. This is based on a representation by Godefroy and Saphar [25] of the bidual $\mathcal{K}(X,Y)^{**}$.

**Proposition 4.3.** If $X$ is a reflexive Banach space, then

$$\mathcal{A}(X) \cap \mathcal{K}(X) = \mathcal{A}(X).$$

**Proof.** Let $X$ be any reflexive Banach space. We require the fact that, up to isometric isomorphism, we may identify

$$(4.1)\quad \mathcal{A}(X)^{**} = \overline{\mathcal{A}(X)}^{\mathcal{L}(X)} = \mathcal{C}(X),$$

where the closure is taken in $\mathcal{L}(X)$ with respect to the topology $\tau$ of uniform convergence on compact subsets $K \subset X$. This fact is essentially contained in the proofs of Proposition 1.1, Corollaries 1.2 and 1.3 in [25].

Namely, by [25, Proposition 1.1] we have $\mathcal{K}(X)^{**} = \overline{\mathcal{K}(X)}^{w^*}$ up to isometric isomorphism, where the $w^*$-topology denotes the one induced in $\mathcal{L}(X)$ from the duality $(X \hat{\otimes}_\pi X^*)^* = \mathcal{L}(X)$. In fact, let $(T_\alpha) \subset \mathcal{K}(X)^{*}$ be an arbitrary net and $T \in \mathcal{K}(X)^{**}$. According to [25, Proposition 1.1] there is for reflexive spaces $X$ a quotient map $Q : X \hat{\otimes}_\pi X^* \to \mathcal{K}(X)^*$ such that $Q^*$ defines an isometric embedding $\mathcal{K}(X)^{**} \to \mathcal{L}(X)$. In particular,

$$(z,Q^*T_\alpha - Q^*T) = \langle Qz,T_\alpha - T \rangle \text{ for all } z \in X \hat{\otimes}_\pi X^*.$$ 

Moreover, by [25, Corollary 1.3] we may identify $\mathcal{K}(X)^{**} = \overline{\mathcal{K}(X)}^{\mathcal{L}(X)}$ in $\mathcal{L}(X)$. By standard duality we also have $\mathcal{A}(X)^{**} = \mathcal{A}(X)^{1\perp} \subset \mathcal{K}(X)^{**}$, where $\mathcal{A}(X)^{1\perp}$ denotes the biannihilator.
of $A(X)$ in $K(X)^{**}$. Since the $w^*$-topology of $L(X)$ and the $\tau$-topology have the same closed convex sets in $L(X)$, see the proof of [23, Corollary 1.2], we deduce that (1.1) holds.

Recall next that if $Z$ is a Banach space and $M \subset Z$ is a closed subspace, then a lemma of Valdivia [54, pp. 107-108] (see also [26, Lemma A.5.1]) implies that $M^{**} \cap Z = M$, where $w^*$ denotes the $w^*$-topology of $Z^{**}$. By applying these facts to $A(X) \subset K(X)$ we obtain that

$$CA(X) \cap K(X) = A(X)^{**} \cap K(X) = A(X)$$

which completes the argument. □

Remarks 4.4. (i) Proposition 4.3 does not extend to the class of Banach spaces that has the Radon-Nikodým property (RNP), since there are direct sums $W = X \oplus Y \oplus Z$ with the RNP among those of Example 4.2. We refer to e.g. [17, Chapter III] for the definition of the RNP and the facts that separable dual spaces as well as reflexive spaces have the RNP.

Namely, $X$ can be chosen a separable dual space in Example 4.2 and $Y$ a reflexive space by applying the DFJP-factorisation theorem (see e.g. [57, Theorem 2.g.11] or [26, Theorem 3.2.1]). In condition (ii) of Example 4.2 the subspace $Z \subset \ell^p$ is reflexive for $p > 1$, while there are also reflexive reflexive spaces $Z$ which satisfy condition (i) by [55, Propositions 3 and 4].

(ii) In the direct sum $W = X \oplus Y \oplus Z$ of Example 4.2 it is also possible to choose $Z$ reflexive so that $A_Z \neq \{0\}$. Then Proposition 4.3 implies that $CA(A) \cap K(A) = A(A) \subset K(A)$, and hence $Z(W) = CA(W) \cap K(W) \subset K(W)$. However, such a choice does not by itself guarantee that $K(W)/I(W)$ is a non-nilpotent quotient algebra, which is also obtained in Example 4.2.

