Infrared Behaviour and Running Couplings in Interpolating Gauges in QCD

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(Dated: April 26, 2005)

We consider the class of gauges that interpolates between Landau- and Coulomb-gauge QCD, and show the non-renormalisation of the two independent ghost-gluon vertices. This implies the existence of two RG-invariant running couplings, one of which is interpreted as an RG-invariant gauge parameter. We also present the asymptotic infrared limit of solutions of the Dyson-Schwinger equations in interpolating gauges. The infrared critical exponents of these solutions as well as the resulting infrared fixed point of one of the couplings are independent of the gauge parameter. This coupling also has a fixed point in the Coulomb gauge limit and constitutes a second invariant charge besides the well known colour-Coulomb potential.

PACS numbers: 12.38.Aw 14.70.Dj 12.38.Lg 11.15.Tk 02.30.Rz

I. INTRODUCTION

The precise mechanism responsible for the confinement of the coloured states of QCD is still elusive. Lattice calculations of the gauge-invariant Wilson loop have provided clear evidence for a linear rising potential between static colour charges. The underlying long range interaction, however, is provided by gauge dependent objects. It thus seems possible that the confinement mechanism itself looks different, depending on the gauge one is studying. Indeed, recent investigations of the infrared behaviour of QCD in Coulomb and in Landau gauge support this picture.

In Coulomb gauge the confinement of two static external colour charges is related to the colour-Coulomb potential \( V_{\text{coul}}(R) \) [3]. This quantity is a renormalisation-group invariant and couples universally to colour charge. It has been demonstrated that the potential is long ranged and provides an upper bound for the Wilson potential \( V(R) \) [3]. Thus a necessary condition for the Wilson potential to be confining is that \( V_{\text{coul}}(R) \) be confining.

In terms of Green’s functions, the colour-Coulomb potential is given by the instantaneous part of the 00-component of the dressed gluon propagator in minimal Coulomb gauge,

\[
g^2 D_{00}(x, x_0) = \langle g A_0^a(x, x_0) g A_0^b(0, 0) \rangle = V_{\text{coul}}(|x|) \delta(x_0) + \text{(non-inst.)}. \tag{1}
\]

It is a renormalisation group invariant and can be calculated explicitly via [4]

\[ V_{\text{coul}}(x - y) \delta^{ab} = g^2 \langle [M^{-1}(A)(-\partial^2)M^{-1}(A)]_{zy}^{ab} \rangle, \tag{2} \]

where \( M(A) = -\partial_i D_i(A) \) is the Faddeev-Popov operator, and the gauge-covariant derivative is defined by \( [D_i(A) \omega]^a = \partial_i \omega^a - g f^{abc} \omega^b A_i^c \). The expectation value of the inverse Faddeev-Popov operator is the ghost propagator, \( i \Delta(x - y) \delta^{ab} = \langle M(A)^{-1} \rangle_{zy}^{ab} \), and the long range properties of the Coulomb-potential are related to the infrared behaviour of \( \Delta(x - y) \). Moreover \( V_{\text{coul}}(R) \) has been calculated both analytically [5] and numerically [6], and found to be (almost) linearly rising at large \( R \). A running coupling \( \alpha_{\text{coul}}(k) \) may be introduced by [4]

\[
\tilde{V}_{\text{coul}}(k) = \frac{4 \pi \alpha_{\text{coul}}(k)}{k^2} \frac{12 N_c}{11 N_c - 2 N_f}, \tag{3}
\]

which for a linearly rising \( V_{\text{coul}}(R) \) is then proportional to \( 1/k^2 \); a clear instance of the notion of infrared slavery.

The situation in Landau gauge, on the other hand, appears to be much more intricate. Compared to Coulomb-gauge there is no quantity that bears an obvious correspondence to the colour-Coulomb potential. A renormalisation-group invariant running coupling, however, can be defined from either of the primitively divergent vertices [7]. A particularly simple example of such a coupling is given by the one obtained from the ghost-gluon-vertex,

\[
\alpha(k^2) = \frac{q^2}{4 \pi} k^2 \Delta^2(k^2) D(k^2), \tag{4}
\]

which involves the ghost propagator \( \Delta \) and the scalar part \( D \) of the gluon propagator, but not the vertex itself [8, 9, 10]. This fact can easily be traced back to the transversal structure of the gluon propagator in Landau gauge and will be discussed in more detail below. The infrared behaviour of the running coupling [3] has been determined analytically, and an infrared fixed point at \( \alpha(0) \approx 8.915/N_c \) has been found [11]. On a qualitative level this fixed point behaviour has been demonstrated also for the couplings defined from the three- and four-gluon vertices [3]. Numerical solutions of the Dyson-Schwinger (DS) equations for the ghost and gluon propagators furthermore show that the fixed point is continuously connected to the well known perturbative coupling in the ultraviolet momentum regime [12].

The aim of this paper is to explore possible connections between the Coulomb and Landau gauge results described above. To this end we consider a class of gauges that interpolates between Landau and Coulomb gauge. In section II we will demonstrate the existence of
two renormalisation-group invariant running couplings in these gauges. One of these couplings will later be shown to be an RG-invariant gauge parameter. The other one, \( \alpha_I(k) \), is an analogue of the Landau gauge expression. We derive the Dyson-Schwinger equations for the ghost and gluon propagators in the interpolating gauges in section III, and provide their asymptotic infrared solutions in section IV. We find that there is an instability in the solution such that, for each value of the interpolating gauge parameter \( \eta \) that appears in the local action, there is a class of solutions parametrised by a single parameter \( \eta \). It turns out that \( \alpha_I(k) \) has a gauge-independent infrared fixed point \( \alpha_I(0) \) at a value identical to the one in Landau gauge. As shown in section V, such a fixed point can also be found in Coulomb gauge, although we are not able to determine the exact value in this limit. In section VI we show that the degeneracy parameter \( \eta \) that characterizes the solution of the DSE in a given gauge may be identified with an RG-invariant gauge parameter. We summarise and discuss our results further in section VII.

