Critique of the Wheeler-DeWitt equation

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Abstract

The Wheeler-DeWitt equation is based on the use of canonical quantization rules that may be inconsistent for constrained dynamical systems, such as minisuperspaces subject to Einstein’s equations. The resulting quantum dynamics has no classical limit and it suffers from the infamous “problem of time.” In this article, it is shown how a dynamical time (an internal “clock”) can be constructed by means of a Hamilton-Jacobi formalism, and then used for a consistent canonical quantization, with the correct classical limit.

1 Introduction

Classical field theories describe physical phenomena by means of field variables subject to partial differential equations. It often happens that the number of field variables exceeds that of the physical degrees of freedom: there is no unambiguous way of prescribing the values of the field variables that correspond to a given physical situation. In the mathematical structure of the theory, this property is reflected by the existence of a gauge group, that allows transformations of the field variables while the physical situation remains unchanged.

The peculiar feature of Einstein’s theory of gravitation, which sets it quite apart from ordinary field theories, is that its gauge group consists of arbitrary distortions of the space-time coordinates, and thus cannot be disentangled from the structure of space-time itself. In particular, the time evolution of the gravitational field is locally indistinguishable from a gauge transformation—namely a local distortion of the space-time coordinates. As a consequence, the Hamiltonian density $\mathcal{H}$, which generates the time evolution of the field variables, is weakly equal to zero [1]: namely, although $\mathcal{H}$ is a nontrivial function of the field variables (so that it can generate a nontrivial evolution), its numerical value is constrained to vanish.

This Hamiltonian constraint does not cause any difficulty in the classical canonical theory. The numerical value of the Hamiltonian is only an initial value constraint. It is the functional form of the Hamiltonian that is needed for deriving the equations of motion. In quantum theory, however, if the gravitational field equations are quantized according to the standard canonical rule, namely $\pi^m = -i\hbar \delta / \delta g^m$, the resulting Wheeler-DeWitt equation [2, 3] leads to a dilemma known as “the problem of time.” The difficulty is that, when the constraint $\mathcal{H}\Psi = 0$ is imposed on the state vector $\Psi$, the latter is “frozen.” There

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cannot be wavepackets moving along classical trajectories, in accordance with Ehrenfest’s theorem [4]. That is, the quantum equations do not lead to the expected classical limit.

The problem was investigated long ago by Arnowitt, Deser, and Misner [5] who showed that there is an infinite number of possible coordinate conditions that may be used to put the theory in canonical form. The imposition of these coordinate conditions is equivalent to the introduction of “intrinsic” coordinates, defined by the dynamical variables of the physical system. The ADM method [5] and similar ones [6] provide, in principle, a completely general solution to the problem. Unfortunately, it is difficult to actually implement such a solution for a given, specific, physical situation.

In particular, many authors have been interested in the properties of highly symmetric cosmological models, for which there is a reasonable hope of obtaining an explicit solution. The trouble is that these symmetric situations have fewer (if any) dynamical degrees of freedom to which the ADM conditions can be applied, and the problem of time arises again. It would be impossible to mention here all the attempts that were made to solve that problem. Only a few randomly chosen references are listed below [7–11], with apologies to the authors of many similar works.

Why is general relativity special? The reason is the totalitarian nature of Einstein’s theory: all physical phenomena are coupled to gravitation, for the simple reason that all phenomena occur in space-time, and the properties of space-time are determined by the gravitational field. Any stresses, or any other physical forces, are themselves sources of the gravitational field, by virtue of the Einstein equations. (Electromagnetic theory, as a counterexample, is compatible with the existence of forces of non-electromagnetic nature, that have no electromagnetic field of their own, and can cause electric charges to move in arbitrary ways.)

On the other hand, quantum theory, unlike general relativity, is not a “theory of everything” [12]. Its mathematical formalism can be given a consistent physical interpretation only by arbitrarily dividing the physical world into two parts: the system under study, represented by vectors and operators in a Hilbert space, and the observer (and the rest of the world), for which a classical description is used. This point was emphasized long ago by Bohr [13]:

The necessity of discriminating in each experimental arrangement between those parts of the physical system which are to be treated as measuring instruments and those which constitute the objects under investigation may be said to form a principal distinction between classical and quantum-mechanical description of physical phenomena . . . The place within each measuring procedure where this discrimination is made is largely a matter of convenience.

