A Critique of Kumar’s “Necessary and Sufficient Condition for Satisfiability of a Boolean Formula in CNF and Its Implications on P versus NP problem.”

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Abstract

In this paper, we analyze the argument made by Kumar in the technical report “Necessary and Sufficient Condition for Satisfiability of a Boolean Formula in CNF and Its Implications on P versus NP problem” [Kum21]. The paper claims to present a polynomial-time algorithm that decides CNF-SAT. We show that the paper’s analysis is flawed and that the fundamental underpinning of its algorithm requires an exponential number of steps on infinitely many inputs.

1 Introduction

We provide a summary and critique of the third version of Manoj Kumar’s technical report “Necessary and Sufficient Condition for Satisfiability of a Boolean Formula in CNF and Its Implications on P versus NP problem” [Kum21]. The paper claims to have constructed an algorithm that decides CNF-SAT in polynomial time. In order to understand the significance of this claim, one must first understand the significance of NP-complete problems.

Securing cryptographic systems relies on the assumption that solving NP problems is intractable. For example, consider the problem of integer factorization, which has a language version in NP ∩ coNP. The security of RSA encryption relies on the suspicion that integer factorization is intractable, even in the typical case. On the other hand, problem modeling using SAT (another NP problem) is very common in several areas of artificial intelligence. As a result, finding a polynomial-time algorithm for SAT would benefit those areas tremendously. The question of whether all NP problems can be solved in polynomial time is commonly referred to as the P vs. NP problem and is considered the most important unsolved problem in computational complexity theory, and arguably, in all of applied mathematics. NP-complete problems are significant because showing that one is in P is enough to imply that P = NP [Kar72], thereby resolving the P vs. NP problem. Since CNF-SAT is an NP-complete problem [Kar72], Kumar’s purported algorithm to

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decide CNF-SAT in polynomial time would prove $P = NP$. However, the paper’s analysis is deeply flawed and we will show that its algorithm runs in exponential time on an infinite number of inputs.

In Section 2 we give the preliminaries necessary to understand Kumar’s paper. In Section 3, we summarize its definitions and theorems and present the paper’s central theorem and algorithm. Finally, in Section 4 we expose the flaw in the paper’s algorithm, present an infinite family $\mathcal{F}$ of counterexamples that exploit the flaw, and review some optimizations to the algorithm (as proposed in Kumar’s paper) and show that they have no significant impact on the algorithm’s runtime if the input is in $\mathcal{F}$.

2 Preliminaries

Let $x$ be a Boolean variable. We call $x$ and $\overline{x}$ literals. If the variable $x$ has value true (false), then the literal $x$ also has value true (false), while the literal $\overline{x}$ has value false (true). The two literals associated to a variable always have complementary values and hence are called complementary literals. A clause is a disjunction of literals and values (true and false). For example, $C = (x_1 \lor x_2 \lor x_3)$ is a clause. We say that a variable $x$ occurs in or appears in a clause if one of the two literals $x$ and $\overline{x}$ appears in the clause. A Boolean formula in conjunctive normal form (CNF) is a conjunction of clauses. Such formulas are also called CNF formulas. For example, $F = (x_1 \lor x_2) \land (x_3 \lor \overline{x_2} \lor x_1) \land (x_1)$ is a CNF formula. We say that a variable $x$ occurs in or appears in a CNF formula $F$ if $x$ occurs in at least one clause in $F$. If $C$ is a clause, $V$ is a set of Boolean variables, and all the variables of $C$ appear in $V$, then $C$ is said to be a clause over $V$. A CNF formula $F$ is said to be a formula over $V$ if all of $F$’s clauses are over $V$. For example, if $V = \{x_1, x_2, x_3\}$, and $F = (x_1 \lor x_2) \land (x_3 \lor \overline{x_2} \lor x_1) \land (x_1)$, then $F$ is a Boolean formula over $V$.

