Shadow and weak gravitational lensing of a rotating regular black hole in a non-minimally coupled Einstein-Yang-Mills theory in the presence of plasma

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Abstract The null geodesics of the regular and rotating magnetically charged black hole in a non-minimally coupled Einstein-Yang-Mills theory surrounded by a plasma medium is studied. The effect of magnetic charge and Yang-Mills parameter on the effective potential and radius of photon orbits has investigated. We then study the shadow of a regular and rotating magnetically charged black hole along with the observables in the presence of the plasma medium. The presence of plasma medium affects the apparent size of the shadow of a regular rotating black hole in comparison with vacuum case. Variation of shadow radius and deformation parameter with Yang-Mills and plasma parameter has examined. Furthermore, the deflection angle of the massless test particles in weak field approximation around this black hole spacetime in the presence of homogeneous plasma medium is also investigated. Finally, we have compared the obtained results with Kerr-Newman and Schwarzschild black hole solutions in general relativity (GR).

1 Introduction

Black holes (BHs) those make an appearance as the solution of Einstein’s field equations (EFEs) are one of the most striking compact objects predicted by general relativity (GR) and other alternative theories of gravity.\textsuperscript{[1–3]} The first direct image of a BH in the centre of galaxy M87 has been observed recently by event horizon telescope (EHT) collaboration using a very long baseline interferometer (VLBI)\textsuperscript{[4–9]}. The discussions on theoretical aspects of a BH shadow began with Synge who introduced the concept of escape cone\textsuperscript{[10]}. Synge has investigated the shape and size of a shadow for a non-rotating BH and also obtained the formula of angular radius assuming a static observer at infinity\textsuperscript{[11]}. Unlike a non-rotating BH, the shadow of a rotating BH does not remain circular and it could be deformed in the presence of a rotation parameter. Bardeen was the first person who extended the concept of shadow to a rotating BH and obtained an accurate procedure to quantify the effect of rotation on a shadow considering Kerr BH. So far, the number of investigations has been performed to enhance our understanding of BH shadows in GR as well as other modified theories of gravity over the past four decades\textsuperscript{[12–41]}.

The study of complex environments outside the BHs, such as plasma medium, the jets or the accretion disk, has become the field of crucial interest for researchers in recent years. Plasma is a dispersive medium and the light rays are usually refracted by the dispersive medium before they reached our eyes. In this context, it is interesting to investigate the effect of plasma medium in the background of BHs as well as other compact objects. The study of the shadow of Schwarzschild and Kerr BH has been established a remarkable examination to probe their properties with a plasma medium\textsuperscript{[42,43]} by applying the Synge method\textsuperscript{[10]}. Further, the behavior of the plasma medium of the same was investigated using a specific approach\textsuperscript{[44]}. In order to investigate whether the presence of plasma around any BH leads to any observational effects, several investigations have been carried out time and again\textsuperscript{[45–53]}.

Gravitational lensing (GL) can provide useful information about the properties of a BH spacetime. Weak GL in which the geometry of spacetime is less favorable has been proven to be one of the powerful tools in astrophysics and cosmology, in testing theories of gravity and in exploring interaction beyond the Einstein-Maxwell theory\textsuperscript{[54–60]}. Synge was the first person to develop a self-consistent approach to the light propagation in the gravitational field, in the presence of a plasma medium\textsuperscript{[61]}. The application of Synge’s general relativistic Hamiltonian theory for the geometric optics has investigated by Bićak and Hadrava\textsuperscript{[62]}. Thereafter, Perlick implements Synge’s theory and introduces a ray optics method in a plasma medium, and obtained the deflection angle of Schwarzschild BH and Kerr BH in presence of spherically distributed plasma medium\textsuperscript{[42]}. Bisnovatyi-Kogan and Tsupko further
investigated nonlinear effects connected with the combined action of gravity and plasma medium by using Synge’s theory [63,64]. Moreover, the effect of plasma medium around BHs on lensing effects has been studied in [45,66,65–77] and references therein. The study of modified theories of gravity (MGT) has been of great interest because it provides numerous static and spherically symmetric BH solutions [18,78–83]. However, due to the complexity of nonlinear partial differential equations, an exact solution of a rotating BH by solving the coupled field equations in any MGT model is still not achieved. In particular, one can obtain the metric of stationary and axis-symmetric BHs by applying the Newman-Janis algorithm (NJA) [84] and its implementation by starting with any static and spherically symmetric spacetime [85]. The non-complexification procedure can be obtained by implementing these modifications to NJA. Subsequently, this method has been extensively used to obtain rotating BH solutions [52,86–95].

