Fully doubly nonlinear evolution inclusion of second order: A fixed point argument

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Doubly nonlinear evolution inclusion of second order: A fixed point method

Abstract

The well-posedness of the abstract Cauchy problem for the doubly nonlinear evolution inclusion equation of second order

\[
\begin{align*}
  u''(t) + \partial \Psi (u'(t)) + B(t, u(t)) & \ni f(t), & t \in (0, T), \ T > 0, \\
  u(0) = u_0, \quad u'(0) = v_0
\end{align*}
\]

in a real separable Hilbert space \( \mathcal{H} \), where \( u_0 \in H, v_0 \in D(\Psi), f \in L^2(0, T; \mathcal{H}) \). The functional \( \Psi : \mathcal{H} \to (\mathbb{R}, +\infty] \) is supposed to be proper, lower semi-continuous, and convex and the nonlinear operator \( B : [0, T] \times \mathcal{H} \to \mathcal{H} \) is supposed to satisfy a (local) Lipschitz condition. Existence and uniqueness of strong solutions \( u \in H^2(0, T^*; \mathcal{H}) \) as well as the continuous dependence of solutions from the data re shown by employing the theory of nonlinear semigroups and the Banach fixed-point theorem. If \( B \) satisfies a local Lipschitz condition, then the existence of strong local solutions are obtained.

1 Introduction

In this article, we investigate the abstract Cauchy problem

\[
\begin{align*}
  u''(t) + \partial \Psi (u'(t)) + B(t, u(t)) & \ni f(t), & \text{for a.e. } t \in (0, T), \ T > 0, \\
  u(0) = u_0, \quad u'(0) = v_0
\end{align*}
\]

on a a real separable Hilbert space \((\mathcal{H}, ||\cdot||, \langle \cdot, \cdot \rangle)\) equipped with the inner product \( \langle \cdot, \cdot \rangle \) and the induced norm \( ||\cdot|| := (\langle \cdot, \cdot \rangle)^{1/2} \). Before we state the main result, we collect all assumptions concerning the functional \( \Psi : \mathcal{H} \to (\mathbb{R}, +\infty] \) and the nonlinear operator \( B : [0, T] \times \mathcal{H} \to \mathcal{H} \).

We recall that for a proper, lower semi continuous and convex functional \( f : \mathcal{H} \to (\mathbb{R}, +\infty] \), the subdifferential of \( f \) in \( u \in D(f) \) is given by

\[
\partial f(u) = \{ \xi \in \mathcal{H} : f(u) - f(v) \leq \langle \xi, u - v \rangle \}.
\]
Assumption A

Let $\Psi : \mathcal{H} \to (-\infty, +\infty]$ be a proper, lower semicontinuous, and convex functional such that $\partial \Psi(0) \neq \emptyset$.

Assumption B

1) Let $B : [0, T] \times \mathcal{H} \to \mathcal{H}$ satisfy the following local Lipschitz condition: For all $R > 0$, there exists a function $\alpha_R \in L^2(0, T; \mathbb{R}^+_0)$ such that

$$|B(t, u) - B(t, v)| \leq \alpha_R(t)|u - v| \quad \text{for all } u, v \in B(0, R),$$

(1.1)

and almost all $t \in [0, T]$, where $B(0, R)$ denotes the closed ball in $\mathcal{H}$ with radius $R > 0$ and center 0 $\in \mathcal{H}$. Furthermore, there exists a function $g \in L^2(0, T; \mathbb{R}^+_0)$, such that

$$|B(t, 0)| \leq g(t) \quad \text{for a.a. } t \in [0, T].$$

2) For all strongly measurable $v : [0, T] \to \mathcal{H}$, the map $t \mapsto B(t, v(t))$ is strongly measurable as a mapping from $[0, T]$ to $\mathcal{H}$.

Remark 1.1 From Assumption A, it follows in particular that the subdifferential $\partial \Psi$ is a maximal monotone operator in the sense of Brézis, see [3].

Remark 1.2 From the Lipschitz continuity and the square-integrability of $\alpha$, we infer that the map $t \mapsto B(t, u)$ for all $u \in \mathcal{H}$.

