Propositional satisfiability in declarative programming

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Abstract

Answer-set programming (ASP) paradigm is a way of using logic to solve search problems. Given a search problem, to solve it one designs a theory in the logic so that models of this theory represent problem solutions. To compute a solution to a problem one needs to compute a model of the corresponding theory. Several answer-set programming formalisms have been developed on the basis of logic programming with the semantics of stable models. In this paper we show that also the logic of predicate calculus gives rise to effective implementations of the ASP paradigm, similar in spirit to logic programming with stable model semantics and with a similar scope of applicability. Specifically, we propose two logics based on predicate calculus as formalisms for encoding search problems. We show that the expressive power of these logics is given by the class NP-search. We demonstrate how to use them in programming and develop computational tools for model finding. In the case of one of the logics our techniques reduce the problem to that of propositional satisfiability and allow one to use off-the-shelf satisfiability solvers. The language of the other logic has more complex syntax and provides explicit means to model some high-level constraints. For theories in this logic, we designed our own solver that takes advantage of the expanded syntax. We present experimental results demonstrating computational effectiveness of the overall approach.

1 Introduction

Logic is most commonly used in declarative programming and computational knowledge representation as follows. To solve a problem, we represent its general constraints and relevant background knowledge as a theory. We express a specific instance of the problem as a formula. We then use proof techniques to decide whether this formula follows from the theory. A proof of the formula or, more precisely, variable substitutions used by the proof, determine a solution which, in most cases is represented by a ground term. This use of logic in programming and computing stems from the pioneering work by Robinson [Rob65], Green [Gre69] and Kowalski [Kow74]. It led to the establishment of logic programming as, arguably, the most

1Parts of this paper appeared in Proceedings of AAAI-2000 [ET00] and in Proceedings of KI-2001 [ET01].
prominent and most broadly accepted logic-based declarative programming formalism, and to the development of Prolog as its implementation by Colmerauer and his group [CKPR73].

Recently, researchers proposed an alternative way in which logic can be used in computation [MT99, Nie99]. Answer-set programming (ASP) is an approach to declarative programming in which one represents a computational problem as a theory in some logic so that models of this theory, and not proofs or variable substitutions, represent problem solutions. In ASP, finding models rather than proofs is a primary computational task and serves as a foundation for a uniform processing mechanism.

ASP was first explicitly identified as a declarative modeling and computational paradigm in knowledge representation applications of normal logic programs (programs with negation in the bodies) [MT99, Nie99]. That work suggested that in order to model a problem, a programmer should write a normal logic program so that some distinguished models of the program represent solutions. The most broadly accepted 2-valued semantics of normal logic programs is the semantics of stable models [GL88]. Thus, under that proposal, a normal logic program solves a problem if answers to the problem are represented by stable models of the program and can be recovered from them. We refer to this variant of logic programming as stable logic programming (SLP). It is clear that SLP is a specific instance of the ASP paradigm described earlier.

In general, stable models are infinite and, unless one can devise for them some finitary representation schema, they cannot be computed. To overcome this difficulty, it is common to restrict attention in SLP to finite DATALOG\^\neg programs, that is, finite programs without function symbols. A stable model of a logic program is, by definition, an Herbrand model. Thus, if a program is a finite DATALOG\^\neg program, its stable models are finite. Moreover, there are algorithms to compute stable models of finite DATALOG\^\neg programs, as well as fast implementations of these algorithms, including smodels [NS00], dlv [ELM+98], cmodels [BL02] and assat [LZ02]. These implementations compute stable models of DATALOG\^\neg programs in two steps. First, an input program is grounded, that is, replaced by a program consisting of ground clauses only (hence, it is a propositional program) that has the same stable models as the original one. Second, stable models of the ground program are computed by means of search algorithms similar to the Davis-Putnam algorithm for satisfiability (SAT) testing [NS00, ELM+98] or, after some additional preprocessing, by SAT solvers [BL02, LZ02].

SLP is a declarative programming formalism particularly well suited for representing and solving search problems in the class NP-search, as it provides a uniform solution to each search problem in that class [MR01]. Specifically, for every search problem \Pi in the class NP-search there is a finite DATALOG\^\neg program \Pi_H, and an encoding schema that represents every instance I of \Pi as a finite collection data(I) of ground atoms from the Herbrand base of \Pi_H, such that stable models of the program \Pi_H \cup \text{data}(I) specify all solutions to the instance I of the problem \Pi. Problems such as scheduling, planning, diagnosis, abductive reasoning, product configuration, versions of bounded model checking, as well as a broad spectrum of combinatorial problems, are members of the class NP-search and so, admit a uniform solution in SLP. A recent research monograph [Bar03] presents SLP solutions for several of these problems, especially those that appear in knowledge representation. The combination of uniform encoding, high expressive power and fast algorithms for computing stable models makes SLP

\footnote{In fact, \texttt{dlv} implements an algorithm to compute stable models of disjunctive logic programs, a more general task.}
an attractive and effective declarative programming formalism. Extensions of the language of DATALOG with explicit representations for constraints involving cardinalities and other aggregates, a corresponding generalization of the notion of a stable model, and modifications in algorithms to compute (generalized) stable models resulted in even more effective programming and computing systems \[NS00, SNS02\].

While the notion of the answer-set programming paradigm has first explicitly appeared in the context of SLP, it is clear that the way in which propositional SAT solvers are used fits well the ASP paradigm. For instance, in the satisfiability planning \[KMS96\], problems are encoded as propositional theories so that models determine valid plans. SAT solvers are then used to compute them. Other similar uses of SAT solvers abound. Our goal in this paper is to extend this general idea and show that predicate logic with the Herbrand-model semantics, together with SAT solvers and their extensions as processing engines, gives rise to effective implementations of the ASP paradigm similar in spirit to SLP and with a similar scope of applicability. A specific logic we propose to this end is a modification of the logic of propositional schemata \[KMS96\]. The key concept is that of a data-program pair \((D, P)\) to represent separately a search problem \(\Pi\) by the program component \(P\) of \((D, P)\), and concrete instances of \(\Pi\) by the data part \(D\) of \((D, P)\). To define the semantics of data-program pairs, we restrict the class of Herbrand models of the theory \(D \cup P\) to those Herbrand models that satisfy a version of Reiter’s Closed-World Assumption. We refer to our logic as the logic of propositional schemata with Closed-World Assumption or, simply, as the logic of propositional schemata. We denote this logic by \(PS\).

The logic \(PS\) offers only basic logical connectives to help model problem constraints. We extend logic \(PS\) to support direct representation of constraints involving cardinalities. Examples of such constraints are: ”at least \(k\) elements from the list must be in the model” or ”exactly \(k\) elements from the list must be in the model”. They appear commonly in statements of constraint satisfaction problems. We also extend the language of the logic \(PS\) with Horn rules and use them as means to compute consequences of collections of ground facts (in particular, to compute transitive closures of binary relations). We refer to this new logic as extended logic of propositional schemata (with Closed-World Assumption) and denote it by \(PS+\).

In the paper we study basic properties of the logic \(PS\) and observe that they extend to the logic \(PS+\), as well. We show that the logic \(PS\) is nonmonotonic, identify sources of nonmonotonicity and its implications. We demonstrate the use of the logic \(PS\) as a representation language be developing programs for several search problems. We characterize the class of problems that can be solved by programs in the logic \(PS\). To this end, we define a formal setting for the study of the expressive power of ASP formalisms. We establish that the expressive power of the logic \(PS\) is equal to the class \(NP\)-search. In particular, it is the same as the expressive power of SLP.

As we pointed out, in the logic \(PS\), to solve a problem for a particular instance we represent the problem and the instance by a data-program pair so that Herbrand models of the data-program pair correspond to problem solutions. Consequently, the basic computational task is that of computing Herbrand models. It can be accomplished in a similar two-step process to that used in computing stable models. Given a finite data-program pair, we first ground it (compute its equivalent propositional representation) and then find models of the ground theory obtained.

For grounding, we implemented a program, \textit{psgrnd} that, given a data-program pair produces
an equivalent (with the same models) propositional theory. If the input data-program pair is in the language of the basic logic $PS$, one of the options of $psgrnd$ creates a theory in the DIMACS format and allows one to use for solving “off-the-shelf” SAT solvers. In this way, our logic $PS$ and our program $psgrnd$ provide a programming front-end for SAT solvers greatly facilitating their use.

If a data-program pair contains higher-level constructs proper to the logic $PS+$, one still can use propositional solvers for processing by first compiling cardinality and closure constraints to propositional logic and then using SAT solvers to compute models. However, propositional representations of constraints involving cardinalities or closure operators are usually very large and the sizes of the compiled theories limit the effectiveness of satisfiability checkers, even the most advanced ones, as processing engines. Thus, we argue for an alternative approach to design solvers specifically tailored to the syntax of the logic $PS+$. To this end, we propose a “target” propositional logic for the logic $PS+$. In this logic, cardinality and closure constraints have explicit representations and, therefore, do not need to be compiled any further. We develop a satisfiability solver, $aspps$, for the propositional logic $PS+$ and use it as the processing back-end for the logic $PS+$ and $psgrnd$. Our solver is designed along the same lines as most satisfiability solvers implementing the Davis-Putnam algorithm but it takes a direct advantage of the cardinality and closure constraints explicitly present in the language.

Experimental results on the performance of the overall approach are highly encouraging. On one hand, we demonstrate the ease and effectiveness of using off-the-shelf SAT solvers to attack search problems. On the other hand, we show that significant gains in performance can be obtained by developing more general solvers, such as $aspps$, capable of directly processing some classes of more complex constraints than those that can be expressed as clauses. In particular, our solver $aspps$ is competitive with current SLP solvers such as $smodels$ and with complete SAT solvers such as $zchaff$ and $satz$. In fact, in several instances we considered, it was faster. Our work demonstrates that building propositional solvers capable of processing high-level constraints is a promising research direction for the area of propositional satisfiability.

Several interrelated factors motivate us in this work. First, we want to provide an effective programming front-end that would capitalize on dramatic improvements in the performance of satisfiability solvers and would facilitate their use as computational tools. In recent years, researchers have developed several fast implementations of the basic Davis-Putnam method such as $satz$ [LA97], $relsat$ [BS97] and, most recently, $chaff$ [MMZ+01a, MMZ+01b]. A renewed interest in local-search techniques resulted in highly effective (albeit incomplete) satisfiability checkers such as $WALKSAT$ [SKC94], capable of handling large CNF theories, consisting of millions of clauses. These advances make computing Herbrand models of predicate-logic theories feasible. Second, the use of propositional semantics also makes it easy to expand the basic language with constructs to explicitly represent high-level constraints and to exploit ideas developed in the area of propositional satisfiability to design algorithms for computing models of ground theories in expanded languages. Third, the semantics of Herbrand models of predicate theories, being essentially the semantics of propositional logic, is broadly known and much less complex than the semantics of stable models of logic programs. Consequently, the task of programming may in many cases be simpler and the overall approach may gain broader acceptance.
2 Basic logic PS

In this section, we introduce the logic PS that provides a theoretical basis for a declarative programming front-end to satisfiability solvers and facilitates their use. Syntactically, the logic PS is a fragment of first-order logic without function symbols (or, in other words, predicate logic). Specifically, the language of the logic PS consists of:

1. infinite denumerable sets $R$, $C$ and $V$ of relation, constant and variable symbols
2. symbols $\bot$ and $\top$ (later interpreted always as falsity and truth)
3. boolean connectives $\land$, $\lor$ and $\rightarrow$, the universal and existential quantifiers, and punctuation symbols ‘(’, ‘)’ and ‘,’.

In the paper, following the example of logic programming, we adopt the convention that upper-case letters denote variables and lower-case letters stand for constants.

Constant and variable symbols are the only terms of the language. Constants are the only ground terms and they form the Herbrand universe of the language. Atoms are expressions of the form $p(t_1, \ldots, t_n)$, where $p$ is an $n$-ary relation symbol from $R$ and $t_i$, $1 \leq i \leq n$, are terms. An atom $p(t_1, \ldots, t_n)$ is ground if all its terms are ground. The set of all ground atoms forms the Herbrand base of the language.

In the logic PS, we restrict the use of existential quantifiers. Let us consider a tuple of terms $(t_1, \ldots, t_n)$ and let $X_1, \ldots, X_k$ be pairwise distinct variables such that each $X_i$, $1 \leq i \leq k$, appears in the tuple $(t_1, \ldots, t_n)$ exactly once. An expression of the form

$$\exists X_1, \ldots, X_k \ p(t_1, \ldots, t_n)$$

is an e-atom. For instance, the expression $\exists X, Z \ p(X, Y, Z, c)$ is an e-atom while the expression $\exists X, Z \ p(X, X, Z, c)$ is not. In the logic PS, existential quantifiers appear exclusively in e-atoms.