II. NON-RENNORMALISATION OF GHOST-GLUON VERTICES AND RUNNING COUPLING

The interpolating gauges we shall consider are specified by the (unrenormalised) gauge condition

\[
\partial'_\mu A_\mu = a \partial_0 A_0 + \nabla \cdot A = 0, \tag{5}
\]

with \( \partial'_\mu = (a \partial_0, \nabla) \), and bare gauge parameter \( 0 < a \leq 1 \). The values \( a = 0, 1 \) correspond to Coulomb- and Landau-gauge respectively. By means of an extended BRST-formalism, including gauge and Lorentz transformations, this class of gauges has been shown to be renormalisable, and physical observables in these gauges have the same expectation values as in covariant gauges. This class of gauges has also been studied numerically. The partition function

\[
S_{FP}(A, c, \bar{c}) = S_{YM} + \int d^4x (2\xi)^{-1}[\partial'_\mu A_\mu]^2 + \partial'_c \cdot D(A)c, \tag{6}
\]

where \( D(A) \) is the gauge covariant derivative. The class of interpolating gauges that we shall be concerned with is obtained in the limit \( \xi \to 0 \). The gluon propagator then satisfies the transversality condition

\[
k'_\lambda D_{\lambda\mu} = ak_0 D_{0\mu}(k) + k_i D_{i\mu}(k) = 0, \tag{7}
\]

which determines the 3-dimensional longitudinal components of \( D_{\lambda\mu} \) in terms of \( D_{0\mu} \).

\[
D_{0\mu}(k) = -\frac{ak_0k_\mu}{k^2} D_{00}(k), \tag{8}
\]

\[
D_{i\mu}(k) = \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) D^{ii}(k) + \frac{a^2 k_i^2}{k^2} k_j D_{00}(k). \tag{9}
\]

Thus in the interpolating gauge there are two scalar functions \( D^{ii}(k) \) and \( D_{00}(k) \) that characterise the gluon propagator. They and the ghost propagator \( \Delta(k) \) are functions of the two scalar variables \( |k| \) and \( k_0, D^{ii}(k) = D^{ii}(|k|, k_0) \), etc.

A useful property of the dressed ghost-gluon vertex \( \Gamma^{abc} = \Gamma_\mu(p, q)f^{abc}, \) in the (transverse) interpolating gauges considered here is the factorisation of both the incoming and the outgoing ghost momenta.

\[
\Gamma_\mu(p, q) = p'_\mu + p_A F_{\lambda\mu\nu}(p, q)q'_\nu, \tag{10}
\]

where \( p'_\mu = (ap_0, p_i) \) and the function \( F_{\lambda\mu\nu}(p, q) \) is a Feynman integral in every order of perturbation theory. This factorisation of momenta is familiar in Landau gauge, \( a = 1 \), and holds in the interpolating gauges considered here for the same reason. Indeed, factorisation of the outgoing ghost momentum \( p' \) is immediate, and occurs for any value of the gauge parameter \( \xi \). Factorisation of the incoming ghost momentum \( q' \) can be seen easily from its Dyson-Schwinger equation, Fig. 1 and the transversality of the gluon propagator \( D_{\mu\nu} \), which holds only in the limit \( \xi \to 0, l'_{\nu} D_{\mu\nu}(l-q') = q'_\nu D_{\mu\nu}(l-q) \). Factorisation of both ghost momenta depresses the ultraviolet divergence of \( \Gamma_\mu(q, p) \) by a power, so \( \tilde{Z}_1 \) is finite, and may be normalised to \( \tilde{Z}_1 = 1 \). This implies the non-renormalisation of the two ghost-gluon vertices \( g f^{a\mu\nu} \partial_\mu A^a_\nu \) and \( g a f^{a\mu\nu} \partial_\mu A^a_\nu \). (Here \( \tilde{Z}_1 \) represents the renormalisation of the pair of these vertices; in a more complete notation, each vertex has its own renormalisation constant.)

An important consequence is the possibility of deriving expressions for two invariant running couplings that are renormalisation-group invariant, one for each of the propagator functions \( D^{ii} \) and \( D_{00} \). It has been shown in Ref. [14] that the interpolating gauges are multiplica-
tively renormalisable,

\[ A = A_R Z_3^{1/2}; \quad A_0 = A_{0,R} Z_0^{1/2}; \quad (c, \bar{c}) = (c_R, \bar{c}_R) \bar{Z}_3^{1/2} \]

\[ g = g_R Z_g; \quad a = Z_0^{-1/2} Z_3^{1/2} a_R, \]  \hspace{1cm} (11)

with \( Z_3 \neq Z_0 \) for \( a \neq 1 \). The renormalisation of the gauge parameter \( a \) preserves the transversality relation \( a_R \partial_0 A_{R,0} + \partial_i A_{R,i} = 0 \). Gauge invariance furthermore enforces the Slavnov-Taylor identity

\[ 1 = \bar{Z}_1 = Z_g \bar{Z}_3 Z_3^{1/2}. \]  \hspace{1cm} (12)

(In the notation of \( \bar{Z}_3 = Z_g Z_c \), and \( \bar{c} \) is chosen to renormalise contragrediently to \( A_1 \), like the source \( K_i \), of \( D_{i,c} \), and the last equation reads simply \( Z_0 Z_c = 1 \).) The dependence of the renormalised gauge parameter \( a_R(\mu) \) on the renormalisation mass \( \mu \) satisfies the RG flow equation,

\[ a_R(\mu) \frac{\partial a_R}{\partial \mu} = \gamma a_R, \]  \hspace{1cm} (13)

where \( \gamma = \frac{1}{3} \mu \frac{\partial \ln(Z_0/Z_3)}{\partial \mu} \) is a power series in \( g_R \) with finite coefficients.

