F-THRESHOLDS AND BERNSTEIN-SATO POLYNOMIALS

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INTRODUCTION

We introduce and study invariants of singularities in positive characteristic called F-thresholds. They give an analogue of the jumping coefficients of multiplier ideals in characteristic zero. Unlike these, however, the F-thresholds are not defined via resolution of singularities, but via the action of the Frobenius morphism.

We are especially interested in the connection between the invariants of an ideal \( a \) in characteristic zero and the invariants of the different reductions mod \( p \) of \( a \). Our main point is that this relation depends on arithmetic properties of \( p \). We present several examples, as well as some questions on this topic. In a slightly different direction, we describe a new connection between invariants mod \( p \) and the roots of the Bernstein-Sato polynomial.

We will restrict ourselves to the case of an ambient smooth variety, when our invariants have a down-to-earth description. Let \((R, \mathfrak{m})\) be a regular local ring of characteristic \( p > 0 \). We want to measure the singularities of a nonzero ideal \( a \subseteq \mathfrak{m} \). For every ideal \( J \subseteq \mathfrak{m} \) containing \( a \) in its radical, and for every \( e \geq 1 \), we put
\[
\nu_a^J(p^e) := \max\{r | a^r \nsubseteq J[p^e]\},
\]
where \( J[p^e] = (f^{p^e} | f \in J) \). One can check that the limit
\[
c^J(a) := \lim_{e \to \infty} \frac{\nu_a^J(p^e)}{p^e}
\]
exists and is finite. We call this limit the F-threshold of \( a \) with respect to \( J \). When \( J = \mathfrak{m} \), we simply write \( c(a) \) and \( \nu_a(p^e) \). The invariant \( c(a) \) was introduced in [TW] under the name of F-pure threshold.

In the first section we define these invariants and give their basic properties. The second section is devoted to the connection with the generalized test ideals introduced by Hara and Yoshida in [HY]. More precisely, we show that our invariants are the jumping coefficients for their test ideals. As it was shown in [HY] that the test ideals satisfy similar properties with the multiplier ideals in characteristic zero, it is not surprising that the F-thresholds behave in a similar way with the jumping coefficients of the multiplier ideals from [ELSV]. Such an analogy was also stressed in [TW], where it was shown that the smallest F-threshold \( c(a) \) behaves in the same way as the smallest jumping coefficient in characteristic zero (known as the log canonical threshold).
We point out that it is not known whether the analogue of two basic properties of jumping coefficients of multiplier ideals hold in our setting: whether \( c^f(a) \) is always a rational number and whether the set of all F-thresholds of \( a \) is discrete.

There are very interesting questions related to the invariants attached to different reductions mod \( p \) of a characteristic zero ideal \( a \). We discuss these in \( \S 3 \). For simplicity, we assume that \( a \) and \( J \) are ideals in \( \mathbb{Z}[X_1, \ldots, X_n] \), contained in \( (X_1, \ldots, X_n) \) and such that \( a \) is contained in the radical of \( J \). Let us denote by \( a_p \) and \( J_p \) the localizations at \( (X_1, \ldots, X_n) \) of the images of \( a \) and \( J \), respectively, in \( \mathbb{F}_p[X_1, \ldots, X_n] \). We want to compare our invariants mod \( p \) (which we write as \( \nu^J_d(p^e) \) and \( c^f(a_p) \)) with the characteristic zero invariants of \( a \) (more precisely, with the invariants around the origin of the image \( a_q \) of \( a \) in \( \mathbb{Q}[X_1, \ldots, X_n] \)).

First, let us denote by \( \text{lct}_0(a) \) the log canonical threshold of \( a_q \) around the origin. It follows from results of Hara and Watanabe (see [HW]) that if \( p \gg 0 \) then \( c(a_p) \leq \text{lct}_0(a) \) and \( \lim_{p \to \infty} c(a_p) = \text{lct}_0(a) \). Moreover, results of Hara and Yoshida from [HY] allow the extension of these formulas to higher jumping numbers (see Theorems 3.3 and 3.4 below for statements).

It is easy to give examples in which \( c(a_p) \neq \text{lct}_0(a) \) for infinitely many \( p \). On the other hand, one conjectures that there are infinitely many \( p \) with \( c(a_p) = \text{lct}_0(a) \). We give examples in which more is true: there is a positive integer \( N \) such that for \( p \equiv 1 \pmod{N} \) we have equality \( c(a_p) = \text{lct}_0(a) \). Moreover, in these examples one can find rational functions \( R_i \in \mathbb{Q}(t) \) associated to every \( i \in \{1, \ldots, N-1\} \) relatively prime to \( N \), such that \( c(a_p) = R_i(p) \) whenever \( p \gg 0 \) satisfies \( p \equiv i \pmod{N} \). It would be interesting to understand better when such a behavior holds. As the example of a cone over an elliptic curve without complex multiplication shows, this can’t hold in general. On the other hand, motivated by our examples one can speculate that the following holds: there is always a number field \( K \) such that whenever the prime \( p \) is large enough and completely split in \( K \), then \( c(a_p) = \text{lct}_0(a) \).

A surprising fact is that our invariants for \( a_p \) are related to the Bernstein-Sato polynomial \( b_{a,0}(s) \) of \( a \). More precisely, we show that for all \( p \gg 0 \) and for all \( e \), we have \( b_{a,0}(\nu^J_d(p^e)) \equiv 0 \pmod{p} \). We show on some examples in \( \S 4 \) how to use this to give roots of the Bernstein-Sato polynomial (and not just roots mod \( p \)).

In these examples we will see the following behavior: given some ideal \( J \) containing \( a \) in its radical, and \( e \geq 1 \), we can find \( N \) such that for all \( i \in \{1, \ldots, N-1\} \) relatively prime to \( N \) there are polynomials \( P_i \in \mathbb{Q}[t] \) of degree \( e \) satisfying \( \nu^J_d(p^e) = P_i(p) \) for all \( p \gg 0 \), with \( p \equiv i \pmod{N} \). The previous observation implies that \( b_{a,0}(P_i(0)) \) is divisible by \( p \) for every such \( p \). By Dirichlet’s Theorem we deduce that \( P_i(0) \) is a root of \( b_{a,0} \).

An interesting question is which roots can be obtained by the above method. It is shown in [BMS1] that for monomial ideals the functions \( p \to \nu^J_d(p^e) \) behave as described above, and moreover, all roots of the Bernstein-Sato polynomial are given by this procedure. On the other hand, Example 4.1 below shows that in some cases there are roots which can not be given by our method.
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1. F-thresholds

Let $(R, \mathfrak{m}, k)$ be a regular local ring of dimension $n$ and of characteristic $p > 0$. Since $R$ is regular, the Frobenius morphism $F : R \to R$, $F(x) = x^p$ is flat.

In what follows $q$ denotes a positive power of $p$, and if $I = (y_1, \ldots, y_s)$ is an ideal in $R$, then

$$I^{[q]} := (y^q | y \in I) = (y_1^q, \ldots, y_s^q).$$

We will use below the fact that as $R$ is regular, every ideal $I$ is equal with its tight closure (see, for example [HH]). This means that if $u, f \in R$ are such that $uf^q \in I^{[q]}$ for all $q \gg 0$, and if $u \neq 0$, then $f \in I$. This is easy to see: by the flatness of the Frobenius morphism we have $(I^{[q]} : f^q) = ((I : f)^{[q]})$. Therefore $u$ lies in $\bigcap_q (I : f)^{[q]}$, which is zero if $f$ is not in $I$.

Let $a$ be a fixed ideal of $R$, such that $(0) \neq a \subseteq \mathfrak{m}$. To each ideal $J$ of $R$ such that $a \subseteq \text{Rad}(J) \subseteq \mathfrak{m}$, we associate a threshold as follows. For every $q$, let

$$\nu^J_a(q) := \max \{r \in \mathbb{N} | a^r \not\subseteq J^{[q]} \}.$$

As $a \subseteq \text{Rad}(J)$, this is a nonnegative integer.

**Lemma 1.1.** For every $a$, $J$ and $q$ as above, we have $\nu^J_a(pq) \geq p \cdot \nu^J_a(q)$.

**Proof.** The inequality is a consequence of the fact that if $u \not\in J^{[q]}$, then $u^p \not\in J^{[pq]}$. □

It follows from the above lemma that

$$\lim_{q \to \infty} \frac{\nu^J_a(q)}{q} = \sup_q \frac{\nu^J_a(q)}{q}.$$

We call this limit the $F$-threshold of the pair $(R, a)$ (or simply of $a$) with respect to $J$, and we denote it by $c^J(a)$.

**Remark 1.2.** The above limit is finite. In fact, if $a$ is generated by $r$ elements, and if $a^N \subseteq J$, then

$$a^{N(r(p^e-1)+1)} \subseteq (a^{[p^e]})^N = (a^N)^{[p^e]} \subseteq J^{[p^e]}.$$

Therefore $\nu^J_a(p^e) \leq N(r(p^e - 1) + 1) - 1$. Dividing by $p^e$ and taking the limit gives $c^J(a) \leq Nr$.

We also have $c^J(a) > 0$. More precisely, as $a \neq (0)$, Krull’s Intersection Theorem shows that we can find $e$ such that $a \not\subseteq J^{[e]}$, so $c^J(a) \geq 1/p^e$.

We make the convention $c^R(a) = 0$.

**Example 1.3.** If $J$ is an ideal generated by a regular sequence $y_1, \ldots, y_r$ in $R$, then $\nu^J_a(q) = r(q-1)$ for all $q$. Therefore $c^J(J) = r$.

