Abstract. We prove that the class of Banach spaces $Y$ such that the pair $(\ell_1,Y)$ has the Bishop-Phelps-Bollobás property for operators is stable under finite products when the norm of the product is given by an absolute norm. We also provide examples showing that previous stability results obtained for that property are optimal.

1. Introduction

This paper is motivated by recent research on extensions of the so-called Bishop-Phelps-Bollobás theorem for operators. Bishop-Phelps theorem [8] states that every continuous linear functional on a Banach space can be approximated (in norm) by norm attaining functionals. Before to state precisely a “quantitative version” of that result proved by Bollobás [9] we recall some notation. We denote by $B_X$, $S_X$ and $X^*$ the closed unit ball, the unit sphere and the topological dual of a Banach space $X$, respectively. If $X$ and $Y$ are both real or both complex Banach spaces, $L(X,Y)$ denotes the space of (bounded linear) operators from $X$ to $Y$, endowed with its usual operator norm.

**Bishop-Phelps-Bollobás Theorem** (see [10, Theorem 16.1], or [12, Corollary 2.4]).

Let $X$ be a Banach space and $0 < \varepsilon < 1$. Given $x \in B_X$ and $x^* \in S_{X^*}$ with $|1 - x^*(x)| < \frac{\varepsilon^2}{2}$, there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y) = 1$, $\|y - x\| < \varepsilon$ and $\|y^* - x^*\| < \varepsilon$.

A lot of attention has been devoted to extending Bishop-Phelps theorem to operators and interesting results have been obtained about that topic (see for instance [19] and [11]). In [2] the reader may find most of the results on the topic known until 2006 and some open questions on the subject. The survey paper [20] contains updated results for Bishop-Phelps property for the space of compact operators. It deserves to point out that in general the subset of norm attaining compact operators between two Banach spaces is not dense in the corresponding space of compact operators [21, Theorem 8].

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In 2008 the study of extensions of Bishop-Phelps-Bollobás theorem to operators was initiated by Acosta, Aron, García and Maestre \[3\]. In order to state some of these extensions it will be convenient to recall the following notion.

**Definition 1.1** (\[3, Definition 1.1\]). Let $X$ and $Y$ be either real or complex Banach spaces. The pair $(X, Y)$ is said to have the Bishop-Phelps-Bollobás property for operators (BPBp) if for every $0 < \varepsilon < 1$ there exists $0 < \eta(\varepsilon) < \varepsilon$ such that for every $T \in S_{L(X,Y)}$, if $x_0 \in S_X$ satisfies $\|T(x_0)\| > 1 - \eta(\varepsilon)$, then there exist an element $u_0 \in S_X$ and an operator $S \in S_{L(X,Y)}$ satisfying the following conditions

$$\|S(u_0)\| = 1, \quad \|u_0 - x_0\| < \varepsilon \quad \text{and} \quad \|S - T\| < \varepsilon.$$

In the paper already mentioned it is shown that the pair $(X, Y)$ has the BPBp whenever $X$ and $Y$ are finite-dimensional spaces \[3, Proposition 2.4\]. The same result also holds true in case that $Y$ has a certain isometric property (called property $\beta$ of Lindenstrauss), for every Banach space $X$ \[3, Theorem 2.2\]. For instance, the spaces $c_0$ and $\ell_\infty$ have such geometric property. It is known that every Banach space admits an equivalent norm with the property $\beta$. In case that the domain is $\ell_1$ there is a characterization of the Banach spaces $Y$ such that $(\ell_1, Y)$ has the BPBp \[3, Theorem 4.1\]. The geometric property appearing in the previous characterization was called the almost hyperplane series property (in short AHSp) (see Definition 2.5).

In general there are a few results about stability of the BPBp under direct sums both on the domain or on the range. For instance, it was shown in \[6, Proposition 2.4\] that the pairs $(X, (\oplus \sum_{n=1}^\infty Y_n)_{c_0})$ and $(X, (\oplus \sum_{n=1}^\infty Y_n)_{\ell_\infty})$ satisfy the Bishop-Phelps-Bollobás property for operators whenever all pairs $(X, Y_n)$ have the Bishop-Phelps-Bollobás property for operators “uniformly”. On the other hand, on the range the BPBp is not stable under $\ell_p$-sums for $1 \leq p < \infty$ (see \[15, Theorem, p. 149\] and \[1, Theorem 2.3\]). Indeed it is a long-standing open question if for every Banach space $X$, the subset of norm attaining operators from $X$ into the euclidean space $\mathbb{R}^2$ is dense in the corresponding space of operators.

In case that the domain is $\ell_1$, there are some more known results for the stability of the class of Banach spaces $Y$ such that $(\ell_1, Y)$ has the BPBp. In view of the characterization already mentioned, we will list some known results of stability of the AHSp.

As a consequence of \[3, Theorem 4.1\] and \[6, Proposition 2.4\], if the family of Banach spaces $\{Y_n : n \in \mathbb{N}\}$ has AHSp “uniformly”, then the spaces $(\bigoplus_{n=1}^\infty Y_n)_{c_0}$ and $(\bigoplus_{n=1}^\infty Y_n)_{\ell_\infty}$ have AHSp. Also it was proved the stability of AHSp under finite $\ell_p$-sums for every $1 \leq p < \infty$ \[4, Theorems 2.3 and 2.6\]. Recently this result was
extended to any absolute sum of two summands (see Definition 2.3) [5, Theorem 2.6]. The paper [5] also contains some stability result for \( \left( \sum_{n=1}^{\infty} Y_n \right)_E \), where \( E \) is a Banach sequence space satisfying certain additional assumptions [5, Theorem 2.10].

The goal of this paper is to obtain some more stability results. Now we briefly describe the content of the paper. In section 2 we recall the definition of absolute norm on \( \mathbb{R}^N \), the class of norms induced on a finite product of normed spaces by absolute norms and some properties that will be used later. We also provide an example showing that, in general, an absolute norm on \( \mathbb{R}^3 \) cannot be written in terms of two absolute norms on \( \mathbb{R}^2 \) (see Example 2.7 for details).

Later in section 3, we prove that AHSp is stable under products of any finite number of Banach spaces with the same property, when the product is endowed with an absolute norm. Notice that the proof of this general result is far from the one for the case of the product of two spaces. We will provide more detailed arguments in section 3 for that assertion. Let us just mention now that a simple induction argument does not work in view of Example 2.7. It is worth to notice that in general the product of two spaces with AHSp does not necessarily has such property.

In section 4 we show the parallel stability result for AHp (see Definition 4.1). Let us mention that AHp is a property stronger than AHSp. Finally we provide a simple example showing that AHSp is not preserved in general by an infinite product in case that the norm is given by a Banach lattice sequence, even in the case that all the factors have AHp uniformly. This example shows that the stability result proved in [5, Theorem 2.10] is optimal.

2. Definitions and notation

In this section we recall the notions of absolute norm on \( \mathbb{R}^N \), the norm endowed by an absolute norm on a finite product of normed spaces and some main properties that we will use later. We also recall the notion of approximate hyperplane series property that will be essential in this paper.

The notion of an absolute norm for \( \mathbb{C}^2 \) was introduced in [10, §21], where the reader can find some properties of these norms. In different contexts this class of norms has been used in order to study geometric properties of the direct sum of Banach spaces (see for instance [25], [23] and [26]). Although we will use properties of absolute norms that are well known we recall the notion that we use and state properties useful to our purpose.
The following notion is a particular case of the one used in [18, Section 2]. It suffices for our purpose.

**Definition 2.1.** A norm \( f \) on \( \mathbb{R}^N \) is called *absolute* if it satisfies that
\[
    f((x_i)) = f(|x_i|), \quad \forall (x_i) \in \mathbb{R}^N.
\]

An absolute norm \( f \) is said to be *normalized* if \( f(e_i) = 1 \) for every \( 1 \leq i \leq N \), where \( \{e_i : 1 \leq i \leq N\} \) is the canonical basis of \( \mathbb{R}^N \).

