Crossingless matchings and the cohomology of \((n, n)\) Springer varieties

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1 Introduction

We constructed in [6] a family of rings \(H^n, n \geq 0\), a new invariant of tangles, and a conjectural invariant of tangle cobordisms. The invariant of a tangle is a complex of \((H^n, H^m)\)-bimodules (up to chain homotopy equivalences). This paper relates \(H^n\) to the Springer variety \(B_{n,n}\) of complete flags in \(\mathbb{C}^{2n}\) stabilized by a fixed nilpotent operator with two Jordan blocks of size \(n\).

**Theorem 1** The center of \(H^n\) is isomorphic to the cohomology ring of \(B_{n,n}\):

\[ Z(H^n) \cong H^*(B_{n,n}, \mathbb{Z}). \]

Both rings have natural gradings and the isomorphism is grading-preserving.

Theorem 1 is proved in a roundabout way, by finding generators and defining relations for both rings. Cohomology ring of the Springer variety \(B_\lambda\),
for a partition $\lambda$ of $n$, is well-understood. In Section 2 we use a presentation of $B_\lambda$ via generators and relations obtained by de Concini and Procesi [4] to prove

**Theorem 2** The cohomology ring of $B_{n,n}$ is isomorphic, as a graded ring, to the quotient of the polynomial ring $R = \mathbb{Z}[X_1, \ldots, X_{2n}]$, deg$X_i = 2$, by the ideal $R_1$ with generators

$$X_i^2, \quad i \in [1, 2n];$$

$$\sum_{|I|=k} X_I, \quad k \in [1, 2n];$$

where $X_I = X_{i_1} \ldots X_{i_k}$ for $I = \{i_1, \ldots, i_k\}$ and the sum is over all cardinality $k$ subsets of $[1, 2n]$.

Generators (2) are elementary symmetric polynomials in $X_1, \ldots, X_{2n}$. The quotient of $R$ by the ideal generated by (2) is isomorphic to the cohomology ring of the variety $B$ of complete flags in $\mathbb{C}^{2n}$. The inclusion $B_{n,n} \subset B$ induces a surjection of cohomology rings $H^*(B, \mathbb{Z}) \twoheadrightarrow H^*(B_{n,n}, \mathbb{Z})$. It turns out that by adding relations $X_i^2 = 0$ we get a presentation for the cohomology ring of $B_{n,n}$.

We recall several notations and definitions from [6], including that of $H^n$. Let $A = H^*(S^2, \mathbb{Z})$ be the cohomology ring of the 2-sphere, $A \cong \mathbb{Z}[X]/(X^2)$. The trace form

$$\text{tr} : A \to \mathbb{Z}, \quad \text{tr}(1) = 0, \quad \text{tr}(X) = 1,$$

makes $A$ into a commutative Frobenius algebra. We assign to $A$ a 2-dimensional topological quantum field theory $\mathcal{F}$, a functor from the category of oriented cobordisms between 1-manifolds to the category of abelian groups. $\mathcal{F}$ associates

- $A \otimes k$ to the disjoint union of $k$ circles,
- the multiplication map $A \otimes A \to A$ to the ”pants” cobordism (three-holed sphere viewed as a cobordism from two circles to one circle),
- the comultiplication

$$\Delta : A \to A \otimes A, \quad \Delta(1) = 1 \otimes X + X \otimes 1, \quad \Delta(X) = X \otimes X$$

to the ”inverted pants” cobordism,
either the trace or the unit map to the disk (depending on whether we consider the disk as a cobordism from one circle to the empty manifold or vice versa).

Let $B^n$ be the set of crossingless matchings of $2n$ points. Equivalently, $B^n$ is all pairings of integers from 1 to $2n$ such that there is no quadruple $i < j < k < l$ with $(i, k)$ and $(j, l)$ paired. Most of the time $n$ is fixed, and we denote $B^n$ simply by $B$. Figure 1 depicts elements of $B^2$.

![Figure 1: crossingless matchings $\{(12), (34)\}$ and $\{(14), (23)\}$](image)

Figure 1: crossingless matchings $\{(12), (34)\}$ and $\{(14), (23)\}$

For $a, b \in B$ denote by $W(b)$ the reflection of $b$ about the horizontal axis, and by $W(b)a$ the closed 1-manifold obtained by gluing $W(b)$ and $a$ along their boundaries, see figure 2.

![Figure 2:](image)

$F(W(b)a)$ is an abelian group isomorphic to $A^\otimes I$, where $I$ is the set of connected components of $W(b)a$. For $a, b, c \in B$ there is a canonical cobordism from $W(c)bW(b)a$ to $W(c)a$ given by "contracting" $b$ with $W(b)$, see figure 3.

This cobordism induces a homomorphism of abelian groups

$$F(W(c)b) \otimes F(W(b)a) \rightarrow F(W(c)a).$$ (3)

Let

$$H^n \overset{\text{def}}{=} \bigoplus_{a,b \in B} b(H^n)_a, \quad b(H^n)_a \overset{\text{def}}{=} F(W(b)a).$$

Homomorphisms (3), over all $a, b, c$, define an associative multiplication in $H^n$ (we let the products $d(H^n)_c \otimes b(H^n)_a \rightarrow d(H^n)_a$ be zero if $b \neq c$).
Figure 3: The contraction cobordism

\( a(H^n)_a \) is a subring of \( H^n \), isomorphic to \( \mathcal{A}^{\otimes n} \). Its element \( 1_a \overset{\text{def}}{=} 1^{\otimes n} \in \mathcal{A}^{\otimes n} \) is an idempotent in \( H^n \). The sum \( \sum a \) is the unit element of \( H^n \). Notice that \( b(H^n)_a = 1_{bH^n}1_a \).

Cohomological grading of \( \mathcal{A} \) (deg(1) = 0, deg(\( X \)) = 2) gives rise to a grading of \( H^n \), see [6] for details.