2 Well-posedness of the problem

In this section, we establish the existence and uniqueness of solutions as well as the continuous dependence of the solution from the data specified below.

Theorem 2.1 Let $\Psi : \mathcal{H} \to (-\infty, +\infty]$ and $B : [0, T] \times \mathcal{H} \to \mathcal{H}$ be given and satisfy Assumption A and Assumption B. Then, for every initial values $u_0 \in \mathcal{H}, v_0 \in D(\Psi)$ and every external source term $f \in L^2(0, T; \mathcal{H})$, there exists a unique local in time strong solution to (1), i.e., there exists $T > 0$ and functions $u \in H^2(0, T; \mathcal{H})$ and a $\eta : [0, T] \to \mathcal{H}$ strongly measurable such that

$$u''(t) + \eta(t) + B(t, u(t)) = f(t) \quad \text{for a.e. } t \in (0, \check{T}),$$

(2.1)

$$\eta(t) \in \partial \Psi(u'(t)) \quad \text{for a.e. } t \in (0, \check{T}),$$

(2.2)

and the initial conditions $u(0) = u_0$ and $u'(0) = v_0$ are fulfilled. If $B$ satisfies the global Lipschitz condition: There exists a function $\alpha \in L^2(0, T)$ such that

$$|B(t, v) - B(t, w)| \leq \alpha(t)|v - w| \quad \text{for all } v, w \in \mathcal{H},$$

and almost every $t \in (0, T)$, then there exists a unique global solution $u \in H^2(0, T; \mathcal{H})$ to (1). Furthermore, the solution depends continuously on the data, i.e., let $u_1$ and $u_2$ be the solution to (1) associated with the data $(f, u_0^1, v_0^1)$ and $(g, u_0^2, v_0^2)$ from $L^2(0, T; \mathcal{H}) \times \mathcal{H} \times D(\Psi)$, respectively. Then, there exists a constant $M > 0$ such that

$$\|u_1 - u_2\|^2_{C([0, T]; \mathcal{H})} \leq M \int_0^T \alpha(t) dt \left( |u_0^1 - u_0^2|^2 + |v_0^1 - v_0^2|^2 + \|f - g\|^2_{L^2(0, T; \mathcal{H})} \right).$$
Proof. The main idea consists in rewriting the evolution equation as two coupled first order evolutions equations

\begin{align}
  v'(t) + \partial\Psi(v(t)) + B(t, u(t)) &\ni f(t) \quad \text{for a.e. } t \in (0, T), \\
  u'(t) &= v(t) \quad \text{for all } t \in (0, T), \\
  u(0) &= u_0, \quad v(0) = v_0
\end{align}

(2.3)

and considering for fixed \( u \in C([0, T]; \mathcal{H}) \) the auxiliary problem

\begin{align}
  \begin{cases}
    \tilde{v}'(t) + \partial\Psi(\tilde{v}(t)) \ni \tilde{f}(t) &\quad \text{for a.e. } t \in (0, T), \\
    \tilde{v}(0) = v_0,
  \end{cases}
\end{align}

(2.5)

where \( \tilde{f}(t) = f(t) - B(t, u(t)) \), \( t \in [0, T] \). We notice that since \( u \) is continuous and Assumption B holds, \( \tilde{f} \in L^2(0, T; \mathcal{H}) \) is ensured.

Now, existence and uniqueness of strong solutions \( \tilde{v} \in H^1([0, T], \mathcal{H}) \) for every initial value \( v_0 \in \overline{D(\Psi)} \) as well as stability of the solutions are well known for the auxiliary problem (2.5), see for instance Brézis [3], where the results is essentially based on a Moreau-Yosida approximation of the subdifferential \( \partial\Psi \) due to Yosida.

