Negative refraction of inhomogeneous waves in lossy isotropic media

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Abstract

We theoretically study negative refraction of inhomogeneous waves at the interface of lossy isotropic media. We obtain explicit (up to the sign) expressions for the parameters of a wave transmitted through the interface between two lossy media characterized by complex permittivity and permeability. We show that the criterion of negative refraction that requires negative permittivity and permeability can be used only in the case of a homogeneous incident wave at the interface between a lossless and lossy media. In a more general situation, when the incident wave is inhomogeneous, or both media are lossy, the criterion of negative refraction becomes dependent on an incidence angle. Most interestingly, we show that negative refraction can be realized in conventional lossy materials (such as metals) if their interfaces are properly oriented.

Keywords: negative refraction, inhomogeneous waves, optics of lossy media

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1. Introduction

Negative refraction is a very interesting phenomenon which nowadays attracts a lot of attention. Intuitively, negative refraction can be imagined as the phenomenon at the interface between two media when the refracted and incident beams of light remain on the same side of the interface normal. Most frequently, the work by Veselago [1] is cited as the one which represents the early efforts on studies of negative refraction. For a brief history of related studies performed before [1], the reader may refer, for example, to [2].

It is quite often believed that negative refraction appears at the interface of artificial media, such as metamaterials [3–5] or photonic crystals [6–8]. However, negative refraction has also been observed at the interfaces of conventional media, for example, anisotropic materials [9–11], or metallic wedges [12, 13].

The work by Veselago [1] considers the most straightforward way to obtain negative refraction. According to Veselago, if a medium has both negative permittivity, \( \varepsilon < 0 \), and negative permeability, \( \mu < 0 \), the resulting refractive index is also negative, \( n = -\sqrt{\varepsilon \mu} \), thus providing negative refraction at the interface with a medium whose index \( n \) is positive. This idea is utilized in metamaterials, where one uses the overlapping of permittivity and permeability resonances to achieve simultaneously negative \( \varepsilon \) and \( \mu \) [14]. The obvious disadvantage of such an approach is that, as with any resonant media, metamaterials are highly absorptive. Therefore, theoretical studies of negative refraction in metamaterials should consider the propagation of light in lossy media.

From many standard textbooks (for example, [15, 16]) we know that, unlike lossless media, the waves transmitted into lossy media are, in general, inhomogeneous (or nonuniform), that is, their planes of constant phase and constant amplitude are not parallel. Thus as a first step to understand negative refraction in metamaterials we need to consider the formation of inhomogeneous waves at the interface of lossy media, characterized by complex permittivity and permeability.

Inhomogeneous waves in conventional conducting media (with complex \( \varepsilon \) and \( \mu = 1 \)) have been thoroughly studied and discussed in many classical textbooks. Some generalizations of the classical theory were made in [17, 18]. However, the classical theory of inhomogeneous waves does not consider negative refraction. The authors of [19] made an attempt...
to apply the formalism of inhomogeneous waves in order to explain the results of numerical simulations of negative refraction in a metamaterial prism and slab. Unfortunately, they used the formulas of the classical theory, where they formally substituted the complex refractive index of the metamaterial by a negative value $n$, defined as $n^2 = \varepsilon \mu$. However, in an inhomogeneous wave propagating in a lossy medium, the actual refractive index and attenuation coefficient depend on the incidence angle and, therefore, the value $n$ loses its meaning of a complex refractive index [15, 16]. This means that the criterion of negative refraction proposed by Veselago [1] and based on negative $n$ should be reconsidered.

In order to extend the classical theory of inhomogeneous waves, in this work we study negative refraction of inhomogeneous waves at the interface of two lossy isotropic media. Unlike the previous authors [17, 18] we use a parametrization of inhomogeneous waves which allows us to find explicitly all the necessary parameters of transmitted waves, up to their signs. In [17], to obtain these parameters, one needs to find a unique solution of a system of four nonlinear equations with multivalued inverse trigonometric and hyperbolic functions. In [18], the parameters of transmitted waves are given only in a quadratic form. Additionally, unlike [17, 18], we consider a more general problem, where both media at the interface are, in addition to a complex permittivity $\varepsilon$, characterized by a complex permeability $\mu$.

After the formal derivation of all the necessary equations, we show that the inhomogeneity of transmitted waves affects negative refraction. We demonstrate that the recently reported criterion of negative refraction that requires negative real parts of $\varepsilon$ and $\mu$ [14] is applicable only in the case of a homogeneous incident wave at the interface between a lossless and lossy media. In a more general situation, when the incident wave is inhomogeneous, or both media are lossy, the criterion of negative refraction becomes dependent on an incidence angle. In particular, we show that negative refraction can be realized in conventional lossy materials if their interfaces are properly oriented.

### 2. General properties of inhomogeneous waves

Let us consider a plane monochromatic wave propagating in a lossy isotropic medium with complex permittivity $\varepsilon = \varepsilon' - i\varepsilon''$ and permeability $\mu = \mu' - i\mu''$. The electric, $E$, and magnetic, $H$, field vectors of such a wave are given by

$$E = e e^{j(\omega t - k \cdot r)}, \quad H = h e^{j(\omega t - k \cdot r)},$$

where $e$ and $h$ are complex amplitude vectors, $k$ and $r$ are the wavevector and a position vector, respectively, with $\omega$ and $t$ being the frequency and time. In a lossy medium the wavevector $k$ is complex, that is, $k = k' - ik''$, where $k'$ and $k''$ are real phase and attenuation vectors, respectively. If the vectors $k'$ and $k''$ are not parallel, the planes of constant phase and constant amplitude (defined as $k' \cdot r = \text{const}$ and $k'' \cdot r = \text{const}$, respectively) are not parallel either, and the corresponding wave is inhomogeneous.

