Construction of bosonic string theory on infinitely curved Anti-de Sitter space

Adam Clark*, Andreas Karch†, Pavel Kovtun‡, and Daisuke Yamada§
Department of Physics, University of Washington, Seattle, Washington 98195-1560
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Abstract

Free scalar field theory in the sector with a large number of particles can be interpreted as bosonic string theory on anti-de Sitter space of vanishing radius. Different ways of writing the field theory Hamiltonian translate to different ways of reparametrizing the world-sheet $\sigma$ coordinate. Adding a mass term in the field theory corresponds to cutting off the warped AdS direction, with cut-off inversely proportional to the mass. The string theory has neither tachyon, nor critical dimension.

* abc@u.washington.edu
† karch@phys.washington.edu
‡ pkovtun@u.washington.edu
§ dyamada@u.washington.edu
I. INTRODUCTION

The holographic AdS/CFT duality conjecture states the equivalence between string/M theories on various background spaces, and gauge theories, formulated on the boundary of those spaces. In the context of the duality, a correspondence exists between important dimensionless parameters, which control the dynamics. A string theory has two such parameters. One is the ratio of string tension, $T = 1/(2\pi\alpha')$, to the characteristic curvature of the background space. Another is the string coupling constant, which sets the probability of string splitting and joining. On the field theory side, these parameters correspond to the 't Hooft coupling and to the number of colors $N$ in gauge theory. Thus, the $1/N$ expansion in gauge theory corresponds to the genus expansion of string world-sheets, while finite 't Hooft coupling effects correspond to stringy corrections to the effective low-energy string theory. In particular, field theories at infinite 't Hooft coupling are dual to theories of classical supergravity.

Following the duality recipe, field theories at small coupling must correspond to string theories formulated on backgrounds whose characteristic radius of curvature is much smaller than the string length. Since geometry is supposed to be dynamical in string theory, it is not clear what the duality means in this weak-coupling regime. The problem is, of course, that string theory lacks a (non-perturbative) definition. Turning things around, one can choose to use the duality to define string theory in the limit of small tension. It is an important unresolved problem how to construct a general definition.

In this note we consider bosonic string theory on the background of Anti-de Sitter space (rather, its Poincare patch), whose radius of curvature $R$ is exactly zero. In this limit, the classical string Hamiltonian simplifies considerably, and it is not difficult to quantize the theory in the light-cone gauge.\footnote{Such simplification occurs only because the limit $R^2/\alpha' \to 0$ (which gets rid of certain derivative terms) is taken in the classical string Hamiltonian. In general, we do not expect that the same limit, when taken in the quantum Hamiltonian (whatever it is) will produce the same result. In other words, there is no good reason to believe that the zero-radius solution is the correct starting point to define string theory perturbatively in $R^2/\alpha'$. For attempts to construct a systematic large-curvature expansion in string theory, see \cite{[2]}.} What we find is a very natural correspondence between string theory in this singular limit and a free matrix-valued scalar field theory in the sector of single-trace states with a large number of particles. The correspondence was first noted in \cite{[3]}, but the interpretation there was not clear. One can make several observations about the correspondence:

- Both Hamiltonians and states are easily mapped between string and field theory.
- Part of the world-sheet reparametrization invariance of string theory corresponds to the freedom to define different continuum limits in the field theory.
The field theory Hamiltonian provides a discretization of the string theory Hamiltonian. For states with a large number of particles, the jagged string world-sheet is a good approximation.

By checking the closure of the Lorentz algebra in string theory, one finds that there is no critical dimension. This is in natural correspondence with the fact that free field theories exist in any number of spacetime dimensions.

The spectrum of the string Hamiltonian does not contain a tachyon, naturally reflecting the stability of the field theory.

Adding mass \( m \) to the scalar particles corresponds to imposing a hard cutoff equal to \( 2\pi/m \) in the warped direction of the Poincare patch. This is very similar to the standard (super)gravity construction of confining gauge theories (in the large ’t Hooft coupling limit of the AdS/CFT correspondence). Mode number in the bulk maps to particle number on the boundary.

The large \( N \) expansion in the field theory (which in the absence of interactions becomes just the combinatorics of organizing Wick contractions) is mapped to the “genus expansion” in string theory.

The very idea of explicitly building string theories from large-\( N \) field theories, possibly with some sort of discretization, is rather old \[4, 5, 6, 7\]. Recently, significant progress has been made in constructing string world-sheets from large-\( N \) planar diagrams \[8\]. In the approach we take in this note, we are not trying to construct a formulation which specifically requires smooth world-sheet embeddings. The string theory we are discussing has essentially zero tension, and therefore neighboring bits on the string have no correlation in space-time. Rather, our goal is to make a connection between free scalar field theory, and the naive small-radius limit of string theory on Anti-de Sitter space. The paper is organized as follows. In the next section we review quantization of a free scalar field on the light-front, and show how, for single-trace states, a picture of a discretized string emerges. We also discuss different continuum limits for the free theory Hamiltonian in the sector with a large number of particles. In Section \[III\] we show that the same Hamiltonian arises as the lightcone Hamiltonian of bosonic string theory on zero radius AdS space with a hard cut-off on the warped dimension. The number of particles is interpreted as the discrete mode number of the warped dimension. In Section \[IV\] we check that the string theory on zero radius AdS space passes the most crucial test a consistent string theory in light cone gauge has to pass: the Lorentz anomaly vanishes, independent of the number of dimensions. In Section \[V\] we conclude and discuss directions for future work.
II. FREE SCALAR FIELD THEORY ON THE LIGHT-CONE

We start with the following action for a free scalar field theory in flat Minkowski space:

\[
S = \int d^d x \left( -\frac{1}{2} \text{tr}(\partial \Phi)^2 - \frac{1}{2}m^2 \text{tr}\Phi^2 \right). \tag{1}
\]