Denoting with \( J : C([0, T]; \mathcal{H}) \to H^1([0, T]; \mathcal{H}) \) the solution operator which maps the function \( u \mapsto J(u) \) to the unique solution of (2.5), we obtain for \( u_1, u_2 \in B_{C([0,T];\mathcal{H})}(u_0, R) \) for fixed \( R > 0 \), the inequality

\[
  |J(u_1)(t) - J(u_2)(t)|^2 \leq 2 \int_0^t |B(r, u_1(r)) - B(r, u_2(r))||J(u_1)(r) - J(u_2)(r)| \, dr \\
  \leq \int_0^t |J(u_1)(r) - J(u_2)(r)|^2 \, dr + \int_0^t \alpha_R(r)|u_1(r) - u_2(r)|^2 \, dr
\]

for all \( 0 \leq t \leq T \). Then, with Gronwall’s lemma, we obtain

\[
  |J(u_1)(t) - J(u_2)(t)|^2 \leq e^{\int_0^t \alpha_R(r) \, dr} |u_1(0) - u_2(0)|^2 \quad \text{for all } t \in [0, T].
\]

(2.6)

Then, in order to show the existence solutions to the initial value problem (1), it suffices to show that the map \( \mathcal{F} : B_{C([0, T];\mathcal{H})}(u_0, R) \to B_{C([0, T];\mathcal{H})}(u_0, R) \) with

\[
  \mathcal{F}(u)(t) := u_0 + \int_0^t J(u)(s) \, ds, \quad t \in [0, T]
\]

(2.7)

possesses a fix point for a time-point \( 0 < \hat{T} \leq T \), where \( B_{C([0, T];\mathcal{H})}(u_0, R) \) denotes the closed ball in \( C([0, \hat{T}]; \mathcal{H}) \) of radius \( R > 0 \) and center \( u_0 \) which can be seen as constant function in \( C([0, \hat{T}]; \mathcal{H}) \). As soon as existence of a fixed point \( u \) of \( \mathcal{F} \) is shown, it follows

\[
  u(t) = \mathcal{F}(u)(t) = u_0 + \int_0^t J(u)(s) \, ds = u_0 + \int_0^t \tilde{v}(s) \, ds, \quad t \in [0, \hat{T}],
\]

(2.8)

i.e., relation (2.4) holds and it follows \( u \in H^2(0, \hat{T}; \mathcal{H}) \). Since the operator \( J \) maps the fixed point to the unique solution of the auxiliary problem (2.5), the differential inclusion (2.3) holds as well. Finally, taking into account that the initial conditions are also satisfied, we deduce the existence of a strong solution to (1). We notice that since the resolvent operator \( J \) maps continuous functions into (absolutley) continuous functions, the operator \( \mathcal{F} \) itself maps continuous functions into continuous functions.
Uniqueness:

Before showing the existence of strong solutions, we establish uniqueness of solutions on the whole interval $[0, T]$. For this, we assume there are two solutions $u_1, u_2 \in H^2(0, T; \mathcal{H})$ of (1) to the same initial data. Then, by making use of (2.6) and (2.8), we obtain

$$\sup_{s \in [0, t]} |u_1(t) - u_2(t)|^2 \leq \sup_{s \in [0, t]} \left| \int_0^s J(u_1)(\tau) - J(u_2)(\tau) d\tau \right|^2$$

$$\leq \left( \int_0^t |J(u_1)(s) - J(u_2)(s)| ds \right)^2$$

$$\leq \sqrt{T} \int_0^t |J(u_1)(s) - J(u_2)(s)|^2 ds$$

$$\leq \sqrt{T} \int_0^t e^s \int_0^s 2\alpha_R(\tau)|u_1(\tau) - u_2(\tau)|^2 d\tau ds$$

$$\leq \sqrt{T} e^T \int_0^t \|u_1 - u_2\|_{C([0, s]; \mathcal{H})}^2 \int_0^s 2\alpha(\tau) d\tau ds,$$

where $R := \sup_{t \in [0, T]} (|u_1(t)| + |u_2(t)|)$. Defining $a(t) := \|u_1 - u_2\|_{C([0, t]; \mathcal{H})}^2$ and $\lambda(t) := \int_0^t 2\alpha_R(\tau) d\tau$, there holds $a, \lambda \in L^\infty(0, T)$ with $\lambda \geq 0$ a.e. in $(0, T)$ such that

$$a(t) \leq \int_0^t \lambda(s) a(s) ds \quad \text{for all } t \in [0, T].$$

GRONWALLS Lemma yields immediately $a \equiv 0$ on $[0, T]$ so that $u_1 = u_2$.