**Question 1.4.** Is it true that for all nonzero ideals $a$ and $J$ with $a \subseteq \text{Rad}(J) \subseteq \mathfrak{m}$, the $F$-threshold $c^J(a)$ is a rational number?
Remark 1.5. The F-pure threshold $c(a)$ was defined in [TW] (under the assumption that the Frobenius morphism $F$ on $R$ is finite) as the supremum of those $t \in \mathbb{Q}_+$ such that the pair $(R, a^t)$ is F-pure. Under this extra assumption on $F$, since $R$ is regular, the pair $(R, a^t)$ is F-pure if and only if for $q \gg 0$ we have $a^{t(q-1)} \not\subseteq m^{[q]}$ (see Lemma 3.9 in [1a]). Here we use the notation $[\alpha]$ for the largest integer $\leq \alpha$.

The above condition is equivalent with $\nu_a^m(q) \geq [t(q-1)]$ for $q \gg 0$. It follows from our definition that if $(R, a^t)$ is F-pure, then $t \leq c^m(a)$, and that if $t < c^m(a)$, then $(R, a^t)$ is F-pure. Therefore the F-pure threshold $c(a)$ is equal to the F-threshold $c^m(a)$ of $a$ with respect to the maximal ideal. We will keep the notation $c(a)$ for $c^m(a)$, and moreover, we will put $\nu_a(q) := \nu_a^m(q)$.

Note that the F-pure threshold was defined in [TW] without the regularity assumption on $R$, but in what follows we will work under this restrictive hypothesis.

Remark 1.6. In characteristic zero, the only analogue of $J^{[q]}$ which does not depend on the choice of generators for $J$ is the usual power $J^q$. If we imitate the definition of the F-pure threshold in this setting, replacing $m^{[q]}$ by $m^q$, then we get $1/\text{mult}_0(a)$, where $\text{mult}_0(a)$ is the largest power of $m$ containing $a$.

Here are a few properties of F-thresholds. When $J = m$ these have been proved in [TW] in a more general setting.

Proposition 1.7. Let $a, b, J \subseteq m$ be nonzero ideals, such that $a$ and $b$ are contained in the radical of $J$.

1. If $a \subseteq b$, then $c^J(a) \leq c^J(b)$.
2. $c^J(a^s) = \frac{c^J(a)}{s}$ for every positive integer $s$.
3. If $a \subseteq J$ and $J$ can be generated by $m$ elements, then $c^J(a) \leq m/s$. If $a \not\subseteq m^{s+1}$ and $i J \subseteq m^\ell$, then $c^J(a) \geq \ell/s$.
4. If $\overline{a}$ is the integral closure of $a$, then $c^J(a) = c^J(\overline{a})$.
5. For every $q$, we have $\frac{\nu_a(q)}{q} < c^J(a)$.
6. We have $c^J(a + b) \leq c^J(a) + c^J(b)$.

Proof. The first assertion is trivial: since $a \subseteq b$, we get $\nu_a^J(q) \leq \nu_b^J(q)$ for all $q$. Hence $c^J(a) \leq c^J(b)$.

Given $s$ and $q$, we have $(a^s)^r \not\subseteq J^{[q]}$ if and only if $rs \leq \nu_a^J(q)$. Hence $\nu_a^J(q) = \lfloor\nu_a^J(q)/s\rfloor$, which after dividing by $q$ and passing to limit gives (2).

If $a \subseteq J$, then by (1) and (2) we have $c^J(a) \leq c^J(J^s) = \frac{c^J(J)}{s} \leq \frac{m}{s}$. The last inequality follows from Remark 1.2.

Suppose now that $a \not\subseteq m^{s+1}$. If $c^J(a) < \ell/s$, then by taking $q$ large enough we can find $r$ such that

$$c^J(a) < \frac{r}{q} < \frac{\ell}{s}.$$
The first inequality shows that $a^r \subseteq J^{[q]} \subseteq m^{q\ell}$. As by hypothesis $a^r \not\subseteq m^{rs+1}$, we deduce $rs + 1 > q\ell$. This contradicts the second inequality in [2].

For (4) note first that $c^J(a) \leq c^J(\overline{a})$ follows from (1). For the reverse inequality, recall that by general properties of the integral closure, there is a fixed positive integer $s$ such that $\overline{a}^{s+1} \subseteq a^\ell$ for all $\ell$. Hence we have $\nu^J_a(q) \geq \nu^J_a(q) - s$ for every $q$, which implies $c^J(a) \geq c^J(\overline{a})$.

In order to prove (5) suppose that for some $q$ we have $\nu^J_a(q)/q = c^J(a)$. If $\nu^J_a(q) = r$, this implies that $\nu^J_a(qq') = rq'$ for all $q'$. Therefore $a^{rq' + 1} \subseteq J^{[qq']} + 1$ for all $q'$. As $J^{[q]}$ is equal to its tight closure, this gives $a^r \subseteq J^{[q]}$, a contradiction.

We prove now (6). If $(a + b)^r \not\subseteq J^{[q]}$, then there are $\ell_1$ and $\ell_2$ such that $\ell_1 + \ell_2 = r$ and $a^{\ell_1} \not\subseteq J^{[q]}$, $b^{\ell_2} \not\subseteq J^{[q]}$. Therefore $\nu^J_{a+b}(q) \leq \nu^J_a(q) + \nu^J_b(q)$ for all $q$, which gives (6). \[ \square \]

As pointed out in [TW], the F-pure threshold can be considered as an analogue of the log canonical threshold. Similarly, the F-thresholds play the role of the jumping coefficients from [ELSV]. We will see more clearly this analogy in the next sections.

In what follows we fix the ideal $a$, and study the F-thresholds which appear for various $J$. We record in the next proposition some easy properties which deal with the variation of $J$.

**Proposition 1.8.**  
(1) If $a$ and $J_1$, $J_2$ are as above with $J_1 \subseteq J_2$, then $c^J(a) \leq c^{J_2}(a)$. In particular, the F-pure threshold $c(a)$ is the smallest (nonzero) F-threshold of $a$.

(2) If $J = \bigcap_{\lambda \in \Gamma} J^{[q]}_{\lambda}$, then $c^J(a) = \sup_{\lambda \in \Gamma} c^{J_{\lambda}}(a)$.

(3) We have $c^{J^{[q]}}(a) = q \cdot c^J(a)$ for every $q$.

**Proof.** The first assertion is straightforward, as we have $\nu^{J_2}_a(q) \leq \nu^{J_1}_a(q)$ for all $q$. For the second assertion, note that since the Frobenius morphism is flat, we have $J^{[q]} = \bigcap_{\lambda} J^{[q]}_{\lambda}$, so $\nu^J_a(q) = \max_{\lambda} \nu^{J_{\lambda}}_a(q)$ which gives the formula for $c^J(a)$. The equality in (3) is trivial, as in the definition of $c^J(a)$ we have a limit. \[ \square \]

When the ideal $a$ is generated by one element, then we can say more. The next proposition shows that in this case the F-threshold determines the numbers $\nu^J_a(q)$ for all $q$. If $a = (f)$, we simply write $\nu^J_f(q)$ and $c^J(f)$. We denote by $\lceil \alpha \rceil$ the smallest integer $\geq \alpha$.

**Proposition 1.9.** Let $J \subseteq m$ be an ideal whose radical contains $f \neq 0$. For every $q$ we have

$$\frac{\nu^J_f(pq) + 1}{pq} \leq \frac{\nu^J_f(q) + 1}{q},$$

so $c^J(f) = \inf_q \frac{\nu^J_f(q+1)}{q}$. Moreover, we have $\nu^J_f(q) = \lceil c^J(f)q \rceil - 1$ for all $q$. 
Proof. For the first assertion it is enough to note that if $f^{\nu_f(q)+1}$ lies in $J[q]$ then $f^{p(\nu_f(q)+1)}$ is in $J^{[pq]}$. The last statement follows from

$$\frac{\nu_f^J(q)}{q} < c^J(f) \leq \frac{\nu_f^J(q)}{q} + 1.$$ 

□

We clearly have $c^J(f) = 1$, so 1 is always an F-threshold for principal ideals. The next proposition shows that moreover, in this case it is enough to understand the thresholds in $(0, 1)$.

**Proposition 1.10.** If $J$ is an ideal containing the nonzero $f$ in its radical, then

(3) $c^J(f) = c^f(f) + 1$, $c^{(J:f)}(f) = \max\{c^f(f) - 1, 0\}$.

In particular, a nonnegative $\lambda$ is an F-threshold of $\mathfrak{a}$ if and only if $\lambda + 1$ is.

Proof. The proof is straightforward. The only thing to notice is that since the Frobenius morphism is flat, we have $(J : f)^{[q]} = J^{[q]} : f^q$ for all $q$. □

**Remark 1.11.** It follows easily from Proposition 1.9 that when $\mathfrak{a}$ is principal, $c^J(\mathfrak{a})$ is a rational number if and only if the function $e \rightarrow \tau^J(\mathfrak{a}) := \nu^J_{\mathfrak{a}}(p^e) - p\nu^J_{\mathfrak{a}}(p^{e-1})$ is eventually periodic. Furthermore, this is equivalent with the fact that the series

$$P^J_{\mathfrak{a}}(t) = \sum_{e \geq 1} \nu^J_{\mathfrak{a}}(p^e)t^e$$

is a rational function.

One could ask more generally whether for any $\mathfrak{a}$ the above series is a rational function (again, this would imply that $c^J(\mathfrak{a})$ is rational). It follows from [BMS] that this stronger assertion holds for monomial ideals. In fact, in this case it is again true that the function $\tau^J_{\mathfrak{a}}$ is eventually periodic.

