TILTING BRAUER GRAPH ALGEBRAS I: CLASSIFICATION OF TWO-TERM TILTING COMPLEXES

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Abstract. Using only the combinatorics of its defining ribbon graph, we classify the two-term tilting complexes, as well as their indecomposable summands, of a Brauer graph algebra. As an application, we determine precisely the class of Brauer graph algebras which are tilting-discrete. In particular, this allows us to write down a bijection between tilting complexes of two tilting-discrete Brauer graph algebras with the same underlying ribbon graph.

1. Introduction

The derived category of a non-semisimple symmetric algebra is a beast. For example, a simple classification of perfect objects usually does not exist, which makes the task of finding derived equivalent algebras - one of the central themes in modular representation theory of finite groups, extremely difficult. The Okuyama-Rickard construction gives an easy way to calculate non-trivial tilting complexes. Therefore, one would naturally hope that all tilting complexes can be obtained by applying such construction repeatedly, and use this to determine, say, the derived equivalence class. Roughly speaking, an algebra is tilting-connected if such a hope holds.

One of the main results in [AI] is that there is a partial order structure on the set of tilting complexes. Using this translation, tilting-connectedness simply means that the Hasse quiver of this partially ordered set is connected. One could attempt to exploit this combinatorial property to better understand, for instance, the derived Picard group(oid) of the derived category. See [Zv2] for a fruitful result in this direction.

Determining tilting-connectedness is not easy - at least with the current technology. So far, the only known tilting-connected symmetric algebras are the local ones [AI], and the representation-finite ones [Ai1]. The “proof” for tilting-connectedness of local symmetric algebras in fact is the answer we wanted originally - a classification of tilting complexes. More precisely, a tilting complex of a local algebra is precisely a stalk complex given by a finite direct sum of the (unique) indecomposable projective module. On the other hand, the proof for the representation-finite case does not generalise to an arbitrary finite dimensional symmetric algebra.

A symmetric algebra is said to be tilting-discrete if the set of n-term tilting complexes is finite for any natural number n. In such case, the algebra will be tilting-connected. This notion is introduced in [Ai1] in order to find a more computable approach to determine tilting-connectedness. To see if this approach can be successful for non-representation-finite non-local symmetric algebras, we use the Brauer graph algebras as a testing ground. We remark here that the tilting-discreteness, along with its applications, of the preprojective algebras is also studied in a parallel work of the second author and Mizuno [AM].

The representation-finite Brauer graph algebras (the Brauer tree algebras) was discovered by Brauer in the forties during the dawn of modular representations of finite groups. The generalisation - Brauer graph algebras - is the main subject of interest in this paper. The Brauer
graph algebras are tame symmetric algebras which are arguably the easiest class of symmetric algebras one can play with. This is because the structure of such an algebra is encoded entirely in a simple combinatorial object, called the Brauer graph. It often turns out that one can replace many algebraic and homological calculations into simple combinatorial games on the Brauer graph. These results in turn would give inspirations to develop techniques and theories for larger class of algebras, such as group algebras or tame symmetric algebras. We hope to apply this philosophy to flourish the notions of tilting-connectedness and tilting-discreteness.

The first homological calculation which will be turned into pure combinatorics is the determination of two-term (pre)tilting complexes (Theorem 4.6). Since the combinatorics is entirely new, we avoid giving the statement of the theorem here. For readers with no knowledge about Brauer graphs, we briefly remind that a Brauer graph is a graph with an ordering of edges around each vertex, and a positive number called multiplicity associated to each vertex. Forgetting the multiplicities, one obtains an orientable ribbon graph (or fatgraph) - combinatorial object which appears in many other areas of mathematics, such as dessin d’enfants, Teichmüller and moduli space of curves, homological mirror symmetry, etc.

Our second result is the classification of tilting-discrete Brauer graph algebras. We achieve this by using the “two-term combinatorics” and the tilting mutation theory for Brauer graph algebras. Since most of the tilting-discrete Brauer graph algebras are neither representation-finite nor local, we have a new class of tilting-connected symmetric algebras. Apart from using tilting-discreteness to determine derived equivalence classes, another application is to relate the tilting complexes of Brauer graph algebras with the same underlying ribbon graph.

**Theorem 1.1.** Let $\mathcal{G}$ be a Brauer graph, and $\Lambda_{\mathcal{G}}$ its associated Brauer graph algebra.

1. (Theorem 6.10) $\Lambda_{\mathcal{G}}$ is tilting-discrete if, and only if, $\mathcal{G}$ contains at most one cycle of odd length, and no cycle of even length.

2. (Corollary 6.15) In the case of (1), any algebra derived equivalent to $\Lambda_{\mathcal{G}}$ is also a Brauer graph algebra $\Lambda_{\mathcal{G}'}$. Moreover, $\mathcal{G}'$ is flip equivalent to $\mathcal{G}$ in the sense of [Ai2].

3. (Theorem 6.18) In the case of (1), the partially ordered set of basic tilting complexes of $\Lambda_{\mathcal{G}}$ is independent of the multiplicities of the vertices in $\mathcal{G}$.

Note that the derived equivalence class in (2) is not entirely new; for slightly more details, see the discussions at the end of Section 6.2.

The classification and description of two-term tilting complexes (and their indecomposable summands) of the Brauer star algebras, i.e. (representation-finite) symmetric Nakayama algebras, have already been studied in [SZ1], and implicitly in [Ada]. In the case of the Brauer tree algebras, these tasks were carried out in [AZ, Zv1]. We note that the result in [SZ1] actually classifies more than just two-term tilting complexes. In all the mentioned articles, as well as ours, the indecomposable summands of two-term tilting complexes are classified first. Then one gives the conditions on how they can be sum together to form tilting complexes. In contrast to [SZ1, Zv1], we will not calculate the corresponding endomorphism algebra in this article, but leave it for our sequel. For Brauer tree algebras, the classification in [Zv1] is described in terms of the Ext-quiver of the algebra, whereas ours are described by combinatorics on the defining ribbon graph.

The combinatorial language we use is heavily influenced by the ribbon graph theory. The use of this language for Brauer graph algebras is slightly different from the traditional approach used in, for example, [Rog, Kau], but it is also not new - we learn it from the paper of Marsh and Schroll [MS], which relates Brauer graphs with surface triangulation (and $n$-angulation) and cluster theory. The key advantage of adopting this approach is that we can clear out many ambiguities when there is a loop in the Brauer graph. Moreover, while writing up this article, this new language gives us a glimpse of a connection between geometric intersection theory and
the tilting theory of Brauer graph algebras (see Remark 2.10). We hope to address this issue in a subsequent paper.

Our approach to the homological calculations draws results and inspirations from [Al, Ai1, AIR]. The key technique which allows the classification of two-term (pre)tilting complex possible is Lemma 5.1. This is a particular case of a vital lemma in relating $\tau$-tilting theory with two-term silting/tilting complexes in [AIR]. This homological property was also used in [AZ, Zv1]. While we will not mention and introduce $\tau$-tilting theory formally, a partial motivation of this work is in fact to obtain a large database of calculations for $\tau$-tilting theory, and hopefully to inspire further development in the said theory. The idea of investigating tilting-discreteness using only knowledge about two-term tilting complexes in this article (and the investigation of similar vein in [AM]) is also inspired by the elegance of $\tau$-tilting theory.

This article is structured as follows. We will first go through in Section 2 the essential ribbon (and Brauer) graph combinatorics needed for this paper. There is no mathematical prerequisite in order to understand this section, although knowledge of basic notions in ordinary graph theory would be helpful to find intuitions. Section 3 is devoted to recall known results and notions needed to understand the algebraic and homological side of the first main theorem. A brief reminder on the basic tilting theory is separated out in Subsection 3.1 whereas the review on Brauer graph algebras and their modules can be found in Subsection 3.2. In Section 4, we will first explain some elementary observations on the two-term tilting complexes of a Brauer graph algebras. This allows us to write down maps between the homological objects (two-term tilting complexes and its direct summands) and the combinatorial objects, which form the statement of our first main result (Theorem 4.6). We will spend the entire Section 5 to prove this result. Section 6 will be devoted to explore the applications of this dictionary between homological algebra and combinatorics. This section is further divided into three subsections - preliminaries on tilting-connectedness and related notions in 6.1, determination of tilting-discrete Brauer graph algebras in 6.2, and multiplicity-independence phenomenon of tilting theory in 6.3.

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Convention

The following assumptions and conventions will be imposed throughout the article.

1. We treat sequences and subsequences as strings (of symbols) and substrings respectively. In particular, a subsequence $s’$ of a sequence $s$ is always assumed to be a consecutive part of $s$, in contrast to its usual meaning in analysis. The reason for not using the term string is to avoid confusion with the string modules of Brauer graph algebras.

2. The composition of maps $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ is $gf : X \rightarrow Z$.

3. All algebras are assumed to be basic, indecomposable, and finite dimensional over an algebraically closed field $k$. We will often use $\Lambda$ to denote such an algebra.

4. We always work with finitely generated right modules, and use $\text{mod} \Lambda$ to denote the category of finitely generated right $\Lambda$-modules.

5. We denote by $\text{proj} \Lambda$ the full subcategory of $\text{mod} \Lambda$ consisting of all finitely generated projective $\Lambda$-modules. The bounded homotopy category of $\text{proj} \Lambda$ is denoted by $\text{K}^b(\text{proj} \Lambda)$.

6. We sometimes write $\Lambda = \mathbb{k}Q/I$, where $Q$ is a finite quiver with relations $I$. 
(7) A simple (respectively, indecomposable projective) $\Lambda$-module corresponding to a vertex $i$ of $Q$ is denoted by $S_i$ (respectively, by $P_i$); an arrow of $Q$ is identified to a map between indecomposable projective $\Lambda$-modules.

(8) For an object $X$ (in mod $\Lambda$ or $K^b(\text{proj} \Lambda)$), we denote by $|X|$ the number of isomorphism classes of the indecomposable direct summands of $X$.

### 2. Ribbon graphs combinatorics

**Definition 2.1.** A ribbon graph is a datum $G = (V, H, s, \cdot, \sigma)$, where

1. $V$ is a finite set, where elements are called **vertices**.
2. $H$ is a finite set, where elements are called **half-edges**.
3. $s : H \rightarrow V$ specifies the vertex $s(e)$ for which a half-edge $e$ emanating from.
4. $\cdot : H \rightarrow H$ is a fixed-point free involution, i.e. $e \neq \cdot(e)$ and $\cdot(\cdot(e)) = e$ for all $e \in H$.
5. $\sigma : H \rightarrow H$ is a permutation on $H$ such that all half-edges emanating from the same vertex lie in the same cycle in the cycle decomposition of $\sigma$, i.e. if $k = |s^{-1}(v)|$, then any $e \in s^{-1}(v)$ forms a $k$-cycle under $\sigma$. This cycle is called **cyclic ordering around** $v$, and is denoted by $(e, \sigma(e), \ldots, \sigma^k(e))_v$ to clarify the vertex involved.

The geometric realisation of $G$ is $(H \times [0, 1]) / \sim$, where $\sim$ is the equivalence relation defined by $(e, \ell) \sim (\cdot(e), 1 - \ell)$ and $(e, 0) \sim (\sigma(e), 0)$. Intuitively, we “glue” $e$ and $\cdot(e)$ together to form a graph-theoretic edge (geometric line segment) incident to $s(e)$ and $s(\cdot(e))$. Because of this, we call an unordered pair $\{e, \cdot(e)\}$ as an edge of $G$, and the vertices (or vertex) $s(e)$ and $s(\cdot(e))$ as the endpoint(s) of the edge $E$. We also define the *valency* of a vertex $v$ as $\text{val}(v) := |s^{-1}(v)|$. Note that a ribbon graph (or its geometric realisation) is also called a locally embedded graph or fatgraph in the literature.

A **Brauer graph** is a ribbon graph equipped with a **multiplicity function** $m : V \rightarrow \mathbb{Z}_{>0}$ which assigns a positive integer, called **multiplicity**, to each vertex of $G$.

We will impose the following assumptions and conventions on ribbon graphs throughout.

- A Brauer/ribbon graph is always assumed to be connected, i.e. its geometric realisation is connected.
- We will always notate half-edges by small alphabets such as $e \in H$, and we will notate the corresponding edge $\{e, \cdot(e)\}$ by the same letter in capital, which is $E$ here.
- Due to the definition of Brauer graph algebras (see Definition 3.3), we usually present $G$ graphically using its geometric realisation with the cyclic ordering of (half-)edges around each vertex presented in the counter-clockwise direction.
- Having said that, whenever we present a local structure such as

  $\vcenter{\hbox{\includegraphics{ribbon_graph.png}}}$

then the two lines ($e_1$ and $e_2$) emanating from $v$ are regarded as half-edges emanating from $v$.

