CONDUCTORS OF $\ell$-ADIC REPRESENTATIONS

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Abstract. We give two formulas for the Artin conductor of an $\ell$-adic representation of the Weil group of a local field of residue characteristic $p \neq \ell$. Both formulas are surely well known, but seem not to appear in completely satisfactory form in the literature.

1. Introduction

Our aim in this note is to fill a small gap in the literature by giving two formulas for the Artin conductor of an $\ell$-adic representation of the Galois group of a non-archimedean local field of residue characteristic $p \neq \ell$, see Theorem 1 in Section 7. Both are surely well-known, and we make no claim of priority, but neither seems to appear in completely satisfactory form in the literature. We hope having a clear reference will be useful to the community. See below for more details on antecedents.

Throughout, $F$ will be a non-archimedean local field with residue field of characteristic $p$ and cardinality $q$. We write $\overline{F}$ for a separable closure of $F$, $G_F$ for the Galois group $\text{Gal}(\overline{F}/F)$, and $\Phi$ for a geometric Frobenius element, i.e., an element of $G_F$ which induces the $q^{-1}$-power Frobenius automorphism of the residue field of $\overline{F}$. We write $W_F$ for the Weil group of $F$, the subgroup of $G_F$ inducing integral powers of the $q$-power Frobenius on the residue field, and we write $I_F$ for the inertia subgroup of $G_F$, the subgroup acting as the identity on the residue field.

Let $\ell$ be a prime number distinct from $p$ and let $E$ be a finite extension of $\mathbb{Q}_\ell$, the $\ell$-adic numbers. Let

$$\rho_\ell : W_F \to \text{GL}_n(E)$$

be a continuous representation where $\text{GL}_n(E)$ is given the topology induced by the metric ($\ell$-adic) topology on $E$. We are concerned with (the exponent of) the Artin conductor of $\rho_\ell$, which we denote $a(\rho_\ell)$ and call simply the conductor.

The conductor of a representation $\rho$ of $W_F$ depends only on the restriction of $\rho$ to the inertia group $I_F$, and it is defined in the first instance (e.g., in [Ser79, Ch. VI]) only for representations which factor through finite quotients of $W_F$. Since the image of an $\ell$-adic representation restricted to inertia need not be finite, further discussion is required. In the canonical references [Del73] and [Tat79], the conductor is defined rather indirectly in terms of Weil-Deligne representations. (See Section 4 below for the definition.) Luckily, it may also be calculated directly in terms of $\rho_\ell$.

Our main result, Theorem 1 below, gives two expressions for $a(\rho_\ell)$ in terms of simple invariants of $\rho_\ell$ and conductors of representations whose restriction to inertia has finite image. The first appears in [Tat79], but unfortunately there is a typographical error there. The second appears in the as-yet unpublished lecture
notes [Wie12], but this reference does not include a proof that the formula agrees with the definition given in [Del73].

After setting up the necessary definitions and notation in Sections 2 through 6 we give the statement and (easy) proof of Theorem 1 in Section 7. Section 8 gives an application which was our original motivation for thinking about this question.

The notes [Wie12] were pointed out to us by Romyar Sharifi after we re-discovered the second formula in Theorem 1, and we thank him for this reference. We also thank David Rohrlich for his comments and encouragement.

2. Ramification groups

In this section, we review the lower and upper ramification filtrations on Galois groups. See [Ser79, Ch. IV] for more details.

Let $K$ be a finite Galois extension of $F$ with group $G = \text{Gal}(K/F)$. We write $\mathcal{O}_K$ for the ring of integers of $K$, $\pi_K$ for a generator of the maximal ideal of $\mathcal{O}_K$, and $v_K$ for the valuation of $K$ with $v_K(\pi_K) = 1$.

The ramification filtration on $G$ in the lower numbering is defined by the requirement that

$$
\sigma \in G_i \iff v_K(\sigma(x) - x) \geq i + 1 \quad \forall x \in \mathcal{O}_K
$$

for $i$ an integer $\geq -1$. Clearly $G_{-1} = G$, $G_0$ is the inertia subgroup of $G$, and $G_i = 0$ for all sufficiently large $i$. By convention, if $r \geq -1$ is a real number, we set $G_r = G_i$ where $i$ is the smallest integer $\geq r$.

