LEVY APPROXIMATION OF IMPULSIVE RECURRENT PROCESS WITH
MARKOV SWITCHING.

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Abstract. In this paper, the weak convergence of impulsive recurrent process with Markov switching in the scheme of Lévy approximation is proved. For the relative compactness, a method proposed by R. Liptser for semimartingales is used with a modification, where we apply a solution of a singular perturbation problem instead of an ergodic theorem.

1. Introduction

Lévy approximation is still an active area of research in several theoretical and applied directions. Since Lévy processes are now standard, Lévy approximation is quite useful for analyzing complex systems (see, e.g., [1, 8]). Moreover they are involved in many applications, e.g., risk theory, finance, queueing, physics, etc. For a background on Lévy process see, e.g., [1, 8, 3].

In particular in [5] it has been studied the following impulsive process as partial sums in a series scheme

$$\xi^\varepsilon(t) = \xi_0^\varepsilon + \sum_{k=1}^{\nu(t)} \alpha_k^\varepsilon(x_{k-1}^\varepsilon), \quad t \geq 0,$$

(1)

the random variables $\alpha_k^\varepsilon(x), k \geq 1$ are supposed to be independent and perturbed by the jump Markov process $x(t), t \geq 0$.

We propose to study generalization of the problem (1):

$$\xi^\varepsilon(t) = \xi_0^\varepsilon + \sum_{k=1}^{\nu(t)} \alpha_k^\varepsilon(x_{k-1}^\varepsilon), \quad t \geq 0.$$

(2)

Here the random variables $\alpha_k^\varepsilon(u, x), k \geq 1$ depend on the process $\xi^\varepsilon(t)$.

We propose to study convergence of (2) using a combination of two methods. The method proposed by R. Liptser in [6], based on semimartingales theory, is combined with a solution of singular perturbation problem instead of ergodic theorem. So, the method includes two steps.

In the first step we prove the relative compactness of the semimartingales representation of the family $\xi^\varepsilon, \varepsilon > 0$, by proving the following two facts as proposed in Liptser [7]:

$$\lim_{\varepsilon \to 0} \sup_{\varepsilon \leq t_0} \mathbb{P}\{\sup_{t \leq T} |\xi^\varepsilon(t)| > c\} = 0,$$

known as the compact containment condition, and

$$\mathbb{E}(|\xi^\varepsilon(t) - \xi^\varepsilon(s)|^2) \leq k|t - s|,$$

for some positive constant $k$.

In the second step we prove convergence of two components of Markov process $\xi^\varepsilon(t), a^\varepsilon_t := a(t/\varepsilon^2)$ by using singular perturbation technique as presented in [5].

Finally, we apply Theorem 6.3 from [5].

The paper is organized as follows. In Section 2 we present the time-scaled impulsive process (2) and the switching Markov process. In the same section we present the main results of Lévy approximation. In Section 3 we present the proof of the theorem.
2. Main results

Let us consider the space $\mathbb{R}^d$ endowed with a norm $|\cdot|$ ($d \geq 1$), and $(E, \mathcal{E})$, a standard phase space, (i.e., $E$ is a Polish space and $\mathcal{E}$ its Borel $\sigma$-algebra). For a vector $v \in \mathbb{R}^d$ and a matrix $c \in \mathbb{R}^{d \times d}$, $v^*$ and $c^*$ denote their transpose respectively. Let $C_0(\mathbb{R}^d)$ be a measure-determining class of real-valued bounded functions, such that $g(u)/|u|^2 \to 0$, as $|u| \to 0$ for $g \in C_3(\mathbb{R}^d)$ (see [4, 5]).

