ACCESSIBLE ∞-COSMOI

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Abstract. We introduce the notion of an accessible ∞-cosmos and prove that these include the basic examples of ∞-cosmoi and are stable under the main constructions. A consequence is that the vast majority of known examples of ∞-cosmoi are accessible. By the adjoint functor theorem for homotopically enriched categories which we proved in an earlier paper, joint with Lukáš Vokřínek, it follows, for instance, that all such ∞-cosmoi have flexibly weighted homotopy colimits.

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1. Introduction

The theory of $\infty$-categories has experienced an explosion of interest in recent years, due in part to the needs of researchers in various areas of geometry, topology, logic, and mathematical physics. Multiple approaches have appeared, leading to multiple definitions of $(\infty, 1)$-category, or multiple models in the usual parlance, since each of these is seen as being only some sort of presentation of the "true" notion. Prominent examples of these models include quasicategories, complete Segal spaces, and Segal categories. Each of these, as well as various others, has its own distinct flavour, coming with various resulting advantages and disadvantages.

Substantial progress has been made in the comparison between these models — see [2] for a recent survey — but it is also natural to hope for a "model independent" approach. Over a number of years Riehl and Verity have been developing one such approach, under the name of $\infty$-cosmos, and their theory has now reached a high level of power and sophistication. For an introduction to many aspects of this theory, see their book [14].

The theory of $\infty$-cosmoi is very much homotopical in nature. In an earlier paper [5] joint with Lukáš Vokřínek, we proved a very general homotopical adjoint functor theorem for enriched categories. This actually included Freyd’s General Adjoint Functor Theorem (GAFT) as the special case of $\text{Set}$-enriched (i.e. unenriched) homotopically trivial categories, but the main motivation was the study of $\infty$-cosmoi, which are in fact certain simplicially enriched categories.

In the case of ordinary (unenriched, homotopically trivial) categories, the solution set condition which appears in the GAFT can be simplified using the theory of accessible categories [11, 1]. We found similar simplifications were available in the enriched homotopical setting of [5], and included versions of our main results which were formulated using (enriched) accessible categories.

Among other things, we proved the following:

**Theorem 1.1.** Let $U: \mathcal{L} \to \mathcal{K}$ be an accessible cosmological functor between $\infty$-cosmoi which are accessible simplicially enriched categories. Then $U$ has a homotopical left adjoint.
Theorem 1.2. Let \( \mathcal{K} \) be an \( \infty \)-cosmos which is accessible as a simplicially enriched category. Then \( \mathcal{K} \) has flexibly weighted homotopy colimits.

The second of these is in fact a straightforward consequence of the first (much as in the usual unenriched, homotopically trivial case). As such it is an analogue of the classical result that an accessible category which is complete is also cocomplete; indeed these are both special cases of a single theorem [5, Theorem 8.9].

It is significant since the definition of \( \infty \)-cosmos involves the existence of various sorts of limit, but not of any colimits. Colimits can be used for various things such as the formation of Kleisli objects and localizations, while left adjoints as in the first theorem could be used for example to construct free completions of \( \infty \)-categories under some class of limits or colimits.

Of course the interest in the two theorems quoted would be slight if there were not a good supply of \( \infty \)-cosmoi which were accessible in the relevant sense. We described some examples in [5], all firmly based in the world of quasicategories, but promised to expand this list in a future paper. This is what we shall do here. In fact we introduce a notion of accessibility for \( \infty \)-cosmoi which is stronger than that of the earlier paper; we do this because of the good stability properties it enjoys, which allow us to construct many new examples of accessible \( \infty \)-cosmoi from any given one.

An \( \infty \)-cosmos is a universe in which one can develop the theory of \( \infty \)-categorical structures, much as a (suitably endowed) 2-category is a universe in which one can develop the theory of categorical structures. Given an \( \infty \)-cosmos \( \mathcal{K} \), whose objects are referred to as \( \infty \)-categories, there are further \( \infty \)-cosmoi of

- \( \infty \)-categories with limits or colimits of some type (or a combination of both)
- isofibrations of \( \infty \)-categories (analogous to the usual “arrow categories”)
- various flavours of fibration of \( \infty \)-categories (cartesian or cocartesian, 1-sided or 2-sided, discrete or not).

These constructions of new \( \infty \)-cosmoi from old are all described in [14]; what we do here is show that if the original \( \infty \)-cosmos is accessible, in our sense, then so is each of the resulting ones.

In particular, we could take as our starting \( \infty \)-cosmos that consisting of the quasicategories, the complete Segal spaces, or the Segal categories: each of these is accessible.
A precursor to the present work is [4], which deals with 2-categories of categorical structures rather than ∞-cosmoi of ∞-categorical structures. In that setting, accessibility is seen to be closely related to weakness — for instance, the 2-category of monoidal categories and strong monoidal functors is accessible, but the full sub-2-category consisting of strict monoidal categories is not. One of the guiding ideas in the present work is that since in the ∞-categorical world we are primarily interested in weak structures, the vast majority, if not all, the examples of interest should form accessible ∞-cosmoi.

We now turn to an outline of the paper. We begin in Section 2 with a brief review of the necessary background on ∞-cosmoi and accessible categories, both ordinary and simplicial. Then in Section 3 we introduce our main concept of accessible ∞-cosmos, and show that these include the basic examples arising from suitable simplicially enriched model categories. In Section 4, we study a first raft of closure properties of accessible ∞-cosmoi, including ∞-cosmoi of isofibrations, slices and duals of ∞-cosmoi, and pullbacks of cosmological embeddings. The technical heart of the paper is Section 5, where we show that for an accessible ∞-cosmos $\mathcal{K}$, the ∞-cosmos $\text{Rari}(\mathcal{K})$ of left adjoint left inverses in $\mathcal{K}$ is also accessible. In Section 6 we prove the corresponding fact about trivial fibrations in $\mathcal{K}$, with further results on equivalences. In Section 7 we use the results of the previous three sections to deduce all our remaining closure properties for accessible ∞-cosmoi.

2. Preliminaries

In this section, we run through the key concepts needed later in the paper.

2.1. The Joyal model structure. Of importance in the theory of ∞-cosmoi is the Joyal model structure on the category $sSet$ of simplicial sets. This is a combinatorial model structure whose cofibrations are the monomorphisms and whose fibrant objects are the quasicategories, and makes $sSet$ into a cartesian closed model category.

The model structure has generating cofibrations the boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$. The fibrations between quasicategories will be called isofibrations of quasicategories and can be characterised as those morphisms having the right lifting property with respect to the inner horn inclusions $\Lambda_k^n \to \Delta^n$ together with the endpoint inclusions $1 \to \mathbb{I}$, where $\mathbb{I}$ is the nerve of the free-living isomorphism. The weak equivalences between quasicategories will often be referred to as equivalences of quasicategories.
2.2. **∞-cosmoi.** An ∞-cosmos is a simplicially enriched category $\mathcal{K}$ together with a class of morphisms called isofibrations, denoted $A \to B$, closed under composition and containing the isomorphisms, such that

1. each hom $\mathcal{K}(A, B)$ is a quasicategory;
2. the morphism $\mathcal{K}(A, p) : \mathcal{K}(A, B) \to \mathcal{K}(A, C)$ is an isofibration of quasicategories for each isofibration $p : B \to C$;
3. $\mathcal{K}$ has products, powers by simplicial sets, pullbacks along isofibrations, and limits of countable towers of isofibrations;
4. the class of isofibrations is closed under these limits, under Leibniz powers by monomorphisms of simplicial sets, and contains all maps with terminal codomain.

Following [14], we call the various limits appearing in (3) above *cosmological limits.*

**Examples 2.1.** (a) A simple example of an ∞-cosmos is the 2-category $\text{Cat}$ of (small) categories. Like any 2-category, this can be viewed as a simplicially enriched category by taking the nerves of its hom-categories. Isofibrations between categories are those functors $F : C \to \mathcal{D}$ having the isomorphism lifting property: namely, given an isomorphism $f : A \to FB \in \mathcal{D}$, there exists an isomorphism $f' : A' \to B \in C$ such that $Ff' = f$. As explained in Proposition 1.2.11 of [14], with this choice of isofibrations $\text{Cat}$ becomes a ∞-cosmos.

(b) A fundamental example is the ∞-cosmos $\text{qCat}$ of quasicategories, which is the full simplicially enriched subcategory of $\text{SSet}$ with objects the quasicategories, and with isofibrations as described in Section 2.1 — see Proposition 1.2.10 of [14].

Each ∞-cosmos $\mathcal{K}$ has a *homotopy 2-category* $h\mathcal{K}$. This has the same objects as $\mathcal{K}$ and hom-categories $h\mathcal{K}(A, B) = \pi(\mathcal{K}(A, B))$ where $\pi$ is the left adjoint to the nerve functor. An important example is the 2-category of quasicategories [13], which is the homotopy 2-category $h\text{qCat}$.

A morphism $f : B \to C$ in $\mathcal{K}$ is said to be an *equivalence* if the induced $\mathcal{K}(A, f) : \mathcal{K}(A, B) \to \mathcal{K}(A, C)$ is an equivalence of quasicategories for all $A \in \mathcal{K}$, and a trivial fibration if it is both an equivalence and an isofibration.

We write $\mathcal{K}_0$ for the underlying ordinary category of a simplicially enriched category $\mathcal{K}$. We write $\mathcal{K}^2$ for the simplicially enriched category of arrows in $\mathcal{K}$. We define full subcategories of $\mathcal{K}^2$ as follows:

- $\mathcal{K}^1$ consists of the isofibrations
- $\text{Equiv}(\mathcal{K})$ consists of the equivalences
- $\text{TF}(\mathcal{K}) = \text{Equiv}(\mathcal{K}) \cap \mathcal{K}^1$ consists of the trivial fibrations.
A cosmological functor \( F: \mathcal{K} \rightarrow \mathcal{L} \) between \( \infty \)-cosmoi is a simplicially enriched functor which preserves isofibrations and cosmological limits.

2.3. Weighted limits, colimits, and their homotopy versions. In addition to cosmological limits, we now run through the various kinds of (weighted homotopy) limits and colimits that we encounter in the present paper.

Let \( \mathcal{C} \) be a small simplicially enriched category and consider the enriched functor category \( [\mathcal{C}, \mathbf{SSet}] \), whose objects are called weights. In larger diagrams, we will sometimes denote hom-objects \([\mathcal{C}, \mathbf{SSet}](F, G)\) by \((F, G)\). Let \( W \in [\mathcal{C}, \mathbf{SSet}] \) be a weight. Given a diagram \( S: \mathcal{C} \rightarrow \mathcal{K} \) in a simplicially enriched category \( \mathcal{K} \) a weighted limit \( L \) is defined by a cone \( \eta: W \rightarrow \mathcal{K}(L, S-) \) for which the induced morphism

\[
\mathcal{K}(A, L) \longrightarrow [\mathcal{C}, \mathbf{SSet}](W, \mathcal{K}(A, S-))
\]  

is invertible. A weighted colimit is a weighted limit in \( \mathcal{K}^{\text{op}} \).

2.3.1. Flexible limits. Flexible limits and cofibrantly-weighted limits are those whose defining weights are flexible or cofibrant. To understand them, we observe that the Joyal model structure on \( \mathbf{SSet} \) induces the enriched projective model structure on \([\mathcal{C}, \mathbf{SSet}]\) by Proposition A.3.3.2 and Remark A.3.3.4 of [10], and this has generating cofibrations

\[
\mathcal{I} = \{ \partial \Delta^n \times \mathcal{C}(X, -) \rightarrow \Delta^n \times \mathcal{C}(X, -): n \in \mathbb{N}, X \in \mathcal{C} \}
\]

Riehl and Verity’s flexible weights\(^1\) are precisely the \( \mathcal{I} \)-cellular weights, and so — in particular — cofibrant weights. Flexibly weighted limits in an \( \infty \)-cosmos, or just flexible limits, are of importance since, by Proposition 6.2.8(i) of [14], each \( \infty \)-cosmos \( \mathcal{K} \) admits them.

