ON HOMOLOGY OF FINITE TOPOLOGICAL SPACES

NICOLÁS CIANCI AND MIGUEL OTTINA

Abstract. We develop a new method to compute homology groups of finite topological spaces (or equivalently of finite partially ordered sets) by means of spectral sequences giving a complete and simple description of the corresponding differentials. Our method turns out to be quite powerful and involves much fewer computations than the standard one. Moreover, we derive many applications of our results which include homological Morse theory and computation of the Möbius function of posets.

1. Introduction

The interaction between topology and combinatorics has proved to be very fruitful. Examples of this interaction are simplicial homology, discrete Morse theory [9, 14], the celebrated proof of Kneser’s conjecture given by Lovász [12], the subsequent developments in the study of graph properties by means of topological methods [11] and the theory of finite topological spaces, which has grown considerably in the last years from works by Barmak and Minian [2, 3, 4, 5, 6].

The theory of finite topological spaces is based in the well-known correspondence between finite posets and finite \(T_0\)-spaces given by Alexandroff [1] and in the works of Stong [16] and McCord [13] who study finite spaces from totally different perspectives. Stong studies the homotopy types of finite topological spaces by an elementary move which consists of removing a single point of a finite space. Surprisingly, a sequence of these simple moves is enough to determine whether two given finite spaces have the same homotopy type. On the other hand, McCord associates a simplicial complex \(K(X)\) to each finite space \(X\) together with a weak homotopy equivalence \(K(X) \rightarrow X\) and proves that the weak homotopy types of compact polyhedra are in one to one correspondence with the weak homotopy types of finite topological spaces.

Barmak and Minian delve deeply into this theory and obtain many interesting results, among which we mention the introduction of an elementary move in finite \(T_0\)-spaces which corresponds exactly with the elementary collapses of simple homotopy theory of compact polyhedra [1] and a generalization of McCord’s result on the weak equivalence between a compact polyhedron and its order complex [5]. Moreover, they use the theory of finite spaces to study Quillen’s conjecture on the poset of non-trivial \(p\)-subgroups of a group and Andrews–Curtis’ conjecture (see [2]).

As it is shown by the recent works of Barmak and Minian, finite topological spaces can be used in several different situations to develop new tools and techniques to study topological and combinatorial problems. Moreover, in many of them the finite space approach is simpler, more adequate or more tractable than the one given by simplicial

\textbf{Key words and phrases.} Homology groups, Finite Topological Spaces, Posets, Spectral Sequences, Homological Morse Theory, Möbius function.

Research partially supported by grant M015 of SeCTyP, UNCuyo.
completes and polyhedra. In a similar way, problems regarding homotopy invariants of finite topological spaces, which can be tackled by the simplicial complex approach, can be dealt with in a more direct and natural way in their own context.

It is this idea which is exploited in this article, where we develop a new method to compute homology of finite topological spaces by means of spectral sequences. We give not only spectral sequences which converge to the homology groups of a given finite space but also an explicit description of the differentials of all the pages of those spectral sequences. This method turns out to be quite powerful and involves much fewer computations than the standard method of computing the simplicial homology groups of the order complex of the finite space.

Moreover, we apply our results to generalize a result of Minian on homological Morse theory for posets [14], largely extending the class of posets for which it is valid, with a very nice and conceptual proof. We also apply our techniques to obtain different formulas to compute the Möbius function of posets which include an alternative proof to a result of Björner and Walker [7].

2. Preliminaries

If $X$ is a finite $T_0$ topological space and $x \in X$, $U_x$ denotes the minimal open set which contains $x$, that is, the intersection of all the open sets of $X$ which contain $x$. In a similar way, $F_x$ denotes the minimal closed set which contains $x$, that is, $F_x = \{x\}$.

An order relation can be defined in a finite $T_0$–space $X$ as follows: $x \leq y$ if and only if $U_x \subseteq U_y$. Conversely, if $P$ is a finite poset then the subsets $\{x \in P / x \leq a\}$, $a \in P$, form a basis for a topology on $P$. These applications are mutually inverse and give a one-to-one correspondence between finite $T_0$–spaces and finite posets [1] (see also [2]).

Hence, from now on, we will see any finite $T_0$–space as a finite poset and any finite poset as a finite $T_0$–space without further notice.

If $X$ is a finite $T_0$–space then $U_x = \{a \in X / a \leq x\}$ and $F_x = \{a \in X / a \geq x\}$. It is standard to define $\hat{U}_x = \{a \in X / a < x\}$, $\hat{F}_x = \{a \in X / a > x\}$, $C_x = U_x \cup F_x$ and $\hat{C}_x = C_x - \{x\}$. In case several topological spaces are considered at the same time, we will denote $\hat{U}_x$ by $\hat{U}_x^X$ to indicate the space in which the minimal open set is considered. We will use similar notations for $F_x$, $C_x$, $\hat{U}_x$, $\hat{F}_x$ and $\hat{C}_x$.

Let $X$ be a finite $T_0$–space and let $x \in X$. The point $x$ is an up beat point of $X$ if the subposet $\hat{F}_x$ has a minimum. The point $x$ is a down beat point of $X$ if the subposet $\hat{U}_x$ has a maximum. The point $x$ is a beat point of $X$ if it is an up beat point or a down beat point. Stong proves in [16] that if $x$ is a beat point of $X$ then $X - \{x\}$ is a strong deformation retract of $X$. Moreover, he gives a simple criterion to decide whether two given finite topological spaces are homotopy equivalent. Using the results of Stong it is easy to prove that if $X$ is a finite $T_0$–space and $x \in X$ then $C_x$ is contractible.

The order complex of a finite poset $X$ is the simplicial complex $\mathcal{K}(X)$ of the non-empty chains of $X$. McCord proves in [13] that there exists a weak homotopy equivalence $|\mathcal{K}(X)| \rightarrow X$ (where $|\mathcal{K}(X)|$ denotes the geometric realization of $\mathcal{K}(X)$).

The non-Hausdorff suspension of a topological space $X$ is the space $\mathcal{S}(X)$ whose underlying set is $X \cup \{+, -\}$ and whose open sets are those of $X$ together with $X \cup \{+\}$, $X \cup \{-\}$ and $X \cup \{+, -, -\}$. This definition was introduced by McCord in [13], where he proves that for every space $X$ there exists a weak homotopy equivalence between the suspension of $X$ and $\mathcal{S}(X)$.
A finite model of a topological space $Z$ is a finite $T_0$–space $X$ such that $|\mathcal{K}(X)|$ is homotopy equivalent to $Z$. For example, if $D_2$ is the discrete space of two points and $n \in \mathbb{N}$ then $S^n D_2$ is a finite model of the $n$–sphere $S^n$.

If $X$ is a poset $X^{\text{op}}$ will denote the poset $X$ with the inverse order.

Recall that a poset is \textit{homogeneous} of dimension $n$ if all its maximal chains have cardinality $n + 1$.

A poset $X$ is \textit{graded} if $U_x$ is homogeneous for all $x \in X$. In this case, the degree of $x$ is the dimension of $U_x$ and is denoted by $\text{deg}(x)$.

The following definitions were introduced by Minian in [14].

\begin{definition}
\begin{itemize}
\item A finite poset $X$ is called \textit{$h$–regular} if for every $x \in X$, the order complex of $\hat{U}_x$ is homotopy equivalent to $S^{n-1}$ where $n$ is the maximum of the cardinality of the chains in $\hat{U}_x$.
\item A \textit{cellular} poset is a graded poset $X$ such that for every $x \in X$, $\hat{U}_x$ has the homology of a $(p-1)$–sphere, where $p = \text{deg}(x)$.
\end{itemize}
\end{definition}

In a similar way we will say that a finite $T_0$–space is \textit{cellular} if its associated poset is cellular.

If $n \in \mathbb{N}_0$, an $n$–chain of $P$ is a chain of $P$ of cardinality $n + 1$. The empty chain will be regarded as a $(-1)$–chain. We will use the notation $[v_0, \ldots, v_n]$ for an $n$–chain $\{v_0, \ldots, v_n\}$ of $P$ with $v_{j-1} < v_j$ for all $j \in \{1, 2, \ldots, n\}$. Also, if $n \in \mathbb{N}_0$ and $s = [v_0, \ldots, v_n]$ is an $n$–chain and $k \in \{0, \ldots, n\}$ then $s_k$ will denote the $(n - 1)$–chain $[v_0, \ldots, \hat{v}_k, \ldots, v_n]$.

\textbf{Notation.} If $X$ is a poset, $\text{Ch}(X)$ will denote the set of chains of $X$ and, for $n \in \mathbb{N}_0$, $\text{Ch}_n(X)$ will denote the set of $n$–chains of $X$.

