WIGNER- AND MARCHENKO-PASTUR-TYPE LIMIT THEOREMS FOR JACOBI PROCESSES

MARTIN AUER, MICHAEL VOIT, JEANNETTE H.C. WOERNER

Abstract. We study Jacobi processes \((X_t)_{t \geq 0}\) on the compact spaces \([-1, 1]^N\) and on the non-compact spaces \([1, \infty]^N\) which are motivated by the Heckman-Opdam theory for the root systems of type BC and associated integrable particle systems. These processes depend on three positive parameters and degenerate in the freezing limit to solutions of deterministic dynamical systems.

In the compact case, these models tend for \(t \to \infty\) to the distributions of the \(\beta\)-Jacobi ensembles and, in the freezing case, to vectors consisting of ordered zeros of one-dimensional Jacobi polynomials.

Representing these processes by stochastic differential equations, we derive almost sure analogues of Wigner’s semicircle and Marchenko-Pastur limit laws for \(N \to \infty\) for the empirical distributions of the \(N\) particles on some local scale. We there allow for arbitrary initial conditions, which enter the limiting distributions via free convolutions. These results generalize corresponding stationary limit results in the compact case for \(\beta\)-Jacobi ensembles and, in the deterministic case, for the empirical distributions of the ordered zeros of Jacobi polynomials by Dette and Studden. The results are also related to free limit theorems for multivariate Bessel processes, \(\beta\)-Hermite and \(\beta\)-Laguerre ensembles, and the asymptotic empirical distributions of the zeros of Hermite and Laguerre polynomials for \(N \to \infty\).

1. Introduction

By classical results, the empirical distributions of \(\beta\)-Hermite, \(\beta\)-Laguerre, and \(\beta\)-Jacobi ensembles of dimension \(N\) tend for \(N \to \infty\) to semicircle, Marchenko-Pastur as well as Kesten-McKay and Wachter distributions respectively after suitable normalizations; see e.g. [CC, DN, J, RS, W] and references therein. Moreover, in the Hermite and Laguerre cases, there are dynamical versions of these results in terms of Bessel processes \((X_t^N)_{t \geq 0}\) of dimension \(N\) for the root systems of types A and B; see [CGY, RV1] for the background on these processes. Namely, let \(\mu\) be some starting distribution on \(\mathbb{R}\) or \([0, \infty]\), and let for \(N \in \mathbb{N}\), \(x_N\) be starting vectors in \(\mathbb{R}^N\) such that the empirical distributions of the \(x_N\) tend to \(\mu\). If we consider the Bessel processes \((X_t^N)_{t \geq 0}\) with start in these points \(x_N\), then under mild additional conditions and with an appropriate scaling, the empirical distributions of the components of the \(X_t^N\) tend for \(N \to \infty\) almost surely weakly to measures \(\mu_t\) for all \(t \geq 0\). In the \(\beta\)-Hermite case, i.e., for Bessel processes of type A, one has \(\mu_t = \mu \boxplus \mu_{sc,2\sqrt{t}}\), where \(\mu_{sc,2\sqrt{t}}\) denotes the semicircle distribution with radius \(2\sqrt{t}\) and \(\boxplus\) the usual additive free convolution; see Section 4.3 of [AGZ] and [VW1] for different approaches. Moreover, for the \(\beta\)-Laguerre case, i.e. for Bessel processes of type B, there are corresponding results for \(\mu_t\) in terms of Marchenko-Pastur distributions and a more complicated construction involving the
usual additive free convolution in \([VW1]\). This construction may also be described in terms of the rectangular free convolutions of Benaych-Georges \([B1, B2]\). Furthermore, these results for Bessel processes of types A and B can be transferred to stationary Ornstein-Uhlenbeck-type versions of these processes as indicated in the end of Section 2 of \([VW1]\). For the background on stochastic analysis we recommend the monographs \([P, RW]\), and \([AGZ, NS]\) for free probability in our context.

In this paper we show that Ornstein-Uhlenbeck-type limit results also appear for certain \(N\)-dimensional Jacobi processes on \([-1, 1]^N\) for \(N \to \infty\). These Jacobi processes were introduced and studied from different points of views by Doumerc \([Do]\), Demni \([De1, De2]\), Remling and R"osler \([RR1, RR2]\), and \([V]\). They depend on 3 parameters and may be described in different ways. One possibility, motivated by the theory of special functions associated with root systems of Heckman and Opdam \([HS, HO]\), is to describe these processes as time-homogeneous diffusions on the alcoves

\[ \tilde{A}_N := \{ \theta \in [0, \pi] : 0 \leq \theta_1 \leq \ldots \leq \theta_N \leq \pi \} \]

with the Heckman-Opdam Laplacians

\[
L_{\text{trig}, k} f(\theta) := \Delta f(\theta) + \sum_{i=1}^{N} \left( k_1 \cot(\theta_i/2) + 2k_2 \cot(\theta_i) + k_3 \sum_{j \neq i} \left( \coth(\theta_i - \theta_j) + \coth(\theta_i + \theta_j) \right) \right) f_{\theta_i}(\theta).
\]

of type BC as generators with the multiplicity parameters \(k_1, k_2, k_3 \in \mathbb{R}, k_1 > 0\) with \(k_2 \geq 0\) and \(k_1 + k_2 \geq 0\) where we assume reflecting boundaries. It is convenient to transform the processes and their generators in the trigonometric form via the transform \(x_i := \cos \theta_i \ (i = 1, \ldots, N)\) into an algebraic form; see e.g. also \([De2, V]\). We then obtain time-homogeneous diffusions on the alcoves

\[ A_N := \{ x \in \mathbb{R}^N : -1 \leq x_1 \leq \ldots \leq x_N \leq 1 \} \]

with the algebraic Heckman-Opdam Laplacians

\[
L_k f(x) := \sum_{i=1}^{N} (1-x_i^2) f_{x_i,x_i}(x) + \sum_{i=1}^{N} \left( -k_1 - (1 + k_1 + 2k_2)x_i + 2k_3 \sum_{j \neq i} \frac{1-x_j^2}{x_i-x_j} \right) f_{x_i}(x). \tag{1.1}
\]

as generators with the multiplicities \(k_1, k_2, k_3\) with reflecting boundaries. The eigenfunctions of the \(L_k\) may be described via Heckman-Opdam Jacobi polynomials, and the transition probabilities of the Jacobi processes can be expressed via series expansions in terms of these polynomials; see \([De2, RR1, RR2]\). On the other hand, these processes \((X_t = (X_{t,1}, \ldots, X_{t,N}))_{t \geq 0}\) admit a description as a unique strong solution of the stochastic differential equation (SDE)

\[
dX_{t,i} = \sqrt{2(1-X_{t,i}^2)} dB_{t,i} + \left( -k_1 - (1 + k_1 + 2k_2)X_{t,i} + 2k_3 \sum_{j \neq i} \frac{1-X_{t,j}^2}{X_{t,i}-X_{t,j}} \right) dt \tag{1.2}
\]

for \(i = 1, \ldots, N\) with an \(N\)-dimensional Brownian motion \((B_t)_{t \geq 0}\). The paths of \((X_t)_{t \geq 0}\) are reflected on \(\partial A_N\) and we start in some point in the interior of \(A_N\); see Theorem 2.1 of \([De2]\). It is also possible to start the processes satisfying these SDEs on the boundary. This is not shown precisely for this type of Jacobi processes on compact alcoves in the literature, but it may be shown in a similar way as for multivariate Bessel and Jacobi processes on non-compact domains in \([GM, Sch1, Sch2]\).

Following \([De2]\), we introduce the parameters

\[
\kappa := k_3 > 0, \quad q := N - 1 + \frac{1 + 2k_1 + 2k_2}{2k_3} > N - 1, \quad p := N - 1 + \frac{1 + 2k_2}{2k_3} > N - 1, \tag{1.3}
\]
and rewrite (1.2) as
\[
\begin{align*}
\frac{dX_{t,i}}{dt} &= \sqrt{2(1-X_{t,i}^2)} dB_{t,i} + \kappa \left( (p-q) + (2(N-1)-(p+q))X_{t,i} ight) \\
&\quad + 2 \sum_{j:j\neq i} \frac{1-X_{t,j}^2}{X_{t,i}-X_{t,j}} dt \\
&= \sqrt{2(1-X_{t,i}^2)} dB_{t,i} + \kappa \left( (p-q) - (p+q)X_{t,i} ight) + 2 \sum_{j:j\neq i} \frac{1-X_{t,i}X_{t,j}}{X_{t,i}-X_{t,j}} dt 
\end{align*}
\]
for \(i = 1, \ldots, N\) and \(t > 0\). It is known (see e.g. [De2, Do]) that for \(\kappa \geq 1\) and \(p, q \geq N + 1 + 2/\kappa\), the process does not meet \(\partial A_N^\kappa\) almost surely.

It is useful, also to consider the transformed processes \((\tilde{X}_t := X_{t/\kappa})_{t \geq 0}\) which satisfy
\[
\begin{align*}
\frac{d\tilde{X}_{t,i}}{dt} &= \sqrt{\frac{2}{\kappa}} \sqrt{1-\tilde{X}_{t,i}^2} d\tilde{B}_{t,i} + \left( (p-q) - (p+q)\tilde{X}_{t,i} + 2 \sum_{j:j\neq i} \frac{1-\tilde{X}_{t,i}\tilde{X}_{t,j}}{\tilde{X}_{t,i}-\tilde{X}_{t,j}} \right) dt 
\end{align*}
\]
for \(i = 1, \ldots, N\). For \(\kappa = \infty\) and \(p, q > N - 1\), these SDEs with start in \(x_0 \in A_N\) degenerate to the ODE
\[
\begin{align*}
\frac{dx_i(t)}{dt} &= (p-q) - (p+q)x_i(t) + 2 \sum_{j:j\neq i} \frac{1-x_i(t)x_j(t)}{x_i(t)-x_j(t)}, \quad i = 1, \ldots, N, \; t > 0, \\
x(0) &= x_0.
\end{align*}
\]
This ODE is interesting for itself and is closely related to the classical one-dimensional Jacobi polynomials \((P_N^{(\alpha,\beta)})_{N \geq 0}\) on \([-1,1]\) with the parameters
\[
\alpha := q - N > -1, \quad \beta := p - N > -1.
\]
These polynomials are orthogonal w.r.t. the weights \((1-x)^\alpha(1+x)^\beta\) on \([-1,1]\) as usual; see Ch. 4 of [Sz]. All essential information about (1.6) are collected in the following theorem which will be proved in the appendix in Section 6:

**Theorem 1.1.** Let \(N \in \mathbb{N}\) and \(p, q > N - 1\). Then for each \(x_0 \in A_N\) the ODE (1.6) has a unique solution \(x(t)\) for all \(t \geq 0\) in the following sense: If \(x_0\) is in the interior of \(A_N\), then \(x(t)\) exists also in the interior of \(A_N\) for all \(t \geq 0\). Moreover, for \(x_0 \in \partial A_N\), there is a unique continuous function \(x : [0, \infty) \to A_N\) with \(x(0) = x_0\) and \(x(t)\) being in the interior of \(A_N\) for \(t > 0\) such that \(x(t)\) satisfies (1.6) for \(t > 0\).

Furthermore, for each \(x_0 \in A_N\), the solution satisfies \(\lim_{t \to \infty} x(t) = z\) where the coordinates of the vector \(z\) in the interior of \(A_N\) are the ordered roots of the Jacobi polynomial \(P_N^{(q-N,p-N)}\). This vector \(z\) is the only stationary solution of (1.6) in \(A_N\).

The stationary solution \(z \in A_N\) in the deterministic case is the freezing limit for \(\kappa \to \infty\) of the stationary distributions of the corresponding Jacobi processes with fixed parameters \(p, q\); see e.g. [HV] for more details. These stationary distributions are just the distributions of the \(\beta\)-Jacobi (or \(\beta\)-MANOVA) ensembles on \(A_N\) having the Lebesgue densities
\[
c(k_1, k_2, k_3) \cdot \prod_{i=1}^{N} (1-x_i)^{k_1+k_2-1/2}(1+x_i)^{k_2-1/2} \cdot \prod_{i<j} (x_i-x_j)^{2k_3}
\]
(1.7)
with known Selberg-type norming constants \( c(k_1, k_2, k_3) \). We here recapitulate that, possibly after some affine linear-transformation and taking some cosine in all coordinates, these probability measures appear as the distributions of the ordered eigenvalues of the tridiagonal models in \( \mathbb{K} \) and in some log gas models on \([-1, 1]\); see [2]. Moreover, for certain parameters, these distributions and the corresponding Jacobi processes have an interpretation as invariant distributions and Brownian motions respectively on compact Grassmann manifolds of rank \( N \) over the fields \( \mathbb{F} = \mathbb{R}, \mathbb{C} \) and the quaternions by the now classical connection between the Heckman-Opdam theory and spherical functions; see [HO, HS, RR1, RR2, De2]. We also point out that, even more generally for some parameters, these distributions and the corresponding Jacobi processes appear as the ordered eigenvalues of matrices \( B^*B \) for upper left blocks \( B \) of size \( M \times N \) of Haar distributed random variables and Brownian motions in the unitary group \( U(\mathbb{R}, \mathbb{F}) \) respectively with the dimension parameters \( R > M > N \); see [De1, De2] for the details.

We now turn to the main content of this paper. We here follow the approach in [VW1] for Bessel processes and derive several almost sure limit theorems as \( N \to \infty \) for the empirical distributions of the rescaled Jacobi processes \( (\tilde{X}_t^N)_{t \geq 0} \) and their deterministic freezing limits for \( \kappa = \infty \) which satisfy the ODE (1.6), which are related in their flavour to mean field limits of Serfaty [Se]. Considering three involved parameters \( p, q, \kappa \) and their dependence on \( N \), lead after suitable affine-linear transformations to different limiting distributions. The different cases are motivated by the stationary deterministic case, where we just have the empirical distributions of the classical Jacobi polynomials. In this setting several limiting regimes with semicircle, Marchenko-Pastur and Wachter distributions were derived by Dette and Studden [DS]. We thus follow their decomposition of the cases and investigate the deterministic case with the ODE (1.6) first. For this we derive recurrence relations for the moments as well as PDEs for the Stieltjes and the R-transforms of the corresponding Jacobi processes have an interpretation as in variant distributions and Brownian motions respectively on compact Grassmann manifolds of rank \( N \) of the type \( U_N(\mathbb{C}) \). We shall do this in Section 3 for two regimes where semicircle and Marchenko-Pastur distributions appear. In the semicircle case we shall obtain the following result where \( \mu_{\text{sc}, \tau} \) denotes the semicircle law with support \([\tau, \tau]\) for \( \tau \geq 0 \) and \( \mu_{\text{sc}, 0} = \delta_0 \):

**Theorem 1.2.** Consider sequences \((p_N)_{N \in \mathbb{N}}, (q_N)_{N \in \mathbb{N}} \subset ]0, \infty[\) with \( \lim_{N \to \infty} p_N/N = \infty \) and \( \lim_{N \to \infty} q_N/N = \infty \) such that \( C := \lim_{N \to \infty} p_N/q_N \geq 0 \) exists. Define

\[
a_N := \frac{q_N}{\sqrt{N} p_N}, \quad b_N := \frac{p_N - q_N}{p_N + q_N} \quad (N \in \mathbb{N}).
\]

Let \( \mu \in M^1(\mathbb{R}) \) be a probability measure satisfying some moment condition (see Theorem 3.1 for the details), and let \((x_N)_{N \in \mathbb{N}} = ((x_1^N, \ldots, x_N^N))_{N \in \mathbb{N}} \) be a sequence of starting vectors \( x_N \in A_N \) such that all moments of the empirical measures

\[
\mu_{N, 0} := \frac{1}{N} \sum_{i=1}^{N} \delta_{b_N(x_i^N - b_N)}
\]

tend to those of \( \mu \) for \( N \to \infty \). Let \( x_N(t) \) be the solutions of the ODEs (1.4) with \( x_N(0) = x_N \) for \( N \in \mathbb{N} \). Then for all \( t > 0 \), all moments of the empirical measures

\[
\mu_{N, t/(p_N+q_N)} = \frac{1}{N} \sum_{i=1}^{N} \delta_{a_N(x_i^N(t/(p_N+q_N))-b_N)}
\]
tend to those of the probability measures
\[ \mu_t := (e^{-t}\mu) \boxplus \left( \sqrt{1-e^{-2t}\mu_{sc,4(1+C)-3/2}} \right). \]

This in particular implies that the \( \mu_{N,t/(p_N+q_N)} \) tend weakly to the \( \mu_t \).

