Unitarity of induced representations from coisotropic quantum subgroups

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Abstract

We study unitarity of the induced representations from coisotropic quantum subgroups which were introduced in \cite{3}. We define a real structure on coisotropic subgroups which determines an involution on the homogeneous space. We give general invariance properties for functionals on the homogeneous space which are sufficient to build a unitary representation starting from the induced one. We present the case of the one-dimensional quantum Galilei group, where we have to use in all generality our definition of quasi-invariant functional.

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1 Introduction

Induced representations are the fundamental tool in the representation theory of Lie groups. Starting with a unitary representation of a generic closed subgroup $H$ of a given Lie group $G$ there is a standard way to induce a unitary representation of the whole group, realizing it on the space of $L^2$ sections of a homogeneous vector bundle on $G/H$. Whenever the homogeneous space is compact, the unitarity of the induced representation is due to the existence of an invariant measure on it.

For a non compact homogeneous space such an invariant measure does not necessarily exists. Nevertheless one can fix a quasi invariant measure $\mu$, \textit{i.e.} a measure
equivalent to its translates. A quasi invariant measure always exists and is unique only up to equivalence. Once such a measure has been fixed, it is possible to modify accordingly the definition of induced representation, using a weight function, to obtain a unitary representation. It is finally possible to show that equivalent measures give rise to equivalent unitary representations.

For quantum groups induction of representations was up to now analyzed either in the algebraic [10] or in the compact case [5]. In both cases, the authors start from corepresentations of quantum subgroups. However it was realized from the beginning that quantum subgroups are very rare. The more general notion of quantum coisotropic subgroup is sufficient to obtain all embeddable quantum homogeneous spaces. The semiclassical limit of such subgroups has a natural interpretation in terms of Poisson structures: indeed one can observe that the subgroup obtained by conjugation of a Poisson Lie subgroup is no more Poisson Lie but only Poisson coisotropic and the conjugated subgroups determine inequivalent Poisson structures on the corresponding homogeneous space, see [2]. After quantization, this situation gives rise to a family of inequivalent quantum homogeneous spaces, which are not obtained by a quantum subgroup: it is therefore rather natural to induce representations starting from quantum coisotropic subgroups. This program was started in [3], where the basic algebraic properties were proved.

The purpose of the present work is to study unitarity in the same general setting. We propose a definition of quasi invariant functional on a quantum homogeneous space which is quite general. We then fix a quasi invariant functional and we prove a general procedure to define unitary representations from the induced ones.

The exposition given in this paper is mainly based on the coalgebra properties: strictly speaking such an approach is completely suited only for the compact case. The lack of a general definition of non compact quantum group and quantum homogeneous space does not allow to develop a meaningful general theory. Nevertheless many concrete examples exist, in which unitary representations on Hilbert spaces are built starting from $U_q$-module $*$-algebras, like, for example, the $E_6$ case in [6], $SU_q(1,1)$ in [9] and the quantum Galilei group in [1]. We will explicitly study the induced representations of the quantum Galilei Group showing how our results extend to $U_q$-module $*$-algebras. In [1] physical motivations lead to study its unitary representations with respect to a non standard involution. When the involution is chosen to be standard, as we will do here, induced representations must be studied using a functional which is quasi invariant according to our definition.

The paper is structured as follows. In section 2 we introduce coisotropic quantum subgroups, focusing our attention on reality conditions, which were only briefly mentioned in [3]. This is necessary to give an involution on the homogeneous space and to define unitary corepresentations for the subgroup.

In section 3 we define quasi invariant functionals on homogeneous spaces. We will call essentially invariant those functionals already defined in [8] and we show that an essentially invariant functional is always equivalent to an invariant one.

In section 4 we define a sesquilinear form on the representation space and we show
that there is one definite choice according to the fact that the coisotropic subgroup is left or right (see Lemma [3]). We then construct the unitary representation with respect to this sesquilinear form and we finally show that equivalent quasi invariant functionals give rise to equivalent representations.

In section 5 the example of quantum Galilei group $\mathcal{U}_q(G(1))$ is worked out. Dealing with a non compact quantum homogeneous space we are forced to describe it as a $\mathcal{U}_q$-module $\ast$-algebra. We construct its unitary representations by means of a functional which is quasi invariant but not essentially invariant.

2 Real Coisotropic Subgroups

In this section we will give the definition and main properties of real coisotropic quantum subgroups and their associated homogeneous spaces. With respect to [4], to which we refer for a more detailed discussion, the emphasis here is on the $\ast$ structure on the subgroup and on the corresponding homogeneous space.

**Definition 2.1** Given a real quantum group $(\mathcal{F}_q(G), \ast)$ we will call real coisotropic quantum right (left) subgroup $(\mathcal{F}_q(K), \tau_K)$ a coalgebra, right (left) $\mathcal{F}_q(G)$-module $\mathcal{F}_q(K)$ such that:

(i) there exists a surjective linear map $\pi: \mathcal{F}_q(G) \to \mathcal{F}_q(K)$, which is a morphism of $\mathcal{F}_q(G)$-modules (where $\mathcal{F}_q(G)$ is considered as a module on itself via multiplication) and of coalgebras;

(ii) there exists an antilinear map $\tau_K: \mathcal{F}_q(K) \to \mathcal{F}_q(K)$ such that $\tau_K \circ \pi = \pi \circ \tau$, where $\tau = \ast \circ S$.

