EQUATION PROBLEM OVER CENTRAL EXTENSIONS
OF HYPERBOLIC GROUPS

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Abstract. The Equation Problem in finitely presented groups asks if there exists an algorithm which determines in finite amount of time whether any given equation system has a solution or not. We show that the Equation Problem in central extensions of hyperbolic groups is solvable.

1. Introduction

Let $G$ be a finitely presented group and let $C$ be a generating set for $G$. Denote a set of variables by $U$. An equation system is a finite collection of equations $w_i = 1$ where $w_i \in (U \cup C)^*$. Let $\mathcal{E} = \{w_i = 1 \mid i = 1 \cdots n\}$ be an equation system in $G$. A solution of $\mathcal{E}$ in $G$ is a map $f : U \to G$ such that the induced monoid homomorphism (sending each $c \in C$ to itself) $\bar{f} : (U \cup C)^* \to G$ maps $w_i$ to 1 for $1 \leq i \leq n$.

Let $\mathcal{C}$ be a class of finitely presented groups. The Equation Problem in $\mathcal{C}$ asks the following question: Is there an algorithm which takes as input a presentation of a group $G \in \mathcal{C}$ and a finite system of equations with constants in $G$ and which decides whether there exists a solution or not? When the answer is positive, we say that the Equation Problem in $\mathcal{C}$ is solvable.

The Equation problem is a vast generalization of the word problem, the conjugacy problem and the simultaneous conjugacy problem. A direct consequence is the existence of groups with unsolvable equation problem. In fact there exists a free 3-step nilpotent group of rank 2 with unsolvable equations problem (15, see also 13). Therefore the Equation Problem is strictly harder than the word problem and the conjugacy problem since these two problems are solvable in finitely generated nilpotent groups.

The most famous breakthrough in solving equations over groups is the solution of the Equation Problem in free groups by Makanin [8]. In [3] Dahmani and Guirardel and independently in [7] Lohrey and Senizergues give an algorithm for solving equations and inequations with rational constraints in virtually free groups. One of the most
successful methods of solving the Equation Problem for groups is to reduce it to the Equation Problem over a (virtually) free group. Rips and Sela [12] reduce the Equation Problem in torsion free hyperbolic groups to the Equation Problem in free groups. Dahmani and Guirardel [3] reduce the problem in hyperbolic groups (possibly with torsion) to solving equations with rational constraints (See Def 2.5) over virtually free groups. Diekert and Muscholl [4] reduce the Equation Problem in right-angled Artin groups to free groups.

Besides being a natural algorithmic problem in groups, the equation problem has the following interpretation in terms of homomorphisms between groups: Let $K, G$ be finitely presented groups and $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_i \rangle$ be a presentation of $K$. Suppose one’s favorite elements in $K$ are $w_1, \ldots, w_l$. Here $w_i$ are words in $x_i$. Let $c_1, \ldots, c_l$ be elements of $G$. Here is a natural question one might want to ask: Is there a homomorphism from $K$ to $G$ sending $w_i$ to $c_i$? The question is equivalent to asking whether the equation system $E = \{ r_i = 1, w_j = c_j \mid 1 \leq i \leq m; 1 \leq j \leq l \}$ has a solution in $G$. Under this interpretation $x_i$ become variables and note that $r_i$ are words in $x_i$. Hence a positive solution to the Equation Problem for $G$ gives a way to determine algorithmically whether particular kind of homomorphisms to $G$ exist.

In this paper, we consider the Equation Problem in central extensions of hyperbolic groups. We prove the following theorem.

**Theorem 1.1.** The Equation Problem is solvable in central extensions of hyperbolic groups.

A source of interesting examples of central extensions of hyperbolic groups is 3-manifold theory. The fundamental groups of a large and interesting class of Seifert fibered spaces are central extensions of hyperbolic groups. All these Seifert fibered spaces are aspherical and the fundamental group is a complete topological invariant. Hence information about maps between fundamental groups of these spaces gives essentially all information up to homotopy about maps between these spaces. As pointed out above the Equation Problem is an important tool for studying maps between groups. We believe there will be applications of Theorem 1.1 that lead to new understanding of maps between Seifert fibered spaces.

The organization of the paper is as follow. In Section 2 we recall important definitions and key results from [3] and [10], which are the main tools we use in this paper, and describe the idea of the proof of Theorem 1.1. In Section 3 we construct the finite state automata that
we use in the proof of Theorem 1.1. In Section 4 we prove Theorem 1.1.

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2. Preliminaries

In the section we recall important definitions and key results from [3] and [10].

2.1. Equation Problem in hyperbolic groups. Let $\delta > 0$. A geodesic triangle in a metric space is said to be $\delta$-slim if each of its sides is contained in the $\delta$-neighbourhood of the union of the other two sides. A geodesic space $X$ is said to be hyperbolic there exists some $\delta > 0$ if every triangle in $X$ is $\delta$-slim. A finitely generated group is hyperbolic if its Cayley graph is hyperbolic.

Let $\Gamma$ be a hyperbolic group, $X$ be a finite symmetric generating set of $\Gamma$, $K_{\Gamma}$ be the Cayley graph of $\Gamma$ with respect to $X$ and $\delta$ be a hyperbolic constant for $K_{\Gamma}$. Consider the Rips complex of $K_{\Gamma}$, denoted by $R_{50\delta}(K_{\Gamma})$, whose set of vertices is $\Gamma$ and whose simplices are subsets of $\Gamma$ of diameter at most $50\delta$ in $K_{\Gamma}$. Denote the 1-skeleton of the barycentric subdivision of $R_{50\delta}(K_{\Gamma})$ by $K$.

The action of $\Gamma$ on $K_{\Gamma}$ extends to an action on $K$. The quotient $K/\Gamma$ is a finite graph and can be given the structure of a finite graph of finite groups (vertices and edges are decorated by stabilizers of their preimages in $K$). Hence $\pi_1(K/\Gamma)$ is virtually free. A finite presentation of $\pi_1(K/\Gamma)$ can be computed.

As in [3], let $V$ be the set of paths $v$ in $K$ which start at the identity and end at a vertex of $K_{\Gamma}$ up to homotopy relative to endpoints. Let $\pi$ denote the natural homomorphism from $V$ to $\Gamma$ sending each path to its endpoint. We give $V$ a group structure by defining $vv'$ to be the homotopy class of the concatenation $v \cdot (\pi(v)v')$ (where $\pi(v)v'$ is the translate of $v'$ by $\pi(v) \in \Gamma$).

Dahmani and Guirardel prove the following [3, Lemma 9.9]

**Lemma 2.1.** The group $V$ is virtually free. More precisely, it is isomorphic to the fundamental group of the finite graph of finite groups $K/\Gamma$. In particular, a presentation of $V$ is computable from a presentation of $\Gamma$.

We need a slightly stronger statement about $V$. 
Lemma 2.2. A finite presentation \( \langle Y | R \rangle \) of \( V \) is computable. Moreover one can compute each \( y \in Y \) as an explicit path in \( K \).

Note that a finite presentation of \( V \) is computable by Lemma 2.1. The point of Lemma 2.2 is that one can compute each \( y \in Y \) as an explicit path in \( K \).

Proof. First we note that any finite neighborhood of \( K \) can be constructed since the word problem of \( \Gamma \) is solvable. Let \( \theta : T \to K \) be the universal cover of \( K \). By [3, proof of Lemma 9.9] we have:

1. \( V \) acts on \( T \) by isometries;
2. \( \theta \) is \( \pi \)-equivariant;
3. \( T/V \) (given a graph of group structure as in [14, page 54]) is isomorphic to \( K/\Gamma \) as graph of groups. In particular they are isomorphic as graphs.

Starting with a vertex \( t_0 \in T \) which is mapped to \( 1_\Gamma \in K \) by \( \theta \), one can construct any finite neighborhood of \( t_0 \) in \( T \) and compute the map \( u \) in that neighborhood. (This is because one can construct any finite neighborhood of \( 1_\Gamma \in K \) and one can construct any finite ball of the universal cover of a locally finite graph.)

Since the quotient map from \( K \) to \( K/\Gamma \) can be computed in any finite neighborhood of \( 1_\Gamma \), one can compute the quotient map from \( T \) to \( T/V \) explicitly in any finite neighborhood of \( t_0 \).

Given any \( v \in V \) explicitly as a path in \( K \) and any point \( t \) in \( T \) in an explicitly constructed finite neighborhood of \( t_0 \), one can compute \( v \cdot t \) explicitly. (Here we assume that the explicitly constructed finite neighborhood of \( t_0 \) also contains \( v \cdot t \).)

One can compute elements of the vertex groups explicitly as paths in \( K \): vertex groups are stabilizers of vertices of \( T \). One can find those vertices explicitly. For any given element of \( V \), one can check whether it fixes any given vertex. Hence one can compute all elements in the stabilizer of a given vertex by checking each element of \( V \) according to their lengths (in \( K \)). One knows when to stop checking because one knows the size of each stabilizer.

One can also compute all the \( \gamma_y \) (as in [14]) explicitly because given any two vertices of \( T \) in the same \( V \)-orbit, one can find an explicit element of \( V \) that takes one to the other.

Since the presentation of \( V \) given by \( \pi_1(T/V) \) has elements of the vertex groups of \( (G, T/V) \) and \( \gamma_y \)'s as generators, the proof is complete. \( \square \)

We now recall how Dahmani and Guirardel reduce an equation system in \( \Gamma \) to finitely many equation systems in \( V \).
By introducing new variables, one can find an equivalent triangular equation system for any equation of length greater than three. (The picture behind this is that one can cut a polygon into triangles.) For equation of length two, say \( x^2 = 1 \), one can replace it by the triangular equation with \( x^2 1 = 1 \). Therefore it suffice to consider only triangular equation systems.

Denote a set of variables (unknowns) by \( U \) and a finite set of words in \( X \) by \( C \). Let \( \mathcal{F} = \{ \gamma_{i,1} \gamma_{i,2} \gamma_{i,3} = 1, i = 1, \ldots, n \} \) be a triangular equation system over \( \Gamma \), where \( \gamma_{i,j} \in U \cup C \).

Let \( \mu_0 = 8 \) and \( \lambda_0 = 400\delta m_0 \), where \( m_0 \) is a bound on the cardinality of balls of radius \( 50\delta \) in \( K_\Gamma \). Consider \( (\lambda_1, \mu_1) = (\lambda_0, \mu_0 + 2 + 2/\lambda_0) \), so that any concatenation of a \( (\lambda_0, \mu_0) \)-quasi-geodesic with a path of length 1 at each extremity is a \( (\lambda_1, \mu_1) \)-quasi-geodesic.

**Definition 2.3.** \( QG(V) \subset V \) is the set of elements such that the corresponding reduced path in \( K \) is \( (\lambda_1, \mu_1) \)-quasi-geodesic.

Denote by \( V_{\leq l} \) the set of elements of \( V \) whose corresponding reduced path in \( K \) has length at most \( l \).

Dahmani and Guirardel use Rips and Sela’s canonical representatives [12] to prove the following key proposition [3, Proposition 9.10], which allows them to reduce any equation system in \( \Gamma \) to finitely many equation systems in \( V \).

