On generation of metric perturbations during preheating

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We consider the generation of the scalar mode of the metric perturbations during preheating stage in a two field model with the potential

\[ V(\phi, \chi) = \frac{m^2 \phi^2}{2} + \frac{g^2 \phi^2 \chi^2}{2} \].

We discuss two possible sources of such perturbations: a) due to the coupling between the perturbation of the matter field \( \delta \chi \) and the background part of the matter field \( \chi_0(t) \), b) due to non-linear fluctuations in a condensate of “particles” of the field \( \chi \). Both types of the metric perturbations are assumed to be small, and estimated using the linear theory of the metric perturbations. We estimate analytically the upper limit of the amplitude of the metric perturbations for all scales in the limit of so-called broad resonance, and show that the large scale metric perturbations are very small, and taking them into account does not influence the standard picture of the production of the metric perturbations in inflationary scenario.

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I. INTRODUCTION

In the modern scenario of creation of the matter in the Universe there is a stage called preheating, when the matter (usually imitating by a massless scalar field) is generating from vacuum fluctuations due to the effect of parametric resonance [1,2,3]. Namely it is assumed that the matter field \( \chi \) is coupled with inflaton field \( \phi \), and the coupling constant \( g \) is large. During inflation and right after that period the matter field has a large effective “mass” determined by this coupling, and each mode of the matter field with sufficiently small wavenumber evolves by a standard manner, oscillating with a frequency proportional to this "mass" and adiabatically decreasing amplitude. The inflaton starts to oscillate itself after inflation, and near the moments of time when \( \phi(t) \approx 0 \) the matter field effectively loses its “mass”, the adiabatic approximation breaks and a possibility of resonant growth of the matter field amplitude and the corresponding occupation numbers of \( \chi \) “particles” appears. In fact, this effect may lead to exponential growth of the occupation numbers, and the resulting distribution of \( \chi \) “particles” is strongly nonthermal.

After some moment of time \( t_\ast \) (end of the first stage of preheating) the energy density of the \( \chi \) “particles” becomes comparable with the energy density of the inflaton, and back reaction processes become influence the dynamics of inflaton field \( \phi \) and the Universe. It is believed that during some short period of time \( \delta t \) after \( t_\ast \) the \( \chi \) particles are thermalized by rescattering effects, and after thermalization the energy density of these “particles” evolves according to the standard picture of the Hot Big Bang model. Such scenario of the first stage of preheating is called the scenario of broad resonance [2,3].

As was mentioned by a number of authors (e.g [4]), the characteristic value of the growth rate of the matter field modes does not depend significantly on the value of wavenumber \( k \) for the modes with sufficiently small \( k \). At first glance this fact gives a very interesting possibility of amplification of the matter field modes and metric perturbations at exponentially large scales, say, corresponding to the scale of the present horizon. In this paper we consider the simplest model of preheating with two scalar fields, the inflaton field \( \phi \) and the matter field \( \chi \) and the potential

\[ V(\phi, \chi) = \frac{m^2 \phi^2}{2} + \frac{g^2 \phi^2 \chi^2}{2} \],

and show that in this model the value of metric perturbations at the scale of present horizon is suppressed by an extremely small factor, and therefore taking into account the stage of preheating does not lead to any modifications of the standard picture of generation of cosmological perturbations in inflationary scenario. We have two basic arguments supporting this conclusion. At first, the field \( \chi \) has a large effective mass before the stage of preheating (much larger than the Hubble parameter at the end of inflation), and as a consequence the spectrum
of initial field fluctuations $\delta \chi_k$ is strongly suppressed at small $k$ at the time $t_{in}$ of the beginning of the preheating stage, the rms value of the field fluctuations $\delta \chi_{rms} \propto k^{3/2}$ (also [5]). During preheating the field modes may grow exponentially fast, but the characteristic growth rate $G = \frac{\delta \chi_k}{\delta \chi_k}$ is constrained by some maximal value $\mu_{max}/m$, where numerical constant $\mu_{max} = \frac{1}{4} \ln (1 + \sqrt{2}) \approx 0.28$ [3].

The first stage of preheating ends at the time $t_\ast \approx 50 / 100 m^{-1}$, and this estimate does not depend strongly on the parameters of the theory (see [3], and also the eqns. (22), (23) below). Thus, at the end of the first stage of preheating, the r.m.s value of the field fluctuations contains a factor

$$\delta \chi_{rms}(t_\ast) < G \delta \chi_{rms}(t_{in}) \sim e^{-3N/2 + \mu_{max} t_\ast} = e^{-47} e^{-1.5 (N - 50) + 0.28 (mt_\ast - 100)},$$

where the $N$ is the number of e-folds, and we choose the standard value $N = 50$ to represent the scale of present horizon. Here we express the amplitude of the field perturbation in terms of the natural Planck units.

