The Extended UCB Policies for Frequentist Multi-armed Bandit Problems

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\textbf{ABSTRACT}

The multi-armed bandit (MAB) problem is a widely studied model in the field of operations research for sequential decision making and reinforcement learning. This paper mainly considers the classical MAB model with the heavy-tailed reward distributions. We introduce the extended robust UCB policy, which is an extension of the pioneering UCB policies proposed by Bubeck et al. \cite{Bubeck2011} and Lattimore \cite{Lattimore2011}. The previous UCB policies require the knowledge of an upper bound on specific moments of reward distributions or a particular moment to exist, which can be hard to acquire or guarantee in practical scenarios. Our extended robust UCB generalizes Lattimore’s seminal work (for moments of orders $p = 4$ and $q = 2$) to arbitrarily chosen $p$ and $q$ as long as the two moments have a known controlled relationship, while still achieving the optimal regret growth order $O(\log T)$, thus providing a broadened application area of the UCB policies for the heavy-tailed reward distributions.

\textbf{KEYWORDS}

frequentist multi-armed bandit problems; the light-tailed reward distributions; the heavy-tailed reward distributions; upper confidence bounds

\section{Introduction}

The multi-armed bandit problem is an important topic in operations research where a player aims to identify and select the best arm to operate such that the long-term expected reward is maximized. The player’s action at each time involves balancing the trade-off between exploration and exploitation, \textit{i.e.}, choosing an arm to learn its statistics vs. choosing an arm appearing to be best based on past observations. Under resource constraints, the way we acquire knowledge becomes crucial for the efficiency in maximizing the long-term reward in modern AI systems \cite{Liu2024}. While knowledge can be generally treated as data (usually not free in a strict sense), the struggle between exploration (sampling) and exploitation (consuming) as the prototypical dilemma in reinforcement learning is expressly reflected in an MAB model \cite{Bubeck2011, Lattimore2011, Zha2021, Zhang2022}. Specifically, the original stochastic model of the multi-armed bandit problem considers a
bandit machine with several arms; each arm $i$ offers a random reward according to an unknown distribution $X_i$ when pulled by the player (or, agent). Each reward distribution has a mean, say $\theta_i$. The player who has limited knowledge about the underlying reward distributions (no knowledge about the means $\{\theta_i\}$), has to choose only one of these arms to play and accrue a reward at each time step, given the reward history so far observed. The target of the player is to maximize the rate of gaining reward as the time horizon $T$ grows. In many applications of online decision-making problems, such as Internet advertising and recommendation systems, the MAB model is widely assumed with numerous algorithms proposed for efficient reinforcement learning, e.g., Zhao [38], Lattimore and Szepesvári [22]. In a recommendation system, the vendor desires to know which item an online customer is looking for with the most interest. When we suggest a product and the customer makes a purchase, we receive a positive reward. Conversely, if the customer decides not to buy the recommended item, our reward may be either zero or negative, reflecting a potential waste of resources. Therefore the target of a recommendation system is to guess which items are most wanted by each customer, and the problem can be translated into the MAB model directly.

To address the key issue in the conflict between exploration and exploitation, we note that it is impossible for the player to exactly know the expected values of the reward distributions within any finite horizon of time. The only information the player possesses is the (local) reward history, and the player has to learn from this to make a decision at each time step. Based on the limited historical information, the player has to choose between two options. The first option is to explore the arms with a lower average reward, as there is always a possibility that the best arm may not give the best average reward in the past sample path. The second option is to exploit the arm with the highest average reward offered so far. Robbins [28] considered a simple two-armed bandit problem and showed that there is a policy to achieve the best average reward asymptotically as time goes to infinity. Many years later, Lai and Robbins [20] proposed a much stronger performance measure referred to as regret, which represents the expected total loss compared to the benchmark case that the player already knows which arm has the highest reward mean from the very beginning. Given a set of $K$ arms with reward distributions $\mathcal{F} = \{F_1, F_2, \ldots, F_K\}$ with means $\{\theta_i\}_{i=1}^K$ and a policy $\pi$, the regret $R^\mathcal{F}_\pi(T)$ is defined as (without loss of generality, we can assume that $\theta_1 \geq \theta_i$ for $1 \leq i \leq K$)

$$R^\mathcal{F}_\pi(T) = \sum_{\theta_i < \theta_1} E[s_i(T)] \Delta_i,$$

(1)

where

$$\Delta_i = \theta_1 - \theta_i,$$

(2)

and $s_i(T)$ denotes the number of times that arm $i$ was chosen during time steps 1 to $T$ under policy $\pi$. Certainly, a lower regret growth rate indicates better performance of a policy, and also a higher efficiency of learning. Furthermore, any sub-linear regret order implies the best expected time-average reward as $T \to \infty$.

Strikingly, Lai and Robbins [20] proved an asymptotic lower bound on regret, with the growth rate of $O(\log T)$, under the assumption that the probability density/mass function of each reward distribution has the form $f(\cdot; \theta)$, where $f$ is known but parameter $\theta$ is unknown. They also proposed an asymptotically optimal policy for a family
of distributions, under which the best regret can be achieved asymptotically, both the logarithmic order and the leading coefficient. Later Burnetas and Katehakis [6] proved a generalized result under non-parametric settings. Suppose $\mathcal{H}$ is a set of distributions on $\mathbb{R}$. We call a bandit $\mathcal{F}$ (considered as a set of reward distributions of all arms) non-trivial, if there is at least one distribution (re-index it to 1 if necessary) $F_1 \in \mathcal{F}$ satisfying $\theta_1 > \theta_i$ for all $i \in \{2, \ldots, K\}$. They proved that for any policy $\pi$ s.t. for each fixed nontrivial $\mathcal{F} \subset \mathcal{H}$ (this policy condition is called consistency over $\mathcal{H}$)

$$R_\pi^\mathcal{F}(T) = o(T^a), \quad \forall a > 0,$$

we have that

$$\liminf \frac{\mathbb{E}[s_i(T)]}{\log T} \geq \sup \left\{ \frac{1}{\text{KL}(F_1, Y)} : Y \in \mathcal{H} \text{ and } \mathbb{E}Y > \theta_i \right\},$$

where $\text{KL}(\cdot, \cdot)$ denotes the Kullback-Leibler distance between two distributions. This theorem demonstrates that for any consistent strategy under a given family of distributions, the regret will not be lower than the logarithmic growth. Auer et al. [3] proposed a class of policies named and featured by UCB (upper confidence bound), achieving logarithmic regret if the reward distributions have finite support on a closed interval $[a, b] \subset \mathbb{R}$ with $a, b \in \mathbb{R}$ known to the player. Their UCB1 and UCB1-Tuned policies perform very well for the classical MAB problem with a theoretically proven regret bound for any finite time horizon. The UCB policies are widely applied in machine learning development, e.g., the recommendations systems [16, 32], and in various reinforcement learning algorithms such as Monte Carlo Tree Search (MCTS) [19], which is frequently used in gaming AIs [27] and forms a key step in the design of AlphaGo Zero [15].

### 1.1. Heavy-tailed MAB Model

The class of heavy-tailed distributions is a concept in contrast to the light-tailed case. Light-tailed (or, sub-Gaussian) distributions usually have only a relatively small probability of producing a significantly shifted random sample from its expectation, while heavy-tailed distributions have a relatively high probability of producing a large deviation. There are several popular light-tailed probability distributions, such as Bernoulli, Gaussian, Laplacian, and Exponential. Specifically, the class of light-tailed distributions requires the (local) existence of the moment-generating function of the associated random variable and is therefore referred to as the locally sub-Gaussian distributions [10]. Formally, a random variable $X$ is called light-tailed if there exists some $u_0 > 0$ such that its moment-generating function is well-defined on $[-u_0, u_0]$ [10]:

$$M(u) := \mathbb{E}[\exp(uX)] < \infty, \quad \forall |u| \leq u_0. \quad (5)$$

The above condition is equivalent to [10]

$$\mathbb{E}[\exp(u(X - \mathbb{E}X))] \leq \exp(\zeta u^2/2), \quad \forall |u| \leq u_0, \quad \forall \zeta \geq \sup_{|u| \leq u_0} M^{(2)}(u), \quad (6)$$

where $M^{(2)}(\cdot)$ denotes the second derivative of $M(\cdot)$. Note that the above upper bound on $M(u)$ has a form of the moment generating function of the Gaussian distribution,
thus the name “sub-Gaussian” has been adopted.

