Periodic and localized waves in parabolic-law media with third- and fourth-order dispersions

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We study the propagation of femtosecond light pulses inside an optical fiber medium exhibiting higher-order dispersion and cubic-quintic nonlinearities. Pulse evolution in such a system is governed by a higher-order nonlinear Schrödinger equation incorporating second-, third-, and fourth-order dispersions as well as cubic and quintic nonlinearities. Novel classes of periodic wave solutions are identified for the first time by means of an appropriate equation method. Results presented indicate the potentially rich set of periodic waves in the system under the combined influence of higher-order dispersive effects and cubic-quintic nonlinearity. Solitary waves of both bright and dark types are also obtained as a limiting case for appropriate periodic solutions. It is found that the velocity of these structures is uniquely dependent on all orders of dispersion. Conditions on the optical fiber parameters for the existence of these stable nonlinear wave-forms are presented as well.

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I. INTRODUCTION

A soliton in an optical fiber medium can form when the group velocity dispersion is exactly balanced by self-phase modulation. This localized pulse is found in two distinct types called bright and dark solitons which are existent in the anomalous and normal dispersion regimes, respectively. The unique property of optical solitons, either bright or dark, is their particle-like behavior in interaction \cite{1}. Because of their robust nature, such wave packets have been successfully utilized as the information carriers (optical bits) to transmit digital signals over long propagation distances.

Studies of soliton formation in the femtosecond time scale is an important direction of research in nonlinear optics. This because femtosecond duration pulses are required for a wide-ranging potential applications such as ultra-high-bit-rate optical communication systems, optical sampling systems, infrared time-resolved spectroscopy, and ultrafast physical processes \cite{2, 3}. But when these ultrashort pulses are injected in a fiber medium, several higher-order nonlinear effects come into play along with dispersive effects which may significantly change the physical features and stability of optical soliton propagation. Important higher-order effects include third-order dispersion, self-steepening, and self-frequency shift which become important if light pulses are shorter than 100 fs \cite{2}. Taking into account the influence of various processes appearing in the femtosecond regime, the description of signal propagation through an optical fiber medium can be achieved by use of the NLS family of equations incorporating additional higher-order terms. Compared with solitons in Kerr-like media, solitary waves supported by higher-order nonlinear and dispersive effects when they exist can demonstrate much richer dynamics as they propagate through the system. The contribution of these higher-order effects can also lead to the formation of novel structures in optical media, including for example dipole solitons \cite{4}, W-shaped solitons \cite{5}, and multipole solitons \cite{6}. So the higher-order effects play a critical role in the formation of femtosecond solitons.

Recently, attention has been focused on analyzing the dynamic behavior of soliton pulses in optical fibers exhibiting second-, third-, and fourth-order dispersions \cite{7, 8}. In addition to localized pulses, periodic waves play a significant role in the analysis of the data transmission in fiber-optic telecommunications links \cite{10}. Because of their structural stability with respect to the small input profile perturbations and collisions \cite{11}, this kind of nonlinear waves serves as a model of pulse train propagation in optics fibers \cite{10}. It is relevant to mention that the occurrence of periodic waves is not only restricted to optical fibers \cite{12, 13}, but also to other physical systems such as Bose-Einstein condensates \cite{14, 15}, nonlinear negative index materials \cite{16, 17}, and nonlocal media \cite{18}.

It is of interest to further search for new exact periodic and localized wave solutions within the higher-order NLSE framework. The exact nature of the nonlinear waves may be advantageously exploited in designing the optimal fiber system experiments. Moreover, obtaining such structures is helpful to recognize various physical phenomena described by the envelope equation. We should note that exact analytic solutions are also desired for determining certain important quantities and for simulations. In this paper, we present, to our knowledge for the first time, a complete map of the existence and propagation properties of periodic and solitary waves in an optical fiber medium exhibiting all orders of dispersion up to the fourth order as well as cubic and quintic nonlinearities. We will introduce
a special procedure, whereby it becomes possible to derive novel periodic and localized wave solutions of the envelope equation explicitly, and to determine the conditions under which these structures exist. Importantly, the physically relevant nonlinear wave solutions presented below indicate the richness of the fiber medium which is often measured by the variety of nonlinear structures that it can support. We especially note that the finding new localized waves is greatly desired as these pulses are ideal instruments for data transmission over fiber-optic communications lines.

This paper is organized as follows. Section II presents the method used for obtaining traveling wave solutions of the higher-order NLSE that governs the propagation of femtosecond light pulses through a highly dispersive optical cubic-quintic medium. In Sec. III, we identify novel classes of periodic wave solutions based on an appropriate differential equation. We also find the solitary wave solutions of the model in the long wave limit and present the conditions on the optical fiber parameters for their existence. Finally, we summarize our work in Sec. IV.

