The Problem of Dynamic Programming on a Quantum Computer

Pooya Ronagh

Institute for Quantum Computing, Waterloo, ON
Perimeter Institute for Theoretical Physics, Waterloo, ON
Department of Physics and Astronomy, University of Waterloo, Waterloo, ON
1QB Information Technologies (1QBit), Vancouver, BC
(Dated: July 30, 2021)

We discuss the problem of finite-horizon dynamic programming (DP) on a quantum computer. We introduce a query model for studying quantum and classical algorithms for solving DP problems, and provide example oracle constructions for the travelling salesperson problem, the minimum set-cover problem, and the edit distance problem. We formulate open questions regarding quadratic quantum speedups for DP and discuss their implications. We then prove lower bounds for the query complexity of quantum algorithms and classical randomized algorithms for DP, and show that no greater-than-quadratic speedup can be achieved for solving DP problems.

I. Introduction

Solving NP-hard problems efficiently on a quantum computer has been a challenging endeavour for quantum computation. Grover’s search algorithm [1] provides a path to achieving quadratic speedups for some NP-hard problems such as the Boolean satisfiability problem (SAT). While this quantum speedup is much more moderate than what is anticipated from the exponential computing resources of quantum computers, its existence is of theoretical and practical significance for understanding the power and limitations of quantum algorithms. On the other hand, achieving a similar type of speedup for other NP-hard problems, such as the travelling salesperson problem (TSP), has been a long-standing open problem for quantum computing.

For SAT, the exponential time hypothesis speculates that no classical algorithm can essentially perform better than exhaustive search. However, for TSP, the best classical algorithm is much more sophisticated than naïve exhaustive search. For a TSP problem of size $n$, exhaustive search will require enumeration of $n!$ possible solutions, whereas an algorithm of [2, 3] based on dynamic programming (DP) solves the problem in $O^*(2^n)$ [4]. Perhaps this is the reason demonstrating a quadratic quantum speedup for TSP has appeared to be difficult. Recently, [4] studied quantum algorithms for a collection of NP-hard problems for which the best known classical algorithms are exponentially expensive DP solutions and showed a slightly weaker speedup (e.g., $O^*(\alpha^n)$ with the base $\alpha$ being a number less than 2 but greater than $\sqrt{2}$). However, these results assume coherent QRAM access to solutions of smaller DP problems and require exponentially large amounts of classical memory.

In this paper, we introduce and study the problem of dynamic programming on a quantum computer. A DP problem is defined by a finite set of states $S$, a finite set of possible actions (decisions) $A$ at each state, and a set of time epochs $T$. Performing an action at a given state results in a reward (or cost) and a transition to a new state. The goal is to find an optimal policy for an agent at every state. Here, the measure of optimality is the future reward the agent collects should it pursue the actions prescribed by a policy. The cumulative future reward is often called the value function.
Query complexity of dynamic programming. We consider algorithms for solving DP problems that have query access to an oracle that, given a state, action, and time $s \in S, a \in A,$ and $t \in T,$ simulate the effect of performing action $a$ on state $s$ at a given point $t$ in time. The goal is to solve the problem with the fewest queries to this oracle. This model addresses the complexity of generic algorithms for dynamic programming, that is, those that do not possess further information about the oracle or the particular instances of DP problems solved by them.

We focus on the case of finite horizon DP problems, that is, those for which $T$ is a finite set. Finite-horizon DP problems are of great interest in many famous applications in discrete and combinatorial algorithms, including TSP and the minimum set-cover problem (MSC). Despite many efforts in computer science, the best known algorithms for these problems are exponentially expensive DP solutions that have been known for many decades.

Finite-horizon DP problems are closely related to their infinite-horizon counterparts. Infinite-horizon DP problems are the deterministic special cases of Markov decision problems (MDP). The latter is the subject of study in reinforcement learning (RL). In fact, many problems solved using RL do not possess stochastic kernels and reward functions. This includes, for instance, classic computer games and deterministic optimal control problems. The query complexity of infinite-horizon DP problems is therefore of significant practical interest in optimal control and RL. \[5\] proves lower bounds for randomized classical algorithms solving MDPs using various data structures to provide the probability distribution functions associated to the MDP transition kernels.

Summary of our contributions. In Section II, we introduce a general framework for studying finite-horizon DP problems on a quantum computer. We introduce a query model for studying bounded-error quantum algorithms that make coherent queries to an oracle representative of the DP problem. We also provide an analogous classical query setting to compare such bounded-error quantum algorithms against classical randomized algorithms. We then state several open problems pertaining to the potential existence of quadratic quantum speedups in solving DP problems in Section III and discuss their implications. In Section IV we provide several example constructions for the above-mentioned DP oracle in the case of TSP, MSC, and the edit distance problem.

We then prove lower bounds for the query complexity of quantum algorithms and classical randomized algorithms for solving these problems, and show that a greater-than-quadratic speedup in the number of state–action pairs cannot be achieved using quantum algorithms. In Section V we provide a quantum query complexity lower bound of $\Omega(\sqrt{|S||A|})$ for solving DP problems using the generalized relational adversary method \[6\]. Furthermore, in Section VI we apply similar ideas from adversary methods to a classical query complexity setting to prove a lower bound of $\Omega(|S||A|)$ on the query complexity of classical bounded-error randomized algorithms for these problems. We conclude that the discussed potential quadratic speedups would result in optimal quantum algorithms, up to polylogarithmic factors. In particular, we rule out the possibility of achieving exponential quantum speedups for DP.

II. Dynamic Programming

Let $S$ and $A$ be two given finite sets of states and actions, respectively. The actions are taken at points in a discrete set of time epochs $T = \{0, \ldots, T - 1\}$. In this case, the DP problem is said to have a finite horizon, which all DP problems considered in this paper have. The following structure is given:

(a) Finite sets $S$ and $A$, and a transition kernel or law of motion

$$a_t : S \rightarrow S \quad \forall t \in T, \forall a \in A;$$
A reward structure which is a bounded, deterministic, possibly time-dependent function of states, actions, and time epochs, and for simplicity takes values in the set of non-negative integers

\[ r_t = r_t(s, a) : S \times A \rightarrow \mathbb{Z}_{\geq 0} \quad \forall t \in T. \]

The boundedness condition allows us to define a positive integer denoted by \( [r] \in \mathbb{N} \) as an upper bound on reward values. We note that, without loss of generality (and by a constant shift of all rewards if needed), we assume a lower bound of 0 for the reward structure.