Given a variable set $V$ and an assignment of values to the variables in $V$, a clause (over $V$) is said to evaluate to true if at least one of its disjuncts has value true under the assignment. For example, given assignment $x_1 = false$, $x_2 = false$, the clause $C = (x_1 \lor \overline{x_2})$ evaluates to true. On the other hand, given the assignment $x_1 = false$, $x_2 = true$, $C$ does not evaluate to true. A CNF formula is satisfiable if there is an assignment of values to its variables such that, under the assignment, all of the formula’s clauses evaluate to true. For example, the formula $F$ defined above is satisfiable as all of its clauses evaluate to true under the assignment $x_1 = true$, $x_2 = false$, $x_3 = true$. Notice that a formula can have multiple satisfying assignments. The set of all satisfiable CNF formulas is called CNF-SAT, and the Satisfiability Problem, in this context, is the task of deciding whether a given input is a satisfiable CNF formula.

Kumar’s paper views CNF formulas and clauses as sets. A clause is viewed as a set of literals and values, and a CNF formula is a set of clauses. For example, the formula $F$ defined above, would be written as $\{\{x_1, x_2\}, \{x_3, \overline{x_2}, x_1\}, \{x_1\}\}$. In this paper, we treat all CNF formulas and clauses as sets, just like Kumar’s paper does.

Finally, we let $\mathbb{N} = \{0, 1, 2, \ldots\}$ i.e., the set of all natural numbers (including zero), and let $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$ i.e., the set of positive natural numbers.
3 Understanding the Paper’s Argument

Kumar’s paper relates the problem of satisfiability to that of computing a set with a specific property, which we discuss in Section 3.2. We present the key theorems[1] that lead to this result in Section 3.1 and then provide an overview of the paper’s algorithm that purportedly decides CNF-SAT in polynomial time.

3.1 Definitions and Concepts

For the sake of simplicity, Kumar’s paper ignores trivially satisfiable clauses, such as \( C_1 \cup \{ \text{true} \} \), where \( C_1 \) is a clause, or \( C_2 \cup \{x, \overline{x}\} \), where \( C_2 \) is a clause and \( x \) is a variable, because those are easily recognizable in polynomial time. The paper refers to such clauses as tautology clauses.

The set of all nontautology clauses over a fixed variable set is called a complete formula. For example, if \( V = \{x_1, x_2\} \), then the complete formula is \( F_2 = \{\{x_1, x_2\}, \{x_1, \overline{x}_2\}, \{\overline{x}_1, x_2\}, \{\overline{x}_1, \overline{x}_2\}, \{x_1\}, \{\overline{x}_1\}, \{x_2\}, \{\overline{x}_2\}, \emptyset\} \) [Kum21]. Note that \( \emptyset \) is used to refer to the null clause, which can never be satisfied and can be viewed as equivalent to \( \{\text{false}\} \). Additionally, a clause \( C \) over a variable set \( V \) is said to be fully populated over \( V \) if every variable in \( V \) occurs in \( C \).

Kumar’s paper also defines the cardinality of a formula \( F \), denoted by \(|F|\), to be the number clauses in the formula. Finally, given a set \( S \), \( \mathcal{P}(S) \) denotes the powerset of \( S \). We list below, Theorems 1 and 2, which are used in the proof of Theorem 3.

Theorem 1 ([Kum21], Corollary 10.3). Let \( V \) be a variable set and \( S \) be the set of all fully populated clauses over \( V \). Then the complete Boolean formula \( F \) (over \( V \)) can be written as \( F = \bigcup_{C \in S} \mathcal{P}(C) \).

This theorem implies that given a variable set \( V \), every clause over \( V \) is a subset of some fully populated clause over \( V \).