We finally exhibit a quite dramatic example, where the compact-by-approximable algebra contains an uncountable family of closed ideals having the reverse lattice structure of the partially ordered power set $(P(\mathbb{N}), \subset)$ of the natural numbers $\mathbb{N}$. This example has a very special form and in a sense it is the most elementary one contained here.

Example 4.5. Fix $1 < p < \infty$ and let $X$ be any Banach space such that $X$ has the A.P., but $X^*$ fails to have the A.P. By [36, Theorem 1.e.5] there is a Banach space $Y_0$ such that

$$A(X, Y_0) \subset K(X, Y_0),$$

but where $K(Y_0, X) = A(Y_0, X)$ and $K(X) = A(X)$, since $X$ has the A.P. By replacing $Y_0$ with the direct $\ell^p$-sum $Y = (\oplus_{n \in \mathbb{N}} Y_0)_p$, we may assume that the quotient space $K(X, Y)/A(X, Y)$ is infinite-dimensional. In fact, let $S \in K(X, Y_0)$ satisfy $\|S\| = 1$ and $dist(S, A(X, Y_0)) > 1/2$. Define $S_j : X \to Y$ by

$$S_j x = (0, \ldots, 0, Sx, 0, \ldots), \quad x \in X \text{ and } j \in \mathbb{N},$$

where the vector $Sx$ belongs to the $j$:th component of $Y$. It is straightforward to check that $dist(S_i - S_j, A(X, Y)) \geq 1/2$ for $i \neq j$, so that $K(X, Y)/A(X, Y)$ is infinite-dimensional. Note that we still have $K(Y, X) = A(Y, X)$.

We consider the direct sum

$$Z = (\oplus_{k=0}^{\infty} X_k)_p,$$

where we put $X_0 = Y$ and $X_k = X$ for $k \in \mathbb{N}$ for unicity of notation. Let $P_k \in L(Z)$ be the natural projection map onto $X_k$ and let $J_k : X_k \to Z$ be the corresponding natural inclusion map for $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We extend our operator matrix notation to operators on $Z$, and write $U = (U_{r,s}) \in K(Z)$, where

$$U_{r,s} = (U)_{r,s} =: P_r U J_s \in K(X_s, X_r) \text{ for } r, s \in \mathbb{N}_0.$$
(It should be noticed that if we are given \( U_{r,s} \in \mathcal{K}(X_s, X_r) \) for all \( r, s \in \mathbb{N}_0 \), then it is a separate question whether the formal operator matrix \( U = (U_{r,s}) \) defines a compact or even a bounded operator on \( Z \).)

For any given subset \( A \subset \mathbb{N} \) we define

\[
\mathcal{I}_A = \{ U = (U_{r,s}) \in \mathcal{K}(Z) : U_{0,0} \in \mathcal{A}(Y) \text{ and } U_{0,s} \in \mathcal{A}(X,Y) \text{ for all } s \in A \}.
\]

Observe that if \( U = (U_{r,s}) \in \mathcal{K}(Z) \), then \( U_{r,s} \in \mathcal{A}(X_s, X_r) \) whenever \( r \geq 1 \), since \( \mathcal{K}(Y,X) = \mathcal{A}(Y,X) \) and \( \mathcal{K}(X) = \mathcal{A}(X) \) by construction. Hence only the components \( U_{0,s} \in \mathcal{K}(X,Y) \) for \( s \in \mathbb{N} \) will play an explicit role in the definition of \( \mathcal{I}_A \).

**Claim:** the family \( \{ \mathcal{I}_A : A \subset \mathbb{N} \} \) has the following properties:

(i) \( \mathcal{I}_A \) is a closed ideal of \( \mathcal{K}(Z) \) satisfying \( \mathcal{A}(Z) \subseteq \mathcal{I}_A \not\subseteq \mathcal{K}(Z) \) for all subsets \( \emptyset \neq A \subset \mathbb{N} \).

In addition, \( \mathcal{I}_N = \mathcal{A}(Z) \) and right multiplication on \( \mathcal{I}_A \) satisfies \( US \in \mathcal{A}(Z) \) for \( U \in \mathcal{I}_A \) and \( S \in \mathcal{K}(Z) \). In particular, the quotient ideals \( q(\mathcal{I}_A) \) are nilpotent in \( \mathfrak{K}_Z \) whenever \( \emptyset \neq A \subset \mathbb{N} \).

