Sliced Mutual Information: 
A Scalable Measure of Statistical Dependence

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Abstract

Mutual information (MI) is a fundamental measure of statistical dependence, with a myriad of applications to information theory, statistics, and machine learning. While it possesses many desirable structural properties, the estimation of high-dimensional MI from samples suffers from the curse of dimensionality. Motivated by statistical scalability to high dimensions, this paper proposes sliced MI (SMI) as a surrogate measure of dependence. SMI is defined as an average of MI terms between one-dimensional random projections. We show that it preserves many of the structural properties of classic MI, while gaining scalable computation and efficient estimation from samples. Furthermore, and in contrast to classic MI, SMI can grow as a result of deterministic transformations. This enables leveraging SMI for feature extraction by optimizing it over processing functions of raw data to identify useful representations thereof. Our theory is supported by numerical studies of independence testing and feature extraction, which demonstrate the potential gains SMI offers over classic MI for high-dimensional inference.

1 Introduction

Mutual information (MI) is a universal measure of dependence between random variables, defined as

$$I(X;Y) := \int_{X \times Y} \log \left( \frac{dP_{X,Y}}{dP_X \otimes P_Y} \right) dP_{X,Y},$$

(1)

where $(X,Y) \sim P_{X,Y}$ and $dP/dQ$ is the Radon-Nikodym derivative of $P$ with respect to (w.r.t.) $Q$. It possesses many desirable properties, such as meaningful units (bits or nats), nullification if and only if (iff) $X$ and $Y$ are independent, convenient variational forms, and invariance to bijections. In fact, MI can be obtained axiomatically as a unique (up to a multiplicative constant) functional satisfying several natural ‘informativeness’ conditions [1]. As such, it found a variety of applications in communications, information theory, and statistics [1,2]. More recently, it was adopted as a figure of merit for studying [3,4,5,6,7,8] and designing [9,10,11,12,13] machine learning models.

MI is a functional of the joint distribution $P_{X,Y}$ of the considered random variables. In practice, this distribution is often not known and only samples from it are available, thereby necessitating estimation of $I(X;Y)$. While this topic has received considerable attention [14,15,16,17,18], MI is fundamentally hard to estimate in high-dimensional settings due to an exponentially large (in dimension) sample complexity [19]. Motivated by statistical efficiency in high dimensions and inspired by recent slicing techniques of statistical divergences [20,21,22,23], this paper introduces sliced MI (SMI) as a surrogate notion of informativeness. We show that SMI inherits many of the properties of classic MI, while allowing for efficient estimation. Furthermore, in certain aspects, SMI is more compatible with modern machine learning practice than classic MI. In particular, deterministic transformations of the random variables can increase SMI, e.g., if the resulting codomains have
more informative slices (in classic MI sense). This enables using SMI as a figure of merit for feature extraction by identifying transformation (e.g., NN parameters) that maximize it.

1.1 Contributions

SMI is defined as the average of MI terms between one-dimensional random projections. Namely, if \( S^{d-1} \) denotes the \( d \)-dimensional sphere (whose surface area is designated by \( S_{d-1} \)), we define

\[
SI(X;Y) := \frac{1}{S_{d-1} S_{d-1}} \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} I(\theta^T X; \phi^T Y) d\theta d\phi. \tag{2}
\]

We may similarly define a max-sliced version by maximizing over projection directions, as opposed to averaging over them. Despite the projection of \( X \) and \( Y \) to a single dimension, SMI preserves many of the properties of MI. For instance, we show that SMI nullifies iff random variables are independent, it satisfies a chain rule, can be represented as a reduction in sliced entropy, admits a variational form, etc. Further, SMI between (jointly) Gaussian variables is tightly related to their canonical correlation coefficient \([24]\). This is in direct analogy to the relation between classic MI of Gaussian variables \((X,Y)\) and their correlation coefficient, where \( I(X;Y) = \frac{1}{2} \log (1 - \rho^2(X,Y)) \).

SMI is well-positioned for statistical estimation in high-dimensional setups, where one estimates \( SI(X;Y) \) from \( n \) i.i.d. samples of \( P_{X,Y} \). While the error of standard MI (or entropy) estimates, e.g., those in \([16, 25, 26]\), scales as \( n^{-1/d} \) when \( d \) is large, the same estimators admit statistical efficiency when \( d = 1, 2 \), converging at (near) parametric rates. Combining such estimators with Monte-Carlo (MC) sampling to approximate the integral over the unit sphere, we prove that the overall error scales (up to log factors) as \( m^{-1/2} + n^{-1/2} \), where \( m \) is the number of MC samples and \( n \) is the size of the high-dimensional dataset. We validate our theory on synthetic experiments, demonstrating that SMI is a scalable alternative to classic MI when dealing with high-dimensional data.

A notable contrast between classic and sliced MI involves the data processing inequality (DPI). Classic MI cannot grow as a result of processing the involved variables, namely, \( I(X;Y) \geq I(f(X);Y) \) for any deterministic function \( f \). This is since MI encodes arbitrarily fine details about \((X,Y)\) as variables in the ambient space, and transforming them cannot reveal anything that was not already there. SMI, on the other hand, only considers one-dimensional projections of \( X \) and \( Y \), some of which can be more correlated than others. Consequently, SMI can grow as a result of deterministic transformations, i.e., \( SI(X;Y) < SI(f(X);Y) \) is possible if \( f \) of \( X \) are more informative about \( Y \) than those of \( X \) itself. We show theoretically and demonstrate empirically that SMI is increased by projecting the data on more informative directions, highlighting its compatibility with feature extraction tasks.

2 Preliminaries and Background

We take \( \mathcal{P}(\mathbb{R}^d) \) as the class of all Borel probability measures on \( \mathbb{R}^d \). Elements of \( \mathcal{P}(\mathbb{R}^d) \) are denoted by uppercase letters, with subscripts to indicate the associated random variables, e.g., \( P_X \) or \( P_{X,Y} \). The support of \( P_X \) is \( \text{supp}(P_X) \). Our focus throughout is on absolutely continuous random variables; we use lowercase letters, such as \( p_X \) or \( p_{X,Y} \), to denote probability density functions (PDFs). For a function \( f : \mathbb{R}^d \to \mathbb{R}^d \) and a distribution \( P_X \in \mathcal{P}(\mathbb{R}^d) \), we write \( f_* P_X \) for the pushforward measure of \( P_X \) through \( f \), i.e., \( f_* P_X(A) = P_X(f^{-1}(A)) \). The \( L^p(\mathbb{R}^d) \) norm of \( f \) is denoted by \( \|f\|_{p,d} \). The \( d \)-dimensional unit sphere is \( S^{d-1} \), and its surface area is \( S_{d-1} = 2\pi^{d/2}/\Gamma(d/2) \), with \( \Gamma \) as the gamma function. We also define slicing along \( \theta \) as \( \pi^\theta(x) := \theta^T x \).

**Mutual information and entropy.** Information measures, such as MI and entropy, are ubiquitous in information theory and machine learning. MI is defined in \([1]\) and can be equivalently written in terms of the Kullback-Leibler (KL) divergence as \( I(X;Y) := D_{KL}(P_{X,Y} \parallel P_X \otimes P_Y) \). \( I(X;Y) \) thus quantifies how far, in the KL sense, \((X,Y) \sim P_{X,Y} \) are from being independent. The differential entropy of a continuous random variable \( X \) with density \( p_X(X) \) is \( H(X) := -\mathbb{E} \left[ \log p_X(X) \right] \), quantifying a notion of uncertainty associated with \( X \). For a pair \((X,Y) \sim P_{X,Y} \), the conditional entropy of \( X \) given \( Y \) is \( H(X|Y) := \int_y H(X|Y = y) dP_Y(y) \), where \( H(X|Y = y) \) is computed w.r.t. \( P_{X|Y = y} \). Conditional MI is similarly defined as \( I(X;Y|Z) := \int_z I(X;Y|Z = z) dP_Z(z) \). With
these definitions, one can represent MI as
\[ I(X; Y) = H(X) - H(X | Y) = H(Y) - H(Y | X), \]
thus interpreting MI as the reduction in the uncertainty regarding one variable as a result of observing the other. Another useful decomposition is the MI chain rule
\[ I(X; Y; Z) = I(X; Z) + I(Y; Z | X). \]

**Data processing inequality.** The DPI states that
\[ I(X; Y) \geq I(X; Z), \]
if \( X \leftrightarrow Y \leftrightarrow Z \) forms a Markov chain. This inequality is a cornerstone for many information theoretic derivations and is natural when there are no computational restrictions on the model. However, given a restricted computational budget, processing the input may aid inference. Deep neural network classifiers are an excellent example: they generate a hierarchy of processed representations of the input that are increasingly useful (although not more informative in the Shannon MI sense) for inferring the label. The incompatibility between the DPI and deep learning practice was previously observed in [27], motivating their definition of a computationally restricted MI variant that can be grown from processing. As we show in Section 3.2, SMI also shares this property.

### 3 Sliced Mutual Information

Our goal is to define a surrogate notion of MI that is more scalable for computation and estimation from samples in high dimensions. We propose SMI as defined next.

**Definition 1 (Sliced MI).** Fix \((X, Y) \sim P_{X,Y} \in P(\mathbb{R}^{d_x} \times \mathbb{R}^{d_y})\). Let \( \Theta \sim \text{Unif}(\mathbb{S}^{d_x-1}) \) and \( \Phi \sim \text{Unif}(\mathbb{S}^{d_y-1}) \) be independent of each other and of \((X, Y)\). The SMI between \( X \) and \( Y \) is
\[
SI(X; Y) := \frac{1}{S_{d_x-1}S_{d_y-1}} \int_{\mathbb{S}^{d_x-1}} \int_{\mathbb{S}^{d_y-1}} I(\Theta^\top X; \Phi^\top Y | \Theta, \Phi) \, d\Theta \, d\Phi. \tag{3}
\]

In words, the SMI between two high-dimensional variables is defined as an average of MI terms between their one-dimensional projections. By the DPI, \( SI(X; Y) \leq I(X; Y) \) so we inherently introduce information loss. Nevertheless, we will show that SMI inherits key properties of MI such as discrimination between dependence and independence, chain rule, entropy decomposition, etc.

**Remark 1 (One-dimensional variables).** If \( d_x = 1 \) then \( \mathbb{S}^0 = \{ \pm 1 \} \) and we have \( SI(X; Y) = I(X; Y) \), which follows by invariance of MI to bijections and the independence of \((X, Y, \Phi)\) and \( \Theta \). Similarly, when \( d_x = d_y = 1 \) we have \( SI(X; Y) = I(X; Y) \).