Denote by \( S \) the 2-sphere \( S^2 \). Consider the direct product

\[ S^{\times 2n} \overset{\text{def}}{=} S \times S \times \cdots \times S \quad (2n \text{ terms}) \]

and a submanifold \( S_a \in S^{\times 2n} \), for \( a \in B \), which consists of sequences
\((x_1, \ldots, x_{2n}), x_i \in S \) such that \( x_i = x_j \) whenever \((i, j)\) is a pair in \( a \). This submanifold is diffeomorphic to \( S^{\times n} \). Let \( \tilde{S} = \cup_{a \in B} S_a \), a subspace in \( S^{\times 2n} \). For example, if \( n = 2 \) then \( \tilde{S} \) is homeomorphic to two copies of \( S \times S \) glued together along their diagonals.

\( H^n \) and \( \tilde{S} \) are constructed along similar lines: the cohomology ring of \( S_a \) is canonically isomorphic to the ring \( a(H^n)_a \), while the abelian group \( a(H^n)_b \) is canonically isomorphic to \( H^*(S_a \cap S_b, \mathbb{Z}) \). These isomorphisms are our starting point in the proof (Section 3) of

**Theorem 3** The center of \( H^n \) is isomorphic to the cohomology ring of \( \tilde{S} \):

\[ Z(H^n) \cong H^*(\tilde{S}, \mathbb{Z}). \]

In Section 4 we prove

**Theorem 4** Cohomology ring of \( \tilde{S} \) is isomorphic, as a graded ring, to the quotient of the polynomial ring \( R = \mathbb{Z}[X_1, \ldots, X_{2n}], \deg X_i = 2, \) by relations

\[ X_i^2 = 0, \quad i \in [1, 2n]; \quad (4) \]
\[ \sum_{|I|=k} X_I = 0, \quad k \in [1, 2n]. \quad (5) \]
Theorems 2, 3 and 4 imply Theorem 1. They also show that spaces \( \tilde{S} \) and \( B_{n,n} \) have isomorphic cohomology rings. These spaces have similar combinatorial structure. Irreducible components of \( B_{n,n} \), just like those of \( \tilde{S} \), are enumerated by crossingless matchings. Each component \( K_a \subset B_{n,n} \) is an iterated \( \mathbb{P}^1 \)-bundle over \( \mathbb{P}^1 \) (see [5]), and homeomorphic to \( S_a \subset \tilde{S} \). Moreover, \( K_a \cap K_b \) and \( S_a \cap S_b \) are homeomorphic. We expect that there is a compatible family of homeomorphisms and suggest

Conjecture 1 \( B_{n,n} \) and \( \tilde{S} \) are homeomorphic.

Warning: \( \tilde{S} \) can be naively upgraded to an algebraic variety, by changing the 2-sphere \( S \) to \( \mathbb{P}^1 \) in the definition of \( \tilde{S} \). With this structure, however, \( \tilde{S} \) is not isomorphic to \( B_{n,n} \) as an algebraic variety, since irreducible components of \( B_{n,n} \) are nontrivial iterated \( \mathbb{P}^1 \) bundles over \( \mathbb{P}^1 \), while those of \( \tilde{S} \) are just direct products of \( \mathbb{P}^1 \).

Let \( B_\kappa \) be the Springer variety of complete flags in \( \mathbb{C}^n \) stabilized by a fixed nilpotent operator with Jordan decomposition \( \kappa = (k_1, \ldots, k_m) \). The cohomology ring of \( B_\kappa \) admits a natural action of the symmetric group, see [3, Section 3.6] and references therein. In particular, \( S_{2n} \) acts on the cohomology ring of \( B_{n,n} \). In view of Theorem 1 it therefore acts on the center of \( H^n \). Explicitly, the action is by permutations of \( X_i \)'s. It does not come from any action of \( S_{2n} \) on \( H^n \).

In Section 5 we present an intrinsic construction of this action. The 2
\[ \text{stranded braid group acts on the category} \ K \text{ of complexes of} \ H^n \text{-modules modulo chain homotopies, as follows from [6]. This action descends to the braid group action on centers of} \ K \text{ and} \ H^n \text{ and factors through to the symmetric group action on the center of} \ H^n. \]

In Section 6 we discuss conjectural isomorphisms between centers of parabolic blocks of the highest weight category for the Lie algebra \( \mathfrak{sl}_n \) and cohomology algebras of Springer varieties.

Acknowledgments: Ideas relating rings \( H^n \) and \((n,n)\) Springer varieties appeared during discussions between Paul Seidel and the author, and we are planning a joint paper on various aspects of this correspondence [7]. The present work can be viewed as a side result of [7]. I am grateful to Ragnar-Olaf Buchweitz and Ivan Mirkovic for useful consultations. This work was partially supported by NSF grant DMS-0104139.
2 Proof of Theorem 2

De Concini and Procesi [4] found a presentation for the cohomology ring of the Springer variety associated to a partition. We describe their result specialized to the \((n,n)\) partition.

Start with the ring \( R = \mathbb{Z}[X_1, \ldots, X_{2n}] \). For \( I \subset [1,2n] \) let \( X_I = \prod_{i \in I} X_i \) and let \( e_k(I) \) be the elementary symmetric polynomial of order \( k \) in variables \( X_i \), for \( i \in I \):

\[
e_k(I) = \sum_{|J|=k, J \subseteq I} X_J.
\]

**Proposition 1** (see [4]) The cohomology ring of \( B_{n,n} \) is isomorphic to the quotient ring of \( R = \mathbb{Z}[X_1, \ldots, X_{2n}] \) by the ideal \( R_2 \) generated by \( e_k(I) \) for all \( k + |I| = 2n + 1 \), \( X_I \) for all \( |I| = n + 1 \), and \( X_i^2 \) for \( i \in [1,2n] \).

**Remark:** Defining relations are expressed in [4] in terms of complete symmetric functions. Complete and elementary symmetric functions coincide modulo the ideal generated by \( X_i^2, i \in [1,2n] \).

We want to show the equality of ideals \( R_1 = R_2 \), where \( R_1 \) was defined in Theorem 1. Let \( R_3 \) be the ideal of \( R \) generated by \( X_i^2, i \in [1,2n] \). Let \( e_k = e_k([1,2n]) \in R_1 \).