**Proposition 2.4** (Dahmani-Guirardel). Then there exists a computable constant \( \kappa_1 \) (depending on \( \Gamma \) and \( \mathcal{F} \)) such that for any solution \( (g_u) \in \Gamma^U \) we have the following:

For each \( u \in U \cup \tilde{U} \), there exists \( \tilde{g}_u \in QG(V) \) with \( \tilde{g}_u = (g_u)^{-1} \) and \( \pi(\tilde{g}_u) = g_u \). For each \( \gamma_{i,1} \gamma_{i,2} \gamma_{i,3} = 1 \) in \( \mathcal{F} \) and for each \( j \in \{1, 2, 3 \mod 3\} \), there exists \( l_i,j \in QG(V) \) and \( c_i,j \in V_{\leq \kappa_1} \) such that:

1. \( \tilde{g}_{\gamma_{i,j}} = l_i,j c_{i,j} (l_{i,j+1})^{-1} \) in \( V \);
2. \( \pi(c_{i,1} c_{i,2} c_{i,3}) = 1 \) in \( \Gamma \).

Conversely, given any family of elements of \( V : (\tilde{g}_u)_{u \in U \cup \tilde{U}} ; (l_{i,j})_{1 \leq i \leq n, 1 \leq j \leq 3} \) and \( (c_{i,j})_{1 \leq i \leq n, 1 \leq j \leq 3} \) satisfying \( \tilde{g}_u = (\tilde{g}_u)^{-1} \), (1) and (2), respectively, the family \( g_u = \pi(\tilde{g}_u) \) is a solution of \( \mathcal{F} \).

The \( \tilde{g}_u \)'s are called Canonical Representatives.

For each tuple \( \tilde{c} \) of \( c_{i,j} \in V_{\leq \kappa_1} \) satisfying (2), define an equation system \( \mathcal{F}(\tilde{c}) \) in \( V \) as follows: Let \( \{v_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq 3\} \) be a set of variables such that \( v_{i,j} \) and \( v_{i',j'} \) are the same variable if and only if \( \gamma_{i,j} \) and \( \gamma_{i',j'} \) are the same. Let \( \{p_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq 3\} \) be a set of
of distinct variables. We define
\[ F(\bar{c}) = \begin{cases} 
  p_{i,1}c_{i,1}(p_{i,2})^{-1} = v_{i,1} \\
  p_{i,2}c_{i,2}(p_{i,3})^{-1} = v_{i,2} \\
  p_{i,3}c_{i,3}(p_{i,1})^{-1} = v_{i,3} 
\end{cases} \quad 1 \leq i \leq n; 1 \leq j \leq 3; \]

We call \( F(\bar{c}) \) tripod equation system associated with \( \bar{c} \).

Suppose that \( \{ \bar{v}_{i,j}, \bar{p}_{i,j} \} \) is a solution of \( F(\bar{c}) \) in \( V \). Let \( \bar{v}_{i,j} = \pi(\bar{v}_{i,j}) \).
Then since \( c_{i,j} \) satisfy Condition (2) in Proposition 2.4 we have that
\[ \bar{v}_{i,1}\bar{v}_{i,2}\bar{v}_{i,3} = 1, \quad i = 1, \ldots, n. \]

With these equations we are almost ready to say that \( \{ \bar{v}_{i,j} \} \) is a solution of \( F \). The one extra thing we need is that \( \bar{v}_{i,j} \) should equal \( \gamma_{i,j} \) whenever \( \gamma_{i,j} \) is a constant in \( \Gamma \). This is ensured by using Equation system with rational constraints, which is defined below.

**Definition 2.5.** A rational subset of a finitely presented group \( G \) is the image of a regular language of some generating set under the canonical projection.

Let \( C \) be a generating set for \( G \). Denote a set of variables by \( U \). An equation system with rational constraints is an a finite collection of equations \( w_i = 1 \) where \( w_i \in (U \cup C)^* \) with finite set of pairs \( (u, R_u) \), where \( u \in U \) and \( R_u \) is a rational subset of \( G \).

Let \( \mathcal{E} = \{ w_i = 1 \mid i = 1 \cdots n \} \cup \{ (u, R_u) \mid u \in U \} \) be an equation system with rational constraints in \( G \). A solution of \( \mathcal{E} \) in \( G \) is a map \( f : U \rightarrow G \) such that \( f(u) \in R_u \) and the induced monoid homomorphism (sending each \( c \in C \) to itself) \( \bar{f} : (U \cup C)^* \rightarrow G \) maps \( w_i \) to 1 for \( 1 \leq i \leq n \).

**Convention 2.6.** We will use the notation \( u \in R_u \) instead of \( (u, R_u) \) when we write rational constraints in equation systems.

We add rational constraints to the tripod equations system \( F(\bar{c}) \) as follow: If \( \gamma_{i,j} \) is a constant in \( \Gamma \), let \( L_{\gamma_{i,j}} \) be the rational subset (of \( V \)) \( \{ \bar{g} \in QG(V) \mid \pi(\bar{g}) = \gamma_{i,j} \} \). Otherwise let \( L_{\gamma_{i,j}} = V \). We define:
\[ F_R(\bar{c}) = \begin{cases} 
  p_{i,1}c_{i,1}(p_{i,2})^{-1} = v_{i,1} \\
  p_{i,2}c_{i,2}(p_{i,3})^{-1} = v_{i,2} \\
  p_{i,3}c_{i,3}(p_{i,1})^{-1} = v_{i,3} 
\end{cases} \quad 1 \leq i \leq n; 1 \leq j \leq 3; \]

\( v_{i,j} \in L_{\gamma_{i,j}} \)

Suppose \( \{ \bar{v}_{i,j}, \bar{p}_{i,j} \} \) to denote a solution of \( F_R(\bar{c}) \) in \( V \). When \( \gamma_{i,j} \) is a constant, the rational constraint \( v_{i,j} \in L_{\gamma_{i,j}} \) implies that \( \bar{v}_{i,j} = \pi(\bar{v}_{i,j}) = \gamma \). Therefore \( \{ \bar{v}_{i,j} \} \) is a solution of \( F \).

On the other hand, by Proposition 2.4 if \( F \) has a solution in \( \Gamma \) then \( F_R(\bar{c}) \) has a solution in \( V \) for some tuple \( \bar{c} \) of \( c_{i,j} \in V_{\leq \kappa_1} \) satisfying
(2) in Proposition 2.4. All such tuples \(\tilde{c}\) can be computed since each \(c_{i,j}\) has bounded length. Note that the length bound \(\kappa_1\) on \(c_{i,j}\) given by Proposition 2.4 is determined by \(\Gamma\) and \(\mathcal{F}\). Hence the set of tuples \(\tilde{c}\) of \(c_{i,j} \in V_{\leq \kappa_1}\) satisfying (2) in Proposition 2.4 is determined by \(\Gamma\) and \(\mathcal{F}\). Since there are finitely many \(\tilde{c}\), any equation system in any hyperbolic group can be reduced to finitely many equation systems \(\mathcal{F}_R(\tilde{c})\) in some virtually free group, which are solvable by the next theorem [3, Theorem 3]:

**Theorem 2.7** (Dahmani-Guirardel). There exists an algorithm which takes as input a presentation of a virtually free group \(G\), and a system of equations with constants in \(G\), together with a set of rational constraints, and which decides if there exists a solution or not.

Rational constraints play a very important role in our proof of Theorem 1.1 as we explain in the next subsection.

2.2. Idea of the proof of Theorem 1.1. Let \(E\) be a central extension of a hyperbolic group by a finitely generated abelian group. Given a finite presentation of \(E\), by [2, Proposition 1.1] one can compute the finite presentations of all terms in a short exact sequence

\[1 \rightarrow A \rightarrow E \rightarrow \Gamma \rightarrow 1\]

where \(\Gamma\) is a hyperbolic group and \(A\) is a finitely generated abelian group. Denote the inclusion from \(A\) to \(E\) by \(i\) and the projection from \(E\) to \(\Gamma\) by \(p\).

Let \(\mathcal{E} = \{e_{i,1}e_{i,2}e_{i,3} = 1, i = 1, \cdots, n\}\) be a triangular equation system in \(E\). Denote by \(\mathcal{F}\) the equation system \(p(\mathcal{E}) = \{p(e_{i,1})p(e_{i,2})p(e_{i,3}) = 1, i = 1, \cdots, n\}\) in \(\Gamma\), where \(p(e_{i,j}) = e_{i,j}\) if \(e_{i,j}\) is a variable. Here is an attempt to check whether \(\mathcal{E}\) has a solution in \(E\):

Use the algorithm in Theorem 2.7 to solve all the tripod equation systems with rational constraints \(\mathcal{F}_R(\tilde{c})\) associated to \(\mathcal{F}\) and \(\Gamma\).

If none of the \(\mathcal{F}_R(\tilde{c})\) has a solution, then \(\mathcal{F}\) has no solution in \(\Gamma\) and hence \(\mathcal{E}\) has no solution in \(E\).

Suppose some \(\mathcal{F}_R(\tilde{c})\) has a solution. Let \(\{\tilde{v}_{i,j}, \tilde{p}_{i,j}\}\) be a solution. Then \(\{\pi(\tilde{v}_{i,j})\}\) is a solution of \(\mathcal{F}\) in \(\Gamma\). Let \(s : \Gamma \rightarrow E\) be a section. In general, \(\{s(\pi(\tilde{v}_{i,j}))\}\) is not a solution of \(\mathcal{E}\). But we know that \(s(\pi(\tilde{v}_{i,1}))s(\pi(\tilde{v}_{i,2}))s(\pi(\tilde{v}_{i,3})) \in A\) for all \(1 \leq i \leq n\). We set up the following equation system in \(A\):

\(\mathcal{W}(\tilde{c}, \tilde{v}_{i,j}, \tilde{p}_{i,j}) = \{w_{i,1}w_{i,2}w_{i,3} = -s(\pi(\tilde{v}_{i,1}))s(\pi(\tilde{v}_{i,2}))s(\pi(\tilde{v}_{i,3})) | 1 \leq i \leq n\} \)

where \(w_{i,j}\) and \(w_{i,j'}\) represent the same variable if \(e_{i,j}\) and \(e_{i,j'}\) are the same variable in \(\mathcal{E}\) and \(w_{i,j} = e_{i,j} - s(p(\tilde{v}_{i,j}))\) is a constant in \(A\) when \(e_{i,j}\) is a constant in \(E\). Suppose \(\{\tilde{w}_{i,j}\}\) is a solution of \(\mathcal{W}(\tilde{c}, \tilde{v}_{i,j}, \tilde{p}_{i,j})\).
It is easy to check that \( \{ \bar{e}_{i,j} = s(\pi(\bar{v}_{i,j}))\bar{w}_{i,j} \} \) is a solution of \( \mathcal{E} \). Linear algebra can be used to solve \( \mathcal{W}(\bar{e}, \bar{v}_{i,j}, \bar{p}_{i,j}) \) since \( A \) is finitely generated and abelian. We check all \( \mathcal{W}(\bar{e}, \bar{v}_{i,j}, \bar{p}_{i,j}) \) to see if they have a solution in \( A \). If at least one does, then \( \mathcal{E} \) has a solution.

We point out that if \( \mathcal{E} \) has a solution, then the process above will detect it. (To see this, suppose \( \mathcal{E} \) has a solution \( \{ \bar{e}_{i,j} \} \), then \( \{ p(\bar{e}_{i,j}) \} \) is a solution of \( \mathcal{F} \). Hence there is some \( \bar{c} \) such that \( \mathcal{F}(\bar{c}) \) has a solution \( \{ \bar{v}_{i,j}, \bar{p}_{i,j} \} \) with \( \pi(\bar{v}_{i,j}) = p(\bar{e}_{i,j}) \). It is easy to check that \( \{ \bar{w}_{i,j} = \bar{e}_{i,j} - s(p(\bar{v}_{i,j})) \} \) is a solution of \( \mathcal{W}(\bar{c}, \bar{v}_{i,j}, \bar{p}_{i,j}) \). Hence if the above process terminates before any solution is found, then \( \mathcal{E} \) has no solution in \( E \).