Clearly the usual norms on \( \mathbb{R}^N \) are absolute norms. The following statement gathers some properties of absolute norms. Proofs can be found for instance in [18, Remark 2.1]. Since we consider finite dimensional spaces next assertions can be also checked by using a similar argument to the one used in [10, Lemmas 21.1 and 21.2]

**Proposition 2.2.** Let \( f \) be an absolute normalized norm on \( \mathbb{R}^N \). The following assertions hold

a) If \( x, y \in \mathbb{R}^N \) and \( |x_i| \leq |y_i| \) for each \( i \leq N \) then \( f(x) \leq f(y) \).

b) It is satisfied that
\[
    \|x\|_\infty \leq f(x) \leq \|x\|_1, \quad \forall x \in \mathbb{R}^N.
\]

c) If \( x, y \in \mathbb{R}^N \) and \( |x_i| < |y_i| \) for each \( i \leq N \) then \( f(x) < f(y) \).

Of course, the topological dual of \( \mathbb{R}^N \) can be identified with \( \mathbb{R}^N \) and the identification is given by the mapping \( \Phi : \mathbb{R}^N \rightarrow (\mathbb{R}^N)^* \) defined by
\[
    \Phi(y)(x) = \sum_{i=1}^{N} y_i x_i, \quad \forall y, x \in \mathbb{R}^N.
\]

Under this identification, by defining the mapping
\[
    f^*(y) = \max\{\Phi(y)(x) : x \in \mathbb{R}^N, f(x) \leq 1\},
\]
it is immediate that \( f^* \) is also an absolute normalized norm in case that \( f \) is an absolute normalized norm on \( \mathbb{R}^N \) and \( \Phi \) is a surjective linear isometry from \( (\mathbb{R}^N, f^*) \) to the dual of the space \( (\mathbb{R}^N, f) \).

Next concept is standard and has been used in the literature very frequently for the product of two spaces (see for instance [7], [22], [23], [24] and [17]).

**Definition 2.3.** Let \( N \) be a nonnegative integer, \( X_i \) a Banach space for each \( i \leq N \) and \( f : \mathbb{R}^N \rightarrow \mathbb{R} \) be an absolute norm. Then the mapping \( \| \|_f : \prod_{i=1}^{N} X_i \rightarrow \mathbb{R} \)
given by
\[ \|x\|_f = f(\|x_i\|), \quad \forall x = (x_i) \in \prod_{i=1}^{N} X_i \]
is a norm on \( \prod_{i=1}^{N} X_i \). In what follows, we denote \( Z = \prod_{i=1}^{N} X_i \), endowed with the norm \( \| \|_f \).

The following result describes the dual and the duality mapping of the space \( Z \), that is essentially well known. In any case there is a proof in [16, Proposition 3.3].

**Proposition 2.4.** Under the previous setting the dual space \( Z^* \) can be identified with the space \( \prod_{i=1}^{N} X_i^* \), endowed with the absolute norm \( f^* \). More precisely, the mapping \( \psi : \prod_{i=1}^{N} X_i^* \rightarrow Z^* \) given by
\[ \Psi((x_i^*)) (x_i) = \sum_{i=1}^{N} x_i^*(x_i), \quad \forall (x_i) \in \prod_{i=1}^{N} X_i, \ (x_i^*) \in \prod_{i=1}^{N} X_i^* \]
is a surjective linear isometry from \( \prod_{i=1}^{N} X_i^* \) to the topological dual of \( Z \), where we consider in \( \prod_{i=1}^{N} X_i^* \) the norm associated to \( f^* \), that is,
\[ \| (x_i^*) \|_{f^*} = f^*(\|x_i^*\|), \quad \forall (x_i^*) \in \prod_{i=1}^{N} X_i^*. \]
Moreover, if \( z^* = \psi((x_i^*)) \in S_{Z^*} \) and \( z = (x_i) \in S_Z \), then \( z^*(z) = 1 \) if and only if
\[ x_i^*(x_i) = \|x_i^*\| \|x_i\|, \quad \forall i \leq N. \]

In what follows by a convex series we mean a series \( \sum \alpha_n \) of nonnegative real numbers such that \( \sum_{n=1}^{\infty} \alpha_n = 1 \). Now we recall other notion essential in our paper which is related to the Bishop-Phelps-Bollobás property for operators.

**Definition 2.5** ([3, Remark 3.2]). A Banach space \( X \) has the approximate hyperplane series property (AHSp) if for every \( \varepsilon > 0 \) there exist \( \gamma_X(\varepsilon) > 0 \) and \( \eta_X(\varepsilon) > 0 \) with \( \lim_{\varepsilon \to 0} \gamma_X(\varepsilon) = 0 \) such that for every sequence \( \{x_n\} \) in \( S_X \) and every convex series \( \sum_{n} \alpha_n \) with
\[ \left\| \sum_{k=1}^{\infty} \alpha_k x_k \right\| > 1 - \eta_X(\varepsilon), \]
there exist a subset \( A \subset \mathbb{N} \) and a subset \( \{z_k : k \in A\} \subset S_X \) satisfying the following conditions
1) \( \sum_{k \in A} \alpha_k > 1 - \gamma_X(\varepsilon), \)
2) \( \|z_k - x_k\| < \varepsilon \) for all \( k \in A \) and
3) there is \( x^* \in S_{X^*} \) such that \( x^*(z_k) = 1 \) for all \( k \in A \).

Finite-dimensional spaces, uniformly convex spaces, the classical spaces \( C(K) \) (\( K \) is a compact and Hausdorff space) and \( L_1(\mu) \) (\( \mu \) is a positive measure) have AHSp (see [3, Section 3]).

It is convenient to recall the following characterization of AHSp.

**Proposition 2.6** ([4, Proposition 1.2]). Let \( X \) be a Banach space. The following conditions are equivalent.

a) \( X \) has the AHSp.

b) For every \( 0 < \varepsilon < 1 \) there exist \( \gamma_X(\varepsilon) > 0 \) and \( \eta_X(\varepsilon) > 0 \) with \( \lim_{\varepsilon \to 0} \gamma_X(\varepsilon) = 0 \) such that for every sequence \( \{x_n\} \) in \( B_X \) and every convex series \( \sum_n \alpha_n \) with
\[
\left\| \sum_{k=1}^{\infty} \alpha_k x_k \right\| > 1 - \eta_X(\varepsilon),
\]
there are a subset \( A \subseteq \mathbb{N} \) with \( \sum_{k \in A} \alpha_k > 1 - \gamma_X(\varepsilon) \), an element \( x^* \in S_{X^*} \), and \( \{z_k : k \in A\} \subseteq (x^*)^{-1}(1) \cap B_X \) such that \( \|z_k - x_k\| < \varepsilon \) for all \( k \in A \).

c) For every \( 0 < \varepsilon < 1 \) there exists \( 0 < \eta < \varepsilon \) such that for any sequence \( \{x_n\} \) in \( B_X \) and every convex series \( \sum_n \alpha_n \) with \( \left\| \sum_{k=1}^{\infty} \alpha_k x_k \right\| > 1 - \eta \), there are a subset \( A \subseteq \mathbb{N} \) with \( \sum_{k \in A} \alpha_k > 1 - \varepsilon \), an element \( x^* \in S_{X^*} \), and \( \{z_k : k \in A\} \subseteq (x^*)^{-1}(1) \cap B_X \) such that \( \|z_k - x_k\| < \varepsilon \) for all \( k \in A \).

d) The same statement holds as in (c) but for every sequence \( \{x_n\} \) in \( S_X \).

Acosta, Mastyło and Soleimani-Mourchehkhorti proved that the AHSp is stable under product of two spaces, endowed with an absolute norm [5, Theorem 2.6]. The argument for extending that result for more summands is not obvious. Next we provide an example of an absolute norm on \( \mathbb{R}^3 \) that cannot be expressed in terms of two absolute norms on \( \mathbb{R}^2 \). As a consequence, induction cannot be applied directly to prove the stability result of AHSp under absolute norms.

