Vacuum material properties and Cherenkov radiation in logarithmic electrodynamics

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Abstract We study some observational signatures of non-linearities of the electromagnetic field. First to all we show the vital role played by nonlinearities in triggering a material behavior of the vacuum with \((\varepsilon > 0, \mu < 0)\), which corresponds to a ferrimagnetic material. Secondly, the permittivity and susceptibility induced by nonlinearities are considered in order to obtain the refractive index via the dispersion relation for logarithmic electrodynamics. Finally, we consider the electromagnetic radiation produced by a moving charged particle interacting with a medium characterized by nonlinearities of the electromagnetic field. To this end we consider logarithmic electrodynamics. The result shows that the radiation is driven by the medium through which the particle travels like the one that happens in the Cherenkov effect.

1 Introduction

The physical manifestations of vacuum electromagnetic non-linearities have been a fascinating topic of research since the discovery by Euler and Heisenberg [1] of a striking prediction of quantum electrodynamics (QED), that is, the light-by-light scattering arising from the interaction of photons with virtual electron-positron pairs. As is well known, the physical consequences of this crucial finding, such as vacuum birefringence and vacuum dichroism, have been largely considered from different points of view [2–5]. However, despite remarkable progress [6–11], this prediction has not yet been confirmed.

It is appropriate to remark, in this context, that recently the ATLAS and CMS collaborations at the Large Hadron Collider (LHC) have reported on the high energy gamma-gamma pair emission from virtual gamma-gamma scattering in ultra-peripheral Pb-Pb collisions [12,13]. However, as emphasized in [16], in these results there is no modification of the optical properties of the vacuum. In addition, the coming of laser facilities has given rise to various proposals to probe quantum vacuum nonlinearities [14,15]. An interesting example is provided by the experiment (DeLLight project) [16], which exploits the change in the index of refraction due to nonlinear electrodynamics.

In this connection, it may be recalled that different nonlinear electrodynamics of the vacuum may have significant contributions to photon-photon scattering such as Born-Infeld [17] and Lee-Wick [18] theories. As is well known, these electrodynamics were introduced in order to avoid the divergences inherent in the Maxwell theory at short distances.

With these ideas in mind, in previous works [19–22], we have considered the physical effects presented by different models of \((3 + 1)\)-D nonlinear electrodynamics in vacuum. Evidently, this has helped us to gain insights into the peculiarities of quantum vacuum nonlinearities in different contexts. For example, the Generalized Born-Infeld, and Logarithmic Electrodynamics the field energy of a point-like charge is finite, which also exhibit the vacuum birefringence phenomenon. As well as we have studied the lowest-order modifications of the static potential within the framework of the gauge-invariant but path-dependent variables formalism, which is an alternative to the Wilson loop approach.

We further note that recently an interesting study on vacuum Cherenkov radiation in Euler–Heisenberg-like nonlinear electrodynamics has been considered in [23]. As is well known, a charged particle moving in a medium under an external electromagnetic field emits Cherenkov radiation when its velocity of light exceeds the phase velocity in that medium.

It is worth recalling, at this stage, that any variation of the velocity of light with respect to \(c = 1/\sqrt{\varepsilon_0 \mu_0}\), where \(\varepsilon_0\) and \(\mu_0\) are the vacuum permittivity and the vacuum permeability respectively, is due to that light propagates in a medium.

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In this manner, we have to introduce the constants \( \varepsilon \) and \( \mu \) which characterize the medium. As is well known, the velocity of light in a medium is less than that of light in vacuum by a factor (index of refraction) \( n = \sqrt{\frac{\varepsilon \mu}{\varepsilon_0 \mu_0}} \). From the previous remark it follows that both \( \varepsilon \) and \( \mu \) are positives. Nevertheless, as was first hypothesized in \([24]\), considerable attention has been paid recently to the \( \varepsilon < 0 \) and \( \mu < 0 \) case. The interest in studying this case is mainly due to the laboratory construction of an exotic form of dielectric, metamaterial, with both \( \varepsilon \) and \( \mu \) negatives. As it offers a valuable observational window on the constitutive parameters \( \varepsilon \) and \( \mu \), electrodynamics with metamaterial features has stimulated lots of experimental works \([25]\).

In this context it is particularly important to notice that, as emphasized in \([25]\), the four possible sign combinations in the pair \((\varepsilon, \mu)\) are \((+, +), (+, -), (-, -)\) and \((-,-)\). Evidently, this last combination corresponds to Veselago’s materials. Thus, in this work we focus on the interesting possibility if non-linearities can induce any of the above combinations, excepting the first one. In other words, one of our goals is to understand what might be another observational signature of nonlinearities.

Another of our goals is devoted to study the stability of the above radiation scenario in the presence of other nonlinear electrodynamics. Of special interest will be to check the effects of the ”medium” on the production of this radiation. To do this, we will work out the radiated energy for logarithmic electrodynamics following the conventional path of calculating the Poynting vector. As we will see, our analysis renders manifest the vital role played by vacuum electromagnetic nonlinearities in triggering the radiated energy for logarithmic electrodynamics.

Our work is organized as follows. In Sect. 2, we describe the vital role played by nonlinearities in triggering a material with \((\varepsilon > 0, \mu < 0)\), which corresponds to a ferrimagnetic material. Subsequently, the permittivity and susceptibility induced by nonlinearities are considered in order to obtain the refractive index via the dispersion relation for logarithmic electrodynamics. This would not only provide the theoretical setup for our subsequent work, but also fix the notation. In Sect. 3, we consider the calculation of the electromagnetic radiation. A summary of our work is the subject of Sect. 4. In our conventions the signature of the metric is \((+1, -1, -1, -1)\).

