The Hodge Dual Symmetry of the Green-Schwarz Superstring in $AdS_5 \otimes S^5$

Chuan-Hua Xiong$^{ab}$ *

$^a$Institute of Modern Physics, Northwest University, Xi’an, 710069, China

$^b$Department of Modern Physics University of Science and Technology of China, Hefei, Anhui, 230026, P. R. China

Abstract

The hidden symmetry and an infinite set non-local conserved currents of the Green-Schwarz superstring on $AdS_5 \otimes S^5$ have been pointed out by Bena et al. In this paper, we shown that the Hodge dual between the Maurer-Cartan equation and the equation of motion gives the hidden symmetry in the moduli space of Green-Schwarz superstring. Thus by twisty transforming the vielbeins, we can express the currents of the paper[2] as the Lax connections by a unique spectral parameter.

PACS:11.25.-Tq; 11.15.-q

Keywords: Green-Schwarz superstring, Hodge dual, Hidden symmetry

*E-mail: chxiong@nwu.edu.cn
1 Introduction

The study of the type IIB superstring theory on $AdS_5 \otimes S^5$ give us many new results that are useful for further understanding the AdS/CFT correspondence\textsuperscript{1}. Bena et al\textsuperscript{2} found the hidden symmetry and an infinite set currents of classically conserved current for the Green-Schwarz superstring\textsuperscript{3} on $AdS_5 \otimes S^5$, such that it may be exact solvable. Dolan, Nappi and Witten\textsuperscript{4} have describe the equivalences between this integrable structure and the Yangian symmetry of the nonlocal currents as Bernard’s paper\textsuperscript{5}.

For a sigma model on coset space $G/H$, there exists Hodge dual symmetry between equations of motion and Maurer-Cartan equations. In term of the Hodge dual symmetry, one can find the flat connection $A(\lambda)$ which depend on the Lorentz boost parameter $\lambda = \exp \varphi$. Thus the flat connection $A(\lambda)$ satisfy

$$\partial_\mu U(\lambda) = A_\mu(\lambda)U(\lambda). \quad (1)$$

The Green-Schwarz superstring in $AdS_5 \otimes S^5$ is described by the nonlinear sigma model with WZW term where fields take value in the coset superspace: $\frac{SU(2,2|4)}{SO(4,1)\otimes SO(5)}$\textsuperscript{6}. However, this model is differ from a simply sigma model because of the $\kappa$ symmetry in the Green-Schwarz superstring. It is the $\kappa$ symmetry which guarantee the Hodge duality of the odd vielbeins between Maurer-Cartan equation and equation of motion although the cosetspace $\frac{SU(2,2|4)}{SO(4,1)\otimes SO(5)}$ is not a symmetric space.

In this paper, we investigate the Hodge duality between the Maurer-Cartan equations and the equations of motion and obtain the Lax-matrix by using the twisted dual transformation which represents a dressing symmetry for Green-Schwarz string embedding into $AdS_5 \otimes S^5$. 


2 Hodge dual symmetry of the nonlinear model on symmetric space $G/H$

Given the group $G$ and defined $H$ to be its stability subgroup, we can obtain the coset space, $G/H$

$$M = \frac{G}{H}$$  \hspace{1cm} (2)

and $M$ should be a Riemannian manifold on which $G$ act by isometries. If $\mathcal{G}$ is the Lie algebra of $G$ and $\mathcal{H} \subset \mathcal{G}$ denotes the Lie algebra of $H \subset G$, we have the following direct decomposition:

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{K}$$  \hspace{1cm} (3)

In particular, $H$ invariance of this decomposition implies

$$[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}, \quad [\mathcal{H}, \mathcal{K}] \subset \mathcal{K}. \quad (4)$$

If

$$[\mathcal{K}, \mathcal{K}] \subset \mathcal{H}, \quad (5)$$

the coset space $M$ is called symmetric space.