**Example 2.7.** Consider the function on \( \mathbb{R}^3 \) given by
\[
|(x, y, z)| = \max\{\sqrt{x^2 + y^2}, |x| + |z|\} \quad ((x, y, z) \in \mathbb{R}^3).
\]

Then \( | \cdot | \) is an absolute normalized norm on \( \mathbb{R}^3 \) and there are no absolute norms \( f \) and \( g \) on \( \mathbb{R}^2 \) satisfying any of the following three assertions

i) \( |(x, y, z)| = f(g(y, z), x), \forall (x, y, z) \in \mathbb{R}^3. \)

ii) \( |(x, y, z)| = f(g(x, z), y), \forall (x, y, z) \in \mathbb{R}^3. \)
iii) \(|(x, y, z)| = f(g(x, y), z)|, \ \forall (x, y, z) \in \mathbb{R}^3.\)

Proof. It is immediate to check that \(| |\) is an absolute normalized norm on \(\mathbb{R}^3.\)

i) Assume that it is satisfied the equality
\[|(x, y, z)| = f(g(y, z), x), \ \forall (x, y, z) \in \mathbb{R}^3.\]

Since \(|e_2| = |e_3| = 1\) we have that
\[1 = f(g(1,0),0) = g(1,0)f(1,0), \quad 1 = f(g(0,1),0) = g(0,1)f(1,0)\]

and so
\[g(1,0) = g(0,1).\]

As a consequence we obtain that
\[\sqrt{2} = |(1,1,0)| = f(g(1,0),1) = f(g(0,1),1) = |(1,0,1)| = 2,\]

which is a contradiction. So condition i) cannot be satisfied.

ii) Assume now that it is satisfied
\[|(x, y, z)| = f(g(x, z), y), \ \forall (x, y, z) \in \mathbb{R}^3.\]

So
\[(2.1) \quad |x| + |z| = |(x, 0, z)| = f(g(x, z), 0) = g(x, z)f(1,0), \quad \forall (x, z) \in \mathbb{R}^2.\]

Hence we obtain that
\[\sqrt{x^2 + y^2} = |(x, 0, y)| = f(g(x, 0), y) = f\left(\frac{x}{f(1,0)}, y\right), \quad \forall (x, y) \in \mathbb{R}^2.\]

That is,
\[f(x, y) = \sqrt{(f(1,0)x)^2 + y^2}, \quad \forall (x, y) \in \mathbb{R}^2.\]

As a consequence, in view of the previous equality and (2.1) we deduce that
\[f(g(x, z), y)) = \sqrt{(f(1,0)g(x, z))^2 + y^2} = \sqrt{|x| + |z|}^2 + y^2, \quad \forall (x, y, z) \in \mathbb{R}^3.\]

But the last equality contradicts the assumption of ii).

iii) Assume now that
\[|(x, y, z)| = f(g(x, y), z), \ \forall (x, y, z) \in \mathbb{R}^3.\]

Hence we get that
\[(2.2) \quad \sqrt{x^2 + y^2} = f(g(x, y), 0) = g(x, y)f(1,0), \quad \forall (x, y) \in \mathbb{R}^2.\]

As a consequence we have that
\[|x| + |z| = |(x, 0, z)| = f(g(x, 0), z) = f\left(\frac{x}{f(1,0)}, z\right), \quad \forall (x, z) \in \mathbb{R}^2,\]
that is,
\[(2.3) \quad f(x, z) = f(1, 0)|x| + |z|, \quad \forall (x, z) \in \mathbb{R}^2.\]

For each \((x, y, z) \in \mathbb{R}^3\), in view of (2.3) and (2.2) we obtain that
\[
\max\{\sqrt{x^2 + y^2}, |x| + |z|\} = f(g(x, y), z) = f(1, 0)g(x, y) + |z| \quad \forall (x, y, z) \in \mathbb{R}^3
\]
which is a contradiction. So \(| \cdot |\) cannot satisfy condition iii).

\[\square\]

3. Stability result of the approximate hyperplane series property

As we already mentioned in the introduction, the goal of this section is to prove that the AHSp is stable under finite products in case that the norm of the product is given by an absolute norm. For product of two spaces that result was proved in [5, Theorem 2.6].

In the proof of the stability of AHSp for the product of two spaces Lemma 2.5 in [5] plays an essential role. But the statement of that result does not hold in case that we replace \(\mathbb{R}\) by \(\mathbb{R}^2\). For instance, this is the case of the absolute norm on \(\mathbb{R}^3\) whose closed unit ball is the convex hull of the set given by
\[
\{(x, y, 0) : x^2 + y^2 \leq 1\} \cup \{(x, 0, z) : x^2 + z^2 \leq 1\}
\]
\[
\cup \{(0, y, z) : y^2 + z^2 \leq 1\} \cup \left\{\frac{1}{\sqrt{2}}(r, s, t) : r, s, t \in \{1, -1\}\right\}.
\]

The following result is a consequence of [3, Lemma 3.3].

**Lemma 3.1.** Let \(\{z_k\}\) be a sequence of complex numbers with \(|z_k| \leq 1\) for any nonnegative integer \(k\), and let \(0 < \eta < 1\) and \(\sum \alpha_k\) be a convex series such that\(\text{Re} \sum_{k=1}^{\infty} \alpha_k z_k > 1 - \eta^2\). If we define \(A := \{k \in \mathbb{N} : \text{Re} z_k > 1 - \eta\}\) then
\[
\sum_{k \in A} \alpha_k > 1 - \eta.
\]

The next statement is a refinement of [3, Lemma 3.4] that will be very useful.

**Lemma 3.2.** Assume that \(|\cdot|\) is a norm on \(\mathbb{R}^N\). Then for every \(\varepsilon > 0\), there is \(\delta > 0\) such that whenever \(a^* \in S_{(\mathbb{R}^N)^*}^+\), there exists \(b^* \in S_{(\mathbb{R}^N)^*}^+\) satisfying \(\text{dist}(a, F(b^*)) < \varepsilon\) for all \(a \in \{z \in S_{\mathbb{R}^N} : a^*(z) > 1 - \delta\}\), where \(F(b^*) := \{y \in S_{\mathbb{R}^N} : b^*(y) = 1\}\) and also \(b^*(e_i) = 0\) for every \(i \leq N\) such that \(a^*(e_i) = 0\).
Proof. For a subset $G \subset \{k \in \mathbb{N} : k \leq N\}$ we define

$$Z_G := \{z^* \in S(\mathbb{R}^N)^* : z^*(e_i) = 0, \forall i \in G\}.$$  

It is clear that $Z_G$ is a compact set of $(\mathbb{R}^N)^*$.

We argue by contradiction. So assume that there is a set $G \subset \{k \in \mathbb{N} : k \leq N\}$, some positive real number $\varepsilon_0$ such that for each $\delta > 0$ there is $a^*_\delta \in Z_G$ such that for each $b^* \in Z_G$ there is some element $a \in \{z \in S_{\mathbb{R}^N} : a^*_\delta(z) > 1 - \delta\}$ such that $\text{dist}(a, F(b^*)) \geq \varepsilon_0$.