By the above definition of the reward structure, we have implicitly assumed that all actions in set \( A \) are admissible for all states in \( S \). For a DP in which this condition is not naturally satisfied by the model (i.e., some actions are not allowed at certain states), we may, without loss of generality, let an originally inadmissible action \( a \) at a state \( s \) map \( s \) to a null state additionally defined.

Value function. A (deterministic) policy consists of the choice of a single action at every state and every point in time:

\[ \pi_t : S \rightarrow A \quad \forall t \in T. \]

To a policy \( \pi = (\pi_t)_{t \in T} \), we associate a possibly time-dependent value function \( V^\pi_t : S \rightarrow \mathbb{Z}_{\geq 0} \) defined via

\[ V^\pi_t(s) = \sum_{i \geq t} r_i(s_i, a_i), \]

where \( s_0 = s \) is an initial state, and all subsequent actions are chosen according to the policy \( \pi \). That is, \( a_t = \pi_t(s_t) \) and \( s_{t+1} = a_t(s_t) \). We assume that a marked initial state \( s_0 \in S \) is given. The goal of DP is to find an optimal policy at \( s_0 \) at time \( t = 0 \), that is, to find

\[ \pi^* = \arg\max_{\pi} V^\pi_0(s_0). \tag{1} \]

Bellman’s optimality criteria. Bellman’s optimality criteria [7], for the value function states that an optimal policy \( \pi^* = (\pi^*_t) \) is associated to the (unique) optimal value function \( V^*_t(s) := V^{\pi^*}_t(s) \) satisfying

\[ V^*_t(s) = \max_a \left\{ r_t(s, a) + V^*_{t+1}(a_t(s)) \right\} \quad \forall t \in T \tag{2} \]

and the boundary condition that no reward can be accumulated after the final time epoch. That is, \( V^*_T(s) = 0 \) for all states \( s \in S \).

It is easy to verify that if the optimal value function \( (V^*_t) \) is known, an optimal action at \( s_0 \) at time \( t = 0 \) can be found by finding the action \( a \in A \) that maximizes \( r_0(s_0, a) + V^*_1(a(s_0)) \). Alternatively, an optimizer of \( V^* \) in (1), that is, the \( s_0 \) component of an optimal policy, may be directly calculated. Such an algorithm can iteratively be used \( O(T) \) times at subsequent states visited by the agent until a complete optimal policy for the DP problem has been traversed along the time horizon \( T \).

Query model. We consider quantum and (randomized) classical algorithms that make queries to the transition kernel and reward structure in order to solve a DP problem. The quantum algorithms are considered to make coherent queries to

\[ U_{DP} : \{s\} |a\} |t\} |x\} |y\} \mapsto \{s\} |a\} |t\} |x \oplus a_t(s)\} |y \oplus r_t(s, a)\}. \tag{3} \]
For classical algorithms, the oracle is similar but queried classically:

\[ O_{\text{DP}} : (s, a, t) \mapsto (a_t(s), r_t(s, a)) . \] (4)

We note that, in many practical scenarios, any one of the transition kernel, the reward structure, or the policies may be independent of time. In this case they are called \textit{time homogeneous}.

Based on Bellman’s recursion, we consider two algorithms for solving problem (1). We first define the \textit{value iteration operator} \( \mathcal{F}^{(t)} : \mathbb{Z}_{\geq 0}^{\lvert S \rvert} \rightarrow \mathbb{Z}_{\geq 0}^{\lvert S \rvert} \) for all \( t \in T \) via

\[ \mathcal{F}^{(t)} : v_s \mapsto \max_{a \in A} \{ r_t(s, a) + v_{a_t(s)} \} \] (5)

and consider their recursive applications

\[ v^{(T-t-1)} = \mathcal{F}^{(T-t-1)}(v^{(T-t)}) \quad t \in T, \]

starting with the initial vector of all zeroes, \( v^{(T)} = 0 \), for all \( s \in S \). It is easy to see via induction that \( v^{(T-k)} \) attains the optimal value function at time \( T - k \):

\[ V^{*}_{T-k}(s) = \mathcal{F}^{(T-k)} \circ \ldots \circ \mathcal{F}^{(T-1)}(0). \]

Therefore, in order to find the optimal action at \( s_0 \) at time \( t = 0 \), it suffices to find \( v^{(1)} \) and find the maximizer

\[ \arg\max_{a \in A} \left[ r_0(s_0, a) + v^{(1)}_{a_0(s_0)} \right]. \]

**Proposition 1.** Value iteration (Algorithm 1) solves a DP problem in \( O(|S||A|T) \) queries to the oracle (4).

```
1: procedure VALUEITERATION(s0)
2:     Initialize an array \( v[s] \leftarrow 0 \) for all \( s \in S \)
3:     for \( t = T - 1, T - 2, \ldots, 1 \) do
4:         for \( s \in S \) do
5:             \( w[s] \leftarrow \max_a \{ r_t(s, a) + v[a_t(s)] \} \)
6:         end for
7:     end for
8:     return \( \arg\max_a \{ r_0(s_0, a) + v[a_0(s_0)] \} \)
9: end procedure
```

Algorithm 1. Value iteration

We let \( S_t \subseteq S \) be the set of all states that are reachable at time \( t \in T \). We call a DP problem \textit{time ordered} whenever the sets \( S_t \) form a partitioning of \( S \), \( S = \bigsqcup_{t \in T} S_t \), that is, a state \( s \in S_t \) is only reachable at a time epoch \( t \). We note that a DP problem that is not readily time ordered can be turned into one that is by replicating every state \( s \in S \) to at most \( O(T) \) copies \( (s, t) \in S \times T \). Nevertheless, working with time-ordered DP problems allows us to simplify the query complexity of solving them using Bellman’s criteria via the following algorithm.

