HERMITIAN-EINSTEIN INEQUALITIES AND
HARDER-NARASIMHAN FILTRATIONS

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ABSTRACT. Unstable holomorphic bundles can be described algebraically by Harder-Narasimhan filtrations. In this note we show how such filtrations correspond to the existence of special metrics defined by Hermitian-Einstein inequalities.

1. Introduction

By a theorem of Uhlenbeck and Yau (cf. [10]), the stability of a holomorphic bundle over a closed Kähler manifold can be detected by the existence of a Hermitian bundle metric which satisfies the Hermitian-Einstein equation. In [6], Guan showed how a modification of this equation can be used to quantify deviations from stability for non-stable bundles. Such failure of stability can also be measured algebraically, namely by means of a Harder-Narasimhan filtration. These two methods naturally invite comparison, and that is the primary goal of this note. We also give some conditions on non-stable bundles which are sufficient for the existence of solutions to the modified Hermitian-Einstein equations.

Let \((X, \omega)\) be a closed Kähler manifold, and let \(E \rightarrow X\) be a rank \(R\) holomorphic bundle over \(X\). Denote the underlying smooth bundle by \(E\). Then \(E\) corresponds to \(E\) together with an integrable \(\overline{\partial}\)-operator, which we denote by \(\overline{\partial}_E\). To avoid unnecessary extra notation, we will use \(E\) to refer both to the bundle and the corresponding locally free coherent analytic sheaf. We define the slope of any coherent analytic subsheaf \(\mathcal{E}' \subset \mathcal{E}\) in the usual way, i.e. by

\[
\mu(\mathcal{E}') = \frac{\int_X c_1(\mathcal{E}') \wedge \omega^{n-1}}{\text{rank}(\mathcal{E}')}.
\]

The bundle is stable (respectively semistable) if for all coherent subsheaves with \(0 < \text{rank}(\mathcal{E}') < R\) we have

\[
\mu(\mathcal{E}') < \mu(\mathcal{E}) \quad (\mu(\mathcal{E}') \leq \mu(\mathcal{E})).
\]

The Hermitian-Einstein equation is an equation for an Hermitian bundle metric. It is given in terms of the curvature of the unique connection compatible with both
The metric and the bundle’s holomorphic structure. If we denote the metric by $H$, and the curvature of the associated connection by $F_{\bar{\partial}E,H}$, then the equation reads

$$\frac{\sqrt{-1}}{2\pi} \Lambda F_{\bar{\partial}E,H} = \mu(E)I.$$ 

Here $I$ is the identity bundle automorphism, and $\Lambda F_{\bar{\partial}E,H}$ denotes the bundle endomorphism obtained by taking the contraction of $F_{\bar{\partial}E,H}$ against the Kähler form. Thus $\Lambda F_{\bar{\partial}E,H} \wedge \omega^n = F_{\bar{\partial}E,H} \wedge \omega^{n-1}$.

**Theorem 10.** An indecomposable holomorphic bundle $E$ is stable if and only if it supports a Hermitian metric satisfying the Hermitian-Einstein equation.

In the modification considered by Guan, a parameter is introduced in the right hand side of the Hermitian-Einstein equation. Because of the Chern-Weil homomorphism, which relates $\frac{\sqrt{-1}}{2\pi} \Lambda F_{\bar{\partial}E,H}$ to the first Chern class of $E$, there can be no solutions to the equality $\frac{\sqrt{-1}}{2\pi} \Lambda F_{\bar{\partial}E,H} = mI$ unless $m = \mu(E)$. This topological constraint will however permit solutions if the equality is replaced by an inequality. Notice that the expressions on both sides of such a condition are Hermitian bundle endomorphisms. One can make sense of an inequality between two such endomorphisms by interpreting $A \leq B$ to mean that $A - B$ is negative semi-definite (with similar meaning for the inequality $A \geq B$). Guan’s result is

**Theorem 1 [6].** Let $E \rightarrow X$ be a holomorphic bundle over $X$. Let $m$ (resp. $m'$) be a real number and suppose that $E$ supports a metric $H$ such that

$$(1) \quad \frac{\sqrt{-1}}{2\pi} \Lambda F_{\bar{\partial}E,H} \leq mI \quad \text{(resp.} \quad \frac{\sqrt{-1}}{2\pi} \Lambda F_{\bar{\partial}E,H} \geq m'I) \),$$

i.e. such that $\frac{\sqrt{-1}}{2\pi} \Lambda F_{\bar{\partial}E,H} - mI$ (resp. $m'I - \frac{\sqrt{-1}}{2\pi} \Lambda F_{\bar{\partial}E,H}$) is a negative semidefinite (Hermitian) bundle endomorphism. Then

$$(2) \quad \mu(E') \leq m \quad \left( \text{resp.} \quad \mu(E/E') \geq m' \right)$$

for all subsheaves $E' \subset E$.

