A Monte-Carlo study of meanders

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I. INTRODUCTION

The concept of folding has an important place in polymer physics. Considering a polymer chain made of $n$ identical constituents (the monomers), the entropy of such a system can be obtained by counting the number of inequivalent ways of folding the chain onto itself. If the model of polymer does not take self-avoidance into account, it is then equivalent to the well-known Brownian motion. Several more involved models have been proposed which study the compact folding of a self-avoiding polymer as a Hamiltonian cycle (i.e. a closed, self-avoiding walk which visits each vertex) on a regular lattice. They can also be defined for some kinds of random lattices, where each configuration is now described by a system of non-intersecting arches which connect the pairs of monomers, which are neighbors in the real space.

In the present paper, the compact folding of a polymer chain is modeled by a folded strip of stamps, with a complete piling of the strip on top of one stamp. It is then equivalent to another model of non-intersecting arches, the so-called meander problem, which can be summarized by this simple question: in how many ways $M_n$ can a road cross a river through $n$ bridges, and possibly wind around the source.

A related problem can be defined by forbidding the winding around the source (i.e. the river is infinite at the both ends). It is now equivalent to enumerate the “simple alternating transit mazes” of depth $n$; it was also investigated in connection with Hilbert 16’th problem, namely the enumeration of ovals of planar algebraic curves.

By analogy with some models of statistical mechanics like random walks or self-avoiding walks, the meanders can be described in the language of critical phenomena. In particular, the asymptotic behavior of meanders when $n$ is large can be characterized by “critical” exponents. However the exact enumeration of meanders is particularly complicated: there is no known formula for $M_n$ in terms of $n$. By generating all possible configurations, by hand or with a computer, the beginning of the sequence $M_n$ can be computed exactly. As $M_n$ increases exponentially with $n$, the limits of computers are reached for $n \sim 30$ and the estimates of the exponents are too inaccurate to validate (or invalidate) some conjectures.

One should mention that several exact results, for arbitrarily large $n$, which are unfortunately not helpful to determine the values of the exponents, have been obtained with other techniques: random matrix model methods and an algebraic approach using the Temperley-Lieb algebra or the Hecke algebra.

Many models in statistical physics can be studied by Monte-Carlo (or stochastic) methods. With these algorithms, only a small set of configurations among all the possible ones are generated. In principle, the expectation of physical quantities (like energy or magnetization) with a given law of probability (like the Boltzmann law involving an external temperature) can be estimated from these randomly generated samples, if their probabilities of generation are known. For example, with the Metropolis algorithm for classical spin systems, the probability of generation is built to be equal to the Boltzmann law and the average is done over the generated configurations with equal weights. To bypass some difficulties (for example when the phase space has many local minima with high free energy barriers between them), it is possible, in principal, to generate the random configurations with another more adapted law and to correct this bias when the average is done.

But for the meanders problem, the situation is quite different: the phase space is not easy to built because the number $M_n$ of configurations is unknown and the only known efficient method to draw a meander of size $n$ is a recurrence over $n$. Moreover the naive way to use this recurrence gives a distribution of meanders which is not flat, and this default increases exponentially with $n$. This paper presents a Monte-Carlo method which explores the meanders with an almost flat distribution law. Furthermore the bias is known and can be corrected exactly. Therefore, the average can be done over meanders.
with equal probabilities. In particular, better estimates of critical exponents are obtained.

After Section 2 devoted to the definitions, and Section 3 which explains the building of meanders by recurrence, Section 4 of this paper describes the Monte-Carlo method. The results are presented and discussed in Section 5.

II. DEFINITIONS

A meander of size \( n \) is a planar configuration of a non-self-intersecting loop (road) crossing a half line (semi-infinite river with a source) through \( n \) points (bridges).

Two meanders are considered as equivalent if their roads can be continuously deformed into each other, keeping the bridges fixed: this is therefore a topological equivalence. We call an arch each section of road between two consecutive bridges. So a meander of size \( n \) has \( n \) bridges and \( n \) arches.

The number of different meanders of size \( n \) is denoted by \( M_n \). For example, \( M_1 = 1 \), \( M_2 = 1 \), \( M_3 = 2 \), \( M_4 = 4 \).

In Fig. 1, the 4 meanders of size 4 are drawn. The \( M_n \)'s, up to \( n = 29 \), can be found in Ref. [4].

In previous articles [3][4][19], these objects were called semi-meanders, to distinguish them from the case where the line is infinite (river without source). In this paper, the river is always a half-line and the word meanders is used for convenience.

We can define meanders with \( k \) connected components, i.e. made of one river and \( k \) non-intersecting roads. But, in this work, we do not include this generalization and we keep \( k = 1 \). However the Monte-Carlo method, used in this article, can be adapted without difficulties to an arbitrary fixed \( k \), and even for varying \( k \) with a fugacity \( q^k \).

As explained with many details in Ref. [4], the meander problem is absolutely equivalent to the problem of the compact folding of a strip of stamps because each meander of size \( n \) can be continuously deformed in such a way that the “road” becomes a vertical line and the “river” becomes a folded strip of \( n - 1 \) stamps. We prefer to present our results with the meander representation because the main recursion relation, described later, seems more “natural” in this picture.

The meander problem has certain similarities with two-dimensional self-avoiding walks: a meander is obtained by intersecting a closed self-avoiding walk by a half-line and keeping only the topological aspect. By analogy, it is expected [10] that

\[
M_n \sim c \frac{R^n}{n^\gamma},
\]

where the estimates given in Ref. [14] are \( R = 3.50(1) \) and \( \gamma \approx 2 \).

