Sound Mode Hydrodynamics from Bulk Scalar Fields

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Abstract

We study the hydrodynamic sound mode using gauge/gravity correspondence by examining a generic black brane background’s response to perturbations. We assume that the background is generated by a single scalar field, and then generalize to the case of multiple scalar fields. The relevant differential equations obeyed by the gauge invariant variables are presented in both cases. Finally, we present an analytical solution to these equations in a special case; this solution allows us to determine the speed of sound and bulk viscosity for certain special metrics. These results may be useful in determining sound mode transport coefficients in phenomenologically motivated holographic models of strongly coupled systems.
1 Introduction

The plasma created in heavy ion collisions at RHIC appears to be both strongly coupled, and well described by hydrodynamics \[1, 2, 3\]. Hydrodynamics is an effective theory which describes the behavior of a fluid on length and time scales which are much longer than any microscopic scale. The hydrodynamic stress-energy tensor is constructed from conserved quantities, and is built so that it respects equilibrium thermodynamics and the symmetries in the problem. One can then add plane wave type perturbations \(~e^{i(qz-wt)}\), and use the conservation equation \(\partial_\mu T^{\mu\nu}\) to analyze the normal modes of the system - see \[4, 5\] for reviews. For the shear mode, the dispersion relation between the energy of the perturbation \(w\) and the momentum \(q\) is found to be

\[ w(q)_{\text{shear}} = -i \frac{\eta}{\epsilon + P} q^2, \tag{1} \]

while for the sound mode it is

\[ w(q)_{\text{sound}} = v_s q - i \frac{\eta}{\epsilon + P} \left( \frac{p - 1}{p} + \frac{\zeta}{2\eta}\right) q^2. \tag{2} \]

Here, \(p\) is the number of spatial dimensions, \(\epsilon\) and \(P\) denote the equilibrium pressure and energy density, respectively, \(v_s\) is the speed of sound and \(\eta\) and \(\zeta\) are the shear viscosity and bulk viscosity. It is desirable to calculate these latter three quantities which are used in hydrodynamic models of heavy ion collisions, but in the regime of strong coupling, perturbative calculations are unreliable.

The AdS/CFT correspondence \[6, 7, 8\], (or more generally, the ‘gauge/gravity duality’ or ‘holography’) has become an indispensable tool for describing strongly coupled systems, and has enjoyed arguably its greatest success calculating hydrodynamic transport coefficients. Within this framework, hydrodynamic dispersion relations are calculated by computing correlation functions of the stress-energy tensor, and then either examining the poles of such quantities, or by using Kubo relations \[9, 10, 11, 12, 13, 14\]. Alternatively, one can examine behavior of a background under perturbations, and determine the dispersion relation by imposing appropriate boundary conditions \[15, 16\]. The resulting dispersion relations from all of these methods coincide as demonstrated in \[15\]. After computing the dispersion relation, one can compare with the formulas \(1\), \(2\), and read off the appropriate transport coefficients.

Another method which has been employed to access such quantities is the ‘membrane paradigm’, in which one computes the hydrodynamic properties of a black hole’s ‘stretched horizon’. These hydrodynamic quantities agree with the AdS/CFT calculations in all cases tested so far \[17, 18, 19, 20, 21\]. The details of the membrane paradigm approach are not relevant for this paper, but we mention these works for historical reasons. In \[17\], a formula for the shear viscosity to entropy density ratio \(\eta/s\) was derived which is applicable to a wide variety of gravity duals. Application of this formula to known gravity duals always resulted in \(\eta/s = 1/4\pi\), and eventually led to the celebrated viscosity bound conjecture that \(\eta/s \geq 1/4\pi\) for all physical substances. While the membrane paradigm approach was first
used to derive this general formula, it can also be derived by examining the quasi-normal mode spectrum as shown in [22]. Clearly, a similar general formula for the bulk viscosity would also be desirable in the hopes that other universal behavior might be discovered. Such a formula would also be useful when attempting to phenomenologically fit the results from the lattice [23, 24].

This work is motivated by the goal of developing such a formula, though such a formula will not be presented here. Instead, this work should be viewed as a first step towards this goal. Here, we consider sound mode fluctuations of a dual gravity theory supported by a single scalar field, and then generalize to the case of multiple scalar fields. We attempt to be as general as possible by not specifying the background profiles of the scalar field(s).

The recent works [25, 26] also explore the sound mode in phenomenologically motivated single scalar models. The work presented here is complementary to these papers, though we allow for the possibility of multiple scalar fields, and also do not restrict ourselves to five dimensions.

In section 1, we write the relevant equations that must be obeyed by the background fields. In section 2, we introduce the standard sound mode perturbations and explicitly compute the linearized Einstein equations. In section 3, we reduce these Einstein equations to two gauge invariant equations. In section 4, we illustrate how one uses these equations to solve for the hydrodynamical dispersion relation and thus compute the speed of sound and bulk viscosity. The gauge invariant equations are rather complicated, so we only consider a simple special case in this section. The results of this section are generalizations of the results of [16]. Finally, in section 5, we generalize our gauge invariant equations by including an arbitrary number of scalar fields. We conclude and mention some prospects for further investigation in section 6.

