Some Aspects of Higher-Page Non-Kähler Hodge Theory
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Abstract. The main thrust of this work is to extend some basic results in Hodge Theory to the higher pages of the Frölicher spectral sequence. For an arbitrary nonnegative integer \( r \), we introduce the class of page-\( r \)-\( \partial \overline{\partial} \)-manifolds by requiring the analogue of the Hodge decomposition to hold on a compact complex manifold \( X \) when the usual Dolbeault cohomology groups \( H^{p,q}_\partial(X) \) are replaced by the spaces \( E^{p,q}_{r+1}(X) \) featuring on the \((r+1)\)-st page of the Frölicher spectral sequence of \( X \). The class of page-\( r \)-\( \partial \overline{\partial} \)-manifolds increases as \( r \) increases and coincides with the usual class of \( \partial \overline{\partial} \)-manifolds when \( r = 0 \). We investigate various properties of these manifolds and show that they are analogous to those of \( \partial \overline{\partial} \)-manifolds with some noteworthy exceptions. We also point out a number of examples. For instance, all complex parallelisable nilmanifolds, including the Iwasawa manifold and a 5-dimensional analogue thereof, are page-1-\( \partial \overline{\partial} \)-manifolds, although they are seldom \( \partial \overline{\partial} \)-manifolds. The deformation properties of page-1-\( \partial \overline{\partial} \)-manifolds are also investigated and a general notion of essential small deformations is introduced for Calabi-Yau manifolds. We also introduce higher-page analogues of the Bott-Chern and Aeppli cohomologies and highlight their relations to the new class of manifolds. On the other hand, we prove analogues of the Serre duality for the spaces featuring in the Frölicher spectral sequence and for the higher-page Bott-Chern and Aeppli cohomologies.

1 Introduction

Let \( X \) be an \( n \)-dimensional compact complex manifold. Recall the following notion that goes back to Deligne-Griffiths-Morgan-Sullivan [DGMS75] in the form (equivalent to that in [DGMS75]) and with the name given in [Pop14, Definition 1.6].

Definition 1.1. A compact complex manifold \( X \) is said to be a \( \partial \overline{\partial} \)-manifold if for any \( d \)-closed pure-type form \( u \) on \( X \), the following exactness properties are equivalent:

\[
\text{u is } d\text{-exact } \iff \text{u is } \partial\text{-exact } \iff \text{u is } \overline{\partial}\text{-exact } \iff \text{u is } \partial \overline{\partial}\text{-exact}.
\]

The classical \( \partial \overline{\partial} \)-lemma asserts that every compact Kähler manifold is a \( \partial \overline{\partial} \)-manifold. More generally, thanks to [AB93], every class \( \mathcal{C} \) manifold (i.e. every compact complex manifold bimeromorphically equivalent to a compact Kähler manifold) is a \( \partial \overline{\partial} \)-manifold. However, there exist many \( \partial \overline{\partial} \)-manifolds that are not of class \( \mathcal{C} \) (see e.g. [Pop14, Observation 4.10], [FOU14, Theorem 5.2], [Fri17]).

On the other hand, every \( \partial \overline{\partial} \)-manifold admits a canonical Hodge decomposition and a canonical Hodge symmetry (accounting for the fact that some authors call these manifolds cohomologically Kähler). In other words, \( \partial \overline{\partial} \)-manifolds are those manifolds that support a standard Hodge theory. In particular, the Frölicher spectral sequence (FSS) of any \( \partial \overline{\partial} \)-manifold degenerates at the first page. However, the converse fails. (Indeed, as is well known, any non-Kähler compact complex surface provides a counter-example to the converse.) Meanwhile, \( \partial \overline{\partial} \)-manifolds also have good deformation and modification properties. For a review of these and some further properties of \( \partial \overline{\partial} \)-manifolds, see e.g. [Pop14].
One of our main goals in this work is to relax the notion of $\partial \bar{\partial}$-manifold to accommodate a Hodge theory involving the higher pages of the Frölicher spectral sequence (FSS) of $X$ when degeneration does not occur at the first page. As a result, we enhance the class of $\partial \bar{\partial}$-manifolds to classes of manifolds that seem worthy of further attention. This is motivated by the existence of many well-known non-Kähler and even non-$\partial \bar{\partial}$ compact complex manifolds, such as the Iwasawa manifold and higher-dimensional analogues thereof, whose Frölicher spectral sequence degenerates only at the second page or later. Thus, we aim for a unified Hodge theoretical treatment of as large a class of compact complex manifolds as possible.

This approach enables us to get exact analogues of the usual Hodge decomposition and symmetry properties for what we call page-$(r - 1)$-$\partial \bar{\partial}$-manifolds using the spaces $E^{p,q}_r(X)$ featuring on the $r$-th page of the FSS, for some given $r \geq 2$, rather than the usual spaces $E^{p,q}_1(X) = H^{p,q}_{\bar{\partial}}(X)$ of the Dolbeault cohomology. The standard notion of $\partial \bar{\partial}$-manifold coincides with that of page-0-$\partial \bar{\partial}$-manifold, while the class of page-$r$-$\partial \bar{\partial}$-manifolds, our main find in this work, grows as $r \in \mathbb{N}$ increases.

The following statement and the ensuing terminology explainer sum up several results and definitions of section 3. They are cast in a language compatible with a description of the FSS given by Cordero-Fernández-Gray-Ugarte in [CFGU97] (see Proposition 2.3 for details) according to which the classes in $E^{p,q}_r(X)$ are represented by those pure-type $(p, q)$-forms that we call $E^r$-closed, rather than by not necessarily pure-type forms $u$ with pure-type components of holomorphic degrees $\geq p$ (i.e. $u \in \mathcal{F}^p$) such that $du \in \mathcal{F}^{p+r}$ that are used in the standard definition of the FSS. All pure-type $d$-closed forms are trivially $E^r$-closed, but the converse fails. Requiring the existence of $d$-closed pure-type representatives for the $E^r$-classes in (1) of the next statement is a key hypothesis.

**Theorem and Definition 1.2.** Let $X$ be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. Fix an arbitrary $r \in \mathbb{N}^*$. The following statements are equivalent.

1. For every bidegree $(p, q)$, every class $\{\alpha^{p,q}\}_{E_r} \in E^{p,q}_r(X)$ can be represented by a $d$-closed $(p, q)$-form and for every $k$, the linear map

$$\bigoplus_{p+q=k} E^{p,q}_r(X) \ni \sum_{p+q=k} \{\alpha^{p,q}\}_{E_r} \mapsto \left\{ \sum_{p+q=k} \alpha^{p,q} \right\}_{DR} \in H^k_{DR}(X, \mathbb{C})$$

is well-defined by means of $d$-closed pure-type representatives and bijective.

In this case, $X$ is said to have the $E_r$-Hodge Decomposition property.

2. The Frölicher spectral sequence of $X$ degenerates at $E_r$ and the De Rham cohomology of $X$ is pure.

3. For all $p, q \in \{0, \ldots, n\}$ and for every form $\alpha \in C^\infty_{p,q}(X)$ such that $d\alpha = 0$, the following equivalences hold:

$$\alpha \in \text{Im} d \iff \alpha \text{ is } E_r\text{-exact} \iff \alpha \text{ is } \overline{E}_r\text{-exact} \iff \alpha \text{ is } E_r\overline{E}_r\text{-exact}.$$ 

4. For all $p, q \in \{0, \ldots, n\}$, the canonical linear maps

$$E^{p,q}_{r,BC}(X) \to E^{p,q}_r(X) \quad \text{and} \quad E^{p,q}_r(X) \to E^{p,q}_{r,A}(X)$$
are isomorphisms, where \( E_{p,q}^{r,BC}(X) \) and \( E_{p,q}^{r,A}(X) \) are the \( E_r \)-Bott-Chern, respectively the \( E_r \)-Aeppli, cohomology groups of bidegree \((p, q)\) introduced in Definition 3.40.

(5) For all \( p, q \in \{0, \ldots, n\} \), the canonical linear map \( E_{p,q}^{r,BC}(X) \to E_{p,q}^{r,A}(X) \) is injective.

A compact complex manifold \( X \) that satisfies any of the equivalent conditions (1)–(5) is said to be a \( \text{page-}(r-1)\)-\( \bar{\partial}\partial \)-manifold.

Let us explain the terminology used in the above statement. See section 3 for details.

(a) In (1), by well-definedness we mean that the map is independent of the choices of \( d \)-closed representatives of the \( E_r \)-cohomology classes involved and that the images of the spaces \( E_{p,q}^{r}(X) \) in \( H^{k}_{DR}(X, \mathbb{C}) \) are in a direct sum.

(b) The De Rham cohomology of \( X \) is said to be pure if, for every \( k \), the vector subspaces \( H^{p,q}_{DR}(X, \mathbb{C}) \) of \( H^{k}_{DR}(X, \mathbb{C}) \), consisting of De Rham cohomology classes representable by pure-type \((p, q)\)-forms with \( p + q = k \), are in a direct sum and if they fill out \( H^{k}_{DR}(X, \mathbb{C}) \). See Definition 3.4, which has been present in the literature for some time (cf. [Del71], [LZ09]). Some authors call this property of the De Rham cohomology pure and full.

(c) A \((p, q)\)-form \( \alpha \) is said to be \( E_r \)-exact if \( \alpha \) represents the zero \( E_r \)-cohomology class on the \( r \)-th page of the Frölicher spectral sequence of \( X \). Also, \( \alpha \) is said to be \( E_r \)-exact if \( \bar{\alpha} \) is \( E_r \)-exact. When \( r = 1 \), these notions coincide with \( \bar{\partial} \)-exactness, respectively \( \partial \)-exactness.

The notion of \( E_r E_r \)-exactness is introduced in (iii) of Definition 3.37 as a weakening, inspired by the characterisations of \( E_r \)-exactness and \( E_r \)-exactness given in Proposition 2.3, of the standard notion of \( \partial \bar{\partial} \)-exactness. When \( r = 1 \), the notion of \( E_1 E_1 \)-exactness is defined as \( \partial \bar{\partial} \)-exactness.

The notion of \( E_r E_r \)-closedness is introduced in (i) of Definition 3.37 as a strengthening of the standard notion of \( \partial \bar{\partial} \)-closedness. When \( r = 1 \), the notion of \( E_1 E_1 \)-closedness is defined as \( \partial \bar{\partial} \)-closedness.

All the exactness notions become weaker and weaker as \( r \) increases, in contrast to the closedness conditions that become stronger and stronger.

(d) The \( E_r \)-Bott-Chern, respectively the \( E_r \)-Aeppli, cohomology groups of bidegree \((p, q)\) are introduced in Definition 3.40 by quotienting out

- the smooth \( d \)-closed \((p, q)\)-forms by the \( E_r E_r \)-exact ones;
- respectively, the smooth \( E_r E_r \)-closed \((p, q)\)-forms by those lying in \( \text{Im} \partial + \text{Im} \bar{\partial} \).

When \( r = 1 \), these cohomology groups coincide with the standard Bott-Chern, respectively Aeppli, cohomology groups.

In section 3, we also provide examples of page-\( r \)-\( \partial \bar{\partial} \)-manifolds, with \( r \geq 1 \), that are not page-0-\( \partial \bar{\partial} \)-manifolds (i.e. \( \partial \bar{\partial} \)-manifolds in the usual sense). Foremost among these are the complex parallelisable nilmanifolds that are proved in Theorem 3.31 to be page-1-\( \partial \bar{\partial} \)-manifolds. However, they are not \( \partial \bar{\partial} \)-manifolds unless they are complex tori. This large class of manifolds includes the complex 3-dimensional Iwasawa manifold \( I^{(3)} \) and its 5-dimensional counterpart \( I^{(5)} \) discussed
in §3.4.2, §5.1, §5.3 and §7. Recall that a compact complex manifold is said to be complex parallelisable if its holomorphic tangent bundle $T^{1,0}X$ is trivial.

Using the Hodge theory based on pseudo-differential Laplacians introduced in [Pop16] and [Pop19], we prove in §2, respectively §3.5, the analogues of the Serre duality for the $E_r$-cohomology, respectively the $E_r$-Bott-Chern and $E_r$-Aeppli cohomologies, for all $r \geq 2$. The case $r = 1$ is standard. These results can be summed up as follows (cf. Theorem 2.1, Corollaries 2.4 and 2.9, Theorem 3.47).

**Theorem 1.3.** Let $X$ be a compact complex manifold with dim$_C X = n$. Fix an arbitrary $r \in \mathbb{N}^*$. For every $p, q \in \{0, \ldots, n\}$, the canonical bilinear pairings

$$E^{\mu, \nu}_r(X) \times E^{\mu-n, \nu-q}_r(X) \to \mathbb{C}, \quad (\{\alpha\}_{E_r}, \{\beta\}_{E_r}) \mapsto \int_X \alpha \wedge \beta,$$

and

$$E^{\mu, \nu}_{r, BC}(X) \times E^{\mu-n, \nu-q}_{r, A}(X) \to \mathbb{C}, \quad (\{\alpha\}_{E_{r, BC}}, \{\beta\}_{E_{r, A}}) \mapsto \int_X \alpha \wedge \beta,$$

are well defined and non-degenerate.

This means that every space $E^{\mu-n, \nu-q}_r(X)$ can be viewed as the dual of $E^{\mu, \nu}_r(X)$ and every space $E^{\mu, \nu}_{r, BC}(X)$ can be viewed as the dual of $E^{\mu-n, \nu-q}_{r, A}(X)$.

We use these dualities to obtain numerical information on the dimensions of these cohomology groups and a necessary, but surprisingly not sufficient, numerical condition for a compact complex manifold to be page-$(r-1)\bar{\partial}$-manifold. We denote by $h^{\mu, \nu}_r, \bar{h}^{\mu, \nu}_r$ and $e^{\mu, \nu}_r$ the respective dimensions of the $\mathbb{C}$-vector spaces $H^{\mu, \nu}_{r, BC}(X), H^{\mu, \nu}_{r, A}(X)$ and $E^{\mu, \nu}_r(X)$, while $h^k_r, \bar{h}^k_r$ and $e^k_r$ stand for the dimensions of the spaces obtained thereof as direct sums over $p + q = k$. As usual, $b_k$ is the $k$-th Betti number of $X$.

Corollaries 3.56 and 3.57 can be summed up as follows.

**Proposition 1.4.** Let $X$ be a compact complex manifold with dim$_C X = n$.

(i) For every $r \in \mathbb{N}^*$ and every $k \in \{0, \ldots, 2n\}$, the following inequalities hold:

$$h^k_{r, BC} + \bar{h}^k_{r, A} \geq 2e^k_r \geq 2b_k.$$

(ii) If $X$ is a page-$(r-1)\bar{\partial}$-manifold, then $h^k_{r, BC} + \bar{h}^k_{r, A} = 2b_k$ for all $k \in \{0, \ldots, 2n\}$.

The last numerical identity implies the degeneration at $E_r$ of the Frölicher spectral sequence, but it does not imply the page-$(r-1)\bar{\partial}$-property when $r \geq 2$, as the example of certain Calabi-Eckmann manifolds (see Example 3.58) shows. This is in stark contrast with the case $r = 1$ and the Angella-Tomassini result of [AT13] characterising the standard $\bar{\partial}$-property of compact complex manifolds by the numerical identities corresponding to $r = 1$ in (ii) of the above Proposition 1.4.

In section 4, we prove several stability properties of page-$r$-$\bar{\partial}$-manifolds under various geometric operations. The most vivid of these statements is the following one (see Theorem 4.2 for extra properties), in which part (i) has long been known in the standard case $r = 0$, but whose part (ii) has been proved only recently even in that case (c.f. [Ste18, Cor. 28]).
Theorem 1.5. Let $\tilde{X} \to X$ be the blow-up of a compact complex manifold $X$ along a submanifold $Z \subset X$ with $\text{codim}_X Z \geq 2$.

(i) If $\tilde{X}$ is page-$r$-$\partial\bar{\partial}$ for some $r \in \mathbb{N}$, so are $X$ and $Z$.

(ii) Conversely, if $X$ is page-$r_1$-$\partial\bar{\partial}$ for some $r_1 \in \mathbb{N}$ and $Z$ is page-$r_2$-$\partial\bar{\partial}$ for some $r_2 \in \mathbb{N}$, then $\tilde{X}$ is page-$r$-$\partial\bar{\partial}$, where $r = \max\{r_1, r_2\}$.

The proof relies heavily on the theory of the indecomposable double complexes, called squares and zigzags, associated with a double complex, developed in the second-named author’s thesis (see [Ste18]), a rundown of which is given in §3.3 (which contains a characterisation of our page-$r$-$\partial\bar{\partial}$-property in this language) and in parts of §3.4.2 and §3.5.

In section 5, we begin to investigate the role of the new class of page-1-$\partial\bar{\partial}$-manifolds in the theory of deformations of complex structures and in the new approach to Mirror Symmetry, extended to the possibly non-Kähler context, proposed in [Pop18].

The notion of essential deformations of the Iwasawa manifold $I^{(3)}$ was put forward in [Pop18] as consisting of those small deformations of $I^{(3)}$ that are not complex parallelisable. Indeed, $I^{(3)}$ itself is complex parallelisable, so removing from its Kuranishi family its complex parallelisable small deformations, which have the same geometry as $I^{(3)}$, does not induce any loss of geometric information. Meanwhile, the remaining, essential, small deformations turn out to be parametrised by the $E_2$-cohomology space $E^{2,1}_2(X)$ featuring on the second page of the Frölicher spectral sequence (FSS) of $I^{(3)}$ and to have much better Hodge-theoretical properties than the space of all small deformations, parametrised by the bigger Dolbeault cohomology space $H^{2,1}_\partial(X) = E^{2,1}_1(X)$ featuring on the first page. Indeed, the FSS degenerates at $E_2$, rather than at $E_1$, in the case of $I^{(3)}$ and an analogue of the Hodge decomposition and symmetry for $I^{(3)}$ was observed in [Pop18] when the traditional first page is replaced by the second page of the FSS.

We take up this issue again in §5.3 in the general context of compact Calabi-Yau page-1-$\partial\bar{\partial}$-manifolds. It turns out that for some complex parallelisable nilmanifolds, such as the 5-dimensional analogue $I^{(5)}$ of the Iwasawa manifold, the non-complex parallelisable small deformations no longer coincide with those parametrised by $E^{n-1,1}_2(X)$ even when this space injects in a natural way into $E^{n-1,1}_1(X)$. We call the small deformations of the latter type essential deformations of a compact Calabi-Yau page-1-$\partial\bar{\partial}$-manifold $X$ (see the general Definition 5.9).

The main result we get in this direction is a generalisation to Calabi-Yau page-1-$\partial\bar{\partial}$-manifolds of the following classical Bogomolov-Tian-Todorov theorem (see [Tia87], [Tod89]).

The Kuranishi family of a compact Kähler Calabi-Yau manifold is unobstructed.

The Kähler assumption can be weakened to the $\partial\bar{\partial}$-assumption (see e.g. [Pop13]). In Theorem 5.13, we show that, under certain cohomological conditions, a similar result holds when the $\partial\bar{\partial}$-assumption is further weakened to the page-1-$\partial\bar{\partial}$-assumption.

This undertaking is justified by the fact that unobstructedness of the Kuranishi family occurs for some well-known compact complex manifolds that are not $\partial\bar{\partial}$-manifolds but are page-1-$\partial\bar{\partial}$-manifolds, such as $I^{(3)}$ and $I^{(5)}$. The point we will make is that $I^{(3)}$ and $I^{(5)}$ are not isolated examples, but they are part of a pattern.
2 Serre-type duality for the Frölicher spectral sequence

Let $X$ be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. For every $r \in \mathbb{N}$, we let $E_{r}^{p,q}(X)$ stand for the space of bidegree $(p, q)$ featuring on the $r^{th}$ page of the Frölicher spectral sequence of $X$.

As is well known, the first page of this spectral sequence is given by the Dolbeault cohomology of $X$, namely $E_{1}^{p,q}(X) = H_{\bar{\partial}}^{p,q}(X)$ for all $p, q$. Moreover, the classical Serre duality asserts that every space $H_{\bar{\partial}}^{p,q}(X)$ is the dual of $H_{\partial}^{n-p,n-q}(X)$ via the canonical non-degenerate bilinear pairing

$$
H_{\bar{\partial}}^{p,q}(X) \times H_{\partial}^{n-p,n-q}(X) \rightarrow \mathbb{C}, \quad ([\alpha]_{\bar{\partial}}, [\beta]_{\partial}) \mapsto \int_{X} \alpha \wedge \beta.
$$

In this section, we will extend this duality to all the pages of the Frölicher spectral sequence. For the sake of perspicuity, we will first treat the case $r = 2$ and then the more technically involved case $r \geq 3$.

2.1 Serre-type duality for the second page of the Frölicher spectral sequence

The main ingredient in the proof of the next statement is the Hodge theory for the $E_{2}$-cohomology introduced in [Pop16] via the construction of a pseudo-differential Laplace-type operator $\tilde{\Delta}$.

**Theorem 2.1.** For every $p, q \in \{0, \ldots, n\}$, the canonical bilinear pairing

$$
E_{2}^{p,q}(X) \times E_{2}^{n-p,n-q}(X) \rightarrow \mathbb{C}, \quad \{\alpha\}_{E_{2}}, \{\beta\}_{E_{2}} \mapsto \int_{X} \alpha \wedge \beta, \quad (1)
$$

is well defined (i.e. independent of the choices of representatives of the cohomology classes involved) and non-degenerate.

**Proof.** To prove well-definedness, let $\{\alpha\}_{E_{2}} \in E_{2}^{p,q}(X)$ and $\{\beta\}_{E_{2}} \in E_{2}^{n-p,n-q}(X)$ be arbitrary classes in which we choose arbitrary representatives $\alpha, \beta$. Thus, $\partial \alpha = 0$, $\bar{\partial} \alpha \in \text{Im} \, \bar{\partial}$ (since $[\alpha]_{\bar{\partial}} \in \ker d_{1}$) and $\beta$ has the analogous properties. In particular, $\bar{\partial} \beta = \bar{\partial} v$ for some $(n-p+1, n-q-1)$-form $v$. Any other representative of the class $\{\alpha\}_{E_{2}}$ is of the shape $\alpha + \partial \eta + \bar{\partial} \zeta$ for some $(p-1, q)$-form $\eta \in \ker \bar{\partial}$ and some $(p, q-1)$-form $\zeta$. (Indeed, $[\partial \eta]_{\bar{\partial}} = d_{1}( [\eta]_{\bar{\partial}})$.) We have

$$
\int_{X} (\alpha + \partial \eta + \bar{\partial} \zeta) \wedge \beta = \int_{X} \alpha \wedge \beta + (-1)^{p+q} \int_{X} \eta \wedge \bar{\partial} \beta + (-1)^{p+q} \int_{X} \zeta \wedge \partial \beta \quad \text{(by Stokes)}
$$

$$
= \int_{X} \alpha \wedge \beta + (-1)^{p+q} \int_{X} \eta \wedge \bar{\partial} v \quad \text{(since } \bar{\partial} \beta = \bar{\partial} v \text{ and } \bar{\partial} \beta = 0)\n$$

$$
= \int_{X} \alpha \wedge \beta + \int_{X} \bar{\partial} \eta \wedge v = \int_{X} \alpha \wedge \beta \quad \text{(by Stokes and } \bar{\partial} \eta = 0).\n$$

Similarly, the integral $\int_{X} \alpha \wedge \beta$ does not change if $\beta$ is replaced by $\beta + \partial a + \bar{\partial} b$ with $a \in \ker \bar{\partial}$.
To prove non-degeneracy for the pairing (1), we fix an arbitrary Hermitian metric $\omega$ on $X$ and use the pseudo-differential Laplacian associated with $\omega$ introduced in [Pop16, §1.1]:

$$\tilde{\Delta} := \partial p'' \partial^* + \partial^* p'' \partial + \partial \tilde{\partial}^* + \tilde{\partial}^* \partial : C^\infty_{p,q}(X) \rightarrow C^\infty_{p,q}(X), \quad p, q = 0, \ldots, n,$$

where $p'' : C^\infty_{p,q}(X) \rightarrow \mathcal{H}^{p,q}_{\Delta}(X) := \ker \Delta''$ is the orthogonal projection w.r.t. the $L^2$ inner product defined by $\omega$ onto the $\Delta''$-harmonic space in the standard $3$-space decomposition

$$C^\infty_{p,q}(X) = \mathcal{H}^{p,q}_{\Delta}(X) \oplus \text{Im} \tilde{\partial} \oplus \text{Im} \partial^*.$$

Recall that $\Delta'' = \tilde{\partial} \partial^* + \partial^* \tilde{\partial} : C^\infty_{p,q}(X) \rightarrow C^\infty_{p,q}(X)$ is the usual $\tilde{\partial}$-Laplacian induced by $\omega$ and the above decomposition is $L^2_{\omega}$-orthogonal. It was proved in [Pop16, Theorem 1.1.] that for every $p, q \in \{0, \ldots, n\}$, the linear map

$$\mathcal{H}^{p,q}_{\Delta}(X) := \ker(\tilde{\Delta} : C^\infty_{p,q}(X) \rightarrow C^\infty_{p,q}(X)) \rightarrow E^p_{\alpha} := \bigoplus_{ii} \mathbb{C}^2,$$

is an isomorphism. This is a Hodge isomorphism showing that every double class $\{\alpha\} \in E^p_{\alpha}$ contains a unique $\tilde{\Delta}$-harmonic representative.

Claim 2.2. For every $\alpha \in C^\infty_{p,q}(X)$, the equivalence holds: $\tilde{\Delta} \alpha = 0 \iff \tilde{\Delta}(\star \tilde{\alpha}) = 0$, where $\star = \star_\omega$ is the Hodge-star operator associated with $\omega$.

Suppose for a moment that this claim has been proved. To prove non-degeneracy for the pairing (1), let $\{\alpha\} \in E^p_{\alpha}$ be an arbitrary non-zero class whose unique $\tilde{\Delta}$-harmonic representative is denoted by $\alpha$. So, $\alpha \neq 0$ and $\star \tilde{\alpha} \in \mathcal{H}^{n-p,n-q}_{\Delta}(X) \setminus \{0\}$. In particular, $\star \tilde{\alpha}$ represents an element in $E_{2}^{n-p,n-q}(X)$ and the pair $(\{\alpha\}, \{\star \tilde{\alpha}\})$ maps under (1) to $\int_X \alpha \wedge \tilde{\alpha} = \int_X |\alpha|_{\omega}^2 dV = ||\alpha||^2_{L^2_{\omega}} \neq 0$.

Since $p, q$ and $\alpha$ were arbitrary, we conclude that the pairing (1) is non-degenerate.

Proof of Claim 2.2. Since $\tilde{\Delta}$ is a sum of non-negative operators of the shape $A^*A$, we have

$$\ker \tilde{\Delta} = \ker(p'' \partial) \cap \ker(p'' \partial^*) \cap \ker \partial \cap \ker \partial^*.$$

Thus, the orthogonal $3$-space decomposition recalled above yields the following equivalence:

$$\alpha \in \ker \tilde{\Delta} \iff (i) \partial \alpha \in \text{Im} \partial \cap \text{Im} \partial^* \quad (\text{ii}) \partial^* \alpha \in \text{Im} \partial \cap \text{Im} \partial^* \quad \text{and} \quad (\text{iii}) \alpha \in \text{ker} \partial \cap \text{ker} \partial^*.$$

Let $\alpha \in \ker \tilde{\Delta}$. Since $\star : \Lambda^p q T^*X \rightarrow \Lambda^{n-q,n-p} T^*X$ is an isomorphism, the well-known identities $\star = (-1)^{p+q}$ on $(p, q)$-forms, $\partial^* = -\star \tilde{\partial} \star$ and $\partial^* = -\star \partial \star$ yield:

$$\partial \alpha = 0 \iff \partial \tilde{\alpha} = 0 \iff \partial^* (\star \tilde{\alpha}) = 0 \quad \text{and} \quad \partial^* \alpha = 0 \iff \partial^* \tilde{\alpha} = 0 \iff \partial^* (\star \tilde{\alpha}) = 0.$$

Thus, $\alpha$ satisfies condition (iii) if and only if $\star \tilde{\alpha}$ satisfies condition (iii).

Meanwhile, $\alpha$ satisfies condition (ii) if and only if there exist forms $\xi, \eta$ such that $\partial^* \alpha = \partial \xi + \partial^* \eta$. The last identity is equivalent to

$$\partial^* \tilde{\alpha} = \partial^* \xi + \partial^* \tilde{\eta} \iff -(\star \tilde{\alpha}) = \pm \partial^* (\star \xi) \pm \star (\tilde{\partial} \star \tilde{\eta}) \iff \partial^* (\star \tilde{\alpha}) = \pm \partial^* (\star \xi) \pm \partial (\star \eta).$$

Thus, $\alpha$ satisfies condition (ii) if and only if $\star \tilde{\alpha}$ satisfies condition (i).

Similarly, $\alpha$ satisfies condition (i) if and only if there exist forms $u, v$ such that $\partial \alpha = \partial u + \tilde{\partial} v$. The last identity is equivalent to

$$\partial \tilde{\alpha} = \partial \tilde{u} + \partial^* \tilde{v} \iff -\star \partial \star (\star \tilde{\alpha}) = -\star \partial \star (\star \tilde{u}) - \partial \star (\star \tilde{v}) \iff \partial^* (\star \tilde{\alpha}) = \tilde{\partial} \star (\star \tilde{u}) + \tilde{\partial} (\star \tilde{v}).$$

Thus, $\alpha$ satisfies condition (i) if and only if $\star \tilde{\alpha}$ satisfies condition (ii).

This completes the proof of Claim 2.2 and implicitly that of Theorem 2.1. \qed
2.2 Serre-type duality for the pages $r \geq 3$ of the Frölicher spectral sequence

In this section, we construct elliptic pseudo-differential operators $\overline{\Delta}_{(r)}^{(\omega)}$ associated with any given Hermitian metric $\omega$ on $X$ whose kernels are isomorphic to the spaces $E_r^{p,q}(X)$ in every bidegree $(p, q)$. This extends to arbitrary $r \in \mathbb{N}^*$ the construction performed in [Pop16] for $r = 2$. We then apply this construction to prove the existence of a (non-degenerate) duality between every page $E_1^{p,q}(X)$ and the page $E_r^{n-p,n-q}(X)$ that extends to every page in the Frölicher spectral sequence the classical Serre duality (corresponding to $r = 1$).

Let $X$ be an arbitrary compact complex $n$-dimensional manifold. Fix $r \in \mathbb{N}$ and a bidegree $(p, q)$ with $p, q \in \{0, \ldots, n\}$. A smooth $\mathbb{C}$-valued $(p, q)$-form $\alpha$ on $X$ will be said to be $E_r$-closed if it represents an $E_r$-cohomology class, denoted by $\{\alpha\}_{E_r} \in E_r^{p,q}(X)$, on the $r^{th}$ page of the Frölicher spectral sequence of $X$. Meanwhile, $\alpha$ will be said to be $E_r$-exact if it represents the zero $E_r$-cohomology class, i.e. if $\{\alpha\}_{E_r} = 0 \in E_r^{p,q}(X)$. The $\mathbb{C}$-vector space of $C^\infty$ $E_r$-closed (resp. $E_r$-exact) $(p, q)$-forms will be denoted by $Z^{p,q}(X)$ (resp. $\mathcal{C}^{p,q}(X)$). Of course, $\mathcal{C}^{p,q}(X) \subset Z^{p,q}(X)$ and $E_r^{p,q}(X) = Z^{p,q}(X)/\mathcal{C}^{p,q}(X)$.

The following statement was proved in [CFGU97]. It renders explicit the $E_r$-closedness and $E_r$-exactness conditions. In particular, it gives a more concrete description, equivalent to the more formal standard one, of the spaces $E_r^{p,q}(X)$ and the differentials $d_r$ featuring in the Frölicher spectral sequence. It was also used in [Pop19].

**Proposition 2.3.** (i) Fix $r \geq 2$. A form $\alpha \in C^\infty_{p,q}(X)$ is $E_r$-closed if and only if there exist forms $u_l \in C^\infty_{p+l,q-l}(X)$ with $l \in \{1, \ldots, r-1\}$ satisfying the following tower of $r$ equations:

\[
\begin{align*}
\bar{\partial}\alpha &= 0 \\
\partial\alpha &= \bar{\partial}u_1 \\
\partial u_1 &= \bar{\partial}u_2 \\
& \quad \vdots \\
\partial u_{r-2} &= \bar{\partial}u_{r-1}.
\end{align*}
\]

We say in this case that $\bar{\partial}\alpha = 0$ and $\partial\alpha$ runs at least $(r-1)$ times.

(ii) Fix $r \geq 2$. The map $d_r : E_r^{p,q}(X) \to E_r^{p+r,q-r+1}(X)$ acts as $d_r(\{\alpha\}_{E_r}) = \{\partial u_{r-1}\}_{E_r}$ for every $E_r$-class $\{\alpha\}_{E_r}$, any representative $\alpha$ thereof and any choice of forms $u_l$ satisfying the above tower of $E_r$-closedness equations for $\alpha$.

(iii) Fix $r \geq 2$. A form $\alpha \in C^\infty_{p,q}(X)$ is $E_r$-exact if and only if there exist forms $\zeta \in C^\infty_{p-1,q}(X)$ and $\xi \in C^\infty_{p,q-1}(X)$ such that

\[
\alpha = \partial \zeta + \bar{\partial} \xi,
\]

with $\xi$ arbitrary and $\zeta$ satisfying the following tower of $(r-1)$ equations:

\[
\begin{align*}
\bar{\partial}\zeta &= \partial v_{r-3} \\
\bar{\partial}v_{r-3} &= \partial v_{r-4} \\
& \quad \vdots \\
\bar{\partial}v_1 &= \partial v_0 \\
\bar{\partial}v_0 &= 0,
\end{align*}
\]
for some forms $v_0, \ldots, v_{r-3}$. (When $r = 2$, $\zeta_{r-2} = \zeta_0$ must be $\bar{\partial}$-closed.)

We say in this case that $\bar{\partial} \zeta$ reaches 0 in at most $(r - 1)$ steps.

(iv) The following inclusions hold in every bidegree $(p, q)$:

$$
\cdots \subset C_{r}^{p,q}(X) \subset C_{r+1}^{p,q}(X) \subset \cdots \subset Z_{r}^{p,q}(X) \subset Z_{r+1}^{p,q}(X) \subset \cdots,
$$

with $\{0\} = C_{r}^{p,q}(X) \subset C_{r+1}^{p,q}(X) = (\text{Im } \bar{\partial})^{p,q}$ and $Z_{r}^{p,q}(X) = (\text{ker } \bar{\partial})^{p,q} \subset Z_{r+1}^{p,q}(X) = C_{r+1}^{p,q}(X)$.

Proof. See [CFGU97]. □

The immediate consequence that we notice is the well-definedness of the pairing that parallels on any page of the Frölicher spectral sequence the classical Serre duality.

**Corollary 2.4.** Let $X$ be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. For every $r \in \mathbb{N}^{*}$ and every $p, q \in \{0, \ldots, n\}$, the canonical bilinear pairing

$$
E_{r}^{p,q}(X) \times E_{r}^{n-p,n-q}(X) \longrightarrow \mathbb{C}, \quad (\{\alpha\}_{E_r}, \{\beta\}_{E_r}) \mapsto \int_{X} \alpha \wedge \beta,
$$

is well defined (i.e. independent of the choices of representatives of the $E_{r}$-classes involved).

