Constructions of complementary sequence sets and complete complementary codes by 2-level autocorrelation sequences and permutation polynomials

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Abstract

In this paper, a recent method to construct complementary sequence sets and complete complementary codes by Hadamard matrices is deeply studied. By taking the algebraic structure of Hadamard matrices into consideration, our main result determine the so-called $\delta$-linear terms and $\delta$-quadratic terms. As a first consequence, a powerful theory linking Golay complementary sets of \( p \)-ary (\( p \) prime) sequences and the generalized Reed-Muller codes by Kasami et al. is developed. These codes enjoy good error-correcting capability, tightly controlled PMEPR, and significantly extend the range of coding options for applications of OFDM using \( p^n \) subcarriers. As another consequence, we make a previously unrecognized connection between the sequences in CSSs and CCCs and the sequence with 2-level autocorrelation, trace function and permutation polynomial (PP) over the finite fields, which provides an answer to the open problem proposed by Paterson and Tarokh in 2000.

Index Terms Complementary sequence set, Complete complementary codes, Permutation polynomial, Generalized Reed-Muller codes, 2-level auto-correlation.

1 Introduction

The concept of Golay sequence pair (GSP) was first introduced by Golay \cite{14, 15}, and it was extended later to complementary sequence set (CSS) for binary case \cite{52} and polyphase case \cite{49}, where the aperiodic autocorrelations of all the sequences in a CSS are summed to zero except at zero shift. The concept of complete mutually orthogonal complementary set (CMOCS) or complete complementary code (CCC) was proposed in \cite{50}, which can be regarded as a collection of CSSs with the additional aperiodic cross-correlation property. CSSs and CCCs have been applied in diverse areas of digital communications, including channel measurement, synchronisation, spread spectrum communications and power control

1
for multi-carrier wireless transmission. In addition, CCCs are source to design zero correlation zone (ZCZ) sequences, which have been shown in [24, 41, 43].

Among these applications, orthogonal frequency division multiplexing (OFDM) has recently seen rising popularity in international standards including coming 5G cellular systems, and one attractive method for power control in OFDM [26] is coding across the subcarriers and selecting codewords with lower peak-to-mean envelope power ratio (PMEPR). These codewords can be a linear code or codewords drawn from cosets of a linear code [51, 39]. It has been shown in [40] that the use of sequences in CSSs as codewords results in OFDM signals with low PMEPR.

An effective way of combining the coding approach and the use of Golay sequences was established by Davis and Jedwab [8] by showed that the Golay sequences can be obtained from specific second-order cosets of the first-order generalized Reed-Muller (GRM) code [30]. Form then on, a large volume of research works on the constructions of CSSs [37, 38, 44, 45, 5, 48, 55] and CCCs [6, 27, 42] have been done along this line that the sequences are all described by the generalized Boolean functions (GBFs) and fall into cosets of the GRM code.

In addition to the above direct constructions of CSSs and CCCs based on GBFs, there exists a recursive approach to construct CSSs and CCCs based on para-unitary (PU) matrices, such as [52, 51, 50, 54, 2, 3, 7]. Recently, we proposed a framework [53] on the construction of PU matrices by Hadamard matrices, which established a connection between the aforementioned GBF-based constructions and PU-based constructions, and a great number of the new CSSs and CCCs are constructed. The key theoretical result of this method is to extract functions from the so called generalized seed matrices, from which the $q$-ary sequences of length $N^m$ in CSSs and CCCs of size $N$ can be represented by the $\delta$-linear terms and $\delta$-quadratic terms, where the word linear and quadratic are with respect to the Kronecker-delta functions from $\mathbb{Z}_N$ to $\mathbb{Z}_q$. We have shown some realized constructions by such a method in [53] by Bruce-force computation, which is very heavy even for $q = N = 4$.

In this paper, by taking the algebraic structure of the Butson-type Hadamard (BH) matrix into consideration, the structure of the $\delta$-linear terms and $\delta$-quadratic terms is explicit given, which avoid the heavy computation from the basis of Kronecker-delta functions. In particular, we make an previously unrecognized connection between the sequences in CSSs and CCCs and the sequence with 2-level autocorrelation, trace function and permutation polynomial (PP) over the finite fields.

By taking the Discrete Fourier transform (DFT) matrix of order $N$ as a BH matrix, we obtained that the $\delta$-quadratic terms can be represented by the product of any two permutation functions over $\mathbb{Z}_N$. For $N = 2^n$ being a power of 2, such permutation functions can be realized by the bijective GBFs. For $N = p$ prime, such permutation functions can be realized by PPs over finite field $\mathbb{F}_p$. A powerful theory linking $p$-ary sequence of length $p^m$ in CSSs of size $p$ and the GRM codes proposed by Kasami, Lin and Peterson [21] is developed. The reader should note that there are two types of nonbinary
generalizations of the classical RM codes. The first type directly generalize alphabets of classical RM codes from binary to $q$-ary, which have been made a connection with sequences in CSSs of size $2^n$, such as [8,37]. Here the GRM code is the another type of generalization proposed by Kasami et al. in 1968. For $p = 2$, the constructed sequences agree with the binary Golay sequences [5]. For $p$ odd prime, the $\delta$-linear terms, which contains $p^m(p−1)+1$ elements, can be treated as a linear subcode of the $(p−1)$th order second type GRM codes, and we explicitly identify $\frac{1}{2}m!(((p−1))^{m−1}((p−2))^{m−1}$ cosets within this specified linear subcode from the $\delta$-quadratic terms. Moreover, their error correction capability is given by showing that Hamming distance of any two distinct sequence is at least $3p^{m−2}$.

By taking the Hadamard matrix of order $p^n$ over the finite field $F_{p^n}$ as a BH matrix, we obtained that the $\delta$-quadratic terms can be represented by the trace function of product of any two PPs over $F_{p^n}$. Note that there have been numerous books and papers on the study of PPs over finite fields. A wealth of results covering different periods in the development of this active area can be found in [22,25,33,19]. Nevertheless, this is the first time to use PPs over the finite field to construct CSSs and CCCs.

Every $p$-ary ($p$ prime) sequence with 2-level auto correlation of period $p^n−1$ determine a BH matrix of order $p^n$, where each entry can be represented by the trace representation of the sequence. Then the $\delta$-quadratic terms can be represented by the trace representation of the 2-level auto correlation sequence with the product of any two PPs over $F_{p^n}$ as variable. Note that there is a large volume research on the construction of the sequences with 2-level autocorrelation, which correspond to the cyclic Hadamard difference sets, such as $m$ sequences, GMW sequences, WG sequences and so on. A collection of the results in this area can be found in [16]. We showed that $m$ sequences yield the same result as Hadamard matrix over the finite field $F_{p^n}$, and other 2-level autocorrelation sequences produce new CSSs and CCCs. Nevertheless, this is the first time to use the trace representation of any sequence with 2-level auto correlation to construct CSSs and CCCs.

The coset representatives of sequences proposed in this paper are presented by the trace function, so the results in this paper provide an answer to the open problem proposed by K.G. Paterson and V. Tarokh [39] in 2000: *It may be possible to obtain significant reductions in PMEPR by using such offsets, and we leave the analytical determination of good offsets for trace codes as a difficult open problem.*

The rest of our paper is organized as follows. In the next section, we introduce the definition and notations, revisit the framework on the construction of PU matrices by Hadamard matrices in [53]. In Section 3, we study on the $\delta$-linear terms and $\delta$-quadratic terms. In Section 4, we take DFT matrices as BH matrices in $\delta$-quadratic terms, and link $p$-ary sequence of length $p^n$ in CSSs of size $p$ with the GRM codes proposed in [21]. In Section 5, we take Hadamard matrices over the finite field as BH matrices in $\delta$-quadratic terms, and construct CSSs and CCCs by PPs and trace function over finite fields. Section 6 make a connection of the constructions of CSSs and CCCs with the sequences with
2-level autocorrelation. We conclude the paper in Section 7.

2 Preliminaries

In this section, we introduce some basic definitions and notations of CSSs, CCCs, and two different type generalizations of the classical Reed-Muller codes. A framework on the construction of CSSs and CCCs by Hadamard matrices [53] are also revisited.

2.1 Sequences, CSS and CCC

Let \( \mathbb{Z}_p \) be a residue class ring modulo \( p \). If \( p \) is a prime, we use the finite field \( \mathbb{F}_p \) instead of the ring \( \mathbb{Z}_p \). A \( q \)-ary sequence \( f \) of length \( L \) is defined as

\[
    f = (f(0), f(1), \cdots, f(L-1))
\]

for each entry \( f(t) \in \mathbb{Z}_q \) (\( t \in \mathbb{Z}_L \)).

**Definition 1** For two \( q \)-ary sequences \( f_1 \) and \( f_2 \) of length \( L \), the aperiodic cross-correlation of \( f_1 \) and \( f_2 \) at shift \( \tau \) \((-L < \tau < L)\) is defined by

\[
    C_{f_1,f_2}(\tau) = \begin{cases} 
    \sum_{t=0}^{L-1-\tau} \omega_{q}^{f_1(t+\tau)-f_2(t)}, & 0 \leq \tau < L, \\
    \sum_{t=0}^{L-1+\tau} \omega_{q}^{f_1(t)-f_2(t-\tau)}, & -L < \tau < 0,
    \end{cases}
\]

where \( \omega \) is a \( q \)th root of unity. If \( f_1 = f_2 = f \), the aperiodic autocorrelation of sequence \( f \) at shift \( \tau \) is denoted by

\[
    C_f(\tau) = C_{f,f}(\tau).
\]

**Definition 2** A set of sequences \( S = \{f_0, f_1, \cdots, f_{N-1}\} \) is called a complementary sequence set (CSS) of size \( N \) if

\[
    \sum_{u=0}^{N-1} C_{f_u}(\tau) = 0 \text{ for } \tau \neq 0.
\]

If the set size \( N = 2 \), such a set is called a Golay sequence pair (GSP). Each sequence in GSP is called a Golay sequence.