We can formally define the refractive index $m'$ and attenuation coefficient $m''$ of an inhomogeneous wave through the lengths of $k'$ and $k''$ as $|k'| = m'\omega/c_0$ and $|k''| = m''\omega/c_0$, where $c_0$ is the speed of light in vacuum. Such a definition retains the physical meaning of the refractive index and attenuation coefficient; namely, $m'$ equals the ratio of $c_0$ to the phase velocity and $m''$ determines the distance at which the amplitude is reduced to $1/e$ times. Note that both $m'$ and $m''$ are real and positive by definition, since we determine them through the length of a real vector.

By separately equating the real and imaginary parts in the dispersion equation $k^2 = \varepsilon \mu \omega^2/c_0^2 = n^2\omega^2/c_0^2$, where the complex value $n = n' + in''$ is defined as $n^2 = \varepsilon \mu$, we find [15, 16]

$$m'^2 - m''^2 = \varepsilon' \mu' - \varepsilon'' \mu'' = n^2 - n''^2, \quad (2a)$$

$$m' m'' \cos \vartheta = \frac{1}{2}(\varepsilon' \mu'' + \varepsilon'' \mu') = n'n'', \quad (2b)$$

in which $\vartheta$ is the angle between $k'$ and $k''$. Note that inhomogeneous waves with $m'' \neq 0$ can exist in lossless media ($\varepsilon'' = \mu'' = 0$) if $\vartheta = 90^\circ$ in (2b); that is, if $k' \perp k''$. For example, such a kind of inhomogeneous wave describes total internal reflection, where a wave at the boundary of two lossless media propagates along the boundary and attenuates in the direction perpendicular to the interface. Additionally, (2a) and (2b) indicate that $n' = m'$ and $n'' = m''$ only if $\vartheta = 0^\circ$; that is, if $k' \parallel k''$ (homogeneous damped waves). Thus the complex quantity $n$ loses its traditional meaning of refractive index for inhomogeneous waves ($\vartheta \neq 0^\circ$). Therefore, a natural question occurs: 'How should we reinterpret the criterion of negative refraction that is commonly believed to be $n < 0$?'

### 3. Generalized laws of reflection and refraction

To answer this question, we examine the formation of inhomogeneous waves at the interface of two lossy isotropic media: the first medium is characterized by a complex permittivity $\varepsilon_1 = \varepsilon_1' - i\varepsilon_1''$ and permeability $\mu_1 = \mu_1' - i\mu_1''$, and the second medium by $\varepsilon_2 = \varepsilon_2' - i\varepsilon_2''$ and $\mu_2 = \mu_2' - i\mu_2''$. The incident wave comes from the first medium, and a unit interface normal $\hat{q}$ points to the second medium (see figure 1).
The wavevectors $k_i$, $k_t$, and $k_r$ of the incident, reflected and transmitted waves can be written as the sum of two vectors that are parallel and perpendicular to the interface [15, 16]:

$$ k_i = p + q \hat{q}, \quad k_t = p + q' \hat{q'}, \quad k_r = p + q \hat{q}. \quad (3) $$

The parallel components $p$ are continuous across the interface:

$$ p = [\hat{q} \times [k_i \times \hat{q}]] = [\hat{q} \times [k_r \times \hat{q}]] = [\hat{q} \times [k_t \times \hat{q}]]. \quad (4) $$

The normal components have magnitudes $q_t = (k_t \cdot \hat{q})$ and $q_r = (k_r \cdot \hat{q})$, where

$$ q_t = -q_t, \quad q_r^2 = \varepsilon_2 \mu_2 \omega^2/c_0^2 - p^2. \quad (5) $$

For the most general case, all three wavevectors $k_{\alpha}$ (with $\alpha = i, r, t$) are complex, that is, $k_{\alpha} = k'_{\alpha} - ik''_{\alpha}$. Therefore,

$$ p = p' - ip'' \text{ and } q_{\alpha} = q'_{\alpha} - iq''_{\alpha}, $$

where

$$ p' = [\hat{q} \times [k'_i \times \hat{q}]], \quad q'_{\alpha} = (k'_t \cdot \hat{q}), \quad q''_{\alpha} = (k''_t \cdot \hat{q}). \quad (6a) $$

By separately equating the real and imaginary parts in (3), we obtain two sets of equations for the phase vectors $k'_{\alpha}$ and attenuation vectors $k''_{\alpha}$:

$$ k'_i = p' + q'_i \hat{q}, \quad k''_i = p'' + q''_i \hat{q}, \quad (7a) $$

$$ k'_t = p' + q'_t \hat{q}, \quad k''_t = p'' + q''_t \hat{q}, \quad (7b) $$

$$ k'_r = p' + q'_r \hat{q}, \quad k''_r = p'' + q''_r \hat{q}. \quad (7c) $$

According to (7a)–(7c) the phase vectors $k'_{\alpha}$ and attenuation vectors $k''_{\alpha}$, in general, lie in two different planes (see figure 1). In other words, at the plane interface of two lossy isotropic media, there are, in general, two incidence planes: the incidence plane for the phase vectors (spanned by $k'_i$ and $\hat{q}$ with the normal $s' = [k'_i \times \hat{q}]$) and the incidence plane for the attenuation vectors (spanned by $k''_i$ and $\hat{q}$ with the normal $s'' = [k''_i \times \hat{q}]$) [18].