**Example 2.2.** Let $G$ be the ribbon graph:

$$(\{v\}, \{e, \cdot(e), \cdot^2(e)\}, \text{ s } \equiv v, \text{ s } \equiv v, (e, \cdot(e), \cdot^2(e))_v),$$
Then the geometric realisation of $G$ is:

![Diagram](image)

**Definition 2.3** (Half-walk, walk, and signed walk). (a) A non-empty sequence of half-edges $(e_1, \ldots, e_l)$ such that $s(e_{i+1}) = s(e_i)$ for all $i = 1, \ldots, l - 1$ is called a half-walk. Define $\overline{w} = (e_i, \ldots, e_l)$ makes $\tau$ an involution on the set of half-walks.

(b) A walk on a ribbon graph $G$ is the unordered pair $W = \{w, \overline{w}\}$ of half-walks. For exposition convenience, by “a walk $W$ given by the half-walk $w$” we mean that $W = \{w, \overline{w}\}$.

(c) For a half-walk $w = (e_1, \ldots, e_l)$, we define $s(w) := s(e_1)$ and $t(w) := s(\overline{w}) = s(e_l)$ as the endpoint(s) of $W = \{w, \overline{w}\}$. We say a vertex $v$ (resp. half-edge $e$) is in $W$ if $v = s(\overline{w})$ or $v = s(e_i)$ (resp. $e = e_i$ or $e_i$) for some $i \in \{1, \ldots, l\}$.

(d) A signature on a walk $W$ is an assignment of signs $\epsilon_W(e) = \epsilon_W(\overline{e}) \in \{+, -\}$ on half-edges in $W$ such that $\epsilon_W(e_i) \neq \epsilon_W(e_{i+1})$ for all $i \in \{1, \ldots, l - 1\}$. A walk equipped with a signature is called a signed walk. A signed half-walk is a half-walk $w$ equipped with a signature on $W = \{w, \overline{w}\}$. We often notate a signed half-walk by $(w; e)$ or $(\epsilon_1^{e_1}, \epsilon_2^{e_2}, \ldots, \epsilon_l^{e_l})$. Denote by $SW(G)$ the set of signed walks of $G$.

The combinatorial gadget is named in the spirit of the Green’s walk around Brauer tree, which is essentially a combinatorial description of the minimal projective resolution of a simple module of a Brauer tree algebra. We will see that the signed walks will coincide with the minimal projective presentation of certain modules of a Brauer graph algebra.

Another reason we use the name walk in the above definition is that they can be visualised as walks in graph theory, when we regard (the geometric realisation of) $G$ as a graph of undirected edges. However, they are not exactly the same as graph-theoretic walks, as we can see with the following example.

**Example 2.4.** Let $G$ be the ribbon graph in the previous example. There are four signed walks induced by a graph-theoretic walk (with signs) $(v, E^+, v, F^-, v)$. They are given by half-walks $w_1 = (e^+, f^-), w_2 = (e^+, \overline{f}^-), w_3 = (\overline{e}^+, f^-), w_4 = (\overline{e}^+, \overline{f}^-)$.

Note that one can not always find a signature for a walk. For example, the walk $(e, \overline{e})$ in this example cannot have a signature. On the other hand, if one can define a signature on a walk, then there are exactly two choices of signature.

Recall our aforementioned convention on the word subsequence requires it to describe a consecutive part of a sequence; this convention is vital in the following definition.

**Definition 2.5** (Common subwalk). Suppose $w$ is a half-walk. Denote by $z \subset w$ if $z$ is a non-empty subsequence of $w$. We denote by $w \cap w'$ the set of half-walks $z$ such that $z \subset w, z \subset w'$, and there is no $z' \neq z$ with $z \subset z' \subset w$ and $z \subset z' \subset w'$.

A subwalk of $W$ is a walk $Z = \{z, \overline{z}\}$ with $z \subset w$ for some $w \in W$. We denote this situation by $Z \subset W$. A common subwalk of walks $W, W'$ is a subwalk $Z$ of both $W$ and $W'$. It is said to be maximal, if there is no other common subwalk $Z' \neq Z$ of $W$ and $W'$ with $Z \subset Z'$. This is equivalent to having $Z$ given by a $z \in w \cap w'$ for some $w \in W$ and $w' \in W'$. We will denote the set of maximal common subwalks of $W$ and $W'$ by $W \cap W'$.
To ease our burden of explaining the combinatorics for various definitions and proofs, we will attach some extra data for a signed half-walk $w = (e_1, \ldots, e_{\ell}; e_W) \in W$, which are uniquely determined by the signature of $W$.

**Definition 2.6.** A *virtual (half-)*edge is an element in the set $\{v_-((e), v_+(e) \mid e \in H\}$. We can augment $s$ on the virtual edges so that $s(v_+(e)) = s(e)$. Let $v$ be $s(e)$ and suppose $(e_1, \ldots, e_k)_v$ is the cyclic ordering around $v$. We define the cyclic ordering (around $v$) accounting the virtual edges as

$$(v_-(e_1), e_1, v_+(e_1), v_-(e_2), e_2, v_+(e_2), \ldots, v_-(e_k), e_k, v_+(e_k))_v.$$  

Suppose a signed walk $W$ is given by $w = (e_1, \ldots, e_{\ell}; e_W)$. We define the following virtual edges attached to $W$:

- $e_0 = \overline{e_0} := v_-v_W(e_1)$
- $e_{\ell+1} = \overline{e_{\ell+1}} := v_-v_W(e_{\ell})$

We also define $e_W(e_0) = e_W(\overline{e_0}) = -e_W(e_1)$ and $e_W(e_{\ell+1}) = e_W(\overline{e_{\ell+1}}) = -e_W(e_{\ell})$.

From now on, one should always bear in mind the following convention:

A cyclic ordering around an endpoint of a (half-)walk involving $e_0$ and/or $e_{\ell+1}$ is always regarded as the cyclic ordering accounting the virtual edges.

Unless otherwise specified, we fix $W$ and $W'$ as two (not necessarily distinct) signed walks given by half-walks $w = (e_1, \ldots, e_{\ell}; e_W)$ and $w' = (e'_1, \ldots, e'_{\ell}; e_W')$ respectively. Moreover, it is automatically understood what we mean by $e_0, e_{\ell+1}, e'_0, e'_{\ell+1}$, etc. from the definition of virtual edges.

**Definition 2.7 (Sign condition).** We say that a signed walk $W$ (or a signed half-walk $w \in W$) satisfies the sign condition if $e_W(e_1) = e_W(e_m)$ whenever $s(e_1) = s(e_m)$. In general, we say that two walks $W, W'$ satisfy the sign condition when $e_W(e_1) = e_W'(e'_1)$ if $s(e_1) = s(e'_1)$, and $e_W(e_m) = e_W'(e'_m)$ if $s(e_m) = s(e'_m)$.

To improve readability of various definitions, statements, and proofs, we only write down the half-edges required in the cycle of the cyclic ordering around their emanating vertex. In other words, if, for instance, the half-edges $e, f$, and $g$ are the only important half-edges around $v$ in a definition (like the following one), then we will write $(e, f, g)_v$ instead of $(e, \ldots, f, g, \ldots, g)$.

**Definition 2.8 (Non-crossing condition at a maximal common subwalk).** Let $Z$ be a maximal common subwalk of $W$ and $W'$ given by $z = (t_1, t_2, \ldots, t_{\ell}) \in w \cap w'$ with endpoints $u = s(t_1)$ and $v = s(t_{\ell})$, so that $t_k = e_{i+k-1} = e'_{j+k-1}$ for all $k \in \{1, \ldots, \ell\}$. We say that $W$ and $W'$ are non-crossing at $Z$ if the following holds:

1. $e_W(t_k) = e_W'(t_k)$ for each $k \in \{1, \ldots, \ell\}$.
2. With the exception of $i = j = 1$ and/or $m + 1 - i - \ell = n + 1 - j - \ell = 0$ (i.e. $u$ and/or $v$ being the endpoint of both walks), the cyclic orderings around $u$ and $v$ being either $(t_1, e_{i-1}, e'_{j-1})_u$ and $(u, e_{i-1}, e'_{j-1})_v$ respectively,
   or $(t_1, e_{i-1}, e'_{j-1})_u$ and $(u, e_{i-1}, e'_{j-1})_v$ respectively,

Visualising the two cases locally:
In the future, we will refer to this pair of cyclic orderings as the (pair of) neighbourhood cyclic orderings.

Suppose there is a vertex \( v = s(e_i) = s(e'_j) \) for some \( i \in \{1, \ldots, m+1 \} \) and \( j \in \{1, \ldots, n+1 \} \). Let \( a, b, c, d \) be half-edges emanating from \( v \) given by:

\[
\{a, b\} := \{e_{i-1}, e_i\}, \quad \{c, d\} := \{e_{j-1}', e'_j\}.
\]

We say that \( v \) is an intersecting vertex of \( W \) and \( W' \) if \( a, b, c, d \) are pairwise distinct. Let \( W \cap W' \) be the set of all intersecting vertices of \( W \) and \( W' \). We call the above two sets of half-edges as the neighbourhood of \( v \) in \( W \) and \( W' \) respectively. Note that an intersecting vertex can have multiple different neighbourhoods in \( W \) (or \( W' \)).

**Definition 2.9** (Non-crossing condition at an intersecting vertex and admissible walks). We say that \( W \) and \( W' \) is non-crossing at the intersecting vertex \( v \in W \cap W' \) if the following condition is satisfied.

\( (NC3) \) If \( \{a, b\} \) and \( \{c, d\} \) are neighbourhoods of \( v \) in \( W \) and \( W' \) respectively, and at most one of \( a, b, c, d \) is virtual, then the cyclic ordering around \( v \) and the signature is

\[
\text{either } (a^+, b^-, c^+, d^-)_v, \quad \text{or } (a^+, b^-, c^-, d^+)_v.
\]

The local structure around \( v \) for these two conditions can be visualised as

\[
\begin{array}{c}
| & a^+ & \text{and} & b^- & \text{v} & c^- & d^+ \\
\text{b}^- & \text{v} & a^+ & \text{v} & c^- & \text{v} & d^+
\end{array}
\]

respectively.

We say that two walks \( W, W' \) are non-crossing, or they satisfy the non-crossing conditions, if they are non-crossing at all maximal common subwalks and all intersecting vertices.

If in addition \( W = W' \), we may specify that \( W \) is self-non-crossing. An admissible walk is a self-non-crossing (signed) walk which also satisfies the sign condition. Denote by \( AW(G) \) the subset of \( SW(G) \) consisting of all admissible walks.

**Remark 2.10.** While it is easy to see why \( (NC2) \) is called non-crossing, it is not apparent that why \( (NC1) \) is a non-crossing condition but not a sign condition, and why such signatures on half-edges around \( v \) are required in \( (NC3) \) to make them non-crossing. Although we will not use this representation of signed walks in this article, the correct graphical (geometrical) realisation is as follows. When we go along the signed (half-)walk, say \( (e_i, e_{i+1}) \subset w \) with \( i \neq 0 \) and \( i \neq m+1 \), instead of visualising the situation as a line passing through a vertex:

\[
\overline{v e_{i+1}}
\]

we think of the vertex \( v = s(e_{i+1}) \) as lying below (resp. above) the line (relative to this presentation) if \( \epsilon_W(e_{i+1}) = - \) (resp. \( \epsilon_W(e_{i+1}) = + \)).

This “correct” visualisation is in fact a generalisation of the technique used in [KS, ST] - from a disc to compact orientable Riemann surfaces with marked point and boundaries (specified by \( G \)). In this geometric setting, our signed walks become certain type of curves, and the non-crossing condition translates into requiring a curve to have no self-intersection. We hope to explain this geometric setting with more details in a sequel paper.

Checking non-crossing condition is in practice much easier than the way it is defined here - draw the walk around the geometric realisation of \( G \) in a non-crossing way and put signs on the edges to check \( (NC3) \). We briefly explain how to check non-crossing-ness algorithmically here,
and recommend the reader to carefully go through the proof of the next proposition in order to familiarise the procedure.

First, fix some \( w \in W \) and \( w' \in W' \). Start with a vertex, say \( v = s(e_i) = s(e'_j) \), which is in both \( w \) and \( w' \). In the special case of \( W = W' \), one can simply start with a vertex which appears at least twice in (any) \( w \in W \). If the half-edges \( e_{i-1}, e_i, e'_{j-1}, e'_j \) are pairwise distinct, then \( v \in W \cap W' \), and one can verify if the signed cyclic ordering required by (NC3) is satisfied simultaneously. Otherwise, we have a common subwalk given by the coinciding half-edge, and one can expand this half-edge to some \( z \in w \cap w' \) or \( z \in w \cap w' \). Iterate this procedure for all possible pair of \( i, j \) with \( s(e_i) = s(e'_j) \).

