Model theory of the field of $p$-adic numbers expanded by a multiplicative subgroup.

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Abstract

Let $G$ be a multiplicative subgroup of $\mathbb{Q}_p$. In this paper, we describe the theory of the pair $(\mathbb{Q}_p, G)$ under the condition that $G$ satisfies Mann property and is small as subset of a first-order structure. First, we give an axiomatisation of the first-order theory of this structure. This includes an axiomatisation of the theory of the group $G$ as valued group (with the valuation induced on $G$ by the $p$-adic valuation). If the subgroups $G[n]$ of $G$ have finite index for all $n$, we describe the definable sets in this theory and prove that it is NIP. Finally, we extend some of our results to the subanalytic setting.

Let $\mathcal{M} = (M, \cdots)$ be a $\mathcal{L}$-structure and $A$ be a subset of $M$. In various context, people have studied the theory of the pair $(\mathcal{M}, A)$ (in the language $\mathcal{L}$ expanded by a unary predicate interpreted by $A$). This problem has been discussed in a pure abstract setting e.g. [2] or in particular cases of $\mathcal{M}$ and $A$. Well-known examples are the pairs of fields: $\mathcal{M}$ is a field and $A$ is an elementary substructure e.g. [20]. An other example that has been studied by many authors is when $K$ is a field and $A$ is a multiplicative subgroup of $K^*$. The first instances of this problem are L. van den Dries [22] where he works on the theory of the pair $({\mathbb{R}}, {2^Z})$ or B. Zilber [24] where the pair $({\mathbb{C}}, U)$ is considered ($U$ is the group of roots of unity). Both these results have been generalised to the case where $K$ is a real or algebraically closed field in [9] and $A$ is a small multiplicative subgroup.

In this paper, we consider the same problem where $K$ is now the field of $p$-adic numbers $\mathbb{Q}_p$. A special case of this problem is presented in [17] ($G$ is $n^Z$ with $n \in \mathbb{N}$). The aim of this paper is first to generalise the results proved there (working with a general subgroup $G$) and second to discuss classical problems that were not considered previously (extension to subanalytic setting and NIPness of the theory). In section 2 we axiomatise the theory of the pair $(\mathbb{Q}_p, G)$. We will see in this section that $G$ can be reduced to two main parts (definably in a suitable language): a discrete cyclic group and a subgroup $D$ that is dense in an open subgroup $1 + p^n\mathbb{Z}_p$. On the discrete group, the $p$-adic valuation induces a structure of ordered group. This part is a $p$-adic equivalent to $\mathbb{Q}_p$. If we add a function symbol $\lambda$ interpreted by a function that sends an element of the field to an element of the group with same valuation, we get quantifier elimination. On the dense part $D$, the valuation induces a structure of valued group. The theory of this valued group is part of the theory of the pair

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structure of exponential field on $\mathbb{Q}$

In section 1 we will study the model theory of $D$ as valued group. We will axiomatise the theory of this group, prove that its theory admits quantifier elimination in a natural language and that it is NIP.

The axiomatisation of these two cases is part of the axiomatisation of the theory $(\mathbb{Q}_p, G)$. Two extra properties are required in order to obtain a complete theory. First, we will assume that $G$ has Mann property (this gives us control on the field $\mathbb{Q}(G)$). This property was used in [24] and it is quite natural to use it in the same way in our context. Finally we will assume that $G$ is small in some sense (this is the case if $G$ is finitely generated for instance).

In section 2 we describe the definable sets in models of our theory assuming that $G^{[n]} := \{g^n : g \in G\}$ has finite index in $G$ for all $n$. We also describe the subsets of $G$ that are definable in $(\mathbb{Q}_p, G)$ in the case where the discrete part of $G$ is trivial.

In section 3, we describe the definable sets in models of our theory assuming that $G$ is a discrete group (e.g. $G = \hat{\mathbb{Z}}$). In that case, we prove that the theory of $(\mathbb{Q}_p, G)$ admits the elimination of quantifiers in the language $\mathcal{L}_{an}^G(\lambda, A)$ where $\lambda$ is as defined before.

We apply this result to the study of the $p$-adic Iwasawa logarithm. Let us recall that the logarithm defined by the usual power series $\log_p(1+x) = \sum_{n>0} (-1)^{n+1} x^n$ is convergent in $\mathbb{Q}_p$ only for the elements with positive valuation. On the other hand, as $\mathbb{Q}_p^* = p^\mathbb{Z} \times \mu_{p-1} \times (1 + p\mathbb{Z}_p)$ (where $\mu_{p-1}$ is the group of roots of unity of order $p - 1$), one can define a morphism of groups

$$\log : (\mathbb{Q}_p^*, \cdot) \to (\mathbb{Q}_p, +)$$

that extends $\log_p(1+x)$. Furthermore, this map is unique up to the choice of the value of $p$. Set $\log(p) = 0$. Then for all $x \in \mathbb{Q}_p$, $x = p^n \xi y \in \mathbb{Q}_p$ for some unique $n \in \mathbb{Z}$, $\xi \in \mu_{p-1}$ and $y \in 1 + p\mathbb{Z}_p$. So, because $\log$ is a morphism of groups,

$$\log(x) = n\log(p) + \log(\xi) + \log(y) = \log_p(y).$$

This logarithm map is called the $p$-adic Iwasawa logarithm. The expansion $(\mathbb{R}, e^{xp})$ of $\mathbb{R}$ has been extensively studied by various authors. In the $p$-adic context, the usual exponential is convergent only on $p\mathbb{Z}_p$. There is no natural structure of exponential field on $\mathbb{Q}_p$. On the other hand the $p$-adic Iwasawa logarithm induces a structure of logarithmic field. This structure is canonical up the the choice of $\log(p)$ if we require the logarithm to be analytic on $1 + p\mathbb{Z}_p$.

In [6] and many papers afterwards, the theory of $\mathbb{Q}_p$ with restricted analytic structure has been studied. On the other hand, there is no proper expansion of $\mathbb{Q}_p$ by a global function that has been studied in the context of model theory.

The expansion by the Iwasawa Logarithm is an example of such a structure. As an application of our result on $(\mathbb{Q}_p, G)$, we obtain that the theory of $(\mathbb{Q}_p, \log)$ admits the elimination of quantifier in the language $\mathcal{L}_{an}^G(\lambda, A, \xi)$. In section 5 we prove that in some (countable) reduct, this theory is model-complete using techniques from [16].

In the last section 6 we prove that the following theories are NIP: (1) $Th(\mathbb{Q}_p, p^\mathbb{Z})$ in the language $\mathcal{L}_{an}^G(\lambda, \xi)$ and (2) $Th(\mathbb{Q}_p, G)$ with $G$ a dense subgroup of $1 + p\mathbb{Z}_p$. This is an application of the abstract setting in [2] where it is proved that the theory of $Th(M, A)$ is NIP whenever it admits elimination of quantifiers up to bounded formulas, $Th(M)$ is NIP and the theory of the structure induced on $A$ is NIP. In our case, the first and last points are essentially...
proved in section 3 and the second hypothesis is well-known. As consequence of (1), \( Th(\mathbb{Q}_p, \text{LOG}) \) is NIP.

**Notation.** Let \( K \) be a valued, we will denote by \( v_K \) or \( v \) its valuation, by \( O_K \) its valuation ring, by \( vK \) its value group, by \( res \) \( K \) its residue field and by \( res \) the residue map. The \( p \)-adic valuation will be denoted \( v_p \). Let \( K^h \) denote the henselisation of \( K \). Let \( A \) be a ring, we denote by \( A^\times \) the set of nonzero elements and by \( A^\times \) the set of units. We will denote by \( L_{Mac} \) the language of \( p \)-adically closed fields \((+, -, 0, 1, P_n(n \in \mathbb{N}))\) where \( P_n \) is interpreted in \( \mathbb{Q}_p \) by the set of \( n \)-th powers. If \( G \) is a group, \( G_{tor} \) denotes its torsion part.

1 Elementary properties of abelian \( p \)-valued groups

Let \( G \) be a subgroup of \((\mathbb{Z}_p, +, 0)\). Then, \( v_p \) induces a map on \( G \) such that \((G, v_p)\) is a valued group. In this section, we study the theory of the structure \((G, v_p)\). In a first time, we axiomatise the theory of this group and prove a result of quantifier elimination. At the end of this section, we will also prove that \( Th(G, v_p) \) is NIP if the index \([G : nG]\) is finite for all \( n \). The special case \( G = \mathbb{Z} \) has been discussed in [3][17].

**Definition 1.1.** Let \((G, +, 0_G)\) be an abelian group and \( V : G \rightarrow \Gamma \cup \{\infty\} \) where \( \Gamma \) is a totally ordered set with discrete order and no largest element and \( \infty \) is an element such that \( \infty > \gamma \) for all \( \gamma \in \Gamma \). We say that \((G, V)\) is a \( p \)-valued group if for all \( x, y \in G \) and for all \( n \in \mathbb{Z} \),

- \( V(x) = \infty \) iff \( x = 0_G \);
- \( V(nx) = V(x) + v_p(n) \);
- \( V(x + y) \geq \min\{V(x), V(y)\} \);

where \( v_p \) is the \( p \)-adic valuation, \( nx = x + \cdots + x \) \((n \text{ times})\), \( (-n)x = -(nx) \) for all \( n > 0, 0x = 0_G \) and if \( x \in G \), \( V(x) + k \) denotes the \( k \text{th successor of } V(x) \) in \( \Gamma \cup \{\infty\} \) (by convention the successor of \( \infty \) is \( \infty \)).

**Remark.** The above axioms implies that if \( V(x) \neq V(y) \) then \( V(x + y) = \min\{V(x), V(y)\} \).

Let \( G \) be an abelian group. We denote by \([n]G\) the index of \( nG \) in \( G \). If it is not finite, we set \([n]G = \infty \) with no distinction between cardinalities. Let \( L_{pV} \) be the two-sorted language \( \{(+, -, 0, 1, \equiv_n (n \in \mathbb{N}), c_{ij}), (\prec, S_0r, \infty), V\} \) where the first sort corresponds to the group \( G \) and the second to the ordered set \( V(G) \). \( c_{ij} \) is a collection of constant symbols interpreted by representatives for the cosets of \( iG \). We fix \((q_n, n \in \mathbb{N})\) an enumeration of the prime numbers. Let \( N = (t_1, t_2, \cdots) \) be a sequence in \( \mathbb{N} \cup \{\infty\} \) where the index \( t_n \) associated to \( p \) is nonzero (we will assume \( n = 1 \)). By convention, we set \( q_{0_n}^\infty := \infty \). Let \( T_{pV, N} \) be the \( L_{pV} \)-theory axiomatised as follows: Let \((G, +, -, 0, 1, \equiv_n (n \in \mathbb{N}), c_{ij}(0 \leq j \leq q_i^1)), (V G \cup \{\infty\}, \prec, S_0r, \infty), V\) be a model of \( T_{pV, N} \) then

- \((G, +, -, 0)\) is an abelian group, \( x \equiv_y y \) iff \( \exists g \in G, x = y + ng \) and \([q_n]G = q_n^{\infty}, c_{ij} \neq c_{ik} \) for all \( i \) and for all \( k \neq j, c_{i0} = 0 \);
Remark. 1. If \([n]G \neq 1\), any nonzero element has a predecessor and there is no last element, \(\infty\) is an element such that \(\gamma < \infty\) for all \(\gamma \in VG\) and \(S(\infty) := \infty\);

2. \((G, V)\) is a \(p\)-valued group;

3. For all \(x, y \in G\), if \(V(x) = V(y)\), then there is a unique \(0 < i < p\) such that \(V(x - iy) > V(x)\);

4. \(G\) is regularly dense i.e. for all \(n, nG\) is dense in \(\{x \in G \mid V(x) \geq v_p(n)\}\) (where \(v_p(n)\) denotes the \(v_p(n)\)th successor of \(0_T\) in \(VG\)) i.e. for all \(n\)

\[\forall x \in G \ V(x) \geq v_p(n) \rightarrow \left[\forall \gamma \geq v_p(n) \in VG \exists y \in nG \ V(x - y) \geq \gamma\right];\]

5. \(G\) is equidistributed i.e. for all \(n\) such that \([n]G\) is infinite, for all \(\gamma \in VG\), for all \(c \in G\), the set \(\{x \mid V(x - c) \geq \gamma\}\) contains representatives for infinitely many classes of \([n]G\).

Remark. 1. If \([n]G = k\), then it is written as

\[\Psi_{n,k} \equiv \exists x_1, \ldots, x_k \in G \bigcap_{i \neq j} x_i \not\equiv_n x_j \land \forall y \in G \bigcup_{i=1}^k y \equiv_n x_i.\]

If \([n]G = \infty\), then it is written as the scheme of formulas \(\Phi_{n,k}\) (for all \(k \in \mathbb{N}\)) where

\[\Phi_{n,k} \equiv \exists x_1, \ldots, x_k \in G \bigcap_{i \neq j} x_i \not\equiv_n x_j.\]

Similarly, equidistribution is described by the collection of the following axioms: for all \(n\) such that \([n]G = \infty\) and for all \(m\)

\[\varphi_{n,m} \equiv \forall \gamma \in VG \ \forall c \in G \exists x_1, \ldots, x_m \bigcap_i V(x_i - c) \geq \gamma \bigcap_{i \neq j} x_i \not\equiv_n x_j.\]

2. \(t_n = 1\) for all \(n\), then \(T_{pV,N}\) is the theory of \(p\)-valued \(\mathbb{Z}\)-group that was studied in [17] and [3].

3. The last axiom is true for subgroups of \(\mathbb{Z}_p\) by the pigeon-hole principle and by compactness of \(\mathbb{Z}_p\). Note that if \(G\) is a subgroup of \(\mathbb{Z}_p\) and \(t_n\) is the value such that \([G : q_nG] = q_n^{t_n}\) then \((G, v_p) \equiv T_{pV,N}\).

4. If \(M'\) is a \(p\)-valued group, then it is torsion-free. In particular, any model of \(T_{pV,N}\) is torsion-free.

**Theorem 1.2.** The theory \(T_{pV,N}\) is consistent and admits the elimination of quantifiers.

First, let us prove the consistency of the theory. By the above remark it is sufficient to find a subgroup \(\mathbb{Z}_p\) with the right indices. Fix \(q\) a prime number, say \(q\) is the \(n\)th prime number in our list. We assume that \(t_n \geq 1\).
Let \( x_1, \ldots, x_{n-1} \) be elements of \( \mathbb{Z}_p \) algebraically independent over \( \mathbb{Q} \). Let 
\[
G_1 = \mathbb{Z} \oplus \bigoplus_{i \in I} x_i \mathbb{Z}(p, q) \subset \mathbb{Z}_p
\]
with 
\[
\mathbb{Z}(p, q) = \{ a/b \mid a, b \in \mathbb{Z} \text{ and } p, q \text{ does not divide } b \}.
\]
Then, for all \( q' \) prime not equal to \( p, q \), \([G_1 : q'G_1] = q'\). Indeed, any element of \( G_1 \) can be written as 
\[
x = n + \sum (a_i/b_i)x_i = n + q' \sum (a_i/b_i q')x_i
\]
where \( n \in \mathbb{Z}, n_i \in \mathbb{Z}(p, q) \) i.e. \( x \in \mathbb{Z} + q'\mathbb{Z}(p, q) \). So, \( 0, \ldots, q' - 1 \) form a set of representatives of the cosets of \( q'G_1 \) (by algebraic independence of the \( x_i \)'s).

On the other hand, \([G_1 : pG_1] = p^n\) and \([G_1 : qG_1] = q^n\). Indeed, the collection \( \{ n + \sum n_i x_i : 0 \leq n, n_i < q, n_i \in \mathbb{Z} \} \) forms a set of representatives of the cosets of \( qG_1 \). Similarly for the prime number \( p \).

In order to reduce \([p]G_1\), we define \( y_{ij} := \frac{1}{p} \sum_{k \geq 1} x_{ikp^k} \in \mathbb{Z}_p \) where \( x_i = \sum_{k \geq 0} x_{ikp^k} \) with \( x_{ij} \in \{0, \ldots, p - 1\} \). Note that \( y_{00} = x_i \). Let 
\[
G_2 := G_1 \bigoplus \bigoplus_{j \in \mathbb{N}_0} y_{ij} \mathbb{Z}(p, q),
\]
\[
= \{ n + \sum_{0 \leq i, j < R} n_{ij} y_{ij} \mid n \in \mathbb{Z}, n_{ij} \in \mathbb{Z}(p, q) \}.
\]
Then for all \( q' \) prime not equal to \( q \), \([G_2 : q'G_2] = q'\). Indeed, for \( q' \) not equal to \( p \), we argue like above. For \( q' = p \), we notice that \( y_{ij} = x_{ij} + py_{ij+1} \in \mathbb{Z} + pG_2 \). So, \( 0, \ldots, p - 1 \) form a set of representatives of the cosets of \( pG_2 \).

