A Cut-Free Gentzen-Style Sequent Calculus for the Modal Logic S5 *

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Abstract

We present the system G3S5, a Gentzen-style sequent calculus system for the modal propositional logic S5, which in a sense has the subformula property. We formulate the rules of G3S5 in the system G3S5 which has the subformula property and prove the admissibility of the weakening, contraction and cut rules for it.

1 Introduction

Sequent calculus systems for the modal logic S5 have been widely studied for a long time. Several authors have proposed many sequent calculus for S5, however, each of them presents some difficulties (see e.g. [12, 13, 19, 16]). There are also many extensions of the sequent calculus notably they are labelled sequent calculus (see e.g. [4, 10, 9, 7]), display calculus (see e.g. [2, 20]), hypersequent calculus (see e.g. [14, 15, 8]), deep inference system (see e.g. [17]) and nested sequent (see [5]). These extensions are departing from the Gentzen-style sequent calculus. Among these extensions, labelled and display sequent calculus are syntactically impure because of using explicit semantic parameters.

However, modal logics weaker than S5 have cut-free sequent calculus system. For example, a Gentzen-style sequent calculus for S4 is presented in [18]. It is obtained by extending G3c (The
sequent calculus for classical logic in which the structural rules of weakening and contraction are admissible) with the following rules:

\[
\frac{\Gamma, A, \Box A \vdash \Delta}{\Gamma, \Box A \vdash \Delta} \quad \text{L}\Box \\
\frac{\Box \Gamma, A \vdash \Diamond A}{\Gamma, \Box A, \Diamond A \vdash \Delta} \quad \text{R}\Diamond.
\]

In this paper, we provide a Gentzen-style sequent calculus system for S5, and call it G3s5. Similar to the sequent calculus for S4, G3s5 is obtained by extending G3c with the following rules:

\[
\frac{\Gamma, A, \Box A \vdash \Delta}{\Gamma, \Box A \vdash \Delta} \quad \text{L}\Box \\
\frac{M, \Diamond \land (P, \neg Q) \vdash N, A}{M, P \vdash Q, N, \Box A} \quad \text{R}\Box
\]

where \(M\) and \(N\) are multisets of modal formulas, and \(P\) and \(Q\) are multisets of atomic formulas. In the following we prove a simple sequent to show details of this rules.

\[
\frac{p, \Box p \vdash p}{\neg p, \Box p \vdash \Diamond \neg p} \quad \text{L}\Box \\
\frac{\Diamond \neg p, \Box p \vdash \Diamond \neg p}{\neg p, \Box \neg p \vdash \Diamond p} \quad \text{R}\Box
\]

The rule L\Diamond and R\Box does not have subformulas property, but the formulas in the premises are constructed from the atomic formulas in the conclusions, in this sense the system G3s5 has the subformula property. Moreover, for convenience we rewrite this system using semicolon (;), which not only has the subformula property in the strict sense but also help us to prove the admissibility of the weakening, contraction and cut rules.

**Organization.** This paper is organized as follows. In Section 2, we recall axioms of S5 and its kripke models. In Section 3, we present Gentzen-style sequent calculus G3s5, and also show that if one uses some simpler versions of the rules L\Diamond and R\Box, then the completeness, invertibility of the rules, and admissibility of the cut rule will not be satisfied. In addition in Subsection 3.1, for convenience, we formulate the rules of G3s5 by semicolon (;), in which the system G3s5 enjoys the subformula property. Using this notation, in Section 4, we prove the admissibility of the weakening, contraction, the general versions of the rules L\Diamond and R\Box, and some other properties of the G3s5. In Section 5, besides the cut rule, a new version of cut rule is introduced, where the admissibility of each of them concludes the admissibility of the other. We prove the admissibility of them by induction simultaneously.
2 Modal logic S5

In this section, we recall the axiomatic formulation and the Kripke semantic of modal logic S5.

The language of modal logic S5 is obtained by adding to the language of propositional logic the two modal operators \( \square \) and \( \Diamond \). Atomic formulas are denoted by \( p, q, r \), and so on. Formulas, denoted by \( A, B, C, \ldots \), are defined by the following grammar:

\[
A := \bot \mid \top \mid p \mid \neg A \mid A \land A \mid A \lor A \mid A \rightarrow A \mid \Diamond A \mid \square A.
\]

where \( \bot \) is a constant for falsity, and \( \top \) is a constant for truth.

Modal logic S5 has the following axiom schemes:

- All propositional tautologies,
- (Dual) \( \square A \leftrightarrow \neg \Diamond \neg A \),
- (K) \( \square (A \rightarrow B) \rightarrow (\square A \rightarrow \square B) \),
- (T) \( \square A \rightarrow A \),
- (5) \( \Diamond A \rightarrow \square \Diamond A \).

Instead of (5) we can use:

- (4) \( \square A \rightarrow \square \square A \),
- (B) \( A \rightarrow \square \Diamond A \).

The proof rules are Modus Ponens and Necessitation:

\[
\frac{A}{A \rightarrow B} \quad \text{MP}, \quad \frac{A}{\square A} \quad \text{N}.
\]

The rule Necessitation, can be applied only to premises which are derivable in the axiomatic system. If \( A \) is derivable in S5 from the hypotheses \( \Gamma \), we write \( \Gamma \vdash_{S5} A \).

A Kripke model \( \mathcal{M} \) for S5 is a triple \( \mathcal{M} = (W, R, V) \) where \( W \) is a set of states, \( R \) is an equivalence relation on \( W \), and \( V : \Phi \rightarrow \mathcal{P}(W) \) is a valuation function, where \( \Phi \) is the set of propositional variables. Suppose that \( w \in W \). We inductively define the notion of a formula \( A \) being satisfied in \( \mathcal{M} \) at state \( w \) as follows:

- \( \mathcal{M}, w \models p \iff w \in V(p) \), where \( p \in \Phi \),
- \( \mathcal{M}, w \models \neg A \iff \mathcal{M}, w \not\models A \),
- \( \mathcal{M}, w \models A \lor B \iff \mathcal{M}, w \models A \text{ or } \mathcal{M}, w \models B \),
- \( \mathcal{M}, w \models A \land B \iff \mathcal{M}, w \models A \text{ and } \mathcal{M}, w \models B \),
- \( \mathcal{M}, w \models A \rightarrow B \iff \mathcal{M}, w \not\models A \text{ or } \mathcal{M}, w \models B \),
• \( M, w \models \Diamond A \iff M, v \models A \) for some \( v \in W \) such that \( R(w, v) \),
• \( M, w \models \Box A \iff M, v \models A \) for all \( v \in W \) such that \( R(w, v) \).

Formula \( A \) is S5-valid iff it is true in every state of every S5-model.

Note that from the point of view of the Kripke semantics, S5 is sound and complete with respect to two different classes of frames which are equivalent. In the first class, the accessibility relation between states of a Kripke frame is reflexive, transitive, and symmetric (equivalently it is reflexive and Euclidean). In the second class, the accessibility relation is absent (equivalently each two states of a Kripke frame are in relation); for more details see [1, 3].

Lemma 2.1. Let \( M = (W, R, V) \) be a Kripke model for S5.

1. \( M, w \models \Box A \iff M, w' \models \Box A \), for all \( w' \in W \), where \( wRw' \).
2. \( M, w \models \Diamond A \iff M, w' \models \Diamond A \), for all \( w' \in W \), where \( wRw' \).
3. If \( M, w \models A \), then \( M, w' \models \Diamond A \), for all \( w' \in W \), where \( wRw' \).

Proof. The proof clearly follows from the definition of satisfiability and equivalence of \( R \). \( \Box \)

3 The Gentzen system G3ss

The concept of a sequent is defined as usual (see, e.g. [6, 18, 11]). We use the notation \( \Gamma \vdash \Delta \) for sequent, where both \( \Gamma \) and \( \Delta \) are multisets of formulas. The sequent \( \Gamma \vdash \Delta \) is S5-valid if \( \bigwedge \Gamma \rightarrow \bigvee \Delta \) is S5-valid. The notation \( \Gamma \vdash_n \Delta \) means that \( \Gamma \vdash \Delta \) is derivable with a height of derivation at most \( n \).

Notation 3.1. Throughout this paper, we use the following notations.

• The multisets of arbitrary formulas are denoted by \( \Gamma, \Gamma', \Gamma_1, \Gamma_2 \) and \( \Delta, \Delta', \Delta_1, \Delta_2 \). The multisets of modal formulas are denoted by \( M, M', M_1, M_2 \) and \( N, N', N_1, N_2 \). The multisets of atomic formulas are denoted by \( P, P', P_1, P_2, P_3 \) and \( Q, Q', Q_1, Q_2, Q_3 \).

• The union of multisets \( \Gamma \) and \( \Delta \) is indicated simply by \( \Gamma, \Delta \). The union of a multiset \( \Gamma \) with a singleton multiset \( \{ A \} \) is written \( \Gamma, A \).

• \( \Box \Gamma = \{ \Box A : A \in \Gamma \} \), \( \Diamond \Gamma = \{ \Diamond A : A \in \Gamma \} \) and \( \neg \Gamma = \{ \neg A : A \in \Gamma \} \).

The system G3ss is given in Table 1, which in the rules \( R\Box \) and \( L\Diamond \), \( M \) and \( N \) are multisets of modal formulas, and \( P \) and \( Q \) are multisets of atomic formulas. The formulas \( \Diamond \bigwedge (P, \neg Q) \) in the antecedent and \( \Box \bigvee (\neg P, Q) \) in the succedent of the premises in these rules have the same role in derivations, and equivalently can be exchanged, or be taken both of them; taking each
of them, one can prove the admissibility of the others. In a bottom-up proof search, these formulas work as storage for \( P \) in the antecedent and \( Q \) in the succedent, that in final steps they may be used to get axioms (using the rules R\( \Box \)) or L\( \Diamond \) with \( \Diamond \land (P, \neg Q) \) or \( \Box \lor (\neg P, Q) \) as principal formulas, which are probably followed by the rule R\( \Diamond \) or L\( \Box \), see Example 3.3). The rules R\( \Box \) and L\( \Diamond \) are valid for each \( \Gamma \) and \( \Delta \) instead of \( P \) and \( Q \), and in Lemma 4.13, we show that the general versions of these rules are admissible in G3s5. Since these rules do not have the subformula property in the strict sense, we restrict them in the system to multisets \( P \) and \( Q \) of atomic formulas. Because of this restriction, as mentioned above, these formulas are used as principal formulas in final steps of a bottom-up proof search.

### Table 1: The Gentzen system G3s5 for the modal logic S5

| Rule | Precedence |
|------|------------|
| \( p, \Gamma \vdash \Delta, p \) | \( \text{Ax} \) |
| \( \Gamma \vdash \Delta, A \) | \( \neg A, \Gamma \vdash \Delta \) | \( \text{L}\neg \) |
| \( \Gamma \vdash \Delta, A \) | \( A, B, \Gamma \vdash \Delta \) | \( \text{L}\land \) |
| \( \Gamma \vdash \Delta, A \) | \( A \lor B, \Gamma \vdash \Delta \) | \( \text{L}\lor \) |
| \( \Gamma \vdash \Delta, A \) | \( A \rightarrow B, \Gamma \vdash \Delta \) | \( \text{L}\rightarrow \) |
| \( \Gamma \vdash \Delta, A \) | \( A, M \vdash \Box \lor (\neg P, Q), N \) | \( \text{L}\Box \) |
| \( \Gamma \vdash \Delta, A \) | \( \Diamond A, M, P \vdash Q, N \) | \( \text{L}\Diamond \) |
| \( \Gamma \vdash \Delta, A \) | \( \Diamond A, M, P \vdash Q, N, \Box A \) | \( \text{L}\Box \) |

Note that the premises in the rules L\( \Diamond \) and R\( \Box \) can be constructed from the conclusions. In this sense, G3s5 has the subformula property.