Two CSSs \( S_1 = \{f_{1,0}, f_{1,1}, \cdots, f_{1,N-1}\} \) and \( S_2 = \{f_{2,0}, f_{2,1}, \cdots, f_{2,N-1}\} \) are said to be mutually orthogonal if

\[
    \sum_{u=0}^{N-1} C_{f_{1,u},f_{2,u}}(\tau) = 0 \text{ for } \forall \tau.
\]
It is known that the number of CSSs which are pairwise mutually orthogonal is at most equal to \( N \), the number of sequences in a CSS.

**Definition 3** Let \( S_u = \{ f_{u,0}, f_{u,1}, \ldots, f_{u,N-1} \} \) be CSSs of size \( N \) for \( 0 \leq u < N \), which are pairwise mutually orthogonal. Such a collection of \( S_u \) is called complete mutually orthogonal complementary sets (CMOCS) or complete complementary codes (CCC).

The concept of CCC is better to view through a matrix whose \( u \)th row is given by \( S_u \), i.e.,

\[
\begin{bmatrix}
  f_{0,0} & f_{0,1} & \cdots & f_{0,N-1} \\
  f_{1,0} & f_{1,1} & \cdots & f_{1,N-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  f_{N-1,0} & f_{N-1,1} & \cdots & f_{N-1,N-1}
\end{bmatrix}
\]

(3)

### 2.2 CCA, CAS and PU Matrix

The concept of GSP was generalized to Golay array pair (GAP) in [29]. Moreover, a powerful three-stage process was presented in [13] by showing that all known standard [8] and non-standard [11, 12, 23] Golay sequences of length \( 2^m \) can be derived from the seed GAPs. Inspired by the excellent idea for array pair [13, 36], the concepts of CSS and CCC were generalized from sequence to array in [53]. Here we introduce them from the viewpoint of the generating function.

An \( m \)-dimensional \( q \)-ary array of size \( p \times p \times \cdots \times p \) can be represented by a corresponding function from \( \mathbb{Z}_p^m \) to \( \mathbb{Z}_q \):

\[
f(\mathbf{x}) = f(x_0, x_1, \cdots, x_{m-1}): \mathbb{Z}_p^m \rightarrow \mathbb{Z}_q,
\]

where \( \mathbf{x} = (x_0, x_1, \cdots, x_{m-1}) \) and \( x_k \in \mathbb{Z}_p \). Let \( \mathbf{z} = (z_0, z_1, \cdots, z_{m-1}) \) and \( \mathbf{z}^{-1} = (z_0^{-1}, z_1^{-1}, \cdots, z_{m-1}^{-1}) \).

The generating function of an array \( f(\mathbf{x}) \) is defined by

\[
F(\mathbf{z}) = \sum_{x_0=0}^{p-1} \sum_{x_1=0}^{p-1} \cdots \sum_{x_{m-1}=0}^{p-1} \omega^{f(\mathbf{x})x_0x_1\cdots x_{m-1}} z_0^{x_0}z_1^{x_1}\cdots z_{m-1}^{x_{m-1}}.
\]

(4)

**Definition 4** A set of arrays \( \{ f_0(\mathbf{x}), f_1(\mathbf{x}), \cdots, f_{N-1}(\mathbf{x}) \} \) from \( \mathbb{Z}_p^m \) to \( \mathbb{Z}_q \) is called a complementary array set (CAS) of size \( N \) if their generating functions \( \{ F_0(\mathbf{z}), F_1(\mathbf{z}), \cdots, F_{N-1}(\mathbf{z}) \} \) satisfy

\[
\sum_{u=0}^{N-1} F_u(\mathbf{z}) \cdot F_u(\mathbf{z}^{-1}) = N \cdot p^m.
\]

(5)
Two CASs $S_1 = \{f_{1,0}(x), f_{1,1}(x), \cdots, f_{1,N-1}(x)\}$ and $S_2 = \{f_{2,0}(x), f_{2,1}(x), \cdots, f_{2,N-1}(x)\}$ are said to be mutually orthogonal if their generating functions $\{F_{u,0}(z), \cdots, F_{u,N-1}(z)\}$ $(u = 1, 2)$ satisfy

$$
\sum_{v=0}^{N-1} F_{1,v}(z) F_{2,v}^*(z^{-1}) = 0.
$$

**Definition 5** Let $S_u = \{f_{u,0}(x), f_{u,1}(x), \cdots, f_{u,N-1}(x)\}$ $(0 \leq u < N)$ be CASs of size $N$, which are pairwise mutually orthogonal. We call such a collection of $S_u$ $(0 \leq u < N)$ a complete mutually orthogonal array set or a complete complementary arrays (CCA).

Let $\tilde{M}(x)$ be a matrix where the $u$th row is given by $S_u$, i.e.,

$$
\tilde{M}(x) = \begin{bmatrix}
  f_{0,0}(x) & f_{0,1}(x) & \cdots & f_{0,N-1}(x) \\
  f_{1,0}(x) & f_{1,1}(x) & \cdots & f_{1,N-1}(x) \\
  \vdots & \vdots & \ddots & \vdots \\
  f_{N-1,0}(x) & f_{N-1,1}(x) & \cdots & f_{N-1,N-1}(x)
\end{bmatrix}.
$$

(7)

The generating functions of the entries in matrix $\tilde{M}(x)$ can be presented by a matrix $M(z)$ with each entry given by $M_{u,v}(z) = F_{u,v}(z)$, the generating function of $f_{u,v}(x)$, i.e.,

$$
M(z) = \begin{bmatrix}
  F_{0,0}(z) & F_{0,1}(z) & \cdots & F_{0,N-1}(z) \\
  F_{1,0}(z) & F_{1,1}(z) & \cdots & F_{1,N-1}(z) \\
  \vdots & \vdots & \ddots & \vdots \\
  F_{N-1,0}(z) & F_{N-1,1}(z) & \cdots & F_{N-1,N-1}(z)
\end{bmatrix}.
$$

(8)

$M(z)$ is called the generating matrix of $\tilde{M}(x)$. If $\tilde{M}(x)$ is a CCA, it is necessary that its generating matrix $M(z)$ is a multivariate para-unitary (PU) matrix, i.e., $M(z) \cdot M^\dagger(z^{-1}) = c \cdot I_N$, where $(\cdot)^\dagger$ denotes the Hermitian transpose, $I_N$ is the identity matrix of order $N$, and $c$ is a real constant.

For an array (function) $f(x) = f(x_0, x_1, \cdots, x_{m-1}) : \mathbb{Z}_p^m \rightarrow \mathbb{Z}_q$, by ordering of the elements $x$ in $\mathbb{Z}_p^m$, $f(x)$ can be associated with a sequence $f$ of length $L = p^m$, where

$$
f(t = \sum_{k=0}^{m-1} x_k \cdot p^k) = f(x).
$$

In this paper, we say that the sequence $f(t)$ is evaluated by the function $f(x)$. CSSs and CCCs can be constructed from a single CCA.
**Fact 1** ([53]) For a given CCA $	ilde{M}(x)$ in the matrix form (7) and an arbitrary permutation $\pi$ of symbol $\{0, 1, \cdots, m-1\}$, let $\tilde{M}(\pi \cdot x)$ be a matrix with entry $\tilde{M}_{u,v}(\pi \cdot x) = f_{u,v}(\pi \cdot x)$, where $\pi \cdot x = (x_{\pi(0)}, x_{\pi(1)}, \cdots, x_{\pi(m-1)})$. Then we have

1. Sequences evaluated by functions in $\tilde{M}(\pi \cdot x)$ form a CCC.
2. Sequences evaluated by functions in each row (or column) of $\tilde{M}(\pi \cdot x)$ form a CSS.

### 2.3 Algebraic Normal Form and Generalized Reed Muller Codes

We are particularly interested in the case that the function $f(x)$ from $\mathbb{Z}^m_p$ to $\mathbb{Z}^q$ can be realized by *algebraic normal form* (ANF).

For $p = 2$, the function $f(x)$ can be realized by *generalized Boolean function* (GBF) from $\mathbb{F}^m_2$ to $\mathbb{Z}^q$. Every such function can be written in ANF as a sum of monomials of the form $x_0^{i_0}x_1^{i_1}\cdots x_m^{i_m}$ over $\mathbb{Z}^q$, where $i_k = 0$ or 1 for $0 \leq k \leq m - 1$.

For $q = p$ prime, the function $f(x)$ can be realized by function from $\mathbb{F}^m_p$ to $\mathbb{F}^q_p$. Every such function can be written in ANF as a sum of monomials of the form $x_0^{i_0}x_1^{i_1}\cdots x_m^{i_m}$ over $\mathbb{F}^q_p$, where $0 \leq i_k \leq p - 1$ for $0 \leq k \leq m - 1$.

The $r$th-order classical Reed-Muller code $\text{RM}(r,m)$ ([30]) is defined to be the binary code whose codewords are (the sequences evaluated by) the Boolean functions of degree at most $r$ in variables $x_0, x_1, \cdots, x_{m-1}$. The code $\text{RM}(r,m)$ is linear, has minimum Hamming distance $2^{m-r}$. There are two types of nonbinary generalizations of the classical Reed-Muller codes.

The first type of generalization ([8,37]), denoted by $\text{RM}_q(r,m)$, directly generalize alphabets of classical Reed-Muller codes from binary to $q$-ary case. $\text{RM}_q(r,m)$ is defined to be a linear code over $\mathbb{Z}^q$ comprised of all the $q$-ary sequences evaluated by ANF of GBFs from $\mathbb{F}^m_2$ to $\mathbb{Z}^q$ with degree less than or equal to $r$. It is known that $\text{RM}_q(r,m)$ also has minimum Hamming distance $2^{m-r}$.

The another type of generalization, denoted by $\text{GRM}_p(r,m)$ in this paper, was studied in [21,9]. $\text{GRM}_p(r,m)$ is defined to be a linear code over $\mathbb{F}^m_p$ comprised of all the $p$-ary sequences evaluated by ANF of functions from $\mathbb{F}^m_p$ to $\mathbb{F}^m_p$ with degree less than or equal to $r$. It is known in [21] that $\text{GRM}_p(r,m)$ has minimum Hamming distance $(R + 1) \cdot p^Q$, where $R$ is the remainder and $Q$ the quotient resulting from dividing $m(p - 1) - r$ by $p - 1$.