In addition to Einstein-Maxwell theory of gravity where the $U(1)$ electromagnetic fields are used to describe the matter [96], a more generalized and geometrically rich picture is however emerged in Einstein-Yang-Mills (EYM) theory of gravity where the solutions of EYM equations with non-Abelian gauge fields (viz SU(2), SU(3)) to describe matter are studied in diverse contexts [97–100]. There also exist a number of BH solutions in EYM theory of gravity [101–105] with an essentially non-Abelian gauge structure. Furthermore, the gauge group SU(2), a shred of observational evidence about the existence of various BH solutions to the EYM theory for any event horizon, is provided in [106]. The numerically rotating BHs in the minimally coupled EYM theory along with non-static spherically symmetric EYM BHs and the slowly rotating non-Abelian BHs have been studied previously in [107–110]. Introducing NJA formalism, a new regular rotating BH solution has been also derived recently with a YM electromagnetic field [111]. Our aim here is to study the shadow and WL to observe the effect of various BH parameters involved on the shadow and lensing in the background of a homogeneous plasma medium.

The organization of the present paper is as follows. In Sect. 3, we have obtained the equations of motion by the Hamilton-Jacobi method around a regular and rotating magnetically charged BH in a non-minimally coupled EYM theory surrounded by a plasma medium. In Sect. 4, we have examined the effective potential and radius of the photon sphere with their pictorial representation. We have obtained the necessary analytic expression for the radius of shadow and obtained various shadow images from different inclination angles and also examined observables in Sect. 5. The analysis of the energy emission rate by taking into account a non-rotating charged BH in EYM theory is also incorporated in subsection of Sect. 5. Section 6, includes a detailed analysis of the deflection angle in a weak-field approximation. Finally, in Sect. 7, we have concluded our results obtained.

2 Properties of a rotating regular BH in non-minimally coupled EYM theory

In Boyer-Lindquist coordinates, a regular and rotating magnetically charged BH with a YM electromagnetic source in the non-minimal EYM theory is characterized by the line element [111],

$$ds^2 = -c^2 \left(1 - \frac{2P(r)r}{\Sigma}\right)dt^2 - 2ac\sin^2\theta \frac{2P(r)r}{\Sigma}drd\phi + \frac{\Sigma}{\Delta}d\sigma^2 + \Sigma d\theta^2 + \frac{\left[(r^2 + a^2)^2 - a^2\Delta\sin^2\theta\right]\sin^2\theta}{\Sigma}d\phi^2,$$

(1)

where

$$P(r) = \frac{r(1 - g(r))}{2},$$

(2)

$$\Sigma = r^2 + a^2\cos^2\theta,$$

(3)

g(r) = 1 + \left(\frac{r^4}{r^4 + 2\lambda}\right)\left(-\frac{2GM}{c^2r} + \frac{GQ^2}{4\pi\epsilon_0 c^2 r^2}\right),$$

(4)

and

$$\Delta(r) = r^2 - \left(\frac{r^4}{r^4 + 2\lambda}\right)\left(-\frac{2GM}{c^2r} + \frac{GQ^2}{4\pi\epsilon_0 c^2 r^2}\right) + a^2.$$  

(5)

Here $\lambda = \sigma Q^2$ is the YM parameter, while $Q$ is the magnetic charge parameter. The parameters $M$ and $a$ represent the total mass and spin of the BH, respectively. The BH metric in Eq. 1 represents an exact solution of the EFEs in Einstein YM theory [111] and reduced to the Kerr-Newman BH with a magnetic charge if $\lambda = 0$, and it further reduces to Schwarzschild BH in GR if $\lambda = 0$ and $Q = 0$. The horizons of this BH spacetime can be obtained by solving $\Delta(r) = 0$, and its ergo-surface can be calculated via $g_{tr} = 0$. The behavior of event horizon and ergo-surface of this BH spacetime recently has been studied detail in [111]. The YM parameter and charge parameter have an evident influence on horizon and ergoregion of the BH. However, it has observed that this solution represents a compact object without horizons and singularities at the centre since beyond the critical value of angular momentum parameter; the horizons of BH no longer exist.