There is a obvious problem about solving \( \mathcal{E} \) this way: \( \mathcal{F} = p(\mathcal{E}) \) can have infinitely many solutions in \( \Gamma \) even if \( \mathcal{E} \) has no solution. In this case at least of the tripod equation system with rational constraints \( \mathcal{F}_R(\bar{c}) \) has infinitely many solutions. So there are infinitely many \( \mathcal{W}(\bar{c}, \bar{v}_{i,j}, \bar{p}_{i,j}) \) to check. But none of them has a solution and so the process of checking will never terminate. Here is an explicit example of this phenomenon:

**Example 2.8.** Let \( S \) be the genus two surface and \( T^1 S \) be the unit tangent bundle of \( S \). The following short exact sequence defines \( \pi_1(T^1 S) \) as a central extension of \( \pi_1(S) \), which is hyperbolic.

\[
1 \to \mathbb{Z} \to \langle a, b, c, d, z \mid [a, b][c, d] = z^{-2}, z \text{ central} \rangle \to \langle a, b, c, d \mid [a, b][c, d] = 1 \rangle \to 1
\]

Let \( x \) be a variable. The equation \([a, b][x, d] = 1\) has no solution in \( \pi_1(T^1 S) \) but its projection in \( \pi_1(S) \) has infinitely many solutions: \( x = cd^n \) for all \( n \in \mathbb{Z} \). To see this, first solve \([a, b][x, d] = 1\) in \( \pi_1(S) \), where we have \([a, b][x, d] = [a, b][c, d]\). By simply algebraic manipulation, we got \( x^{-1}cd = dx^{-1}c \). But in \( \pi_1(S) \), the only elements that commute with \( d \) are of the form \( d^n \) for \( n \in \mathbb{Z} \). Hence we have that \( x = cd^n \) for all \( n \in \mathbb{Z} \) are the only solutions. Now any solution of \([a, b][x, d] = 1\) in \( \pi_1(T^1 S) \) must project down to a solution of its projection in \( \pi_1(S) \). Hence if there is a solution, then it has the form \( x = cd^n z^m \). Plug it into \( x \), we have \([a, b][c, d] = 1\), which does not hold in \( \pi_1(T^1 S) \). Therefore \([a, b][x, d] = 1\) has no solution in \( \pi_1(T^1 S) \).

To deal with the above problem, we use rational constraints.

For each \( \mathcal{F}_R(\bar{c}) \) we will have finitely many ways to add more rational constraints to it. We denote the resulting equation systems with rational constraints by \( \mathcal{F}_R^k(\bar{c}) \). We will prove the following:

1. If \( \{ \bar{v}_{i,j}, \bar{p}_{i,j} \} \) is solution of \( \mathcal{F}_R(\bar{c}) \) then it is a solution of \( \mathcal{F}_R^k(\bar{c}) \) for some \( k \).
2. If \( \{ \bar{v}_{i,j}, \bar{p}_{i,j} \} \) and \( \{ \bar{v}'_{i,j}, \bar{p}'_{i,j} \} \) are solutions of \( \mathcal{F}_R^k(\bar{c}) \), then we have
   \[
   s(\pi(\bar{v}_{i,1}))s(\pi(\bar{v}_{i,2}))s(\pi(\bar{v}_{i,3})) = s(\pi(\bar{v}'_{i,1}))s(\pi(\bar{v}'_{i,2}))s(\pi(\bar{v}'_{i,3}))
   \]
and one can compute this value from the rational constraints in \( F_R^k(\bar{c}) \). We denote the above value by \( s(\bar{c}, k) \).

Now instead of solving possibly infinitely many equation systems \( W(\bar{c}, \bar{v}_{i,j}, \bar{p}_{i,j}) \), we only need to solve, for each \( \bar{c} \) and \( k \), the corresponding pair of equation systems \( F_R^k(\bar{c}) \) and

\[
W^k(\bar{c}) = \{ w_{i,1}w_{i,2}w_{i,3} = -s(\bar{c}, k) \mid 1 \leq i \leq n \}.
\]

By (1), (2) we have that \( E \) has a solution in \( E \) if and only if \( F_R^k(\bar{c}) \) has a solution in \( V \) and \( W^k(\bar{c}) \) has a solution in \( A \) for some \( \bar{c} \) and \( k \).

Since there are finitely many tuples \( \bar{c} \) and for each \( \bar{c} \) there are finitely many \( k \), there are finitely many pairs of \( F_R^k(\bar{c}) \) and \( W^k(\bar{c}) \). So any equation system in \( E \) can be reduced to finitely many equation systems with rational constraints in \( V \) and finitely many equation systems in \( A \). Therefore we know that the Equation Problem in central extensions of hyperbolic groups is solvable.

To define the rational constraints in \( F_R^k(\bar{c}) \) we need tools from [10], which we recall in the next subsection.

### 2.3. Central extensions of hyperbolic groups.

In this subsection, we recall some definitions and facts from [10]. We include the proof of some of these facts since they are not in [10].

Recall that \( K_\Gamma \) is the Cayley graph of \( \Gamma \) with respect to \( X \). Denote by \( L \) the language (over \( X \)) of all \((\lambda, \nu)\)-quasi-geodesic words in \( K_\Gamma \). Note that \( L \) is regular (See [3]). Let \( N \) a finite-state automaton accepting \( L \). Then \( L \) is an asynchronous biautomatic structure of \( \Gamma \) (See [5] and [9]).

**Definition 2.9 (L-rational).** Let \( \pi : X^* \to \Gamma \) be the canonical projection. A subset \( T \) of \( \Gamma \) is called \( L \)-rational if the language \( \{ w \in L \mid \pi(w) \in T \} \) is regular.

Let \( s : \Gamma \to E \) be a section and \( \sigma_s : \Gamma \times \Gamma \to A \) be the cocycle defined by \( s \).

**Definition 2.10.** The cocycle \( \sigma_s \) is \( L \)-regular if

1. The sets \( \sigma_s(g, \Gamma) \) and \( \sigma_s(\Gamma, g) \) are finite for each \( g \in \Gamma \).
2. For each \( h \in \Gamma \) and \( a \in A \) the subset \( \{ g \in \Gamma \mid \sigma_s(g, h) = a \} \) is an \( L \)-rational subset of \( \Gamma \).

**Theorem 2.11 (Neumann-Reeves).** For any central extensions of hyperbolic groups \( E \) defined by \( 1 \to A \to E \to \Gamma \to 1 \), there exists a section \( \rho : \Gamma \to E \) such that \( \sigma_\rho \) is \( L \)-regular, where \( L \) is any biautomatic structure of \( \Gamma \).
Remark 2.12. Note that given a presentation of $E$, the $L$-rational structure of $\{g \in \Gamma \mid \sigma_\rho(g, h) = a\}$ can be computed (i.e. a finite state automaton which accepts all $L$-words representing elements of $\{g \in \Gamma \mid \sigma_\rho(g, h) = a\}$ can be constructed explicitly). One can see this by examining the proof of the above theorem from [10] and using the fact that any finite balls of the Cayley graphs of $E$ and $\Gamma$ can be constructed.

Let $\rho : \Gamma \to E$ be a section such that $\sigma_{\rho}$ is $L$-regular. Hence we know that $\{g \in \Gamma \mid \sigma_{\rho}(g, x) = a\}$ is $L$-rational for $x \in X$ and $a \in A$. We also need the fact that $\{g \in \Gamma \mid \sigma_{\rho}(x, g) = a\}$ is $L$-rational for $x \in X$ and $a \in A$. Since [10] doesn’t include a proof, we give a proof.

**Lemma 2.13.** $\{g \in \Gamma \mid \sigma_{\rho}(x, g) = a\}$ is $L$-rational for $x \in X$ and $a \in A$.

**Proof.** Consider the finite subset $D = \{\rho(x)i(-\sigma_{\rho}(g, x)) \mid g \in \Gamma, x \in X\}^{\pm 1}$ of $E$ as in [9] Proposition 2.2. If $w = x_1 \cdots x_n \in X^*$ then there exists $w' \in D^*$ whose initial segments have values $\rho(x_1), \rho(x_1x_2), \ldots, \rho(x_1 \cdots x_n)$. Let $L'$ be the language $L' = \{w' \mid w \in L\}$. Then by [9] Proposition 2.2 $L'$ is a regular language. Let $Z$ be a generating set for $A$. It is easy to see that $D \cup i(Z)$ is a generating set of $E$. Let $K_E$ be the Cayley graph of $E$ with respect to $D \cup i(Z)$.

For $x \in X$ and $a \in A$ consider the set $F = \{(w_1, w_2) \in L' \times L' \mid \rho(x)w_1 = w_2i(a)\}$. Since $L$ is a biautomatic structure on $\Gamma$, we can apply [9] Proposition 2.2 and see that there exist $K$ such that $w_1$ and $w_2$ $K$-fellow-travel in $K_E$ if $(w_1, w_2) \in F$. Hence $F$ is the language of a two-tape finite state automata. Therefore $F_1 = \{w_1 \in L' \mid \exists w_2 \in L', \rho(x)w_1 = w_2i(a)\}$, which is the projection of $F$ to the first factor, is regular. Let $F'_1 = \{v \in L \mid \sigma_\rho(x, v) = a\}$. One can use the finite-state automaton accepting $F_1$ to read words in $L$ and $F'_1$ is the language accepted by it. Therefore $F'_1$ is regular, which is equivalent to $\{g \in \Gamma \mid \sigma_{\rho}(x, g) = a\}$ being $L$-rational.

**Convention 2.14** (s-coordinates). For any section $s : \Gamma \to E$ and any $g \in \Gamma$, $a \in A$, we denote by $(g, a)_s$ the element $s(g)i(a)$ in $E$ and call $(g, a)$ the s-coordinates of $s(g)i(a)$. For simplicity we omit the subscript when the section under consideration is clear.
We will use a symmetric section to lift the solutions of $E$ in $\Gamma$ to $E$. The symmetric section $q$ associated to $\rho$ lands in an extension of $E$, which we now define.

Since $A$ is finitely generated and abelian, it is isomorphic to $\mathbb{Z}^n \oplus \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_m}$. We identify $A$ with $\mathbb{Z}^n \oplus \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_m}$ by fixing an isomorphism between them. Let $A' = \mathbb{Z}^n \oplus \mathbb{Z}_{2d_1} \oplus \cdots \oplus \mathbb{Z}_{2d_m}$.

Let $\iota_1$ be the injective homomorphism from $A$ to $A'$ defined by

$$\iota_1(a_1, \ldots, a_n, b_1, \ldots, b_m) = (2a_1, \ldots, 2a_n, 2b_1, \ldots, 2b_m).$$

This map determines a pushout extension $1 \to A' \to E' \to \Gamma \to 1$ in the following sense: Let $E' = \Gamma \times A'$ be the direct product of $\Gamma$ and $A'$ as sets. The map from $A'$ to $E'$ is the inclusion from $A'$ to $\{1\} \times A'$ and the map from $E'$ to $\Gamma$ is the projection of $E'$ to the first factor. Make $E'$ into a group by defining

$$(g_1, a_1)(g_2, a_2) = (g_1g_2, a_1 + a_2 + \iota_1(\sigma_\rho(g_1, g_2))).$$

Let $\iota_2$ denote the natural inclusion from $E$ to $E'$, which maps $(g, a)$ to $(g, \iota_1(a))$. Let $\rho'$ be the section from $\Gamma$ to $E'$ defined by $\rho' = \iota_2\rho$. Hence an element $(g, a')$ in $E'$ has $\rho'$-coordinates $(g, a')$ and we have $\sigma_{\rho'}(g, h) = \iota_1(\sigma_\rho(g, h))$ for any $g, h \in \Gamma$.