### 2.4 Hadamard Matrices and Generalized Seed PU Matrices

A complex matrix $H$ of order $N$ is called *Butson-type Hadamard* (BH) matrix [4] if $H \cdot H^\dagger = N \cdot I_N$ and all the entries of $H$ are $q$th roots of unity. For given $N$ and $q$, the set of all BH matrices is denoted by $H(q,N)$.
Two BH matrices, $H_1, H_2 \in H(q, N)$ are called equivalent, denoted by $H_1 \simeq H_2$, if there exist diagonal unitary matrices $Q_1, Q_2$ where each diagonal entry is a $q$th root of unity and permutation matrices $P_1, P_2$ such that $H_1 = P_1 \cdot Q_1 \cdot H_2 \cdot Q_2 \cdot P_2$.

For a BH matrix $H \in H(q, N)$, define its phase matrix $\hat{H}$ by $\hat{H}_{i,j} = s$ if $H_{i,j} = \omega^s$. Suppose that $S_H(q, N)$ is a set containing all the phase matrix of the representatives of BH matrices in $H(q, N)$ with respected to the equivalence relation.

In this and the next subsections, we revisit and extend a method proposed in [53] to construct the so-called generalized seed PU matrices. Note that the order of the seed PU matrices and the generalized seed PU matrices in [53] are $N = p$ and $N = 2^n$, respectively, while the order of the generalized seed PU matrices here is straightforwardly extended to $N = p^n$ for arbitrary $p$.

The delay matrix $D(z)$ of order $p$ is defined by $D(z) = diag\{z^0, z^1, z^2, \ldots , z^{p-1}\}$. And the generalized delay matrix $D(z)$ with multi-variables $z = (z_0, z_1, \cdots , z_{n-1})$ is defined by the Kronecker product of $D(z_j)$ for $0 \leq j < n$, i.e.,

$$D(z) = D(z_{n-1}) \otimes \cdots \otimes D(z_1) \otimes D(z_0).$$

By mathematical induction, it is straightforward to show the generalized delay matrix $D(z)$ can be explicitly expressed by a diagonal matrix $D(z) = diag\{\phi_0(z), \phi_1(z), \cdots , \phi_{p^n-1}(z)\}$ where $\phi_y(z) = \prod_{j=0}^{n-1} z_j^{x_j}$ for $y = \sum_{j=0}^{n-1} x_j \cdot p^j$.

Let $H^{(k)}$ be arbitrary BH matrices chosen from $H(q, N)$ for $0 \leq k \leq m$ and $D(z_k)$ be the generalized delay matrices with $z_k = (z_{kn}, z_{kn+1}, \cdots , z_{kn+n-1})$ for $0 \leq k < m$. Let $z = (z_0, z_1, \cdots , z_{nm-1})$. It has been shown in [53] that a multivariate polynomial matrix $M(z)$, defined by

$$M(z) = H^{(0)} \cdot D(z_0) \cdot H^{(1)} \cdot D(z_1) \cdots H^{(m-1)} \cdot D(z_{m-1}) \cdot H^{(m)},$$

must be the generating matrix of a CCA, denoted by $\tilde{M}(x)$ where $x = (x_0, x_1, \cdots , x_{mn-1})$. $M(z)$ is called the generalized seed PU matrix.

### 2.5 Extracting Functions from Generalized Seed PU Matrices

Let $x_k = (x_{kn}, x_{kn+1}, \cdots , x_{kn+n-1}) \in \mathbb{Z}_p^n$ be the $p$-ary expansion of $y_k$ and $y = (y_0, y_1, \cdots , y_{m-1})$.

Each entry of $\tilde{M}(x)$ extracted from the generalized seed PU matrix $M(z)$ can be represented by a function $f_{u,v}(y)$ from $\mathbb{Z}_p^m$ to $\mathbb{Z}_q$ determined by the formula

$$\omega f_{u,v}(y) = H^{(0)}_{u,y_0} \cdot \left( \prod_{k=1}^{m-1} H^{(k)}_{y_k-1,y_k} \right) \cdot H^{(m)}_{y_{m-1},v}.$$
The function $f_{u,v}(y)$ can be alternatively presented by $f_{u,v}(x)$, which is an array from $\mathbb{Z}_p^m$ to $\mathbb{Z}_q$. The general form of the entries of $\tilde{M}(x)$, denoted by $f(x)$ (or $f(y)$), is given by introducing a basis of the functions from $\mathbb{Z}_p^n$ to $\mathbb{Z}_q$.

**Definition 6** Let $\delta_\alpha(y)$ be a function: $\mathbb{Z}_p^n \rightarrow \mathbb{Z}_q$ such that $\delta_\alpha(\beta) = \delta_{\alpha,\beta}$ where $\delta_{\alpha,\beta}$ is the Kronecker-delta function, i.e.,

\[
\delta_\alpha(\beta) = \begin{cases} 
1, & \text{if } \alpha = \beta, \\
0, & \text{if } \alpha \neq \beta.
\end{cases}
\]

The function $f(y)$ can be explicitly represented by the combination of the linear terms and the quadratic terms with respect to the functions $\delta_\alpha(y_k)$ for $\alpha \in \mathbb{Z}_p^n$ and $0 \leq k \leq m - 1$.

**Definition 7** The $\delta$-linear terms are the linear combinations of $\delta_\alpha(y_k)$ over $\mathbb{Z}_q$ for $\alpha \in \mathbb{Z}_p^n$ and $0 \leq k \leq m - 1$. The collection of the $\delta$-linear terms is denoted by

\[
\delta_L(q,p^n) = \left\{ c_{\alpha,k} \cdot \delta_\alpha(y_k) \mid c_{\alpha,k} \in \mathbb{Z}_q \right\}, \quad (11)
\]

It has been shown in [53, Lemma 7] that the $\delta$-linear terms can be represented by

\[
\delta_L(q,p^n) = \left\{ \sum_{k=0}^{m-1} \sum_{\alpha=0}^{p^n-1} c_{\alpha,k} \cdot \delta_\alpha(y_k) + c' \mid c_{\alpha,k}, c' \in \mathbb{Z}_q \right\}, \quad (12)
\]

which is a free $\mathbb{Z}_q$-submodule of dimension $m(p^n - 1) + 1$ with basis $\{\delta_\alpha(y_k), 1|0 \leq k \leq m - 1, 1 \leq \alpha \leq q^n - 1\}$.

Let $\chi$ be a permutation of symbols $\{0,1,\cdots,p^n-1\}$ and $\delta(y) = (\delta_0(y), \delta_1(y), \cdots, \delta_{p^n-1}(y))$. Denote the permutation of the vector function $\delta(y)$ by

\[
\delta_\chi(y) = (\delta_\chi(0)(y), \delta_\chi(1)(y), \cdots, \delta_\chi(p^n-1)(y)). \quad (13)
\]

The $\delta$-quadratic terms can be obtained from the phase matrices $\tilde{H} \in S_{\tilde{H}}(q,p^n)$.

**Definition 8** The $\delta$-quadratic terms are quadratic forms:

\[
g_{\chi_L}(y_0)\tilde{H}g_{\chi_R}(y_1)^T, \quad (14)
\]

where $\chi_L, \chi_R$ are permutations of symbols $\{0,1,\cdots,p^n-1\}$ and $\tilde{H} \in S_{\tilde{H}}(q,p^n)$. The collection of the $\delta$-quadratic terms is denoted by $\delta_Q(q,p^n)$. 

9
Fact 2 All the functions extracted from the generalized seed PU matrices can be represented in a general form

\[ f(x) = \sum_{k=1}^{m-1} h_k(y_{k-1}, y_k) + l(y), \quad (15) \]

where \( h_k(\cdot, \cdot) \in \delta_Q(q, p^n)(1 \leq k \leq m - 1) \) and \( l(y) \in \delta_L(q, p^n) \).

Fact 3 Let \( f(x) \) be a function (or array of size \( p \times p \times \cdots \times p \)) with the form (15) and \( h_0(\cdot, \cdot), h_m(\cdot, \cdot) \in \delta_Q(q, p^n) \). Then the arrays

\[ f_u(x) = f(x) + h_0(u, y_0), \quad u \in \mathbb{Z}_{p^n} \]

form a CAS of size \( p^n \), and the arrays

\[ f_{u,v}(x) = f(x) + h_0(u, y_0) + h_m(y_m, v), \quad u, v \in \mathbb{Z}_{p^n} \]

form a CCA.

3 δ-Linear Terms and δ-Quadratic Terms

CSSs and CCCs are constructed by Facts 1-3, the kernel of which are the δ-linear terms and δ-quadratic terms. However, it have been shown in [53] that these terms are calculated by heavy computation even for \( p = n = 2 \). In this section, we will continue our study on δ-linear terms and δ-quadratic terms to avoid the computation of the Kronecker-delta functions in Definition 6.

3.1 δ-Linear Terms

From the definition of the function \( \delta_\alpha(\cdot) \), any function \( g(y): \mathbb{Z}_{p^n} \rightarrow \mathbb{Z}_q \) can be represented by

\[ g(y) = \sum_{\alpha \in \mathbb{Z}_{p^n}} g(\alpha) \delta_\alpha(y). \]

Then it is clear that the function \( \sum_{\alpha \in \mathbb{Z}_{p^n}} c_{\alpha,k} \cdot \delta_\alpha(y_k) \) in (11) can be expressed by a function \( l_k(y_k): \mathbb{Z}_{p^n} \rightarrow \mathbb{Z}_q \) such that \( l_k(\alpha) = c_{\alpha,k} \).