The curvature invariants play an important role to understand the geometric properties of BH spacetime. In order to investigate the curvature singularity in this BH spacetime, we obtain the curvature invariants of a rotating regular BH in a non-minimally coupled EYM theory. The curvature invariant is thus given as
Fig. 1 The variation of Ricci Scalar with radial distance and YM parameter for (a) $Q = 0.3$ and (b) $Q = 0.6$

$$R = \frac{8\lambda r^2 \left[ Q^2 (5r^4 - 6\lambda) + M \left( -6r^5 + 20 (5r^4 - 6\lambda r) \lambda \right) \right]}{(r^2 + a^2 \cos^2 \theta) (r^4 + 2\lambda)^3},$$

(6)

and

$$R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} = \frac{P (r, \cos^2 \theta, M, Q^2, a^2, \lambda)}{(r^2 + a^2 \cos^2 \theta)^6 (r^4 + 2\lambda)^6}.$$

(7)

Here, $P$ represents a finite polynomial of its arguments. Further, the behavior of Ricci scalar with radial distance and YM parameter is shown in Fig. 1, for two distinct values of charge parameter. From Fig. 1, it is clearly seen that the curvature scalar is regular and BH spacetime is singularity free for $\lambda > 0$.

In order to investigate the effect of various parameters on the event horizon, one can analyze the structure of horizon and static limit surface. The corresponding horizon radius of this BH spacetime can be obtained by calculating roots of the following equation

$$\Delta_1(r) = r^2 - \left( \frac{r^4}{r^4 + 2\lambda} \right) \left( -\frac{2M}{r} + \frac{Q^2}{r^2} \right) + a^2 = 0,$$

(8)

and the corresponding static limit surface can also be obtained by $g_{tt} = 0$, i.e.,

$$r^2 g(r) + a^2 \cos^2 \theta = 0.$$

(9)

In Fig. 2, the behavior of $\Delta(r)$ is depicted with radial distance for different values of YM, plasma, charge and spin parameter, respectively. From Fig. 2a, it is clearly observed that for a fix value of $Q$ and $a$ one can obtain two horizons if $\lambda < \lambda_c$ (critical value). However, one can have an extremal BH if the both horizons coincide to each other i.e., $\lambda = \lambda_c$. One of the important aspects of BH geometry is the singularity and in case of going beyond to critical value, the horizon no longer exists and singularity is observed at the centre. Similarly, the effect of other parameters and there corresponding critical value can also examined from the pictorial representation. It is interesting to see that apart from spin parameter beyond any fix value of radial distance, there is no variation observed for different values of $\lambda$ and $Q$. The variation of static limit surface with radial distance for different values of various parameter is depicted in Fig. 3. It turns out that the behavior of static limit surface is almost similar to the horizon.

3 Photon motion around the rotating regular BH in non-minimally coupled EYM theory in the presence of plasma

In this section, we have considered a static inhomogeneous plasma for our further calculations with a refractive index $n$ and the expression for this refractive index was formulated by Synge [11] as expressed below,

$$n^2 = 1 + \frac{p_\mu p^\mu}{(p_\mu u^\mu)^2},$$

(10)

here, $p_\mu$ and $u^\nu$ correspond to the four momentum and four velocity of the massless test particle, respectively. All these expressions reduce to vacuum case when we takes $n = 1$. In order to find out the null geodesic equations for this BH spacetime surrounded by plasma medium, we have used the Hamilton-Jacobi equation below,
Fig. 2 Variation of $\Delta(r)$ with radial distance for different values of parameters involved in the BH spacetime

$$H(x^\mu p_\mu) = \frac{1}{2} \left[ g^{\mu\nu} p_\mu p_\nu - (n^2 - 1)(p_0^2)^2 \right].$$

(11)

Now the equation of motions of the photons for a given spacetime geometry can be defined by the Hamilton-Jacobi equation which is given as

$$H(x^\mu p_\mu) = \frac{1}{2} \left[ g^{\mu\nu} p_\mu p_\nu - (n^2 - 1)(p_0^2)^2 \right].$$

(12)

The equations $\dot{x}^\mu = \partial H/\partial p_\mu$ and $\dot{p}^\mu = \partial H/\partial x_\mu$ define the trajectories of massless particles in the plasma medium as below,

$$\Sigma \frac{dt}{d\sigma} = \frac{r^2 + a^2}{\Delta} \left[ n^2 E (r^2 + a^2) - aL \right] - a(an^2 E sin^2 \theta - L),$$

(13)

$$\Sigma \frac{d\phi}{d\sigma} = \frac{a}{\Delta} \left[ E(r^2 + a^2) - aL \right] - \left( aE - \frac{L}{sin^2 \theta} \right),$$

(14)

$$\Sigma \frac{dr}{d\sigma} = \pm \sqrt{R(r)},$$

(15)

and

$$\Sigma \frac{d\theta}{d\sigma} = \pm \sqrt{\Theta(\theta)},$$

(16)

where

$$R(r) = [X(r)E - aL]^2 - \Delta(r) \left[ K + (L - aE)^2 \right] + X^2(r)(n^2 - 1)E^2.$$ 

(17)
Fig. 3 The variation of static surface limit with radial distance for different values of parameters involved in the BH spacetime

\[
\Theta(\theta) = K + (n^2 - 1)a^2E^2 - L^2cot^2\theta, \tag{18}
\]

with \(X(r) = (r^2 + a^2)\). Here, \(K\) is the separation constant popularly known as Carter constant, while the function \(\Delta(r)\) is defined by Eq. 5. The conserved quantities \(E\) and \(L\), along the axis of symmetry, represent the energy and angular momentum of massless particle at infinity, respectively.