The requirement that each $X_i$, $1 \leq i \leq k$, appears in the tuple $(t_1, \ldots, t_n)$ exactly once is not essential and can be lifted. We adopt it as it allows us to simplify the notation for e-atoms. Namely, we write an e-atom

$$\exists X_1, \ldots, X_k \ p(t_1, \ldots, t_n)$$

as

$$p(t'_1, \ldots, t'_n),$$

where $t'_i = t_i$ if $t_i$ is not one of the variables $X_1, \ldots, X_k$, and $t_i = \_\_$ (underscore), otherwise. For instance, we write an e-atom $\exists X, Z \ p(X, Y, Z, c)$ as $p(\_\_, Y, \_\_, c)$.

The only formulas we allow in the logic PS are rules, that is, formulas

$$\forall X_1, \ldots, X_k (A_1 \land \ldots \land A_m \rightarrow B_1 \lor \ldots \lor B_n),$$

where all $A_i$, $1 \leq i \leq m$, and $B_j$, $1 \leq j \leq n$, are atoms, none of $A_i$’s is an e-atom (in other words, e-atoms do not appear in the antecedents of clauses) and $X_1, \ldots, X_k$ are the free variables appearing in $A_1, \ldots, A_m$ and $B_1, \ldots, B_n$. If $m = 0$, we replace the conjunct in the antecedent of the clause with the symbol $\top$. If $n = 0$, we replace the empty disjunct in the consequent of the clause with the symbol $\bot$. 

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As usual, we drop the universal quantifiers from the rule notation. For instance, to denote the rule

\[ \forall X, Y, Z (p(X, Y) \land p(Y, Z) \rightarrow q(\_ X) \lor q(\_ Y) \lor r(Z)), \]

which is already a shorthand for

\[ \forall X, Y, Z (p(X, Y) \land p(Y, Z) \rightarrow \exists W q(W, X) \lor \exists W q(Y, W) \lor r(Z)), \]

we write

\[ p(X, Y) \land p(Y, Z) \rightarrow q(\_ X) \lor q(\_ Y) \lor r(Z). \]

This notation is reminiscent of that commonly used for clauses in predicate logic. There is a key difference, though. Some of the atoms in the consequent of a rule may be e-atoms (as it is the case in the example just given). Thus, unlike in the case of clauses, a rule of the logic PS may contain the existential quantifier in the consequent.

The last syntactic notion we need is that of a theory. A theory in the logic PS (or, a PS theory) is any finite collection of rules that contains at least one occurrence of a constant symbol.

To recap the discussion of the syntax of the logic PS, it is essentially a fragment of the syntax of the first-order logic with the following restrictions and caveats: (1) function symbols are not allowed, (2) rules are the only formulas, (3) theories are finite and contain at least one constant symbol, and (4) through the use of notational conventions, the quantifiers are only implicitly present in the language.

The difference between the logic PS and the corresponding fragment of the first-order logic is in the way we interpret theories. Namely, we view a theory \( T \) as a representation of a certain class of models of \( T \) and not as a representation of logical consequences of \( T \). In fact, due to the way we use the logic PS, the concept of provability plays virtually no role in logic PS. This is an essential departure from the classical first-order logic perspective.

Specifically, we assign to a PS theory \( T \) a collection of its Herbrand models. The concepts of an Herbrand interpretation, of truth in an interpretation and of an Herbrand model, that we use in the paper, are standard (for details, we refer to any text in logic, for instance, [NS93]). Here we will only introduce some necessary notation. Let \( T \) be PS theory. We denote by \( HU(T) \) the Herbrand universe of \( T \), that is, in our case, the set of all constants that appear in \( T \). By the definition of a PS theory, this set is not empty and finite. We denote by \( HB(T) \) the Herbrand base of \( T \), that is, the collection of all ground atoms \( p(c_1, \ldots, c_n) \), where \( p \) is an \( n \)-ary relation symbol appearing in \( T \) and \( c_i \in HU(T), 1 \leq i \leq n \). Following a standard practice, we identify Herbrand interpretations of \( T \) with subsets of \( HB(T) \).

The restriction to Herbrand interpretations is important. In particular, it implies that the logic PS is nonmonotonic. Indeed, if \( T_1 \subseteq T_2 \) are two PS theories, it is not necessary that every Herbrand model of \( T_2 \) is a Herbrand model of \( T_1 \). For example, let \( T_1 = \{ \top \rightarrow p(\_), p(a) \rightarrow \bot \} \), and let \( T_2 = T_1 \cup \{ \top \rightarrow p(b) \} \). It is easy to see that \( M = \{ p(b) \} \) is a Herbrand model of \( T_2 \) and that \( T_1 \) has no Herbrand models. Indeed, \( HU(T_1) = \{ a \} \) and the only Herbrand model satisfying the first rule is \( M' = \{ p(a) \} \). This model, however, does not satisfy the second rule. In contrast, classical first-order logic is monotone: for every two collections of sentences \( T_1 \subseteq T_2 \), if \( M \) is a model of \( T_2 \) then it is a model of \( T_1 \), as well. In the example discussed above, the Herbrand model specified by the subset \( \{ p(b) \} \) of \( HB(T_2) \) is a model of \( T_1 \) but not a Herbrand model of \( T_1 \).
The restriction to Herbrand models allows us also to develop algorithms to compute them. Namely, as in the case of stable logic programming, Herbrand models of a PS theory $T$ can be computed in two steps. First, we ground $T$ to a propositional theory that has the same Herbrand models as $T$. Next, we compute models of $T$ by computing models of the ground theory. This latter task can be accomplished by off-the-shelf propositional satisfiability solvers.

The concept of grounding is similar to that used in the context of universal theories in first-order logic or programs in logic programming. The only difference comes from the fact that rules in PS theories may include e-atoms in the consequents. We will now discuss the task of grounding in detail. Let $p(t)$ be an atom that occurs in $T$. If $p(t)$ is not an e-atom, we define $p^d(t) = p(t)$. If $p(t)$ is an e-atom (we assume that all e-atoms in $T$ are given in the “underscore” notation), we define $p^d(t)$ as the disjunction of all atoms of the form $p(t')$, where $t'$ is obtained from $t$ by replacing all occurrences of the underscore symbol $\_\_$ in $t$ with constants from $HU(T)$ (that is, constants that appear in $T$). For example, if $a$ and $b$ are the only two constants in $T$ and $p(\_X,\_a)$ is an e-atom in $T$, we have

$$p^d(\_X,\_a) = p(a, X, a, a) \lor p(a, X, b, a) \lor p(b, X, a, a) \lor p(b, X, b, a).$$

Further, for a rule $r \in T$, where

$$r = A_1 \land \ldots \land A_m \rightarrow B_1 \lor \ldots \lor B_n,$$

we define

$$r^d = A_1 \land \ldots \land A_m \rightarrow B_1^d \lor \ldots \lor B_n^d.$$  

We note that $T^d = \{r^d : r \in T\}$ contains no occurrences of the underscore symbol.

Let $\vartheta$ be a ground substitution, that is, a mapping whose domain is a finite subset of the set of variables from the language, and which assigns constants from the language to variables in its domain. By an expression we mean a term, a list of terms, a formula without any occurrence of the underscore symbol, or a set of formulas without any occurrence of the underscore symbol. A ground substitution is defined for an expression $E$ if every variable appearing in $E$ belongs to the domain of $\vartheta$. Let $E$ be an expression and let $\vartheta$ be a ground substitution that is defined for $E$. By $E\vartheta$ we denote the expression obtained from $E$ by replacing all variables occurring in $E$ with their images under $\vartheta$. We call an expression of the form $E\vartheta$ a ground instance of $E$. For an expression $E$, by $gr(E)$ we denote the set of all ground instances of $E$. Finally, we extend the notion of grounding to PS theories which are not, in general, expressions as they may contain the underscore symbol. Namely, for PS theory $T$, we set $gr(T) = gr(\{r^d : r \in T\})$.

We will illustrate the concepts we introduced with an example. Let $T$ be a PS theory that consists of the following two clauses:

$$C_1 = q(b, c) \rightarrow p(a)$$

$$C_2 = p(X) \rightarrow q(X, \_).$$

To compute $gr(T)$ we need to compute all ground instances of $C_2$ ($C_1$ is already in the ground form). First, we compute the formula $C_2^d$:

$$C_2^d = p(X) \rightarrow q(X, a) \lor q(X, b) \lor q(X, c).$$

To obtain all ground instances of $C_2$ (or $C_2^d$), we replace $X$ in $C_2^d$ with $a$, $b$ and $c$ and obtain the following three clauses:
\[ p(a) \rightarrow q(a, a) \vee q(a, b) \vee q(a, c) \]
\[ p(b) \rightarrow q(b, a) \vee q(b, b) \vee q(b, c) \]
\[ p(c) \rightarrow q(c, a) \vee q(c, b) \vee q(c, c). \]

These three clauses together with \( C_1 \) form \( gr(T) \).

The following proposition establishes the adequacy of the concept of grounding in the study of models of \( PS \) theories. It also demonstrates that satisfiability provers can be used to compute models of \( PS \) theories. The proof of the proposition is simple and reflects closely the corresponding argument in the first-order case. Thus, we omit it.

**Proposition 2.1** Let \( T \) be a \( PS \) theory. A set \( M \subseteq HB(T) \) is a Herbrand model of \( T \) if and only if \( M \) is a propositional model of \( gr(T) \).

### 3 Equality and arithmetic in the logic \( PS \)

In the following sections, we will often consider a version of the logic \( PS \) in which some relation symbols are given a prespecified interpretation. They are quality, inequality and basic arithmetic relations such as \( \leq, <, >, +, *, -, / \), etc. Inclusion of these relation symbols in the language is important as they greatly facilitate the task of programming (modeling knowledge, representing constraints as rules). We will use standard symbols to represent them, as well as the standard infix notation. In particular, we will write \( t_1 = t_2 \) rather than \( = (t_1, t_2) \), and \( t = t_1 + t_2 \) (or \( t_1 + t_2 = t \)) rather than \( + (t_1, t_2, t) \). We will denote the set of these relation symbols by \( EA \).

In this section, we will define the semantics for this variant of the logic \( PS \). The idea is to interpret all symbols in the set \( EA \) according to their intended meaning. Specifically, let \( C \) be the set of constant symbols. We define a theory \( =_C \) to consist of all clauses of the form

1. \( \top \rightarrow = (t, t) \) (we will write them as \( \top \rightarrow (t = t) \)), for every \( t \in C \), and
2. \( = (s, t) \rightarrow \bot \) (we will write them as \( (s = t) \rightarrow \bot \)), for all \( s, t \in C \) such that \( s \neq t \).

Next, we define a theory \( +_C \) to consist of all clauses of the form

1. \( \top \rightarrow + (t, u, s) \) (we will write them as \( \top \rightarrow (s = t + u) \)), for every integers \( s, t, u \in C \) such that \( s = t + u \), and
2. \( + (t, u, s) \rightarrow \bot \) (we will write them as \( (s = t + u) \rightarrow \bot \)), for every \( s, t, u \in C \) such that at least one of \( s, t, u \) is not an integer, or \( s, t, u \) are integers and \( s \neq t + u \).

In the same way we define theories \( p_C \) for other relation symbols in \( EA \) such as \( \leq, -, \ast, \) etc. All these theories provide explicit intended definitions of the corresponding relation symbols. We will often refer to the relation symbols in \( EA \) as *predefined* since their interpretation is fixed.

Let \( T \) be a \( PS \) theory in the language containing distinguished relation symbols from the set \( EA \). Let \( C \) be the set of constants appearing in \( T \) (that is, \( C = \text{HU}(T) \)). A set \( M \) of ground atoms in the language is a *model* of \( T \) if \( M \) is a model of the theory \( T \cup \{ p_C : p \in EA \} \) as defined above.
It is clear that, with the help of additional variables, we can express in the logic $PS$ arbitrary arithmetic expressions. For instance, we will write

$$q((X + Y) * Z, a) \land B \rightarrow H$$

and interpret this expression as

$$(T_1 = X + Y) \land (T_2 = T_1 * Z) \land q(T_2, a) \land B \rightarrow H$$

Similarly, we will interpret

$$B \rightarrow q((X + Y) * Z, a) \lor H$$

as the clause

$$B \land (T_1 = X + Y) \land (T_2 = T_1 * Z) \rightarrow q(T_2, a) \lor H.$$  

In each case, variables $T_1$ and $T_2$ are different from all other variables appearing in the rules. In order to obtain uniqueness of the interpretation, when decomposing arithmetic expressions, we follow the standard order in which they are evaluated. Let us emphasize that arithmetic expressions are simply notational shortcuts and not elements of the language.

In the remainder of the paper, we will always assume that the language contains predefined relation symbols. Since their extensions are already fully specified, we will omit the corresponding ground atoms when describing models of theories.