Let $\varphi : [-1, \infty) \to [-1, \infty)$ be the continuous, piecewise linear function with $\varphi(-1) = -1$, slope 1 on $[-1,0)$, and slope $1/[G_0 : G_i]$ on $(i - 1, i)$. Let $\psi = \varphi^{-1}$, the inverse function. The upper numbering of the ramification filtration on $G$ is given by

$$
G^\varphi = G_{\varphi(s)} \quad \text{and} \quad G^{\psi(r)} = G_r.
$$

Note that the breaks in the upper numbering (i.e., the values $s$ so that $G^{s+\epsilon} \neq G^s$ for all $\epsilon > 0$) are in general rational numbers, not necessarily integers.

The upper numbering is adapted to quotients in the following sense: if $L/F$ is a Galois extension with $L \subset K$ and $H = \text{Gal}(L/F)$, so that $H$ is a quotient of $G$, then the upper numbering satisfies

$$
H^s = \text{Im}(G^s \to H).
$$

This property allows us to define a ramification filtration on $G_F = \text{Gal}(\overline{F}/F)$ by declaring that

$$
G_F^s = \{ \sigma \in G_F \mid \sigma|_K \in \text{Gal}(K/F)^s \quad \forall K \}\}
$$

where $K$ runs through all finite Galois extensions of $F$. Clearly we have $G_F^{-1} = G_F$ and $G_F^0 = I_F$.

We define

$$
G_F^{\geq 0} = \cup_{\epsilon > 0} G_F^\epsilon
$$

where the union is over all positive real numbers $\epsilon$. We also write $P_F$ for $G_F^{> 0}$ and call this the wild inertia group of $F$. It is known to be a pro-$p$ group and the quotient $I_F/P_F$ is isomorphic as a profinite group to $\prod_{\ell \neq p} \mathbb{Z}_\ell$. 

3. Conductors

In this section we review the definition of the Artin conductor of a representation of $G = \text{Gal}(K/F)$ where $K/F$ is a finite Galois extension. See [Ser79, Ch. VI] for more details.

Let $\rho: G \to \text{GL}_n(E)$ be a representation where $E$ is a field of characteristic zero. We write $V$ for the space where $\rho$ acts, namely $E^n$, and for a subgroup $H$ of $G$ we write $V^H$ for the invariants under $H$:

$$V^H = \{ v \in V \mid \rho(h)(v) = v \ \forall h \in H \}. $$

Recall the ramification subgroups $G_i$ of the previous section. For a subspace $W$ of $V$, we write $\text{codim} W$ for the codimension of $W$ in $V$, i.e., $\dim V - \dim W$. With these notations, we define the Artin conductor of $\rho$ as

$$a(\rho) := \sum_{i=0}^{\infty} \text{codim} V^{G_i} \frac{[G_0 : G_i]}{[G_0 : G_i]}. $$

Note that this is in fact a finite sum and that it depends only on the restriction of $\rho$ to $G_0$ the inertia subgroup of $G$. It true but not at all obvious that $a(\rho)$ is an integer; see [Ser79, Ch. VI, §2, Thm. 1'].

Because the definition of $a(\rho)$ depends only on $\rho$ restricted to inertia, we may extend it to representations $\rho$ which are only assumed to have finite image after restriction to inertia.

We give two alternate expressions for $a(\rho)$ which will be useful in what follows. First, we have

$$a(\rho) = \int_{-1}^{\infty} \text{codim} V^{G_r} \frac{[G_0 : G_r]}{[G_0 : G_r]} dr$$

because the integrand is constant on intervals $(i - 1, i)$ and the corresponding Riemann sum for the integral is exactly the sum defining $a(\rho)$. Second,

$$a(\rho) = \int_{-1}^{\infty} \text{codim} V^{G_s} ds.$$ 

This follows from the previous expression and the definition of the function $\varphi$ relating the upper and lower numberings. Indeed, if $s = \varphi(r)$, then $ds = \varphi'(r) dr$ and $\varphi'(r) = 1/[G_0 : G_i]$ for $r \in (i - 1, i)$.

