Euclidean scalar Green functions near the black hole and black brane horizons

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Abstract

We discuss approximations of the Riemannian geometry near the horizon. If a $D+1$ dimensional manifold $N$ has a bifurcate Killing horizon then we approximate $N$ by a product of the two dimensional Rindler space $\mathcal{R}_2$ and a $D-1$ dimensional Riemannian manifold $\mathcal{M}$. We obtain approximate formulas for scalar Green functions. We study the behaviour of the Green functions near the horizon and their dimensional reduction. We show that if $\mathcal{M}$ is compact then the Green function near the horizon can be approximated by the Green function of the two-dimensional quantum field theory. The correction term is exponentially small away from the horizon. We extend the results to black brane solutions of supergravity in ten and eleven dimensions. The near horizon geometry can be approximated by $N = AdS_p \times S_q$. We discuss the Euclidean Green functions on $N$ and their behaviour near the horizon.

1 Introduction

The Hawking radiation shows that quantum phenomena accompanying the motion of a quantum particle in a neighborhood of a black hole require a description in the framework of relativistic quantum theory of many particle systems. It seems that quantum field theory supplies a proper method for a treatment of a varying number of particles. Quantum field theory can be defined by means of Green functions. In the Minkowski space the locality and Poincare invariance determine the Green functions and allow a construction of free quantum fields. In the curved space the Green function is not unique. The non-uniqueness can be interpreted as a non-uniqueness of the physical vacuum [1][2]. There is less ambiguity in the definition of the Green function on the Riemannian manifolds
(instead of the physical pseudo-Riemannian ones). The Euclidean approach appeared successful when applied to the construction of quantum fields on the Minkowski space-time [3]. We hope that such an approach will be fruitful in application to a curved background as well. In contradistinction to the Minkowski space-time an analytic continuation of Euclidean fields to quantum fields from the Riemannian metric to the pseudoRiemannian one can be achieved only if the manifold has an additional reflection symmetry [4](for a possible physical relevance of the reflection symmetry see [5]).

The black hole can be defined in a coordinate independent way by the event horizon. The event horizon is a global property of the pseudoRiemannian manifold. It is not easy to see what is its counterpart after an analytic continuation to the Riemannian manifold. Nevertheless, there is a proper substitute: the bifurcate Killing horizon. As proved in [6] a manifold with the Killing horizon can be extended to the manifold with a bifurcate Killing horizon. Moreover, there always exists an extension with the wedge reflection symmetry [7] which seems crucial for an analytic continuation between pseudoRiemannian and Riemannian manifolds. The bifurcate Killing horizon is a local property which can be treated in local coordinates [8]. In local static coordinates close to the bifurcate Killing horizon the metric tensor $g$ tends to zero at the horizon [9]. This property is preserved after a continuation to the Riemannian metric. In sec.2 we approximate the Riemannian manifold $\mathcal{N}$ with the bifurcate Killing horizon as $\mathcal{N} = \mathcal{R}_2 \times M_{D-1}$, where $\mathcal{R}_2$ is the two-dimensional Rindler space and $M$ enters the definition of the bifurcate Killing horizon as an intersection of past and future horizons. There is the well-known example of the approximation in the form of a product: the four-dimensional Schwarzschild solution can be approximated near the horizon by a product of the Rindler space and the two-dimensional sphere. However, we do not restrict ourselves to metrics which are solutions of Einstein gravity.

The Euclidean quantum fields are defined by Green functions. For an approximate metric near the bifurcate Killing horizon we consider an equation for the scalar Green functions. We expand the solution into eigenfunctions of the Laplace-Beltrami operator on $M$. If $M$ is compact without a boundary then the Laplace-Beltrami operator has a discrete spectrum starting from 0 (the zero mode). We show in sec.3 that the higher modes are damped by a tunneling mechanism. As a consequence, the position of the point on the manifold $M$ becomes irrelevant. The Green function near the bifurcate Killing horizon can be well approximated by the Green function of the two-dimensional free field. The splitting of the Green function near the horizon into a product of the two-dimensional function and a function on $M$ has been predicted by Padmanabhan [10]. However, we obtain its exact form. In sec.4 we discuss an application of our earlier results [11] on the dimensional reduction of the Green functions. In sec.5 we apply the method to black brane solutions of string theory [12][13] in 10 and 11 dimensional supergravity which at the horizon have the geometry of $AdS_p \times S_q$. We show that Euclidean quantum field theory on the brane can
be approximated by that on the hyperbolic space (the Euclidean version of AdS). The propagator on the background manifold of the black brane can be applied for a construction of supergravity with the black brane as the vacuum state. Anti-de-Sitter space is the homogeneous space of the conformal group. Hence, at the level of the two-point functions we derive the relation between supergravity and conformal field theory on the boundary of of $AdS_p$ (which is $S_{p-1}$)[14].

2 An approximation at the bifurcate Killing horizon

We consider a $D + 1$ dimensional Riemannian manifold $\mathcal{N}$ with a metric $\sigma_{AB}$ and a bifurcate Killing horizon. This notion assumes a symmetry generated by the Killing vector $\xi$. Then, it is assumed that the Killing vector is orthogonal to a (past oriented) $D$ dimensional hypersurface $\mathcal{H}_A$ and a (future oriented) hypersurface $\mathcal{H}_B$ [8]. The Killing vector $\xi^A$ is vanishing (i.e., $\xi^A \xi_A = 0$) on an intersection of $\mathcal{H}_A$ and $\mathcal{H}_B$ defining a $D - 1$ dimensional surface $\mathcal{M}$ (which can be described as the level surface $f = \text{const}$). The bifurcate Killing horizon implies that the space-time has locally a structure of an accelerated frame, i.e., the structure of the Rindler space [15]. In [6] it is proved that the space-time with a Killing horizon can be extended to a space-time with the bifurcate Killing horizon. Padmanabhan [10] describes such a bifurcate Killing horizon as a transformation from a local Lorentz frame to the local accelerated (Rindler) frame. In [6] it is shown that the extension can be chosen in such a way that the "wedge reflection symmetry" [7] is satisfied. In the local Rindler coordinates the reflection symmetry is $X = (x_0, y, z) \rightarrow \tilde{X} = (x_0, -y, z)$. The symmetry means that the metric $\sigma_{AB}$ splits into a block form (we denote coordinates on $\mathcal{N}$ and its indices by capital letters)

$$ds^2 = \sigma_{AB} dX^A dX^B = g_{00} dx^0 dx^0 + g_{11} dy dy + 2g_{01} dx^0 dy + \sum_{j,k \geq 1} g_{jk} dz^j dz^k$$

