Sieve functions in arithmetic bands

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Abstract. An arithmetic function \( f \) is called a sieve function of range \( Q \), if its Eratosthenes transform \( g \) is supported in \( [1, Q] \cap \mathbb{N} \), where \( g(q) \ll \varepsilon q^\varepsilon \) (\( \forall \varepsilon > 0 \)). Here, we study the distribution of \( f \) over short arithmetic bands \( \bigcup_{1 \leq a \leq H} \{n \in [N, 2N] : n \equiv a \pmod{q}\} \), with \( H = o(N) \), and give applications to both the correlations and to the so-called weighted Selberg integrals of \( f \), on which we have concentrated our recent research.

1. Introduction and statement of the results

An arithmetic function \( f : \mathbb{N} \to \mathbb{C} \) is called a sieve function of range \( Q \), if

\[
f(n) = \sum_{d \mid n} g(d),
\]

where \( g : \mathbb{N} \to \mathbb{C} \) is essentially bounded, namely \( g(d) \ll_{\varepsilon} d^\varepsilon \), \( \forall \varepsilon > 0 \). As usual, \( \ll \) is Vinogradov’s notation, synonymous to Landau’s \( O \)-notation. In particular, \( \ll_{\varepsilon} \) means that the implicit constant might depend on an arbitrarily small and positive real number \( \varepsilon \), which might change at each occurrence. When \( f \) is the convolution product of \( g \) and the constantly 1 function, i.e.

\[
f(n) = (g * 1)(n) = \sum_{d \mid n} g(d),
\]

we say, with Wintner \([W]\), that \( g \) is the Eratosthenes transform of \( f \). Observe that \( f = g * 1 \) is a sieve function whenever it is assumed that \( g \) is essentially bounded and vanishes outside \( [1, Q] \) for some \( Q \in \mathbb{N} \), that is to say, the Eratosthenes transform of \( f \) is the restriction \( g_Q \) \( \overset{\text{def}}{=} g \cdot 1_{[1, Q]} \) (hereafter, \( 1_B \) denotes the indicator function of the set \( B \cap \mathbb{Z} \)). Moreover, by the Möbius inversion formula it turns out that \( f = g * 1 \) is essentially bounded if and only if so is \( g \).

Sieve functions are ubiquitous in analytic number theory. For example, the truncated divisor sum \( \Lambda_R \), exploited by Goldston, Pintz and Yıldırım for their recent outstanding results \([GPY]\), is a linear combination of sieve functions of range \( R \) (see \( \S 5 \)). Compare also \([C2]\) for more examples of sieve functions. However, the reader is cautioned that by a sieve function some authors simply mean any sieve-related function that often arises within the theory of sieve methods (see \([DH]\)).

The first author has intensively investigated symmetry properties of sieve functions in short intervals through the study of their correlations and the associated Selberg integrals (\([C1]\), \([C2]\) and \([C3]\)). Here we wish to relate such a study to the distribution of a sieve function in modular arithmetic short bands. More precisely, for given positive integers \( q, N, H \) we search for non-trivial bounds on the total (balanced) value of \( f \) in arithmetic bands modulo \( q \) defined as

\[
T_f(q, N, H) \overset{\text{def}}{=} \sum_{a \leq H} \sum_{n \sim N \atop n \equiv a \pmod{q}} f(n) - \frac{H}{q} \sum_{n \sim N} f(n),
\]

where \( n \sim N \) means that \( n \in (N, 2N] \cap \mathbb{N} \) (hereafter, we omit \( a \geq 1 \) in sums like \( \sum_{a \leq H} \)). In particular, given any \( N, H \in \mathbb{N} \), we prove that (see the remark after Theorem 1) for every real sieve function \( f \) of range \( Q \ll N \) and every \( q \ll N \) one has

\[
T_f(q, N, H) \ll_{\varepsilon} N^{\varepsilon}(N/q + q + Q). \tag{1}
\]

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It transpires from our method that similar bounds can be immediately established for weighted versions of the above problem, namely

$$T_{w,f}(q, N, H) \overset{\text{def}}{=} \sum_{0 \leq |a| \leq H} w(a) \sum_{n \equiv a \pmod{q}} f(n) - \frac{1}{q} \sum_{0 \leq |h| \leq H} w(h) \sum_{n \sim N} f(n),$$

whenever $w : \mathbb{R} \to \mathbb{R}$ is a piecewise-constant weight. Indeed, it is plain that $T_f(q, N, H) = T_{w,f}(q, N, H)$ involves the unit step weight

$$u(h) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } h > 0 \\ 0 & \text{otherwise}. \end{cases}$$

However, we give more general conditions on $w$ to treat $T_{w,f}(q, N, H)$. First, let us set

$$w_H(h) \overset{\text{def}}{=} w \cdot 1_{[-H, H]}(h) = \begin{cases} w(h) & \text{if } h \in [-H, H] \cap \mathbb{Z} \\ 0 & \text{otherwise}, \end{cases}$$

and state our first result.

**Theorem 1.** Let $q, N, H, Q \in \mathbb{N}$ such that $q \ll N$ and $Q \ll N$, as $N \to \infty$. For every sieve function $f : \mathbb{N} \to \mathbb{R}$ of range $Q$ and every weight $w : \mathbb{R} \to \mathbb{R}$ one has

$$T_{w,f}(q, N, H) \ll \varepsilon N^r \left( \frac{N}{q} + q + Q \right) \max_{\ell \mid q} L_{\ell}^1(\tilde{w}_H).$$

**Remark.** By taking $w = u$ and recalling $\|r\| \overset{\text{def}}{=} \min_{n \in \mathbb{Z}} |r - n|$, $\forall r \in \mathbb{R}$, we have $\forall \ell > 1$, (see [Da], Ch.25),

$$L_{\ell}^1(\tilde{w}_H) = \frac{1}{\ell} \sum_{(j, \ell) = 1} \left| \sum_{h \leq H} e \left( \frac{h j}{\ell} \right) \right| \ll \frac{1}{\ell} \sum_{(j, \ell) = 1} \frac{1}{\|j/\ell\|} \ll \sum_{j \ell \leq \ell/2} \frac{1}{\ell} \ll \log \ell.$$

Therefore, (1) follows immediately from Theorem 1.

