CLASSIFICATION OF SPACES OF CONTINUOUS FUNCTION ON ORDINALS

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Abstract. We conclude the classification of spaces of continuous functions on ordinals carried out by Gorak [5]. This gives a complete topological classification of the spaces \( C_p([0, \alpha]) \) of all continuous real-valued functions on compact segments of ordinals endowed with the topology of pointwise convergence. Moreover, this topological classification of the spaces \( C_p([0, \alpha]) \) completely coincides with their uniform classification.

2010 MSC: 54C35

Keywords: spaces of continuous function, pointwise topology, homeomorphisms, uniform homeomorphisms, ordinal numbers.

1. Introduction

Our terminology basically follows [4]. In particular, we understand cardinals as initial ordinals, compare [4], page 6. A segment of the ordinals \([0, \alpha]\) is endowed with a standard order topology. The symbol \( C_p([0, \alpha]) \) denotes the set of all continuous real-valued functions defined on \([0, \alpha]\) and endowed with the topology of pointwise convergence.

A complete linear topological classification of Banach spaces \( C([0, \alpha]) \) was carried out in [7] and independently in [8] (for the initial part of this classification, see also [3] and [9]). Similar complete linear topological classification for \( C_p([0, \alpha]) \) can be found in [6, 2].

The topological classification of the spaces \( C_p([0, \alpha]) \) is carried out in the Gorak’s paper [5], in which the question whether the spaces \( C_p([0, \alpha]) \) and \( C_p([0, \beta]) \) are homeomorphic is solved for all ordinals \( \alpha \) and \( \beta \) with except for the case \( \alpha = k^+ \cdot k, \beta = k^+ \cdot k^+ \), where \( k \) is
the initial ordinal, and \( k^+ \) is the smallest initial ordinal greater than \( k \). We note that an ordinal of the form \( k^+ \) is always regular ordinal.

In this paper we prove the following theorem.

**Theorem 1.** Let \( \tau \) be an arbitrary initial regular ordinal, \( \sigma \) and \( \lambda \) be initial ordinals satisfying the inequality \( \omega \leq \sigma < \lambda \leq \tau \). Then the space \( C_p([0, \tau \cdot \sigma]) \) is not homeomorphic to the space \( C_p([0, \tau \cdot \lambda]) \).

If we combine this result with the results of [5], we get a complete topological classification of the spaces \( C_p([0, \alpha]) \) (which coincides with the uniform classification). We can write it in the form of the following theorem.

**Theorem 2.** Let \( \alpha \) and \( \beta \) be ordinals and \( \alpha \leq \beta \).

(a) If \( |\alpha| \neq |\beta| \), then \( C_p([0, \alpha]) \) and \( C_p([0, \beta]) \) are not homeomorphic.

(b) If \( \tau \) is an initial ordinal, \( |\alpha| = |\beta| = \tau \) and either \( \tau = \omega \) or \( \tau \) is a singular ordinal or \( \beta \geq \alpha \geq \tau^2 \), then the spaces \( C_p([0, \alpha]) \) and \( C_p([0, \beta]) \) are (uniformly) homeomorphic.

(c) if \( \tau \) is a regular uncountable ordinal and \( \alpha, \beta \in [\tau, \tau^2] \), then the space \( C_p([0, \alpha]) \) is (uniformly) homeomorphic to the space \( C_p([0, \beta]) \) if and only if \( \tau \cdot \sigma \leq \alpha \leq \beta < \tau \cdot \sigma^+ \), where \( \sigma \) is the initial ordinal, \( \sigma < \tau \), and \( \sigma^+ \) is the smallest initial ordinal, exceeding \( \sigma \).

2. **Proof of Theorem 1.**

We need some notation and auxiliary statements. For an arbitrary ordinal \( \alpha \) and the initial ordinal \( \lambda \leq \alpha \) we set

\[
A_{\lambda, \alpha} = \{ t \in [0, \alpha] : \chi(t) = |\lambda| \},
\]

where \( \chi(t) \) is the character of the point \( t \in [0, \alpha] \). In particular, \( A_{\omega, \alpha} \) is the set of all limit points of \( t \in [0, \alpha] \), having a countable base of neighborhoods.