**Claim:** \( R_2 \subset R_1 \). Since both \( R_1 \) and \( R_2 \) are stable under the permutation action of \( S_{2n} \) on \( R \), it suffices to show that \( e_k([1,2n - k + 1]) \) and \( X_{[1,n+1]} \) lie in \( R_1 \). Indeed,

\[
e_k([1,2n-k+1]) \equiv \sum_{i=0}^{k-1} (-1)^i e_i([2n-k+2,2n]) e_{k-i} \pmod{R_3}, \quad (6)
\]

\[
X_{[1,n+1]} \equiv \sum_{i=0}^{n-1} (-1)^i e_i([n+2,2n]) e_{n+1-i} \pmod{R_3}. \quad (7)
\]

The right hand sides lie in \( R_1/R_3 \) and the claim follows. □

**Claim:** \( R_1 \subset R_2 \). Equalities (6) and induction on \( k \) imply that \( e_k \in R_2 \) for all \( k \leq n \). Moreover, \( e_k \in R_2 \) for \( k > n \) since \( X_I \in R_2 \) for any \( |I| > n \). □

Therefore, \( R_1 = R_2 \) and the cohomology ring of \( B_{n,n} \) is isomorphic to \( R/R_1 \).

□
3 Proof of Theorem 3

$H(Y)$ will denote the cohomology ring of a topological space $Y$ with integer coefficients.

There is a canonical isomorphism of rings $H(S_a) \cong \mathcal{A}^I \cong aH_a$, where $I$ is the set of arcs of $a$. Similarly, there are natural abelian group isomorphisms $H(S_a \cap S_b) \cong \mathcal{A}^I \cong aH_b$, where $I$ is the set of connected components of $W(a)b$. These isomorphisms allow us to make $aH_b$ into a ring. $a1_b \overset{\text{def}}{=} 1 \otimes k \in \mathcal{A}^k$ is the unit of this ring.

Inclusions $S_b \supset (S_a \cap S_b) \subset S_a$ induce ring homomorphisms

$$\psi_{a,a,b} : H(S_a) \rightarrow H(S_a \cap S_b), \quad \psi_{b,a,b} : H(S_b) \rightarrow H(S_a \cap S_b).$$

Likewise, maps

$$\gamma_{a,a,b} : a(H^n)_a \rightarrow a(H^n)_b, \quad \gamma_{b,a,b} : b(H^n)_b \rightarrow a(H^n)_b,$$

given by $x \mapsto x a 1_b$ and $x \mapsto a 1_b x$ are ring homomorphisms. The following diagram made out of these homomorphisms and automorphisms commutes:

\[
\begin{array}{ccc}
H(S_a) & \longrightarrow & H(S_a \cap S_b) \\
\downarrow{\cong} & & \downarrow{\cong} \\
a(H^n)_a & \longrightarrow & a(H^n)_b
\end{array}
\]

\[\begin{array}{ccc}
\cong & & \cong \\
\end{array}\]

\[\begin{array}{ccc}
b(H^n)_b & \longleftarrow & b(H^n)_b
\end{array}\]

\[
(8)
\]

Suppose given finite sets $I$ and $J$, rings $A_i, i \in I$ and $B_j, j \in J$, and ring homomorphisms $\beta_{i,j} : A_i \rightarrow B_j$ for some pairs $(i, j)$. Let

$$\beta = \sum_{i \in I} \beta_{i,j}, \quad \beta : \prod_{i \in I} A_i \rightarrow \prod_{j \in J} B_j.$$

Define the equalizer of $\beta$ (denoted $\text{Eq}(\beta)$) as the subring of $\prod_{i \in I} A_i$ which consist of $\times a_i$ such that $\beta_{i,j} a_i = \beta_{k,j} a_k$ whenever $\psi_{i,j}$ and $\psi_{k,j}$ are defined.

Diagrams (8) give rise to a commutative diagram of ring homomorphisms

\[
\begin{array}{ccc}
\text{Eq}(\psi) & \longrightarrow & \prod_{a} H(S_a) \\
\downarrow{\cong} & & \downarrow{\cong} \\
\text{Eq}(\gamma) & \longrightarrow & \prod_{a} a(H^n)_a
\end{array}
\]

\[\begin{array}{ccc}
\cong & & \cong \\
\end{array}\]

\[\begin{array}{ccc}
\prod_{a} H(S_a) & \rightarrow & \prod_{a \neq b} H(S_a \cap S_b) \\
\downarrow{\cong} & & \downarrow{\cong} \\
\prod_{a} a(H^n)_a & \rightarrow & \prod_{a \neq b} a(H^n)_b
\end{array}\]

\[\begin{array}{ccc}
\cong & & \cong \\
\end{array}\]

\[\begin{array}{ccc}
(9)
\end{array}\]
where
\[ \psi = \sum_{a \neq b} (\psi_{a,a,b} + \psi_{b,a,b}) \quad \text{and} \quad \gamma = \sum_{a \neq b} (\gamma_{a,a,b} + \gamma_{b,a,b}). \]

For an element \( z \in H \) write \( z = \sum_{a,b} b z_a \) where \( b z_a \in \mu H_a \). If \( z \) is central, \( b z_a = 0 \) if \( a \neq b \), since \( 0 = z 1_b 1_a = 1_b z_a = b z_a \). Thus, \( z = \sum_{a} a z_a \). Denote \( a z_a \) by \( z_a \). Clearly, \( z = \sum_{a} z_a \) is central iff \( z_a 1_b = a 1_b z_b \) for all \( a, b \) such that \( a \neq b \). Therefore, \( Z(H^n) \cong \text{Eq}(\gamma) \).

Inclusions \( S_a \subset \tilde{S} \) induce ring homomorphisms \( H(\tilde{S}) \longrightarrow \prod_a S_a \) which factor through \( \text{Eq}(\psi) \). Putting everything together, we obtain the following diagram

\[
\begin{array}{ccccccc}
H(\tilde{S}) & \xrightarrow{\tau} & \text{Eq}(\psi) & \longrightarrow & \prod_a H(S_a) & \xrightarrow{\psi} & \prod_{a \neq b} H(S_a \cap S_b) \\
\downarrow{\cong} & & & & \downarrow{\cong} & & \downarrow{\cong} \\
Z(H^n) & \xrightarrow{\cong} & \text{Eq}(\gamma) & \longrightarrow & \prod_a (H^n)_a & \xrightarrow{\gamma} & \prod_{a \neq b} (H^n)_b
\end{array}
\]

Theorem 3 will follow from

**Proposition 2** \( \tau \) is an isomorphism.

Proof of this proposition occupies the rest of this section.