Theorem 1 The collection of the δ-linear terms can be represented in an alternative form:

\[ \delta_L(q, p^n) = \left\{ \sum_{k=0}^{m-1} l_k(y_k) \mid \forall l_k(y_k): \mathbb{Z}_{p^n} \rightarrow \mathbb{Z}_q \right\}. \]
We are particularly interested in the case that the functions in $\delta_L(q, p^n)$ can be realized by ANF introduced in Subsection 2.3. Recall that $x_k = (x_{kn}, x_{kn+1}, \ldots, x_{kn+n-1}) \in \mathbb{Z}_p^n$ be the $p$-ary expansion of $y_k \in \mathbb{Z}_{p^n}$.

**Corollary 1** For the case $x_k$ introduced in Subsection 2.3. Recall that where

$$
\begin{align*}
\delta_L(q, p^n = 2^n) &= \left\{ \sum_{k=0}^{m-1} \sum_{i=1}^{2^n-1} \left( c_{k,i} \prod_{j=0}^{n-1} x_{kn+j}^{i_j} \right) + c' \left| c_{k,i}, c' \in \mathbb{Z}_q \right. \right\},
\end{align*}
$$

where $(i_0, i_1, \ldots, i_{n-1})$ is the binary expansion of integer $i$.

**Corollary 2** For the case $q = p$ prime, the variables $y_k$ over $\mathbb{Z}_{p^n}$ can be replaced by $x_k \in \mathbb{F}_p^n$, and any function $l_k(y_k)$ from $\mathbb{Z}_{p^n}$ to $\mathbb{Z}$ can be represented by a GFB from $\mathbb{F}_p^n$ to $\mathbb{Z}_q$. Then we have

$$
\begin{align*}
\delta_L(q = p, p^n) &= \left\{ \sum_{k=0}^{m-1} \sum_{i=1}^{p^n-1} \left( c_{k,i} \prod_{j=0}^{n-1} x_{kn+j}^{i_j} \right) + c' \left| c_{k,i}, c' \in \mathbb{F}_p \right. \right\},
\end{align*}
$$

where $(i_0, i_1, \ldots, i_{n-1})$ is the $p$-ary expansion of integer $i$.

**Remark 1** Both Corollaries 1 and 2 change the basis from the Kronecker-delta functions shown in Definition 6 to classical basis of monomials shown in Subsection 2.3. Moreover, from the basis of monomials, it is obvious that $\delta_L(q, p^n = 2^n)$ and $\delta_L(q = p, p^n)$ are $\mathbb{Z}_q$-submodule of dimension $m(2^n - 1) + 1$ and $m(p^n - 1) + 1$, respectively, which agree with the results in [23] Lemma 7.

### 3.2 $\delta$-Quadratic Terms

Recall that the $\delta$-quadratic terms are quadratic forms: $g_{\chi_L}(y_0) \overline{H} g_{\chi_R}(y_1)^\top$. We will show in the rest of paper that, if we take the algebraic structure of BH matrices into consideration, the computation of the ANF of $\delta$-quadratic terms can be significantly simplified.

**Theorem 2** Let $H$ be a BH matrix with entry $H_{u,v} = \omega^{h(u,v)}$ where $h(u,v)$ is a function from $\mathbb{Z}_{p^n}^2$ to $\mathbb{Z}_q$ for $u, v \in \mathbb{Z}_{p^n}$, and $g(\cdot), g'(\cdot)$ be arbitrary permutation functions over $\mathbb{Z}_{p^n}$. We have

$$
\begin{align*}
h(g(y_0), g'(y_1)) \in \delta_Q(q, p^n).
\end{align*}
$$

**Proof** We take $H$ as a representative of its equivalence class of BH matrices. Then the entry of its phase matrix $\overline{H}$ is given by $\overline{H}_{u,v} = h(u, v)$ for $u, v \in \mathbb{Z}_{p^n}$. We have

$$
\begin{align*}
g_{\chi_L}(y_0) \overline{H} g_{\chi_R}(y_1)^\top = \sum_{u \in \mathbb{Z}_{p^n}} \sum_{v \in \mathbb{Z}_{p^n}} h(u, v) \delta_{\chi_L(u)}(y_0) \delta_{\chi_R(v)}(y_1),
\end{align*}
$$
where $\chi_L, \chi_R$ are permutations of symbols $\{0, 1, \cdots, p^n - 1\}$. We define the Kronecker-delta function $\delta_{\alpha, \beta}(y_0, y_1)$ from $\mathbb{Z}_p^2$ to $\mathbb{Z}_q$ such that $\delta_{\alpha, \beta}(y_0, y_1) = \delta_{\alpha}(y_0) \cdot \delta_{\beta}(y_1)$, i.e.,

$$\delta_{\alpha, \beta}(y_0, y_1) = \begin{cases} 1, & \text{if } (\alpha, \beta) = (y_0, y_1), \\ 0, & \text{if } (\alpha, \beta) \neq (y_0, y_1). \end{cases}$$

Then any function $h(y_0, y_1) : \mathbb{Z}_2^{2n} \to \mathbb{Z}_q$ can be represented by

$$h(y_0, y_1) = \sum_{\alpha \in \mathbb{Z}_p^n} \sum_{\beta \in \mathbb{Z}_p^n} h(\alpha, \beta) \cdot \delta_{\alpha, \beta}(y_0, y_1).$$

Thus, we have

$$g(\chi_L(y_0)) \tilde{H} g(\chi_R(y_1))^T = \sum_{u \in \mathbb{Z}_p^n} \sum_{v \in \mathbb{Z}_p^n} h(u, v) \cdot \delta_{\chi_L(u), \chi_R(v)}(y_0, y_1)$$

$$= \sum_{u \in \mathbb{Z}_p^n} \sum_{v \in \mathbb{Z}_p^n} h(\chi_L^{-1}(u), \chi_R^{-1}(v)) \cdot \delta_{u, v}(y_0, y_1)$$

$$= h(\chi_L^{-1}(y_0), \chi_R^{-1}(y_1)).$$

Let $\chi_L$ and $\chi_R$ be the inverse functions of the permutation functions $g(\cdot)$ and $g'(\cdot)$ over $\mathbb{Z}_p^n$, respectively. We complete the proof.

4 Constructions from DFT Matrices over $\mathbb{Z}_N$

In this section, we assume $N = p^n = q$. Discrete Fourier transform (DFT) matrix of order $N$ is a BH matrix with entry $H_{u,v} = w^{uv}$ for $u, v \in \mathbb{Z}_N$. Then the entry of its phase matrix is given by $\tilde{H}_{u,v} = u \cdot v$. We have

$$g(y_0)g'(y_1) \in \delta_Q(N, N)$$

for arbitrary permutation functions $g(\cdot), g'(\cdot)$ over $\mathbb{Z}_{p^n}$. These terms are called the $\delta$-quadratic terms determined by DFT matrices.

**Remark 2** From the arguments on the equivalence of $\delta$-quadratic terms in [53], there are totally $\varphi(N) \cdot ((N - 1)!)^2$ $\delta$-quadratic terms determined by DFT matrices, where $\varphi(N)$ is the Euler function of integer $N$. These functions will be explicitly given in this section for $N$ prime or a power of 2.

Recall that the set of the $\delta$-linear terms in Theorem 1 is given by

$$\delta_L(q = N, p^n = N) = \left\{ \sum_{k=0}^{m-1} l_k(y_k) \mid \forall l_k(y_k) : \mathbb{Z}_N \to \mathbb{Z}_N \right\}. \quad (17)$$
We obtain the following results immediately by Facts 2 and 3.

**Theorem 3** Let \( g_k(\cdot), g_k'(\cdot) \) are arbitrary permutation functions over \( \mathbb{Z}_N \) for \( 0 \leq k \leq m \), \( l(\mathbf{y}) \) arbitrary functions in the set \( \delta_L(N, N) \), and \( f(\mathbf{y}) \) an \( N \)-ary function with the form

\[
    f(\mathbf{y}) = \sum_{k=1}^{m-1} g_k(y_{k-1}) \cdot g_k'(y_k) + l(\mathbf{y}).
\]

(1) The following functions from \( \mathbb{Z}_N \) to \( \mathbb{Z}_N \) form a CAS of size \( N \):

\[
    f_u(\mathbf{y}) = f(\mathbf{y}) + u \cdot g_0(y_0) \text{ for } u \in \mathbb{Z}_N.
\]

(2) The following functions from \( \mathbb{Z}_N \) to \( \mathbb{Z}_N \) form a CCA of size \( N \):

\[
    f_u(\mathbf{y}) = f(\mathbf{y}) + u \cdot g_0(y_0) + v \cdot g_0'(y_{m-1}) \text{ for } u \in \mathbb{Z}_N.
\]

In the rest of the section, we study the case that the functions \( f(\mathbf{y}) \) can be realized by ANF shown in Subsection 2.3.

### 4.1 Constructions from PPs over \( \mathbb{F}_p \)

In this subsection, we will set \( N = q = p \) prime and \( n = 1 \). Then we have \( g_k(\cdot), g_k'(\cdot) \) are arbitrary PPs over \( \mathbb{F}_p \), \( y_k = x_k \) and \( \mathbf{x} = (x_0, x_1, \ldots, x_{m-1}) \). From Corollary 2, the set of the \( \delta \)-linear terms is given by

\[
    \delta_L(q = p, p^n = p) = \left\{ \sum_{k=0}^{m-1} \sum_{i=1}^{p-1} c_{k,i} x_k^i + c' \mid c_{k,i}, c' \in \mathbb{F}_p \right\}.
\]

**Theorem 4** Let \( g_k(\cdot), g_k'(\cdot) \) are arbitrary PPs over \( \mathbb{F}_p \) for \( 0 \leq k \leq m \), \( \pi \) arbitrary permutation of symbols \( \{0, 1, \ldots, m-1\} \), and \( \forall l(\mathbf{x}) \in \delta_L(p, p) \). For any \( p \)-ary function \( f(\mathbf{x}) \) with the form

\[
    f(\mathbf{x}) = \sum_{k=1}^{m-1} g_k(x_{k-1}) \cdot g_k'(x_k) + l(\mathbf{x}),
\]

we have the following results.