The connectivity \( R \) can be reinterpreted as the average number of ways of adding a bridge close to the source by deforming an arch of a given large meander. Then \( \ln(R) \) is the entropy per bridge. The configuration exponent \( \gamma \) is sensitive to the boundary conditions, for example whether the road is closed or open, whether the river is infinite or semi-infinite, straight or forked, whether the meander is drawn on a planar surface, a sphere or a surface with higher genus. Conversely, we expect that \( R \) remains the same for all these boundary conditions.

It is similar in on-lattice self-avoiding walks problem where the connectivity depends on the type of lattice (square, honeycomb . . . ) and not on the boundary conditions, whereas the “universal” configuration exponent depends on the boundary conditions, but is not sensitive to the small scale details of the lattice. For these reasons, we think that the numerical value of \( R \) is valid only for this particular model of meanders. But \( \gamma \) is expected to be more “universal” and to keep its value in other variants of the meander problem. Unfortunately, the numerical determination of \( \gamma \) is less precise than \( R \), because \( n^\gamma \) describes the correction to the leading exponential asymptotic behavior \( R^n \).

For a given meander \( m \), the winding number \( w(m) \) of the road around the source of the river can be defined as the minimal number of intersections between the road and a half (semi-infinite) line starting at the source and extending the river on the opposite side. For an example, see Fig. 4. We define \( w_n \) as the average of the winding number

\[
w_n = \frac{1}{M_n} \sum_{m=1}^{M_n} w(m)
\] (2)

over all the meanders \( m \) of size \( n \). We can see the winding number as the topological end-to-end distance between the source (right end of the river) and the infinite (left end of the river). Here the distance between two points is simply the minimal number of arches which must be crossed to go from one point to the other. By analogy with the end-to-end exponent of self-avoiding walk, we expect that
FIG. 2. The height $h(i)$ (resp. $h(-i)$) is the number of arches over (resp. below) the segment $i$. For this meander of size 5, \( h(i) = \{0, 1, 2, 3, 2, 1, 0, 1, 2, 1, 0\} \) for $i = -5, \ldots, 5$. The area is $A = 13$.

\[
w_n \sim n^\nu, \tag{3}\]

where the estimate given in Ref. [14] is $\nu = 0.52(1)$.

If we study the meanders by leaving free the number $k$ of connected components, the problem is equivalent [14] to a random walk on a semi-infinite line and can be studied with usual methods of combinatorics. In particular, $\gamma = 3/2, R = 4$ and $\nu = 1/2$ is the Brownian exponent. But, by fixing $k = 1$, the problem is drastically more difficult and the above values are, to our knowledge, not yet known exactly.

For a given meander $m$ of size $n$, we label by $i = 0, \ldots, n$ each segment of river in-between two consecutive bridges, from right to left. So the rightmost segment (with source) is labeled 0, and the leftmost (semi-infinite) segment is labeled $n$. We define the height $h(i, m)$ as the number of arches over the segment $i$, and $h(-i, m)$ as the number of arches below the segment $i$. An example is given in Fig. 3.

For the case $i = 0$, the both definitions $h(+0, m)$ and $h(-0, m)$ are equivalent and equal to the winding number: $h(0, m) = w(m)$. From the definition, we have $h(n, m) = h(-n, m) = 0$, $h(i, m) \geq 0$ and $h(i, m) = h(i + 1, m) \pm 1$.

For a given meander $m$ of size $n$, we define the area $A(m)$ as

\[
A(m) = \sum_{i=-n}^{n} h(i, m). \tag{4}\]

For meanders of size $n$, it can be proved that the maximal area is $(n-1)^2 + 1$ for the two meanders with a snail shape (where $\{h(i)\} = 0, 1, 2, \ldots, n-2, n-1, n-2, \ldots, 2, 1, 0, 1, 0\} \pm$ the symmetric meander), and the minimal area is $2n-2$ (resp. $2n-1$) when $n$ is even (resp. odd) for the $2n^{1/2}-1$ (resp. $2(n^{-1/2})$) snake shaped meanders characterized by $h(i) + h(-i) = 2$ for $0 < i < n$. As in the case of the winding number, we will consider the average profile height

\[
h_n(i) = \frac{1}{M_n} \sum_{m=1}^{M_n} h(i, m) \tag{5}\]

and the average area

\[
A_n = \frac{1}{M_n} \sum_{m=1}^{M_n} A(m) \tag{6}\]

over all the meanders $m$ of size $n$.

III. RECURSION RELATION FOR MEANDERS

In this section, we describe a recursive algorithm to enumerate and build all meanders of a given size $n$. Though it was described in Ref. [13,14], we prefer to recall it in details, because the Monte-Carlo method is based on this recursion.

We have different ways to build a meander of size $n+1$, starting from a meander of size $n$. Our method consists in adding a bridge on the left most part of the river (opposite to the source) and changing the road to cross this new bridge. To keep this change minimal, only an exterior arch is modified (an arch is exterior when no other arch surrounds it).

Take a meander of size $n$ (the parent) and choose (or label) one of its exterior arch. By the process described in Fig. 3, a meander of size $n+1$ (the child) is built. The parent has as many different children as exterior arches. By inverting this process, it appears that each meander of size $n+1$ has one and only one parent. More precisely, it is a one-to-one mapping between the meanders of size $n+1$, and the meanders of size $n$ with one labeled exterior arch.

The starting point of the recursion is the unique meander of size 1. By $n-1$ successive applications of the
IV. THE MONTE-CARLO METHOD

As explained in the previous section, the exponential growth of the computations with an exact enumeration method limits the size of meanders around \( n \sim 30 \). To reach bigger \( n \), it is then natural to try to study this problem with a Monte-Carlo (or stochastic) method.

As the set of the \( M_n \) meanders of a given size \( n \) is too large to be fully explored, the general idea is to randomly select a small subset. Then, the measurements are done and averaged on the selected meanders. It gives an estimator of the exact (but unknown) result, with an unknown error. This error has two components: statistical fluctuations and bias.