**Remark 2 (Single projection direction).** When slicing statistical divergences, like the Wasserstein distance [20], one typically considers a single slicing direction. Namely, given that \( X \) and \( Y \) are of the same dimension \( d \), they are both projected onto the same \( \theta \in \mathbb{S}^{d-1} \) and the distances between \( \theta^\top X \) and \( \theta^\top Y \) are then averaged over the unit sphere. While this approach is possible also in the context of SMI, we chose to define it using two slicing directions, \( \theta \) and \( \phi \), for several reasons:

1. this definition is invariant to rotations of the spaces in which \( X \) and \( Y \) take values—with a single direction, rotating either space would change the SMI, seemingly an undesirable property for an information measure;
2. it gives rise to a crucial property of SMI, that \( SI(X; Y) = 0 \iff X \) and \( Y \) are independent (see Proposition 7), which does not hold with a single slicing direction 3
3. it fares naturally with variables of different dimensions (although one can use zero padding to circumvent this issue in the single-direction version); and
4. it is inspired by the canonical correlation coefficient [24], that also uses two projection directions.

To later establish a chain rule and entropy-based decompositions, we define SMI between more than two random variables, conditional SMI, and sliced entropy.

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1Assuming that the appropriate PDFs exist.

2Indeed, let \( X_1, X_2 \sim \mathcal{N}(0, 1) \) be independent, set \( X = (X_1, X_2)^\top \) and \( Y = (X_2, -X_1)^\top \). As 2-dimensional vectors, \( X \) and \( Y \) are dependent, but one readily verifies that \( \text{cov}(\theta^\top X, \theta^\top Y) = 0 \), for any \( \theta \in \mathbb{S}^1 \).

3This implies independence of \( \theta^\top X \) and \( \theta^\top Y \), hence the single slicing direction SMI would nullify in this case.
Definition 2 (Joint and conditional SMI). Let \((X, Y, Z) \sim P_{X,Y,Z} \in \mathcal{P}(\mathbb{R}^{d_x} \times \mathbb{R}^{d_y} \times \mathbb{R}^{d_z})\) and take \(\Theta \sim \text{Unif}(\mathbb{S}^{d_x-1})\), \(\Phi \sim \text{Unif}(\mathbb{S}^{d_y-1})\), and \(\Psi \sim \text{Unif}(\mathbb{S}^{d_z-1})\) mutually independent. The SMI between \((X, Y)\) and \(Z\) is defined as

\[
\text{SI}(X, Y; Z) := I(\Theta^\top X, \Phi^\top Y; \Psi^\top Z|\Theta, \Phi, \Psi).
\] (4a)

The conditional SMI between \(X\) and \(Y\) given \(Z\) is

\[
\text{SI}(X; Y|Z) := I(\Theta^\top X, \Phi^\top Y|\Theta, \Phi, \Psi, \Psi^\top Z).
\] (4b)

The expression in (4a) extends Definition 1.Conditional SMI is the sliced information given access to another projected random variable along with its projection direction. Accordingly, conditional SMI is in the spirit of the original definition of \(\text{SI}(X; Y)\), incorporating only projected data without introducing additional uncertainty about the direction.

Remark 3 (Extensions). Joint and conditional SMI have natural multivariate extensions. For example, \(\text{SI}(X_1, \ldots, X_n; Y_1, \ldots, Y_m) := I(\Theta^\top_n X_1, \ldots, \Theta^\top_n X_n; \Phi^\top_1 Y_1, \ldots, \Phi^\top_m Y_m|\Theta, \Phi)\), where \(\Theta = (\Theta_1, \ldots, \Theta_n)\) and \(\Phi = (\Phi_1, \ldots, \Phi_m)\). The extension of conditional SMI is similar.

Definition 3 (Sliced entropy). Let \((X, Y) \sim P_{X,Y} \in \mathcal{P}(\mathbb{R}^{d_x} \times \mathbb{R}^{d_y})\) and take \(\Theta \sim \text{Unif}(\mathbb{S}^{d_x-1})\) and \(\Phi \sim \text{Unif}(\mathbb{S}^{d_y-1})\) to be independent. The sliced entropy of \(X\) is \(\text{SH}(X) := H(\Theta^\top X|\Theta)\), while the conditional sliced entropy of \(X\) given \(Y\) is \(\text{SH}(X|Y) := H(\Theta^\top X|\Theta, \Phi^\top Y)\).

Sliced entropy is interpreted as the average uncertainty in one-dimensional projections of the considered random variable. Conditional sliced entropy is the remaining uncertainty when a projected version and the projection direction of another random variable is revealed.

The following proposition shows that SMI retains many of the properties of classic MI.

Proposition 1 (SMI properties). The following properties hold:

1. **Non-negativity**: \(\text{SI}(X; Y) \geq 0\) with equality iff \(X\) and \(Y\) are independent.

2. **Bounds**: \(\inf_{\theta \in \mathbb{S}^{d_x-1}} I(\theta^\top X; \phi^\top Y) \leq \text{SI}(X; Y) \leq \sup_{\phi \in \mathbb{S}^{d_y-1}} I(\theta^\top X; \phi^\top Y) \leq I(X; Y)\).

3. **KL divergence**: We have \(\text{SI}(X; Y) = \mathbb{E}_{\Theta, \Phi} \left[ D_{\text{KL}}(\pi_{\Theta}^\phi, \pi_{\Phi}^\theta) P_{X,Y} \| \pi_{\Theta}^\phi P_X \otimes \pi_{\Phi}^\theta P_Y \right] \).

4. **Chain rule**: For any random variables \(X_1, \ldots, X_n, Y, Z\), we have the decomposition \(\text{SI}(X_1, \ldots, X_n; Y, Z) = \text{SI}(X_1; Y) + \sum_{i=2}^{n} \text{SI}(X_i; Y|X_1, \ldots, X_{i-1})\). In particular, \(\text{SI}(X; Y, Z) = \text{SI}(X; Z) + \text{SI}(Y; Z|X)\).

5. **Tensorization**: Suppose that \((X_1, Y_1), \ldots, (X_n, Y_n)\) are mutually independent. Then \(\text{SI}(X_1, \ldots, X_n; Y_1, \ldots, Y_n) = \sum_{i=1}^{n} \text{SI}(X_i; Y_i)\).

The proof of Proposition 1 is given in Appendix A.1

Remark 4 (SMI versus MI). Proposition 1 shows that SMI inherits many of the favorable properties of classic MI. Nevertheless, we stress that SMI is posed as a new measure of dependence that (although closely related) is different from MI. In particular, the gap between MI and SMI may not be bounded\(^3\). SMI thus should not be treated as a proxy of MI, but rather as an alternative figure of merit. The premise of the SMI framework is that its meaningful structure does not translate into computational or statistical inefficiency. Indeed, Section 3.1 shows that SMI can be efficiently estimated with parametric rate (up to logarithmic factors).

Similarly to MI, the sliced version simplifies when the variables are jointly Gaussian.

Example 1 (Gaussian SMI). If \(X \sim \mathcal{N}(0, \Sigma_X)\) and \(Y \sim \mathcal{N}(0, \Sigma_Y)\) are jointly Gaussian with cross-covariance \(\Sigma_{XY}\), then

\[
\text{SI}(X; Y) = \frac{1}{2 \pi^{d_x-1} \pi^{d_y-1}} \int_{\mathbb{S}^{d_x-1}} \int_{\mathbb{S}^{d_y-1}} \log \left( \frac{1}{1 - \rho^2(\theta^\top X, \phi^\top Y)} \right) \, d\theta \, d\phi,
\]

\(^3\)See the Example from the beginning of Section 3.2 and note that under that setup, for any \(0 < \alpha < \infty\), we have \(I(g_{\alpha}(X); Y) = \infty\) while \(\text{SI}(g_{\alpha}(X); Y)\) is finite.
where $\rho(\theta^T X, \phi^T Y) := \frac{\theta^T \Sigma X \phi}{\sigma(\theta^T \Sigma X \phi)^{1/2}}$ is the correlation coefficient of $\theta^T X$ and $\phi^T Y$. Denoting by $p_{CCA}(X, Y) := \sup_{(\theta, \phi) \in \mathbb{S}^{d_x-1} \times \mathbb{S}^{d_y-1}} \rho(\theta^T X, \phi^T Y)$ the canonical correlation coefficient, we get $\text{SI}(X; Y) \leq -0.5 \log (1 - p_{CCA}^2(X, Y))$.

The Gaussian distribution is also special for sliced entropy, where, as for classic entropy, it maximizes $\text{SH}(X)$ under a fixed (mean and) covariance constraint.

**Proposition 2** (Gaussian maximizes sliced entropy). Let $P_1(\mu, \Sigma) := \{ P \in \mathcal{P}(\mathbb{R}^d) : \text{supp}(P) = \mathbb{R}^d, \ E_P[X] = \mu, E[(X - \mu)(X - \mu)^T] = \Sigma \}$. Then $\arg \max_{P \in P_1(\mu, \Sigma)} \text{SH}(P) = \mathcal{N}(\mu, \Sigma)$, i.e., the normal distribution maximizes sliced entropy inside $P_1(\mu, \Sigma)$.

Lastly, SMI admits a variational form in the spirit of the Donsker-Varadhan representation of MI.

**Proposition 3** (Variational form). Let $\Theta \sim \text{Unif}(\mathbb{S}^{d_x-1})$, $\Phi \sim \text{Unif}(\mathbb{S}^{d_y-1})$ be independent of each other and of $(X, Y) \sim P_{X,Y} \in \mathcal{P}(\mathbb{R}^{d_x} \times \mathbb{R}^{d_y})$, and set $(\tilde{X}, \tilde{Y}) \sim P_X \otimes P_Y$. We have

$$\text{SI}(X; Y) = \sup_{g: \mathbb{S}^{d_x-1} \times \mathbb{S}^{d_y-1} \times \mathbb{R}^2 \to \mathbb{R}} \mathbb{E}[g(\Theta, \Phi, \Theta^T X, \Phi^T Y)] - \log \mathbb{E}[e^{g(\Theta, \Phi, \Theta^T X, \Phi^T Y)}],$$

where the supremum is over all measurable functions for which both expectations are finite.

This representation is leveraged in Section 4.3 to implement a feature extractor based on SMI neural estimation. The proof is found in Appendix A.3.

### 3.1 Estimation

A main virtue of SMI is that its estimation from samples is much easier than classic MI. One may combine any MI estimator between scalar variables with an MC integrator to estimate SMI between high-dimensional variables without suffering from the curse of dimensionality. This gain is expected as SMI is defined as an average of low dimensional MI terms.

