THE INTEGRAL OF THE SUPREMUM PROCESS OF BROWNIAN MOTION

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Abstract

In this paper we study the integral of the supremum process of standard Brownian motion. We present an explicit formula for the moments of the integral (or area) \( A(T) \) covered by the process in the time interval \([0, T]\). The Laplace transform of \( A(T) \) follows as a consequence. The main proof involves a double Laplace transform of \( A(T) \) and is based on excursion theory and local time for Brownian motion.

Keywords: Brownian motion; supremum process; local time; Brownian areas

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1. Introduction

Let \( B(t), t \geq 0, \) be a standard Brownian motion. Consider the following associated processes: the supremum process \( S(t) = \max_{0 \leq s \leq t} B(s) \) and the local time \( L(t) \), which can be regarded as a measure of the time \( B(t) \) spends at 0 in the interval \([0, t]\); see [10, Chapter VI] for details. It is well known that these two processes, although pathwise quite different, have the same distribution [10, Chapter VI.2],

\[
\{S(t)\}_{t \geq 0} \overset{\text{d}}{=} \{L(t)\}_{t \geq 0},
\]

where \( \overset{\text{d}}{=} \) denotes equality in distribution.

The purpose of this paper is to study the distribution of the area under \( S(t) \) or, equivalently, \( L(t) \) over a given time interval \([0, T]\). That is, the integral

\[
A(T) := \int_0^T S(t) \, dt \overset{\text{d}}{=} \int_0^T L(t) \, dt.
\]  

(1.1)

For ease of notation, let \( A := A(1) \).

The area (1.1) appeared as a random parameter when analysing displacements for linear probing hashing. The Laplace transform of \( A \), which is presented in Corollary 2.1, provided the means to prove one of the main theorems in [9].

Note that the usual Brownian scaling

\[
\{B(Tt)\}_{t \geq 0} \overset{\text{d}}{=} \{T^{1/2} B(t)\}_{t \geq 0} \quad \text{for any } T > 0,
\]

implies the corresponding scaling for the supremum process,

\[
\{S(Tt)\}_{t \geq 0} \overset{\text{d}}{=} \{T^{1/2} S(t)\}_{t \geq 0}.
\]

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Thus, for $T > 0$, \[ A(T) = T \int_0^1 S(Tt) \, dt = T^{3/2} \mathcal{A}, \] (1.2) and it is enough to study $\mathcal{A}$.

2. Results

Let $\psi(s) := \mathbb{E}(e^{-sA})$ denote the Laplace transform of $\mathcal{A}$. An essential part of this paper is devoted to proving the following formula for the Laplace transform of a variation of $\psi$, or in other words, a double Laplace transform of $\mathcal{A}$. Such formulae have already been derived for the integral of $|B(t)|$ and other similar integrals of processes related to Brownian motion; see [8] and the survey [3].

**Theorem 2.1.** Let $\psi$ be the Laplace transform of $\mathcal{A}$. For all $\alpha, \lambda > 0$, \[ \int_0^\infty \psi(\alpha s^{3/2}) e^{-\lambda s} \, ds = \int_0^\infty \left( 1 + \frac{3\alpha s}{2\sqrt{2\lambda}} \right)^{-2/3} e^{-\lambda s} \, ds. \]

**Remark 2.1.** One of the parameters $\alpha$ and $\lambda$ in Theorem 2.1 can be eliminated (by setting it equal to 1, for instance) without loss of generality. In fact, for any $\beta > 0$, the formula is preserved by the substitutions $\lambda \mapsto \beta \lambda$, $\alpha \mapsto \beta^{3/2} \alpha$, and $s \mapsto \beta^{-1} s$.

The proof is given in Section 5. It is based on the excursion theory for Brownian motion and is inspired by similar arguments for other Brownian areas; see [8].