Let \( \alpha \) be a limit ordinal. The smallest order type of sets \( A \subset [0, \alpha] \) cofinal in \([0, \alpha]\), is called cofinality of the ordinal \( \alpha \) and denoted by \( \text{cf}(\alpha) \).

It is easy to see that \( |\text{cf}(\alpha)| = \chi(\alpha) \) for the limit ordinal \( \alpha \). The initial ordinal \( \alpha \) is called regular if \( \text{cf}(\alpha) = \alpha \). Otherwise, the initial ordinal is called singular.

The symbol \( D(x) \) denotes the set of points of discontinuity of the function \( x \).

The proof of the following two lemmas is standard (see Example 3.1.27 in [4]).
Lemma 1. Let $\alpha$ be an arbitrary ordinal and let $\tau$ be an initial ordinal such that $\omega < \tau \leq \alpha$, $t_0 \in A_{\tau, \alpha}$ and a function $x: [0, \alpha] \to \mathbb{R}$ is continuous at all points of the set $A_{\omega, \alpha}$. Then there is an ordinal $\gamma < t_0$ such that $x|_{[\gamma, t_0)} = \text{const}$. □

Lemma 2. If a function $x: [0, \alpha] \to \mathbb{R}$ is continuous at all points of the set $A_{\omega, \alpha}$, then the set $D(x)$ is at most countable. □

For the function $x \in \mathbb{R}^{[0, \alpha]}$ and the initial ordinal $\lambda \leq \alpha$ the symbol $G_\lambda(x)$ denotes the family

$$G_\lambda(x) = \left\{ \bigcap_{s \in S} V_s : V_s \text{ is standard nbhd of } x \text{ in } \mathbb{R}^{[0, \alpha]} \text{ and } |S| = |\lambda| \right\}.$$  

The elements of the family $G_\lambda(x)$ will be called $\lambda$-neighborhoods of the function $x$.

For a regular ordinal $\tau \geq \omega_1$ and a initial ordinal $\sigma \leq \tau$ we put

$$M_{\tau, \sigma} = \left\{ x \in \mathbb{R}^{[0, \tau \cdot \sigma]} : x \text{ is continuous at those points } t \in [0, \tau \cdot \sigma], \right.$$ 
for which $\text{cf}(t) < \tau \left\}.$$  

It is clear, that $C([0, \tau \cdot \sigma]) \subseteq M_{\tau, \sigma}$.

Lemma 3. Let $\tau \geq \omega_1$ be an initial regular ordinal and let $\sigma$ be an initial ordinal such that $\sigma \leq \tau$. Then

$$M_{\tau, \sigma} = \left\{ x \in \mathbb{R}^{[0, \tau \cdot \sigma]} : V \cap C_p([0, \tau \cdot \sigma]) \neq \emptyset \text{ for every } V \in G_\lambda(x) \right.$$  
and each $\lambda < \tau \left\}.$$  

Proof. We denote by $L_{\tau, \sigma}$ the right-hand side of the equality and assume that $x \notin M_{\tau, \sigma}$, that is, $x$ is discontinuous at some point $t_0$ for which $\text{cf}(t_0) < \tau$. Since $|\text{cf}(t_0)| = \chi(t_0)$, there exists a base $\{U_j(t_0)\}_{j \in J}$ of neighborhoods of the point $t_0$ such that $|J| < \tau$. Since $x$ is discontinuous at $t_0$, there exists a number $\varepsilon_0 > 0$ such that for each $j \in J$ there is a point $t_j \in U_j(t_0)$ such that $|x(t_j) - x(t_0)| \geq \varepsilon_0$. Let $V = \bigcap \{V(x, t_j, t_0, 1/n) : j \in J, n \in \mathbb{N}\}$, where $V(x, t_j, t_0, 1/n)$ is the standard neighborhood of the function $x$ in the space $\mathbb{R}^{[0, \tau \cdot \sigma]}$. If $y \in V$, then $y(t_j) = x(t_j)$ and $y(t_0) = x(t_0)$. Hence, the function $y$ is discontinuous at the point $t_0$ and then $y \notin C_p([0, \tau \cdot \sigma])$. Thus, $V \cap C_p([0, \tau \cdot \sigma]) = \emptyset$, that is, $x \notin L_{\tau, \sigma}$.

