The Hochschild Cohomology ring of preprojective algebras of type $L_n$

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ABSTRACT

We compute the Hochschild Cohomology of a finite-dimensional preprojective algebra of generalized Dynkin type $L_n$ over a field of characteristic different from 2. In particular, we describe the ring structure of the Hochschild Cohomology ring under the Yoneda product by giving an explicit presentation by generators and relations.

Keywords: preprojective algebra, periodic algebra, Hochschild cohomology ring.

Classification Code: 16Gxx ; Representation theory of rings and algebras.

1 Introduction

Given a nonoriented finite graph $\Delta$, with $\Delta_0$ and $\Delta_1$ as sets of vertices and edges, respectively, the preprojective algebra of (type) $\Delta$, denoted $P(\Delta)$, is the algebra given by quiver and relations as follows. The quiver $Q := Q_\Delta$ of $P(\Delta)$ has the same vertices and the same loops as $\Delta$. Then, for each edge $i \to j$ in $\Delta$ which is not a loop, $Q$ will have two opposite arrows $a : i \to j$ and $\bar{a} : j \to i$. Convening that $\bar{a} = a$ whenever $a$ is a loop, the algebra is subject to as many relations as vertices in $\Delta$, namely, one relation $\sum_{i(a) = j} a\bar{a} = 0$ per each $i \in Q_0 = \Delta_0$ (here $i(a)$ denotes the initial vertex of $a$).

The notion of preprojective algebra first appeared in the late 70s in the work of Gelfand and Ponomarev [22] to study the representation theory of a finite quiver without oriented cycles. They found their first applications in classification problems of algebras of finite type ([10], [11]) and have been linked to universal enveloping algebras and cluster algebras ([20], [21]). They also occur in very diverse parts of mathematics. For instance, they play a special role in Lusztig’s perverse sheaf approach to quantum groups ([33], [34]) and have been used to tackle differential geometry problems [28] or to study non-commutative deformations of Kleinian singularities [8].

It is well known that $P(\Delta)$ is finite-dimensional if and only if $\Delta$ is a disjoint union of generalized Dynkin graphs, $A_n, D_n, E_6, E_7, E_8$ or $L_n$, where $L_n$ is the graph:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{L_n_graph.png}
\caption{$(n \geq 1$ vertices$)$}
\end{figure}

The aim of this paper is to determine the structure of the classical and stable Hochschild cohomology rings of a preprojective algebra of type $L_n$ over an algebraically closed field $K$, provided that $\text{Char}(K)$ is different from 2, and the structure as graded modules over them of the classical and stable Hochschild homology. For preprojective algebras of Dynkin type, the structure of the Hochschild cohomology ring is known for type $A_n$ in arbitrary characteristic ([14], [15]) and, in the case of a field of characteristic zero, for types $D_n$ and $E$ ([17]).
An important common feature of the preprojective algebras of generalized Dynkin type is that, except for $\Delta = k_1$, $P(\Delta)$ is $(\Omega-\text{periodic of period})$ at most 6 (thus self-injective) where $\Omega$ is the Heller's syzygy operator (see [36] and [16] for the Dynkin cases and [3] for the case $L_n$). The multiplicative structure of the Hochschild cohomology ring $HH^*(\Lambda)$ for a self-injective finite dimensional algebra $\Lambda$ is of great interest in connection with the study of varieties of modules and with questions about its relationship with the Yoneda algebra of $\Lambda$. This is the graded algebra $E(\Lambda) = Ext^*_\Lambda(\Lambda/J, \Lambda/J)$, where $J = J(\Lambda)$ denotes the Jacobson radical of $\Lambda$. Indeed, with inspiration from modular representation theory of finite groups, where the theory of varieties of modules had been developed by Carlson ([6], [7]), Benson ([2]) and others, Snashall and Solberg ([37], see also [12]) started the study of varieties of modules over arbitrary finite dimensional algebras, replacing the group cohomology ring $HH^*(G, K)$ by the Hochschild cohomology ring $HH^*(\Lambda)$ of the considered algebra $\Lambda$. For the new theory to be satisfactory it is generally required that $\Lambda$ is self-injective and it is necessary that the algebra $HH^*(\Lambda)$ satisfies some finite generation conditions, which are always satisfied when $\Lambda$ is periodic. On the other side, there is a canonical homomorphism of graded algebras $\varphi : HH^*(\Lambda) \to E(\Lambda)$ whose image is contained in the graded center $Z^{gr}(E(\Lambda))$ of $E(\Lambda)$ ([37]). It is known that, in case $\Lambda$ is a Koszul algebra, one has $Im(\varphi) = Z^{gr}(E(\Lambda))$ ([5]), but little else is known on the inclusion $Im(\varphi) \subseteq Z^{gr}(E(\Lambda))$ for general algebras. Related with this question, there is an intriguing open problem ([24]) which asks whether $\Omega$-periodicity of $\Lambda/J$ as a $\Lambda$-module implies that $\Lambda$ is a periodic algebra.

The above questions suggest that finding patterns of behaviour of the homogeneous elements of $HH^*(\Lambda)$ with respect to the Yoneda product, in particular cases where the multiplicative structure of $HH^*(\Lambda)$ is computable, can help to give some hints on how to tackle them.

The two main results of the paper are the following, from which all the desired structures (classical and stable Hochschild homology and cohomology) are described (see Corollaries 3.10 and 3.11).

**Theorem 1.1.** Let $\Lambda = P(L_n)$ and suppose that $Char(K) \neq 2$ and $Char(K)$ does not divide $2n+1$. Then $HH^*(\Lambda)$ is the graded commutative algebra given by

a) **Generators:** $x_0, x_1, \ldots, x_n, y, z_1, \ldots, z_n, \gamma, h$ with degrees $\deg(x_i) = 0$, $\deg(y) = 1$, $\deg(z_j) = 2$, $\deg(\gamma) = 4$ and $\deg(h) = 6$.

b) **Relations:**

i) $x_i = 0$, for each $i = 1, \ldots, n$ and each generator $\xi$.

ii) $x_0^2 = x_0 z_j = 0$ (for $j = 1, \ldots, n$)

iii) $z_j = (-1)^{k-j+1}(2j-1)(n-k+1)x_0^{n-1}\gamma$ for $1 \leq j \leq k \leq n$.

iv) $z_j \gamma = (-1)^{j}(n-j+1)x_0^{n-1}h$ (for $j = 1, \ldots, n$)

v) $\gamma^2 = z_1 h$

**Theorem 1.2.** Let $\Lambda = P(L_n)$ and suppose that $Char(K)$ divides $2n+1$. Then $HH^*(\Lambda)$ is the graded commutative algebra given by

a) **Generators:** $x_0, x_1, \ldots, x_n, y, z_1, \ldots, z_n, t_1, t_2, \ldots, t_{n-1}, \gamma, h$ with degrees $\deg(x_i) = 0$, $\deg(y) = 1$, $\deg(t_k) = 3$, $\deg(\gamma) = 4$ and $\deg(h) = 6$.

b) **Relations:**

i) $x_i = 0$, for each $i = 1, \ldots, n$ and each generator $\xi$.

ii) $x_0^2 = y^2 = x_0 z_j = x_0 t_i = 0$ (for $j = 1, \ldots, n$, $i, k = 1, \ldots, n-1$)

iii) $z_j z_k = (-1)^{k-j+1}(2j-1)(n-k+1)x_0^{n-1}\gamma$ for $1 \leq j \leq k \leq n$.

iv) $z_j \gamma = (-1)^{j}(n-j+1)x_0^{n-1}h$

v) $\gamma^2 = z_1 h$

vi) $yz_j = (-1)^{j-1}(2j-1)y z_1$

vii) $z_k t_j = \delta_{jk} x_0^{n-1}y \gamma$ (for $k = 1, \ldots, n$, $j = 1, \ldots, n-1$)
\[
\text{Remark 1.3.} \text{ We recently learnt about the preprint [18], where the multiplicative and the Batalin-Vilkovisky structure of } HH^*(\Lambda) \text{ is calculated in characteristic zero (actually over the complex numbers). We do not look at the Gerstenhaber bracket in this paper. On what concerns the multiplicative structure, the techniques used in our paper are valid for all characteristics } \neq 2 \text{ and detect a subtle difference of behaviour between the cases when } \text{char}(K) \text{ divides or not } 2n + 1, \text{ where } n \text{ is the number of vertices. All through the paper, we will make frequent comments on the similarities and dissimilarities of our results and those in [18].}
\]

The organization of the paper is as follows. In Section (2) we recall some general facts concerning self-injective algebras which are needed through paper. Special emphasis is put on the behaviour of the classical and stable Hochschild homology and cohomology of these algebras, revisiting some results of Eu and Schedler [19] concerning the Calabi-Yau Frobenius algebras. We also show that the stable Hochschild cohomology ring is a localization of the classical one for periodic algebras (see Proposition 2.15). We introduce the concept of dualizable basis and give conditions for its existence (Lemma 2.4). In section (3) we introduce the preprojective algebra of type \( Q \) (see Proposition 2.15). We introduce the concept of dualizable basis and give conditions for its existence (Lemma 2.4). In section (3) we introduce the preprojective algebra of type \( Q \). In [18], the term 'of type T' instead of 'of type L' is used and different relations are used to present the algebra. We prove that the algebra has a dualizable basis and is hence symmetric. We then give a concrete cochain complex which computes the Hochschild cohomology (Proposition 3.7. The graded Frobenius condition of a concrete cochain complex which computes the Hochschild cohomology (Proposition 3.7).

\[ \text{The algebra has a dualizable basis and is hence symmetric. We then give a concrete cochain complex which computes the Hochschild cohomology (Proposition 3.7).} \]

\[ \text{The graded Frobenius condition of } HH^*(\Lambda) \text{ follows from the symmetric and periodic condition of } \Lambda. \text{ Excepting this result all other results in that section are characteristic-free and will be applied in a forthcoming paper to tackle the case of characteristic } 2. \text{ In Section (4) we explicitly calculate the dimensions of the Hochschild homology and homology spaces, and also those of the cyclic homology spaces in characteristic zero (Theorem 4.4 and Corollary 4.6). Actually, by identifying the structure of each } HH^i(\Lambda) \text{ as a module over } Z(\Lambda) = HH^0(\Lambda), \text{ we give a canonical basis of each } HH^i(\Lambda) \text{ (cf. Proposition 4.11). The final section (5) studies the multiplication in } HH^*(\Lambda), \text{ giving auxiliary results leading to the proof of the two main theorems, from which we also derive a presentation } HH^*(\Lambda) \text{ by generators an relations (Corollary 5.7).} \]

\[ \text{2 Preliminaries} \]

We will fix an algebraically closed field \( K \) all through the paper. No condition on its characteristic is required in this section, all through which \( \Lambda \) will be a finite dimensional algebra of the form \( \Lambda = KQ/I \), where \( Q \) is a finite quiver, \( KQ \) is its path algebra and \( I \) is an admissible ideal, i.e., \( I \) is generated by set of linear combinations of paths of length \( \geq 2 \), called relations, and \( I \) contains all paths of length \( m \), for some \( m \geq 2 \). We shall denote by \( Q_0 \) and \( Q_1 \) the sets of vertices and arrows of \( Q \), respectively, and \( i(a) \) and \( t(a) \) will denote the origin and the end of a given \( a \in Q_1 \). For each \( i \in Q_0 \), we will denote by \( e_i \) the associated idempotent element of \( \Lambda \).

After section 3.1 we shall concentrate on the preprojective algebra of type \( \mathbb{L}_n \) but, for the moment, we need some preliminaries in this general context. All through the paper unadorned tensor products are considered over the field \( K \). The word 'module' will mean 'left module' unless otherwise stated and we shall denote by \( \Lambda \).Mod the category of \( \Lambda - \Lambda \)-bimodules (abbreviated \( \Lambda \)-bimodules). If \( \Lambda^e := \Lambda \otimes \Lambda^{op} \) denotes the enveloping algebra of \( \Lambda \) and \( M \) is a \( \Lambda \)-bimodule, then we will view \( M \) either as a left \( \Lambda^e \)-module, with multiplication \( (a \otimes b)m = amb \), or as a right \( \Lambda^e \)-module, with multiplication \( m(a \otimes b') = bma \), for all \( a, b \in \Lambda \) and all \( m \in M \). In this way, we identify \( \Lambda^e \).Mod = \( \Lambda \).Mod = \( \Lambda \).Mod. Whenever \( M \) and \( N \) are \( \Lambda \)-bimodules, we shall denote by \( \text{Hom}_{\Lambda^e}(M, N) \) the corresponding space of morphisms.

\[ \text{2.1 Equivalences of categories induced by automorphisms} \]

It is well-known that, given any automorphism \( \sigma \in Aut(\Lambda) \), each \( \Lambda \)-module \( M \) admits a twisted version \( _{\sigma}M \), where the underlying vector space is \( M \) and the multiplication by elements of \( \Lambda \) is given by \( a \cdot m = \sigma(a)m \), for all \( a \in \Lambda \) and \( m \in M \). It is also well-known that the assignment \( M \mapsto \sigma^\ast M \) underlines and equivalence of categories (which acts as the identity on morphisms) \( \Lambda \).Mod \( \rightarrow \Lambda \).Mod with a quasi-inverse taking \( M \mapsto \sigma^\ast M \).
Suppose now that \( \sigma, \tau \in Aut(\Lambda) \). Then we get an automorphism of the enveloping algebra, \( \sigma \otimes \tau^op : \Lambda \otimes \Lambda^op \rightarrow \Lambda \otimes \Lambda^op \), which takes \( (a \otimes b^o) \mapsto \sigma(a) \otimes \tau(b)^o \). If \( M \) is a \( \Lambda \)-bimodule, which we view as a left \( \Lambda^e \)-module, the previous paragraph gives a new left \( \Lambda^e \)-module \( \sigma \otimes \tau^e M \). In the usual way, we interpret it as a \( \Lambda \)-bimodule \( \tau M \), and then the multiplications by elements of \( \Lambda \) are given by \( a \cdot m \cdot b = \sigma(a) m \tau(b) \). In particular, the assignment \( M \mapsto \sigma M \) underlies an equivalence of categories \( \Lambda \text{Mod}_\Lambda \cong \Lambda \text{Mod}_\Lambda \).

We are specially interested in the case of the previous paragraph when \( \sigma = 1_\Lambda \), and denote by \( F_\tau : \Lambda \text{Mod}_\Lambda \rightarrow \Lambda \text{Mod}_\Lambda \) the equivalence taking \( M \mapsto \tau M \). We will need an alternative description of the selfequivalence of categories induced by \( F_\tau \) on the full subcategory \( \Lambda \text{Proj}_\Lambda \) of \( \Lambda \text{Mod}_\Lambda \) consisting of the projective \( \Lambda \)-bimodules. We still denote by \( F_\tau : \Lambda \text{Proj}_\Lambda \rightarrow \Lambda \text{Proj}_\Lambda \) the mentioned selfequivalence.

**Lemma 2.1.** Let \( \tau \in Aut(\Lambda) \) be an automorphism which fixes the vertices and consider the \( K \)-linear functor \( G_\tau : \Lambda \text{Proj}_\Lambda \rightarrow \Lambda \text{Proj}_\Lambda \) identified by the following data:

1. \( G_\tau(P) = P \), for each projective \( \Lambda \)-bimodule \( P \)
2. \( G_\tau \) preserves coproducts
3. If \( f : \Lambda e_i \otimes e_j \Lambda \rightarrow \Lambda e_k \otimes e_l \Lambda \) is a morphism in \( \Lambda \text{Proj}_\Lambda \), then \( f_\tau := G_\tau(f) \) is the only morphism of \( \Lambda \)-bimodules \( f_\tau : \Lambda e_i \otimes e_j \Lambda \rightarrow \Lambda e_k \otimes e_l \Lambda \) taking \( e_i \otimes e_j \mapsto \sum_{1 \leq r \leq m} a_r \otimes \tau^{-1}(b_r) \), provided that \( f(e_i \otimes e_j) = \sum_{1 \leq r \leq m} a_r \otimes b_r \).

Then \( G_\tau \) is naturally isomorphic to the selfequivalence \( F_\tau = \tau_1 (-) : \Lambda \text{Proj}_\Lambda \rightarrow \Lambda \text{Proj}_\Lambda \).

**Proof.** It is clear that the given conditions determine a unique \( K \)-linear functor \( G_\tau : \Lambda \text{Proj}_\Lambda \rightarrow \Lambda \text{Proj}_\Lambda \) since each projective \( \Lambda \)-bimodule is a coproduct of bimodules of the form \( \Lambda e_i \otimes e_j \Lambda \). In order to give the desired natural isomorphism \( \psi : G_\tau \cong F_\tau \) it will be enough to define it on the indecomposable projective \( \Lambda \)-bimodules \( P = \Lambda e_i \otimes e_j \Lambda \). Indeed, for such a \( P \), we define \( \psi_P : G_\tau(P) = P \rightarrow_1 \tau_1 P = F_\tau(P) \) by the rule \( \psi_P(a \otimes b) = a \otimes \tau(b) \). It is clear that \( \psi_P \) is an isomorphism of \( \Lambda \)-bimodules. Finally, it is routine to see that if \( f : \Lambda e_i \otimes e_j \Lambda \rightarrow Q = \Lambda e_k \otimes e_l \Lambda \) is a morphism between indecomposable objects of \( \Lambda \text{Proj}_\Lambda \), then

\[
F_\tau(f) \circ \psi_P = \psi_Q \circ f = \psi_Q \circ G_\tau(f),
\]

which shows that the \( \psi_P \) define a natural isomorphism \( \psi : G_\tau \cong F_\tau \) as desired.

### 2.2 The Yoneda product of extensions

For the convenience of the reader we recall the definition of \( HH^*(\Lambda) \) and of the Yoneda product. By classical theory of derived functors, for each pair \( M, N \) of \( \Lambda \)-modules, one can compute the \( K \)-vector space \( Ext^*_\Lambda(M, N) \) of \( n \)-extensions as the \( n \)-th cohomology space of the complex \( Hom_\Lambda(P^*, N) \), where

\[
P^* : \cdots \rightarrow P^{n-1} \rightarrow P^n \rightarrow \cdots \rightarrow P^1 \rightarrow P^0 \rightarrow M \rightarrow 0
\]

is a projective resolution of \( M \).

Suppose that \( L, M, N \) are \( \Lambda \)-modules, that \( P^* \) and \( Q^* \) are projective resolutions of \( L \) and \( M \), respectively, and that \( m, n \) are natural numbers. If \( \delta \in Ext^*_\Lambda(L, M) \) and \( \epsilon \in Ext^*_\Lambda(M, N) \), then we can choose a \( \delta \in Hom_\Lambda(P^n, M) \), belonging to the kernel of the transpose map \( (d^{n-1})^* : Hom_\Lambda(P^{n-1}, M) \rightarrow Hom_\Lambda(P^n, M) \) of the differential \( d^{n-1} : P^{n-1} \rightarrow P^n \) of \( P^* \), which represents \( \delta \). Similarly we can choose an \( \epsilon \in Hom_\Lambda(Q_m, N) \) which represents \( \epsilon \). Due to the projectivity of the \( P^i \), there is a non-unique sequence of morphisms of \( \Lambda \)-modules \( \delta^{−k} : P^{n−k} \rightarrow Q^{−k} \), \( k = 0, 1, \ldots, \) making commute the following diagram:

\[
\begin{array}{ccccccc}
\cdots & P^{−n−k} & \cdots & P^{−n−1} & P^{−n} & \cdots & P^0 & \rightarrow L & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & Q^{−k} & \cdots & Q^{−1} & Q^0 & \rightarrow M & \rightarrow 0 \\
\end{array}
\]
Then the composition $\tilde{c} \circ \delta^{-m} : P^{-m-n} \rightarrow N$ is in the kernel of $(d^{-m-n})^*$ and, thus, it represents an element of $\text{Ext}^m_{\Lambda}(L, N)$. This element is denoted by $\epsilon \delta$ and does not depend on the choices made. It is called the Yoneda product of $\epsilon$ and $\delta$. It is well-known that the map

$$\text{Ext}^n_{\Lambda}(M, N) \times \text{Ext}^n_{\Lambda}(L, N) \rightarrow \text{Ext}^{n+n}_{\Lambda}(L, N) \quad ((\epsilon, \delta) \mapsto \epsilon \delta)$$

is $K$-bilinear.

When $M = N$ in the above setting, the vector space $\text{Ext}^n_{\Lambda}(M, M) = \oplus_{i \geq 0} \text{Ext}^i_{\Lambda}(M, M)$ inherits a structure of graded $K$-algebra, where the multiplication of homogeneous elements is the Yoneda product. In this paper we are specifically interested in a particular case of this situation. Namely, when we replace $\Lambda$ by its enveloping algebra $\Lambda^e$ and replace $M$ by $\Lambda$, viewed as $\Lambda^e$-module (i.e. as a $\Lambda$-bimodule). Then $HH^i(\Lambda) := \text{Ext}^i_{\Lambda^e}(\Lambda, \Lambda)$ is called the $i$-th Hochschild cohomology space, for each $i \geq 0$. The corresponding graded algebra $\text{Ext}^*_\Lambda(\Lambda, \Lambda)$ is denoted by $HH^*(\Lambda)$ and called the Hochschild cohomology ring (or algebra) of $\Lambda$. By a celebrated result of Gerstenhaber ([23]), we know that $HH^*(\Lambda)$ is graded commutative. That is if $\epsilon \in HH^i(\Lambda)$ and $\delta \in HH^j(\Lambda)$ are homogeneous elements then $\epsilon \delta = (-1)^{ij} \delta \epsilon$.

### 2.3 Some facts about self-injective algebras

In this paragraph we assume $\Lambda$ to be self-injective. It is well-known (see section 2 in [21] and [26]) that there is an automorphism $\eta \in \text{Aut}(\Lambda)$, called the Nakayama automorphism and uniquely determined up to inner automorphism, such that $D(\Lambda)$ is isomorphic to the twisted bimodule $\Lambda \otimes Q_0$. The automorphism $\eta$ is also identified by the fact that the Nakayama functor

$$D\text{Hom}_\Lambda(-, \Lambda) : \Lambda \text{Mod} \rightarrow \Lambda \text{Mod}$$

is naturally isomorphic to the self-equivalence of $\Lambda \text{Mod}$ which takes $M \mapsto \eta_{-1}(M)$.

The automorphism $\eta$ can be chosen to permute the vertices of $Q$ and the corresponding permutation $\nu$ of $Q_0$ is called the Nakayama permutation. This permutation is identified by the fact that $\text{Soc}(e_i\Lambda) \cap e_i\Lambda e_{\nu(i)} \neq 0$.

Any isomorphism $f : \Lambda_{\eta} \cong D(\Lambda)$ yields a nondegenerate, but not necessarily symmetric, bilinear form $(-, -) : \Lambda \times \Lambda \rightarrow K$ defined by $(a, b) = f(b)(a)$. Then one gets $(ac, b) = (a, cb)$ for all $a, b, c \in \Lambda$. A bilinear form $(-, -) : \Lambda \times \Lambda \rightarrow K$ such that $(ac, b) = (a, cb)$, for all $a, b, c \in \Lambda$, will be called a Nakayama form. It always comes from a Nakayama automorphism of $\Lambda$ in the just described way. To see that, note that $bf(1) = f(b) = f(1)\eta^{-1}(b)$, and so

$$(a, b) = f(b)(a) = (bf(1))(a) = (f(1)\eta^{-1}(b))(a) = f(1)(\eta^{-1}(b)a) = (af(1))(\eta^{-1}(b)) = f(a)(\eta^{-1}(b)) = (\eta^{-1}(b), a),$$

for all $a, b \in \Lambda$. This tells us that we recuperate $\eta$ from $(-, -)$ by the rule that $(a, b) = (\eta^{-1}(b), a)$, using the nondegeneracy condition on $(-, -)$.

