CARLEMAN ESTIMATES AND CONTROLLABILITY OF STOCHASTIC DEGENERATE PARABOLIC HEAT EQUATIONS

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Abstract. This paper concerns the null controllability for a class of stochastic degenerate parabolic equations. We first establish a global Carleman estimate for a linear forward stochastic degenerate equation with multiplicative noise. Using this estimate we prove the null controllability of the backward equation and obtain a partial result for the controllability of the forward equation. Also, using a new Carleman estimate for backward equation with weighted function which does not vanish at time \( t = 0 \) and the duality method HUM we get the null controllability of a forward stochastic degenerate equation under the action of two controls.

1. Introduction and main result

The purpose of this paper is to study the controllability of linear stochastic parabolic degenerate equations by using Carleman estimates. For this reason, we consider the following uncontrolled forward degenerate equation with multiplicative noise:

\[
\begin{aligned}
    d_t y - \left[ (a(x)y_x)_x + by(t,x) \right] dt &= cydW(t) \quad \text{in } Q_T, \\
    Cy &= 0 \quad \text{on } \Sigma_T, \\
    y(0,\cdot) &= y_0(\cdot) \quad \text{in } (0,1),
\end{aligned}
\]

where \( y \) is the state of the system. Let \( T > 0 \), put \( Q_T = (0,T) \times (0,1) \), \( \Sigma_T = (0,T) \times \{0,1\} \), \( \omega \subset (0,1) \) and \( \omega_T = (0,T) \times \omega \). Let \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) be a complete filtered probability space, on which a one-dimensional standard Brownian motion \( W(t)_{t \geq 0} \) is defined, so that \( \mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0} \) is the natural filtration generated by \( W(\cdot) \) augmented by all \( \mathbb{P} \)-null sets in \( \mathcal{F} \).

The function \( a \) is a diffusion coefficient which can degenerate at 0 (i.e. \( a(0) = 0 \)). The weak (WD) and strong (SD) degenerate cases depend on the diffusion coefficient \( a \) and are specified as follows:

- The diffusion coefficient \( a \) satisfies a weak degeneracy (WD), i.e.,
  \[
  \begin{align*}
  (i) & \quad a \in C([0,1]) \cap C^1((0,1]), a > 0 \text{ in } (0,1], a(0) = 0, \\
  (ii) & \quad \exists K \in [0,1) \text{ such that } xa'(x) \leq Ka(x) \text{ for all } x \in [0,1],
  \end{align*}
  \]

- Or, \( a \) satisfies a strong degeneracy (SD), i.e.,
  \[
  \begin{align*}
  (i) & \quad a \in C^1([0,1]), a > 0 \text{ in } (0,1], a(0) = 0, \\
  (ii) & \quad \exists K \in (1,2) \text{ such that } xa'(x) \leq Ka(x) \text{ for all } x \in [0,1], \\
  (iii) & \quad \exists \theta \in (1,K] : x \mapsto \frac{a(x)}{x^\theta} \text{ is nondecreasing near 0, if } K > 1, \\
  & \quad \exists \theta \in (0,1) : x \mapsto \frac{a(x)}{x^\theta} \text{ is nondecreasing near 0, if } K = 1.
  \end{align*}
  \]

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In both degeneracies (WD) and (SD), the following function \( x \mapsto \frac{x^2}{a(x)} \) is non-decreasing on \((0, 1]\), and \(b, c\) are given functional satisfying some properties to be specified later. The boundary operator \(C\) is either the trace operator in the weak degenerate case (WD) or the associated Neumann boundary condition in the strongly degenerate case (SD).

In the deterministic case, the null controllability of parabolic equations and systems has been widely studied during the last half-century and there have been a great number of results, among which we can cite G. Lebeau and L. Robbiano [12], A. V. Fursikov and O. Y. Imanuvilov [10], Ammar-Khodja et al. [5] and references therein. Likewise, the controllability of degenerate parabolic equations has been the subject of numerous results [1, 2, 3, 4, 9]. The previous list is not at all exhaustive, highlighting that the technique used for this purpose is essentially based on Carleman-type estimates.

In the stochastic case, the literature dealing with backward stochastic differential equations and connecting to control theory has been extensively studied. Barbu et al [6] gave a positive result of the null controllability for backward stochastic heat equation and Tang and Zhang [17] studied the null controllability of both forward and backward stochastic heat equations. Victor Hernández-Santamaría et al [8] studied Global null-controllability for stochastic semilinear parabolic equations and used the Hilbert Uniqueness Method (HUM) to prove the null controllability of a forward stochastic parabolic equation with two controls. Having in mind that the previous technique was developed by J.L. Lions [13] and widely used by many authors later.

On the other side, in their paper [14], Xu Liu and Yongyi Yu studied some stochastic degenerate parabolic equations and established Carleman estimates in the particular case where the diffusion coefficient \(a(x) = x^\alpha\) with \(\alpha \in (0, 2)\) and they used also the duality technique HUM to establish Carleman estimates for a backward stochastic degenerate problem.

Our first main question is to establish Carleman estimates with suitable weighted functions for the stochastic equation (FSDE), and use these estimates in order to prove the null-controllability for the following backward equation:

\[
\begin{aligned}
&dz + (a(x)z_x)_x \, dt + bz(t, x)dt + c\, kdt = 1_\omega vdt + k\, dW(t) \quad \text{in } Q_T, \\
&Cz = 0, \quad \text{on } \Sigma_T, \\
&z(T, \cdot) = z_T, \quad \text{in } (0, 1),
\end{aligned}
\]

where the couple \((z, k)\) is the solution of (BSDE) in the sens of [11] and [15], \(\omega\) is an open subset of \((0, 1]\), \(1_\omega\) is the characteristic function of the subset \(\omega\) and \(v\) is the control acting on \(\omega\).

For this purpose, we give a new proof of Carleman estimates with one weighted function for forward stochastic degenerate equation (FSDE) in general case where the diffusion coefficient \(a\) satisfies the assumptions (WD) or (SD) (see Theorem 3.10). Using the previous estimates, we establish an observability inequality for the problem (FSDE). Therefore, we deduce the unique continuation property for forward stochastic degenerate equation (FSDE) and finally, we prove the following first main result.

**Theorem 1.1** (Null controllability of the backward stochastic degenerate equation (BSDE)).

For every final datum \(z_T\) in \(L^2(\Omega, F_T; L^2(0, 1))\) there exists a control \(v\) in \(L^2_{reg}(\Omega; L^2(0, T; L^2(0, 1)))\) such that \(z(0, x) = 0\) for almost every \(x \in (0, 1)\).

In the second main result, we adopt the duality technique HUM to study the null controllability of the following stochastic forward degenerate parabolic equation with two controls

\[
\begin{aligned}
&dy = [(a(x)y_x)_x + F + 1_\omega h] \, dt + (G + H) \, dW(t), \quad \text{in } Q_T, \\
&Cy = 0, \quad \text{on } \Sigma_T, \\
&y(0, \cdot) = y_0(\cdot) \quad \text{in } (0, 1),
\end{aligned}
\]

where \((h, H)\) is the pair of controls. This leads us to establish a global Carleman estimates with weighted function which does not vanish at \(t = 0\). After then, the null controllability of the forward stochastic degenerate parabolic equation (FSDE2) under the action the pair \((h, H)\) of controls is obtained by the following theorem:
1.2. For any \( y_0 \in L^2(\Omega, F_0, L^2(0,1)) \) and any couple \((F, G) \in \mathcal{S}_s\). There exists a pair of controls \((\tilde{h}, \tilde{H}) \in L^2_\mathbb{F}(0, T; L^2(\omega)) \times L^2_\mathbb{F}(0, T; L^2(0,1))\) such that the associated solution \( \tilde{y} \) to the system (FSDE2) satisfies \( \tilde{y}(T) = 0 \) a.s. Moreover, we have the following estimate

\[
\mathbb{E} \int_{Q_T} e^{2s\varphi} \tilde{y}^2 dx dt + \mathbb{E} \int_{0}^{T} \int_{\omega} e^{2s\varphi} s^{-3}\theta^{-3} \tilde{H}^2 dx dt + \mathbb{E} \int_{Q_T} e^{2s\varphi} s^{-2}\theta^{-3} \tilde{H}^2 dx dt
\]

\[
\leq C \left( \|y_0\|^2_{L^2(0,1)} \right) + C\|(F, G)\|^2_{\mathcal{S}_s}, \quad (1.3)
\]

where \( \mathcal{S}_s \) is the following space

\[
\mathcal{S}_s = \{(F, G) \in L^2_\mathbb{F}(0, T; L^2(0,1)): \left( \mathbb{E} \int_{Q_T} e^{2s\varphi} s^{-3}\theta^{-3} \frac{\partial}{\partial x} F^2 dx dt \right) + \left( \mathbb{E} \int_{Q_T} e^{2s\varphi} s^{-2}\theta^{-3} G^2 dx dt \right) < \infty \}
\]

endowed with the canonical norm.

