An $sl(2, \mathbb{R})$ current algebra from $AdS_3$ gravity

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Abstract: We provide a set of chiral boundary conditions for three-dimensional gravity that allow for asymptotic symmetries identical to those of two-dimensional induced gravity in light-cone gauge considered by Polyakov. These are the most general boundary conditions consistent with the boundary terms introduced by Compère, Song and Strominger recently. We show that the asymptotic symmetry algebra of our boundary conditions is an $sl(2, \mathbb{R})$ current algebra with level given by $c/6$. The fully non-linear solution in Fefferman–Graham coordinates is also provided along with its charges.
1 Introduction

In his seminal 1987 paper [1], Polyakov provides a solution to the two-dimensional induced gravity theory [2],

\[ S = \frac{c}{96\pi} \int d^2x \sqrt{-g} R \frac{1}{\nabla^2} R, \quad (1.1) \]

by working in a light-cone gauge. The gauge choice puts the metric into the form

\[ ds^2 = -dx^+ dx^- + F(x^+, x^-)(dx^+)^2. \quad (1.2) \]

Polyakov shows that the quantum theory for the dynamical field \( F(x^+, x^-) \) admits an \( sl(2,\mathbb{R}) \) current algebra symmetry with level \( k = c/6 \). In this note, we present the three-dimensional bulk theory that is dual to this two-dimensional theory.

2 Chiral boundary conditions in \( AdS_3 \) gravity

The action of three-dimensional gravity with negative cosmological constant [3] is given by

\[ S = -\frac{1}{16\pi G} \int d^3x \sqrt{-g} \left( R - \frac{2}{l^2} \right) - \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^2x \sqrt{-\gamma} \Theta + \frac{1}{8\pi G} S_{\text{ct}}(\gamma_{\mu\nu}), \quad (2.1) \]

where \( \gamma_{\mu\nu} \) is the induced metric and \( \Theta \) is trace of the extrinsic curvature of the boundary. Varying the action yields

\[ \delta S = \int_{\partial\mathcal{M}} d^2x \sqrt{-\gamma} \frac{1}{2} T^{\mu\nu} \delta\gamma_{\mu\nu}, \quad (2.2) \]

where

\[ T^{\mu\nu} = \frac{1}{8\pi G} \left[ \Theta^{\mu\nu} - \Theta \gamma^{\mu\nu} + \frac{2}{\sqrt{-\gamma}} \delta S_{\text{ct}} \delta\gamma_{\mu\nu} \right]. \quad (2.3) \]

The variational principle is made well-defined by imposing \( \delta\gamma_{\mu\nu} = 0 \) (Dirichlet) or \( T^{\mu\nu} = 0 \) (Neumann) at the boundary (see [4] for a recent discussion).

Recently Côme, Song and Strominger (CSS) [5, 6] and Troessaert [7] proposed new sets of boundary conditions for three-dimensional gravity, which differ from the well-known Dirichlet-type Brown–Henneaux boundary conditions [8].\(^1\) Before delving into specifics, let us discuss the general strategy employed by [6]. One begins by adding a term of the type

\[ S' = -\frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^2x \sqrt{-\gamma} \frac{1}{2} T^{\mu\nu} \gamma_{\mu\nu} \quad (2.4) \]

\(^1\)In fact, the boundary conditions of [7] subsume those of [8].
for a fixed \((\gamma_{\mu\nu}-\text{independent})\) symmetric boundary tensor \(T_{\mu\nu}\). The variation of this term is

\[
\delta S' = -\frac{1}{8\pi G} \int_{\partial M} d^2x \sqrt{-\tilde{\gamma}} \tilde{T}^{\mu\nu} \delta \gamma_{\mu\nu},
\]

where \(\tilde{T}^{\mu\nu} = T^{\mu\nu} - \frac{1}{2}(T^{\alpha\beta}\gamma_{\alpha\beta}) \gamma^{\mu\nu}\). The variation of the total action then gives

\[
\delta S + \delta S' = \frac{1}{8\pi G} \int_{\partial M} d^2x \sqrt{-\gamma} (T^{\mu\nu} - \tilde{T}^{\mu\nu}) \delta \gamma_{\mu\nu}.
\]

Now the boundary conditions consistent with the variational principle depend on \(\tilde{T}^{\mu\nu}\). Generically, this leads to “mixed” type boundary conditions. If for a given class of boundary conditions some particular component of \(T^{\alpha\beta} - \tilde{T}^{\alpha\beta}\) vanishes sufficiently fast in the boundary limit such that its contribution to the integrand in (2.6) vanishes, then the corresponding component of \(\gamma_{\alpha\beta}\) can be allowed to fluctuate. Since we want the boundary metric to match (1.2), we would like Neumann boundary conditions for \(\gamma_{++}\). Therefore we choose \(T_{\mu\nu}\) such that the leading term of \(T^{++}\) equals \(\tilde{T}^{++}\) in the boundary limit.