Theorem 2 ([Kum21], Theorem 7.2). Let \( V \) be a variable set. If \( C_1 \) and \( C_2 \) are two sibling clauses that are over \( V \), then \((\forall D_1 \in \mathcal{P}(C_1) - \mathcal{P}(C_2)) (\exists V' \subseteq V) (\exists D_2 \in \mathcal{P}(C_2)) [D_1 \text{ and } D_2 \text{ are sibling clauses that are over } V'] \).

Informally, it means that given two sibling clauses \( C_1 \) and \( C_2 \), one can construct a pair of sibling clauses, \( D_1 \subseteq C_1 \) and \( D_2 \subseteq C_2 \), such that \( D_1 \) and \( D_2 \) have no common literal.

3.2 The Argument

By combining the above-mentioned definitions and theorems, the following theorem is proved:

Theorem 3 ([Kum21], Theorem 10.8). Let \( V \) be a variable set. A Boolean formula \( F \) (over \( V \)) is satisfiable if and only if there exists fully populated clause \( C \) (over \( V \)) such that \((\forall E \in \mathcal{P}(C))[E \notin F] \).

Proof summary. The “if” part is proved by way of contradiction, using the assumption that the consequent does not hold. The proof then demonstrates that, given a formula \( F \) that satisfies the antecedent, (1) for each assignment \( \alpha \) over \( V \), there is a fully populated clause \( C_\alpha \) that evaluates

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[1] We note that we do not state the paper’s theorems as they originally appear. Rather, we provide equivalent statements using more common notation.
to false under \( \alpha \), (2) by the assumption, it follows that for each fully populated clause \( \tilde{C} \), there is a subset \( E \) of \( \tilde{C} \) such that \( E \in F \), and (3) given an assignment \( \beta \) and a clause \( \tilde{C} \), if \( \tilde{C} \) evaluates to false under \( \beta \), then every subset of \( \tilde{C} \) also evaluates to false under \( \beta^2 \). For each assignment \( \gamma \) over \( V \), let \( C_\gamma \) be a clause as described in (1). It then follows from (2) and (3) that there is a subset of \( C_\gamma \) that evaluates to false and is a clause of \( F \). Therefore, \( F \) is not satisfiable, which is a contradiction.

The “only if” part is more complicated and makes use of Theorems 1 and 2, given a clause \( C \) that satisfies the antecedent, to show that for each clause \( D \in F \), there is a clause \( E \in \mathcal{P}(C) \) such that \( D \) and \( E \) are sibling clauses. The next argument is that, given an assignment \( \alpha \) and two sibling clauses, at least one of the two clauses evaluates to true under assignment \( \alpha \). By picking an assignment \( \beta \) such that \( C \) evaluates to false, it follows that every \( E \in \mathcal{P}(C) \) also evaluate to false under \( \beta \) and that each \( D \), which is a sibling clause of some \( E \in \mathcal{P}(C) \), must then evaluate to true under \( \beta \), thereby proving that \( F \) is satisfiable.

This theorem asserts that to show satisfiability of a Boolean formula \( F \), one only needs to find a fully populated clause \( C \) such that \((\forall E \in \mathcal{P}(C))[E \not\in F]\). Failure to find such a clause implies that \( F \) is not satisfiable. To tackle this clause-finding problem, Kumar’s paper constructs a tree based on the input CNF formula and searches for the clause in that tree. We reproduce the paper’s algorithm to build and search a tree, given a CNF formula, as Algorithm 1. To assist in this search, and in an attempt to prevent the tree from growing exponentially large, Kumar’s paper also implements a pruning algorithm. We have also modified some of the paper’s figures to produce Figure 1 to exemplify how the tree is constructed and pruned.