Another remarkable instance concerns the correlation of $w_H$, that is defined as

$$W_H(a) \overset{\text{def}}{=} \sum_{0 \leq |h_1| \leq H} \sum_{0 \leq |h_2| \leq H} w_H(h_1) w_H(h_2) = \sum_{0 \leq |h| \leq H} w(h) w(h - a).$$

Observe that $W_H$ vanishes outside $[-2H, 2H]$. Moreover, uniformly in $\beta \in [0, 1]$,

$$\hat{W}_H(\beta) = \sum_{0 \leq |h| \leq 2H} W_H(h)e(h\beta) = \sum_{m \equiv n \equiv h} w_H(m) w_H(n)e(h\beta) = \left| \sum_r w_H(r)e(r\beta) \right|^2 = |\tilde{w}_H(\beta)|^2.$$

Besides revealing that not all the weights are correlations of other weights, this yields

$$\hat{W}_H(0) = \tilde{w}_H(0)^2 \ll H^2,$$
Since Corollary 1.

proof of Theorem 1. if it satisfies (2). Thus, the following result is immediately established in a completely analogous way to the

sign above, other remarkable examples of good weights are the

\( w \)

Accordingly to the terminology of [CL1], an uniformly bounded weight \( w \) (as \( H \to \infty \)) is said to be good, if it satisfies (2). Thus, the following result is immediately established in a completely analogous way to the proof of Theorem 1.

Corollary 1. Let \( q, N, H, Q \in \mathbb{N} \) such that \( q \ll N \) and \( Q \ll N \), as \( N \to \infty \). For every sieve function \( f : \mathbb{N} \to \mathbb{R} \) of range \( Q \) and every good weight \( w : \mathbb{R} \to \mathbb{R} \) one has

\[
\sum_{n=-N}^{n=N} W_H(a) \sum_{n \equiv a \pmod{Q}} f(n) = \frac{W_H(0)}{q} \sum_{n=-N}^{n=N} f(n) + \mathcal{O}_\varepsilon \left( N^2 H \left( \frac{N}{q} + q + Q \right) \right),
\]

where \( W_H \) is the correlation of \( w_H \).

Remarks.
1. Though analogous definitions can be easily formulated for a complex weight \( w \) (with the only exception of \( W_H \), whose definition has to be modified by taking the complex conjugate of \( w_H(h_2) \)), here we stick to real weights and real sieve functions for simplicity.

2. From [CL1] (see Propositions 2 and 3 there) it turns out that, beyond the unit step function \( u \) defined above, other remarkable examples of good weights are the sign function and the Cesaro weight, respectively defined as

\[
\text{sgn}(h) \equiv \begin{cases} 0 & \text{if } h = 0, \\ h/|h| & \text{otherwise}, \end{cases} \quad C_H(h) \equiv \begin{cases} 1 - |h|/H & \text{if } |h| \leq H, \\ 0 & \text{otherwise}. \end{cases}
\]

Since

\[
C_H(h) = \frac{1}{H} \sum_{t \leq H - |h|} 1 = \frac{1}{H} \sum_{m, n \leq H} 1,
\]

then \( HC_H \) is the correlation of \( u_H \), and consequently \( \widehat{C_H}(0) = \widehat{u_H}(0)^2/H = H \). We conclude that Corollary 1 is non-trivial for \( w_H = u_H \), yielding

\[
\sum_{n=-N}^{n=N} C_H(a) \sum_{n \equiv a \pmod{Q}} f(n) = \frac{H}{q} \sum_{n=-N}^{n=N} f(n) + \mathcal{O}_\varepsilon \left( N^2 \left( \frac{N}{q} + q + Q \right) \right).
\]

3. The main terms in the formulæ furnished by Theorem 1 and Corollary 1 can be explicitly related to the Eratosthenes transform of \( f = g_Q * 1 \), with \( Q \ll N \). Indeed,

\[
\sum_{n=-N}^{n=N} f(n) = \sum_{n=-N}^{n=N} \sum_{d \mid n} g_Q(d) = \sum_{d \leq Q} g(d) \sum_{n=-N}^{n=N} 1 = \sum_{d \leq Q} g(d) \sum_{n=-N/d}^{n=N/d} 1 = \sum_{d \leq Q} g(d) \left( \frac{N}{d} q + O(1) \right) = \frac{N}{q} \sum_{d \leq Q} g(d) \left( d/q \right) + \mathcal{O}_\varepsilon \left( Q^{1+\varepsilon} \right).
\]

In particular, for the long intervals we get the formula

\[
\sum_{n=-N}^{n=N} f(n) = R_1(f) N + \mathcal{O}_\varepsilon \left( Q^{1+\varepsilon} \right),
\]
where the so-called first Ramanujan coefficient $R_1(f)$ is the mean value of $f$ (see §2):

$$R_1(f) \overset{df}{=} \sum_{d \leq Q} \frac{g(d)}{d} = \lim_{x \to \infty} \left( \sum_{d \leq Q} \frac{g(d)}{d} + \frac{1}{x} \sum_{d \leq Q} O(|g(d)|) \right) = \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} f(n).$$

On the other side, by taking $F$ as the Dirichlet series generating $f$, one has

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} f(n) = \text{Res}_{s=1} F(s) x^{s-1} / s.$$ 

Since $f = g_\eta \ast 1$ is a sieve function, then $F$ can be expressed in terms of the Riemann zeta function $\zeta$ and the Dirichlet polynomial generating its Eratosthenes transform, namely

$$F(s) \overset{df}{=} \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \zeta(s) \sum_{d \leq Q} \frac{g(d)}{d^s}.$$ 