Two incidence planes result in two sets of equalities for Snell’s law, for the phase and attenuation vectors, respectively. We find them by equating the magnitudes of the real and imaginary parts in (4) and using the definitions of $m'$ and $m''$

$$ m' \sin \theta' = m'' \sin \theta'' = m'_i \sin \theta'_i, $$

$$ m'' \sin \theta'' = m'_i \sin \theta'_i. \quad (8a) $$

where $\theta'_i$ and $\theta''_i$ are the angles between the unit normal $\hat{q}$ and the associated phase vectors $k'_{\alpha}$ and attenuation vectors $k''_{\alpha}$, respectively.

Before we focus on the transmitted wave, we make a simple remark on the reflected wave. The reflected wave has the same inhomogeneity as that of the incident wave, so that $m'_i = m'_r$ and $m''_i = m''_r$ (the reflection angles are $\theta'_i = \pi - \theta'_r$ and $\theta''_i = \pi - \theta''_r$). This implies that if the first medium is lossless and the incident wave is homogeneous, then the reflected wave is also homogeneous, even if the second medium is lossy [15, 16].

Now we examine the transmitted wave whose phase and attenuation vectors are given by (7c). In (7c) we need to determine the projections $q'_t$ and $q''_t$, which are the real and imaginary parts of the complex projection $q_t = q'_t - iq''_t$. Note that we cannot uniquely determine $q_t$ using (5), since $q_t$ is in the quadratic form. Standard textbooks choose a positive sign for $q_t$, or equivalently $q'_t = (k'_t \cdot \hat{q}) > 0$ and $q''_t = (k''_t \cdot \hat{q}) > 0$, because the phase vector, and consequently the phase velocity, are outgoing from the interface only for this choice of sign. We now know, however, that the phase velocity can be incoming towards the interface if negative refraction takes place. Thus a careful study is necessary to unambiguously determine the sign of $q_t$. For this purpose we introduce a complex dimensionless parameter $\xi = \xi' - i\xi''$, defined as

$$ \xi = q'_t/c_0^2. \quad (9) $$

For its real and imaginary parts, we find $\xi' = (q'_t - q''_t)/c_0^2/\omega^2$ and $\xi'' = 2q'_t q''_t/c_0^2/\omega^2$. Therefore, taking into account that $q'_t$ and $q''_t$ are real by definition, we obtain $q'_t = (|\xi| + \xi')/\omega^2/2c_0^2$ and $q''_t = (|\xi| - \xi')/\omega^2/2c_0^2$, where $|\xi| = \sqrt{\xi'^2 + \xi''^2}$. To find the signs of $q'_t$ and $q''_t$ we express them as

$$ q'_t = s' \alpha \sqrt{(|\xi| + \xi')/\omega^2}, \quad q''_t = s'' \alpha \sqrt{(|\xi| - \xi')/\omega^2}, \quad (10) $$

where $s' = \pm 1$ and $s'' = \pm 1$ are their signs. Then, since $\xi'' < 0$, specific values of $s'$ and $s''$ can be selected only by applying an additional constraint: that the energy flux in the second medium must be directed away from the interface. Mathematically we write this constraint as $P_i \cdot \hat{q} \geq 0$, meaning that the projection of the time-averaged Poynting vector $P_i$ of the transmitted wave on the interface normal must be non-negative.

To calculate $q'_t$ and $q''_t$ by (10), we need to express $\xi'$ and $\xi''$ through some known values such as permittivities, permeabilities, incidence angles, etc. To do so we substitute (5) into (9) and then separate the real and imaginary parts. As a result we obtain

$$ \xi' = (\varepsilon'_2 \mu'_2 - \varepsilon''_2 \mu'_2) - (p'^2 - p''^2)/c_0^2, \quad (12a) $$

$$ \xi'' = (\varepsilon'_2 \mu'_2 + \varepsilon''_2 \mu'_2) - 2(p' \cdot p'')/c_0^2. \quad (12b) $$

Using (6a), we express the terms with $p'$ and $p''$ in (12a) and (12b) through the parameters of the incident wave:

$$ p'^2 = m'^2 \omega^2 \sin^2 \theta'_i, \quad p''^2 = m''^2 \omega^2 \sin^2 \theta''_i, \quad (13a) $$

$$ p' \cdot p'' = m'_i m''_i (\cos \theta_i - \cos \theta'_i \cos \theta''_i), \quad (13b) $$

where the angle $\theta_i$ between $k'_i$ and $k''_i$ can be found from (2b) for the incident wave. Then we obtain

$$ \xi' = (\varepsilon'_2 \mu'_2 - \varepsilon''_2 \mu'_2) - (m'^2 \sin^2 \theta'_i - m''^2 \sin^2 \theta''_i), \quad (14a) $$