**Proposition 2.** Bellman’s recursion (Algorithm 2) solves a time-ordered DP problem in \( O(|S||A|) \) queries to the oracle (4).

```
1: procedure BELLMANREC(S0)
2:     Initialize an array \( v[s] \leftarrow 0 \) for all \( s \in S \)
3:     for \( t = T - 1, T - 2, \ldots, 1 \) do
4:         for \( s \in S \) do
5:             \( w[s] \leftarrow \max_a \{ r_t(s, a) + v[a_t(s)] \} \)
6:         end for
7:     end for
8:     return \( \arg\max_a \{ r_0(s_0, a) + v[a_0(s_0)] \} \)
9: end procedure
```

Algorithm 2. Bellman’s recursion
implemented using $\tilde{v}$ where that receives a register prepared in the superposition of a set of indices $s$. See Section III.A for a summary of recent attempts.

Achieving such a speedup for solving DP problems has appeared to be a challenging open problem.



Algorithm 2. Bellman’s recursion

III. Open Problems for Quantum Computation

As will be apparent from the examples presented in Section IV, typically the number of states $|S|$ is exponentially larger than the time horizon $T$ and the number of actions $|A|$. Therefore, providing quantum speedups in terms of $|S|$ is of particular interest to us, and computational complexity factors of the form $\text{poly}(|A|, T)$ are considered negligible. The oracle above can be efficiently constructed using $\text{polylog}(|S|)$ qubits and the same order of customary preliminary gate sets (e.g., the Clifford+T set) in practical cases of interest. In view of Proposition 1 we now present the following problem.

**Problem A.** Does there exist a bounded-error quantum algorithm that returns the $s_0$ component of the solution to (1) using $\tilde{O}(\sqrt{|S|}\text{poly}(|A|, T))$ queries to the oracle (3)?

Achieving such a speedup for solving DP problems has appeared to be a challenging open problem. See Section III.A for a summary of recent attempts.

A quantum variant of the value iteration operator (5) can be viewed as a unitary transformation that receives a register prepared in the superposition of a set of indices $s \in S_t$ and a set of associated values $v(s)$ in the computational basis, and performs the transformation

$$U_t : \sum_{s \in S_t} |s\rangle |v(s)\rangle |0\rangle \mapsto \sum_{s \in S_t} |s\rangle |v(s)\rangle |w(s)\rangle,$$

where $w(s) = \max_{a_t} (r_t(s, a) + v(a_t(s)))$ for a given $t \in T$. Given Algorithm 1 if $U_t$ can be implemented using $\tilde{O}(\sqrt{|S_t|}\text{poly}(|A|))$ queries to the oracle $\mathcal{O}$, then the answer to Problem A is positive. More abstractly, we ask the following question.

**Problem B.** Let $f : X \to \mathbb{Z}_{\geq 0}$ be an integer-valued function on a discrete finite domain $X$. Let $Y = \{\sigma : X \to X\}$ be a finite set of mappings from $X$ to itself. Does there exist a unitary transformation

$$U : \sum_{x \in X} |x\rangle |f(x)\rangle |0\rangle \mapsto \sum_{x \in X} |x\rangle |f(x)\rangle |\max_{\sigma} (f(\sigma(x)))\rangle,$$

that uses $\tilde{O}(\sqrt{|X|}\text{poly}(|Y|))$ queries to the oracle $\mathcal{O} : |x\rangle |\sigma\rangle |z\rangle \mapsto |x\rangle |\sigma\rangle |z \oplus \sigma(x)\rangle$?
We note that the quadratic scaling in Problem B is with respect to the size of the domain $X$ as opposed to the size of $Y$, over which the optimization is performed. This is unlike the behaviour expected from amplitude amplification. Value iteration implies that a scaling better than $\tilde{O}(\sqrt{|X|})$ in Problem B would contradict the lower bounds proven in Section V. That is, an algorithm solving Problem B will solve the following problem as well.

**Problem C.** Does there exist a bounded-error quantum algorithm that returns the $s_0$ component of the solution to (1) using $\tilde{O}(\sum_{t\in T} \sqrt{|S_t| \text{poly}(|A|)})$ queries to the oracle (3)?

We note that for time-ordered DP problems $\sum_{t\in T} \sqrt{|S_t|} \geq \sqrt{|S|}$. Therefore, it is useful to distinguish the claim of this problem from the following stronger claim.

**Problem D.** Does there exist a bounded-error quantum algorithm that returns the $s_0$ component of the solution to (1) for a time-ordered DP problem using $\tilde{O}(\sqrt{|S| \text{poly}(|A|)})$ queries to the oracle (3)?

A quantum algorithm solving Problem C also solves Problem A, and a quantum algorithm solving either of Problems B and D also solves Problem C.

### III.A. Related problems

**Linear programming with high precision.** We can write a linear program (LP) that is equivalent to the functional equation (2). The value function depends on the time epochs $t \in \{0, \ldots, T\}$ and states $s \in S$. For each value $V^*(s)$ of the value function, we assign a real variable $v_{s,t}$ and, for consistency, write the constants $r_{t}(s,a)$ as $r_{s,a,t}$. The linear programming formulation is as follows:

$$\begin{align*}
\min & \quad v_{s_0,0} \\
\text{s.t.} & \quad v_{s,t} \geq r_{s,a,t} + v_{a(s),t+1} \quad \forall a \in A, s \in S, t \in T \\
& \quad v_{s,t} \geq 0 \quad \forall s \in S, t \in T \cup \{T\}
\end{align*}$$

(8)

It is easy to check that the above LP is feasible and attains a unique solution. In this unique solution, $v_{s,T} = 0$ for all $s \in S$.

Intuitively, the LP can be thought of as the formulation of a network flow problem wherein the inward flow of each node $(s,t)$ must match the largest outward flow of it toward the states $(a(s),t+1)$ for all $a \in A$ with the addition of a flow bias in the amount of $r_{s,a,t}$. We would like to find the smallest required inward flow from the initial node $(s_0,0)$.