To illustrate these concepts, let us consider the following $PS$ theory $T$:

$$T = \{ \top \rightarrow p(1), \top \rightarrow p(2), \ p(X) \rightarrow q(X, X + 1) \}.$$  

This theory represents the following $PS$ theory $T'$:

$$T' = \{ \top \rightarrow p(1), \top \rightarrow p(2), \ p(X) \land (Y = X + 1) \rightarrow q(X,Y) \}.$$  

The theory $gr(T')$ consists of the first two rules (they are already ground) and the following four instantiations of the third rule:

$$p(1) \land (1 = 1 + 1) \rightarrow q(1, 1)$$

$$p(1) \land (2 = 1 + 1) \rightarrow q(1, 2)$$

$$p(2) \land (1 = 2 + 1) \rightarrow q(2, 1)$$

$$p(2) \land (2 = 2 + 2) \rightarrow q(2, 2).$$

Models of this theory are (by the definition) models of the theory $T$. For instance, $\{p(1), p(2), q(1, 2)\}$ is a model of $T$. We point out that, according to our convention, we omitted from the model description the atom $2 = 1 + 1$ (or, more formally, the atom $= (1, 1, 2)$).

4 Programming with logic $PS$

The logic $PS$, described in the previous section, can be used as a basis for a declarative programming formalism based on the paradigm of answer-set programming [MT99]. To this end, we need to introduce the concepts of input data and a program. We follow the approach proposed and studied in the area of relational databases [Ull88]. A relational database can be viewed as a collection of ground atoms of some logic language. We often use (for instance,
in the context of DATALOG and its variants) the term extensional database to refer to a collection of ground atoms specifying a relational database. Queries are finite theories, often of special form, in this logic language (for instance, definite Horn theories without function symbols serve as queries in the case of DATALOG). Queries define new properties (relations) in terms of those relations that are explicitly specified by the underlying (extensional) database.

Guided by these intuitions, we define a data-program pair to be a pair \( (D, P) \), where \( D \) is a finite set of ground atoms in a language of the logic \( PS \) and \( P \) is a finite collection of \( PS \) rules. We use data-program pairs to represent specific computational problem instances. We view \( D \) as an encoding of relevant input data and \( P \) as a declarative specification of the computational task in question. Accordingly, given a data-program pair \( (D, P) \), we refer to \( D \) as a data set and to \( P \) as a program. We use the term data predicate for all relation symbols appearing in \( D \). We use the term program predicate to refer to all relation symbols that appear in \( P \) and are neither data predicates nor predefined predicates from \( EA \). Intuitively, \( D \) is a counterpart of an extensional database and \( P \) is a counterpart of a database query.

We will now introduce a semantics for data-program pairs. To this end, we will encode a data-program pair \( (D, P) \) as a theory in the logic \( PS \). Since \( D \) is a set of ground atoms representing the problem instance (input data), we assume that \( D \) provides a complete specification of the input. That is, we assume that no other ground atoms built of predicates appearing in \( D \) are true. Since the only formulas in the logic \( PS \) are rules, we encode the information specified by \( D \) as a set of rules \( cl(D) \) defined as follows. For every relation symbol \( p \) appearing in \( D \) and for every ground tuple \( t \) (with constants from the Herbrand universe of \( D \cup P \)) of appropriate arity, if \( p(t) \in D \), we include in \( cl(D) \) the clause

\[
\top \rightarrow p(t).
\]

Otherwise, if \( p(t) \notin D \), we include in \( cl(D) \) the clause

\[
p(t) \rightarrow \bot.
\]

It is clear that the set \( cl(D) \) can be regarded as the result of applying Reiter’s Closed-World Assumption to \( D \).

We represent a data-program pair \( (D, P) \) by a \( PS \) theory \( cl(D) \cup P \). We say that a set \( M \) of ground atoms, \( M \subseteq HB(cl(D) \cup P) \), is a model of a data-program pair \( (D, P) \) if it is a model of \( cl(D) \cup P \). We denote the set of all models of a data-program pair \( (D, P) \) by \( \text{Mod}(D, P) \).

In separating data and program predicates and in adopting the closed-world assumption for the treatment of data atoms we are guided by the intuition that data predicates are intended to represent input data. Their extensions should not be affected by the computation. The effects of the computation should be reflected in the extensions of program predicates only.

We designed the logic \( PS \) and introduced the concept of a data-program pair to model computational problems. To illustrate this use of our formalism, we show how to encode several well-known search problems by means of data-program pairs. We assume that the language contains predefined relation symbols to represent equality and arithmetic relations.

We start with the graph \( k \)-colorability problem: given an undirected graph and a set of \( k \) colors, the objective is to find an assignment of colors to vertices so that no two identically colored vertices are joined with an edge (or to determine that no such coloring exists).

We set

\[
D_{gcl}(G, k) = \{vtx(v) : v \in V\} \cup \{edge(v, w) : \{v, w\} \in E\} \cup \{color(i) : 1 \leq i \leq k\}.
\]
The set of atoms $D_{gcl}$ represents an instance of the coloring problem. The predicates $vtx$, $edge$ and $color$ are data predicates. Their extensions define vertices and edges of an input graph, and the set of available colors.

Next, we construct a program, $P_{gcl}$, encoding the constraints of the problem. It involves predicates $vtx$, $edge$ and $color$, specified in the data part, and defines a new relation $clrd$ that models assignments of colors to vertices.

C1: $clrd(X,C) \rightarrow vtx(X)$
C2: $clrd(X,C) \rightarrow color(C)$
C3: $vtx(X) \rightarrow clrd(X,\bot)$
C4: $clrd(X,C) \land clrd(X,D) \rightarrow (C = D)$
C5: $edge(X,Y) \land clrd(X,C) \land clrd(Y,C) \rightarrow \bot$.

The condition (C1) states that the only objects that get colored are vertices. Indeed, by the definition, a model of the theory $(D_{gcl}(G,k), P_{gcl})$ contains an atom $vtx(x)$ if and only if $x$ is a vertex of an input graph. Thus, if $clrd(v,c)$ belongs to a model of $(D_{gcl}(G,k), P_{gcl})$, then $vtx(v)$ belongs to the model and, so, $v$ is a vertex. Similarly, (C2) states that the only objects assigned by the predicate $clrd$ to a vertex are colors. (C3) states that each vertex $X$ gets assigned at least one color. (C4) enforces that each vertex is assigned at most one color. (C5) ensures that two vertices connected by an edge are assigned different colors. These clauses correctly capture the constraints of the coloring problem.

**Proposition 4.1** Let $G = (V,E)$ be an undirected graph and let $k$ be a positive integer. An assignment $f:V \rightarrow \{1,\ldots,k\}$ is a $k$-coloring of $G$ if and only if $M = D_{gcl}(G,k) \cup \{clrd(v,f(v)) : v \in V\}$ is a model of the data-program pair $(D_{gcl}(G,k), P_{gcl})$.

Proof: $(\Rightarrow)$ Let us assume that $f:V \rightarrow \{1,\ldots,k\}$ is a $k$-coloring of $G$. We will show that $M = D_{gcl}(G,k) \cup \{clrd(v,f(v)) : v \in V\}$ is a model of $(D_{gcl}, P_{gcl})$, that is, it is a model of $gr(cl(D_{gcl}(G,k)) \cup P_{gcl})$. From the definition of $M$, it follows that $M$ satisfies all rules in $cl(D_{gcl}(G,k))$. We will now show that $M$ satisfies all rules in $gr(cl(D_{gcl}(G,k)) \cup P_{gcl})$ that are obtained by grounding rules in $P_{gcl}$.

First, we consider an arbitrary ground instance of rule (C1), say, $clrd(x,c) \rightarrow vtx(x)$, where $x$ and $c$ are two constants of the language. It is clear from the definition of $M$ that if $clrd(x,c) \in M$, then $x \in V$ and, consequently, $vtx(x) \in M$. Thus all ground instances of (C1) are satisfied by $M$.

Next, we consider a ground instance $r$ of rule (C3), say,

$$r = vtx(x) \Rightarrow \bigvee \{clrd(x,c) : c \in V \cup \{1,\ldots,k\}\},$$

where $x \in V \cup \{1,\ldots,k\}$. If $vtx(x) \in M$, then $x \in V$. Since $f(x) \in \{1,\ldots,k\}$, and $clrd(x,f(x)) \in M$, it follows that $r$ is satisfied by $M$. All other rules can be dealt with in a similar way.

$(\Leftarrow)$ We will now assume that $M$ is a model of $(D_{gcl}, P_{gcl})$. By the definition of a model, we have (1) $vtx(x) \in M$ if and only if $x \in V$, (2) $edge(x,y) \in M$ if and only if $\{x,y\} \in E$, and (3) $color(i) \in M$ if and only if $i \in \{1,\ldots,k\}$.

Now, we observe that since $M$ satisfies all ground instances of (C1), if $clrd(x,c) \in M$, then $x \in V$. Similarly, since $M$ satisfies all ground instances of (C3), for every $x \in V$ there is at least one constant $c$ such that $clrd(x,c) \in M$. On the other hand, since $M$ satisfies all ground
instances of (C2), for each such constant \( c \), \( c \in \{1, \ldots, k\} \). Next, we have that \( M \) satisfies all ground instances of (C4). Consequently, for every \( x \in V \) there is exactly one \( c \in \{1, \ldots, k\} \) such that \( \text{clrd}(x, c) \in M \). Let us denote by \( f \) the function that assigns to each \( x \in V \) the unique \( c \in \{1, \ldots, k\} \) such that \( \text{clrd}(x, c) \in M \). It follows that \( M = D \cup \{ \text{clrd}(v, f(v)) : v \in V \} \). Moreover, as \( M \) satisfies all ground instances of (C5), \( f \) is a \( k \)-coloring of \( G \).

Let us note that the proof of Proposition 4.1 implies, in fact, that the correspondence between models and colorings is a bijection.

Next, we will describe a data-program pair encoding an instance of the vertex-cover problem for graphs. Let \( G = (V, E) \) be a graph. A set \( W \subseteq V \) is a vertex cover of \( G \) if for every edge \( \{x, y\} \in E \), \( x \) or \( y \) (or both) are in \( W \). The vertex-cover problem is defined as follows: given a graph \( G = (V, E) \) and an integer \( k, k \leq |V| \), decide whether \( G \) has a vertex cover with no more than \( k \) vertices.

For the vertex-cover problem the input data is described by the following set of ground atoms:

\[
D_{vc}(G, k) = \{ \text{vtx}(v) : v \in V \} \cup \{ \text{edge}(v, w) : \{v, w\} \in E \} \cup \{ \text{index}(i) : i = 1, \ldots, k \}.
\]

This set of atoms specifies the set of vertices and the set of edges of an input graph. It also provides a set of \( k \) indices which we will use to select a subset of no more than \( k \) vertices in the graph, a candidate for a vertex cover of cardinality at most \( k \).

The vertex cover problem itself is described by the program \( P_{vc} \). It introduces a new relation symbol \( \text{vc} \). Intuitively, we use \( \text{vc} \) to represent the fact that a vertex has been selected to a candidate set.

\[
\begin{align*}
\text{VC1:} & \quad \text{vc}(I, X) \rightarrow \text{vtx}(X) \\
\text{VC2:} & \quad \text{vc}(I, X) \rightarrow \text{index}(I) \\
\text{VC3:} & \quad \text{index}(I) \rightarrow \text{vc}(I, \_)
\end{align*}
\]

\[
\begin{align*}
\text{VC4:} & \quad \text{vc}(I, X) \land \text{vc}(I, Y) \rightarrow X = Y \\
\text{VC5:} & \quad \text{edge}(X, Y) \rightarrow \text{vc}(\_ X) \lor \text{vc}(\_ Y).
\end{align*}
\]

(VC1) and (VC2) ensure that \( \text{vc}(i, x) \) is false if \( i \) is not an integer from the set \( \{1, \ldots, k\} \) or if \( x \) is not a vertex (that is, if \( \text{vc}(i, x) \) is true, \( i \in \{1, \ldots, k\} \) and \( x \in V \)). The rules (VC3) and (VC4) together impose the requirement that every index \( i \) has exactly one vertex assigned to it. It follows that the set of ground atoms \( \text{vc}(i, x) \) that are true in a model of the data-program pair \( (D_{vc}(G, k), P_{vc}) \) defines a subset of \( V \) with cardinality at most \( k \). Finally, (VC5) ensures that each edge has at least one end vertex assigned by \( \text{vc} \) to an index from \( \{1, \ldots, k\} \) (in other words, that vertices assigned to indices \( 1, \ldots, k \) form a vertex cover). The correctness of this encoding is formally established in the following result. Its proof is similar to that of Proposition 4.1 and we omit it.

**Proposition 4.2** Let \( G = (V, E) \) be an undirected graph and let \( k \) be a positive integer. If \( W \subseteq V \) is a vertex cover of \( G \) and \( |W| \leq k \), then for every sequence \( w_1, \ldots, w_k \) that enumerates all elements in \( W \) (possibly with repetitions), \( M = D_{vc}(G, k) \cup \{ \text{vc}(i, w_i) : i = 1, \ldots, k \} \) is a model of the data-program pair \( (D_{vc}(G, k), P_{vc}) \). Conversely, if \( M \) is a model of \( (D_{vc}(G, k), P_{vc}) \) then the set \( W = \{ w \in V : \text{vc}(i, w) \in M, \text{ for some } i = 1, \ldots, k \} \) is a vertex cover of \( G \) with \( |W| \leq k \).