Construction begins with the empty tree that contains only the root node, which is a special node with only one child pointer. All the other nodes in the tree have two child pointers. When those pointers are not assigned to a node, they can have one of two values: open or null. The former indicates that a child can be added at that location, while the latter indicates that no child can ever be added there. By default, when a node is created, its child pointers are set to open. Each node is labeled with a variable name. We shall say a variable \( x \) is in the tree if there is a node in the tree with label \( x \). The left pointer out of a node labeled \( x \) represents the literal \( x \), while the right pointer represents literal \( \overline{x} \). Kumar’s paper treats these representations (of literals) as the labels of the pointers. Now, consider a path \( p \) from the root downwards (optionally including an open pointer, if there is one). The set of labels on the pointers in \( p \) describes a unique clause. Let \( V \) be the set of variables appearing in the tree. Then, each path in the tree (from the root to a leaf) uniquely identifies a clause in the complete formula over \( V \). The main loop of the algorithm proceeds as follows. For each \( C \) in the input formula \( F \), let the variables of \( C \) be denoted by \( V_C \). For each \( x \in V_C \), if \( x \) is not in the tree, then each open pointer is assigned to a new node labeled with \( x \). Once the iteration over \( V_C \) ends, the tree is pruned.

Figure 1b (1d) shows how the tree in Figure 1a (1c) is pruned by Algorithm 1. After a clause \( C \) has been added to the tree, pruning consists of removing all paths representing supersets of \( C \) from the tree. Given a clause \( C \), to remove all of its supersets from the tree, it suffices to find the path from the root that describes \( C \). Let the last edge in that path be \( e \) and the corresponding pointer be \( q \). Delete everything that is connected downwards of \( q \), by setting \( q \) to null. In the figure, the first clause we look at is \( \{ \overline{x_1} \} \). The path that describes it is false, \( \overline{x_1} \). Thus the left subtree of node \( x_1 \) (along with that edge) is deleted and the left pointer of that node is set to null. The same

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Theorem 1 and 2 are rather easy to see so we don’t prove them here.

Kumar’s paper actually proves (1) and (3), but they’re rather easy to see so we don’t prove them here.
Figure 1: The construction and pruning of the tree when $F = \{ \{x_1\}, \{x_1, \overline{x}_2\} \}$.  

(a) The subtree constructed from processing clause $\{\overline{x}_1\}$.

(b) The result of pruning the previous subtree.

(c) The resulting subtree after processing clause $\{x_1, \overline{x}_2\}$.

(d) The resulting subtree after pruning the previous subtree.

process is used to prune the tree further upon encountering clause $\{x_1, \overline{x}_2\}$. Since the complete clause $\{x_1, x_2\}$ “survives” the pruning process, $F$ is satisfiable.

We note, before addressing the error in the algorithm, that while the explanatory text and examples in Kumar’s paper only prune the tree after it has been completely constructed, the actual code that is provided prunes the tree after each clause has been processed.

4 Identifying the Error

The algorithm iterates over every clause $C \in F$, rejecting if $C = \emptyset$ and ignoring $C$ if it is a tautology clause. If neither case is true, then the algorithm iterates over all the variables in $C$ that are not in the tree $S$. For each variable $x$, if there are no open pointers in $S$, then the algorithm rejects. Otherwise, $x$ is added to the tree. It’s only on line 16 that the algorithm starts to prune supersets of
Algorithm 1 Algorithm to purportedly decide CNF-SAT in polynomial time.

Ensure: Input $F$ is a CNF formula.

1: $V \leftarrow$ set of variables appearing in $F$.
2: $S \leftarrow$ root node with its pointer set to open.
3: for all $C \in F$ do
4:  if $C = \emptyset$ then
5:    Reject.
6:  end if
7:  if $C$ is not a tautology clause then
8:    for all variables $v \in C$ do
9:      if $S$ does not contain a node labeled $v$ then
10:        if $S$ contains no open pointers then
11:          Reject.
12:        end if
13:        Add a distinct node labeled $v$ to each open pointer.
14:      end if
15:    end for
16:    for all pointers $p$ in $S$ do
17:      if the clause represented by the path from the root to $p$ superset $C$ then
18:        Set $p$ to null and delete all nodes below it.
19:      end if
20:    end for
21:  end if
22: end for
23: if $S$ contains no open pointer then
24:  Reject.
25: else
26:  Accept.
27: end if

$C$—this is the paper’s crucial error. Since the subtree construction takes place in full (for a given clause) before pruning, there is the possibility that an exponentially large tree will be produced before the algorithm has a chance to prune. To make matters worse, during the pruning step the algorithm iterates over every pointer in the tree, potentially iterating over an exponential number of pointers. We construct an infinite family of counterexamples below in which both issues occur.