We introduce a family of random sequences $\alpha_k(x), k = 1, 2, \ldots, x \in E$, where $E$ is a non-empty set, indexed by the small parameter $\varepsilon > 0$. For any $\varepsilon > 0$, and any sequence $z_k, k \geq 0$, of elements of $\mathbb{R}^d \times E$, the random variables $\alpha_k^\varepsilon(z_k-1), k \geq 1$ are supposed to be independent. Let us denote by $G_{u,x}^\varepsilon$ the distribution function of $\alpha_k^\varepsilon(x)$, that is,

$$G_{u,x}^\varepsilon(dv) := P(\alpha_k^\varepsilon(u, x) \in dv), k \geq 0, \varepsilon > 0, x \in E, u \in \mathbb{R}^d.$$

The switching Markov process $x(t), t \geq 0$ on the standard phase space $(E, \mathcal{E})$, is defined by the generator

$$Q\varphi(x) = q(x) \int_E P(x, dy)[\varphi(y) - \varphi(x)], \quad (3)$$

where $q(x), x \in E$, is the intensity of jumps function of $x(t), t \geq 0$, and $P(x, dy)$ the transition kernel of the embedded Markov chain $x_n, n \geq 0$, defined by $x_n = x(\tau_n), n \geq 0$, with $0 = \tau_0 \leq \tau_1 \leq \ldots \leq \tau_n \leq \ldots$ the jump times of $x(t), t \geq 0$. Corresponding counting processes of jumps $\nu(t) := \max\{k \geq 0 : \tau_k \leq t\}$.

We make natural assumptions for the counting process $\nu(t)$, namely:

$$\int_0^t E[\varphi(s)dv(s)] < l_1 \int_0^t E(\varphi(s))ds \tag{4}$$

for any nonnegative, increasing $\varphi(s)$ and $l_1 > 0$.

Now we define a family of jump Markov processes $x^\varepsilon(t) := x(t/\varepsilon^2), t \geq 0$, with embedded Markov renewal process $x_k^\varepsilon, \tau_k^\varepsilon, k \geq 0$, and counting processes of jumps $\nu^\varepsilon(t) = \nu(t/\varepsilon^2), t \geq 0$. Thus, times $\tau_k^\varepsilon, k \geq 0$, are jump times, $x_k^\varepsilon := x^{t/\varepsilon^2}, \tau_k^\varepsilon$, and $\nu^\varepsilon(t) := \max\{k \geq 0 : \tau_k^\varepsilon \leq t\}$.

The impulsive processes $\xi^\varepsilon(t), t \geq 0, \varepsilon > 0$ on $\mathbb{R}^d$ in the series scheme with small series parameter $\varepsilon \to 0, (\varepsilon > 0)$ are defined by the sum ([5 Section 9.2.1])

$$\xi^\varepsilon(t) = \xi_0^\varepsilon + \sum_{k=1}^{\nu(t/\varepsilon^2)} \alpha_k^\varepsilon(\xi_{k-1}^\varepsilon, x_k^\varepsilon), \quad t \geq 0. \tag{5}$$

Here

$$\xi_n^\varepsilon := \xi(\varepsilon^2 \tau_n) = \xi_0^\varepsilon + \sum_{k=1}^n \alpha_k^\varepsilon(\xi_{k-1}^\varepsilon, x_k^\varepsilon).$$

It is worth noticing that the coupled process $\xi^\varepsilon(t), x^\varepsilon(t), t \geq 0$, is a Markov additive process (see, e.g., [5 Section 2.5]).

The Skorokhod limit of Markov impulsive process ([5]) is considered under the following conditions.

C1:: The Markov process $x(t), t \geq 0$ is uniformly ergodic with $\pi(B), B \in \mathcal{E}$ as stationary distribution.

C2:: Lévy approximation. The family of impulsive processes $\xi^\varepsilon(t), t \geq 0$ satisfies the Skorokhod approximation conditions ([5 Section 7.2.3]).