**Convention 2.15.** Let $\iota_3$ be the map (not a homomorphism) from $A$ to $A'$ defined by

$$\iota_3(a_1, \ldots, a_n, b_1, \ldots, b_m) = (a_1, \ldots, a_n, b_1, \ldots, b_m).$$

To simplify notation, in the rest of the paper, if $\sigma_\rho(\cdot, \cdot)$ appears in the second component of the $\rho'$-coordinates of an element in $E'$, it represents the element $\iota_3(\sigma_\rho(\cdot, \cdot))$ in $A'$.

**Definition 2.16.** The symmetric section $q : \Gamma \to E'$ is defined by:

$$q(g) = (g, -\sigma_\rho(g, g^{-1}))$$

in $\rho'$-coordinate of $E'$.

**Lemma 2.17.** $q$ is symmetric, i.e., $q(g)q(g^{-1}) = 1$.

**Proof.** In $\rho'$-coordinates of $E'$ we have

$$q(g)q(g^{-1}) = (g, -\sigma_\rho(g, g^{-1}))(g^{-1}, -\sigma_\rho(g^{-1}, g))$$

$$= (1, -\sigma_\rho(g, g^{-1}) - \sigma_\rho(g^{-1}, g) + 2\sigma_\rho(g, g^{-1}))$$

$$= (1, \sigma_\rho(g, g^{-1}) - \sigma_\rho(g^{-1}, g))$$

One the other hand, in $\rho$-coordinate of $E$ we have

$$(g, 0)(g^{-1}, -\sigma_\rho(g, g^{-1})) = (1, -\sigma_\rho(g, g^{-1}) + \sigma_\rho(g, g^{-1})) = (1, 0).$$
Lemma 2.18. $\sigma_q$ is $L$-regular.

Proof. Let $x \in X$ and $g \in \Gamma$. In the $\rho'$-coordinate we have

$$q(g)q(x) = (g, -\sigma_\rho(g, g^{-1}))(x, -\sigma_\rho(x, x^{-1}))$$

$$= (gx, -\sigma_\rho(g, g^{-1}) - \sigma_\rho(x, x^{-1}) + 2\sigma_\rho(g, x))$$

On the other hand, we have $q(gx) = (gx, -\sigma_\rho(gx, (gx)^{-1}))$. Therefore we have

$$\sigma_q(g, x) = \sigma_\rho(gx, (gx)^{-1}) - \sigma_\rho(g, g^{-1}) - \sigma_\rho(x, x^{-1}) + 2\sigma_\rho(g, x)$$

$$= \sigma_\rho(g, g^{-1}) - \sigma_\rho(g, x) - \sigma_\rho(x^{-1}, g^{-1}) + \sigma_\rho(x, x^{-1})$$

$$- \sigma_\rho(g, g^{-1}) - \sigma_\rho(x, x^{-1}) + 2\sigma_\rho(g, x)$$

$$= \sigma_\rho(g, x) - \sigma_\rho(x^{-1}, g^{-1}).$$

From the above equation we know that $\sigma_q(\Gamma, x)$ is finite since both $\sigma_\rho(\Gamma, x)$ and $\sigma_\rho(x^{-1}, \Gamma)$ are finite. Similarly, one can show that $\sigma_q(x, \Gamma)$ is finite. Let $a \in A'$. From the above computation we have

$$\{g \in \Gamma \mid \sigma_q(g, x) = a\}$$

$$= \bigcup_{a_1 - a_2 = a} (\{g \in \Gamma \mid \sigma_\rho(g, x) = a_1\} \cap \{g \in \Gamma \mid \sigma_\rho(x^{-1}, g^{-1}) = a_2\})$$

Since $\sigma_\rho$ is $L$-regular, the left side of the above equation is a finite union and $\{g \in \Gamma \mid \sigma_\rho(g, x) = a_1\}$ is an $L$-rational set. By Lemma 3.1 and the fact that the reverse of a regular language is a regular language, $\{g \in \Gamma \mid \sigma_\rho(x^{-1}, g^{-1}) = a_2\}$ is a $L$-rational set. Therefore $\{g \in \Gamma \mid \sigma_q(g, x) = a\}$ is $L$-rational; so $\sigma_q$ is regular.

Lemma 2.19. $\{g \in \Gamma \mid \sigma_q(x, g) = a\}$ is $L$-rational for $x \in X$ and $a \in A'$.

Proof. Reverse the roles of $g$ and $x$ in the proof of the last lemma.
3. Future Predicting Automata and Parity Predicting Automata

In this section we define Future Predicting Automata and Parity Predicting Automata. They allow us to define the rational constraints we put on the variables of the Tripod equation systems in $V$.

3.1. Lifting rational constraints to $V$. Both Future Predicting Automata and Parity Predicting Automata define rational subsets of $\Gamma$. We explain how to lift these rational subsets to $V$ first. This allows us to determine and fix the constants $\lambda$ and $\nu$ which define $L$.

Recall that $Y$ is a finite generating set of $V$. We define a morphism $\phi$ from $Y^*$ to $X^*$ to lift rational constraints to $V$.

By Lemma 2.2 we know each $y \in Y$ as explicit path in $K$. For each $y \in Y$ we choose and fix a $K_{\Gamma}$-geodesic word $w_y$ representing $\pi(y)$. Let $\phi : Y^* \rightarrow X^*$ be the monoid homomorphism inducde by the map from $Y$ to $X$ sending $y$ to $w_y$. For any regular language $K \subset X^*$, it is a standard fact that $\phi^{-1}(K) = \{w \in Y^* | \phi(w) \in K\}$ is a regular language over $Y$. (See [L] Theorem 4.2.4 for a proof.) Let $\mathcal{QG}(V)$ be as in Definition 2.3.

Lemma 3.1. There exist $\lambda, \nu$ depending only on $\Gamma$ such that the following is true: For any $v \in \mathcal{QG}(V)$, there exists $w \in Y^*$ representing $v$ such that $\phi(w)$ represents a $(\lambda, \nu)$-quasi-geodesics in $K_{\Gamma}$.

Proof. For any adjacent vertices $s, t \in K$, let $[s, t]$ denote the unique edge between them. If $s = t$, let $[s, t]$ be $s$ considered as a constant path.

Suppose the length of $v$ (considered as a path in $K$) is $n$. Let $s_0, \ldots, s_n$ be the vertices along $v$. By the construction of $K$, there are two types of vertices in $K$: vertices of $K_{\Gamma}$ and barycenters. If $s_i$ is a barycenter of some simplex of $R_{508}(K_{\Gamma})$, let $s'_i$ be a vertex of this simplex. If $s_i$ is a vertex of $K_{\Gamma}$, let $s'_i = s_i$. Then $s'_0, \ldots, s'_n$ determine a path $v'$ in $K$. Here the path between $s'_{i-1}$ and $s'_i$ consists of the edges $[s'_{i-1}, s'_{i-1}]$, $[s_{i-1}, s_i]$ and $[s_i, s'_i]$ and we denote this path by $[s'_i, s'_i]$. Note that $v'$ equals $v$ in $V$.

Since $s'_i$ is a vertex in $K_{\Gamma}$, it also represent an element of $\Gamma$. Let $v_i = (s'_{i-1})^{-1}[s'_{i-1}, s'_i]$. Then $v_i$ are element of $V$ and $v = v' = v_1 \cdots v_n$.

For each $v_i$, pick a $Y$-geodesic word $w_i$ representing $v_i$. Let $w = w_1 \cdots w_n$. Then $w$ is a $Y$-word representing $v$. Since there are finitely many elements of $V$ of length less than four, there is an upper bound $K_1$ on the length of $w_i$. Here $K_1$ depends only on $\Gamma$. Hence $w$ has length at most $nK_1$. 


There is an upper bound $K_2$ depending only on $\Gamma$ on the length of $\phi(y)$ for all $y \in Y$ since $Y$ is finite. Let $n'$ be the length of $\phi(w)$. Then we have

$$n' \leq nK_1K_2$$

Let $d$ be the distance in $\mathcal{K}$ between the end points of $v$. Let $d_X$ be the distance in $K_\Gamma$ between the end points of $v$.

Since $\mathcal{K}$ is quasi-isometric to $K_\Gamma$, we have

$$\frac{1}{K_0}d - C \leq d_X$$

for some $K_0$ and $C$ depending only on $\Gamma$.

We know that $v$ is a $(\lambda_1, \nu_1)$-quasi-geodesic in $\mathcal{K}$ since $v \in QG(V)$, hence we have

$$\frac{1}{\lambda_1}n - \nu_1 \leq d.$$ 

From the three equations above, we have

$$\frac{1}{K_0K_1K_2}\lambda_1n' - \nu_1 - C \leq d_X.$$ 

Let $\lambda = K_0K_1K_2\lambda_1$ and $\nu = \nu_1 + C$. Then the above equation implies that $\phi(w)$ is a $(\lambda, \nu)$-quasi-geodesic in $K_\Gamma$. Note that $(\lambda, \nu)$ depends only on $\Gamma$. \hfill $\square$

### 3.2. Future Predicting Automata.

We are now ready to define the Future Predicting Automaton. We will first define the Future Predicting Automaton, which accepts exactly the language $L$. The Future Predicting Automaton has many accepting states. The languages defined by each of these accepting states give a partition of $L$ and these subsets of $L$ will be used to define the rational constraints we need in the next section.

For the rest of the paper, we use $L$ to denote the regular language of words over $X$ representing $(\lambda, \nu)$-quasi-geodesic in $K_\Gamma$, where $\lambda$ and $\nu$ are given by Lemma 3.1.

By Lemma 2.18 $\{g \in \Gamma \mid \sigma_q(g, x) = a\}$ is $L$-rational for $x \in X$, $a \in A'$ and

$$A_x = \{\sigma_q(g, x) \in A' \mid g \in \Gamma\}.$$

is finite. For each $x \in X$, $a \in A_x$ we choose and fix a finite state automaton $M_{x,a}$ which accepts exactly the $L$-words representing elements in $\{g \in \Gamma \mid \sigma_q(g, x) = a\}$. Denote the set of states and the transition map of $M_{x,a}$ by $S_{x,a}$ and $F_{x,a}$, respectively.

**Definition 3.2 (FPA).** The Future Predicting Automaton, denoted by $M$, is defined as follows:
States: \( S = \prod_{x \in X, a \in A_x} S_{x,a}. \)

Transition function: \( F : S \times X \to S \) is defined by
\[
F((s_{x,a})_{x \in X, a \in A_x}, x') = (F_{x,a}(s_{x,a}, x'))_{x \in X, a \in A_x}.
\]

Initial state: \( (I_{x,a})_{x \in X, a \in A_x} \), where \( I_{x,a} \) is the initial state of \( M_{x,a} \).

Accepting state: The set of accepting states \( T \) consists of states \( (s_{x,a})_{x \in X, a \in A_x} \in S \) satisfying: for all \( x' \in X \) there exists a unique \( a' \in A_{x'} \) such that \( s_{x',a'} \) is an accepting state of \( M_{x',a'} \).