Given any basis $B$ of $\Lambda$, one obtains a (right) dual basis $B^* = \{b^* : b \in B\}$ identified by the property that $(b, c^*) = \delta_{bc}$, for all $b, c \in B$.

The following result shows that essentially all Nakayama forms can be constructed from suitable bases of $\Lambda$ in particular way.

**Proposition 2.2.** Let $\Lambda$ be a selfinjective algebra, let $(-, -) : \Lambda \times \Lambda \rightarrow K$ be a bilinear form and consider the following assertions:

1. $(-, -)$ is a Nakayama form
2. There is a basis $B = \bigcup_{i,j \in Q_0} e_iB e_j$ of $\Lambda$ containing the vertices and a basis $\{w_i : i \in Q_0\}$ of $\text{Soc}(\Lambda)$ such that $(x, y) = \sum_{i,e_i} \lambda_i$ for all $x, y \in \Lambda$, where $\lambda_i$ is the coefficient of $w_i$ in the expression of $xy$ as a linear combination of the elements of $B$.

Then $2) \implies 1)$ and, in case $(e_i, e_i) = 0$ for all $i \in Q_0$, the inverse implication is also true.
Let $\Lambda$ be a self-injective algebra with $\eta$ as Nakayama automorphism. The following assertions hold:

1. $\Lambda^c = \Lambda \otimes \Lambda^{op}$ is a self-injective algebra with Nakayama automorphism $\eta \otimes (\eta^{-1})^{op}$.

2. $M^* := \text{Hom}_\Lambda(M, \Lambda^c)$ is isomorphic to $\eta D(M)_{\eta^{-1}}$, for each $\Lambda$-bimodule $M$, and the isomorphism is natural on $M$.

3. $\text{Hom}_\Lambda(\Lambda, \Lambda^c) \cong \Lambda_{\eta^{-1}}$.

Proof. 1) It is well-known that the tensor product of self-injective algebras is again self-injective. Moreover, if $A$ and $B$ are self-injective algebras, then the map

$$< -, - > : (A \otimes B) \times (A \otimes B) \rightarrow K$$

Proof. 2) \implies 1) Fix a basis $B$ as in assertion 2, where, without loss of generality, we assume that $w_i = e_i, w_i^{*} = e_i^*$ for all $i \in Q_0$, where $\nu$ is the Nakayama permutation. Then we clearly have $(x, y) = (x, ay)$, for all $a, x, y \in \Lambda$. Suppose now that $x \in \Lambda$ is an element such that $(x, y) = 0$, for all $y \in \Lambda$. Then one also has $(e_i, x, y) = 0$, for all $y \in \Lambda$, so that, in order to prove the (left) nondegeneracy of $(-, -)$, we can assume that $x = e_i x$ for a (unique) $i \in Q_0$. Since $\text{Soc}(\Lambda)$ intersects nontrivially any nonzero (left or right) ideal, in case $x \neq 0$, we have $0 \neq x \Lambda \cap \text{Soc}(\Lambda) \subseteq e_i \text{Soc}(\Lambda) = e_i \text{Soc}(\Lambda)e_i(\nu(i))$. Then there is $y \in \Lambda$ such that $xy = \lambda w_i$, with $0 \neq \lambda \in K$. This implies that $(x, y) = \lambda \neq 0$, which is a contradiction. Therefore $(-, -)$ is left nondegenerate and right nondegeneracy follows dually.

1) \implies 2) We know that the map $f : \Lambda \rightarrow D(\Lambda)$, given by $f(b) = (-, b) : a \mapsto (a, b)$, is an isomorphism of left $\Lambda$-modules. Moreover, we have $(a, b) = (\eta(b) \cdot a, 1)$. If now $b \in \text{Soc}(\Lambda)$ and $a \in J(\Lambda)$, then $f(b) \in \text{Soc}(\Lambda D(\Lambda))$ and so $af(b) = 0$, and hence $(a, b) = 0$. It follows that $J(\Lambda) \subseteq \text{Soc}(\Lambda)$ and a comparison of dimensions, using the nondegeneracy of $(-, -)$, gives that $J(\Lambda) = \perp \text{Soc}(\Lambda)$. By a ‘symmetric’ argument, one gets that $\text{Soc}(\Lambda)^{\perp} = J(\Lambda)$. It follows that $(-, -)$ defines by restriction nondegenerate bilinear forms

$$(-, -) : KQ_0 \times \text{Soc}(\Lambda) \rightarrow K$$

$$(-, -) : \text{Soc}(\Lambda) \times KQ_0 \rightarrow K$$

If we denote by $\{e_i^* : i \in Q_0\}$ and $\{e_i^* : i \in Q_0\}$ the right and left dual basis of $Q_0 \equiv \{e_i : i \in Q_0\}$, respectively, with respect to the these nondegenerate forms $w_i = e_i$. The equality $(a, b) = (\eta^{-1}(b), a)$, where $\eta$ is the Nakayama automorphism, implies that $e_i^* = \eta^{-1}(e_{\nu(i)})$, for all $i \in Q_0$. Therefore, up to the ordering of its elements, the left and the right dual bases of $Q_0$ coincide. We put in the sequel $w_i = e_i^*$, for each $i \in Q_0$, and $B' = \{w_i : i \in Q_0\}$ is the basis of $\text{Soc}(\Lambda)$ that we choose. We next put $W := KQ_0 \oplus \text{Soc}(\Lambda)$ and claim that $\Lambda = W + W^\perp$. Due to the nondegeneracy of $(-, -)$, it will be enough to prove that $W \cap W^\perp = 0$. Indeed if $v \in W \cap W^\perp$ and we write it as $v = u + w$, with $u \in KQ_0$ and $w \in \text{Soc}(\Lambda)$, then we have $0 = (u + w, w_i) = (u, w_i)$, for all $i \in Q_0$. The nondegeneracy of $(-, -) : KQ_0 \times \text{Soc}(\Lambda) \rightarrow K$ gives that $u = 0$, and hence $u = w$. But then $w \in W^\perp \subseteq (KQ_0)^{\perp}$, which contradicts the nondegeneracy of $(-, -) : KQ_0 \times \text{Soc}(\Lambda) \rightarrow K$, and thus settles our claim.

Our desired basis $B$ will be the union (in this order) of the basis $Q_0 \equiv \{e_i : i \in Q_0\}$ of $KQ_0$, of any basis of $W^\perp$ contained in $\bigcup_{i, j \in Q_0} e_i^* A^*_j$ and of the basis $\{w_i : i \in Q_0\}$ of $\text{Soc}(\Lambda)$.

It only remains to prove that if $x, y \in B$, then $(x, y)$ is as in the statement. In case $x = e_i$, we have that $(x, y) = (e_i, y) = 0$ unless $y = w_i$, and clearly $(x, y)$ is the sum of the coefficients of the $w_j$ in $y$. In case $x \not\in Q_0$, then we have $(x, y) = (e_i, xy)$ for $(-, -)$ is a Nakayama form. But if $xy = \sum_{b \in B} \lambda_b b$, with $\lambda_b \in K$, then $(e_i, xy) = \sum_{b \in B} \lambda_b (e_i, b)$ and the result follows from the case $x = e_i$ already studied.

Definition 1. Let $B$ be a basis of $\Lambda$ such that $B = \bigcup_{i, j \in Q_0} e_i^* A^*_j$. $B$ contains the vertices of $Q$ and contains a basis $\{w_i : i \in Q_0\}$ of $\text{Soc}(\Lambda)$, with $w_i \equiv e_i A^*$ for each $i \in Q_0$. The Nakayama form $(-, -) : \Lambda \times \Lambda \rightarrow K$ given by the above proposition is called the Nakayama form associated to $B$.

The basis $B$ will be called dualizable when its associated Nakayama form is symmetric.

The following is a useful property:

Lemma 2.3. Let $\Lambda$ be a self-injective algebra with $\eta$ as Nakayama automorphism. The following assertions hold:

1. $\Lambda^c = \Lambda \otimes \Lambda^{op}$ is a self-injective algebra with Nakayama automorphism $\eta \otimes (\eta^{-1})^{op}$.

2. $M^* := \text{Hom}_\Lambda(M, \Lambda^c)$ is isomorphic to $\eta D(M)_{\eta^{-1}}$, for each $\Lambda$-bimodule $M$, and the isomorphism is natural on $M$.

3. $\text{Hom}_\Lambda(\Lambda, \Lambda^c) \cong \Lambda_{\eta^{-1}}$.
given by \( <a \otimes b, a' \otimes b' >= (a, a')(b, b') \) is a Nakayama form for \( A \otimes B \), from which it easily follows that \( \eta_A \otimes \eta_B \) is a Nakayama automorphism for \( A \otimes B \).

We just need to check now that \((\eta^{-1})^\circ \) is a Nakayama automorphism for \( \Lambda^{op} \). But note that we have an obvious Nakayama form \(-, - : \Lambda^{op} \times \Lambda^{op} \to K \) defined by the rule \( <x^\circ, y^\circ > = (y, x) \), where the second member of the equality is given by a fixed Nakayama form \((-,-) \) of \( \Lambda \). Then we get:

\[
< \eta(b)^\circ, a^\circ > = (a, \eta(b)) = (\eta^{-1}(\eta(b)), a) = (b, a) = < a^\circ, b^\circ >
\]

from which it follows that the assignment \( b^\circ \mapsto \eta^{-1}(b)^\circ \) is a Nakayama automorphism for \( \Lambda^{op} \).

We have that \( DHom_{\Lambda^e}(-, \Lambda^e) \) is the Nakayama functor \( \Lambda^e - \text{mod} \to \Lambda^e - \text{mod} \). It follows that we have natural isomorphisms \( DHom_{\Lambda^e}(-, \Lambda^e) \cong [\_]_{\eta \otimes (\eta^{-1})} \) \((-) \cong \eta \otimes (-)_{\eta} \) of functors \( \Lambda^{Mod}_\Lambda \to \Lambda^{Mod}_\Lambda \). We then get \( Hom_{\Lambda^e}(M, \Lambda^e) \cong D(\eta^{-1} \oplus \eta^0 M) \cong D(\eta^{-1} \oplus \eta^0)(M) \) \( \text{it is easy to see that if } A \text{ is an algebra and } \sigma \in Aut(A), \text{ then } D(\sigma N) \cong D(N)_{\sigma} \) for each left \( A \)-module \( N \).

But the right structure of \( \Lambda^e \)-module on a \( A \)-bimodule \( X \) is given by \( x(a \otimes b^\circ) = bx(a) \). It follows that \( D(\Lambda)_{\eta^{-1}} \cong D(\Lambda)_{\eta^{-1}} \).

By assertion 2, we have \( Hom_{\Lambda^e}(\Lambda, \Lambda^e) \cong D(\Lambda)_{\eta^{-1}} \). But \( D(\Lambda)_{\eta^{-1}} \cong \Lambda^e \) and then we have \( \eta D(\Lambda)_{\eta^{-1}} \cong \Lambda^e \).

\( \square \)

We look now at the case when \( I \) is a homogeneous ideal of \( KQ \) with respect the length grading of \( KQ \). In such case we get an induced grading on \( \Lambda \) in the obvious way. We shall call it the natural grading. The following lemma gives a handy criterion for a basis to be dualizable.

**Lemma 2.4.** Let \( \Lambda = \frac{KQ}{I} \) be a graded self-injective algebra such that its Nakayama permutation is the identity and \( \text{dim}(e_i \Lambda e_j) \leq 1 \), for all \( i, j \in Q_0 \) and all natural numbers \( n \). Let \( B \) be a basis of \( \Lambda \) consisting of paths and negative paths and containing the vertices and a basis of \( \text{Soc}(\Lambda) \). If \( (-,-) : \Lambda \times \Lambda \to K \) is the Nakayama form associated to \( B \), then the following assertions are equivalent:

1. \( a^* = \omega_i(a) \), for all \( a \in Q_1 \).
2. \( b^{**} = b \), for each \( b \in B \).
3. \((-,-) \) is symmetric, i.e., \( B \) is a dualizable basis.

**Proof.** Let \( b \in e_i Be_j \) any element of the basis. We claim that there is a unique \( \tilde{b} \in B \) such that \( (b, \tilde{b}) \neq 0 \). Clearly we have \( (b, c) = 0 \) when \( c \notin e_i Be_j \). If \( \omega_i \in e_i \text{Soc}(\Lambda) \) in \( B \), then we also have \( (b, c) = 0 \) whenever \( c \in e_j Be_i \) but \( \text{deg}(b) + \text{deg}(c) \neq \text{deg}(\omega_i) \). Therefore \( (b, c) \neq 0 \) implies that \( c \in e_j B_i e_i \) and \( r = \text{deg}(\omega_i) - \text{deg}(b) \). Since \( \text{dim}(e_i \Lambda e_i) \leq 1 \) and \((-,-) \) is non-degenerate our claim is settled by choosing \( \tilde{b} \) to be the unique element in \( e_j Be_i \), with \( r = \text{deg}(\omega_i) - \text{deg}(b) \).

The last paragraph implies that the assignment \( b \mapsto \tilde{b} \) gives an involutive bijection \( B \to B \) and that \( b^{**} = (b, \tilde{b})^{-1} \tilde{b} \), for each \( b \in B \). In particular, if \( B^{**} \) is the (right) dual basis of \( B^* \) then \( \lambda^* = \lambda \), for some \( \lambda \in \Lambda^* \) which can be explicitly calculated. Namely, we have

\[
1 = (b^*, b^{**}) = ((b, \tilde{b})^{-1} \tilde{b}, \lambda \tilde{b}) = \lambda \tilde{b}, \tilde{b})^{-1} \tilde{b}, \tilde{b})
\]

and so \( \lambda^* = \lambda \). It follows that \( b^{**} = b \) if and only if \( (b, \tilde{b}) = (\tilde{b}, b) \). From this the equivalence of assertions 2 and 3 is immediate.

Note that our hypotheses guarantee that the nonzero homogeneous elements of \( \Lambda \) are precisely the scalar multiples of the elements of \( B \). We denote by \( H \) the set of those nonzero homogeneous elements. Then an alternative description of \( b^* \) is that it is the unique element of \( H \) such that \( bb^* = \omega_i(b) \). We can then extend \( (-)^* \) to a bijective map \( (-)^* : H \to H \), so that \( h^* \) is the unique element of \( H \) such that \( hh^* = \omega_i(h) \). It is then clear that \( (\lambda h)^* = \lambda^{-1} h^* \), for all \( h \in H \) and \( \lambda \in K^* \).

1. \( \implies \) 2. Note that if \( h_1, h_2 \in H \) are such that \( h_1 h_2(h_1 h_2)^* = \omega_i(h_1) \) implies that \( h_2(h_1 h_2)^* = h_1^* \).
We next prove that if \( a \in Q_1 \) and \( h \in H \) are such that \( ah \neq 0 \), then \((ah)^\ast a = h^\ast\). We proceed by induction on \( \deg(h) \), the case \( \deg(h) = 0 \) being a direct consequence of the hypothesis. Since \( h \) is a scalar multiple of an element of \( B \), we can assume without loss of generality that \( h \) is a path in \( Q \), say, \( h = \alpha_1 \cdots \alpha_r \). Then we have

\[
h[(ah)^\ast a] = \alpha_1 \cdots \alpha_r(\alpha_1 \cdots \alpha_r)^\ast a = \alpha_1 \cdots \alpha_r-1(\alpha_r(\alpha_1 \cdots \alpha_{r-1})^\ast a
\]

By the induction hypothesis, the last term is equal to \( \alpha_1 \cdots \alpha_{r-1}(\alpha_1 \cdots \alpha_{r-1})^\ast = \omega_{i(h)} \). It follows that \((ab)^\ast a = h^\ast\).

We finally prove by induction on \( \deg(h) \) that \( h^\ast h = \omega_{i(h)}\), for all \( h \in H \), which implies that \( h^\ast h = \omega_{i(h)} \). Hence ends the proof. The cases of \( \deg(h) = 0,1 \) are clear. So we assume that \( \deg(h) > 1 \) and, again, assume that \( h = \alpha_1 \cdots \alpha_r \) is a path \((r > 1)\). Then

\[
h^\ast h = [\alpha_1(\alpha_2 \cdots \alpha_r)^\ast \alpha_1 \cdots \alpha_r = (\alpha_2 \cdots \alpha_r)^\ast \alpha_2 \cdots \alpha_r
\]

and, by the induction hypothesis, the last term is equal to \( \omega_{i(\alpha_r)} = \omega_{i(h)} \). □

### 2.4 Stable and Absolute Hochschild (Co)Homology of Self-injective Algebras

In this section we briefly recall some results from [19] which will be needed through the paper. All through the section \( A \) is a basic finite dimensional self-injective algebra.

**Definition 2.** Let \( M \) be any \( A \)-module. A complete projective resolution of \( M \) is an acyclic complex of projective \( A \)-modules

\[
P : \cdots \longrightarrow P^{-2} \xrightarrow{d^{-2}} P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} P^2 \longrightarrow \cdots
\]

such that \( Z^1 := \text{Ker} (d^1) = M \). Such a complete resolution is called minimal in case the induced morphism \( P^n \longrightarrow Z^{n+1} := \text{Im} (d^n) \) is a projective cover, for each \( n \in \mathbb{Z} \).

**Proposition 2.5.** Let \( M \) be an \( A \)-module. A complete projective resolution of \( A \) is unique, up to isomorphism in the homotopy category \( \mathcal{H}A \). A minimal complete projective resolution is unique, up to isomorphism in the category \( \text{CA} \) of (cochain) complexes of \( A \)-modules.

**Proof.** The first assertion is a consequence of a more general fact (see, e.g., [31] Proposition 7.2 and Example 7.16) which states that the assignment \( P \mapsto Z^1(P) = \text{Ker} (d^1) \) gives an equivalence of triangulated categories \( \mathcal{H}_{ac}(A - \text{Inj}) \cong A - \text{Mod} \), where \( \mathcal{H}_{ac}(A - \text{Inj}) \) is the full subcategory of \( \mathcal{H}A \) consisting of acyclic complexes of injective (=projective) \( A \)-modules.

The final assertion of the proposition is a direct consequence of the uniqueness of the projective cover up to isomorphism. □

**Notation 2.6.** Under the hypothesis of definition [2] let \( M, N \) be left \( A \)-modules and \( X \) be a right \( A \)-module, and let \( P \) be a complete projective resolution of \( M \). For each \( i \in \mathbb{Z} \), we put

1. \( \text{Ext}_A^i(M, N) = H^i(\text{Hom}_A(P, N)) \)
2. \( \text{Tor}_A^i(X, M) = H^{-i}(X \otimes_A P) \),

where \( H^i(\cdot) \) denotes the \( i \)-th homology space of the given complex.

We call \( \text{Ext}_A^i(\cdot, \cdot) \) and \( \text{Tor}_A^i(\cdot, \cdot) \) the stable \( \text{Ext} \) and the stable \( \text{Tor} \), respectively. Their definition does not depend on the complete resolution \( P \) that we choose.

We clearly have \( \text{Ext}_A^i(M, N) = \text{Ext}_A^i(M, N) = \text{Ext}_A^i(M, N) \) and \( \text{Tor}_A^i(X, M) = \text{Tor}_A^i(X, M) \), for all \( i > 0 \). In particular, we have canonical isomorphisms of graded \( K \)-vector spaces

\[
\text{Ext}_A^i(M, N) = \bigoplus_{i \geq 0} \text{Ext}_A^i(M, N) \overset{\lambda_{M,N}}{\longrightarrow} \bigoplus_{i \in \mathbb{Z}} \text{Ext}_A^i(M, N) =: \text{Ext}_A^\ast (M, N)
\]
and
\[
\text{Tor}^A_r(X, M) = \oplus_{i \in \mathbb{Z}} \text{Tor}^A_r(X, M)_{\mu_X, M} \oplus \oplus_{i > 0} \text{Tor}^A_r(X, M) = \text{Tor}^A_r(X, M)
\]
where \(\text{Ker}(\lambda_{M, N})\) and \(\text{Coker}(\mu_{M, N})\) are concentrated in degree 0. Actually, \(\text{Ker}(\lambda_{M, N}) = \mathcal{P}(M, N) = \{f \in \text{Hom}_A(M, N) = \text{Ext}^0_A(M, N) : f\) factors through a projective \(A\)-module\} and \(\text{Coker}(\mu_{X, N})\) is isomorphic to the image of the morphism \(1_X \otimes j_M : X \otimes M \to X \otimes \mathcal{P}^1\), where \(j : M \to \mathcal{P}^1\) is the injective envelope of \(M\). The following fact for \(\text{Ext}\) is well-known. Finally, note that \(\text{Ext}_A^r(M, N) \cong \text{Hom}_A(\Omega^r_M(M), N)\) for all \(i \in \mathbb{Z}\). In particular, for \(M = N\) we get a structure of graded algebra on \(\text{Ext}_A^r(M, M)\) induced from that of \(\oplus_{i \in \mathbb{Z}} \text{Hom}_A(\Omega_i^r(M), M)\), which is defined by the rule
\[
g \cdot f = g \circ \Omega^r_M(f),
\]
whenever \(f \in \text{Hom}_A(\Omega_i^r(M), M)\) and \(g \in \text{Hom}_A(\Omega_j^r(M), M)\). In particular, the multiplication on \(\text{Ext}_A^r(M, M)\) extends the Yoneda product defined in subsection 2.2. Then the next result follows in a straightforward way.

**Proposition 2.7.** Let \(M, N\) be left \(A\)-modules. The space \(\text{Ext}_A^r(M, M)\) has a canonical structure of graded algebra over which \(\text{Ext}_A^r(M, N)\) is a graded right module. Moreover the map
\[
\lambda_{M, N} : \text{Ext}_A^r(M, M) \to \text{Ext}_A^r(M, M)
\]
is a homomorphism of graded algebras and the following diagram is commutative, where the horizontal arrows are the multiplication maps:

\[
\begin{array}{ccc}
\text{Ext}_A^r(M, N) \otimes \text{Ext}_A^r(M, M) & \xrightarrow{\text{Yoneda}} & \text{Ext}_A^r(M, N) \\
\downarrow{\lambda_{M, N} \otimes \lambda_{M, M}} & & \downarrow{\lambda_{M, N}} \\
\text{Ext}_A^r(M, N) \otimes \text{Ext}_A^r(M, M) & \xrightarrow{} & \text{Ext}_A^r(M, N)
\end{array}
\]

Consider now \(\text{Tor}^A_{-\ast}(X, M)\) as a graded \(K\)-vector space, but taking as \(n\)-homogeneous component precisely \(\text{Tor}^A_{-\ast}(X, M)_{\mu_X, M}\). For that reason we shall write \(\text{Tor}^A_r(X, M)\). If now \(P\) is a fixed minimal complete projective resolution of \(M\), then \(P\) is canonically differential graded (dg) \(A\)-module, i.e., an object in \(\mathcal{C}_{dg} A\) using the terminology of [29]. Then \(B = \text{End}_{\mathcal{C}_{dg} A}(P)\) is a dg algebra which acts on \(X \otimes A P\) in the obvious way, making \(X \otimes A P\) into a dg left \(B\)-module. As a consequence, \(H^\ast(X \otimes A P) = \text{Tor}^A_{-\ast}(X, M)\) is a graded left module over the cohomology algebra \(H^\ast B = \oplus_{n \in \mathbb{Z}} \text{Hom}_{\mathcal{C}_{dg}}(P, P[n])\), where \(\mathcal{C}_{dg}\) is the homotopy category of \(A\) (see [29]). But, due to the fact that the 1-cocycle functor gives an equivalence of triangulated categories \(Z^1 : \mathcal{C}_{A} (\text{Inj} - A) \to A - \text{Mod}\), we have canonical isomorphisms of graded algebras \(H^\ast B \cong \oplus_{n \in \mathbb{Z}} \text{Hom}_{\mathcal{C}_{dg}}(M, \Omega^{-n}_A M) \cong \text{Ext}_A^r(M, M)\). It then follows the desired structure of \(\text{Tor}^A_{-\ast}(X, M)\) as a graded \(\text{Ext}^r_A(M, M)\)-module.