The terminology “heavy-tailed”, opposite to “light-tailed”, implies that the reward distribution might take large values with higher probabilities, thus its moment-generating function does not exist (even locally). Compared to the light-tailed distributions, the heavy-tailed are harder to learn in terms of the rank of their means. Bubeck et al. [5] proposed the robust UCB algorithm, under the assumption that there exists a known \( p > 1 \) such that \( \mathbb{E}[|X_i|^p] \) exists and is upper bounded by a known parameter, for all \( 1 \leq i \leq K \). Robust UCB shows significant progress for the heavy-tailed MAB model by achieving the optimal logarithmic order of regret growth. Later on, Lattimore [21] developed a new UCB policy, which is referred to as the scale free algorithm. This algorithm removes the general assumption of a known upper bound on the \( p \)-th moment. Instead, it allows the variance and the squared fourth moment to scale freely as long as their ratio (kurtosis) is bounded by a known parameter. But this assumption restricts \( p \) to be no less than 4 (thus cannot be too heavy-tailed). In this paper, we extend Lattimore’s work to any \( p > 1 \) with a tighter regret upper bound in the case of \( p = 4 \) and low discrimination where arm means are close and learning the arm rank becomes challenging.

From this point on, we assume that the player knows a constant \( C_{p,q} \) for some \( p \) and \( q \) (\( p > q > 1 \)) such that

\[
\mathbb{E}[|X - \mathbb{E}X|^p] \leq C_{p,q} (\mathbb{E}[|X - \mathbb{E}X|^q])^{p/q}.
\]  

(7)

The above inequality is a direct generalization of the assumption on kurtosis in Lattimore [21] where \( p = 4 \) and \( q = 2 \). For a specific example, suppose that \( X \) has a Pareto distribution of type III with cumulative distribution function [2]

\[
F(x) = \begin{cases} 
1 - \left(1 + \left(\frac{x-a}{\sigma}\right)^{1/\gamma}\right)^{-1}, & x > a \\
0, & x \leq a
\end{cases},
\]

where \( 0 < \gamma < 1 \) and \( \sigma > 0 \). Simple computations show that

\[
\mathbb{E}[|X - \mathbb{E}X|^p] = C(p, q, \gamma) (\mathbb{E}[|X - \mathbb{E}X|^q])^{p/q}
\]

for \( 1 < q < p < 1/\gamma \), where

\[ I(p, \gamma) := \int_0^{+\infty} \left(x - \frac{\gamma \pi}{\sin \gamma \pi}\right)^p \frac{x^{1/\gamma}}{\gamma x(1 + x^{1/\gamma})^2} dx, \]

\[ C(p, q, \gamma) = I(p, \gamma)/I(q, \gamma)^{p/q}. \]

However, the moment under this distribution is a function of \( \sigma \); when \( \sigma \) increases, the moment of any fixed order also increases, if it exists. The robust UCB policy requires the knowledge of \( p \) and an upper bound of the \( p \)-th moment [5]. Therefore, for an arm with an unknown \( \sigma \), we cannot apply the robust UCB policy. Furthermore, the learning efficiency can be improved if a larger \( p \) can be used (tighter bound on the tail, but the corresponding moment bound becomes harder to know or compute a priori), it is desired to eliminate the need for any prior knowledge regarding the specific values or bounds of the moments. This motivates Lattimore [21] to consider a scale free algorithm. After we generalize Lattimore’s work, we will give a correction to
a minor flaw in a lemma of Bubeck et al. \cite{5} which subsequently caused a small error in the computation of regret upper bound in Lattimore \cite{21}.

### 1.2. Related Works

The theorems of the frequentist MAB have been developed for a long time and various milestones have been made since Lai and Robbins \cite{20}. In contrast to the Bayesian model, the frequentist MAB does not assume a priori knowledge of the initial probability or state of the system but learns the core parameters solely through the past sampling history, \textit{e.g.}, computing an upper confidence bound as a function of the frequency that an arm is selected and the values of the observed samples. Such a way of decision-making in balancing exploration with exploitation contrasts to the Bayesian MAB where the tradeoff is usually addressed by dynamic programming for the time-varying characteristics of the system or the observation model \cite{12}. After the UCB1 policy by Auer et al. in \cite{3}, many other policies of the UCB-type have been proposed in the literature. Maillard et al. \cite{26} directed their attention to the prospect of KL divergence between probability distributions. They introduced the KL-UCB algorithm, which notably enhanced the regret bound. This improvement was achieved under the assumption of rewards being bounded and having a known upper bound. Furthermore, KL-UCB policy achieves the asymptotic lower bound of regret growth for this class of rewards given in Burnetas and Katehakis \cite{6}. Kaufmann et al. \cite{18} combined the Bayesian method with the UCB class and proposed Bayes-UCB for bounded rewards with a known bound, achieving a theoretical regret bound similar to that of KL-UCB. Numerical experiments demonstrated the high efficiency of learning by Bayes-UCB for Gaussian reward distributions. A survey of such UCB policies can be found in Burtini et al. \cite{7}. These policies were neither implemented nor proved to achieve the logarithmic regret growth in the case of unbounded reward distributions. In 2011, the first author of this paper proposed UCB1-LT with a proven bound on the regret growth of the logarithmic order for the class of light-tailed reward distributions \cite{24}, filling the vacancy for the case of unbounded light-tailed reward distributions. Together with the extended robust UCB for the heavy-tailed class focused in this paper, we provide a relatively complete picture of UCB policies for general reward distributions. After 2011, Bubeck and Cesa-Bianchi \cite{4} extended UCB1-LT to $(\alpha, \psi)$-UCB under a more general assumption that the moment-generating function is bounded by a convex function $\psi$ for all $u \in \mathbb{R}$.

For the case of heavy-tailed reward distributions, we have mentioned that Bubeck et al. \cite{5} proposed the robust UCB policy under the assumption that an upper bound of the moments is known. The robust UCB policy uses different mean estimators rather than the empirical one and remarkably achieves the logarithmic regret growth rate. Under certain regularity conditions, the robust UCB policy may adopt three different mean estimators: the truncated mean estimator, the median-of-means estimator, and the Catoni mean estimator \cite{8}. The truncated mean estimator requires the knowledge of an upper bound of origin moments, which is not very effective in the case that the reward mean is far away from zero. The median-of-means estimator considers the (central) moment and thus patches up the deficiency of the truncated mean estimator: the mean estimation will not be affected when the reward distributions are changed by translations. The Catoni mean estimator is more complicated and can only be used when variances of the reward distributions exist ($p \geq 2$). But when applied in the robust UCB policy, it gives a much smaller regret coefficient than the other
two estimators. Recently Chen et al. [11] showed a result of applying Catoni mean estimator to the case that \( p < 2 \), but the performance was not proven to be better than the truncated mean and the median-of-means estimators. Finer estimations when extending the Catoni mean estimator to the case of \( p < 2 \) are interesting for future investigations. Several recent papers also considered the heavy-tailed MAB model and made significant progresses. Wei and Srivastava [36] proposed the robust MOSS policy with logarithmic regret growth. Agrawal et al. [1] established a policy called KL-inf-UCB, which achieves a logarithmic upper bound of regret very close to the asymptotic lower bound proved by Burnetas and Katehakis [6]. For other non-classical MAB models, the heavy-tailed model has also been considered [20, 37]. However, all these policies do not discard the knowledge requirement of an upper bound of moments, which has been removed from the extended robust UCB policy proposed in this paper.

There are also policies not following the idea of UCB. The \( \varepsilon \)-greedy policy is well-studied, and many policies are proposed using its idea, e.g., the constant \( \varepsilon \)-decreasing policy [34], GreedyMix [9] and \( \varepsilon_n \)-greedy [3]. The Deterministic Sequencing of Exploration and Exploitation (DSEE) policy [25, 33] achieves the logarithmic regret for both cases of single player and multiple players (with the knowledge on moment upper bounds). Other recent progresses on variants of MAB, such as adversarial bandits, contextual bandits and linear bandits, can be found in Lattimore and Szepesvári [13, 17, 22].

2. The Heavy-Tailed Distribution

2.1. The Extended Robust UCB Policy

Consider \( K \) arms offering random rewards \( \{X_1, X_2, \ldots, X_K\} \) with distributions \( F = \{F_1, F_2, \ldots, F_K\} \). In contrast with the light-tailed class, the heavy-tailed distribution may yield a very high (or low) reward realization with a high probability. Consequently, the empirical mean estimator \( \bar{X}_{i,s} = \frac{1}{s} \sum_{j=1}^{s} X_{i,j} \) for a sequence of i.i.d. random variables \( \{X_{i,j}\}_{j=1}^{s} \) may not work well as an estimation to the expectation of \( X_i \) for arm \( i \).

The appendix of Bubeck et al. [5] has shown that under the assumption of heavy-tailed distributions, with probability at least \( 1 - \delta \),

\[
\frac{1}{s} \sum_{j=1}^{s} X_{i,j} \leq \theta_i + \left( \frac{3E|X_i - \theta_i|^r}{\delta s^{r-1}} \right)^{\frac{1}{r}},
\]

where \( 1 < r \leq 2 \). This bound is too loose to obtain a UCB policy achieving the logarithmic regret growth. To remedy this issue of divergence, we adopt the median-of-means estimator proposed in Bubeck et al. [5].