**Theorem 2.2.** The $n$th moment of $\mathcal{A}$ is \[ \mathbb{E}(\mathcal{A}^n) = \frac{n! \Gamma(n + 2/3)}{\Gamma(2/3) \Gamma(3n/2 + 1)} \left( \frac{3\sqrt{2}}{4} \right)^n, \quad n \in \mathbb{N}. \]

**Proof.** Set $\lambda = 1$ in Theorem 2.1 and denote the left- and right-hand sides by \[ I(\alpha) := \int_0^\infty \psi(\alpha s^{3/2}) e^{-s} \, ds \]
and \[ J(\alpha) := \int_0^\infty \left( 1 + \frac{3\alpha s}{2\sqrt{2}} \right)^{-2/3} e^{-s} \, ds. \]

The integrand of $I(\alpha)$ and all its derivatives with respect to $\alpha$ are dominated by functions of the form $s^k e^{-s}$, uniformly in $\alpha > 0$. Differentiation of $I(\alpha)$ is therefore allowed indefinitely due to dominated convergence. The same argument applies to $J(\alpha)$.

Also, the dominated convergence theorem shows that integration (with respect to $s$) can be interchanged with taking the limit $\alpha \to 0^+$. Thus, \[ \lim_{\alpha \to 0^+} \frac{d^n I(\alpha)}{d\alpha^n} = \lim_{\alpha \to 0^+} \int_0^\infty \frac{d^n}{d\alpha^n} \psi(\alpha s^{3/2}) e^{-s} \, ds \]
\[ = \int_0^\infty \lim_{\alpha \to 0^+} \frac{d^n}{d\alpha^n} (-s^{3/2})^n \mathbb{E}(\mathcal{A}^n \exp(-\alpha s^{3/2} \mathcal{A})) e^{-s} \, ds \]
\[ = (-1)^n \mathbb{E}(\mathcal{A}^n) \int_0^\infty s^{3n/2} e^{-s} \, ds \]
\[ = (-1)^n \Gamma \left( \frac{3n}{2} + 1 \right) \mathbb{E}(\mathcal{A}^n). \]
The integral of the supremum process of Brownian motion

Table 1: The first four moments of $A$.

| $n$ | $E(A^n)$ |
|-----|----------|
| 1   | $4/3\sqrt{2\pi}$ |
| 2   | $\frac{5}{\pi}$ |
| 3   | $64/63\sqrt{2\pi}$ |
| 4   | $\frac{11}{24}$ |

and

$$
\frac{d^n J(\alpha)}{d\alpha^n} = \lim_{\alpha \to 0^+} \int_0^\infty \frac{d^n}{d\alpha^n} \left( 1 + \frac{3\alpha s}{2\sqrt{2}} \right)^{-2/3} e^{-s} ds
$$

$$
= \int_0^\infty \lim_{\alpha \to 0^+} \frac{\Gamma(n + 2/3)}{\Gamma(2/3)} \left( \frac{-3s}{2\sqrt{2}} \right)^n \left( 1 + \frac{3\alpha s}{2\sqrt{2}} \right)^{-n-2/3} e^{-s} ds
$$

$$
= \frac{\Gamma(n + 2/3)}{\Gamma(2/3)} \left( -\frac{3\sqrt{2}}{4} \right)^n n!
$$

The fact that $I(\alpha) = J(\alpha)$ completes the proof.

The first four moments of $A$ are listed in Table 1. Furthermore, Stirling’s formula provides the asymptotic relation

$$
E(A^n) \sim \frac{2\sqrt{3\pi}}{3\Gamma(2/3)} n^{1/6} \left( \frac{n}{3e} \right)^{n/2}, \quad n \to \infty.
$$

(2.1)

**Corollary 2.1.** The Laplace transform of $A$ is

$$
\psi(s) = \frac{1}{\Gamma(2/3)} \sum_{n=0}^{\infty} \frac{\Gamma(n + 2/3)}{\Gamma(3n/2 + 1)} \left( \frac{-3\sqrt{2}s}{4} \right)^n
$$

**Proof.** The corollary follows from the identity

$$
\psi(s) = \sum_{n=0}^{\infty} \frac{(-s)^n}{n!} E(A^n).
$$

Note that the sum converges absolutely for every complex $s$.