2.3.2. Weighted colimits that are homotopy colimits. Let us now turn to the question of when colimits are homotopy colimits — here we emphasise the colimit point of view which will be our primary interest. Consider a weight \( W: \mathcal{C}^{\text{op}} \rightarrow \mathbf{SSet} \), not necessarily cofibrant, but suppose now that \( \mathcal{K} \) is locally fibrant and \( S: \mathcal{C} \rightarrow \mathcal{K} \), and let \( p: Q \rightarrow W \) be a cofibrant replacement of \( W \). The weighted colimit \( W \star S \) is said to be a homotopy colimit if the induced morphism

\[
\mathcal{K}(W \star S, A) \xrightarrow{\cong} (W, \mathcal{K}(S-, A)) \xrightarrow{p^*} (Q, \mathcal{K}(S-, A))
\]

\(^1\)This differs from the usage in 2-category theory, where “flexible” is taken to mean cofibrantly-weighted.
is an equivalence of quasicategories. Note that this is equally to say that the second component

\[ p^* : [C^{\text{op}}, \text{SSet}](W, \mathcal{K}(S-, A)) \to [C^{\text{op}}, \text{SSet}](Q, \mathcal{K}(S-, A)) \]

is an equivalence of quasicategories. Since \( \mathcal{K} \) is locally fibrant it follows that the property of being a homotopy colimit is independent of the choice of cofibrant replacement so that, in testing for homotopy colimits, we are free to assume that \( Q \) is flexible and that \( p : Q \to W \) is a trivial fibration.

Of particular interest in this paper is the case where \( C \) is a small \( \lambda \)-filtered category and \( W = \Delta 1 \) is the terminal weight, so that \( W * S \) is the \( \lambda \)-filtered colimit of \( S \). Specialising the above situation, we can thus speak of \( \lambda \)-filtered colimits being homotopy colimits. Dually, a limit \( \{ W, S \} \) is a homotopy limit if it is a homotopy colimit in \( \mathcal{K}^{\text{op}} \).

2.4. Accessible categories. We now turn to some basic results about accessible categories. For further information, see [1] or [11].

A category \( \mathcal{K} \) is \( \lambda \)-accessible, for a regular cardinal \( \lambda \), just when it is the free completion under \( \lambda \)-filtered colimits of a small category. More concretely, this will be the case when \( \mathcal{K} \) has \( \lambda \)-filtered colimits, and there is a small full subcategory \( \mathcal{G} \) whose objects are \( \lambda \)-presentable and such that every object of \( \mathcal{K} \) is a \( \lambda \)-filtered colimit of objects in \( \mathcal{G} \).

A category is accessible if it is \( \lambda \)-accessible for some regular cardinal \( \lambda \).

An accessible category is complete if and only if it is cocomplete, in which case it is said to be a locally presentable category.

A functor between accessible categories is said to be accessible if it preserves \( \lambda \)-filtered colimits for some regular cardinal \( \lambda \).

Example 2.2. (See [1, Proposition 2.3].) Any left or right adjoint between accessible categories is an accessible functor.

The Makkai-Paré Limit Theorem [11, Theorem 5.1.6] asserts that the 2-category of accessible categories, accessible functors, and natural transformations has bicategorical limits (bilimits), and these are formed at the level of underlying ordinary categories. For our purposes, important examples of bilimits include products, powers by small categories (functor categories in \textbf{Cat}) as well as comma objects, of which special cases are slice categories. We shall also apply the Makkai-Paré Limit Theorem in the following special case.
Proposition 2.3. Consider a pullback of categories

\[
\begin{array}{ccc}
A & \xrightarrow{P} & B \\
Q \downarrow & & \downarrow F \\
C & \xrightarrow{G} & D
\end{array}
\]

in which \(F\) and \(G\) are accessible functors between accessible categories, and \(F\) is an isofibration of categories. Then \(A\) is an accessible category and \(P\) and \(Q\) are accessible functors.

Proof. The assumption that \(F\) is an isofibration means that this pullback is also a bipullback [7], and now the result follows by the Limit Theorem. \(\square\)

In this paper, we are primarily interested in simplicially enriched categories. A simplicially enriched category is \(\lambda\)-accessible just when it is the free completion of a small (simplicially enriched) category under (enriched) \(\lambda\)-filtered colimits\(^2\). In the simplicially enriched categories of interest to us each object \(A\) moreover has a power (also known as cotensor) \(X \otimes A\) by each simplicial set \(X\). In this context there are simpler descriptions of accessibility, as described in the following proposition, which follows immediately from [5, Proposition 8.11].

Proposition 2.4. For a simplicially enriched category \(K\) with powers, the following are equivalent:

1. \(K\) is \(\lambda\)-accessible as a simplicially enriched category;
2. the underlying ordinary category \(K_0\) is \(\lambda\)-accessible, and the hom-functor \(K(A, -) : K_0 \to \mathbf{SSet}\) is \(\lambda\)-accessible for each \(\lambda\)-presentable object of \(K_0\);
3. \(K_0\) is an accessible category and \(\Delta[n] \otimes - : K_0 \to K_0\) is an accessible functor for each \(n \in \mathbb{N}\).

For such a \(K\), the functors \(K(A, -) : K_0 \to \mathcal{V}_0\) and \(X \otimes - : K_0 \to K_0\) are accessible for all \(A \in K\) and \(X \in \mathcal{V}\).

3. ACCESSIBLE \(\infty\)-COSMOI

Definition 3.1. An \(\infty\)-cosmos \(K\) is said to be accessible if

1. \(K\) is accessible as a simplicially enriched category;
2. \(K_0^1\) is accessible and accessibly embedded in \(K_0^2\);\(^3\)

\(^2\)For general \(\mathcal{V}\) another notion of enriched accessibility is considered in [3]. However, in the special case of simplicial enrichment, it is equivalent to the notion described above by [9, Theorem 3.14].
(3) There exists a regular cardinal $\lambda$ such that $\lambda$-filtered colimits exist in $\mathcal{K}$ and are homotopy colimits.

A cosmological functor between accessible $\infty$-cosmoi is accessible if its underlying functor is accessible.

**Remark 3.2.** In [5] an $\infty$-cosmos was said to be accessible when it is so as a simplicially enriched category, our Condition (1). As mentioned in the introduction, and as anticipated in a footnote in [5, Section 9.4], we have strengthened the definition here so that the class of accessible $\infty$-cosmoi has better stability properties, such as Proposition 4.1 below. In fact Proposition 4.1 and the other results in Section 4 would all hold if we added only Condition (2); it is in order to prove the accessibility of $\text{Rari}(\mathcal{K})$ for an accessible $\infty$-cosmos $\mathcal{K}$ that we include Condition (3) in the definition. For further comments on Condition (3), see Remark 6.5.

**Remark 3.3.** In this paper we are largely avoiding the question of indices of accessibility, but perhaps a few words are appropriate. If $\mathcal{K}$ is $\lambda$-accessible as a simplicially-enriched category, it follows by the characterization in Proposition 2.4 that it is also $\lambda'$-accessible for any $\lambda' \triangleright \lambda$, in the sense of [1, Definition 2.12]. Similarly, if $\mathcal{K}_0^\perp$ is $\lambda$-accessible closed in $\mathcal{K}_0^\perp$ under $\lambda$-filtered colimits, then the same is true for any $\lambda' \triangleright \lambda$. On the other hand, if $\mathcal{K}$ satisfies Condition (3) for a given $\lambda$, then it does so for any $\lambda' > \lambda$. By the Uniformization Theorem [1, Theorem 2.19], if $\mathcal{K}$ is an accessible $\infty$-cosmos, there are arbitrarily large $\lambda$ for which $\mathcal{K}$ is $\lambda$-accessible as an enriched category, $\mathcal{K}_0^\perp$ is accessible and accessibly embedded in $\mathcal{K}_0^\perp$, and Condition (3) holds for the given $\lambda$.

In addition to the accessibility of powering functors, further exactness properties are easily seen to hold in an accessible $\infty$-cosmos.

**Lemma 3.4.** Let $\mathcal{K}$ be an $\infty$-cosmos for which $\mathcal{K}$ is $\lambda$-accessible as a simplicially enriched category. Then pullbacks of isofibrations and finite products commute with $\lambda$-filtered colimits in $\mathcal{K}$.

**Proof.** Let $J : \mathcal{K}_\lambda \to \mathcal{K}$ be the inclusion of the full subcategory consisting of the $\lambda$-presentable objects. The induced functor $\mathcal{K}(J-,1) : \mathcal{K} \to [\mathcal{K}_\lambda^\text{op}, \text{SSet}]$ is fully faithful, preserves $\lambda$-filtered colimits, and preserves any existing limits. Since it reflects isomorphisms, $\mathcal{K}(J-,1)$ also reflects any commutativities between existing limits and $\lambda$-filtered colimits that hold in $[\mathcal{K}_\lambda^\text{op}, \text{SSet}]$. Since $\text{SSet}$ is locally finitely presentable, both pullbacks and finite products commute with filtered colimits in $\text{SSet}$ and so in $[\mathcal{K}_\lambda^\text{op}, \text{SSet}]$, as required. \qed

In the following sections, we will show that various constructions applied to accessible $\infty$-cosmoi yield new accessible $\infty$-cosmoi. But for
Proposition 3.5. Let $\mathcal{M}$ be a simplicially enriched category, equipped with a combinatorial model structure which is enriched with respect to the Joyal model structure on $\text{SSet}$. Suppose that every fibrant object of $\mathcal{M}$ is also cofibrant. Then the full subcategory $\mathcal{M}_{\text{fib}}$ of $\mathcal{M}$ consisting of the fibrant objects is an accessible $\infty$-cosmos, in which the isofibrations and equivalences are the fibrations and weak equivalences (between fibrant objects in each case) of the model structure. Moreover, the inclusion $\mathcal{M}_{\text{fib}} \hookrightarrow \mathcal{M}$ is an accessible embedding.

**Proof.** $\mathcal{M}_{\text{fib}}$ is an $\infty$-cosmos by [14, Proposition E.1.1] and accessible as a simplicially enriched category by [5, Proposition 9.1]. The same result shows that $\mathcal{M}_{\text{fib}} \hookrightarrow \mathcal{M}$ is an accessible embedding.

Now the full subcategories $\mathcal{F}, \mathcal{W} \hookrightarrow \mathcal{M}^2$ of fibrations and weak equivalences are accessible and accessibly embedded. Therefore, by Proposition 2.3, the pullback $\mathcal{F} \cap \mathcal{M}_{\text{fib}} \hookrightarrow \mathcal{M}_{\text{fib}}^2$ is accessible and accessibly embedded, establishing Condition (2).

Now since $\mathcal{F}$ and $\mathcal{W}$ are accessibly embedded, there exists a regular cardinal $\lambda$ such that both $\mathcal{F}$ and $\mathcal{W}$ are closed in $\mathcal{M}^2$ under $\lambda$-filtered colimits; this closure property of $\mathcal{F}$ also ensures that $\mathcal{M}_{\text{fib}} \hookrightarrow \mathcal{M}$ is closed under $\lambda$-filtered colimits. We will use these assumptions to prove that $\lambda$-filtered colimits in $\mathcal{M}_{\text{fib}}$ are homotopy colimits, thereby establishing Condition (3).

To this end, let $\mathcal{C}$ be a small $\lambda$-filtered category. Since $\mathcal{M}$ is complete as an enriched category, we can consider the weighted colimit functor

$$- \star - : [\mathcal{C}^{\text{op}}, \text{SSet}]_0 \times [\mathcal{C}, \mathcal{M}]_0 \to \mathcal{M}_0$$

Both $\text{SSet}$, equipped with the Joyal model structure, and $\mathcal{M}$ are combinatorial model categories. Therefore $[\mathcal{C}^{\text{op}}, \text{SSet}]_0$ and $[\mathcal{C}, \mathcal{M}]_0$ each admit both the projective and injective model structure. The key result for us is that when one of these is equipped with the projective model structure and the other with the injective model structure, the weighted colimit functor becomes a left Quillen bifunctor. (This is a special case of Theorem C.3.13 of [14], which generalises Gambino’s result [6] for the Kan-Quillen model structure.)