Our general relativistic conventions are those of [27]. We use $R_{\mu\nu}$, $G_{\mu\nu}$ and $T_{\mu\nu}$ to denote the Ricci, Einstein, and stress-energy tensors respectively. We will also make use of the shorthand notation

\[
\begin{align*}
R^0_0 & \equiv F_0 \\
R^x_x & \equiv F_x \\
R^r_r & \equiv F_r.
\end{align*}
\]

(3)

In Appendix A one can find explicit forms of these functions in terms of the metric components. $\nabla_\mu$ denotes the covariant derivative, and $\square \equiv \nabla_\mu \nabla^\mu$. Throughout the paper, we also use the notation $D_L$ to denote the logarithmic derivative, namely

\[
D_L [X(r)] \equiv \frac{X'(r)}{X(r)}.
\]

(4)

2 Background Fields

We wish to examine hydrodynamic fluctuations on the following $p + 2$ dimensional gravitational background

\[
d s^2 = g_{00}(r) d t^2 + g_{xx}(r) d x^j d x^j + g_{rr}(r) d r^2
\]

(5)
where \( j = 1, 2 \ldots p \), and \( r \) is the extra-dimensional coordinate. We will often make use of the definition

\[
f(r) \equiv \sqrt{-g_{00}(r)g^{xx}(r)}. \tag{6}
\]

We assume the position of a horizon at \( r = r_0 \), and that the behavior of the metric components near the horizon is

\[
\begin{align*}
g_{00}(r) &\approx -\gamma_0(r - r_0) + \mathcal{O}(r - r_0)^2 \\
g_{rr}(r) &\approx \frac{\gamma_r}{r - r_0} + \mathcal{O}(1) \\
g_{xx}(r) &\approx g_{xx}(r_0).
\end{align*} \tag{7}
\]

The quantities \( \gamma_0, \gamma_r \) and \( g_{xx}(r_0) \) are independent of \( r \). The Hawking temperature is given by

\[
T = \frac{1}{4\pi} \sqrt{\frac{\gamma_0}{\gamma_r}}. \tag{8}
\]

Let us make the simplest assumption, that this metric is created by a single, minimally coupled scalar field. In other words, we assume the action is of the form

\[
S = \frac{1}{16\pi G_{p+2}} \int d^{p+2}x \sqrt{-g} \left( R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - U(\phi) \right). \tag{9}
\]

Then, the background equations are

\[
\begin{align*}
G_{\mu\nu} &= -8\pi G_{p+2} T_{\mu\nu} \tag{10} \\
\Box \phi &= \frac{dU}{d\phi}. \tag{11}
\end{align*}
\]

Here we ignore any subtleties regarding boundary terms that come from integration by parts. Such terms can be taken care of by adding additional boundary terms to the action, though we have not explicitly written such terms above.

The energy-momentum tensor derived from the action is

\[
\begin{align*}
8\pi G_{p+2} T_{\mu\nu} &= \frac{1}{2} \left( \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \mathcal{L}_\phi \right) \tag{12} \\
\mathcal{L}_\phi &= \frac{1}{2} \partial_\lambda \phi \partial^\lambda \phi + U(\phi). \tag{13}
\end{align*}
\]

Because the background metric only depends on the extra dimensional coordinate \( r \), it is clear that the field \( \phi \) is also only a function of \( r \). In the case of multiple scalar fields, it might be possible to have fields which depend on the other coordinates, provided that all such dependence cancels out in the combination that appears in \( T_{\mu\nu} \). We will not consider such special cases, and will always assume that the scalar fields only depend on \( r \).

In this case, we can get a more explicit form of the potential:

\[
\frac{dU(\phi(r))}{d\phi} = \Box \phi = \frac{1}{\sqrt{-g}} \partial_r \left[ \sqrt{-g} g^{rr} \phi'(r) \right]. \tag{14}
\]
Throughout this work, the prime denotes derivative with respect to \( r \). It will also be useful to express the second derivative of the potential as

\[
\frac{d^2U(\phi(r))}{d\phi^2} = \frac{1}{\phi'(r)} \partial_r \left[ \frac{dU(\phi(r))}{d\phi} \right].
\] (15)

One can write the field \( \phi(r) \) in terms of the metric components by noting that

\[
g^{00}G_{00} - g^{rr}G_{rr} = F_0(r) - F_r(r)
\] (16)

and

\[
8\pi G_{p+2} \left( g^{00}T_{00} - g^{rr}T_{rr} \right) = \frac{1}{2} \left[ -L_{\phi} - \left( g^{rr} \phi'(r)^2 - L_{\phi}(r) \right) \right]
\] (17)

\[
= -\frac{1}{2} g^{rr} \phi'(r)^2.
\] (18)

Thus,

\[
\phi'(r)^2 = 2 g_{rr} (F_0(r) - F_r(r)).
\] (19)

It is also noteworthy that this background has the special property \( F_0 = F_x \) as can be seen by considering

\[
g^{00}G_{00} - g^{xx}G_{ii} = -8\pi G_{p+2} \left( g^{00}T_{00} - g^{xx}T_{ii} \right)
\] (20)

\[
F_0(r) - F_x(r) = L_{\phi}(r) - L_{\phi}(r) = 0.
\] (21)

Because of this fact, and the general theorem given in \[28\], all backgrounds we consider have \( \eta/s = 1/4\pi \).