Proof. By symmetry, it suffices to prove that $\int_{X} \alpha \wedge \beta = 0$ whenever $\alpha \in C_{r}^{\infty}(X)$ is $E_{r}$-exact and $\beta \in C_{n-p,n-q}^{\infty}(X)$ is $E_{r}$-closed. By Proposition 2.3, these conditions are equivalent to

$$
\bar{\partial} \beta = 0, \quad \partial \beta = \bar{\partial} u_1, \quad \partial u_1 = \bar{\partial} u_2, \ldots, \partial u_{r-2} = \bar{\partial} u_{r-1},
$$

for some forms $u_j$ and to $\alpha = \partial \zeta + \bar{\partial} \zeta$ for some form $\zeta$ satisfying

$$
\bar{\partial} \zeta = \partial v_{r-3}, \quad \bar{\partial} v_{r-3} = \partial v_{r-4}, \ldots, \bar{\partial} v_1 = \partial v_0, \bar{\partial} v_0 = 0
$$

for some forms $v_k$. We get

$$
\int_{X} \alpha \wedge \beta = \int_{X} \partial \zeta \wedge \beta + \int_{X} \bar{\partial} \zeta \wedge \beta.
$$

Every integral on the r.h.s. above is seen to vanish by repeated integration by parts. Specifically, $\int_{X} \bar{\partial} \zeta \wedge \beta = \pm \int_{X} \xi \wedge \partial \beta = 0$ since $\bar{\partial} \beta = 0$, while for every $l \in \{1, \ldots, r - 2\}$ we have

$$
\int_{X} \partial \zeta \wedge \beta = \pm \int_{X} \zeta \wedge \partial \beta = \pm \int_{X} \zeta \wedge \bar{\partial} u_1 = \pm \int_{X} \bar{\partial} \zeta \wedge u_1 = \pm \int_{X} \partial v_{r-3} \wedge u_1
$$

$$
= \pm \int_{X} v_{r-3} \wedge \partial u_1 = \pm \int_{X} v_{r-3} \wedge \bar{\partial} u_2 = \pm \int_{X} \bar{\partial} v_{r-3} \wedge u_2 = \pm \int_{X} \partial v_{r-4} \wedge u_2
$$

$$
\vdots
$$

$$
= \pm \int_{X} v_0 \wedge \partial u_{r-2} = \pm \int_{X} v_0 \wedge \bar{\partial} u_{r-1} = \pm \int_{X} \bar{\partial} v_0 \wedge u_{r-1} = 0,
$$

9
We will now prove that the above pairing is also non-degenerate, thus defining a Serre-type duality on every page of the Frölicher spectral sequence. Much of the following preliminary discussion appeared in [Pop19, §2.2 and Appendix], so we will only recall the bare bones.

Let us fix an arbitrary Hermitian metric \( \omega \) on \( X \). For every bidegree \((p, q)\), \( \omega \)-harmonic spaces (also called \( E_r \)-harmonic spaces):

\[
\cdots \subset H_{r+1}^{p,q} \subset H_r^{p,q} \subset \cdots \subset H_1^{p,q} \subset C_{p,q}^\infty(X)
\]

were inductively constructed in [Pop17, §3.2, especially Definition 3.3. and Corollary 3.4.] such that every subspace \( H_r^{p,q} = H_r^{p,q}(X, \omega) \) is isomorphic to the corresponding space \( E_r^{p,q}(X) \) on the \( r \)-th page of the Frölicher spectral sequence.

Moreover, these spaces fit into the inductive construction described in the next

Proposition 2.5. Let \((X, \omega)\) be a compact Hermitian manifold with \( \dim \mathbb{C} X = n \).

(i) For every bidegree \((p, q)\), the space \( C_{p,q}^\infty(X) \) splits successively into mutually \( L^2_\omega \)-orthogonal subspaces as follows:

\[
C_{p,q}^\infty(X) = \text{Im} d_0 \oplus \underbrace{H_1^{p,q}}_{\text{Im } d_0^{(\omega)} \oplus H_2^{p,q} \oplus \text{Im } (d_1^{(\omega)})^*} \oplus \text{Im } d_0^*
\]

\[
\text{Im } d_1^{(\omega)} \oplus H_2^{p,q} \oplus \text{Im } (d_1^{(\omega)})^* \oplus \text{Im } d_0^*
\]

\[
\text{Im } d_{r-1}^{(\omega)} \oplus H_r^{p,q} \oplus \text{Im } (d_{r-1}^{(\omega)})^* \oplus \text{Im } d_r^{(\omega)}
\]

\[
\text{Im } d_r^{(\omega)} \oplus H_{r+1}^{p,q} \oplus \text{Im } (d_r^{(\omega)})^* \oplus \text{Im } d_{r+1}^{(\omega)}
\]

where, for \( r \in \mathbb{N}^* \), the operators \( d_r^{(\omega)} \) are defined as

\[
d_r^{(\omega)} = d_r^{(\omega)p,q} = p_r \partial D_{r-1} p_r : H_r^{p,q} \rightarrow H_r^{p+r,q-r+1}
\]

using the \( L^2_\omega \)-orthogonal projections \( p_r = p_r^{p,q} : C_{p,q}^\infty(X) \rightarrow H_r^{p,q} \) onto the \( \omega \)-harmonic spaces \( H_r^{p,q} \) and where we inductively define

\[
D_{r-1} := ((\tilde{\Delta})^{(1)})^{-1} \bar{\partial}^* \partial \ldots ((\tilde{\Delta})^{(r-1)})^{-1} \bar{\partial}^* \partial \quad \text{and} \quad D_0 = \text{Id}.
\]
(So, \( p_1 = p'' \).) See (iii) below for the inductive definition of the pseudo-differential Laplacians \( \tilde{\Delta}^{(r)} \).

Thus, the triples \((p_r, d_r^{(\omega)}, \mathcal{H}_r^{(p,q)})\) are defined by induction on \( r \in \mathbb{N}^* \): once the triple \((p_{r-1}, d_{r-1}^{(\omega)}, \mathcal{H}_{r-1}^{(p,q)})\) has been constructed for all the bidegrees \((p, q)\), it induces \( p_r \), which induces \( d_r^{(\omega)} \), which induces \( \mathcal{H}_r^{(p,q)} \) defined as the \( L^2_{\omega} \)-orthogonal complement of \( \text{Im} \, d_r^{(\omega)} \) in \( \ker \, d_r^{(\omega)} \).

The operators \( d_r^{(\omega)} \) can also be considered to be defined on the whole spaces of smooth forms:

\[
d_r^{(\omega)} = p_r \partial D_{r-1} : C^\infty_p(X) \to C^\infty_{p+r,q-r+1}(X).
\]

(ii) The above definition of \( d_r^{(\omega)} \) follows from the requirement that the following diagram be commutative:

\[
\begin{array}{ccc}
E_r^{p,q}(X) & \xrightarrow{d_r} & E_r^{p+r,q-r+1}(X) \\
\downarrow \cong & & \downarrow \cong \\
\mathcal{H}_r^{p,q} & \xrightarrow{d_r^{(\omega)} = p_r \partial D_{r-1}} & \mathcal{H}_r^{p+r,q-r+1},
\end{array}
\]

where the maps \( d_r : E_r^{p,q}(X) \to E_r^{p+r,q-r+1}(X) \) are the differentials on the \( r \)-th page of the Frölicher spectral sequence. Thus, the maps \( d_r^{(\omega)} \) are the metric realisations, at the level of the harmonic spaces, of the canonical maps \( d_r \).

(iii) For every \( r \in \mathbb{N}^* \), the adjoint of \( d_r^{(\omega)} \) is

\[
(d_r^{(\omega)})^* = p_r \partial^* D_{r-1} \partial^* p_r : \mathcal{H}_r^{p+r,q-r+1} \to \mathcal{H}_r^{p,q}.
\]

It induces the “Laplacian”

\[
\tilde{\Delta}^{(\omega)}_{(r+1)} = (d_r^{(\omega)})^* + (d_r^{(\omega)})^* d_r^{(\omega)} : \mathcal{H}_r^{p,q} \to \mathcal{H}_r^{p,q}
\]

given by the explicit formula

\[
\tilde{\Delta}^{(\omega)}_{(r+1)} = p_r \left[ (\partial D_{r-1} p_r) (\partial D_{r-1} p_r)^* + (p_r \partial D_{r-1})^* (p_r \partial D_{r-1}) + \tilde{\Delta}^{(r)} \right] p_r,
\]

which is the restriction and co-restriction to \( \mathcal{H}_r^{p,q} \) of the pseudo-differential Laplacian

\[
\tilde{\Delta}^{(r+1)} := (\partial D_{r-1} p_r) (\partial D_{r-1} p_r)^* + (p_r \partial D_{r-1})^* (p_r \partial D_{r-1}) + \tilde{\Delta}^{(r)} : C^\infty_{p,q}(X) \to C^\infty_{p,q}(X).
\]

(iv) For every \( r \in \mathbb{N}^* \) and every bidegree \((p, q)\), the following orthogonal 3-space decomposition holds:

\[
\mathcal{H}_r^{p,q} = \text{Im} \, d_r^{(\omega)} \oplus \mathcal{H}_r^{p,q} \oplus \text{Im} \, (d_r^{(\omega)})^*,
\]

where \( \ker \, d_r^{(\omega)} = \text{Im} \, d_r^{(\omega)} \oplus \mathcal{H}_r^{p,q} \). In particular, this confirms that \( \mathcal{H}_r^{p,q} \) is the orthogonal complement for the \( L^2_{\omega} \)-inner product of \( \text{Im} \, d_r^{(\omega)} \) in \( \ker \, d_r^{(\omega)} \). Moreover,

\[
\mathcal{H}_r^{p,q} = \ker \, \tilde{\Delta}^{(r+1)} = \ker \, d_r^{(\omega)} \cap \ker \, (d_r^{(\omega)})^* \simeq E_r^{p,q}(X),
\]

for every \( r \in \mathbb{N} \) and all \( p, q \in \{0, \ldots, n\} \).
Proof. The verification of the details of these statements was done in [Pop19, §2.2 and Appendix]. □

We saw in (i) of Proposition 2.3 how the $E_r$-closedness property of a differential form is characterised in explicit terms. We will now define by analogy the property of $E^*_r$-closedness when a Hermitian metric has been fixed.

**Definition 2.6.** Let $(X, \omega)$ be an $n$-dimensional compact complex Hermitian manifold. Fix $r \geq 1$ and a bidegree $(p, q)$. A form $\alpha \in C^\infty_{p,q}(X)$ is said to be $E^*_r$-closed with respect to the metric $\omega$ if and only if there exist forms $v_l \in C^\infty_{p-l,q+l}(X)$ with $l \in \{1, \ldots, r-1\}$ satisfying the following tower of $r$ equations:

\[
\bar{\partial}^* \alpha = 0 \\
\partial^* \alpha = \bar{\partial}^* v_1 \\
\partial^* v_1 = \bar{\partial}^* v_2 \\
\vdots \\
\partial^* v_{r-2} = \bar{\partial}^* v_{r-1}.
\]

We say in this case that $\bar{\partial}^* \alpha = 0$ and $\partial^* \alpha$ runs at least $(r-1)$ times.

We can now use the $E_r$-closedness and $E^*_r$-closedness properties to characterise the $H_r$-harmonicity property defined above.

**Proposition 2.7.** Let $(X, \omega)$ be an $n$-dimensional compact complex Hermitian manifold. Fix $r \geq 1$ and a bidegree $(p, q)$. For any form $\alpha \in C^\infty_{p,q}(X)$, the following equivalence holds:

\[ \alpha \in H^p_{r, q} \iff \alpha \text{ is } E_r \text{-closed and } E^*_r \text{-closed}. \]

**Proof.** We know from Proposition 2.5 that $\alpha \in H^p_{r+1}$ if and only if $\alpha \in H^p_{r}$ and $\alpha \in \ker (d^\omega_r) \cap \ker (d^\omega_r)^*$. Now, for $\alpha \in H^p_{r}$, the definition of $d^\omega_r$ shows that $\alpha \in \ker (d^\omega_r)$ if and only if $\alpha \in \ker (p_r \partial D_{r-1})$ and this last fact is equivalent to $\alpha$ being $E_{r+1}$-closed. Similarly, for $\alpha \in H^p_{r}$, the definition of $(d^\omega_r)^*$ shows that $\alpha \in \ker (d^\omega_r)^*$ if and only if $\alpha \in \ker (\partial D_{r-1}p_r)^*$ and this last fact is equivalent to $\alpha$ being $E^*_{r+1}$-closed. □

**Corollary 2.8.** In the setting of Proposition 2.7, the following equivalence holds:

\[ \alpha \text{ is } E_r \text{-closed } \iff \star \bar{\alpha} \text{ is } E^*_r \text{-closed}. \]

**Proof.** We know from (i) of Proposition 2.3 that $\alpha$ is $E_r$-closed if and only if there exist forms $u_l \in C^\infty_{p+l, q-l}(X)$ for $l = 1, \ldots, r-1$ such that

\[ (-\star \partial^\ast) \star \bar{\alpha} = 0, \ (-\star \bar{\partial}^\ast) \star \bar{\alpha} = (-\star \partial^\ast) \star \bar{u}_1, \ldots, (-\star \bar{\partial}^\ast) \star \bar{u}_{r-2} = (-\star \partial^\ast) \star \bar{u}_{r-1}. \]

Indeed, we have transformed the $E_r$-closedness condition of (i) in Proposition 2.3 by conjugating and applying the Hodge star operator several times. Since $-\star \partial^\ast = \bar{\partial}^\ast$ and $-\star \bar{\partial}^\ast = \partial^\ast$, the above
conditions are equivalent to \( \star \bar{\alpha} \) being \( E_r \)-closed (with the forms \( \star \bar{u}_t \) playing the part of the forms \( v_t \)).

An immediate (and new to our knowledge) consequence of this discussion is the analogue on every page \( E_r \) of the Frölicher spectral sequence of the classical **Serre duality**. The well-definedness was proved in Corollary 2.4. The case \( r = 1 \) is the Serre duality, while the case \( r = 2 \) was proved in Theorem 2.1.

**Corollary 2.9.** Let \( X \) be a compact complex manifold with \( \dim \mathbb{C} X = n \). For every \( r \in \mathbb{N}^\star \) and every \( p, q \in \{0, \ldots, n\} \), the canonical bilinear pairing

\[
E_r^{p,q}(X) \times E_r^{n-p,n-q}(X) \longrightarrow \mathbb{C}, \quad (\{\alpha\}_{E_r}, \{\beta\}_{E_r}) \mapsto \int_X \alpha \wedge \beta,
\]

is non-degenerate.

**Proof.** Let \( \{\alpha\}_{E_r} \in E_r^{p,q}(X) \setminus \{0\} \). If we fix an arbitrary Hermitian metric \( \omega \) on \( X \), we know from Proposition 2.5 that the associated harmonic space \( H_r^{p,q} \) is isomorphic to \( E_r^{p,q}(X) \) and that the class \( \{\alpha\}_{E_r} \) contains a (unique) representative \( \alpha \) lying in \( H_r^{p,q} \). By Proposition 2.7, this is equivalent to \( \alpha \) being both \( E_r \)-closed and \( E_{\star r} \)-closed, while by Corollary 2.8, this is further equivalent to \( \star \bar{\alpha} \) being both \( E_{\star r} \)-closed and \( E_r \)-closed, hence to \( \star \bar{\alpha} \) lying in \( H_r^{n-p,n-q} \).

In particular, \( \star \bar{\alpha} \) represents a non-zero class \( \{\star \bar{\alpha}\}_{E_r} \in E_r^{n-p,n-q}(X) \). We have

\[
(\{\alpha\}_{E_r}, \{\star \bar{\alpha}\}_{E_r}) \mapsto \int_X \alpha \wedge \star \bar{\alpha} = ||\alpha||^2 > 0,
\]

where \( || \cdot || \) stands for the \( L_2^2 \)-norm. This shows that for every non-zero class \( \{\alpha\}_{E_r} \in E_r^{p,q}(X) \), the map \( (\{\alpha\}_{E_r}, \cdot) : E_r^{n-p,n-q}(X) \longrightarrow \mathbb{C} \) does not vanish identically, proving the non-degeneracy of the pairing. \( \square \)

**Remark 2.10.** The numerical version of Cor. 2.9 was proved in [Pop17] and the present version and its method of proof were already announced and used in various works by the first and third named authors. See, e.g. [BP18, §3.4]. Subsequently, Cor 2.9 was also proved via a different method in [Ste18], reducing it to the classical Serre duality for the first page of the Frölicher spectral sequence. Quite recently, A. Milivojevic [Mil19] has also found a third proof, by yet another method.

### 3 Page-\( r \)-\( \partial \bar{\partial} \)-manifolds

In this section, we give the main definitions and some of the basic properties of the new class of manifolds that we introduce herein. Unless otherwise stated, \( X \) will stand for an \( n \)-dimensional compact complex manifold.

#### 3.1 Preliminaries

We start by recalling some well-known facts in order to fix the setup and to spell out the relationship between the Frölicher spectral sequence and the \( \mathbb{C} \)-De Rham cohomology when the latter is pure in a
sense that will be specified.

(I) For all non-negative integers \( k \leq 2n \) and \( p \leq \min\{k, n\} \), it is standard to put

\[
\mathcal{F}^pC_k^\infty(X, \mathbb{C}) := \bigoplus_{i \geq p} C_{i,k-1}^\infty(X) \subset C_k^\infty(X, \mathbb{C})
\]

and get a filtration of \( C_k^\infty(X, \mathbb{C}) \) for every \( k \):

\[
\{0\} \subset \cdots \subset \mathcal{F}^{p+1}C_k^\infty(X, \mathbb{C}) \subset \mathcal{F}^pC_k^\infty(X, \mathbb{C}) \subset \cdots \subset C_k^\infty(X, \mathbb{C}).
\]

It is equally standard to put

\[
\mathcal{F}^pH_{DR}^k(X, \mathbb{C}) := \frac{\mathcal{F}^pC_k^\infty(X, \mathbb{C}) \cap \ker d}{\mathcal{F}^pC_k^\infty(X, \mathbb{C}) \cap \text{Im } d} \subset H_{DR}^k(X, \mathbb{C}),
\]

the subspace of De Rham cohomology classes of degree \( k \) that are representable by forms in \( \mathcal{F}^pC_k^\infty(X, \mathbb{C}) \). It can be easily seen that \( \mathcal{F}^pH_{DR}^k(X, \mathbb{C}) \) coincides with the image of the following canonical map induced by the identity:

\[
\frac{\ker (d : \mathcal{F}^pC_k^\infty(X, \mathbb{C}) \to \mathcal{F}^pC_{k+1}^\infty(X, \mathbb{C}))}{\text{Im } (d : \mathcal{F}^pC_{k-1}^\infty(X, \mathbb{C}) \to \mathcal{F}^pC_k^\infty(X, \mathbb{C}))} \to H_{DR}^k(X, \mathbb{C}),
\]

which, for every form \( u = \sum_{i \geq p} u^{i,k-i} \in \ker d \cap \mathcal{F}^pC_k^\infty(X, \mathbb{C}) \), maps the class of \( u \) modulo \( \text{Im } d \) to the De Rham class \( \{\sum_{i \geq p} u^{i,k-i}\}_{DR} \) of \( u \).

It is standard that in this way we get a filtration of \( H_{DR}^k(X, \mathbb{C}) \) for every \( k \):

\[
\{0\} \subset \cdots \subset \mathcal{F}^{p+1}H_{DR}^k(X, \mathbb{C}) \subset \mathcal{F}^pH_{DR}^k(X, \mathbb{C}) \subset \cdots \subset H_{DR}^k(X, \mathbb{C}).
\]

Let us now recall the following standard result (see e.g. [Dem97, chapter IV, Theorem 10.6]).

**Theorem 3.1.** Let \( X \) be an \( n \)-dimensional compact complex manifold. For every \( p, q \in \{0, \ldots, n\} \), the vector space \( E_{\infty}^{p,q}(X) \) of type \( (p, q) \) on the degenerating page of the Frölicher spectral sequence of \( X \) is canonically isomorphic to the relevant graded module associated with the filtration (5):

\[
E_{\infty}^{p,q}(X) \simeq G_pH_{DR}^{p+q}(X, \mathbb{C}) := \frac{F^pH_{DR}^{p+q}(X, \mathbb{C})}{F^{p+1}H_{DR}^{p+q}(X, \mathbb{C})},
\]

where the canonical isomorphism \( G_pH_{DR}^{p+q}(X, \mathbb{C}) \simeq E_{\infty}^{p,q}(X) \) is induced by the projection

\[
F^pH_{DR}^{p+q}(X, \mathbb{C}) \ni \{\sum_{i \geq p} u^{i,p+q-i}\}_{DR} \mapsto \{u^{p,q}\}_{E_{\infty}} \in E_{\infty}^{p,q}(X).
\]

The following statement is immediate to prove.
Lemma 3.2. The following relations hold:

\[ C_k^\infty(X, \mathbb{C}) = \mathcal{F}^p C_k^\infty(X, \mathbb{C}) \oplus \mathcal{F}^{k-p+1} C_k^\infty(X, \mathbb{C}) \quad \text{for all } 0 \leq p \leq \min\{k, n\}; \quad (6) \]

\[ C_{p,q}^\infty(X) = \mathcal{F}^p C_k^\infty(X, \mathbb{C}) \cap \mathcal{F}^q C_k^\infty(X, \mathbb{C}) \quad \text{for all } p, q \text{ such that } p + q = k. \quad (7) \]

(II) On the other hand, for all \( p, q \in \{0, \ldots, n\} \), let us consider the following space of De Rham cohomology classes of degree \( p + q \) that are representable by pure-type \((p, q)\)-forms:

\[ H_{DR}^{p,q}(X) := \left\{ \alpha \in H_{DR}^{p,q}(X, \mathbb{C}) \mid \exists \beta \in C_{p,q}^\infty(X) \cap c \right\} \subset H_{DR}^{p,q}(X, \mathbb{C}). \]

This definition makes it obvious that the analogue of the Hodge symmetry for the spaces \( H_{DR}^{p,q}(X) \) always holds. In other words, the conjugation induces an isomorphism

\[ H_{DR}^{p,q}(X) \in \{ \alpha \}_DR \mapsto \{\overline{\alpha}\}_DR \in H_{DR}^{q,p}(X) \quad \text{for all } 0 \leq p, q \leq n. \quad (8) \]

The following analogue in cohomology of identity (7), resp. of one of the inclusions defining the filtration (3) of \( C_k^\infty(X, \mathbb{C}) \), can be immediately proved to hold.

Lemma 3.3. The following relations hold:

\[ H_{DR}^{p,q}(X) = F^p H_{DR}^k(X, \mathbb{C}) \cap F^q H_{DR}^k(X, \mathbb{C}) \quad \text{for all } p, q \text{ such that } p + q = k; \quad (9) \]

\[ H_{DR}^{i,k-i}(X) \subset F^p H_{DR}^k(X, \mathbb{C}) \quad \text{for all } i \geq p \text{ and all } p \leq k. \quad (10) \]

Proof. Everything is obvious, except perhaps the inclusion “\( \supset \)” in (9) which can be proved as follows. Let \( \{\alpha\}_DR = \{\beta\}_DR \in F^p H_{DR}^k(X, \mathbb{C}) \cap F^q H_{DR}^k(X, \mathbb{C}) \) with \( \alpha = \sum_{i \geq p} \alpha_i^{i,k-i} \in \mathcal{F}^p C_k^\infty(X, \mathbb{C}) \) and \( \beta = \sum_{s \leq p} \beta_s^{s,k-s} \in \mathcal{F}^q C_k^\infty(X, \mathbb{C}) \). Since \( \alpha \) and \( \beta \) are De Rham-cohomologous, there exists a form \( \sigma \in C_{k-1}^\infty(X, \mathbb{C}) \) such that \( \alpha - \beta = d\sigma \). This identity implies, after equating the terms with a holomorphic degree \( p \) on either side, the second identity below:

\[ \alpha - \alpha^{p,q} = \sum_{i > p} \alpha_i^{i,k-i} = d(\sum_{j \geq p} \sigma_j^{j,k-1-j}) - \overline{d}\sigma^{p,q-1}, \]

which, in turn, implies that \( \{\alpha\}_DR = \{\alpha^{p,q} - \overline{d}\sigma^{p,q-1}\}_DR \). Since \( \alpha^{p,q} - \overline{d}\sigma^{p,q-1} \) is a \((p, q)\)-form, we get \( \{\alpha\}_DR \in H_{DR}^{p,q}(X) \) and we are done. \( \square \)

Note that, with no assumption on \( X \), the subspaces \( H_{DR}^{i,k-i}(X) \) may have non-zero mutual intersections inside \( H_{DR}^k(X, \mathbb{C}) \), so they may not sit in a direct sum.

Let us now introduce the following

Definition 3.4. Let \( X \) be an \( n \)-dimensional compact complex manifold. The De Rham cohomology of \( X \) is said to be pure if

\[ H_{DR}^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H_{DR}^{p,q}(X) \quad \text{for all } k \in \{0, \ldots, 2n\}. \]
Note that the above definition requires all the subspaces $H_{DR}^{p,q}(X)$ of $H_{DR}^k(X, \mathbb{C})$ with $p + q = k$ to form a direct sum and to add up to the total space $H_{DR}^k(X, \mathbb{C})$.

**Note on terminology 3.5.** Some authors call this property **complex-$C^\infty$-pure-and-full** in degree $k$ (cf. [LZ09]). It was remarked in [AT11] that the **complex-$C^\infty$-full property** in degree $k$ (i.e. the sum of the $H_{DR}^{p,q}(X)$’s is not necessarily direct but it fills out $H_{DR}^k(X, \mathbb{C})$) implies the **complex-$C^\infty$-pure property in degree** $(2n - k)$ (i.e. the sum of the $H_{DR}^{p,q}(X)$’s is direct but it may not fill out $H_{DR}^{2n-k}(X, \mathbb{C})$). We will show further down that the converse is also true, i.e. the **complex-$C^\infty$-pure property in degree** $k$ implies the **complex-$C^\infty$-full property in degree** $(2n - k)$. Therefore, compact complex manifolds satisfying either the **complex-$C^\infty$-full property in every degree** $k$ or the **complex-$C^\infty$-pure property in every degree** $k$ are of pure De Rham cohomology in the sense of our Definition 3.4.

**Proposition 3.6.** Suppose $X$ is an $n$-dimensional compact complex manifold whose De Rham cohomology is pure. Then

$$F_p H_{DR}^k(X, \mathbb{C}) = \bigoplus_{i \geq p} H_{DR}^{i,k-i}(X) \quad \text{for all} \quad p \leq k. \quad (11)$$

In particular, the spaces $E_{DR}^{p,q}(X)$ in the Frölicher spectral sequence of $X$ are given by

$$E_{DR}^{p,q}(X) \simeq H_{DR}^{p,q}(X) \quad \text{for all} \quad p, q \in \{0, \ldots, n\}, \quad (12)$$

where $\simeq$ stands for the canonical isomorphism induced by the identity.

**Proof.** Inclusion $\supseteq$ in (11) follows at once from (10) and from the De Rham purity assumption.

To prove inclusion $\subseteq$ in (11), let $\{\alpha\} = F_p H_{DR}^k(X, \mathbb{C})$ with $\alpha = \sum_{r \geq p} \alpha^{r,k-r} \in \ker d$. Since $F_p H_{DR}^k(X, \mathbb{C}) \subset H_{DR}^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H_{DR}^{p,q}(X)$ (the last identity being due to the purity assumption), there exist pure-type $d$-closed forms $\beta^{r,k-r}$ such that $\{\alpha\} = \{\sum_{0 \leq r \leq k} \beta^{r,k-r}\}_D$. Hence, there exists a $(k-1)$-form $\sigma$ such that $\alpha - \sum_{0 \leq r \leq k} \beta^{r,k-r} = -d\sigma$, which amounts to

$$\alpha^{r,k-r} - \beta^{r,k-r} + d\sigma^{r-1,k-r} + \bar{\partial}\sigma^{r,k-r} = 0, \quad r \in \{0, \ldots, k\},$$

with the understanding that $\sigma^{r,k-r} = 0$ whenever $r > p$.

Therefore, $\beta^{r,k-r} = \partial\sigma^{r-1,k-r} + \bar{\partial}\sigma^{r,k-r}$ whenever $r < p$. Since every $\beta^{r,k-r}$ is $d$-closed (hence also $\partial$- and $\bar{\partial}$-closed), we infer that $\sigma^{r-1,k-r}$ and $\sigma^{r,k-r}$ are $\partial\bar{\partial}$-closed for every $r < p$. Hence

$$\sum_{r=0}^{k} \beta^{r,k-r} - d((\sum_{r<p} \sigma^{r,k-r-1}) = \sum_{r<p} (\beta^{r,k-r} - \partial\sigma^{r-1,k-r} - \bar{\partial}\sigma^{r,k-r}) - \partial\sigma^{p-1,k-p} + \sum_{r \geq p} \beta^{r,k-r} = \sum_{r \geq p} \beta^{r,k-r} - \partial\sigma^{p-1,k-p}. \quad (13)$$

Note that from the identity $\beta^{p-1,k-p+1} = \partial\sigma^{p-2,k-p+1} + \bar{\partial}\sigma^{p-1,k-p}$ and the $d$-closedness of $\beta^{p-1,k-p+1}$ we infer that $\partial\sigma^{p-1,k-p} \in \ker d$.

Thus, (13) shows that the $k$-form $\sum_{r \geq p} \beta^{r,k-r} - \partial\sigma^{p-1,k-p} \in F_p C_k^{\infty}(X, \mathbb{C})$, whose all pure-type components are $d$-closed, is De Rham-cohomologous to $\sum_{0 \leq r \leq k} \beta^{r,k-r}$, hence to $\alpha$. Consequently, we have

$$\{\alpha\} = \left\{ \sum_{r \geq p} \beta^{r,k-r} - \partial\sigma^{p-1,k-p} \right\}_{DR} \in \bigoplus_{i \geq p} H^{i,k-i}_{DR}(X).$$
The proof of (11) is complete.

Identity (12) follows at once from (11) and from Theorem 3.1. □

We end these preliminaries by noticing an immediate consequence of a standard fact.

**Observation 3.7.** Let $X$ be a compact complex manifold $X$ with $\dim\mathbb{C} X = n$. Then, the following inequality holds between the dimensions of the Bott-Chern cohomology space $H^{p,q}_{BC}(X)$ of $X$ and of $H^{p,q}_{DR}(X)$:

$$h^{p,q}_{BC} \geq h^{p,q}_{DR} \quad \text{for all } 0 \leq p, q \leq n. \quad (14)$$

Moreover, if the De Rham cohomology of $X$ is pure, then

$$\sum_{p+q=k} h^{p,q}_{BC} \geq b_k \quad \text{for all } 0 \leq k \leq 2n, \quad (15)$$

where $b_k := \dim\mathbb{C} H^k_{DR}(X, \mathbb{C})$ is the $k$-th Betti number of $X$.

**Proof.** It is standard (and immediate to check) that, for every bidegree $(p, q)$, the canonical map $H^{p,q}_{BC}(X) \ni \{\alpha\}_{BC} \mapsto \{\alpha\}_{DR} \in H^{p,q}_{DR}(X, \mathbb{C})$ is well defined (i.e. independent of the choice of representative $\alpha$ of the Bott-Chern class $\{\alpha\}_{BC}$). This map need not be either injective, or surjective, but its image is, obviously, $H^{p,q}_{DR}(X)$. Hence inequality (14).

Inequality (15) follows at once from (14) and from the De Rham purity assumption. □

### 3.2 Definition of page-$r$-$\bar{\partial}$-manifolds

Recall that $X$ is a fixed $n$-dimensional compact complex manifold and $E^{p,q}_r(X)$ stands for the space of bidegree $(p, q)$ on the $r$-th page of the Frölicher spectral sequence of $X$.

**Definition 3.8.** Fix $r \in \mathbb{N}^*$ and $k \in \{0, \ldots, 2n\}$. We say that the identity induces an isomorphism between $\bigoplus_{p+q=k} E^{p,q}_r(X)$ and $H^k_{DR}(X, \mathbb{C})$ if the following two conditions are satisfied:

(a) for every bidegree $(p, q)$ with $p + q = k$, every class $\{\alpha^{p,q}\}_{E_r} \in E^{p,q}_r(X)$ contains a $d$-closed representative $\alpha^{p,q};$

(b) the linear map

$$\bigoplus_{p+q=k} E^{p,q}_r(X) \ni \sum_{p+q=k} \{\alpha^{p,q}\}_{E_r} \mapsto \left\{ \sum_{p+q=k} \alpha^{p,q} \right\}_{DR} \in H^k_{DR}(X, \mathbb{C})$$

is well-defined (in the sense that it does not depend on the choices of $d$-closed representatives $\alpha^{p,q}$ of the classes $\{\alpha^{p,q}\}_{E_r}$) and bijective.

Moreover, if, for a fixed $r \in \mathbb{N}^*$, the identity induces an isomorphism $\bigoplus_{p+q=k} E^{p,q}_r(X) \simeq H^k_{DR}(X, \mathbb{C})$ for every $k \in \{0, \ldots, 2n\}$, we say that the manifold $X$ has the $E_r$-Hodge Decomposition property.
Note that whenever the identity induces a well-defined (not necessarily injective) linear map $E_r^{p,q}(X) \to H^{k}_{DR}(X, \mathbb{C})$, the image of this map is $H^{p,q}_{DR}(X)$. Indeed, one inclusion is obvious. The reverse inclusion follows from the observation that any $d$-closed $(p, q)$-form defines an $E_r$-cohomology class (i.e. it is $E_r$-closed in the terminology of [Pop19]). Further note that whenever $X$ has the $E_r$-Hodge Decomposition property, the Frölicher spectral sequence of $X$ degenerates at $E_r$ (at the latest).

**Definition 3.9.** Fix $r \in \mathbb{N}^*$ and $p, q \in \{0, \ldots, n\}$. We say that the conjugation induces an isomorphism between $E_r^{p,q}(X)$ and the conjugate of $E_r^{p,q}(X)$ if the following two conditions are satisfied:

(a) every class $\{\alpha^{p,q}\}_{E_r} \in E_r^{p,q}(X)$ contains a $d$-closed representative $\alpha^{p,q}$;

(b) the linear map

$$E_r^{p,q}(X) \ni \{\alpha^{p,q}\}_{E_r} \mapsto \overline{\{\alpha^{p,q}\}_{E_r}} \in E_r^{\overline{p},\overline{q}}(X)$$

is well-defined (in the sense that it does not depend on the choice of $d$-closed representative $\alpha^{p,q}$ of the class $\{\alpha^{p,q}\}_{E_r}$) and bijective.

Moreover, if, for a fixed $r \in \mathbb{N}^*$, the conjugation induces an isomorphism $E_r^{p,q}(X) \simeq E_r^{\overline{p},\overline{q}}(X)$ for every $p, q \in \{0, \ldots, n\}$, we say that the manifold $X$ has the $E_r$-Hodge Symmetry property.

We shall now see that the $E_r$-Hodge Decomposition property implies the $E_r$-Hodge Symmetry property. This follows from the characterisation of the former property.

**Theorem and Definition 3.10.** Let $X$ be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. Fix an arbitrary $r \in \mathbb{N}^*$. Then, the following two conditions are equivalent:

(i) $X$ has the $E_r$-Hodge Decomposition property;

(ii) the Frölicher spectral sequence of $X$ degenerates at $E_r$ (we will denote this by $E_r(X) = E_\infty(X)$) and the De Rham cohomology of $X$ is pure.

A compact complex manifold $X$ that satisfies any of the equivalent conditions (i) and (ii) is said to be a page-$(r-1)$-$\partial\bar{\partial}$-manifold.

**Proof.** (i) $\implies$ (ii) We have already noticed that the $E_r$-Hodge Decomposition property implies $E_r(X) = E_\infty(X)$ and that the image of each $E_r^{p,q}(X)$ in $H_{DR}^{p+q}(X, \mathbb{C})$ under the map induced by the identity is $H_{DR}^{p+q}(X)$. We get (iii).

(ii) $\implies$ (i) Since the De Rham cohomology of $X$ is supposed pure, we know from Proposition 3.6 that $E_\infty^{p,q}(X) \simeq H_{DR}^{p,q}(X)$ (isomorphism induced by the identity) for all bidegrees $(p, q)$. On the other hand, $E_\infty^{p,q}(X) = E_r^{p,q}(X)$ for all bidegrees $(p, q)$ since we are assuming that $E_r(X) = E_\infty(X)$. Combined with the De Rham purity assumption, these facts imply that $X$ has the $E_r$-Hodge Decomposition property.