We point out that Theorem 1.2 is a local limit theorem on the behaviour of the particle systems with \( N \) particles around the starting points \( b_N \in [-1,1] \) for small times on the space scale \( 1/a_N \) for large \( N \). We also mention the slightly astonishing fact that this local result preserves the asymptotic stationarity of the global systems. In fact there are local limit results on different time and space scales in Section 3 where this asymptotic stationarity does not appear; see e.g. Theorem 3.4. Besides these two results and further variants with Wigner-type limits in Section 3, we shall also derive local limit results with Marchenko-Pastur type limits in neighbourhoods of the boundary points \( \pm 1 \) in Section 3; see for instance Theorem 3.8 below. In the proof of this theorem we again solve the associated PDE for the R-transforms explicitly. We point out that a modification of this PDE in the Marchenko-Pastur setting appears also in [CG].

There are further limit regimes where Kesten-MacKay and Wachter distributions are involved, and which are also motivated by [DS] and corresponding limit results for \( \beta \)-Jacobi ensembles e.g. in [DN, W]. In these cases it can be also shown that under corresponding conditions on the initial conditions, the empirical measures \( \mu_{N,t/(p_N+q_N)} \) also converge to some probability measures \( \mu_t \) for \( t \geq 0 \). However, the details of the description of the limits are more involved here and will be published in the future separately.

The results of Section 3 on the compact, deterministic case will be extended in Section 4 to almost sure versions for Jacobi processes with fixed parameter \( \kappa \) in the compact setting. It turns out the limiting distributions stay the same for the rescaled processes. Hence as for Bessel processes the form of the limiting distribution is already determined by the frozen process.

Furthermore, in Section 5 we transfer some of our Wigner- and Marchenko-Pastur type results in the Sections 2-4 to a noncompact setting. For some parameters, these results have interpretations in terms of Brownian motions on the noncompact Grassmann manifolds over \( \mathbb{R} \), \( \mathbb{C} \), and the quaternions. It will turn out that in these hyperbolic cases, some results remain valid up to some kind of time inversion. However, it seems that here no analogue to the stationary results like Theorem 1.2 are available, as the the initial conditions do not fit to the conditions on the parameters \( p_N, q_N, a_N, b_N \) in this theorem. Finally, as mentioned above, we prove Theorem 1.1 and its noncompact analogue in Section 6.

2. Moments of the empirical distributions in the deterministic case

In this section we study the solutions \( x^N(t) \) of the ODEs (1.0) for suitable initial conditions \( x^N_0 \in A_N \) for \( N \in \mathbb{N} \) and suitable parameters \( p = p_N, q = q_N > N - 1 \) where we are interested in the case \( N \to \infty \) which implies that also \( p = p_N, q = q_N \to \infty \) holds. It will turn out that there are several limit regimes for the empirical measures
\[ \frac{1}{N} (\delta_{x^N_1(t)} + \ldots + \delta_{x^N_N(t)}) \in M^1([-1,1]) \]
for \( N \to \infty \) and all \( t \geq 0 \) under the condition that a corresponding limit holds for the initial conditions at time \( t = 0 \). For some of these limit results we have to transform the data in an affine-linear way in all coordinates depending on \( N \). For this we introduce suitable sequences \( (a_N)_{N \in \mathbb{N}} \subset [0,\infty[ \) and \( (b_N)_{N \in \mathbb{N}} \subset \mathbb{R} \) which will be specified later in several specific situations. We
consider the transformed solutions \( \tilde{x}^N(t) = (\tilde{x}_1^N(t), \ldots, \tilde{x}_N^N(t)) \) with
\[
\tilde{x}_i^N(t) := a_N(x_i^N(t) - b_N) \quad (1 \leq i \leq N)
\]
as well as the transformed empirical distributions
\[
\mu_{N,t} := \frac{1}{N}(\delta_{\tilde{x}_1^N(t)} + \ldots + \delta_{\tilde{x}_N^N(t)}) = \frac{1}{N}(\delta_{a_N(x_1^N(t) - b_N)} + \ldots + \delta_{a_N(x_N^N(t) - b_N)}).
\]
(2.1)
In order to determine possible weak limits of the measures \( \mu_{N,t} \), we shall study the moments
\[
S_{N,l}(t) := \int_{[-1,1]} y^l \, d\mu_{N,t}(y) = \frac{d^l}{dt^l} N \sum_{i=1}^N (x_i^N(t) - b_N)^l = \frac{1}{N} \sum_{i=1}^N \tilde{x}_i^N(t)^l,
\]
of these measures for \( t \in \mathbb{N}_0, \ t \geq 0, \) and \( N \in \mathbb{N} \). In particular we have \( S_{N,0} \equiv 1 \). To study higher moments, we rewrite the ODE (2.2) as an ODE for \( \tilde{x}_i^N \) by
\[
\frac{d}{dt} \tilde{x}_i^N(t) = a_N(p - q - b_N(p + q)) - (p + q)\tilde{x}_i^N(t) \quad (i,j)
\]
where we always agree that a summation over \( j : j \neq i \) means that we sum over all \( j \neq i \) from 1 to \( N \). In the following we also suppress the dependence of \( S_{N,l} \) and \( \tilde{x}_i^N \) on \( t \). (2.3) yields the following ODEs for the \( S_{N,l} \) for \( l \in \mathbb{N} \):
\[
\frac{d}{dt}S_{N,l} = \frac{1}{N} \sum_{i=1}^N (\tilde{x}_i^N)^l - 1 \left( \frac{d}{dt} \tilde{x}_i^N \right) \quad (2.4)
\]
which for \( l = 1 \),
\[
\frac{d}{dt}S_{N,1} = a_N(p - q - b_N(p + q)) - (p + q)S_{N,1}. \quad (2.5)
\]
Moreover, for \( l \geq 2 \) we first observe that
\[
2 \sum_{i,j: i \neq j} (a_N^2(1 - b_N^2) - \tilde{x}_i^N \tilde{x}_j^N) \frac{(\tilde{x}_i^N)^{l-1}}{\tilde{x}_i^N - \tilde{x}_j^N} = 2 \sum_{i,j: i < j} (a_N^2(1 - b_N^2) - \tilde{x}_i^N \tilde{x}_j^N) \frac{(\tilde{x}_i^N)^{l-1}}{\tilde{x}_i^N - \tilde{x}_j^N}
\]

\[
= \sum_{k=0}^{l-2} \sum_{i,j: i \neq j} (a_N^2(1 - b_N^2) - \tilde{x}_i^N \tilde{x}_j^N) \frac{(\tilde{x}_i^N)^{l-2-k} \tilde{x}_j^N}{\tilde{x}_i^N - \tilde{x}_j^N} = a_N^2(1 - b_N^2) \left( N^2 \sum_{k=0}^{l-2} S_{N,k}S_{N,l-2-k} - (l - 1)NS_{N,l-2} \right)
\]

\[
- N^2 \sum_{k=0}^{l-2} S_{N,k+1}S_{N,l-1-k} + (l - 1)NS_{N,l}.
\]
Furthermore, with the usual convention for empty sums,

\[ 2 \sum_{i,j: i \neq j} \frac{(\hat{x}_i^N)^l + (\hat{x}_j^N)^{l-1} \hat{x}_j^N}{\hat{x}_i^N - \hat{x}_j^N} = 2 \sum_{i,j=1: i \neq j} \frac{(\hat{x}_i^N)^l - (\hat{x}_j^N)^l}{\hat{x}_i^N - \hat{x}_j^N} + 2 \sum_{i,j: i \neq j} \hat{x}_i^N \hat{x}_j^N \frac{(\hat{x}_i^N)^{l-2} - (\hat{x}_j^N)^{l-2}}{\hat{x}_i^N - \hat{x}_j^N} \]

\[= \sum_{k=0}^{l-1} \sum_{i,j: i \neq j} (\hat{x}_i^N)^k (\hat{x}_j^N)^{l-1-k} + \sum_{k=0}^{l-3} \sum_{i,j: i \neq j} (\hat{x}_i^N)^{k+1} (\hat{x}_j^N)^{l-2-k} \]

\[= N^2 \sum_{k=0}^{l-1} S_{N,k} S_{N,l-1-k} - N! S_{N,l-1} + N^2 \sum_{k=0}^{l-3} S_{N,k+1} S_{N,l-2-k} - N(l-2) S_{N,l-1} \]

\[= N^2 \sum_{k=0}^{l-1} S_{N,k} S_{N,l-1-k} + N^2 \sum_{k=0}^{l-3} S_{N,k+1} S_{N,l-2-k} - 2N(l-1) S_{N,l-1} \]

\[= N^2 \left[ \sum_{k=0}^{l-2} S_{N,k} S_{N,l-1-k} + S_{N,l-1} + \sum_{k=0}^{l-2} S_{N,k+1} S_{N,l-2-k} - S_{N,l-1} \right] - 2N(l-1) S_{N,l-1} \]

\[= 2N^2 \sum_{k=0}^{l-2} S_{N,k} S_{N,l-1-k} - 2N(l-1) S_{N,l-1}. \]

Therefore, for \( l \geq 2 \), and \( p = p_N, q = q_N \),

\[
\frac{d}{dt} S_{N,l} = l \left[ (p - q - b_N(p + q - 2(l - 1))) a_N S_{N,l-1} - (p + q - (l - 1)) S_{N,l} 
- a_N^2 (1 - b_N^2)(l - 1) S_{N,l-2} + N a_N^2 (1 - b_N^2) \sum_{k=0}^{l-2} S_{N,k} S_{N,l-2-k} 
- N \sum_{k=0}^{l-2} S_{N,k+1} S_{N,l-1-k} - 2a_N b_N N \sum_{k=0}^{l-2} S_{N,k} S_{N,l-1-k} \right]. 
\tag{2.6}
\]

In summary, we have the recursion \(2.6\) together with

\[
\frac{d}{dt} S_{N,0} = 0, \quad \frac{d}{dt} S_{N,1} = -(p + q) S_{N,1} + a_N (p - q - b_N(p + q)). \tag{2.7}
\]

In the next step we consider the Cauchy transforms of the measures \( \mu_{N,t} \). For this we recapitulate that for \( \mu \in M^1(\mathbb{R}) \) the Cauchy transform is given by

\[ G_{\mu}(z) := \int_{\mathbb{R}} \frac{1}{z - x} d\mu(x) \quad (z \in \{ z \in \mathbb{C} : \Im(z) > 0 \}). \]

We set \( G^N(t, z) := G_{\mu_{N,t}}(z) \). For \( |z| \) sufficiently large we can write \( G^N \) as

\[ G^N(t, z) = \sum_{l=0}^{\infty} z^{-(l+1)} S_{N,l}(t). \tag{2.8} \]

We now consider the partial derivatives \( G^N_t(t, z) := \partial_t G^N(t, z) \) and \( G^N_z(t, z) := \partial_z G^N(t, z) \) and similarly for higher orders. \(2.8\) thus leads to

\[ G^N_t(t, z) = \sum_{l=0}^{\infty} z^{-(l+1)} \frac{d}{dt} S_{N,l}(t) = \sum_{l=1}^{\infty} z^{-(l+1)} \frac{d}{dt} S_{N,l}(t). \tag{2.9} \]
We now calculate this series by using (2.6) and (2.7). For this we use the following equations:

\[
- \sum_{l=1}^{\infty} z^{-(l+1)} \{ (p + q - (l - 1)) S_{N,l} = - (p + q) \sum_{l=1}^{\infty} z^{-(l+1)} l S_{N,l} + \sum_{l=1}^{\infty} z^{-(l+1)} l (l - 1) S_{N,l} \} \quad (2.10)
\]

\[
= (p + q) z^{N^2} (t, z) + (p + q) G^N (t, z) + \partial_{z z} \left( z^2 G^N (t, z) \right),
\]

\[
\sum_{l=1}^{\infty} l z^{-(l+1)} a_N (p - q - b_N ((p + q) - 2(l - 1))) S_{N,l-1}
\]

\[
= - a_N (p - q - b_N (p + q)) G^N_z (t, z) + 2 a_N b_N \partial_{z z} \left( z G^N (t, z) \right),
\]

\[
- \sum_{l=2}^{\infty} z^{-(l+1)} l (l - 1) S_{N,l-2} = - G^N_z (t, z), \quad \sum_{l=2}^{\infty} z^{-(l+1)} l N \sum_{k=0}^{l-2} S_{N,k} S_{N,l-2-k} = - 2 N G^N (t, z) G^N_z (t, z),
\]

\[
- \sum_{l=2}^{\infty} z^{-(l+1)} l N \sum_{k=0}^{l-2} S_{N,k+1} S_{N,l-1-k} = 2 N (z^2 G^N (t, z) G^N_z (t, z) + z (G^N (t, z))^2 - z G^N_z (t, z) - G^N (t, z)),
\]

and

\[
- \sum_{l=2}^{\infty} z^{-(l+1)} \sum_{k=0}^{l-2} S_{N,k} S_{N,l-1-k} = \partial_z \left[ \sum_{l=2}^{\infty} z^{l} \sum_{k=0}^{l-2} S_{N,k} S_{N,l-1-k} \right]
\]

\[
= \partial_z \left[ \sum_{l=1}^{\infty} z^{-(l+1)} \sum_{k=0}^{l-1} S_{N,k} S_{N,l-1-k} \right]
\]

\[
= \partial_z \left[ \sum_{l=1}^{\infty} z^{-(l+1)} l \sum_{k=0}^{l-2} S_{N,k} S_{N,l-1-k} - \sum_{l=1}^{\infty} z^{-(l+1)} l S_{N,l} \right]
\]

\[
= \partial_z \left[ \sum_{l=1}^{\infty} z^{-(l+1)} l \sum_{k=0}^{l} S_{N,k} S_{N,l-1-k} - z^{-1} - G^N + z^{-1} \right]
\]

\[
= \partial_z \left[ z (G^N)^2 - G^N = (G^N)^2 + 2 z G^N_G - G^N \right],
\]

(2.13)

If we combine (2.10)–(2.13) with (2.6), (2.7), and (2.10), we finally obtain the PDE

\[
G^N_t (t, z)
\]

\[
= (p + q) z^{N^2} (t, z) + (p + q) G^N (t, z) + \partial_{z z} \left( z^2 G^N (t, z) \right) - a (p - q - b (p + q)) G^N_z (t, z)
\]

\[
+ 2 ab \partial_{z z} \left( z G^N (t, z) \right) - (1 - b^2) a G^N_z (t, z) - 2 N a (1 - b^2) G^N (t, z) G^N_z (t, z)
\]

\[
+ 2 N \left[ z^2 G^N (t, z) G^N_z (t, z) + z (G^N (t, z))^2 - z G^N_z (t, z) - G^N (t, z) \right]
\]

\[
+ 2 b N a (G^N)^2 + 2 z G^N_G - G^N \right),
\]

(2.15)

for the Cauchy transforms \( G^N (t, z) \) of the measures \( \mu_{N,t} \). This PDE can be used to derive limit theorems for the \( \mu_{N,t} \) under different assumptions on the parameters \( p = p_N, q = q_N, a_N, b_N \) for \( N \to \infty \) and \( t \geq 0 \). We present such limit results in the next section where in the limit roughly free sums of the limit initial distributions with Wigner- and Marchenko-Pastur distributions appear.
3. Wigner- and Marchenko-Pastur-type limit theorems in the deterministic case

In this section we study several conditions on the parameters $p_N, q_N, a_N, b_N$ above leading to limit results for the measures $\mu_{N,t}$ which involve semicircle and Marchenko-Pastur distributions. In both cases, we consider $a_N \to \infty$ which implies that we must work possibly with measures with noncompact supports. We thus need some condition on the moment $s$. We recapitulate e.g. from [A] that a probability measure $\mu$ in both cases, we consider $\mu$.

It is well-known that $\mu \in M^1(\mathbb{R})$ satisfies the Carleman condition if the moments $c_l = \int_{\mathbb{R}} x^l \, d\mu(x)$ ($l \in \mathbb{N}$), of $\mu$ satisfy

$$\sum_{l=1}^{\infty} c_{2l} \gamma^{2l} = \infty. \quad (3.1)$$

By [A], a probability measure with the Carleman condition is determined uniquely by its moments.