The rest of this paper is organized as follows. In section 2, we give some weighted spaces related to the diffusion coefficient \( a \) and some assumptions on the both problems (FSDE) and (BSDE) in order to discuss their well-posedness. Section 3 is devoted to establish Carleman estimates with one weighted function for the forward stochastic degenerate parabolic equation with multiplicative noise (FSDE). In section 4, we establish the unique continuation of the equation (FSDE) and we prove the null controllability of the backward stochastic degenerate equation with multiplicative noise (BSDE). In section 5, we give the proof of the second main theorem 1.2. Finally, in the appendix we give the proof of Caccioppoli’s inequality.

All along the article, we use generic constants for the estimates, whose values may change from line to line.

2. Preliminaries

Let us introduce the following weighted spaces related to the diffusion coefficient \( a \). In the (WD) case:

\[
H^2_a = \{ u \in L^2(0,1)/u \text{ absolutely continuous in } [0,1], \sqrt{a}u_x \in L^2(0,1) \text{ and } u(1) = u(0) = 0 \}
\]

and

\[
H^1_a = \{ u \in H^2_a / au_x \in H^1(0,1) \}.
\]

In the (SD) case:

\[
H^2_a = \{ u \in L^2(0,1)/u \text{ absolutely continuous in } (0,1], \sqrt{a}u_x \in L^2(0,1) \text{ and } u(1) = 0 \}
\]

and

\[
H^1_a = \{ u \in H^2_a / au_x \in H^1(0,1) \}\]

\[
= \{ u \in L^2(0,1)/u \text{ absolutely continuous in } (0,1], au \in H^3(0,1), au_x \in H^1(0,1) \text{ and } (au_x)(0) = 0 \}.
\]

In both cases, the norms are defined as follow

\[
\| u \|^2_{H^1_a} = \| u \|^2_{L^2(0,1)} + \| \sqrt{a}u_x \|^2_{L^2(0,1)}, \quad \| u \|^2_{H^2_a} = \| u \|^2_{H^1_a} + \| (au_x)_x \|^2_{L^2(0,1)}.
\]

(2.1)

In the sequel, we will be interested by the equations (FSDE) and (BSDE) with the following assumptions:

1) \( \omega \subset (0,1) \) is an open non-empty subset of the interval \( (0,1) \) such that \( \mathcal{F} \subset (0,1) \).

2) \( \{W(t); t \geq 0\} \) is a standard, one dimensional Brownian motion on a complet probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) endowed with the filtration \( \mathcal{F}_t = \sigma(W(s); s \in [0,t]) \cup \{A \in \mathcal{F}, \mathbb{P}(A) = 0\} \).

3) The coefficients \( b, c \) are adapted processes with values in \( W^{1,\infty}(0,1) \) and \( b, c \in L^\infty(\Omega \times [0,T]; W^{1,\infty}(0,1)) \).

Moreover, given a Banach space \( X \), we denote by \( L^2_\mathbb{F}(\Omega; L^2(0,T;X)) \) the Banach space consisting of all \( X \)-valued \( \{F_t\}_{t \geq 0} \)-adapted processes \( z \) such that \( \mathbb{E}(\|z\|^2_{L^2(0,T;X)}) < \infty \) with the canonical norm, \( L^2_\mathbb{F}(\Omega; C([0,T];X)) \) the Banach space consisting of all \( X \)-valued \( \{F_t\}_{t \geq 0} \)-adapted continuous processes \( z \) such that \( \mathbb{E}(\|z\|^2_{C([0,T];X)}) < \infty \) with the canonical norm, and by \( L^\infty_\mathbb{F}(\Omega;X) \) the Banach space consisting of all \( X \)-valued \( \{F_t\}_{t \geq 0} \)-adapted essentially bounded processes.

Let consider the forward stochastic equation

\[
\begin{align*}
dy + [a(x)y_x]_x + b_0 y(t,x) \; dt = f(t,x)dt + c_0 ydW(t) & \quad \text{in } Q_T, \\
cy = 0 & \quad \text{on } \Sigma_T, \\
y(0,\cdot) = y_0(\cdot) & \quad \text{in } (0,1).
\end{align*}
\]

(2.2)
where
\[
\begin{align*}
(1) & \ f \in L^2_T(\Omega; L^2(0, T; L^2(0, 1))), \\
(2) & \ b_0, c_0 \in L^\infty_T(\Omega; L^\infty(0, T; L^\infty(0, 1))), \\
(3) & \ y_0 \in L^2(\Omega, F_0, P; L^2(0, 1)).
\end{align*}
\]

It is well known, see [6, 7], that under such assumptions there exists a unique solution \( y \) of (2.2) belonging to the space \( L^2_T(\omega \in \mathbb{C}([0, T]; L^2(0, 1))) \cap L^2_T(\Omega; L^2(0, T; H^1_0)) \), with for a suitable constant \( C \) independent of \( y_0 \) and \( f 

\[
E \left[ \sup_{t \in [0, T]} \|y(t)\|^2_{L^2(0, 1)} + \int_0^T \|y(t)\|^2_{H^1_0} \right] \leq C \left( \|y_0\|^2_{L^2(0, 1)} + E \int_0^T \|f\|^2_{L^2(0, 1)} \right).
\]

Furthermore, if \( y_0 \in H^1_0 \) then \( y \in L^2_T(\omega \in \mathbb{C}([0, T]; H^1_0)) \cap L^2_T(\Omega; L^2(0, T; H^1_0)). 

We also consider the backward equation
\[
\begin{align*}
(1) & \ G \in L^2_T(\Omega; L^2(0, T; L^2(0, 1))), \\
(2) & \ z_T \in L^2(\Omega; F_T, P; L^2(0, 1)).
\end{align*}
\]

Likewise in [16, 18], under (2.5) the backward equation (2.4) has a unique solution \((z, k)\) with \( z \in L^2_T(\omega \in \mathbb{C}([0, T]; L^2(0, 1))) \cap L^2_T(\Omega; L^2(0, T; H^1_0)) \) and \( k \in L^2_T(\Omega; L^2(0, T; L^2(0, 1))) \).

3. CARLEMAN ESTIMATE FOR FORWARD STOCHASTIC PARABOLIC DEGENERATE EQUATION

Let us now consider the following non controlled stochastic problem for which the diffusion coefficient \( a \) degenerates at zero (i.e. \( a(0) = 0 \) and \( a > 0 \) in \((0, 1))
\[
\begin{align*}
(1) & \ dy(t, x) = (a(x)y_x)_x dt + f(t, x)dt + g(t, x)dW(t), \text{ in } Q_T, \\
(2) & \ C_y = 0, \text{ on } \Sigma_T, \\
(3) & \ y(0, \cdot) = y_0(\cdot),
\end{align*}
\]

where the term \( f \) is in \( L^2_T(\Omega; L^2(0, T; L^2(0, 1))) \) and \( g \) is in \( L^2_T(\Omega; L^2(0, T; H^1_0)). \)

To define the weight functions we proceed as follows: we consider \( \omega = (a, b) \in (0, 1) \) a nonempty open subset of \((0, 1) \) and \( \omega_1 = (a_1, b_1) \in \omega = (a, b). \) Let also consider \( \xi \in C(\mathbb{R}) \) such that
\[
\xi(x) = \begin{cases} 
1 & \text{if } x \in [0, a_1], \\
0 & \text{if } x \in [b_1, 1].
\end{cases}
\]

Define \( \phi(x) = d - \int_0^x \frac{V}{a(v)} dv, \) where the real \( d \) is chosen such that \( \phi > 0 \) on \((0, 1), \) let \( \rho \) be a smooth function defined by \( \rho(x) = \int_x^1 \frac{s}{a(s)} ds, \) for all \( x \in (0, 1). \)

We define \( \psi(x) = e^{2\beta(x)} - e^\beta(x) \) and \( \beta(x) = \xi(x)\phi(x) + (1 - \xi(x))\psi(x). \) Notice that \( \beta \geq 0 \) by construction. Finally, we set
\[
\varphi(t, x) = \theta(t)\beta(x),
\]

where \( \theta(t) = \frac{1}{t^4(T-t)^2} \) on \((0, T). \)

Now, we need to recall some useful properties of the previous function \( \theta \) (see [2]).

