Remarks on Gauge-Invariant Variables and Interaction Energy in QED

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Abstract

The calculation of the interaction energy in pure QED and Maxwell-Chern-Simons gauge theory is re-examined by exploiting the path dependence of the gauge-invariant variables formalism. In particular, we consider a spacelike straight line which leads to the Poincaré gauge. Subtleties related to the problem of exhibiting explicitly the interaction energies are illustrated.

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I. INTRODUCTION

It is well known that the development of an analytical understanding of non-perturbative aspects of quantum chromodynamics (QCD) is a subject of intense study. As a step towards this goal there is a renewed recent interest in formulations of QCD in which gauge invariant variables are explicitly constructed \[1,2\]. For instance, quarks are considered as dressed objects, where this dressing can be viewed as surrounding the quark with a cloud of gauge fields; all this in connection with the confinement problem.

On the other hand, this last remark about the gauge-invariant or dressing field is no new. It was Dirac \[3\] who originally proposed to use for the electron the gauge-invariant field

$$\Psi = \exp \left( ie \frac{\partial_i A^i}{\nabla^2} \right) \psi,$$

as a physical variable. Consequently, \(\Psi\) describes creation and annihilation of charges together with their proper electric field, where this field gives rise to the static interaction that obeys Coulomb’s law. It is worthwhile remarking at this point that Dirac’s expression for the electron field is both non-local and non-covariant. Hence the physical electron in QED is dressed, and as we mentioned before, this dressing looks like a cloud of gauge fields surrounding the charged particles with the characteristic feature that this cloud spreads out over the whole space, giving rise to a non-local object. Furthermore, we recognize that Dirac’s electron field is written in the Coulomb gauge, whence one can appreciate the link between the physical or dressed variables and gauge fixing. Recently, by using a gauge-invariant but path dependent formalism in abelian gauge theories, we illustrated how the gauge fixing procedure corresponds, in this formalism, to a path choice. We therefore developed a path-dependent but physical QED where a consistent quantization directly in the path space was carried out \[4\].

Thus, our concern in this Brief Report is to reconsider the calculation of the interaction energy between point-like sources in QED, paying due attention to the structure of the fields that surround the charges by exploiting the path dependence of the gauge-invariant
variables formalism. In Sec.II we reexamine this calculation in pure QED or, more precisely, electrodynamics of massive charges, and in Maxwell-Chern-Simons gauge theories.

II. INTERACTION ENERGY

A. PURE QED

In order to calculate the energy of the external fields of static charges, we shall begin by recalling the expression for a physical electron \[3\,4\],

\[
\Psi(y) = \exp \left( -ie \int_{C_{\xi y}} dz^\mu A_\mu(z) \right) \psi(y), \tag{2}
\]

where the integral is taken along some path or contour \(C_{\xi y}\) connecting \(\xi\) and \(y\). As already expressed, the main point is that we can choose a particular gauge condition by selecting a specific path or contour. Thus we now choose the path as a spacelike straight line which leads to the Poincaré gauge \[4\]:

\[
z^k = \xi^k + \alpha(y - \xi)^k, \quad \text{where} \quad 0 \leq \alpha \leq 1
\]

is the parameter describing the contour and \(\xi\) is a fixed point (reference point). There is no essential loss of generality if we restrict our considerations to \(\xi^k = 0\). Accordingly, in the Poincaré gauge, (2) becomes

\[
\Psi(y) = \exp \left( -ie \int_0^y dz^k A_k(z) \right) \psi(y) \tag{3}
\]

Two remarks are pertinent at this point. First, Eq.(3) does not follow from Eq.(2) in the temporal gauge. It was shown in Ref. \[4\] that to attain the temporal gauge we have to choose a timelike straight line:

\[
z^0 = t^0 + \alpha (x^0 - t^0), \quad \text{where} \quad t^0 \text{ is a arbitrary parameter.}
\]

Second, it should be noted that with respect to gauge transformations the physical electron (3) acquire a phase factor \(\exp (-ie\Lambda(\xi = 0))\); as well as, since the point \(\xi = 0\) is assumed to be fixed, the translational invariance of the physical electron is broken. These drawbacks are avoided by letting to point \(\xi\) go to infinity.

