LOCAL STRUCTURE THEOREMS FOR SMOOTH MAPS OF FORMAL SCHEMES

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Abstract. We continue our study on infinitesimal lifting properties of maps between locally noetherian formal schemes started in [3]. In this paper, we focus on some properties which arise specifically in the formal context. In this vein, we make a detailed study of the relationship between the infinitesimal lifting properties of a morphism of formal schemes and those of the corresponding maps of usual schemes associated to the directed systems that define the corresponding formal schemes. Among our main results, we obtain the characterization of completion morphisms as pseudo-closed immersions that are flat. Also, the local structure of smooth and étale morphisms between locally noetherian formal schemes is described: the former factors locally as a completion morphism followed by a smooth adic morphism and the latter as a completion morphism followed by an étale adic morphism.

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Introduction

Formal schemes have always been present in the backstage of algebraic geometry but they were rarely studied in a systematic way after the foundational [6, §10]. It has become more and more clear that the wide applicability of formal schemes in several areas of mathematics require such study. Let us cite a few of this applications. The construction of De Rham cohomology for a scheme $X$ of zero characteristic embeddable in a smooth scheme $P$, studied by Hartshorne [10] (and, independently, by Deligne), is defined as the hypercohomology of the completion of the De Rham complex of the formal completion of $P$ along $X$. Formal schemes play a key role in $p$-adic cohomologies (crystalline, rigid . . .) and are also algebraic models of rigid analytic spaces. These developments go back to Grothendieck with further elaborations by Raynaud, in collaboration with Bosch and Lütkebohmert, and later work by Berthelot and de Jong. In a different vein, Strickland [16] has pointed out the importance of formal schemes in the context of (stable) homotopy theory.

A particular assumption that it is almost always present in most earlier works on formal schemes is that morphisms are adic, i.e. that the topology of the sheaf of rings of the initial scheme is induced by the topology of the base formal scheme. This hypothesis on a morphism of formal schemes guarantees that its fibers are usual schemes, therefore an adic morphism between formal schemes is, in the terminology of Grothendieck’s school, a relative scheme over a base that is a formal scheme. But there are important examples of maps of formal schemes that do not correspond to this situation. The first example that comes into mind is the natural map $\text{Spf}(A[[X]]) \to \text{Spf}(A)$ for an adic ring $A$. This morphism has a finiteness property that had not been made explicit until [1] (and independently, in [17]). This property is called pseudo-finite type.\footnote{In [17] the terminology formally finite type is used.} The fact that pseudo-finite type morphisms need not be adic allows fibers that are not usual schemes, and the structure of these maps is, therefore, more complex than the structure of adic maps. The study of smoothness and, more generally, infinitesimal lifting properties in the context of noetherian formal schemes together with this hypothesis of finiteness was embraced in general in our previous work [3]. We should mention a preceding study of smooth morphisms under the restriction that the base is a usual scheme in [17] and also the overlap of several results in [3] and a set of results in [11, §2], based on Nayak’s 1998 thesis.

In [3] we studied the good properties of these definitions and the agreement of their properties with the corresponding behavior for usual noetherian schemes, obtaining the corresponding statement of Zariski’s Jacobian criterion for smoothness. Now we concentrate on studying properties which make sense specifically in the formal context getting information about the infinitesimal lifting properties from information present in the structure of
a formal scheme. This study continues by the third author in [15] where a deformation theory for smooth morphisms is developed.

This paper can be structured roughly into three parts. The first, formed by sections 1, 2 and 3, includes preliminaries, introduces the notion of quasi-covering and the study of completion morphisms. We know of no previous reference about these matters, so we include all the needed details. They will be indispensable to state our results. The second part encompasses three sections (4, 5 and 6). We show that there exists a close relationship between the infinitesimal lifting properties of an adic morphism and the infinitesimal lifting properties of the underlying morphism of ordinary schemes \( f_0 \). The third part (section 7) treats the structure theorems, which are the main results of this work. We characterize open immersions and completion morphisms in terms of the \( \acute{e} \)tale property. We classify \( \acute{e} \)tale adic coverings of a noetherian formal scheme. Finally, we give local structure theorems for unramified, \( \acute{e} \)tale and smooth maps, that show that it is possible to factor them locally into simpler maps.

Let us discuss in greater detail the contents of every section. Our framework is the category of locally noetherian formal schemes. In this category a morphism \( f: \mathcal{X} \to \mathcal{Y} \) can be expressed as a direct limit

\[
f = \lim_{n \in \mathbb{N}} f_n
\]

of a family of maps of ordinary schemes using appropriate ideals of definition. The first section sets the basic notations and recalls some definitions that will be used throughout the paper. The second section deals with morphisms between locally noetherian formal schemes expressed as before as a limit in which every map \( f_n \) is a closed immersion of usual schemes. It is a true closed immersion of formal schemes when \( f \) is adic. We treat radicial maps of formal schemes and see that the main results are completely similar to the case of usual schemes. On usual schemes, quasi-finite maps play a very important role in the understanding of the structure of \( \acute{e} \)tale maps. In the context of formal schemes there are two natural generalizations of this notion. The simplest one is pseudo-quasi-finite (Definition 2.7) — in a few words: “of pseudo-finite type with finite fibers”. The key notion though is that of quasi-covering (Definition 2.8). While both are equivalent in the context of usual schemes, the latter is a basic property of unramified and, therefore, \( \acute{e} \)tale maps between formal schemes (cf. Corollaries 4.7 and 6.6). In section 3 we discuss flat morphisms in the context of locally noetherian formal schemes. Next, we study morphisms of completion in this setting. They form a class of flat morphisms that are closed immersions as topological maps. Such maps will be essential for the results of the last section.

Expressing a morphism \( f: \mathcal{X} \to \mathcal{Y} \) between locally noetherian formal schemes as a limit as before, it is sensible to ask about the relation that exists between the infinitesimal lifting properties of \( f \) and the infinitesimal lifting properties of the underlying morphisms of usual schemes \( \{f_n\}_{n \in \mathbb{N}} \).
This is one of the main themes of the next three sections. The case of unramified morphisms is simple: \( f \) is unramified if and only if \( f_n \) are unramified \( \forall n \in \mathbb{N} \) (Proposition 4.1). Another characterization is that \( f \) is unramified if and only if \( f_0 \) is and the fibers of \( f \) and of \( f_0 \) agree (Corollary 4.10). A consequence of this result is a useful characterization of pseudo-closed immersion as those unramified morphisms such that \( f_0 \) is a closed immersion (Corollary 4.13). Smooth morphisms are somewhat more difficult to characterize. An adic morphism \( f \) is smooth if and only if \( f_0 \) is and \( f \) is flat (Corollary 5.6). For a non adic morphism, one cannot expect that the maps \( f_n \) are going to be smooth when \( f \) is smooth as it is shown by example 5.7. On the positive side, there is a nice characterization of smooth closed subschemes (Proposition 5.11). Also, the matrix jacobian criterion holds for formal schemes, see Corollary 5.13 for a precise statement. In section 6 we combine these results to obtain properties of étale morphisms. It is noteworthy to point out that a smooth pseudo-quasi-finite map need not be étale (Example 6.7).

The last section contains our main results. First we recover in our framework the classical fact for usual schemes [9, (17.9.1)] that an open immersion is a map that is étale and radicial (Theorem 7.3). We also characterize completion morphisms as those pseudo-closed immersions that are flat. This and other characterizations are given in Proposition 7.5. Writing a locally noetherian formal scheme \( \mathcal{Y} \) as

\[
\mathcal{Y} = \lim_{\longrightarrow} Y_n
\]

with respect to an ideal of definition, Proposition 7.7 says that there is an equivalence of categories between étale adic \( \mathcal{Y} \)-formal schemes and étale \( Y_0 \)-schemes. A special case already appears in [17, Proposition 2.4]. In fact, this result is a reinterpretation of [9, (18.1.2)]. The factorization theorems are based on Theorem 7.11 that says that an unramified morphism can be factored locally into a pseudo-closed immersion followed by an étale adic map. As consequences we obtain Theorem 7.12 and Theorem 7.13. They state that every smooth morphism and every étale morphism factor locally as a completion morphism followed by a smooth adic morphism and an étale adic morphism, respectively. These results explain the local structure of smooth and étale morphisms of formal schemes. It has been remarked by Lipman, Nayak and Sastry in [11, p. 132] that this observation may simplify some developments related to Cousin complexes and duality on formal schemes.

1. Preliminaries

We denote by \( \mathbf{NFS} \) the category of locally noetherian formal schemes and by \( \mathbf{NFS}_{af} \) the subcategory of locally noetherian affine formal schemes. We write \( \mathbf{Sch} \) for the category of ordinary schemes.
We assume that the reader is familiar with the basic theory of formal schemes as is explained in [6, §10]: formal spectrum, ideal of definition of a formal scheme, fiber product of formal schemes, functor $M \rightsquigarrow M^\Delta$ for modules over adic rings, completion of a usual scheme along a closed subscheme, adic morphisms, separated morphisms, etc.

From now on and, except otherwise indicated, every formal scheme will belong to NFS. Every ring under consideration will be assumed to be noetherian. So, every complete ring and every complete module will be separated under the corresponding adic topology.

1.1. Henceforth, the following notation [6, §10.6] will be used:

1. Given $X \in \text{NFS}$ and $J \subset O_X$ an ideal of definition for each $n \in \mathbb{N}$ we put $X_n := (X, O_X/J^{n+1})$ and we indicate that $X$ is the direct limit of the schemes $X_n$ by

$$X = \lim_{\longrightarrow} X_n.$$ 

The ringed spaces $X$ and $X_n$ have the same underlying topological space, so we will not distinguish between a point in $X$ or $X_n$.

2. If $f : X \to Y$ is in NFS, $J \subset O_X$ and $K \subset O_Y$ are ideals of definition such that $f^*(K)O_X \subset J$ and $f_n : X_n := (X, O_X/J^{n+1}) \to Y_n := (Y, O_Y/K^{n+1})$ is the morphism induced by $f$, for each $n \in \mathbb{N}$, then $f$ is expressed as

$$f = \lim_{\longrightarrow} f_n.$$ 

3. Furthermore, given $f : X \to Y$ a morphism in NFS and $K \subset O_Y$ an ideal of definition, there exist $J \subset O_X$ an ideal of definition such that $f^*(K)O_X \subset J$. Such a pair of ideals of definition will be called $f$-compatible.

1.2. Let $f : X \to Y$ be a morphism in NFS and let $J \subset O_X$ and $K \subset O_Y$ be $f$-compatible ideals of definition. The morphism $f$ is of pseudo-finite type (pseudo-finite) [1, p.7] if $f_0$ (and in fact any $f_n$) is of finite type (finite, respectively). Moreover, if $f$ is adic we say that $f$ is of finite type (finite) [6, 10.13.3] ([7, (4.8.2)], respectively). Note that these definitions do not depend on the choice of ideals of definition.

1.3. [3, Definition 2.1 and Definition 2.6] A morphism $f : X \to Y$ in NFS is smooth (unramified, étale) if it is of pseudo-finite type and satisfies the following lifting condition:

For all affine $Y$-schemes $Z$ and for each closed subscheme $T \subset Z$ given by a square zero ideal $I \subset O_Z$ the induced map

$$\text{Hom}_Y(Z, X) \to \text{Hom}_Y(T, X)$$

is surjective (injective or bijective, respectively).
 Moreover, if \( f \) is in addition adic we say that \( f \) is smooth adic (unramified adic or étale adic, respectively).

We say that \( f \) is smooth (unramified or étale) at \( x \) if there exists an open subset \( U \subset X \) with \( x \in U \) such that \( f|_U \) is smooth (unramified or étale, respectively). It holds that \( f \) is smooth (unramified or étale) if and only if \( f \) is smooth (unramified or étale, respectively) at \( x, \forall x \in X \) (cf. [3, Proposition 4.3, 4.1]).

1.4. (cf. [3, §3]) Given \( f : X \to Y \) in NFS the differential pair of \( X \) over \( Y \), \( (\hat{\Omega}^1_{X/Y}, \hat{d}_{X/Y}) \), is locally given by \( (\hat{\Omega}^1_{A/B}, \hat{d}_{A/B}) \) for all open sets \( \mathfrak{U} = \text{Spf}(A) \subset X \) and \( \mathfrak{V} = \text{Spf}(B) \subset Y \) with \( f(\mathfrak{U}) \subset \mathfrak{V} \). The \( \mathcal{O}_X \)-Module \( \hat{\Omega}^1_{X/Y} \) is called the module of 1-differentials of \( X \) over \( Y \) and the continuous \( Y \)-derivation \( \hat{d}_{X/Y} \) is called the canonical derivation of \( X \) over \( Y \).

1.5. [6, p. 442] A morphism \( f : Z \to X \) in NFS is a closed immersion if it factors as \( Z \xrightarrow{g} X' \xrightarrow{j} X \) where \( g \) is an isomorphism of \( Z \) into a closed subscheme \( X' \hookrightarrow X \) of the formal scheme \( X \) ([6, (10.14.2)]). Recall from [7, (4.8.10)] that a morphism \( f : Z \to X \) in NFS is a closed immersion if it is adic and, given \( \mathcal{K} \subset \mathcal{O}_X \) an ideal of definition of \( X \) and \( J = f^*(\mathcal{K})\mathcal{O}_Z \), the corresponding ideal of definition of \( Z \), the induced morphism \( f_0 : Z_0 \to X_0 \) is a closed immersion, equivalently, the induced morphisms \( f_n : Z_n \to X_n \) are closed immersions for all \( n \in \mathbb{N} \).

A morphism \( f : Z \to X \) in NFS is an open immersion if it factors as \( Z \xrightarrow{g} X' \xrightarrow{j} X \) where \( g \) is an isomorphism of \( Z \) into an open subscheme \( X' \hookrightarrow X \).

**Definition 1.6.** Let \( X \) be in NFS, let \( \mathcal{J} \subset \mathcal{O}_X \) be an ideal of definition and \( x \in X \). We define the topological dimension of \( X \) at \( x \) as

\[
\dim_{\text{top}} x = \dim_x X_0.
\]

It is easy to see that the definition does not depend on the chosen ideal of definition of \( X \). We define the topological dimension of \( X \) as

\[
\dim_{\text{top}} X = \sup_{x \in X} \dim_{\text{top}} x = \sup_{x \in X} \dim_x X_0 = \dim X_0.
\]

Given \( A \) an \( I \)-adic noetherian ring, put \( X = \text{Spec}(A) \) and \( X = \text{Spf}(A) \), then \( \dim_{\text{top}} X = \dim A/I \). While the only “visible part” of \( X \) in \( X = \text{Spec}(A) \) is \( V(I) \), it happens that \( X \setminus V(I) \) has a deep effect on the behavior of \( X \) as we will see along this work. So apart from the topological dimension of \( X \), it is necessary to consider another notion of dimension that expresses part of the “hidden” information: the algebraic dimension.

**Definition 1.7.** Let \( X \) be in NFS and let \( x \in X \). We define the algebraic dimension of \( X \) at \( x \) as

\[
\dim_x X = \dim \mathcal{O}_{X,x}.
\]
Definition 1.10. Let \( X \).
Moreover, if \( y \) is the point\( f \)
affine formal
Example 1.11. Let \( \mathcal{A} \)
In general, for \( X \) scheme does not measure the dimension of the underlying topological space.

\[ \dim X = \sup_{x \in X} \dim A_p x, \]

from which it follows the equality.

Proposition 1.8. If \( \mathcal{X} = \text{Spf}(A) \) with \( A \) an \( I \)-adic noetherian ring then \( \dim \mathcal{X} = \dim A \).

Proof. For each \( x \in X \), if \( p_x \) is the corresponding open prime ideal in \( A \)
we have that \( \dim_x X = \dim A_{\{p_x\}} = \dim A_{p_x} \) since \( A_{p_x} \hookrightarrow A_{\{p_x\}} \) is a flat extension of local rings with the same residue field (cf. [12, (24.D)]). Since \( I \subset A \) is in the Jacobson radical, it holds that \( \dim A = \sup_{x \in X} \dim A_{p_x} \),

Example 1.9. Given \( A \) an \( I \)-adic noetherian ring and \( T = T_1, T_2, \ldots, T_r \)
a finite number of indeterminates, the affine formal space of dimension \( r \)
over \( A \) is \( \mathcal{A}^r_{\text{Spf}(A)}(\mathcal{A}(\mathcal{T})) \) and the formal disc of dimension \( r \)
over \( A \) is \( \mathcal{D}^r_{\text{Spf}(A)}(\mathcal{T}) \) (see [3, Example 1.6]). It holds that

\[ \text{dimtop } \mathcal{A}^r_{\text{Spf}(A)} = \dim \mathcal{A}^r_{\text{Spec}(A/I)} = \dim A/I + r \]

\[ \text{dimtop } \mathcal{D}^r_{\text{Spf}(A)} = \dim \mathcal{A}^r_{\text{Spec}(A/I)} = \dim A/I \]

and

\[ \dim \mathcal{A}^r_{\text{Spf}(A)} = \dim A(\mathcal{T}) = \dim A + r = \dim \text{Spf}(A) + r \]

\[ \dim \mathcal{D}^r_{\text{Spf}(A)} = \dim A[[\mathcal{T}]] = \dim A + r = \dim \text{Spf}(A) + r. \]

From these examples, we see that the algebraic dimension of a formal scheme does not measure the dimension of the underlying topological space. In general, for \( \mathcal{X} \) in NFS, \( \dim_x \mathcal{X} \geq \dim \text{top}_x \mathcal{X} \), for any \( x \in \mathcal{X} \) and, therefore

\[ \dim \mathcal{X} \geq \dim \text{top}_x \mathcal{X}. \]

Moreover, if \( \mathcal{X} = \text{Spf}(A) \) with \( A \) an \( I \)-adic ring then \( \dim \mathcal{X} \geq \dim \text{top}_x \mathcal{X} + \text{ht}(I) \).