Here \( \Phi \equiv \Phi_{ij}(x^0, x^1, \ldots, x^{d-1}) \) is an \( N \times N \) hermitian matrix field, and we use the mostly plus metric. The action has a \( U(N) \) symmetry, which transforms \( \Phi \rightarrow U\Phi U^\dagger \). The theory can be quantized on the light-front, as an alternative to equal-time canonical quantization, see e.g. [9, 10]. To do so, one introduces light-cone coordinates \( x^\pm = (x^1 \pm x^0)/\sqrt{2} \), and commutation relations for fields \( \Phi(x^+, x^-, x^2, \ldots, x^{d-1}) \equiv \Phi(x^+, x^-, \mathbf{x}^\perp) \) are imposed at equal \( x^+ \), rather than at equal \( x^0 \):

\[
[\partial_\pm \Phi_{ij}(x^+, x^-, \mathbf{x}^\perp), \Phi_{kl}(x^+, y^-, \mathbf{y}^\perp)] = -\delta_{ik} \delta_{jl} \frac{i}{2} \delta(x^- - y^-) \delta(\mathbf{x}^\perp - \mathbf{y}^\perp) \tag{2}
\]

where \( \partial_\pm \equiv \partial/\partial x^\pm \). Mode expansions for “Schrödinger picture” operators now become

\[
\Phi_{ij}(x^-, \mathbf{x}^\perp) = \int_0^\infty \frac{dp_-}{2\pi} \int \frac{d^{d-2}p_\perp}{(2\pi)^{d-2}} \left[ a_{ij}(p_-, \mathbf{p}_\perp)e^{ip_-x^- + ip_\perp \cdot \mathbf{x}^\perp} + a_{ij}^\dagger(p_-, \mathbf{p}_\perp)e^{-ip_-x^- - ip_\perp \cdot \mathbf{x}^\perp} \right] \tag{3}
\]

and the light-cone Hamiltonian \( H_{LC} \) generates translations in \( x^+ \). The commutation relations for the creation/annihilation operators are

\[
[a_{ij}(\mathbf{p}), a_{kl}^\dagger(\mathbf{k})] = \delta_{ik} \delta_{jl} 2p_- (2\pi)^{d-1} \delta(\mathbf{p} - \mathbf{k}) \tag{4}
\]

\[
[a_{ij}(\mathbf{p}), a_{kl}(\mathbf{k})] = [a_{ij}^\dagger(\mathbf{p}), a_{kl}^\dagger(\mathbf{k})] = 0. \tag{5}
\]

The vacuum state \( |0 \rangle \) is annihilated by all \( a_{ij}(\mathbf{p}) \), and a general \( M \)-particle state is

\[
a_{i_1j_1}(\mathbf{p}_1) a_{i_2j_2}^\dagger(\mathbf{p}_2) \ldots a_{i_Mj_M}^\dagger(\mathbf{p}_M) |0 \rangle \tag{6}
\]

where we use \( \mathbf{p} \) as short for \( (p_-, \mathbf{p}_\perp) \). The single-particle dispersion relation takes the form

\[
\mathcal{E}_{LC}(\mathbf{p}) = -p_+ = \frac{p_\perp^2 + m^2}{2p_-} \tag{7}
\]

and thus positivity of energy requires \( p_- > 0 \). For states with a given number \( M \) of particles, the light-cone Hamiltonian takes the usual second-quantized form:

\[
H_{LC} = \sum_\mathbf{p} N_\mathbf{p} \mathcal{E}_{LC}(\mathbf{p}) = \sum_\mathbf{p} N_\mathbf{p} \frac{p_\perp^2 + m^2}{2p_-}, \tag{8}
\]

where \( N_\mathbf{p} \) is the number operator, and \( \sum_\mathbf{p} N_\mathbf{p} = M \).
A. Continuum limit with constant density of particles

Instead of summing over momenta, the same Hamiltonian can be rewritten as a sum over particles. Knowing the number of particles $N_p$ with given momentum, one can assign momentum $p(n)$ to every integer $n$ between 1 and $M$. Such an assignment is, of course, not unique, because the labeling of particles is arbitrary. Then the Hamiltonian (8) becomes

$$H_{LC} = \sum_{n=1}^{M} \frac{p^2_\perp(n) + m^2}{2p_(n)}, \quad (9)$$

where $p_-(n)$, $p_\perp(n)$ are momenta of the $n$-th particle. A natural assignment of $p(n)$ can be made, when one considers single-trace $M$-particle states:

$$|\Psi_M \rangle \sim \text{tr} \left[ a^\dagger(p_1)a^\dagger(p_2)\ldots a^\dagger(p_M) \right] |0\rangle \quad (10)$$

$$\sum_{n=1}^{M} p_{n-} = P_- , \quad \sum_{n=1}^{M} p_{n\perp} = P_\perp \quad (11)$$

For such states, there is a natural way to label the particles, up to cyclic permutations. The states carry definite momenta, and can be labeled as

$$|(p_-, p_\perp)_{n_1}, (p_-, p_\perp)_{n_2}, \ldots (p_-, p_\perp)_{n_M} \rangle. \quad (12)$$

In other words, one can think of these states as strings of particles, created by the string of operators inside the trace. One way to visualize these states is shown in Fig. 1. When the number of particles $M$ is large, the summation over particle number in the Hamiltonian (9) can be represented as an integral over positions of the particles:

$$H_{LC} = \sum_{n=1}^{M} \frac{p^2_\perp(n) + m^2}{2p_(n)} \rightarrow \frac{1}{a} \int_0^l dx \, \frac{p^2_\perp(x) + m^2}{2p_-(x)} \quad (13)$$

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2 Restricting to the subsector of single-trace states is consistent at large $N$, when the overlap of single-trace and multi-trace states is suppressed by powers of $1/N$. As usual, single-trace states will be interpreted as single-string states, multi-traces as multi-string states, and at large $N$ splitting and joining string interactions do not happen.
FIG. 2: The same $M = 4$ particles with total momentum $P_- = \sum_{n=1}^{M} p_-(n)$

where $a \equiv l/M$ is a constant “lattice spacing”, and $1/a$ is particle density on the string. The cyclic symmetry of the trace implies that $p_-(x)$ and $p_\perp(x)$ are periodic functions of $x$ on the interval $(0, l)$. In other words, the string of particles is closed. The functions $p_-(x)$ and $p_\perp(x)$ are integrable, but not smooth, because particle momenta in $|\Psi^M\rangle$ are arbitrary.

