NORMALIZED INTERTWINING OPERATORS AND NILPOTENT
ELEMENTS IN THE LANGLANDS DUAL GROUP

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Abstract. Let $F$ be a local non-archimedian field and $G$ be a split reductive
group over $F$ whose derived group is simply connected. Set $G = G(F)$. Let also
$\psi : F \to \mathbb{C}^\times$ be a non-trivial additive unitary character of $F$. For two parabolic
subgroups $P$ and $Q$ in $G$ with the same Levi component $M$ we construct an explicit
unitary isomorphism $F_{P,Q,\psi} : L^2(G/\lfloor P, P \rfloor) \cong L^2(G/\lfloor Q, Q \rfloor)$ commuting with the
natural actions of the group $G \times M/[M,M]$ on both sides. In some special cases
$F_{P,Q,\psi}$ is the standard Fourier transform. The crucial ingredient in the definition
is the action of the principal $\mathfrak{sl}_2$-subalgebra in the Langlands dual Lie algebra $\mathfrak{m}^\vee$
on the nilpotent radical $\mathfrak{u}^\vee$ of the Langlands dual parabolic.

For $M$ as above and using the operators $F_{P,Q,\psi}$ we define a Schwartz space
$\mathcal{S}(G,M)$. This space contains the space $C_c(G/\lfloor P, P \rfloor)$ of locally constant compactly
supported functions on $G/\lfloor P, P \rfloor$ for every $P$ for which $M$ is a Levi component (but
doesn’t depend on $P$). We compute the space of spherical vectors in $\mathcal{S}(G,M)$ and
study its global analogue.

Finally we apply the above results in order to give an alternative treatment
of automorphic $L$-functions associated with standard representations of classical
groups (thus reproving the results of [10] using the same method as [9]).

1. Introduction

1.1. Notation. Let $F$ be a local non-archimedian field. Throughout this paper we
denote algebraic varieties over $F$ by boldface letters (e.g. $X$, $G$ etc.) and their sets of
$F$-points by the corresponding ordinary letters ($G$, $X$ etc). If $X$ is a smooth algebraic
variety over $F$ and $\omega$ is a top-degree differential form on $X$, we denote by $|\omega|$ the
corresponding complex-valued measure on $X$ (cf. for example [19]).

For $X$ as above we denote by $C(X)$ the space of locally constant functions on $X$. We
also denote by $C_c(X)$ the space of locally constant functions with compact support.

Let $p : X \to Y$ be a map of algebraic varieties. Then for a distribution $\eta$ on $X$ we
let $p_!(\eta)$ denote its direct image to $Y$ (it is well-defined if the corresponding integral
is convergent).

1.2. Some results from [3]. The main part of this paper may be viewed as a continu-
ation of [3]. In [3] we considered the following situation. Let $G$ be a split reductive
algebraic group over $F$ such that its derived group is simply connected. Let also
Let \( U \subseteq G \) be the maximal unipotent subgroup and set \( X = G / U \). Then \( X \) admits the natural action of \( G \times T \) where \( T \) is the (abstract) Cartan group of \( G \).

The variety \( X \) admits unique (up to multiplication by a constant) \( G \)-invariant top-degree differential form \( dx \). Let \( L^2(X) \) be the space of \( L^2 \)-functions on \( X \) with respect to the measure \( |dx| \).

In \([3]\) we considered (following Gelfand and Graev) the natural unitary action of the Weyl group \( W \) on the space \( L^2(X) \). For every \( w \in W \) we denote by \( F_w \) the corresponding operator. The operators \( F_w \) are given by certain explicit integral formulas which generalize the standard Fourier transform. The main property of this \( W \)-action is that it commutes with the natural \( G \) action on \( L^2(X) \) and is compatible with the natural action of \( W \) on \( T \). Since the space \( L^2(X) \) is the direct integral of all representations induced from unitary characters of a Borel subgroup \( B \) it follows that one can think about the operators \( F_w \) as the "universal" normalized intertwining operators.

In \([3]\) we also study the Schwartz space \( S(X) \) of functions on on \( X \). By the definition, it is the sum \( \sum_{w \in W} F_w(C_c(X)) \) (the sum is taken in \( L^2(X) \)). It is shown in \([3]\) that \( S(X) \) consists of locally constant functions. In \([3]\) we also compute explicitly the space of spherical vectors in \( S(X) \).

1.3. Generalization to the parabolic case. In this paper we consider the following generalization of the above results. Let \( M \subseteq G \) be a Levi subgroup defined over \( F \) and let \( M^{ab} = M/[M,M] \). For every parabolic subgroup \( P \subseteq G \) containing \( M \) we may consider the quotient \( X_P = G / [P,P] \). Then \( X_P \) is naturally a \( G \times M^{ab} \)-variety. Moreover, \( X_P \) admits a \( G \)-invariant top-degree differential form which is unique up to multiplication by a constant. Hence it makes sense to consider \( L^2(X_P) \).

The group \( G \) acts naturally on \( L^2(X_P) \) since it acts on \( X_P \). Let \( \delta_P : M^{ab} \to \mathbb{G}_m \) be the determinant of the action of \( M \) on the Lie algebra of the unipotent radical \( U_P \) of \( P \) (by definition this action is the differential of \( u \mapsto m^{-1}um \)). We define an action of \( M^{ab} \) on \( L^2(X_P) \) by setting

\[
m(f)(x) = f(xm)|\delta_P(m)|^{\frac{1}{2}}
\]

Thus \( f \mapsto m(f) \) is a unitary operator for each \( m \).

Similarly, if \( \eta \) is a distribution on \( M^{ab} \) then we set

\[
\eta(f) = \int_{M^{ab}} \eta(m)m(f).
\]

The following theorem is one of the main results of this paper (we are going to make it more precise in the next subsection).

**Theorem 1.4.** Let \( \psi : F \to \mathbb{C}^\times \) be a non-trivial character. Then the following hold.
Let $P, Q$ be any two parabolic subgroups of $G$ which contain $M$ as their Levi factor. Then there exists canonical unitary isomorphism
\[
\mathcal{F}_{P,Q,\psi} : L^2(X_P) \rightarrow L^2(X_Q)
\]
commuting with the above convolutions.
\[
\text{Remark. In fact, for (1.4) to make sense we must choose $G$-invariant measures on $G$ and $X_Q$ (both are well defined up to a constant). The choice of normalization is explained in Section 4.}
\]

We shall omit the subscript $\psi$ in $\mathcal{F}_{P,Q,\psi}$ when it does not lead to a confusion.

1.5. $\mathcal{F}_{P,Q,\psi}$ and the action of the principal nilpotent in $m^\vee$. Let us give the precise formula for the operator $\mathcal{F}_{P,Q,\psi}$. In this introduction we are going to ignore all convergence issues. The rigorous treatment is given in Section 2 and Section 4.

Recall that for $P$ and $Q$ as above there exists a non-normalized intertwining operator $\mathcal{R}_{P,Q}$ acting from $\mathcal{C}_c(X_P)$ to $\mathcal{C}(X_Q)$ which commutes with the action of $G \times M^{ab}$. This operator is given by the following formula. Let $Z_{P,Q} \subset X_P \times X_Q$ be the image of $G$ in $X_P \times X_Q$. Then for every $f \in \mathcal{C}_c(X_P)$
\[
\mathcal{R}_{P,Q}(f)(y) = \int_{(x,y) \in Z_{P,Q}} f(x) dx
\]

Remark. In fact, for (1.4) to make sense we must choose $G$-invariant measures on $G$ and $X_Q$ (both are well defined up to a constant). The choice of normalization is explained in Section 4.

We would like now to correct this operator in order to get a unitary operator from $L^2(X_P)$ to $L^2(X_Q)$. More precisely we want to construct a distribution $\eta_{P,Q,\psi}$ on $M^{ab}$ such that
\[
\mathcal{F}_{P,Q,\psi}(f) = \eta_{P,Q,\psi}(\mathcal{R}_{P,Q}(f)).
\]

Assume that $T$ is a split torus over $F$. Let $T^\vee$ denote the Langlands dual torus. Set $\Lambda_+(T) = \text{Hom}(\mathbb{G}_m, T) = \text{Hom}(T^\vee, \mathbb{G}_m)$. Let $L = \oplus L_i$ be a graded finite dimensional representation of $T^\vee$. Assuming that a certain technical condition on $L$ is satisfied (cf. Section 2) we can associate to $L$ a distribution $\eta_{L,\psi}$ on $T$ in the following way. Choosing a homogeneous $T^\vee$-eigen-basis for $L$ we may identify $L$ with a collection $\lambda_1, ..., \lambda_k$ of elements of $\Lambda_+(T)$ (with multiplicities) with integers $n_1, ..., n_k$ (the corresponding degrees) attached to them. Set $s_i = n_i \frac{\text{char}(F)}{2}$. Then $\eta_{L,\psi}$ is the convolution of distributions $(\lambda_i)(\psi(t)|t|^{s_i} dt)$ (the technical condition mentioned above guarantees the convergence of the above convolutions).