4 Effective potential and photon sphere

Here, first we introduce the refractive index parameter in terms of plasma frequency since the frequency of plasma medium affects the geodesics of a photon passing by a compact object. The refractive index \(n\) is related to the plasma frequency \(\omega_p\) in a specific form which reads as,

\[
n^2 = 1 - \left(\frac{\omega_p}{\omega}\right)^2, \tag{19}
\]

where \(\omega_p\) has the form

\[
\omega_p = \frac{4\pi e^2 N(r)}{m_{e}}. \tag{20}
\]
In this equation, \( e, N(r) \) and \( m_e \) represent the charge, number density and mass of electron in plasma medium, respectively. One can obtain the physically relevant form of \( N(r) \) by the implication of radial law density which is given as,

\[
\left( \frac{\alpha P}{\alpha} \right)^2 = \frac{k}{r^h}, \quad k \geq 0.
\]

The refractive index may therefore read as

\[
n = \sqrt{1 - \frac{k}{r^h}}.
\]

It is well known that the power of radial distance characterizes the different properties of the plasma medium, but for the simplicity, here we choose to work with \( h = 1 \). So that the expression of refractive index takes the simple form

\[
n = \sqrt{1 - \frac{k}{r}}.
\]

The effective radial potential can be directly evaluated by radial photon motion, and it is written as,

\[
V_{\text{eff}} = \frac{1}{r^5} \left[ \Delta(r) \left( K + (L - aE)^2 \right) - X^2(r)(n^2 - 1)E^2 - [X(r)E - aL]^2 \right].
\]

The condition for the unstable circular orbits is given by

\[
V_{\text{eff}}(r)_{r=r_p} = 0, \quad V_{\text{eff}}''(r)_{r=r_p} = 0.
\]

The condition for maximizing \( V_{\text{eff}}(r) \) can however be interpreted as,

\[
V_{\text{eff}}'(r)_{r=r_p} < 0.
\]

The first condition in Eq. 21 leads to,

\[
\left[ \Delta(r) \left( K + (L - aE)^2 \right) - X^2(r)(n^2 - 1)E^2 - [X(r)E - aL]^2 \right]_{r=r_p} = 0,
\]

while the second condition leads to

\[
\frac{8}{r^5} \left[ \Delta(r) \left( K + (L - aE)^2 \right) - 4X^2(r)(n^2 - 1)E^2 - 4[X(r)E - aL]^2 \right]
\]

\[
- \frac{[X(r)E - aL]X'(r) + \Delta'(r) \left( K + (L - aE)^2 \right)}{r^4}
\]

\[
- \frac{2X(r)X'(r)(n^2 - 1)E^2 - 2nn'X^2(r)E^2}{r^4}_{r=r_p} = 0.
\]

The solution of above equation provides the radii of photon orbits. However, the analytical solution of this equation might be a bit cumbersome, so we have solved the above-mentioned equation by numerical method to investigate the effect of plasma on the photon sphere.

The pictorial representation of the variation of effective potential with radial distance for different values of various BH parameters is depicted in Fig. 4. It is observed that the effective potential attains its maximum value with increasing values of the spin and charge parameters. YM parameter also shows the similar behavior; however, the effective potential decreases with decrease in the value of \( \lambda \). One of the interesting results is that the effective potential attains its maximum value such that the turning point does not depend on the different values of spin, charge as well as \( \lambda \). However, in the last case, it can be observed that after the potential attains maxima, as the value of plasma parameter decreases, also the turning point shifts toward the left. Figure 5 represents the behavior of radius of photon orbits with spin and charge parameter for different values of plasma parameter. We have examined that the radii of photon sphere decreases with an increase in spin parameter and same effect has also observed in case of charge parameter. However, in case of charge parameter, the variation of photon orbit is observed maximum as compared to spin parameter.