**Remark 1.12.** In the study of singularities in characteristic zero, one can often reduce the invariant of an arbitrary ideal $\mathfrak{a}$ to that of a principal ideal $(f)$ by taking $f$ general in $\mathfrak{a}$. This does not work in our setting. For example, let $\mathfrak{a} = \mathfrak{m}[p]$. We have $c(\mathfrak{a}) = c(\mathfrak{m}^p) = n/p$, but for every $f \in \mathfrak{a}, \nu_f(p) = 0$, so $c(f) \leq 1/p$.

2. **F-thresholds as jumping coefficients**

Test ideals are a very useful tool in tight closure theory. In [HY] Hara and Yoshida introduced a generalization of test ideals in the setting of pairs. These ideals enjoy properties similar to those of multiplier ideals in characteristic zero. In fact, there is a strong connection between the test ideals and the multiplier ideals via reduction mod $p$ (see [HY], and also the next section). We start by reviewing their definition in our particular setting, in order to describe the connection between test ideals and F-thresholds.

Let us fix first some notation. Let $E = E(R) := H^m_n(R)$ be the top local cohomology module of $R$, so $E$ is isomorphic to the injective hull of $k$. If $x_1, \ldots, x_n$ form a regular
system of parameters in $R$, then the completion $\hat{R}$ of $R$ is isomorphic to the formal power series ring $k[[X_1, \ldots, X_n]]$ such that $x_i$ corresponds to $X_i$. Note that we have

$$(4) \quad E(R) \simeq E(\hat{R}) \simeq R_{x_1, \ldots, x_n} / \sum_{i=1}^{n} R_{x_1, \ldots, \hat{x}_i, \ldots, x_n}.$$ 

Whenever working in $E$ we will assume we have fixed such a regular system of parameters, so via the above isomorphism we may represent each element of $E$ as the class $[u/(x_1 \ldots x_n)^d]$ for some $u \in R$ and some $d$.

We will use freely Matlis duality: $\text{Hom}(-, E)$ induces a duality between finitely generated $\hat{R}$-modules and Artinian $\hat{R}$-modules (which are the same as the Artinian $R$-modules). See, for example [BH] for more on local cohomology and Matlis duality.

On $E$ we have a Frobenius morphism $F_E$ which via the isomorphism in (4) is given by

$$F_E([u/(x_1 \ldots x_n)^d]) = [u^p/(x_1 \ldots x_n)^{pd}].$$

$F_E$ is injective. Moreover, if $a \in R \setminus \{0\}$ is such that $aF_E(w) = 0$ for all $e$, then $w = 0$. Indeed, this is an immediate consequence of the fact that every ideal is equal with its tight closure.

Let $a \subseteq m$ be a fixed ideal. For every $r \geq 0$ and $e \geq 1$ we put

$$Z_{r,e} := \ker(a^rF_E^e) = \{w \in E | hF_E^e(w) = 0 \text{ for all } h \in a^r\}.$$  

**Lemma 2.1.** If $r < s$, then $Z_{r,e} \subseteq Z_{s,e}$. We have $E = \bigcup_r Z_{r,e}$. Moreover, $Z_{pr,e+1}$ is contained in $Z_{r,e}$.

**Proof.** The first assertions are clear, and the last one follows from the injectivity of $F_E$ and the fact that $F_E(hw) = h^pF_E(w)$ for all $h \in \hat{R}$ and $w \in E$. \qed

**Definition 2.2.** ([HY]) If $a \subseteq m$ is a nonzero ideal, and if $c \in \mathbb{R}_+$, the test ideal of $a$ of exponent $c$ is

$$\tau(a^c) := \text{Ann}_R \left( \bigcap_{e \geq 1} Z_{\lfloor cp^e \rfloor, e} \right).$$

As $E$ is Artinian, it follows from Lemma 2.1 that $\tau(a^c) = \text{Ann}_R Z_{\lfloor cp^e \rfloor, e}$ if $e \gg 0$.

For every $c > 0$, let $Z_c := \bigcap_e Z_{\lfloor cp^e \rfloor, e}$. Note that $Z_c \neq E$. Indeed, if $m \geq c$ is an integer and if $h$ is a nonzero element of $a$, then $Z_c \subseteq Z_{\lfloor mp^e \rfloor, e} \subseteq \ker(h^{mp^e}F_E^e)$, which is equal to the kernel of the multiplication by $h^m$ on $E$ (this follows from the injectivity of $F_E$). Therefore $Z_c$ is a proper submodule of $E$.

If we replace $R$ by $\hat{R}$ and $a$ by $a\hat{R}$, then $Z_c$ remains the same. For the basic properties of test ideals we refer the reader to [HY]. We prove only the following Lemma which we will need in the next section. See [HY], [HW] and [Smi] for related stronger statements.

**Lemma 2.3.** For every $c > 0$, the submodule $Z_c$ is the unique maximal proper submodule of $E$ invariant by all $hF_E^e$, where $e \geq 1$ and $h \in a^{\lfloor cp^e \rfloor}$.
Proof. It is clear that $Z_c$ is invariant under $hF^e_E$ as above, as $hF^e_E(Z_{[cp^e],e}) = 0$ by definition. Since $Z_c$ does not change when we pass to the completion, we may assume that $R$ is complete.

In this case every proper submodule of $E$ has nonzero annihilator. Therefore in order to finish the proof it is enough to show that if $g \in R$ is a nonzero element, and if 

$w \in E$ is such that $ga^{[cp^e]}F^e_E(w) = 0$ for all $e \geq 1$, then $w \in Z_c$. Fix $e'$ and $h \in a^{[cp^e]}$. For every $e$ we have $gF^e_E(hF^{e'}_E(w)) = gh^{p^e}F^{e+e'}_E(w) = 0$, as $h^{p^e} \in a^{p^e[cp^e]} \subseteq a^{[cp^e+e']}$.$\,\Box$

Our goal is to show that the F-thresholds we have introduced in the previous section can be interpreted as jumping coefficients for the test ideals. We start by interpreting the function $\nu^I_a$ in terms of the Frobenius morphism on $E$.

**Lemma 2.4.** Let $a$ and $J \subseteq m$ be nonzero ideals, with $a$ contained in the radical of $J$. If $M$ is a submodule of $E$ such that $J = \text{Ann}_R(M)$, then $\nu^I_a(p^e)$ is the largest $r$ such that $M \nsubseteq Z_{r,e}$.

**Proof.** For every $w \in M$ we put $J_w = \text{Ann}_Rw$, so $J = \bigcap_{w \in M} J_w$. If $w = [u/(x_1 \ldots x_n)^d]$, then $J_w = (x_1^{e_1}, \ldots, x_n^{e_n}) : w^{p^e}$. For every $w$, we see that $\nu^I_a(w^{p^e})$ is the largest $r$ such that $w \nsubseteq Z_{r,e}$. As $\nu^I_a = \max_{w \in M} \nu^I_a(w)$, we get the assertion in the lemma. $\Box$

**Remark 2.5.** By Matlis duality, we may take in the above Lemma $M = \text{Ann}_E(J)$.

For future reference we include also the next lemma whose proof is immediate from definition.

**Lemma 2.6.** If $a \subseteq b$, then $\tau(a^c) \subseteq \tau(b^c)$ for every $c \in \mathbb{R}_+$. If $c_1 < c_2$, then $\tau(a^{c_2}) \subseteq \tau(a^{c_1})$ for every ideal $a$.

**Proposition 2.7.** If $a \subseteq m$ is a nonzero ideal contained in the radical of $J$, then

$$\tau(a^{c^J(a)}) \subseteq J.$$

Going the other way, if $\alpha \in \mathbb{R}_+$, then $a$ is contained in the radical of $\tau(a^{\alpha})$ and

$$c^{\tau(a^{\alpha})}(a) \leq \alpha.$$

Therefore the maps $J \rightarrow c^J(a)$ and $\alpha \rightarrow \tau(a^{\alpha})$ give a bijection between the set of test ideals of $a$ and the set of F-thresholds of $a$.

**Proof.** For the first statement, let $M = \text{Ann}_E J$, so by Matlis duality we need to prove that $M \subseteq Z_{[c^J(a)p^e],e}$ for all $e$. This follows from Lemma 2.4 and the fact that $[c^J(a)p^e] > \nu^I_a(p^e)$.

We show now that for $\alpha \in \mathbb{R}_+$, we have $a \subseteq \text{Rad}(\tau(a^{\alpha}))$. Let $e \gg 0$ be such that $\tau(a^{\alpha}) = \text{Ann}_R Z_{[\alpha p^e],e}$. If $m \geq \alpha$ is an integer, it follows from the injectivity of $F_E$ that $a^m \subseteq \tau(a^{\alpha})$. 

We deduce now from Lemma 2.8 and from the definition of $\tau(a^n)$ that
\[ \nu_a^{\tau(a^n)}(p^e) \leq \lceil \alpha p^e \rceil - 1 < \alpha p^e. \]
Dividing by $p^e$ and taking the limit gives the required inequality.

The last statement is a formal consequence of the first two assertions. \qed

**Remark 2.8.** It follows from Proposition 2.7 that if we have an $F$-threshold $c$ of $a$, then there is a unique minimal ideal $J$ such that $c^J(a) = c$. Indeed, this is $\tau(a^\ell)$. Moreover, if $c_1$ and $c_2$ are such $F$-thresholds, then $c_1 < c_2$ if and only if $\tau(a^{c_2})$ is strictly contained in $\tau(a^{c_1})$.

**Remark 2.9.** As $R$ is Noetherian, it follows from the previous remark that there is no strictly decreasing sequence of $F$-thresholds of $a$.