Now let $x \in M_{\tau, \sigma}$, i.e. the function $x$ can be discontinuous only at the points of the set $A_{\tau, \tau, \sigma}$. It is easy to see that the set $A_{\tau, \tau, \sigma}$ has
the form

\[ A_{\tau,\tau} = \{ \tau \cdot (\xi + 1) : 0 \leq \xi < \sigma \}, \]

or

\[ A_{\tau,\tau} = \{ \tau \cdot (\xi + 1) : 0 \leq \xi < \tau \} \cup \{ \tau \cdot \tau \}, \]

if \( \sigma = \tau \).

By Lemma 2, the set \( D(x) \) is at most countable and therefore

\[ A_{\tau,\tau} \cap D(x) = \{ \tau \cdot (\xi_n + 1) : \xi_n < \sigma, n \in \mathbb{N} \}, \]

\[ A_{\tau,\tau} \cap D(x) = \{ \tau \cdot (\xi_n + 1) : \xi_n < \tau, n \in \mathbb{N} \} \cup \{ \tau \cdot \tau \}, \]

if \( \sigma = \tau \).

Let \( \lambda < \tau \) and \( V(x) = \bigcap\{ U(x, \eta, 1/n) : \eta \in S, n \in \mathbb{N} \} \) be a \( \lambda \)-neighbourhood of the point \( x \). Then \( |S| < |\tau| \).

Since the countable set \( A_{\tau,\tau} \cap D(x) \) is not cofinal in the regular ordinal \( \tau \geq \omega_1 \), for each \( n \in \mathbb{N} \) there is an ordinal \( \gamma_n \) such that \( n \geq \gamma_n \cdot (\xi_n + 1) \) and \( (\gamma_n, \tau(\xi_n + 1)) \cap S = \emptyset \). In the case \( \sigma = \tau \) there is also an ordinal \( \tau_0 < \tau^2 \), such that \( (\gamma_0, \tau_0^2) \cap S = \emptyset \) and \( (\gamma_0, \tau_0^2) \cap \{ \tau(\xi_n + 1) \}_n^\infty = \emptyset \).

Consider the function

\[ \bar{x}(t) = \begin{cases} 
  x(\tau(\xi_n + 1)), & \text{if } t \in (\gamma_n, \tau(\xi_n + 1)); \\
  x(\tau^2), & \text{if } t \in (\gamma_0, \tau^2); \\
  x(t), & \text{otherwise}.
\end{cases} \]

It is not difficult to see that the function \( \bar{x} \) is continuous at all points \( t \in [0, \tau \cdot \sigma] \), and since \( \bar{x}|S = x|S \), \( \bar{x} \in V(x) \), that is, \( V(x) \cap C_p([0, \tau \cdot \sigma]) \neq \emptyset \) and therefore \( x \in L_{\tau \sigma} \).

If \( X \) is a Tikhonoff space, then the symbol \( \nu X \) denotes the Hewitt completion of the space \( X \). The proof of the following lemma can be found in [4], p. 218.

**Lemma 4.** If \( \varphi : X \to Y \) is a homeomorphism of Tikhonoff spaces, then there exists a homeomorphism \( \tilde{\varphi} : \nu X \to \nu Y \) such that \( \tilde{\varphi}(x) = \varphi(x) \) for each \( x \in X \).