From McCord’s theorem it is clear that $H_n(X) = H_n(|\mathcal{K}(X)|)$. Hence, the homology groups of a finite topological space can be computed from the simplicial chain complex associated to $\mathcal{K}(X)$. This fact can be expressed entirely in terms of the finite space $X$ since the simplices of $\mathcal{K}(X)$ are the chains of $X$. Thus, making the translation to the context of posets, we can introduce the following definition which will be useful for our work.

\begin{definition}
Let $X$ be a finite $T_0$–space. The \textit{$f$–chain complex associated to $X$} is the chain complex $C^f(X) = (C^f_n(X), d^f_n)_{n \in \mathbb{Z}}$ defined by

$$
C^f_n(X) = \begin{cases} 
\bigoplus_{\text{Ch}_n(X)} \mathbb{Z} & \text{if } n \geq 0 \\
0 & \text{if } n < 0
\end{cases}
$$

and where, for $n \in \mathbb{N}$, the morphisms $d^f_n : C^f_n(X) \rightarrow C^f_{n-1}(X)$ are defined by

$$
d^f_n([v_0, \ldots, v_n]) = \sum_{i=0}^{n} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_n]
$$

for all $[v_0, \ldots, v_n] \in \text{Ch}_n(X)$.

It is clear that the chain complex $C^f(X)$ is isomorphic to the simplicial chain complex of $\mathcal{K}(X)$.
Definition 2.3. Let $X$ be a finite $T_0$-space. For $n \in \mathbb{Z}$ we define $H^f_n(X)$ as the $n$-th homology group of the chain complex $C^f(X)$. The group $H^f_n(X)$ will be called the $n$-th $f$-homology group of $X$.

It follows that if $X$ is a finite $T_0$-space then $H_n(X) = H^f_n(X)$ for all $n \in \mathbb{Z}$.

A similar translation can be made for the relative case. We include below the notations and definitions for this case.

Definition 2.4. Let $X$ be a finite $T_0$-space and let $A \subseteq X$. An $n$-chain of $(X, A)$ is an $n$-chain of $X$ which is not included in $A$.

Definition 2.5. Let $X$ be a finite $T_0$-space and let $A \subseteq X$. We define the $f$-relative chain complex associated to $(X, A)$ as the chain complex $C^f(X, A) = C^f(X)/C^f(A)$. We also define, for $n \in \mathbb{Z}$, $H^f_n(X, A)$ as the $n$-th homology group of the chain complex $C^f(X, A)$.

Note that, for $n \in \mathbb{N}_0$, $C^f_n(X, A) \cong \bigoplus_{C_n(X, A)} \mathbb{Z}$.

Clearly, the chain complex $C^f(X, A)$ is isomorphic to the relative simplicial chain complex of $(\mathcal{K}(X), \mathcal{K}(A))$ and hence $H^f_n(X, A) = H_n(X, A)$ for all $n \in \mathbb{Z}$.

In a similar way, one can define reduced $f$-homology groups. In this case, the group in degree $-1$ of the corresponding chain complex will be the free abelian group generated by the empty chain.

3. Main theorem

In this section we will develop a spectral sequence which converges to the homology groups of a given finite space and we will provide an explicit description of the differentials of all the pages of this spectral sequence. Moreover, we will prove that our result generalizes a cellular-type method of Minian [14].

Notation. Let $X$ be a finite $T_0$-space, let $A \subseteq X$ and let $n \in \mathbb{N}_0$. If $\sigma \in C^f_n(X)$ then we will write $\overline{\sigma}^A$ (or simply $\overline{\sigma}$) for the class of $\sigma$ in $C^f_n(X, A)$.

If $x \in X$ and $\sigma \in C^f_n(C_x)$, we will write $\overline{\sigma}^x$ for the class of $\sigma$ in $C^f_n(C_x, \hat{C}_x)$.

The class of the $n$-cycle $\sigma$ in $H^f_n(X)$ will be denoted by $[\sigma]$, and the class of the $n$-relative cycle $\overline{\sigma}^A$ in $H^f_n(X, A)$ will be denoted by $[\overline{\sigma}^A]$.

If $I$ is a set, $H$ is a group, $\{G_i\}_{i \in I}$ is a collection of groups and $\{f_i : G_i \to H\}_{i \in I}$ is a collection of group homomorphisms, we define $\bigcup_{i \in I} f_i : \bigoplus_{i \in I} G_i \to H$ by

$$\left(\bigcup_{i \in I} f_i\right)(\langle x_i \rangle_{i \in I}) = \sum_{i \in I} f_i(x_i).$$

The following lemma, which will be used to prove proposition 3.2, contains a simple idea which will be very important in our method.

Lemma 3.1. Let $X$ be a finite $T_0$-space and let $D \subseteq X$ be an antichain. For $x \in D$ and $n \in \mathbb{N}_0$ we define:

- $i^+_n : C^f_n(C_x, \hat{C}_x) \to C^f_n(X, X - D)$ by $i^+_n(\overline{\sigma}^x) = \overline{\sigma}^{X-D}$ for every $\sigma \in C^f_n(C_x)$.
\[ \rho_n^x : C^f_n(X, X - D) \longrightarrow C^f_n(C_x, \hat{C}_x) \] as the group homomorphism that satisfies
\[ \rho_n^x(sX - D) = \begin{cases} s & \text{if } x \in s \\ 0 & \text{if } x \notin s \end{cases} \]
for every \( n \)-chain \( s \) in \( X \).

\[ \phi_n : C^f_n(X, X - D) \longrightarrow \bigoplus_{x \in D} C^f_n(C_x, \hat{C}_x) \]
as the group homomorphism that satisfies
\[ \phi_n(sX - D) = (\rho_n^x(sX - D))_{x \in D} \]
for every \( n \)-chain \( s \) in \( X \).

Then, \( \phi_n \) is a group isomorphism and \( \phi_n^{-1} = \bigcup_{x \in D} i_n^x \) for every \( n \in \mathbb{N}_0 \).

**Proof.** It is easy to check that the morphisms of above are well-defined.

For each \( n \in \mathbb{N}_0 \) and for each \( x_0 \in D \), let \( i_n^{x_0} : C^f_n(C_{x_0}, \hat{C}_{x_0}) \longrightarrow \bigoplus_{x \in D} C^f_n(C_x, \hat{C}_x) \) be the inclusion.

Let \( n \in \mathbb{N}_0 \). It is clear that \( \bigcup_{x \in D} i_n^x \) is an epimorphism since every \( n \)-chain in \( (X, X - D) \) is an \( n \)-chain in \( (C_x, \hat{C}_x) \) for some \( x \in D \). Besides, \( \phi_n i_n^x(s) = i_n^x(s) \) for every \( x \in D \) and for every \( n \)-chain \( s \) in \( (C_x, \hat{C}_x) \) since every \( n \)-chain in \( (X, X - D) \) must have exactly one element of \( D \), as \( D \) is an antichain. Therefore, \( \phi_n i_n^x = i_n^x \).

Then, \( \phi_n \circ \bigcup_{x \in D} i_n^x = \bigcup_{x \in D} (\phi_n i_n^x) = \bigcup_{x \in D} i_n^x = \text{Id} \). Hence, \( \bigcup_{x \in D} i_n^x \) is a monomorphism.

Thus, \( \bigcup_{x \in D} i_n^x \) is an isomorphism with inverse \( \phi_n \). \( \square \)

**Proposition 3.2.** Let \( X \) be a finite \( T_0 \)-space and let \( D \) be an antichain in \( X \). Then \( H_n(X, X - D) = \bigoplus_{x \in D} \hat{H}_{n-1}(\hat{C}_x) \) for every \( n \in \mathbb{Z} \).

**Proof.** For each \( n \in \mathbb{Z} \) we have a commutative diagram

\[
\begin{array}{ccc}
C^f_n(X, X - D) & \xrightarrow{\overline{d}_n^f} & C^f_{n-1}(X, X - D) \\
\bigcup_{x \in D} i_n^x & & \bigcup_{x \in D} i_{n-1}^x \\
\bigoplus_{x \in D} C^f_n(C_x, \hat{C}_x) & \xrightarrow{\bigoplus_{x \in D} (\overline{d}_n^f)_x} & \bigoplus_{x \in D} C^f_{n-1}(C_x, \hat{C}_x) \\
\end{array}
\]

where for every \( x \in D \) the morphism \( (\overline{d}_n^f)_x \) is the restriction of \( \overline{d}_n^f \) to \( C^f_n(C_x, \hat{C}_x) \).