Finally, \([G_2 : qG_2] = q^n\). Let us show that \( \{ n_0 + \sum n_i x_i : 0 \leq n_k < q \} \subset G_1 \) forms a set of representatives of \( qG_2 \). For it is sufficient to prove that 
\[
n_0 + \sum n_i x_i \notin qG_2 \text{ (if nonzero) and that for all } i, j > 0, y_{ij} \in G_1 + qG_2.
\]
The first part is true as \( n_0 + \sum_{i \leq t_{n-1}} n_i x_i \in qG_2 \) iff 
\[
n_0 + \sum_{i \leq t_{n-1}} n_i x_i = qn' + \sum_{i, j \leq s} qn'_{ij} y_{ij}
\]
for some \( n' \in \mathbb{Z}, n'_{ij} \in \mathbb{Z}(p, q) \) and \( s \in \mathbb{N} \). The right hand side can be rewritten as 
\[
qn_0 + \sum qn_i x_i \text{ for some } n_k \in \mathbb{Z}(q) \text{ as } y_{ij} = (x_i - \sum_{k \leq j} x_{ikp^k})/p^j \in x_i \mathbb{Z}(q) + \mathbb{Z}(q).
\]
As the \( x_i \)'s are \( \mathbb{Q} \)-algebraically independent, \( n_0 + \sum n_i x_i = qn_0 + \sum qn_i x_i \) iff \( n_k = qn_k \) for all \( k \) iff \( n_k = 0 \) for all \( k \) (as \( 0 \leq n_k < q \)). For the last claim, let us remark that \( py_{ij} = r + x_i \) for some \( r \in \mathbb{Z} \). Let \( a, b \in \mathbb{Z} \) such that \( qaq + bpr = 1 \). Then, \( bp^j y_{ij} = br + bx_i \). So, \( y_{ij} = (aq + bp^j) y_{ij} = br + bx_i + qay_{ij} \in G_1 + qG_2 \).

Write \( G_2 \) as \( \mathbb{Z} \oplus G_2' \). We define similarly \( G_2'' \) for all \( q' = q_n \) prime number such that \( t_n \) is nonzero (we choose the \( x_i, q_n \)'s all distinct among a transcendence basis of \( \mathbb{Z}_p \) over \( \mathbb{Q} \); that is an uncountable set). Note that for \( q' = p \), it is actually sufficient to take \( G_2'_{2, p} := G_1_{p} \). Then,
\[
G := \mathbb{Z}(q_i : i \in I) \bigoplus \bigoplus_{(q_i, a \notin I)} G_2'_{2, q_i}.
\]
is a model of our theory where \( I = \{ i : t_i = 0 \} \) (where the valuation on \( G \) is induced by the \( p \)-adic valuation). The axioms of indices are satisfied by construction: by definition, \( qG_2''_{2, q} = G_2''_{2, q} \) if \( q \neq q' \) and as above \( \{ n_0 + \sum n_k x_{ikp^k} : 0 \leq n_0, n_k < q_n \} \) forms a set of representatives of the cosets of \( q_n G \). The other axioms are true as \( G \) is a subgroup of \( \mathbb{Z}_p \).
Definition 1.3. Let $M'$ be a model of $T_{PV,N}$ and $M$ be a pure subgroup of $M'$. Let $c \in M' \setminus M$. We define

$$M(c) = \{ x \in M' \mid nx = mc + y \text{ where } y \in M, m, n \in \mathbb{Z} \}.$$ 

It is not hard to see that $M(c)$ is a pure subgroup of $M'$.

Let $\square$ denote either $<$, $>$ or $\geq$.

Lemma 1.4. Let $M, M'$ be models of $T_{PV,N}$ and $M_0$ be a pure common subgroup of $M$ and $M'$. Let $\forall \pi \subset M_0, \pi, \overline{\pi} \subset \mathbb{Z}$ and $\pi \subset VM_0 \cup \{\infty\}$. Let

$$\alpha(X) \equiv \bigwedge_i \left\{ V(X - a_i) + n_i \square_i V(X - a_j) + m_j \right\}$$

where $\square_i := \square_i \in \{<, \leq \}$.

Then, $M \models \exists x \alpha(x)$ iff $M' \models \exists x \alpha(x)$.

Proof. Let $x \in M$ be a solution of $\alpha(X)$. Note that if the formula $V(X - a_i) \geq \infty$ occurs in $\alpha$ then $x = a_i$ so we are done. Furthermore, $V(X - a_i) \leq \infty$ is always true. So, we may assume that $\gamma_i \neq \infty$ for all $i$. If $x \in M_0$, we are done. Otherwise, we consider different cases:

(a) First, if $V(x - a_i) \in VM_0$ for all $i$, then we find that $\alpha(X)$ is realised in $M_0$. For we may assume that $V(x - a_1)$ is maximal among the $V(x - a_i)$'s. If $V(x) > V(a_1)$, $V(x - a_i) = V(a_i)$ for all $i$. So any $y \in M_0$ such that $V(y) > V(a_1)$ is solution of $\alpha(X)$. Otherwise, by the axioms of $p$-valued groups, there is $0 < t < p$ such that $V(x - ta_1) > V(x - a_1)$. Take $y = ta_1$. Then for all $i$, $V(y - a_i) = \min\{V(x-y), V(x-a_i)\} = V(x-a_i)$ i.e. $y$ is solution of $\alpha(X)$.

(b) There is $i$ such that $V(x - a_i) \notin VM_0$. Let $j \neq i$. Then, $V(x - a_i) = \min\{V(x-a_i), V(a_i - a_j)\}$ as $V(x - a_i) \neq V(a_i - a_j) \in VM_0$. Let $\tilde{\alpha}(X)$ be the system where we substitute in $\alpha(X)$ the variable $V(X-a_j)$ by $V(X-a_i)$ if $V(x-a_i) < V(a_i-a_j)$ and by $V(a_i-a_j)$ if $V(x-a_i) > V(a_i-a_j)$ and in this way we add the inequalities $V(x-a_i) < V(a_i-a_j), V(x-a_i) < V(a_i-a_j)$ according to the case where $j$ falls. Then, $y \in M'$ is such that $\tilde{\alpha}(y)$ then $V(y - a_j) = \min\{V(a_i - a_j), V(y - a_i)\}$. So, $M' \models \tilde{\alpha}(y)$.

Let $t := V(x - a_i)$. Let $\tilde{\alpha}_B(t)$ be the system of quantifier-free formulas in the language of discrete ordered sets obtained when we replace $V(X - a_i)$ by $t$. Let us recall that in our language, $\text{Th}(VM) = \text{Th}(VMP)$ admits the elimination of quantifiers. So, there is $t' \in VM'$ such that $M' \models \tilde{\alpha}(t')$. Let $y \in M'$ such that $V(y - a_i) = t'$. Then, $M' \models \tilde{\alpha}(y)$.

Lemma 1.5. Let $M'$ be a model of $T_{PV,N}$ and $M$ be a structure that is a pure subgroup of $M'$. Let $\pi, \overline{\pi} \subset \mathbb{Z}$ and $\pi, \overline{\pi} \subset \mathbb{Z}$, $\pi, \overline{\pi} \subset M$. Let

$$\beta(X) \equiv \bigwedge_i k_i X \equiv_n a_i \land \bigwedge_j l_j X \equiv_m b_j.$$ 

If $M' \models \exists X \beta(X)$ then $M \models \exists X \beta(X)$. Furthermore, if $c \in M$ is so that $M \models \beta(c)$, then there is $t \in \mathbb{N}$ such that for all $y \in M$ with $y \equiv t$, $M \models \beta(y)$.

Remark. For all $n$, $[n]M' = [n]M$. If $[n]M' < \infty$, this follows from the pureness assumption. If $[n]M' = \infty$, $[n]M = \infty$ as $M$ contains the constants $c_{nk}$ and by definition of our theory there are infinitely many $k \neq l$ such that $c_{nk} \neq n c_{nl}$.
Proof. Let $K = \prod k_i l_i$. Then, $k_i X \equiv_{n_i} a_i$ is equivalent to $K X \equiv_{n_i(K/k_i)} (K/k_i)a_i$; similarly for the incongruences. We proceed to a change of variables $Y = K X$ and we extend $\beta$ by $Y \equiv_K 0$. So, up to this transformation, we may assume $k_i = l_i = 1$. Next, if there is no incongruences, then by the following fact we are done.

**Fact 1.** $\bigwedge_i X \equiv_{n_i} a_i$ has a solution in $M'$ iff $\gcd(n_i,n_j)$ divides $(a_i - a_j)$. In this case the set of solution is of the form $X \equiv_n a$ for some $a \in M$ and $n = \text{lcm}(n_1, \cdots, n_k)$.

This is a classical result for system of congruences in $\mathbb{Z}$. Its proof can be easily adapted in our theory. So, $\beta$ is equivalent to

$$X \equiv_n a \land \bigwedge_j \neg X \equiv_{m_j} b_j$$

for some $n \in \mathbb{N}$ and $a \in M$. Next, we may assume that $[m_j]M = \infty$. For assume $[m_1]M$ finite. Then let $y$ be a solution of $\beta$ in $M'$. As $[m_1]M$ finite and $M$ pure in $M'$, there is $c \in M$ such that $c \equiv_{m_1} y$. It is sufficient to prove that the system $X \equiv_n a \land X \equiv_{m_j} c \land \bigwedge_{j \neq 1} \neg X \equiv_{m_j} b_j$ has a solution in $M$. We can repeat the argument for each $j$ such that $[m_j]M$ is finite then apply again the above fact.

We deal now with incongruences of subgroups with infinite index. Fix $j$. By the last fact, $X \equiv_n a \land X \equiv_{m_j} b_j$ is equivalent to $X \equiv_{\text{lcm}(m_j,n)} b_j'$ for some $b_j' \in M$. So, $X \equiv_n a \land \neg X \equiv_{m_j} b_j$ is equivalent to $X \equiv_n a \land \neg X \equiv_{\text{lcm}(m_j,n)} b_j'$.

So, we may assume that our system is of the type

$$X \equiv_n a \land \bigwedge_j \neg X \equiv_{m_j} b_j \quad (\beta')$$

where $n$ divides $m_j$ and $[nM : m_j M] = \infty$.

The lemma is now a corollary of [18] Lemma 4.1:

**Fact 2.** [Neumann [18] Lemma 4.1] Let $G$ be a group and $C_1, \cdots, C_n$ be subgroups of $G$. Then if for some $g_i \in G$, $G = \bigcup_i C_i g_i$, then at least one of the $C_i$ has finite index.

As $[nM : m_j M] = \infty$, by the above fact, $(\beta')$ has a solution $c$ in $M$. The last statement is immediate: Let $t$ be the least common multiple of $n, m_1, \cdots, m_k$ then any $x \in M$ such that $x \equiv_t c$ is solution of $(\beta)$.

**Proposition 1.6.** Let $M', N'$ be models of $T_{p,V,N}$ and $M, N$ be pure subgroups of $M'$ and $N'$. Assume that $N'$ is $[M]$-saturated. Let $h : M \to N$ be an isomorphism of $p$-valued groups. Let $a \in M' \setminus M$. Then, there is $b \in N'$ such that for all $n \in \mathbb{N}$, $k, l, r \in \mathbb{Z}$, $x \in M$ and $\gamma \in VM$,

$$V(\gamma a - x) + r [\gamma] \iff V(\gamma b - h(x)) + r [h] h(\gamma).$$

$$(- \gamma)ka - x \equiv_n 0 \iff (-\gamma)kb - h(x) \equiv_n 0$$

Furthermore, $h$ can be extended to an isomorphism of $p$-valued groups $\tilde{h} : M(a) \to N(b)$ such that $\tilde{h}(a) = b$. 

7
Proof. Let $p(X)$ be the partial type given by formulas
\[ V(lX - h(x)) + r \square h(\gamma) \]
\[ (\neg)kX - h(x) \equiv_n 0 \]
where $M' \models (\neg)\exists a \mathrel{\equiv} x \equiv_n 0$ and $M' \models V(a - x) \square \gamma$ for all $x \in M$ and $\gamma \in VM$. It is sufficient by saturation to prove that the type is finitely consistent. Let $\Phi(X)$ be a conjunction of formulas in $p(X)$. Let $\alpha(X)$ and $\beta(X)$ be the subformula of $\Phi$ such that $\alpha$ involves only valutational (in)equalities, $\beta$ only involves (in)equalities and $\Phi \equiv \alpha \land \beta$. Let us remark that $p(X)$ does not contain a formula of the type $V(lX - h(y)) = \infty$: otherwise, $la - x = 0$ in $M'$, so as $M$ is a pure, torsion-free subgroup of $M'$, $a \in M$; contradiction. Next, we may assume that $k = l = 1$. Indeed, if we multiply each equation by a suitable integer, we may assume $k = l$. Then replace in each equation $kX$ by $X$ and add a congruence $X \equiv_n 0$ in the formula $\Phi$.

As in the proof of Lemma 1.4 we may assume that $\alpha$ is:
\[ X \equiv_n c \land \bigwedge_j \neg X \equiv_{m_j} b_j \]
where $c, b_j \in N$, $n$ divides $m_j$ and $[nN' : m_jN'] = \infty$. By regular density, $c + nN'$ is a dense subset of $B := B(c, v_p(n))$ (the ball of centre $c$ and valuative radius $v_p(n)$). So any solution of $\Phi(X)$ is in $B$. Let $\alpha'(X) \equiv \alpha(X) \land X \in B$. Let $\alpha'_M$ be the system obtained from $\alpha'$ when we apply $h^{-1}$ to all parameters. This system has a solution in $M'$ therefore $\alpha'$ has a solution $y$ in $N'$ by Lemma 1.4. Furthermore, there is $\gamma \in VN'$ such that $B' = B(y, \gamma)$ is a set of solutions of $\alpha'$ in $N'$ (take $\gamma$ be larger than any valuation that appears previously). Let us remark that $B'$ is a subset of $B$. By equidistribution, there is a solution $y'$ of $\beta$ in $B'$. For let $B'' = B(0, v_p(n)) \supset B(y - c, \gamma)$. Apply Fact 2 with $G = nN' \cap B''$ (this is a group) and $G' = m_iN' \cap B''$. By equidistribution, $[G : G'] = \infty$. Therefore, by Fact 2 and equidistribution there is $y' \in B(y - c, \gamma)$, $y' \equiv_n 0$ and $y' \not\equiv m_j b_j - c$ for all $j$. So $y' + c$ is a realisation of $\alpha \land \beta$ in $N'$. This proves that the type $p(X)$ is finitely consistent in $N'$.

Let $b$ be a realisation of this type in $N'$. Then $\tilde{h}$ is an isomorphism. Indeed it is immediate that this is a isomorphism of groups. It remains to prove that it is a morphism of valued groups. For it is sufficient to show that for all $l, l' \in \mathbb{N}$, $r, r' \in \mathbb{Z}$, $x, x' \in M$,
\[ V(la - x) + r \bigcirc V(l'(a - x') + r' \text{ iff } V(lb - h(x)) + r \bigcirc V(l'a - h(x')) + r' \]
We replace $V(la - x) + r \bigcirc V(l'(a - x') + r'$ by $V(l'l'a - l'x') + r' + V(l')$. So, we may assume $l = l'$. If $\gamma = V(la - x) \in VM$, then
\[ V(la - x) + r \bigcirc V(la - x') + r' \text{ iff } V(la - x) = \gamma \land \gamma + r \bigcirc V(la - x') + r' \]
Then by choice of $b$, $V(lb - h(x)) = h(\gamma) \land h(\gamma) + r \bigcirc V(lb - h(x')) + r'$. So we are done. The case $V(la - x') \in VM$ is similar. If $V(la - x) \notin VM$ and $V(la - x') \notin VM$. Then, $V(la - x') = \min\{V(la - x), V(x - x')\} = V(la - x)$ as $V(x - x') \in VM$. So,
\[ V(la-x)+r\bigcirc V(l'a-x')+r' \text{ iff } r' \land V(la-x') \land V(la-x) < V(x-x') \land V(la-x) < V(x-x'). \]
Then, by choice of $b$, $V(lb - h(x')) < V(h(x) - h(x')) \wedge V(lb - h(x)) < V(h(x) - h(x'))$. So, $V(lb - h(x)) + r \square V(lb - h(x')) + r'$ holds as $V(lb - h(x)) = V(lb - h(x'))$.