This last formula for $a(\rho)$ turns out to be best as it generalizes $\text{telle quelle}$ to $\ell$-adic representations. It also makes evident the fact that if $\rho$ factors through $H = \text{Gal}(L/F)$ for some subextension $L \subset K$, then the conductor of $\rho$ as a representation of $G$ is the same as the its conductor as a representation of $H$.

For use later we note that the first term in the sum for $a(\rho)$ and the first part of the integrals for it are all equal:

$$\int_{-1}^{0} \text{codim} V^{G_r} ds = \int_{-1}^{0} \frac{\text{codim} V^{G_r}}{[G_0 : G_r]} dr = \text{codim} V^{G_0} = \text{codim} V^{I_F} \quad (3.1)$$

4. Weil-Deligne representations

In this section we review (with the minimum of details) the notion of a Weil-Deligne representation. See [Del73, §8] or [Tat79, 4.1] for more details, and [Roh94] for a motivated introduction aimed at arithmetic geometers.

We write $\| \cdot \|$ for the homomorphism $W_F \to \mathbb{Q}$ which sends $\Phi$ to $q^{-1}$ and which is trivial on $I_F$. 

Let \( V \) be a vector space over a field of characteristic 0. We define a Weil-Deligne representation of \( W_F \) on \( V \) as a pair \((\rho, N)\) where \(\rho : W_F \to \text{GL}(V)\) is a homomorphism continuous with respect to the trivial topology on \( V \) and \( N : V \to V \) is an endomorphism satisfying

\[
\rho(w)N\rho(w)^{-1} = ||w||N.
\]

Continuity of \(\rho\) implies that is has finite image restricted to inertia, and the displayed formula implies that \( N \) is nilpotent (because its eigenvalues are stable under multiplication by \( q \)).

Because \(\rho\) has finite image when restricted to \( I_F \), its conductor is defined by the formulas of the preceding section. We define the Artin conductor of a Weil-Deligne representation \((\rho, N)\) as

\[
a(\rho, N) := a(\rho) + \dim V^{I_F} - \dim V_N^{I_F}.
\]

Here \( V_N \) is the kernel of \( N \) on \( V \), so that

\[
V_N^{I_F} = \{ v \in V | N(v) = 0, \rho(w)(v) = v \forall w \in I_F \}.
\]

5. \( \ell \)-adic representations

We define an \( \ell \)-adic representation to be a continuous homomorphism

\[
\rho_\ell : W_F \to \text{GL}_n(E)
\]

where \( E \) is a finite extension of \( \mathbb{Q}_\ell \) and \( \text{GL}_n(E) \) is given the \( \ell \)-adic topology. A primary source is of such representations is \( \ell \)-adic cohomology. More precisely, if \( X \) is a variety over \( F \), then the \( \ell \)-adic étale cohomology groups \( H^i(X \times \overline{F}, \mathbb{Q}_\ell) \) (and variants) are equipped with continuous actions of \( G_F \) and we may restrict to \( W_F \) to obtain \( \ell \)-adic representations as defined above.

Because \( P_F \) is a pro-\( p \) group and \( I_F/P_F \cong \prod_{\ell \neq p} \mathbb{Z}_\ell \), there is a non-zero homomorphism \( t_\ell : I_F \to \mathbb{Q}_\ell \) which is unique up to a scalar. It satisfies \( t_\ell(w\sigma w^{-1}) = ||w||t_\ell(\sigma) \) for all \( w \in W_F \).

The structure of the inertia group \( I_F \) briefly alluded to above leads to a description of the behavior of \( \rho_\ell \) restricted to \( I_F \). Namely, the Proposition of [ST68, App.], attributed to Grothendieck, says that there is a unique nilpotent linear transformation \( N : E^n \to E^n \) such that for all \( \sigma \) in some finite index subgroup of \( I_F \)

\[
\rho_\ell(\sigma) = \exp(t_\ell(\sigma)N)
\]

as automorphisms of \( E^n \). Here \( \exp \) is defined by the usual series \( 1+ x + x^2/2! + \cdots \) and \( \exp(t_\ell(\sigma)N) \) is in fact a finite sum because \( N \) is nilpotent.