Note that the zeta function forces $F$ to have a simple pole at $s = 1$, the Dirichlet polynomial being an entire function of $s$. Thus, if $f = g_\eta \ast 1$ is gauged by a weight $w$ in the short interval $[x-H, x+H]$ (i.e., $H = o(N)$, as $N \to \infty$), then it is natural to take the expected mean value of $w_H(n-x)f(n)$ for $N < x \leq 2N$ to be (compare [CL])

$$\widehat{w}_H(0) R_1(f) = \sum_a w_H(a) \sum_{d \leq Q} \frac{g(d)}{d}, \quad \text{(that is independent of $x$)}.$$ 

Indeed, a basic tool for the study of the distribution of the sieve function $f$ in short intervals is its weighted Selberg integral

$$J_{w,f}(N, H) \overset{df}{=} \sum_{x \sim N} \left| \sum_n w_H(n-x)f(n) - \widehat{w}_H(0) R_1(f) \right|^2,$$

whose non-trivial bounds might lead to results on the distribution of $f$ in almost all short intervals $[x-H, x+H]$, i.e., with $o(N)$ possible exceptions $x \in (N, 2N] \cap \mathbb{N}$. Observe that the trivial bound for $J_{w,f}(N, H)$ is $N^{1+\varepsilon} H^2$, because $f$ is essentially bounded. In [CL] and [CL1] we have investigated and exploited the link between $J_{w,f}(N, H)$ and the correlation

$$\mathcal{E}_f(a) \overset{df}{=} \sum_{n \sim N} f(n)f(n-a),$$

in order to get non-trivial bounds under suitable conditions on $f$ and a good weight $w$.

As a consequence of Theorem 1, we obtain a further result on such a link with a slight generalization. Let us define the correlation of real arithmetic functions $f_1$ and $f_2$ as

$$\mathcal{E}_{f_1, f_2}(a) \overset{df}{=} \sum_{n \sim N} f_1(n)f_2(n-a).$$

In such a context, we might refer to $\mathcal{E}_f = \mathcal{E}_{f, f}$ as the autocorrelation of $f$. Since here the shift $a$ is confined to $a \ll H$, the conditions $n \sim N$ and $H = o(N)$ clearly yield $\max(n, n-a) \leq 2N + |a| \leq 3N$. Moreover, if $f_1$ and $f_2$ are essentially bounded, then trivially $\mathcal{E}_{f_1, f_2}(0) \ll N^{1+\varepsilon}$, and for any $a \ll H$ one has

$$\mathcal{E}_{f_1, f_2}(a) = \sum_{n_1 \sim N} \sum_{n_2 \sim N} f_1(n_1)f_2(n_2) + O\left( N^2 H \right),$$

(that should be compared to the previous definition of the correlation of a weight).
Correspondingly, the *mixed* weighted Selberg integrals associated to the pair \((f_1, f_2)\) is (compare [C])

\[
J_{w, (f_1, f_2)}(N, H) \overset{\text{def}}{=} \sum_{x \sim N} \prod_{j=1,2} \left( \sum_{n} w(n-x) f_j(n) - \overline{w}(0) R_1(f_j) \right).
\]

By applying Theorem 1 we obtain a *first generation* formula (consistently with the terminology of [CL]) for the correlation of the sieve functions \(f_1\) and \(f_2\), together with an estimate of the mixed weighted Selberg integral when these functions are gauged by a good weight \(w\).

**Corollary 2.** Let \(N, H, Q_1, Q_2 \in \mathbb{N}\) with \(Q_1 \leq Q_2 \ll N\), as \(N \to \infty\). For any real and essentially bounded arithmetic functions \(g_1\) and \(g_2\) supported in \([1, Q_1]\) and \([1, Q_2]\), respectively, one has

\[
\sum_{a \leq H} \mathbb{C}_{f_1, f_2}(a) = R_1(f_1)R_1(f_2)NH + O_{\varepsilon}(N^\varepsilon(N + Q_2^2 + Q_1H)),
\]

where \(f_j = g_j \ast \mathbf{1}\) for \(j = 1, 2\). Furthermore, if \(H = o(N)\), as \(N \to \infty\), and \(w : \mathbb{R} \to \mathbb{R}\) is a good weight, then

\[
J_{w, (f_1, f_2)}(N, H) \ll_{\varepsilon} N^\varepsilon (NH + Q_2H^2 + Q_2^2H + H^3).
\]

**Remark.** For every real sieve function \(f\) of range \(Q \ll N\), this corollary gives

\[
\sum_{a \leq H} \mathbb{C}_{f}(a) = R_1^2(f)NH + O_{\varepsilon}(N^\varepsilon(N + Q^2 + QH)),
\]

\[
J_{w, f}(N, H) \ll_{\varepsilon} N^\varepsilon (NH + QH^2 + Q^2H + H^3).
\]

We stress that such a bound for the weighted Selberg integral has been already established in Theorem 3 of [CL1]. In §3 we propose a much simpler proof through the new approach of the *arithmetic bands* formulæ provided by Theorem 1.

Furthermore, from such an approach we find an important relation between weighted Selberg integrals and the *total (weighted) content* of a sieve function \(f\) of range \(Q \ll N\), namely (see Lemma 2 and the proof of Corollary 2)

\[
J_{w, f}(N, H) \ll_{\varepsilon} N^\varepsilon \sum_{q \leq Q} |TW_f(q, N, H)| + N^\varepsilon H^2(Q + H),
\]

where for the correlation of \(w_H\) we set

\[
TW_f(q, N, H) \overset{\text{def}}{=} \sum_{a} \sum_{n \sim N \atop n \equiv a \pmod{q}} f(n) - \frac{\overline{W}(0)}{q} \sum_{n \sim N} f(n).
\]