**Remark 2.10.** There are arbitrarily large thresholds: take, for example $p^e c(a) = c^{\lfloor pe \rfloor}(a)$ for $e \geq 1$. Note also that a sequence of thresholds $\{c_m\}_m$ of $a$ is unbounded if and only if $\bigcap_m \tau(a^{c_m}) = (0)$. The only thing one needs to check is that $\bigcap_{c \in \mathbb{R}_+} \tau(a^c) = (0)$. This follows since for every integer $\ell \geq n$, we have $\tau(a^\ell) \subseteq \tau(m^\ell) \subseteq m^{\ell-n+1}$.

The jumping coefficients for multiplier ideals are discrete. We do not know if the analogous assertion is true for the $F$-thresholds.

**Question 2.11.** Given an ideal $(0) \neq a \subseteq m$, could there exist finite accumulation points for the set of $F$-thresholds of $a$?

**Remark 2.12.** Given a test ideal $J$ corresponding to $a$, the set of those $\alpha \in \mathbb{R}_+$ such that $\tau(a^\alpha) = J$ is an interval of the form $[a, b)$. Indeed, if $a = c^J(a)$, it follows from Lemma 2.6 and Proposition 2.7 that for $\lambda < a$, $\tau(a^\lambda)$ strictly contains $J = \tau(a^a)$. On the other hand, if $J'$ is the largest test ideal strictly contained in $J$, and if $b = c^{J'}(a)$, then it is clear that $b = \sup\{\lambda|\tau(a^\lambda) = J\}$, which gives our assertion.

**Example 2.13.** Consider the case when $a$ is a monomial ideal, i.e. $a$ is generated by monomials in the localization of $k[X_1, \ldots, X_n]$ at $(X_1, \ldots, X_n)$. It is shown in [HY] that for every $\alpha$ we have $\tau(a^\alpha) = \mathcal{I}(a^\alpha)$, where $\mathcal{I}(a^\alpha)$ is the multiplier ideal of $a$ with exponent $\alpha$. It follows from this and from Proposition 2.7 that the set of $F$-thresholds of $a$ coincides with the set of jumping coefficients of the multiplier ideals of $a$.

Let us recall the description of multiplier ideals for monomial ideals from [Ho2]. Consider the Newton polyhedron $P_a$ of $a$: this is the convex hull in $\mathbb{R}_n$ of $\{u \in \mathbb{N}_n|X^u \in a\}$, where for $u = (u_1, \ldots, u_n)$ we put $X^u = X_1^{u_1} \ldots X_n^{u_n}$. If we put $e = (1, \ldots, 1)$, then

\[ \mathcal{I}(a^\alpha) = (X^u|u + e \in \text{Int}(\alpha \cdot P_a)). \]

It follows that each jumping coefficients $\alpha$ of the multiplier ideals is associated to some $b = (b_i)$ with all $b_i$ positive integers, where $\alpha$ is such that $b$ lies in the boundary of $\alpha \cdot P_a$. Of course, several distinct $b$ can give the same $\alpha$.

In fact, one can show that in order to compute all the $F$-thresholds of $a$ it is enough to consider only ideals $J$ of the form $(X_1^{b_1}, \ldots, X_n^{b_n})$ with $b_i$ positive integers. Moreover, one can check directly that if $J$ is the above ideal, then $c^J(a) = \alpha$, where $\alpha$ is associated to $b = (b_1, \ldots, b_n)$ as above (see [BMS1] for this approach).
We end this section by considering in more detail the case when \(a = (f)\) is a principal ideal. One can easily check that for such \(a\) we have \(Z_{r,e} = Z_{pr,e+1}\). This shows that any two \(Z_{r_1,e_1}\) and \(Z_{r_2,e_2}\) are comparable. As \(E\) is Artinian, we may consider the submodules of \(E\) defined inductively as follows: let \(M_0 := \{0\}\) be the minimal module in \(M := \{Z_{r,e}\,|\,r, e\}\), and for \(m \geq 1\), let \(M_m\) be the unique minimal module in \(M \setminus \{M_0, \ldots, M_{m-1}\}\). It follows that \(M_m\) is properly contained in \(M_{m+1}\). In addition, given any \(r\) and \(e\), either \(M_m \subseteq Z_{r,e}\), or \(Z_{r,e} \in \{M_0, \ldots, M_{m-1}\}\).

**Proposition 2.14.** With \(f \in m\) as above, we put for every \(i\), \(J_i = \text{Ann}_R(M_i)\), and \(c_i = c^{J_i}(f)\).

(1) For \(i \geq 1\), let \(\nu_i(e)\) be the largest \(r\) such that \(Z_{r,e} \subseteq M_{i-1}\). Then

\[
\lim_{e \to \infty} \frac{\nu_i(e)}{p^e}.
\]

(2) Every \(J_i\) is a test ideal of \(a\), and if \(J\) is any test ideal different from all \(J_i\), then \(J\) is contained in all these ideals.

**Proof.** Note that by definition \(\nu_i(e)\) is the largest \(r\) such that \(M_i \not\subseteq Z_{r,e}\). By Lemma 2.4, we get \(\nu_i(e) = \nu_i^J(p^e)\) and this proves (1).

We show that \(J_i\) is a test ideal by proving that \(\tau(f^{c_i}) = J_i\). By Lemma 2.7, it is enough to show that \(J_i \subseteq \tau(f^{c_i})\) for \(i \geq 1\). This follows from \([c_i p^e] = \nu_i(p^e) + 1\) (see Proposition 1.9) which implies \(\bigcap_e Z_{[c_i p^e],e} = M_i\).

For the last statement it is enough to show that for all \(i \geq 0\) and for \(c \in [c_i, c_{i+1}]\) we have \(\tau(f^c) = J_i\) (with the convention \(c_0 = 0\)). If \(e \gg 0\), we have \([cp^e] < \nu_{i+1}(e)\), so \(\bigcap_e Z_{[cp^e],e} \subseteq M_i\). This implies \(J_i \subseteq \tau(f^c)\), and the other inclusion is clear, as we have seen that \(\tau(f^{c_i}) = J_i\).

**Remark 2.15.** Note that in the case of a principal ideal the set of F-thresholds of \((f)\) is discrete if and only if \(\lim_{m \to \infty} c_m = \infty\). This is equivalent with the fact that \(\bigcup_m Z_m = E\). Note also that by the periodicity of the F-thresholds (see Lemma 1.10), this is further equivalent with the finiteness of the set of F-thresholds in \((0, 1)\).

It follows from Proposition 1.9 that \(c(f) = 1\) if and only if \(\nu_f(p^e) = p^e - 1\) for every \(e\). The following proposition based on an argument of Fedder shows that in fact, it is enough to check this for only one \(e \geq 1\).

**Proposition 2.16.** ([Fe]) If \(f\) is a nonzero element in \(m\), then \(c(f) = 1\) if and only if there is \(e\) such that \(\nu_f(p^e) = p^e - 1\). Moreover, this is the case if and only if the action of the Frobenius morphism on \(H^{n-1}_m(R/(f))\) is injective.

**Proof.** The exact sequence

\[
0 \to R \to R \to R/(f) \to 0
\]

induces an isomorphism of \(H^{n-1}_m(R/(f))\) with the annihilator of \(f\) in \(E\). Moreover, via this identification the Frobenius morphism on \(H^{n-1}_m(R/(f))\) is given by \(\tilde{F}(u) = f^{p-1}F_E(u)\).
We see that \( \tilde{F}^e \) is injective if and only if \( Z_{p^e-1,p^e} = (0) \). This is the case if and only if \( \nu_f(p^e) = p^e - 1 \). Since \( F \) is injective if and only if \( \tilde{F}^e \) is, this completes the proof. \( \square \)

3. Reduction mod \( p \) and the connection with the Bernstein polynomial

In this section we study the way our invariants behave for different reductions mod \( p \) of a given ideal. Everything in this section works in the usual framework for reducing mod \( p \) which is used in tight closure theory (see for example [HY]). In order to simplify the presentation as well as the notation, we prefer to work in the following concrete setup. The interested reader should have no trouble translating everything to the general setting.

Let \( A \) be the localization of \( \mathbb{Z} \) at some nonzero integer. We fix a nonzero ideal \( a \) of \( A[X] = A[X_1, \ldots, X_n] \), such that \( a \subseteq (X_1, \ldots, X_n) \). Let \( \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \). We want to relate the invariants attached to \( a_Q := a \cdot Q[X] \) around the origin with those attached to the localizations of the reductions mod \( p \),

\[ a_p := a \cdot \mathbb{F}_p[X](X_1, \ldots, X_n), \]

where \( p \) is a large prime. We will use the same subscripts whenever tensoring with \( Q \) or reducing (and localizing) mod \( p \). Note that since we are interested only in large primes, we are free to further localize \( A \) at any nonzero element.

Let us consider a log resolution of \( a_Q \) defined over \( Q \): this is a proper birational morphism \( \pi_Q : Y_Q \longrightarrow A^n_Q \), with \( Y_Q \) smooth, such that the product between \( \pi_Q^{-1}(a_Q) \) and the ideal defining the exceptional locus of \( \pi_Q \) is principal, and it defines a divisor with simple normal crossings. Such a resolution exists by [Hir]. After further localizing \( A \), we may assume that \( \pi_Q \) is obtained by extending the scalars from a morphism \( \pi : Y \longrightarrow A^n \) with analogous properties.