Then, the chain complexes \( C^f(X, X - D) \) and \( \bigoplus_{x \in D} C^f(C_x, \hat{C}_x) \) are isomorphic and therefore
\[ H_n(X, X - D) = H_n^f(X, X - D) = \bigoplus_{x \in D} H_n^f(C_x, \hat{C}_x) = \bigoplus_{x \in D} H_n(C_x, \hat{C}_x) \]
for every \( n \in \mathbb{Z} \).

Now, since \( C_x \) is contractible, \( H_n(C_x, \hat{C}_x) = \hat{H}_{n-1}(\hat{C}_x) \). The result follows. \( \square \)
Remark 3.3. In the previous lemma we allow $\hat{C}_x$ to be empty, in which case, $\hat{H}_{-1}(\hat{C}_x) = \mathbb{Z}$ and $\hat{H}_n(\hat{C}_x) = 0$ for $n \neq -1$.

Definition 3.4. Let $X$ be a finite $T_0$-space. Let $n \in \mathbb{N}_0$ and let $x \in X$. Let $s = [y_0, \ldots, y_n]$ be an $n$–chain in $(C_x, \hat{C}_x)$. Let $k^*_x$ denote the only integer $i \in \{0, \ldots, n\}$ such that $y_i = x$. We define the sign of $x$ in $s$ by $\text{sgn}_s(x) = (-1)^{k^*_x}$.

Lemma 3.5. Let $X$ be a finite $T_0$–space, let $x \in X$ and let $n \in \mathbb{N}_0$.

Let $\partial : H_n(C_x, \hat{C}_x) \rightarrow H_{n-1}(\hat{C}_x)$ be the connection homomorphism of the long exact sequence associated to the finite chain complex of $(C_x, \hat{C}_x)$.

Let $\sigma = \sum_{i=1}^{l} \alpha_i s_i + \sum_{j=1}^{m} \beta_j t_j \in C'_n(C_x)$, where $l, m \in \mathbb{N}$, $\alpha_i \in \mathbb{Z}$ for every $i = 1, \ldots, l$, $\beta_j \in \mathbb{Z}$ for every $j = 1, \ldots, m$ and where for every $i = 1, \ldots, l$, $s_i$ is an $n$–chain in $C_x$ such that $x \in s_i$, and for every $j = 1, \ldots, m$, $t_j$ is an $n$–chain of $C_x$ such that $x \notin t_j$.

If $\bar{\sigma} \in \ker \partial_n$, then $\partial(\bar{\sigma}) = \left[ \sum_{i=1}^{l} \alpha_i \text{sgn}_{s_i}(x)(s_i - \{x\}) \right]$.

Proof. Let $\tau = \sum_{i=1}^{l} \alpha_i s_i$. Note that

$$\bar{\sigma} = \sum_{i=1}^{l} \alpha_i s_i + \sum_{j=1}^{m} \beta_j t_j = \sum_{i=1}^{l} \alpha_i s_i = \tau$$

in $C'_n(C_x, \hat{C}_x)$ since $\sum_{j=1}^{m} \beta_j t_j \in C'_n(\hat{C}_x)$.

By the proof of the Snake Lemma, $d_n^f(\tau) \in C'_n(\hat{C}_x)$ and $\partial(\bar{\sigma}) = \partial(\tau)$ is the class of $d_n^f(\tau)$ in $H^f_{n-1}(\hat{C}_x)$. On the other hand

$$d_n^f(\tau) = \sum_{i=1}^{l} \alpha_i \left( \sum_{k=0}^{n} (-1)^k(s_i)_k \right) = \sum_{i=1}^{l} \alpha_i \text{sgn}_{s_i}(x)(s_i - \{x\}) + \sum_{i=1}^{l} \alpha_i \left( \sum_{k \neq k^*_x} (-1)^k(s_i)_k \right).$$

Now, note that $\sum_{i=1}^{l} \alpha_i \text{sgn}_{s_i}(x)(s_i - \{x\}) \in C'_n(\hat{C}_x)$, since it is a sum of $(n-1)$–chains in $C_x$ that do not contain $x$. Since $d_n^f(\sigma) \in C'_{n-1}(\hat{C}_x)$, then

$$\sum_{i=1}^{l} \alpha_i \left( \sum_{k \neq k^*_x} (-1)^k(s_i)_k \right) \in C'_{n-1}(\hat{C}_x)$$

But for every $i = 1, \ldots, l$, $k \neq k^*_x$ implies that $x \notin (s_i)_k$. Then, $\sum_{k \neq k^*_x} (-1)^k(s_i)_k$ is a sum of chains that contain $x$. Since $C'_{n-1}(C_x)$ is a free abelian group, $\sum_{i=1}^{l} \alpha_i \left( \sum_{k \neq k^*_x} (-1)^k(s_i)_k \right) = 0$. Hence,

$$d_n^f(\tau) = \sum_{i=1}^{l} \alpha_i \text{sgn}_{s_i}(x)(s_i - \{x\})$$,
and therefore,
\[ \partial(\sigma) = [d_i^f(\tau)] = \left[ \sum_{i=1}^{l} \alpha_i \text{sgn}(x)(s_i - x) \right]. \]

\[ \square \]

**Definition 3.6.** Let \( X \) be a finite \( T_0 \)-space and let \( \mathcal{F} = \{ X_p : p \in \mathbb{Z} \} \) be a filtration of \( X \). We say that the filtration \( \mathcal{F} \) is induced by antichains if \( X_{-1} = \emptyset \) and \( X_n - X_{n-1} \) is an antichain for every \( n \in \mathbb{N} \).

Note that the subposet \( X_0 \) needs not be an antichain.

The following is the main theorem of this article.

**Theorem 3.7.** Let \( X \) be a finite \( T_0 \)-space and let \( \{ X_p : p \in \mathbb{Z} \} \) be a filtration of \( X \) which is induced by antichains. For each \( p \in \mathbb{N} \), let \( D_p = X_p - X_{p-1} \).

Then there is a spectral sequence \( \{ (E_{pq}^{r}, (d_{r}^{p,q})_{p,q \in \mathbb{Z}}) \}_{r \in \mathbb{N}} \) that converges to \( H_{\ast}(X) \) such that:

- \( E_{p,q}^{1} = 0 \) for every \( p \leq -1 \).
- \( E_{0,q}^{1} = H_{q}(X_0) \).
- \( E_{1,q}^{1} = \bigoplus_{x \in D_p} H_{p+q-1}(\tilde{C}^{X_p}_{x}) \) for \( p \geq 1 \).

The morphisms \( d_{p,q}^{1} : \bigoplus_{x \in D_p} H_{p+q-1}(\tilde{C}^{X_p}_{x}) \rightarrow H_{p+q-2}(\tilde{C}^{X_{p-1}}_{y}) \) are defined by:

\[ d_{p,q}^{1}(\sum_{x \in D_p} [\sigma_x]) = \sum_{x \in D_p} [\sigma_x] \]

- If \( p \geq 1 \) and \( q \leq -p \), then \( d_{p,q}^{1} \) is the trivial homomorphism.
- If \( p \geq 2 \) and \( q \geq 1 - p \), then \( d_{p,q}^{1} : \bigoplus_{x \in D_p} H_{p+q-1}(\tilde{C}^{X_p}_{x}) \rightarrow \bigoplus_{y \in D_{p-1}} H_{p+q-2}(\tilde{C}^{X_{p-1}}_{y}) \)

is defined by:

\[ d_{p,q}^{1} \left( \left[ \sum_{i=1}^{l_x} a_i^{x} s_i^{x} \right] \right) = \left[ \sum_{x \in D_p} \sum_{y \in D_{p-1}} a_i^{x} \text{sgn}(y)(s_i^{x} - y) \right]. \]

where for every \( x \in D_p \), \( l_x \in \mathbb{N} \), and for every \( i \in \{1, \ldots, l_x\} \), \( a_i^{x} \in \mathbb{Z} \) and \( s_i^{x} \in \tilde{C}_{p+q-1}(\tilde{C}^{X_{p-1}}_{y}). \)

**Proof.** Let \( \{ (E_{pq}^{r}, (d_{r}^{p,q})_{p,q \in \mathbb{Z}}) \}_{r \in \mathbb{N}} \) be the bigraded spectral sequence associated to the filtration \( \{ X_p \}_{p \in \mathbb{Z}} \) of \( X \), that is,

- \( E_{p,q}^{1} = H_{p+q}(X_p, X_{p-1}) \) for every \( p, q \in \mathbb{Z} \)
- \( d_{p,q}^{1} = j_{\ast} \partial \)

where \( j_{\ast} \) is the homomorphism induced in the homology groups by the projection \( j : C_{p+q-1}(X_{p-1}) \rightarrow C_{p+q-1}(X_{p-1}, X_{p-2}) \) and where \( \partial : H_{p+q}(X_p, X_{p-1}) \rightarrow \tilde{H}_{p+q-1}(X_{p-1}) \) is the connection homomorphism of the long exact sequence associated to the pair \( (X_p, X_{p-1}) \).
Since $X$ is finite and $X_p = \emptyset$ for every $p \leq -1$, it follows that conditions $(i)$ and $(ii)$ of proposition 1.2 of [10] are satisfied. Then $\{(\hat{E}_p)_{p,q} \in \mathbb{Z}, (\hat{d}_p)_{p,q} \in \mathbb{Z}\}_{r \in \mathbb{N}}$ converges to $H_\ast(X)$. 