Definition 1.10. Let \( f : \mathcal{X} \to \mathcal{Y} \) be in NFS and \( y \in \mathcal{Y} \). The fiber of \( f \) at the point \( y \) is the formal scheme

\[ f^{-1}(y) = \mathcal{X} \times_{\mathcal{Y}} \text{Spec}(k(y)). \]

For example, if \( f : \mathcal{X} = \text{Spf}(B) \to \mathcal{Y} = \text{Spf}(A) \) is in NFS$_{af}$ we have that \( f^{-1}(y) = \text{Spf}(B \hat{\otimes}_A k(y)) \).

Example 1.11. Let \( \mathcal{Y} = \text{Spf}(A) \) be in NFS$_{af}$ and let \( \mathcal{T} = T_1, T_2, \ldots, T_r \)
be a set of indeterminates. If \( p : \mathcal{X}_{\mathcal{Y}} \to \mathcal{Y} \) is the canonical projection of the affine formal \( r \)-space over \( \mathcal{Y} \), then for all \( y \in \mathcal{Y} \) we have that

\[ p^{-1}(y) = \text{Spf}(A(\mathcal{T}) \hat{\otimes}_A k(y)) = \text{Spec}(k(y)[\mathcal{T}]) = \mathcal{A}^r_{\text{Spec}(k(y))}. \]

If \( q : \mathcal{D}^r_{\mathcal{Y}} \to \mathcal{Y} \) is the canonical projection of the formal \( r \)-disc over \( \mathcal{Y} \), given \( y \in \mathcal{Y} \), it holds that

\[ q^{-1}(y) = \text{Spf}(A[[\mathcal{T}]] \hat{\otimes}_A k(y)) = \text{Spf}(k(y)[[\mathcal{T}]]) = \mathcal{D}^r_{\text{Spec}(k(y))}. \]
1.12. Let $f : X \to \mathfrak{Y}$ be in NFS and let us consider $\mathcal{J} \subset \mathcal{O}_X$ and $\mathcal{K} \subset \mathcal{O}_\mathfrak{Y}$ $f$-compatible ideals of definition. According to 1.1,

$$f = \lim_{n \in \mathbb{N}} (f_n : X_n \to Y_n).$$

Then, by [6, (10.7.4)], it holds that

$$f^{-1}(y) = \lim_{n \in \mathbb{N}} f_n^{-1}(y)$$

where $f_n^{-1}(y) = X_n \times_{Y_n} \text{Spec}(k(y))$, for each $n \in \mathbb{N}$.

If $f$ is adic, by base-change (cf. [3, 1.3]) we deduce that $f^{-1}(y) \to \text{Spec}(k(y))$ is adic so, $f^{-1}(y)$ is an ordinary scheme and $f^{-1}(y) = f_n^{-1}(y)$, for all $n \in \mathbb{N}$.

1.13. We establish the following convention. Let $f : X \to \mathfrak{Y}$ be in NFS, $x \in X$ and $y = f(x)$ and assume that $\mathcal{J} \subset \mathcal{O}_X$ and $\mathcal{K} \subset \mathcal{O}_\mathfrak{Y}$ are $f$-compatible ideals of definition. From now and, except otherwise indicated, whenever we consider the rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{\mathfrak{Y},y}$ we will associate them the $\mathcal{J}\mathcal{O}_{X,x}$ and $\mathcal{K}\mathcal{O}_{\mathfrak{Y},y}$-adic topologies, respectively. And we will denote by $\widehat{\mathcal{O}_{X,x}}$ and $\widehat{\mathcal{O}_{\mathfrak{Y},y}}$ the completion of $\mathcal{O}_{X,x}$ and $\mathcal{O}_{\mathfrak{Y},y}$ with respect to the $\mathcal{J}\mathcal{O}_{X,x}$ and $\mathcal{K}\mathcal{O}_{\mathfrak{Y},y}$-adic topologies, respectively. Note that these topologies do not depend on the choice of ideals of definition of $X$ and $\mathfrak{Y}$.

**Definition 1.14.** Let $f : X \to \mathfrak{Y}$ be in NFS. Given $x \in X$ and $y = f(x)$, we define the **relative algebraic dimension of $f$ at $x$** as

$$\dim_x f = \dim_x f^{-1}(y).$$

If $\mathcal{J} \subset \mathcal{O}_X$ and $\mathcal{K} \subset \mathcal{O}_\mathfrak{Y}$ are $f$-compatible ideals of definition, then

$$\dim_x f = \dim \mathcal{O}_{f^{-1}(y),x} = \dim \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{\mathfrak{Y},y}} k(y) = \dim \widehat{\mathcal{O}_{X,x}} \otimes_{\widehat{\mathcal{O}_{\mathfrak{Y},y}}} k(y).$$

If the topology in $\widehat{\mathcal{O}_{X,x}} \otimes_{\widehat{\mathcal{O}_{\mathfrak{Y},y}}} k(y)$ is the $\mathcal{J}\widehat{\mathcal{O}_{X,x}}$-adic then $\widehat{\mathcal{O}_{X,x}} \otimes_{\widehat{\mathcal{O}_{\mathfrak{Y},y}}} k(y) = \widehat{\mathcal{O}_{X,x}} \otimes_{\widehat{\mathcal{O}_{\mathfrak{Y},y}}} k(y)$.

1.15. Given an adic morphism $f : X \to \mathfrak{Y}$ in NFS and $f$-compatible ideals of definition $\mathcal{J} \subset \mathcal{O}_X$ and $\mathcal{K} \subset \mathcal{O}_\mathfrak{Y}$, then $\dim_x f = \dim_x f_0$ for every $x \in X$.

For example:

1. If $p : \mathbb{A}^r_{\mathfrak{Y}} := \mathbb{A}^r_\mathbb{Z} \times_{\mathbb{Z}} \mathfrak{Y} \to \mathfrak{Y}$ is the canonical projection of the affine formal $r$-space over $\mathfrak{Y}$, given $x \in \mathbb{A}^r_{\mathfrak{Y}}$ we have that

$$\dim_x p = \dim k(y)[[T]] = r,$$

where $y = p(x)$. In contrast, if $q : \mathbb{D}^r_{\mathfrak{Y}} := \mathbb{D}^r_\mathbb{Z} \times_{\mathbb{Z}} \mathfrak{Y} \to \mathfrak{Y}$ is the canonical projection of the formal $r$-disc over $\mathfrak{Y}$, $x \in \mathbb{D}^r_{\mathfrak{Y}}$ and $y = q(x)$ it holds that

$$\dim_x q = \dim k(y)[[[T]]] = r > \dim k(y) = 0.$$
(2) If $X$ is a usual noetherian scheme and $X'$ is a closed subscheme of $X$, recall that the morphism of completion of $X$ along $X'$, $\kappa : X/X' \to X$ ([6, (10.8.5)]) is not adic, in general. Note however that
\[ \dim_x \kappa = \dim k(x) = 0 \]
for all $x \in X/X'$.

2. PSEUDO-CLOSED IMMERSIONS AND QUASI-COVERINGS

**Definition 2.1.** A morphism $f : \mathcal{X} \to \mathcal{Y}$ in NFS is a pseudo-closed immersion if there exists $\mathcal{J} \subset \mathcal{O}_X$ and $\mathcal{K} \subset \mathcal{O}_\mathcal{Y}$ $f$-compatible ideals of definition such that the induced morphisms of schemes $\{f_n : X_n \to Y_n\}_{n \in \mathbb{N}}$ are closed immersions.

Note that if $f : \mathcal{X} \to \mathcal{Y}$ is a pseudo-closed immersion, $f(\mathcal{X})$ is a closed subset of $\mathcal{Y}$.

Let us show that this definition does not depend on the choice of ideals of definition. Being a local question, we can assume that $f : \mathcal{X} = \text{Spf}(A) \to \mathcal{Y} = \text{Spf}(B)$ is in NFS_{af} and that $\mathcal{J} = J^\Delta$, $\mathcal{K} = K^\Delta$ for ideals of definition $J \subset A$ and $K \subset B$ such that $KA \subset J$. Then, given another pair of ideals of definition $J' \subset A$ and $K' \subset B$ satisfying that $f^*(K')\mathcal{O}_X \subset J'$, there exists $n_0 > 0$ such that $J^{n_0} \subset J'$ and $K^{n_0} \subset K'$. The morphism $B \to A$ induces the following commutative diagrams
\[ B/K^{n_0(n+1)} \longrightarrow A/J^{n_0(n+1)} \]
\[ \downarrow \quad \quad \downarrow \]
\[ B/K^{n+1} \longrightarrow A/J^{n+1} \]
and it follows that $B/K^{n+1} \longrightarrow A/J^{n+1}$ is surjective, for all $n \in \mathbb{N}$. Then, using 1.5, it follows that the morphism $(\mathcal{X}, \mathcal{O}_X/J^{n+1}) \to (\mathcal{Y}, \mathcal{O}_\mathcal{Y}/K^{n+1})$ is a closed immersion, for all $n \in \mathbb{N}$.

**Example 2.2.** Let $X$ be a noetherian scheme and let $X' \subset X$ be a closed subscheme defined by an ideal $I \subset \mathcal{O}_X$. The morphism of completion $X/X'$ of $X$ along $X'$ ([6, (10.8.5)]) is expressed as
\[ \lim_{n \in \mathbb{N}} \left( (X', \mathcal{O}_X/I^{n+1}) \xrightarrow{\kappa_n} (X, \mathcal{O}_X) \right), \]
therefore, it is a pseudo-closed immersion.

Notice that an adic pseudo-closed immersion is a closed immersion (cf. 1.5). However, to be a pseudo-closed immersion is not a topological property:

**Example 2.3.** Given $K$ a field, let $p : \mathbb{D}^1_{\text{Spec}(K)} \to \text{Spec}(K)$ be the canonical projection. If we consider the ideal of definition $\langle T \rangle^\Delta$, of $\mathbb{D}^1_{\text{Spec}(K)}$ then $p_0 = 1_{\text{Spec}(K)}$ is a closed immersion. However, the morphisms
\[ p_n : \text{Spec}(K[T]/\langle T \rangle^{n+1}) \to \text{Spec}(K) \]

are not adic.
are not closed immersions, for all $n > 0$ and, thus, $p$ is not a pseudo-closed immersion.

**Proposition 2.4.** Let $f : X \to Y$ and $g : Y \to S$ be two morphisms in NFS. It holds that:

1. If $f$ and $g$ are pseudo-closed immersions then $g \circ f$ is a pseudo-closed immersion.
2. If $f$ is a pseudo-closed immersion, given $h : Y' \to Y$ in NFS we have that $X_{Y'} = X \times_Y Y'$ is in NFS and that $f' : X_{Y'} \to Y'$ is a pseudo-closed immersion.

**Proof.** As for (1) let $J \subset O_X$, $K \subset O_Y$ and $L \subset O_S$ be ideals of definition such that $J$ and $K$ are $f$-compatible, $K$ and $L$ are $g$-compatible and consider the corresponding expressions for $f$ and $g$ as direct limits:

$$f = \lim_{n \in \mathbb{N}} (X_n \xrightarrow{f_n} Y_n) \quad g = \lim_{n \in \mathbb{N}} (Y_n \xrightarrow{g_n} S_n)$$

Since

$$g \circ f = \lim_{n \in \mathbb{N}} g_n \circ f_n$$

the assertion follows from the stability under composition of closed immersions in Sch. Let us show (2). Take $K' \subset O_{Y'}$ an ideal of definition with $h^*(K)O_{Y'} \subset K'$ and such that, by 1.1,

$$h = \lim_{n \in \mathbb{N}} (h_n : Y'_n \to Y_n).$$

Then by [6, (10.7.4)] we have that

$$X_{Y'} \xrightarrow{f'} Y' \xrightarrow{h} X \xrightarrow{f} Y$$

$$\xrightarrow{\sim} \quad \lim_{n \in \mathbb{N}} \quad \left( X_n \times_Y Y'_n \xrightarrow{f'_n} Y'_n \right) \xrightarrow{h'_n} \left( X_n \xrightarrow{f_n} Y_n \right)$$

By hypothesis, $f_n$ is a closed immersion and since closed immersions in Sch are stable under base-change we have that $f'_n$ is a closed immersion of noetherian schemes, $\forall n \in \mathbb{N}$. Finally, since $f$ is a morphism of pseudo-finite type, from [3, Proposition 1.8.(2)] we have that $X_{Y'}$ is in NFS. □

Next we turn to the study of radicial morphisms in the context of formal schemes. This notion will allow us later (Theorem 7.3) to give a characterization of open immersions in terms of étale morphisms.

**Definition 2.5.** A morphism $f : X \to Y$ in NFS is radicial if given $J \subset O_X$ and $K \subset O_Y$ $f$-compatible ideals of definition the induced morphism of schemes $f_0 : X_0 \to Y_0$ is radicial.
Given $x \in X$, the residue fields of the local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X_0,x}$ agree and analogously for $\mathcal{O}_{Y,f(x)}$ and $\mathcal{O}_{Y_0,f_0(x)}$. Therefore the definition of radicial morphisms does not depend on the chosen ideals of definition of $X$ and $Y$.

2.6. From the sorites of radicial morphisms in $\text{Sch}$ it follows that:

(1) Radicial morphisms are stable under composition and noetherian base-change.

(2) Every monomorphism is radicial. So, open immersions, closed immersions and pseudo-closed immersions are radicial morphisms.

The notion of quasi-finite morphism of usual schemes (see [6, Definition (6.11.3)]) is based on the equivalence between several conditions for morphisms between schemes (Corollaire (6.11.2) in loc. cit.) that are no longer equivalent in the full context of formal schemes. Specifically, we study two notions that generalize that of quasi-finite morphism of usual schemes. They will play a basic role in understanding the structure of unramified and étale morphisms in $\text{NFS}$.

**Definition 2.7.** Let $f : X \to Y$ be a pseudo-finite type morphism in $\text{NFS}$. We say that $f$ is pseudo-quasi-finite if there exist $\mathcal{J} \subset \mathcal{O}_X$ and $\mathcal{K} \subset \mathcal{O}_Y$ $f$-compatible ideals of definition such that $f_0$ is quasi-finite. And $f$ is pseudo-quasi-finite at $x \in X$ if there exists an open neighborhood $x \in U \subset X$ such that $f|_U$ is pseudo-quasi-finite.

Notice that if $f : X \to Y$ is a pseudo-quasi-finite morphism (in $\text{NFS}$) then, for all $f$-compatible ideals of definition $\mathcal{J} \subset \mathcal{O}_X$ and $\mathcal{K} \subset \mathcal{O}_Y$, the induced morphism of schemes $f_0 : X_0 \to Y_0$ is quasi-finite.

As an immediate consequence of the analogous properties in $\text{Sch}$ we have that:

(1) The underlying sets of the fibers of a pseudo-quasi-finite morphism are finite.
(2) Closed immersions, pseudo-closed immersions and open immersions are pseudo-quasi-finite.
(3) Pseudo-finite and finite morphisms are pseudo-quasi-finite.
(4) If $f : X \to Y$ and $g : Y \to S$ are pseudo-quasi-finite morphisms, then so is $g \circ f$.
(5) If $f : X \to Y$ is pseudo-quasi-finite, given $h : Y' \to Y$ a morphism in $\text{NFS}$ we have that $f' : X_{Y'} \to Y'$ is pseudo-quasi-finite.

In $\text{Sch}$ it is the case that a morphism is étale if and only if it is smooth and quasi-finite. However, we will show that in $\text{NFS}$ not every smooth and pseudo-quasi-finite morphism is étale (see Example 6.7). That is why we introduce a stronger notion than pseudo-quasi-finite morphism and that also generalizes quasi-finite morphisms in $\text{Sch}$: the quasi-coverings.

**Definition 2.8.** Let $f : X \to Y$ be a pseudo-finite type morphism in $\text{NFS}$. The morphism $f$ is a quasi-covering if $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,f(x)}} k(f(x))$ is a finite type $k(f(x))$-module, for all $x \in X$. We say that $f$ is a quasi-covering at $x \in X$ if there exists an open $U \subset X$ with $x \in U$ such that $f|_U$ is a quasi-covering.
We reserve the word covering for a dominant (i.e. with dense image) quasi-covering. These kind of maps will play no role in the present work but they are important, for instance, in the study of finite group actions on formal schemes.

