Characterization of the degree of Musical non-Markovianity

Maria Mannone\textsuperscript{1}\textsuperscript{,2}
\textsuperscript{1}IRCAM, 1 Place Igor Stravinsky, 75004 Paris, France
\textsuperscript{2}UPMC Paris VI Sorbonne, 4 Place Jussieu, 75005 Paris, France

Musical compositions could be characterized by a certain degree of memory, that takes into account repetitions and similarity of sequences of pitches, durations and intensities (the patterns). The higher the quantity of variations, the lower the degree of memory. This degree has never quantitatively been defined and measured. In physics, mathematical tools to quantify memory (defined as non-Markovianity) in quantum systems have been developed. The aim of this paper is to extend these mathematical tools to music, defining a general method to measure the degree of memory in musical compositions. Applications to some musical scores give results that agree with the expectations.

I. INTRODUCTION

Methods to mathematically quantify specific aspects of musical compositions have been recently developed in computational musicology \cite{1}. These methods are used both as an aid for musical analysis, and to investigate some aspects of samples of compositions \cite{2}.

In such a way, the following have been studied: musical theory in a geometrical way \cite{3}, common patterns in sounds amplitude \cite{4}, and chords succession \cite{5}. In addition, algorithms have been developed to find the most repeated sequences \cite{6}.\cite{7}, \cite{8}, algorithms to compose music \cite{9}\cite{11} and improvise in real time \cite{12}\cite{14}.

Because of the importance of repetition in music in order to characterize sequences as something of definite characteristic \cite{15}\cite{20}, algorithms have been used either to generate music or to detect repeated sequences.

Markov chains or hidden Markov chains \cite{24} have been used in the techniques of score following \cite{12}\cite{13}, to analyzing chord sequences in jazz improvisation \cite{21}, to study spectral similarity \cite{22}, to composing Markovian stochastic music \cite{23}.

Typically, standard music is not Markovian: it presents a memory of repeated sequences. However, compositions are characterized by a different degree of memory. Is it possible to quantify this degree of memory in existing compositions? While algorithms to find the theme have been developed, the degree of memory in musical composition has never been studied.

In order to define quantitatively the degree of memory in music, a formalism equivalent to the one used in Quantum Mechanics, defined in Information Theory \cite{25}\cite{20} will be utilized.

In physics, non-Markovianity is defined as the conservation of information of an initial state of a quantum system. Two quantum states, distinguishable at zero time, when both subjected to the same Markovian dynamics, progressively lose their characteristics (then lose their memory) in an exponential decay. After a certain interval $\tau$ they become indistinguishable.

Systems without memory are called Markovian, and ones with memory are called non-Markovian \cite{27}.

Quantification of the degree of memory in open systems is still an open problem. Recently various criteria have been developed to quantify the presence of memory, and to measure its total amount. These methods either directly utilize the master equation \cite{28}, the distance between two states \cite{29}, or the separability of the dynamical map \cite{30}. The difficulty when one tries to quantify memory is the request of independence of the results from the initial states of a chosen quantum system, or from specific mathematical models. Moreover, the utilization of a specific one of the above criteria can be, in given conditions, difficult to apply while another can be simpler, while the opposite can appear for a different one. Recently the relationship between criteria has started to be studied in detail \cite{31}.

As described above, the concept of Markovianity has been applied to Music \cite{12}\cite{13}\cite{21}\cite{21}\cite{32}, but the concept of non-Markovianity has never found application.

It is, however, useful to remark that the acceptation of memory, and then the definition of non-Markovianity conceived expressly for music, is different to the definition considered in the quantum case, where the comparisons are made between different states at the same temporal instant. However, in a musical composition we compare sequences of finite duration (starting in different time instants) of the same musical piece. The common aspect is the idea of memory as conservation of characteristics which make a pattern distinguishable from another, like a state from another.

The aim of this paper is to develop a general method to define and measure the Musical non-Markovianity degree of musical compositions, adopting concepts and mathematical techniques used in Quantum Physics. Among the various criteria to measure the degree of memory, the directly applicable one in the case of musical composition is the one that utilizes the distance between matrices \cite{29}.

As an application, the memory degree of three different compositions has been calculated, written respectively by V. Bellini \cite{33}, B. Maderna \cite{34} and P. Glass \cite{35}, that present a very different degree of memory by listening:

\footnote{Electronic address: maria.mannone@ircam.fr}
the results obtained fit well with the listeners’ expectations.

The interest of our research is to give to musicians and musicologists an additional quantitative method to classify compositions.