We give an example on verifying the sign condition and the non-crossing condition:

**Example 2.11.** Let \( G \) be a ribbon graph whose geometric realisation is given on the left-hand side of the following picture. For readability, we label the half-edges with numeral instead of alphabets and place the labelling next to their respective emanating vertex.

\[
\begin{array}{c}
\begin{tikzpicture}
\node (v1) at (0,0) [circle,draw] {1};
\node (v2) at (1,1) [circle,draw] {2};
\node (v3) at (1,-1) [circle,draw] {3};
\node (v4) at (2,0) [circle,draw] {4};
\node (v5) at (-1,1) [circle,draw] {5};
\node (v6) at (-1,-1) [circle,draw] {6};
\draw (v1) -- (v2);
\draw (v2) -- (v3);
\draw (v3) -- (v4);
\draw (v4) -- (v5);
\draw (v5) -- (v6);
\draw (v6) -- (v1);
\end{tikzpicture}
\end{array}
\]

Define \( w_1 \) as the following signed half-walk.

\[
w_1 = (1^+, 2^-, 3^+, 6^-, 5^+, 4^-, 3^+, 2^-, 1^+, 4^-).
\]

Applying the involution on them we get:

\[
\overline{w}_1 = (4^-, 1^+, 2^+, 3^+, 4^-, 5^+, 6^-, 3^+, 2^-, 1^+).
\]

The walk \( W_1 \) is shown on the right-hand side of the above picture. We represent the virtual edges as dashed lines to indicate their relative position with respect to the (augmented) cyclic ordering around \( u \) and \( v \). From this picture, one then expect \( W_1 \) is self-crossing. We show how this can be checked properly.

We have the following sets:

\[
w_1 \cap w_1 = \{w_1\}, \quad w_1 \cap \overline{w}_1 = \{z_1 = (1, 2, 3), \overline{z}_1, z_2 = (4), \overline{z}_2\},
\]

which gives us the following sets of maximal common subwalks

\[
W_1 \cap W_1 = \{W_1, Z_1 = \{z_1, \overline{z}_1\}, Z_2 = \{z_2, \overline{z}_2\}\}.
\]

It is easy to see that (NC1) is satisfied in all the cases. In the above picture, \( Z_1 \) is represented by the overlapping upper triangle, whereas \( Z_2 \) is represented by the overlapping edge in the lower triangle. Focusing on the adjacent half-edges around these overlapping parts, one can see that (NC2) does not hold at both \( Z_1 \) and \( Z_2 \).

We will be more precise here. The cyclic orderings of the half-edges emanating from \( s(z_1) \) and \( s(\overline{z}_1) \) in the walk \( z_1 \) are

\[
(1, 4, \text{vr}_-(1))_{v = s(z_1)} \quad \text{and} \quad (3, 4, 6)_{v = s(\overline{z}_1)} \quad \text{respectively.}
\]

For \( z_2 \), we have instead

\[
(4, 1, \overline{3})_{v = s(z_2)} \quad \text{and} \quad (\overline{4}, \text{vr}_+(\overline{4}), 5)_{u = s(\overline{z}_2)}.
\]

Therefore, (NC2) does not hold in both cases - as we have claimed.
From the picture above, it is easy to see that \( W_1 \cap W_1 = \{ v \} \), and we left it for the reader to confirm the claim. There are four neighbourhoods of \( v \):

\[ \{ v_r(1), 1 \}, \{ 1, 4 \}, \{ 3, 4 \}, \text{ and } \{ 3, 6 \} \]

Then, to check (NC3), we only need to observe the three cases \{ v_r(1), 1, 3, 5 \}, \{ v_r(1), 1, 3, 4 \}, and \{ 1, 4, 3, 6 \}. These cyclic orderings with signatures are given by

\[ (3^+, 6^-, v_r(1)^-, 1^+)_v, (3^+, 4^-, v_r(1)^-, 1^+)_v, \text{ and } (1^+, 3^+, 4^-, 6^-)_v \]

Thus, while the first two cases comply with (NC3), the third one does not.

**Proposition 2.12.** Let \( G \) be a ribbon graph. Then the following are equivalent.

1. \( AW(G) \) is a finite set.
2. \( G \) consists of at most one odd cycle and no even cycle.

**Remark 2.13.** A cycle of length \( n \) in \( G \) is an \( n \)-gon embedded in \( G \), i.e. there is a walk \((e_1, \ldots, e_k)\) in \( G \) with endpoints being the same, and no repeating vertices along the walk. An odd cycle (resp. even cycle) is a cycle of odd (resp. even) length. Note that a 1-gon (cycle of length 1) is a loop. We denote a cycle in \( G \) by a sequence \( C = (v, E_1, v_1, E_2, \ldots, v_{k-1}, E_k, v) \), so that \( s(e_1) = v = s(E_k) \), \( s(e_i) = v_{i-1} = s(E_{i+1}) \).

For ease of exposition, regardless of whether the entries of a sequence are half-walk or half-edges, or a mixture of both, the sequence is understood as the half-walk given by concatenating all the data in the obvious way.

**Proof.** (2)\( \Rightarrow \) (1): Suppose \( G \) is a graph containing at most one odd cycle and no even cycle. Since sign alternates as we go along a signed walk, the same edge never appears more than once. In particular, the length \( \ell \) of the sequence defining a signed walk is less than or equal to the number of edges in \( G \). Moreover, for a given \( \ell \), there is only finitely many sequences (of half-edges) of length \( \ell \), so \( SW(G) \) is finite. Hence, the subset \( AW(G) \) of \( SW(G) \) is also finite.

(1)\( \Rightarrow \) (2): Assume that \( G \) has an even cycle \( C = (v, E_1, v_1, E_2, \ldots, v_{k-1}, E_k, v) \), which has the following geometric realisation in \( G \):

\[
\begin{array}{c|cc}
& E_k & v_{k-1} \\
\hline
v & & \\
E_1 & & \\
v_1 & \cdots & v_{k-2}
\end{array}
\]

Note that none of the edges is a loop. We label the half-edges \( e_i \) so that \( s(e_i) = v_{i-1} \). For better readability, we put all the labellings of half-walks and walks in bold face.

Let \( s \) be the signed half-walk \((e_1^+, e_2^-, \ldots, e_k^-)\). For a positive integer \( p \), we define a signed half-walk \( w_p \) and show that its corresponding signed walk \( W_p \) is admissible. For clarity, we will note some of the vertices (as well as indexing them) along the walk.

Let \( w_p := (v(0), s, v(1), s, \ldots, s, v(p), e_1^+, v_1) \) with \( v(i) = v \) for each \( i \in \{ 0, 1, \ldots, p \} \), i.e. concatenation of \( p \) copies \( s \) and a copy of \( e_1 \). By construction, \( W_p \cap W_p \) is empty. The set \( w_p \cap w_p \) consists of \( w_p \) and the following half-walks.

\[
\begin{align*}
z_1 : & \quad (v(0), e_1^+, v_1) = (v(p), e_1^+, v_1) \\
z_2 : & \quad (v(0), s, v(1), e_1^+, v_1) = (v(p-1), s, v(p), e_1^+, v_1) \\
z_3 : & \quad (v(0), s, v(1), s, v(2), e_1^+, v_1) = (v(p-2), s, v(p-1), s, v(p), e_1^+, v_1) \\
& \quad \vdots \\
z_p : & \quad (v(0), s, v(1), s, \ldots, s, v(p-1), e_1^+, v_1) = (v(1), s, v(2), s, \ldots, s, v(p), e_1^+, v_1)
\end{align*}
\]

Since the set \( w_p \cap w_p \) is empty, \( W_p \cap W_p = \{ Z_1, \ldots, Z_p, W_p \} \).

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Now we can see that the sign condition and the non-crossing conditions are satisfied at \( Z_i \) for all \( i \in \{1, \ldots, p\} \). Hence, \( W_p \) is admissible for all \( p \geq 1 \). In particular, \( AW(G) \) is an infinite set.

Suppose now that \( G \) has two odd cycles \( C = (u, E_1, u_1, E_2, \ldots, u_{m-1}, E_m, u) \) and \( C' = (v, F_1, v_1, F_2, \ldots, v_{n-1}, F_n, v) \). First note that we can assume that vertices in \( \{u_i, v_j \mid i = 1, \ldots, m - 1; j = 1, \ldots, n - 1\} \) are pairwise distinct. Otherwise, there will be an even cycle subgraph, which reverts us to the previous case, or there is/are odd cycle(s) of short length(s) which satisfies the assumption on vertices being pairwise distinction. Therefore, we have the following two possible geometric realisations:

\[
\text{or}
\]

where \( D_1, \ldots, D_l \) are non-loop edges connecting \( C \) and \( C' \). Note that \( l \) can be zero, i.e. \( u = v \).

We label the half-edges as follows. In the first case, we require the cyclic ordering around \( v \) is given by \((e_1, f_n, e_m, f_1)\), where the second case, we require cyclic orderings \((e_1, e_1^+, f_1, f_n)\) if \( l > 0 \), or \((e_1, e_m, f_1, f_n)\) if \( l = 0 \). The remaining labellings are then fixed (uniquely) in the way that the following signed half-walks in \( G \) can be defined:

\[
d := (d_1^-, d_2^+, \ldots, d_{l-1}^-, d_l^+)
\]

\[
e := (e_1^+, e_2^-, \ldots, e_{m-1}^-, e_m^+)
\]

\[
f := (f_1^-, f_2^+, \ldots, f_{n-1}^-, f_n^+)
\]

where \( \epsilon = + \) in the first case, and \( \epsilon \) is the sign of \((-1)^l\) in the second. Here \( e \) and \( f \) are half-walks given by walking along respective cycles.

Let us concentrate in the first case for now. For any positive integer \( p \), define the following signed half-walk:

\[
w_p := (u^{(0)}, e, u^{(1)}, f, u^{(2)}, e, u^{(3)}, f, u^{(4)}, \ldots, e, u^{(2p-1)}, f, u^{(2p)}, e, u^{(2p+1)}),
\]

where \( u^{(i)} = v \) for all \( i \in \{0, 1, \ldots, 2p + 1\} \).

First note that a vertex not equal to \( v \) in \( C \) must be distinct from a vertex not equal to \( v \) in \( C' \), so we only need to consider \( v \) to determine for self-intersecting vertex of \( W_p \). There are only four possible neighbourhoods of \( v \) in \( W_p \), given by \( v \) at \( v = u^{(x)} \) with \( x \in \{0, 1, 2, 2p + 1\} \). This means that we can see when does \( v \) appear as intersecting vertex by considering \( u^{(a)} = u^{(b)} \) with \( a, b \in I \) and \( a < b \).

Now one can simply draw out the neighbourhoods (like in the definition of (NC3)) to see that \( W_p \) intersects itself at \( v = u^{(a)} = u^{(b)} \) if and only if \( b - a \) is odd. One can simultaneously check that the signed cyclic ordering in (NC3) holds for all \( (a, b) \) apart from \( (0, 2p + 1) \). Since the case \( (a, b) = (0, 2p + 1) \) gives us the comparison of neighbourhood of endpoints of \( W_p \), we need only to check sign condition but not the signed cyclic ordering from (NC3).

To determine maximal common subwalks, (a generalisation of) our previous observation says that we only need to consider \( a, b \in \{0, 1, \ldots, 2p + 1\} \) with \( a < b \) and \( b - a \) even. Set \( s := (e, f) \) so that we can write \( w_p \) in the form \((v^{(0)}, s, v^{(1)}, s, \ldots, s, v^{(p)}, e, v^{(p+1)})\), where \( v^{(i)} = u^{(2i)} \) for all \( i \in \{0, 1, \ldots, p\} \), and \( v^{(p+1)} = u^{(2p+1)} \). Then with the new labellings, the maximal (self-)common subwalks obtained by expanding \( u^{(a)} = v^{(b)} \) are exactly \( z_1, \ldots, z_p, w_p \) after replacing \( e_1 \) by \( e \), and \( v_1 \) by \( v^{(p+1)} \) in the proof for the even cycle case. These are all the maximal common subwalk. So \( W_p \) is admissible, which implies the infiniteness of \( AW(G) \) for the first double odd cycle case.
Finally, we look at the last case. We define $s$ as the signed half-walk $(d, f, \bar{d})$. For every odd number $p \geq 5$, we define the following signed half-walk $w_p$, and show that it is admissible.

$$w_p := (u^{(0)}, e, u^{(1)}, s, u^{(2)}, e, u^{(3)}, s, \ldots, s, u^{(p-1)}, \bar{e}, u^{(p)}),$$

where $u^{(i)} = u$ for any $i \in \{0, 1, 2, \ldots, p\}$.

Observe carefully that if one starts with a common vertex not equal to $u$, then the maximal common subwalk containing such vertex also contains $u$. To be more precise, we can start by picking $u^{(r)} = u^{(s)}$ for some $r, s \in \{0, 1, \ldots, p\}$.

Using the visualisation below, we can classify the possible cases into the following list:

1. $1 \leq r < p$ is odd and $2 \leq s < p - 1$ is even.
2. $r = p - 1$ and $1 \leq s < p$ is odd.
3. $r = p$ and $0 \leq s < p - 1$ is even.
4. $1 \leq r < s < p$ are odd.
5. $0 \leq r < s \leq p - 1$ are even.
6. $1 \leq r < p$ is odd and $s \in \{0, p\}$.
7. $r = p - 1$ and $s \in \{0, p\}$.