5 Shadow in the presence of plasma

The position of light source and observer plays a crucial role to understand the shape of the shadow cast by any compact object. The BH shadow is defined as the reason of observer’s sky which is left completely dark if there are light sources continuously distributed everywhere. This property of BH shadow remains unchanged when we examine the same effect from behind the observer \[112\]. One can consider the light rays coming from a source, and an observer is situated at the coordinate \((r_0, \theta_0)\), where \( \theta_0 \) is the inclination angle between the rotation axis of the BH and the line of sight of the observer with limit \( r_0 \rightarrow \infty \). If we define the conserved quantities \( i.e., \) angular momentum and energy parameter along with the separation constant \( K \), one can introduce the parameters
Fig. 4  Variation of effective potential with radial distance for different values of various parameters for $L = 4$ and $E = 0.9$.

Fig. 5  Variation of photon orbit with spin parameter (left panel) and charge parameter (right panel) for different values of plasma parameter.
\( \xi = L/E \) and \( \eta = K/E^2 \). By using the condition for unstable circular orbits, i.e., \( \mathcal{R}(r) = 0 \) and \( d\mathcal{R}(r)/dr = 0 \), we have the following expressions,

\[
[X(r) - a\xi]^2 - \Delta(r) \left[ \eta + (\xi - a)^2 \right] + X^2(r)(a^2 - 1) = 0, \tag{29}
\]

\[
2[X(r) - a\xi]X'(r) - \Delta'(r) \left[ \eta + (\xi - a)^2 \right] + 2X(r)X'(r)(a^2 - 1) + 2n'nX^2(r) = 0. \tag{30}
\]

By solving these two equations simultaneously, one can easily obtain the parameters \( \xi \) and \( \eta \) as,

\[
\xi = \frac{\mathcal{F} - \mathcal{G}}{\mathcal{H}} + \frac{\sqrt{\mathcal{G}^2 + \mathcal{I}}}{\mathcal{H}}, \tag{31}
\]

\[
\eta = \frac{[X(r) - a\xi]^2 + X^2(r)(a^2 - 1) - \Delta(r)(\xi - a)^2}{\Delta(r)}, \tag{32}
\]

where the notations are defined as follows,

\[
\mathcal{H} = \frac{a^2}{\Delta(r)}, \tag{33}
\]

\[
\mathcal{F} = \frac{a^2}{\Delta(r)} X(r), \tag{34}
\]

\[
\mathcal{G} = \frac{a^2}{\Delta(r)} \frac{\Delta(r)X'(r)}{\Delta'(r)}, \tag{35}
\]

and

\[
\mathcal{I} = \frac{n^2X(r)}{\Delta(r)} + \frac{2n^2X(r) - nnX^2(r)}{\Delta'(r)}. \tag{36}
\]

Here, prime denotes the differentiation with respect to radial coordinate \( r \). One can obtain the apparent shape of the BH shadow by using the celestial coordinates as defined below,

\[
\alpha = \lim_{r_0 \to \infty} \left( -r_0^2 \sin \theta_0 \frac{d\phi}{dr}(r_0, \theta_0) \right), \tag{37}
\]

\[
\beta = \lim_{r_0 \to \infty} \left( r_0^2 \frac{d\theta}{dr}(r_0, \theta_0) \right), \tag{38}
\]

where \((r_0, \theta_0)\) are the position coordinates of the observer’s plane. After taking on account the limit, we obtain,

\[
\alpha = -\frac{\xi}{n \sin \theta_0}, \tag{39}
\]

\[
\beta = \pm \sqrt{n + a^2 - n^2a^2 \sin^2 \theta_0 - \xi^2 \cot^2 \theta_0}, \tag{40}
\]

These equations represent a direct relationship between celestial coordinates \((\alpha, \beta)\) and the parameters \( \xi, \eta \). We have studied the variation of BH shadow for the particular choices of parameters involved in this BH spacetime which can be visualized in Figs. 6, 7 and 8, respectively. The behavior of BH shadow from different inclination angle and different values of spin and charge parameter in the presence of plasma medium is represented in Figs. 6 and 7, respectively. From these figures, it is observed that the presence of plasma medium affects the shape and size of BH shadow to be increased gradually. The radius of shadow also depends on the inclination angle, and we have examined that the shadow radius increases with the decrease in inclination angle. In Fig. 7, we have simultaneously shown the effect of spin, magnetic charge and YM parameter by varying these parameters for vacuum case as well as for different values of plasma parameter. It is also investigated that the shadow radius decreases with the increase in spin and magnetic charge parameter, while with the increase of YM parameter, the shadow radius decreases monotonically.

