Entropy from Carnot to Bekenstein

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Abstract

The second half of this essay tells the story of the genesis and early development of the notion of black hole entropy, in the style of an after dinner talk (apart from a long technical footnote). The first half sketches the development of the concept of entropy, beginning with Carnot, passing via Thomson, Clausius, Boltzmann, Planck, von Neumann, and Shannon on the way to Bekenstein, and the essay ends with a retrospective on Lemaître. The central theme is the fault-tolerant way that profound insights have emerged from simple (yet subtle) thermodynamic reasoning.

1 Introduction

Thermodynamics is a mysterious subject. Powerful conclusions can materialize out of scant inputs, and it is forgiving of fundamental misconceptions about the underlying details. Before recounting the tale of Jacob Bekenstein’s discovery of black hole entropy and the generalized second law, I want to take a few steps further back, to review the origins of thermodynamics and the concept of entropy, and how it led to quantum physics and quantum field theory. This will help to place in perspective the sort of process by which radical insights into new physics can come about through the study of thermodynamics.

2 Heat Engines: Carnot, Thomson and Clausius

To begin with, it is amazing that Sadi Carnot in 1824 was able to infer that there is a universal upper bound to the efficiency of a heat engine, even while using an
A heat engine produces a quantity of work when drawing heat out of a thermal reservoir and depositing heat into a colder thermal reservoir. Carnot reasoned that the most efficient such engines could in principle come arbitrarily close to being reversible. He argued that all reversible heat engines, operating between the same two temperatures, must produce the same amount of work from the same amount of heat flow out of the hotter reservoir. Were that not the case, some of the work extracted from the more efficient engine could be injected to run the less efficient engine in reverse, while retaining the balance of the work for other purposes. If heat is a conserved “caloric” fluid, then when the less efficient engine is run backwards, the caloric flow from the colder to the hotter reservoir exactly cancels the flow that occurred in the more efficient engine. The result, as Carnot put it, would be “not only perpetual motion, but an unlimited creation of motive power without consumption either of caloric or of any other agent,” which he rejected as being “inadmissible”.

While Carnot’s conclusion was correct, his argument contained a single deep flaw: heat is not by itself conserved! More heat flows out of the hot reservoir than flows into the colder reservoir, the difference being the work extracted. And less heat flows into the cold reservoir with the more efficient engine than with the less efficient engine. But now, if the less efficient engine is run backwards, the cold reservoir is no longer restored to its initial state: more heat is drawn out than went in. The leftover work, then, is not produced from nothing, but rather from the heat drawn out of the colder reservoir. While not as inadmissible as Carnot’s result, this is nevertheless inadmissible. Its impossibility is Kelvin’s version of the second law of thermodynamics. Alternatively, all of the work from the more efficient engine could be used to run the less efficient engine backwards, in which case the net result would be spontaneous (but engineered) heat flow from the colder reservoir to the hotter one, in violation of Clausius’ version of the second law.

William Thomson (later Lord Kelvin) proposed in 1848 an absolute thermometric scale based on Carnot’s theory, still using the concept of heat as a conserved caloric fluid that could not be converted to mechanical energy. After being convinced of the mechanical equivalence of heat by James Joule’s experiments,
Thomson in 1851 revised his proposal. The new version was based on the fact
that, for reversible heat engines, the ratio \( Q_h/Q_l \) of the heat \( Q_h \) drawn from the
hotter reservoir to the heat \( Q_l \) deposited in the colder reservoir must be not only
independent of the heat engine for a given \( Q_h \) (as required by the corrected Carnot
argument), but also independent of \( Q_h \). (Were the latter not the case, one could di-
vide the operation into sub-processes and obtain reversible engines with different
efficiencies.) The ratio \( Q_h/Q_l \) is therefore a property of the two reservoirs, inde-
pendent of the construction of the engine, and so it serves to define in an absolute
fashion the ratio of the temperatures of the two reservoirs, \( T_h/T_l := Q_h/Q_l \). This
is the absolute thermodynamic temperature scale, which is defined up to an overall
arbitrary constant multiple.