Let $M^\vee$ be the Langlands dual group of $M$. Then $(M^{ab})^\vee = Z(M^\vee)$ (the center of $M^\vee$). Let $u^\vee_p$ and $u^\vee_q$ denote the nilpotent radicals of the parabolic subalgebras of $g^\vee$ dual to $P$ and $Q$. Let $u^\vee_{P,Q} = u^\vee_p \cap u^\vee_q$. Let $e, h, f$ be a principal $sl_2$-triple inside $m^\vee$. Consider $u^\vee_{P,Q}$. This is naturally a representation of $Z(M^\vee)$ graded by the eigenvalues of $h$. Set $\eta_{P,Q,\psi} = \eta_{u^\vee_{P,Q}}^{\psi, \psi}$. 

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Theorem 1.6.  
(1) There exists a subspace \( C_0^c(X_P) \subset C_c(X_P) \) which is dense in \( L^2(X_P) \) such that for every \( f \in C_c(X_P) \) we have \( R_{P,Q}(f) \in C_c(X_Q) \).

(2) For any \( f \in C_0^c(X_P) \) define
\[
F_{P,Q,\psi}(f) = \eta_{P,Q,\psi}(R_{P,Q}(f)).
\]

Then \( F_{P,Q,\psi} \) extends to a unitary operator from \( L^2(X_P) \) to \( L^2(X_Q) \) which satisfies all the requirements of Theorem 1.4.

1.7. Example. Let \( V \) be a vector space over \( F \) of dimension \( n \) and let \( \overline{V} \) denote the corresponding algebraic variety over \( F \). Consider \( G = \text{SL}(V) \). Let \( P \subset G \) be the stabilizer of a line \( l \) in \( V \) and let \( Q \) be the stabilizer of a line in \( V^* \) (which is in generic position with respect to \( l \)). In this case \( X_P = V \setminus \{0\} \) and \( X_Q = V^* \setminus \{0\} \). Hence we have
\[
L^2(X_P) = L^2(V) \quad \text{and} \quad L^2(X_Q) = L^2(V^*).
\]

In this case \( M^{ab} = \mathbb{G}_m \) and it is easy to see that \( \eta_{P,Q,\psi}(t) = \psi(t)|\frac{2\pi}{t}| \) where \( n = \dim V \). Also
\[
Z_{P,Q} = \{(v,v^*) \in V \times V^* \mid \langle v, v^* \rangle = 1\}.
\]

It is easy to check that \( F_{P,Q,\psi} \) is just the Fourier transform (corresponding to \( \psi \)) acting from \( L^2(V) \) to \( L^2(V^*) \).

1.8. The space \( S(G,M) \). For a Levy subgroup \( M \) of \( G \) let us denote by \( \mathcal{P}(M) \) the set of all parabolic subgroups of \( G \) containing \( M \). It follows from Theorem 1.4 that for any \( P, Q \in \mathcal{P}(M) \) we may identify the spaces \( L^2(X_P) \) and \( L^2(X_Q) \). Thus we may regard all of them as one vector space which we shall denote by \( L^2(G,M) \). We define
\[
S(G,M) = \sum_{P \in \mathcal{P}(M)} C_c(X_P) \subset L^2(G,M).
\]

In the situation of Section 1.7 one can prove that \( S(G,M) \) is equal to the space \( C_c(V) \) (which is isomorphic to \( C_c(V^*) \) by means of Fourier transform). In the general case we don’t have such a nice “local” description of \( S(G,M) \).

We study \( S(G,M) \) in Section 3 in some detail. In particular we compute its subspace of spherical vectors (i.e. vectors invariant with respect to a standard maximal compact subgroup).

Remark. The formula for the spherical vectors in \( S(G,M) \) is “essentially equivalent” to the formula for the intersection cohomology sheaf on Drinfeld’s compactification of the moduli space of \( \mathbb{P} \)-bundles on a smooth projective algebraic curve (this intersection cohomology sheaf is studied in [1] (cf. also [6] for the Borel case)). We don’t have a geometric explanation for this phenomenon (the main reason for this is that at the moment we don’t have an algebro-geometric analogue of the operators \( F_{P,Q,\psi} \)).
1.9. Contents. This paper is organized as follows. In Section 2 we collect some auxiliary results about distributions on a torus that will be used later. In Section 3 we recall some results from [7], [12] and [3] and reformulate them in a little different language. Section 4 is devoted to the proof of Theorem 1.6. In Section 5 we study the Schwartz space $S(G, M)$ and compute its subspace of spherical vectors. In Section 6 we study the analogue of $S(G, M)$ when $F$ is replaced by a global field $K$. We also formulate and prove certain analogue of the Poisson summation formula for the operators $F_{P, Q, \psi}$ (in the situation of Section 1.7 it becomes the standard Poisson summation formula for the Fourier transform).

Finally in Section 7 we sketch how the above results may be applied in order to define and study the (local and automorphic) L-functions associated with the standard representation of every classical group, generalizing directly the method of [9] where this is done for $GL(n)$. These L-functions were studied in [10] by a different method and in [17] by a method which is essentially equivalent to ours. However, the language of [17] was more complicated since the operators $F_{P, Q, \psi}$ and the above mentioned Poisson summation formula were not used there explicitly. Thus one should think about Section 7 as a reformulation of [17] using the results of this paper.

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2. The distributions $\eta_{L, \psi}$

2.1. Good distributions on $F^\times$. Let $\eta$ be a distribution on $F^\times$. We say that $\eta$ is good if the following conditions hold:

1) For every open compact subgroup $K$ of $O^\times$ there exists a positive real number $a = a(K)$ such that the integral

$$\int_{t \in K} \eta(tx)$$

vanishes for all $x$ such that $||x|| > a$.

2) For every character $\chi : F^\times \to \mathbb{C}^\times$ the Laurent power series

$$\sum_{n=-\infty}^{\infty} z^n \int_{||x||=q^{-n}} \eta(x)\chi(x)$$

converges to a rational function $m_{\eta, \chi}(z)$. Note that the above series is indeed a Laurent power series if we assume that 1 holds.

Let $\widehat{F}^\times$ denote the group of all characters of $F^\times$. This set has a natural structure of an algebraic variety over $\mathbb{C}$ which is isomorphic to a disjoint union of infinitely many copies of $\mathbb{C}^\times$ indexed by $\text{Hom}(O^\times, \mathbb{C}^\times)$.

Conditions 1 and 2 above imply that every good distribution $\eta$ on $F^\times$ defines a rational function $M(\eta)$ on $\widehat{F}^\times$ (namely for generic $\chi$ we have $M(\eta)(\chi) = m_{\eta, \chi}(1)$).
In this way $\mathcal{M}$ becomes an isomorphism between the space of good distributions on $F^\times$ and the space of rational functions on $\hat{F}^\times$.

The basic example of a good distribution is the following. Let $\psi : F \to \mathbb{C}^\times$ as before be a non-trivial additive character of $F$. Let also $s$ be any complex number. Consider the distribution
\[ \eta_s^\psi = \psi(x)|x|^s|dx|. \]

**Lemma 2.2.** $\eta_s^\psi$ is good.

The proof is left to the reader.

**Remark.** In this paper we are always going to normalize the measure $|dx|$ in such a way that the Fourier transform
\[ f(x) \mapsto \int f(x)\psi(xy)|dx| \]

is a unitary operator (of course such normalization depends on the choice of $\psi$).

### 2.3. Good distributions on a torus.

We now generalize the above definitions to the case of an arbitrary split torus over $F$. Let $T$ be such a torus. We denote by $\Lambda_+(T)$ and $\Lambda^+(T)$ the corresponding coweight and weight lattices. These lattices are naturally dual to each other. Note that $T$ is canonically defined over $\mathbb{Z}$ and $T(O)$ is its maximal compact subgroup.

We have the natural valuation map $v : T \to \Lambda_+(T)$ defined as follows. Let $t \in T$. Then for every $\lambda \in \Lambda^+(T)$ we define
\[ v(t)(\lambda) = \text{the valuation of } \lambda(t) \in F^\times. \]

In this way $v$ defines an isomorphism between the quotient $T/T(O)$ and $\Lambda_+(T)$. For $\gamma \in \Lambda_+(T)$ we set $T_\gamma = v^{-1}(\gamma)$.

Let $t_\mathbb{R} = \Lambda_+(T) \otimes \mathbb{R}$ and let $K \subset t_\mathbb{R}$ be a closed cone satisfying the following conditions:

a) $K$ is generated by finitely many elements of $\Lambda_+(T)$.
b) The interior of $K$ is open in $t_\mathbb{R}$.
c) $K$ contains no straight lines.

Let also $\hat{T} = \text{Hom}(T, \mathbb{C}^\times)$. Then $\hat{T}$ has a natural structure of an algebraic variety over $\mathbb{C}$ which is isomorphic to a disjoint union of infinitely many copies $(\mathbb{C}^\times)^{\dim T}$ indexed by $\text{Hom}(T(O), \mathbb{C}^\times)$.

