A FACTORIZATION CONSTANT FOR $l_p^n$, $0 < p < 1$

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In this paper we are concerned with factoring the identity operator on an $n$-dimensional quasi-normed space $X$ through a space $\ell^K$. We seek a good lower bound $\lambda(x)$ for $\|P\| \|T\|$ over all factorizations $\text{Id}_X = PT$, with $T : X \rightarrow \ell^K$, $P : \ell^K \rightarrow X$. When $X$ is $\ell_p^n$, $1 \leq p$, the constant is known: see [5, Theorem 32.9] and the references given for that theorem.

For $p < 1$, we will obtain the lower estimate $\lambda(l_p^n) \geq Cn^{\frac{1}{p} - \frac{1}{2}}(\log n)^{-\frac{1}{2}}$. (A $T$ and $P$ with $\|P\| \|T\| \leq Cn^{\frac{1}{p} - \frac{1}{2}}$ are easily obtained.) Throughout, $C$ denotes a constant, which may vary from one occurrence to the next, but which is independent of $n$.

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Lemma 1. Let $(r_i)$, $1 \leq i \leq n$, be the first $n$ Rademacher functions and let $\alpha_1, \ldots, \alpha_n$ be real. Letting $m$ denote Lebesgue measure on $(0,1)$, we have

$$m \left\{ \left| \sum_{i=1}^{n} \alpha_i r_i \right| > \alpha \sqrt{\log n \sum \alpha_i^2} \right\} \leq n^{-C\alpha^2},$$

for any positive $\alpha$.

Proof. This is well known; for completeness, we sketch a proof, following a suggestion of R. Kaufman.
We can assume $\sum_{i=1}^{n} \alpha_i^2 = 1$. Put $f = \sum_{i=1}^{n} \alpha_i r_i$. By Khintchine’s inequality, for some constant $C$ and all $p \geq 1$, $\|f\|_p \leq C p^{\frac{1}{2}}$. From this, $m\{|f| > \lambda\} \leq (C/\lambda)^p p^{\frac{p}{2}}$ with $K = C/\lambda$.

Now minimize $K p^{\frac{p}{2}}$ in $p$. At the minimizer, we find $p = K^{-2}e$, from which $K p^{\frac{p}{2}} = \exp(-\lambda^2/\epsilon C^2)$. (Note that $p > 1$ for $\lambda > C$.) Finally, put $\lambda = \alpha \sqrt{\log n}$ to get the conclusion.

The space $\ell_\infty$ is not of type 2; but the conclusion of the next lemma will suffice for our purposes.

**Lemma 2.** Let $Y$ be an $n$-dimensional subspace of $L_1(0,1)$ and let $f_1 \ldots f_n$ be elements of $Y^*$, of norm at most 1. Then for some $\overline{\sigma}$ in $(0,1)$,

$$\sup_{\|y\| \leq 1} \left| \sum_{i=1}^{n} r_i(\overline{\sigma}) f_i(y) \right| \leq C \sqrt{n \log n}.$$ 

**Proof.** For any $0 < \epsilon < \frac{1}{3}$, a result of Schechtman [3] implies that there are an $N \leq C \epsilon^{-2} \log(\epsilon^{-1}) n^2$ and an isomorphism $U : Y \rightarrow \ell_1^N$ such that $\|U\| \|U^{-1}\| \leq 1 + \epsilon$. See also the results of Bourgain–Lindenstrauss–Milman [1] and Talagrand [4]. In particular, taking $\epsilon = \frac{1}{4}$, say, we obtain the corresponding $U$; we can assume $\|U\| = 1$, so that $\|U^{-1}\| \leq \frac{5}{3}$ and $N \leq C n^2$ (after changing $C$) $\leq n^3$, if $n$ is sufficiently large.

Let $e_1, \ldots e_N$ be the unit basis vectors in $\ell_1^N$. For each $i$ let $\Phi_i$ be a Hahn–Banach extension to $\ell_1^N$ of $f_i U^{-1}$ on $U(Y)$, with $\|\Phi_i\| \leq \frac{5}{4}$; then for each $j$, $\|\Phi_i(e_j)\| \leq \frac{5}{4}$.