**Example 2.9.** If $X$ is a locally noetherian scheme and $X' \subset X$ a closed subscheme the morphism of completion $\kappa : X = X/X' \to X$ is a quasi-covering. In fact, for all $x \in X$ we have that

$$\mathcal{O}_{X,x} \hat{\otimes}_{\mathcal{O}_{X,\kappa(x)}} k(\kappa(x)) = k(\kappa(x)).$$

**Lemma 2.10.** We have the following:

1. Closed immersions, pseudo-closed immersions and open immersions are quasi-coverings.
2. If $f : X \to Y$ and $g : Y \to S$ are quasi-coverings, the morphism $g \circ f$ is a quasi-covering.
3. If $f : X \to Y$ is a quasi-covering, and $h : Y' \to Y$ a morphism in NFS, then $f' : X_{Y'} \to Y'$ is a quasi-covering.

**Proof.** Immediate. \hfill \Box

**Proposition 2.11.** If $f : X \to Y$ is a quasi-covering in $x \in X$ then:

$$\dim_x f = 0$$

**Proof.** It is a consequence of the fact that $\mathcal{O}_{X,x} \hat{\otimes}_{\mathcal{O}_{Y,f(x)}} k(f(x))$ is an artinian ring. \hfill \Box

**Remark.** Observe that given $J \subset \mathcal{O}_X$ and $K \subset \mathcal{O}_Y$ ideals of definition such that $f^*(J) \mathcal{O}_X \subset K$, for all $x \in X$ it holds that

$$\mathcal{O}_{X,x} \hat{\otimes}_{\mathcal{O}_{Y,f(x)}} k(f(x)) = \lim_{n \in \mathbb{N}} \mathcal{O}_{X_n,x} \otimes_{\mathcal{O}_{Y_n,f_n(x)}} k(f(x)).$$

Over usual schemes quasi-coverings and pseudo-quasi-finite morphisms are equivalent notions. More generally we have the following.

**Proposition 2.12.** Let $f : X \to Y$ be a morphism in NFS. If $f$ is a quasi-covering, then it is pseudo-quasi-finite. Furthermore, if $f$ is adic then the converse holds.

**Proof.** Suppose that $f$ is a quasi-covering and let $J \subset \mathcal{O}_X$ and $K \subset \mathcal{O}_Y$ be $f$-compatible ideals of definition. Given $x \in X$ and $y = f(x)$, $\mathcal{O}_{X,x} \hat{\otimes}_{\mathcal{O}_{Y,y}} k(y)$ is a finite $k(y)$-module and, therefore,

$$\frac{\mathcal{O}_{X_0,x}}{m_{Y_0,y} \mathcal{O}_{X_0,x}} = \frac{\mathcal{O}_{X,x}}{J \mathcal{O}_{X,x}} \otimes_{\mathcal{O}_{Y_0,y}} k(y)$$

is $k(y)$-finite, so it follows that $f$ is pseudo-quasi-finite.

If $f$ is an adic morphism, $f^{-1}(y) = f_0^{-1}(y)$ for each $y \in Y$, so

$$\mathcal{O}_{X_0,x}/m_{Y_0,y} \mathcal{O}_{X_0,x} = \mathcal{O}_{X,x} \hat{\otimes}_{\mathcal{O}_{Y,y}} k(y)$$

for all $x \in X$ with $y = f(x)$. If $f$ is moreover pseudo-quasi-finite, it follows from [6, Corollaire (6.11.2)] that $f$ is a quasi-covering. \hfill \Box
Corollary 2.13. Every finite morphism $f : \mathcal{X} \to \mathcal{Y}$ in NFS is a quasi-covering.

Proof. Finite morphisms are adic and pseudo-quasi-finite. Therefore the result is consequence of the last proposition. □

Nevertheless, by the next example, not every pseudo-finite morphism is a quasi-covering and, therefore, pseudo-quasi-finite does not imply quasi-covering for morphisms in NFS.

Example 2.14. For $r > 0$, the canonical projection $p : \mathbb{D}^r_X \to \mathcal{X}$ is not a quasi-covering since
\[ \dim_x p = r > 0 \quad \forall x \in \mathcal{X}. \]

But considering an appropriate pair of ideals of definition, the scheme map $p_0 = 1_{X_0}$ is finite.

2.15. In short, we have the following diagram of strict implications (with the conditions that imply adic morphism in italics):
\[
\begin{array}{ccc}
\text{closed immersion} & \Rightarrow & \text{finite} & \Rightarrow & \text{quasi-covering} \\
\downarrow & & \downarrow & & \downarrow \\
\text{pseudo-closed immersion} & \Rightarrow & \text{pseudo-finite} & \Rightarrow & \text{pseudo-quasi-finite}
\end{array}
\]

3. Flat morphisms and completion morphisms

In the first part of this section we discuss flat morphisms in NFS. Whenever a morphism
\[ f = \lim_{n \in \mathbb{N}} f_n \]
is adic, the local criterion of flatness for formal schemes (Proposition 3.3) relates the flat character of $f$ and that of the morphisms $f_n$, for all $n \in \mathbb{N}$. In absence of the adic hypothesis this relation does not hold, though (Example 3.2). In the second part, we study the morphisms of completion in NFS, a class of flat morphisms that are pseudo-closed immersions (so, they are closed immersions as topological maps). Even though the construction of the completion of a formal scheme along a closed formal subscheme is a natural one, it has not been systematically developed in the basic references about formal schemes. Morphisms of completion will be an essential ingredient in the main theorems of Section 7, namely, Theorems 7.11, 7.12 and 7.13.

3.1. A morphism $f : \mathcal{X} \to \mathcal{Y}$ is flat at $x \in \mathcal{X}$ if $\mathcal{O}_{\mathcal{X},x}$ is a flat $\mathcal{O}_{\mathcal{Y},f(x)}$-module. We say that $f$ is flat if it is flat at $x$, for all $x \in \mathcal{X}$.

Given $\mathcal{F} \subset \mathcal{O}_\mathcal{X}$ and $\mathcal{K} \subset \mathcal{O}_\mathcal{Y}$ $f$-compatible ideals of definition, by [5, III, §5.4, Proposition 4] the following are equivalent:

1. $f$ is flat at $x \in \mathcal{X}$.
2. $\mathcal{O}_{\mathcal{X},x}$ is a flat $\mathcal{O}_{\mathcal{Y},f(x)}$-module.
3. $\mathcal{O}_{\mathcal{X},x}$ is a flat $\mathcal{O}_{\mathcal{Y},f(x)}$-module.
Example 3.2. Let $K$ be a field let $\mathbb{A}_K^1 = \text{Spec}(K[T])$ and consider the closed subset $X' = V(\langle T \rangle) \subset \mathbb{A}_K^1$. The canonical morphism of completion of $\mathbb{A}_K^1$ along $X'$

$$D^1_K \xrightarrow{\kappa} \mathbb{A}_K^1$$

is flat but, the morphisms

$$\text{Spec}(K[T]/\langle T \rangle^{n+1}) \xrightarrow{\kappa_n} \mathbb{A}_K^1$$

are not flat, for every $n \in \mathbb{N}$.

Proposition 3.3. (Local flatness criterion for formal schemes.) Given an adic morphism $f : \mathfrak{X} \to \mathfrak{Y}$ in NFS, and an ideal of definition $K \subset \mathcal{O}_\mathfrak{Y}$, then $J = f^*(K)\mathcal{O}_\mathfrak{X} \subset \mathcal{O}_\mathfrak{X}$ is an ideal of definition. Let $\{f_n : X_n \rightarrow Y_n\}_{n \in \mathbb{N}}$ be the morphisms induced by $f$ through $K$ and $J$. The following assertions are equivalent:

1. The morphism $f$ is flat.
2. The morphism $f_n$ is flat, for all $n \in \mathbb{N}$.
3. The morphism $f_0$ is flat.

Proof. We may suppose that $f : \mathfrak{X} = \text{Spf}(A) \to \mathfrak{Y} = \text{Spf}(B)$ is in NFS. Then if $\mathcal{K} = K^\Delta$ for an ideal of definition $K \subset B$, we have that $J = (\mathcal{K}\mathcal{A})^\Delta$ and the proposition is a consequence of [1, Lemma 7.1.1] and of the local flatness criterion for rings (cf. [13, Theorem 22.3]).

Associated to a (usual) locally noetherian scheme $X$ and a closed subscheme of $X' \subset X$ there is a locally noetherian formal scheme $X/X'$, called completion of $X$ along $X'$ and, a canonical morphism $\kappa : X/X' \rightarrow X$ ([6, (10.8.3) and (10.8.5)]). Next, we define the completion of a formal scheme $\mathfrak{X}$ along a closed formal subscheme $\mathfrak{X}' \subset \mathfrak{X}$.

Definition 3.4. Let $\mathfrak{X}$ be in NFS and let $\mathfrak{X}' \subset \mathfrak{X}$ be a closed formal subscheme defined by a coherent ideal $\mathcal{I}$ of $\mathcal{O}_\mathfrak{X}$. Given an ideal of definition $\mathcal{J}$ of $\mathfrak{X}$ we define the completion of a sheaf $\mathcal{F}$ on $\mathfrak{X}$ over $\mathfrak{X}'$, denoted by $\mathcal{F}/\mathfrak{X}'$, as the restriction to $\mathfrak{X}'$ of the sheaf

$$\left\lim_{\substack{n \in \mathbb{N}}} \frac{\mathcal{F}}{\langle \mathcal{J} + \mathcal{I} \rangle^{n+1} \mathcal{F}}.$$

The definition does not depend neither on the chosen ideal of definition $\mathcal{J}$ of $\mathfrak{X}$ nor on the coherent ideal $\mathcal{I}$ that defines $\mathfrak{X}'$.

We define the completion of $\mathfrak{X}$ along $\mathfrak{X}'$, and it will be denoted $\mathfrak{X}/\mathfrak{X}'$, as the topological ringed space whose underlying topological space is $\mathfrak{X}'$ and whose sheaf of topological rings is $\mathcal{O}_{\mathfrak{X}/\mathfrak{X}'}$.

It is easy to check that $\mathfrak{X}/\mathfrak{X}'$ satisfies the hypothesis of [6, (10.6.3) and (10.6.4)], from which we deduce that:

1. The formal scheme $\mathfrak{X}/\mathfrak{X}'$ is locally noetherian.
(2) The ideal \((\mathcal{I} + \mathcal{J})/\mathcal{X} \subset \mathcal{O}_{\mathcal{X}/\mathcal{X}'}\) defined by the restriction to \(\mathcal{X}'\) of the sheaf
\[\lim_{n \in \mathbb{N}} \frac{\mathcal{J} + \mathcal{I}}{(\mathcal{J} + \mathcal{I})^{n+1}}\]
is an ideal of definition of \(\mathcal{X}/\mathcal{X}'\).

(3) It holds that \(\mathcal{O}_{\mathcal{X}/\mathcal{X}'}/(\mathcal{I} + \mathcal{J})^{n+1}\) agrees with the restriction to \(\mathcal{X}'\) of the sheaf \(\mathcal{O}_{\mathcal{X}}/((\mathcal{J} + \mathcal{I})^{n+1})\) for every \(n \in \mathbb{N}\).

3.5. With the above notations, if \(Z_n = (\mathcal{X}', \mathcal{O}_{\mathcal{X}}/((\mathcal{J} + \mathcal{I})^{n+1}))\) for all \(n \in \mathbb{N}\), by 1.1 we have that
\[\mathcal{X}/\mathcal{X}' = \lim_{n \in \mathbb{N}} Z_n\]
For each \(n \in \mathbb{N}\), let \(X_n = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{J}^{n+1})\) and \(X_n' = (\mathcal{X}', \mathcal{O}_{\mathcal{X}}/\mathcal{J}^{n+1} + \mathcal{I})\).

The canonical morphisms
\[\mathcal{O}_{\mathcal{X}}/\mathcal{J}^{n+1} \rightarrow \mathcal{O}_{\mathcal{X}}/((\mathcal{J} + \mathcal{I})^{n+1}) \rightarrow \mathcal{O}_{\mathcal{X}}/\mathcal{J}^{n+1} + \mathcal{I}\]
provide the closed immersions of schemes \(X_n' \xrightarrow{j_n} Z_n \xrightarrow{\kappa_n} X_n\), such that the diagram, whose vertical maps are the obvious closed immersions,
\[
\begin{array}{ccc}
X_m' & \xrightarrow{j_m} & Z_m \\
\downarrow & & \downarrow \\
X_n' & \xrightarrow{j_n} & Z_n \\
\end{array}
\xrightarrow{\kappa_n} \xrightarrow{\kappa_n} X_m \\
\begin{array}{ccc}
\downarrow & & \downarrow \\
\kappa_n & & \kappa_n \\
\end{array}
\]
commutes, for all \(m \geq n \geq 0\). Then by 1.1 we have the canonical morphisms in NFS
\[\mathcal{X}' \xrightarrow{j} \mathcal{X}/\mathcal{X}' \xrightarrow{\kappa} \mathcal{X}\]
where \(j\) is a closed immersion (see 1.5). The morphism \(\kappa\) as topological map is the inclusion and it is called morphism of completion of \(\mathcal{X}\) along \(\mathcal{X}'\).

Remark. Observe that \(\kappa\) is adic only if \(\mathcal{I}\) is contained in a ideal of definition of \(\mathcal{X}\), in which case \(\mathcal{X} = \mathcal{X}/\mathcal{X}'\) and \(\kappa = 1_{\mathcal{X}}\).

3.6. If \(\mathcal{X} = \text{Spf}(A)\) is in NFS\(_{af}\) with \(A\) a \(J\)-adic noetherian ring, and \(\mathcal{X}' = \text{Spf}(A/I)\) is a closed formal subscheme of \(\mathcal{X}\), then
\[A = \lim_{n \in \mathbb{N}} \frac{A}{(J + I)^{n+1}} =: \hat{A}\]
and from [6, (10.2.2) and (10.4.6)] we have that \(\mathcal{X}/\mathcal{X}' = \text{Spf}(\hat{A})\) and the morphisms \(\mathcal{X}' \xrightarrow{j} \mathcal{X}/\mathcal{X}' \xrightarrow{\kappa} \mathcal{X}\) correspond to the natural continuous morphisms \(A \rightarrow \hat{A} \rightarrow A/I\).

Proposition 3.7. Given \(\mathcal{X}\) in NFS and \(\mathcal{X}'\) a closed formal subscheme of \(\mathcal{X}\), the morphism of completion \(\kappa : \mathcal{X}/\mathcal{X}' \rightarrow \mathcal{X}\) is a pseudo-closed immersion and \(\acute{e}tale\) (and therefore, from [3, Proposition 4.8], it is flat).
Proof. With the notations of 3.5 we have that
\[ \kappa = \lim_{n \to \infty} \kappa_n. \]
Since \( \kappa_n \) is a closed immersion for all \( n \in \mathbb{N} \), it follows that \( \kappa \) is a pseudo-closed immersion. In order to prove that \( \kappa \) is an étale morphism we may suppose that \( X = \text{Spf}(A) \) and \( X' = \text{Spf}(A/I) \), where \( A \) is a \( J \)-adic noetherian ring. Note that \( X/X' = \text{Spf}(\hat{A}) \) where \( \hat{A} \) is the completion of \( A \) for the \( (J + I) \)-adic topology and, therefore, is étale over \( A \). By [3, 2.2], \( \kappa \) is an étale morphism. \( \square \)

Remark. In Theorem 7.5 we will see that the converse holds: every flat pseudo-closed immersion is a morphism of completion.

3.8. Given \( f : X \to Y \) in NFS, let \( X' \subset X \) and \( Y' \subset Y \) be closed formal subschemes given by ideals \( I \subset O_X \) and \( L \subset O_Y \) such that \( f^*(L)O_X \subset I \), that is, \( f(X') \subset Y' \). If \( J \subset O_X \) and \( K \subset O_Y \) are \( f \)-compatible ideals of definition, let us denote for all \( n \in \mathbb{N} \)
\[ X_n = (X, O_X/J^{n+1}), \quad Y_n = (Y, O_Y/K^{n+1}), \]
\[ Z_n = (X', O_X/(J + I)^{n+1}), \quad W_n = (Y', O_Y/(K + L)^{n+1}) \]
\[ X'_n = (X', O_{X'/}(J^{n+1} + I)), \quad Y'_n = (Y', O_{X'/}(K^{n+1} + L)). \]
Then the morphism \( f \) induces the following commutative diagram of locally noetherian schemes where the oblique maps are the canonical immersions:

\[
\begin{array}{ccc}
X_n & \xrightarrow{f_n} & Y_n \\
\downarrow \kappa_n & & \downarrow \kappa_n \\
X_m & \xrightarrow{f_m} & Y_m \\
\downarrow \kappa_m & & \downarrow \kappa_m \\
Z_n & \xrightarrow{\hat{f}_n} & W_n \\
\downarrow j_n & & \downarrow j_n \\
X'_n & \xrightarrow{f'_n} & Y'_n \\
\downarrow \kappa'_n & & \downarrow \kappa'_n \\
X'_m & \xrightarrow{f'_m} & Y'_m \\
\end{array}
\]
for all \( m \geq n \geq 0 \). Note that
\[ f' = \lim_{n \to \infty} f'_n \]
is the restriction \( f|_{X'} : X' \to Y' \). Applying the direct limit over \( n \in \mathbb{N} \) we obtain a morphism
\[ \hat{f} : X/X' \to Y/Y' \]
in NFS, such that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\
\uparrow \kappa & & \uparrow \kappa' \\
\mathcal{X}/\mathcal{X}' & \xrightarrow{\hat{f}} & \mathcal{Y}/\mathcal{Y}' \\
\uparrow & & \uparrow \\
\mathcal{X}' & \xrightarrow{f|_{\mathcal{X}'}} & \mathcal{Y}'
\end{array}
\]

We will call \(\hat{f}\) the completion of \(f\) along \(\mathcal{X}'\) and \(\mathcal{Y}'\).