Secondly, the suppression factor for the metric perturbations may be even much smaller than the estimate (1) if one uses the standard theory. In this theory the contribution of the second field $\chi$ in the scalar mode of metric perturbations is determined by the terms containing multiplication of homogeneous “background” part of the field $\chi_0(l)$ and the perturbed part $\delta \chi$, and their derivatives (see eqns. (25,26) below). The amplitude of homogeneous part of the field behaves like the mode $\delta \chi_k$ with wavenumber $k = 0$, and is constrained by the fact that the field $\chi_0$ cannot contribute to the total energy density during the last stage of inflation. Assuming that during last $N$ e-folds the dynamics of the Universe has been controlled by the inflaton field $\phi$, after the end of the first stage of preheating the amplitude $\chi_0$ should contain the same factor (1). Therefore the rms amplitude of the metric perturbations $\delta_{rms}$ contains the factor (1) squared:

$$\delta_{rms}(t_\ast) < e^{-94} e^{-3(N - 50) + 0.56(mt_\ast - 100)},$$

and is suppressed by enormously small factor $e^{-94}$ for the typical parameters of the theory. This estimate is however changed in a more self-consistent approach to the evolution of the perturbations during preheating. In the theory of preheating the role of background is effectively played by a condensate of $\chi$ particles with relatively large wavenumbers ($\chi_k$ is always larger than the characteristic wavenumber corresponding to the scale of cosmological horizon during the preheating stage, see [3] and also the eq. (13)). The fluctuations of energy-momentum tensor of such condensate can give rise to additional metric perturbations, which are second order with respect to the perturbations of the matter field, and therefore cannot be obtained in the frameworks of the standard perturbation theory. The estimate shows that the fluctuations of the energy-momentum tensor decrease with scale proportional to $k^{3/2}$ (similar to the rms value of $\delta \chi$ field, see eq. (47)), and is of order of unity at the scale corresponding to the critical wavenumber $k_{crit}$ at the time $t_\ast$. Since in the large scale limit the metric fluctuations are of the order of the energy density fluctuations, the estimate $\delta_{rms}(t_\ast) \sim \delta \chi_{rms}(t_\ast)$ is a more reliable estimate in the more realistic approach to the dynamics of preheating, and this estimate still contains a very small number $e^{-47}$ for the scale corresponding to the present horizon. In fact, it can be shown (see Section 3) that the metric perturbations of this kind take their maximal value at the scale corresponding to the horizon size at the time of the end of preheating, and this value is smaller than $\sim 10^{-3}$ for the broad resonance case [4]. Therefore in the simplest models the transition from inflation to the hot Big Bang proceeds smoothly, without significant generation of the metric perturbations.

We present our arguments in a more rigorous form in the next two Sections. In Section 2 we introduce the basic ideas of preheating theory, and estimate the time $t_\ast$ of duration of the first stage of preheating. In Section 3 we discuss the application of the theory of cosmological perturbations to our case of two interacting fields, and estimate the upper limit on the metric perturbations.

We use below the natural system of units, and set the Plank mass $M_{pl} = \sqrt{8\pi}$.  

II. PREHEATING IN THE REGIME OF BROAD RESONANCE

The theory of initial stage of preheating has been developed in the paper [3], and we will closely follow this paper. For our purposes we need to know the evolution of both fields $\phi$ and $\chi$, and also the evolution of the scale factor $a(t)$.

\[1\] Note, that when estimating the metric perturbations induced by the non-linear terms, we do not take into account the contribution of such fluctuations in the dynamics of the background model. This contribution is of order of the leading term at the end of preheating, and can change our estimates on a numerical factor. We believe that this factor is of order of unity, and cannot change our results significantly.
As usually we divide both fields on background parts and perturbations \( \phi = \phi_0(t) + \delta \phi(t, \vec{x}), \chi = \chi_0(t) + \delta \chi(t, \vec{x}) \), and apply Fourier transform to the perturbed parts. We temporary neglect the influence of the metric perturbations in this Section, and consider this effect later on. Assuming that the contribution of the field \( \chi \) in the energy density and the pressure is negligible, we can use the standard expression describing the evolution of the scale factor and the field \( \phi_0 \) in the theory of massive scalar field (see e.g. [6]):

\[
a(t) \approx a_0 \tau^{2/3},
\]

\[
\phi_0(t) \approx \phi_{in} \frac{\sin \tau}{\tau},
\]

where \( \phi_{in} = 2\sqrt{\frac{2}{g}} \) is a characteristic value of the field in the beginning of preheating stage, \( a_0 \) is an “initial” value of the scale factor, dimensionless time \( \tau = mt \), and the preheating stage begins when approximately \( \tau = \tau_n \approx 1 \).