4.1 Counterexample

We now define the infinite family $\mathcal{F}$ of counterexamples to the proposed polynomial runtime of Algorithm 1. Without loss of generality, let the set of all variables be $\mathcal{V} = \{x_i \mid i \in \mathbb{N}^+\}$. For each $n \in \mathbb{N}^+$, we define $F_n = \{x_j \in \mathcal{V} \mid j \in \mathbb{N}^+ \land j \leq n\}$ and let $\mathcal{F} = \{F_k \mid k \in \mathbb{N}^+ \land k > 1\}$ be our family of counterexamples. Note that all Boolean formulas in $\mathcal{F}$ are satisfiable (although this has no bearing on the runtime of the algorithm). We will now show how $\mathcal{F}$ precludes Algorithm 1.

\footnote{We note in passing that, technically, Kumar’s algorithm accepts the empty formula (i.e., $F = \emptyset$) when it should not. However, correcting this error is rather trivial so we do not give that error further consideration.}
from being a polynomial-time algorithm.

Fix $F_n \in \mathcal{F}$. Since $\|F_n\| = 1$, the main loop of the algorithm only iterates once and solely inspects the single clause in $F_n$, $C$. Because $C \neq \emptyset$ and $C$ is not a tautology clause, the algorithm will proceed to loop over all variables $x \in C$ on line 8. By design, the algorithm won’t begin pruning until line 16, at which point all variables in $C$ will have been added to the tree. $C$ will be the first clause the algorithm sees as $F_n$ contains only one clause. Since pointers are only set to null during pruning, and the algorithm has yet to prune, all pointers originating from leaf nodes will be open and the check on line 10 will fail. As $\|C\| = n$, the loop on line 8 only runs $n$ times. At the $i$th iteration of that loop, the algorithm will insert $2^{i-1} + 1$ new nodes. Hence, after the loop has run $n$ times, the resulting tree will contain $1 + \sum_{i=1}^{n} 2^{i-1} = 2^n$ nodes (including the root node).

Intuitively, because no pruning takes place, after the algorithm has looped through all $x \in C$ the tree will contain a branch representing every possible assignment to $C$. Hence, the size of this tree will be $2^n$.

Furthermore, since $C$ contains all the variables in $V$, the only paths that can represent supersets of $C$ are those that also contain all the variables in $V$. In fact, only one path in the tree can represent a superset of $C$: the path representing $C$ itself. This is because every other path must represent either a subset of $C$ or a sibling clause of $C$. As a result, only the pointer at the end of the path representing $C$ would be set to null during pruning and the size of the tree would still be exponential (specifically $2^n - 1$). Even worse, to accomplish this the pruning step iterates over every pointer in the tree. Since the tree is exponentially large, line 16 will explore $2^{n+1} - 1$ pointers.

We now argue that there is no polynomial that upper bounds the runtime of Algorithm 1. Without loss of generality, we can assume that there is a polynomial, say $g$, from $\mathbb{N}$ to $\mathbb{N}$ such that for each $n \in \mathbb{N}^+$, the length of $F_n \in \mathcal{F}$ when given as an input to the algorithm is upper bounded by $g(n)$. This is because one only needs roughly $\log_2(n)$ bits to represent each of the $n$ literals and a constant number of extra bits to denote the separation between each pair of literals. Let $q : \mathbb{N} \to \mathbb{N}^+$ be the hypothesized polynomial that upper bounds the runtime of Algorithm 1. From our observations we have seen that, for each $n \in \mathbb{N}^+$ Algorithm 1 performs at least $2^{n+1} - 1$ steps on input $F_n \in \mathcal{F}$. However, there exists a sufficiently large $n_0 \in \mathbb{N}^+$, such that for each $n \geq n_0$, $2^{n+1} - 1 > q(g(n))$. Therefore, $q$ does not upper bound Algorithm 1’s runtime and so there is no polynomial that upper bounds the runtime of Algorithm 1.