$$ \xi'' = (\varepsilon'_2 \mu'_2 + \varepsilon''_2 \mu'_2) - 2m'_i m''_i (\cos \theta_i - \cos \theta'_i \cos \theta''_i). \quad (14b) $$
Having clarified the sign issue of the complex projection $q_t$, we now turn to the refractive index $m'_t$ and attenuation coefficient $m''_t$. From (7c) and (10) we have $k''_m = p''_t + q''_t = \frac{1}{2}(|\xi| + \xi' + 2p''_t c_0^2 / \omega^2 c_0^2)$, which is equal to $k''_m = m''_t c_0^2 / c_0^2$. Therefore, we obtain $m'_t = \sqrt{|(\xi| + \xi' + 2p''_t c_0^2 / \omega^2 c_0^2 / 2)^{1/2}}$, where we have chosen the ‘+’ sign since $m'_t$ is positive by definition. Similarly, we obtain $m''_t = \sqrt{|(|\xi| - \xi' + 2p''_t c_0^2 / \omega^2 c_0^2 / 2)^{1/2}}$. With the help of (13a), we find that $m'_t$ and $m''_t$ can be written as

$$m'_t = \sqrt{(|\xi| + \xi' + 2m''_t \sin^2 \theta'_t / 2)}$$

$$m''_t = \sqrt{(|\xi| - \xi' + 2m''_t \sin^2 \theta''_t / 2)}$$

Equations (15a) and (15b) clearly show that the refractive index $m'_t$ and attenuation coefficient $m''_t$ depend not only on the material properties of the second medium, but also on the incidence angles $\theta'_t$ and $\theta''_t$.

To summarize the results of this section, we describe how one can find the parameters of the transmitted wave if the corresponding parameters of the incident wave are known: (i) calculate $\xi'$ and $\xi''$ from (14a) and (14b) ($\theta_t$ can be found from (2b)); (ii) find the refractive index $m'_t$ and attenuation coefficient $m''_t$ from (15a) and (15b); (iii) with the obtained $m'_t$ and $m''_t$ calculate the transmission angles $\theta'_t$ and $\theta''_t$ using (8a) and (8b); (iv) find the signs of $q'_t$ and $q''_t$ in (10), utilizing (11) with the additional constraint $P'_t \cdot \hat{q} \geq 0$, where the time-averaged Poynting vector of the transmitted wave, $P'_t$, should be calculated separately.

Before concluding this section, we wish to say a few words about the limitations of the model. It is known that, at optical frequencies, the permeability $\mu(\omega)$ loses its usual physical meaning [20]. To overcome this problem the authors of [21, 22] proposed describing the linear response of a negatively dispersive medium by a spatially dispersive permittivity tensor $\varepsilon_{ij}(\omega, k)$. However, they showed that, as long as the spatial dispersion is restricted to terms $\propto k^2$, the formal description of a medium by $\varepsilon(\omega)$ and $\mu(\omega)$ is adequate, if an effective permeability is introduced, even if this permeability loses its original physical meaning. Thus our model implies a weak spatial dispersion and effective permeabilities $\mu_1$ and $\mu_2$ that do not have a direct physical meaning. In addition, we note that the spatial dispersion is formally present in our result, because the obtained refractive index and attenuation coefficient depend on an incidence angle (see (15a) and (15b)) and, consequently, on a wavevector of the incident wave.

4. Specific examples of interface problems

4.1. Single interface

As a first example, we consider a single interface where the first medium is a lossless dielectric ($\varepsilon'_1 = \mu'_1 = 0$) and the second medium is lossy and isotropic (see figure 2). We further assume that the incident wave is a homogeneous plane wave, which is characterized by the real wavevector $k'_t = k'_1$ ($k'_1 = 0$ and $m''_1 = 0$), refractive index $m'_1 = \sqrt{\varepsilon'_1 \mu'_1} = n_1$, and incidence angle $\theta'_1 = \theta_t$. According to (6a) and (7c), the condition $k''_1 = 0$ means that $p''_t = 0$ and $k''_2 = q''_1 \hat{q}$. As a consequence the transmission angle $\theta''_1 = 0$. Thus the attenuation vector $k''_2$ of the transmitted wave is normal to the interface for any incidence angle. In other words, equi-amplitude planes of the transmitted wave are always parallel to the interface [15, 16].

For the case of a single interface (14a) and (14b) can be simplified as

$$\xi' = (\varepsilon'_2 \mu'_2 - \varepsilon''_2 \mu''_2) - n_1^2 \sin^2 \theta_t, \quad \xi'' = \varepsilon''_2 \mu''_2 + \varepsilon''_2 \mu''_2.$$

With these $\xi'$ and $\xi''$ we can calculate the refractive index $m'_t$ and attenuation coefficient $m''_t$ of the transmitted wave using (15a) and (15b), and then the transmission angle $\theta''_1$ using (8a).

To find the orientation of $k'_t$ and $k''_2$, we need to determine the signs of their projections $q'_t$ and $q''_t$ in (7c). In appendix A we obtain (A.4), which says that, in any lossy medium, the projection of an attenuation vector onto a time-averaged Poynting vector is always positive. In our case (A.4) reads as $k''_2 \cdot P'_t > 0$, where $P'_t$ is the time-averaged Poynting vector of the transmitted wave. By substituting $k''_2$ from (7c) into (A.4) and taking into account that $p''_t = 0$, we find $q''_t = (P'_t \cdot \hat{q}) > 0$. Since the condition for the energy flux to be directed away from the interface is $P'_t \cdot \hat{q} \geq 0$, we eventually find that $q''_t$ must be positive. That is, the attenuation vector $k''_2$ of the transmitted wave is outgoing from the interface.