The last two cases give us all the intersecting vertices. For the other cases, the resulting maximal common walk containing $w_{(r)}$ and $u^{(s)}$ is given by:

Case (1): $(u^{(r)} = u^{(s)}, d, v) \in w_p \cap \bar{w}_p$.

Case (2), (3): $(v, d, u^{(p-1)}, \bar{e}, u^{(p)}) \in w_p \cap \bar{w}_p$.

Case (4), (5):

$$(u^{(0)}, e, u^{(1)}, s, \ldots, s, u^{(p-1)-(s-r)}) = (u^{(s-r)}, e, u^{(s-r+1)}, s, \ldots, s, u^{(p-1)}) \in w_p \cap \bar{w}_p.$$

It is easy to see via the visualisation above that the non-crossing conditions are satisfied at each of these cases. The sign condition is easy to check. Therefore, $\mathcal{AW}(\mathbb{G})$ is an infinite set. \hfill \square

3. Homological and Algebraic Preliminaries

3.1. Tilting theory. We will work in the bounded homotopy category $K^b(\text{proj} \Lambda)$ in this subsection. Without loss of generality, each indecomposable complex takes the form $T = (T_i, d_i)_{i \in \mathbb{Z}}$ with $d_i$ lies in the Jacobson radical of $\text{proj} \Lambda$ for all $i$.

**Definition 3.1.** Let $T$ be a complex in $K^b(\text{proj} \Lambda)$.

1. We say that $T$ is **pretilling** if it satisfies $\text{Hom}_{K^b(\text{proj} \Lambda)}(T, T[n]) = 0$ for all non-zero integers $n$.
2. We say that $T$ is **tilting** if it is pretilling and generates $K^b(\text{proj} \Lambda)$ by taking direct sums, direct summands, mapping cones and shifts.
3. A pretilling complex is said to be **partial** if it is a direct summand of some tilting complex. We denote by $\text{tilt} \Lambda$ the set of isomorphism classes of basic tilting complexes of $\Lambda$.

For a complex $T$ in $K^b(\text{proj} \Lambda)$, the $n$-th term of $T$ is denoted by $T^n$. A complex $T$ in $K^b(\text{proj} \Lambda)$ is called

- **stalk** if it is of the form $(0 \rightarrow T^n \rightarrow 0)$ for some $n \in \mathbb{Z}$;
- **$n$-term** if it is of the form $(0 \rightarrow T^{-n+1} \rightarrow \ldots \rightarrow T^{-1} \rightarrow T^0 \rightarrow 0)$ for a non-negative integer $n$. Note that this is different from simply requiring $T$ to concentrate on $n$ consecutive degrees.

We denote by $2ipt \Lambda$ the set of isomorphism classes of indecomposable two-term pretilling complexes of $\Lambda$. The subset of $\text{tilt} \Lambda$ consisting of two-term complexes is denoted by $2$-$\text{tilt} \Lambda$. 

Proposition 3.2. Let $\Lambda$ be a symmetric algebra and $T$ a two-term pretilting complex of $\Lambda$. Then the following hold:

(i) $T$ satisfies $\text{add } T^0 \cap \text{add } T^{-1} = 0$.
(ii) $T$ is partial tilting. In fact, it is a direct summand of a two-term tilting complex.
(iii) It is (two-term) tilting if and only if $|T| = |\Lambda|$.

3.2. Brauer graph algebras and their modules. For convenience, we say that the half-edge $e$ is truncated if $\sigma(e) = e$ and $m(s(e)) = 1$.

Definition 3.3. Let $G$ be a Brauer graph whose multiplicity function is denoted by $m$.

If $G$ is the connected graph (tree) with one edge $E = \{e, \overline{e}\}$ and two vertices (endpoints of $E$), i.e., $\circ - \circ$, with multiplicity $m \equiv 1$, then we define the Brauer tree algebra of $G$ as $k[\alpha_e]/(\alpha_2^2)$. Otherwise, we define the Brauer graph algebra $\Lambda_G$ of $G$ by giving its quiver and relations as follows. Define finite quiver $Q_G$ as follows.

- The set of vertices of $Q_G$ is in bijection with the set of edges in $G$. For an edge $E = \{e, \overline{e}\}$ in $G$, we denote the trivial path corresponding to $E$ by $[e]^0$ and also by $[\overline{e}]^0$. The reason for such notation will become clear soon.
- If $e$ is not truncated, then there is an arrow from $E' := \{\sigma(e), \overline{\sigma(e)}\}$ to $E = \{e, \overline{e}\}$, denoted by $\alpha_{e, \sigma(e)}$.

In general, suppose $e' := \sigma^k(e)$ for some $k \leq \text{val}(s(e))$ such that $e$ is not truncated, we define $\alpha_{e, e'}$ to be the path $\alpha_{e, \sigma(e)}\alpha_{\overline{\sigma(e)}, \sigma^2(e)} \cdots \alpha_{\overline{\sigma^k(e)}, e'}$. We call this path short if $k \leq \text{val}(s(e))$. Otherwise, the path $\alpha_{e, e'}$ will be called a Brauer cycle. For better readability, sometimes we write $(e', e)$ instead of $\alpha_{e', e}$, and $[e]$ instead of $\alpha_{e, e}$.

The relations of $\Lambda_G$ are generated by the following three types of Brauer relations.

- If both $e$ and $\overline{e}$ are not truncated, then $\alpha_{e, e} m(s(e)) - \alpha_{\overline{e}, \overline{e}} m(s(\overline{e})) = 0$.
- If $\overline{e}$ is truncated, then $\alpha_{\overline{e}, e} m(s(e)) \alpha_{e, \sigma(e)} = 0$.
- $\alpha_{e, \sigma^{-1}(e), e} \alpha_{\overline{e}, \sigma(e)} = 0$ for any $e$ with the indicated arrows in $Q_G$.

Remark 3.4. (1) For flexibility, if $E = \{e, \overline{e}\}$ is an edge of $G$, then the notations $P_E, P_e, P_{\overline{e}}$ all means the same indecomposable projective module - the one corresponding to $E$.

(2) Recall our convention of taking right modules and identifying maps with arrows of Ext-quiver. It says that a short path $\alpha_{e', e}$ can be regarded as a map $P_e \rightarrow P_{e'}$ given by multiplication of $\alpha_{e', e}$ on the left. Consequently, we call such a map as short map.

It is well-known that every Brauer graph algebra is a symmetric special biserial algebra and vice versa. In particular, an indecomposable non-projective module of a Brauer graph algebras falls into either one of the two sub-classes, called string modules and band modules; see, for example, [Erd, WW].

An indecomposable module $M$ (of a biserial algebra) is a band module if, and only if, it is $\tau$-invariant. Here $\tau$ is the Auslander-Reiten translation, and well-known to be isomorphic to the second syzygy functor $\Omega^2$ for symmetric algebras. For reason to be clear later (see Lemma 5.1), we will not give any more details about band modules, and concentrate only on the string modules.

We avoid introducing the string combinatorics for biserial algebras here, and instead use the following equivalent definition.

Definition 3.5. An indecomposable non-projective $\Lambda_G$-module $M$ is a string module if, and only if, its minimal projective presentation $P_M$ can be written in one of the following forms:
with each $d_{i,j}$ given by a left-multiplication of a path from $F_j$ to $E_i$.

**Remark 3.6.** The definition is presented with a “chosen direction” to keep it short. In practice (as well as in our forthcoming proofs), one can reorder $E_i$’s and $F_j$’s “upside-down”, i.e. swap $E_i$ with $E_{m-i+1}$ and swap $F_j$ with $F_{n-j+1}$. Under such reordering, the diagram of (2) is reflected along a horizontal in the middle.

The definition can also be rephrased as saying that $P_M = (P_M^{-1} \xrightarrow{d_M} P_M^0)$ can be written so that

- $P_M^0 = \bigoplus_{i=1}^m P_{E_i}$ and $P_M^{-1} = \bigoplus_{j=1}^n P_{F_j}$, with $m \in \{n-1, n, n+1\}$;
- $d_M = (d_{i,j})_{i,j}$ is an upper-triangular or lower-triangular matrix of maps (possibly after a reordering of $E_i$’s and $F_j$’s) such that $d_{i,j} = 0$ if and only if $|i - j| > 2$;
- every non-zero $d_{i,j}$ is given by left-multiplying a (non-zero) non-trivial path from $F_j$ to $E_i$.

Note that by appropriately choosing the representing half-edges $e_i$’s and $f_j$’s for $E_i$’s and $F_j$’s, $d_M = (d_{i,j})$ can be written in the following forms respectively to diagram (1), (2), (3):

**Diagram (1):**

\[
\begin{pmatrix}
[e_1]^{k_1}(e_1, \overline{f_1}) & [e_1]^{k_1}(e_1, f_2) \\
[e_2]^{k_2}(e_2, \overline{f_2}) & \cdots \\
& \cdots \\
& \cdots \\
& \cdots \\
&e_m]^{k_m}(e_m, \overline{f_{m-1}}) & [e_m]^{k'_m}(e_m, f_n)
\end{pmatrix}
\]

**Diagram (2):**

\[
\begin{pmatrix}
[e_1]^{k_1}(e_1, \overline{f_1}) & [e_1]^{k'_1}(e_1, f_2) \\
[e_2]^{k_2}(e_2, \overline{f_2}) & \cdots \\
& \cdots \\
& \cdots \\
& \cdots \\
& \cdots \\
& \cdots \\
& \cdots \\
&e_{m-1}]^{k_{m-1}}(e_{m-1}, \overline{f_{m-1}}) & [e_{m-1}]^{k'_{m-1}}(e_{m-1}, f_n) \\
&e_m]^{k_m}(e_m, \overline{f_n}) & [e_m]^{k'_m}(e_m, f_n)
\end{pmatrix}
\]

**Diagram (3):**

\[
\begin{pmatrix}
[e_1]^{k'_1}(e_1, f_1) & [e_2]^{k_2}(e_2, \overline{f_1}) \\
[e_2]^{k_2}(e_2, f_2) & \cdots \\
& \cdots \\
& \cdots \\
& \cdots \\
& \cdots \\
& \cdots \\
& \cdots \\
&e_{m-1}]^{k'_{m-1}}(e_{m-1}, f_n) \\
&e_m]^{k_m}(e_m, \overline{f_n})
\end{pmatrix}
\]

Here $k_i$ and $k'_j$ are non-negative integers strictly less than the associating Brauer cycles’ multiplicities, for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.

The choice for the representing half-edges $e_i$’s and $f_j$’s is unique unless one reorders the indecomposable projective modules upside-down. In which case, the representing half-edges
become $\overline{e_i}$ and $\overline{f_j}$. Also, note that when reflecting in case (2), the upper-triangular matrix will be turned into a lower-triangular matrix.

We will always assume a minimal projective presentation of a string module is of the form described above. If a complex of projective $\Lambda_G$-modules is (after possibly reordering the projective components) in one of the three forms above, then it is called a two-term string complex.

This is the analogue of complexes which are called homotopy string used in, for example, [BM] for studying indecomposable objects in the derived category of a gentle algebra. Since we will only study two-term complexes in this paper, we will drop the adjective “two-term” for string complexes.

A particular example of string module is a hook module:

**Definition 3.7.** A hook (module) is a (string) module of the form $H_e := [e]^0 / \alpha e, e \sigma(e) \Lambda$, whose minimal projective presentation is given by $P_\sigma(\tau) \xrightarrow{(\sigma, \tau(e))} P_e$. We also call the module $H_e$ as the hook module with top $e$.

**Remark 3.8.** Let $(e_1, e_2, \ldots, e_k)_v$ be the cyclic ordering around the vertex $v$ in the Brauer graph $\mathcal{G}$. For each $i \in \{1, \ldots, k\}$, there is a hook module $H_{e_i}$ with basis

$$\{[e_i^l], [e_i^l] (e_i, e_{i+1}), \ldots, [e_i^l] (e_i, e_{i-1}) \mid l = 0, \ldots, m(v) - 1\}.$$  

The composition factor multiplicity $[H_{e_i} : S_{E}]$ is $m(v)$ if $E_j$ is not a loop, or $2m(v)$ otherwise. Note that if $E = \{e, \overline{e}\}$, then $H_e \neq H_{\overline{e}}$ unless $\mathcal{G}$ has only a single edge. In particular, our hook module is slightly different from the traditional definition [Erd]. Also, if $e$ is a truncated half-edge, then $H_e$ is just the simple module $S_e = S_{E}$.

### 4. Two-term (pre)tilting complexes via ribbon combinatorics

Let $\mathcal{G}$ be a Brauer graph and $\Lambda := \Lambda_G$ be the associating Brauer graph algebra. In this section, we study the relationship between two-term tilting complexes and walks.

Let $T$ be a complex in $K^b(\text{proj} \Lambda)$. If the homology of $T$ is non-trivial at two consecutive degrees then $T$ is isomorphic to some shift of a two-term complex given by the minimal projective presentation of $T^{n+1}(T)$; and also dually, some shift of the minimal injective (co)presentation of $T^n(T)$. In particular, we will assume that every two-term non-stalk complex in $K^b(\text{proj} \Lambda)$ takes such a form.