For this purpose we begin by considering a generic Lagrangian density:

\[
\mathcal{L} = \mathcal{L}(\mathcal{F}, \mathcal{G}),
\]

where the arguments of \( \mathcal{L} \) are the usual electromagnetic field invariants, that is, \( \mathcal{F} \equiv -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} \left( \frac{E^2}{c^2} - B^2 \right) \) and \( \mathcal{G} \equiv -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{E}{c} \cdot B \).

Next, after splitting \( F^{\mu\nu} \) in the sum of a classical background, \( F_B^{\mu\nu} \), and a small fluctuation, \( f^{\mu\nu} \), the corresponding linearized field equations read

\[
\partial_\mu \left( C_1 f^{\mu\nu} + C_2 \tilde{f}^{\mu\nu} \right) - \frac{1}{2} \partial_\mu \left( k_B^{\mu\nu\kappa\lambda} f_{\kappa\lambda} + t_B^{\mu\nu\kappa\lambda} \tilde{f}_{\kappa\lambda} \right) - \frac{1}{4} \partial_\mu \left( \varepsilon^{\mu\nu\kappa\lambda} t_B^{\kappa\lambda\rho\sigma} f^{\rho\sigma} \right) = -\partial_\mu \left( C_1 F_B^{\mu\nu} + C_2 \tilde{F}_B^{\mu\nu} \right) + j^{\mu},
\]

where \( k_B^{\mu\nu\kappa\lambda} = D_1 F_B^{\mu\nu} F_B^{\kappa\lambda} + D_2 \tilde{F}_B^{\mu\nu} \tilde{F}_B^{\kappa\lambda} \) and \( t_B^{\mu\nu\kappa\lambda} = D_3 F_B^{\mu\nu} F_B^{\kappa\lambda} \).

Whereas \( C_1 = \frac{\varepsilon B^2}{\varepsilon c^2} \bigg|_B, C_2 = \frac{\varepsilon B^2}{\mu c^2} \bigg|_B, D_1 = \frac{\varepsilon^2 B^2}{\mu c^2} \bigg|_B, D_2 = \frac{\varepsilon^2 B^2}{\varepsilon c^2} \bigg|_B \) and \( D_3 = \frac{\varepsilon^2 B^2}{\mu c^2} \bigg|_B \).

At this stage, we are not bound to consider a constant and uniform electromagnetic background, so that the coefficients \( C_1, C_2, D_1, D_2, D_3 \) are, in principle, space-time dependent. This is why the field equations take the form of Eq. (2). However, in what follows, we adopt space-time constancy of the background, so that the coefficients above are not acted upon by the space-time derivatives, giving rise to the constitutive tensors to be present soon below.

However, in what follows we will write the equations of motion in the case \( j^{\mu} = 0 \), in the presence of a constant background with both electric and magnetic fields \((\mathbf{E}, \mathbf{B})\). We thus find

\[
\nabla \cdot \mathbf{d} = 0,
\]

where \( d_i = \varepsilon_{ij} e_j + \xi_{ij} b_j \).

Whereas, \( \varepsilon_{ij} \) and \( \xi_{ij} \) are given by

\[
\varepsilon_{ij} = \delta_{ij} + \alpha_i E_j + \beta_i B_j,
\]

and

\[
\xi_{ij} = -c^2 \alpha_i B_j + \beta_i E_j,
\]

here we have used the notation \( \alpha \equiv \frac{1}{c^2} \left( D_1 B + D_2 E \right) \) and \( \beta \equiv \frac{1}{c^2} \left( D_2 B + D_1 E \right) \). Note that the tensors \( \varepsilon \) and \( \xi \) are completely determined by the electromagnetic background. Throughout, \( \mathbf{e} \) and \( \mathbf{b} \) are the electric and magnetic fields arising from the fluctuation \( f^{\mu\nu} \).
On can now further observe that
\[ \nabla \times \mathbf{h} = \frac{1}{c^2} \frac{\partial}{\partial t} \mathbf{e}, \]  
(6)
where \( h_i \equiv \mu^{-1}_{ij} b_j + \eta_{ij} e_j \).

In this case, \( \mu^{-1}_{ij} \) and \( \eta_{ij} \) are given by
\[ \mu^{-1}_{ij} \equiv \delta_{ij} - B_i \gamma_j - E_i \Delta_j, \]  
(7)
and
\[ \eta_{ij} \equiv B_i \alpha_j - \frac{1}{c^2} E_i \beta_j = -\frac{1}{c^2} \xi_{ij}, \]  
(8)
where we have defined \( \mathbf{\gamma} \equiv \frac{1}{C_1} (D_1 \mathbf{B} - D_3 \mathbf{E}) \) and \( \mathbf{\Delta} \equiv \frac{1}{C_1} \left( -\frac{D_2}{C_1} \mathbf{B} + \frac{D_3}{C_1} \mathbf{E} \right) \).

Incidentally, we would like to point out that, in a recent paper [26], the authors carry out a detailed study of bi-isotropic and bi-anisotropic material media in terms of a general class of constitutive tensors. In our case, we stress that it is the vacuum—subject to strong external electromagnetic fields—that acts as the material medium, with permittivity and permeability tensors completely determined by the electromagnetic background, according to the equations cast previously. In the particular cases of a purely electric or a purely magnetic external field, the constitutive tensors naturally arise as symmetric 3 \times 3-matrices.