**Definition 1.** For a finite i.i.d. sequence of random variables \( \{X_{i,j}\}_{j=1}^{s} \) drawn from a distribution \( F_i \), define the median-of-means estimator \( \hat{\mu}(\{X_{i,j}\}_{j=1}^{s}, k) \) with \( k \) \((k \leq s)\) bins as the median of \( k \) empirical means

\[
\left\{ \frac{1}{N} \sum_{j=lN+1}^{(l+1)N} X_{i,j} \right\}_{l=0}^{k-1},
\]

where \( N = \lfloor s/k \rfloor \).
The following lemma offers an approximation of the proximity between the median-of-means estimator, utilizing a specific number of bins, and the actual mean. It should be noted that this lemma is similar to Lemma 2 in Bubeck et al. [5]. However, there is a minor flaw in the proof of the latter, which will be explained later. Moreover, there exist differences between the two lemmas, prompting us to restate and establish their distinctions in the following exposition.

**Lemma 1.** Given an arbitrary \( \varepsilon > 0 \). Suppose that

\[
\log \delta^{-1} > 1/\varepsilon, \quad s \geq (8 + \varepsilon) \log \delta^{-1} + \frac{1}{\varepsilon} \log(\delta^{-1}) - 1.
\]

and \( v = \mathbb{E}[|X_i - \theta_i|^p] < \infty \), for some \( p \) such that \( p > 1 \). Then with probability at least \( 1 - \delta \), the median-of-means estimator

\[
\hat{\mu}(\{X_{i,j}\}_{j=1}^s, [8 \log \delta^{-1}]) \geq \theta_i - (12v)^{1/p} (\frac{(8 + \varepsilon) \log(\delta^{-1})}{s})^{(s-1)/p}.
\]

Also, with a probability of at least \( 1 - \delta \),

\[
\hat{\mu}(\{X_{i,j}\}_{j=1}^s, [8 \log \delta^{-1}]) \leq \theta_i + (12v)^{1/p} (\frac{(8 + \varepsilon) \log(\delta^{-1})}{s})^{(s-1)/p}.
\]

**Proof.** Observe that \( s \geq [8 \log \delta^{-1}] \) holds under the given condition in the lemma. For simplicity of presentation, we introduce some symbols as below. Let

\[
\begin{align*}
    k &:= [8 \log \delta^{-1}], \quad \chi := \frac{(8 + \varepsilon) \log \delta^{-1}}{s}, \\
    \eta &:= (12v)^{1/p} \chi^{(p-1)/p}, \\
    \hat{\mu}_l &:= \frac{1}{N} \sum_{j=lN+1}^{(l+1)N} X_{i,j}, \quad l = 0, 1, \ldots, k - 1.
\end{align*}
\]

According to the appendix of Bubeck et al. [5],

\[
\xi := \mathbb{P}(\hat{\mu}_l > \theta_i + \eta) \leq \frac{3v}{N^{p-1}p^p} = \frac{1}{4N^{p-1}p^p}.
\]

Note that \( N = \lfloor s/[8 \log \delta^{-1}] \rfloor \). Since \( s \geq (8 + \varepsilon) \log \delta^{-1} + \frac{1}{\varepsilon} \log(\delta^{-1}) - 1 \), a direct computation yields \( \xi \leq 1/4 \). Then using Hoeffding’s inequality for the tail of a binomial distribution [14], we have

\[
\mathbb{P}(\hat{\mu}(\{X_{i,j}\}_{j=1}^s, k) > \theta_i + \eta) \leq \mathbb{P}\left( \sum_{l=0}^{k-1} \mathbb{I}(\hat{\mu}_l \geq \theta_i + \eta) \geq \frac{k}{2} \right) \leq \exp(-2k(1/2 - \xi)^2) \leq \delta.
\]

Similarly, the other inequality also holds by symmetry. \( \square \)
With the help of the median-of-means estimator, we can now define the upper confidence bound for the extended robust UCB policy. As mentioned above, we assume that for any arm distribution $X_i \in \mathcal{F}$ with mean $\theta_i$, there is a moment control coefficient $C_{p,q}$ ($1 < q < p$) such that

$$E[|X_i - \theta_i|^p] \leq C_{p,q} E[|X_i - \theta_i|^q]^r,$$  \tag{12}

where $r = p/q$.

**Definition 2.** Recall the notations introduced by (8) and (9). Given an arbitrary $\varepsilon > 0$, a moment order $p > 1$, another moment order $1 < q < p$, and the moment control coefficient $C_{p,q}$, define the upper confidence bound of the extended robust UCB as (value of the fraction is set to $+\infty$ if divided by 0)

$$\tilde{\mu}(\{X_{i,j}\}_{j=1}^{s}, \delta) := \sup \left\{ \theta \in \mathbb{R} : \theta \leq \tilde{\mu}(\{X_{i,j}\}_{j=1}^{s}, k) + \left( \frac{12\tilde{\mu}(\{X_{i,j} - \theta\}_{j=1}^{s}, k)}{\max \{0, 1 - C'\chi^{(p-q)/p}\}} \right)^{1/q} \chi^{(q-1)/q} \right\},$$

where

$$C' = (12(C_{p,q} + 1))^{q/p}.$$ \tag{13}

Note that this upper confidence bound always exists, since the left-hand side of the above inequality tends to $-\infty$ as $\theta \to -\infty$, while the right-hand side tends to $+\infty$.

**Lemma 2.** Suppose that the conditions in Lemma 2 are satisfied, then with probability at least $1 - 2\delta$, we have

$$\tilde{\mu}(\{X_{i,j}\}_{j=1}^{s}, \delta) \geq \theta_i.$$

**Proof.** Recall the notations introduced by (8), (9) and (13). If $\tilde{\mu}(\{X_{i,j}\}_{j=1}^{s}, \delta) = +\infty$, the inequality holds with probability 1. Assume that $\tilde{\mu}(\{X_{i,j}\}_{j=1}^{s}, \delta) < +\infty$, i.e.

$$1 - C'\chi^{(p-q)/p} > 0.$$

Define

$$Y_i := |X_i - \theta_i|^q,$$
$$Y_{i,j} := |X_{i,j} - \theta_i|^q,$$
$$v_q := \mathbb{E}Y_i,$$
$$v_p := \mathbb{E}[|X_i - \theta_i|^p].$$ \tag{14}
From Lemma 1, we have

\[
P \left( \frac{\theta_i - \hat{\mu}(\{X_{i,j}\}_{j=1}^s) - (12\nu_q)^{1/q}}{v_q - \hat{\mu}(\{Y_{i,j}\}_{j=1}^s, k)} \leq \chi^{(q-1)/q} \right) \geq 1 - 2\delta.
\]

(15)

From Lemma 1 and by the definition of \(C_{p,q}\), we have

\[
E[|Y_i - v_q|^{p/q}] \leq E[|X_i - \theta_i|^p + v_q^{p/q}] = v_p + v_q^{p/q} \leq (C_{p,q} + 1)v_q^{p/q}.
\]

(16)

Merging the two inequalities (15) and (16), with probability at least \(1 - 2\delta\), we have

\[
\theta_i \leq \hat{\mu}(\{X_{i,j}\}_{j=1}^s) + \left( \frac{12\hat{\mu}(\{|X_{i,j} - \theta_i|^{p/q}\}_{j=1}^s) + \max\{0, 1 - C'\chi^{(p-q)/p}\}}{\chi^{(q-1)/q}} \right)
\]

The proof is thus completed by the definition of \(\hat{\mu}(\{X_{i,j}\}_{j=1}^s, \delta)\).

Corollary 1. Under the conditions of Lemma 1, with probability at least \(1 - \delta\), we have

\[
\hat{\mu}(\{Y_{i,j}\}_{j=1}^s) - v_q \leq C' v_q \chi^{(p-q)/p}.
\]

Proof. This corollary is a direct consequence of Lemma 1 and the inequality in (16).

Finally, with the upper confidence bound estimator \(\hat{\mu}(\{X_{i,j}\}_{j=1}^s)\), we complete the extended robust UCB policy as follows.

\begin{algorithm}
\textbf{Algorithm 1:} The Extended Robust UCB Policy
\begin{algorithmic}[1]
\State Input: \(\varepsilon > 0, K\): the number of arms
\State Initialize: \(s_i \leftarrow 0\) and \(\{X_{i,j}\}_{j=1}^s\) an empty sequence for each arm \(i, 1 \leq i \leq K\)
\For {\(t \leftarrow 1\) to \(T\)}
\For {\(i \leftarrow 1\) to \(K\)}
\If {\(s_i = 0\)}
\State Set arm \(i\)'s upper confidence bound to \(+\infty\)
\EndIf
\Else
\State Compute arm \(i\)'s upper confidence bound as \(\hat{\mu}(\{X_{i,j}\}_{j=1}^s, t^{-2})\)
\EndIf
\EndFor
\State Choose an arm \(j\) that maximizes the upper confidence bound and obtain reward \(x\)
\State Update: \(s_j \leftarrow s_j + 1\) and append \(x\) to sequence \(\{X_{j,k}\}_{k=1}^s\)
\EndFor
\end{algorithmic}
\end{algorithm}
Here we present the main theorem that the extended robust UCB policy achieves the logarithmic order of regret growth.