The second part of Theorem 1.2 follows from the above proposition:

**Proof.** Let $M', N'$ be saturated models of $T_{pV,N}$ and $M, N$ be small substructures and $h$ be an isomorphism between $M$ and $N$. By the choice of our language $M$ and $N$ are pure subgroups of $M'$ and $N'$. Let $a \in M' \setminus M$. Let $p$ be its quantifier-free type over $M$. First remark that $p$ does not contain a formula of the form $kX - b = 0$ with $k \in \mathbb{Z}$ and $b \in M$. Indeed, if this is not the case then $ka = b$ and as $M$ is a pure subgroup of $M'$ and $M'$ is torsion-free, $a \in M$: contradiction. Next, we observe that any formula in $p$ is equivalent to a formula like in Proposition 1.6, indeed, a formula of the form $\neg V(kX - b) \leq \gamma$ is equivalent to $V(kX - b) > \gamma$. Similarly for any formula that involves a negation attached to a valuation (in)equality. Finally, $-kX - b = 0$ is equivalent to $V(kX - b) < \infty$.

So, by Proposition 1.6 there is $b \in N'$ a realisation of the image of $p$ by $h$ and $h$ extends to an isomorphism between $M(a)$ and $N(b)$. As $M(a)$ and $N(b)$ are small pure subgroup of $M$ and $N$, we obtain a back-and-forth system. This completes the proof of quantifier elimination.

Before we end this section, we prove that the theory of $p$-valued groups is NIP whenever all indices $[n]G$ are finite. We refer to [21] for definitions and properties of NIP theories.

**Theorem 1.7.** For all $N \in \mathbb{N}^\infty$, $T_{pV,N}$ is NIP. In particular, $Th((\mathbb{Z}, +, -, 0, 1, \equiv_n), (\mathbb{N} \cup \{\infty\}, 0, \infty, S), v_p)$ is NIP.

The case $G = \mathbb{Z}$ has been proved by F. Guignot using an argument of coheir counting in [8]. We propose here a proof using sequences of indiscernibles, which is essentially the same as the case of valued fields with NIP theory (see [21] Appendix A for instance).

**Proof.** Let $\mathcal{M} = (M, vM)$ be a model of the theory. Let $(x_i, i < \omega)$ be a sequence of indiscernibles. We have to prove that for all $\underline{\gamma} \subseteq \mathcal{M}$ and for all $\Phi$ formula, either $\mathcal{M}$ satisfies $\Phi(x_i, \underline{\gamma})$ for all $i$ large enough or $\mathcal{M} \models \neg \Phi(x_i, \underline{\gamma})$ for all $i$ large enough. By quantifier elimination and properties of NIP theory, we may assume that $\Phi$ is of the type

1. $\varphi(x_i, \underline{\gamma})$ a $(+, -, 0, 1, \equiv_n)$-formula and $x_i, \underline{\gamma} \subset M$;

2. $\varphi(x_i, \underline{\gamma})$ a $(0, \infty, S, <)$-formula and $x_i, \underline{\gamma} \subset VM$;

3. $\varphi(V(P_1(x_i, \underline{\gamma})), \cdots, V(P_n(x_i, \underline{\gamma})), \underline{\gamma})$ a $(0, \infty, S, <)$-formula and $x_i, \underline{\gamma} \subset M$, $\underline{\gamma} \subset VM$ and $P_i$ is a $\mathbb{Z}$-linear combination.

In case (1) and (2), we are done because the theory of each sort is NIP. We reduce now case (3) to case (2):

Let $P(X, \underline{\gamma}) = nX + \sum m_i Y_i$ with $n, m_i \in \mathbb{Z}$. We claim that there is $(a_i < \omega)$ sequence of indiscernible in the second sort $VM$ such that for all $i$ large enough, $V(P(x_i, \underline{\gamma})) = S^k(a_i)$ for some $k \in \mathbb{Z}$ (which depends only on $n$) or $V(P(x_i, \underline{\gamma}))$ is eventually constant: Set $B = \sum m_i y_i$. Then $V(P(x_i, \underline{\gamma})) = V(nx_i + B)$.
• First, if \( v(x_i) \) is not constant, then by indiscernibility, it is either strictly increasing or strictly decreasing. So, for all \( i \) large enough, \( V(nx_i + B) = V(nx_i) \) or \( V(nx_i + B) = V(B) \). Take \( a_i = V(x_i) \).

• Otherwise, let \( t_i = x_i - x_0 \). If \( V(t_i) \) is not constant, it has to be decreasing. For assume it is strictly increasing. Then, for all \( i > 1 \), \( V(x_i - x_1) = V(t_i - t_1) = V(t_1) \). Therefore, the sequence \( V(x_i - x_1) \) is constant while the sequence \( V(x_i - x_0) \) is increasing: this contradicts the indiscernibility of the sequence \( (x_i) \). Now, \( V(nx_1 + B) = V(nx_1 - nx_0 + B - nx_0) \). The latter is equal to \( V(B - nx_0) \) or to \( V(nx_1 - nx_0) = V(nt_i) \) for all \( i \) large enough. Take \( a_i = V(x_i - x_0) \) in the second case.

• Finally, assume that \( V(t_i) \) is constant. Then, by properties of \( p \)-valued \( \mathbb{Z} \)-groups, \( V(at_i - t_1) > V(t_i) \) for some \( 1 \leq a < p \) (independent of \( i \) by indiscernibility). As \( V(at_i - t_1) > V(t_1) \) and \( V(at_i - t_1) > V(t_1) \),

\[
V(t_i - t_j) = V(at_i - at_j) \geq \min\{V(at_i - t_1), V(at_j - t_1)\} > V(t_1) = V(t_i).
\]

Let \( x_\omega \) such that \( (x_i, i \leq \omega) \) is indiscernible. As \( V(x_2 - x_1) = V(t_2 - t_1) > V(t_2) = V(x_2 - x_0) \), the sequence \( V(x_\omega - x_i) \) is increasing by indiscernibility. Take \( a_i = V(x_\omega - x_i) \) like in the second case.

In all cases, either \( V(nx + B) \) is constant or it is equal to \( a_i + v_p(n) = S^{v_p(n)}(a_i) \).

\[\square\]

2 Expansion of \( \mathbb{Q}_p \) by a multiplicative subgroup

In this section, we assume \( p \neq 2 \) (the case \( p = 2 \) is similar but require technical changes as in the remark below).

**Lemma 2.1.** Let \( G < \mathbb{Q}_p^* \). Then \( G = T^D \times D \), where \( T \in G \), \( D < \mathbb{Z}_p^* \) and \( D^s \) is either trivial or dense in \( 1 + p^n\mathbb{Z}_p \) for some \( n \in \mathbb{N} \) and \( s \) that divides \( p - 1 \).

**Proof.** First, let \( T \in G \) with minimal positive valuation among the valuations of the elements of \( G \). If such an element does not exist, then \( G < \mathbb{Z}_p^* \) and we take \( T = 1 \). If \( T \neq 1 \), for all \( t \in G \), there is \( n \in \mathbb{Z} \) such that \( v_t(t) = nv(T) \). Otherwise, if \( t \) has positive valuation, there is \( n \) such that \( v_t(T) < v_t(t) < (n + 1)v(T) \) i.e. \( 0 < v(t/T^n) < v(T) \): this contradicts the minimality of \( v(T) \). If \( t \) has negative valuation, we replace \( t \) by \( t^{-1} \). Then, \( t = T^n u \) with \( u \in G \cap \mathbb{Z}_p^* \), \( n \in \mathbb{Z} \) (take \( u = t/T^n) \).

Now, we may assume that \( G < \mathbb{Z}_p^* \). Let us recall that \( \mathbb{Z}_p^* \cong \mu_{p-1} \times (1 + p\mathbb{Z}_p) \) where \( \mu_{p-1} \) is the set of \( (p - 1) \)th roots of unity (that is isomorphic to \( \mathbb{F}_p^* \) via the residue map \( \text{res} \)). So, \( \text{res}(G) \) is a subgroup of \( \mathbb{F}_p^* \). Let \( s \) be its order. Then, for all \( g \in G \), \( \text{res}(g^s) = 1 \) i.e. \( G^s < 1 + p\mathbb{Z}_p \).

Finally, let us remind that \( 1 + p^n\mathbb{Z}_p \) is isomorphic to \( p^n\mathbb{Z}_p \) (as groups):

\[\exp_p : p^n\mathbb{Z}_p \to 1 + p^n\mathbb{Z}_p : x \mapsto \sum_k x^k/k! \] is an isomorphism for all \( n > 0 \) whose inverse is determined by the \( p \)-adic logarithm map \( \log_p \) (given by the usual power series \( \sum_k (-1)^{k+1} x^k/k! \)). Furthermore, \( \log_p : (1 + p^n\mathbb{Z}_p,.) \to (p^n\mathbb{Z}_p,+) \) is a bicontinuous isomorphism of groups. So, \( G' := \log_p G \) is an additive subgroup of \( \mathbb{Z}_p \). If \( G' = \{0\} \) then \( G' = \{1\} \) and \( G = \mu_p \). Otherwise, let \( n \) be the minimal valuation of the elements of \( G' \). Then, let \( g \in G' \) with \( v(g) = n \). As \( \mathbb{Z} \) is dense in \( p^n\mathbb{Z}_p \), \( G' \) is a dense subset of \( p^n\mathbb{Z}_p \) (by minimality of \( n \)). So, \( G = \exp_p(G') \) is a dense subset of \( 1 + p^n\mathbb{Z}_p \).

\[\square\]
Remark. If \( p = 2 \), \( \mathbb{Z}_2^\times = 1 + 2\mathbb{Z}_2 \). The exponential is well-defined on \( 1 + 2\mathbb{Z}_2 \). So, we obtain that \( D^2 \) either trivial or dense in \( 1 + 2\mathbb{Z}_2 \).

First, we deal with the discrete case i.e. when \( D \) is a finite group i.e. \( D = \mu_6 < \mathbb{F}_p^* \). This case is similar to \( (\mathbb{R}, Q^2) \) in [22]. Let \( G = T^\mathbb{Z} \times D < \mathbb{Q}_p^* \) with \( v_p(T) > 0 \). Then, the valuation induces an isomorphism between \( T^\mathbb{Z} \) and \( v((\mathbb{Q}_p^*)^*) \) where \( k = v_p(T) \). We denote by \( \lambda \) the map \( \mathbb{Q}_p [k] \to T^\mathbb{Z} : x \mapsto T^x \) where \( n = v_p(x)/k \) and \( \mathbb{Q}_p [k] \) is the set of elements in \( \mathbb{Q}_p^* \) with valuation divisible by \( k \). We work in the language \( L_G = L_{Mac} \cup \{ A, \gamma_T, r, \lambda \} \) where \( A \) will be interpreted in \( \mathbb{Q}_p \) by the subgroup \( G, \gamma_T \) by \( T, r \) by a (fixed) \( s \)th root of unity \( \xi \) and \( \lambda \) by the above map. Let \( K \) be a \( p \)-adically closed field. Then, let \( K[k] \) denote the subgroup of \( K^* \) of elements of valuation divisible by \( k \). Let us remark that this group is \( L_{Mac} \)-definable.

Theorem 2.2. \( T_d(G) = Th(\mathbb{Q}_p, +, -, \cdot, 0, 1, P_n(n \in \mathbb{N}), G, T, \xi, \lambda) \) admits the elimination of quantifiers. Furthermore, \( T_d(G) \) is axiomatised by: if \( (K, +, -, \cdot, 0, 1, P_n(n \in \mathbb{N}), A, \gamma_T, r, \lambda) \) is a model of \( T_d(G) \) then

\[
\begin{align*}
&\bullet (K, +, -, \cdot, 0, 1, P_n(n \in \mathbb{N})) \text{ is a } p \text{-adically closed field;} \\
&\bullet (A, \cdot, 1) \text{ is a subgroup of } K[k], \gamma_T \in A, A_{tor} = \mu_s = \{1, r, \ldots, r^{s-1}\}; \\
&\bullet \forall x \in K[k] (x \neq 0 \rightarrow \exists y \in Av_K(x) = v_K(y) \wedge (\forall z \in Av_K(z) = v_K(x) \rightarrow v_0 \cdot 1 < z = r^i y)) \\
&\text{where } k = v_p(T); \\
&\bullet \lambda : K[k] \to A \text{ is a morphism of groups, } \lambda(p^k) = \gamma_T, \text{ for all } z \in K[k]v(\lambda(z)) = v(z) \text{ and } A = \lambda(A) \times A_{tor}, A_{tor} \cap \lambda(A) = \{1\}; \\
&\bullet tp(\gamma_T / \mathbb{Q}) = tp(T/\mathbb{Q}), \ r^s = 1, \ res r = res \xi.
\end{align*}
\]

Remark. In the above theorem, the valuation \( v_K \) is not part of the language but the above axioms are first-order as for \( p \)-adically closed fields: \( v_K(x) \geq v_K(y) \) iff \( y^2 + px^2 \) is a square.

This theorem is a corollary of Theorem 1.1 and Theorem 1.2 in [17]. These theorems give an axiomatisation and quantifier elimination in the special case \( T_d(\mathbb{Z}^2) \). As \( G = \mu_s T^\mathbb{Z} \) is definable in \( (\mathbb{Q}_p, +, \cdot, 0, 1, T^\mathbb{Z}, \langle P_n \rangle_{n \in \mathbb{N}}) \), we obtain the above result. Note that the proof of the theorem of the given if \( K, L \vdash T_d(G) \) have isomorphic value groups, then there is an isomorphism of \( L_G \)-structures between \( \mathbb{Q}(A(K))^h \) and \( \mathbb{Q}(A(L))^h \) (note that both are models of \( T_d(G) \)). See also Theorem 1.1 in Section 4.

The above theorem treats the case where \( D \) is finite. Let us now deal with the case \( s = 1 \) i.e. \( G = T^\mathbb{Z} \times D \) for some \( D \) dense subgroup of \( 1 + p^n \mathbb{Z}_p \). So, \( log_p(D) \) is an additive subgroup of \( \mathbb{Z}_p \) and therefore \( log_p(D), v_p \) is a abelian \( p \)-valued group. On the other hand, \( v_p(exp_p(x) - 1) = v_p(x) \) and \( v_p(log_p(1 + x)) = v_p(x) \). We define \( V : D \to \mathbb{N} \cup \{ \infty \} : x \mapsto v_p(x) - 1 - n \). Then, \( (D, V) \) is an abelian \( p \)-valued group isomorphic (as valued group) to \( (log_p(D), v_p) \). We will give now an axiomatisation of \( Th(\mathbb{Q}_p, G) \). First, let us introduce some notions required for our axiomatisation.

Let \( K \) be a field of characteristic zero and \( G \) be a subgroup of \( K^* \). Let \( a_1, \ldots, a_n \in \mathbb{Q} \) nonzero. We consider the equation

\[ a_1 x_1 + \cdots + a_n x_n = 1. \]
A solution \((g_1, \cdots, g_n)\) of this equation in \(G\) is called nondegenerate if \(\sum_{i \in I} a_ig^i \neq 0\) for all \(I \subset \{1, \cdots, n\}\) nonempty. We say that \(G\) has the Mann property if for any equation like above there is finitely many nondegenerate solutions in \(G\).

Examples of groups with Mann property are the roots of unity in \(\mathbb{C}\) [15] or any group of finite rank in a field of characteristic zero [7][14][23]. In particular, any subgroup of \(\mathbb{Q}_p^*\) of finite rank has the Mann property.

Let \(G < K^*\) be a group with the Mann property. Then the Mann axioms are axioms in the language of rings expanded by constant symbols \(\gamma_g\) for the elements of \(G\) and a unary predicate \(A\) for \(G\). Let \(a_1, \cdots, a_n \in \mathbb{Q}^*\). As \(G\) has the Mann property, there is a collection of \(n\)-uples \(\mathcal{G}_i = (g_{1i}, \cdots, g_{ni})\) \(1 \leq i \leq l\) in \(G^n\) so that these \(n\)-uples are the nondegenerate solutions of the equation \(a_1x_1 + \cdots + a_nx^n = 1\). The corresponding Mann axiom express that there are no extra nondegenerate solutions in \(A\) i.e.

\[
\forall \mathcal{G} \left( \bigwedge_{i} A(y_i) \land \sum_{i=1}^{n} a_iy_i = 1 \land \bigwedge_{\ell \subset \{1, \cdots, n\}} \sum_{i \in \ell} a_iy_i \neq 0 \rightarrow \bigwedge_{k=0}^{l} \mathcal{G} = \mathcal{G}_{y_k} \right).
\]

The main consequence of Mann axioms that we will use is the following:

**Lemma 2.3** (Lemmas 5.12 and 5.13 in [9]). Let \(K\) be a field of characteristic zero, let \(G\) be a subgroup of \(K^*\) and let \(\Gamma\) be a subgroup of \(G\) such that for all \(a_1, \cdots, a_n \in \mathbb{Q}^*\) the equation \(a_1x_1 + \cdots + anx^n\) has the same nondegenerate solutions in \(\Gamma\) as in \(G\). Then, for all \(g, g_1, \cdots, g_n \in G\)

- if \(g\) is algebraic over \(\mathbb{Q}(\Gamma)\) of degree \(d\) then \(g^d \in \Gamma\);
- if \(g_1, \cdots, g_n\) are algebraically independent over \(\mathbb{Q}(\Gamma)\) then they are multiplicatively independent over \(\Gamma\).