Now consider a diagram $S : \mathcal{C} \to \mathcal{M}$ taking values among the fibrant objects. Since $\mathcal{M}_{\text{fib}}$ is closed under $\lambda$-filtered colimits, the colimit $\Delta 1 \star S$ is also fibrant. Let $p : Q\Delta 1 \to \Delta 1$ in $[\mathcal{C}^{\text{op}}, \text{SSet}]$ and $q : QS \to S$ in $[\mathcal{C}, \mathcal{M}]$ be projective cofibrant replacements. We then have a
Now since $S$ is pointwise fibrant, it is pointwise cofibrant and therefore injectively cofibrant. Hence $q: QS \to S$ is a weak equivalence between injectively cofibrant objects. Since $Q\Delta 1$ is projectively cofibrant and $-\ast -$ a left Quillen bifunctor with respect to the (projective, injective) model structures, it follows that $Q\Delta 1 \ast QS \to Q\Delta 1 \ast S$ is a weak equivalence between cofibrant objects.

Similarly, since all objects are cofibrant in $\mathbf{SSet}$, $p: Q\Delta 1 \to \Delta 1$ is a weak equivalence between injectively cofibrant objects, whilst $QS$ is projectively cofibrant. Therefore taking the (injective, projective) choice, it follows that the left leg of the square $p \ast QS: Q\Delta 1 \ast QS \to \Delta 1 \ast QS$ is a weak equivalence of cofibrant objects.

Since weak equivalences are closed under $\lambda$-filtered colimits in $\mathcal{M}$, it is also true that $\Delta 1 \ast q: \Delta 1 \ast QS \to \Delta 1 \ast S$ is a weak equivalence. Therefore, by 2-from-3 on the above square, the morphism $p \ast S: Q\Delta 1 \ast S \to \Delta 1 \ast S$ is a weak equivalence of cofibrant objects.

Since $\mathcal{M}$ is an enriched model category, if $A \in \mathcal{M}$ is fibrant the map $\mathcal{M}(p \ast S, A) : \mathcal{M}(\Delta 1 \ast S, A) \to \mathcal{M}(Q\Delta 1 \ast S, A)$ is a weak equivalence, whence so is the isomorphic $[\mathcal{C}^{\mathcal{op}}, \mathbf{SSet}](\Delta 1, \mathcal{K}(S- , A)) \to [\mathcal{C}^{\mathcal{op}}, \mathbf{SSet}](Q\Delta 1, \mathcal{K}(S- , A))$ as required.

Example 3.6. The $\infty$-cosmos $\mathbf{qCat}$ of quasicategories arises in this way as $\mathcal{M}_{\text{fib}}$, where $\mathcal{M}$ is $\mathbf{SSet}$ equipped with the Joyal model structure. The $\infty$-cosmos $\mathbf{CSS}$ of complete Segal spaces arises in this way as $\mathcal{M}_{\text{fib}}$, where $\mathcal{M}$ is a simplicial model structure on the category of bisimplicial sets due to Rezk [12, Theorem 7.2]. The proposition can also be applied to the model category $\mathbf{Cat}$, with the “natural” model structure: see [14, Proposition 1.2.11]; in this case, of course, $\mathbf{Cat}_{\text{fib}}$ is just $\mathbf{Cat}$. For further examples, including various models for $(\infty, n)$-categories, see [14, Appendix E].
Example 3.7. An example of a non-accessible $\infty$-cosmos is $\mathbf{Cat}^\text{op}$. As explained in [14, Example E.1.6], this is an $\infty$-cosmos, with the injective-on-objects functors as isofibrations. But it is not accessible, since the underlying category is not accessible; indeed, if a category and its opposite are both accessible then the category must be a preorder: see [1, Theorem 1.64].

4. First stability properties of accessible $\infty$-cosmoi

In Section 6 of [14], Riehl and Verity show that a given $\infty$-cosmos gives rise to many others, such as the $\infty$-cosmos of isofibrations and slice $\infty$-cosmoi. In the present section, we investigate a first group of these constructions, showing that they lift to the world of accessible $\infty$-cosmoi.

4.1. The $\infty$-cosmos of isofibrations. If $\mathcal{K}$ is an $\infty$-cosmos, then $\mathcal{K}^\perp$ becomes one too [14, Proposition 6.1.1] on defining a commutative square

\[
\begin{array}{ccc}
A & \longrightarrow & A' \\
\downarrow^p & & \downarrow^p' \\
B & \longrightarrow & B'
\end{array}
\]

(4.1)

to be an an isofibration just when both the lower horizontal $B \to B'$ and the induced map $A \to B \times_{B'} A'$ are isofibrations in $\mathcal{K}$.

Proposition 4.1. If $\mathcal{K}$ is an accessible $\infty$-cosmos, so is $\mathcal{K}^\perp$, and the inclusion $\mathcal{K}^\perp_0 \to \mathcal{K}^\perp_0 + \mathcal{K}^\perp_0$ is an accessible functor.

Proof. The fact that $\mathcal{K}^\perp_0$ and the inclusion $\mathcal{K}^\perp_0 \to \mathcal{K}^\perp_0 + \mathcal{K}^\perp_0$ are accessible is part of the definition of $\mathcal{K}$ being an accessible $\infty$-cosmos. Accessibility of the power functors holds because it holds in $\mathcal{K}$, and powers and sufficiently-filtered colimits in $\mathcal{K}^\perp$ are computed pointwise.

Next, we need to show that $(\mathcal{K}^\perp_0)^\perp_0$ is accessible and accessibly embedded in $(\mathcal{K}^\perp_0)^2_0$. To see this, consider the pullback

\[
\begin{array}{ccc}
(\mathcal{K}^\perp_0)^{\perp_0} & \longrightarrow & (\mathcal{K}^\perp_0)^2_0 \\
\downarrow & & \downarrow^{(\text{lh}, \text{pb})} \\
\mathcal{K}^\perp_0 \times \mathcal{K}^\perp_0 & \longrightarrow & \mathcal{K}^\perp_0 \times \mathcal{K}^\perp_0
\end{array}
\]

in which the horizontal maps are the inclusions, and $(\text{lh}, \text{pb})$ is the map sending a commutative square (4.1) to the pair consisting of the lower horizontal $B \to B'$ and the induced map $A \to B \times_{B'} A'$. 
The bottom leg of the pullback square is a product of two copies of the accessible isofibration of categories $\mathcal{K}^\dagger_0 \to \mathcal{K}^2_0$, and so is an accessible isofibration of categories. Therefore, the claim will follow from Proposition 2.3 if we can show that the right leg 

$$(lh, pb): (\mathcal{K}^\dagger_0)^2 \to (\mathcal{K})^2_0 \times (\mathcal{K})^2_0$$

is accessible. Using Lemma 3.4 we choose $\lambda$ such that, first, $\mathcal{K}^\dagger_0 \to \mathcal{K}^2_0$ is closed under $\lambda$-filtered colimits; and second, pullbacks of isofibrations commute with $\lambda$-filtered colimits in $\mathcal{K}$. The first assumption ensures that $lh$ preserves $\lambda$-filtered colimits whilst the second assumption ensures that $pb$ does so too; hence so does $(lh, pb)$.

In addition to the above properties of $\lambda$, we now further assume that $\lambda$-filtered colimits are homotopy colimits in $\mathcal{K}$. We will show that the same property holds in $\mathcal{K}^\dagger$. To this end, let $C$ be $\lambda$-filtered and consider $S: C \to \mathcal{K}^\dagger_0$, and let $Q \to \Delta^1 \in [\mathcal{C}^{op}, \mathbb{SSet}]$ be a projective cofibrant replacement. We must prove that

$$p^*: [\mathcal{C}^{op}, \mathbb{SSet}](\Delta^1, \mathcal{K}^\dagger(S^-, A)) \to [\mathcal{C}^{op}, \mathbb{SSet}](Q, \mathcal{K}^\dagger(S^-, A)) \quad (4.2)$$

is an equivalence of quasicategories. Now $S$ is specified by its source and target components $S_0, S_1: C \to \mathcal{K}$ plus a natural pointwise isofibration $s: S_0 \to S_1$, whilst $A$ is a single isofibration $a: A_0 \to A_1$, and by definition of $\mathcal{K}^\dagger$ we have a pullback square

$$\begin{array}{ccc}
\mathcal{K}^\dagger(S^-, A) & \longrightarrow & \mathcal{K}(S_0^-, A_0) \\
\downarrow & & \downarrow a^* \\
\mathcal{K}(S_1^-, A_1) & \longrightarrow & \mathcal{K}(S_0^-, A_1)
\end{array}$$

in $[\mathcal{C}^{op}, \mathbb{SSet}]$ whose right leg is a pointwise isofibration. Both the representatives $[\mathcal{C}^{op}, \mathbb{SSet}](\Delta^1, -)$ and $[\mathcal{C}^{op}, \mathbb{SSet}](Q, -)$ preserve pullbacks, so that (4.2) is in fact the unique induced map between the pullbacks of the two horizontal rows below.

$$(\Delta^1, \mathcal{K}(S_1^-, A_1)) \xrightarrow{(\Delta^1, d^*)} (\Delta^1, \mathcal{K}(S_0^-, A_1)) \xrightarrow{(\Delta^1, a^*)} (\Delta^1, \mathcal{K}(S_0^-, A_0))$$

$$p^*: (Q, \mathcal{K}(S_1^-, A_1)) \xrightarrow{(Q, d^*)} (Q, \mathcal{K}(S_0^-, A_1)) \xrightarrow{(Q, a^*)} (Q, \mathcal{K}(S_0^-, A_0)) \quad (4.3)$$

Let us first observe that since $\lambda$-filtered colimits are homotopy colimits in $\mathcal{K}$, each of the three vertical morphisms is an equivalence of quasicategories — in particular, all of the objects in the diagram are
fibrant in the Joyal model structure. Therefore, to prove that the induced map between the pullbacks is an equivalence of quasicategories, it will suffice by Proposition C.1.13 of [14] to show that the two left-pointing morphisms are isofibrations.

Examining first the lower left-pointing morphism, observe that since $Q$ is projectively cofibrant and $a_*: K(S_0-, A_1) \to K(S_0-, A_0)$ is a pointwise isofibration, it follows that $(Q, a_*)$ is an isofibration, as required. The upper left-pointing morphism $(\Delta 1, a_*)$ is isomorphic to $K(\text{colim } S_0, a): K(\text{colim } S_0, A_1) \to K(\text{colim } S_0, A_0)$, which is an isofibration since $a$ is one, completing the proof. □

4.2. Slice constructions. Slice categories can be a very convenient tool for expressing various universal properties. This remains true in the $\infty$-cosmos setting, but here the slice construction is based on isofibrations with given codomain rather than arbitrary morphisms.

For a simplicially enriched category $K$ and an object $A \in K$, we write $K \downarrow A$ for the enriched slice category: an object is a morphism $p: B \to A$, while if $q: C \to A$ is also an object then the corresponding hom is given by the pullback

\[ (K \downarrow A)(p, q) \to K(B, C) \]

\[ \downarrow_{K(B, q)} \]

\[ 1 \to K(B, A). \]

In particular, a morphism in $K \downarrow A$ is just a commutative triangle. If now $K$ is an $\infty$-cosmos, we write $K/A$ for the full subcategory of $K \downarrow A$ consisting of those $p: B \to A$ which are isofibrations. This $K/A$ is also an $\infty$-cosmos [14, Proposition 1.2.22] on defining a morphism from $p: B \to A$ to $q: C \to A$ to be an isofibration in $K/A$ just when the corresponding morphism $B \to C$ is one in $K$.

Proposition 4.2. If $K$ is an accessible $\infty$-cosmos then so is $K/A$ for each $A \in K$, and the inclusion $K/A \to K \downarrow A$ is an accessible functor.

Proof. In the pullback below left

\[ (K/A)_0 \xrightarrow{I_0} K^\perp_0 \]

\[ \downarrow \]

\[ 1 \xrightarrow{A} K_0 \]

\[ (K/A)^\perp_0 \xrightarrow{(K^\perp)^\perp_0} (K^\perp)_0 \]

the right vertical is an accessible isofibration between accessible categories, and the lower horizontal an accessible functor between accessible
categories. It follows by Proposition 2.3 that \((\mathcal{K}/A)_0\) is accessible and \(I_0\) an accessible functor.

Now consider the pullback above right. Since \(I_0\) is accessible, the Makkai-Paré Limit Theorem ensures that the lower horizontal \(I^2_0\) is also accessible. The right vertical inclusion is an accessible functor between accessible categories by Proposition 4.1 and moreover an isofibration. Hence, by Proposition 2.3 once again, the left vertical is an accessible functor between accessible categories, verifying Condition (2) in the definition of accessible \(\infty\)-cosmos.