**Theorem 1.** Suppose $p > 1$ such that the $p$-th order moments exist for all $F_i \in \mathcal{F}$, $1 < q < p$ and $C_{p,q}$ such that \[12\] holds. Then for any $\varepsilon > 0$, the regret $R_{\pi^*}^F(T)$ of the extended robust UCB policy $\pi^*$ has the logarithmic order with respect to the time horizon $T$.

To prove Theorem 1, we establish the following core lemma to bound the time $s_i$ spent on each arm $i$ by the extended robust UCB policy.

**Lemma 3.** The expected number of times $E[s_i(T)]$ that a sub-optimal arm is chosen by the extended robust UCB has an upper bound of the logarithmic order with $T$.

**Proof.** In addition to notations \[8\], \[9\] and \[13\], we introduce two new notations:

\[
\tau(x) := \frac{1}{1 - C' x^{(p-q)/p}}.
\]

\[
\zeta := 12 \tau(\chi) \hat{\mu} \left(|\{X_{i,j} - \hat{\mu}(\{X_{i,j}\}_{j=1}^s, \delta)|^q, k\right).
\]

By the definition of $\hat{\mu}(\{X_{i,j}\}_{j=1}^s, \delta)$, we have

\[
\hat{\mu}(\{X_{i,j}\}_{j=1}^s, \delta) = \hat{\mu}(\{X_{i,j}\}_{j=1}^s, k) + \zeta^{1/q} \chi^{(q-1)/q}.
\]

We need to consider the probability of the following event:

\[
\hat{\mu}(\{X_{i,j}\}_{j=1}^s, \delta) - \theta_i \leq \Delta_i,
\]

where $\Delta_i$ is defined in \[2\]. First we estimate an upper bound of $\zeta$. Suppose that $\tau(\chi) > 0$, we have

\[
|X_{i,j} - \hat{\mu}(\{X_{i,j}\}_{j=1}^s, \delta)|^q \leq 2^{q-1}|X_{i,j} - \theta_i|^q + 2^{q-1}|\theta_i - \hat{\mu}(\{X_{i,j}\}_{j=1}^s, \delta)|^q.
\]

Recall that $Y_{i,j} = |X_{i,j} - \theta_i|^q$ defined in \[14\]. Thus

\[
\zeta \leq 12 \tau(\chi) \left(2^{q-1} \hat{\mu}(\{Y_{i,j}\}_{j=1}^s, k) + 2^{q-1}|\theta_i - \hat{\mu}(\{X_{i,j}\}_{j=1}^s, \delta)|^q\right).
\]

\[
\leq 12 \tau(\chi) \left(2^{q-1} \hat{\mu}(\{Y_{i,j}\}_{j=1}^s, k) + 2^{q-2} \left(|\theta_i - \hat{\mu}(\{X_{i,j}\}_{j=1}^s, \delta)|^q + |\hat{\mu}(\{X_{i,j}\}_{j=1}^s, k) - \hat{\mu}(\{X_{i,j}\}_{j=1}^s, \delta)|^q\right)\right).
\]

\[
= 12 \tau(\chi) \left(2^{q-2} \hat{\mu}(\{Y_{i,j}\}_{j=1}^s, k) + 2^{q-2} \left(|\theta_i - \hat{\mu}(\{X_{i,j}\}_{j=1}^s, \delta)|^q + \zeta^{q-1}\right)\right).
\]

The inequality above uses \[19\] and also the fact that

\[
|a + b|^q \leq 2^{q-1}(|a|^q + |b|^q)
\]

by the convexity of $|x|^q$ ($q > 1$) and Jensen’s inequality \[29\]. Suppose that

\[
1 > 3 \cdot 2^{2q} \chi^{q-1} \tau(\chi),
\]

\[21\]
then
\[ \zeta \leq \frac{12\tau(\chi) \left( 2^{q-1}\hat{\mu}(\{Y_{i,j}\}_{j=1}^s, k) + 2^{2q-2}\theta_i - \hat{\mu}(\{X_{i,j}\}_{j=1}^s, k) \right)}{1 - 3 \cdot 2^{2q\chi_q - 1}\tau(\chi)}. \]

Suppose that \( \delta \) and \( s \) meet the conditions in Lemma 1. Then with probability at least \( 1 - 2\delta \), we have
\[ |\hat{\mu}(\{X_{i,j}\}_{j=1}^s, k) - \theta_i| \leq (12v_q)^{1/q} \chi^{(q-1)/q}. \]  
Using (22) and Corollary 1, by defining a function \( B(x) \) for simplicity as
\[ B(x) := \frac{6 \cdot 2^q\tau(x)}{1 - 3 \cdot 2^{2q\chi_q - 1}\tau(x)} \left( 1 + C'(x)^{(p-q)/p} \right) + \frac{3 \cdot 2^{2q}\tau(x) \cdot 12x^{(q-1)}}{1 - 3 \cdot 2^{2q\chi_q - 1}\tau(x)}, \]  
then with probability at least \( 1 - 3\delta \), we have
\[ \zeta \leq v_q B(\chi). \]  

Now we go back to the upper bound estimation of \( \hat{\mu}(\{X_{i,j}\}_{j=1}^s, \delta) - \theta_i \). Using Lemma 1 again and (24), with probability at least \( 1 - 3\delta \), we have
\[ \hat{\mu}(\{X_{i,j}\}_{j=1}^s, \delta) - \theta_i = (\hat{\mu}(\{X_{i,j}\}_{j=1}^s, \delta) - \hat{\mu}(\{X_{i,j}\}_{j=1}^s, k)) + (\hat{\mu}(\{X_{i,j}\}_{j=1}^s, k) - \theta_i) \]
\[ \leq v_q^{1/q} B(\chi)^{1/q} \chi^{(q-1)/q} + 12(12v_q)^{1/q} \chi^{(q-1)/q} \]
\[ = v_q^{1/q} \chi^{(q-1)/q} B(\chi)^{1/q} + 12^{1/q}. \]

Therefore, if
\[ v_q^{1/q} \chi^{(q-1)/q} B(\chi)^{1/q} + 12^{1/q} \leq \Delta_i \]  
holds, then (20) is true with probability at least \( 1 - 3\delta \).

Again, we need to emphasize the assumptions under which (20) is true with probability at least \( 1 - 3\delta \): \( \tau(\chi) > 0 \), (21) and (25). Equivalently, the following three inequalities are required for \( x = \chi^q \):
\[ 1 > C'(x)^{(p-q)/p}, \]  
\[ 1 > 3 \cdot 2^{2q\chi_q - 1}\tau(x), \]  
\[ \Delta_i \geq v_q^{1/q} x^{(q-1)/q} B(x)^{1/q} + 12^{1/q}. \]  

Observe that both \( \tau(x) \) and \( B(x) \) decrease as \( x \) decreases and
\[ \lim_{x \downarrow 0} x^{(q-1)/q} B(x)^{1/q} + 12^{1/q} = 0. \]

So one can always choose a \( \chi_M > 0 \) such that (26), (27) and (28) hold for \( x \leq \chi_M \).
Note that we also need the conditions \( \log \delta^{-1} > 1/\varepsilon \) and
\[
s \geq (8 + \varepsilon) \log \delta^{-1} + 1 + \frac{8 \log \delta^{-1} + 1}{\varepsilon \log \delta^{-1} - 1}
\]
to apply Lemma 1.