(ii) If \( A \subset B \), then \( \mathcal{I}_B \subset \mathcal{I}_A \). Moreover, if \( A \subset B \), then \( \mathcal{I}_A / \mathcal{I}_B \) is infinite-dimensional.

(iii) \( \mathcal{I}_A \cap \mathcal{I}_B = \mathcal{I}_{A \cap B} \) and \( \mathcal{I}_A + \mathcal{I}_B = \mathcal{I}_{A \cup B} \) for all subsets \( A, B \subset \mathbb{N} \). In particular, \( \mathcal{I}_A + \mathcal{I}_B \) is closed in \( \mathcal{K}(Z) \) for all \( A, B \subset \mathbb{N} \).

Hence the family \( \{ \mathcal{I}_A : \emptyset \neq A \subset \mathbb{N} \} \) of non-trivial closed ideals of \( \mathcal{K}(Z) \) does not have a smallest or a largest element, and \( \mathcal{I}_{\{k\}} \) are maximal elements of this family for each \( k \in \mathbb{N} \).

By Proposition 2.1 the family \( \{q(\mathcal{I}_A) : \emptyset \neq A \subset \mathbb{N} \} \) is an uncountable family of non-trivial closed ideals of \( \mathfrak{K}_Z \) having the reverse partial order structure of \( (\mathcal{P}(\mathbb{N}), \subset) \).

**Proof.** (i) For \( s \in \mathbb{N}_0 \) define the bounded linear map \( \psi_s : \mathcal{K}(Z) \to \mathcal{K}(X_s, Y) \) by \( \psi_s(S) = P_0 S J_s \) for \( S \in \mathcal{K}(Z) \). Deduce that

\[
\mathcal{I}_A = \bigcap_{s \in A \cup \{0\}} \psi^{-1}_s(\mathcal{A}(X_s,Y))
\]

is a closed linear subspace of \( \mathcal{K}(Z) \).

Let \( U = (U_{r,s}) \in \mathcal{I}_A \) and \( S = (S_{r,s}) \in \mathcal{K}(Z) \) be arbitrary. We next claim that \( SU \in \mathcal{I}_A \) and \( US \in \mathcal{I}_A \), which means that \( \mathcal{I}_A \) is a closed ideal of \( \mathcal{K}(Z) \). Note first that \( SU \in \mathcal{K}(Z) \) so that the components \( (SU)_{r,s} \in \mathcal{K}(X_s, X_r) \) for all \( r, s \in \mathbb{N}_0 \), and \( (SU)_{r,s} \in \mathcal{K}(X_s, X_r) = \mathcal{A}(X_s, X_r) \) for \( r \in \mathbb{N} \). Similar facts hold for the components \( (US)_{r,s} \).

In order to verify that \( (SU)_{0,s} \in \mathcal{A}(X_s,Y) \) for \( s \in A \cup \{0\} \), observe that the finite sums

\[
\sum_{k=0}^N S_{0,k} U_{k,s} = \sum_{k=0}^N P_0 S J_k P_k U J_s \in \mathcal{A}(X_s,Y)
\]

for all \( N \in \mathbb{N} \), since \( U_{0,s} = P_0 U J_s \in \mathcal{A}(X_s,Y) \) for \( s \in A \) and \( U_{0,0} \in \mathcal{A}(Y) \) by assumption. Moreover,

\[
\|P_0 SU J_s - \sum_{k=0}^N P_0 S J_k P_k U J_s\| = \|P_0 S (U - \sum_{k=0}^N J_k P_k U) J_s\| \leq \|S\| \cdot \|U - \sum_{k=0}^N J_k P_k U\| \to 0
\]

