STRICTLY CONVEX WULFF SHAPES
AND $C^1$ CONVEX INTEGRANDS

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Abstract. Let $\gamma : S^n \to \mathbb{R}_+$ be a continuous function and let $\mathcal{W}_\gamma$ be the Wulff shape associated with $\gamma$. We show that Wulff shape $\mathcal{W}_\gamma$ is strictly convex if and only if convex integrand of $\mathcal{W}_\gamma$ is of class $C^1$. We also show that if the boundary of $\mathcal{W}_\gamma$ is a $C^1$ submanifold, then $\gamma$ must be the convex integrand of $\mathcal{W}_\gamma$.

1. Introduction

Let $n$ be a positive integer. Given a continuous function $\gamma : S^n \to \mathbb{R}_+$ where $S^n \subset \mathbb{R}^{n+1}$ is the unit sphere and $\mathbb{R}_+$ is the set consisting of positive real numbers, the Wulff shape associated with $\gamma$, denoted by $\mathcal{W}_\gamma$, is the following intersection

$$\mathcal{W}_\gamma = \bigcap_{\theta \in S^n} \Gamma_{\gamma,\theta}.$$ 

Here, $\Gamma_{\gamma,\theta}$ is the following half-space:

$$\Gamma_{\gamma,\theta} = \{x \in \mathbb{R}^{n+1} \mid x \cdot \theta \leq \gamma(\theta)\}.$$ 

Figure 1. A Wulff shape $\mathcal{W}_\gamma$.

By definition, the Wulff shape $\mathcal{W}_\gamma$ is a convex body such that the origin of $\mathbb{R}^{n+1}$ is an interior point of $\mathcal{W}_\gamma$. The notion of Wulff shape was first introduced by G. Wulff in [9]. Let $Id : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \times \{1\}$ be the map defined by $Id(x) = (x, 1)$. Denote the point $(0, \ldots, 0, 1) \in \mathbb{R}^{n+2}$ by $N$. The set $S^n \cap H(-N)$ is denoted by $N$.

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$S_{N,+}^{n+1}$. Let $\alpha_N : S_{N,+}^{n+1} \to \mathbb{R}^{n+1} \times \{1\}$ be the central projection relative to $N$, namely, $\alpha_N$ is defined as follows for any $P = (P_1, \ldots, P_{n+1}, P_{n+2}) \in S_{N,+}^{n+1}$ (see Figure 2):

$$\alpha_N(P_1, \ldots, P_{n+1}, P_{n+2}) = \left(\frac{P_1}{P_{n+2}}, \ldots, \frac{P_{n+1}}{P_{n+2}}, 1\right).$$

![Diagram of central projection $\alpha_N$.](image)

**Figure 2.** The central projection $\alpha_N$.

Next, we consider the mapping $\Psi_N : S^{n+1} - \{\pm N\} \to S_{N,+}^{n+1}$ (see Figure 3), defined by

$$\Psi_N(P) = \frac{1}{\sqrt{1 - (N \cdot P)^2}}(N - (N \cdot P)P).$$

The mapping $\Psi_N$ was introduced in [5], has the following intriguing properties:

![Diagram of mapping $\Psi_N$.](image)

**Figure 3.** $P \cdot \Psi_N(P) = 0$.

(1) For any $P \in S^{n+1} - \{\pm N\}$, the equality $P \cdot \Psi_N(P) = 0$ holds,

(2) for any $P \in S^{n+1} - \{\pm N\}$, the property $\Psi_N(P) \in \mathbb{R}N + \mathbb{R}P$ holds,

(3) for any $P \in S^{n+1} - \{\pm N\}$, the property $N \cdot \Psi_N(P) > 0$ holds,

(4) the restriction $\Psi_N|_{S_{N,+}^{n+1} - \{N\}} : S_{N,+}^{n+1} - \{N\} \to S_{N,+}^{n+1} - \{N\}$ is a $C^\infty$ diffeomorphism.

For any point $P \in S^{n+1}$, let $H(P)$ be the closed hemisphere centered at $P$, namely,

$$H(P) = \{Q \in S^{n+1} | P \cdot Q \geq 0\},$$
where the dot in the center stands for the scalar product of two vectors \( P, Q \in \mathbb{R}^{n+2} \).