3.9. Suppose that \(f : \mathcal{X} = \text{Spf}(A) \rightarrow \mathcal{Y} = \text{Spf}(B)\) is in NFS\(_{af}\) and that \(\mathcal{X}' = \text{Spf}(A/I)\) and \(\mathcal{Y}' = \text{Spf}(B/L)\) with \(LA \subset I\). If \(J \subset A\) and \(K \subset B\) are ideals of definition such that \(KA \subset J\), the morphism \(\hat{f} : \mathcal{X}/\mathcal{X}' \rightarrow \mathcal{Y}/\mathcal{Y}'\) corresponds to the morphism induced by \(B \rightarrow \hat{A}\)

\[
\hat{B} \rightarrow \hat{A}
\]

(cf. [6, (10.4.6)]) where \(\hat{A}\) is the completion of \(A\) for the \((I+J)\)-adic topology and \(\hat{B}\) denotes the completion of \(B\) for the \((K+L)\)-adic topology.

**Proposition 3.10.** Given \(f : \mathcal{X} \rightarrow \mathcal{Y}\) in NFS, let \(\mathcal{Y}' \subset \mathcal{Y}\) be a closed formal subscheme and \(\mathcal{X}' = f^{-1}(\mathcal{Y}')\). Then,

\[
\mathcal{X}/\mathcal{X}' = \mathcal{Y}/\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X}.
\]

**Proof.** We may restrict to the case in which \(\mathcal{X} = \text{Spf}(A), \mathcal{Y} = \text{Spf}(B)\) and \(\mathcal{Y}' = \text{Spf}(B/L)\) are affine formal schemes and \(J \subset A\) and \(K \subset B\) are ideals of definition such that \(KA \subset J\). By hypothesis, \(\mathcal{X}' = \text{Spf}(A/LA)\), so \(\mathcal{X}/\mathcal{X}' = \text{Spf}(\hat{A})\) where \(\hat{A}\) is the completion of \(A\) for the \((J+LA)\)-adic topology. On the other hand, \(\mathcal{Y}/\mathcal{Y}' = \text{Spf}(\hat{B})\) where \(\hat{B}\) denotes the completion of \(B\) for the \((K+L)\)-adic topology and it holds that

\[
\hat{B} \otimes_B A = \hat{B} \otimes_B A = \hat{A},
\]

since \(J + (K + L)A = J + KA + LA = J + LA\), the result follows. \(\square\)

**Proposition 3.11.** Given \(f : \mathcal{X} \rightarrow \mathcal{Y}\) in NFS, let us consider closed formal subschemes \(\mathcal{X}' \subset \mathcal{X}\) and \(\mathcal{Y}' \subset \mathcal{Y}\) such that \(f(\mathcal{X}') \subset \mathcal{Y}'\).

1. Let \(\mathcal{P}\) be one of the following properties of morphisms in NFS:
   - pseudo-finite type, pseudo-finite, pseudo-closed immersion,
   - pseudo-quasi-finite, quasi-covering, flat, separated, radicial,
   - smooth, unramified, étale.

   If \(f\) satisfies \(\mathcal{P}\), then so does \(\hat{f}\).

2. Moreover, if \(\mathcal{X}' = f^{-1}(\mathcal{Y}')\), let \(\mathcal{Q}\) be one of the following properties of morphisms in NFS:
   - adic, finite type, finite, closed immersion, smooth adic, unramified adic, étale adic.
Then, if $f$ satisfies $Q$, then so does $\hat{f}$.

Proof. Suppose that $f$ is flat and let us prove that $\hat{f}$ is flat. The question is local so we may assume $f : X = \text{Spf}(A) \to Y = \text{Spf}(B)$ in NFS$_{af}$, $X' = \text{Spf}(A/I)$ and $Y' = \text{Spf}(B/L)$ with $L \subset I$. Let $J \subset A$ and $K \subset B$ be ideals of definition such that $KA \subset J$ and $\hat{A}$ and $\hat{B}$ the completions of $A$ and $B$ for the topologies given by $(I + J) \subset A$ and $(K + L) \subset B$, respectively. By [5, III, §5.4, Proposition 4] we have that the morphism $\hat{B} \to \hat{A}$ is flat and, from 3.9 and [1, Lemma 7.1.1] it follows that $\hat{f}$ is flat.

Suppose that $f$ satisfies any of the other properties $P$ and let us prove that $\hat{f}$ inherits them using the commutativity of the diagram

$$
\xymatrix{
X \ar[r]^f \ar[d]_\kappa & Y \\
X'/Y' \ar[r]_{\hat{f}} & Y'/Y
}
$$

where the vertical arrows are morphisms of completion. Since all of these properties $P$ are stable under composition and a morphism of completion satisfies $P$ (Proposition 3.7) we have that $P$ holds for $f \circ \kappa = \kappa' \circ \hat{f}$. If $P$ is smooth, unramified or étale the result is immediate from [3, Proposition 2.13].

If $P$ is any of the other properties, then closed immersions verify $P$ and $P$ is stable under composition and under base-change in NFS. Therefore, since $\kappa' \circ \hat{f}$ has $P$ and $\kappa'$ is separated (Proposition 3.7), by the analogous argument in NFS to the one in Sch [6, (5.2.7), $i$, $ii \Rightarrow iii$]) we get that $\hat{f}$ also satisfies $P$.

Finally, if $f$ is adic, from Proposition 3.10 and from [3, 1.3], we deduce that $\hat{f}$ is adic. Then, if $Q$ is any of the properties in statement (2) and $f$ satisfies $Q$, by (1) so does $\hat{f}$. □

4. Unramified morphisms

Let $f : X \to Y$ be a morphism of locally noetherian formal schemes. Given $\mathcal{J} \subset O_X$ and $\mathcal{K} \subset O_Y$ $f$-compatible ideals of definition, express $f$ as a limit

$$f : X \to Y \cong \lim_{n \in \mathbb{N}} (f_n : X_n \to Y_n).$$

We begin relating the unramified character of $f : X \to Y$ and that of the underlying ordinary scheme morphisms $\{f_n\}_{n \in \mathbb{N}}$.

**Proposition 4.1.** With the previous notations, the morphism $f$ is unramified if and only if $f_n : X_n \to Y_n$ is unramified, for all $n \in \mathbb{N}$.

**Proof.** Notice that both conditions in the statement imply that $f$ is a pseudo-finite type morphism. Applying [3, Proposition 4.6] we have to show that $\hat{\Omega}^1_{X/Y} = 0$ is equivalent to $\Omega^1_{X_n/Y_n} = 0$, for all $n \in \mathbb{N}$. If $\hat{\Omega}^1_{X/Y} = 0$, $\hat{\Omega}^1_{X/Y}$.
by the Second Fundamental Exact Sequence ([3, Proposition 3.13]) for the morphisms

$$X_n \hookrightarrow \mathfrak{X} \xrightarrow{f_n} \mathfrak{Y},$$

we have that $\Omega^1_{X_n/\mathfrak{Y}} = 0$, for all $n \in \mathbb{N}$. From the First Fundamental Exact Sequence ([3, Proposition 3.10]) associated to the morphisms

$$X_n \xrightarrow{f_n} Y_n \hookrightarrow \mathfrak{Y},$$

it follows that $\Omega^1_{X_n/Y_n} = 0$. The converse follows from the identification

$$\hat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} = \lim_{\leftarrow n} \Omega^1_{X_n/Y_n}$$

(cf. [3, §1.9]).

**Corollary 4.2.** With the previous notations, if the morphisms $f_n : X_n \to Y_n$ are immersions for all $n \in \mathbb{N}$, then $f$ is unramified.

In the class of adic morphisms in NFS the following proposition provides a criterion, stronger than the last result, to determine when a morphism $f$ is unramified.

**Proposition 4.3.** Let $f : \mathfrak{X} \to \mathfrak{Y}$ be an adic morphism in NFS and let $K \subset O_{\mathfrak{Y}}$ be an ideal of definition. Write

$$f = \lim_{n\in\mathbb{N}} f_n$$

by taking ideals of definition $K \subset O_{\mathfrak{Y}}$ and $J = f^*(K)O_{\mathfrak{X}} \subset O_{\mathfrak{X}}$. The morphism $f$ is unramified if and only if the induced morphism $f_0 : X_0 \to Y_0$ is unramified.

**Proof.** If $f$ is unramified by Proposition 4.1 we have that $f_0$ is unramified. Conversely, suppose that $f_0$ is unramified and let us prove that $\hat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} = 0$.

The question is local so we may assume that $f : \mathfrak{X} = \text{Spf}(A) \to \mathfrak{Y} = \text{Spf}(B)$ is in NFS$_{af}$ and that $J = J^\triangle$, with $J \subset A$ an ideal of definition. By hypothesis $\Omega^1_{X_0/Y_0} = 0$ and thus, since $f$ is adic it holds that

$$(4.3.1) \quad \hat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} \otimes_{O_{\mathfrak{X}}} O_{X_0} = \Omega^1_{X_0/Y_0} = 0.$$

Then by the equivalence of categories [6, (10.10.2)], the last equality says that $\hat{\Omega}^1_{A/B} / J\hat{\Omega}^1_{A/B} = 0$. Since $A$ is a $J$-adic ring it holds that $J$ is contained in the Jacobson radical of $A$. Moreover, [3, Proposition 3.3] implies that $\hat{\Omega}^1_{A/B}$ is a finite type $A$-module. From Nakayama’s lemma we deduce that $\hat{\Omega}^1_{A/B} = 0$ and therefore, $\hat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} = (\hat{\Omega}^1_{A/B})^\triangle = 0$. Applying [3, Proposition 4.6] it follows that $f$ is unramified. \qed

The following example illustrates that in the non adic case the analogous of the last proposition does not hold.
Example 4.4. Let $K$ be a field and $p : \mathbb{D}_K^1 \to \text{Spec}(K)$ be the projection morphism of the formal disc of dimension 1 over $\text{Spec}(K)$. By [3, Example 3.14] we have that $\hat{\Omega}^1_p = (K[[T]]dT)^\wedge$ and therefore, $\mathbb{D}_K^1$ is ramified over $K$ ([3, Proposition 4.6]). However, given the ideal of definition $\langle T \rangle \subset K[[T]]$ the induced morphism $p_0 = 1_{\text{Spec}(K)}$ is unramified.

Let us consider for a morphism $f : X \to Y$ in NFS the notation established at the beginning of the section. In view of the example, our next goal will be to determine when the morphism $f$ such that $f_0$ is unramified but $f$ itself is not necessarily adic, is unramified (Corollary 4.10). In order to do that, we will need some results that describe the local behavior of unramified morphisms. Next, we provide local characterizations of unramified morphisms in NFS, generalizing the analogous properties in the category of schemes (cf. [9, (17.4.1)]).

Proposition 4.5. Let $f : X \to Y$ be a morphism in NFS of pseudo-finite type. For $x \in X$ and $y = f(x)$ the following conditions are equivalent:

1. $f$ is unramified at $x$.
2. $f^{-1}(y)$ is an unramified $k(y)$-formal scheme at $x$.
3. $m_{X,x} \hat{\mathcal{O}}_{X,x} = m_{Y,y} \hat{\mathcal{O}}_{X,x}$ and $k(x) | k(y)$ is a finite separable extension.
4. $\hat{\Omega}^1_{X,x}/O_{Y,y} = 0$.
4′. $(\hat{\Omega}^1_{X/y})_x = 0$.
5. $\mathcal{O}_{X,x}$ is a formally unramified $\mathcal{O}_{Y,y}$-algebra for the adic topologies.
5′. $\hat{\mathcal{O}}_{X,x}$ is a formally unramified $\hat{\mathcal{O}}_{Y,y}$-algebra for the adic topologies.

Proof. Keep the notation from the beginning of this section and write

$$f : X \to Y = \lim_{n \in \mathbb{N}} \left( f_n : X_n \to Y_n \right)$$

(1) $\iff$ (2) By Proposition 4.1, $f$ is unramified at $x$ if and only if all the morphisms $f_n : X_n \to Y_n$ are unramified at $x$. Applying [9, (17.4.1)], this is equivalent to $f_n^{-1}(y)$ being an unramified $k(y)$-scheme at $x$, for all $n \in \mathbb{N}$, which is also equivalent to

$$f^{-1}(y) = \lim_{n \in \mathbb{N}} f_n^{-1}(y)$$

being an unramified $k(y)$-formal scheme at $x$.

(1) $\Rightarrow$ (3) The assertion (1) is equivalent to $f_n : X_n \to Y_n$ being unramified at $x$, for all $n \in \mathbb{N}$, and from [9, loc. cit.] it follows that $k(x) | k(y)$ is a finite separable extension, and that $m_{X_n,x} = m_{Y_n,y} \mathcal{O}_{X_n,x}$, for all $n \in \mathbb{N}$. Hence,

$$m_{X,x} \hat{\mathcal{O}}_{X,x} = \lim_{n \in \mathbb{N}} m_{X_n,x} = \lim_{n \in \mathbb{N}} m_{Y_n,y} \mathcal{O}_{X_n,x} = m_{Y,y} \hat{\mathcal{O}}_{X,x}.$$
(4) \Leftrightarrow (4') By [3, Proposition 3.3] it holds that \((\Omega^1_{X/k})_x\) is a finite type \(O_{X,x}\)-module and therefore,

\[
\Omega^1_{O_{X,x}/O_{\mathfrak{p},y}} = (\Omega^1_{X/k})_x = (\Omega^1_{X/k})_x \otimes_{O_{X,x}} O_{X,x}.
\]

Then, since \(\widehat{O}_{X,x}\) is a faithfully flat \(O_{X,x}\)-algebra, \(\Omega^1_{O_{X,x}/O_{\mathfrak{p},y}} = 0\) if and only if \((\Omega^1_{X/k})_x = 0\).

(3) \Rightarrow (4) Since \(k(x)|k(y)\) is a finite separable extension we have that \(\Omega^1_{k(x)/k(y)} = 0\) and from [3, Proposition 3.3] \(\Omega^1_{O_{X,x}/O_{\mathfrak{p},y}} = (\Omega^1_{X/k})_x\) is a finite type \(\widehat{O}_{X,x}\)-module. Therefore, it holds that

\[
\Omega^1_{O_{X,x}/O_{\mathfrak{p},y}} \otimes \widehat{O}_{X,x} k(x) = \Omega^1_{(O_{X,x} \otimes O_{\mathfrak{p},y} k(y))/k(y)} = \Omega^1_{k(x)/k(y)} = 0.
\]

By Nakayama’s lemma, \(\Omega^1_{O_{X,x}/O_{\mathfrak{p},y}} = 0\).

(4) \Leftrightarrow (5) It is straightforward from [8, (0, 20.7.4)].

(5) \Leftrightarrow (5') Immediate.

(4') \Rightarrow (1) Since \(\Omega^1_{X/k} \in \text{Coh}(X)\) ([3, Proposition 3.3]), assertion (4') implies that there exists an open subset \(U \subseteq X\) with \(x \in U\) such that \((\Omega^1_{X/k})_U = 0\) and therefore, by [3, Proposition 4.6] we have that \(f\) is unramified at \(x\). \hfill \(\square\)

**Corollary 4.6.** Let \(f : X \to \mathfrak{Y}\) be a pseudo-finite type morphism in NFS. The following conditions are equivalent:

1. \(f\) is unramified.
2. For all \(x \in X\), \(f^{-1}(f(x))\) is an unramified \(k(f(x))\)-formal scheme at \(x\).
3. For all \(x \in X\), \(m_{X,x}\widehat{O}_{X,x} = m_{\mathfrak{p},f(x)}\widehat{O}_{X,x}\) and \(k(x)|k(f(x))\) is a finite separable extension.
4. \(\Omega^1_{O_{X,x}/O_{\mathfrak{p},f(x)}} = 0\), for all \(x \in X\).
4' For all \(x \in X\), \((\Omega^1_{X/k})_x = 0\).
5. For all \(x \in X\), \(O_{X,x}\) is a formally unramified \(O_{\mathfrak{p},f(x)}\)-algebra for the adic topologies.
5' For all \(x \in X\), \(\widehat{O}_{X,x}\) is a formally unramified \(O_{\mathfrak{p},f(x)}\)-algebra for the adic topologies.

**Corollary 4.7.** Let \(f : X \to \mathfrak{Y}\) be a pseudo-finite type morphism in NFS. If \(f\) is unramified at \(x \in X\), then \(f\) is a quasi-covering at \(x\).

