A POSSIBLE ORIGIN OF LOGNORMAL DISTRIBUTIONS IN GAMMA-RAY BURSTS

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ABSTRACT

We show that if the intrinsic break energy of gamma-ray bursts (GRBs) is determined by the product of more than three random variables, the observed break energy distribution becomes almost lognormal including the redshift effect because of the central limit theorem. The excess from the lognormal distribution at the low break energy is possibly due to the high-redshift GRBs. The same argument may also apply to the pulse duration, the interval between pulses, and so on.

Subject headings: gamma rays: bursts — gamma rays: theory

1. INTRODUCTION

Among the statistical properties of the observed quantities, lognormal distributions are frequently seen in gamma-ray bursts (GRBs). The lognormal distribution may be defined as the distribution of a random variable \( x \) whose logarithm is normally distributed as

\[
f(x)dx = \begin{cases} 
\frac{1}{\sqrt{2\pi} \sigma} \exp\left[-\frac{(\log x - \mu)^2}{2\sigma^2}\right] d\log x & \text{if } x > 0, \\
0 & \text{if } x \leq 0,
\end{cases}
\]

where \( f(x) \) is the probability density function for \( x \), and \( \mu \) and \( \sigma \) are the sample mean and the variance of \( \log x \), respectively.\(^{1}\) Crow & Shimizu (1988). This distribution is unimodal and positively skewed. McBreen et al. (1994) pointed out that the total duration of the long and short bursts and the time interval between pulses are consistent with the lognormal distributions. Li & Fenimore (1996) showed that the pulse fluence and the pulse interval distributions within each burst are consistent with lognormal distributions. Nakar & Piran (2001) found that the pulse durations also have a lognormal distribution. The break energy distribution is also lognormal (Preece et al. 2000; see § 2).

Lloyd, Fryer, & Ramirez-Ruiz (2001) suggested that \( \sim 10\% \) of GRBs might have a redshift larger than 6, so the redshift distribution might be wide. Therefore, it is quite strange that the observed break energy distribution and the duration distribution are lognormal, since it does not seem that the observed lognormal distribution reflects the redshift distribution of the GRBs. Even if the break energy distribution is lognormal at the source, the observed break energy should be smaller than the intrinsic one by a factor of \( 1 + z \), while the observed duration should be longer than the intrinsic one by a factor of \( 1 + z \). These factors change by order of unity between \( z = 0 \) and \( \sim 6 \). In this Letter we consider a possible origin of the observed lognormal distributions in GRBs from the viewpoint of the central limit theorem.\(^2\)

2. BREAK ENERGY DISTRIBUTIONS

Figure 1 shows the histogram of the break energy \( E_b \) taken from the electronic edition of Preece et al. (2000).\(^3\) The \( \chi^2 \) test of all data gives the probability of \( 1.4 \times 10^{-85} \) (the reduced \( \chi^2 \) is 16.5 with 66 degrees of freedom [dof]) that the data were taken from the lognormal distribution. Therefore, the null hypothesis that the break energy distribution is lognormal fails. However, if we exclude the data in the high- and low-energy ends, the fit becomes good, as shown in Figure 1. The \( \chi^2 \) test of the data between 70.8 and 708 keV gives the probability of 0.497 (the reduced \( \chi^2 \) is 0.963 with 17 dof) that the data were taken from the lognormal distribution, with \( \mu = 2.38 \pm 0.004 \) (\( E_b \simeq 238 \) keV) and \( \sigma = 0.240 \pm 0.004 \) (1 sigma width is between 137 and 413 keV). The improvement of the lognormal fit to the break energy distribution excluding the high- and low-energy ends may suggest that the soft and hard bursts originate from a different class of GRBs or emission mechanisms. Anyway, we hereafter assume that the observed break energy distribution is lognormal.

Now let us assume that the intrinsic break energy distribution is lognormal. Next, we numerically calculate the observed break energy distribution assuming that the redshift distribution has the form

\[
f(z)dz = \begin{cases} 
\frac{A(1 + z)^{\alpha - 1}(1 + z_0)^{\beta - 1} dz}{A(1 + z_0)^{\alpha - 1}(1 + z)^{\beta - 1} dz} & \text{if } 0 < z < z_0, \\
0 & \text{if } z_0 < z,
\end{cases}
\]

where \( \beta < 0 \). This redshift distribution rises proportional to \( (1 + z)^{\alpha - 1} \) to a redshift of \( z_0 \) and then declines proportional to \( (1 + z)^{\beta - 1} \); it is similar to that in Figure 8 of Lloyd et al. (2001). To mimic Figure 1, we generate 155 bursts, with each burst having 35 spectra for each realization. We take the mean and the variance of the intrinsic break energy distribution so that the mean and the variance of the observed break energy are close to those in Figure 1.