**Remark:** If $\mu(E) \neq 0$, then one can write $m = t\mu(E)$, $m' = t'\mu(E)$, which is the way the results are presented in [6].

A natural question to consider is for which values (if any) of $m$ or $m'$ the equations in (1) have a solution. That is, if

$$\mathcal{M} = \{ m : \frac{\sqrt{-1}}{2\pi} \Lambda F_{\bar{\partial}E,H} \leq mI \text{ for some metric } H \},$$

$$\mathcal{M}' = \{ m' : \frac{\sqrt{-1}}{2\pi} \Lambda F_{\bar{\partial}E,H} \geq m'I \text{ for some metric } H \},$$

what can one say about $\mathcal{M}$ and $\mathcal{M}'$?

It is immediately clear that if non-empty, the sets $\mathcal{M}$ and $\mathcal{M}'$ are half-infinite intervals, with $\mu(E)$ being a lower bound for $\mathcal{M}$ and an upper bound for $\mathcal{M}'$. In fact one can be a bit more precise, even without any further work. We must first recall that for a holomorphic bundle over a closed Kähler manifold, the following is true (cf. [9]).
Theorem 2 (Harder-Narasimhan Filtrations). Given a holomorphic bundle \( E \) over a closed Kähler manifold \((X, \omega)\), there is a unique filtration (called the Harder-Narasimhan filtration) by subsheaves

\[
(3a) \quad 0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \mathcal{E}_k = \mathcal{E},
\]

(where \( \mathcal{E} \) is the sheaf associated to \( E \)) such that \( \mathcal{E}_i/\mathcal{E}_{i-1} \) is the unique maximal semistable subsheaf of \( \mathcal{E}/\mathcal{E}_i \), for \( 1 \leq i \leq k \). In particular, the slope of the quotients are ordered such that

\[
(3b) \quad \mu(\mathcal{E}_1) > \mu(\mathcal{E}_2/\mathcal{E}_1) > \ldots > \mu(\mathcal{E}_k/\mathcal{E}_{k-1}).
\]

If \( \mathcal{E} \) is semistable, then there is a filtration by subsheaves (called the Seshadri filtration)

\[
(3c) \quad 0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \mathcal{E}_k = \mathcal{E},
\]

such that the quotients \( \mathcal{E}_i/\mathcal{E}_{i-1} \) are all stable bundles and have slope \( \mu(\mathcal{E}_i/\mathcal{E}_{i-1}) = \mu(E) \). Moreover,

\[
(3d) \quad \text{Gr}(\mathcal{E}) = \mathcal{E}_1 \oplus \mathcal{E}_2/\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}/\mathcal{E}_{k-1}
\]

is uniquely determined up to an isomorphism.

If \( X \) is a Riemann surface, then the terms in both of these filtrations are locally free, i.e. are subbundles of \( \mathcal{E} \).

Definition 3. Define \( \mu_1(\mathcal{E}) \) by

\[
\mu_1(\mathcal{E}) = \mu(\mathcal{E}_1),
\]

where \( \mathcal{E}_1 \subset \mathcal{E} \) is the first term in the Harder-Narasimhan filtration for \( E \).

Thus \( \mu_1(\mathcal{E}) \) is the least upper bound for the slopes of all subsheaves \( \mathcal{E}' \subset \mathcal{E} \). It follows that \( \inf(\mathcal{M}) \geq \mu_1(\mathcal{E}) \). Similarly, by using the correspondence between quotients of \( \mathcal{E} \) and subsheaves of the dual bundle \( \mathcal{E}^* \), one sees that \( \sup(\mathcal{M}') \leq -\mu_1(\mathcal{E}^*) \).

2. OVER A COMPACT RIEMANN SURFACE

In the case where the base manifold \( X \) is a closed Riemann surface, we can considerably strengthen the connection between the Hermitian-Einstein inequalities and the Harder-Narasimhan filtrations. To be precise:

Theorem 4. Let \( \mathcal{E} \rightarrow X \) be a holomorphic bundle over a closed Riemann surface, and let \( \mathcal{M}, \mathcal{M}' \) be as above. Then the sets \( \mathcal{M} \) and \( \mathcal{M}' \) are non-empty, and

1. \( \inf(\mathcal{M}) = \mu_1(\mathcal{E}) \),
2. \( \sup(\mathcal{M}') = -\mu_1(\mathcal{E}^*) \).