Now fix $\alpha$ with $C\alpha^2 > 3$, where $C$ is the constant in the conclusion of lemma 1;
then $n^3n^{-C\alpha^2} \leq \frac{1}{4}$ if $n$ is sufficiently large. Since
\[
m \left\{ \left| \sum_{i} \phi_i(e_j)r_i \right| > \frac{5}{4} \alpha \sqrt{n \log n} \right\} \leq m \left\{ \left| \sum_{i} \phi_i(e_j)r_i \right| > \alpha \sqrt{n \sum \phi_i^2(e_j)} \right\} \leq n^{-C\alpha^2}, \quad \text{for each } j,
\]
there is a set $A$, $m(A) > \frac{3}{4}$, such that $\left| \sum_{i=1}^{n} \phi_i(e_j)r_i(\overline{s}) \right| \leq \frac{5}{4} \alpha \sqrt{n \log n}$ for each $j$ and each $\overline{s}$ in $A$.

Now if $y \in Y$ and $\|y\| \leq 1$, $\|Uy\| \leq 1$. Write $Uy = \sum_{j=1}^{N} \alpha_j e_j$, $\sum_{j=1}^{N} |\alpha_j| \leq 1$; applying the above inequality to each $j$ and recalling that $f_{i} = f_{i}U^{-1}U$ on $Y$, we have
\[
\left| \sum_{i=1}^{n} f_{i}(y)r_{i}(\overline{s}) \right| \leq \frac{5}{4} \alpha \sqrt{n \log n} \leq C \sqrt{n \log n}
\]
for each $\overline{s}$ in $A$.

\textbf{Notation.} Let $\mathcal{A}$ be an algebra of measurable subsets of $(0,1)$. For $0 < p \leq \infty$, $L_p(\mathcal{A})$ is the space of functions in $L_p(0,1)$ which are $\mathcal{A}$-measurable. For ease of argument, we deal with an $L_\infty(\mathcal{A})$ with “homogeneous” $\mathcal{A}$, rather than $\ell_\infty^K$.

\textbf{Theorem.} Let $\mathcal{A}$ be a finite subalgebra of measurable subsets of $(0,1)$ containing the dyadic intervals $(\frac{j-1}{2^n}, \frac{j}{2^n})$, $1 \leq j \leq 2^n$, and assume the atoms of $\mathcal{A}$ all have the same measure. Let $X$ be an $n$-dimensional vector space. Let $T : X \rightarrow L_\infty(\mathcal{A})$ be a linear map; and let $(e_i)_{i=1}^{n}$ be elements of $X$ such that $\|T(e_i)\|_\infty \leq 1$, $1 \leq i \leq n$. Let $P : L_\infty(\mathcal{A}) \rightarrow X$ be a linear operator such that $PT = Id_X$. Then there is $w$ in $L_\infty(\mathcal{A})$ with $\|w\|_\infty \leq C \sqrt{n \log n}$ such that $P(w) = \sum_{i=1}^{n} r_{i}(\overline{s})e_{i}$, for some $\overline{s}$ in $(0,1)$. 

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Proof. Let \((I_j)_{j=1}^k\) be the atoms of \(\mathcal{A}\), and let \(z_1, \ldots, z_{k-n}\) be a basis for \(\ker P\).

Define a \((k-n) \times k\) matrix by \(z_{i,j} = \text{constant value of } z_i \text{ on the atom } I_j\).

Now row-reduce the matrix \((z_{i,j})\). In \(k-n\) of the columns there will be one 1 with all other entries 0; denote the atoms corresponding to the \(n\) remaining “distinguished” columns by \(I_{s_1}, \ldots I_{s_n}\). Enlarge the matrix \((z_{i,j})\) to a \(k \times k\) matrix \((y_{i,j})\) by adding \(n\) rows of zeros in each of rows \(s_1\) through \(s_n\).