The bifurcate Killing horizon distinguishes a two-dimensional subspace of the tangent space. At the bifurcate Killing horizon the two-dimensional metric tensor $g_{ab}$ is degenerate. In the adapted coordinates such that $\xi = \partial_{x_0}$ we have $g_{10} = 0$ and the metric does not depend on $x_0$. Then, $\det[g_{ab}] \rightarrow 0$ at the horizon means that $g_{00}(y = 0, z) = 0$ or $g_{11}(y = 0, z) = 0$. We assume $g_{00}(y = 0, z) = 0$. As $g_{00}$ is non-negative its Taylor expansion must start with $y^2$. Hence, if we neglect the dependence of the two-dimensional metric $g_{ab}$ on $x_0$ and on $z$ then we can write it in the form

$$ds_g^2 = -y^2 (dx^0)^2 + dy^2 + \sum_{j,k \geq 2} g_{jk}(y, z) dz^j dz^k$$

$$\equiv y^2 \left[-(dx^0)^2 + y^{-2}(dy^2 + ds_{D-1}^2)\right] \quad (1)$$
If we neglect the dependence of $g_{jk}$ on $y$ near the horizon then the metric $ds^2_{D-1}$ (denoted $ds^2_M$) can be considered as a metric on the $D - 1$ dimensional surface $\mathcal{M}$ being the common part of $\mathcal{H}_A$ and $\mathcal{H}_B$. Hence, in eq.(1) $\mathcal{N} = \mathcal{R}_2 \times \mathcal{M}$ where $\mathcal{R}_2$ is the two-dimensional Rindler space. As an example of the approximation of the geometry of $\mathcal{N}$ we could consider the four dimensional Schwarzschild black hole when $\mathcal{N} \simeq \mathcal{R}_2 \times S^2$ (quantum theory with such an approximation is discussed in [16]).

We shall work with Euclidean version of the metric (1)

$$ds^2 = y^2(dx^0)^2 + dy^2 + \sum_{j,k \geq 2} g_{jk}(0,z) dz^j dz^k$$

(2)

We consider the Laplace-Beltrami operator

$$\Delta_N = \frac{1}{\sqrt{\sigma}} \partial_A \sigma^{AB} \sqrt{\sigma} \partial_B$$
on $\mathcal{N}$ ($\sigma = \text{det}(\sigma_{AB})$).

In the approximation (2) we have (if $g_{jk}$ is independent of $y$)

$$\Delta_N = y^{-2} \partial_0^2 + y^{-1} \partial_y y \partial_y + \Delta_M = \Delta_R + \Delta_M$$

(3)

where $\Delta_R$ is the Laplace-Beltrami operator on the two-dimensional Rindler space and $\Delta_M$ is the Laplace-Beltrami operator for the metric

$$ds^2_M = \sum_{j,k} g_{jk}(0,z) dz^j dz^k$$

We are interested in the calculation of the Green functions

$$(-\Delta_N + m^2) G^m_N = \frac{1}{\sqrt{\sigma}} \delta$$

(4)

Then, eq.(4) for $\mathcal{N} = \mathcal{R}_2 \times \mathcal{M}$ reads

$$-(\partial_0^2 + y \partial_y y \partial_y + y^2 \Delta_M - y^2 m^2) G^m_N = y \frac{1}{\sqrt{\sigma_M}} \delta$$

(5)

After an exponential change of coordinates

$$y = \exp x_1$$

(6)

eq.(5) takes the form

$$\left(- \partial_0^2 - \partial_1^2 - \exp(2x_1)(\Delta_M - m^2)\right) G^m_N = g_M \delta(x_0 - x'_0) \delta(x_1 - x'_1) \delta(z - z')$$

(7)

If $\mathcal{M}$ is approximated by $R^{D-1}$ then the metric (1) is conformally related to the hyperbolic metric. This relation has its impact on the form of the Green functions (5) as will be seen in secs.3 and 4.
3 Green functions near the bifurcate Killing horizon

We investigate in this section the Green function (5) in $D + 1$ dimensions under the assumption that $\mathcal{M}$ is $D - 1$ dimensional compact manifold without a boundary. We introduce the complete basis of eigenfunctions in the space $L^2(dx_0 dx_1)$ of the remaining two coordinates

$$(-\partial_0^2 - \partial_1^2 + \omega_k^2 \exp(2x_1))\phi_k^E(x_0, x_1) = E\phi_k^E(x_0, x_1)$$

(8)

where

$$\omega_k^2 = \epsilon_k + m^2$$

(9)

In eq.(8) $E$ denotes the set of all the parameters the solution $\phi^E$ depends on. The solutions $\phi$ satisfy the completeness relation

$$\int d\nu(E)\phi_k^E(x_0, x_1)\phi_k^E(x_0', x_1') = \delta(x_0 - x_0')\delta(x_1 - x_1')$$

(10)

with a certain measure $\nu$ and the orthogonality relation

$$\int dx_0 dx_1 \phi_k^E(x_0, x_1)\phi_k'^E(x_0, x_1) = \delta(E - E')$$

(11)

where again the $\delta$ function concerns all parameters characterizing the solution.

If $\mathcal{M}$ is a compact manifold without a boundary then $-\Delta_M$ has a complete discrete set of orthonormal eigenfunctions [17]

$$-\Delta_M u_k = \epsilon_k u_k$$

(12)

satisfying the completeness relation

$$\sum_k \pi_k(z)u_k(z') = g^{-\frac{1}{2}}\delta(z - z')$$

Solutions of Eq.(12) always have a zero mode $u_k = 1$ (we normalize the Riemannian volume element of $\mathcal{M}$ to 1) corresponding to $\epsilon_k = 0$. We expand $\mathcal{G}$ (we drop the index $N$ in $\mathcal{G}$ because there is only one Green function in this section) distinguishing the contribution of the zero mode

$$\mathcal{G}^m(x_0, x_1; z; x_0', x_1') - g_0^m(x_0, x_1; x_0', x_1') = \sum_{k\neq 0} g_k^m(x_0, x_1; x_0', x_1')\pi_k(z)u_k(z')$$