One can easily verify that for an external field \( \mathbf{B} (\mathbf{E} = 0) \) we have \( \alpha_i = \frac{D_1}{C_1} B_i, \beta_i = \frac{D_2}{C_1} B_i, \gamma_i = \frac{D_3}{C_1} B_i \) and \( \Delta_i = -\frac{D_1}{C_1} B_i = -\alpha_i. \) We thus find that,
\[ \varepsilon_{ij} = \delta_{ij} + \frac{D_2}{C_1} B_i B_j, \]  
(9)
\[ \xi_{ij} = -\frac{D_3}{C_1} B_i B_j, \]  
(10)
\[ \mu^{-1}_{ij} = \delta_{ij} - \frac{D_1}{C_1} B_i B_j, \]  
(11)
and
\[ \eta_{ij} = \frac{D_3}{C_1} B_i B_j = -\frac{1}{c^2} \xi_{ij}. \]  
(12)

By defining the matrix \( \mathbf{B}_{ij} \equiv B_i B_j, \) which is a symmetric one with eigenvalues 0, 0, and \( \mathbf{B}^2, \) the expressions for \( d \) and \( h, \) in matrix notation, become
\[ d = e + \frac{D_2}{C_1} \mathbf{B} e - c \frac{D_3}{C_1} \mathbf{B} b, \]  
(13)
\[ h = b - \frac{D_1}{C_1} \mathbf{B} b + \frac{D_3}{c C_1} \mathbf{B} e. \]  
(14)

Whereas
\[ \nabla \cdot \mathbf{d} = 0, \]  
(15)
\[ \nabla \times \mathbf{h} = \frac{1}{c^2} \frac{\partial}{\partial t} \mathbf{d}. \]  
(16)

After the matrix \( \mathbf{B} \) is diagonalized, we then have
\[ \partial_i d_i = 0, \]  
(17)
\[ \varepsilon_{ijk} \partial_j h_k = \frac{1}{c^2} \frac{\partial}{\partial t} d_i, \]  
(18)

where the matrix \( \mathbf{B} = \begin{pmatrix} B^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \).

An immediate consequence of this result is that \( d_\varepsilon = \left( 1 + \frac{D_2}{C_1} B^2 \right) e_\varepsilon - c \frac{D_3}{C_1} \mathbf{B}^2 b_\varepsilon, d_y = e_y \) and \( d_z = e_z. \) We also find that \( h_\varepsilon = \left( 1 - \frac{D_2}{C_1} B^2 \right) b_\varepsilon + \frac{D_1}{C_1} \mathbf{B}^2 e_\varepsilon, h_y = b_y \) and \( h_z = b_z. \)

This leads to the following expressions for \( \varepsilon_{xx} \) and \( \mu_{xx}, \)
\[ \varepsilon_{xx} = 1 + \frac{D_2}{C_1} B^2, \]  
(19)
and
\[ \mu_{xx}^{-1} = 1 - \frac{D_1}{C_1} \mathbf{B}^2. \]  
(20)

It should be further noted that, in principle, the positivity of the above expressions is not assured. However, from Eq. (20) it is evident that
\[ \mu_{xx} = \frac{1}{1 - \frac{D_1}{C_1} \mathbf{B}^2}, \]  
(21)
this then implies that \( \mu_{xx} < 0 \) if \( \frac{D_1}{C_1} \mathbf{B}^2 > 1. \) We see, therefore, a remarkable feature of nonlinearities of the electromagnetic field.

The preceding considerations clearly show that the nonlinearities induce the second combination mentioned above (\( \varepsilon > 0, \mu < 0, \)), which corresponds to a ferrimagnetic material in the classification given in [25].

So far our treatment is completely general. However, for the specific case of logarithmic electrodynamics, the permittivity and susceptibility induced by nonlinearities will be considered in the next subsection.

Indeed, in this specific case, as we shall explicitly compute, \( \varepsilon_{xx}, \) as given by the Eq. (19) above, comes out positive-definite, \( \varepsilon_{xx} > 1. \) On the other hand, \( \mu_{xx} \) may become negative for a range of external magnetic fields. In so being, the refractive index, \( n, \) is purely imaginary, \( n^2 < 0. \) Therefore,
from $n = \frac{|k|}{w}$, $w^2 < 0$; consequently, $\varepsilon_{xx} > 0$ with $\mu_{xx} < 0$ corresponds to an evanescent wave propagation.

2.2 Some features of logarithmic electrodynamics

We now proceed to explore other relevant aspects on nonlinearities. Let us commence our undertaking by considering logarithmic electrodynamics [20]:

$$\mathcal{L} = -\beta^2 \ln \left[ 1 - \frac{\mathcal{F}}{\beta^2} - \frac{\mathcal{G}^2}{2\beta^4} \right].$$

(22)

recalling again that $\mathcal{F} = \frac{1}{2} \left( \frac{\mathcal{E}^2}{c^2} - \mathbf{B}^2 \right)$ and $\mathcal{G} = \frac{\mathcal{E}}{c} \cdot \mathbf{B}$. We further note that the parameter $\beta$ measures the nonlinearity of the theory and in the limit $\beta \to \infty$ the Lagrangian (22) reduces to the Maxwell theory.

As already stated, in this subsection we will be mainly interested in the dispersion relations for the electrodynamics under consideration. The first step in this direction is to consider a generic Lagrangian density $\mathcal{L} = \mathcal{L}(\mathcal{F}, \mathcal{G})$, in the presence of a constant background with both $(\mathbf{E}, \mathbf{B})$. As before, after splitting $F^{\mu \nu}$ in the sum of a classical background $F_{B}^{\mu \nu}$, and a small fluctuation, $f^{\mu \nu}$, the corresponding linearized equations of motion read

$$\nabla \cdot d = \partial_i d_i = \varepsilon_{ij} \partial_i j + \xi_{ij} \partial_i b_j = 0,$$

$$\nabla \times e = -\frac{\partial}{\partial t} \mathbf{b} \Rightarrow \varepsilon_{ijk} \partial_j e_k = -\frac{\partial}{\partial t} b_i,$$

$$\nabla \cdot b = 0,$$

$$\nabla \times h = \frac{1}{c^2} \frac{\partial}{\partial t} e$$

$$\Rightarrow \varepsilon_{ijk} \mu_{kl}^{-1} \partial_i b_l + \varepsilon_{ijk} \nu_{kl}^{-1} \partial_j e_i = \frac{1}{c^2} \varepsilon_{ij} \partial_i e_j + \frac{1}{c^2} \xi_{ij} \partial_i b_j.$$

(23)

Throughout, $\mathbf{e}$ and $\mathbf{b}$ are the electric and magnetic fields arising from the fluctuation $f^{\mu \nu}$.

