MAC LANE METHOD IN THE INVESTIGATION OF MAGNETIC
TRANSLATION GROUPS

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Central extensions of the three-dimensional translation group \( T \cong \mathbb{Z}^3 \) by the
unitary group \( U(1) \) (a group of factors) are considered within the frame of the
Mac Lane method. All nonzero vectors \( t \in T \) are considered to be generators of
\( T \). This choice leads to very illustrative relations between the Mac Lane method
and Zak’s approach to magnetic translation groups. It is shown that factor systems
introduced by Zak and Brown can be realized only for the unitary group \( U(1) \) and
for some of its finite subgroups.

1. Introduction

Much attention is paid recently to two-dimensional systems due their relations with
high-\( T_c \) superconductors, anyons, the Hall effect etc. (see e.g. [1]). The behaviour of electrons in periodic potentials and a constant magnetic field is one of the most important
problems. Brown and Zak showed independently [1–3], that the magnetic field has to
be aligned along a vector of the crystal lattice and the translations in the plane perpen-
dicular to this vector do not commute. Hence, it is enough to consider two-dimensional
translation groups (lattices) to determine properties of the so-called magnetic translation
groups (MTG’s). However, this requirement (\( H \parallel t \)) follows from the periodic boundary
conditions imposed on the three-dimensional infinite lattice. In this paper, the infinite
(three-dimensional) lattice is considered.

It can be shown that the MTG is a central extension of the translation group \( T \cong \mathbb{Z}^3 \)
by a group of factors \( G \) (see, e.g., [1]); here the case \( G = U(1) \) is considered but
finite (cyclic) groups can be also taken into account. Determination of all nonequivalent
(abelian) extensions for given groups, \( T \) and \( G \) in the considered case, is equivalent to the
determination of the second cohomology group \( H^2(T,G) \), which can be done be means
of the Mac Lane method [6–8]. The form of factor systems \( m: T \times T \to \mathbb{Z} \) obtained by
this method depends strongly on the assumed set of generators of \( T \). In this paper, the
maximal set of generators \( A = T \setminus \{0\} \) is considered and relations with Zak’s definition
of MTG are stressed.
2. Choice of Generators

Zak \cite{Zak} introduced the MTG’s as pairs consisting of a vector \( t \in T \) and a path \( P \) joining the point \( O \) (the crystal lattice origin) with the point defined by the vector \( t \); this path is constructed from \( n \) vectors, \( P = (t_1, t_2, \ldots, t_n) \) such that \( \sum_{j=1}^{n} t_j = t \).

With each pair \((t \mid P)\) the following operator (in the space of functions \( \psi : \mathbb{R}^3 \to \mathbb{C} \)) is associated

\[
\tau(t \mid P) = \exp[-(i/\hbar) t \cdot (p - (e/c)A)] \exp[-(i/\hbar) \Phi(t_1, t_2, \ldots, t_n)],
\]

where \( A \) is the vector potential of the magnetic field and \( \Phi(t_1, t_2, \ldots, t_n) \) is the magnetic flux through the polygon enclosed by the vectors \( t_1, \ldots, t_n, -t \) (the symmetric gauge \( A = (H \times r)/2 \) is assumed).

Such a definition is closely related to the notion of free groups: paths can be interpreted as words written in an alphabet \( X \), which is in a one-to-one correspondence with the set of nonzero vectors in \( T \). Moreover, the multiplication rule for the operators \( \tau \)

\[
\tau(t \mid P) \tau(t' \mid P') = \tau(t + t' \mid P \bullet P'),
\]

where ‘\( \bullet \)’ denotes the concatenation of words, i.e.

\[ P \bullet P' = (t_1, t_2, \ldots, t_n) \bullet (t'_1, t'_2, \ldots, t'_n) = (t_1, t_2, \ldots, t_n, t'_1, t'_2, \ldots, t'_n), \]

is an analogue of the multiplication rule in a free group.

On the other hand, the Mac Lane method consists in replacing an exact sequence

\[
\{1\} \to U(1) \to \text{MTG} \to T \to \{0\},
\]

by the following one

\[
\{1\} \to R \to F \to T \to \{0\},
\]

where \( R \) is the kernel of a homomorphism \( M : F \to T \) and \( F \) is a free group. Therefore, one may expect that there are closer relations between the Mac Lane method and the operators \( \tau \) introduced by Zak.