The consistency of this hybrid quantum-classical formalism can formally be proved, under suitably restrictive assumptions on the properties of the classical world [14]. That is, as foreseen by Bohr, the precise location of the boundary between the classical and quantum parts of the system is irrelevant for well posed problems.

General relativity and quantum theory therefore appear to be fundamentally incompatible. Nevertheless, as will be shown in this article, they can be combined in a consistent way, by a careful choice of the dynamical variables. This is shown below by means of
two simple examples, with a few degrees of freedom. Each example starts by specifying a Lagrangian. Indeed it is known that canonical commutation relations are compatible with specified equations of motion only if the latter are equivalent to the Euler-Lagrange equations derived from some Lagrangian [15]. In both examples, the Lagrangians are chosen in such a way that the resulting dynamics are afflicted by the infamous “problem of time,” just as in canonical quantum gravity. Yet, canonical quantization is possible, provided that one degree of freedom is kept classical, so that it can be used as a clock.

The general method for solving this type of problem, for an arbitrary Lagrangian, is briefly discussed. It involves the solution of a first order partial differential equation, similar to the Hamilton-Jacobi equation, but with a very different physical meaning.

2 A simple example of constrained dynamics

As our first example, consider a dynamical system with three degrees of freedom, $x, y, z$, and with a Lagrangian

$$L = \frac{1}{2} \left( \frac{\dot{x}^2}{z} - zx^2 - \frac{\dot{y}^2}{z} + zy^2 \right).$$

The canonical momenta are $p_x = \dot{x}/z$, $p_y = -\dot{y}/z$, and $p_z = 0$. The equations of motion thus are

$$\dot{p}_x = d(\dot{x}/z)/dt = \partial L/\partial x = -zx,$n

$$\dot{p}_y = d(-\dot{y}/z)/dt = \partial L/\partial y = zy,$n

$$\dot{p}_z = 0 = \partial L/\partial z = \frac{1}{2} (-p_x^2 - x^2 + p_y^2 + y^2).$$

Equations of motion with a similar behavior occur in a cosmological model with a homogeneous scalar field, as in the Weinberg-Salam theory, but without a Higgs potential [16].

It is now convenient to introduce an auxiliary time, $\tau$, by means of $d\tau = zdt$, so that $p_x = dx/d\tau$ and $p_y = -dy/d\tau$. Together with this new time, we also have a new Lagrangian, given by $L_{\tau}d\tau = Ldt$, so that the action remains the same. That is,

$$L_{\tau} = L/z = \frac{1}{2} \left[ \left( \frac{dx}{d\tau} \right)^2 - x^2 - \left( \frac{dy}{d\tau} \right)^2 + y^2 \right].$$

The corresponding Hamiltonian is

$$H_{\tau} = \frac{1}{2} (p_x^2 + x^2 - p_y^2 - y^2),$$

and it is easy to derive from it the equations of motion of the two harmonic oscillators, $x$ and $y$. It follows from Eq. (5) that $H_{\tau} = 0$. This causes no difficulty at the classical level. The Hamilton equations of motion are derived from the functional form of the Hamiltonian, irrespective of its numerical value, and they are equivalent to Eqs. (2) and (3) above.

3
Trouble arises, however, if we attempt to quantize such a system by introducing a wave function $\psi(x, y)$ that satisfies $H\psi = 0$. Separation of variables readily leads to the general solution

$$\psi = \sum_n c_n u_n(x) u_n(y),$$

(7)

where the $c_n$ are arbitrary constants, and the $u_n$ are harmonic oscillator eigenfunctions corresponding to energy $E_n = (n + \frac{1}{2})\hbar$. Obviously, this state is time independent. Nothing moves. If we try to get a semi-classical solution by using large values of $n$, we find that the amplitude of the wave function is large in the vicinity of the four corners of a square, $x, y \simeq \pm \sqrt{2E_n}$, and it is of course fixed in time.