For an object \(p: B \to A\) of \(\mathcal{K}/A\), the corresponding hom-functor \(\mathcal{K}/A((B,p),-)\): \(\mathcal{K}/A \to \mathbf{SSet}\) can be constructed as a pullback

\[
\begin{array}{ccc}
\mathcal{K}/A((B,p),-) & \longrightarrow & \mathcal{K}(B, \text{dom } -) \\
\downarrow & \quad & \downarrow \\
1 & \quad & \mathcal{K}(B,A)
\end{array}
\]

of functors \(\mathcal{K}/A \to \mathbf{SSet}\), where the objects 1 and \(\mathcal{K}(B,A)\) are seen as constant functors, while \(\mathcal{K}(B, \text{dom } -)\) is the functor sending \(q: C \to A\) to \(\mathcal{K}(B,C)\), and the right vertical has component at \(q: C \to A\) given by \(\mathcal{K}(B,q): \mathcal{K}(B,C) \to \mathcal{K}(B,A)\). This is a pullback of accessible functors, so is itself accessible, since pullbacks commute with \(\lambda\)-filtered colimits in \(\mathbf{SSet}\), for any infinite cardinal \(\lambda\). It follows by Proposition 2.4 that Condition (1) in the definition of accessible \(\infty\)-cosmos holds.

The verification of Condition (3) is identical in form to the corresponding verification in the proof of Proposition 4.1, the main difference is that the left vertical in (4.3) is replaced by the identity \(1 \to 1\).

Finally, accessibility of the inclusion follows from the fact that there is a pullback

\[
\begin{array}{ccc}
(\mathcal{K}/A)_0 & \longrightarrow & (\mathcal{K}^\downarrow)_0 \\
\downarrow & \quad & \downarrow \\
(\mathcal{K} \downarrow A)_0 & \longrightarrow & (\mathcal{K}^2)_0
\end{array}
\]

of accessible categories and accessible functors, in which the right vertical is an isofibration. \(\square\)

Later on, we will use the following simple result in our applications, and so record it now.

**Proposition 4.3.** If \(F: \mathcal{L} \to \mathcal{K}\) is an accessible cosmological functor, then so is the induced \(F/A: \mathcal{L}/A \to \mathcal{K}/F_A\) for any \(A \in \mathcal{L}\).
Proof. There is a commutative square

\[
\begin{array}{ccc}
L/A & \xrightarrow{F/A} & K_{/FA} \\
\downarrow & & \downarrow \\
L & \xrightarrow{F_{\downarrow A}} & K \downarrow FA
\end{array}
\]

in which the vertical maps are the fully faithful inclusions. These are accessible by Proposition 4.2, while \( F \downarrow A \) is so since it is the induced map between comma-categories (or comma objects) in the 2-category of accessible categories and accessible functors. Thus \( F_{/A} \) is also accessible. \( \square \)

4.3. Dual \( \infty \)-cosmoi. As described in Definition 1.2.25 of [14], each \( \infty \)-cosmos \( K \) has a dual \( \infty \)-cosmos \( K^{co} \). This has the same underlying category as \( K \), with simplicial homs given by \( K^{co}(A,B) = K(A,B)^{op} \), and with the same isofibrations as in \( K \). Powers in \( K^{co} \) by \( X \) are given by powers in \( K \) by the opposite simplicial set \( X^{op} \).

Proposition 4.4. If \( K \) is an accessible \( \infty \)-cosmos, so is its dual \( K^{co} \).

Proof. The only condition left to be verified is Condition (3), for which purpose we will investigate weighted colimits in \( K \). Let \( W : C^{op} \to \text{SSet} \) a weight and \( S : C \to K \) a diagram, which corresponds to a diagram \( S^{co} : C^{co} \to K^{co} \). For simplicity, let us suppose that \( C \) is merely a category, so that \( C^{co} = C \). Applying the involution \((-)^{op} : \text{SSet} \to \text{SSet} \) levelwise gives an involution \((-)^{op} : [C^{op}, \text{SSet}]_0 \to [C^{op}, \text{SSet}]_0 \).

We then have an isomorphism

\[
\varphi_{W,A} : [C^{op}, \text{SSet}](W, K(S-, A))^{op} \cong [C^{op}, \text{SSet}](W^{op}, K^{co}(S^{co}-, A))
\]

natural in \( W \) and \( A \). Suppose now that \( C \) is \( \lambda \)-filtered, that \( \lambda \)-filtered colimits exist and are homotopy colimits in \( K \), and that \( p : Q \to \Delta 1 \) is a cofibrant replacement with \( p \) a trivial fibration. By the above, we have a commuting square

\[
\begin{array}{ccc}
[C^{op}, \text{SSet}](\Delta 1, K(S-, A))^{op} & \xrightarrow{(p^*)^{op}} & [C^{op}, \text{SSet}](Q, K(S-, A))^{op} \\
\varphi_{\Delta 1,A} & & \varphi_{Q,A} \\
[C^{op}, \text{SSet}](\Delta 1^{op}, K^{co}(S^{co}-, A)) & \xrightarrow{(p^{op})^*} & [C^{op}, \text{SSet}](Q^{op}, K^{co}(S^{co}-, A))
\end{array}
\]

with vertical maps isomorphisms. The upper horizontal is an equivalence since the opposite of an equivalence is an equivalence, so that the lower horizontal is one too. Since this is induced by precomposition with \( p^{op} : Q^{op} \to \Delta 1^{op} = \Delta 1 \), we will have verified Condition (3) if
we can show that this is a cofibrant replacement of $\Delta 1$. Indeed, this follows easily from the fact that the involution $(-)^{op}: [C^{op}, SSet]_0 \to [C^{op}, SSet]_0$ leaves the projective trivial fibrations unchanged, and so leaves the projectively cofibrant objects unchanged too. □

4.4. **Cosmological embeddings.** Let $\mathcal{K}$ be an $\infty$-cosmos. A (not necessarily full) simplicial subcategory $\mathcal{L}$ of $\mathcal{K}$ is said to be replete [14, Definition 6.3.1] if:

1. each object of $\mathcal{K}$ equivalent to one in $\mathcal{L}$ belongs to $\mathcal{L}$;
2. each equivalence in $\mathcal{K}$ between objects of $\mathcal{L}$ belongs to $\mathcal{L}$;
3. each 0-arrow of $\mathcal{K}$ isomorphic to one in $\mathcal{L}$ belongs to $\mathcal{L}$;
4. the inclusion is full on positive-dimensional arrows.

We also say that the inclusion $\mathcal{L} \to \mathcal{K}$ is replete.

**Remark 4.5.** In the presence of Condition (4), the other conditions amount to the fact that the 2-functor $h\mathcal{L} \to h\mathcal{K}$ is a fibration for the model structure of [8] for 2-categories, there called an equiv-fibration.

If moreover $\mathcal{L}$ is closed in $\mathcal{K}$ under cosmological limits, then it can be made into an $\infty$-cosmos [14, Proposition 6.3.3] by defining a morphism in $\mathcal{L}$ to be an isofibration if and only if it is one in $\mathcal{K}$. The inclusion $\mathcal{L} \to \mathcal{K}$ is then said to be a cosmological embedding.

**Lemma 4.6.** If $J: \mathcal{L} \hookrightarrow \mathcal{K}$ is a cosmological embedding, then each $J_{X,Y}: \mathcal{L}(X,Y) \to \mathcal{K}(X,Y)$ is an isofibration of quasicategories.

**Proof.** We must show that the $J_{X,Y}$ have the right lifting property with respect to the inner horn inclusions and the inclusion $1 \to I$, where $I$ denotes the (nerve of the) free-living isomorphism. The inner horn inclusions are bijective on vertices, while $J_{X,Y}$ is fully faithful on positive dimensional arrows, and these two classes are orthogonal, so there are in fact unique liftings. As for $1 \to I$, we have a lifting problem

$$
\begin{array}{ccc}
1 & \xrightarrow{f} & \mathcal{L}(X,Y) \\
\downarrow & & \downarrow J_{X,Y} \\
I & \xrightarrow{1} & \mathcal{K}(X,Y)
\end{array}
$$

which amounts to giving a 0-arrow $f \in \mathcal{L}(X,Y)$ and a map $I \to \mathcal{K}(X,Y)$ which sends the isomorphism $0 \cong 1$ to an isomorphism $f \to g$ in $\mathcal{K}(X,Y)$; then by Condition (3) in the definition of repleteness it follows that also $g \in \mathcal{L}(X,Y)$. This shows that $I \to \mathcal{K}(X,Y)$ factorizes through $\mathcal{L}(X,Y)$ on 0-simplices; since $J_{X,Y}$ is full on positive dimensional simplices it factorizes in all dimensions, as required. □
Lemma 4.7. Suppose that $\mathcal{L}$ and $\mathcal{K}$ are locally fibrant simplicially enriched categories, and that $\mathcal{L} \hookrightarrow \mathcal{K}$ has each $\mathcal{L}(X,Y) \to \mathcal{K}(FX, FY)$ an isofibration, injective in each dimension. If $\mathcal{L}$ and $\mathcal{K}$ have $W$-weighted colimits and they are homotopy colimits in $\mathcal{K}$, then they are also homotopy colimits in $\mathcal{L}$. There is also a dual result involving limits.

Proof. Given $S: C \to \mathcal{L}$, we can form $W \ast S \in \mathcal{L}$. For $p: Q \to W$ a cofibrant replacement, with $p$ a pointwise trivial fibration, we will prove that $p^* : [\mathcal{C}^{\text{op}}, S\text{Set}](\Delta 1, \mathcal{L}(S-, A)) \to [\mathcal{C}^{\text{op}}, S\text{Set}](Q, \mathcal{L}(S-, A))$ is a weak equivalence. Consider the commutative square

\[
\begin{array}{ccc}
[C^{\text{op}}, S\text{Set}](W, \mathcal{L}(S-, A)) & \xrightarrow{p^*} & [C^{\text{op}}, S\text{Set}](Q, \mathcal{L}(S-, A)) \\
(J_{S-, A})_* & & (J_{S-, A})_* \\
[C^{\text{op}}, S\text{Set}](W, \mathcal{K}(JS-, JA)) & \xrightarrow{p^*} & [C^{\text{op}}, S\text{Set}](Q, \mathcal{K}(JS-, JA))
\end{array}
\]

in $S\text{Set}$. The map $J_{S-, A}: \mathcal{L}(S-, A) \to \mathcal{K}(JS-, JA)$ is a (pointwise) monomorphism and $p: Q \to W$ is a regular epimorphism since it is a pointwise split epimorphism. Thus by the (enriched) orthogonality of regular epimorphisms and monomorphisms, the above square is a pullback.

Since $J_{S-, A}: \mathcal{L}(S-, A) \to \mathcal{K}(JS-, JA)$ is a pointwise isofibration between pointwise fibrant objects, and $Q$ is cofibrant in the projective model structure, it follows that the right vertical arrow in the square is an isofibration of fibrant objects. Moreover, since the lower left object $[C^{\text{op}}, S\text{Set}](W, \mathcal{K}(JS-, JA))$ is isomorphic to $\mathcal{K}(W \ast JS, JA)$, it is fibrant too, and the lower horizontal is thus a weak equivalence of fibrant objects. Now the pullback of a weak equivalence between fibrant objects along a fibration between fibrant objects is always a weak equivalence — see Lemma A.2.4.3 of [10], for example — and so the upper horizontal is also a weak equivalence, as required. \hfill $\Box$

Proposition 4.8. Suppose that $\mathcal{L} \hookrightarrow \mathcal{K}$ is a cosmological embedding with $\mathcal{K}$ an accessible $\infty$-cosmos. If $\mathcal{L}_0 \hookrightarrow \mathcal{K}_0$ is an accessible functor between accessible categories, then $\mathcal{L}$ is also an accessible $\infty$-cosmos in such a way that the inclusion $\mathcal{L} \hookrightarrow \mathcal{K}$ is an accessible cosmological embedding.