**Lemma 3.1.** We have
\[
\lim_{t \to 0^+} \theta(t) = \lim_{t \to T^-} \theta(t) = +\infty, \quad \theta(t) \geq c_1, \quad |\dot{\theta}| \leq c_2\theta^2, \quad |\ddot{\theta}| \leq c_3\theta^3,
\]
where \( c_1 = (\frac{2}{T})^8, \) \( c_2 = 8(\frac{T}{2})^7 \) and \( c_3 = 80(\frac{T}{2})^{14}. \)

Moreover, we have \( |\dot{\theta}(t)| \leq c_4\theta^\frac{3}{2} \) and \( |\ddot{\theta}(t)| \leq c_5\theta^2 \) with \( c_4 = T^3 \) and \( c_5 = 80(\frac{T}{2})^6. \)
Let us recall that the unbounded operator \( \mathcal{M} : \mathcal{D}(\mathcal{M}) \subset L^2(0,1) \rightarrow L^2(0,1) \) defined by
\[
\mathcal{M}y = (a(x)y_x)_x, \quad \mathcal{D}(\mathcal{M}) = H^2_0
\]
generates a contraction strongly continuous semi-group \((T(t))_{t \geq 0}\). The following result is a weighted identity for the forward stochastic degenerate parabolic operator \( dy - \mathcal{M}y dt \). For the proof see [14].

**Lemma 3.2.** Let \( y \) be a \( H^2_0 \)-valued continuous semimartingale, and set \( z = e^{-s\varphi}y \). Then, for a.e. \((t,x) \in Q_T \) and \( \mathbb{P}\)-a.s. \( \kappa \in \Omega \), one has the following weighted identity:
\[
e^{-s\varphi} [Az - (az_z)_x] [dy - (ay_y)_x dt] = \int e^{-s\varphi} [Az - (az_z)_x]^2 dt + d \left( \frac{1}{2} A z^2 + \frac{1}{2} (az_z)^2 \right) - \frac{1}{2} A(dx)^2 - \frac{1}{2} ((az_z) dx)_x
\]
\[
+ \left\{ a(a\varphi_x)_x z_x z - s A a\varphi_x z^2 - \frac{1}{2} sa(a\varphi_x)_{xx} z^2 + sa^2 \varphi_x z_x^2 \right\} dt
\]
\[
+ \left\{ s(Aa\varphi_x)_x - s A(a\varphi_x)_x + \frac{1}{2} [sa(a\varphi_x)_{xx} x] - \frac{1}{2} A t \right\} z^2 dt
\]
\[
- \left\{ sa[(a\varphi_x)_x + a\varphi_{xx}] \right\} z^2 dt,
\]
where \( A = s \varphi_t - s^2 a\varphi_x \).

Let \( y \) be any solution of (3.1) and notice that \( e^{-s\varphi}(0,x) = e^{-s\varphi}(T,x) = 0 \) in \([0,1]\]. Integrating (3.2) on \( Q_T \) and taking expectation, one obtains that
\[
E \int_{Q_T} e^{-s\varphi} [Az - (az_z)_x] [dy - (ay_y)_x dt] dx = \int_{Q_T} \left[ e^{-s\varphi} (Az - (az_z)_x)^2 + \frac{1}{2} E \int_{Q_T} A e^{-2s\varphi} g^2 dx dt - \frac{1}{2} E \int_{Q_T} a(e^{-s\varphi} g)_x^2 dx dt \right]
\]
\[
+ \left\{ s(Aa\varphi_x)_x - s A(a\varphi_x)_x + \frac{1}{2} [sa(a\varphi_x)_{xx} x] - \frac{1}{2} A t \right\} z^2 dx dt
\]
\[
- \left\{ sa[(a\varphi_x)_x + a\varphi_{xx}] \right\} z^2 dx dt.
\]

To estimate terms on the right of the expression (3.3), let us write
\[
\begin{align*}
Q_1 &= -\frac{1}{2} E \int_{Q_T} A e^{-2s\varphi} g^2 dx dt, \quad Q_2 = -\frac{1}{2} E \int_{Q_T} a(e^{-s\varphi} g)_x^2 dx dt, \quad Q_3 = - E \int_{Q_T} (az_z)_x dx dt, \\
Q_4 &= E \int_{Q_T} \left[ sa(a\varphi_x)_x z_x z - s A a\varphi_x z^2 - \frac{1}{2} sa(a\varphi_x)_{xx} z^2 + sa^2 \varphi_x z_x^2 \right] dx dt, \\
Q_5 &= E \int_{Q_T} \left\{ s(Aa\varphi_x)_x - s A(a\varphi_x)_x + \frac{1}{2} [sa(a\varphi_x)_{xx} x] - \frac{1}{2} A t \right\} z^2 dx dt, \\
Q_6 &= - E \int_{Q_T} \left\{ sa[(a\varphi_x)_x + a\varphi_{xx}] \right\} z^2 dx dt.
\end{align*}
\]

We then have the following lemmas

**Lemma 3.3.**
\[
Q_1 \geq - CE \int_{Q_T} s^2 \theta^2 e^{-2s\varphi} g^2 dx dt.
\]

**Proof.** It suffices to see that \( A = s \varphi_t - s^2 a\varphi_x = \sqrt{\beta} - s^2 a \theta^2 \varphi_x^2 \leq C s^2 \theta^2 \).

**Lemma 3.4.**
\[
Q_2 = - \frac{1}{2} E \int_{Q_T} a(e^{-s\varphi} g)_x^2 dx dt \geq - CE \int_{Q_T} \left( s \sqrt{\theta} x^2 e^{-2s\varphi} g^2 + a e^{-2s\varphi} g_x^2 \right) dx dt.
\]
Proof. Since $[e^{-s\varphi}g]_s = -s\varphi e^{-s\varphi}g + e^{-s\varphi}g_s$, then

$$Q_2 = -\frac{1}{2} \E \int_{Q_T} \left[ sa(a\varphi)_x x z_s - sa(a\varphi)_x x^2 z_s - \frac{1}{2} sa(a\varphi)_x x z_s^2 + sa^2 \varphi a z_s^2 \right]_{x=0}^{x=1} dt.$$

On the interval $[0, a_1]$ we have $a\varphi a = \theta^2 \frac{x^2}{a}$, then we can bound $a\varphi a$ over $[0, 1]$ by $C\theta^2 \frac{x^2}{a}$. □

Lemma 3.5.

$$Q_3 = 0.$$

Proof. $Q_3 = -\E \int_0^T [az_s z_s]_{x=0}^{x=1} dt$. In the weak degenerate case (WD) we have $z(t, 0) = z(t, 1) = 0$ for all $t \in (0, T)$, thus $Q_3 = 0$. Whereas in the strong degenerate case (SD) we have $ay_s(t, 0) = z(t, 1) = 0$ for all $t \in (0, T)$ and $z_s = s\theta^2 e^{-s\varphi} y + e^{-s\varphi} y_s$ on the interval $[0, a_1]$. Hence,

$$Q_3 = \E \int_0^T az_s dz |_{x=0} dt = \E \int_0^T s\theta x e^{-s\varphi} y dz |_{x=0} dt = 0.$$

Lemma 3.6.

$$Q_4 \geq 0.$$

Proof. We notice that $\varphi_x = \theta^2 \frac{x}{a} e^{\rho(x)}$ for all $x$ in $[0, 1]$ and $\varphi_x = -\frac{\rho}{a}$, $(a\varphi_x)_x = 0$ for all $x$ in $[0, a_1]$. Then, in the weak degenerate case (WD) we have

$$Q_4 = \E \int_0^T \left[ sa(a\varphi)_x x z_s - sa(a\varphi)_x x^2 z_s + sa^2 \varphi x z_s^2 \right]_{x=0}^{x=1} dt$$

$$= \E \int_0^T \left[ sa^2 \varphi x z_s^2 \right]_{x=0}^{x=1} dt = \E \int_0^T s\theta x e^{\rho(x)} x z_s^2 |_{x=0}^{x=1} dt \geq 0.$$

Whereas in the strong degenerate case (SD), we get

$$Q_4 = \E \int_0^T sa^2 \varphi x z_s^2 |_{x=0}^{x=1} dt - \E \int_0^T -sa(a\varphi)_x x z_s^2 = -\frac{1}{2} sa(a\varphi)_x x z_s^2 + sa^2 \varphi x z_s^2 |_{x=0}^{x=1} dt$$

$$= \E \int_0^T s\theta x e^{\rho(x)} x z_s^2 |_{x=0}^{x=1} dt = \E \int_0^T s\theta x e^{\rho(x)} x z_s^2 |_{x=0}^{x=1} dt \geq 0.$$