For $p, q \in \mathbb{Z}$ we define:

- $E_{p,q}^1 = 0$ if $p \leq -1$.
- $E_{0,q}^1 = H_q(X_0)$.
- $E_{p,q}^1 = \bigoplus_{x \in D_p} \hat{H}_{p+q-1}(\hat{C}_x^X)$ if $p \geq 1$.

Since the filtration $\{X_p : p \in \mathbb{Z}\}$ is induced by antichains, $D_p$ is an antichain for every $p \in \mathbb{N}$. Thus, by the proof of 3.2 for each $p \geq 1$ and for all $q \in \mathbb{Z}$ we have isomorphisms

$$
\left( \bigcup_{x \in D_p} i_n^x \right)_* \circ \left( \bigoplus_{x \in D_p} \partial^x \right)^{-1} : \bigoplus_{x \in D_p} \tilde{H}_{p+q-1}(\hat{C}_x^X) \rightarrow H_{p+q}(X_p, X_{p-1})
$$

where $\partial^x : H_{p+q}(C_x^X, \hat{C}_x^X) \rightarrow \tilde{H}_{p+q-1}(\hat{C}_x^X)$ is the connection homomorphism of the corresponding long exact sequence.

On the other hand, we have that $E_{p,q}^1 = \hat{E}_{p,q}^1$ for $p \leq 0$. So we have group isomorphisms $\theta_{p,q} : E_{p,q}^1 \rightarrow \hat{E}_{p,q}^1$ for every $p, q \in \mathbb{Z}$. We define $d_{p,q}^1 = \theta_{p-1,q}^{-1} \circ d_{p,q}^1 \circ \theta_{p,q}$ for all $p, q \in \mathbb{Z}$.

Let $n \in \mathbb{Z}$. We have a diagram

$$
\begin{array}{ccc}
H_n(X_1, X_0) & \xrightarrow{\partial} & H_{n-1}(X_0) \\
\left( \bigcup_{x \in D_1} i_n^x \right)_* \bigoplus_{x \in D_1} H_n(C_x^X, \hat{C}_x^X) & \xrightarrow{\left( \bigcup_{x \in D_1} \tau_n^x \right)_*} & \bigoplus_{x \in D_1} \tilde{H}_{n-1}(\hat{C}_x^X) \\
\bigoplus_{x \in D_1} H_n(C_x^X, \hat{C}_x^X) \xrightarrow{\left( \bigoplus_{x \in D_1} \partial^x \right)^{-1}} & & \bigoplus_{x \in D_1} \tilde{H}_{n-1}(\hat{C}_x^X) \xrightarrow{d_{1,n-1}^1} H_{n-1}(X_0)
\end{array}
$$

where the maps $(\iota^x_n)_*$ are defined as in 3.1 and where $\tau_n^x : C_{n-1}^X(\hat{C}_x^X) \rightarrow C_{n-1}^X(\hat{X}_0)$ are the inclusion homomorphisms.

It is clear that $\partial((\iota^x_n)_*) = (\tau_n^x)_* \partial^x$ for every $x \in D_1$ whenever $n \neq 1$ by the naturality of the long exact sequence. Moreover, since $\partial : H_1(X_1, X_0) \rightarrow \hat{H}_0(X_0)$ is the (range) restriction of $\partial : H_1(X_1, X_0) \rightarrow H_0(X_0)$, then we have that $\partial((\iota^x_1)_*) = (\tau_0^x)_* \partial^x$ for every $x \in D_1$ as well. Thus the left square in the last diagram commutes and therefore $d_{1,n-1}^1 = \left( \bigcup_{x \in D_1} \tau_{n-1}^x \right)_*$.

Now, let $[\sigma] \in \bigoplus_{x \in D_1} \tilde{H}_{n-1}(\hat{C}_x^X)$. Then $[\sigma] = ([\sigma_x])_{x \in D_1}$ with $[\sigma_x] \in \tilde{H}_{n-1}(\hat{C}_x^X)$ for each $x \in D_1$, and therefore $d_{1,n-1}^1([\sigma]) = \sum_{x \in D_1} [\sigma_x]$. 

Let \( p \geq 2 \) and let \( n \in \mathbb{N} \). Consider the following diagram

\[
\begin{array}{ccc}
H_n(X_p, X_{p-1}) & \xrightarrow{\partial} & H_{n-1}(X_{p-1}) \\
\bigoplus_{x \in D_p} H_n(C_x^{X_p}, \hat{C}_x^{X_p}) & \xrightarrow{\bigoplus_{x \in D_p} \partial^x} & \bigoplus_{x \in D_p} \hat{H}_{n-1}(C_x^{X_p}) \\
\bigoplus_{x \in D_p} \hat{H}_{n-1}(C_x^{X_p}) & \xrightarrow{\bigoplus_{y \in D_{p-1}} \partial^y} & \hat{H}_{n-2}(C_y^{X_{p-1}})
\end{array}
\]

where, as above, the maps \((i^x_n)_*\) are defined as in \ref{inclusion} and where \( \tau_{n-1}^n : C_{n-1}^{f}(\hat{C}_x^{X_p}) \to C_{n-1}^{f}(X_{p-1}) \) are the inclusion homomorphisms.

We have that

\[
d_{p,n-p}^1 = \left( \bigoplus_{y \in D_{p-1}} \partial^y \right) (\phi_{n-1})_* j_* \left( \bigoplus_{x \in D_p} i^x_n \right) \left( \bigoplus_{x \in D_p} \partial^x \right)^{-1}
\]

As before, the left square of the diagram commutes for every \( n \in \mathbb{N} \). Therefore,

\[
d_{p,n-p}^1 = \left( \bigoplus_{y \in D_{p-1}} \partial^y \right) (\phi_{n-1})_* j_* \left( \bigoplus_{x \in D_p} \tau_{n-1}^x \right)_* = \left( \bigoplus_{y \in D_{p-1}} \partial^y \right) \left( \bigoplus_{x \in D_p} \phi_{n-1} j_\tau_{n-1}^x \right)_*
\]

Now we will find an explicit formula for \( d_{p,n-p}^1 \). Let \( \sigma = (\sigma_x)_{x \in D_p} \) with \( \sigma_x \in C_{n-1}^{f}(\hat{C}_x^{X_p}) \) for each \( x \in D_p \). Then

\[
\left( \bigcup_{x \in D_p} \phi_{n-1} j_\tau_{n-1}^x \right)(\sigma) = \sum_{x \in D_p} \phi_{n-1} j_\sigma_x = \sum_{x \in D_p} \phi_{n-1}(\sigma_x^{X_{p-2}}) = \left( \sum_{x \in D_p} \rho_{n-1}^y(\sigma_x^{X_{p-2}}) \right)_{y \in D_{p-1}}
\]

Now, for each \( x \in D_p \) we write \( \sigma_x = \sum_{i=1}^{l_x} a_i^x s_i^x \) with \( l_x \in \mathbb{N} \), \( a_i^x \in \mathbb{Z} \) and \( s_i^x \) an \((n-1)\)-chain in \( \hat{C}_x^{X_p} \) for each \( i \in \{1, \ldots, l_x\} \). Then we have that

\[
\rho_{n-1}^y(\sigma_x^{X_{p-2}}) = \sum_{i=1}^{l_x} a_i^x \rho_{n-1}^y(s_i^x) = \sum_{s_i^x \geq y} a_i^x s_i^y
\]

for every \( x \in D_p \).
Then,
\[
\left( \bigoplus_{x \in D_p} \phi_{n-1} j_n^x \right) (\sigma) = \left( \sum_{x \in D_p} \sum_{s_i^x \ni y} a_i^x \bar{s}_i^y \right)_{y \in D_{p-1}}
\]

Finally, by lemma 3.5 we see that
\[
d_{p,n-p}^1(\sigma) = \left( \bigoplus_{y \in D_{p-1}} \partial^y \left( \left( \sum_{x \in D_p} \sum_{s_i^x \ni y} a_i^x \bar{s}_i^y \right)_{y \in D_{p-1}} \right) \right) =
\]
\[
= \left( \partial^y \left( \sum_{x \in D_p} \sum_{s_i^x \ni y} a_i^x \bar{s}_i^y \right) \right)_{y \in D_{p-1}} =
\]
\[
= \left( \sum_{x \in D_p} \sum_{s_i^x \ni y} a_i^x \text{sgn} s_i^x (y)(s_i^x - \{ y \}) \right)_{y \in D_{p-1}}
\]

**Remark 3.8.** It is not difficult to prove that the morphisms of all the pages of the spectral sequence of the previous theorem can be computed in the same way as those of the first page, since for all \( p, q \in \mathbb{Z} \) the groups \( E^r_{p,q} \), \( r \in \mathbb{N} \), are subquotients of the group \( E^1_{p,q} \) and the morphisms of the \( r \)-th page of the spectral sequence are induced by the exact couple obtained from the long exact sequences in homology associated to the topological pairs \( (X_p, X_{p-1}) \).