On the other hand, there is no such difficulty with the Heisenberg equations of motion, for example, $dx/d\tau = [x, H]/i\hbar$. These are formally identical to the classical oscillator equations of motion, and they lead to a nontrivial motion of the Heisenberg operators. We thus see that the Ehrenfest theorem [17] and, more generally, the correspondence principle, are not valid for such a dynamical system.

In order to find a quantum counterpart to the dynamical system that is represented, in classical physics, by the Hamiltonian (6), we must proceed more carefully. One of the harmonic oscillators, for example $y$, will serve us as a clock, and then the other one can be quantized in the usual way. We thus perform, still at the classical stage, a canonical transformation from $y$ and $p_y$ to new canonical variables,

$$Q^0 = \tan^{-1}(p_y/y),$$

(8)

$$P_0 = -(p_y^2 + y^2)/2.$$

(9)

There is no corresponding unitary transformation in quantum mechanics (since the spectrum is not invariant), but in classical mechanics, such a canonical transformation is perfectly possible.

It is easily seen that $[Q^0, H]/_{PB} = 1$, so that

$$dQ^0/d\tau = 1.$$

(10)

We can thus write

$$H = P_0 + H = 0,$$

(11)

where $H = \frac{1}{2}(p_x^2 + x^2)$ is the ordinary Hamiltonian of the $x$-oscillator. Its equations of motion are $dx/d\tau = p_x$ and $dp_x/d\tau = -x$. Thanks to Eq. (10), we can also write them as

$$dx/dQ^0 = p_x,$$

(12)

$$dp_x/dQ^0 = -x.$$

(13)

Finally, if we replace $p_x$ by $-i\hbar \partial / \partial x$, and $P_0$ by $-i\hbar \partial / \partial Q^0$, as usual, Eq. (11) becomes the standard Schrödinger equation for a harmonic oscillator.
However, at this point, we must be careful: the wave function \( \psi(x, Q^0) \) should be normalized according to
\[
\int |\psi(x, Q^0)|^2 \, dx = 1,
\]
without any further integration \( \int \cdots dQ^0 \). This follows from our decision of keeping the clock time \( Q^0 \) classical, so that it can play the ordinary role of time in Schrödinger’s equation. In this way, we have obtained a simple, consistent formalism, with the correct classical limit. Obviously, there are many other possible consistent formalisms, that are not equivalent to each other, and yet give the same classical limit. Quantization is possible, but it is not a unique process.

3 Definition of a dynamical time

It will be now be shown how a similar quantization process can be performed for any classical dynamical system with a constrained Hamiltonian,
\[
H_\tau(q, p) = 0.
\]
Here, \( q \) and \( p \), without indices, mean \( \{q^1 \ldots q^n\} \) and \( \{p_1 \ldots p_n\} \), respectively, and \( \tau \) is an arbitrary, convenient time parameter, in terms of which the problem has been formulated. In the case of Einstein’s gravitational field equations, there is an infinite number of dynamical variables and of constraints (there are twelve canonical variables, \( g_{mn} \) and \( \pi^{mn} \), and four constraints per space point). In the present article, however, my main interest is in solving the “problem of time” for minisuperspaces. I shall therefore assume that there is only a finite number, \( n \), of degrees of freedom, and a single Hamiltonian constraint, Eq. (15).

Following the method illustrated in the preceding section, let us seek a canonical transformation from \( \{q^k\} \) and \( \{p_k\} \) to new canonical variables, \( \{Q^\mu\} \) and \( \{P_\mu\} \), with \( \mu = 0, \ldots, n-1 \), such that
\[
\frac{dQ^0}{d\tau} = [Q^0, H_\tau]_{PB} = 1.
\]
It follows from Eq. (16) that
\[
H_\tau = P_0 + H(Q^0 \ldots Q^{n-1}, P_1 \ldots P_{n-1}).
\]
The latter equation defines an effective Hamiltonian \( H \). Note that \( H \) does not depend on \( P_0 \) so that, after we replace \( P_0 \) by \(-i\hbar \partial / \partial Q^0 \), the classical equation \( H_\tau = 0 \) becomes a Schrödinger equation for the new time \( Q^0 \) and the \((n-1)\) dynamical variables, \( Q^1 \ldots Q^{n-1} \).