**Proof.** By assertion (3) of Proposition 4.5 we have that

\[
O_{X,x} \otimes_{O_{\mathfrak{p},f(x)}} k(f(x)) = k(x)
\]

with \(k(x)|k(f(x))\) a finite extension and therefore, \(f\) is a quasi-covering at \(x\) (see Definition 2.8).  \hfill \(\square\)
Corollary 4.8. Let $f: \mathfrak{X} \to \mathfrak{Y}$ be a pseudo-finite type morphism in NFS. If $f$ is unramified at $x \in \mathfrak{X}$, then $\dim_x f = 0$.

Proof. It is straightforward from the previous Corollary and Proposition 2.11.

Proposition 4.9. Let $f: \mathfrak{X} \to \mathfrak{Y}$ be a pseudo-finite type morphism in NFS. Given $x \in \mathfrak{X}$ and $y = f(x)$ the following conditions are equivalent:

1. $f$ is unramified at $x$.
2. $f_0: X_0 \to Y_0$ is unramified at $x$ and $\hat{O}_{\mathfrak{X},x} \otimes_{\hat{O}_{\mathfrak{Y},y}} k(y) = k(x)$.

Proof. If $f$ is unramified at $x$, then $f_0$ is unramified at $x$ (Proposition 4.1). Moreover, assertion (3) of Proposition 4.5 implies that $\hat{O}_{\mathfrak{X},x} \otimes_{\hat{O}_{\mathfrak{Y},y}} k(y) = k(x)$ so $(1) \Rightarrow (2)$ holds. Let us prove that $(2) \Rightarrow (1)$. Since $f_0$ is unramified at $x$ we have that $k(x) \otimes k(y)$ is a finite separable extension (cf. [9, (17.4.1)]). From the equality $\hat{O}_{\mathfrak{X},x} \otimes_{\hat{O}_{\mathfrak{Y},y}} k(y) = k(x)$ we deduce that $m_{x,x} \hat{O}_{\mathfrak{X},x} = m_{x,y} \hat{O}_{\mathfrak{X},x}$. Thus, the morphism $f$ and the point $x$ satisfy assertion (3) of Proposition 4.5 and it follows that $f$ is unramified at $x$.

Now we are ready to state the non adic version of Proposition 4.3:

Corollary 4.10. Given $f: \mathfrak{X} \to \mathfrak{Y}$ a morphism in NFS of pseudo-finite type let $\mathcal{J} \subset O_\mathfrak{X}$ and $\mathcal{K} \subset O_\mathfrak{Y}$ be $f$-compatible ideals of definition and let $f_0: X_0 \to Y_0$ be the induced morphism. The following conditions are equivalent:

1. The morphism $f$ is unramified.
2. The morphism $f_0$ is unramified and, for all $x \in \mathfrak{X}$, $f^{-1}(y) = f_0^{-1}(y)$ with $y = f(x)$.

Proof. Suppose that $f$ is unramified and fix $x \in \mathfrak{X}$ and $y = f(x)$. By Proposition 4.9 we have that $f_0$ is unramified and that $\hat{O}_{\mathfrak{X},x} \otimes_{\hat{O}_{\mathfrak{Y},y}} k(y) = k(x)$. Therefore, $\mathcal{J}(\hat{O}_{\mathfrak{X},x} \otimes_{\hat{O}_{\mathfrak{Y},y}} k(y)) = 0$ and applying Lemma 4.11 we deduce that $f^{-1}(y) = f_0^{-1}(y)$. Conversely, suppose that (2) holds and let us show that given $x \in \mathfrak{X}$, the morphism $f$ is unramified at $x$. If $y = f(x)$, we have that $f_0^{-1}(y)$ is an unramified $k(y)$-scheme at $x$ (cf. [9, (17.4.1)]) and since $f^{-1}(y) = f_0^{-1}(y)$, from Proposition 4.5 it follows that $f$ is unramified at $x$.

Lemma 4.11. Let $A$ be a $J$-adic noetherian ring such that for all open prime ideals $p \subset A$, $J_p = 0$. Then $J = 0$ and therefore, the $J$-adic topology in $A$ is the discrete topology.

Proof. Since every maximal ideal $m \subset A$ is open for the $J$-adic topology, we have that $J_m = 0$, for all maximal ideal $m \subset A$, so $J = 0$.

4.12. As a consequence of Corollary 4.10 it holds that:
• If \( f: \mathcal{X} \to \mathcal{Y} \) is an unramified morphism in NFS then \( f^{-1}(y) \) is a usual scheme for all \( x \in \mathcal{X} \) where \( y = f(x) \).

• In Corollary 4.6 assertion (2) may be written:

(2') For all \( x \in \mathcal{X}, y = f(x), f^{-1}(y) \) is a unramified \( k(y) \)-scheme at \( x \).

From Proposition 4.5 we obtain the following result, in which we provide a description of pseudo-closed immersions that will be used in the characterization of completion morphisms (Theorem 7.5).

**Corollary 4.13.** Given \( f: \mathcal{X} \to \mathcal{Y} \) in NFS, let \( J \subset O_\mathcal{X} \) and \( K \subset O_\mathcal{Y} \) be \( f \)-compatible ideals of definition and express

\[
f = \lim_{n \in \mathbb{N}} f_n.
\]

The morphism \( f \) is a pseudo-closed immersion if and only if \( f \) is unramified and \( f_0: X_0 \to Y_0 \) is a closed immersion.

**Proof.** If \( f \) is a pseudo-closed immersion, by Corollary 4.2 it follows that \( f \) is unramified. Conversely, suppose that \( f \) is unramified and that \( f_0 \) is a closed immersion and let us show that \( f_n: X_n \to Y_n \) is a closed immersion, for each \( n \in \mathbb{N} \). By [6, (4.2.2.(ii))] it suffices to prove that, for all \( x \in \mathcal{X} \) with \( y = f(x) \), the morphism \( O_{Y_n,y} \to O_{X_n,x} \) is surjective, for all \( n \in \mathbb{N} \). Fix \( x \in \mathcal{X}, y = f(x) \in \mathcal{Y} \) and \( n \in \mathbb{N} \). Since \( f_0 \) is a closed immersion, by [6, loc. cit.], we have that \( O_{Y_0,y} \to O_{X_0,x} \) is surjective and therefore, \( \text{Spf}(O_{X,x}) \to \text{Spf}(O_{\mathcal{Y},y}) \) is a pseudo-finite morphism, so, the morphism \( O_{Y_n,y} \to O_{X_n,x} \) is finite. On the other hand, the morphism \( f \) is unramified therefore by Proposition 4.1 we get that \( f_n \) is unramified and applying Proposition 4.5 we obtain that \( m_{Y_n,y}O_{X_n,x} = m_{X_n,x} \). Then by Nakayama’s lemma we conclude that \( O_{Y_n,y} \to O_{X_n,x} \) is a surjective morphism. \( \square \)

## 5. Smooth morphisms

The contents of this section can be structured in two parts. In the first part we study the relationship between the smoothness of a morphism

\[
f = \lim_{n \in \mathbb{N}} f_n
\]

in NFS and the smoothness of the ordinary scheme morphisms \( \{ f_n \}_{n \in \mathbb{N}} \). In the second part, we provide a local factorization for smooth morphisms (Proposition 5.9). In this section we also prove in Corollary 5.13 the matrix Jacobian criterion, that is a useful explicit condition in terms of a matrix rank for determining whether a closed subscheme of the affine formal space or of the affine formal disc is smooth or not.

**Proposition 5.1.** Given \( f: \mathcal{X} \to \mathcal{Y} \) in NFS, let \( J \subset O_\mathcal{X} \) and \( K \subset O_\mathcal{Y} \) be \( f \)-compatible ideals of definition and write

\[
f = \lim_{n \in \mathbb{N}} f_n.
\]
If \( f_n : X_n \rightarrow Y_n \) is smooth, for all \( n \in \mathbb{N} \), then \( f \) is smooth.

**Proof.** By [3, Proposition 4.1] we may assume that \( f \) is in \( \text{NFS}_{af} \). Let \( Z \) be an affine scheme, consider a morphism \( w : Z \rightarrow \mathfrak{Y} \), a closed \( \mathfrak{Y} \)-subscheme \( T \hookrightarrow Z \) given by a square zero ideal and a \( \mathfrak{Y} \)-morphism \( u : T \rightarrow X \). Since \( f \) and \( w \) are morphisms of affine formal schemes we find an integer \( m \geq 0 \) such that \( w^* (K^m+1)O_Z = 0 \) and \( u^* (J^m+1)O_T = 0 \) and therefore \( u \) and \( w \) factors as \( T \xrightarrow{i_m} X_m \xrightarrow{\iota_m} X \) and \( Z \xrightarrow{w_m} Y_m \xrightarrow{\iota_m} \mathfrak{Y} \), respectively. Since \( f_m \) is formally smooth, there exists a \( \mathfrak{Y} \)-morphism \( v_m : Z \rightarrow X_m \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
T & \xrightarrow{u_m} & Z \\
\downarrow & & \downarrow \\
X_m & \xrightarrow{f_m} & Y_m \\
\downarrow & & \downarrow \\
\mathfrak{X} & \xrightarrow{f} & \mathfrak{Y}
\end{array}
\]

Thus the \( \mathfrak{Y} \)-morphism \( v := i_m \circ v_m \) satisfies that \( v|_T = u \) and then, \( f \) is formally smooth. Moreover, since \( f_0 \) is a finite type morphism, it holds that \( f \) is of pseudo-finite type and therefore, \( f \) is smooth. \( \square \)

**Corollary 5.2.** Let \( f : \mathfrak{X} \rightarrow \mathfrak{Y} \) be an adic morphism in \( \text{NFS} \) and consider \( \mathcal{K} \subset O_{\mathfrak{Y}} \) an ideal of definition. The morphism \( f \) is smooth if and only if all the scheme morphisms \( \{f_n : X_n \rightarrow Y_n\}_{n \in \mathbb{N}} \), determined by the ideals of definition \( \mathcal{K} \subset O_{\mathfrak{Y}} \) and \( \mathcal{J} = f^*(\mathcal{K})O_{\mathfrak{X}} \), are smooth.

**Proof.** If \( f \) is adic, by [6, (10.12.2)], we have that for each \( n \in \mathbb{N} \), the diagram

\[
\begin{array}{ccc}
\mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \\
\downarrow & & \downarrow \\
X_n & \xrightarrow{f_n} & Y_n
\end{array}
\]

is a cartesian square. Then by base-change ([3, Proposition 2.9 (2)]) we have that \( f_n \) is smooth, for all \( n \in \mathbb{N} \). The converse follows from the previous proposition. \( \square \)

Next example shows us that the converse of Proposition 5.1 does not hold in general.

**Example 5.3.** Let \( K \) be a field and \( \mathbb{A}^1_K = \text{Spec}(K[T]) \). For the closed subset \( X = V(\langle T \rangle) \subset \mathbb{A}^1_K \), Proposition 3.7 implies that the canonical completion morphism

\[
\mathbb{D}^1_K \xrightarrow{\alpha} \mathbb{A}^1_K
\]
of $\mathbb{A}_k^1$ along $X$ is étale. However, picking in $\mathbb{A}_k^1$ the ideal of definition $0$, the morphisms
\[
\text{Spec}(K[T]/(T)^{n+1}) \xrightarrow{\kappa_n} \mathbb{A}^1_k
\]
are not flat, whence it follows that $\kappa_n$ can not be smooth for all $n \in \mathbb{N}$ (see [3, Proposition 4.8]).

Our next goal will be to determine the relation between smoothness of a morphism
\[
f = \lim_{\overline{n} \in \mathbb{N}} f_n
\]
and that of $f_0$ (Corollaries 5.6 and 5.8). In order to do that, we need to characterize smoothness locally.

**Proposition 5.4.** Let $f : X \to Y$ be a pseudo-finite type morphism in NFS. Given $x \in X$ and $y = f(x)$ the following conditions are equivalent:

1. The morphism $f$ is smooth at $x$.
2. $\mathcal{O}_{X,x}$ is a formally smooth $\mathcal{O}_{Y,y}$-algebra for the adic topologies.
3. $\hat{\mathcal{O}}_{X,x}$ is a formally smooth $\hat{\mathcal{O}}_{Y,y}$-algebra for the adic topologies.
4. The morphism $f$ is flat at $x$ and $f^{-1}(y)$ is a $k(y)$-formal scheme smooth at $x$.

**Proof.** The question is local and $f$ is of pseudo-finite type, so we may assume that $f : X = \text{Spf}(A) \to \mathfrak{X} = \text{Spf}(B)$ is in NFS$_{af}$, with $A = B\{T_1, \ldots, T_r\}[[Z_1, \ldots, Z_s]]/I$ and $I \subset B' := B\{T_1, \ldots, T_r\}[[Z_1, \ldots, Z_s]]$ an ideal ([3, Proposition 1.7]). Let $p \subset A$ be the open prime ideal corresponding to $x$, let $q \subset B'$ be the open prime such that $p = q/I$ and let $r \subset B$ be the open prime ideal corresponding to $y$.

(1) $\Rightarrow$ (3) Replacing $X$ by a sufficiently small open neighborhood of $x$ we may suppose that $A$ is a formally smooth $B$-algebra. Then, by [8, (0, 19.3.5)] we have that $A_p$ is a formally smooth $B_c$-algebra and [8, (0, 19.3.6)] implies that $\hat{\mathcal{O}}_{X,x} = A_p$ is a formally smooth $\hat{\mathcal{O}}_{Y,y}$-algebra.

(2) $\Leftrightarrow$ (3) It is a consequence of [8, (0, 19.3.6)].

(3) $\Rightarrow$ (1) By [8, (0, 19.3.6)], assertion (3) is equivalent to $A_p$ being a formally smooth $B_c$-algebra. Then Zariski’s Jacobian criterion ([3, Proposition 4.14]) implies that the morphism of $\hat{A}_p$-modules
\[
\frac{I}{I^2} \to \Omega^1_{B'_c/B_c} \hat{\otimes}_{B'_c} A_p
\]
is right invertible. Since $\hat{A}_p$ is a faithfully flat $A_{\{p\}}$-algebra and the $A_{\{p\}}$-module$(\hat{\Omega}^1_{B'_c/B_c} \otimes_{B'_c} A)_{\{p\}}$ is projective (see [3, Proposition 4.8]), it holds that the morphism
\[
\left( \frac{I}{I^2} \right)_{\{p\}} \to (\hat{\Omega}^1_{B'_c/B_c} \otimes_{B'_c} A)_{\{p\}}
\]
is right invertible by \([8, (0, 19.1.14.(ii))]\). From the equivalence of categories \([6, (10.10.2)]\) we find an open subset \(U \subset X\) with \(x \in U\) such that the morphism

\[
\left(\frac{I}{I^2}\right)^{\Delta} \rightarrow \hat{\Omega}_{B_{q}^1 / B_{q}}^{1} \otimes_{\hat{O}_{B_{q}^1 / B_{q}}} \hat{O}_{X}
\]

is right invertible over \(U\). Now, by Zariski’s Jacobian criterion for formal schemes \([3, \text{Corollary 4.15}]\) it follows that \(f\) is smooth in \(U\).

\((3) \Rightarrow (4)\) By \([8, (0, 19.3.8)]\) we have that \(\hat{O}_{X,x}\) is a formally smooth \(\hat{O}_{Y,y}\)-algebra for the topologies given by the maximal ideals. Then it follows from \([8, (0, 19.7.1)]\) that \(\hat{O}_{X,x}\) is \(\hat{O}_{Y,y}\)-flat and by 3.1, \(f\) is flat at \(x\). Moreover from \([8, (0, 19.3.5)]\) we deduce that \(\hat{O}_{X,x} \otimes_{\hat{O}_{Y,y}} k(y)\) is a formally smooth \(k(y)\)-algebra for the adic topologies or, equivalently, by \((3) \Leftrightarrow (1)\), \(f^{-1}(y)\) is a \(k(y)\)-formal scheme smooth at \(x\).

