**Digital measuring precision depending on not only resolutions but also sampling frequencies**

**Y S Hao**  
North China Electric University, Baoding, Hebei 071051, China  
E-mail: y.s.hao@foxmail.com

**Abstract.** Digital measurements have widely been applied to observing signals in the world. Powers and frequencies are both main characteristic of signals. It is obvious that power precision of the signals is affected by resolutions of ADCs. There will be frequency errors if sampling frequencies are less than their lower limits, which was found by Nyquist at 1928. It will be proved true that there would be frequency errors too if sampling frequencies are larger than their supper limits in this paper, which are proportional to precisions in S-domain, and inversely proportional to resolutions of ADCs.

1. **Introduction**  
Digital measurements (DM) have been applied to observing quantum, universe, biology, medicine and engineering. It is necessary to improve precisions of DM to increase the resolutions of analog-digital-converters (ADCs).  

Powers as main characteristic of signals are affected in precisions by resolutions of ADCs. Frequency components are another characteristic of signals. There would be frequency errors if sampling frequencies \((f_s)\) are less than double of cut frequencies \((f_c)\) of the signals early in 1928 [1], which was called as Nyquist’s Sampling Theorem (NST) by Shannon [2] and well known in information, science and technology, etc.

Difference equation (DCE) is approximation of differential equation (DTE) in mathematics. The approximation would be better if sampling intervals \((T_s)\) were smaller. So, \(f_s\) go faster and faster in DM.

There are problems if \(f_s\) are too fast in applications. The fast Fourier transform (FFT) goes uncertain [3]. Controllers are possibly unstable in digital which worked well in analog [4,5]. There are too big errors to identify parameters of n-order systems where \(n\) is greater than 2 [6,7].

On the other hand, divisor cannot be equal to zero in mathematics and errors will be expanded when divisor less than one. So there would be a minimum of \(T_s\).

It will be proved true in this paper that there are upper limitation of sampling frequencies \(f_{sh}\) and frequency errors of signals will be significant if \(f_s\) is over the \(f_{sh}\).

2. **The upper limitations of sampling frequencies**  
The world is in a 3-dimentional continuous space with time as parameter. Dynamic models are DTEs.

It is necessary firstly to sample signals of the world by ADCs to digits to let computers know the signal. There will secondly be digital treatments, such as digital filter (DF), digital analysis (DA), digital controls (DC) and decisions (DD). Frequency characteristic analysis (FCA) has to be done at least in most applications.
2.1. About sampling

Sampling analog signals to digital ones by ADCs in physics is a transform from time-domain to Z-domain in digital control theory [4]. H(z) is the transfer function in Z-domain.

\[ \delta_z = \delta_{ad} \]  

(1)

Where \( \delta \) means here relative error and \( \Delta \) absolute error. Noise levels of applications should be less than the grids in Z-domain \( \Delta_z \) which will be constant if ADCs are selected.

So Z-domain is discontinuous, and points in Z-domain must be on the grids as shown in figure 1.

![Figure 1. Grids in Z-domain.](image)

2.2. Grids in S-domain

2.2.1. Transform of S-domain and Z-domain.

FCA is done in S-domain. G(s) is transfer function in S-domain.

Transform from S-domain to Z-domain is  

\[ z = e^{sT_f} \]

The inverse transform from Z-domain to S-domain is  

\[ s = f_s \cdot \ln(z) \]  

(4)

The inverse transform is,

\[ \begin{align*}
    r &= e^{x/f_s} \\
    \theta &= y/f_s
\end{align*} \]  

(2)

Since \( \theta \) is within \( (-\pi, \pi) \), \( y \) will be within \( (-\pi \cdot f_s, \pi \cdot f_s) \) according to equation (3).

Since \( x \leq 0 \) in S-domain, \( r \leq 1 \) according to equation (2).

So, Left half plane of the S-domain is transferred into unit circle in Z-domain [3]. In fact, area away from the imaginary axis will be no use. So, common area \( (A_0) \) shown in figure 2 is a hollow ring in Z-domain.

Since \( y \) as \( \omega \) is within \( (-\pi \cdot f_s, \pi \cdot f_s) \), \( f = \omega / (2 \cdot \pi) \) will be within \( (-f_s/2, f_s/2) \). So \( f < f_s/2 \).

Cut frequency \( f_c \) is the biggest frequency of the system studied. Hence, \( f_s > 2 \cdot f_c \), which is known as the famous NST.
Another area, \( A_1 \) as in figure 3, will be transformed into \( A_0 \) in Z-domain too, which will lead to one point in Z-domain corresponding to multi-points in S-domain, which are called frequency errors mixed as the inference of the NST.

\[
\begin{align*}
\frac{dx}{f_s} & = \frac{dr}{r} \\
\frac{dy}{f_s} & = d\theta
\end{align*}
\] (4)

\[
\begin{align*}
\frac{dx}{f_s} & \approx dr \\
\frac{dy}{f_s} & = d\theta
\end{align*}
\] (5)

2.2.2. Grids in Z-domain into S-domain. According to equation (3).
Hence, grids in S-domain are,
\[ \delta_s = f_s \cdot \delta_z \]  \hspace{1cm} (6)

It means that the distance of the grids in S-domain is \(f_s\) times one in Z-domain.

2.2.3. \textit{Illustrations}. A hollow ring in Z-domain with inner radius 0.368 and outer radius 1 shown in figure 4(a) is transferred into rectangle from (-10 -31.42) to (0, 31.42) in S-domain shown in figure 4(b) when \(f_s\) is equal to 10 Hz. The rectangle will be expanded to (-30, -94) - (0, 94) in figure 4(c) when \(f_s\) is increased to 30 Hz.