When re-expressed as \( Q_h/T_h = Q_l/T_l \), Thomson’s definition of temperature
is directly connected to the concept of entropy which Rudolph Clausius developed
over the decade from 1854 to 1865, when he finally gave entropy its name. If the
entropy extracted from the hotter reservoir is defined as \( Q_h/T_h \), and the entropy
added to the colder reservoir is defined as \( Q_l/T_l \), then a reversible engine can be
described as one for which there is no net change of entropy: the entropy extracted
from the hotter reservoir is exactly equal to that deposited in the colder one. At
one level, this statement is virtually empty, being nothing but a restatement of the
definition of thermodynamic temperature in different words. However, because the
absolute temperature can be shown to be equivalent to the ideal gas thermometer scale,
the criterion of zero entropy change has nontrivial implications for the macroscopic state variables. Moreover, this definition of entropy change can be
integrated along a sequence of equilibrium states, to determine the difference in
entropies of the initial and final equilibrium states. The result is independent of
the sequence of states chosen to connect the two equilibria, because if it were to
depend on the sequence, one could contrive a means to pump heat from a colder to
a hotter reservoir, without the input of any net work, in violation of the second law.
A concept of the entropy of an equilibrium state is thus defined, up to an overall
additive constant, and up to the multiplicative constant inherent in the definition
of absolute temperature. In a non-reversible engine, the ratio \( Q_l/Q_h \) is larger, so
that more entropy enters the colder reservoir than leaves the hotter one, and thus
entropy irreversibly increases.

3 Atoms and Radiation: Boltzmann and Planck

Clausius’ entropy clearly had some relation to an underlying mechanical disorder, a
relation to which Ludwig Boltzmann devoted three decades of thought, beginning
at age 22 in 1866. By 1871 he had obtained the expression \( -\int \rho \ln \rho \) for the entropy of a dynamical system (not necessarily in equilibrium) as a phase space integral, the distribution \( \rho \propto \exp(-\beta H) \) being the probability density that the Hamiltonian has the value \( H \) at the temperature \( \beta^{-1} \). The logarithm is defined only up to an additive constant, since a unit of phase space volume must be chosen in order to render the argument of the log dimensionless. This ambiguity was no cause for concern, because in any case Clausius’ entropy was defined only up to an additive constant.

Boltzmann introduced in 1877 the notion that the probability of a macrostate of a gas could be identified with the number of compatible microstates, equilibrium being the configuration that maximizes this number, and entropy being its logarithm. To count the microstates he divided the velocity space of a molecule into small cells of uniform volume, and described a microstate, or “complexion”, by an assignment of cells to the atoms. He argued that the cell size is unimportant, as long as it can be chosen large enough to contain the velocities of a large number of atoms, while at the same time being small enough so that the relative variation of this number from one cell to the next is small. The cell size then affects significantly only the overall additive constant in the entropy, which plays no role in thermodynamics.

At this point the stage is set for the work of Max Planck. Planck set out in 1894 to establish the second law of thermodynamics as an exact statement, like the conservation of energy. He initially thought the absorption and emission of electromagnetic radiation would be a fundamentally irreversible process that could account for an absolute formulation of the second law. At some point he realized this was simply inconsistent with classical electrodynamics, but nevertheless this is the concern that led him to the blackbody problem. Despite what we are told in textbooks and colloquia, the “ultraviolet catastrophe” was not the problem he set out to solve. (That phrase was introduced in 1911 by Paul Ehrenfest who was, by the way, a student of Boltzmann.) Following Boltzmann, Planck believed that entropy is well-defined out of equilibrium, and that it is maximized in equilibrium, which led him to the problem of understanding the blackbody spectrum from first principles.

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4For an authoritative history of Boltzmann’s work see Atoms, Mechanics, and Probability. Ludwig Boltzmann’s Statistico-Mechanical Writings. An Exegesis, by Olivier Darrigol (Oxford: Oxford University Press, 2018).

5The motivation and evolving reasoning of Planck, as well as the reactions of his contemporaries to his work, was penetratingly analyzed by Thomas Kuhn in his masterpiece, Blackbody Theory and the Quantum Discontinuity, 1894-1912. The following comments are based on Kuhn’s analysis, and on the forthcoming book by Anthony Duncan and Michel Janssen, How We Got to Quantum Mechanics.
Planck developed over a period of five years the theory of oscillators (“resonators”) driven by a stochastic electromagnetic field and damped by radiation reaction. One product of this work was the equilibrium relation $\rho_\nu = \frac{8\pi\nu^2 U_\nu}{c^3}$, between the energy density per unit frequency interval $\rho_\nu$ of the radiation field, and the average energy $U_\nu$ of a single damped oscillator with frequency $\nu$. This reduced the problem to the one of finding the equilibrium distribution of $U_\nu$ values.