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\(^{3}\) The lognormal distribution of the time interval between pulses and the pulse fluence might be reproduced by a fine tuning of model parameters (Spada, Panaitescu, & Mészáros 2000).

\(^{4}\) It is not known whether the paucity of the soft and hard bursts is real or not because harder bursts have fewer photons (Cohen, Piran, & Narayan 1998; Lloyd & Petrosian 1999; but see Brainerd et al. 1999), and there may exist relatively many soft bursts with low luminosities, so-called X-ray–rich GRBs (or X-ray flashes or fast X-ray transients; Strohmayer et al. 1998; Heise et al. 2002; Kippen et al. 2001). Here we assume that the selection effect is small.
For each realization, we make a $\chi^2$ fit to obtain the probability that the data are taken from the lognormal distribution. As in Figure 1, we use the data between 70.8 and 708 keV for the $\chi^2$ test. From this simulation, we have found that the average probability can reach ~0.5 even when the variance of the redshift distribution is comparable to that observed ($\sigma^2 \sim 0.240^2$), although it takes a slightly stronger condition to preserve the lognormal form than $\sigma^2 < 0.240^2$. At first glance it appears strange that the simulations do not reflect the redshift distribution, contrary to the argument in § 1. In § 3 we show that it is not actually strange but natural owing to the central limit theorem.

Figure 2 shows the histogram of the observed break energy for one experimental realization with $(z_0, a, b) = (3, 3, -3)$. The $\chi^2$ test of the data between 70.8 and 708 keV gives the probability of 0.128 (the reduced $\chi^2$ is 1.39 with 17 dof) that the data were taken from the lognormal distribution, with $\mu = 2.37 \pm 0.004$ ($E_b \approx 235$ keV) and $\sigma = 0.256 \pm 0.004$ (1 $\sigma$ width is between 130 and 423 keV). It is interesting to note the excess of soft bursts relative to the lognormal fit as in Figure 1. The average redshifts of these soft bursts are relatively high.

3. LOGNORMAL DISTRIBUTIONS

The standard model of the GRB emission is the optically thin synchrotron shock model (e.g., Piran 1999). A similar discussion in the following is applied to the inverse Compton model. Let us consider a slow (rapid) shell with a Lorentz factor $\gamma_s$ ($\gamma_r$), a mass $m_s$ ($m_r$), and a width $l_s$ ($l_r$). When a separation between two shells is $L$, the collision takes place at a radius of $R_c \approx 2L\gamma_r^2$. At the collision, a forward and reverse shock are formed. Here we consider the reverse shock propagating into the rapid shell. The discussion for the forward shock is similar. We assume that a fraction of electrons $\xi$ is accelerated in the shock to a power-law distribution of Lorentz factor $\gamma_w$, $N(\gamma_w) d\gamma_w \propto \gamma_w^{-p} d\gamma_w$, for $\gamma_w \geq \gamma_{min} = ((p - 2)/(p - 1))\xi \gamma_r \gamma_r^2 n_m m e^2$, where $n'$ and $u'$ are the number density and the internal energy density in the local frame, respectively, $\gamma \equiv 2$, and we assume that a fraction $\xi_e$ of the internal energy goes into the electrons. We also assume that a fraction $\xi_{int}$ of the internal energy goes into the magnetic field, $B^2 = 8\pi \xi_{int} u'$. The local frame quantities, $u'$ and $n'$, can be calculated using the shock jump conditions (Blandford & McKee 1976; Sari & Piran 1995). We assume that the unshocked shells are cold and the shocked shells extremely hot. If the Lorentz factor of the shocked region is $\gamma_s$, the relative Lorentz factor of the unshocked and shocked regions is given by $\gamma_{rel} = (\gamma_s/\gamma_t + \gamma_t/\gamma_s)/2 \approx \gamma_t/\gamma_s$, so $\gamma_t = (\gamma_{rel} - 1) \gamma s \gamma_t m e^2 \approx \gamma_{rel} n' m e^2$. The number densities of the unshocked and shocked regions are given by $n' s = m_s/4\pi \gamma s R_c^2 L^{2}, n' s = m_s/16\pi \gamma s L^{2} \gamma_t^2 L_t^{2}$, and $n' t = (4\gamma_{rel} + 1) n' s = 4\gamma_{rel} n' s$, respectively. The characteristic synchrotron energy is given by