One can rewrite the Hamiltonian (13) in terms of momentum densities

$$p_-(x) = \frac{\pi_\perp(x)}{a}$$

$$p_\perp(x) = \frac{\pi_\perp(x)}{a}$$

$$\pi_Y \equiv m/a$$

on the string:

$$H_{LC} = \int_{0}^{l} dx \frac{\pi_\perp^2(x) + \pi_Y^2}{2 \pi_-(x)}$$

(14)

Momentum densities are functions of $x$, but particle density on the string is $x$-independent.

B. Continuum limit with constant momentum density

One can look at the same multiparticle states from another point of view. With the natural ordering in the single-trace states, particles can be arranged on the momentum axis. A simple example is shown in Fig. 2. For a given total momentum $P_-$ of the state, when the number of particles $M$ becomes large, the summation over particle number in the Hamiltonian (9) can be represented as an integral over momenta of the particles:

$$H_{LC} = \frac{1}{2k_-} \int_{0}^{l} dx \left( \frac{\pi_\perp^2(x)}{\tilde{a}(x)^2} + \frac{\pi_Y^2}{\tilde{a}(x)^2} \right)$$

(15)

Here we chose the position-dependent “lattice spacing” to be $\tilde{a} = p_-(n)/k_- \to p_-(x)/k_-\), where $k_- = P_-/l$ is a constant, whose physical meaning is $p_-$-momentum density on the string. One can rewrite the Hamiltonian (13) in terms of momentum densities $\tilde{\pi}_Y(x) \equiv m/\tilde{a}(x)$, $\tilde{\pi}_\perp(x) \equiv p_\perp(x)/\tilde{a}(x)$ on the string:

$$H_{LC} = \frac{1}{2k_-} \int_{0}^{l} dx \left( \tilde{\pi}_\perp^2(x) + \tilde{\pi}_Y^2(x) \right)$$

(16)

In this approach, $p_-$-momentum density is constant, but particle density on the string, $1/\tilde{a}$, is a function of $x$.

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3 Open strings of particles naturally appear when the theory contains fields, which transform as vectors under the $U(N)$ symmetry. If $b_j^i$ are creation operators for such fields, then open string states have the form $|\Psi_M\rangle \sim b_j^1(p_1)a_{j12}^i(p_2)a_{j23}^i(p_3)\ldots a_{jM-2,M-1}^i(p_{M-1})b_j^M(p_M)|0\rangle$.

4 It is convenient to choose the arbitrary length $l$ to be the same in (15) and (16).
III. STRING THEORY ON ANTI-DE SITTER SPACE OF VANISHING RADIUS

A. Classical string theory on a given background

To maintain self-consistency of our presentation, we derive in this subsection the classical light-cone Hamiltonian for bosonic string theory on AdS space \cite{11,12}. One starts with the Polyakov action for the bosonic string on a general \((d + 1)\) dimensional background space:

\[
S = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma \sqrt{-\gamma} \gamma^{ab} \partial_a Z^\mu \partial_b Z^\nu \kappa_{\mu\nu}(Z). \tag{17}
\]

Indices \(a, b\) label world-sheet coordinates \(\tau, \sigma\), and indices \(\mu, \nu\) run from 0 to \(d\). \(\kappa_{\mu\nu}(Z)\) is the background metric with signature \(- + ... +\), and \(\gamma_{ab}\) is the world-sheet metric. It will be convenient to make use of \(h_{ab} = \sqrt{-\gamma} \gamma^{ab}\), which satisfies the identity \((h_{\tau\tau})^2 - h_{\tau\sigma} h_{\sigma\sigma} = 1\).

Canonical momentum densities for the coordinates \(Z^\mu\),

\[
\pi_\mu = \frac{\delta L}{\delta \dot{Z^\mu}} = -\frac{1}{2\pi\alpha'} \kappa_{\mu\nu} \left( h_{\tau\tau} \dot{Z}^\nu + h_{\tau\sigma} Z^\nu \right), \tag{18}
\]

give rise to the following Hamiltonian density:

\[
\mathcal{H} = \frac{-\pi\alpha'}{h_{\tau\tau}} \left[ G^{\mu\nu} \pi_\mu \pi_\nu + \frac{1}{(2\pi\alpha')^2} G_{\mu\nu} Z^\nu Z^\nu \right] - \frac{h_{\tau\sigma}}{h_{\tau\tau}} \pi_\mu Z^\mu, \tag{19}
\]

where dot and prime denote derivatives with respect to \(\tau\) and \(\sigma\), respectively. Components of the world-sheet metric are non-dynamical fields, and therefore the equations of motion for \(h_{\tau\tau}\) and \(h_{\tau\sigma}\) become constraint equations for the coordinates \(Z^\mu\):

\[
G^{\mu\nu} \pi_\mu \pi_\nu + \frac{1}{(2\pi\alpha')^2} G_{\mu\nu} Z^\nu Z^\nu = 0, \tag{20}
\]

\[
\pi_\mu Z^\mu = 0. \tag{21}
\]

It is convenient to introduce light-cone coordinates

\[
Z^\pm = \frac{1}{\sqrt{2}} \left( Z^1 \pm Z^0 \right). \tag{22}
\]

We will label the remaining \(Z^\mu\) coordinates by small Latin indices \(i, j, ...\) which run from 2 to \(d\). Quantities with lower \(\pm\) indices are defined similarly.\(^5\) We will take the background space to be such that \(G_{\pm i} = G_{++} = G_{--} = 0\), which implies \(G^{+} = (G_{++})^{-1}\).