It follows from this (see [Del73, §8]) that there exists a unique Weil-Deligne representation \((\rho, N)\) on \( V = E^n \) such that for all \( m \in \mathbb{Z} \) and all \( \sigma \in I_F \)

\[
\rho_\ell(\Phi^m \sigma) = \rho(\Phi^m \sigma) \exp(t_\ell(\sigma)N).
\] (5.1)

Conversely, given a Weil-Deligne representation \((\rho, N)\) on \( V \), the displayed formula defines an \( \ell \)-adic representation. This correspondence gives a bijection on isomorphism classes. (The correspondence \(\rho_\ell \leftrightarrow (\rho, N)\) depends on the choices of \( t_\ell \) and \( \Phi \), but after passing to isomorphism classes it is independent of these choices, see [Del73].)

The point of introducing Weil-Deligne representations is that their definition uses only the trivial topology on \( V \), so is convenient for shifting between different ground fields (such as \( \mathbb{Q}_\ell \) for varying \( \ell \) and \( \mathbb{C} \)).
Using this correspondence, we define

$$a(\rho) := a(\rho, N).$$

We note that $\rho(I_F)$ is finite if and only if the corresponding $N = 0$, and in this case the definition above reduces to that of Section 3.

We note also that $t_\ell$ is trivial on the wild inertia group $P_F = G^1_F$, so $\rho_\ell$ and $\rho$ are equal on $G^1_F$.

6. Semi-simplification

Fix an $\ell$-adic representation $\rho_\ell : W_F \to \text{GL}_n(E)$ and let $(\rho, N)$ be the corresponding Weil-Deligne representation. To keep the various actions distinct, we adopt the following notation: We write $V_\ell$ for $E^n$ with its action of $W_F$ via $\rho_\ell$, and we write $V$ for $E^n$ with its action of $W_F$ via $\rho$ and its nilpotent endomorphism $N$.

We will also consider $V_{ss}$ and $\rho_{ss}$, the semi-simplifications of $V_\ell$ and $\rho_\ell$, defined as the direct sum of the Jordan-Hölder factors of $V_\ell$ as a $W_F$ module.

We note that $\rho$ restricted to inertia has finite image, so must be semi-simple. On the other hand, equation (5.1) shows that $\rho$ and $\rho_\ell$ have the same character, which is also the character of $\rho_{ss}$. It follows that $\rho$ restricted to inertia and $\rho_{ss}$ restricted to inertia are isomorphic representations.

7. Two formulas for $a(\rho_\ell)$

We can now state the main result of this note. The first equality of the theorem is the definition, and the next two are the formulas of the title of this section. The second appears in [Tat79, 4.2.4], but is missing the exponent $I_F$ on the last term. The third appears in [Wie12] as Def. 3.1.27. This reference seems to include everything needed to prove that Def. 3.1.27 agrees with the standard definition, but the proof is not given there.

**Theorem 1.** Let $\rho_\ell$ be an $\ell$-adic representation of $W_F$ on $V_\ell$ with corresponding Weil-Deligne representation $(\rho, N)$ on $V$ and semisimplification $\rho_{ss}$ on $V_{ss}$. Let $a(\rho_\ell)$ be Artin conductor of $\rho_\ell$, defined as in Section 3 by

$$a(\rho_\ell) = a(\rho) + \dim V^{IF} - \dim V^{IF}_N. \quad (7.1)$$

Then we have

$$a(\rho_\ell) = a(\rho_{ss}) + \dim V^{IF}_{ss} - \dim V^{IF}_\ell \quad (7.2)$$

and

$$a(\rho_\ell) = \int_{-1}^{\infty} \text{codim} V^G_\ell^* ds. \quad (7.3)$$

**Proof.** Since $\rho$ and $\rho_{ss}$ are isomorphic when restricted to inertia, $a(\rho) = a(\rho_{ss})$ and $\dim V^{IF} = \dim V^{IF}_{ss}$. On the other hand, it is easy to see from equation (5.1) (with $m = 0$) that $V^{IF}_N = V^{IF}_\ell$. Thus the right hand sides of (7.1) and (7.2) are equal.