The shadow radius and distortion parameter are two astronomical observables those are useful to analyze the shape of BH shadow in detail. The approximate size of the shadow of BH can be described with the help of shadow radius, while distortion parameter measures the distortion appearing in shadow. In order to examine the shadow radius, we assume a circle passing through the different points, \((\alpha_1, \beta_1), (\alpha_2, \beta_2)\) and \((\alpha_r, 0)\), respectively. The representation of all these points is depicted in Fig. 9. The radius of BH shadow can be calculated through

\[
R_s = \frac{(\alpha_1 - \alpha_r)^2 + \beta_1^2}{2|\alpha_1 - \alpha_r|}, \tag{41}
\]

while the distortion parameter as given as

\[
\delta_s = \frac{(\alpha_\mu - \alpha_r)}{R_s}. \tag{42}
\]
Fig. 6  Shape of shadows casted by a rotating regular BH in a non-minimally coupled EYM theory surrounded by plasma medium for the different values of rotation parameter and the refractive index of homogeneous plasma. The solid (blue) lines represent the vacuum case, while the dotted (black) and dashed (red) lines correspond to plasma parameter $k = 0.5$ and $k = 1.0$, respectively.

In Fig. 9, the points $(\tilde{\alpha}_p, 0)$ and $(\alpha_p, 0)$ intersect the coordinate plane at the opposite side of $(\alpha_r, 0)$. However, the distance between reference circle and the left point of the shadow has represented by the point $(\tilde{\alpha}_p - \alpha_p)$. The variation of the observables $R_s$ and $\delta_s$ with plasma parameter and YM parameter can be seen in Fig. 10. It is observed that the radius of shadow increases with an increase of plasma parameter and the distortion is reduced with the increase of plasma parameter. On other side, in case of YM parameter, the shadow radius decreases with increasing distortion parameter. Furthermore, in Fig. 8, if $\lambda = 0$ then the results automatically reduce to the Kerr-Newman BH case with magnetic charge instead of electric charge and the significant change in shadow radius is clearly visible.
Fig. 7 Shape of shadows casted by a rotating regular BH in a non-minimally coupled EYM theory surrounded by plasma medium for the different values of charge parameter and the refractive index of homogeneous plasma. The solid (blue) lines represent the vacuum case, while the dotted (black) and dashed (red) lines correspond to plasma parameter $k = 0.5$ and $k = 1.0$, respectively.

5.1 Energy emission rate of a non-rotating BH in EYM theory

In this section, we study the energy emission rate of a regular and rotating magnetically charged BH with a YM electromagnetic source in the non-minimal coupled EYM theory. The expression of energy emission rate reads as,

$$\frac{d^2 Z(\omega)}{d\omega dt} = \frac{2\pi^2 \sigma_{\text{lim}}}{\exp\left(\frac{\omega}{T_H}\right) - 1} \omega^3,$$

(43)

where the parameters $Z(\omega)$, $\omega$ and $T_H$ represent the energy, frequency and Hawking temperature, respectively, corresponding to the BH. The expression of limiting constant value $\sigma_{\text{lim}}$ for rotating charged accelerating BH can be expressed as
Fig. 8 Variation of radius of a BH shadow for different values of various parameters. The left panel represents the vacuum case (k=0), middle panel represents plasma medium with $k=0.5$, and right panel represents plasma medium with $k=1$.

$$\sigma_{lim} = \pi R_s^2,$$

(44)

where, $R_s^2$ is the shadow radius of the BH. Therefore, the expression of energy emission rate for this BH spacetime becomes

$$\frac{d^2 Z(\omega)}{d\omega dt} = \frac{2\pi^3 R_s^2}{\omega} \exp\left(\frac{2\pi}{\omega}\right) - 1 \omega^3.$$

(45)

Figure 11 shows the variation of energy emission rate with frequency for different values of charge and plasma parameter. It has been observed that as plasma parameter increases, the rate of energy emission decreases which marks the opposite behavior of plasma when we compare it to Kerr-Newman BH spacetime case in GR. This opposite behavior of rate of energy emission is due to the presence of magnetic charge unlike the usual Kerr-Newman BH which contains electric charge. Further, we have also observed that in the absence of charge parameter the energy emission rate of BH is maximum, and when we consider the charge parameter,
Fig. 9 Illustration of the observables of the BH shadow [47]

Fig. 10 Variation of radius of a BH shadow (left panel) and deformation parameter (right panel) with plasma and YM parameter, respectively

it decreases with an increase in the charge parameter. Apart from these, one of the noticeable point observed here is that the energy emission rate of BH is unaffected due to the presence of YM parameter.