□

Lemma 3.7. There exist two positive $C'$ and $C''$ such that for every choice of small scalar $\varepsilon > 0$ we have

$$Q_5 \geq C' \E \int_{Q_T} \frac{s^3 \theta \frac{x^2}{a}}{2} z^2 dx dt - C'' \E \int_{Q_T} \frac{s^3 \theta \frac{x^2}{a}}{2} z^2 dx dt - \varepsilon \E \int_{Q_T} (s\theta a x^2 + s^3 \theta \frac{x^2}{a} z^2) dx dt. \quad (3.4)$$

Proof. Since $A_r = s\varphi_{\|} - 2s a\varphi x \varphi x t$ and $A_x = s\varphi_x - s^3 (a\varphi_x)_x$, then

$$s(Aa\varphi_x) - s A(a\varphi)_x + \frac{1}{2} [sa(a\varphi)_x]_x^2 - \frac{1}{2} A_t$$

$$= s A a \varphi_x + \frac{1}{2} sa(a\varphi_x)_x + \frac{1}{2} sa(a\varphi)_x x x - \frac{1}{2} A_t$$

$$= \frac{1}{2} sa(a\varphi_x)_x + \frac{1}{2} sa(a\varphi)_x x x + 2s a\varphi x \varphi x t - s^3 (a\varphi_x)_x a\varphi_x - \frac{1}{2} s\varphi_{\|}$$

$$= \frac{1}{2} sa(a\varphi_x)_x + \frac{1}{2} sa(a\varphi)_x x x + 2s a\varphi x \varphi x t - s^3 a\varphi^2 ((a\varphi)_x + a\varphi x) - \frac{1}{2} s\varphi_{\|}.$$

Hence,

$$Q_5 = \E \int_{Q_T} \left( \frac{1}{2} sa(a\varphi)_x x x + \frac{1}{2} sa(a\varphi)_x x x + 2s a\varphi x \varphi x t - \frac{1}{2} s\varphi_{\|} \right) z^2 dx dt$$

$$- \E \int_{Q_T} s^3 a\varphi^2 ((a\varphi)_x + a\varphi x) z^2 dx dt = J_1 + J_2.$$

For the first integral $J_1$ we have

$$J_1 = \E \int_{Q_T} \frac{1}{2} s (a x a\varphi x) x x + a (a\varphi) x x x x) z^2 dx dt + \E \int_{Q_T} 2s^2 a\varphi x \varphi x t z^2 dx dt - \E \int_{Q_T} \frac{1}{2} s\varphi_{\|} z^2 dx dt$$
Thus, for \( x \) on the interval \([0, a_1]\), we can bound \( J_1^1 \) as follows

\[
J_1^1 = \frac{1}{2} \mathbb{E} \int_{Q_T} s (a_x(a_x')_{xx} + a(a_{xx})_{xx}) z^2 dxdt \leq C \int_0^T \int_{[a_1, 1]} s \theta z^2 dxdt
\]

\[
\leq CE \int_0^T \int_{[a_1, 1]} s \theta^2 z^2 dxdt
\]

\[
\leq \varepsilon \mathbb{E} \int_{Q_T} s^3 \theta^2 z^2 dxdt,
\]

(3.5)

for \( \varepsilon > 0 \) and \( s \) large enough.

To estimate \( J_1^2 \), notice that

\[
J_1^2 = 2\mathbb{E} \int_{Q_T} s^2 a\theta \beta_x^2 z^2 dxdt
\]

\[
\leq CE \int_0^T \int_{[0, a_1]} s^2 \theta^2 x^2 dxdt + CE \int_0^T \int_{[a_1, 1]} s^2 a\theta \beta_x^2 z^2 dxdt
\]

\[
\leq CE \int_0^T \int_{[0, a_1]} s^2 \theta^2 x^2 dxdt + CE \int_0^T \int_{[a_1, 1]} s^2 \theta^3 z^2 dxdt.
\]

Once again, since the functions \( x \mapsto \frac{a}{x} \) and \( x \mapsto \frac{a}{x^2} \) do not vanish on \([a_1, 1]\), for a suitable constant we have

\[
J_1^2 \leq CE \int_0^T \int_{[0, a_1]} s^2 \theta^2 x^2 dxdt + CE \int_0^T \int_{[a_1, 1]} s^2 \theta^3 z^2 dxdt.
\]

Thus, for \( s \) sufficiently large we obtain

\[
|J_1^2| \leq \varepsilon \mathbb{E} \int_{Q_T} s^3 \theta^2 z^2 dxdt.
\]

(3.6)

Now, we estimate \( J_1^3 \). Using Lemma 3.1 and Hardy-Poincaré inequality we get

\[
|J_1^3| \leq CE \int_{Q_T} s \theta^2 z^2 dxdt
\]

\[
\leq CE \int_{Q_T} (s \theta^2 \frac{\sqrt{z}}{x})(s \theta^2 \frac{x}{\sqrt{z}}) dxdt
\]

\[
\leq CE \int_{Q_T} s^2 \theta \frac{a}{x^2} z^2 dxdt + CE \int_{Q_T} s^2 \theta^3 z^2 dxdt
\]

\[
\leq CE \int_{Q_T} s^2 \theta a z^2 dxdt + CE \int_{Q_T} s^2 \theta^3 z^2 dxdt.
\]

Thus, for \( s \) sufficiently large we have

\[
|J_1^3| \leq \varepsilon \mathbb{E} \int_{Q_T} (s \theta a z^2 + s^3 \theta^3 \frac{z^2}{a} z^2) dxdt.
\]

(3.7)

Finally, for the integral \( J_2 \), note that

\[
(a_{xx})_{xx} + a_{xx} = \mathbb{E}_{[0, a_1]} \theta[(a\phi')' + a\phi''] + \mathbb{E}_{[0, 1]} \theta[(a\psi')' + a\psi''] + \mathbb{E}_{[a_1, b_1]} \theta k(x)
\]

\[
= -\theta \left[ \frac{2a - x' a}{a} \right] \mathbb{E}_{[0, a_1]} - \theta \left[ \frac{2a - x' a}{a} + \frac{2x^2}{a} \right] e^{\theta x} \mathbb{E}_{[a_1, 1]} + \mathbb{E}_{[a_1, b_1]} \theta k(x),
\]

(3.8)

where \( k \) is a bounded function on \([a_1, b_1]\). Thus,

\[
J_2 = -\mathbb{E} \int_{Q_T} s^3 a\phi_x^2 \left\{ -\theta \left[ \frac{2a - x' a}{a} \right] \mathbb{E}_{[0, a_1]} - \theta \left[ \frac{2a - x' a}{a} + \frac{2x^2}{a} \right] e^{\theta x} \mathbb{E}_{[a_1, 1]} + \mathbb{E}_{[a_1, b_1]} \theta k(x) \right\} z^2 dxdt
\]

\[
= \mathbb{E} \int_0^T \int_{[0, a_1]} s^3 \theta a^2 \left[ \frac{2a - x' a}{a} \right] z^2 dxdt + \mathbb{E} \int_0^T \int_{[b_1, 1]} s^3 \theta a^2 \left[ \frac{2a - x' a}{a} + \frac{2x^2}{a} \right] e^{\theta x} z^2 dxdt
\]

\[
- \mathbb{E} \int_0^T \int_{[a_1, b_1]} s^3 \theta a^2 \left( k(x) \right) z^2 dxdt
\]
Lemma 3.8. There exist two positive constants $C$ and $C_1$ such that
\[
Q_6 \geq CE \int_{[a,b]} sa\theta z_2^2 dxdt - C_1 E \int_0^T \int_{[a,b]} sa\theta z_2^2 dxdt.
\]

Proof. Likewise in the previous Lemma, taking into account (3.8) we have
\[
Q_6 = -E \int_{Q_T} \{sa[(a\varphi_x)_x + a\varphi_x)]\} z_2^2 dxdt
\]
\[
= -E \int_{Q_T} \{sa\theta[\frac{2a - xa'}{a}] a_{[0,a]} - sa\theta[\frac{2a - xa'}{a}] + \frac{2\pi^2}{a} e^{\rho(x)} a_{[b,1]} + a_{[a,b]} sa\theta k(x)\} z_2^2 dxdt
\]
\[
= E \int_{[0,T]} \int_{[a,b]} sa\theta[\frac{2a - xa'}{a}] z_2^2 dxdt + E \int_{[a,b]} sa\theta[\frac{2a - xa'}{a}] + \frac{2\pi^2}{a} e^{\rho(x)} z_2^2 dxdt
\]
\[
- E \int_0^T \int_{[a,b]} sa\theta k(x) z_2^2 dxdt
\]
\[
\geq CE \int_{Q_T} sa\theta z_2^2 dxdt - C_1 E \int_0^T \int_{[a,b]} sa\theta z_2^2 dxdt.
\]