Let $\eta$ be a distribution on $T$. We say that $\eta$ is $K$-good if the the following conditions hold:

1) For every open compact subgroup $K$ of $T$ there exists $\gamma_0 \in \Lambda_+(T)$ such that the integral
\[ \int_{t \in K} \eta(tx) \]
vanishes for every \( x \in T \) such that \( v(x) \not\in \gamma_0 + \mathcal{K} \). We denote by \( \text{Av}_K(\eta) \) the \( K \)-invariant distribution on \( T \) given by the above integral (where \( x \) is considered as a variable). We also denote by \( \text{Av}_K(\eta)_\gamma \) its restriction to \( T_\gamma \).

2) \( \text{Av}_K(\eta) \) has polynomial growth, i.e. there exists a polynomial function \( p \in \mathbb{R} \) and a functional \( \lambda \in t_\mathbb{R}^* \) such that for every \( \gamma \in \Lambda_*(T) \) we have

\[
||\text{Av}_K(\eta)_\gamma||_{L^1} \leq |p(\gamma)|q^{\lambda(\gamma)},
\]

where \( ||\text{Av}_K(\eta)||_{L^1} \) denotes the \( L^1 \)-norm of \( \text{Av}_K(\eta)_\gamma \).

Condition 2 implies that there exists an open subset of every connected component of \( \hat{T} \) such that for every \( \chi \) in this subset the integral

\[
\int_T \eta(t)\chi(t) = \sum_{\gamma \in \Lambda_*(T)} \int_{T_\gamma} \eta(t)\chi(t)
\]

is absolutely convergent. Thus we can impose the following (last) condition on \( \eta \):

3) \( \eta \) defines a rational function on \( \hat{T} \), i.e. there exists a rational function \( M(\eta) \) on \( \hat{T} \) such that for any \( \gamma \) in this subset the integral

\[
\int_T \eta(t)\chi(t) = \sum_{\gamma \in \Lambda_*(T)} \int_{T_\gamma} \eta(t)\chi(t)
\]

is absolutely convergent. Thus we can impose the following (last) condition on \( \eta \):

We denote the space of \( \mathcal{K} \)-good distributions by \( \mathcal{D}_K(T) \).

Given \( \eta_1, \eta_2 \in \mathcal{D}_\sigma(T) \) the convolution \( \eta_1 \ast \eta_2 \) makes sense. Indeed, for every open compact subgroup \( K \) of \( T \) the convolution \( \text{Av}_K(\eta_1) \ast \text{Av}_K(\eta_2) \) makes sense because it is defined by a proper integral. Let now \( \phi \) be a test function, i.e. a compactly supported function on \( T \) which is invariant under some maximal compact subgroup \( K \subset T \). Then

\[
(\eta_1 \ast \eta_2)(\phi) = \text{Av}_K(\eta_1) \ast \text{Av}_K(\eta_2)(\phi)
\]

It is easy to see that

\[
\mathcal{M}(\eta_1 \ast \eta_2) = \mathcal{M}(\eta_1) \cdot \mathcal{M}(\eta_2).
\]

It is clear from this formula that \( \eta_1 \ast \eta_2 \in \mathcal{D}_K(T) \).

2.4. Example. Let \( \lambda_1, ..., \lambda_k \) be a collection of non-zero elements of \( \mathcal{K} \). Let also \( s_1, ..., s_k \) be some complex numbers. In this case we define the distribution

\[
\eta^{s_1, ..., s_k}_{\lambda_1, ..., \lambda_k, \psi} = (\lambda_1)! (\eta_{\psi, s_1}) \ast ... \ast (\lambda_k)! (\eta_{\psi, s_k}).
\]

If \( T = \mathbb{G}_m \) and all \( \lambda_i \) are equal to the standard character of \( \mathbb{G}_m \) then we shall just write \( \eta^{s_1, ..., s_k}_{\psi} \) for the above distribution.

Let \( L = \oplus L_i \) be a graded representation of \( T^\vee \). Choosing a homogeneous \( T^\vee \)-eigenbasis we may identify \( L \) with a collection \( \lambda_1, ..., \lambda_k \) of elements of \( \Lambda_*(T) \) with integers \( n_1, ..., n_k \) attached to them. Set \( s_j = \frac{n_j}{2} \).

We say that \( L \) is \( \mathcal{K} \)-good if \( \lambda_i \in \mathcal{K} \) for every \( i \). In this case we define the distribution \( \eta_{L, \psi} \) by setting

\[
\eta_{L, \psi} = \eta^{s_1, ..., s_k}_{\lambda_1, ..., \lambda_k, \psi} \tag{2.1}
\]
It follows from Lemma 2.2 and from the fact that $L$ is $\mathcal{K}$-good that $\eta_{L,\psi}$ is well-defined and belongs to $\mathcal{D}_\mathcal{K}(T)$.

2.5. Unitarity properties. Let $X$ be an algebraic variety over $F$ endowed with a free $T$-action. Let $dx$ be top-degree differential form on $X$. Assume that there exists a character $\delta : T \to \mathbb{G}_m$ such that $d(t^{-1} \cdot x) = \delta(t)dx$. In this case we define an action of $T$ on functions on $X$ by setting

$$t(f)(x) = |\delta(t)|^{1/2} f(t^{-1}x).$$

Thus every $t \in T$ acts on the space $L^2(X, |dx|)$ as a unitary operator.

As before for a distribution $\eta$ on $T$ we write

$$\eta(f) = \int_T t(f)\eta(t).$$

In what follows we say that some statement holds for generic $f \in C_c(X)$ if it there exists a subspace $C^0_c(X)$ which is dense in $L^2(X)$ such that the above statement holds for any $f \in C^0_c(X)$.

We say that a distribution $\eta$ is unitary if for generic $f \in C_c(X)$ we have

$$||\eta(f)|| = ||f||.$$

Lemma 2.6. Let $T = \mathbb{G}_m$. Then $\eta^s_\psi$ is unitary if and only if $s = -\frac{1}{2}$. Also, $\eta^{s_1,s_2}_\psi$ is unitary on $L^2(X)$ if and only if $s_1 + s_2 = -1$.

Proof. Because of our normalizations it is enough to assume that $X = \mathbb{G}_m$ with the multiplicative measure $d^*x = \frac{|dx|}{|x|}$ on $X$. Thus we have

$$\eta^s_\psi(x) = \psi(x)|x|^s|dx| = \psi(x)|x|^{s+1}d^*x.$$

Let $A_s$ denote the operator of convolution with $\eta^s_\psi$. Define also an operator $B : C_c(F^\times) \to C(F^\times)$ by

$$B_s(f)(y) = \int_{F^\times} f(x)\overline{\psi(xy^{-1})}|x|^{s+1}|y|^{s+1}d^*x.$$ 

It is easy to see that $B_s$ is the Hermitian conjugate of $A_s$. More precisely we claim that there exists a subspace $C^0_c(F^\times)$ such that

a) $C^0_c(F^\times)$ which is dense in $L^2(F^\times)$

b) For every $f \in C^0_c(F^\times)$ both $A_s(f)$ and $B_s(f)$ lie in $C_c(F^\times)$.

c) For any $f, g \in C^0_c(F^\times)$ we have

$$\langle A_s(f), g \rangle = \langle f, B_s(g) \rangle.$$

To prove Lemma 2.6 it is enough to show that that $B_{s_2} \circ A_{s_1} = \text{Id}$ if $s_1 + s_2 = -1$ (the “only if” statement thus follows automatically).

However,
We have the Fourier transform

\[ F_X(\eta) \]

The latter integral is clearly equal \( f(z) \) if \( s_1 + s_2 = -1 \).

**Corollary 2.7.** Let \( X \) be as above. Let \( \lambda \in \Lambda_s(T) \) be any non-zero element. Then \( \eta_{\lambda,\psi} \) is unitary if and only if \( s = -\frac{1}{2} \). Also \( \eta_{s_1, s_2} \) is unitary if and only if \( s_1 + s_2 = -1 \).

**Corollary 2.8.** Let \( L' \) be a finite-dimensional \( sl(2) \times T^\vee \)-module and and \( L \) be its quotient by the space of highest weight vectors. Consider the grading on \( L \) by the eigenvalues of \( h \). Then \( \eta_{L, \psi} \) induces a unitary operator on the space \( L^2(X, |dx|) \).

**Proof.** It is easy to see that for every \( \lambda \in \Lambda_s(T) \) the multiplicity of \( n \) as an eigenvalue of \( h \) in \( L_\lambda \) is equal to the multiplicity of \( 2 - n \). Thus the proof follows from Corollary 2.7. \( \Box \)

2.9. Example: Fourier transform vs. Radon transform. Let \( p : X \to Y \) be a rank \( n \) vector bundle over a smooth variety \( Y \) defined over \( F \) and let \( p^\vee : X^\vee \to Y \) be the dual vector bundle. Let \( dy \) be a volume form on \( Y \). Let \( \omega \) be a non-vanishing section of \( \text{det} X^\vee \) on \( Y \) and let \( \omega^\vee \) be the corresponding section of \( \text{det} X \). Then \( \omega \otimes dy \) makes sense as a volume form on \( X \) which we denote by \( dx \). Similarly we define \( dx^\vee \).