To proceed with the dynamics of the fields \(Z^\mu\), we go to the light-cone gauge: one can use \(\tau, \sigma\) reparametrization invariance of \(S_P\) to fix

\[
Z^+ = \tau, \tag{23}
\]

\[
h_{\tau\sigma} = 0. \tag{24}
\]

\(^5\) For the metric, which lowers \(\pm\) indices, one finds \(G_{++} = \frac{1}{2} (G_{00} + 2G_{01} + G_{11})\), \(G_{+-} = G_{-+} = \frac{1}{2} (G_{11} - G_{00})\), \(G_{--} = \frac{1}{2} (G_{00} - 2G_{01} + G_{11})\), \(G_{\pm i} = G_{i\pm} = \frac{1}{\sqrt{2}} (G_{1i} \pm G_{0i})\). Similar relations hold for \(G^{++}, G^{-}, \text{ and } G^{+-}\).
Using the gauge-fixing conditions (23), (24), the Lagrangian density becomes

\[ \mathcal{L}_{LC} = -\frac{h^{\tau\tau}}{4\pi\alpha'} \left( 2G_{+-}\dot{Z}^- + G_{ij}\dot{Z}^i\dot{Z}^j - \frac{1}{(h^{\tau\tau})^2}G_{ij}Z^{i'}Z^{j'} \right) \]  

(25)

The equations of motion which follow from this Lagrangian are to be supplemented by the constraint equations (20), (21), which in the light-cone gauge read

\[ 2G^{+-}\pi_+\pi_- + G^{ij}\pi_i\pi_j + \frac{1}{(2\pi\alpha')^2}G_{ij}Z^{i'}Z^{j'} = 0 \]  

(26)

\[ \pi_-\dot{Z}^- + \pi_iZ^{i'} = 0 , \]  

(27)

where \( \pi_+, \pi_- \) are conjugate momenta for the fields \( Z^+, Z^- \). The equation of motion for \( Z^- \) implies that its canonical momentum is a function of \( \sigma \) only: \( \pi_- = \pi_- (\sigma) \). The single-string light-cone Hamiltonian, which one obtains from the Lagrangian (25), is

\[ H_{LC} = \int_0^l d\sigma \mathcal{H}_{LC} = \int_0^l d\sigma \left( G^{+-}\pi_+\pi_- + \frac{1}{(2\pi\alpha')^2}G_{ij}Z^{i'}Z^{j'} \right) \]  

(28)

and using the constraint (26), it is easy to see that \( \mathcal{H}_{LC} \) is just the negative of \( \pi_+ \):

\[ \mathcal{H}_{LC} = -G_{+-}G^{+-}\pi_+ = -\pi_+ . \]  

(29)

This form of the Hamiltonian is very convenient, when the background space is \((d + 1)\)-dimensional Anti-de Sitter space in Poincare coordinates:

\[ ds_{AdS}^2 = \frac{R^2}{Y^2} \left( 2dZ^+dZ^- + dZ^i dZ^j \delta_{ij} \right) . \]  

(30)

Here \( R \) is \((Z^\mu\text{-independent})\) curvature radius of the space, and \( Y \equiv Z^{\mu=d} \). The “warped coordinate” \( Y \) must take non-negative values, while the rest of the \( Z^\mu \) can be of any sign. The boundary of the space is at \( Y = 0 \). With the AdS metric (30), the light-cone Hamiltonian (28) takes the simple form

\[ H_{LC}^{\text{(AdS)}} = \int_0^l d\sigma \left( \pi_i\pi_i + \frac{1}{(2\pi\alpha')^2} \frac{R^4}{Y^4}Z^{i'}Z^{j'} \right) . \]  

(31)

B. The choice of \( \sigma \) coordinate

The Hamiltonian (31) can be further simplified by using some leftover reparametrization invariance – the reparametrizations of \( \sigma \) only. One convenient way to fix the \( \sigma \) coordinate is to measure it in units of \( \pi_- (\sigma) \), while keeping the total string length \( l \) fixed. This is a natural choice, because \( \pi_- (\sigma) \sim -h^{\tau\tau} = \gamma_{\sigma\sigma}/\sqrt{-\gamma} \), and therefore it transforms under reparametrizations \( \sigma \rightarrow \tilde{\sigma}(\sigma) \) as

\[ \tilde{\pi}_-(\tilde{\sigma})d\tilde{\sigma} = \pi_-(\sigma)d\sigma . \]  

(32)
Let us choose the new $\sigma$ coordinate as
\[
\tilde{\sigma} = \frac{1}{p_-} \int_0^\sigma \pi_-(\sigma')d\sigma',
\]
where $p_- \equiv \frac{1}{l} \int_0^l \pi_-(\sigma')d\sigma'$. We will refer to this choice of $\sigma$ as the $\pi_- = \text{const}$ gauge. Then the light-cone Hamiltonian (31) becomes
\[
H_{LC}^{(AdS)} = \frac{1}{2p_-} \int_0^l d\sigma \left( \pi_i \pi_i + \frac{1}{(2\pi \alpha')^2} \frac{R^4}{Y^4} Z_i' Z_i' \right).
\]
This Hamiltonian can be compared with the Hamiltonian of string theory on a flat $d + 1$ dimensional background:
\[
H_{LC}^{(\text{flat})} = \frac{1}{2p_-} \int_0^l d\sigma \left( \pi_i \pi_i + \frac{1}{(2\pi \alpha')^2} Z_i' Z_i' \right).
\]
Thus, strings in Anti-de Sitter space can be thought of as having a variable tension, $\frac{1}{2\pi \alpha'} R^2$, which increases as one approaches the boundary of the space.

When the radius of curvature is exactly zero, the AdS Hamiltonian takes the simple form
\[
H_{LC} = \frac{1}{2p_-} \int_0^l d\sigma \pi_i \pi_i \equiv \frac{1}{2p_-} \int_0^l d\sigma \left( \pi_\perp(\sigma)^2 + \pi_Y(\sigma)^2 \right),
\]
where $\pi_Y \equiv \pi_{i=d}$, and momentum densities are written as $\pi_i = (\pi_\perp, \pi_Y)$. This limit corresponds to vanishing tension, and therefore describes the situation when each piece of the string moves independently. The Hamiltonian (36) is precisely the Hamiltonian of the free field theory, which describes a collection of free particles. A particle in field theory corresponds to a freely moving bit of a tensionless string.