We consider the nonlinear coset model where the field take the values in the coset $M = G/H$. Define the left-invariant current

$$j = G^{-1}dG = h + k, \quad (6)$$

we also have

$$[k, k] = h, \quad [h, h] = h \quad (7)$$

and

$$[h, k] = k. \quad (8)$$

The action of this coset model is described by

$$L \propto \text{Tr}(k_\mu k^\mu). \quad (9)$$
From the action, we obtain the equation of motion
\[ D_{\mu}k^{\mu} = 0 \] (10)
where
\[ D_{\mu} = \partial_{\mu} + h_{\mu}. \] (11)
In the view of the geometry, there is the inherent structure equation, i.e. Maurer-Cartan equation:
\[ dj + j \wedge j = 0. \] (12)
Inducing this equation to the worldsheet, we obtain
\[ D_{\mu}k_{\nu} - D_{\nu}k_{\mu} = 0. \] (13)
It equal to
\[ \epsilon^{\mu\nu} D_{\mu}k_{\nu} = 0. \] (14)
Defined the Hodge dual as
\[ *k_{\mu} = \epsilon^{\mu\nu}k_{\nu}, \] (15)
then the eq.(14) becomes
\[ D_{\mu} *k^{\mu} = 0. \] (16)
It is easy to find that the equation of motion becomes Maurer-Cartan equation with the \( k^{\mu} \) replacing by \( *k^{\mu}. \)
We can combine the \( k \) and the \( *k \) with the \( \varphi \)-depended parameter owing to the Lorentz invariance:
\[ \begin{pmatrix} \tilde{k}^{\mu} \\ *\tilde{k}^{\mu} \end{pmatrix} = \begin{pmatrix} \cosh \varphi & \sinh \varphi \\ \sinh \varphi & \cosh \varphi \end{pmatrix} \begin{pmatrix} k^{\mu} \\ *k^{\mu} \end{pmatrix}. \] (17)
It is easy to check that \( \tilde{k}^{\mu} \) and \( *\tilde{k}^{\mu} \) also satisfy the Maurer-Cartan equation and the equation of motion respectively.
If we define the projection operator of worldsheet as
\[ P^{\mu\nu}_{\pm} = \frac{1}{2}(g^{\mu\nu} \pm \frac{1}{\sqrt{g}} \epsilon^{\mu\nu}), \] (18)
then the $\tilde{K}^\mu$ and $\ast \tilde{K}^\mu$ can be rewritten as follow:

\[
\tilde{k}^\mu = \exp \varphi P^\mu_+ k^\nu + \exp(-\varphi) P^\mu_- k^\nu \\
= \cosh \varphi k^\nu + \sinh \varphi \ast k^\nu
\]  
(19)

\[
\ast \tilde{k}^\mu = \exp \varphi P^\mu_- k^\nu + \exp(-\varphi) P^\mu_+ k^\nu \\
= \sinh \varphi k^\nu + \cosh \varphi \ast k^\nu
\]  
(20)

Thus, we can construct the one-parameter families flat connections $a(\varphi)$

\[
a(\varphi) = h + \tilde{k}(\varphi)
\]  
(21)

which satisfy:

\[
da + a \wedge a = 0
\]  
(22)

Given the flat connections, we have the integrable equation:

\[
dU = a(\varphi) U
\]  
(23)

3 The worldsheet Hodge Dual between Maurer-Cartan equation and equation of motion of Green-Shwarz superstring in $AdS^5 \otimes S^5$

3.1 Maurer-Cartan structure equation of coset superspace:

\[
\frac{PSU(2,2|4)}{SO(4,1) \otimes SO(5)}
\]

The $AdS_5 \otimes S^5$ is a coset space $\frac{SO(4,2)}{SO(4,1)} \otimes \frac{SO(6)}{SO(5)}$. It also preserves the full supersymmetry of the SUGRA and corresponds to the maximally supersymmetric background vacuum of IIB SUGRA. Combining the bosonic $SO(4, 2) \otimes SO(6)$ isometry symmetry with the full supersymmetry, the symmetry turns to be the $PSU(2, 2|4)$ acting on the super coset space $\frac{PSU(2,2|4)}{SO(4,1) \otimes SO(5)}$. In what follows, we adapt the conventions introduced by [6].
The left-invariant Cartan 1-forms

\[ L^A = dX^M L^A_M, \quad X^M = (x, \theta) \]  

(24)

are given by

\[ G^{-1}dG = L^A T_A \equiv L^a P_a + L^{a'} P_{a'} + \frac{1}{2} L^{ab} J_{ab} + \frac{1}{2} L^{a'b'} J_{a'b'} + L^{a'\alpha} Q_{\alpha a'} \]  

(25)

where \( G = G(x, \theta) \) is a coset representative in \( PSU(2,2|4) \). The indices \( a, b \) are index of \( AdS_5 \) and \( a', b' \) are the indices of \( S^5 \).