**Corollary 3.11.** Any page-$(r-1)$-$\partial\bar{\partial}$-manifold has the $E_r$-Hodge Symmetry property.

**Proof.** We have already noticed in (8) that the conjugation (trivially) induces an isomorphism between any space $H_{DR}^{p,q}(X)$ and the conjugate of $H_{DR}^{\overline{p},\overline{q}}(X)$. Meanwhile, we have seen that the page-$(r-1)$-$\partial\bar{\partial}$-assumption implies that the identity induces an isomorphism between any space $E_r^{p,q}(X)$
and $H^{p,q}_{DR}(X)$. Hence, the conjugation induces an isomorphism between any space $E^{p,q}_r(X)$ and the conjugate of $E^{p,q}_r(X)$. \hfill \Box

Another obvious consequence of (ii) of Theorem and Definition 3.10 is that the page-$r$-$\partial \bar{\partial}$-property becomes weaker and weaker as $r$ increases.

**Corollary 3.12.** Let $X$ be a compact complex manifold. Then, for every $r \in \mathbb{N}^*$, the following implication holds:

$$X \text{ is a page-$r$-$\partial \bar{\partial}$-manifold} \implies X \text{ is a page-$(r+1)$-$\partial \bar{\partial}$-manifold}.$$ 

Indeed, the purity of the De Rham cohomology is independent of $r$, while the property $E_r(X) = E_\infty(X)$ obviously implies $E_{r+1}(X) = E_\infty(X)$ for every $r \in \mathbb{N}$.

### 3.3 Characterisation in terms of squares and zigzags

The goal of this section is to relate the page-$r$-$\partial \bar{\partial}$-property to structural results about double complexes. Specifically, we work here with arbitrary double complexes, i.e. bigraded vector spaces $A = \bigoplus_{p,q \in \mathbb{Z}} A^{p,q}$ with endomorphisms $\partial_1, \partial_2$ of bidegrees $(1,0)$, resp. $(0,1)$, satisfying $d^2 = 0$ for $d = \partial_1 + \partial_2$. This degree of generality has the advantage of emphasising which aspects of the theory are purely algebraic. Even if one is only interested in the complex $A_X := (C^\infty_{p,q}(X), \partial, \bar{\partial})$ of $\mathbb{C}$-valued forms on a complex manifold $X$, in the more general setting one can consider certain finite-dimensional subcomplexes on an equal footing.

There are now two Fröhlicher-style spectral sequences, starting from column, i.e. $(\partial_2)$- resp. row, i.e. $(\partial_1)$-cohomology and converging to the total (De Rham) cohomology of $(A, d)$. We denote them by

$$iE^{p,q}_r(A) \implies (H^{p+q}_r(A), F_i) \quad i = 1, 2.$$ 

In the case $A = A_X$, the case $i = 1$ is the Fröhlicher spectral sequence and $i = 2$ its conjugate.

The following is a minor extension to general double complexes of the definition (based on its second characterisation) of the page-$(r-1)$-$\partial \bar{\partial}$-property of manifolds.

**Definition 3.13.** A double complex $A$ is said to satisfy the page-$(r-1)$-$\partial_1 \partial_2$-property if both Fröhlicher spectral sequences degenerate at page $r$ and the De Rham cohomology is pure.

Just as before, one can also see that this property is equivalent to the statement that for $i = 1, 2$, every $iE^{p,q}_r(X)$-class contains a $(\partial_1 + \partial_2)$-closed representative and the corresponding map

$$\bigoplus_{p+q=k} iE^{p,q}_r(A) \rightarrow H^k_{DR}(A)$$

induced by the identity is well-defined and bijective.

The following observation will motivate the subsequent considerations.

**Observation 3.14.** The Fröhlicher spectral sequences, as well as $H_{DR}, H_A$ and $H_{BC}$, are compatible with direct sums. In particular, if $A = B \oplus C$, then $A$ satisfies the page-$r$-$\partial_1 \partial_2$-property if and only if $B$ and $C$ do.
Recall that a (nonzero) double complex $A$ is called \textbf{indecomposable} if there exists no nontrivial decomposition $A = B \oplus C$ into subcomplexes $B, C$, while $A$ is called \textbf{bounded} if $A^{p,q} \neq 0$ for only finitely many bidegrees $(p,q)$.

\textbf{Theorem 3.15.} ([Ste18, KQ19]) For every bounded double complex over a field $K$, there exists an isomorphism

$$A \cong \bigoplus_C C^{\oplus \text{mult}_C(A)},$$

where $C$ runs over a set of representatives for the isomorphism classes of bounded \textbf{indecomposable} double complexes and $\text{mult}_C(A)$ are cardinal numbers uniquely determined by $A$.

Moreover, each bounded \textbf{indecomposable} double complex is isomorphic to a complex of one of the following types:

\begin{itemize}
  \item \textbf{square}: a double complex generated by a single pure-$(p,q)$-type element $a$ in a given bidegree with no further relations:
    $$\langle \partial_2 a \rangle \longrightarrow \langle \partial_2 \partial_1 a \rangle$$
    $$\uparrow \hspace{1cm} \uparrow$$
    $$\langle a \rangle \longrightarrow \langle \partial_1 a \rangle.$$
  
  \item \textbf{even-length zigzag of type 1 and length $2l$}. This is a complex generated by elements $a_1, \ldots, a_l$ and their differentials such that $\partial_2 a_1 = 0$ and $\partial_1 a_1 = -\partial_2 a_2$, $\partial_1 a_2 = -\partial_2 a_3$, $\ldots$, $\partial_1 a_{l-1} = -\partial_2 a_l$, $\partial_1 a_l \neq 0$. It is of the shape:
    $$\langle a_1 \rangle \longrightarrow \langle \partial_1 a_1 \rangle$$
    $$\uparrow$$
    $$\langle a_2 \rangle \longrightarrow \ldots$$
    $$\uparrow$$
    $$\langle a_l \rangle \longrightarrow \langle \partial_1 a_l \rangle.$$
\end{itemize}

Here, as in all the following examples, the length of a zigzag is the number of its vertices.
• even-length zigzag of type $2$ and length $2l$. This is a complex of the shape:

\[
\langle \partial_2 a_1 \rangle \\
\uparrow \\
\langle a_1 \rangle \rightarrow \langle \partial_1 a_1 \rangle \\
\uparrow \\
\langle a_2 \rangle \rightarrow \ldots \\
\uparrow \\
\langle a_l \rangle.
\]

• odd-length zigzag of type $M$ and length $2l + 1$. This is a complex generated by elements $a_1, \ldots, a_{l+1}$ such that $\partial_1 a_i = -\partial_2 a_{i+1}$, $\partial_2 a_1 = 0$ and $\partial_1 a_{l+1} = 0$. It has the shape:

\[
\langle a_1 \rangle \rightarrow \langle \partial_1 a_1 \rangle \\
\uparrow \\
\langle a_2 \rangle \rightarrow \ldots \\
\uparrow \\
\langle a_{l+1} \rangle.
\]

The special case where $l = 0$ is also called a dot.

• odd-length zigzag of type $L$ and length $2l + 1$ ($l > 0$). This is a complex generated by elements $a_1, \ldots, a_l$ such that both $\partial_2 a_1 \neq 0 \neq \partial_1 a_l$ and $\partial_1 a_i = -\partial_2 a_{i+1}$. It has the shape:

\[
\langle \partial_2 a_1 \rangle \\
\uparrow \\
\langle a_1 \rangle \rightarrow \langle \partial_1 a_1 \rangle \\
\ldots \\
\langle \partial_2 a_l \rangle \\
\uparrow \\
\langle a_l \rangle \rightarrow \langle \partial_1 a_l \rangle.
\]
It is a useful exercise to work out which indecomposable complexes satisfy the page-$r$-$∂_1∂_2$-property. Doing it and combining it with Observation 3.14, one gets

**Theorem 3.16.** Let $A$ be a bounded double complex over a field $K$. The following are equivalent:

1. $A$ satisfies the page-$r$-$∂_1∂_2$-property.

2. There exists an isomorphism between $A$ and a direct sum of squares, even-length zigzags of length $\leq 2r$ and odd-length zigzags of length one (i.e. dots).

**Proof.** It follows at once from [Ste18, Thm C, Prop. 6, Cor. 7] as pointed out above.\[\square\]

**Remark 3.17.** This theorem also gives a quick alternative proof to Prop. 3.25 (equivalence of page-0-$∂_1∂_2$ with the usual $∂_1∂_2$-property) of the next subsection.

Indeed, the page-0-$∂_1∂_2$-property means that there is a decomposition of $A$ into squares and dots. Obviously, both satisfy the usual $∂_1∂_2$-property. Conversely, in any zigzag of length $\geq 2$, there is a closed element (‘form’) of pure type, which is $∂_1$- or $∂_2$-exact, but no nonzero element in a zigzag is $∂_1∂_2$-exact. Hence, if $A$ satisfies the usual $∂_1∂_2$-property, in any decomposition of $A$ into elementary complexes only squares and length-one zigzags can occur.

**Definition 3.18.** A map $A \longrightarrow B$ of double complexes is an $E_r$-isomorphism if $iE_r(f)$ is an isomorphism for $i \in \{1, 2\}$.

One writes $A \simeq_r B$ if there exist such an $E_r$-isomorphism. The usefulness of this notion stems from the following

**Lemma 3.19.** ([Ste18, Prop. 12]) If $H$ is a linear functor from the category of double complexes to the category of vector spaces which maps squares and even-length zigzags of length $\leq 2r$ to 0, then $H(f)$ is an isomorphism for any $E_r$-isomorphism $f$.

**Lemma 3.20.** ([Ste18, Prop. 11]) For two double complexes $A, B$ one has $A \simeq_1 B$ if and only if ‘the same zigzags occur in $A$ and $B’$, i.e. $\text{mult}_Z(A) = \text{mult}_Z(B)$ for all zigzags $Z$.

**Example 3.21.** Examples of functors $H$ satisfying the hypotheses of Lemma 3.19 are provided by $H_{DR}, H^{p,q}_{BC}, E^{p,q}_r$, as well as by $E^{p,q}_{r,BC}$ and $E^{p,q}_{r,A}$ that will be introduced further down. This, together with the following duality Lemma, opens up a second way of proving duality statements for these cohomologies, but we will not pursue this further here.

By their explicit description above, one sees that an indecomposable double complex $C$ is determined up to isomorphism by its shape $S(C) = \{(p, q) \in \mathbb{Z}^2 \mid C^{p,q} \neq 0\}$. Abusing notation slightly, it is sometimes convenient to write $\text{mult}_S(A)$ instead of $\text{mult}_C(A)$, when $S = S(C)$.

We will need the following duality results in the special case $A = A_X$, which follow from the real structure and the Serre duality.

**Lemma 3.22.** ([Ste18, Ch. 4]) Let $A = A_X$ for a compact complex manifold $X$ and define the conjugate complex by $\bar{A}^{p,q} = A^{p,q}$ and the dual complex $DA$ by $DA^{p,q} = \text{Hom}(A^{n-p,n-q}, \mathbb{C})$, for all $p, q$.  

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Then, conjugation $\omega \mapsto \overline{\omega}$ and integration $\omega \mapsto \int_X \omega \wedge -$ define an isomorphism, resp. an $E_1$-isomorphism: $A \cong \bar{A}$, resp. $A \to \bar{D}A$.

In particular, the set of zigzags occurring in $A_X$ is symmetric under reflection along the diagonal and the antidiagonal. More precisely, for any zigzag shape $S$, $\text{mult}_S(A) = \text{mult}_{rS}(A) = \text{mult}_{dS}(A)$, where $rS = \{(p, q) \in \mathbb{Z}^2 \mid (q, p) \in S\}$ and $dS := \{(p, q) \in \mathbb{Z}^2 \mid (n-p, n-q) \in S\}$.

As a consequence, we obtain

**Proposition 3.23.** Fix arbitrary integers $0 \leq k \leq 2n$. A compact complex manifold $X$ of dimension $n$ satisfies the complex-$C^\infty$-pure property in degree $k$ if and only if it satisfies the complex-$C^\infty$-full property in degree $2n-k$.

**Proof.** Let $Z$ be a zigzag with $H^k_{DR}(Z) \neq 0$. The sum of the subspaces $H^{p,q}_{DR}(Z)$ with $p+q = k$ is not direct if and only if $Z$ is of odd length and of type $L$. Meanwhile, the sum of the subspaces $H^{p,q}_{DR}(Z)$ with $p+q = k$ is strictly contained in $H^k_{DR}(Z)$ if and only if $Z$ is of odd length $> 1$ (i.e. not a dot) and of type $M$. (See [Ste18, Prop. 6, Cor. 7] for both of these statements.)

Hence, $X$ is complex-$C^\infty$-pure in degree $k$ if and only if $\text{mult}_Z(A_X) = 0$ for all odd zigzags $Z$ of type $L$ with $H^k_{DR}(Z) \neq 0$ and $X$ is complex $C^\infty$-full in degree $k$ if and only if $\text{mult}_Z(A_X) = 0$ for all odd zigzags $Z$ of type $M$ and length $> 1$ with $H^k_{DR}(Z) \neq 0$.

The result then follows from Lemma 3.22 and Lemma 3.20 since zigzags of type $L$ and those of type $M$ and length greater than 1 are exchanged when forming the dual complex. \hfill $\square$

**Corollary 3.24.** For a compact complex manifold $X$, the following statements are equivalent:

1. $X$ satisfies the complex-$C^\infty$-pure property in all degrees;
2. $X$ satisfies the complex-$C^\infty$-full property in all degrees;
3. The De Rham cohomology of $X$ is pure (in the sense of Definition 3.4).

### 3.4 Examples of page-$r$-$\partial\bar{\partial}$-manifolds and counterexamples

We shall organise our examples in several classes, each flagged by a specific heading.

#### 3.4.1 A general fact

The first observation is the following rewording of (5.21) in [DGMS75].

**Proposition 3.25.** For any compact complex manifold $X$, the following equivalence holds:

$$X \text{ is a } \partial\bar{\partial} \text{-manifold} \iff X \text{ is a page-0-}\partial\bar{\partial} \text{-manifold}.$$  

**Proof.** The implication “$\implies$” is standard and its verification is left to the reader.

To prove the implication “$\impliedby$”, suppose that $X$ is a page-0-$\partial\bar{\partial}$-manifold and fix a $d$-closed $(p, q)$-form $\alpha$ for an arbitrary bidegree $(p, q)$. Put $k = p + q$. We will expand on the arguments in the proof of the implication (iii) $\implies$ (i) of Proposition (5.17) of [DGMS75].

- Let us first prove the equivalence:
  $$\alpha \in \text{Im } \partial \iff \alpha \in \text{Im } d.$$
Since \( \alpha \) is \( d \)-closed and of pure type, \( \alpha \) is also \( \bar{\partial} \)-closed. Thus, \( \alpha \) defines classes in both \( E_1^{p,q}(X) \) and \( H_{DR}^{p,q}(X) \). Meanwhile, \( X \) has the \( E_1 \)-Hodge Decomposition property (by the page-0-\( \partial \bar{\partial} \)-assumption), so, in particular, the identity induces a linear injection \( E_1^{p,q}(X) \hookrightarrow H_{DR}^{p,q}(X) \) whose image is \( H_{DR}^{p,q}(X) \). Therefore, \( \{ \alpha \}_{E_1} = 0 \) implies \( \{ \alpha \}_{DR} = 0 \) (because the image of 0 under a linear map is 0) and \( \{ \alpha \}_{DR} = 0 \) implies \( \{ \alpha \}_{E_1} = 0 \) (by injectivity of this map). This proves the above equivalence.

Since the above equivalence has been proved in every bidegree, by conjugation we also get the equivalence \( u \in \text{Im} \partial \iff u \in \text{Im} d \) in every bidegree for every \( d \)-closed pure-type form \( u \).

- Let us now prove the equivalence:

\[
\alpha \in \text{Im} (\partial \bar{\partial}) \iff \alpha \in \text{Im} \bar{\partial}.
\]

The implication \( \implies \) being trivial, we will prove the implication \( \impliedby \). Suppose there exists \( \beta \in C_{p,q-1}^\infty(X) \) such that \( \alpha = \bar{\partial} \beta \). Then \( \alpha - d \beta = -\partial \beta \in C_{p+1,q-1}^\infty(X) \), hence

\[
\bigoplus_{i \leq p} H_{DR}^{i,k-i}(X) = F^q H_{DR}^k(X, \mathbb{C}) \in \{ \alpha \}_{DR} = \{ -\partial \beta \}_{DR} \in F^{p+1} H_{DR}^k(X, \mathbb{C}) = \bigoplus_{i \geq p+1} H_{DR}^{i,k-i}(X),
\]

where the De Rham purity assumption has been used. Since \( \bigoplus_{i \geq p+1} H_{DR}^{i,k-i}(X) \cap \bigoplus_{i \leq p} H_{DR}^{i,k-i}(X) = \{0\} \), we conclude that \( \{ \alpha \}_{DR} = 0 \), a fact that also follows from the assumption on \( \alpha \) and the equivalence \( (i) \).

Recall that \( F^p H_{DR}^k(X, \mathbb{C}) \) is the image of the map

\[
\frac{\ker (d : F^p C_{k-1}^\infty(X, \mathbb{C}) \to F^p C_k^\infty(X, \mathbb{C}))}{\text{Im} (d : F^p C_{k-1}^\infty(X, \mathbb{C}) \to F^p C_k^\infty(X, \mathbb{C}))} \to H_{DR}^k(X, \mathbb{C}).
\]

We claim that this map is injective for all \( p, k \) if and only if the Frölicher spectral sequence of \( X \) degenerates at \( E_1 \). Indeed, it is injective if and only if

\[
\text{Im} (d : F^p C_{k-1}^\infty(X, \mathbb{C}) \to F^p C_k^\infty(X, \mathbb{C})) = F^p C_k^\infty(X, \mathbb{C}) \cap \text{Im} d,
\]

or in other words, if and only if the differential respects the filtration \( \mathcal{F} \) strictly. This, in turn, is known to be equivalent to the degeneration at \( E_1 \) of the spectral sequence (see Proposition 1.3.2 in [Del71], or for a more detailed argument, Lemma 1.3 in [SB18]).

Using the injectivity of the above map, the vanishing of the class \( \{ \alpha \}_{DR} \in F^p H_{DR}^k(X, \mathbb{C}) \) implies that \( \alpha = du \) for some form \( u \in F^p C_{k-1}^\infty(X, \mathbb{C}) \). From the analogous argument for \( F^q H_{DR}^k(X, \mathbb{C}) \) we infer that the vanishing of the class \( \{ \alpha \}_{DR} \in F^q H_{DR}^k(X, \mathbb{C}) \) implies that \( \alpha = dv \) for some form \( v \in F^q C_{k-1}^\infty(X, \mathbb{C}) \).

In particular, \( u - v \in \ker d \cap (F^p C_{k-1}^\infty(X, \mathbb{C}) + F^q C_{k-1}^\infty(X, \mathbb{C})) \), hence \( u - v \) defines a class

\[
\{ u - v \}_{DR} \in F^p H_{DR}^{k-1}(X, \mathbb{C}) + F^q H_{DR}^{k-1}(X, \mathbb{C}).
\]

Therefore, there exist forms \( u_1 \in F^p C_{k-1}^\infty(X, \mathbb{C}) \cap \ker d \) and \( v_1 \in F^q C_{k-1}^\infty(X, \mathbb{C}) \cap \ker d \) such that

\[
u - v = u_1 - v_1 + d\eta \]
for some form $\eta \in C^\infty_{k-2}(X, \mathbb{C})$. Equating the terms of bidegree $(p - 1, q)$, we get $v^{p-1,q} = v_1^{p-1,q} - \partial \eta^{p-2,q} - \bar{\partial} \eta^{p-1,q-1}$. Hence,

$$\alpha = dv = \partial v^{p-1,q} = \partial v_1^{p-1,q} - \partial \bar{\partial} \eta^{p-1,q-1}.$$ 

Now, since $v_1 \in \mathcal{F}^2C^\infty_{k-1}(X, \mathbb{C}) \cap \ker d$, we get $0 = dv_1 = \partial v_1^{p-1,q} + \text{forms of holomorphic degrees} \geq q + 1$. Thus, $\partial v_1^{p-1,q} = 0$, so $\alpha = \partial \bar{\partial} \eta^{p-1,q-1} \in \Im (\partial \bar{\partial})$. \hfill \square

### 3.4.2 Case of the Iwasawa manifold and its small deformations

Recall that the Iwasawa manifold $I^{(3)}$ is the nilmanifold of complex dimension 3 obtained as the quotient of the Heisenberg group of $3 \times 3$ upper triangular matrices with entries in $\mathbb{C}$ by the subgroup of those matrices with entries in $\mathbb{Z}[^i]$.

It is well known that the Iwasawa manifold is not a $\partial \bar{\partial}$-manifold. In fact, its Frölicher spectral sequence is known to satisfy $E_1 \neq E_2 = E_\infty$. On the other hand, it is known that the De Rham cohomology of the Iwasawa manifold can be generated in every degree by De Rham classes of ($d$-closed) pure-type forms. (See e.g. [Ang14].) Together with Cor. 3.24 this yields

**Proposition 3.26.** The Iwasawa manifold is a page-1-$\partial \bar{\partial}$-manifold.

However, the situation is more complex for the small deformations of the Iwasawa manifold, all of which are already known to not be $\partial \bar{\partial}$-manifolds. The following result shows, in particular, that unlike the $\partial \bar{\partial}$-property, the page-1-$\partial \bar{\partial}$-property is not deformation open.

**Proposition 3.27.** Let $(X_t)_{t \in B}$ be the Kuranishi family of the Iwasawa manifold $X_0$. For every $t \in B$, we have:

(i) $X_t$ is a page-1-$\partial \bar{\partial}$-manifold if and only if $X_t$ is complex parallelisable (i.e. lies in Nakamura’s class (i));

(ii) if $X_t$ lies in one of Nakamura’s classes (ii) or (iii), the De Rham cohomology of $X_t$ is not pure, so $X_t$ is not a page-$r$-$\partial \bar{\partial}$-manifold for any $r \in \mathbb{N}$.

**Proof.** That deformations in Nakamura’s class (i) are page-1-$\partial \bar{\partial}$-manifolds can be proved in the same way as the Iwasawa manifold was proved to have this property in Proposition 3.26. This fact also follows from the far more general Proposition 3.31 since all the small deformations $X_t$ of $X_0$ are nilmanifolds.

To show (ii), we will actually prove a slightly more general result. Calculations of Angella [Ang14] show that the hypotheses of the next Lemma are satisfied in this case. \hfill \square

**Lemma 3.28.** Let $X$ be a compact complex manifold with $b_1 = 4$, $h^{1,0}_\partial = h^{0,1}_\partial = 2$ and $h^{1,0}_A = 3$. Then, either $H^1_{DR}(X, \mathbb{C})$ or $H^2_{DR}(X, \mathbb{C})$ is not pure.

**Proof.** The proof is combinatorial. We first give a proof using the notation of [Ste18] and then a more pictorial proof that is hopefully clearer without having read all of [Ste18]. We will exploit the fact that the De Rham, Dolbeault and Aeppli cohomologies of indecomposable complexes are computable. This is spelt out in detail in [Ste18]. Summarised briefly, an even-length zigzag has a nonzero differential in the Frölicher spectral sequence or its conjugate, but has no De Rham cohomology. Meanwhile, odd-length zigzags have no differentials in the Frölicher spectral sequence, but
have a nonzero De Rham cohomology and $h_A^{p,q}$ counts the zigzags that have a nonzero component in degree $(p,q)$ with possibly outgoing but no incoming arrows.

**The formal proof.** Denote by $A = (C_{p,q}^\infty(X), \partial, \bar{\partial})$ the double complex of $\mathbb{C}$-valued forms on $X$. We investigate for which zigzag shapes $S$ with $(1,0) \in S$ or $(0,1) \in S$, one can have $\text{mult}_S(A) \neq 0$. Assume $H^1_{\text{DR}}(X, \mathbb{C})$ is pure. This is equivalent to $\text{mult}_{S_1^p,q}^p(A) = 0$ unless $p+q = 1$, i.e. $(p,q) = (1,0)$ or $(p,q) = (0,1)$, in which case $h_{\partial}^{1,0} + h_{\bar{\partial}}^{0,1} = b_1$, there are no differentials in the Frölicher spectral sequences starting or ending in $(1,0)$ or $(0,1)$, i.e., one has $\text{mult}_S(A) = 0$ for all even-length zigzag shapes such that $(1,0) \in S$ or $(0,1) \in S$. The only possible zigzag shapes containing $(1,0)$ or $(0,1)$ that are left are $S_2^{1,1}, S_2^{2,1}$ and $S_2^{2,2}$. One of the latter two has to occur since otherwise we would have $h_A^{1,0} = 2$. But this means that $H^2_{\text{DR}}(X, \mathbb{C})$ is not pure!

**The pictorial proof.** Choose any decomposition of the double complex $A$ into squares and zigzags and assume that $H^1_{\text{DR}}(X, \mathbb{C})$ is pure. This means that any zigzag contributing to the De Rham cohomology $H^1_{\text{DR}}(X, \mathbb{C})$ is of length one, i.e. drawing only the odd zigzags and leaving out the squares and the even zigzags (which do not contribute to the De Rham cohomology), the lower part of the double complex looks like this:

![Diagram showing zigzag shapes](image)

Here, a • denotes a dot, i.e. a zigzag of length one and multiplicity one. The symmetry along the diagonal comes from the real structure of $A$ given by complex conjugation. A priori, there may be other zigzags passing through $(1,0)$ and $(0,1)$. Schematically, these would all arise by choosing some connected subgraph with at least one arrow of the diagram

![Subdiagram](image)

They could either be of even-length or of odd-length and not contributing to $H^1_{\text{DR}}(X, \mathbb{C})$ but contributing to $H^2(X, \mathbb{C})$. Note that the subdiagram
is not allowed since this would give rise to a non-pure class in $H_{DR}^1(X, \mathbb{C})$, i.e. a class which does not lie in $H_{DR}^{1,0}(X) + H_{DR}^{0,1}(X))$. But we have already ruled this out by purity.

However, since $h_\partial^{1,0} + h_\partial^{0,1} = b_4$, there can be no differentials in the Frölicher spectral sequence starting or ending in degree $(1, 0)$ or $(0, 1)$. In terms of zigzags, this means no even-length zigzag passes through these bidegrees. This rules out the zigzags

\[
\begin{array}{c}
\bullet^{0,1} \xrightarrow{\partial} \bullet \\
\downarrow \partial \\
\bullet^{1,0} \xrightarrow{\partial} \bullet
\end{array}
\]

and their reflections along the diagonal (which have to occur with the same multiplicity since $A$ is equipped with a real structure). So, the only options for zigzags passing through $(1, 0)$ that are left are

\[
\begin{array}{c}
\bullet^{0,1} \xrightarrow{\partial} \bullet \\
\downarrow \partial \\
\bullet^{1,0} \xrightarrow{\partial} \bullet
\end{array}
\]

or

\[
\begin{array}{c}
\bullet^{1,0} \xrightarrow{\partial} \bullet \\
\downarrow \partial \\
\bullet^{0,1} \xrightarrow{\partial} \bullet
\end{array}
\]

and one of these has to occur since otherwise $H_{DR}^1$ would be of dimension 2, contradicting the assumptions. But the occurrence of either one implies that $H_{DR}^1(X, \mathbb{C})$ is not pure. \hfill \Box

3.4.3 Case of 3-dimensional complex nilmanifolds

Let us now consider complex nilmanifolds $X = (M, J)$ of complex dimension 3, i.e. $M = \Gamma \backslash G$ is a compact quotient of a simply-connected nilpotent real Lie group $G$ of dimension 6 by a co-compact, discrete subgroup $\Gamma$. Let $J$ be a complex structure on $M$ that is supposed to be invariant, i.e. it stems from a left-invariant complex structure on the Lie group $G$. Any invariant complex structure on $M$ identifies with a complex structure on the Lie algebra of the Lie group $G$.

We denote by $N$ the real 6-dimensional nilmanifold underlying the Iwasawa manifold. The Lie algebra underlying the nilmanifold $N$ is isomorphic to the nilpotent Lie algebra

\[
\mathfrak{n} = (0, 0, 0, 0, 13 + 42, 14 + 23).
\]

This notation means that $\mathfrak{n}$ is generated by $\{e_i\}_{i=1}^6$ satisfying the bracket relations $[e_1, e_3] = -[e_2, e_4] = -e_5$, $[e_1, e_4] = [e_2, e_3] = -e_6$, or equivalently there exists a basis $\{\alpha^i\}_{i=1}^6$ of the dual $\mathfrak{n}^*$ such that $d\alpha^1 = d\alpha^2 = d\alpha^3 = d\alpha^4 = 0$, $d\alpha^5 = \alpha^1 \wedge \alpha^3 - \alpha^2 \wedge \alpha^4$, $d\alpha^6 = \alpha^1 \wedge \alpha^4 + \alpha^2 \wedge \alpha^3$. 

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It is well known that complex nilmanifolds are not $\partial\bar{\partial}$-manifolds, except when they are complex tori. The following result identifies which 6-dimensional simply connected Lie groups with left-invariant complex structures give rise to page-$(r - 1)$-$\partial\bar{\partial}$-nilmanifolds for some $r \in \mathbb{N}^*$, after quotienting out a lattice.

**Proposition 3.29.** Let $X = (\Gamma \backslash G, J)$ be a complex nilmanifold of complex dimension 3, different from a torus. If there exists $r \in \mathbb{N}^*$ such that $X$ is a page-$(r - 1)$-$\partial\bar{\partial}$-manifold, then the Lie algebra $\mathfrak{g}$ of $G$ is isomorphic to $\mathfrak{n}$, and the invariant complex structure $J$ is equivalent to the complex parallelisable structure of $I^{(3)}$ or to the complex structure $\bar{J}$ defined by the structure equations

$$d\tau_1 = 0, \quad d\tau_2 = 0, \quad d\tau_3 = \tau_1 \wedge \bar{\tau}_2 := \tau_{12}.$$ 

In both cases $r = 2$, i.e. both of these manifolds are page-1-$\partial\bar{\partial}$-manifolds.

**Proof.** We already know that $I^{(3)}$ is a page-1-$\partial\bar{\partial}$-manifold. Let us consider the complex nilmanifold $\tilde{X} = (\tilde{N}, \bar{J})$ defined by the above equations. By Nomizu’s theorem [Nom54], the De Rham cohomology groups of $\tilde{X}$ are:

$$H^1_{DR}(\tilde{X}, \mathbb{C}) = \langle [\tau_1], [\tau_2] \rangle \oplus \langle [\tau_1], [\bar{\tau}_2] \rangle,$$

$$H^2_{DR}(\tilde{X}, \mathbb{C}) = \langle [\tau_{12}], [\tau_{13}] \rangle \oplus \langle [\tau_{11}], [\tau_{22}], [\tau_{23}], [\tau_{32}] \rangle \oplus \langle [\tau_{12}], [\tau_{13}] \rangle,$$

$$H^3_{DR}(\tilde{X}, \mathbb{C}) = \langle [\tau_{123}] \rangle \oplus \langle [\tau_{123}, [\tau_{131}], [\tau_{321}]], [\tau_{231}], [\tau_{321}] \rangle \oplus \langle [\tau_{123}], [\tau_{132}] \rangle \oplus \langle [\tau_{123}], [\tau_{132}] \rangle \oplus \langle [\tau_{123}], [\tau_{132}] \rangle,$$

and

$$H^5_{DR}(\tilde{X}, \mathbb{C}) = \langle [\tau_{123}, i\tau_{123}] \rangle \oplus \langle [\tau_{1312}, \bar{\tau}_{1312}] \rangle \oplus \langle [\tau_{1312}, \bar{\tau}_{1312}] \rangle.$$

This proves that the De Rham cohomology of $\tilde{X}$ is pure.

On the other hand, it is proved in [LU15] (where the Lie algebra $\mathfrak{n}$ is denoted by $\mathfrak{h}_8$) that any other invariant complex structure (i.e. not equivalent to $\bar{J}$ or to the complex parallelisable structure of $I^{(3)}$) fails to be pure in degree 4 or 5, that is, the direct sum decomposition of Definition 3.4 is not satisfied for $k = 4$ or $k = 5$ (or both).

Finally, to see that $\tilde{X} = (N, \bar{J})$ is a page-1-$\partial\bar{\partial}$-manifold, it remains to prove that its Frölicher spectral sequence degenerates at $E_2$. Since the complex structure $\bar{J}$ is invariant, we can compute the Dolbeault cohomology groups of $\tilde{X}$ (and hence the spaces $E^{p,q}_2(\tilde{X})$ in the FSS) by means of invariant forms (see e.g. [LU15] and references therein for more details). From the structure equations of $\tilde{X}$, we have the following Dolbeault cohomology groups:

$$E^{1,0}_1(\tilde{X}) = \langle \{\tau_1\}_{E_1}, \{\tau_2\}_{E_1} \rangle,$$

$$E^{0,1}_1(\tilde{X}) = \langle \{\tau_1\}_{E_1}, \{\tau_2\}_{E_1}, \{\tau_3\}_{E_1} \rangle,$$

$$E^{2,0}_1(\tilde{X}) = \langle \{\tau_{12}\}_{E_1}, \{\tau_{13}\}_{E_1} \rangle,$$

$$E^{1,1}_1(\tilde{X}) = \langle \{\tau_{11}\}_{E_1}, \{\tau_{13}\}_{E_1}, \{\tau_{21}\}_{E_1}, \{\tau_{22}\}_{E_1}, \{\tau_{23}\}_{E_1}, \{\tau_{32}\}_{E_1} \rangle,$$

$$E^{0,2}_1(\tilde{X}) = \langle \{\tau_{12}\}_{E_1}, \{\tau_{13}\}_{E_1}, \{\tau_{23}\}_{E_1} \rangle,$$

$$E^{3,0}_1(\tilde{X}) = \langle \{\tau_{123}\}_{E_1} \rangle.$$
Lemma 3.30.

The only Dolbeault cohomology classes for which the map $d_1$ is not zero are the following:

- $d_1((\tau_3)_{E_1}) = \{\partial \tau_3\}_{E_1} = \{\tau_{21}\}_{E_1} \neq 0$ in $E_1^{1,1}(\tilde{X})$;
- $d_1((\tau_{13})_{E_1}) = \{\partial \tau_{13}\}_{E_1} = \{\tau_{121}\}_{E_1} \neq 0$ in $E_1^{2,1}(\tilde{X})$;
- $d_1((\tau_{23})_{E_1}) = \{\partial \tau_{23}\}_{E_1} = \{\tau_{212}\}_{E_1} \neq 0$ in $E_1^{1,2}(\tilde{X})$;
- $d_1((\tau_{133})_{E_1}) = \{\partial \tau_{133}\}_{E_1} = \{\tau_{1231}\}_{E_1} \neq 0$ in $E_1^{3,1}(\tilde{X})$.

By the definition of the spaces $E_2^{p,q}$ and using the duality for the FSS proved in Section 2, we get

$$H^k_{DR}(\tilde{X}, \mathbb{C}) \cong \bigoplus_{p+q=k} E_2^{p,q}(\tilde{X})$$

for every $k$.

According to [Pop15] and [PU18], a compact complex manifold $X$ is called an **sGG manifold** if every Gauduchon metric $\omega$ on $X$ is sG, i.e. $\partial\omega^{n-1}$ is $\bar{\partial}$-exact. For instance, the Iwasawa manifold is sGG (see [PU18]).

By the numerical characterisation proved in [PU18, Theorem 1.6], a compact complex manifold is sGG if and only if $b_1 = 2h^{0,1}_{\bar{\partial}}$. From the proof of Proposition 3.29, we get

$$b_1 = 4 \neq 6 = 2h^{0,1}_{\bar{\partial}}(\tilde{X}),$$

so $\tilde{X}$ is not an sGG-manifold.