The very first, and very important, step in the Mac Lane method is the choice of the generators of \( T \). In this way, the alphabet \( X \) of \( F \) is determined by the condition:

\[ M \text{ restricted to } X \text{ is a bijection.} \]

There are (infinitely) many sets \( A \) being generators of \( T \cong \mathbb{Z}^3 \). If \( a_j, j = 1, 2, 3 \), denote the basis vectors of the crystal lattice then the most natural way is to assume that \( A = \{a_1, a_2, a_3\} \) and \( X = \{a_1, a_2, a_3\} \) with \( M(a_j) = a_j \) for \( j = 1, 2, 3 \). This choice (the smallest \( A \)) can be easily generalized to any dimension of crystal lattice and is very convenient in further calculations (cf. \cite{Zak}). However, all paths constructed from vectors \( a_j \) consist of sections, which lie along edges of a crystal lattice, whereas in Zak’s approach segments lie along any vectors of a lattice. So one may decide to consider \( A \) being the set of all nonzero vectors, i.e.

\[
A := T \setminus \{0\},
\]
where \( 0 \) denotes the zero vector in \( T \). A letter of the alphabet \( X \) corresponding to a given vector \( t \in T \) will be denoted by \( t \), so \( M(t) = t \). Since \( A \) contains all nonzero vectors, \( X \) also contains the letter \( (-t) \) such that \( M(-t) = -t \). On the other hand, \( F \) being a free group generated by the alphabet \( X \) contains the inversions of all letters \( t \in X \), i.e. elements \( \bar{t} \) such that \( M(\bar{t}) = \bar{t} \). Since \( A \) contains all nonzero vectors, \( X \) also contains the letter \( (−t) \) such that \( M(−t) = −t \). On the other hand, \( F \) being a free group generated by the alphabet \( X \) contains the inversions of all letters \( t \in X \), i.e. elements \( \bar{t} \) such that \( \bar{t}t = t\bar{t} = 1 \) for all \( t \in X \).

Hence, the words \( (−t)t \) belong to \( F \), in spite of the fact that they correspond to trivial loop \( (−t,t) \) — the area enclosed by such a loop is simply 0. To avoid these loops one may consider a slightly smaller set of generators containing only one vector from each pair \( \{t,−t\} \), \( t \neq 0 \). If a vector \( t \) is represented by \( t \in X \) then \( −t \) is represented by \( \bar{t} \in \bar{X} \) and the words \( (−t)t \) are excluded. However, it occurs that in further calculations sets \( X \) and \( \bar{X} \) have to be considered separately and in each case all four possible products, \( tt', \bar{t}t', \bar{t}t', \) and \( t\bar{t}' \) must be taken into account. This leads to a rather cumbersome calculations and, therefore, the set \( A \) of the translation group \( T \) generators is chosen to be \( T \setminus \{0\} \).

Those inconvenient loops, corresponding to two antiparallel vectors \( t \) and \( −t \), will be excluded in another way.

3. Factor Systems

In the next step of the Mac Lane method one chooses representatives of right cosets \( Rf \). It is important to do it in such a way that these representatives form the so-called Schreier set \( \{f, \bar{f} \} \). In other words, one defines a mapping \( \psi: T \to F \) such that \( M \circ \psi = \text{id}_T \). In the case considered the simplest solution is to put \( \psi(t) = t \) and \( \psi(0) = 1 \).

The mapping \( \psi \) determines also the so-called choice function

\[ \beta = \psi \circ M, \]

where \( M \) is the above introduced homomorphism, which calculates the value of a given word \( f \in F \) in the group \( T \). Of course, words in \( F \) are in one-to-one correspondence with the paths \( P \) introduced by Zak, whereas elements of \( R = \{ f \mid M(r) = 0 \} \) correspond to loops \( L \).