A $\ast$-Hopf algebra $\mathcal{F}_q(K)$ is said to be a real quantum subgroup if there exists a $\ast$-Hopf algebra epimorphism $\pi: \mathcal{F}_q(G) \to \mathcal{F}_q(K)$.

**Remark 2.2** A coisotropic quantum subgroup is not a $\ast$-coalgebra but it has only $\tau_K$ defined on it. It is easy to verify that if a coisotropic quantum subgroup is also a $\ast$-coalgebra and $\pi \circ \ast = \ast \circ \pi$, then it is possible to complete the structure so to have a quantum subgroup.

In the following we will always use subgroup to mean real subgroup. Coisotropic quantum subgroups are easily characterized by the following proposition.
Proposition 2.3 There exists a bijective correspondence between coisotropic quantum right (left) subgroups and \( \tau \)-invariant two-sided coideals, right (left) ideals in \( \mathcal{F}_q(G) \). \( \blacksquare \)

Coisotropic quantum subgroups give in a canonical way embeddable quantum homogeneous spaces.

Definition 2.4 A \(*\)-algebra \( B \) is said to be an embeddable quantum left (right) \( \mathcal{F}_q(G) \)-homogeneous space if there exists a coaction \( \delta : B \rightarrow B \otimes \mathcal{F}_q(G) \), \( (\delta : B \rightarrow \mathcal{F}_q(G) \otimes B) \) and an injective morphism of \(*\)-algebras \( i : B \rightarrow \mathcal{F}_q(G) \) such that \( \Delta \circ i = (i \otimes \text{id}) \circ \delta \) and \( \Delta \circ i = (\text{id} \otimes i) \circ \delta \).

In the following we will identify left (right) embeddable quantum homogeneous spaces with \(*\)-subalgebras and right (left) coideals of \( \mathcal{F}_q(G) \).

Proposition 2.5 Let \((\mathcal{F}_q(K), \tau_K)\) be a left coisotropic quantum subgroup of \((\mathcal{F}_q(G), \ast)\) then
\[
B_\pi = \{ a \in \mathcal{F}_q(G) \mid (\pi \otimes \text{id}) \Delta a = \pi(1) \otimes a \}
\]
is a left embeddable quantum homogeneous space.

If \((\mathcal{F}_q(K), \tau_K)\) is a right coisotropic quantum subgroup of \((\mathcal{F}_q(G), \ast)\) then
\[
B^\ast_\pi = \{ a \in \mathcal{F}_q(G) \mid (\text{id} \otimes \pi) \Delta a = a \otimes \pi(1) \}
\]
is a right embeddable quantum homogeneous space.

Proof. Let \( \mathcal{F}_q(K) \) be a left coisotropic quantum subgroup. As a consequence of the properties of \( \pi \) it is easy to see that if \( a, b \in B_\pi \) then \((\pi \otimes \text{id}) \Delta ab = \pi(1) \otimes ab \) and \((\pi \otimes \text{id} \otimes \text{id}) (\Delta \otimes \text{id}) \Delta a = \pi(1) \otimes \Delta(a) \), which means that \( B_\pi \) is a subalgebra and right coideal of \( \mathcal{F}_q(G) \). We have to show that \( B_\pi \) is also \(*\)-invariant.

If \( a \in B_\pi \), then
\[
\epsilon(a) \pi(1) = \sum_{(a)} \pi(S^{-1}(a_{(2)})a_{(1)}) = \sum_{(a)} S^{-1}(a_{(2)}) \pi(a_{(1)}) = \sum_{(a)} \epsilon(a_{(1)}) S^{-1}(a_{(2)}) \pi(1) = \pi(S^{-1}(a)),
\]
where we used the property of \( \pi \) with respect to the left module structure of \( \mathcal{F}_q(K) \) and the structure of right coideal of \( B_\pi \). Applying \( \tau_K \) and using commutativity with \( \pi \), we conclude that
\[
\pi(a^\ast) = \epsilon(a^\ast) \pi(1).
\]
Because \( B_\pi^\ast \) is a subalgebra and right coideal of \( \mathcal{F}_q(G) \) then, using Proposition 4.7 of \([2]\) we have that \((\pi \otimes \text{id}) \Delta a^\ast = \pi(1) \otimes a^\ast \) and so \( B_\pi^\ast = B_\pi \).

The proof for right coisotropic quantum subgroups is very similar. \( \blacksquare \)

Remark 2.6 A left coisotropic quantum subgroup cannot give rise in general to a right homogeneous space (referring to the proof of Proposition 2.3, note that \( \pi(S(a)) \neq \pi(a) \), for \( a \in B_\pi \)).
3 Quasi-invariant Functionals

We denote by the same symbol $X.a$ the left regular representation of $\mathcal{U}_q$ on $\mathcal{F}_q(G)$ and its restriction to the left quantum homogeneous space $B_\pi$, i.e. $X.a = (id \otimes X)\Delta a \in B_\pi$ if $a \in B_\pi$. Analogously we denote by $a.X$ the restriction of the right regular representation of $\mathcal{U}_q$ on $\mathcal{F}_q(G)$ to the right quantum homogeneous space $B_\pi$, i.e. $a.X = (X \otimes id)\Delta a \in B_\pi$ if $a \in B_\pi$.