The evolution equation for the perturbed part of the field \( \chi \) can be conveniently written in the form:

\[
X_k + \omega_k^2 X_k = 0,
\]

where \( X_k \) is the rescaled field amplitude: \( X_k = a^{3/2} \delta \chi_k \), \( \omega_k \) is the effective frequency:

\[
\omega_k^2 = \frac{k^2}{a^2} + g^2 \phi_{in}^2 \left( \frac{\sin \tau}{\tau} \right)^2 + \Delta,
\]

and the correction \( \Delta = -(\frac{3}{2} H^2 + \frac{3}{2} \frac{\dot{a}}{a}) \). Hereafter \( H = \frac{\dot{a}}{a} \) is the expansion rate. The positive frequency solutions of this equation \( X_+ \) determine a vacuum state, and must have a form:

\[
X_+ = \frac{1}{(2\pi)^{3/2} \sqrt{2 \omega_k}} e^{-i \theta}
\]

at the moments of time sufficiently close to \( \tau_n \). Here \( \theta = \int^t dt' \omega \). The solution of the equation (5) can also be represented in another form by introducing two complex functions \( \alpha, \beta \), which are constrained by the normalization condition \(|\alpha|^2 - |\beta|^2 = 1 \):

\[
X_+ = \frac{1}{(2\pi)^{3/2}} \left( \frac{\alpha(t)}{\sqrt{2\omega}} e^{-i \theta} + \frac{\beta(t)}{\sqrt{2\omega}} e^{i \theta} \right).
\]

To reconcile this representation with (7), we should set \( \alpha(\tau \sim \tau_n) = 1 \), and \( \beta(\tau \sim \tau_n) = 0 \). Obviously, the representation (8) is specially convenient if the solution of eq. (5) is close to its adiabatic approximation, and \( \alpha, \beta \approx const \). For the general case, the evolution of the functions \( \alpha, \beta \) follows from the eq. (5):

\[
\dot{\alpha} = \frac{\dot{\omega}}{2\omega} e^{2i \theta} \beta, \quad \dot{\beta} = \frac{\dot{\omega}}{2\omega} e^{-2i \theta} \alpha.
\]

If the adiabatic condition \( \frac{\dot{\omega}}{\omega} \ll 1 \) is satisfied, the functions \( \alpha, \beta \) are approximately constant, and no additional “particles” of the field \( \chi \) are produced, the field oscillates with the frequency \( \approx \omega \), and the field amplitude decays as \( a^{-3/2} \sim \tau^{-1} \). The adiabatic approximation breaks when the field \( \phi \) is close to zero, and the time \( \tau \) is close to \( \tau_j = \pi j \) (the integer index \( j \) must be much larger than unity for the validity of the approximate equations (3, 4)). Rewriting the adiabatic condition near the points \( \tau = \tau_j \), we have

\[
\frac{\dot{\omega}}{\omega^2} \approx \frac{n \tau_j}{g \phi_{in} \Delta \tau^2} = \frac{1}{2q(\tau_j)^{1/2} \Delta \tau^2},
\]

\(^2\)A care should be taken when specifying the positive frequency solution for the modes with wavelengths larger than the horizon scale. Strictly speaking the vacuum state should be the standard Bunch-Davies vacuum state for a massive field, but a more accurate expression give essentially the same result.

\(^3\)Wherever it is possible we will drop the index \( k \) in our expressions.
where $\Delta \tau = \tau - \tau_j$, and the parameter $q(\tau)$ characterizes the strength of the resonance \[3\]:

$$q = \frac{2}{\pi} \left( \frac{g}{m} \right)^2 \frac{1}{\tau^2}. \tag{11}$$

We will also use $q_0 = q(\tau = 1)$ to parameterize our expressions. The regime of broad resonance corresponds to very large values of $q_0$, see below.

Thus when the parameter $q$ is very large, the functions $\alpha$ and $\beta$ are approximately constant between the moments of time $\tau_j$, but when the time is very close to $\tau_j$, the functions $\alpha$ and $\beta$ can be changed. The rule of change can be written as an iterative mapping between the functions $\alpha_{j-1}$, $\beta_{j-1}$ and $\alpha_j$, $\beta_j$ corresponding to the time periods $\tau_{j-1} < \tau < \tau_j$ and $\tau_j < \tau < \tau_{j+1}$ \[3\]:

$$
\begin{pmatrix}
\alpha_j \\
\beta_j
\end{pmatrix} =
\begin{pmatrix}
a & b \\
b^* & a^*
\end{pmatrix}
\begin{pmatrix}
\alpha_{j-1} \\
\beta_{j-1}
\end{pmatrix},
$$