L1:: Initial value condition

$$\sup_{\varepsilon > 0} E|\xi^\varepsilon_0| \leq C < \infty.$$

L2:: Approximation of the mean values:

$$a^\varepsilon(u; x) = \int_{\mathbb{R}^d} vG_{u,x}^\varepsilon(dv) = \varepsilon a_1(u; x) + \varepsilon^2 [a(u; x) + \theta^a_\varepsilon(u; x)],$$

and

$$c^\varepsilon(u; x) = \int_{\mathbb{R}^d} vG_{u,x}^\varepsilon(dv) = \varepsilon^2 [c(u; x) + \theta^c_\varepsilon(u; x)],$$

where functions $a_1, a$ and $c$ are bounded.
L3:: Poisson approximation condition for intensity kernel (see [4]),
\[ G_g^* (u; x) = \int_{\mathbb{R}^d} g(v) G_{u,x}^* (dv) = \varepsilon^2 [G_g (u; x) + \theta_g^* (u; x)] \]
for all \( g \in C_3 (\mathbb{R}^d) \), and the kernel \( G_g (u; x) \) is bounded for all \( g \in C_3 (\mathbb{R}^d) \), that is,
\[ |G_g (u; x)| \leq G_g \quad \text{(a constant depending on } g). \]

Here
\[ G_g (u; x) = \int_{\mathbb{R}^d} g(v) G_{u,x} (dv), \quad g \in C_3 (\mathbb{R}^d). \]

The above negligible terms \( \theta_u^*, \theta_{u,c}^*, \theta_g^* \) satisfy the condition
\[ \sup_{x \in E} |\theta^*_u (u; x)| \to 0, \quad \varepsilon \to 0. \]

L4:: Balance condition.
\[ \int_{E} \rho (dx) a_1 (u; x) = 0. \]

In addition the following conditions are used:

C3:: Uniform square-integrability.
\[ \lim_{\varepsilon \to \infty} \sup_{x \in E} \int_{|v| > \varepsilon} vv^* G_{u,x} (dv) = 0, \]
where the kernel \( G_{u,x} (dv) \) is defined on the measure determining class \( C_3 (\mathbb{R}^d) \) by the relation (6).

C4:: Linear growth: there exists a positive constant \( L \) such that
\[ |a (u; x)| \leq L (1 + |u|), \quad \text{and} \quad |c (u; x)| \leq L (1 + |u|^2), \]
and for any real-valued non-negative function \( f (x), x \in \mathbb{R}^d \), such that \( \int_{\mathbb{R}^d \setminus \{0\}} (1 + f (x)) |x|^2 dx < \infty \), we have
\[ |G_{u,x} (v)| \leq L f (v) (1 + |u|). \]

The main result of our work is the following.

THEOREM 1. Under conditions C1 – C4 the weak convergence
\[ \xi^* (t) \Rightarrow \xi^0 (t), \quad \varepsilon \to 0 \]
takes place.

The limit process \( \xi^0 (t), t \geq 0 \) is a Lévy process defined by the generator \( L \) as follows
\[ L \varphi (u) = (\hat{a} (u) - \hat{a}_0 (u)) \varphi' (u) + \frac{1}{2} \sigma^2 (u) \varphi'' (u) + \lambda (u) \int_{\mathbb{R}^d} [\varphi (u + v) - \varphi (u)] G_{u}^0 (dv), \quad (7) \]
with \( \sigma^2 (u) \geq 0 \), where:
\[ \hat{a} (u) = q \int_{E} \rho (dx) a (u; x), \quad \hat{a}_0 (u) = \int_{E} v G_{u} (dv), \quad G_{u} (dv) = q \int_{E} \rho (dx) G_{u,x} (dv), \]
\[ \sigma^2 (u) = 2 \int_{E} \pi (dx) [\hat{a}_1 (u; x) R_0 \hat{a}_1^* (u; x)], \quad \hat{a}_1 (u; x) := q (x) \int_{E} P (x, dy) a_1 (u; x) \]
\[ \lambda (u) = q G_{u} (\mathbb{R}^d), \quad G_{u}^0 (dv) = G_{u} (dv) / G_{u} (\mathbb{R}^d). \]
3. Proof of Theorem 1

The proof of Theorem 1 is based on the semimartingale representation of the impulsive process [5]. We split the proof of Theorem 1 in the following two steps.