To determine an alphabet \( Y \) of the kernel \( R \) one has to consider all products

\[ \psi(t)\psi(t')\psi(t+t') = \varphi(t,t'), \]

where the last symbol denotes a factor belonging to \( R \). Except for general case

\[ \varphi(t,t') = tt'\beta(tt'), \]

there are two special cases:
• $t$ or $t'$ is equal to 0, so that
  \[ \varrho(t, 0) = \varrho(0, t) = 1_F, \]
  which means that the factor system $\varrho$ is normalized;

• $t' = -t$, so that
  \[ \rho(t, -t) = t(-t). \]

The words determined by the relations (8) and (9) form the alphabet $Y$ of the kernel $R$, i.e. each word in $R$ can be written as a (finite) product of such elements. It is easy to notice that these letters are determined by two nonzero vectors $t$ and $t'$, which define the following triangle:

\[
\begin{array}{c}
\textbf{t} \\
\downarrow \\
-(t + t')^* \\
\downarrow \\
\textbf{t'}
\end{array}
\]

where an asterisk denotes that a vector $-t$ is the image (under the homomorphism $M$) of $t$ and not of $(-t)$. In a similar way, $t^*$ denotes the image of $(-t)$ to distinguish it from the image of $t$. Of course, for $t' = -t$ this triangle is trivial. Each triangle $T(t, t')$ encloses an area $(t \times t')/2$ and each loop in $T$ can be constructed from such triangles (in many ways) by adding pairs $(t, -t^*)$.

Factor systems $m: T \times T \rightarrow U(1)$ are images of $\varrho$ under the so-called operator homomorphisms $\phi: R \rightarrow U(1)$, which satisfy the condition

\[ \phi(y) = \phi(fy\bar{f}), \quad \forall y \in Y, f \in F. \]  

This requirement results in the following restrictions:

1. $\phi(t(-t)) = \phi((-t)t)$ for any $t \in X$;

2. $\phi(t(-t))$ is determined by the values of $\phi$ for two triangles:
   \[ T(t', t) \quad \text{and} \quad T(t' + t, -t) \]
   for any $t' \in A$, namely
   \[ \phi(t(-t)) = \phi(t't\bar{\beta(t't)})\phi(\beta(t't)(-t)\bar{t}). \]

3. $\phi(t't\bar{\beta(t't')})$ is symmetric under any cyclic permutation of the arguments:
   \[ \phi(t't\bar{\beta(t't')}) = \phi(t'\beta(t't)\bar{t}) = \phi(\beta(t't)t\bar{t}). \]

\[ ^{1} \]To determine all possible operator homomorphisms $\phi$ it is enough to consider $f = t \in X$. 

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\[  \]

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\[  \]
4. For any three letters \(x, t, t' \in X\) the following condition has to be satisfied
\[
\phi(tt' \beta(tt')) = \phi(xt \beta(xt)) \phi(\beta(xt) t' \beta(xtt')) \phi(x \beta(tt') \beta(xtt')) ;
\]

If this result is illustrated by vectors of the crystal lattice, it is easy to notice that this equation corresponds to the relation between areas, considered as vectors, of a tetrahedron walls.

Let \(m_\phi := \phi \circ \rho\), i.e.
\[
m_\phi(t, t') = \phi(g(t, t'))
\]
be a factor system determined by the operator homomorphism \(\phi\). If \(\gamma: F \rightarrow U(1)\) is a homomorphism (i.e. it is determined by the values of \(\gamma(t), t \in X\)) then the operator homomorphisms \(\phi\) and \(\phi \gamma\), \((\phi \gamma)(r) = \phi(r) \gamma(r)\), determine equivalent factor systems \(\gamma\). Choosing \(\gamma\) one can obtain factor systems in the most convenient form. If \(\phi\) is an operator homomorphism with \(\phi(t(-t)) = \exp(i \alpha t)\) then taking \(\gamma\) determined by \(\gamma(t) = \gamma((-t)) = \exp(-i \alpha t/2)\) one obtains a new operator homomorphism\(^2\) (a new factor system) with \((\phi \gamma)(t(-t)) = 1\), i.e.
\[
m_{\phi \gamma}(t, -t) = m_{\phi \gamma}(-t, t) = 1 .
\]

In this way, the “nonphysical” loops \(t(-t)\) are “removed” from consideration and for such factor systems one may always add a letter \(t(-t) \in Y \subset R\) to any element \(r \in R\), since
\[
(\phi \gamma)(rt(-t)) = (\phi \gamma)(r)(\phi \gamma)(t(-t)) = (\phi \gamma)(r) .
\]