II. MODEL AND TRAVELING WAVES

Ultrashort light pulse propagation in a highly dispersive optical fiber exhibiting a parabolic nonlinearity law obeys the following high dispersive cubic-quintic NLSE [19]:

\[
i \frac{\partial \psi}{\partial z} = \alpha \frac{\partial^2 \psi}{\partial \tau^2} + i \nu \frac{\partial^3 \psi}{\partial \tau^3} - \beta \frac{\partial^4 \psi}{\partial \tau^4} - \gamma |\psi|^2 \psi + \mu |\psi|^4 \psi,
\]

where \(\psi(z, \tau)\) is the complex field envelope, \(z\) represents the distance along direction of propagation, and \(\tau = t - \beta_1 z\) is the retarded time in the frame moving with the group velocity of wave packets. Also \(\alpha = \beta_2 / 2\), \(\nu = \beta_3 / 6\), and \(\nu = \beta_4 / 24\), with \(\beta_k = (d^k \beta / d \omega^k)_{\omega = \omega_0}\) denotes the k-order dispersion of the optical fiber with \(\beta(\omega)\) is the propagation constant depending on the optical frequency. Parameters \(\gamma\) and \(\mu\) govern the effects of cubic and quintic nonlinearity, respectively.

For relatively long optical pulses having width more than 10ps, all three parameters of third- and fourth-order dispersions and quintic nonlinearity are so small that the model (1) reduces to the standard NLSE which is completely integrable by the inverse scattering method [20]. In the absence of quintic nonlinearity \((\mu = 0)\), solitonlike solution having a sech\(^2\) shape and dipole soliton solution of Eq. (1) have been found by employing a regular method [7, 8]. In practice, however, the quintic nonlinearity plays a significant role in the response of many optical materials and can therefore affect the temporal evolution of optical fields. We therefore analyze the situation in which the effect of quintic nonlinearity is important and should be taken into account along with all orders of dispersion up to the fourth order, as described by the underlying equation (1). It is worthy to mention here that optical media featuring the quintic nonlinearity in the effective index of refraction include for example chalcogenide glasses [21], organic polymers [22], ferroelectrics [23], and semiconductor-doped glasses [24].

In order to determine the exact traveling wave solutions of Eq. (1), we consider a solution of the form,

\[
\psi(z, \tau) = u(\xi) \exp[i(\kappa z - \delta \tau + \theta)],
\]

where \(u(\xi)\) is a real amplitude function which depends on the variable \(\xi = \tau - qz\), with \(q = v^{-1}\) is the inverse velocity. Also the real parameters \(\kappa\) and \(\delta\) represent the wave number and frequency shift respectively, while \(\theta\) represents the phase of the pulse at \(z = 0\).

From substitution of the representation (2) into Eq. (1), one finds the following system of ordinary differential equations,

\[
\nu \frac{d^4 u}{d \xi^4} - (\alpha + 3\rho \delta + 6\nu \delta^2) \frac{d^2 u}{d \xi^2} + \gamma u^3 - \mu u^5 - (\kappa - \alpha \delta^2 - \rho \delta^3 - \nu \delta^4) u = 0,
\]

\[
(\rho + 4\nu \delta^2) \frac{d^3 u}{d \xi^3} + (q - 2\alpha \delta - 3\rho \delta^2 - 4\nu \delta^3) \frac{du}{d \xi} = 0.
\]

Then from Eq. (4), we find that nontrivial solutions for Eqs. (3) and (4) with \(\nu \neq 0\) can exist for values of the frequency shift \(\delta\) and inverse velocity \(q\) satisfying the relations:

\[
\delta = -\frac{\rho}{4\nu}, \quad q = 2\alpha \delta + 3\rho \delta^2 + 4\nu \delta^3.
\]

We can make use of the parameters in (5) to determine the wave velocity \(v = q^{-1}\) as

\[
v = \frac{8\nu^2}{\rho(\rho^2 - 4\alpha \nu)}.
\]
Relation (6) shows that the velocity of propagating waves is uniquely dependent on the parameters of second-, third-, and fourth-order dispersions and it does not depend upon the nonlinearity parameters. Therefore, a natural way to control the velocity of a pulse is to vary various dispersion parameters in the fiber.