Proposition 3.9. There exist two positive constants $C$ and $s_0$ such that, for all $y_0 \in L^2(0,1)$, the solution $y$ of (3.1) satisfies
\[
E \int_{Q_T} sa\theta y_2^2 e^{-2sx} dxdt + E \int_{Q_T} s^3 \theta^3 x^2 a y^2 e^{-2sx} dxdt
\]
\[
\leq CE \int_{Q_T} \left( f^2 e^{-2sx} + s^2 \theta^2 \frac{x^2}{a} e^{-2sx} g + ae^{-2sx} g^2 \right) dxdt + CE \int_0^T \int_{[a,b]} s^3 \theta^3 y^2 e^{-2sx} dxdt,
\] (3.10)
for all $s > s_0$.

Proof. By the formula (3.3) and Lemmas 3.3-3.8, for any solution $y$ of (3.1), one has the following estimate
\[
E \int_{Q_T} sa\theta z_2^2 dxdt + E \int_{Q_T} s^3 \theta^3 x^2 a z_2^2 dxdt \leq E \int_{Q_T} e^{-sx} [Az + (az)_x] \{dy - (ay)_x\} dx
\]
\[
+ CE \int_{Q_T} s^2 \theta^2 x^2 a e^{-2sx} g^2 dxdt + CE \int_{Q_T} ae^{-2sx} g^2 dxdt
\]
\[
+ CE \int_0^T \int_{[a,b]} s^3 \theta^3 x^2 a z_2^2 dxdt + CE \int_0^T \int_{[a,b]} sa\theta z_2^2 dxdt.
\]
We thus get
\[
E \int_{Q_T} sa\theta z_2^2 dxdt + E \int_{Q_T} s^3 \theta^3 x^2 a z_2^2 dxdt \leq CE \int_{Q_T} f^2 e^{-2sx} + s^2 \theta^2 \frac{x^2}{a} e^{-2sx} g^2 + ae^{-2sx} g^2 dxdt
\]
\[
+ CE \int_0^T \int_{[a,b]} s^3 \theta^3 x^2 a z^2 + sa\theta z_2^2 dxdt.
\]
Using the Caccioppoli’s inequality and taking into account that $z = e^{-sx} y$ and $z_x = -s\varphi_x e^{-sx} y + e^{-sx} y_x$, we obtain the desired estimate.
\[\square\]
Now, we gave the following global Carleman estimate related to the equation (FSDE).

**Theorem 3.10.** There exist two positive constants $C$ and $s_0$ such that, for all $y_0 \in L^2((0,1))$, the solution $y$ of (FSDE) satisfies

$$E \int_{QT} sa\theta y_x^2 e^{-2\varphi} dxdt + E \int_{QT} s^3 \theta^3 \frac{x^2}{a} e^{-2\varphi} dxdt \leq CE \int_0^T \int_{Q} s^3 \theta^3 \frac{y^2}{a} e^{-2\varphi} dxdt,$$

(3.11)

for all $s > s_0$.

**Proof.** We consider a particular case of equation (3.1) with $f(t, x) = g(t, x)$ and $g(t, x) = y(t, x)$. By applying the Proposition 3.9 to the equation (FSDE), we get

$$E \int_{QT} sa\theta y_x^2 e^{-2\varphi} dxdt + E \int_{QT} s^3 \theta^3 \frac{x^2}{a} e^{-2\varphi} dxdt \leq CE \int_0^T \int_{Q} s^3 \theta^3 \frac{y^2}{a} e^{-2\varphi} dxdt,$$

for all $s > s_0$.

Notice that the term $E \int_{QT} y^2 e^{-2\varphi} dxdt$ can be bounded by $E \int_{QT} s^2 \theta^2 \frac{x^2}{a} e^{-2\varphi} y^2 dxdt$ for $s$ large enough. Consequently,

$$CE \int_0^T \left( y^2 e^{-2\varphi} + s^2 \theta^2 \frac{x^2}{a} e^{-2\varphi} y^2 + ae^{-2\varphi} y_x^2 \right) dxdt \leq \frac{1}{2} E \int_{QT} sa\theta y_x^2 e^{-2\varphi} dxdt + E \int_{QT} s^3 \theta^3 \frac{x^2}{a} y{2} e^{-2\varphi} dxdt.$$

□

4. CONTROLLABILITY RESULTS

We start by the following statement which is of big interest to prove the unique continuation property and the main theorem.

**Proposition 4.1.** Under the same assumptions of the Theorem 3.10, there exist two positive constants $C$ and $s_0$ such that

$$E \int_0^1 y^2(T, x) dx \leq CE \int_0^T \int_{Q} s^3 \theta^3(t) y^2(t, x) e^{-2\varphi} dxdt,$$

(4.1)

for all $s \geq s_0$.

**Proof.** Differentiating $d_s \| y \|^2_{L^2((0,1))}$, integrating on $[s, t]$ (with $0 \leq s \leq t \leq T$) and taking the mean value, we have

$$E \int_0^1 y^2(t, x) dx \leq E \int_0^1 y^2(s, x) dx + CE \int_s^t \int_0^1 y^2(t, x) dx dt.$$

Tanks to Gronwall’s lemma, we get

$$E \int_0^1 y^2(t, x) dx \leq e^{C(t-s)} E \int_0^1 y^2(s, x) dx \leq e^{CT} E \int_0^1 y^2(s, x) dx.$$

Knowing that $\theta(t)e^{-\varphi(s,x)} \geq \frac{2}{T} e^{-\varphi(t)}$ for all $(s, x) \in [0, T] \times (0, 1)$, where $D = \max_{x \in [0, 1]} \beta(x)$.

Hence, integrating on $[0, T]$, using Hardy-Poincaré inequality and Theorem 3.10, we obtain

$$\left( \int_0^T \frac{2}{T} e^{-2\varphi} \right) \left( E \int_0^1 y^2(T, x) dx \right) \leq e^{CT} E \int_0^T \int_0^1 e^{-2\varphi} y^2(s, x) dx dt$$

$$\leq e^{CT} E \int_0^T \int_0^1 \left( e^{-2\varphi} \theta^2 \frac{y^2(s, x)}{x^2} \right) \left( e^{-2\varphi} \theta^2 \frac{y^2(s, x)}{x^2} \right) dx dt$$

$$\leq \frac{e^{CT}}{2} E \int_0^T \int_0^1 \left( e^{-2\varphi} \theta^2 \frac{y^2(s, x)}{x^2} + e^{-2\varphi} \theta^2 \frac{y^2(s, x)}{x^2} \right) dx dt$$

$$\leq C \theta_0 e^{CT} E \int_0^T \int_0^1 \left( e^{-2\varphi} \theta^2 y^2(s, x) + e^{-2\varphi} \theta^2 y^2(s, x) \right) dx dt$$

$$\leq CE \int_0^T \int_0^1 \mu^2 \theta^2(t) y^2(t, x) e^{-2\varphi} dx dt.$$
for a suitable constant $\tilde{C}$. \hfill \Box

Therefore, the unique continuation of the forward stochastic degenerate equation (FSDE) is an immediate consequence of Proposition 4.1.

**Corollary 4.2.** Let $y$ be a solution of (FSDE) satisfying $y(t, x) = 0$ \(\mathbb{P}\text{-a.s.}\), for all $t$ in a right neighborhood of 0 and almost every $x \in \omega$. Then, $y(t, x) = 0$ \(\mathbb{P}\text{-a.s.}\), for all $t \in [0, T]$ and almost every $x \in (0, 1)$.

In order to give the proof of the Theorem 1.1, we will need the following well-known functional analysis lemma (See \cite[Theorem 2.2, p. 208]{19}).

**Lemma 4.3.** Let $X, Y, Z$ be three Hilbert spaces, $X^*, Y^*, Z^*$ their dual spaces and $F \in \mathcal{L}(X, Z)$, $G \in \mathcal{L}(Y, Z)$. Assume that $Y$ is separable. Then $\text{Range}(F) \subseteq \text{Range}(G)$ if and only if there exists a constant $C > 0$ such that

$$\|F^* z\|_{X^*} \leq C\|G^* z\|_{Y^*}, \ z \in Z^*,$$

where $F^*$ and $G^*$ are the adjoint operators.