(13)

where

$$g_k^m(x_0, x_1; x_0', x_1') = \int d\nu(E)E^{-1}\phi_k^E(x_0, x_1)\phi_k^E(x_0', x_1')$$

(14)

g_k^m$ is the kernel of the inverse of the operator

$$H = -\partial_0^2 - \partial_1^2 + \omega_k^2 \exp(2x_1)$$
We have moved the zero mode in eq.(13) from the rhs to the lhs. \( g_0^m \) is the solution of the equation

\[
(-\partial_0^2 - \partial_1^2 + m^2 \exp(2x_1))g_0^m = \delta(x_0 - x_0')\delta(x_1 - x_1')
\]  
(16)

Taking the Fourier transform in \( x_0 \) we express eq.(16) in the form

\[
\tilde{H}\tilde{g}_k^m(p_0, x_1; x_1') = \delta(x_1 - x_1')
\]  
(17)

where

\[
\tilde{H} = p_0^2 - \partial_1^2 + \omega_k^2 \exp(2x_1)
\]  
(18)

In order to obtain an approximate estimate on the behaviour of \( \tilde{g}_k^m \) we write \( \tilde{g}_k^m \) in the WKB form

\[
\tilde{g}_k = \exp(-\omega_k W)
\]  
(19)

Then, from eq.(15) for large \( x_1 \)

\[
\omega_k^2(\partial_1 W)^2 = \omega_k^2 \exp(2x_1) + p_0^2
\]  
(20)

Hence, if \( \omega_k \neq 0 \) then \( W \simeq \exp(x_1) \) for large \( x_1 \) showing that \( G^m - g_0^m \) is decaying exponentially fast away from the horizon (if \( m = 0 \) then \( \omega_k > 0 \) if \( \epsilon_k > 0 \)). We study this phenomenon in more detail now. First, we write the solution of eq.(8) in the form

\[
\phi^E_k = \exp(ip_0 x_0)\phi^p_1 k(x_1)
\]  
(21)

where

\[
(-\partial_1^2 + \omega_k^2 \exp(2x_1))\phi^p_1 k = p_1^2 \phi^p_1 k
\]  
(22)

Now, \( E = p_0^2 + p_1^2 \) and \( d\nu = dp_0 dp_1 \) in eqs.(10)-(11). When \( \omega_k = 0 \) then the solution of eq.(22) is the plane wave

\[
\phi^p_1 k = (2\pi)^{-\frac{1}{2}} \exp(ip_1 x_1)
\]

The normalized solution of eq.(22, which behaves like a plane wave with momentum \( p_1 \) for \( x_1 \to -\infty \) and decays exponentially for \( x_1 \to +\infty \), reads

\[
\phi^p_1 k = N_{p_1} K_{i\nu}(\omega_k \exp(x_1))
\]  
(23)

where \( K_{i\nu} \) is the modified Bessel function of the third kind of order \( \nu \) [18].

This solution is inserted into the formulas (13)-(14) for the Green function with the normalization (11)

\[
\int_{-\infty}^{\infty} dx_1 \phi^{p_1}_{k}(x_1)\phi^{' p_1}_{k}(x_1) = \delta(p_1 - p_1')
\]  
(24)
Hence (see [1][19][20]),
\[ N^2_{pi} = p_1 \sinh(\pi p_1) \frac{2}{\pi^2} \]  
(25)

Then, performing the integral over \( p_0 \) in eq.(14)
\[ \int d p_0 \exp(i p_0(x_0 - x'_0))(p_0^2 + p_1^2)^{-1} = \pi|p_1|^{-1} \exp(-|p_1||x_0 - x'_0|) \]
we obtain
\[ G(x_0, x_1, z; x'_0, x'_1) - g_0^m(x_0, x_1; x'_0, x'_1) = \frac{\delta}{\pi^2} \int_0^{\infty} dp_1 \sinh(\pi p_1) \exp(-p_1|x_0 - x'_0|) \sum_{k \neq 0} K_{ip_1}(\omega_k \exp(x_1))K_{ip_1}(\omega_k \exp(x'_1))\pi_k(z)u_k(z') \]
(26)

We expect that the rhs of eq.(26) is decaying fast to zero away from the horizon. Each term on the rhs of eq.(26) is decaying exponentially fast when \( x_1 \to +\infty \) because the Bessel function \( K_\nu \) is decreasing exponentially. We wish to show that the sum is decreasing exponentially as well. This is not a simple problem because we need some estimates on eigenfunctions and eigenvalues of the Laplace-Beltrami operator on \( \mathcal{M} \).

Let us consider the simplest example \( \mathcal{M} = S^2_0 \) (the circle of radius \( a \) which can be related to the three-dimensional BTZ black hole [21]). Then, \( u_k(x_2) = (2\pi a)^{-\frac{1}{2}} \exp(\frac{\delta}{a}kx_2) \) and \( \epsilon_k = a^{-2}k^2 \) (here we denote the coordinate \( z \) of eq.(26) by \( x_2 \)). We assume \( m = 0 \) for simplicity of the argument. Then, the sum over \( k \) can be performed by means of the representation of the Bessel function (if \( m \neq 0 \) then we are unable to do the summation exactly but for our estimates this is not necessary because only large eigenvalues are relevant for large eigenvalues \( \omega_k \simeq \epsilon_k \))
\[ K_{\nu}(u) = \int_0^{\infty} dt \exp(-u \cosh t) \cos(\nu t) \]
(27)

We have
\[ \sum_{k=1}^{\infty} \exp \left( -ku \cosh(t) - ku' \cosh(t') \right) \cos(k(x_2 - x'_2)) = \\
\Re \left( \exp \left( -u \cosh(t) - u' \cosh(t') + \frac{a}{\delta}(x_2 - x'_2) \right) \right) \left( 1 - \exp(-u \cosh(t) - u' \cosh(t') + \frac{a}{\delta}(x_2 - x'_2)) \right)^{-1} \]
(28)

Inserting (28) in eq.(26) and approximating the denominator in eq.(28) by 1 we obtain an asymptotic estimate for large positive \( x_1 \) and \( x'_1 \) (large \( y \) and \( y' \); this is eq.(26) neglecting \( |k| > 1 \))
\[ G(x_0, x_1, x_2; x'_0, x'_1, x'_2) - g_0^m(x_0, x_1; x'_0, x'_1) \simeq \frac{8}{\delta \pi} \cos \left( \frac{x_2}{a} - \frac{x'_2}{a} \right) \int_0^{\infty} dp_1 \sinh(\pi p_1) \exp(-p_1|x_0 - x'_0|)K_{ip_1}(\frac{1}{a} \exp(x_1))K_{ip_1}(\frac{1}{a} \exp(x'_1)) \]
(29)