For \( a, b \in B \) we will write \( a \rightarrow b \) if there is a quadruple \( i < j < k < l \) such that \((i, j)\) and \((k, l)\) are pairs in \( a \), \((i, l)\) and \((j, k)\) are pairs in \( b \), and otherwise \( a \) and \( b \) are identical (see Figure 4). Figure 5 depicts all arrow relations for \( n = 3 \).

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Figure 4: a → b
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Introduce a partial order on \( B \) by \( a < b \) iff there is a chain of arrows \( a \rightarrow a_1 \rightarrow \cdots \rightarrow a_m \rightarrow b \). We extend the partial order \( < \) to a total order on \( B \) in an arbitrary way and denote it by \( < \).
Define the distance $d(a, b)$ between $a$ and $b$ as the minimal length $m$ of a sequence $(a = a_0, a_1, \ldots, a_m = b)$ such that for each $i$ either $a_i \to a_{i+1}$ or $a_{i+1} \to a_i$. One geometric interpretation of the distance: the diagram $W(b)a$ has $n - d(a, b)$ circles.

Lemma 1 For any $a, b \in B$ there is $c$ such that $d(a, b) = d(a, c) + d(c, b)$ and $a \succ c \prec b$.

Proof is left to the reader. □

Lemma 2 If $d(a, c) = d(a, b) + d(b, c)$ then

$$S_a \cap S_c = S_a \cap S_b \cap S_c.$$ 

□

Let $S_{<a} = \bigcup_{b < a} S_b$ and $S_{\leq a} = \bigcup_{b \leq a} S_b$. Note that if $c$ is the next element after $a$ in the total order $<$ on $B$ then $S_{<c} = S_{\leq a}$.

Lemma 3

$$S_{<a} \cap S_a = \bigcup_{b \to a} (S_b \cap S_a).$$

Follows from the previous lemma and lemma □.

Lemma 4 $S_{<a} \cap S_a$ has cohomology in even degrees only. The inclusion $(S_{<a} \cap S_a) \subset S_a$ induces a surjective homomorphism of cohomology rings $\text{H}(S_a) \twoheadrightarrow \text{H}(S_{<a} \cap S_a)$.

Proof: We construct a cell decomposition of $S_a$. Let $I$ be the set of arcs of $a$. There is a canonical homeomorphism $S_a \cong S^{\times I}$. Let $\Gamma$ be the graph with $I$ as the set of vertices and $y, z \in I$ are connected by an edge iff there
exist \( b \rightarrow a \) such that \( b \) is obtained from \( a \) by erasing \( y, z \) and reconnecting their endpoints in a different way. See Figure 6 for an example.

\( \Gamma \) is a forest (a disjoint union of trees). Let \( E \) be the set of edges of \( \Gamma \). Mark a vertex in each connected component of \( \Gamma \) and denote by \( M \) the set of marked vertices. Note that \( |E| + |M| = n \).

Fix a point \( p \in S \). For each \( J \subset (E \sqcup M) \) let \( c(J) \) be the subset of \( S \times I \) consisting of points \( \{x_i\}_{i \in I}, x_i \in S \) such that

\[
\begin{align*}
  x_i &= x_j \quad \text{if } (i, j) \in J, \\
  x_i &\neq x_j \quad \text{if } (i, j) \notin J, \\
  x_i &= p \quad \text{if } i \in M \cap J, \\
  x_i &\neq p \quad \text{if } i \in M, i \notin J.
\end{align*}
\]

Clearly, \( S \times I = \sqcup_J c(J) \) and \( c(J) \) is homeomorphic to \( \mathbb{R}^{2(n-|J|)} \). We obtain a decomposition of \( S_a \cong S \times I \) into even dimensional cells. It restricts to a cell decomposition of \( S \cap S_a \), the latter a union of cells \( c(J) \) such that \( J \cap E \neq \emptyset \). The lemma follows, since these decompositions give us cochain complexes with zero differentials that describe cohomology groups of \( S_a \) and \( S \cap S_a \).

\begin{lemma}
Homomorphism
\[
\begin{array}{c}
H(S_{<a} \cap S_a) \longrightarrow \bigoplus_{b \rightarrow a} H(S_b \cap S_a)
\end{array}
\]
induced by inclusions \( (S_b \cap S_a) \subset (S_{<a} \cap S_a) \) is injective.
\end{lemma}

\begin{proof}
It suffices to check that

\[
\begin{array}{c}
H(S_{<a} \cap S_a) \longrightarrow \bigoplus_{b \rightarrow a} H(S_b \cap S_a)
\end{array}
\]

is injective. The cell decomposition of \( S_{<a} \cap S_a \) constructed above restricts to a cell decomposition of \( S_b \cap S_a \), for each \( b \rightarrow a \). Since \( S_{<a} \cap S_a = \sqcup_{b \rightarrow a} (S_b \cap S_a) \), the lemma follows.
\end{proof}
Note that $S_{\leq a} = S_{< a} \cup S_a$. Consider the Mayer-Vietoris sequence for $(S_{< a}, S_a)$:

$$\rightarrow H^m(S_{\leq a}) \rightarrow H^m(S_{< a}) \oplus H^m(S_a) \rightarrow H^m(S_{< a} \cap S_a) \rightarrow$$

**Proposition 3** $S_{\leq a}$ has cohomology in even degrees only. The Mayer-Vietoris sequence for $(S_{< a}, S_a)$ breaks down into short exact sequences

$$0 \rightarrow H^{2m}(S_{\leq a}) \rightarrow H^{2m}(S_{< a}) \oplus H^{2m}(S_a) \rightarrow H^{2m}(S_{< a} \cap S_a) \rightarrow 0 \quad (10)$$

for $0 \leq m \leq n$.