The structure of this paper is as follows. In chapter II we introduce non-Markovianity criteria used in Quantum Mechanics, giving more details about which ones are conceptually applicable to musical cases, and in which way. In chapter III we define musical non-Markovianity. In chapter IV we give technical details about the musical matrices defined and the algorithm developed to find them and then we apply our method to fictitious examples. In chapter V we apply the same method to existing compositions, giving numerical results. In chapter VI we give some conclusions; in the appendix there is an example of distribution matrices.

II. NON-MARKOVIANITY CRITERIA

The aim of this paragraph is the quantitative definition of Musical non-Markovianity, following the analysis of non-Markovianity idea in Quantum Mechanics, and its mathematical formalism, the reason being that the definition of memory must be different to the one used in the physics of open quantum systems (OQS).

Schematically:

- in the OQS case, if two states (distinct at the initial time) became progressively indistinguishable, this is taken as an indication of the loss of memory: i.e. the lower the distinguishability, the lower the memory;

- in the music case, in a musical score memory must be associated to repetitions of patterns. To higher number of repetitions, must correspond higher amount of memory, i.e. the lower the distinguishability among sequences, the higher the memory.

To quantify memory in music, one can introduce an object similar to quantifier of memory (non-Markovianity), similarly to those used in OQS, but defined differently.

Quantification of non-Markovianity in the theory of OQS had been addressed recently [27], introducing different quantum criteria to measure memory. The application of a particular quantifier may result in being simpler than the application of another, and this justifies the adoption of more than one criterion. It remains to understand which of them is more adapted to describe memory in music.

Different quantifiers are associated to the different criteria. One [29] utilizes the distinguishability of quantum states, by quantifying the variation of their distances with time, the increase or decrease of this quantifier being associated to persistence or loss of presence of memory (non-Markovianity or Markovianity).

Others are associated to the so called separability of the map [29], one looking directly to the behavior of this quantity, via [30], the dynamical equation satisfied by the density matrix that describes the state (master equation), another thus looking at the signs of the coefficients appearing in the master equation [29].

However, it has been proved that, for particular dynamics, non-Markovianity criteria do not always agree on attributing the presence of memory revivals in different time regions [36], and the comparison among them is an open problem [37].

For practical purposes, the use of a given criterion can be very difficult (if not impossible), either for the difficulty of calculations, or for the absence of required mathematical objects.

In particular, if the distance between states depends on a particular chosen pair of initial states, a maximization over all possible pairs of states in the system, to obtain the total amount of non-Markovianity, is required, which in general can be computationally very difficult. In some cases, we may have either the dynamical map or the master equation, but not the map. And although it is possible to obtain the master equation from the map and vice versa, in general it requires a long and difficult mathematical procedure [37].

The criterion that can be extended to the case of our interest, that is the quantification of memory in music, is one of the above criteria that directly utilizes matrices. This because we do not have anything corresponding to a map or a master equation for a musical score, while it is instead possible to represent a musical sequence as a matrix.

The measure of distance between two states represented by the matrices $\rho_1$ and $\rho_2$ is the trace distance defined as [29]:

$$D(\rho_1, \rho_2) = \frac{1}{2} \text{tr}|\rho_1 - \rho_2|,$$

where, if $A$ is an Hermitian operator, $|A| = \sqrt{A^\dagger A}$. The rate of variation of the distance $D$ is defined as

$$\sigma = \frac{d}{dt}D(\rho_1, \rho_2).$$

The sign of $\sigma$ determines if the distance is increasing, thus the distinguishability between states is preserved; or if the distance decreases and the memory is progressively lost.

A requirement that must be satisfied by $\sigma$ in order to use it as a quantifier, is that it does not have to depend on the initial states. Otherwise, a maximization over all couples of initial states is required. The maximization over all possible couples of states $\rho_1$ and $\rho_2$, and the integration over all times gives the degree of non-Markovianity $\mathcal{N}$ [29]:

$$\mathcal{N} = \max_{\rho_1, \rho_2(0)} \int_{\mathbb{R}^+} dt \sigma(t, \rho_{1, 2}(0)).$$

This quantity, however, does not give any information about the revivals.
III. MUSICAL NON-MARKOVIANITY

To define non-Markovianity in music, we adopt a criterion similar to the one proposed in [29]. In fact, the maximization procedure is not applicable over all initial states, because we intend to define the memory degree for each singular musical piece, thus for a single initial state, not several possible different initial states.

Most musical composition are divisible into sections. The structure of the entire musical piece is determined by the structure of sections.