We also recapitulate the R-transform of $\mu \in M^1(\mathbb{R})$ from [AGZ], which is given by $R_\mu(z) := \sum_{n=0}^{\infty} k_{n+1}(\mu)z^n$ with the $n$-th free cumulants $k_n(\mu)$ of $\mu$. It is related to the Cauchy transform by

$$R_\mu(G_\mu(z)) = z - 1/G_\mu(z). \quad (3.2)$$

Furthermore, the R-transform satisfies $R_{\mu \boxplus \nu} = R_\mu + R_\nu$ for $\mu, \nu \in M^1(\mathbb{R})$ for the free additive convolution $\boxplus$.

We shall also use the following notation: We denote the image of some probability measure $\mu \in M^1(\mathbb{R})$ under some continuous mapping $f$ by $f(\mu)$. We use this notation in particular for the maps $x \mapsto |x|$ and $x \mapsto x^2$ and write $|\mu|$ and $\mu^2$. Moreover, for a constant $v \in \mathbb{R} \setminus \{0\}$ let $v\mu$ the image of $\mu$ under the map $x \mapsto vx$. Finally, for a probability measure $\mu$ on $[0,\infty[$, let $\mu_{\text{even}}$ the unique even probability measure on $\mathbb{R}$ with $|\mu_{\text{even}}| = \mu$.

With these notations we have $G_{v\mu}(z) = v^{-1}G_\mu(z/v)$ and thus, by (3.2),

$$R_{v\mu}(z) = vR_\mu(vz). \quad (3.3)$$

We now turn to the first limit case where semicircle laws $\mu_{sc,\lambda} \in M^1(\mathbb{R})$ with radius $\lambda > 0$ appear. We recapitulate that the Wigner law $\mu_{sc,\lambda}$ with radius $\lambda > 0$ has the Lebesgue density

$$\frac{2}{\pi \lambda^2} \sqrt{\lambda^2 - x^2} 1_{[-\lambda,\lambda]}(x).$$

It is well-known that $R_{\mu_{sc,\lambda}}(z) = \lambda^2 z^2$; see Section 5.3 of [AGZ]. We have the following first result:

**Theorem 3.1.** Consider sequences $(p_N)_{N \in \mathbb{N}}, (q_N)_{N \in \mathbb{N}} \subset [0,\infty[$ with $\lim_{N \to \infty} p_N/N = \infty$ and $\lim_{N \to \infty} q_N/N = \infty$ such that $C := \lim_{N \to \infty} p_N/q_N \geq 0$ exists. Define

$$a_N := \frac{q_N}{\sqrt{p_N}}, \quad b_N := \frac{p_N - q_N}{p_N + q_N} \quad (N \in \mathbb{N}).$$

Let $\mu \in M^1(\mathbb{R})$ be a probability measure such that its moments $c_l$ satisfy $|c_l| \leq (\gamma l)^l$ for $l \in \mathbb{N}_0$ with some constant $\gamma > 0$. Moreover, let $(x_N)_{N \in \mathbb{N}} = ((x_1^N, \ldots, x_N^N))_{N \in \mathbb{N}}$ be a sequence of starting vectors $x_N \in A_N$ such that all moments of the empirical measures

$$\mu_{N,0} := \frac{1}{N} \sum_{i=1}^{N} \delta_{a_N(x_i^N-b_N)}$$

tend to those of $\mu$ for $N \to \infty$. Let $x_N(t)$ be the solutions of the ODEs (1.6) with $x_N(0) = x_N$ for $N \in \mathbb{N}$. Then for all $t > 0$, all moments of the empirical measures

$$\mu_{N, t/(p_N + q_N)} := \frac{1}{N} \sum_{i=1}^{N} \delta_{a_N(x_i^N(t/(p_N + q_N)-b_N)}$$
tend to those of the probability measures

\[ \mu_t := (e^{-t} \mu) \boxplus \left( \sqrt{1 - e^{-2t} \mu_{sc, A(1+C)^{-3/2}}} \right). \]  

**Proof.** Using the recurrence relations (2.6), (2.7) together with the initial conditions for \( t = 0 \) and our choice of \( b_N \), we see that the moments \( \tilde{S}_{N,l}(t) := S_{N,l}(t/(p_N + q_N)) \) of \( \mu_{N,t/(p_N + q_N)} \) satisfy

\[ \tilde{S}_{N,0} \equiv 1, \quad \tilde{S}_{N,1}(t) = e^{-t}S_{N,1}(0) \]

and, for \( l \geq 2 \),

\[ \tilde{S}_{N,l}(t) = \exp\left((-t + \frac{l(l-1)}{p_N + q_N})t\right) \left[S_{N,l}(0) + \frac{l}{p_N + q_N} \int_0^t \exp\left((-t + \frac{l(l-1)}{p_N + q_N})s\right) \left(2a_Nb_N(l-1)\tilde{S}_{N,l-1}(s) - (1 - b_N^2)a_N^2(l-1)\tilde{S}_{N,l-2}(s) + Na_N^2(l-1) \sum_{k=0}^{l-2} \tilde{S}_{N,k}(s)\tilde{S}_{N,l-2-k}(s) - N \sum_{k=0}^{l-2} \tilde{S}_{N,k+1}(s)\tilde{S}_{N,l-1-k}(s) - 2b_NNa_N \sum_{k=0}^{l-2} \tilde{S}_{N,k}(s)\tilde{S}_{N,l-1-k}(s)\right) ds \right]. \]

As the starting moments \( S_{N,l}(0) \) converge to the corresponding moments of \( \mu \) for \( N \to \infty \), we conclude by induction on \( l \), that the \( \tilde{S}_{N,l}(t) \) converge to some functions \( S_l(t) \) for \( l \geq 0 \) and \( t \geq 0 \). Moreover, these limits satisfy

\[ S_0 \equiv 1, \quad S_1(t) = S_1(0)e^{-t}, \quad S_l(t) = e^{-lt} \left(S_l(0) + 4l(1+C)^{-3} \int_0^t e^{ls} \sum_{k=0}^{l-2} S_k(s)S_{l-2-k}(s) ds \right) \]

for \( l \geq 2 \). We will now prove that the \( S_l(t) \) satisfy the Carleman condition \( (3.6) \) for \( t > 0 \) so that, by the moment convergence theorem, there exist unique \( \mu_t \in M^1(\mathbb{R}) \) with \( (S_l(t))_t \) as sequences of moments. For this we show that there exists an \( R > 1 \) such that \( |S_l(t)| \leq (Rt)^l \) for all \( t \geq 0 \) and \( l \in \mathbb{N}_0 \). Clearly this holds for \( l \in \{0,1\} \) for \( R \) sufficiently large. Moreover, by induction we have for \( l \geq 2 \) and \( t \geq 0 \) that

\[ |S_l(t)| \leq e^{-lt}|S_l(0)| + e^{-lt}4l(1+C)^{-3} \int_0^t e^{ls} \sum_{k=0}^{l-2} |S_k(s)||S_{l-2-k}(s)| ds \]

\[ = e^{-lt}|S_l(0)| + 4l(1+C)^{-3} \int_0^t e^{-ls} \sum_{k=0}^{l-2} |S_k(t-s)||S_{l-2-k}(t-s)| ds \]

\[ \leq (\gamma l)^l + 4(1+C)^{-3}(Rl)^l \leq (\gamma l)^l + R^l - 2l^l. \]

For \( R \) large enough (depending on \( \gamma \)) we can bound the RHS of \( (3.7) \) by \( (Rl)^l \) as claimed. We thus see that \( (S_l(t))_{t \in \mathbb{N}_0} \) satisfies the Carleman condition for \( t \geq 0 \). We thus conclude that the measures \( \mu_{N,t/(p_N + q_N)} \) tend weakly to some probability measures \( \mu_t \).

To identify the \( \mu_t \) we employ a PDE for the corresponding Cauchy and R-transforms. We set

\[ G(t, z) := G_{\mu_t}(z) = \lim_{N \to \infty} G_{\mu_{N,t/(p_N + q_N)}}(z). \]
We now use the PDEs \(2.15\) and interchange derivatives w.r.t. \(t, z\) with the limits \(N \to \infty\). This interchangeability can be proved via the Laurent series for \(G, G_N\) as in Proposition 2.9 of [VW1]. In this way we obtain that \(G\) satisfies the PDE

\[
G_t(t, z) = zG_z(t, z) + G(t, z) - 8(1 + C)^{-3}G(t, z)G_z(t, z), \quad G(0, z) = G_\mu(z).
\]

Using the transformation rules

\[
R(t, G(t, z)) = z - 1/G(t, z)
\]

\[
R_z(t, G(t, z)) = 1/G_z(t, z) + 1/G^2(t, z)
\]

for the \(R\)-transforms \(R(t, z) := R_\mu(z)\), we see that

\[
R_t(t, z) = -R(t, z) + 8(1 + C)^{-3}z - R_z(t, z)z, \quad R(0, z) = R_\mu(z).
\]

As the solution of (3.9) is given by

\[
R(t, z) = e^{-t}R_\mu(z) + 4(1 + C)^{-3}(1 - e^{-2t})z,
\]

it follows from (3.3) and the further properties of the \(R\)-transform mentioned above that

\[
\mu_t = (e^{-t} \mu) \boxplus \left( \sqrt{1 - e^{-2t} \mu_{sc,4(1+C)^{-3/2}}} \right)
\]

as claimed. \(\square\)

**Remark 3.2.** The exchange of the \(p_N, q_N\) in our dynamical systems corresponds to a sign change (and thus a reverse numbering) of all particles in \([-1, 1]\). In this way we may assume w.l.o.g. that \(C := \lim_{N \to \infty} p_N/q_N \in [0, 1]\) holds in Theorem 3.1. Moreover, the degenerated case \(C = \infty\) corresponds to the degenerated case \(C = 0\) and is thus also included in Theorem 3.1 in principle.

In order to understand the meaning of Theorem 3.1, consider the following example:

**Example 3.3.** Let \(p_N, q_N, a_N, b_N\) be given as in Theorem 3.1 and take \(x_i^N := b_N\) for all \(i, N\). Then all \(\mu_{N,0} = \delta_0\) and \(\rho_0 = \delta_0\). In this case the measures \(\mu_t\) from (3.3) are the semicircle laws \(\mu_t = \sqrt{1 - e^{-2t} \mu_{sc,4(1+C)^{-3/2}}}\) for \(t > 0\). These measures describe the deviation of the particles \(x_i^N(t)\) at time \(t/(p_N + q_N)\) from the numbers \(b_N \in [-1,1]\) locally w.r.t. to the space scalings \(a_N\).

Notice that this even makes sense for the degenerated case \(C = 0\) where \(\lim_{N \to \infty} b_N = -1\) holds.

In summary, Theorem 3.1 is a local limit theorem which describes the behaviour of the system around the numbers \(b_N\) for small times. It is therefore astonishing that in the limit (3.4) a stationary behaviour appears which is available on the global scale of the particle processes on \([-1, 1]\). This picture appears also in a variant of Theorem 3.1 in the degenerated case \(C = 0\) in the following Theorem 3.4. However, this stationarity disappears if we use scalings in space and time of higher orders than in Theorem 3.1 see Theorem 5.5 below.

**Theorem 3.4.** Consider sequences \((p_N)_{N \in \mathbb{N}}, (q_N)_{N \in \mathbb{N}} \subset [0, \infty)\) with \(\lim_{N \to \infty} p_N/N = \infty\) and \(\lim_{N \to \infty} q_N/N = \infty\) and \(C := \lim_{N \to \infty} p_N/q_N = 0\). Define

\[
a_N := \frac{\sqrt{q_N}}{\sqrt{N}}, \quad b_N := \frac{p_N - q_N}{p_N + q_N} \quad (N \in \mathbb{N}).
\]

Let \(\mu \in M^1(\mathbb{R})\) be a starting measure and \((x_N)_{N \in \mathbb{N}}\) starting vectors \(x_N \in A_N\) as in Theorem 3.1. Let \(x_N(t)\) be the solutions of the ODEs (1.1) with \(x_N(0) = x_N\) for \(N \in \mathbb{N}\). Then for all \(t > 0\), all moments of the measures \(\mu_{N,t/(p_N+q_N)}\) as in Theorem 3.1 tend to those of the measure \(e^{-t} \mu\).
Proof. The proof is completely analogue to that of Theorem \ref{thm:3.1}. We thus skip the proof. We only point out that the limit can be interpreted as \((e^{-t}\mu) \boxplus (\sqrt{1-e^{-2t}}\mu_{sc,0})\) where the semicircle law degenerates into \(\mu_{sc,0} = \delta_0\).

We next consider a further variant of Theorem \ref{thm:3.1} with a different scaling in space and time where the limit loses its stationary behaviour, and where the limit corresponds to the results for the Bessel processes of type A and their frozen versions in Sections 2 and 3 of \cite{VW1}. We point out that here the conditions on the parameters \(p_N, q_N, b_N\) are much more flexible, and that this result admits an analogue for Jacobi processes on noncompact spaces; see Section 5.

**Theorem 3.5.** Consider sequences \((p_N)_{N \in \mathbb{N}}, (q_N)_{N \in \mathbb{N}} \subset [0, \infty[\) with \(p_N, q_N > N - 1\) for \(N \geq 1\). Let \((b_N)_{N \in \mathbb{N}} \subset [-1, 1]\) be any sequence such that \(B := \lim b_N \in [-1, 1]\) exists. Let \((s_N)_{N \in \mathbb{N}} \subset [0, \infty[\) be a sequence of time scalings with

\[
\lim_{N \to \infty} \frac{p_N + q_N}{\sqrt{N s_N}} = 0,
\]

and define the space scalings \(a_N := \sqrt{s_N/N}\).

Let \(\mu \in M^1(\mathbb{R})\) be a starting measure and \((x_N)_{N \in \mathbb{N}}\) starting vectors as in Theorem \ref{thm:3.1}. Let \(x_N(t)\) be the solutions of the ODEs (\ref{eq:1.6}) with \(x_N(0) = x\) for \(N \in \mathbb{N}\). Then for all \(t > 0\), all moments of the empirical measures

\[
\mu_{N,t/s_N} = \frac{1}{N} \sum_{i=1}^{N} \delta_{a_N(x_N^{N}(t/s_N) - b_N)}
\]

tend to those of \(\mu \boxplus \mu_{sc,3\sqrt{2(1-B^2)}}\).

**Proof.** The proof is again analog to that of Theorem \ref{thm:3.1}. In fact, the recurrence relations (\ref{eq:2.6}), (\ref{eq:2.7}) show that here the moments \(\tilde{S}_{N,l}(t) := S_{N,l}(t/s_N)\) of \(\mu_{N,t/s_N}\) satisfy

\[
\frac{d}{dt} \tilde{S}_{N,1} = -\frac{p_N + q_N}{s_N} \tilde{S}_{N,1} + \frac{a_N(p_N - q_N - b_N(p_N + q_N))}{s_N} \to 0
\]

for \(l \geq 2,
\]

\[
\frac{d}{dt} \tilde{S}_{N,l} = l \left[ \frac{(p_N - q_N - b_N(p_N + q_N - 2(l - 1)))a_N}{s_N} \tilde{S}_{N,l-1} - \frac{p_N + q_N - (l - 1)}{s_N} \tilde{S}_{N,l}(t) - \frac{a_N^2}{s_N}(1 - b_N^2)(l - 1) \tilde{S}_{N,l-2}(t) + \sum_{k=0}^{l-2} \frac{N a_N^2}{s_N} \tilde{S}_{N,k}(t) \tilde{S}_{N,l-2-k}(t) \right]
\]

\[
\tilde{S}_{N,0} = 1,
\]

\[
\tilde{S}_{N,1}(t) = \left[ (p_N - q_N - b_N(p_N + q_N - 2(l - 1)))a_N \right] \tilde{S}_{N,l-1} - \frac{p_N + q_N - (l - 1)}{s_N} \tilde{S}_{N,l}(t)
\]

\[
\tilde{S}_{N,l} \to \tilde{S}_{N,l}(t) = l(1 - B^2) \sum_{k=0}^{l-2} \tilde{S}_{N,k}(t) \tilde{S}_{N,l-2-k}(t).
\]

Our starting conditions and induction show that the \(\tilde{S}_{N,l}(t)\) tend to some functions \(S_l(t)\) with

\[
S_0 \equiv 1, \quad S_1(t) = S_1(0), \quad S_l(t) = S_l(0) + l(1 - B^2) \int_0^t \sum_{k=0}^{l-2} S_k(s) S_{l-2-k}(s) ds
\]

for \(l \geq 2\) and \(t \geq 0\). The computations in Section 2 of \cite{VW1} (see in particular the proofs of Lemma 2.4 and Theorem 2.10 there) now yield the claim similar to the proof of Theorem \ref{thm:3.1} \hfill \(\square\)
Furthermore, with a slight modification in the assumptions:

**Theorem 3.6.** Consider sequences \((p_N)_{N \in \mathbb{N}}, (q_N)_{N \in \mathbb{N}} \subset [0, \infty]\) with \(\lim_{N \to \infty} (p_N + q_N)/N = \infty\). Let \((b_N)_{N \in \mathbb{N}} \subset [-1, 1]\) be any sequence such that \(B := \lim b_N \in [-1, 1]\) exists. Let \((s_N)_{N \in \mathbb{N}} \subset [0, \infty]\) be a sequence of time scalings with

\[
\lim_{N \to \infty} \frac{p_N + q_N}{\sqrt{N} s_N} \geq 0,
\]

and define the space scalings \(a_N := \frac{q_N}{\sqrt{N} p_N}\), \(b_N := \frac{p_N - q_N}{p_N + q_N}\) \((N \in \mathbb{N})\).