Beyond the estimate given by Corollary 1 when \(w\) is a good real weight, more in general, if \(f\) is essentially bounded, then by means of (4) a non-trivial bound, like

\[
\sum_{q \leq Q} |TW_f(q, N, H)| \ll N^{1-\delta} H^2 \text{ for some real } \delta > 0,
\]

might yield a non-trivial bound of the same type for \(J_{w, f}(N, H)\) (but not necessarily with the same *gain* \(N^\delta\)). Analogous considerations hold for mixed weighted Selberg integrals. Rather surprisingly, in spite of the fact that the presence of absolute values in the total content seems to prevent it from further possible cancellation, next theorem makes it clear that there are non-trivial bounds for (weighted) Selberg integrals, involving a sieve function \(f\) of range \(Q \ll N^{1-\delta}\) for some \(\delta > 0\), if and only if there are non-trivial results on the distribution of \(f\) in short arithmetic bands.
Theorem 2. Let \( f : \mathbb{N} \to \mathbb{R} \) be a sieve function of range \( Q \ll N^{1-\delta} \), for some \( \delta > 0 \), and let \( w : \mathbb{R} \to \mathbb{R} \) be such that \( w_H \) is uniformly bounded for any \( H \ll N^{1-\delta} \), as \( N \to \infty \).

I) The following three assertions are equivalent:

i) a non-trivial bound holds for \( |T_{W,f}(q, N, H)| \)

ii) a non-trivial bound holds for \( J_{w,f}(N, H) \)

iii) a non-trivial bound holds for \( J_{w,f_1,f_1}(N, H) \), where \( f_1 \) is any sieve function of range \( Q \).

II) If \( N^{\delta/2} \ll H \ll N^{1-\delta} \), as \( N \to \infty \), then the following assertions are equivalent:

iv) a non-trivial bound holds for \( \sum_{q \leq Q} |T_f(q, N, H)| \)

v) a non-trivial bound holds for the Selberg integral

\[
J_f(N, H) \overset{def}{=} \sum_{x \sim N} \left| \sum_{x < n \leq x + H} f(n) - R_1(f)H \right|^2.
\]

Note that in iv) a non-trivial bound is meant to be of the type \( N^{1-\delta}H \) for some \( \delta > 0 \).

After a short section on some further notation and basic formulæ, in §3 we give the necessary lemmata for our theorems and Corollary 2, whose proofs constitute the fourth section, whereas we omit the proof of Corollary 1, it being completely analogous to the proof of Theorem 1. In §5 we explicitly specialize the results of the present article to the aforementioned function \( \Lambda_R \). The last section is devoted to a comparison between classical results in arithmetic progressions and ours in arithmetic bands.

2. Further notation, conventions and standard properties

As already mentioned, we omit \( a \geq 1 \) in sums like \( \sum_{a \leq X} \). For the same sake of brevity, somewhere we write \( n \equiv a \ (q) \) in place of \( n \equiv a \ (\text{mod} \ q) \). Thus, the well-known orthogonality of additive characters,

\[ e_q(r) \overset{def}{=} e(r/q) = e^{2\pi ir/q}, \ (q \in \mathbb{N}, r \in \mathbb{Z}), \]

can be written as

\[
\frac{1}{q} \sum_{j \ (q)} e_q(j(n - m)) = \begin{cases} 1 & \text{if } n \equiv m \ (\text{mod } q), \\
0 & \text{otherwise}
\end{cases}
\]

since the sum is over a complete set of residue classes \( j \ (\text{mod } q) \).

We write \( \sum_{j(q)}^* \) to mean that the sum is over a complete set of reduced residue classes \( (\text{mod } q) \), i.e. the set \( \mathbb{Z}_q^* \) of \( 1 \leq j \leq q \) such that \( (j, q) = 1 \). In particular, the Ramanujan sum is written as

\[ c_q(n) \overset{def}{=} \sum_{j(q)}^* e_q(jn). \]

Without further references, we will appeal to the well-known inequality (see [Da], Ch.25)

\[ \sum_{V_1 < v \leq V_2} e(v\alpha) \ll \min \left( V_2 - V_1, \frac{1}{\|\alpha\|} \right). \]

Recalling that \( 1(n) \overset{def}{=} 1, \ \forall n \in \mathbb{N} \), we set

\[ 1_{d|n} \overset{def}{=} \begin{cases} 1 & \text{if } d|n \\
0 & \text{otherwise.}
\end{cases} \]

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Consequently, the aforementioned orthogonality of characters becomes

\[ 1_{d|n} = \frac{1}{d} \sum_{j'(d)} e_d(j'n) = \frac{1}{d} \sum_{l|d} \sum_{j'(d)} e_d(j'n) = \frac{1}{d} \sum c_l(n). \]

Therefore, one has the following Ramanujan expansion of a sieve function \( f = g_Q * 1 \):

\[ f(n) = \sum_{d|n} g_Q(d) = \sum_{d \leq Q} g(d) 1_{d|n} = \sum_{d \leq Q} \frac{g(d)}{d} \sum_{l|d} c_l(n) = \sum_{\ell \leq Q} \frac{g(d)}{d} c_\ell(n) = \sum_{\ell \leq Q} R_\ell(f)c_\ell(n), \]

where we have introduced the so-called \( \ell \)-th Ramanujan coefficient of \( f \), i.e.

\[ R_\ell(f) \overset{\text{def}}{=} \sum_{d \equiv 0(\ell)} \frac{g_Q(d)}{d}. \]

The hypothesis that \( g \) is essentially bounded yields the bound

\[ R_\ell(f) \ll 1 \sum_{m \leq \frac{Q}{\ell}} \frac{|g(\ell m)| m}{m} \ll \varepsilon \sum_{m \leq \frac{Q}{\ell}} \frac{1}{m} \ll \varepsilon \frac{Q^\varepsilon}{\ell}. \]

We refer the reader to [ScSp] and [W] for more extensive accounts on the theory of the Ramanujan expansions.