The graph of $\psi(s)$ is shown in Figure 1.

**Remark 2.2.** The Laplace transform of $A$ can also be expressed in terms of generalised hypergeometric functions:

$$
\psi(s) = \text{$_1F_1$} \left( \frac{5}{6}, \frac{4}{6}, \frac{s^2}{6} \right) - \frac{4s}{3\sqrt{2\pi}} \text{$_2F_2$} \left( \frac{6}{6}, \frac{7}{6}, \frac{9}{6}, \frac{s^2}{6} \right).
$$
3. Tail asymptotics

Tauberian theorems by Davies [1] and Kasahara [7] (see [4, Theorem 4.5] for a convenient version) show that the moment asymptotics (2.1) imply the estimate \( \ln P(A > x) \sim -3x^2/2 \) for the tail of the distribution function. Thus, the following corollary is obtained.

**Corollary 3.1.** \( A \) has the tail estimate

\[
P(A > x) = \exp \left\{ -\frac{3x^2}{2} + o(x^2) \right\}, \quad x \to \infty.
\]

(This result can also be proved by large deviation theory; cf. similar results in [2].)

It seems difficult to obtain more precise tail asymptotics from the moment asymptotics, but it is natural to make a conjecture.

**Conjecture 3.1.** \( A \) has a density function \( f_A(x) \) satisfying

\[
f_A(x) \sim 2 \cdot 3^{1/3} \exp \left\{ -\frac{3x^2}{2} \right\}, \quad x \to \infty.
\]

In fact, if \( A \) has a density with \( f_A(x) \sim ax^b \exp(-cx^d) \) for some constants \( a, b, c, \) and \( d \), then it is the only possible choice that yields the moment asymptotics (2.1); cf. [5].

Conjecture 3.1 may be compared with similar results for several Brownian areas in [5]; see also [3]. Note that in these results for Brownian areas, the exponent of \( x \) is always an integer (0, 1, or 2). It is therefore of little surprise that here the exponent seems to be \( \frac{2}{3} \), corresponding to the power \( n^{1/6} \) in (2.1).

4. Preliminaries on point processes

Let \( \mathcal{S} \) be a measurable space. (In this paper, \( \mathcal{S} \) is either an interval of the real line or the product of two such intervals.) Although a point process \( \Xi \) will be regarded as a random set \( \{\xi_i\} \subset \mathcal{S} \), it is technically convenient to formally define it as an integer-valued random
measure $\sum \delta_{\xi_i}$. Hence, $\Xi(A)$ denotes the number of points $\xi_i$ that belong to a (measurable) subset $A \subseteq \mathcal{S}$. Also, $x \in \Xi$ is equivalent to $\Xi(\{x\}) > 0$. For further details, see, e.g. [6].

A Poisson process with intensity $d\mu$, where $d\mu$ is a measure on $\mathcal{S}$, is a point process $\Xi$ such that $\Xi(A)$ has a Poisson distribution with mean $\mu(A)$ for every measurable $A \subseteq \mathcal{S}$ and $\Xi(A_1), \ldots, \Xi(A_k)$ are independent for every family $A_1, \ldots, A_k$ of disjoint measurable sets. Lemma 4.1, below, is a standard formula for Laplace functionals; see, e.g. [6, Lemma 12.2(i)].

**Lemma 4.1.** If $\Xi$ is a Poisson process with intensity $d\mu$ on a set $\mathcal{S}$ and $f : \mathcal{S} \to [0, \infty)$ is a measurable function, then

$$E\left(\exp\left\{-\sum_{\xi \in \Xi} f(\xi)\right\}\right) = \exp\left\{-\int_{\mathcal{S}} (1-e^{-f(x)}) d\mu(x)\right\}.$$ 

Lemma 4.2, below, is a standard formula for Laplace functionals; see, e.g. [6, Lemma 12.2(i)].