### 3 Hydrodynamic Fluctuations

We now introduce fluctuations of the fields on this background \( g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu} \) and \( \phi \to \phi + \delta \phi \), and assume the usual time dependence

\[
\delta g_{\mu\nu}(t, z, r) = e^{i(qz - wt)} h_{\mu\nu}(r),
\] (22)

\[
\delta \phi(t, z, r) = e^{i(qz - wt)} \delta \phi(r).
\] (23)

Here we use the coordinate \( z \) to denote one of the spatial coordinates: \( z \equiv x_p \), and \( w \) and \( q \) are the energy and momentum of the perturbation. For the sound mode, in the gauge where \( h_{\mu r} = 0 \), the only non-zero fluctuations are \[11 \, 16\]

\[
h_{00}(r) \equiv g_{00}(r)A(r),
\] (24)

\[
\frac{1}{p-1} \sum_{i=1}^{p-1} h_{ii}(r) \equiv g_{xx}(r)B(r),
\] (25)

\[
h_{zz}(r) \equiv g_{xx}(r)C(r),
\] (26)

\[
h_{0z}(r) \equiv g_{00}(r)D(r),
\] (27)

\[
\delta \phi(r).
\] (28)

Turning on these perturbations, and expanding the background equations of motion to first order in the perturbation, we end up with a set of 8 differential equations for the perturbations. The equations are presented in full in Appendix \[13\].
4 Gauge Invariant Equations

There is still residual gauge freedom in these equations under the infinitesimal diffeomorphisms

\[
\begin{align*}
    h_{\mu\nu} &\rightarrow h_{\mu\nu} - \nabla^{(0)}_\mu \xi_\nu - \nabla^{(0)}_\nu \xi_\mu, \\
(\delta \phi) &\rightarrow (\delta \phi) - \xi_\mu (\partial^\mu \phi)
\end{align*}
\]

(29)

(30)

for any vector \( \xi_\mu = \xi_\mu (r)e^{i(qz - wt)} \). The following gauge invariant combinations given in [16] transform only into themselves under such a diffeomorphism. (i.e. \( Z_0 \rightarrow Z_0 \) and \( Z_\phi \rightarrow Z_\phi \)).

\[
Z_0 (r) = - f(r)^2 \left( q^2 A(r) + 2 qw D(r) \right) + w^2 C(r) - \left( q^2 g_{00} (r) g_{xx} (r) + w^2 \right) B(r)
\]

(31)

\[
Z_\phi (r) = \delta \phi (r) - \frac{\phi'(r)}{D_L [g_{xx}(r)]} B(r)
\]

(32)

By taking an appropriate combination of the equations \( (72 - 79) \), one can arrive at two coupled second order equations for the gauge invariant combinations. The particular combination of the Einstein equations used, as well as additional details regarding this derivation, are presented in Appendix C. A combination which appears frequently in these equations is

\[
\alpha (r) \equiv q^2 \left( (p - 1) + \frac{D_L [g_{00}(r)]}{D_L [g_{xx}(r)]} \right) - pw \frac{f(r)}{f(r)}^2.
\]

(33)

The following is the most compact form of the gauge invariant equations that we have been able to find.

\[
\frac{g_{rr}}{\sqrt{-g}} \alpha^2 f^4 \partial_r \left[ \frac{\sqrt{-gg_{rr}^r}}{\alpha^2 f^4} Z'_0 \right] + Z_0 \left( D_L [f^2] D_L [f^2 \alpha] - g_{rr} \left( w^2 g_{00} + q^2 g_{xx} \right) \right) +
\]

\[
2 Z_\phi \phi' f^2 \left( \alpha \partial_r \left[ \frac{1}{f^2} \frac{w^2 - q^2}{f^2} \right] + \frac{q^2 D_L [f^2]}{p D_L [g_{xx}]} D_L \left[ \sqrt{-gg_{rr}^r} \phi' \right] \right) = 0
\]

(34)

\[
\frac{g_{rr}}{\sqrt{-g}} \partial_r \left[ \frac{\sqrt{-gg_{rr}^r}}{\alpha^2 f^4} Z'_\phi \right] + \frac{2}{\alpha} \partial_r \left[ \frac{\phi'}{D_L [g_{xx}]} \right] \left\{ \frac{1}{f^2} \partial_r \left[ \frac{Z_0}{f} \right] - \left( w^2 g_{00} + q^2 g_{xx} \right) \phi' g_{xx} Z_\phi \right\}
\]

\[
- Z_\phi \left\{ g_{rr} \left( q^2 g_{xx} + w^2 g_{00} \right) + \frac{\left( D_L [g_{xx}] \right)^2}{f^2 \phi'} \partial_r \left[ \frac{f^2 \phi'}{\left( D_L [g_{xx}] \right)^2} D_L \left[ \sqrt{-gg_{rr}^r} \phi' \right] \right] \right\} = 0
\]

(35)

Of course, one can write these equations only in terms of the metric components because of the relation \([19]\).

To determine the dispersion relation, one needs to solve the above equations perturbatively in \( w, q \), and then apply the incoming wave and Dirichlet boundary conditions \([15]\). We will illustrate this procedure in the following section. Some terms can be neglected when considering these equations to \( \mathcal{O}(w, q) \), but one should keep in mind that \( Z_0 \) is \( \mathcal{O}(q^2) \), \( Z_\phi \) is \( \mathcal{O}(1) \), and \( \alpha \) is \( \mathcal{O}(q^2) \).
5 Example Application

As a simple example of how to apply these equations, consider the special case of \( \phi(r) = C_1 \log [g_{xx}(r)] \) where \( C_1 \) is a constant. In this special case, the equation (35) reduces to

\[
\frac{g_{rr}}{\sqrt{-g}} \partial_r \left[ \sqrt{-g} g^{rr} \phi' \right] - Z_\phi g_{rr} \left( q^2 g^{xx} + w^2 g^{00} \right) = 0.
\]

This can be more easily seen by re-writing

\[
\partial_r \left[ \frac{f^2 \phi'}{(DL[g_{xx}])^2} DL \left[ \sqrt{-g} g^{rr} \phi' \right] \right] = C_1 \partial_r \left[ \frac{f^2}{DL[g_{xx}]} DL \left[ \sqrt{-g} g^{rr} DL[g_{xx}] \right] \right] \]

\[
\partial_r \left[ \frac{2g_{rr} f^2}{DL[g_{xx}]} (F_0 - F_x) - f^2 \partial_r \left[ \frac{2g_{rr} (F_0 - F_r)}{p (DL[g_{xx}])^2} \right] \right] = 0.
\]

