Abstract

Submodular optimization has numerous applications such as crowdsourcing and viral marketing. In this paper, we study the fundamental problem of non-negative submodular function maximization subject to a \( k \)-system constraint, which generalizes many other important constraints in submodular optimization such as cardinality constraint, matroid constraint, and \( k \)-extendible system constraint. The existing approaches for this problem achieve the best-known approximation ratio of \( k + 2\sqrt[k]{k+2} + 3 \) (for a general submodular function) based on deterministic algorithmic frameworks. We propose several randomized algorithms that improve upon the state-of-the-art algorithms in terms of approximation ratio and time complexity, both under the non-adaptive setting and the adaptive setting. The empirical performance of our algorithms is extensively evaluated in several applications related to data mining and social computing, and the experimental results demonstrate the superiorities of our algorithms in terms of both utility and efficiency.

1. Introduction

Submodular optimization is an active research area in machine learning due to its wide applications such as crowdsourcing (Singla et al., 2016; Han et al., 2018a), clustering (Gomes & Krause, 2010; Han et al., 2019), viral marketing (Kempe et al., 2003; Han et al., 2018b), and data summarization (Badanidiyuru et al., 2014; Iyer & Bilmes, 2013). A lot of the existing studies in this area aim to maximize a submodular function subject to a specific constraint, and it is well known that these problems are generally NP-hard. Therefore, extensive approximation algorithms have been proposed, with the goal of achieving improved approximation ratios or lower time complexity.

Formally, given a ground set \( \mathcal{N} \) with \( |\mathcal{N}| = n \), a constrained submodular maximization problem can be written as:

\[
\max\{f(S) : S \in \mathcal{I}\}
\]

where \( f : 2^\mathcal{N} \mapsto \mathbb{R}_{\geq 0} \) is a submodular function satisfying \( \forall X, Y \subseteq \mathcal{N} : f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \), and \( \mathcal{I} \subseteq 2^\mathcal{N} \) is the set of all feasible solutions. For example, if \( \mathcal{I} = \{X : X \subseteq \mathcal{N} \land |X| \leq d\} \) for a given \( d \in \mathbb{N} \), then \( S \in \mathcal{I} \) represents a cardinality constraint. We also call \( f(\cdot) \) “monotone” if it satisfies \( \forall X \subseteq Y \subseteq \mathcal{N} : f(X) \leq f(Y) \), otherwise \( f(\cdot) \) is called “non-monotone”.

Although some application problems only have simple constraints like a cardinality constraint, many others have to be cast as submodular maximization problems with more complex “independence system” constraints such as matroid, \( k \)-matchoid, and \( k \)-system constraint. Among these constraints, the \( k \)-system constraint is the most general one, and a strict inclusion hierarchy of them is: cardinality \( \subset \) matroid \( \subset \) intersection of \( k \) matroids \( \subset k \)-matchoid \( \subset k \)-extendible \( \subset k \)-system (Mestre, 2006). Due to the generality of \( k \)-system constraint, it can be used to model a lot of constraints in various applications, such as graph matchings, spanning trees and scheduling (Feldman et al., 2020; Mirzasoleiman et al., 2016).

It is recognized that submodular maximization with a \( k \)-system constraint is one of the most fundamental problems in submodular optimization (Calinescu et al., 2011; Feldman et al., 2017; 2020), so a lot of efforts have been devoted to it since

1School of Computer Science and Technology / SuZhou Research Institute, University of Science and Technology of China; 2Data Science and Analytics Thrust, The Hong Kong University of Science and Technology; 3School of Computer Science and Technology, Soochow University. Correspondence to: Kai Han <hankai@ustc.edu.cn>.

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The Power of Randomization: Efficient and Effective Algorithms for Constrained Submodular Maximization

In the 1970s, and the state-of-the-art approximation ratios are $k + 1$ (Fisher et al., 1978) and $k + 2\sqrt{k + 2} + 3$ (Feldman et al., 2020) for monotone $f(\cdot)$ and non-monotone $f(\cdot)$, respectively. Feldman et al. (2020) also showed that, by weakening their approximation ratio by a factor of $(1 - 2\epsilon)^{-2}$, their algorithm can be implemented under time complexity of $O(\frac{n}{\epsilon^2} \log(\frac{k}{\epsilon}))$.

Surprisingly, all the existing algorithms with nice approximation ratios for this problem are intrinsically deterministic. Therefore, it is an interesting open problem whether the “power of randomization” can be leveraged to achieve better approximation ratios or better efficiency, as randomized algorithms are known to outperform the deterministic ones in many other problems.

It is noted that the utility function $f(\cdot)$ is assumed to be deterministic in Problem (1). However, in many applications such as viral marketing and sensor placement, the utility function could be stochastic and is only submodular in a probabilistic sense. To address these settings, Golovin & Krause (2011a) introduced the concept of adaptive submodular maximization, where each element $u \in \mathcal{N}$ is assumed to have a random state and the goal is to find an optimal adaptive policy that can select a new element based on observing the realized states of already selected elements. Based on this concept, they also investigated the adaptive submodular maximization problem under a $k$-system constraint and provide an approximation ratio of $k + 1$ (Golovin & Krause, 2011b), but this ratio only holds when the utility function is adaptive monotone (a property similar to the monotonicity property under the non-adaptive case). However, it still remains as an open problem whether provable approximation ratios can be achieved for this problem when the considered utility function is more general (i.e., not necessarily adaptive monotone).

In this paper, we provide confirmative answers to all the open problems mentioned above, by presenting novel randomized algorithms for the problem of (not necessarily monotone) submodular function maximization with a $k$-system constraint. Our algorithms advance the state-of-the-art under both the non-adaptive setting and the adaptive setting. More specifically, our contributions include:

- Under the non-adaptive setting, we present a randomized algorithm dubbed RANDOMMULTIGREEDY that achieves an approximation ratio of $(1 + \sqrt{k})^2$ under time complexity of $O(nr)$, where $r$ is the rank of the considered $k$-system. We also show that RANDOMMULTIGREEDY can be accelerated to achieve an approximation ratio of $(1 + \epsilon)(1 + \sqrt{k})^2$ under nearly-linear time complexity of $O(\frac{n}{\epsilon^2} \log(\frac{k}{\epsilon}))$. Therefore, our algorithm outperforms the state-of-the-art algorithm in (Feldman et al., 2020) in terms of both approximation ratio and time complexity. Furthermore, we show that RANDOMMULTIGREEDY can also be implemented as a deterministic algorithm with better performance bounds than the existing algorithms.

- Under the non-adaptive setting, we also propose a randomized algorithm dubbed BATCHEDRANDOMGREEDY which achieves a slightly worse approximation ratio of $(1 + \epsilon)^2(1 + \sqrt{k + 1})^2$ but can be implemented in poly-logarithmic “adaptive rounds”. This result greatly improves upon the best-known approximation ratio of $1 + \epsilon(1 + \sqrt{k + 2(1 + 1)} + 5)$ achieved by the state-of-the-art algorithm with poly-logarithmic adaptivity proposed in (Quinzian et al., 2021).

- Under the adaptive setting, we provide a randomized policy dubbed ADAPTRANDOMGREEDY that achieves an approximation ratio of $(1 + \sqrt{k + 1})^2$ when the utility function is not necessarily adaptive monotone. To the best of our knowledge, ADAPTRANDOMGREEDY is the first adaptive algorithm to achieve a provable performance ratio under this case.

- We test the empirical performance of the proposed algorithms in several applications including movie recommendation, image summarization and social advertising with multiple products. The extensive experimental results demonstrate that, RANDOMMULTIGREEDY achieves approximately the same performance as the best existing algorithm in terms of utility, while its performance on efficiency is much better than that of the fastest known algorithm; besides, ADAPTRANDOMGREEDY can achieve better utility than the non-adaptive algorithms by leveraging adaptivity.

For the fluency of description, the proofs of all our lemmas/theorems are deferred to the supplementary file.

2. Related Work

There are extensive studies on submodular maximization such as (Chekuri & Quanrud, 2019; Balkanski et al., 2019; Lee et al., 2010; Han et al., 2021; Kuhnle, 2019). For example, Kuhnle (2019) addressed a simple cardinality constraint using a nice “interlaced greedy” algorithm, where two candidate solutions are considered in a compulsory round-robin way; but it is unclear whether this algorithm can handle more complex constraints. In the sequel, we only review the studies most closely related to our work.
Krause (2011a) initiated the study on adaptive submodular maximization and also provided several algorithms under the best performance bounds under a system constraint in (Golovin & Krause, 2011b). After that, an Unconstrained Submodular Maximization (USM) algorithm (e.g., (Buchbinder et al., 2015)) is called to find $S_j' \subseteq S_j$ for all $j \in [t]$. Finally, the set in $\{S_j, S_j': j \in [t]\}$ with the maximum utility is returned. Note that the USM algorithm is only used as a “black-box” oracle and can be any deterministic/randomized algorithm, so this algorithmic framework is intrinsically deterministic. Gupta et al. (2010) showed that, by setting $\ell = k + 1$, REPEATEDGREEDY can achieve an approximation ratio of $3k + 6 + 3k^{-1}$ under $O(nrk)$ time complexity. However, through a more careful analysis, Mirzasoleiman et al. (2016) proved that REPEATEDGREEDY actually has an approximation ratio of $2k + 3 + k^{-1}$. Subsequently, Feldman et al. (2017) further revealed that REPEATEDGREEDY can achieve an approximation ratio of $k + 2\sqrt{k} + 3 + \frac{6}{\sqrt{k}}$ under $O(nr/\sqrt{k})$ time complexity by setting $\ell = \lceil \sqrt{k} \rceil$.

Han et al. (2020) proposed a different “simultaneous greedy search” framework, where two disjoint candidate solutions $S_1$ and $S_2$ are maintained simultaneously, and the algorithm always greedily selects a pair $(e, S_i)$ such that adding $e$ into $S_i$ brings the maximum marginal gain. By incorporating a “thresholding” method akin to that in (Badanidiyuru & Vondrák, 2014), Han et al. (2020) proved that their algorithm achieves $(2k + 2 + \epsilon)$-approximation under $O(\frac{k}{\epsilon} \log k)$ time complexity. Feldman et al. (2020) also proposed an elegant algorithm where $\lceil 2 + k \rceil$ disjoint candidate solutions are maintained. By leveraging a thresholding method similar to (Badanidiyuru & Vondrák, 2014; Han et al., 2020), Feldman et al. (2020) proved that their algorithm can achieve $(1 - 2\epsilon)^{-2}(k / 2\sqrt{k} + 2 + 3)$-approximation under $O(\frac{2k}{\epsilon} \log k)$ time complexity.

On the hardness side, Feldman et al. (2017) proved that no algorithm making polynomially many queries to the value and independence oracles can achieve an approximation better than $k + 0.5 - \epsilon$. For clarity, we list the performance bounds of the closely related algorithms mentioned above in Table 1.

Recently, Balkanski & Singer (2018) proposed the concept of “adaptivity” for submodular optimization algorithms: an algorithm has $T$ adaptivity if it can be implemented in $T$ “adaptive rounds”, where the algorithm is allowed to make polynomial number of independent queries to the function value of $f(\cdot)$ in each adaptive round. Based on this concept, a lot of studies (e.g., (Balkanski et al., 2019; Chekuri & Quanrud, 2019; Fahrbach et al., 2019b; Quinzan et al., 2021)) have proposed algorithms with low-adaptivity under various constraints. Among these studies, (Quinzan et al., 2021) achieved the best performance bounds under a $k$-system constraint. More specifically, the REP-SAMPLING algorithm in (Quinzan et al., 2021) achieves an approximation ratio of $\frac{1 + \epsilon}{(1 - \epsilon)^2}(k + 2\sqrt{2(k + 1)} + 5)$ with $O(\sqrt{\frac{k}{\epsilon}} \log k \log n \log r)$ adaptivity, while incurring at most $O(\sqrt{\frac{k}{\epsilon}} \log \frac{k}{\epsilon} \log n \log r)$ queries to the function value of $f(\cdot)$ in expectation. Compared to REP-SAMPLING, our BATCHED-RANDOMGREEDY algorithm achieves a much better approximation ratio of $(1 + \epsilon)^2(1 + \sqrt{k + 1})^2$, under nearly the same adaptivity of $O(\sqrt{\frac{k}{\epsilon}} \log \frac{k}{\epsilon} \log n \log r)$ and query complexity of $O(\sqrt{\frac{k}{\epsilon}} \log \frac{k}{\epsilon} \log n \log r)$.

Adaptive Algorithms: We then provide a brief review on the related studies on adaptive submodular maximization. Golovin & Krause (2011a) initiated the study on adaptive submodular maximization and also provided several algorithms under cardinality or knapsack constraints. They also studied the more general $k$-system constraint in (Golovin & Krause, 2011b) and provided a $(k + 1)$-approximation. Recently, Esfandiari et al. (2021) proposed adaptive submodular maximization algorithms with fewer adaptive rounds of observation. There also exist many other studies on adaptive optimization under $O(knr)$ time complexity, while RANDOMMULTIGREEDY achieves an approximation ratio of $(1 + \sqrt{k})^2$ under $O(nr)$ time complexity.