as \( N \to \infty \). Here \( \|U - \sum_{k=0}^N J_k P_k U\| \to 0 \) as \( N \to \infty \), because the sequence \( \sum_{k=0}^N J_k P_k \to I_Z \) pointwise on \( Z \) as \( N \to \infty \) and \( U \in \mathcal{K}(Z) \) is compact. In a similar manner one deduces that \( (US)_{0,s} \in \mathcal{A}(X_s,Y) \) for all \( s \in \mathbb{N}_0 \). In this case \( \sum_{k=0}^N U_{0,k} S_{k,s} \in \mathcal{A}(X_s,Y) \) for all \( s \in \mathbb{N}_0 \) and all \( N \), since we also have \( U_{0,0} \in \mathcal{A}(Y) \).
If \( U = (U_{r,s}) \in \mathcal{I}_{\mathbb{N}} \) is arbitrary, then \( U_{r,s} \in \mathcal{A}(X_s, X_r) \) for all \( r, s \in \mathbb{N}_0 \). It follows that \( U \in \mathcal{A}(Z) \) from the general fact proved separately below in Lemma 4.6 so that \( \mathcal{I}_{\mathbb{N}} = \mathcal{A}(Z) \). Thus the above argument says that \( US \in \mathcal{I}_{\mathbb{N}} = \mathcal{A}(Z) \) whenever \( U \in \mathcal{I}_{\mathbb{N}}, S \in \mathcal{K}(Z) \), and \( \emptyset \neq A \subset \mathbb{N} \). In particular, \( UV \in \mathcal{A}(Z) \) whenever \( U, V \in \mathcal{I}_{\mathbb{N}} \), so that \( q(\mathcal{I}_{\mathbb{N}}) \) defines a nilpotent ideal in \( \mathfrak{R}_Z \).

Finally we verify that \( \mathcal{A}(Z) \not\subseteq \mathcal{I}_{\mathbb{N}} \not\subseteq \mathcal{K}(Z) \) for \( \emptyset \neq A \subset \mathbb{N} \). First fix \( s \in A \) and pick \( U_{0,s} \in \mathcal{K}(X_s, Y) \setminus \mathcal{A}(X_s, Y) \). Define \( U = (U_{r,s}) \in \mathcal{K}(Z) \) by \( U_{r,s} = 0 \) for \( (r, s) \neq (0, s) \), so that \( U \notin \mathcal{I}_{\mathbb{N}} \). Next fix \( t \in A^c \) and pick \( V_{0,t} \in \mathcal{K}(X_t, Y) \setminus \mathcal{A}(X_t, Y) \). If we define \( V = (V_{r,s}) \in \mathcal{K}(Z) \) by \( V_{r,s} = 0 \) for \( (r, s) \neq (0, t) \), then \( V \in \mathcal{I}_{\mathbb{N}} \setminus \mathcal{A}(Z) \).

(ii) Clearly \( \mathcal{I}_B \subset \mathcal{I}_A \) if \( A \subset B \). Since the quotient \( \mathcal{K}(X_k, Y)/\mathcal{A}(X_k, Y) \) is infinite-dimensional by construction, there is a normalised sequence \( (S_j^i) \subset \mathcal{K}(X_k, Y) \) such that

\[
\text{dist}(S^j_i - S^i_i, \mathcal{A}(X_k, Y)) > 1/2 \quad \text{for all } i \neq j.
\]

Define a normalised sequence \( (V_j) \subset \mathcal{K}(Z) \) such that the components of \( V_j = (V^j_{r,s}) \) satisfy \( V^j_{0,k} = S^j_i \) and \( V^j_{r,s} = 0 \) for \( (r, s) \neq (0, k) \). Then \( V_j \in \mathcal{I}_A \setminus \mathcal{I}_B \) for all \( j \in \mathbb{N} \). Let \( S = (S_{r,s}) \in \mathcal{I}_B \) be arbitrary. It follows that

\[
\|V_j - V_i - S\| \geq \|P_0(V_j - V_i - S)_j\| = \|S^j_i - S^i_i - S_{0,k}\| > 1/2
\]

for all \( i \neq j \), since \( S_{0,k} \in \mathcal{A}(X_k, Y) \). Conclude that \( \text{dist}(V_j - V_i, \mathcal{I}_B) \geq 1/2 \) for \( i \neq j \), so that the quotient \( \mathcal{I}_A/\mathcal{I}_B \) is infinite-dimensional.

(iii) Let \( A, B \subset \mathbb{N} \) be non-empty subsets. If \( U = (U_{r,s}) \in \mathcal{I}_A \cap \mathcal{I}_B \), then \( U_{0,k} \in \mathcal{A}(X_k, Y) \) for all \( k \in A \cup B \), so that \( \mathcal{I}_A \cap \mathcal{I}_B \subset \mathcal{I}_{A \cup B} \). Conversely, \( \mathcal{I}_{A \cup B} \subset \mathcal{I}_A \) and \( \mathcal{I}_{A \cup B} \subset \mathcal{I}_B \) by monotonicity.