In an earlier preprint [1], the author attempted to solve this LP using the multiplicative weight update method (MWUM). This technique was previously used in [9, 10] to solve semidefinite and linear programming problems. It turned out that the scaling of the method in the precision parameter of the solution prohibits the providing of a quadratic quantum advantage. Ignoring other factors, the MWUM requires $O(1/\epsilon^2)$ queries to return an $\epsilon$-feasible solution (a point that is $\epsilon$ away from the feasible domain of the LP in the $L^1$ norm). This scaling in precision is the main drawback of MWUM. In particular, the proof of [8, Theorem III.5] cannot be reduced to the case of basic feasible solutions of the LP. The fractional approximate solutions of the LP can incur exponentially many small amounts of error and result in the readout of a suboptimal solution. In the network flow analogy, this amounts to $O(|S|)$ of the nodes of the graph incurring an $\epsilon = O(1/|S|)$ deficit in the outward flow they are supposed to generate. This adds up to an $O(1)$ error in the approximation of $v_{a(s_0),1}$, which is enough to disguise the optimal action at $s_0$.

We note that, assuming $|A|$ and $T$ are polylogarithmic in $|S|$, the number of variables $n$ and the number of constraints $m$ in the LP (8) are both $\tilde{O}(|S|)$. In the context of MWUM, the primal
width $\ell$ of (8) (i.e., a bound on the optimal value of the objective of the LP) and its dual width $L$ (i.e., a bound on the slack of the constraints of the LP) are both $O(T|\tau|)$, where $|\tau|$ is an upper bound on the reward structure, as introduced in Section II.

For generic algorithms for solving LPs, the parameters $\ell$, $L$, and $\frac{1}{\epsilon}$ are related such that for equivalent LPs the quantity $\eta = \frac{\ell}{L} \epsilon$ is invariant. For (8), we have $\eta = O(T^2|\tau|^2|S|)$. Therefore, for a generic LP solver to provide a quadratic speedup in solving (8), a scaling of $O(\sqrt{\max\{n, m\}} \text{polylog}(\eta))$ is required. However, [10] shows that any generic quantum LP solver with sublinear dependence on $n$ or $m$ has to depend at least polynomially on $\eta$. Therefore, the desired polylog($\eta$) dependence is not possible.

**Coherent computation of convex conjugates of functions.** Another attempt at solving DP problems using quantum computation is reported in [11, 12], wherein the authors’ aim was to demonstrate a quadratic quantum speedup for DP problems for which the value functions are convex.

Let $f : D \rightarrow \mathbb{R}$ be a convex function defined on a bounded real domain. The argument of [11] relies on the existence of a unitary transformation that evolves a register prepared in the superposition of the values of $f$ to the superposition of the values of the convex conjugate $f^* : K \rightarrow \mathbb{R}$ of this function defined on a dual bounded domain $K$ via $f^*(s) = \sup_{x \in D} (\langle s, x \rangle - f(x))$. Solving convex DP problems is thus reduced to the efficient implementation of the evolution

$$\sum_{x \in D} |x\rangle |f(x)\rangle \mapsto \sum_{y \in K} |y\rangle |f^*(y)\rangle$$

using $\text{polylog}(|D|, |K|)$ quantum gates. However, the existence of such a unitary is an open problem. We note that such a transformation resembles the evolution (7) in Problem B.

**IV. Examples**

**IV.A. The travelling salesperson problem**

Let $G$ be a fully connected graph with vertices $V = \{1, \ldots, n\}$. We let 1 be a fixed starting vertex and $c_{ij}$ be the cost of travelling from vertex $i$ to vertex $j$. The goal is to find a Hamiltonian cycle (a cycle that visits each vertex of the graph exactly once) starting and ending at 1, while incurring the lowest total cost. The best known classical algorithm for TSP is due to Bellman [2] and Held and Karp [3] (BHK), and performs DP with a runtime of $O(n^2 2^n)$.

We define a state to be a pair $(H, i)$, where $i \in H$ and $H \subseteq V$. An action at a state $(H, i)$ corresponds to the choice of a vertex $j \in H \setminus \{i\}$. The instantaneous cost of travelling from state $(H, i)$ to $(H \setminus \{i\}, j)$ is the cost of travelling from vertex $j$ to $i$, that is, $c_{ji}$. The cost function $C(H, i)$ represents the minimum total cost of a Hamiltonian path starting at 1, entering $H$ immediately, traversing $H$, and ending at $i$. Bellman’s optimality criteria may now be written as

$$C(H, i) = \min_{j \in H \setminus \{i\}} \left[ C(H \setminus \{i\}, j) + c_{ji} \right].$$

Note that it is trivial to move from a cost-minimizing formulation to a reward-maximizing one by assigning $r_{ij} = |c| + 1 - c_{ij}$, where $|c|$ is an upper bound on the edge weights $c_{ij}$. The definition of states $(H, i)$ can be extended to allow $i \not\in H$ and the definition of the action of $j$ on $(H, i)$ can be extended to allow $j \not\in H \setminus \{i\}$. For every singleton $H = \{i\}$, any action $j$ maps $(\{i\}, i)$ to $(\emptyset, j)$ with reward $c_{1i}$. Otherwise, when $j \not\in H \setminus \{i\}$ or if $i \not\in H$, then the action of $j$ maps the state $(H, i)$...
to the state \((H \setminus \{i\}, j)\) with reward 0. We may now rewrite the DP problem as the problem of solving the functional equation

\[
V^*(H, i) = \max_{j \in H \setminus \{i\}} \left[ V^*(H \setminus \{i\}, j) + r_{ji} \right]
\]

with boundary condition \(V^*(\emptyset, j) = 0\) for all \(j\).

**Remark IV.1.** This DP problem is time ordered. It includes \(|S| = O(n2^n)\) states, \(|A| = O(n)\) actions, and a time horizon of \(T = O(n)\). Therefore, the BHK algorithm has a time complexity of \(O(n^22^n) = O(|S||A|)\).

**Oracle construction.** We begin by assuming an oracle \(O\) made from \(O\) actions, and a time horizon of \(T\).

\[
IV.1
\]

**Remark** that makes \(O\) solving the functional equation \(V\) with boundary condition to the state \((H, i)\).

\[
IV.3
\]

The registers in \(U\) set \(c\) \(O\) the affirmative to Problems C and D would provide similar quantum speedups.