**Lemma 3.2.** Sequents of the form \( A \vdash A \), with arbitrary formula \( A \), are derivable in G3s5.

**Proof.** The proof is routine by induction on the complexity of the formula \( A \). \( \square \)

**Example 3.3.** The following sequents are derivable in G3s5.

1. \((\Diamond \Box (\neg p \lor q) \land p) \rightarrow q\)
2. \( p \rightarrow (q \lor \Box \Diamond (p \land \neg q))\)
3. \(\Diamond (p \rightarrow \Box p)\)

4. \(\Box (\neg p \lor p) \rightarrow \Box (\neg p \lor \Box p)\)

**Proof.**

\[
\begin{array}{c}
\text{D}_1 \\
1. \quad \begin{array}{c}
\Box (\neg p \lor q) \vdash \Box (\neg p \lor q) \\
\Diamond \Box (\neg p \lor q), p \vdash q & \text{(L)} \\
\Box (\neg p \lor q), p \vdash q & \text{(L)} \\
\vdash (\Diamond \Box (\neg p \lor q) \land p) \rightarrow q & \text{(R)} \\
\end{array} \\
\text{D}_2 \\
2. \quad \begin{array}{c}
\Diamond (p \land \neg q) \vdash \Diamond (p \land \neg q) \\
p \vdash q, \Box \Diamond (p \land \neg q) & \text{(R)} \\
p \vdash q \lor \Box \Diamond (p \land \neg q) & \text{(R)} \\
\vdash p \rightarrow (q \lor \Box \Diamond (p \land \neg q)) & \text{(R)} \\
\end{array}
\end{array}
\]

where \(\text{D}_1\) and \(\text{D}_2\) are used for a derivation by Lemma 3.2.

\[
\begin{array}{c}
p, \Diamond p \vdash p, \Box p, \Diamond (p \rightarrow \Box p) & \text{Ax} \\
\Diamond p \vdash p, p \rightarrow \Box p, \Diamond (p \rightarrow \Box p) & \text{R} \\
\Diamond p \vdash p, \Diamond (p \rightarrow \Box p) & \text{R} \\
\vdash \Diamond (p \rightarrow \Box p) & \text{R} \\
\end{array}
\]

\[
\begin{array}{c}
p, \Box \neg p, \Box (\Box \neg p \lor p) \vdash \Box p, p & \text{Ax} \\
\neg p, \Box \neg p, \Box (\Box \neg p \lor p) \vdash \neg p, \Box p & \text{L} \\
\neg p, \Box (\Box \neg p \lor p) \vdash \neg p, \Box p & \text{L} \\
\vdash \Box (\Box \neg p \lor p) \rightarrow \Box (\neg p \lor \Box p) & \text{R} \\
\end{array}
\]

where \(\text{D}_4\) is as follows

\[
\begin{array}{c}
\Box \neg p, \Box (\Box \neg p \lor p), p, p \vdash \Box p, p & \text{Ax} \\
\Box \neg p, \Box (\Box \neg p \lor p), p, p \vdash \Box p, p & \text{L} \\
\Box \neg p, \Box (\Box \neg p \lor p), p \land p \vdash \Box p, p & \text{L} \\
\Box \neg p, \Diamond (\Box \neg p \lor p), \Diamond (p \land p) \vdash p & \text{Ax} \\
\Box (\Box \neg p \lor p), \Diamond (p \land p) \vdash p & \text{L} \\
\Box (\Box \neg p \lor p), \Diamond (p \land p) \vdash p & \text{L} \\
\Box (\Box \neg p \lor p), \Diamond (p \land p) \vdash p & \text{L} \\
\end{array}
\]

\(\Box\)

In the following remark, some cases of the rules L\(\Box\) and R\(\Box\) are expressed that if they are used instead of the rules L\(\Box\) and R\(\Box\) in G3\(\Box\), then the system is not complete.

**Remark 3.4.** Consider the following cases of the rules L\(\Box\) and R\(\Box\).
(1) \[
\begin{array}{c}
A, M, \Diamond P \vdash \Box Q, N \\
\Diamond A, M, P \vdash Q, N
\end{array}
\]
\[\text{L\Diamond} \quad \text{R\Box}\]

(2) \[
\begin{array}{c}
A, M, \Diamond \Gamma \vdash \Box \Delta, N \\
\Diamond A, M, \Gamma \vdash \Delta, N
\end{array}
\]
\[\text{L\Diamond} \quad \text{R\Box}\]

(3) \[
\begin{array}{c}
A, M, \Diamond \land P \vdash \Box \lor Q, N \\
\Diamond A, M, P \vdash Q, N
\end{array}
\]
\[\text{L\Diamond} \quad \text{R\Box}\]

(4) \[
\begin{array}{c}
A, M, \Diamond \land \Gamma \vdash \Box \lor \Delta, N \\
\Diamond A, M, \Gamma \vdash \Delta, N
\end{array}
\]
\[\text{L\Diamond} \quad \text{R\Box}\]

If we use each of the above cases instead of the rules L\Diamond and R\Box in G3s, then we do not have completeness, invertibility of the rules, and admissibility of the cut rule. For example, the items 1 and 2 of Example 3.3 are not provable in all cases. For more details, consider the following examples:

(a) \[\vdash p \to (q \to (\Box \Diamond (p \land q)))\]

(b) \[\vdash (p \land q) \to \Box \Diamond (p \land q)\]

(c) \[\vdash \Diamond \Box (p \lor q) \to (p \lor q)\]

In the case (1), Examples (a), (b) and (c) are not provable, and the rule

\[\begin{array}{c}
p \land q \vdash \Box \Diamond (p \land q) \\
\Diamond (p \land q) \vdash \Box \Diamond (p \land q)
\end{array}\]

\[\text{L\Diamond}\]

is not invertible, thus we do not have the cut rule

\[\begin{array}{c}
p \land q \vdash \Box \Diamond (p \land q) \\
\Diamond (p \land q) \vdash \Box \Diamond (p \land q)
\end{array}\]

\[\text{R\Diamond} \quad \text{R\Box} \quad \text{Cut.}\]

In the case (2), Example (a) is not provable but (b) and (c) are provable, and the rules L\land and R\lor:

\[\begin{array}{c}
p, q \vdash \Box \Diamond (p \land q) \\
\Diamond (p \lor q) \vdash \Box \Diamond (p \lor q)
\end{array}\]

\[\text{L\Diamond} \quad \text{R\Diamond} \quad \text{R\Box} \quad \text{Cut.}\]

are not invertible thus we do not have the cut rules

\[\begin{array}{c}
p, q \vdash \Box \Diamond (p \land q) \\
\Diamond (p \lor q) \vdash \Box \Diamond (p \lor q)
\end{array}\]

and

\[\begin{array}{c}
\Diamond \Box (p \lor q) \vdash p \lor q \\
p \lor q \vdash \Box \Diamond (p \lor q)
\end{array}\]

In the cases (3) and (4), Examples (a), (b) and (c) are provable, and since \[\Diamond \Box (\neg p \lor q), p \vdash q\] is not provable in them but we have
\[\Diamond(\neg p \lor q) \vdash \neg p, q\]

and

\[
\begin{align*}
\Diamond(\neg p \lor q) & \vdash \Diamond(\neg p \lor q) \\
\Diamond \neg p & \vdash \neg p, q \\
\end{align*}
\]

Thus R\neg is not invertible and the cut rule is not admissible.

### 3.1 The system G3s\textsuperscript{\text{5}}

In this subsection, we present G3s\textsuperscript{\text{5}} another version of G3s which is a rewriting of sequents by using semicolon (;). This system not only has the subformula property in the strict sense but also helps us to prove the admissibility of the weakening, contraction and cut rules.

Let \( P \) and \( Q \) be multisets of atomic formulas, \( \Gamma \) and \( \Delta \) be multisets of arbitrary formulas, and let \( P = \{X_1, \ldots, X_n, Y_1, \ldots, Y_m\} \) be a partition of \( P \cup Q \), and \( X'_i = \{p \in P : p \in X_i\} \cup \{\neg q : q \in Q, q \in X_i\} \), \( Y'_j = \{\neg p : p \in P, p \in Y_j\} \cup \{q \in Q : q \in Y_j\} \), for \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \).

We denote the sequent \( \Gamma, \Diamond \bigwedge X'_1, \ldots, \Diamond \bigwedge X'_n \vdash \Box \bigvee Y'_1, \ldots, \Box \bigvee Y'_m, \Delta \) by \( \Gamma \vdash^P Q; \Delta \), or by \( \Gamma; P \vdash^P Q; \Delta \).

We say that two occurrences of formulas in the middle part (between two semicolons) are related if they are in the same \( X_i \) or \( Y_j \) in the partition. A relatedness of the middle part of the sequent \( \Gamma; P \vdash^P Q; \Delta \) is an equivalent relation corresponding to a partition of \( P \cup Q \) as above. The corresponding formula of the formulas in \( X_i \) is \( \Diamond \bigwedge X'_i \), and the corresponding formula of the formulas in \( Y_j \) is \( \Box \bigvee Y'_j \).

Note that a multiset \( X \) is called a submultiset of \( \Gamma \) if \( M_X(A) \leq M_\Gamma(A) \) for all \( A \in X \), where \( M_X(A) \) and \( M_\Gamma(A) \) are the multiplicities of \( A \) in multisets \( X \) and \( \Gamma \), respectively.

If \( P \) or \( Q \) is empty, we will avoid to write its corresponding semicolon.

Each sequent \( \Gamma; \vdash^P \Delta \) can be written variously in the form \( \Gamma; P \vdash^P Q; \Delta \). For example, the sequent \( \Gamma, \Diamond q, \Diamond(q \land \neg r) \vdash \Box(\neg q \lor r), \Box(\neg p \lor r), \Box r, \Delta \) is denoted by each of the following

1. \( \Gamma, \Diamond q; q \vdash r; \Box(\neg q \lor r), \Box(\neg p \lor r), \Box r, \Delta \)
2. \( \Gamma, \Diamond q; q \vdash r; \Box(\neg p \lor r), \Box r, \Delta \)
3. \( \Gamma, \Diamond q; p; q \vdash r, r, r; \Box r, \Delta \)
4. \( \Gamma; q, p, q, q \vdash r, r, r; \Delta \)

In the sequent (1), \( P = \{X_1\}, X_1 = \{q, r\}, X'_i = \{q, \neg r\} \), \( q \) and \( r \) in the middle part are related, and their corresponding formula, \( \Diamond \bigwedge X'_1 \), is \( \Diamond(q \land \neg r) \).