Next, we bound \( \mathbb{E}[s_i(T)] \) for any \( i \) such that \( \theta_i < \theta_1 \) and prove the logarithmic regret growth by (1). Define
\[
I(X) := \begin{cases} 1, & X \text{ occurs} \\ 0, & X \text{ doesn’t occur} \end{cases}
\]
to denote the characteristic function of an event \( X \). Let \( \delta = t^{-2} \). We use the notation \( A_t = i \) to indicate that arm \( i \) is chosen at time \( t \). For any \( i \) such that \( \theta_i \neq \theta_1 \), note that \( A_t = i \) implies \( \hat{\mu}(\{X_{1,j}\}_{j=1}^{s_{i}(t-1)}, t^{-2}) \leq \hat{\mu}(\{X_{i,j}\}_{j=1}^{s_{i}(t-1)}, t^{-2}) \). We have
\[
s_i(T) = \sum_{t=1}^{T} I(A_t = i) \leq \sum_{t=1}^{T} \mathbb{I}(\hat{\mu}(\{X_{1,j}\}_{j=1}^{s_{i}(t-1)}, t^{-2}) \leq \theta_1) + \sum_{t=1}^{T} \mathbb{I}(\hat{\mu}(\{X_{i,j}\}_{j=1}^{s_{i}(t-1)}, t^{-2}) \geq \theta_1).
\]  \( (29) \)

We will bound the first part of the right hand side of (29) by Lemma 2. Note that \( \log \delta^{-1} > 1/\varepsilon \) is equivalent to \( t > \exp(\frac{1}{2\varepsilon}) \). Write
\[
l_0 := \max \left\{ \left\lceil \exp\left(\frac{1}{2\varepsilon}\right) \right\rceil , \left\lceil \frac{(8 + \varepsilon) \log T^2 + 1}{\varepsilon \log T^2 - 1} \right\rceil \right\}.
\]  \( (30) \)

Also note that for sufficiently large \( T \), the function \( x^2 \frac{8 \log x^2 + 1}{\varepsilon \log x^2 - 1} \) is increasing for \( x \in \left[l_0 + 1, +\infty\right) \). We thus have
\[
\sum_{t=1}^{T} \mathbb{I}(\hat{\mu}(\{X_{1,j}\}_{j=1}^{s_{i}(t-1)}, t^{-2}) \leq \theta_1) \leq l_0 + \sum_{t=l_0+1}^{T} \mathbb{I}(\hat{\mu}(\{X_{1,j}\}_{j=1}^{s_{i}(t-1)}, t^{-2}) \leq \theta_1 \text{ and } s_{i}(t-1) \geq (8 + \varepsilon) \log T^2 \frac{8 \log T^2 + 1}{\varepsilon \log T^2 - 1}) \leq l_0 + \sum_{t=1}^{T} \mathbb{I}(\mu(\{X_{1,j}\}_{j=1}^{s_{i}(t-1)}, t^{-2}) \leq \theta_1 \text{ and } s_{i}(t-1) \geq (8 + \varepsilon) \log t^2 \frac{8 \log t^2 + 1}{\varepsilon \log t^2 - 1}).
\]

By Lemma 2, we have
\[
\mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{I}(\hat{\mu}(\{X_{1,j}\}_{j=1}^{s_{i}(t-1)}, t^{-2}) \leq \theta_1) \right] \leq l_0 + \sum_{t=1}^{\infty} \frac{2}{t^2} \leq l_0 + \frac{\pi^2}{3}.
\]
Now we bound the second part of the right hand side of (29). Define

\[ l_1 := \max \left\{ l_0, \left[ \frac{8 + \varepsilon}{\chi M} \log T^2 \right] \right\} . \]  

(31)

For sufficiently large \( T \), we have

\[
\sum_{t=1}^{T} \mathbb{I} \left( \tilde{\mu}(\{X_{i,j}\}_{j=1}^{s_i(t-1)}, t^{-2}) \geq \theta_1 \right) \\
\leq l_1 + \sum_{t=l_1+1}^{T} \mathbb{I} \left( \tilde{\mu}(\{X_{i,j}\}_{j=1}^{s_i(t-1)}, t^{-2}) - \theta_i \geq \Delta_i \text{ and } s_i(t-1) \geq l_1 \right) .
\]

(32)

By the definition of \( \chi \) in (9) and the choice of \( \chi M \), using (20), we have

\[
\mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{I} \left( \tilde{\mu}(\{X_{i,j}\}_{j=1}^{s_i(t-1)}, t^{-2}) \geq \theta_1 \right) \right] \leq l_1 + \frac{\pi^2}{2}.
\]

Finally, we have

\[
\mathbb{E}[s_i(T)] \leq l_0 + l_1 + \frac{5\pi^2}{6}.
\]

For sufficiently large \( T \), we can assume that \( l_0 = \left[ (8 + \varepsilon) \log T^2 \frac{8 \log T^2 + 1}{\varepsilon \log T^2 - 1} \right] \). Thus the above inequality becomes

\[
\mathbb{E}[s_i(T)] \leq \left[ (8 + \varepsilon) \log T^2 \frac{8 \log T^2 + 1}{\varepsilon \log T^2 - 1} \right] + \max \left\{ \left[ (8 + \varepsilon) \log T^2 \frac{8 \log T^2 + 1}{\varepsilon \log T^2 - 1} \right], \left[ \frac{8 + \varepsilon}{\chi M} \log T^2 \right] \right\} + \frac{5\pi^2}{6} .
\]

(33)

A Monte Carlo simulation example based the Pareto distribution of type III (see Sec. 1.1) is shown in Fig. 1. Note that the regret rate grows linearly at the beginning to prepare the ground for ranking until at least one arm has a finite UCB (i.e., \( \chi \) defined in (9) becomes sufficiently small such that, in the sup expression of \( \tilde{\mu}(\{X_{i,j}\}_{j=1}^{s_i(t-1)}, t^{-2}) \), both the denominator is nonzero and the growth rate of the right-hand side with \( \theta \) becomes small).
2.2. Further Analysis and Comparison of the Extended Robust UCB

Now we consider scenarios where specific values of $p$ and $q$ are chosen to further improve the efficiency of the extended robust UCB policy.

Case 1. Suppose that the $p'$-th order of moments of reward distributions exist with $p' > 2$ for all arms. Then we can choose $p, q$ such that $p = 2q \leq p'$. The inequality (16) can be refined to

$$E[|Y_i - v_q|^{p/q}] = EY_i^2 + v_q^2 - 2EY_i = EY_i^2 - v_q^2 = v_p - v_q^2 \leq (C_{p,q} - 1)v_q^{p/q}. \quad (34)$$

Hence, we can refine $C'$ from (13) to

$$C' = \sqrt{12(C_{p,q} - 1)}, \quad (35)$$

which can be much smaller than the original definition (13).

Let’s come back to the definition of $\chi_M$, which guarantees the three inequalities (26), (27) and (28) to hold. The refinement of $C'$ leads to a smaller right-hand side for all these inequalities, i.e., the constant $\chi_M$ will be larger and the regret upper bound will be smaller, which further implies that the policy may converge to the best arm faster.

Case 2. In addition to the assumption in Case 1, we add an assumption that the 4-th order moments of reward distributions exist as in Lattimore [21]. Thus we can choose $p = 4$ and $q = 2$. In this case, our UCB in Definition 2 can be rewritten as

$$\hat{\mu}((X_{i,j})_{j=1}^s, \delta) := \sup \left\{ \theta \in \mathbb{R} : \theta \leq \hat{\mu}((X_{i,j})_{j=1}^s), k + \sqrt{\frac{12\hat{\mu}((X_{i,j} - \theta)^2)_{j=1}^s, k}{\max \{0^+, 1 - C' \sqrt{\chi}\}}} \right\}. \quad 14$$
In this case $C_{p,q}$ degenerates to the upper bound on kurtosis. The function $B(x)$ defined in (23) becomes

$$ B(x) = \frac{24\tau(x)}{1 - 48x\tau(x)} \left(1 + C'\sqrt{x}\right) + \frac{576\sqrt{x}\tau(x)}{1 - 48x\tau(x)}, $$

and the three inequalities (26), (27) and (28) are refined to

1. $1 > C'\sqrt{x}$, \hspace{1cm} (36)
2. $1 > 48x\tau(x)$, \hspace{1cm} (37)
3. $\Delta_i \geq \sqrt{v_2}B(x) + 2\sqrt{3}$, \hspace{1cm} (38)

Under these refined estimations for $\chi_M$, the leading coefficient of $\log T$ in (33) will be significantly decreased. Now we compare the extended robust UCB and the scale free algorithm proposed in Lattimore [21], under the same assumptions specified at the beginning of Case 2.

We first address a flaw within Lemma 2 of Bubeck et al. [5]. This lemma, crucial in Lattimore’s analysis [21] for deriving the regret upper bound of the scale free algorithm, is examined more closely below. For clarity and coherence with the notations employed in this paper, we present the following adapted version.

A Flawed Statement [5]: Let $\delta \in (0, 1)$ and $p \in (1, 2]$. Let $\{X_{i,j}\}_{j=1}^s$ be i.i.d. random variables with mean $\mathbb{E}X_i = \theta_i$ and (centered) $p$-th moment $\mathbb{E}|X_i - \theta_i|^p = v_p$. Let $k' = \lfloor \min\{8\log(e^{1/8}\delta^{-1}), s/2\}\rfloor$. Then with probability at least $1 - \delta$,

$$ \hat{\mu}(\{X_{i,j}\}_{j=1}^s, k') \leq \theta_i + (12v_p)^{1/p} \left(\frac{16\log(e^{1/8}\delta^{-1})}{s}\right)^{(p-1)/p}. $$

The last inequality in the proof of this statement in Bubeck et al. [5] assumes that $\exp(-k'/8) \leq \delta$, and this is equivalent to $k' \geq 8\log \delta^{-1}$. However, the definition of $k'$ in this statement is

$$ k' = \lfloor \min\{8\log(e^{1/8}\delta^{-1}), s/2\}\rfloor. $$

To ensure that the inequality $\exp(-k'/8) \leq \delta$ in their proof to hold, it is required that

$$ \lfloor s/2 \rfloor \geq 8\log \delta^{-1}, $$

which is not assumed in the statement. As the statement is applied to the robust UCB, under the choice $\delta = t^{-2}$ and the definition $s := s_i(t - 1)$ as the number of times that arm $i$ was chosen before time $t$, Assumption [39] becomes

$$ \lfloor s_i(t - 1)/2 \rfloor \geq 16\log t. $$

Hence the regret upper bound in Theorem 3 of Bubeck et al. [5] is at least $32\Delta_i\log T$.