The proof of Theorem 4 follows from a more general result, namely:
**Theorem 5.** Let $X$ be a closed Riemann surface, and suppose that the holomorphic bundle $\mathcal{E} \to X$ has Harder-Narasimhan filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_k = \mathcal{E}.$$  

Let the ranks and slopes of $Q_i = \mathcal{E}_i/\mathcal{E}_{i-1}$ be given by $(r_i, \mu_i(\mathcal{E}))$, $i = 1, 2, \ldots, k$.

Then given any $\epsilon > 0$ there is a complex bundle automorphism, $g$ of $E$, and an Hermitian metric $H$ on $E$ such that

$$-\epsilon I \leq \frac{-1}{2\pi} \Lambda F_{g(\overline{\nabla}_E),H} - \begin{pmatrix} \mu_1 I_{r_1} & 0 & \cdots & 0 \\ 0 & \mu_2 I_{r_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \mu_k I_{r_k} \end{pmatrix} \leq \epsilon I$$

A solution corresponding to $\epsilon = 0$ can be found if the smooth decomposition $\mathcal{E} = \bigoplus_{i=1}^k \mathcal{E}_i/\mathcal{E}_{i-1}$ is a holomorphic decomposition, and each quotient $Q_i = \mathcal{E}_i/\mathcal{E}_{i-1}$ is a polystable bundle.

**Proof.** Using Seshadri filtrations, we can refine the filtration for $\mathcal{E}$ to get a filtration in which all quotients are stable bundles. We may thus assume that all the $Q_i$ are stable bundles, and hence admit Hermitian-Einstein metrics. Denote these by $K_i$, for $i = 1, 2, \ldots, k$. We thus get a metric, say $K$, on the graded object $Gr(\mathcal{E}) = \bigoplus Q_i$ for which

$$\frac{-1}{2\pi} \Lambda F_{g(\overline{\nabla}_E),K} = \begin{pmatrix} \mu_1 I_{r_1} & 0 & \cdots & 0 \\ 0 & \mu_2 I_{r_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \mu_k I_{r_k} \end{pmatrix},$$

where $\overline{\nabla}_E$ denotes the holomorphic structure on $Gr(\mathcal{E})$. We now show how to construct a bundle automorphism $g$ such that the curvature $\frac{-1}{2\pi} \Lambda F_{g(\overline{\nabla}_E),K}$ is arbitrarily close to $\frac{-1}{2\pi} \Lambda F_{g(\overline{\nabla}_E),K}$. Such arguments are well known, and can be found in [1] (cf. section 8), and also [5]. The basic idea can be illustrated in the simplest non-trivial case, i.e. the case where $\mathcal{E}$ is given as an extension of stable bundles, say

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0.$$  

In this case we can pick Hermitian-Einstein metrics, $K_i$, on the bundles $\mathcal{E}_i$. Thus we have

$$\frac{-1}{2\pi} \Lambda F_{g(\overline{\nabla}_E),K} = \begin{pmatrix} \mu_1 I_{r_1} & 0 \\ 0 & \mu_2 I_{r_2} \end{pmatrix},$$

where $\overline{\nabla}_E$ denotes the holomorphic structure on $\mathcal{E}_1 \oplus \mathcal{E}_2$, and $K = K_1 \oplus K_2$. Now with respect to the orthogonal splitting determined by $K$, the holomorphic structure on $\mathcal{E}$ is given by

$$\overline{\nabla}_E = \begin{pmatrix} \overline{\nabla}_1 & \beta \\ 0 & \overline{\nabla}_2 \end{pmatrix},$$

where $\overline{\nabla}_i$ gives the holomorphic structure on $\mathcal{E}_i$ and $\beta \in \Omega^{0,1}(Hom(\mathcal{E}_2, \mathcal{E}_1))$ is the second fundamental form of the inclusion $\mathcal{E}_1 \hookrightarrow \mathcal{E}$. A straightforward computation gives

$$\frac{-1}{2\pi} \Lambda F_{g(\overline{\nabla}_E),K} = \begin{pmatrix} \mu_1 I_{r_1} - \frac{-1}{2\pi} \Lambda \beta \wedge \beta^* \sqrt{-1} \Lambda d\beta \\ \sqrt{-1} \Lambda d\beta^* \sqrt{-1} \Lambda \beta \wedge \beta \end{pmatrix}.$$
where $d$ denotes covariant differentiation determined by the metric connections on $E_1$ and $E_2$.