We also get that \( \mathcal{I}_A \subset \mathcal{I}_{A \cap B} \) and \( \mathcal{I}_B \subset \mathcal{I}_{A \cap B} \) by monotonicity, so that \( \mathcal{I}_{A \cap B} \subset \mathcal{I}_{A \cup B} \), because \( \mathcal{I}_{A \cap B} \) is a closed ideal. Conversely, let \( U = (U_{r,s}) \in \mathcal{I}_{A \cap B} \) be arbitrary. We define the bounded operator \( \theta_A \in \mathcal{L}(Z) \) by

\[
\theta_A z = (\chi_A(k)z_k) \quad \text{for } z = (z_k) \in Z,
\]

where \( \chi_A \) is the characteristic function of \( A \subset \mathbb{N}_0 \). Thus \( U\theta_A \in \mathcal{K}(Z) \) and

\[
U = (U - U\theta_A) + U\theta_A
\]

so it will be enough to verify that \( U\theta_A \in \mathcal{I}_B \) and \( U - U\theta_A \in \mathcal{I}_A \). In fact, this immediately yields that \( \mathcal{I}_A + \mathcal{I}_B = \mathcal{I}_{A \cap B} \), and hence that \( \mathcal{I}_A + \mathcal{I}_B \) is a closed ideal of \( \mathcal{K}(Z) \).

Let \( k \in B \) be arbitrary. If \( k \in A \cap B \), then \( (U\theta_A)_{0,k} = U_{0,k} \in \mathcal{A}(X_k, Y) \) by assumption. On the other hand, if \( k \in B \setminus A \), then \( (U\theta_A)_{0,k} = 0 \). We deduce that \( U\theta_A \in \mathcal{I}_B \) as \( (U\theta_A)_{0,0} = 0 \). Moreover, if \( k \in A \) is arbitrary, then

\[
(U - U\theta_A)_{0,k} = U_{0,k} - U_{0,k} = 0.
\]

We conclude that \( U - U\theta_A \in \mathcal{I}_A \), since also \( (U - U\theta_A)_{0,0} = U_{0,0} \in \mathcal{A}(Y) \). \( \square \)

The following technical result was used in the proof of Claim (i) of Example 4.5. It is a vector-valued analogue of a well-known fact for scalar operator matrices.

**Lemma 4.6.** Suppose that \( X_k \) are Banach spaces for \( k \in \mathbb{N} \), and let

\[
Z = \bigoplus_{k=1}^{\infty} X_k
\]
be the corresponding direct $\ell^p$-sum, where $1 < p < \infty$. Let $P_k$ be the natural projection onto $X_k \subset Z$ and $J_k : X_k \to Z$ be the natural inclusion for $k \in \mathbb{N}$. Assume that $T = (T_{r,s}) \in \mathcal{K}(Z)$ is a compact operator such that the components $T_{r,s} = P_r J_s : \mathcal{A}(X_s, X_r)$ for all $r,s \in \mathbb{N}$. Then $T = (T_{r,s}) \in \mathcal{A}(Z)$ is an approximable operator.

**Proof.** Let $Q_n = \sum_{k=1}^n J_k P_k$ for $k \in \mathbb{N}$, that is, $Q_n z = (z_1, \ldots, z_n, 0, \ldots)$ for $z = (z_k) \in Z$, is the natural projection of $Z$ onto the closed linear subspace $\oplus_{k=1}^n X_k \subset Z$.

We know by definition that $Q_n = \sum_{k=1}^n J_k P_k \to I_Z$ pointwise on $Z$ as $n \to \infty$. It follows that

$$\|T - Q_n T\| = \|(I - Q_n) T\| \to 0 \text{ as } n \to \infty,$$

since $T$ is a compact operator on $Z$. Moreover, by standard duality $Q_n^*$ is the natural projection of $Z^* = \left( \oplus_{k=1}^\infty X_k^* \right)_{\ell^p}$ onto the closed subspace $\oplus_{k=1}^n X_k^* \subset Z^*$ for $n \in \mathbb{N}$, where $\ell^p \in (1, \infty)$ is the dual exponent of $p$. Hence

$$\|T - T Q_n\| = \|T^* - Q_n^* T^*\| \to 0 \text{ as } n \to \infty,$$

since also $Q_n^* \to I_{Z^*}$ pointwise on $Z^*$ as $n \to \infty$ and $T^* \in \mathcal{K}(Z^*)$. (Here we use that $1 < p' < \infty$.)