Below we give the corrected version of the above statement. Again, we will adopt notations introduced in this paper for consistency.

Lemma 4. Let $\delta \in (0, 1)$ and $p \in (1, 2]$. Let $\{X_{i,j}\}_{j=1}^s$ be i.i.d. random variables with mean $\mathbb{E}X_i = \theta_i$ and (centered) $p$-th moment $\mathbb{E}|X_i - \theta_i|^p = v_p$, where $s$ is chosen such
that \(|s/2| \geq 8 \log \delta^{-1}\). Let \(k' = \lfloor \min\{8 \log(e^{1/8} \delta^{-1}), s/2\}\rfloor\). Then with probability at least \(1 - \delta\),

\[
\hat{\mu} \left( \{X_{i,j}\}_{j=1}^s, k' \right) \leq \theta_i + (12v_p)^{1/p} \left( \frac{16 \log(e^{1/8} \delta^{-1})}{s} \right)^{(p-1)/p}.
\]

Now we look into the results in Lattimore [21], whose Lemma 4 is a special version of the flawed Lemma 2 in Bubeck et al. [5]. Lattimore’s Lemmas 5 and 6 in [21] both use the flawed version to estimate the regret upper bound as in the proof of their Theorem 2. We do not intend to give details about fixing all of them, but it is important to point out that, after applying the correct version (i.e., adding Assumption (39)) as in Lemma 4, the upper bound estimation of \(E[s_i(T)]\) in Lattimore [21] is at least (the constants introduced in their paper are explicitly calculated with notations translated here):

\[
3648 \max \left\{ \frac{(C')^2}{12} \frac{v_2}{\Delta_i^2} \right\} \log(e^{1/8} \delta^{-1}) + 16 \log \delta^{-1}.
\]

The next theorem shows that for properly chosen parameter \(\varepsilon > 0\), the upper bound given in (33) is also tighter than (40) over long-run if \(\Delta_i\) is small, i.e., arms are hard to be distinguished (the low-discrimination case).

**Theorem 2.** Choose any \(8 < \varepsilon < \frac{280 - 16\sqrt{2}}{2\sqrt{2} + 3} \approx 44.158\) and let \(\delta = t^{-2}\). The upper bound given in (40) of \(E[s_i(T)]\) is greater than that given in (33) for sufficiently large \(T\) and sufficiently small \(\Delta_i\). That is, by dividing both the upper bound estimations by \(\log \delta^{-1}\) and letting \(T \to +\infty\), we have

\[
(8 + \varepsilon) \left( \frac{8}{\varepsilon} + \max \left\{ \frac{8}{\varepsilon}, \frac{1}{\chi_M} \right\} \right) < 3648 \max \left\{ \frac{(C')^2}{12} \frac{v_2}{\Delta_i^2} \right\} + 16
\]

for sufficiently small \(\Delta_i\). The regret upper bound of the scale free algorithm proposed in Lattimore [21] is thus larger than that of the extended robust UCB policy for \(\varepsilon \in \left(8, \frac{280 - 16\sqrt{2}}{2\sqrt{2} + 3}\right)\).

**Proof.** First note that, since \(\Delta_i\) is sufficiently small and \(\varepsilon, C'\) are not related to \(\Delta_i\), we only need to show (noticing that \(\varepsilon > 8\) implies \((8 + \varepsilon)^{8/7} \leq 16)\)

\[
(8 + \varepsilon) \frac{1}{\chi_M} < 3648 \frac{v_2}{\Delta_i^2}.
\]

Equivalently, it is sufficient to show that

\[
x' := \frac{(8 + \varepsilon)\Delta_i^2}{3648v_2} < \chi_M
\]

for sufficiently small \(\Delta_i\). In order to prove the above, we need to show that (36), (37) and (38) are all true for \(x = x'\) based on the definition of \(\chi_M\). For (36), let \(x = x'\), we
have $1 > \frac{C\Delta_i}{\delta} \sqrt{\frac{\varepsilon + 8}{37\nu_2}}$, which is true for small $\Delta_i$. Furthermore, (37) becomes

$$1 > 6(8 + \varepsilon)\Delta_i^2 \frac{1}{456\nu_2 - \sqrt{57}C'\Delta_i\sqrt{(8 + \varepsilon)\nu_2}},$$

which also holds for sufficiently small $\Delta_i$. For (38), let $x = x'$ and the inequality becomes

$$(\varepsilon + 8) \left( \frac{B(x') + 24}{456\nu_2} \frac{\sqrt{\varepsilon + 8} + 456}{\sqrt{57}C'\Delta_i\sqrt{(8 + \varepsilon)\nu_2}} - 6\Delta_i^2(8 + \varepsilon)\right) < 912. \quad (44)$$

Taking the limit $\Delta_i \to 0$, (44) becomes $(6\sqrt{2} + 9)(\varepsilon + 8) < 912$, which holds for $\varepsilon < \frac{280 - 16\sqrt{2}}{2\sqrt{2} + 3}$. By continuity, we conclude that (43) holds for sufficiently small $\Delta_i$ if $\varepsilon < \frac{280 - 16\sqrt{2}}{2\sqrt{2} + 3}$.

Last, we show that the regret of the extended robust UCB deviates from the theoretical lower bound by only a constant factor and a constant term as in the scale free algorithm of Lattimore [21] (see (40)). First, we restate the lower bound derived in Lattimore [21] as follows.

**Theorem 3.** Let $H_{\kappa_0}$ be the set of distributions that has kurtosis less than $\kappa_0$. Assuming $\Delta > 0$ and $\kappa_0 \geq 7/2$. Suppose that $X \in H_{\kappa_0}$ has a mean of $\mu$, a positive variance of $\sigma^2$, and a kurtosis of $k$. Then

$$\inf\{\text{KL}(X, X') : X' \in H_{\kappa_0} \text{ and } E[X'] > \mu + \Delta\} \leq \begin{cases} \min\{-\log(1 - p), \frac{C_0\Delta^2}{\sigma^2}\}, & \text{if } C_0\sqrt{k}(k + 1)\frac{\Delta}{\sigma} < \kappa_0, \\ -\log(1 - p), & \text{otherwise} \end{cases} \quad (45)$$

where $C_0, C_1 > 0$ are universal constants and $p = \min\{\Delta/\sigma, 1/\kappa_0\}$.

Based on this, we can draw the following conclusion:

**Theorem 4.** Let $\delta = t^{-2}$. Under case 2, the regret of the extended robust UCB policy applied on $H_{\kappa_0}$ ($\kappa_0 \geq 7/2$) differs from the lower bound of the regret by a constant factor and a constant term.

**Proof.** If $\Delta_i$ of arm $i$ is large, that is, if (36) and (37) directly lead to (38), then $E[s_i(T)]/\log T$ is upper bounded by a constant not related to $\Delta_i$. We are interested in the case where $\Delta_i$ is small, i.e., when (38) becomes the main restraint of $\chi_M$:

$$\Delta_i = \sqrt{\nu_2\chi_M} \left( \sqrt{B(\chi_M)} + 2\sqrt{3} \right). \quad (46)$$

By (36), since $B(x)$ is an increasing function of $x$, $B(\chi_M)$ is bounded by a constant only related to $C'$. We have

$$\frac{1}{\chi_M} \leq C_B \frac{\nu_2}{\Delta_i^2}. \quad (47)$$
where \( C_B = B(1/C^2) \) is a constant. Note that

\[
\limsup \frac{\mathbb{E}[s_i(T)]}{\log T} \leq 2(8 + \varepsilon) \left( 1 + \max \left\{ \frac{8}{\varepsilon}, \frac{1}{\chi M} \right\} \right) \\
\leq 2(8 + \varepsilon) \left( 1 + \max \left\{ \frac{8}{\varepsilon}, C_B \frac{v_2}{\Delta_i} \right\} \right).
\]  

(48)

With the conclusion of (4) and Theorem 3, we finished the proof. \(\square\)

3. Light-Tailed Reward Distributions

In this section, we formally and briefly discuss the past work of the first author of this paper for light-tailed reward distributions [24] and its comparisons with some subsequent work following this line.

