Transition from Markovian to Glassy Dynamics in Damped Kuramoto-Sivashinsky Turbulence

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Abstract. In this research, we study Damped Kuramoto-Sivashinsky (DKS) equation, one of the famous nonlinear differential equation which exhibits turbulence. To solve DKS equation, we use Exponential Time Differencing (ETD) scheme which is combined with the Pseudospectral method. To know the effect of damping factor term, we analyze the resulting dynamics using Lyapunov exponent and total autocorrelation function. From that analysis, we elucidate the transition of Markovian to glassy dynamics in DKS equation. We conclude that there is a kind of mixing of several modes in the dynamics of the DKS equation which resulting the glassy dynamics in this system.

1. INTRODUCTION

One of the most famous and ubiquitous nonlinear phenomena is turbulence. It is observed in nature as well as in the experimental setup. The well-known experimental system showing turbulence are Rayleigh-Benard convection [1], electro convection in liquid crystal [2] and turbulence in chemical reaction [3]. The study of turbulence has also been conducted using mathematical modeling, i.e. by the nonlinear partial differential equations (PDEs). One of the fundamental models in the research of nonlinear phenomenon is Kuramoto-Sivashinsky (KS) equation.

In the present research, we study KS equation with damping factor as follows [4][5][6]

\[ \frac{\partial u}{\partial t} = ru - u - 2\frac{\partial u}{\partial x} - \frac{\partial ^2 u}{\partial x^2} - \frac{\partial u}{\partial t}, \tag{1} \]

where \( r \) is damping factor. Using a perturbation \( u(x,t) \approx \exp[\omega t + ikx] \) in the linear term, one will obtain the dispersion relation of \( \omega(k) = (r - 1)k + 2k^2 - k^3 \). From Fig.1, we see the shift of curve due to the damping factor \( r \) indicating that the wave number involves are decreased.
Several studies have been done to solve DKS equation using numerical treatment. Elder et al. [4] had examined the transition to spatiotemporal chaos in DKS equation. The transition to spatiotemporal chaos can be seen from the shift of energy spectrum. Brunet [7] using DKS equation as a model of secondary instabilities and dynamics in the cellular flow. Other research has completed in two-dimensional DKS equation, such as Gomes and Paris [6]. Here we report an evident of transition from Markovian to glassy dynamics at a certain value of $r$ which hasn’t been reported in the previous research.

2. Computational Methods

In this research, we solve the DKS equation using pseudo spectral method which combined with Exponential Time Differencing (ETD) scheme [8][9][10] and periodic boundary condition and using traveling wave as initial function. Note that the initial function in the present study arbitrary selected because it will not change the resulted solution. The ETD scheme is one of the numerical methods to solve differential equation especially stiff differential equation [8]. First, we perform the Fourier series of $u$ as the following

$$u(x,t) = \sum_{n=1}^{N} \hat{u}_n \exp[iK_n x],$$

where $K_n = 2n\pi / L$. Using equation (2), the equation (1) can be rewritten as

$$\frac{d\hat{u}}{dt} = \left(\left((r-1)K_n + 2K_n^2 - K_n^4\right)\hat{u} - \frac{iK_n}{2} \sigma_n\right).$$

Where $\sigma_n$ is Fourier Transform of $u^2$.

To solve the first term of equation (1) the ETD2 written by

$$u_{n+1} = u_n e^{\Delta t} + F_s \frac{(1+hc)e^{hc} - 1 - 2hc}{hc^2} + F_{n+1} \frac{(-e^{hc} + 1 + hc)}{hc^2}$$

where $c = \left((r-1)K_n + 2K_n^2 - K_n^4\right)$ and $F_n = -\frac{iK_n}{2} \sigma_n$.

3. RESULT AND DISCUSSION

It is well known that the solution of KS equation depends on system size used ($L$) [11][12][13], in the present research $L=256$ and discretization number $N=512$. The maximum computational process is $t_{\text{max}} = 1000$
with the time discretization of $h = 0.01$. The traveling wave equation $u(x,t=0) = 0.5\sin(x)$ was used as an initial function. From such initial function, the solution the DKS equation shown by spatiotemporal plots in the Fig. 2.