The most common form, in classical compositions, is A - B - A'. Let us consider a simplified structure A - B. Every section contains several sequences; let us suppose $A_1, A_2, A_3$ for the section A, and $B_1, B_2, B_3$ for the section B.

The natural succession in time of the sequences in a score is thus the following:

$$A_1, A_2, A_3, B_1, B_2, B_3 \rightarrow t$$

To compare these sequences to apply non-Markovianity criterion, we can reorder them in time.

To compare sections A and B:

$$t \downarrow A_1, B_1 \downarrow A_2, B_2 \downarrow A_3, B_3$$

Time is taken to flow downward.

For each sequence, we assign a set of two-dimensional matrices. Each matrix represent the distribution of a couple of variables for every note (for example frequency and mean duration). The exact procedure we have developed and followed to construct these matrices will be explained in next chapter. Here we will describe the general method used to apply the criterion to a generic musical composition.

Let us consider for example the couple of frequency and duration. For each sequence, we can obtain a matrix: $\rho^A_i$ for $A_1$, $\rho^B_i$ for $B_1$, and so on. In general, the sequences have different length; but we construct time normalized distribution matrices. Therefore, we will have the following structure:

$$t \downarrow \rho^A_1, \rho^B_1 \downarrow \rho^A_2, \rho^B_2 \downarrow \rho^A_3, \rho^B_3$$

Now, we can evaluate the trace distance defined in eq. [30] between simultaneous sequences. Our matrices are not calculated at each time instant, but at each time interval that corresponds to the normalized length of the sequence.

We indicate as $D_i$ the distance $D_i(\rho^A_i, \rho^B_i)$ between the sequences $\rho^A_i$ and $\rho^B_i$. The variation rate $\sigma$ in the musical case can be defined as $\frac{\Delta D}{\Delta t}$ (we are considering finite time intervals).

Because of the normalization, all time intervals $\Delta t$ are equal, and we may consider only $\Delta D$. We then define the rates $\sigma_i = D_{i+1} - D_i$. We consider only positive values, which represent the case of increasing distance (as in eq. [3]).

In music there is memory if the distance between matrices is zero, constantly different to zero or decreasing. If the distance is increasing, the musical thematic memory is progressively lost. To equal musical sequences correspond equal matrices.

We observe that in the OQS case, there is non-Markovianity when the distinguishability between couples of states is preserved, i.e. when the distance is constant or increasing (and the rate is zero or positive).

Consequently, in physics, the lower the distinguishability, the lower the memory, while in music, the lower the distinguishability, the higher the memory.

So, while in the physical case non-Markovianity degree is given by $\mathcal{N}$ (eq. [2]), in the musical case we define a correspondingly quantity as $\mathcal{M} = 1 - \mathcal{N}$. We shall use the sum of positive rates $\sigma_i$ as indicator of non-Markovianity.

We normalize $\mathcal{M}$ so that $\mathcal{M}$ is equal to 1 when the thematic memory is the maximum, i.e. when in a musical composition there are only repetitions of the same sequence, and 0 when the memory is the minimum. To evaluate the different degrees of memory we subtract the quantity $m = \sum \sigma_i$ from the maximum value of memory, 1. The same quantity $m$ must then be normalized between 0 and 1 as $\frac{m}{1 + m} [30].$

The quantifier will be

$$1 - \frac{\sum \sigma_i}{1 + \sum \sigma_i},$$

and then the quantifier we may adopt is

$$\mathcal{M} = \frac{1}{1 + \sum \sigma_i}$$

This definition however does not differentiate cases with an identical sum of positive rates $\sum \sigma_i$, because it does not give any information about the total number of rates, positive and negative and null. For example, let us consider a score with ten total rates, with only two positive ones, with sum $\sum \sigma_i$: the positive contribution has the proportion of 2 over 10. Let us consider another composition, with only three total rates with two positive ones, with identical sum $\sum \sigma_i$: the positive contribution in this case is 2 over 3. The information given by eq. [4] is the same for the two compositions, and does not take into account the different proportions (2/10 vs 2/3).

This problem can be corrected by introducing a correction factor, $r = \frac{n_+}{n_+ i}$, defined as the ratio between the
number of positive rates $n_+$ and the number of total rates $n_T$. So we defined a new quantifier

$$M_C = \frac{1}{1 + r \sum_i (if \sigma_i > 0) \sigma_i}.$$  \hfill (5)

The result is more coherent with the musical structure.