**Lemma 4.2.** If $\lambda > 0$ then

$$\int_{0}^{\infty} (1-e^{-\lambda x})x^{-3/2} \, dx = 2\sqrt{\pi \lambda}.$$ 

### 5. Proof of Theorem 2.1

The set $\{t : B(t) = 0\}$ is almost surely (a.s.) closed and unbounded, so its complement $\{t : B(t) \neq 0\}$ is an infinite union of finite open intervals, denoted by $I_v = (g_v, d_v)$, $v = 1, 2, \ldots$, in some order. (The intervals cannot be ordered by appearance, since there is a.s. an infinite number of them in, say, $[0, 1]$. Fortunately, the order does not matter.) The restrictions of $B(t)$ to these intervals are called the excursions of $B(t)$. Let $\hat{e}_v$ be the excursion during $I_v$.

The local time $L(t)$ is constant during each excursion. Let $\tau_v$ be the local time during $\hat{e}_v$ and let $\ell_v := d_v - g_v$ be the length of $\hat{e}_v$. It is well known (see [10, Chapter XII]) that the collection of pairs $\{(\tau_v, \ell_v)\}_{v=1}^{\infty}$ forms a Poisson process in $[0, \infty) \times (0, \infty)$ with intensity

$$d\Lambda = (2\pi \ell^3)^{-1/2} \, d\tau \, d\ell.$$

Note also that, a.s., if the excursion $\hat{e}_v$ comes before $\hat{e}_{v1}$ then $\tau_{v1} < \tau_{v2}$.

Next, consider a Poisson process $\{T_i\}_{i=1}^{\infty}$ on $[0, \infty)$ with intensity $\lambda \, d\tau$, independent of $\{B(t)\}$. Assume that the points are ordered with $0 < T_1 < T_2 < \cdots$. Then $T_1, T_2 - T_1, \ldots$ are i.i.d. $\text{Exp}(\lambda)$ random variables with density function $\lambda e^{-\lambda t}$. Furthermore, $T_i$ is independent of $\{B(t)\}$ and, thus, of $\{\mathcal{A}(T)\}$. It follows from (1.2) that $\mathcal{A}(T_i) \overset{\text{d}}{=} T_i^{3/2} \mathcal{A}$ and, consequently,

$$E(\exp\{-\alpha \mathcal{A}(T_i)\}) = E(\exp\{-\alpha T_i^{3/2}(\mathcal{A})\}) = E(\exp(\alpha T_i^{3/2})) = \lambda \int_{0}^{\infty} e^{-\lambda s} \psi(\alpha s^{3/2}) \, ds. \quad (5.1)$$

The times $T_i$ are called marks, and an excursion is called marked if it contains at least one of the marks $T_i$. The marks $\{T_i\}$ are placed by first constructing $\{B(t)\}$ and then adding marks according to independent Poisson processes with intensities $\lambda \, dt$ in each excursion. Thus, given the excursions $\{\hat{e}_v\}$, each excursion $\hat{e}_v$ is marked with probability $1 - \exp\{-\lambda \ell_v\}$, independently of the other excursions. The Poisson process $\Xi := \{(\tau_v, \ell_v)\}$ defined by the excursions can be written as the union $\Xi' \cup \Xi''$, where

$$\Xi' := \{(\tau_v, \ell_v) : \hat{e}_v \text{ is unmarked}\}, \quad \Xi'' := \{(\tau_v, \ell_v) : \hat{e}_v \text{ is marked}\}.$$
By the general independence properties of Poisson processes, \( \Xi' \) and \( \Xi'' \) are independent Poisson processes with intensities
\[
d\Lambda' := e^{-\lambda \ell} \, d\Lambda = (2\pi)^{-1/2} \ell^{-3/2} e^{-\lambda \ell} \, d\tau \, d\ell \tag{5.2}
\]
and
\[
d\Lambda'' = (1 - e^{-\lambda \ell}) \, d\Lambda = (2\pi)^{-1/2} \ell^{-3/2} (1 - e^{-\lambda \ell}) \, d\tau \, d\ell,
\]
respectively. In particular, if the lengths are ignored, the local times of the marked excursions form a Poisson process \( \tilde{\Xi}' \) on \((0, \infty)\) with intensity
\[
\int_\ell=0^\infty (1 - e^{-\lambda \ell}) \, d\Lambda = \tilde{\lambda} \, d\tau,
\]
where, using Lemma 4.2,
\[
\tilde{\lambda} = \int_0^\infty (2\pi)^{-1/2} \ell^{-3/2} (1 - e^{-\lambda \ell}) \, d\ell = \sqrt{2\lambda}.
\tag{5.3}
\]