When $R^2/\alpha' = 0$, other choices of parametrizing $\sigma$ are also useful. Namely, when the radius is zero, equations of motion require all transverse momenta $\pi_i$ to be $\tau$-independent. All of them are proportional to $h^{\tau\tau}$, and therefore change under reparametrizations just like $\pi_-$ in (32). Measuring the $\sigma$ coordinate in units of $|\pi_Y(\sigma)|$ will be particularly convenient; in this case the Hamiltonian becomes
\[
H_{LC} = \int_0^l d\sigma \sum_{i=2}^{d-1} \frac{\pi_i \pi_i + p_Y^2}{2\pi_-(\sigma)} \equiv \int_0^l d\sigma \frac{\pi_\perp(\sigma)^2 + p_Y^2}{2\pi_-(\sigma)},
\]
where $p_Y = \frac{1}{l} \int_0^l |\pi_Y(\sigma)| d\sigma$ is $\sigma$-independent. Again, this is precisely the field theory Hamiltonian (14), when the continuum limit is taken with constant density of particles. This choice of parametrization will be referred to as $\pi_Y = \text{const}$ gauge. For the rest of the paper, we restrict ourselves to the case $R^2/\alpha' = 0$.

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6. $\pi_- = -h^{\tau\tau} G_{+\tau}/(2\pi \alpha')$ is positive because both $\gamma_{\sigma\sigma}$ and $G_{+\tau}$ are positive. Therefore, changing $\sigma$ to $\tilde{\sigma}$ involves no back-tracking.

7. When the radius of curvature is non-zero, the second term in the Hamiltonian (34) becomes important as one approaches the boundary at $Y = 0$. The second term also contains derivatives of the fields $Z^i$, and therefore can not in general be treated as a small perturbation, even when $R^2/\alpha'$ is small.
C. Mode expansions

The description of the dynamics becomes particularly simple when distance along the string is measured in units of $\pi_-(\sigma)$. Canonical momenta become $\pi_+ = p_-\dot{Z}^-$, $\pi_i = p_-\dot{Z}^i$, and canonical equations of motion reduce to $\ddot{Z}^i(\sigma, \tau) = 0$. The independent degrees of freedom are $Z^i$ and $\pi_i$; once $Z^i$ are known, one can determine $Z^-(\sigma, \tau)$ (up to a constant) from the constraints (26), (27):

$$\dot{Z}^- = -\frac{1}{2p_-^2} \pi_i \pi_i,$$  
(38)

$$Z'^- = -\frac{1}{p_-} \pi_i Z'^i.$$  
(39)

Since the transverse momenta are conserved, $\dot{\pi}_i = 0$, the first of the constraints implies that $Z^-$ satisfies the same equation as the transverse coordinates: $\ddot{Z}^- (\sigma, \tau) = 0$.

A general solution to the equations of motion, which satisfies Neumann boundary conditions $Z^i(\sigma = 0, \tau) = 0 = Z^i(\sigma = l, \tau)$ (open string with free ends) can be written as

$$Z^i(\sigma, \tau) = \sum_{n=-\infty}^{\infty} \left( z^i_n + \frac{p^i_n}{l} \tau \right) \cos \left( \frac{n \pi \sigma}{l} \right)$$  
(40)

$$\pi_i = p_- \dot{Z}^i = p_- \sum_{n=-\infty}^{\infty} p^i_n \cos \left( \frac{n \pi \sigma}{l} \right)$$  
(41)

where $z_n^i = z_{-n}^i$, $p_n^i = p_{-n}^i$. From (39) it follows that, if $Z^i$ satisfies Neumann boundary conditions, then so does $Z^-$. This tells us that $Z^-$ has a mode expansion, which is identical in form to the expansion of the transverse coordinates:

$$Z^-(\sigma, \tau) = \sum_{n=-\infty}^{\infty} \left( z^-_n + \frac{p^-_n}{l} \tau \right) \cos \left( \frac{n \pi \sigma}{l} \right)$$  
(42)

Comparing this expansion with (38), (39) determines $z^-_n$ and $p^-_n$ to be\(^8\)

$$z^-_n = -\frac{1}{l} \sum_{m=-\infty}^{\infty} z^i_{n-m} p^i_m \left( 1 - \frac{m}{n} \right)$$  
(43)

$$p^-_n = -\frac{1}{2l} \sum_{m=-\infty}^{\infty} p^i_{n-m} p^i_m$$  
(44)

The coefficient $z^-_0$ is left undetermined, which corresponds to the fact that the constraints (38), (39) determine $Z^-(\sigma, \tau)$ only up to an overall additive constant. The rest of the mode expansion coefficients satisfy $z^-_{-n} = z^-_n$, $p^-_{-n} = p^-_n$. The light-cone Hamiltonian (36) can be expressed in terms of the mode expansion coefficients:

$$H_{LC} = \frac{1}{2p_-} \int_0^l d\sigma \pi_i \pi_i = \frac{p_-}{2l} \sum_{n=-\infty}^{\infty} p^i_n p^i_n$$  
(45)

\(^8\) The identity $\sum_{n=-\infty}^{\infty} \left( \frac{1}{n} - n \right) p^i_{-n} p^i_n = 0$ is of help.
For periodic boundary conditions $Z^i(\sigma, \tau) = Z^i(\sigma + l, \tau)$ (closed string), a general solution to the equations of motion can be written as

$$Z^i(\sigma, \tau) = \sum_{n=-\infty}^{\infty} \left( z^i_n + \frac{p^i_n}{l} \right) e^{\frac{2\pi in\sigma}{l}},$$

(46)

$$\pi_j = p_- \dot{Z}^j = p_- \sum_{n=-\infty}^{\infty} \frac{p^j_n}{l} e^{\frac{2\pi in\sigma}{l}}$$

(47)

Reality of $Z^i(\sigma, \tau)$ requires $z^i_n = (z^i_n)^*$, $p^i_n = (p^i_n)^*$. The periodicity of $Z^-$ does not follow from the constraint: relation (39) tells one only that $Z^{-'}$ is periodic; $Z^-$ itself might have a term linear in $\sigma$, which violates periodicity. Thus, we must impose periodicity of $Z^-$ as an extra requirement.\textsuperscript{9} The mode expansion for $Z^-(\sigma, \tau)$ again has an analogous form:

$$Z^-(\sigma, \tau) = \sum_{n=-\infty}^{\infty} \left( z^i_n + \frac{p^i_n}{l} \right) e^{\frac{2\pi in\sigma}{l}}$$