Definition 3.1 We say that the linear functional $h : B_\pi \to \mathbb{C}$ (resp. $h : B_\pi \to \mathbb{C}$) is real if $h(a^*) = h(a)$; furthermore if $h(a^*a) \geq 0$ for each $a \in B_\pi$ (resp. $a \in B_\pi$) $h$ is said to be positive. Two such real functionals $h_1, h_2$ are $\xi$ equivalent if there exists an invertible $\xi \in B_\pi$ such that

$$h_1(a) = h_2(\xi^*a\xi) .$$

It is easily verified that Definition 3.1 defines an equivalence relation.

Definition 3.2 Let $\phi : \mathcal{U}_q \to B_\pi$ be a linear application which satisfies the following properties:

$$\phi[XY] = \sum_{(X)} X(1).\phi[Y] \phi[X(2)]; \quad \phi[1_\mathcal{U}_q] = 1_{B_\pi},$$

(1)

and $h : B_\pi \to \mathbb{C}$ a real functional. We say that $h$ is quasi invariant with weight $\phi$ if the following relation is valid for each $X \in \mathcal{U}_q$

$$h(X.a) = \sum_{(X)} h(\phi[X(1)]^*a\phi[S(X(2))]).$$

(2)

If there exists an invertible $\xi \in B_\pi$ such that $\phi[X] = X.\xi \xi^{-1}$ then $h$ is said to be essentially invariant.

Remark 3.3 (i) If $\xi$ is group like and $\tau(\xi) = \xi$ then the essentially invariant functional with weight $\phi[X] = \langle X, \xi \rangle 1_{B_\pi}$ is invariant, i.e. $h(X.a) = \epsilon(X)h(a)$.

(ii) Definition 3.2 is well posed: indeed it preserves reality of $h$ and is compatible with the regular representation.

(iii) An essentially invariant functional with a group-like $\xi$ is quasi invariant according to (i).

Lemma 3.4 A real functional $h : B_\pi \to \mathbb{C}$ is quasi-invariant if and only if there exists a linear application $\phi : \mathcal{U}_q \to B_\pi$ which satisfies (1) and such that, for each $X \in \mathcal{U}_q$, we have

$$\sum_{(X)} h(X(1).a \phi[X(2)]) = h(\phi[X^*]^*a) .$$

(3)
Proof. Let \( h \) verify (\( \mathbb{I} \)), then
\[
\sum_X h(\phi[X(1)]*a\phi[S(X(2))]) = \sum_X h(X(1).a\phi[S(X(3))].\phi[X(2)])
= \sum_X h(X(1).a\phi[S(X(3))].\phi[X(2)])
= h(X.a),
\]
where we applied (\( \mathbb{I} \)) at the first line and (\( \mathbb{I} \)) to obtain the last one. Let now \( h \) be quasi-invariant, then
\[
\sum_X h(X(1).a\phi[X(2)]) = \sum_X h(X(1).aS(X(3)).\phi[X(4)]\phi[X(2)])
= h(\phi[X(1)] a),
\]
where we used trivial properties of the regular representation in the first line, relation (\( \mathbb{I} \)) in the second one and (\( \mathbb{I} \)) in the last one.

We give the equivalent definition and lemma (without proof) for right homogeneous spaces.

**Definition 3.5** Let \( \psi: \mathcal{U}_q \to B^\pi \) be a linear application which satisfies the following properties:
\[
\psi[XY] = \sum_{(X)} \psi[Y(1)]\psi[X].Y(2); \quad \psi[1_{\mathcal{U}_q}] = 1_{B^\pi},
\]
and \( h: B^\pi \to \mathbb{C} \) a real functional. We say that \( h \) is quasi invariant if the following relation is valid for each \( X \in \mathcal{U}_q \)
\[
h(a.X) = \sum_{(X)} h(\psi[S(X(1))].a\psi[X(2)]^*).
\]
The functional \( h \) is said to be essentially invariant if, for each \( X \in \mathcal{U}_q \), \( \psi[X] = \xi^{-1}\xi.X \) for some invertible \( \xi \in B^\pi \).

**Lemma 3.6** A real functional \( h: B^\pi \to \mathbb{C} \) is quasi-invariant if and only if there exists a linear application \( \psi: \mathcal{U}_q \to B^\pi \) which satisfies (\( \mathbb{I} \)), and such that
\[
\sum_{(X)} h(\psi[X(1)].a.X(2)) = h(a\psi[X(1)]^*).
\]

**Remark 3.7** Definitions 3.2 and 3.5 don’t require the coalgebra structure of \( B_\pi \) and \( B^\pi \). They can be used in the general setting of \( \mathcal{U}_q \)-module left and right \( * \)-algebras. In the left case they are algebras and left \( \mathcal{U}_q \)-modules such that \( X(ab) = \sum_{(X)} X(1).aX(2).b \), \( X.1 = \epsilon(X)1 \) and \( (X.a)^* = \tau(X).a^* \); analogously for the right case.
Until the end of this section we will consider only left homogeneous spaces. Similar statements hold in the right case with obvious modifications.

In the following Proposition we list some properties of quasi invariant functionals.