\((4) \Rightarrow (3)\) By 3.1 we have that \(A_p\) is a flat \(B_r\)-module and therefore, it holds that

\[(5.4.1)\quad 0 \rightarrow \frac{I_q}{I^2_q} \rightarrow \frac{B'_{q}}{\tau B'_{q}} \rightarrow \frac{A_p}{\tau A_p} \rightarrow 0\]

is an exact sequence. On the other hand, since \(f^{-1}(y)\) is a \(k(y)\)-formal scheme smooth at \(x\), from \((1) \Rightarrow (2)\) we deduce that \(\hat{O}_{X,x} \otimes_{\hat{O}_{Y,y}} k(y)\) is a formally smooth \(k(y)\)-algebra for the adic topologies or, equivalently by \([8, (0, 19.3.6)]\), \(A_p/\tau A_p\) is a formally smooth \(k(\tau)\)-algebra for the adic topologies. Applying Zariski’s Jacobian criterion \([3, \text{Proposition 4.14}]\), we have that the morphism

\[
\frac{I_q}{I^2_q} \otimes_{B_r} k(\tau) \rightarrow (\hat{\Omega}_{B'_{q} / B_q}^{1})_{q} \otimes_{B'_{q}} A_p \otimes_{B_r} k(\tau)
\]

is right invertible. Now, since \((\hat{\Omega}_{B'_{q} / B_q}^{1})_{q}\) is a projective \(B'_{q}\)-module (see \([3, \text{Proposition 4.8}]\) by \([6, (0, 6.7.2)]\) we obtain that

\[
\frac{I_q}{I^2_q} \rightarrow \hat{\Omega}_{B'_{q} / B_q}^{1} \otimes_{B'_{q}} \hat{A}_p
\]

is right invertible. Again, by the Zariski’s Jacobian criterion, \(A_p\) is a formally smooth \(B_r\)-algebra for the adic topologies or, equivalently by \([8, (0, 19.3.6)]\), \(A_p\) is a formally smooth \(\hat{B}_r\)-algebra.

\[\square\]

**Corollary 5.5.** Let \(f : X \rightarrow Y\) be a pseudo-finite type morphism in \(\text{NFS}\). The following conditions are equivalent:

1. The morphism \(f\) is smooth.
2. For all \(x \in X\), \(\hat{O}_{X,x}\) is a formally smooth \(\hat{O}_{Y,f(x)}\)-algebra for the adic topologies.
For all \( x \in X \), \( \hat{O}_{X,x} \) is a formally smooth \( \hat{O}_{Y,f(x)} \)-algebra for the adic topologies.

(4) The morphism \( f \) is flat and \( f^{-1}(f(x)) \) is a \( k(f(x)) \)-formal scheme smooth at \( x \), for all \( x \in X \).

**Corollary 5.6.** Let \( f : \mathfrak{X} \to \mathfrak{Y} \) be an adic morphism in NFS and let \( J \subset O_{\mathfrak{X}} \) be an ideal of definition. Put

\[
f = \lim_{n \in \mathbb{N}} f_n
\]

using the ideals of definition \( K \subset O_{\mathfrak{Y}} \) and \( J = f^*(K)O_X \subset O_X \). Then, the morphism \( f \) is smooth if and only if it is flat and the morphism \( f_0 : X_0 \to Y_0 \) is smooth.

**Proof.** Since \( f \) is adic, the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & \mathfrak{Y} \\
\uparrow & & \uparrow \\
X_0 & \xrightarrow{f_0} & Y_0 \\
\end{array}
\]

is a cartesian square ([6, (10.12.2)]). If \( f \) is smooth, by base-change it follows that \( f_0 \) is smooth. Moreover by [3, Proposition 4.8] we have that \( f \) is flat. Conversely, if \( f \) is adic, by 1.12, we have that \( f^{-1}(f(x)) = f_0^{-1}(f(x)) \), for all \( x \in X \). Therefore, since \( f_0 \) is smooth, by base-change it holds that \( f^{-1}(f(x)) \) is a \( k(f(x)) \)-scheme smooth at \( x \), for all \( x \in X \) and applying Corollary 5.5 we conclude that \( f \) is smooth. \( \square \)

The upcoming example shows that the last result is not true without assuming the adic hypothesis for the morphism \( f \).

**Example 5.7.** Given \( K \) a field, let \( \mathbb{P}^n_K \) be the \( n \)-dimensional projective space and \( X \subset \mathbb{P}^n_K \) a closed subscheme that is not smooth over \( K \). If we denote by \( (\mathbb{P}^n_K)_X \) the completion of \( \mathbb{P}^n_K \) along \( X \), by Proposition 3.11 we have that the morphism

\[
(\mathbb{P}^n_K)_X \xrightarrow{f} \text{Spec}(K)
\]

is smooth but \( f_0 : X \to \text{Spec}(K) \) is not smooth.

**Corollary 5.8.** Given \( f : \mathfrak{X} \to \mathfrak{Y} \) a morphism in NFS let \( J \subset O_{\mathfrak{X}} \) and \( K \subset O_{\mathfrak{Y}} \) be \( f \)-compatible ideals of definition. Write

\[
f = \lim_{n \in \mathbb{N}} f_n.
\]

If \( f \) is flat, \( f_0 : X_0 \to Y_0 \) is a smooth morphism and \( f^{-1}(f(x)) = f_0^{-1}(f(x)) \), for all \( x \in \mathfrak{X} \), then \( f \) is smooth.

**Proof.** Since \( f_0 \) is smooth and \( f^{-1}(y) = f_0^{-1}(y) \) for all \( y = f(x) \) with \( x \in \mathfrak{X} \), we deduce that \( f^{-1}(y) \) is a smooth \( k(y) \)-scheme. Besides, by hypothesis \( f \) is flat and Corollary 5.5 implies that \( f \) is smooth. \( \square \)
Example 5.7 illustrates that the converse of the last corollary does not hold.

Every smooth morphism \( f : X \to Y \) in \( \text{Sch} \) is locally a composition of an étale morphism \( U \to \mathbb{A}^r_Y \) and a projection \( \mathbb{A}^r_Y \to Y \). Proposition 5.9 generalizes this fact for smooth morphisms in \( \text{NFS} \). The same result has already appeared stated in local form in [17, Proposition 1.11]. We include it here for completeness.

**Proposition 5.9.** Let \( f : X \to Y \) be a pseudo-finite type morphism in \( \text{NFS} \).

The morphism \( f \) is smooth at \( x \in X \) if and only if there exists an open subset \( U \subset X \) with \( x \in U \) such that \( f|_U \) factors as

\[
U \xrightarrow{g} \mathbb{A}^n_Y \xrightarrow{p} Y
\]

where \( g \) is étale, \( p \) is the canonical projection and \( n = \text{rg}(\Omega^1_{X,y}/\mathcal{O}_Y) \).

**Proof.** As this is a local question, we may assume that \( f : X = \text{Spf}(A) \to Y = \text{Spf}(B) \) is a smooth morphism in \( \text{NFS}_{af} \). By [3, Proposition 4.8] and by [6, (10.10.8.6)] we have that \( \tilde{\Omega}^1_{A/B} \) is a projective \( A \)-module of finite type and therefore, if \( p \subset A \) is the open prime ideal corresponding to \( x \), there exists \( h \in A \setminus p \) such that \( \Gamma(D(h), \tilde{\Omega}^1_{X,y}) = \tilde{\Omega}^1_{A(h)/B} \) is a free \( A_{(h)} \)-module of finite type. Put \( U = \text{Spf}(A_{(h)}) \). Given \( \{d_1, \ldots, d_n\} \) a basis of \( \tilde{\Omega}^1_{A(h)/B} \) consider the morphism of \( Y \)-formal schemes

\[
U \xrightarrow{g} \mathbb{A}^n_Y = \text{Spf}(B\{T_1, \ldots, T_n\})
\]

defined by the continuous morphism of topological \( B \)-algebras

\[
B\{T_1, \ldots, T_n\} \twoheadrightarrow A_{(h)}
\]

See [6, (10.2.2) and (10.4.6)]. The morphism \( g \) satisfies that \( f|_U = p \circ g \).

Moreover, we deduce that \( g^*\tilde{\Omega}^1_{\mathbb{A}^n_Y/Y} \cong \tilde{\Omega}^1_{X,y} \) (see the definition of \( g \)) and by [3, Corollary 4.13] we have that \( g \) is étale. \( \square \)

**Corollary 5.10.** Let \( f : X \to Y \) be a smooth morphism at \( x \in X \) and \( y = f(x) \). Then

\[
\dim_x f = \text{rg}(\tilde{\Omega}^1_{X,x}/\mathcal{O}_{X,y}).
\]

**Proof.** Put \( n = \text{rg}(\tilde{\Omega}^1_{X,x}/\mathcal{O}_{X,y}) \). By Proposition 5.9 there exists \( U \subset X \) with \( x \in U \) such that \( f|_U \) factors as \( U \xrightarrow{g} \mathbb{A}^n_Y \xrightarrow{p} Y \) where \( g \) is an étale morphism and \( p \) is the canonical projection. Applying [3, Proposition 4.8] we have that \( f|_U \) and \( g \) are flat morphisms and therefore,

\[
\dim_x f = \dim \frac{\tilde{\mathcal{O}}_{X,x} \otimes \tilde{\mathcal{O}}_{Y,y}}{\mathcal{O}_{X,y}} k(y) = \dim \tilde{\mathcal{O}}_{X,x} - \dim \tilde{\mathcal{O}}_{Y,y}
\]

\[
\dim_x g = \dim \frac{\tilde{\mathcal{O}}_{X,x} \otimes \tilde{\mathcal{O}}_{\mathbb{A}^n_Y,g(x)}}{\mathcal{O}_{\mathbb{A}^n_Y,g(x)}} k(g(x)) = \dim \tilde{\mathcal{O}}_{X,x} - \dim \tilde{\mathcal{O}}_{\mathbb{A}^n_Y,g(x)}.
\]
Now, since $g$ is unramified by Corollary 4.8 we have that $\dim_x g = 0$ and therefore $\dim_x f = \dim \mathcal{O}_{X, g(x)} - \dim \mathcal{O}_{\mathfrak{g}, y} = n$. \hfill \Box

**Proposition 5.11.** Let $f : X \to \mathfrak{Y}$ be a morphism of pseudo-finite type and let $X' \hookrightarrow X$ be a closed immersion given by the ideal $\mathcal{I} \subset \mathcal{O}_X$ and put $f' = f|_{X'}$. If $f$ is smooth at $x \in X'$, $n = \dim_x f$ and $y = f(x)$ the following conditions are equivalent:

1. The morphism $f'$ is smooth at $x$ and $\dim_x f'^{-1}(y) = n - m$.
2. The natural sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{I} \mathcal{I}/\mathcal{I}^2 \to \widehat{\Omega}^1_{X/\mathfrak{Y}} \otimes_{\mathcal{O}_X} \mathcal{O}_{X'} \to \widehat{\Omega}^1_{X'/\mathfrak{Y}} \to 0$$

is exact\(^2\) at $x$ and, on a neighborhood of $x$, the displayed $\mathcal{O}_{X'}$-Modules are locally free of ranks $m$, $n$ and $n - m$, respectively.

**Proof.** Since $f : X \to \mathfrak{Y}$ is a smooth morphism at $x$, replacing $X$, if necessary, by a smaller neighborhood of $x$, we may assume that $f : X = \text{Spf}(A) \to \mathfrak{Y} = \text{Spf}(B)$ is a morphism in NFS\(_{af}\) smooth at $x$ and that $X' = \text{Spf}(A/I)$. Therefore, applying [3, Proposition 4.8] and Corollary 5.10 we have that $\widehat{\Omega}^1_{X/\mathfrak{Y}}$ is a locally free $\mathcal{O}_X$-Module of rank $n$.

Let us prove that (1) $\Rightarrow$ (2). Replacing $X'$ with a smaller neighborhood of $x$ if necessary, we may also assume that $f' : X' \to \mathfrak{Y}$ is a smooth morphism. Then, by an argument along the lines of the previous paragraph, it follows that $\widehat{\Omega}^1_{X'/\mathfrak{Y}}$ is a locally free $\mathcal{O}_{X'}$-Module of rank $n - m$. Zariski’s Jacobian criterion for formal schemes ([3, Corollary 4.15]) implies that the sequence

$$0 \to \mathcal{I} \mathcal{I}/\mathcal{I}^2 \to \widehat{\Omega}^1_{X/\mathfrak{Y}} \otimes_{\mathcal{O}_X} \mathcal{O}_{X'} \to \widehat{\Omega}^1_{X'/\mathfrak{Y}} \to 0$$

is exact and split, from which we deduce that $\mathcal{I}/\mathcal{I}^2$ is a locally free $\mathcal{O}_{X'}$-Module of rank $m$.

Conversely, applying [6, (0, 5.5.4)] and [3, Proposition 3.13] we deduce that there exists an open formal subscheme $\mathcal{U} \subset X'$ with $x \in \mathcal{U}$ such that

$$0 \to \left( \mathcal{I}/\mathcal{I}^2 \right) |_{\mathcal{U}} \to (\widehat{\Omega}^1_{X/\mathfrak{Y}} \otimes_{\mathcal{O}_X} \mathcal{O}_{X'})|_{\mathcal{U}} \to (\widehat{\Omega}^1_{X'/\mathfrak{Y}})|_{\mathcal{U}} \to 0$$

is exact and split. From Zariski’s Jacobian criterion it follows that $f'|_{\mathcal{U}}$ is smooth and therefore, $f'$ is smooth at $x$. \hfill \Box

**Remark.** The natural sequence in Proposition 5.11 is the Second Fundamental Exact Sequence associated to the morphisms $X' \hookrightarrow X \overset{f}{\to} X$ ([3, Proposition 3.13]).

\(^2\)Let $(X, \mathcal{O}_X)$ be a ringed space. We say that the sequence of $\mathcal{O}_X$-Modules $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is exact at $x \in X$ if and only if $0 \to \mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x \to 0$ is an exact sequence of $\mathcal{O}_{X,x}$-modules.
Locally, a pseudo-finite type morphism \( f : X \to Y \) factors as \( \mathcal{U} \xrightarrow{j} \mathbb{D}_{s_{\mathcal{Y}}}^{r} \xrightarrow{p} Y \) where \( j \) is a closed immersion (see [3, Proposition 1.7]). In Corollary 5.13 we provide a criterion in terms of a matrix rank that tells whether \( \mathcal{U} \) is smooth over \( Y \) or not.

5.12. Let \( \mathcal{Y} = \text{Spf}(A) \in \text{NFS}_{\text{af}} \). Consider \( X \subset \mathbb{D}_{s_{\mathcal{Y}}}^{r} \), a closed formal subscheme given by an ideal \( I = I^{\triangle} \), with \( I = \langle g_1, g_2, \ldots, g_l \rangle \subset A\{T\}[\![Z]\!] \) where \( T = T_1, T_2, \ldots, T_r \) and \( Z = Z_1, Z_2, \ldots, Z_s \) are two sets of of indeterminates. From [3, 3.14] we have that

\[
\{dT_1, \ldots, dT_r, dZ_1, \ldots, dZ_s\}
\]
is a basis of \( \hat{\Omega}^1_{A\{T\}[\![Z]\!]/A} \) and also that given \( g \in A\{T\}[\![Z]\!] \) it holds that:

\[
\hat{d}g = \sum_{i=1}^{r} \frac{\partial g}{\partial T_i} \hat{d}T_i + \sum_{j=1}^{s} \frac{\partial g}{\partial Z_j} \hat{d}Z_j,
\]

where \( \hat{d} \) is the complete canonical derivation of \( A\{T\}[\![Z]\!] \) over \( A \). For any \( g \in A\{T\}[\![Z]\!], w \in \{dT_1, \ldots, dT_r, dZ_1, \ldots, dZ_s\} \) and \( x \in X \), denote by \( \frac{\partial g}{\partial w}(x) \) the image of \( \frac{\partial g}{\partial w} \in A\{T\}[\![Z]\!] \) in \( k(x) \). We will call

\[
\text{Jac}_{X/\mathcal{Y}}(x) = \begin{pmatrix}
\frac{\partial g_1}{\partial T_1}(x) & \ldots & \frac{\partial g_1}{\partial T_r}(x) & \ldots & \frac{\partial g_1}{\partial Z_1}(x) & \ldots & \frac{\partial g_1}{\partial Z_s}(x) \\
\frac{\partial g_2}{\partial T_1}(x) & \ldots & \frac{\partial g_2}{\partial T_r}(x) & \ldots & \frac{\partial g_2}{\partial Z_1}(x) & \ldots & \frac{\partial g_2}{\partial Z_s}(x) \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\frac{\partial g_l}{\partial T_1}(x) & \ldots & \frac{\partial g_l}{\partial T_r}(x) & \ldots & \frac{\partial g_l}{\partial Z_1}(x) & \ldots & \frac{\partial g_l}{\partial Z_s}(x)
\end{pmatrix}
\]

the Jacobian matrix of \( X \) over \( \mathcal{Y} \) at \( x \). This matrix depends on the chosen generators of \( I \) and therefore, the notation \( \text{Jac}_{X/\mathcal{Y}}(x) \) is not completely accurate.

**Corollary 5.13.** (Jacobian criterion for the affine formal space and the affine formal disc.) With the previous notations, the following assertions are equivalent:

1. The morphism \( f : X \to \mathcal{Y} \) is smooth at \( x \) and \( \dim_x f = r + s - l \).
2. There exists a subset \( \{g_1, g_2, \ldots, g_l\} \subset \{g_1, g_2, \ldots, g_k\} \) such that \( \mathcal{I}_x = \langle g_1, g_2, \ldots, g_l \rangle \mathcal{O}_{X,x} \) and \( \text{rg}(\text{Jac}_{X/\mathcal{Y}}(x)) = l \).