In particular, if \(\Gamma\) is a pure subgroup of \(G\), then the extension \(\mathbb{Q}(G)\) over \(\mathbb{Q}(\Gamma)\) is purely transcendental.

Let \(\mathcal{M} = (M, \cdots)\) be a \(\mathcal{L}\)-structure. Let \(A \subset M\) and \(\mathcal{L}_A\) be the expansion of \(\mathcal{L}\) by a unary predicate that will be interpreted by \(A\) in \(M\). We denote by \(f : X \rightarrow Y\) a map from \(X\) to the subsets of \(Y\) of size at most \(n\). We say that \(A\) is large in \(M\) if there is a \(\mathcal{L}_A\)-definable map \(f : M^m \rightarrow M\) such that \(f(A) = \bigcup_{x \in A^m} f(x) = M\). We say that \(G\) is small if it is not large. If \(G\) is a subgroup of \(\mathbb{Q}_p^*\) with finite rank then it is small (because of the respective cardinalities of \(\mathbb{Q}_p^*\) and \(G\)). Let us remark that smallness can be written as a scheme of first-order sentences in the language \(\mathcal{L}_A\).

Let \(G < \mathbb{Q}_p^*\) of the type \(T^Z \times D\) where \(T \subset G\) and \(D\) is dense in \(1 + p^\alpha \mathbb{Z}_p\) as in Lemma 2.4 (case \(s = 1\)). We assume that \(G\) has the Mann property and is small (for instance \(G\) is finitely generated). We will give an axiomatisation of \(Th(\mathbb{Q}_p, G)\). We assume that \(T \neq 1\). The changes that have to be made in the case \(T = 1\) are obvious. Let \(\mathcal{L}_G = \mathcal{L}\text{Mac} \cup \{A, \lambda, \equiv_n (n \in \mathbb{N}), \gamma_g (g \in G)\}\). We define the theory \(T_G\) as follow: Let \((K, +, - ,\cdot, 0, 1, P_n(n \in \mathbb{N}), A, \lambda, \equiv_n (n \in \mathbb{N}), \gamma_g (g \in G))\) be a model of \(T_G\), then

- \((K, +, - ,\cdot, 0, 1, P_n(n \in \mathbb{N}))\) is a \(p\)-adically closed field;
- \(A\) is a multiplicative subgroup of \(K^*\);
Let $k = v_p(T)$. Then, for all $x \in K[k]$, there is $a \in A$ such that $v_K(x) = v_K(a)$. Let $K[x] \to A : x \mapsto a$ with $v_K(x) = v_K(a)$ is a morphism groups, $\lambda(x) = 1$ for all $x$ with $v(x) = 0$ and $\lambda(p^k) = \gamma_T$.

- For all $x \in A$, there is a unique $x' \in (A \cap O_K) =: A_V$ such that $x = \lambda(x)x'$.
- $(K, +, -, 0, 1, P_n(n \in \mathbb{N}), \lambda(A), \gamma_T, 1, \lambda)$ is a model of $T_d(T^2)$.

**Remark.** As the valuation is interpretable in our language, we can write the condition that $A_V \equiv D$ (as valued group). For note that the theory $T_{p,V,N}$ as defined in section 13 is interpretable and its expansion by the diagram of $D$ is complete by Theorem 1 of [1].

**Theorem 2.4.** $T_G$ is complete.

The proof is similar to Theorem 7.1 in [9] or Theorem 3.3 in [17]. As in these proofs, we use the notions of free, linearly disjoint and regular extension of fields. We refer to [15] for their definitions and elementary properties.

**Proof.** Let $(K, A)$ and $(L, B)$ be two saturated models of $T_G$ of same cardinality (the saturation is at least $|G|$). Let $\text{Sub}(K, A)$ be the collection of $\mathcal{L}_{Mac} \cup \{A\}$-substructures $(K', A')$ where

- $K'$ is a $p$-adically closed field, $|K'| < |K|$;
- $A'$ is a pure subgroup of $A$;
- $K'$ and $\mathbb{Q}(A)$ are free over $\mathbb{Q}(A')$.

We define $\text{Sub}(L, B)$ similarly. Let $\Gamma$ be the set of $\mathcal{L}_{Mac} \cup \{A\}$-isomorphisms of $K' \in \text{Sub}(K, L)$ and $L' \in \text{Sub}(L, B)$ that extends to an isomorphism of valued fields between $K'(\lambda(K))^b$ and $L'(\lambda(L))^b$. Note that this latter isomorphism is also an isomorphism of $\mathcal{L}_G$-structures.

**Claim 2.5.** $\Gamma$ is nonempty.

For let us remark that $(\mathbb{Q}(G)^b, G)$ belongs to the sets $\text{Sub}(K, A)$ and $\text{Sub}(L, B)$ (this follows from the axioms and Lemma 2.3). Now let us remark that $(\mathbb{Q}(G)^b, \lambda(G))$, $(K, \lambda(K))$ and $(L, \lambda(L))$ are all models of $T_d(\lambda(G))$. As $(\mathbb{Q}(G)^b, \lambda(G))$ is a substructure of $(K, \lambda(K))$ and $(L, \lambda(L))$ resp., by the proof of Theorem 2.2, there is an isomorphism between $\mathbb{Q}(G)(\lambda(K))^b$ and $\mathbb{Q}(G)(\lambda(L))^b$ (extending the identity).
Claim 2.6. Let $K'$ $p$-adically closed fields with $|K'| < |K|$. Let $x \in K \setminus K'$. Then the type $tp(x/K'(\lambda(K))^h)$ is realised in any $|K'|$-saturated expansion of $K'$.

For as $K$ is an immediate expansion of $K'\lambda(K))^h$, the type of $x$ over $K'(\lambda(K))^h$ is determined by a pseudo-Cauchy sequences over $K'(\lambda(K))$. So, it is sufficient to find a sequence with same pseudo-limit and indexed over a set of size at most $|K'|$.

Let $K(n) = K'\lambda(K'(x))$ and by induction $K(n)\lambda := K(n)\lambda(K'(n)))$. Set $K(n) = \bigcup_n K(n)\lambda$. Let us remark that $|K(n)\lambda| = |K|$.

Now it is sufficient to prove that $\{v(x - b) : b \in K(n)\lambda\}$ is cofinal in $\{v(x - f(\pi, \gamma)) : \pi \in K, \gamma \subset \lambda(K)\}$ for all $f \in \pi[X, \gamma]$. Let $f(\gamma) := f_0 + \sum f_1\gamma$ with $f_1 \in K(n)\lambda$ (in particular, any polynomial $f(\pi, \gamma)$ in the last set has this form with $n = 1$). We prove by induction on the number $d$ of nonzero $f_1$ that either $v(x - f_0 - f_1|\gamma|) \leq v(x - b)$ for some $b \in K(n)\lambda$ or there $d$ are non zero $f_1 \in K(n)\lambda$ such that $f_0 + \sum f_1\gamma = f_0 + \sum f_1\gamma$. This proves our claim by induction on $d$ the degree of $f$.

If $f(\gamma)$ has only one monomial $f_1\gamma$ then $x - f(\gamma) = x - f_0 - f_1\gamma$. Then either $v(x - f_0) \neq v(f_1\gamma)$ so $v(x - f_0 - f_1\gamma) = \min\{v(x - f_0), v(f_1|\gamma|)\} \leq v(x - f_0)$.

In that case, take $b = f_0$. Or $v(x - f_0) = v(f_1|\gamma|)$. Therefore, $x - f_0 - f_1\gamma = x - f_0 - f_1\lambda\gamma^{-1}x - f_0). As f_0 + f_1\lambda\gamma^{-1}x - f_0) \in K(n)\lambda$ we are done: take $b = f_0 + f_1\lambda\gamma^{-1}x - f_0)$.

For the inductive step, let $f(\gamma) = f_0 + \sum f_1\gamma$ with $d$ nonzero factors $f_1\gamma$. First let us remark that if $v(f_1\gamma) \neq v(f_1\gamma)$ and $v(x - f_0) \neq v(f_1\gamma)$ for all $\gamma \neq J$ with $f_1, f_1$ nonzero then $v(x - f_0 + \sum f_1\gamma) = \min\{v(x - f_0), v(f_1|\gamma|)\} \leq v(x - f_0)$. In that case, we take $b = f_0$. Otherwise, if there is $I \neq J$ such that $f_I, f_J \neq 0$ and $v(f_I|\gamma|) = v(f_I|\gamma|)$ i.e. $\lambda(f_I|\gamma|) = \lambda(f_I|\gamma|)$. Set $f_I = f_1 + f_2\lambda(f_I|\gamma|^{-1})$, $f_J = 0$ and $f_K = f_K$ for $f \neq I, J$.

Then, $f_0 + \sum f_1\gamma = f_0 + \sum f_1\gamma$ for all $I$ and there are $d - 1$ nonzero $f_I$. Finally, if $v(x - f_0) = v(f_1|\gamma|)$, we proceed similarly substituting $f_0$ by $f_0 + f_1\lambda(x - f_0)\gamma^{-1}x - f_0)$. This completes the proof of Claim 2.6.

We will prove that $\Gamma$ forms a back-and-forth system. So, let $\iota : (K', A') \rightarrow (L', B')$ in $\Gamma$ and $\iota$ its extension to $K'(\lambda(A))$. Let $\alpha \in K \setminus K'$. We construct an extension of $\iota$ that contains $\alpha$ in its domain and compatible with $\iota$. There are different cases according to the choice of $\alpha$.

1. If $\alpha \in A$ then $\alpha = \lambda(\alpha)$ for some unique $\alpha' \in A \cap O_K$. So, we may assume that

(a) $\alpha = \lambda(\alpha)$. As $\alpha \notin K', v_K(\alpha) \notin v(K')$. Let $l \in v_K(L)$ such that it realises the same cut as $v_K(\alpha)$ over $v_K(K')$ and the same congruence classes in $v_K(L)$ (I exists by saturation and quantifier elimination for $\exists$). Then, as $v_K(\alpha)$ is $k$-divisible, $l$ is $k$-divisible. Let $x \in L$ such that $v_L(x) = l$. Take $\beta = \lambda(x)$. Let $K'' = K'(\alpha)^h, A'' = A \cap K''$, $L'' = L'(\beta)^h, B'' = B \cap L''$. $\iota$ extends to an isomorphism between $(K'', A'')$ and $(L'', B'')$ with $\alpha \mapsto \beta$ (that is the reduct of $\iota$). Also, by Lemma 2.3 and properties of $p$-adically closed fields, $(K'', A'') \subset Sub(K, A)$ and $(L'', B'') \subset Sub(L, B)$. This follows from the following fact:
Fact 3. For all $t_1, \ldots, t_n \in K'$ algebraically independent over $A$,
\[ acl(t_1, \ldots, t_n, \alpha, A') \cap A = A'(\alpha) \]
(here acl is taken in the language of rings, $A'(\alpha)$ is the pure closure of $A'(\alpha)$ in $A$ as defined in Definition 1.3).

This is a consequence of Mann property (in particular, Lemma 2.3) and exchange property for acl, see [H] proof of Lemma 4.2 for instance. In particular, for all $t \in K''$, $t \in A$ iff $t \in A'(\alpha)$ (so, $A'' = A'(\alpha)$).

(b) $\alpha \in (A \cap \mathcal{O}_K) =: A_V$. We want to find $\beta$ in $B_V$ that realises the same type as $\alpha$ over $A'_V$ (type as in the theory of $p$-valued group) and the same type as $\alpha$ over $K'(\lambda(K))$ (this latter is determined by the set of formulas $v_K(x - a) \sqcap v_K(t)$ where $a, t$ are in a subfield of $K'(\lambda(K))$ of size $|K'|$). In that case, $A'_V(\alpha) \cong B'_V(\beta)$ as valued field and $K'(\lambda(A))\langle h \rangle \cong L'(\lambda(B))\langle h \rangle$ as valued fields. Let us prove that the union of these type is finitely consistent i.e. that given finitely many conditions $V(x - g) \sqcap \gamma, x \equiv_n g$ with $g \in A'$ and $\gamma \in V A'$ and a ball $B$ in $K'(\lambda(K'(\alpha)))$ the image under $\tilde{\iota}$ of these formulas is realised. Without loss of generality, we may assume that $B$ is a subset of the set of $x \in K'$ such that $V(x - g) \sqcap \gamma$. By Proposition 1.6 and its proof, there is $\beta \in B_V$ that realises the formulas $V(x - i(g)) \sqcap \gamma$ and $x \equiv_n i(g)$. By density of $B_V$ in $1 + p^n\mathcal{O}_K$, we may even assume that such realisation is in $\tilde{\iota}(B)$. So the image by $\tilde{\iota}$ of the type is consistent. Let $\beta$ be one of its realisation in $L$. We conclude like in case 1.(a) setting $K'' = K'(\alpha)^h$, $A'' = A \cap K''$, $L'' = L'(\beta)^h$, $B'' = B \cap L''$.

2. If $\alpha \in K'(\alpha_1, \ldots, \alpha_n)^h$ where $\alpha_i \in A$, then apply case 1. $n$ times.

3. If $\alpha \notin K'(A)^h$. Consider the cut of $\alpha$ over $K'(\lambda(K))$. By saturation, Claim 2.3 and smallness of $A$, we can find $\beta \in L \setminus L'(B)^h$ which realises the corresponding cut (under $\tilde{\iota}$). Then, $K'(\lambda(K))\langle \alpha \rangle \cong L'(\lambda(L))\langle \beta \rangle$. Take $K'' = K'(\alpha)^h$ and $L'' = L'(\beta)^h$. Note that as $Q(A)$ is a regular extension of $Q(A')$, $K'$ and $Q(A)$ are linearly disjoint over $Q(A')$. By linear disjointness, $\iota$ extends to an isomorphism between $(K'', A')$ and $(L'', B')$ that sends $\alpha$ on $\beta$. The freeness of $K''$ and $Q(A)$ over $Q(A')$ follows from the assumption that $\alpha \notin K'(A)^h$. 

Finally, let us return the case: let $G$ be a subgroup of $Q_p$. In Lemma 2.1 we have seen that $G = T^p \times D$ where $D^p$ is dense in $1 + p^n\mathbb{Z}_p$. Let us remark that any $t \in D$ is uniquely determined by the pair $(t^p, res t)$. Indeed, for all $h \in D^p$ and $\xi \in \mathbb{F}_p^\times$, there is by Hensel's Lemma at most one $t \in D$ such that $t^p = h$ and $res t = \xi$. Furthermore, if $t' \in D$ is such that $t'^p = h$ then $t' = t\omega$ for some $\omega \in G_{tor}$.

Let us return to the back-and-forth system of Theorem 2.4. The general case does not change neither case 1. (a), case 2. nor case 3. It remains to deal with case 1.(b). For let us remark that given $\alpha \in A_V$, we can find $\beta \in B'_V$ such that $A'_V(\alpha^h) \cong B'_V(\beta^h)$ (where the pure closure is taken in $A^*$, $B^*$ resp.) and $K''(\alpha^h) \cong L''(\beta^h)$ as valued fields (this latter isomorphism is compatible with $\tilde{\iota}$). This follow from the proof of case 1.(b). It remains to see if this isomorphism is an isomorphism of $\mathcal{L}_G$-structures. For it is sufficient to prove that for all $x \in K$, $x \in A'_V(\alpha)$ iff $\iota(x) \in B'_V(\beta)$. For the above remark, $x \in A'_V(\alpha)$ is determined by the properties $x^p \in A'_V(\alpha^h)$ and $res x = \xi$. If $x \in A'_V(\alpha^h)$ say $x^m = aa^m$ for some $a \in A'_V$ and $n, m \in \mathbb{N}$. Then $\alpha$ satisfies $\alpha^m = a^{-1}D_n(\xi)$ where $D_n(\xi) = \{ t \in D^* : t \equiv_n 0, \exists y \in D \cdot y^p = t \text{ and } res y = \xi \}$ (where the
congruence is in the group $D^*).$ If $n = 1$, we set $D_n(x) := D(x)$. Therefore, for $i$ to be an isomorphism of $L_D$-structures, it is necessary that $y^m = i^{-1}D_n(x)$ (where $D_n$ is now interpreted in $L$). This will be guaranteed by a scheme of axioms expressing the density of $D_n(x)$ and adding this property in the type of $\beta^\prime$.