By considering the plane waves

$$e_i = e_{0i} e^{i(k \cdot x - wt)}, \quad b_i = b_{0i} e^{i(k \cdot x - wt)},$$

(24)

from the Eq. (23) it follows that

$$M_{in}(w, k; \mathbf{E}_B, \mathbf{B}_B) e_n = 0,$$

(25)

where

$$M_{in} = \frac{w^2}{c^2} \varepsilon_{in} + \frac{w}{c^2} \xi_{ij} \varepsilon_{lmn} \mu_{kl}^{-1} k_m k_l + w \varepsilon_{ijk} \eta_{kl} k_j.$$  

(26)

in which we have used

$$\varepsilon_{ij} = \delta_{ij} + \alpha_i E_j + \beta_i B_j,$$

(27)

$$\xi_{ij} = -c^2 \alpha_i B_j + \beta_i E_j,$$

(28)

$$\mu_{ij}^{-1} = \delta_{ij} - B_i \gamma_j - E_i \Delta_j,$$

(29)

$$\eta_{ij} = -\frac{1}{c^2} \xi_{ji} = \alpha_j B_i - \frac{1}{c^2} \beta_j E_i.$$  

(30)

In passing we recall that $\alpha_i, \beta_i, \gamma_i$ and $\Delta_i$ are given in terms of $C_1, D_1, D_2$ and $D_3$.

It is of interest to note that in the particular case of an external electric field $\mathbf{E} = 0$, Eq. (25) takes the particularly simple form

$$\left( \frac{w^2}{c^2} \varepsilon_{in} + \varepsilon_{ijk} \varepsilon_{lmn} \mu_{kl}^{-1} k_m k_l \right) e_n = 0.$$  

(31)

Here we have used that $D_3 = 0$ ($\mathcal{G}_{\text{background}} = 0$), whereas $\xi_{ij} = 0$ and $\eta_{ij} = 0$. We accordingly express Eq. (31) in the form

$$\left[ \frac{w^2}{c^2} \varepsilon_{in} + \varepsilon_{ijk} \varepsilon_{lmn} \mu_{kl}^{-1} k_m k_l \right] e_n = 0,$$

(32)

where

$$\varepsilon_{in} = \delta_{in} + D_3 \frac{C_1}{C_1} B_i B_n \equiv \varepsilon_{ni},$$

(33)

$$\mu_{ij}^{-1} = \delta_{ij} - D_1 \frac{C_1}{C_1} B_i B_j \equiv \mu_{ji}^{-1},$$

(34)

$$tr \mu^{-1} = 3 - D_1 \frac{C_1}{C_1} \mathbf{B}^2.$$  

(35)

Making use of these relations, we find that Eq. (32) reduces to

$$\left[ \frac{w^2}{c^2} \mu_{mi} \varepsilon_{in} + k^2 \delta_{mn} - k_m k_n + \left( tr \mu^{-1} \right) k_i k_m \right] e_n = 0,$$

(36)

where $\mu_{ij} = \delta_{ij} + \frac{D_1}{1 - \frac{D_1}{C_1} B_i B_j} B_i B_j$.

It is also important to observe that in the configuration space we have $w = i \partial_t$, $k_i = -i \partial_i$ and $\mathbf{k}^2 = -\nabla^2$. Hence, we readily verify that Eq. (36) can be brought to the form

$$\left[ -\frac{1}{c^2} \delta^2 \mu_{mi} \varepsilon_{in} - \delta_{mn} \nabla^2 \varepsilon_{in} + \left( tr \mu^{-1} \right) \mu_{mi} \partial_i \partial_n \right] e_n = 0.$$  

(37)
Thus, finally we end up with
\[
\left[ \mu_{m} \varepsilon_{m} \frac{1}{c^2} \partial_t^2 - \left( \mu_{mn} \varepsilon_{r} \mu^{-1} - \delta_{mn} \right) \nabla^2 - \partial_m \partial_n \right] \\
+ \left( tr \mu^{-1} \right) \mu_{m} \partial_t \partial_n + \left( \mu_{mn} \varepsilon_{i} \mu^{-1} - \mu_{mi} \mu^{-1} \right) \partial_i \partial_j \\
\times \varepsilon_{n} = 0.
\] (38)

Before proceeding our analysis of the dispersion relation, we call attention to the fact that
\[
\varepsilon_{n} = \frac{1}{2} \mu_{n} \varepsilon_{r} \mu^{-1} - \delta_{nn}.
\]

Restricting our considerations to the \( E = 0 \) case, we have \( F = -\frac{1}{2} B^2 \) and \( G = 0 \). We thus find
\[
C_1 = \frac{1}{\beta^2}, \quad C_2 = 0, \quad C_3 = \frac{1}{\beta^2}, \quad D_1 = \frac{1}{\beta^2} B, \quad D_2 = \frac{1}{\beta^2} C, \quad D_3 = 0.
\]

From this equation it is clear that the determinant has the form
\[
\det \left[ \begin{array}{c}
\hat{\mathbf{F}} \\
\hat{\mathbf{G}} \\
\hat{\mathbf{H}}
\end{array} \right] = 0.
\]

For logarithmic electrodynamics this eigenvalue reduces to \( 1 + B^2 / \beta^2 \). Similarly, from \( \mu_{ij} = \delta_{ij} + \frac{D_{1}}{C_1} B_i B_j \), we again have two eigenvalues 1 and \( 1 + \frac{D_{2}}{C_1} B^2 \).