**Lemma 5.** Let \( \alpha \) be an arbitrary ordinal. Then

\[ \nu(C_p([0, \alpha])) = \{ x \in \mathbb{R}^{[0, \alpha]} : x \text{ is continuous at all points of the set } A_{\omega,\alpha} \}. \]

**Proof.** It is known ([10], p. 382) that for an arbitrary Tikhonov space \( X \) the space \( \nu(C_p(X)) \) coincides with the set of all strictly \( \aleph_0 \)-continuous functions from \( X \) to \( \mathbb{R} \). In this case, the function \( f \in \mathbb{R}^X \) is called strictly \( \aleph_0 \)-continuous ([11]), if for any countable set \( A \subset X \) there is a continuous function \( g \in \mathbb{R}^X \) such that \( f|_A = g|_A \).

Since for each countable set \( A \subset [0, \omega] \), its closure \( \bar{A} \) is also countable, by the Tietze-Uryson theorem we obtain that the set of all
strictly $\aleph_0$-continuous functions in $[0, \alpha]$ in $\mathbb{R}$ coincides with the set of all those functions that are continuous on each countable subset $A \subset [0, \alpha]$. It is easy to see that these are precisely all those functions that are continuous at all points of the set $A_{\omega, \alpha}$.

**Corollary 6.** If $\tau \geq \omega_1$ is the initial regular ordinal and $\sigma \leq \tau$ is the initial ordinal, then $M_{\tau\sigma} \subset \nu(C_p([0, \tau \cdot \sigma]))$.

For the initial ordinal $\sigma$ we denote by $\Gamma_\sigma$ the discrete space of cardinality $|\sigma|$ and consider the space

$$c_0(\Gamma_\sigma) = \{ x \in \mathbb{R}^{\Gamma_\sigma} : \{ t \in \Gamma_\sigma : |x(t)| \geq \varepsilon \} \text{ is finite for any } \varepsilon > 0 \}.$$  

**Lemma 7.** Let $\tau \geq \omega_1$ be an initial regular ordinal, $\sigma \leq \tau$ be an initial ordinal. Then there exists a homeomorphic embedding $f : c_0(\Gamma_\sigma) \to M_{\tau\sigma}$ such that $f(0) = 0$ and $f(x) \in M_{\tau\sigma} \setminus C_p([0, \tau \cdot \sigma])$, if $x \neq 0$.

**Proof.** We enumerate the points of the set $\Gamma_\sigma$ by the ordinals $t \in [0, \sigma)$. Then $\Gamma_\sigma = \{ t_\xi \}_{\xi \in [0, \sigma)}$. For each characteristic function $\chi_{\{t_\xi\}} \in c_0(\Gamma_\sigma)$ we put $f(\chi_{\{t_\xi\}}) = \chi_{\{\tau(\xi+1)\}}$. It is obvious that $\chi_{\{\tau(\xi+1)\}} \in M_{\tau\sigma} \setminus C_p([0, \tau \cdot \sigma])$. It remains to extend the map $f$ in the standard way to the space $c_0(\Gamma_\sigma)$.

**Lemma 8.** Let $\tau \geq \omega_1$ be an initial regular ordinal, $\sigma, \lambda$ be an initial ordinals and $\omega \leq \lambda < \sigma \leq \tau$. If $f : c_0(\Gamma_\sigma) \to M_{\tau\lambda}$ is an injective mapping such that $f(0) = 0$ and $f(x) \in M_{\tau\lambda} \setminus C_p([0, \tau \cdot \lambda])$ for $x \neq 0$, then the map $f$ is not continuous.

**Proof.** Suppose that there exists a continuous map $f : c_0(\Gamma_\sigma) \to M_{\tau\lambda}$ with the above-mentioned properties. As in Lemma 7, let $\Gamma_\sigma = \{ t_\xi \}_{\xi \in [1, \sigma)}$. Since the space $c_0(\Gamma_\sigma)$ is considered in the topology of pointwise convergence, any sequence of the form $\chi_{\{t_{\xi_n}\}}$ converges to zero in this space. Consequently, at each point $\gamma \in [0, \tau \cdot \lambda]$ only a countable number of functions $f(\chi_{\{t_\xi\}})$ is nonzero. Since by the condition $f(\chi_{\{t_\xi\}}) \in M_{\tau\lambda} \setminus C_p([0, \tau \cdot \lambda])$, each function $f(\chi_{\{t_\xi\}})$ is discontinuous at some point of the set $A_{\tau, \tau\lambda} \subset [0, \tau \cdot \lambda]$.