For $P_{A,B} \in \mathcal{P}(\mathbb{R} \times \mathbb{R})$, let $(A_1, B_1), \ldots, (A_n, B_n)$ be pairwise i.i.d. from $P_{A,B}$. Consider an MI estimator $\hat{l} : A^n \times B^n \to \mathbb{R}_{\geq 0}$ that attains $\delta(n)$ absolute error uniformly over a class $\mathcal{F}$ of distributions:

$$\sup_{P_{A,B} \in \mathcal{F}} \mathbb{E} \left[ |\hat{l}(A^n, B^n) - l(A; B)| \right] \leq \delta(n). \quad (5)$$

We use $\hat{l}$ to construct an estimator of SMI. Given high-dimensional pairwise i.i.d. samples $(X_1, Y_1), \ldots, (X_n, Y_n)$ from $P_{X,Y} \in \mathcal{P}(\mathbb{R}^{d_x} \times \mathbb{R}^{d_y})$, first note that $(\theta^T X_i, \phi^T Y_i)$, for $\theta \in \mathbb{S}^{d_x-1}$ and $\phi \in \mathbb{S}^{d_y-1}$, is distributed according to $(\pi^\theta, \pi^\phi) P_{X,Y}$. Thus, we can convert $\{(X_i, Y_i)\}_{i=1}^n$ into pairwise i.i.d. samples of the projected variables. Let $\Theta_1, \ldots, \Theta_m$ and $\Phi_1, \ldots, \Phi_m$ be i.i.d. according to $\text{Unif}(\mathbb{S}^{d_x-1})$ and $\text{Unif}(\mathbb{S}^{d_y-1})$, respectively, set $(\Theta_j^T X) := (\Theta_1^T X_1, \ldots, \Theta_j^T X_n)$ for $j = 1, \ldots, m$, and similarly define $(\Phi_j^T Y)^n$. We consider the following SMI estimator:

$$\hat{S}_{I_{n,m}} = \hat{S}_{I_{n,m}}(X^n, Y^n, \Theta^m, \Phi^m) := \frac{1}{m} \sum_{i=1}^m \hat{l}((\Theta_j^T X)^n, (\Phi_j^T Y)^n). \quad (6)$$

Pseudocode and computational complexity for $\hat{S}_{I_{n,m}}$ can be found in Appendix B.

#### 3.1.1 Non-asymptotic performance guarantees

We now present convergence guarantees for the estimator (6) over the following class of distributions:

$$\mathcal{F}_{d_x,d_y}(M) := \left\{ P_{X,Y} \in \mathcal{P}(\mathbb{R}^{d_x} \times \mathbb{R}^{d_y}) : \sup_{(\theta, \phi) \in \mathbb{S}^{d_x-1} \times \mathbb{S}^{d_y-1}} l(\theta^T X, \phi^T Y) \leq M, (\pi^\theta, \pi^\phi) P_{X,Y} \in \mathcal{F}, \forall (\theta, \phi) \in \mathbb{S}^{d_x-1} \times \mathbb{S}^{d_y-1} \right\},$$

i.e., the class of all $P_{X,Y}$ with bounded MI and projections that belong to the class $\mathcal{F}$ from (5).
Theorem 1 (Convergence rate). The following uniform error bound over $\mathcal{F}_{d_x,d_y}(M)$ holds:

$$\sup_{p_{X,Y} \in \mathcal{F}_{d_x,d_y}(M)} \mathbb{E} \left[ |\text{SI}(X;Y) - \hat{\text{SI}}_{n,m}| \right] \leq \frac{M}{2\sqrt{m}} + \delta(n).$$

(7)

See Appendix A.4 for the proof.

Remark 5 (Instance-dependent bound). An inspection of the proof of Theorem 7 reveals that (7) can be converted into the instance-dependent bound

$$\mathbb{E} \left[ |\text{SI}(X;Y) - \hat{\text{SI}}_{n,m}| \right] \leq \frac{1}{2} \sup_{(\theta,\phi) \in \mathbb{S}^{d_x-1} \times \mathbb{S}^{d_y-1}} I(\theta^\top X;\phi^\top Y) \cdot m^{-1/2} + \delta(n),$$

so long that $P_{X,Y} \in \mathcal{F}_{d_x,d_y}(M)$ for some $M$.

Theorem 1 applies to classes of distribution with uniformly bounded per-slice MI. Since this boundedness may be hard to verify in practice, we present a primitive sufficient condition for (7) to hold. Specifically, when $P_{X,Y}$ is log-concave and symmetric, it is enough to require that the canonical correlation coefficient of $(X,Y)$ is bounded. Recall that a probability measure $P \in \mathcal{P}(\mathbb{R}^d)$ is called log-concave if for any compact Borel sets $A$ and $B$ and $0 < \lambda < 1$, we have $P(\lambda A + (1 - \lambda)B) \geq P(A)^\lambda P(B)^{1 - \lambda}$. Let

$$\mathcal{F}_{d_x,d_y}^{(\text{LC})}(M) := \{ P_{X,Y} \in \mathcal{P}(\mathbb{R}^{d_x} \times \mathbb{R}^{d_y}) : P_{X,Y} \text{ is symmetric and log-concave}, \int_{\mathbb{R}^d} \rho_{\text{CCA}}(X,Y) \leq M \}.$$

The following error bound is proved in Appendix A.5.

Corollary 1 (Convergence for log-concave class). The following uniform error bound over $\mathcal{F}_{d_x,d_y}^{(\text{LC})}(M)$ holds:

$$\sup_{p_{X,Y} \in \mathcal{F}_{d_x,d_y}^{(\text{LC})}(M)} \mathbb{E} \left[ |\text{SI}(X;Y) - \hat{\text{SI}}_{n,m}| \right] \leq \left( \frac{1}{2} \log \left( \frac{\pi^2}{8} \frac{1}{1 - M} \right) \frac{1}{m} \right)^{-1/2} + \delta(n).$$

(8)

3.1.2 End-to-end SMI estimation guarantees over Lipschitz balls

To provide a concrete SMI estimator with guarantees, we instantiate the low-dimensional MI estimator from [26] for densities in the generalized Lipschitz class.

Definition 4 (Generalized Lipschitz class). For $d \in \mathbb{N}$, $p \in [2, \infty)$, $s > 0$, and $L \geq 0$, let $\text{Lip}_{s,p,d}(L)$ be the class of probability density functions $f : [0,1]^d \to \mathbb{R}$ with $\|f\|_{\text{Lip}_{s,p,d}} \leq L$, where

$$\|f\|_{\text{Lip}_{s,p,d}} := \|f\|_{p,d} + \sup_{t > 0} t^{-s} \sup_{e \in \mathbb{R}^d, \|e\| \leq 1} \|\Delta_t^s f\|_{p,d}$$

(9)

and $\Delta_t^s f(x) := \sum_{k=0}^r (-1)^{r-k} \binom{r}{k} f(x + \left( k - \frac{s}{2} \right) h)$. We note that the norm of $\Delta_t^s f$ is taken over the whole Euclidean space to ensure a smooth decay of $f$ at the boundary. Consequently, the $\text{Lip}_{s,p,d}(L)$ class includes, e.g., densities whose derivatives up to order $[s] - 1$ all vanish at the boundary, where $s$ is the smoothness parameter.

Differential entropy estimation over $\text{Lip}_{s,p,d}(L)$ was considered in [26], where an optimal estimator based on best polynomial approximation and kernel density estimation techniques was proposed. Adhering to their setup, for $A \sim P_A \in \mathcal{P}([0,1]^d)$ with density $p_A \in \text{Lip}_{s,p,d}(L)$, we denote the aforementioned entropy estimate based on $n$ i.i.d. samples $A^n$ by $\hat{\text{H}}(A^n)$.

The SMI estimator from (6) employs a MI estimate between scalar variables $(A,B)$. Assume their joint density is $p_{A,B} \in \text{Lip}_{s,p,2}(L)$ and let $(A^n, B^n)$ be i.i.d. samples. To estimate $\text{I}(A;B)$, consider

$$\hat{\text{I}}_{\text{Lip}}(A^n;B^n) := \hat{\text{H}}(A^n) + \hat{\text{H}}(B^n) - \hat{\text{H}}(A^n, B^n).$$

(10)

Plug (10) into (6) and let $\hat{\text{SI}}_{n,m}^{(\text{Lip})}$ be the resulting SMI estimate. We next state the effective estimation rate over $\mathcal{F}_{s,p,d_x,d_y}^{(\text{Lip})}(L,M) := \{ p_{X,Y} \in \text{Lip}_{s,p,d_x,d_y}(L) : \sup_{(\theta,\phi) \in \mathbb{S}^{d_x-1} \times \mathbb{S}^{d_y-1}} I(\theta^\top X;\phi^\top Y) \leq M \}.
We now discuss how SMI can increase via processing. In contrast to the DPI of classic MI, we show that SMI can be grow by extracting linear features of bound and scalar MI estimation error. This scalability to high dimensions is the expected gain from Corollary 2, i.e., that depends only on $d_x$, $d_y$, $p$, $s$, and $L$.

The proof of Corollary 2 (Appendix A.6) shows that densities in the generalized Lipschitz class have the property that all their projections are also in that class (with different parameters). We then bound $\delta(n)$ using [26] Theorem 4] to control the error of each differential entropy estimate in (10).

Remark 6 (SMI versus MI estimation rates). The SMI estimation rate from Corollary 2 is considerably faster than the $n^{-1/(d_x+d_y)}$ rate attainable when estimating classic MI [26]. Our bound shows that $n$ and $(d_x, d_y)$ are decoupled in the SMI convergence rate, unlike their interleaved dependence in MI estimation. The ambient dimension still enters the bound via the constant $C$, but its effect is expected to be much milder than in the classic case. As Theorem 7 shows, $(d_x, d_y)$ can only enter via $M$ and $\delta(n)$, both of which correspond to scalar MI terms (namely, the uniform per-sliced MI bound and scalar MI estimation error). This scalability to high dimensions is the expected gain from slicing.

Remark 7 (Optimal rate for smooth densities). Restricting attention to densities of maximal smoothness in Corollary 2 i.e., $s = 2$, the resulting rate is $0.5Mm^{-1/2} + Cn^{-1/2}(1 + (\log n)^{3/2(1-1/p)})$. Equating the number of MC and data samples, $m$ and $n$, the rate is parametric, up to polylog factors.

3.2 Extracting Sliced Information

We now discuss how SMI can increase via processing. In contrast to the DPI of classic MI, we show that SMI can grow by extracting linear features of $X$ and $Y$ that are informative of each other.

To illustrate the idea we begin with a simple example. Let $X = (X_1, X_2) \sim \mathcal{N}(0, I)$, $Y = X_1$, and consider $SI(X; Y)$ ($Y$ is a scalar and it is thus not sliced). For any $\theta = (\theta_1 \theta_2) \in S^1$ with $\theta_1 \neq 0$, we have $I(\theta^T X; Y) = I(X_1 + (\theta_2/\theta_1) X_2; X_1) = \frac{1}{2} \log (1 + (\theta_1/\theta_2)^2)$, where the last step uses the independence of $X_1$ and $X_2$. Consider the function $g_a : \mathbb{R}^2 \to \mathbb{R}$ given by $g_a(x_1, x_2) = (x_1 ax_2)^T$, for some $0 < a < 1$. Following the same procedure, we have $\theta^T g_a(X) = \theta_1 Y + \theta_2 X_2$ and $I(\theta^T g_a(X); Y) = \frac{1}{2} \log \left(1 + (\theta_1/(a \theta_2))^2\right) > I(\theta^T X; Y)$, for almost all $\theta \in S^1$, and consequently, $SI(X; Y) < SI(g_a(X); Y)$. Generally, this shows that by varying $a$ one can both create and diminish sliced information by processing via $g_a$.