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1For simplicity, we only list the performance bounds of the accelerated versions of Feldman et al. (2020)’s algorithm and RANDOMMULTIGREEDY in Table 1. Without acceleration, Feldman et al. (2020) can achieve an approximation ratio of $(1 + \sqrt{k + 2})^2$ under $O(knr)$ time complexity, while RANDOMMULTIGREEDY achieves an approximation ratio of $(1 + \sqrt{k})^2$ under $O(nr)$ time complexity.
3. Preliminaries and Notations

It is well known that all the structures including matroid, $k$-matchoid, $k$-extendible system and $k$-set system are set systems obeying the "down-closed" property captured by the concept of an independence system:

**Definition 1 (independence system).** Given a finite ground set $\mathcal{N}$ and a collection of sets $\mathcal{I} \subseteq 2^{\mathcal{N}}$, the pair $(\mathcal{N}, \mathcal{I})$ is called an independence system if it satisfies: (1) $\emptyset \in \mathcal{I}$; (2) if $X \subseteq Y \subseteq \mathcal{N}$ and $Y \in \mathcal{I}$, then $X \in \mathcal{I}$.

Given an independence system $(\mathcal{N}, \mathcal{I})$ and any two sets $X \subseteq Y \subseteq \mathcal{N}$, $X$ is called a base of $Y$ if $X \in \mathcal{I}$ and $X \cup \{u\} \notin \mathcal{I}$ for all $u \in Y \setminus X$. We also use $r$ to denote the rank of $(\mathcal{N}, \mathcal{I})$, i.e., $r = \max\{|X| : X \in \mathcal{I}\}$. A $k$-system is a special independence system defined as:

**Definition 2 (k-system).** An independence system $(\mathcal{N}, \mathcal{I})$ is called a k-system ($k \geq 1$) if $|X_1| \leq k|X_2|$ holds for any two bases $X_1$ and $X_2$ of any set $Y \subseteq \mathcal{N}$.

### 3.1. Non-adaptive Setting

Under the non-adaptive setting, our problem is to identify an optimal solution $O$ to Problem (1) given a $k$-system $(\mathcal{N}, \mathcal{I})$ and a (not necessarily monotone) submodular function $f(\cdot)$. For convenience, we use $f(X | Y)$ as a shorthand for $f(X \cup Y) - f(Y)$ for all $X, Y \subseteq \mathcal{N}$. It is well known that any non-negative submodular function $f(\cdot)$ satisfies the "diminishing returns" property: $\forall X \subseteq Y \subseteq \mathcal{N}, x \in \mathcal{N} \setminus Y : f(x | Y) \leq f(x | X)$. Following the existing studies, we assume that the values of $f(S)$ and $1_Z(S)$ can be got by calling oracle queries, and use the number of oracle queries to measure time complexity unless otherwise stated.

Following some related work such as (Balkanski et al., 2019; Chekuri & Quanrud, 2019; Fahrbach et al., 2019b; Quinzan et al., 2021), we define the "adaptivity" of an algorithm under the non-adaptive setting as the minimum number of "adaptive rounds" needed by the algorithm, such that the algorithm can make polynomially-many independent queries to the value of the objective function $f(\cdot)$ in each "adaptive round".

### 3.2. Adaptive Setting

Under the adaptive setting, each element $u \in \mathcal{N}$ is associated with an initially unknown state $\Phi(u) \in Z$, where $Z$ is the set of all possible states. A realization is any function $\phi: \mathcal{N} \rightarrow Z$ mapping every element $u \in \mathcal{N}$ to a state $z \in Z$. Therefore, $\Phi$ is the true realization and we follow (Golovin & Krause, 2011a) to assume that $\Pr[\Phi = \phi]$ is known for any possible realization $\phi$. In adaptive optimization problems, an adaptive policy $\pi$ is allowed to sequentially select elements in $\mathcal{N}$, and the true state $\Phi(u)$ of any $u \in \mathcal{N}$ can only be observed after $u$ is selected. In such a case, the utility of $\pi$ depends on not only the selected elements but also their states, so we re-define the utility function as $f: 2^\mathcal{N} \times Z^\mathcal{N} \rightarrow \mathbb{R}_{\geq 0}$. Let $\mathcal{N}(\pi, \phi)$ denote the set of elements selected by $\pi$ under any realization $\phi$, the expected utility of policy $\pi$ is defined as

$$f_{avg}(\pi) := \mathbb{E}[f(\mathcal{N}(\pi, \phi), \Phi)],$$

where the expectation is taken over both the randomness of $\Phi$ and the internal randomness (if any) of $\pi$.

Given any $M \subseteq \mathcal{N}$, a mapping $\psi: M \rightarrow Z$ is called a partial realization, and $\text{dom}(\psi) = M$ is called the domain of $\psi$. Therefore, a partial realization is $\psi$ is also a realization when $\text{dom}(\psi) = \mathcal{N}$. Intuitively, a partial realization can be used to record the already selected elements and the observed states of them during the execution of an adaptive policy. We also abuse the notations a little by regarding $\psi$ as the set $\{(u, \psi(u)): u \in \text{dom}(\psi)\}$. Given two partial realizations $\psi$ and $\psi'$, we say $\psi$ is a subrealization of $\psi'$ (denoted by $\psi' \sim \psi$) if $\psi \subseteq \psi'$. With these definitions, we follow Golovin & Krause (2011a) to define the concept of adaptive submodularity:

**Definition 3.** Given a partial realization $\psi$ and an element $u$, the expected marginal gain of $u$ conditioned on $\psi$ is defined as

$$\Delta(u | \psi) = \mathbb{E}[f(\text{dom}(\psi) \cup \{u\}, \Phi) - f(\text{dom}(\psi), \Phi) | \Phi \sim \psi].$$

A function $f: 2^{\mathcal{N}} \times Z^{\mathcal{N}} \rightarrow \mathbb{R}_{\geq 0}$ is called adaptive submodular if it satisfies $\forall \psi \subseteq \psi', u \in \mathcal{N}\setminus \text{dom}(\psi'): \Delta(u | \psi) \geq \Delta(u | \psi')$. 


The utility function $f(\cdot)$ is also called adaptive monotone if $\Delta(u \mid \psi) \geq 0$ for any $u \in \mathcal{N}$ and any partial realization $\psi$ satisfying $\Pr[\Phi = \psi] > 0$. However, in this paper we consider the case that $f(\cdot)$ is not necessarily adaptive monotone. In such a case, all the the current studies (e.g., (Amanatidis et al., 2020; Gotovos et al., 2015)) assume that $f(\cdot)$ is also pointwise submodular, whose definition is given below:

**Definition 4.** A function $f: 2^\mathcal{N} \times Z^\mathcal{N} \mapsto \mathbb{R}_{\geq 0}$ is pointwise submodular if $f(\cdot, \phi)$ is submodular for any realization $\phi$ satisfying $\Pr[\Phi = \phi] > 0$.

Given a $k$-system $(\mathcal{N}, \mathcal{I})$ and an adaptive and pointwise submodular function $f: 2^\mathcal{N} \times Z^\mathcal{N} \mapsto \mathbb{R}_{\geq 0}$, our problem is to identify an optimal policy $\pi_{opt}$ to the following adaptive optimization problem:

$$\max\{f_{avg}(\pi): \mathcal{N}(\pi, \phi) \in \mathcal{I} \text{ for all realization } \phi\}. \quad (2)$$

### 4. A Randomized Algorithm under the Non-Adaptive Setting

In this section, we propose an algorithm dubbed **RANDOMMULTIGREEDY**, as shown by Algorithm 1. **RANDOMMULTIGREEDY** iterates for $T$ steps to construct $\ell$ candidate solutions $S_1, S_2, \ldots, S_\ell$. At each step $t$, it greedily finds a pair $(u_t, i_t)$ such that $S_{i_t} \cup \{u_t\} \in \mathcal{I}$ and $f(u_t \mid S_{i_t})$ is maximized. If $f(u_t \mid S_{i_t}) > 0$, then Algorithm 1 adds $u_t$ into $S_{i_t}$ with probability $p$ and discard $u_t$ with probability $1 - p$. After that, $u_t$ is removed from $\mathcal{N}$. The iterations stop immediately when the pair $(u_t, i_t)$ cannot be found or $f(u_t \mid S_{i_t}) \leq 0$.

For convenience, we introduce the following notations. Let $\mathcal{U} = \{u_1, \ldots, u_T\}$ denote the set of all elements that have been considered to be added into $\cup_{i \in [\ell]} S_i$. For any $u \in \mathcal{N}$, let $S_i^u(u)$ denote the set of elements already in $S_i$ at the moment that $u$ is considered by the algorithm, and let $S_i^u(u) = S_i$ if $u$ is never considered by the algorithm.

Although the design of **RANDOMMULTIGREEDY** is quite simple, its performance analysis is highly non-trivial due to the complex relationships between the elements in $S_1, \ldots, S_\ell$ and the randomness of the algorithm. To address these challenges, we first classify the elements in $\mathcal{O}$ as follows:

**Definition 5.** Let $D_j$ denote the set of elements in $\mathcal{U}$ that have been considered to be added into $S_j$ but are discarded due to
Line 9. For any \(i, j \in \{\ell\}\) satisfying \(i \neq j\), we define:

\[
\begin{align*}
O_{i}^{+} & = \{ u \in O \cap S_{j} : S_{i}^{\leq}(u) \cup \{ u \} \in \mathcal{I} \}; \\
O_{i}^{-} & = \{ u \in O \cap S_{j} : S_{i}^{\leq}(u) \cup \{ u \} \notin \mathcal{I} \}; \\
\hat{O}_{i}^{+} & = \{ u \in O \cap D_{j} : S_{i}^{\leq}(u) \cup \{ u \} \in \mathcal{I} \}; \\
\hat{O}_{i}^{-} & = \{ u \in O \cap D_{j} : S_{i}^{\leq}(u) \cup \{ u \} \notin \mathcal{I} \}; \\
O_{i}^{-} & = \{ u \in O \setminus U : S_{i} \cup \{ u \} \notin \mathcal{I} \wedge f(u \mid S_{i}) > 0 \};
\end{align*}
\]

Note that both \(O_{i}^{+}\) and \(O_{i}^{-}\) are disjoint subsets of \(O \cap S_{j}\). Intuitively, each element \(u \in O_{i}^{+}\) (resp. \(u \in O_{i}^{-}\)) can (resp. cannot) be added into \(S_{i}\) without violating the feasibility of \(\mathcal{I}\) at the moment that \(u\) is added into \(S_{j}\). The sets \(\hat{O}_{i}^{+}\), \(\hat{O}_{i}^{-}\) are also defined similarly for the elements in \(O \cap D_{j}\). Based on Definition 5, it can be seen that, when Algorithm 1 terminates, all the elements in \(O_{i}^{-}\), \(O_{i}^{-}\) and \(\hat{O}_{i}^{-}\) \((\forall j \neq i)\) cannot be added into \(S_{i}\) due to the violation of \(\mathcal{I}\). Note that these elements together with the elements in \(O \cap S_{j}\) all belong to \(O\). So we can map them to the elements in \(S_{i}\) using a method similar to that in (Calinescu et al., 2011; Han et al., 2020) based on the definition of \(k\)-system, as shown by Lemma 1:

**Lemma 1.** For each \(i \in \{\ell\}\), let \(Q_{i} = \cup_{j \in \{\ell\} \setminus \{i\}} (O_{j}^{-} \cup \hat{O}_{j}^{-}) \cup (O \cap S_{j}) \cup O_{i}^{-}\). There exists a mapping \(\sigma_{i} : Q_{i} \rightarrow S_{i}\) satisfying: (1) The element \(\sigma_{i}(u)\) can be added into \(S_{i}^{\leq}(\sigma_{i}(u))\) without violating the feasibility of \(\mathcal{I}\) for all \(u \in Q_{i}\); (2) The number of elements in \(Q_{i}\) mapped to the same element in \(S_{i}\) by \(\sigma_{i}(\cdot)\) is no more than \(k\); and (3) there have \(\forall u \in O \cap S_{i} : \sigma_{i}(u) = u\).

The purpose for creating the mapping in Lemma 1 is to bound the value of \(f(u \mid S_{i})\) for all \(u \in Q_{i}\). For example, given any element \(u \in O_{i}^{-}\), we can map it to an element \(v = \sigma_{i}(u)\) satisfying \(S_{i}^{\leq}(v) \cup \{ u \} \notin \mathcal{I}\), which implies that the value of \(f(u \mid S_{i})\) is no more than \(f(v \mid S_{i}^{\leq}(v))\), because otherwise \(u\) should have been added into \(S_{i}\) instead of \(v\) according to the greedy rule of Algorithm 1. Based on this intuition, a more careful analysis reveals that:

**Lemma 2.** For any \(u \in U\) where \(t \in [T]\), we define \(\delta(u) = \sum_{i=1}^{t} \mathbb{1}\{i = j\} \cdot f(u \mid S_{i}^{\leq}(u))\). Given any \(i, j \in \{\ell\}\) satisfying \(i \neq j\), we have

\[
\begin{align*}
\forall u \in O_{i}^{+} \cup \hat{O}_{i}^{+} : f(u \mid S_{i}) & \leq \delta(u); \\
\forall u \in Q_{i} : f(u \mid S_{i}) & \leq \delta(\pi_{i}(u));
\end{align*}
\]

where \(Q_{i}\) is defined in Lemma 1.

Using Lemma 2, we can prove Lemma 3, which provides an upper bound of \(\sum_{i \in \{\ell\}} f(O \mid S_{i})\):

**Lemma 3.** For any \(u \in \mathcal{N}\), define \(X_{u} = 1\) if \(u \in (U \cap O) \cup \cup_{i=1}^{t} S_{i}\), otherwise define \(X_{u} = 0\). Given any integer \(\ell \geq 2\), we have

\[
\sum_{i \in \{\ell\}} f(O \mid S_{i}) \leq \ell(k + \ell - 2) f(S^{\ast}) + \ell \sum_{u \in \mathcal{N}} X_{u} \cdot \delta(u) \tag{5}
\]

The proof idea of Lemma 3 is roughly explained as follows. As the elements in \(O \setminus S_{i}\) can be classified using the sets defined in Definition 5, we can leverage Lemma 1 and Lemma 2 to bound \(f(u \mid S_{i})\) for all \(u \in O \setminus S_{i}\) using the marginal gains of the elements in \(\cup_{i=1}^{t} S_{i}\). These marginal gains are further grouped in a subtle way such that their summation can be bounded by the RHS of Eqn. (5).