Recall that the class $[\beta] \in H^1(Hom(E_2, E_1))$ determines the isomorphism class of (5) as an extension, while it is the corresponding point in $\mathbb{P}(H^1(Hom(E_2, E_1)))$ which gives the isomorphism class of the bundle $E$. Thus if we define

$$\overline{\sigma}_t = \left( \begin{array}{cc} \overline{\sigma}_1 & t\beta \\ 0 & \overline{\sigma}_2 \end{array} \right),$$

we get a 1-parameter family of extensions, all of which are isomorphic to $E$ as bundles. In fact, the holomorphic structures $\overline{\sigma}_t$ and $\overline{\sigma}_1 = \overline{\sigma}_E$ are related by the complex gauge transformation

$$g_t = \left( \begin{array}{cc} I_{r_1} & 0 \\ 0 & tI_{r_2} \end{array} \right).$$

If we pick $\beta$ to be the harmonic representative in its cohomology class, then we find

$$\sqrt{-1} \Lambda F_{\overline{\sigma}_t, K} - \left( \begin{array}{ccc} \mu_1(1)I_{r_1} & 0 & \ldots & 0 \\ 0 & \mu_2(1)I_{r_2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \mu_k(1)I_{r_k} \end{array} \right) \leq \epsilon I,$$

where the $r_i$'s and $\mu_i(1)$'s are the ranks and degrees of the quotients in the filtration for $E_1$. By exactly the same argument as above, we see that we can find a 1-parameter family of complex bundle automorphism, $g_t$, such that

$$\sqrt{-1} \Lambda F_{\overline{\sigma}_t, K} - \left( \begin{array}{cc} \mu_1(1)I_{r_1} & 0 \\ 0 & \mu_2(1)I_{r_2} \\ \ldots & \ldots & \ldots & \mu_k(1)I_{r_k} \end{array} \right) = t^2 \left( \begin{array}{ccc} -\sqrt{-1} \Lambda \beta \wedge \beta^* & 0 \\ 0 & -\sqrt{-1} \Lambda \beta^* \wedge \beta \end{array} \right).$$

Combining this with (6) shows that the theorem then applies to $E$. We may thus apply this method, one step at a time, to the filtration for $E$. Since the gauge transformation at stage $j$ leaves unchanged the filtration up to $E_j$, the composition of all the gauge transformations produces the required result to prove the theorem. □

**Remark.** Another way to obtain this result is by considering the gradient flow for the Yang-Mills functional on $C$, the space of holomorphic structure for $E$. This goes back to the work of [1], with some important analytic details being supplied by [4]. Using a fixed background metric, $K$, we can define the functional $f : C \rightarrow \mathbb{R}$ by

$$f(\overline{\sigma}_E) = \|\sqrt{-1} \Lambda F_{\overline{\sigma}_E, K} - \mu(E)I\|_{2, \beta}^2.$$
Via the identification of $\mathcal{C}$ and the space of unitary connections, this is essentially the same as the Yang-Mills functional $\mathcal{M}(D) = ||F_D||_{L^2}^2$. The critical points correspond to reducible holomorphic structures of the form $E = \bigoplus E_i$, where each summand is stable. The results of [1] (especially section 8) and [4] show that under the gradient flow, each holomorphic bundle $E$ converges to a critical point corresponding precisely to the graded object $Gr(E)$. One then uses the fact that the gradient flow preserves the isomorphism class of $E$, and that at the critical points $\frac{\sqrt{-1}}{2\pi} \Lambda F_{\bar{\partial}_{E,K}}$ has the required diagonal form.