Applications of Grover’s search to solve this problem in \(O\) where the equality follows from Stirling’s approximation \(\sum_{k=0}^{n} \sqrt{\binom{n}{k}} \leq n^{1/2} \sum_{k=0}^{n} \sqrt{\binom{n}{k}} \leq n^{3/2} \sqrt{n/2} = O \left( n^{3/2} \left( \frac{2}{\sqrt{n\pi}} \right)^{1/4} 2^{n/2} \right) \),

where the equality follows from Stirling’s approximation \(\binom{n}{n/2} \sim \frac{2}{\sqrt{n\pi}} 2^{n/2}\). Therefore, answers in the affirmative to Problems \([\mathcal{C}]\) and \([\mathcal{D}]\) would provide similar quantum speedups.

**Remark IV.2.** Problem \([\mathcal{A}]\), therefore, asks whether there exists a quantum algorithm for TSP that makes \(O^* (\sqrt{2^n})\) queries to the oracle \([9]\). We note that

\[
\sum_{k=0}^{n} \sqrt{k\binom{n}{k}} \leq n^{1/2} \sum_{k=0}^{n} \sqrt{\binom{n}{k}} \leq n^{3/2} \sqrt{\binom{n}{n/2}} = O \left( n^{3/2} \left( \frac{2}{2^{n\pi}} \right)^{1/4} 2^{n/2} \right)
\]

**Proposition 3.** The oracle \([9]\) can be constructed using \(O(n^2 \text{polylog}(n, [c]))\) qubits and a similar order of elementary quantum gates.

**Remark IV.3.** \([4]\) shows a bounded-error quantum algorithm for solving TSP that uses recursive applications of Grover’s search to solve this problem in \(O^*(1.728^n)\). However, this algorithm requires QRAM access to the classical BHK algorithm on graphs of size \(0.24n\), run in superposition.

**IV.B.** The minimum set-cover problem

Consider a set \(U\), called the *universe*, with \(n\) elements, and a family \(\mathcal{F} = \{V_1, \ldots, V_m\}\) of \(m\) subsets \(V_i \subseteq U\). The minimum set-cover problem (MSC) is the problem of finding the minimum number of these subsets required to cover the entire universe. That is, the goal is to find the minimum cardinality \(F \subseteq \mathcal{F}\) such that \(\bigcup_{V \in F} V = U\). We will use the notation \(\bar{F}\) to denote the union of all elements of the members of \(F\), so \(\bar{F} = \bigcup_{V \in F} V\).
We define a DP problem as follows. The states are the subsets $S \subseteq U$ of the universe. There are only two actions $A = \{u, v\}$ where the transition from $S$ via $u$ at time $t$ is the inclusion of $V_{t+1}$ and the transition via $v$ skips this inclusion. Hence,

$$u_t : S \mapsto S \cup V_{t+1} \quad \forall S \subseteq U \quad \text{and} \quad v_t : S \mapsto S \quad \forall S \subseteq U.$$ 

The transition via $v$ occurs at no additional cost, whereas transition via $u$ adds a new set to the candidate set cover. To remain in a reward-maximizing framework, we therefore define the reward for transition via $v$ as 1 and the reward for transition via $u$ as 0. The actions $u_n$ and $v_n$ send any state $S$ to itself with a reward of 0 if $S \neq U$ and a reward of $m + 1$ for the state $U$. We mark an initial state $s_0 = \emptyset$ at time $t = 0$. It is straightforward to see that the value function at $s_0$ is maximized by a policy that constructs a minimum set cover.

Remark IV.4. The DP problem has a time horizon $T = O(m)$, $|A| = 2 = O(1)$ actions, and $|S| = O(2^n)$ states. The best known classical algorithm for MSC is the above DP solution \[13\]. The runtime is $O(n m 2^n)$, consisting of $O(m 2^n) = O(T |S|)$ queries to the classical oracle \[4\] and the oracle itself contributing an additional $O(n)$ factor (for set operations).

Remark IV.5. This DP problem is not time ordered. However, one can replace the definition of states from subsets $S \subseteq U$ to pairs $(S, k)$ of a subset $S \subseteq U$ and an integer $k = 0, \ldots, m$. Then, the DP problem becomes time ordered with $|S| = O(m 2^n)$.

The family $\mathcal{F}$ can be prepared using $O(m n)$ qubits by encoding any set $V_i \subseteq U$ using a binary string of size $n$. Forming unions and set comparisons can be done using $O(n)$ elementary quantum gates. This suffices for efficient construction of an oracle

$$U_{\text{MSC}} : |S\rangle |t\rangle |a\rangle |x\rangle |y\rangle \mapsto |S\rangle |t\rangle |a\rangle |x \oplus a_t(S)\rangle |y \oplus r_{s,a,t}\rangle,$$

where $a \in \{u, v\}$.

Proposition 4. The oracle $U_{\text{MSC}}$ can be constructed using $O(m n)$ qubits and the same order of elementary gate operations.

Remark IV.6. Problem \[A\] asks whether MSC can be solved in $O^*(\sqrt{2^m \text{poly}(m)})$ queries to the oracle \[10\], while Problems \[C\] and \[D\] ask for query complexities $O^*(m \sqrt{2^n})$ and $O^*(\sqrt{m 2^n})$, respectively.

Remark IV.7. \[4\] shows a bounded-error quantum algorithm for solving MSC that uses recursive applications of Grover’s search to solve this problem in $O(1.728^n \text{poly}(m, n))$ using QRAM.

IV.C. The edit distance problem

Given two strings $x$ and $y$, find the small sequence of edit operations that will transform $x$ to $y$. The edit operations consist of substitution of one character for another, the removal of a character, and the insertion of a new character. Let $n = |x|$ and $m = |y|$ be the original sizes of the strings. We define a state $s(i, j)$ for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$. Each state represents the pair of strings $(x[1 : i], y[1 : j])$. So, the initial state is $(n, m)$ and there are three actions, $A = \{\rho, \iota, \delta\}$, acting via

$$\rho : (i, j) \mapsto (i - 1, j), \quad \iota : (i, j) \mapsto (i, j - 1), \quad \delta : (i, j) \mapsto (i - 1, j - 1),$$

respectively representing the removal of the last character from $x'$, the insertion of the last character in $y'$, and the change of the last character of $x'$ to the last character of $y'$. The cost of these actions
is time homogeneous and is defined as 1 for ρ, 1 for ι, and c = 0 for δ when the last characters of 
x’ and y’ are the same, and c = 2 when the last characters are different. We will switch around 
the costs 0 and 2 to achieve a reward of r = 2 − c for these actions, and a reward-maximizing 
formulation for the DP problem:

\[ r((i, j), \rho, t) = 2\delta(x[i], y[j]) \quad \forall t \in T, \]

where δ is the Kronecker delta on the set of characters.