In the sequent (2), \( P = \{X_1, Y_1\}, Y_1 = \{q, r\}, Y'_i = \{\neg q, r\} \), the new occurrences of \( q \) and \( r \) in the middle part are related and their corresponding formula, \( \Box \bigvee Y'_1 \), is \( \Box(\neg q \lor r) \). Similarly in
(3), (4) and (5), for \( \mathcal{P} = \{X_1, X_2, Y_1\} \), \( \mathcal{P} = \{X_1, X_2, Y_1, Y_2\} \) and \( \mathcal{P} = \{X_1, X_2, Y_1, Y_2, Y_3\} \), where \( X_2 = \{q\} \), \( Y_2 = \{\neg p, r\} \) and \( Y_3 = \{r\} \), we have the corresponding formulas \( \Diamond q \), \( \Box (\neg p \lor r) \), and \( \Box r \), respectively. The premises and conclusion of every rule can be rewritten by this notation, and the correspondence between the original and rewritten sequents and the relatedness between formulas in the middle parts in the premises can be determined from the correspondence between them and the relatedness in the conclusion. Thus by determining the correspondence and relatedness in the root of a derivation (usually we take the middle part as the empty set), all correspondences and relatednesses during a derivation are determined, and we will not encounter any ambiguity.

Therefore, we rewrite the rules of \( \text{G3s5} \) in Table 2, where the premise and conclusion of any rule except \( \text{L}\Diamond \) and \( \text{R}\Box \) have the same formulas in the middle parts and the same corresponding formulas and relatedness. By applying the rules \( \text{L}\Diamond \) and \( \text{R}\Box \), the atomic formulas in the conclusion move to the middle part in the premise. For example, the corresponding formulas in the rule \( \text{L}\Diamond \) are as

\[
\frac{M, A, \Diamond \bigwedge X_1, \ldots, \Diamond \bigwedge X_n \vdash \Box \bigvee Y_1, \ldots, \Box \bigvee Y_m, \Box \bigvee (\neg P_1, Q_1), N}{M, P_1, \Diamond A, \Diamond \bigwedge X_1, \ldots, \Diamond \bigwedge X_n \vdash \Box \bigvee Y_1, \ldots, \Box \bigvee Y_m, Q_1, N} \text{ L}\Diamond,
\]

where \( X_1, \ldots, X_n \) and \( Y_1, \ldots, Y_m \) are submultisets of \( P_2, \neg Q_2 \) and \( \neg P_2, Q_2 \), respectively, and \( X_1', \ldots, X_n', Y_1', \ldots, Y_m' \) is the partition of \( P_2, Q_2 \), the middle part of the conclusion. The partition of the middle part of the premise is \( X_1', \ldots, X_n', Y_1', \ldots, Y_m', Y_{m+1}' \), where \( Y_{m+1}' = \neg P_1, Q_1 \).

The formulas in \( P_1, Q_1 \) are related in the premise with corresponding formula \( \Box \bigvee (\neg P_1, Q_1) \), and the relatedness between formulas in \( P_2, Q_2 \) and their corresponding formulas in the premise is the same as in the conclusion.

The rule \( \text{R}\Box \) is similar, except we have \( X_{n+1} = P_1, \neg Q_1 \) with corresponding formula \( \Diamond \bigwedge (P_1, \neg Q_1) \) in the premise.
The two last rules, L◊ and R□, seem to be the same, but they are versions of the rules L◊ and L□ in the L◊ rule the principal formula is ◊(P₂, ¬Q₂), and in R□ the principal formula is □(¬P₂, Q₂), which are the corresponding formulas for P₂ and Q₂ in the middle parts, as

\[
M, P₂, ◊ \bigwedge X'_1, \ldots, ◊ \bigwedge X'_n \vdash \Box \bigvee Y'_1, \ldots, \Box \bigvee Y'_m, \Box \bigvee (¬P₁, Q₁), Q₂, N
\]

\[
M, \bigwedge (P₂, ¬Q₂), ◊ \bigwedge X'_1, \ldots, ◊ \bigwedge X'_n \vdash \Box \bigvee Y'_m, \bigvee (¬P₁, Q₁), N
\]

and

\[
M, P₂, ◊ \bigwedge (P₁, ¬Q₁), ◊ \bigwedge X'_1, \ldots, ◊ \bigwedge X'_n \vdash \Box Y''_1, \ldots, \Box Y''_m, Q₂, N
\]

\[
M, P₁, ◊ \bigwedge X'_1, \ldots, ◊ \bigwedge X'_n \vdash \Box Y''_1, \ldots, \Box Y''_m, Q₁, □ \bigvee (¬P₂, Q₂), N
\]

where X₁, ..., Xₙ, Y₁, ..., Yₘ is the partition of P₃ ∪ Q₃.

In the rule L◊, the partition of P₂, P₃, Q₃, Q₂, the middle part of the conclusion, is

\[
X₁, \ldots, Xₙ, X_{ₙ₊₁}, Y₁, \ldots, Yₘ,
\]

where Xₙ₊₁ = P₂ ∪ Q₂. The partition of the middle part of the premise is

\[
X₁, \ldots, Xₙ, Y₁, \ldots, Yₘ, Y_{ₘ₊₁},
\]
where $Y_{m+1} = P_1 \cup Q_1$. The formulas in $P_1$, $Q_1$ are related in the premise with corresponding formula $\Box V(\neg P_1, Q_1)$, and the relatedness between formulas in $P_3, Q_3$ and their corresponding formulas in the premise is the same as in the conclusion.

In the rule $R\Box$: the partition of $P_2, P_3, Q_3, Q_2$, the middle part of the conclusion, is

$$X'_1, \ldots, X'_n, Y'_1, \ldots, Y'_m, Y'_{m+1},$$

where $Y_{m+1} = \neg P_2, Q_2$. The partition of the middle part of the premise is

$$X'_1, \ldots, X'_n, X'_{n+1}, Y'_1, \ldots, Y'_m,$$

where $X_{n+1} = P_1, \neg Q_1$. The formulas in $P_1, Q_1$ are related in the premise with corresponding formula $\Diamond \land (P_1, \neg Q_1)$, and the relatedness between formulas in $P_3, Q_3$ and their corresponding formulas in the premise is the same as in the conclusion.

**Remark 3.5.** We can consider the system $G3s5$: primary. As mentioned above the partition and relatedness of formulas in the middle part of the premises can be determined from theirs in the conclusion of the rules. Then there is no ambiguity in determining the partition of sequents from the partition in the root, that is usually empty. Therefore it is possible to remove the partitions $\mathcal{P}$ during a derivation.

For example, we prove $\vdash (r \land p) \rightarrow (q \rightarrow \Box(\Diamond(p \land q) \land \Diamond r))$ in this system as follows.

$$\begin{array}{c}
p, q, r \vdash \Diamond(p \land q), p \quad \text{Ax} \\
p, q, r \vdash \Diamond(p \land q), q \quad \text{Ax} \\
p, q, r \vdash \Diamond(p \land q), p \land q \quad \text{R}\Diamond \\
p, q, r \vdash \Diamond(p \land q) \quad \text{R}\Diamond \\
r, p, q \vdash \Box(p \land q) \quad \text{R}\Box \\
r, p, q \vdash \Box(p \land q) \land \Diamond r \quad \text{R}\land \\
r \land p \vdash q \rightarrow \Box(\Diamond(p \land q) \land \Diamond r) \quad \text{R}\rightarrow \\
\vdash (r \land p) \rightarrow (q \rightarrow \Box(\Diamond(p \land q) \land \Diamond r))
\end{array}$$

The Case 4 of Example 3.3 is proved as follows:

$$\begin{array}{c}
\Box \neg p, \Box(\neg p \lor p), p \vdash p; p \quad \text{Ax} \\
\neg p, \Box(\neg p \lor p), p \vdash p; p \quad \text{L}\neg \\
\Box(\neg p \lor p), p \vdash p; p \quad \text{L}\lor \\
\Box(p \lor p); p \vdash p; p \quad \text{Ax} \\
\Box(\neg p \lor p); p \vdash p; p \quad \text{L}\Box.
\end{array}$$
Note that in the above derivations, going upward, r, p, q together in the first and p in the latter move into the middle parts in the rules R□ and move out in the rules L◊.

Using this notation, we prove the admissibility of the cut rule (Theorem 5.1) and its special cases the weakening rule (Lemma 4.2) and the contraction rule (Lemma 4.12).

**Theorem 3.6** (Soundness). If Γ ⊢ Δ is provable in G3s5, then it is S5-valid.

**Proof.** The proof is by induction on the height of the derivation of Γ ⊢ Δ in G3S5. Initial sequents are obviously valid in every Kripke model for S5. We only check the induction step for the rule R□, the other cases can be verified similarly.

Suppose that the sequent Γ ⊢ Δ is M, P; P; P | Q; Q; Q | N, □A, the conclusion of rule R□, with the premise M; P; P; Q; Q; Q | N, A, and assume by the induction hypothesis that the premise is valid in every Kripke model for S5. By contradiction assume that the conclusion is not S5-valid. We assume without loss of generality that M; P; P; Q; Q; Q | N, □A stands for M; P; ♦ (∧ (P; Q; Q)), N, □A. Therefore, there is a Kripke model M = (W, R, V) such that

\[ M, w ⊨ \bigwedge (M, P; P; Q; Q) \]
\[ M, w ∉ \bigvee (Q; Q, □A), \]

for some \( w ∈ W \). Also, since the premise is valid, in all kripke models like M we have

\[ \text{If } M, w ⊨ \bigwedge (M, ♦ (∧ (P; Q; Q))) \text{ then } M, w ⊨ \bigvee (∧ P; Q; Q, N, A). \]

By (1) and (3), we have the following cases:

(a) \( M, w ⊨ □ (∧ P; Q; Q) \)

(b) \( M, w ⊨ ∨ N \)

(c) \( M, w ⊨ A \)

The case (a) is a contradiction with (1) and (2). The cases (b) will contradicts with (2). Finally, if \( M, w ⊨ A \), then we will show that it is also a contradiction with (2). Since M is the multiset of modal formulas by Lemma 2.1, \( M, w' ⊨ ∧ (∧ (P; Q; Q)) \) for all \( w' ∈ W, wRw' \).

Then by (3), we have

\[ M, w' ⊨ ∧ (∧ (∧ P; Q; Q), N, A) \]

If \( M, w' ⊨ ∧ (∧ P; Q; Q, N) \), then since N is the multiset of modal formulas again by Lemma 2.1

\[ M, w ⊨ ∧ (∧ P; Q; Q, N), \]

which contradicts with (1) and (2). So \( M, w' ⊨ A \) for all \( w' ∈ W, wRw' \), hence \( M, w ⊨ □ A \) which also contradicts with (2).
4 Structural properties

In this section, we show that weakening and contraction rules are admissible in G3S5. We remove the proofs of some lemmas since they are easy or similar to the proofs in [11, 18].

A rule of G3S5 is said to be height-preserving invertible if whenever an instance of its conclusion is derivable in G3S5 with height $n$, then so is the corresponding instance of its premise(s).

Lemma 4.1 (Inversion Lemma). All rules of G3S5, with the exception of $L\Diamond$ and $R\Box$, are height-preserving invertible.

Proof. The proof is by induction on the height of derivations.

Lemma 4.2. The rule of weakening,

$$
\frac{\Gamma; P \vdash Q; \Delta}{\Gamma'; \Gamma; P' \vdash Q'; \Delta', \Delta'} \quad W,
$$

is admissible, where $\Gamma'$ and $\Delta'$ are multisets of arbitrary formulas, and $P'$ and $Q'$ are multisets of atomic formulas.