Proof of Theorem 4. Part (1) : Recall that $\mu_1(E) = \mu_1 > \mu_2 > \cdots > \mu_k$. Thus

$$
\begin{pmatrix}
\mu_1 I_{n_1} & 0 & \cdots & 0 \\
0 & \mu_2 I_{n_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \mu_k I_{n_k}
\end{pmatrix} \leq \mu_1(E) I.
$$

It then follows from Theorem 5 that for any $\epsilon > 0$ there is a complex bundle automorphism, $g$ of $E$, and an Hermitian metric $H$ on $E$ such that

$$
\frac{\sqrt{-1}}{2\pi} \Lambda F_{\bar{\partial}_{E,H}} \leq (\mu_1(E) + \epsilon) I.
$$

Instead of using $g$ to transform the $\bar{\partial}$-operator, we can use it to change the metric on $E$. If the new metric is $H_g = Hg^* g$, we have the relation

$$
\frac{\sqrt{-1}}{2\pi} \Lambda F_{\bar{\partial}_{E,H}} = g \circ \frac{\sqrt{-1}}{2\pi} \Lambda F_{\bar{\partial}_{E,H_g}} \circ g^{-1}.
$$

It follows then from (4) that

$$
\frac{\sqrt{-1}}{2\pi} \Lambda F_{\bar{\partial}_{E,H_g}} \leq (\mu_1(E) + \epsilon) I.
$$

Together with the observation that $\inf(\mathcal{M}) \geq \mu_1(E)$, this proves part (1)

Part (2) : Given a connection $D$ on $E$, there is a corresponding induced connection $D^*$ on $E^*$. Furthermore, if $H^*$ is the Hermitian metric on $E^*$ dual to the metric $H$ on $E$, then the metric connection $D_{H^*}$ on $E^*$ is precisely the connection corresponding to the metric connection $D_H$ on $E$. Suppose that with respect to a local holomorphic frame for $E$, the curvature of $D_H$ is $F_H$, and that with respect to the dual holomorphic frame for $E^*$ the curvature of $D_{H^*}$ is $F_{H^*}$. Then

$$
i \Lambda F_{H^*} = -i \Lambda F^t_H,$$

where $F^t_H$ denotes the transpose of $F_H$. It follows that

$$i \Lambda F_H \geq m'I \iff -i \Lambda F_{H^*} \geq m'I \iff i \Lambda F_{H^*} \leq -m'I$$

The result now follows from part (1). \qed
3. Over higher dimensional Kähler manifolds

For bundles over closed Kähler manifolds of dimension greater than two, the filtrations of holomorphic bundles are by subsheaves, rather than subbundles. The above arguments thus do not apply. For related reasons, the proof of Theorem 4 based on the Morse theory of the Yang-Mills functional also fails, with the method breaking down because of the failure of the Yang-Mills gradient flow to converge. Such failure to converge is caused by the “bubbling” phenomenon on the space of connections.

In the case where \( X \) is not a Riemann surface, some information about \( \inf(M) \) and \( \sup(M') \) can be obtained by relating the Hermitian-Einstein inequality to the equality given by the \( \tau \)-Vortex equation. This is an equation for a metric on a holomorphic bundle with a prescribed holomorphic section, and has the form

\[
\frac{\sqrt{-1}}{2\pi} \Lambda F_{H,\partial E} + \frac{1}{2\pi} \phi \otimes \phi^* = \tau I.
\]

Here \( \tau \) is a real parameter which must lie in the range \( (\mu(E_1), \frac{R-1}{R} \mu(E)) \) (cf[2]). Since the bundle endomorphism \( -\phi \otimes \phi^* \) is non-positive, it is apparent that

**Lemma 7.** Let \( E \to X \) be a holomorphic bundle over a closed Kähler manifold of dimension \( n \geq 1 \). Fix the parameter \( \tau \). Then there is a solution to the equation

\[
\frac{\sqrt{-1}}{2\pi} \Lambda F_H \leq \tau I
\]

if for some choice of section \( \phi \in H^0(X, E) \) there is a solution to the \( \tau \)-Vortex equation

\[
\frac{\sqrt{-1}}{2\pi} \Lambda F_{\partial E, H} + \frac{1}{2\pi} \phi \otimes \phi^* = \tau I.
\]

For a given pair \( (E, \phi) \), there is a necessary and sufficient condition for the \( \tau \)-vortex equation to have a solution. This is the \( \tau \)-stability condition for a holomorphic pair. We recall the result from [2]

**Definition 8.** Given a real number \( \tau \), we say that the pair \( (E, \phi) \) is \( \tau \)-stable (resp. \( \tau \)-semistable) if the following two conditions hold:

1. \( \mu(E') < \tau \) (resp. \( \leq \tau \)), for every holomorphic subbundle \( E' \subset E \);
2. \( \mu(E/E_\phi) > \tau \) (resp. \( \geq \tau \)), for every proper holomorphic subbundle \( E_\phi \subset E \) such that \( \phi \) is a section of \( E_\phi \).