We have the Fourier transform \( F_X : L^2(X, |dx|) \to L^2(X^\vee, |dx^\vee|) \).

Let \( X_0 \) (resp. \( X^\vee_0 \)) be the complement to the zero section in \( X \) (resp. in \( X^\vee \)). Note that \( L^2(X, |dx|) = L^2(X_0, |dx|) \). We let the group \( G_m \) act freely on \( X_0 \) and \( X^\vee_0 \) by setting

\[ t \cdot x = t^{-1}x \quad t \cdot x^\vee = tx^\vee \]

(where in the right hand side of the above equalities we mean usual action of scalars on the fibers of a vector bundle). The reasons for such normalization will (hopefully) become clear in the next section.

Let

\[ Z_X = \{ (x, x^\vee) \in X \times X^\vee | \langle x, x^\vee \rangle = 1 \} \]

and let \( \pi \) : \( Z_X \to X, \pi^\vee : Z_X \to X^\vee \) be the natural projections. There is a natural fiberwise volume form along the fibers of either \( \pi \) or \( \pi^\vee \). Hence the operations \( \pi_1 \) and \( \pi_1^\vee \) make sense on compactly supported functions.

For \( f \in C_c(X_0) \) let

\[ \mathcal{R}_X(f) = \pi_1^\vee \pi_1(f). \]

It is easy to see that the intersection of the support of \( \mathcal{R}_X(f) \) with every \( F^\times \)-orbit is compact. Hence the convolution of \( \mathcal{R}_X(f) \) with any distribution on \( F^\times \) is well-defined.

Let \( \eta(t) = \psi(t)|t|^{-\frac{\dim X}{2}} |dt| \).

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**Lemma 2.10.**

\[ \mathcal{F}_X(f) = \eta(\mathcal{R}_X(f)). \]

3. **Digression on the Borel case**

In this section we review the case when \( M = T \) is a maximal split torus in \( G \), i.e. all the parabolic subgroups in question are Borel subgroups. In this case the formula for \( \mathcal{F}_{P,Q} \) is essentially due to Gelfand and Graev.

Let \( B, B' \) be two Borel subgroups of \( G \). As is well-known, to such a pair one can canonically associate an element \( w \) of the Weyl group. It is clear that the collection of operators \( \mathcal{F}_{B,B'} \) satisfying the conditions of Theorem 1.4 is uniquely determined by those for which \( w \) is a simple reflection. Below we give a formula for \( \mathcal{F}_{B,B'} \) in that case.

3.1. **The case of a simple reflection.** The construction explained below is a reformulation of the construction of [1] (cf. also [12] and [3]).

Let \( \alpha \) be a simple root of \( G \) and assume that \( B \) and \( B' \) are in position \( w = s_\alpha \). Let \( \mathcal{P} \) be the minimal parabolic subgroup containing both \( B \) and \( B' \). Let \( p : X_B \to X_P \) (resp. \( p' : X_{B'} \to X_P \)) be the natural projections. Let also \( \overline{p} : X_B \to X_P \) (resp. \( \overline{p}' : X_{B'} \to X_P \)) be their affine completions (i.e. \( \overline{p} \) is the affine morphism corresponding to the sheaf of algebras \( p_* \mathcal{O}_{X_B} \) on \( X_P \)). Then \( \overline{p} \) and \( \overline{p}' \) are mutually dual vector bundles over \( X_P \). Let us explain how the natural pairing \( \kappa : X_B \times X_{B'} \to \mathbb{A}^1 \) looks like. Let \( \alpha^\vee : \mathbb{G}_m \to T \) be the simple coroot corresponding to \( \alpha \). Then \( \kappa \) is uniquely characterized by the following two requirements:

1) For every \( g \in G \) we have

\[ \kappa(g \mod U, g \mod U') = 1. \quad (3.1) \]

2) For every \( (x, y) \in X_B \times X_{B'} \) and every \( t \in T \) one has

\[ \kappa(\alpha^\vee(t)x, y) = \kappa(x, \alpha^\vee(t^{-1}y)) = t \cdot \kappa(x, y). \quad (3.2) \]

In what follows we choose \( G \)-invariant measures on \( G \) and \( X_B \) (for all Borel subgroups \( B \) defined over \( F \)) in such a way that the measure of the image of \( G(\mathcal{O}) \) is equal to 1. This measure extends naturally to \( \overline{X}_B \).

Note that \( L^2(X_B) = L^2(\overline{X}_B) \). We define \( \mathcal{F}_{B,B',\psi} \) to be the Fourier transform in the fibers of the bundle \( \overline{p} \). It makes sense as a unitary operator acting from \( L^2(X_B) \) to \( L^2(X_{B'}) \).

3.2. **The general case.** Let \( B, B' \) be any two Borel subgroups of \( G \). Then there exists a sequence \( (B_0 = B, B_1, \ldots, B_n = B') \) of Borel subgroups defined over \( F \) such that \( B_i \) and \( B_{i+1} \) are in position \( s_\alpha \), where \( \alpha \) is a simple root of \( G \) (for every \( i \)). We define

\[ \mathcal{F}_{B,B',\psi} = \mathcal{F}_{B_{n-1},B_n,\psi} \circ \ldots \circ \mathcal{F}_{B_0,B_1,\psi}. \]
It is shown in [12] that $\mathcal{F}_{B,B',\psi}$ does not depend on the choice of the sequence $(B_0, B_1, ..., B_n)$. It is clear that the operators $\mathcal{F}_{B,B',\psi}$ are unitary and satisfy the requirements of Theorem 1.4.

3.3. The distribution $\eta_{B,B',\psi}$. Let $B, B'$ be as above and assume that they are in position $w \in W$. We choose a maximal torus $T \subset B \cap B'$.

Let $\Pi_w$ denote the set of all positive with respect to $B$ coroots of $T$ which are made negative by $w$. Let also $K \subset t_\mathbb{R}$ be the cone of positive coroots.

Thus for every $\alpha \in \Pi_V$ the distribution $\eta_{\alpha,\psi}$ is $K$-good. Therefore, their convolution makes sense. We define $\eta_{B,B',\psi}$ to be equal to the convolution of $\eta_{\alpha,\psi}$ for all $\alpha \in \Pi_w$. The reader will readily check that this definition of $\eta_{B,B',\psi}$ coincides with the one given in Section 1.3.

3.4. Proof of Theorem 1.6 in the Borel case. Let $B, B'$ as before be two Borel subgroups defined over $F$. Thus we may consider the operator $R_{B,B'}$ defined by (1.4). The operator $R_{B,B'}$ is well defined as an operator from $\mathcal{C}_c(X_B)$ to $\mathcal{C}(X_{B'})$. Moreover, it is easy to see that for every $f \in \mathcal{C}_c(X_B)$ the intersection of $\text{supp} R_{B,B'}(f)$ with any orbit of $T$ is compact. Thus $\eta(R_{B,B'}(f))$ is well-defined for any distribution $\eta$ on $T$. We claim that for every $f \in \mathcal{C}_c(X_B)$ we have

$$\mathcal{F}_{B,B',\psi}(f) = \eta_{B,B',\psi}(R_{B,B'}(f)).$$

Let $B, B', B''$ be three Borel subgroups. Let $w_1$ be the relative position of $B$ and $B'$ and let $w_2$ be the relative position of $B'$ and $B''$. Assume that $l(w_2w_1) = l(w_2) + l(w_1)$ (here $l$ denotes the length function on $W$). It is easy to see that in this case we have

$$R_{B',B''} \circ R_{B,B'} = R_{B,B''} \quad \text{and} \quad \eta_{B',B'',\psi} \ast \eta_{B,B',\psi} = \eta_{B,B'',\psi}.$$ 

Hence it is enough to check (3.3) when the relative position of $B$ and $B'$ is a simple reflection. In this case it follows from Lemma 2.10.

4. Proof of Theorem 1.4 and Theorem 1.6

4.1. Let $M$ be a Levi subgroup of $G$ and let $P, Q \in \mathcal{P}(M)$. Choose a Borel subgroup $B$ contained in $P$ and let $B'$ another Borel subgroup such that:

1) $B'$ is contained in $Q$.

2) The relative position $w$ of $B$ and $B'$ has minimal length subject to the first requirement.

Let $\pi : X_B \to X_P$, $\pi' : X_{B'} \to X_Q$ denote the natural projections. Since all the spaces in question are endowed with natural measures the operations $\pi_l$ and $\pi'_l$ may be applied to functions.