**Proof:** Induction on $a$ with respect to the total order $<$. Induction base is obvious. Induction step: let $e$ be the element before $a$ relative to $<$ and assume the proposition holds for $e$. Then spaces $S_{< e}, S_a, S_{< a} \cap S_a$ have cohomology in even degrees only (the last one by lemma 4) and the Mayer-Vietoris sequence degenerates into exact sequences

$$0 \rightarrow H^{2m}(S_{\leq a}) \rightarrow H^{2m}(S_{< a}) \oplus H^{2m}(S_a) \rightarrow H^{2m}(S_{< a} \cap S_a) \rightarrow \rightarrow H^{2m+1}(S_{\leq a}) \rightarrow 0.$$

By lemma 4 the map $H(S_a) \rightarrow H(S_{< a} \cap S_a)$ is surjective, so that the last term of the sequence is zero. \( \square \)

**Proposition 4** The following sequence is exact

$$0 \rightarrow H(S_{\leq a}) \xrightarrow{\phi} \bigoplus_{b \leq a} H(S_b) \xrightarrow{\psi^-} \bigoplus_{b \leq c \leq a} H(S_b \cap S_c),$$

where $\phi$ is induced by inclusions $S_b \subset S_{\leq a}$, while

$$\psi^- \overset{\text{def}}{=} \sum_{b \leq c \leq a} (\psi_{b,c} - \psi_{c,b}),$$

where

$$\psi_{b,c} : H(S_b) \rightarrow H(S_b \cap S_c)$$

is induced by the inclusion $(S_b \cap S_c) \subset S_b$.

**Proof:** Induction on $a$. The induction base, $a$ is minimal relative to $<$, is obvious. Induction step: assume $e$ precedes $a$ relative to $<$ and the claim is true for $e$. Lemma 4 allows us to substitute $\bigoplus_{b \leq a} H(S_b \cap S_a)$ for $H(S_{< a} \cap S_a)$.
in the sequence (10) while maintaining exactness everywhere but in the last term. Thus,

\[ 0 \rightarrow H(S_{\leq a}) \rightarrow H(S_{<a}) \oplus H(S_a) \rightarrow \bigoplus_{b<a} H(S_b \cap S_a) \] (12)

is exact. Moreover, \( S_{<a} = S_{\leq e} \). By induction hypothesis

\[ 0 \rightarrow H(S_{\leq e}) \rightarrow \bigoplus_{f \leq e} H(S_f) \rightarrow \bigoplus_{f < g \leq e} H(S_f \cap S_e) \]

is exact. Substituting in (12), and using standard properties of complexes, we conclude that (11) is exact. \( \square \)

When \( a \) is the maximal element of \( B \), Proposition 4 tells us that the sequence

\[ 0 \rightarrow H(\tilde{S}) \xrightarrow{\phi} \bigoplus_{b} H(S_b) \xrightarrow{\psi} \bigoplus_{b < c} H(S_b \cap S_c) \] (13)

is exact. This is equivalent to Proposition 4. \( \square \)

Remark: A similar method establishes an isomorphism between the quotient of \( H^n \) by its commutant subspace and homology of \( \tilde{S} \) with integer coefficients:

\[ H^n/[H^n, H^n] \cong H_*(\tilde{S}, \mathbb{Z}). \]

If \( \Lambda \) is a symmetric ring, the center of \( \Lambda \) is dual to \( \Lambda/[\Lambda, \Lambda] \). Ring \( H^n \) is symmetric [6, Section 6.7].

4 Proof of Theorem 4

Lemma 6 \( H(\tilde{S}) \) is a free abelian group of rank \( \binom{2n}{n} \).

Proof The cell decomposition of \( S_a \) defined in the proof of Lemma 4 restricts to a cell decomposition of \( S_a \cap S_{<a} \). Hence, a cell partition of \( \tilde{S} \) can be obtained starting with the cell decomposition of \( S_a \), for the minimal \( a \in B \), and then adding the cells of \( S_a \setminus S_{<a} \), over all \( a \) in \( B \) following the total order \( < \). Note that this is a cell partition of \( \tilde{S} \), not a cell decomposition, since the closure of a cell is not, in general, a union of cells. Nevertheless, since all cells are even-dimensional and the boundary of each cell has codimension 2 relative to the cell, \( H(\tilde{S}) \) is a free abelian group with a basis consisting of delta functions of these cells.
For $a \in B$ let $t(a)$ be the number of ”bottom” arcs of $a$, that is, arcs with no arcs below them. $t(a)$ is also the number of connected components of the graph $\Gamma$, defined in the proof of Lemma $4$. For instance, Figure $1$ diagrams have two and one bottom arcs (the left diagram has two). Our decomposition of $S_a \setminus S_{<a}$ has $2^{t(a)}$ cells. Therefore, the cell partition of $\widetilde{S}$ has $\sum_{a \in B} 2^{t(a)}$ cells. It is easy to see that this sum equals $\binom{2n}{n}$. Lemma follows. □

Recall that $X$ denotes a generator of $H^2(S)$. The inclusion $\iota : \widetilde{S} \subset S \times 2n$ induces a homomorphism $\iota^* : H(S \times 2n) \to H(\widetilde{S})$. Let $\phi_i : S \times 2n \to S$ be the projection on the $i$-th component. Consider the composition $\phi_i \circ \iota$ and let $X_i \overset{\text{def}}{=} (-1)^i \iota^* \circ \phi_i^*(X)$, $X_i \in H(\widetilde{S})$.

We denote by $[1,2n]$ the set of integers from $1$ to $2n$. For $I \subset [1,2n]$ let $X_I = \prod_{i \in I} X_i$.

**Proposition 5** The cohomology ring of $\widetilde{S}$ is generated by $X_i$, $i \in [1,2n]$ and has defining relations

\begin{align*}
X_i^2 &= 0, \quad i \in [1,2n]; \quad (14) \\
\sum_{|I|=k} X_I &= 0, \quad k \in [1,2n]. \quad (15)
\end{align*}

**Proof:** First we show that these relations hold. (14) is obvious. Let $j_a : S_a \subset \widetilde{S}$ and $j_a^*$ be the induced map on cohomology. (15) will follow if we check that

\[ \sum_{|I|=k} j_a^*(X_I) = 0 \quad (16) \]

for all $a \in B$, since

\[ \sum_{a \in B} j_a^* : H(\widetilde{S}) \longrightarrow \bigoplus_{a \in B} H(S_a) \]

is an inclusion. If $(i,i')$ is a pair in $a$ then $j_a^*(X_iX_{i'}) = 0$ and $j_a^*(X_i + X_{i'}) = 0$ (because of the term $(-1)^i$ in the definition of $X_i$, and since $i+i' \equiv 1(\text{mod } 2)$). Therefore,

\[ \sum_{|I|=k, \{i,i'\} \cap I \neq \emptyset} j_a^*(X_I) = 0 \]
where the sum is over all subsets of cardinality $k$ that intersect $\{i, i'\}$ non-trivially. To take care of the remaining terms in the L.H.S. of (16),

$$\sum_{|I|=k, \{i, i'\} \cap I = \emptyset} j_a(X_I),$$

pick another pair $(r, r')$ in $a$ and apply the same reduction to it. After $\frac{n-k}{2} + 1$ iterations all cardinality $k$ subsets will be accounted for. (16) follows.