**Theorem 9.** [2] Suppose that \( (E, \phi) \) is \( \tau \)-stable for a given value of the parameter \( \tau \). Then the \( \tau \)-Vortex equation

\[
\frac{\sqrt{-1}}{2\pi} \Lambda F_{\partial E, H} + \frac{1}{2\pi} \phi \otimes \phi^* = \tau I
\]

considered as an equation for the metric \( H \), has a unique smooth solution.

Conversely, suppose that for a given value of \( \tau \) there is a Hermitian metric \( H \) on \( E \) such that the \( \tau \)-vortex equation is satisfied on \( (E, \phi) \). Then \( E \) splits holomorphically as \( E = E_\phi \oplus E_s \), where

1. \( E_s \), if not empty, is a direct sum of stable bundles, each of slope \( \tau \cdot \frac{\text{Vol}(X)}{4\pi} \);
2. \( E_s \) contains the section \( \phi \) and \( (E_s, \phi) \) is \( \tau \)-stable.
Notice that the split case \( \mathcal{E} = \mathcal{E}_\phi \oplus \mathcal{E}_s \) cannot occur unless \( \tau \cdot \frac{Vol(X)}{4\pi} \) corresponds to the slope of a subbundle, i.e. unless \( \tau \cdot \frac{Vol(X)}{4\pi} \) is a rational number with denominator less than the rank of \( \mathcal{E} \). Hence, for generic values of \( \tau \) the summand \( \mathcal{E}_s \) is empty, and \( \tau \)-stability is the necessary and sufficient condition for the existence of solutions to the \( \tau \)-vortex equation.

For our present purposes it is convenient to define the following parameter.

**Definition 10.** Given a holomorphic pair \( (\mathcal{E}, \phi) \), let

\[
inf(\mathcal{E}, \phi) = \text{Min}\{\mu(\mathcal{E}/\mathcal{E}_\phi) : \mathcal{E}_\phi \subset \mathcal{E}, \phi \in H^0(X, \mathcal{E}_\phi)\}.
\]

We then have the result

**Proposition 11.** Let \( (\mathcal{E}, \phi) \) be a holomorphic pair. Let \( \mu_1(\mathcal{E}) \) be the slope of the first subbundle in the Harder-Narasimhan filtration for \( \mathcal{E} \). Then the pair is \( \tau \)-stable for some value of \( \tau \) if and only

\[
\mu_1(\mathcal{E}) < inf(\mathcal{E}, \phi).
\]

In that case, \( \tau \) lies in the interval \((\mu_1(\mathcal{E}), inf(\mathcal{E}, \phi))\).

This gives us the corollary

**Corollary 12.** Let \( \mathcal{E} \to X \) be a holomorphic bundle over a closed Kähler manifold of dimension \( n \geq 1 \).

1. If there is a section \( \phi \in H^0(X, \mathcal{E}) \) such that \( \mu_1(\mathcal{E}) < inf(\mathcal{E}, \phi) \), then for all \( m > \mu_1(\mathcal{E}) \), there is a solution to

\[
\sqrt{-1} \Lambda F_{\sigma_{E,H}} \leq m \mathbf{I}.
\]

2. If there is a section \( \phi^* \in H^0(X, \mathcal{E}^*) \) such that \( \mu_1(\mathcal{E}) < inf(\mathcal{E}^*, \phi^*) \), then for all \( m > \mu_1(\mathcal{E}^*) \), there is a solution to

\[
\sqrt{-1} \Lambda F_{\sigma_{E,H}} \geq -m \mathbf{I}.
\]

**Proof.**

1. Given a section \( \phi \in H^0(X, \mathcal{E}) \) such that \( \mu_1 < inf(\mathcal{E}, \phi) \), the pair \((\mathcal{E}, \phi)\) is \( \tau \)-stable for any \( \mu_1 < \tau < inf(\mathcal{E}, \phi) \). The result thus follows from Lemma 9.

2. Replace \( \mathcal{E} \) by \( \mathcal{E}^* \) in the proof of part (1) \( \square \)

This result can be rephrased in an interesting way by using the interpretation in [7], [8] of the vortex equations as a dimensional reduction of the Hermitian-Einstein equations. As shown in [8], a holomorphic pair \( (\mathcal{E}, \phi) \) over \( X \) can be identified with a holomorphic extension over \( X \times \mathbb{P}^1 \) of the form

\[
0 \to p^* \mathcal{E} \to \mathcal{E} \to q^* \mathcal{O}(2) \to 0.
\]