Wien had introduced a theoretical distribution law for $\rho_\nu$ which fit the experiments reasonably well at the time, and in 1899 Planck set out to account for Wien’s distribution using thermodynamics. Taking Wien’s law as given, Planck inferred the corresponding temperature dependence of $U_\nu$, and from that he inferred that the entropy $S(U)$ of one resonator, as a function of its energy $U$, must satisfy the relation $S''(U) \propto -1/U$. He further showed that this relation is implied by the entropy maximization principle, Wien’s displacement law, and the assumption that the entropy of a collection of $N$ resonators of a given frequency depends only on their total energy, and not on $N$. This was all classical physics.

Soon came more data, however, extending the spectrum further into the infrared, and differing from the Wien distribution. This led Planck to realize that his assumption of the $N$-independence of the entropy was unjustified, and he quickly found the simplest modification of his formula for $S''(U)$ that would agree with the observed fact that, for each frequency, at sufficiently high temperature $T(= 1/S'(U))$, the radiance is proportional to temperature. This led him directly to $S''(U) = -a/(U(1 + bU))$, which produced a spectral density that fit all of the data very well; in fact it produces precisely the Planck distribution! The equilibrium distribution of radiation was thus determined by two fundamental physical constants, $a$ and $b$. A giant step had been taken, but Planck as yet had no derivation of the new entropy formula from first principles.

For this he turned to Boltzmann’s notion that entropy corresponds to the logarithm of the number of microstates compatible with a macrostate, and that entropy is maximized in equilibrium. Adapting Boltzmann’s combinatorial counting of the microstates of gases, Planck set out to enumerate the number of ways that a given total energy can be distributed among a collection of resonators of different frequencies. The distribution of energy with respect to frequency that maximizes the

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6 The same two constants had already entered the Wien distribution, which is after all just the high frequency limit of the Planck distribution, and Planck had enthusiastically noted that, together with Newton’s constant and the speed of light, these allowed the construction of a natural system of units. Two years later in 1901, after having found his statistical derivation of the Planck distribution, he showed that consistency of equilibrium between radiation and ideal gas implied that these radiation constants must also determine the Avogadro constant $N_A$ and the gas constant $R$ (or Boltzmann’s constant $k = R/N_A$), and vice versa. This allowed him to express the elementary electronic charge, which is related to $N_A$ and $R$ via electrochemical measurements, in terms of the measured radiation constants, and he found agreement with the available estimates.
total number of such “complexions” would then correspond to the equilibrium dis-
tribution. To this end it was necessary to adopt what could be called an energy ‘bin
size’ (though Planck certainly did not refer to it this way), in order to discretize
the counting. To maintain agreement with Wien’s displacement law, he took the
bin size to be proportional to the natural frequency $\nu$ of the resonators, $\Delta E = h\nu$,
which required him to introduce a proportionality factor $h$ with dimensions of ac-
tion. This quickly led to the desired result for the entropy, at the end of 1900. As
Kuhn convincingly argues, contrary to what we are taught in school, and to what
his early readers like Lorentz and Ehrenfest thought he was doing, Planck was most
certainly not assuming that the energy of a resonator was itself quantized. In fact,
he did not initially think that a break with classical physics was required.

Although Planck of course recognized that the observed radiation distribution
depended on the constant $h$ he had introduced to specify the bin sizes, he resisted
the conclusion that classical physics was unable to account for that formula, despite
the fact that the classical physics was complete without that constant. Eventually,
he was convinced in 1908, by Lorentz (who had himself just come around, and who
was not the only nor the first to note), that classical electromagnetic theory cannot
produce a sensible equilibrium distribution. In the meantime, only a few people,
beginning with Ehrenfest and Einstein, saw that a break with classical physics was
required. And since classical equipartition of electromagnetic energy, or maxi-
mization of the field entropy, led inevitably not to the Planck distribution but to the
Jeans distribution at all frequencies, there must have been a flaw in Planck’s rea-
soning if taken classically. In fact, Einstein had first noted the problem in 1906: the
Planck distribution entailed a significant variation of the probability across one en-
ergy bin when $h\nu/kT$ is not much smaller than unity. By the end of the decade the
collection was widely appreciated: only by quantizing the energy of the modes of
the radiation field, and maximizing their entropy at fixed energy, could the Planck
distribution be consistently derived. To account for a sensible theory of thermal
equilibrium of electromagnetic radiation, quantum mechanics and quantum field
theory had to be discovered!