(1) The \( p \)-ary sequences evaluated by functions from \( \mathbb{F}_p^n \) to \( \mathbb{F}_p \):

\[
    f_u(\mathbf{x}) = f(\pi \cdot \mathbf{x}) + u \cdot g_0(x_{\pi(0)}) \text{ for } u \in \mathbb{F}_p,
\]

form a CSS of size \( p \).
(2) The $p$-ary sequences evaluated by functions from $\mathbb{F}_p^{m}$ to $\mathbb{F}_p$: 
\begin{equation}
    f_{u,v}(x) = f(\pi \cdot x) + u \cdot g_0(x_{\pi(0)}) + v \cdot g_0(x_{\pi(m-1)}) \quad \text{for } u, v \in \mathbb{F}_p,
\end{equation}
form a CCC.

The above theorem is valid, since $g_0(x_0) \cdot g_1(x_1) \in \delta_Q(p, p)$ is a $\delta$-quadratic term if $g_0(\cdot)$ and $g_1(\cdot)$ are PPs over $F_p$. However, different $\delta$-quadratic terms may result in the same sequence if their difference is a $\delta$-linear term \[33\]. To avoid duplication of the sequences in Theorem \[4\] we introduce the following PPs over finite field $F_{p^n}$ for $p$ prime.

**Definition 9** A polynomial $g(x)$ over $\mathbb{F}_{p^n}$ is called a semi-normalized PP if $g(x)$ is a monic PP and $g(0) = 0$. The collection of all semi-normalized PPs over $\mathbb{F}_{p^n}$ are denoted by $S^{(p^n)}$ in this paper.

It is obvious that the number of the semi-normalized PPs is $(p^n - 2)!$. If $g(x)$ is a PP and $a \neq 0, b \neq 0, c \in \mathbb{F}_{p^n}$, then $g_1(x) = a \cdot g(bx + c)$ is also a PP. By suitably choosing $a, b, c$, we can arrange to have $g_1(x)$ in normalized form so that $g_1(x)$ is monic, $g_1(0) = 0$, and when the degree $d$ of $g_1(x)$ is not divisible by $p$, the coefficient of $x^{d-1}$ is $0$. Normalized PPs are well-studied in the literature. For example, a list of all normalized PPs of degree at most $5$ can be found in \[33\] Ch.8], and all normalized PPs of degree $6$ was tabulated in \[46\]. For any semi-normalized PP $g(x)$, it is obvious that there exists normalized PP $g_1(x)$ and $c, e \in \mathbb{F}_{p^n}$ such that $g(x) = g_1(x + c) + e$, so all the semi-normalized PPs in Definition \[9\] can be obtained by the well-studied normalized PPs over $\mathbb{F}_{p^n}$.

**Example 1** For $p = 5$ and $n = 1$, there are two normalized PPs from \[33\] Ch.8]: $x, x^3$. Then $3! = 6$ semi-normalized PPs in $S^{(5)}$ can be obtained as follows.

\[
S^{(5)} = \{x, x^3, (x + 1)^3 + 4, (x + 2)^3 + 2, (x + 3)^3 + 3, (x + 4)^3 + 1\} = \{x, x^3, x^3 + 3x^2 + 3x, x^3 + x^2 + 2x, x^3 + 4x^2 + 2x, x^3 + 2x^2 + 3x\}.
\]

It is obvious that $dg_0(x_0)g_1(x_1) \in \delta_Q(p, p)$ is a $\delta$-quadratic term for $d \in \mathbb{F}_p^*$ and $g_0(\cdot), g_1(\cdot) \in S^{(p)}$. Moreover, it is easy to check that the difference of $dg_0(x_0)g_1(x_1)$ and $d'g_0'(x_0)g_1'(x_1)$ cannot be a $\delta$-linear term if $(d, g_0(\cdot), g_1(\cdot)) \neq (d', g_0'(\cdot), g_1'(\cdot))$. Then sequences in CSSs determined by DFT over $\mathbb{F}_p$ and the enumeration are given as follows.

**Corollary 3** For $\forall g_k(\cdot), g_k'(\cdot) \in S^{(p)} \ (0 \leq k \leq m - 1)$, $d_k \in \mathbb{F}_p^*$, $l(x) \in \delta_L(p, p)$ and permutation $\pi$, the $p$-ary sequence evaluated by
\begin{equation}
    f(x) = \sum_{k=1}^{m-1} d_k \cdot g_k(x_{\pi(k-1)}) \cdot g_k'(x_{\pi(k)}) + l(x)
\end{equation}
lies in a CSS of size $p$. Moreover, formula (25) explicitly determines

$$\frac{1}{2} m! ((p-1)!)^{m-1} ((p-2)!)^{m-1} p^{m(p-1)+1}$$

distinct $p$-ary sequences.

**Proof.** In formula (25), there are $(p-1)$ choices of $d_k$, $(p-2)!$ choices of $g_k(\cdot)$ and $g_k'(\cdot)$ respectively for a fixed $k$, so there are totally $(p-1)!(p-2)! \delta$-quadratic terms determined by DFT matrix of order $p$. There are $m!$ choices of permutations $\pi$, and $p^m(p-1)+1$ choices of function $l(x)$. Moreover, for two functions with the form (25):

$$f(j)(x) = \sum_{k=1}^{m-1} d_k^{(j)} \cdot g_k^{(j)}(x^{(j)(k-1)}) \cdot g_k'(x^{(j)(k)}) + l^{(j)}(x) \quad (j = 1, 2),$$

we have $f^{(1)}(x) = f^{(2)}(x)$ if and only if

1. $\pi^{(1)} = \pi^{(2)}$, $d_k^{(1)} = d_k^{(2)}$, $g_k^{(1)} = g_k^{(2)}$, $l^{(1)}(x) = l^{(2)}(x)$, or

2. $\pi^{(1)}(k) = \pi^{(2)}(m-k)$, $d_k^{(1)} = d_k^{(2)}$, $g_k^{(1)} = g_k^{(2)}$, $g_k'(1) = g_k'(m-k)$, $l^{(1)}(x) = l^{(2)}(x)$.

The proof is completed. □

We firstly give an example for $p = 2$ to illustrate the constructions in Theorem 4 and Corollary 3.

**Example 2** For $p = 2$, $S^{(2)}$ contains only one PP: $g(x) = x$. The functions in Corollary 3 can be expressed by

$$f(x) = \sum_{k=1}^{m-1} x^{(k-1)} x^{(k)} + \sum_{k=0}^{m-1} c_k x_k + c',$$

for $c_k, c' \in \mathbb{F}_2$. From Theorem 4, the sequences evaluated by

$$\begin{cases} f(x), \\ f(x) + x^{(0)} \end{cases}$$

form a binary Golay complementary pair.

The results in Example 2 coincide with the known binary standard Golay sequences, which have been well studied in [8]. In the rest of this subsection, we discuss the case for odd prime $p$. Any $p$-ary function from $\mathbb{F}_p^m$ to $\mathbb{F}_p$ can be treated as a vector over $\mathbb{F}_p$ of dimension $p^m$. Then $\delta_L(p, p)$ is a subspace
(or a code) of dimension \( m(p - 1) + 1 \). The collection of the sequences in Corollary 3 is actually the union of the cosets of \( \delta_L(p, p) \) with coset leaders:

\[
\sum_{k=1}^{m-1} d_k \cdot g_k(x_{\pi(k-1)}) \cdot g_k'(x_{\pi(k)}). \tag{26}
\]

On the other hand, every sequence in Corollary 3 lies in \( \text{GRM}_{p}(2(p-2), m) \) \cite{21}, since the degree of the PPs over \( \mathbb{F}_p \) is no more than \( p - 2 \). The Hamming distance of \( \text{GRM}_{p}(r, m) \) has been shown in \cite{21}.

Then a lower bound of the Hamming distance of the union of these cosets is obtained immediately.

**Corollary 4** The Hamming distance of any two distinct sequences in Corollary 3 is at least \( 3p^{m-2} \) for odd prime \( p \).

**Proof** It was shown in \cite{21} Theorem 5] that \( \text{GRM}_{p}(r, m) \) has minimum Hamming distance \((R+1) \cdot p^Q\), where \( R \) is the remainder and \( Q \) the quotient resulting from dividing \( m(p - 1) - r \) by \( p - 1 \). If we set \( r = 2(p - 2) \), we have \( Q = m - 2 \) and \( R = 2 \) for \( p > 3 \), and we have \( Q = m - 1 \) and \( R = 1 \) for \( p = 3 \). Both two cases result in minimum Hamming distance \( 3p^{m-2} \). \(\square\)

We continue to give examples for \( p = 3 \) and 5.

**Example 3** For \( p = 3 \), \( S^{(3)} \) contains only one PP: \( g(x) = x \). Then the functions in Corollary 3 can be expressed by

\[
f(x) = \sum_{k=1}^{m-1} d_k x_{\pi(k-1)} x_{\pi(k)} + \sum_{k=0}^{m-1} c_{k,2} x_k^2 + \sum_{k=0}^{m-1} c_{k,1} x_k + c',
\]

for \( d_k \in \mathbb{F}_3^* \) and \( c_{k,2}, c_{k,1}, c' \in \mathbb{F}_3 \). From Theorem 4, the sequences evaluated by

\[
\begin{cases}
f(x), \\
f(x) + x_{\pi(0)}, \\
f(x) + 2x_{\pi(0)}
\end{cases}
\]

form a ternary CSS of size 3. The sequences evaluated by

\[
f(x) \cdot J_3 + x_{\pi(0)} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix} + x_{\pi(m-1)} \cdot \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}
\]

form a CCC, where \( J_N \) is the all 1 matrix of order \( N \). These results coincide with ternary case in \cite{53} Construction 2].
Table 1: The Comparisons of Code Rate With Known Results

| Codes               | PMEPR at most | # of subcarriers | info. rate | code rate |
|---------------------|---------------|------------------|------------|-----------|
| Binary [37]         | 4             | 128              | 0.180      | 0.180     |
| Quaternary [37]     | 4             | 128              | 0.296      | 0.148     |
| Octary [37]         | 4             | 128              | 0.405      | 0.135     |
| Quinary in this paper | 5         | 125              | 0.369      | 0.159     |
| Binary [37]         | 8             | 128              | 0.172      | 0.172     |

Example 4  For \( p = 5 \), \( S^{(5)} \) has been given in Example 1. The functions in Corollary 3 can be expressed by

\[
f(x) = \sum_{k=1}^{m-1} d_k \cdot g_k(x_{\pi(k-1)}) \cdot g'_k(x_{\pi(k)}) + \sum_{k=0}^{m-1} (c_k,4x^4_k + c_k,3x^3_k + c_k,2x^2_k + c_k,1x_k) + c',
\]

where \( g_k(\cdot), g'_k(\cdot) \in S^{(5)} \), \( d_k \in \mathbb{F}_5^* \) and \( c_{k,i}, c' \in \mathbb{F}_5 \).