The statistical fluctuations can be reduced by independently repeating the procedure many times. Then, we obtain a histogram of the estimator, with an average and a variance. Under the hypothesis of finite variance, the statistical fluctuations of the average can be estimated by usual formulas of statistics.

The bias is the difference between the exact result and the mathematical expectation of the estimator. If it can be exactly calculated, we subtract it from the estimator. But, in general, an unknown part remains, which can not be reduced by a better statistics. As explained below, by adjusting parameters of the simulation, the bias can be reduced to become smaller than the statistical fluctuations.

In this section, we will first introduce the simplest algorithm, the one-squirrel method. We will see that its statistical fluctuations grow exponentially with \( n \) and they are too big for \( n \approx 30 \). Then, we present an algorithm, the multi-squirrel method, for which the fluctuations increase less rapidly.

A. The one-squirrel method

The method is based on the recursion relation (see Fig. 3), with which the set of meanders is organized as a tree (see Fig. 4). The Monte-Carlo squirrel has the following stochastic behavior. It starts at the root of the tree (the meander of size \( n = 1 \)). It climbs into the tree. At each level \( n \), it stands on a node and makes some measurements concerning the meander of size \( n \), represented by this node. Then the squirrel goes to the level \( n + 1 \) by choosing at random one of the \( b_n \) branches, starting from this node. The squirrel stops at a prefixed level \( n = n_{\text{max}} \). This process constitutes one simulation.

The probability that the squirrel reaches a given meander of size \( n \) is \( 1/\prod_{l<n} b_l \). This probability law is not flat because the sequence of \( b_l \) depends on the visited nodes. As seen in Fig. 4, for \( n = 5 \), the two meanders on the left side have a probability 1/8 and the three on the right side have 1/12. So, to correct this bias between meanders, the squirrel has a weight...
\[ q_n = \prod_{i=1}^{n-1} b_i, \]  
which is calculated during its climbing.

By noting \( \langle \cdot \rangle \) the mathematical expectation (over all possible simulations), it is then obvious that \( \langle q_n \rangle = M_n \), because the sum runs over all paths and the contribution of a given path is \( q_n \), with probability \( \frac{1}{q_n} \).

More generally, for some quantity \( Z \) (for example the winding number), we wish to determine

\[ Z = \sum_{m=1}^{M_n} Z(m), \tag{8} \]

where \( Z(m) \) is the value of \( Z \) on the \( m \)-th meander of size \( n \). Each simulation gives, at level \( n \), a measurement \( Z(s) \) on the meander \( s \) reached by the squirrel and \( \langle q_n \cdot Z(s) \rangle = Z \) which is the generalization of \( \langle q_n \rangle = M_n \) obtained with \( Z = 1 \). Then

\[ z = q_n \cdot Z(s) \tag{9} \]

is an unbiased estimator of \( Z \) (i.e. \( \langle z \rangle = Z \)).

As usual in Monte-Carlo methods, several simulations are made independently and we hope that the average \( \bar{z} \) of all the measurements \( z \) is close to \( Z \). Unfortunately, this method does not work in practice, because the weight \( q_n \) is the product of \( b_i \). Although the distribution of each \( b_i \) is regular, the product of many random variables is not self-averaging.

As the sum of \( \ln(b_i) \) is self-averaging (i.e. the observed result is closed to its mathematical expectation when \( n \) goes to infinite), most of the observed \( q_n \) are not closed to \( \langle q_n \rangle \) and

\[ \frac{\langle q_n \rangle}{q_n(\text{observed})} \sim \exp \sum_{i<n} (\ln(b_i) - \langle \ln b_i \rangle) \tag{10} \]

increases like an exponential. Then the averages with weight \( q_n \) are dominated by exponentially rare events and the statistical fluctuations become large. To keep the observed average close to the mathematical expectation, the number of simulations must increase exponentially with \( n \) and fluctuations become too big for \( n \approx 30 \) or 35. As the difficulties increase exponentially with \( n \) (as for exact enumeration), is is useless to increase the power of the computer. We need a new algorithm.

### B. Multi-squirrels method

We generalize the one-squirrel method, but now with a population of \( S \) squirrels, which reproduce and die. \( S \) is a fixed parameter during all the simulations. It is more simple to choose \( S = M_{n_0} \), with at the starting point, a squirrel staying at each node of level \( n_0 \) (meanders of size \( n_0 \)). In this work, \( n_0 = 17 \) and we use the up-down symmetry to reduce the population to \( S = M_{17}/2 = 1664094 \) squirrels.

The population evolves from level \( n \) to level \( n+1 \) by the following process. Each squirrel \( i \) lives on a node \( s_i \) on level \( n_i \), connected to \( b_i \) nodes on level \( n_i+1 \). It reproduces and has \( b_i \) children and each child lives on each one of these \( b_i \) nodes. The total number of children \( S' = \sum_{i=1}^S b_i \) is calculated. The ratio \( B_n = S'/S \) is an estimate of \( M_{n+1}/M_n \): the average of the number of children per parent. To prevent an exponential growth of the population and of the needed computer memory and time, the total population is keep constant by decimating the children: only \( S \) among the \( S' \) children survive. The choice is made at random with uniform distribution. Then the probability of surviving is \( 1/B_n \). This decimation is the single Monte-Carlo step of the algorithm.

This process is iterated up to reach a prefixed level \( n = n_{\text{max}} \): it gives one simulation. Many independent simulations are done and averaged.