This condition has been imposed in [6], with the addition of an extra boundary term (2.4) with\(^2\)

\[
T^{\mu\nu} = -\frac{1}{2r^4} N^2 \delta^\mu_+ \delta^\nu_+,
\]

and the following boundary conditions are imposed on the metric:

\[
g_{rr} = \frac{l^2}{r^2} + \mathcal{O}(r^{-4}), \quad g_{rx} = \mathcal{O}(r^{-3}),
\]

\[
g_{+-} = -\frac{r^2}{2} + \mathcal{O}(r^0), \quad g_{++} = r^2 f(x^+) + \mathcal{O}(r^0), \quad g_{--} = -\frac{l^2}{4} N^2 + \mathcal{O}(r^{-1}),
\]

where \(f(x^+)\) is a dynamical field and \(N^2\) is fixed constant.\(^3\) These boundary conditions give rise to an asymptotic symmetry algebra: a chiral \(U(1)\) current algebra with level determined by \(N\). These also ensure that \(T_{--}\) is held fixed in the variational problem, whereas \(g_{++}\) is allowed to fluctuate as long as its boundary value is independent of \(x^-\).

In what follows, we show that (2.8) are not the most general boundary conditions consistent with the variational principle and the extra boundary term given by (2.7). For this, we introduce a weaker set of consistent boundary conditions that enhance the asymptotic symmetry algebra to an \(sl(2,\mathbb{R})\) current algebra whose level is independent of \(N\).

### 2.1 New boundary conditions

In the new boundary conditions, the class of allowed boundary metrics coincides with that of (1.2). Since we want to allow \(\gamma_{++}\) to fluctuate, we keep \(T_{--}\) fixed in our asymptotically

\(^2\)The induced metric \(\gamma_{\mu\nu}\) differs from \(g^{(0)}_{\mu\nu}\) of [6] by a factor of \(r^2\).

\(^3\)To relate to the notation in [6], set \(N^2 = -\frac{16G\Delta}{l^2}\) and \(f(x^+) = l^2 \partial_+ P(x^+)\).
of the boundary to be \( R \). We impose periodic boundary conditions on \( g \). This involves constructing the non-linear solution in an expansion in inverse powers of \( x \). One must, of course, check the consistency of these conditions with the equations of motion. This involves constructing the non-linear solution in an expansion in inverse powers of \( r \). Working to the first non-trivial order, one finds the following condition on \( F(x^+, x^-) \):

\[
N^2 \partial_- F(x^+, x^-) + \partial^3_x F(x^+, x^-) = 0,
\]

which forces \( F(x^+, x^-) \) to take the form

\[
F(x^+, x^-) = f(x^+) + g(x^+) e^{iN x^-} + \bar{g}(x^+) e^{-iN x^-}
\]

where \( f(x^+) \) is a real function and \( \bar{g}(x^+) \) is the complex conjugate of \( g(x^+) \).

Let us note that this is directly analogous to the form of \( F(x^+, x^-) \) derived in [1]. Throughout our discussion we think of \( \phi = \frac{x^+ - x^-}{2} \) as \( 2\pi \)-periodic (and \( \tau = \frac{x^+ + x^-}{2} \) as the time coordinate), and therefore we restrict our consideration to \( N \in \mathbb{Z} \). Similarly, we impose periodic boundary conditions on \( f(x^+) \) and \( g(x^+) \). If one takes the spatial part of the boundary to be \( \mathbb{R} \) instead of \( S^1 \), there are no such restrictions and one may even consider \( N^2 < 0 \) like in [6].

### 2.2 The non-linear solution

One can write a general non-linear solution of \( AdS_3 \) gravity in Fefferman–Graham coordinates [9] as:

\[
ds^2 = \frac{dr^2}{r^2} + r^2 \left[ g_{ab}^{(0)} + \frac{r^2}{\ell^2} g_{ab}^{(2)} + \frac{r^4}{\ell^4} g_{ab}^{(4)} \right] dx^a dx^b.
\]

The full non-linear solution with our boundary conditions is obtained when

\[
\begin{align*}
g_{++}^{(0)} &= f(x^+) + g(x^+) e^{iN x^-} + \bar{g}(x^+) e^{-iN x^-}, \quad g_{+-}^{(0)} = -\frac{1}{2}, \quad g_{-+}^{(0)} = 0, \\
g_{++}^{(2)} &= \kappa(x^+) + \frac{1}{2} N^2 \left[ g^2(x^+) e^{2iN x^-} + \bar{g}^2(x^+) e^{-2iN x^-} \right] \\
&\quad + \frac{i}{2} N \left[ g'(x^+) e^{iN x^-} - \bar{g}'(x^+) e^{-iN x^-} \right], \\
g_{+-}^{(2)} &= \frac{1}{4} N^2 \left[ f(x^+) - g(x^+) e^{iN x^-} - \bar{g}(x^+) e^{-iN x^-} \right], \quad g_{-+}^{(2)} = -\frac{1}{4} N^2, \\
g_{ab}^{(2)} &= \frac{1}{4} g_{ab}^{(2)} g_{cd}^{(2)},
\end{align*}
\]