Bellman’s recursion is therefore written as

\[ V^*(i, j) = \max(V^*(i - 1, j) + 1, V^*(i, j - 1) + 1, V^*(i - 1, j - 1) + 2\delta(x[i], y[j])), \]

with the boundary conditions \( V^*(i, 0) = i \) and \( V^*(0, j) = j \). Alternatively, we can extend the 
definitions of the actions and their rewards by

\[
\begin{align*}
\rho : (i, j) &\mapsto (\max(0, i - 1), j), \quad \text{with reward } 1 - \delta(i, 0); \\
\iota : (i, j) &\mapsto (i, \max(0, j - 1)), \quad \text{with reward } 1 - \delta(j, 0); \quad \text{and} \\
\delta : (i, j) &\mapsto (\max(0, i - 1), \max(0, j - 1)), \quad \text{with reward } 2\delta(x[i], y[j]) - \delta(i, 0) + \delta(j, 0).
\end{align*}
\]

We have \( |S| = O(nm) \), \( |A| = O(1) \), and \( T = O(n + m) \). The transition kernel and reward structure 
are both time homogeneous; therefore, Bellman’s recursion succeeds in \( O(|S|) \) queries \([14]\).

When \( n = m \), the DP runtime is \( O(n^2) \) and, under the strong exponential time hypothesis, the 
problem cannot be solved in a time of \( O(n^{2-\epsilon}) \). We also note that, for \( m < n \), \([15]\) provides a 
quantum query complexity lower bound of \( \Omega((\sqrt{nm})^{1-\epsilon}) \).

**Remark IV.8.** We can achieve a time-ordered formulation by replicating each state at all points 
\( t \) in time from which that state is accessible. A state \((i, j)\) is reachable in the window of time 
\( t = \max(i, j), \ldots, i + j \). Therefore, the cardinality of the set 

\[ S_t = \{ s : s \text{ is accessible at time } t \} = \{(i, j) : i, j \leq t, i + j \geq t \} \]

is \( O(t^2) \). We note that

\[
\sum_{t=0}^{n+m} \sqrt{|S_t|} = 2 \sum_{t=1}^{m} \sqrt{\frac{t^2}{2} + (n - m)\sqrt{\frac{m^2}{2}}} = O(m^2) + O((n - m)m) = O(nm),
\]

whereas

\[
\sqrt{\sum_{t=0}^{n+m} |S_t|} = \sqrt{2 \sum_{t=1}^{m} \frac{t^2}{2} + (n - m)\frac{m^2}{2}} = \sqrt{O(m^3) + O((n - m)m^2)} = O(\sqrt{nm}).
\]

Therefore, Problem \([C]\) does not provide a quantum advantage but an answer in the affirmative to 
Problem \([D]\) would close the gap with the lower bound provided in \([15]\).

**V. Quantum Complexity Lower Bound**

We now investigate the quantum query complexity of solving DP problems using the adversary method of \([8]\). Our construction follows ideas from \([5]\). Consider two families of DP problem 
instances \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \), depicted in Fig. \([1]\). The two families share the same state space 
\( S = S_\top \sqcup S_1 \sqcup S_2 \sqcup S_{\perp} \), the same action space \( A \), and the same time horizon \( T \in \mathbb{N} \). We let
\( S_1 = S_2 = [n] = \{1, \ldots, n\} \) and assume that \(|A| > 2\). The set \( S_\perp \) is a singleton \(|S_\perp| = 1\). For all instances in \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \), every action maps \( s \in S_\perp \) to itself with a reward of 2 and every \( s \in S_2 \) to itself with a reward of 0.

The structure of \( S_T \) is also common between DP problem instances in \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \). It contains the initial state \( s_0 \in S_T \). Let \( a_L, a_R \in A \) be two fixed actions. The states in \( S_T \) form a binary tree with \( s_0 \) as the root. The role of \( S_T \) is to make every state in \( S_1 \) accessible from \( s_0 \) in \( \lceil \log n \rceil \) steps. The actions \( a_L \) and \( a_R \) map every parent state to its left and right children (which might coincide) with a reward of 0, and every action \( a \in A \setminus \{a_L, a_R\} \) maps every state in \( S_T \) to itself with a reward of 1. It is easy to see that \(|S_T| \leq 2^n \) and thus \(|S| = O(n)\).

For any \( M_1 \in \mathcal{M}_1 \), every \( a \in A \) maps every \( s \in S_1 \) to some \( a(s) \in S_2 \) with a reward of 0. Therefore, the optimal value function for \( M_1 \) at \( s_0 \) is \( v^*_{M_1}(s_0) = T \) and any action \( a \neq a_L, a_R \) is optimal. The instances \( M_2 \in \mathcal{M}_2 \) differ from those in \( \mathcal{M}_1 \) only in a special state–action pair \((\bar{s}, \bar{a}) \in S_1 \times A\) for which \( \bar{a}(\bar{s}) \) is the single element of \( S_\perp \) with a reward of \( \bar{r} = 2 \). So long as \( T > 2\lceil \log(n) \rceil \), the optimal action at \( s_0 \) is one of \( a_L \) and \( a_R \), depending on the choice of \( \bar{s} \). We note that in the argument that follows we could instead assume \( T > \lceil \log(n) \rceil \) but use \( \bar{r} = T \). However, this would impose a scaling constraint of \( \lceil \bar{r} \rceil = \Omega(\log(n)) \) on the reward structure.

Now, consider a function \( f : \{0,1\}^* \rightarrow \{0,1\} \) that receives a binary string describing the transition kernel of a problem instance in \( \mathcal{M}_1 \cup \mathcal{M}_2 \) and returns 0 if and only if the optimal action at \( s_0 \) is in \( \{a_L, a_R\}\).