Step 1. In this step we establish the relative compactness of the family of processes \( \xi^\varepsilon(t), t \geq 0, \varepsilon > 0 \) by using the approach developed in [6]. Let us remind that the space of all probability measures defined on the standard space \((E, \mathcal{E})\) is also a Polish space; so the relative compactness and tightness are equivalent.

First we need the following lemma.

**LEMMA 1.** Under assumption C4 there exists a constant \( k > 0 \), independent of \( \varepsilon \) and dependent on \( T \), such that

\[
E \sup_{t \leq T} |\xi^\varepsilon(t)|^2 \leq k_T.
\]

**COROLLARY 1.** Under assumption C4, the following compact containment condition (CCC) holds:

\[
\lim_{c \to \infty} \sup_{c \leq \varepsilon \leq c_0} \mathbb{P} \{ \sup_{t \leq T} |\xi^\varepsilon(t)| > c \} = 0.
\]

**Proof:** The proof of this corollary follows from Kolmogorov’s inequality.

\[
\square
\]

**Proof of Lemma 1:** (following [6]). The impulsive process [6] has the following semimartingale representation

\[
\xi^\varepsilon(t) = u + B^\varepsilon_t + M^\varepsilon_t,
\]

where \( u = \xi^\varepsilon_0 \); \( B^\varepsilon_t \) is the predictable drift

\[
B^\varepsilon_t = \sum_{k=1}^{\nu(t/\varepsilon^2)} a^\varepsilon(\xi^\varepsilon_{k-1}, x^\varepsilon_{k-1}) = A^\varepsilon_1(t) + A^\varepsilon(t) + \theta^\varepsilon(t),
\]

where

\[
A^\varepsilon_1(t) := \varepsilon \sum_{k=1}^{\nu(t/\varepsilon^2)} a_1(\xi^\varepsilon_{k-1}, x^\varepsilon_{k-1}), A^\varepsilon(t) := \varepsilon^2 \sum_{k=1}^{\nu(t/\varepsilon^2)} a(\xi^\varepsilon_{k-1}, x^\varepsilon_{k-1}).
\]

\[
\langle M^\varepsilon \rangle_t = \varepsilon^2 \sum_{k=1}^{\nu(t/\varepsilon^2)} c(\xi^\varepsilon_{k-1}, x^\varepsilon_{k-1}) + \varepsilon^2 \sum_{k=1}^{\nu(t/\varepsilon^2)} \int_{R^d \setminus \{0\}} \nu \nu^* G(\xi^\varepsilon_{k-1}, dv; x^\varepsilon_{k-1}) + \theta^\varepsilon(t),
\]

and for every finite \( T > 0 \)

\[
\sup_{0 \leq t \leq T} |\theta^\varepsilon(t)| \to 0, \varepsilon \to 0.
\]

To verify compactness of the process \( \xi^\varepsilon(t) \) we split it at two parts. The first part of order \( \varepsilon \)

\[
A^\varepsilon_1(t) = \varepsilon \sum_{k=1}^{\nu(t/\varepsilon^2)} a_1(\xi^\varepsilon_{k-1}, x^\varepsilon_{k-1}),
\]

can be characterized by the martingale

\[
\tilde{\mu}^\varepsilon = \varphi^\varepsilon(A^\varepsilon_1(t)) + \varphi^\varepsilon(A^\varepsilon_1(0)) - \int_0^t L^\varepsilon \varphi^\varepsilon(A^\varepsilon_1(s)) ds.
\]

Thus (see, for example Theorem 1.2 in [5]), it has quadratic characteristic

\[
< \tilde{\mu}^\varepsilon >_t = \int_0^t \left[ L^\varepsilon (\varphi^\varepsilon(A^\varepsilon_1(s)))^2 - 2 \varphi^\varepsilon(A^\varepsilon_1(s))L^\varepsilon \varphi^\varepsilon(A^\varepsilon_1(s)) \right] ds.
\]