In the section [V], we compare the results of eq. 4 and 5 obtained applying our method to three compositions, that seem to present to listeners a very different degree of thematic memory. The composition are the vocal parts of Dolente Immagine [33] by Vincenzo Bellini, the first piece of the suite Solo [34] by Bruno Maderna, and Metamorphosis 3 [35] by Philip Glass. In the next chapter we will explain how to construct musical matrices that contain the information written in each sequence of a score.

IV. REALIZATION

A musical composition is divisible into sequences, and each sequence can be reduced to numerical parameters (frequencies, intensities, times of start and durations). The matrices we propose represent the distribution of every note in each sequence around the mean value of these parameters.

In order to construct musical matrices, we must convert into numbers the symbolic information contained in a musical score. The parameter considered in our analysis is frequencies, times of start, duration, intensities. We ignore the timbre.

We used the following scales.

For the heights of notes, we do not use Hz, but differences in Hz. In particular we have chosen differences of semitones respect to the middle C.

For the intensities, we use dimensionless numbers to indicate relative intensity indications in musical scores, in particular we choose 90 for fff, 80 for ff, 70 for f, 60 for f, 50 for mf, 40 for mp, 30 for p, and so on.

For times, we have not used seconds but dimensionless units: multiples of a measured time unity (specified by the metronomic indication) are indicated in a score. In particular, for the durations we have chosen 1 for semiquavers, 2 for quavers, and so on.

The duration of a rest will be counted as the time before the start of the following note.

The parameters that we have chosen to characterize our matrices are the couples duration-intensity, frequency-start, start-intensity, frequency-duration and frequency-intensity.

For each musical sequence, we have first calculated the mean values of the parameters. Then, we have evaluated the distance (normalized between 0 and 1) of the parameters of each note in the sequence respect to the mean values.

The range of distances from the first parameter and the range of distance from the second one has been divided into equal parts.

The matrices we propose represent the distribution of each sequence can be reduced to numerical parameters that contain the information written in each sequence of a score.

V. MUSICAL NON-MARKOVIANITY OF REAL COMPOSITIONS

We have applied the methods of the previous chapters, and the technique described in chapter realization to find  

![ FIG. 1: Trivial succession of two identical bars: for the parameters frequency-start, the memory is 1, the maximum value.](image1)

![ FIG. 2: The first bar is a quasi-random sequence of pitches and durations, while the second one is totally different: in this case the memory for the parameters frequency-start is 0.3 (the minimum possible value is 0).](image2)
the non-Markovianity degree of different compositions.

A classic vocal Italian composition, an avant-garde Italian composition and a contemporary minimalist one have been considered. The chosen compositions are, respectively, the vocal part of *Dolente Immagine* by Vincenzo Bellini, the first part of the oboe suite *Solo* by Bruno Maderna, and the piano piece *Metamorphosis 3* by Philip Glass. The compositions seem to present, by listening, a very different degree of memory.

The compositions will be analyzed using the Musical non-Markovianity degree $M$ defined in eq. 4 and the $MC$ of eq. 5, where has been utilized the concept of corrective factor $r$. We will see that the use of $r$ allows a better characterization of cases as the Glass’ one.

The informations contained in the scores will be plotted in tridimensional graphs. Identical colors has been used for identical sequences. The use of tridimensional graphs to study characteristics of sounds is prosed by I. Xenakis [23], and the use to graphically study orchestra\-tion of musical scores is proposed by M. Betta [38].

Here we will describe briefly the structure of the chosen compositions, in order to find the optimal subdivision into sequences.

**Bellini.** The vocal score has a structure of type A - B - A’, and can be divided into seven periods: sections A has three periods, section B only one, and section A’ three ones. The motivation of the subdivision into three sections is due to the tonality change in period 4, and the reprise of theme and of its tonality in period 5. The subdivision into sequences as discussed is due to reasons of musical analysis of a classical model. There are some identical parts between sections A and A’. This fact induces to expect a high degree of memory.

A matrix has been associated (for each couple of parameters, as discussed in previous chapters) to every sequence, that in the case of Bellini is therefore naturally corresponding to a period.

The graph of fig. 3 shows some repeated patterns. The intensity written in the score is constant, and then the complete development of the composition can be represented in time-frequency plane (fig. 6).