Taking the ratios \( D_1 / C_1 \) and \( D_2 / C_1 \) given above to \( \varepsilon_{xx} \) and \( \mu_{xx} \) of Eqs. (19) and (21), respectively, we get:
\[
\varepsilon_{xx} = 1 + \frac{B^2}{\beta^2} > 1, \quad \mu_{xx} = \frac{2 \beta^2 + B^2}{2 \beta^2 - B^2}.
\]

\( \mu_{xx} < 0 \) whenever \( |B| > \sqrt{2} \beta \). As anticipated at the end of subsection (IIA), \( |B| > \sqrt{2} \beta \) yields a complex refractive index \( n \), and the electromagnetic signal becomes an evanescent wave. Consequently, in this situation the vacuum mimics a ferrimagnetic material.

Next, by making use of \( \varepsilon_{in}, \mu_{in}^{-1} \) and \( tr \mu^{-1} \) in the dispersion matrix \( (M_{in}) \), we can write the corresponding matrix as:
\[
M_{in} = \left( \frac{w^2}{c^2} - k^2 \right) \delta_{in} + \frac{D_1}{C_1} \left( B^2 k^2 - (B \cdot k)^2 \right) \delta_{in} \\
+ \left( \frac{D_2}{C_1} \right) k_i k_n + \left( 1 - \frac{D_1}{C_1} B^2 \right) k_i k_n.
\]

We are now in position to examine the condition, \( \det M = 0 \), in order to obtain the dispersion relations. We also recall that the index of refraction is given by \( n \equiv \frac{k \ell}{w} \), and after some manipulations, it follows that
\[
M_{in} = \left[ 1 + \frac{D_1}{C_1} \left( B^2 - (B \cdot \hat{k})^2 \right) \right] \delta_{in} \\
+ \left( \frac{D_2}{C_1} - \frac{D_1}{C_1} B_i B_n + \left( 1 - \frac{D_1}{C_1} B^2 \right) \right) \\
\times \hat{k}_i \hat{k}_n + \frac{D_1}{C_1} \left( B \cdot \hat{k} \right) \left( \hat{k}_i B_n + \hat{k}_n B_i \right).
\]

where \( \hat{k} \equiv \frac{k}{|k|} \).

This last expression clearly shows that the \( M \)-matrix does not depend on \( |k| = 2 \pi / \lambda \). Then the refractive index, \( n \), arising from the condition, \( \det M = 0 \), does not depend on \( \lambda \), but of the relative direction between the propagation vector \( \hat{k} \) and the external field. In this manner, we obtain an effective refractive index \( n \equiv n \left( \mathbf{B}, \hat{k} \right) \).

To further elaborate on the comparative features of the index of refraction, we shall examine two different situations. First, we consider \( n \) if \( \hat{k} \perp B \). In this case, the condition, \( \det M = 0 \), reads
\[
\det \left[ \begin{array}{c}
\left( 1 - \frac{D_1}{C_1} B^2 \right) \\
\left( 1 - \frac{D_1}{C_1} B^2 \right) - \frac{D_1}{C_1} B_i B_n \end{array} \right] = 0.
\]

From this equation it is clear that the determinant has the form \( \det \left( a \delta_{ij} + b u_i u_j + c v_i v_j \right) \), whose solution is given by \( a \left[ (a + bu^2) (a + cv^2) - bc (u - v)^2 \right] \). It is a simple matter to verify that the condition, \( \det M = 0 \), becomes
\[
\frac{1}{n^2} \left[ \frac{1}{n^2} - 1 + \frac{D_1}{C_1} B^2 \right] = 0.
\]

Thus, we finally obtain two modes associated to the direction of propagation \( \hat{k} \), that is,
\[
n_1^2 = \frac{1}{1 - \frac{D_1}{C_1} B^2}, \quad n_2^2 = 1 + \frac{D_2}{C_1} B^2.
\]
For logarithmic electrodynamics we have $D_t = C_1 \beta^2 = \frac{1}{\beta^2 + \frac{B^2}{2}}$, and $D_2 = \frac{1}{\beta^2}$. Hence we see that the two modes take the form

$$n^2_\perp = \frac{\beta^2 + \frac{B^2}{2}}{\beta^2 - \frac{B^2}{2}}, \quad (48)$$

and

$$n^2_\parallel = 1 + \frac{B^2}{\beta^2}. \quad (49)$$

Second, we consider $n_\parallel$ if $\hat{k} \parallel \mathbf{B}$. By using $\mathbf{B} = \xi |\mathbf{B}| \hat{k}$ and $\mathbf{B} \cdot \hat{k} = \xi |\mathbf{B}|$, where $\xi = \pm 1$ stands parallel or anti-parallel to the propagation direction. After some manipulations the condition $\det M = 0$ becomes

$$\frac{1}{n^2} \left( \frac{1}{n^2} - 1 \right)^2 \left( 1 + \frac{D_2}{C_1} B^2 \right) = 0. \quad (50)$$

In this case, the corresponding mode associated to the direction of propagation $\mathbf{k}$ becomes $n_\parallel = 1$.

In summary then, we easily verify that the previous electromagnetic vacuum acts like a birefringent medium with two indices of refraction determined by the relative direction between the propagation vector $\hat{k}$ and the external field. More recently, this has also helped us to gain insights into the peculiarities about vacuum nonlinearities such as calculating the bending of light [27].

It should, however, be noted here that Eqs. (48) and (49) express vacuum birefringence in the situation the wave propagation is orthogonal to the external magnetic field. In the next section, we shall work out the expressions for the power of the Cherenkov radiation in either case of refractive index. On the other hand, the situation in which $\mathbf{k}$ is parallel to $\mathbf{B}$, Eq. (50) corresponds to $n = n_\parallel = 1$; therefore, in this particular case, there is no re-emitted wave as Cherenkov radiation in the direction of the external magnetic field.