The standard way of applying the incoming wave boundary conditions is to make the ansatz \([15, 16]\)

\[
Z_0(r) = f(r)^{-\frac{iw}{2\pi T}} (Y_0(r) + qY_1(r) + ...)
\]

\[
Z_\phi(r) = f(r)^{-\frac{iw}{2\pi T}} (Y_\phi_0(r) + qY_\phi_1(r) + ...)
\]

\[
w(q) = w_1 q + w_2 q^2 + ...
\]

with the condition that all \( Y \) functions are regular at the horizon. Inserting this ansatz into (36), expanding the result in powers of \( q \), and neglecting terms of \( O(q^2) \) and higher, we find the following equation:

\[
\partial_r \left[ \sqrt{-g} g^{rr} Y_\phi' \right] + q \partial_r \left[ \sqrt{-g} g^{rr} \left( Y_\phi' - \frac{iw_1}{2\pi T} DL(f) Y_\phi_0 \right) \right].
\]

Solving this order by order in \( q \) is now quite simple. The solution for \( Y_\phi_0 \) can be written in terms of an integral

\[
Y_\phi_0(r) = c_0 + c_1 \int_{r_0}^\infty \frac{g_{rr}(r')}{\sqrt{-g(r')}} dr',
\]

but this integral is logarithmically divergent at the horizon by (7). Thus, the assumption of regularity on the \( Y \) functions leads to \( Y_\phi_0 = c_0 \). Finally, this constant must be set to zero by the Dirichlet boundary condition at infinity. Plugging \( Y_\phi_0 = 0 \) into the next order equation, one also finds \( Y_\phi_1 = 0 \).

Next, one must solve the equation for \( Z_0 \) using these boundary conditions with the knowledge that \( Z_\phi = 0 \). But before doing so, it is useful to pause and ask what type of
metrics these results will be applicable to. There are 3 unknown metric functions, \( g_{00}, g_{xx}, g_{rr} \), but we have two constraints on them, namely (19) and (21). It can be shown using these constraints, and the assumption of the scalar field profile \( \phi \sim \log(g_{xx}) \) (as at the beginning of this section), that we are considering metrics which satisfy \( F_0 = F_x \) and the following constraint:

\[
g_{00}(r) = \frac{a_0}{a_2 - p} g_{xx}(r) + a_1 g_{xx}(r)^{a_2 - p + 1}. \tag{47}
\]

Here, \( a_0, a_1, a_2 \) are independent of \( r \), and we have defined the coefficients as above for future convenience. Clearly, \( a_1 \) can be determined in terms of \( a_0 \) and \( a_2 \), by requiring \( g_{00} \) vanish at the horizon.

Returning now to the equation for \( Z_0 \), going through the same steps of inserting the incoming wave ansatz, expanding in powers of \( q \), solving order by order in \( q \), and applying the boundary conditions leads to the dispersion relation

\[
w(q) = \sqrt{\frac{a_0 - a_2}{p} q - i \frac{a_2}{2\pi T} q^2} + \mathcal{O}(q^3). \tag{48}
\]

The details of this derivation are presented in the Appendix D. Comparing this dispersion relation with the expected hydrodynamic dispersion relation (2) yields the relations for the speed of sound and bulk viscosity:

\[
\begin{align*}
    v_s &= \sqrt{\frac{a_0 - a_2}{p}} \\
    \zeta &= \frac{2(2a_2 - p + 1)}{p}.
\end{align*} \tag{49}
\]

Let us now explicitly check these results agree with known cases. Consider the ‘Schwarzschild AdS Black Hole’ metric in \( p+2 \) dimensions,

\[
ds^2 = \frac{r^2}{L^2} \left[ -f(r)^2 dt^2 + dx_j dx^j \right] + \frac{L^2 dr^2}{r^2 f(r)^2} \tag{50}
\]

\[
f(r)^2 = 1 - \left(\frac{r_0}{r}\right)^{p+1} \tag{51}
\]

This is equivalent to the metric (17) with the choices \( a_0 = (p+1)/2, a_2 = (p-1)/2 \). Inserting these into (49) yields

\[
\begin{align*}
    v_s &= \frac{1}{\sqrt{p}} \\
    \zeta &= 0
\end{align*} \tag{52}
\]

which are in perfect agreement with our expectations, since the metric is conformal, and is thus dual to a conformal field theory.
Next, consider the ‘Dp-Brane metric’ in the Einstein frame, which can be reduced to

$$ ds^2 = \left( \frac{r}{L} \right)^{\frac{2-p}{p}} \left[ -f(r)dt^2 + dx_i dx^i \right] + \left( \frac{r}{L} \right)^{\frac{p^2-8p+9}{p}} \frac{dr^2}{f(r)^2}, \quad (53) $$

$$ f(r)^2 = 1 - \left( \frac{r_0}{r} \right)^{7-p}. \quad (54) $$

This is the same as (47) with the choices $a_0 = (7p - p^2)/(9 - p)$ and $a_2 = 2p/(9 - p)$. Inserting these into (49) gives

$$ v_s = \sqrt{\frac{5 - p}{9 - p}}, \quad (55) $$

which is in agreement with the result of [16].