Observe next that $Q_n T Q_n = \sum_{k,l=1}^n J_k P_l T J_l P_l = \sum_{k,l=1}^n J_k T_{l,k} P_l \in \mathcal{A}(Z)$ for all $n \in \mathbb{N}$ by our assumption on $T$. Deduce from the above facts that

$$\|T - Q_n T Q_n\| \leq \|T - Q_n T\| + \|Q_n\| \cdot \|T - T Q_n\| \to 0$$

as $n \to \infty$, which yields that $T \in \mathcal{A}(Z)$. \hfill $\square$

We also formulate a version of Example 4.5 for finite direct sums. Let $N \geq 2$ and consider $Z_N = \left( \oplus_{k=0}^N X_k \right)_p$, where $X_0 = Y$ and $X_k = X$ for $k = 1, \ldots, N$, and the spaces $X$ and $Y$ are those of Example 4.5. Put $\{N\} = \{1, \ldots, N\}$.

**Example 4.7.** Let $Z_N = \left( \oplus_{k=0}^N X_k \right)_p$ be as above for $N \geq 2$ and define $\mathcal{I}_A \subset \mathcal{K}(Z_N)$ as in Example 4.5 for subsets $A \subset \{N\}$. Then the family

$$\{\mathcal{I}_A : \emptyset \neq A \subset \{N\}\}$$

contains $2^N - 2$ closed ideals of $\mathcal{K}(Z_N)$ that satisfy $\mathcal{A}(Z_N) \subsetneq \mathcal{I}_A \subsetneq \mathcal{K}(Z_N)$. Moreover, this family of closed ideals have the reverse order structure of the power set $\mathcal{P}(\{N\}, \subset)$. \hfill $\square$

**Proof.** The argument is a simpler variant of that of Example 4.5. In this case $U = (U_{r,s})_{r,s=0}^N \in \mathcal{K}(Z_N)$ is a $(N + 1) \times (N + 1)$ operator matrix, and there are no convergence issues with the operators on $Z_N$ or in the verification of the ideal properties. We leave the details to the interested reader. \hfill $\square$

**Remark 4.8.** We note that in Example 4.5 the closed ideals $\mathcal{I}_A$ of $\mathcal{K}(Z)$ are not ideals of $\mathcal{L}(X)$ for any $\emptyset \neq A \subset \mathbb{N}$.

In fact, fix $r \notin A \cup \{0\}$ and $s \in A$. By construction we may pick $T \in \mathcal{K}(X_r, Y) \setminus \mathcal{A}(X_r, Y)$. Define $U = (U_{k,l}) \in \mathcal{I}_A$ through $U_{0,r} = T$ and $U_{k,l} = 0$ for $(k,l) \neq (0,r)$, and the operator $V = (V_{k,l}) \in \mathcal{L}(Z)$ by $V_{r,s} = I_X$ and $V_{k,l} = 0$ for $(k,l) \neq (r,s)$. It follows that

$$(UV)_{0,s} = U_{0,r} V_{r,s} = T \notin \mathcal{A}(X_r, Y),$$

so that $UV \notin \mathcal{I}_A$.

We also draw attention to a few problems suggested by our results.
Question 4.9. Let \( X \subset \ell^p \) be a closed subspace such that \( \mathfrak{A}_X \neq \{0\} \). Is it always possible to find a non-trivial closed ideal
\[
\mathcal{A}(X) \subsetneq \mathcal{J} \subsetneq \mathcal{K}(X) \?
\]
By Example 3.12 there is such a closed subspace \( X \subset \ell^p \) for \( 4 < p < \infty \). Note that for \( 1 \leq p < 2 \) one has \( \bigcap_{p>1} \mathcal{A}(X) = \mathcal{A}(X) \) for any \( X \), so new classes are needed.

There remains combinations of \( (p,q) \) for which examples of a strict inclusion \([3.1]\) does not appear to be known.

Question 4.10. Let \( 1 \leq p < 2 < q < \infty \). Are there closed subspaces \( X \subset \ell^p \) and \( Y \subset \ell^q \) such that \( \mathcal{A}(X,Y) \subsetneq \mathcal{K}(X,Y) \)? Note that there are closed subspaces \( X \subset \ell^p \), where \( p \in [1, \infty) \) and \( p \neq 2 \), and \( Z \subset c_0 \) for which
\[
\mathcal{A}(X,Z) \subsetneq \mathcal{K}(X,Z) \quad \text{and} \quad \mathcal{A}(Z,X) \subsetneq \mathcal{K}(Z,X).
\]
Namely, by Facts \([2.5]\)ii) there is a closed subspace \( X \subset \ell^p \), for which there is an operator \( T \in \mathcal{K}(X) \setminus \mathcal{A}(X) \). By Terzioglu’s compact factorisation theorem \([32]\) there is a closed subspace \( Z \subset c_0 \) and a factorisation \( T = BA \), where \( A \in \mathcal{K}(X,Z) \) and \( B \in \mathcal{K}(Z,X) \). Here \( A \) and \( B \) cannot be approximable operators.