Applying the operator \( L^\varepsilon = \varepsilon^{-2} Q + \varepsilon^{-1} A_1 \) to test-function \( \varphi^\varepsilon = \varphi + \varepsilon \varphi_1 \) (here \( A_1(u; x) \varphi(v) = a_1(u; x) \varphi(v) \)) we obtain the integrand of the view

\[
Q \varphi^2 - 2 \varphi_1 Q \varphi.
\]

It is independent of \( \varepsilon \) and limited. The boundedness of the quadratic characteristic provides \( \tilde{\mu}^\varepsilon \) is compact. Thus, \( \varphi(A^\varepsilon_1(t)) \) is compact too and bounded uniformly by \( \varepsilon \).
Now we should study the second part of order $\varepsilon^2$.
For a process $y(t), t \geq 0$, let us define the process $y^\dagger(t) = \sup_{s \leq t} |y(s)|$, then from (5) we have
\[
((\xi^\varepsilon(t))^\dagger)^2 \leq 4|a|^2 + ((A^\varepsilon(t))^\dagger)^2 + ((M^\varepsilon_s(t))^\dagger)^2.
\] (10)

Now we may apply the result of Section 2.3 [5], namely
\[
\sum_{k=1}^{n(t)} a(\xi^\varepsilon_{k-1}, x^\varepsilon_{k-1}) = \int_0^t a(\xi^\varepsilon(s), x^\varepsilon(s))d\nu(s).
\]

Condition C4 implies that for sufficiently large $\varepsilon$
\[
(A^\varepsilon(t))^\dagger = \varepsilon^2 \int_0^{t/\varepsilon^2} a(\xi^\varepsilon(s), x^\varepsilon(s))d\nu(s) \leq L\varepsilon^2 \int_0^{t/\varepsilon^2} (1 + (\xi^\varepsilon(s))^\dagger)d\nu(s)
\] (11)

Now, by Doob’s inequality (see, e.g., [7, Theorem 1.9.2]),
\[
\mathbb{E}((M^\varepsilon(t))^\dagger)^2 \leq 4\mathbb{E}(M^\varepsilon(t))^2,
\]
and condition C4 we obtain
\[
|\langle M^\varepsilon \rangle_t| = \varepsilon^2 \int_0^{t/\varepsilon^2} c(\xi^\varepsilon(s); x^\varepsilon(s)) + \varepsilon^2 \int_{\mathbb{R}^d \setminus \{0\}} vv^* G(\xi^\varepsilon(s), dv; x^\varepsilon) d\nu(s) \leq 2L(1 + r_1)\varepsilon^2 \int_0^{t/\varepsilon^2} [1 + (\xi^\varepsilon(s))^\dagger]^2 d\nu(s),
\] (12)
where $r_1 = \int_{\mathbb{R}^d \setminus \{0\}} |x|^2 f(x)dx$.

Inequalities (10)-(12), condition C4 and Cauchy-Bunyakovskv-Schwarz inequality, $(\int_0^t |\varphi(s)| ds)^2 \leq t \int_0^t \varphi^2(s) ds$, imply
\[
\mathbb{E}(\langle M^\varepsilon(t)^\dagger \rangle)^2 \leq k_1 + k_2\varepsilon^2 \int_0^{t/\varepsilon^2} \mathbb{E}((\xi^\varepsilon(s))^\dagger)^2 d\nu(s) \leq k_1 + k_2 l_1 \varepsilon^2 \int_0^{t/\varepsilon^2} \mathbb{E}(\langle M^\varepsilon(s) \rangle)^2 ds = k_1 + k_2 l_1 \int_0^t \mathbb{E}(\langle M^\varepsilon(s) \rangle)^2 ds,
\]
where $k_1, k_2$ and $l_1$ are positive constants independent of $\varepsilon$.