Let

$$B_\gamma = \{ f(\chi_{\{t_\xi\}}) : \gamma \leq \lambda \} \text{ is discontinuous at a point } \tau(\gamma + 1) \in A_{\tau, \tau\lambda}.$$  

Since $\bigcup_{\gamma<\lambda} B_\gamma = f(\{ \chi_{\{t_\xi\}} : \xi < \sigma \})$ and $|\lambda| = |A_{\tau, \tau\lambda}| < |\sigma|$, there is a point $\gamma_0 < \lambda$, such that $|B_{\gamma_0}| = |\sigma|$. Since at the point $\tau(\gamma_0 + 1)$
only a countable number of functions from $B_{\gamma_0}$ are nonzero, without loss of generality we can assume that all functions from $B_{\gamma_0}$ at the point $\tau(\gamma_0 + 1)$ are equal to zero. By Lemma 1, for each function $f(\chi(t_n)) \in B_{\gamma_0}$ there exists an ordinal $\gamma_x < \tau(\gamma_0 + 1)$ such that $f(\chi(t_n))|_{[\gamma_x, \tau(\gamma_0 + 1))] = \text{const} = C_\xi$. Since $|B_{\gamma_0}| = |\sigma| > \omega$, in $B_{\gamma_0}$ there is an uncountable family of functions for which $|C_\xi| \geq \varepsilon_0$. Consider the sequence $\{f(\chi(t_n))\}_{n=1}^\infty$ of such functions and put $\gamma_0 = \sup\{\gamma_x : n = 1, 2, \ldots\}$. Since $\text{cf}(\tau(\gamma_0 + 1)) > \omega$, $\gamma_0 < \tau(\gamma_0 + 1)$ and therefore $|f(\chi(t_n))(t)| \geq \varepsilon_0$ for each $t \in (\gamma_0, \tau(\gamma_0 + 1))$. But this contradicts the fact that the sequence $\{f(\chi(t_n))\}_{n=1}^\infty$ converges pointwise to zero. □

Proof of Theorem 1. Suppose that there exists a homeomorphism $\varphi : C_p([0, \tau \cdot \sigma]) \rightarrow C_p([0, \tau \cdot \lambda])$. We can assume that $\varphi(0) = 0$. By Lemma 4, there exists a homeomorphism $\tilde{\varphi} : \nu(C_p([0, \tau \cdot \sigma])) \rightarrow \nu(C_p([0, \tau \cdot \lambda]))$ such that $\tilde{\varphi}(C_p([0, \tau \cdot \sigma])) = C_p([0, \tau \cdot \lambda])$. By Corollary 6 $M_{\tau \sigma} \subset \nu(C_p([0, \tau \cdot \sigma]))$, and by Lemma 3 $\tilde{\varphi}(M_{\tau \sigma}) = M_{\tau \lambda}$. By Lemma 7 the mapping $\tilde{\varphi} \cdot f : c_0(\Gamma_\sigma) \rightarrow M_{\tau \lambda}$ is continuous, $(\tilde{\varphi} \cdot f)(0) = 0$ and $(\tilde{\varphi} \cdot f)(M_{\tau \sigma}) \subset M_{\tau \lambda} \setminus C_p([0, \tau \cdot \lambda])$ for $x \neq 0$. In this case, the map $\tilde{\varphi}|_{c_0(\Gamma_\sigma)} : c_0(\Gamma_\sigma) \subset M_{\tau \sigma}$ onto the subspace $M_{\tau \lambda}$ such that $\tilde{\varphi}(0) = 0$ and $\tilde{\varphi}(x) \subset M_{\tau \lambda} \setminus C_p([0, \tau \cdot \lambda])$ for $x \neq 0$. But this is impossible by Lemma 8. □

The authors are grateful to the anonymous referee for helpful comments and suggestions to improve the manuscript.

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CLASSIFICATION OF SPACES OF CONTINUOUS FUNCTION ON ORDINALS

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