$$E_b = \frac{\hbar \gamma \gamma_{min}^2}{m_e c(1 + z)} \approx 260 \frac{(p - 2)^2}{p - 1} \times \varepsilon_{\gamma}^{1/2} \varepsilon_{\gamma}^{1/2} \gamma_{rel}^{-2} L^{-1/2} \gamma_t^{-2} \gamma_t^{-2} \gamma_{rel}^{-2} (1 + z)^{-1} \text{keV},$$

where we assume that the source is at a redshift $z$. Note that the relative Lorentz factor of the unshocked and shocked regions $\gamma_{rel}$ depends on the relative Lorentz factor of the rapid and slow shell, $\gamma_{rel} = (\gamma_s/\gamma_t + \gamma_t/\gamma_s)/2 \approx \gamma_t/\gamma_s$, and the ratio between the number densities in these shells $f = n_1/n_2 = m_1 l_2 / m_2 l_1$, (Sari & Piran 1995). For the ultrarelativistic shock case $\gamma_{rel} \gg f$, $\gamma_{rel} = (\gamma_s/\gamma_t + \gamma_t/\gamma_s)/2 = (m_1 l_2 / m_2 l_1)^{1/2}/2$.

Equation (3) shows that the break energy is written in the form of a product of many variables. For such a variable made
from the product of many variables, the lognormal distribution may have a very simple origin, i.e., the central limit theorem (Crow & Shimizu 1988; Montroll & Shlesinger 1982). Let a variable \( q \) be written in the form of a product of variables,

\[
q = x_1 x_2 \ldots x_n.
\]  

(4)

Then,

\[
\log q = \log x_1 + \log x_2 + \ldots + \log x_n.
\]  

(5)

When the individual distributions of \( \log x_i \) satisfy certain weak conditions that include the existence of second moments, the central limit theorem is applicable to the variable \( \log q \), so the distribution function of \( \log q \) tends to the normal distribution as \( n \) tends to infinity.

As an example, we numerically generated random variables \( x_i \) \( (i = 1, 2, \ldots) \) whose logarithms are uniformly distributed between 0 and 1. Figure 3 shows the histogram of the product of these three variables, \( q = x_1 x_2 x_3 \), for \( 10^4 \) experimental realizations. The distribution of \( q \) agrees with the lognormal distribution quite well. It is surprising that the \( \chi^2 \) test gives the probability of 0.483 (the reduced \( \chi^2 = 1.00 \) with 278 dof) that the distribution of \( q \) is taken from the lognormal distribution. This example shows that the lognormal distributions may be achieved by a relatively small number of variables (Yonetoku & Murakami 2001). Note that when the number of variables is two, i.e., \( q = x_1 x_2 \), the probability that the distribution is taken from the lognormal distribution is only \( 1.6 \times 10^{-3} \), so a product of only one more variable may make a distribution lognormal.

Therefore, the lognormal distribution of the break energy may be a natural result from the central limit theorem. We might say, "Astrophysically, not but gives the lognormal probability of \( 0.483 \) (the reduced \( \chi^2 = 1.00 \) with 278 dof) that the distribution of \( q \) is taken from the lognormal distribution. This may suggest the existence of a product of only one more variable may make a distribution lognormal.

4. PULSE FLUENCE/DURATION/INTERVAL DISTRIBUTIONS

Let us consider the lognormal distributions in other quantities related to GRBs. When the rapid shell catches up with the slow one in the internal shock using the conservation of the energy and momentum, the Lorentz factor of the merged shell \( \gamma_m \) and the internal energy \( E_{in} \) produced by the collision are given by

\[
\gamma_m = [(m_\gamma + m_m)(m_\gamma + m_m + m_\gamma)\sqrt{1 - (m_\gamma + m_m)/m_\gamma} + (m_m + m_\gamma)/\gamma_m], \quad E_{in} = m_\gamma c^2 + m_m c^2 - m_m^2 c^2/\gamma_m^2,
\]

respectively (e.g., Piran 1999). If we assume that a fraction \( \epsilon_e \) of the internal energy goes into the electrons and a fraction \( \epsilon_m \) of the energy radiated by the electrons is within the gamma-ray band, the observed energy is given by

\[
E_{obs} = \epsilon_e \epsilon_m E_{in} (1 + z)^{-1} \sim \epsilon_e \epsilon_m m_\gamma c^2 (1 + z)^{-1}.
\]  