3.1. The UCB1-LT Policy

The empirical mean is adopted in estimating the upper confidence bound for the class of light-tailed reward distributions. As before, let \( \overline{X}_{i,s} \) denote the empirical mean \( \frac{1}{s} \sum_{j=1}^{s} X_{i,j} \) for arm \( i \), where \( \{X_{i,j}\}_{j=1}^{s} \) forms the i.i.d. random reward sequence drawn from an unknown distribution \( X_i \in \mathcal{F} \).

Lemma 5. (Bernstein-type bound) For i.i.d. random variables \( \{X_{i,j}\}_{j=1}^{s} \) drawn from a light-tailed distribution \( X_i \) (with mean \( \theta_i \)) with a finite moment-generating function \( M(u) \) over range \( u \in [-u_0, u_0] \) for some \( u_0 > 0 \). We have, \( \forall \varepsilon > 0 \),

\[
\mathbb{P}(\overline{X}_{i,s} - \theta_i \geq \varepsilon) \leq \begin{cases} 
\exp\left(-\frac{s \varepsilon^2}{2} \right), & \varepsilon < \zeta u_0 \\
\exp\left(-\frac{su_0^2}{2}\varepsilon \right), & \varepsilon \geq \zeta u_0 
\end{cases},
\]  

(49)

where \( \zeta > 0 \) satisfies \( \zeta \geq \sup|u| \leq u_0 M^{(2)}(u) \). A similar bound holds for \( \mathbb{P}(\overline{X}_{i,s} - \theta_i \leq -\varepsilon) \) by symmetry.

The proof of the above lemma follows a similar argument as in Vershynin [35]. Using the Bernstein-type bound by Lemma 5, we propose the UCB1-LT policy as follows.

The UCB1-LT policy considers two upper confidence bounds and alternatively uses one of them according to values of \( t \) and \( s_i(t-1) \). To minimize the theoretical regret upper bound, we can choose \( a_1 = 8\zeta \) and \( a_2 = \frac{8}{u_0} \). The following theorem shows that UCB1-LT achieves the logarithmic regret growth for light-tailed reward distributions.

Theorem 5. For light-tailed reward distributions, the regret of the UCB1-LT policy satisfies the following inequality:

\[
\mathcal{R}_\pi^F(T) \leq \sum_{i: \theta_i < \theta_1} \Delta_i \left( \max \left\{ \frac{4a_1}{\Delta_i^2}, \frac{2a_2}{\Delta_i} \right\} \log T + 1 + \frac{\pi^2}{3} \right),
\]  

(50)
Algorithm 2: The UCB1-LT Policy

1. **Input:** $a_1 \geq 8\zeta$, $a_2 \geq a_1/(\zeta u_0)$, $K$: the number of arms
2. Initialize: $s_i \leftarrow 0$ and $\bar{X}_i \leftarrow 0$ for each arm $i$, $1 \leq i \leq K$
3. for $t \leftarrow 1$ to $T$ do
4.   for $i \leftarrow 1$ to $K$ do
5.     if $s_i = 0$ then
6.       Assign $+\infty$ to the upper confidence bound
7.     end
8.     else
9.       if $\sqrt{\frac{a_1 \log t}{s_i}} < \zeta u_0$ then
10.      Compute the upper confidence bound as $\bar{X}_i + \sqrt{\frac{a_1 \log t}{s_i}}$
11.     end
12.    else
13.      Compute the upper confidence bound as $\bar{X}_i + \frac{a_2 \log t}{s_i}$
14.    end
15. end
16. end
17. Choose arm $j$ that maximizes the upper confidence bound and obtain
18. $\bar{X}_j \leftarrow \frac{x + s_j X_j}{s_j + 1}$ and $s_j \leftarrow s_j + 1$
19. end

**Proof.** Define

$$c(t, s) := \begin{cases} \sqrt{\frac{a_1 \log t}{s}}, & \sqrt{\frac{a_1 \log t}{s}} < \zeta u_0 \\ \sqrt{\frac{a_2 \log t}{s}}, & \sqrt{\frac{a_1 \log t}{s}} \geq \zeta u_0 \end{cases}$$

Similar to the procedure in Auer et al. [3], for any integer $L > 0$ and $n$ such that $\theta_i < \theta_1$, we have

$$\mathbb{E}[s_i(T)] \leq L + \sum_{t=1}^{T} \mathbb{P}\{X_i,t + c(t, s_i(t-1)) \geq \bar{X}_{1,t} + c(t, s_1(t-1)) \text{ and } s_i(t-1) > L\}$$

$$\leq L + \sum_{t=1}^{\infty} \sum_{s=1}^{t-1} \sum_{k=L}^{t-1} \mathbb{P}\left(\frac{1}{k} \sum_{j=1}^{k} X_{i,j} + c(t, k) \geq \frac{1}{s} \sum_{j=1}^{s} X_{1,j} + c(t, s)\right)$$

$$\leq L + \sum_{t=1}^{\infty} \sum_{s=1}^{t-1} \sum_{k=L}^{t-1} \mathbb{P}\left(\frac{1}{k} \sum_{j=1}^{k} X_{i,j} \geq \theta_i + c(t, k)\right)$$

$$+ \mathbb{P}\left(\frac{1}{s} \sum_{j=1}^{s} X_{1,j} \leq \theta_i - c(t, s)\right) + \mathbb{P}(\theta_i + 2c(t, k) > \theta_1).$$
Choose $L_0 = \max\{\frac{4a_1 \log T}{(\theta_1 - \theta_i)^2}, \frac{2a_2 \log T}{\theta_1 - \theta_i}\}$, $\forall k \geq L_0$, we have

$$c(t, k) \leq \max \left\{ \sqrt{\frac{a_1 \log t}{k}}, \frac{a_2 \log t}{k} \right\} \leq \max \left\{ \sqrt{\frac{a_1 \log t}{L_0}}, \frac{a_2 \log t}{L_0} \right\} \leq \max \left\{ \sqrt{a_1 \log t} (\theta_1 - \theta_i)^2, \frac{a_2 \log t}{\theta_1 - \theta_i} \right\} \leq \frac{\theta_1 - \theta_i}{2}.$$  

Then

$$\mathbb{E}[s_i(T)] \leq L_0 + \sum_{t=1}^{\infty} \sum_{s=1}^{t-1} \sum_{k=1}^{t-1} \mathbb{P} \left( \frac{1}{k} \sum_{j=1}^{k} X_{i,j} \geq \theta_i + c(t, k) \right) + \mathbb{P} \left( \frac{1}{s} \sum_{j=1}^{s} X_{1,j} \leq \theta_1 - c(t, s) \right).$$

Now we bound the probabilities by the Bernstein-type bound [49]. If

$$\sqrt{\frac{a_1 \log t}{k}} < \zeta u_0,$$

then

$$\mathbb{P} \left( \frac{1}{k} \sum_{j=1}^{k} X_{i,j} \geq \theta_i + c(t, k) \right) = \mathbb{P} \left( \frac{1}{k} \sum_{j=1}^{k} X_{i,j} \geq \theta_i + \sqrt{\frac{a_1 \log t}{k}} \right) \leq \exp \left( -\frac{k}{2\zeta} \left( \sqrt{\frac{a_1 \log t}{k}} \right)^2 \right) \leq t^{-4};$$

otherwise

$$\mathbb{P} \left( \frac{1}{k} \sum_{j=1}^{k} X_{i,j} \geq \theta_i + c(t, k) \right) = \mathbb{P} \left( \frac{1}{k} \sum_{j=1}^{k} X_{i,j} \geq \theta_i + \frac{a_2 \log t}{k} \right) \leq \exp \left( -\frac{ku_0 a_2 \log t}{2k} \right) \leq t^{-4}.$$

The same argument also applies to $\mathbb{P} \left( \frac{1}{s} \sum_{j=1}^{s} X_{1,j} \leq \theta_1 - c(t, s) \right)$. Together we have

$$\mathbb{E}[s_i(T)] \leq L_0 + 2 \sum_{t=1}^{\infty} \sum_{s=1}^{t-1} \sum_{k=1}^{t-L_0} t^{-4} \leq \max \left\{ \frac{4a_1 \log T}{(\theta_1 - \theta_i)^2}, \frac{2a_2 \log T}{\theta_1 - \theta_i} \right\} + 1 + \frac{\pi^2}{3},$$

A direct substitution of the above into [1] completes the proof. \qed
3.2. From UCB1-LT to \((\alpha, \psi)\)-UCB

After the establishment of UCB1-LT by Liu and Zhao [24], Bubeck and Cesa-Bianchi [4] subsequently proposed \((\alpha, \psi)\)-UCB under a more general assumption on the moment generating functions. Specifically, the \((\alpha, \psi)\)-UCB policy assumes that the distribution of reward \(X_i \in \mathcal{F}\) satisfies the following condition: there exists a convex function \(\psi\) defined on \(\mathbb{R}\) such that for all \(\lambda > 0\),

\[
\log \mathbb{E} \left[ e^{\lambda(X_i - \theta_i)} \right] \leq \psi(\lambda) \quad \text{and} \quad \log \mathbb{E} \left[ e^{\lambda(\theta_i - X_i)} \right] \leq \psi(\lambda).
\] (51)