We can estimate the integral over \( p_1 \) if \( |x_0 - x'_0| > \pi \) (if this condition is not satisfied then the estimates are much more difficult because we must estimate
the behaviour of $K_{ip_1}(\exp(x_1))$ simultaneously for large $p_1$ and large $x_1$. In such a case inserting the asymptotic expansion of the Bessel function we obtain

$$G^0(x_0, x_1, x_2; x_0', x_1', x_2') - g^0_0(x_0, x_1; x_0', x_1')$$

\[
\simeq \frac{1}{\alpha^2\sqrt{2\pi}} \exp\left(-\frac{x_1}{2} - \frac{1}{2}x_1'\right) \cos\left(\frac{x_2}{a} - \frac{x_2'}{a}\right)(|x_0 - x_0'| - \pi)^{-1} \exp\left(-\frac{1}{a}\exp(x_1) - \frac{1}{a}\exp(x_1')\right) \tag{30}\]

where (for $m = 0$)

$$g^0_0(x_0, x_1; x_0', x_1') = -\frac{1}{4\pi} \ln((x_0 - x_0')^2 + (x_1 - x_1')^2)$$

is the solution of the equation

$$(-\partial^2_0 - \partial^2_1)g^0_0 = \delta(x_0 - x_0')\delta(x_1 - x_1') \tag{31}\]

It follows that $G^0$ close to the horizon tends exponentially fast to the Green function for the two-dimensional quantum field theory.

In a similar approach we treat the case $M = S^2_\infty$, where $S^2_\infty$ is the two-dimensional sphere with radius $a$. This case is interesting because it describes the near horizon geometry of the four-dimensional Schwarzschild black hole [22]. The approximate near horizon geometry of the four-dimensional Schwarzschild black hole is $\mathcal{N} = \mathcal{R}_2 \times S^2_\infty$ (in $D + 1$ dimensions this is $\mathcal{N} = \mathcal{R}_2 \times S^2_{D-1}$) where $\mathcal{R}_2$ is the two-dimensional Rindler space. Now, the formula (26) reads

$$G^w_{\mathcal{N}}(x_0, x_1, \theta, \phi; x_0', x_1', \theta', \phi') - g^0_0(x_0, x_1; x_0', x_1') = \frac{1}{\pi^2} \int_0^\infty dp_1 \sinh(p_1) \exp(-p_1|x_0 - x_0'|)$$

\[
\sum_{l \neq 0} K_{ip_1}(\omega_l \exp(x_1))K_{ip_1}(\omega_l \exp(x_1'))(2l + 1)P_l(\cos^{1/2}_2) \tag{32}\]

where $\omega^2_l = l(l+1)a^{-2} + m^2$, $P_l$ is the Legendre polynomial and $\sigma$ is the geodesic distance on $S^2_\infty$. In the spherical angles $(\theta, \phi)$

$$(\cos^2_a)(\theta, \phi; \theta', \phi') = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi') \tag{33}\]

A finite number of terms on the rhs of eq.(32) is decaying exponentially in $y = \exp x_1$ (because $K_{ip_1}$ is decaying exponentially). Hence, it is sufficient to estimate the sum in eq.(32) for $l \geq L$. For large $l$ we can use the approximation $\omega_l \approx \frac{1}{a}$. Applying the representation of the Legendre polynomials

$$P_l(\cos \theta) = \frac{1}{\pi} \int_0^\pi \left(\cos \theta + i \sin \theta \cos \phi\right)^l d\phi \tag{34}\]

and the representation (27) of the Bessel functions we obtain

$$\sum_{l \geq L} \gamma^l(2l + 1) \exp\left(-\frac{l}{a}\exp(x_1) \cosh(t) - \frac{l}{a}\exp(x_1') \cosh(t')\right)$$

\[
= \Gamma^L\left(2L + 1 - (2L - 1)\Gamma\right)(1 - \Gamma)^{-2} \tag{35}\]

where
\[ \Gamma = \gamma \exp \left( - \frac{1}{a} \exp(x_1) \cosh(t) - \frac{1}{a} \exp(x_1') \cosh(t') \right) \]

and
\[ \gamma = \cos \sigma + i \sin \frac{\sigma}{a} \cos \phi \]  
\[ (36) \]

If in \((1 - \Gamma)^{-2}\) we neglect \(\Gamma\) (or expand it in a power series of \(\Gamma\)) and apply the asymptotic expansion for the Bessel functions then we can conclude that the sum starting from \(L\) is decaying as \(\exp(-\frac{L}{a} \exp(x_1) - \frac{L}{a} \exp(x_1'))\) for large positive \(x_1\). Therefore, the behaviour of the rhs of eq.(26) for large positive \(x_1\) is determined by the lowest non-zero eigenvalue. Taking only the term with the lowest non-zero eigenvalue we obtain similarly as in eq.(30) (if \(|x_0 - x_0'| > \pi\)) the approximation
\[ g^m(x_0, x_1, \theta, \phi; x'_0, x'_1, \theta', \phi') = g^m_0(x_0, x_1; x'_0, x'_1) \]
\[ \approx \frac{2a}{\pi} \exp(-\frac{1}{2}x_1 - \frac{1}{2}x_1') \]
\[ (|x_0 - x'_0| - \pi)^{-1} \exp \left( - \sqrt{m^2 + \frac{2}{a^2} (\exp(x_1) + \exp(x_1'))} \right) \cos \frac{2\pi}{a} \]  
\[ (37) \]

where \(g^m_0\) is the solution of eq.(16).