It has been conjectured [29] that the relationship

$$ \frac{\zeta}{\eta} \geq 2 \left( \frac{1}{p} - v_s^2 \right) \quad (56) $$

should hold for a strongly coupled plasma in $p$ spatial dimensions. From our formulas, one can see that this relation is satisfied provided

$$ a_0 + a_2 \geq p. \quad (57) $$

Both special cases considered above have $a_0 + a_2 = p$, and thus saturate the conjectured bound. At first sight it appears that it would be possible to violate the conjectured bound with an appropriate choice of metric, but whether or not such a gravity dual could be embedded into string theory, and whether or not such a model would provide a reasonable description of a strongly coupled plasma should perhaps be investigated in the future.

This completes the example calculation for the single scalar field profile chosen. In the process of this example, we have generated equation (49), which is applicable to metrics which obey $F_0 = F_x$, and (47). Equations (49) are generalizations of formulas given in [16].

This is only one (particularly simple) application of the gauge invariant equations in the previous section. In the next section we increase the generality of the gauge invariant equations by allowing for multiple scalar fields.

### 6 Multiple Scalar Fields

Consider now adding additional scalar fields, so that the total number is $n$.

$$ S = \frac{1}{16\pi G_{p+2}} \int d^{p+2}x \sqrt{-g} \left( R - \frac{1}{2} \sum_{k=1}^{n} \partial_{\mu} \phi_k \partial^{\mu} \phi_k - U(\phi_1, \phi_2...\phi_n) \right). \quad (58) $$
There is now a background equation for each scalar field. As before, assume each field only depends on the radial coordinate \( r \).

\[
\Box \phi_k = \frac{1}{\sqrt{-g}} \partial_r \left[ \sqrt{-g} g^{rr} \phi'_k(r) \right] = \frac{\partial U(\phi_1, \phi_2, \ldots \phi_n)}{\partial \phi_k}.
\] (59)

Adding these all together gives

\[
\sum_{k=1}^{n} \Box \phi_k = \sum_{k=1}^{n} \frac{1}{\sqrt{-g}} \partial_r \left[ \sqrt{-g} g^{rr} \phi'_k(r) \right] = \sum_{k=1}^{n} \frac{\partial U(\phi_1, \phi_2, \ldots \phi_n)}{\partial \phi_k}.
\] (60)

In addition, the energy momentum tensor now becomes

\[
8\pi G_{p+2} T_{\mu\nu} = \frac{1}{2} \sum_{k=1}^{n} (\partial_{\mu} \phi_k \partial_{\nu} \phi_k - g_{\mu\nu} \mathcal{L}_{\phi_k})
\] (61)

\[
\mathcal{L}_{\phi_k} = \frac{1}{2} \sum_{k=1}^{n} \partial^\lambda \phi_k \partial^\nu \phi_k + U(\phi_1, \phi_2, \ldots \phi_n)
\] (62)

and the constraint equation (19) becomes

\[
\sum_{k=1}^{n} \phi'_k(r)^2 = 2g_{rr}(F_0(r) - F_r(r)).
\] (63)

It is fairly straightforward to generalize the gauge invariant equations to include more scalar fields. The linearized Einstein equations are modified due to the presence of other fields, and one needs to introduce an additional gauge invariant variable for each additional scalar field. We use the index \( i \) to denote a particular scalar field with \( i = 1, 2, \ldots n \).

\[
Z_{\phi_i}(r) = \delta \phi_i(r) - \frac{\phi'_i(r)}{D_L[g_{xx}(r)]} B(r)
\] (64)

One then gets additional terms in the differential equation for \( Z_0 \). It can be written:

\[
\frac{g_{rr}}{\sqrt{-g}} \frac{\alpha^2 f^4}{\alpha f^4} \partial_r \left[ \frac{\sqrt{-g} g^{rr}}{\alpha f^4} Z'_0 \right] + Z_0 \left( D_L[f^2] D_L[f^2 \alpha] - g_{rr} \left( w^2 g^{00} + q^2 g^{xx} \right) \right) + \\
\sum_{k=1}^{n} \left\{ 2Z_{\phi_k} \phi_k' f^2 \left( \alpha \partial_r \left( \frac{1}{\alpha} \left( \frac{w^2}{f^2} - q^2 \right) \right) + \frac{q^2 D_L[f^2]}{p D_L[g_{xx}]} D_L \left[ \sqrt{-g} g^{rr} \phi_k' \right] \right) \right\} = 0
\] (65)

There are also \( n \) gauge invariant equations, one for each of the scalar fields. Each of these equations takes the form

\[
\frac{g_{rr}}{\sqrt{-g}} \partial_r \left[ \sqrt{-g} g^{rr} Z'_{\phi_i} \right] - Z_{\phi_i} g_{rr} \left( w^2 g^{00} + q^2 g^{xx} \right) - g_{rr} \sum_{k=1}^{n} Z_{\phi_k} \frac{\partial^2 U}{\partial \phi_i \partial \phi_k} \\
- \frac{2 \phi_i'}{p D_L[g_{xx}] \alpha} \left\{ \sum_{k=1}^{n} Z_{\phi_k} \phi_k' \left( \alpha D_L \left[ \sqrt{-g} g^{rr} \phi_k' \right] + p g_{xx} D_L \left[ \frac{\phi_i'}{D_L[g_{xx}]} \right] \left( w^2 g^{00} + q^2 g^{xx} \right) \right) \right\} \\
+ \frac{2 \phi_i'}{\alpha f} \partial_r \left[ \frac{\phi_i'}{D_L[g_{xx}]} \right] \partial_r \left[ \frac{Z_0}{f} \right] = 0.
\] (66)
At first sight, this equation appears slightly different than that of a single scalar field \((35)\). There are two reasons for this. First, in the case of a single scalar field, the term \(\frac{\partial^2 U}{\partial \phi_i \partial \phi_k}\) can be written in terms of the metric only due to \((15)\); one cannot do this if the potential depends on more than one variable. Secondly, in the case of a single scalar field we removed a term which vanishes by the equations of motion, but we have not done so here. The term in question appears in \((35)\).