The first step thus is to find a suitable clock time \( Q^0(q, p) \). This can easily be done, as least in a restricted domain of phase space, as shown in Fig. 1. We start with an arbitrary \((2n-1)\)-dimensional hypersurface \( \mathcal{K} \), oriented in such a way that the flow lines \( dq/d\tau = \partial H_\tau/\partial p \) and \( dp/d\tau = -\partial H_\tau/\partial q \) are nowhere tangent to \( \mathcal{K} \). That is, all the flow lines lie on the same side of \( \mathcal{K} \), for \( \tau \) positive and short enough. Then, at least
for some finite time, these flow lines will not intersect—as long as they do not reach a critical point—and they will not reenter \( K \) from the other side (however, if the motion is bounded, for example if it is periodic, reentry must obviously happen after enough time has elapsed). Anyway, for a finite time, each flow line that originates from \( K \) ascribes a unique set of \( q \) and \( p \) to each value of \( \tau \) and, conversely, in a finite domain of phase space, there is a unique \( \tau \) for each set of \( \{q,p\} \), say \( \tau = f(q,p) \).

There still is here a formidable technical difficulty, because generic dynamical problems are not integrable: the number of constants of motion is usually less than the number of degrees of freedom, and the function \( f(q,p) \) defined above cannot be obtained in closed form and does not exist globally. For that reason, some authors [18] take the liberty of “fixing the gauge” by an arbitrary choice of the function \( f(q,p) \), leading to a form which is convenient for further work. It is not clear to me why this is permitted. This is also not necessary, because there do exist approximation methods for performing a sequence of canonical transformations which reduce the Hamiltonian to a normal form [19, 20]. These give approximate constants of motion, which are represented in phase space by “vague tori” and are useful for describing the dynamics over extended time periods. These tori remnants [21, 22] become important in quantum theory because, if their missing parts are small compared to \( 2\pi\hbar \), the quantum system behaves as if it were regular, with ordinary selection rules.

Let us now return to the “problem of time.” What we need is a canonical transformation such that \( Q^0 = f(q,p) \) is a prescribed function. As explained above, \( f(q,p) \) is chosen in such a way that

\[
[f(q,p), H_{r}]_{PB} = 1,
\]

and therefore \( dQ^0/d\tau = 1 \). At this point, it is natural to ask what would happen if we
had chosen another initial hypersurface, say \( K' \), leading us to a different time function, \( f'(q, p) \), say. We would then have

\[
[f(q, p) - f'(q, p), H_\tau]_{PB} = 0, \tag{19}
\]

so that \((f - f')\) has to be a constant of the motion. Either it is a function of \( H_\tau \) or, if there are other, nontrivial constants of the motion, \((f - f')\) can be a function of them. In particular, if \( H_\tau = 0 \) is the only constant of motion, \((f - f')\) can only be a mere number. Anyway, it does not matter for the sequel whether \( f(q, p) \) is uniquely defined, up to a numerical constant, or can be modified by adding a nontrivial constant of the motion (thus effectively giving a different version of the theory).

Our task thus is is to find explicitly a canonical transformation that leads to the decomposition (17). This can be done, in principle, by the solution of a first order partial differential equation of the same type as the Hamilton-Jacobi equation. It is easiest to use a generating function [23] of type \( F_1 \), that we shall write as \( S(q, Q) \). We have

\[
p_k = \partial S/\partial q^k, \tag{20}
\]

\[
P_\mu = -\partial S/\partial Q^\mu. \tag{21}
\]

Since \( S \) is time-independent (there is no explicit appearance of \( \tau \) in \( S \)), the new Hamiltonian is numerically equal to the old one, as in Eq. (17).