For general compact manifolds \(\mathcal{M}\) we must apply some approximations in order to estimate the infinite sums. We estimate the rhs of eq.(13) for large \(x_1\) and \(x'_1\) by means of a simplified argument applicable when \(z = z'\), \(x_1 = x'_1\) and \(|u(z)| \leq C\). Then,
\[ \left| G^m(x_0, x_1, z; x'_0, x_1, z) - g^m_0(x_0, x_1; x'_0, x_1) \right| \]
\[ \leq C^2 \frac{4}{\pi} \int_0^\infty dp_1 \sinh(\pi p_1) \exp(-p_1|x_0 - x'_0|) \left| K_{ip_1}(\omega_k \exp(x_1)) \right|^2 \]  
\[ (38) \]

Let \(\epsilon_1\) be the lowest non-zero eigenvalue. The finite sum on the rhs of eq.(38) is decreasing as \(\exp(-2\sqrt{m^2 + \epsilon_1 \exp(x_1)})\). For this reason we can begin the sum starting from large eigenvalues. For large eigenvalues \((\epsilon_k \geq n)\) with \(n\) sufficiently large we can apply the Weyl approximation for the eigenvalues distribution [23] with the conclusion
\[ \left| G^m(x_0, x_1, z; x'_0, x_1, z) - g^m_0(x_0, x_1; x'_0, x_1) \right| \]
\[ \leq C^2 \frac{4}{\pi} \int_0^\infty dp_1 \sinh(\pi p_1) \exp(-p_1|x_0 - x'_0|) \sum_{\delta \leq \epsilon_k \leq \eta} \left| K_{ip_1}(\omega_k \exp(x_1)) \right|^2 \]
\[ + R \int_0^\infty dp_1 \sinh(\pi p_1) \exp(-p_1|x_0 - x'_0|) \int_{|k| \geq \sqrt{m}} dk \left| K_{ip_1}(\sqrt{|k|^2 + m^2} \exp(x_1)) \right|^2 \]  
\[ (39) \]

The finite sum as well as the integral on the rhs of eq.(39) are decaying exponentially in \(x_1\) with the rate \(\sqrt{\epsilon_1 + m^2}\) whereas for \(g^m_0\) we can get the estimate
\[ \left| g^m_0(x_0, x_1; x'_0, x_1) \right| \leq K \exp \left( -m|x_0 - x'_0| - m \exp(x_1) \right) \]
for non-negative \(x_1\). Hence, for any \(m \geq 0\) the rhs of eq.(13) is decreasing to zero faster than both \(G^m\) and \(g^m_0\).
4 Green functions on a product manifold

In secs. 2 and 3 we have approximated a manifold with a bifurcate Killing horizon by a product manifold and discussed the Green functions in such an approximation. In this section we consider Green functions on product manifolds in a formulation based on our earlier paper [11]. Here, we emphasize some methods which have applications to the black brane solutions to be discussed in the next section.

Let us consider a manifold in the form of a product
\[ N = K \times M \]
where \( K \) has \( D - d + 1 \) dimensions and \( M \) is a \( d \) dimensional manifold. The metric on \( N \) takes the form
\[ ds^2 = \sigma_{AB} dX^A dX^B = g_{ab}(w) dw^a dw^b + h_{jk}(z) dz^j dz^k \]  (40)
where the coordinates on \( N \) are denoted by the capital \( X = (w, z) \), the ones on \( K \) by \( w \) and the coordinates on \( M \) are denoted by \( z \). A solution of eq.(4) can be expressed by the fundamental solution of the diffusion equation
\[ \frac{\partial}{\partial \tau} P^N_\tau = \frac{1}{2} \Delta_N P^N_\tau \]  (41)
with the initial condition \( P_0(X, X') = \sigma^{-\frac{1}{2}} \delta(X - X') \). Then
\[ G^m_N(X, X') = \frac{1}{2} \int_0^\infty d\tau \exp(-\frac{1}{2} m^2 \tau) P^N_\tau \]  (42)
We may write eq.(4) in the form (here \( h_M = \det(h_{jk}) \) and \( g = \det(g_{ab}) \))
\[ (-\Delta_N + m^2) G^m_N = (-\Delta_M + m^2 - \Delta_K) G^m_N = h_M^{-\frac{1}{2}} g^{-\frac{1}{2}} \delta(X - X') \]  (43)
From eq.(43) we have a simple formula (in the sense of a product of semigroups)
\[ P^N_\tau = P^K_\tau P^M_\tau \]  (44)
where the upper index of the heat kernel denotes the manifold of its definition. Hence
\[ G^m_N(X, X') = \frac{1}{2} \int_0^\infty d\tau \exp(-\frac{1}{2} m^2 \tau) P^K_\tau (w, w') P^M_\tau (z, z') \]  (45)

We expand the Green function (distinguishing the zero mode) in eigenfunctions \( u_k (12) \) of the Laplace-Beltrami operator \( \Delta_M \)
\[ G^m_N(X, X') - g^m_0(w, w') = \sum_{k \neq 0} g^m_k(w, w') \bar{u}_k(z) u_k(z') \]  (46)
\( g^m_k \) is a solution of the equation
\[ A_k g^m_k(w, w') = \left( \sqrt{g} \omega_k^2 - \partial_a g^{ab} \sqrt{g} \partial_b \right) g^m_k = \delta(w - w') \]  (47)
where $\omega_k$ is defined in eq.(9). The zero mode $g^m_0$ is the solution of the equation
\[
(- \partial_a g^{ab} \sqrt{g} \partial_b + m^2 \sqrt{g}) g^m_0 = \delta(w - w')
\] (48)

As in sec.3 we ask the question whether $G^m_N$ can be approximated by $g^m_0$. As a first rough approximation for large distances we apply the WKB representation expressing the Green function $g_k$ (47) in the form
\[
g^m_k(w, w') = \exp(-\omega_k W(w, w'))
\] (49)

Assuming that $W$ is growing uniformly in each direction we obtain in the leading order for large distances the equation
\[
1 = g^{ab} \partial_a W \partial_b W
\] (50)

for $W$. Eq.(50) is an equation for the geodesic distance $\sigma_K$ on the manifold $K$ with the metric $g_{ab}$ [24]. Hence, the geodesic distance $W(w, w') = \sigma_K(w, w')$ is the solution of eq.(50) which is symmetric under the exchange of the points and satisfies the boundary condition $W(w, w) = 0$. We insert the approximate solutions $g^m_k$ (49) into the sum (46). Then, we can express the sum by the heat kernel of $\sqrt{-\triangle_M + m^2}$ or the heat kernel of $-\triangle_M + m^2$

\[
\mathcal{G}^m_N(X, X') - g_0(w, w') \simeq \sum_k \exp(-\omega_k W(w, w')) \overline{u}_k(z) u_k(z') \\
= \exp(-\sqrt{-\triangle_M + m^2} W(w, w'))(z, z') \\
= (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} d\theta \exp\left(-\frac{\theta^2}{2}\right) \exp\left(-\frac{W^2(w, w')}{2\theta^2} (-\triangle_M + m^2)\right) (z, z')
\] (51)

Eq.(51) gives a better approximation than eq.(49) (together with eq.(50)) of the Green function on $N$ because the sum over eigenvalues $\epsilon_k$ has been performed. If the heat kernel on $M$ is known (to be discussed in the next section) then from eq.(51) we can obtain an approximation for $G^m_N$.