The spatial gluon propagator renormalises according to

\[ D_{ij}(k, a, \Lambda) = D_{R,ij}(k, a_R, \mu) Z_3 \]  \hspace{1cm} (14)

which fixes the renormalisation of the two gluon scalar propagators

\[ D_{ij}(k, a, \Lambda) = D_{R,ij}(k, a_R, \mu) Z_3, \]

\[ D_{00}(k, a, \Lambda) = D_{R,00}(k, a_R, \mu) Z_0, \]

\[ \Delta(k, a, \Lambda) = \Delta_R(k, a_R, \mu) \bar{Z}_3, \]

\[ \Gamma_i(p, q, a) = \Gamma_{R,i}(p, q, a_R), \]

\[ \Gamma_0(p, q, a) = \Gamma_{R,0}(p, q, a_R) Z_0^{-1/2} Z_3^{1/2}, \]  \hspace{1cm} (15)

where we have also written the renormalisation of the ghost propagator and the two ghost-gluon vertices. Here \( \mu \) denotes the renormalisation mass, and \( \Lambda \) is a cutoff scale.

The Slavnov-Taylor identity (12) implies that the multiplicative renormalisation constants cancel in the two products

\[ g^2(\Lambda) \Delta^2(k, a, \Lambda) D^{\text{inst}}(k, a, \Lambda) = g_R^2(\mu) \Delta_R^2(k, a_R, \mu) D_R^{\text{inst}}(k, a_R, \mu), \]  \hspace{1cm} (16)

\[ g^2(\Lambda) \Delta^2(k, a, \Lambda) \Delta^2(k, a, \Lambda) = g_R^2(\mu) \Delta_R^2(k, a_R, \mu) \Delta_R^2(k, a_R, \mu) D_{R,00}(k, a_R, \mu), \]  \hspace{1cm} (17)

where it is understood that \( a_R = a_R(\mu) \) runs with \( \mu \), as does the suppressed argument \( g_R = g_R(\mu) \), and correspondingly for \( a = a(\Lambda) \) and \( g = g(\Lambda) \). Since the l.h.s. is independent of the renormalisation mass \( \mu \), the r.h.s. is also. These products define two RG-invariant running couplings. However because Lorentz invariance is not explicit, these products depend on two scalar variables \(|k|\) and \( k_0 \). It is useful to define running couplings that are dimensionless and that depend on a single scalar variable. This may be done in various ways. For later convenience we define an RG-invariant running coupling \( \alpha_I(|k|) \), and an RG-invariant running gauge-type parameter \( \alpha_I(\mu) \) by

\[ \alpha_I(|k|) \equiv |k|^5 \frac{16}{3} g_R^2 \int \frac{dk_0}{2\pi} \left( \frac{\Delta^2_R}{D_R^{\text{inst}}}(|k|, k_0) \right), \]

\[ a_I^2(|k|) \alpha_I(|k|) \equiv |k|^5 \frac{32}{5} g_R^2 \bar{g}^2 \int \frac{dk_0}{2\pi} \left( \frac{\Delta^2_R}{D_R^{\text{inst}}}(|k|, k_0) \right), \]  \hspace{1cm} (18)

The second invariant vanishes in the Coulomb-gauge limit, \( a_R = 0 \). However it was shown in [2, 13] that in Coulomb gauge an RG-invariant running coupling can be defined by

\[ \alpha_{\text{coul}}(|k|) \equiv |k|^5 \frac{g_R^2}{4\pi} D_{R,00}^{\text{inst}}(|k|), \]  \hspace{1cm} (19)

where \( D_{R,00}^{\text{inst}}(|k|) \) is the instantaneous part of \( D_{R,00}(k) \). This is consistent with eqs. (17), (18) provided that

\[ \lim_{\mu \to 0} \frac{a_R \Delta_R}{a \Delta} = 1, \]  \hspace{1cm} (20)

and this is true if \( Z_3^{1/2} Z_0^{-1/2} \bar{Z}_3 = 1 \) holds in the Coulomb-gauge limit. In fact it was shown [2, 13] that in Coulomb gauge this identity does hold, in addition to the Slavnov-Taylor identities (12). We conclude that also in Coulomb gauge there are two RG-invariant couplings, \( \alpha_{\text{coul}}(|k|) \), previously known, and a second one, \( \alpha_I(|k|) \), defined above, that to our knowledge has not been considered before.

### III. DS EQUATIONS

It is of interest to solve the Dyson-Schwinger (DS) equations in the interpolating gauge, represented by Fig. 2, in order to connect the Landau and Coulomb gauges which, as discussed above, have very different behaviour. The unrenormalised DS equation for the ghost propagator reads

\[ \Delta^{-1}(p) = p'_\mu p_\mu - N g^2 \left( \frac{2\pi}{|k|} \right)^4 \int dk' p'_{\mu} D_{\mu \nu}(k) \times \Delta(p + k) \Gamma_{\nu}(p + k, p), \]  \hspace{1cm} (21)

where \( p'_\mu = (a p_0, p_i) \).