In particular,
\[
\phi(tt' \beta(tt')) = \phi(tt' \beta(tt') \beta(tt')(t(-t'))) = \phi(tt'(-t'')) ,
\]
where \((-t'') = \psi(-t'')\) and \(t'' = t + t' = M(tt')\). It means that depicting elements of \(R\) by loops in the crystal lattice one may omit an asterisk if: (i) only values of an operator homomorphism are relevant and (ii) a letter \(t \in X\) occurs at the beginning or at the end of a word \(r \in R\). The first condition is obviously satisfied here, where the second follows from the fact that each \(r \in R\) can be written using the letters \(t(-t)\) (with \(\phi(t(-t)) = 1\)) and \((tt' \beta(tt'))\) (moreover, one may permute arguments of \(\phi\) in the latter case).

Further analysis is rather tedious and consists of considering colinear (parallel), coplanar and general (non-coplanar) triples of lattice vectors \(t_1, t_2, t_3\) in order to show that for each operator homomorphism there exists an equivalent homomorphism given by
\[
\phi(tt' \beta(tt')) = \exp\{-i(t \times t') \cdot h/2\} ,
\]
where \(\times\) denotes the vector product
\[
(t \times t')_j = \varepsilon_{jkl} t_k t'_l ,
\]
\(^2\)Note that for a finite subgroups of \(U(1)\) with an even number of elements it may be impossible to determine such a homomorphism \(\gamma\).
and \( h \in \mathbb{R}^3 \) is any three-dimensional vector. Please note that operator homomorphisms (so factor systems, too) determined by different \( h \in \mathbb{R}^3 \) may be equivalent (see discussion in the next section). The factor 1/2 was introduced to stress that it is possible to write each factor system (each operator homomorphism) in such a form if and only if the equation \( \alpha = 2\beta \) can be solved (in other words, the equation \( e^{i\alpha} = e^{i2\beta} \)). Of course, it is possible in \( U(1) \) but may be impossible in its (finite) subgroups. Comparing this with the formula (9) in [3] (or (14) in [3]) which reads

\[
T(t)T(t') = \exp[-i/2(t \times t') \cdot (eH/\hbar c)],
\]

one can see that the parameter \( h \) should be interpreted as

\[
h = \frac{eH}{\hbar c}.
\]  

(14)

4. Final Remarks

It has been shown that the magnetic translation group introduced by Zak [3] is in fact a free group \( F \) generated by nonzero vectors of the (ordinary) translation group \( T \equiv \mathbb{Z}^3 \), namely a path \( \mathcal{P} \) associated with a vector \( t \in T \) is a word written in this free group. The homomorphism \( M: F \rightarrow T \) “calculates” the value of this word in \( T \), giving the vector \( t \). One-letter words \( t = \psi(t) \) can be chosen to form the alphabet of \( F \) and, together with the unit element \( 1_F = \psi(0) \), form the set of (right) coset representatives. This group is too big for physics applications since factor systems (labelled by \( h \)) depend on the area of a polygon enclosed by vectors \( t_1, t_2, \ldots, t_n, -t \), belonging to the path \( \mathcal{P} \), in the plane perpendicular to \( h \). On the other hand, the central extension of \( T \) by \( U(1) \) introduced in this paper can be also too big in some cases. Let \( \langle \exp(i\alpha), t \rangle \) denotes an element of this extension. The multiplication rule reads

\[
\langle \exp(i\alpha), t \rangle \langle \exp(\delta \alpha), t' \rangle = \langle \exp[i(\alpha + \alpha' - (t \times t') \cdot h)], t + t' \rangle.
\]

Since

\[
t \times t' = \frac{V}{2\pi}(k_1a_1^* + k_2a_2^* + k_3a_3^*),
\]

where \( a_j^* \) are vectors of the reciprocal lattice, \( V \) is the volume of the unit cell, and all the \( k_j \) are integers, then for

\[
h = \frac{4\pi}{V}(q_1a_1 + q_2a_2 + q_3a_3),
\]

where \( q_j \) are rational numbers, one obtains

\[
\frac{1}{2}(t \times t') \cdot h = 2\pi(k_1q_1 + k_2q_2 + k_3q_3).
\]  