where $a = \sqrt{1 + e^{-\pi^2 q^4 \phi^2}}$, and $b = i e^{-\pi^2 q^4 + 2i \phi}$, and $\theta_j = \int_{\tau_{j-1}}^{\tau_j} dt \omega$, $t_j = \tau_j/m$, $\phi = \arg \Gamma(1 + i q_0^4) + \pi^2 (1 + 2 \ln \frac{2}{\pi})$, and $\kappa = k/k_*$, $k_*$ is a characteristic value of comoving momentum: $k_* = 2^{1/2} a(t) q^{1/4} m$. If the wavenumber is much larger than a critical cutoff wavenumber:

$$k_{\text{crit}} = \frac{k_*}{\pi} = \sqrt{\frac{2}{\pi}} a(t) q^{1/4} m = \sqrt{\frac{2}{\pi}} a_0 q_0^{1/4} q^{1/6} m, \tag{13}$$

the effect of particle creation is exponentially damped. It is important to note that the bulk of the produced particles has values of wavenumbers close to the cutoff value \(13\), the corresponding wavelengths of these particles are well inside the cosmological horizon. The particles occupation numbers $n_j = |\beta_j|^2$ can also be expressed by an iterative way:

$$n_j \approx (1 + 2 e^{-\pi^2 q^4} - 2 \sin \theta_{\text{tot}} e^{-\pi^2 q^4 \sqrt{1 + e^{-\pi^2 q^4}}}) n_{j-1}, \tag{14}$$

where $\theta_{\text{tot}} = 2 \theta_j - \phi + \arg(\beta_j - \text{arg}(\alpha_j - 1))$, and the limit of large occupation numbers $n_{j-1} \gg 1$ is assumed hereafter. In particular, from the eq. \(14\) it follows that the number of “particles” can either increase or decrease depending on the value of $\theta_{\text{tot}}$. The evolution of $\theta_{\text{tot}}$ is rather complicated in the limit of large $q$, but for our purposes it is sufficient to note that the occupation number cannot increase more than $3 + 2 \sqrt{2}$ times, and the amplitude of the field $\sim \sqrt{\eta_j}$ cannot increase more than $1 + \sqrt{2}$ times. It is convenient to characterize the change of the amplitude by the growth rate:

$$\mu_j = \frac{1}{2 \pi} \ln \frac{n_j^2}{n_{j-1}}, \tag{15}$$

and also average the growth rate over the time:

$$\mu_{\text{eff}} = \frac{\pi}{\tau} \sum_j \mu_j. \tag{16}$$

From the eqns. \(14-16\) it follows that neither $\mu_j$ nor $\mu_{\text{eff}}$ can exceed the maximal value $\mu_{\text{max}} = \frac{1}{2} \ln (1 + \sqrt{2}) \approx 0.28$. In fact, the effective growth rate $\mu_{\text{eff}}$ is about two times smaller than its maximal value. The numerical calculations give $\mu_{\text{eff}}$ in the interval $0.1 - 0.18$ with an average value of order of 0.14 for the coupling constant $g$ in the interval from $0.9 \cdot 10^{-4}$ to $10^{-3}$ \[3\].

The background field $\chi_0(t)$ obeys the same equation \(5\) provided the wavenumber $k$ is set to zero. Therefore we can obtain the expression for the evolution of the background field combining the positive and negative frequency solutions $\delta \chi_+$, $\delta \chi_- = \delta \chi_+^\pm$. Let denote the value of the field $\chi_0$ at the beginning of the preheating stage as $\chi_{in}$, and its time derivative as $\dot{\chi}_{in}$. The values of these quantities are determined by their evolution at the previous stage of inflation. Assuming that the contribution of the field $\chi$ in the potential has been negligible during last $N$ e-folds, we estimate the initial amplitude as $\chi_{in} \approx \frac{g}{\sqrt{3}} e^{-3N/2}$, and its time derivative as $\dot{\chi}_{in} \approx \omega (\tau - \tau_{in}) \chi_{in} \approx me^{-3N/2}$. Let introduce a characteristic field amplitude and angle:

$$\chi_* = \sqrt{\chi_{in}^2 + \frac{g^2}{2} \dot{\chi}_{in}^2}, \quad \phi_0 = \arctan \frac{g \chi_{in}}{\dot{\chi}_{in}}. \tag{17}$$
Then
\[
\chi_0(t) = \frac{(2\pi a_0)^{3/2}}{i\sqrt{2}} \omega_0^{1/2} \chi^*(e^{i\phi_0} \delta \chi^- - e^{-i\phi_0} \delta \chi^+) = \\
\frac{\chi^*(q_0 a)}{2i} \left(\frac{\omega_0}{\omega}\right)^{1/2} \left(e^{-i\phi_0} (\beta^* e^{i\phi_0} - \alpha e^{-i\phi_0}) + e^{i\theta} (\alpha^* e^{i\phi_0} - \beta e^{-i\phi_0})\right),
\]
where the wavenumber \( k \) is set to zero in the expressions for \( \alpha \) and \( \beta \), and \( \omega_0 = \omega(\tau = \tau_{in}) \sim g \).