So there are sequences $(r_n) \to 1$, $(a^*_n) \subset Z_G$ such that for all $b^* \in Z_G$, $\{a \in S_{\mathbb{R}^N} : a^*_n(a) > r_n\} \cap \{a \in S_{\mathbb{R}^N} : \text{dist}(a, F(b^*)) \geq \varepsilon_0\} \neq \emptyset$. By compactness of $Z_G$, we may assume that $(a^*_n) \to a^*$ for some $a^* \in Z_G$. By the previous condition there is a sequence $(a_n)$ in $S_{\mathbb{R}^N}$ satifying $r_n < a^*_n(a_n) \leq 1$ for each $n$ and such that

$$(3.1) \quad \text{dist}(a_n, F(a^*)) \geq \varepsilon_0, \quad \forall n \in \mathbb{N}.$$  

By passing to a subsequence, if needed, we also may assume that $(a_n)$ converges to some $a \in S_{\mathbb{R}^N}$. Since $(a^*_n(a_n)) \to 1$ and both sequences are convergent, it follows that $a^*(a) = 1$; that is, $a \in F(a^*)$. As a consequence we obtain that $\text{dist}(a_n, F(a^*)) \leq \|a_n - a\|$ for every $n$. Since $(a_n)$ converges to $a$, the previous inequality contradicts $(3.1)$. \hfill \Box

Theorem 3.3. Assume that $\|\cdot\|$ is an absolute normalized norm on $\mathbb{R}^N$ and $\{X_i : i \leq N\}$ are Banach spaces having the AHSp, then $Z = \prod_{i=1}^N X_i$ has the AHSp, where $Z$ is endowed with the norm given by

$$\|(x_1, \ldots, x_N)\| = \left(\|x_1\|, \ldots, \|x_N\|\right), \quad (x_i \in X_i, \forall i \leq N).$$  

Proof. For a set $G \subset \{k \in \mathbb{N} : k \leq N\}$ we define $P_G : Z \longrightarrow Z$ by

$$P_G(z)(i) = z_i \text{ if } i \in G \quad \text{and} \quad P_G(z)(i) = 0 \text{ if } i \in \{1, 2, \ldots, N\} \setminus G.$$  

For each $i \leq N$ we denote by $Q_i(z) = z_i$ for every $z \in Z$.

We can clearly assume that $X_i \neq \{0\}$ for each $i \leq N$. We will prove the result by induction on $N$. For $N = 1$ the result is trivially satisfied. So we assume that $N \geq 2$ and the result is true for the space $\prod_{i \in G} X_i$ for any subset $G \subset \{k \in \mathbb{N} : k \leq N\}$ such that $|G| \leq N - 1$. We will prove the result for $G = \{k \in \mathbb{N} : k \leq N\}$. To this end we use that in view of [3, Proposition 3.5] finite-dimensional spaces have AHSp.

Assume that $0 < \varepsilon < 1$ and let $\eta : [0, 1[ \longrightarrow [0, 1[$ be a function such that

a) the pair $(\varepsilon, \eta(\varepsilon))$ satisfies condition c) in Proposition 2.6 for the space $(\mathbb{R}^N, \|\cdot\|)$ and for the Banach spaces $\prod_{i \in G} X_i$ for each $G \subset \{k \in \mathbb{N} : k \leq N\}$ such that $|G| \leq N - 1$,
b) the pair $(\varepsilon, \eta(\varepsilon))$ satisfies Lemma 3.2 for $\delta = \eta(\varepsilon)$ and 
c) $\eta(\varepsilon) < \varepsilon$ for every $\varepsilon \in ]0, 1[$.

We will show that $Z$ satisfies condition d) in Proposition 2.6 for
$$\eta' = \left(\frac{\eta(\frac{\varepsilon}{4N})}{2N}\right)^8.$$ 
Assume that $(u_k)$ is a sequence in $S_Z$ and $\sum \alpha_k$ is a convex series such that
$$\left\|\sum_{k=1}^{\infty} \alpha_k u_k\right\| > 1 - \eta'.$$

By Hahn-Banach theorem there is a functional $u^* = (u_1^*, u_2^*, \ldots, u_N^*) \in S_{Z^*}$ such that

$$1 - \eta' < \text{Re} \left(\sum_{k=1}^{\infty} \alpha_k u_k\right) = \text{Re} \left(\sum_{k=1}^{\infty} \alpha_k \left(\sum_{i=1}^{N} u_i^*(u_k(i))\right)\right).$$

Now we define the set $F \subset \{k \in \mathbb{N} : k \leq N\}$ by
$$F = \left\{i \leq N : \|u_i^*\| > \sqrt[8]{\eta'}\right\} \quad \text{and} \quad F^c = \{i \in \mathbb{N} : i \leq N\} \setminus F.$$

Since $u^* \in S_{Z^*}$, in view of Proposition 2.4 and assertion b) in Proposition 2.2 we obtain that $F \neq \emptyset$. We consider two cases.

**Case 1.** Assume that $|F| < N$.

Notice that
$$1 - \eta' < \text{Re} \sum_{k=1}^{\infty} \alpha_k u_k^*(u_k)$$
$$= \text{Re} \sum_{k=1}^{\infty} \alpha_k \left(\sum_{i \in F} u_i^*(u_k(i))\right) + \text{Re} \sum_{k=1}^{\infty} \alpha_k \left(\sum_{i \in F^c} u_i^*(u_k(i))\right)$$
$$\leq \text{Re} \sum_{k=1}^{\infty} \alpha_k \left(\sum_{i \in F} u_i^*(u_k(i))\right) + \sum_{k=1}^{\infty} \alpha_k \left(\sum_{i \in F^c}^{\sqrt[8]{\eta'}} u_i^*(u_k(i))\right)$$
$$\leq \text{Re} \sum_{k=1}^{\infty} \alpha_k \left(\sum_{i \in F} u_i^*(u_k(i))\right) + N \sqrt[8]{\eta'}$$
$$\leq \text{Re} \sum_{k=1}^{\infty} \alpha_k \left(\sum_{i \in F} u_i^*(u_k(i))\right) + \frac{\eta(\frac{\varepsilon}{4N})}{2}.$$
So

\[(3.3) \quad \Re \sum_{k=1}^{\infty} \alpha_k \left( \sum_{i \in F} u_i^* (u_k(i)) \right) > 1 - \eta' - \eta \left( \frac{\varepsilon}{4N} \right) > 1 - \eta \left( \frac{\varepsilon}{4N} \right). \]

By assumption the space $\prod_{i \in F} X_i$ has AHSp, and in view of a) there is a set $A \subset \mathbb{N}$ and $v^* = (v^*_i)_{i \in F} \in S(\prod_{i \in F} X_i)^*$ such that

\[(3.4) \quad \sum_{k \in A} \alpha_k > 1 - \eta \left( \frac{\varepsilon}{4N} \right) > 1 - \frac{\varepsilon}{4N} > 1 - \varepsilon
\]

and for every $k \in A$ there is $v_k \in S_{\prod_{i \in F} X_i}$ such that

\[(3.5) \quad v^*(v_k) = \sum_{i \in F} v_i^* (v_k(i)) = 1, \quad \forall k \in A
\]

and

\[(3.6) \quad \|v_k - P_F(u_k)\| < \eta \left( \frac{\varepsilon}{4N} \right) < \frac{\varepsilon}{4N} < \frac{\varepsilon}{4}, \quad \forall k \in A.
\]

Now we define $G$ as follows

\[G = \{i \in F : \exists k \in A, v_i^* (v_k(i)) \neq 0\}.
\]

By (3.5) we have that

\[(3.7) \quad \sum_{i \in G} v_i^* (v_k(i)) = \sum_{i \in F} v_i^* (v_k(i)) = v^*(v_k) = 1, \quad \forall k \in A.
\]

As a consequence $(v_i^*)_{i \in G} \in S(\prod_{i \in G} X_i)^*$. In view of Proposition 2.4 we have that

\[(3.8) \quad v_i^* (v_k(i)) = \|v_i^*\| \|v_k(i)\|, \quad \forall i \in F, k \in A.
\]

Now we define the element $w^* \in Z^*$ as follows

\[w_i^* = \begin{cases}
  v_i^* & \text{if } i \in G \\
  0 & \text{if } i \in \{j \in \mathbb{N} : j \leq N\} \setminus G.
\end{cases}
\]

It is trivially satisfied that $w^* \in S_{Z^*}$ and by (3.7) we have

\[(3.9) \quad w^*(v_k) = \sum_{i \in G} w_i^* (v_k(i)) = \sum_{i \in G} v_i^* (v_k(i)) = 1, \quad \forall k \in A.
\]