**Proof of the Theorem 1.1.** Let us consider the following operators

$$S_0 : \begin{array}{c} \mathcal{L}^2(\Omega, \mathcal{F}_T, L^2(0, 1)) \rightarrow L^2(0, 1) \\ \eta \rightarrow \int_0^1 \eta(t) y(T, x) dt \end{array}$$

where $z^{0, \eta}$ is the solution of (BSDE) with final datum $\eta$ and control $v = 0$, and

$$L_0 : \begin{array}{c} \mathcal{L}^2(\Omega, L^2(0, T; L^2(0, 1))) \rightarrow L^2(0, 1) \\ v \rightarrow \int_0^1 \int_0^T \eta(t) y(s, x) ds dt \end{array}$$

where $z^{v, 0}$ is the solution of (BSDE) with control $v$ and final datum $\eta = 0$. Hence, Theorem 1.1 holds if $\text{Range}(S_0) \subseteq \text{Range}(L_0)$. Let $z$ be a solution of (BSDE) and $y$ a solution of (FSDE), differentiating by Itô rule $d_s(z(s), y(s))_{L^2(0, 1)}$, integrating in $[0, T]$ and computing the mean, value we obtain

$$\mathbb{E} \int_0^1 \eta(t) y(T, x) dt = \mathbb{E} \int_0^1 \int_0^T \eta(t) y(s, x) ds dt.$$

Consequently,

$$S_0^* y_0(x) = y(T, x) \quad \text{and} \quad L_0^* y_0(t, x) = -1_\omega y(t, x).$$

Proposition 4.1 implies that there exists a constant $C > 0$ such that

$$\|S_0^* y_0\|_{\mathcal{L}^2(\Omega, \mathcal{F}_T, L^2(0, 1))} \leq C\|L_0^* y_0\|_{\mathcal{L}^2(\Omega, L^2(0, T; L^2(0, 1)))}.$$

By Lemma 4.3, we get $\text{Range}(S_0) \subseteq \text{Range}(L_0)$ which completes the proof. \hfill \Box

5. A controllability result for a linear forward parabolic degenerate stochastic equation with two controls

5.1. Carleman estimate for backward stochastic equation. In this subsection, we establish Carleman estimate for backward stochastic equation using weighted function which does not vanish at $t = 0$. Let consider $\bar{\varphi}(t, x) = \bar{\theta}(t) \beta(x)$, where

$$\bar{\theta}(t) = \begin{cases} \left(\frac{t}{T}\right)^8 & \text{if } t \in [0, \frac{T}{2}] \\ \frac{1}{t(2-t)} & \text{if } t \in \left[\frac{T}{2}, T\right] \end{cases}$$

Let us consider the following uncontrolled backward stochastic problem

$$\begin{cases} dy = \left[-(a(x) \varphi_x)_x + F\right] dt + \bar{y} dW(t), & \text{in } Q_T \\ Cy = 0, & \text{on } \Sigma_T \\ y(T, \cdot) = y_T(\cdot) & \text{in } (0, 1). \end{cases} \quad (5.1)$$

Consider $z = e^{-s\Delta} y$ likewise in Lemma 3.2 replacing $a$ by $-a$, we get

$$e^{-s\Delta} [A z + (az_x)_x] [dy + (ay_x)_x dt] = [Az + (az_x)_x]^2 dt + d \left[\frac{1}{2} Az^2 - \frac{1}{2} az_x^2\right] - \frac{1}{2} A dz_x^2 + \frac{1}{2} a(dz_x)^2 + (az_x dz)_x$$

$$+ \left[sa(a\bar{\varphi}_x)_{xx} z + saa\bar{\varphi}_x z^2 - \frac{1}{2} sa(a\bar{\varphi}_x)_{xx} z^2 + sa^2 \varphi_{xx} z_x^2\right] dt.$$
we get

where \( A = s \tilde{\varphi}_t + s^2 a \tilde{\varphi}_x \). From the definition of \( \tilde{\theta} \) we have \( e^{-s\tilde{\varphi}}(T, x) = 0 \) and \( e^{-s\tilde{\varphi}}(0, x) \neq 0 \).

We have the following estimate related to the solution \((y, \tilde{y})\) of the backward stochastic equation (5.1).

**Proposition 5.1.** There exist two positive constants \( C \) and \( s_0 \) such that, for any \( y_t \in L^2(\Omega, F_T, P; L^2(0, 1)) \) and any \( F \in L^2(0, T; L^2(0, 1)) \). The solution \((y, \tilde{y})\) to (5.1) satisfies

\[
E \int_0^1 A(0) e^{-2sx}(0)^2 \frac{dy}{dx} + E \int_0^1 s^2 \theta^2 e^{-2sx} \tilde{y}^2 dx dt + E \int_0^1 s^3 \theta^3 \tilde{y}^2 dx dt \\
\leq C \int_0^1 F^2 e^{-2sx} dx dt + C \int_0^1 s^2 \theta^2 e^{-2sx} \tilde{y}^2 dx dt + C \int_0^1 s^3 \theta^3 e^{-2sx} \tilde{y}^2 dx dt.
\]

for all \( s > s_0 \).

**Proof.** Integrating (5.2) and taking the expectation we get

\[
E \int_0^1 A(0) e^{-s\tilde{\varphi}}(0)^2 \frac{dy}{dx} + E \int_0^1 s^2 \theta^2 e^{-2s\tilde{\varphi}} \tilde{y}^2 dx dt + E \int_0^1 s^3 \theta^3 e^{-2s\tilde{\varphi}} \tilde{y}^2 dx dt \\
\leq C \int_0^1 F^2 e^{-2s\tilde{\varphi}} \tilde{y}^2 dx dt + C \int_0^1 s^2 \theta^2 e^{-2s\tilde{\varphi}} \tilde{y}^2 dx dt + C \int_0^1 s^3 \theta^3 e^{-2s\tilde{\varphi}} \tilde{y}^2 dx dt.
\]

It’s not difficult to check that the terms \( Q_1, Q_4, Q_5 \) and \( Q_6 \) are non negative. So it remains to estimate the other terms. Let \( \delta > 0 \) a chosen small scalar, for \( s \) large enough we have

\[
Q_2 \geq -\delta \left( \int_0^1 s^3 \theta^3 \frac{x^2}{a} \tilde{y}^2 dx dt + \int_0^1 s \tilde{\theta} \tilde{y}_x^2 dx dt \right).
\]

Arguing as in Lemma 3.3 we get

\[
Q_3 \geq -C \int_0^1 s^2 \theta^2 e^{-2s\tilde{\varphi}} \tilde{y}^2 dx dt.
\]

By similar arguments as in Lemma 3.7 we obtain

\[
Q_7 \geq C' \int_0^1 s^3 \theta^3 \frac{x^2}{a} \tilde{y}^2 dx dt - C'' \int_0^1 s^2 \theta^2 \frac{x^2}{a} \tilde{y}^2 dx dt - \delta E \int_0^1 s \tilde{\theta} \tilde{y}_x^2 dx dt.
\]

where \( C' \) and \( C'' \) are two positive constants. As \( A \) is a deterministic function, then

\[
\int_0^1 A_t \tilde{y}_x^2 dt = \left[ A \tilde{y}_x \right]_0^1 - \int_0^1 A \tilde{y}_x^2 dt = -A(0) \tilde{y}(0) - \int_0^1 A \tilde{y}_x^2 dt.
\]

Therefore

\[
Q_8 = \frac{1}{2} \int_0^1 A(0) \tilde{y}(0) dx + \frac{1}{2} \int_0^1 A \tilde{y}_x^2 dx.
\]

(5.8)
And as in Lemma 3.8, there exist two positive constants $C$ and $C_1$ such that
\[ Q_0 \geq CE \int_Q s\theta z^2 dxdt - C_1E \int_0^T \int_{[a_1,b_1]} s\theta z^2 dxdt. \] (5.9)

From (5.4)–(5.9) we infer
\[
\frac{1}{2}E \int_0^1 A(0)z^2(0)dx + E \int_Q s\theta z^2 dxdt + E \int_Q s^3\theta^3 \frac{z^2}{a} dxdt
\leq E \int_Q e^{-s\varphi} [A z + (az)_x] dy + (ay)_x dt dx + CE \int_Q s^3\theta^3 e^{-2s\varphi} \hat{y}^2 dxdt
\]
\[ + CE \int_0^T \int_{[a_1,b_1]} s^3\theta^3 \frac{z^2}{a} dxdt + CE \int_0^T \int_{[a_1,b_1]} s\theta z^2 dxdt. \]

The rest of the proof is identical to that of Proposition 3.9. \(\Box\)

5.2. Controllability for a linear forward parabolic degenerate stochastic equation with two controls. In this subsection we give the proof of the second main Theorem 1.2 which relies on a classical duality method called Hilbert Uniqueness Method (HUM).