If we denote by \( D \) the effective divisor defined by \( \pi^{-1}(a) \) and if \( K \) is the relative canonical divisor of \( \pi \) (i.e. the effective divisor defined by the Jacobian of \( \pi \)), then for all \( \alpha \in \mathbb{R} \)

\[ \mathcal{I}(a^\alpha) := H^0(Y, O_Y(K - [\alpha D])). \]

Here \( [\alpha D] \) denotes the integral part of the \( \mathbb{R} \)-divisor \( \alpha D \). Note that \( \mathcal{I}(a^\alpha)_Q \) is the multiplier ideal of \( a_Q \) of with exponent \( \alpha \). We refer for the theory of multiplier ideals to [Laz].

The jumping numbers at \((0, \ldots, 0)\) introduced in [ELSV] are the numbers \( \lambda \) such that \( \mathcal{I}(a^\lambda)_Q \) is strictly contained in \( \mathcal{I}(a^{\lambda-\epsilon})_Q \) in a neighborhood of the origin, for every \( \epsilon > 0 \). The smallest positive such number is the log canonical threshold \( \text{lct}_Q(a) \): it is the first \( \lambda \) such that \( \mathcal{I}(a^\lambda)_Q \) is different from the structure sheaf around the origin. In order to simplify the notation we will drop the subscript \( Q \) whenever considering the invariants associated to \( a_Q \).

In our setting, by taking \( p \gg 0 \), we may assume that the above resolution induces a log resolution \( \pi_p \) for \( a_p \). Over \( Q \) we have \( R^i\pi_*(O_Y)_Q = 0 \) for \( i \geq 1 \). This remains true for the reductions mod \( p \) if \( p \gg 0 \). From now on we assume that \( p \) is large enough, so these
conditions are satisfied. We define $I(\alpha^p)$ by a formula similar to (6), using $\pi_p$. Note that for a fixed $\alpha$, we have $I(\alpha^p) = I(\alpha^0)$ if $p \gg 0$.

We recall two results which describe what is known about the connection between multiplier ideals and test ideals. The first one is proved in more generality in [HY], based on ideas from [HW]. We include the proof as it is quite short in our context.

**Theorem 3.1.** With the above notation, if $p \gg 0$, then for every $\alpha$ we have $\tau(\alpha^p) \subseteq I(\alpha^0)$.

**Proof.** Let $R$ be the localization of $\mathbb{F}_p[X]$ at $(X_1, \ldots, X_n)$, and let $m$ be its maximal ideal. We denote by $W \subseteq Y_p$ the subset defined by $\pi_p^{-1}(m)$. We will use the notation from the previous section.

If $E = H^n_W(\mathcal{O}_{Y_p})$, using the fact that the higher direct images of $\mathcal{O}_{Y_p}$ are zero, and the long exact sequence for local cohomology we get $E \simeq H^n_W(\mathcal{O}_{Y_p}(\lfloor \alpha D_p \rfloor))$. A version of Local Duality shows that if $\delta : H^n_W(\mathcal{O}_{Y_p}) \longrightarrow H^n_W(\mathcal{O}_{Y_p}(\lfloor \alpha D_p \rfloor))$ is the surjective morphism induced by the natural inclusion of sheaves, then $I(\alpha^0) = \text{Ann}_R(\ker \delta)$. By Lemma 2.3 it is therefore enough to show that if $h \in \mathfrak{a}^{\lfloor \alpha^p \rfloor}$, then $h F^p(\ker \delta) \subseteq \ker \delta$.

The Frobenius morphism on local cohomology is induced by the Frobenius morphism $F$ on the fraction field of $R$. As the inclusion $h F^p(\mathcal{O}_{Y_p}) \subseteq \mathcal{O}_{Y_p}$ is clear, in order to finish it is enough to show also that $h F^p(\mathcal{O}_{Y_p}(\lfloor \alpha D_p \rfloor)) \subseteq \mathcal{O}_{Y_p}(\lfloor \alpha D_p \rfloor)$. This is an immediate consequence of the definitions. \qed

The proof of the next Theorem is more involved, so we refer the reader to [HY].

**Theorem 3.2.** With the above notation, if $\alpha$ is given and if $p \gg 0$ (depending on $\alpha$), then $\tau(\alpha^p) = I(\alpha^0)$.

We reformulate the above results in terms of thresholds. In order to do this we index the jumping coefficients of $\mathfrak{a}_Q$ at the origin by analogy with the F-thresholds, as follows. Suppose that $J \subseteq (X_1, \ldots, X_n)A[X]$ is an ideal containing $\mathfrak{a}$ in its radical. We define $\lambda_J^0(\mathfrak{a}) := \min\{\alpha > 0 \mid I(\alpha^0) \subseteq J \text{ around } 0\}$. It is clear that this is a jumping coefficient of $\mathfrak{a}_Q$ around the origin, and that every such coefficient appears in this way for a suitable $J$. For example, if $J = (X_1, \ldots, X_n)$, then $\lambda_J^0(\mathfrak{a}) = \text{lc}_0(\mathfrak{a})$.

Using Proposition 2.7 we may reformulate the above results as follows. We will denote the invariants of $\mathfrak{a}_p$ with respect to $J_p$, which we have introduced in §1, simply by $c^J(\mathfrak{a}_p)$ and $\nu^J_d(p^r)$.

**Theorem 3.3.** If $p \gg 0$, then for every ideal $J$ as above we have $c^J(\mathfrak{a}_p) \leq \lambda_J^0(\mathfrak{a})$. In particular, we have the following inequality between the F-pure threshold and the log canonical threshold: $c(\mathfrak{a}_p) \leq \text{lc}_0(\mathfrak{a})$. 
Theorem 3.4. Given an ideal $J$ as above, we have
$$\lim_{p \to \infty} c'(a_p) = \lambda_0^J(a).$$
In particular, we have $\lim_{p \to \infty} c(a_p) = lc_0(a)$.

Remark 3.5. The fact that in Theorem 3.2 $p$ depends on $\alpha$ is reflected in Theorem 3.4
in that we may have $c'(a_p) < \lambda_0^J(a)$ for infinitely many $p$. This is a very important point,
and we will see examples of such a behavior (for $J = m$) in the next section.

We discuss now possible further connections between the invariants over $\mathbb{Q}$ and those
of the reductions mod $p$. We formulate them in the case $J = m$ and we will give some examples in §4. However, note that similar questions can be asked for arbitrary $J$.

Conjecture 3.6. Given the ideal $a$, there are infinitely many primes $p$ such that $c(a_p) = lc_0(a)$.

Problem 3.7. Given the ideal $a$, give conditions such that there is a positive integer $N$
with the following property: for every prime $p$ with $p \equiv 1 \pmod{N}$ we have $c(a_p) = lc_0(a)$.

Problem 3.8. Give conditions on an ideal $a$ such that there is a positive integer $N$, and
rational functions $R_i \in \mathbb{Q}(t)$ for every $i \in \{0, \ldots, N-1\}$ with $\gcd(i, N) = 1$ with the
following property: $c(a_p) = R_i(p)$ whenever $p \equiv i \pmod{N}$ and $p$ is large enough.

These problems are motivated by the examples we will discuss in the next section.
We will see that the behavior described in the problems is satisfied in many cases. On
the other hand, Example 4.6 below shows that one can not expect for such a behavior to
hold in general. We will see in this example that the failure is related to subtle arithmetic
phenomena.

However, note that if $p$ is an odd prime, then one can reinterpret the condition $p \equiv 1 \pmod{N}$ in Problem 3.7 as saying that $p$ is completely split in the cyclotomic field of the
$N$th roots of unity (see [Neu], Cor. 10.4). We will see that somethings similar happens
in Example 4.6 below: there is a number field $K$ such that if $p$ splits completely in $K$,
then the log canonical threshold is equal to the corresponding F-pure threshold. This
motivates the following

Question 3.9. Given an ideal $a$ as above, is there a number field $K$ such that whenever
the prime $p \gg 0$ splits completely in $K$, we have $c(a_p) = lc(a)$ ?

Note that by Čebotarev’s Density Theorem (see [Neu], Cor. 13.6), given a number
field $K$ there are infinitely many primes $p$ which split completely in $K$. Therefore a
positive answer to Question 3.9 would imply Conjecture 3.6.

We include here another problem with a similar flavor, on the behavior of the functions $\nu_f^J(p^e)$ when we vary $p$. The interest in this problem comes from the fact that
whenever we can prove that such a behavior holds, one can use this to give roots of the
Bernstein-Sato polynomial of $a_Q$ (see Remark 3.13 below). The Conjecture is proved for
monomial ideals in [BMS1]. For other examples, see the next section.
Problem 3.10. Find conditions on an ideal $\mathfrak{a}$ such that the following holds. Given an ideal $J$ as above, and $e \geq 1$, there is a positive integer $N$, and polynomials $P_j \in \mathbb{Q}[t]$ of degree $e$, for every $j \in \{1, \ldots, N-1\}$ with $\gcd(j, N) = 1$, such that $\nu_a^j(p^e) = P_j(p)$ for every $p \gg 0$, $p \equiv j \pmod{N}$. When could $N$ be chosen independently on $J$ and $e$?

We turn now to a different connection between invariants which appear in characteristic zero and the ones we have defined in §1. The characteristic zero invariants we will consider are the roots of the Bernstein-Sato polynomial, whose definition we now recall.