(15)

Therefore, the factors take only a finite number of different values and, moreover, these values form a group \( G \subset U(1) \) generated by three complex numbers

\[
\exp(2\pi i q_j), \quad j = 1, 2, 3.
\]
In such a case it is enough to consider an extension of \( T \) by this group. Moreover, the formula (13) shows that factor systems are periodic with respect to the parameters \( q_j \) (even if they are not rational) with identical periods equal to 1, i.e. the periods of \( h \) are equal to \((4\pi/V)|a_j|\). Substituting it into (14) one finds that the physical properties of the considered system are periodic with respect to the magnetic \( H \) field with periods

\[
H_j = \frac{2hc}{V} |a_j|.
\]

which agrees with the results of Brown [3] and Zak [4]. The quantity \( hc/e \) is the elementary quantum of magnetic flux (fluxon), and the above formulae show that a finite group \( G' \) may be considered if all the \( q_j \) are rational numbers, which means that the magnetic flux through areas enclosed by any vectors of the Bravais lattice is equal to a rational multiplicity of the fluxon.

It is interesting to consider a factor system, which is equivalent to the previous one, given by (13)

\[
m'(t, t') = \exp\{-i(t \wedge t') \cdot h\},
\]

where

\[
t \wedge t' = t_2t_3a_1^* + t_3t_1a_2^* + t_1t_2a_3^*.
\]

These systems are equivalent since they differ by trivial factor system

\[
\theta(t, t') = \exp\{-i(t \vee t') \cdot h/2\},
\]

where

\[
t \vee t' = (t_2t_3 + t_3t_2)a_1^* + (t_3t_1 + t_1t_3)a_2^* + (t_1t_2 + t_2t_1)a_3^*
\]

which is generated by a mapping \( \eta: T \to U(1) \)

\[
\eta(t) := \exp\{i(t \wedge t) \cdot h/2\}
\]

according to the well-known formula

\[
\theta(t, t') = \eta(t)\eta(t')/\eta(t + t')
\]

Three important facts have to be stressed:

(i) This equivalence of factor systems can be established only for the group \( U(1) \) and its subgroups, in which the equation \( e^{i\alpha} = e^{2i\beta} \) can be solved. The factor system \( m' \) is more general, since this condition need not be satisfied. (This factor system is a general one, in a sense: it can be introduced for any \( G \subset U(1) \), whereas the second one only for groups fulfilling the above condition.)

(ii) It leads to nontrivial factors for pairs \( (t, -t) \) (words \( t(-t) \)) since

\[
m'(t, -t) = \exp\{i(t \wedge (-t)) \cdot h/2\} = \exp\{-i(t_2t_3a_1^* + t_3t_1a_2^* + t_1t_2a_3^*) \cdot h\}.
\]

(iii) However, in both cases one obtains

\[
\langle 1, t \rangle \langle 1, t' \rangle = \langle 1, t' \rangle \langle 1, t \rangle \exp\{-i(t \times t') \cdot h\}.
\]
This means that these factor systems correspond to two different decompositions of
the above commutator, expressed by the vector product $\times$, into two parts. The first
decomposition is (anti)symmetric, whereas the second is asymmetric and resembles the
Landau gauge. Recalling Brown’s work [2], in which the ray (projective) representation
of $T$ was introduced, one can see why his approach is appropriate only in the case of the
symmetric gauge $A = (H \times r)/2$ (a more general condition is given by formula (12) in [3]):
Two ray representations with different factor systems are nonequivalent and one has to
define another ray representation of $T$ to cover other gauges (other vector potentials $A$).

Finally, it should be emphasised that all the above considerations can be applied to a
crystal lattice of any dimension (which was partially done in [6]) and with application of
tensor algebra introducing in a proper way co- and contravariant tensors, contractions,
tensor products, polyvectors (multivectors) etc. As was mentioned at the beginning, one
may limit oneself to two-dimensional lattices (with the magnetic field perpendicular to
the crystal plain, so it can be considered as a scalar), but it is interesting and important
to learn more about the algebraic structure of magnetic translation groups for the sake
of a proper understanding of the physical consequences.

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