Proposition 4.2. Recall that $u_{\mathfrak{p},q}^\vee = u_{\mathfrak{p}}^\vee / u_{\mathfrak{p}}^\vee \cap u_{\mathfrak{q}}^\vee$; this space is graded by the eigenvaules of $h$ where $(e, h, f) \in \mathfrak{m}^\vee$ is a principal $\mathfrak{sl}_2$-triple.

1) For generic $\phi \in \mathcal{C}_c(X_P)$ we have $R_{P,Q}(\phi) \in \mathcal{C}_c(X_Q)$ and $R_{Q,P} \circ R_{P,Q}(\phi) \in \mathcal{C}_c(X_P)$.  

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(2) There exists unique unitary operator $G_{P,Q,\psi} : L^2(X_P) \to L^2(X_Q)$ such that for every $f \in C_c(X_B)$ we have

$$G_{P,Q,\psi}(\pi(f)) = \pi'_f(\mathcal{F}_{B,B',\psi}(f)). \quad (4.1)$$

(3) We have

$$G_{P,Q,\psi}(\phi) = \eta_{\psi,\phi}'(\mathcal{R}_{P,Q}(\phi)) \quad (4.2)$$

for generic $\phi \in C_c(X_P)$.

Proof. Let us prove 1. For a character $\chi : M^{ab} \to \mathbb{C}^\times$ we denote by $C_c(X_P)_{\chi}$ the corresponding space of $(M, \chi)$-coinvariants in $C_c(X_P)$ (this space is dual to the corresponding induced representation). It is well-known that for generic (i.e. lying in a Zariski dense subset) $\chi$ the operators $\mathcal{R}_{P,Q}$ and $\mathcal{R}_{Q,P}$ give rise to isomorphisms $C_c(X_P)_{\chi} \cong C_c(X_Q)_{\chi}$. Let now $D \subset \hat{M}^{ab}$ denote the complement to the above Zariski dense subset. Then it is clear that 1 holds for every $\phi$ whose image in $C_c(X_P)_{\chi}$ is equal to 0 for every $\chi \in D$.

Let us now prove 2 and 3. Let $\phi \in C_c(X_P)$. Then there exists a function $f \in C_c(X_B)$ such that $\pi(f) = \phi$. Therefore $G_{P,Q,\psi}(\phi)$ is uniquely defined by (4.1). On the other hand it is easy to see that for every $\phi \in C_c(X_P)$ and $f$ as above the right hand side of (4.1) is equal to (4.2). This shows that $G_{P,Q,\psi}(\phi)$ is well-defined as an operator $C_0(X_P) \to C_0(X_Q)$.

Let $\phi \in C_c(X_P), \phi' \in C_c(X_Q)$. Then it follows from (3.3) that

$$\langle G_{P,Q,\psi}(\phi), \phi' \rangle = \langle \phi, G_{Q,P,\psi}^{-1}(\phi') \rangle.$$

On the other hand, since $\mathcal{F}_{B,B',\psi}^{-1} = \mathcal{F}_{B',B,\psi}'$ it follows that that $G_{P,Q,\psi}^{-1} = G_{Q,P,\psi}'$. Thus the inverse of $G_{P,Q,\psi}$ is equal to its hermitian conjugate which means that $G_{P,Q,\psi}$ is unitary.

4.3. Proof of Theorem 1.6. We have to prove that the operator $F_{P,Q,\psi}$ defined as in Theorem 1.6 is unitary.

Let $(u_{p,q})^{junk} = u_{p,q}'/(u_{p,q}')^e$ (with grading induced by the eigenvalues of $h$). Let $\eta_{P,Q,\psi}^{junk} = \eta_{(u_{p,q}')^{junk}}$. Thus

$$\eta_{\psi,\phi}' = \eta_{(u_{p,q}')^{junk}} \eta_{P,Q,\psi}.$$

Hence it follows from Proposition 1.2 that for every $\phi \in C_c(X_P)$ we have

$$G_{P,Q,\psi} = \eta_{P,Q,\psi}^{junk}(\mathcal{F}_{P,Q,\psi}(\phi)).$$

Since $G_{P,Q,\psi}$ is a unitary operator it follows that to prove the unitarity of $F_{P,Q,\psi}$ it is enough to prove the unitarity of the operator of convolution with $\eta_{P,Q,\psi}^{junk}$. This, however, follows immediately from Corollary 2.8.

We have to prove now the assertion of Theorem 1.4(2). However, it is enough to do it in the following 2 cases:

1) $R = P$.
2) $\dim u_{p,c}' = \dim u_{p,q} + \dim u_{q,r}$.
In case 1 the statement follows immediately from the unitarity of $\mathcal{F}_{P,Q,\psi}$ and in case 2 this is obvious from the definitions.

### 4.4. Example.
Consider the case when $G = \text{SL}(n)$ and $M = \text{GL}(n-1)$ embedded into $G$ in the standard way:

\[(a_{ij})_{i,j=1}^n \mapsto \begin{pmatrix}
a_{11} & \cdots & a_{1,n-1} & 0 \\
\vdots & \ddots & \vdots & \vdots \\
a_{n-1,1} & \cdots & a_{n-1,n-1} & 0 \\
0 & \cdots & 0 & \det(a_{ij})^{-1}
\end{pmatrix}\]

There are two parabolic subgroups $\mathcal{P}$ and $\overline{\mathcal{P}}$ containing $M$. Namely, we set

\[
\mathcal{P} = \text{stab}(\text{span} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix})
\]

and $\overline{\mathcal{P}}$ is the corresponding opposite parabolic.

Let $V$ denote the defining representation of $G$. Then $X_\mathcal{P} = V \setminus \{0\}$ and $X_{\overline{\mathcal{P}}} = V^* \setminus \{0\}$. Hence we have $L^2(X_\mathcal{P}) = L^2(V)$ and $L^2(V^*)$ (where the measure on $V$ and $V^*$ is a Haar measure with respect to addition). It follows from Lemma 2.10 that $\mathcal{F}_{P,\overline{\mathcal{P}},\psi}$ is equal to the Fourier transform $\mathcal{F}_{V,\psi}$.

### 5. The space $\mathcal{S}(G,M)$

In this section we assume that the character $\psi$ is trivial on $\mathcal{O}$ and that it is non-trivial on $\pi^{-1} \mathcal{O}$.

5.1. Let $M \subset G$ be a Levi subgroup. It follows from the result of the previous section that for every two parabolic subgroups $\mathcal{P}$ and $\mathcal{Q}$ for which $M$ is the Levi factor the spaces $L^2(X_\mathcal{P})$ and $L^2(X_\mathcal{Q})$ are canonically isomorphic. Hence we may regard it as one space which we shall denote by $L^2(G,M)$.

For every $\mathcal{P}$ as above we denote by $C_c(X_\mathcal{P})$ the space of compactly supported locally constant functions on $X_\mathcal{P}$. We have the natural embedding $C_c(X_\mathcal{P}) \subset L^2(G,M)$. We define

\[
\mathcal{S}(G,M) = \sum C_c(X_\mathcal{P}) \subset L^2(G,M)
\]

(5.1)

(the sum is being taken over all parabolic subgroups in which $M$ is a Levi factor). Clearly, $\mathcal{S}(G,M)$ is a representation of $G \times M^{\text{ab}}$. In the case $M = T$ this space has been studied in [3].
5.2. Example. Assume that we are in the situation of Section 4.4.

Lemma 5.3. In this case we have

\[ \mathcal{S}(G, M) = \mathcal{C}_c(V). \]

Proof. Since \( \mathcal{C}_c(V) \) is invariant under Fourier transform, it follows that \( \mathcal{S}(G, M) \) is a subspace of \( \mathcal{C}_c(G, M) \). Since \( \mathcal{C}_c(X_P) = \mathcal{C}_c(V \setminus \{0\}) \) has codimension 1 in \( \mathcal{C}_c(V) \), in order to prove the opposite inclusion it is enough to find \( f \in \mathcal{C}_c(V^* \setminus \{0\}) \) whose Fourier transform does not vanish. Let \( K = G(\mathcal{O}) \) be the standard maximal compact subgroup of \( G \).

In the case of arbitrary \( G \) and \( M \) we don’t know any ”nice” description of \( \mathcal{S}(G, M) \). Let us, however, discuss some simple properties of its elements.

For every \( f \in \mathcal{S}(G, M) \) we denote by \( f_P \) the corresponding function on \( X_P \).

Lemma 5.4. Let \( f \in \mathcal{S}(G, M) \). Then \( f_P \) is a locally constant function on \( X_P \) for every \( P \in P(M) \).

Proof. Clearly it is enough to show the following: let \( P, Q \in P(M) \) and let \( h \in \mathcal{C}_c(X_P) \). Then \( F_{P,Q,\psi}(h) \) is locally constant on \( X_Q \).

To prove this let us note that \( h \) is fixed by some compact subgroup \( C \) of \( G \). Since \( F_{P,Q} \) commutes with the action of \( G \) the same is true for \( F_{P,Q}(h) \). Since \( G \) acts transitively on \( X_Q \) this implies that \( h \) is locally constant.