**Definition 4.1.** Suppose $T := (T^{-1} \xrightarrow{d} T^0)$ is a two-term complex in $K^b(\text{proj} \Lambda)$ such that $T = P_M$ for a string module $M$, where $d$ is written in one of the forms shown in Definition 3.5. Then we call $d$ (resp. $T$, resp. $M$) a short string map (resp. complex, resp. module). We denote by $2\text{scx} \Lambda$ the set of indecomposable stalk complexes of projective modules concentrated in degree 0 or −1, and two-term short string complexes $T := (T^{-1} \xrightarrow{d} T^0)$ which satisfies $\text{add} T^0 \cap \text{add} T^{-1} = 0$.

For a signed half-walk $w = (e_1, \ldots, e_n; e)$, define $T^0 := \bigoplus_{\epsilon(e_a) = +} P_{e_a}$ and $T^{-1} := \bigoplus_{\epsilon(e_b) = -} P_{e_b}$.

By the definition of signed half-walks, two consecutive half-edges in $w$ have different signs, and so we can define a map $T^{-1} \xrightarrow{d} T^0$ with entries $d_{ab} : P_{e_b} \to P_{e_a}$ given by

$$d_{ab} := \begin{cases} \alpha_{e_a, e_b} & \text{if } b = a - 1; \\ \alpha_{e_b, e_a} & \text{if } b = a + 1; \\ 0 & \text{otherwise.} \end{cases}$$

The following properties are almost immediate from the construction.

**Lemma 4.2.** (1) $T_w = T_{\overline{w}}$ holds. In particular, for a signed walk $W = \{w, \overline{w}\}$, we can define $T_W := T_w = T_{\overline{w}}$. 
(2) $T_W$ is in $2\text{scx} \Lambda$.

Proof. (1) This is clear by construction.

(2) This is clear for $w = (e)$ with $e$ a half-edge. Otherwise, we have short maps $d_{ab}$, so $T^{-1} \xrightarrow{d} T^0$ is a minimal projective presentation of the (short string) module $H^0(T)$. Also, reappearing edges in a signed walk have the same sign, it follows that $\text{add} T^{-1} \cap \text{add} T^0 = 0$, and so is (2).

Lemma 4.3. The map $\text{SW}(G) \to 2\text{scx}(\Lambda)$ given by $W \mapsto T_W$ is bijective.

Proof. We prove this by finding the inverse of the map. We define first a map from $2\text{scx}(\Lambda) \to \text{SW}(G)$. Note that it is sufficient to define for a given $T$ a half-walk $w_T$ along with signatures on the half-edges in $w_T$, as we then have $T \mapsto W_T = \{w_T, \overline{w_T}\}$.

If $T$ is a stalk complex $P_E[s]$ with $s \in \{0, 1\}$, then take $w_T = (e)$ with signature being the sign of $(-1)^s$. If $T$ is not a stalk complex, then it is a minimal projective presentation of a short string module $M = H^0(T)$. Using the notations in the matrices below Definition 3.5, we obtain a half-walk of the form $(e_1, f_1, e_2, \ldots)$ (or $(f_1, e_1, f_2, \ldots)$). As discussed therein, this walk is uniquely defined up to applying $\tau$. Since $\text{add} T^0 \cap \text{add} T^{-1}$ and $P_e = P_f = P_E$, one can define a signature $\epsilon$ by $\epsilon(e_i) = +$ and $\epsilon(f_j) = -$.

Observe that, as a short map $\alpha_{e,T}$ corresponds uniquely to an ordered pair $(e, \overline{f})$ of half-edges incident to the same vertex, the assignment above gives a well-defined injective map. Lemma 4.2 (2) implies that $W \mapsto T_W$ is the inverse of this map. □

The following says that the bijection given above can be improved to give a combinatorial model describing the indecomposable two-term pretilting complexes.

Lemma 4.4. Every non-stalk indecomposable two-term pretilting complex $T := (T^{-1} \xrightarrow{d} T^0)$ is a short string complex. In particular, $2\text{ipt} \Lambda$ is a subset of $2\text{scx} \Lambda$.

Proof. Suppose $T^{-1} = \bigoplus_{j=1}^n P_{F_j}$ and $T^0 = \bigoplus_{i=1}^m P_{E_i}$. Since $T$ is pretilting, it follows that $\text{add} T^0 \cap \text{add} T^{-1} = 0$ by Proposition 3.2. In particular, we have $E_i \neq F_j$ for all $i, j$.

Suppose on the contrary that $d_{a,b} : P_{F_a} \to P_{E_b}$ is not short. As $E_a \neq F_b$, we have a short map $(e_a, \overline{f_b})$. Let $\alpha : T \to T[1]$ be a map of complexes given by mapping $P_{F_a}$ in $T^{-1}$ to $P_{E_a}$ in $T^0 = (T[1])^{-1}$ via $(e_a, \overline{f_b})$. Non-shortness of $d_{a,b}$ means that we can write it as $[e_a]^k(e_a, \overline{f_b})$ for some $k \geq 1$. It is easy to see that $\alpha$ is not null-homotopy, or one will have $h \cdot [e_a]^k(e_a, \overline{f_b}) = (e_a, \overline{f_b})$ for some $h \in \text{Hom}(P_{E_a}, P_{E_a}) \cong [e_a]^0\Lambda G[e_a]^0$, which is not possible. We now have a non-zero map $\alpha \in \text{Hom}_{K^b(\text{proj}\Lambda G)}(T, T[1])$, contradicting the pretilting-ness of $T$. □

We define one more notion before stating our main result.

Definition 4.5. A collection $\mathcal{W}$ of signed walks is admissible if for any pair of (not necessarily distinct) walks $W$ and $W'$ in $\mathcal{W}$, they are non-crossing and satisfy the sign condition. An admissible collection is called complete if any admissible collection containing $\mathcal{W}$ is $\mathcal{W}$ itself. Denote by $\text{CW}(G)$ the set of all complete admissible collections of (admissible) signed walks.

Theorem 4.6. The correspondence in Lemma 4.3 induces the following bijections:

$$2\text{ipt} \Lambda \leftrightarrow \text{AW}(G) \quad \text{and} \quad 2\text{-tilt} \Lambda \leftrightarrow \text{CW}(G).$$

Before we move on to proving the theorem, we remark on two properties of a complete admissible collection, which are far from apparent from the definition but follows immediately from the theorem.

Firstly, there is a priori no limitation to the number of signed walks in a complete collection. However, as $|T| = |\Lambda|$ for any tilting complex $T$, the theorem implies that a complete collection must have exactly the same number of signed walks as the number of edges in $G$. 

The other property is that each edge in $G$ appears in at least one signed walk in a complete collection. This follows from the fact that a tilting complex induces a basis of the Grothendieck group $K_0(K^b(proj\Lambda_G))$, whose canonical basis is the isomorphism classes of projective indecomposable $\Lambda_G$-module, which is in bijection with edges of $G$.

5. Proof of Theorem 4.6

In this section, we give a proof of Theorem 4.6. Let $T := (T^{-1} \xrightarrow{d} T^0)$ be a two-term complex in $K^b(proj\Lambda)$. We define two modules $M_T$ and $N_T$ as follows:

$$M_T := H^0(T), \quad N_T := H^{-1}(T) = \begin{cases} \Omega^2 M_T & \text{if } T^0 \neq 0 \\ T^{-1} & \text{if } T^0 = 0 \end{cases}$$

The following lemma is the central trick in our proof.

**Lemma 5.1.** Let $\Lambda$ be a symmetric algebra. Let $T$ and $T'$ be indecomposable two-term complexes in $K^b(proj\Lambda)$. Then $\text{Hom}_{K^b(proj\Lambda)}(T', T[1]) = 0$ if and only if $\text{Hom}_\Lambda(M_T, N_T) = 0$.

**Proof.** Assume that $T^0 \neq 0$. Then, by [AIR, Lemma 3.4], we have that $\text{Hom}_\Lambda(M_T, \Omega^2 M_{T'}) = 0$ if and only if $\text{Hom}_{K^b(proj\Lambda)}(T', T[1]) = 0$. Assume that $T^0 = 0$. Then we have

$$\text{Hom}_\Lambda(M_T, T'^{-1}) = 0 \iff \text{Hom}_\Lambda(T'^{-1}, M_T) = 0$$

$$\iff \text{Hom}_{K^b(proj\Lambda)}(T', T[1]) = 0.$$

Hence the assertion follows. \hfill $\Box$

This lemma gives the reason why we are not interested in the band modules. Indeed, a band module $M$ is $\tau$-invariant (recall that $\tau \cong \Omega^2$ here), so $\text{id}_M \in \text{Hom}_{\Lambda_G}(M, \Omega^2 M) = \text{Hom}_{\Lambda_G}(M, M)$.

In other words, indecomposable two-term pretilting complexes cannot be isomorphic to a minimal projective presentation of a band module.

Let $T = (T^{-1} \xrightarrow{d} T^0)$ and $T' = (T'^{-1} \xrightarrow{d'} T'^0)$ be complexes in $2\text{soc}\Lambda$. Let $W := W_T$ and $W' := W_{T'}$ be signed walks with signatures $\epsilon$ and $\epsilon'$ respectively, corresponding to $T$ and $T'$ under the bijection in Lemma 4.3. $\text{Hom}_\Lambda(M_T, N_{T'}) \neq 0$ if and only if there is a string module $L$ such that it is (isomorphic to) both a factor module of $M_T$ and a submodule of $N_{T'}$. In the sequel, we find the combinatorial incarnation of this homological phenomenon, which will prove the theorem. Note that $M_T$ is by definition a short string module or an indecomposable projective module, whereas $N_T$ can be an indecomposable projective module or a string module which is not necessarily short.

We may assume that $T^0 \neq 0$ and $T'^{-1} \neq 0$, otherwise $M_T$ and $N_{T'}$ will be zero in respective cases. Note that $L$ is a string module of one of the following forms:

(A) $L$ is simple.

(B) $L$ is non-simple uniserial.

(C) $L$ is non-uniserial.

The following lemma will be used extensively to analyse the structure of $M_T, N_{T'}$, and $L$.

**Lemma 5.2.** Let $W$ be a signed walk. Then there is a multiset isomorphism between the summands of the top($H^0(T_W)$) and the positively signed (half-)edges, and a multiset isomorphism between the summands of soc($H^{-1}(T_W)$) and the negatively signed (half-)edges.

**Proof.** Since $T_W$ is the minimal projective (resp. injective) presentation of $H^0(T_W)$ (resp. $H^{-1}(T_W)$), the two correspondence follows respectively. \hfill $\Box$

**Lemma 5.3.** The following are equivalent:

(A1) $L = S_E$ is a simple module corresponding to an edge $E = \{e, \bar{e}\}$.

(A2) There is an edge $E = \{e, \bar{e}\}$ such that $\epsilon(e) = +$ and $\epsilon'(e) = -$. 

Proof. (A1) is equivalent to $S_E \in \text{add} \left( \text{top } M_T \right) \cap \text{add} \left( \text{soc } N_{T'} \right)$, which is then equivalent to (A2) by Lemma 5.2. \[ \square \]

**Lemma 5.4.** The following are equivalent:

(B1) $L$ is non-simple uniserial module with $\text{top } L = S_E$ and $\text{soc } L = S_{E'}$.

(B1') $L$ can be chosen as a non-simple uniserial short string module with $\text{top } L = S_E$ and $\text{soc } L = S_{E'}$.

(B2) There is a choice $w = \left( e_1, \ldots, e_m \right) \in W$ and $w' = \left( e'_1, \ldots, e'_n \right) \in W'$, such that for some $i \in \{0, \ldots, m\}$ and $j \in \{0, \ldots, n\}$, the pair of neighbourhoods around $v = s(e_{i+1}) = s(e'_{j+1})$ in $W$ and $W'$ can be visualised in one of the following forms.

\[ e'_{i+1} \quad (i) \quad e^-_{i+1} \quad \sigma_j^+ \quad (ii) \quad e^-_{i+1} \quad e'_{i+1} \quad (iii) \quad e'_j = e'^+_{j+1} \]

Proof. (B1) $\iff$ (B1'): (B1') implies (B1) is trivial. Suppose $L$ is a non-simple uniserial module which is not short. Then $L$ has basis \[
\left\{ [e], [e, \sigma(e)], [e, \sigma^2(e)], \ldots, [e, \sigma^k(e)] \mid i = 0, \ldots, l \right\}
\]
for some $k, l > 0$. The short string quotient of $L$ (hence a quotient of $M_T$) with basis \[
\left\{ [e]^0, [e, \sigma(e)], \ldots, [e, e'] \right\}
\]
is also a submodule of $L$, hence a submodule of $N_{T'}$.