Furthermore, the Cartan 1-forms satisfy the Maurer-Cartan (MC) equation, i.e. the structure equation of basic one forms on the superspace \( \frac{PSU(2,2|4)}{SO(4,1) \otimes SO(5)} \)

\[ d(G^{-1}dG) + (G^{-1}dG) \wedge (G^{-1}dG) = 0. \]  

(26)

Then the super Gauss equations of the induced curvatures \( F^{ab} \) and \( F^{a'b'} \) defined by \( F = dH + H \wedge H \) are

\[ F^{ab} \equiv dL^{ab} + L^{ac} \wedge L^{cb} = -L^a \wedge L^b + \epsilon^{IJ} L^I \gamma^{ab} \wedge L^J, \]  

(27)

\[ F^{a'b'} \equiv dL^{a'b'} + L^{a'c'} \wedge L^{c'b'} = L^{a'} \wedge L^{b'} - \epsilon^{IJ} L^I \gamma^{a'b'} \wedge L^J. \]  

(28)

The super Coddazi equation for the even beins are

\[ dL^a + L^b \wedge L^{ba} = -iL^I \gamma^a \wedge L^I, \quad dL^{a'} + L^{b'} \wedge L^{b'a'} = L^I \gamma^{a'} \wedge L^I, \]  

(29)

and the super Coddazi equation for the odd beins are

\[ dL^I - \frac{1}{4} \gamma^{ab} L^I \wedge L^{ab} - \frac{1}{4} \gamma^{a'b'} L^I \wedge L^{a'b'} = -\frac{1}{2} \gamma^a \epsilon^{IJ} L^J \wedge L^a + \frac{1}{2} \epsilon^{IJ} \gamma^{a'} L^J \wedge L^{a'}. \]  

(30)

In the super Gauss equations and the Coddazi equations, the terms on the left hand side are the usual gauge covariant exterior derivative \( d + H \wedge \), while the right hand side include the contributions of curvature and torsion by the fermions.

To embed the IIB superstring into the super coset space \( M \), we should pull back the Cartan form down to the world sheet \( \Sigma(\sigma, \tau) \) as

\[ L^A = L^A_M dx^M = L^A_M \partial_i x^M d\sigma^i = L^A_i d\sigma^i \equiv L^A_1 d\tau + L^A_2 d\sigma . \]  

(31)
Then the Maurer-Cartan 1-form becomes
\[ G^{-1} \partial_i G = L^A_i P_A = L^{a'}_i P_a + L^a_i P_{a'} + \frac{1}{2} (L_i^{ab} J_{ab}^{\omega} + L_i^{a'b'} J_{a'b'}) + L_i^{a'\omega} Q_{a'\omega}, \] (32)
and e.g. the super Coddazi equations for the vector 5-beins (29) become
\[
\begin{align*}
\epsilon^{ij} (\partial_i L^a_j + L_i^{ab} L^b_j) + i \epsilon^{ij} \bar{L}_i^I \gamma^a L^I_j &= 0, \quad (33) \\
\epsilon^{ij} (\partial_i L^a_{j'} + L_i^{a'b'} L^b_{j'}) - \epsilon^{ij} \bar{L}_i^I \gamma^{a'} L^I_{j'} &= 0. \quad (34)
\end{align*}
\]

The Maurer-Cartan equations for the vielbeins describes the geometric behavior for the embedding of the type IIB string world-sheet into the target space \( AdS_5 \otimes S^5 \).

### 3.2 The Equation of Motion of Green-Schwarz superstring in \( AdS_5 \otimes S^5 \)

The \( AdS_5 \otimes S^5 \) Green-Schwarz superstring action is given as a nonlinear sigma model on the coset superspace \( SU(2,2|4)/SO(4,1)\otimes SO(5) \)
\[
I = -\frac{1}{2} \int_{\partial M_3} d^2 \sigma \sqrt{g} g^{ij} (L^a_i L^a_j + L^{a'}_i L^{a'}_j) + i \int_{M_3} s^{IJ} (L^a \wedge \bar{L}^I \gamma^a \wedge L^J + i L^{a'} \wedge \bar{L}^I \gamma^{a'} \wedge L^J). \] (35)