By Gronwall inequality (see, e.g., [2, p. 498]), we obtain
\[
\mathbb{E}(\langle M^\varepsilon(t)^\dagger \rangle)^2 \leq k_1 \exp(k_2 l_1 t).
\]

Thus, both parts of $\xi^\varepsilon(t)$ are compact and bounded, so
\[
\mathbb{E} \sup_{t \leq T} |\xi^\varepsilon(t)|^2 \leq k_T.
\]

Hence the lemma is proved. □

**Lemma 2.** Under assumption C4 there exists a constant $k > 0$, independent of $\varepsilon$ such that
\[
\mathbb{E}|\xi^\varepsilon(t) - \xi^\varepsilon(s)|^2 \leq k|t - s|.
\]

**Proof:** In the same manner with (10), we may write
\[
|\xi^\varepsilon(t) - \xi^\varepsilon(s)|^2 \leq 2|B^\varepsilon_t - B^\varepsilon_s|^2 + 2|M^\varepsilon_t - M^\varepsilon_s|^2.
\]

By using Doob’s inequality, we obtain
\[
\mathbb{E}|\xi^\varepsilon(t) - \xi^\varepsilon(s)|^2 \leq 2\mathbb{E}\{|B^\varepsilon_t - B^\varepsilon_s|^2 + 8|(M^\varepsilon)_t - (M^\varepsilon)_s|\}.
\]

Now (12) and condition C4 and assumption C4 imply
\[
|B^\varepsilon_t - B^\varepsilon_s|^2 + 8|(M^\varepsilon)_t - (M^\varepsilon)_s| \leq k_3 [1 + (\langle M^\varepsilon \rangle)^\dagger]^2 |t - s|,
\]
where $k_3$ is a positive constant independent of $\varepsilon$.

From the last inequality and Lemma 1 the desired conclusion is obtained. □
The conditions proved in Corollary 2 and Lemma 2 are necessary and sufficient for the compactness of the family of processes $\xi^\varepsilon(t), t \geq 0, \varepsilon > 0$.

Step 2. At the next step of proof we apply the problem of singular perturbation to the generator of the process $\xi^\varepsilon(t)$. To do this, we mention the following theorem. $C^2_d(\mathbb{R}^d \times E)$ is the space of real-valued twice continuously differentiable functions on the first argument, defined on $\mathbb{R}^d \times E$ and vanishing at infinity, and $C(\mathbb{R}^d \times E)$ is the space of real-valued continuous bounded functions defined on $\mathbb{R}^d \times E$.

**THEOREM 2.** (\cite[Theorem 6.3]{5}) Let the following conditions hold for a family of Markov processes $\xi^\varepsilon(t), t \geq 0, \varepsilon > 0$:

- **CD1:** There exists a family of test functions $\varphi^\varepsilon(u, x)$ in $C^2_d(\mathbb{R}^d \times E)$, such that
  $$\lim_{\varepsilon \to 0} \varphi^\varepsilon(u, x) = \varphi(u),$$
  uniformly on $u, x$.

- **CD2:** The following convergence holds
  $$\lim_{\varepsilon \to 0} L^\varepsilon \varphi^\varepsilon(u, x) = L\varphi(u),$$
  uniformly on $u, x$. The family of functions $L^\varepsilon \varphi^\varepsilon, \varepsilon > 0$ is uniformly bounded, and $L\varphi(u)$ and $L^\varepsilon \varphi^\varepsilon$ belong to $C(\mathbb{R}^d \times E)$.