A new element arises when writing the DS equation for the gluon in interpolating gauge, as compared to Landau gauge. We treat the transversality condition on-shell, so projectors onto the transverse subspace appear on the right hand side. In interpolating gauges the transverse
subspace is orthogonal to the vector \( k'_\mu = (ak_0, k) \), so the projector onto the transverse subspace depends upon the gauge parameter \( a \) which gets renormalised. To avoid this complication, we multiply the DS equation for the gluon on the left and right by \( D_{\kappa\lambda}(p) \), and obtain

\[
D_{\mu\nu}(p) = \frac{N g^2}{(2\pi)^4} D_{\mu\lambda}(p) \int d^4 k' D_{\kappa\lambda}(p) \times \Gamma_\kappa(k, p + k) \Delta(p + k) D_{\kappa\nu}(p) + ..., \tag{22}
\]

where the ellipses represent the tree-level term and gluon loops.

If one substitutes in renormalised quantities, using \( \Delta_R^{-1}(p) = p'_\mu p_\mu \bar{Z}_3 - \frac{N g^2_R}{(2\pi)^4} \int d^4 k' p'_\mu D_{\kappa\lambda}(k) \times \Delta_R^0(k) \Gamma_R(p, k) D_{\kappa\nu}(p) + ... \), \( \Delta_R^{-1}(p) \), these equations read

\[
\Delta_R^{-1}(p) = p'_\mu p_\mu \bar{Z}_3 - \frac{N g^2_R}{(2\pi)^4} \int d^4 k' p'_\mu D_{\kappa\lambda}(k) \times \Delta_R^0(k) \Gamma_R(p, k) D_{\kappa\nu}(p) + ..., \tag{23}
\]

\[
D_{\mu\nu}(p) = \frac{N g^2}{(2\pi)^4} D_{\mu\lambda}(p) \int d^4 k' k'_{\kappa\lambda} \Delta_R(k) \times \Gamma_{\kappa\sigma}(k, p + k) \Delta_R(p + k) D_{\kappa\nu}(p) + ... , \tag{24}
\]

where \( p' = (a p_0, p_i) \) and \( p'_R = (a R p_0, p_i) \). To obtain a finite ghost equation, without divergent integrals and without divergent renormalisation constants, we use the factorisation of the incoming ghost momentum to write \( \Gamma_{\kappa\nu}(p + k, p) = \Gamma_{R,\kappa\nu}(p + k, p) p'_R \Gamma_{\lambda\nu} \), and we add and subtract the part of the loop integral that is quadratic in \( p \),

\[
\Delta_R^{-1}(p) = p_\mu c_{\mu\nu} p_\nu + \frac{N g^2_R}{(2\pi)^4} \int d^4 k' p'_\mu D_{\kappa\lambda}(k) \times \Delta_R^0(k) V_R(p, k) \Delta_R(p + k) D_{\kappa\nu}(p) + ... , \tag{25}
\]

where

\[
p_{\mu} c_{\mu\nu} p_\nu = p'_\mu p_\mu \bar{Z}_3 - \frac{N g^2_R}{(2\pi)^4} \int d^4 k' p'_\mu D_{\kappa\lambda}(k) \times \Delta_R^0(k) V_R(p, k) \Delta_R(p + k) p'_\nu. \tag{26}
\]

The subtracted integral in \( \Delta_R \) is of higher order in \( p \). It is also finite due to the factorisation of external ghost momenta, and only finite renormalised quantities appear in the ghost DSE, so the remaining term \( p_{\mu} c_{\mu\nu} p_\nu = c_1 p^2 + c_2 p_0^2 \) which is quadratic in \( p \) (and rotationally invariant) must also be finite.

We recall the discussion of \( \Delta_R \), and refer to it for more details. To avoid Gribov copies, the functional integral in \( A \)-space gets cut off at the (first) Gribov horizon which occurs where the Faddeev-Popov operator \( M \) has a vanishing eigenvalue. The Faddeev-Popov determinant vanishes on this boundary, and with it the integrand of the functional integral. There is therefore no boundary contribution to the DS equations, which are consequently unchanged by the cut-off at the Gribov horizon.

Moreover it has been shown in lattice gauge theory [18] that, as a consequence of the cut-off of the functional integral at the Gribov horizon, the ghost propagator \( \Delta(p) \) is more singular than the free propagator.\(^2\) Thus the condition that the functional integral be cut-off at the Gribov horizon is imposed by choosing a solution \( \Delta_R(p) \) which is more singular than the free propagator \( 1/(p' \cdot p) \) at \( p = 0 \). We therefore require that the term quadratic in \( p \) on the r.h.s. of \( \Delta_R \) vanish, \( p_{\mu} c_{\mu\nu} p_\nu = 0 \). This gives the finite renormalised ghost DSE, with the horizon condition imposed,

\[
[\Delta_R^{-1}(p) = \frac{N g^2_R}{(2\pi)^4} \int d^4 k' p'_\mu D_{\kappa\lambda}(k) \times \Delta_R^0(k) V_R(p, k) \Delta_R(p + k) p'_\nu. \tag{27}
\]

[As a by-product we note that the vanishing of \( p_{\mu} c_{\mu\nu} p_\nu = c_1 p^2 + c_2 p_0^2 = 0 \) on the l.h.s. of \( \Delta_R \) gives formulas for the renormalisation constants,

\[
\bar{Z}_3 = \frac{N g^2_R}{(2\pi)^4} \int d^4 k \frac{1}{3} \sum_{i=1}^{3} D_{R,\nu}(k) V_R(k, 0) \Delta_R(k) \tag{28}
\]

\[
\frac{1}{Z_0 Z_0^{1/2}} = \frac{N g^2_R}{(2\pi)^4} \int d^4 k D_{R,\nu}(k) V_R(k, 0) \Delta_R(k) a_R. \tag{29}
\]

\(^2\) A simple, hand-waving argument why this should be so is that the ghost propagator \( \Delta = (M^{-1}(A)) \) blows up on the Gribov horizon, and entropy favours configurations near it, which leads to an enhancement of the ghost propagator in momentum space at \( p = 0 \).
where an ultraviolet cut-off is understood, and we used \(Z_3^{1/2} \bar{Z}_3 = 1/Z_g\). The last equation is of interest because in Coulomb gauge it was shown that \(Z_3 Z_0^{1/2} = 1\).