This action is invariant with respect to the local \( \kappa \)-transformations in terms of \( \delta x^a \equiv \delta X^M L^a_M, \delta x^{a'} \equiv \delta X^M L^{a'}_M, \delta \theta^I \equiv \delta X^M L^I_M \)
\[
\begin{align*}
\delta_{\kappa} x^a &= 0, \quad \delta_{\kappa} x^{a'} = 0, \quad \delta_{\kappa} \theta^I = 2(L_1^a \gamma^a - i L_1^{a'} \gamma^{a'}) \kappa^I \\
\delta_{\kappa} (\sqrt{g} g^{ij}) &= -16i \sqrt{g} (P_-^{jk} \bar{L}_k \kappa_1^I + P_+^{jk} \bar{L}_k \kappa_2^I). \quad (37)
\end{align*}
\]

Here \( P_{ij}^{\pm} \equiv \frac{1}{2} (g^{ij} \pm \frac{1}{\sqrt{2}} g^{ij}) \) are the projection operators, and 16-component spinor \( \kappa^I \) (the corresponding 32-component spinor has opposite chirality to that of \( \theta \)) satisfy the (anti) self duality constraints
\[
P_-^{ij} \kappa_1^I = \kappa_1^i, \quad P_+^{ij} \kappa_2^I = \kappa_2^i, \quad (38)
\]
which can be written as \( \frac{1}{\sqrt{2}} \epsilon^{ij} \kappa_1^I = -\kappa_1^i, \quad \frac{1}{\sqrt{2}} \epsilon^{ij} \kappa_2^I = \kappa_2^i, \) i.e. \( \frac{1}{\sqrt{2}} \epsilon^{ij} \kappa_1^I = -S^{ij} \kappa^I. \)
From the variation of action \([35]\), the equations of motion (EOM) are obtained

\[\sqrt{g}g^{ij}(\nabla_{i}L_{j}^{a} + L_{i}^{ab}L_{j}^{b}) + i\epsilon^{ij}S^{IJ}L_{I}^{i}L_{j}^{J} = 0,\]  
(39)

\[\sqrt{g}g^{ij}(\nabla_{i}L_{j}^{a'} + L_{i}^{a'b'}L_{j}^{b'}) - \epsilon^{ij}S^{IJ}L_{I}^{i}L_{j}^{J} = 0,\]  
(40)

\[(\gamma^{a}L_{i}^{a} + i\gamma^{a'}L_{i}^{a'})(\sqrt{g}g^{ij}S^{IJ} - \epsilon^{ij}S^{IJ})L_{j}^{J} = 0,\]  
(41)

where \(\nabla_{i}\) is the \(g_{ij}\)-covariant derivative on the world-sheet \(\Sigma(\sigma, \tau)\).

In term of the relation

\[\nabla_{i}L_{i}^{ai} = \frac{1}{\sqrt{g}}\partial_{i}(\sqrt{g}L_{i}^{ai}),\]  
(42)

the above equations of motion \(39\) and \(40\) can be rewritten as

\[g^{ij}(\partial_{i}(\sqrt{g}L_{j}^{a}) + L_{i}^{ab}L_{j}^{b}) + i\epsilon^{ij}S^{IJ}L_{I}^{i}L_{j}^{J} = 0,\]  
(43)

\[g^{ij}(\partial_{i}(\sqrt{g}L_{j}^{a'}) + L_{i}^{a'b'}L_{j}^{b'}) - \epsilon^{ij}S^{IJ}L_{I}^{i}L_{j}^{J} = 0.\]  
(44)

Note that

\[\frac{1}{\sqrt{g}}\epsilon^{ij}L_{j}^{1} = -L_{1}^{i}, \quad \frac{1}{\sqrt{g}}\epsilon^{ij}L_{j}^{2} = L_{2}^{i},\]  
(45)

the equations become

\[\partial_{i}(\sqrt{g}L_{j}^{ia}) + L_{i}^{ab}\sqrt{g}L_{j}^{b} - iL_{i}^{i}\gamma^{a}\sqrt{g}L_{j}^{i} = 0,\]  
(46)

\[\partial_{i}(\sqrt{g}L_{j}^{ia'}) + L_{i}^{a'b'}\sqrt{g}L_{j}^{b'} + L_{i}^{i}\gamma^{a'}\sqrt{g}L_{j}^{i} = 0.\]  
(47)