**Theorem 5.** Any quantum algorithm that computes the function \( f \) above uses \( \Omega(\sqrt{|S||A|}) \) queries.

**Proof.** We consider the relation \( R \) between instances \( M_1 \in \mathcal{M}_1 \) and \( M_2 \in \mathcal{M}_2 \) to be defined as \((M_1, M_2) \in R\) if and only if their transition kernel differs in exactly a single pair \((\bar{s}, \bar{a})\). We now use [6, Theorem 2]. We note the following:

- Each instance in \( \mathcal{M}_1 \) is in relation \( R \) with \(|S_1||A|\) instances in \( \mathcal{M}_2\);
- Each instance in \( \mathcal{M}_2 \) is in relation \( R \) with \(|S_2|\) instances in \( \mathcal{M}_1\);
- For every instance in \( \mathcal{M}_1 \) and every pair \((s, a) \in S \times A\) there is at most 1 instance in \( \mathcal{M}_2 \) with a different transition kernel \((s, a) \mapsto (a(s), r(s, a))\); and

![FIG. 1. Schematics of instances in \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \). A pair \( M_1 \in \mathcal{M}_1 \) and \( M_2 \in \mathcal{M}_2 \) of DP instances is depicted that are in relation \( R \), as their transition kernels differ in a single state–action pair \((\bar{s}, \bar{a}) \in S_1 \times A\).](image-url)
problems with Corollary 7. A bounded-error quantum algorithm solving Problem A is optimal in horizon DP problems with time horizon $T$.

Proposition 8. A bounded-error quantum algorithm solving time-ordered finite-horizon DP problems with states $S$, actions $A$, and time horizon $T = \Omega(\log(|S|))$ makes $\Omega(\sqrt{|S||A|})$ queries to the oracle (3).

Proof. We note that the DP families $M_1$ and $M_2$ are not time ordered since the actions $a \neq a_L, a_R$ map the states in $S_T$ to themselves. However, this can be rectified by the addition of $O(\log(n))$ states $\{u_t : t = 1, \ldots, \log(n)\}$ to $S_T$. The role of state $u_t$ is to “absorb” the actions of $a \neq a_L, a_R$ at time $t - 1$ from all states in $S_T$. It is easy to see that this modification turns $M_1$ and $M_2$ into time-ordered DP problems while the argument of Theorem 5 remains valid.

Corollary 9. A bounded-error quantum algorithm solving Problem D for time-ordered finite-horizon DP problems with time horizon $T = \Theta(\log(|S|))$ is optimal in $|S|$, and dependence on a $\text{poly}(\sqrt{|A|})$ factor is inevitable.

VI. Classical Complexity Lower Bound

We now investigate the computational complexity of solving DP problems classically in an analogous but classical oracle setting. Once again, we borrow techniques from adversary methods [4, 6, 16], but this time apply them to bounded-error classical randomized algorithms. As in Section V, we define families of DP instances $M_1$ and $M_2$ that share the same state and action spaces. We then show that, if a randomized algorithm solves DP problems with high probability, there should be a deterministic algorithm $\mu$ that also succeeds in distinguishing a large fraction of the instances in the two families.

The family $M = M_1 \cup M_2$ of DP instances is defined as in Section V and Fig. 1. By a similar argument to that in the previous section, it is obvious that an algorithm that finds an optimal action at $s_0$ is able to distinguish instances between $M_1$ and $M_2$. Let $m = n|A| = |S_1||A|$. It is straightforward to see that $|M_1| = |M_2| = mn^m$.

Let $\Pi_Q$ be the set of all the deterministic algorithms which, for an instance $M \in M$, make at most $Q$ queries to the oracle (4) given by $(s, a, t) \mapsto (a_t(s), r_t(s, a))$ before returning an optimal action at $s_0$. A randomized algorithm running at most $Q$ steps is a distribution $\mu$ on $\Pi_Q$. Let $P(\Pi_Q)$ be the set of all probability measures on $\Pi_Q$ and $a^*_M$ be the action returned by $\mu$ on
input $M$. Suppose there exists a randomized algorithm $\mu \in \mathcal{P}(\Pi_Q)$ that, when run on every $M \in \mathcal{M}$, correctly returns an optimal action $a^*_M \in \pi_M^*(s_0)$ with high probability. That is to say,

$$\max_{\mu \in \mathcal{P}(\Pi_Q)} \min_{M \in \mathcal{M}} P_{a \sim \mu(M)} \left( a^*_M \in \pi_M^*(s_0) \right) \geq 1 - \xi,$$

(11)

which by Yao’s minimax principle implies

$$\min_{D \in \mathcal{P}(\mathcal{M})} \max_{\mu \in \Pi_Q} P_{M \sim D} \left( a^*_M \in \pi_M^*(s_0) \right) \geq 1 - \xi,$$

(12)

where $D$ is a distribution on $\mathcal{M}$.

Let $D_1$ and $D_2$ be uniform distributions on $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, and let $D$ be the uniform mixture of the two. Now let $\mu \in \Pi_Q$ be a deterministic algorithm which fails to return an optimal $a^*_M \in \pi_M^*(s_0)$ with a probability of at most $\xi$ on inputs from $D$. This implies that $\mu$ fails with a probability of at most $2\xi$ if the instance is drawn from either of $D_1$ or $D_2$ considered individually.

We define $C_i \subseteq \mathcal{M}_i$ as the sets of instances for which $\mu$ succeeds. It is obvious that

$$|C_i| \geq (1 - 2\xi)|\mathcal{M}_i| = (1 - 2\xi)mn^m.$$

We call $M_1 \in \mathcal{M}_1$ and $M_2 \in \mathcal{M}_2$ a twin if their transition kernels are identical except that the reward for taking action $\bar{a}$ at state $\bar{s}$ is $r_i$ for $i = 1, 2$. We let $E(A_1, A_2)$ denote the number of twins where the $i$-th component of the twin is in $\mathcal{A}_i$ for $i = 1, 2$. The number of twins on which $\mu$ succeeds is lower bounded by

$$E(C_1, C_2) \geq E(C_1, \mathcal{M}_2) - E(\mathcal{M}_1, \mathcal{M}_2 \setminus C_2) \geq (1 - 2\xi)mn^m - 2\xi mn^m = (1 - 4\xi)mn^m.$$

Setting $\xi = \frac{1}{8}$ guarantees that $\mu$ distinguishes at least $\frac{1}{2}mn^m$ twins of the DP instances. The key observation now is that, for any twin, $\mu$ has to query $(\bar{s}, \bar{a})$, that is, the special state-action pair associated to the twin; otherwise, $\mu$ cannot distinguish the twin. We now define a new problem.