When $a = 0$, we have $SI(g_a(X); Y) = \infty$, yielding $SI(g_a(X); Y) = \sup_{\theta} I(\theta^T X; Y)$. Thus, maximizing SMI by varying $a$ extracts the most informative feature $X_1$ and deletes the uninformative feature $X_2$. We next generalize this observation (see Appendix A.7 for the proof).

Proposition 4. Let $(X, Y) \sim \mathcal{P}(\mathbb{R}^{d_x} \times \mathbb{R}^{d_y})$ and consider optimizing the SMI between linear processing of $X$ and $Y$ in the following scenarios:

1. Arbitrary linear processing: For matrices $A_x, A_y$ and vectors $b_x, b_y$ of the appropriate dimension, we have
   \[\sup_{A_x, A_y, b_x, b_y} SI(A_x X + b_x; A_y Y + b_y) = \sup_{A_x, A_y} SI(A_x X; A_y Y) = \sup_{\theta, \phi} I(\theta^T X; \phi^T Y).\] (11)
   Furthermore, if $(\theta^*, \phi^*) \in \arg \max \{I(\theta^T X; \phi^T Y)\}$, then an optimal pair of matrices $A_x^*$ and $A_y^*$ have $\theta^*$ or $\phi^*$, respectively, in their first rows and zeros otherwise.

2. Rank constrained linear processing: For $d_1, d_2, r \in \mathbb{N}$ and $c > 0$, let $\mathcal{M}_{d_1, d_2}(r, c) := \{ A \in \mathbb{R}^{d_1 \times d_2} : \frac{1}{c} \leq \sigma_i(A) \leq \ldots \leq \sigma_1(A) \leq c\}$, where $\sigma_i(A)$ is the $i$th largest singular value of $A$. We have
   \[\sup_{A_x \in \mathcal{M}_{d_1, d_2}(r_x, c_x), A_y \in \mathcal{M}_{d_1, d_2}(r_y, c_y)} SI(A_x X; A_y Y) = \sup_{B_x \in \mathcal{M}_{d_1, d_2}(r_x, c_x), B_y \in \mathcal{M}_{d_1, d_2}(r_y, c_y)} SI(B_x X; B_y Y), \quad \forall c_x, c_y > 0.\]
   Furthermore, if $(B_x^*, B_y^*)$ are maximizers of the RHS, then $B_1$ (resp., $B_2$) has the first $r_x$ (resp., $r_y$) rows span the top $r_x$ (resp., $r_y$) scalar MI slicing directions and the remaining rows are zero.
The proposition suggests that SMI can be used as an objective for extracting informative linear features. The setup in Case 2 precludes reduction to Case 1. Indeed, if eigenvalues can shrink or grow without bound, it is always better to consider a maximizing slice than to average several slices.

**Remark 8 (Processing one variable).** Similar results hold when only one of the arguments \((X\) or \(Y)\) is processed. In this case, rather than the maximum being a projected MI term, it would be an SMI where the slicing is only in the opposite argument. For example, (11) becomes

\[
\sup_{A_x, b_x} SI(A_xX + b_x; Y) = \sup_{A_x} SI(A_xX; Y) = \sup_{\theta} I(\theta^T X; \Phi^T Y|\Phi), \text{ for } \Phi \sim \text{Unif}(S^{d_y-1}) \text{ independent of } (X, Y).
\]

Proposition 4 accounts for linear processing but the argument readily extends to nonlinear processing. For simplicity, the following corollary states the result for a shallow (single hidden layer) neural network (see Appendix A.8 for the proof).

**Corollary 3 (Shallow neural network).** Let \((X, Y) \sim P_{X,Y}\). For any scaling matrices \(A_x, A_y\), weight matrices \(W_x, W_y\), and bias vectors \(b_x, b_y\) of appropriate dimension, define \(X_{nn} := A_x\sigma(W_x^T X + b_x), Y_{nn} := A_y\sigma(W_y^T Y + b_y)\), where \(\sigma\) is a scalar, continuous, and monotonically increasing nonlinearity (e.g. sigmoid, tanh) and the hidden dimension is arbitrary. Then

\[
\sup_{A_x, A_y, W_x, W_y, b_x, b_y} SI(X_{nn}; Y_{nn}) = \sup_{\theta, \phi, W_x, W_y, b_x, b_y} I(\theta^T \sigma(W_x^T X + b_x); \phi^T \sigma(W_y^T Y + b_y)).
\]

![Figure 1: Convergence of the SMI estimator versus the number of data samples \(n\) and/or slice samples \(m\): (a) \(d = 3\), \(m_{\text{fixed}} = n_{\text{fixed}} = 10^4\); (b) \(d = 10\), \(m_{\text{fixed}} = n_{\text{fixed}} = 10^4\); (c) \(d = 10\), \(m, n\) varied independently.](image)

4 Empirical Results

4.1 Convergence of the SMI estimator

We validate the empirical convergence rates for SMI estimation derived in Section 3.1. Consider densities with smoothness parameter \(s = 2\) in the setup of Corollary 2 the expected convergence rate (up to log factors) is \(n^{-1/2} + m^{-1/2}\). While the theoretical results use the optimal estimator of [26] to obtain the tightest bounds, in our experiments we implemented the simpler Kozachenko–Leonenko
estimator. The justification for doing so comes from [25], who showed that this estimator achieves the same rate (up to log factors) as the optimal one from [26].

Figure 1 shows convergence of the estimated SMI RMSE for the case where $X$ and $Y$ are overlapping subsets of a standard normal random vector $Z \sim \mathcal{N}(0, I_d)$. For $d = 3$, we set $X = Z[1:3] := (Z_1 \ Z_2 \ Z_3)^\top$, $Y = Z[2:4]$ (i.e., 2 coordinates overlap). For $d = 10$, we take $X = Z_{1:10}, Y = Z_{5:15}$ (5 coordinates overlap). Convergence is shown when both $n$ and $m$ grow together (i.e., $n = m$), and when one is fixed to a large value and the other varies. The large value is chosen so that the error term corresponding to the fixed parameter is negligible compared to the varying term. For $d = 10$, we also plot results for $m, n$ varying independently. Appendix C provides corresponding MI estimation results.

4.2 Independence testing

Proposition 1 states that $\text{SI}(X; Y) = 0$ if and only if $X$ and $Y$ are independent. This implies that, like MI, we may use SMI for independence testing. MI-based independence tests of high-dimensional continuous variables can be burdensome due to slow convergence of MI estimation [26]. We show that, as our theory implies, SMI is a scalable alternative.

Figure 2 shows independence testing results for a variety of relationships between $X, Y$ pairs. The figure shows the area under the curve (AUC) of the receiver operating characteristic (ROC) for independence testing via SMI (or MI) thresholding as a function of the number of samples $n$ from the joint distribution. Both the SMI and MI were computed using the Kozachenko–Leonenko estimator [13]; the MC step for SMI estimation (see (6)) uses 1000 random slices, and the AUC ROC curves are computed from 100 random trials. The joint distribution of $(X, Y)$ in each case of Figure 2 is:

(a) One feature (linear): $X, Z \sim \mathcal{N}(0, I_d)$ i.i.d. and $Y = \frac{1}{\sqrt{d}} (\frac{1}{\sqrt{d}} (1^\top X) 1 + Z)$, where $1 := (1 \ldots 1)^\top \in \mathbb{R}^d$.

(b) One feature (sinusoid): $X, Z \sim \mathcal{N}(0, I_d)$ i.i.d. and $Y = \frac{1}{\sqrt{d}} (\frac{1}{\sqrt{d}} \sin(1^\top X) 1 + Z)$.

(c) Two features: $X, Z \sim \mathcal{N}(0, I_d)$ i.i.d. and $Y_i = \frac{1}{\sqrt{2}} \begin{cases} \frac{1}{\sqrt{d}} (1_{d/2}^\top 0 \ldots 0^\top) X + Z_i, & i \leq \frac{d}{2} \\ \frac{1}{\sqrt{d}} (0^\top \ldots 01_{d/2})^\top X + Z_i, & i > \frac{d}{2} \end{cases}$.

(d) Low rank common signal: $Z_1, Z_2 \sim \mathcal{N}(0, I_d)$ and $V \sim \mathcal{N}(0, I_2)$ are independent; $X = P_1 V + Z_1$ and $Y = P_2 V + Z_2$, where $P_1, P_2 \in \mathbb{R}^{d \times 2}$ are projection matrices (realized at the beginning of each iteration by drawing i.i.d. standard normal entries).

(e) Independent coordinates: $X, Z \sim \mathcal{N}(0, I_d)$ i.i.d. and $Y = \frac{1}{\sqrt{2}} (X + Z)$.

Note that in all the cases with underlying lower-dimensional structure, SMI scales well with dimension while MI does not; in the independent case of subfigure (d), both perform similarly and rather well. While the SMI is always on par or better than MI in these experiments, the results suggest that SMI performs best in structured (specifically, low rank) settings. This is because in these settings the MI term associated with random $(\Theta, \Phi)$ slices has lower variance. This is not the case for the unstructured setting of Figure 2(d). There, when dimension is high, random slices carry relatively little information compared to the maximum MI over slices (which attained between the $i$th coordinates of $X$ and $Y$ for any $i$). Since SMI is an average quantity, in this case it offers little gain over classic MI.

4.3 Feature extraction

In the above, we focused on a nonparametric estimator for which we derived tight bounds. In practice, applying neural estimators (à la MINE [29]) is more compatible with modern optimizers. The SMI neural estimator (S-MINE) relies on the variational representation from Proposition 3.

4

For every $n$, we generate 50 datasets comprising $n$ positive samples (i.e., drawn from the joint distribution) and 50 more dataset of negative samples in each setting. SMI and MI are then estimated over each dataset, the ROC curve is found, and the area under it computed. The ROC curve plots test performance (precision and recall) as the threshold is varied over all possible values. The AUC ROC quantifies the test’s discriminative ability: an omniscient classifier has AUC ROC 1, while random tests have AUC ROC 0.5.
We next combine S-MINE with independence testing, looking to maximize the SMI using trans-formations \(A_X, A_Y\), where \(X, Y\) are samples from a random MNIST class (either 0 or 1) and \(A \in \mathbb{R}^{10 \times d}\). Specifically, we choose a class \(C \sim \text{Ber}(0.5)\) and then sample \(X, Y\) uniformly from that class’ training dataset. In this setup, \(X, Y\) share up to 1 bit of information, i.e., \(I(X;Y) \leq H(C) = \log 2\). This suggests that maximizing the SMI between \(AX\) and \(AY\) will find an \(A\) that extracts information about \(C\), revealing whether \(X\) and \(Y\) are in the same class. Optimizing \(A\) yields an estimated SMI of 0.680 bits (compare this to, e.g., 0.0752 SMI achieved by a matrix \(A\) with i.i.d. standard normal entries). To confirm \(A\) is not being overfit, we also optimized the SMI over \(A\) when \(X, Y\) are drawn independently, i.e., no longer sharing a class, yielding 0.0289 estimated SMI (the ground truth is 0 here). These results indicate that (a) an SMI-based independence test would be successful at detecting dependence between \(X, Y\) and (b) optimizing \(A\) not only succeeds at significantly (almost 10\%\) increasing the SMI, but also comes relatively close to the true MI of 1 bit. Heatmaps of two rows of \(A\) rearranged into the MNIST image dimensions are shown in Figure 4. Observe the \(A_i\) visually correspond to the numeral 0, which, from a matched filter perspective, yields an \(AX\) (resp., \(AY\)) that is informative of whether \(X\) (resp., \(Y\)) is in class zero or not. This, in turn, conveys information of whether \(X, Y\) share a class (since 0 and 1 are the only options), as desired.