Note that Eqn. (5) holds for every random output of Algorithm 1. So the inequality still holds after taking expectation. Furthermore, we introduce Lemma 4 to bound the expectations of the LHS and RHS of Eqn. (5). The proof of Lemma 4 leverages the property that each element in \(\mathcal{N}\) is only accepted with probability of at most \(p\).

**Lemma 4.** For any \(p \in (0, 1]\), we have

\[
\begin{align*}
\mathbb{E}\left[ \sum_{i \in \{\ell\}} f(O \cup S_{i}) \right] & \geq (\ell - p) f(O) \tag{6} \\
\mathbb{E}\left[ \sum_{u \in \mathcal{N}} X_{u} \cdot \delta(u) \right] & \leq \frac{1 - p}{p} \mathbb{E}\left[ \sum_{i \in \{\ell\}} f(S_{i}) \right] \tag{7}
\end{align*}
\]
By combining Lemma 3, Lemma 4 and the fact that \( \forall i \in [\ell]: f(S_i) \leq f(S^*) \), we can immediately get the approximation ratio of Algorithm 1 as follows:

**Theorem 1.** For any \( \ell \geq 2 \) and \( p \in (0, 1] \), the \textsc{RandomMultiGreedy}(\( \ell, p \)) algorithm outputs a solution \( S^* \) satisfying

\[
f(O) \leq \frac{\ell(k + \ell - 1)}{\ell - p} \mathbb{E}[f(S^*)]
\]

**Discussion of Theorem 1:** From Theorem 1, it can be seen that the approximation ratio of Algorithm 1 can be optimized by choosing proper values of \( \ell \) and \( p \). Indeed, the ratio can be minimized to \( (1 + \sqrt{k})^2 \) by setting \( \ell = 2 \), \( p = \frac{2}{1+\sqrt{k}} \). Besides, if we set \( \ell = \lceil \sqrt{k} \rceil + 1 \), \( p = 1 \), then the approximation ratio turns into \( k + \sqrt{k} + \lceil \sqrt{k} \rceil + 1 \). Clearly, setting \( \ell = 2 \) implies faster running time as only two candidate solutions are maintained, while setting \( p = 1 \) implies a deterministic algorithm.

### 4.1. Acceleration

It can be seen that \textsc{RandomMultiGreedy} has time complexity of \( O(\ell nr) \). This time complexity can be further reduced by implementing Lines 2-5 using a “lazy evaluation” method inspired by (Minoux, 1978; Ene & Nguyen, 2019). More specifically, for each solution set \( S_i \), we maintain an ordered list \( A_i \) which is initialized to \( N \). Each element \( u \in A_i \) has a weight \( w_i(u) = f(u | S_i) \) and the elements in \( A_i \) are always sorted according to the non-increasing order of their weights. When \( S_i \) changes, we pop out the top element \( u \) from \( A_i \) and discard \( u \) if \( S_i \cup \{u\} \notin \mathcal{I} \). If \( S_i \cup \{u\} \in \mathcal{I} \) and \( f(u | S_i) \) has not been computed, then we update the weight of \( u \) and set \( v_i = u \) if the new weight of \( u \) is at least \( (1 + \epsilon)^{-1} \) fraction of its old weight (otherwise \( u \) is re-inserted into \( A_i \) and we pop out the next element). During this process, any element in \( A_i \) is removed from \( A_i \) immediately when its weight has been updated for more than \( O\left(\frac{\epsilon}{2} \log \frac{\epsilon}{2} \right) \) times. Using this method, we can guarantee that \( f(v_i | S_i) \) is at least \( \frac{1}{1+\epsilon} \) fraction of the marginal gain of the best element in \( A_i \), that can be added into \( S_i \), and the total number of incurred value and independence oracles is no more than \( O\left(\frac{\epsilon}{2} \log \frac{\epsilon}{2} \right) \) for each \( S_i : i \in [\ell] \). Combining these results with Theorem 1, we can get:

**Theorem 2.** For the problem of submodular maximization subject to a \( k \)-system constraint, there exist: (1) a randomized algorithm with an approximation ratio of \( (1 + \epsilon)(1 + \sqrt{k})^2 \) under \( O\left(\frac{\epsilon}{2} \log \frac{\epsilon}{2} \right) \) time complexity, and (2) a deterministic algorithm with an approximation ratio of \( (1 + \epsilon)(k + \sqrt{k} + \lceil \sqrt{k} \rceil + 1) \) under \( O\left(\frac{\epsilon}{2} \log \frac{\epsilon}{2} \right) \) time complexity.

**Remark:** From Theorem 2, it can be seen that Algorithm 1 actually can be regarded as a “universal algorithm” that achieves the best-known performance bounds under different settings, as explained in the following. First, if \( f(\cdot) \) is non-monotone, then Algorithm 1 outperforms the state-of-the-art algorithm of (Feldman et al., 2020) in terms of both approximation ratio and time efficiency, no matter Algorithm 1 is implemented as a randomized algorithm or as a deterministic algorithm; moreover, when \( (\mathcal{N}, \mathcal{I}) \) is a matroid (i.e., \( k = 1 \)), Algorithm 1 achieves an approximation ratio of \( 4 + \epsilon \) under \( O\left(\frac{\epsilon}{2} \log \frac{\epsilon}{2} \right) \) time complexity, matching the performance bounds of the fastest algorithm in (Han et al., 2020) for a matroid constraint. Second, if the considered submodular function \( f(\cdot) \) is monotone, it can be easily seen that the standard greedy algorithm proposed in (Fisher et al., 1978) equals to \textsc{RandomMultiGreedy}(1, 1), so Algorithm 1 also achieves the best-known approximation ratio of \( k + 1 \) under this case.

### 5. A Randomized Non-Adaptive Algorithm with Lower Adaptivity

It can be easily seen that the \textsc{RandomMultiGreedy} algorithm has \( O(r) \)-adaptivity, as only one element is added into the candidate sets at each time. As \( r \) can be in the order of \( O(n) \), \textsc{RandomMultiGreedy} has a large adaptivity. In this section, we propose a new algorithm dubbed \textsc{BatchedRandomGreedy} with lower adaptivity, as shown by Algorithm 3.

In contrast to \textsc{RandomMultiGreedy}, the \textsc{BatchedRandomGreedy} algorithm maintains only one candidate solution \( S \), and uses a threshold \( \tau \) ranges from \( \tau_{\text{max}} \) to \( \tau_{\text{min}} \) to control the quality of the elements added into \( S \). Given a threshold \( \tau \), \textsc{RandomMultiGreedy} first finds a set \( C \) of elements which have not been considered yet, such that each element in \( C \) can be added into \( S \) without violating \( \mathcal{I} \) to bring a marginal gain no less than \( \tau \), and then runs in iterations and and tries to add a batch of elements into \( S \) in each iteration, as shown by Lines 4-11. More specifically, in each iteration, Algorithm 3 first selects a random sequence \( \{a_1, \ldots, a_d\} \) by calling the \textsc{RandomSeq} procedure, and then uses binary search to find a sub-sequence \( \{a_1, \ldots, a_j\} \) of \( \{a_1, \ldots, a_d\} \) such that at least a \( \left(\frac{1}{2j} \right) \)-fraction of the elements in \( C \) can be dropped if \( \{a_1, \ldots, a_j\} \) is added into \( S \); finally, it adds \( \{a_1, \ldots, a_j\} \) into \( S \) with probability of \( p \) and updates \( C \) accordingly. These
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Algorithm 2 `RANDSEQ(S, C)`

Initialize: \( A \leftarrow \emptyset \)

1: while \( C \neq \emptyset \) do
2: \( s \leftarrow \max \{i \in [\lvert C \rvert]: S \cup A \cup \{z_1, \ldots, z_s\} \in I\} \)
3: \( A \leftarrow A \cup \{z_1, \ldots, z_s\}; \)
4: \( C \leftarrow \{u : u \in C \setminus A \land S \cup A \cup \{u\} \in I\}; \)
5: end while
6: return \( A \)

Algorithm 3 `BATCHEDRANDOMGREEDY(p, \epsilon)`

Initialize: \( \tau_{\text{max}} \leftarrow \max \{f(u \mid \emptyset): u \in N\}; \tau_{\text{min}} \leftarrow \epsilon \cdot \tau_{\text{max}}/r; S \leftarrow \emptyset; U \leftarrow \emptyset \)

1: for \((\tau \leftarrow \tau_{\text{max}}; \tau \geq \tau_{\text{min}}; \tau \leftarrow \frac{\tau}{1 + \epsilon})\) do
2: \( C \leftarrow \{u \in N \setminus U : S \cup \{u\} \in I \land f(u \mid S) \geq \tau\} \)
3: while \( C \neq \emptyset \) do
4: \( \{a_1, a_2, \ldots, a_d\} \leftarrow \text{RandSEQ}(S, C) \)
5: Define \( C_i = \{u : u \in C \land S \cup \{a_1, \ldots, a_i, u\} \in I \land f(u \mid S \cup \{a_1, \ldots, a_i\}) \geq \tau\} \) for each \( i \in \{0, \ldots, d\} \)
6: Using binary search to find \( j = \min \{i \in [d] : \lvert C_i \rvert < \frac{\lvert C_i \rvert}{1 + \epsilon}\} \)
7: \( U \leftarrow U \cup \{a_1, \ldots, a_j\} \)
8: with probability \( p \) do
9: \( S \leftarrow S \cup \{a_1, \ldots, a_j\}; C \leftarrow C_j; \)
10: otherwise
11: \( C \leftarrow C_0 \setminus U; \)
12: end while
13: end for
14: return \( S \)

iterations terminates when \( C = \emptyset \), and then `BATCHEDRANDOMGREEDY` reduces \( \tau \) by a factor of \( \frac{1}{1 + \epsilon} \) and repeats the above process.

Note that the `RANDSEQ` procedure is inspired by Karp et al. (1988), whose goal is to find a subset \( A \) of \( C \) with maximum cardinality such that \( S \cup A \in I \). It adopts a simple idea of repeatedly adding a random sequence of elements into \( A \) until no more elements can be added without violating \( I \). Karp et al. (1988) have proved that `RANDSEQ` can terminate by repeating the while-loop in it for at most \( O(\sqrt{n}) \) times.

5.1. The Approximation Ratio of `BATCHEDRANDOMGREEDY`

In this section, we analyze the approximation ratio of the `BATCHEDRANDOMGREEDY` algorithm. For clarity, we first introduce the following notations. When the algorithm finishes, let \( \{e_1, e_2, \ldots, e_r\} \) denote the set of elements sequentially added into \( S \), and let \( S_i \) denote the set \( \{e_1, \ldots, e_i\} \) for any \( i \in [r] \). As it is possible that \( |S| < r \), we define \( e_i \) as a “dummy element” satisfying \( f(e_i \mid S_{i-1}) = 0 \) if \( |S| < t \leq r \). Similar to that in Sec. 4, for any \( u \in U \), we use \( S^r(u) \) to denote the set of elements already added into \( S \) before considering \( u \) (note that \( U \) is defined in Algorithm 3, denoting the set of all elements considered by the algorithm to be added into \( S \)). For any \( \tau \) considered in the for-loop of Algorithm 3, let \( S^\tau \subseteq S \) denote the current set of elements in \( S \) right after the while-loop using \( \tau \) in Algorithm 3 is finished. Finally, we use \( O^- \) to denote the set \( \{u \in O \setminus U : S \cup \{u\} \notin I \land f(u \mid S) > 0\} \).

With the above definitions, we first introduce the following lemma, which implies that `BATCHEDRANDOMGREEDY` is intrinsically a greedy algorithm that iteratively selects an element with the (approximately) maximal marginal gain in expectation:

**Lemma 5.** We have

\[
\forall i \in [k] : \quad (1 + \epsilon) \mathbb{E}[f(e_i \mid S_{i-1}) \mid S_{i-1}] \geq \max \{f(u \mid S_{i-1}) \mid u \in N \setminus U \land S_{i-1} \cup \{u\} \in I\}
\]  

(9)
With Lemma 5, we can prove the approximation ratio of BatchedRandomGreedy using the ideas described as follows. When the BatchedRandomGreedy algorithm terminates, any element \( u \in O \setminus S \) must belong to one of the following three categories: (1) \( u \in O \); (2) \( u \in (U \cap O) \setminus S \); (3) \( f(u \mid S) \leq \tau_{\text{min}} \). So we prove the approximation ratio by bounding the total marginal gains of the elements in each category with respect to \( S \). More specifically, for the elements in the first category, this bounding can be done by creating a mapping similar to that in Lemma 1; for the elements in the second category, we can bound their marginal gains by leveraging the property that these elements are discarded with probability of \( 1 - p \); the marginal gains of the elements in the third categories can be bounded by using the definition of \( \tau_{\text{min}} \). Based on these ideas, we prove the following theorem:

**Theorem 3.** The BatchedRandomGreedy algorithm returns a solution \( S \) satisfying

\[
f(O) \leq \frac{(1 + \epsilon)^2 k + 1}{1 - p} + \epsilon E[f(S)],
\]

which implies that it achieves an approximation ratio of \((1 + \epsilon)^2 (1 + \sqrt{k + 1})^2 \) if \( p \) is set to \((1 + \sqrt{k + 1})^{-1}\).