**Definition 1** (Function distinction). Let $f, g : X \to \{0, \ldots, n\}$ be two integer-valued functions defined on a discrete domain $X = \{1, \ldots, m\}$. We say that a deterministic algorithm is able to distinguish $f$ from $g$ if it queries a witness point $x \in X$ for which $f(x) \neq g(x)$.

We say $f$ and $g$ as given in the above definition form a twin if $f$ takes only nonzero values and $g$ differs from $f$ in exactly one point $x \in X$ at which $g(x) = 0$. We note that each DP instance in the families $\mathcal{M}_1$ and $\mathcal{M}_2$ corresponds uniquely to a function $S_1 \times A \to S_2 \sqcup S_\perp$ and therefore to a function $X \to \{0, \ldots, n\}$. Therefore, an algorithm $\mu$ as given above that distinguishes twins of DP instances is equivalent to an algorithm that distinguishes twins of functions.

**Proposition 10.** Any deterministic algorithm $\mu$ that performs vector differentiation needs $\Omega(m)$ queries to distinguish at least $\frac{1}{2}mn^m$ twins of functions.

**Proof.** We view the queries of $\mu$ as a decision tree. At every node of the tree, $\mu$ queries its input function at a certain point in the domain. The root of the tree is the beginning of the algorithm at which no queries have yet been made. We say this node is at depth 0. A node at which a $k$-th query to the vector is made is called a depth-$k$ node. It is obvious that a depth-$k$ node can distinguish at most $n^{m-k}$ pairs of functions. Let $(f, g)$ be a twin, with $f$ and $g$ distinguishable at a depth-$k$ node. This means that all previous $k-1$ queries to $f$ and $g$ have returned the same integers. The $k$-th query has resulted in a nonzero integer for one of the functions and 0 for the other. There are $m - k$ remaining entries and $f$ and $g$ have to coincide for all of them. This means that there are $n^{m-k}$ ways to complete $f$ and $g$ into twins.

On the other hand, there are at most $n^k$ nodes at a depth of $k$. Therefore, the depth-$k$ nodes can in total distinguish at most $n^m$ twins of functions. In order for $\mu$ to distinguish $\frac{1}{2}mn^m$ twin functions, the total depth of the decision tree of $\mu$ has to be at least $\frac{1}{2}m$. This proves the claim. 

\[\square\]
Corollary 11. Any classical randomized algorithm that solves a DP problem at a marked initial state and a time horizon \( T = \Omega(\log(|S|)) \) via queries to the oracle \((4)\) has to make at least \( \Omega(|S||A|) \) queries to that oracle.

VII. Acknowledgement

The author thanks Ronald de Wolf, Artur Scherer, Seyed Saeed Changiz Rezaei, Yichen Chen, Ryuhei Mori, Yoichi Iwata, Jevgēnijs Vihrovs, Krišjānis Prūsis, Jānis Iraids, Martins Kokainis, and Scott Aaronson for useful technical discussions. The author further thanks Marko Bucyk for his careful review and editing of this manuscript. The author acknowledges the support of 1QBit, the Government of Ontario, and Innovation, Science and Economic Development Canada.

[1] L. K. Grover, A fast quantum mechanical algorithm for database search, in *Proceedings of the twenty-eighth annual ACM symposium on Theory of computing* (1996) pp. 212–219.
[2] R. Bellman, Dynamic programming treatment of the travelling salesman problem, Journal of the ACM (JACM) 9, 61 (1962).
[3] M. Held and R. M. Karp, A dynamic programming approach to sequencing problems, Journal of the Society for Industrial and Applied Mathematics 10, 196 (1962).
[4] A. Ambainis, K. Balodis, J. Iraids, M. Kokainis, K. Prūsis, and J. Vihrovs, Quantum speedups for exponential-time dynamic programming algorithms, in *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms* (SIAM, 2019) pp. 1783–1793.
[5] Y. Chen and M. Wang, Lower bound on the computational complexity of discounted markov decision problems, arXiv:1705.07312 (2017).
[6] A. Ambainis, Quantum lower bounds by quantum arguments, Journal of Computer and System Sciences 64, 750 (2002).
[7] R. Bellman, *Dynamic programming* (Princeton University Press, 1957).
[8] P. Ronagh, Quantum algorithms for solving dynamic programming problems, arXiv:1906.02229v2 (2019).
[9] F. G. Brandao and K. M. Svore, Quantum speed-ups for solving semidefinite programs, in *Foundations of Computer Science (FOCS)*, 2017 IEEE 58th Annual Symposium on (IEEE, 2017) pp. 415–426.
[10] J. van Apeldoorn, A. Gilyén, S. Gribling, and R. de Wolf, Quantum sdp-solvers: Better upper and lower bounds, in *Foundations of Computer Science (FOCS)*, 2017 IEEE 58th Annual Symposium on (IEEE, 2017) pp. 403–414.
[11] D. Sutter, G. Nannicini, T. Sutter, and S. Woerner, Quantum legendre-fenchel transform, arXiv:2006.04823 (2020).
[12] D. Sutter, G. Nannicini, T. Sutter, and S. Woerner, Quantum speedups for convex dynamic programming, arXiv:2011.11654 (2020).
[13] F. V. Fomin and P. Kaski, Exact exponential algorithms, Communications of the ACM 56, 80 (2013).
[14] R. A. Wagner and M. J. Fischer, The string-to-string correction problem, Journal of the ACM (JACM) 21, 168 (1974).
[15] A. Ambainis, K. Balodis, J. Iraids, K. Khadiev, V. Klevickis, K. Prūsis, Y. Shen, J. Smotrovs, and J. Vihrovs, Quantum lower and upper bounds for 2d-grid and dyck language, arXiv:2007.03402 (2020).
[16] S. Aaronson, Lower bounds for local search by quantum arguments, SIAM Journal on Computing 35, 804 (2006).