At the same time (3) represents charged particles with a static electric field on a line or, more precisely, on a tube. To see this more clearly let \(|E\rangle\) be an eigenvector of the electric field operator \(E_i(x)\), with eigenvalue \(\varepsilon_i(x)\) :
Next we will consider the state $\Psi(y) \langle E \rangle$. Using (4) we obtain

$$E_i(x) \Psi(y) \langle E \rangle = \Psi(y) E_i(x) \langle E \rangle + [E_i(x), \Psi(y)] \langle E \rangle. \quad (5)$$

Using the Hamiltonian formalism developed in Ref. [4], we find that

$$E_i(x) \Psi(y) \langle E \rangle = \left(\varepsilon_i(x) + e \int_0^1 d\alpha \delta^{(3)}(\alpha y - x)\right) \Psi(y) \langle E \rangle. \quad (6)$$

We have thus verified that the operator $\Psi(y)$ is the dressing operator of creation of an electron together with their proper electric field (or the operator of absorption of a positron). Moreover, the above result clearly proves that we have a static electric field on a line (tube), because the integral in (6) is nonvanishing only on the contour of integration. It is perhaps worth mentioning at this stage that if we consider a modified form for the physical electron in the Poincaré gauge (3), which is equivalent to the Coulomb gauge [4], that is,

$$\Psi(y) = \exp \left(-i e \int_{y_0}^y dz A_k^L(z)\right) \psi(y), \quad (7)$$

where $A_k^L$ refers to the longitudinal part of $A_k$, we would obtain that the field $\Psi$ dresses the charge $\psi$ with the static Coulomb electric field, in other words,

$$E_i(x) \Psi(y) \langle E \rangle = \left(\varepsilon_i(x) + e \frac{x_i - y_i}{4\pi |x - y|^3}\right) \Psi(y) \langle E \rangle. \quad (8)$$

For more details about the comparison between the Poincaré and Coulomb gauges we refer to [3].

Now we compute the energy of the external field of static charges, where a fermion is localized at $y'$ and an antifermion at $y$. So we proceed to calculate the mean value of the electromagnetic energy operator $H$ in the physical state $\langle \Omega \rangle$, which we will denote by $\langle H \rangle_\Omega$.

From the Hamiltonian framework presented in Ref. [4], $\langle H \rangle_\Omega$ is given by

$$\langle H \rangle_\Omega = \langle \Omega | \int d^3x \left(-\frac{1}{2}\pi^i \pi^i + \frac{1}{4}F_{ij}F^{ij}\right) | \Omega \rangle. \quad (9)$$

As mentioned before, the fermions are taken to be infinitely massive (static), which means that there is no magnetic field. In such a case the expression (9) then becomes
\[ \langle H \rangle_\Omega = \langle \Omega | \frac{1}{2} \int d^3x \ E^2(x) | \Omega \rangle. \]  

As it has been established by Dirac [3], the physical states \(| \Omega \rangle\) are characterized by the gauge invariant ones. Thus a state which has a fermion at \(y'\) and an antifermion at \(y\) is given by

\[ | \Omega \rangle \equiv | \bar{\Psi}(y)\Psi(y') \rangle = \bar{\psi}(y) \exp \left( -ie \int_{y}^{y'} dz^i A_i(z) \right) \psi(y') | 0 \rangle, \]  

where \(| 0 \rangle\) is the physical vacuum state. Following steps similar to those leading to Eq.(6) we obtain

\[ E_i(x) | \bar{\Psi}(y)\Psi(y') \rangle = \langle \bar{\Psi}(y)\Psi(y') | H | 0 \rangle + e \int_{y}^{y'} dz_i \delta^{(3)}(z - z') | \bar{\Psi}(y)\Psi(y') \rangle \]  