We will see in the next lemma that such set $a^{-1}D_n(x)$ is dense in a finite union of ball in $1 + p^nZ_p$ (possibly empty). Furthermore, we can add the information $a^\prime \in a^{-1}D_n(x)$ to the type defined in case 1.(b) and still guarantee that this type is finitely satisfied in $L$. In that case we are done.

**Lemma 2.7.** If $D(x)$ is nonempty, then it is dense in $1 + p^nZ_p$. Furthermore, for all $x_1, \ldots, x_m \in 1 + p^nZ_p$, for all $k_1, \ldots, k_m, n_1, \ldots, n_m \in \mathbb{N}$ if there is $y \in D$ such that $y_{k_i} \in x_i^{-1}D_{n_i}(x)$ for all $i$, then the set of $z \in D$ such that $z_{k_i} \in x_i^{-1}D_{n_i}(x)$ for all $i$ is dense in a finite union of (multiplicative) cosets of $1 + p^nZ_p$ in $1 + p^nZ_p$ for some $r \geq n$ where $r$ is independent of $x_i$. If $x_i \in D^*$ the cosets depend only on the congruence classes of $x_i$ modulo $s + \sum n_i$ for some $s$.

Proof. Let us prove the first part of the lemma. Let $y \in D(x)$ i.e. $y = x^s$ for some $x \in D$ with res $x = \xi$. First let us remark that $yD^s \subset D(x)$. For $z \in D^s$, then $t = tz$ for some $t \in D^s$. So $yz = (xt)^s$ and res $xt = res x \cdot res t = res \xi$ (as res $t = 1$ since $t \in D^* \subset 1 + p^nZ_p$) i.e. $yz \in D(x)$. Now, it remains to see that $yD^s$ is a dense subset of $1 + p^nZ_p$. That is $\log_p(yD^s) = \log_p(y) + s \log D^s$ is a dense subset of $p^nZ_p$. This is the case as $\log_p(s) = 0$, $\log_p(y) \geq 0$ and $\log D^s$ is dense in $p^nZ_p$.

The general case is similar. Let $X := \{ y \in D^* : \wedge_i y_{k_i} \in x_i^{-1}D_{n_i}(x) \}$. We assume that $X$ is nonempty. Let $y_0 \in X$. For all i, $y_{0k_i} = x_i^{-1}(t_{0i})_{n_i}$ with res $t_{0i} = \xi_i$. Let $X_0 := \{ z \in D^* : z_{k_i} \in D_{n_i}(1) \}$. Then $X = y_0X_0$. For let $y \in X$ i.e. $y_{k_i} = x_i^{-1}(t_{i})_{n_i}$ with res $t_i = \xi_i$. Then $(y/y_0)_{k_i} = (t_{i}/t_{0i})_{n_i}$ with res $t_i = \xi_i$. Conversely, let $z \in X$ i.e. $z_{k_i} = (t_{i})_{n_i}$ with res $t_i = 1$. Then $(y_0z)_{k_i} = x_i^{-1}((t_{0i}/t_i)^{s})_{n_i}$ with res $t_{0i}/t_i = \xi_i$, i.e. $y_0z \in X$.

Let us prove that $X_0$ is a dense subset of a finite union of (multiplicative) cosets of $1 + p^nZ_p$ in $1 + p^nZ_p$ for some $r \geq n$. Let $z \in X_0$ i.e. for all $i$, $z_{k_i} = (t_{i})_{n_i}$ where res $t_i = 1$. Then $zD^s \subset X_0$. For if $y = t^s \Pi \subset X_0$. If $y = (t_{i})_{n_i}$ then $(y)_{k_i} = (t_{i}/t_{0i})_{n_i}$ and res $t_i = \xi_i$, res $t_{0i} = 1$ as $D^s \subset 1 + p^nZ_p$. Let us remind that $D^s \subset X_0$ is a dense subset of $1 + p^nZ_p$ where $r = s + v_p(s \Pi \sum n_i)$. Then, $X_0$ is dense in the coset $(1 + p^nZ_p)$. If there is no $y \in X_0$ such that $y \not \in z(1 + p^nZ_p)$, then $X_0$ is a dense subset of $z(1 + p^nZ_p)$. Otherwise, let $y \in X_0$, $y \not \in z(1 + p^nZ_p)$. Then by the same argument as before $X_0$ is dense in the coset $y(1 + p^nZ_p)$. As $(1 + p^nZ_p)$ has finitely many coset in $1 + p^nZ_p$, $X_0 \subset (1 + p^nZ_p)$, we are done.

Now, we have that $X = y_0X_0$ is a dense subset of finitely many cosets $y_0c(1 + p^nZ_p)$ where $c, r$ does not depend on $x_i$. It remains to prove that these cosets only depends on the on the congruence classes of $z_i$ modulo $s + \sum n_i$. For it is sufficient to prove that for all $z_i \in D^s$ such that $z_i = x_i \mod (s + \sum n_i)D^s$ (i.e. $z_i = x_i b_{i} \Pi n_i$ (for some $b_i \in D^s$)), if $X(\xi) = \{ y \in D^* : \wedge y_{k_i} \in x_i^{-1}D_{n_i}(\xi_i) \}$. Then $X = X(\xi)$. Let $w \in X(\xi)$. Then $w_{k_i} = (t_{i}b_{i} \Pi n_i)^{-1}((v_{i})_{n_i})$, with res $v_i = \xi_i$. So

$v_{k_i} = (x_i b_{i} \Pi n_i)^{-1}((x_i)^{-1}((v_{i} b_{i} \Pi x_i n_i))^{s})_{n_i}$, where $v_{i} b_{i} \Pi x_i n_i = \xi_i$ as $b_{i} \Pi x_i n_i \in D^s \subset 1 + p^nZ_p$. This proves that $w \in X$. By symmetry, $X = X(\xi)$. [End of proof]
Let \((K, A)\) be any model of our theory. Let \(A_V := (A \cap O_K^*)^\times\). Let \(x_i \in A_V^\times\) and \(X := \{y \in A_V^\times : \land y^{k_i} \in x_i^{-1}D_n(\xi_i)\}\). Then by the above lemma, \(X\) is either empty or dense in a finite union of \(c(1 + p'\mathcal{O}_K)\) where \(r\) is independent of \(x_i\) and \(c\) only depends on the congruence classes of \(x_i\) modulo \(s + \sum n_j\). If all indices \([n]D^s\) are finite, then the congruence classes of \(x_i\) modulo \(s + \sum n_j\) have representatives in \(G\) (or in any pure subgroup of \(A\)). In that case, the dense axioms of \(D_n(\xi)\) is an scheme of axioms that expresses in our language when \(X\) is empty or in which coset of \((1 + p'\mathcal{O}_K)\) it is dense.

Let \(G < \mathbb{Q}_p^p\) be a small group with Mann property and finite index \([n]G\) (for instance a group of finite rank). We fix \(T \in G\) such that \(T^2 \times D\), where \(D^s\) is dense in \(1 + p\mathbb{Z}_p\) as in Lemma 2.1. We will give an axiomatisation of \(Th(Q_p, G)\). Again, we assume that \(T \neq 1\). Let \(L_G = L_{Mac} \cup \{A, \lambda, \equiv_n (n \in \mathbb{N}), \gamma_g(g \in G)\}\). We define the theory \(T_G\) as follow: let \((K, +, -, \cdot, 0, 1, P_n(n \in \mathbb{N}), A, \lambda, \equiv_n (n \in \mathbb{N}), \gamma_g(g \in G)\) be a model of \(T_G\) then

- \((K, +, -, \cdot, 0, 1, P_n(n \in \mathbb{N}))\) is a \(p\)-adically closed field;
- \(A\) is a multiplicative subgroup of \(K^*\), \(A_{Tor} = G_{Tor}\);
- Let \(k = v_p(T)\). Then, for all \(x \in K[k]\), there is \(a \in A\) such that \(v_k(x) = v_k(a)\). \(\lambda : K[k] \rightarrow A : x \mapsto a\) with \(v_k(x) = v_k(a)\) is a morphism groups, \(\lambda(x) = 1\) for all \(x\) with \(v(x) = 0\) and \(\lambda(p^k) = \gamma_T\);
- For all \(a \in A\) there is a unique \(a' \in O_K^*\) such that \(a = a'\lambda(a);\)
- \((A_V^\times, +, 1, \gamma_c, \equiv_n (n \in \mathbb{N}), (V K_{\geq 0}, <, S, 0, \infty), \gamma_g(g \in G), V)\) is elementary equivalent to \((D^s, (\gamma_g)_{g \in G}, V)\) as abelian \(p\)-valued groups where \(V : A_V^\times \rightarrow vK_{\geq 0}^* \cup \{\infty\} : a \mapsto v_K(a - 1) - n\) (surjective map), \(\gamma_c\) is a fixed element of \(G\) with minimal \(V\)-valuation and \(a \equiv_n b\) iff there is \(z \in A\) such that \(a = bz^n;\)
- \(A_V^\times\) is a dense subset of \(1 + p\mathbb{O}_K;\)
- \(A\) satisfies the dense axioms of \(D_n(\xi);\)
- \(A\) satisfies the same Mann axioms as \(G;\)
- \(A\) is a small subset of \(K;\)
- \(A \vDash Diag(G/\mathbb{Q}).\)

**Theorem 2.8.** \(T_G\) is complete.

**Proof.** With the same notations as in Theorem 2.4 it remains to deal with the case 1. (b) where \(a \in A_V\). For as discussed before it is sufficient to prove that the type

\[tp(\alpha^s / K'(\lambda(A))) , (\alpha^s)^k \in a^{-1}D_n(\xi) : k, n \in \mathbb{Z}, \xi \in \{0, \ldots, p - 1\}, a \in A^s_V\]  

is realised in \(L\) where the first type is in the language \(L_{Mac}\) and the second in the language of \(p\)-valued groups. For by Claim 2.6 it is sufficient to prove that it is finitely consistent. By the axioms of density \(D_n(\xi)\) this is the case. Let \(\beta^s\) be one of its realisation. We conclude as in Theorem 2.4 case 1.(a) setting \(K'' = K'(\alpha)^h, A'' = A \cap K'', L'' = L'(\beta)^h, B'' = B \cap L''.\)
Definable sets in \((K, A)\)

In this section and for the rest of the paper, we assume \([n]G\) finite for all \(n\).

Let \(G \leq Q_p^\ast\) be a small subgroup with Mann property. Then, \(G = \lambda(G) \times D\) where \(D = G \cap \mathbb{Z}_p^\ast\). Let us remark that the structures \((Q_p^\ast, +, \cdot, (P_n)_n \in \mathbb{N}, G, \lambda, (\equiv_n)_{n \in \mathbb{N}}, \gamma_g)\) and \((Q_p^\ast, +, \cdot, (P_n)_n \in \mathbb{N}, \lambda(G), D, \lambda, (\equiv_n)_{n \in \mathbb{N}}, (\gamma_g)_{g \in G})\) are interdefinable. By Theorem 2.4, \(T_g(\lambda(G))\) admits the elimination of quantifier. It remains to describe the definable sets of the expansion of \((Q_p^\ast, \lambda(G))\) by a predicate for \(D\). By Lemma 2.1, there is \(s, n\) such that \(D^s\) is a dense subgroup of \(1 + p^n \mathbb{Z}_p\). We work in the structure \((Q_p^\ast, \lambda(G), D)\). The description of definable sets is similar to the one in Theorem 7.5 in [9]. We define the language \(\mathcal{L}_{D, \lambda} = \mathcal{L}_{\text{Mac}} \cup \{\lambda(G), G_V, \lambda, (\equiv_n)_{n \in \mathbb{N}}, (\gamma_g)_{g \in G}\}\) where \(\lambda G\) is interpreted by \(\lambda(G)\) and \(G_V\) by \(D\).

Let \(\mathcal{L}_{pV}\) be the language of \(p\)-valued groups as in Section 1. Let \(\mathcal{L}_{pV}(\Sigma) = \mathcal{L}_{pV} \cup \{\Sigma \subseteq \mathbb{Z}\}\) the expansion of the language of \(p\)-valued groups by predicates \(\Sigma\) interpreted in \(D^s\) by

\[
D^s = \Sigma(\vec{y}) \text{ iff } \sum_i k_i g_i = 0.
\]

We denote by \(D(\Sigma)\) the \(\mathcal{L}_{pV}(\Sigma)\)-structure with underlying set \(D^s\). If \(\Phi\) is a \(\mathcal{L}_{pV}(\Sigma)\)-formula, we define \(\Phi_G\) a \(\mathcal{L}_{D, \lambda}\)-formula defined as follows (by induction on the complexity of \(\Phi\)):

- If \(\Phi\) is atomic, \(\Phi_G \equiv \Phi\) (where the language \(\mathcal{L}_{pV}(\Sigma)\) is interpreted in \(\mathcal{L}_{D, \lambda}\));
- If \(\Phi \equiv \lnot \Psi, \Phi_G \equiv \lnot \Psi_G\);
- If \(\Phi \equiv \Psi \land \theta, \Phi_G \equiv \Psi_G \land \theta_G\);
- If \(\Phi \equiv \exists x \Psi(x), \Phi_G \equiv \exists x \in D^s \Phi_G(x)\).

A special formula is a formula of the type

\[
\exists y \land, y_i \in G_V \land \Phi_G(\vec{y}^s) \land \theta(\vec{x}, \vec{y})
\]

where \(\Phi\) is a \(\mathcal{L}_{pV}(\Sigma)\)-formula and \(\theta\) is a \(\mathcal{L}_{\text{Mac}} \cup \{\lambda(A), \lambda\}\)-formula.

Lemma 3.1. Any \(\mathcal{L}_{D, \lambda}\)-formula is equivalent in \(T_G\) to a boolean combination of special formulas.
Proof. Let \((K, A), (L, B)\) be two saturated models of \(T_G\) of same cardinality. Let \((K, \lambda(A), D_A)\) and \((L, \lambda(B), D_B)\) be the corresponding \(L_{D, \lambda}\)-structure. So, \(A = \lambda(A) \times D_A\) and \(B = \lambda(B) \times D_B\). Let \(\overline{\sigma} \in K^n\) and \(\overline{\beta} \in L^n\) that satisfy the same special formulas. We have to prove that \(tp_{(K, \lambda(A), D_A)}(\overline{\sigma}) = tp_{(L, \lambda(B), D_B)}(\overline{\beta})\).

For it is sufficient to find \(\epsilon\) isomorphism in the back-and-forth system defined in the proof of Theorem 2.4 that sends \(\overline{\sigma}\) to \(\overline{\beta}\).

Claim 3.2. \(\text{trdeg } Q(A)(\overline{\sigma}) = \text{trdeg } Q(B)(\overline{\beta})\).