### 5.2. Complexity Analysis on BatchedRandomGreedy

Roughly speaking, BatchedRandomGreedy has a low adaptivity due to the following reasons. First, only independent value oracle queries to \( f(\cdot) \) are needed to find each batch of elements that BatchedRandomGreedy tries to add into \( S \). Second, the number of batches of elements considered by BatchedRandomGreedy is logarithmic with respect to \( n \), as BatchedRandomGreedy repeatedly drops a \((\frac{1}{1 + \epsilon})\)-fraction of the candidate elements. So we get the following theorem:

**Lemma 6.** The BatchedRandomGreedy algorithm can be implemented in \( O(\frac{\sqrt{k}}{\epsilon^2} \log \frac{k}{\epsilon} \log n \log r) \) adaptive rounds in expectation, and incurs an expected number of \( O(\frac{\sqrt{k}n}{\epsilon} \log \frac{k}{\epsilon} \log n \log r) \) oracle queries to the function value of \( f(\cdot) \).

**Proof.** Note that there are at most \( O(\frac{1}{\epsilon^2} \log \frac{k}{\epsilon}) \) different values of \( \tau \) considered in the for-loop of BatchedRandomGreedy. Besides, for each tested threshold \( \tau \), the size of the set \( C \) of candidate elements gets smaller by at least a factor of \( \frac{1}{1 + \epsilon} \) in at most \( \sum_{i=1}^{\infty} i \cdot (1 - p)^{i-1} \cdot p = p^{-1} \) adaptive rounds in expectation (due to the reason that the considered sequence can be dropped with a probability of \( p \)), which implies that there are at most \( O(\frac{1}{\epsilon^2} \log n) \) iterations in the while-loop for threshold \( \tau \). Note that each iteration of the while-loop of BatchedRandomGreedy can be implemented in \( O(\log r) \) adaptive rounds due to the binary search process in Line 6. So the claimed complexity on adaptivity follows by combining all the above results and by setting \( p = (1 + \sqrt{k + 1})^{-1} \) (the same value as that in Theorem 3). Finally, the claimed query complexity holds due to the reason that at most \( O(n) \) value oracle queries are incurred in each adaptive round of BatchedRandomGreedy.

Note that BatchedRandomGreedy also incurs independence oracle queries (i.e., query the value of \( 1_{\tau}(X) \) for any given set \( X \)). These independence oracle queries can also be implemented in parallel if they are mutually independent. So we can investigate the number of “adaptive rounds on independence oracle queries” of BatchedRandomGreedy in a way similar to that in Lemma 6, as shown by the following lemma:

**Lemma 7.** In expectation, the BatchedRandomGreedy algorithm can be implemented in \( O(\frac{\sqrt{k}}{\epsilon^2} \log \frac{k}{\epsilon} \log n \log r) \) adaptive rounds on independence oracle queries, and incurs no more than \( O(\frac{n^{3/2} \sqrt{k}}{\epsilon^2} \log \frac{k}{\epsilon} \log n \log r) \) independence oracle queries.

**Proof.** In Lemma 6, we have proved that the while-loop of BatchedRandomGreedy has at most \( O(\frac{\sqrt{k}}{\epsilon^2} \log \frac{k}{\epsilon} \log n) \) iterations in expectation. In each iteration of the while-loop, the RANDSEQ procedure is called once, which can be implemented in at most \( O(\sqrt{n}) \) adaptive rounds for independence oracle queries according to Karp et al. (1988); after that, the binary search process in Line 6 can be implemented in at most \( O(\log r) \) adaptive rounds. Finally, it is noted that at most \( O(n) \) independence oracle queries can be incurred in each adaptive round. So the lemma follows by combining all the above results.

### 6. Adaptive Optimization

The framework of Algorithm 1 can be naturally extended to address the adaptive case (i.e., Problem (2)), as shown by Algorithm 4. For convenience, we use \( \pi_{\text{RE}} \) to denote the adaptive policy adopted by Algorithm 4. Algorithm 4 runs in
Although the framework of Algorithm 4 looks similar to RANDOMMULTIGREEDY, its performance analysis is very different, as there does not exist a fixed optimal solution set under the adaptive setting, and we have to compare the average performance of \( \pi_A \) with that of an optimal policy \( \pi_{opt} \). To address this problem, we first build a relationship between \( \pi_A \) and \( \pi_{opt} \) as follows:

**Lemma 8.** Given any two adaptive policy \( \pi_1 \) and \( \pi_2 \), let \( \pi_1 \oplus \pi_2 \) denote a new policy that first execute \( \pi_1 \) and then execute \( \pi_2 \) without any knowledge about \( \pi_1 \). So we have

\[
 f_{avg}(\pi_1 \oplus \pi_2) = f_{avg}(\pi_{opt} \oplus \pi_A) \geq (1-p) \cdot f_{avg}(\pi_{opt})
\]

Lemma 8 implies that we may get an approximation ratio by further bounding \( f_{avg}(\pi_A \oplus \pi_{opt}) \) using \( f_{avg}(\pi_A) \). Given any \( u \in \mathcal{N} \) and any realization \( \phi \), let \( \psi_u(\phi) \) denote the partial realization observed by \( \pi_A \) right before \( u \) is considered by Lines 7-10 of Algorithm 4; if \( u \) is never considered, then let \( \psi_u(\phi) \) denote the observed partial realization at the end of \( \pi_A \). Based on this definition, we can get:

**Lemma 9.** The value of

\[
 f_{avg}(\pi_A \oplus \pi_{opt}) - f_{avg}(\pi_A)
\]

is no more than

\[
 \mathbb{E}_{\pi_A, \phi} \left[ \sum_{u \in \mathcal{N}(\pi_{opt}, \phi) \setminus \mathcal{N}(\pi_A, \phi)} \Delta(u | \psi_u(\phi)) \right],
\]

where the expectation is taken with respect to both the randomness of \( \Phi \) and the randomness of \( \pi_A \).

Next, we try to establish some quantitative relationships between \( f_{avg}(\pi_A) \) and the upper bound found in Lemma 9. Given any realization \( \phi \), Note that \( \mathcal{N}(\pi_{opt}, \phi) \setminus \mathcal{N}(\pi_A, \phi) \) denotes the set of elements that are selected by \( \pi_{opt} \) but not \( \pi_A \) under the realization \( \phi \). The elements in this set can be partitioned into three disjoint sets \( O_1(\phi), O_2(\phi) \) and \( O_3(\phi) \), where \( O_2(\phi) \) denotes the set of elements that have been considered by \( \pi_A \) in Lines 7-10 but discarded (due to the probability \( p \)); \( O_3(\phi) \) denotes the set of elements satisfying \( \Delta(u | \psi_u(\phi)) \leq 0 \) for all \( u \in O_3(\phi) \); and the rest elements are all in \( O_1(\phi) \). It can be seen that each element \( u \) in \( O_1(\phi) \) must satisfy \( \text{dom}(\psi_u(\phi)) \cup \{u\} \notin \mathcal{I} \). Therefore, by using a similar method as that under the non-adaptive case, we can map the elements in \( O_1(\phi) \) to the elements selected by \( \pi_A \) under realization \( \phi \), and hence prove:

**Lemma 10.** We have

\[
 \mathbb{E}_{\pi_A, \phi} \left[ \sum_{u \in O_1(\phi)} \Delta(u | \psi_u(\phi)) \right] \leq k \cdot f_{avg}(\pi_A)
\]

Now we try to bound the “utility loss” caused by \( O_2(\phi) \). Note that although these elements are discarded (with probability \( 1-p \)), they got a chance to be selected by \( \pi_A \) with probability \( p \). So the ratio of the total expected (conditional) marginal gain of these elements to \( f_{avg}(\pi_A) \) should be no more than \( (1-p)/p \), which is proved by the following lemma:
Lemma 11. We have
\[ \mathbb{E}_{\pi_A, \Phi} \left[ \sum_{u \in O_2(\Phi)} \Delta(u \mid \psi_u(\Phi)) \right] \leq \frac{1-p}{p} \cdot f_{\text{avg}}(\pi_A) \]

Combining all the above lemmas, we can get the approximation ratio of \textsc{AdaptRandomGreedy} as follows:

**Theorem 4.** \textsc{AdaptRandomGreedy} achieves an approximation ratio of \( \frac{p+1}{p(1-p)} \) (i.e., \( f_{\text{avg}}(\pi_A) \geq \frac{p(1-p)}{p+1} \cdot f_{\text{avg}}(\pi_{\text{opt}}) \)) under time complexity of \( O(nr) \). The ratio is minimized to \((1 + \sqrt{k+1})^2 \) when \( p = (1 + \sqrt{k+1})^{-1} \).

**Remark:** When the objective function \( f(\cdot) \) is monotone, it can be easily seen that \textsc{AdaptRandomGreedy}(1) can achieve an approximation ratio of \((k+1)\) -- the same ratio as that in (Golovin & Krause, 2011b). Therefore, \textsc{AdaptRandomGreedy}(\(p\)) can also be considered as a “universal algorithm” for both non-monotone and monotone submodular maximization.

### 7. Performance Evaluation

In this section, we compare our algorithms with the state-of-the-art algorithms for submodular maximization subject to a k-system constraint, using the metrics of both utility and the number of oracle queries to the objective function. We implemented five algorithms in the experiments: (1) the accelerated version of our \textsc{RandomMultiGreedy} algorithm (as described in Sec. 4.1), abbreviated as “RAMG”; (2) the \textsc{RepeateGreedy} algorithm presented in (Feldman et al., 2017), abbreviated as “REPG”; (3) the \textsc{TwinGreedyFast} algorithm proposed in (Han et al., 2020), abbreviated as “TGF”; (4) the \textsc{FastSGS} algorithm proposed in (Feldman et al., 2020), abbreviated as “FSGS”; and (5) our \textsc{AdaptRandomGreedy} algorithm, abbreviated as “ARG”. Note that the three baseline algorithms REPG, TGF and FSGS achieve the best-known performance bounds among the related studies, as illustrated in Table 1. In all experiments, we adopt the optimal settings of each implemented algorithm such that their theoretical approximation ratio is minimized (e.g., setting \( \ell = 2, p = \frac{2}{1+\sqrt{k}} \) for RAMG), and we set \( \epsilon = 0.1 \) whenever \( \epsilon \) is an input parameter for the considered algorithms. The implemented algorithms are tested in three applications, as elaborated in the following.

#### 7.1. Movie Recommendation

This application is also considered in (Mirzasoleiman et al., 2016; Feldman et al., 2017; Haba et al., 2020), where there are a set \( N \) of movies and each movie is labeled by several genres chosen from a predefined set \( G \). The goal is to select a subset \( S \) of movies from \( N \) to maximize the utility

\[ f(S) = \sum_{u \in N} \sum_{v \in S} M_{u,v} - \sum_{u \in S} \sum_{v \in S} M_{u,v}, \]  

under the constraint that the number of movies in \( S \) labeled by genre \( g \) is no more than \( m_g \) for all \( g \in G \) and \( |S| \leq m \), where \( m_g : g \in G \) and \( m \) are all predefined integers. Intuitively, by using \( M_{u,v} \) to denote the “similarity” between movie \( u \) and movie \( v \), the first and second factors in Eqn. (11) encourage the “coverage” and “diversity” of the movie set \( S \), respectively. It is indicated in (Mirzasoleiman et al., 2016; Feldman et al., 2017) that the function \( f(\cdot) \) is submodular and the problem constraint is essentially a k-system constraint with \( k = |G| \). In our experiments, we use the MovieLens dataset (Haba et al., 2020) containing 1793 movies, where each movie \( u \) is associated with a 25-dimensional feature vector \( t_u \) calculated from user ratings. We set \( M_{u,v} = e^{-\lambda \text{dist}(t_u, t_v)} \) where \( \text{dist}(t_u, t_v) \) denotes the Euclidean distance between \( t_u \) and \( t_v \), and \( \lambda \) is set to 0.2. There are three genres “Adventure”, “Animation” and “Fantasy” in MovieLens, and we set \( m_g = 10 \) for all genres.

In Fig. 1(a)-(b), we scale the the total number of movies allowed to be selected (i.e., \( m \)) to compare the performance of the implemented algorithms. It can be seen from Fig. 1(a) that RAMG and REPG achieve almost the same utility, while both of them outperform TGF and FSGS. Moreover, Fig. 1(b) shows that RAMG incurs much fewer oracle queries than all the baseline algorithms, and TGF is more efficient than FSGS. This can be explained by the reason that, FSGS maintains more candidate solutions than TGF, while the acceleration method adopted by RAMG is more efficient than the “thresholding” method adopted by TGF in practice.

#### 7.2. Image Summarization

This application is also considered in (Mirzasoleiman et al., 2016; Fahrbach et al., 2019a), where there is a set \( N \) of images classified into several categories, and the goal is to select a subset \( S \) of images from \( N \) to maximize the utility
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Figure 1. Movie Recommendation

\[ f(S) = \sum_{u \in N} \max_{v \in S} s_{u,v} - \frac{1}{|N|} \sum_{u \in S} \sum_{v \in S} s_{u,v} \]

(where \( s_{u,v} \) denotes the similarity between image \( u \) and image \( v \)), under the constraint the the numbers of images in \( S \) belonging to every category and the total number of images in \( S \) are all bounded. It can be verified that such a constraint is a matroid (i.e., 1-system) constraint. We perform the experiment using the CIFAR-10 dataset (Krizhevsky et al., 2009) containing ten thousands 32 × 32 color images. The similarity \( s_{u,v} \) is computed as the cosine similarity of the 3,072-dimensional pixel vectors of images \( u \) and \( v \). We restrict the selection of images from three categories: Airplane, Automobile and Bird, and the number of images selected from each category is bounded by 5.