## 5 Green functions for approximate geometries of black holes and black branes

In sec.3 we have derived an approximation for the Riemannian metric near the bifurcate Killing horizon. We could see that at the horizon $N \simeq R_2 \times M$. In the case of the Schwarzschild black hole in four dimensions we have $N \simeq R_2 \times S^2$. We are interested in this section also in the black brane solutions in 10 dimensional supergravity which have the near horizon geometry $N \simeq AdS_5 \times S^5$ and the black brane solutions in 11 dimensional supergravity with the near horizon geometry $N \simeq AdS_7 \times S^4$ or $N \simeq AdS_4 \times S^7$ [13]. These manifolds can be considered as solutions to the string theory compactification problem.
We can see that for a description of the black hole (black brane) Green functions the formulas for the heat kernel on the Rindler space and AdS space will be useful. We shall set the radius of the black hole (brane) \( a = 1 \) in this section. The radius can be inserted in our formulas by a restoration of proper dimensionality of the numbers entering these formulas (as will be indicated further on). For the Rindler space the heat kernel equation reads

\[
\partial_\tau P^R_\tau = \frac{1}{2} \Delta_R P^R_\tau = \frac{1}{2}(y^{-2}\partial_0^2 + y^{-1}\partial_y y\partial_y)P^R_\tau
\]

If we take the Fourier transform in \( x_0 \) in eq.(52) then we obtain an equation for the Bessel function \( I_{|p_0|} \) [18]. Hence, we can see that the solution of the heat kernel for \( R^2 \) with the initial condition \((yy')^{-\frac{1}{2}}\delta\) can be expressed in the form

\[
P^R_\tau(x, y; x', y') = \frac{1}{\pi \tau} (2\pi)^{-\frac{1}{2}} \int dp_0 \exp(ip_0(x_0-x'_0))I_{|p_0|}(\tau^{-1}yy') \exp\left(-\frac{1}{2\tau}(y^2+y'^2)\right)
\]

Eq.(52) for the heat kernel \( P^2_\tau \) on \( R^2 \) coincides with the heat equation on the plane but expressed in cylindrical coordinates, i.e., if

\[
w_0 = y \cos x_0, w_1 = y \sin x_0
\]

then

\[
\Delta_R = \frac{\partial^2}{\partial w_0^2} + \frac{\partial^2}{\partial w_1^2}
\]

As a consequence the heat kernel \( P^{2\pi}_\tau \) with periodic boundary conditions imposed on \( x_0 \) (with the period \( 2\pi \) ) must coincide with the heat kernel on the plane \( R^2 \) (see also [25]). Hence,

\[
P^{2\pi}_\tau = (2\pi\tau)^{-1} \exp\left(-\frac{1}{2\tau}|w-w'|^2\right)
\]

where

\[
|w-w'|^2 = (w_0-w'_0)^2 + (w_1-w'_1)^2
\]

First, we apply the methods of sec.4 to the case discussed already in another way in sec.3. The approximate near horizon geometry of the Schwarzschild black hole in four dimensions is \( N = R^2 \times S^2 \) (in \( D + 1 \) dimensions this is \( N = R^2 \times S^{D-1} \)). We may apply eq.(45) in order to express the Green function by the heat kernels. For this purpose the eigenfunction expansion on \( S^2 \) is useful

\[
P^S_\tau(\sigma) = \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l+1)P_l(\cos \sigma) \exp\left(-\frac{\tau}{2}l(l+1)\right)
\]

where \( P_l \) is the Legendre polynomial and the geodesic distance \( \sigma \) is defined in eq.(33). Applying the representation (34) of the Legendre polynomials we can
sum up the series (56) and express it by an integral

\[ P^{S_+}_\tau(\sigma) = \frac{1}{4\pi^2}(2\pi\tau)^{-\frac{1}{2}} \int_0^\pi d\phi \int_{-\infty}^\infty du \exp\left(-\frac{u^2}{2\tau}\right) \]

\[ (1 + \Omega)(1 - \Omega)^{-2} \]

(57)

where

\[ \Omega = \exp(iu - \frac{\tau}{2})(\cos \sigma + i \sin \cos \phi) \]

(58)

If we expand \((1 - \Omega)^{-2}\) in \(\Omega\) then we obtain the expansion (56) of the heat kernel.

Applying eqs. (45), (54) and (57) we can represent the scalar Green function on the four-dimensional black hole of temperature \(\beta = 2\pi\) (this is the conventional Hawking temperature in dimensionless units; the \(x_0\) coordinate is made an angular variable in order to make the singular conical manifold regular [26][22][27][28])

\[ G^m_{2\tau}(w, \theta; w', \theta') - g^m_{00}(w, w') = \frac{1}{2\pi} \int_0^\pi d\phi \int_0^{\infty} d\tau (2\pi\tau)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}m^2\tau\right) \]

\[ \exp\left(-\frac{1}{\tau}(|w - w'|^2 + u^2)\right)\Omega(1 - \Omega)^{-2} \]

(59)

where

\[ (\frac{\partial^2}{\partial w_0^2} - \frac{\partial^2}{\partial w_1^2} + m^2)g^m_{00} = \delta(w - w') \]

We obtain an exponential decay of the rhs of eq. (59) if we make the approximation \((1 - \Omega)^{-2} \simeq 1\) and estimate the correction to this approximation. In this way we reach the approximation (37) of sec. 3 but now at the Hawking temperature. For zero temperature or a temperature different from the Hawking temperature it would be difficult to obtain a useful approximation for the Green function from eq. (45) because the formula (53) for the heat kernel on the Rindler space is rather implicit.