Here we have used the property of the \(\kappa\) symmetry:

\[P_{ij}^{i}L_{j}^{1} = L_{1}^{i}, \quad P_{ij}^{i}L_{j}^{2} = L_{2}^{i}.\]  
(48)

### 3.3 The worldsheets Hodge dual transformation between Maurer-Cartan equation (MCE) and equation of motion (EOM) of superstring on \(AdS_{5} \otimes S^{5}\)

In order to disclose the duality between the MCE and the EOM, we first describes the Hodge dual of bosonic and fermionic forms.
As usual, the Hodge dual of the coordinates of world-sheet is given by
\[
* (d\sigma^i) = \frac{-1}{\sqrt{g}} \epsilon^{ij} dz_j, \quad (d\sigma^1) = d\tau, \quad (d\sigma^2) = d\sigma, \quad \epsilon_{12} = -\epsilon_{21} = \epsilon^{21} = -\epsilon^{12} = 1 .
\] (49)

Thus, the Hodge dual of the even beins \( L^{\hat{a}} \) given by
\[
* L^{\hat{a}} = -\frac{\epsilon^{ij}}{\sqrt{g}} L_j^\hat{a}, \quad *L_i^\hat{a} = \epsilon_{ij} \sqrt{g} L_j^I .
\] (50)

For the odd beins \( L^I (I = 1, 2) \), we have
\[
* L^{I1} = \frac{1}{\sqrt{g}} \epsilon^{ij} L_j^{I1}, \quad (51)
\]
\[
* L^{I2} = \frac{1}{\sqrt{g}} \epsilon^{ij} L_j^{I2} . \quad (52)
\]

In term of these duality relations, the MCE (33) and (34) can be rewritten as
\[
\partial_i (\sqrt{g} * L^{Ia}) + L^{ab}_i (\sqrt{g} * L^{ib}) - i\tilde{L}_i^I \gamma^a (\sqrt{g} * L^{Ii}) = 0 ,
\] (53)
\[
\partial_i (\sqrt{g} * L^{Ia'}) + L^{a'b'}_i (\sqrt{g} * L^{Ib'}) + \tilde{L}_i^I \gamma^{a'} (\sqrt{g} * L^{Ii}) = 0 . \quad (54)
\]

Applying the Hodge dual transformation:
\[
* L^{i\hat{a}} \longleftrightarrow L^{i\hat{a}}, \quad *L^{Ii} \longleftrightarrow L^{Ii} \quad (55)
\]
to these equations, we have
\[
\partial_i (\sqrt{g} L^{Ia}) + L^{ab}_i \sqrt{g} L^{ib} - i\tilde{L}_i^I \gamma^a (\sqrt{g} L^{Ii}) = 0 ,
\] (56)
\[
\partial_i (\sqrt{g} L^{Ia'}) + L^{a'b'}_i \sqrt{g} L^{Ib'} + \tilde{L}_i^I \gamma^{a'} (\sqrt{g} L^{Ii}) = 0 . \quad (57)
\]

This show that the Maurer-Cartan equation dual to equation of motion with the transformation (55).

It is clear that the GS string action is invariant under the above dual transformation.

There exists no dual between MC eq.(30) and EOM eq.(41), because the \( L^I \) only
appears in the Wess-Zumino-Witten term and has no dynamical contribution to the action. Under the dual transformation, the 3rd EOM changes into

\[(\gamma^a * L_i^a + i \gamma^a * L_i^{a'}) (\sqrt{g} g^{ij} \delta^{I J} - \epsilon^{ij} s^{I J}) L_j^J = 0.\] (58)

Namely, only the first factor takes the dual form.

For the \(L^{\hat{a}}\), it does not change under duality since it is not dynamical and does not appear in the Green-Schwarz string action.

## 4 The twisted dual and integrality

Now we introduce the twisted dual transformation of vielbeins as follows. The duality discussed in previous section will be included as a special case of it. On the world sheet \(\Sigma(\sigma, \tau)\), it is the re-parametrization transformations along the two directions of the positive and negative light-cone \(\tau \pm \sigma\) with the scale factors \(\lambda = e^\varphi\) and \(\lambda^{-1}\) correspondently.