(48)

and the coefficients are given by the same formulas (43), (44). As in the open string case, $z^0_-$ is undetermined. Note that $z^i_- = (z^i_-)^*$, $p^i_- = (p^i_-)^*$. The periodicity of $Z^-$ becomes the condition $n z^i_-|_{n=0} = 0$, which translates to

$$\sum_{m=-\infty}^{\infty} m z^i_- m p^i_m = 0.$$  

(49)

The mode expansion of the Hamiltonian is:

$$H_{LC} = \frac{p_-}{2l} \sum_{n=-\infty}^{\infty} p^i_n p^i_n = \frac{p_-}{2l} \sum_{n=-\infty}^{\infty} p^i_n (p^i_n)^*.$$  

(50)

In the gauge where distance along the string is measured in units of $|\pi_Y|$, the description of dynamics looks different. The independent degrees of freedom now are $Z^\perp \equiv Z^i |_{i=2..d-1}$, $\pi^\perp$, $Z^-$, and $\pi_-$, while $Y$ is determined by the constraints

$$\dot{Y} = \frac{p_Y}{\pi_-},$$

(51)

$$Y' = -\frac{\pi_- Z^{-'} + \pi^\perp \cdot Z^\perp}{p_Y}.$$  

(52)

The constraints fix $Y$ only up to an overall additive constant, and therefore the zeromode $y_0$ of $Y(\sigma, \tau)$ is left undetermined, analogously to $z^0_-$ in the discussion of the $\pi_- = const$ gauge.

\textsuperscript{9} This can be viewed as fixing the last gauge freedom of $\sigma$ shifts by a constant (which is present only in the closed string case). In quantum theory, periodicity of $Z^-$ is equivalent to the condition that one works in the subspace of the Hilbert space which has total momentum along the closed string equal to zero.
D. String quantization

To quantize, one imposes canonical commutation relations on the independent variables (including the unconstrained zeromodes). In the $\pi = \text{const}$ gauge,

$$\left[Z_j(\tau, \sigma), \pi_k(\tau, \sigma')\right] = i \delta(\sigma - \sigma') \delta^j_k \tag{53}$$

For the closed string, commutation relations (53) give

$$\left[z^j_n, p^k_m\right] = \frac{i}{p_-} \delta^{jk} \delta_{n,-m} \tag{54}$$

For the open string we will take $p^i_{-n} = p^i_n$, $z^i_{-n} = z^i_n$, and treat only mode expansion coefficients with $n \geq 0$ as independent degrees of freedom. With the normalization of the $p^i_n$s above, the mode expansion coefficients satisfy

$$\left[z^j_n, p^k_m\right] = \frac{i}{2p_-} \delta_{nm} \delta^{jk}, \quad n, m \geq 0 \tag{55}$$

For both open and closed strings,

$$\left[z^-_0, p_-\right] = i/l \tag{56}$$

and both $p_-$ and $z^-_0$ commute with all transverse mode coefficients. The Hamiltonians (45), (50) contain only momenta, and therefore have no ordering ambiguity. The eigenstates of the Hamiltonians have positive energy, and thus there is no tachyon in the spectrum.

In the $\pi_Y = \text{const}$ gauge, one has

$$\left[Z^j(\sigma, \tau), \pi_k(\sigma', \tau)\right] = i \delta(\sigma - \sigma') \delta^j_k, \quad j, k = 2..d-1 \tag{57}$$

$$\left[Z^-(\sigma), \pi_-(\sigma')\right] = i \delta(\sigma - \sigma') \tag{58}$$

The zeromode of the fixed momentum $\pi_Y$ commutes with all independent degrees of freedom, but does not commute with the zeromode of $Y$:

$$\left[y_0, p_Y\right] = i/l \tag{59}$$

E. Cutoff in the warped direction

So far we saw that single-string Hamiltonians of string theory on AdS space exactly reproduce the Hamiltonians of free scalar field theory in the sector of single-trace states with a large number of particles. Let us now show that a non-zero mass of particles in the field theory can be viewed as coming from a cutoff in the $Y$ direction, that is, from the requirement that $Y$ is finite-ranged. This is very similar to the supergravity construction of confining gauge theories in the strong coupling limit of the AdS/CFT correspondence.

In that limit, one finds the following picture. A fundamental string, whose ends on the boundary represent external source particles in field theory, is gravitationally pulled away
from the boundary, in the direction of increasing \( Y \). Since the string can not move past the cutoff, the potential energy between the external sources is linearly proportional to their separation at large distances. Alternatively, one can solve the wave equation for the corresponding supergravity field with the boundary conditions that the field vanishes at the cutoff and is normalizable at the boundary. In this case the boundary conditions make the spectrum of the wave equation discrete, and the eigenvalues are interpreted as masses of the particles created by the operator which couples to the corresponding supergravity field. An analogous situation takes place in the zero-coupling limit we consider here.

To see this, it is convenient to work in the \( \pi_Y = \text{const} \) gauge. Let us impose the cutoff in the \( Y \) direction, so that \( 0 < Y < Y_{\text{max}} \). The \( Y \)-momentum zeromode commutes with both \( \pi_\perp \) and \( \pi_- \), and therefore in the Schrödinger equation with Hamiltonian \( \mathcal{H} \), the \( y_0 \)-dependent part of the wave function can be factored out. The commutation relation (59) implies that \( p_Y \) has a simple representation as \( p_Y = -i/l \partial/\partial y_0 \), and therefore one has to solve
\[
-\frac{1}{l^2} \frac{d^2}{dy_0^2} \psi(y_0) = E_Y \psi(y_0) ,
\]
where \( 0 < y_0 < Y_{\text{max}} \), and \( E_Y \) is the eigenvalue of \( p_Y^2 \). By analogy with the strong-coupling case we impose Dirichlet boundary conditions \( \psi(y_0 = 0) = \psi(y_0 = Y_{\text{max}}) = 0 \), which gives
\[
E_Y = \frac{4\pi^2 n^2}{l^2 Y_{\text{max}}^2} ,
\]
where \( n = 0, 1, 2 \ldots \) is the mode number. Thus for the eigenstates \( \psi \) the string Hamiltonian \( \mathcal{H} \) becomes
\[
H_{LC} = \int_0^l d\sigma \frac{1}{2 \pi_-(\sigma)} \left( \pi_\perp(\sigma)^2 + \left( \frac{2\pi}{Y_{\text{max}}} \right)^2 n^2 \right) ,
\]
This is to be compared to the continuum version of the field theory Hamiltonian with constant density of particles, \( \mathcal{H} \), in the limit when the number of particles \( M \) is large:
\[
H_{LC} = \int_0^l dx \frac{1}{2 \pi_-(x)} \left( \pi_\perp^2(x) + m^2 M^2 \right) .
\]
Thus, one can identify the mass of the free scalar as
\[
m = \frac{2\pi}{Y_{\text{max}}} ,
\]
and the number of particles \( M \) of the field theory state as the mode number \( n \) of the zeromode wavefunction. Massless field theory gives rise to string theory on the AdS space with cutoff removed, \( Y_{\text{max}} \rightarrow \infty \). The freedom to impose the cutoff on the AdS space is the freedom to add a mass term to the free scalar.
IV. CRITICAL DIMENSION OF THE BOSONIC STRING