**Proof.** Assume (1). By Proposition 5.11 we have that the sequence

\[
0 \to \frac{I}{I^2} \to \hat{\Omega}^1_{\mathbb{D}_{s_{\mathcal{Y}}}^{r}/\mathcal{Y}} \otimes \mathcal{O}_{\mathbb{D}_{s_{\mathcal{Y}}}^{r}} \mathcal{O}_X \to \hat{\Omega}^1_{X/\mathcal{Y}} \to 0
\]
is exact at \( x \) and the corresponding \( \mathcal{O}_X \)-Modules are locally free, in a neighborhood of \( x \), of ranks \( l, r + s \) and \( r + s - l \), respectively. Therefore,

\[
(5.13.1) \quad 0 \to \frac{I}{I^2} \otimes \mathcal{O}_X k(x) \to \hat{\Omega}^1_{\mathbb{D}_{s_{\mathcal{Y}}}^{r}/\mathcal{Y}} \otimes \mathcal{O}_{\mathbb{D}_{s_{\mathcal{Y}}}^{r}} k(x) \to \hat{\Omega}^1_{X/\mathcal{Y}} \otimes \mathcal{O}_X k(x) \to 0
\]
is an exact sequence of $k(x)$-vector spaces of dimension $l$, $r + s$, $r + s - l$, respectively. Thus, there exists a set \{g_1, g_2, \ldots, g_l\} such that \{g_1(x), g_2(x), \ldots, g_l(x)\} provides a basis of $\mathcal{I}/\mathcal{I}^2 \otimes \mathcal{O}_x k(x)$ at $x$. By Nakayama’s lemma it holds that $\mathcal{I}_x = \langle g_1, g_2, \ldots, g_l \rangle \mathcal{O}_{X,x}$. Besides, from the exactness of the sequence (5.13.1) and from the equivalence of categories [6, (10.10.2)] we deduce that the set

$$\{ \hat{d}g_1(x), \hat{d}g_2(x), \ldots, \hat{d}g_l(x) \} \subset \hat{\Omega}^1_{A\{T\}|[Z]|/A} \otimes A\{T\|[Z]\} k(x)$$

is linearly independent. Therefore, \(\text{rg}(\text{Jac}_{X/Y}(x)) = l\).

Conversely, from the Second Fundamental Exact Sequence associated to the morphisms $X \hookrightarrow \mathbb{D}_{K_{\mathbb{A}^1}}$ [3, Proposition 3.13] we get the exact sequence

$$\frac{\mathcal{I}}{\mathcal{I}^2} \otimes \mathcal{O}_x k(x) \to \hat{\Omega}^1_{\mathbb{D}_{K_{\mathbb{A}^1}}/\mathcal{Y}} \otimes \mathcal{O}_{\mathbb{D}_{K_{\mathbb{A}^1}} \mathcal{Y}} k(x) \to \hat{\Omega}^1_{X/\mathcal{Y}} \otimes \mathcal{O}_x k(x) \to 0.$$ 

Since \(\text{rg}(\text{Jac}_{X/Y}(x)) = l\), we have that

$$\{ \hat{d}g_1(x), \hat{d}g_2(x), \ldots, \hat{d}g_l(x) \} \subset \hat{\Omega}^1_{A\{T\}|[Z]|/A} \otimes A\{T\|[Z]\} k(x)$$

is a linearly independent set. Extending this set to a basis of the vector space \(\hat{\Omega}^1_{A\{T\}|[Z]|/A} \otimes A\{T\|[Z]\} k(x)\), by Nakayama’s lemma we find a basis \(\mathcal{B} \subset \hat{\Omega}^1_{A\{T\}|[Z]|/A}\) such that \(\{ \hat{d}g_1, \hat{d}g_2, \ldots, \hat{d}g_l \} \subset \mathcal{B}\) and therefore

$$\{ \hat{d}g_1, \hat{d}g_2, \ldots, \hat{d}g_l \} \subset \hat{\Omega}^1_{A\{T\}|[Z]|/A} \otimes A\{T\|[Z]\} A\{T\|[Z]\} / I$$

is a linearly independent set at $x$. Thus the set \(\{g_1, g_2, \ldots, g_l\}\) provides a basis of $\mathcal{I}/\mathcal{I}^2$ at $x$ and by the equivalence of categories [6, (10.10.2)] we have that the sequence of $\mathcal{O}_X$-Modules

$$0 \to \frac{\mathcal{I}}{\mathcal{I}^2} \to \hat{\Omega}^1_{\mathbb{D}_{K_{\mathbb{A}^1}}/\mathcal{Y}} \otimes \mathcal{O}_{\mathbb{D}_{K_{\mathbb{A}^1}} \mathcal{Y}} \mathcal{O}_x \to \hat{\Omega}^1_{X/\mathcal{Y}} \to 0$$

is split exact at $x$ of locally free Modules of ranks $l$, $r + s$ and $r + s - l$, respectively. Applying Proposition 5.11 it follows that $f$ is smooth at $x$ and $\dim_x f = r + s - l$. \(\square\)

Notice that the matrix form of the Jacobian criterion for the affine formal space and the affine formal disc (Corollary 5.13) generalize the usual matrix form of the Jacobian criterion for the affine space in $\text{Sch}$ ([4, Ch. VII, Theorem (5.14)])

6. Étale morphisms

The main results of this section are consequences of those obtained in Sections 4 and 5. They will allow us to characterize in Section 7 two important classes of étale morphisms: open immersions and completion morphisms.

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Proposition 6.1. Given \( f : X \to Y \) in NFS let \( J \subset \mathcal{O}_X \) and \( K \subset \mathcal{O}_Y \) be \( f \)-compatible ideals of definition. Using these ideals, set
\[
f = \lim_{n \in \mathbb{N}} f_n.
\]
If \( f_n : X_n \to Y_n \) is étale, \( \forall n \in \mathbb{N} \), then \( f \) is étale.

Proof. The sum of Proposition 4.1 and Proposition 5.1. \( \square \)

Corollary 6.2. Let \( f : X \to Y \) be an adic morphism in NFS and let \( K \subset \mathcal{O}_Y \) be an ideal of definition. Consider \( \{f_n\}_{n \in \mathbb{N}} \) the direct system of morphisms of schemes associated to the ideals of definition \( K \subset \mathcal{O}_Y \) and \( J = f^*(K)\mathcal{O}_X \subset \mathcal{O}_X \). The morphism \( f \) is étale if and only if the morphisms \( f_n : X_n \to Y_n \) are étale \( \forall n \in \mathbb{N} \).

Proof. It follows from Proposition 4.1 and Corollary 5.2. \( \square \)

Proposition 6.3. Let \( f : X \to Y \) be an adic morphism in NFS and let \( f_0 : X_0 \to Y_0 \) be the morphism of schemes associated to the ideals of definition \( K \subset \mathcal{O}_Y \) and \( J = f^*(K)\mathcal{O}_X \subset \mathcal{O}_X \). Then, \( f \) is étale if and only if \( f \) is flat and \( f_0 \) is étale.

Proof. Put together Proposition 4.3 and Corollary 5.6. \( \square \)

Note that Example 5.3 shows that in the non adic case the last two results do not hold and also that, in general, the converse of Proposition 6.1 is not true.

Proposition 6.4. Let \( f \) be a pseudo-finite type morphism in NFS and choose \( J \subset \mathcal{O}_X \) and \( K \subset \mathcal{O}_Y \) \( f \)-compatible ideals of definition. Write
\[
f = \lim_{n \in \mathbb{N}} f_n.
\]
If \( f_0 : X_0 \to Y_0 \) is étale, \( f \) is flat and \( f^{-1}(f(x)) = f_0^{-1}(f(x)) \), for all \( x \in X \), then \( f \) is étale.

Proof. It follows from Corollary 4.10 and Corollary 5.8. \( \square \)

Example 5.3 shows that the converse of the last result is not true. Next Proposition gives us a local characterization of étale morphisms.

Proposition 6.5. Let \( f : X \to Y \) be a morphism in NFS of pseudo-finite type, let \( x \in X \) and \( y = f(x) \), the following conditions are equivalent:

1. \( f \) is étale at \( x \).
2. \( \mathcal{O}_{X,x} \) is a formally étale \( \mathcal{O}_{Y,y} \)-algebra for the adic topologies.
3. \( \mathcal{O}_{X,x} \) is a formally étale \( \mathcal{O}_{Y,y} \)-algebra for the adic topologies.
4. \( f \) is flat at \( x \) and \( f^{-1}(y) \) is a \( k(y) \)-formal scheme étale at \( x \).
5. \( f \) is flat and unramified at \( x \).
6. \( f \) is flat at \( x \) and \( (\Omega^1_{X/Y})_x = 0 \).
7. \( f \) is smooth at \( x \) and a quasi-covering at \( x \).
Proof. Applying Proposition 4.5 and Proposition 5.4 we have that
\[(5) \iff (1) \iff (2) \iff (2') \iff (3) \iff (4) \iff (4').\]

Let \(C := \widehat{O}_{X,x} \otimes \widehat{O}_{Y,y} k(y).\) To show \((4) \implies (5),\) by Corollary 4.7 it is only left to prove that \(f\) is smooth at \(x.\) By hypothesis, we have that \(f\) is unramified at \(x\) and by Proposition 4.5, it follows that \(C = k(x)\) and \(k(x)|k(y)\) is a finite separable extension, therefore, formally étale. Since \(f\) is flat at \(x,\) by Proposition 5.4 we conclude that \(f\) is smooth at \(x.\)

To prove that \((5) \implies (1),\) it suffices to check that \(f\) is unramified at \(x\) or, equivalently by Proposition 4.5, that \(C = k(x)\) and \(k(x)|k(y)\) is a finite separable extension. As \(f\) is smooth at \(x,\) applying Proposition 5.4, we have that \(\widehat{O}_{X,x}\) is a formally smooth \(\widehat{O}_{Y,f(x)}\)-algebra for the adic topologies. Then by base-change it holds that \(C\) is a formally smooth \(k(y)\)-algebra. By \([8, (0, 19.3.8)]\) we have that \(C\) is a formally smooth \(k(y)\)-algebra for the topologies given by the maximal ideals and from \([13, \text{Lemma 1, p. 216}]\) it holds that \(C\) is a regular local ring. Besides, by hypothesis we have that \(C\) is a finite \(k(y)\)-module, therefore, an artinian ring, so \(C = k(x).\) Since \(k(x) = C\) is a formally smooth \(k(y)\)-algebra we have that \(k(x)|k(y)\) is a separable extension (cf. \([8, (0, 19.6.1)]\)).

\[\Box\]

Corollary 6.6. Let \(f : X \to Y\) be a pseudo-finite type morphism in \(\text{NFS}.\)

The following conditions are equivalent:

\begin{enumerate}
  \item \(f\) is étale.
  \item For all \(x \in X,\) \(O_{X,x}\) is a formally étale \(O_{Y,f(x)}\)-algebra for the adic topologies.
  \item For all \(x \in X,\) \(\widehat{O}_{X,x}\) is a formally étale \(\widehat{O}_{Y,f(x)}\)-algebra for the adic topologies.
  \item For all \(x \in X,\) \(f^{-1}(f(x))\) is a \(k(f(x))\)-formal scheme étale at \(x\) and \(f\) is flat.
  \item \(f\) is flat and unramified.
  \item \(f\) is flat and \(\Omega^1_{X/Y} = 0.\)
  \item \(f\) is smooth and a quasi-covering.
\end{enumerate}

Example 6.7. Given a field \(K,\) the canonical morphism \(\mathbb{D}^1_K \to \text{Spec}(K)\) is smooth, pseudo-quasi-finite but it is not étale.

In \(\text{Sch}\) a morphism is étale if and only if it is smooth and quasi-finite. The previous example shows that in \(\text{NFS}\) there are smooth and pseudo-quasi-finite morphisms that are not étale. That is why we consider quasi-coverings in \(\text{NFS}\) (see Definition 2.8) as the right generalization of quasi-finite morphisms in \(\text{Sch}.\)

7. Structure theorems of the infinitesimal lifting properties

We begin with two results that will be used in the proof of the remainder results of this section.
Proposition 7.1. Consider a formally étale morphism \( f : \mathcal{X} \to \mathcal{Y} \) and a morphism \( g : \mathcal{S} \to \mathcal{Y} \), both in \( \text{NFS} \). Take \( \mathcal{L} \subset \mathcal{O}_\mathcal{S} \) an ideal of definition of \( \mathcal{S} \) and write

\[
\mathcal{S} = \lim_{\longrightarrow} S_n.
\]

If \( h_0 : S_0 \to \mathcal{X} \) is a morphism in \( \text{NFS} \) that makes the diagram

\[
\begin{array}{ccc}
S_0 & \xleftarrow{h_0} & \mathcal{S} \\
\downarrow & & \downarrow g \\
\mathcal{X} & \xrightarrow{f} & \mathcal{Y}
\end{array}
\]

commutative, where \( S_0 \hookrightarrow \mathcal{S} \) is the canonical closed immersion, then there exists a unique \( \mathcal{Y} \)-morphism \( l : \mathcal{S} \to \mathcal{X} \) in \( \text{NFS} \) such that \( l|_{S_0} = h_0 \).

Proof. By induction on \( n \) we are going to construct a collection of morphisms \( \{h_n : S_n \to \mathcal{X}\}_{n \in \mathbb{N}} \) such that the diagrams

\[
\begin{array}{ccc}
S_{n-1} & \xrightarrow{h_{n-1}} & S_n \\
\downarrow & & \downarrow h_n \\
\mathcal{X} & \xrightarrow{f} & \mathcal{Y}
\end{array}
\]

commute. For \( n = 1 \), by [3, 2.4] there exists a unique morphism \( h_1 : S_1 \to \mathcal{X} \) such that \( h_1|_{S_0} = h_0 \) and \( g|_{S_1} = f \circ h_1 \). Now let \( n \in \mathbb{N}, n > 1 \) and suppose we already have for all \( 0 < k < n \) morphisms \( h_k : S_k \to \mathcal{X} \) such that \( h_k|_{S_{k-1}} = h_{k-1} \) and \( g|_{S_k} = f \circ h_k \). Then by [3, loc. cit.] there exists a unique morphism \( h_n : S_n \to \mathcal{X} \) such that \( h_n|_{S_{n-1}} = h_{n-1} \) and \( g|_{S_n} = f \circ h_n \). It is straightforward that

\[
l := \lim_{\longrightarrow} h_n
\]

is a morphism of formal schemes and is the unique one such that the diagram

\[
\begin{array}{ccc}
S_0 & \xleftarrow{h_0} & \mathcal{S} \\
\downarrow & & \downarrow g \\
\mathcal{X} & \xrightarrow{f} & \mathcal{Y}
\end{array}
\]

commutes. \( \square \)

Corollary 7.2. Let \( f : \mathcal{X} \to \mathcal{Y} \) be an étale morphism in \( \text{NFS} \) and \( \mathcal{J} \subset \mathcal{O}_{\mathcal{X}} \) and \( \mathcal{K} \subset \mathcal{O}_{\mathcal{Y}} \) \( f \)-compatible ideals of definition such that the corresponding morphism \( f_0 : X_0 \to Y_0 \) is an isomorphism. Then \( f \) is an isomorphism.
Proof. By Proposition 7.1 there exists a (unique) morphism $g : Y \to X$ such that the following diagram is commutative

$$
\begin{array}{ccc}
Y_0 & \subset & Y \\
\downarrow^{f_0^{-1}} & & \downarrow^g \\
X_0 & \subset & X \\
\downarrow^f & & \downarrow^{1_X} \\
X & \subset & X
\end{array}
$$

Then, by [3, Proposition 2.13] it follows that $g$ is an étale morphism. Thus, applying Proposition 7.1 we have that there exists a (unique) morphism $f' : X \to Y$ such that the following diagram is commutative

$$
\begin{array}{ccc}
X_0 & \subset & X \\
\downarrow^{f_0} & & \downarrow^{f'} \\
Y_0 & \subset & Y \\
\downarrow^g & & \downarrow^{1_Y} \\
Y & \subset & Y
\end{array}
$$

From $f \circ g = 1_Y$ and $g \circ f' = 1_X$ we deduce that $f = f'$ and therefore $f$ is an isomorphism. \qed

In $\textbf{Sch}$ open immersions are characterized as being those étale morphisms that are radicial (see [9, (17.9.1)]). In the following theorem we extend this characterization and relate open immersions in formal schemes with their counterparts in schemes.

**Theorem 7.3.** Let $f : X \to Y$ be a morphism in $\textbf{NFS}$. The following conditions are equivalent:

1. $f$ is an open immersion.
2. $f$ is adic, flat and if $\mathcal{K} \subset \mathcal{O}_Y$ is an ideal of definition such that $\mathcal{J} = f^*(\mathcal{K})\mathcal{O}_X \subset \mathcal{O}_X$, the associated morphism of schemes $f_0 : X_0 \to Y_0$ is an open immersion.
3. $f$ is adic étale and radicial.
4. There are $\mathcal{J} \subset \mathcal{O}_X$ and $\mathcal{K} \subset \mathcal{O}_Y$ $f$-compatible ideals of definition such that the morphisms $f_n : X_n \to Y_n$ are open immersions, for all $n \in \mathbb{N}$.