For the even vielbein forms \(L^{\hat{a}}\), they will be Lorentz rotate by \(\pm \varphi\) oppositely

\[
\begin{pmatrix}
(L^{\hat{a}}) \\
(*L^{\hat{a}})
\end{pmatrix}
= \begin{pmatrix}
cosh \varphi & \sinh \varphi \\
\sinh \varphi & \cosh \varphi
\end{pmatrix}
\begin{pmatrix}
L^{\hat{a}} \\
(*L^{\hat{a}})
\end{pmatrix}.
\] (59)

Thus, we have

\[
L^{i \hat{a}}(\lambda) = \cosh \varphi L^{i \hat{a}} + \sinh \varphi * L^{i \hat{a}}
\]

\[
= \frac{1}{2}(\lambda + \lambda^{-1}) L^{i \hat{a}} + \frac{1}{2}(\lambda - \lambda^{-1}) * L^{i \hat{a}}
\]

\[
= \lambda P^{ij}_+ L^{\hat{a}}_j + \lambda^{-1} P^{-ij}_- L^{\hat{a}}_j.
\] (60)

Where we have used \(\lambda = \exp \varphi\) and \(P^{ij}_\pm \equiv \frac{1}{2}(g^{ij} \pm \frac{1}{\sqrt{g}} \epsilon^{ij})\).

The odd vielbein forms \(L^I\) will rotate oppositely by \(\pm \frac{\varphi}{2}\) together with \(\theta^I\) and \(\kappa^I\)

\[
\begin{pmatrix}
(L^I) \\
(*L^I)
\end{pmatrix}
= \begin{pmatrix}
cosh \frac{\varphi}{2} & \sinh \frac{\varphi}{2} \\
\sinh \frac{\varphi}{2} & \cosh \frac{\varphi}{2}
\end{pmatrix}
\begin{pmatrix}
L^I \\
(*L^I)
\end{pmatrix}, I = 1, 2.
\] (61)
The transformations of odd vielbeins are

\[
\mathcal{L}^{ij} (\lambda) = \cosh \frac{\varphi}{2} L^{ij} + \sinh \frac{\varphi}{2} * L^{ij} \\
= \frac{1}{2} (\lambda^2 + \lambda^{-2}) L^{ij} + \frac{1}{2} (\lambda^2 - \lambda^{-2}) * L^{ij} \\
= \lambda \frac{i}{2} P_{ij}^+ L_j^I + \lambda^{-\frac{i}{2}} P_{ij}^- L_j^I.
\] (62)

Using the \( \kappa \) symmetry

\[
P_{ij}^+ L_j^1 = L^{i1}, \quad P_{ij}^+ L_j^2 = L^{i2},
\] (63)

we have

\[
\mathcal{L}^{i1} (\lambda) = \lambda^{-\frac{i}{2}} L^{i1},
\] (64)

\[
\mathcal{L}^{i2} (\lambda) = \lambda^{\frac{i}{2}} L^{i2}.
\] (65)

It should be pointed out that the Hodge twisted dual symmetry is not the symmetry of Metsaev-Tseytlin’s action [6]. Actually it is the hidden symmetry in the moduli space, which is described by the continuous spectral parameter \( \lambda \).

Now we can construct the Lax connection \( A_i(\lambda) \) with the spectral parameter \( \lambda \) as

\[
A_i(\lambda) = H + \mathcal{K}(\lambda) + \mathcal{F}(\lambda) \\
= \frac{1}{2} L_i^{ab} J_{ab} + \mathcal{L}_i^a (\lambda) P_a + \mathcal{L}_i^{a\alpha'1} (\lambda) Q_{a\alpha'1} \\
= \frac{1}{2} (L_i^{a'b} J_{ab} + L_i^{a'\alpha'\prime} J_{a'b'} + [L^{a'\alpha'\prime} P_{a'} + L_i^{a'a'} P_{a'}]) \\
+ \frac{1}{2} (\lambda - \lambda^{-1}) \left[ * (L_i^{a'}) P_{a'} + * (L_i^{a'}) P_{a'} \right] \\
+ \lambda \frac{i}{2} L_i^{a\alpha'1} Q_{a\alpha'1} + \lambda^{\frac{i}{2}} L_i^{a\alpha'2} Q_{a\alpha'2},
\] (66)

which looks like the original Cartan form with beins replaced by \( \mathcal{L}(\lambda) \). Such an O(2) transformation, should be defined in the same covariantly shifted moving frame, (the same gauge) with covariant constant \( N(x) \). Thus the \( H \) including in the covariant derivative will not be twisted. Obviously if \( \lambda = 1, \phi = 0 \), it is the original Cartan form [25]. On the “wick rotated” world-sheet we may take

\[
\lambda = \exp \varphi = i.
\] (67)
Then
\[ \mathcal{L}^{i\bar{a}} = i \ast L^{i\bar{a}}, \]  
(68)