To all the evidence discussed by Kuhn, and by Duncan
& Janssen, I would add Planck’s Nobel Prize lecture in 1920,
https://www.nobelprize.org/prizes/physics/1918/planck/lecture/
In that lecture, Planck nowhere mentions having quantized the energies of the oscillators. Rather, he
refers to “‘elementary regions’ or ‘free rooms for action’”. He does describe the difficulties fitting
his derivation into classical physics, but there he is (I believe) referring to subsequent developments.
4 Quantum Mechanics and Information Theory

Along with quantum mechanics came a radical shift in the nature of the probabilities that figure in the statistical definition of entropy, but it wasn’t until a quarter century after Planck’s groundbreaking work that this became fully clear. Quantum mechanics demanded a new role for probability in physics: that of governing indeterminism. Classical states were replaced by quantum state vectors, whose squared amplitudes give fundamental probabilities (whatever those are). Moreover, because the state spaces of composite systems combine by tensor product, a generic quantum subsystem cannot be assigned a definite quantum state vector. This phenomenon of nonseparability, or entanglement of bipartite quantum states, was noted very early on by Lev Landau in 1927 (at the ripe old age of 19), in a paper on radiation damping of an electric dipole. The opening lines of the paper read: “In wave mechanics, a system can not be uniquely defined; we always have to do with a probabilistic ensemble (statistical conception). If the system is coupled with another one, its behavior has a double indeterminacy.” Landau then proceeds to explain algebraically what is meant. The same year, John von Neumann introduced the entropy $S = -\text{Tr} \rho \ln \rho$ associated with the density matrix $\rho$ of a quantum system, in his paper “Thermodynamics of Quantum Mechanical Ensembles”. Five years later, in The Mathematical Foundations of Quantum Mechanics, von Neumann attributed to Landau’s paper the notion that, in quantum mechanics, a subsystem of a pure quantum state can have entropy.

Our last station on the road to black hole entropy is information theoretic entropy, introduced by Claude Shannon in 1948. For practical reasons, it was of interest to Shannon to quantify the information that can be transferred via a communication channel. The connection to thermodynamic entropy is that the microstate of a physical system can be viewed as a message whose information content is greater when the entropy of the corresponding macrostate is greater. Although quantum mechanics played no role in Shannon’s considerations, the shift of focus from the state of the system to the state of knowledge of the communicator is in a sense required by quantum mechanics, at least if quantum probabilities are conceived as referring to information possessed by an observer. But the information theoretic interpretation of entropy further widens the application of this concept, by detaching it from the particulars of the thermodynamics, phase space, or Hilbert space of a specific physical system, and focusing on the information acquisition of an agent who interacts with that system. As a result, the concept of entropy can be applied even when the nature of the system is only partially known.

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*In his Nobel Prize lecture [https://www.nobelprize.org/prizes/physics/1954/born/lecture/](https://www.nobelprize.org/prizes/physics/1954/born/lecture/), Max Born gives an overview of the emergence of the statistical interpretation of quantum mechanics. For an in depth historical look, focusing on the approaches of Jordan and von Neumann, see Ref. [2].*
It is presumably for this generality, as well as its relation to the information acquired by an observer of a quantum system, that Jacob Bekenstein in 1972 reached for the information theoretic definition of entropy when trying to determine the entropy that should be assigned to a black hole.

5 Black Holes and Bekenstein

The genesis of black hole entropy has been traced to the famous story of John Wheeler and his graduate student, Jacob, discussing teacups and black holes. As Wheeler recalled it, the conversation took place in his office. In his 1990 book, *A Journey into Gravity and Spacetime*, he writes, “I told him [Jacob] of the concern I always feel when a hot cup of tea exchanges energy with a cold cup of tea.”

I wanted to get to the bottom of this tea thing.

I found another reference, Wheeler’s 1998 book: *Geons, Black Holes, and Quantum Foam: A Life in Physics*, written with Ken Ford. An Amazon.com search in the book yielded 8 passages in which the word tea appears — one being the index entry for the Fine Hall tea at Princeton. The first passage was about breaking for tea after discussions with Bohr, and mentioned that Bohr would lift the edge of the rug in his office and kick the chalk bits under the carpet, to avoid being scolded by the janitor. Several were about tea in Fine Hall, and discussions there with physics luminaries. Wheeler says, “Bohr and I were regulars at the afternoon teas.” He quotes Oppenheimer as saying that “Tea is where we explain to each other what we do not understand.” Another entry mentions that Einstein invited Wheeler’s general relativity class to tea at his house, during the last two years of Einstein’s life. So it is clear that tea played a very important role in Wheeler’s scientific life...