In this case, we do not have a one-to-one correspondence between models and vertex covers of cardinality at most \( k \). It is so because we represent sets by means of sequences.
Next, we will consider the Hamiltonian-cycle problem in a directed graph. To represent an input graph $G = (V, E)$ we use the following set of ground atoms:

$$D_{hc}(G) = \{vtx(x): x \in V\} \cup \{edge(x, y): (x, y) \in E\} \cup \{index(i): i = 1, \ldots, |V|\}.$$ 

The set of indices is introduced as part of input because we will represent a Hamiltonian cycle by a bijective sequence of vertices such that every two consecutive vertices in the sequence, as well as the last and the first, are connected with an edge. To represent such sequences we use a relations symbol $hc\_perm$. The program, $P_{hc}$, defining “Hamiltonian” sequences $hc\_perm(i, x)$ looks as follows. In this example we assume that $\oplus$ denotes a predefined relation of addition modulo $n$ defined on the set of integers $\{1, \ldots, n\}$ (thus, in particular, $n \oplus 1 = 1$).

HC1: $hc\_perm(I, X) \rightarrow index(I)$
HC2: $hc\_perm(I, X) \rightarrow vtx(X)$
HC3: $index(I) \rightarrow hc\_perm(I, \_)$
HC4: $hc\_perm(I, X) \land hc\_perm(I, Y) \rightarrow X = Y$
HC5: $hc\_perm(I, X) \land hc\_perm(J, X) \rightarrow I = J$
HC6: $hc\_perm(I, X) \land hc\_perm(I \oplus 1, Y) \rightarrow edge(X, Y)$.

The first two rules ensure that if $hc\_perm(i, x)$ is true in a model of $(D_{hc}, P_{hc})$ then $i$ is an integer from the set $\{1, \ldots, |V|\}$ and $x \in V$. The rules (HC3) - (HC5) together enforce the constraint that $hc\_perm$ defines a permutation of vertices. Finally, the last rule imposes the Hamiltonicity constraint that from every vertex in the sequence to the next one (and from the last one to the first one, too) there is an edge in the graph. Formally, we have the following result (the correspondence it establishes is not one to one as a Hamiltonian cycle can be represented by $|V|$ different permutations, each being a cyclic shift of another).

**Proposition 4.3** Let $G = (V, E)$ be a directed graph with $n$ vertices. A permutation $v_1, \ldots, v_n$ of all vertices of $V$ is a Hamiltonian cycle if and only if $M = D_{hc}(G) \cup \{hc\_perm(i, v_i): i = 1, \ldots, n\}$ is a model of the data-program pair $(D_{hc}(G), P_{hc})$.

We will next consider the $n$-queens problem, that is, the problem of placing $n$ queens on a $n \times n$ chess board so that no queen attacks another. The representation of input data specifies the set of row and column indices:

$$D_{nq}(n) = \{index(i): i = 1, \ldots, n\}.$$ 

The problem itself is described by the program $P_{nq}$. The predicate $q$ describes a distribution of queens on the board: $q(x, y)$ is true precisely when there is a queen in the position $(x, y)$.

nQ1: $q(R, C) \rightarrow index(R)$
nQ2: $q(R, C) \rightarrow index(C)$
nQ3: $index(R) \rightarrow q(R, \_)$
nQ4: $q(R, C1) \land q(R, C2) \rightarrow (C1 = C2)$
nQ5: $q(R1, C) \land q(R2, C) \rightarrow (R1 = R2)$
nQ6: $q(R, C) \land q(R + I, C + I) \rightarrow \perp$
nQ7: $q(R, C) \land q(R + I, C - I) \rightarrow \perp$

The first two rules ensure that if $q(r, c)$ is true in a model of $(D_{nq}, P_{nq})$ then $r$ and $c$ are integers from the set $\{1, \ldots, n\}$. The rules (nQ3) - (nQ5) together enforce the constraint that
each row and each column contains exactly one queen. Finally, the last two rules guarantee that no two queens are placed on the same diagonal. As in the other cases, we can formally state and prove the correctness of this encoding. The proof is again quite similar to that of Proposition 4.1 and so we omit it.

**Proposition 4.4** Let \( n \) be a positive integer. A set of points on an \( n \times n \) board, \( \{(r_i, c_i) : i = 1, 2, \ldots, n\} \), is a solution to the \( n \)-queens problem if and only if the set \( M = D_{nq}(n) \cup \{q(r_i, c_i) : i = 1, 2, \ldots, n\} \) is a model of the data-program pair \((D_{nq}(n), P_{nq})\).

As in the case of the graph-coloring problem, the correspondence between models and valid arrangements of queens on the board is a bijection.

For the last example in this section, we consider computing the transitive closure of a finite directed graph \( G = (V, E) \), where \( V \) is a set of vertices and \( E \) is a set of directed edges (we will assume that \( G \) has no loops). We recall that the transitive closure of the graph \( G = (V, E) \) is the directed graph \( (V, E') \) such that an edge \( (x, y) \) belongs to \( E' \) if and only if there is in \( G \) a directed path from \( x \) to \( y \) of length at least 1.

We will now describe the representation of data instances and give a \( PS \) program solving the transitive closure problem. The data instance consists of a specification of an input graph \((V, E)\) and of a collection of integers, \( \{1, 2, \ldots, |V|\} \) that will allow us to count edges in the paths. Thus, we set

\[
D_{tc}(G) = \{vtx(v) : v \in V\} \cup \{edge(v, w) : \{v, w\} \in E\} \cup \{index(i) : 1 \leq i \leq k\}.
\]

Next, we construct a program, \( P_{tc} \), encoding the constraints of the problem. Our encoding uses an auxiliary 4-ary relation symbol \( path \). The intended meaning of \( path(X, Y, Z, I) \) is that it is true precisely when there is a directed path from \( X \) to \( Y \) such that \( Z \) is the immediate predecessor of \( Y \) on the path and the path length is at most \( I \). In \( P_{tc} \) we define the relation \( path \) and use it to specify the relation \( tc \) that represents the transitive closure of the input graph.

TC1: \( path(X, Y, Z, I) \rightarrow vtx(X) \)
TC2: \( path(X, Y, Z, I) \rightarrow vtx(Y) \)
TC3: \( path(X, Y, Z, I) \rightarrow vtx(Z) \)
TC4: \( path(X, Y, Z, I) \rightarrow index(I) \)
TC5: \( tc(X, Y) \rightarrow vtx(X) \)
TC6: \( tc(X, Y) \rightarrow vtx(Y) \)
TC7: \( path(X, Y, X, 1) \rightarrow edge(X, Y) \)
TC8: \( edge(X, Y) \rightarrow path(X, Y, X, 1) \)
TC9: \( path(X, Y, Z, 1) \rightarrow X = Z \)
TC10: \( path(X, Y, Z, I + 1) \rightarrow path(X, Z, \_ , I) \)
TC11: \( path(X, Y, Z, I + 1) \rightarrow edge(Z, Y) \)
TC12: \( path(X, Z, W, I) \land e(Z, Y) \rightarrow path(X, Y, Z, I + 1) \)
TC13: \( tc(X, Y) \rightarrow path(X, Y, \_, \_) \)
TC14: \( path(X, Y, Z, I) \rightarrow tc(X, Y) \).

The first four rules enforce that if an atom \( path(x, y, z, i) \) is in a model of the data-program pair \((D_{tc}(G), P_{tc})\) then \( x, y \) and \( z \) are vertices \( (vtx(x), vtx(y) \) and \( vtx(z) \) hold) and \( i \) is an index \( (index(i) \) holds). The effect of the next two rules is similar but they concern the relation symbol
The rules (TC7) - (TC9) enforce conditions that atoms \(\text{path}(x,y,z,1)\) must satisfy to be in a model. The rules (TC10) - (TC12) enforce recursive conditions that atoms \(\text{path}(x,y,z,i), i \geq 2\), must satisfy in order to be in the model. Finally, the rules (TC13) - (TC14) define the relation symbol \(tc\) in terms of the relation \(path\).

The following result can now be proved by an easy induction.

**Proposition 4.5** Let \(G\) be a directed graph. The data-program pair \((D_{tc}(G), P_{tc})\) has a unique model that consists of (1) all atoms in \(D_{tc}(G)\), (2) all atoms \(\text{path}(x,y,z,i)\) such that there is a path in \(G\) from \(x\) to \(y\) of length \(i\) and with \(z\) being the last but one vertex on this path, and (3) of all atoms \(tc(x,y)\) such that there is a directed path of positive length from \(x\) to \(y\) in \(G\).

We have chosen to discuss in detail the question of the transitive closure since it is well known that this property is not definable in first-order logic [AV91]. We can define it in our logic \(PS\) because our notion of definability is different: data-program pairs define concepts as special Herbrand models. A more detailed discussion of these issues follows in the next section.

5 Expressive power of the logic \(PS\)

Our discussion in the previous sections demonstrated the use of the logic \(PS\) as a tool to represent computational problems. In this section, we will study the expressive power of the logic \(PS\), that is, we will identify a class of computational problems that can be represented by means of finite \(PS\) programs.

We first recall some database terminology [Ull88]. Let \(Dom\) be a fixed infinite set (for instance, the set of all natural numbers). A relational schema over a domain \(Dom\) is a nonempty sequence \(R = (r_1, \ldots, r_k)\) of relation symbols. Each relation symbol \(r_i\) comes with integer arity \(a_i > 0\). An instance of a relation schema \(R\) is a nonempty and finite set of ground atoms, each of the form \(r_i(u_1, \ldots, u_{a_i})\), where \(1 \leq i \leq k\) and \(u_1, \ldots, u_{a_i} \in Dom\). By \(I(R)\) we denote the set of all instances of a relational schema \(R\). Since \(Dom\) is fixed, form now on we will not explicitly mention it. We also emphasize that, unlike in standard presentations, we require that instances of a relation schema be nonempty.

Relational schemas provide a framework for a precise definition to a class of computational problems known as search problems. Let \(R\) and \(S\) be two disjoint relational schemas. A search problem (over relational schemas \(R\) and \(S\)) is a recursive relation \(\Pi \subseteq I(R) \times I(S)\). The set \(I(R)\) is the set of instances of \(R\). Given an instance \(I \in I(R)\), the set \(\{J \in I(S) : (I,J) \in \Pi\}\) is the set of solutions to \(\Pi\) for the instance \(I\).

Search problems abound. It is clear that the graph problems and the \(n\)-queens problem considered earlier in the paper are examples of search problems. More generally, all constraint satisfaction problems over discrete domains, including such basic AI problems as planning, scheduling and product configuration, can be cast as search problems.

A search language or (language for short) is a set \(L\) of expressions and a function \(\mu\) such that for every expression \(e \in L\), \(\mu(e)\) is a search problem. We call \(\mu\) the interpretation function for \(L\). By the expressive power of a language \(L\) we mean the class of search problems defined by expressions from \(L\): \(\{\mu(e) : e \in L\}\).

We note that the concept of a search problem extends that of a database query [Var82], which is defined as a partial recursive function from \(I(R)\) to \(I(S)\). Consequently, fragments of search languages consisting of those expressions that define partial functions are, in particular,
database query languages. In fact, one can regard a search problem as a second-order query — a mapping from the set of instances of some relational schema $R$ into the power set of the set of instances of another (disjoint) relational schema $S$. Pushing the analogy further, a search language can be viewed as a second-order database query language — an expression in such a language defines, given an instance of a relational schema $R$, a collection of instances of a relation schema $S$ rather than a single instance.

We will show that the logic $PS$ gives rise to a search language and establish its expressive power. An expression is a pair $(P, R, S)$, where $P$ is a $PS$ program, and $R$ and $S$ are disjoint nonempty sets of relation symbols in $P$. We will show that $(P, R, S)$ can be viewed as a specification of a search problem over relational schemas $R$ and $S$. Namely, let $D \in \mathcal{I}(R)$. For every set $M \subseteq HB(D \cup P)$, by $M[S]$ we denote the set of all those atoms in $M$ that are built by means of relation symbols from $S$. We define the interpretation function $\mu$ as follows:

$$\mu(P, R, S) = \{(D, F) : D \in \mathcal{I}(R), \text{ and } F = M[S], \text{ where } M \in Mod(D, P)\}.$$ 

It is clear that $\mu(P, R, S) \subseteq \mathcal{I}(R) \times \mathcal{I}(S)$. Consequently, the set of $PS$ expressions together with the function $\mu$ is a search language.

In a similar way we can view as a search language the language of DATALOG$^-$ (logic programming without function symbols) with the semantics of Herbrand models, supported models \cite{Cla78,Apt90} or stable models \cite{GL88}. Since the expressive power of DATALOG$^-$ with the supported-model semantics will play a role in our considerations, we will recall relevant notions and results.