If we take now the nonnegatively graded subalgebra \(\text{Ext}^{\geq 0}_A(M, M) := \oplus_{n \geq 0} \text{Ext}^r_A(M, M)\) of \(\text{Ext}^r_A(M, M)\), then \(\text{Tor}^A_{-\ast}(X, M) = \oplus_{j < 0} \text{Tor}^A_r(X, M)\) is a graded \(\text{Ext}^{\geq 0}_A(M, M)\)-module of \(\text{Tor}^A_{-\ast}(X, M)\) and the quotient \(\text{Tor}^A_{-\ast}(X, M) / \text{Tor}^A_{-\ast}(X, M)_{\geq 0}\), which is isomorphic to \(\text{Tor}^A_{-\ast}(X, M)\) as a graded \(K\)-vector space, is a graded left \(\text{Ext}^{\geq 0}_A(M, M)\)-module. That is, \(\oplus_{i \geq 0} \text{Tor}^A_r(X, M)\) has a canonical structure of graded left \(\text{Ext}^{\geq 0}_A(M, M)\)-module, where \(\text{Tor}^A_r(X, M)\) is the component of degree \(-i\), for all \(i \geq 0\). Since we have a surjective morphism of graded algebras \(\text{Ext}^r_A(M, M) \twoheadrightarrow \text{Ext}^{\geq 0}_A(M, M)\), we get a structure of graded left \(\text{Ext}^r_A(M, M)\)-module on \(\text{Tor}^A_{-\ast}(X, M)\) given by restriction of scalars.

We can now provide \(\text{Tor}^A_{-\ast}(X, M)\) (that is just \(\text{Tor}^A_r(X, M)\), but with \(\text{Tor}^A_r(X, M)\) in degree \(-i\), for all \(i \geq 0\)) with a structure of graded left \(\text{Ext}^r_A(M, M)\)-module of which \(\text{Tor}^A_{-\ast}(X, M)\) is a graded submodule. Indeed, the product \(\text{Ext}^r_A(M, M) \cdot \text{Tor}^A_{-\ast}(X, M) = \text{Ext}^r_A(M, M) \cdot \text{Tor}^A_{-\ast}(X, M)\) is clear when \(j = 0\). For \(j = 0\) we put
\[
\text{Ext}^r_A(M, M) \cdot \text{Tor}^A_{0}(X, M) = 0 \text{ if } i > 0,
\]
and for $i = 0$ the multiplication is identified by the following diagram,

$$
\begin{align*}
\text{Ext}_A^0(M, M) \times \text{Tor}_0^A(X, M) & \longrightarrow \text{Tor}_0^A(X, M) \\
\downarrow & \\
\text{End}_A(M) \times (X \otimes_A M) & \longrightarrow X \otimes_A M 
\end{align*}
$$

where the bottom horizontal arrow is the canonical map $(f, x \otimes m) \mapsto x \otimes f(m)$.

These comments prove the following correspondent of proposition 2.7 for $\text{Tor}$.

**Proposition 2.8.** Let $X$ and $M$ be a right and a left $A$-modules, respectively. Then $\text{Tor}_0^A(X, M)$ (resp. $\text{Tor}_0^A((X, M))$ has a canonical structure of graded left $\text{Ext}_A^*(M, M)$- (resp. $\text{Ext}_A^*(M, M)$-)module. Moreover, the following diagram is commutative

$$
\begin{align*}
\text{Ext}_A^*(M, M) \times \text{Tor}_0^A(X, M) & \xrightarrow{\lambda_{M, X, M} \times 1} \text{Ext}_A^*(M, M) \times \text{Tor}_0^A(X, M) \xrightarrow{\text{mult.}} \text{Tor}_0^A(X, M) \\
\downarrow & \\
\text{Ext}_A^*(M, M) \times \text{Tor}_0^A(X, M) & \xrightarrow{\mu_{X, M}} \text{Tor}_0^A(X, M) 
\end{align*}
$$

We are specially interested in the particular case of the two previous propositions in which $A = \Lambda e = \Lambda \otimes \Lambda^o$, for a self-injective algebra $\Lambda$ and $M = \Lambda$ viewed as $\Lambda^e$-module. In that case, we put $HH^*(\Lambda, N) = \text{Ext}_\Lambda^*(\Lambda, N)$ and $HH_*(\Lambda, N) = \text{Tor}_\Lambda^*(\Lambda, N)$ and call then the $n$-th stable Hochschild cohomology and $n$-th stable Hochschild homology space of $\Lambda$ with coefficients in $N$.

Putting $HH^*(\Lambda, N) = \oplus_{n \in \mathbb{Z}} HH^n(\Lambda, N)$, $HH^*(\Lambda) = HH^*(\Lambda, \Lambda)$, $HH_*(\Lambda, N) = \oplus_{n \in \mathbb{Z}} HH_n(\Lambda, N)$ and $HH_*(\Lambda) = HH_*(\Lambda, \Lambda)$, we have the following straightforward consequence of Propositions 2.7 and 2.8. The graded commutativity of $HH^*(\Lambda)$ was showed by Gerstenhaber ([23]) and that of $HH_*(\Lambda)$ can be found in [19].

**Corollary 2.9.** In the situation above, $HH^*(\Lambda)$ (resp. $HH^*(\Lambda)$) has a canonical structure of graded commutative algebra over which $HH^*(\Lambda, N)$ (resp. $HH^*(\Lambda, N)$) is a graded right module and $HH_*(\Lambda, N)$ (resp. $HH_*(\Lambda, N)$) is a graded left module. Moreover, the graded algebra structure on $HH^*(\Lambda)$ and the graded module structures on $HH^*(\Lambda, N)$ and $HH_*(\Lambda, N)$ are determined by its stable correspondent, except in degree zero.

**Remark 2.10.** If $B = \oplus_{i \in \mathbb{Z}} B_i$ is a graded commutative algebra, then any graded left $B$-module $V = \oplus_{i \in \mathbb{Z}} V_i$ may be viewed as a graded right $B$-module by defining $vb = (-1)^{\deg(b) \deg(v)}bv$, for all homogeneous elements $b \in B$ and $v \in V$. In particular, we shall view in this way $HH_*(\Lambda, M)$ as graded right $HH^*(\Lambda)$-module, for each $\Lambda$-bimodule $M$. We proceed similarly with $HH_*(\Lambda, M)$ over $HH^*(\Lambda)$.

Note that if $Q$ and $M$ are a projective and an arbitrary $\Lambda$-bimodule, then $D(Q \otimes_{\Lambda^e} M) \cong \text{Hom}_{\Lambda^e}(Q, D(M))$ by adjunction. If now $P$ is the complete minimal projective resolution of $\Lambda$ as a bimodule (equivalently, as a right $\Lambda^e$-module), we have an isomorphism of complexes $D(P \otimes_{\Lambda^e} M) \cong \text{Hom}_{\Lambda^e}(P, D(M)) = \text{Hom}_{\mathcal{C}_d}(P, D(M))$, convening that $D(T)^i = D(T^{-i})$ for each complex (or
each graded vector space) $T$ and each $i \in \mathbb{Z}$. It is routine to see that the last isomorphism preserves the structures of right dg modules over the dg algebra $B := \text{End}_{C[A]}(P, P)$. It then follows easily:

**Remark 2.11.** If $\Lambda$ is a finite dimensional self-injective algebra, then

1. $HH_{-\bullet}(\Lambda, M) \cong D(HH^{\bullet}(\Lambda, D(M)))$ as graded $HH^{\bullet}(\Lambda)$-modules.

2. $HH_{-\bullet}(\Lambda, M) \cong D(HH^{\bullet}(\Lambda, D(M)))$ as graded $HH^{\bullet}(\Lambda)$-modules.

### 2.5 The Calabi-Yau Property

If $\Lambda$ is a self-injective algebra, then $\Lambda$-mod is a Frobenius abelian category (i.e., it has enough projectives and injectives and the projective objects coincide with the injective ones). As a consequence, $\Lambda - \text{mod}$ is a triangulated category with $\Omega\Lambda^{-1} : \Lambda - \text{mod} \to \Lambda - \text{mod}$ as its suspension functor (cf. [25], Chapter 1).

Recall (see [27] and [30]) that a $\text{Hom}$ finite triangulated $K$-category $\mathcal{T}$ with suspension functor $\sum : \mathcal{T} \to \mathcal{T}$ is called Calabi-Yau when there is a natural number $n$ such that $\sum^n$ is a Serre functor (i.e. $D\text{Hom}_\mathcal{T}(X, -)$ and $\text{Hom}_\mathcal{T}(-, \sum^n X)$ are naturally isomorphic as cohomological functors $\mathcal{T}^{op} \to K-\text{mod}$). In such a case, the smallest natural number $m$ such that $\sum^m$ is a Serre functor is called the Calabi-Yau dimension (CY-dimension for short) of $\mathcal{T}$.

In case $\Lambda$ is a self-injective algebra, Auslander formula (see [1], Chapter IV, Section 4) says that one has a natural isomorphism $D\text{Hom}_\Lambda(X, -) \cong \text{Ext}_\Lambda^1(X, -)$, where $\tau : \Lambda - \text{mod} \to \Lambda - \text{mod}$ is the Auslander-Reiten (AR) translation. Moreover $\tau = \Omega^2 N$, where $N = D\text{Hom}_\Lambda(-, \Lambda) \cong D(\Lambda) \otimes \Lambda$ and $\Lambda - \text{mod}$ is the Nakayama functor (see [1]). As a consequence, as shown in [13], $\Lambda$-mod is $m$-CY if and only if $m$ is the smallest natural number such that $\Omega\Lambda^{-m-1} \cong N \cong \eta^{-1}(-)$ as triangulated functors $\Lambda$-mod $\to \Lambda$-mod, where $\eta$ is the Nakayama automorphism.

In [19], an algebra is called Calabi-Yau Frobenius algebras of dimension $m$ if it is self-injective and $m$ is the smallest natural number such that $\Omega^m\Lambda^{-1} \cong N \cong \eta^{-1}(-)$ as triangulated functors $\Lambda$-mod $\to \Lambda$-mod, where $\eta$ is the Nakayama automorphism.

**Corollary 2.12.** If $\Lambda$ is a Calabi-Yau Frobenius algebra of dimension $m$, then $\Lambda - \text{mod}$ is $d$-Calabi-Yau, for some natural number $d \leq m$. In case $\Lambda$ is symmetric, $d + 1$ is a divisor of $m + 1$.

**Proof.** For each $k \geq 0$ the functors

$$\Omega^k_\Lambda : \Lambda - \text{mod} \to \Lambda - \text{mod}$$

and

$$\Omega^k_{\Lambda^e} : \Lambda - \text{mod} \to \Lambda - \text{mod}$$

are naturally iso morphic as triangulated functors. If $\Omega^{m+1}_\Lambda(\Lambda) \cong \Lambda_{\eta^{-1}}$, then $\Omega^{m+1}_{\Lambda^e} \cong \Lambda_{\eta^{-1}} \otimes \Lambda - \cong \eta(\Lambda)$ as triangulated functors $\Lambda - \text{mod} \to \Lambda - \text{mod}$. By taking quasi-inverse functors, we then have $\Omega^{m-1}_\Lambda \cong \eta^{-1}(\Lambda)$.

In case $\Lambda$ is symmetric (i.e. $\eta$ is inner), the CY dimension of $\Lambda - \text{mod}$ is $d$ when $d + 1$ is the order of $\Omega\Lambda$ in the stable Picard group of $\Lambda$, i.e., in the group of natural isoclasses of triangulated self-equivalences $\Lambda - \text{mod} \to \Lambda - \text{mod}$. The fact that $\Omega^{m-1}_\Lambda \cong \eta^{-1}(\Lambda) \cong \text{id}\Lambda - \text{mod}$ implies that $d + 1$ divides $m + 1$.

**Remark 2.13.** To the best of our knowledge, it is not known whether the Frobenius CY dimension of $\Lambda$ coincides with the CY dimension of $\Lambda - \text{mod}$. Recall that if $B = \oplus_{n \in \mathbb{Z}} B_n$ is a graded algebra, then a graded (left or right) $B$-module $V = \oplus_{n \in \mathbb{Z}} V_n$ is said to be locally finite dimensional graded left (resp. right) $B$-modules is denoted by $B$-lfgr ($B$-lfgr). We have the canonical duality $D : B - \text{lfgr} \to \text{lfgr} B$, which is inverse of itself. Slightly diverting from the terminology of [19], we say that the graded algebra $B$ is graded Frobenius if the category $B$-lfgr is a Frobenius category, which is equivalent to say that the injective and the projective objects coincide in $B$-lfgr. Clearly, a graded Frobenius algebra in the sense of [19] is graded Frobenius in our sense.
Theorem 2.14. (Eu-Schedler) Let $\Lambda$ be a Calabi-Yau Frobenius algebra of dimension $m$ and let $M$ be any $\Lambda$-bimodule. There are isomorphisms of graded right $\HH^*(\Lambda)$-modules:

1. $\HH_{-s}(\Lambda, M)[-m] \cong \HH^s(\Lambda, M)$
2. $\HH^*(\Lambda, M) \cong D(\HH^*(\Lambda, D(M)))[-m] = D(\HH^*(\Lambda, D(M))[m])$
3. $\HH^*(\Lambda) \cong D(\HH^*(\Lambda))[-2m - 1] = D(\HH^*(\Lambda)[2m + 1])$

In particular $\HH^*(\Lambda)$ is a graded Frobenius algebra.

Proof. Fix a complete minimal projective resolution $P$ of $\Lambda$ as $\Lambda$-bimodule. We put

$$P^* : \cdots \longrightarrow \text{Hom}_{\Lambda^e}(P^1, \Lambda^c) \longrightarrow \text{Hom}_{\Lambda^e}(P^0, \Lambda^c) \longrightarrow \text{Hom}_{\Lambda^e}(P^{-1}, \Lambda^c) \longrightarrow \cdots$$

and

$$D(P) : \cdots \longrightarrow D(P^1) \longrightarrow D(P^0) \longrightarrow D(P^{-1}) \longrightarrow \cdots$$

1. $P[-m - 1]$ is a minimal complete projective resolution of $\Omega^{m+1}_\Lambda(\Lambda)$ and we have $Z^0(P^*) = \text{Hom}_{\Lambda^e}(\Lambda, \Lambda^c) \cong \text{Hom}^1_{\Lambda^e}(\Lambda, \Lambda^c)$ (see Lemma 2.3) due to the self-injective condition of $\Lambda^e$. Therefore $P^*[-1]$ is also a minimal complete projective resolution of $\Omega^{m+1}_\Lambda(\Lambda)$. The uniqueness of the minimal complete projective resolution gives that $P[-m - 1] \cong P^*[m]$, hence $P \cong P^*[m]$ in the category $\text{C}(\Lambda^e)$ of complexes of $\Lambda$-bimodules. That gives isomorphisms of complexes of $K$-vector spaces $\text{Hom}_{\Lambda^e}(P, M) \cong \text{Hom}_{\Lambda^e}(P^*[m], M) \cong \text{Hom}_{\Lambda^e}(P^*, M)[-m] \cong P \otimes_{\Lambda^e} M[-m]$, the last one due to the fact that there is a natural isomorphism $\text{Hom}_{\Lambda^e}(P, M) \cong Q \otimes_{\Lambda^e} M$.

Note that the above isomorphisms of complexes are really isomorphisms of left dg modules over the dg algebra $B := \text{End}_{c_{\Lambda^e}(\Lambda^e)}(P)$, whose homology algebra $H^*(B)$ is isomorphic to $\HH^*(\Lambda)$ (see the paragraph following Proposition 2.7). Just as a sample, we do the last isomorphism. We view $P$ as a complex of left $\Lambda^e$-module The complex $P^* = \text{Hom}_{\Lambda^e}(P, \Lambda^c)$ is a dg right $B$-module with multiplication given by $f \beta := f \circ \beta$, for all homogeneous elements $f \in P^*$ and $\beta \in B$. It has also a structure of right $\Lambda^e$-module given by $f(a \otimes b^c) := p \rightarrow f(p)(a \otimes b^c)$, for all homogeneous elements $p \in P$, $f \in P^*$ and all $a, b \in \Lambda$. One readily sees that we have an equality $(f \beta)(a \otimes b^c) = [f(a \otimes b^c)] \beta$. By looking now at a right $\Lambda^e$-module as a left $\Lambda^e$-module in the usual way, this means that $P^*$ has a structure of dg $\Lambda^e - B$-bimodule. As a consequence, when the $\Lambda$-bimodule $M$ is viewed as a left $\Lambda^e$-module, the the complex of $K$-vector spaces $\text{Hom}_{\Lambda^e}(P^*, M)$ is a left dg $B$-module, with multiplication given by the rule $(\beta \psi)(f) = \psi(f \beta) = \psi(f \circ \beta)$, for all homogeneous elements $\psi \in \text{Hom}_{\Lambda^e}(P^*, M)$, $\beta \in B$ and $f \in P^*$. It is routinary to see that the canonical isomorphism of complex of $K$-vector spaces

$$\Psi : P \otimes_{\Lambda^e} M \cong \text{Hom}_{\Lambda^e}(P^*, M),$$

identified by the formula $\Psi_{p \otimes m}(f) = f(p)m$, preserves the left multiplication by elements of $B$.

We then get isomorphisms of graded left $\HH^*(\Lambda)$-modules:

$$\HH^*(\Lambda, M) = H^*(\text{Hom}_{\Lambda^e}(P, M)) \cong H^*((P \otimes_{\Lambda^e} M)[-m]) = H^{*-m}(P \otimes_{\Lambda^e} M) \cong$$

$$\HH^-s+m(\Lambda, M) = \HH^-s-(s-m)(\Lambda, M) = \HH^-s(\Lambda, M)[-m]$$

2. It follows from 1) and from the isomorphism of graded $\HH^*(\Lambda)$-module $D(\HH^*(\Lambda, D(M))) \cong \HH^-s(\Lambda, M)$ (see remark 2.11).
3. Since $\Lambda$ is $m$-Calabi-Yau Frobenius we have $\Omega^{m+1}_\Lambda(\Lambda) \cong \text{Hom}_\Lambda(\Lambda, \Lambda^e) \cong_1 \Lambda_{\eta-1}$, from which we get $\Omega^{m-1}_\Lambda(\Lambda) \cong_1 \Lambda_{\eta} \cong D(\Lambda)$. The isomorphism of 2 for $M = \Lambda$ gives then

$$\text{HH}^*(\Lambda) \cong D(\text{HH}^*(\Lambda, \Omega^{m-1}_\Lambda(\Lambda))[m]) = D(\text{HH}^{*-m+1}(\Lambda)[m]) = D(\text{HH}^*(\Lambda)[2m+1])$$

The fact that $\text{HH}^*(\Lambda)$ is graded Frobenius is a direct consequence of the isomorphism in 3.

\[\square\]

**Definition 3.** A finite dimensional algebra $\Lambda$ is said to be periodic of period $m > 0$ if $\Omega^m(\Lambda) \cong \Lambda$ in the category of $\Lambda$-bimodules and $m$ is minimal with that property.

It is well-known that any periodic algebra is self-injective (see [4]). In case $R$ is a graded commutative ring and $f \in R$ is a homogeneous element which is not nilpotent, we will denote by $R(f)$ the localization of $R$ with respect to the multiplicative subset $\{1, f, f^2, \ldots \}$. It is a graded commutative ring where $\deg(\frac{g}{f}) = \deg(g) - n \cdot \deg(f)$, for all homogeneous elements $g \in R$ and all $n \geq 0$. If $M$ is a graded $R$-module we will denote by $M(f)$ the localization of $M$ at $\{1, f, f^2, \ldots \}$.

**Proposition 2.15.** Let $\Lambda$ be a periodic algebra of period $s$ and let $h \in \text{HH}^*(\Lambda)$ be any element represented by an isomorphism $\Omega^s_\Lambda(\Lambda) \overset{\sim}{\to} \Lambda$. Suppose that $M$ is a $\Lambda$-bimodule. The following assertions hold:

1. $\text{HH}^*(\Lambda, M) \cong \text{HH}^*(\Lambda, M)[s]$ and $\text{HH}^*(\Lambda, M) \cong \text{HH}^*(\Lambda, M)[s]$ as graded $\text{HH}^*(\Lambda)$-modules.
2. $h$ is an element of $\text{HH}^*(\Lambda)$ which is not nilpotent and $\text{HH}^*(\Lambda)$ is isomorphic, as a graded algebra, to $\text{HH}^*(\Lambda)(h)$.
3. $\text{HH}^*(\Lambda, M)$ is isomorphic to $\text{HH}^*(\Lambda, M)(h)$ as a graded $\text{HH}^*(\Lambda)$-module.

**Proof.** We have already seen in the previous comments that $\text{HH}^*(\Lambda)$ is isomorphic to the graded algebra $@_{n \in \mathbb{Z}} \text{Hom}_\Lambda(\Omega^s_\Lambda(\Lambda, \Lambda), \Lambda)$, where the multiplication of homogeneous elements on this algebra is given by $g \cdot f = g \circ \Omega^s_\Lambda(f)$. Now $\hat{h} : \Omega^s_\Lambda(\Lambda) \overset{\sim}{\to} \Lambda$ is an isomorphism representing $h$, then $\Omega^s_\Lambda(\hat{h}^{-1}) : \Omega^s_\Lambda(\Lambda) \to \Lambda$ represents an element $h' \in \text{HH}^{*-s}(\Lambda)$. But then $h' \cdot h = 1$ since $h' \cdot h$ is represented by $\Omega^s_\Lambda(\hat{h}^{-1}) \circ \Omega^s_\Lambda(h) = \Omega^s_\Lambda(\hat{h}^{-1} \hat{h}) = \Omega^s_\Lambda(1_{\Omega^s_\Lambda(\Lambda)}) = 1_{\Lambda}$.

The above paragraph shows that $h$ is invertible (of degree $s$) in $\text{HH}^*(\Lambda)$, from which it follows that multiplication by $h$ gives an isomorphism $Y \overset{\sim}{\to} Y[s]$, for each graded $\text{HH}^*(\Lambda)$-module $Y$ (here we have used that, for $\text{Char}(K) \neq 2$, the period $s$ is even, cf. [19] Theorem 2.3.47).