To obtain \( S \) explicitly, we have to solve

\[
Q^0 = f \left( q, \frac{\partial S(q, Q)}{\partial q} \right). \tag{22}
\]

The various \( Q^\mu \), with \( \mu > 0 \), are unspecified integration constants in the solution of Eq. (22). As in the Hamilton-Jacobi case, there is no guarantee that (22) has well behaved global solutions. However, it is always possible to achieve arbitrarily close approximations in a finite domain. An example is given in the next section.

Once we have obtained \( S(q, Q) \), we get \( P_\mu(q, Q) \) from Eq. (21). We can then invert these equations, in principle, and find \( q(Q, P) \). Likewise, Eq. (20) gives us \( p = p(q, Q) \), and since \( q(Q, P) \) is already known, this gives \( p = p(Q, P) \). All these results are then substituted in \( H_\tau \) so as to obtain the explicit form of Eq. (17). Finally, that equation can be quantized in the usual way, replacing \( Q^0 \) by a new variable, \( t \) (recall that \( dQ^0/d\tau = 1 \)), and \( P_0 \) by \(-i\hbar \partial /\partial t \). Note, however, that the new parameter \( t \) is not a function of the space-time coordinates: it is a function, \( f(q, p) \), of the phase-space coordinates. This is a meaningful dynamical time, not a meaningless (gauge dependent) coordinate-time.

4 Quantization of a minisuperspace

Let us finally return to general relativity. As the simplest example, consider a spatially flat Friedmann-Lemaître universe [24], with metric

\[
ds^2 = N^2(t) \, dt^2 - a^2(t) \left( dx^2 + dy^2 + dz^2 \right). \tag{23}
\]
The matter source is a massless scalar field \( \phi \), for which the energy density and pressure are

\[
\rho = p = \frac{1}{2} \dot{\phi}^2, \tag{24}
\]

where natural units have been used: \( c = 8\pi G = 1 \).

The Einstein field equations, for the above metric and sources, become ordinary differential equations for the three variables \( N(t) \), \( a(t) \), and \( \phi(t) \). In order to obtain a quantum version of this theory, the above differential equations must be obtainable as the Euler-Lagrange equations resulting from a Lagrangian [15]. It is easily found that a suitable Lagrangian, giving the correct equations, is

\[
L = \left( \frac{1}{2} a^3 \dot{\phi}^2 - 3a \dot{a}^2 \right)/N. \tag{25}
\]

Note that \( \dot{N} \) does not appear in \( L \), so that

\[
p_N \equiv \partial L/\partial \dot{N} = 0, \tag{26}
\]

and therefore

\[
\dot{p}_N = \partial L/\partial N = -L/N^2 = 0. \tag{27}
\]

The fact that \( L = 0 \) is an initial value constraint imposed on the variables \( a, \dot{a}, \) and \( \dot{\phi} \).

Likewise, we have

\[
p_\phi \equiv \partial L/\partial \dot{\phi} = a^3 \dot{\phi}/N, \tag{28}
\]

This is a constant of the motion, because \( \partial L/\partial \phi = 0 \). Finally,

\[
p_a \equiv \partial L/\partial \dot{a} = -6a \dot{a}/N, \tag{29}
\]

and

\[
\dot{p}_a = \partial L/\partial a = 3 \left( \frac{1}{2} a^2 \dot{\phi}^2 - \dot{a}^2 \right)/N. \tag{30}
\]

As in Sect. 2, it is convenient to introduce an auxiliary time \( \tau \) by means of \( d\tau = N dt \). We then have a new Lagrangian, given by \( L_\tau d\tau = L dt \), so that the action remains invariant. Furthermore, it is convenient to introduce, instead of the radial scale variable \( a \), a new variable, \( v(t) = a^3(t) \), which scales the volume element. We then have

\[
L_\tau = \frac{v}{2} \left( \frac{d\phi}{d\tau} \right)^2 - \frac{1}{3v} \left( \frac{dv}{d\tau} \right)^2, \tag{31}
\]

from which we obtain

\[
p_\phi = v \frac{d\phi}{d\tau}, \tag{32}
\]
\[ p_v = -\frac{2}{3v} \frac{dv}{d\tau}. \]  