3. Lemmata

Here we state and prove two lemmas that are interesting in their own right. The first lemma is required to prove Theorem 1, while the second one is invoked within the proofs of Corollary 2 and Theorem 2. To this end, analogously to the exponential sums for the weights already introduced in §1, we set

\[ \hat{f}(\alpha) \overset{\text{def}}{=} \sum_{n \sim N} f(n)e(n\alpha), \quad (\alpha \in \mathbb{R}). \]

Notice that now we can write

\[ T_w,f(q, N, H) = \sum_a W_H(a) \sum_{n \sim N, n \equiv a \mod q} f(n) - \hat{w}_H(0) \hat{f}(0), \]

\[ T_W,f(q, N, H) = \sum_a W_H(a) \sum_{n \sim N, n \equiv a \mod q} f(n) - \hat{w}_H(0) \hat{f}(0), \]

while the formula (3) becomes

\[ \hat{f}(0) = R_1(f)N + O_\varepsilon(Q^1+\varepsilon). \]

The first lemma gives a similar relation between the \( \ell \)-th Ramanujan coefficient of \( f \) and \( \hat{f}(\alpha) \), when \( \alpha = j/\ell \) is any non-integer rational with \( (j, \ell) = 1 \). Note that such a formula is not a straightforward consequence of Wintner’s criterion (see VIII.2 of [ScSp]).

**Lemma 1.** Let \( f \) be a sieve function of range \( Q \ll N \), with \( Q, N \in \mathbb{N} \). Then

\[ \hat{f}(j/\ell) = R_1(f)N + O_\varepsilon((\ell Q)^\varepsilon(Q + \ell)), \quad \forall \ell > 1, \forall j \in \mathbb{Z}_\ell^*. \]
PROOF. By assuming that \( f = g_q * 1 \) with an essentially bounded \( g \), we write
\[
\hat{f}(j/\ell) = \sum_d g_q(d) \sum_{v \sim \ell} e_v(jd) = \sum_{d \equiv 0 (\ell)} g_q(d) \left( \frac{N}{d} + O(1) \right) + O \left( \sum_{d \not\equiv 0 (\ell)} \frac{|g_q(d)|}{|jd/\ell|} \right).
\]
Since
\[
\sum_{d \equiv 0 (\ell)} g_q(d) \left( \frac{N}{d} + O(1) \right) = R_\ell(f)N + O_{\varepsilon} \left( Q^\varepsilon \left( \frac{Q}{\ell} + 1 \right) \right),
\]
then the lemma is proved whenever we show that
\[
\sum_{d \leq Q \atop d \not\equiv 0 (\ell)} \frac{1}{|jd/\ell|} \leq \varepsilon (Q + \ell).
\]
To this end, it suffices to observe that
\[
\sum_{d \leq Q \atop d \not\equiv 0 (\ell)} \frac{1}{|jd/\ell|} \leq \sum_{0 < \epsilon \leq \ell/2} \sum_{d \equiv 0 (\ell)} \frac{1}{|r/\ell|} \leq \varepsilon \ell \sum_{r \leq \ell/2} \frac{Q}{\ell} \left( \frac{Q}{\ell} + 1 \right).
\]
The lemma is completely proved. \( \square \)

REMARK. Note that the formula of the above lemma is non-trivial when \( \ell, Q \ll N^{1-\delta} \), for some \( \delta > 0 \). Moreover, it is easy to see that it holds uniformly with respect to \( j \in \mathbb{Z}_* \).

Let us turn our attention to next lemma. As already mentioned in \( \S 1 \), by means of an elementary dispersion method, in [CL], Lemma 7, we established a link between weighted Selberg integrals and autocorrelations of an arithmetic function \( f \) gauged by a weight \( w \) such that \( w_n \) is bounded, as \( H \to \infty \). Under the further hypothesis that the sieve function \( f \) and the weight \( w \) are real, the formula of the aforementioned lemma becomes
\[
J_{w,f}(N,H) = \sum_{0 \leq |a| \leq H} W_H(a) \Phi_f(a) + \sum_{x \sim N} |\tilde{w}_H(0) R_1(f)|^2 \\
-2 \tilde{w}_H(0) R_1(f) \sum_{n \leq 3N} f(n) \sum_{x \sim N} w_H(n-x) + O_{\varepsilon} \left( H^3 N^\varepsilon \right).
\]
Similarly, for the mixed weighted Selberg integral of sieve functions \( f_1, f_2 \) we have
\[
J_{w,(f_1,f_2)}(N,H) = \sum_a W_H(a) \Phi_{f_1,f_2}(a) - \tilde{W}_H(0) R_1(f_1) R_1(f_2) N \\
- \tilde{w}_H(0) \left( R_1(f_1) \sum_{x \sim N} \Delta_2(x) + R_1(f_2) \sum_{x \sim N} \Delta_1(x) \right) + O_{\varepsilon} \left( H^3 N^\varepsilon \right),
\]
where we set \( \Delta_j(x) \overset{def}{=} \sum_n w_H(n-x) f_j(n) - \tilde{w}_H(0) R_1(f_j) \). By using such a formula we prove the next lemma, where \( J_{w,(f_1,f_2)}(N,H) \) is expressed in terms of arithmetic bands of \( f_1 \) or \( f_2 \).

LEMMA 2. Let \( g_1 \) and \( g_2 \) be real and essentially bounded arithmetic functions supported in \([1, Q_1]\) and \([1, Q_2]\), respectively, with \( Q_1, Q_2 \in \mathbb{N} \) such that \( Q_1 \leq Q_2 \ll N \), as \( N \to \infty \). If \( w : \mathbb{R} \to \mathbb{R} \) is such that \( w_n \) is uniformly bounded, as \( H \to \infty \), then one has
\[
J_{w,(f_1,f_2)}(N,H) = \sum_{q \leq Q_1} g_1(q) T_{W,f_1}(q,N,H) + O_{\varepsilon} \left( N^\varepsilon H^2(Q_2 + H) \right) = \\
= \sum_{q \leq Q_2} g_2(q) T_{W,f_1}(q,N,H) + O_{\varepsilon} \left( N^\varepsilon H^2(Q_2 + H) \right),
\]
where we set \( f_j = g_j \ast 1 \), and \( W_H \) is the correlation of \( w_H \).