Existence:

In order to prove existence of local solutions, we make use of the BANACH fixed-point theorem which provides the existence of a (unique) solution on a possibly small time-interval, i.e., we show existence of local solutions. Then, by iterating this procedure and making sure that the time interval do not minimize in each iteration step, global solution can be constructed. Therefore, we need to check that the conditions of the BANACH fixed-point theorem are fulfilled. Primarily, we show that for fixed $R > 0$ the map $\mathfrak{F} : B_{C([0, \tilde{T}], \mathcal{H})}(u_0, R) \rightarrow B_{C([0, \tilde{T}], \mathcal{H})}(u_0, R)$ is well defined for sufficiently small $\tilde{T} > 0$, i.e., it maps the closed ball in $C([0, T]; \mathcal{H})$ of radius $R > 0$ and center $u_0$ into itself. In order to do that, we need the following a priori estimate:

$$\frac{1}{2} \frac{d}{dt} |J(u)(t)|^2 = \frac{1}{2} \frac{d}{dt} |v(t)|^2$$

$$= (v'(t), v(t)) + (\xi(t) - \eta, v(t))$$

$$= (f(t) - B(t, u(t) - \eta, v(t))$$

$$\leq (|f(t)| + |B(t, u(t))| + |\eta|)|v(t)|$$

$$\leq (|f(t)| + \alpha_R(t)|u(t)| + g(t) + |\eta|)|J(u)(t)|$$

for a.e. $t \in [0, T]$,

where we have tested the auxiliary problem (2.3) with its unique solution $v = J(u)$. This
again yields by Gronwall’s Lemma

\[ \sup_{s \in [0,T]} |J(u)(t)| \leq |v_0| + \int_0^t 2(|f(s)| + \alpha(s)|u(s)| + g(s) + |\eta|)ds \]

\[ = C + \int_0^t 2(|f(s)| + \alpha(s)|u(s)| + g(s))ds \quad \text{for all } t \in [0, T]. \quad (2.9) \]

where we used the fact that \( J(u)(0) = \tilde{v}(0) = v_0 \) and defined \( C := (|v_0| + T|\eta|) \). Employing (2.6), we obtain for \( u \in B_{C([0,\tilde{T}];\mathcal{X})}(u_0, R) \)

\[ \|\mathfrak{F}(u) - u_0\|_{C([0,\tilde{T}];\mathcal{X})} \leq \int_0^{\tilde{T}} |J(u)(t)|dt \]

\[ \leq \int_0^{\tilde{T}} C + 2 \int_0^t (|f(s)| + \alpha(s)|u(s)| + g(s))dsdt \]

\[ \leq \tilde{T}(C + 2(\|f\|_{L^1(0,T)} + \|\alpha\|_{L^1(0,T)}\|u\|_{C([0,\tilde{T}];\mathcal{X})} + \|g\|_{L^1(0,T)})) \]

\[ \leq \tilde{T}(C + 2(\|f\|_{L^1(0,T)} + \|\alpha\|_{L^1(0,T)}(R + |u_0|) + \|g\|_{L^1(0,T)})) \]

\[ \leq R \]

with \( \tilde{T} \leq T_1 := R(|v_0| + 2(\|f\|_{L^1(0,T)} + \|\alpha\|_{L^1(0,T)}(R + |u_0|) + \|g\|_{L^1(0,T)}))^{-1} > 0 \). Second, we show that for sufficiently small \( \tilde{T} > 0 \), the map \( \mathfrak{F} \) is also a contraction. Let \( u, v \in B_{C([0,\tilde{T}];\mathcal{X})}(u_0, R) \). Then, doing the same calculations as in the uniqueness part, we obtain

\[ \|\mathfrak{F}(u) - \mathfrak{F}(v)\|_{C([0,\tilde{T}];\mathcal{X})} = \sup_{t \in [0,\tilde{T}]} \left| \int_0^t J(u)(s) - J(v)(s)ds \right| \]

\[ \leq \int_0^{\tilde{T}} |J(u)(s) - J(v)(s)|ds \]