For any non-empty subset \( \hat{W} \subset S^{n+1} \), the spherical polar set of \( \hat{W} \), denoted by \( \hat{W}^\circ \), is defined as follows:

\[
\hat{W}^\circ = \bigcap_{P \in \hat{W}} H(P).
\]

for details on spherical polar set, see for instance [1, 6]

**Proposition 1** ([6]). Let \( \gamma : S^n \to \mathbb{R}_+ \) be a continuous function. Let \( \text{graph}(\gamma) = \{ (\theta, \gamma(\theta)) \in \mathbb{R}^{n+1} - \{0\} \mid \theta \in S^n \} \), where \((\theta, \gamma(\theta))\) is the polar plot expression for a point of \( \mathbb{R}^{n+1} - \{0\} \). Then, \( \mathcal{W}_\gamma \) is characterized as follows:

\[
\mathcal{W}_\gamma = \text{Id}^{-1} \circ \alpha_N \left( \left( \Psi_N \circ \alpha_N^{-1} \circ \text{Id}(\text{graph}(\gamma)) \right)^\circ \right).
\]

**Proposition 2** ([6]). For any Wulff shape \( \mathcal{W}_\gamma \), the following set, too, is a Wulff shape:

\[
\text{Id}^{-1} \circ \alpha_N \left( \left( \alpha_N^{-1} \circ \text{Id}(\mathcal{W}_\gamma) \right)^\circ \right).
\]

**Definition 1** ([6]). Let \( \mathcal{W}_\gamma \) be a Wulff shape. The Wulff shape given in Proposition 2 is called the dual Wulff shape of \( \mathcal{W}_\gamma \).

A Wulff shape \( \mathcal{W}_\gamma \) said to be self-dual Wulff shape if the equality \( \mathcal{W}_\gamma = \text{Id}^{-1} \circ \alpha_N \left( \left( \alpha_N^{-1} \circ \text{Id}(\mathcal{W}_\gamma) \right)^\circ \right) \) holds, for details on self-dual Wulff shapes, see for instance [4].

The mapping \( \text{inv} : \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}^{n+1} - \{0\} \), defined as follows, is called the inversion with respect to the origin of \( \mathbb{R}^{n+1} \).

\[
\text{inv}(\theta, r) = \left( -\theta, \frac{1}{r} \right).
\]

Let \( \Gamma_\gamma \) be the boundary of the convex hull of \( \text{inv}(\text{graph}(\gamma)) \). If the equality \( \Gamma_\gamma = \text{inv}(\text{graph}(\gamma)) \) is satisfied, then \( \gamma \) is called a convex integrand. The notion of convex integrand was firstly introduced by J. Taylor in [8].

2. Main Results

**Theorem 1** ([2]). Let \( W \subset \mathbb{R}^{n+1} \) be a convex body containing the origin of \( \mathbb{R}^{n+1} \) as an interior point of \( W \). Then, \( W \) is strictly convex if and only if its convex integrand \( \gamma_W \) is of class \( C^1 \).

**Theorem 2** ([3]). Let \( \gamma : S^n \to \mathbb{R}_+ \) be a continuous function and let \( \mathcal{W}_\gamma \) be the Wulff shape associated with \( \gamma \). Suppose that the boundary of \( \mathcal{W}_\gamma \) is a \( C^1 \) submanifold. Then, \( \gamma \) must be the convex integrand of \( \mathcal{W}_\gamma \).

3. Applications of Theorem 1

Since the boundary of the convex hull of a \( C^1 \) closed submanifold is a \( C^1 \) closed submanifold (for instance, see [7, 10]), as a corollary of Theorem 1, we have the following:

**Corollary 1** ([2]). Let \( \gamma : S^n \to \mathbb{R}_+ \) be a function of class \( C^1 \). Then, \( \mathcal{W}_\gamma \) is strictly convex.

In particular, we have the following:
Corollary 2 ([6], Theorem 1.3). Let $\gamma : S^n \to \mathbb{R}_+$ be a function of class $C^1$. Then, $W_\gamma$ is never a polytope.