The condition $q''_t > 0$ means that we should choose $s'' = +1$ in (11). As a result, we obtain that the sign of the projection $q'_t$ is given by $s' = \text{sgn}(\xi')$. Thus, if $\xi'' < 0$ we have $q'_t < 0$ and the phase vector is incoming towards the interface, and hence negative refraction takes place. Therefore, the criterion of negative refraction is $\xi'' < 0$, which, according to (16), is equivalent to

$$\varepsilon'_2 \mu''_2 + \varepsilon''_2 \mu''_2 < 0.$$

Thus, starting from the most general case of light wave transmission through two lossy isotropic media, we have rediscovered the well-known criterion for negative refraction, which implies negative $\varepsilon'_2$ and $\mu'_2$ [14].

We note that the angle $\theta_t$ between $k'_t$ and $k''_2$ is equal to $\theta'_t$, since $\theta''_t = 0$. Therefore, if the condition of negative refraction is satisfied and $q'_t = (k'_t \cdot \hat{q}) < 0$, we have $\theta'_t > \theta_t > 90^\circ$. In other words, negative refraction is accompanied by the formation of inhomogeneous waves with an obtuse angle between equi-amplitude and equi-phase planes (see figure 2(b)). Using this argument, we can now clarify the relation between negative refraction and the condition $n < 0$: at an angle $\theta > 90^\circ$ the cosine in (2b) becomes negative, which results
in a negative \( n' \) on the right-hand side of the equation (\( n'' \) is positive in lossy media). In particular, in the case of negative refraction at normal incidence \( \theta = 180^\circ \) and \( \cos \theta = -1 \) in (2b). As a result, we find \( m' = |n'| \) and \( m'' = |n''| \). In this case the choice between a negative cosine and negative \( n' \) is a matter of preference; hence our theory does not violate the commonly used criterion for negative refraction; namely, that \( n' \) is negative. But our theory predicts more than that: it says that, if an incidence angle is different from zero, the use of \( n' \) and \( n'' \) instead of \( m' \) and \( m'' \) can result in an error.

To be more quantitative we now assume that the first medium is vacuum and the second medium is the negative-index metamaterial reported as ‘structure 3’ in [23]. To retrieve the permittivity and permeability of this metamaterial we digitized the relevant data from [23] and then approximated it by the Drude model using parameter fitting [24]. According to (14) the retrieved parameters, the value of \( \xi'' \) is negative in the spectral region between 710 and 800 nm, irrespective of the incidence angle. Figures 3(a)–(c) show the change of \( m'_t \), \( m''_t \) and \( \theta'_t \), calculated by (15a), (15b) and (8a), as functions of the wavelength and the incidence angle \( \theta_i \). At normal incidence, as was mentioned before, the spectral dependencies of \( m'_t \) and \( m''_t \) (black curves in figure 3) reproduce the absolute values of the corresponding experimental data from [23]. For example, the spectral dependence of \( m'_t \) at normal incidence is the modulus of a well-known bell-gap shape which one usually observes in experiments (see figure 3 in [23]). We see that both \( m'_t \) and \( m''_t \) increase as \( \theta'_t \) increases. From figure 3(b) we find that the absorption is minimum at \( \theta_i = 0^\circ \). As \( \theta_i \) increases from zero, the transmission angle \( \theta'_t \) very quickly approaches almost 90°, which means that \( \hat{k}'_t \) becomes nearly parallel to the interface (see figure 3(c)).

Unfortunately, at incidence angles different from zero, we cannot directly compare the above example with the experimental data, since the metamaterial in [23] is anisotropic. Instead, we consider \( m'_t \), \( m''_t \) and \( \theta'_t \) in figure 3 as characteristic parameters of some hypothetical isotropic metamaterial whose permittivity and permeability are represented by the Drude model. Since the majority of existing metamaterials are highly anisotropic, the assumption of such isotropic metamaterials in this work may raise questions. However, the first isotropic metamaterials have been fabricated very recently [5, 25].

### 4.2. Metallic prism

In section 3 we have shown that the sign of the \( \xi'' \) parameter determines the directions of the phase and attenuation vectors in the transmitted wave (see (11)). In particular, for a single interface, we have shown that, if \( \xi'' < 0 \), then the phase vector of the transmitted wave points towards the interface. That is, \( \xi'' < 0 \) is the criterion of negative refraction.

According to (14b), in general, \( \xi'' \) consists of two terms. The first term is associated with the material properties of a medium below the interface, while the second term depends on the incidence angles and thus is associated with the geometry of the problem. Hereinafter we will refer to the first and second terms of \( \xi'' \) in (14b) as the material and geometric terms, respectively. We have already seen that for a single interface the geometric term is zero. Since the sign of \( \xi'' \) affects the criterion of negative refraction, we expect that this criterion will change if we manage to find a case where the geometric term is nonzero.