### IV. ASYMPTOTIC SOLUTION IN INFRARED

In Landau gauge the ghost propagator \(\Delta(k)\) is enhanced at small \(k\) whereas the gluon propagator \(\tilde{D}_{\mu\nu}(k)\) is suppressed at small \(k\). Consequently the ghost loop, written explicitly in [21], dominates the DS equation of the gluon in the infrared, the remaining terms represented by the ellipses, including the tree level term, being subdominant [7]. We seek a solution where the ghost loop is also dominant in the gluon DSE in the infrared for some anomalous dimension \(\kappa_D\), and likewise for \(\Delta(p)\). More precisely, the infrared asymptotic propagators are defined by

\[
\tilde{D}(p) = \lim_{\lambda \to 0} \lambda^{2+2\kappa_D} D_R(\lambda p)
\]

\[
\tilde{\Delta}(p) = \lim_{\lambda \to 0} \lambda^{2+2\kappa_D} \Delta_R(\lambda p).
\]

They satisfy the infrared asymptotic DS equations,

\[
\tilde{D}_{\mu\nu}(p) = \frac{N g^2}{(2\pi)^4} \tilde{D}_{\mu\nu}(p) \int d^4 k' R_{\mu\nu} \tilde{\Delta}(k') R_{\mu\nu} \tilde{\Delta}(p + k') \tilde{D}_{\mu\nu}(p),
\]

\[
\tilde{\Delta}^{-1}(p) = \frac{N g^2}{(2\pi)^4} \int d^4 k' R_{\mu\nu} \tilde{\Delta}(k') R_{\mu\nu} \tilde{\Delta}(p + k') \tilde{D}_{\mu\nu}(p) [\tilde{\Delta}(k) - \tilde{\Delta}(p + k)],
\]

We have dropped the subdominant terms represented by the ellipses in the gluon equation [21]. We have also truncated the DS equations by replacing the dressed ghost-gluon vertex by the tree-level vertex, which has recently been shown to be a good approximation [11, 13, 21]. It is easy to verify, using [11, 12] and [15], that these equations are form-invariant under a renormalisation transformation.

We now exhibit a change of variables that transforms the infrared asymptotic DS equations from the interpolating gauge to Landau gauge. The gauge parameter \(a_R\) appears in the DSE only in the expressions of the type \(p' R_{\mu\nu} \tilde{D}_{\mu\nu}(k) = p' D_{\mu\nu} + p' a_R D_{\mu\nu}\). The gauge parameter may be absorbed by a rescaling of \(D_{\mu\nu}\) or of the coordinate \(p_0\) or both. For this purpose we introduce a factorisation of \(a_R = \theta \eta\), where \(\eta\) is an arbitrary constant in the interval \(0 < \eta < 1\), and \(\theta \equiv a_R \eta^{-1}\). We define the matrices \(H \equiv \text{diag}(1, 1, 1, \eta)\) and \(\Theta \equiv \text{diag}(1, 1, 1, a_R \eta^{-1})\), so \(p' R_{\mu\nu} \tilde{D}_{\mu\nu}(k) = [p H \Theta D_R(H)]_{\mu\nu}\). We make the change of variable

\[
\tilde{\mu} = (H \mu) = (\eta p_0, p)
\]

\[
\tilde{D}_{\mu\nu}(\tilde{k}) = \eta^{-1} (\Theta \tilde{D}(\Theta))_{\mu\nu}
\]

\[
\tilde{\Delta}(\tilde{k}) = \tilde{\Delta}(k),
\]

In the new variables, the transversality condition reads \(\tilde{\rho}_\mu \tilde{D}_{\mu\nu}(\tilde{p}) = 0\), and the ghost and gluon DS equations become

\[
\tilde{\Delta}^{-1}(\tilde{p}) = \frac{N g^2}{(2\pi)^4} \int d^4 k \tilde{\rho}_\mu \tilde{D}_{\mu\nu}(\tilde{k}) \tilde{\rho}_\nu \times [\tilde{\Delta}(\tilde{k}) - \tilde{\Delta}(\tilde{p} + \tilde{k})],
\]

\[
\tilde{D}_{\mu\nu}(\tilde{p}) = \frac{N g^2}{(2\pi)^4} D_{\mu\nu}(\tilde{p}) \int d^4 k \tilde{k}_{\lambda} \tilde{\Delta}(\tilde{k}) \times [\tilde{\Delta}(\tilde{p} + \tilde{k})] \tilde{k}_{\sigma} \tilde{D}_{\sigma\nu}(\tilde{p}).
\]

The change of variable has brought the transversality condition, the ghost equation, and the ghost-loop term in the gluon equation from the interpolating gauge to the Landau gauge.

