Effects of degree distribution in mutual synchronization of neural networks

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We study the effects of the degree distribution in mutual synchronization of two-layer neural networks. We carry out three coupling strategies: large-large coupling, random coupling, and small-small coupling. By computer simulations and analytical methods, we find that couplings between nodes with large degree play an important role in the synchronization. For large-large coupling, less couplings are needed for inducing synchronization for both random and scale-free networks. For random coupling, cutting couplings between nodes with large degree is very efficient for preventing neural systems from synchronization, especially when subnetworks are scale-free.

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Synchronization phenomena in neural systems have attracted much attention. These phenomena are thought to be important for functioning neural system, such as neural coding, visual information processing, sleeping, and memory in brain \[1\].\[2\].\[3\].\[4\]. Besides the complete synchronization, which results in identical states of all neurons in a uniform population, more subtle forms of synchronization should be present for examining the brain functions. The brain is essentially a system of interacting neural networks and the activity pattern of different networks may become synchronized while retaining their complex spatiotemporal dynamics. Therefore, it is interesting to investigate mutual synchronization in ensembles of coupled neural networks \[5\]. This problem has been studied on fully connected networks \[6\] and on random networks \[7\].

Recently, it was suggested that connectivity in neural systems is more complex \[8\]. The effects of complicated topologies on network dynamics have been confirmed in some theoretical studies. For instance, small-world neural networks give rise to fast system response with coherent oscillations \[9\]. Scale-free Hopfield networks can recognize blurred pattern efficiently \[10\]. Complex networks have both sensitivity and robustness as responding to different stimuli \[11\]. On the other hand, synchronization is not always desired in neural systems. For instance, several neurological diseases such as Parkinson’s disease and epilepsy are caused by synchronized firing of neural oscillators \[12\]. So it is also important to study desynchronization and instability of synchronized motion of neural systems.

In this paper, we investigate the efficiency of scale-free topology in inducing synchronization and preventing the system from synchronization in two-layer neural networks. We carry out three coupling strategies between subnetworks: large-large coupling (couplings built between nodes with large degree); random coupling (couplings built between randomly selected nodes); small-small coupling (couplings built between nodes with small degree). Computer simulations reveal that couplings between nodes with large degree play an important role either in inducing synchronization or preventing the system from synchronization. An analytical treatment confirms the numerical simulation result.

We consider a neural network model that consists of \(N\) neurons \(x_i(t) \in (-1, 1), i = 1, \cdots, N\). The topology of networks was represented by symmetric adjacency matrix \(A\) whose entry \(a_{ij} (i, j = 1, \cdots, N)\) is equal to 1 when neuron \(i\) connects to neuron \(j\), and zero otherwise. Each link has a weight \(J_{ij}\) which is a random number distributed uniformly in the interval between \(-1\) and 1. The system considered is composed of two identical neural networks, and a part of corresponding nodes in different subsystems is coupled together. The dynamics of the system is described by the following equations \[6\].\[7\].

\[
\begin{align*}
    x_i^1(t+1) &= (1-\varepsilon_i)\Theta(h_i^1(t)) + \varepsilon_i\Theta(h_i^1(t) + h_i^2(t)), \\
    x_i^2(t+1) &= (1-\varepsilon_i)\Theta(h_i^2(t)) + \varepsilon_i\Theta(h_i^1(t) + h_i^2(t)).
\end{align*}
\]

In the equation, \(\varepsilon_i\) represents the coupling strength between the nodes \(i\) in different networks. When a pair of nodes is coupled, the coupling strength between them is equal to a constant \(\varepsilon_i = \varepsilon\), otherwise \(\varepsilon_i = 0\). For large-large (small-small) coupling we choose a group of nodes which consists of the nodes of greatest (smallest) degree in a subnetwork, and take \(\varepsilon_i = \varepsilon\) if node \(i\) is in the group. Here \(h_i^l(t)\) is the local field of the \(l\)th neuron and is expressed by

\[
h_i^l = \sum_{j=1}^{N} a_{ij} J_{ij} x_j^l(t), \ l = 1, 2
\]

denoting the signal arriving at neuron \(i\) at time \(t\) from neurons of the same network. The local field is determined by the network topology. The non-coupled nodes in each subnetwork interact indirectly with another subnetworks through the local field. The activity function \(\Theta(r)\) is defined by

\[
\Theta(r) = [1 + \tanh(\beta r)]/2,
\]

where \(\beta = 1/T\) characterizes a measure of the inverse magnitude of the amount of noise affecting this neuron, performing the role of reciprocal temperature in analogy to thermodynamic systems. For convenience, we set \(\beta = 10\) through
the small-small coupling is the most inefficient method for coupling, respectively. This implies that the large-large coupling can be regarded as removing couplings which link nodes occupying the position \( k \) of the network to synchronize in the case of large-large coupling. Moreover, given the coupling strength \( \varepsilon \) and the degree of coupled nodes influence the efficiency of the system, and the scale-free topology is more efficient than random network for inducing synchronization or preventing the system from synchronized states.