On the other hand, all the sGG nilmanifolds of complex dimension 3 are identified in [PU18, Theorem 6.1]. In particular, there exist such complex nilmanifolds different from the Iwasawa manifold and $\tilde{X}$ which are sGG, so they are not page-1-$\partial\bar{\partial}$-manifolds by Proposition 3.29.

Therefore, the page-1-$\partial\bar{\partial}$ and the sGG properties of compact complex manifolds are unrelated.

### 3.4.4 Case of complex parallelisable nilmanifolds

We will now prove that all complex parallelisable nilmanifolds are page-1-$\partial\bar{\partial}$-manifolds. On the one hand, this generalises one implication in (i) of Proposition 3.27. On the other hand, it provides a large class of page-1-$\partial\bar{\partial}$-manifolds that are not $\partial\bar{\partial}$-manifolds. Indeed, it is known that a nilmanifold $\Gamma \backslash G$ is never $\partial\bar{\partial}$ (or even formal in the sense of [DGMS75]) unless it is Kähler (i.e. a complex torus, or equivalently, the Lie group $G$ is abelian).

Recall that a compact complex parallelisable manifold $X$ is a manifold whose holomorphic tangent bundle is trivial. By Wang’s theorem [Wan54], $X$ is a quotient $\Gamma \backslash G$ of a complex Lie group $G$ by a co-compact, discrete subgroup $\Gamma$. When $G$ is nilpotent, the manifold $X$ is a complex parallelisable nilmanifold. The Iwasawa manifold and the 5-dimensional Iwasawa manifold are examples of this type.

We first need an algebraic result.

**Lemma 3.30.** Let $(A^*, d_A)$ and $(B^*, d_B)$ be two complexes of vector spaces and $C = A \otimes B$ their tensor product, considered as a double complex, i.e.:

$$C^{p,q} := A^p \otimes B^q$$

$$\partial_1(a \otimes b) := d_A a \otimes b$$

$$\partial_2(a \otimes b) := (-1)^{|a|}a \otimes d_B b$$
Then $C$ satisfies the page-1-$\partial_1\partial_2$-property.

**Proof.** First, we compute the first and second pages of the column Frölicher spectral sequence. (We only treat the column case, the row case being analogous.) The first page is the column cohomology:

$$(1E^{•,•}_1, \partial_1) = (H^q(C^{p•}, \partial_2), \partial_1)$$

Since $\partial_2$ is, up to sign, $\text{Id}_A \otimes d_B$, one has $H^q(C^{p•}, \partial_2) = A^p \otimes H^q(B, d_B)$ and $d_1 = d_A \otimes \text{Id}_{H(B)}$. Therefore, $2E^{p,q}_2 = H^p(A, d_A) \otimes H^q(B, d_B)$. Now, for every $d_A$-closed element $a \in A^p$ and every $d_B$-closed element $b \in B^q$, the element $a \otimes b \in C^{p+q}$ is $d = \partial_1 + \partial_2$ closed. Similarly, if one of the two is $d_A$ or $d_B$ exact, the form $a \otimes b$ will be $d$-exact. Hence we get a natural map $\bigoplus_{p+q=k} H^p(A, d_A) \otimes H^q(B, d_B) \to H^k_{dR}(C)$. Since we are working over a field, the Künneth formula tells us that this is an isomorphism. \hfill \Box

Given a complex nilmanifold $\Gamma \backslash G$, let $\mathfrak{g}$ be the (real) Lie algebra of $G$, and denote by $J : \mathfrak{g} \to \mathfrak{g}$ the endomorphism induced by the complex structure of the Lie group $G$. Then $J^2 = -\text{Id}$ and

$$[Jx, y] = J[x, y], \quad (17)$$

for all $x, y \in \mathfrak{g}$. Let $\mathfrak{g}_C^*$ be the dual of the complexification $\mathfrak{g}_C$ of $\mathfrak{g}$ and denote by $\mathfrak{g}^{1,0}$ (respectively $\mathfrak{g}^{0,1}$) the eigenspace of the eigenvalue $i$ (resp. $-i$) of $J$ considered as an endomorphism of $\mathfrak{g}_C^*$. Condition (17) is equivalent to $[\mathfrak{g}^{0,1}, \mathfrak{g}^{1,0}] = 0$ which is equivalent to $d(\mathfrak{g}^{1,0}) \subset \Lambda^2(\mathfrak{g}^{1,0})$, i.e. there is no component of bidegree $(1, 1)$. Therefore, $\partial_1$ is identically zero on $\Lambda^p(\mathfrak{g}^{1,0})$, and $\partial$ is identically zero on $\Lambda^q(\mathfrak{g}^{0,1})$, that is,

$$\partial_{|\Lambda^p(\mathfrak{g}^{1,0})} = d_{|\Lambda^p(\mathfrak{g}^{1,0})}, \quad \partial_{|\Lambda^p(\mathfrak{g}^{0,1})} = 0, \quad \partial_{|\Lambda^q(\mathfrak{g}^{0,1})} = 0, \quad \partial_{|\Lambda^q(\mathfrak{g}^{1,0})} = d_{|\Lambda^q(\mathfrak{g}^{0,1})}. \quad (18)$$

**Theorem 3.31.** Complex parallelisable nilmanifolds are page-1-$\partial\bar{\partial}$-manifolds.

**Proof.** Sakane [Sak76] showed that the inclusion of the double complex $(\Lambda^{•,•} \mathfrak{g}_C^*, \partial, \bar{\partial})$ as left invariant forms into the complex of all forms on $\Gamma \backslash G$ induces an isomorphism of the respective first pages of the corresponding Frölicher spectral sequences (hence of all later pages). But the equations (18) mean that the double complex $(\Lambda^{•,•} \mathfrak{g}_C^*, \partial, \bar{\partial})$ is the tensor product of the simple complexes $(\Lambda• \mathfrak{g}^{1,0}, d)$ and $(\Lambda• \mathfrak{g}^{0,1}, d)$, so we can apply Lemma 3.30. \hfill \Box

### 3.4.5 Relations with the $E_r$-sG manifolds

Let $X$ be an $n$-dimensional compact complex manifold and let $\omega$ be a Gauduchon metric on $X$. This means that $\omega$ is a smooth, positive definite $(1, 1)$-form on $X$ such that $\partial\bar{\partial}\omega^{n-1} = 0$. Gauduchon metrics were introduced and proved to always exist in [Gau77]. It was noticed in [Pop19] that $\partial\omega^{n-1}$ is $E_r$-closed for every $r \in \mathbb{N}^*$ and the following definition was introduced.

**Definition 3.32.** ([Pop19]) Let $r \in \mathbb{N}^*$. A Gauduchon metric $\omega$ on $X$ is said to be $E_r$-sG if $\partial\omega^{n-1}$ is $E_r$-exact.

We say that $X$ is an $E_r$-sG manifold if an $E_r$-sG metric $\omega$ exists on $X$. 

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The $E_1$-sG property coincides with the strongly Gauduchon (sG) property introduced in [Pop09]. The following implications and their analogues for $X$ are obvious:

$$\omega \text{ is } E_1\text{-sG } \implies \omega \text{ is } E_2\text{-sG } \implies \omega \text{ is } E_3\text{-sG}$$

and so is the fact that, for bidegree reasons, the $E_r$-sG property coincides with the $E_3$-sG property for all $r \geq 4$.

The link between the page-$(r-1)$-$\bar{\partial}\partial$ and the $E_r$-sG properties is spelt out in the following

**Proposition 3.33.** Let $r \in \mathbb{N}^*$ and let $X$ be a page-$(r-1)$-$\bar{\partial}\partial$-manifold. Then, every Gauduchon metric on $X$ is $E_r$-sG. In particular, $X$ is an $E_r$-sG manifold.

**Proof.** Let $\omega$ be a Gauduchon metric on $X$. Then, $\bar{\partial}\omega^{n-1}$ is $\bar{\partial}$-closed and $\partial$-closed, hence $d$-closed. It is also $\partial$-exact, or equivalently, $E_1$-exact, hence also $E_r$-exact.

Now, thanks to Theorem 3.48, the page-$(r-1)$-$\bar{\partial}\partial$-property of $X$ implies the equivalence between $E_r$-exactness and $E_r$-exactness for $d$-closed pure-type forms. Consequently, $\partial\omega^{n-1}$ must be $E_r$-exact, so $\omega$ is an $E_r$-sG metric. $\square$

Let $X_{u,v}$ be a Calabi-Eckmann manifold, i.e. any of the complex manifolds $C^\infty$-diffeomorphic to $S^{2u+1} \times S^{2v+1}$ constructed by Calabi and Eckmann in [CE53]. Recall that the $X_{0,v}$’s and the $X_{u,0}$’s are Hopf manifolds. By [Pop14], $X_{u,v}$ does not admit any sG metric. However, we now prove the existence of $E_2$-sG metrics when $uv > 0$.

**Proposition 3.34.** Let $X_{u,v}$ be a Calabi-Eckmann manifold of complex dimension $\geq 2$. Let $u \leq v$.

(i) If $u > 0$, then $X_{u,v}$ does not admit sG metrics, but it is an $E_2$-sG manifold.

(ii) If $u = 0$, $X_{u,v}$ does not admit $E_r$-sG metrics for any $r$.

**Proof.** By Borel’s result in [Hir78, Appendix Two by A. Borel], we have

$$H^{\bullet,\bullet}_\bar{\partial}(X_{u,v}) \cong \mathbb{C}[x_{1,1}] / (x_{u+1,1}) \otimes \bigwedge (x_{v+1,v}, x_{0,1}).$$

In other words, a model for the Dolbeault cohomology of the Calabi-Eckmann manifold $X_{u,v}$ is provided by the CDGA (see [NT78])

$$(V \langle x_{0,1}, x_{1,1}, y_{u+1,u}, x_{v+1,v}, \bar{\partial} \rangle),$$

with differential

$$\bar{\partial}x_{0,1} = 0, \quad \bar{\partial}x_{1,1} = 0, \quad \bar{\partial}y_{u+1,u} = x_{u+1,1}, \quad \bar{\partial}x_{v+1,v} = 0.$$  

Thus, if $u > 0$, we have a minimal model. (For $u = 0$ we have only a cofibrant model in the sense of [NT78].)

Moreover, $\partial$ acts on generators as follows [NT78]:

$$\partial x_{0,1} = x_{1,1} \quad \text{(hence } \partial x_{1,1} = 0), \quad \partial y_{u+1,u} = 0, \quad \partial x_{v+1,v} = 0.$$  

Next we determine the spaces $E^n_{r\cdot n-1}$, for any $r \geq 1$, where $n = u + v + 1$.  

Lemma 3.35. Let $r \in \mathbb{N}$ that the Dolbeault cohomology groups $H_{\bar{\partial}}^{n-1,n-1}(X_{u,v})$ and $H_{\bar{\partial}}^{n,n-1}(X_{u,v})$. They are given by

$$H_{\bar{\partial}}^{u+v,u+v}(X_{u,v}) = \langle x_{0,1}, x_{1,1}^{u-1}, x_{v+1,v} \rangle, \quad H_{\bar{\partial}}^{u+v+1,u+v}(X_{u,v}) = \langle x_{1,1}^{u}, x_{v+1,v} \rangle.$$  

Now we consider

$$H_{\bar{\partial}}^{u+v,u+v}(X_{u,v}) \xrightarrow{\bar{\partial}} H_{\bar{\partial}}^{u+v+1,u+v}(X_{u,v}) \to 0. $$

Since $\partial(x_{0,1}, x_{1,1}^{u-1}, x_{v+1,v}) = \partial(x_{0,1}, x_{1,1}^{u-1}, x_{v+1,v}) = x_{1,1}^{u}, x_{v+1,v}$, the first map is surjective. Therefore, $E_2^{n,n-1}(X_{u,v}) = 0$. Thus, any Gauduchon metric on $X_{u,v}$ is an $E_2$-sG metric.

Next we focus on the case (ii), i.e. $u = 0$, so $v \geq 1$. In this case $H_{\bar{\partial}}^{u,v}(X_{0,v}) = \langle x_{v+1,v} \rangle$. Notice that the Dolbeault cohomology groups $H_{\bar{\partial}}^{u-v+1,v+r-1}(X_{0,v})$ are all zero for every $r \geq 1$. Therefore, $E_r^{u-v+1,v+r-1}(X_{0,v}) = \{0\}$ for every $r \geq 1$. Meanwhile, from

$$\{0\} = E_r^{u-v+1,v+r-1}(X_{0,v}) \xrightarrow{d_r} E_r^{u,v+1}(X_{0,v}) \to 0,$$

we get $E_r^{u,v+1}(X_{0,v}) = H_{\bar{\partial}}^{u,v+1}(X_{0,v})$ for every $r \geq 2$. So, the existence of an $E_r$-sG metric on $X_{0,v}$ would imply the existence of an $sG$ metric, which would contradict [Pop14].

\[\square\]

As a by-product of Borel’s description of the Dolbeault cohomology of the Calabi-Eckmann manifolds $X_{u,v}$ used in the above proof, one gets that the Frölicher spectral sequence of $X_{u,v}$ satisfies $E_1 \neq E_2 = E_\infty$ when $u > 0$, whereas it degenerates at $E_1$ when $u = 0$. This latter fact implies that no Hopf manifold $X_{0,v}$ can have a pure De Rham cohomology, hence cannot be a page-$r$-$\partial \bar{\partial}$-manifold for any $r$. Indeed, if the De Rham cohomology were pure, then $X_{0,v}$ would be a $\partial \bar{\partial}$-manifold, a fact that is trivial to contradict.

By the previous result, all the Calabi-Eckmann manifolds that are not Hopf manifolds are $E_2$-sG manifolds. However, the next observation shows that they are not page-$r$-$\partial \bar{\partial}$-manifolds for any $r \in \mathbb{N}$.

**Lemma 3.35.** Let $u, v \geq 0$ and let $X = S^{2u+1} \times S^{2v+1}$ be equipped with any of the Calabi-Eckmann complex structures. Assume that either $u \neq v$ or $u = v = 1$.

Then, the De Rham cohomology of $X$ is not pure.

**Proof.** When $u \neq v$, this was proved in [Ste18, p.29ff] as a consequence of the computation of the Hodge numbers of $X$ by Borel in [Hir78, Appendix Two by A. Borel]. As explained in [Ste18], when $u = v$, the only consequence that one can draw from the numerical information given by Borel is that the only possible zigzags passing through the middle degree $2u + 1$ are either two dots or two length-three zigzags situated in the following bidegrees:

![Bidegrees](image.png)

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In the first case, the De Rham cohomology is pure, while in the second one it is not. They cannot be distinguished by the Hodge numbers. However, they may be distinguished by the Bott-Chern numbers. Specifically, $h_{BC}^{u+1,u+1} = 0$ in the former case and $h_{BC}^{u+1,u+1} = 1$ in the latter. (Recall that $h_{BC}^{p,q}$ counts ‘top right’ corners of zigzags in bidegree $(p, q)$, i.e. those which have no outgoing edges). A calculation in [TT17] shows that the latter case occurs when $u = v = 1$.

Remark 3.36. It appears to be very likely that this Lemma also holds for arbitrary $u = v > 1$. To settle this issue, it suffices to determine whether $h_{BC}^{v+1,u+1} = 1$ for the higher-dimensional Calabi-Eckmann manifolds with $u = v$.

3.5 Higher-page Bott-Chern and Aeppli cohomologies: definition, Hodge theory and duality

Let $X$ be an $n$-dimensional compact complex manifold. Fix an arbitrary positive integer $r$ and a bidegree $(p, q)$. In §2.2, we defined the notions of $E_r$-closedness and $E_r$-exactness for forms $\alpha \in C_{\infty}^{p,q}(X)$ as higher-page analogues of $\bar{\partial}$-closedness (that can now be called $E_1$-closedness) and $\bar{\partial}$-exactness (that can now be called $E_1$-exactness). We then gave these notions explicit descriptions in Proposition 2.3.

In the same vein, we say that $\alpha$ is $E_r$-closed if $\bar{\alpha}$ is $E_r$-closed and we say that $\alpha$ is $E_r$-exact if $\bar{\alpha}$ is $E_r$-exact. In particular, characterisations of $E_r$-closedness and $E_r$-exactness are obtained by permuting $\partial$ and $\bar{\partial}$ in the characterisations of $E_1$-closedness and $E_1$-exactness of Proposition 2.3.

Moreover, we can take our cue from Proposition 2.3 to define higher-page analogues of $\partial\bar{\partial}$-closedness and $\partial\bar{\partial}$-exactness in the following way.

Definition 3.37. Suppose that $r \geq 2$.

(i) We say that a form $\alpha \in C_{\infty}^{p,q}(X)$ is $E_r\overline{E}_r$-closed if there exist smooth forms $\eta_1, \ldots, \eta_{r-1}$ and $\rho_1, \ldots, \rho_{r-1}$ such that the following two towers of $r - 1$ equations are satisfied:

\[
\begin{align*}
\partial \alpha &= \bar{\partial} \eta_1 \\
\partial \eta_1 &= \bar{\partial} \eta_2 \\
\vdots \\
\partial \eta_{r-2} &= \bar{\partial} \eta_{r-1}, \\
\bar{\partial} \alpha &= \partial \rho_1 \\
\bar{\partial} \rho_1 &= \partial \rho_2 \\
\bar{\partial} \rho_{r-2} &= \partial \rho_{r-1}.
\end{align*}
\]

(ii) We refer to the properties of $\alpha$ in the two towers of $(r - 1)$ equations under (i) by saying that $\partial \alpha$, resp. $\bar{\partial} \alpha$, runs at least $(r - 1)$ times.

(iii) We say that a form $\alpha \in C_{\infty}^{p,q}(X)$ is $E_r\overline{E}_r$-exact if there exist smooth forms $\zeta, \xi, \eta$ such that

\[
\alpha = \partial \zeta + \bar{\partial} \xi + \bar{\partial} \eta
\]

and such that $\zeta$ and $\eta$ further satisfy the following conditions. There exist smooth forms $v_{r-3}, \ldots, v_0$.
and \( u_{r-3}, \ldots, u_0 \) such that the following two towers of \( r - 1 \) equations are satisfied:

\[
\begin{align*}
\bar{\partial} \zeta &= \partial v_{r-3} \\
\bar{\partial} v_{r-3} &= \partial v_{r-4} \\
\vdots &= \vdots \\
\bar{\partial} v_0 &= 0,
\end{align*}
\]

\[
\begin{align*}
\partial \eta &= \bar{\partial} u_{r-3} \\
\partial u_{r-3} &= \bar{\partial} u_{r-4} \\
\vdots &= \vdots \\
\partial u_0 &= 0.
\end{align*}
\]

(iv) We refer to the properties of \( \bar{\partial} \zeta \), resp. \( \partial \eta \), in the two towers of \( (r - 1) \) equations under (iii) by saying that \( \bar{\partial} \zeta \), resp. \( \partial \eta \), reaches 0 in at most \( (r - 1) \) steps.

When \( r - 1 = 1 \), the properties of \( \bar{\partial} \zeta \), resp. \( \partial \eta \), reaching 0 in \( (r - 1) \) steps translate to \( \bar{\partial} \zeta = 0 \), resp. \( \partial \eta = 0 \).

To unify the definitions, we will also say that a form \( \alpha \in C_{p,q}^\infty(X) \) is \( E_1 \bar{E}_1 \)-closed (resp. \( E_1 \bar{E}_1 \)-exact) if \( \alpha \) is \( \partial \bar{\partial} \)-closed (resp. \( \bar{\partial} \partial \)-exact).

As with \( E_r \) and \( \bar{E}_r \), it follows at once from Definition 3.37 that the \( E_r \bar{E}_r \)-closedness condition becomes stronger and stronger as \( r \) increases, while the \( E_r \bar{E}_r \)-exactness condition becomes weaker and weaker as \( r \) increases. In other words, the following inclusions of vector spaces hold:

\[
\{ \partial \bar{\partial} \text{-exact forms} \} \subset \cdots \subset \{ E_r \bar{E}_r \text{-exact forms} \} \subset \{ E_{r+1} \bar{E}_{r+1} \text{-exact forms} \} \subset \cdots
\]

\[
\cdots \subset \{ E_{r+1} \bar{E}_{r+1} \text{-closed forms} \} \subset \{ E_r \bar{E}_r \text{-closed forms} \} \subset \cdots \subset \{ \partial \bar{\partial} \text{-closed forms} \}.
\]

The following statement collects a few other immediate relations among these notions.

**Lemma 3.38.** Fix an arbitrary \( r \in \mathbb{N}^* \).

(i) A pure-type form \( \alpha \) is simultaneously \( E_r \)-closed and \( \bar{E}_r \)-closed if and only if \( \alpha \) is simultaneously \( \partial \)-closed and \( \bar{\partial} \)-closed. This is further equivalent to \( \alpha \) being \( d \)-closed.

(ii) If \( \alpha \) is \( E_r \bar{E}_r \)-exact, then each of the classes \( \{ \alpha \}_E \) and \( \{ \alpha \}_{\bar{E}} \) contains a \( \partial \bar{\partial} \)-exact form and \( \alpha \) is both \( E_r \)-exact and \( \bar{E}_r \)-exact.

(iii) Fix any bidegree \( (p, q) \) and let \( \alpha \in C_{p,q}^\infty(X) \). If \( \alpha \) is \( E_r \bar{E}_r \)-exact for some \( r \in \mathbb{N}^* \), then \( \alpha \) is \( d \)-exact.

**Proof.** (i) is obvious. To see (ii), let \( \alpha = \partial \zeta + \bar{\partial} \xi + \bar{\partial} \eta \) be \( E_r \bar{E}_r \)-exact, with \( \zeta \) and \( \eta \) satisfying the conditions under (ii) of Definition 3.37. Then

\[
\{ \alpha \}_E = \{ \alpha - \partial \zeta - \bar{\partial} \eta \}_E = \{ \partial \bar{\partial} \xi \}_E \quad \text{and} \quad \{ \alpha \}_{\bar{E}} = \{ \alpha - \partial \zeta - \bar{\partial} \eta \}_{\bar{E}} = \{ \partial \bar{\partial} \xi \}_{\bar{E}},
\]

while \( \alpha = \partial \zeta + \bar{\partial}(-\partial \xi + \eta) \) is \( E_r \)-exact and \( \alpha = \partial(\zeta + \bar{\partial} \xi) + \bar{\partial} \eta \) is \( \bar{E}_r \)-exact.

To prove (iii), let \( \alpha = \partial \zeta + \bar{\partial} \xi + \bar{\partial} \eta \), where \( \zeta \) and \( \eta \) satisfy the conditions in the two towers under (ii) of Definition 3.37. Going down the first tower, we get

\[
\partial \zeta = d(\zeta - v_{r-3}) = \cdots = d(\zeta - v_{r-3} + \cdots + (-1)^r v_0)
\]

In particular, \( \partial \zeta \) is \( d \)-exact.

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Similarly, going down the second tower, we get
\[ \bar{\partial}\eta = d(\eta - u_{r-3} + \cdots + (-1)^r u_0). \]

In particular, \( \bar{\partial}\eta \) is \( d \)-exact.

Since \( \partial \bar{\partial}\xi \) is also \( d \)-exact, we infer that \( \alpha \) is \( d \)-exact. Explicitly, we have
\[ \alpha = d[(\zeta + \eta) + \bar{\partial}\xi - w_{r-3} + \cdots + (-1)^r w_0], \]
where \( w_j := u_j + v_j \) for all \( j \).

The main takeaway from Lemma 3.38 is that \( E_r \bar{E}_r \)-exactness implies \( E_r \)-exactness, \( \bar{E}_r \)-exactness and \( d \)-exactness. Let us now pause briefly to notice a property involving the spaces \( C^{p,q}_{r,\infty}(X) \) of \( E_r \)-exact \((p, q)\)-forms, resp. \( \bar{C}^{p,q}_r \) of \( \bar{E}_r \)-exact \((p, q)\)-forms.

**Lemma 3.39.** Fix an arbitrary \( r \in \mathbb{N}^* \). For any bidegree \( (p, q) \), the following identity of vector subspaces of \( C^{p,q}_{r,\infty}(X) \) holds:
\[ C^{p,q}_r + \bar{C}^{p,q}_r = \text{Im} \partial + \text{Im} \bar{\partial}. \]

**Proof.** For any bidegree \( (p, q) \), consider the vector spaces (see (iv) of Definition 3.37 for the terminology):
\[ E^{p,q}_{\partial, r} := \{ \alpha \in C^{p,q}_{\infty}(X) \mid \partial\alpha \text{ reaches 0 in at most } r \text{ steps} \}, \]
\[ E^{p,q}_{\bar{\partial}, r} := \{ \beta \in C^{p,q}_{\infty}(X) \mid \bar{\partial}\beta \text{ reaches 0 in at most } r \text{ steps} \}. \]

From the definitions, we get:
\[ C^{p,q}_{r,\infty} = \partial(E^{p,q}_{\partial, r-1}) + \text{Im} \bar{\partial} \text{ and } \bar{C}^{p,q}_r = \text{Im} \partial + \bar{\partial}(E^{p,q}_{\bar{\partial}, r-1}). \]
This trivially implies the contention. \( \square \)

We now come to the main definitions of this subsection.

**Definition 3.40.** Let \( X \) be an \( n \)-dimensional compact complex manifold. Fix \( r \in \mathbb{N}^* \) and a bidegree \( (p, q) \).

(i) The \( E_r \)-**Bott-Chern** cohomology group of bidegree \( (p, q) \) of \( X \) is defined as the following quotient complex vector space:
\[ E^{p,q}_{r,BC}(X) := \frac{\{ \alpha \in C^{p,q}_{\infty}(X) \mid d\alpha = 0 \}}{\{ \alpha \in C^{p,q}_{\infty}(X) \mid \alpha \text{ is } E_r \bar{E}_r \text{-exact} \}}. \]

(ii) The \( E_r \)-**Aeppli** cohomology group of bidegree \( (p, q) \) of \( X \) is defined as the following quotient complex vector space:
\[ E^{p,q}_{r,A}(X) := \frac{\{ \alpha \in C^{p,q}_{\infty}(X) \mid \alpha \text{ is } E_r \bar{E}_r \text{-closed} \}}{\{ \alpha \in C^{p,q}_{\infty}(X) \mid \alpha \in \text{Im} \partial + \text{Im} \bar{\partial} \}}. \]
When \( r = 1 \), the above groups coincide with the standard Bott-Chern, respectively Aeppli, cohomology groups (see [BC65] and [Aep62]). Note that, by (i) of Lemma 3.38, the representatives of \( E_r \)-Bott-Chern classes can be alternatively described as the forms that are simultaneously \( E_r \)-closed and \( \overline{E}_r \)-closed, while by Lemma 3.39, the \( E_r \)-Aeppli-exact forms can be alternatively described as those forms lying in \( \overline{C}_r^{p,q} + \overline{C}_r^{p,q} \).

Also note that the inclusions of vector spaces spelt out just before Lemma 3.38 and their analogues for the \( E_r \)- and \( \overline{E}_r \)-cohomologies imply the following inequalities of dimensions:

\[
\cdots \leq \dim E_{p,q}^{r,BC}(X) \leq \dim E_{p,q}^{r-1,BC}(X) \leq \cdots \leq \dim E_{1,q}^{1,BC}(X) = \dim H_{q,BC}(X)
\]

and their analogues for the \( E_r \)-Aeppli cohomology spaces.

The first step towards extending to the higher pages of the Frölicher spectral sequence the standard Serre-type duality between the classical Bott-Chern and Aeppli cohomology groups of complementary bidegrees is the following

**Proposition 3.41.** Let \( X \) be a compact complex manifold with \( \dim C^*X = n \). For every \( r \in \mathbb{N}^* \) and all \( p, q \in \{0, \ldots, n\} \), the following bilinear pairing is well defined:

\[
E_{p,q}^{r,BC}(X) \times E_{n-p,n-q}^{r,A}(X) \rightarrow \mathbb{C}, \quad (\{\alpha\}_{E_{r,BC}}, \{\beta\}_{E_{r,A}}) \mapsto \int_X \alpha \wedge \beta,
\]

in the sense that it is independent of the choice of representative of either of the classes \( \{\alpha\}_{E_{r,BC}} \) and \( \{\beta\}_{E_{r,A}} \).

**Proof.** The proof consists in a series of integrations by parts (mathematical ping-pong).

- To prove independence of the choice of representative of the \( E_r \)-Bott-Chern class, let us modify a representative \( \alpha \) to some representative \( \alpha + \partial \zeta + \partial \overline{\partial} \xi + \overline{\partial} \eta \) of the same \( E_r \)-Bott-Chern class. This means that \( \partial \zeta + \partial \overline{\partial} \xi + \overline{\partial} \eta \) is \( E_r \overline{E}_r \)-exact, so \( \zeta \) and \( \eta \) satisfy the towers of \( r - 1 \) equations under (ii) of Definition 3.37. We have

\[
\int_X (\alpha + \partial \zeta + \partial \overline{\partial} \xi + \overline{\partial} \eta) \wedge \beta = \int_X \alpha \wedge \beta \pm \int_X \zeta \wedge \partial \beta \pm \int_X \xi \wedge \partial \overline{\partial} \beta \pm \int_X \eta \wedge \overline{\partial} \beta.
\]

Since \( \beta \) is \( E_r \overline{E}_r \)-closed, it is also \( \partial \overline{\partial} \)-closed (see (i) of Lemma 3.38), so the last but one integral above vanishes.

Using the \( r - 1 \) equations in the first tower under (i) of Definition 3.37 (with \( \beta \) in place of \( \alpha \)) and the first tower under (ii) of the same definition, we get:

\[
\int_X \zeta \wedge \partial \beta = \int_X \zeta \wedge \overline{\partial} \eta_1 = \pm \int_X \overline{\partial} \zeta \wedge \eta_1 = \pm \int_X \partial v_{r-3} \wedge \eta_1 = \pm \int_X v_{r-3} \wedge \partial \eta_1 = \pm \int_X v_{r-3} \wedge \partial \eta_1 = \pm \int_X v_{r-3} \wedge \partial \eta_1 = \pm \int_X v_{r-4} \wedge \partial \eta_2 \\
\vdots \\
\int_X v_0 \wedge \partial \eta_{r-1} = \pm \int_X \overline{\partial} v_0 \wedge \eta_{r-1} = 0.
\]
where the last identity follows from $\bar{\partial}v_0 = 0$.

Playing the analogous mathematical ping-pong while using the second tower under both (i) and (ii) of Definition 3.37, we get:

$$\int_X \eta \wedge \bar{\partial} \beta = \int_X \eta \wedge \partial \rho_1 = \pm \int \partial \eta \wedge \rho_1 = \pm \int u_{r-3} \wedge \rho_1 = \pm \int u_{r-3} \wedge \bar{\partial} \rho_1$$

$$\vdots$$

$$= \pm \int u_0 \wedge \partial \rho_{r-1} = \pm \int \partial u_0 \wedge \rho_{r-1} = 0,$$

where the last identity follows from $\partial u_0 = 0$.

We conclude that

$$\int_X (\alpha + \partial \zeta + \bar{\partial} \xi + \bar{\partial} \eta) \wedge \beta = \int_X \alpha \wedge \beta.$$

• To prove independence of the choice of representative of the $E_r$-Aeppli class, let us modify a representative $\beta$ to some representative $\beta + \partial \zeta + \bar{\partial} \xi$ of the same $E_r$-Aeppli class. So, $\zeta$ and $\xi$ are arbitrary forms. We get:

$$\int_X \alpha \wedge (\beta + \partial \zeta + \bar{\partial} \xi) = \int_X \alpha \wedge \beta \pm \int \partial \alpha \wedge \zeta \pm \int \bar{\partial} \alpha \wedge \xi = 0,$$

where the last identity follows from $\partial \alpha = 0$ and $\bar{\partial} \alpha = 0$. $\square$

We now take up the issue of the non-degeneracy of the above bilinear pairing. For the sake of expediency, we start by defining the dual notion to the $E_r$-$\bar{E}_r$-closedness of Definition 3.37 after we have fixed a metric.

**Definition 3.42.** Let $(X, \omega)$ be a compact complex Hermitian manifold. Fix an integer $r \geq 2$ and a bidegree $(p, q)$.

We say that a form $\alpha \in C^\infty_{p,q}(X)$ is $E_r E_r^*$-closed with respect to the Hermitian metric $\omega$ if there exist smooth forms $a_1, \ldots, a_{r-1}$ and $b_1, \ldots, b_{r-1}$ such that the following two towers of $r-1$ equations are satisfied:

$$\partial^* \alpha = \bar{\partial}^* a_1$$
$$\partial^* a_1 = \bar{\partial}^* a_2$$
$$\vdots$$
$$\partial^* a_{r-2} = \bar{\partial}^* a_{r-1},$$

$$\bar{\partial}^* \alpha = \partial^* b_1$$
$$\bar{\partial}^* b_1 = \partial^* b_2$$
$$\vdots$$
$$\bar{\partial}^* b_{r-2} = \partial^* b_{r-1}.$$

That this notion is indeed dual to the $E_r E_r^*$-closedness via the Hodge star operator $\ast = \ast_\omega$ associated with the metric $\omega$ is the content of the following analogue of Corollary 2.8.

**Lemma 3.43.** In the setting of Definition 3.42, the following equivalence holds for every form $\alpha \in C^\infty_{p,q}(X)$:

$$\alpha \text{ is } E_r E_r^* \text{-closed } \iff \ast \alpha \text{ is } E_r E_r^* \text{-closed.}$$
Proof. Thanks to conjugations, to the fact that $\star = \pm \text{Id}$ (with the sign depending on the parity of the total degree of the forms involved) and to $\star$ being an isomorphism, the two towers of $r - 1$ equations that express the $E_r \bar{E}_r$-closedness of $\alpha$ (cf. (i) of Definition 3.37) translate to

\[
(- \star \bar{\partial} \star)(\star \bar{\alpha}) = (- \star \partial \star)(\star \bar{\eta}) \\
(- \star \partial \star)(\star \bar{\eta}) = \cdots = (- \star \partial \star)(\star \bar{\eta}_{r-2}) \quad \text{and} \quad (- \star \partial \star)(\star \bar{\eta}_{r-1}) = (- \star \partial \star)(\star \bar{\eta}_{r-1}).
\]

Now, put $a_j := \star \bar{\eta}_j$ and $b_j := \star \bar{\rho}_j$ for all $j \in \{1, \ldots, r - 1\}$. Since $- \star \bar{\partial} \star = \partial$ and $- \star \partial \star = \bar{\partial}$, these two towers amount to $\star \bar{\alpha}$ being $E_r^* \bar{E}_r^*$-closed. (See Definition 3.42).

We now come to two crucial lemmas from which Hodge isomorphisms for the $E_r$-Bott-Chern and the $E_r$-Aeppli cohomologies will follow. Based on the terminology introduced in (ii) of Definition 3.37, we define the vector spaces:

\[
F^{p,q}_{\bar{\partial}, r} := \{ \alpha \in C^\infty_{p,q}(X) \mid \bar{\partial} \alpha \text{ runs at least } r \text{ times} \},
\]

\[
F^{p,q}_{\partial, r} := \{ \beta \in C^\infty_{p,q}(X) \mid \partial \beta \text{ runs at least } r \text{ times} \}
\]

and their analogues $F^{p,q}_{\bar{\partial}, r}$ and $F^{p,q}_{\partial, r}$ when $\partial$ is replaced by $\partial^*$ and $\bar{\partial}$ is replaced by $\bar{\partial}^*$. Note that the space of $E_r^* \bar{E}_r^*$-closed $(p, q)$-forms defined in Definition 3.42 is precisely the intersection $F^{p,q}_{\bar{\partial}, r} \cap F^{p,q}_{\partial, r}$. 