We can obviously regard \(y_{i,j}\) as an \(\mathcal{A} \times \mathcal{A}\)-measurable function \(y(s,t)\) on \((0,1) \times (0,1)\), which satisfies these properties:

1. if \(s \notin UI_{s_i}\) and if \(s\) and \(t\) are in the same atom, \(y(s,t) = 1\);
2. if \(s \notin UI_{s_i}\) and if \(s\) and \(t\) are in different atoms, \(y(s,t) = 0\);
3. if \(t \notin UI_{s_i}\), \(y(s,t) = 0\) for all \(s\);
4. if we define \(y_t\) on \((0,1)\) by \(y_t(s) = y(s,t)\), then \(y_t \in \ker P\);
5. a function \(f\) in \(L_\infty(\mathcal{A})\) is in \(\ker P\) if and only if there is a function \(\beta(t)\) in \(L_\infty(\mathcal{A})\) so that \(f(s) = \int y_t(s)\beta(t)\,dt\) for all \(s\).

Properties (1) – (5) are evident from the description of \(\ker P\) and the properties of a row-reduced matrix.

Let \(s^*_i\) be a point of the atom \(I_{s_i}\), \(1 \leq i \leq n\), and let \(g_i = T(e_i), 1 \leq i \leq n\); then \(\|g_i\|_\infty \leq 1\).

Now define

\[
R(s,t) = \sum_{i=1}^n r_i(s)g_i(t),
\]

and for a function \(y(t)\), define

\[
\psi_s(y) = \int R(s,t)y(t)\,dt = \sum_{i=1}^n \left(\int y(t)g_i(t)\,dt\right)r_i(s).
\]
Let $Y$ be the span of the functions $y_t(s^*_i)$, $1 \leq i \leq n$, regarded as functions of $t$. Since $y \mapsto \int y(t)g_i(t)dt$ is of norm at most 1, Lemma 2 implies that there is a set $A$ of measure $\geq \frac{3}{4}$ such that for any $\bar{s}$ in $A$,

$$|\psi_{\bar{s}}(y)| \leq C\sqrt{n \log n} \|y\|_1$$

for all $y$ in $Y$.

Take any norm-preserving extension of $\psi_s$ to all of $L_1(.,dt)$; then there is a function $h_{\bar{s}}(t)$ in $L_\infty(.,dt)$ with $\|h_{\bar{s}}\|_\infty \leq C\sqrt{n \log n}$ and such that

$$\psi_{\bar{s}}(y) = \int y(t)h_{\bar{s}}(t)dt$$

for all $y$ in $Y$. Restating this,

$$(6) \quad \int (R_{\bar{s}}(t) - h_{\bar{s}}(t))y(t)dt = 0$$

for all $y$ in $Y$. Now put

$$\alpha_{\bar{s}}(s) = \frac{1}{m} \int (R_{\bar{s}}(t) - h_{\bar{s}}(t))y_t(s)dt,$$

where $m$ is the measure of an atom of $A$. Then $\alpha_{\bar{s}} \in \ker P$ by property (5), $\alpha_{\bar{s}}(s) = R_{\bar{s}}(s) - h_{\bar{s}}(s)$, for $s$ not in $\bigcup_{i=1}^n I_{s^*_i}$, by (1) and (2), and $\alpha_{\bar{s}}(s_i) = 0$, $1 \leq i \leq n$, by (6).

The set $A$ in the proof of Lemma 2 has measure $\geq \frac{3}{4}$. Applying Lemma 1 again, we can require that

$$|R_{\bar{s}}(s^*_i)| \leq C\sqrt{n \log n}$$

for each $i$, $1 \leq i \leq n$, for all $\bar{s}$ in a set $B$ with $m(B) \geq \frac{3}{4}$. Thus $A \cap B$ has positive measure, so choose $\bar{s}$ in $A \cap B$. 