\[ \leq \int_0^{\tilde{T}} \|u - v\|_{C([0,\tilde{T}];\mathcal{X})} \int_0^t 2\alpha(\tau)d\tau ds \]

\[ \leq \int_0^{\tilde{T}} \|u - v\|_{C([0,\tilde{T}];\mathcal{X})}2\alpha\|_{L^1(0,\tilde{T})} d\tau ds \]

\[ \leq \tilde{T}(2\|\alpha\|_{L^1(0,\tilde{T})})\|u - v\|_{C([0,\tilde{T}];\mathcal{X})} \]

\[ \leq L\|u - v\|_{C([0,\tilde{T}];\mathcal{X})}, \]

where \( L := \tilde{T}2\|\alpha\|_{L^1(0,T)} < 1 \) for \( \tilde{T} < T_2 := (2\|\alpha\|_{L^1(0,T)})^{-1} > 0 \). Thus, by the Banach fixed-point theorem, there exists a unique solution \( u \in C([0,\tilde{T}], \mathcal{X}) \) to (1) on the time interval \([0, \tilde{T}]\) with \( 0 < \tilde{T} < \min\{T_1, T_2\} \). We assumed here without loss of generality that \( \alpha \neq 0 \) in \( L^2(0, \tilde{T}), \) otherwise \( B \) would be constant almost everywhere and the assertion would be trivial.

Now, there are two possibilities to show global existence of solutions in the case where \( B \) satisfies the global Lipschitz condition. The first possibility is to show the boundedness of the derivative of a solutions on the whole interval, such that blow ups of not only the solution itself but also of its derivative in finite time can not occur. This would lead to an interval of existence independent of the initial values. Then, applying successively the Banach fixed-point theorem to the new initial value problem where the initial values are determined by the solution of the previous step, so that this procedure would cover the whole interval. Another possibility is to define the operator \( \mathfrak{F} \) on the whole space \( C([0,T];\mathcal{X}) \) equipped with a norm equivalent to the standard one and employ again
the Banach fixed point theorem, where we need the equivalent norm to ensure the contractivity of $\mathcal{F}$. We tackle the problem with the latter option and define the operator $\mathcal{F} : C([0, T], \mathcal{H}) \to C([0, T], \mathcal{H})$ as in (2.5), where we equip the space $C([0, T], \mathcal{H})$ with the norm $\|v\|_x := \sup_{t \in [0, T]} e^{-Lt}\|v(t)\|$ with $L = 2\|\alpha\|_{L^1(0, T)}$. Since $\mathcal{F}$ is obviously a self map, it remains to show that $\mathcal{F}$ is a contraction:

$$
\|\mathcal{F}(u) - \mathcal{F}(v)\|_x = \sup_{t \in [0, T]} e^{-Lt} \left| \int_0^t J(u)(s) - J(v)(s) ds \right|
\leq \sup_{t \in [0, T]} e^{-Lt} \int_0^t |J(u)(s) - J(v)(s)| ds
\leq \sup_{t \in [0, T]} 2e^{-Lt} \int_0^t \int_0^s \alpha(\tau)|u(\tau) - v(\tau)| d\tau ds
\leq \sup_{t \in [0, T]} 2e^{-Lt} \int_0^t \sup_{\tau \in [0, s]} |u(\tau) - v(\tau)| \int_0^s \alpha(\tau) d\tau ds
\leq \sup_{t \in [0, T]} 2e^{-Lt} \|\alpha\|_{L^1(0, T)} \int_0^t \sup_{\tau \in [0, s]} e^{Lt} e^{-Lt} |u(\tau) - v(\tau)| ds
\leq \sup_{t \in [0, T]} 2e^{-Lt} \|\alpha\|_{L^1(0, T)} \int_0^t e^{-Lt} ds \|u - v\|_x
= \sup_{t \in [0, T]} 2e^{-Lt} \|\alpha\|_{L^1(0, T)} \frac{e^{Lt} - 1}{L} \|u - v\|_x
= \sup_{t \in [0, T]} (1 - e^{-Lt}) \|u - v\|_x
= (1 - e^{-LT}) \|u - v\|_x
$$

Therefore, the map $\mathcal{F}$ is a contraction on $C([0, T], \mathcal{H})$ and by the Banach fixed point theorem there exists a unique fixed point $u \in C([0, T], \mathcal{H})$ which is a solution to (1).