5 Summary and Concluding Remarks

Motivated to address the computational and statistical unscalability of MI to high dimensions, this paper proposed an alternative notion of informativeness dubbed sliced mutual information.
Figure 3: \((A^+_x, A^+_y)\) optimizers for S-MINE over a Gaussian dataset using S-MINE: (a) matrix \(A^+_x\); (b) matrix \(A^+_y\). Both have correctly recovered the unit vector \(e_1\) as their rows.

Figure 4: Solution for optimized \(A\)-transformed SMI of the 0-1 MNIST setup using S-MINE. Rows 0 and 1 of \(A\) are shown in (a) and (b), respectively.

(SMI). SMI projects high-dimensional random variables to scalars and averages over random projections. We showed that SMI shares many of the structural properties of classic MI, while enjoying efficient empirical estimation. We also showed that, as opposed to classic MI, SMI can be increased by processing the variables. This observation was quantified for linear and nonlinear SMI-based feature extractors. Experiments validating the theoretical study were provided, demonstrating dimension free empirical convergence rates, statistical efficiency for independence testing, and feature extraction examples.

Our results pose SMI as a favorable figure of merit for information quantification between high-dimensional random variables. We expect it to turn useful for a variety of applications in inference and machine learning, although a large-scale empirical exploration is beyond the scope of this paper. In particular, SMI seems well adapted for representation learning via the (sliced) InfoMax principle [30, 31], and we plan to test this hypothesis in future work. On the theoretical side, appealing future directions are abundant, including convergence guarantees for the neural SMI estimator used in Section 4.3, operational channel/source coding settings for which SMI characterizes the information-theoretic fundamental limit, and a statistical analysis of SMI-based independence testing.

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Also recall that independence is equivalent to zero classic mutual information. Denote \(\phi := \mathbb{E}[e^{itX + isY}]\), which means
\[
\phi(t, s) = \mathbb{E}[e^{itX}] \mathbb{E}[e^{isY}], \quad \forall t, s \in \mathbb{R}.
\]
Indeed, as \(\phi(t, s)\) decomposes into a product, i.e.,
\[
\phi(t, s) = \varphi_X(t)\varphi_Y(s) = \mathbb{E}[e^{itX}] \mathbb{E}[e^{isY}], \quad \forall t, s \in \mathbb{R},
\]

Also recall that independence is equivalent to zero classic mutual information. Denote \(X_\theta := \theta^TX\) and \(Y_\phi := \phi^TY\) and observe that \(\text{SI}(X;Y) = 0\) is equivalent to
\[
\int_{S_{d_x}^{d-1}} \int_{S_{d_y}^{d-1}} l(X_\theta;Y_\phi)d\theta d\phi = 0.
\]
but this is the same as
\[
\varphi_{X,Y}(\theta, \phi) = \varphi_X(t^\theta)\varphi_Y(s^\phi), \quad \forall t, s \in \mathbb{R}, \quad \theta \in S_{d_x}^{d-1}, \quad \phi \in S_{d_y}^{d-1}.
\]
Changing variables \(t' = t\theta\) and \(s' = s\phi\), the last equality holds iff
\[
\varphi_{X,Y}(t', s') = \varphi_X(t')\varphi_Y(s'), \quad \forall t' \in \mathbb{R}^{d_x}, \quad s' \in \mathbb{R}^{d_y},
\]
which means \(X\) and \(Y\) are independent.
We further note that the infimum in the lower bound is always attained, as is thus a minimum. This concludes the proof by noting that

$$I(X;Y) = \inf_{\theta \in \mathbb{S}_{d_x-1}, \phi \in \mathbb{S}_{d_y-1}} I(\theta^TX;\phi^TY) \leq \mathcal{I}(X;Y) \leq \sup_{\theta \in \mathbb{S}_{d_x-1}, \phi \in \mathbb{S}_{d_y-1}} I(\theta^TX;\phi^TY).$$

By the DPI for classic MI we further upper bound the right-hand side (RHS) by $I(X;Y)$.

We further note that the infimum in the lower bound is always attained, as is thus a minimum. This is because for any $(\theta_n, \phi_n)$, $(\theta, \phi) \in \mathbb{S}_{d_x-1} \times \mathbb{S}_{d_y-1}$ with $\theta_n \to \theta$ and $\phi_n \to \phi$, we have that $(\theta_n^TX, \phi_n^TY)$ converge to $(\theta^TX, \phi^TY)$ almost surely (in fact, surely) and therefore in distribution. Since MI is weakly lower semicontinuous, it attains a minimum on the compact set $\mathbb{S}_{d_x-1} \times \mathbb{S}_{d_y-1}$.

To attain the supremum one must impose additional regularity on the Lebesgue density of $P_{X,Y}$ to ensure that MI is continuous in the weak topology; see, e.g., [32, Theorem 1].

**Proof of Proposition 5**

This follows because conditional mutual information can be expressed as

$$I(X;Y|Z) = \mathbb{E}_Z \left[ D_{KL}(P_{X,Y|Z}|P_{X|Z}P_{Y|Z}) \right],$$

and because the joint distribution of $(\Theta^TX, \Phi^TY)$ given $\{\Theta = \theta, \Phi = \phi\}$ is $(\pi^\theta, \pi^\phi) P_{X,Y}$, while the corresponding conditional marginals are $\pi^\theta P_X$ and $\pi^\phi P_Y$, respectively.

**Proof of Proposition 2**

We only prove the small chain rule; generalizing to $n$ variables is straightforward. Consider:

$$\mathcal{I}(X,Y|Z) = I(\Theta^TX, \Phi^TY; \Psi^T|\Theta, \Phi, \Psi)$$

$$= I(\Theta^TX; \Psi^T|\Theta, \Phi, \Psi) + I(\Phi^TY; \Psi^T|\Theta, \Phi, \Psi, \Theta^TX),$$

where the last equality is the regular chain rule. Since $(X,Y,Z,\Theta,\Psi)$ are independent of $\Phi$, we have

$$I(\Theta^TX; \Psi^TZ|\Theta, \Phi, \Psi) = I(\Theta^TX; \Psi^TZ|\Theta, \Psi) = \mathcal{I}(X;Z).$$

We conclude the proof by noting that

$$I(\Phi^TY; \Psi^TZ|\Theta, \Phi, \Theta^TX) = \frac{1}{\mathbb{S}_{d_x-1}} \int_{\mathbb{S}_{d_x-1}} I(\Phi^TY; \Psi^TZ|\Theta = \theta, \Phi, \Theta^TX)d\theta$$

$$= \frac{1}{\mathbb{S}_{d_x-1}} \int_{\mathbb{S}_{d_x-1}} I(\Phi^TY; \Psi^TZ|\Phi, \Theta^TX)d\theta$$

$$= \mathcal{I}(Y;Z|X),$$

where the penultimate equality is because $(X,Y,Z,\Phi,\Psi)$ are independent of $\Theta$.

**Proof of Proposition 2**

By Definition 2 we have

$$\mathcal{I}(X_1, \ldots, X_n; Y_1, \ldots, Y_n) = I(\Theta_1^TX_1, \ldots, \Theta_n^TX_n; \Phi_1^TY_1, \ldots, \Phi_n^TY_n|\Theta_1, \ldots, \Theta_n, \Phi_1, \ldots, \Phi_n),$$

where the $\Theta_i$, $\Phi_i$ are all independent and uniform on their respective spheres. Now by mutual independence of the $\Theta_i$, $\Phi_i$ and $(X_i, Y_i)$ across $i$,

$$I(\Theta_1^TX_1, \ldots, \Theta_n^TX_n; \Phi_1^TY_1, \ldots, \Phi_n^TY_n|\Theta_1, \ldots, \Theta_n, \Phi_1, \ldots, \Phi_n) = \sum_{i=1}^n I(\Theta_i^TX_i; \Phi_i^TY_i|\Theta_i, \Phi_i)$$

$$= \sum_{i=1}^n \mathcal{I}(X_i; Y_i).$$

This concludes the proof.

**A.2 Maximum Sliced Entropy and Proof of Proposition 2**

In this section we prove the extended claim stated next, which includes Proposition 2 as the first item.

**Proposition 5** (Max sliced entropy). The following max sliced differential entropy statements hold.
1. **Mean and covariance:** Let $\mathcal{P}_1(\mu, \Sigma) := \{ P \in \mathcal{P}(\mathbb{R}^d) : \text{supp}(P) = \mathbb{R}^d, \mathbb{E}_P[X] = \mu, \mathbb{E}[(X - \mu)(X - \mu)^\top] = \Sigma \}$ be the class of probability measures supported on $\mathbb{R}^d$ with fixed mean and covariance. Then

$$
\arg\max_{P \in \mathcal{P}_1(\mu, \Sigma)} \mathcal{S}H(P) = \mathcal{N}(\mu, \Sigma),
$$

i.e. the normal distribution maximizes sliced entropy inside $\mathcal{P}_1(\mu, \Sigma)$.

2. **Support contained in a ball:** Let $\mathcal{P}_2(c, r) := \{ P \in \mathcal{P}(\mathbb{R}^d) : \text{supp}(P) \subseteq B_d(c, r) \}$ be the class of probability measures supported inside a $d$-dimensional ball centered at $c \in \mathbb{R}^d$ of radius $r > 0$ (denoted by $B_d(c, r)$). Then

$$
\arg\max_{P \in \mathcal{P}_2(c, r)} \mathcal{S}H(P) = \text{Unif}(S^{d-1}(c, r)),
$$

i.e. the uniform distribution on the surface of $B_d(c, r)$ maximizes sliced entropy inside $\mathcal{P}_2(c, r)$.

3. **Expected absolute deviation:** Let $\mathcal{P}_3(\mu, a) := \{ P \in \mathcal{P}(\mathbb{R}^d) : \text{supp}(P) = \mathbb{R}^d, \mathbb{E}_P[X] = \mu, \mathbb{E}_P[|\theta^T(X - \mu)|] = a, \forall \theta \in S^{d-1} \}$ be the class of probability measures supported on $\mathbb{R}^d$ with fixed mean and expected absolute deviation of the slice marginals from their mean. Then the sliced entropy inside $\mathcal{P}_3$ is maximized by a $d$-dimensional symmetric multivariate Laplace distribution $[28]$ with characteristic function

$$
\Phi(t; \mu, b) = \frac{e^{i \mu^\top t}}{1 + \frac{1}{2}bt^\top t},
$$

for some $b$ depending on $a$.