In Fig. 2, we plot the experimental results by scaling the number of images allowed to be selected. It can be seen from Fig. 2(a) that RAMG and REPG achieve approximately the same utility and outperform FSGS and TGF again. Besides, TGF performs much worse than FSGS on utility in this application, as it uses a more rigorous stopping condition in its thresholding method and hence neglects many elements with small marginal gains. The results in Fig. 2(b) show that the superiority of RAMG on efficiency still maintains, while REPG outperforms FSGS significantly. This can be explained by the fact that, the marginal gains of the unselected elements diminish vastly after a new element is selected in the image summarization application, so the performance of the thresholding method adopted in FSGS deteriorates to be close to a naive greedy algorithm, which results in its worse efficiency as FSGS maintains more candidate solutions than the other algorithms. In contrast, the performance of RAMG on efficiency is more robust against the variations of underlying data distribution.

7.3. Social Advertising with Multiple Products

This application is also considered in (Mirzasoleiman et al., 2016; Fahrbach et al., 2019a; Amanatidis et al., 2020). We are given a social network \( G = (N, E) \) where each node represents a user and each edge \( (u, v) \in E \) is associated with a weight \( w_{u,v} \), denoting the “strength” that \( u \) can influence \( v \). Suppose that there are \( d \) kinds of products and an advertiser needs to select a “seed” set \( H_i \subseteq N \) for each \( i \in [d] \), such that the total revenue can be maximized by presenting a free sample of product with type \( i \) to each node in \( H_i \). We also follow (Mirzasoleiman et al., 2016; Amanatidis et al., 2020) to assume that the valuation of any user for a product is determined by the neighboring nodes owning the product with the same type, and the total revenue of \( H_i \) is defined as

\[ f_i(H_i) = \sum_{u \in N \setminus H_i} \alpha_{u,i} \sqrt{\sum_{v \in H_i} w_{v,u}} \]

(12)
where $\alpha_{u,i}$ is a random number with known distributions. Suppose that each node $u \in \mathcal{N}$ can serve as a seed for at most $q$ types of products, and the total number of free samples available for any type of product is no more than $m$. The goal of the advertiser is to identify the seed sets $H_1, \cdots, H_d$ to maximize the expected value of $\sum_{i \in [d]} f_i(H_i)$ under the constraints described above. It is indicated in (Mirzasoleiman et al., 2016) that this problem is essentially a submodular maximization problem with a 2-system constraint.

We use the LastFM Social Network (Barbieri & Bonchi, 2014; Aslay et al., 2017) with 1372 nodes and 14708 edges, and the edge weights in the network are randomly generated from the uniform distribution $\mathcal{U}(0,1)$. We adopt the same settings of (Amanatidis et al., 2020) to assume that, the parameter $\alpha_{u,i}$ follows a Pareto Type II distribution with $\lambda = 1, \alpha = 2$ for all node $u$ and product $i$; and the parameters of $u$’s neighboring nodes can be observed after $u$ is selected under the adaptive setting. The values of $d$ and $q$ are set to 5 and 3, respectively. Following a comparison method in (Amanatidis et al., 2020), we also implement a variation of RAMG (dubbed RAMG+) where the input parameter $p$ is randomly sampled from $(0.9,1)$.

To test the performance of ARG, we randomly generate 20 realizations of the problem instance described above, and plot the average utility/number of queries of ARG on all the generated realizations.

We study the performance of all algorithms in Fig. 3 by scaling the number of items of each product available for seeding (i.e., $m$). It can be seen from Fig. 3(a) that RAMG+, TGF and REPG achieve approximately the same utility, while the performance of RAMG and FSGS is slightly weaker. Note that RAMG+ has a weaker theoretical approximation ratio than RAMG. However, it is well known that approximation ratio is only a worst-case performance guarantee. Fig. 3(a) also reveals that ARG performs the best on utility, which is not surprising as it can take advantage on side observation. In Fig. 3(b), we compare the efficiency of all implemented algorithms and the results are qualitatively similar to those in Figs. 1-2. Note that RAMG+ performs almost the same with RAMG on efficiency, which implies that it improves the utility performance of RAMG “for free”.

8. Conclusion

We have proposed the first randomized algorithms for submodular maximization with a $k$-system constraint, under both the non-adaptive setting and the adaptive setting. Our algorithms outperform the existing algorithms in terms both approximation ratio and time complexity, and their superiorities have also been demonstrated by extensive experimental results on several applications related to data mining and social computing.

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A. Missing Proofs from Section 4

A.1. Proof of Lemma 1

Proof. The proof is constructive and is inspired by (Calinescu et al., 2011; Han et al., 2020). For clarity, we provide a procedure to construct $\sigma_i(\cdot)$, as shown by Algorithm 5. Suppose that the elements in $S_i$ are $\{z_1, \ldots, z_q\}$ (listed according to the order that they are added into $S_i$). Algorithm 5 finds a series of sets $J_0 \subseteq J_1 \subseteq \cdots \subseteq J_q = Q_i$ such that all the elements in $M_i = J_1 \setminus J_{i-1}$ is mapped to $z_i$ by $\sigma_i(\cdot)$ for any $t \in \{1, 2, \ldots, q\}$. From Algorithm 5, it can be easily seen that $\sigma_i(\cdot)$ satisfies the conditions required by the lemma. The only problem left is to prove that all the elements in $Q_i$ is mapped by $\sigma_i(\cdot)$, i.e., to prove $J_0 = \emptyset$. Indeed, we can prove a stronger result $\forall t \in \{0, 1, \ldots, q\} : |J_t| \leq kt$ by induction:

- When $t = q$, we will prove $|J_q| \leq kq$ by showing that $S_i$ is a base of $Q_i \cup S_i$. It is obvious that each element $u \in O_i^{-}$ satisfies $S_i \cup \{u\} \notin \mathcal{I}$ according to the definition of $O_i^{-}$. Moreover, for any element $u \in \cup_{j \in [t]} (O_j^{i-} \cup \hat{O}_j^{i-})$, we must have $S_i \cup \{u\} \notin \mathcal{I}$, because otherwise we have $S_i^{<}(u) \cup \{u\} \in \mathcal{I}$ due to $S_i^{<}(u) \subseteq S_i$ and the down-closed property of independence systems, contradicting the definition of $O_j^{i-}$ and $\hat{O}_j^{i-}$. These reasoning implies that $S_i$ is a base of $Q_i \cup S_i$. Note that $Q_i \subseteq O_i$. So we can get $|J_q| = |Q_i| \leq k|S_i| = kq$ according to the definition of $k$-systems.

- Suppose that $|J_i| \leq kt$ holds, we will prove $|J_{t-1}| \leq k(t-1)$. If the set $C_t$ determined in Line 2 of Algorithm 5 has a cardinality larger than $k$, then we have $|M_t| = k$ according to Algorithm 5 and hence $|J_{t-1}| = |J_t| - k \leq k(t-1)$. If $|C_t| \leq k$, then $\{z_1, \ldots, z_{t-1}\}$ must be a base of $\{z_1, \ldots, z_{t-1}\} \cup J_{t-1}$, because there does not exist $u \in J_{t-1} \setminus \{z_1, \ldots, z_{t-1}\}$ such that $\{z_1, \ldots, z_{t-1}\} \cup \{u\} \in \mathcal{I}$ according to Algorithm 5. So we also have $|J_{t-1}| \leq k(t-1)$ according to $J_{t-1} \in \mathcal{I}$ and the definition of $k$-systems. From the above reasoning we know $J_0 = \emptyset$. So the lemma follows.

Algorithm 5 CONSTRUCTING THE MAPPING $\sigma_i(\cdot)$

Initialize: Denote the elements in $S_i$ as $\{z_1, \ldots, z_q\}$, where elements are listed according to the order that they are added into $S_i$; $J_q \leftarrow Q_i$.

for $t = q$ to 0 do
  $C_t \leftarrow \{e \in J_t \setminus \{z_1, \ldots, z_{t-1}\} : \{z_1, \ldots, z_{t-1}, e\} \in \mathcal{I}\}$
  if $|C_t| \leq k$ then
    $M_t \leftarrow C_t$
  else
    if $z_t \in C_t$ then
      Find a subset $M_t \subseteq C_t$ satisfying $|M_t| = k$ and $z_t \in M_t$
    else
      Find a subset $M_t \subseteq C_t$ satisfying $|M_t| = k$
  end if
end if
Let $\sigma_i(z) = z_t$ for all $z \in M_t$; $J_{t-1} \leftarrow J_t \setminus M_t$
end for

A.2. Proof of Lemma 2

Proof. We first prove Eqn. (3). According to the definitions of $O_j^{i+}$ and $\hat{O}_j^{i+}$, any element $u \in O_j^{i+} \cup \hat{O}_j^{i+}$ can also be added into $S_i$ without violating the feasibility of $\mathcal{I}$ when $u$ is inserted into $S_j$. Therefore, according to the greedy rule of RANDOMMULTIGREEDY and the submodularity of $f(\cdot)$, we must have

$$\forall u \in O_j^{i+} \cup \hat{O}_j^{i+} : f(u \mid S_i) \leq f(u \mid S_j^{<}(u))) \leq f(u \mid S_j^{<}(u))) = \delta(u)$$

(13)

Now we prove Eqn. (4). Recall that $Q_i = \cup_{j \in [t]} (O_j^{i-} \cup \hat{O}_j^{i-}) \cup (O \cap S_i) \cup O_i^{-}$. According to Lemma 1, any element $u \in O_j^{i-} \cup \hat{O}_j^{i-}$ can be added into $S_j^{<}(\pi_i(u))$ without violating the feasibility of $\mathcal{I}$. Moreover, $u$ must have not been
considered by the algorithm at the moment that \( \pi_i(u) \) is added to \( S_i \), because otherwise we have \( S_i^< (u) \subseteq S_i^< (\pi_i(u)) \) and hence \( S_i^< (u) \cup \{u\} \in I \) due to the definition of independence systems, which contradicts the definitions of \( O_j^- \) and \( \tilde{O}_j^- \). Therefore, according to the greedy rule of RANDOMMULTIGREEDY and submodularity, we can get

\[
\forall u \in O_j^- \cup \tilde{O}_j^- : f(u \mid S_i) \leq f(u \mid S_i^< (u))) \leq f(u \mid S_i^< (\pi_i(u))) \leq f(\pi_i(u) \mid S_i^< (\pi_i(u))) = \delta(\pi_i(u))
\] (14)

By similar reasoning, we can also prove \( \forall u \in O_j^- : f(u \mid S_i) \leq f(u \mid S_i^< (\pi_i(u))) \leq \delta(\pi_i(u)) \). Finally, \( f(u \mid S_i) \leq \delta(\pi_i(u)) \) trivially holds for all \( u \in O \setminus S_i \) as \( \pi_i(u) = u \) due to Lemma 1. So the lemma follows. \( \square \)

A.3. Proof of Lemma 3

As the proof of Lemma 3 is a bit involved, we first introduce Lemma 12, and then use Lemma 12 to prove Lemma 3.

Lemma 12. We have

\[
\sum_{i \in [\ell]} \left( \sum_{j \in [\ell] \setminus \{i\}} \left( \sum_{u \in O_j^+} \delta(u) + \sum_{u \in O_j^- \cup \tilde{O}_j^-} \delta(\pi_i(u)) \right) + \sum_{u \in O_i^-} \delta(\pi_i(u)) \right) \leq \ell(\ell - 2) f(S^*)
\] (15)

Proof. For any \( i \in [\ell] \), let \( \lambda(i) = (i \mod \ell) + 1 \). So we have

\[
\sum_{i \in [\ell]} \sum_{j \in [\ell] \setminus \{i\}} \sum_{u \in O_{j}^{\lambda(i)}} \delta(u) = \sum_{i \in [\ell]} \sum_{j \in [\ell] \setminus \{i, \lambda(i)\}} \sum_{u \in O_j^+} \delta(u) + \sum_{u \in O_j^- \cup \tilde{O}_j^-} \delta(u) = \sum_{i \in [\ell]} \sum_{j \in [\ell] \setminus \{i\}} \sum_{u \in O_j^+} \delta(u) + \sum_{u \in O_j^- \cup \tilde{O}_j^-} \delta(u),
\] (16)

where the inequality is due to \( O_{j}^{\lambda(i)} \subseteq O \cap S_j \) and \( \forall u \in S_j : \delta(u) > 0 \). So we can get

\[
\sum_{i \in [\ell]} \sum_{j \in [\ell] \setminus \{i\}} \sum_{u \in O_{j}^{\lambda(i)}} \delta(u) = \sum_{i \in [\ell]} \left( \sum_{j \in [\ell] \setminus \{i, \lambda(i)\}} \sum_{u \in O_j^+} \delta(u) + \sum_{u \in O_j^- \cup \tilde{O}_j^-} \delta(u) \right) \\
\leq \sum_{i \in [\ell]} \sum_{j \in [\ell] \setminus \{i, \lambda(i)\}} \sum_{u \in O_j^+} \delta(u) + \sum_{u \in O_j^- \cup \tilde{O}_j^-} \delta(u)
\] (17)

\[
\leq \ell(\ell - 2) f(S^*) + \sum_{i \in [\ell]} \sum_{u \in O_j^- \cup \tilde{O}_j^-} \delta(\pi_i(u))
\] (18)

where we leverage Eqn. (16) to derive Eqn. (17), and Eqn. (18) is due to \( \sum_{u \in O \cap S_j} \delta(u) \leq \sum_{u \in S_j} \delta(u) \leq f(S_j) \leq f(S^*) \). Moreover, we can get

\[
\sum_{i \in [\ell]} \left( \sum_{j \in [\ell] \setminus \{i\}} \left( \sum_{u \in O_j^+} \delta(u) + \sum_{u \in O_j^- \cup \tilde{O}_j^-} \delta(\pi_i(u)) \right) + \sum_{u \in O_i^-} \delta(\pi_i(u)) \right)
\]

\[
= \sum_{i \in [\ell]} \sum_{j \in [\ell] \setminus \{i\}} \sum_{u \in O_j^+} \delta(u) + \sum_{i \in [\ell]} \left( \sum_{j \in [\ell] \setminus \{i\}} \sum_{u \in O_j^- \cup \tilde{O}_j^-} \delta(\pi_i(u)) + \sum_{u \in O_i^-} \delta(\pi_i(u)) \right)
\]

\[
\leq \ell(\ell - 2) f(S^*) + \sum_{i \in [\ell]} \sum_{u \in O_j^- \cup \tilde{O}_j^-} \delta(\pi_i(u))
\]

\[
= \ell(\ell - 2) f(S^*) + k \sum_{i \in [\ell]} \sum_{u \in O_i} \delta(u)
\]

\[
\leq \ell(\ell - 2) f(S^*) + k \sum_{i \in [\ell]} f(S_i) \leq \ell(k + \ell - 2) f(S^*)
\] (21)
where \( Q_i = \bigcup_{j \in [\ell] \setminus \{i\}} (O_{j}^+ \cup O_{j}^-) \cup (O \cap S_i) \cup O_{i}^- \) is defined in Lemma 1; Eqn. (19) is due to Eqn. (18); and Eqn. (20) is due to Lemma 1. So the lemma follows.