We also need the evolution of the field \( \delta \phi \), but we discuss it in the next Section.

Since our order-of-magnitude estimates (1), (2) are exponentially sensitive to the value of the moment of time \( t_* \) of the end of the first stage of preheating, it is very important to obtain a reliable estimate of \( t_* \). Following [3] we assume that the first stage of preheating ends when the vacuum expectation value for \( \chi^2 \) gives a contribution to the potential of order of the leading classical term \( \sim \frac{m^2 \delta^2}{2} \).

\[
\langle \chi^2 \rangle (t_*) = \frac{m}{g}.
\]

Let estimate the dependence of \( \langle \chi^2 \rangle \) on time. Using the definition of the function \( \beta \), we have
\[
\langle \chi^2 \rangle (t) = \frac{1}{2\pi^2 a^3 \omega} \int_0^{k_{crit}} dk k^2 |\beta|^2,
\]
where \( k_{crit} \) is given by the eq. (13). In order to calculate the integral in the eq. (20) we assume that each mode with a given \( k \) grows with its average growth rate \( \mu^{eff} \approx 0.14 \) when \( k < k_{crit}(t) \), and the growth rate for the modes with \( k > k_{crit}(t) \) is zero. We have:
\[
\langle \chi^2 \rangle = \frac{1}{12\pi^2} \frac{q_0^{1/4} m^2}{\mu^{eff} t^{1/2}} e^{2\mu^{eff} t},
\]
where we approximate the value of \( \omega \) as \( \omega \approx \frac{2\phi_0}{t} \). Note that our estimate is slightly different from the estimate [3] due to a different approximation in the calculation of the integral in (20). Substituting the eq. (21) in (19), we obtain:
\[
\tau_* = m t_* = \frac{1}{2\mu^{eff}} \ln \left( \frac{2}{3} \frac{12\pi^2 \mu^{eff} t^{1/2}}{m^2 q_0^{3/4}} \right).
\]
An approximate solution of the eq. (21) can be written as:
\[
\tau_* \approx \mu_{0.14}^{-1} \left(107 + 3.57 \ln F\right),
\]
where \( F = m^{-2} q_4^{-3/4} \mu_{0.14}^{1/2} \), and \( m_{-6} = \frac{m}{10^{-6}}, q_4 = \frac{q_0}{10^4}, \mu_{0.14} = \frac{\mu^{eff}}{0.14} \). The time \( \tau_* \) decreases with increase of the parameter \( q_0 \), and is bounded by some minimal and maximal values \( \tau_{min}, \tau_{max} \). For example, if the mass \( m \) has its “canonical” value \( \sim 10^{-6} \), the parameter \( q_0 \) cannot exceed the maximal value \( q_{max} \sim m^{-2} = 10^{12} \), and we have \( t_{min} \approx 55 \). On the other hand, the parameter \( q(t_*) \) cannot be smaller than unity for applicability of our theory. \(^4\) Since \( q \sim \tau^{-2} \), we have \( q_{min} \sim 10^4 \) as a minimal possible value of \( q_0 \). Therefore, the time of the end of the first stage of preheating is constrained by a condition: \( 55 < \tau_* < 107 \) for \( \mu^{eff} \approx 0.14 \) and \( m \sim 10^{-6} \).

### III. THE METRIC PERTURBATIONS DURING PREHEATING

Now let us take into account the metric perturbations and consider the self-consistent theory of perturbations. The shape of the metric perturbations depends on gauge. In many applications the gauge-independent formalism is used,

\(^4\) Of course preheating can go on when \( q(\tau) \ll 1 \). However, in that case the growth rate \( \sim q \) is small, and preheating is inefficient. In the two fields model the metric perturbations in the limit of small \( q \) have been discussed in the paper [7]. The similar one field model has been considered in the paper [8].
where all perturbed quantities are expressed in terms of their gauge-independent combinations. These combinations are naturally linked to some preferable coordinate system. The most commonly used preferable coordinate systems are so-called comoving and Newtonian coordinate systems. However, both these systems are not convenient in our case because the dynamical equations written in these systems have a singularity when $\phi_0 = 0$. Therefore we use a synchronous coordinate system, and to specify the gauge we assume that our synchronous system is comoving at the moment of time $\tau = \tau_m$. For the simplicity we also neglect the metric perturbations generated during inflation. In a synchronous coordinate system the metrics has the following form:

$$ds^2 = dt^2 - a^2(t)(A\delta_0^2 + B_{\alpha\beta})dx_\alpha dx^\beta,$$

where the Greek indices run from 1 to 3, and the summation rule is used hereafter. The prime stands for differentiation with respect to $x^\alpha$. The perturbed equations of motion are (e.g. [9,10]):