6 Weak field lensing in the presence of plasma

In this section, the study of lensing in weak field limit is performed by considering the non-rotating analogue of this BH spacetime in the weak field approximation which is expressed by the following relation,
Fig. 11 Variation of energy emission rate with frequency for different values of plasma and charge parameter. Here, $\lambda = 0$ for left panel and $\lambda = 0.5$ for right panel

\[
\frac{dE}{d\omega} = g_{\mu\nu} + h_{\mu\nu}, \quad \eta_{\mu\nu} = (-1, 1, 1, 1), \quad h_{\mu\nu} << 1, \quad h_{\mu\nu} \to 0 \text{ under } x^l \to \infty. \tag{46}
\]

Here, $g_{\mu\nu}$ is the metric tensor defined as, $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ and $h_{\mu\nu}$ is the flat spacetime metric $(-1, 1, 1, 1)$ with a small perturbation. The contravariant tensors of the metric are given as

\[
g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \quad h^{\mu\nu} = h_{\mu\nu}. \tag{47}
\]

In the weak plasma strength, taking into account the weak field approximation for propagation of photons along $z$ direction, one can introduce the deflection angle in the following form

\[
\delta_p = -\frac{1}{2} \int_{-\infty}^{\infty} \left( h_{33} + \frac{h_{00} \omega^2}{\omega^2 - \omega_0^2} \right) d\zeta. \tag{48}
\]

In the limit of large radial distance, the metric for BH spacetime can be written as,

\[
ds^2 = ds_0^2 + \left( \frac{2M}{r} - \frac{Q^2}{r^2} + \frac{2(2Mr - Q^2)\lambda}{r^6} \right) dt^2 + \left( \frac{2M}{r} - \frac{Q^2}{r^2} + \frac{2(2Mr - Q^2)\lambda}{r^6} \right) dr^2,
\]

where the flat part of above metric is $ds_0^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$. The components $h_{\mu\nu}$ in terms of Cartesian coordinates are written as,

\[
h_{00} = \left( \frac{R_g}{r} - \frac{Q^2}{r^2} + \frac{2(R_g r - Q^2)\lambda}{r^6} \right),
\]
\[
h_{ik} = \left( \frac{R_g}{r} - \frac{Q^2}{r^2} + \frac{2(R_g r - Q^2)\lambda}{r^6} \right) n_i n_k,
\]
The photon frequency at large radial distance is given as,

\[ \omega^2 = \left(1 - \frac{\omega_\infty^2}{\omega^2}\right)^{-1} \simeq 1 + \frac{4\pi e^2 N_0\rho_0}{m\omega_\infty^2 r} - \frac{4\pi e^2 N_0\rho_0 R_g}{m\omega_\infty^2 r^2}. \]  

(54)

Here, \( \omega_\infty \) represents the photon frequency in terms of asymptotic value. One can expand the expression of refractive index in series on the powers of \( 1/r \) within the approximation of the YM, charge and large distance; we have,

\[ n^2 = \left(1 - \frac{\omega_\infty^2}{\omega^2}\right)^{-1} \simeq 1 + \frac{4\pi e^2 N_0\rho_0}{m\omega_\infty^2 r} - \frac{4\pi e^2 N_0\rho_0 R_g}{m\omega_\infty^2 r^2}. \]  

(55)

Using the above approximation, one can calculate the deflection angle of light for this BH spacetime in the presence of plasma medium as,

\[ \delta_{EYM(P)} = \frac{2 R_g}{b} \left[ 1 + \frac{\pi^2 e^2 N_0\rho_0}{m\omega_\infty^2 b} - \frac{4\pi e^2 N_0\rho_0 R_g}{m\omega_\infty^2 b^2} \right] - \frac{Q^2}{4b^2} \left[ 3\pi + \frac{4\pi e^2 N_0\rho_0}{m\omega_\infty^2 b} \left( 8 - \frac{3\pi R_g}{b} \right) \right] \]

\[ + \frac{\lambda(Q^2 - R_g)}{8b^2} \left[ 5\pi + \frac{4\pi e^2 N_0\rho_0}{m\omega_\infty^2 b^2} \left( 15 - \frac{5\pi R_g}{b} \right) \right]. \]  

(56)