(B1') $\implies$ (B2): Suppose condition (B1') holds. Let $E = \{ e, e' \}$, $E' = \{ e', e' \}$ so that $L$ has basis \[
\{ e, (e, \sigma(e)), \ldots, (e, e') \}. \]
We use the diagram

\[
\begin{array}{c}
\text{e} \\
\downarrow \\
\text{e'}
\end{array}
\]

to represent this uniserial module (reflecting its Loewy structure). Let $v$ be the vertex emanating $e$ and $e'$. $L$ being non-simple means that $e \neq e'$. We point out that the choice of half-edges $e$ and $e'$ is uniquely determined by the structure of $L$ - a non-apparent yet crucial point in this proof.

Now combining Lemma 5.2 with the description of string modules, $M_T$ (resp. $N_{T'}$) have the following “Loewy structure”:

\[ (a) \quad M_T = P_e \quad \text{(b) } M_T: \text{non-projective} \quad (c) \quad N_{T'} = P_{e'} \quad (d) \quad N_{T'}: \text{non-projective} \]

where $M_1, M_2, N_1, N_2$ are string modules or zero. We have chosen the labelling in the way that $f, f'$ also incident to the vertex $v$.

It is easy to see that the half-walks obtained by the former construction are $w_T = (e^+)$ in case (a) and $w_{T'} = (e'^-)$ in case (c). The other cases are more complicated. There are two subcases for (b).
(b1): Suppose $M_2 \neq 0$. Then the corresponding signed half-walk is of the form $w = (\ldots, T^-, e^+, \ldots)$ or $\ldots, e^+, T^-, \ldots)$. Since $M_T$ is short string, all half-edges in the cyclic-ordering around $v$ appear at most once. In particular we have the cyclic-ordering $(e, e', f)_v$.

(b2): Suppose $M_2 = 0$. Since $M_T$ is a short string and $L$ is non-simple, its uniserial quotient $U$ with top $e$ and socle $f$ is either a short string module, or a (non-simple) hook module with top $e$. To see this, if $e \in E_m$ in Definition 3.5 (2), or $e \in E_1$ or $E_m$ in Definition 3.5 (3), then we have the hook module case; otherwise we get the short string case.

In the short string case, we have $w = (\sigma(f)^-, e^+, \ldots)$ and cyclic ordering $(e, e', f, \sigma(f))_v$ if $f \neq e'$; otherwise $(e, e', \sigma(f))_v$. If $U$ is a hook, then $f = \sigma^{-1}(e)$ and $s(e)$ is the endpoint of $w = (e^+, g^-, \ldots)$. The cyclic ordering accounting virtual edges is $(e, e', v_{-1}(e))_v$. Summarising:

| $M_2$ | uniserial subquotient | $w$ | cyclic ordering |
|-------|-----------------------|-----|----------------|
| (b1)  | non-zero              | short | $\ldots, f^-, e^+, \ldots$ | $(e, e', f)_v$ |
| (b2)  | 0                     | short | $(\sigma(f)^-, e^+, \ldots)$ | $(e, e', \sigma(f))_v$ |
| (b3)  | 0                     | hook  | $(e^+, g^-, \ldots)$ | $(e, e', v_{-1}(e))_v$ |

Write $w \in W$ as $(e_1, \ldots, e_m)$ and let $i$ be the integer so that $e$ in the above analysis appears as $e_{i+1}$, then the cyclic ordering around $v$ with signs can be expressed as $(e^+_{i+1}, e^-_i, e^-_i)_v$; visually:

$$e = e^+_{i+1} \quad e^-_i \quad e^-_i \quad v \quad e'$$

The argument for (d) goes dually to (b). We summarise the data from such deduction as follows:

| $N_1$ | uniserial subquotient | $w'$ | cyclic ordering |
|-------|-----------------------|-----|----------------|
| (d1)  | non-zero              | short | $(\ldots, f^+, e^-, \ldots)$ | $(e, e', f')_v$ |
| (d2)  | 0                     | short | $(\sigma^{-1}(f)^+, e^-, \ldots)$ | $(e, e', \sigma^{-1}(f'))_v$ |
| (d3)  | 0                     | hook  | $(e^-, g^+, \ldots)$ | $(e, e', v_{+1}(e'))_v$ |

Similarly, we can visualise the local structure as follows:

$$e^+ \quad \overline{e^+_j} \quad v \quad e^+ = e^+_j$$

Combine the two local structure and we get (B2).

(B2)⇒(B1'): This is essentially reading the previous part of the proof in reverse. We give a brief guide to how this goes. Set $e, e'$ as the half-edges $e_{i+1}, e'_{j+1}$ shown in the diagrams respectively. For each case of the given (signed) cyclic orderings, consider $(e, e', \overline{e_i})_v$ and $(e, e', \overline{e_j})_v$ separately. These two cyclic orderings can be matched to two unique entries in the last column of the two tables above respectively. This allows one to reconstruct $w$ and $w'$.

Suppose $w = (e^+)$ (resp. $w' = (e'^-)$. We obtain $M_T$ (resp. $N_T$) shown in case (a) (resp. (c)). Suppose $w$ (resp. $w'$) is a longer sequence. Then it takes the form as shown in one of the entries in the penultimate column of the corresponding table. One then obtain the condition of $M_2$ (resp. $N_1$) and the uniserial subquotient forming $M_T$ (resp. $N_T$) in case (b) (resp. (d)). From these “Loewy pictures”, one can determine $L$ easily.

Suppose $L$ is a non-uniserial module. Then (the Loewy structure of) $L$ has one of the following forms. In all of the following diagrams, $E_i$’s ($i = 0, 1, \ell, \ell + 1$) are edges (simple modules) and $L_0$ is a short string (non-hook) module.
One can regard the diagram on the right as having $L_0 = 0$ and $\ell = 2$ in the diagram on the left.

One can regard the diagram on the right as having $L_0 = 0$ and $\ell = 1$ in the diagram on the left.

We remark that, even though these modules look like the picture of string complexes rotated, simply rotating a diagram does not transfer you from the module to its projective/injective presentation, and vice versa.

Lemma 5.5. The following are equivalent:

(C1) $L$ is of type (LZ) (resp. (LM), resp. (LW)).

(C2) There is a $z = (e_1^+, \ldots, e_\ell^-, \ldots, e_\ell^+, \ldots, e_1^-)$ in $w \cap w'$ (resp. $z = (e_1^+, \ldots, e_\ell^+)$, resp. $z = (e_1^-, \ldots, e_\ell^-)$) with $(e, z, f) \subset w$ and $(e', z, f') \subset w'$ such that the neighbourhood orderings are $(e', \tau, e_1)_s(e_1)$ and $(e_\ell, f', f)_s(e_\ell)$.

Proof. $L$ is of type (LZ) if and only if $M_T$ and $N_{T'}$ have the following Loewy structures respectively.

Here $A, A'$ are non-zero composition factors, and each of the modules $M_1, M_2, N_1, N_2$ can be a string module or zero.

Let $U$ be the uniserial subquotient of $M$ with top $E_1$ and socle $A$, and let $U'$ be the uniserial subquotient of $N$ with top $A'$ and socle $E_\ell$. Again, combining Lemma 5.2 with the description of string modules, one can see that condition (2) holds if and only if (after choosing the half-edges appropriately) $W_T$ and $W_{T'}$ have the following local structures labelled in solid lines:
To show (C1) implies (C2) in the (LZ) case, we make the following choices:

\[
\overline{f} = \begin{cases} 
    a & \text{if } M_1 \neq 0, \\
    \sigma(a) & \text{if } M_1 = 0 \text{ and } U \text{ short string}, \\
    \nu_r(e_1) & \text{if } M_1 = 0 \text{ and } U \cong H_{e_1};
\end{cases}
\]

and

\[
\overline{f}' = \begin{cases} 
    a' & \text{if } N_2 \neq 0, \\
    \sigma^{-1}(a') & \text{if } N_2 = 0 \text{ and } U' \text{ short string}, \\
    \nu_r^{+}(e_1) & \text{if } N_2 = 0 \text{ and } U' \cong H_{e_1};
\end{cases}
\]

where \(a \in A\) and \(a' \in A'\) are the half-edges emanating from \(u\) and \(v\) respectively.

To show (C2) implies (C1) for case (LZ), we can construct the diagram of \(M\) and \(N\) as shown above, where \(A\) and \(A'\) are uniquely determined by \(\overline{f}\) and \(\overline{f}'\) respectively using the above formulae after replacing the word “if” by “then set”. This finishes the proof for (LZ) case.

The statement in the case of (LM) and (LW) can be proofed similarly after appropriately modifying the diagrams. The only part that the proof for case (LZ) did not cover is when \(\ell = 1\). Then \(M\) (resp. \(N\)) can be an indecomposable projective module in the case of (LM) (resp. (LW)). Nevertheless, the argument is similar to those presented in the case when \(L\) is non-simple uniserial, so we refrain from giving the details and leave it as an exercise for the reader.

The following is the final piece needed to prove Theorem 4.6.

**Proposition 5.6.** (1) Retaining the notations used so far, \(\text{Hom}_{\mathbb{K}^b(\text{proj} \Lambda_G)}(T', T[1]) = 0\) is equivalent to having none of the situations in (A2), (B2), (C2) for \(W = W_T\) and \(W' = W'_T\).

(2) For two not necessarily distinct complexes \(U, V \in 2\text{scx} \Lambda_G\), the hom-spaces \(\text{Hom}_{\mathbb{K}^b(\text{proj} \Lambda_G)}(U, V[1])\) and \(\text{Hom}_{\mathbb{K}^b(\text{proj} \Lambda_G)}(V, U[1])\) are simultaneously zero if, and only if \(W_U, W_V\) are non-crossing and satisfy the sign condition.

Note that the right-hand side of (2) is not equivalent to saying \(\{W_U, W_V\}\) is admissible, as it does not require \(W_U\) and \(W_V\) to be admissible.

**Proof.** (1) This follows from Lemma 5.3, 5.4, and 5.5.

(2) First, we clarify in the following table the relation between the non-crossing and signs conditions and those in (A2), (B2), (C2). In each row of the table, the condition in the left entry fails for the walks \(W_U, W_V\) is equivalent to the condition in the right entry holds for \((W, W') = (W_U, W_V)\) or \((W, W') = (W_V, W_U)\).

| The following fails for \(\{W_U, W_V\}\) | The following holds for \((W, W')\), where \((W, W') = (W_U, W_V)\) or \((W, W') = (W_V, W_U)\) |
|------------------------------------------|--------------------------------------------------|
| (NC1)                                   | (A2) or (B2)(iii)                                 |
| (NC2)                                   | (C2)                                             |
| (NC3)                                   | (B2)(i) or (B2)(ii)                              |
| sign condition                          | (A2) or (B2)(i) or (B2)(iii)                     |

So now the statement follows by applying (1) to \((T, T') = (U, V)\) and to \((T, T') = (V, U)\) simultaneously.

**Proof of Theorem 4.6.** Suppose \(U, V\) are (possibly the same) complexes in \(2\text{scx} \Lambda_G\). By definition, \(U \oplus V\) is pretilting means that the four hom-spaces \(\text{Hom}(U, U[1]), \text{Hom}(U, V[1]), \text{Hom}(V, U[1]), \text{Hom}(V, V[1])\),
Hom(V, V[1]), Hom(V, U[1]) are all zero. By Lemma 5.6, this is equivalent to \{W_U, W_V\} being an admissible set. This implies that a (basic) two-term pretilting complex correspond to precisely an admissible set of signed walks. The first bijection is just the special case \( U = V \).

Since any two-term pretilting complex \( T \) is partial, if \( T \oplus U \) is a two-term pretilting complex with \( T \) tilting, then \( U \in \text{add} \, T \). Translating this to the combinatorial side, we obtain that a (basic) tilting complex correspond precisely to a complete admissible collection of signed walks.

\[ \square \]

6. Tilting-discrete Brauer graph algebras

6.1. Preliminaries. We first recall some results about tilting mutation theory from [AI, AI1, AIR]. Throughout this subsection, \( \Lambda \) is assumed to be a finite dimensional symmetric algebra. Most of the facts stated here have analogues for general finite dimensional algebras by replacing the word “tilting” to “silting”; see loc. cit. for the details.

Let \( \mathcal{A} \) be a full additive subcategory of \( \mathcal{C} = K^b(\text{proj} \, \Lambda) \) or \( \text{mod} \, \Lambda \). A map \( f : X \to Y \) in \( \mathcal{C} \) is left minimal if all maps \( g : Y \to W \) with \( gf = f \) are isomorphisms. A map \( f : U \to X \) in \( \mathcal{C} \) is called a left \( \mathcal{A} \)-approximation of \( U \) if \( X \) belongs to \( \mathcal{A} \) and \( \text{Hom}_\mathcal{C}(f, C) \) is surjective for any \( C \) in \( \mathcal{A} \). We say that \( \mathcal{A} \) is covariantly finite in \( \mathcal{C} \) if for all \( U \) in \( \mathcal{C} \), there exists a left \( \mathcal{A} \)-approximation. Dually, we define right minimality, right \( \mathcal{A} \)-approximations, and contravariantly finite subcategories in \( \mathcal{C} \). A full subcategory is called functorially finite if it is covariantly and contravariantly finite in \( \mathcal{C} \).