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where in the last line $g^{c(0)}_{cd}$ is $g^{(0)}_{cd}$ inverse. As above, demanding that the solution respects the periodicity of $\phi$-direction requires $N$ to be an integer and the functions $f(x^+), g(x^+)$ and $\kappa(x^+)$ to be periodic. This solution reduces to the one given in [6] when $g(x^+) = \bar{g}(x^+) = 0$.

As mentioned in the previous subsection one can take $N$ to be purely imaginary when the boundary spatial coordinate is not periodic. In this case too the non-linear solution (2.12) continues to be a valid solution with $g(x^+)$ and $\bar{g}(x^+)$ treated as two real and independent functions. However, we will not consider this case further here.

### 3 Charges, algebra and central charges

It is easy to see that vectors of the form

$$\xi^\prime = -\frac{1}{2} [B'(x^+) + iN A(x^+) e^{iN x^-} - iN \bar{A}(x^+) e^{iN x^-}] r + \mathcal{O}(r^0)$$

satisfy the criteria of [10], which allow us to construct corresponding asymptotic charges. If, on the other hand, one demands that the asymptotic symmetry generators $\xi$ leave the space of boundary conditions invariant, one finds the same vectors but with the first subleading terms appearing at one higher order for each component. For either set of vectors, the Lie bracket algebra closes to the same order as one has defined the vectors.

Here, $B(x^+)$ and $A_0(x^+)$ are real and $A(x^+)$ is complex; therefore, there are four real, periodic functions of $x^+$ that specify this asymptotic vector. We take the following basis for the modes of the vector fields:

$$L_n = ie^{i n x^+} [\partial_+ - \frac{i}{2} n r \partial_r] + \cdots$$

$$T_n^{(0)} = \frac{i}{N} e^{i n x^+} \partial_- + \cdots$$

$$T_n^{(+)} = \frac{i}{N} e^{i (n x^+ + N x^-)} [\partial_- - \frac{i}{2} N r \partial_r - \frac{N^2}{2 r^2} \partial_+ ] + \cdots$$

$$T_n^{(-)} = \frac{i}{N} e^{i (n x^+ - N x^-)} [\partial_- + \frac{i}{2} N r \partial_r - \frac{N^2}{2 r^2} \partial_+ ] + \cdots,$$

which satisfy the Lie bracket algebra

$$[L_m, L_n] = (m - n) L_{m+n}, \quad [L_m, T_n^{(a)}] = -n T_{m+n}^{(a)};$$

$$[T_m^{(0)}, T_n^{(\pm)}] = \mp T_{m+n}^{(\pm)}, \quad [T_m^{(+)}, T_n^{(-)}] = 2 T_m^{(0)};$$

Thus, the classical asymptotic symmetry algebra is a Witt algebra and an $sl(2, \mathbb{R})$ current algebra.

We use the Brandt–Barnich–Compère (BBC) formulation [10, 11] for computing the corresponding charges of our geometry. We find that the charges are integrable over the solution space if $\delta N = 0$ with
\[ \$Q_\xi = \frac{1}{8\pi G} \delta \int d\phi \left\{ B(x^+) \left[ \kappa(x^+) + N^2(\frac{1}{2}f^2(x^+) - g(x^+)\bar{g}(x^+)) \right. \right. \\
+ \left. \left. \frac{N^2}{2}(g^{iN\gamma} - g(x^+) + e^{-iN\gamma}\bar{g}(x^+)) \right] \\
+ \frac{i}{2}N \partial_+ [B(x^+)(e^{iN\gamma} - g(x^+) - e^{-iN\gamma}\bar{g}(x^+)]) \right\} \\
- \frac{1}{8\pi G} \delta \int d\phi N^2 \left[ \frac{1}{2}A_0(x^+)f(x^+) - (g(x^+)A(x^+) + \bar{g}(x^+)\bar{A}(x^+)) \right]. \] 