**Proof.** First, let us write

\[
\sum_{x \sim N} \sum_{n} w_H(n-x)f_j(n) = \sum_{n \sim N} f_j(n) \sum_{n-H \leq x \leq n+H} w(n-x) + O_\varepsilon (N^\varepsilon H^2) = \]

\[
= \hat{w}_H(0) \sum_{n \sim N} f_j(n) + O_\varepsilon (N^\varepsilon H^2). \]

Then, by arguing as (3) and recalling that \( R_1(f_j) \leq Q_2 \), we get

\[
\sum_{x \sim N} \Delta_j(x) = \hat{w}_H(0) \left( \sum_{n \sim N} f_j(n) - R_1(f_j)N \right) + O_\varepsilon (N^\varepsilon H^2) \leq N^\varepsilon H(Q_j + H). \]

Thus, since \( W_H \) is even and \( Q_1 \leq Q_2 \ll N \), the above formula (6) yields the equalities

\[
J_{w,(f_1,f_2)}(N,H) = \sum_a W_H(a)\epsilon_{f_1,f_2}(a) - \hat{W}_H(0)R_1(f_1)R_1(f_2)N + O_\varepsilon (N^\varepsilon H^2(Q_2 + H)) = \]

\[
= \sum_a W_H(a)\epsilon_{f_2,f_1}(a) - \hat{W}_H(0)R_1(f_1)R_1(f_2)N + O_\varepsilon (N^\varepsilon H^2(Q_2 + H)). \]

Whence we can stick to the first equality and apply (3) to \( f_1 \), in order to write

\[
\sum_a W_H(a)\epsilon_{f_1,f_2}(a) - \hat{W}_H(0)R_1(f_1)R_1(f_2)N = \]

\[
= \sum_a W_H(a) \sum_{n \sim N} f_1(n) \sum_{q \sim Q_2} \frac{g_2(q)}{q} \hat{W}_H(0) \sum_{n \sim N} f_1(n) \frac{g_2(q)}{q} + O_\varepsilon (Q_2^{1+\varepsilon} H^2) = \]

\[
= \sum_{q \sim Q_2} g_2(q) \left( \sum_a W_H(a) \sum_{n \sim N} f_1(n) - \hat{W}_H(0) \frac{f_1(0)}{q} \right) + O_\varepsilon (Q_2^{1+\varepsilon} H^2). \]

The lemma is completely proved. \( \square \)

**4. Proofs of Theorem 1, Corollary 2 and Theorem 2**

**Proof of Theorem 1.** By the orthogonality of additive characters we get

\[
T_{w,f}(q,N,H) = \frac{1}{q} \sum_a w_H(a) \sum_{n \sim N} f(n) \sum_{j^\prime \leq q} e_q(j^\prime(a-n)) - \frac{\hat{w}_H(0)}{q} \hat{f}(0) = \]

\[
= \frac{1}{q} \sum_{j^\prime \leq q} \sum_a w_H(a) e_q(j^\prime a) \hat{f}(-j^\prime/q) = \]

\[
= \frac{1}{q} \sum_{\ell \geq 1} \sum_{j \ell | q} \hat{f}(-j/\ell) \hat{w}_H(j/\ell), \]

where we have set \( \ell = q/(j^\prime,q) \). By applying Lemma 1 and the bound (5) we see that

\[
T_{w,f}(q,N,H) \ll_\varepsilon \frac{1}{q} \sum_{\ell \geq 1} \left( |R_\ell(f)|N + (\ell Q)^\varepsilon (Q + \ell) \right) \sum_{j \ell | q} \left| \hat{w}_H \left( \frac{j}{\ell} \right) \right| \ll_\varepsilon \]

9
\[
\llz \frac{Q^\varepsilon}{q} \sum_{\ell \leq q} \left( \frac{N}{\ell} + Q\ell^\varepsilon + \ell^{1+\varepsilon} \right) \ell \mathcal{L}_\ell^1(\omega_H) \llz N^\varepsilon \left( \frac{N}{q} + Q + q \right) \max_{\ell \geq 1} \mathcal{L}_\ell^1(\omega_H).
\]

This concludes the proof of Theorem 1. \(\square\)

**Proof of Corollary 2.** As already noticed in the proof of Lemma 2, we can write
\[
\mathcal{E}_{f_1, f_2}(a) = \sum_{n \sim N} f_1(n) f_2(n - a) = \sum_{n \sim N} f_1(n \sum_{q \mid n - a} g_2(q) = \sum_{q \leq Q_2} g_2(q) \sum_{n \sim N} f_1(n).
\]

Thus, the formula (3) and Theorem 1 yield
\[
\sum_{a \leq H} \mathcal{E}_{f_1, f_2}(a) = \sum_{q \leq Q_2} g_2(q) \left( \frac{H}{q} \hat{f}_1(0) + T_{f_1}(q, N, H) \right) =
\]
\[
= H \left( \sum_{q \leq Q_2} \frac{g_2(q)}{q} \right) \left( R_1(f_1) N + O_\varepsilon \left( Q_1^{1+\varepsilon} \right) \right) + O_\varepsilon \left( \sum_{Q_2 \leq q} \left( \frac{N}{q} + Q_1 \right) \right) =
\]
\[
= R_1(f_1) R_1(f_2) N H + O_\varepsilon \left( Q_1^2 + Q_1 + Q_1 H \right),
\]
that is the first formula of Corollary 2. In order to prove the stated inequality for the mixed weighted Selberg integral, it is enough to observe that Lemma 2 and the hypothesis \(Q_1 \leq Q_2 \ll N\) imply
\[
J_{w, f_1, f_2}(N, H) = \sum_{q \leq Q_2} g_2(q) T_{W, f_1}(q, N, H) + O_\varepsilon \left( N^\varepsilon H^2 (Q_2 + H) \right) \llz N^\varepsilon \sum_{q \leq Q_2} \left| T_{W, f_1}(q, N, H) \right| + N^\varepsilon H^2 (Q_2 + H).
\]

Whence the conclusion follows immediately from Corollary 1. \(\square\)

Before going to the proof of Theorem 2, let us remark explicitly that (4) is plainly a particular case of the latter inequality. Moreover, it transpires from the previous proof that, for every real and essentially bounded arithmetic function \(g\) supported in \([1, Q]\), with \(Q \ll N\), one has
\[
(7) \quad \sum_{a \leq H} \mathcal{E}_f(N) = R_1(f)^2 N H + \sum_{q \leq Q} g(q) T_f(q, N, H) + O_\varepsilon \left( N^\varepsilon Q H \right),
\]
where we set \(f = g * 1\).