Let $I \subseteq \mathbb{C}[X_1, \ldots, X_n]$ be a nonzero ideal, and let $f_1, \ldots, f_r$ be nonzero generators of $I$. We introduce indeterminates $s_1, \ldots, s_r$ and the Bernstein-Sato polynomial $b_I$ is the monic polynomial in one variable of minimal degree such that we have an equation

\[ b_I(s_1 + \ldots + s_r) \prod_{i=1}^r f_i^{s_i} = \sum_c \mathcal{P}_c(s, X, \partial_X) \cdot \prod_{j, c_j < 0} \binom{s_j}{-c_j} \prod_{i=1}^r f_i^{s_i+c_i}. \]

Here the sum varies over finitely many $c \in \mathbb{Z}^r$ such that $\sum_j c_j = 1$, for every such $c$ we have the nonzero differential operator $\mathcal{P}_c \in \mathbb{C}[s_j, X_i, \partial_{X_i}]$ for $j \leq r, i \leq n$, and as usual $\binom{s_j}{-c_j} = s_j(s_j-1) \ldots (s_j+c_j+1)/(-c_j)$. Note that $\partial_X$ denotes the action of a differential operator. Equation (7) is understood formally, but if we let $s_i = m_i \in \mathbb{N}$, then it has the obvious meaning. If we require (7) to hold only in some neighborhood of the origin in $\mathbb{C}^n$, then we get the local Bernstein-Sato polynomial $b_{I,0}(s)$.

Note that if $r = 1$, i.e. if $I = (f)$ is a principal ideal, then (7) takes the more familiar form

\[ b_f(s)^f = P(s, X, \partial_X) \cdot f^{s+1}. \]

We refer to [B] for some basic properties properties of the Bernstein-Sato polynomial of principal ideals, and to [BMS2] for the general case.

In the case of principal ideals, there is an extensive literature on connections between this polynomial and other invariants of singularities (see [Mal], [Ka2], [Ig] and [Kol]). Some of these results have been extended to arbitrary ideals in [BMS2].

Here are a few properties which are relevant to our study. First, it is proved in [BMS2] that this polynomial does not depend on the choice of generators. All the roots of $b_{I,0}$ are negative rational numbers, the largest one is $-\text{lcm}(I)$, and for every jumping coefficient around the origin $\lambda$ of $I$, if $\lambda \in [\text{lcm}(I), \text{lcm}(I) + 1)$, then $-\lambda$ is a root of $b_{I,0}$. For these facts, see [Ka2], [Kol] and [ELSV] for principal ideals and [BMS2] for the general case.

We return now to our setting. The extension of our ideal $\mathfrak{a}$ to $\mathbb{C}$ defines a Bernstein-Sato polynomial around the origin, which we simply denote by $b_{\mathfrak{a},0}$. Consider the defining equation (7) and let $B$ be a subalgebra of $\mathbb{C}$, finitely generated over $\mathbb{Z}$ and containing all the coefficients of $b_{\mathfrak{a},0}$ and of the $\mathcal{P}_c$. Moreover, we may assume that for all $c$ which appear in (7) and for all $j$ such that $c_j < 0$, $(-c_j)!$ is invertible in $B$.

It is clear that there is $M$ such that for every prime $p \geq M$, there is a maximal ideal $P$ of $B$ with $p \mathfrak{a} = P \cap \mathfrak{a}$. For such $p$ and $P$, let $R_p$ and $S_P$ be the localizations of
\[ \mathbb{F}_p[X] \text{ and } (B/P)[X], \] respectively, at the ideal generated by the variables. Suppose now that \( J \subseteq (X_1, \ldots, X_n)A[X] \) is an ideal containing \( a \) in its radical. We will denote by \( J_p \) and \( J_P \) the image of \( J \) in \( R_p \) and \( S_P \), respectively.

Note that since \( S_P \) is flat over \( R_p \) and since the Frobenius morphism is flat, it follows that for every \( e \) we have \( J_p^{[p^e]} \cap R_p = J_p^{[p^e]} \). In particular, we have

\[ \nu_a^J(p^e) = \nu_a^{J_p}(p^e). \]

**Proposition 3.11.** If \( a \subseteq (X_1, \ldots, X_n)A[X] \) is a nonzero ideal, then for every prime \( p \gg 0 \) and for every \( J \subseteq (X_1, \ldots, X_n)A[X] \) containing \( a \) in its radical we have

\[ b_{a,0}(\nu_a^J(p^e)) = 0 \text{ in } \mathbb{F}_p \]

for all \( e \).

**Remark 3.12.** Recall that all roots of \( b_{a,0} \) are rational, so \( b_{a,0} \in \mathbb{Q}[s] \). After localizing \( A \) at a suitable element, we may assume that \( b_{a,0} \in A[s] \). Therefore for every \( m \in \mathbb{Z} \), \( b_{a,0}(m) \) has a well-defined class in \( \mathbb{F}_p \).

**Proof of Proposition 3.11.** We use the above notation and let \( m = \nu_a^J(p^e) \). Recall that we have generators \( f_1, \ldots, f_r \) of \( a \). It follows from (8) that there are nonnegative integers \( \ell_1, \ldots, \ell_r \) such that \( \sum \ell_i = m \) and \( \prod f_i^\ell_i \notin J_p^{[p^e]} \). On the other hand, for every nonnegative integers \( \ell'_1, \ldots, \ell'_r \) with \( \sum \ell'_i = m + 1 \), we have \( \prod f_i^\ell'_i \in J_p^{[p^e]} \).

Note that (17) holds in \( S_P \) if \( s_i = \ell_i \) for all \( i \). If \( c \) and \( i \) are such that \( \ell_i + c_i < 0 \), then \( \ell_i(\ell_i - 1) \ldots (\ell_i + c_i + 1) = 0 \), so this term does not appear in the corresponding equality. As \( J_p^{[p^e]} \) is invariant under the action of operators in \( B/P[X_i, \partial X_i | i \leq n] \), we deduce that \( b_{a,0}(m) \) is zero in \( R_p \), hence in \( \mathbb{F}_p \). \( \square \)

**Remark 3.13.** Note that whenever we can show that \( \nu_a^J(p^e) \) has \( \nu_a^J(p^e) \) we get roots of \( b_{a,0} \). More precisely, suppose that for some \( J \) as above and for some \( e \), there is a positive number \( N \) and polynomials \( P_j \in \mathbb{Q}[t] \) for every \( j \) with \( \gcd(j, N) = 1 \) such that \( \nu_a^J(p^e) = P_j(p) \) for every prime \( p \gg 0 \) with \( p \equiv j \pmod{N} \). In this case, the above proposition shows that \( b_{a,0}(P_j(p)) \) is divisible by \( p \), so \( p \) divides \( b_{a,0}(P_j(0)) \). By the Dirichlet Theorem there are infinitely many such \( p \), and therefore \( P_j(0) \) is a root of \( b_{a,0} \).

**Remark 3.14.** Let \( a \) be a principal ideal generated by \( f \). Suppose that the analogue of the setup in Problem 3.10 holds for a jumping coefficient \( \mu \in (0, 1] \) (around the origin) of \( f_0 \). More precisely, suppose that \( J \) and \( N \) are such that if \( p \equiv 1 \pmod{N} \), then \( c^J(f_p) = \mu \) (a natural choice for such a \( J \) is \( J = I(f^\mu) \)). We may choose such \( p \) so that \( \mu(p - 1) \) is an integer. Since \( \mu \leq 1 \), it follows from Proposition 1.3 that in this case we have \( \nu_a^J(p^e) = \mu(p^e - 1) \) for all \( e \geq 1 \). Remark 3.13 implies now that \( -\mu \) is a root of \( b_{f,0}(s) \). As we have already mentioned, this is proved in [BLSV], but this would provide an “explanation” from our point of view.

**Remark 3.15.** It is an interesting question which roots of \( b_{a,0} \) can be given by the procedure in Remark 3.13. It is proved in [BMS1] that this is the case for all the roots if \( a \) is a monomial ideal. On the other case, Example 1.1 below shows that some roots may not come from our approach.
4. Examples

Example 4.1. Let $n \geq 3$ and $f = X_1X_2 + X_3^2 + \ldots + X_n^2$, so its Bernstein-Sato polynomial is given by $b_f(s) = (s + 1)(s + \frac{3}{2})$ (see [Kal], Example 6.19, but this is actually one of the few examples which can be computed directly). We will see that we can not account for the root $-\frac{3}{2}$ by the procedure described in Remark 3.13.

We claim that for every $p$ and for every $e \geq 1$, we have $\nu_f(p^e) = p^e - 1$. To see this note that if over $\mathbb{F}_p$, we have $f^e \in (X_1^{p^e}, \ldots, X_n^{p^e})$, then $(X_1X_2)^e \in (X_1^{p^e}, \ldots, X_n^{p^e})$, as follows by choosing a monomial order on the polynomial ring such that in$(f)$ is contained in the convex hull of $\alpha_i \lambda_i$ for every $i$. We will see that we can not account for equality.

This shows that the smallest nonzero F-threshold is $c(f_p) = 1$. Proposition 1.10 shows that if $\lambda$ is an F-threshold of $f_p$ which is not an integer, then the fractional part of $\lambda$ gives an F-threshold in $(0, 1)$, a contradiction. Therefore the set of F-thresholds of $f_p$ consists of the set of positive integers.

Using Proposition 1.9 it follows that for every ideal $J$ contained in $(X_1, \ldots, X_n)$ and such that $f \in \text{Rad}(J)$, and for every $p$, there is a positive integer $m$ such that $\nu_f(p) = mp - 1$ for all $e$. Therefore the only root of $b_f(s)$ we get by the procedure described in Remark 3.13 is $-1$.