And then there is the legendary discussion with Jacob. Wheeler writes:

*The idea that a black hole has no entropy troubled me, but I didn’t see any escape from this conclusion. In a joking mood one day in my*... 

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*The following part of this article is closely based in part on an after-banquet speech given on the occasion of the conference, *40 Years of Black Hole Thermodynamics*, held in honor of Jacob Bekenstein at the Institute for Advanced Studies, Jerusalem, September, 2012. Here I have maintained the tone and style that was appropriate to that occasion. In particular, some of the text is written in first person.*
office, I remarked to Jacob Bekenstein that I always feel like a criminal when I put a cup of hot tea next to a glass of iced tea and then let the two come to a common temperature. My crime, I said to Jacob, echoes down to the end of time, for there is no way to erase or undo it. But let a black hole swim by and let me drop the hot tea and the cold tea into it. Then is not all evidence of my crime erased forever? This remark was all that Jacob needed. He took it seriously and went away to think about it.

There are several notable and, frankly, puzzling things about this passage. First, Wheeler’s 1990 “concern” has escalated into “feeling like a criminal” because of his facilitation of an increase of entropy. I read this passage to my wife, who is not a scientist. She asked me why did he feel like a criminal, if he was not breaking any laws? Good question. In fact, the second law of thermodynamics was fully upheld when the heat flowed from hot to cold, and the entropy increased. The crime, if there were one, would have been dropping the teacups into the black hole—then he might be breaking a law, namely the second law. Next, notice that the cold cup of tea has now become a glass of iced tea, further complicating the situation. I mean, wouldn’t the iced tea really absorb more heat from the room temperature air than from the hot tea? From a physics point of view, however, the most interesting thing to me is that Wheeler said that he and others thought at the time that, since a black hole has zero temperature, it must have zero entropy. I guess Wheeler was thinking a zero temperature black hole was like a zero temperature material system which, if it had a unique ground state, would have zero entropy. But, unlike a material system, the black hole could absorb energy and still remain in equilibrium at zero temperature. One should therefore have thought it would have infinite entropy, because any time it absorbed heat $Q$ its entropy would rise by the infinite amount $Q/0$.

Had I been there with Wheeler in his office that day, I think I would have said “What’s the problem? The entropy went down into the black hole, so what?” And I may have tried to argue that since the black hole has zero temperature, it has infinite entropy, and therefore the second law, including the black hole entropy, is upheld. But, as Jacob appreciated, the problem is deeper than that: what good is the second law if you can’t use it? It presumably applies to the accessible world, and nothing that goes on inside the black hole is accessible to those who remain outside. Moreover, Jacob believed that, like all other entropy, black hole entropy must be a measure of missing information. If a black hole really had infinite changes of entropy, then in any process an infinite amount of information would be lost, which sounds absurd. It should be possible to lose a finite amount of information, that is, to add a finite amount of entropy. This argument suggests that a black hole
therefore must have a nonzero temperature. But Jacob didn’t make the argument this way.

In an interview in the Israeli newspaper Haaretz [from June, 2012] Jacob spoke of his feelings about the “laws and order of physics”. He is quoted there as saying: “I get a sense of security that not everything is random and that I can actually understand and not be surprised by things.” Thermodynamics is the order that emerges from underlying randomness. I guess that in seizing upon Wheeler’s question, Jacob was on a quest to tame the randomness of the world. He insisted on the second law, even when black holes are present, somehow knowing in his heart that the law must be truly fundamental.

The only hope for saving the second law in the presence of a black hole was to assign a finite entropy to the black hole itself. Roger Penrose had shown in 1969 that rotational energy can be extracted from a black hole [6][7], and together with R. M. Floyd found in 1971 that, exploiting the fissioning of a body into two fragments, one with negative Killing energy and the other with positive Killing energy, such energy extraction always results in an increase of the horizon area [8]. In parallel, Demetrios Christodoulou had shown in 1970 that in such processes a quantity he called the irreducible mass—which is in fact proportional to the horizon area—cannot decrease [9]. And in 1971 Stephen Hawking proved a general theorem demonstrating that the horizon area cannot decrease [10].