Note also that one can develop a similar spectral sequence to compute relative homology groups.

The spectral sequence of the previous theorem will get a much simpler form in the case the homology of \( \hat{U}_x \) is concentrated in some degree for all \( x \in X \). This leads to the following definition.

**Definition 3.9.** Let \( X \) be a finite \( T_0 \)-space. We say that \( X \) is quasicellular if there is an order preserving map \( p : X \to \mathbb{N}_0 \), which will be called *quasicellular morphism for \( X \)*, such that

1. The set \( \{ x \in X : p(x) = n \} \) is an antichain for every \( n \in \mathbb{N}_0 \).
2. For every \( x \in X \), the reduced homology of \( \hat{U}_x \) is concentrated in degree \( p(x) - 1 \).

Note that if \( X \) is a quasicellular finite \( T_0 \)-space, \( p \) is a quasicellular morphism for \( X \) and \( x, y \in X \), then \( x < y \) implies that \( p(x) < p(y) \).

Note also that cellular spaces are quasicellular and that \( h \)-regular posets are quasicellular. But both inclusions are strict since the non-Hausdorff suspension of the discrete space of three points is quasicellular but neither cellular nor \( h \)-regular.

Also, since an \( h \)-regular poset is not necessarily graded (see [14, example 2.4]) we conclude that a poset might be quasicellular and non-graded.

**Corollary 3.10.** Let \( X \) be a quasicellular finite space and let \( h \) be a quasicellular morphism for \( X \). Let \( C(X) = (C_n(X), d_n) \) be the chain complex defined by

- \( C_n(X) = \bigoplus_{h(x) = n} \hat{H}_{n-1}(\hat{U}_x) \) for each \( n \in \mathbb{N}_0 \).
• \(d_n\) is the group homomorphism \(d_{n+1,-1}^1\) of theorem 3.7 for each \(n \in \mathbb{N}\).

Then, \(H_n(X) = H_n(C(X))\) for all \(n \in \mathbb{N}\).

**Proof.** Consider the filtration \(\{X_p\}_{p \in \mathbb{Z}}\) given by
\[
X_p = \{x \in X : h(x) \leq p - 1\}
\]
for each \(p \in \mathbb{Z}\). Clearly, the filtration \(\{X_p\}_{p \in \mathbb{Z}}\) is induced by antichains, and thus theorem 3.7 applies. Since \(h\) is a quasicellular morphism for \(X\), \(\hat{C}_X^p = \hat{U}_p\) and hence it follows that \(q = -1\) is the only non-trivial row of the first page of the spectral sequence, and therefore, the homology groups of \(X\) are the homology groups of the chain complex determined by the groups and group homomorphisms on this row. By theorem 3.7, this chain complex is precisely \(C(X)\). \(\square\)

As a corollary of 3.10, we obtain theorem 3.7 of [14]:

*Theorem 3.11 (Minian).* Let \(X\) be a cellular space and let \(C(X)\) be its cellular chain complex [14, definition 3.6]. Then \(H_p(X) = H_p(C(X))\) for each \(p \in \mathbb{Z}\).

4. Applications

4.1. Computation of homology groups of posets.

Just as a first and simple example consider the following:

**Example 4.1 (Non-Hausdorff suspension).** Let \(X\) be a finite \(T_0\)-space.

Consider the filtration \(\{X_p\}_{p \in \mathbb{Z}}\) of \(S^X\) given by \(X_0 = U_+\) and \(X_1 = S^X\). Note that this filtration is induced by antichains. Since \(U_+\) is contractible and \(\hat{U}_+ = X\), by theorem 3.7 we have a bigraded spectral sequence \((E, d)\) that converges to \(H_*(S^X)\) whose first page is

\[
\begin{array}{ccc}
q & \vdots & \vdots \\
0 & \hat{H}_2(X) & 0 \\
0 & \hat{H}_1(X) & 0 \\
\mathbb{Z} & \hat{H}_0(X) & 0 \\
& \vdots & \vdots
\end{array}
\]

It is not hard to see that \(d_{1,0}^1 : \hat{H}_0(X) \rightarrow \mathbb{Z}\) is trivial. Hence, \(H_0(S^X) = \mathbb{Z}\) and \(H_n(S^X) = \hat{H}_{n-1}(X)\) for all \(n \in \mathbb{N}\).

In the next example we will apply theorem 3.7 to compute the homology groups of a more complicated finite space, which turns out to be a finite model of the real projective plane. This finite space was constructed by Barmak and Minian in [5].
Example 4.2 (Finite model of the real projective plane). Let $X$ be the finite $T_0$-space whose Hasse diagram is

Consider the filtration $\{X_p\}_{p \in \mathbb{Z}}$ of $X$ given by $X_0 = F_a$, $X_1 = F_a \cup \{h, i\}$ and $X_2 = X$. Note that this filtration is induced by antichains.

By [3,7] there exists a bigraded spectral sequence $\{E^r_{s,t}\}_{r \in \mathbb{N}}$ that converges to $H_*(X)$. Let $Z_a = E^1_{0,0} = H_0(F_a) \cong \mathbb{Z}$. Note that $Z_a$ is generated by $[a]$.

Observe that $E^1_{1,0} = \tilde{H}_0(\tilde{F}_b) \oplus \tilde{H}_0(\tilde{F}_c) = Z_h \oplus Z_i$, where $Z_h = \tilde{H}_0(\tilde{F}_b) \cong \mathbb{Z}$ and $Z_i = \tilde{H}_0(\tilde{F}_c) \cong \mathbb{Z}$. Also note that $Z_h$ is generated by $[l] - [j]$ and $Z_i$ is generated by $[m] - [k]$.

Similarly, since $\tilde{F}_b$ and $\tilde{F}_c$ are finite models for $S^1$, we have that $E^1_{2,0} = \tilde{H}_1(\tilde{F}_b) \oplus \tilde{H}_1(\tilde{F}_c) = Z_b \oplus Z_c$, where $Z_b = \tilde{H}_1(\tilde{F}_b) \cong \mathbb{Z}$ and $Z_c = \tilde{H}_1(\tilde{F}_c) \cong \mathbb{Z}$. In this case we see that $Z_b$ and $Z_c$ are generated by

$$g_0 = \{d, j\} + \{h, l\} + \{l, e\} + \{e, k\} + \{k, i\} + \{i, m\} + \{m, d\}$$

and

$$g_1 = \{f, j\} + \{h, l\} + \{l, m\} + \{g, m\} + \{m, i\} + \{i, k\} + \{k, f\}$$

respectively. Now, it is easy to see that the first page of our spectral sequence is, in fact, a chain complex:

$$\cdots \rightarrow 0 \rightarrow Z_a \xrightarrow{\alpha} Z_h \oplus Z_i \xrightarrow{\beta} Z_b \oplus Z_c \rightarrow 0 \rightarrow \cdots$$

Using theorem [3,7] it is clear that $\alpha = 0$. On the other hand, a quick calculation shows that $\beta(g_0) = ([l] - [j], [m] - [k])$ and $\beta(g_1) = ([l] - [j], [k] - [m])$. It follows that

$$E^2_{1,0} = Z_h \oplus Z_i / \text{Im} \beta \cong \mathbb{Z}_2.$$

Thus, $H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}_2$ and $H_n(X) = 0$ for $n \geq 2$.

It is interesting to observe that the chain complex of above is much simpler than the simplicial chain complex of $\mathcal{K}(X)$.

4.2. Homological Morse theory of posets. G. Minian introduced in [14] a discrete version of Morse theory for posets and develops a homological variant of this theory showing that given a homologically admissible Morse matching on a cellular poset $X$, the homology groups of $X$ can be computed from a chain complex which in degree $p$ consists of the free abelian group generated by the critical points of $X$ of degree $p$. Using our techniques we
will give here a generalization of his result with a completely different and more conceptual proof.