Equation 56 represents the deflection angle of EYM BH surrounded by a plasma medium. In this equation, the first term is an additive correction to the gravitational deflection due to plasma medium, while in the absence of plasma medium, the first term gives vacuum gravitational deflection. The second and third terms correspond to the charge parameter and YM parameter in the presence of plasma parameter, respectively. The expression of deflection angle (Eq. 56) reduces to the usual Reissner-Nordström BH case in GR with a magnetic charge in the prescribed limit \( \lambda = 0 \), and it further reduces to Schwarzschild BH if \( \lambda = 0 \) & \( Q = 0 \). It can be observed from Table 1 that the value deflection angle decreases with increase in the value of \( \lambda \). The comparison of these BHs through pictorial representation is depicted in Fig. 12d. It can be easily seen that the presence of additional parameters i.e., YM and charge parameter, in plasma medium leads to a decrease in the deflection angle when we compared it to the Schwarzschild BH in plasma medium. The dependence of the deflection angle on the impact parameter for different values of charge, YM and plasma parameter is demonstrated in Fig. 12. In the upper left panel, the deflection angle increases with an increase in plasma parameter and the deviation of light i.e., photons is smaller in the vacuum case (\( k = 0 \)). In the upper right panel and lower left panel, the deflection

| \( \lambda \) | 0    | 0.1  | 0.2  | 0.3  | 0.4  | 0.5  |
|------|------|------|------|------|------|------|
| \( \delta \) | 1.14286 | 1.07874 | 1.01463 | 0.950515 | 0.886401 | 0.822286 |
| \( \delta \) | 1.13516 | 1.07233 | 1.0095 | 0.946668 | 0.883836 | 0.821004 |
| \( \delta \) | 1.11208 | 1.0531 | 0.994112 | 0.935127 | 0.876142 | 0.817157 |
| \( \delta \) | 1.07361 | 1.02104 | 0.968467 | 0.915893 | 0.86332 | 0.810746 |
| \( \delta \) | 1.01976 | 0.97616 | 0.932563 | 0.889965 | 0.845368 | 0.80177 |

(56)
angle for different values of charge and YM parameters is depicted, respectively. In both cases, we have observed that the deflection angle decreases with an increase in charge as well as the YM parameter. Further, the small difference of the YM parameter shows the large deviation in deflection angle which is the effect of magnetic charge and indicates the photons are reflected abruptly in the plasma medium.

### 7 Conclusions

In this paper, we have studied the effect of homogeneous plasma medium on the shadow and of a rotating BH in a non-minimally coupled EYM theory. First, we have obtained the null geodesic equations by solving the Hamilton-Jacobi equation. The effective potential and radius of the photon sphere are calculated, and their typical variation with different parameters is depicted in Fig. 4. Further, we have derived the necessary analytical expression to obtain the shadow of BH in the plasma medium. The location of the observer is an important aspect of the BH shadow, and we have examined the images of BH shadow for different inclination angles and it is observed that if the observer locates near the equatorial plane, the size of shadow decreases, while the deformation parameter ($\delta_s$) increases. Furthermore, it is observed from Fig. 7 that the presence of plasma medium makes the size of shadow radius enhanced. However, shadow radius decreases with an increase in charge parameter and shows the opposite behavior in the
case of a spin parameter (see Fig. 8). The effect of the YM parameter on BH shadow is depicted in Fig. 9 which has a trivial effect, and in absence of the YM parameter, our results reduce to Kerr-Newman BH case. We have also obtained from the observables that shadow radius decreases with the increase of the YM parameter and the deformation parameter increases significantly. We have further observed that the energy emission rate increases with a decrease in the value of plasma parameter and the emission rate attains maximum value for the vacuum case i.e., $k = 0$. It is noticeable that the appearance of the YM parameter does not affect significantly the rate of energy emission. Finally, we have computed the deflection angle of massless particles by considering the weak field limit around this BH spacetime in the presence of the homogeneous plasma medium and represented them graphically in Fig. 12. It is also investigated that the presence of homogeneous plasma parameter increases the deflection angle along with the impact parameter attaining its saturation, while on the other hand, the deflection angle decreases with an increase in the charge and YM parameter and deviation of photons due to both of these parameters are higher than in the case of plasma medium. The obtained results in Table 2 also reduce to well-known BHs in GR within the prescribed limit as mentioned previously, and the difference between them can be seen in Fig. 12d. In the presence of homogeneous plasma, the deflection angle of this particular BH spacetime in the prescribed limits is compared with other well-known BH spacetime in the GR. The present investigations comprise a more general view of the optical studies of this BH carried out in past in the presence of the plasma medium. Since the astronomical BHs in the universe are generally surrounded by the gases accreting around them, this study will provide a mathematical model to learn about the behavior of photons in these extreme conditions, i.e., in the background of the plasma. This model can further be improvised by comparing the results obtained with the observed data of supermassive BH in M87 galaxy, and it is a matter of separate discussion. We intend to report on this issue in near future.

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Data Availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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