By using (12) we then evaluate the energy in presence of the static charges, yielding

\[ \langle H \rangle_\Omega = \langle H \rangle_0 + \frac{e^2}{2} \int_{y}^{y'} dz^i \int_{y}^{y'} dz'_i \delta^{(3)}(z - z'), \]  

where \( \langle H \rangle_0 = \langle 0 | H | 0 \rangle \). We emphasize that the second term in the r.h.s. of (13) corresponds to an interaction energy of the external fields of static charges, after substracting the divergent energies associated with the single particle states. Remembering that the integrals over \(z_i\) and \(z'_i\) are zero except on the contour of integrations, one obtains the following interaction energy

\[ V = \frac{e^2}{2} k | y - y' | , \]  

where \( k = \delta^{(2)}(0) \).

Special care has to be exercised since the physical interpretation of (14) is not clear in the literature and its discussion must be amended [7]. For instance, it has been argued [8] that the energy (14) is unstable, that is, it breaks down into electromagnetic radiation and the Coulomb field of two opposite charges. Such a picture, in our opinion, is certainly debatable because the sources of the external fields are stationary. In fact, we
now show that although the Coulomb interaction does not appear explicitly in the quantity \( \frac{e^2}{2} \int d^3x \left( \oint_y d^3z_i \delta^{(3)}(x - z) \right)^2 \), this expression is nothing but the Coulomb interaction plus an infinite self-energy term. For this, we focus our attention to

\[
V = \frac{e^2}{2} \int d^3x \left( \oint_y d^3z_i \delta^{(3)}(x - z) \right)^2. \tag{15}
\]

As already expressed, in order to carry out this calculation we write the path as \( z = y + \alpha(y' - y) \), where \( \alpha \) is the parameter describing the contour. In this case we have that

\[
\int_y^{y'} dz_i \delta^{(3)}(x - z) = (y' - y) \int_0^1 d\alpha \delta^{(3)}(y - x + \alpha(y' - y)). \tag{16}
\]

This can be conveniently written using spherical coordinates as

\[
\int_y^{y'} dz_i \delta^{(3)}(x - z) = \frac{(y' - y)}{|y' - y|} \frac{1}{|y' - y| - |y - x|} \sum_{l,m} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi), \tag{17}
\]

hence expression (17) reduces to

\[
\int_y^{y'} dz_i \delta^{(3)}(x - z) = \frac{1}{|y' - y|} \sum_{l,m} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi). \tag{18}
\]

According to (18), the expression for the interaction energy (15) reads

\[
V = \frac{e^2}{2} \int d^3x \left( \frac{y' - y}{|y' - y|} \frac{1}{|y' - y|} \sum_{l,m} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) \right)^2. \tag{19}
\]

Introducing the integration variable \( r = y - x \), and using usual properties for the spherical harmonics, we obtain

\[
V = \frac{e^2}{2} \int d^3x \left( \frac{y' - y}{|y' - y|} \frac{1}{|y' - y|} \sum_{l,m} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) \right)^2. \tag{20}
\]

with \( r = |r| \). In other words, we find that

\[
V = -\frac{e^2}{4\pi} \frac{1}{|y' - y|} \tag{21}
\]

after substracting the self-energy term.
To end this subsection we also draw attention to the fact that with the path choice stated in (3) (modified Poincaré gauge) which is equivalent to the Coulomb gauge, and from the formalism developed in [4], we can write a scalar potential as

$$A_\nu(\perp, x) = - \int_0^1 d\alpha \, x \cdot E^L(t, \alpha x),$$

which, by employing the Hamiltonian structure of pure QED, may be rewritten as

$$A_\nu(\perp, x) = \int_0^1 d\alpha \, x^i E^L_i(t, \alpha x) = - \int_0^1 d\alpha \, \frac{x^i \partial^\alpha x^i}{\nabla^2 \alpha} J^0(\alpha x),$$

where the superscript $L$ refers to the longitudinal part and $J^0$ is the external source. As a consequence of this, the static potential $V$ for a pair of static point-like opposite charges located at $y$ and $y'$, that is, $J^0(t, x) = e \left( \delta^{(3)}(x - y) - \delta^{(3)}(x - y') \right)$, is given by

$$V = e \left( A_\nu(y) - A_\nu(y') \right) = -\frac{e^2}{4\pi} \frac{1}{2} \frac{1}{|y' - y|},$$

after subtracting the self-energy term. Let us point out that (24) is the same expression as in Ref. [6], which find here an independent derivation.