Here \( p, q \) are the projections from \( X \times \mathbb{P}^1 \) onto \( X \) and \( \mathbb{P}^1 \) respectively, and \( \mathcal{O}(2) \) is the degree two line bundle over \( \mathbb{P}^1 \). The extension class of \( \mathcal{E} \) is related to a section \( \phi \in H^0(X, \mathcal{E}) \) by the isomorphism

\[
H^1(X \times \mathbb{P}^1, p^* \mathcal{E} \otimes q^* \mathcal{O}(-2)) \cong (H^0(X, \mathcal{E}) \otimes H^1(\mathbb{P}^1, \mathcal{O}(-2)))
\]

\[
\cong H^0(X, \mathcal{E}).
\]
To define stability or the Hermitian-Einstein equations on $E$, we need to fix a Kähler metric on $X \times \mathbb{P}^1$. We consider the 1-parameter family of such metrics corresponding to the Kähler forms

$$\Omega_\sigma = p^* \omega + \sigma q^* \omega_{\mathbb{P}^1}. $$

Here $\omega$ and $\omega_{\mathbb{P}^1}$ are the Kähler forms on $X$ and $\mathbb{P}^1$ respectively, and $\sigma$ is a positive real parameter. Notice that there is a natural $SU(2)$ action on $X \times \mathbb{P}^1$ (which is trivial on $X$). The Kähler forms $\Omega_\sigma$ are invariant under this action, and there is a natural lift of the action to $E$. Garcia-Prada has shown

**Theorem 13** [7]. Let $\sigma$ and $\tau$ be related by

$$\sigma = \frac{2Vol(X)}{(\text{rank}(E) + 1)\tau - \deg(E)}. $$

Under the above correspondence between holomorphic pairs on $X$ and holomorphic extensions on $X \times \mathbb{P}^1$, the following are equivalent:

1. There is a metric on $E$ satisfying the $\tau$-vortex equation,
2. There is an $SU(2)$-invariant metric on $E$ satisfying the Hermitian-Einstein equation with respect to $\Omega_\sigma$.

As a result of Theorem 9, and the correspondence between stability and Hermitian-Einstein metrics (cf. [10]), this leads to the following

**Theorem 14** [7]. Let $\sigma$ and $\tau$ be related as in Theorem 13. Under the above correspondence between holomorphic pairs on $X$ and holomorphic extensions on $X \times \mathbb{P}^1$, the following are equivalent:

1. The pair $(E, \phi)$ is $\tau$-stable,
2. The extension $E$ is stable with respect to $\Omega_\sigma$.

Corollary 12 (1) can thus be rephrased as

**Corollary 15.** Let $E \to X$ be a holomorphic bundle over a closed Kähler manifold of dimension $n \geq 1$. Suppose that there is a choice of $\sigma$ and an extension

$$0 \to p^* E \to E \to q^* \mathcal{O}(2) \to 0$$

such that $E$ is stable with respect to $\Omega_\sigma$. Then for all $m > \mu_1(E)$, there is a metric on $E$ satisfying $\sqrt{-1} \Lambda F_{E,H} \leq m \mathbf{1}$.

**Proof.** If $E$ is stable with respect to $\Omega_\sigma$, then the corresponding pair $(E, \phi)$ is $\tau$-stable, where $\tau$ and $\sigma$ are related as in (6). The result now follows as before. □

A similar rephrasing of Corollary 12(2) is also possible.

In the special case of rank two bundles, we can be even more explicit. We can use the fact that any section $\phi \in H^0(X, E)$ generates a rank one subsheaf with torsion free quotient, and that this is the only proper subsheaf which contains the section and has torsion free quotient. Denoting this subsheaf by $[\phi]$, we thus get that

$$\inf f(E, \phi) = \mu(E/[\phi]) = 2\mu(E) - \mu([\phi]). $$

The holomorphic pair $(E, \phi)$ will then be $\tau$-stable for some value of $\tau$ if and only if $\mu([\phi]) \neq \mu(E)$, i.e., $\mu([\phi]) < \mu(E)$. Combining this with Corollary 13, we get

$$\inf f(E, \phi) = 2\mu(E) - \mu([\phi]). $$
Corollary 16. Let $\mathcal{E} \to X$ be a rank 2 holomorphic bundle over a closed Kähler manifold of dimension $n \geq 1$.