Let $\mathcal{L}$ be a language of predicate logic. A DATALOG$^-$ clause is an expression $r$ of the form

$$r = p(X) \leftarrow q_1(X_1), \ldots, q_m(X_m), \neg q_{m+1}(X_{m+1}), \ldots, \neg q_{m+n}(X_{m+n}),$$

where $p, q_1, \ldots, q_{m+n}$ are relation symbols and $X, X_1, \ldots, X_{m+n}$ are tuples of constant and variable symbols with arities matching the arities of the corresponding relation symbols. We call the atom $p(X)$ the head of the clause $r$ and denote it by $h(r)$. If a clause has empty body, we represent it by its head (thus, atoms can be regarded as clauses). For a clause $r$ we also set

$$B(r) = q_1(X_1) \land \ldots \land q_m(X_m) \land \neg q_{m+1}(X_{m+1}) \land \ldots \land \neg q_n(X_n).$$

A DATALOG$^-$ program is a collection of DATALOG$^-$ clauses. Let $P$ be a DATALOG$^-$ program. As usual, we call relation symbols that appear in the heads of clauses in $P$ intentional. We refer to all other relation symbols in $P$ as extensional. We denote the sets of intentional and extensional relation symbols of a DATALOG$^-$ program $P$ by $I(P)$ and $E(P)$, respectively. Next, for a relation symbol $p$ that appears in $P$, we denote by $Def(p)$ the set of all clauses in $P$ whose head is of the form $p(t)$, for some tuple $t$ of constant and variable symbols. In other words, $Def(p)$ consists of all clauses that define $p$.

In the paper we restrict our attention to DATALOG$^-$ programs of special form, called I/O programs, providing a clear separation of data facts (ground atom representing data) from clauses (definitions of intentional relation symbols). To this end, we define first a class of pure programs. We say that a DATALOG$^-$ program $P$ is pure if

\footnote{All concepts related to DATALOG$^-$ that we mention here can be defined in a more general setting of logic programming languages that include function symbols. For an in-depth discussion of logic programming, we refer the reader to \cite{Apt90}.}
1. for every relation symbol \( p \in I(P) \), all clauses in \( \text{Def}(p) \) have the same head of the form \( p(X) \), where \( X \) is a tuple of distinct variables

2. \( P \) contains no occurrences of constant symbols

3. \( E(P) \neq \emptyset \).

Pure programs are, in particular, in the so-called normal form as they satisfy condition (1)\cite{Apt90}. An I/O program is a DATALOG\(^{-}\) program of the form \( D \cup P \), where \( P \) is a pure program and \( D \in \mathcal{I}(E(P)) \) (that is, \( D \) is a nonempty and finite set of ground atoms built of relation symbols in \( E(P) \)). To simplify the discussion, we define supported models for I/O programs only. It does not cause any loss of generality. Indeed, one can show that for every logic program containing at least one constant symbol there is an I/O program with the same intentional relation symbols and such that supported models of both programs, when restricted to intentional ground atoms, coincide (that is, under the semantics of supported models, both programs define the same relations).

Let \( P \) be a pure program and let \( D \in \mathcal{I}(E(P)) \). For a predicate \( p \) from \( I(P) \), we define its (Clark’s) completion \( \text{cc}(p) \) as

\[
\text{cc}(p) = p(X) \iff \bigvee \{ \exists Y_r \ B(r) : r \in \text{Def}(p) \},
\]

where \( X \) is a tuple of variables and \( Y_r \) is the tuple of distinct variables occurring in the body of \( r \) but not in the head of \( r \) (we exploit the normal form of \( P \) here) \cite{Cla78}. We define the (Clark’s) completion of \( P \), \( \text{CC}(P) \), by setting

\[
\text{CC}(P) = \{ \text{cc}(p) : p \in \text{Pr} \}.
\]

Finally, we define a set of ground atoms \( M \subseteq \text{HB}(D \cup P) \) to be a supported model of an I/O program \( D \cup P \) if it is a Herbrand model of \( \text{cl}(D) \cup \text{CC}(P) \), where \( \text{cl}(D) \) is defined as in Section 2. We denote by \( \text{Sup}(D \cup P) \) the collection of all supported models of \( D \cup P \).

Let \( P \) be a pure program and let \( S \subseteq I(P) \). We define

\[
\nu(P,S) = \{(D,F) : D \in \mathcal{I}(E(P)), \text{ and } F = M[S], \text{ where } D \neq \emptyset \text{ and } M \in \text{Sup}(D \cup P)\}.
\]

Since \( E(P) \), \( I(P) \) and \( S \) can be regarded as relational schemas, \( \nu(P,S) \) is a search problem. Thus, the set of expressions \( (P,S) \), where \( P \) is a pure program and \( S \) is a subset of \( I(P) \), together with the function \( \nu \) form a search language.

The expressive power of this language is known. A search problem \( \Pi \) over relational schemas \( R \) and \( S \) is in the class \( \text{NP-search} \) if there is a nondeterministic Turing Machine \( TM \) such that

1. \( TM \) runs in polynomial time

2. for every instance \( I \) of the schema \( R \) (input instance of \( \Pi \)), the set of strings left on the tape when accepting computations for \( I \) terminate is precisely the set \( \{ J \in \mathcal{I}(S) : (I,J) \in \Pi \} \), that is, the set of solutions to \( \Pi \) for the input \( I \).

The class NP-search is precisely the class of search problems captured by finite DATALOG\(^{-}\) programs with the supported-model semantics.
Theorem 5.1 ([MR01]) For every finite pure program $P$ and every $S \subseteq I(P)$, $\nu(P,S)$ is a search problem in the class NP-search. Conversely, for every problem $\Pi$ in the class NP-search there is a pure program $P$ and a set $S \subseteq I(P)$ such that $\nu(P,S) = \Pi$.

We will now show that the expressive powers of $PS$ and of DATALOG$^-$ with supported model semantics are the same. Namely, we will prove the following result.

Theorem 5.2 For every finite pure program $P$ and every set $S \subseteq I(P)$, there is a finite $PS$ program $P'$ such that $E(P) \cup I(P)$ are among the relation symbols appearing in $P'$ and $\nu(P,S) = \mu(P',E(P),S)$. Conversely, for every finite $PS$ program $P'$ and every nonempty and disjoint sets $R$ and $S$ of relation symbols appearing in $P'$, there is a finite pure program $P$ such that $R = E(P)$, $S \subseteq I(P)$ and $\mu(P',R,S) = \nu(P,S)$.

Proof: Let $P$ be a pure program. We will consider the completion $CC(P)$ of $P$ and construct its equivalent representation in terms of $PS$ rules (we recall that $PS$ rules are just special formulas from the language of predicate logic).

We build this representation of $CC(P)$ as follows. Let $p$ be a predicate symbol in $I(P)$. Let us assume that $p(X)$, where $X$ is a tuple of distinct variables, is the common head of all clauses in $Def(p)$. Let us consider a clause $r \in Def(p)$, say

$$ r = p(X) \leftarrow q_1(X_1), \ldots, q_m(X_m), \neg q_{m+1}(X_{m+1}), \ldots, \neg q_n(X_n), $$

and let $Y_r$ be a tuple of distinct variables that appear in the body of $r$ but not in its head. We introduce a new predicate symbol $d_r$, of the arity $|X| + |Y_r|$ and define the following $PS$ rules

$$ \psi_i(r) = d_r(X,Y_r) \rightarrow q_i(X_i), \quad i = 1, \ldots, m $$
$$ \psi_i(r) = d_r(X,Y_r) \land q_i(X_i) \rightarrow \bot, \quad i = m+1, \ldots, n $$
$$ \psi_0(r) = q_1(X_1) \land \ldots \land q_m(X_m) \rightarrow d_r(X,Y_r) \lor q_{m+1}(X_{m+1}) \lor \ldots \lor q_n(X_n). $$

We define $\Psi(r) = \{\psi_0(r), \psi_1(r), \ldots, \psi_n(r)\}$. It is clear that $\Psi(r)$ entails (in the first-order logic) the universal sentence $d_r(X,Y_r) \leftrightarrow B(r)$ (intuitively, $\Psi(r)$ specifies $d_r(X,Y_r)$ so that it can be regarded as an abbreviation for $B(r)$).

We will now use atoms $d_r(X,Y_r)$ to define $PS$ rules that form an equivalent representation to the formula $cc(p)$. Let us recall that

$$ cc(p) = p(X) \iff \forall Y_r B(r) : r \in Def(p). $$

Thus, we define the following $PS$ rules:

$$ cc'_r(p) = d_r(X,Y_r) \rightarrow p(X), \quad r \in Def(p) $$
$$ cc'(p) = p(X) \rightarrow \forall Y_r d_r(X,Y_r) : r \in Def(p). $$

It is clear (by first-order logic tautologies) that

$$ \Phi(p) = \{\Psi(r) : r \in Def(p)\} \cup \{cc'_r(p) : r \in Def(p)\} \cup \{cc'(p)\} $$

and $cc(p)$ have the same first-order models (modulo new relation symbols $d_r$).

Let us define $P' = \bigcup\{\Phi(p) : p \in I(P)\}$. Clearly, $P'$ is a $PS$ program and every relation symbol in $E(P) \cup I(P)$ in $P'$. Moreover, by the comment made above, for every instance $D$
of the schema $E(P)$, $cl(D) \cup P$ and $cl(D) \cup P'$ have the same models and, in particular, the same Herbrand models (again modulo new relation symbols). Thus, $\nu(P, S) = \mu(P, E(P), S)$.

We will now prove the second part of the assertion. Let $P'$ be a $PS$ program and let $R$ and $S$ be nonempty and disjoint sets of relation symbols appearing in $P'$. By Theorem 5.1, it is enough to show that the search problem $\mu(P', R, S)$ belongs to the class NP-search. This is, however, straightforward. A nondeterministic Turing machine $M$ (as defined in \cite{GJ79}) for solving $\mu(P', R, S)$ can be described as follows:

1. Given an instance $D \in I(R)$, $M$ grounds (in a deterministic way) the data-program pair $(D, P')$. Since $P'$ is fixed, the task can accomplished in polynomial time with respect to the size of $D$ (measured as the total number of symbols in $D$).

2. $M$ generates in a nondeterministic fashion (using its guessing module) a subset of the Herbrand base. This task involves the number of guesses that is not greater than $|HB(D \cup P')|$, again a polynomial in the size of $D$.

3. Next, $M$ checks (deterministically) that the subset that was guessed is a model of the ground theory. This task can be accomplished in time that is polynomial in the size of the grounding of the data-program pair $(D, P)$ which, as we already pointed out, is polynomial in the size of $D$.

4. If the subset that was guessed is not a model, $M$ moves to halting state NO. Otherwise, $M$ rewrites the contents of the tape so that only these ground atoms of the supported model that are built of relation symbols in $S$ are left, and moves to halting state YES.

It is clear that tape contents for accepting computations are precisely projections of models of the data-program pair $(D, P)$ onto $S$. That is, $M$ solves $\Pi$ nondeterministically in polynomial time. It follows that $\mu(P', R, S)$ is in the class NP-search.

**Corollary 5.3** A search problem $\Pi$ is in the class NP-search if and only if there is a finite $PS$ program $P$ and nonempty disjoint sets $R$ and $S$ of relation symbols appearing in $P$ such that $\Pi = \mu(P, R, S)$.

Decision problems can be viewed as special search problems. Thus, in particular, every decision problem in the class NP can be expressed by means of a finite $PS$ program (and two nonempty disjoint sets of relation symbols appearing in it). This observation is a counterpart to a result by Schlipf concerning DATALOG\cite{Sch95}.

**Corollary 5.4** A decision problem $\Pi$ is in the class NP if and only if there is a finite $PS$ program $P$ and nonempty disjoint sets $R$ and $S$ of relation symbols appearing in $P$ such that $\Pi = \mu(P, R, S)$.

## 6 Extensions of the logic $PS$

From the programming point of view, the logic $PS$ provides a limited repertoire of modeling means: constraints must be represented as rules (essentially, standard clauses of predicate logic). We will now present ways to enhance the effectiveness of logic $PS$ as a programming formalism. Namely, we will introduce extensions to the basic formalism of the logic $PS$ to provide direct support representations of some common “higher-level” constraints. We denote this extended logic $PS$ by $PS+$.
6.1 Adding cardinality atoms

When considering the PS theories developed for the n-queens and vertex-cover problems one observes that these theories could be simplified if the language of the logic PS contained direct means to capture constraints such as: “exactly one element is selected” or “at most k elements are selected”.

We already noted in the introduction that extensions of the language of DATALOG with explicit constructs to model such constraints and the corresponding modifications in the algorithms to compute stable models resulted in significant performance improvements. These gains can be attributed to the fact that programs in the extended language are usually much more concise, their ground versions use fewer variables and have smaller sizes. Thus, the search space of candidate models is also smaller.

It is natural to expect that similar gains are also possible in the case of our formalism. With this motivation in mind, we extend the language of the logic PS by cardinality atoms.

With this in mind, we extend the language of the logic PS by cardinality atoms. We first consider a propositional language specified by a set of atoms \( \text{At} \). By a propositional cardinality atom (propositional c-atom, for short), we mean any expression of the form \( m\{p_1,\ldots,p_k\}n \) (one of \( m \) and \( n \), but not both, may be missing), where \( m \) and \( n \) are non-negative integers and \( p_1,\ldots,p_k \) are atoms from \( \text{At} \). The notion of a rule generalizes in an obvious way to the case when propositional c-atoms are present in the language. Namely, a c-rule is an expression of the form

\[
C = A_1 \land \ldots \land A_s \rightarrow B_1 \lor \ldots \lor B_t,
\]

where all \( A_i \) and \( B_i \) are (propositional) atoms or c-atoms.