Proof of Theorem 1.2. For $\varepsilon > 0$, let us consider
\[ \tilde{\theta}_\varepsilon(t) = \left\{ \begin{array}{ll} \frac{1}{2(t^2 + t)^{\frac{1}{2}}} & \text{if } t \in [0, \frac{T}{2}] \\ \frac{1}{\varepsilon t^2 (T-t^2)^{\frac{1}{2}}} & \text{if } t \in [\frac{T}{2}, T] \end{array} \right. \]
and $\tilde{\varphi}_\varepsilon = \tilde{\theta}_\varepsilon \beta$. We introduce the functional
\[ J_\varepsilon(h, H) = \frac{1}{2}E \int_Q e^{-2s\tilde{\varphi}_\varepsilon} \hat{y}^2 dxdt + \frac{1}{2}E \int_{-T}^T e^{-2s\tilde{\varphi}_\varepsilon} s^{-3} \tilde{\theta}_\varepsilon h^2 dxdt
\]
\[ + \frac{1}{2}E \int_Q e^{-2s\tilde{\varphi}_\varepsilon} s^{-2} \tilde{\theta}_\varepsilon^3 H^2 dxdt + \frac{1}{2}\varepsilon E \int_{(0,1)} |y(T)|^2 dx \]
let us consider the following optimal control
\[
\begin{aligned}
\{ & \min_{(h, H) \in \mathcal{H}} J_\varepsilon(h, H) \\
& \text{subject to equation (FSDE2)} \}
\end{aligned} \tag{5.10}
\]
where
\[ \mathcal{H} = \{ (h, H) \in L^2(0, T; L^2(0, 1))^2 : \left( E \int_{-T}^T e^{-2s\tilde{\varphi}_\varepsilon} s^{-3} \tilde{\theta}_\varepsilon |h|^2 dxdt \right) \]
\[ + \left( E \int_Q e^{-2s\tilde{\varphi}_\varepsilon} s^{-2} \tilde{\theta}_\varepsilon^3 |H|^2 dxdt \right) < \infty \}. \tag{5.11} \]
The functional $J_\varepsilon$ is continuous, strictly convex and coercive. So, the problem (5.10) admits a unique optimal solution $(h_\varepsilon, H_\varepsilon)$, which can be characterized (see [13]) as
\[
\begin{aligned}
h_\varepsilon &= -\frac{1}{2}\varepsilon e^{-2s\tilde{\varphi}_\varepsilon} s^3 \tilde{\theta}_\varepsilon z_\varepsilon \text{ in } Q_T \ a.s \\
H_\varepsilon &= -e^{-2s\tilde{\varphi}_\varepsilon} s^2 \tilde{\theta}_\varepsilon Z_\varepsilon \text{ in } Q_T \end{aligned} \tag{5.12}
\]
where the pair $(z_\varepsilon, Z_\varepsilon)$ satisfies the following backward stochastic equation
\[
\begin{aligned}
dz_\varepsilon &= \left[ -\left( a(x)z_\varepsilon \right)_x - e^{2s\varphi} y_\varepsilon \right] dt + Z_\varepsilon dW(t), \text{ in } Q_T \\
Cz_\varepsilon &= 0, \text{ on } \Sigma_T \\
z_\varepsilon(T, \cdot) &= \frac{1}{s} y_\varepsilon(T, \cdot) \text{ in } (0, 1) \tag{5.13}
\end{aligned}
\]
and where $y_\varepsilon$ is the solution to the problem (FSDE2) with controls $h = h_\varepsilon$ and $H = H_\varepsilon$.

Let us remark as $y_\varepsilon \in L^2_2(\Omega, C([0, T]; L^2(0, 1)))$ the problem is well-posed for any $\varepsilon > 0$.

Now, differentiating $d(y_\varepsilon, z_\varepsilon)_{L^2(0,1)}$; integrating on $[0, T]$ and taking the expectation we get
\[ E \int_{(0,1)} y_\varepsilon(T) z_\varepsilon(T) dx = E \int_{(0,1)} y_\varepsilon(0) z_\varepsilon(0) dx + E \int_{Q_T} (a(x)y_\varepsilon)_x + F + \frac{1}{2} h_\varepsilon)(z_\varepsilon) dxdt \]
From (FSDE2), (5.12) and (5.13) we infer
\[
E \int_0^T \int_\Omega e^{-2\phi \cdot \phi} s^3 \partial^3 z^2_{t,x} \, dx \, dt + E \int_0^T e^{-2\phi \cdot \phi} s^2 \partial^1 Z^2_{t,x} \, dx \, dt + E \int_0^T e^{2\phi \cdot \phi} y^2_{t,x} \, dx \, dt \\
+ \frac{1}{\varepsilon} E \int_0^1 \left| y_{e}(T) \right|^2 \, dx = E \int_0^1 y_{e}(0) \, dx + E \int_0^T F_{z_t} \, dt + E \int_0^T G Z_{t,x} \, dx dt
\]  
(5.15)

Now, we apply the Proposition 5.1 to the problem (5.13) with \( F = -e^{2\phi \cdot \phi} y_{e} \) and \( H = Z_{e} \), we remove some unnecessary terms and we add an integral of \( Z_{e} \) on the left hand side we get
\[
E \int_0^T \int_\Omega e^{-2\phi \cdot \phi} z^2_{t,x} \, dx \, dt + E \int_0^T \int_\Omega s^3 \partial^3 z^2_{t,x} e^{-2\phi \cdot \phi} \, dx \, dt + E \int_0^T s^2 \partial^1 Z^2_{t,x} e^{-2\phi \cdot \phi} \, dx \, dt \\
\leq C E \int_0^T y_{e} e^{-2\phi \cdot \phi} \, dx \, dt + C E \int_0^T s^2 \partial^3 z^2_{t,x} \, dx \, dt + C E \int_0^T s^2 \partial^1 Z^2_{t,x} \, dx \, dt + \frac{1}{\varepsilon} E \int_0^1 \left| y_{e}(T) \right|^2 \, dx
\]  
(5.16)

Using Cauchy-Schwarz and Young inequalities in the right hand side of (5.15)
\[
E \int_0^T \int_\Omega e^{-2\phi \cdot \phi} s^3 \partial^3 z^2_{t,x} \, dx \, dt + E \int_0^T e^{-2\phi \cdot \phi} s^2 \partial^1 Z^2_{t,x} \, dx \, dt + E \int_0^T e^{2\phi \cdot \phi} y^2_{t,x} \, dx \, dt + \frac{1}{\varepsilon} E \int_0^1 \left| y_{e}(T) \right|^2 \, dx \\
\leq \delta \left( E \int_0^1 A(0) e^{-2\phi \cdot \phi} z^2_{t,x} \, dx + E \int_0^T s^2 \partial^3 z^2_{t,x} \, dx \, dt + E \int_0^T S^2 \partial^3 Z^2_{t,x} \, dx \, dt \right) \\
+ C \delta \left( E \int_0^T \frac{e^{2\phi \cdot \phi}}{A(0)} y^2_{t,x} \, dx \, dt + E \int_0^T s^2 \partial^3 \frac{a}{x^2} e^{2\phi \cdot \phi} F^2 \, dx \, dt + E \int_0^T s^{-2} \partial^3 e^{2\phi \cdot \phi} G^2 \, dx \, dt \right)
\]  
(5.17)

for any \( \delta > 0 \). Using inequality (5.16) to estimate the right-hand side of (5.17) and the fact that \( e^{-2\phi \cdot \phi} \leq 1 \) for all \((t,x) \in Q_T \), we obtain, after taking \( \delta > 0 \) small enough, that
\[
E \int_0^T \int_\Omega e^{-2\phi \cdot \phi} s^3 \partial^3 z^2_{t,x} \, dx \, dt + E \int_0^T e^{-2\phi \cdot \phi} s^2 \partial^1 Z^2_{t,x} \, dx \, dt + E \int_0^T e^{2\phi \cdot \phi} y^2_{t,x} \, dx \, dt + \frac{1}{\varepsilon} E \int_0^1 \left| y_{e}(T) \right|^2 \, dx \\
\leq C \left( E \int_0^1 \frac{e^{2\phi \cdot \phi}}{A(0)} y^2_{t,x} \, dx \, dt + E \int_0^T s^{-2} \partial^3 \frac{a}{x^2} e^{2\phi \cdot \phi} F^2 \, dx \, dt + E \int_0^T s^{-2} \partial^3 \frac{a}{x^2} e^{2\phi \cdot \phi} G^2 \, dx \, dt \right)
\]  
(5.18)