Remark 3.3. Note that \( M \) has finitely many states since \( X, A_x \) and \( S_{x,a} \) are finite for all \( x \in X, a \in A_x \).

Lemma 3.4. The language accepted by the Future Predicting Automaton is \( L \).

Proof. Suppose \( w \) is accepted by \( M \). Then \( w \) ends up in a state in \( T \). By the definition of \( T \), for any \( x \in X \), there exists a unique \( a \in A_x \), such that \( s_{x,a} \) is an accepting state of \( M_{x,a} \). This together with the definition of \( M \) implies that \( w \) is accepted by \( M_{x,a} \). Then by the definition of \( M_{x,a} \), we know that \( w \) is an \( L \)-word representing an element in \( \{ g \in \Gamma \mid \sigma_g(g, x) = a \} \). In particular, \( w \) is in \( L \).

Now suppose \( w \) is in \( L \). Use \( M \) to read \( w \). Suppose it ends at the state \( (s_{x,a})_{x \in X, a \in A_x} \). Note that \( s_{x,a} \) is an accepting state of \( M_{x,a} \) if and only if we have \( \sigma_g(w, x) = a \), where \( w \) is interpreted as the element of \( \Gamma \) it represents. Hence for each \( x' \in X \), there is a unique \( a' \in A_{x'} \) such that \( s_{x',a'} \) is an accepting state of \( M_{x',a'} \). Therefore \( (s_{x,a})_{x \in X, a \in A_x} \) is in \( T \). So \( w \) is accepted by \( M \).

Definition 3.5. Let \( \bar{s} = (s_{x,a}) \in T \). The Future Predicting Automaton associated with \( \bar{s} \), denoted by \( M(\bar{s}) \), is the finite state automaton having the same states, transition function and initial state as the Future Predicting Automaton \( M \), but \( \bar{s} \) as the only accepting state.

Lemma 3.6. \( \{ L(\bar{s}) \mid \bar{s} \in T \} \) is a finite partition of \( L \).

For any \( \bar{s} \in T \), let \( M_\bar{s} \) be the finite state automaton which has the same states, transition map and accepting states as \( M \), but has \( \bar{s} \) as the initial state. Note that \( M_\bar{s} \) is different from \( M(\bar{s}) \).

Definition 3.7. A word \( v \in X^* \) is compatible with \( \bar{s} \in T \) if \( v \) is accepted by \( M_\bar{s} \).
Note that \( v \) is compatible with \( \bar{s} \) simply means that for any \( w \in L(\bar{s}) \), the word \( wv \) is in \( L \) (or equivalently accepted by \( M \)). The following fact is clear from Definition 3.7.

**Lemma 3.8.** For any \( \bar{s} \in T \), the language of all words compatible with \( \bar{s} \) is regular.

**Convention 3.9.** We interpret any \( w \in X^* \) as the element of \( \Gamma \) represented by the word \( w \) when any cocycle is applied to \( w \).

The next lemma is the key property of the Future Predicting Automata.

**Proposition 3.10.** Suppose that \( w_1, w_2 \in L(\bar{s}) \). Then \( \sigma_q(w_1, v) = \sigma_q(w_2, v) \) provided \( v \in X^* \) is compatible with \( \bar{s} \in T \).

**Proof.** Let \( v = x_1x_2 \cdots x_l \). We argue by induction on \( l \).

Suppose \( l = 1 \). Since \( w_1, w_2 \in L(\bar{s}) \), they both end at the same state of \( M(\bar{s}) \). Hence for \( a \in A_{x_1} \), we have that \( w_1 \) and \( w_2 \), when read by \( M_{x_1,a} \), end up in the same states. By the definition of FPA there exists a unique \( a_1 \in A_{x_1} \) such that \( M_{x_1,a_1} \) accepts both \( w_1 \) and \( w_2 \). Therefore by the definition of \( M_{x_1,a_1} \), we have \( \sigma_q(w_1, x_1) = \sigma_q(w_2, x_1) = a_1 \).

By the cocycle condition of \( \sigma_q \), for \( i = 1, 2 \) we have

\[
\sigma_q(w_i, x_1x_2 \cdots x_l) = \sigma_q(w_i, x_1x_2 \cdots x_{l-1}) + \sigma_q(w_i, x_1x_2 \cdots x_{l-1}, x_l) - \sigma_q(x_1x_2 \cdots x_{l-1}, x_l)
\]

By induction the first term does not depend on \( i \). The third term clearly does not depend on \( i \). Showing that the second term does not depend on \( i \) will complete the proof.

Since \( \bar{s} \) and \( v \) are compatible, we have that \( w_tv = w_ix_1x_2 \cdots x_l \) is in \( L \). Therefore we know that \( w_ix_1x_2 \cdots x_{l-1} \) is in \( L \) for \( i = 1, 2 \) since \( L \) is closed under taking subword. In fact both \( w_1x_1x_2 \cdots x_{l-1} \) and \( w_2x_1x_2 \cdots x_{l-1} \) end up in the same accepting state when read by \( M \) since that is true for \( w_1 \) and \( w_2 \). So there exists a unique \( a_l \in A_{x_l} \) such that \( M_{x_l,a_l} \) accepts \( w_ix_1x_2 \cdots x_{l-1} \) for \( i = 1, 2 \). Hence \( \sigma_q(w_1, x_1x_2 \cdots x_{l-1}, x_l) = a_l \), which does not depend on \( i \). \( \square \)

The above lemma explains the name “Future Predicting Automaton” since all we need to know to “predict” the value of \( \sigma_q(w, v) \) is where \( w \) ends when read by the Future Predicting Automaton. By Proposition 3.10 the following definition makes sense.

**Definition 3.11.** Let \( \bar{s} \in T \) and \( v \in X^* \). Suppose \( \bar{s} \) and \( v \) are compatible. Define \( \sigma_q(\bar{s}, v) \) to be \( \sigma_q(w, v) \) for any \( w \in L(\bar{s}) \).
3.3. **Parity Predicting Automata.** Recall that $q : \Gamma \to E'$ is the symmetric section (Definition 2.16). We are lifting solutions of equation system in $\Gamma$ by $q$. We need to define appropriate rational constraints on variables of the Tripod Equation systems so that we can predict whether their lifts land in $E$ or not. The Parity Predicting Automata define these rational constraints.

We first define the Left Future Predicting Automaton (LFPA) and the Right Future Predicting Automaton (RFPA), which we will use to define the Parity Predicting Automaton.

Let $\rho$ be the section given by Theorem 2.11. Hence we know that 
\[ \{ g \in \Gamma \mid \sigma_\rho(g, x) = a \} \] is $L$-rational for all $x \in X$ and $a \in A$ and $A^1_x = \{ \sigma_\rho(g, x) \mid g \in \Gamma \}$ is finite.

For each $x \in X$ and $a \in A^1_x$ choose and fix a finite state automaton $M^1_{x,a}$ which accepts exactly the $L$-words representing elements in $\{ g \in \Gamma \mid \sigma_\rho(g, x) = a \}$. Denote by $S^1_{x,a}$ and $F^1_{x,a}$ the set of states and the transition function of $M^1_{x,a}$, respectively.

The following definition is almost the same as the definition of the Future Predicting Automaton $M$, except that we are now considering the section $\rho$ instead of $q$.

**Definition 3.12.** The Left Future Predicting Automaton $M_1$ over $X$ is defined as follows:

- **States:** $S_1 = \prod_{x \in X, a \in A^1_x} S^1_{x,a}$
- **Transition function:** $F_1 : S_1 \times X \to S_1$ is defined by 
  \[ F_1((s^1_{x,a}), x') = (F^1_{x,a}(s^1_{x',a}), x') \] for $x' \in X, a \in A^1_{x'}$.
- **Initial state:** $(I^1_{x,a})_{x \in X, a \in A^1_x}$, where $I^1_{x,a}$ is the initial states of $M^1_{x,a}$.
- **Accepting states:** The set of accepting states $T_1$ consists of states $(s^1_{x,a})_{x \in X, a \in A^1_x}$ satisfying that for all $x' \in X$ there exists a unique $a' \in A^1_{x'}$ such that $s^1_{x',a'}$ is the accepting state of $M^1_{x',a'}$.

One can prove the following lemma the same way as we prove Lemma 3.4.

**Lemma 3.13.** The language accepted by $M_1$ is $L$.

**Definition 3.14.** Suppose $s_1 = (s^1_{x,a})_{x \in X, a \in A^1_x}$ is an accepting state of $M_1$. For any $x' \in X$, define $\sigma_\rho(s_1, x') = a'$, where $a'$ is the unique element of $A^1_{x'}$ such that $s^1_{x',a'}$ is the accepting state of $M^1_{x',a'}$.

The next lemma follows directly from the above definition and the definition of $M^1_{x,a}$. 
Lemma 3.15. Suppose \( w \in X^* \) ends up in the state \( \bar{s}_1 \in T_1 \) when read by \( M_1 \). Then \( \sigma_\rho(w,x) = \sigma_\rho(\bar{s}_1,x) \)

By Lemma 2.13 the set of all the \( L \)-words representing elements in \( \{ g \in \Gamma \mid \sigma_\rho(x,g) = a \} \) is a regular language. Hence its reverse is also a regular language. Let \( M_2^{x,a} \) be the finite state automaton accepting the reverse language. Let \( S_2^{x,a} \) and \( F_2^{x,a} \) be the set of states and the transition function of \( M_2^{x,a} \), respectively. Let \( A_2^x = \{ \sigma_\rho(x,g) \mid g \in \Gamma \} \). Note that \( A_2^x \) is finite by Theorem 2.11.

Definition 3.16. The Right Future Predicting Automaton \( M_2 \) over \( X \) is defined as follows:

- **States:** \( S_2 = \prod_{x \in X, a \in A_2^x} S_2^{x,a} \).
- **Transition function:** \( F_2 : S_2 \times X \to S_2 \) is defined as follow:
  \[
  F_2((s_2^{x,a})_{x \in X, a \in A_2^x}, x') = (F_2^{x,a}(s_2^{x,a}, (x')^{-1}))_{x \in X, a \in A_2^x}.
  \]
- **Initial state:** \( (I_2^{x,a})_{x \in X, a \in A_2^x} \), where \( I_2^{x,a} \) is the initial states of \( M_2^{x,a} \).
- **Accepting states:** The set of accepting states \( T_2 \) consists of states \( (\bar{s}_2^{x,a})_{x \in X, a \in A_2^x} \) satisfying that for all \( x' \in X \) there exists a unique \( a' \in A_2^x \) such that \( s_2^{x,a'} \) is the accepting state of \( M_2^{x,a'} \).

One can prove the following lemma the similar way as we prove Lemma 3.14.

Lemma 3.17. The language accepted by \( M_2 \) is

\[
L^{-1} = \{ w^{-1} = w_n^{-1} \cdots w_1^{-1} \mid w = w_1 \cdots w_n \in L \}.
\]

Definition 3.18. Suppose \( \bar{s}_2 = (s_2^{x,a})_{x \in X, a \in A_2^x} \) is an accepting state of \( M_2 \). Define \( \sigma_\rho(x', \bar{s}_2) = a' \), where \( a' \) is the unique element of \( A_1^x \), such that \( s_2^{x,a'} \) is the accepting state of \( M_2^{x,a'} \).

The next lemma follows directly from the above definition and the definition of \( M_2^{x,a} \).