Further substitution of Eq. (5) into Eq. (3), we obtain an evolution equation for \( u(\xi) \) as

\[
\frac{d^4u}{d\xi^4} + \lambda_0 \frac{d^2u}{d\xi^2} + \lambda_1 u + \lambda_2 u^3 + \lambda_3 u^5 = 0,
\]

where the parameters \( \lambda_n \) \((n = 0, \ldots, 3)\) are defined by

\[
\lambda_0 = \frac{3\rho^2}{8\nu^2} - \frac{\alpha}{\nu}, \quad \lambda_1 = -\frac{\kappa}{\nu} - \frac{\rho^2}{16\nu^3} \left( \frac{3\rho^2}{16\nu} - \alpha \right),
\]

\[
\lambda_2 = \frac{\gamma}{\nu}, \quad \lambda_3 = -\frac{\mu}{\nu}.
\]

It is critically important to find exact analytical localized and periodic solutions of the amplitude equation (7) in most general case, when all parameters of Eq. (1) have nonzero values and no constraint for them. This enables us to examine the individual influence of each type of dispersive and nonlinear effects on the characteristics of propagating nonlinear waves. It is interesting to point out that the finding of such closed form solutions is greatly desired to experiments as they give a precise formulation of the existing solitary and periodic pulses.

We observe that the nonlinear differential equation (7) includes two coexisting cubic \( u^3 \) and quintic \( u^5 \) nonlinear terms in addition to two even-order derivative terms. In general, it would be very difficult to find solutions in analytic form for such equation. In the present study, we have been able to find new types of periodic and localized wave solutions by using an appropriate equation method. Remarkably, we have found that integration of Eq. (7) leads to physically relevant solutions satisfying the following equation,

\[
\left( \frac{du}{d\xi} \right)^2 = a + bu^2 + cu^4,
\]

The corresponding second- and fourth-order differential equations for \( u(\xi) \) read,

\[
\frac{d^2u}{d\xi^2} = bu + 2cu^3,
\]

\[
\frac{d^4u}{d\xi^4} = (b^2 + 12ac)u + 20bcu^3 + 24c^2u^5.
\]

The substitution of Eqs. (11) and (12) to Eq. (7) leads to the system of algebraic equations as

\[
b^2 + 12ac + \lambda_0 b + \lambda_1 = 0,
\]

\[
20bc + 2\lambda_0 c + \lambda_2 = 0, \quad 24c^2 + \lambda_3 = 0.
\]

The solution of these algebraic equations yields the parameters for Eq. (10) in an explicit form as

\[
c = \pm \frac{1}{2} \sqrt{\frac{\mu}{6\nu}}, \quad b = \frac{\alpha}{10\nu} - \frac{3\rho^2}{80\nu^2} \mp \frac{\kappa}{10\nu} \sqrt{\frac{6\nu}{\mu}},
\]

\[
a = \pm \frac{1}{6} \sqrt{\frac{6\nu}{\mu} (\lambda_1 + \lambda_0 b + b^2)}.
\]

Thus the parameters \( c, b \) and \( a \) have two different forms with the top and bottom signs respectively.

We present below a number of novel periodic (or elliptic) solutions of the model (1) based on solving the nonlinear differential equation (10). These closed form solutions are expressed in terms of Jacobean elliptic functions of modulus \( k \). We further show that special limiting cases of these families include the bright and dark and solitary wave solutions.
III. PERIODIC AND SOLITARY WAVE SOLUTIONS

Before discussing the precise nature of periodic and solitary wave solutions of the model (1), we first consider the transformation of Eq. (10) based on new function \( y(\xi) \) as
\[
  u^2(\xi) = -\frac{1}{4c}y(\xi).  \tag{17}
\]
Thus, we have found the nonlinear differential equation as
\[
  \left(\frac{dy}{d\xi}\right)^2 = f(y), \quad f(y) = \sigma_1 y + \sigma_2 y^2 - y^3,  \tag{18}
\]
where \( \sigma_1 = -16ac \) and \( \sigma_2 = 4b \). The function \( f(y) \) can also be written in the form \( f(y) = -y(y-y_-(y-y_+) \) which yields the nonlinear differential equation,
\[
  \left(\frac{dy}{d\xi}\right)^2 = -y(y-y_-(y-y_+).  \tag{19}
\]
The polynomial \( f(y) \) has tree roots as
\[
y_0 = 0, \quad y_\pm = 2(b \pm g), \quad g = \sqrt{b^2 - 4ac}.  \tag{20}
\]
1. Periodic \((A + Bcn^2)^{1/2}\)-waves

We can order the roots of polynomial \( f(y) \) as \( y_1 < y_2 < y_3 \) where \( y_1 = y_0, \ y_2 = y_-, \ y_3 = y_+ \). In this case Eqs. (17) and (18) yield the periodic solution as
\[
u(\xi) = \pm[A + Bcn^2(w(\xi - \xi_0), k)]^{1/2}.  \tag{21}\]
The parameters of this solution are
\[
  A = \frac{g - b}{2c}, \quad B = -\frac{g}{c},  \tag{22}
\]
\[
w = \frac{1}{2}\sqrt{2(b + g)}, \quad k = \sqrt{\frac{2g}{b + g}}.  \tag{23}
\]
Here \( cn(w(\xi - \xi_0), k) \) is Jacobi elliptic function where the modulus \( k \) belongs the interval \( 0 < k < 1 \). It follows from this solution the conditions for parameters as \( b > g, \ c < 0 \) and \( b^2 > 4ac \).