These equations have been solved, [11, 17], with the result, \(\tilde{D}_{\mu\nu}(\tilde{p}^2) = D_L(\tilde{p}^2) (\delta_{\mu\nu} - \tilde{p}_\mu \tilde{p}_\nu/\tilde{p}^2)\),

\[
\tilde{\Delta}(\tilde{p}^2) = \frac{\tilde{c} \mu^{2\kappa_D}}{(\tilde{p}^2)^{1+\kappa_D}},
\]

where \(\kappa_D = (93 - \sqrt{1201})/98 \approx 0.595353\), \(\kappa_D = -2\kappa_G\), with dimensionless coefficients \(\tilde{b}\) and \(\tilde{c}\). There is an infrared fixed point

\[
\alpha(p \to 0) = (\tilde{p}^2)^3 \frac{g^2}{4\pi} D_L(\tilde{p}^2) [\tilde{\Delta}(\tilde{p}^2)]^2 \frac{\tilde{c} \tilde{b}^2}{4\pi} = \frac{2\pi}{3N_c} \Gamma(3 - 2\kappa) \Gamma(3 + \kappa) \Gamma(1 + \kappa) \approx 8.915/N_c,
\]

where \(\kappa \equiv \kappa_D\).

In terms of the original variables, this solution reads

\[
\Delta_R(k) \to \tilde{\Delta}(k) = \frac{\tilde{b} \mu^{2\kappa_D}}{(k^2 + \eta^2 k_0^2)^{1+\kappa_D}}
\]

\[
g_R^2 \tilde{D}_{R; R}^{\mu\nu} \to \tilde{g}_R^2 \tilde{D}_{R; R}^{\mu\nu}(k) = \frac{\eta \tilde{c} \mu^{2\kappa_D}}{(k^2 + \eta^2 k_0^2)^{1+\kappa_D}}
\]

\[
g_R^2 \tilde{D}_{R; R}^{\mu\nu} \to \tilde{g}_R^2 \tilde{D}_{R; R}^{\mu\nu}(k) = \frac{\tilde{c} \mu^{2\kappa_D}}{(k^2 + \eta^2 k_0^2)^{1+\kappa_D}}
\]

where \(\to\) means asymptotic infrared limit defined in [30]. For all \(\eta\) in the range \(0 < \eta < 1\), the infrared asymptotic solution [35] is self consistent because counting of powers of momenta remains the same as in Landau gauge, so the terms in the gluon equation that were neglected remain subdominant by power counting.
The $\mu$-dependence nicely factors out of the RG-invariant products [16] and [17]

\[
g_R^2 \hat{D}^{ir} \hat{\Delta}^2 = \frac{1}{(k^2 + \eta^2 k_0^2)^3} \frac{\eta \, \bar{b}^2}{4\pi}, \\
a_R^2 g_R^2 \hat{D}_{00} \hat{\Delta}^2 = \frac{k^2}{(k^2 + \eta^2 k_0^2)^4} \frac{\eta^3 \, \bar{b}^2}{4\pi},
\]

(39)
because $\kappa_D + 2\kappa_\Delta = 0$. The RG-invariant running couplings, eqs. [18], become in the asymptotic infrared limit $\alpha_I(0) = \lim_{\lambda \to \infty} \alpha(\lambda|k|)$, $\alpha_I(0) = \lim_{\lambda \to \infty} a(\lambda|k|)$,

\[
\alpha_I(0) = \frac{16}{3} \int \frac{dk_0}{2\pi} \frac{|k|^5}{(k^2 + \eta^2 k_0^2)^3} \frac{\eta \, \bar{b}^2}{4\pi}, \\

\alpha_I^2(0) \alpha_I(0) = \frac{32}{5} \int \frac{dk_0}{2\pi} \frac{|k|^7}{(k^2 + \eta^2 k_0^2)^4} \frac{\eta^3 \, \bar{b}^2}{4\pi},
\]

(40)
where we have also rescaled $k_0 \to \lambda k_0$. This gives

\[
\alpha_I(0) = \frac{\bar{c} \, \bar{b}^2}{4\pi} \approx \frac{8.915}{N_c}. \\
\alpha_I(0) = \eta.
\]

The RG-invariant running coupling $\alpha_I(|k|)$ goes to the same infrared fixed point, $8.915/N_c$, in all interpolating gauges as in Landau gauge. On the other hand the RG-invariant parameter $a_R(\mu)$ is useless for this purpose, being subject to arbitrary renormalisation. Indeed under a finite renormalisation $A_{R,4} \to z_3^{1/2} A_{R,4}$ and $A_{R,0} \to z_0^{1/2} A_{R,0}$ it changes according to $a_R \to a_R z_3^{1/2} z_0^{-1/2}$.

The solution [18] has some attractive features. In the Coulomb-gauge limit $\eta \to 0$, the ghost propagator becomes independent of $k_0$, as it should. The infrared anomalous dimensions $\kappa_\Delta$ and $\kappa_D$ of the ghost and gluon propagators, and the infrared fixed point $\alpha_I(0)$ are independent of the gauge parameter $\eta$ in the range $0 < \eta \leq 1$, and may have some gauge-invariant meaning.\footnote{It seems also that the anomalous dimensions are independent of the gauge parameter $\xi$ of linear covariant gauges.}

It is somewhat surprising that the transverse gluon propagator $D_R^{tr}(k)$ is non-analytic at the unphysical gauge-dependent point $k^2 + \eta^2 k_0^2 = 0$, whereas it is singular at the physical point $k^2 + k_0^2 = 0$ in every order of perturbation theory. However according to the gluon DSE, [21], the non-analyticity of the gluon propagator is determined by the $\eta$-dependent singularity of the ghost propagator which gives the dominant term in the infrared. It seems that by imposing the horizon condition, which is non-perturbative, the solution is forced into a confined phase in which $D_R^{tr}(k)$ is non-analytic at an unphysical point. Note that $D_R^{tr}(k)$ actually vanishes at this point, because $1 + \kappa_D < 0$, so this non-analyticity does not contradict the Nielsen identity which states that poles in physical channels are gauge-independent.