Proof. The category $\mathcal{L}_0$ is accessible by assumption. Compatibility of powers and sufficiently filtered colimits holds in $\mathcal{L}_0$ since these are both calculated as in $\mathcal{K}_0$. 

By definition of the isofibrations in $\mathcal{K}$, we have a pullback square

\[
\begin{array}{ccc}
\mathcal{L}_0 & \longrightarrow & \mathcal{L}_0^2 \\
\downarrow & & \downarrow \\
\mathcal{K}_0 & \longrightarrow & \mathcal{K}_0^2
\end{array}
\]

of categories. The right vertical and lower horizontal are accessible functors between accessible categories, and the lower horizontal is also an isofibration, thus the upper horizontal is an accessible functor between accessible categories by Proposition 2.3.

For Condition (3), let $\lambda$ be such that $\mathcal{L}$ has $\lambda$-filtered colimits preserved by the inclusion to $\mathcal{K}$ and such that $\lambda$-filtered colimits are homotopy colimits in $\mathcal{K}$. Then by Lemma 4.7, $\lambda$-filtered colimits are also homotopy colimits in $\mathcal{L}$. \(\square\)

**Proposition 4.9.** Suppose that $J: \mathcal{L} \hookrightarrow \mathcal{K}$ is an accessible cosmological embedding and $F: \mathcal{K}' \rightarrow \mathcal{K}$ is an accessible cosmological functor. Then in the pullback

\[
\begin{array}{ccc}
\mathcal{L}' & \longrightarrow & \mathcal{L} \\
\downarrow^{G} & & \downarrow^{J} \\
\mathcal{K}' & \longrightarrow & \mathcal{K}
\end{array}
\]

of simplicially enriched categories, $\mathcal{L}'$ is an accessible $\infty$-cosmos, while $J': \mathcal{L}' \rightarrow \mathcal{K}'$ is an accessible cosmological embedding, and $G: \mathcal{L}' \rightarrow \mathcal{L}$ is an accessible cosmological functor.

**Proof.** By Proposition 6.3.12 of [14], $G: \mathcal{L}' \rightarrow \mathcal{L}$ is a cosmological embedding of $\infty$-cosmoi and $J'$ a cosmological functor.

A replete inclusion such as $J$ is in particular an isofibration at the level of underlying categories. By Proposition 2.3, it follows that $(\mathcal{L}')_0$ is an accessible category and $G_0$ and $J'_0$ are accessible functors. Now $\mathcal{L}'$ is an accessible $\infty$-cosmos by Proposition 4.8, and it follows immediately that $J'$ is an accessible cosmological embedding and $G$ is an accessible cosmological functor. \(\square\)

### 5. Left adjoint left inverses

A more exotic construction of $\infty$-cosmoi than those seen so far is the $\infty$-cosmos of $\infty$-categories with limit of a given shape — see Section 6.3 of [14]. As described therein, such examples involving $\infty$-categorical structures with universal properties, are naturally understood using the $\infty$-cosmos of lalis. The present section adapts $\infty$-cosmoi of lalis...
to the accessible setting, and our results here make full use of all the
axioms of an accessible $\infty$-cosmos.

A morphism $f : A \to B$ in a 2-category $\mathcal{K}$ is said to be a left adjoint
left inverse (lali) if it admits a right adjoint $u$ for which the counit
$\varepsilon : fu \Rightarrow 1_B$ is invertible — the right adjoint $u$ is then called a right
adjoint right inverse (rari). In particular, a morphism $f$ is a lali just
when it admits a rari, and vice versa.

A commutative square

$$
\begin{array}{ccc}
A & \xrightarrow{r} & A' \\
\downarrow f & & \downarrow f' \\
B & \xrightarrow{s} & B'
\end{array}
$$

with $f$ and $f'$ lalis is said to be a morphism of lalis just when it also
commutes with the right adjoints in the sense that its mate $ru \Rightarrow u's$ is
invertible. Note that this is independent of the choice of right adjoints.

Now if $\mathcal{K}$ is an $\infty$-cosmos, a morphism $f : A \to B$ is said to be a lali/rari when it is one in the homotopy 2-category $h\mathcal{K}$. Likewise, a
commuting square $(r, s) : f \to f'$ is said to be a morphism of lalis if it
is so in $h\mathcal{K}$.

In Proposition 6.3.10 of [14], Riehl and Verity construct a cosmologi-
cally embedded $\infty$-cosmos

$$
\text{Rari}(\mathcal{K}) \hookrightarrow \mathcal{K}^\perp
$$

whose objects are the isofibrations that are lalis (in other words, admit
a rari) and with morphisms the morphisms of lalis. The fact that it
is a cosmological embedding fully determines the remaining structure:
the inclusion reflects isofibrations and is full on positive-dimensional
arrows.

Let us mention an important point: by Lemma 3.6.9 of [14], if an
isofibration $f : A \to B$ is a lali, then the right adjoint $u : B \to A$ can be
chosen so that it is a section of $f$ and so that the counit is the identity
$fu = 1$ in $h\mathcal{K}$.

The goal of this section is to prove that if $\mathcal{K}$ is an accessible $\infty$-cosmos
then so is $\text{Rari}(\mathcal{K})$ with, moreover, the inclusion $\text{Rari}(\mathcal{K}) \hookrightarrow \mathcal{K}^\perp$ an
accessible cosmological embedding. This result, whose proof makes full
use of all of the axioms for an accessible $\infty$-cosmos, is essential for our
later applications.

In moving towards this result, we begin by showing that lalis and
their morphisms are representable notions.

**Proposition 5.1.** Let $\mathcal{K}$ be an $\infty$-cosmos.
(1) An isofibration \( p: A' \twoheadrightarrow A \) in \( \mathcal{K} \) is a lali if and only if
(a) \( \mathcal{K}(C, p): \mathcal{K}(C, A') \to \mathcal{K}(C, A) \) is a lali in \( \mathbf{qCat} \) for each \( C \in \mathcal{K} \);
(b) the square
\[
\begin{array}{ccc}
\mathcal{K}(D, A') & \xrightarrow{K(C,A')} & \mathcal{K}(C, A') \\
\mathcal{K}(D, p) & \downarrow & \mathcal{K}(C, p) \\
\mathcal{K}(D, A) & \xrightarrow{K(C,A)} & \mathcal{K}(C, A)
\end{array}
\]
defines a morphism of lalis in \( \mathbf{qCat} \) for each \( C \to D \) in \( \mathcal{K} \).

In fact the cases \( C = A \) and \( C = A' \) in (a), and \( c = p \) in (b) suffice.

(2) Similarly, if \( p: A' \twoheadrightarrow A \) and \( q: B' \twoheadrightarrow B \) are lalis in \( \mathcal{K} \) then a morphism
\[
\begin{array}{ccc}
A' & \xrightarrow{f'} & B' \\
\downarrow p & & \downarrow q \\
A & \xrightarrow{f} & B
\end{array}
\]
in \( \mathcal{K}^1 \) is a morphism of lalis in \( \mathcal{K} \) if and only if
\[
\begin{array}{ccc}
\mathcal{K}(C, A') & \xrightarrow{K(C,f')} & \mathcal{K}(C, B') \\
\mathcal{K}(C, p) & \downarrow & \mathcal{K}(C, q) \\
\mathcal{K}(C, A) & \xrightarrow{K(C,f)} & \mathcal{K}(C, B)
\end{array}
\]
is one in \( \mathbf{qCat} \); and in fact the case \( C = A \) suffices.

Proof. Any cosmological functor preserves lalis and morphisms of lalis, and so in particular each representable \( \mathcal{K}(C, -): \mathcal{K} \to \mathbf{qCat} \) does so. Similarly simplicially enriched natural transformations between cosmological functors induce morphisms of lalis. Applying these facts to the cosmological functors \( \mathcal{K}(C, -): \mathcal{K} \to \mathbf{qCat} \) and the natural transformations \( \mathcal{K}(c, -): \mathcal{K}(D, -) \to \mathcal{K}(C, -) \) gives the “only if” parts of the proposition. We now turn to the converses.

Suppose then that \( \mathcal{K}(C, p) \) is a lali in \( \mathbf{qCat} \) if \( C = A \) or \( C = A' \), and also that
\[
\begin{array}{ccc}
\mathcal{K}(A, A') & \xrightarrow{K(p,A')} & \mathcal{K}(A', A') \\
\mathcal{K}(A, p) & \downarrow & \mathcal{K}(A', p) \\
\mathcal{K}(A, A) & \xrightarrow{K(p,A)} & \mathcal{K}(A', A)
\end{array}
\]
is a morphism of lalis. Since $K(A, p): K(A, A') \to K(A, A)$ is a lali, the right adjoint will send $1: A \to A$ to some $s: A \to A'$ with $ps = 1$, such that $p$ induces a bijection between 2-cells $x \to s$ and $px \to 1$ in $hK$ for any $x: A \to A'$.

Since the above square is a morphism of lalis, $p$ also induces a bijection between 2-cells $y \to sp$ and $py \to p$, for any $y: A' \to A'$. In particular, the identity $p \to p$ corresponds to some $\sigma: 1 \to sp$ with $p\sigma = 1$; on the other hand, the images of $\sigma s, 1_s: s \Rightarrow s$ under $p$ are equal to the identity, so that $\sigma s = 1_s$. This proves that $p$ is a lali, giving the “if” part of (1).

As for (2), suppose that $\sigma: 1 \to sp$ and $\sigma': 1 \to s'q$ exhibit $p$ and $q$ as lalis, and that

\[
\begin{array}{c}
K(A, A') \xrightarrow{K(f', f)} K(A, B') \\
\downarrow \ K(A, p) \downarrow \ K(A, q) \\
K(A, A) \xrightarrow{K(f, f)} K(A, B)
\end{array}
\]

is a morphism of lalis in $q\text{Cat}$. We are to show that the induced

\[
f's \xrightarrow{\sigma'f's} s'qf's \xrightarrow{s'fps} s'f
\]

is invertible, but this is just the component at $1_A$ of the induced

\[
hK(A, f')hK(A, s) \to hK(A, s')hK(A, f)
\]

which is invertible by assumption. \hfill \Box

Consider a cosmological embedding $J: \mathcal{L} \to \mathcal{K}$ of $\infty$-cosmoi and a diagram $S: \mathcal{C} \to \mathcal{L}$ such that $\{W, JS\}$ exists in $\mathcal{K}$. Let us say that the cosmological embedding creates the weighted limit if $\{W, JS\} \in \mathcal{L}$ and we have a pullback square

\[
\begin{array}{c}
\mathcal{L}(A, \{W, JS\}) \xrightarrow{[\mathcal{C}, \text{SSet}](W, \mathcal{L}(A, S-))} \\
\downarrow J \downarrow J_* \\
\mathcal{K}(A, \{W, JS\}) \xrightarrow{[\mathcal{C}, \text{SSet}](W, \mathcal{K}(A, JS-))} \\
\end{array}
\]

natural in $A$. This says precisely that the unit $W \to K(\{W, JS\}, JS-)$ factorizes through $J$ as $W \to \mathcal{L}(\{W, JS\}, S-)$ and exhibits $\{W, JS\}$ as the weighted limit $\{W, S\}$ in $\mathcal{L}$.

**Lemma 5.2.** Each cosmological embedding $J: \mathcal{L} \to \mathcal{K}$ of $\infty$-cosmoi creates any weighted limits that are homotopy limits in $\mathcal{K}$. 