**Lemma 3.44.** Let $(X, \omega)$ be a compact complex Hermitian manifold. Fix an integer $r \geq 2$, a bidegree $(p, q)$ and a form $\alpha \in C^\infty_{p,q}(X)$.

The following two statements are equivalent.

(i) $\alpha$ is $E_r^* \bar{E}_r^*$-closed (w.r.t. $\omega$); 

(ii) $\alpha$ is $L_\omega^2$-orthogonal to the space of smooth $E_r \bar{E}_r$-exact $(p, q)$-forms.

**Proof.** “(i) $\implies$ (ii)” Suppose that $\alpha$ is $E_r^* \bar{E}_r^*$-closed. This means that $\alpha$ satisfies the two towers of $(r - 1)$ equations in Definition 3.42. Let $\beta = \partial \zeta + \bar{\partial} \bar{\zeta} + \bar{\eta}$ be an arbitrary $E_r \bar{E}_r$-exact $(p, q)$-form. So, $\zeta$ and $\eta$ satisfy the respective towers of $r - 1$ equations under (ii) of Definition 3.37. For the $L_\omega^2$-inner product of $\alpha$ and $\beta$, we get:

\[
\langle \langle \alpha, \beta \rangle \rangle = \langle \langle \partial^* \alpha, \zeta \rangle \rangle + \langle \langle \bar{\partial}^* \alpha, \zeta \rangle \rangle + \langle \langle \bar{\partial}^* \alpha, \eta \rangle \rangle.
\]

Since $\bar{\partial}^* \partial^* \alpha = \bar{\partial}^* \partial^* a_1 = 0$, the middle term on the r.h.s. of (20) vanishes. 

For the first term on the r.h.s. of (20), we use the towers of equations satisfied by $\alpha$ and $\zeta$ to get:

\[
\langle \langle \partial^* \alpha, \zeta \rangle \rangle = \langle \langle \bar{\partial}^* a_1, \zeta \rangle \rangle = \langle \langle a_1, \partial \zeta \rangle \rangle = \langle \langle a_1, \partial v_{r-3} \rangle \rangle = \langle \langle \partial^* a_1, v_{r-3} \rangle \rangle = \langle \langle \partial^* a_2, v_{r-3} \rangle \rangle = \langle \langle a_2, \partial v_{r-3} \rangle \rangle = \langle \langle a_2, \partial v_{r-4} \rangle \rangle = \cdots = \langle \langle a_{r-1}, \partial v_0 \rangle \rangle = 0
\]
forms are of the shape \( \beta \) where the last identity followed from the property \( \bar{\partial}v_0 = 0 \).

For the last term on the r.h.s. of (20), we use the towers of equations satisfied by \( \alpha \) and \( \eta \) to get:

\[
\langle \langle \bar{\partial}^* \alpha, \eta \rangle \rangle = \langle \langle \bar{\partial}^* b_1, \eta \rangle \rangle = \langle \langle b_1, \bar{\partial} \eta \rangle \rangle = \langle \langle b_1, \bar{\partial} u_{r-3} \rangle \rangle = \langle \langle \bar{\partial}^* b_1, u_{r-3} \rangle \rangle = \langle \langle \bar{\partial}^* b_2, u_{r-3} \rangle \rangle = \langle \langle b_2, \bar{\partial} u_{r-3} \rangle \rangle = \langle \langle b_2, \bar{\partial} u_{r-4} \rangle \rangle
\]

\[
\vdots
\]

\[
= \langle \langle b_{r-1}, \bar{\partial} u_0 \rangle \rangle = 0,
\]

where the last identity followed from the property \( \bar{\partial}w_0 = 0 \).

“(ii) \implies (i)” Suppose that \( \alpha \) is orthogonal to all the smooth \( E_r \overline{E}_r \)-exact \((p, q)\)-forms \( \beta \). These forms are of the shape \( \beta = \bar{\partial} \xi + \bar{\partial} \bar{\partial} \xi + \bar{\partial} \eta \), where \( \xi \) is subject to no condition, while \( \zeta \in \mathcal{E}_{\bar{\partial}, r-1}^{p-1, q} \) and \( \eta \in \mathcal{E}_{\bar{\partial}, r-1}^{p, q-1} \). (See notation introduced in the proof of Lemma 3.39).

The orthogonality condition is equivalent to the following three identities:

\[
(a) \langle \langle \bar{\partial}^* \alpha, \xi \rangle \rangle = 0, \quad (b) \langle \langle \bar{\partial}^* \alpha, \zeta \rangle \rangle = 0, \quad (c) \langle \langle \bar{\partial}^* \alpha, \eta \rangle \rangle = 0
\]

holding for all the forms \( \zeta, \xi, \eta \) satisfying the above conditions.

Since \( \xi \) is subject to no condition, (a) amounts to \( \bar{\partial}^* \partial^* \alpha = 0 \). This means that \( \partial^* \alpha \in \ker \bar{\partial}^* \) and \( \bar{\partial} \alpha \in \ker \bar{\partial}^* \). Condition (b) requires \( \partial^* \alpha \perp \mathcal{E}_{\bar{\partial}, r-1}^{p-1, q} \), while (c) requires \( \bar{\partial}^* \alpha \perp \mathcal{E}_{\bar{\partial}, r-1}^{p, q-1} \).

- **Unravelling condition (b).** The forms \( \zeta \in \mathcal{E}_{\bar{\partial}, r-1}^{p-1, q} \) are characterised by the existence of forms \( v_{r-3}, \ldots, v_{0} \) satisfying the first tower of \((r - 1)\) equations in (iii) of Definition 3.37. That tower imposes the condition \( v_{r-j} \in \mathcal{E}_{\bar{\partial}, r-j+1} \cap \mathcal{F}_{\partial, j-2} \) for every \( j \in \{3, \ldots, j\} \). (We have dropped the superscripts to lighten the notation.)

Now, every form \( \zeta \in \ker \Delta'' \) satisfies the condition \( \bar{\partial} \zeta = 0 \), hence \( \zeta \in \mathcal{E}_{\bar{\partial}, 1} \subseteq \mathcal{E}_{\bar{\partial}, r-1} \). From condition (b), we get \( \partial^* \alpha \perp \ker \Delta'' \). Since \( \ker \bar{\partial}^* \) (to which \( \partial^* \alpha \) belongs by condition (a)) is the orthogonal direct sum between \( \ker \Delta'' \) and \( \text{Im} \bar{\partial}^* \), we get \( \partial^* \alpha \in \text{Im} \bar{\partial}^* \), so

\[
\partial^* \alpha = \bar{\partial}^* a_1 \tag{21}
\]

for some form \( a_1 \). Condition (b) becomes:

\[
0 = \langle \langle \partial^* \alpha, \zeta \rangle \rangle = \langle \langle a_1, \bar{\partial} \zeta \rangle \rangle = \langle \langle a_1, \bar{\partial} v_{r-3} \rangle \rangle = \langle \langle \partial^* a_1, v_{r-3} \rangle \rangle \quad \text{for all} \quad v_{r-3} \in \mathcal{E}_{\bar{\partial}, r-2} \cap \mathcal{F}_{\partial, 1}.
\]

In other words, \( \partial^* a_1 \perp (\mathcal{E}_{\bar{\partial}, r-2} \cap \mathcal{F}_{\partial, 1}) \).

We will now use the 3-space decomposition (53) of \( C_{p, q}^\infty (X) \) for the case \( r = 2 \). (See Proposition 6.2 in Appendix one.) It is immediate to check the inclusion \( \mathcal{E}_{\bar{\partial}, r-2} \cap \mathcal{F}_{\partial, 1} \supseteq \mathcal{H}_2 \oplus (\text{Im} \bar{\partial} + \bar{\partial} (\mathcal{E}_{\bar{\partial}, 1})) \). Therefore, condition (b) implies that \( \partial^* a_1 \perp (\mathcal{H}_2 \oplus (\text{Im} \bar{\partial} + \bar{\partial} (\mathcal{E}_{\bar{\partial}, 1}))) \). Since the orthogonal complement of \( \mathcal{H}_2 \oplus (\text{Im} \bar{\partial} + \bar{\partial} (\mathcal{E}_{\bar{\partial}, 1})) \) is \( \partial^*(\mathcal{E}_{\bar{\partial}, 1}) + \text{Im} \bar{\partial}^* \) by the 3-space decomposition (53) for \( r = 2 \), we infer that \( \partial^* a_1 \in \partial^*(\mathcal{E}_{\bar{\partial}, 1}) + \text{Im} \bar{\partial}^* \). Therefore, there exist forms \( b_1 \in \ker \bar{\partial}^* \) and \( a_2 \) such that

\[
\partial^* a_1 = \bar{\partial}^* b_1 + \bar{\partial}^* a_2 \tag{22}
\]

Since \( \bar{\partial}^* b_1 = 0 \), equations (21) and (22) yield:

\[
\partial^* \alpha = \bar{\partial}^* (a_1 - b_1) \quad \text{and} \quad \partial^* (a_1 - b_1) = \bar{\partial}^* a_2. \tag{23}
\]

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Thus, condition (b) becomes:
\[
0 = \langle \langle \partial^* \alpha, \zeta \rangle \rangle = \langle \langle \bar{\partial}^*(a_1 - b_1), \zeta \rangle \rangle = \langle \langle a_1 - b_1, \partial v_{r-3} \rangle \rangle = \langle \langle \partial^*(a_1 - b_1), v_{r-3} \rangle \rangle = \langle \langle \partial^* a_2, v_{r-3} \rangle \rangle = \langle \langle a_2, \partial v_{r-4} \rangle \rangle = \langle \langle \partial^* a_2, v_{r-4} \rangle \rangle \text{ for all } v_{r-4} \in E_{\bar{\partial}, r-3} \cap F_{\bar{\partial}, 2}.
\]
In other words, \( \partial^* a_2 \perp (E_{\bar{\partial}, r-3} \cap F_{\bar{\partial}, 2}) \).

Now, it is immediate to check the inclusion \( E_{\bar{\partial}, r-3} F_{\bar{\partial}, 2} \subset \mathcal{H}_3 \oplus (\text{Im } \bar{\partial} + \partial (E_{\bar{\partial}, 2})) \). Since the orthogonal complement of \( \mathcal{H}_3 \oplus (\text{Im } \bar{\partial} + \partial (E_{\bar{\partial}, 2})) \) is \( \partial^* (E_{\bar{\partial}, r-2}) + \text{Im } \partial^* \) by the 3-space decomposition (53) for \( r = 3 \), we infer that \( \partial^* a_2 \in \partial^* (E_{\bar{\partial}, r-2}) + \text{Im } \partial^* \). Therefore, there exist forms \( b_2 \in E_{\bar{\partial}, r-2} \) and \( a_3 \) such that
\[
\partial^* a_2 = \partial^* b_2 + \bar{\partial}^* a_3.
\]
Since the condition \( b_2 \in E_{\bar{\partial}, r-2} \) translates to the equations
\[
\bar{\partial}^* b_2 = \partial^* c_1 \quad \text{and} \quad \bar{\partial}^* c_1 = 0,
\]
for some form \( c_1 \), equations (23) and (24) yield:
\[
\begin{align*}
\partial^* \alpha &= \bar{\partial}^* (a_1 - b_1 - c_1) \\
\partial^* (a_1 - b_1 - c_1) &= \bar{\partial}^* (a_2 - b_2) \\
\partial^* (a_2 - b_2) &= \bar{\partial}^* a_3.
\end{align*}
\]
Continuing in this way, we inductively get the following tower of \((r - 1)\) equations:
\[
\begin{align*}
\partial^* \alpha &= \bar{\partial}^* (a_1 - b_1 - c_1 - c_1^{(3)} - \cdots - c_1^{(r-2)}) \\
\partial^* (a_1 - b_1 - c_1) &= \bar{\partial}^* (a_2 - b_2 - c_2^{(3)} - \cdots - c_2^{(r-2)}) \\
\vdots \\
\partial^* (a_{r-2} - b_{r-2}) &= \bar{\partial}^* a_{r-1},
\end{align*}
\]
where \( b_j \in E_{\bar{\partial}, r-j} \) for all \( j \in \{1, \ldots, r-2\} \), so \( b_j \) satisfies the following tower of \( j \) equations:
\[
\begin{align*}
\bar{\partial}^* b_j &= \partial^* c_{j-1}^{(j)} \\
\bar{\partial}^* c_{j-1}^{(j)} &= \partial^* c_{j-2}^{(j)} \\
\vdots \\
\bar{\partial}^* c_2^{(j)} &= \partial^* c_1^{(j)} \\
\bar{\partial}^* c_1^{(j)} &= 0,
\end{align*}
\]
for some forms \( c_{i}^{(j)} \).

Consequently, conditions (a) and (b) to which \( \alpha \) is subject imply that \( \alpha \in F_{\partial^*, r-1} \) (cf. tower (26)), which is the first of the two conditions required for \( \alpha \) to be \( E_{\partial^*} E^*_{r} \)-closed under Definition 3.42.

- **Unravelling condition (c).** Proceeding in a similar fashion, with \( \partial^* \) and \( \bar{\partial}^* \) permuted, we infer that conditions (a) and (c) to which \( \alpha \) is subject imply that \( \alpha \in F_{\bar{\partial}^*, r-1} \), which is the second of the two conditions required for \( \alpha \) to be \( E_{\partial^*} E^*_{r} \)-closed under Definition 3.42.
- We conclude that \( \alpha \) is indeed \( E_{\partial^*} E^*_{r} \)-closed. \( \square \)

The immediate consequence of Lemma 3.44 is the following *Hodge isomorphism* for the \( E_r \)-Bott-Chern cohomology.
Corollary and Definition 3.45. Let \((X, \omega)\) be a compact complex Hermitian manifold. For every bidegree \((p, q)\) and every \(r \in \mathbb{N}^+\), every \(E_r\)-Bott-Chern cohomology class \(\{\alpha\}_{E_r,BC} \in E_{r,BC}^{p,q}(X)\) can be represented by a unique form \(\alpha \in C_{p,q}^{\infty}(X)\) satisfying the following three conditions:

\(\alpha\) is \(\partial\)-closed, \(\bar{\partial}\)-closed and \(E_r^r\)-closed.

Any such form \(\alpha\) is called \(E_r\)-Bott-Chern harmonic with respect to the metric \(\omega\).

There is a vector-space isomorphism depending on the metric \(\omega\):

\[
E_{r,BC}^{p,q}(X) \simeq H_{r,BC}^{p,q}(X),
\]
where \(H_{r,BC}^{p,q}(X) \subset C_{p,q}^{\infty}(X)\) is the space of \(E_r\)-Bott-Chern harmonic \((p, q)\)-forms associated with \(\omega\).

Of course, the above isomorphism maps every class \(\{\alpha\}_{E_r,BC} \in E_{r,BC}^{p,q}(X)\) to its unique \(E_r\)-Bott-Chern harmonic representative.

The analogous statement for the \(E_r\)-Aeppli cohomology follows at once from standard material. Indeed, it is classical that the \(L_2^2\)-orthogonal complement of \(\text{Im} \partial\) (resp. \(\text{Im} \bar{\partial}\)) in \(C_{p,q}^{\infty}(X)\) is \(\ker \partial^*\) (resp. \(\ker \bar{\partial}^*\)). The immediate consequence of this is the following Hodge isomorphism for the \(E_r\)-Aeppli cohomology.

Corollary and Definition 3.46. Let \((X, \omega)\) be a compact complex Hermitian manifold. For every bidegree \((p, q)\), every \(E_r\)-Aeppli cohomology class \(\{\alpha\}_{E_r,A} \in E_{r,A}^{p,q}(X)\) can be represented by a unique form \(\alpha \in C_{p,q}^{\infty}(X)\) satisfying the following three conditions:

\(\alpha\) is \(E_r\)-\(E_r\)-closed, \(\partial^*\)-closed and \(\bar{\partial}^*\)-closed.

Any such form \(\alpha\) is called \(E_r\)-Aeppli harmonic with respect to the metric \(\omega\).

There is a vector-space isomorphism depending on the metric \(\omega\):

\[
E_{r,A}^{p,q}(X) \simeq H_{r,A}^{p,q}(X),
\]
where \(H_{r,A}^{p,q}(X) \subset C_{p,q}^{\infty}(X)\) is the space of \(E_r\)-Aeppli harmonic \((p, q)\)-forms associated with \(\omega\).

Of course, the above isomorphism maps every class \(\{\alpha\}_{E_r,A} \in E_{r,A}^{p,q}(X)\) to its unique \(E_r\)-Aeppli harmonic representative.

We can now conclude from the above results that there is a Serre-type canonical duality between the \(E_r\)-Bott-Chern cohomology and the \(E_r\)-Aeppli cohomology of complementary bidegrees.

Theorem 3.47. Let \(X\) be a compact complex manifold with \(\dim_{\mathbb{C}} X = n\). For all \(p, q \in \{0, \ldots, n\}\), the following bilinear pairing is well defined and non-degenerate:

\[
E_{r,BC}^{p,q}(X) \times E_{r,A}^{n-p,n-q}(X) \rightarrow \mathbb{C}, \quad \left(\{\alpha\}_{E_r,BC}, \{\beta\}_{E_r,A}\right) \mapsto \int_X \alpha \wedge \beta.
\]
Proof. The well-definedness was proved in Proposition 3.41. The non-degeneracy is proved in the usual way on the back of the above preliminary results, as follows.

Let \( \{ \alpha \}_{E_{r,BC}} \subseteq E_{r,BC}^{p,q}(X) \) be an arbitrary non-zero class. Fix an arbitrary Hermitian metric \( \omega \) on \( X \) and let \( \alpha \) be the unique \( E_r \)-Bott-Chern harmonic representative (w.r.t. \( \omega \)) of the class \( \{ \alpha \}_{E_{r,BC}} \) (whose existence and uniqueness are guaranteed by Corollary and Definition 3.45). In particular, \( \alpha \) is \( E_r \)-Aeppli harmonic. In particular, \( \star \alpha \) represents an \( \alpha \)-Aeppli harmonicity given in Corollaries and Definitions 3.45 and 3.46, Lemma 3.43 and the standard equivalences \( \leftrightarrow \) (whose existence and uniqueness are guaranteed by Corollary and Definition 3.45). In particular, \( \alpha \neq 0 \).

Based on the characterisations of the \( E_r \)-Bott-Chern and \( E_r \)-Aeppli harmonicities given in Corollaries and Definitions 3.45 and 3.46, Lemma 3.43 and the standard equivalences \( \leftrightarrow \) and \( \alpha \) is \( E_r \)-Aeppli harmonic. In particular, \( \star \alpha \) represents an \( E_r \)-Aeppli class \( \{ \star \alpha \}_{E_{r,A}} \subseteq E_{r,A}^{n-p,n-q}(X) \). Moreover, pairing \( \{ \alpha \}_{E_{r,BC}} \) with \( \{ \star \alpha \}_{E_{r,A}} \) yields \( \int_X \alpha \wedge \star \alpha = ||\alpha||^2 \neq 0 \), where \( || \cdot || \) stands for the \( L^2_w \)-norm.

Similarly, starting off with a non-zero class \( \{ \beta \}_{E_{r,A}} \subseteq E_{r,A}^{n-p,n-q}(X) \) and selecting its unique \( E_r \)-Aeppli harmonic representative \( \beta \), we get that \( \beta \neq 0 \), \( \star \beta \) is \( E_r \)-Bott-Chern harmonic (hence it represents a class in \( E_{r,BC}^{p,q}(X) \)) and the classes \( \{ \star \beta \}_{E_{r,BC}} \) and \( \{ \beta \}_{E_{r,A}} \) pair to \( \pm ||\beta||^2 \neq 0 \). \( \Box \)

3.6 Characterisation in terms of exactness properties

We now state and prove the following higher-page analogue of Proposition 3.25.

**Theorem 3.48.** Let \( X \) be a compact complex manifold with \( \dim \mathbb{C}X = n \). Fix an arbitrary integer \( r \geq 1 \). The properties \( (A) \) and \( (B) \) below are equivalent.

(\( A \)) \( X \) is a page-\( (r-1) \)-\( \partial \bar{\partial} \)-manifold.

(\( B \)) For all \( p,q \in \{0, \ldots ,n\} \) and for every form \( \alpha \in C^\infty_{p,q}(X) \) such that \( d\alpha = 0 \), the following equivalences hold:

\[
\alpha \in \text{Im}d \iff \alpha \text{ is } E_r \text{-exact} \iff \alpha \text{ is } E_r \bar{E}_r \text{-exact} \iff \alpha \text{ is } E_r \bar{E}_r \text{-exact.} \tag{27}
\]

Except for one step, the proof is purely algebraic, so let us first do the algebra in the following

**Lemma 3.49.** Let \( (A, \partial_1, \partial_2) \) be a bounded double complex of vector spaces. Then, property \( (B) \) in Theorem 3.48 (with \( \partial_1 \) in place of \( \partial \) and \( \partial_2 \) in place of \( \bar{\partial} \)) is equivalent to \( A \) being isomorphic to a direct sum of squares, even-length zigzags of length \( < 2r \) and odd-length zigzags of length \( 2l + 1 \leq 2r - 1 \) and type \( M \).

**Proof.** Since \( (A, \partial_1, \partial_2) \) is always isomorphic to a sum of indecomposable complexes, it suffices to check each possible indecomposable summand separately. We refer to Theorem 3.15 for diagrams of all possibilities and will use the same notation as the one introduced there.

- **Case of squares**
  Every \( d \)-closed pure-type form is a multiple of \( \partial_2 \partial_1 a = -\partial_1 \partial_2 a \) and the latter form is easily seen to be exact in all four ways appearing in \( (B) \), so the properties are equivalent in this case.

- **Case of even-length zigzags**
  In the type 1 case, every \( d \)-closed pure-type form is a multiple of some \( \partial_1 a_i \), so we have to investigate these forms more closely. They have the following properties.
1. All the $\partial_1 a_i$'s are $d$-exact (indeed, $\partial_1 a_i = d(a_1 + \cdots + a_i)$) and $\partial_1$-(hence $\overline{E}_r$)-exact.

2. Using the tower of equations in the definition of $E_r$-exactness, one sees that $\partial_1 a_i$ is $E_r$-exact if and only if $i + 1 \leq r$.

3. Since for this double complex $\partial_1 \partial_2 = 0$ and a nontrivial tower of equations for $\overline{E}_r$-exactness can never be found, $\partial_1 a_i$ is $E_r$-exact if and only if it is $E_r \overline{E}_r$-exact.

Hence, one sees that for even-length zigzags, all four properties are equivalent if and only if $l < r$. The type 2 case is handled analogously.

- **Case of odd-length zigzags**
  For type $M$, the pure-type $d$-closed forms are again the $\partial_1 a_i$'s. Their exactness properties are as follows.

  1. Each $\partial_1 a_i$ is $d$-, $\partial_1$- and $\partial_2$-exact, so in particular also $E_r$- and $\overline{E}_r$-exact.

  2. Using the towers of equations in the definition of $E_r \overline{E}_r$-exactness, we see that $\partial_1 a_i$ is $E_r \overline{E}_r$-exact if and only if $i < r$ or $l - i + 1 < r$.

    Indeed, if $\partial_1 a_i$ is viewed as $\partial_1$-exact with potential $a_i$, it is $E_r \overline{E}_r$-exact if and only if $\partial_2 a_i$ reaches 0 in at most $r - 1$ steps. Since $\partial_2 a_i = -\partial_1 a_{i-1}, \partial_2 a_{i-1} = -\partial_1 a_{i-2}, \ldots, \partial_2 a_2 = -\partial_1 a_1, \partial_2 a_1 = 0$, this is the case if and only if $i \leq r - 1$.

    Meanwhile, if $\partial_1 a_i = -\partial_2 a_{i+1}$ is viewed as $\partial_2$-exact with potential $a_{i+1}$, it is $E_r \overline{E}_r$-exact if and only if $\partial_1 a_{i+1}$ reaches 0 in at most $r - 1$ steps. Since $\partial_1 a_{i+1} = -\partial_2 a_{i+2}, \partial_1 a_{i+2} = -\partial_2 a_{i+3}, \ldots, \partial_1 a_{i+1} = 0$, this is the case if and only if $l - i + 1 \leq r - 1$.

    Hence, one sees that in this type of zigzags the exactness properties are equivalent for all the bidegrees if and only if $l + 1 < r$. (In particular, this is always true for $l = 0$).

    It is left to check type $L$: the $d$-closed pure forms are the $\partial_1 a_i$. None of these is $d$-exact, but all except $\partial_1 a_1$ are $\partial_2$-exact, hence $E_r$-exact and all but $\partial_2 a_1$ are $\partial_1$-exact, so $\overline{E}_r$-exact. In particular, the exactness properties under $(B)$ are never equivalent. \hfill $\Box$

*Proof of Theorem 3.48.* The symmetry of occuring zigzags along the antidiagonal $p + q = n$ stated in Lemma 3.22 exchanges, among the odd-length zigzags of length $> 1$, those of type $M$ with those of type $L$. It also and sends dots to dots. Thus, by Lemma 3.49, in the case of $A = A_X = (C^\infty, (X, \mathbb{C}), \partial, \overline{\partial})$ for a compact connected complex manifold $X$, Property $(B)$ in Theorem 3.48 is equivalent to the existence of a decomposition of $A_X$ into squares, odd-length zigzags of length one (giving rise to pure De Rham classes) and even-length zigzags of length $2r$ (responsible for possible differentials in early pages). This, in turn, has already been seen to be equivalent to the page-$(r - 1)$-$\partial \overline{\partial}$-property of $X$. \hfill $\Box$

### 3.7 Characterisation in terms of maps to and from higher-page Bott-Chern and Aeppli cohomology groups

Let $X$ be an $n$-dimensional compact complex manifold. Fix $r \in \mathbb{N}^*$ and a bidegree $(p, q)$. Recall that $\mathcal{Z}^{p,q}_r$ and $\mathcal{C}^{p,q}_r$ stand for the space of $E_r$-closed, resp. $E_r$-exact, smooth $(p, q)$-forms on $X$. (See section 2.2.) Let $\mathcal{D}^{p,q}_r$ stand for the space of $E_r \overline{E}_r$-exact smooth $(p, q)$-forms on $X$.  

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Lemma 3.50. (i) The following inclusions of vector subspaces of $C^\infty_{p+1,q}(X)$ hold:

$$\text{Im}(\partial \bar{\partial}) \subset \partial(Z^p_r) \subset D^p+1,q_r \subset C^p+1,q_r \cap \ker d \cap \text{Im} d$$ (28)

(ii) Every $E_r$-class $\{\alpha\}_{E_r} \in E^p,q_r(X)$ can be represented by a $d$-closed form if and only if $\partial(Z^p_r) \subset \text{Im}(\partial \bar{\partial})$. In other words, this happens if and only if the first inclusion in (28) is an equality.

Proof. (i) To prove the first inclusion, it suffices to show that every $\bar{\partial}$-exact $(p, q)$-form is $E_r$-closed. Let $\alpha = \bar{\partial} \beta$ be a $(p, q)$-form. Then, $\partial \alpha = 0$ and $\partial \alpha = \bar{\partial}(-\partial \beta)$. Putting $u_1 := -\partial \beta$, we have $\partial u_1 = 0$, so we can choose $u_2 = 0, \ldots, u_{r-1} = 0$ to satisfy the tower of equations under (i) of Proposition 2.3. This shows that $\alpha$ is $E_r$-closed.

To prove the second inclusion, let $\alpha \in Z^p_r$. By (i) of Proposition 2.3, this implies that $\bar{\partial} \alpha = 0$, so if we write $\partial \alpha = \partial \zeta + \bar{\partial} \xi + \partial \eta$ with $\zeta = \alpha$, $\xi = 0$ and $\eta = 0$, we satisfy the conditions under (ii) of Definition 3.37 with $v_j = 0$ and $u_j = 0$ for all $j \in \{0, \ldots, r-3\}$. This proves that $\partial \alpha$ is $E_r$-$\bar{\partial}$-exact.

The third inclusion on the first row is a consequence of (iii) and (iv) of Lemma 3.38, while the “vertical” inclusion is a translation of (iv) of the same lemma.

(ii) Let $\{\alpha\}_{E_r} \in E^p,q_r(X)$ be an arbitrary class and let $\alpha$ be an arbitrary representative of it. Then, $\{\alpha\}_{E_r} \in E^p,q_r(X)$ can be represented by a $d$-closed form if and only if there exists an $E_r$-exact form $\rho = \partial a + \bar{\partial} b$, with $a$ satisfying the conditions $\bar{\partial} a = \partial c_{r-3}$, $\bar{\partial} c_{r-3} = \partial c_{r-4}$, $\ldots$, $\bar{\partial} c_0 = 0$ for some forms $c_j$, such that $\partial(\alpha - \rho) = 0$. This last identity is equivalent to $\partial \bar{\partial} b = \partial \alpha$.

Thus, the class $\{\alpha\}_{E_r}$ contains a $d$-closed form if and only if the form $\partial \alpha$, which already lies in $\partial(Z^p_r)$, is $\partial \bar{\partial}$-exact.

This proves the contention. \hfill \Box

Theorem 3.51. Let $X$ be a compact complex manifold with dim $\subset X = n$. Fix an arbitrary integer $r \geq 2$. The following properties are equivalent.

(A) $X$ is a page-$(r-1)$-$\partial \bar{\partial}$-manifold.

(C) For all $p, q \in \{0, \ldots, n\}$, the following identities of vector subspaces of $C^\infty_{p+1,q,r}(X)$ hold:

$$\begin{align*}
(i) \quad & \text{Im}(\partial \bar{\partial}) = \partial(Z^p_r) \\
(ii) \quad & C^p,q_r \cap \ker d = \text{Im} \ d.
\end{align*}$$ (29)

Proof. By (ii) of Lemma 3.50, identity (i) in (C) is equivalent to every $E_r$-class of type $(p, q)$ being representable by a $d$-closed form. On the other hand, if this is the case, then identity (ii) in (C) is equivalent to the map $E^p,q_r(X) \ni \{\alpha\}_{E_r} \mapsto \{\alpha\}_{DR} \in H^p,q_{DR}(X, \mathbb{C})$ (with $\alpha \in \ker d$) being well defined and injective.

The contention follows from Theorem and Definition 3.10. \hfill \Box

As an aside, note that identity (ii) in (C) of Theorem 3.51 is a reformulation of the first equivalence in (B) of Theorem 3.48.
We will now relate the page-$(r - 1)$-$\partial\bar{\partial}$-property of compact complex manifolds to the $E_r$-Bott-Chern and $E_r$-Aeppli cohomologies introduced in §3.5. The study will reveal an analogy with the standard $\partial\bar{\partial}$-property in relation to the standard Bott-Chern and Aeppli cohomologies (corresponding to the case $r = 1$).

**Lemma 3.52.** Let $X$ be a compact complex manifold with $\dim \mathbb{C}X = n$. Fix any integer $r \geq 2$.

(i) There are well-defined canonical linear maps induced by the identity:

\[
E^{p,q}_{r,BC}(X) \xrightarrow{T^{p,q}_{r}} E^{p,q}_{r}(X) \xrightarrow{S^{p,q}_{r}} E^{p,q}_{r,A}(X) \quad \text{and} \quad E^{p,q}_{r,BC}(X) \longrightarrow H^{p,q}_{DR}(X) \longrightarrow E^{p,q}_{r,A}(X)
\]

\[
\{\alpha\}_{E_r,BC} \mapsto \{\alpha\}_{E_r} \mapsto \{\alpha\}_{E_r,A}
\]

(ii) The following equivalences hold:

\[
T^{p,q}_{r} \ \text{is injective} \iff \mathcal{D}^{p,q}_{r} \supset \mathcal{C}^{p,q}_{r} \cap \ker d \iff \mathcal{D}^{p,q}_{r} = \mathcal{C}^{p,q}_{r} \cap \ker d
\]

\[
T^{p,q}_{r} \ \text{is surjective} \iff \text{Im}(\partial\bar{\partial}) \supseteq \partial(\mathcal{Z}^{p,q}_{r}) \iff \text{Im}(\partial\bar{\partial}) = \partial(\mathcal{Z}^{p,q}_{r})
\]

\[
S^{p,q}_{r} \ \text{is injective} \iff \mathcal{C}^{p,q}_{r} \cap \ker d \subset \mathcal{C}^{p,q}_{r}
\]

\[
S^{p,q}_{r} \ \text{is surjective} \iff \text{Im}(\partial\bar{\partial}) \supseteq \bar{\partial}(\mathcal{Z}^{p,q}_{r}) \iff \text{Im}(\partial\bar{\partial}) = \bar{\partial}(\mathcal{Z}^{p,q}_{r})
\]

where $\mathcal{Z}^{p,q}_{r}$ stands for the space of smooth $E_r\mathbb{E}_r$-closed $(p, q)$-forms.

(iii) If the map $T^{p,q}_{r}$ is bijective, the identity induces a well-defined surjection

\[
E^{p,q}_{r}(X) \longrightarrow H^{p,q}_{DR}(X),
\]

in the sense that every class $\{\alpha\}_{E_r} \in E^{p,q}_{r}(X)$ contains a $d$-closed representative and the linear map $E^{p,q}_{r}(X) \ni \{\alpha\}_{E_r} \mapsto \{\alpha\}_{DR} \in H^{p,q}_{DR}(X)$ is independent of the choice of $d$-closed representative $\alpha$ of the class $\{\alpha\}_{E_r} \in E^{p,q}_{r}(X)$.

Proof. It consists of immediate verifications based on the definitions and is left to the reader. \qed

We now come to the main result of this subsection.

**Theorem 3.53.** Let $X$ be a compact complex manifold with $\dim \mathbb{C}X = n$. Fix an arbitrary integer $r \geq 1$. The following statements are equivalent.

(A) $X$ is a page-$(r - 1)$-$\partial\bar{\partial}$-manifold.

(D) For all $p, q \in \{0, \ldots, n\}$, the canonical linear maps $T^{p,q}_{r} : E^{p,q}_{r,BC}(X) \longrightarrow E^{p,q}_{r}(X)$ and $S^{p,q}_{r} : E^{p,q}_{r}(X) \longrightarrow E^{p,q}_{r,A}(X)$ are isomorphisms.

(E) For all $p, q \in \{0, \ldots, n\}$, the canonical linear map $S^{p,q}_{r} \circ T^{p,q}_{r} : E^{p,q}_{r,BC}(X) \longrightarrow E^{p,q}_{r,A}(X)$ is injective.

Proof. “(A) $\implies$ (D)” Suppose that $X$ is a page-$(r - 1)$-$\partial\bar{\partial}$-manifold. Thanks to Theorem 3.51, the page-$(r - 1)$-$\partial\bar{\partial}$-property of $X$ is equivalent to the first inclusion on the left in (28) being an identity (which is further equivalent to $T^{p,q}_{r}$ being surjective) and to the last space on the right in
(28) being equal to $\text{Im} \, d$ (which, after conjugation of its occurence in bidegree $(q, p)$, implies that $S^p_\beta$ is injective).

On the other hand, the equivalence “$\alpha$ is $E_r$-exact $\iff$ $\alpha$ is $E_r \overline{E}_r$-exact”, ensured for $d$-closed forms $\alpha$ by characterisation (B) of the page-$(r - 1) - \partial \bar{\partial}$-property given in Theorem 3.48, is equivalent to the third inclusion on the first row in (28) being an identity. Thanks to (ii) of Lemma 3.52, this is further equivalent to $T^{p,q}_r$ being injective. Thus, $T^{p,q}_r$ is bijective.