We can obtain simple formulas for the Schwarzschild black hole in an odd dimension \(d + 2\). For an odd \(d\) the formula for the heat kernel on \(S_d\) reads [29]

\[ P^S_{\tau}(\sigma) = \frac{1}{4\pi^2}(2\pi\tau)^{-\frac{1}{2}} \int_0^\pi d\phi \int_{-\infty}^\infty du \exp\left(-\frac{u^2}{2\tau}\right) \]

\[ (1 + \Omega)(1 - \Omega)^{-2} \]

\[ \sum_{2n > d - 1} K_0\left((n^2 - \frac{1}{4}(d - 1)^2 + \frac{m^2}{2} |w - w'|)\cos(n\sigma)\right) \]

(60)

(61)
We can use the representation (27) of the Bessel function

\[ K_0(u) = \int_0^\infty dt \exp(-u \cosh(t)) \]

in order to sum the series (61) for large \( n \geq \Lambda \). Then, we can use the approximation \( (n^2 + \frac{1}{2}(d-1)^2 + m^2)^{\frac{d}{2}} \simeq |n| \). In such a case

\[ G^m_{2n}(w, w'; \sigma) - g^0_{2n}(w, w') \simeq \frac{1}{2}(\frac{1}{\pi})^{\frac{d-1}{2}} \left( \frac{\partial}{\partial \cosh \sigma} \right)^{\frac{d+1}{2}} \]

\[ \sum_{2n > d-1} n^\Lambda \cos(n\sigma) K_0 \left( n^2 + \frac{1}{2}(d-1)^2 + m^2 \right)^{\frac{d}{2}} |w - w'| \cos(n\sigma) \]

\[ + \int_0^\infty dt \left( \exp(i\Lambda \sigma - \Lambda |w - w'| \cosh(t)) \left( 1 - \exp(i\sigma - |w - w'| \cosh(t)) \right)^{-1} \right. \]

\[ + \exp(-i\Lambda \sigma - \Lambda |w - w'| \cosh(t)) \left( 1 - \exp(-i\sigma - |w - w'| \cosh(t)) \right)^{-1} \]

(62)

The finite number of terms on the rhs of eq.(62) is decreasing exponentially because \( K_0(u) \) is decreasing exponentially for large \( u \). Then, the integral over \( t \) is decreasing exponentially as in the definition of the Bessel function (27). In order to prove this we make the approximation

\[ \left( 1 - \exp(i\sigma - |w - w'| \cosh(t)) \right)^{-1} \simeq 1 \]

and subsequently estimate the correction to this approximation. We can conclude that the rhs of eq.(62) is decreasing as \( \exp(-\sqrt{m^2 + a^2} |w - w'|) \)

(as \( \exp(-\sqrt{m^2 + a^2} |w - w'|) \) after an insertion of the radius \( a \) of the sphere).

We obtain explicit formulas for scalar Green functions on black brane solutions of supergravity in ten dimensions which near the horizon have the \( AdS_5 \times S_5 \) geometry [12][30][13]. The heat kernel on the 2k + 3 dimensional hyperbolic space \( \mathcal{H}_{2k+3} \) (Euclidean \( AdS_{2k+3} \)) is [31][32]

\[ p_{\mathcal{H}}^{(k+1)}(\sigma_H) = (-2\pi)^{-k-1} \exp(-\frac{(k+1)^2}{2} \tau + \frac{1}{2} \tau) \]

\[ \left( \frac{d}{d \cosh \sigma_H} \right)^k p_{\mathcal{H}}^{(1)}(\sigma_H) \]

(63)

with

\[ p_{\mathcal{H}}^{(1)}(\sigma_H) = (2\pi)^{-\frac{d}{2}} \sigma_H \left( \sinh \sigma_H \right)^{-1} \exp(-\frac{\tau}{2} - \frac{\sigma_H^2}{2\tau}) \]

(64)

It is a function of the Riemannian distance \( \sigma_H \). In the Poincare coordinates \((y, x)\)

\[ \cosh \sigma_H = 1 + (2gy')^{-1}(x - x')^2 + (y - y')^2 \]

The integral (45) over \( \tau \) can be calculated using eqs.(60) and (63). Then, for \( \mathcal{H}_5 \times S_5 \) we obtain

\[ G^m_{5\Sigma}(\sigma_H, \sigma) - g^0_{5\Sigma}(\sigma_H) = \frac{1}{4\pi^2} (2\pi)^{-\frac{d}{2}} \left( \frac{\partial}{\partial \cosh \sigma_H} \right)^2 \frac{\partial}{\partial \cosh \sigma_H} \left( \sinh \sigma_H \right)^{-1} \sum_{n>2} (n^2 + m^2)^{\frac{d}{2}} \exp \left( -\frac{(n^2 + m^2)^{\frac{d}{2}} \sigma_H}{2\tau} \right) \cos(n\sigma) \]

(65)
where $g_{05}^m$ is the Green function on the hyperbolic space $\mathcal{H}_5$. In the Poincare coordinates $g_{05}^m$ is the solution of the equation

$$(-y^2\partial_y^2 + 3y\partial_y - y^2\Delta_x + m^2)g_{05}^m = y^5\delta$$

We have

$$g_{05}^m(\sigma_H) = -\frac{1}{2\pi} \frac{\partial}{\partial \cosh \sigma_H} (\sinh \sigma_H)^{-1} \exp \left( -\sqrt{4 + m^2 \sigma_H} \right) \quad (66)$$

If the sphere has radius $a$ and the hypersphere the radius $b$ then the exponential factor in eq.(66) reads

$$\exp \left( -b\sqrt{4a^{-2} + m^2 \sigma_H} \right)$$

We can sum on the rhs of eq.(65) a finite number of terms and subsequently approximate the remaining series replacing the square root in the argument of the exponential by $|n|$ for large $n$ (the procedure which we applied at eq.(62)). If the mass $m = 0$ then the sum in eq.(65) can be calculated exactly with the result

$$G_{5+5}(\sigma_H, \sigma) - g_{05}^m(\sigma_H) = \left(2\pi^{-2}\right)^{-\frac{m}{2}} \frac{\partial}{\partial \cosh \sigma_H} \left(\frac{\partial}{\partial \cosh \sigma_H}\right)^2 (\sinh \sigma_H)^{-1} \sum_{l>0} \left(l(l+1) + m^2 + 9\right)^\frac{m}{2} \exp \left( -\left(l(l+1) + m^2 + 9\right)^\frac{m}{2} \sigma_H \right) P_l(\sigma) \quad (67)$$