1. Suppose that there is a section $\phi \in H^0(X, \mathcal{E})$ such that $\deg([\phi]) < \mu_1(\mathcal{E})$. Then for all $m > \mu_1(\mathcal{E})$, there is a solution to

$$\sqrt{-1} \Lambda F_{\mathcal{E}, H} \leq m \mathbf{1}.$$

2. Suppose that there is a section $\phi^* \in H^0(X, \mathcal{E}^*)$ such that $\deg([\phi^*]) < \mu_1(\mathcal{E}^*)$. Then for all $m > \mu_1(\mathcal{E}^*)$, there is a solution to

$$\sqrt{-1} \Lambda F_{\mathcal{E}, H} \geq -m \mathbf{1}.$$

Remark. The vortex equations and the definition of stable pairs can be generalized in a way which conveniently takes into account the duality between the two cases covered by Theorem 1. The new equations are coupled equations for metrics on two bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ over $X$, and have the form

$$\frac{\sqrt{1}}{2\pi} \Lambda F_{\mathcal{E}_1, H_1} + \frac{1}{2\pi} \Phi \otimes \Phi^* = \tau \mathbf{1},$$

$$\frac{\sqrt{1}}{2\pi} \Lambda F_{\mathcal{E}_2, H_2} - \frac{1}{2\pi} \Phi^* \otimes \Phi = \tau' \mathbf{1}.$$

The section $\Phi$ is now a section of $H^0(X, \text{Hom}(\mathcal{E}_2, \mathcal{E}_1))$, and the constants $\tau$ and $\tau'$ are related by the constraint

$$r_1 \tau + r_2 \tau' = \deg(\mathcal{E}_1) + \deg(\mathcal{E}_2),$$

where $r_1$ is the rank of $\mathcal{E}_1$ etc.

These equations where introduced in [7]. It is clear that the solutions to these coupled vortex equations provide solutions to the inequalities $\sqrt{-1} \Lambda F_{\mathcal{E}_1, H_1} \leq \tau \mathbf{1}$ on $\mathcal{E}_1$, and $\sqrt{-1} \Lambda F_{\mathcal{E}_2, H_2} \geq \tau' \mathbf{1}$ on $\mathcal{E}_2$. In [7] and [3] it is shown that the existence of such solutions is related to a stability criterion (also called $\tau$-stability) for the triple $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$. The definition of $\tau$-stability is a slope condition on subtriples, i.e. on $(\mathcal{E}'_1, \mathcal{E}'_2, \Phi')$ where for $i = 1, 2$ $\mathcal{E}'_i$ is a rank $r'_i$ subsheaf of $\mathcal{E}_i$ and $\Phi' \in H^0(X, \text{Hom}(\mathcal{E}'_2, \mathcal{E}'_1))$ is such that the obvious diagram commutes (cf. [7]). For all such subtriples let

$$\theta_\tau(\mathcal{E}'_1, \mathcal{E}'_2) = (\mu(\mathcal{E}'_1 \oplus \mathcal{E}'_2) - \tau) - \frac{r'_2 r_1 + r_2}{r'_1 + r'_2} (\mu(\mathcal{E}_1 \oplus \mathcal{E}_2) - \tau),$$

and define the triple to be $\tau$-stable if $\theta_\tau(\mathcal{E}'_1, \mathcal{E}'_2) < 0$ for all proper subtriples. Then (cf. [3]) $\tau$-stability implies the existence of metrics on $\mathcal{E}_1$ and $\mathcal{E}_2$ satisfying the coupled vortex equations, and the following observation can then be made:

Proposition 14. Let $\mathcal{E} \to X$ be a holomorphic bundle over a closed Kähler manifold of dimension $n \geq 1$.

1. Suppose there is a bundle $\mathcal{F} \to X$, a holomorphic section $\Phi \in H^0(X, \mathcal{E} \otimes \mathcal{F}^*)$, and a real number $\tau$ such that $(\mathcal{E}, \mathcal{F}, \Phi)$ is a $\tau$-stable triple. Then there is a solution to the inequality $\sqrt{-1} \Lambda F_{\mathcal{E}, H} \leq \tau \mathbf{1}$ on $\mathcal{E}$.

2. Suppose there is a bundle $\mathcal{F} \to X$, a holomorphic section $\Phi \in H^0(X, \mathcal{F} \otimes \mathcal{E}^*)$, and a real number $\tau$ such that $(\mathcal{F}, \mathcal{E}, \Phi)$ is a $\tau$-stable triple. Then there is a solution to the inequality $\sqrt{-1} \Lambda F_{\mathcal{E}, H} \geq \tau' \mathbf{1}$ on $\mathcal{E}$, where $\tau$ and $\tau'$ are related as above.
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