A Limiting Absorption Principle for the three-dimensional Schrödinger equation with $L^p$ potentials

M. Goldberg, W. Schlag

1 Introduction

Agmon’s fundamental work [Agm] establishes the bound, known as the limiting absorption principle,

$$\sup_{\lambda > \lambda_0, \varepsilon > 0} \| (− \Delta + V − (\lambda^2 + i\varepsilon))^{-1} \|_{L^{2,\sigma}(\mathbb{R}^d) \to L^{2,-\sigma}(\mathbb{R}^d)} < \infty$$

provided that $\lambda_0 > 0$, $(1 + |x|)^{1+}|V(x)| \in L^\infty$ and $\sigma > \frac{1}{2}$. Here

$$L^{2,\sigma}(\mathbb{R}^d) = \{ (1 + |x|)^{-\sigma} f : f \in L^2(\mathbb{R}^d) \}$$

is the usual weighted $L^2$. The bound (1) is obtained from the same estimate for $V = 0$ by means of the resolvent identity. This bound for the free resolvent is related to the so called trace lemma, which refers to the statement that for every $f \in L^{2,\frac{d}{2}+}$ there is a restriction of $\hat{f}$ to any (compact) hypersurface, and this restriction belongs to $L^2$ relative to surface measure. Note that this fact does not require any curvature properties of the hypersurface - in fact, it is proved by reduction to flat surfaces. Another fundamental restriction theorem is the Stein-Tomas theorem, see [Ste]. It requires the hypersurfaces $S \subset \mathbb{R}^d$ with $d \geq 2$ to have non vanishing Gaussian curvature, and states that

$$\int_S |\hat{f}(\omega)|^2 \sigma(d\omega) \leq C\|f\|^2_{L^p(\mathbb{R}^d)} \text{ where } p = \frac{2d + 2}{d + 3}.$$ 

It is not hard to see that the related estimate for the free resolvent in $\mathbb{R}^3$ is given by

$$\|R_0(\lambda^2 + i0)\|_{\frac{4}{5} \to 4} = C \lambda^{-\frac{2}{3}} \text{ for } \lambda > 0.$$ 

This fact depends on the oscillation in the resolvent, i.e., on the exponential in

$$R_0(\lambda^2 + i0)(x, y) = \frac{e^{i\lambda|x−y|}}{4\pi|x−y|}.$$ 

In contrast, using the denominator alone one obtains that

$$\sup_{\lambda} \|R_0(\lambda^2 + i0)\|_{\frac{6}{5} \to 6} \leq C$$

via fractional integration. In analogy with Agmon’s work, it is natural to ask for which potentials (3) can be extended to the perturbed operators $H = −\Delta + V$. In this paper we show that this is the case for real-valued $V \in L^p(\mathbb{R}^3) \cap L^{\frac{d}{2}}(\mathbb{R}^3), p > \frac{3}{2}$, and suggest two possible extensions.
Theorem 1. Let $V \in L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3), p > \frac{3}{2}$ be real-valued. Then for every $\lambda_0 > 0$, one has

\begin{equation}
\sup_{0 < \varepsilon < 1, \lambda \geq \lambda_0} \left\| (-\Delta + V - (\lambda^2 + i\varepsilon))^{-1} \right\|_{\frac{1}{2}, \infty} \leq C(\lambda_0, V) \lambda^{-\frac{1}{2}}.
\end{equation}

In particular, the spectrum of $-\Delta + V$ is purely absolutely continuous on $(0, \infty)$.

We also formulate dynamical consequences of this result, in particular the existence and completeness of the wave operators. This theorem is the analogue of the classical Kato-Agmon-Kuroda theorem, see [ReesSim], Theorem XIII.33. It of course requires the absence of imbedded eigenvalues. In the classical context one uses Kato’s theorem for that purpose. Here we wish to use a result on the absence of imbedded eigenvalues that only requires an integrability condition on $V$. One such result was obtained by Ionescu and Jerison [IonJer], namely:

Theorem 2. Let $V \in L^2(\mathbb{R}^3)$. Suppose $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^3)$ satisfies $(-\Delta + V)u = \lambda^2 u$ where $\lambda \neq 0$ in the sense of distributions. If, moreover, $\|(1 + |x|)^{\frac{1}{2} - \frac{4}{p}} u\|_2 < \infty$ for some $\delta > 0$, then $u \equiv 0$.