$$H\dot{h} = \dot{\phi}_0\delta\phi + \chi_0\delta\chi + \frac{\partial V}{\partial\phi}\delta\phi + \frac{\partial V}{\partial\chi}\delta\chi,$$

$$\dot{A} = -(\dot{\phi}_0\delta\phi + \chi_0\delta\chi),$$

$$\ddot{\phi}_i + 3H\dot{\phi}_i + \frac{\dot{\phi}_i}{2} + \frac{\partial^2 V}{\partial\phi_0\partial\phi_j}\delta\phi_j = 0,$$

where $h = 3A + \Delta B$, $i, j = 1, 2$, for definiteness index 1 stands for the fields $\phi_0$, $\delta\phi$, and 2 for the fields $\chi_0$, $\delta\chi$. We neglect the terms proportional to $k^2$ in the eqns (25 – 27), assuming that the wavenumbers of the perturbations are very small. The quantities $A$ and $B$ are related as (e.g. [10])

$$B = \int^t dt' \frac{dt'}{a^2} \int dt'' (aA).$$

As we discussed above the background field $\chi_0$ is very small, and does not contribute to the expansion law, and in that case the equations of motion of the fields perturbations (27) can be rewritten in a simplified manner. The equation for the field $\delta\chi$ is reduced to the eq. (5), and the equation for the field $\delta\phi$ has a form

$$\dddot{\phi} + (3H + \frac{\dot{\phi}_0^2}{2H})\dot{\phi} + \frac{\partial^2 V}{\partial\phi\partial\chi}\delta\chi = J(\phi, \chi_0, \delta\chi),$$

$$J(\phi, \chi_0, \delta\chi) = -\frac{\partial^2 V}{\partial\phi\partial\chi} + \frac{\dot{\phi}_0}{2H} \frac{\partial V}{\partial\chi}\delta\chi + \frac{\dot{\phi}_0\chi_0\delta\dot{\chi}}{2H}.$$  

In this approximation the fields $\chi_0$, $\delta\chi$ enter only in the source term $J(\phi, \chi_0, \delta\chi)$, and the formal solution of the eq. (29) can easily be obtained if the source term is known as a function of time $J(\phi(t), \chi_0(t), \delta\chi(t)) = J(t)$:

$$\delta\phi \sim \int^t d\xi J(\xi) \frac{\dot{\phi}_0(t)\phi_0(\xi)H}{a^3} \int^\xi \frac{d\xi'}{a^3H},$$

Then one can substitute the solution (31) and the expressions for $\chi_0(t)$ and $\delta\chi(t)$ into the eqns. (25), (26), and find the metric perturbations by integration. Unfortunately, this program is analytically very complicated. Even the numerical solution is not trivial for all possible parameters of the theory. Still, not everything is lost. At first we need to estimate only the upper limit on the metric perturbations, and therefore we can assume that both fields: the background field $\chi_0$ and the perturbation $\delta\chi$ grow with the maximal possible rate $\mu_{max}$. In this approximation

5Obviously, when approaching the time $t_*$ the back reaction of the produced $\chi$ “particles” influences the motion of our system much more significantly than the field $\chi_0$, and comparable with the contribution of the field $\phi_0(t)$. We cannot take into account the back reaction without a very significant complication of our consideration, but we hope that this cannot change order-of-magnitude estimates. Therefore, our results should be treated as semi-qualitative only.
at the time close to $\tau_*$ the growth rate is much larger than the expansion rate $H = \frac{2m^2}{\lambda + m^3}$, and we can take into account the source term (30) only during the last Hubble epoch before the end of preheating $\frac{\lambda^*}{m^2} \ll 1, \tau_* \gg 1$. Secondly, we should take into account only the leading terms in the expansion on powers of the small parameter $q^{-1}(\tau)$. Using the first approximation, we neglect the terms caused by the expansion of the Universe in the equation of motion of the field $\phi$, and write:

$$\dot{\phi}^2 + m^2 \phi = - \frac{\partial^2 V}{\partial \phi \partial \chi} \delta \chi = -2g^2 \phi \sin \frac{\tau}{\tau} \chi \delta \chi,$$

where the term $\chi \delta \chi$ can be written as:

$$\chi \delta \chi = \frac{x_s}{2\sqrt{2}(2\pi)^{3/2}} \frac{a_0}{a} \frac{3/2}{\omega_0} \omega^{1/2} (s_0^j + j \omega e^{-2i\theta} + j \omega e^{2i\theta}),$$

where

$$s_0^j = (2n_j + 1)e^{i\phi_0} - 2a_j \beta \omega e^{-i\phi_0}, \quad j^2 = \beta (\alpha^* \omega e^{i\phi_0} - \beta \omega e^{-i\phi_0}), \quad s_0^j = \alpha (\beta^* \omega e^{i\phi_0} - \alpha \omega e^{-i\phi_0}),$$