Now we provide the proof of Lemma 3:

**Proof.** Let \( G_i = [\bigcup_{j \in [\ell] \setminus \{i\}} (O_{j}^+ \cup O_{j}^- \cup \hat{O}_{j}^+ \cup \hat{O}_{j}^-)] \cup O_{i}^- \cup [O \cap D_i] \) for all \( i \in [\ell] \). It is not hard to see that \( G_i \subseteq O \setminus S_i \) and \( \forall u \in O \setminus (S_i \cup G_i) : f(u \mid S_i) \leq 0 \). Therefore, we can get

\[
\sum_{i \in [\ell]} \left( f(O \cup S_i) - f(S_i) \right) \leq \sum_{i \in [\ell]} \left( \sum_{j \in [\ell] \setminus \{i\}} \left( \sum_{u \in O_{j}^+ \cup \hat{O}_{j}^+} f(u \mid S_i) + \sum_{u \in O_{j}^- \cup \hat{O}_{j}^-} f(u \mid S_i) \right) + \sum_{u \in O^-} f(u \mid S_i) + \sum_{u \in O \cap D_i} f(u \mid S_i) \right) \tag{22}
\]

\[
\leq \sum_{i \in [\ell]} \left( \sum_{j \in [\ell] \setminus \{i\}} \left( \sum_{u \in O_{j}^+ \cup \hat{O}_{j}^+} \delta(u) + \sum_{u \in O_{j}^- \cup \hat{O}_{j}^-} \delta(\pi_i(u)) \right) + \sum_{u \in O^-} \delta(\pi_i(u)) + \sum_{u \in O \cap D_i} \delta(u) \right) \tag{23}
\]

\[
= \sum_{i \in [\ell]} \left( \sum_{j \in [\ell] \setminus \{i\}} \left( \sum_{u \in O_{j}^+} \delta(u) + \sum_{u \in O_{j}^- \cup \hat{O}_{j}^-} \delta(\pi_i(u)) \right) + \sum_{u \in O^-} \delta(\pi_i(u)) \right) \\
+ \sum_{i \in [\ell]} \left( \sum_{j \in [\ell] \setminus \{i\}} \sum_{u \in \hat{O}_{j}^+} \delta(u) + \sum_{u \in O \cap D_i} \delta(u) \right) \tag{24}
\]

where Eqn. (22) is due to submodularity of \( f(\cdot) \); Eqn. (23) is due to Lemma 2 and submodularity; and Eqn. (24) is due to Lemma 12. Moreover, we can get

\[
\sum_{i \in [\ell]} \left( \sum_{j \in [\ell] \setminus \{i\}} \sum_{u \in O_{j}^+} \delta(u) + \sum_{u \in O \cap D_i} \delta(u) \right) \leq \sum_{i \in [\ell]} \left( \sum_{j \in [\ell] \setminus \{i\}} \sum_{u \in O \cap D_i} \delta(u) + \sum_{u \in O \cap D_i} \delta(u) \right) \tag{25}
\]

\[
= \sum_{i \in [\ell]} \sum_{j \in [\ell] \setminus \{i\}} \sum_{u \in O \cap D_j} \delta(u) = \ell \sum_{u \in \mathcal{N}} X_u \cdot \delta(u) \tag{26}
\]

where Eqn. (25) is due to \( \hat{O}_{j}^+ \subseteq O \cap D_j \) and \( \forall u \in D_j : \delta(u) > 0 \). Combining Eqn. (24) and Eqn. (26) finishes the proof of Lemma 3.

### A.4. Proof of Lemma 4

We first quote the following lemma presented in (Buchbinder et al., 2014):

**Lemma 13. (Buchbinder et al., 2014)** Given a ground set \( \mathcal{N} \) and any non-negative submodular function \( g(\cdot) \) defined on \( 2^\mathcal{N} \), we have \( \mathbb{E}[g(Y)] \geq (1 - p)g(\emptyset) \) if \( Y \) is a random subset of \( \mathcal{N} \) such that each element in \( \mathcal{N} \) appears in \( Y \) with probability of at most \( p \) (not necessarily independently).

With the above lemma, Lemma 4 can be proved as follows:

**Proof.** We first prove Eqn. (6). Note that \( S_1, S_2, \ldots, S_\ell \) are disjoint sets. Using submodularity, we have

\[
\sum_{i=1}^\ell f(S_i \cup O) \geq f(O) + f(S_1 \cup S_2 \cup O) + \sum_{i=3}^\ell f(S_i \cup O) \geq 2f(O) + f(S_1 \cup S_2 \cup S_3 \cup O) + \sum_{i=4}^\ell f(S_i \cup O) \geq \cdots \geq (\ell - 1)f(O) + f(\cup_{i=1}^\ell S_i \cup O) \tag{27}
\]
Let $g : 2^N \rightarrow \mathbb{R}_{\geq 0}$ be a non-negative submodular function defined as: $\forall S \subseteq N : g(S) = f(S \cup O)$. As each element in $N$ appears in $\bigcup_{i=1}^{t} S_i$ with probability of no more than $p$, we can use Lemma 13 to get

$$E[f(\bigcup_{i=1}^{t} S_i \cup O)] = E[g(\bigcup_{i=1}^{t} S_i)] \geq (1-p)g(\emptyset) = (1-p)f(O) \quad (28)$$

Combining Eqn. (27) and Eqn. (28) finishes the proof of Eqn. (6).

Next, we prove Eqn. (7). For any $u \in N$, let $Y_u = 1$ if $u \in \bigcup_{i=1}^{t} S_i$ and $Y_u = 0$ otherwise; let $E_u$ be an arbitrary event denoting all the random choices of RANDOMMULTIGREEDY up until the time that $u$ is considered to be added into a candidate solution, or denoting all the randomness of RANDOMMULTIGREEDY if $u$ is never considered. Note that we have $\sum_{u \in N} Y_u \cdot \delta(u) \leq \sum_{i=1}^{t} f(S_i)$. Therefore, by the law of total probability, we only need to prove

$$\forall u \in N : \frac{1-p}{p} E[Y_u \cdot \delta(u) | E_u] \geq E[X_u \cdot \delta(u) | E_u] \quad (29)$$

for any event $E_u$ defined above. Note that we have $X_u = 0$ and hence Eqn. (29) clearly holds if $u \notin O$ or $u$ is never considered by the algorithm. Otherwise we have $E[Y_u \cdot \delta(u) | E_u] = p \cdot \delta(u)$ and $E[X_u \cdot \delta(u) | E_u] = (1-p) \cdot \delta(u)$ due to the reason that $u$ is accepted with probability of $p$ and discarded with probability of $1-p$. Combining all these results completes the proof of Eqn. (7).

\[\square\]

A.5. Proof of Theorem 2

For clarity, we first provide the detailed design of the accelerated version of RANDOMMULTIGREEDY, as shown by Algorithm 7. In the $t$-th iteration, Algorithm 7 calls a procedure CHOOSE to greedily find an candidate element $v_i$ for $S_i$ satisfying $f(v_i \mid S_i) > 0$ and $S_i \cup \{v_i\} \in \mathcal{I}$ for each $i \in [t]$. The CHOOSE procedure also returns an index $i_t$ same to that in Algorithm 1. After that, Algorithm 7 runs similarly as Algorithm 1, i.e., it inserts $v_{i_t}$ into $S_{i_t}$ with probability $p$, and then enters the $(t+1)$-th iteration. Note that the elements $v_1, \cdots, v_t$ and $v_{i_t}$ found in the $t$-th iteration are also used to call CHOOSE in the $(t+1)$-th iteration, so that CHOOSE need not to identify a new $v_i$ for all $i \in [t] : v_i \neq v_{i_t}$ (as $S_t$ does not change for these $i$'s) and hence time efficiency can be improved. Finally, Algorithm 7 returns the optimal set among $S_1, \cdots, S_t$ and $S_0$, where $S_0$ is the singleton set with the maximum utility.

Next, we provide a brief description on the CHOOSE procedure. As explained in Sec. 4.1, CHOOSE maintains $\ell$ sets $A_1, A_2, \cdots, A_\ell$ such that $v_i$ can be selected from $A_i$. At the first time that CHOOSE is called, CHOOSE assigns each element $u \in A_i$ a weight $w_i(u) = f(u \mid \emptyset)$ and an integer $\tau_i(u)$ indicating how many times $w_i(u)$ has been updated (Lines 3–7). Afterwards, CHOOSE runs as that described in Sec. 4.1 and finds $v_i$ for each $i \in [\ell]$. Finally, CHOOSE identifies $v_{i_t}$ from $\{v_i : i \in [\ell]\}$ which has the maximum marginal gain, and it also removes $v_{i_t}$ from all $A_i : i \in [t]$ because $v_{i_t}$ will be used as $v_\ell$. Algorithm 7.

Note that Algorithm 7 differs from Algorithm 1 in two points: (1) the element $u_t$ found in the $t$-th iteration is only an $\frac{1}{1+\epsilon}$-approximate solution; (2) there are elements removed from $A_i$ due to “too many updates”. Based on this observation, we can slightly modify the proofs for Algorithm 1 to prove Theorem 2, as presented below:

**Proof.** Let $L_i$ denote the set of all elements removed from $A_i$ due to Line 25 of Algorithm 6. We can slightly modify Definition 5 to re-define the sets $O_{i_t}^+, O_{i_t}^-, \tilde{O}_{i_t}^+, \tilde{O}_{i_t}^-$ as follows:

$$O_{i_t}^+ = \{u \in O \cap S_i : S_i^-(u) \cup \{u\} \in \mathcal{I}\} \setminus L_i;$$

$$O_{i_t}^- = \{u \in O \cap S_i : S_i^-(u) \cup \{u\} \notin \mathcal{I}\} \setminus L_i;$$

$$\tilde{O}_{i_t}^+ = \{u \in O \cap D_j : S_i^-(u) \cup \{u\} \in \mathcal{I}\} \setminus L_i;$$

$$\tilde{O}_{i_t}^- = \{u \in O \cap D_j : S_i^-(u) \cup \{u\} \notin \mathcal{I}\} \setminus L_i;$$

$$O_{i}^- = \{u \in O \setminus U : S_i \cup \{u\} \notin \mathcal{I} \land f(u \mid S_i) > 0\} \setminus L_i;$$

With this new definition, it can be easily verified that each element $u$ in $O_{i_t}^+ \cup O_{i_t}^- \cup \tilde{O}_{i_t}^+ \cup \tilde{O}_{i_t}^-$ is still a candidate considered for $S_i$ in the CHOOSE procedure when the algorithm tries to insert $u$ into $S_j$. Therefore, according to the greedy rule of RANDOMMULTIGREEDY and the $(1+\epsilon)^{-1}$-approximation ratio of CHOOSE, we can use similar reasoning as that
Algorithm 6 \textsc{Choose}(S_1, S_2, \cdots, S_t, v_1, \cdots, v_t, v^*)