**Maderna.** Looking at the score of the first piece of the oboe suite *Solo*, and looking at the tridimensional representations of the score in fig. 4 it is clear that the structure is very different from Bellini’s composition, since the quantity of repetitions is clearly lower, there are not thematic or relevant patterns, but some fragments. So in this case the expectation is a lower degree of memory. Since there were not clearly thematic or tonality motivations as in Bellini’s case, it was impossible in this case to talk about periods, but only sequences separated by the taking of breath and slurs, the most natural criterion in this case. We have then divided the score into eleven sequences.

**Glass.** The piano composition *Metamorphosis 3* is an example of the minimalist style, where there are repetitions with the lower number of pattern changes and variations. The expected degree of memory is therefore higher with respect to the Maderna’s and Bellini’s cases.

We have chosen this example also to verify the usefulness of the corrective factor $r$, since the greater duration of the examined composition implies a greater number of coefficients $\sigma_i$ in the calculus of the degree of non-Markovianity (eq. 5). Due to its regularity, the composition has been divided into twenty-two sequences, each sequence containing four bars (bar 1 of the refrain has...
FIG. 5: (Color online) Tridimensional representation of the piano composition *Metamorphosis 3* by Philip Glass. In this graph, we chose the value 1 for the eight note (in other pieces, 2 for the eight note). The calculations of non-Markovianity are unaffected of these variations, because are important distributions towards mean value.

FIG. 6: (Color online) The projection of the graph of fig. 3 in the plane time-frequency. The intensity written in the score is constantly equal to p (piano, equal to 30 in our scale), and then the plane time-frequency contains the entire develop of the vocal score. It is evident the presence of repeated patterns.

Calculating musical matrices and analyzing the memory degree $\mathcal{M}$ and $\mathcal{M}_C$, as respectively defined in eqq. 4 and 5, we obtain the following results.

|        | Bellini | Maderna | Glass |
|--------|---------|---------|-------|
| $\mathcal{M}$ | 0.97    | 0.73    | 0.55  |
| $\mathcal{M}_C$ | 0.99    | 0.77    | 0.97  |

|        | Bellini | Maderna | Glass |
|--------|---------|---------|-------|
| $\mathcal{M}$ | 0.75    | 0.40    | 0.38  |
| $\mathcal{M}_C$ | 0.82    | 0.53    | 0.78  |

|        | Bellini | Maderna | Glass |
|--------|---------|---------|-------|
| $\mathcal{M}$ | 0.86    | 0.54    | 0.51  |
| $\mathcal{M}_C$ | 0.81    | 0.59    | 0.86  |

|        | Bellini | Maderna | Glass |
|--------|---------|---------|-------|
| $\mathcal{M}$ | 0.75    | 0.56    | 0.55  |
| $\mathcal{M}_C$ | 0.82    | 0.62    | 0.79  |

|        | Bellini | Maderna | Glass |
|--------|---------|---------|-------|
| $\mathcal{M}$ | 0.85    | 0.55    | 0.40  |
| $\mathcal{M}_C$ | 0.94    | 0.67    | 0.57  |

The results apparently correspond to the empirical expectations: in fact, Bellini’s and Glass’ compositions have a medium degree of memory higher than Maderna’s one, and Glass’ memory is higher than Bellini’s one (using the corrective factor). Therefore, we have seen that the degree of memory $\mathcal{M}_C$ is better than the uncorrected one $\mathcal{M}$, since it is more adequate to describe cases such as that of Glass. In some cases the results are not significantly different, but in other cases, for example depending on the length of the composition, the correction...
V is decisive. Therefore, the results obtained fit well in the comparison with the limiting cases of chapter IV.

Therefore, it seems that the method can be extended to analyze the degree of memory of musical compositions using a larger set of data.

It could be useful to use automatisation for the entire process, that is also for the reduction into numbers of the parameters contained in the score, and for the subdivision into sequences.

VI. CONCLUSIONS

What is memory in musical compositions? It is not the only amount of repetitions of a theme, since there are cases in which there is not a theme, but various fragments of patterns. In our work we have indicated as memory an idea closer to the reciprocal similarity of the sequences into which musical compositions can be divided. And the thematic memory? It is necessary to remark that the thematic memory implies the memory, but the contrary is not true. A musical composition may present a degree of memory, but it could not have a clearly defined theme, but only some thematic cells, where cell is a term used by musicologists to indicate some repeated rhythmic and melodic design that can be isolated, or can as well constitute a part in a thematic context, or not. Therefore there exists something that, by listening, can not have relevance, for example some secondary element, but that becomes decisive in a detailed analysis context of the score, and then also in computational analysis.