The corresponding Hamiltonian is

\[ H_{\tau} \equiv p_\phi \frac{d\phi}{d\tau} + p_v \frac{dv}{d\tau} - L_{\tau} = \frac{1}{2v} p_\phi^2 - \frac{3v}{4} p_v^2. \]  

Note that both \( L_{\tau} \) and \( H_{\tau} \) vanish weakly, as a consequence of (27). The non-essential dynamical variable \( N(t) \) has thus been eliminated, but it has left a remnant, which is the initial value constraint, \( H_{\tau} = 0 \). The equations of motion resulting from the new Hamiltonian are:

\[ p_\phi = \text{const.}, \quad \text{Eq. (33)}, \]  

and

\[ \frac{dp_v}{d\tau} = -\frac{\partial H_{\tau}}{\partial v} = \frac{p_\phi^2}{2v^2} + \frac{3p_v^2}{4}. \]  

Our task now is to find a dynamical time function, \( Q^0 = f(v, p_v, p_\phi) \), such that \( dQ^0/d\tau = 1 \). (Obviously, \( f \) is not a function of the cyclic variable \( \phi \), since the latter does not appear explicitly in the equations of motion.) In other words, we want a function \( f(v, p_v, p_\phi) \) that satisfies

\[ [f(v, p_v, p_\phi), H_{\tau}]_{\text{PB}} = 1. \]  

For this, we have to solve the equations of motion explicitly.

Substitution of (33) into the right hand side of (34) gives, after some rearrangement,

\[ \frac{1}{2} \left( \frac{dv}{d\tau} \right)^2 + \frac{3}{4} H_{\tau} v = \frac{3}{4} p_\phi^2. \]  

This looks like the elementary energy equation for free fall of a particle of unit mass, height \( v \), and total energy \( \frac{3}{4} p_\phi^2 \), in a gravity field \( g = \frac{3}{4} H_{\tau} \). The solution is

\[ v = -\frac{3}{4} H_{\tau} \tau^2 \pm \sqrt{\frac{3}{2}} p_\phi \tau, \]  

where the integration constant was set so that \( v = 0 \) when \( \tau = 0 \) (in other words, the \( \mathcal{K} \) hypersurface is given by \( v = 0 \)). Since by definition \( v \geq 0 \), the \( \pm \) sign in (38) has to be the same as the sign of \( p_\phi \). Note that we are not allowed to set \( H_{\tau} = 0 \) at this stage: consistency of the method that was proposed in the preceding section requires that the equations of motion be valid for the entire phase space, not only for the orbits with initial conditions that satisfy \( H_{\tau} = 0 \).

It is possible to solve directly (38) for \( \tau \), and then to substitute (34) in the result. However, it is simpler to proceed as follows. From (38), we have

\[ \frac{dv}{d\tau} = -\frac{3}{2} H_{\tau} \tau \pm \sqrt{\frac{3}{2}} p_\phi, \]  

whence, thanks to Eq. (33),

\[ v \ p_v = -\frac{2}{3} \frac{dv}{d\tau} = H_{\tau} \tau \mp \sqrt{\frac{3}{2}} p_\phi. \]
Thus, (38) becomes
\[ v = -\frac{3}{4} \tau \left( H_\tau \tau \mp \sqrt{\frac{2}{3}} p_\phi \right) = -\frac{3}{4} \tau \left( v p_v \mp \sqrt{\frac{2}{3}} p_\phi \right), \] (41)
and therefore
\[ \tau \equiv f(v, p_v, p_\phi) = \frac{v}{-\frac{3}{4} v p_v \pm \sqrt{\frac{2}{3}} p_\phi}. \] (42)

It is easy to verify directly that Eq. (36) indeed holds.