Let \(\mu \in M^1(\mathbb{R})\) be a starting measure and \((x_N)_{N \in \mathbb{N}}\) starting vectors as in Theorem 3.1. Let \(x_N(t)\) be the solutions of the ODEs (1.6) with \(x_N(0) = x_N\) for \(N \in \mathbb{N}\). Then for all \(t > 0\), all moments of the empirical measures

\[
\tilde{\mu}_{N,t/s_N} = \frac{1}{N} \sum_{i=1}^{N} \delta_{a_N(x_N(t/s_N) - b_N)}
\]

tend to those of \(\mu \boxplus \mu_{sc,2 \sqrt{2(1-B^2)}t} \boxplus \delta_0\).

In the next step we use the ideas of the proof of Theorem 3.1 in combination with Theorem 1.1 which says that the vectors with the ordered zeros of corresponding Jacobi polynomials form stationary solutions of the ODEs (1.6). This leads to the following limit result on the empirical measures of the zeros of the Jacobi polynomials which was derived in [DS] by different methods:

**Theorem 3.7.** Consider sequences \((p_N)_{N \in \mathbb{N}}, (q_N)_{N \in \mathbb{N}} \subset [0, \infty]\) with \(\lim_{N \to \infty} p_N/N = \infty\) and \(\lim_{N \to \infty} q_N/N = \infty\) such that \(C := \lim_{N \to \infty} p_N/q_N \geq 0\) exists. Define

\[
a_N := \frac{q_N}{\sqrt{N} p_N}, \quad b_N := \frac{p_N - q_N}{p_N + q_N} \quad (N \in \mathbb{N}).
\]

Let \(-1 < z_N^1 < \ldots < z_N^N < 1\) be the ordered zeros of the Jacobi polynomials \(P_N^{(q_N-N,p_N-N)}\). Then all moments of the empirical measures

\[
\tilde{\mu}_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{a_N(z_N^i - b_N)}
\]

tend to those of \(\mu_{sc,4(1+C)^{-3/2}}\). In particular, the \(\tilde{\mu}_N\) tend weakly to \(\mu_{sc,4(1+C)^{-3/2}}\).

**Proof.** Consider the solutions of the ODEs (1.6) as in Theorem 3.1 with the initial conditions \(x_N := (b_N, \ldots, b_N) \in A_N\), i.e., with \(\mu = \delta_0\) and \(S_N, l(0) = 0\) for \(l \geq 1\). We show that for the moments \(\hat{S}_N(t)\) from the proof of Theorem 3.1 the limits \(\hat{S}_N(\infty) := \lim_{t \to \infty} \hat{S}_N(t)\) exist. In fact,
this is clear for \( l = 0, 1 \), and (3.5) and dominated convergence show inductively for \( l \geq 2 \) that
\[
\hat{S}_{N,l}(\infty) = \frac{l}{p_N + q_N} \lim_{t \to \infty} \int_0^t \exp\left( -\left( l - \frac{l(l-1)}{p_N + q_N} \right)(t-s) \right) H_{N,l}(s) \, ds \\
= \frac{l}{p_N + q_N} \lim_{t \to \infty} \int_0^t \exp\left( -\left( l - \frac{l(l-1)}{p_N + q_N} \right)s \right) H_{N,l}(t-s) \, ds \\
= \frac{1}{p_N + q_N - l + 1} \left( 2a_N b_N (l-1) \hat{S}_{N,l-1}(\infty) \\
- (1 - b_N^2) a_N^2 (l-1) \hat{S}_{N,l-2}(\infty) + N a_N^2 (1 - b_N^2) \sum_{k=0}^{l-2} \hat{S}_{N,k}(\infty) \hat{S}_{N,l-2-k}(\infty) \\
- N \sum_{k=0}^{l-2} \hat{S}_{N,k+1}(\infty) \hat{S}_{N,l-1-k}(\infty) - 2b_N a_N \sum_{k=0}^{l-2} \hat{S}_{N,k}(\infty) \hat{S}_{N,l-1-k}(\infty) \right)
\]
where \( H_{N,l}(s) \) is the term in the big brackets in the last 3 lines of (3.5). On the other hand, as this is just the recurrence for the Catalan numbers up to some rescaling (see e.g. Section 2.1.1 of [AGZ]), it follows readily that the \( \hat{S}_{N,l}(\infty) \) are the moments of the empirical measures \( \tilde{\mu}_N \). Furthermore, similar to (3.3), we see that for all \( l \) the limits \( S_l(\infty) := \lim_{N \to \infty} \hat{S}_{N,l}(\infty) \) exist with \( S_0(\infty) = 1 \), \( S_1(\infty) = 0 \), and
\[
S_l(\infty) = \frac{4l}{(1 + C)^3} \sum_{k=0}^{l-2} S_k(\infty) S_{l-2-k}(\infty) \quad (l \geq 2).
\]
As this is just the recurrence for the Catalan numbers up to some rescaling (see e.g. Section 2.1.1 of [AGZ]), it follows readily that the \( S_l(\infty) \) are the moments of \( \mu_{sc,4(1+C)^{3/2}} \). \( \square \)

We next turn to the second limit case which concerns Marchenko-Pastur distributions, and which is motivated by Corollary 2.5 of [DS]. We here assume that the sequences \((p_N)_{N \in \mathbb{N}}, (q_N)_{N \in \mathbb{N}}\) satisfy
\[
\lim_{N \to \infty} p_N / N =: \tilde{p} \in [1, \infty[, \quad \lim_{N \to \infty} q_N / N =: \infty.
\]
We then choose the norming constants
\[
b_N := -1, \quad a_N := q_N / N.
\]
In this regime we will obtain a limit theorem which involves Marchenko-Pastur distributions. For this we recall that for \( c \geq 0, t > 0 \), the Marchenko-Pastur distribution \( \mu_{MP,c,t} \in M^1([0, \infty]) \) is the probability measure with \( \mu_{MP,c,t} = \tilde{\mu} \) for \( c \geq 1 \) and \( \mu_{MP,c,t} = (1 - c) \delta_0 + c \tilde{\mu} \) for \( 0 \leq c < 1 \), where for \( x_\pm := t(\sqrt{c} \pm 1)^2 \), the measure \( \tilde{\mu} \) on \([x_-, x_+]\) has the density
\[
\frac{1}{2\pi t} \sqrt{(x_+ - x)(x - x_-)}.
\]
We also recall (see Exercise 5.3.27 of [AGZ]) that the R-transforms of the Marchenko-Pastur distributions are given by
\[
R_{MP,c,t}(z) = \frac{ct}{1 - tz}.
\]
This in particular implies the following well-known relation
\[
\mu_{MP,a,t} \boxplus \mu_{MP,b,t} = \mu_{MP,a+b,t} \quad (a, b, t > 0).
\]
Theorem 3.8. Consider $p_N, q_N, a_N, b_N$ as in (3.14) and (3.15). Let $\mu \in M^1([0, \infty])$ be a probability measure such that its moments $c_l$ satisfy $|c_l| \leq (\gamma l)!$ for $l \in \mathbb{N}_0$ with some constant $\gamma > 0$. Moreover, let $(x_N)_{N \in \mathbb{N}} = ((x_1^N, \ldots, x_N^N))_{N \in \mathbb{N}}$ be an associated sequence of starting vectors $x_N \in A_N$ as described in Theorem 3.1.

Let $x_N(t)$ be the solutions of the ODEs (1.10) with start in $x_N(0) = x_N$ for $N \in \mathbb{N}, t \geq 0$. Then for all $t > 0$, all moments of the empirical measures

$$\mu_{N,t/(p_N+q_N)} = \frac{1}{N} \sum_{i=1}^{N} \delta_{\alpha_N(x_N^N(t/(p_N+q_N))-b_N)}$$

tend to those of the probability measures

$$\mu(t) := \left(\mu_{SC,2\sqrt{2(1-e^{-t})}} \boxplus (e^{-t}\mu)_{ev} \right)^2 \boxplus \mu_{MP,\hat{\theta}_{0,1,2}(1-e^{-t})}, \quad t > 0. \tag{3.19}$$

Proof. As in the proof of Theorem 3.1 we see that the recurrence relations (2.6), (2.7) together with the initial conditions for $t = 0$ and our choice of $b_N$, that the moments $\tilde{S}_{N,l}(t) := S_{N,l}(t/(p_N+q_N))$ of $\mu_{N,t/(p_N+q_N)}$ satisfy

$$\tilde{S}_{N,0} = 1, \quad \tilde{S}_{N,1}(t) = e^{-t} \left( S_{N,1}(0) - \frac{2a_N p_N}{(p_N+q_N)} \right) + \frac{2a_N p_N}{(p_N+q_N)}$$

and, for $l \geq 2$,

$$\tilde{S}_{N,l}(t) = e^{-t} \left( S_{N,l}(0) \right) \left( 1 + \frac{l(l-1)}{p_N + q_N} \right) + \frac{l}{p_N + q_N} \int_0^t \exp\left( \left( l - \frac{l(l-1)}{p_N + q_N} \right) s \right) \left( 2a_N(p_N - 2(l-1)) \tilde{S}_{N,l-1}(s) - N \sum_{k=0}^{l-2} \tilde{S}_{N,k+1}(s) \tilde{S}_{N,l-1-k}(s) + 2a_N \sum_{k=0}^{l-2} \tilde{S}_{N,k}(s) \tilde{S}_{N,l-1-k}(s) \right) \, ds \right]. \tag{3.20}$$

As the starting moments $S_{N,l}(0)$ ($l \geq 0$) converge to the corresponding moments of $\mu$ for $N \to \infty$, we conclude by induction on $l$, that the $\tilde{S}_{N,l}(t)$ converge to some functions $S_l(t)$ for $l \geq 0$ and $t \geq 0$. Moreover, these limits satisfy

$$S_0 = 1, \quad S_1(t) = e^{-t} (S_1(0) - 2\hat{p}) + 2\hat{p},$$

$$S_l(t) = e^{-t} \left( S_l(0) + 2l \int_0^t e^{ls} \left( \hat{p}S_{l-1}(s) + \sum_{k=0}^{l-2} \tilde{S}_{k}(s)S_{l-1-k}(s) \right) \, ds \right), \quad l \geq 2. \tag{3.21}$$

Analogously to the the proof of Theorem 3.1 one can show that the $S_l(t)$ satisfy the Carleman condition (3.1) for $t > 0$. Thus, by the moment convergence theorem there exist unique $\mu_t \in M^1(\mathbb{R})$ with $(S_l(t))_l$ as sequences of moments.

To identify the $\mu_t$ we again derive a PDE for the Cauchy and R-transforms of the $\mu_t$. We set

$$G(t, z) := G_{\mu_t}(z) = \lim_{N \to \infty} G_{\mu_{N,t/(p_N+q_N)}}(z).$$

The PDEs (3.15) here lead to the PDE

$$G_t(t, z) = z G_z(t, z) + G(t, z) - 2(G(t, z)^2 + 2zG(t, z)G_z(t, z) - G_z(t, z)) - 2\hat{p}G_z(t, z)$$

$$= (z - 2(\hat{p} - 1) - 4zG(t, z))G_z(t, z) + G(t, z) - 2G(t, z)^2. \tag{3.22}$$
Using (3.8), we obtain

\[-R(t, G(t, z)) = \frac{G_t(t, z)}{G_z(t, z)} = (R(t, G(t, z)) + \frac{1}{G(t, z)})(1 - 4G(t, z)) - 2(\hat{p} - 1)
\]

\[+ (G(t, z) - 2G(t, z)^2)(R_z(t, G(t, z)) - \frac{1}{G(t, z)^2})\]

\[= - 4G(t, z)R(t, G(t, z)) - 2(\hat{p} - 1) - 2 + (G(t, z) - 2G(t, z)^2)R_z(t, G(t, z)) + R(t, G(t, z))\]

and thus

\[0 = R_t(t, z) - (2z^2 - z)R_z(t, z) - (4z - 1)R(t, z) - 2\hat{p}.\]

If \(\phi(z) := R(0, z)\), the method of characteristics (see e.g. [SV]) leads to the solution

\[R(t, z) = e^{-t}(1 - 2z(1 - e^{-t}))^{-2}\phi(e^{-t}z(1 - 2z(1 - e^{-t})^{-1})) + \frac{2(1 - e^{-t})}{1 - 2(1 - e^{-t})z} + \frac{2(\hat{p} - 1)(1 - e^{-t})}{1 - 2(1 - e^{-t})z}.
\]

(3.23)

The third summand on the RHS of this equation corresponds to the second \(\oplus\)-summand in (3.19).

We thus only have to investigate the first two summands on the RHS of (3.23). For this we fix \(s > 0\) and define the function \(\hat{\phi}(z) := e^{-s}\phi(e^{-s}z)\). We also define

\[f(t, z) := (1 - tz)^{-2}\hat{\phi}\left(\frac{z}{1 - tz}\right) + \frac{t}{1 - tz} \quad (z \in \mathbb{C} \setminus \mathbb{R}, t > 0).
\]

With the abbreviation \(\hat{z} := \frac{z}{1 - tz}\) we then obtain

\[f_t(t, z) = 2z(1 - tz)^{-3}\hat{\phi}(\hat{z}) + \frac{z^2}{(1 - tz)^4}\hat{\phi}'(\hat{z}) + \frac{1}{(1 - tz)^2}
\]

\[= \frac{2z(1 - tz) + 2tz^2}{(1 - tz)^3}\hat{\phi}(\hat{z}) + \frac{z^2}{(1 - tz)^4}\hat{\phi}'(\hat{z}) + \frac{2zt(1 - tz) + t^2z^2 + (1 - tz)^2}{(1 - tz)^2}
\]

\[= \frac{2tz^2}{(1 - tz)^3}\hat{\phi}(\hat{z}) + \frac{z^2}{(1 - tz)^4}\hat{\phi}'(\hat{z}) + \frac{t^2z^2}{(1 - tz)^2} + \frac{2z}{(1 - tz)^2}\hat{\phi}(\hat{z}) + \frac{2zt}{1 - tz} + 1
\]

\[= z^2f_z(t, z) + 2zf(t, z) + 1.
\]

Therefore, our \(f\) solves the PDE

\[f_t(t, z) = 1 + 2zf(t, z) + z^2f_z(t, z), \quad f(0, z) = R_{\exp(-s)\mu}(z).
\]

(3.24)

Theorem 4.8 in [VWT] and (3.8) now imply that

\[f(t, z) = R_{\mu_{S,C,2\nu\Theta}}(\sqrt{\exp(-s)\mu})_{\text{even}}(z) \quad \text{for } t > 0.
\]

This and the formula \(R_{\mu_{\oplus}0} = R_\mu + R_\nu\) for the R-transform now complete the proof. \(\square\)

Remark 3.9. If we take the starting distribution \(\mu = \mu_{MP,r,s}\) for \(r \geq 0, s > 0\), then, with the notations of the preceding proof, \(\hat{\phi}(z) = R_{\mu_{MP,r,s}}(z) = \frac{r}{1 - rz}\). A partial fraction decomposition here leads to

\[f(t, z) = (1 - tz)^{-2}\hat{\phi}\left(\frac{z}{1 - tz}\right) + \frac{t}{1 - tz} = \frac{rs}{(1 - tz)(1 - (t + s)z)} + \frac{t}{1 - tz} = \frac{r(t + s)}{1 - (t + s)z} + \frac{(1 - r)t}{1 - tz}.
\]
This leads to

\[ R_{\mu_{SC,\sqrt{\varphi (p_{MP,b})}_{even}}}(z) = \frac{r(t+s)}{1-(t+s)z} + \frac{(1-r)t}{1-tz}, \quad r, s, t \geq 0 \]

which generalizes (4.14) in [VW1] slightly.