Lemma 3.19. Suppose \( w \in X^* \) ends up in the state \( \bar{s}_2 \in T_2 \) when read by \( M_2 \). Then \( \sigma_\rho(x, w^{-1}) = \sigma_\rho(x, \bar{s}_2) \)

We now define a homomorphism \( Pa \), which makes what we mean by “parity” precise. Denote the natural project from \( \mathbb{Z} \) to \( \mathbb{Z}_2 \) by \( P_2 \). Recall that the finite generated abelian group \( A \) central in \( E \) is identified with \( \mathbb{Z}^n \oplus \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_m} \). The homomorphism \( Pa : A \to \mathbb{Z}_2^n \oplus \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_m} \) is defined by

\[
Pa(a_1, \ldots, a_n, b_1, \ldots, b_m) = (P_2(a_1), \ldots, P_2(a_n), b_1, \ldots, b_m).
\]

The parity of an element \( a \in A \) is define to be \( Pa(a) \).
Let \( a \in A \). Recall that \( \iota_3 \) and \( \iota_1 \) are defined in Convention 2.15 and the paragraph before it, respectively. In general \( \iota_3(a) \) does not lie in \( \iota_1(A) \). But once we know the parity of \( a \). We can use it to “move” \( \iota_3(a) \) to something in \( \iota_1(A) \). We now make this precise: Define the map (not a homomorphism) \( \iota_4 : \mathbb{Z}_n^r \oplus \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_m} \to A' = \mathbb{Z}_n^r \oplus \mathbb{Z}_{2d_1} \oplus \cdots \oplus \mathbb{Z}_{2d_m} \) by

\[
(a_1, \ldots, a_n, b_1, \ldots, b_m) = (a_1, \ldots, a_n, b_1, \ldots, b_m)
\]

Then the following fact is clear.

**Lemma 3.20.** For any \( a \in A \), we have \( \iota_3(a) + \iota_4(Pa(a)) \in \iota_1(A) \).

Now we define the Parity Predicting Automaton.

**Definition 3.21 (PPA).** The Parity Predicting Automaton, denoted by \( D \), is defined as follows:

- **States:** \( S_D = S_1 \times S_2 \times (\mathbb{Z}_n^r \oplus \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_m}) \cup \{\emptyset\} \).
- **Transition functions:** \( F_D : S_D \times X \to S_D \) is defined as follows:
  
  If \( \bar{s}_1 \in T_1 \) and \( \bar{s}_2 \in T_2 \), then we define

  \[
  F_D((\bar{s}_1, \bar{s}_2, b), x) = \left(F_1(\bar{s}_1, x), F_2(\bar{s}_2, x), b'\right),
  \]

  where

  \[
  b' = b + Pa\left(\sigma_\rho(x, x^{-1}) - \sigma_\rho(\bar{s}_1, x) - \sigma_\rho(x^{-1}, \bar{s}_2)\right).
  \]

  Otherwise we define

  \[
  F_D((\bar{s}_1, \bar{s}_2, b), x) = \emptyset.
  \]

For all \( x \in X \), we define

\[
F_D(\emptyset, x) = \emptyset.
\]

The initial state: \( (I_1, I_2, 0) \). Here \( I_1 \) and \( I_2 \) are the initial states of \( M_1 \) and \( M_2 \) respectively.

The accepting states: All states \((\bar{s}_1, \bar{s}_2, b)\) so that \( \bar{s}_1 \in T_1 \) and \( \bar{s}_2 \in T_2 \).

The following fact is clear from the above definition.

**Lemma 3.22.** The language accepted by \( D \) is \( L \).

Let \( d \in \mathbb{Z}_2^r \oplus \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_m} \).

**Definition 3.23.** The Parity Predicting Automaton associated to \( d \), denoted by \( D(d) \), is the same as the Parity Predicting Automaton with the extra requirement that the third component of any accepting state is \( d \).
Lemma 3.25. \( \{ L(d) \mid d \in \mathbb{Z}_2^n \oplus \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_m} \} \) is a finite partition of \( L \).

The following lemma is the key property of the Parity Predicting Automata.

Lemma 3.25. Let \( w \in L(d) \). Then \( P\alpha(\sigma_\rho(w, w^{-1})) = d \).

Proof. We proceed by induction on the length of \( w \). The base case is when \( w \) has length 0. In this case, \( w \) is the identity. Hence \( \sigma_\rho(w, w^{-1}) = 0 \). Since \( D \)'s initial state has 0 as its last component, the lemma is true in this case.

Suppose the lemma is true for words of length less then \( l \). Let \( w = x_1 \cdots x_l \). By the cocycle condition of \( \sigma_\rho \) we have

\[
\sigma_\rho(w, w^{-1}) = \sigma_\rho(x_1 \cdots x_l, x_l^{-1} \cdots x_1^{-1}) = \sigma_\rho(x_1 \cdots x_l, x_l^{-1}) + \sigma_\rho(x_1 \cdots x_{l-1}, x_{l-1}^{-1} \cdots x_1^{-1}) - \sigma_\rho(x_l^{-1}, x_l^{-1} \cdots x_1^{-1}) + \sigma_\rho(x_1 \cdots x_{l-1}, x_{l-1}^{-1} \cdots x_1^{-1}) - \sigma_\rho(x_l^{-1}, x_l^{-1} \cdots x_1^{-1})
\]

Suppose \( x_1 \cdots x_{l-1} \) ends at a state \((\bar{s}_1, \bar{s}_2, b_{l-1})\). Note that \( \bar{s}_1 \in T_1 \) and \( \bar{s}_2 \in T_2 \) (otherwise \( w \) wouldn’t be accepted by \( D \)). By the induction hypothesis, \( P\alpha(\sigma_\rho(x_1 \cdots x_{l-1}, x_{l-1}^{-1} \cdots x_1^{-1})) = b_{l-1} \). Then by the equation above we have

\[
P\alpha(\sigma_\rho(w, w^{-1})) = P\alpha(\sigma_\rho(x_1 \cdots x_{l-1}, x_{l-1}^{-1} \cdots x_1^{-1})) + P\alpha(\sigma_\rho(x_l, x_l^{-1}) - \sigma_\rho(x_1 \cdots x_{l-1}, x_l) - \sigma_\rho(x_l^{-1}, x_{l-1}^{-1} \cdots x_1^{-1})) = b_{l-1} + P\alpha(\sigma_\rho(x_l, x_l^{-1}) - \sigma_\rho(x_1 \cdots x_{l-1}, x_l) - \sigma_\rho(x_l^{-1}, x_{l-1}^{-1} \cdots x_1^{-1}))
\]

On the other hand, by the definition of \( D \), \( x_1 \cdots x_l \) ends at the state \((F_1(\bar{s}_1, x_l), F_2(\bar{s}_2, x_l^{-1}), b)\), where

\[
b = b_{l-1} + P\alpha(\sigma_\rho(x_l, x_l^{-1}) - \sigma_\rho(\bar{s}_1, x_l) - \sigma_\rho(x_l^{-1}, \bar{s}_2))
\]

Therefore it is enough to show that

\[
\sigma_\rho(x_1 \cdots x_{l-1}, x_l) = \sigma_\rho(\bar{s}_1, x_l)
\]

and

\[
\sigma_\rho(x_l^{-1}, x_{l-1}^{-1} \cdots x_1^{-1}) = \sigma_\rho(x_l^{-1}, \bar{s}_2).
\]
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Since \( x_1 \cdots x_{l-1} \) ends at \((s_1, s_2, b_{l-1})\), we know that \( x_1 \cdots x_{l-1} \) ends at \( s_1 \) when read by \( M_1 \) and it ends at \( s_2 \) when read by \( M_2 \). Hence by Lemma 3.15 and Lemma 3.19 the above two equations follow and the proof of the lemma is completed. \( \square \)

4. PROOF OF THEOREM 1.1

Let \( U \) be a finite set of variables and \( C \subset E \) be a finite set of constants in \( E \). Recall from Section 2 that it is enough to consider triangular equation systems. Let \( \mathcal{E} = \{e_{i,1}e_{i,2}e_{i,3} = 1, i = 1, \ldots, n\} \) be a triangular equation system where \( e_{i,j} \in U \cup C \).

Now we construct equation systems \( \mathcal{V}_i \) over \( V \) and \( \mathcal{W}_i \) over \( A \) where \( t \) runs over some finite set \( \Theta \). The size of \( \Theta \) depends on \( \mathcal{E} \) and \( \Gamma \). Then we show that \( \mathcal{E} \) has a solution in \( E \) if and only if there is some \( t \in \Theta \) such that \( \mathcal{V}_i \) has a solution in \( V \) and \( \mathcal{W}_i \) has a solution in \( A \).

First we describe the finite set \( \Theta \) over which the subscript \( t \) of \( \mathcal{V}_i \) and \( \mathcal{W}_t \) runs. The index set \( \Theta \) consists of tuples \((c_{i,j}), (s_{i,j}), (b_{i,j}), (d_{i,j})\) satisfying the following 4 conditions:

Recall that \( V_{<l} \) is the set of elements of \( V \) whose corresponding reduced path in \( \mathcal{K} \) has length at most \( l \). Let \( \kappa_1 \) be as in Proposition 2.4. There exists \( \kappa_2 \) such that all elements of \( V_{<\kappa_1} \) are represented by some \( Y \)-words of word length as most \( \kappa_2 \). Recall that \( \phi \) is the monoid homomorphism from \( Y^* \) to \( X^* \) defined in the Section 3.1.

**Condition 1:** For each \( 1 \leq i \leq n, 1 \leq j \leq 3, c_{i,j} \) is a \( Y \)-word of word length at most \( \kappa_2 \) and for each \( i \) we have \( \pi(c_{i,1}c_{i,2}c_{i,3}) = 1 \) in \( \Gamma \).

**Condition 2:** For \( 1 \leq i \leq n, 1 \leq j \leq 3, s_{i,j} \in T \) is an accepting state of the Further Predicting Automaton so that \( s_{i,j} \) and \( \phi(c_{i,j}) \) are compatible.

Recall that for any \( \bar{s} \in T, M_i \) is the finite state automaton which has the same states, transition map and accepting states as \( M \), but has \( \bar{s} \) as the initial state. Let \( s_{i,j}' \) be where \( \phi(c_{i,j}) \) ends when read by \( M_{s_{i,j}} \). Denote the language of words in \( X \) that are compatible with \( s_{i,j}' \) by \( L(s_{i,j}, c_{i,j}) \). Let \( A(s_{i,j}, c_{i,j}) = \{\sigma_q(s_{i,j}', w) \mid w \in L(s_{i,j}, c_{i,j})\} \).

**Condition 3:** For all \( 1 \leq i \leq n, 1 \leq j \leq 3, b_{i,j} \in A(s_{i,j}, c_{i,j}) \).

**Condition 4:** For all \( 1 \leq i \leq n, 1 \leq j \leq 3, d_{i,j} \in \mathbb{Z}_2^n \oplus \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_m} \).

Lemma 4.1. \( \Theta \) is finite.

Proof. Since the lengths of \( c_{i,j} \)'s are bounded, there are finitely many tuples \( (c_{i,j}) \) satisfying Condition 1. We know that \( T \) is finite for it is
the set of accepting states of a finite state automaton (FSA). Hence there are finitely many tuples \((\bar{s}_{i,j})\) satisfying Condition 2.