The particular case \( S = 1 \) gives the previous method with one squirrel. But, as for \( S = 1 \), for every value of \( S \), the probability that a given meander is reached, is not uniform. The nodes with small number of "brothers" or "cousins" have always a small advantage. But this bias becomes smaller when the population \( S \) is large. That is the main improvement of this method. The limit \( S = \infty \) corresponds to the exact enumeration.

To correct the bias, each simulation has a weight

\[ q_n = \prod_{i=n_0}^{n-1} B_i \tag{11} \]

and the averages runs over all the simulations with their weight. More exactly, for some quantity \( Z \), by keeping the notations of Eq. (8), one simulation with \( S \) squirrels gives \( S \) measurements \( \{Z(s_i)\} \) for \( i \in [1,S] \) with a weight \( q_n \) and the estimator

\[ z = q_n \cdot \sum_{i=1}^S Z(s_i) \tag{12} \]

is unbiased, i.e.

\[ \langle z \rangle = Z. \tag{13} \]

We note that the case \( Z = 1 \) gives \( S \cdot \langle q_n \rangle = M_n \).

In order to prove Eq. (13), we define the operator \( \delta_m \) characterizing a given meander \( m \) by \( \delta_m(m') = 1 \) when \( m = m' \) and 0 otherwise. Then every operator \( Z \) can be split up into \( Z = \sum_m Z(m) \cdot \delta_m \). As the expectation of a sum is always the sum of expectations, we have to prove Eq. (13) for the operators \( \delta_m \) only, which becomes

\[ \langle q_n \cdot \Delta_m \rangle = 1, \]

where \( \Delta_m = 1 \) if the \( m \)-th meander is occupied by a squirrel and 0 otherwise. Let \( p \) represent the parent of \( m \) in the tree at level \( n-1 \). The probability that \( \Delta_m = 1 \) (i.e. \( m \) is occupied) is the product of the probability \( \Delta_p \) that \( p \) was occupied at the level \( n-1 \) and
the probability $1/B_{n−1}$ that its child $m$ survives after the
decimation process $(n − 1 → n)$.

By using Eq. (13) and averaging on the random deci-
mation $(n − 1 → n)$ only,

$$
\langle \Delta m \prod_{l=n_0}^{n-1} B_l \rangle = \langle \Delta p \prod_{l=n_0}^{n-2} B_l \rangle .
$$

It is a recursion relation between a given meander of size $n$
and its parent of size $n−1$. By iterating, we go down to
the ancestor at the starting level $n_0$ for which $\Delta = 1$ and
the empty product of $B_l$ is 1. It proves that $\langle q_n, \Delta_n \rangle = 1$
for every meander $m$ of size $n$, and Eq. (13) is valid for
every operator $Z$.

As usual, many simulations are done and the measure-
ment are averaged with their respective weight. A priori,
it seems that this method has the same defect as the
one-squirrel method because the weight $q_n$ is the product
of many $B_l$, not self-averaging when $n$ becomes large. The
ratio

$$
\frac{\langle q_n \rangle}{q_n (\text{observed})} \sim \exp \sum_{l=n_0}^{n-1} (\ln(B_l) - \langle \ln B_l \rangle)
$$

between the mathematical expectation and the most fre-
cently observed $q_n$ increases like an exponential. But,
the main improvement of the multi-squirrel method is
that the distribution of $B_l$ becomes narrow when the
population $S$ of squirrels is large. As $B_l = S'/S$, the
fluctuations of $B_l$ are of order $O(1/\sqrt{S})$ because the
number $S' = \sum_i b_i$ of children is the sum of $S$ ran-
dom variables. A Taylor expansion of $\ln B_l$ shows that
$\ln(B_l) - \langle \ln B_l \rangle = O(1/S)$

Then, with these simple arguments, we can hope that
the ratio (13) grows like $1 + O(n/S)$ and that problems
appear only when $n$ becomes on the same order than $S$.
In our simulations, we observe that the fluctuations grow
with $n$ faster than this optimistic prediction $O(n/S)$. In
fact, the $B_l$’s are not independent and the exponential
function accentuates all deviations. Then we supervised
carefully the distribution of $q_n$. When $n$ is small, we
see a regular bell-shaped curve. But, when $n$ increases,
the distribution becomes asymmetric, with a long and
irregular tail for the large $q_n$.

For example, for $n = 400$, with $S = 1664094$ squirrels,
the width $\sigma$ of the distribution is only 12%, but we ob-
served rare events with a value of $q_n$ as big as three times
the average. However, in this case, their contribution to
the average and fluctuations is not yet problematic. But,
if we let $n$ increase without control, rare events will dom-
inate and the results will become hazardous.

How to choose $n$ and $S$ ? The naïve point of view is
to take $n$ as bigger as possible. But, to make $N_s$ inde-
pendent simulations with $S$ squirrels of size $n$, the need
for computer memory is of order $O(n \cdot S)$ and the need
for computer time is of order $O(n^2 \cdot S \cdot N_s)$. If $S$ is large
enough, the fluctuations are Gaussian and the error bars
are of order $O(1/\sqrt{S \cdot N_s})$. As $n$ is always limited, we will
extrapolate to study the asymptotic behavior. For that,
it is of no help to have large values of $n$ if the error bars
are too big. So for a fixed computer time, we prefer ac-
cumulate good statistics by limiting $n \leq 400$. Finally for
a fixed product $S \cdot N_s$, we prefer to take $S = 1664094$ as
bigger as permitted by the memory computer to avoid
the problem of rare but large fluctuations.

\section{C. Bias for non-linear observables}

In the previous section, we have seen how to obtain
unbiased Monte-Carlo estimates of the sum $Z$ over all
the meanders of size $n$ of some quantity $Z$ (see Eq. (8)
and its notations). However we are more interested by
the average $Z/M_n$ over all the meanders of size $n$. For
example, the average winding number $w_n$ (see Eq. (8))
is obtained when $Z$ counts the winding. To evaluate $R$
of Eq. (8), we can analyze $M_{n+1}/M_n$; in this case, $Z$
counts the exterior arches. More generally, we want to use non-
linear combinations of $Z$ and $M_n$.