The obvious analogy of these results with the second law of thermodynamics suggested to Jacob that a black hole be assigned an entropy proportional to its horizon area [11]. But entropy is dimensionless (when temperature is assigned units of energy), so it was necessary to divide the horizon area by a universal constant with dimensions of area, and for this only one candidate presented itself: the (tiny) squared Planck length, $\hbar G/c^3$, which is equal to about $(10^{-33}\text{cm})^2$. Jacob remarked that the appearance of $\hbar$ in the entropy “is not totally unexpected”, since “the underlying states of any system are always quantum in nature”, and he proposed the black hole entropy formula

$$S = \frac{\eta A}{(\hbar G/c^3)},$$

where $\eta$ is a dimensionless proportionality constant. As to the value of $\eta$, he wrote that

...it would be somewhat pretentious to attempt to calculate the precise value of the constant $\eta$ without a full understanding of the quantum reality which underlies a “classical” black hole.

Instead, he provided an estimate of $\eta$, using an argument which also provided an independent derivation of the entropy formula [11], using quantum mechanics and the information theoretic interpretation of entropy.
Jacob reasoned this way: The entropy of a black hole should correspond to the information it is hiding. When one additional bit of information is added, the entropy should increase by $\ln 2$, and there must therefore be a smallest nonzero amount of area that can be added with certainty to a black hole horizon, corresponding to one bit of information. In classical physics there is no lower limit. But he showed that, in view of the interplay of the gravitational redshift and the Heisenberg uncertainty relation, there is indeed in quantum physics a smallest area that can be added with certainty, and it has always the universal value $\sim \hbar G/c^3$, regardless of the mass, spin and charge of the black hole. Having thus determined that one bit of additional entropy corresponds to one square Planck length of area, he had established (1) from “first principles”, and deduced that $\eta$ is a number of order unity. He could then compute, using the Clausius relation, the “effective” temperature that should be assigned to a black hole. Were it not for quantum mechanics, the entropy would have been infinite and the temperature would have been zero. With quantum mechanics, the entropy was finite but huge, and the temperature was non-zero but tiny—indeed proportional to $\hbar$ (and in fact equal to the Hawking temperature up to an unknown order unity factor).

With this understanding in hand, Jacob proposed the generalized second law (GSL) of thermodynamics: the sum of the ordinary entropy outside the black hole

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$^{10}$ Let me briefly explain in this technical footnote the argument just described. It made use of the first law of black hole mechanics which, as far as I know, was first derived by Jacob. He found that law simply by varying the mass, angular momentum, and charge parameters in the formula for the horizon area of a Kerr-Newman black hole. The result, with zero charge variation, can be written as $\delta A = (8\pi G/\kappa)(\delta M - \Omega_H \delta J)$, in units with $c = 1$. Here $M$ is the mass, $J$ is the angular momentum, $\Omega_H$ is the angular velocity of the horizon, and $\kappa$ is the surface gravity. Jacob had a formula for $\kappa$ in terms of the mass, spin and charge of the black hole, but he did not then know that this quantity is the surface gravity, which is an “intensive variable” that is constant on the horizon. That was shown by Bardeen, Carter and Hawking in 1973 [12], who called it the “Zeroth law”. To minimize $\delta A$, one should thus minimize $\delta M - \Omega_H \delta J$, which is the variation of the conserved quantity corresponding to the horizon-generating Killing vector $\chi = \partial_t + \Omega_H \partial_\phi$. I’ll call this conserved quantity the “boost energy”, because on the horizon this Killing vector behaves like a Lorentz boost on a Rindler horizon in Minkowski spacetime. Jacob thus considered a particle of mass $m$ outside the horizon, with 4-velocity parallel to $\chi$, where its boost energy is equal to $|\chi| m$, with $|\chi|$ the norm of $\chi$. This vanishes if the mass vanishes, or if the particle sits on the horizon, where $|\chi| = 0$. But the uncertainty relation precludes using vanishing boost energy to make $\delta A$ vanish: If the particle is localized a distance $d$ from the horizon (along the hypersurface orthogonal to $\chi$), then $d \gtrsim h/m$. If the mass is very small, then it is totally delocalized and cannot be known to have entered the black hole. On the other hand, if $h/m$ is much smaller than the black hole, we can use the approximation $|\chi| \approx \kappa d$, so that the conserved quantity is $(\kappa d)m \gtrsim \hbar c$. Referring back to the first law, this implies that $\delta A \gtrsim (8\pi G/\kappa)(\hbar c) = 8\pi \hbar G = 8\pi (\text{Planck length})^2$. The smallest area increase is thus independent of the mass and spin of the black hole. The black hole entropy is therefore given by $S \sim A \ln 2/8\pi \hbar G$, corresponding to $\eta \sim \ln 2/8\pi \ln h$. The temperature is $T_{bh} \sim \hbar c/\ln 2$ (which is $2\pi/\ln 2$ times the Hawking temperature, $T_H = \hbar c/2\pi$). Notice that the minimal boost energy that can be added with certainty to the black hole coincides with $\sim T_{bh}$. 