So by (3.6) for each $k \in A$ we have that

\[\Re w^*(u_k) = \Re w^*(P_F(u_k)) \geq \Re w^*(v_k) - \|v_k - P_F(u_k)\| > 1 - \eta \left( \frac{\varepsilon}{4N} \right).
\]
That is, for each $k \in A$ it is satisfied that

$$1 - \eta \left( \frac{\varepsilon}{4N} \right) < \Re \sum_{i=1}^{N} w_i^* (u_k(i)) \leq \sum_{i=1}^{N} \| w_i^* \| \| u_k(i) \| = \sum_{i \in G} \| v_i^* \| \| u_k(i) \|.$$ 

By using condition b) there exists $s = (s_1, s_2, \ldots, s_N) \in S_{(\mathbb{R}^N)^*}$ such that for every $i \in \{ k \in \mathbb{N} : k \leq N \} \setminus G$, $s_i = 0$ and for every $k \in A$ there exists $r_k = (r_k(i))_{i \leq N} \in S_{\mathbb{R}^N}$ such that

$$\sum_{k=1}^{N} s_i r_k(i) = 1, \quad \left| (r_k(i))_{i \leq N} - (\| u_k(i) \|)_{i \leq N} \right| < \frac{\varepsilon}{4N} < \frac{\varepsilon}{4}.$$ 

Finally we define $z^* = (z^*_i)_{i \leq N} \in Z^*$ as follows

$$z^*_i = \begin{cases} s_i \frac{v_i^*}{\| v_i^* \|} & \text{if } i \in G \\ 0 & \text{if } i \in \{ j \in \mathbb{N} : j \leq N \} \setminus G. \end{cases}$$

By Proposition 2.4 we have that $\| z^* \| = \| (s_i)_{i \leq N} \|_{(\mathbb{R}^N)^*} = 1$, so $z^* \in S_{Z^*}$.

Notice that for every $i \in G$ there exists $k_0^i \in A$ such that $v_i^*(v_{k_0^i}(i)) \neq 0$. For every $i \in \{ j \in \mathbb{N} : j \leq N \} \setminus G$ we choose $x_i \in S_{X_i}$ and for every $k \in A$ we define $z_k \in S_Z$ as follows

$$z_k(i) = \begin{cases} r_k(i) \frac{v_k(i)}{\| v_k(i) \|} & \text{if } i \in F \text{ and } v_k(i) \neq 0 \\ r_k(i) \frac{v_{k_0}(i)}{\| v_{k_0}(i) \|} & \text{if } i \in G \text{ and } v_k(i) = 0 \\ r_k(i) x_i & \text{if } i \in F \setminus G \text{ and } v_k(i) = 0 \\ r_k(i) \frac{u_k(i)}{\| u_k(i) \|} & \text{if } i \in \{ j \in \mathbb{N} : j \leq N \} \setminus F \text{ and } u_k(i) \neq 0 \\ r_k(i) x_i & \text{if } i \in \{ j \in \mathbb{N} : j \leq N \} \setminus F \text{ and } u_k(i) = 0. \end{cases}$$

Since $\| z_k \| = \left| (r_k(i))_{i \leq N} \right| = 1$ we have that $z_k \in S_Z$ for every $k \in A$. By (3.8) and (3.10), taking into account that $s_i = 0$ for each $i \in \{ j \leq N \} \setminus G$, it is also satisfied that

$$z^*(z_k) = \sum_{i \in G} s_i r_k(i) = \sum_{i=1}^{N} s_i r_k(i) = 1, \quad \forall k \in A.$$
Let us fix \( k \in A \). For \( i \in F \) it is clear that
\[
\| z_k(i) - u_k(i) \| \leq \| z_k(i) - v_k(i) \| + \| v_k(i) - u_k(i) \|
= |r_k(i) - \| v_k(i) \| | + \| v_k(i) - u_k(i) \|
\leq |r_k(i) - \| u_k(i) \| | + 2 \| v_k(i) - u_k(i) \|.
\]
As a consequence, by using also (3.10) and (3.6) we obtain that
\[
\| P_F(z_k) - P_F(u_k) \| \leq |(r_k(i))_{i \leq N} - (\| u_k(i) \|)_{i \leq N}| + 2 \| P_F(v_k) - P_F(u_k) \|
< \varepsilon + \frac{2 \varepsilon}{4} = \frac{3 \varepsilon}{4}.
\] (3.12)

For \( i \in \{ j \in \mathbb{N} : j \leq N \}\setminus F \) we have that \( \| z_k(i) - u_k(i) \| = |r_k(i) - \| u_k(i) \| | \), so in view of (3.10) we obtain that
\[
\| P_{F^c}(z_k) - P_{F^c}(u_k) \| \leq |(r_k(i))_{i \leq N} - (\| u_k(i) \|)_{i \leq N}| < \frac{\varepsilon}{4}.
\] (3.13)

From (3.12) and (3.13) we conclude that \( \| z_k - u_k \| < \varepsilon \) for every \( k \in A \). Since we know that \( z^* \in S_{Z^*} \) and by (3.11) and (3.4) the proof is finished in case 1.

**Case 2.** Assume now that \( F = \{ i \in \mathbb{N} : i \leq N \} \). We define the set \( B \) by
\[
B = \left\{ k \in \mathbb{N} : \Re u^*(u_k) > 1 - \sqrt{\eta} \right\}.
\]
In view of (3.2) and Lemma 3.1 we obtain that
\[
\sum_{k \in B} \alpha_k > 1 - \sqrt{\eta}.
\] (3.14)

In view of Proposition 2.4, for every \( k \in B \) we have that
\[
1 - \eta \left( \eta \left( \frac{\varepsilon}{4N} \right) \right) < 1 - \sqrt{\eta}
= \Re u^*(u_k)
\]
\[
\leq \Re \left( \sum_{i=1}^{N} u_i^*(u_k(i)) \right)
\]
\[
\leq \sum_{i=1}^{N} \| u_i^* \| \| u_k(i) \| \leq 1.
\] (3.15)
By condition b) there is \( s = (s_1, s_2, \ldots, s_N) \in S_{(\mathbb{R}^N)^*} \), and for every \( k \in B \) there is \( (r_k(i))_{i \leq N} \in S_{\mathbb{R}^N} \) such that
\[
\sum_{k=1}^{N} s_k r_k(i) = 1, \quad \left| (r_k(i))_{i \leq N} - \left( \|u_k(i)\| \right)_{i \leq N} \right| < \eta \left( \frac{\varepsilon}{4N} \right) < \frac{\varepsilon}{4N},
\]
where we also used that \( \eta \) satisfies condition c). From (3.15) for each \( k \in B \) we have
\[
\text{Re} u_i^* (u_k(i)) \geq \|u_i^*\| \|u_k(i)\| - \sqrt{\eta'}, \quad \forall 1 \leq i \leq N.
\]
Now for each \( i \leq N \) we define the set \( C_i \subset B \) as follows
\[
C_i = \left\{ k \in B : \|u_k(i)\| > \sqrt{\eta'} \right\}.
\]
Since \( F = \{ i \in \mathbb{N} : i \leq N \} \), for each \( i \leq N \) we know that \( \|u_i^*\| > \sqrt{\eta'} \). Hence for each \( i \leq N \) such that \( C_i \neq \emptyset \) from (3.17) we obtain that
\[
\frac{\text{Re} u_i^* (u_k(i))}{\|u_i^*\| \|u_k(i)\|} \geq 1 - \frac{\sqrt{\eta'}}{\sqrt{\eta'} \sqrt{\eta'}} = 1 - \frac{1}{\sqrt{\eta'}} > 1 - \eta \left( \frac{\varepsilon}{4N} \right), \quad \forall k \in C_i.
\]
Since \( X_i \) has AHSp, by using a) there is a set \( D_i \subset C_i \) such that
\[
\sum_{k \in D_i} \alpha_k \geq \left( 1 - \eta \left( \frac{\varepsilon}{4N} \right) \right) \sum_{k \in C_i} \alpha_k
\]
and there is \( v_i^* \in S_{(X_i)^*} \) and for every \( k \in D_i \) there is \( v_k(i) \in S_{X_i} \) such that
\[
v_i^* (v_k(i)) = 1, \quad \left| v_k(i) - \frac{u_k(i)}{\|u_k(i)\|} \right| < \eta \left( \frac{\varepsilon}{4N} \right).
\]
In case that \( C_i = \emptyset \) for some \( i \leq N \) we take \( D_i = \emptyset \). Now we define the set \( E \subset B \) by \( E = \bigcap_{i=1}^{N} (D_i \cup (B \setminus C_i)) \). Notice that for every \( 1 \leq i \leq N \) we have
\[
\sum_{k \in D_i \cup (B \setminus C_i)} \alpha_k = \sum_{k \in D_i} \alpha_k + \sum_{k \in B \setminus C_i} \alpha_k
\geq \left( 1 - \eta \left( \frac{\varepsilon}{4N} \right) \right) \sum_{k \in C_i} \alpha_k + \sum_{k \in B \setminus C_i} \alpha_k \quad \text{(by (3.18))}
\geq \sum_{k \in B} \alpha_k - \eta \left( \frac{\varepsilon}{4N} \right)
\geq 1 - \sqrt{\eta'} - \frac{\varepsilon}{4N} \quad \text{(by (3.14) and condition c))}.
\]
From the definition of $E$ and the previous chain of inequalities it follows that