Owing to the fact that \( B(T_1) \neq 0 \) a.s., there exists a unique excursion \( \hat{e}_{\nu^*} \) that contains the first mark \( T_1 \), i.e. \( T_1 \in I_{\nu^*} \). Let \( \zeta := L(T_1) = \tau_{\nu^*} \) be the local time at \( T_1 \) (and, thus, during \( \hat{e}_{\nu^*} \)). Since \( \hat{e}_{\nu^*} \) is the first marked excursion, its local time \( \zeta \) is the first of the points in the Poisson process \( \tilde{\Xi} \) and, hence,
\[
\zeta \sim \text{Exp}(\sqrt{2\lambda}). \tag{5.4}
\]

The restriction of \( B(t) \) to the interval \([0, T_1]\) consists of all excursions \( \hat{e}_{\nu} \) with local time \( \tau_{\nu} < \tau_{\nu^*} = \zeta \) and the part of \( \hat{e}_{\nu^*} \) on \((g_{\nu^*}, T_1)\), plus the set
\[
[0, T_1] \setminus \bigcup_{\nu} I_{\nu} = \{ t \leq T_1 : B(t) = 0 \},
\]
which, a.s., has measure 0 and, thus, may be ignored. Consequently, since \( L(t) = \tau_{\nu} \) on \( I_{\nu} \),
\[
A(T_1) := \int_0^{T_1} L(t) \, dt
= \sum_{\nu} \int_{\nu} \int_b L(t) \, dt + \int_{g_{\nu^*}}^{T_1} L(t) \, dt
= \sum_{\nu} \int_{\nu} \tau_{\nu} \ell_{\nu} + \zeta (T_1 - g_{\nu^*})
= A' + A''.
\]

The sum defined as \( A' = \sum_{\nu} \tau_{\nu} \ell_{\nu} \) only contains terms for unmarked excursions \( \hat{e}_{\nu} \). Thus,
\[
A' = \sum_{(\tau_{\nu}, \ell_{\nu}) \in \Xi' : \tau_{\nu} < \zeta} \tau_{\nu} \ell_{\nu}.
\]

Recall that \( \zeta \) is determined by \( \Xi'' \) (as the smallest \( \tau \) with \((\tau, \ell) \in \Xi'' \) for some \( \ell \)) and that \( \Xi' \) and \( \Xi'' \) are independent. Hence, \( \Xi' \) and \( \zeta \) are independent. It follows from Lemma 4.1, with \( \mathcal{S} = (0, \zeta) \times (0, \infty) \) and \( f((\tau, \ell)) = \alpha \tau \ell \), that
\[
E(e^{-\alpha A' | \zeta}) = \exp \left\{ -\int_\tau=0^\zeta \int_{\ell=0}^\infty (1 - e^{-\alpha \tau \ell}) \, d\Lambda'(\tau, \ell) \right\}.
\]
Thus, (5.5) and (5.6) yield

\[
\begin{aligned}
\int_{\tau=0}^{\xi} \int_{\ell=0}^{\infty} (1 - e^{-\alpha \ell \tau}) \, d\Lambda'(\tau, \ell) &= \int_{\tau=0}^{\xi} \int_{\ell=0}^{\infty} (1 - e^{-\alpha \ell \tau})(2\pi)^{-1/2} \ell^{-3/2} e^{-\lambda \ell} \, d\ell \, d\tau \\
&= (2\pi)^{-1/2} \int_{\tau=0}^{\xi} \int_{\ell=0}^{\infty} (e^{-\lambda \ell} - e^{-(\lambda + \alpha) \ell}) \ell^{-3/2} \, d\ell \, d\tau \\
&= \int_{\tau=0}^{\xi} \sqrt{2(\lambda + \alpha \tau - \sqrt{\lambda})} \, d\tau \\
&= \frac{2\sqrt{2}}{3\alpha} ((\lambda + \alpha \xi)^{3/2} - \lambda^{3/2}) - \sqrt{2\lambda} \xi,
\end{aligned}
\]