Let \( \Lambda_*(M) \) denote the lattice of cocharacters of \( M^{ab} \). Let also \( \Lambda_* = \Lambda_*(T) \) be the coroot lattice of \( G \). We have the natural restriction map \( \Lambda_* \to \Lambda_*(M) \).

Fix \( \gamma \in \Lambda_*(M) \). Let \( \tilde{\gamma} \) be any lift of \( \gamma \) to an element of \( \Lambda_*(M) \). It is easy to see that the \( K \)-orbit of \( \tilde{\gamma}(\pi) \mod [P, P] \) in \( X_P \) depends only on \( \gamma \). We denote this orbit by \( X_\gamma^P \).

The following lemma is well-known:

Lemma 5.5. The assignment \( \gamma \mapsto X_\gamma^P \) is a one-to-one correspondence between \( \Lambda_*(M) \) and the set of \( K \)-orbits on \( X_P \).

Let \( \Lambda_+ \) denote the set of all linear combinations of positive coroots of \( G \) with non-negative coefficients. We say that \( \gamma \in \Lambda_*(M) \) is positive if it is equal to the image of some element of \( \Lambda^+ \). We denote by \( \Lambda_*(M)^+ \) the set of positive elements in \( \Lambda_*(M) \). We say that a function \( f \) on \( X_P \) has bounded support if there exists \( \gamma \in \Lambda_*(M) \) such that

\[ \text{supp } f \subset \bigcup_{\gamma' \in \gamma + \Lambda_*(M)^+} X_{\gamma'}^P. \] (5.2)

Conjecture 5.6. Let \( f \in \mathcal{S}(G, M) \). Then for every \( P \in P(M) \) the function \( f_P \) on \( X_P \) has bounded support.

We don’t know how to prove this conjecture in general. In [3] we proved it for \( M = T \). It follows also from Theorem 5.10 below that Conjecture 5.6 holds also for \( K \)-invariant elements in \( \mathcal{S}(G, M) \).
5.7. $S(G, M)$ as a module over $\mathcal{H}(M^{ab})$. Let $\mathcal{H}(M^{ab})$ denote the Hecke algebra of $M^{ab}$. We have the natural decomposition

$$\mathcal{H}(M^{ab}) = \bigoplus_{\sigma \in \text{Hom}(M^{ab}(O)\to \mathbb{C}^\times)} \mathcal{H}(M^{ab})_{\sigma}. \quad (5.3)$$

For each $\sigma$ the algebra $\mathcal{H}(M^{ab})_{\sigma}$ is non-canonically isomorphic to the algebra of Laurent polynomials $\mathbb{C}[t_1, \ldots, t_l, t_1^{-1}, \ldots, t_l^{-1}]$ where $l = \dim M^{ab}$.

The space $S(G, M)$ is naturally a module over $M^{ab}$ and hence over $\mathcal{H}(M^{ab})$. Let $S(G, M)_{\sigma}$ denote the part of $S(G, M)$ on which $M^{ab}(O)$ acts by means of the character $\sigma$.

The following lemma is never used in the sequel but we think that it gives some intuition about the space $S(G, M)$:

**Lemma 5.8.** $S(G, M)_{\sigma}$ is a free module over $\mathcal{H}(M^{ab})_{\sigma}$ (as a $G$-module).

**Proof.** First of all by a theorem of Quillen it is enough to prove that $S(G, M)_{\sigma}$ is locally free over $\mathcal{H}(M^{ab})_{\sigma}$.

It is clear that for each $P$ the $G$-module $C_c(X_P)_{\sigma}$ is free over $\mathcal{H}(M^{ab})_{\sigma}$. Let now $\chi : M^{ab} \to \mathbb{C}^\times$ be a character such that $\chi|_{M^{ab}(O)} = \sigma$. We denote by $S(G, M)_{\chi}$ (resp. $C_c(X_P)_{\chi}$) the $G$-module of $(M^{ab}, \chi)$-coinvariants on $S(G, M)$ (resp. on $C_c(X_P)$). It is now easy to see that for every $\chi$ there exists $P \in \mathcal{P}(M)$ such that the inclusion $C_c(X_P)_{\chi} \hookrightarrow S(G, M)_{\chi'}$ for every $\chi'$ in some neighbourhood of $\chi$ which implies what we need. \hfill $\square$

5.9. $K$-invariant vectors. For every $\gamma \in \Lambda_*(M)$ we denote by $\delta_{P, \gamma}$ the following function on $X_P$:

$$\delta_{P, \gamma}(x) = \begin{cases} q^{\langle \gamma, \rho - \rho_M \rangle}, & \text{if } x \in X_P^\gamma \\ 0, & \text{otherwise.} \end{cases}$$

Let $L = \bigoplus L_i$ be a graded representation of $\mathbb{Z}(M^\vee)$. Assume that for each $\gamma \in \Lambda_*(M)$ the multiplicity of $\gamma$ in $L$ is finite. Then we define a function $\phi^L_P \in C(X_P)^K$ by setting

$$\phi^L_P |_{X_P^\gamma} = \sum \dim L_i \gamma^{-i} q^{-i \langle \gamma, \rho - \rho_M \rangle} \quad (5.4)$$

Similarly, for every $\mu \in \Lambda_*(M)$ we define

$$\phi^L_{P, \mu} |_{X_P^\gamma} = \sum \dim L_i \gamma^{-i} q^{-i \langle \gamma, \rho - \rho_M \rangle} \quad (5.5)$$

In other words,

$$\phi^L_{P, \mu} = \sum \dim L_i \gamma^{-i} q^{-i} \delta_{P, \gamma}. \quad (5.5)$$

Let $e, f, h \in \mathfrak{m}^\vee$ be a principal $sl_2$-triple. Take

$$L = \text{Sym}(\mathfrak{u}_e^\vee)^e).$$

Clearly $L$ has a natural action of $\mathbb{Z}(M^\vee)$. Also the $L$ carries a natural action of $h$. \hfill 15
Define a grading on $L$ in the following way. Let $x \in \text{Sym}^k((u_p^\vee)^f)$ Assume that $x$ has eigenvalue $j$ with respect to $h$. Then we say that $x$ has grading $k + j$.

We set $c_{P,\mu} = \phi_{P,\mu}^L$ for $L$ as above.

We would like to understand the structure of $S(G, M)^K$. Since every $K$-orbit in $X_P$ is $M^{ab}(O)$-invariant it follows that every element of $S(G, M)^K$ is automatically $M^{ab}(O)$-invariant. Thus $M^{ab}$-action on $S(G, M)$ reduces to an $M^{ab}/M^{ab}(O) = \Lambda_*(M)$-action on $S(G, M)$.

**Theorem 5.10.**

1. For every $P$ and $\mu$ as above we have $c_{P,\mu} \in S(G, M)$.
2. For every $P \in \mathcal{P}(M)$ the functions $c_{P,\mu}$ (where $\mu$ runs over all elements of $\Lambda_*(M)$) form a basis in $S(G, M)^K$.
3. For every $P, Q \in \mathcal{P}(M)$ we have $\mathcal{F}_{P,Q,\psi}(c_{P,\mu}) = c_{Q,\mu}$.

**Proof.** We argue along the lines of the proof of Theorem 3.13 in [3].

First of all we claim that $c_{P,\mu} \in L^2(X_P)$. Recall that we denote by $L$ the space $\text{Sym}((u_p^\vee)^e)$ with the grading discussed above. For every $\gamma \in \Lambda_*(M)$ let

$$K_P(\gamma) = \sum \dim L_i q^{-i}.$$ 

Then to prove that $c_{P,\mu} \in L^2(X_P)$ we must show that the series

$$\sum_{\gamma \in \Lambda_*(M)} K_P(\gamma)^2$$

is convergent. However, it is easy to see that the series

$$\sum_{\gamma \in \Lambda_*(M)} K_P(\gamma)$$

is convergent to $\prod_{i=1}^{\infty} (1 - q^{-i})^{\dim(u_p^\vee)^e}$. Hence (5.7) is convergent too.

Since $c_{P,\mu} \in L^2(X_P)$ it follows that $\mathcal{F}_{P,Q,\psi}(c_{P,\mu})$ is well-defined. Let us show that point 3 of Theorem 5.10 implies points 1 and 2.

We have the natural isomorphism $\mathbb{C}[Z(M^\vee)] \simeq \mathbb{C}[\Lambda_*(M)]$ where $\mathbb{C}[Z(M^\vee)]$ denotes the algebra of regular functions on $Z(M^\vee)$ and $\mathbb{C}[\Lambda_*(M)]$ denotes the group algebra of $\Lambda_*(M)$. Let $\mathcal{W}$ denote the span of $c_{P,\mu}$ for all $\mu \in \Lambda_*(M)$. It follows from 3 that $\mathcal{W}$ does not depend on $P$ as a subspace of $L^2(G, M)$. We may identify $\mathcal{W}$ with $\mathbb{C}[Z(M^\vee)]$ by identifying $c_{P,\mu}$ with $\mu$ (again, this doesn’t depend on $P$). For every $P \in \mathcal{P}(M)$ we set $\mathcal{V}_P = \mathcal{C}_c(X_P)^K$. To prove 1 and 2 we need to show that $\mathcal{V}_P \subset \mathcal{W}$ for every $P$ and that $\mathcal{W} = \text{span}\{\mathcal{V}_P\}_{P \in \mathcal{P}(M)}$.