With our Monte-Carlo method, we have seen that one
simulation gives a measurement $z$ which is an unbiased
estimator: $\langle z \rangle = Z$. With $N_s$ independent simulations,
we call $\bar{z}$ the usual average of the $N_s$ measurements $z$;
its fluctuations are $\sqrt{N_s}$ times smaller. The bar over the
symbols distinguishes the average of observed values by
Monte-Carlo method, from the (unknown) mathematical
expectation, marked with $\langle \ldots \rangle$. The same work can be
done with $q_n$ which is a unbiased estimator of $M_n$.

We must be careful with the Monte-Carlo estimate of
$Z/M_n$. For example, the average of the ratio $z/q_n$ gives
bad results. It is better to compute the ratio of the av-
ergages $\bar{z}/\bar{q}_n$. Indeed, with a Taylor expansion of $z$ and
$q_n$ around their mathematical expectations $Z$ and $M_n$,
the bias (defined as the difference of the mathematical
expectation $\langle \bar{z}/\bar{q}_n \rangle$ and the target $Z/M_n$)

$$
\langle \bar{z}/\bar{q}_n \rangle - Z/M_n = O(1/N_s).
$$

It can be neglected if it is smaller than the stochastic
fluctuations. Usually, in Monte-Carlo simulations, this
problem disappears because several millions of indepen-
dent measurements are done. But, in this work, the
situation is quite different. In fact, each simulation is a
complex process involving millions of squirrels, and the
number $N_s$ of simulations is small.

Of course, we can not compute this bias exactly, oth-
ereise we would have already subtract it from measure-
ments. But we can estimate it by the following process.
The set of simulations is divided into $N_s/2^p$ subsets, with
$2^p$ simulations each. In each subset, $\bar{z}/\bar{q}_n$ is computed.
We obtained $N_s/2^p$ independent values, one for each sub-
set; by usual formulæ of statistics, we compute their av-
gerage $E^{(p)}$ and the error bars. This work is done for all
integer $p$ between $p = 0$ (each subset contains only one
simulation) and $p = \log_2 N_s$ (only one set with all the $N_s$ simulations).

Which value of $p$ is the best? Following Eq. (16), the bias of $E^{(p)}$ is expected to decrease like $1/2^p$. For small values of $p$, we observe really a dependency of $E^{(p)}$ on $p$: the bias is visible. But, for $p > 5$, variations become smaller than statistical error bars: the size $2^p$ of subsets is sufficiently large to neglect the bias. But, if $p$ is close to its maximum, the number of subsets becomes very small and the error bars are not properly estimated. As in our work $N_s = 8192 = 2^{13}$, we finally keep $p = 7$: the estimator $E^{(7)}$ is computed with 64 independent subsets of 128 simulations each. This method was used for all quantities presented below. It is valid, not only for ratios like $Z/M_n$, but also for non-linear functions like $\ln(M_{n+1}/M_n)$. It could also be possible to use more complex estimators. For example, the combination $2E^{(p)} - E^{(p-1)}$ cancels the order $1/2^p$ of the bias.

V. RESULTS

In this section, we describe the results obtained by our Monte-Carlo multi-squirrels method. After several tests, we used a population of $S = 1664094 (= M_{17}/2)$ squirrels for meanders with size up to $n = 400$. To do $N_s = 8192$ independent simulations, we have used during 8 days a parallel computer (the Cray T3E of the Cea-Grenoble) with 128 processors (Dec-alpha at 375 MHz) and 13 Gigabytes of total memory, equivalent to 24000 hours of workstation cpu time.

We have verified that the results are stable when $S$ (the population) increases. More exactly, the tests with smaller $S$ have larger error bars, but are compatible with results and error bars obtained with the largest $S$. As explained in the Sect. IV B, we have carefully checked that $S = 1664094$ is sufficiently large to explore sizes of meander up to $n = 400$.

A. Enumeration

We want to measure $R$ and $\gamma$, which describe the asymptotic behavior of the number of meanders $M_n \sim c R^n/n^\gamma$ for large $n$. The entropy $\ln R$ can be estimated by $\ln(M_n/M_{n-1})$. But it appears that the sub-sequences $M_{2n}$ and $M_{2n+1}$ have an alternating sub-leading correction. We have estimated it to be $u(-1)^n/(n \ln n)$ with $u = 0.5(1)$. This alternating effect is dramatically amplified by the ratio $M_n/M_{n-1}$. So it is better to consider

$$L_n = \frac{1}{2} \ln(M_n/M_{n-2}), \quad (17)$$

with a jump from $n - 2$ to $n$. But even with this precaution, the reader can still see on the following figures a small parity effect. To estimate $L_n$, we have used the procedure described in Sect. IV C.

![FIG. 5. $\ln R$ and $\gamma$: plot of the Monte-Carlo estimate of $L_n - \ln 3.5 + 2/n$, for $n$ between 50 and 400, versus $1/n$. The limit when $x$ goes to 0 is $\ln(R/3.5)$, and the (negative) slope is $2 - \gamma$. The error bars are not drawn; their maximum is $10^{-5}$, then they are smaller than the symbols. A parity effect, between the odd and even $n$, is visible. A linear extrapolation gives $R = 3.5019(2)$ and $\gamma = 2.056(10)$.

![FIG. 6. $\ln R$ and $\alpha$: under the hypothesis $\gamma = 2$: the same plot as Fig. 8 but the x-axis is $1/(n \ln n)$. The (negative) slope is $-\alpha$. A linear extrapolation gives $R = 3.5017(2)$ and $\alpha = 0.25(5)$.