1: if $\cup_{i=1}^t S_i = \emptyset$ then
2:   Let $A_i \leftarrow \{u \in N: \{u\} \cap f(u | \emptyset) > 0\}$ for all $i \in [t]$;
3:   for all $i \in [t]$ do
4:     Let $w_i(u) \leftarrow f(u | \emptyset)$ and $\tau_i(u) \leftarrow 0$ for all $u \in A_i$;
5:     Store $A_i$ as a priority list according to the non-increasing order of $w_i(u): u \in A_i$ for all $i \in [t]$;
6:     Let $v_i \leftarrow \arg\max_{u \in A_i} w_i(u)$;
7:   end for
8: else
9:   $C \leftarrow [t]\setminus \{j \in [t]: (v_j \neq v^*) \lor (v_j = \text{NULL})\}$
10:  for all $i \in C$ do
11:     Let $v_i \leftarrow \text{NULL}$ and remove all elements in $A_i$ with non-positive weights;
12:     while $A_i \neq \emptyset$ do
13:       pop out the top element $u$ from $A_i$;
14:       if $f(u | S_i)$ has been computed then
15:         $v_i \leftarrow u$: exit while;
16:     end if
17:     if $S_i \cup \{u\} \notin \mathcal{I}$ then
18:       continue;
19:     end if
20:     old $\leftarrow w_i(u)$; $\tau_i(u) \leftarrow \tau_i(u) + 1$;
21:     Compute $f(u | S_i)$ and let $w_i(u) \leftarrow f(u | S_i)$;
22:     if $w_i(u) \geq \frac{\text{old}}{1+\epsilon}$ then
23:       $v_i \leftarrow u$: exit while;
24:     else
25:       if $\tau_i(u) \leq \lceil \log_{1+\epsilon} \frac{\ell}{\epsilon} \rceil$ then
26:         re-insert $u$ into $A_i$ and resort the elements in $A_i$;
27:       end if
28:     end if
29:   end while
30: end if
31: end if
32: Let $i^* \leftarrow \arg\max_{i \in [t] \setminus \{\text{NULL}\}} f(v_i | S_i)$ and remove $v_{i^*}$ from $A_i$ for all $i \in [t]$
33: Output: $v_1, v_2, \cdots, v_t, i^*$

for Lemma 2 to prove

$$\forall u \in O_j^{++} \cup \hat{O}_j^{++}: f(u | S_i) \leq (1 + \epsilon)\delta(u);$$

$$\forall u \in \cup_{j \in [t]} (O_j^{--} \cup \hat{O}_j^{--}) \cup (O \cap S_i) \cup O_r: f(u | S_i) \leq (1 + \epsilon)\delta(\pi_i(u));$$

With the above results, we can use similar reasoning as that in Lemma 3 to prove:

$$\frac{1}{1+\epsilon} \sum_{i \in [t]} f(O | S_i) \leq \ell(k + \ell - 2)f(S^*) + \ell \sum_{u \in N} X_u \cdot \delta(u) + \sum_{i \in [t]} \sum_{u \in L \cap O} f(u | S_i)$$

(32)

Moreover, we have

$$\sum_{u \in L \cap O} f(u | S_i) \leq \sum_{u \in L \cap O} f(u | \emptyset)(1 + \epsilon)^{-\lceil \log_{1+\epsilon} \frac{\ell}{\epsilon} \rceil} \leq \sum_{u \in L \cap O} \frac{\epsilon}{\ell\tau} f(u) \leq \epsilon f(S^*)/\ell$$

(33)

where the first inequality is due the reason that the weight of each element $u \in L_i$ have been updated in \textsc{Choose} procedure for more than $\lceil \log_{1+\epsilon} \frac{\ell}{\epsilon} \rceil$ times and it diminishes by a factor of $\frac{1}{1+\epsilon}$ for each update. Combining Eqn. (32), Eqn. (33) and Lemma 4, we can prove

$$f(O) \leq \left[ (1 + \epsilon) \frac{\ell(k + \ell - 1)}{\ell - p} - \frac{(\ell - 1)\epsilon - \epsilon^2}{\ell - p} \right] \mathbb{E}[f(S^*)]$$

(34)
Algorithm 7 RANDOMMULTIGREEDY($\ell, p$) 

Initialize: $\forall i \in [\ell]: S_i \leftarrow \emptyset; v_i \leftarrow \text{NULL}; \; t \leftarrow 1; u_0 \leftarrow \text{NULL};$

1: repeat
2: $(v_1, v_2, \ldots, v_\ell, i) \leftarrow \text{CHOOSE}(S_1, \ldots, S_\ell, v_1, \ldots, v_\ell, u_0)$
3: if $\exists j \in [\ell]: v_j \neq \text{NULL}$ then
4: $u_t \leftarrow v_i$;
5: With probability $p$ do $S_{i_t} \leftarrow S_{i_t} \cup \{u_t\}$
6: $t \leftarrow t + 1$
7: end if
8: until $(\forall i \in [\ell]: v_i = \text{NULL})$
9: $u^* \leftarrow \max_{u \in \mathcal{N} \setminus \{u\}} f(u); S_0 \leftarrow \{u^*\}$
10: $S^* \leftarrow \max_{S \in \mathcal{I}} f(S); T \leftarrow t - 1$
11: Output: $S^*, T$

Therefore, the approximation ratio of the accelerated RANDOMMULTIGREEDY algorithm is at most $(1 + \epsilon)(1 + \sqrt{k})^2$ when $\ell = 2, p = \frac{2}{1 + \sqrt{k}}$ (for a randomized algorithm), or at most $(1 + \epsilon)(k + \sqrt{k} + \lceil \sqrt{k} \rceil + 1)$ when $\ell = \lceil \sqrt{k} \rceil + 1, p = 1$ (for a deterministic algorithm). Finally, it can be seen that the CHOOSE procedure incurs at most $O(\log(1 + \epsilon) \ell^2)$ value and independence oracle queries for each element in each $A_i : i \in [\ell]$. So the total time complexity of the accelerated RANDOMMULTIGREEDY algorithm is at most $O(\ell n \log_2 \frac{\ell \ell}{\epsilon}) = O(\frac{\ell n}{\epsilon} \log \frac{\ell \ell}{\epsilon})$, which completes the proof.

B. Missing Proofs from Section 5

B.1. Proof of Lemma 5

Proof. Consider the while-loop of Algorithm 3 in which $e_i$ is added into $S$. Let $\{a_1, a_2, \ldots, a_d\}$ denote the random sequence returned by RANDSEQ and the threshold considered in that iteration, respectively. Suppose that $e_i = a_q \in \{a_1, a_2, \ldots, a_d\}$. According to Line 6 of Algorithm 3, we have

$$E[f(e_i | S_{i-1}) | S_{i-1}] \geq \frac{|C_{q-1}|}{|C|} \cdot \tau \geq \tau/(1 + \epsilon) \quad (35)$$

Suppose by contradiction that there exists $u \in \mathcal{N} \setminus \mathcal{U}$ satisfying $S_{i-1} \cup \{u\} \in \mathcal{I}$ and $f(u | S_{i-1}) > (1 + \epsilon)^2 E[f(e_i | S_{i-1}) | S_{i-1}]$. Then we must have $f(u | S_{i-1}) \geq (1 + \epsilon)\tau$ according to Eqn. (35) and hence $\tau < \tau_{\text{max}}$. Using the down-closed property of independence systems and submodularity, this implies that $S^{(1+\epsilon)\tau} \cup \{u\} \in \mathcal{I}$ and $f(u | S^{(1+\epsilon)\tau}) > (1 + \epsilon)\tau$. However, this contradicts the fact that, given any threshold $\tau$ considered by Algorithm 3, no elements with marginal gain larger than $\tau$ can be added into $S^\tau$ without violating $\mathcal{I}$ when the while-loop using $\tau$ in Algorithm 3 is finished.

B.2. Proof of Theorem 3

To prove the theorem, we first introduce the following lemma:

Lemma 14. We have $E[f(O^- | S)] \leq (1 + \epsilon)^2 k \cdot E[f(S)]$

Proof. For every possible pair of $(S, O^-)$, we can use similar reasoning as that in Lemma 1 to find a mapping $\sigma : O^- \mapsto S$ such that at most $k$ elements are mapped to the same element in $S$ and $\{u\} \cup S^\subseteq(\sigma(u)) \in \mathcal{I}$ for all $u \in O^-$. Recall that the elements in $S$ are denoted by $\{e_1, \ldots, e_r\}$. For any $i \in [r]$, let $\sigma^{-1}(e_i)$ denote the set of elements in $O^- \setminus S$ that are mapped to $e_i$ by $\sigma(\cdot)$ and let $\sigma^{-1}(e_i) = \emptyset$ if $e_i$ is a dummy element. So we have $|\sigma^{-1}(e_i)| \leq k$ for all $i \in [r]$. Using Lemma 5 and
submodularity, we get

\[
\mathbb{E}[f(O^- | S)] \leq \mathbb{E}\left[\sum_{u \in O \setminus S} f(u | S)\right] \leq \mathbb{E}\left[\sum_{u \in O \setminus S} f(u | S^\subset(\sigma(u)))\right]
\]

(36)

\[
\leq \mathbb{E}\left[\sum_{j \in [r]} \sum_{u \in \sigma^{-1}(e_j)} f(u | S_{i-1})\right] \leq k \cdot \mathbb{E}\left[\sum_{j \in [r]} \max_{u \in \sigma^{-1}(e_j)} f(u | S_{i-1})\right]
\]

(37)

\[
\leq (1 + \epsilon)^2 k \cdot \mathbb{E}\left[\sum_{i \in [r]} \mathbb{E}[f(e_i | S_{i-1}) | S_{i-1}]\right]
\]

(38)

\[
= (1 + \epsilon)^2 k \cdot \sum_{i \in [r]} \mathbb{E}[\mathbb{E}[f(e_i | S_{i-1}) | S_{i-1}]]
\]

(39)

\[
= (1 + \epsilon)^2 k \cdot \sum_{i \in [r]} \mathbb{E}[f(e_i | S_{i-1})] = (1 + \epsilon)^2 k \cdot \mathbb{E}\left[\sum_{i \in [r]} f(e_i | S_{i-1})\right]
\]

(40)

\[
\leq (1 + \epsilon)^2 k \cdot \mathbb{E}[f(S)]
\]

(41)

So the lemma follows. □

Now we use Lemma 14 to prove Theorem 3:

**Proof.** For any \( u \in \mathcal{N} \), define \( X_u = 1 \) if \( u \in (U \cap O) \setminus S \), otherwise define \( X_u = 0 \). Note that any element \( u \in O \setminus S \) must satisfy one of the following conditions: (1) \( u \in O^- \); (2) \( u \in (U \cap O) \setminus S \); (3) \( f(u | S) \leq \tau_{\min} \). So we can use submodularity, Lemma 14 and \( \tau_{\max} \leq f(S) \) to get

\[
\mathbb{E}[f(O | S)] \leq r \tau_{\min} + \mathbb{E}[f(O^- | S)] + \mathbb{E}\left[\sum_{u \in \mathcal{N}} X_u \cdot f(u | S^\subset(\sigma(u)))\right]
\]

(42)

\[
\leq \epsilon \cdot \mathbb{E}[f(S)] + (1 + \epsilon)^2 k \cdot \mathbb{E}[f(S)] + \mathbb{E}\left[\sum_{u \in \mathcal{N}} X_u \cdot f(u | S^\subset(\sigma(u)))\right]
\]

(43)

Meanwhile, as each element in \( \mathcal{N} \) appears in \( S \) with probability of no more than \( p \), we can use similar reasoning as that in the proof of Lemma 4 to get

\[
\mathbb{E}[f(S \cup O)] \geq (1 - p)f(O);
\]

(44)

\[
\mathbb{E}\left[\sum_{u \in \mathcal{N}} X_u \cdot f(u | S^\subset(\sigma(u)))\right] \leq \frac{1 - p}{p} \mathbb{E}[f(S)]
\]

(45)

Combining the above equations completes the proof. □

**C. Missing Proofs from Section 6**

**C.1. Proof of Lemma 8**

*Proof.** Given any element set \( Y \subseteq \mathcal{N} \) and any realization \( \phi, \) let \( g(Y, \phi) := f(Y \cup \mathcal{N}(\pi_{\text{opt}}, \phi), \phi) \). It is easy to verify that the non-negative function \( g(\cdot, \phi) \) is submodular. Thus, given a fixed realization \( \phi, \) by Lemma 13, we know that

\[
\mathbb{E}_{\pi, \mathcal{A}}[g(\mathcal{N}(\pi, \phi), \phi)] \geq (1 - p)g(\emptyset, \phi)
\]

(46)

Therefore, we have

\[
f_{\text{avg}}(\pi_{\text{opt}}@\pi, \mathcal{A}) = \mathbb{E}_\Phi[\mathbb{E}_{\pi, \mathcal{A}}[g(\mathcal{N}(\pi, \Phi), \Phi)]] \geq \mathbb{E}_\Phi[(1 - p)g(\emptyset, \Phi)] = (1 - p)f_{\text{avg}}(\pi_{\text{opt}}),
\]

(47)

which completes the proof. □
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C.2. Proof of Lemma 9

Proof. We first give an equivalent expression of the expected utility by a function of conditional expected marginal gains. Given a deterministic policy \( \pi \) and a realization \( \phi \), for each \( u \in \mathcal{N} \), let \( Y_u(\phi) \) be a boolean random variable such that \( Y_u(\phi) = 1 \) if \( u \in \mathcal{N}(\pi, \phi) \) and \( Y_u(\phi) = 0 \) otherwise. Further, denote by \( \psi_u^\pi(\phi) \) the partial realization observed by \( \pi \) right before considering \( u \) under realization \( \phi \), and denote by \( \Psi_u^\pi \) a random partial realization right before considering \( u \) by \( \pi \). We also use \( Y_u(\psi_u^\pi(\phi)) \) to represent \( Y_u(\phi) \), since the partial realization \( \psi_u^\pi(\phi) \) suffices to determine whether \( u \) is added to the solution under realization \( \phi \). Thus,

\[
\mathbb{E}_\phi[f(\mathcal{N}(\pi, \Phi), \Phi)] = \mathbb{E}_\Phi\left[ \sum_{u \in \mathcal{N}} (Y_u(\Phi) \cdot (f(\text{dom}(\psi_u^\pi(\Phi)) \cup \{u\}, \Phi) - f(\text{dom}(\psi_u^\pi(\Phi)), \Phi))) \right]
\]

\[
= \sum_{u \in \mathcal{N}} \mathbb{E}_{\Psi_u^\pi} \mathbb{E}_\Phi\left[ Y_u(\Phi) \cdot (f(\text{dom}(\Psi_u^\pi(\Phi)) \cup \{u\}, \Phi) - f(\text{dom}(\Psi_u^\pi(\Phi)), \Phi)) \mid \Phi \sim \Psi_u^\pi \right]
\]

\[
= \sum_{u \in \mathcal{N}} \mathbb{E}_{\Psi_u^\pi} \left[ Y_u(\Psi_u^\pi(\Phi)) \cdot \Delta(u \mid \Psi_u^\pi(\Phi)) \right] = \sum_{u \in \mathcal{N}} \mathbb{E}_\Phi \left[ \mathbb{E}_{\Psi_u^\pi} \left[ Y_u(\Psi_u^\pi(\Phi)) \cdot \Delta(u \mid \Psi_u^\pi(\Phi)) \mid \Phi \sim \Psi_u^\pi \right] \right]
\]