For each choice of \((c_{i,j})\) and \((\bar{s}_{i,j})\), note that \(A(\bar{s}_{i,j}, c_{i,j}) = \{\sigma_q(u_{i,j}, w) \mid w \in L(\bar{s}_{i,j}, c_{i,j})\}\) for any \(u_{i,j} \in L(\bar{s}_{i,j})\). Hence \(A(\bar{s}_{i,j}, c_{i,j})\) is finite by Lemma 2.18. Therefore there are finitely many choices for \((b_{i,j})\). At last, possibilities of \((\bar{d}_{i,j})\) are bounded since \(\mathbb{Z}_n^* \ominus \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_m}\) is finite.

For each \(t = ((c_{i,j}), (\bar{s}_{i,j}), (\bar{b}_{i,j}), (\bar{d}_{i,j}))\), we have the following setups:

Let \(L(\bar{s}_{i,j}) \subset X^*\) be the regular language associated to \(\bar{s}_{i,j}\).

Let \(L(b_{i,j}) = \{w \in L(\bar{s}_{i,j}, \sigma_q) \mid \sigma_q(\bar{s}_{i,j}, w) = b_{i,j}\}\).

**Lemma 4.2.** \(L(b_{i,j})\) is regular.

**Proof.** Pick and fix \(u_{i,j} \in L(\bar{s}_{i,j})\). Note that \(L(b_{i,j}) = \{w \in L \mid \sigma_q(u_{i,j}, w) = b_{i,j}\} \cap L(\bar{s}_{i,j}, c_{i,j})\). By Lemma 2.18 \(\{w \in L \mid \sigma_q(u_{i,j}, w) = b_{i,j}\}\) is regular and \(L(\bar{s}_{i,j}, c_{i,j})\) is regular by Lemma 3.8. Hence \(L(b_{i,j})\) is a regular language.

Let \(L(d_{i,j}) \subset X^*\) be the regular language associated to \(d_{i,j}\).

If \(e_{i,j}\) is a constant in \(E\), let \(L(e_{i,j})\) denote the regular language of all \(\Gamma\)-representatives of \(p(e_{i,j})\); otherwise, let \(L(e_{i,j}) = L\).

**Convention 4.3.** For any regular language \(K\), we use \(\phi^{-1}(K)\) to denote the rational subset of \(V\) defined by \(\phi^{-1}(K) \subset Y^*\).

We now define \(\mathcal{V}_t\) as follows:

Let \(\{v_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq 3\}\) be a set of variables such that \(v_{i,j}\) and \(v_{i', j'}\) are the same variable if and only if \(e_{i,j}\) and \(e_{i', j'}\) are the same. Let \(P = \{p_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq 3\}\) be another set of distinct variables.

\[
\mathcal{V}_t = \left\{ \begin{array}{l}
p_{i,1}c_{i,1}(p_{i,2})^{-1} = v_{i,1} \\
p_{i,2}c_{i,2}(p_{i,3})^{-1} = v_{i,2} \\
p_{i,3}c_{i,3}(p_{i,1})^{-1} = v_{i,3} \\
p_{i,j} \in \phi^{-1}(L(\bar{s}_{i,j})) & 1 \leq i \leq n; 1 \leq j \leq 3; 3 + 1 = 1 \\
p_{i,j+1} \in \phi^{-1}(L(b_{i,j})) \\
v_{i,j} \in \phi^{-1}(L(d_{i,j})) \\
v_{i,j} \in \phi^{-1}(L(e_{i,j})) \\
\end{array} \right. 
\]

Let \(a_{i,j} = \sigma_q(\bar{s}_{i,j}, \phi(c_{i,j}))\). Recall that \(\pi : V \to \Gamma\) sends each \(v \in V\) (which is a path in \(K\)) to its terminal point.

**Lemma 4.4.** Suppose \(\{\bar{v}_{i,j}, \bar{p}_{i,j}\}\) is a solution of \(\mathcal{V}_t\). The following are true:
Hence we can prove any statement about $\sigma$ we have

(1) $\sigma_q(\pi(\tilde{p}_{i,j}), \pi(c_{i,j})) = a_{i,j}$;

(2) $\sigma_q(\pi(\tilde{p}_{i,j}c_{i,j}), \pi((\tilde{p}_{i,j+1})^{-1})) = b_{i,j}$;

(3) $Pa(\sigma_{\rho}(\pi(\tilde{v}_{i,j}), \pi((\tilde{v}_{i,j})^{-1}))) = d_{i,j}$.

(4) $\pi(\tilde{v}_{i,j}) = p(e_{i,j})$ if $e_{i,j}$ is a constant.

Proof. First note for $v_i \in V$ and $w_i \in Y^*$ represents $v_i$ for $i = 1, 2$, we have

$$\sigma_q(\pi(v_1), \pi(v_2)) = \sigma_q(\phi(w_1), \phi(w_2))$$

by the following commutative diagram

$$\begin{array}{ccc}
Y^* & \xrightarrow{\phi} & X^* \\
\downarrow & & \downarrow \\
V & \xrightarrow{\pi} & \Gamma
\end{array}$$

Hence we can prove any statement about $\sigma_q(\pi(v_1), \pi(v_2))$ by proving the same statement about $\sigma_q(\phi(w_1), \phi(w_2))$.

Since $\tilde{p}_{i,j} \in \phi^{-1}(L(\tilde{s}_{i,j}))$, there exists $p'_{i,j} \in Y^*$ representing $\tilde{p}_{i,j}$ such that $\phi(p'_{i,j}) \in L(\tilde{s}_{i,j})$. Hence by Lemma 3.10 and the definition of $a_{i,j}$, we have $\sigma_q(\phi(p'_{i,j}), \phi(c_{i,j})) = a_{i,j}$, which proves (1).

For (2), since $(\tilde{p}_{i,j+1})^{-1} \in \phi^{-1}(L(b_{i,j}))$, there exists $p'_{i,j+1} \in Y^*$ representing $\tilde{p}_{i,j+1}$ such that $\phi((p'_{i,j+1})^{-1}) \in L(b_{i,j})$. By the definition of $L(b_{i,j})$ we know that $\sigma_q(u_{i,j}, \phi((p'_{i,j+1})^{-1})) = b_{i,j}$ and that $\phi((p'_{i,j+1})^{-1})$ is compatible with $\tilde{s}_{i,j}$. We know that $\phi(p'_{i,j}c_{i,j}) \in L(s'_{i,j})$ by the definition of $s'_{i,j}$. We have $u_{i,j} \in L(s'_{i,j})$. Hence by Lemma 3.10 we have

$$\sigma_q(\phi(p'_{i,j}c_{i,j}), \phi((p'_{i,j+1})^{-1})) = \sigma_q(u_{i,j}, \phi((p'_{i,j+1})^{-1})) = b_{i,j}$$

For (3), since $\tilde{v}_{i,j} \in \phi^{-1}(L(d_{i,j}))$, there exists $v'_{i,j} \in Y^*$ representing $\tilde{v}_{i,j}$ such that $\phi(v'_{i,j}) \in L(d_{i,j})$. Hence by Lemma 3.25 we have

$$Pa\left(\sigma_{\rho}(\phi(v'_{i,j}), (\phi(v'_{i,j}))^{-1})\right) = d_{i,j}.$$ 

(4) directly follows from the definition of $L(e_{i,j})$. \qed

We now define the equation system $\mathcal{W}_t$ in $A$ correspond to $\mathcal{V}_t$. Let \{t_{w_{i,j}} \mid 1 \leq i \leq n, 1 \leq j \leq 3\} be a set of constants and variables satisfying the follows:

(1) $w_{i,j} = w_{r,j'}$ if and only if $e_{i,j} = e_{r,j'}$.
(2) If $e_{i,j}$ is a variable in $E$, then $w_{i,j}$ is a variable in $\mathcal{W}_t$.
(3) When $e_{i,j}$ is a constant in $E$, we define

$$w_{i,j} = t_{i,j}^{-1}(t_{2}(e_{i,j}) \cdot (qp(e_{i,j}))^{-1} \cdot t_{4}(d_{i,j}))$$
Note that in the last case $w_{i,j}$ may not be well defined since $\iota_1$ is not surjective. If this happens, we define $W_t$ to have no solution.

The equation system $W_t$ over $A$ is defined as follow:

$$W_t = \{ \sum_{j=1}^{3} w_{i,j} = \iota_1^{-1}\left( \sum_{j=1}^{3} (a_{i,j} + b_{i,j} + \nu_4(d_{i,j})) - \sigma_q(\pi(c_{i,1}), \pi(c_{i,2})) \right) : 1 \leq i \leq n \}$$

Note that the right hand sides of the equations above might not be well defined since $\iota_1$ is not surjective. In that case, we define $W_t$ to have no solution.

**Theorem 4.5.** $E$ has a solution in $E$ if and only if $E_t = V_t \cup W_t$ constructed above has a solution for some $t \in \Theta$.

**Proof.** Suppose $E_t$ has a solution for $t = ((c_{i,j}), (\bar{s}_{i,j}), (b_{i,j}), (d_{i,j}))$, i.e. $V_t$ has a solution in $V$ and $W_t$ has a solution in $A$. Let $\{\tilde{v}_{i,j}, \tilde{p}_{i,j}\}$ be a solution of $V_t$ and $\{\tilde{w}_{i,j}\}$ be a solution of $W_t$.

Recall that $i$ is the inclusion from $A'$ to $E'$, $\iota_1$ is embedding of $A$ into $A'$ and $\iota_2$ is the embedding of $E$ into $E'$. We will show that

$$\tilde{e}_{i,j} = q(\pi(\tilde{v}_{i,j}))i(\iota_1(\tilde{w}_{i,j}) - \nu_4(d_{i,j}))$$

is a solution of $E$ in $E'$. Here we think of $E$ as an equation system in $E'$ by replacing all constants by their image under $\iota_2$.

To simplify notion let $v_{i,j} = \pi(\tilde{v}_{i,j})$, $p_{i,j} = \pi(\tilde{p}_{i,j})$ and $c_{i,j} = \pi(c_{i,j})$.

First note that $\tilde{e}_{i,j}$ has $q$-coordinates $\left(v_{i,j}, \iota_1(\tilde{w}_{i,j}) - \nu_4(d_{i,j})\right)$. By direct computation, we have:

$$\tilde{e}_{i,1}\tilde{e}_{i,2}\tilde{e}_{i,3} = (v_{i,1}, \iota_1(\tilde{w}_{i,1}) - \nu_4(d_{i,1}))(v_{i,2}, \iota_1(\tilde{w}_{i,2}) - \nu_4(d_{i,2}))(v_{i,3}, \iota_1(\tilde{w}_{i,3}) - \nu_4(d_{i,3}))$$

$$= (v_{i,1}v_{i,2}v_{i,3}, \sigma_q(v_{i,1}, v_{i,2}) + \sigma_q(v_{i,1}v_{i,2}, v_{i,3}) + \sum_{j=1}^{3}(\iota_1(\tilde{w}_{i,j}) - \nu_4(d_{i,j})))$$

By the definition of $V_t$ and the fact that $\{\tilde{v}_{i,j}\}$ is a solution of $V_t$, we have $v_{i,1}v_{i,2}v_{i,3} = 1$ in $\Gamma$.

Now we consider the second component of (1). The following claim reduces the first two terms into something we have control over.