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and the black hole entropy \( S \) can never decrease. To support this proposal, he argued that the black hole entropy should represent the inaccessible information about how the black hole might have formed, so that if a body with common entropy \( S_c \) is dropped into the black hole, \( S_{bh} \) must increase by at least \( S_c \), implying that the generalized entropy must increase. This argument makes sense provided its premise, that \( S_{bh} \) represents the inaccessible information, is indeed valid. Of course he had no justification for that premise other than his deep insight and intuition that black hole thermodynamics should make sense. But he then tested the GSL, without assuming that premise. The first test involved a nonrelativistic oscillator enclosed in a box at finite temperature and dropped into a black hole, and he showed that the generalized entropy increases. The second test involved a beam of thermal radiation aimed at the black hole.

Now the story takes a surprising twist. Of course the black hole absorbs radiation from the thermal beam and grows, increasing its entropy. But there is a problem: If the beam temperature is lower than \( T_{bh} \), then heat flows from cold to hot! The GSL is violated because the beam entropy decreases by more than the black hole entropy increases.

In retrospect, we can see that to save the GSL there was only one way out: to suppose that the “effective” temperature \( T_{bh} \) of the black hole is in fact an actual temperature, so that the black hole must emit thermal radiation, and therefore entropy. However, Jacob did not make that argument. In his PRD paper he instead pointed out that, if the bath were colder than the black hole, the entropy-decreasing process of absorbing radiation would involve stochastic fluctuations, because the wavelength of the thermal radiation would be much larger than the black hole. And then came a crucial error: he asserted that in the domain of such fluctuations “the second law is irrelevant.” His deeply felt, and correct, conviction—that the generalized second law must hold—led him to dismiss a counter example using a spurious argument.

Why did Jacob not instead infer that a black hole must have a real temperature, proportional to \( h \)? I am fascinated and puzzled by this. I’m sure he understood these matters very well. While the second law does not hold at the level of individual fluctuations, it holds on average, as he himself stated in the very same paper. So, why did he not predict black hole radiation in order to save the GSL?

Jacob stated in the PRD article that to assign a black hole a real temperature “can easily lead to all sorts of paradoxes” . . . but he doesn’t say what those paradoxes might be. He states in his book, Buchi Neri (apparently published only in Italian), that he wrote up the article in between lectures at the famous Les Houches summer school in 1972. He recalls a long talk he had with Brandon Carter there, writing:
He stressed that since a black hole absorbs radiation perfectly, there can be no option but to assign it zero temperature, which contradicted my claim that the black hole has a real temperature. I almost lost my confidence then, and this accounts for the somewhat weak statement about black hole temperature I make in my paper...

But while it is perhaps true that nothing can escape from a black hole, there is another, overlooked possibility: radiation could escape from the outside, with no causality violation, as Hawking discovered in 1974 [13].

It seems that what was missing from Jacob’s thoughts at the time is just one fundamental point: even in “empty space,” a black hole is immersed in the vacuum of quantum fields. I conjecture that, although he thought a lot about field quanta falling into a black hole, he never thought about the quantum field vacuum falling into a black hole. Had that thought crossed his mind, one day in 1972, I venture to guess that he would have immediately realized that the vacuum fluctuations give the black hole a real temperature, which could uphold the GSL in all settings, even when fluctuations dominate. I’d also guess that Jacob would have gone on to conceive the information-theoretic notion that black hole entropy arises from quantum entanglement of those vacuum fluctuations on either side of the horizon, an idea only suggested ten years later by Rafael Sorkin [14]. As it happened, Hawking discovered this real temperature two years later while investigating other matters entirely. Because Hawking’s calculation doesn’t involve “the quantum reality which underlies a classical black hole,” but rather only the quantum reality of test fields on a classical black hole background, it turned out that Jacob’s pessimism regarding the possibility of computing the proportionality constant $\eta$ was unfounded. Indeed, Hawking’s calculation showed that $\eta = 1/4$.