Taking into account the characterization (5.12) we deduce
\[
E \int_0^T \int_\Omega e^{2\phi \cdot \phi} s^{-3} \partial^3 h^2_{t,x} \, dx \, dt + E \int_0^T e^{2\phi \cdot \phi} s^{-2} \partial^3 H^2_{t,x} \, dx \, dt + E \int_0^T e^{2\phi \cdot \phi} y^2_{t,x} \, dx \, dt + \frac{1}{\varepsilon} E \int_0^1 \left| y_{e}(T) \right|^2 \, dx \\
\leq C \left( E \int_0^1 \frac{e^{2\phi \cdot \phi}}{A(0)} y^2_{t,x} \, dx \, dt + E \int_0^T s^{-3} \partial^3 \frac{a}{x^2} e^{2\phi \cdot \phi} F^2 \, dx \, dt + E \int_0^T s^{-2} \partial^3 \frac{a}{x^2} e^{2\phi \cdot \phi} G^2 \, dx \, dt \right)
\]  
(5.19)

Knowing that the right hand side of (5.19) is uniform with respect to \( \varepsilon \) we can easily deduce that there exists \((\hat{h}, \hat{H}, \hat{y})\) such that
\[
\begin{cases} 
  h_e \rightharpoonup \hat{h} & \text{weakly in } L^2(\Omega \times (0, T); L^2(\omega_T)) \\
  H_e \rightharpoonup \hat{H} & \text{weakly in } L^2(\Omega \times (0, T); L^2(Q_T)) \\
  y_e \rightharpoonup \hat{y} & \text{weakly in } L^2(\Omega \times (0, T); L^2(\omega_T))
\end{cases}
\]  
(5.20)

Now, we prove that \( \hat{y} \) is the solution to (FSDE2) associated to \((\hat{h}, \hat{H})\). Indeed, let \( \bar{y} \) the unique solution in \( L^2(\Omega; C([0,T]; L^2(0,1))) \) to (FSDE2) with controls \((\hat{h}, \hat{H})\). For any \( m \in L^2(0, T; L^2(0, 1)) \) consider the unique solution to the backward problem
\[
\begin{cases} 
  dz = \left[ - (a(x) z_e) - m \right] dt + ZdW(t), & \text{in } Q_T \\
  Cz = 0, & \text{on } \Sigma_T \\
  z(T, \cdot) = 0 & \text{in } (0,1)
\end{cases}
\]  
(5.21)
Thanks to Itô’s formula, from the duality between (5.21) and (FSDE2) associated to \((h_{\varepsilon}, H_{\varepsilon})\) and \((\hat{h}, \hat{H})\) respectively we have

\[
-\mathbb{E} \int_{(0,1)} y_0(0)z(0)dx = -\mathbb{E} \int_{Q_T} y_0 mdxdt + \mathbb{E} \int_{Q_T} Fzdxdt
+ \mathbb{E} \int_{Q_T} h_{\varepsilon} zdx + \mathbb{E} \int_{Q_T} H_{\varepsilon} zdxdt \tag{5.22}
\]

and

\[
-\mathbb{E} \int_{(0,1)} y_0(0)z(0)dx = -\mathbb{E} \int_{Q_T} \tilde{y} mdxdt + \mathbb{E} \int_{Q_T} Fzdxdt
+ \mathbb{E} \int_{Q_T} \tilde{h} zdx + \mathbb{E} \int_{Q_T} \tilde{H} zdxdt \tag{5.23}
\]

Therefore, using (5.20) in (5.22) to pass to the limit \(\varepsilon \to 0\) and subtracting the result from (5.23), we get \(\tilde{y} = \tilde{y} \) in \(Q_T\), a.s.

Moreover, from (5.19) we infer \(\tilde{y}(T) = 0\) in \(Q_T\), a.s. Taking into account (5.20), from Fatou’s lemma and the uniform estimate (5.19) we deduce (1.3). This completes the proof.

6. Appendix

In this section, we give a new proof of the Caccioppoli’s inequality in the stochastic case. Let \(\omega'' := (x_1'', x_2'') \in \omega'\) an open nonempty subset of \(\omega\). We consider the following equation

\[
d_t y(t, x) = (a(x)y_x)_x dt + by(t, x)dt + f(t, x) \quad \text{in } [0, T] \times \omega',
\]

where \(b \in L^\infty([0, T] \times \omega'), a \in \mathcal{C}^1(\omega'), \min a(x) > 0\) and \(f \in L^2([0, T] \times \omega').

Lemma 6.1 (Caccioppoli’s inequality).

There exists a constant positive \(C\) such that every solution \(v\) of (6.1) satisfies

\[
\mathbb{E} \int_0^T \int_{\omega'} v_x^2 e^{2\mu v} \leq C \left( \mathbb{E} \int_0^T \int_{\omega'} \mu^2 v^2 e^{2\mu v} + \mathbb{E} \int_0^T \int_{\omega'} f^2 e^{2\mu v} \right). \tag{6.2}
\]

Proof. Likewise in the deterministic case and without lose of generality, let \(\eta : \mathbb{R} \to \mathbb{R}\) be a smooth function satisfying \(\frac{\eta^2}{\eta} \in L^\infty(\mathbb{R})\) such that \(0 \leq \eta \leq 1\), \(\eta \equiv 1\) on \((x_1', x_2')\) and \(\eta \equiv 0\) on \([x_1, x_1'] \cup (x_2', x_2'')\).

Let \(v\) be a solution of equation (6.1). Again differentiating \(\eta \int_0^T \eta v^2 e^{2\mu v(t, x)}dx\), integrating in \([0, T]\) and computing the mean value, we obtain

\[
0 = -2\mathbb{E} \int_0^T \int_{\omega'} \eta a v_x v e^{2\mu v} - 2\mathbb{E} \int_0^T \int_{\omega'} \eta a v_x e^{2\mu v} - 4\mathbb{E} \int_0^T \int_{\omega'} \eta \mu v_x v e^{2\mu v} + 2\mathbb{E} \int_0^T \int_{\omega'} \eta v e^{2\mu v} d + 2\mathbb{E} \int_0^T \int_{\omega'} \mu \eta v_x v e^{2\mu v},
\]

Taking into account the Young inequality, we have for every \(\varepsilon > 0\)

\[
\mathbb{E} \int_0^T \int_{\omega'} \eta v x e^{2\mu v} \leq \frac{\varepsilon}{2} \mathbb{E} \int_0^T \int_{\omega'} \eta a v_x e^{2\mu v} + \frac{1}{2\varepsilon} \mathbb{E} \int_0^T \int_{\omega'} \eta^2 a v_x e^{2\mu v},
\]

and

\[
\mathbb{E} \int_0^T \int_{\omega'} \eta \mu v_x v e^{2\mu v} \leq \frac{\varepsilon}{4} \mathbb{E} \int_0^T \int_{\omega'} \eta a v_x e^{2\mu v} + \frac{1}{4\varepsilon} \mathbb{E} \int_0^T \int_{\omega'} \eta^2 v_x e^{2\mu v},
\]

and

\[
\mathbb{E} \int_0^T \int_{\omega'} \eta v f e^{2\mu v} \leq \frac{1}{2} \mathbb{E} \int_0^T \int_{\omega'} \eta v^2 e^{2\mu v} + \frac{1}{2} \mathbb{E} \int_0^T \int_{\omega'} \eta^2 v e^{2\mu v}.
\]

Choosing \(\varepsilon\) small enough, we obtain that

\[
\mathbb{E} \int_0^T \int_{\omega'} \eta a v_x e^{2\mu v} \leq C \left( \mathbb{E} \int_0^T \int_{\omega'} k_\eta \mu^2 v_x^2 e^{2\mu v} + \mathbb{E} \int_0^T \int_{\omega'} f^2 e^{2\mu v} \right),
\]

where \(k_\eta\) is a bounded function with support in \(\omega'\) and \(\mu\) large enough. Since the diffusion coefficient \(a\) does not vanish in \(\omega'\), we infer the desired inequality (6.2).
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