We say that a subset $I$ of $[1, 2n]$ is admissible if $I \cap [1, m]$ has at most $\frac{m}{2}$ elements for each $m \in [1, 2n]$.

**Lemma 7** There are $\binom{2n}{n}$ admissible subsets.

*Proof* is left to the reader. $\Box$

**Lemma 8** $X_J$, for any $J \subset [1, 2n]$, is a linear combination of $X_I$, over admissible $I$.

*Proof:* let $y(J) = \sum_{j \in J} j$. Assume the lemma is false, and find such a non-admissible $J$ with the minimal possible $y(J)$. Take the smallest possible $m$ such that $|J \cap [1, m]| > \frac{m}{2}$. Then $m$ is odd, $m = 2r + 1$ and $|J \cap [1, m]| = r + 1$. Arguments in the proof of Theorem 2 in Section 2 imply that $e_{r+1}((J \cap [1, m]) \cup [m + 1, 2n]) = 0$. Therefore, $X_J \cap [1, m] \cup [m + 1, 2n]$ is a linear combination of $X_K$ with $K \subset (J \cap [1, m]) \cup [m + 1, 2n]$ and $y(K) > y(J)$. Since $X_J = X_J \cap [1, m] \cdot X_{J \cap [m+1, 2n]}$, this contradicts minimality of $y(J)$. $\Box$

**Lemma 9** $\{X_I\}$, over all admissible $I$, are linearly independent in $H(\widetilde{S})$.

*Sketch of proof:* Induction on $n$, use homomorphism $H(\widetilde{S}_n) \rightarrow H(\widetilde{S}_{n-1})$ induced by the inclusion $\widetilde{S}_{n-1} \subset \widetilde{S}_n$ (where $\widetilde{S}_n$ is what we usually call $\tilde{S}$). Details are left to the reader. $\Box$

Lemmas 6, 7, 8 and 9 imply Proposition 5 and the following results.

**Corollary 1** $H(\widetilde{S})$ has a basis $\{X_I\}$, over all admissible $I$.

**Corollary 2** The inclusion $\widetilde{S} \subset S \times 2^n$ induces a surjective ring homomorphism $H(S \times 2^n) \rightarrow H(\widetilde{S})$. 

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5 Symmetric group action on the center of $H^n$

The center of a category

The center of a category is defined as the commutative monoid of natural transformations of the identity functor. If the category is pre-additive (Hom$(X,Y)$ is an abelian group for any objects $X,Y$, and the composition of morphisms is bilinear) then the center is a commutative ring. A down-to-earth example: the center of the category of modules over a ring $A$ is isomorphic to the center of $A$.

Let $F$ be a functor in a category $\mathcal{C}$. The center of $\mathcal{C}$ acts in two ways on the set $\text{End}(F)$ of endomorphisms of $F$, since we can compose $\alpha \in Z(\mathcal{C})$ and $\beta \in \text{End}(F)$ on the left or on the right:

\[
\alpha \circ \beta : F \cong \text{Id} \circ F \xrightarrow{\alpha \circ \beta} \text{Id} \circ F \cong F, \\
\beta \circ \alpha : F \cong F \circ \text{Id} \xrightarrow{\beta \circ \alpha} F \circ \text{Id} \cong F.
\]

Assume that $F$ is invertible. Then any endomorphism of $F$ has the form $\text{Id}_F \circ \alpha$, for a unique $\alpha \in Z(\mathcal{C})$, as well as $\alpha' \circ \text{Id}_F$, for a unique $\alpha' \in Z(\mathcal{C})$. Thus, $F$ defines an automorphism of the monoid $Z(\mathcal{C})$ which takes $\alpha$ to $\alpha'$.

Suppose group $G$ acts weakly on $\mathcal{C}$, meaning that there are functors $F_g : \mathcal{C} \rightarrow \mathcal{C}$ for each $g \in G$, such that $F_1 \cong \text{Id}$ and $F_g F_h \cong F_{gh}$ (we do not impose compatibility conditions on these isomorphisms). Each $F_g$ is invertible and gives rise to an automorphism of the center of $\mathcal{C}$. We get an action of $G$ on $Z(\mathcal{C})$. Therefore, a weak group action on a category descends to an action on the center of the category.

Centers of triangulated and derived categories

Define the center of a triangulated category $\mathcal{D}$ as the set of natural transformations of the identity functor that commute with the shift functor $[1]$. Let $\mathcal{C}$ be an abelian category and $\widehat{\mathcal{C}}$ one of the triangulated categories associated to $\mathcal{C}$ (for instance, the bounded derived category of $\mathcal{C}$, or the category of bounded complexes of objects of $\mathcal{C}$ modulo chain homotopies). There are ring homomorphisms

\[
Z(\mathcal{C}) \xrightarrow{f} Z(\widehat{\mathcal{C}}) \xrightarrow{g} Z(\mathcal{C})
\]

whose composition is the identity. $f$ extends $\alpha \in Z(\mathcal{C})$ termwise to complexes of objects of $\mathcal{C}$. Homomorphism $g$ is induced by the inclusion of categories $\mathcal{C} \subset \widehat{\mathcal{C}}$. 

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$f$ and $g$ are not always isomorphisms, as observed by Jeremy Rickard. If $\mathcal{C}$ is the category of modules over the exterior algebra in one generator, then $f$, respectively $g$, has a nontrivial cokernel, respectively kernel.