### 3 Electromagnetic radiation

As already mentioned, our immediate objective is to compute the electromagnetic radiation produced by a moving charged particle interacting with a medium characterized by nonlinearities of the electromagnetic field. With this in view, the starting point are the Maxwell equations for a moving charged particle in a medium characterized by logarithmic electrodynamics:

$$\nabla \cdot \mathbf{E} = \frac{4\pi}{c} \rho_{\text{ext}},$$

$$\nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{E} = \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \times \mathbf{B} = \frac{4\pi \mu}{c} \mathbf{j}_{\text{ext}}. \quad (51)$$

where $\rho_{\text{ext}}$ and $\mathbf{j}_{\text{ext}}$ denote the external charge and current densities. Whereas $\mathbf{d} = \epsilon \mathbf{e}$ and $\mathbf{b} = \mu \mathbf{h}$. Here we have simplified our notation by setting $\mathbf{E}_p = \mathbf{e}$ and $\mathbf{B}_p = \mathbf{b}$. In passing we note that the subscript $p$ in $\mathbf{E}_p$ and $\mathbf{B}_p$ indicates propagating fields.

It is straightforward to see that the foregoing equations can be written alternatively in the form

$$\nabla^2 \mathbf{b} - \frac{\epsilon \mu}{c^2} \frac{\partial^2 \mathbf{b}}{\partial t^2} = -\frac{4\pi \mu}{c^2} \nabla \times \mathbf{j}_{\text{ext}}, \quad (52)$$

and

$$\nabla^2 \mathbf{e} - \frac{\epsilon \mu}{c^2} \frac{\partial^2 \mathbf{e}}{\partial t^2} = \frac{4\pi \mu}{c^2} \frac{\partial \mathbf{j}_{\text{ext}}}{\partial t} + \frac{4\pi}{\epsilon} \nabla \rho_{\text{ext}}, \quad (53)$$

where the external charge and current densities are given by: $\rho_{\text{ext}}(t, \mathbf{x}) = Q \delta(\mathbf{x}) \delta(z - vt)$ and $\mathbf{j}_{\text{ext}}(t, \mathbf{x}) = Q v \delta(\mathbf{x}) \delta(z - vt) \hat{\mathbf{e}}_z$. In passing we note that, for simplicity, we are considering the $z$ axis as the direction of the moving charged particle.

Next, in order to solve Eqs. (52) and (53), we shall begin by performing a Fourier transform to momentum space via

$$f(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{-i\mathbf{k} \cdot \mathbf{x}} \mathcal{F}(w, \mathbf{k}), \quad (54)$$

where $f$ stands for the electric and magnetic fields. Then, the corresponding electric and magnetic fields read:

$$\mathbf{b}(w, \mathbf{k}) = -i \frac{4\pi \mu}{c} \mathbf{k} \times \mathbf{j}_{\text{ext}}(w, \mathbf{k}) \mathcal{O}, \quad (55)$$

and

$$\mathbf{e}(w, \mathbf{k}) = i \frac{4\pi}{\epsilon} \frac{\rho_{\text{ext}}(w, \mathbf{k})}{\mathcal{O}} - i \frac{4\pi \mu \mathbf{w}}{c^2} \mathbf{j}_{\text{ext}}(w, \mathbf{k}) \mathcal{O}, \quad (56)$$

where

$$\mathcal{O} = \frac{w^2}{c^2} - \mathbf{k}^2, \quad \frac{1}{c^2} \equiv \frac{\epsilon \mu}{c^2}. \quad (57)$$

In the same way, the external charge and current densities, in the Fourier space, take the form: $\rho_{\text{ext}}(w, \mathbf{k}) = 2\pi Q \delta(w - \mathbf{k} \cdot \mathbf{v})$ and $\mathbf{j}_{\text{ext}}(w, \mathbf{k}) = 2\pi Q v \delta(w - k_z v) \hat{\mathbf{e}}_z$. 

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From the above we can proceed to obtain \( \mathbf{b}(w, x) \) and \( \mathbf{e}(w, x) \). It is clear now that \( \mathbf{b}(w, x) \) is given by

\[
\mathbf{b}(w, x) = \int \frac{d^3k}{(2\pi)^3} e^{ik\cdot x} \mathbf{b}(w, k) .
\]

(58)

We may now take advantage of the axial symmetry of the problem under consideration. If we take cylindrical coordinates, \( \mathbf{b}(w, x) \) becomes

\[
\mathbf{b}(w, x) = \frac{-i\mu Q v}{\pi c} e^{iwz/v} \int_0^\infty dk_T k_T \nonumber \\
\times \left[ \int_{-\infty}^{2\pi} d\alpha \frac{e^{ik_T x_T \cos\alpha}}{\mathcal{O}^{1}_{k_T=x_T/v}} \left( k_T \sin\alpha \hat{\rho} - k_T \cos\alpha \hat{\phi} \right) \right],
\]

(59)

where \( \hat{\rho} \) and \( \hat{\phi} \) are unit vectors normal and tangential to the cylindrical surface, respectively. While the subscript \( T \) designates transversal to the \( z \) direction, in cylindrical coordinates.

In passing we recall that \( \int_0^{2\pi} d\theta e^{i\theta} \sin\theta = 0 \) and \( \int_0^{2\pi} d\theta e^{i\theta} \cos\theta = 2\pi i J_\alpha(1) \), which implies

\[
\mathbf{b}(w, x) = -\frac{2\mu Q v}{c} e^{iwz/v} \int_0^\infty dk_T k_T^2 \frac{J_1(k_T x_T)}{\mathcal{O}^{1}_{k_T=x_T/v}} \hat{\phi},
\]

(60)

where \( J_1(k_T x_T) \) is a Bessel function of the first kind.