7 Conclusion

In this work we have presented a set of gauge invariant equations for sound mode perturbations on a generic black brane type background. The equations \((65-66)\) are the main results of this paper. These equations can be used to determine the speed of sound and bulk viscosity for any metric which can be generated by a set of minimally coupled scalar fields. In order to determine the dispersion relation, one must solve these equations perturbatively in \(q\), applying the incoming wave boundary condition at the horizon, and Dirichlet boundary condition at \(r = \infty\).

The gauge invariant equations are quite complicated, and so far a general analytic solution has eluded us, though we did present a solution for a particular class of metrics. Metrics which obey \((17)\) and \(F_0 = F_x\) have speed of sound and bulk viscosity given by \((49)\). These results are a generalization of the results of \([16]\), and include both Dp-brane, and Schwarzschild AdS black hole metrics.

In the future, we hope to report further on the possibility of an analytic solution of these equations. In the event such a solution is not possible, the equations can be solved numerically for a specified metric and set of scalar field profiles. One could also increase the generality of the gauge invariant equations by including other types of matter such as gauge fields.

We believe that the equations derived here may be useful for phenomenologically based models such as those of \([30, 31]\). It would be interesting to see what these models (which were chiefly developed to match the meson mass spectrum) have to say about hydrodynamics.

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References

[1] I. Aresene, et al. (BRAHMS Collaboration), Nucl. Phys. A757, 1 (2005); B.B. Back, et al. (PHOBOS Collaboration), Nucl. Phys. A757, 28 (2005); J. Adams, et al. (STAR Collaboration), Nucl. Phys. A A757, 102 (2005); K. Adcox, et al. (PHENIX Collaboration), Nucl. Phys. A A757, 184 (2005);

[2] D. Molnár, M. Gyulassy, Nucl. Phys. A697, 495 (2002), erratum - ibid A703, 893 (2002).

[3] P. Huovinen, P.F. Kolb, U.W. Heinz, P.V. Ruuskanen, S.A. Voloshin, Phys. Lett B503, 58 (2001); P. Huovinen in Quark-Gluon Plasma 3 eds. R.C. Hwa and X.N. Wang, World Scientific, Singapore (2004).

[4] D.T. Son, A.O. Starinets, Ann.Rev.Nucl.Part.Sci. 57, 95-118 (2007).

[5] P. Kovtun and L. G. Yaffe, Phys. Rev. D 68, 025007 (2003).

[6] J. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998).

[7] E. Witten, Adv. Theor. Math. Phys. 2, 505 (1998).

[8] S.S. Gubser, I.R. Klebanov, A.M. Polyakov, Phys. Lett. B 428, 105 (1998)

[9] G. Policastro, D. T. Son and A. O. Starinets, Phys. Rev. Lett. 87, 081601 (2001).

[10] D. T. Son and A. O. Starinets, JHEP 09, 042 (2002).

[11] G. Policastro, D. T. Son and A. O. Starinets, JHEP 09, 043 (2002).

[12] G. Policastro, D. T. Son and A. O. Starinets, JHEP 12, 054 (2002).

[13] C. P. Herzog, JHEP 0212, 26 (2002).

[14] C. P. Herzog, Phys. Rev. D 68, 024013 (2003).

[15] P.K. Kovtun and A. O. Starinets, Phys. Rev. D 72, 086009 (2005).

[16] J. Mas and J. Tarrío, JHEP 0705, 036 (2007).

[17] P. Kovtun, D. T. Son and A. O. Starinets, JHEP 10, 064 (2003).

[18] O. Saremi, arXiv:hep-th/0703170

[19] J.I. Kapusta, T. Springer, Phys. Rev. D 78, 066017 (2008)

[20] M. Natsuume and T. Okamura, Phys. Rev. D 77, 066014 (2008).

[21] N. Iqbal, H. Liu, arXiv:0809.3808 [hep-th]
A Background Ricci Tensor

An explicit computation of the Ricci tensor for our background gives

$$F_{0}(r) \equiv g^{00}R_{00} = \frac{1}{2\sqrt{-g}} \partial_{r}(\sqrt{-g}g^{rr}D_{L}[g_{00}]),$$ (67)

$$F_{x}(r) \equiv g^{xj}R_{ij} = \frac{1}{2\sqrt{-g}} \partial_{r}(\sqrt{-g}g^{rr}D_{L}[g_{xx}]),$$ (68)

$$F_{r}(r) \equiv g^{rr}R_{rr} = \frac{1}{4g_{00}'} \partial_{r} \left( g_{00}g^{rr}D_{L}[g_{00}]^{2} \right) + \frac{p}{4g_{xx}} \partial_{r} \left( g_{xx}g^{rr}D_{L}[g_{xx}]^{2} \right).$$ (69)

Here $i$ denotes the spatial coordinates $x_1...x_p$, and we are using the notation $D_{L}$ to denote the logarithmic derivative as defined in the text $[1]$. Combinations which appear frequently in the text are

$$F_{0} - F_{x} = \frac{1}{\sqrt{-g}} \partial_{r} \left( \sqrt{-g}g^{rr}D_{L}[f] \right),$$ (70)

and

$$F_{0} - F_{r} = \frac{p}{2} g^{rr}D_{L}[g_{xx}]D_{L} \left[ f \sqrt{g_{rr}} \right] \frac{D_{L}[g_{xx}]}{D_{L}[g_{xx}]},$$ (71)

where $f$ is defined as in the text $[6]$. 