Proof. Consider $W: \mathcal{C} \to \mathbf{SSet}$ and $S: \mathcal{C} \to \mathcal{L}$ such that $\{W, JS\}$ exists in $\mathcal{K}$ and is a homotopy limit. Let $p: Q \to W$ be a flexible cofibrant replacement of $W$. Since $\infty$-cosmoi have flexible limits and cosmological functors preserve them, $\{Q, S\}$ exists and is preserved by $J$; since $\{W, JS\}$ is a homotopy limit, the canonical comparison $\lambda: \{W, JS\} \to \{Q, JS\} = \{Q, S\}$ in $\mathcal{K}$ is an equivalence. Therefore, by repleteness, both $\{W, JS\}$ and $\lambda$ belong to $\mathcal{L}$. This allows us to consider the commutative diagram below.

\[
\begin{array}{ccc}
\mathcal{L}(A, \{W, JS\}) & \xrightarrow{\lambda_*} & \mathcal{L}(A, \{Q, JS\}) \\
J & \downarrow & J \\
\mathcal{K}(A, \{W, JS\}) & \xrightarrow{\lambda_*} & \mathcal{K}(A, \{Q, JS\})
\end{array}
\]

The right square is a pullback since its two horizontal components are isomorphisms. The verticals in the left square are full on positive-dimensional arrows. Therefore to show that the left square is a pullback, it suffices to show that if $f: A \to \{W, JS\}$ in $\mathcal{K}$ has $\lambda \circ f: A \to \{Q, JS\}$ in $\mathcal{L}$, then $f$ is also in $\mathcal{L}$. Now by repleteness of $\mathcal{L}$, the equivalence-inverse $\lambda^{-1}$ is in $\mathcal{L}$, which so is the composite $\lambda^{-1} \circ \lambda \circ f$, and so finally the isomorphic $f$. In particular the left square is a pullback, so that the outer square is a pullback. Now its lower composite horizontal coincides with the corresponding morphism in the diagram below.

\[
\begin{array}{ccc}
\mathcal{L}(A, \{W, JS\}) & \xrightarrow{\exists t_A} & (W, \mathcal{L}(A, S-)) \\
J & \downarrow & J_* \\
\mathcal{K}(A, \{W, JS\}) & \xrightarrow{\cong} & (W, \mathcal{K}(A, JS-))
\end{array}
\]

In this diagram, the right square is a pullback by the orthogonality of the regular epimorphism $p: Q \to W$ and the monomorphism $J_{A, S-}: \mathcal{L}(A, S-) \to \mathcal{K}(A, S-)$. Therefore, by the universal property of the right pullback square, we obtain a unique morphism $t_A$ to the pullback, making the left square a pullback and such that the upper horizontals of the two diagrams coincide. Naturality and invertibility of the $t_A$ follows from the uniqueness of their construction. \[\square\]

Using the lemma we next show that, in the accessible case, we can test for lalis and their morphisms using small objects.
Proposition 5.3. If $\mathcal{K}$ is an accessible $\infty$-cosmos, for any sufficiently large $\lambda$, the canonical square

$$
\begin{array}{ccc}
\text{Rari}(\mathcal{K})_0 & \longrightarrow & [\mathcal{G}^{\text{op}}, \text{Rari}(\text{qCat})]_0 \\
\downarrow & & \downarrow \\
\mathcal{K}_0^{\perp} & \longrightarrow & [\mathcal{G}^{\text{op}}, \text{qCat}]_0
\end{array}
$$

is a pullback, where $\mathcal{G} = \mathcal{K}_\lambda$.

Proof. The lower horizontal sends $p: A_0 \to A_1$ to $\mathcal{K}(J-, p): \mathcal{G}^{\text{op}} \to \text{qCat}^{\perp}$, where $J: \mathcal{G} \to \mathcal{K}$ is the inclusion, and this lifts along the forgetful vertical functors to the upper horizontal by virtue of Proposition 5.1.

To show that it is a pullback we need to show that, in the characterization of Proposition 5.1, it suffices to consider the case where the objects $C$ and morphisms $c: C \to D$ lie in $\mathcal{G}$.

Choose $\lambda$ such that

- $\mathcal{K}$ is $\lambda$-accessible as a simplicially enriched category;
- $\lambda$-filtered colimits are homotopy colimits in $\mathcal{K}$

and let $\mathcal{G} = \mathcal{K}_\lambda$.

Suppose then that $p: A_0 \to A_1$ is an isofibration in $\mathcal{K}$ such that each $\mathcal{K}(G, p)$ is a lali and each square

$$
\begin{array}{ccc}
\mathcal{K}(H, A_0) & \xrightarrow{\mathcal{K}(g, A_0)} & \mathcal{K}(G, A_0) \\
\mathcal{K}(H, p) & & \mathcal{K}(G, p) \\
\mathcal{K}(H, A_1) & \xrightarrow{\mathcal{K}(g, A_1)} & \mathcal{K}(G, A_1)
\end{array}
$$

is a morphism of lalis, for $g: G \to H$ in $\mathcal{G}$.

For an arbitrary $C \in \mathcal{K}$, we may write $C$ as a $\lambda$-filtered colimit $\text{colim}(S: J \to \mathcal{K})$ of a diagram taking values in $\mathcal{G}$. Then, homming into $p$, we obtain $\mathcal{K}(C, p) = \text{lim}(\mathcal{K}(S-, p)): J^{\text{op}} \to \text{SSet}^2$ and since this diagram lifts to $\mathcal{K}(S-, p): J^{\text{op}} \to \text{qCat}^{\perp}$, and moreover since $\mathcal{K}(C, f)$ belongs to $\text{qCat}^{\perp}$, it is also true that $\mathcal{K}(C, p) = \text{lim}(\mathcal{K}(S-, p)): J^{\text{op}} \to \text{qCat}^{\perp}$). We claim that this limit is in fact a homotopy limit.

To this end, let $p: Q \to \Delta 1 \in [J^{\text{op}}, \text{SSet}]$ be a flexible cofibrant replacement. By Proposition 6.2.8(i) of [14], each $\infty$-cosmos admits flexible limits, so the limit $\{Q, \mathcal{K}(S-, p)\}$ exists in $\text{qCat}^{\perp}$. Therefore, to show that $\mathcal{K}(C, p)$ is the homotopy limit is equivalently to show that the induced map

$$
\mathcal{K}(C, p) \cong \{\Delta 1, \mathcal{K}(S-, p)\} \to \{Q, \mathcal{K}(S-, p)\}
$$
is an equivalence in $\text{qCat}^\perp$. Since equivalences in $\text{qCat}^\perp$ are pointwise as in $\text{qCat}$, and since the above limits are pointwise — the projections to $\text{qCat}$ being cosmological — this is equally to show that

$$\{\Delta 1, \mathcal{K}(S-, A_i)\} \to \{Q, \mathcal{K}(S-, A_i)\}$$

is an equivalence of quasicategories for $i = 0, 1$, but this is simply

$$[\mathcal{J}^{\text{op}}, \text{SSet}](\Delta 1, \mathcal{K}(S-, A_i)) \to [\mathcal{J}^{\text{op}}, \text{SSet}](Q, \mathcal{K}(S-, A_i))$$

which is an equivalence since $\lambda$-filtered colimits are homotopy colimits in $\mathcal{K}$.

Thus $\mathcal{K}(C, p)$ is a homotopy limit in $\text{qCat}^\perp$ of lalis and morphisms of lalis. Therefore, by Lemma 5.2, $\mathcal{K}(C, p)$ is itself a lali and a limit in $\text{Rari}(\mathcal{K})$: this means that the cone projections $\mathcal{K}(C, p) \to \mathcal{K}(S_j, p)$ are morphisms of lalis and jointly reflect morphisms of lalis.

To complete the proof that $p$ is a lali, suppose now that $c: C \to D$ is a morphism in $\mathcal{K}$, and write $D$ as a $\lambda$-filtered colimit $\text{colim}_j H_j$ of a diagram in $\mathcal{G}$. Each pre-composite $G_i \to C \to D$ of $c$ by a cocone inclusion $G_i \to C$ factorizes through some $H_j \to D$, and now in the resulting diagram

$$\mathcal{K}(D, p) \to \mathcal{K}(C, p) \to \mathcal{K}(C, q)$$

$$\mathcal{K}(H_j, p) \to \mathcal{K}(G_i, p)$$

the vertical morphisms are morphisms of lalis since they are cone projections as above, whilst the lower horizontal is a morphism of lalis by assumption. Hence the composite from top left to bottom right is a morphism of lalis. Since the cone projections on the right vertical jointly reflect morphisms of lalis, it follows that the upper horizontal is a morphism of lalis too, as required. Thus $p$ is a lali by Proposition 5.1.

Finally suppose that $p \to q$ is a morphism in $\mathcal{K}^\perp$ where $p$ and $q$ are lalis, and that the image of the square under $\mathcal{K}(C, -)$ is a morphism of lalis for each $C \in \mathcal{G}$. Write $C = \text{colim}_j D_j$ as a $\lambda$-filtered colimit of objects of $\mathcal{G}$. Then for each cocone inclusion $D_j \to C$ we have the commutative square

$$\mathcal{K}(C, p) \to \mathcal{K}(C, q) \to \mathcal{K}(D_j, p) \to \mathcal{K}(D_j, q)$$

in which the vertical cone projections are morphisms of lalis and jointly detect morphisms of lalis. By assumption the lower horizontal is a
morphism of lalis, and it follows as before that the upper horizontal is a morphism of lalis too. Therefore, by Proposition 5.1, the original square is in fact a morphism of lalis in \( \mathcal{K} \), completing the proof. □

This last result essentially allows us to reduce to the case \( \mathcal{K} = \text{qCat} \), to which we now turn. For this, we need to work more “analytically”, using a characterization of lalis in \( \text{qCat} \) from [14].

**Definition 5.4.** Let \( p: A' \to A \) be an isofibration in \( \text{qCat} \). Say that \( a' \in A' \) is \( p \)-universal, or just universal if \( p \) is understood, if for every diagram as in the solid part of

\[
\begin{array}{ccc}
1 & \rightarrow & A' \\
\downarrow & & \downarrow \scriptstyle{p} \\
\partial\Delta[n] & \rightarrow & A
\end{array}
\]

there exists a dotted arrow making the diagram commute.

The following proposition illustrates the usefulness of this notion.

**Proposition 5.5.** Consider a morphism

\[
\begin{array}{ccc}
A' & \xrightarrow{f'} & B' \\
\downarrow \scriptstyle{p} & & \downarrow \scriptstyle{q} \\
A & \xrightarrow{f} & B
\end{array}
\]

in \( \text{qCat}^+ \).

(1) \( p \) is a lali if and only if for every \( a: 1 \to A \) there is a universal \( a': 1 \to A' \) with \( pa' = a \);

(2) if \( p \) and \( q \) are lalis, the square is a morphism of lalis if and only if \( f' \) sends \( p \)-universal elements to \( q \)-universal elements.

**Proof.** (1) This is [14, Lemma F.3.1].

(2) Suppose that \( p \) and \( q \) are lalis, and that \( \sigma: 1 \to sp \) and \( \tau: 1 \to tq \) exhibit \( s \) and \( t \) as right adjoints to \( p \) and \( q \).

Consulting the proof of [14, Lemma F.3.1], one sees that \( s: A \to A' \) can be constructed in such a way that each \( sa \) is universal, and indeed any choice of universal lifts \( sa \) of each \( a \in A \) can be assembled into an \( s \). Thus it follows that \( \sigma a': a' \to spa' \) is invertible if and only if \( a' \) is universal.

The square will be a morphism of lalis just when \( \tau f's: f's \to tqf's = tfps = tf \) is invertible; and this in turn will be the case if and only if
τf'sa: f'sa → tfa is invertible for each a ∈ A, which by the previous paragraph amounts to the requirement that f' preserve universals. □

Proposition 5.6. \( \text{Rari}(\text{qCat}) \rightarrow \text{qCat} \uparrow \) is an accessible cosmological embedding.

Proof. It is a cosmological embedding by [14, Proposition 6.3.10]. Thus by Proposition 4.8 it will suffice to show that \( \text{Rari}(\text{qCat})_0 \rightarrow \text{qCat}_0 \) is accessible.

We know that \( \text{qCat}_1 \) is accessible and accessibly embedded in \( \text{SSet}_0^2 \), so it will suffice to show that \( \text{Rari}(\text{qCat})_0 \) is an accessible category and the inclusion \( \text{Rari}(\text{qCat})_0 \rightarrow \text{SSet}_0^2 \) is an accessible functor. We do so by showing that \( \text{Rari}(\text{qCat})_0 \) is a small injectivity class in \( \text{SSet}_0^2 \), using techniques similar to those in Section 9 of [5].

Consider the category \( 1_{\Delta^0}[\text{SSet}_0^2] \) in which an object is a simplicial map \( p: X' \rightarrow X \), equipped with a subset \( S \subseteq X'_0 \) of “marked objects”, denoted \( p: (X',S) \rightarrow X \); and a morphism is a commutative square

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow p & & \downarrow q \\
X & \xrightarrow{f} & Y
\end{array}
\]

such that \( f' \) sends marked objects to marked objects.