Proof. The proof is by induction on the height of derivation of the premise. Let $D$ be a derivation of $\Gamma; P \vdash Q; \Delta$. We consider only the case when the last rule in $D$ is $R\Box$, since $L\Diamond$ is treated symmetrically. For the remaining rules it is sufficient to apply the induction hypothesis to the premise(s) and then use the same rule to obtain conclusion.

Let the last rule be $R\Box$. The proof is by subinduction on the complexity of formulas in $\Gamma'$ and $\Delta'$. We just prove the case when $\Gamma'$ and $\Delta'$ contains only atomic and modal formulas, since derivations of other cases are constructed by this case. So suppose that $\Gamma' = M_1', P_1'$ and $\Delta' = N_1', Q_1'$, where $M_1'$ and $N_1'$ are multisets of modal formulas, and $P_1'$ and $Q_1'$ are multisets of atomic formulas. There are two subcases, according to the position of the principal formula.

Subcase 1. Let $\Gamma = M_1, P_1$ and $\Delta = Q_1, N_1, \Box A$, and let $\Box A$ be the principal formula. Assume $D$ is as

$$
\frac{M_1; P_1, P \vdash Q, Q_1; N_1, A}{M_1, P_1, P \vdash Q, Q_1, N_1, \Box A} \quad R\Box.
$$

Then we have

$$
\frac{M_1'; M_1; P_1, P \vdash Q, Q_1; N_1, A}{M_1', M_1', P_1, P', P' \vdash Q, Q', Q_1', Q_1'; N_1, A, N_1'} \quad \text{IH}
$$

$$
\frac{M_1'; M_1', P_1, P', P \vdash Q, Q'; Q_1, N_1, \Box A, Q_1', N_1'}{M_1', P_1, M_1, P_1, P' \vdash Q, Q'; Q_1, N_1, \Box A, Q_1', N_1'} \quad R\Box.
$$

Subcase 2. Let $\Gamma = M_1, P_1$ and $\Delta = Q_1, N_1$, and let $\Diamond \land (P_2, \neg Q_2)$ be the principal formula, where $P = P_2, P_3$ and $Q = Q_3, Q_2$. Assume $D$ is as
Then we have

\[
\begin{array}{c}
\mathcal{D}_1 \\
M_1, P_2; P_1, P_3 \vdash Q_3, Q_1, Q_2; N_1, N_2 \quad \mathbf{IH} \\
M_1, P_2; P_1, P_3 \vdash Q_3, Q_1; Q_2; N_1, N_2, N_3 \quad \mathbf{R} \square .
\end{array}
\]

In order to prove the admissibility of the contraction and the cut rules, we need to state some properties in the following lemmas which are also required to prove the admissibility of the general versions of the rules \(\mathbf{R} \square\) and \(\mathbf{L} \lozenge\) where the constriction of atomic for formulas has been removed.

**Lemma 4.3.** The following rules are admissible.

1. \[
\frac{(A \lor B) \land C, \Gamma \vdash \Delta}{A \land C, \Gamma \vdash \Delta}
\]
2. \[
\frac{\Gamma \vdash \Delta, \neg(A \lor B) \lor C}{\Gamma \vdash \Delta, \neg A \lor C}
\]
3. \[
\frac{(A \lor B) \land C, \Gamma \vdash \Delta}{B \land C, \Gamma \vdash \Delta}
\]
4. \[
\frac{\Gamma \vdash \Delta, \neg(A \lor B) \lor C}{\Gamma \vdash \Delta, \neg B \lor C}
\]
5. \[
\frac{\neg(A \lor B) \land C, \Gamma \vdash \Delta}{\neg A \land B \land C, \Gamma \vdash \Delta}
\]
6. \[
\frac{\neg(A \land B) \land C, \Gamma \vdash \Delta}{\neg A \land B \land C, \Gamma \vdash \Delta}
\]
7. \[
\frac{\neg(A \lor B) \land C, \Gamma \vdash \Delta}{\neg A \land C, \Gamma \vdash \Delta}
\]
8. \[
\frac{\Gamma \vdash \Delta, (A \land B) \lor C}{\Gamma \vdash \Delta, A \lor C}
\]
9. \[
\frac{\neg(A \lor B) \land C, \Gamma \vdash \Delta}{\neg B \land C, \Gamma \vdash \Delta}
\]
10. \[
\frac{\Gamma \vdash \Delta, (A \lor B) \lor C}{\Gamma \vdash \Delta, B \lor C}
\]
11. \[
\frac{(A \rightarrow B) \land C, \Gamma \vdash \Delta}{B \land C, \Gamma \vdash \Delta}
\]
12. \[
\frac{\neg(A \lor B) \land C, \Gamma \vdash \Delta}{\neg A \land C, \Gamma \vdash \Delta}
\]
13. \[
\frac{\neg(A \rightarrow B) \land C, \Gamma \vdash \Delta}{(A \land \neg B) \land C, \Gamma \vdash \Delta}
\]
14. \[
\frac{\neg(A \lor B) \land C, \Gamma \vdash \Delta}{\neg A \land C, \Gamma \vdash \Delta}
\]
15. \[
\frac{(A \rightarrow B) \land C, \Gamma \vdash \Delta}{\neg A \land C, \Gamma \vdash \Delta}
\]
Lemma 4.4. The following rules are admissible.

\[
\begin{align*}
(1) & \quad \Diamond((A \lor B) \land C), \Gamma \vdash \Delta \\
(2) & \quad \Gamma \vdash \Delta, \Box(\neg(A \lor B) \lor C) \\
(3) & \quad \Diamond((A \lor B) \land C), \Gamma \vdash \Delta \\
(4) & \quad \Gamma \vdash \Delta, \Box(\neg(A \lor B) \lor C) \\
(5) & \quad \Diamond((A \lor B) \land C), \Gamma \vdash \Delta \\
(6) & \quad \Gamma \vdash \Delta, \Box(\neg(A \lor B) \lor C) \\
(7) & \quad \Diamond((A \lor B) \land C), \Gamma \vdash \Delta \\
(8) & \quad \Gamma \vdash \Delta, \Box((A \lor B) \lor C) \\
(9) & \quad \Diamond((A \rightarrow B) \land C), \Gamma \vdash \Delta \\
(10) & \quad \Gamma \vdash \Delta, \Box((A \lor B) \lor C) \\
(11) & \quad \Diamond((A \rightarrow B) \land C), \Gamma \vdash \Delta \\
(12) & \quad \Gamma \vdash \Delta, \Box((A \lor B) \lor C) \\
(13) & \quad \Diamond((A \rightarrow B) \land C), \Gamma \vdash \Delta \\
(14) & \quad \Gamma \vdash \Delta, \Box((A \lor B) \lor C) \\
(15) & \quad \Diamond((A \lor B) \land C), \Gamma \vdash \Delta \\
(16) & \quad \Gamma \vdash \Delta, \Box((A \lor B) \lor C) \\
(17) & \quad \Diamond((A \lor B) \land C), \Gamma \vdash \Delta \\
(18) & \quad \Gamma \vdash \Delta, \Box((A \lor B) \lor C) \\
(19) & \quad \Diamond((A \lor B) \land C), \Gamma \vdash \Delta \\
(20) & \quad \Gamma \vdash \Delta, \Box((A \lor B) \lor C) \\
(21) & \quad \Diamond((A \lor B) \land C), \Gamma \vdash \Delta \\
(22) & \quad \Gamma \vdash \Delta, \Box((A \lor B) \lor C) \\
(23) & \quad \Diamond((A \lor B) \land C), \Gamma \vdash \Delta \\
(24) & \quad \Gamma \vdash \Delta, \Box((A \lor B) \lor C)
\end{align*}
\]

Proof. The proof is by induction on the height of derivation of the premise in each case. As a typical example, we prove (1). If \( \Diamond((A \lor B) \land C), \Gamma \vdash \Delta \) is an axiom, then \( \Diamond((A \lor B) \land C) \) is not principal, and \( \Diamond(A \land C), \Gamma \vdash \Delta \) is an axiom. If \( \Diamond((A \lor B) \land C) \) in not principal, we apply the induction hypothesis to the premise and then use the same rule to obtain deductions of \( \Diamond(A \land C), \Gamma \vdash \Delta \). For example if \( \Delta = N, Q, \Box D, \Gamma = M, P \) and the last rule is

\[
\Diamond((A \lor B) \land C), M; P \vdash Q; N, D \quad \text{R}\Box,
\]

then we have

\[
\begin{align*}
\Diamond((A \lor B) \land C), M; P \vdash Q; N, D & \quad \text{IH} \\
\Diamond((A \lor B) \land C), M; P \vdash Q; N, D & \quad \text{R}\Box,
\end{align*}
\]

If on the other hand \( \Diamond((A \lor B) \land C) \) is principal, the last rule is:
then we have

\[
\frac{(A \lor B) \land C, M; P \vdash Q; N}{\Box((A \lor B) \land C), M, P \vdash Q, N}
\]

Lemma 4.3

\[
\frac{(A \lor B) \land C, M; P \vdash Q; N}{A \land C, M; P \vdash Q; N}
\]

\[
\Box(A \land C), M, P \vdash Q, N
\]

Lemma 4.5. The following rules are admissible.

(I) \[
\frac{\Diamond A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}
\]

(II) \[
\frac{\Gamma \vdash \Delta, \Box A}{\Gamma \vdash \Delta, A}
\]

Before we prove this lemma, we need to state some properties in the following.

Similar to the propositional logic, every formula \( A \) has an equivalent disjunctive normal form (DNF) and an equivalent conjunctive normal form (CNF), such that each conjunction in DNF and disjunction in CNF are respectively as follows:

\[
(p_1 \land \cdots \land p_n \land \neg q_1 \land \cdots \land \neg q_m \land B_1 \land \cdots \land B_k \land \neg C_1 \land \cdots \land \neg C_l)
\]

and

\[
(p_1 \lor \cdots \lor p_n \lor \neg q_1 \lor \cdots \lor \neg q_m \lor B_1 \lor \cdots \lor B_k \lor \neg C_1 \lor \cdots \lor \neg C_l)
\]

where \( p_1, \ldots, p_n \) and \( q_1, \ldots, q_m \) are atomic, and \( B_1, \ldots, B_k \) and \( C_1 \ldots, C_l \) are modal formulas.

For these conjunctions and disjunctions we have the following lemma.

Lemma 4.6. Let \( A \) be a formula, and let

\[
(p_1 \land \cdots \land p_n \land \neg q_1 \land \cdots \land \neg q_m \land B_1 \land \cdots \land B_k \land \neg C_1 \land \cdots \land \neg C_l)
\]

be a conjunction in the DNF and

\[
(p_1 \lor \cdots \lor p_n \lor \neg q_1 \lor \cdots \lor \neg q_m \lor B_1 \lor \cdots \lor B_k \lor \neg C_1 \lor \cdots \lor \neg C_l)
\]

be a disjunction in the CNF, where \( p_1, \ldots, p_n \) and \( q_1, \ldots, q_m \) are atomic, and \( B_1, \ldots, B_k \) and \( C_1 \ldots, C_l \) are modal formulas. Then the following rules are admissible.