For example, if \(X_i\) is bounded in \([0, 1]\) as assumed in UCB1, one can choose \(\psi(\lambda) = \frac{\lambda^2}{2}\) and (51) becomes the well-known Hoeffding’s lemma [4]. Since \(+\infty\) is allowed in the range of \(\psi\), this assumption is more general than that in UCB1-LT. By using the Legendre-Fenchel transform \(\psi^*(\varepsilon)\) of \(\psi\), defined as

\[
\psi^*(\varepsilon) := \sup_{\lambda \in \mathbb{R}} \lambda \varepsilon - \psi(\lambda).
\]

The upper confidence bound of \((\alpha, \psi)\)-UCB at time \(t\) is defined as

\[
I_i(t) := \mathbb{X}_{i,s_i(t-1)} + (\psi^*)^{-1}\left(\frac{\alpha \log t}{s_i(t-1)}\right).
\] (52)

**Theorem 6** ([4]). Assume that the reward distributions satisfy (51). Then \((\alpha, \psi)\)-UCB with \(\alpha > 2\) achieves

\[
\mathcal{R}^F_{\pi}(T) \leq \sum_{\theta_i < \theta_1} \left(\frac{\alpha \Delta_i}{\psi^*(\Delta_i/2)} \log T + \frac{\alpha}{\alpha - 2}\right). \] (53)

For the UCB1-LT policy, we can choose

\[
\psi(\lambda) = \begin{cases} 
\frac{\lambda^2}{2}, & \lambda \leq u_0 \\
+\infty, & \lambda > u_0
\end{cases}.
\] (54)

Then (51) becomes the same assumption as in UCB1-LT (see (6)). In this case, we have

\[
\psi^*(\varepsilon) = \begin{cases} 
\frac{\varepsilon^2}{2}, & |\varepsilon| < \zeta u_0 \\
u_0(|\varepsilon| - \frac{\zeta u_0}{2}), & |\varepsilon| \geq \zeta u_0
\end{cases}.
\]

Thus the inverse \((\psi^*)^{-1}(x)\) of \(\psi^*\) for \(x \geq 0\) will be

\[
(\psi^*)^{-1}(x) = \begin{cases} 
\sqrt{2x}, & 0 \leq x < \frac{\zeta u_0^2}{2} \\
x + \frac{\zeta u_0}{2}, & x \geq \frac{\zeta u_0^2}{2}
\end{cases}.
\]

Furthermore, if we choose \(\alpha = 4\), then the upper confidence bound defined in (52)
becomes
\[ I_i(t) = \tilde{X}_{i,s_i(t-1)} + \begin{cases} \sqrt{8\zeta \log t / s_i(t-1)}, & \log t / s_i(t-1) < \frac{\zeta u_0^2}{8} \\ \frac{\log t}{u_0 s_i(t-1)}, & \frac{\log t}{s_i(t-1)} \geq \frac{\zeta u_0^2}{8} \end{cases}, \]

(55)

Notice that, if we choose \(a_1 = 8\zeta\) and \(a_2 = \frac{8}{u_0}\) in UCB1-LT, we get the same index function under the condition \(\log t / s_i(t-1) < \frac{\zeta u_0^2}{8}\). For the case \(\log t / s_i(t-1) \geq \frac{\zeta u_0^2}{8}\), the upper confidence bound of \((\alpha, \psi)-UCB\) is not larger than that of UCB1-LT and thus may achieve a smaller theoretical upper bound of regret as follows:

\[ R^\pi_f(T) \leq \sum_{\theta_i < \theta_1} (r(\Delta_i) \log T + 2), \]

(56)

where

\[ r(\Delta_i) = \begin{cases} \frac{32\zeta}{\Delta_i}, & \Delta_i < 2\zeta u_0 \\ \frac{8\zeta}{\Delta_i}, & \Delta_i \geq 2\zeta u_0 \end{cases}. \]

Note that this regret bound is better than (50) in Theorem 5 for UCB1-LT as \(T\) becomes sufficiently large, as shown in the following lemma.

**Lemma 6.** Choose the parameter \(\alpha\) in \((\alpha, \psi)-UCB\) such that \(\alpha \leq 4\), and assume that Assumption (6) is satisfied and thus \(\psi(\lambda)\) is chosen as in (54). Then the logarithmic regret upper bound in Theorem 6 has a leading constant no larger than that in Theorem 5, i.e.,

\[ \sum_{\theta_i < \theta_1} \frac{\alpha \Delta_i}{\psi^*(\Delta_i/2)} \leq \sum_{\theta_i < \theta_1} \Delta_i \max \left\{ \frac{4a_1}{\Delta_i}, \frac{2a_2}{\Delta_i} \right\}. \]

**Proof.** We only need to show that

\[ \frac{\alpha x}{\psi^*(x/2)} \leq x \max \left\{ \frac{4a_1}{x^2}, \frac{2a_2}{x} \right\} \]

for any \(x > 0\). From (54), we have

\[ \frac{\alpha x}{\psi^*(x/2)} = \begin{cases} \frac{8a\zeta}{x}, & 0 < x < 2\zeta u_0 \\ \frac{2ax}{u_0(x-\zeta u_0)}, & x \geq 2\zeta u_0 \end{cases}. \]

From UCB1-LT, we have \(a_1 \geq 8\zeta\) and \(a_2 \geq a_1 / (\zeta u_0) \geq 8 / u_0\), then

\[ x \max \left\{ \frac{4a_1}{x^2}, \frac{2a_2}{x} \right\} \geq \max \left\{ \frac{32\zeta}{x}, \frac{16}{u_0} \right\} \]

\[ = \begin{cases} \frac{32\zeta}{x}, & 0 < x < 2\zeta u_0 \\ \frac{16}{u_0}, & x \geq 2\zeta u_0 \end{cases}. \]

Since \(\frac{8a\zeta}{x} \leq \frac{32\zeta}{x}\) as \(\alpha \leq 4\) and \(\frac{2ax}{u_0(x-\zeta u_0)} \leq \frac{16}{u_0}\) as \(x \geq 2\zeta u_0\), (57) is proved. \(\square\)
Now we compare the actual performance between \((\alpha, \psi)\)-UCB and UCB1-LT through numerical experiments. Assume that the arms are offering rewards according to 19 normal distributions. The means of reward distributions are 0.5, 1.0, 1.5, \ldots, 9.5 with standard deviations all set to 60. A Monte Carlo (MC) simulation with 100 runs and time period \(T = 50000\) is shown below. The horizontal axis in both figures is the time step \(t\). The first figure only shows \(T\) from 10000 to 50000 for better display, while the second one ranges from 1 up to 50000. The vertical axis of the left figure is the logarithmic regret averaged from the 100 MC runs (to approximate the expectation), while of the right one is the approximately expected time-average reward. The parameters for UCB1-LT and \((\alpha, \psi)\)-UCB are respectively \(u_0 = 1\) and \(\alpha = 2.5\) or 4 with \(\zeta = 3600\) as their common parameter.

These results reveal an interesting phenomenon: the actual performance of UCB-LT is similar to that of \((\alpha, \psi)\)-UCB which has a better theoretical regret bound over long-run for \(\alpha = 4\). However, a smaller choice of \(\alpha\) slightly improves the performance of \((\alpha, \psi)\)-UCB over long-run as consistent with the theoretical bound in \cite{53} whose leading constant of the logarithmic order decreases as \(\alpha\) decreases.

![The comparison of regrets](image1.png)

![The comparison of time-average rewards](image2.png)

Figure 2.: Performance comparison between UCB1-LT and \((\alpha, \psi)\)-UCB

4. Conclusions

In this paper, we have proposed two UCB policies, namely the extended robust UCB and UCB1-LT, dealing with the heavy-tailed and light-tailed reward distributions in the frequentist multi-armed bandit problems, respectively.

The extended robust UCB policy aims at the class of heavy-tailed arm distributions and performs well in achieving the optimal logarithmic regret growth order. The policy takes advantage of a controlled relationship between different moments and discards the prerequisite knowledge of an upper bound of the moments. It is a generalization of the scale free algorithm proposed by Lattimore \cite{21} with the assumption that \(p = 4\). In scenarios characterized by low discrimination and when \(p = 4\), our extended robust UCB policy exhibits a tighter regret upper bound in the long run. The achieved regret is merely a constant factor plus a constant term away from the established lower bound.

The UCB1-LT is efficient for the class light-tailed reward distributions where the optimal logarithmic regret growth order is attained. Through numerical evaluations, we observe a comparable actual performance to its extension, \((\alpha, \psi)\)-UCB. Since the upper confidence bound in UCB1-LT is easy to compute (with \(O(1)\) time complexity),
it provides a practical archetype of UCB policies for light-tailed reward distributions.

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