13
B Linearized Perturbation Equations

Hydrodynamic behavior of the dual gauge theory is accessed by introducing perturbations on top of the black brane background. Turning on the sound mode perturbations \( \delta \phi \), and expanding the background equations of motion to first order in the perturbation results in the following set of equations. We use the superscript “1” to denote quantities that are \( \mathcal{O}(h_{\mu\nu}) \) or \( \mathcal{O}(\delta \phi) \).

\[
G_{00}^{(1)} = -8\pi G_{p+2} T_{00}^{(1)}
\]

\[
g_{rr} \frac{f}{\sqrt{-g}} \partial_r \left[ \sqrt{-g} g^{rr} f A' \right] + \frac{g_{rr}}{\sqrt{-g}} \partial_r \left[ \sqrt{-g} g^{rr} ((p - 1) B' + C') \right] + g_{rr} \frac{dU}{d\phi} (\delta \phi) + \phi' (\delta \phi)' - (p - 1) q^2 g_{rr} g^{xx} B = 0 
\]

\[
\sum_{i=1}^{p-1} G_{ii}^{(1)} = \sum_{i=1}^{p-1} \left( -8\pi G_{p+2} T_{ii}^{(1)} \right)
\]

\[
g_{rr} \frac{f}{\sqrt{-g}} \partial_r \left[ \sqrt{-g} g^{rr} f A' \right] + \frac{g_{rr}}{\sqrt{-g}} \partial_r \left[ \sqrt{-g} g^{rr} ((p - 2) B' + C') \right] + g_{rr} \frac{dU}{d\phi} (\delta \phi) + \phi' (\delta \phi)'
\]

\[
-(p - 1) w^2 g_{rr} g^{00} B = 0
\]

\[
G_{zz}^{(1)} = -8\pi G_{p+2} T_{zz}^{(1)}
\]

\[
g_{rr} \frac{f}{\sqrt{-g}} \partial_r \left[ \sqrt{-g} g^{rr} f A' \right] + \frac{g_{rr}}{\sqrt{-g}} \partial_r \left[ \sqrt{-g} g^{rr} (p - 1) B' \right] + g_{rr} \frac{dU}{d\phi} (\delta \phi) + \phi' (\delta \phi)'
\]

\[
-2 g_{rr} (F_0 - F_x) D + qw(p - 1) g_{rr} g^{00} B = 0
\]

\[
G_{0z}^{(1)} = -8\pi G_{p+2} T_{0z}^{(1)}
\]

\[
g_{xx} \frac{(d_{xx})^{p+1}}{f^2} \partial_r \left[ \frac{(d_{xx})^{p+1}}{\sqrt{-g}} \partial_r \left( f^2 D \right) \right] - 2 g_{rr} (F_0 - F_x) D + qw(p - 1) g_{rr} g^{00} B = 0
\]

\[
G_{rr}^{(1)} = -8\pi G_{p+2} T_{rr}^{(1)}
\]

\[
\mathcal{D}_L [(g_{xx})^p] A' + \mathcal{D}_L \left[ g_{00} (g_{xx})^{p-1} \right] ((p - 1) B' + C') - 2 \phi' (\delta \phi)' + 2 g_{rr} \frac{dU}{d\phi} (\delta \phi)
\]

\[
-2 g_{rr} \left( w^2 g^{00} ((p - 1) B + C) + q^2 g^{xx} (A + (p - 1) B) + 2qw g^{xx} D \right) = 0
\]

\[
G_{or}^{(1)} = -8\pi G_{p+2} T_{0r}^{(1)}
\]

\[
w f \partial_r \left[ \frac{1}{f} ((p - 1) B + C) \right] - q f^2 D' + w \phi' (\delta \phi) = 0
\]
\[ G_{zz}^{(1)} = -8\pi G_{p+2} T_{zz}^{(1)} \]

\[
\frac{q}{f} \partial_r [f A] + \frac{w}{f^2} \partial_r [f^2 D] + q(p - 1)B' + q\phi' \delta(\phi) = 0
\] (78)

\[
\Box(\phi) = \delta \frac{df}{d\phi}
\]

\[
\frac{g_{rr}}{\sqrt{-g}} \partial_r \left[ \sqrt{-g} g^{rr}(\delta \phi) \right] + \frac{1}{2} \phi' H' - g_{rr} \left( w^2 g^{00} + q^2 g^{xx} + \frac{d^2 U}{d\phi^2} \right) (\delta \phi) = 0
\] (79)

where we have defined

\[
H(r) \equiv A(r) + (p - 1)B(r) + C(r).
\] (80)

C Derivation of Gauge Invariant Equations

Here we present the explicit combination of the Einstein equations that lead to the gauge invariant equations. Consider the following combination of the Einstein equations (72 - 79):

\[
f^2 \left\{ \frac{\alpha}{p} \left[ (72) - (73) + (74) + \frac{1}{2} (76) \right] + \left[ (72) + \frac{1}{2} (76) \right] \left( \frac{w^2}{f^2} \right) - 2qw(75) \right\} + 2qf^2 D_L[\alpha](77) - w D_L[\alpha f^2](77).\]

(81)

Here \(\alpha\) is defined as in the text (33). After a long calculation, one can show that this reduces to

\[
Z_1 - 2g_{rr}(F_0 - F_x) \left( w^2 (B - C) + Z_0 \right) - Bq^2 f^2 \Delta_1 = 0
\] (82)

where \(Z_1\) is the left side of (34), and

\[
\Delta_1 = 2D_L[g_{xx}] \alpha^2 \partial_r \left[ \frac{g_{rr}}{\alpha^2 D_L[g_{xx}]^2} (F_0 - F_x) \right] - \frac{4g_{rr}}{pD_L[g_{xx}]^2} (F_0 - F_x) (\phi')^2
\]

(83)

Clearly, \(\Delta_1\) vanishes by the background equations of motion (21) and (19), leaving a differential equation involving only the gauge invariant variables, namely, (34).