(i) \[
\frac{\Diamond A, \Gamma \vdash \Delta}{B_1, \ldots, B_k, \Diamond(p_1 \land \cdots \land p_n \land \neg q_1 \land \cdots \land \neg q_m), \Gamma \vdash \Delta, C_1 \ldots, C_l}
\]

(ii) \[
\frac{\Gamma \vdash \Delta, \Box A}{C_1 \ldots, C_l, \Gamma \vdash \Delta, B_1, \ldots, B_k, \Box(p_1 \lor \cdots \lor p_n \lor \neg q_1 \lor \cdots \lor \neg q_m)}
\]

Proof. This easily follows from Lemma 4.4. \( \blacksquare \)
Lemma 4.7. The following rule is admissible:

\[
\Gamma; P \models Q; \Delta \quad \frac{}{\Gamma, P \models Q, \Delta}
\]

where \( \Gamma \) and \( \Delta \) are multisets of arbitrary formulas, and \( P \) and \( Q \) are multisets of atomic formulas.

**Proof.** The proof is by induction on the height of derivation of the premise. If the premise is an axiom, then the conclusion is an axiom. For the induction step, we consider only the cases where the last rule is \( \Diamond \), since the rule \( \Box \) is treated symmetrically and for the remaining rules it suffices to apply the induction hypothesis to the premise and then use the same rule to obtain deduction of \( \Gamma, P \models Q, \Delta \).

Case 1. Let \( \Gamma = \Diamond A, M, P_1 \) and \( \Delta = Q_1, N \) and let \( \Diamond A \) be the principal formula.

\[
\frac{A, M; P_1 \models Q, Q_1; N}{\Diamond A, M, P_1; P \models Q, Q_1, N} \quad \text{L\Diamond}.
\]

We have

\[
\frac{A, M; P_1 \models Q, Q_1; N}{\Diamond A, M, P_1; P \models Q, Q_1, N} \quad \text{L\Diamond}.
\]

Case 2. Let \( \Gamma = M, P_1 \) and \( \Delta = Q_1, N \), and let \( \Diamond \land (P_2, \neg Q_2) \) be the principal formula, where \( P = P_2, P_3 \) and \( Q = Q_3, Q_2 \).

\[
\frac{M, P_2; P_1, P_3 \models Q_3, Q_1; Q_2, N}{M, P_1; P_2, P_3 \models Q_3, Q_1; Q_2, N} \quad \text{L\Diamond}.
\]

In this case, by induction hypothesis on the premise we are done. \( \square \)

The following corollary which is a special case of Lemma 4.5 and is used in its proof can be concluded from Lemma 4.7.

**Corollary 4.8.** The following rules are admissible

\[
\begin{align*}
(i) & \quad \Diamond (p_1 \land \cdots \land p_n \land \neg q_1 \land \cdots \land \neg q_m), \Gamma \models \Delta \\
& \quad \frac{p_1, \ldots, p_n, \Gamma \models \Delta, q_1, \ldots, q_m}{p_1, \ldots, p_n, \Gamma \models \Delta, q_1, \ldots, q_m} \\
(ii) & \quad \Box (p_1 \lor \cdots \lor p_n \lor \neg q_1 \lor \cdots \lor \neg q_m), \Gamma \models \Delta, q_1, \ldots, q_m \\
& \quad \frac{\Gamma \models \Delta, \Box (p_1 \lor \cdots \lor p_n \lor \neg q_1 \lor \cdots \lor \neg q_m)}{\Gamma \models \Delta, \Box (p_1 \lor \cdots \lor p_n \lor \neg q_1 \lor \cdots \lor \neg q_m)}
\end{align*}
\]

**Lemma 4.9.**

(i) If \( B_1, \ldots, B_k, p_1, \ldots, p_n, \Gamma \models \Delta, q_1, \ldots, q_m, C_1, \ldots, C_l \), for each conjunction

\[
(p_1 \land \cdots \land p_n \land \neg q_1 \land \cdots \land \neg q_m \land B_1 \land \cdots \land B_k \land \neg C_1 \land \cdots \land \neg C_l)
\]

in the DNF of \( A \), then \( A, \Gamma \models \Delta \).
(ii) If \( C_1, \ldots, C_l, q_1, \ldots, q_m, \Gamma \vdash \Delta, B_1, \ldots, B_k, p_1, \ldots, p_n \), for each disjunction
\[
(p_1 \lor \cdots \lor p_n \lor \neg q_1 \lor \cdots \lor \neg q_m \lor B_1 \lor \cdots \lor B_k \lor \neg C_1 \lor \cdots \lor \neg C_l)
\]

in the CNF of \( A \), then \( \Gamma \vdash \Delta, A \).

We now deduce Lemma 4.5 from Lemma 4.6, Corollary 4.8 and Lemma 4.9.

**Proof of Lemma 4.5.** For the proof of part (I), we have
\[
\diamond A, \Gamma \vdash \Delta
\]

for every conjunction \( (p_1 \land \cdots \land p_n \land \neg q_1 \land \cdots \land \neg q_m), \Gamma \vdash \Delta, C_1, \ldots, C_l \)

Similarly, for the proof of part (II), we have
\[
\Gamma \vdash \Delta, \Box A
\]

for every disjunction \( (p_1 \lor \cdots \lor p_n \lor \neg q_1 \lor \cdots \lor \neg q_m \lor B_1 \lor \cdots \lor B_k \lor \neg C_1 \lor \cdots \lor \neg C_l) \) in the CNF of \( A \). Again we are done by applying Lemma 4.9.

**Corollary 4.10.** The following rule is admissible.
\[
\frac{\diamond \Gamma_1, \Gamma; P \vdash Q; \Delta, \Box \Delta_1}{\Gamma_1, \Gamma; P \vdash Q; \Delta, \Delta_1}
\]

where \( \Gamma_1 \) and \( \Delta_1 \) are multisets of arbitrary formulas.

In Lemma 4.5 and Corollary 4.10, the deduction of the conclusion sequent is produced by permutations of the rules in the deduction of the premise sequent. The following example is provided to show this permutation.

**Example 4.11.** Let \( \mathcal{D} \) be a derivation of \( M, P, \diamond \diamond r, \diamond (p \rightarrow \Box \diamond q) \vdash \Box (\diamond s \lor t), Q, N \). We get a derivation \( \mathcal{D}' \) for
\[
M, P, \diamond r, \diamond (p \rightarrow \Box \diamond q) \vdash \diamond s \lor t, Q, N.
\]

Suppose \( \mathcal{D} \) is as

\[
\begin{align*}
\mathcal{D}_1 & \quad \mathcal{D}_2 \\
M, \diamond \diamond r; P \vdash Q, p; \Box (\diamond s \lor t), N & \quad M, \diamond q, \diamond \diamond r; P \vdash Q; \Box (\diamond s \lor t), N \quad \text{R\Box} \\
M, \diamond \diamond r; P \vdash Q; \Box (\diamond s \lor t), p, N & \quad M, \Box \diamond q, \diamond \diamond r; P \vdash Q; \Box (\diamond s \lor t), N \quad \text{L\Box} \\
M, \diamond \diamond r, p \rightarrow \Box \diamond q; P \vdash Q; \Box (\diamond s \lor t), N & \quad M, P, \diamond \diamond r, \diamond (p \rightarrow \Box \diamond q) \vdash \Box (\diamond s \lor t), Q, N \quad \text{L\Box, W}
\end{align*}
\]

where \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) are respectively as follows:
Thus by permutation of the rules L→ and the rules L◊ and R∨ we get \( D' \) as

\[
\begin{align*}
\text{D}_{11} & : M, \Diamond r; P \vdash Q; \Diamond s, p, N \\
\text{D}_{12} & : M, \Diamond r; P \vdash Q; t, p, N
\end{align*}
\]

and

\[
\begin{align*}
\text{D}_{21} & : M, \Diamond q, \Diamond r; P \vdash Q; \Diamond s, N \\
\text{D}_{22} & : M, \Diamond q, \Diamond r; P \vdash Q; t, N
\end{align*}
\]

and then by applying the rule R∨ to derive \( M, P, \Diamond (p \to \Diamond q) \vdash \Diamond s \lor t, Q, N \).

We now prove the admissibility of the contraction rule that is required for the proof of the admissibility of the cut rule (Theorem 5.1).

**Lemma 4.12.** The rules of contraction,

\[
\begin{align*}
\Gamma; p, p, P & \vdash n, Q; \Delta \quad \text{LC}^c \\
\Gamma; P & \vdash n, Q; \Delta \\
A, A, \Gamma; P & \vdash Q; \Delta \\
A, \Gamma; P & \vdash Q; \Delta \\
\Gamma; P & \vdash Q, A, A \\
\Gamma; P & \vdash Q; \Delta, A \\
\end{align*}
\]

are admissible, where \( A \) is an arbitrary formula, \( p \) and \( q \) are atomic formulas, and both \( p \)'s in the rule LC\(^c\) as well as both \( q \)'s in the rule RC\(^c\) have the same corresponding formulas \( \Diamond \bigwedge X_i \) or \( \Box \bigvee Y_j \) as in Subsection 3.1.

**Proof.** All cases are proved simultaneously by induction on complexity of \( A \) with subinduction on the height of derivation of the premises. We only consider some cases that \( p \) is in the principal formula or \( A \) is a principal formula, the other cases are proved by a similar argument. In all cases, if the premise is an axiom, then the conclusion is an axiom too.

For the rule LC\(^c\), let \( \Gamma = M_1, P_1 \) and \( \Delta = Q_1, N_1 \), and let \( P = P_2, P_3 \) and \( Q = Q_3, Q_2 \). If the last rule is
where \( R \) is \( \Box \) or \( \lozenge \) with principal formula \( \Box (p, p, P, Q_2) \) or \( \Box (\neg p, \neg p, P, Q_2) \), then the conclusion is obtained by applying induction hypothesis to the middle part which preserves height then to the first part, and then by applying the rule \( R \) on the formula \( \Box (p, P, Q_2) \) or \( \Box (\neg p, \neg p, P, Q_2) \):

\[
\frac{\mathcal{D}}{M_1, p, p, P, P_3 \vdash Q_3, Q_1; Q_2, N_1} \quad R,
\]

For the rule LC, let \( \mathcal{D} \) be a derivation of \( A, A, \Gamma; P \vdash Q; \Delta \), and let \( \Gamma = M_1, P_1 \) and \( \Delta = Q_1, N_1 \). There are some cases according to the complexity of \( A \).

Case 1. \( A = \Box B \) and the last rule is \( \Box \):

\[
\frac{\mathcal{D}'}{B, \Box B, M_1; P_1, P \vdash Q, Q_1; N_1} \quad \Box,
\]

we have

\[
\frac{\mathcal{D}'}{B, \Box B, M_1; P_1, P \vdash Q, Q_1; N_1} \quad \text{IH}
\]

\[
\frac{\mathcal{D}'}{B, \Box B, M_1; P_1, P \vdash Q, Q_1; N_1} \quad \text{IH}
\]

Case 2. \( A = \Box B \) and the last rule is \( \Box \):

\[
\frac{\mathcal{D}'}{B, \Box B, \Box B, \Gamma; P \vdash Q; \Delta} \quad \Box,
\]

we have

\[
\frac{\mathcal{D}'}{B, \Box B, \Box B, \Gamma; P \vdash Q; \Delta} \quad \text{IH}
\]

Case 3. \( A = B \rightarrow C \) and the last rule is \( \rightarrow \):

\[
\frac{\mathcal{D}_1}{B \rightarrow C, \Gamma; P \vdash Q; \Delta, B} \quad \frac{\mathcal{D}_2}{C, B \rightarrow C, \Gamma; P \vdash Q; \Delta} \quad \rightarrow,
\]

By the inversion lemma applied to the first premise, \( \Gamma; P \vdash Q; \Delta, B, B \), and applied to the second premise, \( C, C, \Gamma; P \vdash Q; \Delta \). We then use the induction hypothesis and obtain \( \Gamma; P \vdash Q; \Delta, B, A \) and \( C, \Gamma; P \vdash Q; \Delta \). Thus by \( \rightarrow \), we have \( B \rightarrow C, \Gamma; P \vdash Q; \Delta \).