Similarly to Theorem 3.5 we now consider a variant of Theorem 3.8 with a different scaling in space and time where the limit loses its stationary behaviour, and where the limit corresponds to the results for the Bessel processes of type B and their frozen versions in Sections 4 and 5 of [VW1].

**Theorem 3.10.** Consider sequences \((p_N)_{N \in \mathbb{N}}, (q_N)_{N \in \mathbb{N}} \subset \mathbb{R}_+\) with \(p_N, q_N > N - 1\) for \(N \geq 1\) and \(\lim_{N \to \infty} p_N/N = \hat{p}\). Let \((s_N)_{N \in \mathbb{N}} \subset \mathbb{R}_+\) be a sequence of time scalings with \(\lim_{N \to \infty} (p_N + q_N)/s_N = 0\). Define the space scalings \(a_N := s_N/N, b_N := -1 (N \in \mathbb{N})\). Let \(\mu \in M^1([0, \infty[)\) be a starting measure and \((x_N)_{N \in \mathbb{N}}\) starting vectors as in Theorem 3.8. Let \(x_n(t)\) be the solutions of the ODEs \((\ref{eq:ode})\) with \(x_N(0) = x_N\) for \(N \in \mathbb{N}\). Then for all \(t > 0\), all moments of the empirical measures

\[ \mu_{N,t/s_N} = \frac{1}{N} \sum_{i=1}^{N} \delta_{\mu_N(x_N^i(t/s_N) - b_N)} \]

tend to those of \(\mu_{SC, 2\sqrt{\hat{p}}} \circ (\sqrt{\hat{p}})_{even}^2 \circ \mu_{MP,b-1,2\hat{p}}\).

**Proof.** The proof is analogous to that of Theorem 3.8. In fact, the recurrence relations \((\ref{eq:recurrence1}), (\ref{eq:recurrence2})\) show that here the moments \(\tilde{S}_{N,0}(t) := S_{N,0}(t/s_N)\) of \(\mu_{N,t/s_N}\) satisfy

\[
\tilde{S}_{N,0}(t) = 1, \quad \frac{d}{dt} \tilde{S}_{N,1}(t) = -\frac{p_N + q_N}{s_N} \tilde{S}_{N,1}(t) + \frac{a_N(p_N - q_N - b_N(p_N + q_N))}{s_N} \tilde{S}_{N,1}(t) \to 2\hat{p}
\]

and, for \(l \geq 2\),

\[
\frac{d}{dt} \tilde{S}_{N,l}(t) = \left[ \frac{(p_N - q_N - b_N(p_N + q_N) - 2(l - 1))a_N}{s_N} \tilde{S}_{N,l-1}(t) - \frac{p_N + q_N - (l - 1)}{s_N} \tilde{S}_{N,l-1}(t) \right] + \frac{N a_N^2 (1 - b_N^2)}{s_N} (l - 1) \tilde{S}_{N,l-2}(t) + \frac{N a_N^2}{s_N} (1 - b_N^2) \sum_{k=0}^{l-2} \tilde{S}_{N,k}(t) \tilde{S}_{N,l-2-k}(t)
\]

\[
- \frac{N}{s_N} \sum_{k=0}^{l-2} \tilde{S}_{N,k+1}(t) \tilde{S}_{N,l-1-k}(t) - 2a_N b_N N \sum_{k=0}^{l-2} \tilde{S}_{N,k}(t) \tilde{S}_{N,l-1-k}(t)
\]

\[ N \to \infty 2l\hat{p} \tilde{S}_{N,l-1}(t) + 2l \sum_{k=0}^{l-2} \tilde{S}_{N,k}(t) \tilde{S}_{N,l-1-k}(t). \quad (3.25)\]

Our starting conditions and induction show that the \(\tilde{S}_{N,l}(t)\) tend to some functions \(S_l(t)\) with

\[
S_0 \equiv 1, \quad S_1(t) = S_1(0) + 2\hat{p}t, \quad S_l(t) = S_l(0) + 2l \int_0^t \left[ \hat{p} S_{l-1}(s) + \sum_{k=0}^{l-2} S_k(s) S_{l-1-k}(s) \right] ds \quad (3.26)
\]

for \(l \geq 2\) and \(t \geq 0\). The computations in Section 4 of [VW1] (see in particular the proofs of Lemma 4.3 and Theorem 4.8 there) now yield the claim similar to the proof of Theorem 3.1. \(\square\)

A slight modification of the proof of Theorem 3.8 in combination with the assertion about the stationary case in Theorem 3.1 leads to the following limit result on the zeros of the Jacobi
polynomials which was derived in [DS] by different methods. As the modification is completely analogous to the relations between Theorems 3.7 and 3.1, we skip the proof.

**Theorem 3.11.** Consider sequences $p_N, q_N$ with
\[
\lim_{N \to \infty} p_N/N =: \hat{p} \in [1, \infty], \quad \lim_{N \to \infty} q_N/N = \infty,
\]
and define the norming constants $b_N := -1$, $a_N := q_N/N$.

Let $-1 < z_1^N < \ldots < z_N^N < 1$ be the ordered zeros of the Jacobi polynomials $P_{N}^{(q_N^N, p_N^N)}$. Then all moments of the empirical measures
\[
\tilde{\mu}_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{a_N(z_i^N - b_N)}
\]
tend to those of
\[
\left(\mu_{SC, 2\sqrt{2}}\right)^2 \ast \mu_{MP, \hat{p} - 1, 2} = \mu_{MP, \hat{p}, 2}. \tag{3.27}
\]

In particular, the $\tilde{\mu}_N$ tend weakly to $\mu_{MP, \hat{p}, 2}$.

4. Almost sure limit theorems for Jacobi processes

In this section we study the empirical measures of the renormalized Jacobi processes $(\tilde{X}_t)_{t \geq 0}$ on $A_N$ from the introduction. Recall that these processes satisfy
\[
d\tilde{X}_{t,i} = \frac{\sqrt{2}}{\sqrt{\kappa}} \sqrt{1 - \tilde{X}_{t,i}^2} dB_{t,i} + \left((p_N - q_N) - (p_N + q_N)\tilde{X}_{t,i} + 2 \sum_{j: j \neq i} \frac{1 - \tilde{X}_{t,i} \tilde{X}_{t,j}}{\tilde{X}_{t,i} - \tilde{X}_{t,j}}\right) dt \tag{4.1}
\]
for $i = 1, \ldots, N$ with fixed $\kappa > 0$.

Let $a_N \subset ]0, \infty]$ and $b_N \subset \mathbb{R}$. As in Section 3 we investigate the empirical measures
\[
\mu_{N, t} := \frac{1}{N} \sum_{i=1}^{N} \delta_{a_N(x_i^N - b_N)}
\]
for $N \to \infty$ for appropriate scalings $a_N, b_N, s_N$. We begin with the following almost sure version of Theorem 3.1.

**Theorem 4.1.** Consider sequences $(p_N)_{N \in \mathbb{N}}, (q_N)_{N \in \mathbb{N}} \subset ]0, \infty]$ with $\lim_{N \to \infty} p_N/N = \infty$ and $\lim_{N \to \infty} q_N/N = \infty$ such that $C := \lim_{N \to \infty} p_N/q_N \geq 0$ exists. Define
\[
a_N := \frac{q_N}{\sqrt{Np_N}}, \quad b_N := \frac{p_N - q_N}{p_N + q_N} \quad (N \in \mathbb{N}).
\]

Let $\mu \in M^1(\mathbb{R})$ be a probability measure such that its moments $c_l$ satisfy $|c_l| \leq (\gamma l)!$ for $l \in \mathbb{N}_0$ with some constant $\gamma > 0$. Moreover, let $(x_N)_{N \in \mathbb{N}} = ((x_1^N, \ldots, x_N^N))_{N \in \mathbb{N}}$ be a sequence of starting vectors $x_N \in A_N$ such that all moments of the empirical measures
\[
\mu_{N, 0} := \frac{1}{N} \sum_{i=1}^{N} \delta_{a_N(x_i^N - b_N)}
\]
Furthermore, for tend to those of the probability measures \((\tilde{X}_t)_{i\geq 0}\) with start in \(X(0) = x\) for \(N \in \mathbb{N}, t \geq 0\). Then for all \(t > 0\), all moments of the empirical measures

\[
\mu_{N,t/(p_N+q_N)} = \frac{1}{N} \sum_{i=1}^{N} \delta_{(X_t^{N}_{i/(p_N+q_N)}),i} - b_N
\]

tend to those of the probability measures \((e^{-t} \mu) \boxplus \left( \sqrt{1 - e^{-2t}} \mu_{sc,4(1+C)^{-3/2}} \right)\) almost surely.

Before proving this theorem with the specific scaling there, we first proceed as in Section 2 and investigate arbitrary affine shifts of \(\tilde{X}_t\) first. For this, define \(Y_t := a_N(\tilde{X}_t/((p_N+q_N)) - b_N)\) and

\[
\mu_{N,t} = \frac{1}{N} \sum_{i=1}^{N} \delta_{Y_{t,i}}, \quad S_{N,t}(t) = \frac{1}{N} \sum_{i=1}^{N} Y_{t,i}^t
\]

which fits to the notation in our theorem. For abbreviation, we now suppress the dependence of \(p, q, a, b\) on \(N\). Then by Itô’s formula

\[
dY_{t,i} = \sqrt{\frac{2}{\kappa(p+q)}} \sqrt{a^2 - (Y_{t,i} + ab)^2} \, dB_{t,i} + \left[ a \left( \frac{p-q}{p+q} - b \right) - Y_{t,i} + \frac{2}{p+q} \sum_{j: j \neq i} a^2 \frac{(1-b^2) - Y_{s,j} Y_{t,i} - ab(Y_{s,i} + Y_{t,j})}{Y_{s,i} - Y_{t,j}} \right] \, dt. \tag{4.2}
\]

Furthermore, for \(l \in \mathbb{N}\) we define

\[
M_{l,t} := \frac{1}{N} \sqrt{\frac{2}{\kappa(p+q)}} \int_0^t \sum_{i=1}^{N} Y_{(s,i)}^{t-1} \sqrt{a^2 - (Y_{s,i} + ab)^2} \, dB_{s,i}. \tag{4.3}
\]

Note that all \((M_{l,t})_{l \geq 0}\) are continuous martingale (w.r.t. the usual filtration) since \(|Y_{t,i}| \leq a(1+|b|)\) holds for all \(i, t\). The first empirical moment now satisfies

\[
S_{N,1}(t) - S_{N,1}(0) = \sqrt{\frac{2}{\kappa(p+q)}} \frac{1}{N} \sum_{i=1}^{N} \int_0^t \sqrt{a^2 - (Y_{s,i} + ab)^2} \, dB_{s,i} + \frac{1}{N} \sum_{i=1}^{N} \int_0^t \left[ a \left( \frac{p-q}{p+q} - b \right) - Y_{s,i} + \frac{2}{p+q} \sum_{j: j \neq i} a^2 \frac{(1-b^2) - Y_{s,j} Y_{s,i} - ab(Y_{s,i} + Y_{s,j})}{Y_{s,i} - Y_{s,j}} \right] \, ds
\]

\[
= \int_0^t -S_{N,1}(s) + a \left( \frac{p-q}{p+q} - b \right) \, ds + M_{1,t}. \tag{4.4}
\]

This is a linear stochastic differential equation of the form

\[
f(t) - f(0) = \int_0^t (\lambda f(s) + g(s)) \, ds + h(t), \tag{4.4}
\]

where, in our case,

\[\lambda = -1, \quad f(t) = S_{N,1}(t), \quad g(t) = a \left( \frac{p-q}{p+q} - b \right), \quad h(t) = M_{1,t}.\]
As the solution of (4.4) is given by
\[
f(t) = e^{\lambda t} \left( f(0) + \int_0^t e^{-\lambda s} \left( g(s) + \lambda h(s) \right) \, ds \right) + h(t),
\]
we have
\[
S_{N,1}(t) = e^{-t} \left( S_{N,1}(0) + \int_0^t e^s \left( a \left( \frac{p-q}{p+q} - b \right) - M_{1,s} \right) \, ds \right) + M_{1,t}.
\]
By another application of Itô’s formula the higher empirical moments satisfy
\[
S_{N,l}(t) - S_{N,l}(0) = M_{l,t} + \int_0^t F_l(s) \, ds - \frac{2(l-1)}{\kappa(p+q)} \int_0^t S_{N,l}(s) + 2abS_{N,l-1}(s) - a^2(1-b^2)S_{N,l-2}(s) \, ds,
\]
where by the calculations in (2.10) and (2.11)
\[
F_l = -l \left[ \left( 1 - \frac{l-1}{p+q} \right) S_{N,l} - a \left( \frac{p-q}{p+q} - b \left( 1 - 2 \frac{l-1}{p+q} \right) \right) \right] S_{N,l-1}
\]
\[
+ \frac{a^2(1-b^2)(l-1)}{p+q} S_{N,l-2} - \frac{N a^2(1-b^2)}{p+q} \sum_{k=0}^{l-2} S_{N,k} S_{N,l-2-k}
\]
\[
+ \frac{N}{p+q} \sum_{k=0}^{l-2} S_{N,k+1} S_{N,l-1-k} + \frac{2bNa}{p+q} \sum_{k=0}^{l-2} S_{N,k} S_{N,l-1-k}.
\]
Rearranging (4.7) we obtain
\[
S_{N,l}(t) - S_{N,l}(0) = \int_0^t C_l S_{N,l}(s) + f_l(S_{N,1}(s), \ldots, S_{N,l-1}(s)) \, ds + M_{l,t},
\]
with
\[
C_l := -l \left( 1 + \frac{l-1}{p+q} \left( \frac{2}{\kappa} - 1 \right) \right)
\]
and
\[
f_l(S_{N,1}, \ldots, S_{N,l-1}) = -l \left( -a \left( \frac{p-q}{p+q} - b \left( 1 + \frac{2(l-1)}{p+q} \left( \frac{2}{\kappa} - 1 \right) \right) \right) + \frac{N a^2(1-b^2)(l-1)}{p+q} \right) S_{N,l-2}
\]
\[
- \frac{N a^2(1-b^2)}{p+q} \sum_{k=0}^{l-2} S_{N,k} S_{N,l-2-k} + \frac{N}{p+q} \sum_{k=0}^{l-2} S_{N,k+1} S_{N,l-1-k} + \frac{2bNa}{p+q} \sum_{k=0}^{l-2} S_{N,k} S_{N,l-1-k}.
\]
Moreover, by the triangle inequality and Jensen’s inequality,

\[ S_{N,l}(t) = e^{C_l t} \left( S_{N,l}(0) + \int_0^t e^{-C_l s} \left( f_1(S_{N,1}(s), \ldots, S_{N,l-1}(s)) + C_l M_{l,t} \right) \, ds \right) + M_{l,t}. \]  

(4.9)

For the proof of Theorem 4.1 and further limit theorems the following observation is crucial.

**Lemma 4.2.** Let \( T > 0 \). Let \( p, q, a, b \) as in Theorem 4.1 or Theorem 4.3 below. Assume that \( \lim_{N \to \infty} S_{N,l}(0) \) exists for all \( l \in \mathbb{N} \). Then for all \( l \in \mathbb{N} \) the martingales \((M_{l,t})_{t \geq 0}\) from (4.8) converge uniformly to 0 on \([0, T]\) a.s.

**Proof.** In a first step we show that the sequence \((E[|S_{N,l}(t)|])_{N \in \mathbb{N}}\) is uniformly bounded on \([0, T]\).