FIG. 5. $\ln R$ and $\gamma$: plot of the Monte-Carlo estimate of $L_n - \ln 3.5 + 2/n$, for $n$ between 50 and 400, versus $1/n$. The limit when $x$ goes to 0 is $\ln(R/3.5)$, and the (negative) slope is $2 - \gamma$. The error bars are not drawn; their maximum is $10^{-5}$, then they are smaller than the symbols. A parity effect, between the odd and even $n$, is visible. A linear extrapolation gives $R = 3.5019(2)$ and $\gamma = 2.056(10)$.

![FIG. 6. $\ln R$ and $\alpha$: under the hypothesis $\gamma = 2$: the same plot as Fig. 8 but the x-axis is $1/(n \ln n)$. The (negative) slope is $-\alpha$. A linear extrapolation gives $R = 3.5017(2)$ and $\alpha = 0.25(5)$.}
As we expect \( L_n \sim \ln R - \gamma/n \) for large \( n \), by plotting \( y = L_n \) versus \( x = 1/n \), we can estimate \( \ln R \) (limit when \( x \) goes to 0) and the exponent \( \gamma \) (asymptotic slope). In Fig. 6, we have plotted the Monte-Carlo estimate of \( L_n - \ln 3.5 + 2/n \) versus \( 1/n \) for \( n \) between 50 and 400. We have arbitrarily subtracted the linear function \( y = \ln 3.5 - 2x \), to reduce the amplitude of \( y \); we obtain a figure where the small quantities \( 2 - \gamma \) (remaining slope), \( \ln(R/3.5) \) (limit when \( x \) goes to 0) and curvature (deviation to the expected linear behavior) are more visible. The curvature remains small and a linear extrapolation gives a limit between 0.0005 and 0.0006 with an estimated slope 0.056(10). Then
\[
R = 3.5019(2) \quad \text{and} \quad \gamma = 2.056(10).
\]

With the assumption that the asymptotic behavior is \( M_n \sim c R^n/n^\gamma \), the conjecture \( \gamma = 2 \) is incompatible with these simulations. But, we can try another asymptotic shape, for example
\[
M_n \sim c R^n/n^\gamma \frac{1}{\ln^n n},
\]
by introducing a new exponent \( \alpha \). In Fig. 6, we have plotted \( nL_n - \ln 3.5 + 2/n \) (as in Fig. 6), but now versus \( 1/(n \ln n) \). With this transformation of the \( x \)-axis, a linear behavior corresponds to \( \gamma = 2 \) and the slope measures \(-\alpha\). A linear extrapolation gives
\[
R = 3.5017(2), \quad \gamma = 2 \quad \text{and} \quad \alpha = 0.25(5).
\]
with \( \alpha \) compatible with the simple fraction 1/4.

How shall we choose between both results Eq. (18) or Eq. (20)? We notice that the quality of the alignment of points is the same in Fig. 6 and Fig. 6. In fact, it is very difficult to distinguish between a logarithmic law and a power law with such a small exponent, with statistical error bars and when the amplitude of \( \ln n \) is small. In fact, for \( n \) close to \( n' \), the term \( \alpha/(n \ln n) \) looks like \( \alpha'/n \) with \( \alpha'/\alpha = 1/\ln n + 1/n^2 \). Then, for any choice of \( \alpha \) not too large, \( \gamma = 2.056 - 0.22\alpha \) gives a class of acceptable behaviors.

We have tried to use more sophisticated extrapolation methods. For example, with a fixed jump \( i \), \((nL_n - (n - i)L_{n-i})/i\) gives theoretically the same limit \( \ln R \) but by removing the term \( \gamma/n \). Thus \( n^2(L_n - L_{n-i})/i \) gives a direct estimate of \( \gamma \). Unfortunately these kinds of derivative amplify statistical errors, and the results are compatible but less precise than the previous estimates.

So, with our numerical simulations, we cannot say if a logarithmic factor is present or not. However, \( R = 3.5018(3) \), and we can exclude \( R = 3.5 \) and the conjecture \( \gamma = 2 \) without logarithmic factor (\( \alpha = 0 \)).

### B. Winding

We will present the Monte-Carlo results of the exponent \( \nu \), which describe the asymptotic behavior of the average winding number \( w_n \sim n^\nu \). To avoid problem with bias, we have used the procedure described in Sect. 4.1.1. By plotting \( \ln(w_n) \) versus \( \ln(n) \), the asymptotic slope will be a measurement of \( \nu \). In Fig. 7, we have plotted \( y = \ln(w_n + 1) - 1/2 \ln n \) versus \( x = \ln n \). We have arbitrarily considered \( \ln(w_n + 1) \) and not \( \ln(w_n) \) because \( w_n + 1 \) is less sensitive than \( w_n \) to the finite size effects [4]. As the main question is to know whether \( \nu = 1/2 \) or not, we have arbitrarily subtracted the linear function \( y = x/2 \). Then the variation of \( y \) is reduced and we obtain a figure where \( \nu = 1/2 \) (residual slope) and the curvature are more visible. We see that the curvature is small and a linear extrapolation gives \( \nu = 0.518 \). As it is difficult to estimate the errors with the data of Fig. 7, we have also tried more sophisticated quantities like
\[
G_i(n) = \frac{n}{i} \ln \left( \frac{w_n + 1}{w_{n-i} + 1} \right)
\]
which are discrete derivatives of \( \ln(w_n + 1) \) with step \( i \). They give a direct value for \( \nu \), but unfortunately the statistical fluctuations are amplified by this differentiation and the uncertainty over \( \nu \) is of order 0.002.