The examples of Theorem \([2.6]\) point to a number of further questions.

Questions 4.11. (i) Let \( 1 \leq p < 2 < q < \infty \), and suppose that \( X \subset \ell^p \) and \( Y \subset \ell^q \) are closed subspaces such that \( \mathfrak{A}_X \neq \{0\} \) and \( \mathfrak{A}_Y \neq \{0\} \). How may one construct further closed ideals
\[
\mathcal{A}(X \oplus Y) \subsetneq \mathcal{J} \subsetneq \mathcal{K}(X \oplus Y)
\]
in addition to those contained in Theorem \([2.6]\) and Example \([3.15]\)? It is straightforward to check that if \( \mathcal{A}(X,Y) \subsetneq M \subset \mathcal{K}(X,Y) \) is a closed linear subspace such that
\[
SA \in M \quad \text{and} \quad BS \in M \quad \text{for all} \quad S \in M, \quad A \in \mathcal{K}(X), \quad B \in \mathcal{K}(Y),
\]
then
\[
\mathcal{K}_M := \begin{pmatrix} \mathcal{A}(X) & \mathcal{A}(Y,X) \\ M & \mathcal{A}(Y) \end{pmatrix}
\]
defines a new non-trivial closed ideal of \( \mathcal{K}(X \oplus Y) \). We do not have examples of such non-trivial ideal components \( M \subset \mathcal{K}(X,Y) \) (this is also related to Question 4.10 above).

(ii) Is it possible to iterate the construction of Theorem \([2.6]\) for finite direct sums \( \oplus_{k=1}^n X_k \) with \( n \geq 3 \) ?

Our concluding remarks point to some research in parallel directions.

Remarks 4.12. (i) Let \( S \) be the class of strictly singular operators, which defines a closed Banach operator ideal contained in the class \( \mathcal{R} \) of the inessential operators. By Proposition \([2.4]\), the quotient algebra
\[
\mathfrak{S}_X^\mathcal{K} =: S(X)/\mathcal{K}(X)
\]
is a radical Banach algebra for both real and complex scalars. By a simple modification of Proposition \([2.4]\) the closed ideals of \( \mathfrak{S}_X^\mathcal{K} \) correspond to the closed ideals \( \mathcal{J} \subset \mathcal{S}(X) \). Much more is known about such ideals for classical Banach spaces \( X \). For instance, there is a continuum of closed ideals \( \mathcal{K}(X) \subset \mathcal{J} \subset \mathcal{S}(X) \) in the following cases:

(a) \( X = \ell^p \oplus \ell^q \) for \( 1 < p < q < \infty \) and \( X = L^p \) for \( 1 < p < \infty \) and \( p \neq 2 \) by Schlumprecht and Zsák \([47]\).
(b) $X = L^1$ and $X = C(0,1)$ by Johnson, Pisier and Schechtman [31].

In addition, Tarbard [51] has constructed for each $k \geq 2$ a real Banach space $X_k$ having a Schauder basis such that $\dim(\mathcal{S}_{X_k}) = k - 1$. A similar remark also applies to the radical quotient algebras $\mathcal{S}(X)/\mathcal{A}(X)$, since $\mathcal{K}(X) = \mathcal{A}(X)$ for these results.

This line of research concerns classical spaces that have the A.P., and we refer to the introductions of [47], [31], and [3] for many additional results and further references about closed ideals of $L(X)$.

(ii) Let $(I, | \cdot |_I)$ be a Banach operator ideal such that $I \subset \mathcal{K}$. Properties of the generalised compact-by-approximable algebras

$$\mathfrak{A}_X^I = : I(X)/\mathcal{F}(X)^{|x|}$$

are studied in [52] for Banach spaces $X$. The generalised setting of $\mathfrak{A}_X^I$ allows for new phenomena and for examples of different structure of closed ideals.

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