**Proposition 3.8** Let $B_π$ be a left homogeneous space. Then

(i) if two real functionals $h_i$, $i = 1, 2$, are $ξ$-equivalent and $h_2$ is quasi invariant with weight $φ_2$ then $h_1$ is quasi invariant with weight

$$\phi_1[X] = \sum_{(X)} X_{(1)}ξφ_2[X_{(2)}]ξ^{-1} ;$$

(ii) if $h$ is a quasi invariant real functional with weight $φ$ and $k ∈ U_q$ such that $∆(k) = k ⊗ k$ and $τ(k) = k$ then the translated functional $h_k(a) = h(k.a)$ is an equivalent real functional, which is quasi invariant with weight

$$φ_k[X] = φ[X S(k)] S(k).φ[k]$$

and is positive if $h$ is positive;

(iii) a real functional is essentially invariant if and only if it is equivalent to an invariant one.

Proof. Let us prove item (i):

$$h_1(X.a) = h_2(ξ^∗ X.a ξ)$$

$$= \sum_{(X)} h_2(X_{(2)}). (S^{-1}(X_{(1)}).ξ^∗ a S(X_{(3)}).ξ)$$

$$= \sum_{(X)} h_2((X_{(1)}^∗ ξφ_2[X_{(2)}^∗])^∗ a S(X_{(3)}).ξ φ_2[S(X_{(3)})])$$

$$= \sum_{(X)} h_2((φ_1[X_{(1)}]ξ)^∗ aφ_1[S(X_{(2)})]ξ)$$

$$= \sum_{(X)} h_1(φ_1[X_{(1)}]^∗ aφ_1[S(X_{(2)})]).$$

Let’s prove the item (ii). The property for $h_k$ to be a real functional (and positive if $h$ is positive) directly follows from reality of $k$ and from standard duality relations. From the property of $h$ of being quasi invariant we have that $h_k(a) = h(ξ^∗ a ξ)$ with $ξ = φ[k^∗]$. From the first item $h_k$ is quasi invariant and its weight is obtained by direct computation.

As a consequence of item (i) a functional which is equivalent to an invariant one is essentially invariant. Let now $h_ξ$ be essentially invariant with weight $φ[X] = X.ξ ξ^{-1}$. Then always from item (i) $h(a) = h_ξ((ξ^{-1})^∗ aξ^{-1})$ is quasi invariant with weight $ε(X)$.

**Remark 3.9** (i) Let $F_q(G)$ be a compact quantum group according to [4]. On every quantum homogeneous space of a compact quantum group there exists a unique invariant real functional $h$ such that $h(1) = 1$. 7
(ii) Definitions for weight functions can be rephrased in a cohomological language. Let’s define $C^0(B_\pi) = \{ \xi \in B_\pi | \exists \xi^{-1} \in B_\pi \}$ and $C^k(B_\pi) = \{ \phi : \otimes^k U_q \to B_\pi \}$ for $k \geq 1$. Then, let $d_0 : C^0 \to C^1$, $d_1 : C^1 \to C^2$ as 

$$d_0(\xi)[X] = X.\xi \xi^{-1}; \quad d_1(\phi)[X \otimes Y] = \phi[XY] - \sum_{(x)} X(1).\phi[Y] \phi[X(2)].$$

It is easy to see that $d_1 \circ d_0 = 0$. We can define as usual $H^1(B_\pi) = \text{Ker} \ d_1/\text{Im} \ d_0$. If $H^1(B_\pi) = \{0\}$ then every quasi invariant real functional is essentially invariant.

## 4 Induced Corepresentations

We summarize in this section general properties of induced corepresentations and we start to study their unitarity. We refer to [3] for a detailed study of induced corepresentations from a purely algebraic point of view.

Let $(\mathcal{F}_q(K), \tau_K)$ be a coisotropic quantum left (right) subgroup of $(\mathcal{F}_q(G), \ast)$ and let $B_\pi$ ($B_\pi^\ast$) be the associated left (right) quantum homogeneous space according to Proposition 2.3. Let $\rho_R : V \to V \otimes \mathcal{F}_q(K)$ be a right corepresentation of $\mathcal{F}_q(K)$; if $\{e_i\}$ is an orthonormal basis of $V$ with respect to a scalar product $\langle , \rangle$ then $\rho_R(e_i) = \sum_j e_j \otimes a_{ji}$. Analogously, let $\rho_L : V \to \mathcal{F}_q(K) \otimes V$ be a left corepresentation of $\mathcal{F}_q(K)$, i.e. $\rho_L(e_i) = \sum_j b_{ji} \otimes e_j$. We say that $\rho_R$ ($\rho_L$) is unitary if $\tau_K(a_{ij}) = a_{ji}$ ($\tau_K(b_{ij}) = b_{ji}$).

Let $L = (\pi \otimes \text{id})\Delta : \mathcal{F}_q(G) \to \mathcal{F}_q(K) \otimes \mathcal{F}_q(G)$ and $R = (\text{id} \otimes \pi)\Delta : \mathcal{F}_q(G) \to \mathcal{F}_q(G) \otimes \mathcal{F}_q(K)$. We define

$$\text{ind}_K^G(\rho_R) = \{ F \in V \otimes \mathcal{F}_q(G) | (\text{id} \otimes L)F = (\rho_R \otimes \text{id})F \},$$

$$\text{ind}_K^G(\rho_L) = \{ F \in \mathcal{F}_q(G) \otimes V | (R \otimes \text{id})F = (\text{id} \otimes \rho_L)F \}.$$

It is easy to prove the following lemma.