Remark: Ragnar-Olaf Buchweitz pointed out to me that a triangulated category $\mathcal{D}$ also has extended center (or "Hochschild cohomology"),

$$\bigoplus_{m \in \mathbb{Z}} \operatorname{Hom}_{\operatorname{Fun}(\mathcal{D})} (\operatorname{Id}, \operatorname{Id}[m]),$$

(only natural transformations that supercommute with the shift functor are included). The extended center of the derived category of $\Lambda$-mod, for an algebra $\Lambda$, contains the Hochschild cohomology algebra $\operatorname{Ext}_{\Lambda^e}(\Lambda, \Lambda)$.

An action of $G$ on $\hat{\mathcal{C}}$ induces an action on $Z(\hat{\mathcal{C}})$. If $\ker(g)$ is $G$-stable, the action descends to $Z(\mathcal{C})$. Let $\Lambda$ be a ring and $D(\Lambda)$ the bounded derived category of $\Lambda$-mod. If a self-equivalence $F$ of $D(\Lambda)$ is given by tensoring with a bounded complex of left and right projective $\Lambda$-bimodules, then it descends to an automorphism of $Z(\Lambda)$ (compare with [8, Proposition 9.2]).

**Symmetric group action**

The symmetric group action on $Z(H^n)$ can be described intrinsically as follows. Let $\mathcal{K}$ be one of triangulated categories associated to $H^n$ (say, the category of bounded complexes of left $H^n$-modules up to chain homotopies). The structures described in [8] lead to a weak action of the braid group with $2n$-strands on $\mathcal{K}$. Diagram $U_i$ (see Figure 7) defines an $H^n$-bimodule $F(U_i)$ together with bimodule homomorphisms

$$F(U_i) \xrightarrow{\alpha} H^n, \quad H^n \xrightarrow{\beta} F(U_i),$$

induced by elementary cobordisms between $U_i$ and $\text{Vert}_{2n}$ (we use notations from [8], note that $H^n \cong F(\text{Vert}_{2n})$).

![Figure 7: Diagrams $U_i$ and $\text{Vert}_{2n}$](image)

Let $\mathcal{R}_i : \mathcal{K} \to \mathcal{K}$ be the functor of tensoring with the complex of bimodules

$$0 \longrightarrow F(U_i) \xrightarrow{\alpha} H^n \longrightarrow 0.$$  \hspace{1cm} (18)

$\mathcal{R}_i$'s are invertible and satisfy the braid group relations.
In Section 4 we defined generators $X_i$ of $H(\tilde{S})$. We now describe the image of $X_i$ (also denoted $X_i$) in $Z(H^n)$ under the isomorphism $H(\tilde{S}) \cong Z(H^n)$ established in Section 3:

$$X_i = \sum_{a \in B} a(X_i)_a, \quad a(X_i)_a \in \mathcal{A} \otimes_{n-1} \mathcal{A},$$

where the separated $\mathcal{A}$ corresponds to the circle in $\mathcal{F}(W(a)a)$ that contains the $i$-th endpoint of $a$, counting from the left.

For a complex $V$ of $H_n$-bimodules and $z \in Z(H_n)$ let $l_z$, respectively $r_z$, be the endomorphism of $V$ given by left, respectively right, multiplication by $z$. Endomorphisms $l_{X_i} - r_{X_{i+1}}$ and $l_{X_{i+1}} - r_{X_i}$ of the complex (18) are homotopic to 0, via the homotopy $\pm \beta : H^n \to \mathcal{F}(U_i)$. Therefore, the braid group action on $K$ descends to the action of the symmetric group $S_{2n}$ on $Z(H^n)$ by permutations of $X_i$'s.

6 Conjectures on centers of highest weight categories

If $e$ is an idempotent in a ring $\Lambda$, there is a homomorphism $Z(\Lambda) \to Z(\mathcal{E} \Lambda \mathcal{E})$ which takes $z \in Z(\Lambda)$ to $ze$.

Let $\mathcal{O}^{n,n}$ be the full subcategory of a regular block of the highest weight category of $\mathfrak{sl}_{2n}$ which consists of locally $U\mathfrak{p}(n,n)$-finite modules, where $\mathfrak{p}(n,n)$ is the parabolic subalgebra in $\mathfrak{sl}_{2n}$ of $(n,n)$ block-upper-triangular matrices. $\mathcal{O}^{n,n}$ is equivalent to the category of perverse sheaves on the Grassmannian of $n$-planes in $\mathbb{C}^{2n}$, constructible relative to the Schubert stratification. There is a unique finite-dimensional $\mathbb{C}$-algebra $\mathcal{A}^{n,n}$ such that

(i) $\mathcal{O}^{n,n}$ is equivalent to the category of finite-dimensional $\mathcal{A}^{n,n}$-modules,

(ii) every irreducible $\mathcal{A}^{n,n}$-module is one-dimensional.

$\mathcal{A}^{n,n}$ was explicitly described by Tom Braden [1]. In [2] we’ll construct an idempotent $e$ in $\mathcal{A}^{n,n}$ and an isomorphism $h : H^n \otimes_{\mathbb{Z}} \mathbb{C} \cong e\mathcal{A}^{n,n}e$.

**Conjecture 2** $Z(\mathcal{A}^{n,n}) \cong Z(e\mathcal{A}^{n,n}e)$, and $h$ induces an isomorphism of centers of $H^n \otimes_{\mathbb{Z}} \mathbb{C}$ and $\mathcal{O}^{n,n}$.

We would like to suggest a more general conjecture relating parabolic highest weight categories and Springer varieties. Let $\kappa = (k_1, \ldots, k_m)$ be a decomposition of $n$, $k_1 + \cdots + k_m = n$ and denote by $\mathfrak{p}(\kappa)$ the corresponding Lie algebra of block upper-triangular $n$-by-$n$ matrices. Let $\mathcal{O}^\kappa$ be the full
subcategory of locally $U^p(\kappa)$-finite modules in a regular block $\mathcal{O}_{\text{reg}}$ of the highest weight category $\mathcal{O}$ for $\mathfrak{sl}_n$. Let $Y^\kappa$ be the partial flag variety associated to $\kappa$. It is known that $\mathcal{O}^\kappa$ is equivalent to the category of perverse sheaves on $Y^\kappa$, smooth along Schubert cells.