\begin{equation}
\sum_{k \in E} \alpha_k \geq 1 - N \sqrt{\eta} - \frac{\varepsilon}{4} > 1 - \varepsilon. \tag{3.20}
\end{equation}

If $i \leq N$ and $C_i \neq \emptyset$ then $D_i \neq \emptyset$ so we can choose an element $k_0^i \in D_i$. In case that $C_i = \emptyset$ we choose $x_i \in S_X$, and $x_i^* \in S_{X_i^*}$ such that $x_i^*(x_i) = 1$. For each $k \in E$ we define $z_k \in S_Z$ as follows

$$z_k(i) = \begin{cases} 
    r_k(i) v_k(i) & \text{if } k \in D_i \\
    r_k(i) v_{k_0^i}(i) & \text{if } k \in B \setminus C_i \text{ and } C_i \neq \emptyset \\
    r_k(i) x_i & \text{if } k \in B \setminus C_i \text{ and } C_i = \emptyset.
\end{cases}$$

Also we define $z^* \in Z^*$ by

$$z^*(i) = \begin{cases} 
    s_i v_i^* & \text{if } C_i \neq \emptyset \\
    s_i x_i^* & \text{if } C_i = \emptyset.
\end{cases}$$

By Proposition \[2.14\] it is clear that $z^* \in S_{Z^*}$ since $\|z^*\| = \|(s_i)_{i \leq N}\|_{(\mathbb{R}N)^*} = 1$.

In view of (3.19) and (3.16) it is satisfied that

\begin{equation}
(3.21) \quad z^*(z_k) = \sum_{i=1}^{N} z_i^*(z_k(i)) = \sum_{i=1}^{N} s_i r_k(i) = 1, \quad \forall k \in E.
\end{equation}

Let us fix $k \in E$. If $k \in D_i$ we have

\begin{align*}
\|z_k(i) - u_k(i)\| &= \|r_k(i) v_k(i) - u_k(i)\| \\
&= \left| r_k(i) v_k(i) - r_k(i) \frac{u_k(i)}{\|u_k(i)\|} \right| + r_k(i) \frac{u_k(i)}{\|u_k(i)\|} - u_k(i) \\
&\leq |r_k(i)| \left| v_k(i) - \frac{u_k(i)}{\|u_k(i)\|} \right| + |r_k(i) - \|u_k(i)\|| \\
&\leq \frac{\varepsilon}{4N} + |r_k(i) - \|u_k(i)\|| \tag{by (3.19) and condition c)}.
\end{align*}

(3.22)
In case that \( k \in B \setminus C_i \) we obtain that
\[
\|z_k(i) - u_k(i)\| \leq \|z_k(i)\| + \|u_k(i)\| = |r_k(i)| + \|u_k(i)\|
\] (3.23)
\[
\leq |r_k(i)| - \|u_k(i)\| + 2\|u_k(i)\| \leq |r_k(i)| - \|u_k(i)\| + 2\sqrt[4]{\eta_i}
\]
\[
\leq |r_k(i)| - \|u_k(i)\| + \frac{\varepsilon}{4N}.
\]
By (3.22) and (3.23) we proved that for every \( k \in E \) we have
\[
\|z_k(i) - u_k(i)\| \leq |r_k(i)| - \|u_k(i)\| + \frac{\varepsilon}{4N}.
\] (3.24)

Taking into account (3.16) for every \( k \in E \) we deduce that
\[
\|z_k - u_k\| \leq \left| \left( r_k(i) \right)_{i \leq N} - \left( \|u_k(i)\| \right)_{i \leq N} \right| + \sum_{i=1}^{N} \frac{\varepsilon}{4N} \leq \frac{\varepsilon}{4N} + \frac{\varepsilon}{4} \leq \frac{\varepsilon}{2} < \varepsilon.
\]
Since \( z^* \in S_{Z^*} \), in view of (3.20), (3.21) and the previous inequality the proof is also finished in case 2. \( \square \)

Let us notice that the converse of Theorem 3.3 also holds. That is, in case that the product space \( Z = \prod_{i=1}^{N} X_i \), endowed with an absolute normalized norm, has the AHSp, then each space \( X_i \) also has the AHSp for \( 1 \leq i \leq N \), a result proved in [14, Theorem 2.3].

4. Stability of the approximate hyperplane property under finite products

The goal of this section is a result that asserts the stability of a property stronger than the approximate hyperplane series property under finite products endowed with an absolute norm. We begin with the following notion that was introduced in [13, Definition 2.1].

**Definition 4.1.** A Banach space \( X \) has the approximate hyperplane property (AHp) if there exists a function \( \delta : [0,1[ \rightarrow [0,1[ \) and a 1-norming subset \( C \) of \( S_{X^*} \), satisfying the following property.

Given \( \varepsilon > 0 \) there is a function \( \Upsilon_{X,\varepsilon} : C \rightarrow S_{X^*} \) with the following condition
\[
x^* \in C, \ x \in S_X, \ \text{Re} \ x^*(x) > 1 - \delta(\varepsilon) \Rightarrow \text{dist} ( x, F(\Upsilon_{X,\varepsilon}(x^*))) < \varepsilon,
\]
where \( F(y^*) = \{ y \in S_X : y^*(y) = 1 \} \) for any \( y^* \in S_{X^*} \).
A family of Banach spaces \( \{X_i : i \in I\} \) has AHp uniformly if every space \( X_i \) has property AHp with the same function \( \delta \).

Clearly we can assume that the 1-norming subset \( C \) in the previous definition satisfies \( T C \subset C \), where \( T \) is the unit sphere of the scalar field.

Let us notice that a similar property to AHp was implicitly used to prove that several classes of spaces have \( \text{AHSp} \) (see [3]). It is known that property AHp implies \( \text{AHSp} \) (see for instance [13, Proposition 2.2]). It is an open question whether or not the converse is true. Examples of spaces having AHp are finite-dimensional spaces, uniformly convex spaces, \( L_1(\mu) \) for every measure \( \mu \) and also \( C(K) \) for every compact Hausdorff topological space \( K \) (see [3, Propositions 3.5, 3.8, 3.6 and 3.7] and also [13, Corollary 2.12]).

**Remark 4.2.** Let us notice that in view of Lemma 3.2 the space \( \mathbb{R}^N \), endowed with any norm, satisfies AHp for the 1-norming set \( S(\mathbb{R}^N)^* \). Moreover if for some \( 1 \leq i \leq N \) and \( a^* \in S(\mathbb{R}^N)^* \), \( a^*(e_i) = 0 \), then \( (\Upsilon_{\mathbb{R}^N,\varepsilon}(a^*))(e_i) = 0 \), where \( \Upsilon_{\mathbb{R}^N,\varepsilon} \) is the mapping appearing in Definition 4.1.