Here we first study the case \( l \in 2\mathbb{N} \). By (4.9) and our assumptions on \( p, q, a, b \) it holds, that there are non-negative bounded sequences \( d_1(N), \ldots, d_5(N) \) of numbers such that

\[ E(S_{N,l}(t)) \leq e^{C_l t} \left( S_{N,l}(0) + \int_0^t e^{-C_l s} \left( d_1 E[|S_{N,l-1}(s)|] + d_2 |S_{N,l-2}(s)| + d_3 \sum_{k=0}^{l-2} E[|S_{N,k}(s)S_{N,l-2-k}(s)|] \right. \right. \]

\[ + \left. \left. d_4 \sum_{k=0}^{l-2} E[|S_{N,k+1}S_{N,l-1-k}(s)|] + d_5 \sum_{k=0}^{l-2} E[|S_{N,k}(s)S_{N,l-1-k}(s)|] \right) \, ds \right). \]

Moreover, by the triangle inequality and Jensen’s inequality,

\[ |S_{N,l-1}(s)| \leq \left( \frac{1}{N} \sum_{i=1}^N Y_{s,i} \right)^{l-1} \leq \left( \frac{1}{N} \sum_{i=1}^N Y_{s,i}^l \right)^{\frac{l-1}{l}} \leq 1 + S_{N,l}(s). \]  

(4.10)

By the same reasons, we also have

\[ |S_{N,k}(s)S_{N,l-1-k}(s)| \leq \left( \frac{1}{N} \sum_{i=1}^N Y_{s,i} \right)^{|l-1|} \left( \frac{1}{N} \sum_{i=1}^N Y_{s,i}^l \right)^{\frac{|l-1|}{l}} \leq 1 + S_{N,l}(s), \]

\[ |S_{N,k}(s)S_{N,l-2-k}(s)| \leq S_{N,l-2}(s) \leq 1 + S_{N,l}(s) \quad \text{and} \quad |S_{N,k+1}(s)S_{N,l-1-k}(s)| \leq S_{N,l}(s). \]

Thus there exist non-negative bounded sequences \( \tilde{d}_1(N), \tilde{d}_2(N) \) of numbers such that

\[ e^{-C_l t} E[S_{N,l}(t)] \leq S_{N,l}(0) + \int_0^t e^{-C_l s} \left( \tilde{d}_1 + \tilde{d}_2 E[S_{N,l}(s)] \right) \, ds. \]

By Gronwall’s inequality we conclude that

\[ e^{-C_l t} E[S_{N,l}(t)] \leq \left( S_{N,l}(0) + \int_0^t \tilde{d}_1 e^{-C_l s} \, ds \right) \cdot \exp \left( \tilde{d}_2 t \right) \]

where the \( C_l \) from (4.8) remain bounded. Thus \((E[|S_{N,l}(t)|])_{N \in \mathbb{N}}\) remains uniformly bounded for \( t \in [0, T] \) in the case of even \( l \). Finally, by (4.10) this also holds for \( l \) odd.

In a second step we now show the claim of the lemma. As the Brownian motions \( B_i, B_j \) are independent for \( i \neq j \), the quadratic variation of \( M_{l,t} \) is given by

\[ [M_{l}]_t = \frac{2}{N^2 \kappa(p + q)} \sum_{i=1}^N \int_0^t l^2 Y_{s,i}^2 a^2 - (Y_{s,i} + ab)^2 \, ds. \]
By the Tchebychev inequality and the Burkholder-Davis-Gundy inequality there is a constant $c > 0$ independent from $N$ such that

$$P \left( \sup_{0 \leq t \leq T} |M_{l,t}| > \epsilon \right) \leq \frac{1}{\epsilon^2} E \left[ \sup_{0 \leq t \leq T} |M_{l,t}|^2 \right] \leq \frac{c}{\epsilon^2} E\left[ |M_t|^r \right].$$

$$= \frac{2c l^2}{N^2 \kappa(p + q)} \sum_{i=1}^{N} \int_{0}^{T} E \left[ Y_{s,i}^{2l-2} (a^2 - (Y_{s,i} + ab)^2) \right] \, ds$$

$$\leq \frac{2c l^2 a^2}{N^2 \kappa(p + q)} \sum_{i=1}^{N} \int_{0}^{T} E \left[ Y_{s,i}^{2l-2} \right] \, ds$$

$$= \frac{2c l^2 a^2}{N \kappa(p + q)} \int_{0}^{T} E \left[ S_{N,2l-2}(s) \right] \, ds.$$  

Note that in the case $b_N \equiv 1$ as in Theorem 4.3 we similarly get the bound

$$P \left( \sup_{0 \leq t \leq T} |M_{l,t}| > \epsilon \right) \leq \frac{4c l^2 a}{N \kappa(p + q)} \int_{0}^{T} E \left[ S_{N,2l-2}(s) \right] \, ds.$$  

If we choose $p, q$ and $a$ as in Theorem 4.1 we have $a^2 N (p + q) \in O(N^{-2})$. If we choose $p, q$ and $a$ as in Theorem 4.3 we have $a N (p + q) \in O(N^{-2})$. By the first part of the proof we thus conclude that in either case $P \left( \sup_{0 \leq t \leq T} |M_{l,t}| > \epsilon \right) \in O(N^{-2})$ for each $\epsilon > 0$. The claim now follows by the Borel-Cantelli lemma. $\square$

We now turn to the specific scaling in Theorem 4.1.

**Proof of Theorem 4.1** To keep formulas short we again suppress the dependence of $p, q, a, b$ on $N$. We define

$$\mu_t := (e^{-t} \mu) \boxplus \left( \sqrt{1 - e^{-2t} \mu_{sc,4(1+C)^{-3/2}}} \right)$$

with the moments $c_l(t) := \int_{\mathbb{R}} x^l \, d\mu_t(x)$. By the proof of Theorem 3.1 we have $c_1(t) = e^{-t} c_1(0)$ and

$$c_l(t) = e^{-lt} \left( c_l(0) + 4l(1+C)^{-3} \int_{0}^{t} e^s \sum_{k=0}^{l-2} c_k(s)c_{l-2-k}(s) \, ds \right), \quad l \geq 2.$$  

By induction we will show that the limits $S_l(t) := \lim_{N \to \infty} \int_{\mathbb{R}} x^l \, d\mu_{N,t/(p+q)}(x)$, $l \in \mathbb{N}$, exist and satisfy the same recursion as the $c_l(t)$.

Let $l = 1$. By (4.6), our choice of $b_N$ and Lemma 4.2 we have $S_1(t) := \lim_{N \to \infty} S_{N,1}(t) = e^{-t} c_l$ locally uniformly in $t$ a.s.
Let $l \geq 2$. Note that $C_l$ in (4.9) converges to $-l$. We now calculate the limit of $f_l(S_{N,1}(t), \ldots, S_{N,l-1}(t))$. 
For this note that
\[
\lim_{N \to \infty} \frac{4l(l-1)ab}{\kappa(p+q)} = 0, \quad \lim_{N \to \infty} \frac{2l(l-1)a^2}{\kappa(p+q)} = 0, \quad \lim_{N \to \infty} a \left( \frac{p-q}{p+q} - b \left( 1 + \frac{2(l-1)}{p+q} \left( \frac{2}{\kappa} - 1 \right) \right) \right) = 0,
\]
\[
\lim_{N \to \infty} \frac{(1-b^2)a^2(l-1)}{p+q} \left( \frac{2}{\kappa} - 1 \right) = 0, \quad \lim_{N \to \infty} N/(p+q) = 0, \quad \lim_{N \to \infty} \frac{2bNa}{p+q} = 0,
\]
\[
\lim_{N \to \infty} Na^2(1-b^2) \frac{N}{p+q} = 4(1+C)^{-3}.
\]

Hence, by our induction assumption, we have a.s. locally uniformly in \( t \) that
\[
\lim_{N \to \infty} f_l(S_{N,1}(t), \ldots, S_{N,l-1}(t)) = 4l(1+C)^{-3} \sum_{k=0}^{l-2} S_k(t)S_{l-2-k}(t).
\]
Thus by (4.9) and Lemma 4.2 the limit \( S_l(t) = \lim_{N \to \infty} S_{N,l}(t) \) exists and satisfies
\[
S_l(t) = e^{-lt} \left( S_l(0) + 4l(1+C)^{-3} \int_0^t e^{ls} \sum_{k=0}^{l-2} S_k(s)S_{l-2-k}(s) \, ds \right)
\]
a.s., so that the \( S_l(t) \) satisfy the same recursion as the \( c_l(t) \).

This proves the claim in the same way as in the proof of Theorem 3.1. □

By using the same technique we also readily get the following stochastic version of Theorem 3.8; please notice that here also Lemma 4.2 is available.

**Theorem 4.3.** Consider \( p_N, q_N, a_N, b_N \) as in (4.14) and (4.15). Let \( \mu \in M^1([0, \infty[) \) be a probability measure such that its moments \( c_l \) satisfy \( |c_l| \leq (\gamma l)^l \) for \( l \in \mathbb{N}_0 \) with some constant \( \gamma > 0 \). Moreover, let \( (x_N^N)_{N \in \mathbb{N}} = ((x_1^N, \ldots, x_N^N))_{N \in \mathbb{N}} \) be an associated sequence of starting vectors \( x_N^N \in A_{N,F} \) as the preceding results.

Let \( \tilde{X}_t^N \) be the solutions of the SDEs (4.11) with start in \( \tilde{X}_N^N(0) = x_N^N \) for \( N \in \mathbb{N}, t \geq 0 \). Then for all \( t > 0 \), all moments of the empirical measures
\[
\mu_{N,l/(p_N+q_N)} = \frac{1}{N} \sum_{i=1}^N \delta_{a_N(\tilde{X}_{t/(p_N+q_N)},i)-b_N}
\]
tend almost surely to those of the probability measures
\[
\left( \mu_{SC,2\sqrt{2(1-e^{-t})}} \right. \left. \otimes \left( \sqrt{e^{-t}} \mu \right)_{\text{even}} \right) \left. \otimes \mu_{MP, \delta_{1,2(1-e^{-t})}} \right), \quad t > 0.
\]

**Remark 4.4.** We point out that by using the methods of the proof as above we also have stochastic versions of Theorems 3.5, 3.6 and 3.10. This means that in these theorems the moment convergence holds a.s. if replacing the solution \( x(t) \) of (1.6) by the rescaled Jacobi process \( X_t \) satisfying (1.1).

For some parameters \( \kappa, p, q \), the solutions \( (\tilde{X}_t)_{t \geq 0} \) of the SDEs (4.11) admit interpretations in terms of dynamic versions of MANOVA-ensembles over the fields \( F = \mathbb{R}, \mathbb{C} \) by Doumerc [16] as follows. Let \( d = 1, 2 \) be the real dimension of \( F \). Consider Brownian motions \( (Z_t^d)_{t \geq 0} \) on the compact groups \( SU(n, F) \) with some suitable time scalings. Now take positive integers \( N, p \) with
$N \leq p \leq n$, and denote the $N \times p$-block of a square matrix $A$ of size $n$ by $\pi_{N,p}(A)$. Moreover, let $\sigma(B)$ be the ordered spectrum of some positive semidefinite matrix $B$. It is shown in [Do] that then
\[
\left( \frac{\dot{X}_t}{\tau} := 2 \cdot \sigma\left( \pi_{N,p}(Z_t^N) \pi_{N,p}(Z_t^N)^* \right) - 1 \right)_{t \geq 0}
\]
is a diffusion on $\mathbb{A}_N$ satisfying the SDE (4.1) with the parameters $p \geq N$, $q := n - p$, and $\kappa = d/2$. Clearly, all of the preceding limit results in Section 4 can be applied in this case for suitable sequences $p_N, n_N$ of dimension parameters depending on $N$.

We point out that this geometric interpretation of some Jacobi processes includes the interpretation for the special case $n = p + N$, i.e., $q = N$, where the Jacobi processes are suitable projections of Brownian motions on the compact Grassmann manifolds with the dimension parameters of type B by
\[
\sigma \left( \pi_{N,p}(Z_t^N) \pi_{N,p}(Z_t^N)^* \right) \leq 1,
\]
for the analytical background.

5. LIMIT THEOREMS IN THE NONCOMPACT CASE

The Jacobi processes on compact alcoves in the preceding section admit analogues in a noncompact setting, namely the so-called Heckman-Opdam Markov processes associated with root systems of type BC introduced in Schapira [Sch1, Sch2]. Due to the close connections with the compact setting, namely the so-called Heckman-Opdam Markov processes associated with root systems of type BC, we shall call these processes in a way which fits to the compact case. We fix some references therein.

end of the preceding section. For the general background we refer to the monographs [HO, HS] and references therein.

We here derive analogues of the main results of the Sections 2–4 in this noncompact setting. For this we first introduce these processes in a way which fits to the compact case. We fix some parameters, these processes are related to Brownian motions on noncompact Grassmann manifolds over $\mathbb{R}, \mathbb{C}$, and the quaternions similar to the comments in the end of the preceding section. For the general background we refer to the monographs [HO, HS] and references therein.

We here derive analogues of the main results of the Sections 2–4 in this noncompact setting. For this we first introduce these processes in a way which fits to the compact case. We fix some dimension $N \geq 2$ and parameters $k_1, k_2 \in \mathbb{R}$ and $k_3 > 0$ with $k_2 \geq 0$ and $k_1 + k_2 \geq 0$. We define the (noncompact) Heckman-Opdam Laplacians of type BC on the Weyl chambers
\[
\tilde{C}_N := \{ w \in \mathbb{R}^N : 0 \leq w_1 \leq \ldots \leq w_N \}
\]
of type B by
\[
L_{\text{trig}, k} f(w) := \Delta f(w) + \sum_{i=1}^{N} \left( k_1 \coth \left( \frac{w_i}{2} \right) + 2k_2 \coth \left( \frac{w_i}{2} \right) \right) f_{x_i}(w) + \sum_{j=1}^{N} \left( \coth \left( \frac{w_i - w_j}{2} \right) + \coth \left( \frac{w_i + w_j}{2} \right) \right) f_{x_i}(w)
\]
for functions $f \in C^2(\mathbb{R}^N)$ which are invariant under the associated Weyl group. By [Sch1, Sch2], the $L_{\text{trig}, k}$ are the generators of Feller diffusions $(W_t)_{t \geq 0}$ on $\tilde{C}_N$ where the paths are reflected on the boundary. We next use the transformation $x_i := \cosh w_i$ ($i = 1, \ldots, n$) with
\[
x \in C_N := \{ x \in \mathbb{R}^N : 1 \leq x_1 \leq \ldots \leq x_N \}.
\]
The diffusions $(W_t)_{t \geq 0}$ on $\tilde{C}_N$ then are transformed into Feller diffusions $(X_t)_{t \geq 0}$ on $C_N$ with reflecting boundaries and, by some elementary calculus, with the generators
\[
L_k f(x) := \sum_{i=1}^{N} \left( x_i^2 - 1 \right) f_{x_i}(x) + \sum_{i=1}^{N} \left( (k_1 + 2k_2 + 2k_3(N-1) + 1)x_i + k_1 + 2k_3 \right) \sum_{j: j \neq i} \frac{x_i x_j - 1}{x_i - x_j} f_{x_i}(x).
\]

(5.2)
As in the introduction, we redefine the parameters by
\[ \kappa := k_3 > 0, \quad q := N - 1 + \frac{1 + 2k_1 + 2k_2}{2k_3}, \quad p := N - 1 + \frac{1 + 2k_2}{2k_3} \] (5.3)
with \( p, q > N - 1 \) and rewrite (5.2) as
\[ L_k f(x) := \sum_{i=1}^{N} (x_i^2 - 1)f_{x_i}(x) + \kappa \sum_{i=1}^{N} \left( (q - p) + (q + p)x_i + 2 \sum_{j:j \neq i} \frac{x_ix_j - 1}{x_i - x_j} \right) f_{x_i}(x). \] (5.4)
Moreover, we also consider the transformed processes \( \tilde{X}_t := (\tilde{X}_{t,i})_{i \geq 0} \) with the generators \( \frac{1}{\kappa} L_k \) which then are the unique strong solutions of the SDEs
\[ d\tilde{X}_{t,i} = \frac{\sqrt{\gamma}}{\sqrt{\kappa}} \sqrt{\tilde{X}_{t,i}^2 - 1} dB_{t,i} + \left( (q - p) + (q + p)\tilde{X}_{t,i} + 2 \sum_{j:j \neq i} \frac{\tilde{X}_{t,i} \tilde{X}_{t,j} - 1}{\tilde{X}_{t,i} - \tilde{X}_{t,j}} \right) dt \] (5.5)
for \( i = 1, \ldots, N \), a Brownian motion \( (B_{t,1}, \ldots, B_{t,N})_{t \geq 0} \) on \( \mathbb{R}^N \), and starting points \( x_0 \) in the interior of \( C_N \).