![Spatiotemporal plots of solution DKS equation using ETD2 from two damping value factor, (a) $r = 1.0$ and (b) $r = 0.9$.](image)

Fig.2 shows that by setting the damping factor with higher value, the fluctuation in the DKS turbulence is suppressed. The smaller the damping factor, the longer the periodic structure in the spatiotemporal plot from $t \approx 25$ at (a) and $t \approx 200$ [arbitrary units] at (b). In Fig.2a the orderly area occurs when $t < 100$, while Fig.2b the order area increase for $t < 200$.

To characterize the transition from simple to the complex dynamic, we plot the orbit in the phase space for many values of $r$ as shown in Fig.3. From Fig 3 we see that the attractor change from unstable periodic (see Fig.3a) to the stable fixed point (see Fig.3d), i.e., there is a transition from complex dynamics to a stable one.

![The orbit in phase space diagram of DKS equation, (a) $r=0.9$, (b) $r=0.85$, (c) $r=0.5$, and $r=0.0$.](image)
For further consideration of transition of dynamics in DKS turbulence, we use dynamical analysis method using Lyapunov exponents $\{\lambda\}$[5][14]. In the present study, the Lyapunov exponent is calculated using an algorithm built by Rosenstein et al. [15].

**FIGURE 4.** (a) Lyapunov Exponents $\{\lambda\}$ of DKS equation for several values of damping factor $r$ that are $r=0.863$(red), $r=0.864$(blue), $r=0.865$(dark-green), $r=0.866$(green), $r=0.867$(magenta), $r=0.868$(aqua), $r=0.869$(yellow), $r=0.87$(orange), and $r=0.871$(skyblue).

Figure 4 shows the Lyapunov exponents $\{\lambda\}$ of DKS equation for several values of damping factor $r$. Fig.4 shows that there is a transition from ordinary to chaotic dynamics of fluctuations in DKS equation at a certain damping factor of $r \approx 0.87$.

Next, to study the statistical properties of DKS equation, we use the total autocorrelation function of the evolution of $u$ as follows

$$Q(x, \tau) = \langle \Delta u(x, t + \tau) \Delta u(x, t) \rangle \langle \Delta u(x, t) \rangle^{-1}. \quad (6)$$

To take into account all points in the simulation, we take the average of all 512 positional points. The autocorrelation function from various damping factor is shown in Fig.5.

**FIGURE 5.** The autocorrelation function of DKS equation for $r=0.865$(red), $r=0.861$(blue), $r=0.865$(dark-green), $r=0.868$(green), $r=0.87$(magenta), $r=0.872$(aqua), $r=0.875$(yellow).

Figure 5 shows that the autocorrelation function is fitted by Kohlrausch–Williams–Watts (KWW) function [19] which fulfills

$$Q(\tau) = m_1 + m_2 \exp \left[-\left(\frac{\tau}{\tau_0}\right)^\beta\right]. \quad (7)$$

where $m_1$, $m_2$, $\beta$, $\tau$ is a constant, normalization parameter, KWW exponent and correlation time, respectively. From parameter in KWW fitting function, we can get the relationship between $\beta$ and $\tau$ with $r$ as shown by Fig.6.
Figure 6 (a) shows that at $r < 0.864$ the value of $\tau^{-1}$ is almost zero which corresponds to order dynamics. This behavior of the relaxation time, while at $r > 0.864$ the value of $\tau^{-1}$ increases indicating that the correlation time decreases meaning that complex dynamics occur. Fig 6 (b) shows the behavior of KWW exponent $\beta$, which is important to know what kind of dynamics produced in the system [16][17][18]. Note that $\beta = 1$ at $r < 0.864$, i.e simple exponential decay of autocorrelation indicates the Markovian dynamics. While the $\beta > 1$ at $r > 0.864$, i.e compressed exponential represents glassy dynamics [17][18]. Thus we conclude that there is a transition from Markovian to glassy dynamics in DKS by increasing damping factor.

4. CONCLUSION

In summary, we have solved DKS equation with damping factor is $r$ with numerical treatment using ETD2 combined with the pseudo spectral method and analyzed the resulting dynamics using Lyapunov exponent and total autocorrelation function. In this research there are two kinds of attractors, the first attractor is an unstable periodic orbit that will generate chaotic dynamics for $r > 0.85$, and the second attractor type is a stable fixed point that produces order or ordinary dynamics for $r < 0.85$. The Lyapunov exponent and autocorrelation of the DKS equation show the transition from Markovian to glassy dynamics in DKS by increasing damping factor. Thus we report that a new kind of dynamics observed in the DKS equation.

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