The interpretation of the $\mathbb{E}_P[|\theta^T(X - \mu)|] = a, \forall \theta \in S^{d-1}$ constraint in 3. as follows. Note that if the constraint were only for $d\theta$s in the cardinal directions (rather than for all $\theta \in S^{d-1}$), the constraint could be satisfied be the product of i.i.d. Laplace distributions. Unfortunately, the product of Laplace distributions is not a spherical distribution, so the condition would not be satisfied in general for non-cardinal $\theta$. To extend to all $\theta$ on the sphere, it is necessary to find some distribution that is spherical but still has Laplace marginals, in other words, a collection of identically distributed Laplace r.v.s that are coupled such that the joint density is spherical. The Symmetric Multivariate Laplace distribution is exactly this distribution.

**Proof.** For any $P \in \mathcal{P}(\mathbb{R}^d)$ and $\theta \in S^{d-1}$, denote the distribution of the corresponding projection by $P_\theta := \pi_\theta^\top P$. For $X \sim P$, we interchangeably write $H(X)$ and $H(P)$ for entropy (similarly, for sliced entropy), and thus express sliced entropy as

$$
\mathcal{S}H(P) = \frac{1}{S_{d-1}} \oint_{S^{d-1}} H(P_\theta) d\theta.
$$

**Proof of 1.** Note that for any $P \in \mathcal{P}_1(\mu, \Sigma)$ and $\theta \in S^{d-1}$, the mean and covariance of $P_\theta$ is $\theta^\top \mu$ and $\theta^\top \Sigma \theta$, respectively. Since the Gaussian distribution maximizes classic entropy over scalar distributions supported on $\mathbb{R}$ with a fixed (mean and) variance, we have $H(P_\theta) \leq H(\mathcal{N}(\theta^\top \mu, \theta^\top \Sigma \theta)) = \frac{1}{2} \log(2\pi e \theta^\top \Sigma \theta)$ for any $\theta \in S^{d-1}$. Consequently,

$$
\mathcal{S}H(P) \leq \frac{1}{S_{d-1}} \oint_{S^{d-1}} \frac{1}{2} \log(2\pi e \theta^\top \Sigma \theta) d\theta, \quad \forall P \in \mathcal{P}_1(\mu, \Sigma). \tag{13}
$$

Take $P^* = \mathcal{N}(\mu, \Sigma) \in \mathcal{P}(\mu, \Sigma)$ and observe that for any $\theta \in S^{d-1}$, we have $P^*_\theta = \mathcal{N}(\theta^\top \mu, \theta^\top \Sigma \theta)$. Therefore $\mathcal{S}H(P^*)$ achieves the upper bound in (13) and is the maximum sliced entropy distribution over $\mathcal{P}_1(\mu, \Sigma)$.

**Proof of 2.** We first show that a maximum entropy distributions over $\mathcal{P}_2(c, r)$ must be rationally invariant and simultaneously maximize the differential entropy associated with each slice. For $X \sim P \in \mathcal{P}(\mathbb{R}^d)$ and an orthogonal matrix $U \in \mathbb{R}^{d \times d}$, denote (with some abuse of notation)
the distribution of UX by $U_2 P$. Since the support constraint and the definition of sliced entropy are rotationally symmetric, if $P \in \mathcal{P}_2(c, r)$ is a maximum sliced entropy distribution, then so is $U_2 P$, for any $U$ orthogonal.

Assume $P \in \mathcal{P}_2(c, r)$ maximizes sliced entropy. For any orthogonal $U \in \mathbb{R}^{d \times d}$ define $A_U \subseteq S^{d-1}$ as the set of $\theta$ vectors for which the distribution of $\theta^T X$ and $\theta^T UX$ are different. Note that if $P$ maximizes $SH$ then the measure of $A_U$ must be zero. Indeed, if this is not the case, consider the mixture distribution $X^\lambda \sim \mathcal{P}^\lambda := \lambda P + (1 - \lambda)U_2 P$, and note that by convexity of entropy

$$H(\theta^T X^\lambda) > \lambda H(\theta^T X) + (1 - \lambda)H(\theta^T UX), \quad \forall \lambda \in (0, 1), \theta \in A_U.$$ 

Now, if $A_U$ has positive measure, by the definition of sliced entropy we get

$$SH(X^\lambda) > \frac{1}{S^{d-1}} \int_{S^{d-1}} (\lambda H(\theta^T X) + (1 - \lambda)H(\theta^T UX))\,d\theta = \lambda SH(X) + (1 - \lambda)SH(UX) = SH(X),$$

violating the assumption that $X \sim P$ is a maximum sliced entropy distribution. Hence $X \sim P$ is rotationally invariant and has $H(\theta^T X)$ invariant with $\theta$, as claimed.

In what follows, we set $c = 0$, the general case is recovered by the translation invariance of entropy. For $d = 3$, by Archimedes’ Hat Box Theorem, the projection of the distribution $\text{Unif}(S^2(0, r))$ onto any $\theta$ yields $\theta^T X \sim \text{Unif}([-r, r])$, the entropy-maximizing distribution for the slice. Thus, $P = \text{Unif}(S^2(0, r))$ maximizes $SH$ for $d = 3$.

For dimensions $d > 3$, by symmetry we may consider the form $(\theta_1, \theta_2, \theta_3, 0 \ldots 0)^T$. Let $X \sim P$ for some rotationally-symmetric distribution $P$. Observe that

$$\theta^T X = (\theta_1, \theta_2, \theta_3)(X_1, X_2, X_3)^T = (\theta_1, \theta_2, \theta_3)\| (X_1, X_2, X_3) \|_2 \left( \frac{(X_1, X_2, X_3)^T}{\| (X_1, X_2, X_3) \|_2} \right).$$

Define $R = \| (X_1, X_2, X_3) \|_2, \bar{\theta} = (\theta_1, \theta_2, \theta_3)^T$, and $\bar{X} = \frac{(X_1, X_2, X_3)^T}{\| (X_1, X_2, X_3) \|_2}$. By the spherical symmetry of $P$, we have that $\bar{X} \sim \text{Unif}(S^2(0, 1))$ and is independent of $R$. Let $\rho$ be the probability distribution of $R$, and recall that $\text{supp}(\rho) = [0, r]$.

For any fixed $\bar{\theta}$ and $R = r$, by Archimedes’ Hat Box Theorem, $r \bar{\theta}^T \bar{X} \sim \text{Unif}([-r, r])$. By independence, the density $g$ of $R \bar{\theta}^T \bar{X}$ is then

$$g(t) = \int_0^1 \frac{1}{(2\alpha)^d} \mathbf{1}_{|\alpha| \leq \alpha} d\rho(\alpha), \quad t \in [-r, r],$$

where $\mathbf{1}_A$ is the indicator of $A$. Observe that $g$ is symmetric about 0 and is monotonically nonincreasing away from 0.

We next show that transporting mass in $\rho$ to larger radii values cannot decrease entropy. Let $\epsilon > 0$ and consider moving mass $\epsilon$ in $\rho$ from location $\alpha$ to $\alpha' > \alpha$, changing $g$ to $g'$. Doing so decreases $g$ by $\epsilon (1/(2\alpha) - 1/(2\alpha'))$ on the interval $t \in (-\alpha, \alpha)$, and increases it by $\epsilon/(2\alpha')$ on the intervals $t \in [-\alpha', -\alpha) \cup (\alpha, \alpha']$. Furthermore, both $g$ and $g'$ monotonically nonincrease away from 0. At $t = \alpha, -\alpha$, set $g = g'$. The corresponding change in entropy is

$$H(g') - H(g) = \int g \log g - g' \log g' \, dt$$

$$= 2 \int_\alpha^{\alpha'} [g \log g - g' \log g'] \, dt + 2 \int_0^\alpha [g \log g - g' \log g'] \, dt$$

We bound these terms separately. Since $g, g'$ are both monotonically non-increasing away from 0,

$$\int_\alpha^{\alpha'} [g \log g - g' \log g'] \, dt \geq \int_\alpha^{\alpha'} \left[ g \log g - g' \left( \log g + \frac{g' - g}{g} \right) \right] \, dt$$

$$= \int_\alpha^{\alpha'} \left[ (g - g') \left( \log g + \frac{g'}{g} \right) \right] \, dt$$

$$= -\frac{\epsilon}{2\alpha'} \int_\alpha^{\alpha'} \left[ \log g + \frac{g'}{g} \right] \, dt$$

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\[\log g' \leq \log g + \frac{(g' - g)}{g}.\] Similarly, we have

\[
\int_0^\alpha [g \log g - g' \log g'] dt \geq \int_0^\alpha \left[ g \log g - g' \left( \log g + \frac{g' - g}{g} \right) \right] dt
\]

\[
= \int_0^\alpha \left[ (g - g') \left( \log g + \frac{g' - g}{g} \right) \right] dt
\]

\[
\geq \epsilon \left( \frac{1}{2\alpha} - \frac{1}{2\alpha'} \right) \int_0^\alpha \left[ \log g + \frac{g' - g}{g} \right] dt
\]

\[
= \epsilon \left( \frac{1}{2\alpha} - \frac{1}{2\alpha'} \right) \alpha \left[ \log g(\alpha) + \frac{g'(\alpha)}{g(\alpha)} \right]
\]

\[
= \epsilon \left( \frac{1}{2\alpha} - \frac{1}{2\alpha'} \right) \alpha \left[ \log g(\alpha) + 1 \right]
\]

(15)

Substituting (15) and (16) into (14) yields

\[H(g') - H(g) \geq 2 \left[ \epsilon \alpha \left( \frac{1}{2\alpha} - \frac{1}{2\alpha'} \right) - \frac{\epsilon}{2\alpha}(\alpha' - \alpha) \right] \left[ \log g(\alpha) + 1 \right] = 0.
\]

Thus, entropy cannot decrease by moving the mass in \(\rho\) to larger \(R\) values. Note that for any spherically symmetric \(X \sim P\) supported in \(S^{d-1}(0, r)\), the transformation \(X' \leftarrow r \frac{X}{\|X\|_2}\) yields \(R' = \| (X_1^2 X_2^2 X_3^2) \|_2 = \frac{r}{\|X\|_2} R\), i.e. since \(\|X\|_2 \leq r\) the transformation uniformly increases \(R\) (and thus \(H(g)\)), with no change to the distribution of \(\bar{X}\). Therefore, \(P = \text{Unif}(S^{d-1}(0, r))\) is the maximum sliced-entropy distribution.