For \( \forall g_0(\cdot) \in S^{(5)} \), the sequences evaluated by

\[
\begin{cases}
f(x), \\
f(x) + g_0(x_{\pi(0)}), \\
f(x) + 2g_0(x_{\pi(0)}), \\
f(x) + 3g_0(x_{\pi(0)}), \\
f(x) + 4g_0(x_{\pi(0)})
\end{cases}
\]

form a quinary CSS of size 5.

For \( \forall g_0(\cdot), g'_0(\cdot) \in S^{(5)} \), the sequences evaluated by

\[
f(x) \cdot J_5 + g_0(x_{\pi(0)}) \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{pmatrix} + g'_0(x_{\pi(m-1)}) \cdot \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}
\]

form a CCC.

A table of parameters including PMEPR bound, code rate, information rate for quinary sequences of length 125 given in Example 4, and for binary, quaternary and octary sequences of length 128 proposed in [37] is shown in Table 1. Note that it is difficult to do a normalized comparison with known results, since the length and the bound of the PMEPR of the previous results are always power of 2.
4.2 Constructions from Bijective GBFs

In this subsection, we will set \( p = 2 \) and \( N = 2^n = q \). In this case, the explicit form of the \( \delta \)-linear terms \( \delta_L(q = 2^n, p^n = 2^n) \) have been shown in Corollary 4. For \( \delta \)-quadratic terms, any permutation function \( g(y) \) over \( \mathbb{Z}_{2^n} \) can be realized a bijective GBF \( g(x) \) from \( \mathbb{F}_2^n \) to \( \mathbb{Z}_{2^n} \). Then the ANF of the functions in CCA and CAS can be given by Theorem 3.

With the same arguments in the previous subsection to avoid the duplication, we define a subset \( S \) of the bijective GBFs from \( \mathbb{F}_2^n \) to \( \mathbb{Z}_{2^n} \) such that

1. \( g(x = 0) = 0 \) for \( \forall g(x) \in S \),
2. \( g_1(x) \neq d \cdot g_2(x) \) for \( \forall g_1(x), g_2(x) \in S \), and \( (d, 2) = 1 \).

Then we have \( dg(x_0)g'(x_1) \in \delta_Q(2^n, 2^n) \) for \( g(x_0), g'(x_1) \in S \), \( d \in \mathbb{Z}_{2^n} \) and \( d \) odd. There are totally \( \frac{(2^n-1)!}{2} \) bijective GBFs in the set \( S \), which leads to \( \frac{(2^n-1)!}{2} \) \( \delta \)-quadratic terms determined by DFT matrices of order \( 2^n \). We give an example for \( n = 2 \) to illustrate it.

**Example 5** There are 3 bijective GBFs from \( \mathbb{F}_2^2 \) to \( \mathbb{Z}_4 \) in the set \( S \), which can be explicitly given by

\[
\begin{align*}
g_1(x_0, x_1) &= x_0 + 2x_1, \\
g_2(x_0, x_1) &= 2x_0 + x_1, \\
g_3(x_0, x_1) &= 2x_0x_1 + x_0 + 3x_1.
\end{align*}
\]

Then there are 18 \( \delta \)-quadratic terms determined by DFT matrices of order 4, which can be presented by

\[
dg(x_0, x_1)g'(x_2, x_3) \quad \text{for } g, g' \in \{g_1, g_2, g_3\}, d = 1, 3.
\]

The collection of these \( \delta \)-quadratic terms is denoted by \( \delta_Q(1)(4, 4) \). We will continue the study of the quadratic terms \( \delta_Q(4, 4) \) in Example 7 by another BH matrix.

5 Constructions from Hadamard Matrices over Finite Fields

In this section, we will set \( p \) prime. We first introduce some notations in this and the next sections.

Recall that \( (x_0, x_1, \cdots, x_{n-1}) \) be the \( p \)-ary expansion of integer \( y \), i.e., \( y = \sum_{j=0}^{n-1} x_j p^j \). Define the mapping \( \gamma \) from \( \mathbb{Z}_{p^n} \) to \( \mathbb{F}_p^n \) by

\[
\gamma(y) = \sum_{j=0}^{n-1} x_j \alpha_j,
\]
where \( \{\alpha_0, \alpha_1, \cdots, \alpha_{n-1}\} \) is a basis of finite field \( \mathbb{F}_{p^n} \) over \( \mathbb{F}_p \). We will directly use \( y \in \mathbb{F}_{p^n} \) instead of \( \gamma(y) \in \mathbb{F}_{p^n} \) if the context is clear in this and the next sections. Then we have variables \( y_k = \sum_{j=0}^{n-1} \alpha_j x_{kn+j} \in \mathbb{F}_{p^n} \) where \( x_k = (x_{kn}, x_{kn+1}, \cdots, x_{kn+n-1}) \in \mathbb{F}_p^n \).

The trace function from \( \mathbb{F}_{p^n} \) to \( \mathbb{F}_p \) is defined by
\[
Tr(\alpha) = \alpha + \alpha^p + \alpha^{p^2} + \cdots + \alpha^{p^{n-1}}.
\]

Theorem 5

5.1 Constructions by the Trace Function and PPs over \( \mathbb{F}_{2^n} \)

In this subsection, we will set \( p = 2 \) and \( q \) even. From Corollary 1, the set of the \( \delta \)-linear terms is given by
\[
\delta_L(q, p^n = 2^n) = \left\{ \sum_{k=0}^{m-1} \sum_{i=1}^{2^n-1} (c_{k,i} \cdot \prod_{j=0}^{n-1} x_{kn+j}) + c' \mid c_{k,i}, c' \in \mathbb{Z}_q \right\},
\]
where \( (i_0, i_1, \cdots, i_{n-1}) \) is the binary expansion of integer \( i \).

Let \( H \) be a Hadamard matrix with entry \( H_{u,v} = (-1)^{Tr(u,v)} = w^q Tr(u,v) \) for \( u, v \in \mathbb{F}_{2^n} \). Then the entry of its phase matrix is given by \( \tilde{H}_{u,v} = \frac{q}{2} Tr(u \cdot v) \). And we have
\[
\frac{q}{2} Tr(g(y_0)g'(y_1)) \in \delta_Q(q, 2^n)
\]
for arbitrary PPs \( g(\cdot), g'(\cdot) \) over \( \mathbb{F}_{2^n} \). These terms are called the \( \delta \)-quadratic terms determined by Hadamard matrix over \( \mathbb{F}_{2^n} \).

Remark 3 \( Tr(g(y_0)g'(y_1)) \) can be expressed by a Boolean function from \( \mathbb{F}_{2^n}^2 \) to \( \mathbb{F}_2 \) with variables \( (x_0, x_1) \). For \( n \geq 2 \), since the degree of the PPs \( g(y) \) over \( \mathbb{F}_{2^n} \) must be not more than \( 2^n - 2 \) with respected to variables \( y \), the degree of the \( Tr(g(y_0)g'(y_1)) \) must be no more than \( 2n - 2 \) with respected to Boolean variables \( (x_0, x_1) \).

Theorem 5 Let \( g_k(\cdot), g'_k(\cdot) \) are arbitrary PPs over \( \mathbb{F}_{2^n} \) for \( 0 \leq k \leq m \), \( \pi \) arbitrary permutation of symbols \( \{0, 1, \cdots, mn - 1\} \), and \( \forall l(x) \in \delta_L(q, 2^n) \). For any \( q \)-ary GBF \( f(x) \) from \( \mathbb{F}_{2^n}^{2m} \) to \( \mathbb{Z}_q \) with the form
\[
f(x) = \frac{q}{2} \sum_{k=1}^{m-1} Tr(g_k(y_{k-1}) \cdot g'_k(y_k)) + l(x),
\]
we have the following results.

1. The following \( q \)-ary GBFs form a CAS of size \( 2^n \):
\[
f_u(x) = f(x) + \frac{q}{2} Tr(u \cdot g_0(y_0)) \text{ for } u \in \mathbb{F}_{2^n}.
\]
Consequently, the \( q \)-ary sequences evaluated by GBFs \( \{f_u(\pi \cdot x) \mid u \in \mathbb{F}_{2^n}\} \) form a CSS of size \( 2^n \).

(2) The following \( q \)-ary GBFs form a CCA:

\[
f_{u,v}(x) = f(x) + \frac{q}{2} \text{Tr}(u \cdot g_0(y_0)) + \frac{q}{2} \text{Tr}(v \cdot g'_0(y_{m-1})) \quad \text{for } u, v \in \mathbb{F}_{2^n}.
\]

Consequently, the \( q \)-ary sequences evaluated by GBFs \( \{f_{u,v}(\pi \cdot x) \mid u, v \in \mathbb{F}_{2^n}\} \) form a CCC.

With the same arguments in the subsection 4.1 to avoid the duplication, recall the semi-normalized PPs in Definition 9.

Corollary 5 For \( \forall g_k(\cdot), g'_k(\cdot) \in S^{(2^n)} (0 \leq k \leq m-1), d_k \in \mathbb{F}_{2^n}, l(x) \in \delta_L(q, 2^n) \) and permutation \( \pi \), the \( q \)-ary sequence evaluated by GBF \( f(\pi \cdot x) \), where

\[
f(x) = \frac{q}{2} \sum_{k=1}^{m-1} \text{Tr}(d_k \cdot g_k(y_{k-1}) \cdot g'_k(y_k)) + l(x),
\]

lies in a CSS of size \( 2^n \).