Proof. Without loss of generality, \(\alpha_1, \ldots, \alpha_r\) are algebraically independent over \(Q(A)\) and the above transcendence degree is \(r\). Assume that \(\beta_1, \ldots, \beta_r\) are algebraically dependent over \(Q(B)\). So there is \(\phi(\overline{x}, \overline{y})\) a \(L_{Mac}\)-formula and \(\overline{h} \subset H\) such that

\[
(L, B) \models \phi(\overline{h}, \beta_1, \ldots, \beta_r) \land \exists x_1 \ldots x_r \Phi(\overline{h}, \beta_1, \ldots, \beta_r, x_1, \ldots, x_r).
\]

Assume that \(h_i = s_i \lambda(h_i)\) where \(s_i \in D_B\). Then,

\[
(L, B) \models \exists \overline{x} \in D_B \exists \overline{y} \in \lambda(B) \Phi(\overline{x}, \beta_1, \ldots, \beta_r) \land \exists x_1 \ldots x_r \Phi(\overline{x}, \beta_1, \ldots, \beta_r, x_1, \ldots, x_r).
\]

As \(\overline{\sigma}\) and \(\overline{\beta}\) satisfy the same special formulas, there is \(\overline{g} \subset D_A\) and \(\overline{\theta} \subset \lambda(A)\) such that

\[
(K, A) \models \Phi(\overline{g}, \alpha_1, \ldots, \alpha_r) \land \exists x_1 \ldots x_r \Phi(\overline{g}, \lambda(h), \alpha_1, \ldots, \alpha_r, x_1, \ldots, x_r).
\]

This is a contradiction with the algebraic independence of \(\alpha_1, \ldots, \alpha_r\) over \(A\). \(\square\)

Let \(D_A' := \{g_1, g_2, \ldots\}\) be a subgroup of \(D_A^*\) of cardinality \(|G|\) such that \(D_A'(\Sigma) \prec D_A'(\Sigma)\) [as valued groups] and \(\overline{\sigma}\) has transcendence degree \(r\) over \(Q(D_A', \lambda(A))\). Let \(\tilde{D}_A = \{g \in A \mid g^* \in D_A'\}\). Let \(\tilde{A}_\lambda\) be the image under \(\lambda\) of the \(\lambda\)-closure of \(Q(D_A, \overline{\sigma})\) (the \(\lambda\)-closure is of a substructure \(M\) is the smallest field that contains \(M\) and is closed under \(\lambda\) - see construction in the proof of Claim 2.4). Let \(\theta_1, \ldots, \theta_n\) be \(L_{pV}\)-formulas and \(\Phi_1, \ldots, \Phi_m\) be \(L_{Mac} \cup \{\lambda(A), \lambda\}\)-formulas such that

\[
D_A(\Sigma)^* \models \theta_i(\overline{g}^*) \quad K \models \Phi_j(\overline{\sigma}, \overline{g})
\]

for some \(\overline{g} \subset \tilde{D}_A\) and for all \(i, j\). Then,

\[
(K, A) \models \exists \overline{g} \in D_A \bigwedge_i \theta_i(\overline{g}^*) \land \Phi_j(\overline{\sigma}, \overline{g}).
\]

As \(\overline{\sigma}\) and \(\overline{\beta}\) satisfy the same special formulas,

\[
(L, H) \models \exists \overline{g} \in D_B \bigwedge_i \theta_i(\overline{g}^*) \land \Phi_j(\overline{\beta}, \overline{g}).
\]

So the following type is consistent:

\[
\{G_V(\overline{g})\} \cup \{\theta_i(\overline{g}^*)\} \cup \{\Phi_j(\overline{\beta}, \overline{g})\},
\]

where \(\theta_i, \Phi_j\) runs over the formulas like above. Let \(\tilde{D}_B = \{h_1, h_2, \ldots\}\) be a realisation of the type in \((L, B)\). Let \(D_B' = \tilde{D}_B^*\) and \(\tilde{B}_\lambda\) be the image under \(\lambda\) of the
\[ \lambda \text{ closure of } \mathbb{Q}(D_B, \overline{\beta}). \] Then \( D'_B(\Sigma) \prec D''_B(\Sigma) \) and the application \( g^*_B \to h^*_B \) induces an isomorphism of valued groups between \( D'_A \) and \( D''_A \). Let us remark that if there is \( y \in D_A \) with \( \text{res } y = \xi \) then there is \( z \in D_B \) such that \( z^* \) is the image of \( y^* \) and \( \text{res } z = \xi \) (for it is expressible by a special formula). This \( z \) is uniquely determined by the pair \( (z^*, \text{res } y) \). Furthermore, \( \overline{\beta} \) has transcendence degree \( r \) over \( \mathbb{Q}(D_B, \lambda(B)) \). Finally, the extension of this morphism that sends \( \overline{\alpha} \) to \( \overline{\beta} \) induces an isomorphism of valued groups, \( \{ Q(\overline{D}_B, \lambda) \} \) and \( \mathbb{Q}(D_B, \overline{B}_\lambda)(\overline{\beta}) \). Let \( K' = \mathbb{Q}(\overline{D}_A, \overline{A}_\lambda)(\overline{\alpha})^h, L' = \mathbb{Q}(\overline{D}_B, \overline{B}_\lambda)(\overline{\beta})^h \). So, by Fact 3 and construction of \( \overline{D}_A \) and \( \overline{D}_B \), \( K' \cap D_A = \overline{D}_A \) and \( L' \cap D_B = \overline{D}_B \). So, \( (K', G') \) and \( (L', H') \) are isomorphic \( \mathcal{L}_G \)-structures. Furthermore, by Theorem 2.2 this isomorphism extends to an isomorphism between \( K'/(\lambda(A))^h \) and \( L'/(\lambda(B))^h \). So, it belongs to the back-and-forth system. 

**Proposition 3.3.** Let \( G = T^Z \times D \preceq \mathbb{Q}_p^* \) be a small subgroup with Mann property and \( [n]D^p \) is finite for all \( n \). Let \( (K, A) \) be a model of \( T_G \). Then, every definable subset of \( (K, A) \) is a boolean combination of subsets of \( K^n \) defined by formulas \( \exists \mathcal{F} \bigcup \{ \lambda(A), \lambda \} \)-structures between \( \exists \mathcal{F} \overline{D}_A, \overline{A}_\lambda(\overline{\alpha}) \) and \( \exists \mathcal{F} \overline{D}_B, \overline{B}_\lambda(\overline{\beta}) \). Let \( K' = \exists \mathcal{F}(\overline{D}_A, \overline{A}_\lambda)(\overline{\alpha})^h, L' = \exists \mathcal{F}(\overline{D}_B, \overline{B}_\lambda)(\overline{\beta})^h \). So, by Fact 3 and construction of \( \overline{D}_A \) and \( \overline{D}_B \), \( K' \cap D_A = \overline{D}_A \) and \( L' \cap D_B = \overline{D}_B \). So, \( K' \cap D_A \) and \( L' \cap D_B \) are isomorphic \( \mathcal{L}_G \)-structures. Furthermore, by Theorem 2.2 this isomorphism extends to an isomorphism between \( K'/(\lambda(A))^h \) and \( L'/(\lambda(B))^h \). So, it belongs to the back-and-forth system. 

**Proof.** By the above lemma, every formula is equivalent to a boolean combination of special formulas in \( \mathcal{F} \) i.e. formula of the type

\[ \Psi(\overline{\alpha}) \equiv \exists \mathcal{F} \bigcup \{ \lambda(A), \lambda \} \cap \phi(\overline{\alpha}). \]

Let \( A_{V,s} = (A \cap \mathcal{O}_K^n)^* \) (s-th powers). By quantifier elimination for abelian \( p \)-valued groups, \( \{ \overline{a} \in A^n_{V,s} \mid A_{V,s} \models \phi(\overline{a}) \} \) is equivalent to a boolean combination of

\begin{align*}
(\text{a}) & \quad \{ \overline{a} \in A^n_{V,s} \mid \xi(\overline{a}) = 1 \}; \\
(\text{b}) & \quad \{ \overline{a} \in A^n_{V,s} \mid V(\xi(\overline{a})) + r \boxplus V(\xi(\overline{a})) + r' \}; \\
(\text{c}) & \quad \{ \overline{a} \in A^n_{V,s} \mid \xi(\overline{a}) \equiv_n i \},
\end{align*}

where \( \xi(\overline{a}) = g_{1}^{k_1} \cdots g_{n}^{k_n} \) with \( k_i \in \mathbb{Z} \). Any of these formulas is definable by a quantifier-free \( \mathcal{L}_{\text{Mac}} \)-formula and existential quantifier over \( A_{V,s} \) (we need that \( [n](G \cap \mathbb{Z}_p) \) is finite for congruences). So, \( \Psi \) is equivalent to

\[ \Psi(\overline{\alpha}) \equiv \exists \mathcal{F} \bigcup \{ \lambda(A), \lambda \} \cap \phi(\overline{\alpha}), \]

with \( \phi' \) a \( \mathcal{L}(\lambda(A), \lambda) \)-formula (it can be assumed quantifier-free by Theorem 2.2). 

Finally, we describe which subsets of \( A \) are definable in the dense case.

**Proposition 3.4.** Let \( G < \mathbb{Q}_p^* \) be a dense subgroup of \( 1 + p^n \mathbb{Z}_p \). Let \( (K, A) \models \text{Th}(\mathbb{Q}_p, G) \). Let \( X \) be a subset of \( A^n \) definable in \( (K, A) \) (with parameters in \( K \)). Then there is \( Y \) definable in \( K \) and \( Z \) definable in \( A \) (in the language of \( p \)-valued groups) such that \( X = Y \cap Z \).
Proof. It is sufficient to prove the following:

Let $(K', A')$, $(L, H)$ be two $|K|$-saturated expansions of $(K, A)$. Let $\overline{g} \in A'^n$ and $\overline{h} \in H^n$ such that $tp_f(\overline{g}/K) = tp_f(\overline{h}/K)$ (type in the language of $p$-adically closed fields) and $tp_g(\overline{g}/A) = tp_g(\overline{h}/A)$ (type in the language of $p$-valued groups) then $(K', A', \overline{g}) \equiv_K (L, H, \overline{h})$.

For it is sufficient to find an element of the back-and-forth system in the proof of Theorem 2.3 whose domain and image contain $\overline{g}$ and $\overline{h}$ respectively. Take $K_1 = K(\overline{g})^h$, $A_1 = K_1 \cap A'$ and $K_2 = K(\overline{h})^K$ and $A_2 = K_2 \cap H$. By choice of $\overline{g}, \overline{h}$ and the proof of Theorem 2.3 (case 1. (b)), the application that sends $\overline{g}$ to $\overline{h}$ induces an isomorphism of $\mathcal{L}(G)$-structures between $(K_1, A_1)$ and $(K_2, A_2)$. Furthermore, this latter belongs to the back-and-forth system (remark that as $\lambda(G)$ is trivial, the steps that involve it can be removed from the proof of 2.3). This completes the proof of the proposition.

4 Expansion for the subanalytic structure

In this section, we consider $\mathbb{Q}_p^a$ the expansion of $\mathbb{Q}_p$ by all restricted analytic functions (in the sense of [6]). Let $\mathbb{Z}_p\{X\}$ be the set of restricted power series i.e. formal power series $f(X) = \sum a_i X^i$ with $a_i \in \mathbb{Z}_p$ and $v_p(a_i) \to \infty$ as $|I| \to \infty$. Let 

$$\mathcal{L}_a = \mathcal{L}_{Mac} \cup \{(f)_{f \in \mathbb{Z}_p\{X_1, \ldots, X_m\}, m \in \mathbb{N}}\}$$

be the expansion of the Macintyre language for $p$-adically closed fields by function predicates for these series where one interprets $f \in \mathbb{Z}_p\{X_1, \ldots, X_n\}$ in $\mathbb{Q}_p$ by

$$f(x) = \begin{cases} 
\sum a_i x^i & \text{if } \overline{x} \in \mathbb{Z}_p^n \\
0 & \text{otherwise.}
\end{cases}$$

Let $G$ be a subgroup of $\mathbb{Q}_p^a$. First, we remark that the case where $G$ is a dense subgroup of $1 + p^n\mathbb{Z}_p$ is beyond the scope of this paper: consider for instance the expansion of $\mathbb{Q}_p^n$ by a predicate for the multiplicative group $(1 + p\mathbb{Z})^\times$. In this expansion, the ring $(\mathbb{Z}, +, 0, 1)$ is definable. Indeed, let $exp_p(X) = \sum \frac{X^n}{n!}$ and $log_p(1 + X) = \sum (-1)^{n+1}X^n/n$ be the exponential and logarithm maps defined by the usual power series. Then, for all $n \in \mathbb{Z}$, $(1 + p)^n = exp_p(log_p(1 + p^n))$ (recall that $log_p(1 + X)$ and $exp_p(X)$ converges on $p\mathbb{Z}_p$ and therefore the last equality is well-defined as $v_p(log_p(1 + p)) = 1$). Therefore the map $exp_p(log_p(1 + p)X) : \mathbb{Z}_p \to 1 + p\mathbb{Z}_p$ establishes an isomorphism of groups between $(\mathbb{Z}, +) \subset \mathbb{Q}_p$ and $(1 + p\mathbb{Z}, +)$ that is $\mathcal{L}_{an}$-definable as $exp_p(log_p(1 + p)X) \in \mathbb{Z}_p\{X\}$. Therefore the ring $(\mathbb{Z}, +, 0, 1)$ is $\mathcal{L}_{an}$-definable. Similarly, for any subgroup of $(1 + p\mathbb{Z}_p, +)$, the above exponential map induces a structure of ring (isomorphic to a subring of $(\mathbb{Z}_p, +, 0, 1)$). We do not push further this case and focus on discrete subgroups of $\mathbb{Q}_p^a$.

Let $G$ be a discrete subgroup of $\mathbb{Q}_p^a$ i.e. $G = T^\times \times \mu_s$ for some $T \in p\mathbb{Z}_p$ and $\mu_s < \mu_{s+1}$. We will treat the case $G := \mu_{s-1} \cdot p^n$, the kernel of the Iwasawa Logarithm. The general case is similar.

Let $\mathcal{L}_a(G) = \mathcal{L}_a \cup \{A, \lambda, \xi\}$ where $\xi$ is interpreted by a given $(p - 1)$th root of unity in $\mathbb{Q}_p$, $\lambda$ is a unary function defined in $\mathbb{Q}_p$ by $\lambda(x) = p^{\lambda_p(x)}$ for all $x \in \mathbb{Q}_p$, and $A$ is a unary predicate interpreted by $G$. Let us recall that for all $x \in \mathbb{Q}_p$, there is $y_1, \ldots, y_p \in A$ with $y_{i+1} = \xi y_i$ and $v(x) = v(y_i)$ for
all $i$ i.e. $x = \lambda(x)^i$ for some $0 \leq i < p - 1$. This is a first-order statement. Therefore, if $(M, A)$ is a model of $Th(Q_p, G)$ in our language, we have a function $\lambda : M^* \to A$ such that for all $x, y \in M^*$, $v(x) = v(\lambda(x))$, $\lambda(xy) = \lambda(x)\lambda(y)$, $\lambda(p) = p$, $\lambda(\xi) = 1$ and for all $a \in A$, $a = \lambda(a)^i$ for some $0 \leq i \leq p - 1$. Then, $\lambda M^*$ is a subgroup of $A$ which contains $\mu^\infty$ and $A = \lambda M^* \times \mu_s$. Let $L^{\infty}_an = L_{an} \cup \{D\}$ and $L^{\infty}_an(G) := L_{an}(G) \cup \{D\}$ where $D$ is interpreted in $Q_p$ by

$$D(x, y) = \begin{cases} xy^{-1} & \text{if } v(x) \geq v(y) \\ 0 & \text{otherwise.} \end{cases}$$

Let $T^an_d(G)$ be the theory of $Q_p$ in the language $L^{\infty}_an(G)$. Then,

**Theorem 4.1.** The theory $T^an_d(G)$ admits the elimination of quantifiers in the language $L^{\infty}_an(G)$.

**Remark.** Theorem 2.2 claims that $Th(Q_p, +, -, 0, 1, (P_n), \mu_{p-1}p^\infty, \lambda)$ admits the elimination of quantifiers. On the other hand, quantifier elimination for $p$-adic analytic structure in $L_{an}$ is a classical result by Denef and van den Dries [6].

**Proof.** Let $\mathcal{M}$ be a $L^{\infty}_an(G)$-structure and $\mathcal{N}$ a model of our theory which contains $M$. Let $\mathcal{M}^*$ be a saturated model expansion of $M$. We have to prove that $\mathcal{N}$ embed in $\mathcal{M}^*$ over $\mathcal{M}$. First, we prove that the substructure generated by $M$ and $\lambda(N)$ embeds in $\mathcal{M}^*$. Let $b \in \lambda(N) \setminus \lambda(M)$. We will denote by $\mathcal{M}(b)$ the structure generated by $M$ and $b$ (i.e. the set of $f(b, \overline{\sigma})$ for all $\overline{\sigma} \in M^*$ and for all $L^{\infty}_an(G)$-term $f$). Then, it is sufficient to prove that $\mathcal{M}(b)$ embed in $\mathcal{M}^*$ over $\mathcal{M}$. Let $\mathcal{M}(b)$ be the field generated by $M$ and $b$. Note that $\mathcal{M}(b)$ is an immediate extension of $\mathcal{M}(b)$. Indeed, the closure of $\mathcal{M}(b)$ under $L_{an}$-terms is an immediate extension (use Weierstrass preparation theorem like in [6] to prove that for all $\overline{\sigma} \in M^*$, for all $f \in \mathbb{Z}_p[X, Y]$, there is $P(X) \in M[X]$ such that $v(f(b, \overline{\sigma})) = v(P(b))$ - see also later in the proof). Then the closure under $\lambda$ and $D$ is also immediate.

As $b \notin \lambda(M)$, $v(\mathcal{M}(b)) \neq v(M)$. Then, $v(\mathcal{M}(b)) = v(M) \oplus v(b)\mathbb{Z}$. Without loss of generality we may assume that $v(b) > 0$.