In eleven dimensions the counterpart of the ten dimensional black brane has the near horizon geometry $\text{AdS}_7 \times S_4$. The heat kernel on $S_4$ can be expressed in the form [29]

$$P_{S^4}(\sigma) = \frac{1}{8\pi^2} \frac{d}{d \cos \sigma} \sum_{l=0}^{\infty} (2l + 1) P_l(\cos \sigma) \exp \left( -\frac{\tau}{2} (l + 1) \right) \quad (68)$$

where $\sigma$ is the geodesic distance on $S_4$. Hence, from eq.(45)

$$G_{7+4}(\sigma_H, \sigma) - g_{07}^m(\sigma_H) = \left(\frac{2\pi}{2\pi}\right)^{-\frac{m}{2}} \frac{\partial}{\partial \cosh \sigma_H} \left(\frac{\partial}{\partial \cosh \sigma_H}\right)^2 (\sinh \sigma_H)^{-1} \sum_{l>0} \left(l(l+1) + m^2 + 9\right)^\frac{m}{2} \exp \left( -\left(l(l+1) + m^2 + 9\right)^\frac{m}{2} \sigma_H \right) P_l(\sigma) \quad (69)$$

where $g_{07}^m$ is the Green function on the seven dimensional hyperbolic space $\mathcal{H}_7$. It is the solution of the equation (in Poincare coordinates)

$$(-y^2\partial_y^2 + 3y\partial_y - y^2\Delta_x + m^2)g_{07}^m = y^7\delta \quad (70)$$

The solution of eq.(70) reads

$$g_{07}^m(\sigma_H) = (2\pi)^{-2} \left(\frac{\partial}{\partial \cosh \sigma_H}\right)^2 (\sinh \sigma_H)^{-1} \exp \left( -\sqrt{9 + m^2 \sigma_H} \right) \quad (71)$$
Again we can obtain good approximation for the Green function summing a finite number of terms on the rhs of eq.(69) and subsequently calculating the infinite sum with the approximation \( \left( l(l+1)+m^2+9\right)^{\frac{1}{2}} \approx l. \)

Finally, let us consider the black brane solution of supergravity in eleven dimensions with the near horizon geometry \( \text{AdS}_4 \times S_7. \) The Green functions have a more involved representation in this case because the heat kernel on the even dimensional hyperbolic space cannot be expressed by elementary functions. We have

\[
p_\tau = -\sqrt{2}(2\pi)^{-1} \frac{d}{d \cosh \sigma_H} \exp(-\frac{9}{8} \tau) (2\pi \tau)^{-\frac{3}{2}} \int_{\sigma_H}^\infty (\cosh r - \cosh \sigma_H)^{-\frac{3}{2}} r \exp(-\frac{r^2}{2\tau}) dr
\]

Hence, from eqs.(45) and (65) we obtain

\[
G_{m+7}^4(\sigma_H, \sigma) - g_{04}^m(\sigma_H) = -\sqrt{2}(2\pi)^{-4} \frac{d}{d \cosh \sigma_H} \left( \frac{d}{d \cosh \sigma} \right)^3 \sum_{n>3} \sqrt{-\frac{27}{4} + m^2 + n^2} \int_{\sigma_H}^\infty dr (\cosh r - \cosh \sigma_H)^{-\frac{3}{2}} \exp\left(-r \sqrt{-\frac{27}{4} + m^2 + n^2}\right)
\]

(73)

where

\[
g_{04}^m(\sigma_H) = -2\sqrt{2}(2\pi)^{-1} \frac{d}{d \cosh \sigma_H} \int_{\sigma_H}^\infty (\cosh r - \cosh \sigma_H)^{-\frac{3}{2}} \exp(-r \sqrt{\frac{9}{4} + m^2}) dr
\]

(74)

The integrals over \( r \) in eqs.(72)-(74) can be expressed by the Legendre functions \( Q_\alpha \) using the integral representation [18]

\[
Q_\alpha(\cosh \sigma) = \int_\sigma^\infty dr (2 \cosh r - 2 \cosh \sigma)^{-\frac{1}{2}} \exp(-\alpha r - \frac{1}{2} r)
\]

6 Discussion

It is known that the Riemannian geometry of the spherically symmetric black hole manifold of \( D + 1 \) dimensions near the horizon can be approximated by \( \mathcal{N} = \mathcal{R}_2 \times S_{D-1}. \) The black brane solutions of supergravity and string theory near the horizon can be approximated by a product \( \mathcal{N} = \text{AdS}_p \times S_q. \) Let us denote these manifolds with a horizon by \( \tilde{\mathcal{N}}. \) A question could be raised to what extent the product manifold \( \mathcal{N} = K \times M \) is a good approximation to \( \tilde{\mathcal{N}} \) in the sense that \( |G_N(X, X') - \tilde{G}_N(X, X')| \) is small in a certain range of \( X \) and \( X'. \) Such problems have been studied for the heat kernels in relation to the parametrix method for diffusion equations [33][34]. By these methods we could estimate \( |P_\tau^N - \tilde{P}_\tau^N| \) and subsequently integrate the estimate over \( \tau \) (in the sense of eq.(42)). However, explicit estimates would be difficult and are beyond the scope of this work. We restricted ourselves in secs.3 and 5 to an answer (by different methods) to a simpler problem: whether the Green function on
a product manifold can be approximated by its zero mode. We have obtained explicit formulas for the correction to the zero mode contribution. The zero mode is the Green function on the non-compact part. We have shown that the scalar propagator on the product manifold can indeed be approximated for large distances by the one on the non-compact part of \( \mathcal{N} \). The contribution of the compact part is decreasing exponentially as a function of the distance. This is in fact an expression of the compactification in the Kaluza-Klein setting.

In the case of the Schwarzschild black hole we could call such a phenomenon a dimensional reduction. The compact degrees of freedom become irrelevant for the scalar quantum field theory defined on the black hole background. The \( \mathcal{N} \simeq AdS_p \times S_q \) approximation is usually discussed \([14][30]\) in relation to an approximation by a conformal field theory on the boundary of \( AdS_p \). The form of the propagator on the product of the non-compact and compact manifolds could be useful for a complete reconstruction of the quantum field theory on \( \mathcal{N} \) from the one on the boundary of \( AdS_p \). Some consequences of the near horizon dimensional reduction in the case of the Schwarzschild black hole have been discussed by Padmanabhan \([10]\).

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