Remark 4 The collection of the sequences in Corollary 4 is the union of the cosets of the first order Reed-Muller code \( RM_q(1, mn) \). Moreover, every sequences in Corollary 4 lies in \( RM_q(2n - 2, mn) \) for \( n \geq 2 \).

We give examples for \( n = 1 \) and \( 2 \) to illustrate the constructions in Theorem 5 and Corollary 5.

Example 6 For \( n = 1 \), \( S^{(2)} \) contains only one PP: \( g(x) = x \). The functions in Corollary 5 can be expressed by

\[
f(x) = \frac{q}{2} \sum_{k=1}^{m-1} x_{\pi(k-1)}x_{\pi(k)} + \sum_{k=0}^{m-1} c_kx_k + c'
\]

for \( c_k, c' \in \mathbb{Z}_q \). From Theorem 5, the sequences evaluated by

\[
\begin{cases} 
  f(x), \\
  f(x) + \frac{q}{2}x_{\pi(0)} 
\end{cases}
\]

form a \( q \)-ary Golay complementary pair. These results coincide with the known \( q \)-ary standard Golay sequences [8]. The sequences evaluated by

\[
f(x) \cdot J_2 + \frac{q}{2} \begin{bmatrix} 0 & x_{\pi(m-1)} \\
  x_{\pi(0)} & x_{\pi(0)} + x_{\pi(m-1)} \end{bmatrix}
\]

form a CCC.
Example 7 For \( n = 2, q = 4 \) and \( \mathbb{F}_4 = \{0, 1, \alpha, \alpha^2 = \alpha + 1\} \), we have \( \{1, \alpha\} \) is a basis of \( \mathbb{F}_2 \) over \( \mathbb{F}_2 \). \( S^{(2^2)} \) contains two semi-normalized PPs: \( g(y) = y \) and \( g(y) = y^2 \).

We first study the Boolean function \( \text{Tr}(d \cdot g(y_0) \cdot g'(y_1)) \) where \( y_0 = x_0 + \alpha x_1, y_1 = x_2 + \alpha x_3, d \in \mathbb{F}_2^* \) and \( g(\cdot), g'(\cdot) \in S^{(2^2)} \). Note that \( \text{Tr}(\beta) = \text{Tr}(\beta^2) \) for \( \forall \beta \in \mathbb{F}_2^* \), then \( \delta \)-quadratic terms determined by Hadamard Matrix over \( \mathbb{F}_2 \) are given by

\[
\delta^{(2)}_Q(4, 4) = \{2\text{Tr}(y_0y_1), 2\text{Tr}(\alpha y_0y_1), 2\text{Tr}(\alpha^2 y_0y_1), 2\text{Tr}(y_0y_1^2), 2\text{Tr}(\alpha y_0y_1^2), 2\text{Tr}(\alpha^2 y_0y_1^2)\}.
\]

Here we take \( \text{Tr}(\alpha y_0y_1) \) and \( \text{Tr}(\alpha^2 y_0y_1^2) \) as two examples to show the calculation process:

\[
\text{Tr}(\alpha y_0y_1) = \text{Tr}(\alpha(x_0 + \alpha x_1)(x_2 + \alpha x_3)) = \text{Tr}(\alpha x_0 x_2 + \alpha^2 x_0 x_3 + \alpha^2 x_1 x_2 + x_1 x_3) = \text{Tr}(\alpha) \cdot x_0 x_2 + \text{Tr}(\alpha^2) \cdot x_0 x_3 + \text{Tr}(\alpha^2) \cdot x_1 x_2 + \text{Tr}(1) \cdot x_1 x_3 = x_0 x_2 + x_0 x_3 + x_1 x_2,
\]

and

\[
\text{Tr}(\alpha^2 y_0y_1^2) = \text{Tr}(\alpha^2(x_0 + \alpha x_1)(x_2 + \alpha x_3)^2) = \text{Tr}(\alpha^2(x_0 + \alpha x_1)(x_2 + \alpha^2 x_3)) = \text{Tr}(\alpha^2 x_0 x_2 + \alpha x_0 x_3 + x_1 x_2 + \alpha^2 x_1 x_3) = \text{Tr}(\alpha^2) \cdot x_0 x_2 + \text{Tr}(\alpha) \cdot x_0 x_3 + \text{Tr}(1) \cdot x_1 x_2 + \text{Tr}(\alpha^2) \cdot x_1 x_3 = x_0 x_2 + x_0 x_3 + x_1 x_3,
\]

By a similar process, we have

\[
\begin{align*}
\text{Tr}(y_0y_1) &= x_1 x_3 + x_0 x_3 + x_1 x_2, \\
\text{Tr}(\alpha^2 y_0y_1) &= \text{Tr}(y_0y_1) + \text{Tr}(\alpha y_0y_1) = x_1 x_3 + x_0 x_2, \\
\text{Tr}(y_0y_1^2) &= x_0 x_3 + x_1 x_2, \\
\text{Tr}(\alpha y_0y_1^2) &= \text{Tr}(y_0y_1^2) + \text{Tr}(\alpha^2 y_0y_1^2) = x_0 x_2 + x_1 x_3 + x_1 x_2.
\end{align*}
\]

The functions in Corollary 8 can be expressed by

\[
f(\mathbf{x}) = \sum_{k=1}^{m-1} h_k(x_\pi(2k-2), x_\pi(2k-1), x_\pi(2k), x_\pi(2k+1)) + \sum_{k=0}^{m-1} e_k x_\pi(2k) x_\pi(2k+1) + \sum_{k=0}^{2m-1} c_k x_k + c' \tag{32}
\]

for \( h_k(\cdot, \cdot) \in \delta^{(2)}_Q(4, 4) \) and \( e_k, c_k, c' \in \mathbb{Z}_4 \).
By applying Theorems 5, the sequences evaluated by
\[
\begin{align*}
&f(x), \\
&f(x) + 2x_{\pi(0)}, \\
&f(x) + 2x_{\pi(1)}, \\
&f(x) + 2x_{\pi(0)} + 2x_{\pi(1)}
\end{align*}
\]
form a quaternary CSS of size 4.

Recall the \(\delta\)-quadratic terms determined by DFT matrix of order 4, which have been shown in Example 5. There are two equivalent classes of the quaternary BH matrix of order 4. One class is equivalent to the DFT matrix of order 4, and the other is equivalent to the Hadamard Matrix over \(\mathbb{F}_2\). Then we have
\[
\delta_Q(4, 4) = \delta_1^{(1)}(4, 4) \cup \delta_2^{(2)}(4, 4),
\]
which exactly contains \(18 + 6 = 24\) \(\delta\)-quadratic terms. Then the function derived from \(\delta_Q(4, 4)\) and \(\delta_Q(4, 4)\) can be expressed by \(f(x)\) in the same form (32) for \(h_k \in \delta_Q(4, 4)\) and \(e_k, c_k, c' \in \mathbb{Z}_4\).

Moreover, The sequences evaluated by
\[
\begin{align*}
&f, \\
&f + 2x_{\pi(0)}, \\
&f + 2x_{\pi(1)}, \\
&f + 2x_{\pi(0)} + 2x_{\pi(1)}
\end{align*}
\]
form a complementary set of size 4.

The results in Example 4 agree with the Constructions 3 and 5 in [53]. However, all constructions in [53] are based on a Brute-force method, where the computation is very heavy. From Example 4 we can see that both the \(\delta\)-quadratic terms and the constructed sequences can be explicit given based on the algebraic structure of BH matrices.

### 5.2 Constructions by the Trace Function and PPs over \(\mathbb{F}_{p^n}\)

In this subsection, we assume \(q = p\) prime and \(N = p^n\). From Corollary 2, the set of the \(\delta\)-linear terms is given by
\[
\delta_L(q = p, p^n) = \left\{ \sum_{k=0}^{m-1} \sum_{i=1}^{p^n-1} c_{k,i} \prod_{j=0}^{n-1} x_{kn+j}^{i} \right\} + \left[ c_{k,i}, c' \in \mathbb{F}_p \right].
\]
where \((i_0, i_1, \cdots, i_{n-1})\) is the \(p\)-ary expansion of integer \(i\).

Let \(H\) be a Hadamard matrix with entry \(H_{u,v} = w^{Tr(u \cdot v)}\) for \(u, v \in \mathbb{F}_{p^n}\). Then we have

\[
Tr(g(y_0)g'(y_1)) \in \delta_Q(p, p^n)
\]

for arbitrary PPs \(g(\cdot), g'(\cdot)\) over \(\mathbb{F}_{p^n}\). These terms are called the \(\delta\)-quadratic terms determined by Hadamard matrix over \(\mathbb{F}_{p^n}\).

**Theorem 6** Let \(g_k(\cdot), g'_k(\cdot)\) are arbitrary PPs over \(\mathbb{F}_{p^n}\) for \(0 \leq k \leq m\), \(\pi\) arbitrary permutation of symbols \(\{0, 1, \cdots mn - 1\}\), and \(l(x) \in \delta_L(p, p^n)\). For any \(p\)-ary function \(f(x)\) from \(\mathbb{F}_{p^n}^{mn}\) to \(\mathbb{F}_p\) with the form

\[
f(x) = \sum_{k=1}^{m-1} Tr(g_k(y_{k-1}) \cdot g'_k(y_k)) + l(x),
\]

we have the following results.

1. The following \(p\)-ary functions form a CAS of size \(p^n\):

\[
f_u(x) = f(x) + Tr(u \cdot g_0(y_0)) \text{ for } u \in \mathbb{F}_{p^n}.
\]

Consequently, the \(p\)-ary sequences evaluated by functions \(\{f_u(\pi \cdot x) \mid u \in \mathbb{F}_{p^n}\}\) form a CSS of size \(p^n\).

2. The following \(p\)-ary functions form a CCA:

\[
f_{u,v}(x) = f(x) + Tr(u \cdot g_0(y_0)) + Tr(v \cdot g'_0(y_{m-1})) \text{ for } u, v \in \mathbb{F}_{p^n}.
\]

Consequently, the \(p\)-ary sequences evaluated by functions \(\{f_{u,v}(\pi \cdot x) \mid u, v \in \mathbb{F}_{p^n}\}\) form a CCC.