The next step is to find explicitly the transformation from the original canonical coordinates to the new ones, that include \( Q^0 \) and \( P_0 \). We have, from Eqs. (22) and (42),
\[ Q^0 = \frac{v}{-\frac{3}{4} v (\partial S/\partial v) \pm \sqrt{\frac{2}{3}} (\partial S/\partial \phi)}, \] (43)
where \( S = S(v, \phi, Q^0, Q^1) \). An obvious way for obtaining a solution is to separate variables, namely,
\[ S = \phi Q^1 + S'(v, Q^0, Q^1), \] (44)
so that
\[ Q^1 = p_\phi. \] (45)

Rearranging Eq. (43), we obtain
\[ \frac{1}{Q^0} = -\frac{3}{4} \frac{\partial S'}{\partial v} \pm \sqrt{\frac{2}{3}} \frac{Q^1}{v}, \] (46)
whose solution is
\[ S' = \frac{4}{3} \left( -\frac{v}{Q^0} \pm \sqrt{\frac{2}{3}} Q^1 \ln v \right). \] (47)

We thus have
\[ p_v = \frac{\partial S'}{\partial v} = \frac{3}{4} \left( -\frac{1}{Q^0} \pm \sqrt{\frac{2}{3}} \frac{p_\phi}{v} \right), \] (48)
in agreement with (41).

Note that
\[ P_0 = -\partial S' / \partial Q^0 = -\frac{4}{3} v /(Q^0)^2; \] (49)
whence
\[ v = -\frac{3}{4} P_0 (Q^0)^2. \] (50)
When these equations for \( v \) and \( p_v \) are substituted into (34), we obtain

\[
H_\tau = P_0 \pm \sqrt{\frac{8}{3}} p_\phi / Q^0.
\] (51)

The reduced Hamiltonian \( H \), defined by Eq. (17), thus is

\[
H = \pm \sqrt{\frac{8}{3}} p_\phi / Q^0.
\] (52)

Recall that the \( \pm \) sign in \( H \) is the same as the sign of \( p_\phi / Q^0 \). Note that if \( Q^0 \) is considered as equivalent to the time \( \tau \), the number of degrees of freedom has been reduced by 2: the variable \( N \) disappeared in the transformation from \( t \) to \( \tau \), and the \( v \) and \( p_v \) variables have been absorbed in the dynamical definition of a “clock-time” \( Q^0 \).

Here, we must be careful and avoid expressing \( Q^0 \), in Eq. (52), by means of the right hand side of (42). This would give

\[
H = \left( p_\phi^2 / v \right) \mp \sqrt{\frac{8}{3}} p_\phi p_v \quad \text{wrong}.
\] (53)

Such a way of writing the Hamiltonian is not correct: it would give the true equations of motion for \( v \) and \( p_v \) only if the initial conditions are set in such a way that \( H_\tau = 0 \) in Eq. (34), namely

\[
p_\phi^2 = \frac{3}{2} (v p_v)^2.
\] (54)

Indeed, we have from (53)

\[
dv/d\tau = [v, H]_{PB} = \mp \sqrt{\frac{8}{3}} p_\phi \quad \text{wrong},
\] (55)

and

\[
dp_v/d\tau = [p_v, H]_{PB} = p_\phi^2 / v^2 \quad \text{wrong},
\] (56)

and these agree with Eqs. (33) and (37) only if (54) is satisfied. Therefore \( H \) in Eq. (53) is not a valid, unconstrained Hamiltonian for this problem. Only \( H \) given by (52) is acceptable. (More generally, the reader may easily verify that \( P_0 = H_\tau - H \) has vanishing Poisson brackets with all the canonical variables only on the hypersurface \( H_\tau = 0 \).)