The other cases are proved by a similar argument.
In the rest of this section, we prove the admissibility of the general versions of the rules $R\Box$ and $L\Box$ that is also required for the proof of the admissibility of the cut rule.

**Lemma 4.13.** The general versions of the rules $R\Box$ and $L\Box$,
\[ M; \Gamma, P \vdash Q, \Delta; N, A \quad R\Box^G \]
\[ A, M; \Gamma, P \vdash Q, \Delta; N \quad L\Box^G \]
are admissible, where $\Gamma$ and $\Delta$ are multisets of arbitrary formulas.

Before we prove this lemma, we need to state the following lemmas.

**Lemma 4.14.** The following rules are admissible, where the formulas $A \circ B$, $\circ \in \{\land, \lor, \to\}$, in the premises, and $A$ and $B$ in the conclusions of the rules have the same related formulas.

\begin{align*}
(1) & \quad \frac{\Gamma; A \land B, P \vdash Q; \Delta}{\Gamma; A, B, P \vdash Q; \Delta} \\
(2) & \quad \frac{\Gamma; P \vdash Q, A \lor B; \Delta}{\Gamma; P \vdash Q, A, B; \Delta} \\
(3) & \quad \frac{\Gamma; A \lor B, P \vdash Q; \Delta}{\Gamma; A, P \vdash Q; \Delta} \\
(4) & \quad \frac{\Gamma; P \vdash Q, A \land B; \Delta}{\Gamma; P \vdash Q, A; \Delta} \\
(5) & \quad \frac{\Gamma; A \lor B, P \vdash Q; \Delta}{\Gamma; B, P \vdash Q; \Delta} \\
(6) & \quad \frac{\Gamma; P \vdash Q, A \land B; \Delta}{\Gamma; P \vdash Q, B; \Delta} \\
(7) & \quad \frac{\Gamma; A \rightarrow B, P \vdash Q; \Delta}{\Gamma; B, P \vdash Q; \Delta} \\
(8) & \quad \frac{\Gamma; P \vdash Q, A \rightarrow B; \Delta}{\Gamma; A, P \vdash Q, B; \Delta} \\
(9) & \quad \frac{\Gamma; A \rightarrow B, P \vdash Q; \Delta}{\Gamma; P \vdash Q, A; \Delta} \\
(10) & \quad \frac{\Gamma; P \vdash Q, \neg A; \Delta}{\Gamma; A, P \vdash Q; \Delta} \\
(11) & \quad \frac{\Gamma; \neg A, P \vdash Q; \Delta}{\Gamma; P \vdash Q, A; \Delta}
\end{align*}

The following lemma states that, the rules of the above lemma are invertible.

**Lemma 4.15.** The following rules are admissible, the formulas $A$ and $B$ in the premises, and $A \circ B$, $\circ \in \{\land, \lor, \to\}$, in the conclusions of the rules have the same related formulas.

\begin{align*}
(1) & \quad \frac{\Gamma; A, B, \Gamma' \vdash \Delta'; \Delta}{\Gamma; A \land B, \Gamma' \vdash \Delta'; \Delta} \\
(2) & \quad \frac{\Gamma; \Gamma' \vdash \Delta', A; \Delta}{\Gamma; \Gamma' \vdash \Delta', A \land B; \Delta} \\
(3) & \quad \frac{\Gamma; A, \Gamma' \vdash \Delta'; \Delta}{\Gamma; A \lor B, \Gamma' \vdash \Delta'; \Delta} \\
(4) & \quad \frac{\Gamma; \Gamma' \vdash \Delta', A, B; \Delta}{\Gamma; \Gamma' \vdash \Delta', A \lor B; \Delta} \\
(5) & \quad \frac{\Gamma; A, \Gamma' \vdash \Delta'; \Delta}{\Gamma; A \rightarrow B, \Gamma' \vdash \Delta'; \Delta} \\
(6) & \quad \frac{\Gamma; A, \Gamma' \vdash \Delta', B; \Delta}{\Gamma; \Gamma' \vdash \Delta', A \rightarrow B; \Delta} \\
(7) & \quad \frac{\Gamma; \Gamma' \vdash \Delta', A; \Delta}{\Gamma; \neg A, \Gamma' \vdash \Delta'; \Delta} \\
(8) & \quad \frac{\Gamma; \Gamma' \vdash \Delta', \neg A; \Delta}{\Gamma; \Gamma' \vdash \Delta', \neg A; \Delta}
\end{align*}

**Proof.** The proof is by induction on the height of derivation in each case. \qed

**Lemma 4.16.** The following rules are admissible.

\begin{align*}
(1) & \quad \frac{\Gamma; \Box A; P \vdash Q; \Delta}{\Gamma; A, P \vdash Q; \Delta} \\
(2) & \quad \frac{\Gamma; P \vdash Q; \Box A, \Delta}{\Gamma; P \vdash Q, A; \Delta} \\
(3) & \quad \frac{\Gamma; \Box A, P \vdash Q; \Delta}{\Gamma; \Box A, P \vdash Q; \Delta} \\
(4) & \quad \frac{\Gamma; P \vdash Q; \Box A, \Delta}{\Gamma; P \vdash Q, \Box A, \Delta} \\
(5) & \quad \frac{\Gamma; \Box A, P \vdash Q; \Delta}{\Gamma; \Box A, P \vdash Q; \Delta} \\
(6) & \quad \frac{\Gamma; P \vdash Q; \Box A, \Delta}{\Gamma; P \vdash Q, \Box A, \Delta}
\end{align*}
Actually Lemma 4.4 is an expression of Lemmas 4.14 and 4.16 in the original notation, and Lemma 4.3 is its correspondence without modals.

We now deduce Lemma 4.13 from Lemmas 4.14 and 4.16.

**Proof of Lemma 4.13.** The proof is by induction on complexity of the formulas in $\Gamma$ and $\Delta$.

We prove the admissibility of the rule $R \Box$, the rule $L \Diamond$ is treated symmetrically.

Case 1. Let $\Gamma = \Gamma_1, B \land C$. We have

\[
\begin{align*}
M; \Gamma_1, B \land C, P \vdash Q, \Delta; N, A & \quad \text{Lemma 4.14} \\
M; \Gamma_1, B, C, P \vdash Q, \Delta; N, A & \quad \text{IH} \\
M; \Gamma_1, B \land C; P \vdash Q; \Delta, N, \Box A & \quad \text{L} \land.
\end{align*}
\]

Case 2. Let $\Gamma = \Gamma_1, B \lor C$. We have

\[
\begin{align*}
M; \Gamma_1, B \lor C, P \vdash Q, \Delta; N, A & \quad \text{4.14} \\
M; \Gamma_1, B, P \vdash Q; \Delta, N, \Box A & \quad \text{IH, 4.14} \\
M; \Gamma_1, C, P \vdash Q; \Delta, N, \Box A & \quad \text{L} \lor.
\end{align*}
\]

Case 3. Let $\Gamma = \Gamma_1, B \rightarrow C$. We have

\[
\begin{align*}
M; \Gamma_1, C, P \vdash Q, \Delta; N, A & \quad \text{4.14} \\
M; \Gamma_1, B \rightarrow C, P \vdash Q, \Delta; N, A & \quad \text{IH, 4.14} \\
M; \Gamma_1, B \rightarrow C, \Gamma_1 \vdash Q; \Delta, N, \Box A & \quad \text{L} \rightarrow.
\end{align*}
\]

Case 4. Let $\Gamma = \Gamma_1, \Diamond B$. We have

\[
\begin{align*}
M; \Gamma_1, \Diamond B, P \vdash Q, \Delta; N, A & \quad \text{Lemma 4.16} \\
M, \Diamond B; \Gamma_1, P \vdash Q; \Delta, N, A & \quad \text{IH} \\
M, \Diamond B, \Gamma_1; P \vdash Q; \Delta, N, A & \quad \text{IH}.
\end{align*}
\]

The other cases are proved similarly.

In the above lemma, similar to Example 4.11 the deduction of the conclusion sequent is produced by permutations of the rules in the deduction of the premise sequent. The following example is provided to show this permutation.

**Example 4.17.** We show that the following rule is admissible.

\[
\begin{align*}
M, \Diamond ((r \land \Box s) \land \neg(r \rightarrow s)) & \vdash p \land (q \lor \Box s), N & \text{R} \Box^G \\
M, r \land \Box s & \vdash \Box r \rightarrow s, \Box(p \land (q \lor \Box s)), N
\end{align*}
\]

where the premise is derived by $D$ as follows:
Corollary 4.18. The following rules are admissible.

\[ D_1 \]
\[
M, □s, □r; r \vdash s; p, N \quad R□;
\]
\[
M, r, □s, □r \vdash p; s, N \quad R\land
\]
\[
M, r \land □s, □r \vdash p; s, N \quad L\land
\]
\[
M, (r \land □s) \land (¬(□r \to s)) \vdash p; N \quad L\lor
\]

\[ D_2 \]
\[
M, □s, □r; r \vdash s; q, □s, N \quad R□;
\]
\[
M, r, □s, □r \vdash q; s, □s, N \quad R\to
\]
\[
M, r \land □s, (¬(□r \to s)) \vdash q; s, N \quad L\land
\]
\[
M, □((r \land □s) \land (¬(□r \to s))) \vdash q \lor □s, N \quad R\lor
\]

Therefore by permutation of the rules we get the following derivation \( D' \) for the conclusion

\[ D_1 \]
\[
M, □s, □r; r \vdash s; p, N \quad R\lor
\]
\[
M, □s, □r; r \vdash s; q \lor □s, N \quad R\land
\]
\[
M, □s, □r \vdash s; p \land (q \lor □s), N \quad R□;
\]
\[
M, □s, □r \vdash s, □(p \land (q \lor □s)), N \quad R\to
\]
\[
M, r \land □s \vdash □r \to s, □(p \land (q \lor □s)), N \quad L\land
\]

Corollary 4.18. The following rules are admissible.

\[ M, \Gamma_2; \Gamma_1, \Gamma_3 \models △_3, △_1; △_2, N \quad R\land̂;
\]
\[ M, □(\Gamma_2, ¬△_2) \]

where \( ▩ \land (\Gamma_2, ¬△_2) \) is the principle formula in \( R□̂ \) and \( ▩ \lor (¬\Gamma_2, △_2) \) is the principle formula in \( L□̂ \).

5 Admissibility of the Cut rule

In this section, we prove the admissibility of the cut rule and the completeness theorem.