Let $B_\kappa$ be the Springer variety of complete flags in $\mathbb{C}^n$ stabilized by a fixed nilpotent operator with Jordan decomposition $(k_1, \ldots, k_m)$.

**Conjecture 3** The center of $\mathcal{O}^\kappa$ is isomorphic to the cohomology algebra of $B_\kappa$:

$$Z(\mathcal{O}^\kappa) \cong H^*(B_\kappa, \mathbb{C}).$$

(19)

Note that the right hand side of the isomorphism (19) depends only on the partition type of $\kappa$, i.e. preserved by permutations of terms $k_1, \ldots, k_m$ of $\kappa$. The category $\mathcal{O}^\kappa$, featured in the left hand side, generally does not possess the same kind of invariance:

**Proposition 6** Categories $\mathcal{O}^{2,1,1}$ and $\mathcal{O}^{1,2,1}$ are inequivalent.

However, we have

**Proposition 7** If decompositions $\kappa$ and $\kappa'$ differ only by a permutation of terms, the categories $\mathcal{O}^\kappa$ and $\mathcal{O}^{\kappa'}$ are derived equivalent.

Propositions 6 and 7 are proved at the end of this section.

If two rings are derived equivalent, their centers are isomorphic ([8, Proposition 9.2]). Since $\mathcal{O}^\kappa$ and $\mathcal{O}^{\kappa'}$ are equivalent to categories of modules over finite-dimensional algebras, and these algebras are derived equivalent, centers of $\mathcal{O}^\kappa$ and $\mathcal{O}^{\kappa'}$ are isomorphic. Thus, the left hand side of (19) also depends only on the partition type of $\kappa$.

We would like conjectural isomorphisms (19) to be compatible with the inclusions of categories $\mathcal{O}^\kappa \subset \mathcal{O}_{\text{reg}}$ and topological spaces $B_\kappa \subset B$, where $B$ is the variety of complete flags in $\mathbb{C}^n$. These inclusion induce ring homomorphisms $Z(\mathcal{O}_{\text{reg}}) \rightarrow Z(\mathcal{O}^\kappa)$ and $H^*(B, \mathbb{C}) \rightarrow H^*(B_\kappa, \mathbb{C})$ which should be a part of the following commutative diagram

$$
\begin{array}{ccc}
\mathbb{H}^*(B, \mathbb{C}) & \longrightarrow & \mathbb{H}^*(B_\kappa, \mathbb{C}) \\
| & & | \\
\cong & & \cong \\
Z(\mathcal{O}_{\text{reg}}) & \longrightarrow & Z(\mathcal{O}^\kappa)
\end{array}
$$

(20)
Let $\Theta_i : \mathcal{O}^\kappa \to \mathcal{O}^\kappa$ be the functor of translation across the $i$-th wall. $\Theta_i$ is the product of two biadjoint functors (translations on and off the $i$-th wall). Let $\alpha_i : \Theta_i \to \text{Id}$ be one of the natural transformations coming from biadjointness. $\mathcal{R}_i$ is a functor in the derived category of $\mathcal{O}^\kappa$. We conjecture that this action descends to a symmetric group action on $Z(\mathcal{O}^\kappa)$, and the ring isomorphism (19) can be made $S_n$-equivariant.

Let $A_\kappa$ be a finite-dimensional algebra such that $A_\kappa\text{-mod} \cong \mathcal{O}^\kappa$. Let $e \in A_\kappa$ be the maximal idempotent such that the left $A_\kappa$-module $A_\kappa e$ is injective.

**Conjecture 4** Inclusion $eA_\kappa e \subseteq A_\kappa$ induces an isomorphism of centers $Z(A_\kappa) \cong Z(eA_\kappa e)$.

**Proof of Proposition 6**: Assume the two categories are equivalent. Any intrinsic homological information about them is identical. The equivalence restricts to a bijection between isomorphism classes of simple objects. The bijection induces isomorphisms between Ext rings of simple objects of these categories.

Simple objects are in a one-to-one correspondence with Schubert cells in partial flag varieties $X_{2,1,1}$ and $X_{1,2,1}$. For a simple object $L$ let $IC(L)$ be the intersection cohomology sheaf on the closure of the Schubert cell associated to $L$. Then $\text{Ext}(L, L) \cong \text{Ext}(IC(L), IC(L))$.

Let us count the number of simple objects $L$ in each category with $\dim(\text{Ext}(L, L)) = 3$. The Schubert cell of such an object is necessarily 2-dimensional and its closure is diffeomorphic to $\mathbb{CP}^2$ (use that the cohomology of the closure of the cell is a direct summand of $\text{Ext}(IC(L), IC(L))$). $X_{2,1,1}$ has only one such cell, while $X_{1,2,1}$ has two. Contradiction. □

**Proof of Proposition 7**: To construct an equivalence between $D^b(\mathcal{O}^\kappa)$ and $D^b(\mathcal{O}^{\kappa'})$ note that these categories are isomorphic to the derived categories of sheaves on partial flag varieties $Y_\kappa$ and $Y_{\kappa'}$, smooth along Schubert stratifications. It suffices to treat the case when $\kappa$ and $\kappa'$ differ by a transposition of adjacent terms, $\kappa = (k_1, \ldots, k_m), \kappa' = (k_1, \ldots, k_{i-1}, k_i, \ldots, k_m)$. Let $U \subset Y_\kappa \times Y_{\kappa'}$ be the set

$$(F_1, \ldots, F_m) \in Y_\kappa, (F'_1, \ldots, F'_m) \in Y_{\kappa'}) | F_j = F'_j \text{ for } j \neq i, F_i \cap F'_i = F_{i-1},$$

of pairs of partial flags. Let $\mathcal{G}$ be sheaf on $Y_\kappa \times Y_{\kappa'}$ which is the continuation by 0 of the constant sheaf on $U$. Convolution with $\mathcal{G}$ is an equivalence of
derived categories of sheaves on $Y_\kappa$ and $Y_{\kappa'}$, and restricts to an equivalence of subcategories of cohomologically constructible (relative to the Schubert startification) complexes of sheaves. The latter categories are equivalent to the derived categories of $\mathcal{O}_\kappa$ and $\mathcal{O}_{\kappa'}$. □

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