The equation for modulus \( k \) in Eq. (23) yields relation as \( a = b^2(1 - k^2)/c(2 - k^2)^2 \). Thus, the wave number \( \kappa \) by Eqs. (15) and (16) is
\[
k = b \left( \frac{3\rho^2}{8\nu} - \alpha \right) - \frac{\rho^2}{16\nu^2} \left( \frac{3\rho^2}{16\nu} - \alpha \right) + \nu b^2 + \frac{12\nu b^2(1 - k^2)}{(2 - k^2)^2}.  \tag{24}
\]
Substitution of the solution (24) into the wave function (2) yields the following family of periodic wave solutions for the high dispersive cubic-quintic NLSE (1):
\[
  \psi(z, \tau) = \pm[A + Bcn^2(w(\xi - \xi_0), k)]^{1/2}\exp[i(\kappa z - \delta \tau + \theta)],  \tag{25}
\]
where modulus \( k \) is an arbitrary parameter in the interval \( 0 < k < 1 \) and \( \xi_0 \) is the position of pulse at \( z = 0 \). We note that in the limiting cases with \( k = 1 \) this periodic wave reduces to a bright-type soliton solution.

Figure 1(a) presents the evolution of the \( cn^2 \)-type periodic wave (24) for the physical parameter values \( \rho = 0.25, \ \alpha = -0.3125, \ \nu = 0.25, \ \gamma = -1.5, \ \text{and} \ \mu = 0.6144 \). To satisfy the parametric conditions \( c < 0 \) and \( b^2 > 4ac \), we considered the case of lower sign in all the parameters given in Eqs. (15) and (16). Also, the value of the elliptic modulus \( k \) is taken as \( k = 0.6 \). As concerns the inverse group velocity \( q = v^{-1} \) of this \( cn^2 \)-type periodic wave, it can be determined from the relation (4) as \( q = 0.1875 \). Additionally, the position \( \xi_0 \) of the periodic waves at \( z = 0 \) is chosen to be equal to zero. As is seen from this figure, the intensity profile presents an oscillating character which makes the wave a setting of light pulse train propagation in optical fibers. A particularly interesting property of this type of periodic waves is that its oscillating behaviour is superimposed at a nonzero background, which is advantageous for a wide range of practical applications.
FIG. 1: Evolution of nonlinear wave solutions (a) $cn^2$-type periodic wave solution (25) with parameters $\rho = 0.25$, $\alpha = -0.3125$, $\nu = 0.25$, $\gamma = -1.5$, $\mu = 0.6144$, $\xi_0 = 0$, and $k = 0.6$ (b) bright solitary wave (28) with parameters $\alpha = 0.4$, $\rho = 1$, $\nu = 0.5$, $\gamma = 0.8$, $\mu = 0.75$, $\xi_0 = 0$ (c) cn-type periodic wave (32) with parameters $\alpha = 0.4$, $\rho = 0.5$, $\nu = -0.5$, $\gamma = -1.375$, $\mu = 0.75$, $\xi_0 = 0$ (d) dark solitary wave (36) with parameters $\alpha = 0.4$, $\rho = 1$, $\nu = 0.5$, $\gamma = 1.075$, $\mu = 0.75$, $\xi_0 = 0$.

2. Bright solitary waves

We consider the limiting case of solution in Eq. (21) with $k = 1$. Thus, we have the soliton solution of Eqs. (17) and (18) as

$$u(\xi) = \pm \left( -\frac{b}{c} \right)^{1/2} \operatorname{sech}(\sqrt{\frac{b}{c}}(\xi - \xi_0)).$$

(26)

The condition $k = 1$ in Eq. (24) leads to the wave number $\kappa$ as

$$\kappa = b \left( \frac{3\rho^2}{8\nu} - \alpha \right) - \frac{\rho^2}{16\nu^2} \left( \frac{3\rho^2}{16\nu} - \alpha \right) + \nu \nu^2.$$

(27)

Thus, the bright solitary wave solution can be obtained for the high dispersive cubic-quintic NLSE (1) using Eqs. (2) and (26) as

$$\psi(z, \tau) = \pm \left( -\frac{b}{c} \right)^{1/2} \operatorname{sech}(\sqrt{\frac{b}{c}}(\xi - \xi_0)) \exp[i(\kappa z - \delta \tau + \theta)],$$

(28)

with $b > 0$ and $c < 0$.