Since the multiplication of homogeneous elements of degree $> 0$ is the same in $\text{HH}^*(\Lambda)$ and in $\text{HH}^*(\Lambda)$ and $h$ in invertible in this latter algebra it follows that $h$ is not nilpotent in $\text{HH}^*(\Lambda)$. On the other hand, the universal property of the module of quotients gives a unique morphism of graded $\text{HH}^*(\Lambda)$-modules

$$\Phi : \text{HH}^*(\Lambda, M)(h) \rightarrow \text{HH}^*(\Lambda, M)$$

which takes the fraction $\frac{n}{m} \rightarrow h'^{m} \eta$, where $h'$ is the inverse of $h$ in $\text{HH}^*(\Lambda)$. It is clear that the homogeneous elements of degree $\geq 0$ are in the image of $\Phi$. On the other hand, if $\xi \in \text{HH}^{*-j}(\Lambda)$, with $j > 0$, then there is a $k > 0$ such that $k s > j$. Fixing such a $k$, we have that $\eta := h^{k} \xi \in \text{HH}^{k s-j}(\Lambda, M) = \text{HH}^{k s-j}(\Lambda, M)$ and, clearly, the equality $\Phi(\frac{n}{m}) = \xi$ holds. Therefore $\Phi$ is surjective. Moreover $\text{Ker}(\Phi)$ consists of those fractions $\frac{n}{m}$ such that $h^{n} \eta = 0 \in \text{HH}^*(\Lambda, M)$. This is in turn equivalent to say that $\eta = 0$ in $\text{HH}^*(\Lambda, M)$ for $h'$ invertible in $\text{HH}^*(\Lambda)$. That is, $\eta$ is in the kernel of the canonical map $\lambda_{h,M} : \text{HH}^*(\Lambda, M) \rightarrow \text{HH}^*(\Lambda, M)$. Hence we get that $\eta \in \mathcal{P}(\Lambda, M)$, which implies that $h \eta = 0 \in \text{HH}^*(\Lambda, M)$. It follows that $\frac{n}{m} = \frac{h^{n} \eta}{h^{n}} = 0$ and so $\Phi$ is also injective. Finally, in case $\Lambda = M$, the map $\Phi$ is a homomorphism of graded algebras, and the proof is complete.

\[\square\]

Note that if $\Lambda$ is symmetric, then $\Lambda$ is periodic of period $s$ exactly when it is $(s-1)$-Calabi-Yau Frobenius. We then have
Corollary 2.16. If $\Lambda$ is a symmetric periodic algebra of period $s$ and $M$ is a $\Lambda$-bimodule, then:

1. The multiplicative structure of $HH^*(\Lambda)$ is determined by that of $HH^*(\Lambda)$.
2. The structures of $HH^*(\Lambda, M)$ and $HH_{-s}(\Lambda, M)$ as graded $HH^*(\Lambda)$-modules and the structure of $HH_{-s}(\Lambda, M)$ as graded $HH^*(\Lambda)$-module are determined by the structure of $HH^*(\Lambda, M)$ as graded $HH^*(\Lambda)$-module.

Proof. Since $\Lambda$ is CY Frobenius, the two assertions are a direct consequence of the last two propositions.

3 The algebra $\Lambda = P(\mathbb{L}_n)$

3.1 A dualizable basis for the algebra

In the rest of the paper, unless otherwise stated, $\Lambda := P(\mathbb{L}_n)$ is the preprojective algebra associated to the generalized Dynkin quiver $\mathbb{L}_n$. Then its quiver $Q$ is

![Quiver Diagram]

In [3] the authors used the fact that $\Lambda$ is self-injective to prove that $\Lambda$ is a periodic algebra. Note that the path algebra $KQ$ admits an obvious involutive anti-isomorphism $(-)^{-}: KQ \to KQ$ which fixes the vertices and the arrow $\epsilon$ and maps $a_i \sim \bar{a}_i$ and $\bar{a}_i \sim a_i$, for all $i = 1, \ldots, n - 1$. It clearly preserves the relations for $\Lambda$, and hence it induces another involutive anti-isomorphism $(-)^{-}: \Lambda \to \Lambda$. We shall call it the canonical (involutive) antiautomorphism of $\Lambda$.

The next proposition shows that we can apply to $\Lambda$ the results in the previous subsection. It also fixes the basis of $\Lambda$ with which we shall work all through the paper.

Proposition 3.1. Let $\Lambda = P(\mathbb{L}_n)$ be the preprojective algebra of type $L$ and put $B = \bigcup_{i,j} e_iBe_j$, where

a) $e_1Be_1 = \{e_1, \epsilon, \epsilon^2, \ldots, \epsilon^{2n-1}\}$

b) $e_1Be_j = \{a_1 \cdots a_{j-1}, a_i \cdots a_j, \epsilon a_1 \cdots a_{j-1}, \epsilon^2 a_1 \cdots a_{j-1}, \ldots, \epsilon^{2(n-j)+1} a_1 \cdots a_{j-1}\}$ in case $j \neq 1$

c) $e_iBe_j = \{a_1 \cdots a_{j-1}, a_i \cdots a_j, \bar{a}_i \cdots \bar{a}_j, a_i \cdots a_{n-1} \bar{a}_{n-1} \cdots \bar{a}_j\}$

U

where $s_{ij} = 0$ for $i \neq j$ and $s_{ij} = \frac{a_{i-1}(i-1)}{2}$, whenever $1 < i \leq j \leq n$ (here we convene that $a_{i-1} a_{j-1} = e_i$ in case $i = j$).

d) $e_iBe_j = \{b : b \in e_jBe_1\}$ in case $i > j$.

Then $B$ is a dualizable basis of $\Lambda$. In particular, $\Lambda$ is a symmetric algebra.

Proof. Note that $e_iBe_j$ contains, at most, one element of a given degree. In order to see that $B$ is a basis of $\Lambda$ we just need to see that all the paths in $e_iBe_j$ are nonzero and that they generate $e_i\Lambda e_j$ as a $K$-vector space. Note that then $\dim(e_i\Lambda e_j) \leq 1$, for all $i, j \in Q_0$ and $k \geq 0$, and Lemma 2.4 can be applied.

Suppose that we have already proved that $B$ is a basis of $\Lambda$. We claim that the condition on the parallel paths holds. Indeed $\Lambda$ is a graded algebra and, given $i, j \in Q_0$ and $n \geq 0$ integer,
there is at most one element in \( e_i Be_j \) of degree \( n \). It follows that any path \( p : i \to ... \to j \) which is not in \( I \) satisfies that \( p - \lambda q \in I \), for some \( 0 \neq \lambda \in K \), where \( q \in e_i Be_j \) is the only element in \( e_i Be_j \) of degree equal to \( \text{length}(p) \). But the shape of the relations which generate \( I \) implies that \( \lambda = (-1)^s \), for some integer \( s \). Therefore, given two parallel paths \( p \) and \( q \) of equal length which do not belong to \( I \), one has that either \( p - q \) or \( p + q \) belong to \( I \).

We now pass to check that all the paths in \( e_i Be_j \) are nonzero and that they generate \( e_i A e_j \) as a \( K \)-vector space. Assume that \( i,j \in Q_0 \) are vertices such that \( i \leq j \). The antiautomorphism \((-)^-1\) given before guarantees that once we have a basis for \( e_i Be_j \), the remaining cases, \( e_j Be_i \), can be described by adding bars to the monomials obtained for \( e_i Be_j \).

Observe that for each vertex \( i \neq 1 \) we have, up to sign, a unique cycle of minimum length, namely \( a_i \bar{a}_i \). However for the vertex \( i = 1 \) we do not only have the cycle \( a_1 \bar{a}_1 = -\epsilon^2 \) but also the loop \( \epsilon \).

Let \( 0 \neq b \) be a monomial of a fixed length starting at \( i \) and ending at \( i + s \). The previous comment tell us that \( b \) contains either an even number or an odd number of arrows of type \( \epsilon \).

In the first case, the equality \((a_{i-1} a_{i-2}) a_{i-1} \cdots a_{i+s-1} = (-1)^i a_{i-1} \cdots a_{i+s-1} (a_i \bar{a}_i a_{i+s})\) shows that \( b \) has at most \( n - i \) non-bar letters and \( n - (s + i) \) bar letters. Thus we can set as a basis element the non-zero path \( b = a_{i-1} \cdots a_{i+s-1} (a_i \bar{a}_i a_{i+s}) \) if \( j \leq n - 1 - i - s \), that is, when all the bar letters are to the right.

On the contrary, if \( b \) contains and odd number of \( \epsilon \) arrows, we have that

\[
b = (a_i \bar{a}_i) (a_{i-1} \cdots a_{2} \epsilon^{2i} a_{1} \cdots a_{i+s-1}) = (-1)^i (a_{i-1} \cdots a_{1} \epsilon^{2i} a_{1} \cdots a_{i+s-1})
\]

which, up to sign, equal to

\[
\bar{a}_{i-1} \cdots \bar{a}_1 \epsilon (a_1 \cdots a_{i} \bar{a}_{i-1} \cdots a_{1} a_{i+s-1})
\]

But notice that the arrows between brackets form a path with an even number of \( \epsilon \) arrows which is in time, up to sign, equal to \( a_1 \cdots a_{i+s-1} (a_i \bar{a}_{i+s-1}) \). Hence we can conclude that \( a_{i-1} \cdots a_{1} \epsilon^{2i+1} a_{1} \cdots a_{i+s-1} \) is a non-zero path if and only if \( 0 \leq t \leq n - (s + i) \). Thus the sets given in the statement are in fact a basis of \( \Lambda \).

It remains to prove that \( B \) is a dualizable basis. This task is reduced to prove that \( a \ast a = \omega_i(a) \), for each \( a \in Q_1 \). We have \( \omega_i(a) = \epsilon^{2n-1} \), hence \( \epsilon^* = \epsilon^{2n-2} \) and we clearly have \( \epsilon^* \epsilon = \omega_i(a) \).

For \( a_i \) \((i=1, \ldots, n)\) we have

\[
a_i [\bar{a}_{i-1} \cdots a_{1} \epsilon^{2(n-i)-1} a_{1} \cdots a_{i-1}] = (-1)^i \bar{a}_{i-1} \cdots a_{1} \epsilon^{2(n-i)+1} a_{1} \cdots a_{i-1} =
\]

\[
(-1)^i [(-1)^{\frac{n-i+1}{2}} \omega_i = (-1)^{\frac{n-i+1}{2}} \omega_i = (-1)^{\frac{n-i+1}{2}} \omega_i(a_i)].
\]

Then \( a_i^* = (-1)^{\frac{n-i+1}{2}} \bar{a}_{i} \cdots \bar{a}_{1} \epsilon^{2(n-i)-1} a_{1} \cdots a_{i-1} \) and therefore

\[
a_i^* a_i = (-1)^{\frac{n-i+1}{2}} \bar{a}_{i} \cdots \bar{a}_{1} \epsilon^{2(n-i)-1} a_{1} \cdots a_{i-1} \bar{a}_{i} = \omega_i(a_i) = \omega_i(a_i) = \omega_i(a_i)
\]

The argument is symmetric for the arrows \( a_i \) and therefore the basis \( B \) is dualizable.

\[ \square \]

**Remark 3.2.** If one modifies the basis \( B \) of proposition 3.1 by putting \( \omega_i = \bar{a}_{i-1} \cdots \bar{a}_{1} \epsilon^{2(n-i)+1} a_{1} \cdots a_{i-1} \) for all \( i = 1, \ldots, n \), then the resulting basis is no longer dualizable. Indeed the proof of the lemma shows that \( a_i^* = (-1)^i \bar{a}_{i-1} \cdots \bar{a}_{1} \epsilon^{2(n-i)-1} a_{1} \cdots a_{i-1} \) in the new situation, and then \( a_i^* a_i = (-1)^i \omega_i(a_i) \).

By [3], we know that the third syzygy of \( \Lambda \) as a bimodule is isomorphic to \( 1 \Lambda e_r \), for some \( r \in \text{Aut}(\Lambda) \) such that \( r^2 = id_{\Lambda} \). Our emphasis on choosing a dualizable basis on \( \Lambda \) comes from the fact that it allows a very precise determination of \( \tau \). Indeed, combining results of [3] and [16], we know that if \( B \) is a dualizable basis, then the initial part of the minimal projective resolution of \( \Lambda \) as a bimodule is:

\[
0 \to N \xrightarrow{\iota} P \xrightarrow{R} Q \xrightarrow{\delta} P \xrightarrow{u} \Lambda \to 0,
\]

where \( P = \oplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda, \quad Q = \oplus_{e \in Q_1} \Lambda e_i(a) \otimes e_i(a) \Lambda \) and \( N \) is the \( \Lambda \)-subbimodule of \( P \) generated by the elements \( \xi_i = \sum_{x \in e_i} B(-1)^{deg(x)} x \otimes x^* \), where \( B \) is any given basis of \( \Lambda \) consisting of paths
and negative of paths which contains the vertices, the arrows and a basis of Soc(Λ). Here i is the inclusion, u is the multiplication map and R and δ are as in proposition 3.0 below.

The following result was proved in [3].

**Lemma 3.3** (see [3], Proposition 2.3). Let B be a dualizable basis of Λ, let N be the Λ-bimodule mentioned above and let τ ∈ Aut(Λ) be the only automorphism of Λ such that τ(e_i) = e_i and τ(a) = -a, for all i ∈ Q_0 and a ∈ Q_1. There is an isomorphism of Λ-bimodules φ : Λτ ∼ N mapping e_i ↦ ξ_i, for each i ∈ Q_0.

**Remark 3.4.** The dualizable basis hypothesis does not appear in the statement of Proposition 2.3 in [3]. However, it is implicitly used in the proof of [3][Lemma 2.4]. From our work with examples it seems that, without that extra hypothesis, the element \( \sum_{x \in e_iB}(-1)^{deg(x)}x \otimes x^* \) need not be in Ker(R).

The dualizable hypothesis seems to be implicitly used also in the argument of [18][Section 7.1], where the corresponding result (with the automorphism τ conveniently modified) is proved. In both cases, the crucial point is to guarantee that if \( x \in B \) is a homogeneous element of the basis B of degree \( \geq 0 \), then, for any arrow \( a \in Q_1 \), the element \( ax^* \) (resp. \( x^*a \)) should again be of the form \( y^* \), for some \( y \in B \), whenever the product is nonzero. This follows immediately in case one has \( a(ya)^* = y^* \) and \( (ay)^*a = y \), for all \( y \in B \) and \( a \in Q_1 \). This is precisely the statement of Lemma 2.4 in [3] and is implicit in the argument of [18][Section 7.1].

Essentially by the proof of our Lemma 2.4 we see that the mentioned crucial point is tantamount to require that B is a dualizable basis and that (-,-) is its associated Nakayama form. If, as in the spirit of [18][Section 6.3], one has from the beginning a symmetric Nakayama form (−,−) such that \( (e_i,e_i) = 0 \), for all \( i \in Q_0 \), and finds a basis B consisting of homogeneous elements which contains the vertices and has the property that the dual elements \( \{w_i := e_i^* : i \in Q_0 \} \) (in B*) belong to \( B \cap Soc(Λ) \), then one readily sees that B is dualizable and (−,−) is its associated Nakayama form.

In the rest of the paper, the basis B will be always that of proposition 3.1. The following properties can be derived in a routinary way. We leave the verifications to the reader.

**Corollary 3.5.** Let \( i,j \in Q_0 \) be vertices. The following holds:

1. The set of possible degrees of the elements in \( e_iBe_j \) is

\[
\{j - i, j - i + 2, j - i + 4, \ldots j - i + 2(n - j) = 2n - (i + j)\} \cup \\
\{j + i - 1, j + i + 1, j + i + 3, \ldots j + i + 2(n - max(i,j)) - 1\}
\]

2. If \( a_{i-1} \cdots a_1^2 a_1 \cdots a_{j-1} \) is a nonzero element of Λ, then \( k \leq n - i - j + 1 \).

3. \( a_1 \cdots a_{j-1} a_j \cdots a_1 = (-1)^{i+1} 2^j e_2 e_1 \cdots e_{j-1} \) for \( j = 2, \ldots, n \).

4. \( a_1 \cdots a_{j-1} a_j = (-1)^j e_2 a_1 \cdots a_{j-2} \)

5. \( a_ia_{i+1} \cdots a_j = (-1)^{i+1} a_{i+1} \cdots a_{j+1} \) whenever \( i \leq j < n \) (convening that \( a_n = 0 \)).

6. \( \dim(Hom_\Lambda(P,Λ)) = \sum_{i=1}^n \dim(e_iΛe_i) = \sum_{i=1}^n [2(n - i) + 2] = n^2 + n \)

7. \( \dim(Hom_\Lambda(Q,Λ)) = \dim(e_1Λe_1) + 2 \sum_{i=1}^{n-1} (e_iΛe_{i+1}) = 2n + 2 \sum_{i=1}^{n-1} [2(n - i - 1) + 1] = 2n^2 \)

8. The Cartan matrix of Λ is given by:

\[
C_{P(L_n)} = \begin{pmatrix}
2n & 2(n-1) & 2(n-2) & \cdots & 2 \\
2(n-1) & \cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
2 & \cdots & \cdots & \cdots & \cdots \\
2 & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

\( C_{P(L_n)} \)
where
\[
C_{P(\ell_2)} = \begin{pmatrix}
4 & 2 \\
2 & 2
\end{pmatrix}
\]

Its determinant is \(\text{det}(C_{P(\ell_2)}) = 2^n\) (see remark 3.3 in [20]).

3.2 The minimal projective resolution of \(\Lambda\)

We are now ready to give all the modules and maps of the minimal projective resolution \(\Lambda = P(\ell_n)\) as a bimodule.

Proposition 3.6. Let \(\Lambda = P(\ell_n)\) be the preprojective algebra of type \(\ell_n\), let \(B\) be the dualizable basis of proposition [3,1] and let \(\tau \in \text{Aut}(\Lambda)\) the algebra automorphism that fixes the vertices and satisfies that \(\tau(a) = -a\), for all \(a \in Q_1\). The chain complex \(\ldots \rightarrow P^{-2} \xrightarrow{d^{-2}} P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{u} \Lambda \rightarrow 0\) is identified by the following properties is a minimal projective resolution of \(\Lambda\) as a bimodule:

a) \(P^{-n} = Q := \bigoplus_{a \in Q_1} \Lambda e_i(a) \otimes e_i(\Lambda)\) if \(n \equiv 1\) (mod 3) and \(P^{-n} = P := \bigoplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda\) otherwise.

b) \(u\) is the multiplication map, \(d^m = (d^m)_\tau\) whenever \(m = n = \pm 3\) and the initial differentials \(d^{-1} =: \delta\), \(d^{-2} =: R\) and \(d^{-3} =: k\) are the only homomorphisms of \(\Lambda\)-bimodules satisfying:

i) \(\delta(e_i(a) \otimes e_i(a)) = a \otimes e_i(a) - e_i(\Lambda) \otimes a\)

ii) \(R(e_i \otimes e_i) = \sum_{a \in Q_1} e_i(a) \otimes a + a \otimes e_i(a)\)

iii) \(k(e_i \otimes e_i) = \sum_{x \in e_i B} (-1)^{\text{deg}(x)} x \otimes x^*\)

for all \(a \in Q_1\) and \(i \in Q_0\).

Proof. By Lemma 3.3 (see [3, Prop. 2.3]) we have an exact sequence of \(\Lambda\)-bimodules:

\[
0 \rightarrow \Lambda \xrightarrow{j} P \xrightarrow{R} Q \xrightarrow{\delta} P \xrightarrow{u} \Lambda \rightarrow 0,
\]

where the map \(j\) satisfies that \(j(e_i) = \sum_{x \in e_i B} (-1)^{\text{deg}(x)} x \otimes x^*\) for each \(i \in Q_0\).

Applying the self-equivalence \(F_\tau : \text{Mod}_\Lambda \rightarrow \text{Mod}_\Lambda\), which acts as the identity on morphisms, and bearing in mind that \(\tau^2 = 1\Lambda\), we get an exact sequence

\[
0 \rightarrow \Lambda \xrightarrow{j} P_{\tau} \xrightarrow{R_{\tau}} Q_{\tau} \xrightarrow{\delta_{\tau}} P_{\tau} \xrightarrow{u_{\tau}} \Lambda_{\tau} \rightarrow 0.
\]

By Lemma 3.1 we then get an exact sequence of \(\Lambda\)-bimodules

\[
0 \rightarrow \Lambda \xrightarrow{j_{\tau}} P_{\tau} \xrightarrow{R_{\tau}} Q_{\tau} \xrightarrow{\delta_{\tau}} P_{\tau} \xrightarrow{u_{\tau}} \Lambda_{\tau} \rightarrow 0.
\]

where, if \(\psi : G_{\tau} \xrightarrow{\approx} F_{\tau}\) denotes the natural isomorphism of lemma 3.1 then \(\tilde{u} = u \circ \psi_P : a \otimes b \mapsto a \tau(b)\) and \(j = \psi_P^{-1} \circ j\) which takes \(e_i \mapsto -\sum_{x \in e_i B} x \otimes x^*\).

The composition \(P \xrightarrow{\tilde{u}} \Lambda \xrightarrow{j} P\) takes \(e_i \otimes e_i \mapsto \sum_{x \in e_i B} (-1)^{\text{deg}(x)} x \otimes x^*\) and, hence, coincides with the morphisms \(k\) given in the statement. Finally, the composition \(P \xrightarrow{u} \Lambda \xrightarrow{j} P\) takes \(e_i \otimes e_i \mapsto \sum_{x \in e_i B} (-1)^{\text{deg}(x)} x \otimes (x^*)\) and \(\tau(x^*) = k \tau(e_i \otimes e_i)\). Therefore \(j \circ u = k\).

The rest of the proof is clear. \(\square\)

3.3 A cochain complex which gives the Hochschild cohomology

Recall that if \(f : \bigoplus_{i=1}^n \Lambda e_i \otimes e_j \Lambda \rightarrow \bigoplus_{i=1}^n \Lambda e_i \otimes e_i \Lambda\) is a morphism of \(\Lambda\)-bimodules, an application of the contravariant functor \(\text{Hom}_{\Lambda^e}(\cdot, \Lambda) : \text{Mod}_\Lambda \rightarrow \text{k Mod}\) gives a \(K\)-linear map

\[
f^* : \text{Hom}_{\Lambda^e}(\bigoplus_{i=1}^n \Lambda e_i \otimes e_i \Lambda) \rightarrow \text{Hom}_{\Lambda^e}(\bigoplus_{i=1}^n \Lambda e_i \otimes e_j \Lambda, \Lambda).
\]

Due to the isomorphism of \(K\)-vector spaces \(\text{Hom}_{\Lambda^e}(\Lambda e_i \otimes e_j \Lambda, \Lambda) \cong e_i \Lambda e_j\), for all \(i, j \in Q_0\), we get an induced map, still denoted the same \(f^* : \oplus_{i=1}^n \oplus_{i=1}^n \oplus_{i=1}^n e_i e_j \Lambda^e \rightarrow \oplus_{i=1}^n e_i e_j \Lambda^e\). As usual we will also denote by \(J = J(\Lambda)\) the Jacobson radical of \(\Lambda\). With this terminology, we get
Proposition 3.7. For each \( n \geq 0 \), \( HH^n(\Lambda) \) is the \( n \)-th cohomology space of the complex

\[
V^* : \cdots \longrightarrow \oplus_{i \in Q_0} e_i \Lambda e_i \overset{\delta^*}{\longrightarrow} \oplus_{a \in Q_1} e_{i(a)} \Lambda e_{t(a)} \overset{R^*}{\longrightarrow} \oplus_{i \in Q_0} e_i \Lambda e_i \overset{k^*}{\longrightarrow} \oplus_{a \in Q_1} e_{i(a)} \Lambda e_{t(a)} \cdots
\]

where \( V^0 = \sum_{i \in Q_0} e_i \Lambda e_i \) and \( V^n = 0 \) \( \forall n < 0 \). Moreover, viewing \( \oplus_{i \in Q_0} e_i \Lambda e_i \) and \( \oplus_{a \in Q_1} e_{i(a)} \Lambda e_{t(a)} \) as subspaces of \( \Lambda \), the differentials of \( V^* \) act as follows for each oriented cycle \( c \) at \( i \) and each path \( p : i(a) \rightarrow \cdots \rightarrow t(a) \):

a) \( \delta^*(c) = a_{i-1}c - c\bar{a}_{i-1} + \bar{a}_i c - ca_i \)

b) \( R^*(p) = \bar{p} + \bar{a}p \)

c) \( k^*(c) = 0 \) (i.e. \( k^* \) is the zero map)

d) \( \delta^*_c(c) = a_{i-1}c + c\bar{a}_{i-1} + \bar{a}_i c + ca_i \)

e) \( R^*_c(p) = p\bar{a} - \bar{a}p \)

f) \( k^*_c(c) = 0 \) if \( c \in e_i e_i \), and \( k^*_c(e_i) = -\sum_{j \in Q_0} \text{dim}(e_i \Lambda e_j) \omega_j \)

where we convene that \( a_0 = \bar{a}_0 = \epsilon \) and \( a_n = \bar{a}_n = 0 \)

Proof. \( HH^n(\Lambda) \) is the \( n \)-th cohomology space of the complex obtained by applying \( Hom_{\Lambda^*}(\_) \) to the minimal projective resolution of \( \Lambda \) as bimodule. The \( K \)-vector spaces of that complexes are precisely those of \( V^* \) and the only nontrivial part is the explicit definition of its differentials.