Example 4.2. Consider $f \in \mathbb{Z}[X_1, \ldots, X_n]$ which we write as $f = \sum_{i=1}^L c_i X_{\alpha_i}$, where all $\alpha_i = (\alpha_{i,1}, \ldots, \alpha_{i,n}) \in \mathbb{N}^n$ and all $c_i \in k$ are nonzero. We assume that $\alpha_1, \ldots, \alpha_r$ are affinely independent, i.e. if $\sum_i \lambda_i \alpha_i = 0$ and $\sum_i \lambda_i = 0$ for $\lambda = (\lambda_i) \in \mathbb{Q}^r$, then $\lambda_i = 0$ for all $i$. We will assume also that for every $j \leq n$ there is $i \leq r$ with $\alpha_{i,j} > 0$ (otherwise we may work in a smaller polynomial ring).

Let $a = (X^{\alpha_i}|1 \leq i \leq r)$. One can check that our condition of $f$ implies that $f$ is generic with respect to $a$ in the following sense. If $P'$ is a compact face of the convex hull $P_a$ of $\{\alpha_i|1 \leq i \leq r\}$, and if $g$ is the sum of the terms in $f$ which correspond to elements in $P'$, then the differential $dg$ does not vanish on $(\mathbb{C}^*)^n$. Under this assumption it is proved in [Ho1] that around the origin we have $I(a^\alpha) = I(f^\alpha)$ for $\alpha < 1$. In particular, we have $\text{lct}(f) = \min\{1, \text{lct}(a)\}$ (note that $\text{lct}(a) = \text{lct}(f_a)$ as $a$ is a monomial ideal).

We start by computing $\nu_f(p)$ when $p \gg 0$. Over $\mathbb{Z}$ we have

$$\nu_f(p) = \frac{m!}{a_1! \ldots a_r!} \sum c_{a_1} \ldots c_{a_r} X^{\sum_i a_i \alpha_i},$$

where the sum is over those $a = (a_i) \in \mathbb{N}^r$ with $\sum_i a_i = m$. Our hypothesis on $f$ implies that if $a \neq b$ and $\sum_i a_i = \sum_i b_i$, then we have $X^{\sum_i a_i \alpha_i} \neq X^{\sum_i b_i \alpha_i}$.

We may assume that $p$ does not divide any of the $c_{a_i}$, so if $m \leq p - 1$, then $p$ does not divide any of the coefficients in (11). Hence a monomial $X^b$ appears in $f_p$ if and only if there are $a_1, \ldots, a_r \in \mathbb{N}$ such that $\sum_i a_i = m$ and $\sum_i a_i \alpha_i = p$. Therefore $f_p \in (X_1^p, \ldots, X_n^p)$ if and only if the following holds: for every $(a_i) \in \mathbb{N}^r$ with $\sum_i a_i \alpha_i \leq p - 1$ for all $j$, we have $\sum_i a_i \leq m - 1$. 
Let
\[ Q := \{(a_1, \ldots, a_r) \in \mathbb{R}_+^r \mid \sum_{i=1}^r a_i \alpha_{ij} \leq 1 \text{ for all } j \}. \]

The above discussion shows that
\[ \nu_f(p) = \min\{p - 1, \max_{b \in (p-1)Q \cap \mathbb{N}^r} \sum_i b_i\}. \tag{11} \]

Compare this with the following formula for \( \text{lc}(a) \) which follows easily from the description of \( \text{lc}(a) \) in [Ho2] (see, for example, Proposition 3.10 in [BMS1]):
\[ \text{lc}(a) = \max_{b \in Q} \sum_i b_i. \tag{12} \]

Note that since for every \( j \leq r \) we have \( \alpha_{i,j} > 0 \) for some \( i, Q \) is bounded.

It is more complicated to compute \( \nu_f(p^e) \) for \( e \geq 2 \). Let \( c = \text{lc}_0(f) \). We show now that \( \nu_f(p^e) = c(p^e - 1) \) for all \( e \) when \( p \equiv 1 \pmod{N} \), where \( N \) will be suitably chosen.

If \( \text{lc}_0(f) = 1 \), choose any \( v \in Q \cap Q^r \) such that \( \sum_i v_i = 1 \), and if \( \text{lc}_0(f) < 1 \), let \( v \) be one of the vertices of \( Q \), such that \( \sum_i v_i = \max_{v \in Q} \sum_i b_i \). We take \( N \) such that \( Nv_i \) is an integer for all \( i \). Note that we have \( c = \sum_i v_i \).

If \( p \equiv 1 \pmod{N} \), then \( a_i := (p^e - 1)v_i \in \mathbb{N} \), and in order to show that \( \nu_f(p^e) \geq c(p^e - 1) \), it is enough to show that \( p \) does not divide \((p^e - 1)c)/a_1! \ldots a_r! \). Therefore it is enough to check that
\[ [c(p^e - 1)/p^e'] = \sum_{i=1}^r [(p^e - 1)v_i/p^e'] \]
for \( 1 \leq e' \leq e \), where \([x]\) is the integral part of \( x \). This follows by an easy computation.

Dividing by \( p^e \) and passing to the limit, we deduce that if \( p \equiv 1 \pmod{N} \), then \( c(f_p) \geq \text{lc}_0(f) \). On the other hand, the reverse inequality holds by Theorem 3.3 (here this follows also from the fact that \( f \in a \) and \( \text{lc}_0(f) = \text{lc}(a) \)). Note that using Proposition 1.9 we deduce that \( \nu_f(p^e) = c(p^e - 1) \) for all \( e \) and all \( p \) as above. In particular, we see that \( c(f_p) \) exhibits the behavior described in Problem 3.7.

**Example 4.3.** Let \( f = x^2 + y^3 \), so \( \text{lc}_0(f) = 1/2 + 1/3 = 5/6 \). Moreover, \( b_{f,0} \) has simple roots \(-5/6, -1, -7/6 \) (see [Ka1], Example 6.19). We give below the list of F-pure thresholds of \( f_p \). We see that for \( p \neq 2, 3 \), the behavior depends on the congruence class of \( p \pmod{3} \). Note that we see the behavior described in Problems 3.7, 3.8 and 3.10 (when \( J = m \)). Moreover, we obtain by our procedure all the roots of \( b_{f,0}(s) \).

1. If \( p = 2 \), then \( c(f_p) = 1/2 \).
2. If \( p = 3 \), we have \( c(f_p) = 2/3 \).
3. If \( p \equiv 1 \pmod{3} \), then \( c(f_p) = 5/6 \). We have \( \nu_f(p^e) = 5/6(p^e - 1) \) for all \( e \), so this gives the root \(-5/6 \) of \( b_{f,0}(s) \).
(4) If $p \equiv 2 \pmod{3}$ and $p \neq 2$, then $c(f_p) = \frac{5}{6} - \frac{1}{6p}$, so

$$
\nu_f(p^e) = \begin{cases} \\
\frac{5}{6}p - \frac{7}{6} & \text{if } e = 1, \\
\frac{5}{6}p^e - \frac{5}{6}p^{e-1} - 1 & \text{if } e \geq 2.
\end{cases}
$$

Therefore we get the roots $-\frac{2}{6}$ and $-1$ of $b_{f,0}(s)$.

**Example 4.4.** Let $f = x^2 + y^7$. The log canonical threshold is given by $\text{lc}_0(f) = \frac{1}{2} + \frac{1}{7} = \frac{9}{14}$. All the roots of $b_{f,0}$ are simple, and they are

$$
-\frac{9}{14}, -\frac{11}{14}, -\frac{13}{14}, -1, -\frac{15}{14}, -\frac{17}{14}, -\frac{19}{14}
$$

(see [Ka1], Example 6.19).

We assume that $p \neq 2, 7$ and we give the description of the F-pure thresholds and of the functions $\nu_f(p^e)$. The behavior depends on the congruence class of $p \mod 7$. We see again the behavior described in Problems 3.7, 3.8 and 3.10. In addition, we get all the roots of $b_{f,0}$ by our procedure.

(1) If $p \equiv 1 \pmod{7}$, then $c(f_p) = \frac{9}{14}$, and $\nu_f(p^e) = \frac{9}{14}(p^e - 1)$ for all $e$. This gives the root $-\frac{9}{14}$ of $b_{f,0}(s)$.

(2) If $p \equiv 2 \pmod{7}$, then $c(f_p) = \frac{9}{14} - \frac{1}{14p^2}$. Hence

$$
\nu_f(p^e) = \begin{cases} \\
\frac{9}{14}p - \frac{11}{14} & \text{if } e = 1, \\
\frac{9}{14}p^2 - \frac{15}{14} & \text{if } e = 2, \\
\frac{9}{14}p^e - \frac{9}{14}p^{e-2} - 1 & \text{if } e \geq 3.
\end{cases}
$$

This gives the roots $-\frac{11}{14}, -\frac{15}{14}$ and $-1$ of $b_{f,0}(s)$.

(3) If $p \equiv 3 \pmod{7}$, then $c(f_p) = \frac{9}{14} - \frac{5}{14p^3}$. Therefore

$$
\nu_f(p^e) = \begin{cases} \\
\frac{9}{14}p - \frac{13}{14} & \text{if } e = 1, \\
\frac{9}{14}p^2 - \frac{11}{14} & \text{if } e = 2, \\
\frac{9}{14}p^3 - \frac{19}{14} & \text{if } e = 3, \\
\frac{9}{14}p^e - \frac{9}{14}p^{e-3} - 1 & \text{if } e \geq 4.
\end{cases}
$$

We get the roots $-\frac{13}{14}, -\frac{11}{14}, -\frac{19}{14}$ and $-1$ of $b_{f,0}(s)$.