and the functions $\alpha_j, \beta_j$ correspond to the period of time $\tau_j < \tau < \tau_{j+1}$. In the same approximation we can set $a = a(t_*) = a_*$. In this case during the time period $\tau_{j-1} < \tau < \tau_j$ the equation (32) is just an oscillator equation with a source, represented by a sum of a constant and an oscillating parts. Assuming $q(t) \gg 1$, it can be easily seen that a partial solution determined by the oscillating parts is small, and the partial solution of the eq. (32) $\delta \phi_p$ is close to a constant:

$$\delta \phi_p \approx K (-1)^j s_0^j,$$

where

$$K = \frac{i(g/m)^2 g^{-1/2} (a_0/a_s)^{3/2}}{\sqrt{2} (2\pi a_s)^{3/2}} \frac{\chi_s}{\omega_0},$$

and we approximate $\omega_0 \sim g$. The full solution can be found from the continuity conditions at the time moments $\tau_j$, and has a form:

$$\delta \phi = K (-1)^j s_0^j + \sum_{l=0}^{l=j} c_l \cos \tau,$$

where

$$c_l = -(s_0^j + s_0^{j-1}).$$

Now we substitute the solutions (4), (36) into the eq. (26), and by integrating the result obtain the perturbation of the scale factor $A$. Note that the term $\chi_0 \delta \chi < O(\tau_{\phi} \frac{4}{\tau} \phi_0 \delta \phi)$ is not taken into account. It can be said that the perturbation of the matter field $\delta \chi$ induces the metric perturbation not directly, but through the excitation of the inflaton perturbation $\delta \phi$. We have:

$$A \approx -\frac{\dot{\phi}_0 K}{\tau_*} e^{2\mu_{max} \tau} R(\tau),$$

where the function of time

$$R(\tau) = \{(-1)^j s_0^j \sin \tau + \frac{1}{2} \sum_{l=0}^{l=j} c_l (\tau - \tau_l + \frac{\sin 2\tau}{2}) \} e^{-2\mu_{max} \tau},$$

is constrained by the condition $|R(\tau)| < O(1)$. The coefficient $B$ is additionally damped with respect to $A$ in the long wave limit, and is not calculated here. The vacuum expectation value for $A^2$ has a form:

$$\langle A^2 \rangle \approx \frac{4\pi \phi_0^2 |K|^2 |R(\tau)|^2}{\tau_*^2} e^{4\mu_{max} \tau} \int k^3 d\ln k \approx \frac{2}{3\pi^2} \frac{g}{m} \frac{|R(\tau)|^2}{\tau_*^2} m^2 e^{-3N + 4\mu_{max} \tau} \int e^{-3n} dn,$$
where \( n = \ln \left( \frac{2\mu_{\text{eff}}}{k} \right) \) is e-folds number, corresponding to the wavenumber \( k \), and we assume that \( \chi_0 \sim \frac{2}{3} e^{-3N/2} \). Let us remind that for the modes with wavelengths of order of the present horizon scale \( n \approx N \approx 50 \). The r.m.s value of the metric perturbation

\[
A_{r.m.s.}(n) \approx \sqrt{\frac{2}{3\pi^2}} \left( \frac{g}{m} \right)^{1/2} \left| \frac{R(\tau)}{\tau^3} \right| m e^{-3/2(N+n)+2\mu_{\text{max}}\tau_*},
\]

is even much smaller than the estimate (2) due to very small factor \(< 10^{-9}\) in the front of the exponent in the eq. (40).

Thus we conclude that in our case the large-scale metric perturbation induced due to the coupling between the matter field perturbations \( \delta \chi \) and the background part of the field \( \chi_0 \) is absolutely negligible.

There is another source of unavoidable metric perturbations induced by fluctuations of the energy density and the pressure of the condensate of the \( \chi \) “particles”. These fluctuations are non-linear, and therefore give rise to the metric perturbations even in the absence of the classical field \( \chi_0 \). The calculation of this effect is even more complicated problem than the calculations in the linear theory, but it is possible again to roughly estimate the characteristic order of magnitude. Similar to the linear case, there is no hope to find some imprints of this effect at super-large scales, but it might play an important role right before the end of preheating, at the horizon scale \( \lambda_0 = H^{-1}(t_*) = \frac{3}{2} \tau_* m^{-1} \). The relatively large metric perturbations at this scale (say, with r.m.s amplitude \( \sim 10^{-2} \div 10^{-1} \)) might lead to the copious production of primordial black holes, and modify the evolution of the Universe right after the end of the preheating stage.\(^6\) An overproduction of the primordial black holes might constrain the parameters of our model.