These can be integrated between the configurations trivially in the solution space from \( f(x^+) = g(x^+) = \kappa(x^+) = 0 \) to general values of these fields to write down the charges

\[ Q_B = \frac{1}{8\pi G} \int_0^{2\pi} d\phi \left[ B(x^+)\left( \kappa(x^+) + \frac{N^2}{2}(f^2(x^+) - 2g(x^+)\bar{g}(x^+)) \right) \right. \\
+ \frac{1}{2} \left( \partial_+ - \partial_- \right) \partial_- [e^{iN\gamma}g(x^+) + e^{-iN\gamma}\bar{g}(x^+)] \right] \\
= \frac{1}{8\pi G} \int_0^{2\pi} d\phi \left[ B(x^+)\kappa(x^+) + \frac{N^2}{2}(f^2(x^+) - 2g(x^+)\bar{g}(x^+)) \right. \\
\left. + \frac{1}{32\pi G} \partial_- \left[ e^{iN\gamma}g(x^+) + e^{-iN\gamma}\bar{g}(x^+) \right] \right] \bigg|_{\phi=0}^{\phi=2\pi}, \] 

\[ Q_A = -\frac{N^2}{8\pi G} \int_0^{2\pi} d\phi \left[ \frac{1}{2}A_0(x^+)f(x^+) - (g(x^+)A(x^+) + \bar{g}(x^+)\bar{A}(x^+)) \right]. \] 

The boundary term in (3.5) vanishes as we assumed \( g(x^+) \) to be periodic and \( N \) to be an integer. The algebra of these charges admits central charges. We find that the central term in the commutation relation between charges corresponding to two asymptotic symmetry vectors \( \xi \) and \( \bar{\xi} \) is given by

\[ (-i)\frac{l}{32\pi G} \int_0^{2\pi} d\phi \left[ B'(x^+)\bar{B}''(x^+) - B(x^+)\bar{B}''''(x^+) \right. \\
+ 2N^2A_0(x^+)\bar{A}_0(x^+) - 4N^2 \left( A(x^+)^2 + \bar{A}(x^+)^2 \right) \left. + A(x^+)\bar{A}(x^+) + \bar{A}(x^+)A(x^+) \right]. \] 

These give rise to the following algebra for the charges\(^4\)

\[ [L_m, L_n] = (m - n) L_{m+n} + \frac{c}{12}m^3 \delta_{m+n,0}, \]
\[ [L_m, T^a_n] = -n T^a_{m+n}, \]
\[ [T^a_m, T^b_n] = f^{ab}_{\quad c} T^c_{m+n} + \frac{k}{2}\eta^{ab}m \delta_{m+n,0}. \]

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\(^4\)The bracket in (3.8) is \( i \) times the Dirac bracket.
with
\[ c = \frac{3l}{2G}, \quad k = \frac{c}{6}, \quad f^0_+ = -1, \quad f^0_- = 1, \quad f^{+-} = 2, \quad \eta^{00} = -1, \quad \eta^{+-} = 2. \]
\[ (3.9) \]

This is precisely the $sl(2, \mathbb{R})$ current algebra found in [1].

4 Conclusion

In this note we have provided boundary conditions for 3-dimensional gravity with negative cosmological constant such that the algebra of asymptotic symmetries is an $sl(2, \mathbb{R})$ current algebra. In the process we showed that the boundary term proposed by CSS [6] admits a more general set of boundary conditions, which enables our result.

It should be noted that our asymptotic symmetry algebra does contain the full isometry algebra of the global $AdS_3$ solution. This feature is similar to Brown–Henneaux [8] though one does not demand that the asymptotic vector fields of interest be asymptotically Killing; instead one uses the more general notion of asymptotic symmetries advocated by BBC [10, 11]. Using the BBC formulation, we computed the algebra of charges and found the level $k$ to be $c/6$, independent of the parameter $N$.

To understand the relation to 2-dimensional induced gravity of Polyakov in light-cone gauge [1] further, it will be interesting to see if the correlation functions of the boundary currents, and the effective action for the dynamical fields of the boundary can also be recovered from the gravity side. See [12] for a discussion on the latter issue. Of course, connections between 3-dimensional gravity with negative cosmological constant and Liouville theory, which arises as a different gauge-fixing of (1.1), are well-known (see e.g. [13], and the recently proposed boundary conditions in [7]).

It will be interesting to see how adding matter to $AdS_3$ gravity would generalize our analysis. The boundary conditions of [6] have been found to be related to string theory solutions of [14] with a warped $AdS_3$ factor. It will be interesting to explore whether the boundary conditions in (2.9) also play a role in some string theory context.

The non-linear solution in (2.12, 2.13) does not contain the conventional positive mass BTZ [15] black hole. The special case of vanishing charges is given by $f(x^+) = g(x^+) = \tilde{g}(x^+), \kappa(x^+) = 0$ which is simply an extremal BTZ but with negative mass (in global $AdS_3$ vacuum). The comments of CSS [6] about the possible existence of ergoregions and instabilities in their solution also apply to (2.12). It will be important to understand these issues better.

Finally, it is intriguing that different ways of gauge-fixing the induced gravity [1] lead to different boundary conditions in the bulk and therefore apparently different holographic duals. It will be important to understand the class of physical theories one can obtain this way and how they are related to each other.
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