(4) If $p \equiv 4 \pmod{7}$, then $c(f_p) = \frac{9}{14} - \frac{1}{14p}$. Hence

$$
\nu_f(p^e) = \begin{cases} \\
\frac{9}{14}p - \frac{15}{14} & \text{if } e = 1, \\
\frac{9}{14}p^e - \frac{9}{14}p^{e-1} - 1 & \text{if } e \geq 2.
\end{cases}
$$

This gives the roots $-\frac{15}{14}$ and $-1$ of $b_{f,0}(s)$.

(5) If $p \equiv 5 \pmod{7}$, then $c(f_p) = \frac{9}{14} - \frac{3}{14p}$. Therefore

$$
\nu_f(p^e) = \begin{cases} \\
\frac{9}{14}p - \frac{17}{14} & \text{if } e = 1, \\
\frac{9}{14}p^e - \frac{3}{14}p^{e-1} - 1 & \text{if } e \geq 2.
\end{cases}
$$

This gives the roots $-\frac{17}{14}$ and $-1$ for $b_{f,0}(s)$. 
(6) If \( p \equiv 6 \pmod{7} \), then \( c(f_p) = \frac{9}{14} - \frac{5}{14p} \). Hence

\[
\nu_f(p^e) = \begin{cases} 
\frac{9}{14}p - \frac{9}{14} & \text{if } e = 1, \\
\frac{9}{14}p^e - \frac{5}{14}p^{e-1} - 1 & \text{if } e \geq 2.
\end{cases}
\]

This gives the roots \(-\frac{9}{14}\) and \(-1\) of \( b_{f,0}(s) \).

**Example 4.5.** Let \( f = x^5 + y^4 + x^3y^2 \). The following are the roots of \( b_{f,0}(s) \) (see [Ya])

\[
-\frac{9}{20}, -\frac{11}{20}, -\frac{13}{20}, -\frac{7}{10}, -\frac{17}{20}, -\frac{9}{20}, -\frac{19}{20}, -1, -\frac{21}{20}, -\frac{11}{20}, -\frac{23}{20}, -\frac{13}{20}, -\frac{27}{20}.
\]

As in Example 4.2, since the exponents of the monomials in \( f \) satisfy that genericity condition, we can compute the jumping coefficients of \( f \) using Howald's description from [Ho2]. The ones in \((0, 1]\) are

\[
\frac{9}{20}, \frac{13}{20}, \frac{7}{10}, \frac{17}{20}, \frac{9}{20}, \frac{19}{20}, 1.
\]

As pointed out in [Sa], the interest in this example comes from the fact that there is a root \( \lambda \in (-1, 0) \) of \( b_{f,0}(s) \) such that \(-\lambda\) is not a jumping coefficients of \( f \) (namely \( \lambda = -\frac{11}{20} \)). Note also that \( f \) has an isolated singularity at the origin.

We show that we can get all roots of \( b_{f,0}(s) \) by the procedure described in Remark 3.13. Note however that it is not enough to consider only \( c^f_J(p^e) \) for \( J = m \). We will use the notation in Proposition 2.14 to index the functions \( \nu^J(-) \). In particular, \( \nu_1(p^e) = \nu_f(p^e) \).

We assume that \( p \neq 2, 5 \) and we compute \( \nu_1(p^e) \), depending on the congruence class of \( p \) mod 20.

1. If \( p \equiv 1 \pmod{20} \), then \( \nu_1(p^e) = \frac{9}{20}(p^e - 1) \) for all \( e \geq 1 \).

2. If \( p \equiv 3 \pmod{20} \), then \( \nu_1(p^e) = \begin{cases} 
\frac{9}{20}p - \frac{27}{20} & \text{if } e = 1, \\
\frac{9}{20}p^e - \frac{7}{20}p^{e-1} - 1 & \text{if } e \geq 2.
\end{cases} \)

3. If \( p \equiv 7 \pmod{20} \), then \( \nu_1(p^e) = \begin{cases} 
\frac{9}{20}p - \frac{23}{20} & \text{if } e = 1, \\
\frac{9}{20}p^e - \frac{3}{20}p^{e-1} - 1 & \text{if } e \geq 2.
\end{cases} \)

4. If \( p \equiv 9 \pmod{20} \), then \( \nu_1(p^e) = \begin{cases} 
\frac{9}{20} - \frac{21}{20} & \text{if } e = 1, \\
\frac{9}{20}p^e - \frac{1}{20}p^{e-1} - 1 & \text{if } e \geq 2.
\end{cases} \)

5. If \( p \equiv 11 \pmod{20} \), then

\[
\nu_1(p^e) = \begin{cases} 
\left( \frac{9}{20}p - \frac{19}{20} \right) \cdot \frac{p^{e+1} - 1}{p - 1} + \left( \frac{19}{20}p - \frac{9}{20} \right) \cdot \frac{p^e - p}{p - 1} & \text{if } e \text{ is odd}, \\
\frac{9}{20}(p^e - 1) & \text{if } e \text{ is even}.
\end{cases} \]

6. If \( p \equiv 13 \pmod{20} \), then \( \nu_1(p^e) = \begin{cases} 
\frac{9}{20}p - \frac{17}{20} & \text{if } e = 1, \\
\frac{9}{20}p^2 - \frac{21}{20} & \text{if } e = 2, \\
\frac{9}{20}p^e - \frac{1}{20}p^{e-2} - 1 & \text{if } e \geq 3.
\end{cases} \)
(7) If \( p \equiv 17 \pmod{20} \), then \( \nu_1(p^e) = \begin{cases} \frac{9}{20}p - \frac{13}{20} & \text{if } e = 1, \\ \frac{9}{20}p^2 - \frac{21}{20} & \text{if } e = 2, \\ \frac{9}{20}p^e - \frac{1}{20}p^{e-1} - 1 & \text{if } e \geq 3. \end{cases} \)

(8) If \( p \equiv 19 \pmod{20} \), then \( \nu_1(p^e) = \left( \frac{9}{20}p - \frac{11}{20} \right) (1 + p + \ldots + p^{e-1}) \) for all \( e \geq 1 \).

We see that in this way we have accounted for the following roots of \( b_{f,0}(s) \): \(-\frac{9}{20}, -\frac{11}{20}, -\frac{13}{20}, -\frac{17}{20}, -\frac{19}{20}, -1, -\frac{21}{20}, -\frac{23}{20}, -\frac{27}{20} \). In order to get the other four roots, we need to compute also some values of \( \nu_3(p) \).

(1) If \( p \equiv 1 \pmod{10} \), then \( \nu_3(p) = \frac{7}{10}(p - 1) \).

(2) If \( p \equiv 3 \pmod{10} \), then \( \nu_3(p) = \frac{7}{10}p - \frac{11}{10} \).

(3) If \( p \equiv 5 \pmod{10} \), then \( \nu_3(p) = \frac{7}{10}p - \frac{9}{10} \).

(4) If \( p \equiv 7 \pmod{10} \), then \( \nu_3(p) = \frac{7}{10}p - \frac{13}{10} \).

Therefore we recover also the roots \(-\frac{7}{10}, -\frac{9}{10}, -\frac{11}{10}, -\frac{13}{10} \) of \( b_{f,0}(s) \).

Note that if \( p \equiv 19 \pmod{20} \), then

\[
c(f_p) = \frac{9p - 11}{20(p - 1)},
\]

so this gives an example when \( c(f_p) \) is not a polynomial in \( \frac{1}{p} \).

**Example 4.6.** Let \( f \) be a homogeneous polynomial of degree \( n \) in \( \mathbb{Z}[X_1, \ldots, X_n] \), defining a smooth hypersurface \( Y \) in \( \mathbb{P}^{n-1} \). It is well-known that in this case \( \text{lc}_0(f) = 1 \) (see [Kol]). On the other hand, it follows from Proposition 2.16 that \( c(f_p) = 1 \) if and only if the action of the Frobenius morphism on \( H^{n-1}_m(R/(f_p)) \) is injective. Here \( R = \mathbb{F}_p[X_1, \ldots, X_n]/(X_1, \ldots, X_n) \).

This action is injective if and only if it is injective on the socle of \( H^{n-1}_m(R/(f_p)) \). We assume that \( p \) is large enough, so \( Y_p \), the reduction mod \( p \) of \( Y \), is smooth. It follows that \( c(f_p) = 1 \) if and only if the action induced by the Frobenius morphism on \( H^{n-2}(Y_p, \mathcal{O}_{Y_p}) \) is injective.

If \( n = 3 \), then \( Y \) is an elliptic curve. In this case we see that \( c(f_p) = 1 \) if and only if \( Y \) is not supersingular. There are two cases: suppose first that \( Y \) has complex multiplication (over \( \mathbb{C} \)). In this case, \( Y_p \) is supersingular if and only if \( p \) is inert in the imaginary quadratic CM field.

On the other hand, if \( Y \) has no complex multiplication then Serre [Se] proved that the set of primes \( p \) for which \( Y_p \) is supersingular has natural density zero in the set of all primes. This clearly suggests that the behaviour of \( c(f_p) \) does not depend on the congruence of \( p \) modulo some \( N \), as in Problems 5.7 and 5.8. It would be interesting to compute explicitly the F-pure thresholds at the primes where the curve is supersingular.

We mention a result of Elkies [El] which is relevant in this setting: it says that for every elliptic curve \( Y \) as above, there are infinitely many primes \( p \) for which \( Y_p \) is supersingular, and therefore \( c(f_p) \neq 1 \).
As B. Conrad and N. Katz pointed out to us, if $K$ is the field obtained by adjoining to $\mathbb{Q}$ all points of order $\ell$ of $Y$ (for some odd prime $\ell$), then for every odd prime $p$ such that $p$ splits completely in $K$, the curve $Y_p$ is not supersingular. This provides an affirmative answer to Question 3.9 in this example.

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