We begin by recalling some definitions from [14]. Let \( X \) be a finite poset and let \( \mathcal{H}(X) \) be its Hasse diagram.

Let \( M \) be a matching on \( \mathcal{H}(X) \) and let \( \mathcal{H}_M(X) \) be the graph obtained from \( \mathcal{H}(X) \) by reversing the orientations of the edges which are not in \( M \). We say that \( M \) is a Morse matching if the graph \( \mathcal{H}_M(X) \) is acyclic.

Given a Morse matching \( M \) on \( \mathcal{H}(X) \) we define the critical points of \( X \) as the points of \( X \) which are not incident to any edge of \( M \).

We say that an edge \((a, b)\) of the Hasse diagram of \( X \) is homologically admissible if the space \( \hat{U}_b - \{a\} \) is acyclic.

We say that a matching \( M \) on \( \mathcal{H}(X) \) is homologically admissible if every edge of \( M \) is homologically admissible.

For our proof we need some lemmas. The first of them is one of the keys of our proof.

**Lemma 4.3.** Let \( X \) be a poset. Let \( b \) be a maximal point of \( X \). Suppose there exists \( a \in X \) such that \( E_a = \{ x \in X / a < x \} = \{ b \} \) and such that the edge \((a, b)\) of the Hasse diagram of \( X \) is homologically admissible. Then \( H_n(X, X - \{a, b\}) = 0 \) for all \( n \in \mathbb{Z} \).

**Proof.** Clearly, \( a \) is upbeat point of \( X \) an hence \( X - \{a\} \) is a strong deformation retract of \( X \). Thus, \( H_n(X, X - \{a\}) = 0 \) for all \( n \in \mathbb{Z} \).

On the other hand, applying 3.2 we obtain that

\[
H_n(X - \{a\}, X - \{a, b\}) = H_{n-1}(C_b^{X-\{a\}}) = \tilde{H}_{n-1}(\hat{U}_b - \{a\}) = 0
\]

for all \( n \in \mathbb{Z} \) since the edge \((a, b)\) is homologically admissible.

Then the result follows from the long exact sequence in homology of the triple \((X, X - \{a\}, X - \{a, b\})\). \( \square \)

**Lemma 4.4.** Let \( X \) be a quasicellular poset and let \( p : X \to \mathbb{N}_0 \) be its quasicellular morphism. Let \((a, b)\) be an homologically admissible edge of the Hasse diagram of \( X \). Then \( p(b) = p(a) + 1 \).

**Proof.** We have that \( \tilde{H}_n(\hat{U}_b - \{a\}) = 0 \) for all \( n \in \mathbb{Z} \) since the edge \((a, b)\) is homologically admissible and \( H_n(\hat{U}_b, \hat{U}_b - \{a\}) = \tilde{H}_{n-1}(\hat{U}_a) \) by 3.2. Then the result follows from the long exact sequence in homology of the pair \((\hat{U}_b, \hat{U}_b - \{a\})\). \( \square \)

Now, we will prove that the homology groups of a quasicellular poset \( X \) can be computed from a chain complex constructed from a homologically admissible Morse matching, generalizing theorem 3.14 of [14]. The idea of the proof is that, by lemma 4.3, under certain hypotheses a homologically admissible edge of a poset can be removed without altering the homology groups of the poset. Thus, we will prove that the edges of a homologically admissible Morse matching can be arranged in such a way that the hypotheses of lemma 4.3 are satisfied and hence we will be able to remove all the edges of the matching to obtain that the homology groups of \( X \) can be computed from the set of critical points.

To put this idea into work, we consider a suitable filtration of the poset \( X \) and construct a spectral sequence which converges to the homology groups of \( X \).

**Theorem 4.5.** Let \( X \) be a quasicellular poset and let \( p : X \to \mathbb{N}_0 \) be its quasicellular morphism. Let \( M \) be a homologically admissible Morse matching on \( \mathcal{H}(X) \). For \( n \in \mathbb{N}_0 \) let \( A_n = \{ x \in X / x \text{ is a critical point of } X \text{ and } p(x) = n \} \). Let \((C_n, d_n)_{n \in \mathbb{N}_0} \) be the chain
complex defined by \( C_n = \bigoplus_{x \in A_n} \hat{H}_{n-1}(\hat{U}_x) \) and where the differentials \( d_n \) are defined as in theorem 3.7. Then the homology of \( X \) coincides with the homology of \((C_n, d_n)_{n \in \mathbb{N}_0}\).

**Proof.** For \( n \in \mathbb{N}_0 \) let
\[
X_n = \{ x \in X / p(x) \leq n \} \cup \{ z / (y, z) \in M \text{ and } p(y) = n \}.
\]
By 4.3, \( X_{n-1} \subseteq X_n \) for all \( n \in \mathbb{N} \) and since \( X \) is a finite space it follows that \((X_n)_{n \in \mathbb{N}_0}\) is a filtration of \( X \).

Let \( X_{-1} = \emptyset \). We will compute now \( H_i(X_n, X_{n-1}) \) for all \( i \in \mathbb{Z} \) and for all \( n \in \mathbb{N}_0 \). Fix \( n \in \mathbb{N}_0 \). Let \( S = \{(y, z) \in M / p(y) = n \} \). Suppose \( S \neq \emptyset \). We define a relation \( \preceq \) in \( S \) as follows: \((a, b) \preceq (a', b')\) if and only if there exist \( l \in \mathbb{N} \) and \((a_0, b_0), \ldots, (a_l, b_l) \in S \) such that \((a_0, b_0) = (a, b), (a_l, b_l) = (a', b')\) and \( a_j \in \hat{U}_{b_{j+1}} \) for all \( j \in \{0, \ldots, l-1\} \).

This relation is clearly reflexive and transitive. We will prove now that it is also antisymmetric. Suppose that \((a, b)\) and \((a', b')\) are distinct elements of \( S \) such that \((a, b) \preceq (a', b')\) and \((a', b') \preceq (a, b)\). Then there exist \( l, m \in \mathbb{N} \) and \((a_0, b_0), \ldots, (a_l, b_l), (a_j, b_j) \in S \) and \((a'_0, b'_0), \ldots, (a'_m, b'_m) \in S \) such that \((a_0, b_0) = (a, b), (a_l, b_l) = (a', b')\) and \( a_j \in \hat{U}_{b_{j+1}} \) for all \( j \in \{0, \ldots, l-1\} \) and \( a'_k \in \hat{U}'_{b'_{k+1}} \) for all \( k \in \{0, \ldots, m-1\} \). In addition, we may suppose that \((a_j, b_j) \neq (a_{j+1}, b_{j+1})\) for all \( j \in \{0, \ldots, l-1\} \) and that \((a'_k, b'_k) \neq (a'_{k+1}, b'_{k+1})\) for all \( k \in \{0, \ldots, m-1\} \). And since \( M \) is a matching, we obtain that \( a_j \neq a_{j+1} \) for all \( j \in \{0, \ldots, l-1\} \) and \( a'_k \neq a'_{k+1} \) for all \( k \in \{0, \ldots, m-1\} \). Now, note that \((a_j, b_{j+1}) \in \mathcal{H}(X) - M \) for all \( j \in \{0, \ldots, l-1\} \) since \( a_j < b_{j+1} \), \( X \) is quasicellular, \( p(a_j) = n = p(b_{j+1}) - 1 \) by the previous lemma and \( M \) is a matching. In a similar way \((a'_k, b'_{k+1}) \in \mathcal{H}(X) - M \) for all \( k \in \{0, \ldots, m-1\} \). Thus,
\[(a'_0, b'_0), (a'_1, b'_1), \ldots, (a'_m, b'_m), (a_0, b_0), (a_1, b_1), \ldots, (a_l, b_l)\]
is a cycle in \( \mathcal{H}_M(X) \) which entails a contradiction since \( M \) is a Morse matching.

Then the relation \( \preceq \) is antisymmetric and hence it is a partial order.

Let \( N = \#S \). Extending the partial order in \( S \) to a linear order we obtain that we may label the elements of \( S \) as \((y_1, z_1), \ldots, (y_N, z_N)\) in such a way that if \((y_j, z_j) \preceq (y_k, z_k)\) then \( j \leq k \).

Note that
\[
X_n = X_{n-1} \cup A_n \cup \bigcup_{j=1}^{N} \{y_j, z_j\},
\]
(recall that \( A_n \) was defined as \( A_n = \{ x \in X / x \text{ is a critical point of } X \text{ and } p(x) = n \} \)).

For \( k \in \{0, 1, \ldots, N\} \) let
\[
B_k = X_{n-1} \cup A_n \cup \bigcup_{j=1}^{k} \{y_j, z_j\}.
\]
Hence, \( B_0 = X_{n-1} \cup A_n \) and \( B_N = X_n \).