In this case, \( \mathcal{O}^{1}_{k_T=x_T/v} = w^2 \left( \frac{1}{c^2} - \frac{1}{v^2} \right) - k_T^2 \). From this last expression it follows that

\[
\mathbf{b}(w, x) = \frac{2\mu Q v}{c} e^{iwz/v} \int_0^\infty dk_T k_T^2 \frac{J_1(k_T x_T)}{\mathcal{O}^{1}_{k_T=x_T/v}} \hat{\phi},
\]

(61)

where \( \sigma^2 = w^2 \left( \frac{1}{c^2} - \frac{1}{v^2} \right) \). We next observe that the previous expression can be brought to the form

\[
\mathbf{b}(w, x) = \frac{2\mu Q v}{c} e^{iwz/v} \int_0^\infty dy y e^{-y^2} \nonumber \\
\times \int_0^\infty dk_T k_T^2 e^{-y k_T^2} J_1(k_T x_T) \hat{\phi},
\]

(62)

From this last expression it follows that

\[
\mathbf{b}(w, x) = \frac{2\mu Q v}{c} e^{iwz/v} \frac{x_T}{4} \int_0^\infty dy y e^{-y^2/4y^2} \hat{\phi} ,
\]

(63)

or, in terms of the modified Bessel function, Eq. (63), becomes

\[
\mathbf{b}(w, x) = \frac{2\mu Q v}{c} e^{iwz/v} \sigma K_1(\sigma x_T) \hat{\phi},
\]

(64)

where, in cylindrical coordinates, \( x_T = \rho \).

Now we come to the calculation of the electric field. From the expression (56), we find that the electric field may be written in the form

\[
\mathbf{e}(w, x) = i 8\pi^2 Q \int \frac{d^3k}{(2\pi)^3} \frac{w}{v} \delta(k_z - w/v) e^{ik\cdot x} \times \left\{ \frac{k_x}{\epsilon} \hat{e}_x + \frac{k_y}{\epsilon} \hat{e}_y + \left( \frac{k_z}{\epsilon} - \frac{\mu w v}{\epsilon c^2} \right) \hat{e}_z \right\}.
\]

(65)

In the same way as was done for the magnetic field, we then get

\[
e_\rho(w, x) = \frac{i Q}{v\pi} \int_0^\infty dk_T k_T e^{iwz/v} \frac{1}{\mathcal{O}^{1}_{k_T=x_T/v}} \nonumber \\
\times \left\{ 2\pi i k_T \left( \frac{1}{v} - \frac{v}{c^2} \right) J_1(k_T x_T) \right\}.
\]

(66)

and

\[
e_z(w, x) = \frac{i Q}{v\pi} \int_0^\infty dk_T k_T e^{iwz/v} \frac{1}{\mathcal{O}^{1}_{k_T=x_T/v}} \nonumber \\
\times \left\{ 2\pi w \left( \frac{1}{v} - \frac{v}{c^2} \right) J_0(k_T x_T) \right\}.
\]

(67)

The integral occurring on the right-hand side of the previous expressions can be as before. We get accordingly

\[
e_\rho(w, x) = \frac{2Q}{v\epsilon} e^{iwz/v} \sigma K_1(\sigma x_T),
\]

(68)

and

\[
e_z(w, x) = -i \frac{2Q}{v\epsilon} e^{iwz/v} \left( \frac{1}{v} - \frac{v}{c^2} \right) K_0(\sigma x_T).
\]

(69)

We are now equipped to compute the corresponding radiated energy in the case under consideration.

In order to accomplish this purpose, let us start by observing that the density of power carried out by the radiation fields across the surface bounding the volume \( V \) is given by the real part of the Poynting vector (time averaged value)

\[
S = \frac{c}{2\pi} \text{Re} \left( \mathbf{e} \times \mathbf{h}^* \right).
\]

(70)

We further recall that we will calculate the energy radiated through the surface [28], that is,

\[
\mathcal{E} = \int_{-\infty}^{\infty} dt \int_S \mathbf{S} \cdot ds.
\]

(71)

It is worth emphasizing that in our case we shall consider a cylinder as the integration surface. Also, it may be mentioned that in order to get a meaningful expression we shall use a cylinder infinitesimally small [29].
Consequently, the power radiated per unit length through the surface then reads
\[
P = \frac{c}{2\pi} \text{Re} \int_0^\infty dw \left\{ 2\pi \rho_0 S_\rho |_{\rho=\rho_0} \right\} + \frac{c}{2\pi} \text{Re} \int_0^\infty dw \left\{ \frac{\partial}{\partial \varepsilon} \int_0^\rho \int_0^{2\pi} S_\varepsilon \rho d\rho d\phi \right\},
\] (72)
where \( S_\rho = -e_\perp h_\perp^\rho, S_\varepsilon = e_\rho h_\rho^\varepsilon \) and \( \rho_0 \to 0 \). Let us also recall here that, in our case, the \( \phi \)-component of the Poynting vector \( \left( S_\phi \right) \) vanishes.

According to Eqs. (64), (68) and (69), the expression for the power radiated per unit length (72) takes the form
\[
P = -\pi \frac{Q^2 v \nu^2}{c^2 \epsilon} \int_0^\infty dw \left( 1 - \frac{c^2}{v^2} \right).
\] (73)
One immediately sees that this expression is similar to that encountered in the Cherenkov radiation theory [28]. This last expression clearly shows the role played by vacuum electromagnetic nonlinearities in triggering the radiated energy. We also point out that in Eq. (73) we have used the asymptotic behavior of the Bessel \( K_\nu (x) \to \frac{\pi}{\sqrt{2x}} e^{-x} \), since we are describing outgoing radiation.