Then by Proposition 9.2 of [5], the category \( 1_{\Delta^0}[\text{SSet}_0^2] \) is locally presentable and the forgetful functor to \( \text{SSet}_0^2 \) accessible, so, as per Corollary 9.3 of [5], the proof will be complete if we can show that \( \text{Rari}(\text{qCat})_0 \) is a small injectivity class in \( 1_{\Delta^0}[\text{SSet}_0^2] \). More precisely, we show that the collection of all those \( p: (X',S) \rightarrow X \) for which \( p \) is both an isofibration and a lali, and \( S \) consists precisely of all the universal objects, is an injectivity class. Now injectivity with respect to the diagrams

\[
\begin{array}{ccc}
(\emptyset,\emptyset) & \xrightarrow{j} & (Z,\emptyset) \\
\downarrow & \downarrow & \downarrow j \\
Y & \rightarrow & Z
\end{array}
\]

for \( j \) an inner horn inclusion or an endpoint inclusion \( 1 \rightarrow \mathbb{I} \) says that \( X \) is a quasicategory, and \( p \) an isofibration (thus \( X' \) is also a quasicategory).
Injectivity with respect to the first of the following diagrams

\[ (\emptyset, \emptyset) \to (1, 1) \quad (\partial \Delta[n], \{n\}) \to (\Delta[n], \{n\}) \quad (\mathbb{I}, \{0\}) \to (\mathbb{I}, \{0, 1\}) \]

\[ \begin{array}{ccc}
\downarrow & \downarrow & \downarrow \\
1 & \Delta[n] & (1, 1) \\
\downarrow & \downarrow & \downarrow \\
\Delta[n] & (1, 1) & (1, 1)
\end{array} \]

says that \( S \) is non-empty; with respect to the second (for all \( n \)) says that elements of \( S \) are universal objects; and with respect to the third says that \( S \) consists of all the universal objects. \( \square \)

We can now put all the pieces together to prove the main result of the section.

**Theorem 5.7.** If \( \mathcal{K} \) is an accessible \( \infty \)-cosmos then so is \( \mathcal{R} \mathcal{a} \mathcal{i} \mathcal{r}(\mathcal{K}) \), and the cosmological embedding \( \mathcal{R} \mathcal{a} \mathcal{i} \mathcal{r}(\mathcal{K}) \to \mathcal{K}^\perp \) is also accessible.

**Proof.** By Proposition 4.8, it will suffice to show that \( \mathcal{R} \mathcal{a} \mathcal{i} \mathcal{r}(\mathcal{K})_0 \) is an accessible category and the inclusion \( \mathcal{R} \mathcal{a} \mathcal{i} \mathcal{r}(\mathcal{K})_0 \to \mathcal{K}^\perp_0 \) is an accessible functor.

By Proposition 5.3 we have a pullback

\[ \begin{array}{ccc}
\mathcal{R} \mathcal{a} \mathcal{i} \mathcal{r}(\mathcal{K})_0 & \to & \mathcal{G}^{\text{op}}, \mathcal{R} \mathcal{a} \mathcal{i} \mathcal{r}(\mathcal{q} \mathcal{C} \mathcal{a} \mathcal{t})_0 \\
\downarrow & \downarrow & \downarrow \\
\mathcal{K}^\perp_0 & \to & \mathcal{G}^{\text{op}}, \mathcal{q} \mathcal{C} \mathcal{a} \mathcal{t}^\perp_0
\end{array} \]

in which the right vertical is an isofibration of categories since the inclusion \( \mathcal{R} \mathcal{a} \mathcal{i} \mathcal{r}(\mathcal{q} \mathcal{C} \mathcal{a} \mathcal{t})_0 \to \mathcal{q} \mathcal{C} \mathcal{a} \mathcal{t}^\perp_0 \) is one. The lower horizontal and right vertical are accessible functors between accessible categories, in the case of the right vertical by Proposition 5.6. It now follows by Proposition 2.3 that the left leg is an accessible functor between accessible categories, as required. \( \square \)

Dually, we have

**Corollary 5.8.** If \( \mathcal{K} \) is an accessible \( \infty \)-cosmos, then so is the \( \infty \)-cosmos \( \mathcal{L} \mathcal{a} \mathcal{i} \mathcal{r}(\mathcal{K}) \) of right adjoint left inverses in \( \mathcal{K} \), and the cosmological embedding \( \mathcal{L} \mathcal{a} \mathcal{i} \mathcal{r}(\mathcal{K}) \to \mathcal{K}^\perp \) is also accessible.

**Proof.** Reversing 2-cells interchanges ralis and lalis — in particular, \( \mathcal{L} \mathcal{a} \mathcal{i} \mathcal{r}(\mathcal{K}) \to \mathcal{K}^\perp \) is just \( \mathcal{R} \mathcal{a} \mathcal{i} \mathcal{r}(\mathcal{K}^{\text{co}}) \to ((\mathcal{K}^{\text{co}})^\perp)^{\text{co}} \). The claim then follows from Theorem 5.7 combined with two applications of the \( (\mathcal{\_})^{\text{co}} \) duality of Proposition 4.4, on noting that since \( (\mathcal{\_})^{\text{co}} \) doesn’t change underlying categories, it also respects accessibility of cosmological functors. \( \square \)
We conclude this section with the observation that dealing with the other two duals — the laris and the raris — would require an alternative approach. While a 2-category $\mathcal{K}$ possesses four duals, namely $\mathcal{K}^{\text{op}}$, $\mathcal{K}^{\text{co}}$, $\mathcal{K}^{\text{co,op}}$, and $\mathcal{K}$ itself, an $\infty$-cosmos $\mathcal{K}$ possesses only two: $\mathcal{K}$ and $\mathcal{K}^{\text{co}}$. Thus one cannot simply define raris in $\mathcal{K}$ to be lalis in $\mathcal{K}^{\text{op}}$. In fact it seems unlikely that one can even define an $\infty$-cosmos of laris or raris in general.

6. TRIVIAL FIBRATIONS AND EQUIVALENCES

Recall that the trivial fibrations in an $\infty$-cosmos are the isofibrations which are also equivalences. They are the objects of a full subcategory $\mathbf{TF}(\mathcal{K})$ of $\mathcal{K}^{\downarrow}$ which, following Proposition 6.1.5(ii) of [14], is a cosmologically embedded $\infty$-cosmos. To understand accessibility in this context, we can now follow the same steps as we did when dealing with lalis. In fact the fullness makes things easier, so we do not give all the details.

**Proposition 6.1.** If $\mathcal{K}$ is an accessible $\infty$-cosmos, for any sufficiently large $\lambda$, the canonical square

$$
\begin{array}{ccc}
\mathbf{TF}(\mathcal{K}) & \longrightarrow & [\mathcal{G}^{\text{op}}, \mathbf{TF}(\mathbf{qCat})] \\
\downarrow & & \downarrow \\
\mathcal{K}^{\downarrow} & \longrightarrow & [\mathcal{G}^{\text{op}}, \mathbf{qCat}^{\downarrow}]
\end{array}
$$

is a pullback, where $\mathcal{G} = \mathcal{K}_\lambda$.

**Proof.** Choose $\lambda$ such that

- $\mathcal{K}$ is $\lambda$-accessible as a simplicially enriched category;
- $\lambda$-filtered colimits are homotopy colimits in $\mathcal{C}$.

The lower horizontal is the fully faithful simplicial functor sending $p: A' \to A$ to $\mathcal{K}(J-, p): \mathcal{G}^{\text{op}} \to \mathbf{SSet}$, where $J: \mathcal{G} \to \mathcal{K}$ is the inclusion. The vertical maps are fully faithful. The upper horizontal exists (and is therefore fully faithful) because an isofibration $p$ in $\mathcal{K}$ is a trivial fibration if and only if $\mathcal{K}(C, p)$ is one for all $C \in \mathcal{K}$. We need to prove that it will be one provided only that $\mathcal{K}(C, p)$ is one for $C \in \mathcal{G}$.

Suppose then that $\mathcal{K}(G, p)$ is a trivial fibration for all $G \in \mathcal{G}$, and let $C \in \mathcal{K}$ be arbitrary. We may write $C$ as a $\lambda$-filtered colimit colim$_i G_i$ of objects in $\mathcal{G}$, and this colimit is also a homotopy colimit. Thus $\mathcal{K}(C, p)$ is a homotopy limit of the trivial fibrations $\mathcal{K}(G_i, p)$, and so is itself a trivial fibration. \qed

Just as in the case of lalis, this last result now allows us to restrict to the case of $\mathbf{qCat}$. 
Proposition 6.2. $\text{TF}(\mathbf{qCat}) \to \mathbf{qCat}^\triangleright$ is an accessible cosmological embedding.

Proof. It is a cosmological embedding by Proposition 6.1. Thus by Proposition 4.8 it will suffice to show that $\text{TF}(\mathbf{qCat})_0 \to \mathbf{qCat}^\triangleright_0$ is accessible, or equivalently that $\text{TF}(\mathbf{qCat})_0 \to \mathbf{qCat}^\triangleright_0$ is so. But this follows from the fact that $\mathbf{qCat}_0$ is accessible, and that the trivial fibrations are the maps with the right lifting property with respect to the boundary inclusions $\partial \Delta[n] \to \Delta[n]$.

Theorem 6.3. If $\mathcal{K}$ is an accessible $\infty$-cosmos then so is $\text{TF}(\mathcal{K})$, and the inclusion $\text{TF}(\mathcal{K}) \to \mathcal{K}^\triangleright$ is an accessible cosmological embedding.

Proof. By Proposition 4.8 it will suffice to show that $\text{TF}(\mathcal{K})_0$ is an accessible category and the inclusion $\text{TF}(\mathcal{K})_0 \to \mathcal{K}^\triangleright_0$ is an accessible functor. This now follows from Propositions 2.3, 6.1, and 6.2, just as in the proof of Theorem 5.7.

An easy consequence of the above and the Brown factorisation lemma is the following result.

Proposition 6.4. Let $\mathcal{K}$ be an accessible $\infty$-cosmos. Then $\text{Equiv}(\mathcal{K})_0$ is accessible and accessibly embedded in $\mathcal{K}^2_0$.

Proof. By [14, Proposition 1.2.19] there is a pullback

$$
\begin{array}{ccc}
\text{Equiv}(\mathcal{K})_0 & \longrightarrow & \text{TF}(\mathcal{K})_0 \\
\downarrow & & \downarrow \\
\mathcal{K}^2_0 & \underset{R}{\longrightarrow} & \mathcal{K}^2_0
\end{array}
$$

where the vertical maps are the (fully faithful) inclusions and $R$ is the functor which sends a morphism $f: A \to B$ to $p_f: Pf \to B$, constructed via the pullback

$$
\begin{array}{ccc}
Pf & \longrightarrow & B^\bagger \\
\downarrow & & \downarrow \\
A \times B & \underset{f \times 1}{\longrightarrow} & B \times B.
\end{array}
$$

As usual, we now apply Proposition 2.3. The right vertical is an isofibration, and is accessible by Proposition 6.2, so we only need to check that $R$ is accessible. But this is constructed using finite limits, and these commute with sufficiently filtered colimits in any accessible category.

\[\square\]
Remark 6.5. This hints at a possible alternative definition of an accessible ∞-cosmos in which we replace Condition (3) concerning homotopy colimits by an axiom asserting that $\text{Equiv}(\mathcal{K})_0$ is accessible and accessibly embedded in $\mathcal{K}^2_0$. The previous result ensures that our usual definition implies this second one. If the second definition was equivalent to our usual one, it would be useful as many proofs concerning the stability of accessible ∞-cosmoi would become shorter. However, we have not been able to prove this, and we leave it as an open problem.

7. Applications to the motivating examples

In this section we apply the results of the previous three sections to show that accessible ∞-cosmoi are stable under a whole host of key further constructions.

7.1. ∞-categories with limits. Let $J$ be a simplicial set. There is a cosmological functor $F_J: \mathcal{K} \to \mathcal{K}^J$ sending $A \in \mathcal{K}$ to the isofibration $A^{J^0} \to A^J$ given by restriction along the inclusion $J \to J^0 = J \sqcup J$ of [14, Notation 4.2.6].