**Proof of Theorem 2.** For simplicity and without loss of generality, let us assume that, whatever the choice of an assertion among \(i)\)-\(v)\) as hypothesis, the gain of the non-trivial bound is always \(N^\delta\).

Part I. \(i) \implies ii)\): as we said, let us suppose that
\[
\sum_{q \leq Q} \left| T_{W, f}(q, N, H) \right| \llz N^{1-\delta} H^2.
\]

Thus, \(ii) \implies iii)\): since we assume that \(J_{w, f}(N, H) \llz N^{1-\delta} H^2\), then by the Cauchy inequality and the trivial bound for \(J_{w, f_1}(N, H)\) we get
\[
J_{w, (f, f_1)}(N, H) \leq \sqrt{J_{w, f}(N, H)} \sqrt{J_{w, f_1}(N, H)} \llz N^\varepsilon \sqrt{N^{1-\delta} H^2} \sqrt{N H^2} \llz N^{1-\delta/3} H^2.
\]
iii) \implies i): after setting 
\[ s_{W,f}(q) \overset{\text{def}}{=} \begin{cases} \text{sgn}(T_{W,f}(q,N,H)) & \text{if } 1 \leq q \leq Q \\ 0 & \text{otherwise}, \end{cases} \]

it is readily seen that \( f_1 = s_{W,f} \ast 1 \) is a sieve function of range \( Q \). Thus, we can write
\[ \sum_{q \leq Q} |T_{W,f}(q,N,H)| = \sum_{q} s_{W,f}(q)T_{W,f}(q,N,H). \]

Now, by taking \( g_1 = s_{W,f} \) and \( f_2 = f \) in Lemma 2 we see that
\[ \sum_{q \leq Q} |T_{f}(q,N,H)| = J_{w,(f,f_1)}(N,H) + O_{\varepsilon}\left( N^\varepsilon H^2(Q + H) \right), \]

where again \( H^2(Q + H) \) is non-trivial. The first part of the theorem is completely proved.

Part II. iv) \implies v): since \( Q /\ll N^{1-\delta} \) and we assume that
\[ \sum_{q \leq Q} |T_{f}(q,N,H)| \ll N^{1-\delta} H, \]

then it is easily seen that the formula (7) yields
\[ \sum_{a \leq t} \mathcal{E}_f(a) = R_1(f)^2 N\left[t\right] + O_{\varepsilon}\left( N^{1-\delta + \varepsilon} t \right) \quad \text{for all } 1 \leq t \leq H, \]

where \( \left[t\right] \) is the integer part of \( t \). Thus, by partial summation we can write
\[ \sum_{1 \leq a \leq H} (H - a)\mathcal{E}_f(a) = \int_{1}^{H} \sum_{a \leq t} \mathcal{E}_f(a) dt = \]
\[ = \int_{1}^{H} \left( R_1(f)^2 N\left[t\right] + O_{\varepsilon}\left( N^{1-\delta + \varepsilon} t \right) \right) dt = \]
\[ = \frac{R_1(f)^2}{2} \cdot NH^2 + O_{\varepsilon}\left( N^{1+\varepsilon} H \right) + O_{\varepsilon}\left( N^{1-\delta + \varepsilon} H^2 \right). \]

Now, since \( \mathcal{E}_f(0) \ll \varepsilon N^{1+\varepsilon} \), and for \( 1 \leq a \leq H \) one has
\[ \mathcal{E}_f(-a) = \sum_{n \sim N} f(n)f(n + a) = \sum_{N + a < m \leq 2N + a} f(m - a)f(m) = \mathcal{E}_f(a) + O_{\varepsilon}\left( N^\varepsilon H \right), \]

then
\[ \sum_{0 \leq |a| \leq H} (H - |a|)\mathcal{E}_f(a) = R_1(f)^2 NH^2 + O\left( N^{1-\delta/3} H^2 \right). \]

By using this formula in (6), where we take \( W_H(a) = HC_H(a) = \max(H - |a|, 0) \) (see Remark 2 after Corollary 1), we immediately obtain \( J_f(N,H) \ll N^{1-\delta/3} H^2. \)

v) \implies iv): we suppose that \( J_f(N,H) \ll N^{1-\delta} H^2 \) and set
\[ s_f(q) \overset{\text{def}}{=} \begin{cases} \text{sgn}(T_f(q,N,H)) & \text{if } 1 \leq q \leq Q \\ 0 & \text{otherwise}, \end{cases} \quad f_1 \overset{\text{def}}{=} s_f \ast 1. \]

Thus, we can write
\[ \sum_{q \leq Q} |T_f(q,N,H)| = \sum_{q} s_f(q) \left( \sum_{a \leq H} \sum_{n \equiv a \mod q} f(n) - \frac{H}{q} \sum_{n \sim N} f(n) \right) = \]
essentially bounded, we apply the Cauchy inequality and the above assumption on where we have applied (3) to both $\Lambda R$ hypotheses on $Q$.

Theorem 2 is completely proved.

Let us recall that the truncated divisor sum \([GPY]\) is defined as

\[
\Lambda_R(n) \overset{\text{def}}{=} \sum_{d \mid n \atop d \leq R} \mu(d) \log(R/d) = (\log R) \sum_{d \mid n} \mu(d) - \sum_{d > R} \mu(d) \log d,
\]

so that $\Lambda_R$ is plainly a linear combination (with relatively small coefficients) of two sieve functions whose Eratosthenes transforms are respectively the restricted M"obius function, $\mu_R \overset{\text{def}}{=} \mu \cdot 1_{[1,R]}$, and $\mu_R \cdot \log$, i.e.