In this paper we have proposed a method to quantify the amount of memory in a musical composition, defining a quantifier (eq. 3) applicable at each musical piece, inspired by a non-Markovianity degree used in physical literature (eq. 5). This is only the first step for a deeper comprehension of the link between mathematical formalism and composition. This method could be developed, by automatizing every step, and applying it to a large number of musical compositions, of different musical genres or periods.

The method proposed can be utilized by musicians and musicologists to connect mathematical/physical to musical concepts, to increase the comprehension of technical aspects of their art, giving them other methods to aid musical analysis. The Musical non-Markovianity degree \( M_C \) can be used to classify different composers in various historical periods by the mean memory level of their compositions. One may ask if \( M_C \) could be used to characterize different artistic phases of the same composer, either if it is possible to assign the same mean value of memory to all composers of the same periods or the same artistic movement.

A mean non-Markovianity degree can be assigned to each musical form (fugue, theme with variations, fantasia...). It is also possible to extend this analysis to the comparison between two similar pieces. For example, a musical piece could contain a musical idea not repeated in the same piece, but present in another composition; then the value of memory increase when we compare the sequences of the two compositions between them.

VII. ACKNOWLEDGEMENTS

I am grateful to the professor Giuseppe Compagno, who has guided me in the Master thesis’ research and who has always encouraged me to pursue the conjunct studies of music-science. In particular, without him this paper would never have been born. I am grateful also to the composer Marco Betta, for fruitful discussions about the concept of memory in music.

VIII. APPENDIX

To obtain musical matrices, we have developed an algorithm, with the following steps: 1. evaluation of the normalized distance, for each value, from the mean value; 2. count of the number of notes with a normalized distance (from the mean value) between 0 and 0.25, 0.25 and 0.5, 0.5 and 0.75, 0.75 and 1. To describe the procedure we have developed to construct our distribution matrices, we choose, as an example, only the couple of parameters frequency (\( \nu \)) vs time of start of each note in the same sequence (\( \tau \)). Let us consider the following short sequence of sounds as an example.

```
\begin{align*}
\text{\( \nu = 10, \nu_2 = 12, \nu_3 = 20 \), and the time of start are (the durations are all equal, and corresponding to a crotchet, i.e. a quarter note, = 4 in our scale):} \\
\tau_1 &= 0, \tau_2 = 4, \nu_3 = 8. \\
\text{The distance from the mean values are} \\
\delta \nu_1 &= 4, \delta \nu_2 = 2, \delta \nu_3 = 6; \delta \tau_1 = 4, \delta \tau_2 = 0, \delta \tau_3 = 4.
\end{align*}
```

To normalize these distances, the algorithm find minimum and maximum distance values:

\[
\delta \nu_{\min} = 2, \delta \nu_{\max} = 6; \delta \tau_{\min} = 0, \delta \tau_{\max} = 4.
\]

The normalized distance are evaluated as (if \( \delta_{\max} \) and \( \delta_{\min} \) are equal, the algorithm does not normalize):

\[
\delta \nu^N_i = \frac{\delta \nu_i - \delta \nu_{\min}}{\delta \nu_{\max} - \delta \nu_{\min}}, \quad \delta \tau^N_i = \frac{\delta \tau_i - \delta \tau_{\min}}{\delta \tau_{\max} - \delta \tau_{\min}}.
\]
where $i = 1, 2, 3$ (in the example considered there are only three notes). In our example we obtain

\[
\begin{align*}
\delta \nu_1^N &= 0.5, \quad \delta \tau_1^N = 1, \\
\delta \nu_2^N &= 0, \quad \delta \tau_2^N = 0, \\
\delta \nu_3^N &= 1, \quad \delta \tau_3^N = 1.
\end{align*}
\]

Now it is possible construct the matrix frequency-start for this sequence. The matrix will contain, in the rows, the number of notes with normalized distance between 0 and 0.25, 0.25 and 0.5, 0.5 and 0.75, 0.75 and 1 from the mean value of start; and, in the columns, the number of notes with normalized distance between 0 and 0.25, 0.25 and 0.5, 0.5 and 0.75, 0.75 and 1 from the mean value of frequency.

To avoid the difficulty of the different length of each sequence (each sequence can contain a different number of notes, the notes can have different durations, the sequence in total can have different durations...), we divide each matrix element by the total number of notes in the considered sequence. In this way, we obtain a distribution matrix, and the sum of each element is 1. So we can then compare easily sequences of different length, and with different numbers of notes. In our simple example, we divide all elements of the matrix by 3 and obtain:

\[
\rho = \begin{pmatrix}
0.33 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0.33
\end{pmatrix}
\]