Denote by \( \psi(\pi_A, \phi) \) the observed partial realization at the end of \( \pi_A \) under realization \( \phi \). Then, similar to the above analysis, we have

\[
f_{\text{avg}}(\pi_A \oplus \pi_{\text{opt}}) = \mathbb{E}_{\pi_A \oplus \pi_{\text{opt}}} \left[ f(\mathcal{N}(\pi_A \oplus \pi_{\text{opt}}, \Phi), \Phi) \right]
\]

\[
= \mathbb{E}_{\pi_A \oplus \pi_{\text{opt}}} \left[ \sum_{u \in \mathcal{N}(\pi_A, \Phi)} \Delta(u \mid \psi_u(\Phi)) + \sum_{u \in \mathcal{N}(\pi_{\text{opt}}, \Phi) \setminus \mathcal{N}(\pi_A, \Phi)} \Delta(u \mid \psi(\pi_A, \Phi) \cup \psi_u(\Phi)) \right]
\]

\[
= f_{\text{avg}}(\pi_A) + \mathbb{E}_{\pi_A \oplus \pi_{\text{opt}}} \left[ \sum_{u \in \mathcal{N}(\pi_{\text{opt}}, \Phi) \setminus \mathcal{N}(\pi_A, \Phi)} \Delta(u \mid \psi(\pi_A, \Phi) \cup \psi_u(\Phi)) \right]
\]

\[
\leq f_{\text{avg}}(\pi_A) + \mathbb{E}_{\pi_A} \left[ \sum_{u \in \mathcal{N}(\pi_{\text{opt}}, \Phi) \setminus \mathcal{N}(\pi_A, \Phi)} \Delta(u \mid \psi_u(\Phi)) \right],
\]

where the inequality is due to adaptive submodularity and \( \psi_u(\Phi) \subseteq \psi(\pi_A, \Phi) \subseteq \psi(\pi_A, \Phi) \cup \psi_u(\Phi) \).

C.3. Proof of Lemma 10

Proof. Since \( f_{\text{avg}}(\pi_A) = \mathbb{E}_{\pi_A} \left[ \mathbb{E}_\Phi \left[ \sum_{u \in \mathcal{N}(\pi_A, \Phi)} \Delta(u \mid \psi_u(\Phi)) \right] \right] \), it suffices to prove

\[
\sum_{u \in O_1(\phi)} \Delta(u \mid \psi_u(\Phi)) \leq k \cdot \sum_{u \in \mathcal{N}(\pi_A, \phi)} \Delta(u \mid \psi_u(\phi)) \tag{49}
\]

for any given realization \( \phi \in Z^N \) and fixed randomness of \( \pi_A \). Given a realization \( \phi \), let \( \hat{u}_i \) be the \( i \)-th element selected by \( \pi_A \) and let \( \hat{S}_i \) be the first \( i \) elements picked, i.e., \( \hat{S}_i = \{\hat{u}_1, \ldots, \hat{u}_i\} \), for \( i = 1, 2, \ldots, h \) where \( h := \lfloor \mathcal{N}(\pi_A, \phi) \rfloor \). Suppose that there exists a partition \( O_{1,1}, O_{1,2}, \ldots, O_{1,h} \) of \( O_1(\phi) \) such that for all \( i = 1, 2, \ldots, h \),

\[
\sum_{u \in O_{1,i}} \Delta(u \mid \psi_u(\phi)) \leq k \cdot \Delta(\hat{u}_i \mid \psi_{\hat{u}_i}(\phi)), \tag{50}
\]

then Eqn. (49) must hold due to

\[
\sum_{u \in O_1(\phi)} \Delta(u \mid \psi_u(\phi)) = \sum_{i=1}^{h} \sum_{u \in O_{1,i}} \Delta(u \mid \psi_u(\phi)) \leq k \cdot \sum_{i=1}^{h} \Delta(\hat{u}_i \mid \psi_{\hat{u}_i}(\phi)) = k \cdot \sum_{u \in \mathcal{N}(\pi_A, \phi)} \Delta(u \mid \psi_u(\phi)). \tag{51}
\]

Therefore, we just need to show the existence of such a desired partition of \( O_1 \), as proved below.
We use the following iterative algorithm to find the partition, which is inspired by (Calinescu et al., 2011). Define \( N_h := O(1) \). For \( i = h, h - 1, \ldots, 2 \), let \( B_i := \{ u \in N_i \mid S_{i-1} \cup \{ u \} \notin \mathcal{I} \} \). If \( |B_i| \leq k \), set \( O_{1,i} = B_i \). Otherwise, pick an arbitrary \( O_{1,i} \subseteq B_i \) with \( |O_{1,i}| = k \). Then, set \( N_{i-1} = N_i \setminus O_{1,i} \). Finally, set \( O_{1,1} = N_1 \). Clearly, \( |O_{1,i}| \leq k \) for \( i = 2, \ldots, h \). We further show that \( |O_{1,1}| \leq k \). We prove it by contradiction and assume \( |O_{1,1}| > k \). If \( |B_2| \leq k \), then we have \( S_1 \cup \{ u \} \notin \mathcal{I} \) for every \( u \in N_1 \) according to the above process. So \( S_1 \) is a base of \( S_1 \cup N_1 \), which implies that \( |N_1| \leq k \cdot |S_1| \), contradicting the assumption that \( |N_1| = |O_{1,1}| > k \). Consequently, it must hold that \( |B_2| > k \) and hence \( |O_{1,2}| = k \) and \( |N_2| > 2k \). Using a similar argument, we can recursively get that \( |B_i| > k \) and hence \( |O_{1,i}| = k \) and \( |N_i| > ik \) for any \( i = 3, \ldots, h \), e.g., \( |N_h| > hk \). However, as \( S_h \) is a base of \( S_h \cup O_{1}(\phi) \), we should have \( |N_h| = |O_{1}(\phi)| \leq hk \), which shows a contradiction. Therefore, we can conclude that \( |O_{1,1}| \leq k \) for all \( i = 1, 2, \ldots, h \).

According to the partition \( O_{1,i} : i \in [h] \) constructed above, it is obvious that for every \( u \in O_{1,i} \), \( S_{i-1} \cup \{ u \} \notin \mathcal{I} \). This implies that for every \( u \in O_{1,i} \), \( u \) cannot be considered before \( u \) is added by \( \pi_A \), i.e., \( \psi_u(\phi) \subseteq \psi_u(\phi) \). Meanwhile, due to the greedy rule of \( \text{AdaptRandomGreedy} \), it follows that \( \Delta(u | \psi_u(\phi)) \geq \Delta(u | \psi_u(\phi)) \) for each \( u \in O_{1,i} \). Hence,

\[
\sum_{u \in O_{1,i}} \Delta(u | \psi_u(\phi)) \leq \sum_{u \in O_{1,i}} \Delta(u | \psi_u(\phi)) \leq \sum_{u \in O_{1,i}} \Delta(u | \psi_u(\phi)) \leq k \cdot \Delta(u | \psi_u(\phi))
\]

holds for any \( i \in [h] \). Combining the above results completes the proof.

\[\Box\]

### C.4. Proof of Lemma 11

**Proof.** Again, since \( f_{\text{avg}}(\pi_A) = \mathbb{E}_{\pi_A} \left[ \mathbb{E}_{\phi} \left[ \sum_{u \in N(\pi_A, \phi)} \Delta(u | \psi_u(\phi)) \right] \right] \), we only need to prove that, for any \( \phi \in Z^N \),

\[
\mathbb{E}_{\pi_A} \left[ \sum_{u \in O_2(\phi)} \Delta(u | \psi_u(\phi)) \right] \leq \frac{1 - p}{p} \cdot \mathbb{E}_{\pi_A} \left[ \sum_{u \in N(\pi_A, \phi)} \Delta(u | \psi_u(\phi)) \right].
\]

Given a realization \( \phi \in Z^N \), for each \( u \in N \), let \( X_u \) be a random variable such that \( X_u = 1 \) if \( u \in O_2(\phi) \) and \( X_u = 0 \) otherwise. So we have

\[
\sum_{u \in O_2(\phi)} \Delta(u | \psi_u(\phi)) = \sum_{u \in N} (X_u \cdot \Delta(u | \psi_u(\phi))).
\]

Similarly, for each \( u \in N \), let \( Y_u \) be a random variable such that \( Y_u = 1 \) if \( u \in N(\pi_A, \phi) \) and \( Y_u = 0 \) otherwise. Thus,

\[
\sum_{u \in N(\pi_A, \phi)} \Delta(u | \psi_u(\phi)) = \sum_{u \in N} (Y_u \cdot \Delta(u | \psi_u(\phi))).
\]

Therefore, it is sufficient to prove:

\[
\forall u \in N : \mathbb{E}_{\pi_A} \left[ X_u \cdot \Delta(u | \psi_u(\phi)) \right] \leq \frac{1 - p}{p} \cdot \mathbb{E}_{\pi_A} \left[ Y_u \cdot \Delta(u | \psi_u(\phi)) \right]
\]

Observe that, for any given \( u \in N \), if \( \Delta(u | \psi_u(\phi)) \leq 0 \) or \( \text{dom}(\psi_u(\phi)) \cup \{ u \} \notin \mathcal{I} \), then we have \( u \notin N(\pi_A, \phi) \) and \( u \notin O_2(\phi) \) by definition, which indicates \( X_u = Y_u = 0 \). Consider the event that \( \Delta(u | \psi_u(\phi)) > 0 \) and \( \text{dom}(\psi_u(\phi)) \cup \{ u \} \in \mathcal{I} \), and denote such an event as \( \mathcal{E}_u \). Since \( \Pr[u \in N(\pi_A, \phi) \mid \mathcal{E}_u] = p \), it is trivial to see that

\[
\mathbb{E}_{\pi_A} \left[ Y_u \cdot \Delta(u | \psi_u(\phi)) \right] = p \cdot \mathbb{E}_{\psi_u(\phi)} \left[ \Delta(u | \psi_u(\phi)) \mid \mathcal{E}_u \right] \cdot \Pr[\mathcal{E}_u],
\]

where the expectation is taken over the randomness of \( \psi_u(\phi) \) (i.e., \( \psi_u(\phi) \sim \mathcal{E}_u \)) due to the internal randomness of algorithm. On the other hand, if \( u \in O(\phi) \), then we have \( \Pr[u \in O_2(\phi) \mid \mathcal{E}_u] = 1 - p \) as \( u \) is discarded with probability of \( 1 - p \), while we also have \( \Pr[u \in O_2(\phi) \mid \mathcal{E}_u] = 0 \) if \( u \notin O(\phi) \). Thus, we know \( \Pr[u \in O_2(\phi) \mid \mathcal{E}_u] \leq (1 - p) \) and hence we can immediately get

\[
\mathbb{E}_{\pi_A} \left[ X_u \cdot \Delta(u | \psi_u(\phi)) \right] \leq (1 - p) \cdot \mathbb{E}_{\psi_u(\phi)} \left[ \Delta(u | \psi_u(\phi)) \mid \mathcal{E}_u \right] \cdot \Pr[\mathcal{E}_u].
\]

The lemma then follows by combining all the above reasoning.

\[\Box\]
C.5. Proof of Theorem 4

Proof. According to Lemmas 9–11, we have

\[
    f_{\text{avg}}(\pi \oplus \pi_{\text{opt}}) - f_{\text{avg}}(\pi, A) \leq E_{\pi, A, \Phi} \left[ \sum_{u \in N(\pi_{\text{opt}}, \Phi) \setminus N(\pi, A, \Phi)} \Delta(u \mid \psi_u(\Phi)) \right]
\]

\[
    \leq E_{\pi, A, \Phi} \left[ \sum_{u \in O_1(\Phi)} \Delta(u \mid \psi_u(\Phi)) + \sum_{u \in O_2(\Phi)} \Delta(u \mid \psi_u(\Phi)) \right]
\]

\[
    \leq \left( k + \frac{1 - p}{p} \right) \cdot f_{\text{avg}}(\pi, A)
\]

where the second inequality is due to the definition of \( O_3(\Phi) \), i.e., \( \Delta(u \mid \psi_u(\Phi)) \leq 0 \) for every \( u \in O_3(\Phi) \). Combining the above result with Lemma 8 gives

\[
    f(\pi_{\text{opt}}) \leq \frac{kp + 1}{p(1 - p)} \cdot f_{\text{avg}}(\pi, A).
\]

Moreover, \( \frac{kp + 1}{p(1 - p)} \) achieves its minimum value of \( (1 + \sqrt{k + 1})^2 \) at \( p = (1 + \sqrt{k + 1})^{-1} \). Finally, the \( \mathcal{O}(nr) \) time complexity is evident, as the algorithm incurs \( \mathcal{O}(n) \) oracle queries for each selected element. \( \square \)