The following result is a version of [5, Lemma 2.9].

**Lemma 4.3.** Assume that \( \| \cdot \| \) is an absolute normalized norm on \( \mathbb{R}^N \) and \( X_i \) is a Banach space for \( 1 \leq i \leq N \). If for each \( 1 \leq i \leq N \), \( A_i \subset B_{X_i}^* \) is a 1-norming set for \( X_i \) such that \( TA_i \subset A_i \), where \( T \) is the unit sphere of the scalar field, then the set

\[
A = \left\{ (r^*_i x^*_i)_{i \leq N} : (r^*_i)_{i \leq N} \in S(\mathbb{R}^N)^* , r^*_i \geq 0, x^*_i \in A_i, \forall 1 \leq i \leq N \right\}
\]

is a 1-norming set for \( Z = \prod_{i=1}^N X_i \), endowed with the absolute norm associated to \( \| \cdot \| \).

**Proof.** Assume that \( (x_i)_{i \leq N} \in S_Z \) and \( 0 < \varepsilon < 1 \). By assumption for each \( 1 \leq i \leq N \) there is an element \( x^*_i \in A_i \) satisfying that

\[
(4.1) \quad \text{Re } x^*_i(x_i) \geq (1 - \varepsilon)\|x_i\| \geq 0.
\]

By Hahn-Banach theorem there is \( (r^*_i)_{i \leq N} \in S(\mathbb{R}^N)^* \) such that

\[
(4.2) \quad \sum_{i=1}^N r^*_i\|x_i\| = 1.
\]
Clearly we can also assume that $r^*_i \in \mathbb{R}_0^+$ for each $1 \leq i \leq N$. As a consequence we have that

$$\text{Re} \left( \sum_{i=1}^{N} r^*_i x^*_i(x_i) \right) \geq (1 - \varepsilon) \sum_{i=1}^{N} r^*_i \|x_i\| \quad \text{(by (4.1))}$$

$$= 1 - \varepsilon \quad \text{(by (4.2))}.$$

**Theorem 4.4.** Assume that $\| \cdot \|$ is an absolute and normalized norm on $\mathbb{R}^N$ and $X_i$ is a Banach space satisfying the approximate hyperplane property for each $1 \leq i \leq N$. Then the space $Z = \prod_{i=1}^{N} X_i$, endowed with the absolute norm associated to $\| \cdot \|$, also has the approximate hyperplane property.

**Proof.** Without loss of generality we can assume that for each $1 \leq i \leq N$, $X_i \neq \{0\}$ and $X_i$ has the AHp with a 1-norming set $A_i \subset S_{X_i^*}$ such that $\mathbb{T}A_i \subset A_i$. By assumption and Remark 4.2 there is a function $\eta : ]0, 1[ \rightarrow ]0, 1[$ satisfying the following three conditions:

**i)** the pair $(\varepsilon, \eta(\varepsilon))$ satisfies the definition of AHp for the Banach space $X_i$ for each $1 \leq i \leq N$.

**ii)** the space $(\mathbb{R}^N, \| \cdot \|)$ satisfies the definition of AHp with the function $\eta$ playing the role of $\delta$ and

**iii)** $\eta(\varepsilon) < \varepsilon$ for each $\varepsilon \in ]0, 1[$.

Now we define $\eta'(\varepsilon) = \eta \left( \frac{\eta^3(\varepsilon)}{4N^2} \right)$. By Lemma 4.3 the set $A$ given by

$$A = \left\{ (r^*_i x^*_i)_{i \leq N} : (r^*_i)_{i \leq N} \in S_{(\mathbb{R}^N)^*}, r^*_i \geq 0, x^*_i \in A_i, \forall 1 \leq i \leq N \right\}$$

is a 1-norming set for $Z = \prod_{i=1}^{N} X_i$. We will show that $Z$ has the AHp with the set $A$ and the function $\eta'$.

We take an element $(r^*_i x^*_i)_{i \leq N} \in Z^*$ satisfying the conditions in the definition of the set $A$ and $(x_i)_{i \leq N} \in S_Z$ such that

$$\text{Re} \left( \sum_{i=1}^{N} r^*_i x^*_i(x_i) \right) > 1 - \eta'(\varepsilon).$$
Define the set $G$ by

$$G = \left\{ i \leq N : r_i^* > \frac{\eta(\frac{\varepsilon}{4N})}{2N} \right\}$$

and write

$$G^c = \{ i \in \mathbb{N} : i \leq N \} \backslash G.$$  

So

$$\sum_{i \in G} r_i^* \|x_i\| \geq \sum_{i \in G} r_i^* \text{Re} x_i^*(x_i)$$

(4.4)

$$> 1 - \eta'(\varepsilon) - \sum_{i \in G^c} r_i^* \text{Re} x_i^*(x_i)$$

$$\geq 1 - \eta'(\varepsilon) - N \eta\left(\frac{\varepsilon}{4N}\right)$$

$$> 1 - \eta\left(\frac{\varepsilon}{4N}\right) > 0.$$

Define $t^* = (t_i^*)_{i \leq N} = \frac{P_G r^*}{\|P_G r^*\|^*} \in S(\mathbb{R}^N)^*$, where we denoted by $| |^*$ the dual norm in $(\mathbb{R}^N, | |)^*$. In view of (4.4) we have that

$$\sum_{i \in G} t_i^* \|x_i\| > 1 - \eta\left(\frac{\varepsilon}{4N}\right) > 0.$$

Now we use that the space $(\mathbb{R}^N, | |)$ has the AHp (see condition ii)) and we write $(s_i^*)_{i \leq N} = \Upsilon_{\mathbb{R}^N, \frac{\varepsilon}{4N}}(t^*)$. In view of Remark 4.2 we know that $s_i^* = 0$ if $i \in G^c$ since $t_i^* = 0$ in this case. So we obtain that

$$\text{dist}\left(\left(\|x_i\|\right)_{i \leq N}, F\left((s_i^*)_{i \leq N}\right)\right) < \frac{\varepsilon}{4N}. $$

(4.5)

So there is $(s_i)_{i \leq N} \in F\left((s_i^*)_{i \leq N}\right) \subset S_{\mathbb{R}^N}$ satisfying

$$| (s_i)_{i \leq N} - (\|x_i\|)_{i \leq N} | < \frac{\varepsilon}{4N}. $$

(4.6)

Notice that

$$ (s_i^*)_{i \leq N}(s_i)_{i \leq N} = \sum_{i \in G} s_i^* s_i = 1. $$

(4.7)

Now for each $1 \leq i \leq N$, we define $z_i \in S_{X_i}$ as follows.