For \( \kappa = \infty \) and \( p, q > N - 1 \), these SDEs degenerate to the ODEs
\[ \frac{dx_i(t)}{dt} = (q - p) + (q + p)x_i(t) + 2 \sum_{j:j \neq i} \frac{x_i(t)x_j(t) - 1}{x_i(t) - x_j(t)} \quad (i = 1, \ldots, N). \] (5.6)
Please notice that the RHS of (5.6) is equal to the negative of the RHS of (1.6) in the compact case where the solutions exist on some different “complementary” domain. Theorem 1.1 here has the following form; it will be proved in the next section.

**Theorem 5.1.** Let \( N \in \mathbb{N} \) and \( p, q > N - 1 \). Then for each each \( x_0 \in C_N \) and \( x(t) \) exists all \( t \geq 0 \) in the following sense: If \( x_0 \in \partial A_N \), then there exists a unique continuous function \( x : [0, \infty) \to C_N \) with \( x(0) = x_0 \) and \( x(t) \) in the interior of \( \bar{C}_N \) if \( t > 0 \), where \( x(t) \) satisfies (5.6).

For the solutions of (5.6) we have the following local Wigner-type limit theorem which is completely analogous to Theorem 3.5.

**Theorem 5.2.** Consider sequences \( (p_N)_{N \in \mathbb{N}}, (q_N)_{N \in \mathbb{N}} \subset [0, \infty[ \) with \( p_N, q_N > N - 1 \) for \( N \geq 1 \). Let \( (b_N)_{N \in \mathbb{N}} \subset [1, \infty[ \) be a sequence such that \( B := \lim b_N \in [1, \infty[ \) exists.

Let \( (s_N)_{N \in \mathbb{N}} \subset [0, \infty[ \) be a sequence of time scalings with
\[ \lim_{N \to \infty} \frac{p_N + q_N}{\sqrt{N}s_N} = 0, \]
and define the space scalings \( a_N := \sqrt{s_N/N} \).

Let \( \mu \in M^1(\mathbb{R}) \) be a starting measure such that its moments \( c_l \) satisfy \( |c_l| \leq (\gamma l) \) for \( l \in \mathbb{N}_0 \) with some constant \( \gamma > 0 \). Let \( (x_N)_{N \in \mathbb{N}} \) be associated starting vectors with \( x_N \in C_N \) as the preceding limit results.

Let \( x_N(t) \) be the solutions of the ODEs (1.3) with \( x_N(0) = x_N \) for \( N \in \mathbb{N} \). Then for all \( t > 0 \), all moments of the empirical measures
\[ \mu_{N,t/(p_N+q_N)} = \frac{1}{N} \sum_{i=1}^{N} \delta_{\delta_N(x_N(t/s_N)-b_N)} \]
solutions of the ODEs \((5.6)\) imply that for \(l \geq 0\) and \(t \geq 0\), the moments \(\tilde{S}_{N,l}(t)\) of the empirical measures \(\mu_{N,t/s_N}\) converge for \(N \to \infty\) to functions \(S_l(t)\) which satisfy

\[
S_0 \equiv 1, \quad S_1(t) = S_1(0), \quad S_l(t) = S_l(0) + l(B^2 - 1) \int_0^t \sum_{k=0}^{l-2} S_k(s) S_{l-2-k}(s) \, ds \quad (l \geq 2).
\]

(5.7)

The claim now follows in the same way as in Theorem 3.5. \(\square\)

The stationary local limit Theorem 3.3 does not seem to have a meaningful analogue in the noncompact setting, as the assumptions on the \(p_N, q_N, a_N, b_N\) in Theorem 3.1 imply that \(b_N \in ]-1,1[\) holds for all \(N\) such that the rescaled empirical measures measures for \(t = 0\) in the assumptions of Theorem 3.1 cannot converge.

On the other hand, we have the following variants of the stationary Theorem 3.8 as well as of the non-stationary Theorem 3.10 both of which involve Marchenko-Pastur distributions. Note that due to the time-inversion also the analogue to Theorem 3.8 is now non-stationary:

**Theorem 5.3.** Consider sequences \((p_N)_{N \in \mathbb{N}}, (q_N)_{N \in \mathbb{N}} \subset ]0, \infty]\) with

\[
\lim_{N \to \infty} p_N/N = \infty \quad \text{and} \quad \lim_{N \to \infty} q_N/N = \tilde{q}.
\]

Define \(a_N := p_N/N, b_N := 1\) \((N \in \mathbb{N})\). Let \(\mu \in M^1([0, \infty])\) be a probability measure such that its moments \(c_l\) satisfy \(|c_l| \leq (\gamma l)^l\) for \(l \in \mathbb{N}_0\) with some constant \(\gamma > 0\). Moreover, let \((x_N)_{N \in \mathbb{N}}\) be an associated sequence of starting vectors \(x_N \in C_N\) as in the preceding limit results. Let \(x_N(t)\) be the solutions of the ODEs \((5.6)\) with start in \(x_N(0) = x_N\) for \(N \in \mathbb{N}, t \geq 0\). Then for all \(t > 0\), all moments of the empirical measures

\[
\mu_{N,t/(p_N+q_N)} = \frac{1}{N} \sum_{i=1}^N \delta_{a_N(x_N^n(t/(p_N+q_N))-b_N)}
\]

tend to those of the probability measures

\[
\mu(t) := \left(\mu_{SC,2\sqrt{(e'-1)}} \boxplus \left(\sqrt{e'}/e\right)^{\text{even}}\right)^2 \boxplus \mu_{MP,\tilde{q}-1,2(e'-1)}, \quad t > 0.
\]

(5.8)

**Proof.** The proof is completely analogous to the one of Theorem 3.8. We just give the main steps. The moments \(\tilde{S}_{N,l}(t)\) of the empirical measures \(\mu_{N,t/(p_N+q_N)}\) converge for \(N \to \infty\) to functions \(S_l(t)\) which satisfy

\[
S_0 \equiv 1, \quad S_1(t) = e^t (S_1(0) - 2\tilde{q}) + 2\tilde{q},
\]

\[
S_l(t) = e^{lt} \left(S_l(0) + 2t \int_0^t e^{-ls} \left(q S_{l-1}(s) + \sum_{k=0}^{l-2} S_k(s) S_{l-2-k}(s)\right) \, ds\right), \quad l \geq 2.
\]

(5.9)

Denote the Cauchy-transform of the limiting measure \(\mu_t := \lim_{N \to \infty} \mu_{N,t/(p_N+q_N)}\) by

\[
G(t, z) := G_{\mu_t}(z) = \lim_{N \to \infty} G_{\mu_{N,t/(p_N+q_N)}}(z).
\]

As for the PDEs \((2.15)\), but with an additional minus sign, this leads to the PDE

\[
G_t(t, z) = -z - 2(\tilde{q} - 1) - 4zG(t, z))G_z(t, z) - G(t, z) - 2G(t, z)^2.
\]

(5.10)
Using (5.6), we obtain for the R-transforms that
\[ 0 = R_t(t, z) - (z + 2z^2)R_z(t, z) - 2\hat{q} - (4z + 1)R(t, z). \]
If we put \( \phi(z) := R(0, z) \), the method of characteristics here leads to
\[ R(t, z) = e^t(1 - 2z(e^t - 1))^{-2}\phi(e^t z(1 - 2z(e^t - 1)^{-1})) + \frac{2(e^t - 1)}{1 - 2z(e^t - 1)} + \frac{2(\hat{q} - 1)(e^t - 1)}{1 - 2z(e^t - 1)}. \] (5.11)
Finally if we set \( \hat{\phi}(z) := e^s\phi(e^sz) \) and
\[ f(t, z) := (1 - tz)^{-2}\hat{\phi}\left(\frac{z}{1-tz}\right) + \frac{t}{1-tz} \quad (z \in \mathbb{C} \setminus \mathbb{R}, t > 0), \]
the claim now follows as in the proof of Theorem 5.3. □

The following result also follows in the same way by the methods of the proof of Theorem 3.10.

**Theorem 5.4.** Consider sequences \( (p_N)_{N \in \mathbb{N}}, (q_N)_{N \in \mathbb{N}} \subset ]0, \infty[ \) with \( p_N, q_N > N - 1 \) for \( N \geq 1 \) and with \( \lim_{N \to \infty} q_N/N = \hat{q} \in ]1, \infty[ \). Let \( (s_N)_{N \in \mathbb{N}} \subset ]0, \infty[ \) be a sequence of time scalings with \( \lim_{N \to \infty} (p_n + q_N)/s_N = 0 \). Define the space scalings \( a_N := s_N/N, b_N := 1 \) \((N \in \mathbb{N})\). Let \( \mu \in M^1([0, \infty[) \) be a starting measure and \( (s_N)_{N \in \mathbb{N}} \) associated starting vectors as before. Let \( x_N(t) \) be the solutions of the ODEs (5.6) with \( x_N(0) = x_N \) for \( N \in \mathbb{N} \). Then for all \( t > 0 \), all moments of the empirical measures
\[ \mu_{N,t/s_N} = \frac{1}{N} \sum_{i=1}^{N} \delta_{a_N(x_N(t/s_N) - b_N)} \]

tend to those of \( \left( \mu_{sc,2\sqrt{2t}} \boxplus (\sqrt{\mu})_{evn} \right)^2 \boxplus \mu_{MP, \hat{q}, -1, 2t} \).

We finally mention that also the stochastic limit results from Section 4, that correspond to the deterministic limit Theorems 5.2, 5.4, can be transferred to the noncompact setting. We here skip the details.

### 6. Appendix: Solutions of the Differential Equations with Start on the Singular Boundary

In this section we prove Theorems 1.1 and 5.4.

We first study the ODE (1.8) which has the form
\[ \frac{d}{dt}x_i(t) = (p - q) - (p + q)x_i(t) + 2 \sum_{j: j \neq i}^{N} \frac{1 - x_i(t)x_j(t)}{x_i(t) - x_j(t)}, \quad i = 1, \ldots, N. \] (6.1)

In order to prove parts of Theorem 1.1 it is useful to interpret this ODE as a gradient system; see e.g. Section 9.4 of [HiS] on the background. However, it can be easily checked that (6.1) is not a gradient system. In order to obtain a gradient system, we use the transformation \( x_i = \cos \tau_i \) with \( \tau \geq \tau_1 \geq \ldots \geq \tau_N \geq 0 \) which is motivated by the theory of Heckman-Opdam hypergeometric functions in [HO] [HiS] in its trigonometric form (see also the introduction), and which is also useful in [HV] for nice covariance matrices in some freezing central limit theorem. In fact, elementary
calculus shows that (6.1) is equivalent to the ODE
\[
\frac{d}{dt} \tau_i(t) = (q - p) \cot \left( \frac{\tau_i(t)}{2} \right) + 2(p + 1 - N) \cot(\tau_i(t)) + \sum_{j:j \neq i} \left( \cot \left( \frac{\tau_i(t) - \tau_j(t)}{2} \right) + \cot \left( \frac{\tau_i(t) + \tau_j(t)}{2} \right) \right)
\]
for \( i = 1, \ldots, N \) which is a gradient system. In fact, if \( V(\tau) := \ln \tilde{V}(\tau) \) with
\[
\tilde{V}(\tau) := \left( \prod_{i=1}^N \sin(\tau_i/2) \right)^{2(q-p)} \cdot \left( \prod_{i=1}^N \sin(\tau_i) \right)^{2(p+1-N)} \cdot \prod_{i,j: i < j} (\sin \left( \frac{\tau_i - \tau_j}{2} \right) \sin \left( \frac{\tau_i + \tau_j}{2} \right)) \right)^2,
\]
then (6.2) has the form \( \frac{d}{dt} \tau(t) = \text{grad} V(\tau(t)) \) with \( \tau = (\tau_1, \ldots, \tau_N) \).

We next search for a maximum of the potential \( V \), i.e., of \( \tilde{V} \). For this we observe that, with some constant \( C \),
\[
\tilde{V}(\tau) = C \cdot \prod_{i=1}^N ((1 - x_i)^{q+1-N}(1 + x_i)^{p+1-N}) \cdot \prod_{i,j: i < j} (x_i - x_j)^2.
\]
A classical result of Stieltjes (see Section 6.7 of [SZ]) now shows that for \( \pi > \tau_1 > \ldots > \tau_N > 0 \), this expression has a unique maximum for \( x = z \) where the vector \( z \) consists of the ordered roots of the Jacobi polynomial \( P_N^{(q-N,p-N)} \). Therefore, Section 9.4 of [HIS] yields the following part of Theorem 1.1.

**Lemma 6.1.** Let \( N \in \mathbb{N} \) and \( p, q > N - 1 \). For each \( x_0 \in \text{int} A_N \) the ODE (6.1) has a unique solution \( x(t) \) with \( x(t) \in \text{int} A_N \) for all \( t \geq 0 \). Moreover, \( \lim_{t \to \infty} x(t) = z \) where \( z \in \text{int} A_N \) is the vector consisting of the ordered roots of \( P_N^{(q-N,p-N)} \).

In order to complete the proof of Theorem 1.1 we still have to prove the following theorem. Its proof is an adaptation of the corresponding results for the Hermite- and Laguerre case in [VW2].

**Theorem 6.2.** Let \( N \in \mathbb{N} \) and \( p, q > N - 1 \). For each \( x_0 \in \partial A_N \) the ODE (6.1) has a unique solution \( x(t) \) for all \( t \geq 0 \) in the following sense: For each \( x_0 \in \partial A_N \) there is a continuous function \( x: [0, \infty) \to A_N \) with \( x(0) = x_0 \) such that \( x(t) \in \text{int} A_N \) for all \( t > 0 \) and \( x: (0, \infty) \to \text{int} A_N \) satisfies (6.1). Moreover, \( \lim_{t \to \infty} x(t) = z \) with \( z \in A_N \) as above.