**B. MAXWELL-CHERN-SIMONS GAUGE THEORY**

We now consider the calculation of the interaction energy between static point-like sources in a topologically massive gauge theory. In such a case the Lagrangian reads [9]:

$$L = -\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{4} \theta \, \varepsilon^{\mu\nu\rho} A_\mu F_{\nu\rho} - A_0 J^0,$$  

where $J^0$ is the external current and $\theta$ is the topological mass.

Before we proceed to work out explicitly the energy, we shall begin by summarizing the canonical quantization of the theory (25) from the Hamiltonian analysis point of view. The canonical momenta are $\pi^\mu = -F^0\mu + \frac{\theta}{2} \varepsilon^{0\mu\nu} A_\nu$ with the only nonvanishing canonical Poisson brackets being

$$\{ A_\mu(t, x), \pi^\nu(t, y) \} = \delta^\nu_\mu \delta^{(2)}(x - y).$$
As we can see there is one primary constraint, $\pi^0 = 0$, and $\pi^i = F^{i0} + \frac{\theta}{2} \varepsilon^{ij} A_j$ ($i, j = 1, 2$). So the canonical Hamiltonian is

$$H_c = \int d^2 x \left( -\frac{1}{2} F_{i0} F^{i0} + \frac{1}{4} F^{ij} F_{ij} + \pi_i \partial^i A_0 - \frac{\theta}{2} \varepsilon^{ij} A_0 \partial_i A_j + A_0 J^0 \right).$$

(27)

The conservation in time of the constraint $\pi^0$ leads to the secondary constraint (Gauss law):

$$\Omega_1(x) = \partial_i \pi^i + \frac{\theta}{2} \varepsilon_{ij} \partial^j A^i - J^0 = 0.$$  

(28)

There are no more constraints in the theory and the two we have found are first class.

The corresponding total (first class) Hamiltonian that generates the time evolution of the dynamical variables is given by

$$H = H_c + \int d^2 x \left( c_0(x) \pi_0(x) + c_1(x) \pi_1(x) \right),$$

(29)

where $c_0(x)$ and $c_1(x)$ are arbitrary functions. Since $\pi^0 = 0$ for all time and $\dot{A}_0(x) = [A_0(x), H] = c_0(x)$, which is arbitrary, we discard $A_0(x)$ and $\pi_0(x)$. In fact, it is redundant to retain the term containing $A_0$ because it can be absorbed by redefining the function $c_1(x)$.

In this case, (29) takes the form

$$H = \int d^2 x \left( -\frac{1}{2} F_{i0} F^{i0} + \frac{1}{4} F^{ij} F_{ij} + c'(x) \left( \partial_i \pi^i + \frac{\theta}{2} \varepsilon_{ij} \partial^j A^i - J^0 \right) \right),$$

(30)

where $c'(x) = c_1(x) - A_0(x)$.

According to the standard procedure we impose one gauge constraint such that the full set of constraints become second-class. Just as for the pure QED case, we write the gauge fixing as follows

$$\Omega_2(x) = \int_0^1 d\alpha \ x^i A_i(\alpha \mathbf{x}) = 0,$$

(31)

where $\alpha$ is the parameter describing a spacelike straight line of integration. In this way, one easily verifies that the fundamental Dirac brackets reads

$$\left\{ A_i(x), A^j(y) \right\}^* = 0 = \left\{ \pi_i(x), \pi^j(y) \right\}^*,$$

(32)
\[
\left\{ A_i(x), \pi^i(y) \right\}^* = \delta^i_j \delta^{(2)}(x - y) - \partial_i^\ast \int_0^1 d\alpha \ x^j \delta^{(2)}(\alpha x - y).
\]