The attenuation vector of any incident wave coming from a lossless medium is zero and therefore has a zero parallel...
component. Since the parallel components of attenuation vectors must be continuous across the interface (see (4)), the attenuation vector of the transmitted wave must be perpendicular to the interface in order to also have a zero parallel component. This argument holds for any number of subsequent interfaces which are parallel to the first one. Therefore, to obtain a nonzero geometric term, we need to consider a case where an incident wave has a nonzero attenuation vector that is tilted relative to the interface. This is most easily realized by a lossy prism (see figure 4) where the attenuation vector of the transmitted wave relative to the interface forms an angle \( \psi \) with respect to the interface. This wave is characterized by a real wavevector \( k''_1 = k''_1 - i \varepsilon''_1 \) with a zero angle \( \theta_1 \) between \( k''_1 \) and \( k''_2 \); refractive index \( n''_1 = n''_2 \) and attenuation coefficient \( \varepsilon''_1 = \varepsilon''_2 \), where \( n'' \) and \( n'' \) are given by \( n = n'' - i n'' = \sqrt{\varepsilon''_2} \) and incidence angles \( \theta'_1 = \theta'_2 = \psi \), where \( \psi \) is the prism angle. We seek to find the orientation of the phase vector \( k'_1 \) and attenuation vector \( k''_1 \) of the transmitted wave relative to the interface. According to (14a) and (14b), the parameter \( \xi = \xi' - i \xi'' \) for the considered interface is given by \( \xi' = \xi'_3 - (m''_2^3 - m''_1^3) \sin^2 \psi \) and \( \xi'' = \xi''_3 - 2 m''_1^3 \sin^2 \psi \), or taking into account (2a) and (2b), by

\[
\xi' = \xi'_3 - \xi'_2 \sin^2 \psi, \quad \xi'' = \xi''_3 - \xi''_2 \sin^2 \psi, \quad (17)
\]

where \( \xi'' \) has the geometric term \(-\xi''_2 \sin^2 \psi\), which is nonzero and negative for any nonzero prism angle \( \psi \).

As we have shown in section 3, the directions of \( k'_1 \) and \( k''_1 \) relative to the interface are specified by the sign of \( \xi'' \) (see (11)). For example, when \( \xi'' > 0 \) such that the sign \( s'' \) of the projection \( \xi'' = (k'_1 \cdot \mathbf{\hat{q}}) \) is negative, then \( k'_1 \) is directed towards the interface, and negative refraction takes place. According to (17), \( \xi'' \) is positive if

\[
\sin^2 \psi \leq \xi''_3 / \xi''_2 \quad (18)
\]

and negative otherwise. Using (18) and (11), we categorize all possible orientations of \( k'_1 \) and \( k''_1 \) into the following cases (see figure 5):

(A) \( \sin^2 \psi \leq \xi''_3 / \xi''_2 \) and \( s' = +1 \). Both \( k'_1 \) and \( k''_1 \) are directed away from the interface. Refraction is positive.

(B) \( \sin^2 \psi \leq \xi''_3 / \xi''_2 \) and \( s' = -1 \). Both \( k'_1 \) and \( k''_1 \) are directed towards the interface. Refraction is negative.
\[
\sin^2 \psi > \epsilon''_e/\epsilon''_s \text{ and } s' = +1 \text{ while } s'' = -1. \text{ Vector } \mathbf{k}'_i \text{ is directed away from the interface, while } \mathbf{k}'_i \text{ points towards the interface. Refraction is positive.}
\]

(D) \[
\sin^2 \psi > \epsilon''_e/\epsilon''_s \text{ and } s' = -1 \text{ while } s'' = +1. \text{ Vector } \mathbf{k}'_i \text{ is directed towards the interface, while } \mathbf{k}'_i \text{ points away from the interface. Refraction is negative.}
\]

It might seem strange that the attenuation vector in cases (B) and (C) points towards the interface. Naively, one can come to the conclusion that in these cases the amplitude of the transmitted wave grows exponentially as the wave moves away from the prism, which violates the energy conservation law. However, the attenuation vector pointing towards the interface merely reflects the fact that the amplitude of the transmitted wave decreases in the direction from the sharp to the blunt end of the prism, since the thicker part of the prism absorbs more. To resolve the confusion, we note that there is always a fraction of the incident wave that propagates without going through the prism (to the left of the prism edge in figure 4). Therefore, the growth of the transmitted wave amplitude is always limited by the amplitude of the incident wave. Thus cases (B) and (C) do not violate the energy conservation law.

Which of (A)–(D) is actually the case depends on the additional condition that requires the time-averaged Poynting vector \( \mathbf{P}_t \) of the transmitted wave to be directed away from the interface. Since at the considered interface the parallel component of \( \mathbf{k}'_i \) is nonzero, we can no longer apply the inequality (A.4) to uniquely determine the sign of \( \psi '' \). Instead, we need to find an expression for \( \mathbf{P}_t \) and directly consider which of the cases (A)–(D) does not violate the inequality \( \mathbf{P}_t \cdot \hat{\mathbf{q}} \geq 0 \). To simplify the task, we assume that the incident wave is p-polarized; that is, in figure 4 the electric field vector of the wave incident on the first prism interface lies in the plane spanned by vectors \( \hat{\mathbf{q}}_1 \) and \( \hat{\mathbf{q}}_2 \) (plane of the figure). We additionally assume that the frequency of the incident wave is less than the plasma frequency of the underlying metal; that is, \( \epsilon'_s < 0 \). In appendix B we show (see (B.6)) that for p-polarized incident wave the condition \( \mathbf{P}_t \cdot \hat{\mathbf{q}} \geq 0 \) is equivalent to \( \psi'' \mathbf{q}'_1 + \epsilon'_s \epsilon''_s \mathbf{q}'_2 \geq 0 \). Taking into account that \( \mathbf{q}'_i = s' |\mathbf{q}'_i| \) and \( \mathbf{q}''_i = s'' |\mathbf{q}''_i| \), while \( \epsilon'_s < 0 \) and \( \epsilon''_s > 0 \), we rewrite the latter inequality as