**Lemma 4.1** If $\mathcal{F}_q(K)$ is a left quantum coisotropic subgroup then the linear space $\text{ind}_K^G(\rho_R)$ is a right $B_\pi$-module, i.e. if $A = \sum_j e_j \otimes A_j \in \text{ind}_K^G(\rho_R)$, $a \in B_\pi$ then $Aa = \sum_j e_j \otimes A_j.a \in \text{ind}_K^G(\rho_R)$.

If $\mathcal{F}_q(K)$ is a right quantum coisotropic subgroup then the linear space $\text{ind}_K^G(\rho_L)$ is a left $B_\pi^\ast$-module, i.e. if $A = \sum_j A_j \otimes e_j \in \text{ind}_K^G(\rho_L)$, $a \in B_\pi^\ast$ then $aA = \sum_j aA_j \otimes e_j \in \text{ind}_K^G(\rho_R)$.

In [3] the following proposition is proven.
Lemma 4.3 The restrictions of \((\text{id} \otimes \Delta)\) to \(\text{ind}^G_K(\rho_R)\) and of \((\Delta \otimes \text{id})\) to \(\text{ind}^G_K(\rho_L)\) respectively define a right and a left corepresentation of \(\mathcal{F}_q(G)\).

The corepresentations thus defined are called induced corepresentations. Induced representations are defined by duality. To obtain unitary representations we must first find an invariant sesquilinear form. The following lemma reduces this problem to that of defining an invariant functional on the corresponding homogeneous space.

Lemma 4.3 Let \(\mathcal{F}_q(K)\) be a coisotropic quantum left subgroup of \(\mathcal{F}_q(G)\). Let \(\rho_R\) be a unitary right corepresentation on \(V\) and \(\langle , \rangle_L\) a sesquilinear map from \(V \otimes \mathcal{F}_q(G)\) to \(\mathcal{F}_q(G)\) given by \(\langle v \otimes a, w \otimes b \rangle_L = \langle v, w \rangle a^*b\). If \(A, B \in \text{ind}^G_K(\rho_R)\) then \(\langle A, B \rangle_L \in B_\pi\).

Let \(\mathcal{F}_q(K)\) be a coisotropic quantum right subgroup of \(\mathcal{F}_q(G)\), \(\rho_L\) a unitary left corepresentation on \(V\) and \(\langle , \rangle_R\) a sesquilinear map from \(\mathcal{F}_q(G) \otimes V\) to \(\mathcal{F}_q(G)\) given by \(\langle a \otimes v, b \otimes w \rangle_R = \langle v, w \rangle ba^*\). If \(A, B \in \text{ind}^G_K(\rho_L)\) then \(\langle A, B \rangle_R \in B_\pi\).

Proof. Let \(\mathcal{F}_q(K)\) be a left coisotropic subgroup. If \(A = \sum_i e_i \otimes A_i \in \text{ind}^G_K(\rho_R)\), then we have that \((\pi \otimes \text{id})\Delta A_i = \sum_j a_{ij} \otimes A_j\). The proof relies on the following identity

\[
\sum_j \sum_{(A_j)} S^{-1}(A_{j(1)})a_{ij} \otimes A_{j(2)} = \pi(1) \otimes A_i. \tag{8}
\]

To prove (8) let’s define \(\epsilon_L : \mathcal{F}_q(G) \rightarrow \mathcal{F}_q(K)\), with \(\epsilon_L(a) = \sum(a) S^{-1}(a_{(2)}) \pi(a_{(1)}) = \epsilon(a) \pi(1)\). Then \((\epsilon_L \otimes \text{id})\Delta A_i = \pi(1) \otimes A_i\). Defining \(\mu_L : \mathcal{F}_q(K) \otimes \mathcal{F}_q(G) \rightarrow \mathcal{F}_q(K)\), \(\mu_L(a \otimes b) = ba\), we have

\[
(\epsilon_L \otimes \text{id})\Delta A_i = (\mu_L \otimes \text{id})(\pi \otimes S^{-1} \otimes \text{id})(\text{id} \otimes \Delta) \Delta A_i
= (\mu_L \otimes \text{id})(\text{id} \otimes S^{-1} \otimes \text{id})(\text{id} \otimes \Delta) \sum_j a_{ij} \otimes A_j
= \sum_j \sum_{(A_j)} S^{-1}(A_{j(1)})a_{ij} \otimes A_{j(2)},
\]

from which we get the result. Finally,

\[
(\pi \otimes \text{id})\Delta \langle A, B \rangle_L = \sum_i (\pi \otimes \text{id}) \Delta (A_i^*B_i)
= \sum_i \Delta A_i^* (\pi \otimes \text{id}) \Delta B_i
= \sum_{ij} \sum_{(A_j)} A_{j(1)}a_{ij} \otimes A_{j(2)}^* B_j
= \sum_{ij} \sum_{(A_j)} S^{-1}(A_{j(1)})a_{ij} \otimes A_{j(2)}^* B_j
= \sum_j \pi(1) \otimes A_j^* B_j = \pi(1) \otimes \langle A, B \rangle_L.
\]