V. COULOMB GAUGE LIMIT

The infrared anomalous dimensions of $D_R^{tr}$ and $D_{R,00}$ are independent of $\eta$, and thus remain constant in the Coulomb-gauge limit $\eta \to 0$. However according to [18], the asymptotic infrared gluon propagators $D^{ir}(k)$ and $D_{00}(k)$ are proportional respectively to $\eta$ and $\eta^3 a_R^2$ which vanish at $\eta = 0$, while the second is ill defined at $a_R = 0$. Because $D^{ir}(k)$ and $D_{00}(k)$ represent the infrared asymptotic limit of $D_R^{tr}(k)$ and $D_{R,00}(k)$, this would indicate that the infrared anomalous dimensions change discontinuously at $\eta = 0$ (while the propagators $D_R^{tr}(k)$ and $D_{00}(k)$ at finite momenta remain well-defined at $\eta = 0$).

Recall that the ghost propagator is given by $\Delta(x-y) = (M^{-1})_{xy}(A)$, where $M = -\partial^\rho p_{\mu} D_{\mu}$ is the Faddeev-Popov operator. In the Coulomb-gauge limit, the inverse Faddeev-Popov operator becomes local in time, $(M^{-1})_{xy}(A) \to (M^{-1})_{xy}(A)\delta(x_0 - y_0)$, so the ghost propagator, when transformed to momentum space becomes independent of $p_0$, $\Delta(p) = \Delta(|p|)$. The DS equation for the gluon is then singular. Indeed in Coulomb gauge the $k_0$ integration that appears in [24] is given by

\[
\int dk_0 \, k_R^I \, \Delta_R(|k|) \Delta_R(|k + p|) \, k_R^I, 
\]

(42)
a catastrophically divergent integral. This is the famous energy divergence of the Coulomb gauge. It is cancelled by a corresponding contribution from the gluon loop which we have not evaluated because it is subdominant in the infrared for $\eta > 0$. Thus the asymptotic infrared solution we have obtained no longer holds at $\eta = 0$.

However we get some information from the ghost equation. In Coulomb gauge the time components of the gluon propagator do not contribute to the ghost DS equation with tree-level vertex, $p_R^\mu D_{R,\mu
u}(k)\bar{p}_R^\nu = p_i D_{R,ij}(k)p_j = [p^2 - (p \cdot k)^2] \, D_R^{tr}(k, k_0)$, by [4]. With $\Delta_R(k) = \Delta_R(|k|)$ independent of $k_0$ in Coulomb gauge, the $k_0$-integral in the ghost DSE, [21], gives the equal-time gluon propagator,

\[
D_R^{tr}(k) = (2\pi)^{-1} \int dk_0 \, D_R^{tr}(k, k_0).
\]

(43)
The ghost equation in Coulomb gauge reduces then to a purely spatial equation relating functions of a single
scalar variable,
\[ \Delta_R^{-1}(|p|) = \frac{N g^2}{(2\pi)^3} \int d^3k \left| p^2 - (p \cdot \hat{k})^2 \right| D^\eta_{ct}(|k|) \times \left[ \Delta_R(|k|) - \Delta_R(|p + k|) \right]. \]  
(44)

In the infrared asymptotic limit, the propagators obey power laws,
\[ \Delta_R(|p|) \rightarrow \frac{b (\mu^2)^{\gamma_A}}{(p^2)^{1+\gamma_A}} \]
\[ g_R^2 D^\eta_{ct}(|p|) \rightarrow \frac{|p| c (\mu^2)^{\gamma_D}}{(p^2)^{1+\gamma_D}}. \]  
(45)

where \( \gamma_A \) and \( \gamma_D \) are infrared anomalous dimensions that characterise the Coulomb gauge. (The extra power of \(|p|\) reflects the dimension of \( D^\eta_{ct} \)) Counting powers of momenta in the ghost DSE \( \ref{eq:44} \) gives the relation between the anomalous dimensions,
\[ \gamma_D = -2\gamma_A. \]  
(46)

The running coupling \( \alpha_I(|k|) \) defined in \( \ref{eq:13} \) is also an RG-invariant in Coulomb gauge where it simplifies to
\[ \alpha_I(|p|) = \frac{16}{3} \frac{g_R^2}{4\pi} |p|^5 \Delta^2(|p|) D^\eta_{ct}(|p|) \]
\[ \rightarrow \frac{4b^2 c}{3\pi}. \]  
(47)

To determine the value of the infrared fixed point \( b^2 c \), one would have to also solve the gluon equation in Coulomb gauge in the asymptotic infrared. However since \( \alpha_I(0) \) is independent of the gauge parameter \( \eta \), it may possibly have the same value in Coulomb gauge, namely \( 4b^2 c/3\pi \approx 8.915/N_c \).

The RG-invariant coupling \( \alpha_I(|p|) \) approaches an infrared fixed point as \( |p| \rightarrow 0 \). This is in sharp contrast to the behaviour of \( \alpha_{\text{coul}}(|p|) \), defined in \( \ref{eq:19} \), which has been approximately determined in Refs. \( \ref{ref:22,23} \) to diverge as
\[ \alpha_{\text{coul}}(p/\Lambda_{\text{QCD}}) \sim \frac{1}{|p|^\beta}, \]  
(48)

with \( \beta \approx 2 \).