- **CD3:** The quadratic characteristics of the martingales that characterize a coupled Markov process $\xi^\varepsilon(t), x^\varepsilon(t), t \geq 0, \varepsilon > 0$ have the representation $\langle \mu^\varepsilon \rangle = \int_0^t \xi^\varepsilon(s)ds$, where the random functions $\xi^\varepsilon, \varepsilon > 0$, satisfy the condition
  $$\sup_{0 \leq \varepsilon \leq T} E[\xi^\varepsilon(s)] \leq c < +\infty.$$

- **CD4:** The convergence of the initial values holds and
  $$\sup_{\varepsilon > 0} E[\xi^\varepsilon(0)] \leq C < +\infty.$$

Then the weak convergence
  $$\xi^\varepsilon(t) \Rightarrow \xi(t), \quad \varepsilon \to 0,$$
  takes place.

We consider the two component Markov process $\xi^\varepsilon(t), x^\varepsilon(t), t \geq 0$ which can be characterized by the martingale
  $$\mu^\varepsilon_t = \varphi(\xi^\varepsilon(t), x^\varepsilon_t) - \int_0^t L^\varepsilon \varphi(\xi^\varepsilon(s), x^\varepsilon_s)ds,$$
  where its generator $L^\varepsilon$ has the following representation \cite[Lemma 9.1]{5}

$$L^\varepsilon \varphi(u, x) = \varepsilon^{-2} q(x) \left[ \int_E P(x, dy) \int_{\mathbb{R}^d} G^\varepsilon_{u, y} (dz) \varphi(u + z, y) - \varphi(u, x) \right]. \quad (13)$$

By analogy with \cite[Lemma 9.2]{5} we may prove the following result:

**LEMMA 3.** The main part in the asymptotic representation of the generator \((13)\) is as follows

$$L^\varepsilon \varphi(u, v, x) = \varepsilon^{-2} Q \varphi(\cdot, \cdot, x) + \varepsilon^{-1} Q_0 a_1(u; x) \varphi'_{u}(u, \cdot, \cdot) + Q_0[a(u; x) - a_0(u; x)] \varphi'(u, \cdot, \cdot) + Q_0 G_{u, x} \varphi(u, \cdot, \cdot).$$

where:

$$Q_0 \varphi(x) := q(x) \int_E P(x, dy) \varphi(y), G_{u, x} \varphi(u) := \int_{\mathbb{R}^d} [\varphi(u + v) - \varphi(u)] G_{u, x}(dv),$$

$$a_0(u; x) = \int_E v G_{u, x}(dv).$$
Proof of this Lemma is analogical to the proof of [5, Lemma 9.2].

The solution of the singular perturbation problem at the test functions \( \varphi^\varepsilon(u, x) = \varphi(u) + \varepsilon \varphi_1(u, x) + \varepsilon^2 \varphi_2(u, x) \) in the form \( L^\varepsilon \varphi^\varepsilon = L \varphi + \theta \varphi \) can be found in the same manner with Lemma 9.3 in [5]. That is

\[
L = \Pi(Q_0(A(x) + G_{u,x}) + Q_0A_1(x)R_0Q_0A_1(x))\Pi,
\]

where

\[
A(x)\varphi(u) := [a(u; x) - a_0(u; x)]\varphi'(u),
\]

\[
A_1(x)\varphi(u) := a_1(u; x)\varphi'(u).
\]

Simple calculations give us (7) from (14).

Now Theorem 2 can be applied.

We see from (13) and (14) that the solution of singular perturbation problem for \( L^\varepsilon \varphi^\varepsilon(u, v; x) \) satisfies the conditions CD1, CD2. Condition CD3 of this theorem implies that the quadratic characteristics of the martingale, corresponding to a coupled Markov process, is relatively compact. The same result follows from the CCC (see Corollary 2 and Lemma 2) by [4]. Thus, the condition CD3 follows from the Corollary 2 and Lemma 2. As soon as \( \xi^\varepsilon(0) = \xi^0(0) \) we see that the condition CD4 is also satisfied. Thus, all the conditions of above Theorem 2 are satisfied, so the weak convergence \( \xi^\varepsilon(t) \to \xi^0(t) \) takes place.

Theorem 1 is proved.

\[\square\]

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