Let $\kappa : SL(2) \to M^\vee$ be the homomorphism corresponding to the $sl_2$-triple $(e, f, h)$ chosen above. Let

$$H_q = \kappa\left(\begin{array}{cc} q & 0 \\ 0 & q^{-1} \end{array}\right).$$
Define $\mathbf{d}_P \in \mathbb{C}[\mathbb{Z}(\mathbf{M}')]$ by

$$
\mathbf{d}_P(z) = \det(1 - H_q^{-1}z)|_{u_\psi}.
$$

Then by definition we have $\mathbf{d}_P(c_{P,\mu}) = \delta_{P,\mu}$. Thus $\delta_{P,\mu} \in \mathcal{W}$, hence $\nu_P \subset \mathcal{W}$ for every $P$. Moreover, as a subspace of $\mathbb{C}[\mathbb{Z}]$ the space $\nu_P$ is equal to the ideal generated by $\mathbf{d}_P$. Applying Hilbert Nullstellensatz we see that points 1 and 2 of Theorem 5.10 follow from the following lemma whose proof is left to the reader.

**Lemma 5.11.** For every $z \in \mathbb{Z}(\mathbf{M}')$ there exists $P \in \mathcal{P}(\mathbf{M})$ such that $\mathbf{d}_P(z) \neq 0$.

Let us now prove 3. For this it is enough to show that

$$
\mathcal{F}_{P,Q,\psi}(\delta_{P,\mu}) = \frac{\mathbf{d}_Q}{\mathbf{d}_P}\delta_{Q,\mu}.
$$

(5.8)

Define

$$
\tilde{\mathbf{d}}_P(z) = \det(1 - H_q^{-1}z)|_{u_\psi}' \quad \text{and} \quad \mathbf{d}_P^{\text{link}} = \det(1 - H_q^{-1}z)|_{u_\psi}/(u_\psi) = \frac{\tilde{\mathbf{d}}_P(z)}{\mathbf{d}_P(z)}.
$$

It follows from formula 3.22 in [3] that

$$
\mathcal{G}_{P,Q,\psi}(\delta_{P,\mu}) = \frac{\tilde{\mathbf{d}}_Q}{\mathbf{d}_P}\delta_{Q,\mu}.
$$

Thus to prove (5.8) we need to show that $\eta_{P,Q,\psi}^{\text{link}}(\delta_{P,\mu}) = \frac{\mathbf{d}_Q^{\text{link}}}{\mathbf{d}_Q}\delta_{P,\mu}$ which is left to the reader.

In particular we see that for every $P, Q \in \mathcal{P}(\mathbf{M})$ we have

$$
\mathcal{F}_{P,Q,\psi}(c_{P,0}) = c_{Q,0}.
$$

In other words there exists canonical element $c \in \mathcal{S}(\mathbf{G}, \mathbf{M})$ such that $c_P = c_{P,0}$ for every $P$.

**Remarks.** 1. It follows from Theorem 7.3 of [1] that the function $c_{P,0}$ is equal (in the appropriate sense) to the function obtained by traces of Frobenius on the stalks of the intersection cohomology sheaves of the Drinfeld compactification $\text{Bun}_P$ of the moduli stack of $P$-bundles on a smooth projective algebraic curve over $\mathbb{F}_q$. This is certainly not unexpected. We don’t know, however, how to prove this directly. Note, however, that the spaces $\text{Bun}_P$ (or at least their singularities) have their local counterparts because of [11] and [5].

2. In [3] we also study the space $\mathcal{S}(\mathbf{G}, \mathbf{T})^I$ where $I \subset \mathbf{G}$ is an Iwahori subgroup. We give two interpretations of this space: one using the periodic Hecke module introduced in [13] and the other using certain equivariant $K$-group. Similar descriptions are probably possible for the space $\mathcal{S}(\mathbf{G}, \mathbf{M})^I$ (the corresponding parabolic periodic Hecke module is studied in [14] and its interpretation using $K$-theory is given in [15] and [16]). We have not checked this precisely.
6. The case of global fields

6.1. Remarks on archimedian places. Let now $G$ be as before but assume that the field $F$ is archimedian. The operators $F_{P,Q,\psi}$ are defined in this case in the same way as for non-archimedian $F$ (it is not difficult to adjust the proof of unitarity of $F_{P,Q,\psi}$ to the archimedian case; we leave it to the reader).

Let $M$ be a Levi subgroup of $P$. For every $P \in \mathcal{P}(M)$ we define $S(X_P)$ in the following way. Let $X_P^\mathbb{R}$ denote the algebraic variety over $\mathbb{R}$ obtained by restriction of scalars from $F$ to $\mathbb{R}$. Let $\mathcal{D}(X_P^\mathbb{R})$ denote the algebra of differential operators on $X_P^\mathbb{R}$ with polynomial coefficients. Clearly $\mathcal{D}(X_P^\mathbb{R})$ acts on $C^\infty(X_P) = C^\infty(X_P(F)) = \mathcal{D}(X_P^\mathbb{R}(\mathbb{R}))$. We define

$$S(X_P) = \{ f \in C^\infty(X_P) \mid d(f) \in L^2(X_P) \text{ for every } d \in \mathcal{D}(X_P^\mathbb{R}) \}.$$

Conjecture 6.2. For every $P,Q \in \mathcal{P}(M)$ we have

$$F_{P,Q,\psi}(S(X_P)) = S(X_Q).$$

Example. Let $G = \text{SL}(n,\mathbb{R})$ and let $P,Q$ be as in Section 1.7. In this case we have $S(X_P) = S(V) = \text{the Schwartz space of rapidly decreasing functions on } V$. Also $S(X_Q)$ is the Schwartz space of $V^*$. Thus in this case the above conjecture reduces to the well-known fact saying that the Fourier transform maps $S(V)$ to $S(V^*)$.

Assuming that the conjecture is true we may define $S_{G,M} = S(X_P)$ for any $P \in \mathcal{P}(M)$ (Conjecture 6.2 thus says that $S(G,M)$ does not depend on the choice of $P$).

We do not know how to prove this conjecture in general. In the sequel we shall choose any $G \times M^\text{ab}$-invariant subspace of $L^2(G,M)$ containing $C_c^\infty(X_P)$ for every $P$ and consisting of $G \times M^\text{ab}$-smooth vectors. We shall denote this subspace by $S(G,M)$.

6.3. The global space $S_K(G,M)$. Let $K$ be a global field and let $\mathcal{V}(K)$ denote its set of places. We now assume that we are given a reductive group $G$ with a Levi subgroup $M$ as above, both defined over $K$. For every $v \in \mathcal{V}(K)$ we let $K_v$ denote the corresponding completion of $K$. We define $S_v(G,M) = S(G(K_v),M(K_v))$. For every finite place $v$ we have canonical spherical function $c_v \in S_v(G,M)$.

Let now $S_K(G,M)$ denote the restricted tensor product of the spaces $S_v(G,M)$ with respect to the functions $c_v$.

Theorem 6.4. There exists a unique $G(K) \times M^\text{ab}(K)$-invariant functional $\varepsilon$ on the space $S_K(G,M)$ such that the following condition holds:

Let $f \in S_K(G,M)$ and assume that for some $v \in \mathcal{V}(K)$ and $P \in \mathcal{P}(M)$ the function $f$ lies in the tensor product

$$C_v(X_P(K_v)) \otimes \bigotimes_{v' \neq v} S_{v'}(G,M).$$
Then
\[ \varepsilon(f) = \sum_{x \in \mathcal{X}_P(K)} f_P(x) \]  

(6.1)

**Proof.** Arguing as in Section 6 of [3] we see that it is enough to prove the following statement: let \(v_1, v_2\) be two places of \(K\) and assume that 
\[ f \in \mathcal{C}_c(\mathcal{X}_P(K_{v_1})) \otimes \bigotimes_{v' \neq v_1} \mathcal{S}_{\varphi}(G, M) \cap \mathcal{C}_c(\mathcal{X}_Q(K_{v_2})) \otimes \bigotimes_{v' \neq v_2} \mathcal{S}_{\varphi}(G, M). \]

Then
\[ \sum_{x \in \mathcal{X}_P(K)} f_P(x) = \sum_{x \in \mathcal{X}_Q(K)} f_Q(x). \]

By using exactly the same argument as in Section 7.22 of [3] we may see that this is equivalent to the functional equation for Eisenstein series (induced respectively from characters of \(P\) and \(Q\)). \(\square\)

### 7. Connection with \(L\)-functions for classical groups

In this section we indicate how one can use the above results in order to give construction of (and prove the standard properties) of \(L\)- and \(\varepsilon\)-functions associated with the standard representation of a classical group. The details will appear elsewhere. These \(L\)-functions were studied in the (nowadays classical) work [10]. The advantage of our approach is that it may be viewed as a “direct” generalization of the work of Godement and Jacquet ([9]) where the case of \(GL(n)\) is studied. A similar approach is discussed in [17]. The main ingredient which makes the presentation of this paper different from [10] and [17] is the space \(\mathcal{S}(G, M)\) which was missing in loc. cit. Because of this in [10] the local zeta-integrals gave the \(L\)-function divided by certain auxiliary denominator (which was equal to some product of abelian \(L\)-functions). This denominator is absent in our formulation.