and it follows that

\[
E(e^{-\alpha A'} \mid \xi) = \exp \left\{ \sqrt{2\lambda} \xi - \frac{2\sqrt{2}}{3\alpha} ((\lambda + \alpha \xi)^{3/2} - \lambda^{3/2}) \right\}. \tag{5.5}
\]

Now consider \(A'' = \xi (T_1 - g_{\nu'})\). Note that \(T_1 - g_{\nu'}\) is the location (relative to the left endpoint of the excursion) of the first mark in the first marked excursion. Since \(\Xi\) is a Poisson process with intensity independent of \(\tau\), the location \(T_1 - g_{\nu'}\) is independent of the local time \(\xi\) of the first marked excursion. Furthermore, the joint distribution of \((\nu', T_1 - g_{\nu'})\) has density

\[
(\tilde{\lambda})^{-1} \lambda e^{-\lambda y} (2\pi)^{-1/2} \ell^{-3/2} \, d\ell \, dy, \quad 0 < y < \ell < \infty,
\]

where the normalisation constant \(\tilde{\lambda}\) is given by (5.3). Consequently,

\[
E(e^{-\alpha A''} \mid \xi) = E(\exp(-\alpha (T_1 - g_{\nu'})) \mid \xi) \\
= \int_{y=0}^{\infty} \int_{\ell=y}^{\infty} e^{-\alpha(\lambda)(\tilde{\lambda})^{-1} \lambda e^{-\lambda y} (2\pi)^{-1/2} \ell^{-3/2} \, d\ell \, dy \\
= \pi^{-1/2} \lambda^{1/2} \int_{y=0}^{\infty} e^{-(\lambda + \alpha \xi)y} y^{-1/2} \, dy \\
= \lambda^{1/2} ((\lambda + \alpha \xi)^{-1/2}. \tag{5.6}
\]

Again, since \(\Xi'\) and \(\Xi''\) are independent, \(A'\) and \(A''\) are conditionally independent given \(\xi\). Thus, (5.5) and (5.6) yield

\[
E(\exp(-\alpha A(T_1)) \mid \xi) = E(e^{-\alpha A'} \mid \xi) E(e^{-\alpha A''} \mid \xi) \\
= \left( \frac{\lambda}{\lambda + \alpha \xi} \right)^{1/2} \exp \left\{ \sqrt{2\lambda} \xi - \frac{2\sqrt{2}}{3\alpha} ((\lambda + \alpha \xi)^{3/2} - \lambda^{3/2}) \right\}.
\]

By (5.4), \(\xi\) has the density \(\sqrt{2\lambda} e^{-\sqrt{2\lambda} x}, \ x > 0\), and it follows that

\[
E(\exp(-\alpha A(T_1))) = \lambda \sqrt{2} \int_{0}^{\infty} (\lambda + \alpha x)^{-1/2} \exp \left\{ -\frac{2\sqrt{2}}{3\alpha} ((\lambda + \alpha x)^{3/2} - \lambda^{3/2}) \right\} \, dx.
\]

Finally, the substitution

\[
\frac{2\sqrt{2}}{3\alpha \lambda} ((\lambda + \alpha x)^{3/2} - \lambda^{3/2}) \mapsto s
\]
provides the slightly simpler formula
\[
E(\exp(-\alpha A(T_1))) = \lambda \int_0^\infty \left( 1 + \frac{3\alpha s}{2\sqrt{2}\lambda} \right)^{-2/3} e^{-s\lambda} \, ds.
\]

The result now follows by a comparison with (5.1).

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