Firstly, we investigate the number of couplings needed between two subnetworks for system synchronization to study the efficiency of the network topology. In Fig. 1 we plot the histograms of the number of couplings built to guarantee synchronization for scale-free and random networks. We use Barabási-Albert (BA) arithmetic to generate the scale-free network and use Erdős-Rényi (ER) arithmetic to construct the random network. According to the parameter setting in \([11]\), we also choose the size \( N = 1000 \) and average degree \( \langle k \rangle = 20 \) for both kinds of networks. When subnetworks are scale-free (see Fig. 1(a)), the mean fraction of couplings is 58.2%, 80.7%, and 95.6%, corresponding to the large-large coupling, random coupling, and small-small coupling, respectively. This implies that the large-large coupling strategy is more efficient than the random coupling and the small-small coupling is the most inefficient method for inducing synchronization. In addition, the small-small coupling can be regarded as removing couplings which link nodes with larger degree from the globally coupled system. Therefore, removing couplings among nodes with larger degree at the initial state can efficiently prevent the system from synchronizing. In contrast to the case of scale-free topology, the peaks corresponding to three coupling methods for the system consisted of random networks are more closer (see Fig. 1(b)), which is caused by the homogeneous distribution of network connectivity. This implies that the topology of subnetworks and the degree of coupled nodes influence the efficiency of the system, and the scale-free topology is more efficient than random network for inducing synchronization or preventing the system from synchronized states.

Figure 2 shows the fraction of couplings needed to induce synchronization versus the coupling strength \( \varepsilon \). Whether subnetworks are scale-free or random, there is a critical point \( \varepsilon_c = 0.44 \) below which partial coupled networks can not synchronize. For \( \varepsilon > \varepsilon_c \), degrees of nodes taking part in interactions between two subnetworks will efficiently affect synchronization or the prevention of networks from synchronization. The larger the parameter \( \varepsilon \) is, the less couplings are needed for system synchronization. Furthermore, given the coupling strength, the scale-free topology is more efficient than the random graph to synchronize in the case of large-large coupling.

We now examine the dependence of the fraction of couplings needed for synchronization on the average links per node \( \langle k \rangle \) in a subnetwork. For small \( \langle k \rangle \), due to few neighbors per node and weak indirect interaction between non-coupled nodes, more couplings are needed for networks to synchronize. For large \( \langle k \rangle \), different from classical results of network synchronization \([15]\), the coupled system still needs many nodes to be coupled to ensure synchronization. Figure 3 displays a surprisingly nonmonotonic dependence of the fraction on \( \langle k \rangle \) for given network size and the coupling strength, i.e., there exists an optimal value of \( \langle k \rangle \) for which the fraction of couplings reaches minimum. With the increase of \( \langle k \rangle \), the interaction of nodes inside a subnetwork is enhanced through the local field. The enhanced inside interaction brings the big indirect interaction between non-coupled nodes, which helps
to the synchronization of non-coupled nodes. So less couplings are needed for system synchronization when \( \langle k \rangle \) increases. On the other hand, the evolution of subsystems with stronger inside interaction is more stable and it needs more couplings to drive their evolutions into synchronization. The indirect interaction between non-coupled nodes and the synchronization of non-coupled nodes is nearly the same as the average degree of the large coupling. When the system is randomly coupled by either scale-free or random subnetworks, the average degree of coupled nodes is nearly the same as the average degree of the subnetworks \( \langle k \rangle \). So the total degree of coupled nodes is

\[
\sum_{i=1}^{F_r} k_i = \langle k \rangle F_r,
\]

where \( k_i \) is the degree of neuron \( i \). For any network topology with the same average degree and size, the fraction of couplings is the same in the case of random coupling. Following ideas developed by Bar-Yam and Epstein, we get the relation between the fraction of couplings for random coupling (denoted by \( f_r \), \( f_r = F_r / N \)) and the fraction of couplings for large-large coupling (denoted by \( f_l \), \( f_l = F_l / N \)).