Figure 1(b) depicts the evolution of the intensity wave profile of the solitary wave solution (27) for the parameter values $\alpha = 0.4$, $\rho = 1$, $\nu = 0.5$, $\gamma = 0.8$, $\mu = 0.75$, and $\xi_0 = 0$. We have also considered the case of lower sign in Eqs. (15) and (16) for the condition $c < 0$ to be fulfilled. It is interesting to see that this localized pulse exhibits a sech-type field profile like the traditional bright solitons of Kerr media, whereas their existence is due to a balance among higher-order effects of different nature.

3. Periodic cn-waves

We can order the roots of polynomial $f(y)$ as $y_1 < y_2 < y_3$ where $y_1 = y_-, y_2 = y_0, y_3 = y_+$. In this case Eqs. (17) and (18) yield the periodic solution as

$$u(\xi) = \pm \Lambda \operatorname{cn}(w(\xi - \xi_0), k),$$

(29)

where the modulus $k \in (0,1)$. Here $\Lambda$ and $w$ are real parameters given by

$$\Lambda = \left( -\frac{b + g}{2c} \right)^{1/2}, \quad w = \sqrt{g}, \quad k = \sqrt{\frac{b + g}{2g}}.$$

(30)
The equation for modulus $k$ in Eq. (30) yields relations as $a = b^2k^2(k^2 - 1)/c(2k^2 - 1)^2$. Thus, the wave number $\kappa$ by Eqs. (15) and (16) is

$$\kappa = b \left( \frac{3\rho^2}{8\nu} - \alpha \right) - \frac{\rho^2}{16\nu^2} \left( \frac{3\rho^2}{16\nu} - \alpha \right) + \nu b^2 + \frac{12\nu b^2k^2(k^2 - 1)}{(2k^2 - 1)^2}.$$  (31)

Substitution of the solution (29) into the wave function (2) yields the following family of periodic wave solutions for the high dispersive cubic-quintic NLSE (1):

$$\psi(z, \tau) = \pm \Lambda cn(w(\xi - \xi_0), k) \exp[i(\kappa z - \delta \tau + \theta)],$$  (32)

where modulus $k$ is an arbitrary parameter in the interval $0 < k < 1$. It follows from this solution the conditions for parameters as $b + g > 0$, $g > b$, $c < 0$ and $b^2 > 4ac$. In the limiting case with $k = 1$ this solution reduces to the soliton solution given by Eq. (28).

In Fig. 1(c), we will have shown the evolution of cn-type periodic wave solution (32) for the parameter values $\alpha = 0.25$, $\rho = 0.5$, $\nu = -0.5$, $\gamma = -0.56875$, $\mu = -0.75$, $\xi_0 = 0$. To satisfy the condition $c < 0$, we have considered the case of lower sign in Eqs. (15) and (16). Unlike the preceding cn$^2$-type periodic wave, the periodic wave in present case propagates on a zero background.

4. Dark solitary tanh-waves

In the case with $g = 0$ or $b^2 = 4ac$ we can order the roots of polynomial $f(y)$ as $y_1 = y_2 < y_3$ where $y_1 = y_2 = y_3 = y_0$. Note that for this case we have the condition $b < 0$. Thus, the solution of Eqs. (17) and (18) have the kink wave solution as

$$u(\xi) = \pm \Lambda \tanh(w(\xi - \xi_0)).$$  (33)

The parameters of this solution are

$$\Lambda = \left( -\frac{b}{2c} \right)^{1/2}, \quad w = \frac{1}{2} \sqrt{-2b},$$  (34)

where $b < 0$ and $c > 0$. The condition $b^2 = 4ac$ yields the wave number $\kappa$ by Eqs. (15) and (16) as

$$\kappa = b \left( \frac{3\rho^2}{8\nu} - \alpha \right) - \frac{\rho^2}{16\nu^2} \left( \frac{3\rho^2}{16\nu} - \alpha \right) + 4\nu b^2.$$  (35)

Hence, one obtains a kink solution for Eq. (11) of the form

$$\psi(z, \tau) = \pm \Lambda \tanh(w(\xi - \xi_0)) \exp[i(\kappa z - \delta \tau + \theta)].$$  (36)

Note that this kink solution has the form of dark soliton for intensity $I = |\psi(z, \tau)|^2 = \Lambda^2 \tanh^2(w(\xi - \xi_0))$.