Then the pullback

\[
\begin{array}{ccc}
\mathcal{K}_{\tau,J} & \longrightarrow & \text{Rari}(\mathcal{K}) \\
\downarrow & & \downarrow \\
\mathcal{K} & \longrightarrow & \mathcal{K}^J.
\end{array}
\]

is, by [14, Proposition 6.3.13] and its proof, the ∞-cosmos $\mathcal{K}_{\tau,J}$ of ∞-categories in $\mathcal{K}$ with $J$-limits, and moreover $\mathcal{K}_{\tau,J} \to \mathcal{K}$ is a cosmological embedding.

Theorem 7.1. If $\mathcal{K}$ is an accessible ∞-cosmos, then so is the ∞-cosmos $\mathcal{K}_{\tau,J}$ of ∞-categories in $\mathcal{K}$ with $J$-shaped limits. The cosmological embedding $\mathcal{K}_{\tau,J} \to \mathcal{K}$ is then accessible as well.

Proof. By Proposition 4.9 and Theorem 5.7, it will suffice to show that the cosmological functor $F_J$ is accessible, or equivalently that its underlying ordinary functor $\mathcal{K}_0 \to (\mathcal{K}^J)_0$ is accessible. Now the fully faithful inclusion $(\mathcal{K}^J)_0 \to \mathcal{K}^2_0$ is accessible, so it will suffice to show that the composite $\mathcal{K}_0 \to \mathcal{K}^2_0$ preserves sufficiently filtered colimits. Since these are formed pointwise in $\mathcal{K}^2_0$, we just need to know that the two functors $\mathcal{K}_0 \to \mathcal{K}_0$ sending $A$ to $A^{J^0}$ and to $A^J$ are accessible. This is true by Proposition 2.4. □
Similar arguments apply to ∞-categories with a set of limit shapes, whilst a dual argument applies to ∞-categories with colimits. One can also combine limits and colimits.

7.2. Discrete objects. An object $A$ of an ∞-cosmos $\mathcal{K}$ is said to be discrete [14, Definition 1.2.26] if the hom-quasicategory $\mathcal{K}(C, A)$ is in fact a Kan complex, for all $C \in \mathcal{K}$. In particular, the discrete objects of $\text{qCat}$ are the Kan complexes.

By [14, Proposition 6.1.6] and its proof, the discrete objects of an ∞-cosmos $\mathcal{K}$ form an ∞-cosmos $\mathcal{K}^\simeq$, whose inclusion into $\mathcal{K}$ is a cosmological embedding.

**Theorem 7.2.** If $\mathcal{K}$ is an accessible ∞-cosmos then so too is $\mathcal{K}^\simeq$, and the (fully faithful) inclusion is an accessible cosmological embedding.

**Proof.** By [14, Lemma 1.2.27], an object $A$ is discrete if and only if the isofibration $A^! \to A^2$ is in fact a trivial fibration. Thus we have a pullback square

$$
\begin{array}{ccc}
\mathcal{K}^\simeq & \longrightarrow & \text{TF}(\mathcal{K}) \\
\downarrow & & \downarrow \\
\mathcal{K} & \longrightarrow & \mathcal{K}^! \\
\end{array}
$$

as in [14, Proposition 6.1.6], in which the vertical maps are fully faithful isofibrations, and $E$ sends $A$ to the projection $A^! \to A^2$. By Proposition 2.4, $E$ is accessible; and by Theorem 6.3, the right leg of the pullback square is an accessible cosmological embedding. Therefore the result follows by Proposition 4.9. $\square$

7.3. Cartesian fibrations. By [14, Proposition 6.3.14], there exists a cosmologically embedded ∞-cosmos $\text{Cart}(\mathcal{K}) \hookrightarrow \mathcal{K}^!$ consisting of those isofibrations which are cartesian fibrations in $\mathcal{K}$, and this can be obtained as the pullback below.

$$
\begin{array}{ccc}
\text{Cart}(\mathcal{K}) & \longrightarrow & \text{Rari}(\mathcal{K}) \\
\downarrow & & \downarrow \\
\mathcal{K}^! & \longrightarrow & \mathcal{K}^! \\
\end{array}
$$

As explained in the proof of [14, Proposition 6.3.14], this $\mathcal{K} : \mathcal{K}^! \to \mathcal{K}^!$ is the cosmological functor sending an isofibration $p : E \to B$ to the
Theorem 7.3. If $\mathcal{K}$ is an accessible $\infty$-cosmos then so is $\text{Cart}(\mathcal{K})$, and the cosmological embedding $\text{Cart}(\mathcal{K}) \to \mathcal{K}^\perp$ is also accessible.

Proof. By Theorem 5.7 once again it will suffice to show that $K: \mathcal{K}^\perp \to \mathcal{K}^\perp$ is accessible. Just as in the proof of Theorem 7.1, this will be the case provided that the two functors $\mathcal{K}^\perp_0 \to \mathcal{K}_0$ sending $p: E \to B$ to $E^2$ and to $B/p$ are accessible.

The first of these is given by the domain functor $\text{dom}: \mathcal{K}^\perp_0 \to \mathcal{K}_0$ followed by $2 \triangleright - : \mathcal{K}_0 \to \mathcal{K}_0$; each of these is accessible, hence so is their composite.

Write $F: \mathcal{K}^\perp_0 \to \mathcal{K}_0$ for the other functor, sending $p$ to $B/p$. It will be accessible if and only if $\mathcal{K}(C, F): \mathcal{K}^\perp_0 \to \text{SSet}$ is so, for each $C \in \mathcal{K}$. And $\mathcal{K}(C, F)$ is the composite of the inclusion $\mathcal{K}^\perp_0 \to \mathcal{K}^2_0$; followed by the hom-functor $\mathcal{K}(C, -): \mathcal{K}^2_0 \to \text{SSet}^2$; followed by the functor $K': \text{SSet}^2 \to \text{SSet}$ sending $q: X \to Y$ in $\text{SSet}$ to the analogous $Y/q$.

Each of these three functors is accessible:

- the inclusion by Condition (2) in the definition of accessible $\infty$-cosmos;
- the hom-functor by Proposition 2.4;
- $K'$ by Example 2.2 and the fact that $K'$ is a right adjoint;

and so their composite is also accessible. \hfill $\square$

Dually (see [14, Proposition 6.3.14] once again) we have:

Theorem 7.4. If $\mathcal{K}$ is an accessible $\infty$-cosmos then so is $\text{coCart}(\mathcal{K})$, and the cosmological embedding $\text{coCart}(\mathcal{K}) \to \mathcal{K}^\perp$ is also accessible.

Similarly we can deal with the $\infty$-cosmoi $\text{DiscCart}(\mathcal{K})$ of discrete cartesian fibrations, and $\text{DisccoCart}(\mathcal{K})$ of discrete cocartesian fibrations.
Theorem 7.5. If $\mathcal{K}$ is an accessible $\infty$-cosmos then so are $\text{DiscCart}(\mathcal{K})$ and $\text{DisccoCart}(\mathcal{K})$, and the inclusions into $\mathcal{K}^\perp$ are accessible cosmological embeddings.

Proof. In the case of $\text{DiscCart}(\mathcal{K})$, this follows from the existence of a pullback

$$
\begin{array}{ccc}
\text{DiscCart}(\mathcal{K}) & \longrightarrow & \text{TF}(\mathcal{K}) \\
\downarrow & & \downarrow \\
\mathcal{K}^\perp & \xrightarrow{K} & \mathcal{K}^\perp
\end{array}
$$

as in the proof of [14, Proposition 6.3.15], where $K$ is the map constructed at the beginning of this section. (Alternatively, this can be deduced from Theorems 7.3 and 7.2, since $\text{DiscCart}(\mathcal{K}) \simeq \text{Cart}(\mathcal{K})^\perp$.)

The case of $\text{DisccoCart}(\mathcal{K})$ is dual. \qed

7.4. Fibrations with fixed base. In this section we consider various flavours of fibration $A' \to A$ for a fixed object $A$ of our $\infty$-cosmos.

In the case of cartesian fibrations, for example, there are pullback squares of simplicially enriched categories

$$
\begin{array}{ccc}
\text{Cart}(\mathcal{K})/A & \longrightarrow & \text{Cart}(\mathcal{K}) \\
\downarrow & & \downarrow \\
\mathcal{K}/A & \xrightarrow{\text{cod}} & \mathcal{K}^\perp \\
\downarrow & & \downarrow \\
1 & \xrightarrow{A} & \mathcal{K}
\end{array}
$$

as explained in the proof of [14, Proposition 6.3.14]. As also explained there, the horizontal maps are neither cosmological nor replete, but the vertical maps are cosmological, and those in the upper square are cosmological embeddings. Furthermore, at the level of underlying categories, the lower horizontal is accessible and the right vertical maps are accessible isofibrations, using Proposition 4.1 and Theorem 7.3 respectively. Therefore both pullback squares consist of accessible categories and accessible functors, and by the universal property of the lower pullback square in the 2-category of accessible categories, the cosmological embedding $\text{Cart}(\mathcal{K})/A \hookrightarrow \mathcal{K}^\perp$ is accessible too. Therefore, by Proposition 4.8, $\text{Cart}(\mathcal{K})/A$ is an accessible $\infty$-cosmos and $\text{Cart}(\mathcal{K})/A \hookrightarrow \mathcal{K}^\perp$ an accessible cosmological embedding.

This proves the first case of the following result, and the proofs for the other three flavours of fibration are similar.
Theorem 7.6. If $\mathcal{K}$ is an accessible $\infty$-cosmos and $A$ an object of $\mathcal{K}$, then each of the following is an accessible $\infty$-cosmos and the inclusion in $\mathcal{K}/A$ is an accessible cosmological embedding:

- $\text{Cart}(\mathcal{K})/A$
- $\text{coCart}(\mathcal{K})/A$
- $\text{DiscCart}(\mathcal{K})/A$
- $\text{DisccoCart}(\mathcal{K})/A$.

7.5. Two-sided fibrations. If $\mathcal{K}$ is an $\infty$-cosmos and $A, B \in \mathcal{K}$, then there is an $\infty$-cosmos $A\backslash\text{Fib}(\mathcal{K})/B$ of 2-sided fibrations from $A$ to $B$, which is cosmologically embedded in $\mathcal{K}/A \times B$. Explicitly, this can be constructed as

$$A\backslash\text{Fib}(\mathcal{K})/B := \text{Cart}(\text{coCart}(\mathcal{K})/A)_{A \times B \to A}$$

as explained in [14, Section 7.2]. Similarly, by [14, Section 7.4], there is another cosmologically embedded $\infty$-cosmos $A\backslash\text{Mod}(\mathcal{K})/B$, given by the discrete objects in $A\backslash\text{Fib}(\mathcal{K})/B$.

Theorem 7.7. Let $\mathcal{K}$ be an accessible $\infty$-cosmos and $A, B \in \mathcal{K}$. Then $A\backslash\text{Fib}(\mathcal{K})/B$ and $A\backslash\text{Mod}(\mathcal{K})/B$ are also accessible $\infty$-cosmoi, and their inclusions into $\mathcal{K}/A \times B$ are accessible cosmological embeddings.

Proof. By Theorem 7.6 we know that $\text{coCart}(\mathcal{K})/A$ is an accessible $\infty$-cosmos, and that $\text{coCart}(\mathcal{K})/A \to \mathcal{K}/A$ is an accessible functor. By the same theorem, applied to the $\infty$-cosmos $\text{coCart}(\mathcal{K})/A$ and the object $A \times B \to A$ therein, we know that the $\infty$-cosmos

$$A\backslash\text{Fib}(\mathcal{K})/B = \text{Cart}(\text{coCart}(\mathcal{K})/A)_{A \times B \to A}$$

is accessible, and that it has an accessible cosmological embedding into $(\text{coCart}(\mathcal{K})/A)_{A \times B \to A}$. So it will suffice to show that the functor

$$(\text{coCart}(\mathcal{K})/A)_{A \times B \to A} \to \mathcal{K}/A \times B$$

is accessible. This follows from Proposition 4.3 applied to the accessible cosmological functor $\text{coCart}(\mathcal{K})/A \to \mathcal{K}/A$ and the object $A \times B \to A$.

The case of $A\backslash\text{Mod}(\mathcal{K})/B$ now follows by Theorem 7.2. □

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