4.2 Attempted Optimizations

Kumar’s paper presents an algorithm to check for the existence of tautology clauses in polynomial time. However, all this algorithm does is check whether there are tautology clauses in the given Boolean formula. Because there is no $F \in \mathcal{F}$ that contains a tautology clause, these optimizations do not affect our family of counterexamples.

The paper also presents the following bounds (see [Kum21, Section 14]), which at first glance seem to help detect those formulas that can result in exponentially large trees. However, as we will show, the bounds fail to cordon off all such Boolean formulas. Before reviewing the proposed bounds, new notation must be introduced. The number of clauses containing the literal $x$ in a Boolean formula $F$ will be denoted by $\#(F, x) = \|\{C \in F \land x \in C\}\|$.

Given a Boolean formula $F$ which contains no tautology clauses, and given that $F$ is over a variable set $V$ containing

\[\text{While this notation is slightly different from Kumar’s, it is equivalent with less room for ambiguity (see [Kum21, Section 13.2]).}\]
\( n \) variables, the proposed bounds are:

1. If \( \|F\| > 3^n - 2^n \), then \( F \) is unsatisfiable.
2. If \( (\exists x \in V)[\min(\#(F, x), \#(F, \overline{x})) > 3^{n-1} - 2^{n-1}] \), then \( F \) is unsatisfiable.
3. If \( (\exists x \in V)[\#(F, x) \leq 3^{n-1} - 2^{n-1} < \#(F, \overline{x})] \), then each satisfying assignment must map \( x \) to false.
4. If \( (\exists x \in V)[\#(F, x) \leq 3^{n-1} - 2^{n-1} < \#(F, \overline{x})] \), then each satisfying assignment must map \( x \) to true.

Fix \( F_n \in \mathcal{F} \) and let \( V \) be the set of variables appearing in \( F_n \). We will show that \( F_n \) is not affected by these bounds. Since \( \|F_n\| = 1 \) and \( n > 1 \), bound 1 does not apply. Note that because \( F_n \) only contains one clause, and because that clause contains all variables in \( V \), for all \( x \in V \) it is always the case that \( \#(F_n, x) = 1 \) and \( \#(F_n, \overline{x}) = 0 \). This means that in bound 2, the min term will always be \( \min(1, 0) = 0 \) which is never greater than the bound of \( 3^{n-1} - 2^{n-1} \) when \( n > 1 \). Thus \( F_n \) is unaffected by bound 2. In bound 3, the inequality will evaluate to \( 1 \leq 3^{n-1} - 2^{n-1} < 0 \) which is never true, hence \( F_n \) is unaffected by bound 3. In bound 4, we have \( 0 \leq 3^{n-1} - 2^{n-1} < 1 \) which is never true as we require \( n > 1 \). Thus \( F_n \) is unaffected by bound 4. Hence, all counterexamples in \( \mathcal{F} \) are unaffected by these proposed bounds and Algorithm 1 still runs in exponential time.

5 Conclusion

Due to the oversight in Kumar’s paper, there is no polynomial that upper bounds the runtime of Algorithm 1 even when the inputs are Boolean formulas as simple as those in \( \mathcal{F} \). Despite several of the paper’s optimizations to the algorithm, the worst-case runtime remains unchanged. Thus Kumar’s paper has not given a polynomial-time algorithm to decide CNF-SAT, and so the paper fails to show \( P = NP \) as claimed.

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