Figure 1(d) displays the intensity profile of the solitary wave solution (36) for the parameter values $\alpha = 0.4$, $\rho = 1$, $\nu = 0.5$, $\gamma = 1.075$, $\mu = 0.75$, $\xi_0 = 0$. To satisfy the conditions $b < 0$ and $c > 0$, we have considered the case of upper sign in Eqs. (15) and (16).

5. Periodic sn/(1 + cn)-waves

We have also found an unbounded periodic solution of Eq. (10) of the form,

$$u(\xi) = \pm \frac{A sn(w(\xi - \xi_0), k)}{1 + cn(w(\xi - \xi_0), k)}.$$  (37)

The parameters for this periodic solution are

$$A = \sqrt{\frac{b}{2c(1 - 2k^2)}}, \quad w = \sqrt{\frac{2b}{1 - 2k^2}}.$$  (38)

This solution takes place for condition $a = b^2 - 4c(1 - 2k^2)^2$ which yields the wave number $\kappa$ by Eqs. (15) and (16) as

$$\kappa = b \left( \frac{3\rho^2}{8\nu} - \alpha \right) - \frac{\rho^2}{16\nu^2} \left( \frac{3\rho^2}{16\nu} - \alpha \right) + 3\nu b^2 \frac{1}{(1 - 2k^2)^2}.$$  (39)
Now, taking into account the representation (2), the higher-order NLSE (1) has the following periodic wave solution:

\[
\psi(z, \tau) = \pm A sn (w(\xi - \xi_0), k) \exp[i(\kappa z - \delta \tau + \theta)],
\]  
(40)

where modulus \(k\) is an arbitrary parameter for intervals \(0 < k < 1/\sqrt{2}\) and \(1/\sqrt{2} < k < 1\).

6. Dark solitary \(\tanh/(1 + \text{sech})\) -waves

The limit \(k \to 1\) in Eq. (40) leads to a solitary wave of the form,

\[
\psi(z, \tau) = \pm A_0 \tanh (w_0(\xi - \xi_0)) \exp[i(\kappa z - \delta \tau + \theta)].
\]  
(41)

The parameters for this solitary wave are

\[
A_0 = \sqrt{-\frac{b}{2c}}, \quad w_0 = \sqrt{-2b},
\]  
(42)

with \(b < 0\) and \(c > 0\) and \(a = b^2/4c\). This solitary wave has the form of dark soliton for intensity \(I = |\psi(z, \tau)|^2\). The wave number \(\kappa\) for this solitary wave follows from Eq. (39) with \(k = 1\):

\[
\kappa = b \left(\frac{3\rho^2}{8\nu} - \alpha\right) - \frac{\rho^2}{16\nu^2} \left(\frac{3\rho^2}{16\nu} - \alpha\right) + 4\nu b^2.
\]  
(43)

![Figure 2](image)

**FIG. 2:** (a) Intensity of the solitary wave profile \(|\psi(0, \tau)|^2\) as a function of \(\tau\) and its (b) evolution as computed from Eq. (41) for the value \(\alpha = 0.4, \rho = 1, \nu = 0.5, \gamma = 1.075, \mu = 0.75, \text{ and } \xi_0 = 0\).

Figure 2(a) presents the intensity profile of the optical solitary wave solution (41) for the parameter values \(\alpha = 0.4, \rho = 1, \nu = 0.5, \gamma = 1.075, \mu = 0.75, \xi_0 = 0\), while Fig. 2(b) shows its evolution. It is interesting to see that this nonlinear waveform is a dark solitary wave, which can be formed in the fiber medium due to a balance among all orders of dispersion up to the fourth order and both third- and fifth-order nonlinearities. Remarkably, the functional form of this solitary wave is different from the simplest dark solitary wave that has the form \(\tanh\).

7. Periodic \(\text{cn}/(1 + \text{sn})\)-waves

We have found the unbounded periodic solution of Eq. (10) of the form,

\[
u(\xi) = \pm \frac{Acn (w(\xi - \xi_0), k)}{1 + \text{sn} (w(\xi - \xi_0), k)},
\]  
(44)