We have two canonical isomorphisms of \( K \)-vector spaces:

\[
\oplus_{j \in Q_0} e_j \Lambda e_j \overset{\sim}{\longrightarrow} Hom_{\Lambda^*}(\oplus_{j \in Q_0} e_j \otimes e_j \Lambda, \Lambda)
\]

\[
\oplus_{a \in Q_1} e_{i(a)} \Lambda e_{t(a)} \overset{\sim}{\longrightarrow} Hom_{\Lambda^*}(\oplus_{a \in Q_1} e_{i(a)} \otimes e_{t(a)} \Lambda, \Lambda)
\]

The first one matches a nonzero oriented cycle \( c \) at \( i \) with the morphism of \( \Lambda \)-bimodules \( \oplus_{i \in Q_0} \Lambda e_j \otimes e_j \Lambda \overset{\cdot c}{\longrightarrow} \Lambda \) taking \( e_j \otimes e_j \rightarrow \delta_{ij} \), where \( \delta \) is the Kronecker symbol. Similarly a nonzero path \( p : i(a) \rightarrow \cdots \rightarrow t(a) \) is matched by the second isomorphism with the morphism of \( \Lambda \)-bimodules \( \oplus_{b \in Q_1} \Lambda e_{i(b)} \otimes e_{t(b)} \Lambda \overset{\cdot \delta bp}{\longrightarrow} \Lambda \) taking \( e_{i(b)} \otimes e_{t(b)} \rightarrow \delta_{ab} p \). With these matches in mind the task of checking that the explicit definition of the differentials is the giving one is routinary and mainly left to the reader. We just do some samples:

a) \( \delta^*(c) \) is the element of \( \oplus_{b \in Q_1} \Lambda e_{i(b)} \Lambda e_{t(b)} \) matched with \( \bar{c} \circ \delta \in Hom_{\Lambda^*}(\oplus_{b \in Q_1} \Lambda e_{i(b)} \otimes e_{t(b)} \Lambda, \Lambda) \).

Then

\[
\delta^*(c) = \sum_{b \in Q_1}(\bar{c} \circ \delta)(e_{i(b)} \otimes e_{t(b)}) = \sum_{b \in Q_1} \bar{c}(b \otimes e_{t(b)} - e_{i(b)} \otimes b) =
\]

\[
\sum_{b \in Q_1} [b \bar{c}(e_{t(b)} \otimes e_{i(b)}) - \bar{c}(e_{t(b)} \otimes e_{i(b)})b] = \sum_{b \in Q_1, t(b) = i} b\bar{c} - \sum_{b \in Q_1, i(b) = i} cb =
\]

\[
a_{i-1}c + \bar{a}_i c - ca_i - \bar{a}_{i-1}
\]

c) \( k^*(c) \) is the element of \( \oplus_{j \in Q_0} e_j \Lambda e_j \) matched with \( \bar{c} \circ k \in Hom_{\Lambda^*}(\oplus_{j \in Q_0} \Lambda e_j \otimes e_j \Lambda, \Lambda) \). Then

\[
k^*(c) = \sum_{j \in Q_0}(\bar{c} \circ k)(e_j \otimes e_j) = \sum_{j \in Q_0} \bar{c}(\sum_{x \in e_j B} (-1)^{\deg(x)} x \otimes x^*) = \sum_{j \in Q_0} \sum_{x \in e_j B_{e_i}} (-1)^{\deg(x)} xx^*
\]

But \( xx^* = 0 \) in case \( \deg(x) > 0 \) because \( xx^* = \omega_j \) is an element in the socle. In case \( c = e_j \) we have \( k^*(e_j) = \sum_{j \in Q_0} \sum_{x \in e_j B_{e_i}} (-1)^{\deg(x)} xx^* \). Bearing in mind that \( xx^* = \omega_j \) for each \( x \in e_j B_{e_i} \) and that the number of elements in \( e_j B_{e_i} \) with even degree is the same as the number of those
with odd degree, we conclude that also \( k^*(e_i) = 0 \). Since \( k^* \) vanishes on all nonzero oriented cycles it follows that \( k^* = 0 \).

f) Arguing similarly with \( k_+^* \) we get that
\[
k_+^*(c) = 0 \text{ if } \deg(c) > 0 \text{ and }
\]
\[
k_+^*(e_i) = -\sum_{j \in Q_0} \sum_{x \in e_j B e_i} xx^* = -\sum_{j \in Q_0} \dim(e_j \Lambda e_i) \omega_j
\]

\[\square\]

**Remark 3.8.** With the adequate change of presentation of the algebra, the complex \( V^\bullet \) should correspond to the sequence of morphisms in [15][Section 7.4], although the there defined differentials seem not to make it into a complex.

**Corollary 3.9.** \( \Lambda \) is a symmetric periodic algebra of period 6 and \( \mathcal{P}(\Lambda, \Lambda) = \text{Soc}(\Lambda) \) when we view the isomorphism \( HH^0(\Lambda) \cong Z(\Lambda) \) as an identification.

**Proof.** By 3.1 we know that \( \Lambda \) is symmetric, and by 3.0 \( \Lambda \) is periodic of period 6.

To see that the isomorphism \( HH^0(\Lambda) \cong Z(\Lambda) \) identifies \( \mathcal{P}(\Lambda, \Lambda) \) with \( \text{Soc}(\Lambda) = \text{Soc}(Z(\Lambda)) \), note that from 3.6 it follows that a minimal complete resolution of \( \Lambda \) is given by

\[
\cdots P^{-2} \longrightarrow P^{-1} \overset{d^{-1}}{\longrightarrow} P^0 \overset{d^0}{\longrightarrow} P^1 \overset{d^1}{\longrightarrow} P^2 \longrightarrow \cdots
\]

where \( P^n = Q = \bigoplus_{a \in Q_0} \Lambda e_{(a)} \Lambda \), when \( n \equiv -1(\text{mod}3) \), and \( P^n = \bigoplus_{e_i} \Lambda e_i \Lambda \) otherwise, and the arrows are given by \( d^m = (d^m)_\tau \) whenever \( m \equiv n(\text{mod}3) \) and \( d^{-1} = \delta, d^{-2} = R \) and \( d^{-3} = k \).

It follows that \( HH^*(\Lambda) \) is the cohomology of the complex

\[
\cdots V^{-2} \overset{R^\tau}{\longrightarrow} V^{-1} \overset{k_+^*}{\longrightarrow} V^0 \overset{\delta^*}{\longrightarrow} V^1 \overset{R^\tau}{\longrightarrow} V^2 \longrightarrow \cdots
\]

In particular, we have \( HH^0(\Lambda) = \frac{Ker(\delta^*)}{\text{Im}(k_+^*)} \). But \( Ker(\delta^*) = HH^0(\Lambda) = Z(\Lambda) \) while \( \text{Im}(k_+^*) = \text{Soc}(\Lambda) \) since the Cartan matrix of \( \Lambda \) is invertible. Note that the isomorphism \( Z(\Lambda) \cong \text{End}_\Lambda(\Lambda) \) identifies \( \text{Im}(k_+^*) \) with \( \mathcal{P}(\Lambda, \Lambda) \).

\[\square\]

**Corollary 3.10.** There are isomorphisms of graded \( HH^*(\Lambda) \)-modules:

\[
HH^*(\Lambda) \cong HH^*(\Lambda)[6]
\]

\[
HH_{-\bullet}(\Lambda) \cong D(HH^*(\Lambda)) \cong HH^*(\Lambda)[5]
\]

and isomorphisms of graded \( HH^*(\Lambda) \)-modules \( HH_{-\bullet}(\Lambda) \cong D(HH^*(\Lambda)) \). Moreover \( HH^*(\Lambda) \) is a graded Frobenius algebra.

**Proof.** \( HH^*(\Lambda) \cong HH^*(\Lambda)[6] \) since \( \Lambda \) is periodic of period 6. On the other hand, \( \Lambda \) is 5-CY Frobenius and, by 2.14 we have

\[
D(HH^*(\Lambda)) \cong HH^*(\Lambda)[11]
\]

\[
HH_{-\bullet}(\Lambda) \cong HH^*(\Lambda)[5]
\]

Then the isomorphisms in the statement follow. The graded Frobenius condition of \( HH^*(\Lambda) \) follows from Theorem 2.14.

On the other hand, we have an isomorphism \( HH_{-\bullet}(\Lambda) \cong D(HH^*(\Lambda, D(\Lambda))) \cong D(HH^*(\Lambda)) \) due to remark 2.11 and the fact that \( D(\Lambda) \cong \Lambda \).

\[\square\]
4 The Hochschild cohomology spaces

In the rest of the paper we assume that \( \text{Char}(K) \neq 2 \).

In this section we will use the complex \( V^* \) of proposition 3.7 to calculate the dimension and an appropriate basis of each space \( HH^*(\Lambda) \). In the proof of the following lemma and in the rest of the paper, the matrix of a linear map is always written by columns.

**Lemma 4.1.** The equality \( \text{Im}(R^*) = \oplus_{i \in \mathbb{Q}_0} \mathbb{E}_i e_i \) holds and \( \text{Im}(R^*_n) \) is a subspace of codimension \( n \) in \( \oplus_{i \in \mathbb{Q}_0} \mathbb{E}_i e_i \). In particular, we have:

\[
\dim(\text{Im}(R^*)) = n^2
\]

\[
\dim(\text{Im}(R^*_n)) = n^2 - n.
\]

**Proof.** We put \( V = \oplus_{a \in \mathbb{Q}} e_i \Lambda e_i(a) \) and \( W = \oplus_{i \in \mathbb{Q}_0} \mathbb{E}_i e_i \) for simplicity and view \( R^* \) and \( R^*_n \) as \( K \)-linear maps \( V \to W \). For each \( 0 < k < 2n \) we denote by \( V_k \) (resp. \( W_k \)) the vector subspace consisting of the elements of degree \( k \). Since both \( R^* \) and \( R^*_n \) are graded maps of degree 1 we have induced \( K \)-linear maps

\[
R^*, R^*_n : V_{k-1} \to W_k
\]

for \( k = 1, \ldots, 2n - 1 \).

It is important now to notice that the canonical antiisomorphism of \( \Lambda, x \mapsto \bar{x} \), is the identity on \( W \). Moreover, we have equalities \( R^*(\bar{p}) = R^*(p) \) and \( R^*_n(\bar{p}) = -R^*_n(p) \). We then get \( R^*(\bar{p}) = R^*(p) \) and \( R^*_n(\bar{p}) = -R^*_n(p) \). This tells us that the images of the maps \( R^*, R^*_n : V \to W \) are the same as those of their restrictions to \( V^* = V \cap (\oplus_{j=0}^{n-1} e_i(a)) \) (convering that \( a_0 = e \)). Those images are in turn the direct sum of the images of the induced maps

\[
R^*, R^*_n : V^+_{k-1} \to W_k \quad (k = 1, \ldots, 2n - 1)
\]

and thus the ones we shall calculate.

Let us denote by \( b^j_i \) the only element in \( e_i B e_i \) of degree \( t \).

We start by considering the case when \( k = 2m \) is even \((1 \leq m \leq n - 1) \). In that situation, a basis of \( W_{2m} \) is given by \( \{b_{1m}^2, b_{2m}^2, \ldots, b_{n-1m}^2\} \) while a basis of \( V_{2m-1}^+ \) is \( \{v, v_{a_1}, \ldots, v_{a_{n-m}}\} \) where \( v = e^{2n-1} \) and \( v_{a_i} = a_i \cdots a_{i+m-1} \bar{a}_{i+m-1} \cdots \bar{a}_{i+1} \) for \( i = 1, \ldots, n-m \). In particular \( \dim(V_{2m-1}^+) = n - m + 1 \) and \( \dim(W_{2m}) = n - m \). Direct computation, using remark 3.5 shows that

\begin{align*}
\text{i) } R^*(v) &= 2b_{1m}^2, & R^*_n(v) &= 0 \\
\text{ii) } R^*(v_{a_1}) &= (-1)^{(m+1)n}b_{1m}^2 + (-1)^mb_{2m}^2, & R^*_n(v_{a_1}) &= (-1)^{(m+1)n}b_{1m}^2 + (-1)^{m+1}b_{2m}^2 \\
\text{iii) } R^*(v_{a_i}) &= b_{2m}^2 + (-1)^{m+1}b_{2m+1}, & R^*_n(v_{a_i}) &= b_{2m}^2 + (-1)^{m+1}b_{2m+1} \\
& \text{(convering that } b_{2m}^2 = 0 \text{ if } j > n - m) \end{align*}

Then in the matrices of \( R^* \) and \( R^*_n \) with respect to the given bases of \( V_{2m-1}^+ \) and \( W_{2m} \), which are both of size \( (n - m) \times (n - m + 1) \), the columns from the \( 2^{nd} \) to the \( (n - m + 1) - \text{th} \) are linearly independent. We then get that the maps \( R^*, R^*_n : V_{2m-1}^+ \to W_{2m} \) are both surjective for each \( m = 1, \ldots, n - 1 \).

We now deal with the case when \( k = 2m - 1 \) is odd, in which case a basis of \( W_{2m-1} \) is \( \{b_{1m-1}^2, \ldots, b_{m-1}^2\} \). On the other hand, a basis of \( V_{2m-2}^+ \) is given by \( \{v', v'_a, \ldots, v'_a\} \), where \( v' = e^{2m-2} \) and \( v'_a = \bar{a}_{i-1} \cdots \bar{a}_{i} e^{2(m-i)-1} a_{1} \cdots a_{i} \) for \( i = 1, \ldots, m-1 \). Direct calculation, using again remark 3.5, shows the following:

\begin{align*}
\text{i) } R^*(v') &= 2b_{1m-1}^2, & R^*_n(v') &= 0 \\
\end{align*}
ii) If $m \neq n$ then
\[ R^*(v'_{a_i}) = (-1)^i b_{i}^{2m-1} + b_{i+1}^{2m-1} \]
\[ R^*_r(v'_{a_i}) = (-1)^i b_{i}^{2m-1} - b_{i+1}^{2m-1} \]

iii) If $m = n$ then
\[ R^*(v'_{a_i}) = (-1)^i b_{i}^{2m-1} + (-1)^i b_{i+1}^{2m-1} = (-1)^i (w_i + w_{i+1}) \]
\[ R^*_r(v'_{a_i}) = (-1)^i b_{i}^{2m-1} - (-1)^i b_{i+1}^{2m-1} = (-1)^i (w_i - w_{i+1}) \]

Therefore, the square matrices of $R^*$ and $R^*_r$ with respect to the given bases of $V_{2m-2}$ and $W_{2m-1}$ are upper triangular. In the case of $R^*$ all its diagonal entries are nonzero while in the case of $R^*_r$ only the entry $(1, 1)$ is zero. It follows:

a) The map $R^*: V_{2m-2} \to W_{2m-1}$ is surjective for all $m = 1, \ldots, n$.

b) The image of the map $R^*_r: V_{2m-2} \to W_{2m-1}$ has codimension $1$ in $W_{2m-2}$ for all $m = 1, \ldots, n$.

The final conclusion is that the map $R^*: V \to W$ is surjective while the image of $R^*_r: V \to W$ has codimension exactly the number of odd numbers in $\{1, 2, \ldots, 2n-1\}$. That is $\dim(W) - \dim(\text{Im}(R^*_r)) = n$.

\[ \square \]

Remark 4.2. The proof of lemma 4.1 gives that if $\omega_j$ is viewed as an element of $Ker(k^*_r) \forall j \in Q_0$, then $\omega_j - \omega_{j+1} \in \text{Im}(R^*_r) \forall j = 1, 2, \ldots, n - 1$.

Lemma 4.3. The center of $\Lambda$ is isomorphic to $K[x_0, x_1, \ldots, x_n]$, where $I$ is the ideal of $K[x_0, x_1, \ldots, x_n]$ generated by $x_0^n$ and all the products $x_i x_j$ with $(i, j) = (0, 0)$. In particular, $\dim(\text{HH}^0(\Lambda)) = 2n$.

Proof. It is well-known that $Z(\Lambda) \subseteq \oplus_{i \in Q_0} e_i e_i e_i$, that $\text{rad}(Z(\Lambda)) = Z(\Lambda) \cap \text{rad}(\Lambda)$ and $\frac{Z(\Lambda)}{\text{rad}(\Lambda)} = K \cdot 1 = K(e_1 + \cdots e_n)$. Since $\Lambda$ is graded one readily sees that the grading on $\Lambda$ gives by restriction a grading on $Z(\Lambda)$.

We claim that if $z \in Z(\Lambda)_{2m-1}$ is an element of odd degree $2m - 1$, then $m = n$ and $z$ is a linear combination of the socle elements $\omega_1, \ldots, \omega_n$. Indeed we have $z = \sum_{i=1}^n \lambda_i b_i^{2m-1}$, with $\lambda_i \neq 0$, for some integer $1 \leq r \leq m$. If $r < n$ then $\lambda_i b_i^{2m-1} a_r = za_r = a_r z = 0$, and hence $0 = b_i^{2m-1} a_r = a_{r-1} \cdots a_1 z^{2(\text{m-r})+1} a_{r} \cdots a_{n-1} a_r$. This only happens when $m = n$, in which case $b_i^{2m-1} = b_i^{2m-1} = \omega_i$. On the other hand, if $r = n$ then $m = n$ and we are done also in this case.

The previous paragraph shows that $Z(\Lambda)_{\text{odd}} := \oplus_{m \geq 0} Z(\Lambda)_{2m-1} = \sum_{i \in Q_0} K \omega_i = \text{Soc}(\Lambda)$ since $\omega_i \in Z(\Lambda)_{2m-1}$ for each $i \in Q_0$. We now want to identify $Z(\Lambda)_{\text{even}} := \oplus_{m > 0} Z(\Lambda)_{2m}$. One easily checks that $x_0 = \sum_{i=1}^{n-1} (-1)^i a_i \bar{a}_i = b_0^2 + \sum_{i=2}^{n-1} (-1)^i b_i^2$ is an element of $Z(\Lambda)_2$. Moreover $b_i^2$ is an even if and only if $0 \leq i \leq n - m$ and $m < n$. In this case we necessarily have an equality $b_i^2 = (-1)^{ki} 2^{-m} k_i$, for some integer exponent $t_i$. In particular $x_0^m \neq 0$ and $x_0^m = \sum_{m=0}^{n-m} \lambda_i b_i^2$, with scalars $\lambda_i$ all nonzero. We claim that if $0 \neq z \in Z(\Lambda)_{2m}$ and we write it as a $K$-linear combination $z = \sum_{i=1}^{n-m} \mu_i b_i^{2m}$, then $\mu_i \neq 0$ for all $i = 1, \ldots, n - m$. Suppose that it is not the case. We first prove that if $\mu_j = 0$ then $\mu_i = 0$ for each $i < j$. For that we can assume $j > 1$ and then we have

\[ 0 = \mu_j a_j - b_j^{2m} = a_j - z a_j = \mu_j b_j^{2m} - a_j - a_j - a_j \]

But $b_j^{2m} a_j - 1 \neq 0$ since $j \leq n - m \leq n - 1$ and so $j - 1 < n - m$. It follows that $\mu_j = 0$ and, by iterating the process, that $\mu_i = 0 \forall i < j$.

We can then write $z = \sum_{r-1}^{n-m} \mu_i b_i^{2m}$ for some $1 \leq r \leq n - m$ and some $\mu_i \neq 0 \forall i = r, \ldots, n - m$. We prove that $r = 1$ and our claim will be settled. Indeed, if $r > 1$ then we have

\[ \mu_r a_r - b_r^{2m} = a_r - z a_r = 0 \]

which implies that $\mu_r = 0$ since $a_r - b_r^{2m} = 0$. This is a contradiction.

Once we know that if $z \in Z(\Lambda)_{2m}\setminus \{0\}$ and $z = \sum_{i=1}^{n-m} \mu_i b_i^{2m}$ then $\mu_i \neq 0 \forall i = 1, \ldots, n - m$, we conclude that any such $z$ is a scalar multiple of $x_0^m$. Then $Z(\Lambda)_{2m} = K x_0^m$, for each $m > 0$.

Putting now $x_i = \omega_i \forall i = 1, \ldots, n$ we clearly have that $x_0, x_1, \ldots, x_n$ generate $Z(\Lambda)$ as an algebra and they are subject to the relations $x_0^m = 0$ and $x_i x_j = 0$ for $(i, j) \neq (0, 0)$.

\[ \square \]
4.1. It follows that \( \dim(\text{HH}_i(\Lambda)) \) isomorphic to \( \dim(\text{HH}_{i+1}(\Lambda)) \) for all \( i > 0 \).

Proof. By the isomorphism \( \text{HH}_i(\Lambda) \cong D(\text{HH}_{i+1}(\Lambda)) \) (see Remark 4.11), it is enough to calculate the dimensions of the Hochschild cohomology spaces.

On the other hand, by Corollary 3.10 we have and isomorphism \( \text{HH}_i(\Lambda) \cong \text{HH}_{i+1}(\Lambda) \). We then get isomorphisms of \( K \)-vector spaces

\[
\text{HH}^0(\Lambda) \cong \text{HH}^0(\Lambda) = \text{HH}^0(\Lambda) = \frac{\text{HH}^0(\Lambda)}{\text{HH}^0(\Lambda)} = \frac{Z(\Lambda)}{\text{soc}(\Lambda)}.
\]

for all \( k > 0 \) and \( i = 1, 2, 3, 4, 5 \).

By the same corollary, we have an isomorphism \( D(\text{HH}^0(\Lambda)) \cong \text{HH}^0(\Lambda)[5] \), which gives isomorphisms of \( K \)-vector spaces:

\[
D(\text{HH}^0(\Lambda)) \cong \text{HH}^5(\Lambda) \\
D(\text{HH}^1(\Lambda)) \cong \text{HH}^4(\Lambda) \\
D(\text{HH}^2(\Lambda)) \cong \text{HH}^3(\Lambda).
\]

Bearing in mind Lemma 4.3, the proof is reduced to check that

\[
\dim(Z(\Lambda)) = \dim(\text{HH}^1(\Lambda)) = \dim(\text{HH}^2(\Lambda)) = n.
\]

That \( \dim(Z(\Lambda)) = n \) follows directly from Lemma 4.3 and its proof. Moreover, we have two exact sequences

\[
0 \rightarrow \text{Ker}(R^*) \rightarrow \oplus_{i \in Q_+} e_{i(a)} \Lambda e_{i(a)} \rightarrow R^* \rightarrow 0 \\
0 \rightarrow Z(\Lambda) \rightarrow \oplus_{i \in Q_+} e_{i(a)} \Lambda e_{i(a)} \rightarrow \text{Im}(\delta^*) \rightarrow 0
\]

From the first one we get \( \dim(\text{Ker}(R^*)) = 2n - 2 = n^2 \) using Lemma 4.1 and Corollary 3.5. From the second sequence we get \( \dim(\text{Im}(\delta^*)) = (n^2 + n) - 2n = n^2 - n \) using lemma 4.3. It follows that \( \dim(\text{HH}^1(\Lambda)) = n \).