In this section we argue that the string theory which one obtains by taking the naive zero-radius limit of the classical Hamiltonian $31$ does not have a critical dimension, as expected from its relation to the free scalar field theory.

A. Critical dimension in light-cone gauge

The critical dimension of string theory arises from the fact that Weyl invariance of the Polyakov action becomes anomalous unless coupled to a matter CFT with the correct central charge. In light-cone gauge this anomaly is subtle to detect: Weyl and reparametrization invariance get completely fixed, and it is spacetime Lorentz invariance that picks up the anomaly. Rotations in the transverse space are manifest in light-cone gauge. Not manifest are the rotations which take $Z^1$ (the spatial coordinate singled out to form $Z^\pm$), to one of the transverse $Z^i$s. Such rotations have to be accompanied by a compensating reparametrization/rescaling of the world-sheet. It is the new $Z^+$ that has to be equal to $\tau$ after the rotation. A failure of reparametrization/rescaling to be a good symmetry shows up as an anomaly in the Lorentz algebra.

This story is well known in flat space $13$. A similar argument can be made for the case of AdS space. With the AdS metric written in Poincare coordinates $30$, the Lorentz invariance of the action $17$ (rotations of $Z^{\mu=0..d-1}$) is manifest. For this Lorentz subgroup of the AdS isometry group, the standard Noether procedure gives rise to the conserved charges

$$J^K_I = \int_0^l d\sigma \left( Z^K \pi_I - \eta_{IJ} \eta^{KL} Z^J \pi_L \right) $$

(large Latin indices run from 0 to $d-1$). By imposing the canonical commutation relations $53$, one finds the usual Lorentz algebra

$$[J^K_I, J^M_N] = i \left( \delta^K_N J^M_I - \delta^K_I J^M_N + \eta_{MI} \eta^{ML} J^K_L - \eta_{NL} \eta^{MK} J^K_I \right).$$

The algebra implies that the light-cone components

$$J^M_\pm = \frac{1}{\sqrt{2}} \left( J^M_1 \pm J^M_0 \right) = \int_0^l d\sigma \left( Z^M \pi_\pm - Z^\mp \pi_L \delta^{LM} \right) $$

must satisfy

$$[J^M_+, J^N_-] = 0.$$ 

A non-zero value for the commutator $68$ signals the presence of the conformal anomaly. In principle it is possible that by some accident the Lorentz algebra will be anomaly free while conformal symmetry will exhibit some anomaly. This is, however, unlikely, since the anomaly is really a failure of diffeomorphisms times Weyl invariance, and the Lorentz generators we are analyzing do require the compensating transformations on the world-sheet.
Scaling the anomaly away

As we saw in Section III B, taking the $\alpha' \to \infty$ limit of the classical string Hamiltonian in flat space gives the AdS Hamiltonian in the limit $R^2/\alpha' \to 0$. Of course, taking $\alpha'$ to infinity is not a very sensible thing to do, because $\alpha'$ is a dimensionful quantity. All this limit means is that we are studying states on time scales very short compared to $\alpha'$. On the other hand, $R^2/\alpha'$ is a dimensionless parameter, and taking it to zero in the classical string Hamiltonian on AdS makes perfect sense. Still, it follows that there is a limit, in which the usual flat space free string spectrum reduces to the zero radius AdS spectrum by taking $\alpha' \to \infty$, while suitably rescaling $\tau$. We show below that in this limit the usual flat-space anomaly scales to zero. All mode expansions will be written in the $\pi_- = \text{const}$ gauge.

For open strings, a general solution to the flat space equations of motion is given by

$$Z^m(\sigma, \tau) = (z^m_0 + \frac{p^m_0}{l}\tau) + i\sqrt{2} \sum_{k \neq 0} \frac{a^m_k}{k} \exp \left[ -\frac{i\pi k c \tau}{l} \right] \cos \left( \frac{\pi k \sigma}{l} \right).$$

(69)

where $c = \frac{1}{2\pi \alpha' p_-}$ is the dimensionless velocity of the wave propagating along the string. Canonical commutation relations (53) imply that the flat space mode expansion coefficients satisfy

$$[a^m_k, a^n_{-k}] = \alpha' k \delta^{mn} \sim \frac{1}{c} \delta^{mn}. \quad (70)$$

Now we can take the $\alpha' \to \infty$ (that is, $c \to 0$) limit. Expanding out the exponential, keeping the constant and the linear terms, and comparing with the AdS result (40)

$$Z^m(\sigma, \tau) = (z^m_0 + \frac{p^m_0}{l}\tau) + 2 \sum_{k>0} \left( z^m_k + \frac{p^m_k}{l}\tau \right) \cos \left( \frac{\pi k \sigma}{l} \right),$$

(71)

we find

$$z^m_k = \frac{i}{\sqrt{2}} \left( \frac{a^m_k}{k} - \frac{a^m_{-k}}{k} \right), \quad (72)$$

$$p^m_k = \frac{\pi c}{\sqrt{2}} \left( a^m_k + a^m_{-k} \right). \quad (73)$$

Keeping these two quantities finite while sending $\alpha' \to \infty$ reduces the flat space solution to the zero radius AdS solution. That is, we take $a^m_k$ (rather, their eigenvalues) to infinity, while keeping their differences finite.