The weighted $L^2$-condition with $\delta > 0$ is natural in view of the Fourier transform of the surface measure of $S^2$, which is a generalized eigenfunction of the free case and decays like $(1 + |x|)^{-1}$. As far as local regularity of the potential is concerned, the requirement that $V \in L^{3/2}_{\text{loc}}$ is essentially optimal. There exist examples of $V \in L^{3/2}_{\text{weak}}$ for which $-\Delta + V$ admits compactly supported eigenfunctions [KoTa]. The necessary decay condition on $V$ is less clearly delineated: Ionescu and Jerison found a smooth real-valued potential $V$ which lies in $L^q(\mathbb{R}^3)$ for all $q > 2$ but such that for $-\Delta + V$ imbedded eigenvalues exist. Their example decays like $r^{-1}$ in some directions, and like $r^{-2}$ in other directions. They further conjectured that their main result (Theorem 2.1 in [IonJer]) remains valid for potentials $V \in L^{3/2}(\mathbb{R}^3)$. Recent work by Koch and Tataru appears to verify this conjecture [KoTa2], and further refinements which allow potentials to exhibit both $L^{3/2}_{\text{loc}}$ singularities and $L^2$ decay seem possible as well. The proof of any such conjecture would immediately increase the scope of Theorem 1 as described below.

Proposition 3. The following inferences are valid:
1. If the conclusion of Theorem 2 holds for all $V \in L^p(\mathbb{R}^3), \frac{3}{2} \leq p < 2$, as is suggested by [KoTa2], then the conclusion of Theorem 1 also holds for all $V \in L^p(\mathbb{R}^3)$.
2. More generally, if the conclusion of Theorem 2 holds for some $V \in L^p(\mathbb{R}^3) + L^q(\mathbb{R}^3), \frac{3}{2} < p, q < 2$, then the conclusion of Theorem 1 also holds for this $V$.

By Kato’s theory of $H$-smoothing operators, see [Kat], it is well-known that the limiting absorption principle for the resolvent gives rise to estimates for the evolution $e^{itH}$ known as smoothing estimates. This is a much studied class of bounds, see [Sjo], [Veg], [ConSan1], [ConSan2], [BenK], [Doi], [Sim]. In fact, the Fourier transform establishes a link between the resolvent and the evolution that in a precise sense allows one to state that a certain class of estimates on the evolution is equivalent to corresponding ones for the resolvent, see [Kat]. In the free case, the $\frac{4}{3} \rightarrow 4$ bound for the resolvent corresponds to the following smoothing bound for the free evolution:

\[ \sup_{\|F\|_4 \leq 1} \int_{-\infty}^{\infty} \left\| F(-\Delta)^{\frac{1}{2}} e^{it\Delta} f \right\|_2 \leq C \|f\|_2. \]
However, this bound is known, see the work of Ruiz and Vega [RuizVeg]. For the perturbed evolution, $H = -\Delta + V$, one can prove similar estimates by means of Theorem [1] but we do not pursue this here. See the work of Ionescu and the second author [IonSch] for statements of this type.

This paper is organized as follows: In Section 2 we prove the bounds on the free resolvent that are needed in order to prove Theorem 1. Our main new bounds involve $R_0(\lambda^2 + i0)$ acting on functions whose Fourier transform vanish on $\lambda S^2$. In Section 3 we apply these bounds in the context of the usual resolvent identity/Fredholm alternative type arguments to deal with $-\Delta + V$. This of course requires Theorem 2. Finally, in Section 4 we return to the free resolvent and prove some end point results.