We also have that \( \text{HH}^2(\Lambda) \cong \text{Coker}(R^*) \) since \( \nu^* = 0 \). But \( \text{Im}(R^*) = \oplus_{i \in Q_0} e_{i} \Lambda e_{i} \), by lemma 4.1. It follows that \( \dim(\text{HH}^2(\Lambda)) = \dim(\oplus_{i \in Q_0} e_{i} \Lambda e_{i}) = n \).

Once we have computed the dimensions of the Hochschild (co)homology spaces of \( \Lambda \), we can do the same for its cyclic homology spaces in characteristic zero, denoted by \( HC_i(\Lambda) \) following the notation used in [32]. We start by recalling the following fact about graded algebras.

**Proposition 4.5.** Suppose \( \text{Char}(K) = 0 \) and let \( \Lambda = \oplus_{i \geq 0} \Lambda_i \) be a positively graded algebra such that \( \Lambda_0 \) is a semisimple algebra. The following assertions hold:

1. As \( K \)-vector spaces \( HC_i(\Lambda_0) \cong \begin{cases} 0 & \text{if } i \text{ is odd} \\ \Lambda_0 & \text{if } i \text{ is even} \end{cases} \)

2. Connes’ boundary map \( B \) induces an exact sequence

\[
0 \rightarrow \Lambda_0 \rightarrow \text{HH}_0(\Lambda) \xrightarrow{B} \text{HH}_1(\Lambda) \xrightarrow{B} \text{HH}_2(\Lambda) \rightarrow \cdots
\]

such that the image of \( B : \text{HH}_1(\Lambda) \rightarrow \text{HH}_{i+1}(\Lambda) \) is isomorphic to \( \frac{HC_i(\Lambda)}{HC_i(\Lambda_0)} \), for all \( n \geq 0 \).
Proof. Assertion 1 is well-known, and is a direct consequence of Connes’ periodicity exact sequence (32, Theorem 2.2.1) and the fact that $HH_i(A_0) = 0$, for all $i > 0$.

On the other hand, by [32, Theorem 4.1.13], we know that Connes’ periodicity exact sequence gives exact sequences:

$$
0 \rightarrow HC_{i-1}(A) \xrightarrow{B} HH_i(A) \xrightarrow{I} HC_i(A) \rightarrow 0
$$

for all $i \geq 0$. Since $HH_i(A_0) = 0$, for $i > 0$, we get an induced $K$-linear map $B \circ I : HH_i(A) \rightarrow HH_{i+1}(A)$ such that $Im(B \circ I) = Im(B) \cong \frac{HC_i(A)}{HC_i(A_0)}$.

Corollary 4.6. If $\Lambda = P(\mathbb{L}_n)$ is the preprojective algebra of type $\mathbb{L}_n$, then

$$
\dim HC_i(\Lambda) = \begin{cases} 0 & \text{if } i \text{ is odd} \\ 2n & \text{if } i \text{ is even} \end{cases}
$$

Proof. Put $B^i := Im(HH_i(\Lambda) \xrightarrow{B} HH_{i+1}(\Lambda))$ where $B$ is Connes’ map. From the previous theorem, we have

$$
\dim(B^0) = \dim HH_0(\Lambda) - \dim(KQ_0) = 2n - n = n
$$

and

$$
\dim(B^i) = \dim HH_i(\Lambda) - \dim(B^{i-1}) = n - \dim(B^{i-1})
$$

, for all $i > 0$.

It follows that $\dim(B^i) = n$, when $i$ is even and zero otherwise.

Then we have

$$
\dim HC_i(\Lambda) - \dim HC_i(KQ_0) = \begin{cases} n & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd} \end{cases}
$$

From this the result follows using the foregoing proposition.

Remark 4.7. In [15, Section 7.5] the author calculates the reduced cyclic homology spaces $HC_i(\Lambda)$ using Connes’ sequence (see Proposition 4.5.2) and, as a byproduct, he also calculates the absolute cyclic homology spaces. However, he states that the equality $HC_i(\Lambda) = \frac{HC_i(A)}{HC_i(A_0)}$ holds, for all $i > 0$. This is not true since $HC_i(\Lambda) = \frac{HC_i(A)}{HC_i(A_0)}$, for all $i > 0$. Therefore the description of the $HC_i(\Lambda)$ in [15, p. 22] is not correct.

Remark 4.8. Due to the fact that $\Lambda$ is a $\Lambda$-co-$Z(\Lambda)$-bimodule, for each $\Lambda$-bimodule $M$, the $K$-vector space $Hom_{\Lambda^e}(M, \Lambda)$ inherits a structure of $Z(\Lambda)$-module. In particular, via the isomorphisms, we have

$$
\bigoplus_{i \in Q_0} e_i \Lambda e_i \xrightarrow{\sim} Hom_{\Lambda^e}(P, \Lambda)
$$

$$
\bigoplus_{i \in Q_1} e_{i(a)} \Lambda e_{i(a)} \xrightarrow{\sim} Hom_{\Lambda^e}(Q, \Lambda)
$$

both $\bigoplus_{i \in Q_0} e_i \Lambda e_i$ and $\bigoplus_{i \in Q_1} e_{i(a)} \Lambda e_{i(a)}$ have a structure of $Z(\Lambda)$-modules. It is routine to see that these structures are given by the multiplication in $\Lambda$ and that the differentials of the complex $V^\bullet$ in Proposition 3.7 are all morphisms of $Z(\Lambda)$-modules.

Lemma 4.9. We view $Soc(\Lambda)$ as an ideal of $Z(\Lambda)$. The following assertions hold.

1) $Soc(\Lambda)HH_j(\Lambda) = 0$ for all $j > 0$.

2) $HH_j(\Lambda)$ is a semisimple $Z(\Lambda)$-module for all $j \equiv 2, 3 \pmod{6}$

3) $HH_j(\Lambda)$ is isomorphic to $\frac{Z(\Lambda)}{Soc(\Lambda)}$ as a $Z(\Lambda)$-module for all $j > 0$, $j \not\equiv 2, 3 \pmod{6}$

Proof. 1) Is a direct consequence of the fact that $\mathcal{P}(\Lambda, \Lambda) = Z(\Lambda)$ and $HH_j(\Lambda) \cong \frac{Hom_{\Lambda^e}(Q_0^j, \Lambda), \Lambda)}{Soc(\Lambda)}$ for all $j > 0$. 

2) Is a direct consequence of the fact that $HH_j(\Lambda) = Z(\Lambda)$ and $HH_j(\Lambda) \cong \frac{Hom_{\Lambda^e}(Q_0^j, \Lambda), \Lambda)}{Soc(\Lambda)}$ for all $j > 0$. 

3) Is a direct consequence of the fact that $HH_j(\Lambda) = Z(\Lambda)$ and $HH_j(\Lambda) \cong \frac{Hom_{\Lambda^e}(Q_0^j, \Lambda), \Lambda)}{Soc(\Lambda)}$ for all $j > 0$.
2) If \( x_0 = \sum_{i=0}^{n-1} (-1)^i a_i \) as in line \( \boxed{2} \), then \( x_0 HH^j(\Lambda) = x_0 \cdot (\oplus \delta \otimes e_i) = 0 \) when \( j \equiv 2 \) (mod6) and \( x_0 HH^j(\Lambda) = x_0 \cdot Soc(\Lambda) = 0 \) when \( j \equiv 3 \) (mod6).

3) We clearly have an isomorphism \( HH^j(\Lambda) \cong HH^{j+6}(\Lambda) \) for all \( j > 0 \), so we only need to prove the claim for \( j = 1, 4, 5, 6 \).

For \( j = 6 \), we take \( h = 1 + Im(k^*_e) \in \frac{Ker(\delta^*)}{Im(k^*_e)} \cong \frac{Z(\Lambda)}{Soc(\Lambda)} \) and one clearly has that \( Z(\Lambda)h = \frac{Z(\Lambda)}{Soc(\Lambda)} = HH^6(\Lambda) \).

For \( j = 1 \) we take the element \( \hat{y} = \sum_{a \in Q_1} a \in \oplus_{a \in Q_1} e_{i(a)} \Lambda e_{t(a)} \). One routine sees that \( R^* (\hat{y}) = 0 \). Then we get an element \( y = \hat{y} + Im(\delta^*) \in HH^1(\Lambda) = \frac{Ker(R^*)}{Im(\delta^*)} \).

We now take the induced morphism of \( Z(\Lambda) \)-modules

\[
\frac{K[x_0]}{(x_0)^n} \cong \frac{Z(\Lambda)}{Soc(\Lambda)} \rightarrow Z(\Lambda) \frac{y}{x}
\]

Its kernel is an ideal of \( \frac{K[x_0]}{(x_0)^n} \), then it is of the form \( \left( x_i \right) \), for some \( k \leq n \).

We claim that if \( k < n \) then \( x_0^k y \neq 0 \). That will imply that \( \frac{K[x_0]}{(x_0)^n} \cong Z(\Lambda)y \) so that \( Z(\Lambda)y = HH^1(\Lambda) \) by a dimension argument.

Suppose that \( k < n \) and \( yx_0^k = 0 \). Then \( \hat{y}x_0^k \in Im(\delta^*) \). Since \( \delta^* \) is a graded map of degree 1 (with respect to length degree) we will have an element \( x = \sum_{i=1}^{n-k} \mu_i b_i^{2k} \) of length-degree \( 2k \) in \( \oplus_{a \in Q_0} e_{i(a)} \Lambda e_{t(a)} \) such that \( \delta^* (x) = \hat{y} x_0^k \). Since this element belongs to \( \oplus_{a \in Q_1} e_{i(a)} \Lambda e_{t(a)} \) we can look at its \( \epsilon \)-component:

\[
\delta^* (x) \epsilon = e x - \epsilon x = \mu_1 b_1^{2k} - \mu_1 b_1^{2k} \epsilon = 0
\]

where \( \lambda_1 \) is the coefficient of \( b_1^{2k} \) in the expression \( x_0^k = \sum_{i=1}^{n-k} \lambda_i b_i^{2k} \). We know from the proof of lemma \( \boxed{3} \) that \( \lambda_1 \neq 0 \), which gives a contradiction since \( \epsilon b_1^{2k} = \epsilon^{2k+1} \neq 0 \).

For \( j = 4 \) we note that \( R^*_e (e_1) = e_1 - e_1 = 0 \) and that \( \delta^*_e (e) = \delta^*_e (e) \), which implies that \( \delta^*_e (x) = \delta^*_e (x) \) for \( x \in \oplus_{a \in Q_0} e_{i(a)} \Lambda e_{t(a)} \). We argue as in the previous paragraph and prove that if \( x_0^k e_1 \in Im(\delta^*_e) \) and \( k \leq n \) then \( k = n \).

If \( x_0^k e_1 \in Im(\delta^*_e) \) then there is \( 1 \leq r \leq k \) and \( \mu_1, \ldots, \mu_r \in K \), with \( \mu_r \neq 0 \), such that \( \delta^*_e (\sum_{i=1}^{n-r+1} \mu_i b_i^{2k-1}) = \epsilon^{2k} \). We look then at the \( a_r \)-component of both members of the equality (i.e. at their image by applying the projection \( \oplus_{a \in Q_1} e_{i(a)} \Lambda e_{t(a)} \rightarrow e_{i(a)} \Lambda e_{t(a)} \)). We then get \( \epsilon^{-1} a_r = 0 \), which is only possible in case \( r = n \), and hence \( k = n \).

As in the previous paragraph, we conclude that \( HH^4(\Lambda) = \frac{Ker(R^*_e)}{Im(\delta^*_e)} = Z(\Lambda) \gamma \), where \( \gamma := e_1 + Im(\delta^*_e) \).

It only remains the case \( j = 5 \). In this case we consider the element \( y \gamma \in HH^1(\Lambda) \cdot HH^4(\Lambda) \subseteq HH^5(\Lambda) \). Via the isomorphism \( \oplus_{a \in Q_1} e_{i(a)} \Lambda e_{t(a)} \rightarrow Hom_{\Lambda^\sigma}(\oplus_{a \in Q_1} e_{i(a)} \Lambda e_{t(a)}, \Lambda) \), the element \( e_1 \) is identified with the morphism of \( \Lambda \)-bimodules \( \oplus_{a \in Q_1} e_{i(a)} \Lambda e_{t(a)} \rightarrow \Lambda \rightarrow \Lambda \) mapping \( e_{i(a)} \otimes e_{t(a)} \rightarrow e_1 \). It obviously lifts to the morphism \( \oplus_{a \in Q_1} e_{i(a)} \Lambda e_{t(a)} \rightarrow \oplus_{i \in Q_0} e_1 \otimes e_1 \Lambda \rightarrow \Lambda \rightarrow \Lambda \) taking \( e_{i(a)} \otimes e_{t(a)} \rightarrow e_1 \otimes e_1 \) and \( e_{i(a)} \otimes e_{t(a)} \rightarrow 0 \), for \( a \neq e \). It is then routine to see that we have a commutative diagram

\[
\begin{array}{ccc}
\oplus_{i \in Q_0} \Lambda e_1 \otimes e_1 \Lambda & \xrightarrow{R_x} & \oplus_{a \in Q_1} \Lambda e_{i(a)} \otimes e_{t(a)} \Lambda \\
\downarrow g & & \downarrow f \\
\oplus_{a \in Q_1} \Lambda e_{i(a)} \otimes e_{t(a)} \Lambda & \xrightarrow{\delta} & \oplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda
\end{array}
\]
where \( g(e_1 \otimes e_1) = e_{i(c)} \otimes e_{i(c)} \) and \( g(e_i \otimes e_i) = 0 \) for \( i \neq 1 \).

On the other hand, via the isomorphism \( \oplus_{a \in Q_1} e_{i(a)} \Lambda e_{t(a)} \cong \text{Hom}_\Lambda(\oplus_{a \in Q_1} e_{i(a)} \otimes e_{t(a)} \Lambda, \Lambda) \) the element \( \tilde{g} \) gets identified with the morphism of \( \Lambda \)-modules \( \oplus_{a \in Q_1} e_{i(a)} \otimes e_{t(a)} \Lambda \rightarrow \Lambda \) such that \( \tilde{g}(e_{i(a)} \otimes e_{t(a)}) = a \) for all \( a \in Q_1 \). By definition of the Yoneda product in \( HH^*(\Lambda) \), the element \( y \gamma \) is represented by the morphism of \( \Lambda \)-bimodules \( \oplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda \rightarrow \Lambda \) which takes \( e_1 \otimes e_1 \mapsto e \) and \( e_i \otimes e_i \mapsto 0 \), for \( i \neq 1 \). Via the isomorphism \( \oplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda \cong \text{Hom}_\Lambda(\oplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda, \Lambda) \) \( \tilde{g} \circ g \) corresponds to \( e \in e_1 \otimes e_1 \subseteq \oplus_{i \in Q_0} e_i \Lambda e_i \).

We apply the argument already used for the cases \( j = 1 \) and \( j = 4 \) and prove that if \( x_0^k \epsilon \in \text{Im}(R^*_\tau) \) and \( 1 \leq k \leq n \) then \( k = n \).

That follows from the proof of lemma 4.11. Indeed, with the same notation, the matrix of the map \( R^*_\tau : V_{2m-2} \rightarrow W_{2m-1} \), with respect to the bases of \( V_{2m-2}^* \) and \( W_{2m-1} \) given there is of the form

\[
\begin{pmatrix}
0 & (-1)^{s_1} & 0 & \ldots & 0 \\
0 & -1 & (-1)^{s_2} & \ldots & \vdots \\
0 & 0 & -1 & 0 & \vdots \\
\vdots & \vdots & \ddots & (-1)^{s_{m-1}} & \vdots \\
0 & 0 & 0 & \ldots & (-1)^{q}
\end{pmatrix}
\]

which implies that no nonzero element of the form \( \begin{pmatrix} * \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \) can be in \( \text{Im}(R^*_\tau) \). Therefore \( x_0^k \epsilon = \epsilon^{2k+1} \in \text{Im}(R^*_\tau) \) implies \( \epsilon^{2k+1} = 0 \) in \( \Lambda \) and, hence, that \( k = n \).

If

\[
\ldots P^{-i} \xrightarrow{d^{-i}} P^{-i+1} \rightarrow \ldots \rightarrow P^{-1} \xrightarrow{d^{-1}} P^{0} \xrightarrow{\nu} \Lambda \rightarrow 0
\]

is the minimal projective resolution of \( \Lambda \) (see proposition 5.0) then, by definition, we have \( HH^i(\Lambda) = \frac{\text{Ker}(d^{-i-1} \circ \nu)}{\text{Im}(d^{-i-1})} \subseteq \frac{(P^{-i})^*}{\text{Im}(d^{-i-1})^*} \) for each \( i > 0 \). Thus any element \( \eta \in HH^i(\Lambda) \) is of the form \( \eta = \tilde{\eta} + \text{Im}(d^{-i-1})^* \), for some \( \tilde{\eta} \in \text{Hom}_\Lambda(P^{-i}, \Lambda) \) such that \( \tilde{\eta} \circ d^{-i-1} = 0 \). We will say that \( \tilde{\eta} \) represents \( \eta \) or that \( \eta \) is represented by \( \tilde{\eta} \). The following is a straightforward consequence of the results of this section and their proofs.

**Proposition 4.10.** The following are bases of the \( HH^i(\Lambda) \), for \( i = 0, 1, \ldots, 6 \):

1. For \( HH^0(\Lambda) = Z(\Lambda) \):
   \[
   \{x_0, x_0^2, \ldots, x_0^{n-1}, x_1, \ldots, x_n\}, \text{ where } x_0 = \sum_{i=1}^{n-1} (-1)^i a_i a_i \text{ and } x_i = \omega_i \text{ for each } i = 1, \ldots, n.
   \]

2. For \( HH^1(\Lambda) = \frac{\text{Ker}(R^*_\tau)}{\text{Im}(\delta^*_1)} \):
   \[
   \{y, y_0 y, y_0^2 y, \ldots, y_0^{n-1} y\}, \text{ where } y = \sum_{a \in Q_1} a + \text{Im}(\delta^*).
   \]
   The element \( y \) is represented by the only morphism \( \tilde{y} : Q \rightarrow \Lambda \) such that \( \tilde{y}(e_{i(a)} \otimes e_{t(a)}) = a \) for each \( a \in Q_1 \).

3. For \( HH^2(\Lambda) = \frac{\text{Ker}(k^*)}{\text{Im}(R^*_\tau)} \):
   \[
   \{z_1, \ldots, z_n\}, \text{ where } z_k = e_k + \text{Im}(R^*_\tau) \text{ for each } k \in Q_0.
   \]
   The element \( z_k \) is represented by the only morphism \( \tilde{z}_k : P \rightarrow \Lambda \) such that \( \tilde{z}_k(e_i \otimes e_i) = \delta_{ik} e_k \).

4. For \( HH^3(\Lambda) = \frac{\text{Ker}(\delta^*_2)}{\text{Im}(k^*)} = \text{Ker}(\delta^*_2) \):
   \[
   \{t_1, \ldots, t_n\}, \text{ where } t_k = w_k \text{ for each } k \in Q_0.
   \]
   The element \( t_k \) is represented by the only morphism \( \tilde{t}_k : P \rightarrow \Lambda \) such that \( \tilde{t}_k(e_i \otimes e_i) = \delta_{ik} \omega_k \).

5. For \( HH^4(\Lambda) = \frac{\text{Ker}(R^*_\tau)}{\text{Im}(\delta^*_2)} \):
   \[
   \{\gamma, x_0 \gamma, \ldots, x_0^{n-1} \gamma\}, \text{ where } \gamma = e_1 + \text{Im}(\delta^*_2).
   \]
   The element \( \gamma \) is represented by the only morphism \( \tilde{\gamma} : Q \rightarrow \Lambda \) such that \( \tilde{\gamma}(e_{i(a)} \otimes e_{t(a)}) = \delta_{ca} e_1 \) for each \( a \in Q_1 \).
6. For $HH^5(\Lambda) = \frac{\ker(k_z)}{\text{Im}(k_z)}: \{y\gamma, x_0y\gamma, \ldots, x_0^{n-1}y\gamma\}$.

7. For $HH^6(\Lambda) = \frac{\ker(k_z)}{\text{Im}(k_z)}: \{h, x_0h, \ldots, x_0^{n-1}h\}$, where $h = 1 + \text{Im}(k_z)$.

The element $h$ is represented by the multiplication map $\tilde{h} = u : \oplus_{i \in Q_0} e_i \otimes e_i \Lambda \rightarrow \Lambda$.

The bases of the $HH^i(\Lambda)$ given in the above proposition will be called canonical bases.

**Remark 4.11.** In [18] the author uses the length grading on $\Lambda$ and looks at the minimal projective resolution of $\Lambda$ as one in the category of graded $\Lambda$-bimodules. With that in mind the Hochschild homology and cohomology spaces become graded vector spaces. Then he calculates this graded structure in terms of three semialgebraic graded vector spaces $R$, $U$ and $K$ (see Theorems 4.0.13 and 4.0.14 in [18]). In our terminology, $R = KQ_0$ (concentrated in degree 0), $U = \frac{Z(\Lambda)}{\text{Soc}(\Lambda)[2]}$ (with the length grading on $\frac{Z(\Lambda)}{\text{Soc}(\Lambda)[2]}$) and $K = HH^2(\Lambda)[2]$ (which is concentrated in degree 0 since $HH^2(\Lambda)$ is concentrated in degree $-2$).

His strategy to prove the mentioned theorems is based on the use of Connes’ exact sequence (see Proposition 4.13) and the fact that the differentials in this sequence are graded maps. Then he is able to describe the graded structure of each $HH_i(\Lambda)$ and, using dualities between the Hochschild homology and cohomology graded spaces obtained in [19], the author also gets the graded structure of each $HH^i(\Lambda)$.

Due to the fact that the dimension of $R$, $U$ and $K$ is $n$, the dimensions of the $HH_i(\Lambda)$ and the $HH^i(\Lambda)$ can be read off from the mentioned theorems 4.0.13 and 4.0.14 of [18], even if they were not explicitly stated in a proposition or corollary. After that and before calculating the ring structure of $HH^*(\Lambda)$, Eu gives explicit bases of the $HH^i(\Lambda)$ using the correspondent of our complex $V^*$, which he considers as a complex of graded vector spaces (see section 8 in [18]).

In our case, we have not used the graded condition of the minimal projective resolution of $\Lambda$. Instead, we have calculated the dimensions of the $HH^i(\Lambda)$ by directly manipulating the complex $V^*$ and using the isomorphisms of Corollary 5.11. The bases of the $HH^i(\Lambda)$ have been obtained in the process of identifying the structure of these spaces as $Z(\Lambda)$-modules, using the process to calculate already some of the products in $HH^*(\Lambda)$ (see the proof of Lemma 4.13).

5 The ring structure of $HH^*(\Lambda)$ and proof of the main theorems

We start by studying the map $\phi_y : HH^2(\Lambda) \rightarrow HH^3(\Lambda)$ given by $\phi_y(u) = yu$ for all $u \in HH^2(\Lambda)$.

**Lemma 5.1.** If $C = (c_{ij})$ is the matrix of $\phi_y$ with respect to the canonical bases of $HH^2(\Lambda)$ and $HH^3(\Lambda)$, then the following conditions hold:

1) $C$ is a symmetric integer matrix.