\textbf{Figure 4.} Grids differences with different sampling frequencies. (a) A hollow ring Image in Z-domain, (b) Photo in S-domain with \(f_s=10\)Hz and (c) and Photo in S-domain with \(f_s=30\) Hz.

The area of the photo in S-domain is expanded 9 times when sampling frequency 3 times. \(f_s\) take functions as zooms in cameras, \(f_s\) bigger, and zoom larger of photo in S-domain with the same image in Z-domain. Distance of grids in S-domain will be larger with the same grids in Z-domain.

2.3. The upper limitation of sampling frequencies
\(\varepsilon_s\) is marked as acceptable error in S-domain. So sampling frequency should satisfy,
\[ f_s \leq \varepsilon_s / \delta_z \]  \hspace{1cm} (7)

It means that the upper limitation of sampling frequencies \(f_{sh}\),
\[ f_{sh} = \varepsilon_s / \delta_z \]  \hspace{1cm} (8)

To include NST, sampling frequencies \(f_s\) should satisfy,
\[ 2 \cdot f_c = f_{sd} < f_s \leq f_{sh} = \varepsilon_s / \delta_z \]  \hspace{1cm} (9)

3. The supper limitations are very small to be over

3.1. \textit{Examples}
Resolutions of ADCs are 12 bits where a bit as sign. Double precision floating point numbers are adopted in analysis. So,
\[ \delta_z = \delta_{adc} = 2^{-11} = 1/2048 = 0.05\% \]  \hspace{1cm} (10)

Precisions of DF in communication (DFC) should be 3 db. So, \(\varepsilon_s = 0.2929\)
\[ f_{sh} = 0.2929 \times 2048 = 600 \]  \hspace{1cm} (11)
\[ \delta_z = \delta_{adc} = 2^{-11} = 1/2048 = 0.05\% \]  \hspace{1cm} (10)
While in DCs, $\varepsilon_s$ will be 0.2, and $f_{sh}$ will be 410. $f_{sh}$ in other applications are shown in table 1.

| App  | $\varepsilon_s$ | $f_{sh}$ |
|------|----------------|----------|
| DFC  | 0.2929         | 600      |
| DCs  | 0.2            | 410      |
| PAs  | 0.05           | 102      |
| DFM  | 0.01           | 20       |

Hence, $f_{sh}$ will vary with applications, be very small to be over.

### 3.2. Ways to increase the upper limitation of sampling frequencies

According to equation (8), $\delta_z$ as grids in Z-domain will be decreased because that $\varepsilon_s$ is constant in order to increase the upper limitation of sampling frequencies in applications. A way to decrease $\delta_z$ is to increase resolutions of ADCs. 14, 16, 18 bits ADCs can be selected. $f_{sh}$ is calculated bellow in table 2.

It should be noticed that noise levels should be less than the resolutions of ADCs.

There is a patent way [8] to satisfy equation (9) if table 2 cannot be met in applications.

| Bits | 12  | 14  | 16  | 18  |
|------|-----|-----|-----|-----|
| DFC  | 600 | 2399| 9598| 38391|
| DCs  | 410 | 1638| 6554| 26214|
| PAs  | 102 | 410 | 1638| 6554 |
| DFM  | 20  | 82  | 328 | 1311 |

### 3.3. Illustration

Take DTE as objects, $G(s)$ as photo, $H(z)$ as image of the object. There is no any distortion between photo $G(s)$ and object DTE. But there will be distortion between the image $H(z)$ and the object DTE.

In order to reduce the distortion, the sampling frequencies should be within $(f_{sl}, f_{sh})$ according to equation (9), where $f_{sl}$ cannot be too slow, too fast either.

### 3.4. Analysis

Take the digital measurements as a whole system, an object to be measured, a measuring tool. So, $\varepsilon_s$ comes from the object, $\delta_z$ is the level of the measuring tool.

DCE is approximation of DTE in t-domain. The approximation needs $T_s$ within $[T_{s_{min}}, T_{s_{max}}]$ in mathematics. Now it is given by equation (9).

### 4. Conclusions

The upper limitation of the sampling frequencies $f_{sh}$ is found in this paper which is proportional to acceptable errors of frequencies and inversely proportional to resolutions in digital measurements where the former are linked with objects to be observed and the latter with levels of measuring equipments. And $f_{sh}$ is very small to be easily crossed over.

### References

[1] Nyquist H 1928 Certain topics in telegraph transmission theory *Trans. AIEE* 47 617-44

[2] Shannon C E 1948 A mathematical theory of communication *The Bell System Technical Journal* 27 379-423
[3] Hill M 2013 The uncertainty principle for Fourier transforms on the real line (Chicago)
[4] Nagrath I J and Gopal M 1982 Control System Engineering (New York: Halsted Press)
[5] Goodwin G C, Salgado M and Ji Y E 2003 Performance limitations for linear feedback systems in the presence of plant uncertainty IEEE Trans. on AC 48(C1) 1312-9
[6] Astrom K J and Wittenmark B 1971 Problems of identification and control J. Math. Anal. Appl. 34 90-113
[7] Hao Y S 2017 Parameter Identifications of Electric Transmission Lines (Beijing, China: China Electric Power Press)
[8] Hao Y S 2016 Data sampling method and system, and application method and system thereof in parameter identification (United States Patent US9438449)