The analogous proof holds when \(\mathcal{F}_q(K)\) is a right coisotropic subgroup. If \(A = \sum_i e_i \otimes A_i \in \text{ind}^G_K(\rho_L)\) then \((\text{id} \otimes \pi)\Delta A_i = \sum_j A_j \otimes b_{ij}\). By defining \(\epsilon_R(a) = \sum(a) \pi(a_{(2)}) S^{-1}(a_{(1)}) = \epsilon(a) \pi(1)\), and by applying \((\text{id} \otimes \epsilon_R)\Delta\) on \(A_i\) we arrive to

\[
\sum_j \sum_{(A_j)} A_{j(1)} \otimes b_{ji} S^{-1}(A_{j(2)}) = A_i \otimes \pi(1), \tag{9}
\]
from which
\[(id \otimes \pi)\Delta \langle A, B \rangle_R = \sum_{ij} (B_j \otimes b_{ji}) \Delta A_i^* = \sum_{ij} \sum_{(A_i^*)} B_j A_i^* \otimes \tau_K(b_{ij}S^{-1}(A_i)) = \sum_j B_j A_j^* \otimes \pi(1) = \langle A, B \rangle_R \otimes \pi(1) .\]

Let’s now fix a left coisotropic subgroup $F_q(K)$ and let $B_\pi$ be the corresponding left homogeneous space. Let $h : B_\pi \to C$ be a quasi invariant functional according to (1). Then, if $A, B \in \text{ind}_K^G(\rho_R)$,
\[
\langle A, B \rangle = h(\langle A, B \rangle_L)
\]
defines a sesquilinear form on $\text{ind}_K^G(\rho_R)$. If $h$ is only quasi invariant and not invariant (i.e. $\phi[X] \neq \epsilon(X)$) the induced representation is not unitary. We can nevertheless define a unitary representation. Let $\rho_I$ be the representation dual to the induced corepresentation on $\text{ind}_K^G(\rho_R)$.

**Proposition 4.4** The application $\tilde{\rho}_I(X)$ defined in the basis $\{e_i\}$ of $V$ by
\[
\tilde{\rho}_I(X)A = \sum_{(X)} \sum_i e_i \otimes X_{(1)}.A_i\phi[X_{(2)}]
\]
is a unitary left representation of $U_q$ on $\text{ind}_K^G(\rho_R)$ with respect to the sesquilinear form (10), i.e.
\[
\langle A, \tilde{\rho}_I(X)B \rangle = \langle \tilde{\rho}_I(X^*)A, B \rangle .
\]

Let $h_1$ and $h_2$ be equivalent quasi-invariant functionals on $B_\pi$ and let $\tilde{\rho}_{I_1}$ and $\tilde{\rho}_{I_2}$ be the corresponding induced unitary representations. Then there is a unitary equivalence between $\tilde{\rho}_{I_1}$ and $\tilde{\rho}_{I_2}$.

**Proof.** From Lemma 4.1 we see that $\tilde{\rho}_I$ maps $\text{ind}_K^G(\rho_R)$ into $\text{ind}_K^G(\rho_R)$ and from (11) we verify that it is a representation of $U_q$.

We prove its unitarity. Using basic properties of coproduct and antipode, we can write $\langle A, \tilde{\rho}_I(X)B \rangle = \sum_{i} \sum_{(X)} h(X_{(2)})(S^{-1}(X_{(1)})A_i^*B_i)\phi[X_{(3)}]$ and applying Lemma 3.4 we obtain
\[
\langle A, \tilde{\rho}_I(X)B \rangle = \sum_{i} \sum_{(X)} h(\phi[X_{(2)}]^*S^{-1}(X_{(1)})A_i^*B_i) .
\]

Since
\[
\langle \tilde{\rho}_I(X^*)A, B \rangle = \sum_{i} \sum_{(X)} h(\phi[X_{(2)}]^*(X_{(1)}A_i)^*B_i)
\]
the result follows from the property $(X^*.a)^* = S^{-1}(X).a^*$. 

10
Concerning the second statement, let $h_1(a) = h_2(\xi^* a \xi)$ and let $F : \text{ind}_K^G(\rho_L) \to \text{ind}_K^G(\rho_L)$ be defined by $F(v \otimes a) = v \otimes a \xi$. If $A \in \text{ind}_K^G(\rho_L)$, then

$$F \circ \tilde{\rho}_I(X)A = \sum_i e_i \otimes X_{(1)} A_i \phi_1[X_{(2)}] \xi$$

$$= \sum_i e_i \otimes X_{(1)} (A_i \xi) \phi_2[X_{(2)}] = \tilde{\rho}_I(X) \circ F(A)$$

where we used (7). By direct computation this equivalence is proved to be unitary.