2) $c_{jk} = (-1)^{k-j+1}(2j-1)(n-k+1)$ whenever $1 \leq j \leq k \leq n$.

3) $\text{rank}(C) = n$, when $\text{Char}(K)$ does not divide $2n+1$, and $\text{rank}(C) = 1$, when $\text{Char}(K)$ divides $2n+1$.

**Proof.** Let $x = \alpha_1 \cdots \alpha_r$ ($r > 0$) be any path in $e_j KQ e_k$ which does not belong to the ideal $I$. We put

$$h_x = \alpha_1 \cdots \alpha_{r-1} \otimes x^* + \alpha_1 \cdots \alpha_{r-2} \otimes \alpha_r x^* + \cdots + e_j \otimes \alpha_2 \cdots \alpha_r x^*$$

which is an element of $\oplus_{a \in Q_1} \Lambda e_{i(a)} \otimes e_{t(a)} \Lambda$. In case $j = k$ we put $h_{e_j} = 0$ and if $p_j = \tilde{a}_{j-1} \cdots \tilde{a}_{j} 2^{(n-j)+1} \tilde{a}_{j+1} \cdots \tilde{a}_{j-1}$ we also define $h_{w_j} = (-1)^{\frac{j(j+1)}{2}p_j}$ (recall that $w_j = (-1)^{\frac{j(j+1)}{2}p_j}$).

In this way we have defined $h_x$ for each $x \in e_j K e_k$ and for all $j, k \in Q_0$.

Direct calculation shows that $\delta(h_x) = x \otimes x^* - e_j \otimes \omega_j$, and hence

$$\delta\left(\sum_{x \in e_j K e_k} (-1)^{\deg(x)}h_x\right) = \sum_{x \in e_j K e_k} (-1)^{\deg(x)}(x \otimes x^* - e_j \otimes \omega_j) = \sum_{x \in e_j K e_k} (-1)^{\deg(x)}x \otimes x^*,$$
bearing in mind that in $e_j B e_k$ there are exactly the same number of odd and even degree.

Now consider $\hat{z}_k : P \to \Lambda$ as in proposition \[4.10\]. It is clear that the morphism of $\Lambda$-bimodules $\hat{z}_k : P \to P$ determined by the rule $\hat{z}_k(e_j \otimes e_i) = \delta_{ik} e_k \otimes e_k$ is a lifting of $\hat{z}_k$ (i.e., $\hat{z}_k = u \circ \hat{z}_k$).

If now $f_k : P \to Q$ is the morphism of $\Lambda$-bimodules determined by the rule $f_k(e_j \otimes e_j) = \sum_{x \in e_j B e_k} (-1)^{\deg(x)} h_x$ then we have a commutative diagram

$$
\begin{array}{ccc}
P & \xrightarrow{f_k} & P \\
\downarrow & & \downarrow \\
Q & \xrightarrow{\hat{z}_k} & P 
\end{array}
$$

and hence $y z_k$ is represented by the morphism

$$
\tilde{y} \circ f_k : P \to \Lambda, \ (e_j \otimes e_j) \mapsto \sum_{x \in e_j B e_k} (-1)^{\deg(x)} \deg(x) \omega_j.
$$

That means that if we put $C_{jk} = \sum_{x \in e_j B e_k} (-1)^{\deg(x)} \deg(x)$ for all $j, k \in Q_0$, then we have $y z_k = \sum_{j \in Q_0} C_{jk} t_j$ (notation as in proposition \[4.10\]). Therefore $C := (C_{jk})$ is the matrix of $\phi_y : HH^2(\Lambda) \to HH^3(\Lambda)$ with respect to the canonical bases of $HH^2(\Lambda)$ and $HH^3(\Lambda)$.

That $C$ is a symmetric integer matrix is clear since the anti-isomorphism $x \mapsto \bar{x}$ gives a bijection between $e_j B e_k$ and $e_j B e_j$ which preserves the term $(-1)^{\deg(x)} \deg(x)$. We then proceed to calculate the entries of this matrix. To do that we should recall the possible degrees of elements in $e_j B e_k$ (see Remark \[5.3\], for $1 \leq j \leq k \leq n$). There are two possibilities.

i) $k \equiv j \pmod{2}$: Then the sum of odd degrees is $\left\lfloor \frac{(k-j)+(k-j)+2(n-k)(n-k+1)}{2} \right\rfloor = (n-j)(n-k+1)$, while the sum of even degree is $\left\lfloor \frac{(k+j-1)+(k+j-1)+2(n-k)(n-k+1)}{2} \right\rfloor = (n+j-1)(n-k+1)$. Therefore we have $C_{jk} = (n-j)(n-k+1) - (n+j-1)(n-k+1) = (1-2j)(n-k+1)$.

ii) $k \not\equiv j \pmod{2}$: In this case $C_{jk}$ is the negative of the number above, i.e., $C_{jk} = (2j-1)(n-k+1)$.

It finally remains to calculate $\text{rank}(C)$. We view each $n \times n$ matrix as a $n$-tuple whose components are its rows. By elementary row transformation one passes from $C = (C_{11}, \ldots, C_{nn})$ to

$$
C' = (C_{11}, 3C_{11}, \ldots, C_{jj} + (-1)^{j} (2j-1) C_{11}, \ldots, C_{nn} + (-1)^{n} (2n-1) C_{11})
$$

so that $\text{rank}(C) = \text{rank}(C')$. We look at the $j$-th row $C'_{j} = C_{jj} + (-1)^{j} (2j-1) C_{11}$ of $C'$. It is straightforward, and thus is left to the reader, to check that for $j \leq k$, then $C'_{jk} = 0$ and if $j > k$, $C'_{jk} = (-1)^{j-k+1} (k-j)(2n+1)$.

Therefore, in case $\text{Char}(K)$ divides $2n + 1$, all rows of $C'$ except the first one are zero. On the other hand, we have $C'_{1n} = C'_{11} = (-1)^{n-1} + (-1)^{n-1} (2 \cdot 1 - 1)(n-n+1) = (-1)^{n}$. It follows that $\text{rank}(C) = 1$ in case $\text{Char}(K)/2n+1$. In case $\text{Char}(K)$ does not divide $2n+1$, we apply the $n$-cycle $(1 \ 2 \ 3 \ \cdots \ n)$ to the rows of $C'$ we obtain a lower triangular matrix with diagonal entries $C'_{21}, C'_{32}, \ldots, C'_{n,n-1}, C'_{1n}$.

We have $C'_{k+1,k} = (-1)^{(k+1)-k+1}(k-k+1)(2n+1) = -(2n+1)$ for $k = 2, \ldots, n$ and $C'_{1n} = (-1)^{n}$. It follows that $\det(C) = \det(C') = (-1)^{2n-1}(2n+1)^{n-1} \not= 0$. Therefore $\text{rank}(C) = n$ in this case.

\[ \square \]

**Remark 5.2.** Given a graph $\Gamma$ without double edges, its adjacency matrix $D = D_\Gamma$ is the symmetric matrix $D = (d_{ij})_{i,j \in \nu_0}$ having $d_{ij} = 1$, in case there is an edge $i \to -j$, and $d_{ij} = 0$ otherwise. In particular, for the graph $\Lambda_n$, one has $d_{11} = 1$, $d_{i,i+1} = d_{i+1,i} = 1$ for $i = 1, \ldots, n-1$, and $d_{ij} = 0$ otherwise. Direct computation shows that the matrix $C$ of Lemma \[5.1\] satisfies the equality $-C(2I_n + D) = (2n+1)I_n$, where $I_n$ is the identity $n \times n$ matrix. Therefore, when $\text{char}(K)$ does not divide $2n+1$, an alternative description of the matrix $C$ is $C = -(2n+1)(2I_n + D)^{-1}$. Up to signs forced by the different presentation of $\Lambda$ and the different choice of the exceptional vertex of $\Lambda_n$, the last equality is that of \[18\] Proposition 9.3.1 (see also \[18\] Theorem 4.0.16)].
Taking into account also the case when \( \text{char}(K) \) divides \( 2n+1 \) is fundamental for the difference of presentations in our two main theorems and is the part of our work where the arguments of [18] cannot be applied.

**Lemma 5.3.** The equality

\[
z_j \gamma = (-1)^j(n - j + 1)x_0^{n-1}h
\]

holds in the ring \( HH^*(\Lambda) \), for each \( j = 1, 2, \ldots, n \).

**Proof.** Let \( \tilde{\gamma} \) be as in proposition[13]. It is clear that the morphism of \( \Lambda \)-bimodules \( \tilde{\gamma} : \oplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda \rightarrow \oplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda \) which maps \( e_i(a) \otimes e_i(a') \rightleftharpoons \delta(e) e_1 \otimes e_1 \) is a lifting of \( \gamma \) (i.e. \( u \circ \tilde{\gamma} = \gamma \)).

If we take now the morphism \( \beta : \oplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda \rightarrow \oplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda \) determined by the rule \( \beta(e_i \otimes e_i) = \delta_1 e_i(e) \otimes e_{t(e)} \), then direct computation shows that \( \gamma \circ R_\tau = \delta \circ \beta \). Our aim is now to define a morphism \( \alpha : \oplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda \rightarrow \oplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda \) completing commutatively the following diagram

\[
\begin{array}{ccc}
P & \xrightarrow{k} & P \\
\uparrow{\alpha} & & \downarrow{\beta} \\
R & \xrightarrow{R_\tau} & Q \\
\downarrow{} & & \downarrow{\delta} \\
P & \xrightarrow{\cdot} & P
\end{array}
\]

Once our goal is attained, the composition

\[
\oplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda \xrightarrow{\alpha} \oplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda \xrightarrow{\tilde{\gamma}} \Lambda
\]

will represent the element \( z_j \gamma \in HH^*(\Lambda) \).

For each vertex \( i \in Q_0 \) and for each integer \( r = 1, 2, \ldots, n - i + 1 \), we put

\[
y_r^i = \sum_{k=0}^{n-i-r+1} \bar{a}_{i-1} \cdots \bar{a}_i \epsilon^{2k} a_1 \cdots a_{r-1} \bar{a}_{r-1} \cdots \bar{a}_1 \epsilon^{2(n-i-r+k+1)} a_1 \cdots a_{i-1},
\]

which is an element of \( e_i \Lambda e_r \otimes e_r \Lambda e_i \). Given \( x = \bar{a}_{i-1} \cdots \bar{a}_1 \epsilon^k e_i e_1 \) (convening that \( \bar{a}_{i-1} \cdots \bar{a}_1 = \epsilon_1 \) when \( i = 1 \)), we put \( x^\dag = \epsilon^2(n-i+1-k) a_1 \cdots a_{i-1} \). Then we have

\[
x x^\dag = \bar{a}_{i-1} \cdots \bar{a}_1 \epsilon^{2(n-i)+1} a_1 \cdots a_{i-1} = (-1)^{(i-1)} \frac{i(i-1)}{2} w_i,
\]

for each \( i \in Q_0 \). Note that this implies that \( x^\dag = (-1)^{(i-1)} \frac{i(i-1)}{2} x^* \).

We claim that if \( (u_n)_{n \geq 1} \) is the sequence of integers \( 0, 0, 1, 1, 0, 0, 1, 1, \ldots \), then the following equality holds

\[
R(\sum_{r=1}^{n-i+1} (-1)^{u_r} y_r^i) = \sum_{x \in e_i B e_1} x \otimes x^\dag.
\]

From this equality, by direct computation the following will follow:

\[
R((-1)^{1+\frac{i(i-1)}{2}} \sum_{r=1}^{n-i+1} (-1)^{u_r} y_r^i) = - \sum_{x \in e_i B e_1} x \otimes x^\dag.
\]

As a consequence, the morphism of \( \Lambda \)-bimodules \( \alpha : \oplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda \rightarrow \oplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda \), determined by the rule \( \alpha(e_i \otimes e_i) = (-1)^{1+\frac{i(i-1)}{2}} \sum_{r=1}^{n-i+1} (-1)^{u_r} y_r^i \), will satisfy the desired equality \( \beta \circ k_\tau = R \circ \alpha \), because \( (\beta \circ k_\tau)(e_i \otimes e_i) = - \sum_{x \in e_i B e_1} x \otimes x^\dag \) for all \( i \in Q_0 \).

To settle our claim we shall prove by induction on \( s = 1, 2, \ldots, n - i + 1 \) the equality

\[
\sum_{r=1}^{s} (-1)^{u_r} R(y_r^i) - \sum_{x \in e_i B e_1} x \otimes x^\dag = \sum_{k=0}^{n-i-s+1} \bar{a}_{i-1} \cdots \bar{a}_1 \epsilon^{2k} a_1 \cdots a_{s-1}(a_s \otimes e_s + e_s \otimes \bar{a}_s) \bar{a}_{s-1} \cdots \bar{a}_1 \epsilon^{2(n-i-k-s+1)} a_1 \cdots a_{i-1}.
\]
The equality, when taken for $s = n - i + 1$, will give:

$$\sum_{r=1}^{n-i+1} (-1)^r R(y_r^i) - \sum_{x \in e_i B e_1} x \otimes x^\dagger = (-1)^{n-i+1}(\bar{a}_{i-1} \cdots \bar{a}_1 \epsilon^2 a_1 \cdots a_{n-i} a_{n-i+1} \otimes$$

$$\bar{a}_{n-i} \cdots \bar{a}_1 \epsilon^2 a_1 \cdots a_{n-i} + \bar{a}_{i-1} \cdots \bar{a}_1 \epsilon^2 a_1 \cdots a_{n-i} \otimes \bar{a}_{n-i+1} \bar{a}_{n-i} \cdots \bar{a}_1 \epsilon^2 a_1 \cdots a_{n-i+1})$$

But the second member of this equality is zero (see remark 3.5(2)) so that our claim will be settled.

For $s = 1$, we have

$$R(y_1^i) - \sum_{x \in e_i B e_1} x \otimes x^\dagger =$$

$$\sum_{k=0}^{n-i} \bar{a}_{i-1} \cdots \bar{a}_1 \epsilon^{2k} (e_1 \otimes e_1 + e_1 \otimes e_1 + e_1 \otimes \bar{a}_1) \epsilon^{2(n-i-k)} a_1 \cdots a_{n-i-1}$$

$$+ \sum_{t=0}^{2(n-i)+1} \bar{a}_{i-1} \cdots \bar{a}_1 \epsilon^t \otimes \epsilon^{2(n-i)+1-t} a_1 \cdots a_{n-i-1}$$

and the desired equality is true in this case.

If now $s > 1$ and the equality holds for $s - 1$ then we get

$$\sum_{r=1}^{s} (-1)^r R(y_r^i) - \sum_{x \in e_i B e_1} x \otimes x^\dagger =$$

$$(-1)^s R(y_s^i) + (-1)^{s-1} \sum_{k=0}^{n-i-s+2} \bar{a}_{i-1} \cdots \bar{a}_1 \epsilon^{2k} a_1 \cdots a_{s-2} (a_{s-1} \otimes e_{s-1} +$$

$$e_{s-1} \otimes \bar{a}_{s-1}) \bar{a}_{s-2} \cdots \bar{a}_1 \epsilon^{2(n-i-k-2)} a_1 \cdots a_{n-i-1}$$

$$+ (-1)^s \sum_{k=0}^{n-i-s+1} \bar{a}_{i-1} \cdots \bar{a}_1 \epsilon^{2k} a_1 \cdots a_{s-1} (a_s \otimes e_s + e_s \otimes \bar{a}_s +$$

$$\bar{a}_{s-1} \otimes e_s + e_s \otimes \bar{a}_{s-1}) \bar{a}_{s-2} \cdots \bar{a}_1 \epsilon^{2(n-i-k-1)} a_1 \cdots a_{n-i-1}$$

$$+ (-1)^{s-1} \sum_{k=0}^{n-i-s+2} \bar{a}_{i-1} \cdots \bar{a}_1 \epsilon^{2k} a_1 \cdots a_{s-2} (a_{s-1} \otimes e_{s-1} + e_{s-1} \otimes \bar{a}_{s-1}) \bar{a}_{s-2} \cdots \bar{a}_1 \epsilon^{2(n-i-k-s+2)} a_1 \cdots a_{n-i-1}$$

Using the equality (4) in remark 4.3 with $j = s$ and the one obtained from it by applying the canonical anti-isomorphism ($\overline{\cdot}$), we can rewrite the second member of the last expression as

$$(-1)^s \sum_{k=0}^{n-i-s+1} \bar{a}_{i-1} \cdots \bar{a}_1 \epsilon^{2k} a_1 \cdots a_{s-1} (a_s \otimes e_s + e_s \otimes \bar{a}_s) \bar{a}_{s-1} \cdots \bar{a}_1 \epsilon^{2(n-i-k-s+1)} a_1 \cdots a_{n-i-1} +$$

$$(-1)^{s-1} \sum_{k=0}^{n-i-s+1} \bar{a}_{i-1} \cdots \bar{a}_1 \epsilon^{2(k+1)} a_1 \cdots a_{s-2} \otimes \bar{a}_{s-1} + a_{s-1} \epsilon^{2(n-i-k-s+1)} a_1 \cdots a_{n-i-1} +$$

$$\bar{a}_{i-1} \cdots \bar{a}_1 \epsilon^{2(k)} a_1 \cdots a_{s-1} \otimes \bar{a}_{s-2} \cdots \bar{a}_1 \epsilon^{2(n-i-k-s+2)} a_1 \cdots a_{n-i-1}$$
\(( -1)^{u_j} \sum_{k=0}^{n-i-s+2} \bar{a}_{i-1} \cdots \bar{a}_{i} \epsilon^{2k} a_1 \cdots a_{s-2}(a_{s-1} \otimes e_{s-1} + e_{s-1} \otimes \bar{a}_{s-1}) \bar{a}_{s-2} \cdots \bar{a}_{1} \epsilon^{2(n-i-k-s+2)} a_1 \cdots a_{i-1}\)

Except two of them, the subsummands of the second and third summands of this expression cancel by pairs due to the fact that \(u_s + s - 1 - u_{s-1}\) is always an odd integer. The two subsummands which do not cancel are

\[\left(-1\right)^{u_j} \bar{a}_{i-1} \cdots \bar{a}_{i} a_1 \cdots a_{s-2} \otimes \bar{a}_{s-1} \cdots \bar{a}_{1} \epsilon^{2(n-i-s+2)} a_1 \cdots a_{i-1}\]

and

\[\left(-1\right)^{u_j} \bar{a}_{i-1} \cdots \bar{a}_{i} \epsilon^{2(n-i-s+2)} a_1 \cdots a_{s-2} a_{s-1} \otimes \bar{a}_{s-2} \cdots \bar{a}_{1} a_1 \cdots a_{i-1}\]

But these two subsummands are zero due to (2) in remark 3.5 and, hence, our claim is settled.

We are finally ready to end the proof. The element \(z_j \gamma\) is represented by \(\tilde{z}_j \circ \alpha\), where \(\alpha : \oplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda \rightarrow \oplus_{i \in Q_0} \Lambda e_i \otimes e_i \Lambda\) is the morphism determined by the rule \(\alpha(e_i \otimes e_i) = (-1)^{1 + \frac{i(i-1)}{2}} \sum_{r=1}^{n-i+1} (-1)^{u_r} \tilde{z}_j(y_i^r)\), for each \(i \in Q_0\). We get:

\[(\tilde{z}_j \circ \alpha)(e_i \otimes e_i) = (-1)^{1 + \frac{i(i-1)}{2}} \sum_{r=1}^{n-i+1} (-1)^{u_r} \tilde{z}_j(y_i^r)\]

This expression is zero in case \(j > n - i + 1\), while it is equal to \((-1)^{1 + \frac{i(i-1)}{2}} (-1)^{u_{j}} \tilde{z}_j(y_j^i)\) otherwise. But, using remark 3.3, we have an equality

\[(-1)^{1 + \frac{i(i-1)}{2}} (-1)^{u_{j}} \tilde{z}_j(y_j^i) = (-1)^{1 + \frac{i(i-1)}{2}} (-1)^{u_{j}} \sum_{k=0}^{n-i-j+1} \bar{a}_{i-1} \cdots \bar{a}_{i} \epsilon^{2k} a_1 \cdots a_{j-1} a_{j-1} \cdots \bar{a}_{1} \epsilon^{2(n-i-j+k+1)} a_1 \cdots a_{i-1}\]

\[= (-1)^{1 + \frac{i(i-1)}{2}} (-1)^{u_{j}} \sum_{k=0}^{n-i-j+1} \bar{a}_{i-1} \cdots \bar{a}_{i} \epsilon^{2k} a_1 \cdots a_{j-1} a_{j-1} \cdots \bar{a}_{1} \epsilon^{2(n-i-j+k+1)} a_1 \cdots a_{i-1}\]

in case \(1 \leq j \leq n - i + 1\). But this is zero unless \(i = 1\) (see (2) in remark 3.5).

In case \(i = 1\), we have \((\tilde{z}_j \circ \alpha)(e_1 \otimes e_1) = (-1)^{1 + \frac{j(j-1)}{2}} (-1)^{u_{j}} \tilde{z}_j(y_j^1)\) otherwise.

It is left as an exercise to see that \(1 + u_{j} + \frac{j(j-1)}{2} \equiv j \pmod{2}\) and the equality \(z_j \gamma = (-1)^{(n-j+1)x_0^{-1}} h\) follows since \(x_0^{-1} h\) is the element \(e^{2n-2} + \text{Im}(k_0) \in \frac{\text{Ker}(\delta^* - 1)}{\text{Im}(k_0)} = HH^0(\Lambda)\).

In order to know the multiplicative structure of \(HH^*(\Lambda)\), the role of \(HH^3(\Lambda)\) is fundamental. The following is a great help.

**Lemma 5.4.** The following assertions hold:

1) \(HH^3(\Lambda) \cdot HH^{2m+1}(\Lambda) = 0\), for all integers \(m \geq 0\).

2) \(z_k \gamma_j = \delta_{kj} x_0^{-1} y_i\), for all \(i, j \in Q_0\).

3) \(t_j \gamma = \delta_{ij} x_0^{-1} y_i\), for all \(j \in Q_0\).

**Proof.** From the proof of lemma 3.3, we know that multiplication by \(h\) gives an isomorphism of \(Z(\Lambda)\)-modules \(\phi_h : HH^k(\Lambda) \simeq HH^{k+6}(\Lambda)\), for all \(k > 0\), and multiplication by \(y\) yields another one \(\phi_y : HH^k(\Lambda) \simeq HH^{k+6}(\Lambda)\). In particular, we get \(HH^{k+6}(\Lambda) = HH^k(\Lambda) \cdot HH^6(\Lambda)\), for all \(k > 0\), and \(HH^1(\Lambda) \cdot HH^3(\Lambda) = HH^6(\Lambda)\). From these considerations we deduce that in order to prove assertion 1, we just need to check that the products \(HH^1(\Lambda) \cdot HH^3(\Lambda)\) and \(HH^3(\Lambda) \cdot HH^3(\Lambda)\) are both zero.