Let \( r \in \{1, \ldots, N\} \). We claim that \( \hat{U}_{z_r}^{X} = \hat{U}_{z_r}^{B_r} \). Indeed, let \( x \in \hat{U}_{z_r}^{X} \). Then \( x < z_r \) and thus \( p(x) < p(z_r) = n + 1 \) by the previous lemma. If \( x \in X_{n-1} \cup A_n \) then \( x \in B_r \). If \( x \notin X_{n-1} \cup A_n \) then there exists \( s \in \{1, \ldots, N\} \) such that \( x = y_s \). Thus, \((y_s, z_s) \preceq (y_r, z_r)\) and hence \( s \leq r \). Then, \( x = y_s \in B_r \). Thus, \( \hat{U}_{z_r}^{X} \subseteq \hat{U}_{z_r}^{B_r} \). The other inclusion is trivial.

Hence, the edge \((y_r, z_r)\) is homologically admissible in \( B_r \).
On the other hand, we claim that $\tilde{F}^{B_r}_{y_r} = \{z_r\}$. Indeed, suppose that $x \in B_r$ satisfies $x > y_r$. Hence, $p(x) > p(y_r) = n$ and thus $x = z_s$ for some $s \in \{1, \ldots, r\}$. But this implies that $y_r \in \tilde{U}_{z_s}$ and hence $(y_r, z_r) \preceq (y_s, z_s)$. Then $r \leq s$ and thus $s = r$ and $x = z_r$. Hence, $F^{B_r}_{y_r} \subseteq \{z_r\}$ while the other inclusion is trivial.

Now, since $p(z_r) = n + 1$ and $B_r \subseteq X_n \subseteq \{x \in X / p(x) \leq n + 1\}$ (by 4.4) it follows that $z_r$ is a maximal point of $B_r$. Hence, we are under the hypotheses of lemma 4.3 and thus we obtain that $H_i(B_r, B_{r-1}) = 0$ for all $i \in \mathbb{Z}$.

Hence, we have proved that $H_i(B_r, B_{r-1}) = 0$ for all $i \in \mathbb{Z}$ and for all $r \in \{1, \ldots, N\}$. It follows that $H_i(B_N, B_0) = 0$ for all $i \in \mathbb{Z}$. Note that this was done under the hypothesis $S \neq \emptyset$, but holds trivially if $S = \emptyset$.

Thus, by 3.2

$$H_i(X_n, X_{n-1}) = H_i(B_N, X_{n-1}) \cong H_i(B_0, X_{n-1}) \cong \bigoplus_{x \in A_n} \tilde{H}_{n-1}(\tilde{U}_x) \text{ if } i = n$$

$$0 \text{ if } i \neq n$$

since $X$ is quasicellular.

From the filtration $(X_n)_{n \in \mathbb{N}_0}$ of $X$ we can construct a spectral sequence in a similar way to the one in the proof of theorem 3.7, which will have in its first page a single nontrivial row and whose differentials can be computed in the same way as in 3.7 by naturality of the long exact sequences since the isomorphisms $H_i(X_n, X_{n-1}) \cong H_i(B_0, X_{n-1})$ are given by the inclusion maps. Thus, the result follows. \hfill \Box

**Remark 4.6.** The previous theorem might not hold if the space $X$ is not quasicellular even if for all $x \in X$ the homology of $\tilde{U}_x$ is concentrated in some degree. For example, let $X$ be defined by the following Hasse diagram

![Hasse diagram](image)

and let $M = \{(c, e), (d, h), (f, g)\}$. It is easy to verify that $M$ is a homologically admissible Morse matching. On the other hand $f$ and $g$ are beat points of $X$ and hence $X$ is homotopy equivalent to $X - \{f, g\}$ which is a finite model for $S^2$. But the set of critical points is $\{a, b\}$ and thus if $(C_n, d_n)_{n \in \mathbb{N}_0}$ is the chain complex of the previous theorem we obtain that $C_0 = \mathbb{Z} \oplus \mathbb{Z}$ and $C_n = 0$ for all $n \in \mathbb{N}$. Clearly, the homology groups of $(C_n, d_n)_{n \in \mathbb{N}_0}$ do not coincide with those of $X$.

### 4.3. M"obius function

Now, we will apply our methods to obtain different formulas to compute the M"obius function of a poset. Recall that the M"obius function of a poset $P$
equals the number of chains of $P$ of odd cardinality minus the number of chains of $P$ of even cardinality (where the empty chain counts as a chain of even cardinality) and is denoted by $\mu(P)$. Clearly, $\mu(P)$ coincides with the reduced Euler characteristic of the order complex of $P$ and thus with the reduced Euler characteristic of $P$ (viewed as a topological space).

Recall also that the Euler characteristic of a finitely generated graded abelian group $G = (G_n)_{n \in \mathbb{Z}}$ is defined as

$$\chi(G) = \sum_{n \in \mathbb{Z}} (-1)^n \text{rg}(G_n)$$

and that the Euler characteristic of a finitely generated bigraded abelian group $E = (E_{p,q})_{p,q \in \mathbb{Z}}$ is defined as

$$\chi(E) = \sum_{p,q} (-1)^{p+q} \text{rg}(E_{p,q}).$$

Clearly, if $\{ (E_{r,p,q})_{p,q \in \mathbb{Z}} \}_{r \in \mathbb{N}}$ is a bigraded spectral sequence such that $E^1$ is finitely generated then $\chi(E^r) = \chi(E^{r+1})$ for all $r \in \mathbb{N}$. Therefore, it follows that if $\{ (E_{r,p,q})_{p,q \in \mathbb{Z}} \}_{r \in \mathbb{N}}$ converges to a graded abelian group $(G_n)_{n \in \mathbb{Z}}$ then $\chi(E^1) = \chi(G)$.

This simple idea suggests that our methods can be effectively applied to compute the Möbius function of a finite poset in several different ways.

If $P$ is a finite poset, $h(P)$ will denote the height of $P$, that is the maximum of the cardinalities of the chains of $P$ minus one.

**Proposition 4.7.** Let $X$ be a finite $T_0$-space (or equivalently a finite poset) and let $V \subseteq X$ be an open subset. Then

$$\mu(X) = \mu(V) - \sum_{x \in X - V} \mu(\hat{U}_x).$$

**Proof.** Let $X_0 = V$. For $p \in \mathbb{N}$, let $X_p = \{ x \in X - X_0 / h(U_x) < p \}$ and let $D_p = X_p - X_{p-1}$. Clearly $(X_p)_{p \in \mathbb{N}_0}$ is a filtration of $X$ which is induced by antichains. Hence, theorem 3.7 applies and we obtain a spectral sequence $\{ (E_{r,p,q})_{p,q \in \mathbb{Z}} \}_{r \in \mathbb{N}}$ that converges to $H_*(X)$ such that

- $E^1_{p,q} = 0$ for every $p \leq -1$.
- $E^1_{0,q} = H_q(X_0)$.
- $E^1_{p,q} = \bigoplus_{x \in D_p} \hat{H}_{p+q-1}(\hat{C}X_x)$ for $p \geq 1$. 

Note that, if \( p \in \mathbb{N} \) and \( x \in D_p \) then \( \hat{C}_x^X = \hat{U}_x \) since \( V \) is an open set. Thus,

\[
\mu(X) = \tilde{\chi}(X) = \chi(X) - 1 = -1 + \chi(E^1) = -1 + \sum_{p \in \mathbb{N}} \sum_{q \in \mathbb{Z}} (-1)^{p+q} \text{rg}(E^1_{p,q}) = -1 + \sum_{q \in \mathbb{Z}} \sum_{p \in \mathbb{N}} (-1)^{p+q} \text{rg}(H_q(X_0)) + \sum_{x \in D_p} \sum_{q \in \mathbb{Z}} (-1)^{p+q} \text{rg}(\hat{H}_{p+q-1}(\hat{C}_x^X)) = -1 + \chi(X_0) + \sum_{x \in D_p} \sum_{q \in \mathbb{Z}} (-1)^{p+q} \text{rg}(\hat{H}_{p+q-1}(\hat{U}_x)) = -1 + \chi(X_0) + \sum_{x \in D_p} \sum_{q \in \mathbb{Z}} (-1)^{p+q} \text{rg}(\hat{H}_{p+q-1}(\hat{U}_x)) = \tilde{\chi}(X_0) - \sum_{x \in X - V} \tilde{\chi}(\hat{U}_x) = \mu(V) - \sum_{x \in X - V} \mu(\hat{U}_x).
\]

\[\square\]

**Corollary 4.8.** Let \( X \) be a finite \( T_0 \)-space (or equivalently a finite poset).

1. Let \( A \subseteq X \) be a contractible open subspace. Then

\[\mu(X) = -\sum_{x \in X - A} \mu(\hat{U}_x).\]

2. Let \( X_0 \) be the set of minimal points of \( X \). Then

\[\mu(X) = \#X_0 - 1 - \sum_{x \in X - X_0} \mu(\hat{U}_x).\]

Note that the second formula of this corollary can also be deduced by partitioning the set of chains of \( X \) according to the maximum element of each chain and noting that the chains of cardinality \( n \) of \( \hat{U}_x \) are in bijection with the chains of cardinality \( n + 1 \) of \( U_x \).