The analogous property is true for a right coisotropic subgroup. Let $h : B^\pi \to C$ a quasi invariant functional on the right homogeneous space. If $A, B \in \text{ind}_K^G(\rho_L)$ then

$$\langle A, B \rangle = h(\langle A, B \rangle_R)$$

defines a sesquilinear form on $\text{ind}_K^G(\rho_L)$. We have:

**Proposition 4.5** The application $\tilde{\lambda}_I(X)$ defined by

$$\tilde{\lambda}_I(X)A = \sum_{(X)} \sum_i \psi[X_{(1)}] A_i X_{(2)} \otimes e_i$$

is a unitary right representation of $U_q$ on $\text{ind}_K^G(\rho_L)$ with respect to the sesquilinear form (11), i.e.

$$\langle A, \tilde{\lambda}_I(X)B \rangle = \langle \tilde{\lambda}_I(X^*)A, B \rangle .$$

Let $h_1$ and $h_2$ be equivalent quasi-invariant functionals on $B^\pi$. and let $\tilde{\lambda}_{I_1}$ and $\tilde{\lambda}_{I_2}$ be the corresponding induced unitary representations. Then there is a unitary equivalence between $\tilde{\lambda}_{I_1}$ and $\tilde{\lambda}_{I_2}$.

5 The Quantum Galilei Group with Imaginary Parameter

The representation theory of the one dimensional quantum Galilei group with real parameter was studied in [1]. In that case physical interpretation required a non standard involution (i.e. $\tau^* \neq \text{id}$). The corresponding results on the unitarity of induced representations cannot be treated in the framework just described. On the contrary the case with imaginary parameter and standard involution is a good example.

The one dimensional Quantum Galilei Group can be introduced by giving its universal enveloping algebra $U_q(G(1))$, which is generated by $\{K, K^{-1}, B, M\}$ and by the relations

$$[K, T] = 0 , \quad KBK^{-1} = B - iwM , \quad [B, T] = i\frac{K - K^{-1}}{2w} ,$$
where $M$ is central and $KK^{-1} = 1 = K^{-1}K$. The deformation parameter is $iw$ with real $w$. The coalgebra is given by

$$
\Delta M = M \otimes K + K^{-1} \otimes M, \quad \Delta K = K \otimes K, \\
\Delta T = T \otimes 1 + 1 \otimes T, \quad \Delta B = B \otimes K + K^{-1} \otimes B
$$

and

$$
S(M) = -M, \quad S(K) = K^{-1}, \quad S(T) = -T, \quad S(B) = -B + iwM; \\
\epsilon(M) = 0, \quad \epsilon(K) = 1, \quad \epsilon(T) = 0, \quad \epsilon(B) = 0.
$$

The Hopf algebra of polynomial functions $F_q(G(1))$ is generated by $\{\mu, x, v, t\}$ and by the relations

$$
[\mu, x] = 2iw\mu, \quad [\mu, v] = -iwv^2, \quad [x, v] = -2iwv,
$$

t being a central element. The structure is completed by

$$
\Delta \mu = \mu \otimes 1 + 1 \otimes \mu + v \otimes x + \frac{1}{2}v^2 \otimes t, \quad \Delta t = t \otimes 1 + 1 \otimes t, \\
\Delta x = x \otimes 1 + 1 \otimes x + v \otimes t, \quad \Delta v = v \otimes 1 + 1 \otimes v; \\
S(\mu) = -\mu + v + \frac{1}{2}v^2 t, \quad S(x) = -x + tv, \quad S(t) = -t, \quad S(v) = -v; \\
\epsilon(\mu) = 0, \quad \epsilon(x) = 0, \quad \epsilon(t) = 0, \quad \epsilon(v) = 0.
$$

The involution is given by defining all the generators of $U_q(G(1))$ and $F_q(G(1))$ to be real, with the exception of $\mu$ for which $\mu^* = \mu - iwv$. Finally, the duality between $U_q(G(1))$ and $F_q(G(1))$ reads

$$
\Omega^\ell K^\ell T^\gamma N^\delta (\mu^\alpha x^\beta t^\gamma v^\delta) = i^{\alpha+\beta+\gamma+\delta} \alpha! \gamma! \delta! (-iw \ell)^\beta \delta_{\alpha,\alpha'} \delta_{\gamma,\gamma'} \delta_{\delta,\delta'}
$$

where $I = K^{-1}M, N = KB$ and $\ell \in \mathbb{Z}$ while the other indices are in $\mathbb{N}$. The involution satisfies the usual property $\tau^2 = \text{id}$ and $X^* (a) = \overline{X(\tau(a))}$.

It is easy to verify that the three primitive and real generators $\{\hat{\mu}, \hat{x}, \hat{t}\}$ with relations $[\hat{\mu}, \hat{x}] = 2iw\hat{\mu}$ and $[\hat{t}, \cdot] = 0$ and the projection $\pi(\mu) = \hat{\mu}, \pi(t) = \hat{t}, \pi(x) = \hat{x}, \pi(v) = 0$ define a real quantum subgroup $F_q(J)$. Finally, $\omega_{m,u} = \exp[-i(m\hat{\mu} + u\hat{t})]$ with $m, u \in \mathbb{R}$ defines a unitary one-dimensional corepresentation of $F_q(J)$.

The space $B_\pi$ is formed by the polynomials in $v$ and it must be extended in order to support the induced representation.