\section{VI. GAUGE EQUIVALENCE AND GAUGE INSTABILITY}

We considered interpolating gauges that are characterised by a gauge parameter \( a_R \) (or \( a \)) that appears in the local action. We found that the space-time character of the solutions of the DSE in the infrared limit is independent of \( a_R \), but depends instead on an RG-invariant parameter \( \eta \). Thus our solutions are characterized by a single parameter, just as gauges are characterized by a single parameter, but it’s a different parameter. This should not come as a surprise. Recall that \( a_R \) satisfies the RG flow equation \( \ref{eq:13} \) with solution \( a_R = a_R(\mu, c) \), where \( c \) is an arbitrary constant of integration. Quantities such as \( g_R^2 D^\eta_{ct} \) that are RG-invariants must be independent of \( \mu \) and thus depend on \( c \) rather than \( a_R \). The gauge dependence thus shifts from \( a_R \to c \). In our solution, the RG-invariant products \( g_R^2 D^\eta_{ct} \Delta^2 \) and \( a_R^2 g_R^2 D_R^{0,0} D^\eta_{ct} \), given in \( \ref{eq:32} \), are found to be independent of \( a_R \) and to depend only upon \( \eta \). This leads us to make the identification \( c = \eta \), which makes \( \eta \) a true (RG-invariant) gauge parameter. It is very encouraging that our asymptotic infrared solutions display exact features of the renormalisation group even though we are far from the perturbative regime and used truncated DS equations.

The possibly novel element is that \( \eta \) appeared in our solution of the DSE as a degeneracy parameter that characterizes different solutions in a fixed gauge, that is, for a fixed value of \( a_R \). This is true even for the Landau gauge \( a_R = 1 \), which is generally thought to be a fixed point of the RG group. However our discussion shows that the Landau gauge is unique only if one restricts solutions of the DSE in Landau gauge to those that are manifestly Lorentz covariant. If one drops this restriction, as we do, then the DSE in Landau gauge also has a continuum of solutions, characterized by the RG-invariant parameter \( \eta \). Thus we may say that each gauge is unstable, having a continuum of solutions, and moreover different interpolating gauges, including the Landau gauge, are equivalent to each other.

If the identification of the degeneracy parameter \( \eta \) with the RG-invariant gauge parameter \( c \) is correct, then it is to be expected that the degeneracy of solutions that we found (for a fixed gauge) holds not only for the infrared asymptotic solutions obtained here, but extends to solutions of the full DSE equations at finite momentum \( k \). Namely for each value of the parameter \( a_R \) that appears in the local action, there is a class of gauge-equivalent solutions parametrised by the RG-invariant gauge parameter \( \eta \), and moreover the dependence on \( a_R \) is trivial. Thus for any value of \( a_R \), including the Landau-gauge value \( a_R = 1 \), there is a one-parameter class of solutions of the DSE that continuously connects the value \( \eta = 1 \), where Lorentz invariance is manifest, to a solution with \( \eta \to 0 \) that approaches the Coulomb gauge, where one has a simple confinement scenario. The observation that the infrared fixed point of the running coupling and the infrared anomalous dimensions of gluon and ghost propagators are all independent of the two gauge parameters \( \eta \) and \( a_R \), and the observation that the infrared fixed point persists in the Coulomb gauge limit, may, hopefully, bring us closer to a unified picture of confinement in Landau and Coulomb gauge.

\section{VII. SUMMARY AND CONCLUSION}

To summarise, we have found two renormalisation-group invariant running couplings, \( \alpha_I \) and \( a_R^2 \alpha_I \), in a
class of gauges that interpolate between Landau gauge and Coulomb gauge. These gauges are conveniently characterised by the RG-invariant, \( \eta = a_I(0) \), that varies between zero and one. We have determined the infrared behaviour of these couplings from a coupled set of Dyson-Schwinger equations for the ghost and gluon propagator. In all interpolating gauges, \( 0 < \eta \leq 1 \), we found an infrared fixed point for the running coupling \( \alpha_I(k) \) at \( \alpha_I(0) = 8.915/N \). This fixed point as well as the infrared anomalous dimensions of the ghost and gluon propagators are independent of the gauge parameter \( \eta \), and coincide with the corresponding well known values in the Landau gauge [9, 10, 11]. Thus they may have some gauge-invariant meaning. Indeed, the violation of the cluster decomposition principle, a necessary condition for confinement, entails that long range correlations must be present in Yang-Mills theory. One is tempted to speculate that these correlations are triggered by the fixed point behaviour of the theory in the infrared. This is manifest in Coulomb gauge, \( \eta = 0 \). Although we cannot determine its exact value in the Coulomb gauge limit, we still find a fixed point for \( \alpha_I \) in the infrared. The other coupling, \( a_2^I \alpha_I \), is replaced by the familiar Coulomb coupling \( \alpha_{\text{coul}} \), eq. [19], which diverges in the infrared and determines the static colour Coulomb potential, eq. [31]. Thus the running coupling, that in Landau gauge merely displays a fixed point in the infrared, bifurcates by gauge instability into two RG-invariant running couplings in interpolating gauges, and these in Coulomb gauge are responsible for fixed point behaviour on one hand and on the other for the long-range colour-Coulomb potential between static colour charges.

**Acknowledgements**

We are grateful to Reinhard Alkofer for valuable discussions and a critical reading of the manuscript. Daniel Zwanziger is grateful for the hospitality of the group at the University of Tübingen and Christian Fischer is grateful for the hospitality of the group at the University of Coimbra where part of this work was done. This work has been supported by the Deutsche Forschungsgemeinschaft (DFG) under contract Fi 970/2-1 and by the National Science Foundation, Grant No. PHY-0099393.

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