\[
-s' |\epsilon'_s| |\mathbf{q}'_i| + s'' |\epsilon''_s| |\mathbf{q}''_i| \geq 0. \tag{19}
\]

Equation (19) is a necessary and sufficient condition for the energy flux in the underlying metal to be directed away from the interface. Thus we need to find under which conditions the cases (A)–(D) do not violate the inequality of (19). Consider each of the cases separately:

(A) \[
\sin^2 \psi \leq \epsilon''_s/\epsilon''_e \text{ and } s' = s'' = +1. \text{ Then (19) gives us } -|\epsilon'_s| |\mathbf{q}'_i| + |\epsilon''_s| |\mathbf{q}''_i| \geq 0 \text{ or }
\]

\[
\frac{\epsilon'_s^2}{\epsilon''_s} \geq \frac{q_{i1}^2}{q_{i2}^2} = \frac{|\epsilon''_s| + \epsilon'}{|\epsilon''_s| - \epsilon'}. \tag{20}
\]

(B) \[
\sin^2 \psi \leq \epsilon''_s/\epsilon''_e \text{ and } s' = s'' = -1. \text{ Then (19) gives us } |\epsilon'_s| |\mathbf{q}'_i| - |\epsilon''_s| |\mathbf{q}''_i| \geq 0 \text{ or }
\]

\[
\frac{\epsilon'_s^2}{\epsilon''_s} \leq \frac{q_{i1}^2}{q_{i2}^2} = \frac{|\epsilon''_s| + \epsilon'}{|\epsilon''_s| - \epsilon'}. \tag{21}
\]

According to (22), to realize negative refraction in case (D), the losses in the prism must be higher than that in the underlying metal, that is, \( \epsilon''_s > \epsilon''_e \). To give a quantitative example, we consider an aluminum prism lying on a silver substrate. The permittivites of both metals we approximate by the Drude model \( \varepsilon(\omega) = 1 - \omega_p^2/(\omega^2 - i\gamma \omega) \), where for aluminum \( \omega_p = 22.9 \times 10^{15} \text{ s}^{-1}, \gamma_s = 0.92 \times 10^{15} \text{ s}^{-1} \) and for silver \( \omega_p = 14 \times 10^{15} \text{ s}^{-1}, \gamma_s = 0.032 \times 10^{15} \text{ s}^{-1} \) [14]. Figure 6 shows the angle \( \psi = \arcsin(\sqrt{\epsilon''_s/\epsilon''_e}) \) as a function of the wavelength \( \lambda \), where \( \epsilon''_s(\lambda) \) and \( \epsilon''_e(\lambda) \) are the imaginary parts of the permittivities for silver and aluminum, respectively. According to (22), negative refraction is realized for any prism angles \( \psi \) that lie above the curve in figure 6. As we see, in order to observe negative refraction in our example, the prism angles must exceed \( 7^\circ \sim 8^\circ \).

To conclude this section, we have shown that negative refraction can be realized in conventional lossy media, if
their interfaces are properly oriented. Please note that what we report is negative refraction of a wavevector, that is, of a phase velocity. This effect should be distinguished from negative refraction of energy flux, that is, of Poynting vector. In anisotropic and lossy media the direction of Poynting vector does not necessarily coincide with the direction of the wavevector [26, 27]. Therefore, negative refraction of energy flux does not automatically imply negative refraction of phase velocity.

Negative refraction of energy flux in conventional media has already been reported for anisotropic materials [9–11] and metallic wedges [12, 13]. It was also predicted for bulk metals in the case of specially polarized incident light [26, 28]. However, as far as we know, negative refraction has never been reported before.

3. Conclusions

In conclusion we have studied the formation of inhomogeneous waves at the interface between two lossy isotropic media. We have found explicit expressions (up to the sign) for the propagation constants, refractive index and attenuation coefficient of transmitted inhomogeneous waves. We have shown that the inhomogeneity of transmitted waves can affect negative refraction. In particular, we have shown that, if a homogeneous wave is incident on the interface between a lossless and a lossy isotropic medium, negative refraction is accompanied by the formation of an inhomogeneous wave with an obtuse angle between the equiphase and equiamplitude planes. In this case, the criterion of negative refraction is equivalent to the well-known one that requires negative real parts of permittivity and permeability. However, in a more general case, when the incident wave is inhomogeneous or both media are lossy, the criterion of negative refraction becomes dependent on an incidence angle. In particular, we have shown that negative refraction can be realized in the system of a metallic prism lying on the surface of another metal. Thus, negative refraction can be realized even in conventional lossy media if their interfaces are properly oriented.