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Now, to finish the proof, set \( w(s) = R_{\pi}(s) - \alpha_{\pi}(s) \). If \( s \notin \bigcup_{i=1}^{n} I_{s_{i}} \), \( w(s) = R_{\pi}(s) - \alpha_{\pi}(s) = h_{\pi}(s) \), so \( |w(s)| \leq C \sqrt{n \log n} \). Also, \( |w(s_{i})| = |R_{\pi}(s_{i})| \leq C \sqrt{n \log n} \); thus \( \|w\|_{\infty} \leq C \sqrt{n \log n} \). Finally, \( Pw = \sum_{i=1}^{n} r_{i}(x)e_{i} \) since \( \alpha_{\pi} \in \ker P \). This completes the proof. 

\[ \text{Corollary 1.} \]

Let \( X \) be \( n \)-dimensional, and let \( X \xrightarrow{T} \ell_{K}^{P} X \) be a factorization of \( \text{Id}_{X} \) through \( \ell_{K}^{P} \). Then \( \|P\| \|T\| \geq C s_{n}(n \log n)^{\frac{1}{2}} \), where 

\[ s_{n} = \sup_{\|T_{e_{i}}\| \leq 1} \inf_{1 \leq i \leq n} \left\| \sum_{i=1}^{n} \pm e_{i} \right\|. \]

In particular, if \( \|T\| \leq 1 \), \( \|P\| \geq C b_{n}(n \log n)^{-\frac{1}{2}} \), where \( b_{n} = \sup_{\|e_{i}\| \leq 1} \inf_{1 \leq i \leq n} \left\| \sum_{i=1}^{n} \pm e_{i} \right\|. \)

(See [2].)

\[ \text{Proof.} \]

For \( K \geq n \), this is an immediate consequence of the Theorem. Assume now that \( K < n \). Let \( j = n - K \), and define \( \tilde{T} : X \rightarrow \ell_{\infty}^{n} = \ell_{\infty}^{K} \otimes \ell_{\infty}^{j} \) by \( \tilde{T}(x) = (T(x), 0) \). Define \( \tilde{P} : \ell_{\infty}^{n} \rightarrow X \) by \( \tilde{P}(w, z) = P(w) \). Note that \( \tilde{P} \tilde{T} = \text{Id}_{X} \) and that \( \|\tilde{P}\| = \|P\|, \|\tilde{T}\| = \|T\| \). The result now follows from the Theorem.

\[ \text{Corollary 2.} \]

Suppose \( \ell_{p}^{n} \xrightarrow{T} \ell_{K}^{P} \ell_{p}^{n} \) is any factorization of the identity on \( \ell_{p}^{n} \) through \( \ell_{K}^{P} \), \( 0 < p \leq 1 \), with \( \|T\| = 1 \). Then \( \|P\| \geq C n^{\frac{1}{p} - \frac{1}{2}}(\log n)^{-\frac{1}{2}} \).}

\[ \text{Proof.} \]

Take \( X = \ell_{p}^{n}, e_{i} \) the usual ith basis vector in \( \ell_{p}^{n}, 1 \leq i \leq n \), for \( 0 < p \leq 1 \). Now apply Corollary 1.

\[ \text{Remark.} \]

For \( 0 < p < 1 \), define \( T : \ell_{p}^{n} \rightarrow L_{\infty}(A) \) by defining \( T(e_{i}) = r_{i}, 1 \leq i \leq n \) and extending linearly. Then \( \|T\| = 1 \). Define \( P : L_{\infty}(A) \rightarrow \ell_{p}^{n} \) by \( T(x) = \sum_{i=1}^{n} (\int x r_{i})e_{i} \). It is easily checked that \( \|T : L_{\infty}(A) \rightarrow \ell_{p}^{n}\| \leq \sqrt{n} \); since \( \|Id : \ell_{1}^{n} \rightarrow \ell_{p}^{n}\| = n^{\frac{1}{p} - \frac{1}{2}} \), it follows that \( \|P\| \leq n^{\frac{1}{p} - \frac{1}{2}} \). Obviously \( PT = Id \), so up to a logarithmic factor, the order of \( \Delta(\ell_{p}^{n}) \) is correct.
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