Using the second integral expression for $a(\rho)$ in Section 3 and equation 8.31, we see that

$$a(\rho_\ell) = \text{codim} V^{IF} + \int_{0}^{\infty} \text{codim} V^G^* ds + \dim V^{IF} - \dim V^{IF}_N$$

$$= \text{codim} V^{IF}_N + \int_{0}^{\infty} \text{codim} V^G^* ds$$
and since $V_{N}^{I} = V_{t}^{I}$ and $\rho = \rho_{t}$ on $G_{F}^{> 0}$, we have
\[
\text{codim } V_{N}^{I} + \int_{0}^{\infty} \text{codim } V_{s}^{G^{s}} \, ds = \text{codim } V_{t}^{I} + \int_{0}^{\infty} \text{codim } V_{s}^{G^{s}} \, ds
\]
\[
= \int_{-1}^{\infty} \text{codim } V_{s}^{G^{s}} \, ds
\]
and thus the right hand sides of (7.1) and (7.3) are equal. This completes the proof of the theorem. □

8. AN APPLICATION TO TWISTING

We give an easy application of the theorem which is the motivation for this work.

Let $\rho_{t} : W_{F} \to \GL_{n}(E)$ be an $\ell$-adic representation and let $\chi : W_{F} \to E^{\times}$ be a character of finite order. We say “$\chi$ is more deeply ramified than $\rho_{t}$” if there exists a non-negative real number $s$ such that $\rho_{t}(G_{F}^{s}) = \{\text{id}\}$ and $\chi(G_{F}^{s}) \neq \{\text{id}\}$. In other words, $\chi$ is non-trivial further into the ramification filtration than $\rho_{t}$ is. Let $s_{0}$ be the largest number such that $\chi$ is non-trivial on $G_{F}^{s_{0}}$. It follows from Section 3 that $a(\chi) = 1 + s_{0}$.

Proposition 1. If $\chi$ is more deeply ramified than $\rho_{t}$, then
\[
a(\rho_{t} \otimes \chi) = \deg(\rho_{t})a(\chi).
\]
Proof. Let $V_{t}$ be the space where $W_{F}$ acts via $\rho_{t}$ and let $V_{t,\chi}$ be the same space where $W_{F}$ acts via $\rho_{t} \otimes \chi$. By the theorem we have
\[
a(\rho_{t} \otimes \chi) = \int_{-1}^{\infty} \text{codim } V_{t,\chi}^{G_{F}^{s}} \, ds.
\]
If $s \leq s_{0}$ then $V_{t,\chi}^{G_{F}^{s}} \subset V_{t}^{G_{F}^{s}}$ and the latter is zero because $\rho_{t}(G_{F}^{s_{0}}) = \{\text{id}\}$ and $\chi(G_{F}^{s_{0}}) \neq \{\text{id}\}$. Thus in this range the integrand is $\dim V_{t} = \deg(\rho_{t})$. On the other hand, if $s > s_{0}$, then $\rho_{t} \otimes \chi(G_{F}^{s}) = \{\text{id}\}$ and the integrand is zero. Thus
\[
\int_{-1}^{\infty} \text{codim } V_{t,\chi}^{G_{F}^{s}} \, ds. = \deg(\rho_{t})(1 + s_{0}) = \deg(\rho_{t})a(\chi)
\]
as desired. □

David Rohrlich points out that under the assumptions that $\rho_{t}$ is irreducible and has finite image, the proposition also follows from \cite{Ser79, p. 103, Ex. 2].

A particularly useful case of the proposition occurs when $\rho_{t}$ is tamely ramified and $\chi$ is wildly ramified, e.g., when $\chi$ is an Artin-Schreier character.

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