Now, there exists $\eta \in v(\mathcal{M}^*)$ which realizes the same type as $v(b)$ over $v(M)$ and furthermore for all $z \in \mathcal{M}^*$ with $v(z) = \eta$ the map $b \mapsto z$ induces an embedding of valued fields of $\mathcal{M}(b)$ into $\mathcal{M}^*$ (see Proposition 4.10B in [10]). Let $b'$ be the element of $\mathcal{M}(\mathcal{M}^*)$ such that $b' = \lambda(z)$ for any $z$ with $v(z) = \eta$. Then, we will prove that the map $\sigma : f(b, \overline{\sigma}) \mapsto f(b', \overline{\sigma})$ for all $\overline{\sigma} \in M^k$ and all $L^{\infty}_an(G)$-term $f$ induces an embedding of $L^{\infty}_an(G)$-structures of $\mathcal{M}(b)$ into $\mathcal{M}^*$.

First, let us remark that for all $f = \sum_i A_i(\overline{X})Y^i \in \mathbb{Z}_p\{Y, \overline{X}\}$ and for all $\overline{\sigma} \in M^m$, either $f(Y, \overline{\sigma}) \equiv 0$ or $v(f(b, \overline{\sigma})) = \min_i\{v(A_i(\overline{\sigma})b')\}$. For by [6] section 1.4, there is $d \in \mathbb{N}$ and $B_{ij} \in p\mathbb{Z}_p(\overline{X})$ such that $\|B_{ij}\| \to 0$ as $i \to \infty$ and for all $i \geq d$

$$A_i(\overline{X}) = \sum_{j<d} A_j(\overline{X})B_{ij}(\overline{X}) \quad (*)$$

If $A_i(\overline{\sigma}) = 0$ for all $i < d$, then $f(Y, \overline{\sigma}) \equiv 0$. Otherwise, there is $i < d$ such that $v(A_i(\overline{\sigma}))$ is minimal among the $v(A_j(\overline{\sigma}))$ and $i$ is the largest index less than $d$ with this property. This can be written as a first-order sentence $\mu_i(\overline{\sigma})$. Let $\overline{\tau} \in \mathbb{Z}_p^m$ such that $v(\overline{\tau}_i) = \mu_i(\overline{\sigma})$. Let $y \in \mathbb{Z}_p$ such that $v_p(y) > 0$ and $v_p(A_k(\overline{\sigma})y^j) \neq v_p(A_k(\overline{\sigma})y^j)$ for all $k \neq i < d$ (such that $A_k(\overline{\sigma}), A_i(\overline{\sigma})$ are nonzero). Let $k < d$
such that \( v_p(A_k(\overline{x})^y) \) is minimal. Note that this can be expressed by a first-order formula, \( \nu_k(\overline{x}, y) \). Then by (⋆), for all \( i \geq d, v_p(A_i(\overline{x})y^i) > v_p(A_k(\overline{x})y^k) \).

So, if \( \theta_{i,k} = \nu_k \land \mu_i \),

\[
Z_p \models \forall \overline{x} y \theta_{i,k}(\overline{x}, y) \rightarrow v(f(\overline{x}, y)) = v(A_k(\overline{x})y^k).
\]

As \( v(M(b)) = v M \oplus v(b) \mathbb{Z} \), for all \( P(X) = \sum p_i X^i \in M[X] \) nonzero, \( v(P(b)) = \min_i \{v(p_i b^i)\} \). For, if \( v(p_i b^i) = v(p_j b^j) \) then \( v(p_i) = v(p_j) = \infty \) or \( v(b) = \frac{v(p_i b^i)}{v(p_j b^j)} \) which is a contradiction with \( v(b) \notin vM \). Therefore, for all \( \overline{v} \in M^n \) such that \( f(\overline{v}, \overline{\pi}) \) is nonzero \( N \models \theta_{i,k}(\overline{v}, b) \) for some \( i, k \).

As we have seen above, for all \( f \in \mathbb{Z}[Y, \overline{X}] \) and \( \overline{\pi} \in M^n \) either \( f(Y, \overline{\pi}) \equiv 0 \) or \( f(b, \overline{\pi}) \neq 0 \) and \( f(b', \overline{\pi}) \neq 0 \). So, \( f(b, \overline{\pi}) = 0 \) iff \( \sigma(f(b, \overline{\pi})) = 0 \). By [11], we know also that \( f(b, \overline{\pi}) \in P_n \) iff \( f(b, \overline{\pi}) \in P_n \) (for the proof of Theorem 1.1 in [6], if \( f(b, \overline{\pi}) \in P_n \) iff \( g(b, \overline{\pi}) \in P_n \), where \( g \) is polynomial in \( b \); then use the fact that \( \sigma \) is an embedding of analytic valued fields). We will now prove that \( f(b, \overline{\pi}) \in A \) iff \( f(b, \overline{\pi}) \in A \). For as we have seen above, if \( f(b, \overline{\pi}) \neq 0 \), then \( v(f(b, \overline{\pi})) = v(A_k(\overline{\pi})b^k) \) for some \( k < d \). Then by definition of \( \lambda \), \( \lambda(f(b, \overline{\pi})) = \lambda(A_k(\overline{\pi})b^k) \).

Let us remark that for all \( \nu, \xi \) order formula, \( \nu \), such that \( \mu \), it follows from above that \( \nu \), depends only on \( \mu \). Therefore, this part follows from the quantifier elimination result in [6]. This concludes the proof of the theorem. 

\[ \square \]
5 Application: Model theory of the $p$-adic Iwasawa Logarithm

Let us remind that the $p$-adic Iwasawa logarithm is defined by $LOG(x) := \log_p(y)$ if $x = p^m \xi y \in \mathbb{Q}_p$ with $\xi \in \mu_{p-1}$ and $y \in 1 + p\mathbb{Z}_p$. Let us remark that as $\log_p(1 + pz) \in \mathbb{Z}_p\{X\}$, $LOG$ is definable in the structure $(\mathbb{Q}_p^{an}, \mu_{p-1}p^\mathbb{Z})$: for all $x \in \mathbb{Q}_p$, $x = y(1 + pz)$ for some (unique) $z \in \mathbb{Z}_p$ and $y \in \mu_{p-1}p^\mathbb{Z}$, then $LOG(x) = \log_p(1 + pz)$. In this section, we will define a sublanguage of $\mathcal{L}_F^n(G)$ for which the theory of $\mathbb{Q}_p$ is model-complete following the strategy of [16]. First, we define the Weierstrass system generated by $\log(1 + px)$: let $F \subset \mathbb{Z}_p\{X\}$ (in our case, we are interested by $F = \{\log(1 + px)\}$). We define by induction on $m$ a family of rings $W_{F,n}^{(m)} \subset \mathbb{Z}_p\{X_1, \ldots, X_n\}$. Let $W_{F,n}^{(0)}$ be the ring generated by $\mathbb{Z}[X_1, \ldots, X_n]$ and the elements of $F$. Assume that $W_{F,n}^{(k)}$ is defined for all $n \in \mathbb{N}$ and all $k \leq m$. Then $W_{F,n}^{(m+1)}$ is the ring generated by:

1. $W_{F,n}^{(m)}$;
2. $f(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$ for all $f \in W_{F,n}^{(m)}$ and $\sigma$ permutation of $\{1, \ldots, n\}$;
3. $f^{-1}$ for all $f \in W_{F,n}^{(m)}$ invertible in $\mathbb{Z}_p\{X_1, \ldots, X_n\}$;
4. $f/g(0)$ for all $f, g \in W_{F,n}^{(m)}$ such that $f$ is divisible by $g(0)$ in $\mathbb{Z}_p\{X\}$;
5. $A_0, \ldots, A_{d-1}Q$ obtained by Weierstrass division: for all $d$ and $f, g \in W_{F,n+1}^{(m)}$ with $f$ regular of order $d$ in $X_{n+1}$, by Weierstrass division, there is $A_0, \ldots, A_{d-1}, Q$ such that

$$g(X) = f(X)Q(X) + [A_0(X) + \cdots + A_{d-1}(X)]X_n^{d+1},$$

where $X = (X_1, \ldots, X_{n+1}), X' = (X_1, \ldots, X_n)$. Then, we require $Q \in W_{F,n+1}^{(m+1)}$ and $A_0, \ldots, A_{d-1} \in W_{F,n}^{(m+1)}$.

**Definition 5.1.** The Weierstrass system generated by $F$ is the collection of $W_{F,n} := \bigcup_m W_{F,n}^{(m)}$ for all $n \in \mathbb{N}$. We denote this system by $W_F$.

**Lemma 5.2.** $W_F$ is a Weierstrass system in the sense of [25] Definition 4.3.5.

**Proof.** From the definition of $W_F$, it is closed under Weierstrass division and therefore, it is a pre-Weierstrass system in the sense of Definition 4.3.3 [5]. It remains to prove condition (c) in Definition 4.3.5: We have to show that for all $f = \sum_I a_I X^I \in W_{F,n}$ then there is a finite set $J \subset \mathbb{N}^n$ such that for all $J \subset \mathcal{J}$, there is $g_J \in W_{F,n} \cap p\mathbb{Z}_p\{X\}$ such that

$$f = \sum_{J \in \mathcal{J}} a_J (1 + g_J)X^J \quad (\ast).$$

We proceed by induction on $n$: If $n = 1$, then $f(X) = a_0 + a_1X + a_2X^2 + \cdots$. As $f \in \mathbb{Z}_p\{X\}$, there is $d \in \mathbb{N}$ such that $\nu(a_i) \geq \nu(a_d)$ for all $i < d$ and $\nu(a_d) > \nu(a_d)$ for all $j > d$. Take $\mathcal{J} = \{1, \ldots, d\}$, $g_i = 0$ for $i < d$ and $g_d = \sum_{j > d} a_j a_d^{-1}X^{d-j}$. It satisfies all the conditions required: Indeed, by
Let \( J \) by induction hypothesis, there are \((1 + g_J)X^i \) such that
- \( v(a_I) \) is minimal among the \( v(a_K) \);
- \( v(a_I) > v(a_I) \) for all \( J = (j_1, \ldots, j_{n+1}) \) with \( j_l > i_l \) for some \( l \leq n + 1 \) and \( j_k \geq i_k \) for all \( k \neq l \);
- \( I \) is maximal for these two properties (with respect to the lexicographical order).

Let \( g_I := \sum_{K \in \mathbb{N}^{n+1}} a_{I+K} X^K \in W_{F,n+1} \). Then \( \sum_{K} a_{I+K} X^K = (1 + g_I)X^I \). Fix \( k \leq n + 1 \) and \( s \leq i_k \). By \( J_{k,s} \) we denote an element of \( \mathbb{N}^{n+1} \) with \( k \)th coordinate \( s \). Then,

\[
\begin{align*}
f_{k,s} &= f'_{k,s} = \sum_{(k',s') < (k,s)} \frac{1}{s!} \frac{\partial^s f_{k',s'}}{\partial X^s_{k}} \\
&= \sum_{j_{k,s}} a_{j_{k,s}} X_1^{j_1} \cdots X_{k-1}^{j_{k-1}} X_{k+1}^{j_{k+1}} \cdots X_{n+1}^{j_{n+1}} \in W_{F,n}.
\end{align*}
\]

Fix some order on the couple \((k, s)\) like above. Then, let

\[
f_{k,s} = f'_{k,s} - \sum_{(k',s') < (k,s)} \frac{1}{s!} \frac{\partial^s f_{k',s'}}{\partial X^s_{k}};
\]

(in \( f_{k,s} \) we only take coefficients \( a_{j_{k,s}} \) in \( f'_{k,s} \) that did not appear previously in some other \( f_{k',s'} \) in order to avoid repetition of coefficients in next equalities). By induction hypothesis, there are \( \mathcal{J}_{k,s} \) and \( g_L \in W_{F,n} \cap p\mathbb{Z}_p \{ X \} \) for all \( L \in \mathcal{J}_{k,s} \) such that

\[
f_{k,s} = \sum_{J \in \mathcal{J}_{k,s}} a_J X_1^{j_1} \cdots X_{k-1}^{j_{k-1}} X_{k+1}^{j_{k+1}} \cdots X_{n+1}^{j_{n+1}} (1 + g_J).
\]

Take \( \mathcal{J} := \bigcup_{k,s} \mathcal{J}_{k,s} \cup \{ I \} \) and \( g_L := g_L X^s \) if \( L \in \mathcal{J}_{k,s} \) and \( g_L = 0 \) otherwise. Then,

\[
f = \sum_{k,s} f_{k,s} X^s + \sum_{K \in \mathbb{N}^{n+1}} a_{I+K} X^{I+K} = \sum_{J \in \mathcal{J}} a_J X^J (1 + g_J).
\]

From this, it follows from [5] that

**Theorem 5.3.** The theory of \( \mathbb{Q}_p \) admits the elimination of quantifiers in the language \((+, -, 0, 1, (P_n)_{n \in \mathbb{N}}, (f)_{f \in W_F}, D)\).

Let \( W_{\log} = W_F \) for \( F = \{ \log_p(1 + px) \} \). A variation of the proof of Theorem [4,1] using the above quantifier elimination shows that

\[
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\]
Corollary 5.4. \( \text{Th}(\mathbb{Q}_p, +, -, \cdot, 0, 1, (P_n)_{n \in \mathbb{N}}, (f)_{f \in W_{\log}}, D, A, \xi, \lambda) \) admits the elimination of quantifiers.

In the above corollary, \( A \) is a unary predicate interpreted by \( \mu_{p-1}V^p \).

In the above theorem, it may not be obvious to describe all the elements in \( W_{\log} \). We will now give a countable language in which our theory is model-complete. The key idea is that in this language the elements of \( W_{\log} \) are existentially definable. First, let us define a family of finite algebraic extensions of \( \mathbb{Q}_p \): We fix \( K_n \) an finite algebraic extension of \( \mathbb{Q}_p \) such that

1. \( K_n \) is the splitting field of \( Q_n \) for some \( Q_n \in \mathbb{Q}[X] \) of degree \( d(n) \);
2. \( K_n = \mathbb{Q}_p(\beta_n) \) for any \( \beta_n \) root of \( Q_n \) and \( V_n := \mathbb{Z}_p[\beta_n] \) is the valuation ring of \( K_n \);
3. \( K_n \) contains \( K_{n-1} \) and all algebraic extension of \( \mathbb{Q}_p \) of degree less than \( n \).

Let \( y \in V_n \). Then, \( y = y_0 + y_1\beta_n + \cdots y_{d(n)-1}\beta_n^{d(n)-1} \) for some \( y_i \in \mathbb{Z}_p \). Let us remark that \( \log_p(1 + pX) \) is convergent on \( V_n \). And,

\[
\log_p(1 + py) = \log_p \left( 1 + p \left( \sum y_i\beta_n^i \right) \right) = c_0, n(\beta) + c_{1, n}(\beta)\beta_n + \cdots c_{d(n)-1, n}(\beta)\beta_n^{d(n)-1},
\]

for some \( c_{i, n} \in \mathbb{Z}_p \). These coefficients determine functions from \( Z_{p}^{d(n)} \) to \( Z_{p}^{d(n)} \). Note that by rearranging formally the power series \( \log_p(1 + p \sum y_i\beta_n^i) \), one shows that \( c_{i, n}(\beta) \in \mathbb{Z}_p \{\beta\} \). If we identify \( V_n \) with its structure of \( \mathbb{Z}_p \)-module, we see that:

Lemma 5.5. The structure \( (V_n, +, -, \cdot, 0, 1, \log_p(1 + pX)) \) is definable in \( (\mathbb{Q}_p, +, -, \cdot, 0, 1, \log_p(1 + pX), (c_{i, n})_{i \leq d(n)}) \).

Even more is true (see [16] Proposition 4.2):

Lemma 5.6. The structure \( (V_n, +, -, \cdot, 0, 1, \log_p(1 + pX), (c_{i, n})_{i \leq d(n)}) \) is definable in \( (\mathbb{Q}_p, +, -, \cdot, 0, 1, \log_p(1 + pX), (c_{i, n})_{i \leq d(n)} \).

It follows from Proposition 5.1 [16] that

Lemma 5.7. Let \( F = \{ \log_p(1 + px), c_{i, n}(X), n \in \mathbb{N}, i \leq d(n) \} \). Then for all \( f \in W_F \), \( f \) is existentially definable (as function from \( Z_p^k \) to \( Z_p \)) in terms of the elements of \( F \) and their derivatives.

As the derivatives of \( \log_p(1 + px) \) are existentially definable, so are the derivatives of \( c_{i, n} \) (using the equality \( (\alpha) \)). Then by the last lemma, the elements of \( W_F \) are definable in terms of the elements of \( W_F^{(0)} \). We combine this with Corollary 5.4 to get

Proposition 5.8. \( \text{Th}(\mathbb{Q}_p, +, -, \cdot, 0, 1, A, \xi, \lambda, \log_p(1 + px), c_{i, n}(X), n \in \mathbb{N}, i \leq d(n)) \) is model-complete.