It remains to show that $S^p_\beta$ is surjective. The duality results of Corollary 2.9 and Theorem 3.47 ensure that the dual map of $S^p_\beta$ is injective, hence $T^{p,q}_r$ is injective, for every $(p, q)$. By (ii) of Lemma 3.52, the injectivity of $T^{p,q}_r$ translates to the following equivalence for all $d$-closed $(p, q)$-forms $\alpha$:

$$
\alpha \text{ is } E_r\text{-exact } \iff \alpha \text{ is } E_r \overline{E}_r\text{-exact}.
$$

Since this holds in every bidegree $(p, q)$, taking conjugates we get the following equivalence for $d$-closed forms $\alpha$ of every bidegree:

$$
\alpha \text{ is } \overline{E}_r\text{-exact } \iff \alpha \text{ is } E_r \overline{E}_r\text{-exact}.
$$

Thanks to characterisation (B) of the page-$(r - 1) - \partial \bar{\partial}$-property given in Theorem 3.48, it remains to prove the implication

$$
\alpha \text{ is } d\text{-exact } \implies \alpha \text{ is } E_r \overline{E}_r\text{-exact}
$$

for every $d$-closed form $\alpha$ of every bidegree $(p, q)$. (Recall that the implication “$\alpha$ is $E_r \overline{E}_r\text{-exact } \implies \alpha$ is $d\text{-exact}$” always holds by (iii) of Lemma 3.38.) Therefore, let $\alpha = dv = \partial v^{p-1,q} + \bar{\partial} v^{p,q-1} \in C^\infty_{p,q}(X)$ be $d$-exact. Then, $\alpha \in \text{Im} \partial + \text{Im} \bar{\partial}$, hence $\{\alpha\}_{E_r,A} = 0 \in E^p_\beta q(X)$. Since $S^p_\beta \circ T^{p,q}_r : E^p_\beta q(X) \rightarrow E^p_\beta q(X)$ is injective, we get $\{\alpha\}_{E_r, BC} = 0 \in E^p_\beta q(X)$, which translates to $\alpha$ being $E_r \overline{E}_r\text{-exact}$.

3.8 Numerical necessary condition for the page-$r - \partial \bar{\partial}$-property

In this subsection, for every fixed $r \in \mathbb{N}^*$, we prove an inequality involving the dimensions of the $E_r$-Bott-Chern, $E_r$-Aeppli and De Rham cohomology groups of any compact complex manifold. It is the exact analogue in the case $r \geq 2$ of (and coincides in the case $r = 1$ with) the inequality given by Angella and Tomassini in [AT13] when $r = 1$. Our presentation will mirror theirs to some extent.

Let $X$ be an $n$-dimensional compact complex manifold. Fix an arbitrary $r \in \mathbb{N}^*$. For every bidegree $(p, q)$, we define $\mathbb{C}$-vector spaces $A^p_\beta q, B^p_\beta q, C^p_\beta q, D^p_\beta q, E^p_\beta q$ and $F^p_\beta q$ that are analogues in the case $r \geq 2$ of (and coincide in the case $r = 1$ with) the vector spaces introduced by Varouchas in [Var86] and used in [AT13], in such a way that we get two exact sequences:

$$
0 \rightarrow D^p_\beta q \hookrightarrow E^p_\beta q(X) \xrightarrow{T^{p,q}_r \overline{BC}} E^p_\beta q(X) \xrightarrow{T^{p,q}_r \overline{BC}} F^p_\beta q \rightarrow 0. 
$$

$$
0 \rightarrow A^p_\beta q \hookrightarrow B^p_\beta q \xrightarrow{T^{p,q}_r \overline{BC}} E^p_\beta q(X) \xrightarrow{T^{p,q}_r \overline{A}} E^p_\beta q(X) \xrightarrow{T^{p,q}_r \overline{A}} C^p_\beta q \rightarrow 0,
$$

(30)
Finally, we compute the kernel of the map $T_{r,BC}^{p,q}$ and the cokernel of $T_{r,A}^{p,q}$ and find

$$D_r^{p,q} := \ker T_{r,BC}^{p,q} = \frac{\partial(E_{\delta,r-1}^{p-1,q}) + \bar{\partial}(\ker(\partial \bar{\partial})))}{D_r^{p,q}}$$

and

$$C_r^{p,q} := \text{coker} T_{r,A}^{p,q} = \frac{Z_{r+r}^{p,q}}{\text{Im} \partial + Z_r^{p,q}},$$

where $D_r^{p,q}$ stands for the space of smooth $E_rE_r$-exact $(p, q)$-forms, while $Z_r^{p,q}$ stands for the space of smooth $E_rE_r$-closed $(p, q)$-forms. Then, we consider their conjugate spaces:

$$B_r^{p,q} := D_r^{\overline{p},\overline{q}} = \frac{\partial(\ker(\partial \bar{\partial})) + \bar{\partial}(E_{\delta,r-1}^{p,q-1})}{D_r^{p,q}}$$

and

$$E_r^{p,q} := C_r^{\overline{p},\overline{q}} = \frac{Z_r^{p,q}}{Z_r^{p,q} + \text{Im} \partial \bar{\partial}}.$$

Finally, we compute the kernel of the map $T_{r,B}^{p,q}$ and the cokernel of $T_{r,E}^{p,q}$ and find

$$A_r^{p,q} := \ker T_{r,B}^{p,q} = \frac{\partial(E_{\delta,r-1}^{p-1,q}) + \bar{\partial}(E_{\delta,r-1}^{p,q-1}) + [\partial(\ker(\partial \bar{\partial})) \cap \bar{\partial}(\ker(\partial \bar{\partial}))]}{D_r^{p,q}}$$

$$F_r^{p,q} := \text{coker} T_{r,E}^{p,q} = \frac{Z_{r+r}^{p,q}}{Z_r^{p,q} + Z_{r+r}^{p,q}}.$$

The computations leading to formulae (32), (33), (34) will be performed as part of the proof of Lemma 3.55. Admitting these formulae for the moment, we sum up the first properties of the vector spaces defined above in the following

**Lemma 3.54.** Let $X$ be a compact complex $n$-dimensional manifold.

(i) The following identities hold in every bidegree $(p, q)$:

$$A_r^{p,q} = A_r^{p,q}$$

and

$$F_r^{p,q} = F_r^{p,q}.$$  $B_r^{p,q} = D_r^{\overline{p},\overline{q}}$  and  $C_r^{p,q} = E_r^{p,q}.$

Moreover, the following inclusions and surjections (defined by the identity map on forms) hold:

$$A_r^{p,q} \subset D_r^{p,q} \subset E_r^{p,q}(X) \quad \text{and} \quad E_r^{p,q}(X) \twoheadrightarrow C_r^{p,q} \twoheadrightarrow F_r^{p,q}$$

in every bidegree $(p, q)$. Likewise, the conjugated relations hold:

$$A_r^{p,q} \subset B_r^{p,q} \subset E_r^{p,q}(X) \quad \text{and} \quad E_r^{p,q}(X) \twoheadrightarrow E_r^{p,q} \twoheadrightarrow F_r^{p,q}.$$  

(ii) The following canonical bilinear pairings are well defined and non-degenerate, hence define dualities:

$$A_r^{p,q} \times F_r^{n-p,n-q} \rightarrow \mathbb{C}, \quad B_r^{p,q} \times \overline{E_r}^{n-p,n-q} \rightarrow \mathbb{C} \quad \text{and} \quad C_r^{p,q} \times D_r^{n-p,n-q} \rightarrow \mathbb{C},$$

for every bidegree $(p, q)$, where in each of the three cases every pair $(\{\alpha\}, \{\beta\})$ of classes of respective bidegrees $(p, q)$ and $(n-p, n-q)$ is mapped to $\int_X \alpha \wedge \beta \in \mathbb{C}.$

(iii) Consequently, denoting by a lower-case letter the dimension of the vector space denoted by the corresponding capital letter, we get the following dimension identities:

$$a_r^{p,q} = a_r^{p,q} = f_r^{n-p,n-q} = f_r^{n-q,n-p} \quad \text{and} \quad b_r^{p,q} = d_r^{p,q} = e_r^{n-p,n-q} = e_r^{n-q,n-p},$$

for every bidegree $(p, q).$
Proof. (i) is obvious. Let us only mention that the inclusion $A^p,q_r \subset D^p,q_r$ follows from the inclusion $E^p,q_{r,-1} \subset \ker(\partial \bar{\partial})$, while the surjection $C^p,q_r \to E^p,q_r$ follows from the inclusion $\operatorname{Im} \bar{\partial} \subset \overline{Z^p,q_r}$.

(ii) Let us first prove the well-definedness (i.e. the independence of the choice of representatives of the cohomology classes involved) of these pairings.

- In the case of $A^p,q_r$ and $F^{n-p,n-q}_r$, let $\alpha = \partial \xi + \bar{\partial} \eta + \rho$ be a $(p, q)$-form representing an $A_r$-class (with $\xi \in E_{\partial, r-1}$, $\eta \in E_{\partial, r-1}$ and $\rho \in \partial(\ker(\partial \bar{\partial})) \cap \bar{\partial}(\ker(\partial \bar{\partial})))$ and let $\beta \in Z_{r+1}$ be an $(n-p, n-q)$-form representing an $F_r$-class.

That $\int_X \alpha \wedge \beta$ remains unchanged when $\alpha$ is modified in the same $A_r$-cohomology class (= the same $E_{r, BC}$-cohomology class) follows from the well-definedness of the duality between $E^p,q_{r, BC}$ and $E^{n-p,n-q}_{r, A}$ proved in Proposition 3.41.

If $\beta$ is modified in the same $F^{n-p,n-q}_r$-cohomology class to some $\beta + u + \bar{v}$, with $u \in Z^{p,q}_r$ and $v \in Z^{q,p}_r$ (hence $\partial u = 0, \bar{\partial} u = \bar{\partial} \rho_1, \partial \rho_1 = \bar{\partial} \rho_2, \ldots, \partial \rho_{r-2} = \bar{\partial} \rho_{r-1}$ and $\bar{\partial} v = 0, \partial v = \partial \xi_1, \bar{\partial} \xi_1 = \bar{\partial} \xi_2, \ldots, \partial \xi_{r-2} = \bar{\partial} \xi_{r-1}$ for some forms $\rho_j$ and $\xi_j$ of the relevant bidegrees), then

$$\int_X \alpha \wedge (\beta + u + \bar{v}) = \int_X \alpha \wedge \beta \pm \int_X \xi \wedge \partial u \pm \int_X \eta \wedge \bar{\partial} u \pm \int_X \xi \wedge \bar{\partial} \bar{v} \pm \int_X \eta \wedge \partial \bar{v}$$

$$= \int_X \alpha \wedge \beta \pm \int_X \xi \wedge \partial \rho_1 \pm \int_X \eta \wedge \bar{\partial} \xi_1 = \int_X \alpha \wedge \beta,$$

where Stokes’s theorem was applied several times and the last identity follows, after two bouts of mathematical ping-pong, from $\xi \in E_{\partial, r-1}$ coupled with the property of $\rho_1$, as well as from $\eta \in E_{\partial, r-1}$ coupled with the property of $\xi_1$.

- In the case of $B^{p,q}_r$ and $\tilde{E}^{n-p,n-q}_r$, let $\alpha = \partial \xi + \bar{\partial} \eta$ be a $(p, q)$-form representing a $B_r$-class (with $\xi \in \ker(\partial \bar{\partial})$ and $\eta \in \overline{E^{p,q}_{\partial, r-1}}$) and let $\beta \in Z_{r+1}$ be an $(n-p, n-q)$-form representing an $\tilde{E}_r$-class.

That $\int_X \alpha \wedge \beta$ remains unchanged when $\alpha$ is modified in the same $B_r$-cohomology class (= the same $E_{r, BC}$-cohomology class) follows from the well-definedness of the duality between $E^p,q_{r, BC}$ and $E^{n-p,n-q}_{r, A}$ proved in Proposition 3.41.

If $\beta$ is modified in the same $\tilde{E}^{p,q}_r$-cohomology class to some $\beta + \bar{u} + v$, with $u \in Z^{p,q}_r$ (hence $\partial u = 0, \partial \bar{u} = \bar{\partial} \rho_1, \partial \rho_1 = \bar{\partial} \rho_2, \ldots, \partial \rho_{r-2} = \bar{\partial} \rho_{r-1}$ for some forms $\rho_j$ of the relevant bidegree) and $v = \bar{\partial} \eta \in \operatorname{Im} \bar{\partial}$, then

$$\int_X \alpha \wedge (\beta + \bar{u} + v) = \int_X \alpha \wedge \beta \pm \int_X \xi \wedge \partial \bar{u} \pm \int_X \eta \wedge \bar{\partial} \bar{u} \pm \int_X \xi \wedge \partial v \pm \int_X \eta \wedge \bar{\partial} v$$

$$= \int_X \alpha \wedge \beta \pm \int_X \eta \wedge \bar{\partial} \rho_1 \pm \int_X \partial \xi \wedge \eta$$

$$= \int_X \alpha \wedge \beta \pm \int_X \partial \eta \wedge \rho_1 = \int_X \alpha \wedge \beta,$$
So, the last space can be called the inclusion space of these pairings.

The case of $C_{r}^{p,q}$ and $D_{r}^{n-p,n-q}$ follows by conjugating the arguments of the case $B_{r}^{p,q}$ and $E_{r}^{n-p,n-q}$.

Now to the proof of the non-degeneracy of these pairings.

The $A_{r}^{p,q}$ duality. Fix an arbitrary Hermitian metric $\omega$ on $X$. Since the elements of $A_{r}^{p,q}$ form a subspace of $E_{r}$-Bott-Chern cohomology classes, each can be represented by a unique $E_{r}$-Bott-Chern harmonic $(p, q)$-form. We know from the $E_{r}$-Bott-Chern/$E_{r}$-Aeppli duality of Theorem 3.47 that a $(p, q)$-form $\alpha$ is $E_{r}$-Bott-Chern harmonic if and only if the $(n - p, n - q)$-form $\star \alpha$ is $E_{r}$-Aeppli harmonic. Since $F_{r}^{n-p,n-q}$ is a quotient space of $E_{r}^{n-p,n-q}(X)$ obtained from the same $E_{r}$-Aeppli-closed forms by enlarging the space that is quotiented out (i.e. by enlarging the classes), we immediately get that every non-zero class (represented by the $E_{r}$-Bott-Chern-harmonic form $\alpha$ it contains) in $A_{r}^{p,q}$, when paired with the class in $F_{r}^{n-p,n-q}$ represented by $\star \alpha$, produces a non-zero (even positive) number $\int_{X} \alpha \wedge \star \alpha = ||\alpha||^{2} > 0$, where $|| \cdot ||$ stands for the $L_{2}^{2}$-norm.

Conversely, to prove that for every non-zero class in $F_{r}^{n-p,n-q}$ there exists a class in $A_{r}^{p,q}$ such that they pair up to a non-zero number, we use the $L_{2}^{-}$-orthogonal decomposition of Proposition 6.2:

$$C_{n-p,n-q}^{\infty}(X) = Z_{r}^{n-p,n-q} \oplus (\partial^{*}(E_{\partial, r-1}^{n-p+1,n-q}) + \text{Im} \partial^{*}).$$  \hspace{1cm} (35)

From this and the definition of $F_{r}^{n-p,n-q}$, we get:

$$F_{r}^{n-p,n-q} \simeq Z_{r}^{n-p,n-q} \cap (Z_{r}^{n-p,n-q} + Z_{\bar{r}}^{n-q,n-p})^{\perp} = Z_{r}^{n-p,n-q} \cap (Z_{r}^{n-p,n-q})^{\perp} \cap (Z_{\bar{r}}^{n-q,n-p})^{\perp}$$

$$= Z_{r}^{n-p,n-q} \cap [\partial^{*}(E_{\partial, r-1}^{n-p+1,n-q}) + \text{Im} \partial^{*}] \cap [\bar{\partial}^{*}(E_{\bar{\partial}, r-1}^{n-p,n-q+1}) + \text{Im} \bar{\partial}^{*}]$$

$$= Z_{r}^{n-p,n-q} \cap [\partial^{*}(E_{\partial, r-1}^{n-p+1,n-q}) + \bar{\partial}^{*}(E_{\bar{\partial}, r-1}^{n-p,n-q+1}) + (\text{Im} \partial^{*} \cap \text{Im} \bar{\partial}^{*})].$$  \hspace{1cm} (36)

So, the last space can be called the $F_{r}$-harmonic space in bidegree $(n - p, n - q)$.

To finish the proof of the non-degeneracy of the $A_{r}^{p,q} \leftrightarrow F_{r}^{n-p,n-q}$ pairing, it suffices to prove the inclusion

$$\bar{x}

\left(Z_{r}^{n-p,n-q} \cap [\partial^{*}(E_{\partial, r-1}^{n-p+1,n-q}) + \bar{\partial}^{*}(E_{\bar{\partial}, r-1}^{n-p,n-q+1}) + (\text{Im} \partial^{*} \cap \text{Im} \bar{\partial}^{*})]\right) \subset$

$$H_{r,BC}^{p,q}(X) \cap \left(\partial(E_{\partial, r-1}^{p-1,q-1}) + \bar{\partial}(E_{\bar{\partial}, r-1}^{p,q-1}) + [\partial(\text{ker}(\partial \bar{\partial})) \cap \bar{\partial}(\text{ker}(\partial \bar{\partial}))]\right),$$  \hspace{1cm} (37)

where $\bar{x}$ is the map $\alpha \mapsto \star \overline{\alpha}$. Note that the last vector space, consisting of the $E_{r}$-Bott-Chern harmonic $A_{r}$-closed $(p, q)$-forms, is isomorphic to $A_{r}^{p,q}$.

Now, the standard identity $\partial^{*} = - \star \partial \star$ and its conjugate imply that

$$\bar{x}(\text{Im} \partial^{*} \cap \text{Im} \bar{\partial}^{*}) = \text{Im} \partial \cap \text{Im} \bar{\partial} = \partial(\text{ker}(\partial \bar{\partial})) \cap \bar{\partial}(\text{ker}(\partial \bar{\partial})),$$

while Lemma 3.43 and Corollary and Definition 3.45 say that

$$\bar{x}(Z_{r}^{n-p,n-q}) = Z_{r}^{p,q} \subset H_{r,BC}^{p,q}(X),$$

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where $\mathcal{Z}^{p,q}_{r,r}$ stands for the space of smooth $E^*_r$-closed $(p, q)$-forms. Meanwhile, using again the standard identity $\partial^* = -\ast \bar{\partial} \ast$ and its conjugate, we easily get the identity:

$$
\ast \mathcal{E}_{\partial^*, r-1}^{n-p+1, n-q} = \mathcal{E}_{\partial, r-1}^{q, p-1}
$$

and its conjugate, hence the inclusions

$$
\bar{\pi}(\partial^*(\mathcal{E}_{\partial^*, r-1}^{n-p+1, n-q})) \subset \partial(\mathcal{E}_{\partial, r-1}^{p-1, q}) \quad \text{and} \quad \bar{\pi}(\bar{\partial}^*(\mathcal{E}_{\partial^*, r-1}^{n-p+1, q+1})) \subset \bar{\partial}(\mathcal{E}_{\partial, r-1}^{p, q-1}).
$$

These relations add up to (37).

- **The $C^{p,q}_r \leftrightarrow D^{n-p,n-q}_r$ duality.** The proof of the non-degeneracy of this pairing follows a similar pattern to the previous one, so we will only point out the main arguments. Decomposition (35) and the definition of $C^{p,q}_r$ yield:

$$
C^{p,q}_r \simeq Z^{p,q}_{r} \cap [\text{Im } \partial + Z^{p,q}_{r}]^\perp = Z^{p,q}_{r} \cap (\text{Im } \partial)^\perp \cap (Z^{p,q}_{r})^\perp \\
= Z^{p,q}_{r} \cap \ker \partial^* \cap [\partial^*(\mathcal{E}_{\partial^*, r-1}^{p+1, q}) + \text{Im } \bar{\partial}^*] \\
= Z^{p,q}_{r} \cap [\partial^*(\mathcal{E}_{\partial^*, r-1}^{p+1, q}) + \bar{\partial}^*(\ker(\partial^*)]].
$$

(38)

So, the last space can be called the $C_r$-harmonic space in bidegree $(p, q)$.

It suffices to prove the inclusion:

$$
\bar{\pi}

\left(

Z^{p,q}_{r} \cap [\partial^*(\mathcal{E}_{\partial^*, r-1}^{p+1, q}) + \bar{\partial}^*(\ker(\partial^*)]]

\right) \subset H^{n-p,n-q}_{r,BC}(X) \cap \left( \partial(\mathcal{E}_{\partial, r-1}^{n-p-1, n-q}) + \bar{\partial}((\ker \partial)) \right),

$$

(39)

where $\bar{\pi}$ is the map $\alpha \mapsto \ast \bar{\alpha}$. Note that the last vector space, consisting of the $E_r$-Bott-Chern harmonic $D_r$-closed $(n-p, n-q)$-forms, is isomorphic to $D^{n-p,n-q}_r$.

As noticed in the previous duality, we have:

$$
\bar{\pi}(Z^{p,q}_{r}) = Z^{n-p,n-q}_{r,r} \subset H^{n-p,n-q}_{r,BC}(X) \quad \text{and} \quad \bar{\pi}(\partial^*(\mathcal{E}_{\partial^*, r-1}^{p+1, q})) \subset \partial(\mathcal{E}_{\partial, r-1}^{n-p-1, n-q}).
$$

Meanwhile, the standard identity $\partial^* = -\ast \bar{\partial} \ast$ and its conjugate yield

$$
\bar{\pi}(\bar{\partial}^*(\ker(\partial^*)]) = \bar{\partial}((\ker \partial)).
$$

Putting these pieces of information together, we get (39).

- **The $B^{p,q}_r \leftrightarrow \tilde{E}^{p,n-q}_r$ duality.** The arguments are obtained by conjugating those of the previous case.

(iii) follows immediately from (i) and (ii).

We now extend to an arbitrary page of the Frölicher spectral sequence an observation of [Var86] that played a key role in [AT13]. The next result bears out the claims made at the beginning of this subsection, in particular the formulae (32), (33) and (34) for the vector spaces $A^{p,q}_r$, $B^{p,q}_r$, $C^{p,q}_r$, $D^{p,q}_r$, $\tilde{E}^{p,q}_r$ and $F^{p,q}_r$ that were originally defined as kernels or cokernels of certain linear maps.
Lemma 3.55. Let $X$ be a compact complex $n$-dimensional manifold. The sequences (30) and (31), in which all the maps are induced by the identity map on forms and easily seen to be well defined, are exact.

Moreover, they are dual to each other in complementary bidegrees (i.e. if (30) is considered in bidegree $(p, q)$, its dual is (31) considered in bidegree $(n - p, n - q)$).

Proof. The duality statement follows from Lemma 3.54, so we only need to prove the exactness statement. Recall that $E^{p,q}_r = Z^{p,q}_r/C^{p,q}_r$.

- We start by computing $\ker T^{p,q}_{r,A} = \{ \alpha \}_{E_r,BC} \in E^{p,q}_{r,BC}(X) \mid \alpha \in \ker \partial \cap \ker \bar{\partial} \cap C^{p,q}_r$. Since $C^{p,q}_r = \partial(E^{p,q-1}_{\partial,r-1}) + \text{Im} \bar{\partial}$, we get:

$$\ker \partial \cap \ker \bar{\partial} \cap C^{p,q}_r = \partial(E^{p,q-1}_{\partial,r-1}) + \bar{\partial}(\ker(\partial \bar{\partial})), $$

yielding the first part of formula (32) for $D^{p,q}_{r}$.

- Next, we compute

$$\text{coker} T^{p,q}_{r,A} = \frac{E^{p,q}_{r,A}(X)}{\text{Im} T^{p,q}_{r,A}} = \frac{Z^{p,q}_r}{(\text{Im} \partial + \text{Im} \bar{\partial})} = \frac{Z^{p,q}_r}{\text{Im} \partial + \text{Im} \bar{\partial} + Z^{p,q}_r} = \frac{Z^{p,q}_r}{\text{Im} \bar{\partial} + Z^{p,q}_r},$$

the last identity being a consequence of the inclusion $\text{Im} \bar{\partial} \subset Z^{p,q}_r$. This proves the latter part of formula (32) for $C^{p,q}_r$.

Now that formula (32) for $D^{p,*}_r$ and $C^{p,*}_r$ has been confirmed, so has formula (33) for $B^{p,*}_r$ and $E^{p,*}_r$. We can move on to the verification of the exactness of the sequences (30) and (31) at the $E^{p,*}_r(X)$ level. By duality, it suffices to prove it in the case of (31).

- To prove that (31) is exact at the $E^{p,q}_r(X)$ level, we first compute $\ker T^{p,q}_{r,A} = \{ \alpha \}_{E_r} \in E^{p,q}_r(X) \mid \alpha \in Z^{p,q}_r(X) \cap [\text{Im} \partial + \text{Im} \bar{\partial}]$. Since

$$Z^{p,q}_r(X) \cap [\text{Im} \partial + \text{Im} \bar{\partial}] = \partial(\ker(\partial \bar{\partial})) + \text{Im} \bar{\partial},$$

we get

$$\ker T^{p,q}_{r,A} = \frac{\partial(\ker(\partial \bar{\partial})) + \text{Im} \bar{\partial}}{C^{p,q}_r} = \frac{\partial(\ker(\partial \bar{\partial}))}{C^{p,q}_r},$$

the last identity being a consequence of the inclusion $\text{Im} \bar{\partial} \subset C^{p,q}_r$. Note that $C^{p,q}_r$ is not a subspace of $\partial(\ker(\partial \bar{\partial}))$; the meaning of the last quotient is that we mod out by $C^{p,q}_r \cap \partial(\ker(\partial \bar{\partial}))$. A similar remark will apply elsewhere in this proof.

On the other hand, taking into account formula (33) for $B^{p,q}_r$, we get

$$\text{Im} T^{p,q}_{r,B} = \frac{\partial(\ker(\partial \bar{\partial})) + \bar{\partial}(E^{p,q-1}_{\partial,r-1})}{C^{p,q}_r} = \frac{\partial(\ker(\partial \bar{\partial}))}{C^{p,q}_r},$$

the last identity being a consequence of the inclusions $\bar{\partial}(E^{p,q-1}_{\partial,r-1}) \subset \text{Im} \bar{\partial} \subset C^{p,q}_r$.

We conclude that $\ker T^{p,q}_{2,A} = \text{Im} T^{p,q}_{2,B}$, so the sequence (31) is exact at the $E^{p,q}_2(X)$ level.

- We now compute $\ker T^{p,q}_{r,B} = \{ \alpha \}_{E_r,BC} \in E^{p,q}_{r,BC}(X) \mid \alpha \in [\partial(\ker(\partial \bar{\partial})) + \bar{\partial}(E^{p,q-1}_{\partial,r-1})] \cap C^{p,q}_r$. 

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Since \([\partial(\text{ker}(\partial\bar{\partial})) + \partial(\mathcal{E}_{p,q-1}^{r-1})] \cap C^{p,q}_r = \partial(\mathcal{E}^{p-1,q}_r) + \partial(\mathcal{E}^{p,q-1}_r) + [\partial(\text{ker}(\partial\bar{\partial})) \cap \text{Im} \bar{\partial}]\) and since \(\partial(\text{ker}(\partial\bar{\partial})) \cap \text{Im} \bar{\partial} = \partial(\text{ker}(\partial\bar{\partial})) \cap \partial(\text{ker}(\partial\bar{\partial}))\), we get the first part of formula (34) for \(A^{p,q}_r\).

- Finally, we compute

\[
\text{coker} T_{r,E}^{p,q} = \frac{\tilde{E}_r^{p,q}}{\text{Im} T_{r,E}^{p,q}} = \frac{\mathcal{Z}_r^{p,q}}{\mathcal{Z}_r^{p,q} + \mathcal{Z}_r^{q,p} + \text{Im} \bar{\partial}} \quad \text{or} \quad \frac{\mathcal{Z}_r^{p,q}}{\mathcal{Z}_r^{p,q} + \mathcal{Z}_r^{q,p}}
\]

the last identity being a consequence of the inclusions \(C^{p,q}_1^{r} = \text{Im} \bar{\partial} \subset C^{p,q}_r \subset \mathcal{Z}^{p,q}_r\). This proves the latter part of formula (34) for \(F^{p,q}_r\).

Putting together Lemmas 3.54 and 3.55, we get the following numerical identity that harks back to [AT13, Theorem A]. The dimensions of the \(\mathbb{C}\)-vector spaces \(E^{p,q}_{r,BC}(X)\) and \(E^{p,q}_{r,A}(X)\) are denoted by \(h^{p,q}_{r,BC}\), resp. \(h^{p,q}_{r,A}\). The other lower-case letters stand for the dimensions of the vector spaces denoted by the corresponding capital letters, as in (iii) of Lemma 3.54.

**Corollary 3.56.** Let \(X\) be a compact complex \(n\)-dimensional manifold. Fix an arbitrary \(r \in \mathbb{N}^*\).

The following inequality holds in every bidegree \((p, q)\):

\[
h^{p,q}_{r,BC} + h^{p,q}_{r,A} \geq e^{p,q}_r + e^{q,p}_r.
\]

In particular, for every \(k \in \{0, \ldots, 2n\}\), we have:

\[
h^{k}_{r,BC} + h^{k}_{r,A} \geq 2e^{k}_r \geq 2b_k,
\]

where \(h^{k}_{r,BC} := \sum_{p+q=k} h^{p,q}_{r,BC}\) and \(h^{k}_{r,A}\) and \(e^{k}_r\) are defined analogously, while \(b_k\) is the \(k\)-th Betti number of \(X\).

**Proof.** From the exact sequence (30) in bidegree \((p, q)\), resp. the exact sequence (31) in bidegree \((q, p)\) (see Lemma 3.55), we infer the identities:

\[
h^{p,q}_{r,BC} = d^{p,q}_r + e^{p,q}_r - \bar{e}^{p,q}_r + f^{p,q}_r \quad \text{and} \quad h^{q,p}_{r,A} = a^{q,p}_r - h^{q,p}_r + e^{q,p}_r + e^{q,p}_r.
\]

Since \(d^{p,q}_r = h^{p,q}_{r,BC}\), summing up the above two identities and using the numerical relations obtained under (iii) of Lemma 3.54 to cancel the terms reoccurring with opposite signs, we get

\[
h^{p,q}_{r,BC} + h^{p,q}_{r,A} = (e^{p,q}_r + e^{q,p}_r) + (a^{p,q}_r + f^{p,q}_r).
\]

The contention follows.  

We are now ready to infer that equality occurring in all the inequalities (40) is a necessary condition for \(X\) to be a page-\((r - 1)\)-\(\partial\bar{\partial}\)-manifold.

**Corollary 3.57.** Let \(X\) be a compact complex \(n\)-dimensional manifold. Fix an arbitrary \(r \in \mathbb{N}^*\).

If \(X\) is a page-\((r - 1)\)-\(\partial\bar{\partial}\)-manifold, then

\[
h^{k}_{r,BC} + h^{k}_{r,A} = 2b_k
\]

for every \(k \in \{0, \ldots, 2n\}\).
Proof. Suppose $X$ is a page-$(r-1)-\partial\bar{\partial}$-manifold. Then, by Theorem and Definition 3.10, the Frölicher spectral sequence of $X$ degenerates at $E_r$. This fact is equivalent to $e_k = b_k$ for all $k$, namely to equality occurring in the latter inequality of (40).

On the other hand, $X$ being a page-$(r-1)-\partial\bar{\partial}$-manifold is equivalent to all the canonical linear maps $T_{r,A}^{p,q} : E_{r,A}^{p,q}(X) \to E_{r,A}^{p,q}(X)$ and $S_{r,A}^{p,q} : E_{r,A}^{p,q}(X) \to E_{r,A}^{p,q}(X)$ being isomorphisms (see Theorem 3.53). This implies that $h_{r,BC}^k + h_{r,A}^k = 2e_k^k$ for all $k$, namely that equality occurs in the former inequality of (40).

Surprisingly, when $r \geq 2$, having equality in all the inequalities (40) does not suffice to infer that $X$ is a page-$(r-1)-\partial\bar{\partial}$-manifold, as the following example shows. This stands in sharp contrast to the case $r = 1$ where the numerical condition is both necessary and sufficient, as shown in [AT13]. However, if all the inequalities (40) are equalities, the weaker property $E_r(X) = E_\infty(X)$ holds since it is equivalent to the latter inequality in (40) being an equality.

Example 3.58. Let $X_{u,v} = S^{2u+1} \times S^{2v+1}$, with $u, v \geq 0$, be equipped with the Calabi-Eckmann complex structure. Then,

$$h_{2,BC}^k + h_{2,A}^k = 2b_k$$

for all $k \in \{0, \ldots, 2n\}$. However, as seen in Lemma 3.35, $X_{u,v}$ is not page-$r-\partial\bar{\partial}$ for any $r$ if $u \neq v$ or $u = v = 1$.

Proof. As shown in [Ste18], the only zigzags that can occur with nonzero multiplicity in $A_{X_{u,v}} = (C^\infty, \partial, \bar{\partial})$ dots, even-length zigzags of length two and odd-length zigzags of length three. Since all the cohomologies involved vanish on squares and commute with direct sums, it suffices to check the equality for these types of zigzags, which is an easy exercise in the spirit of Lemma 3.49. For example, for an odd-length zigzag of type $L$ and length three, consisting of a form $a$ in some bidegree $(p, q)$ with $\partial\bar{\partial}a = 0$, $\partial a \neq 0$, $\bar{\partial}a \neq 0$, one has $h_{2,A}^r = 0$ for all $r$, $h_{2,BC}^r = h_{2,BC}^{r+1} = 1$, $h_{2,BC}^r = 0$ for $(r, s) \notin \{(p + 1, q), (p, q + 1)\}$ and $b_{p+q+1} = 1$ and $b_k = 0$ for $k \neq p + q + 1$. \qed

4 Some geometric properties of page-$r-\partial\bar{\partial}$-manifolds

We start with the following observation that follows at once from standard facts.

Observation 4.1. Fix any $r \in \mathbb{N}$. A compact complex surface is a page-$r-\partial\bar{\partial}$-manifold if and only if it is Kähler.

Proof. It is standard that the Frölicher spectral sequence of any compact complex surface degenerates at $E_1$. It is equally standard that $H^k_{DR}$ is always pure for $k = 0, 2, 4$, while it follows from the Buchdahl-Lamari results (see [Buc99] and [Lam99]) that $H^3_{DR}$ (and hence $H^3_{DR}$) is pure if and only if the surface is Kähler. \qed

Theorem 4.2. Let $X$ and $Y$ be compact complex manifolds.

1. If $X$ is a page-$r-\partial\bar{\partial}$-manifold and $Y$ is a page-$r'-\partial\bar{\partial}$-manifold, the product $X \times Y$ is a page-$\tilde{r}$-$\partial\bar{\partial}$-manifold, where $\tilde{r} = \max\{r, r'\}$.

Conversely, if the product is a page-$r-\partial\bar{\partial}$-manifold, so are both factors.
2. For any vector bundle \( V \) over \( X \), the projectivised bundle \( \mathbb{P}(V) \) is a page-\( r - \partial \bar{\partial} \)-manifold if and only if \( X \) is.

3. Suppose \( X \) is a page-\( r - \partial \bar{\partial} \)-manifold. Let \( f : X \rightarrow Y \) be a surjective holomorphic map and assume there exists a \( d \)-closed \((l,l)\)-current \( \Omega \) on \( X \) (with \( l = \dim X - \dim Y \)) such that \( f^* \Omega \neq 0 \). Then \( Y \) is also a page-\( r - \partial \bar{\partial} \)-manifold.

In particular, this implication always holds when \( \dim X = \dim Y \), e.g. for contractions (take \( \Omega \) to be a constant).

4. Given a a submanifold \( Z \subset X \), denote by \( \widetilde{X} \) the blow-up of \( X \) along \( Z \). If \( X \) is page-\( r - \partial \bar{\partial} \) and \( Z \) is page-\( r' - \partial \bar{\partial} \), then \( \widetilde{X} \) is a page-\( r - \partial \bar{\partial} \) manifold, where \( \tilde{r} = \max\{r, r'\} \).

Conversely, if \( \widetilde{X} \) is page \( r - \partial \bar{\partial} \), so are \( X \) and \( Z \).

5. The page-\( r - \partial \bar{\partial} \)-property of compact complex manifolds is a bimeromorphic invariant if and only if it is stable under passage to submanifolds.