Similarly
\[ \mathcal{L}^{i1} = i^{1\bar{2}} \ast L^{i1}, \quad \mathcal{L}^{i2} = i^{2\bar{1}} \ast L^{i2}, \]  
(69)

here i appears, from the difference of sign of Hodge star in \( M_2 \) and in \( E_2 \). Thus the vierbien \( \mathcal{L}(i) \) becomes simply the Hodge dual of original vierbien on Euclidean world sheet. And it implies the dual symmetry of the MCE and the EOM. It is obvious that the Lax connections \( A_i(\lambda) \) satisfy the zero curvature (flat connection) condition:
\[ \partial_i A_j(\lambda) - \partial_j A_i(\lambda) + [A_i(\lambda), A_j(\lambda)] = 0, \]  
(70)
as the linear combination of MCE and EOM i.e. the system is integrable, and we may introduce the transfer matrices \( U(\lambda, \sigma) \)
\[ \partial_i U(\lambda, \sigma) = A_i(\lambda, \sigma) U(\lambda, \sigma). \]  
(71)

5 Conclusions

Type IIB Green-Schwarz superstring on \( AdS_5 \otimes S^5 \) is the nonlinear Sigma model on the superspace:\[ \text{PSU}(2,2|4)_{SO(4,1) \otimes SO(5)} = AdS_5 \otimes S^5 \times \text{fermionic term.} \] It is well known that there exist the hidden symmetry and an set infinite conserved current in nonlinear sigma model. As the papers\[ 7, 8, 9 \] point out, the hidden symmetry and the integrable structure can be obtained by the Hodge dual transformation. In Green-Schwarz superstring on \( AdS_5 \otimes S^5 \), we find there also exist the dual symmetry between the Maure-Cartan equation and equation of motion because of the \( \kappa \) symmetry. From the dual symmetry, we obtain the flat connection with one-parameter and the integrability of the Green-Schwarz superstring on \( AdS_5 \otimes S^5 \).
Acknowledgments

We would like to thank Bo-Yu Hou, Bo-Yuan Hou, Kang-Jie Shi for helpful discussions. This work is supported in part by funds from National Natural Science Foundation of China with grant No.10575080.

References

[1] J. M. Maldacena, “The Large N Limit of Superconformal Field Theories and Supergravity”, Adv. Theor. Math. Phys. 2 (1998) 231-252; Int. J. Theor. Phys. 38 (1999) 1113-1133, [arXiv:hep-th/9711200].

[2] I. Bena, J. Polchinski, R. Roiban, Phys. Rev. D69 (2004) 046002, [arXiv:hep-th/0305116].

[3] M. B. Green, J. H. Schwarz, Phys. Lett. B 136 (1984) 367-370, Nucl. Phys. B243 (1984) 285.

[4] L. Dolan, C.R. Nappi, E. Witten, ”A Relation Between Approaches to Integrability in Superconformal Yang-Mills Theory”, JHEP 0310 (2003) 017, [arXiv:hep-th/0308089]; ”Yangian Symmetry in D=4 Superconformal Yang-Mills Theory”, arXiv:hep-th/0401243

[5] D. Bernard, Commun. Math. Phys. 137 (1991) 191.

[6] R. R. Metsaev, A. A. Tseytlin, Nucl. Phys. B 533 (1998) 109-126, [arXiv:hep-th/9805028].

[7] B.Yu. Hou, B.Yuan. Hou, “Differential Geometry for Physicists”, World Scientific Publishing Company (April 1, 1997).

[8] Boyu Hou, Yale University reprint YTP 80-29 OCT. 1980, appeared in Commun. theor. phys. Vol. 1 No.3 (1982); B. Y. Hou, M. L. Ge and Y. S. Wu, Phys. Rev. D24 (1981) 2238.
[9] Boyu Hou, *J. Math. Phys.* **25** (1984) 2325.