**Claim 1.**

$$\sigma_q(v_{i,1}, v_{i,2}) + \sigma_q(v_{i,1}v_{i,2}, v_{i,3})$$

$$= \sigma_q(c_{i,1}, c_{i,2}) - \sum_{j=1}^{3}\sigma_q(p_{i,j}, c_{i,j}) - \sum_{j=1}^{3}\sigma_q(p_{i,j}c_{i,j}, p_{i,j+1}^{-1})$$
Proof. Since \{\tilde{v}_{i,j}, \tilde{p}_{i,j}\} is a solution of \mathcal{W}_t. We have
\[
\begin{align*}
\begin{cases}
\tilde{p}_{i,1}c_{i,1}(\tilde{p}_{i,2})^{-1} = \tilde{v}_{i,1} \\
\tilde{p}_{i,2}c_{i,2}(\tilde{p}_{i,3})^{-1} = \tilde{v}_{i,2} \\
\tilde{p}_{i,3}c_{i,3}(\tilde{p}_{i,1})^{-1} = \tilde{v}_{i,3}
\end{cases}
\end{align*}
\]
Project these equations to \Gamma by \pi, we have
\[
\begin{align*}
\begin{cases}
\pi_{i,1}c_{i,1}(\pi_{i,2})^{-1} = \nu_{i,1} \\
\pi_{i,2}c_{i,2}(\pi_{i,3})^{-1} = \nu_{i,2} \\
\pi_{i,3}c_{i,3}(\pi_{i,1})^{-1} = \nu_{i,3}
\end{cases}
\end{align*}
\]
A direct computation using the cocycle condition of \sigma_q and the fact that \nu is symmetric gives the identity in the claim. \qed

By (1) and (2) of Lemma 4.3 we have
\[
\sum_{j=1}^{3} \sigma_q(\pi_{i,j},c_{i,j}) = \sum_{j=1}^{3} a_{i,j}
\]
and
\[
\sum_{j=1}^{3} \sigma_q(\pi_{i,j}c_{i,j},\pi_{i,j+1}^{-1}) = \sum_{j=1}^{3} b_{i,j}.
\]
Now Claim 2 and the fact that \{\tilde{w}_{i,j}\} is a solution of \mathcal{W}_t tell us that the second component of (1) equals
\[
\begin{align*}
\sigma_q(\pi(c_{i,1}),\pi(c_{i,2})) - \sum_{j=1}^{3} (a_{i,j} + b_{i,j}) + \sum_{j=1}^{3} t_1(\tilde{w}_{i,j}) - \sum_{j=1}^{3} t_4(d_{i,j}) \\
= \sigma_q(\pi(c_{i,1}),\pi(c_{i,2})) - \sum_{j=1}^{3} (a_{i,j} + b_{i,j} + t_4(d_{i,j})) + \sum_{j=1}^{3} t_1(\tilde{w}_{i,j}) = 0 \quad (*)
\end{align*}
\]
At this point, we have shown that \tilde{c}_{i,1}\tilde{c}_{i,2}\tilde{c}_{i,3} = 1 in \mathbb{E}'

Claim 2. \tilde{e}_{i,j} = q(\pi(\tilde{v}_{i,j}))i(\nu_{i,j} - t_4(d_{i,j})) is in \nu_2(\mathbb{E})

Proof. We use the \rho'-coordinate for \mathbb{E}'. Note that an element \((g,a)_{\rho'} \in \mathbb{E}'\) is in \nu_2(\mathbb{E}) if and only if \(a \in \nu_1(A)\). By the definition of the symmetric section \nu, we have
\[
\tilde{e}_{i,j} = \left(\pi(\tilde{v}_{i,j}), -t_3(\sigma_{\rho'(\pi(\tilde{v}_{i,j}),\pi(\tilde{v}_{i,j}^{-1}))) + t_1(\tilde{w}_{i,j}) - t_4(d_{i,j}))\right)_{\rho'}.
\]
Hence it is enough to show that \(-t_3(\sigma_{\rho'(\pi(\tilde{v}_{i,j}),\pi(\tilde{v}_{i,j}^{-1}))) = t_4(d_{i,j}) \in \nu_1(A)\). But this follows Lemma 3.20 since we know that
\[
Pa\left(\sigma_{\rho'(\pi(\tilde{v}_{i,j}),\pi(\tilde{v}_{i,j}^{-1})))\right) = d_{i,j}.
\]
by (3) of Lemma 4.3 The proof of the claim is complete. \qed
Therefore we know that $\mathcal{E}$ has a solution in $\iota_2(E)$. Note that $\iota_2(E)$ is isomorphic to $E$. So $\mathcal{E}$ has a solution in $E$. We have completed the proof of the “if” part of the theorem at this point.

Now suppose $\mathcal{E}$ has a solution in $E$. Then $\mathcal{E}$ (with constants replaced by their images under $\iota_2$) has a solution in $\iota_2(E) \subset E'$. Let $\{\tilde{e}_{i,j}\}$ be such a solution. We will show that one of the $\mathcal{E}_t$ we constructed also has a solution.

First note that $\{p(\tilde{e}_{i,j})\}$ is a solution of $\mathcal{E}$ (with the constants replaced by their $p$ images) in $\Gamma$. Then by Proposition 2.4 for some $(c_{i,j})$ satisfying Condition 1, the tripod equation system

$$\mathcal{V}_t^1 = \begin{cases} p_{i,1}c_{i,1}(p_{i,2})^{-1} = v_{i,1} \\ p_{i,2}c_{i,2}(p_{i,3})^{-1} = v_{i,2} \\ p_{i,3}c_{i,3}(p_{i,1})^{-1} = v_{i,3} \end{cases}$$

has a solution $\{\tilde{p}_{i,j}, \tilde{v}_{i,j}\}$ in $V$ such that $\pi(\tilde{v}_{i,j}) = p(\tilde{e}_{i,j})$.

By Proposition 2.4 we know that $\tilde{p}_{i,j}, \tilde{v}_{i,j}$ are $(\lambda_1, \nu_1)$-quasi geodesics in $K$. Hence by Lemma 3.1 there exists $p'_{i,j}, v'_{i,j} \in Y^*$ representing $\tilde{p}_{i,j}, \tilde{v}_{i,j}$ such that $\phi(p'_{i,j}), \phi(v'_{i,j})$ are $(\lambda, \nu)$-quasi geodesics in $K_\Gamma$ and $p'_{i,j}$ is a subword of $v'_{i,j}$. Recall in the definition of the tripod equation system, $v_{i,j}$ and $v_{i',j'}$ are defined to be the same variable if $e_{i,j}$ and $e_{i',j'}$ are the same variable. However we don’t require $v'_{i,j}$ and $v'_{i',j'}$ to be the same $Y$-word even if $v_{i,j}$ and $v_{i',j'}$ are the same variable (hence $\tilde{v}_{i,j} = \tilde{v}_{i',j'}$).

Use the Future Predicting Automaton $M$ to read $\phi(p'_{i,j})$. Suppose it ends at the state $\bar{s}_{i,j}$. Note that $\bar{s}_{i,j}$ and $\phi(c_{i,j})$ are compatible since $\phi(v'_{i,j})$ is in $L$ and $\phi(p'_{i,j})\phi(c_{i,j})$ is a subword of $\phi(v'_{i,j})$. Hence $(\bar{s}_{i,j})$ satisfy Condition 2. With the above choice of $c_{i,j}$ and $\bar{s}_{i,j}$, let $\bar{s}'_{i,j}$ be the state where $\phi(c_{i,j})$ ends when read by $M_{\bar{s}_{i,j}}$.

Note that $\phi(p'_{i,j}c_{i,j}) \in L(\bar{s}'_{i,j})$. Also $\phi(p^{-1}_{i,j}p_{i,j+1}) \in L(\bar{s}_{i,j}, c_{i,j})$ because $\phi(p^{-1}_{i,j}c_{i,j}p_{i,j+1})$ is in $L$. Let $b_{i,j} = \sigma_q(\pi(\bar{s}_{i,j}, c_{i,j}), \pi(\bar{s}'_{i,j}))$. Then we have $b_{i,j} \in A(\bar{s}_{i,j}, c_{i,j}) = \{\sigma_q(\bar{s}_{i,j}, w) \mid w \in L(\bar{s}_{i,j}, c_{i,j})\}$. Therefore $(b_{i,j})$ satisfy Condition 3.

Use the Parity Predicting Automaton $D$ to read $\phi(v'_{i,j})$. Let $d_{i,j} \in \mathbb{Z}_2^2 \oplus \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_m}$ be the last component of the state where it ends.

The above choice of $((c_{i,j}), (\bar{s}_{i,j}), (b_{i,j}), (d_{i,j}))$ satisfies all conditions defining $\Theta$. Let $t = ((c_{i,j}), (\bar{s}_{i,j}), (b_{i,j}), (d_{i,j}))$.

Let $\mathcal{E}_t$ be the system of equations defined by the above $t$. Then it is clear from the construction of $\mathcal{E}_t$ that $\{\tilde{p}_{i,j}, \tilde{v}_{i,j}\}$ is a solution of it.
Let \( \tilde{e}_{i,j} \in A' = \mathbb{Z}^n \oplus \mathbb{Z}_{2d_1} \oplus \cdots \oplus \mathbb{Z}_{2d_m} \) be the second component of the \( \rho' \)-coordinates of \( \tilde{e}_{i,j} \). Since \( \tilde{e}_{i,j} \) lies in \( \iota_2(\mathbb{E}) \), we know that \( \tilde{e}_{i,j}^2 \in \iota_1(A) \).

By the above way of choosing \( d_{i,j} \) and (3) of Lemma 4.4 we have

\[
d_{i,j} = Pa \left( \sigma_\rho \left( \pi(\tilde{v}_{i,j}), \pi(\tilde{v}_{i,j}^{-1}) \right) \right).
\]

Therefore by Lemma 3.20 we know that \( \tilde{e}_{i,j}^2 + \iota_3 \left( \sigma_\rho \left( \pi(\tilde{v}_{i,j}), \pi(\tilde{v}_{i,j}^{-1}) \right) \right) + \iota_4(d_{i,j}) \in A' \) lies in \( \iota_1(A) \). Let \( \tilde{w}_{i,j} \in A \) be the unique element such that

\[
\iota_1(\tilde{w}_{i,j}) = \tilde{e}_{i,j}^2 + \iota_3 \left( \sigma_\rho \left( \pi(\tilde{v}_{i,j}), \pi(\tilde{v}_{i,j}^{-1}) \right) \right) + \iota_4(d_{i,j})
\]

Claim 3. \( \{ \tilde{w}_{i,j} \} \) is a solution of \( W_t \).

Proof. With all the notation above, we have

\[
\tilde{e}_{i,j} = q \left( \pi(\tilde{v}_{i,j}) \right)i \left( \iota_1(\tilde{w}_{i,j}) - d_{i,j} \right)
\]

just as in the proof of the “if” part. But this time, we know that \( \{ \tilde{e}_{i,j} \} \) is a solution of \( E' \) instead of \( \{ \tilde{w}_{i,j} \} \) being a solution of \( W_t \) and everything else is the same. So we can go through the same calculation and when we reach (*), we use the fact that \( \tilde{e}_{i,1} \tilde{e}_{i,2} \tilde{e}_{i,3} = 1 \) to conclude that (*) holds. Therefore \( \{ \tilde{w}_{i,j} \} \) is a solution of \( W_t \) over \( A \). \( \square \)

The proof of the Theorem 4.5 is complete. \( \square \)

Theorem 1.1 follows from the Theorem 4.5 since equation systems with rational constraints in virtually free groups are solvable by the work of Dahmani-Guirardel [3] and equation systems in finitely generated abelian groups can be solved by using linear algebra.

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