The construction is the same described in [1] for the real parameter case to which we refer for details. The algebra $H_0^{\text{irr}}$ of quantum square integrable functions on the homogeneous space is described in terms of two commuting generators $v_0$ and $v_1$ with the relation

$$wmv_0v_1 = v_1 - v_0,$$
and \( v_0^* = v_0, \ v_1^* = v_1 \). An useful basis is given by \( \{ \chi_m^\ell \}_{\ell \in \mathbb{Z}} \), where \( \chi_m = 1 + w m v_1 \) and \( \chi_m^{-1} = 1 - w m v_0 \). We can define the following structure of \( \mathcal{U}_q \)-module *-algebra on \( H_0^{\text{irr}} \):

\[
B. \chi_m^\ell = i w m \ell \chi_m^{\ell+1},
\]

and \( K. f = f, \ T. f = M. f = 0 \) for each \( f \in H_0^{\text{irr}} \). This algebra can be realized in the dual of \( \mathcal{U}_q \). Indeed let \( \varphi_{m,u} = e^{-i(mu+u0)} \), then \( v_0 = v \) and \( v_1 = \varphi_{m,u}^{-1} v \varphi_{m,u} \). The induced representation space is then \( H_{m,u}^{\text{irr}} = \varphi_{m,u} H_0^{\text{irr}} \).

Let’s define the functional \( \nu_w : H_0^{\text{irr}} \rightarrow \mathbb{C} \),

\[
\nu_w(1) = 1, \quad \nu_w(v_0^n) = \frac{1}{(w m)^n}, \quad \nu_w(v_1^n) = \frac{1}{(-w m)^n},
\]

i.e. \( \nu_w(\chi_m^\ell) = \delta_{\ell,0} \); let the corresponding sesquilinear form on \( H_{m,u}^{\text{irr}} \) be \( \langle a, b \rangle = \nu_w(a^* b) \) for \( a, b \in H_{m,u}^{\text{irr}} \). Then, since \( \varphi_{m,u}^* \varphi_{m,u} = \chi_m \), we have that

\[
\langle \varphi_{m,u} \chi_m^\ell, \varphi_{m,u} \chi_m^n \rangle = \nu_w(\chi_m^{\ell+n+1}) = \delta_{\ell+n+1,0}.
\]

(12)

According to [11] we can consider on \( H_0^{\text{irr}} \) the scalar product

\[
(\varphi_{m,u} \chi_m^\ell, \varphi_{m,u} \chi_m^n) = \delta_{\ell,n}.
\]

(13)

If we define \( j : H_{m,u}^{\text{irr}} \rightarrow H_{m,u}^{\text{irr}} \) by \( j(\varphi_{m,u} \chi_m^\ell) = \varphi_{m,u} \chi_m^{-\ell-1} \), we see that

\[
\langle \varphi_{m,u} \chi_m^\ell, \varphi_{m,u} \chi_m^n \rangle = (j(\varphi_{m,u} \chi_m^\ell), \varphi_{m,u} \chi_m^n)
\]

(14)

and \( \langle , \rangle \) is a Minkowski form on the \( j \)-space \( H_0^{\text{irr}} \).

The quasi-invariance properties of \( \nu_w \) are collected in the following proposition.

**Proposition 5.1** Let \( \phi : \mathcal{U}_q \rightarrow H_0^{\text{irr}} \) be defined by

\[
\phi[X] = \epsilon(X) + \sum_{n=1} \ c_n X(v_n^n) \chi_m^n, \quad c_n = \left( \frac{w m}{2} \right)^n \frac{(2n-1)!!}{n!}.
\]

Then \( \nu_w \) is quasi-invariant with weight \( \phi \) but not essentially invariant.

**Proof.** From the definition of \( \phi \) it follows that the cocycle property (11) is verified for \( X = \{M, T, K\} \) and arbitrary \( Y \in \mathcal{U}_q \). From a direct computation we see that (12) for \( X = B \) is a consequence of the following recurrence relation

\[
nc_n = w m \left( n - \frac{1}{2} \right) c_{n-1}.
\]

The result follows from the properties of the coproduct of generators. The property (2) of quasi invariance for \( h \) is trivial for \( X = \{M, T, K\} \) and can be directly checked on \( \chi_m^\ell \) for \( X = B \).  

Define \( \tilde{\varphi}_{m,u}(X) a = \sum_{(X)} X_{(1)} a \phi[X_{(2)}] \), with \( a \in H_{m,u}^{\text{irr}} \). By direct computation and using Proposition 1.4 we obtain the final result.
Proposition 5.2  The representation $\tilde{\rho}_{m,u}$ of $U_q(G(1))$, given by

$$
\tilde{\rho}_{m,u}(K^{\pm1}) \varphi_{m,u} \chi_{\ell}^{m} = \varphi_{m,u} \chi_{m}^{\ell \pm 1}, \quad \tilde{\rho}_{m,u}(B) \varphi_{m,u} \chi_{m}^{\ell} = iwm (\ell + \frac{1}{2}) \varphi_{m,u} \chi_{m}^{\ell}, \\
\tilde{\rho}_{m,u}(T) \varphi_{m,u} \chi_{m}^{\ell} = \varphi_{m,u} \chi_{m}^{\ell} (\frac{1}{2w^2 m} (2 - \chi_{m} - \chi_{m}^{-1}) + u);
$$

is unitary with respect to the form (12).

According to the definitions of $\tilde{\rho}_{m,u}$ is a $j$-representation of the quantum Galilei group.

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