Proof. The proofs are very similar to those in [Ste18, Cor. 28]. We will be using the characterisation of the page-\( r - \partial \bar{\partial} \)-property in terms of occuring zigzags (Theorem 3.16) and \( E_1 \)-isomorphisms (Def. 3.18), in particular Lemma 3.20.

Writing \( A_X \) as shorthand for the double complex \((C_{\infty,\infty}(X, \mathbb{C}, \partial, \bar{\partial}))\) and \( A_X[i] \) for the shifted double complex with bigrading \((A_X[i])^p,q := A_X^{p-i, q-i}\), we have the following \( E_1 \)-isomorphisms:

\[
A_X \otimes A_Y \cong \bigoplus_{i=0}^{\text{rk} V - 1} A_X[i] \quad (42)
\]

\[
A_{\mathbb{P}(V)} \cong \bigoplus_{i=0}^{\dim Y - 1} A_X[i] \quad (43)
\]

\[
A_X \cong A_Y \oplus A_X/f^* A_Y \quad (44)
\]

\[
A_\widetilde{X} \cong A_X \oplus \bigoplus_{i=1}^{\text{codim } Z - 1} A_Z[i] \quad (45)
\]

Since the occurring zigzags get only shifted, \( A_X[i] \) satisfies the page-\( r - \partial \bar{\partial} \)-property if and only if \( A_X \) does. Furthermore, a direct sum of complexes satisfies the page-\( r - \partial \bar{\partial} \)-property if and only if each summand does. So, the second, third and fourth \( E_1 \)-isomorphisms imply 2., 3. and 4.

For the first part of (1), we use the first isomorphism and the fact that one knows how irreducible subcomplexes behave under tensor product (see [Ste18, Prop. 16]). In particular, even-length zigzags do not get longer and the product of two length-one zigzags is again of length one. For the converse, note that \( A_X \) and \( A_Y \) are direct summands in their tensor product, so we can argue as before.

The 'if' statement in the last part of (5) is a direct consequence of (4) and the weak factorization theorem [AKMW02], which says that every bimeromorphic map can be factored as a sequence of

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Footnote: Proved in [Ste18, Sect. 4] and [Ste19], apart from the case of dominations of unequal dimensions, i.e. the maps addressed in the third isomorphism, which is treated in [Meng19b], together with some related work. Cf. also [RYY19], [YY18], [Meng19c] and [ASTT19] for different approaches to the blow-up question in the setting of particular cohomologies.
blow-ups and blow-downs with smooth centres. The ‘only if’ part also follows from 4. (cf. also [Me19a]). Indeed, let $X$ be page-$r$-$\partial\bar{\partial}$ and let $Z \subset X$ be a submanifold. If $Z$ has codimension one, we replace $X$ by $X \times \mathbb{P}^1$ (which is still page-$r$-$\partial\bar{\partial}$ by 1.) and $Z$ by $Z \times \{0\}$. By assumption, the blow-up is still page-$r$-$\partial\bar{\partial}$ and one can apply (4) to infer that the same holds for $Z$.

Since for surfaces and threefolds, the centre of a nontrivial blow-up is a point or a curve, we get

**Corollary 4.3.** Fix any $r \in \mathbb{N}$. The page-$r$-$\partial\bar{\partial}$-property of compact complex surfaces and threefolds is a bimeromorphic invariant.

Since it can be proved by the same methods as above, let us also record the following result, although it will not be used later in this article. According to [Pop19], a compact complex manifold $X$ is a bimeromorphic invariant. According to [Pop19], a compact complex manifold $X$ is an easy generalisation of an observation in [Pop15]:

**Lemma 4.4.** A compact complex manifold $X$ is $E_r$-sGG if and only if $T_r = 0$.

As a consequence of this, we get the bimeromorphic invariance of the $E_r$-sGG property.

**Corollary 4.5.** Let $X$ and $\tilde{X}$ be bimeromorphically equivalent compact complex manifolds.

Then, every Gauduchon metric on $X$ is $E_r$-sG if and only if this is true on $\tilde{X}$.

**Proof.** By the weak factorisation theorem [AKMW02], it suffices again to check this for blow-ups $\tilde{X} \to X$ with $d$-dimensional smooth centers $Z$ of codimension $\geq 2$. After picking any isomorphism realising formula (45), any class $c \in H^{n-1,n-1}_A(\tilde{X})$ can be written as $c = c_X + c_Z$, with $c_X \in H^{n-1,n-1}_A(X)$ and $c_Z \in H^d,1(Z)$. So, $T_r c = T_r c_X + T_r c_Z = T_r c_X$ since $\partial\eta = 0$ for all $(d,d)$-forms on $Z$ for dimension reasons.

Note that the above map $T_r$ is given in all cases by applying $\partial$. Generally speaking, if $A = B \oplus C$, then $H_A(A) = H_A(B) \oplus H_A(C)$ and $E_r(A) = E_r(B) \oplus E_r(C)$ and $T^A_r = T^B_r + T^C_r$. We omitted the superscripts on $T_r$ in the above proof for the sake of simplicity.

## 5 Deformations of page-1-$\partial\bar{\partial}$-manifolds

We first take up some general issues in the theory of deformations of complex structures and then we move on to specific deformation properties of our new class of manifolds introduced in this paper.

### 5.1 Background

Let $X$ be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. Recall that small deformations of the complex structure of $X$ over a base $B$ may be described by smooth $T^{1,0}X$-valued $(0, 1)$-forms $\psi(t) \in C^\infty_0(X, T^{1,0}X)$ depending on $t \in B$ and satisfying the integrability condition

$$\partial\bar{\partial}\psi(t) = \frac{1}{2} [\psi(t), \psi(t)].$$

(46)
In fact, given such a \( \psi \), the space of \((0,1)\)-tangent vectors for the complex structure determined by \( \psi \) is given by \((\text{Id} + \psi)T^{0,1}_X\).

Let \( t = (t_1, \ldots, t_m) \in \mathbb{C}^m \) with \( m = \dim_\mathbb{C} H^{0,1}(X, T^{1,0}X) \). Writing

\[
\psi(t) = \psi_1(t) + \sum_{\nu=2}^{+\infty} \psi_\nu(t)
\]

for the Taylor expansion of \( \psi \) around 0, (so each \( \psi_\nu(t) \) is a homogeneous polynomial of degree \( \nu \) in the variables \( t = (t_1, \ldots, t_m) \)), the integrability condition is easily seen to be equivalent to \( \overline{\partial} \psi_1(t) = 0 \) and the following sequence of conditions:

\[
\overline{\partial} \psi_\nu(t) = \frac{1}{2} \sum_{\mu=1}^{\nu-1} [\psi_\mu(t), \psi_{\nu-\mu}(t)] \quad (\text{Eq. } (\nu)), \quad \nu \geq 2.
\]

The Kuranishi family of \( X \) is said to be unobstructed if there exists a choice \( \{\beta_1, \ldots, \beta_m\} \) of representatives of cohomology classes that form a basis \( \{[\beta_1], \ldots, [\beta_m]\} \) of \( H^{0,1}(X, T^{1,0}X) \) such that the integrability condition is satisfied (i.e. all the equations (Eq. \((\nu)\)) are solvable) for any choice of parameters \( (t_1, \ldots, t_m) \in \mathbb{C}^m \) defining \( \psi_1(t) = t_1 \beta_1 + \cdots + t_m \beta_m \).

By the fundamental result of [Kur62], when this qualitative condition is satisfied, a convergent solution \( \psi(t) \) can be built for small \( t \) through an inductive construction of the \( \psi_\nu(t) \)'s from the given \( \psi_1(t) \) by solving the equations (Eq. \((\nu)\)) and choosing at every step the solution with minimal \( L^2 \) norm for a preassigned Hermitian metric on \( X \). The r.h.s. of each of these equations is \( \overline{\partial} \)-closed, so the only obstruction to solvability is the possible non-\( \overline{\partial} \)-exactness of the r.h.s. The resulting (germ of a) family \( (X_t)_{t \in \Delta} \) of complex structures on \( X \) is called the Kuranishi family of \( X \). (It depends on the metric, but different choices of metrics yield isomorphic families.) If it is unobstructed, its base \( \Delta \) is smooth and can be viewed as an open ball about 0 in the cohomology vector space \( H^{0,1}(X, T^{1,0}X) \).

If, moreover, the canonical bundle \( K_X \) of \( X \) is trivial (and we will call \( X \) a Calabi-Yau manifold in that case), there exists a (unique up to scalar multiplication) smooth non-vanishing holomorphic \((n,0)\)-form \( u \) on \( X \) which induces an isomorphism

\[
H^{0,1}(X, T^{1,0}X) \ni [\theta] \mapsto [\theta, u] \in H^{-n,1}_\overline{\partial}(X) = E^{n-1,1}_1(X),
\]

that we call the Calabi-Yau isomorphism. In particular, \( \Delta \) can be viewed as an open ball around 0 in \( H^{-n,1}_\overline{\partial}(X) \) in this case.

**Example 5.1.** (The Kuranishi family of the 5-dimensional Iwasawa-type manifold)

Let us now consider the specific example of the complex parallelisable nilmanifold \( X = I^{(5)} \) of complex dimension 5. Its complex structure is described by five holomorphic \((1,0)\)-forms \( \varphi_1, \ldots, \varphi_5 \) satisfying the equations

\[
d\varphi_1 = d\varphi_2 = 0, \quad d\varphi_3 = \varphi_1 \wedge \varphi_2, \quad d\varphi_4 = \varphi_1 \wedge \varphi_3, \quad d\varphi_5 = \varphi_2 \wedge \varphi_3.
\]

If \( \theta_1, \ldots, \theta_5 \) form the dual basis of \((1,0)\)-vector fields, then \( [\theta_i, \theta_j] = 0 \) except in the following cases:

\[
[\theta_1, \theta_2] = -\theta_3, \quad [\theta_1, \theta_3] = -\theta_4, \quad [\theta_2, \theta_3] = -\theta_5,
\]

hence also

\[
[\theta_2, \theta_1] = \theta_3, \quad [\theta_3, \theta_1] = \theta_4, \quad [\theta_3, \theta_2] = \theta_5.
\]

In particular, \( H^{0,1}(X, T^{1,0}X) \ni [\varphi_i \otimes \theta_i], [\varphi_2 \otimes \theta_1] \mid i = 1, \ldots, 5 \), so \( \dim_\mathbb{C} H^{0,1}(X, T^{1,0}X) = 10 \).
This manifold is the 5-dimensional analogue of the 3-dimensional Iwasawa manifold $I^{(3)}$. The following fact was observed in [Rol11].

**Proposition 5.2.** The Kuranishi family of the 5-dimensional nilmanifold $I^{(5)}$ is unobstructed.

*Proof.* It was given in [Rol11]. ☐

### 5.2 Cohomological triviality of complex parallelisable deformations of nilmanifolds

For complex parallelisable nilmanifolds $X = G/\Gamma$, where $G$ is a simply connected complex Lie group and $\Gamma \subseteq G$ a lattice, the Dolbeault cohomology can be computed by left invariant forms (cf. [Sak76]). In particular, one has (cf. [Nak75])

$$H^1(X, T^{1,0}_X) \cong H^1(X, \mathcal{O}_X) \otimes g^{1,0} = \ker \bar{\partial} \cap A^{0,1}_g \otimes g^{1,0},$$

where $g$ is the Lie algebra of complex Lie group $G$.

Furthermore, $g$ is actually a complex Lie algebra and $g^{1,0} \subseteq g_\mathbb{C}$ is a complex subalgebra. In fact, one has an identification of complex Lie algebras $g \cong g^{1,0}$ given by $z \mapsto \frac{1}{2}(z - iJz)$. In what follows, we will always tacitly use the above identifications.

Of particular interest are the cohomology classes in $H_{par} := H^1(X, \mathcal{O}_X) \otimes Z(g) = \ker \bar{\partial} \cap A^{0,1}_g \otimes Z(g)$ due to the following theorem.

**Theorem 5.3.** ([Rol11]) Let $X = G/\Gamma$ be a complex parallelisable nilmanifold. Let $\mu \in H^1(X, \Theta_X)$. The following statements are equivalent.

1. $\mu$ defines an ‘infinitesimally complex parallelisable deformation’, i.e. $\mu \in H^1(X, \mathcal{O}_X) \otimes Z(g)$.
2. For all $X, Y \in g$, one has $[X, \mu Y] = 0$.
3. $t\mu$ induces a 1-parameter family of complex parallelisable manifolds for $t$ small enough.

For each such $\mu$, the sequence of equations 5.1 is solvable with $\psi = \psi_1 = \mu$.

We will now show that the cohomology is the same for all the complex parallelisable small deformations of a given complex parallelisable nilmanifold $X = G/\Gamma$.

**Theorem 5.4.** The universal covers of all the small deformations of $X$ corresponding to elements $\mu \in \text{Kur}(X) \cap H^1(X, \mathcal{O}_X) \otimes Z(g)$ are mutually isomorphic as Lie groups with left-invariant complex structures.

*Proof.* It is known that all small deformations of a left-invariant complex structure on a complex parallelisable nilmanifold $X = G/\Gamma$ are again left-invariant (cf. [Rol11, sect. 4]). In particular, they are again of the form $G/\Gamma$, but now with a possibly different, yet still left invariant, complex structure. Thus, differentiably, the universal cover is always $G$, which is determined entirely by $g$, by the Lie-group-Lie-algebra correspondence. The complex structure on $G$ varies with $\mu$ though: Since deformations are again left-invariant, the complex structure is determined by the splitting $g_\mathbb{C}(\mu) = g^{0,1}_\mu \oplus g^{1,0}_\mu$. 

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Claim 5.5. The map of vector spaces

\[ \alpha : \mathfrak{g}_C(0) \longrightarrow \mathfrak{g}_C(\mu) \]

defined as \((\text{Id} + \mu)\) on \(\mathfrak{g}_0^{0,1}\) and \((\text{Id} + \overline{\mu})\) on \(\mathfrak{g}_0^{1,0}\) is an isomorphism of Lie algebras.

Proof of Claim 5.5. We use \([X, \overline{Y}] = 0\) for \(X \in \mathfrak{g}_1^{1,0}, \overline{Y} \in \mathfrak{g}_0^{0,1}\). Since \(\mu \in H^1(X, \mathcal{O}_X) \otimes Z(g)\), one also has \([X, \mu \overline{Y}] = 0\) so \([Z, \mu X] = 0\) for any \(Z \in \mathfrak{g}_C\). So, for \(X, \overline{Y} \in \mathfrak{g}_0^{0,1}\), we have:

\[
\begin{align*}
[\alpha X, \alpha \overline{Y}] &= [X, \overline{Y}] + [\mu X, \mu \overline{Y}] \\
&= [X, \overline{Y}] \\
&= \alpha([X, \overline{Y}])
\end{align*}
\]

Here, the last-but-one equation follows since for a \((0,1)\)-form \(\overline{\eta}\), one has \(\partial \overline{\eta}(X, \overline{Y}) = -\overline{\eta}([X, \overline{Y}])\) and \(\mu \in \ker \partial \cap A^{0,1} \otimes Z(G)\). By a similar argument, \([\alpha X, \alpha Y] = \alpha([X, Y])\) for \(X, Y \in \mathfrak{g}_1^{1,0}\).

Finally, for all \(X \in \mathfrak{g}_1^{1,0}\) and all \(\overline{Y} \in \mathfrak{g}_0^{0,1}\), we have:

\[
\begin{align*}
[\alpha X, \alpha \overline{Y}] &= [X, \overline{Y}] + [\mu X, \mu \overline{Y}] \\
&= 0 \\
&= \alpha([X, \overline{Y}]).
\end{align*}
\]

Summing up, \(\alpha\) is a homomorphism of complex Lie algebras. Since \(\Phi(\mu) = \mu\), \(\alpha\) respects the splitting that defines the complex structure.

This finishes the proof of Claim 5.5 and that of Theorem 5.4. \( \square \)

Corollary 5.6. Let \(X'\) be a complex parallelisable small deformation of a complex parallelisable nilmanifold \(X\). Then there exists an isomorphism between the double complexes of left invariant forms on \(X\) and \(X'\). In particular, there exist isomorphisms \(H(X) \cong H(X')\), where \(H\) denotes any ‘cohomology’ (e.g. Dolbeault and \(E_r\)-pages, De Rham, Bott-Chern, Aeppli and higher page variants).

Proof. The first statement follows from Claim 5.5, since the double complex of left invariant forms can be computed in terms of the Lie-algebra with its complex structure, while the second follows from Lemma 3.19 and the fact that for any nilmanifold \(X = G/\Gamma\), the inclusion of the double complex of left-invariant forms on \(G\) into all forms on \(X\) is an \(E_1\)-isomorphism. (This is conjectured to hold for all complex nilmanifolds and it is known for complex parallelisable ones, cf. again [Sak76]). \( \square \)

5.3 Essential deformations of Calabi-Yau manifolds

The notion of essential deformations was introduced in [Pop18] in the special case of the Iwasawa manifold \(I^{(3)}\). We will now extend it to a larger class of Calabi-Yau manifolds.
Let $X$ be a compact complex manifold with $\dim \mathbb{C}X = n$. Recall that, for every integer $r \geq 1$ and every bidegree $(p, q)$, the vector space of smooth $E_r$-closed (resp. $E_r$-exact) $(p, q)$-forms on $X$ is denoted by $Z^{p, q}_r(X)$ (resp. $C^{p, q}_r(X)$). Let us now define the following vector subspace of $E^{p, q}_1(X)$:

$$E^{p, q}_1(X)_0 := \left\{ \alpha \in C^{p, q}_1(X) \mid \partial \alpha = 0 \text{ and } \partial \alpha \in \text{Im } \bar{\partial} \right\} = \frac{Z^{p, q}_2(X)}{C^{p, q}_1(X)} \subset E^{p, q}_1(X). \quad (49)$$

In other words, $E^{p, q}_1(X)_0 = \ker d_1$ consists of the $E_1$-cohomology classes (i.e. Dolbeault cohomology classes) representable by $E_2$-closed forms of type $(p, q)$.

**Lemma 5.7.** For all $p, q$, the canonical linear map

$$P^{p, q} : E^{p, q}_1(X)_0 \rightarrow E^{p, q}_2(X), \quad \{\alpha\}_{E_1} \mapsto \{\alpha\}_{E_2},$$

is well defined and surjective. Its kernel consists of the $E_1$-cohomology classes representable by $E_2$-exact forms of type $(p, q)$.

In particular, $P^{p, q}$ is injective (hence an isomorphism) if and only if $C^{p, q}_1(X) = C^{p, q}_2(X)$.

**Proof.** Well-definedness means that $P^{p, q}(\{\alpha\}_{E_1})$ is independent of the choice of representative of the class $\{\alpha\}_{E_1} \in E^{p, q}_1(X)_0$. This follows from the inclusion $C^{p, q}_1(X) \subset C^{p, q}_2(X)$. The other three statements are obvious. \qed

Let us now fix a Hermitian metric $\omega$ on $X$. By the Hodge theory for the $E_2$-cohomology introduced in [Pop16] (and used e.g. in §2 above) and the standard Hodge theory for the Dolbeault cohomology, there are Hodge isomorphisms:

$$E^{n-1, 1}_2(X) \simeq \mathcal{H}^{n-1, 1}_2 = \mathcal{H}^{n-1, 1}_{2, \omega} \quad \text{and} \quad E^{n-1, 1}_1(X) \simeq \mathcal{H}^{n-1, 1}_1 = \mathcal{H}^{n-1, 1}_{1, \omega}$$

associating with every $E_2$- (resp. $E_1$-)class its unique $E_2$- (resp. $E_1$-)harmonic representative (w.r.t. $\omega$), where the $\omega$-dependent harmonic spaces are defined by

$$\mathcal{H}^{n-1, 1}_2 := \ker(\tilde{\Delta} : C^{\infty}_{n-1, 1}(X) \rightarrow C^{\infty}_{n-1, 1}(X)) \subset \mathcal{H}^{n-1, 1}_1 := \ker(\Delta'' : C^{\infty}_{n-1, 1}(X) \rightarrow C^{\infty}_{n-1, 1}(X))$$

and $\tilde{\Delta} = \partial p''\partial^* + \partial^* p''\partial^* + \Delta''$ is the pseudo-differential Laplacian introduced in [Pop16] and $\Delta'' = \bar{\partial}\partial^* + \partial^*\bar{\partial}$ is the standard $\bar{\partial}$-Laplacian, both associated with the metric $\omega$. (Recall that $p''$ is the $L^2_\omega$-orthogonal projection onto $\ker \Delta''$.)

**Definition 5.8.** Let $(X, \omega)$ be an $n$-dimensional compact complex Hermitian manifold. The $\omega$-lift of the canonical linear surjection $P^{n-1, 1} : E^{n-1, 1}_1(X)_0 \rightarrow E^{n-1, 1}_2(X)$ introduced in Lemma 5.7 is the $\omega$-dependent linear injection

$$J^{n-1, 1}_\omega : E^{n-1, 1}_2(X) \hookrightarrow E^{n-1, 1}_1(X)_0$$

induced by the inclusion $\mathcal{H}^{n-1, 1}_2 \subset \mathcal{H}^{n-1, 1}_1$, namely the map $J^{n-1, 1}_\omega$ that makes the following diagram commutative:

$$\begin{array}{ccc}
E^{n-1, 1}_2(X) & \xrightarrow{J^{n-1, 1}_\omega} & E^{n-1, 1}_1(X) \\
\simeq \downarrow & & \simeq \downarrow \\
\mathcal{H}^{n-1, 1}_{2, \omega} & \xrightarrow{\text{inclusion}} & \mathcal{H}^{n-1, 1}_{1, \omega}
\end{array}$$

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where the vertical arrows are the Hodge isomorphisms.

It follows from the definitions that the image of the injection $J_{\omega}^{n-1,1} : E_2^{n-1,1}(X) \rightarrow E_1^{n-1,1}(X)$ defined by the above commutative diagram is contained in $E_1^{n-1,1}(X)_0$ and we have

$$P^{n-1,1} \circ J_{\omega}^{n-1,1} = \text{Id}_{E_2^{n-1,1}(X)}.$$  

Thus, every Hermitian metric $\omega$ on $X$ induces a natural injection $J_{\omega}^{n-1,1}$ of $E_2^{n-1,1}(X)$ into $E_1^{n-1,1}(X)$ (and even into $E_1^{n-1,1}(X)_0$). In particular, if a canonical metric $\omega_0$ exists on $X$ (in the sense that $\omega_0$ depends only on the complex structure of $X$ with no arbitrary choices involved in its definition), the associated map $J_{\omega_0}^{n-1,1}$ constitutes a canonical injection of $E_2^{n-1,1}(X)$ into $E_1^{n-1,1}(X)$.

**Definition 5.9.** Let $X$ be a compact complex $n$-dimensional Calabi-Yau page-1-$\partial\bar{\partial}$-manifold. Suppose that $X$ carries a canonical Hermitian metric $\omega_0$.

The space of small essential deformations of $X$ is defined as the image in $E_1^{n-1,1}(X)$ of the canonical injection $J_{\omega_0}^{n-1,1}$, namely

$$E_1^{n-1,1}(X)_{\text{ess}} := J_{\omega_0}^{n-1,1}(E_2^{n-1,1}(X)) \subset E_1^{n-1,1}(X).$$

**Remark 5.10.** If the page-1-$\partial\bar{\partial}$-assumption on $X$ is replaced by a more general one (for example, the page-$r$-$\partial\bar{\partial}$-assumption for some $r \geq 2$ or merely the $E_r(X) = E_\infty(X)$ assumption for a specific $r \geq 2$), one can define a version of essential deformations using higher pages than the second one. The most natural choice is the degenerating page $E_r = E_\infty$ of the FSS if $r > 2$. Since at the moment we are mainly interested in page-1-$\partial\bar{\partial}$-manifolds, we confine ourselves to $E_2$.

**Example 5.11.** (The Iwasawa manifold) If $\alpha$, $\beta$, $\gamma$ are the three canonical holomorphic $(1, 0)$-forms induced on the complex 3-dimensional Iwasawa manifold $X = G/\Gamma$ by $dz_1, dz_2, dz_3 - z_1 dz_2$ from $\mathbb{C}^3$ (the underlying complex manifold of the Heisenberg group $G$), it is well known that $\alpha$ and $\beta$ are $d$-closed, while $d\gamma = \partial\gamma = -\alpha \wedge \beta \neq 0$. It is equally standard that the Dolbeault cohomology group of bidegree $(2, 1)$ is generated as follows:

$$E_2^{2,1}(X) = \left\langle [\alpha \wedge \gamma \wedge \overline{\alpha}]_{\partial}, [\alpha \wedge \gamma \wedge \overline{\beta}]_{\partial}, [\beta \wedge \gamma \wedge \overline{\alpha}]_{\partial}, [\beta \wedge \gamma \wedge \overline{\beta}]_{\partial} \right\rangle \oplus \left\langle [\alpha \wedge \beta \wedge \overline{\alpha}]_{\partial}, [\alpha \wedge \beta \wedge \overline{\beta}]_{\partial} \right\rangle.$$

In particular, we see that every $E_1$-class of bidegree $(2, 1)$ can be represented by a $d$-closed form. Since every pure-type $d$-closed form is also $E_2$-closed (and, indeed, $E_r$-closed for every $r$), we get

$$E_2^{2,1}(X) = E_1^{2,1}(X)_0.$$

It is equally standard that the $E_2$-cohomology group of bidegree $(2, 1)$ is generated as follows:

$$E_2^{2,1}(X) = \left\langle [\alpha \wedge \gamma \wedge \overline{\alpha}]_{E_2}, [\alpha \wedge \gamma \wedge \overline{\beta}]_{E_2}, [\beta \wedge \gamma \wedge \overline{\alpha}]_{E_2}, [\beta \wedge \gamma \wedge \overline{\beta}]_{E_2} \right\rangle.$$

It identifies canonically with the vector subspace

$$H_2^{(2)}(X) = \left\langle [\alpha \wedge \gamma \wedge \overline{\alpha}]_{\partial}, [\alpha \wedge \gamma \wedge \overline{\beta}]_{\partial}, [\beta \wedge \gamma \wedge \overline{\alpha}]_{\partial}, [\beta \wedge \gamma \wedge \overline{\beta}]_{\partial} \right\rangle \cong E_2^{2,1}(X)/\left\langle [\alpha \wedge \beta \wedge \overline{\alpha}]_{\partial}, [\alpha \wedge \beta \wedge \overline{\beta}]_{\partial} \right\rangle.$$  

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of $E^{2,1}_1(X)$ introduced in [Pop18, §4.2] as parametrising the essential deformations defined there for the Iwasawa manifold $X$.

On the other hand, let

$$\omega_0 := i\alpha \wedge \bar{\alpha} + i\beta \wedge \bar{\beta} + i\gamma \wedge \bar{\gamma}$$

be the Hermitian (even balanced) metric on $X$ canonically induced by the complex parallelisable structure of $X$. It can be easily seen that the vector space of small essential deformations coincides with the space $H^{2,1}_{[\nu]}(X)$ of [Pop18]:

$$E^{2,1}_1(X)_{ess} = J_{\omega_0}^2(E^{2,1}_2(X)) = H^{2,1}_{[\nu]}(X) \subset E^{2,1}_1(X) .$$

**Example 5.12.** (The manifold $I^{(5)}$) Let $X = I^{(5)}$ be the complex parallelisable nilmanifold of complex dimension 5 described in Example 5.1 (i.e. the 5-dimensional analogue of the Iwasawa manifold.) It is a page-1-$\partial\bar{\partial}$-manifold by Theorem 3.31.

We will use the standard notation $\varphi_{i_1...i_p}\gamma_{j_1...j_q} := \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_p} \wedge \bar{\varphi}_{j_1} \wedge \cdots \wedge \bar{\varphi}_{j_q}$.

For every $l \in \{3, 4, 5\}$, the linear map

$$T_l : H^{0,1}_l(X, T^{1,0}X) \longrightarrow H^{0,1}_l(X), \quad [\theta] \mapsto [\theta\varphi_l],$$

is well defined. If we set

$$H^{0,1}_{es}(X, T^{1,0}X) := \ker T_3 \cap \ker T_4 \cap \ker T_5 \subset H^{0,1}_l(X, T^{1,0}X),$$

and define $H^{4,1}_{es}(X) \subset H^{4,1}(X)$ to be the image of $H^{0,1}_{es}(X, T^{1,0}X)$ under the Calabi-Yau isomorphism $H^{0,1}_l(X, T^{1,0}X) \longrightarrow H^{4,1}_l(X)$, we get the following description

$$H^{4,1}_{es}(X) = \left< [\varphi_{23451}]_A, [\varphi_{13451}]_A, [\varphi_{23452}]_A, [\varphi_{13452}]_A \right> .$$

Moreover, we have the following identities of $\mathbb{C}$-vector spaces:

$$H^{4,1}_{es}(X) = E^{4,1}_1(X)_{ess} := J_{\omega_0}^4(E^{4,1}_2(X)) \subset E^{4,1}_1(X),$$

where

$$\omega_0 := \sum_{j=1}^5 i\varphi_j \wedge \bar{\varphi}_j,$$

is the canonical metric of $I^{(5)}$.

**5.4 Deformation unobstructedness for page-1-$\partial\bar{\partial}$-manifolds**

We say that the essential Kuranishi family of a Calabi-Yau page-1-$\partial\bar{\partial}$-manifold $X$ is unobstructed if every $E_2$-class in $E^{n-1,1}_2(X)$ admits a representative $\psi_1(t), u$ such that the integrability condition (46) is satisfied (i.e. all the equations (Eq. (v)) of §5.1 are solvable) when starting off with $\psi_1(t) \in C^\infty_{0,1}(X, T^{1,0}X)$, where $u$ is a non-vanishing holomorphic $(n, 0)$-form on $X$. 

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Theorem 5.13. Let $X$ be a Calabi-Yau page-1-$\partial \bar{\partial}$-manifold with $\dim \mathbb{C} X = n$. Fix a non-vanishing holomorphic $(n, 0)$-form $u$ on $X$ and suppose that

$$\psi_1(t, \rho_1(s), u) \in \mathbb{Z}^{n-2, 2}_2$$

for all $\psi_1(t), \rho_1(s) \in C^{\infty}_{0, 1}(X, T^{1, 0}X)$ such that $\psi_1(t)u, \rho_1(s)u \in \ker d \cup \text{Im} \partial$.

(i) Then, the essential Kuranishi family of $X$ is unobstructed.

(ii) If, moreover, $\mathbb{Z}_1^{n-1, 1} = \mathbb{Z}_2^{n-1, 1}$, the Kuranishi family of $X$ is unobstructed.

Before proving this result, we make a few comments. First, we notice an equivalent formulation for the assumption made in (ii). Needless to say, the inclusion $\mathbb{Z}_2^{n-1, 1} \subset \mathbb{Z}_1^{n-1, 1}$ always holds.

Lemma 5.14. Let $X$ be a compact complex page-1-$\partial \bar{\partial}$-manifold with $\dim \mathbb{C} X = n$. Then, $\mathbb{Z}_1^{n-1, 1} = \mathbb{Z}_2^{n-1, 1}$ if and only if every Dolbeault cohomology class of bidegree $(n-1, 1)$ can be represented by a $d$-closed form.

Proof. Let $\alpha \in C^{\infty}_{n-1, 1}(X)$ be an arbitrary $\bar{\partial}$-closed form, i.e. $\alpha \in \mathbb{Z}_1^{n-1, 1}$. The class $\{\alpha\}^{\bar{\partial}}$ can be represented by a $d$-closed form if and only if there exists $\beta$ of bidegree $(n-1, 0)$ such that $\partial(\alpha + \bar{\partial}\beta) = 0$. This is equivalent to $\partial \alpha$ being $\bar{\partial}$-exact, which implies that $\partial \alpha$ is $\bar{\partial}$-exact.

Conversely, since $X$ is a page-1-$\partial \bar{\partial}$-manifold, the $\bar{\partial}$-exactness of $\partial \alpha$ implies its $\bar{\partial}$-exactness. Indeed, $\partial \alpha = 0$ if and $\partial \alpha$ is $\bar{\partial}$-exact, then $\alpha \in \mathbb{Z}_2^{n-1, 1}$, so $\partial \alpha \in \partial(\mathbb{Z}_2^{n-1, 1})$. Now, $\partial(\mathbb{Z}_2^{n-1, 1}) = \text{Im} (\partial \bar{\partial})$ thanks to property (i) in characterisation (C) of the page-1-$\partial \bar{\partial}$-property given in Theorem 3.51 (with $r = 2$). Therefore, $\partial \alpha \in \text{Im} (\partial \bar{\partial})$ whenever $\alpha \in \mathbb{Z}_2^{n-1, 1}$.

Summing up, the class $\{\alpha\}^{\bar{\partial}}$ can be represented by a $d$-closed form if and only if $\partial \alpha$ is $\bar{\partial}$-exact if and only if $\alpha \in \mathbb{Z}_2^{n-1, 1}$. \[\square\]

Second, we notice that both the Iwasawa manifold $I^{(3)}$ and the 5-dimensional Iwasawa manifold $I^{(5)}$ satisfy all the hypotheses of Theorem 5.13. Indeed, $I^{(3)}$ and $I^{(5)}$ are complex parallelisable nilmanifolds, so they are page-1-$\partial \bar{\partial}$-manifolds by Theorem 3.31. They are also Calabi-Yau manifolds, since all nilmanifolds are. Moreover, we have

Lemma 5.15. Let $X$ be either $I^{(3)}$ or $I^{(5)}$ and let $n = \dim \mathbb{C} X \in \{3, 5\}$. Let $u := \varphi_1 \wedge \varphi_2 \wedge \varphi_3 = \alpha \wedge \beta \wedge \gamma \in C^{\infty}_{3, 0}(I^{(3)})$ or $u := \varphi_1 \wedge \cdots \wedge \varphi_6 \in C^{\infty}_{5, 0}(I^{(5)})$ according to whether $X = I^{(3)}$ or $X = I^{(5)}$, a non-vanishing holomorphic $(n, 0)$-form on $X$.

Then, for all $\psi_1(t), \rho_1(s) \in C^{\infty}_{0, 1}(X, T^{1, 0}X)$ such that $\psi_1(t)u, \rho_1(s)u \in \ker d \cup \text{Im} \partial$, we have

$$\psi_1(t)\rho_1(s)u \in \mathbb{Z}_2^{n-2, 2}.$$  \hspace{0.5cm} (51)

Proof. It is given in Appendix two (section 7). \[\square\]

Finally, let us mention that both manifolds $X = I^{(3)}$ and $X = I^{(5)}$ have the property that every Dolbeault cohomology class of type $(n-1, 1)$ can be represented by a $d$-closed form. Indeed, as seen in the proof of Lemma 5.15 spelt out in §8.7, $H^{n-1, 1}_0(X)$ is generated by the classes represented by the $\bar{\varphi}_i \wedge \varphi_1$'s and the $\bar{\varphi}_i \wedge \varphi_2$'s with $i \in \{1, 2, 3\}$ (in the case of $X = I^{(3)}$) and $i \in \{1, \ldots, 5\}$ (in the case of $X = I^{(5)}$). All the forms $\bar{\varphi}_i \wedge \varphi_\lambda$, with $\lambda \in \{1, 2\}$, are $d$-closed.
Note that the hypotheses of Theorem 5.13, all of which are satisfied by \( X = I^{(3)} \) and \( X = I^{(5)} \), have the advantage of being cohomological in nature, hence fairly general and not restricted to the class of nilmanifolds. Indeed, there is no mention of any structure equations in Theorem 5.13.