where \(0 \leq k < 1\). The parameters for this periodic solution are

\[
A = \sqrt{\frac{b(1-k^2)}{2c(1+k^2)}}, \quad w = \sqrt{\frac{2b + \gamma}{1 + k^2}}.
\]  
(45)
with \( b > 0 \) and \( c > 0 \). This solution takes place for condition \( a = b^2(1 - k^2)^2/4c(1 + k^2)^2 \) which yields the wave number \( \kappa \) by Eqs. (15) and (16) as
\[
\kappa = b \left( \frac{3\rho^2}{8\nu} - \alpha \right) - \frac{\rho^2}{16\nu^2} \left( \frac{3\rho^2}{16\nu} - \alpha \right) + \nu b^2 + \frac{3\nu b^2(1 - k^2)^2}{(1 + k^2)^2}.
\]  
(46)
The substitution of solution (44) into Eq. (2) yields the family of periodic bounded solutions for the higher-order NLSE (1) of the form,
\[
\psi(z, \tau) = \pm \frac{A \text{cn}(w(\xi - \xi_0), k)}{1 + \text{dn}(w(\xi - \xi_0), k)} \exp[i(\kappa z - \delta \tau + \theta)],
\]  
(47)
where modulus \( k \) is an arbitrary parameter in the interval \( 0 \leq k < 1 \).

8. Periodic \( \text{sn}/(1 + \text{dn}) \)-waves

We have found the exact periodic bounded solution of Eq. (10) of the form,
\[
u(\xi) = \pm \frac{A \text{sn}(w(\xi - \xi_0), k)}{1 + \text{dn}(w(\xi - \xi_0), k)}.
\]  
(48)
The parameters for this periodic solution are
\[
A = \sqrt{\frac{bk^4}{2c(k^2 - 2)}}, \quad w = \sqrt{\frac{2b}{k^2 - 2}},
\]  
(49)
where \( b < 0 \) and \( c > 0 \). This solution takes place for condition \( a = b^2k^4/4c(k^2 - 2)^2 \) which yields the wave number \( \kappa \) by Eqs. (15) and (16) as
\[
\kappa = b \left( \frac{3\rho^2}{8\nu} - \alpha \right) - \frac{\rho^2}{16\nu^2} \left( \frac{3\rho^2}{16\nu} - \alpha \right) + \nu b^2 + \frac{3\nu b^2k^4}{(k^2 - 2)^2}.
\]  
(50)
Thus, the appropriate periodic bounded solutions of Eq. (1) are
\[
\psi(z, \tau) = \pm \frac{A \text{sn}(w(\xi - \xi_0), k)}{1 + \text{dn}(w(\xi - \xi_0), k)} \exp[i(\kappa z - \delta \tau + \theta)],
\]  
(51)
where modulus \( k \) is an arbitrary parameter in the interval \( 0 \leq k < 1 \). It should be noticed that the limit \( k \to 1 \) in this solution yields the solitary wave solution given in Eq. (11).

Figure 3(a) depicts a typical example of the time evolution of intensity of the periodic wave (40) for the parameters values \( \alpha = 1.64, \rho = -0.3, \nu = 1, \gamma = 0.1, \mu = 1.5 \). Then the velocity of the wave can be determined by using the relation (8) as \( v \approx 4.12 \). The results for the unbounded periodic wave (47) are illustrated in Fig. 3(b) for the values \( \alpha = 0.305, \rho = -0.2, \nu = 0.5, \gamma = -1, \mu = 0.3072 \). The velocity of this wave can be calculated with the help of Eq. (9) resulting in \( v \approx 17.54 \). The intensity profile of the periodic wave solution (51) is shown in Fig. 3(c) for the values \( \alpha = 0.21, \rho = -0.2, \nu = 1.5, \gamma = 1, \mu = 0.0576 \). Accordingly, the velocity of the wave is obtained as \( v \approx 73.77 \). Here we considered the case of upper sign in all the parameters given in Eqs. (15) and (16) and the initial position \( \xi_0 \) of the periodic waves is chosen to be equal to zero. Also, the value of elliptic modulus \( k \) is taken as \( k = 0.6 \). We can see from this figure that the profile of nonlinear waves presents the periodic property as it propagates through the optical fiber. It is also interesting to note that the oscillating behaviour of this kind of periodic waves is superimposed at a zero background.

In view of the above results, we thus see that, in addition to the simplest periodic waves, novel periodic waves taking the forms (25), (40), (47), and (51) can also be formed in the fiber medium in the presence of various higher-order effects. No doubt, this may be helpful for extending the applicability for periodic wave propagation through highly dispersive optical fibers. We should note here that the periodic structures are of increasing interest particularly after the first experimental observation of the evolution of an arbitrarily shaped input optical pulse train to the shape preserving Jacobean elliptic pulse-train corresponding to the Maxwell-Bloch equations (23). The obtained results also showed that the presence of higher-order dispersions and quintic nonlinearity leads to the formation of a novel class of dark-type waves which takes the form given by Eq. (40). Undoubtedly, such ultrashort solitary pulse could find potential applications in optical communication systems since dark solitons are more stable against Gordon-Haus jitters in long communication line, less influenced by noise, and less sensitive to optical fiber loss (26, 27). We emphasis that periodic and solitary wave solutions presented in this section are stable to small perturbations. It can be proved using the analytical method developed in Ref. [8].
IV. CONCLUSION