We thus remain with the reduced Hamiltonian (52), and we may now replace in it \( Q^0 \) by \( \tau \). The only nontrivial equation of motion is

\[
\frac{d\phi}{d\tau} = [\phi, H]_{PB} = \pm \sqrt{\frac{8}{3}} \ln \frac{1}{\tau},
\] (57)

whence \( \phi = \phi_0 \pm \sqrt{\frac{8}{3}} \ln \tau \). Quantization is trivial: the wave function \( \psi(p_\phi, \tau) \) satisfies a Schrödinger equation,

\[
i \hbar \frac{\partial \psi}{\partial \tau} = \pm \sqrt{\frac{8}{3}} \frac{p_\phi}{\tau} \psi,
\] (58)
so that
\[ \psi = F(p_\phi) \exp \left( \pm \frac{i}{\hbar} \sqrt{\frac{2}{3} p_\phi} \ln \tau \right), \]
\[ (59) \]

where \( F(p_\phi) \) is an arbitrary function that takes care of normalization.

Obviously, only a superspace with a larger number of degrees of freedom can give an interesting theory. Unfortunately, “interesting” also means “nonintegrable”: the function \( f(q,p) \) is not in general well behaved (it is not “isolating”) and approximation methods must be used [19–22].

Finally, the question must be raised whether the notion of a minisuperspace is a valid approximation for studying quantum gravity [25]. The arbitrary imposition of symmetry constraints on the gravitational field freezes almost all its dynamical degrees of freedom in a way that appears to be incompatible with the existence of quantum fluctuations. A similar dilemma arises in elementary classical mechanics, when we impose mundane mechanical constraints, such as restricting the motion of a mass to a two-dimensional surface. Classically, such a system is well defined. However, its quantization is not unique and it essentially depends on the nature of the constraining forces [26]. I hope to return to this problem in a future publication.

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References

1. P. A. M. Dirac, Proc. Roy. Soc. (London) A 246 (1958) 333.
2. J. A. Wheeler, in Battelle Rencontres: 1967 Lectures on Mathematical Physics (Benjamin, New York, 1968).
3. B. S. DeWitt, Phys. Rev. 160 (1967) 1113.
4. T. Brotz and C. Kiefer, Nucl. Phys. B 475 (1996) 339.
5. R. Arnowitt, S. Deser, and C. W. Misner, in Gravitation: an Introduction to Current Research, ed. by L. Witten (Wiley, New York, 1962).
6. A. Peres, Phys. Rev. 171 (1968) 1335.
7. W. G. Unruh, Phys. Rev. D 40 (1989) 1048.
8. W. G. Unruh and R. M. Wald, Phys. Rev. D 40 (1989) 2598.
9. C. G. Torre, Phys. Rev. D 46 (1992) 3231.
10. R. M. Wald, Phys. Rev. D 48 (1993).
11. J. D. Brown and K. Kuchař, Phys. Rev. D 51 (1995) 5600.
12. A. Peres and W. H. Zurek, Am. J. Phys. 50 (1982) 807.
13. N. Bohr, Phys. Rev. 48 (1935) 696.
14. A. Peres, Quantum Theory: Concepts and Methods (Kluwer, Dordrecht, 1993) p. 376.
15. S. A. Hojman and L. C. Shepley, J. Math. Phys. 32 (1991) 142.
16. V. N. Pervushin and V. I. Smirichinski, report JINR E2-97-155 (e-print gr-qc/9704078).
17. P. Ehrenfest, Z. Phys. 45 (1927) 455.
18. M. Cavaglià, V. de Alfaro, and A. T. Filippov, Int. J. Mod. Phys. A10 (1995) 611.
19. G. Contopoulos, Astrophys. J. 138 (1963) 1297.
20. F. G. Gustavson, Astronom. J. 71 (1966) 670.
21. C. Jaffé and W. P. Reinhardt, J. Chem. Phys. 77 (1982) 5191.
22. R. B. Shirts and W. P. Reinhardt, J. Chem. Phys. 77 (1982) 5204.
23. H. Goldstein, Classical Mechanics (Addison-Wesley, Reading, MA, 1980) p. 382.
24. C. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation (Freeman, San Francisco, 1973) Chapt. 27.
25. K. V. Kuchař and M. P. Ryan, Jr., Phys. Rev. D 40 (1989) 3982.
26. N. G. van Kampen and J. J. Lodder, Am. J. Phys. 52 (1984) 419.