Stability:

Finally, we want to show the continuous dependence of the solution from the data. Let $u_1$ and $u_2$ be the solution to (1) associated with $(f, u_1^0, v_1^0), (g, u_2^0, v_2^0) \in L^2(0, T; \mathcal{H}) \times \mathcal{H} \times D(\mathcal{F})$, respectively. With the same reasoning as for (2.6), we derive with Gronwall’s lemma

$$
|J(u_1)(t) - J(u_2)(t)|^2 \leq e^t \left( |v_1^0 - v_2^0|^2 + \|f - g\|_{L^2(0, T; \mathcal{H})}^2 + \int_0^t \alpha(r)|u_1(r) - u_2(r)|^2 dr \right)
$$
for all $t \in [0, T]$. Then, continuing as in the uniqueness part, we obtain
\[
\sup_{s \in [0, t]} |u_1(t) - u_2(t)|^2 \leq |u_0^1 - u_0^2|^2 + \sup_{s \in [0, t]} \left| \int_0^s J(u_1)(\tau) - J(u_2)(\tau) d\tau \right|^2 \\
\leq |u_0^1 - u_0^2|^2 + \left( \int_0^t |J(u_1)(s) - J(u_2)(s)| ds \right)^2 \\
\leq |u_0^1 - u_0^2|^2 + \sqrt{t} \int_0^t |J(u_1)(s) - J(u_2)(s)|^2 ds \\
\leq |u_0^1 - u_0^2|^2 + \sqrt{t} \int_0^t e^s \left( |v_0^1 - v_0^2|^2 + \|f - g\|_{L^2(0,T;\mathcal{H})}^2 \right) ds \\
\quad + \int_0^t e^s \int_0^s 2\alpha_R(\tau) |u_1(\tau) - u_2(\tau)|^2 d\tau ds \\
\leq M \left( |u_0^1 - u_0^2|^2 + |v_0^1 - v_0^2|^2 + \|f - g\|_{L^2(0,T;\mathcal{H})}^2 \right) \\
\quad + \sqrt{\pi} e^T \int_0^t \|u_1 - u_2\|^2_{C([0,s];\mathcal{H})} ds \int_0^s 2\alpha(\tau) d\tau ds,
\]
for a constant $M > 0$ independent of the data. Defining again $a(t) := \|u_1 - u_2\|^2_{C([0,t];\mathcal{H})}$ and $\lambda(t) := \int_0^t 2\alpha_R(\tau) d\tau$ as well as $b = M \left( |u_0^1 - u_0^2|^2 + |v_0^1 - v_0^2|^2 + \|f - g\|_{L^2(0,T;\mathcal{H})}^2 \right)$, there holds $a, \lambda \in L^\infty(0,T)$ with $\lambda \geq 0$ a.e. in $(0, T)$ such that
\[
a(t) \leq b + \int_0^t \lambda(s) a(s) ds \quad \text{for all } t \in [0, T].
\]
Again, with the Gronwall lemma, we obtain the desired estimate. \hfill \Box

**Corollary 2.2** In the case, when $B$ satisfies the local Lipschitz condition, there exists a maximal solution, i.e., there exists a time interval $[0, \bar{T}) \subset [0, T]$ and a function $u$ such that for each compact subinterval $[0, S] \subset [0, \bar{T}]$ there holds $u \in H^2(0, S; \mathcal{H})$ and $u$ solves problem (1) pointwise almost every on $(0, \bar{T})$. Furthermore, for every sequence $(t_n) \subset [0, \bar{T})$ with $t_n \nrightarrow \bar{T}$ as $n \to \infty$, there holds $|u(t_n)| \to +\infty$ as $n \to \infty$.

**Remark 2.3** We notice that we did not impose any compactness assumption neither on the sublevels of the dissipation potential $\Psi$ nor on the operator $B$ in order to show existence of solutions.
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