**Proof of Theorem 5.13.** Let \( \{ \eta_i \}_{E_2} \in E_2^{n-1,1}(X) \) be an arbitrary nonzero class. Pick any \( d \)-closed representative \( \eta_1 \in C_{n-1,1}(X) \) of it. A \( d \)-closed representative exists thanks to the page-1-\( \bar{\partial} \)-assumption on \( X \). Under the extra assumption \( Z_1^{n-1,1} = Z_1^{n-1,1}(X) \) of (ii), there is even a \( d \)-closed representative \( \eta_1 \) in every Dolbeault class \( \{ \eta_i \}_{E_1} \in E_1^{n-1,1}(X) \), thanks to Lemma 5.14. So, we choose an arbitrary \( d \)-closed form \( \eta_1 \in C_{n-1,1}(X) \) that represents an arbitrary nonzero class in either \( E_2^{n-1,1}(X) \) or \( E_1^{n-1,1}(X) \) depending on whether we are in case (i) or in case (ii). By the Calabi-Yau isomorphism, there exists a unique \( \psi_1 \in C^\infty_{0,1}(X, T^{1,0}X) \) such that

\[
\psi_1, \omega = \eta_1.
\]

We will prove the existence of forms \( \psi_\nu \in C^\infty_{0,1}(X, T^{1,0}X) \), with \( \nu \in \mathbb{N}^* \) and \( \psi_1 \) being the already fixed such form, that satisfy the equations

\[
\bar{\partial} \psi_\nu = \frac{1}{2} \sum_{\mu=1}^{\nu-1} [\psi_\mu, \psi_{\nu-\mu}] \quad \text{(Eq. (\( \nu - 1 \))), \quad \nu \geq 2},
\]

which, as recalled in §5.1, are equivalent to the integrability condition \( \bar{\partial} \psi(\tau) = (1/2)[\psi(\tau), \psi(\tau)] \) being satisfied by the form \( \psi(\tau) := \psi_1 \tau + \psi_2 \tau^2 + \cdots + \psi_N \tau^N + \cdots \in C^\infty_{0,1}(X, T^{1,0}X) \) for all \( \tau \in \mathbb{C} \) with \( |\tau| \) sufficiently small. The convergence in a Hölder norm of the series defining \( \psi(\tau) \) for \( |\tau| \) small enough is guaranteed by the general Kuranishi theory (cf. [Kur62]), while the resulting \( \psi(\tau) \) defines a complex structure \( \bar{\partial}_\tau \) on \( X \) that identifies on functions with \( \bar{\partial} - \psi(\tau) \) and represents the infinitesimal deformation of the original complex structure \( \bar{\partial} \) of \( X \) in the direction of \( [\psi_1] \in H^{0,1}(X, T^{1,0}X) \).

Since \( \partial(\psi_1, \omega) = \partial \eta_1 = 0 \), the Tian-Todorov lemma ([Tia87], [Tod89]) guarantees that \( [\psi_1, \psi_1], \omega \in \text{Im} \partial \) and

\[
[\psi_1, \psi_1], \omega = \partial(\psi_1, \omega(\psi_1, \omega)).
\]

On the other hand, \( \bar{\partial} \eta_1 = 0 \), hence \( \bar{\partial} \psi_1 = 0 \), hence \( \psi_1, \omega(\psi_1, \omega) \in \ker \bar{\partial} \). We even have the stronger property \( \psi_1, \omega(\psi_1, \omega) \in Z_2^{n-2,2} \) thanks to assumption (50), since \( \psi_1, \omega \in \ker d \). Therefore,

\[
[\psi_1, \psi_1], \omega = \partial(\psi_1, \omega(\psi_1, \omega)) \in \partial(Z_2^{n-2,2}) = \text{Im} (\partial \bar{\partial}),
\]

the last identity being a consequence of the page-1-\( \bar{\partial} \bar{\partial} \)-assumption on \( X \). (See (i) of property (C) in Theorem 3.51.)

Thus, there exists a form \( \Phi_2 \in C^\infty_{n-2,1}(X) \) such that

\[
\bar{\partial} \Phi_2 = \frac{1}{2} [\psi_1, \psi_1], \omega.
\]

If we fix an arbitrary Hermitian metric \( \omega \) on \( X \), we choose \( \Phi_2 \) as the unique solution of the above equation with the extra property \( \Phi_2 \in \text{Im} (\partial \bar{\partial})^* \). This is the minimal \( L^2 \)-norm solution, as follows from the 3-space orthogonal decomposition of \( C^\infty_{n-2,1}(X) \) induced by the Aeppli Laplacian (see [Sch07]). Let \( \eta_2 := \partial \Phi_2 \in C^\infty_{n-1,1}(X) \). Thanks to the Calabi-Yau isomorphism, there exists a unique
\[ \psi_2 \in C_{0,1}^{\infty}(X, T^{1,0}X) \] such that \[ \psi_2 \wedge u = \eta_2. \] In particular, \[ \partial(\psi_2 \wedge u) = 0 \] and \[ (\bar{\partial} \psi_2) \wedge u = \bar{\partial}(\psi_2 \wedge u) = \partial \eta_2 = (1/2) [\psi_1, \psi_1] \wedge u. \] This means that
\[ \bar{\partial} \psi_2 = \frac{1}{2} [\psi_1, \psi_1], \]
so \( \psi_2 \) is a solution of (Eq. 1). Moreover, by construction, \( \psi_2 \) has the extra key property that \( \psi_2 \wedge u \in \text{Im} \partial. \)

Now, we continue inductively to construct the forms \( (\psi^N)_{N \geq 3}. \) Suppose the forms \( \psi_1, \psi_2, \ldots, \psi_{N-1} \in C_{0,1}^{\infty}(X, T^{1,0}X) \) have been constructed as solutions of the equations (Eq. (\( \nu - 1 \))) for all \( \nu \in \{2, \ldots, N - 1 \} \) with the further property \( \psi_{2 \wedge u}, \ldots, \psi_{N-1 \wedge u} \in \text{Im} \partial. \) (Recall that \( \psi_{1 \wedge u} \in \ker d. \)) Since \( \partial(\psi_1 \wedge u) = \partial(\psi_2 \wedge u) = \ldots \partial(\psi_{N-1 \wedge u}) = 0, \) the Tian-Todorov lemma ([Tia87], [Tod89]) guarantees that \([\psi_{\mu}, \psi_{N-\mu}] \wedge u \in \text{Im} \partial \) for all \( \mu \in \{1, \ldots, N - 1 \} \) and yields the first identity below:
\[ \sum_{\mu = 1}^{N-1} [\psi_{\mu}, \psi_{N-\mu}] \wedge u = \partial \left( \sum_{\mu = 1}^{N-1} \psi_{\mu \wedge} (\psi_{N-\mu \wedge} u) \right) \in \partial(\mathbb{Z}_{2,1}^{n-2,2}) = \text{Im} (\partial \bar{\partial}), \]
where the relation “\( \in \)” follows from assumption (50) and the last identity is a consequence of the page-1-\( \partial \bar{\partial} \)-assumption on \( X. \) (See (i) of property (C) in Theorem 3.51.)

Thus, there exists a form \( \Phi_N \in C_{n-2,1}^{\infty}(X) \) such that
\[ \bar{\partial} \partial \Phi_N = \frac{1}{2} \sum_{\mu = 1}^{N-1} [\psi_{\mu}, \psi_{N-\mu}] \wedge u. \]
We choose \( \Phi_N \) to be the solution of minimal \( L^2_{\omega} \)-norm of the above equation, so \( \Phi_N \in \text{Im} (\partial \bar{\partial})^*. \)

Let \( \eta_N := \bar{\partial} \Phi_N \in C_{n-1,1}^{\infty}(X). \) Thanks to the Calabi-Yau isomorphism, there exists a unique \( \psi_N \in C_{0,1}^{\infty}(X, T^{1,0}X) \) such that \( \psi_N \wedge u = \eta_N. \) Hence, \( (\bar{\partial} \psi_N) \wedge u = \bar{\partial}(\psi_N \wedge u) = \bar{\partial} \eta_N = \bar{\partial} \partial \Phi_N, \) so
\[ \bar{\partial} \psi_N = \frac{1}{2} \sum_{\mu = 1}^{N-1} [\psi_{\mu}, \psi_{N-\mu}] \]
which means that \( \psi_N \) is a solution of (Eq. (\( N - 1 \))). Moreover, by construction, \( \psi_N \) has the extra key property that \( \psi_N \wedge u \in \text{Im} \partial. \)

This finishes the induction process and completes the proof of Theorem 5.13. \( \square \)

6 Appendix one

We refer to the appendix of [Pop19] for the details of the inductive construction of a Hodge theory for the pages \( E_r, \) with \( r \geq 3, \) of the Frölicher spectral sequence. Here, we will only remind the reader of the conclusion of that construction and will point out a reformulation of it that was used in the present paper.

Let \( X \) be an \( n \)-dimensional compact complex manifold. We fix an arbitrary Hermitian metric \( \omega \) on \( X. \) As recalled in §2.2, for every bidegree \( (p, q), \) the \( \omega \)-harmonic spaces (also called \( E_r \)-harmonic spaces)
\[ \cdots \subset \mathcal{H}^{p,q}_{r+1} \subset \mathcal{H}^{p,q}_r \subset \cdots \subset \mathcal{H}^{p,q}_1 \subset C_{p,q}^{\infty}(X) \]

were constructed by induction on \( r \in \mathbb{N}^* \) in [Pop17, §3.2] such that every \( \mathcal{H}^{p,q}_r \) is isomorphic to the corresponding space \( E^{p,q}_r(X) \) featuring on the \( r^{th} \) page of the Frölicher spectral sequence of \( X \).

Moreover, pseudo-differential “Laplacians” \( \tilde{\Delta}^{(r+1)} : \mathcal{H}^{p,q}_r \rightarrow \mathcal{H}^{p,q}_r \) were inductively constructed in the appendix to [Pop19] such that

\[ \ker \tilde{\Delta}^{(r)} = \mathcal{H}^{p,q}_r, \quad r \in \mathbb{N}^*, \]

where \( \tilde{\Delta}^{(1)} = \Delta'' = \partial \bar{\partial}^* + \bar{\partial}^* \partial \) is the usual \( \bar{\partial} \)-Laplacian.

The conclusion of the construction in the appendix to [Pop19] was the following statement. It gives a 3-space orthogonal decomposition of each space \( C^{\infty}_{p,q}(X) \), for every fixed \( r \in \mathbb{N}^* \), that parallels the standard decomposition \( C^{\infty}_{p,q}(X) = \ker \Delta'' \oplus \Im \partial \oplus \Im \partial^* \) for \( r = 1 \).

**Proposition 6.1.** (Corollary 4.6 in [Pop19]) Let \((X, \omega)\) be a compact complex \( n \)-dimensional Hermitian manifold. For every \( r \in \mathbb{N}^* \), put \( D_{r-1} := (\tilde{\Delta}^{(1)})^{-1} \partial \bar{\partial} \ldots (\tilde{\Delta}^{(r-1)})^{-1} \partial \bar{\partial} \) and \( D_0 = \Id \).

(i) For all \( r \in \mathbb{N}^* \) and all \((p, q)\), the kernel of \( \tilde{\Delta}^{(r+1)} : C^{\infty}_{p,q}(X) \rightarrow C^{\infty}_{p,q}(X) \) is given by

\[
\ker \tilde{\Delta}^{(r+1)} = \left( \ker(p_r \partial D_{r-1}) \cap \ker(\partial D_{r-1} p_r) \right)^* \cap \left( \ker(p_{r-1} \partial D_{r-2}) \cap \ker(\partial D_{r-2} p_{r-1}) \right)^* \cap \ldots \cap \left( \ker(p_1 \partial) \cap \ker(\partial p_1) \right)^* \cap \left( \ker \partial \cap \ker \partial^* \right).
\]

(ii) For all \( r \in \mathbb{N}^* \) and all \((p, q)\), the following orthogonal 3-space decomposition (in which the sums inside the big parantheses need not be orthogonal or even direct) holds:

\[
C^{\infty}_{p,q}(X) = \ker \tilde{\Delta}^{(r+1)} \oplus \left( \Im \partial + \Im(p_1 \partial) + \Im(\partial D_1 p_2) + \cdots + \Im(\partial D_{r-1} p_r) \right) \oplus \left( \Im \partial^* + \Im(\partial p_1 \partial) + \Im(\partial p_2 D_1) + \cdots + \Im(\partial p_r D_{r-1}) \right)^*, \tag{52}
\]

where \( \ker \tilde{\Delta}^{(r+1)} \oplus (\Im \partial + \Im(p_1 \partial) + \Im(\partial D_1 p_2) + \cdots + \Im(\partial D_{r-1} p_r)) = \ker \partial \cap \ker(p_1 \partial) \cap \ker(p_2 D_1) \cap \cdots \cap \ker(p_r D_{r-1}) \) and \( \ker \tilde{\Delta}^{(r+1)} \oplus (\Im \partial^* + \Im(p_1 \partial^*) + \Im(\partial p_2 D_1) + \cdots + \Im(\partial p_r D_{r-1})^*) = \ker \partial^* \cap \ker(p_1 \partial^*) \cap \ker(\partial p_2 D_1) \cap \cdots \cap \ker(\partial D_{r-1} p_r)^* \).

For each \( r \in \mathbb{N}^* \), \( p_r := p_r^{p,q} \) stands for the \( L_2^\omega \)-orthogonal projection onto \( \mathcal{H}^{p,q}_r \).

We will now cast the 3-space decomposition (52) in the terms used in the present paper. Recall that in the proof of Lemma 3.39, we defined the following vector spaces for every \( r \in \mathbb{N}^* \) and every bidegree \((p, q)\) based on the terminology introduced in (iv) of Definition 3.37:

\[
\mathcal{E}^{p,q}_{\partial, r} := \{ \alpha \in C^{\infty}_{p,q}(X) \mid \partial \alpha \text{ reaches 0 in at most r steps} \}, \quad \mathcal{E}^{p,q}_{\bar{\partial}, r} := \{ \beta \in C^{\infty}_{p,q}(X) \mid \bar{\partial} \beta \text{ reaches 0 in at most r steps} \}.
\]

When a Hermitian metric \( \omega \) has been fixed on \( X \) and the adjoint operators \( \partial^* \) and \( \bar{\partial}^* \) with respect to \( \omega \) have been considered, we define the analogous subspaces \( \mathcal{E}^{p,q}_{\partial^*, r} \) and \( \mathcal{E}^{p,q}_{\bar{\partial}^*, r} \) of \( C^{\infty}_{p,q}(X) \) by replacing \( \partial \) with \( \partial^* \) and \( \bar{\partial} \) with \( \bar{\partial}^* \) in the definitions of \( \mathcal{E}^{p,q}_{\partial, r} \) and \( \mathcal{E}^{p,q}_{\bar{\partial}, r} \).

Part (ii) of Proposition 6.1 can be reworded as follows.
Proposition 6.2. Let \( (X, \omega) \) be a compact complex \( n \)-dimensional Hermitian manifold. For every \( r \in \mathbb{N}^* \) and for all \( p, q \in \{0, \ldots, n\} \), the following orthogonal 3-space decomposition (in which the sums inside the big parantheses need not be orthogonal or even direct) holds:

\[
C_{p, q}^\infty(X) = \mathcal{H}_{r}^{p, q} \oplus \left( \text{Im} \bar{\partial} + \partial(\mathcal{E}_{\partial, r-1}^{p-1, q}) \right) \oplus \left( \partial^*(\mathcal{E}_{\bar{\partial}, r-1}^{p+1, q}) + \text{Im} \bar{\partial}^* \right),
\]

where \( \mathcal{H}_{r}^{p, q} \) is the \( E_{r} \)-harmonic space induced by \( \omega \) (see \S 2.2 and earlier in this appendix) and the next two big parantheses are the spaces of \( E_{r} \)-exact \((p, q)\)-forms, respectively \( E_{r}^* \)-exact \((p, q)\)-forms:

\[
\text{Im} \bar{\partial} + \partial(\mathcal{E}_{\partial, r-1}^{p-1, q}) = \mathcal{C}_{r}^{p, q} \quad \text{and} \quad \partial^*(\mathcal{E}_{\bar{\partial}, r-1}^{p+1, q}) + \text{Im} \bar{\partial}^* = \mathcal{C}_{r}^{p, q*}.
\]

Moreover, we have

\[
\mathcal{Z}_{r}^{p, q} = \mathcal{H}_{r}^{p, q} \oplus \left( \text{Im} \bar{\partial} + \partial(\mathcal{E}_{\partial, r-1}^{p-1, q}) \right) = \mathcal{H}_{r}^{p, q} \oplus \mathcal{C}_{r}^{p, q},
\]

\[
\mathcal{Z}_{r}^{p, q*} = \mathcal{H}_{r}^{p, q} \oplus \left( \partial^*(\mathcal{E}_{\bar{\partial}, r-1}^{p+1, q}) + \text{Im} \bar{\partial}^* \right) = \mathcal{H}_{r}^{p, q} \oplus \mathcal{C}_{r}^{p, q*}
\]

where \( \mathcal{Z}_{r}^{p, q} \) and \( \mathcal{Z}_{r}^{p, q*} \) are the spaces of smooth \( E_{r} \)-closed, resp. \( E_{r}^* \)-closed, \((p, q)\)-forms.

7 Appendix two

In this section, we spell out the proof of Lemma 5.15.

- **Case where \( X = I^{(3)} \)**. We use the notation of Example 5.11, but also put \( \varphi_1 := \alpha, \varphi_2 := \beta \) and \( \varphi_3 := \gamma \). We have: \( d\varphi_1 = d\varphi_2 = 0 \) and \( d\varphi_3 = -\varphi_1 \wedge \varphi_2 \). The dual basis of \((1, 0)\)-vector fields consists of

\[
\theta_1 = \frac{\partial}{\partial z_1}, \quad \theta_2 = \frac{\partial}{\partial z_2} + z_1 \frac{\partial}{\partial z_3}, \quad \theta_3 = \frac{\partial}{\partial z_3},
\]

(actually of the vector fields induced by these ones on \( X \) by passage to the quotient) whose mutual Lie brackets are as follows:

\[
[\theta_1, \theta_2] = -[\theta_2, \theta_1] = \theta_3 \quad \text{and} \quad [\theta_i, \theta_j] = 0 \quad \text{whenever} \quad \{i, j\} \neq \{1, 2\}.
\]

In particular, \( H^{0,1}(X, T^1.0X) = \langle [\varphi_1 \otimes \theta_i], [\varphi_2 \otimes \theta_i] \mid i = 1, \ldots, 3 \rangle \), so \( \text{dim}_\mathbb{C} H^{0,1}(X, T^1.0X) = 6 \).

Note that all the \((2, 1)\)-forms \( (\varphi_1 \otimes \theta_i)_\ast u \) and \( (\varphi_2 \otimes \theta_i)_\ast u \) are \( d \)-closed for \( i \in \{1, 2, 3\} \), so every Dolbeault class in \( H^{2,1}_{\partial}(X) \) can be represented by a \( d \)-closed form.

(a) Let \( \psi_1(t), \rho_1(s) \in C_{0,1}^\infty(X, T^1.0X) \) such that \( \psi_1(t)_\ast u, \rho_1(s)_\ast u \in \ker d \). Then,

\[
\psi_1(t) = \sum_{i=1}^{3} \sum_{\lambda=1}^{2} t_{i\lambda} \theta_i \varphi_\lambda, \quad \text{so} \quad \psi_1(t)_\ast u = \sum_{i=1}^{3} (-1)^{i-1} \sum_{\lambda=1}^{2} t_{i\lambda} \varphi_\lambda \wedge \varphi_i,
\]

\[
\rho_1(s) = \sum_{j=1}^{3} \sum_{\mu=1}^{2} s_{j\mu} \theta_j \varphi_\mu, \quad \text{so} \quad \rho_1(s)_\ast u = \sum_{j=1}^{3} (-1)^{j-1} \sum_{\mu=1}^{2} s_{j\mu} \varphi_\mu \wedge \varphi_j.
\]
where $\bar{\varphi}_j$ stands for $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ with $\varphi_j$ omitted.

Since $\psi_1(t) \cdot u, \rho_1(s) \cdot u \in \ker \bar{\partial}, \psi_1(t)$ and $\rho_1(s)$ are $\bar{\partial}$-closed for the $\bar{\partial}$ of the holomorphic structure of $T^{1,0}X$, hence $\psi_1(t) \cdot (\rho_1(s) \cdot u) \in Z^{1,2}_1$. Moreover, since $\psi_1(t) \cdot u, \rho_1(s) \cdot u \in \ker \bar{\partial}$, the so-called Tian-Todorov Lemma (see [Tia87], [Tod89]) ensures that

$$\bar{\partial}(\psi_1(t) \cdot (\rho_1(s) \cdot u)) = [\psi_1(t) \cdot u, \rho_1(s) \cdot u],$$

where $[\psi_1(t) \cdot u, \rho_1(s) \cdot u]$ is the scalar-valued $(n-1, 2)$-form defined by the identity $[\psi_1(t) \cdot u, \rho_1(s) \cdot u] = [\psi_1(t), \rho_1(s)] \cdot u$. So, we have to show that $[\psi_1(t) \cdot u, \rho_1(s) \cdot u]$ is $\bar{\partial}$-exact. We get:

$$[\psi_1(t), \rho_1(s)] = \sum_{1 \leq i, j \leq 3} \sum_{1 \leq \lambda, \mu \leq 2} t_{i} \lambda \sum_{j \mu} [\theta_i, \theta_j] \varphi_\lambda \wedge \varphi_\mu = D_{t, s} \theta_3 \varphi_1 \wedge \varphi_2,$$

where $D_{t, s} = (t_{11} s_{22} + t_{22} s_{11} - t_{12} s_{21} - t_{21} s_{12})$. Hence,

$$[\psi_1(t), \rho_1(s)] \cdot u = D_{t, s} \varphi_1 \wedge \varphi_2 \wedge \varphi_1 \wedge \varphi_2 = \bar{\partial}(D_{t, s} \varphi_3 \wedge \varphi_3) = \bar{\partial} \partial(D_{t, s} \varphi_3 \wedge \varphi_3) \in \Im \bar{\partial},$$

as desired.

We conclude that $\psi_1(t) \cdot (\rho_1(s) \cdot u) \in Z^{1,2}_1$ and $\bar{\partial}(\psi_1(t) \cdot (\rho_1(s) \cdot u)) \in \Im \bar{\partial}$, hence $\psi_1(t) \cdot (\rho_1(s) \cdot u) \in Z^{1,2}_2$, as desired.

(b) Let $\psi_1(t), \rho_1(s) \in C^\infty_0(X, T^{1,0}X)$ such that $\psi_1(t) \cdot u \in \ker d$ and $\rho_1(s) \cdot u \in \Im \bar{\partial}$. Then, $\psi_1(t) = \sum_{1 \leq i \leq 3} \sum_{1 \leq \lambda \leq 2} t_{i} \lambda \theta_i \varphi_\lambda$ and $\rho_1(s) = (\sum_{1 \leq \mu \leq 3} s_\mu \varphi_\mu) \theta_3$, so

$$\rho_1(s) \cdot u = \sum_{1 \leq \mu \leq 3} s_\mu \varphi_\mu \wedge \varphi_1 \wedge \varphi_2 = \bar{\partial}(\sum_{1 \leq \mu \leq 3} s_\mu \varphi_\mu) \theta_3 \in \Im \bar{\partial}.$$

On the one hand, we get $\psi_1(t) \cdot (\rho_1(s) \cdot u) = \sum_{\lambda = 1}^2 \sum_{\mu = 1}^3 t_{1 \lambda} \sum_{\mu = 1}^3 s_\mu \varphi_\lambda \wedge \varphi_\mu \wedge \varphi_2 = \sum_{\lambda = 1}^2 \sum_{\mu = 1}^3 t_{2 \lambda} \sum_{\mu = 1}^3 s_\mu \varphi_\lambda \wedge \varphi_\mu \wedge \varphi_1$,

$$\text{hence } \bar{\partial}(\psi_1(t) \cdot (\rho_1(s) \cdot u)) = -\sum_{\lambda = 1}^2 t_{1 \lambda} \sum_{\mu = 1}^3 s_\mu \varphi_\lambda \wedge \bar{\partial} \varphi_\mu \wedge \varphi_2 + \sum_{\lambda = 1}^2 t_{2 \lambda} \sum_{\mu = 1}^3 s_\mu \varphi_\lambda \wedge \bar{\partial} \varphi_\mu \wedge \varphi_1 = 0 \text{ because } \bar{\partial} \varphi_3 = -\varphi_1 \wedge \varphi_2. \text{ Thus, } \psi_1(t) \cdot (\rho_1(s) \cdot u) \in \ker \bar{\partial}.$$

On the other hand, since $[\theta_i, \theta_3] = 0$ for all $i$, we get

$$\bar{\partial}(\psi_1(t) \cdot (\rho_1(s) \cdot u)) = [\psi_1(t), \rho_1(s) \cdot u] = \sum_{i = 1}^3 \sum_{\lambda = 1}^2 \sum_{\mu = 1}^3 t_{i \lambda} s_\mu \varphi_\lambda \wedge \varphi_\mu [\theta_i, \theta_3] = 0$$

We conclude that $\psi_1(t) \cdot (\rho_1(s) \cdot u) \in Z^{1,2}_2$.

(c) If $\psi_1(t), \rho_1(s) \in C^\infty_0(X, T^{1,0}X)$ are such that $\psi_1(t) \cdot u$ and $\rho_1(s) \cdot u$ both lie in $\Im \bar{\partial}$, then $\psi_1(t) = (\sum_{1 \leq \lambda \leq 3} t_\lambda \varphi_\lambda) \theta_3$ and $\rho_1(s) = (\sum_{1 \leq \mu \leq 3} s_\mu \varphi_\mu) \theta_3$. We get

$$\psi_1(t) \cdot (\rho_1(s) \cdot u) = -\sum_{1 \leq \lambda \leq 3} t_\lambda \varphi_\lambda \wedge \sum_{1 \leq \mu \leq 3} s_\mu \varphi_\mu \wedge [\theta_3 \cdot (\varphi_1 \wedge \varphi_2)] = 0$$
Since \( \theta_3 \varphi_1 = \theta_3 \varphi_2 = 0 \). In particular, \( \psi_1(t) \cup (\rho_1(s) \cup) \in Z_2^{1,2} \).

- **Case where \( X = I^{(5)} \).** We use the notation of Example 5.1.
  
  (a) Let \( \psi_1(t), \rho_1(s) \in C_0^{\infty}(X, T^{1,0}X) \) such that \( \psi_1(t) \cup, \rho_1(s) \cup \in \ker d \). Then,

\[
\psi_1(t) = \sum_{i=1}^{5} \sum_{\lambda=1}^{2} t_{i\lambda} \psi_{i\lambda}, \quad \text{so} \quad \psi_1(t) \cup = \sum_{i=1}^{5} (-1)^{i-1} \sum_{\lambda=1}^{2} t_{i\lambda} \varphi_{\lambda} \wedge \hat{\varphi}_i,
\]

\[
\rho_1(s) = \sum_{j=1}^{5} \sum_{\mu=1}^{2} s_{j\mu} \theta_{j\mu}, \quad \text{so} \quad \rho_1(s) \cup = \sum_{j=1}^{5} (-1)^{j-1} \sum_{\mu=1}^{2} s_{j\mu} \varphi_{\mu} \wedge \hat{\varphi}_j,
\]

where \( \hat{\varphi}_j \) stands for \( \varphi_1 \wedge \cdots \wedge \varphi_5 \) with \( \varphi_j \) omitted.

Since \( [\theta_i, \theta_j] = 0 \) unless \( \{i, j\} \subset \{1, 2, 3\} \) and given the other values for \( [\theta_i, \theta_j] \), we get:

\[
[\psi_1(t), \rho_1(s)] \cup = -D_3(t, s) \varphi_1 \wedge \varphi_2 \wedge \varphi_4 \wedge \varphi_5 \wedge \varphi_1 \wedge \varphi_2 + D_2(t, s) \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_5 \wedge \varphi_1 \wedge \varphi_2 - D_1(t, s) \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_1 \wedge \varphi_2,
\]

where

\[
D_3(t, s) = \begin{vmatrix}
  t_{11} & t_{12} \\
  s_{21} & s_{22}
\end{vmatrix},
\]

\[
D_2(t, s) = \begin{vmatrix}
  t_{11} & t_{12} \\
  s_{31} & s_{32}
\end{vmatrix},
\]

\[
D_1(t, s) = \begin{vmatrix}
  t_{21} & t_{22} \\
  s_{31} & s_{32}
\end{vmatrix}.
\]

Now, since \( \varphi_1 \wedge \varphi_2 = \partial \varphi_3 \) and \( \varphi_1 \wedge \varphi_2 = \partial \varphi_3 \), using also the other properties of the \( \varphi_i \)'s, we get

\[
\varphi_1 \wedge \varphi_2 \wedge \varphi_4 \wedge \varphi_5 \wedge \varphi_1 \wedge \varphi_2 = \partial \partial (\varphi_3 \wedge \varphi_4 \wedge \varphi_5 \wedge \varphi_3),
\]

\[
\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_5 \wedge \varphi_1 \wedge \varphi_2 = \partial \partial (\varphi_2 \wedge \varphi_4 \wedge \varphi_5 \wedge \varphi_3).
\]

Similarly, since \( \varphi_2 \wedge \varphi_3 = \partial \varphi_5 \) and \( \varphi_1 \wedge \varphi_2 = \partial \varphi_3 \), we get

\[
\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4 \wedge \varphi_1 \wedge \varphi_2 = \partial \partial (\varphi_1 \wedge \varphi_4 \wedge \varphi_5 \wedge \varphi_3).
\]

We conclude that \( \partial (\psi_1(t) \cup (\rho_1(s) \cup)) = [\psi_1(t), \rho_1(s)] \cup \in \text{Im} (\partial \partial) \subset \text{Im} \partial \partial. \) Meanwhile, \( \psi_1(t) \cup (\rho_1(s) \cup) \) is \( \partial \partial \)-closed (because \( \psi_1(t) \cup \) and \( \rho_1(s) \cup \) are), hence \( \psi_1(t) \cup (\rho_1(s) \cup) \in Z_2^{4,1} \), as desired.

(b) Let \( \psi_1(t), \rho_1(s) \in C_0^{\infty}(X, T^{1,0}X) \) such that \( \psi_1(t) \cup \in \ker d \) and \( \rho_1(s) \cup \in \text{Im} \partial \). Then,

\[
\psi_1(t) = \sum_{i=1}^{5} \sum_{\lambda=1}^{2} t_{i\lambda} \psi_{i\lambda}, \quad \text{so} \quad \psi_1(t) \cup = \sum_{i=1}^{5} (-1)^{i-1} \sum_{\lambda=1}^{2} t_{i\lambda} \varphi_{\lambda} \wedge \hat{\varphi}_i,
\]

\[
\rho_1(s) = \sum_{j=3}^{5} s_{j} \theta_{j}, \quad \text{so} \quad \rho_1(s) \cup = \sum_{j=3}^{5} (-1)^{j-1} s_{j} \varphi_{j} \wedge \hat{\varphi}_j.
\]

Indeed, in the case of \( \rho_1(s) \cup \), we have

\[
\hat{\varphi}_3 = \partial (\varphi_3 \wedge \varphi_4 \wedge \varphi_5), \quad \text{so} \quad \varphi_3 \wedge \hat{\varphi}_3 = -\partial (\varphi_3 \wedge \varphi_4 \wedge \varphi_5),
\]

\[
\hat{\varphi}_4 = \partial (\varphi_2 \wedge \varphi_4 \wedge \varphi_5), \quad \text{so} \quad \varphi_3 \wedge \hat{\varphi}_4 = -\partial (\varphi_3 \wedge \varphi_2 \wedge \varphi_4 \wedge \varphi_5),
\]

\[
\hat{\varphi}_5 = \partial (\varphi_1 \wedge \varphi_4 \wedge \varphi_5), \quad \text{so} \quad \varphi_3 \wedge \hat{\varphi}_5 = -\partial (\varphi_3 \wedge \varphi_1 \wedge \varphi_4 \wedge \varphi_5).
\]
and every \( \partial \)-exact \((4,1)\)-form is a linear combination of \( \varphi_3 \wedge \bar{\varphi}_3, \varphi_3 \wedge \bar{\varphi}_4 \) and \( \varphi_3 \wedge \bar{\varphi}_5 \).

On the one hand, we get

\[
\psi_1(t) \wedge (\rho_1(s) \wedge u) = \sum_{i=1}^{5} \sum_{j=3}^{5} \sum_{\lambda=1}^{2} (-1)^{j-1} t_{i\lambda} s_j \varphi_\lambda \wedge \varphi_3 \wedge (\theta_i \wedge \bar{\varphi}_j).
\]

Now, \( \theta_i \wedge \bar{\varphi}_j \) is always \( \bar{\partial} \)-closed because it vanishes when \( i = j \), it equals \( (-1)^{i-1} \bar{\varphi}_{ij} \) when \( i < j \) and it equals \( (-1)^i \bar{\varphi}_{ij} \) when \( i > j \), where \( \bar{\varphi}_{ij} \) stands for \( \varphi_1 \wedge \cdots \wedge \varphi_5 \) with \( \varphi_i \) and \( \varphi_j \) omitted and \( i < j \). All the \( \varphi_i \)'s being \( \bar{\partial} \)-closed, so are all the \( \bar{\varphi}_{ij} \)'s. Meanwhile, \( \bar{\partial}(\varphi_\lambda \wedge \varphi_3) = -\bar{\varphi}_\lambda \wedge \bar{\partial}\varphi_3 = 0 \) for all \( \lambda \in \{1,2\} \), since \( \bar{\partial}\varphi_3 = \varphi_1 \wedge \varphi_2 \). This proves that \( \psi_1(t) \wedge (\rho_1(s) \wedge u) \in \ker \bar{\partial} \).

On the other hand, we get

\[
\partial(\psi_1(t) \wedge (\rho_1(s) \wedge u)) = [\psi_1(t), \rho_1(s)] \wedge u = \sum_{i=1}^{5} \sum_{j=3}^{5} \sum_{\lambda=1}^{2} t_{i\lambda} s_j \varphi_\lambda \wedge \varphi_3 \wedge ([\theta_i, \theta_j] \wedge u)
\]

where the second line followed from the fact that \([\theta_i, \theta_j] = 0\) unless \( i, j \in \{1,2,3\} \) and \( i \neq j \). Given the fact that the summation bears over \( j \in \{3,4,5\} \), this forces \( j = 3 \) and \( i \in \{1,2\} \). Then, we get the second line from \([\theta_1, \theta_3] = -\theta_1 \) and \([\theta_2, \theta_3] = -\theta_5 \).

The facts that \( \psi_1(t) \wedge (\rho_1(s) \wedge u) \in \ker \bar{\partial} \) and \( \partial(\psi_1(t) \wedge (\rho_1(s) \wedge u)) \in \ker \bar{\partial} \) translate to \( \psi_1(t) \wedge (\rho_1(s) \wedge u) \in Z^{3,2}_2 \), as desired.

(c) Let \( \psi_1(t), \rho_1(s) \in C_{0,1}^\infty(X,T^{1,0}X) \) such that \( \psi_1(t) \wedge u, \rho_1(s) \wedge u \in \ker \bar{\partial} \). Then,

\[
\psi_1(t) = \sum_{i=3}^{5} t_i \theta_i \varphi_3, \quad \rho_1(s) = \sum_{j=3}^{5} s_j \theta_j \varphi_3, \quad \text{so} \quad \rho_1(s) \wedge u = \sum_{j=3}^{5} (-1)^{j-1} s_j \varphi_3 \wedge \bar{\varphi}_j.
\]

We get

\[
\psi_1(t) \wedge (\rho_1(s) \wedge u) = \sum_{i=3}^{5} \sum_{j=3}^{5} (-1)^{j-1} t_i s_j \varphi_3 \wedge \varphi_3 \wedge (\theta_i \wedge \bar{\varphi}_j) = 0 \in Z^{3,2}_2,
\]

as desired.

This completes the proof of Lemma 5.15. \( \square \)

**Acknowledgements.** This work has been partially supported by the projects MTM2017-85649-P (AEI/FEDER, UE), and E22-17R “Álgebra y Geometría” (Gobierno de Aragón/FEDER).”

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