Corollary 5.9. \( \text{Th}(\mathbb{Q}_p, +, -, \cdot, 0, 1, A, \xi, \lambda, \text{LOG}, c_{i, n}(X), n \in \mathbb{N}, i \leq d(n)) \) is model-complete.
6 Non-independence property

First, we prove that the theory of $\mathbb{Q}_p$ in the language $\mathcal{L}_a^D(p^2)$ is NIP. Let us recall the following result from [2]:

**Theorem 6.1** (Corollary 2.5 [2]). Let $T$ be a first-order $\mathcal{L}$-theory. Let $M \models T$ and $A \subset M$. Let $T_P$ be the theory of $(M, A)$ where we extend the language by a predicate $P$ interpreted in $M$ by $A$. Assume that $T$ is NIP, $A_{ind}(L)$ is NIP and $T_P$ is bounded. Then $T_P$ is NIP.

In the above theorem, $A_{ind}(L)$ is the structure induced on $A$ by the $\mathcal{L}$-definable sets in $M$. $T_P$ bounded means that any $\mathcal{L} \cup \{P\}$-formula is equivalent to a formula of the form $Q_1 x_1 \in P \cdots Q_n x_n \in P \Psi(\overline{x}, \overline{y})$ with $Q_i \in \{\exists, \forall\}$ and $\Psi$ is a $\mathcal{L}$-formula.

In the case of $T_d^{an}(p^2)$, by our result of quantifier elimination in Section 4, the theory is bounded (in the language $\mathcal{L}_{Mac} \cup \{A\}$ - as the function $\lambda$ is definable by a bounded formula). Furthermore, it is known that $T$ is NIP [10]. It remains to prove that $A_{ind}$ is NIP. For we will show that the induced structure is exactly the Presburger arithmetic.

**Lemma 6.2.** $(A, +, 0, A, (\equiv_n)_{n \in \mathbb{N}}, <)$ is $\mathcal{L}_{Mac}$-definable i.e. $A_{ind}$ is an expansion of Presburger arithmetic.

**Proof.** Let $(M, A, +_M, \cdot_M, 0_M, 1_M)$ be a model of $T_d^{an}(p^2)$. The addition in the group $A$ is given by the multiplication in the field and $0_A = 1_M, 1_A = p_M$. The order is defined by $x < y$ iff $v(x) < v(y)$ for all $x, y \in A$. Finally, note that $x \in A \cap P_n$ if $x$ is $n$ divisible in $A$.

We denote by $\mathcal{L}_{Pres}$ the language of Presburger $(+, -, 0, 1, <, \equiv_n (n \in \mathbb{N}))$. We will prove now that any definable set in $A_{ind}$ is $\mathcal{L}_{Pres}$-definable.

**Lemma 6.3.** Let $f = \sum a_i(\overline{Y})X^i \in \mathbb{Z}_p[\overline{Y}, X]$. Let $(M, A) \models Th(\mathbb{Q}_p^{an}, P^2)$. Then there exists $D(f)$ such that for all $\overline{y} \in M^k$ such that $f(\overline{y}, X)$ is nonzero, for all but finitely many $x \in A^{>0}$, $\lambda(f(\overline{y}), x) = \lambda(a_i(\overline{y}))x^i$ for some $i < D(f)$ (depending on $x$). Furthermore, for fixed $i$, the set of $x \in A^{>0}$ such that $\lambda(f(x)) = \lambda(a_i(x))x^i$ is definable using the order on $A$ (induced by the order on the valuation group).

**Proof.** Let $\overline{y}$ such that $f(\overline{y}, X)$ is nonzero. Then there is $i$ such that $v(a_i(\overline{y})) < \infty$. Let $d = d(\overline{y})$ be the largest index such that $v(a_d(\overline{y}))$ is minimal among the $v(a_i(\overline{y}))$. This number exists and is bounded uniformly over $\overline{y}$ by some $D(f) \in \mathbb{N}$ (see Lemma 1.4 in [6]). Then for all $x \in A^{>0}$ and $i > d$, $v(a_i(\overline{y})x^i) > v(a_d(\overline{y})x^d)$. Let $P(X) = a_0(\overline{y}) + \cdots + a_d(\overline{y})X^d$. First, let us remark that if for all $i \neq j$ $v(a_i(\overline{y})x^i) \neq v(a_j(\overline{y})x^j)$ then $v(P(x)) = \max\{v(a_i(\overline{y})x^i)\}$. In this case, $\lambda(P(x)) = \lambda(a_i(\overline{y})x^i) = \lambda(a_i(\overline{y}))\lambda(x)^i = \lambda(a_i(\overline{y}))x^i$ for the index $i$ such that $v(a_i(\overline{y})x^i)$ reaches the minimum. Then, $v(f(x)) = v(P(x) + \sum_{k>d} a_k(\overline{y})x^k) = v(P(x))$ (this is first-order and true in $\mathbb{Z}_p$ for all parameters $(\overline{y}, x)$ with the same properties, so this holds in $M$). So, $\lambda(f(x)) = \lambda(a_i(\overline{y}))x^i$ for some $i \leq d(\overline{y}) \leq D(f)$.

Assume that $v(a_i(\overline{y})x^i) = v(a_j(\overline{y})x^j) < \infty$ for some $j < i \leq D(f)$. Then, $v(a_i(\overline{y}) - a_j(\overline{y})) = (i - j)v(x)$. This equality can be satisfied by a unique $x \in A$ (as all $x \in A$ have distinct valuation). So, for all $x \in A$ except finitely
many point, either \(a_i(\vec{y}) = 0\) for all \(i < D(f)\) (so, \(f(\vec{y}, X) \equiv 0\)) or there is a unique \(i < D(f)\) such that \(v(a_i(\vec{y}))x^i\) is minimal. This complete the proof of the first part of the lemma.

Let \(i < D(f)\). Let us proof that the set of \(x \in A\) such that \(v(a_i(\vec{y}))\) is the unique minimum is definable using the order. If \(f(\vec{y}, X) \equiv 0\), then any \(x \in A\) satisfies the condition \(\lambda(f(\vec{y}, x)) = \lambda(a_i(\vec{y})x^i) = \lambda(0) := 0\). Otherwise, the minimum is reached. Let us remark that \(v(a_i(\vec{y}))\) is the unique minimum among the \(v(a_k(\vec{y})x^k)\) iff \((v(a_k(\vec{y})) - v(a_i(\vec{y})))/(k - i) > v(x)\) for all \(i < k < D\) and \((v(a_i(\vec{y})) - v(a_k(\vec{y})))/(i - k) < v(x)\) for all \(i > k\). So, our condition is equivalent to an interval (possibly with boundary to infinity) in \(A\).

**Remark.** Like in the above lemma, if \(h\) is a \(L^{\text{an}}\)-term and \(\vec{y} \in M^n\) then for all \(a \in A\) except finitely many, \(\lambda(h(a, \vec{y})) = \lambda(r(\vec{y}))a^i\) for some \(\vec{y} \in M'^n\), \(r\) rational function and \(i \in \mathbb{Z}\). For proceed like in the proof of Theorem 4.11 to prove that \(h(a, \vec{y})\) is the product between a unit and a rational function. Then we argue like in the above lemma where \(P\) is now a rational function.

**Theorem 6.4.** Any definable set in \(A_{\text{ind}}\) is \(L_{\text{Pres}}\)-definable.

**Proof.** First, we show that the definable subsets of \(A\) are \(L_{\text{Pres}}\)-definable. Let \(X\) be such a set. By quantifier elimination, \(X\) is boolean combination of sets of the type:

\[
\{x \in A \mid f(x) \in \mu P_n\} \quad (1)
\]

\[
\{x \in A \mid f(x) = 0\} \quad (2)
\]

\[
\{x \in A \mid f(x) \in A\} \quad (3)
\]

where \(f\) is a term of the language \((+, - , 0, 1, \xi, \lambda, A, (P_n)_{n \in \mathbb{N}}, D, (g)_{g \in \text{Res}(X)})\) (we allow parameters from \(M\)). By Lemma 6.3 and the remark after, there is a finite union of intervals and points such that on each interval \(I\), the set \(X \cap I\) is equivalent to a definable set \(X'\) where any term that appears in the definition of \(X'\) is a term where the function \(\lambda\) applies only on the parameters. So, we may assume that \(X\) is defined by terms of the type \(f(X, \vec{y})\) where \(f\) is a \(L^{\text{an}}\)-term and \(\vec{y} \in M^n\). We may assume that \(f(X, \vec{y})\) is not identically zero (otherwise, the above sets are trivially \(L_{\text{Pres}}\)-definable).

The case (2) is a finite set. In case (3), by Lemma 6.3 we see that \(\lambda(f(x, \vec{y})) = \lambda(a_i(\vec{y})x^i)\) for some \(i\) for all but finitely many \(x \in A\). Furthermore, \(f(x, \vec{y}) \in A\) iff \(\lambda(f(x, \vec{y})) = f(x, \vec{y})\). Then \(f(x, \vec{y}) = \lambda(a_i(\vec{y})x^i)\). Then, \(f(x, \vec{y}) \in A\) iff \(x\) is a root of \(f(X) - \lambda(a_i(\vec{y}))x^i\). If this later series is not identically zero, it has finitely many roots in \(A\) (in particular the set of such \(x\) is definable). Otherwise, the series is identically zero and therefore any points \(x \in A\) such that \(v(a_i(\vec{y}))x^i\) is minimal belongs to (3). This is a \(L_{\text{Pres}}\)-definable condition by Lemma 6.3.

Finally, in the case (1), as by Lemma 6.3 the \(\lambda\) function occurs only in the parameters, we can assume by analytic cell decomposition [3] that the set is of the type

\[
\{x \in A \mid v(\alpha)\square_1 v(x - c)\square_2 v(\beta) \quad x - c \in \mu P_n\},
\]

for some \(\alpha, \beta, c \in M, \mu, n \in \mathbb{N}\). The first condition \(v(\alpha)\square_1 v(x - c)\square_2 v(\beta)\) clearly determines a finite union of intervals (possibly with bounds to infinity) in \(A\) and points: for all \(t \in A\), if \(t \neq \lambda(c)\) then \(v(t - c) = \min\{v(t), v(c)\}\). So, for all \(t\) such that \(v(t) > v(c)\), \(t\) satisfies the first condition iff \(v(\alpha)\square_1 v(c)\square_2 v(\beta)\).
If \( v(t) < v(c) \) then \( t \) satisfies the first condition iff \( v(\alpha) \Box_1 v(t) \Box_2 v(\beta) \) which is an interval. As there is only one \( t \in A \) such that \( v(t) = v(c) \) this last case is equivalent to \( v(\alpha) \Box v(\lambda(c) - c) \Box v(\beta) \): either \( \lambda(c) \) is in \( X \) or not. So, the first condition is \( \mathcal{L}_{\text{Pres}} \)-definable.

It remains to prove that the condition \( x \in A \land x - c \in \mu P_n \) is \( \mathcal{L}_{\text{Pres}} \)-definable. Let us recall that for all \( z \in \mathbb{Q}_p, z \in \mu P_n \) iff \( n \) divides \( v_p(\mu^{-1} z) \) and \( \mu^{-1} z P^{-v(\mu^{-1} z)} \in P_n \). So, for all \( x \in A, x - c \in \mu P_n \) iff \( n \) divides \( v(\mu^{-1} (x - c)) \) and \( \mu^{-1} (x-c) \lambda(\mu^{-1} (x-c))^{-1} \in P_n \). We split this condition into the three possible cases according to whether or not \( v(x) \) is larger than \( v(c) \). Then we will show that each case is \( \mathcal{L}_{\text{Pres}} \)-definable. The three possibilities are:

(a) \( v(x) < v(c) \), \( n \) divides \( v(\mu^{-1} x) \) and \( \mu^{-1} \lambda(\mu) - \mu^{-1} \lambda(\mu) c x^{-1} \in P_n \);

Indeed, in the case where \( v(x) < v(c) \), \( v(\mu^{-1} (x - c)) = v(\mu^{-1} x) \). So, \( n \) divides \( v(\mu^{-1} (x - c)) \) iff \( n \) divides \( v(\mu^{-1} x) \). On the other hand, \( \lambda(\mu^{-1} (x - c)) = \lambda(\mu^{-1}) \lambda(x) = \lambda(\mu^{-1}) x \). Therefore, in this case \( x - c \in \mu P_n \) is equivalent to the above relation.

(b) \( v(x) > v(c) \), \( n \) divides \( v(\mu^{-1} x) \) and \( \mu^{-1} x \lambda(\mu^{-1} c) - \mu^{-1} c \lambda(\mu^{-1} c)^{-1} \in P_n \);

(c) \( x = \lambda(c) \) and \( c - \lambda(c) \in \mu P_n \).

In case (a), \( n \) divides \( v(\mu^{-1} x) \) holds iff \( \lambda(\mu) \equiv_n x \): this is \( \mathcal{L}_{\text{Pres}} \)-definable. Then let us recall that for all \( n \), there is \( K(n) \) such that for all \( a, b \in \mathbb{Q}_p \) with \( v(b) > K(n) \), \( a(1 + b) \in P_n \) iff \( a \in P_n \). Apply this with \( a = \mu^{-1} \lambda(\mu) \) and \( b = -cx^{-1} \). Therefore, for all \( x \in A \) with \( v(x) < v(c) - K(n) \), the second condition of (a) holds iff \( n \) divides \( v(\mu^{-1} c) \): this is independent of \( x \) and so is \( \mathcal{L}_{\text{Pres}} \)-definable. Finally, for all \( x \in A \) with \( v(x) - K(n) \leq v(x) < v(c) \) either \( x \in X \) or \( x \notin X \). As there is only finitely many \( x \) that satisfies the condition \( v(\cdot - K(n) \leq v(\cdot) < v(c) \), the set of such \( x \in X \) corresponds to a finite union of points.

The case (b) is similar: the first condition becomes \( n \) divides \( v(\mu^{-1} x) \) and so does not depends on \( x \). The second condition is equivalent for \( v(x) > v(c) + K(n) \) to \( \mu^{-1} c \lambda(\mu^{-1} c)^{-1} \in P_n \). And for \( v(x) < v(c) + K(n) \) it is satisfied by finitely many \( x \in A \). Finally, case (c) does not depend on \( x \).

We regroup all these conditions and we see that \( X \) is indeed \( \mathcal{L}_{\text{Pres}} \)-definable.

This proves that \( Th(A_{ind}) \) is \( \mathcal{L}_{\text{Pres}} \)-minimal (in the the sense of \([\mathbf{4}]\)). So by Theorem 6 in \([\mathbf{4}]\), any definable set of \( A^n \) is \( \mathcal{L}_{\text{Pres}} \)-definable.

We combine this theorem with Theorem 6.1 to obtain:

**Corollary 6.5.** The theory of \( \mathbb{Q}_p \) in the language \( \mathcal{L}_{\text{an}}^{D}(\mathbb{Z}) \) is NIP.

**Corollary 6.6.** The theory of \( (\mathbb{Q}_p, \text{LOG}) \) is NIP.

**Proof.** This immediate from the above corollary as \((\mathbb{Q}_p, \text{LOG})\) is \( \mathcal{L}_{\text{an}}^{D}(\mathbb{Z}) \)-definable: for all \( x \in \mathbb{Q}_p \), there is a unique \( \xi \in \mu_{p-1}, \alpha \in \mathbb{Z}^p \) such that \( x = a\xi(1 + py) \). This decomposition is \( \mathcal{L}_{\text{an}}^{D}(\mathbb{Z}) \)-definable. So, \( \text{LOG}(x) = \text{LOG}(a) + \text{LOG}(\xi) + \text{LOG}(1 + py) = \log_p (1 + py) \) which is \( \mathcal{L}_{\text{an}} \)-definable.

To conclude this section, we prove that the dense case is also NIP in the finitely generated case.

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Theorem 6.7. The theory of \((\mathbb{Q}_p, G)\) is NIP for all \(G\) finitely generated subgroup of \(1 + p\mathbb{Z}_p\).

Proof. By Theorem 6.1 and Proposition 3.3 it is sufficient to prove that the structure induced on the group \(G\) is NIP. By Proposition 3.4 formulas in \(A_{ind}\) are of the type \(\Phi \land \Psi\) where \(\Phi\) is a formula in the language of \(p\)-adically closed fields (with parameters) and \(\Psi\) is a formula in the language of \(p\)-valued groups. As indiscernible sequences in \(A_{ind}\) are indiscernible in \(K\) (as valued fields) and in \(A\) (as valued groups), as the theory of \(p\)-adically closed fields is NIP and by Theorem 1.7, \(\Phi\) and \(\Psi\) are NIP. Therefore, \(Th(A_{ind})\) is NIP.

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