Now we will apply the previous results to give a different proof of theorem 6.1 of [7]. To this end we need a couple of lemmas.

Recall that a subset \( C \) of a poset \( P \) is convex if \( x < y < z \) and \( x, z \in C \) imply \( y \in C \).

**Lemma 4.9.** Let \( P \) be a finite poset and let \( C \) be a convex subset of \( P \). Let \( a \in C \). Then

\[\mu(\hat{F}_a - C) = \sum_{y \in C \atop y \geq a} \mu(\hat{F}_y).\]

**Proof.** Applying [4,7] with \( X = (\hat{F}_a)^\text{op} \) and \( V = (\hat{F}_a - C)^\text{op} \) yields

\[\mu((\hat{F}_a)^\text{op}) = \mu((\hat{F}_a - C)^\text{op}) - \sum_{x \in (\hat{F}_a)^{\text{op}} \cap C} \mu(\hat{U}_x).\]

(note that \( V \) is an open subset of \( X \) since given \( z \in V \) and \( w \leq z \) we get that \( w \leq z < a \), \( a \in C \) and \( z \notin C \), and thus \( w \notin C \) since \( C \) is convex). Hence,

\[\mu(\hat{F}_a - C) = \mu(\hat{F}_a) + \sum_{y \in \hat{F}_a \cap C} \mu(\hat{F}_y) = \mu(\hat{F}_a) + \sum_{y \in C \atop y \geq a} \mu(\hat{F}_y) = \sum_{y \in C \atop y \geq a} \mu(\hat{F}_y).\]

\[\square\]
If \( X \) and \( Y \) are posets, the non-Hausdorff join \( X \oplus Y \) is the poset whose underlying set is the disjoint union \( X \sqcup Y \) and whose ordering is defined keeping the ordering within \( X \) and \( Y \) and setting \( x \leq y \) for every \( x \in X \) and \( y \in Y \) (cf. [2]).

**Lemma 4.10.** Let \( X \) and \( Y \) be posets. Then \( \mu(X \oplus Y) = -\mu(X)\mu(Y) \).

**Proof.** Let \( \mu_o(X) \) and \( \mu_o(Y) \) denote the number of chains of odd cardinality of \( X \) and \( Y \) respectively and let \( \mu_e(X) \) and \( \mu_e(Y) \) denote the number of chains of even cardinality of \( X \) and \( Y \) respectively. Then
\[
-\mu(X)\mu(Y) = -(\mu_o(X) - \mu_e(X))(\mu_o(Y) - \mu_e(Y)) = \\
(\mu_o(X)\mu_e(Y) + \mu_e(X)\mu_o(Y)) - (\mu_o(X)\mu_e(Y) + \mu_o(X)\mu_o(Y)) = \\
\mu_o(X \oplus Y) - \mu_e(X \oplus Y) = \mu(X \oplus Y).
\]

\( \square \)

Now we state theorem 6.1 of [7] and apply our methods to give an alternative proof of it.

**Theorem 4.11** (Björner–Walker). Let \( P \) be a finite poset and let \( C \) be a convex subset of \( P \). Then
\[
\mu(P) = \mu(P - C) + \sum_{x,y \in C \atop x \leq y} \mu(\hat{U}_x)\mu(\hat{F}_y).
\]

**Proof.** We consider \( P \) as a finite \( T_0 \)-space. Let \( X_0 = P - C \). For \( p \in \mathbb{N} \), let \( X_p = X_0 \cup \{x \in C \mid h(U_x) < p\} \) and let \( D_p = X_p - X_{p-1} \). Clearly \( (X_p)_{p \in \mathbb{N}_0} \) is a filtration of \( P \) which is induced by antichains. Hence, theorem 3.7 applies and we obtain a spectral sequence \( \{E_{p,q}^r\}_{r \in \mathbb{N}} \) that converges to \( H_*(P) \) such that

- \( E_{p,q}^1 = 0 \) for every \( p \leq -1 \).
- \( E_{0,q}^1 = H_q(X_0) \).
- \( E_{p,q}^1 \) is a finite poset and whose ordering is defined keeping the ordering within \( X \) and whose ordering is defined.

Then, proceeding as in the proof of [7] we obtain that
\[
\mu(P) = \mu(P - C) - \sum_{p \in \mathbb{N}} \sum_{x \in D_p} \mu(\hat{C}^X_x).
\]

Note that \( \hat{C}^X_x = \hat{U}_x \oplus \hat{F}_x^{P-C} \) for \( x \in D_p \). Hence, from 4.9 and 4.10 we obtain that
\[
\mu(\hat{C}^X_x) = -\mu(\hat{U}_x)\mu(\hat{F}_x^{P-C}) = -\mu(\hat{U}_x)\sum_{y \in C \atop y \geq x} \mu(\hat{F}_y).
\]

Thus,
\[
\mu(P) = \mu(P - C) - \sum_{p \in \mathbb{N}} \sum_{x \in D_p} \mu(\hat{C}^X_x) = \mu(P - C) + \sum_{p \in \mathbb{N}} \sum_{x \in D_p} \left( \mu(\hat{U}_x) \sum_{y \in C \atop y \geq x} \mu(\hat{F}_y) \right) = \\
= \mu(P - C) + \sum_{x \in C} \left( \mu(\hat{U}_x) \sum_{y \in C \atop y \geq x} \mu(\hat{F}_y) \right) = \mu(P - C) + \sum_{x \in C} \mu(\hat{U}_x)\mu(\hat{F}_y).
\]
Clearly, this technique can be applied to obtain and prove many different formulas of this type for computing the Möbius function of posets.

References

[1] Alexandroff, P. S. Diskrete Räume. Mathematicskii Sbornik (N.S.) 2 (1937) 501–518.
[2] Barmak, J. Algebraic Topology of Finite Topological Spaces and Applications. Lecture Notes in Mathematics Vol. 2032. Springer. 2011. xviii+170 pp.
[3] Barmak, J. Star clusters in independence complexes of graphs. Advances in Mathematics 241 (2013) 33–57.
[4] Barmak, J., Minian, G. Simple homotopy types and finite spaces. Advances in Mathematics 218 (2008) 87–104.
[5] Barmak, J., Minian, G. One-point reductions of finite spaces, h-regular CW-complexes and collapsibility. Algebraic and Geometric Topology 8 (2008) 1763–1780.
[6] Barmak, J., Minian, G. G-colorings of posets, covering maps and computation of low-dimensional homotopy groups. Submitted (2013).
[7] Björner, A., Walker, J. A homotopy complementation formula for partially ordered sets. European Journal of Combinatorics 4 (1983) 11–19.
[8] Dold, A., Thom, R. Quasifaserungen und Unendliche Symmetrische Produkte. Annals of Mathematics 67 (1958) 239–281.
[9] Forman, R. Morse theory for cell complexes. Advances in mathematics 134 (1998) 90–145.
[10] Hatcher, A. Spectral Sequences in Algebraic Topology. Disponible en http://www.math.cornell.edu/~hatcher/SSAT/SSATpage.html.
[11] Kozlov, D. Combinatorial Algebraic Topology. Springer. 2008. xix+389 pp.
[12] Lóvasz, L. Kneser’s conjecture, chromatic number and homotopy. Journal of Combinatorial Theory, series A, 25 (1978) 319–324.
[13] McCord, M. C. Singular homology groups and homotopy groups of finite topological spaces. Duke Mathematical Journal 33 (1966) 465–474.
[14] Minian, G. Some remarks on Morse theory for posets, homological Morse theory and finite manifolds. Topology and its applications 159 (2012) 2860–2869
[15] Osaki, T. Reduction of finite topological spaces. Interdisciplinary Information Sciences 5 (1999) 149–155.
[16] Stong, R. Finite topological spaces. Transactions of the American Mathematical Society 123 (1966) 325–340.

E-mail address: nicocian@gmail.com

E-mail address: emottina@uncu.edu.ar

Facultad de Ciencias Exactas y Naturales, Universidad Nacional de Cuyo, Mendoza, Argentina.