(6)

Equation (6) shows that the observed energy, which is proportional to the pulse fluence, is written in the form of a product of five variables: \( \epsilon_e, \epsilon_m, \gamma_m, \gamma_\gamma, m_\gamma \). Therefore, the lognormal distribution of the pulse fluence may be a natural result of the central limit theorem.

The pulse duration is determined by three timescales: the hydrodynamic timescale, the cooling timescale, and the angular spreading timescale (Kobayashi, Piran, & Sari 1997; Katz 1997; Fenimore, Madras, & Sergei 1996). The cooling timescale is usually much shorter than the other two timescales in the internal shocks (Sari, Narayan, & Piran 1996). The hydrodynamic timescale \( (\sim l/c) \) and the angular spreading timescale determine the rise and decay times of the pulse, respectively. Since most observed pulses rise more quickly than they decay (Norris et al. 1996), we assume that the pulse duration is mainly determined by the angular spreading time, \( \sim R_s/2c\gamma_\gamma^2 \). Then, the pulse duration \( \delta t \) is given by

\[
\delta t = \frac{L}{c} \frac{\gamma_\gamma^2}{\gamma_m} (1 + z).
\]  

(7)

On the other hand, the interval between pulses \( \Delta t \) is determined by the separation between shells,

\[
\Delta t = \frac{L}{c} (1 + z),
\]  

(8)

since all shells are moving toward us with almost the speed of light (Kobayashi et al. 1997; Nakar & Piran 2001).

Equation (8) shows that the pulse interval \( \Delta t \) reflects the separation between shells \( L \), while equation (7) shows that the pulse duration \( \delta t \) is multiplied by one more factor \( (\gamma_\gamma/\gamma_m)^2 \) other than \( (L/c)(1 + z) \). Therefore, if we consider that the distribution of a product of variables tends to the lognormal distribution as the number of the multiplied variables increases, the distribution of the pulse duration \( \delta t \) may be closer to the lognormal distribution than that of the pulse interval \( \Delta t \). In fact, Nakar & Piran (2001) argued that the pulse duration \( \delta t \) has the lognormal distribution while the pulse interval \( \Delta t \) does not, as noticed by Li & Fenimore (1996). The pulse interval has an excess of long intervals relative to the lognormal distribution. This may suggest the existence...
of a different distribution, i.e., quiescent times (long periods with no activity; Nakar & Piran 2001; Ramirez-Ruiz & Merloni 2001). But the central limit theorem may also be responsible for the lognormal distribution of the pulse duration \( \Delta t \).

5. DISCUSSIONS

We considered the possible origin of the lognormal distributions in the break energy, the pulse fluence, and the pulse duration as a result of the central limit theorem. Astrophysically, the lognormal distribution may be achieved by a product of only a few variables. The effect of the redshift is just to add one variable to the product so that the redshift distribution is hidden.

We have no idea about the origin of the lognormal distributions in the pulse interval and the total duration. However, the viewing angle may be one factor to be multiplied to the pulse interval. Recently, we suggested that the luminosity-lag relation could be explained by the variation in the viewing angle \( \theta \) from the axis of the jet (Ioka & Nakamura 2001; Nakamura 2000). The duration of the pulse from the jet also depends on the viewing angle, and according to Figure 2 of Ioka & Nakamura (2001) we have

\[
\Delta t \propto \frac{L}{c} \left( 1 + z \right) \left( 1 + \gamma^2 \theta^2 \right),
\]

when \( \theta \sim \Delta \theta \), where \( \gamma \) is the Lorentz factor of the jet and \( \Delta \theta \) is the opening half-angle of the jet. The multiplied factor \( 1 + \gamma^2 \theta^2 \) may be responsible for the lognormal distribution of the pulse interval \( \Delta t \). Note that the total duration \( \Delta T \) is equal to the lifetime of the central engine and thus does not depend on \( \theta \).

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