We will now prove the three assertions of the lemma by considering the following commutative diagram, for each \(j \in Q_0\):
where the vertical arrows are the only morphisms of $\Lambda$-bimodules satisfying the following properties:

a) $\tilde{t}_j(e_i \otimes e_i) = \frac{1}{2}\delta_{ij}(\omega_j \otimes e_j + e_j \otimes \omega_j)$

b) In case $(j, a) \neq (1, \epsilon)$ we have

$$f_j(e_{i(a)} \otimes e_{t(a)}) = \begin{cases} 
0 & \text{if } j \notin \{i(a), t(a)\} \\
\frac{1}{2}e_{i(a)} \otimes \omega_{i(a)} & \text{if } j = t(a) \\
-\frac{1}{2}\omega_{i(a)} \otimes e_{t(a)} & \text{if } j = i(a)
\end{cases}$$

and, for $(j, a) = (1, \epsilon)$, we have:

$$f_1(e_{i(\epsilon)} \otimes e_{t(\epsilon)}) = \frac{1}{2}(e_{i(\epsilon)} \otimes \omega_{t(\epsilon)} - \omega_{i(\epsilon)} \otimes e_{t(\epsilon)})$$

c) $g_r = \tilde{t}_j$

d) $h_j(e_i \otimes e_i) = \frac{1}{2}\delta_{ij}(e_j \otimes \omega_j - \omega_j \otimes e_j)$

e) In case $(j, a) \neq (1, \epsilon)$ we have

$$l_j(e_{i(a)} \otimes e_{t(a)}) = \begin{cases} 
0 & \text{if } j \notin \{i(a), t(a)\} \\
\frac{1}{2}e_{i(a)} \otimes \omega_{i(a)} & \text{if } j = t(a) \\
\frac{1}{2}\omega_{i(a)} \otimes e_{t(a)} & \text{if } j = i(a)
\end{cases}$$

and, in case $(j, a) = (1, \epsilon)$, we have:

$$l_1(e_{i(\epsilon)} \otimes e_{t(\epsilon)}) = \frac{1}{2}(e_{i(\epsilon)} \otimes \omega_{t(\epsilon)} + \omega_{i(\epsilon)} \otimes e_{t(\epsilon)})$$

Although tedious, checking that these morphisms make commutative the last diagram is easy and routinary. It is left to the reader.

We know have:

1. i) $HH^1(\Lambda) \cdot HH^3(\Lambda) = 0$

   We already know that $\{y, x_0y, \ldots, x_0^{n-1}y\}$ is a basis of $HH^1(\Lambda)$. So we only need to check that $yt_j = 0$, for all $j \in Q_0$. But $yt_j$ is represented by the composition

   $$Q \xrightarrow{f_j} Q \xrightarrow{\tilde{g}} \Lambda.$$ 

   Due to the fact that $a\omega_{t(a)} = 0 = \omega_{i(a)}a \forall a \in Q_1$, one gets that $(\tilde{g} \circ f_j)(e_{i(a)} \otimes e_{t(a)}) = 0 \forall a \in Q_1$, and hence $\tilde{g} \circ f_j = 0$

1. ii) $HH^3(\Lambda) \cdot HH^3(\Lambda) = 0$

   The element $t_k t_j$ of $HH^6(\Lambda)$ is represented by the composition

   $$P \xrightarrow{h_j} P \xrightarrow{i_k} \Lambda.$$ 

   We have $(\tilde{t}_k \circ h_j)(e_i \otimes e_i) = \tilde{t}_k(\frac{1}{2}\delta_{ij}(e_j \otimes \omega_j - \omega_j \otimes e_j))$. This is clearly zero for all $i, j, k \in Q_0$. 

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2] \( z_k t_j = \delta_{jk} x_0^{-n} y \gamma \)

The element \( z_k t_j \in HH^5(\Lambda) \) is represented by the composition

\[
P \xrightarrow{g_j} P \xrightarrow{\delta} \Lambda
\]

Due to the fact that \( g_j(e_i \otimes e_j) = 0 \) for \( i \neq j \) and \( g_j(\Lambda e_j \otimes e_j \Lambda) \subseteq \Lambda e_j \otimes e_j \Lambda \), we readily see that \( \delta_k \circ g_j = 0 \) when \( j \neq k \). Moreover, in case \( j = k \), we have

\[
(\delta_j \circ g_j)(e_i \otimes e_i) = \frac{1}{2} \delta_{ij}(\omega_j + \omega_j) = \delta_{ij} \omega_j
\]

From remark 4.2 we know that, when we view \( \omega_j \) as an element of \( Ker(k^*_\rho) = Soc(\Lambda) \), we have \( \omega_j + Im(R^*_\rho) = \omega_{j+1} + Im(R^*_\rho) \) for all \( j = 1, 2, ..., n - 1 \). But the description of \( y \gamma \) in the proof of lemma 4.9 implies that \( \omega_1 + Im(R^*_\rho) = \epsilon^{2n-1} + Im(R^*_\rho) \) is precisely the element \( x_0^{-n} y \gamma HH(\Lambda) \)

3] \( \gamma t_j = \delta_{ij} x_0^{-n} y \gamma \gamma \) Graded commutativity gives that \( t_j \gamma = \gamma t_j \) and the element \( \gamma t_j \in HH^7(\Lambda) \) is represented by the composition

\[
Q \xrightarrow{l_j} Q \xrightarrow{\delta} \Lambda.
\]

Note that \( l_j(e_{i(a)} \otimes e_{i(a)}) \in \Lambda e_{i(a)} \otimes e_{i(a)} \Lambda \), from which we deduce that \( \delta \circ l_j = 0 \) for \( j \in Q_0/\{1\} \). And for \( j = 1 \) we have

\[
(\delta \circ l_1)(e_{i(a)} \otimes e_{i(a)}) = \begin{cases} 
0 & \text{if } a \neq \epsilon \\
\frac{1}{2}(\omega_1 + \omega_1) = \omega_1 = \epsilon^{2n-1} & \text{if } a = \epsilon
\end{cases}
\]

But, due to the identification \( HH^1(\Lambda) = HH^7(\Lambda) \), which is just multiplication by \( h \), and the proof of lemma 4.9 we know that \( x_0^{-n} y \gamma \) is precisely the element \( \epsilon^{2n-1} + Im(\delta^*) = \frac{Ker(R^*)}{Im(\delta^*)} \).

Therefore we get \( t_j \gamma = \gamma t_j = \delta_{ij} x_0^{-n} y \gamma \), for all \( j \in Q_0 \).

\[ \square \]

**Lemma 5.5.** \( \gamma^2 = z_1 h \)

**Proof.** We have the following commutative diagram of \( \Lambda \)-bimodules:

\[
\begin{array}{ccc}
P & \xrightarrow{R} & Q & \xrightarrow{\delta} & P & \xrightarrow{k_\rho} & Q \\
\beta & & \delta & & \alpha & & \beta \\
Q & \xrightarrow{\delta_\rho} & P & \xrightarrow{k} & P & \xrightarrow{R_\rho} & P \\
\end{array}
\]

where \( \alpha, \beta \) and \( \gamma \) are the morphisms defined in the proof of Lemma 5.3. Indeed the commutativity of the two right squares was proved in that lemma. On the other hand, we have \( \gamma_\tau = \gamma \) and \( \beta_\tau = \beta \) (see Lemma 2.1) since \( \tau \) is fixes the vertices. The commutativity of the left most square is then obtained from the commutativity of the right most square by applying the equivalence \( G_\tau : \Lambda \xrightarrow{\cong} \Lambda \rightarrow \Lambda \rightarrow \Lambda \).

It remains to prove the commutativity of the second square from left to right. We need to prove that \( (\alpha \circ \delta)(e_{i(a)} \otimes e_{i(a)}) = 0 \), when \( a \neq \epsilon \), and that \( (\alpha \circ \delta)(e_{i(\epsilon)} \otimes e_{i(\epsilon)}) = \sum_{x \in e_{i(\epsilon)}} \sum_{y \in e_{i(\epsilon)}} (-1)^{\nu} y^{i+1} \).

For the first equality, we do the case \( a = a_i \), with \( i = 1, ..., n - 1 \), the case \( a = a_i \), being symmetric. We have

\[
(\alpha \circ \delta)(e_{i(a_i)} \otimes e_{i(a_i)}) = a_i \alpha(e_{i+1} \otimes e_{i+1}) - \alpha(e_i \otimes e_i) a_i =
\]

\[
= a_i [(-1)^{1+\frac{i+1}{2}} \sum_{r=1}^{n-i} (-1)^{n-1} y^{i+1}] - [(-1)^{1+i}\sum_{r=1}^{n-i+1} (-1)^n y^{i}] a_i =
\]

\[
= a_i [(-1)^{1+\frac{i+1}{2}} \sum_{r=1}^{n-i} (-1)^{n-1} y^{i+1}] - [(-1)^{1+i}\sum_{r=1}^{n-i+1} (-1)^n y^{i}] a_i =
\]

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\[ = (-1)^{1+\frac{i+1}{2}} \sum_{r=1}^{n-i} (-1)^{u_r} a_i y_r^{i+1} + (-1)^{\frac{r(i-1)}{2}} \sum_{r=1}^{n-i+1} (-1)^{u_r} y_r^i. \] 

Direct calculation shows that

\[ a_i y_r^{i+1} = \sum_{k=0}^{n-i-r} a_i \bar{a}_1 ... \bar{a}_i \epsilon^{2k} a_1 ... a_{r-1} \otimes \bar{a}_r ... \bar{a}_1 \epsilon^{2(n-i-r-k)}a_1 ... a_i. \]

Using remark 3.5(4), we then have

\[ a_i y_r^{i+1} = (-1)^{i} \sum_{k=0}^{n-i-r} a_i \bar{a}_1 ... \bar{a}_i \epsilon^{2(k+1)} a_1 ... a_{r-1} \otimes \bar{a}_r ... \bar{a}_1 \epsilon^{2(n-i-r-k)}a_1 ... a_i = \]

\[ (-1)^{i} \sum_{r=1}^{n-i-r+1} a_i \bar{a}_1 ... \bar{a}_i \epsilon^{2i} a_1 ... a_{r-1} \otimes \bar{a}_r ... \bar{a}_i \epsilon^{2(n-i-r+1)}a_1 ... a_i = \]

\[ (-1)^i [y_i^{r+1} a_i - \bar{a}_1 ... \bar{a}_r \epsilon^{20} a_1 ... a_{r-1} \otimes \bar{a}_r ... \bar{a}_i \epsilon^{2(n-i-r+1)}a_1 ... a_i] = (-1)^i y_i^{r+1} a_i, \]

where the last equality is due to remark 3.5(2).

On the other hand, for \( r = n - i + 1 \), we have the equality

\[ y_{n-i+1}^i a_i = \bar{a}_1 ... \bar{a}_1 a_1 ... a_{n-i-1} \otimes \bar{a}_n ... \bar{a}_1 a_1 ... a_i a_i, \]

which is zero due again to remark 3.5(2). All these considerations allow us to rewrite the equality (\( \star \)) given above as

\[ (\alpha \circ \delta)(e_{i(a)} \otimes e_{t(a)}) = \sum_{r=1}^{n-i} (-1)^{u_r} + (-1)^v r, i] y_i^{r} a_i, \]

where \( u(r, i) = 1 + \frac{i+1}{2} + i + u_r \) and \( v(r, i) = \frac{r(i-1)}{2} + u_r \). Since \( u(r, i) - v(r, i) = 2i + 1 \) we conclude that \((\alpha \circ \delta)(e_{i(a)} \otimes e_{t(a)}) = 0\) as desired.

We next calculate \((\alpha \circ \delta)(e_{i(c)} \otimes e_{t(c)})\). By definition, we have

\[ \alpha(e_1 \otimes e_1) = (-1)^{1+\frac{r(i-1)}{2}} \sum_{r=1}^{n-i} (-1)^{u_r} y_r^{i+1} = \sum_{r=1}^{n-i} (-1)^{u_r+1} y_r^{i}. \]

Then we have

\[ (\alpha \circ \delta)(e_{i(c)} \otimes e_{t(c)}) = \epsilon \alpha(e_1 \otimes e_1) - \alpha(e_1 \otimes e_1) \epsilon = \sum_{r=1}^{n-i} (-1)^{u_r+1} (y_i^{r+1} - y_i^{r} \epsilon). \]

But, with the terminology of the proof of lemma 5.3, we also have the equality

\[ \epsilon y_r^{i+1} - y_i^{r+1} \epsilon = \sum_{k=0}^{n-r} \epsilon^{2k} a_1 ... a_{r-1} \otimes \bar{a}_r ... \bar{a}_1 \epsilon^{2(n-r-k)}a_1 ... a_i - \sum_{k=0}^{n-r} \epsilon^{2k} a_1 ... a_{r-1} \otimes \bar{a}_r ... \bar{a}_1 \epsilon^{2(n-r-k)+1} = \]

\[ \sum_{x \in e_r, B_{e_1}, \ deg(x) \equiv r-1} x \otimes x - \sum_{x \in e_r, B_{e_1}, \ deg(x) \equiv r} x \otimes x = \]

\[ (-1)^{\frac{r(i-1)}{2}} \sum_{x \in e_r, B_{e_1}, \ deg(x) \equiv r-1} x^* \otimes x - \sum_{x \in e_r, B_{e_1}, \ deg(x) \equiv r} x^* \otimes x, \]

where \( \equiv \) means 'congruent modulo 2'. Due to the fact that our basis \( B \) is dualizable and that \( deg(x) + deg(x^*) \equiv 1 \), for all \( x \in B \), we deduce that

\[ \epsilon y_r^{i+1} - y_i^{r+1} \epsilon = (-1)^{\frac{r(i-1)}{2}} \sum_{y \in e_{i(c)}, B_{e_1}, \ deg(y) \equiv r-1} y \otimes y^* - \sum_{y \in e_{i(c)}, B_{e_1}, \ deg(y) \equiv r} y \otimes y^*, \]

where
and hence

\[ (-1)^{r+1}(e^{-r-1} - y^{-r-1}) = (-1)^{r(r-1)/2} \sum_{y \in e_1 \text{Bimod} \, \text{deg}(y) = r} y \otimes y^* - \sum_{y \in e_1 \text{Bimod} \, \text{deg}(y) = r-1} y \otimes y^* \]

for each \( r = 1, \ldots, n \). Since \( r(r-1)/2 + u_r + 1 \equiv r \) it follows that the coefficient of \( y \otimes y^* \) in the last expression is precisely \( (-1)^{\text{deg}(y)} \), for each \( y \in e_1 \text{Bimod} \). From that desired equality holds, consider the morphism of \( \Lambda \)-bimodules \( \tilde{\gamma} = 1 \)-dimensional. In the proof of this lemma it is actually proved that \( \tilde{\gamma} \) divides \( 2^n \), for each \( y \in e_1 \text{Bimod} \). From that desired equality \((\alpha \circ \delta)(e_i(\epsilon) \otimes e_i(\epsilon)) = \sum_{y \in e_1 \text{Bimod} \, \text{deg}(y) \equiv r} y \otimes y^* \) follows.

Once the commutativity of the initital diagram is proved, we get that \( \gamma^2 \) is represented by the morphism of \( \Lambda \)-bimodules \( P \xrightarrow{\delta} P \xrightarrow{\gamma} \Lambda \), which satisfies that \( (\tilde{\gamma} \circ \beta)(e_i \otimes e_i) = \delta_i \epsilon_1 \), for each \( i \in Q_0 \). This morphism represents the element \( e_1 + \text{Im}(R^*) \in \frac{\ker(k_{e_1})}{\text{Im}(R^*)} = HH^2(\Lambda) = HH^4(\Lambda) \), which is precisely the element \( z_1 h \). Therefore we have \( \gamma^2 = z_1 h \) in \( HH^*(\Lambda) \).

We are now ready to prove the two main theorems of the paper.

**PROOF OF THE THEOREMS** 1.1 and 1.2: With the notation used until now, we know from Proposition 4.10 that the set \( \{ y_0, x_1, \ldots, x_n, y, z_1, \ldots, z_n, t_1, \ldots, t_n, \gamma, h \} \) generates \( HH^*(\Lambda) \) as an algebra. If \( \text{Char}(K) \) does not divide \( 2n+1 \), then lemma 5.1 tells us that each \( t_i \) is a \( K \)-linear combination of the \( y z_j \), which allows us to drop all the \( t_i \) from the set of generators. On the other hand, if \( \text{Char}(K) \) divides \( 2n+1 \), then lemma 5.1 tells us that the \( K \)-subspace of \( HH^3(\Lambda) \) generated by \( \{ y z_1, \ldots, y z_n \} \) is \( 1 \)-dimensional. In the proof of this lemma it is actually proved that \( y z_j + (-1)^{(2j-1)y z_j} = 0 \) since this is the relationship between the rows (=columns) of the matrix \( C \). In addition, looking at the first row of this matrix, we see that \( y z_1 = -n t_1 + (n-1)t_2 + \cdots + (-1)^n t_n \), and hence \( \{ t_1, \ldots, t_{n-1}, y z_1 \} \) is a basis of \( HH^3(\Lambda) \). So, in case \( \text{Char}(K) \) divides \( 2n+1 \), we can drop \( t_n \) from the set of generators. This proves that the set of generators given in Theorems 1.1 and 1.2 is correct.

From the fact that \( \text{Soc}(\Lambda)HH^3(\Lambda) = 0 \) \( \forall j > 0 \) one readily obtains the relations in \( \beta \). From lemmas 4.3, 4.9 and 5.3 we obtain the relations in \( \beta \) (in both cases) except \( y^2 = 0 \). To see that this equality also holds, consider the morphism of \( \Lambda \)-bimodules \( \tilde{\gamma} : \oplus_{a \in Q_1} \Lambda e_i(\epsilon) \otimes e_i(\epsilon) \Lambda \rightarrow \Lambda \) defined by \( \tilde{\gamma}(e_i(\epsilon) \otimes e_i(\epsilon)) = a \forall a \in Q_1 \) that represents \( y \in HH^1(\Lambda) \). If \( \tilde{\gamma} : \oplus_{a \in Q_1} \Lambda e_i(\epsilon) \otimes e_i(\epsilon) \Lambda \rightarrow \Lambda e_i \otimes e_i \Lambda \) is the only morphism of \( \Lambda \)-bimodules such that \( \tilde{\gamma}(e_i(\epsilon) \otimes e_i(\epsilon)) = \frac{1}{a} (a \otimes e_i(\epsilon) + e_i(\epsilon) \otimes a) \), for all \( a \in Q_1 \), then we have \( u \circ \tilde{\gamma} = \tilde{\gamma} \), where \( u : \oplus_{a \in Q_1} \Lambda e_i \otimes e_i \Lambda \rightarrow \Lambda \) is the multiplication map.

Note now that we have a commutative diagram in the category of \( \Lambda \)-bimodules:

\[
\begin{array}{ccc}
P & \xrightarrow{R} & Q \\
\eta \downarrow & & \downarrow \tilde{\gamma} \\
Q & \xrightarrow{\delta} & P
\end{array}
\]

where \( \eta \) is the only morphism of \( \Lambda \)-bimodules such that \( \eta(e_i \otimes e_i) = \sum_{a \in Q_1, a(a) = i} e_i(\epsilon) \otimes \tilde{a} \).

The definition of Yoneda product says that \( y^2 \) is represented by the composition

\[
\oplus_{a \in Q_1} \Lambda e_i \otimes e_i \lambda \xrightarrow{\eta} \oplus_{a \in Q_1} \Lambda e_i(\epsilon) \otimes e_i(\epsilon) \Lambda \xrightarrow{\tilde{\gamma}} \Lambda
\]

which takes \( e_i \otimes e_i \simeq \sum_{a \in Q_1, a(a) = i} a \tilde{a} = 0 \).

For the relations in \( \beta \) (iii) note that the proof of lemma 5.9 gives an isomorphism \( \phi_y : HH^4(\Lambda) \xrightarrow{\sim} HH^5(\Lambda) \) \( (\xi \rightsquigarrow y \xi) \). What we shall prove is the equality

\[ z_j(yz_k) = (-1)^{k+j+1}(2j-1)(n-k+1)x_0^{n-1}y^j, \]

from which the desired equality will follow.

Indeed, by lemma 5.1, 1.2 we have that \( yz_k = \sum_{i=1}^n c_i k t_i \) and hence \( z_j(yz_k) = \sum_{i=1}^n c_i k z_j t_i \). From lemma 5.4, 2) we then get
\[ z_j(yz_k) = c_{jk}z_jt_j = (-1)^{k-j+1}(2j-1)(n-k+1)x_0^{n-1}y\gamma \]

Note that lemma \[5.4\] also proves the relations in \ref{vii} for Theorem \ref{1.2}. We claim that it also gives the relations \ref{iv}. Indeed multiplication by \( h \) gives an isomorphism of \( \mathbb{Z}(\Lambda) \)-modules 
\[ HH^1(\Lambda) \sim \rightarrow HH^7(\Lambda). \]
In particular \( yh \) is the canonical generator of \( HH^7(\Lambda) \) as \( \mathbb{Z}(\Lambda) \)-module, which implies that multiplication by \( y \) gives an isomorphism of \( \mathbb{Z}(\Lambda) \)-modules 
\[ HH^0(\Lambda) \sim \rightarrow HH^7(\Lambda). \]
Similar to the previous paragraph, we shall prove the equality
\[ \gamma(yz_j) = (-1)^j(n-j+1)x_0^{n-1}yh, \]
from which the relations in \ref{iv} will follow.

Again we have 
\[ yz_j = \sum_{l=1}^n c_{lj}t_l \]
and so
\[ \gamma(yz_j) = \sum_{l=1}^n c_{lj}\gamma t_l = c_{1j}x_0^{n-1}yh = (-1)^j(n-j+1)x_0^{n-1}yh, \]
using the relations in \ref{vii} which have already been proved in lemma \[5.4\] \ref{3}.

Finally, lemma \[5.3\] gives the relations in \ref{v} while, when \( \text{Char}(K) \) divides \( 2n+1 \), the equality 
\[ yz_1 + (-1)^j(2j-1)yz_1 = 0 \]
mentioned in the first paragraph of this proof gives the relations in \ref{vi} of Theorem \ref{1.2}.

The previous paragraphs show that there is a surjective homomorphism of graded algebras from the algebra given by the mentioned generators and relations to the algebra \( HH^*(\Lambda) \). By looking at dimensions in each degree, it is not difficult to see that it is actually an isomorphism.

**Remark 5.6.** In \cite{18}[Section 9] the graded ring structure of \( HH^*(\Lambda) \) was calculated taking \( \mathbb{C} \) as ground field. However, the arguments and calculations appear to be valid whenever \( \text{char}(K) \neq 2 \) and \( \text{char}(K) \) does not divide \( 2n+1 \). Then, with the suitable changes derived from the different presentations of the algebra, our Theorem \ref{1.1} could be derived from Eu’s work.

Eu’s methods use sometimes direct calculation of the products \( HH^i(\Lambda) \cdot HH^j(\Lambda) \), sometimes the graded condition of the minimal projective resolution of \( \Lambda \) (see \ref{9.2}) and sometimes the matrix Hilbert series \( H_\Lambda(t) \) (see Definition \ref{2.5.2}) together with the equality \( H_\Lambda(t) = (1+t^{2n+1})(1+t^3)I_n - Dt)^{-1} \) proved in \cite{35}, where \( D \) is the adjacency matrix of \( \mathbb{L}_n \) (see the proof of Lemma \ref{9.3.3} and section 6.2 in \cite{18}).

We have not used these tools in our paper and have directly calculated all products \( HH^i(\Lambda) \cdot HH^j(\Lambda) \) working with the bases of Proposition \ref{4.10} which already included in themselves some products.

**Corollary 5.7.** Let us fix the presentation of \( HH^*(\Lambda) \) given by Theorems \ref{1.1} or \ref{1.2}. A presentation of \( HH^*(\Lambda) \) is obtained from it by doing the following:

1. Replace the generators \( x_1, ..., x_n \) by a new generator \( h' \) of degree \(-6\)
2. Replace the relations \ref{i} in the list by a new relation \( hh' = 1 \).
3. Leave the remaining generators and relations unchanged.

**Proof.** It is immediately seen that the graded commutative algebra given by the just described generators and relations is isomorphic to \( HH^*(\Lambda)_{(h)} \), whence isomorphic to \( HH^*(\Lambda) \) (see Proposition \ref{2.15}).

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