6 Cosmology and Lemaître

This story of horizons and the entropy of quantum fluctuations can actually be traced back an additional forty years before Jacob’s paper, to the work of Georges Lemaître, a physicist, mathematician, and Roman Catholic priest from Belgium. Although is it not commonly known, in 1933 Lemaître was the first person to understand the nature of a black hole horizon, long before it was called a black hole. He realized that a black hole horizon is locally equivalent to a cosmological de Sitter horizon [15, 16]. Other than Felix Klein, who had pointed to the globally nonsingular de Sitter hyperboloid, Lemaître was the first to understand that one can fall freely without disaster across a de Sitter horizon. He showed this using

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11 de Sitter, Hermann Weyl, and Einstein all thought de Sitter space was somehow singular at the horizon [17].
a coordinate transformation, and he applied the same type of transformation to understand the black hole horizon. As later shown by Gary Gibbons and Hawking [18], de Sitter horizons have an entropy and a temperature, much as do black hole horizons.

Lemaître was ahead of his time in addressing seriously, with all the currently available physics, the questions raised by cosmology. He deduced the distance-redshift relation, and using available data inferred a value for the constant in “Hubble’s law” for the universal expansion before Hubble did [19, 20, 21, 22], argued that without a cosmological constant the universe would be too young to accommodate the geological age of the earth (given the universal expansion rate, which at that time was incorrectly estimated to be around seven times its actual value), and formulated a theory of the origin and growth of structure. Noting that entropy increases according to the second law as the universe evolves, Lemaître believed that there must have been an initial state of zero entropy. He wrote a visionary paper about this in 1931.

Sir Arthur Eddington had mentioned this issue of zero initial entropy, in a 1931 address to the Mathematical Association, titled “The End of the World: from the Standpoint of Mathematical Physics,” and published in the scientific journal Nature. As the title suggests, the main subject was the infinite future and the so-called heat death of the universe. But he also discussed the beginning of the universe, saying: “We have come to an abrupt end of space-time—only we generally call it the beginning”… and he opined, “Philosophically, the notion of a beginning of the present order of Nature is repugnant to me.”

Lemaître responded to this in a brief letter to Nature (May 9, 1931), less than half a page long. On the same page of the journal with a letter from British Columbia on the chemistry of cheese ripening, and another letter with a photograph of insects found in the gut of a cobra in Malaysia, is Lemaître’s letter, titled: “The Beginning of the World from the Point of View of Quantum Theory”. In the letter he traces the universe back to the ground state of a primeval atom. He writes

... it may be that an atomic nucleus must be counted as a unique quantum, the atomic number acting as a kind of quantum number. If the future development of quantum theory happens to turn in that direction, we could conceive the beginning of the universe in the form of a unique atom, the atomic weight of which is the total mass of the universe. This highly unstable atom would divide in smaller and smaller atoms by a kind of super-radioactive process.

Lemaître goes on to suggest that “Some remnant of this process might, according to Sir James Jeans’s idea, foster the heat of the stars until our low atomic number atoms allowed life to be possible.” We now know that in fact the elements evolved
in the *opposite* direction: the actual source of stellar energy is not radioactive fission, but rather the fusion of light elements into heavier ones. And life could evolve only once stars that forged these elements exploded in supernovae, scattering their nuclear progeny for future chemistry. But the notion that the universe we see originated from the decay of a symmetrical, zero entropy quantum state is perfectly in line with our best current understanding.

Looking back at the beginning of time, Lemaître squarely faces what could be called the cosmic information problem: the initial, pure quantum state of his primordial atom would be unique, with no entropy. Quantum mechanics would preserve this purity, so from whence could the variety of all creation be found? Lemaître had no hesitation answering this question. He wrote:

> Clearly the initial quantum could not conceal in itself the whole course of evolution; but, according to the principle of indeterminacy, that is not necessary. [...] the whole story of the world need not have been written down in the first quantum like a song on the disc of a phonograph. The whole matter of the world must have been present at the beginning, but the story it has to tell may be written step by step.

To my eye, this bears an uncanny resemblance to the modern account of primordial cosmology. The primeval atom of Lemaître’s intuition can be identified with the unstable vacuum of inflationary cosmology. The early universe, according to inflation, was a symmetrical, pure quantum state, the de Sitter vacuum, whose vacuum energy spontaneously decayed into a plasma of matter and radiation. The structure we see today emerged from that zero entropy vacuum as a result of quantum indeterminism. It arose via gravitational instability, seeded by tiny deviations from equilibrium which had descended from nothing but primordial vacuum fluctuations.

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