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Appendix A

In this appendix we show that, in any lossy medium, the projection of an attenuation vector \( k'' \) onto a time-averaged Poynting vector \( P \) is always positive.

Taking the scalar product of (A.1b) with \( E^* \) and (A.1a) with \( H^* \), and summing the results, we obtain

\[
E^* \cdot (H \times k) + [k \times E] \cdot H^* = \varepsilon_0 \varepsilon \omega |E|^2 + \mu_0 \mu \omega |H|^2
\]

or

\[
k \cdot ([E \times H^*] + [E^* \times H]) = \varepsilon_0 \varepsilon \omega |E|^2 + \mu_0 \mu \omega |H|^2.
\]

Since \([E \times H^*] + [E^* \times H] = 2 \text{Re}[[E \times H^*]]\), we rewrite the latter equation as

\[
2 (k \cdot P) = \frac{\varepsilon_0 \varepsilon \omega}{2} |E|^2 + \frac{\mu_0 \mu \omega}{2} |H|^2.
\]

In a lossy medium the right-hand side of (A.3b) is always positive. Therefore, the projection of \( k'' \) onto \( P \) is always also positive:

\[
k'' \cdot P > 0.
\]

Note that, according to (A.3b), inhomogeneous waves in lossless media (where \( \varepsilon'' = \mu'' = 0 \)) have \( k'' \) perpendicular to \( P \).

Appendix B

In this appendix we find the condition under which the energy flux of the transmitted wave is directed away from an interface if the incident wave is p-polarized.

Using (1) we can write the complex time-averaged Poynting vector of the transmitted wave \( S_t = \frac{1}{2} (E_t \times H_t^*) \) as

\[
S_t = \frac{\varepsilon^{*\omega}\varepsilon}{2} [e_t \times h_t^*]. \tag{B.1}
\]

where “*” denotes complex conjugate. Despite the fact that there are two incidence planes in our problem, we still can formally represent the complex amplitude vector \( e_t \) as the sum of s- and p-polarized components [15–17]:

\[
e_t = A_{t\perp}s + A_{||}[s \times k_t], \tag{B.2}
\]

where \( A_{t\perp} = s^{-2} (e_t \cdot s) \) and \( A_{||} = s^{-2} (e_t \cdot \hat{q}) \) are the amplitudes of s- and p-polarized components, respectively, while \( s = k_t \times \hat{q} \). Using (B.2), we can express the complex amplitude vector \( h_t = (\mu_0 \mu_3 \omega)^{-1} [k_t \times e_t] \) in terms of the amplitudes \( A_{t\perp} \) and \( A_{||} \) as

\[
h_t = \varepsilon_0 \varepsilon_3 \omega A_{||}[s - \frac{A_{t\perp}}{\mu_0 \mu_3 \omega} s \times k_t]. \tag{B.3}
\]
Substituting (B.2) and (B.3) into (B.1), we obtain

\[
S_1 = \frac{e^{-2(k_i^0 r)}}{2} \left\{ \frac{|A_{\parallel}|^2}{\mu_0 \varepsilon_0 \omega} [s \times [k_i^0 \times s^*]] + 2k_i^0 \varepsilon_0^2 \omega A_{\parallel} [s \times s^*] \right\} - \frac{A^\ast_{\parallel} A_{\parallel}}{\mu_0 \varepsilon_0^2 \omega} \left[ [k_i^0 \times s] \times [k_i^0 \times s^*] \right].
\]

In the case of a p-polarized incident wave, when \( A_{\parallel} = 0 \), we have

\[
S_1 = \frac{e^{-2(k_i^0 r)}}{2} \varepsilon_0^2 \omega |A_{\parallel}|^2 [s \times [k_i^0 \times s]].
\]

Taking the real part of \( S_1 \), we obtain a real time-averaged vector \( P_t \):

\[
P_t = \frac{e^{-2(k_i^0 r)}}{2} \varepsilon_0 \omega |A_{\parallel}|^2 \left\{ s^2 (\varepsilon_3^0 k_i^0 + \varepsilon_3^0 k_i') \right\} - 2(\hat{q} \cdot [k_i^0 \times k_i']) (\varepsilon_3^0 s'' - \varepsilon_3^0 s').
\]

where \( s' = k_i^0 \times \hat{q} \) and \( s'' = k_i^0 \times \hat{q} \) are the normals to the planes of incidence for the phase and attenuation vectors, respectively. Note that vectors \( P_t \) and \( k_i' \) are non-collinear, meaning the directions of the energy flux and phase velocity in the transmitted wave are different.

From (B.4) we find that the projection \( P_t \cdot \hat{q} \) is given by

\[
P_t \cdot \hat{q} = \frac{e^{-2(k_i^0 r)}}{2} \varepsilon_0 \omega |A_{\parallel}|^2 \left\{ s^2 (\varepsilon_3^0 q_i' + \varepsilon_3^0 q_i'') \right\}.
\]

Since the factor in front of the brackets in the right-hand side of (B.5) is always positive, we find that the condition \( P_t \cdot \hat{q} \geq 0 \) is equivalent to

\[
\varepsilon_3^0 q_i' + \varepsilon_3^0 q_i'' \geq 0.
\]

Equation (B.6) is the condition under which the energy flux of the transmitted wave is directed away from the interface in the case of a p-polarized incident wave.

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