#### 7.1. Let \(H\) denote one of the groups \(GL(n), Sp(2n) \times \mathbb{G}_m,\) or \(GSpin(n)\) (by the definition \(GSpin(n)\) is the quotient of \(Spin(n) \times \mathbb{G}_m\) by the diagonal copy of central \(\mathbb{Z}_2\)). To any \(H\) like that we associate another reductive group \(G\) together with a maximal parabolic subgroup \(P\) in the following way.

1. If \(H = GL(n)\) we set \(G = SL(n^2), P\) is the stabilizer of a line in the standard representation.

2. If \(H = Sp(2n) \times \mathbb{G}_m\) we set \(G = Sp(4n)\) and take \(P\) to consist of all matrices that stabilize a Lagrangian subspace.

3. If \(H = GSpin(n)\) we set \(G = Spin(2n)\) and take \(P\) to be the stabilizer of a maximal isotropic subspace in the standard representation.

In all these cases we denote by \(M\) the corresponding Levi subgroup of \(G\).

For \(H\) as above there is a natural character \(\sigma : H \rightarrow \mathbb{G}_m\) (in case 1 we have \(\sigma = \text{det}\); in cases 2 \(\sigma\) is just the projection to the second multiple; in case 3 if \(g \in GSpin(n)\) is the image of an element \((g', t)\) in \(Spin(n) \times \mathbb{G}_m\) then we let \(\sigma(g) = t^2\).
Let \( \tilde{H}^2 \) denote the set of all pairs \((h_1, h_2) \in H^2\) such that \(\sigma(h_1) = \sigma(h_2)\). Let \(H_1\) denote the kernel of \(\sigma\) in \(H\). We have the natural diagonal embedding \(\Delta : H_1 \rightarrow \tilde{H}^2\). Hence \(H_1\) acts naturally on \(\tilde{H}^2\) (on the right). Moreover the quotient \(\tilde{H}^2/H_1\) is naturally isomorphic to \(H\).

We claim now that in all of the above cases there exists an embedding \(\eta : \tilde{H}^2 \hookrightarrow G \times M^{ab}\) such that the following hold. Let

\[ \tilde{P} = \{(g, m) \in P \times M^{ab} \mid \text{such that the image of } g \text{ in } M^{ab} \text{ is equal to } m\} \]

Thus we have the natural isomorphism

\[ G/[P, P] \simeq G \times M^{ab}/\tilde{P}. \]

Then we require that

(i) \(\eta^{-1}(\tilde{P}) = H_1\).

(ii) The resulting map \(\tilde{H}^2/H_1 \hookrightarrow G/[P, P]\) is an open embedding.

Let us explain the construction of \(\eta\). From now on we shall present all the constructions in cases 1 and 2 only. Case 3 is always similar to Case 2 and we leave it to the reader.

**Case 1.** In this case we identify \(G\) with \(SL(M_n)\) where \(M_n\) denotes the space of \(n \times n\)-matrices and we take \(P\) to be the stabilizer of the line spanned by the identity matrix. Then for all \((g_1, g_2) \in \tilde{H}^2\) the projection of \(\eta(g_1, g_2)\) to \(SL(M_n)\) takes every \(x \in M_n\) to \(g_1 x g_2^{-1}\). The composition of \(\eta\) with the projection to \(M^{ab} \simeq G_m\) is equal to \(\deg g_1^{-1} = \det g_2^{-1}\). In this case we have \(X_P = M_n \setminus \{0\}\) and the resulting embedding \(G \hookrightarrow X_P\) is the natural one.

**Case 2.** Let \(H = Sp(W, \omega) \times G_m\) where \((W, \omega)\) is a symplectic vector space. Set \(V = W \oplus W\) and equip it with the symplectic form \((\omega \oplus (-\omega))\). Let \(\Delta W\) be the diagonal copy of \(W\) in \(V\). This is clearly a Lagrangian subspace of \(V\). Then we may identify \(G\) with \(Sp(V)\) and take \(P\) to be the stabilizer of \(\Delta W\). We define

\[ \eta(g_1, g_2, t) = ((g_1, g_2), t) \]

where \((g_1, g_2)\) is an element of \(Sp(V)\) given by

\[ (g_1, g_2)(w_1, w_2) = (g_1(w_1), g_2(w_2)). \]

The definition of \(\eta\) in Case 3 is analogous to Case 2 and we leave it to the reader.

7.2. **The space \(S(H)\).** We define \(S(H) = S(G, M)\). It follows from Lemma 5.4 that every \(f \in S(H)\) is locally constant.

Let us consider Case 1 above, i.e. the case \(H = GL(n)\). In this case Section 5.2 implies that \(S(H)\) is equal to the space of locally constant compactly supported functions on \(M_n\).
7.3. The “Fourier transform” \( F_{H,\psi} \). Let \( \mathbb{P} \) be a parabolic subgroup opposite to \( P \). We claim that in all the cases 1,2,3 above we can identify \( X_P \) with \( X_{\mathbb{P}} \). Indeed, in Case 1 we just need to identify \( M_n \) with the dual vector space. This can be done by means of the standard bilinear form

\[ (A, B) = \text{tr} AB. \]

In Case 2 both \( X_P \) and \( X_{\mathbb{P}} \) can be identified with the variety of of Lagrangian subspaces of \( V \) equipped with a volume form (note that in this case \( P \) and \( \mathbb{P} \) are conjugate to each other). Case 3 is treated similarly.

Example. Consider Case 1. Then \( F_{P,\mathbb{P},\psi} \) is just the Fourier transform operator on \( M_n \). Namely given a function \( f \in L^2(M_n) \) we have

\[ F_{P,\mathbb{P},\psi}(f)(x) = \int_{M_n} f(y)\psi(\text{tr}(xy))dy. \]

By the definition we have

\[ S(H) = C_c(X_P) + F_{P,\mathbb{P},\psi}(C_c(X_{\mathbb{P}})) \subset L^2(X_P). \]

Hence \( F_{P,\mathbb{P},\psi} \) acts from \( S(H) \) to \( S(H) \). We denote this operator by \( F_{H,\psi} \). The proof of the following lemma is left to the reader.

**Lemma 7.4.** With the above identifications we have \( F_{\mathbb{P},P,\psi} = F_{H,\psi^{-1}} \).

The lemma implies that \( F_{H,\psi} \) and \( F_{H,\psi^{-1}} \) are inverse to each other. Let now \( \pi \) be an irreducible representation of \( H \). The following theorem is proved in [9] in Case 1 above. In other cases the proof may be obtained by a similar analysis. We shall not write the details here.

**Theorem 7.5.** (1) Let \( m \) be any matrix coefficient of \( \pi \) and let \( f \in S(H) \) be any function. Then the integral

\[ Z(f, m, s) = \int_H m(h)f(h)|\sigma(h)|^s dh \]

is absolutely convergent for \( \text{Re}(s) >> 0 \).

(2) \( Z(f, m, s) \) extends meromorphically to the whole of \( \mathbb{C} \) as a rational function of \( q^{-s} \).

(3) For a fixed \( \pi \) the functions \( Z(f, m, s) \) form a fractional ideal \( J_\pi \) in the ring \( \mathbb{C}[q^{-s}] \) of polynomials \( q^{-s} \).

(4) There exists unique polynomial \( P_\pi(t) \) with \( P_\pi(0) = 1 \) such that \( J_\pi \) is generated by \( \frac{1}{P_\pi(q^{-s})} \). We set

\[ L(\pi, s) := \frac{1}{P_\pi(q^{-s})}. \]
In Cases 2 and 3 \( L(\pi, s) \) coincides with the local \( L \)-function of \( \pi \) constructed in \([\text{[10]}]\) (in Case 1 the above definition of \( L(\pi, s) \) is exactly the same as the one in \([3]\)).

One can use the operator \( \mathcal{F}_{H, \psi} \) in order to define the corresponding \( \varepsilon \)-factors (using the idea in \([3]\) where this is done in Case 1). Also, one can show that the generalized Poisson summation formula (6.1) implies that the global version of \( L(\pi, s) \) satisfies the corresponding functional equation.

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