**Proof of 3.** Similar to the Gaussian case of Claim 1, we use the fact that the maximum entropy distribution satisfying \(E|X - \mu| = a\) is the (univariate) Laplace distribution. To maximize the sliced entropy, we thus seek a distribution \(P\) that results in each \(\theta^T X\) having the same Laplace distribution. Since linear projections of the isotropic Symmetric Multivariate Laplace distribution [28] are all univariate Laplace distributions with the same parameter, this is a maximum sliced entropy distribution for the class. Unfortunately we could not find the exact parameter conversion \((b\text{ required to achieve }a)\) in the literature.

\[\square\]

### A.3 Proof of Proposition 3

Denote \(X_\phi := \Theta^T X\) and \(X_{\phi} := \Phi^T X\) and observe that \(P_{X_{\phi}, Y_{\phi}|\Theta, \Phi}(\cdot, \cdot|\theta, \phi) = (\pi^\theta, \pi^\phi) \times P_{X,Y}\) . Consider the following two joint distribution:

\[P_{\Theta, \Phi, X_{\phi}, Y_{\phi}} = P_{\Theta, \Phi} \times P_{X_{\phi}, Y_{\phi}|\Theta, \Phi}\]

\[Q_{\Theta, \Phi, X_{\phi}, Y_{\phi}} = P_{\Theta, \Phi} \times P_{X_{\phi}|\Theta} \times P_{Y_{\phi}|\Phi},\]

where \(P_{\Theta, \Phi} = \text{Unif}(S^{d-1}) \times \text{Unif}(S^{d-1})\), while \(P_{X_{\phi}|\Theta}\) and \(P_{Y_{\phi}|\Phi}\) are the conditional marginals of \(P_{X_{\phi}, Y_{\phi}|\Theta, \Phi}\). By Claim 3 from Proposition [1] we have

\[\text{KL}(X; Y) = D_{\text{KL}}(P_{X_{\phi}, Y_{\phi}|\theta, \phi} \| P_{X_{\phi}|\Theta} \otimes P_{Y_{\phi}|\Phi}) = D_{\text{KL}}(P_{\Theta, \Phi, X_{\phi}, Y_{\phi}} \| Q_{\Theta, \Phi, X_{\phi}, Y_{\phi}}),\]

where the last step using the KL divergence chain rule. The proof is concluded by invoking the Donsker-Varadhan representation for KL divergence [33]

\[D_{\text{KL}}(P \| Q) = \sup_g \mathbb{E}_P[g] - \log \left( \mathbb{E}_Q[e^g] \right).\]
Remark 9 (Max-sliced MI). A similar variational form can be established for max-sliced MI, i.e., \( \sup_{\theta, \phi} I(\theta^T X; \phi^T Y) \). In that case the variation representation is

\[
\sup_{g \in \mathcal{G}_{\text{proj}}} \mathbb{E}[g(X, Y)] - \log \left( \mathbb{E}[e^{g(\hat{X}, \hat{Y})}] \right),
\]

with \( \mathcal{G}_{\text{proj}} := \{ g \circ (\pi^\theta, \pi^\phi) : (\theta, \phi) \in \mathbb{S}^{d_x-1} \times \mathbb{S}^{d_y-1}, g : \mathbb{R}^2 \rightarrow \mathbb{R} \} \) is the class of projecting functions. The derivation is similar and is thus omitted.

A.4 Proof of Theorem 1

Denote \( I_{X, Y}(\theta, \phi) := I(\theta^T X; \phi^T Y) \) and notice that \( \mathbb{E}[I_{X, Y}(\Theta, \Phi)] = \text{SI}(X; Y) \), where \( (\Theta, \Phi) \sim \text{Unif}(\mathbb{S}^{d_x-1}) \otimes \text{Unif}(\mathbb{S}^{d_y-1}) \). By the triangle inequality we have

\[
|\text{SI}(X; Y) - \hat{S}_{l,n,m}| \leq \left| \text{SI}(X; Y) - \frac{1}{m} \sum_{i=1}^m I_{X,Y}(\Theta_i, \Phi_i) \right| + \left| \frac{1}{m} \sum_{i=1}^m I_{X,Y}(\Theta_i, \Phi_i) - \hat{S}_{l,n,m} \right|.
\]

For the first term, since \( \{(\Theta_i, \Phi_i)\}_{i=1}^m \) are i.i.d., we obtain

\[
\mathbb{E}\left[ \left| \text{SI}(X; Y) - \frac{1}{m} \sum_{i=1}^m I_{X,Y}(\Theta_i, \Phi_i) \right| \right] \leq \sqrt{\frac{1}{m} \text{var}(I_{X,Y}(\Theta, \Phi))} \leq \frac{M}{2\sqrt{m}}
\]

uniformly over \( P_{X,Y} \in \mathcal{F}_d(M) \), where the last step follows because 0 \( \leq I_{X,Y}(\Theta, \Phi) \leq I(X; Y) \leq M \) a.s.

For the second term, recall the notation \( X_\theta := \theta^T X \) and \( Y_\phi := \phi^T Y \), and observe that

\[
\mathbb{E}\left[ \left| \frac{1}{m} \sum_{i=1}^m I_{X,Y}(\Theta_i, \Phi_i) - \hat{S}_{l,n,m} \right| \right] \leq \frac{1}{m} \sum_{i=1}^m \mathbb{E}\left[ I_{X,Y}(\Theta_i, \Phi_i) - \hat{I}_{X,Y}(\Theta_i, \Phi_i) \right] \leq \max_{\theta, \phi} \mathbb{E}\left[ I(X_\theta; Y_\phi) - \hat{I}(X_\theta, Y_\phi) \right],
\]

where \( (X_\theta^n, Y_\phi^n) \) are pairwise i.i.d. samples of \((X_\theta, Y_\phi) \sim (\pi^\theta, \pi^\phi)_\sharp P_{X,Y} \). This falls under the MI risk bound from \( \text{(S)} \), yielding a bound of \( \delta(n) \). \( \square \)

A.5 Proof of Corollary 1

The bounded MI assumption in the definition of \( \mathcal{F}_d(M) \) can be relaxed to a bounded the max-SMI, i.e.,

\[
\max_{\theta \in \mathbb{S}^{d_x-1}, \phi \in \mathbb{S}^{d_y-1}} I(\theta^T X; \phi^T Y) \leq M.
\]

We next derive a uniform bound (over \( (\theta, \phi) \in \mathbb{S}^{d_x-1} \times \mathbb{S}^{d_y-1} \)) on

\[
I(\theta^T X; \phi^T Y) = h(\theta^T X) + h(\phi^T Y) - h(\theta^T X, \phi^T Y).
\]

Since the Gaussian distribution maximize sliced (differential) entropy under a second moment constraint, we have

\[
h(\theta^T X) + h(\phi^T Y) \leq \frac{1}{2} \log \left( \frac{(2\pi e)^2(\theta^T \Sigma_X \theta)(\phi^T \Sigma_Y \phi)}{2} \right).
\]

For the joint entropy, recall that log-concavity is preserved under affine transformations of coordinates and marginalization \( \text{(34) Lemma 2.1} \). Therefore \( (\pi^\theta, \pi^\phi)_\sharp P_{X,Y} \) is also log-concave, and by Theorem 4 of \( \text{(35)} \) we obtain

\[
h(\theta^T X, \phi^T Y) \geq \frac{1}{2} \log \left( \frac{\pi^2}{32} \left( (\theta^T \Sigma_X \theta)(\phi^T \Sigma_Y \phi) - (\theta^T \Sigma_X \phi)(\phi^T \Sigma_Y \theta) \right) \right).
\]

Combining the two bounds we obtain

\[
I(\theta^T X; \phi^T Y) \leq \frac{1}{2} \log \left( \frac{\pi^2}{8} \frac{(\theta^T \Sigma_X \theta)(\phi^T \Sigma_Y \phi)}{(\theta^T \Sigma_X \phi)(\phi^T \Sigma_Y \phi) - (\theta^T \Sigma_X \phi)^2} \right).
\]
We now bound the norms that make up the definition of the generalized Lipschitz class. First, consider the inequality. Similarly, denoting by \( \theta \) the density of \( \theta X \), \( \phi Y \) is invariant to rotations of both the \( X \) and \( Y \). Hence, without loss of generality, we may assume that \( \theta \) and \( \phi \) are both canonical unit vectors, e.g., both equal \( (1 \ 0 \ \ldots \ 0) \) of the appropriate dimension. Consequently, \( \theta X = X_1 \) and \( \phi Y = Y_1 \). Denote \( x_2 := (x_2 \ldots x_{d_x}) \) and \( y_2 := (y_2 \ldots y_{d_y}) \) and write

\[
p_{X,Y}(x_1, y_1) = \int_{[0,1]^{d'}} p_{X,Y}(x, y) \, dx_2 \, dy_2,
\]

where \( d' = d_x + d_y - 2 \) and we have used the fact that \( \theta X = X_1 \) and \( \phi Y = Y_1 \). Finally, for each \( x_1, y_1 \in [0,1]^2 \), we denote \( p_{x_1,y_1}(x_2, y_2) := p_{X,Y}(x_1, x_2, y_1, y_2) \).

We now bound the norms that make up the definition of the generalized Lipschitz class. First, consider

\[
\|p_{\theta, \phi}\|_{p,2} = \left\| \int_{[0,1]^{d'}} p(x_2, y_2) \, dx_2 \, dy_2 \right\|_{p,2}
\leq \int_{[0,1]^{d'}} \left( \int_{[0,1]^{d'}} \left( p(x_1, y_1)(x_2, y_2) \right)^p \, dx_2 \, dy_2 \right) \, dx_1 \, dy_1 \right)^{1/p}
= \|p_{X,Y}\|_{p,d_x+d_y},
\]

where the 2nd step follows because \( \int_{[0,1]^{d'}} p(x_1, y_1)(x_2, y_2) \, dx_2 \, dy_2 \leq \|p(x_1, y_1)\|_{p,d'} \) by Jensen’s inequality. Similarly, denoting by \( \epsilon \) the vector that has 1’s in its first and \((d_x + 1)\)th coordinates and 0’s otherwise, for any \((x_1, y_1) \in [0,1]^2\), we have

\[
\left| \Delta_{\epsilon_1}^p p_{\theta, \phi}(x_1, y_1) \right| \leq \int_{[0,1]^{d'}} \left| \Delta_{\epsilon_1}^p p(x_1, y_1)(x_2, y_2) \right| \, dx_2 \, dy_2 \leq \|\Delta_{\epsilon_1}^p p(x_1, x_2)\|_{p,d'},
\]

where the last step uses Jensen’s inequality once more. Having that, we obtain

\[
\|\Delta_{\epsilon_1}^p p_{\theta, \phi}\|_{p,2} \leq \left( \int_{[0,1]^2} \|\Delta_{\epsilon_1}^p p(x_1, y_1)\|_{p,d'} \, dx_1 \, dy_1 \right)^{1/p} = \|\Delta_{\epsilon_1}^p p_{X,Y}\|_{p,d_x+d_y}.
\]

Consequently \( \|p_{\theta, \phi}\|_{\text{Lip}_p,2} \leq \|p_{X,Y}\|_{\text{Lip}_{p,d_x+d_y}} \leq L \), for all \((\theta, \phi) \in \mathbb{S}^{d_x-1} \times \mathbb{S}^{d_y-1}\), as required.