We have studied the femtosecond light pulse propagation in a highly dispersive optical fiber governed by a higher-order nonlinear Schrödinger equation incorporating all orders of dispersion up to the fourth order as well as cubic and quintic nonlinearities. With use of an appropriate equation, novel exact periodic wave solutions have been identified for the model in the presence of various dispersive and nonlinear effects. Solitary waves have been also obtained which includes both bright and dark localized solutions. It is found that the velocity of these structures is uniquely dependent on all orders of dispersion. Moreover, all solutions presented in the paper are stable to small perturbations which follows from appropriate stability analysis. It is apparent that the exact nature of the nonlinear waves presented here can lead to different applications in optical communications.

[1] Yu. S. Kivshar and G. P. Agrawal, Optical Solitons: From Fibers to Photonic Crystals (Academic, New York, 2003).
[2] G. P. Agrawal, Applications of Nonlinear Fiber Optics (Academic, San Diego, 2001).
[3] Alka, A. Goyal, R. Gupta and C. N. Kumar, and T. S. Raju, Phys. Rev. A 84, 063830 (2011).
[4] A. Choudhuri and K. Porsezian, Opt. Commun. 285, 364 (2012).
[5] Z. H. Li, L. Li, H. Tian, and G. S. Zhou, Phys. Rev. Lett. 84, 4096 (2000).
[6] H. Triki, F. Azzouzi, Ph. Grelu, Opt. Commun. 309, 71 (2013).
[7] V. I. Kruglov and J. D. Harvey, Phys. Rev. A 98, 063811 (2018).
[8] H. Triki and V. I. Kruglov, Phys. Rev. E 101, 042220 (2020).
[9] V. I. Kruglov and H. Triki, Phys. Rev. A 102, 043509 (2020).
[10] C. Q. Dai, Y. Y. Wang, C. Yan, Opt. Commun. 283, 1489 (2010).
[11] V.M. Petnikova, V.V. Shuvalov, V.A. Vysloukh, Phys. Rev. E 60, 1009 (1999).
[12] K. W. Chow, K. Nakkeeran, B. A. Malomed, Opt. Commun. 219, 251 (2003).
[13] K. W. Chow, I. M. Merhasin, B. A. Malomed, K. Nakkeeran, K. Senthilnathan, and P. K. A. Wai, Phys. Rev. E 77, 026602 (2008).
[14] F. Kh. Abdullaev, A. M. Kamchatnov, V. V. Konotop, and V. A. Brazhnyi, Phys. Rev. Lett. 90, 230402 (2003).
[15] Z. Y. Yan, K.W. Chow, B. A. Malomed, Chaos, Solitons & Fractals 42, 3013 (2009).
[16] A. Joseph and K. Porsezian, J. Nonlinear Opt. Physics & Materials 19, 177 (2010).
[17] H. Triki and V. I. Kruglov, Opt. Commun. 502, 127409 (2022).
[18] H. Triki and V. I. Kruglov, Chaos, Solitons & Fractals 153, 111496 (2021).
[19] S.L. Palacios and J.M. Fernández-Díaz, Opt. Commun. 178, 457 (2000).
[20] G. P. Agrawal and Clifford Headley III, Phys. Rev. A 46, 1573 (1992).
[21] Y. -F. Chen, K. Beckwitt, F. W. Wise, B. G. Aitken, J. S. Sanghera, and I. D. Aggarwal, J. Opt. Soc. Am. B 23, 347 (2006).
[22] B. L. Lawrence, M. Cha, J. U. Kang, W. Toruellas, G. Stegeman, G. Baker, J. Meth, and S. Etemad, Electron. Lett. 30, 447 (1994).

[23] B. Gu, Y. Wang, W. Ji, and J. Wang, Appl. Phys. Lett. 95, 041114 (2009).

[24] P. Roussignol, D. Ricard, J. Lukasik, and C. Flytzanis, J. Opt. Soc. Am. B 4, 5 (1987); L.H. Acili, A.S.L. Gomes, J.M. Hickmann, and C.B. de Araujo, Appl. Phys. Lett. 56, 2279 (1990); F. Lederer and W. Biehlig, Electron. Lett. 30, 1871 (1994).

[25] J. L. Shultz and G. J. Salamo, Phys. Rev. Lett. 78, 855 (1997).

[26] S. Yu. Kivshar, M. Haelterman, Ph. Emplit, and J. P. Hamaide, Opt. Lett. 19, 19 (1994).

[27] A. Choudhuri and K. Porsezian, Phys. Rev. A 88, 033808 (2013).