The flat space solution for closed string is

$$Z^m(\sigma, \tau) = (z^m_0 + \frac{p^m_0}{l}\tau) + \frac{i}{\sqrt{2}} \sum_{k \neq 0} \left\{ a^m_k \exp \left[ \frac{2\pi i k (\sigma - c \tau)}{l} \right] + \tilde{a}^m_k \exp \left[ -\frac{2\pi i k (\sigma + c \tau)}{l} \right] \right\},$$

(74)

Again, comparing this with the AdS result for the closed string (40)

$$Z^m(\sigma, \tau) = \left( z^m_0 + \frac{p^m_0}{l}\tau \right) + \sum_{k \neq 0} \left( z^m_k + \frac{p^m_k}{l}\tau \right) e^{\frac{2\pi i k \sigma}{l}},$$

(75)
we obtain
\[ z_k^m = \frac{i}{\sqrt{2}} \left( \frac{a_k^m}{k} - \bar{a}_{-k}^m \right), \quad (76) \]
\[ p_k^m = \sqrt{2\pi c} (a_k^m + \bar{a}_{-k}^m). \quad (77) \]

Now it is easy to see that the anomaly vanishes. The commutator of interest is
\[ [J_+^m, J_+^n]_{AdS} = \lim_{c \to 0} [J_+^m, J_+^n]_{flat}. \quad (78) \]

The Lorentz generators \( J_+^m \) in flat space also have the same form as the ones in \( AdS \),
\[ J_+^m = \int_0^l d\sigma \left( Z^m \pi_+ - Z^- \pi \delta^m \right). \]

The constraint equation \( (26) \) implies that \( \pi_+ \) is proportional to \( c^2 \), while \( Z^- \) contains a factor of \( c \). Hence the commutators have the generic form
\[ [J_+, J_+]_{flat} \sim c^4 [aaa, aaa]. \]

Consider the open string case for the concreteness. In flat space, the anomaly arises from the terms quadratic in \( a_k^m \)'s (see e.g. [13]). When one reduces the commutator \([aaa, aaa]\) to \( aa \), the factor of \( 1/c^2 \) is introduced by the commutation relation \( (70) \). Thus the anomaly scales as
\[ [J_+^m, J_+^n]_{AdS} = \lim_{c \to 0} [J_+^m, J_+^n]_{flat} \sim \lim_{c \to 0} \sum_{k=1}^\infty c^2 \Delta_k (a_{-k}^m a_k^n - a_k^m a_{-k}^n) = \]
\[ = \lim_{c \to 0} \sum_{k=1}^\infty \frac{ik}{\pi} \Delta_k (z_k^m p_k^n - z_k^n p_k^m) = 0, \quad (80) \]
where we used \( (72), (73) \) in the penultimate step, and
\[ \Delta_k = k \left( \frac{26 - (d + 1)}{12} \right) + \frac{1}{k} \left( \frac{(d + 1) - 26}{12} - 2(1 - a) \right) \]
is an \( \alpha' \)-independent number [13]. Here \( a \) is undetermined normal ordering constant in \( Z^- \), which does not enter the Hamiltonian. We see that in the limit \( c \to 0 \) Lorentz algebra closes for any values of \( d \) and \( a \).

The closed string case works similarly. Thus we conclude that there is no critical dimension.

V. DISCUSSION

We have shown that states with many particles in a large \( N \) free scalar field theory are well described in terms of a string world-sheet. The string theory is tensionless in the
sense that it has no $\sigma$ derivatives. Thus the string can fall apart into “partons”. Still, it is
described by a Hamiltonian which can be obtained from a non-linear sigma model on a
quite singular target space by light-cone gauge fixing, and it exhibits $\sigma$ reparametrization
invariance.

An interesting question is of course how to turn on interactions. There are two types of
interactions of interest. For one we can look at finite $N$ corrections. Already in the free
scalar theory the combinatorics of the Wick contractions of the free fields give a non-trivial
$1/N$ expansion. In particular only in the large $N$ limit does one get a decoupling of single-
and multi-trace states. The $1/N$ corrections correspond to the $g_s$ expansion of string theory,
allowing strings to split and join. On the field theory side the theory is solvable to all orders
in $N$. One can take this again as a definition of the interacting string theory. One might
then wonder if (at least for states with a large number of partons), this expansion can be
reobtained from purely stringy reasoning, e.g. from light-cone string field theory.

More interesting would be to turn on interactions in the field theory. Since the Hamiltoni-
ans of the two theories are the same, any such perturbation can be mapped to a deformation
of the bulk theory. One interesting question is whether there is any coupling in the field the-
ory that would correspond to turning on finite curvature radius in the bulk. Since the string
perturbation involves $(Z')^2$ terms, whose matrix elements can become arbitrarily large, for
a generic string theory this perturbation will not make sense. This is consistent with the
fact that non-supersymmetric conformal field theories are hard to find. One might hope
that in the case of type IIB string theory on $AdS_5 \times S^5$ and its dual $\mathcal{N} = 4$ supersymmetric
Yang-Mills theory this perturbation is well defined. After all, we know that in the
$\mathcal{N} = 4$ SYM theory the coupling can be turned on smoothly. In fact, while we were finishing this
work, \cite{14} appeared, which, using basically the same ideas in the supersymmetric context,
presents evidence that this perturbation is indeed valid.

Last but not least, another very interesting application of our results is the problem of
closed string tachyon condensation. Flat space is an unstable solution of bosonic or type
0 string theory. It has been a long standing puzzle to determine what happens when the
tachyon rolls down the hill. One natural problem is that the potential energy decreases, so
the most naive guess would be that the true minimum should have a negative cosmological
constant. Our zero radius AdS solutions would be the natural candidate for the endpoint of
tachyon condensation in the bosonic string. So our speculation is that as the closed string
tachyon condenses, the potential for the tachyon becomes arbitrarily negative, while the
dilaton gets frozen at $1/g_s \sim e^{-\Phi} \sim N^2$ for some integer $N$. Very similar backgrounds should
exist in type 0 string theory, with the dual theory being a theory of free vector particles.
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