When subnetworks are random, the total degree of coupled nodes is

\[
\sum_{i=1}^{F_l} k_i = N \sum_{k_l}^{\infty} k(k) e^{-\langle k \rangle} k!/(k!)^2
\]

for large-large coupling, where \( k_l \) denotes the minimum degree of the coupled nodes. Thus the fraction of couplings is

\[
f_l = \sum_{k_l}^{\infty} \langle k \rangle e^{-\langle k \rangle} k!
\]

So the difference of the fraction of couplings for two coupled methods is

\[
f_r - f_l = \langle k \rangle k_{l}^{-1} e^{-\langle k \rangle} / (k_l - 1)!,
\]

which maximum over \( k_l \) is obtained approximately by setting \( k_l = \langle k \rangle + 1/2 \) for a given value \( \langle k \rangle \).

When subnetworks are scale-free, the degree distribution has a power law shape \( P(k) = A k^{-\gamma} \). The total degree of coupled nodes is

\[
\sum_{i=1}^{F_l} k_i = N \int_{k_l}^{\infty} k \langle k \rangle P(k) dk = \frac{1}{\gamma - 2} N A k_l^{2-\gamma}
\]

with the ancillary condition

\[
F_l = N \int_{k_l}^{\infty} \langle k \rangle P(k) dk = \frac{1}{\gamma - 1} N A k_l^{1-\gamma}
\]

for large-large coupling. Normalizing the probability distribution and assuming a sharp cutoff of the distribution at low \( k \), we yield

\[
A = \frac{(\gamma - 2)(\gamma - 1)}{(\gamma - 1)(\gamma - 2)} \langle k \rangle (\gamma - 1).
\]

Combining Eqs. (6), (9) and (11), the relation between \( f_l \) and \( f_r \) is obtained

\[
f_l = f_r^{(\gamma - 1)/(\gamma - 2)}.
\]
For BA model, the degree exponent $\gamma$ is equal to 3 \cite{13}. Thus we have $f_l = f_r^2$.

There also exists a threshold of degree for preventing the system from synchronization when we cut couplings from the globally coupled system. This threshold is equal to the difference between $N\langle k \rangle$ and the threshold of degree for ensuring synchronization. Similarly, the relationship between the fraction of removed couplings for large-large cutting and random cutting also follow Eqs. \eqref{eq:6} and \eqref{eq:12}.

Figure 4(a) shows $f_{l}^{1/2}/f_r$ as a function of coupling strength when subnetworks are scale-free. In the case of building couplings, numerical simulations give that $f_{l}^{1/2}/f_r = 1$, which is consistent with Eq. \eqref{eq:12}. In the case of cutting couplings, the ratio is larger than the analytical prediction when the coupling strength $\epsilon$ is small, which results from the fat tail \cite{16} of BA networks. In other words, the number of nodes with large degree is more than that described by the power-law distribution. As a result, some extra couplings between large degree nodes are removed in simulations and therefore the number of cut couplings predicted by analysis is less than that of simulations. When coupling strength is strong, the number of couplings cut from the coupled system is large, and the simulation results close to the analytical prediction. Figure 4(b) shows the ratio $f_{l}^{1/2}/f_r$ as a function of the average degree $\langle k \rangle$ of scale-free subnetworks. For building couplings, simulation results agree well with the analytical prediction. For cutting couplings, the simulation results are greater than the analytical calculation in the case of either $\langle k \rangle$ is low or large. The deviation results from the small fraction of couplings in these regions as shown in Fig. 4(a). When subnetworks are random, the difference $f_r - f_l$ is shown in Figs. 4(c) and 4(d) as a function of the coupling strength $\epsilon$ and the average degree of subnetworks $\langle k \rangle$. The analytical result of the upper boundary of $f_r - f_l$ gives a good limitation to numerical results. Although building large-large couplings and removing large couplings improve the efficiency in inducing and preventing synchronization, the analytical result of random subnetworks restricts the enhancement of efficiency to a small range which is less than that of scale-free subnetworks.

In summary, we have studied the influences of the degree distribution of networks on mutual synchronization in a two-layer neural networks. We investigated three coupling methods between two subsystems: large-large coupling, random coupling, and small-small coupling. We found that couplings between nodes with large degree nodes play an important role in the synchronization. For large-large coupling, less couplings are needed for inducing synchronization for both random and scale-free networks. For random coupling, cutting couplings between nodes with large degree is very efficient for preventing neural systems from synchronization, especially when subnetworks are scale-free. By assuming that the total degree of coupled nodes in subnetworks determines the system synchronization, the numerical simulation results are interpreted analytically. The analysis reveals that the degree distribution of subnetworks rather than other topological quantities affects the efficiency of systems in synchronization. Although our work is based on a simple model of neural systems, we think that the results found out in this work is proper in more wide and realistic situations in which the dynamics of neurons depend on the mean local field. It would be interesting if Nature takes advantage of the efficiency of the scale-free topology in controlling mutual synchronization of interacted systems.

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