Definition-Theorem 6.1. [AI] Let \( T \) be a basic tilting complex in \( K^b(\text{proj} \, \Lambda) \) with a decomposition \( T = X \oplus M \). A left tilting mutation \( \mu_X(T) \) of \( T \) with respect to \( X \) is a tilting complex given by \( Y \oplus M \), where \( Y \) is the (well-defined) object fitting into the following triangle:

\[
X \xrightarrow{f} M' \longrightarrow Y \longrightarrow X[1]
\]

where \( f \) is a minimal left \( \text{add} \, M \)-approximation of \( X \). Dually, one also has another tilting complex given by right tilting mutation \( \mu_X(T) \) of \( T \) with respect to \( X \).

A tilting mutation means a left or right tilting mutation; it is called irreducible if \( X \) is indecomposable.

Definition-Theorem 6.2. [AI, AI1] Let \( T \) and \( U \) be tilting complexes of \( \Lambda \). We write \( T \geq U \) if \( \text{Hom}_{K^b(\text{proj} \, \Lambda)}(T, U[n]) = 0 \) for any positive integer \( n \). Then \( \geq \) induces partial order structure on \( \text{tilt} \, \Lambda \). Moreover, \( n \)-\text{tilt} \( \Lambda \) is precisely the interval \( \{ T \in \text{tilt} \, \Lambda \mid \Lambda \geq T \geq \Lambda[n-1] \} \) in \( \text{tilt} \, \Lambda \).

Definition 6.3. [AI] The tilting quiver of \( \Lambda \), denoted by \( \mathcal{T}_\Lambda \), is defined as follows:

- The set of vertices is \( \text{tilt} \, \Lambda \).
- Draw an arrow from \( T \to U \) if \( U \cong \mu_X(T) \) for some indecomposable direct summand \( X \) of \( T \).

Note that this is precisely the Hasse quiver of the poset (\( \text{tilt} \, \Lambda, \geq \)). A connected component of \( \mathcal{T}_\Lambda \) is said to be canonical if it contains \( \Lambda \).

We denote by \( \mathcal{T}_\Lambda^{(0, n)} \) the Hasse quiver of the interval \( n \)-\text{tilt} \( \Lambda \) of \( \text{tilt} \, \Lambda \).

Definition 6.4. A symmetric algebra \( \Lambda \) is said to be tilting-connected if the tilting quiver of \( \Lambda \) is connected. We say that \( \Lambda \) is tilting-discrete if for any positive integer \( \ell \), there exist only finitely many tilting complexes \( T \) in \( \text{tilt} \, \Lambda \) satisfying \( \Lambda \geq T \geq \Lambda[\ell] \).

It was shown in [AI] Section 3 that if \( \Lambda \) is tilting-discrete, then it is tilting-connected.

Two classes of tilting-discrete symmetric algebras are found in the second author’s previous works. Namely, the local algebras in [AI] and the representation-finite algebras in [AI1].

We refer to [AI] for more general examples of tilting/silting-connected algebras. On the other hand, in a joint work of the second author with Grant and Iyama, we know that there exist...
non-tilting-connected symmetric algebras. In any case, it is not easy to answer the following question.

**Question 6.5.** When is a symmetric algebra tilting-connected, or even tilting-discrete?

The next subsection is devoted to answering the tilting-discrete part of the question for Brauer graph algebras.

We first look at some properties when the set $2$-tilt $\Lambda$ is finite, or equivalently (by Proposition 3.2 (ii)), when the set $2ipt \Lambda$ is finite.

**Proposition 6.6.** [DIJ] If $2$-tilt $\Lambda$ is a finite set, then every torsion class (i.e. extension-closed quotient-closed full subcategory) in mod $\Lambda$ is functorially finite.

With this proposition in hand, we can reuse some of the tricks in [Ai1], which we recall below, to determine tilting-discreteness, and hence tilting-connectedness.

**Proposition 6.7.** [Ai1, Theorem 3.5] If $2$-tilt $\Lambda$ is a finite set, then any two-term tilting complex can be obtained from $\Lambda$ by iterated irreducible left tilting mutation.

We denote by $M^\perp$ the full subcategory of mod $\Lambda$ consisting of all $\Lambda$-modules $X$ satisfying $\text{Hom}_\Lambda(M, X) = 0$. This is a torsion class in mod $\Lambda$.

**Proposition 6.8.** [AH, Ai1] Let $T$ be an $n$-term tilting complex i.e. $\Lambda \geq T \geq \Lambda[n - 1]$ for a positive integer $n \geq 2$, and put $M := H^0(T)$. If the torsion class $M^\perp$ is functorially finite in mod $\Lambda$, then there exists a two-term tilting complex $P$ of $\Lambda$ satisfying $P \geq T \geq P[n - 2]$.

We provide a result which plays an important role later.

**Proposition 6.9.** Let $\Lambda$ be a symmetric algebra. If for any tilting complex $P$ in the canonical connected component of $\text{tilt}_\Lambda$, the set $2$-tilt $\text{End}_{Kb(\text{proj} \Lambda)}(P)$ is finite, then $\Lambda$ is tilting-discrete. In particular, it is tilting-connected.

This result and its proof in full generality can be found in [AM]; we give the relevant parts of the proof here for the convenience of the reader.

**Proof.** We need to show that there exist only finitely many basic $n$-term tilting complexes $T$ for all positive integers $n$, and we achieve this by induction on $n$. If $n = 1, 2$, then it is obvious. Let $T \in \text{tilt} \Lambda$ with $\Lambda \geq T \geq \Lambda[n - 1]$ for some positive integer $n > 2$. Since $2$-tilt $\Lambda$ is a finite set, it follows from Proposition 6.6 and Proposition 6.8 that there exists a two-term tilting complex $P$ of $\Lambda$ satisfying $P \geq T \geq P[n - 2]$. Thus, we have

$$n\text{-tilt} \Lambda = \{T \in \text{tilt} \Lambda \mid \Lambda \geq T \geq \Lambda[n - 1]\} \subseteq \bigcup_{P \in 2\text{-tilt} \Lambda} \{U \in \text{tilt} \Lambda \mid P \geq U \geq P[n - 2]\}.$$ 

By Proposition 6.7, we see that $P$ belongs to the canonical connected component of $\text{tilt}_\Lambda$. Since the endomorphism algebra $\Gamma := \text{End}_{Kb(\text{proj} \Lambda)}(P)$ of $P$ is derived equivalent to $\Lambda$, the required condition for $\Gamma$ is satisfied. This implies that $\{U \in \text{tilt} \Lambda \mid P \geq U \geq P[n - 2]\} \leftrightarrow (n - 1)\text{-tilt} \Gamma$ is a finite set by the induction hypothesis, whence so is $n\text{-tilt} \Lambda$. \qed

6.2. **Conditions for Tilting-discreteness.** We start by stating the aim of this section.

**Theorem 6.10.** Let $G$ be a Brauer graph. Then the following are equivalent:

(i) $\Lambda_G$ is tilting-discrete;
(ii) $2$-tilt $\Lambda_G$ is a finite set;
(iii) $G$ contains at most one odd cycle and no even cycle.

First of all, we start by recalling the Brauer graph mutation theory from [Ai2].
Definition 6.11 (Mutation/Flip of Brauer graph). Let $G = (V, H, s, \tau, \sigma; m)$ be a Brauer graph and $E$ an edge of $G$. If $G$ has more than one edge, then the left mutation of $G$ at $E$, or the left flip of $G$ at $E$, is a Brauer graph $\mu_{E}^{-}(G)$ given by $(V, H, s', \tau, \sigma'; m)$, where $s'$ and $\sigma'$ are defined as follows. For presentation clarity, we use the shorthand $p_{e} := \sigma^{-1}(e)$ and $s_{e} := \sigma(e)$ for $e \in E$.

| half-edge $h$ | condition | $s'(h)$ | $\sigma'(h)$ |
|---------------|-----------|---------|--------------|
| $e \in E$    | $p_{e} \notin E$, $s_{e} \neq \tau$ | $s(p_{e})$ | $\overline{p_{e}}$ |
| $p_{e} \notin E$, $s_{e} = \tau$ | $s(p_{e})$ | $\tau$ |
| $p_{e} = e$  | $s(e)$ | $e$ |
| $p_{e} = \tau$ | $s(\overline{p_{e}}) = s'(\tau)$ | $\overline{p_{e}}$ |

If $G$ has one edge, then the left flip of $G$ is defined to be itself.

The opposite (Brauer) graph of $G$ is the Brauer graph $G_{op} = (V, H, s, \tau, \sigma^{-1})$. The right mutation of $G$ at $E$, or the right flip of $G$ at $E$, is the Brauer graph $\mu_{E}^{+}(G) := \mu_{E}(G_{op})_{op}$.

The simplest way to present a graphical presentation of the left flip is given below. If $E$ is not a loop, i.e. $u, v$ in the graphs below are distinct vertices:

Here we allow some (or all) of $a, b, v$ to be the same vertex; similarly for $c, d, u$.

If $E$ is a loop, then we have one of the following two cases:

Here $a, b$ can be the same vertex. c.f. [Ai2, Sec 5]

Remark 6.12. (1) We will abuse the notation $\mu_{E}^{\pm}(G)$ to mean “$\mu_{E}^{-}(G)$, and respectively $\mu_{E}^{+}(G)$,” for ease of stating results. Similar abuses of notations will also be adopted for mutations of tilting complexes.

(2) Intuitively, if one draws $G$ on a piece of paper and places a mirror perpendicular to the paper next to $G$, then $G_{op}$ is the reflection of $G$ in the mirror.

(3) The opposite ring $(\Lambda_{G})_{op}$ of $\Lambda_{G}$ is isomorphic to $\Lambda_{G_{op}}$.

(4) One can also define the right flip explicitly, then left flip will be given by $\mu_{E}^{\pm}(G) = \mu_{E}^{\pm}(G_{op})_{op}$.

(5) The left/right flip at $E$ for non-loop $E$ was found by Kauer [Kau], and is termed as Kauer move in [MS]. The terminology “flip” was adopted in [Ai1] to align with the flip of triangulations in cluster mutation theory. Indeed, if the underlying ribbon graph of $G$ is a
triangulation of a Riemann surface, then \( \mu_\pm^E (\mathcal{G}) \) is precisely the flip of the triangulation at the edge \( E \).

**Proposition 6.13.** [Kau, Ai2] Let \( \mathcal{G} \) be a Brauer graph and \( E \) an edge of \( \mathcal{G} \). Then the endomorphism ring of the tilting complex \( \mu_\pm^E (\Lambda_\mathcal{G}) \) is isomorphic to \( \Lambda_\mu_\pm^E (\mathcal{G}) \).

**Lemma 6.14.** Let \( \mathcal{G} \) be a Brauer graph and \( E \) an edge of \( \mathcal{G} \). If \( \mathcal{G} \) contains \( c \) odd cycle with \( c \in \{0, 1\} \), and no even cycle, then so is \( \mu_\pm^E (\mathcal{G}) \).

**Proof.** First note that \( \mu_\pm^E (\mathcal{G}) \) preserves connectedness. In particular, if \( \mathcal{G} \) is a (connected) tree (i.e. \( c = 0 \)), equivalently \( |V| = |H/\tau| + 1 \), then so is \( \mu_\pm^E (\mathcal{G}) \).

Assume now that \( \mathcal{G} \) has an odd cycle and has no even cycle. Now we have \( |V| = |H/\tau| \), and by the same argument, \( \mu_\pm^E (\mathcal{G}) \) must then contain precisely one cycle. We are left to show that the parity of the cycle length remains unchanged after a flip.

Since flipping an edge does not alter the rest of the graph, the odd cycle of \( \mathcal{G} \) stays as the same subgraph if we flip at an edge not in the cycle. If \( E \) is contained in the odd cycle, say of length \( \ell \), of \( \mathcal{G} \), then observe using the graphical presentation of flips that the cycle length is either \( \ell \), or \( \ell - 2 \), or \( \ell + 2 \). \( \square \)

Now we are ready to prove Theorem 6.10.

**Proof of Theorem 6.10.** It is evident that the implication (i) \( \Rightarrow \) (ii) holds. The implications (ii) \( \Leftrightarrow \) (iii) follows from Theorem 4.6 and Proposition 2.12.

We show that the implication (iii) \( \Rightarrow \) (i) holds. Let \( P \) be a tilting complex of \( \Lambda_\mathcal{G} \) in the canonical connected component of \( T_{\Lambda_\mathcal{G}} \). Then it follows from Proposition 6.13 that the endomorphism algebra \( \Gamma := \text{End}_{K\text{b}(\text{proj} \Lambda_\mathcal{G})} (P) \) of \( P \) is a Brauer graph algebra whose Brauer graph \( \mathcal{G}' \) is obtained by a series of left and right flips starting from \( \mathcal{G} \). By Lemma 6.14, \( \mathcal{G}' \) then has the same number (zero or one) of odd cycle as \( \mathcal{G} \), and it also has no even cycle. Combining Theorem 4.6 and Proposition 2.12 we get that 2-\text{tilt} \( \Gamma \) is a finite set, whence \( \Lambda_\mathcal{G} \) is tilting-discrete by Proposition 6.9. \( \square \)

The following corollary is immediate from (the proof of) Theorem 6.10.