On the other hand, the converse of Corollary 1 does not hold in general (see Figure 4).

Combining Theorem 1 and Proposition 1 yields the following:

**Corollary 3 ([2]).** A Wulff shape in $\mathbb{R}^{n+1}$ is strictly convex if and only if the boundary of its dual Wulff shape is $C^1$ diffeomorphic to $S^n$.

In particular, we have the following:

**Corollary 4 ([2]).** A Wulff shape in $\mathbb{R}^{n+1}$ is strictly convex and its boundary is $C^1$ diffeomorphic to $S^n$ if and only if its dual Wulff shape is strictly convex and the boundary of it is $C^1$ diffeomorphic to $S^n$.

It is interesting to compare Corollary 4 and the following proposition:

**Proposition 3 ([6]).** A Wulff shape in $\mathbb{R}^{n+1}$ is a polytope if and only if its dual Wulff shape is a polytope.

Finally, we give an application of Theorem 1 from the view point of pedal.

**Definition 2 ([2]).** Let $p$ (resp., $F : S^n \to \mathbb{R}^{n+1}$) be a point of $\mathbb{R}^{n+1}$ (resp., a $C^1$ embedding). Then, the pedal of $F(S^n)$ relative to $p$ is the mapping $G : S^n \to \mathbb{R}^{n+1}$ which maps $\theta \in S^n$ to the nearest point in the tangent hyperplane to $F(S^n)$ at $F(\theta)$ from the given point $p$.

Let $W$ be a Wulff shape in $\mathbb{R}^{n+1}$. Suppose that $\partial W$ is $C^1$ diffeomorphic to $S^n$. Then, $\partial W$ may be regarded as the graph of a certain $C^1$ embedding $F : S^n \to \mathbb{R}^{n+1}$, and $\gamma_\partial$ is exactly the pedal of $\partial W$ relative to the origin. Theorem 1 gives a sufficient condition for the pedal of $\partial W$ relative to the origin to be smooth:

**Corollary 5 ([2]).** Suppose that a Wulff shape $W$ in $\mathbb{R}^{n+1}$ is strictly convex and its boundary is $C^1$ diffeomorphic to $S^n$. Then, the pedal of $\partial W$ relative to the any interior point of $W$ is of class $C^1$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig4}
\caption{A strictly convex Wulff shape $W_\gamma$ having non smooth support function $\gamma$.}
\end{figure}
References

[1] H. Han and T. Nishimura, *The spherical dual transform is an isometry for spherical Wulff shapes*, preprint (available from arXiv:1504.02845 [math.MG]).

[2] H. Han and T. Nishimura, *Strictly convex Wulff shapes and $C^1$ convex integrands*, preprint (available from arXiv:1507.05162 [math.MG]).

[3] H. Han and T. Nishimura, *Uniqueness of the surface energy density for a Wulff shape with $C^1$ boundary*, preprint (available from arXiv:1509.02786 [math.MG]).

[4] H. Han and T. Nishimura, *Self-dual Wulff shapes and spherical convex bodies of constant width $\pi/2$*, preprint (available from arXiv:1511.04165 [math.MG]).

[5] T. Nishimura, *Normal forms for singularities of pedal curves produced by non-singular dual curve germs in $S^n$*, Geom Dedicata 133(2008), 59–66.

[6] T. Nishimura and Y. Sakemi, *Topological aspect of Wulff shapes*, J. Math. Soc. Japan, 66 (2014), 89–109.

[7] S. A. Robertson and M. C. Romero-Fuster, *The convex hull of a hypersurface*, Proc. London Math. Soc., 50(1985), 370–384.

[8] J. E. Taylor, *Crystalline variational problems*, Bull. Amer. Math. Soc., 84(1978), 568–588.

[9] G. Wulff, *Zur frage der geschwindigkeits der wachstrums und der auflösung der krysalflachen*, Z. Kristallographine und Mineralogie, 34(1901), 449–530.

[10] V. M. Zakalyukin, *Singularities of convex hulls of smooth manifolds*, Functional Anal. Appl., 11(1977), 225–227(1978).

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