Let \( M \subseteq \text{At} \) be a set of atoms. We say that \( M \) satisfies a c-atom \( m\{p_1,\ldots,p_k\}n \) if

\[
m \leq |M \cap \{p_1,\ldots,p_k\}| \leq n.
\]

If \( m \) is missing, we only require that \( |M \cap \{p_1,\ldots,p_k\}| \leq n \). Similarly, when \( n \) is missing, we only require that \( m \leq |M \cap \{p_1,\ldots,p_k\}| \). A set of atoms \( M \) satisfies a c-rule \( C \) if \( M \) satisfies at least one atom \( B_j \) or does not satisfy at least one atom \( A_i \).

For example, if \( \text{At} = \{a,b,c,d\} \), then the expression

\[
a \rightarrow 2\{a,c,d\} \lor d
\]

is a clause. The set \( M = \{a,c\} \) is its model while \( M' = \{a,b\} \) is not.

To generalize the idea of a cardinality atom to the language of predicate calculus, we need a syntax that will facilitate concise representations of sets. To this end, we will adapt the standard set-theoretic notation, where

\[
\{p(x) \mid x \in X \text{ and } q(x)\}
\]

denotes the set of all atoms of the form \( p(x) \) for which \( x \in X \) and \( q(x) \) holds. For instance, in the language used for modeling the n-queens problem an expression

\[
\{q(R,C) \mid \text{index}(C) \text{ and } R \leq C\}
\]

will be interpreted as a template for defining sets. For every ground instantiation \( r \) of its “free” variable \( R \), it gives rise to the set (\( n \) is a constant defined in the data file of the n-queens data-program pair)

\[
\{q(r,c) \mid c = r, r + 1, \ldots, n\}.
\]
Formally, we define a cardinality atom or c-atom, for short, to be any expression

$$l\{S_1; S_2; \ldots; S_k\}u,$$

where \(l\) and \(u\) are terms (constants or variables) and \(S_1, S_2, \ldots, S_k\) are set definitions. Intuitively, the meaning of a c-atom \(l\{S_1; S_2; \ldots; S_k\}u\) is that at least \(l\) and no more than \(u\) of the atoms specified by set definitions \(S_1, \ldots, S_k\) are true. We will now make this intuition precise.

Our definitions are similar to those proposed in \[SN02\] in the context of SLP.

A set definition is an expression of the form \(p(t) : d_1(s_1) \land \ldots \land d_m(s_m)\), where \(p\) is a program relation symbol, \(d_i, 1 \leq i \leq m\), are data or predefined relation symbols, and \(t, s_i, 1 \leq i \leq m\), are tuples of terms. We note that it is possible that \(m = 0\). We also note that this concept is defined only in the context of data-program pairs as in that case there is a clear distinction between data and program predicates. A variable appearing in a set definition as an argument of one of data relation symbols is bound. Other variables appearing in this set definition are free.

Let \(S = p(t) : d_1(s_1) \land \ldots \land d_m(s_m)\) be a set definition appearing in a data-program pair \(T\). By our assumption, \(T\) contains at least one constant. For every ground substitution \(\vartheta\) whose domain contains all free variables in \(S\) and does not contain any bound variables from \(S\), by \(S\vartheta\) we denote the set of atoms defined as follows: \(S\vartheta\) is the set of all atoms of the form \(p(t\vartheta')\), where \(\vartheta'\) is a ground substitution with the domain consisting of all bound variables in \(S\) such that for every \(i, 1 \leq i \leq m, d_i(s_i\vartheta')\) holds (we recall that data and predefined relation symbols are fully specified by a data-program pair and this latter condition can be verified efficiently). We also note that if \(m = 0\), all variables appearing in \(t\) are free and \(S\vartheta = \{p(t\vartheta)\}\).

Let us now consider a c-atom \(A = l\{S_1; \ldots; S_k\}u\) appearing in a theory or a data-program pair \(T\). Without loss of generality, we will assume that sets of bound and free variables appearing in c-atoms in \(T\) are disjoint. Let \(\vartheta\) be a ground substitution whose domain does not contain any bound variables appearing in \(S_1, \ldots, S_k\). We define \(A\vartheta\) as follows:

1. \(A\vartheta = \bot\), if \(l\vartheta\) or \(u\vartheta\) are not integers appearing as constants in \(T\)
2. \(A\vartheta = l\vartheta\{S_1\vartheta \cup \ldots \cup S_k\vartheta\}u\vartheta\), otherwise. In this case, \(l\vartheta\) and \(u\vartheta\) are integer constants appearing in \(T\), and \(S_1\vartheta \cup \ldots \cup S_k\vartheta\) is the set of ground atoms.

We define \(gr(T)\) as in Section \[3\], with the stipulation that c-atoms are grounded as specified above. The ground theory \(gr(T)\) consists of propositional c-rules. We define a set \(M\) of ground atoms to be a model of \(T\) if it is a model of \(gr(T)\).

We will now illustrate these definitions with an example. Let \((D, P)\) be a data-program pair (in the language extended with c-atoms). Let us assume that

\[D = \{d_1(1), d_1(2), d_1(3), d_2(a), d_2(b)\}\]

and that

\[C = d_1(X) \rightarrow X\{p(X, Y) : d_1(Y) \land Y \geq X; q(Z) : d_2(Z)\}\]

is a rule in \(P\). The variables \(Y\) and \(Z\) are bound in \(C\), the variable \(X\) is free. Clearly, for every ground substitution \(\vartheta\) such that \(X\vartheta = a\) or \(b\), both the antecedent and the consequent of the rule ground to \(\bot\) and the rule grounds to
\[ \perp \rightarrow \perp. \]

In every other ground substitution, \( X \) is replaced with 1, 2 or 3. Thus, we get the following three templates for propositional rules:

\[
\begin{align*}
d_1(1) & \rightarrow 1\{p(1,Y); d_1(Y) \land Y \geq 1; q(Z); d_2(Z)\} \\
d_1(1) & \rightarrow 2\{p(2,Y); d_1(Y) \land Y \geq 2; q(Z); d_2(Z)\} \\
d_1(1) & \rightarrow 3\{p(3,Y); d_1(Y) \land Y \geq 3; q(Z); d_2(Z)\}.
\end{align*}
\]

Set definitions in each of these rules specify sets of ground atoms and give rise to the following three ground instances of the rule \( C \):

\[
\begin{align*}
d_1(1) & \rightarrow 1\{p(1,1), d_1(1,2), d_1(3), q(a), q(b)\} \\
d_1(2) & \rightarrow 2\{p(2,2), p(2,3), q(a), q(b)\} \\
d_1(3) & \rightarrow 3\{p(3,3), q(a), q(b)\}.
\end{align*}
\]

From the last of these rules it follows that the atoms \( p(3,3), q(a) \) and \( q(b) \) must be true in every model of \((D, P)\).

In the extended logic \( PS^+ \) we can encode the vertex cover problem in a more straightforward and more concise way. Namely, there is no the need for integers to represent indices as sets are represented directly and not in terms of sequences! In this new representation \((D_{vc}^\prime(G, k), P_{vc}^\prime), D_{vc}^\prime(G, k) \) is given by

\[ D_{vc}^\prime(G, k) = \{vtx(v); v \in V\} \cup \{edge(v, w); \{v, w\} \in E\} \cup \{size(k)\}, \]

and \( P_{vc}^\prime \) consists of the clauses:

- **VC1**: \( invc(X) \rightarrow vtx(X) \)
- **VC2**: \( size(K) \rightarrow \{invc(X); vtx(X)\}K \)
- **VC3**: \( edge(X,Y) \rightarrow invc(X) \lor invc(Y). \)

Atoms \( invc(x) \) that are true in a model of the \( PS \) theory \((D_{vc}^\prime, P_{vc}^\prime)\) define a set of vertices that is a candidate for a vertex cover. \( (VC2) \) guarantees that no more than \( k \) vertices are included. \( (VC3) \) enforces the vertex-cover constraint.

Cardinality atoms also yield alternative encodings to the graph-coloring and \( n \)-queens problems. In both cases, we use the same representation of input data and modify the program component only. In the case of the graph-coloring problem, a single rule, \( (C3) \), directly stating that every vertex is assigned exactly one color, replaces two old rules \((C3) \) and \((C4) \).

\[
\begin{align*}
C'1: & \quad clrd(X,C) \rightarrow vtx(X) \\
C'2: & \quad clrd(X,C) \rightarrow color(C) \\
C'3: & \quad vtx(X) \rightarrow 1\{clrd(X,C); color(C)\}1 \\
C'4: & \quad edge(X,Y) \land clrd(X,C) \land clrd(Y,C) \rightarrow \perp.
\end{align*}
\]

In the case of the \( n \)-queens problem the change is similar. The rules \((nQ3) \) and \((nQ4) \) are replaced with a single rule \((nQ'3) \) and the rules \((nQ5) \) and \((nQ6) \) with a single rule \((nQ'4) \).

\[
\begin{align*}
nQ'1: & \quad q(R, C) \rightarrow index(R) \\
nQ'2: & \quad q(R, C) \rightarrow index(C) \\
nQ'3: & \quad index(R) \rightarrow 1\{q(R,C); index(C)\}1 \\
nQ'4: & \quad index(C) \rightarrow 1\{q(R,C); index(R)\}1
\end{align*}
\]
nQ’5: $\text{index}(R) \rightarrow \{q(R + I - 1, I) : \text{index}(I)\}1$
nQ’6: $\text{index}(C) \rightarrow \{q(I, C + I - 1) : \text{index}(I)\}1$
nQ’7: $\text{index}(R) \rightarrow \{q(R - I + 1, I) : \text{index}(I)\}1$
nQ’8: $\text{index}(C) \rightarrow \{q(n - I + 1, C + I - 1) : \text{index}(I)\}1$.

The rule (nQ’5) enforces the condition that the main ascending diagonal and all ascending diagonals above it contain at most one queen. The rule (nQ’6) enforces the same condition for the ascending main diagonal and all ascending diagonals below it. Finally, the rules (nQ’7) and (nQ’8) enforce the same condition for descending diagonals. In the original encoding we used only two clauses to represent these conditions. We could use them here again. However, the four clauses that we propose here, and that are possible thanks to the availability of c-atoms, result in significantly smaller ground theories. We address this issue in detail in Section 7.

6.2 Adding closure computation to logic $PS+$

In Section 5 we presented programs capturing the concepts of reachability in graphs and of transitive closure of binary relations. These representations are less elegant and, more importantly, less concise than representations possible in SLP. For instance, the transitive closure of a binary relation $r$ can be computed by the following DATALOG program:

TC’1: $tc(X, Y) \leftarrow r(X, Y)$

TC’2: $tc(X, Y) \leftarrow r(X, Z), tc(Z, Y)$.

This encoding capitalizes on the minimality that is inherent in the stable-model semantics (in this case, the program, being a Horn program, has a unique least model). Moreover, the grounding of this program has size linear in the cardinality of the relation $r$.

Constraints involving reachability, transitive closure and other related concepts are quite common. In the problem of existence of a Hamiltonian cycle in a directed graph, we may first constrain candidate sets of edges to those that span collections of disjoint cycles covering all vertices in the graph (for instance, by imposing the restriction that in each vertex exactly one edge from the candidate set starts and exactly one edge from the candidate set ends). Clearly, such a candidate set is a Hamiltonian cycle if and only if it is connected. This requirement can be enforced by the constraint that all graph vertices be reachable, by edges in the candidate set, from some (arbitrary) vertex in the graph.

With this motivation in mind, we will now introduce yet another extension of the basic logic providing, in particular, means to express constraints involving reachability, connectivity, transitive closure and similar related concepts in a way they are used in SLP. To this end, we extend both the syntax and the semantics of the logic $PS+$.

As it is standard, by a Horn rule we mean a $PS$ rule (a rule without cardinality atoms) whose consequent is a single regular atom (that is, not an e-atom). Horn rules play a key role in this extension of the logic. The idea is to split the program component in a data-program pair into three parts. Intuitively, the first of them will describe initial constraints on the space of candidate solutions. The second, consisting of Horn rules, will “close” each candidate generated by the first part. The third component will provide additional constraints that have to be satisfied by the closure.

Formally, by an extended program we mean a triple $(G, H, V)$ such that

1. $G$ and $V$ are collections of (arbitrary) $PS+$ rules, called generating and verifying rules, respectively, and $H$ is a collection of Horn rules
2. No relation symbol appearing in the consequent of a rule in $H$ appears in rules from $G$.

An extended data-program pair is a pair $(D, P)$, where $D$ is a set of ground atoms (data) and $P$ is an extended program. When listing an extended program, we use the following convention. We write Horn rules as in logic programming, starting at the left with the head, followed by the (reversed) arrow ← as the implication connective and, finally, followed by the conjunction of the atoms of the body. There is no need to explicitly distinguish between rules in $G$ and $V$ as the partition is implicitly defined by $H$. Namely, non-Horn rules involving relation symbols appearing in the consequents of Horn rules form the set $V$. All other non-Horn rules form the set $G$.