**Proof.** We use of the elementary symmetric polynomials \( e_n^m \) \( (n = 0, \ldots, m) \) in \( m \) variables which are characterized by
\[
\prod_{j=1}^m (z - x_j) = \sum_{j=0}^m (-1)^m j e_{m-j}^m(x) z^j, \quad z \in \mathbb{C}, \quad x = (x_1, \ldots, x_m).
\]
Consider the map \( e : A_N \to \mathbb{R}^N, e(x) = (e_1^N(x), \ldots, e_N^N(x)) \). Then \( e : A_N \to e(A_N) \) is a homeomorphism, and \( e : \text{int} A_N \to e(\text{int} A_N) \) is a diffeomorphism. We will use the following notation: Let \( x \in \mathbb{R}^N \) and \( S \subseteq \{1, \ldots, N\} \) a nonempty set. Denote by \( x_S \in \mathbb{R}^{|S|} \) the vector with coordinates \( x_i, i \in S \), in the natural ordering on \( S \). With this convention we have
\[
\sum_{i=1}^N e_{k-1}^{N-1}(x_{\{1, \ldots, N\}\setminus\{i\}}) = (N - k + 1) e_{k-1}^N(x), \quad \sum_{i=1}^n e_{k-1}^{N-1}(x_{\{1, \ldots, N\}\setminus\{i\}}) x_i = k e_k^N(x),
\]
and
\[ e_{k-1}^{N-1}(x_{(1,\ldots,N)} \setminus \{i\}) - e_{k-1}^{N-1}(x_{(1,\ldots,N)} \setminus \{j\}) = -(x_i - x_j) e_{k-2}^{N-2}(x_{(1,\ldots,N)} \setminus \{i,j\}) \cdot \]

Hence
\[
\sum_{i,j=1 \atop i \neq j}^{N-1} \frac{e_{k-1}^{N-1}(x_{(1,\ldots,N)} \setminus \{i\})}{x_i - x_j} = \sum_{i,j=1 \atop i < j}^{N-1} \frac{e_{k-1}^{N-1}(x_{(1,\ldots,N)} \setminus \{i\}) - e_{k-1}^{N-1}(x_{(1,\ldots,N)} \setminus \{j\})}{x_i - x_j} = - \sum_{i,j=1 \atop i < j}^{N-2} e_{k-2}^{N-2}(x_{(1,\ldots,N)} \setminus \{i,j\}) = -\frac{(N-k+2)(N-k+1)}{2} e_k^{N}(x) \]

and
\[
\sum_{i,j=1 \atop i \neq j}^{N-1} \frac{e_{k-1}^{N-1}(x_{(1,\ldots,N)} \setminus \{i\})x_i x_j}{x_i - x_j} = \sum_{i,j=1 \atop i < j}^{N-1} \frac{e_{k-1}^{N-1}(x_{(1,\ldots,N)} \setminus \{i\}) - e_{k-1}^{N-1}(x_{(1,\ldots,N)} \setminus \{j\})}{x_i - x_j} x_i x_j = - \sum_{i,j=1 \atop i < j}^{N-2} e_{k-2}^{N-2}(x_{(1,\ldots,N)} \setminus \{i,j\}) x_i x_j = -\frac{k(k-1)}{2} e_k^{N}(x) . \]

By transforming \((6.1)\) with the homeomorphism \(e\) we get the ODEs
\[
\frac{d}{dt} e^N_k(x(t)) = \sum_{i=1}^{N} \frac{d}{dt} x_i(t) = N(p-q) - (p+q)e^N_k(x(t)) ,
\]
\[
\frac{d}{dt} e^N_k(x(t)) = \sum_{i=1}^{N} e^N_{k-1}(x_{(1,\ldots,N)} \setminus \{i\})(t) \left( (p-q) - (p+q)x_i(t) + 2 \sum_{j=1 \atop j \neq i}^{N-1} \frac{1 - x_i(t)x_j(t)}{x_i(t) - x_j(t)} \right) = k(-(p+q) + k - 1)e^N_k(x(t)) + (N-k+1)(p-q)e^N_{k-1}(x(t)) - (N-k+2)(N-k+1)e^N_{k-2}(x(t)) , \quad k \in \{2,\ldots,N\} .
\]

These are linear differential equations of the type \(f'(t) = \lambda f(t) + g(t)\) with the solutions \(f(t) = e^{\lambda t} \left( f(0) + \int_0^t e^{-\lambda s} g(s) \, ds \right)\). Thus,
\[
e^N_k(x(t)) = e^{-(p+q)t} \left( c^N_1(x_0) - \frac{N(p-q)}{p+q} \right) + \frac{N(p-q)}{p+q} \quad \text{and}
\]
\[
e^N_k(x(t)) = e^{c_k t} \left( c^N_1(x_0) + \int_0^t e^{-c_k s} \left( (N-k+1)(p-q)c^N_{k-1}(x(s)) - (N-k+2)(N-k+1)c^N_{k-2}(x(s)) \right) \, ds \right) , \quad c_k = k(-(p+q)+k-1) < 0 , \quad k \in \{2,\ldots,N\} .
\]

where \(c_k = k(-(p+q)+k-1) < 0 , \quad k \in \{2,\ldots,N\} \). By induction we see that each \(e^N_k(x(t))\) is a linear combination of terms of the form \(e^{rt}\), \(r \leq 0\). Thus the limits \(\hat{e}_k := \lim_{t \to \infty} e^N_k(t)\) exist. We claim that \(\hat{e} = e(z)\) holds. To prove this we observe from the limit assertion in Lemma \((6.4)\) that this holds for all starting points \(x_0 \in \text{int} A_N\). Furthermore, as \(\hat{e}\) depends continuously on \(x_0\) by \((6.5)\), we obtain \(\hat{e} = e(z)\) also for \(x_0 \in \partial A_N\).

We now turn to the case \(x_0 \in \partial A_N\). Clearly, as \(e\) is injective there exists at most one solution of \((6.1)\). For the existence of a solution we claim that the inverse mapping of \(e\) transforms solutions of \((6.1)\) back into solutions of \((6.1)\) in the sense of the theorem. For this we prove that for any starting point \(x_0 \in \partial A_N\) in \((1.0)\) and its image \(e(x_0)\) the solution \(\hat{e}(t)\), \(t \geq 0\), of the ODEs \((6.2)\) with
\( \hat{e}(0) = e(x_0) \) satisfies \( \hat{e}(t) \in e(\text{int } A_N) \) for all \( t > 0 \). If this is shown it follows that the preimage of \( (\hat{e}(t))_{t > 0} \) under \( e \) solves (6.1).

To prove this, we recapitulate that for each starting point in \( e(\text{int } A_n) \) the solution \( \hat{e} \) of (6.4) satisfies \( \hat{e}(t) \in e(\text{int } A_n) \) for all \( t \geq 0 \), and that for all fixed \( t \geq 0 \) the solutions \( \hat{e}(t) \) depend continuously on arbitrary starting points in \( R^N \) by a classical result on ODEs. Hence, for each starting point \( \hat{e}(0) \in e(A_N) \) we have \( \hat{e}(t) \in e(A_N) \) for \( t \geq 0 \).

Assume that there is a starting point \( x_0 \in \partial A_N \) and some \( t_0 > 0 \) such that the solution \( (\hat{e}(t))_{t \geq 0} \) of (6.4) with start at \( e(x_0) \) satisfies

\[
\hat{e}(t) \notin e(\text{int } A_n), \quad t \in [0, t_0].
\]

For \( x = (x_1, \ldots, x_n) \in R^N \) we define the discriminant

\[
D(x) := \prod_{i=1}^{N} (1 - x_i^2) \cdot \prod_{i,j=1, i \neq j}^{N} (x_j - x_i).
\]

By (6.6) we thus deduce

\[
\hat{e}(t) \in e(\partial A_N) \subseteq Y := \{y \in R^N : \hat{D}(y) = 0\}, \quad t \in [0, t_0].
\]

We finally turn to the proof of Theorem 5.4 on the ODEs (5.6). We proceed as in the proof of Theorem 1.1 and notice first that the transform \( x_i = \cosh \tau_i \) \((i = 1, \ldots, N)\) transform the ODEs (5.6) again into some gradient system. As for Lemma 6.1 we thus obtain:

**Lemma 6.3.** Let \( N \in \mathbb{N} \) and \( p, q > N - 1 \). For each \( x_0 \in \text{int } C_N \) the ODE (5.6) has a unique solution \( x(t) \) with \( x(t) \in \text{int } C_N \) for \( t \geq 0 \).

**Proof.** We only have to check that the system is not explosive in finite time. For this we again use the elementary symmetric polynomials \( e^*_n \) as well as the homeomorphism \( e : C_N \rightarrow e(C_N) \subset R^N \) with \( e(x) = (e_1^N(x), \ldots, e_N^N(x)) \) as in the proof of Theorem 6.2. As the right hand sides of the ODEs (1.6) and (5.6) are equal up to a sign change, we conclude from the computations in the proof of Theorem 6.2 (see in particular (6.5)) that

\[
e_1^N(x(t)) = e^{(p+q)t} \left( e_1^N(x_0) - N \left( \frac{p - q}{p + q} \right) \right) + N \left( \frac{p - q}{p + q} \right),
\]

\[
e_k^N(x(t)) = e^{ct} \left( e_k^N(x_0) \right)
\]

\[+ \int_{0}^{t} e^{-c_s} \left( (N - k + 1)(p - q)e_{k-1}^N(x(s)) - (N - k + 2)(N - k + 1)e_{k-2}^N(x(s)) \right) ds \),

with \( c_k = k((p + q) + 1 - k) < 0 \) for \( k = 2, \ldots, N \). In summary, \( e(x(t)) \) satisfies some linear ODE and exists thus for all \( t \geq 0 \). The claim now follows by a transfer back to int \( C_N \). \( \square \)

To complete the proof of Theorem 5.4 we prove the following analogue of Theorem 6.2.
Theorem 6.4. Let $N \in \mathbb{N}$ and $p, q > N - 1$. For each starting value $x_0 \in \partial A_N$ the ODE (6.6) has a unique solution $x(t)$ for $t \geq 0$ in the sense as described in Theorem 6.3.

Proof. We use the notations of the proof of Lemma 6.3 and consider some starting point $x_0 \in \partial C_N$. For the existence of a solution we claim that the inverse mapping of $e$ transforms the functions in (6.8) back into solutions of (5.6) in the sense of the theorem, i.e., that $\tilde{e}(t) := (e_1^N(x(t)), \ldots, e_n^N(x(t))) \in e(\text{int} A_N)$ holds for all $t > 0$. To prove this, we have to check that $\tilde{e}(t) \notin e(\partial A_N)$ for $t > 0$.

Assume that for some $x_0 \in \partial A_N$ and $t_0 > 0$ we have $\tilde{e}(t) \notin e(\text{int} A_n)$ for $t \in [0, t_0]$. We now use the discriminant $D$ from (6.7) as well as $\tilde{D}$ there. We conclude from the corresponding methods in in the proof of Theorem 6.2 that $\tilde{D}(\tilde{e}(t)) = 0$ for $t \in [0, t_0]$ implies that $\tilde{D}(\tilde{e}(t)) = 0$ for all $t \in \mathbb{R}$. We now recapitulate that the solutions (6.8) and (6.5) of the corresponding ODEs are equal up to the transform $t \mapsto -t$ for equal starting points $\tilde{e}(0)$, and that these solutions obviously depend analytically from $\tilde{e}(0)$. We thus conclude from the limit assertion in Lemma 6.1 that $\lim_{t \to -\infty} \tilde{e}(t) = e(z)$ holds where $D(z) \neq 0$ holds. As this is a contradiction to $\tilde{D}(\tilde{e}(t)) = 0$ for $t \in \mathbb{R}$, the theorem follows from Lemma 6.3. \qed

References

[A] N.I. Akhiezer, The Classical Moment Problems and Some Related Questions in Analysis. Engl. Translation, Hafner Publishing Co., New York, 1965.

[AGZ] G.W. Anderson, A. Guionnet, O. Zeitouni, An Introduction to Random Matrices. Cambridge University Press, 2010.

[B1] F. Benaych-Georges, Infinitely divisible distributions for rectangular free convolution: classification and matri
cial interpretation. *Probab. Theory Rel. Fields* 139 (2007), 143-189.

[B2] F. Benaych-Georges, Rectangular random matrices, related convolution, *Probab. Theory Rel. Fields* 144 (2009), 471-515.

[CC] M. Capitaine, M. Casalis, Asymptotic freeness by generalized moments for Gaussian and Wishart matrices. Application to beta random matrices. *Indiana Univ. Math. J.* 53 (2004), 397-432.

[CG] T. Cabanal Duvillard, A. Guionnet, Harmonic and stochastic analysis of Dunkl processes, pp. 113-198. Hermann, Paris 2008.

[CGY] O. Chybiryakov, L. Gallardo, M. Yor, Dunkl processes and their radial parts relative to a root system. In: P. Graczyk et al. (eds.), Harmonic and stochastic analysis of Dunkl processes, pp. 113-198. Hermann, Paris 2008.

[De1] N. Demni, Free Jacobi process. *J. Theor. Probab.* 21 (2008), 118-143.

[De2] N. Demni, $\beta$-Jacobi processes. *Adv. Pure Appl. Math.* 1 (2010), 325-344.

[DN] H. Dette, J. Nagel, Some Asymptotic Properties of the Spectrum of the Jacobi Ensemble. *SIAM J. Math. Anal.* 41 (2009), 1491-1507.

[DS] H. Dette, W.J. Studden, Some new asymptotic properties for the zeros of Jacobi, Laguerre, and Hermite polynomials. *Constructive Approx.* 11 (1995), 227-238.

[DV] J.F. van Diejen, L. Vinet, Calogero-Sutherland-Moser Models. CRM Series in Mathematical Physics, Springer, Berlin, 2000.

[Do] Y. Doumerc, Matrix Jacobi process, Ph.D. Thesis, Paul Sabatier University, 2005.

[F] P. Forrester, Log Gases and Random Matrices, London Mathematical Society, London, 2010.

[GM] P. Graczyk, J. Malecki, Strong solutions of non-colliding particle systems. *Electron. J. Probab.* 19 (2014), 21 pp.

[HO] G. Heckman, E. Opdam, Jacobi polynomials and hypergeometric functions associated with root systems. In: Encyclopedia of Special Functions, Part II: Multivariable Special Functions, eds. T.H. Koornwinder, J.V. Stokman, Cambridge University Press, Cambridge, 2021.

[HS] G. Heckman, H. Schlichtkrull, Harmonic Analysis and Special Functions on Symmetric Spaces, Part I. Perspectives in Mathematics, Vol. 16, Academic Press, 1994.

[HV] K. Hermann, M. Voit, Limit theorems for Jacobi ensembles with large parameters. *Tunisian J. Math.* 3-4 (2021), 843–860.
[HiS] M.W. Hirsch, S. Smale, Differential Equations, Dynamical Systems, and Linear Algebra. Academic Press, San Diego, CA, 1974.

[J] T. Jiang, Approximation of Haar distributed matrices and limiting distributions of eigenvalues of Jacobi ensembles. Probability Theory and Related Fields 144 (2009), 221-246.

[K] R. Killip, Gaussian fluctuations for $\beta$ ensembles. Int. Math. Res. Not. 2008, no. 8, Art. ID rnn007, 19 pp.

[KN] R. Killip, I. Nenciu, Matrix models for circular ensembles. Int. Math. Res. Not. 50 (2004), 2665–2701.

[NS] A. Nica, R. Speicher, Lectures on the Combinatorics of Free Probability Theory, Cambridge University Press, Cambridge, 2006.

[P] P.E. Protter, Stochastic Integration and Differential Equations. A New Approach. Springer, Berlin, 2003.

[RR1] H. Remling, M. Rößler, The heat semigroup in the compact Heckman-Opdam setting and the Segal-Bargmann transform. Int. Math. Res. Not. 2011, No. 18, 4200-4225.

[RR2] H. Remling, M. Rößler, Convolution algebras for Heckman-Opdam polynomials derived from compact Grassmannians. J. Approx. Theory 197 (2015), 30-48.

[RS] L.C.G. Rogers, Z. Shi, Interacting Brownian particles and the Wigner law. Probab. Theory Rel. Fields 95 (1993), 555-570.

[RW] L.C.G. Rogers, D. Williams, Diffusions, Markov Processes and Martingales, Vol. 1 Foundations. Cambridge University Press, Cambridge, 2000.

[RV1] M. Rößler, M. Voit, Markov processes related with Dunkl operators. Adv. Appl. Math. 21 (1998), 575-643.

[RV2] M. Rößler, M. Voit, Elementary symmetric polynomials and martingales for Heckman-Opdam processes. To appear in Contemp. Math., arXiv:2108.03228.

[Sch1] B. Schapira, The Heckman-Opdam Markov processes. Probab. Theory Rel. Fields 138 (2007), 495-519.

[Sch2] B. Schapira, Contribution to the hypergeometric function theory of Heckman and Opdam: sharp estimates, Schwarz space, heat kernel. Geom. Funct. Anal. 18 (2008), 222-250.

[Se] S. Serfaty, Mean field limit for Coulomb-type flows. Duke Math. J. 169 (2020), 2887-2935.

[St] W.A. Strauss, Partial Differential Equations: An Introduction. Wiley, 1992.

[Sz] G. Szegö, Orthogonal Polynomials. Colloquium Publications (American Mathematical Society), Providence, 1939.

[TT] H.D. Trinh, K.D. Trinh, Beta Jacobi ensembles and associated Jacobi polynomials. J. Stat. Phys. 185 (2021), No. 1, Paper No. 4, 15 p.

[V] M. Voit, Some martingales associated with multivariate Jacobi processes and Aomoto’s Selberg integral. Indag. Math. 31 (2020), 398-410.

[VW1] M. Voit, J.H.C. Woerner, Limit theorems for Bessel and Dunkl processes of large dimensions and free convolutions. Stoch. Proc. Appl. 143 (2022), 207-253.

[VW2] M. Voit, J.H.C. Woerner, The differential equations associated with Calogero-Moser-Sutherland particle models in the freezing regime. Hokkaido Math. J. 51 (2022), 153–174.

[W] K. W. Wachter, The limiting empirical measure of multiple discriminant ratios. Ann. Stat. 8 (1980), 937–957.

Fakultät Mathematik, Technische Universität Dortmund, Vogelpothsweg 87, D-44221 Dortmund, Germany

Email address: martin.auer@math.tu-dortmund.de, michael.voit@math.tu-dortmund.de, jeannette.woerner@math.tu-dortmund.de