To characterize the energy density fluctuations at some comoving scale \( \sigma \), we introduce coarse-grained energy density operator:

\[
\hat{\epsilon}_{c.g} = \int d^3 y G_{\sigma}(x-y) \hat{\epsilon}(y),
\]

where

\[
\hat{\epsilon} = \frac{1}{2} \left( \dot{\chi}^2 + g^2 \phi_0^2 \chi^2 \right)
\]

is the operator of the energy density of the \( \chi \) field, and the Gaussian window function

\[
G_{\sigma}(\vec{r}) = (2\pi)^{-3/2} \sigma^{-3} e^{-\frac{\vec{r}^2}{2\sigma^2}}.
\]

Consider the relative standard deviation

\[
\delta_\epsilon(\sigma) = \sqrt{\frac{\langle \hat{\epsilon}_{c.g}^2 \rangle(\sigma)}{\langle \hat{\epsilon}^2 \rangle} - 1}.
\]

The calculation of this quantity can be easily done in two limiting cases. Let introduce the wavenumber \( k_\sigma = \sigma^{-1} \) corresponding to the scale \( \sigma \), and also the wavenumber

\[
k_0 = \frac{k_{\text{crit}}}{(2\mu_{\text{eff}}f \tau)^{1/6}} = \sqrt{\frac{2}{\pi}} \left( \frac{q_0}{(2\mu_{\text{eff}}f)} \right)^{1/6} \tau_0 m,
\]

corresponding to the modes which have been amplified during all first stage of preheating: \( k_0 \sim k_{\text{crit}}(\tau_{\text{in}}) \), and the wavenumber corresponding to the horizon scale:

\[
k_h(\tau) = a(\tau) H(\tau) = \frac{2 a_0 m}{3 \tau^{1/3}}.
\]

\(^6\) When calculating the abundance of primordial black holes, one should take into account that the metric perturbations are non-Gaussian in that case. This can lead to a modification of the standard estimates, based on assumption of Gaussian statistics for the perturbations.
Let also average the quantity (42) over many oscillations of the field $\chi$. Then, if $k_0 < k_\sigma < k_{\text{crit}}$, the deviation $\delta_\epsilon$ is close to 1: $\delta_\epsilon \approx 1$. If $k_\sigma < k_0$, we have

$$\delta_\epsilon^2 \approx \frac{3}{2\sqrt{2}} \left( \frac{k_\sigma}{k_0} \right)^3.$$  

The eq. (43) can also be rewritten in terms of ratio $k_\sigma/k_h$:

$$\delta_\epsilon \approx D \left( \frac{k_\sigma}{k_h} \right)^{3/2},$$  

where

$$D = \frac{\pi^{3/4}}{3} \frac{(2\mu_\text{eff})^{1/4}}{q_0^{3/8} \sqrt{\tau_2}} \approx 2 \cdot 10^{-3} \frac{\mu_0.14^{1/4}}{q_4^{3/8} \sqrt{\tau_2}},$$  

and $\tau_2 = \frac{\tau_1}{40}$. For the perturbations with wavelengths smaller than the horizon wavelength, the metric perturbation $A$ is related to the energy density perturbation via Poisson equation: $\frac{\Delta A}{\Omega^2} \approx -\delta_\epsilon$. Assuming that $\delta_\epsilon \sim \delta_\epsilon (\hat{\epsilon})$, and also that at the end of the first stage of preheating the energy density of $\chi$ “particles” influences the expansion law: $H^2 \sim \langle \hat{\epsilon} \rangle$, we have:

$$A(k_\sigma) \sim \left( \frac{k_h(\tau_*)}{k_\sigma} \right)^2 \delta_\epsilon = D \left( \frac{k_h(\tau_*)}{k_\sigma} \right)^{1/2},$$  

for $k_h(\tau_*) < k_\sigma < k_0$. For super-horizon perturbations the dependence of the perturbations on $k$ should be the same as a dependence of a source of the perturbations, and

$$A(k_\sigma) \sim \delta_\epsilon = D \left( \frac{k_\sigma}{k_h(\tau_*)} \right)^{3/2}.$$  

Thus, the characteristic metric perturbation induced by the fluctuations of the energy density takes its maximal value at the horizon scale, and this value is smaller than $\sim 10^{-3}$ if $q_0 > 10^4$.

Similar to the case of linear perturbation, we did not take into account the contribution of $\chi$ ‘particles’ in the dynamics of background model when calculating the amplitude of the metric perturbations induced by the non-linear terms. However, this contribution can change the parameters of the background model (such as, e.g. the expansion rate) only on factor of order two, and we believe this modification cannot change significantly our estimates.  

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Recently, this question as well as the generation of the linear perturbations in the same model has been considered numerically in the paper [11]. The authors came to the similar conclusion that the perturbations of both types are very small at the large scales. I am very grateful to Dr. K. Jedamzik for drawing my attention to that paper.
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