To derive the other gauge invariant equation, the relevant combination is:

\[
\left(73\right) - \left(74\right) - \left(72\right) - \frac{1}{2} \left(76\right) + \frac{2}{\alpha} \partial_r \left[ \frac{\phi'}{D_L[g_{xx}]} \right] \left( \frac{w}{f^2} (77) - q(78) \right).
\] (84)

Another lengthy calculation reduces this combination to

\[
Z_2 - \frac{2}{pD_L[g_{xx}]} \left( (\phi')^2 - 2g_{rr}(F_0 - F_x) \right) D_L \left( \sqrt{-g} g^{rr}\phi' \right) Z_\phi - B\Delta_2 = 0
\] (85)
where $Z_2$ is the left side of (35), and

$$
\Delta_2 = \frac{\phi'}{pD_L[g_{xx}]^2} \left( \frac{g_{rr}}{\sqrt{-g}} \right)^2 \partial_r \left[ \left( \sqrt{-gg''} \right)^2 \left( (\phi')^2 - 2g_{rr}(F_0 - F_r) \right) \right] \\
+ \frac{2}{\alpha D_L[g_{xx}]} \partial_r \left[ \frac{\phi'}{D_L[g_{xx}]} \right] \left( (\phi')^2 - 2g_{rr}(F_0 - F_r) \right) \left( q^2 - \frac{w^2}{f^2} \right) \\
+ \frac{2g_{rr}}{D_L[g_{xx}]} \left( \frac{2q^2}{\alpha} \partial_r \left[ \frac{\phi'}{D_L[g_{xx}]} \right] + \phi' \right) (F_0 - F_x).
$$

(86)

Again, all terms except for $Z_2$ vanish by the background equations of motion, leaving us with (35).

Generalization of these equations to multiple scalar fields is quite straightforward. In this case, the Einstein equations (72 - 79) are modified, but the combinations that lead to the gauge invariant equations remain unchanged. As mentioned in the text, there will be an additional gauge invariant equation for each additional scalar field. In this case, there will be $n$ combinations like (84). Each will have (79) replaced by the analogous equation for each particular scalar field.

## D Special Case Solution

In this section, we present the remainder of the calculation which leads to equations (49). One must go back to (34), insert $Z_\phi = 0$ and the incoming wave ansatz (44), and expand the resulting equation in powers of $q$. In doing so, one should take into account the constraints (21) and (47). The first of these constraints allows one to eliminate $\sqrt{-gg''}$ in favor of $D_L[f^2]$. Once this is done, the lowest order equation for $Y_0$ can be written

$$
\frac{(C_0 - f^2a_2)^2}{a_2f^2} \partial_r \left[ \frac{1}{D_L[f^2]} \left( C_0 - f^2a_2 \right) Y'_0 \right] - \frac{D_L[f^2]}{C_0 - f^2a_2} Y_0 = 0,
$$

(87)

where

$$
C_0 = pw_1^2 - a_0.
$$

(88)

The general solution to that equation contains two arbitrary constants $k_1$ and $k_2$, and can be written

$$
Y_0(r) = (pw_1^2 - a_0 + f(r)^2a_2) \left[ k_0 + k_1 \int_r^\infty \left( \frac{pw_1^2 - a_0 - a_2f(r')^2}{pw_1^2 - a_0 + a_2f(r')^2} \right)^2 \frac{f'(r')}{f(r')} dr' \right],
$$

(89)

as can be found by first guessing a solution $Y_0(r) = b_0 + b_1 f(r)^2$, and then using the technique of reduction of order once this solution is found.

The integral in (89) is logarithmically divergent near the horizon, and thus the assumption of regularity leads to $k_1 = 0$. Finally, applying the Dirichlet boundary condition at $r \to \infty$, leads to

$$
pw_1^2 - a_0 + a_2 = 0,
$$

(90)
where we have assumed that $f(r \to \infty) = 1$. Thus, we find the lowest order term in (48).

Proceeding now to the next order in $q$, and substituting the solutions for $w_1$ and $Y_0$, one finds the following differential equation for $Y_1$:

$$\frac{\partial}{\partial r} \left[ \frac{Y'_1}{D_L[f^2](1 + f^2)^2} \right] + \frac{2ff'}{(1 + f^2)^3} \left( Y_1 - w_1 k_0 \left( \frac{ia_2}{\pi T} + 2pw_2 \right) \right) = 0. \tag{91}$$

The solution to the homogeneous part can be found using the same techniques as for the solution for $Y_0$ listed above. A particular solution to the inhomogeneous equation is obviously $Y_1(r) = \text{Constant}$. This leads to the general solution,

$$Y_1(r) = (f(r)^2 - 1) \left( k_2 + k_3 \int_r^\infty \frac{(f(r')^2 + 1)^2 f'(r')}{(f(r')^2 - 1)^2 f(r')} \, dr' \right) + w_1 k_0 \left( \frac{ia_2}{\pi T} + 2pw_2 \right). \tag{92}$$

The integral above, is again logarithmically divergent, leading to the requirement that $k_3 = 0$. Applying now the Dirichlet boundary condition at $r \to \infty$ leads to

$$w_1 k_0 \left( \frac{ia_2}{\pi T} + 2pw_2 \right) = 0. \tag{93}$$

Equation (48) immediately follows from (90) and (93).