Let $(D, P)$ be an extended data-program pair, where $P = (G, H, V)$. A set of ground atoms from the Herbrand base of $(D, G)$ is a model of $(D, P)$ if

1. $M$ is a model of $(D, G)$

2. the closure of $M$ under $H$, that is, the least Herbrand model of the Horn theory $M \cup H$, satisfies all ground instances of rules in $V$.

The first condition enforces that models of $(D, P)$ satisfy all constraints specified by $G$. Thus, $G$ can be regarded as a generator of the search space, as there are still additional constraints to be satisfied. The second condition eliminates all these models generated by $G$ whose closure under $H$ violates some of the constraints given by $V$. In other words, $H$ computes the closure and $V$ verifies whether the closure has all of the desired properties.

As an illustration of the way this extension of logic $PS+$ can be used we will provide a formal representation of the Hamiltonian-cycle problem, capturing intuitions described above. Let $G = (V, E)$ be a directed graph and let $v_0$ be an arbitrary vertex in $V$. To represent this data we set

$$D'_{hc}(G, v_0) = \{ \text{vtx}(v); v \in V \} \cup \{ \text{edge}(v, w); \{v, w\} \in E \} \cup \{ \text{start}(v_0) \}.$$  

Formally speaking, for the Hamiltonian-cycle problem, there is no need to include $v_0$ in the data set. We do it, as our encoding involves the notion of reachability, for which some arbitrary “starting” point is needed. The (extended) program part, $P'_{hc}$, consists of the following five rules.

- **HC’1**: $hc\_edge(X, Y) \rightarrow edge(X, Y)$
- **HC’2**: $1\{hc\_edge(Y, X); \text{vtx}(Y)\}1$.
- **HC’3**: $1\{hc\_edge(X, Y); \text{vtx}(Y)\}1$.
- **HC’4**: $\text{visit}(Y) \leftarrow \text{visit}(X) \land hc\_edge(X, Y)$
- **HC’5**: $\text{visit}(Y) \leftarrow \text{start}(X)$.
- **HC’6**: $\text{visit}(X)$.

We note the use of our notational convention. Clearly, the rules (HC’4) and (HC’5) form the Horn part (it is indicated by the way they are written). It follows that the rules (HC’1)-(HC’3) are generating and the rule (HC’6) is verifying. Intuitively, the rule (HC’1) guarantees that if an atom $hc\_edge(x, y)$ is true then, $(x, y)$ is an edge (in other words, only edges of the graph can be chosen to form a Hamiltonian cycle). Rule (HC’2) captures the constraint that for every vertex $x$ there is exactly one selected edge that ends in $x$. Similarly, the rule (HC’3) captures the constraint that for every vertex $x$ there is exactly one selected edge that
starts in $x$. Thus, every model of the data-program pair consisting of $D_{hc}(G,v_0)$ and the rules (HC’1)-(HC’3) contains $D_{hc}(G,v_0)$ and a set of atoms $hc\_edge(x,y)$ that describe a particular selection of edges and that span in $G$ disjoint cycles covering all its vertices. Rules (HC’3) and (HC’4) define the relation $visit$ that describes all vertices in $G$ reachable from $v_0$ by means of selected edges. Finally, the last rule verifies that all vertices are reached, that is, that selected edges form, in fact, a Hamiltonian cycle.

6.3 Expressive power of extended logics

We close this section with an observation on the expressive power of the logic $PS+$. Since it is a generalization of the logic $PS$, it can capture all problems that are in the class NP-search. On the other hand, the search problem of computing models of a data-program pair $(D,P)$, where $P$ is a fixed $PS+$ program, is an NP-search problem (a simple modification of the proof of the second assertion of Theorem 5.2 demonstrates that). Thus, it follows that the expressive power of the logics $PS+$ does not extend beyond the class NP-search. In other words, the logic $PS+$ also captures the class NP-search.

7 Computing with $PS+$ theories

In the preceding sections, we focused on the use of the logic $PS+$ as a language for encoding (programming) search problems and established its expressive power. In order to use the logic $PS+$ as a computational problem solving tool we need algorithmic methods for processing data-program pairs and finding their models.

Let us recall that a set $M$ of ground atoms is a model of a data-program pair $(D,P)$ if and only if $M$ is a model of the theory $gr(cl(D) \cup P))$. Thus, to compute models one could proceed in two steps: first, compute $gr(cl(D) \cup P))$ and then, find models of the ground theory. We refer to these steps as grounding and solving, respectively. This two-step approach is used successfully by all current implementations of SLP including, smodels and dlv. We will adhere to it, as well.

It is easy to see that the data complexity of grounding is in the class P. That is, there is an algorithm that, for every data-program pair $(D,P)$ computes $gr(cl(D) \cup P))$ and, assuming that $P$ is fixed, works in time that is polynomial in the size of $D$. For instance, a straightforward enumeration of all substitutions of appropriate arities (determined by the numbers of free variables in program rules) can be adapted to yield a polynomial-time algorithm for grounding.

This straightforward approach can be improved. The size of grounding (although polynomial in the size of the data part) is often very big. To address this potential problem, we note that to compute the models it is not necessary to use $gr(cl(D) \cup P))$. Any propositional theory that has the same models as $gr(cl(D) \cup P))$ can be used instead. In this context, let us note that the truth values of all ground atoms appearing in $gr(cl(D) \cup P))$ that are built of data relation symbols can be computed efficiently by testing whether they are present in $D$. Similarly, we can effectively evaluate truth values of all ground atoms built of predefined relation symbols by, depending on the relation, checking whether two constants are identical, different or, in the case of integer constants, whether one is the sum, product, etc. of two other integer constants.

Thus, the theory $gr(cl(D) \cup P))$ can be simplified by taking into account the truth values of ground atoms built of data and predefined relation symbols. Let $A$ be such a ground atom.
1. If \( A \) appears in the consequent of the clause and is true, we eliminate this clause.

2. If \( A \) appears in the consequent of the clause and is false, we eliminate \( A \) from the consequent of the clause.

3. If \( A \) appears in the body of a clause and is true, we eliminate \( A \) from the body.

4. If \( A \) appears in the body of the clause and is false, we eliminate this clause.

These simplifications may reveal other atoms with forced truth values and the process continues, much in the spirit of unit propagation used in satisfiability solvers. For instance, if we obtain a rule consisting of a single (regular) atom and this atom appears in the consequent of the rule, the atom must be true. If, on the other hand, this single atom appears in the body of the rule, it must be false. Furthermore, if a cardinality atom of the form \( m\{p_1, \ldots, p_k\}n \), is forced to be true and the number of atoms \( p_i \) that have been already assigned value true is \( m \), then all the unassigned atoms \( p_i \) must be false. In addition, if the number of atoms \( p_i \) that have been already assigned value false is \( k - m \), then all unassigned atoms must be true. Similar propagation rules exist for the case when a \( c \)-atom is forced to be false.

We continue the process of simplifying the theory as long as new atoms with forced truth values are discovered. We call the theory that results when no more simplifications are possible the ground core of a data-program pair \((D, P)\). We denote it by \( \text{core}(D, P) \).

We have the following straightforward result (as in the other cases before, we do not explicitly mention ground predefined atoms when specifying models).

**Proposition 7.1** Let \((D, P)\) be a data-program pair. A set \( M \) of ground atom is a model of \((D, P)\) if and only if \( M = D \cup T \cup M' \), where \( T \) is the set of atoms that are forced to be true and \( M' \) is a model of \( \text{core}(D, P) \).

Proposition 7.1 suggests that for the grounding step, it is enough to compute \( \text{core}(D, P) \) rather than \( gr(cl(D) \cup P) \). It is an important observation. The size of the theory \( \text{core}(D, P) \), measured as the total number of symbol occurrences, is usually much smaller when compared to that of \( gr(cl(D) \cup P) \). Following this general idea, we designed and implemented a program, \( \text{psgrnd} \) that, given a data-program pair \((D, P)\), computes its ground equivalent \( \text{core}(D, P) \).

We will now focus on the second step — searching for models of a propositional \( \text{PS}^+ \) theory. First, we will consider the class of theories that are obtained by grounding data-program pairs whose program component does not contain \( c \)-atoms. In this case, the ground core of a data-program pair is a collection of standard propositional clauses (written as implications). The program \( \text{psgrnd} \) provides an option that, in such case, produces the ground core of the input data-program pair in the DIMACS format. Consequently, most of the current implementations of propositional satisfiability (SAT) solvers can be used in the solving step to compute models. Thus, we can view the logic \( \text{PS} \) as a programming tool for modeling problems in terms of propositional constraints and regard \( \text{psgrnd} \) as a front-end facilitating the use of SAT solvers.

If \( c \)-atoms and Horn rules are present in a program, the theory after grounding and simplification is a propositional \( \text{PS}^+ \) theory that contains, in general, (propositional) \( c \)-atoms and propositional Horn rules. Thus, SAT solvers are not directly applicable. One approach in such case is to represent \( c \)-atoms and closure rules by means of equivalent (standard) propositional theories. It is possible since, as we noted earlier, logics \( \text{PS} \) and \( \text{PS}^+ \) have the same expressive power.
We argue, however, that a more promising approach to compute models of data-program pairs is to design solvers for propositional PS+ theories that are direct outcomes of the grounding process and, in general, may contain c-atoms. The reason is that using high-level constraints results in programs whose ground representations are often more concise then those obtained by corresponding programs that do not involve such constraints. We will illustrate this point using programs developed earlier in the paper.

We start with the vertex-cover problem. Let $G$ be an input graph with $n$ vertices and $m$ edges, and let $k$ be an integer $k$ specifying the cardinality of a vertex cover. In the case of the program consisting of rules (VC1) - (VC5), our grounding algorithm results in a propositional theory with $kn$ atoms of the form $vc(i, x)$ and with $\Theta(kn^2)$ rules of total size (measured by the number of atom occurrences) also $\Theta(kn^2)$. On the other hand, grounding of the program consisting of rules (VC′1) - (VC′3) yields a theory with $n$ atoms of the form $invc(x)$ and with $\Theta(m)$ rules of total size $\Theta(n + m)$. Thus, this latter encoding involves fewer atoms (if $k \geq 2$) and has the size that is asymptotically smaller.

Next, we will consider the Hamiltonian-cycle problem. Our first encoding (rules (HC1) - (HC6)) grounds to a theory with $n^2$ atoms and with total size $\Theta(n^2 + n(n^2 - m))$. Our second encoding, involving Horn rules (rules (HC′1) - (HC′7)), grounds to a theory with $n^2 + n$ atoms and the total size of $\Theta(n^2)$. Thus, even though this theory uses slightly more atoms, it has significantly smaller total size (except for the case of “almost complete” graphs, that is, graphs with the number of “missing” edges equal to $o(n^2)$, one-order of magnitude smaller).

For the original encoding of the $n$-queens program, psgrnd produces a propositional theory of size $\Theta(n^3)$. On the other hand, it is easy to see that grounding of the the second encoding (the one involving cardinality atoms), has size $\Theta(n^2)$ — a gain of an order of magnitude.

In the case of the encodings for the graph-coloring problems, we also obtain more concise theories by grounding programs designed with the use of c-atoms. Indeed, the rule (C′3) grounds to a smaller theory than rules (C3) and (C4). The improvement is, in general, by a constant factor and so, it is not asymptotically better.

Since encodings involving c-atoms are usually smaller and define smaller search spaces, it is important to design solvers that can take direct advantage of these small representations. We developed a solver, aspps (short for “answer-set programming with propositional schemata”), that can directly handle c-atoms and closure rules. The aspps solver is an adaptation of the Davis-Putnam algorithm for computing models of propositional CNF theories. That is, it is a backtracking search algorithm whose two key components are unit propagation and branching.

Unit propagation “propagates” through the theory truth values established so far. If there is a rule with all atoms in the antecedent assigned value true and all but one atom in the consequent assigned value false, then the remaining “unassigned” atom in the consequent must be true for the rule to hold. Similarly, if all atoms in the consequent of a rule are false and if all but one atom in the antecedent are true, the only “unassigned” atom in the antecedent must be false. In this way any partial assignment of truth values to atoms forces truth assignments on some additional atoms. When no more atoms can be forced, the second module, branching, selects a way to split search space into separate parts. When search in a part fails, the program backtracks and tries another.

A key difference between aspps and satisfiability algorithms is in how branching is implemented. In satisfiability solvers, in order to branch, we pick an atom, say $a$, and split the search space into two parts. In one of them we assume that the atom $a$ is true. In the other
one we assume that \( a \) is false. Propositional \( PS^+ \) theories may, in general, contain \( c \)-atoms and \( aspps \) considers them too when selecting a way to branch.

To explain the method that \( aspps \) uses, let us observe that the unit propagation may, in particular, assign a truth value to a \( c \)-atom appearing in the theory. That constrains possible truth assignments to unassigned atoms that form the \( c \)-atom.