We have seen that, for the exponent \( \gamma \), a behavior with logarithmic correction is not excluded by the Monte-Carlo data. So we have tried to fit the winding number with
\[
w_n \sim n^{1/2} \ln^\alpha n.
\]
With this hypothesis, a plot of \( y = \ln(w_n + 1) - 1/2 \ln n \), as in Fig. 6, but now versus \( x = \ln \ln n \) would give a straight line with slope \( \alpha \). But the curvature is much stronger than that of Fig. 6. So we dismiss this hypothesis and conclude that
\[ \nu = 0.518(2) \] (23)

With the assumption that the asymptotic behavior is a simple power-law, the Brownian value \( \nu = 1/2 \) is incompatible with these simulations.

C. Probability distribution of winding number

We define the probability distribution \( P_n(w) \) of winding number as the fraction of the meanders of size \( n \) with \( w \) windings. We expect [14] the asymptotic scaling behavior

\[ P_n(w) \approx \frac{2}{w_n + 1} f \left( \frac{w + 1}{w_n + 1} \right) \] (24)

with a scaling function \( f(x) \), where \( w_n \) is the average winding number. With the factor 2, the integral of \( f \) is normalized to 1 because \( w \) and \( n \) are integers with the same parity. In Fig. 8, we plot \( y = (w_n + 1)P_n(w)/2 \) versus \( x = (w + 1)/(w_n + 1) \) for different values of \( n \).

To define the scaling variable \( x \), we prefer to take \( w + 1 \) instead of \( w \) to reduce finite size effects.

We see that the points accumulate on a smooth curve, which represents the scaling function \( f(x) \). By analogy with the end-to-end distribution for polymers, we expect [14] a power-law behavior, \( f(x) \sim x^\theta \), for small \( x \), and a behavior \( f(x) \sim \exp(-\text{const.} x^\delta) \), for large \( x \). Our data give the estimates

\[ \theta = 1.7(1) \quad \text{and} \quad \delta = 2.3(1). \] (25)

To obtain a better precision, it would require to have larger values of \( n \).

D. Height and area

We are interested by the asymptotic behavior, for large size \( n \), of the average area \( A_n \) (see Eq. 14) and average height \( h_n(i) \) (see Eq. 15).

The label \( i \) is the “horizontal” coordinate and varies between \(-n\) and \( n \). So, we introduce the scaled variable \( x = i/n \), with \(-1 \leq x \leq 1\). We expect [14] that

\[ h_n(nx) \sim \frac{A_n}{n} \rho(x) \] (26)

for large \( n \) by fixing \( x \), where \( \rho(x) \) is a scaling function with integral normalized to 1. In Fig. 9, we have plotted \( y = n(h_n(i)+1)/(A_n+2n) \) versus \( x = i/n \) for various size \( n \). Only the positive \( i \) are shown because \( h_n(i) \) is symmetric after summing over all the meanders. More precisely, we have plotted the average of \( \{h_n(i) + h_n(-i)\}/2 \) over the Monte-Carlo samples of meanders, which is equivalent to the average of \( h_n(i) \) over these samples, plus those obtained by the left-right symmetry \( (i \rightarrow -i) \). As in the previous figures, we prefer to take \( (h_n + 1) \) and \( (A_n + 2n) \) to reduce finite size effects.

We see that, for \( 0 < x < 1 \), the points accumulate on a smooth curve, which represents the scaling function \( \rho(x) \). This shape is not a half-circle, as it would be for a random-walk on a semi-infinite line [14].

As \( h_n(0) \) is the winding number \( w_n \) for the particular case \( x = 0 \), Eq. (26) can be valid only if \( h_n(nx) \) scales as \( n^\nu \) with the same exponent \( \nu \) for all \( x \). Consequently, the area \( A_n \) scales as \( n^{\nu+1} \). In a previous section, we have numerically determined \( \nu = 0.518(2) \) by extrapolation of \( w_n = h_n(0) \). The same work with \( h_n(n/2) \) and \( A_n \).
gives $\nu = 0.517$ for both, compatible with the previous estimates, but less precise because the finite size effects are stronger.

In Fig. 10, small deviations appear between the curves for $n=40$ and $400$, at the boundary ($x \simeq \pm 1$) and in the middle of the meander ($x \simeq 0$).

In order to understand why finite-size effects are important near the boundary, Fig. 10 is a plot of $h_n(n-i)$ versus $i$, with various $n$. Clearly, when $n$ is large, curves accumulate on a limiting curve

$$h(i) = \lim_{n \to \infty} h_n(n-i).$$  

(27)

As this function can be fitted by a straight line on this “log-log” plot, we define a new exponent $\phi$ by

$$\tilde{h}(i) \sim i^\phi \quad \text{with} \quad \phi = 0.64(2)$$  

(28)

when $i$ is large. As by construction, $h_n(n) = 0$ and $h_n(n-1) = 1$ for all $n$, $\tilde{h}(0) = 0$ and $\tilde{h}(1) = 1$ satisfy exactly Eq. (28).

If we set $i = ny$, by using Eq. (25) valid when $i$ is fixed, we obtain $h_n(n-i) \sim (ny)^\nu$. On the other hand, by using Eq. (26) which is valid when $y$ is fixed, we obtain now $n^\nu \rho(1-y)$. As the exponents $\nu$ and $\phi$ differs, these two regimes are incompatible and the behavior of $h_n(n-i)$ depends on the order in which $n$ and $i$ go to infinity. In particular, the domain of validity of Eq. (26) is reduced to the single point $x = 1$ for Eq. (24). That explains why the rightmost dots ($i \sim n$) in Fig. 9 for the small size are not superimposed on the curve obtained for the large size.