2 The free resolvent

This section develops some estimates on the free resolvent given by (4). These estimates are motivated on the one hand by the Stein-Tomas theorem (2), and on the other hand, by the applications to the perturbed operator $H = -\Delta + V$, see Theorem 2. For what follows, it will be helpful to keep in mind that for real $\lambda$,

$$[R_0(\lambda^2 + i0) - R_0(\lambda^2 - i0)]f = C(\lambda) \cdot (\sigma_{\lambda S^2} * f),$$

which is exactly of the form $T^*T$, $T$ being the restriction operator to the sphere $\lambda S^2$. Thus $T^*T : L_4^p(\mathbb{R}^3) \to L_4^q(\mathbb{R}^3)$ in view of (2).

We will denote by $\mathbb{H}$ the closed upper half-plane in $\mathbb{C}$, and state most of our results for $\lambda \in \mathbb{H}$. For any positive real number $\lambda$, we have the boundary identities

$$(\lambda + i0)^2 = \lambda^2 + i0 \quad \text{and} \quad (-\lambda + i0)^2 = \lambda^2 - i0,$$

therefore estimates which hold uniformly out to $\partial \mathbb{H}$ are of particular importance.

**Lemma 4.** Let $\lambda \in \mathbb{H}$ be any nonzero element, and $p = \frac{4}{3}$. Then $R_0(\lambda^2) : L^p(\mathbb{R}^3) \to L^q(\mathbb{R}^3)$, with operator norm bounded by $|\lambda|^{-\frac{1}{2}}$.

As suggested above, the proof follows a complex-interpolation argument strongly reminiscent of the proof of (2). For full details see Theorem 2.3 in [KenRuiSog], which establishes this bound for a more general family of inverses of second-order differential operators.

**Lemma 5.** Let $\lambda \in \mathbb{H}$ be any nonzero element. For each pair of exponents $1 < p \leq \frac{4}{3}$, $3p \leq q \leq \frac{3p}{3-2p}$ there exist constants $C_{p,q} < \infty$ such that

$$\|R_0(\lambda^2) f\|_{L^q} \leq C_{p,q} |\lambda|^{3/p-3/q-2} \|f\|_{L^p}$$

For each exponent $\frac{4}{3} \leq p < \frac{3}{2}$, $\frac{p}{3-2p} \leq q \leq \frac{3p}{3-2p}$ there exist constants $C_{p,q} < \infty$ such that

$$\|R_0(\lambda^2) f\|_{L^q^*} \leq C_{p,q} |\lambda|^{3/p-3/q-2} \|f\|_{L^p}$$

**Proof.** The case $p = \frac{4}{3}, q = 4$ is Lemma 4 above. Since $R_0(\lambda^2)$ is realized as a convolution with a kernel satisfying $|K_\lambda(x)| \leq |4\pi x|^{-1}$, the cases $q = \frac{3p}{3-2p}, 1 < p < \frac{3}{2}$ are precisely the Hardy–Littlewood–Sobolev inequality. Note that the scaling exponent for $\lambda$ is zero for these pairs $(p,q)$.
All intermediate cases \((p,q)\) then follow by interpolation. At the endpoint \(p=1, q=3\), we see that \(R^\pm_0(\lambda^2)\) maps \(L^1(\mathbb{R}^3)\) to weak-\(L^3(\mathbb{R}^3)\) uniformly in \(\lambda\), by considering the norm
\[
\|f\|_{L^3_{\text{weak}}(\mathbb{R}^3)} = \sup_{A \subset \mathbb{R}^3, |A| < \infty} |A|^{-\frac{2}{3}} \int_A |f(x)| \, dx
\]
which is equivalent to the usual weak-\(L^3\) “norm” and satisfies a triangle inequality, see Lieb, Loss [LieLos].

Section 4.3 The cases \(q = 3p\), \(1 < p < \frac{4}{3}\) follow by Marcinkiewicz interpolation, and \(q = \frac{p}{p-2p}\), \(\frac{4}{3} < p < \frac{3}{2}\) by duality.

The following results deal with functions whose Fourier transform vanishes on \(S^2\). The first lemma yields a Hölder bound for the \(L^2\) norms of the restrictions to spheres close to \(S^2\).