**Corollary 6.15.** Let \( \mathcal{G} \) be a Brauer graph which contains at most one odd cycle and no even cycle, and \( T \) be a tilting complex of \( \Lambda_\mathcal{G} \). Then the endomorphism algebra of \( T \) in \( K\text{b}(\text{proj} \Lambda_\mathcal{G}) \) is isomorphic to \( \Lambda_{\mathcal{G}'} \) for some \( \mathcal{G}' \) with the same number of odd cycle, even even cycle, and the same multiplicity as \( \mathcal{G} \). In particular, any algebra derived equivalent to \( \Lambda_\mathcal{G} \) is also a Brauer graph algebra.

The class of algebras described in Theorem 6.10 appears in [Ant] as (precisely) the class of Brauer graph algebra whose Cartan matrix is non-degenerate, equivalently the Grothendieck group of the stable module category is finite.

For an arbitrary symmetric algebra \( \Lambda \), we do not know if the non-degeneracy of its Cartan matrix, or the finiteness of the set 2-\text{tilt} \( \Lambda \), is an equivalent condition for tilting-discreteness. We also do not know if the finiteness, or even if the number of elements, of the set 2-\text{tilt} \( \Lambda \) is derived invariant. In fact, one of the original motivations of this work and its sequel is to see if one can take advantages of the rich combinatorics of Brauer trees in order to count the number of two-term tilting complexes for Brauer tree algebras.

Suppose \( \mathcal{G} \) is a Brauer graph such that \( \Lambda_\mathcal{G} \) is tilting-discrete. If we reassign all the multiplicities of vertices to 1, then the associated Brauer graph algebra is of finite type when \( c = 0 \), or of one-parametric Euclidean type when \( c = 1 \). See for example [Sko] for the details.

Also note that the Brauer graph algebras with \( m \equiv 1 \) forms precisely the class of trivial extensions of gentle algebras [Sch]. For readers who are familiar with silting theory [AI] and silting-discreteness [Ai1], the result in [BPP] Prop. 6.8 asserts that all derived-discrete algebras...
of finite global dimension are silting-discrete. However, the trivial extensions of these algebras also consist of (multiplicity-free) Brauer graph algebras which lie outside the class presented in Theorem 6.10. In particular, this shows that silting-discreteness is one of the many properties destroyed by taking trivial extension.

We have now determined the precise condition for a Brauer graph algebra to be silting-discrete, but it is still not known to us if there is a non-silting-discrete Brauer graph algebra which is silting-connected. We remark that, in another on-going work of the second named author with Grant and Iyama, the Brauer graph algebra whose underlying graph is a digon (i.e. cycle of length 2) is neither silting-discrete nor silting-connected.

Our final remark to this result is the problem of classifying derived equivalent classes of algebras. Such classification for Brauer graph algebras is known in the case where the Brauer graph having at most one odd cycle, no even cycle, and at most two vertices with multiplicity higher than 1, see the survey [Sko] for details. So Corollary 6.15 is a slight generalisation of known results but with an entirely different approach.

It is still not known if the derived equivalence class is closed for all Brauer graph algebras, despite many researches on closely related topics - including our investigation here. What we do know is that the endomorphism algebra of a two-term tilting complex for a Brauer graph algebra is also a Brauer graph algebra. However, due to the amount of details and the tedious nature of calculating the endomorphism algebra of a tilting complex, we choose not to present the proof of this result here, and defer it to our next article.

6.3. Multiplicity-independence. Throughout this section, for a Brauer graph G with multiplicity m, we say that it (or its associated algebra \( \Lambda_G \)) has higher multiplicity if there is a vertex \( v \) in G with \( m(v) > 1 \); otherwise it is called multiplicity-free. For a Brauer graph G, we denote by \( G_0 \) the corresponding multiplicity-free Brauer graph.

The aim of this subsection is to compare the tilting quiver of \( \Lambda_G \), where G has higher multiplicity, with that of \( \Lambda_G \). From now on, we always assume G has higher multiplicity.

**Proposition 6.16.** Let G be a Brauer graph. Then there is an isomorphism of the partially ordered sets \( \Psi_{G}^{(2)} : 2\text{-}\text{tilt} \Lambda_G \rightarrow 2\text{-}\text{tilt} \Lambda_{G_0} \).

**Proof.** Note that the set \( \text{AW}(G) \), and hence \( \text{CW}(G) \), does not depend on the multiplicity of G, that is, we have natural bijections \( \text{AW}(G) \rightarrow \text{AW}(G_0) \) and \( \text{CW}(G) \rightarrow \text{CW}(G_0) \). These yield bijections \( 2\text{ipt} \Lambda_G \rightarrow 2\text{ipt} \Lambda_{G_0} \) and \( 2\text{-}\text{tilt} \Lambda_G \rightarrow 2\text{-}\text{tilt} \Lambda_{G_0} \) by Theorem 4.6. By abuse of notation, we name both of them \( \Psi_{G}^{(2)} \).

We are left to show that this map preserves the partial order induced by tilting mutations. Let \( T,T' \in 2\text{-}\text{tilt} \Lambda_G \) with a common direct summand M and \( T \geq T' \). This implies that \( \text{Hom}(X,X'[1]) = 0 \) for all indecomposable summands X and X’ of T/M and T’/M respectively. By definition, the indecomposable summands \( \Psi_{G}^{(2)}(X) \) of \( \Psi_{G}^{(2)}(T) \) (resp. \( \Psi_{G}^{(2)}(X') \) of \( \Psi_{G}^{(2)}(T') \)) correspond also to \( W_X \) (resp. \( W_{X'} \)). Therefore, as Lemma 5.6 (1) is independent of multiplicity, we have \( \text{Hom}(\Psi_{G}^{(2)}(X),\Psi_{G}^{(2)}(X')[1]) = 0 \). This implies \( \Psi_{G}^{(2)}(T) \geq \Psi_{G}^{(2)}(T') \), and so \( \Psi_{G}^{(2)} \) is order-preserving.

We stress that Proposition 6.16 in contrast to the forthcoming results, does not depend on the choice of G.

Our next step is to inductively extend the map \( \Psi_{G}^{(2)} \) to define bijections \( \Psi_{G}^{(n)} : n\text{-}\text{tilt} \Lambda_G \rightarrow n\text{-}\text{tilt} \Lambda_{G_0} \) for all \( n > 2 \), where G is a Brauer graph containing at most one odd cycle and no even cycle. For completeness, we define \( \Psi_{G}^{(1)} : 1\text{-}\text{tilt} \Lambda_G = \{ \Lambda_G \} \rightarrow 1\text{-}\text{tilt} \Lambda_{G_0} \) as the trivial bijection.

**Lemma 6.17.** Suppose G is a Brauer graph containing at most one odd cycle and no even cycle. Then one has a poset isomorphism \( \Psi_{G}^{(n)} : n\text{-}\text{tilt} \Lambda_G \rightarrow n\text{-}\text{tilt} \Lambda_{G_0} \) for every positive integer n.
Suppose \( n \geq 3 \) and \( T \in \text{n-tilt} \Lambda_G \). By Theorem 6.10 and Proposition 6.6, we have that \( H^0(T)^\perp \) is a functorially finite torsion class. Applying Proposition 6.8, there exists a tilting complex \( P_T \in 2\text{-tilt} \Lambda_G \) satisfying \( P_T \geq T \geq P_T[n-2] \). We fix such \( P_T \) for each \( T \in \text{n-tilt} \Lambda_G \setminus (n-1)\text{-tilt} \Lambda_G \), and let \( \tilde{T} := \text{RHom}(P_T,T) \) be the complex in \((n-1)\text{-tilt} \Lambda_G^\ast\) corresponding to \( T \).

By Corollary 6.15, the endomorphism algebra of \( P_T \) is also a Brauer graph algebra, say \( \Lambda_{G^0} \).

Now we have a bijection \( \Psi_{G^0}^{(n-1)} : (n-1)\text{-tilt} \Lambda_G^\ast \to (n-1)\text{-tilt} \Lambda_{G^0}^\ast \) by the induction hypothesis.

On the other hand, the proof of Theorem 6.10 says that the endomorphism algebra of the tilting complex \( \overline{P}_T := (\Psi_G^{(2)}(P_T)) \) is \( \Lambda_{G^0}^\ast \). Let \( \overline{U} \) be the tilting complex \( \Psi_{G^0}^{(n-1)}(\tilde{T}) \). Thus, we get a unique tilting complex \( U \in \text{tilt} \Lambda_{G^0} \) satisfying \( \overline{P}_T \geq U \geq \overline{P}_T[n-2] \) such that \( \overline{U} = \text{RHom}(\overline{P}_T,U) \).

Since \( \overline{P}_T \) is two-term, one can easily check that \( U \) belongs to \( n\text{-tilt} \Lambda_{G^0} \). The schematic of the proof is as follows:

\[
\begin{CD}
\text{n-tilt} \Lambda_G @\psi_G^{(n)}\rightarrow\text{n-tilt} \Lambda_{G_0} \\
P_T, P_T[n-2] @R\text{Hom}(P_T,-)\rightarrow\overline{P}_T, \overline{P}_T[n-2] \\
(n-1)\text{-tilt} \Lambda_{G^0} \psi_{G^0}^{(n-1)}\rightarrow(n-1)\text{-tilt} \Lambda_{G^0}^\ast
\end{CD}
\]

Since we have fixed a \( P_T \) for each \( T \in \text{n-tilt} \Lambda_G \setminus (n-1)\text{-tilt} \Lambda_G \), the map \( \psi_G^{(n)}(T) := U \) is well-defined. Note that this construction is order-preserving as it is obtained by extending the order-preserving map \( \psi_G^{(2)} \) and \( \psi_{G^0}^{(n-1)} \) via poset isomorphisms \( \text{RHom}(P_T,-) : \text{tilt} \Lambda_G \rightarrow \text{tilt} \Lambda_{G^0}^\ast \) and \( \text{RHom}(\overline{P}_T,-) : \text{tilt} \Lambda_{G^0} \rightarrow \text{tilt} \Lambda_{G^0}^\ast \). For surjectivity, starts with \( U \in \text{n-tilt} \Lambda_{G^0} \), then by the same reason as before, mutatis mutandis, one obtains \( T \in \text{n-tilt} \Lambda_G \). In particular, \( \psi_G^{(n)} \) is a poset isomorphism.

Now we can realise the goal of this subsection.

\textbf{Theorem 6.18.} Suppose \( G \) is a Brauer graph containing at most one odd cycle and no even cycle. Then one has an isomorphism of partially ordered sets \( \text{tilt} \Lambda_G \rightarrow \text{tilt} \Lambda_{G^0} \).

\textbf{Proof.} Let \( \text{tilt}_+ \Lambda_G \) be the set of isoclasses of basic tilting complexes concentrated in non-positive degrees. Since we have a sequence of natural poset embeddings

\[
\{ \Lambda_G \} = \text{1-tilt} \Lambda_G \subseteq \text{2-tilt} \Lambda_G \subseteq \cdots \subseteq \text{n-tilt} \Lambda_G \subseteq \cdots ,
\]

tilt\_+ \Lambda_G is just the directed limit of \( n\text{-tilt} \Lambda_G \) over \( n \in \mathbb{Z}_{>0} \) in the category of partially ordered sets (skeletal small categories). In particular, as \( \psi_G^{(n)} \) is a poset isomorphism for every \( n > 0 \), it induces a poset isomorphism \( \psi_G^+ : \text{tilt}_+ (G) \rightarrow \text{tilt}_+ (G_0) \).

A basic tilting complex is by definition \( T[k] \) for some \( T \in \text{tilt}_+ \Lambda_G \) and a non-positive integer \( k \). In particular, the \( k \)-th shift induces a poset isomorphism \( \text{tilt}_+ \Lambda_G[m] \rightarrow \text{tilt}_+ \Lambda_G[m+k] \) for any (non-positive) integers \( m, k \). These give us a directed system where \( \text{tilt} \Lambda_G \) can be realised as the direct limit of \( \text{tilt}_+ \Lambda_G[m] \) over \( m \in \mathbb{Z}_{\leq 0} \). Hence, \( \psi_G^+ \) induces a poset isomorphism between \( \text{tilt} \Lambda_G \) and \( \text{tilt} \Lambda_{G^0} \).

The proof of Lemma 6.17 relies on \( \Lambda_G \) and \( \Lambda_{G^0} \) being tilting-discrete for such class of Brauer graphs. However, we hope that Theorem 6.18 holds without such assumption:
Conjecture. Let $G$ be a Brauer graph. Then one has an isomorphism of partially ordered sets
\[ \text{tilt } \Lambda_G \rightarrow \text{tilt } \Lambda_{G_0}. \]

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