We recall the semi-normalized PPs in definition 10 to avoid the duplication.

**Corollary 6** For \(\forall g_k(\cdot), g'_k(\cdot) \in S^{(p^n)}\) \((0 \leq k \leq m - 1)\), \(d_k \in \mathbb{F}_{p^n}\), \(l(x) \in \delta_L(p, p^n)\) and permutation \(\pi\), the \(p\)-ary sequence evaluated by functions \(f(\pi \cdot x)\), where

\[
f(x) = \sum_{k=1}^{m-1} Tr(d_k \cdot g_k(y_{k-1}) \cdot g'_k(y_k)) + l(x),
\]

lies in a CSS of size \(p^n\).

**Remark 5** Theorem 4 is a special cases of Theorem 6 by restricting \(n = 1\).
6 Constructions from Sequences with 2-Level Autocorrelation

In this section, we assume \( q = p \) prime and \( N = p^n \). Let \( s \) be a \( p \)-ary sequence of length \( N - 1 = p^n - 1 \), given by

\[ s = (s(0), s(1), \ldots, s(p^n - 2)). \]

The \textit{periodic auto-correlation} of sequence \( s \) at shift \( \tau \) \((0 < \tau < N - 1)\) is defined by

\[ \sum_{i=0}^{N-2} \omega^{s(i+\tau) - s(i)}, \]

where \( i + \tau \) is the summation over \( \mathbb{Z}_{N-1} \). We say that \( s \) has (ideal) 2-level autocorrelation if its periodic auto-correlation always equals \(-1\) for \( 0 < \tau < N - 1 \).

6.1 Trace Representation

For any \( p \)-ary sequence \( s \) of length \( p^n - 1 \), there exists a univariate polynomial function, say \( h(y) \) from \( \mathbb{F}_{p^n} \) to \( \mathbb{F}_p \), such that

\[ s(i) = h(\alpha^i), \]

where \( \alpha \) is a primitive element in \( \mathbb{F}_{p^n} \). Such a polynomial function \( h(y) \) can be represented by the sum of the monomial trace term \( Tr(\beta_r y^r) \), where \( Tr(y) \) is the trace function from \( \mathbb{F}_{p^l} \) to \( \mathbb{F}_p \) and \( \beta_r \in \mathbb{F}_{p^l} \) for \( l \) being the coset size of \( r \) and \( l | n \), i.e.,

\[ h(y) = \sum_r Tr(\beta_r y^r), \]

where \( r \)'s are coset leaders modulo \( p^n - 1 \).

Since the sequence \( s \) has period \( p^n - 1 \), there is at least one \( r \) such that the coset containing \( r \) has the full length \( l = n \). Moreover, if \( h(y) \) has only one trace term, it is an \( m \)-sequence, which has been studied in the literature for more than seven decades. The reader is referring to [14] for more details on the trace representation of sequences with period \( p^n - 1 \).

It is obvious that \( s \) is a 2-level autocorrelation sequence if and only if its trace representation \( h(y) \) satisfies

\[ \sum_{y \in \mathbb{F}_{p^n}} \omega^{h(\lambda y) - h(y)} = 0 \text{ for } \forall \lambda \in \mathbb{F}_{p^n}^*. \] (38)
6.2 2-Level Autocorrelation Sequences and BH matrices

For a given 2-level autocorrelation sequence $s$ of period $p^n - 1$, we can construct a BH matrix of order $N = p^n$, say $H = (H_{ij})$, as follows

$$H_{i+1, j+1} = \omega^{s(i+j)}, 0 \leq i, j < p^n - 1,$$

$$H_{0,j} = H_{i,0} = 1, 0 \leq i, j < p^n,$$

where $i + j$ is the summation over $\mathbb{Z}_{N-1}$.

On the other hand, the BH matrix $H$ determined by the 2-level autocorrelation sequence $s$, in the sense of equivalence, can be represented by its trace representation $h(y)$ with entry

$$H_{u,v} = \omega^{h(u-v)} \text{ for } u, v \in \mathbb{F}_{p^n}.$$

**Remark 6** Note that $h(y)$ is the trace representation of sequence $s$, we always have $h(0) = 0$, which leads to $H_{0,v} = H_{u,0} = 1$.

6.3 Known Constructions on 2-level Autocorrelation Sequences

All the known construction on binary 2-level autocorrelation sequences are collected in [16] which remains the record until now. We provide a summary for those constructions for both binary and nonbinary cases of length $p^n - 1$ in the following outlines.

Binary case ($p = 2$):

(1) $m$-sequences (Golomb in 1954).

(2) For Mersenne prime $2^n - 1$, quadratic residue sequences (1932). For Mersenne prime $2^n - 1 = 4a^2 + 27$, Hall’s sextic residue sequences.

(3) For $n \geq 6$, $n$ composite, GMW sequences (Goldon, Mills and Welch [17] in 1962, Scholtz and Welch [47] in 1984).

(4) Hyper-oval construction: Segre case and Glynn I and II cases (Maschietti [32] in 1998).

(5) Dillon-Dobbertin’s Kasami power function construction [10] including conjectured 3-term and 5-term sequences [35] as subclasses, and also proved the case of the WG sequences [34].

Nonbinary case:

(1) For $p > 2$, $m$-sequences (Zieler, 1959), GMW sequences and HG sequences [18].

(2) For $p = 3$,
(a) Lin conjectured sequences (Hu, et al. [20] in 2014, Arasu et al. [1] in 2015).

(b) Conjectured sequences by Ludkovski and Gong [28] in 2000, some cases are proved in [1].

For both binary and nonbinary cases, we also have the subfield constructions: if \( 1 < l < n \) and \( l|n \), \( h_1(y) \) is a function from \( \mathbb{F}_{p^l} \) to \( \mathbb{F}_p \) whose evaluation has 2-level autocorrelation of length \( p^l - 1 \), and \( h_2(y) \) is a GMW function from \( \mathbb{F}_{p^n} \) to \( \mathbb{F}_{p^l} \), then the composition of \( h_1 \) and \( h_2 \) produces a 2-level autocorrelation sequence of length \( p^n - 1 \).

### 6.4 \( \delta \)-Quadratic Terms from Sequences with 2-Level Autocorrelation

Since a \( p \)-ary 2-level autocorrelation sequence \( s \) of period \( p^n - 1 \) can be represented by a trace representation \( h(y) \), which determine a BH matrix with entry \( H_{u,v} = \omega^{h(uv)} \). According to Theorem 2, new \( \delta \)-quadratic terms are obtained.

**Corollary 7** Let \( h(y) \) be the trace representation of a \( p \)-ary sequence with 2-level autocorrelation of period \( p^n - 1 \). Then we have

\[
 h(g(y_0) \cdot g'(y_1)) \in \delta_Q(q = p, p^n),
\]

where \( g(\cdot), g'(\cdot) \) are arbitrary PPs over \( \mathbb{F}_{p^n} \).

**Corollary 8** Let \( h(y) \) be the trace representation of a binary sequence with 2-level autocorrelation of period \( 2^n - 1 \). Then we have

\[
 \frac{q}{2} h(g(y_0) \cdot g'(y_1)) \in \delta_Q(q, 2^n),
\]

where \( g(\cdot), g'(\cdot) \) are arbitrary PPs over \( \mathbb{F}_{2^n} \), and \( q \) is even.

**Example 8** For \( m \)-sequence, it trace representation is given by \( h(y) = Tr(y) \). Then the entry of BH matrix \( H \) determined by \( m \)-sequence is given by \( H_{u,v} = \omega^{Tr(uv)} \), which is the Hadamard matrix over \( \mathbb{F}_{p^n} \) shown in Section 5. Thus, the results in Section 4.1 and Section 5 can be explained from the viewpoint of the \( m \)-sequences.

Other constructions of 2-level autocorrelation sequences yield new \( \delta \)-quadratic terms and new constructions of CSSs and CCCs. We give 3-term sequences to illustrate it.

**Example 9** For \( p = 2 \), odd \( n \geq 5 \), \( n = 2n' + 1 \), the binary 3-term sequence

\[
 s_i = Tr(\alpha^i) + Tr(\alpha^{q_1i}) + Tr(\alpha^{q_2i})
\]

has 2-level auto-correlation, where \( q_1 = 2n' + 1 \) and \( q_1 = 2^{n'} + 2^{n'-1} + 1 \). Its trace representation is given by

\[
 h(y) = Tr(y + y^{q_1} + y^{q_2}).
\]
Then the entry of BH matrix $H$ determined by this three-term sequence is given by

$$H_{u,v} = (-1)^{Tr((uv)^{q_1}+(uv)^{q_2})},$$

And we have

$$\frac{q}{2}Tr(g(y_0) \cdot g'(y_1) + g(y_0)^{q_1} \cdot g'(y_1)^{q_1} + g(y_0)^{q_2} \cdot g'(y_1)^{q_2}) \in \delta_Q(q, 2^n),$$

where $g(\cdot), g'(\cdot)$ are arbitrary PPs over $\mathbb{F}_{2^n}$, and $q$ is even.

In particular, if we set $n = 5$, we have that $H$ with entry $H_{u,v} = (-1)^{Tr((uv)^5+(uv)^7)}$ is a binary BH matrix of order 32, and

$$\frac{q}{2}Tr(g(y_0) \cdot g'(y_1) + g(y_0)^5 \cdot g'(y_1)^5 + g(y_0)^7 \cdot g'(y_1)^7) \in \delta_Q(q, 2^5).$$

7 Concluding Remarks

The theory for $\delta$-quadratic functions in this paper is for arbitrary BH matrices, though we only discuss some special cases such as DFT matrices and BH matrices derived from 2-level autocorrelation sequences. On the other hand, even for the binary case, there are 5 inequivalent BH matrices of order 16, millions of inequivalent BH matrices of orders 32. The future study on these inequivalent BH matrices will produce new CSSs and CCCs, and can exponentially increase the number of sequences with low PMEPR.

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