The admissibility of the cut rule,

\[ \frac{Γ \models △, D}{Γ, Γ' \models △, △'} \text{Cut}, \]

is proved simultaneously with the following rule

\[ M, □(Γ_2, ¬△_2, ¬D) \]

where \( ▩ \land (Γ_2, ¬△_2, ¬D) \) is in the antecedent or \( ▩ \lor (¬Γ_2, D, △_2) \) is in the succedent in the left premise, and \( ▩ \land (D, Γ'_2, ¬△'_2) \) is in the antecedent or \( ▩ \lor (¬D, ¬Γ_2, △_2) \) in the right premise.

Note the permutation of \( Γ_1, Γ_2, Γ'_1 \), of \( Γ'_1, Γ'_2 \), of \( △_1, △_2 \) and of \( △'_1, △'_2 \) in the rule.

We can consider the rule Cut as a new version of the cut rule, where its admissibility is proved by admissibility of the cut rule as follows:
in which the formulas $\Diamond \land (\Gamma_1, \neg \Delta_1)$ and $\Diamond \land (\Gamma_1', \neg \Delta_1')$ are principal in the general version of the rules $R \Box$, respectively. Similarly, the admissibility of the rule $\text{Cut}^+$ concludes the admissibility of the rule cut as follows

$$\frac{\Gamma; \vdash \Delta, D}{\Gamma; \vdash \Delta, D} \quad \frac{\Gamma'; \vdash \Delta', D'}{\Gamma'; \vdash \Delta', D'}$$

**Theorem 5.1.** The rules

$$\frac{\Gamma; P \vdash Q; \Delta, D}{\Gamma, \Gamma'; P, P' \vdash Q, Q'; \Delta, \Delta'}$$

and

$$\frac{M, \Gamma_1; P_2, P_3 \vdash Q_3, Q_2, D; \Delta_1, N}{M', \Gamma_1'; P_2', P_3' \vdash Q_3', Q_2', \Delta_1', N'}$$

are admissible, where all formulas in the middle parts of the premises, specially the formula $D$ in the second cut, are atomic.

**Proof.** The both rules are proved simultaneously by a main induction on the complexity of the cut formula $D$ with a subinduction on the sum of heights of derivations of the two premises (cut-height). We prove the first cut rule, the second is proved similarly except some cases that we consider at the end of the proof.

If both of the premises are axioms, then the conclusion is an axiom, and if only one of the premises is an axiom, then the conclusion is obtained by weakening (see Lemma 4.2).

If one of the last rules in derivations of the premises is neither $L \Box$ nor $R \Box$, then the cut will be transformed to simpler cuts as usual. Note that cut-height can increase in the transformation, but the cut formula is reduced. For example, let $D = \Diamond A$ be the principal formula in the left premise.

$$\frac{\mathcal{D}_1}{\Gamma; P \vdash Q; \Delta, \Diamond A, A} \quad \frac{\mathcal{D}_2}{R \Box \Diamond A, \Gamma'; P' \vdash Q', \Delta'}$$

This cut is transformed into

$$\frac{\mathcal{D}_1}{\Gamma, \Gamma'; P, P' \vdash Q, Q'; \Delta, \Delta', A} \quad \frac{\mathcal{D}_2}{\Diamond A, \Gamma', P' \vdash Q', \Delta'}$$

**Lemma 4.5**

$$\frac{\mathcal{D}_1}{\Gamma, \Gamma'; P, P' \vdash Q, Q'; \Delta, \Delta'} \quad \frac{\mathcal{D}_2}{\Diamond A, \Gamma', P' \vdash Q', \Delta'}$$

$$\frac{\mathcal{D}_1}{\Gamma, \Gamma', P, P' \vdash Q, Q'; \Delta, \Delta'} \quad \frac{\mathcal{D}_2}{\Diamond A, \Gamma', P' \vdash Q', \Delta'}$$

$$\frac{\mathcal{D}_1}{\Gamma, \Gamma', P, P' \vdash Q, Q'; \Delta, \Delta'} \quad \frac{\mathcal{D}_2}{\Diamond A, \Gamma', P' \vdash Q', \Delta'}$$

$$\frac{\mathcal{D}_1}{\Gamma, \Gamma', P, P' \vdash Q, Q'; \Delta, \Delta'} \quad \frac{\mathcal{D}_2}{\Diamond A, \Gamma', P' \vdash Q', \Delta'}$$
where C is used for contraction rules.

Let $D = □A$ be the principal formula in the right premise.

\[
\begin{array}{c}
D_1 & D_2 \\
\Gamma; P \vdash Q; \Delta, □A & A, □A, \Gamma'; P' \vdash Q'; \Delta' \\
\hline
\Gamma, \Gamma'; P, P' \vdash Q, Q'; \Delta, \Delta' & □A, \Gamma'; P' \vdash Q'; \Delta' \\
\text{Cut,}
\end{array}
\]

It is transformed into

\[
\begin{array}{c}
D_1 & \text{Lemma 4.5} & D_1 & D_2 \\
\Gamma; P \vdash Q; \Delta, □A & A, □A, \Gamma'; P' \vdash Q'; \Delta' & A, □A, \Gamma'; P' \vdash Q'; \Delta' & \text{Cut} \\
\hline
\Gamma, \Gamma'; P, P' \vdash Q, Q'; \Delta, \Delta' & □A, \Gamma'; P' \vdash Q'; \Delta' & \text{Cut} \\
\end{array}
\]

If $D = A \land B$, $D = A \lor B$, $D = A \rightarrow B$, or $D = \neg A$, then by Lemma 4.1, we use simpler cuts on $A$ and $B$. Therefore, in the rest of the proof, we just consider the cases in which the last rules are L◊ or R□.

Case 1. The cut formula $D$ is modal.

Subcase 1.1. Let $\Delta = Q_1, N, □A$ and $\Delta' = Q'_1, N', □B$, and let the last rules be R□ with principal formulas □A and □B.

\[
\begin{array}{c}
D_1 & \text{R□} & D_2 \\
M; P_1, P \vdash Q; Q_1; N, A, D & D, M', P'_1, P' \vdash Q'; Q'_1; N', B & \text{R□} \\
\hline
M, P_1, M', P'_1; P, P' \vdash Q, Q'; Q_1, N, Q'_1, N', □A, □B & \text{Cut,}
\end{array}
\]

where $\Gamma = M, P_1$ and $\Gamma' = M', P'_1$. This cut is transformed into

\[
\begin{array}{c}
D_1 & \text{R□} & D_2 \\
M; P_1, P \vdash Q; Q_1; N, A, D & D, M', P'_1, P' \vdash Q'; Q'_1; N', B & \text{R□} \\
\hline
M, M'; P_1, P, P'_1; P' \vdash Q, Q'; Q_1, Q'_1; N, N', A, □B & \text{Cut} \\
M, P_1, M', P'_1; P, P' \vdash Q, Q'; Q_1, N, Q'_1, N', □A, □B & \text{R□.}
\end{array}
\]

Subcase 1.2. Let $\Delta = ◊A, M, P_1$ and $\Delta' = Q'_1, N', □B$, and let the last rules be L◊ and R□ with principal formulas ◊A and □B:

\[
\begin{array}{c}
D_1 & D_2 \\
\text{L◊} & \text{R□} \\
\hline
\Diamond A, M, P_1, P \vdash Q; Q_1; N, D & D, M', P'_1, P' \vdash Q'; Q'_1; N', B & \text{R□} \\
\Diamond A, M, P_1, M', P'_1; P, P' \vdash Q, Q'; Q_1, Q'_1; N, N', □B & \text{Cut,}
\end{array}
\]

where $\Delta = Q_1, N$ and $\Gamma' = M', P'_1$. This cut is transformed into

\[
\begin{array}{c}
D_1 & \text{R□} & D_2 \\
\text{L◊} & \text{R□} \\
\hline
\Diamond A, M, P_1, P \vdash Q; Q_1; N, D & D, M', P'_1, P' \vdash Q'; Q'_1; N'; B & \text{R□} \\
\Diamond A, M, M', P_1, P, P' \vdash Q, Q'; Q_1, Q'_1; N, N', □B & \text{Cut}
\end{array}
\]

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Subcase 1.3. Let \( \Gamma = \Diamond A, M, P_1 \) and \( \Gamma' = \Diamond B, M', P'_1 \), and let the last rules be \( \text{L}\Diamond \) with principal formulas \( \Diamond A \) and \( \Diamond B \):

\[
\begin{align*}
\mathcal{D}_1 & \quad A, M; P_1, P \models Q, Q_1; N, D & \quad \mathcal{D}_2 & \quad D, B, M'; P'_1, P' \models Q', Q'_1; N' \\
\Diamond A, M, P_1; P \models Q; Q_1, N, D & \quad \text{L}\Diamond & \quad D, B, M'; P'_1, P' \models Q', Q'_1; N' & \quad \text{Cut,}
\end{align*}
\]

where \( \Delta = N, Q_1 \) and \( \Delta' = N', Q'_1 \). This cut is transformed into

\[
\begin{align*}
\mathcal{D}_1 & \quad A, M; P_1, P \models Q, Q_1; N, D & \quad \mathcal{D}_2 & \quad D, B, M'; P'_1, P' \models Q', Q'_1; N' \\
\Diamond A, M, P_1; P \models Q; Q_1, N, D & \quad \text{L}\Diamond & \quad D, B, M'; P'_1, P' \models Q', Q'_1; N' & \quad \text{Cut,}
\end{align*}
\]

Subcase 1.4. Let \( \Gamma' = \Diamond B, M', P'_1 \) and \( \Delta = Q_1, N, \Box A \), and let the last rules be \( \text{R}\Box \) and \( \text{L}\Diamond \) with principal formulas \( \Box A \) and \( \Diamond B \):

\[
\begin{align*}
\mathcal{D}_1 & \quad M; P_1, P \models Q, Q_1; N, A, D & \quad \mathcal{D}_2 & \quad D, B, M'; P'_1, P' \models Q', Q'_1; N' \\
M, P_1, P \models Q; Q_1, N, \Box A, D & \quad \text{R}\Box & \quad D, B, M'; P'_1, P' \models Q', Q'_1; N' & \quad \text{L}\Diamond & \quad \text{Cut,}
\end{align*}
\]

where \( \Gamma = M, P_1 \) and \( \Delta' = N', Q'_1 \). This cut is transformed into

\[
\begin{align*}
\mathcal{D}_1 & \quad M; P_1, P \models Q, Q_1; N, A, D & \quad \mathcal{D}_2 & \quad D, B, M'; P'_1, P' \models Q', Q'_1; N' \\
M, \Diamond B, M'; P_1, P, P' \models Q, Q_1, Q', Q'_1; N, A, N' & \quad \text{R}\Box & \quad D, \Diamond B, M'; P'_1, P' \models Q', Q'_1; N' & \quad \text{L}\Diamond & \quad \text{Cut}
\end{align*}
\]

Case 2. The cut formula \( D \) is atomic. In this case we only consider the case that when last rules in derivations of the premises are \( \text{R}\Box \) and \( \text{L}\Diamond \); the other cases are proved by similar argument.