In order to illustrate the discussion, we now write the equations of motion in terms of the magnetic \((B = \varepsilon_{ij} \partial^i A^j)\) and electric \((E^i = \pi^i - \frac{e}{2} \varepsilon_{ij} A_j)\) fields as

\[
\dot{E}_i(x) = -2\theta \varepsilon_{ij} E^j(x) - \varepsilon_{ij} \partial^j B,
\]

\[
\dot{B}(x) = -\varepsilon_{ij} \partial^i E^j.
\]

In the same way, we write the Gauss law as

\[
\partial_i E^i_L + \theta B - J^0 = 0,
\]

where \(E^i_L\) refers to the longitudinal part of \(E^i\).

The following remark deserves to be mentioned. As in the preceding subsection, we will compute the interaction energy between point-like sources in the static approximation (the limit of large fermion masses). Accordingly, from the equations of motion, as opposed to the pure QED case, we have static electromagnetic fields:

\[
B = -\theta \frac{J^0}{\nabla^2 - \theta^2},
\]

\[
E_i(x) = \frac{1}{\theta} \partial_i B,
\]

where \(\nabla^2\) is the two-dimensional laplacian. For \(J^0(t, x) = e\delta^{(2)}(x - y)\), expressions \((37)\) and \((38)\) immediately show that

\[
B(x) = \frac{e\theta}{2\pi} K_0(\theta |x - y|),
\]

\[
E^i(x) = -\frac{e\theta (x - y)^i}{2\pi |x - y|} K_1(\theta |x - y|),
\]

where \(K_0\) and \(K_1\) are modified Bessel’s functions. Their limiting forms for small and large \(\theta |x - y|\) are: \(E^i \sim \frac{1}{|x - y|}\) and \(B \sim \ln(|x - y|)\) for \(|x - y| \ll \frac{1}{\theta}\), while they fall of
exponentially $\sim \exp(-\theta \mid x - y \mid)$ for $\mid x - y \mid \gg \frac{1}{\theta}$. This shows that the electromagnetic fields, while falling off exponentially for $\mid x - y \mid \gg \frac{1}{\theta}$, are localized near the charge. It is worth mentioning that the above expressions were previously derived in Ref. [10] considering the anyon statistics and its variation with wavelength. In that reference it was remarked that an anyon can be viewed as a point charge surrounded by a gauge field cloud of size $\frac{1}{\theta}$.

Let us also point out that the fields (39) and (40) are present at the classical level. For the sake of simplicity, we will now compute the interaction energy via expression (22). Thus we have that

$$A_i(\cup, x) = \int_0^1 d\alpha x_i E_i^1(t, \alpha x) = -\int_0^1 d\alpha \frac{x_i}{\sqrt{\alpha x}} \frac{\partial^\alpha x}{\partial^\alpha x} J^0(\alpha x).$$

(41)

For $J^0(t, x) = e\delta^{(2)}(x - a)$ expression (41) then becomes

$$A_i(\cup, x) = -\frac{e}{2\pi} (K_0(\theta \mid x - a \mid) - K_0(\theta \mid a \mid)).$$

(42)

By means of (41) we evaluate the interaction energy for a pair of static point-like opposite charges at $y$ and $y'$, as

$$V = e (A_i(y) - A_i(y')) = -\frac{e^2}{\pi} K_0(\theta \mid y - y' \mid),$$

(43)

after substracting the constant terms.

Finally, one comment is pertinent in this context. Recently [11], by using bosonization methods, it has been derived the interparticle energy for three-dimensional massive quantum electrodynamics. By starting from the three-dimensional massive QED in the covariant gauge and in the presence of an external source $J^0$, these authors [11] obtain the bosonized version in the large mass limit (quadratic approximation), which is the Maxwell-Chern-Simons theory in the covariant gauge. Next, by using the Feynman propagator in two dimensions, they arrive at the result (43) in the limit for heavy fermions ($m \to \infty$). However, the essential difference between our analysis and that of Ref. [11] lies on the observation that the potential (43) is present at classical level too.
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