**Lemma 6.** Let \(1 \leq p < \frac{4}{3}\) and set \(\gamma = \frac{2}{p} - \frac{2}{3}\). Then for all \(|\delta| < \frac{1}{2}\) one has
\[
\|\hat{f}((1+\delta)\cdot)\|_{L^2(S^2)} \lesssim |\delta|^\gamma \|f\|_{L^p(\mathbb{R}^3)}
\]
for all \(f \in L^p(\mathbb{R}^3)\) with \(\hat{f} = 0\) on \(S^2\).

**Proof.** Let \(\sigma_{(1+\delta)S^2}\) be the normalized measure on \((1+\delta)S^2\). Then one has
\[
\|\hat{f}((1+\delta)\cdot)\|_{L^2(S^2)}^2 = \langle f * \sigma_{(1+\delta)S^2}, f \rangle = \langle f * [\sigma_{(1+\delta)S^2} - \sigma_{S^2}], f \rangle
\]
\[
= \sum_{j=0}^\infty \langle f * K_j, f \rangle
\]
where \(K_j(x) = (\sigma_{(1+\delta)S^2} - \sigma_{S^2}) \chi_j\) and \(\{\chi_j\}_{j \geq 0}\) are a standard dyadic partition of unity. Since \(\|\sigma_{(1+\delta)S^2} - \sigma_{S^2}\|_{\infty} \lesssim \delta\), it follows that
\[
\|K_j\|_{\infty} \lesssim \begin{cases} \delta & \text{if } 2^j < \delta^{-1} \\ 2^{-j} & \text{if } 2^j \geq \delta^{-1} \end{cases}
\]
Thus \(\|K_j\|_{\infty} \lesssim \min(\delta, 2^{-j}) := \alpha_j\). Moreover,
\[
\|\hat{K}_j\|_{\infty} = \|([\sigma_{(1+\delta)S^2} - \sigma_{S^2}] \ast \widehat{\chi}_j)\|_{\infty}
\]
\[
= \left| \int \widehat{\chi}_j(\xi - \eta) \sigma_{(1+\delta)S^2}(d\eta) - \int \widehat{\chi}_j(\xi - \eta) \sigma_{S^2}(d\eta) \right|
\]
\[
= \left| \int \left[ \widehat{\chi}_j(\xi - \eta) - \widehat{\chi}_j(\xi - \eta) \right] \sigma_{S^2}(d\eta) \right|
\]
\[
\lesssim \min(2^{2j} \delta, 2^j) := \beta_j.
\]
If \(1 < p < \frac{4}{3}\), let \(\frac{1}{p} = \frac{\theta}{p} + \frac{1-\theta}{2}\) so that \(\theta > \frac{1}{2}\). Then \(\|K_j * f\|_{p'} \lesssim \alpha_j^{\theta} \beta_j^{1-\theta} \|f\|_p\) for all \(j \geq 0\). Summing over \(j\) yields the desired bound. In the case \(p=1\), the estimate \(\|\sigma_{(1+\delta)S^2} - \sigma_{S^2}\|_{\infty} \lesssim \delta\) mentioned above suffices to show that \(\|\hat{f}((1+\delta)\cdot)\|_{L^2(S^2)} \lesssim \delta^{\frac{1}{2}}\).

\[\square\]
Then (9) supp(\(\hat{\chi}\)) \(\subset \{\xi \in \mathbb{R}^3 : \frac{1}{2} < |\xi| < 2\}\). Let \(\chi\) be a smooth, radial, bump function around zero so that \(\hat{\chi}\) is compactly supported. Let \(R \gg 1\). Then

\[
\|\chi(\frac{\cdot}{R})R_0(1 + i\varepsilon)f\|_2^2 = R^6 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \hat{\chi}(R(\xi - \eta)) \frac{\hat{f}(\eta)}{|\eta|^2 - 1 - i\varepsilon} \, d\eta \int_{\mathbb{R}^3} \hat{\chi}(R(\xi - \eta)) \frac{\overline{\hat{f}(\eta)}}{|\eta|^2 - 1 + i\varepsilon} \, d\eta
\]