Based on the lemma, we may invoke [26, Theorem 2] to obtain error bounds on the estimation of the sliced entropy terms that comprise SMI. We first restate the result of [26]: if \( X \sim p_X \in
\[
\text{Lip}_{p,s,d}(L), \text{ for } d \in \mathbb{N}, s \in (0, 2], p \in [2, \infty), \text{ is } \beta\text{-sub-Gaussian, } \beta > 0, \text{ and satisfies the tail bound } \\
\int_{|x| \geq d} e^{\beta|x|^2} p_X(x) dx \leq L, \text{ then } \]

\[
\left( \mathbb{E} \left[ \left( \tilde{H}(X^n) - H(X) \right)^2 \right] \right)^{\frac{1}{2}} \leq C \left( (n \log n)^{-\frac{1}{2}} (\log n)^{\frac{1}{2}} (1 - \frac{4p}{\beta^2 + 2}) + n^{-\frac{1}{2}} \right),
\]  

(17)

for a constant \( C \) depending only on \( s, p, d, \beta, L \).

Note that \( p_{X^n}, p_{X^n,Y^n}, \) and \( p_{Y^n} \), for any \((\theta, \phi) \in \mathbb{S}_{d_x-1} \times \mathbb{S}_{d_y-1}, \) are compactly supported and hence sub-Gaussian (with a sub-Gaussian constant and tail bound that depend only on \( d \) and \( L \)). Lemma [1] then implies that \( H(\theta^T X), H(\phi^T Y), \) and \( H(\theta^T X, \phi^T Y) \) can all be estimated within the framework of [20] under the error bound from (17). Denoting the respective estimators by adding a hat to the differential entropy notation and letting \( \hat{e}_\theta, \hat{e}_\phi, \) and \( \hat{e}_{\theta,\phi} \) be their \( L_2 \) errors, we obtain

\[
\max \{ \hat{e}_\theta, \hat{e}_\phi, \hat{e}_{\theta,\phi} \} \leq C \left( (n \log n)^{-\frac{1}{2}} (\log n)^{\frac{1}{2}} (1 - \frac{4p}{\beta^2 + 2}) + n^{-\frac{1}{2}} \right), \quad \forall (\theta, \phi) \in \mathbb{S}_{d_x-1} \times \mathbb{S}_{d_y-1}.
\]  

(18)

Here we used the fact that the rate is dominated by the error in estimating the 2-dimensional differential entropy \( H(\theta^T X, \phi^T Y) \). Recall that the considered MI estimator relies on the decomposing

\[
I(\theta^T X^r; \phi^T Y) = H(\theta^T X^r) + H(\phi^T Y) - H(\theta^T X, \phi^T Y)
\]

and estimating each sliced entropy separately. Bounding the MI estimation error via (18) produces the result.

A.7 Proof of Proposition 4

Proof of 1. By Part 2 of Proposition [1] we have

\[
\text{SI}(A_x X + b_x; A_y Y + b_y) \leq \sup_{\theta, \phi} I(\theta^T (A_x X + b_x); \phi^T (A_y Y + b_y)) \leq \sup_{\theta, \phi} I(\theta^T X; \phi^T Y),
\]

where in the last step we have used the DPI of classic MI. Now, let \( \{((\theta_i, \phi_i))\}_{i=1}^\infty \) be a sequence converging to the supremum of \( I(\theta^T X; \phi^T Y) \). Set \( b_y = b_x = 0 \), and consider the sequence \( \{((A_{x,i}, A_{y,i}))\}_{i=1}^n \) where \( A_{x,i} = (\theta_i, 0 \ldots 0)^T \) and \( A_{y,i} = (\phi_i, 0 \ldots 0)^T \). Clearly, for each \( i \), we have

\[
\text{SI}(A_{x,i} X; A_{y,i} Y) = I(\theta_i^T X; \phi_i^T Y),
\]

which implies the first claim.

Proof of 2. Let \( O(d) \) be the set of orthogonal \( d \times d \) real-valued matrices. For \( U \sim \text{Unif}(O(d)) \) and \( \tilde{\Theta} \sim \text{Unif}(\mathbb{S}^{d_x-1}) \) independent, note that \([U]_{1:r,1:r}: \tilde{\Theta} \sim \text{Unif}(\mathbb{S}^{d_x-1})\), where \([U]_{1:r,1:r} \) stands for the first \( r \) columns of \( U \). We therefore have:

\[
\text{SI}(A_x X; A_y Y) = I(\tilde{\Theta}[U_x]_{1:r,c_x}^T A_x X; \tilde{\Phi}[U_y]_{1:r,c_y}^T A_y Y|\tilde{\Theta}, \tilde{\Phi}, U_x, U_y) \leq \sup_{U_x \in O(d_x)} \text{SI}([U_x]_{1:r,c_x}^T A_x X; [U_y]_{1:r,c_y}^T A_y Y),
\]  

(19)

where the last inequality follows by upper bounding the expectation by the supremum and the independence of \((U_x, U_y)\) and \((\tilde{\Theta}, \tilde{\Phi}, X, Y)\).

Note that if \( A_x \in \mathcal{M}_{d_x,d_x}(r_x, c_x) \) and \( A_y \in \mathcal{M}_{d_y,d_y}(r_y, c_y) \), then \([U_x]_{1:r,c_x}^T A_x \in \mathcal{M}_{r_x,d_x}(r_x,c_x), \) \([U_y]_{1:r,c_y}^T A_y \in \mathcal{M}_{r_y,d_y}(r_y,c_y) \) (since the first \( r \) singular values of \( A_x \) and \( A_y \) are inside \([1/c_x, c_x]\) and \([1/c_y, c_y]\), respectively). Using this observation while supremizing the LHS of (19), we obtain

\[
\sup_{A_x \in \mathcal{M}_{d_x,d_x}(r_x,c_x), A_y \in \mathcal{M}_{d_y,d_y}(r_y,c_y)} \text{SI}(A_x X; A_y Y) \leq \sup_{B_x \in \mathcal{M}_{d_x,d_x}(r_x,c_x), B_y \in \mathcal{M}_{d_y,d_y}(r_y,c_y)} \text{SI}(B_x X; B_y Y).
\]

The opposite inequality follows by only considering those matrices \((A_x, A_y)\) whose bottom \( d_x - r_x \) or \( d_y - r_y \) rows are zeros.

\[\text{A d-dimensional random variable } X \text{ is } \beta\text{-sub-Gaussian if } \mathbb{E}[e^{\beta|X|^2}] < \infty.\]
A.8 Proof of Corollary 3

We begin by considering fixed $W_x, W_y, b_x, b_y$. By Part 2 of Proposition 1 we have

$$ \text{SI}(A_x \sigma(W_x^T X + b_x); A_y \sigma(W_y^T Y + b_y)) \leq \sup_{\theta, \phi} I(\theta^T A_x \sigma(W_x^T X + b_x); \phi^T A_y \sigma(W_y^T Y + b_y)) \leq \sup_{\theta, \phi} I(\theta^T (W_x^T X + b_x); \phi^T (W_y^T Y + b_y)), \quad (20) $$

where in the last step we have used the DPI of classic MI. Now, let $\{(\theta_i, \phi_i)\}_{i=1}^\infty$ be a sequence converging to the supremum of $I(\theta^T \sigma(W_x^T X + b_x); \phi^T \sigma(W_y^T Y + b_y))$. Consider the sequence $\{(A_x^i, A_y^i)\}_{i=1}^n$ where $A_x^i = (\theta_i, 0 \ldots 0)^T, A_y^i = (\phi_i, 0 \ldots 0)^T$. Clearly, for each $i$, we have

$$ \text{SI}(A_x^i \sigma(W_x^T X + b_x); A_y^i \sigma(W_y^T Y + b_y)) = I(\theta_i^T \sigma(W_x^T X + b_x); \phi_i^T \sigma(W_y^T Y + b_y)), $$

which implies that equality in (20) can be achieved. Hence the supremum of the LHS over $A_x, A_y$ equals the RHS. Supremizing both sides over $W_x, W_y, b_x, b_y$ then yields the corollary.

B Pseudocode and Complexity of the SMI Estimator

Algorithm 1 shows the pseudocode for our SMI estimator (6), repeated here:

$$ \hat{SI}_{n,m} = \hat{SI}_{n,m}(X^n, Y^n, \Theta^m, \Phi^m) := \frac{1}{m} \sum_{i=1}^m \hat{I}((\Theta_i^T X)^n, (\Phi_i^T Y)^n). $$

It requires as input some 1 dimensional MI estimator $\hat{I}(\cdot, \cdot)$ which takes as input a sample from the joint distribution of two 1-dimensional variables and outputs an estimate of their MI.

**Algorithm 1 SMI Estimator**

**Require:** $n$ (pairs of) samples $(X^n, Y^n)$ i.i.d. according to $P_{X,Y} \in \mathcal{P}(\mathbb{R}^{d_x} \times \mathbb{R}^{d_y})$, a scalar MI estimator $\hat{I}(\cdot, \cdot)$, and a chosen number of slices $m$.

**for** $i = 1 : m$ **do**

- Sample $\Theta_i$ uniform on the sphere $S_{d_x-1}$.
- Sample $\Phi_i$ uniform on the sphere $S_{d_y-1}$.
- Compute the MI estimate: $S_i \leftarrow \hat{I}((\Theta_i^T X)^n, (\Phi_i^T Y)^n)$.

**end for**

$$ \hat{SI}_{n,m} \leftarrow \frac{1}{m} \sum_{i=1}^m S_i $$

Reading off from Algorithm 1, the computational complexity of the estimator is $O((m(d_x + d_y)n + mA(n)))$, where $A(n)$ is the computational complexity of the scalar MI estimator. It can be seen that the computational complexity scales linearly with dimension and the number of slices $m$. The scaling with the number of samples $n$ follows $\max\{n, A(n)\}$.

---

*A uniform sample from $S_{d_x-1}$ can be found by sampling a vector $Z$ from a $d_x$-dimensional isotropic Gaussian and forming $Z/\|Z\|_2$. 

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C MI Convergence Experiment

In Figure 5, we show convergence results of MI estimation using the Kozachenko-Leonenko, EDGE \cite{36}, and MINE \cite{29} estimators. The data is the standard Gaussian vectors with 5 overlapping components as described for the $d = 10$ case in Figure 1(b,c) of the main text. Note that the MI estimators converge slowly in this high dimensional regime, in contrast to the $n^{-1/2}$ convergence rate for SMI estimation seen in Figure 1(b).

Figure 5: Convergence of MI estimation (via Kozachenko-Leonenko, EDGE, and MINE estimators) versus the number of data samples $n$ for $d = 10$ standard Gaussian vectors with 5 overlapping entries. Note that the convergence is significantly slower than that in the SMI estimation experiment from Figure 1(b).