Let \( \Gamma' = \Diamond B, M', P'_1 \) and \( \Delta = Q_1, N, \Box A \), and let the last rules in derivations of the premises be \( \text{L}\Diamond \) and \( \text{R}\Box \) with principal formulas \( \Diamond B \) and \( \Box A \):

\[
\begin{align*}
\mathcal{D}_1 & \quad M; P_1, P \models Q, D, Q_1; N, A & \quad \mathcal{D}_2 & \quad B, M'; P'_1, D, P' \models Q', Q'_1; N' \\
M, P_1, P \models Q; Q_1, N, \Box A, D & \quad \text{R}\Box & \quad B, M', P'_1, D, P' \models Q', Q'_1; N' & \quad \text{L}\Diamond & \quad \text{Cut}
\end{align*}
\]

This is transformed into

\[
\begin{align*}
\mathcal{D}_1 & \quad M; P_1, P \models Q, D, Q_1; N, A & \quad \mathcal{D}_2 & \quad B, M'; P'_1, D, P' \models Q', Q'_1; N' \\
M, P_1, M', P'_1, B, P, P' \models Q, Q'; A, Q_1, N, Q'_1, N' & \quad \text{Cut},
\end{align*}
\]
Finally, for the second rule, we consider some cases.

Case 1. If one of the last rule in derivation of the premises are not modal rules, then the cut is transformed into simpler cut(s) and then by applying Lemma 4.14, the conclusion is obtained.

In the following we consider some cases.

Subcase 1.1. Let $\Gamma' = \Gamma_1' \land, A \land B$, and let $A \land B$ be the principal formula in the right premise.

\[
\frac{D_1}{D_2}
\]

\[
\begin{array}{c}
D_1 \\
M, \Gamma_1; P_2, P_3 \vdash Q_3, Q_2, D; \Delta_1, N \\
\end{array}
\Rightarrow
\begin{array}{c}
D_2 \\
M', \Gamma_1', A, B; D, P_2', P_3' \vdash Q_3', Q_2'; \Delta_1', N' \\
\end{array}
\]

This cut is transformed into

\[
\frac{D_1}{D_2}
\]

\[
\begin{array}{c}
D_1 \\
M, \Gamma_1; P_2, P_3 \vdash Q_3, Q_2, D; \Delta_1, N \\
\end{array}
\Rightarrow
\begin{array}{c}
D_2 \\
M, M', \Gamma_1', A, B; D, P_2', P_3' \vdash Q_3, Q_3', \Delta_1, \Delta_1'; Q_2, Q_2', N, N' \\
\end{array}
\]

Subcase 1.2. Let $\Gamma' = \Gamma_1' \lor, A \lor B$, and let $A \lor B$ be the principal formula.

\[
\frac{D_1}{D_2}
\]

\[
\begin{array}{c}
D_1 \\
M, \Gamma_1; P_2, P_3 \vdash Q_3, Q_2, D; \Delta_1, N \\
\end{array}
\Rightarrow
\begin{array}{c}
D_2 \\
M, M', \Gamma_1', A \lor B; D, P_2', P_3' \vdash Q_3, Q_2, \Delta_1, N' \\
\end{array}
\]

where $D_2$ is as follows

\[
\frac{D_2}{D_2'}
\]

\[
\begin{array}{c}
D_2 \\
M', \Gamma_1'; A; D, P_2', P_3' \vdash Q_3, Q_2, \Delta_1, N' \\
\end{array}
\Rightarrow
\begin{array}{c}
D_2' \\
M', \Gamma_1', A \lor B; D, P_2', P_3' \vdash Q'_3, Q_2', \Delta_1, N' \\
\end{array}
\]

This cut is transformed into two cuts:

\[
\frac{D_1}{D_2}
\]

\[
\begin{array}{c}
D_1 \\
M, \Gamma_1; P_2, P_3 \vdash Q_3, Q_2, D; \Delta_1, N \\
\end{array}
\Rightarrow
\begin{array}{c}
D_2' \\
M, M'; P_2, P_3; \Gamma_1, \Gamma_1'; A, C, P_3' \vdash Q_3, Q_3', \Delta_1, \Delta_1'; Q_2, Q_2', N, N' \\
\end{array}
\]

and

\[
\frac{D_1}{D_2}
\]

\[
\begin{array}{c}
D_1 \\
M, \Gamma_1; P_2, P_3 \vdash Q_3, Q_2, D; \Delta_1, N \\
\end{array}
\Rightarrow
\begin{array}{c}
D_2' \\
M, M', P_2, P_3; \Gamma_1, \Gamma_1'; B, P_3' \vdash Q_3, Q_3', \Delta_1, \Delta_1'; Q_2, Q_2', N, N' \\
\end{array}
\]

Therefore by applying Lemma 4.15 the conclusion is obtained.

Subcase 1.3. Let $\Gamma' = \Gamma_1' \rightarrow, A \rightarrow B$, and let $A \rightarrow B$ be the principal formula.

\[
\frac{D_1}{D_2}
\]

\[
\begin{array}{c}
D_1 \\
M, \Gamma_1; P_2, P_3 \vdash Q_3, Q_2, D; \Delta_1, N \\
\end{array}
\Rightarrow
\begin{array}{c}
D_2 \\
M, M', \Gamma_1', A \rightarrow B; D, P_2', P_3' \vdash Q_3, Q_3', \Delta_1, \Delta_1'; Q_2, Q_2', N, N' \\
\end{array}
\]

where $D_2$ is as follows
This cut is transformed into two cuts:

\[ \begin{align*}
\mathcal{D}_1 & \quad M, \Gamma_1; P_2, P_3 \vdash Q_3, Q_2, D; \Delta_1, N' \\
\mathcal{D}_2 & \quad M', \Gamma_1'; D, P' \vdash Q_3, Q_2, D; \Delta_1', N'
\end{align*} \]

and

\[ \begin{align*}
\mathcal{D}_1 & \quad M, \Gamma_1; P_2, P_3 \vdash Q_3, Q_2, D; \Delta_1, N' \\
\mathcal{D}_2 & \quad M, \Gamma_1; P_2, P_3 \vdash Q_3, Q_2, D; \Delta_1, N'
\end{align*} \]

Therefore by applying Lemma 4.15 the conclusion is obtained.

Case 2. If the last rules in derivation of the premises are modal rules and in one of them, \( D \) does not occur in the principal formula, then the cut is transformed into a simpler cut. As a typical example, let the last rule in derivation of the premise be \( L \hat{\diamond} \) be as follows

\[ \begin{align*}
\mathcal{D}_2 & \quad M, \Gamma_1; P_2, P_3 \vdash Q_3, Q_2, D; \Delta_1, N' \\
\mathcal{D}_2 & \quad M', \Gamma_1'; D, P_2, P_3 \vdash Q_3, Q_2, D; \Delta_1, N'
\end{align*} \]

where \( M' = \hat{\diamond} B, M_1' \) and \( \Gamma_1' = P_1', \Delta_1' = Q_1' \) are atomic. This cut is transformed into

\[ \begin{align*}
\mathcal{D}_2 & \quad M, P_1; P_2, P_3 \vdash Q_3, Q_2, D; Q_1, N' \\
\mathcal{D}_2 & \quad M, P_1; P_2, P_3 \vdash Q_3, Q_2, D; Q_1, N'
\end{align*} \]

where the conclusion in the rule * is a rewriting of its premise.

Case 3. Let the last rules in derivations of the premises be \( L \hat{\diamond} \) or \( R \square \) in which \( D \) occurs in both principal formulas. For example let the following derivation with the principal formulas \( \hat{\diamond} \Lambda (p_2, \neg q_2, \neg d) \) and \( \square \forall (\neg d, \neg p_2', q_2') \), be respectively:

\[ \begin{align*}
\mathcal{D}_1 & \quad M, P_2; P_3 \vdash Q_3, Q_1, Q_2, D, N \\
\mathcal{D}_2 & \quad M', P_2; P_3 \vdash Q_3, Q_1, Q_2, D, N
\end{align*} \]

This cut rule is transformed into the first cut as follows:

\[ \begin{align*}
\mathcal{D}_1 & \quad M, P_2; P_3 \vdash Q_3, Q_1, Q_2, D, N \\
\mathcal{D}_2 & \quad M, P_2; P_3 \vdash Q_3, Q_1, Q_2, D, N
\end{align*} \]

The other cases are proved similarly.

**Theorem 5.2.** The following are equivalent.

\[ \begin{align*}
\text{Theorem 5.2.} & \quad \text{The following are equivalent.}
\end{align*} \]
(1) The sequent $\Gamma \vdash A$ is $S_5$-valid.

(2) $\Gamma \vdash_{S_5} A$.

(3) The sequent $\Gamma \vdash A$ is provable in $G_3 S_5$.

Proof. (1) implies (2) by completeness of $S_5$. (3) implies (1) by soundness of $G_3 S_5$. We show that (2) implies (3). Suppose $A_1, \ldots, A_n$ is an $S_5$-proof of $A$ from $\Gamma$. This means that $A_n$ is $A$ and that each $A_i$ is in $\Gamma$, is an axiom, or is inferred by modus ponens or necessitation. It is straightforward to prove, by induction on $i$, that $\Gamma \vdash A_i$ for each $A_i$.

Case 1. $A_i \in \Gamma$: It is a direct consequence of Lemma 3.2.

Case 2. $A_i$ is an axiom of $S_5$: All axioms of $S_5$ are easily proved in $G_3 S_5$. As a typical example, in the following we prove the axiom 5:

$$
\begin{array}{c}
\Diamond A \vdash \Diamond A \\
\Diamond A \vdash \Box \Diamond A \\
\vdash \Diamond A \rightarrow \Box \Diamond A
\end{array}
$$

Case 3. $A_i$ is inferred by modus ponens: Suppose $A_i$ is inferred from $A_j$ and $A_j \rightarrow A_i$, $j < i$, by use of the cut rule we prove $\Gamma \vdash A_i$:

$$
\begin{array}{c}
\Gamma \vdash A_j \rightarrow A_i \quad \text{IH} \\
\Gamma \vdash A_j \quad \text{IH} \\
A_j \vdash A_i \quad A_i \vdash A_i \quad \text{L} \rightarrow
\end{array}
\begin{array}{c}
A_j, \Gamma \vdash A_i \quad \text{Cut} \\
\Gamma, \Gamma \vdash A_i \quad \text{Cut} \\
\Gamma \vdash A_i \quad \text{LC}.
\end{array}
$$

Case 4. $A_i$ is inferred by necessitation: Suppose $A_i = \Box A_j$ is inferred from $A_j$ by necessitation. In this case, $\vdash_{S_5} A_j$ (since the rule necessitation can be applied only to premises which are derivable in the axiomatic system) and so we have:

$$
\begin{array}{c}
\vdash A_j \quad \text{IH} \\
\vdash \Box A_j \quad \text{R} \Box \\
\Gamma \vdash A_i \quad \text{W}.
\end{array}
$$

Corollary 5.3. $G_3 S_5$ is sound and complete with respect to the $S_5$ Kripke frames.

6 Conclusion

We have presented system $G_3 S_5$, a sequent calculus for $S_5$, this system does not have the subformula property although in a bottom-up proof, the formulas in the premises are constructed by atomic formulas in the conclusions. For convenience we have rewritten the rules of $G_3 S_5$ by using semicolon in system $G_3 S_5$, which enjoys the subformula property. Also we have proved the completeness theorem and admissibility of the weakening, contraction and cut rules in it.
All properties which are proved in G3sSi are also proved in G3sS because the system G3sSi is a rewritten of the system G3sS.

We can consider the system G3sSi primary, since the relatedness of formulas in the middle parts of the premises can be determined from theirs in the conclusion of the rules.

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