(10) \[
= R^3 \int_{\mathbb{R}^6} \rho(R(\eta - \tilde{\eta})) \frac{\hat{f}(\eta)}{|\eta|^2 - 1 - i\varepsilon} \frac{\overline{\hat{f}(\eta)}}{|\tilde{\eta}|^2 - 1 + i\varepsilon} \, d\eta \, d\tilde{\eta},
\]

where we have set

\[
\int_{\mathbb{R}^3} \hat{\chi}(R(\xi - \eta)) \hat{\chi}(R(\xi - \eta)) \, d\xi = R^{-3} \int_{\mathbb{R}^3} \hat{\chi}(\xi - R\eta) \hat{\chi}(\xi - R\tilde{\eta}) \, d\xi
\]

\[
= R^{-3} \int_{\mathbb{R}^3} \hat{\chi}(\xi - R(\eta - \tilde{\eta})) \hat{\chi}(\xi) \, d\xi
\]

\[
=: R^{-3} \rho(R(\eta - \tilde{\eta})).
\]

Note that \(\rho\) is a compactly supported smooth bump function. Introducing polar coordinates in (10) yields uniformly in \(\varepsilon \neq 0\) (recall (9))

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^6} \int_{0}^{\infty} \int_{S^2} \rho(R(\eta - \tilde{\omega})) \frac{\hat{f}(\eta)}{|\eta|^2 - 1 - i\varepsilon} \frac{\overline{\hat{f}(\tilde{\omega})}}{|\tilde{\omega}|^2 - 1 + i\varepsilon} \, d\omega \, d\tilde{\omega} \, d\eta \, d\varepsilon
\]

\[
\lesssim R^3 \int_{\mathbb{R}^3} \int_{|\eta| - R^{-1}}^{R^{-1} + |\eta| - R^{-1}} |\hat{f}(\eta)| |\overline{\hat{f}(\tilde{\omega})}| \, d\omega \, d\tilde{\omega} \, d\eta \, d\varepsilon
\]

\[
\lesssim R^2 \int_{\mathbb{R}^3} |\hat{f}(\eta)| \int_{|\eta| - R^{-1}}^{R^{-1} + |\eta| - R^{-1}} \left( \int_{|S^2(\omega - \frac{\eta}{|\eta|})| < R^{-1}} |\hat{f}(\tilde{\omega})|^2 \, d\omega \right)^{\frac{1}{2}} \, d\tilde{\omega} \, d\eta
\]

\[
\lesssim R^2 \int_{0}^{\infty} \frac{dr}{|r - 1|} \int_{|r - R^{-1}|}^{r + R^{-1}} \left( \int_{|S^2(\omega - \frac{\eta}{|\eta|})| < R^{-1}} |\hat{f}(\tilde{\omega})|^2 \, d\omega \right)^{\frac{1}{2}} \, d\tilde{\omega} \, d\eta
\]
and therefore also

\begin{align}
(10) & \leq R^2 \int_0^\infty \frac{dr}{|r-1|} \int_{r-R^{-1}}^{r+R^{-1}} \frac{d\bar{r}}{|\bar{r}-1|} \left( \int_{S^2} |\hat{f}(\bar{r} \omega)|^2 d\omega \right)^\frac{1}{2} \left( \int_{S^2} \int_{|\bar{\omega}|<R^{-1}} |\hat{f}(\bar{\lambda} \omega)|^2 d\bar{\omega} d\omega \right)^\frac{1}{2} \\
& \leq R \int_\frac{1}{2}^\infty \frac{dr}{|r-1|} \int_{r-R^{-1}}^{r+R^{-1}} \frac{d\bar{r}}{|\bar{r}-1|} |1-r|^\gamma |1-\bar{r}|^p |f|^2_p \\
& \leq R^{1-2\gamma} \|f\|_p^2 = R^2 \|f\|_p^2,
\end{align}

where the last two lines use (7). The lemma now follows by summing over dyadic $R$, at least provided (9) holds. Finally, if

$$\text{supp}(\hat{f}) \subset \{ \xi \in \mathbb{R}^3 : |\xi| \leq \frac{1}{2} \text{ or } |\xi| \geq 2 \},$$

then one notes that

$$\sup_{\varepsilon \neq 0} \|R_0(1 \pm i \varepsilon) f\|_2 \lesssim \|(1 - \Delta)^{-1} f\|_2 \lesssim \|f\|_p$$

by the Sobolev imbedding theorem provided $1 \leq p \leq 2$ and we are done. \(\square\)

In Section 3 we discuss further bounds on the free resolvent which are motivated by the previous proposition.