Proof. The implication (1) $\Rightarrow$ (2) is immediate. Given $\mathcal{K} \subset \mathcal{O}_Y$ an ideal of definition, assume (2) and let us show (3). Since $f_0$ is an open immersion, is radicial, so, $f$ is radicial (see Definition 2.5 and its attached paragraph). Furthermore, $f$ is flat and $f_0$ is an étale morphism then $f$ is étale (see Proposition 6.3). Let us prove that (3) $\Rightarrow$ (4). Given $\mathcal{K} \subset \mathcal{O}_Y$ an ideal of definition and $\mathcal{J} = f^*(\mathcal{K})\mathcal{O}_X$, by Corollary 6.2 the morphisms $f_n : X_n \to Y_n$
are étale, for all $n \in \mathbb{N}$. The morphisms $f_n$ are also radicial for all $n \in \mathbb{N}$ (see Definition 2.5) and thus by [9, (17.9.1)] it follows that $f_n$ is an open immersion, for each $n \in \mathbb{N}$. Finally, suppose that (4) holds and let us see that $f$ is an open immersion. With the notations in (4), there exists an open subset $U_0 \subset Y_0$ such that $f_0$ factors as

$$X_0 \xrightarrow{f_0} U_0 \xrightarrow{i_0} Y_0$$

where $f'_0$ is an isomorphism and $i_0$ is the canonical inclusion. Let $\mathcal{U} \subset \mathcal{Y}$ be the open formal subscheme with underlying topological space $U_0$. Since the open immersion $i : \mathcal{U} \to \mathcal{Y}$ is étale, then Proposition 7.1 implies that there exists a morphism $f' : X \to U$ of formal schemes such that the diagram

$$\begin{array}{ccc}
X & \xrightarrow{f} & \mathcal{Y} \\
\downarrow{f_0} & & \downarrow{i} \\
U_0 & \xrightarrow{i_0} & Y_0
\end{array}$$

is commutative. Since the morphisms $f_n$ are étale, for all $n \in \mathbb{N}$, Proposition 6.1 implies that $f$ is étale. By [3, Proposition 2.13] we have that $f'$ is étale and applying Corollary 7.2, $f'$ is an isomorphism and therefore, $f$ is an open immersion. □

**Corollary 7.4.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a pseudo-finite type morphism in NFS. Then $f$ is unramified if and only if the diagonal morphism $\Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is an open embedding.

**Proof.** Take $\mathcal{J} \subset \mathcal{O}_X$ and $\mathcal{K} \subset \mathcal{O}_Y$ $f$-compatible ideals of definition such that $f$ can be expressed as the limit of maps of usual schemes $f_n : X_n \to Y_n$, $n \in \mathbb{N}$. The morphism $f : \mathcal{X} \to \mathcal{Y}$ is unramified if and only if $f_n$ is unramified for all $n \in \mathbb{N}$ by Proposition 4.1. By [9, Corollaire (17.4.2)] this is equivalent to $\Delta_{f_n} : X_n \to X_n \times_{Y_n} X_n$ being an open embedding for all $n \in \mathbb{N}$. But this, in turn, is equivalent to the fact that $\Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is an open embedding by Theorem 7.3. □

Every completion morphism is a pseudo-closed immersion that is flat (cf. Proposition 3.7). Next, we prove that this condition is also sufficient. Thus, we obtain a criterion to determine whether a $\mathcal{Y}$-formal scheme $\mathcal{X}$ is the completion of $\mathcal{Y}$ along a closed formal subscheme.

**Theorem 7.5.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism in NFS and let $\mathcal{J} \subset \mathcal{O}_X$ and $\mathcal{K} \subset \mathcal{O}_Y$ be $f$-compatible ideals of definition. Let $f_0 : X_0 \to Y_0$ be the corresponding morphism of ordinary schemes. The following conditions are equivalent:
(1) There exists a closed formal subscheme \( Y' \subset Y \) such that \( X = Y / Y' \) and \( f \) is the morphism of completion of \( Y \) along \( Y' \).

(2) The morphism \( f \) is a flat pseudo-closed immersion.

(3) The morphism \( f \) is \( \acute{e} \)tale and \( f_0 : X_0 \to Y_0 \) is a closed immersion.

(4) The morphism \( f \) is a smooth pseudo-closed immersion.

**Proof.** The implication (1) \( \Rightarrow \) (2) is Proposition 3.7. Let us show that (2) \( \Rightarrow \) (3). Since \( f \) is a pseudo-closed immersion, by Corollary 4.13 we have that \( f \) is unramified. Then as \( f \) is flat, Corollary 6.6 establishes that \( f \) is \( \acute{e} \)tale. The equivalence (3) \( \iff \) (4) is consequence of Corollary 4.13. Finally, we show that (3) \( \Rightarrow \) (1). By hypothesis, the morphism \( f_0 : X_0 \to Y_0 \) is a closed immersion. Consider \( \kappa : Y/X_0 \to Y \) the morphism of completion of \( Y \) along \( X_0 \) and let us prove that \( X \) and \( Y/X_0 \) are \( Y \)-isomorphic. By Proposition 3.7 the morphism \( \kappa \) is \( \acute{e} \)tale so, applying Proposition 7.1, we have that there exists a \( Y \)-morphism \( \varphi : X \to Y/X_0 \) such that the following diagram is commutative

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow \varphi & & \downarrow \kappa \\
Y/X_0 & & Y_0 \\
\end{array}
\]

From [3, Proposition 2.13] it follows that \( \varphi \) is \( \acute{e} \)tale and then by Corollary 7.2 we get that \( \varphi \) is an isomorphism. \( \square \)

**Remark.** A consequence of the proof of (3) \( \Rightarrow \) (1) is the following: Given \( Y \) in \( \text{NFS} \) and a closed formal subscheme \( Y' \subset Y \) defined by the ideal \( \mathcal{I} \subset \mathcal{O}_Y \), then for every ideal of definition \( K \subset \mathcal{O}_Y \) of \( Y \), it holds that

\[
Y/Y' = Y_0' = (Y', \mathcal{O}_Y/(\mathcal{I} + K)).
\]

7.6. Given a scheme \( Y \) and a closed subscheme \( Y_0 \subset Y \) with the same topological space, the functor \( X \mapsto X \times_Y Y_0 \) defines an equivalence between the category of \( \acute{e} \)tale \( Y \)-schemes and the category of \( \acute{e} \)tale \( Y_0 \)-schemes by [9, (18.1.2)]. In the next theorem we extend this equivalence to the category of locally noetherian formal schemes. A special case of this theorem, namely when \( Y \) is smooth over a noetherian ordinary base scheme, appears in [17, Proposition 2.4].

**Proposition 7.7.** Let \( Y \) be in \( \text{NFS} \) and \( K \subset \mathcal{O}_Y \) an ideal of definition such that

\[
Y = \lim_{n \in \mathbb{N}} Y_n.
\]
Then the functor

\[
\begin{array}{ccc}
\text{étale adic } \mathcal{Y}\text{-formal schemes} & \overset{F}{\to} & \text{étale } Y_0\text{-schemes} \\
\mathcal{X} & \rightsquigarrow & \mathcal{X} \times_{\mathcal{Y}} Y_0
\end{array}
\]

is an equivalence of categories.

**Proof.** By [14, IV, §4, Theorem 1] it suffices to prove that: (a) $F$ is full and faithful; and (b) Given $X_0$ an étale $Y_0$-scheme there exists an étale adic $\mathcal{Y}$-formal scheme $\mathcal{X}$ such that $F(\mathcal{X}) = \mathcal{X} \times_{\mathcal{Y}} Y_0 \cong X_0$.

The assertion (a) is an immediate consequence of Proposition 7.1.

Let us show (b). Given $X_0$ an étale $Y_0$-scheme in $\text{Sch}$ by [9, (18.1.2)] there exists $X_1$ a locally noetherian étale $Y_1$-scheme such that $X_1 \times_{Y_1} Y_0 \cong X_0$. Reasoning by induction on $n \in \mathbb{N}$ and using [9, loc. cit.], we get a family of schemes $\{X_n\}_{n \in \mathbb{N}}$ such that, for each $n \in \mathbb{N}$, $X_n$ is a locally noetherian étale $Y_n$-scheme and $X_n \times_{Y_n} Y_{n-1} \cong X_{n-1}$, for $n > 0$. Then

\[
\mathcal{X} := \lim_{\longrightarrow} \mathcal{X}_n
\]

is a locally noetherian adic $\mathcal{Y}$-formal scheme (by [6, (10.12.3.1)]),

\[
\mathcal{X} \times_{\mathcal{Y}} Y_0 = \lim_{\longrightarrow} (X_n \times_{Y_n} Y_0) = X_0
\]

and $\mathcal{X}$ is an étale $\mathcal{Y}$-formal scheme (see Proposition 6.1).

**Remark.** It seems plausible that there is a theory of an algebraic fundamental group for formal schemes that classifies adic étale surjective maps onto a noetherian formal scheme $\mathcal{X}$. If this is the case, the previous theorem would imply that it agrees with the fundamental group of $X_0$. We also consider feasible the existence of a bigger fundamental group classifying arbitrary étale surjective maps onto a noetherian formal scheme $\mathcal{X}$, that would give additional information on $\mathcal{X}$.

**Corollary 7.8.** Let $f : \mathcal{X} \to \mathcal{Y}$ be an étale morphism in $\text{NFS}$. Given $\mathcal{J} \subset \mathcal{O}_\mathcal{X}$ and $\mathcal{K} \subset \mathcal{O}_\mathcal{Y}$ $f$-compatible ideals of definition, if the induced morphism $f_0 : X_0 \to Y_0$ is étale, then $f$ is adic étale.

**Proof.** By Proposition 7.7 there is an adic étale morphism $f' : \mathcal{X}' \to \mathcal{Y}$ in $\text{NFS}$ such that $\mathcal{X}' \times_{\mathcal{Y}} Y_0 = X_0$. Therefore by Proposition 7.1 there exists a morphism of formal schemes $g : \mathcal{X} \to \mathcal{X}'$ such that the diagram

![Diagram of formal schemes](image-url)
is commutative. Applying \[3, Proposition 2.13\] we have that \( g \) is étale and from Corollary 7.2 we deduce that \( g \) is an isomorphism and therefore, \( f \) is adic étale.

**Corollary 7.9.** Let \( f : X \to \mathcal{Y} \) be a morphism in NFS. The morphism \( f \) is adic étale if and only if there exist \( f \)-compatible ideals of definition \( J \subset \mathcal{O}_X \) and \( K \subset \mathcal{O}_\mathcal{Y} \) such that the induced morphisms \( f_n : X_n \to Y_n \) are étale, for all \( n \in \mathbb{N} \).

**Proof.** If \( f \) is adic étale, given \( K \subset \mathcal{O}_\mathcal{Y} \) an ideal of definition, take \( J = f^*(K)\mathcal{O}_X \) the corresponding ideal of definition of \( X \). By base change, we have that the morphisms \( f_n : X_n \to Y_n \) are étale, for all \( n \in \mathbb{N} \). The converse is a consequence of Proposition 6.1 and of the previous Corollary. \( \square \)

**Proposition 7.10.** Let \( \mathcal{Y} \) be in NFS and with respect to an ideal of definition \( K \subset \mathcal{O}_\mathcal{Y} \) let us write \( \mathcal{Y} = \lim_{\leftarrow} Y_n \).

Given \( f_0 : X_0 \to Y_0 \) a morphism in \( \text{Sch} \) smooth at \( x \in X_0 \), there exists an open subset \( U_0 \subset X_0 \) such that \( f_0 \mid U_0 \) factors as

\[
U_0 \xrightarrow{f'_0} \mathbb{A}^n_{Y_0} = \text{Spec}(B_0[T]) \xrightarrow{p_0} Y_0
\]

where \( T = T_1, T_2, \ldots, T_r \) is a set of indeterminates, \( f'_0 \) is an étale morphism and \( p_0 \) is the canonical projection. The morphism \( p_0 \) lifts to a projection morphism \( p : \mathbb{A}^n_{\mathcal{Y}} = \text{Spf}(B(T)) \to \mathcal{Y} \) such that the square in the following diagram is cartesian

\[
\begin{array}{ccc}
\mathbb{A}^n_{\mathcal{Y}} & \xrightarrow{p} & \mathcal{Y} \\
\uparrow & & \uparrow \\
U_0 & \xrightarrow{f'_0} & \mathbb{A}^n_{Y_0} \rightarrow Y_0
\end{array}
\]
Applying Proposition 7.7, there exists a locally noetherian étale adic $\mathbb{A}_Y^n$-formal scheme $U$ such that $U_0 \cong U \times_{\mathbb{A}_Y^n} \mathbb{A}_Y^n$. Then $U$ is an smooth adic $\mathcal{Y}$-formal scheme such that $U_0 \cong U \times \mathcal{Y} Y_0$. □

The next theorem transfers the local description of unramified morphisms known in the case of schemes ([9, (18.4.7)]) to the framework of formal schemes.

**Theorem 7.11.** Let $f : \mathfrak{X} \to \mathcal{Y}$ be a morphism in NFS unramified at $x \in \mathfrak{X}$. Then there exists an open subset $U \subset \mathfrak{X}$ with $x \in U$ such that $f|_U$ factors as

$$U \xrightarrow{\kappa} \mathfrak{X}' \xrightarrow{f'} \mathcal{Y}$$

where $\kappa$ is a pseudo-closed immersion and $f'$ is an adic étale morphism.

**Proof.** Let $J \subset \mathcal{O}_{\mathfrak{X}}$ and $K \subset \mathcal{O}_{\mathcal{Y}}$ be ideals of definition. The morphism of schemes $f_0$ associated to these ideals is unramified at $x$ (Proposition 4.1) and by [9, (18.4.7)] there exists an open set $U_0 \subset X_0$ with $x \in U_0$ such that $f_0|_{U_0}$ factors as

$$U_0 \xrightarrow{\kappa_0} X_0' \xrightarrow{f_0'} Y_0$$

where $\kappa_0$ is a closed immersion and $f_0'$ is an étale morphism. Proposition 7.7 implies that there exists an étale adic morphism $f' : \mathfrak{X}' \to \mathcal{Y}$ in NFS such that $\mathfrak{X}' \times \mathcal{Y} Y_0 = X_0'$. Now, if $U \subset \mathfrak{X}$ is the open formal scheme with underlying topological space $U_0$, by Proposition 7.1 there exists a morphism $\kappa : U \to \mathfrak{X}'$ such that the following diagram commutes

Since $f$ is unramified, by [3, Proposition 2.13] it holds that $\kappa$ is unramified. Furthermore, $\kappa_0$ is a closed immersion, then Corollary 4.13 shows us that $\kappa$ is a pseudo-closed immersion. □

As a consequence of the last result we obtain the following local description for étale morphisms.

**Theorem 7.12.** Let $f : \mathfrak{X} \to \mathcal{Y}$ be a morphism in NFS étale at $x \in \mathfrak{X}$. Then there exists an open subset $U \subset \mathfrak{X}$ with $x \in U$ such that $f|_U$ factors as

$$U \xrightarrow{\kappa} \mathfrak{X}' \xrightarrow{f'} \mathcal{Y}$$

where $\kappa$ is a completion morphism and $f'$ is an adic étale morphism.
Proof. By the last theorem we have that there exists an open formal sub-
scheme $\mathcal{U} \subset \mathcal{X}$ with $x \in \mathcal{U}$ such that $f|_{\mathcal{U}}$ factors as

$$\mathcal{U} \xrightarrow{\kappa} \mathcal{X'} \xrightarrow{f'} \mathcal{Y}$$

where $\kappa$ is a pseudo-closed immersion and $f'$ is an adic étale morphism.
Since $f|_{\mathcal{U}}$ is étale and $f'$ is an adic étale morphism we have that $\kappa$ is étale
by [3, Proposition 2.13]. Now, applying Theorem 7.5 it follows that $\kappa$ is a completion morphism. □

**Theorem 7.13.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism in NFS smooth at $x \in \mathcal{X}$.
Then there exists an open subset $\mathcal{U} \subset \mathcal{X}$ with $x \in \mathcal{U}$ such that $f|_{\mathcal{U}}$ factors as

$$\mathcal{U} \xrightarrow{\kappa} \mathcal{X'} \xrightarrow{f''} \mathcal{Y}$$

where $\kappa$ is a completion morphism and $f''$ is an adic smooth morphism.

Proof. By Proposition 5.9 there exists an open formal subscheme $\mathcal{V} \subset \mathcal{X}$ with $x \in \mathcal{V}$ such that $f|_{\mathcal{V}}$ factors as

$$\mathcal{V} \xrightarrow{g} \mathcal{A} \xrightarrow{p} \mathcal{Y}$$

where $g$ is étale and $p$ is the canonical projection. Applying the last Theorem to the morphism $g$ we conclude that there exists an open subset $\mathcal{U} \subset \mathcal{X}$ with $x \in \mathcal{U}$ such that $f|_{\mathcal{U}}$ factors as

$$\mathcal{U} \xrightarrow{\kappa} \mathcal{X'} \xrightarrow{f''} \mathcal{Y},$$

where $\kappa$ is a completion morphism, $f''$ is an adic étale morphism and $p$ is the canonical projection, from where it follows that $f = f'' \circ p$ is adic smooth. □

**Remark.** Lipman, Nayak and Sastry note in [11, pag. 132] that this Theorem may simplify some developments related to Cousin complexes and duality on formal schemes. See the final part of Remark 10.3.10 of loc. cit.

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