**Case 1.** Assume that $i \in G$ and $\|x_i\| > \frac{\eta\left(\frac{\varepsilon}{4N}\right)}{2N}$. 
From (4.3) we obtain that
\[ 1 - \eta'(\varepsilon) < \text{Re} \left( \sum_{i=N}^{\infty} r_i^* x_i^* (x_i) \right) \]
\[ = \text{Re} \sum_{i=1}^{N} r_i^* x_i^* (x_i) \]
\[ \leq \sum_{i=1}^{N} r_i^* \| x_i^* \| \| x_i \| \]
\[ \leq \sum_{i=1}^{N} r_i^* \| x_i \| \leq 1. \]

As a consequence we get that
\[ \text{Re} r_i^* x_i^* (x_i) > r_i^* \| x_i \| - \eta'(\varepsilon), \]
and so
\[ \text{Re} x_i^* \left( \frac{x_i}{\| x_i \|} \right) > 1 - \eta'(\varepsilon) > 1 - \eta \left( \frac{\varepsilon}{4N} \right). \]

Since we assume that \( X_i \) has the AHp with the function \( \eta \) and the subset \( A_i \), we conclude that
\[ \text{dist} \left( \frac{x_i}{\| x_i \|}, F(\Upsilon_{X_i, \frac{1}{4N}}(x_i^*)) \right) < \frac{\varepsilon}{4N}. \]
So there is \( z_i \in F(\Upsilon_{X_i, \frac{1}{4N}}(x_i^*)) \) such that \( \| z_i - \frac{x_i}{\| x_i \|} \| < \frac{\varepsilon}{4N} \). As a consequence we have that \( \| \| x_i \| z_i - x_i \| < \frac{\varepsilon}{4N} \). In view of (4.6) we deduce that have
\[(4.8) \quad \| s_i z_i - x_i \| < \frac{\varepsilon}{2N}. \]

**Case 2.** Assume that \( i \in G \) and \( \| x_i \| \leq \frac{\eta(\varepsilon)}{2N} \).

We choose an element \( z_i \in F(\Upsilon_{X_i, \frac{1}{4N}}(x_i^*)) \). From (4.6) we have
\[(4.9) \quad \| s_i z_i - x_i \| \leq |s_i| + \| x_i \| < \frac{2\varepsilon}{4N} + \frac{\varepsilon}{4N} = \frac{3\varepsilon}{4N}. \]

**Case 3.** Assume that \( i \in G^c \) and \( x_i \neq 0 \).
Define $z_i = \frac{x_i}{\|x_i\|}$. By (4.6) we have

\[(4.10) \quad \|s_i z_i - x_i\| = \left\| s_i \frac{x_i}{\|x_i\|} - x_i \right\| = |s_i - \|x_i\|| < \frac{\varepsilon}{4N}.
\]

**Case 4.** Assume that $i \in G^c$ and $x_i = 0$. In this case we choose any element $z_i \in S_{X_i}$. In view of (4.6) we have

\[(4.11) \quad \|s_i z_i - x_i\| = |s_i| < \frac{\varepsilon}{4N}.
\]

So from (4.8), (4.9), (4.10) and (4.11) we conclude that

\[
\left\| (s_i z_i)_{i \leq N} - (x_i)_{i \leq N} \right\| \leq \sum_{i=1}^{N} \|s_i z_i - x_i\| < \frac{3N \varepsilon}{4N} < \varepsilon.
\]

Notice that

\[
(s_i^* \gamma_{X_i, \frac{\varepsilon}{4N}}(x_i^*))_{i \leq N} \in S_Z, \quad \text{and} \quad (s_i z_i)_{i \leq N} \in S_Z.
\]

From (4.7) we have

\[
(s_i^* \gamma_{X_i, \frac{\varepsilon}{4N}}(x_i^*))_{i \leq N} (s_i z_i)_{i \leq N} = \sum_{i \in G} s_i^* s_i = 1.
\]

So the proof is finished. \[\square\]

In [5, Theorem 2.10] the authors provided a stability result of AHSp under some infinite sums that includes $\ell_p$-sums for $1 \leq p < \infty$. Here we provide a simple example showing that in such stability result some requirement on the Banach lattice sequence used to define the infinite sum of Banach spaces is needed. For that example we need the following easy result.

**Lemma 4.5.** It is satisfied that $\left\| \left( \frac{x_n}{2^n} \right) \right\|_2 \leq \|x\|_1$ for any element $x \in \ell_1$.

**Proof.** If $x \in \ell_1$, it is clear that

\[
\left\| \left( \frac{x_n}{2^n} \right) \right\|_2 \leq \|x\|_\infty \left\| \left( \frac{1}{2^n} \right) \right\|_2 \\
\leq \|x\|_1 \left\| \left( \frac{1}{2^n} \right) \right\|_1 \\
= \|x\|_1.
\]

\[\square\]
We need to recall some notions. In order to do this we denote by $\omega$ the space of all real sequences. A real Banach space $E \subset \omega$ is \textit{solid} whenever $x \in \omega, y \in E$ and $|x| \leq |y|$ then $x \in E$ and $\|x\|_E \leq \|y\|_E$. $E$ is said to be a \textit{Banach sequence lattice} if $E \subset \omega$, $E$ is solid and there exists $u \in E$ with $u > 0$.

Let $E$ be a Banach sequence lattice. For a given sequence $(X_k, \|x_k\|_{X_k})_{k=1}^{\infty}$ of Banach spaces the linear space of sequences $x = (x_k)$, with $x_k \in X_k$ for each $k \in \mathbb{N}$ and satisfying that $(\|x_k\|_{X_k}) \in E$, becomes a Banach space endowed with the norm

$$\|(x_k)\| = \left\| \left( \|x_k\|_{X_k} \right) \right\|_E.$$

We denote the previous space by $(\oplus \sum_{k=1}^{\infty} X_k)_E$. Finally we recall that a Banach lattice $E$ is \textit{uniformly monotone} if for each $\varepsilon > 0$ there is $\delta > 0$ satisfying the following condition

$$x \in S_E, y \in E, x, y \geq 0, \|x + y\| \leq 1 + \delta \Rightarrow \|y\| \leq \varepsilon.$$

\textbf{Example 4.6.} The space $E = \ell_1$, endowed with the norm $\| \|$ given by

$$\|x\| = \|x\|_1 + \left\| \left( \frac{x_n}{2^n} \right) \right\|_2 \quad (x \in E)$$

is a uniformly monotone Banach lattice sequence without the AHSp and so it does not satisfy the AHp.

\textit{Proof.} One can easily check that $E$ is a Banach lattice sequence and $\| \|$ is a strictly convex norm equivalent to the usual norm of $\ell_1$.

Since the norm $\| \|$ is equivalent to the usual norm of $\ell_1$, $E$ is not reflexive and so the norm $\| \|$ is not uniformly convex. By [3, Proposition 3.9] the space $E$ does not have the AHSp and so it cannot satisfy the AHp by [13, Proposition 2.2].

Now we show that the Banach lattice sequence space $E$ is uniformly monotone. Assume $\varepsilon > 0$, $x \in S_E$ and $y \in E$ such that $x, y \geq 0$ and $\|x + y\| \leq 1 + \frac{\varepsilon}{2}$. We will show that $\|y\| \leq \varepsilon$, so $E$ is uniformly monotone.
Since \( x, y \geq 0 \), notice that \( \| x + y \|_1 = \| x \|_1 + \| y \|_1 \) and \( \| \frac{x}{2^n} \|_2 \leq \| \frac{x + y}{2^n} \|_2 \), therefore
\[
1 + \| y \|_1 = \| x \|_1 + \| y \|_1 \\
= \| x \|_1 + \| \frac{x}{2^n} \|_2 + \| y \|_1 \\
\leq \| x \|_1 + \| \frac{x + y}{2^n} \|_2 + \| y \|_1 \\
= \| x + y \|_1 + \| \frac{x + y}{2^n} \|_2 \\
= \| x + y \| \\
\leq 1 + \varepsilon/2
\]
So \( \| y \|_1 \leq \varepsilon/2 \) and by Lemma 4.5 we have that \( \| y \| \leq \varepsilon \).

\[\square\]

**Corollary 4.7.** There exist a uniformly monotone Banach sequence lattice \( E \) and a family of Banach spaces \( \{ X_k : k \in \mathbb{N} \} \) satisfying AHp uniformly such that \( (\oplus \sum_{k=1}^{\infty} X_k)_E \) does not have AHSp, so it does not have AHp.

**Proof.** Assume that \( E \) is the uniformly monotone Banach sequence lattice introduced in the previous example, and we take \( X_k = \mathbb{R} \) for every positive integer \( k \). So the family \( \{ X_k : k \in \mathbb{N} \} \) satisfies AHp uniformly and \( (\oplus \sum_{k=1}^{\infty} X_k)_E = E \). Hence \( (\oplus \sum_{k=1}^{\infty} X_k)_E \) does not have AHSp. Since AHp implies AHSp by [13, Proposition 2.2], \( (\oplus \sum_{k=1}^{\infty} X_k)_E \) does not satisfy AHp. \( \square \)

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