In connection with this last expression (73) a few comments are in order. First, it should be recalled that \( \epsilon \) represents the rate of energy lost due to radiation along the trajectory of the charged particle, \( -\frac{dE}{dt} \), where \( E \) is the energy of the charged particle. Second, we also recall that the \( w \)-integration runs over the frequency range for which \( n(w) > c/v \). Third, we further notice that the vacuum of the electrodynamics studied in this work does not depend on the frequency of the waves, as the Eq. (43), previously derived, explicitly shows. But, this does not prevent us from fixing an upper limit for the integration over \( w \) in (73). For that, let us call into question the physical origin of the Cherenkov effect in a dielectric material. The corresponding emitted (Cherenkov) radiation is produced when the molecules of the polarizable medium re-emit as photons the energy these molecules absorb whenever they are excited by the charged particle that propagates in the material.

From this last remark it follows that the produced Cherenkov radiation corresponds to the energy re-emitted by the excited virtual particles. This naturally defines a frequency, \( \frac{2mc^2}{h} \) (here \( m \) is the mass of the electron/positron), which, we argue, acts as a cutoff frequency for the re-emitted photons. This frequency corresponds to the energy for a pair creation. It is reasonable to think that the charged particle is not able to excite virtual pair production in the QED vacuum. So, the re-emitted energy must be upper-bounded by \( 2mc^2 \) which corresponds to the cutoff \( \Omega = \frac{2mc^2}{h} \) on the frequencies to be integrated over. We emphasize that we are assuming this argument, naturally valid for Euler–Heisenberg, holds true for the Logarithmic model as well. But, let us stress that, in our case, it is an assumption.

In view of the important role which the Cherenkov radiation in its standard form (73) plays in our development, it will be useful to rewrite it in two different cases for logarithmic electrodynamics. First case. From Eqs. (40) and (49) we have \( n^2 \equiv n^2_\perp = 1 + \frac{B^2}{\beta^2} \) and \( \epsilon = 1 + \frac{B^2}{\beta^2} \). In this manner, from Eq. (73) we will calculate the energy lost per unit distance traveled by the charged particle, that is,
\[
P' = -\pi Q^2 \int_0^\Omega dw w \left( 1 - \frac{c^2}{v^2 (1 + \frac{B^2}{\beta^2})} \right),
\] (74)
\[
= -\frac{2\pi Q^2 m^2 e^2}{h^2} \left( 1 - \frac{c^2}{v^2 (1 + \frac{B^2}{\beta^2})} \right).
\] (75)
This result shows that the radiation driven by the nonlinearities of the electromagnetic field is similar to that encountered in a material medium. From the equation above, we can readily conclude that the stronger is the external magnetic field the bigger is the energy re-emitted as Cherenkov radiation, as one should expect for the refractive index grows whenever \( |B| \) increases.

Second case. From Eqs. (40) and (48), we have \( n^2 \equiv n^2_\perp = \frac{\beta^2 + B^2}{\beta^2 - B^2} \) and \( \epsilon = 1 + \frac{B^2}{\beta^2} \). It is of interest also to notice that for \( \beta \gg |B| \), the refractive index becomes \( n^2 \equiv n^2_\perp = \left( 1 + \frac{B^2}{\beta^2} \right) \) to order \( O(1/\beta^2) \). We thus find that the energy lost per unit distance travelled by the charged particle is identical to expression (74), that is,
\[
P' = -\frac{2\pi Q^2 m^2 e^2}{h^2} \left( 1 - \frac{c^2}{v^2 (1 + \frac{B^2}{\beta^2})} \right).
\] (76)
Recalling the interesting results of the work in Ref. [23], where the authors investigate Cherenkov radiation from the quantum vacuum by considering the Euler–Heisenberg nonlinear theory, in this contribution we reassess the issue of Cherenkov radiation by choosing another nonlinear extension of electrodynamics, namely, the birefringent logarithmic model, in presence of a background magnetic field strong enough to probe the vacuum nonlinearities. In our case, we have focused on the connection between the birefringence of the logarithmic model and the properties of the Cherenkov radiation, and the results of our inspection are presented in Eqs. (74) and (75).

Finally, with the foregoing information provided by Eqs. (74) and (75), we can proceed to estimate the energy lost per unit distance travelled for charged particle. Thus, by considering an astrophysical magnetic field of the order of \( 10^6 \) T, and the \( \beta \)-parameter of logarithmic electrodynamics esti-
mated in [20], we obtain $P' = -8.0497 \times 10^{-4} \text{ MeV}^2$. Converting this estimate to appropriate energy per length units they we obtain $P' = -4.0248 \text{ GeV/\mu m}$. 

4 Final remarks

In summary, we have studied some observational signatures of nonlinearities of the electromagnetic field. First, we shown the vital role played by nonlinearities in triggering a material with $(\varepsilon > 0, \mu < 0)$, which corresponds to a ferrimagnetic material. Secondly, the permittivity and susceptibility induced by nonlinearities have been studied in order to obtain the refractive index via the dispersion relation for logarithmic electrodynamics. Finally, we have considered the radiation produced by a moving charged particle (with uniform velocity) interacting in nonlinear medium. Let us also recall here that particles moving with a uniform velocity in vacuum do not lead to radiation. As already mentioned, one is lead to the interesting conclusion that the above radiation is driven by the medium through which the particle travels like the one that happens in the Cherenkov effect. Lastly, we will be focusing efforts to understand in more detail the physical consequences of electrodynamics with metamaterial features, including Cherenkov radiation, in the near future.

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Data Availability Statement The manuscript has associated data in a data repository. [Authors’ comment: This is a theoretical paper, and it does not concern data.] 

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