Here, the exponent $\phi$ is the surface critical exponent, while $\nu$ is the bulk critical exponent. Near the boundary,

$h(i)$ is small and the condition $h(i) \geq 0$ limits appreciably the fluctuations toward the bottom. This effect is so strong that the exponent is changed and $\phi > \nu$. This situation is reminiscent of other critical phenomena [24], like the self-avoiding walk near a surface. Our result is to be contrasted with the case of a random walk on a semi-infinite line for which the surface exponent keeps its Brownian value $1/2$.

Finite-size effects observable near the middle of meander ($x \sim 0$) can be explained with Fig. 11, which is a plot of $h_n(i) - h_n(0)$ versus $i$ for small $i$ with various $n$. Clearly, when $n$ is large, curves accumulate on a limiting curve

$$h(i) = \lim_{n \to \infty} (h_n(i) - h_n(0)).$$  

(29)

If we make the hypothesis that the behavior of $\tilde{h}(i)$ is compatible with Eq. (25) by inverting the limits $n$ large and $x$ small, the consequences would be that $\rho(x)$ has a cusp at $x = 0$ with a infinite derivative $\rho(x) \sim \rho(0) + x^\nu$ when $x$ is small, and $\tilde{h}(i) \sim i^\nu$. But this power law behavior of $\tilde{h}(i)$ is not observed. Then the asymptotic behavior of $h_n(i)$ with $i$ fixed and $n$ large is given by Eq. (26), i.e. $n^\nu \rho(0)$, plus finite corrections of order $\tilde{h}(i)$.

This cusp is an another boundary effect since the point $i = 0$ is the source of the river. Let us consider three consecutive heights $\{h(i-1), h(i), h(i+1), m\}$ for a given meander $m$. By definition, $h(i+1, m) = h(i, m) \pm 1$. Then, for a generic $i$, the couple of its neighbors can have 4 respective values $\{h(i-1), h(i+1)\} = \{h(i) \pm 1, h(i) \pm 1\}$. But, for the special case $i = 0$, the situation $h(-1) = h(1) = h(0) - 1$ happens only if a single arch connects

FIG. 10. Log-log plot of the height $h_n(n-i)$ versus the distance to the boundary $i$ for various sizes $n$. The points accumulate on a limiting curve, which can be fitted by $i^\phi$ with $\phi = 0.64(2)$.

FIG. 11. Plot of the $h_n(i) - h_n(0)$ versus $i$ for small $i$, near the middle of meander, for various size $n$. By symmetry, only the positive $i$ are drawn. As the heights $h_n(i)$ are shifted by $h_n(0)$ and not scaled, we observe effects smaller than the asymptotic behavior $n^\nu \rho(0)$. The points accumulate on a limiting curve $\tilde{h}(i)$. 
the first bridge (near the source) to itself, by drawing a little circle around the source, without visiting the other bridges. This is forbidden if we insist in having only one connected component. In other word, the neighborhood of the source limits the fluctuations of $h(0)$ toward the top. In particular, for every meander, $h(0) \leq \{h(-1) + h(1)\}/2$. That explains why, on average, $h(0) < h(1)$.

To understand why $h(1) < h(2)$, it is mandatory to consider more complex forbidden situations for $\{h(-2), h(-1), h(0), h(1), h(2)\}$. For example, a circle connecting bridges $1$ and $2$ with an upper and a lower arch, with $h(\pm 1) = h(\pm 2) + 1$, is forbidden.

More generally, for a given $i$, the presence of the source forbids the systems of arches connecting the bridges $\{1, 2, \ldots, i\}$ with a closed road without visiting the other bridges $\{i+1, \ldots\}$. Qualitatively, it is a repulsive “force” which favors the connection of bridge $i$ with bridges $j > i$ and gives a concave shape for small $i$.

As the forbidden situations are more and more complex when $i$ becomes large, their statistical effects decrease and this repulsive force has a finite range. In the end, the summation over all forbidden situations gives finally this cusp with a finite amplitude described by $h(i)$.

VI. CONCLUSION

In this paper, we have presented a Monte-Carlo method to investigate a phase space described by a deterministic but irregular tree (i.e. the number of branches at each node is not fixed). With a naive random climbing on the tree (the one-squirrel method), the probability of a path depends on the number of branches encountered at each node. For the meanders problem, the ratio between extremal weights increases exponentially with the size: consequently the most part of the computer time is devoted to generate configurations with small weight, and only a exponentially small number of configurations with high weight contribute efficiently to the average.

With the multi-squirrel method, the distribution becomes almost flat: the bias, i.e. the ratio between extremal weights increases very slowly and never exceeds 3 in our simulations. Moreover, this bias is exactly known during the simulation, then it can be corrected to average over all meanders with a uniform distribution. As usual with Monte-Carlo simulations, results suffer from statistical fluctuations which decrease, in the best case, like the square root of the computer time.

After a simulation on a parallel computer with 3 years of cpu time in single processor units, results with small errors bars have been obtained for meanders up to size $n = 400$. Under some hypothesis inspired by the analogy with random walks problems, large $n$ extrapolation can be done for the enumeration (see Eq. [1] and Fig. [1]), for the distribution (see Eq. [2] and Fig. [2]), and the average of the winding number (see Eq. [3] and Fig. [3]) and the shape (see Section V D) of meanders.

From a Monte-Carlo point of view, the proposed algorithm can be used, in principal, for any combinatorial problem described by a tree: the essential ingredient is to know, for a given node, the number of branches. However algorithms are rare in this kind of problems and better variants or other algorithms could be without doubt invented.

About the meanders, with these estimates, it appears that the critical exponent are not simple fractions as $1/2$ or $7/2$, as conjectured by previous studies [1]. Of course, Monte-Carlo simulations cannot determine the exact values, but can confirm or invalidate analytical proposals, while waiting for a rigorous solution.

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