$\Lambda_R = ( (\log R) \mu_R - \mu_R \cdot \log ) \ast 1$.

After recalling also the well-known relations (see [Da])

\[
\sum_{d=1}^{\infty} \frac{\mu(d) \log d}{d} = -1 \quad \text{and} \quad \sum_{d \leq R} \frac{\mu(d)}{d}, \sum_{d > R} \frac{\mu(d) \log d}{d} \ll \exp \left( -c\sqrt{\log R} \right),
\]

(hereafter, $c > 0$ is an unspecified constant), we see that

\[
R_1(\Lambda_R) = \sum_{d \leq R} \frac{\mu(d) \log(R/d)}{d} =
\]

\[
= (\log R) \sum_{d \leq R} \frac{\mu(d)}{d} - \sum_{d > R} \frac{\mu(d) \log d}{d} = 1 + O\left( \exp \left( -c\sqrt{\log R} \right) \right).
\]

Thus, the mean value formula (3) gives

\[
\sum_{n \sim N} \Lambda_R(n) = N + O\left( N \exp \left( -c\sqrt{\log R} \right) \right) + O(\varepsilon(N^\varepsilon R)),
\]

while, if $R \ll N$, a straightforward application of (1) yields

\[
\sum_{n \ll H} \sum_{n \equiv a \pmod{q}} \Lambda_R(n) = \frac{N H}{q} + O\left( \varepsilon \left( N^\varepsilon \left( \frac{N}{q} + q + R \right) \right) \right) + O\left( N \exp \left( -c\sqrt{\log R} \right) \right).
\]
In case the level $\lambda \overset{\text{def}}{=} (\log R)/(\log N)$ is positive, i.e. $1 > \lambda > \lambda_0 > 0$, we may replace $\log R$ by $\log N$ in the above formulae (where now $c = c(\lambda)$). Assuming that this is the case, Corollary 2 provides the following first generation formula for the correlation of $\Lambda_R$:

$$\sum_{a \leq H} \sum_{n \sim N} \Lambda_R(n)\Lambda_R(n-a) = NH + O(NH \exp(-c\sqrt{\log N})) + O_\varepsilon(N^{\varepsilon}(N + R^2 + RH)).$$

It is worthwhile to remark that by following the classical approach in the literature the remainder term for the single correlation is $\ll_{\varepsilon} N^{\varepsilon} R^2$, that trivially yields a remainder $\ll_{\varepsilon} N^{\varepsilon} R^2 H$ in the first generation formula above, whereas by our method we save $H$.

6. Further comments

The key of the present approach is that the correlation of a real sieve function $f = g_q \ast 1$ can be written as

$$c_f(a) = \sum_{q \leq Q} g(q) \sum_{n \sim N, n \equiv a \pmod{q}} f(n).$$

In the literature (see [Ik], Ch.17), we find several studies of the distribution of an arithmetic function $f$ (not necessarily a sieve function) over primitive residue classes. Most results are focused on non-trivial bounds for the error term

$$E_f(N; q, a) \overset{\text{def}}{=} \sum_{n \sim N, n \equiv a \pmod{q}} f(n) - M_f(N; q, a)$$

for all $(a, q) = 1$, provided $q$ is not too large. Here, $M_f(N; q, a)$ is the expected mean value term. Let us recall two major variants of the problem. The first one concerns the Bombieri-Vinogradov type mean value

$$\sum_{q \leq Q} \max_{(a, q) = 1} |E_f(N; q, a)|,$$

for which we refer the reader to [M]. The second classical variant is the Barban-Davenport-Halberstam type quadratic mean

$$\sum_{q \leq Q} \sum_{\rho \leq q} E_f(N; q, a)^2.$$

The latter has also a short interval version introduced by Hooley [Ho], that is

$$\sum_{q \leq Q} \sum_{\rho \leq q} E_f(N; q, a)^2 \text{ where } \rho \to 0.$$
Consistently with the present notation, the above formula for the correlation of a sieve function becomes
\[ C_f(a) = \sum_{q \leq Q} g(q)M_f(N; q, a) + \sum_{q \leq Q} g(q)E_f(N; q, a), \]
where, by recalling that \( g(q) \ll q^\varepsilon \), one has
\[ \sum_{q \leq Q} g(q)E_f(N; q, a) \ll Q^\varepsilon \sum_{q \leq Q} |E_f(N; q, a)|. \]
Thus, here for each individual residue \( a \) we deal with a sum over \( q \leq Q \) without any further restriction. Then, it is not surprising that a straight asymptotic
\[ C_f(a) \sim \sum_{q \leq Q} g(q)M_f(N; q, a) \]
has been proved for very few interesting instances of \( f \), including the noteworthy case of the divisor function (see the third version of [CL] on arxiv for a brief account on this matter). Better expectations for the first generation of correlation averages,
\[ \sum_{a \leq H} C_f(a), \]
are given substance by Corollary 2 (and by the alternative approach of Lemma 12 in [CL]). Furthermore, note that Theorem 2 concerns the average
\[ \sum_{q \leq Q} \left| \sum_{a \leq H} E_f(N; q, a) \right|, \]
where, unlike the aforementioned means, the sums are taken over all the moduli \( q \leq Q \) and over a short interval of residue classes \( a \), when \( f \) is a sieve function of range \( Q \ll N^{1-\delta} \) and \( H \ll N^{1-\delta} \). The bound for the weighted Selberg integral given in Corollary 2 and its application through Theorem 2 allow \( Q \ll \sqrt{NHN^{-\varepsilon}} \), that is to say, the level might go beyond \( 1/2 \) when we deal with not too short intervals, e.g., \( H \gg N^{3\varepsilon} \).

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