For example, let us consider a propositional \( c \)-atom \( C = 1 \{a, b, c, d\} \) that we know must be true. Let us assume that \( d \) has already been assigned value false and that \( a, b \) and \( c \) have not. There are exactly three ways in which atoms \( a, b \) and \( c \) can be assigned truth values consistent with \( C \) being true:

\[
\begin{align*}
\text{a} &= \text{t}, \text{b} = \text{f}, \text{c} = \text{f} \\
\text{a} &= \text{f}, \text{b} = \text{t}, \text{c} = \text{f} \\
\text{a} &= \text{f}, \text{b} = \text{f}, \text{c} = \text{t}.
\end{align*}
\]

It follows that if a truth value of a \( c \)-atom \( C \) has been forced, we have an additional way to split the search. Namely, we can consider in turn each truth assignment to unassigned atoms appearing in \( C \) that is consistent with the truth value of \( C \). In our example, if \( C \) is true, we could split the search space into three subspaces by assuming first that \( a = \text{t}, b = \text{f} \) and \( c = \text{f} \), then that \( a = \text{f}, b = \text{t} \) and \( c = \text{f} \) and, finally, that \( a = \text{f}, b = \text{f} \) and \( c = \text{t} \).

The choice of the way to branch is of vital importance. To make this selection, the \( aspps \) program approximates the degree to which the atom is constrained. That is, \( aspps \) first assigns to each clause a weight based on its current length. The fewer atoms in a clause the more constraining it is and the greater its weight. The weight of a (regular) atom is defined as the sum of the weights of all clauses containing it. It is this number that \( aspps \) uses to estimate how much the atom is constrained.

When looking for the way to branch, the \( aspps \) program considers all (regular) atoms that have not been assigned a truth value yet. It also considers some \( c \)-atoms. Let \( C \) be a \( c \)-atom that has been forced to be true by earlier choices. Let \( A \) be the set of atoms appearing in \( C \) that have not received a truth value yet. The atom \( C \) is considered as a candidate to define branching if the number of truth assignments to atoms in \( A \) that are consistent with the truth value of \( C \) (the number of branches \( C \) defines) is less than or equal to \(|A|\).

If there are \( c \)-atoms satisfying these conditions, \( aspps \) will select this one among them that maximizes the sum of weights of unassigned atoms that appear in it. Otherwise, \( aspps \) will branch on a regular atom with the maximum weight. If a propositional \( PS^+ \) theory contains Horn clauses, they play no role in the process of selection the next atom for branching. They participate though in the unit propagation step.

The source codes, information on the implementation details for programs \( psgrnd \) and \( aspps \) and on their use is available at [http://www.cs.uky.edu/ai/aspps/](http://www.cs.uky.edu/ai/aspps/).

### 8 Experimental results

Several data-program pairs that we presented in the paper (both with and without \( c \)-atoms and Horn rules) show that the logics \( PS \) and \( PS^+ \) are effective as formalisms for modeling search problems. In this section we will demonstrate computational feasibility of these formalisms when combined either with our native solver \( aspps \), specifically tailored to handle the syntax of the logic \( PS^+ \) (\( c \)-atoms and Horn rules), or with off-the-shelf SAT solvers.
We show that aspps is generally comparable in performance with that of smodels and, in the cases discussed here, even faster. We chose smodels for the comparison since (1) smodels accepts a similar syntax as aspps, (2) in the case of each problem considered here, there is an smodels program essentially identical in its basic structure to the PS+ program presented in the paper, and (3) smodels is at present one of the most advanced implementations of answer-set programming paradigm based on DATALOG with the stable-model semantics.

Next, we show that our language of data-program pairs in the basic logic PS (without c-atoms and Horn rules), together with the program psgrnd, greatly simplifies the use of SAT solvers in computing solutions to search problems.

Finally, we compare the performance of aspps and smodels with that of SAT solvers as engines for solving search problems.

We stress that our experiments did not aim at demonstrating superiority of one solver over another. That would require a much more comprehensive and careful experimental study. Our objective was to demonstrate the feasibility of our approach.

For our test cases we selected problems that we used as examples throughout the paper: the n-queens, graph-coloring, vertex-cover, and Hamiltonian cycle problem. We used psgrnd/aspps and smodels for encodings in the logic PS+ (that is, encodings involving c-atoms and Horn rules). We used the combinations psgrnd/satz and psgrnd/chaff for programs without c-atoms and Horn rules. In the experiments we used the following versions of these programs: zchaff [MMZ01], satz215.2 [L97], lparse-1.0.6 and smodels-2.26 both at [NSS97], aspps.2001.10.18, psgrnd.2001.10.18 and, finally, psgrnd.2002.10.11 (as a front-end for satisfiability solvers) [ET01a]. All our experiments were performed on a Pentium IV 1.7 GHz machine running linux.

In the case of vertex cover, for each \(n = 50, 60, 70, 80\) we randomly generated 100 graphs with \(n\) vertices and \(2n\) edges. For each graph \(G\), we computed the minimum size \(k_G\) for which the vertex cover can be found. We then tested aspps, smodels and satz on all the instances \((G, k_G)\). The results (Table 1) represent the average execution times. Encodings we used for testing aspps and smodels where based on the program (VC’1) - (VC’3). For satisfiability solvers we used encodings based on the clauses (VC1) - (VC5) (as cardinality constraints cannot be handled by satisfiability solvers).

As we observed, the size of the encoding (VC’1) - (VC’3) is, in general, asymptotically smaller than that of (VC1) - (VC5). Thus, satisfiability solvers had to deal with much larger theories (hundreds of thousands of clauses for graphs with 80 vertices as opposed to a few hundred when c-atoms are used). Consequently, they did not perform well.

As concerns smodels versus aspps, in general aspps is somewhat (about three times) faster than smodels and the difference seems to grow with size.

| \(n\) | \(50\) | \(60\) | \(70\) | \(80\) |
|------|------|------|------|------|
| aspps | 0.011 | 0.070 | 0.463 | 1.996 |
| smodels | 0.043 | 0.116 | 1.584 | 8.157 |

Table 1: Timing results (in seconds) for the vertex-cover problem. Average time for each set of 100 random generated graphs; satz and zchaff were halted after 10 minutes on a single instance.

For the n-queens problem, our solver performed exceptionally well. Cardinality constraints play here again a crucial role. Aspps has to work with theories obtained by grounding the
program \((nQ'1) - (nQ'8)\). In contrast, \textit{satz} and \textit{zchaff} have to work with much larger theories obtained by grounding the program \((nQ1) - (nQ7)\). When \(n = 70\) this means the difference between 416 and 562030 rules, respectively (in each case, the number of ground atoms is 4900). Both solvers required therefore much more time than \textit{aspps}. In fact, we stopped \textit{satz} after 10 minutes on the 40-queen instance. \textit{Zchaff} performed much better than \textit{satz} and completed the computation in the case of \(n = 70\) in just under 10 minutes. Given the size of the theory it has to work with, this is quite remarkable. One lesson, we believe, is that there is still a large potential for improvements in the way \textit{aspps} implements search. We also note that \textit{aspps} and \textit{zchaff} solvers both exhibited a somewhat irregular performance growth pattern as the number of queens increased. Lastly, we note that \textit{smodels} did not complete computation for \(n = 40\) in the 10 minutes we allocated, despite the fact that we used a more concise encoding, similar to that processed by \textit{aspps}. Table 2 summarizes these results.

| # of queens | 40  | 50  | 60  | 70  | 80  |
|-------------|-----|-----|-----|-----|-----|
| \textit{aspps} | 0.20 | 0.06 | 2.19 | 0.31 | 0.17 |
| \textit{zchaff} | 158.92 | 157.74 | 283.87 | 558.10 | *** |

Table 2: Timing results (in seconds) for the \(n\)-queen problem; \textit{satz} and \textit{smodels} were halted after 10 minutes on the instance with \(n = 40\).

In the case of the graph colorability problem, as we observed in the previous section, c-atoms do not give rise to significant gains in the size of the ground theory. Given the amount of research devoted to satisfiability solvers and still relatively few efforts to develop fast solvers for logics involving cardinality constraints, it is not surprising that satisfiability solvers outperform both \textit{aspps} and \textit{smodels}. Our results also show \textit{satz} outperforming \textit{zchaff}, which may be attributed to the fact that our test graphs were randomly generated and did not have any significant internal structure that could be capitalized on by \textit{zchaff}. As concerns \textit{aspps} and \textit{smodels}, they show essentially the same performance. We summarize the relevant results in Table 3. The graphs for the 3-colorability problem were generated randomly with vertex/edge ratios such that approximately 1/2 of the graphs were 3-colorable. For each value \(n = 100, 200\) and \(300\), we generated a set of 1000 graphs. The values that we report are the average execution times.

| \(n\) | 100  | 200  | 300  |
|------|------|------|------|
| \textit{aspps} | 0.006 | 0.302 | 13.678 |
| \textit{smodels} | 0.026 | 0.495 | 16.043 |
| \textit{satz} | 0.013 | 0.077 | 1.416 |
| \textit{zchaff} | 0.002 | 0.107 | 7.952 |

Table 3: Timing results (in seconds) for the graph 3-coloring problem.

Our last experiment concerned the problem of computing Hamiltonian cycles. As in other cases when additional constraints (transitive closure computation, in this case) result in much smaller theories, both \textit{smodels} and \textit{aspps} outperform \textit{satz} and \textit{zchaff}. In addition, in this case, \textit{aspps} significantly outperforms \textit{smodels}. In the experiments, we considered graphs with 20, 40, 60, 80 and 100 vertices with the number of edges chosen so that the likelihood of the existence
of a Hamiltonian cycle is close to 0.5. For each set of parameters, we generated 1000 instances. The times given in Table 4 represent average execution times.

| V/E     | 20/75 | 40/180 | 60/300 | 80/425 | 100/550 |
|---------|-------|--------|--------|--------|---------|
| aspps   | 0.000 | 0.001  | 0.002  | 0.003  | 0.005   |
| smodels | 0.006 | 0.034  | 0.117  | 0.255  | 0.456   |
| satz    | 0.122 | ***    | ***    | ***    | ***     |
| zchaff  | 0.144 | 2.277  | 33.405 | ***    | ***     |

Table 4: Timing results (in seconds) for the determining presence of a Hamilton cycle in a graph.

It is clear from these results that our solver aspps is competitive with smodels and SAT solvers such as zchaff and satz as a processing back-end for problems encoded as data-program pairs in the logics PS and PS+.

9 Conclusions

Our work demonstrates that predicate logic and its extensions can support answer-set programming systems in a way in which stable logic programming does. To put it differently, we show that predicate logic can be an effective declarative programming formalism.

In the paper we described logic PS that can be used to uniformly encode search problems. We proved that the expressive power of this logic is given by the class NP-search. Thus, it is the same as the expressive power of DATALOG¬, even though it is conceptually simpler — its semantics is essentially that of propositional logic.

We demonstrated the use of our logic in modeling such search problems as graph coloring, vertex cover, n-queens, Hamiltonian cycle and transitive closure.

We designed a program psgrnd, that given a data-program pair in the logic PS, encoding a search problem, computes its equivalent propositional representation in the DIMACS form. In this way, it becomes possible to compute models of data-program pairs and, consequently, solve the corresponding search problems by means of standard off-the-shelf satisfiability solvers. We demonstrated that the approach is feasible and effective by applying satz and zchaff to propositional theories produced by psgrnd.

We argued that the logic PS can benefit from extensions allowing explicit representations of some commonly used constraint such as cardinality constraints and transitive closure. Encodings of search problems that take advantage of these extensions are usually much smaller and, consequently, could result in smaller search spaces if solvers capable to take advantage of direct representations of high-level constraints were available. We designed one such solver, aspps. Our experimental results are encouraging. Aspps is competitive with smodels, a state-of-the-art processing engine for DATALOG¬ programs extended by cardinality constraints and other constructs. In fact, in several cases, aspps outperforms smodels.

The results of the paper show that programming front-ends for constraint satisfaction problems that support explicit coding of complex constraints facilitate modeling and result in concise representations. They also show that solvers such as aspps that take advantage of those concise encodings and process high-level constraints directly, without compiling them to simpler representations, exhibit very good computational performance. These two aspects are important. Satisfiability checkers often cannot effectively solve problems simply due to
the fact that encodings they have to work with are large. For instance, for the vertex-cover problem for graphs with 80 vertices and 160 edges, aspps has to deal with theories that consist of a few hundred of rules only. In the same time pure propositional encodings of the same problem contain over one million clauses — a factor that undoubtedly is behind much poorer performance of satz and zchaff on this problem.

Our work raises new questions. Further extensions of logic $PS^+$ are possible. For instance, constraints that impose other conditions on set cardinalities than those considered here (such as, the parity constraint) might be included. We will pursue this direction. Similarly, there is much room for improvement in the area of solvers for the propositional logic $PS^+$ and aspps can certainly be improved. There is also a potential for developing local search techniques for the logic $PS^+$. The task seems much easier than in the case of DATALOG$^-$ programs, where finding successful local search algorithms turned out to be hard [DS02].

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