### 3 The perturbed resolvent

The goal of this section is to prove theorem 1. As in [Agm], the proof of Theorem 1 is based on the resolvent identity. This requires inverting the operator $I + R_0(\lambda^2 \pm i 0) V$ on $L^4(\mathbb{R}^3)$. First, we check that this is a compact perturbation of the identity.

**Lemma 8.** Let $V \in L^p(\mathbb{R}^3)$, $\frac{3}{2} \leq p \leq 2$. Then for any nonzero $\lambda \in \mathbb{H}$, the map $A(\lambda) := R_0(\lambda^2) V$ is a compact operator on $L^4(\mathbb{R}^3)$.

**Proof.** Firstly, note that in view of Lemma 3 and because of $V \in L^p$, $A(\lambda)$ is bounded $L^4 \to L^4$. Secondly, observe that we may assume that $V \in L^\infty$ with compact support. Indeed, replace $V$ with $V_n = V \chi_{|V|<n} \chi_{|x|<n}$. Then $\|V - V_n\|_p \to 0$ as $n \to \infty$ implies that $\|A(\lambda) - A_n\|_{4 \to 4} \to 0$ as $n \to \infty$.

If we can show that $A_n := R_0(\lambda^2) V_n$ are compact as operators $L^4 \to L^4$ for each $n$, it therefore follows that $A(\lambda)$ is also compact. So assume that $V$ is bounded, and supported in the ball $\{|x| < R\}$. Fix $\lambda$ and write $A = A(\lambda)$. We first claim that $A : L^4 \to W^{2,4}$. This follows from

$$(-\Delta + 1) A = (-\Delta - \lambda^2) A + (\lambda^2 + 1) A = V + (1 + \lambda^2) A$$

is bounded from $L^4$ to $L^4$. Meanwhile, for $|x| > 2R$ there is the uniform pointwise bound

$$|A f(x)| \lesssim \|V f\|_1 |x|^{-1} \lesssim R^2 \|V\|_\infty \|f\|_4 |x|^{-1}$$

Given $\varepsilon > 0$, we may choose $R_0 \sim R^0 \|V\|_\infty^4 \varepsilon^{-4}$ so that $\|\chi_{|x|>R_0} A f\|_4 < \varepsilon$ for all $\|f\|_4 \leq 1$. 


Let \( \{f_j\}_{j=1}^\infty \subset L^1(\mathbb{R}^3) \) satisfy \( f_j \rightharpoonup 0 \) in \( L^1 \). Since \( \sup_j \|Af_j\|_{W^{2,4}(\mathbb{R}^3)} < \infty \), Rellich’s compactness theorem produces a subsequence \( f_{j_k} \) so that \( Af_{j_k} \rightharpoonup 0 \) in \( L^4(|x| < R_0) \). Thus

\[
\limsup_{k \to \infty} \|Af_{j_k}\|_4 \leq (1 + C_\lambda) \varepsilon.
\]

Sending \( \varepsilon \to 0 \) and passing to the diagonal subsequence finishes the proof. \( \square \)

The following lemma establishes invertibility everywhere except on the imaginary axis.

**Lemma 9.** Let \( V \in L^p(\mathbb{R}^3) \cap L^{2,4}(\mathbb{R}^3) \), \( \frac{2}{3} < p < 2 \) and assume that \( V \) is real-valued. Then for any nonzero \( \lambda \in \mathbb{H} \), the inverse \( (I + R_0(\lambda^2)V)^{-1} : L^4(\mathbb{R}^3) \to L^4(\mathbb{R}^3) \) exists.

**Proof.** By the previous lemma it suffices to show that

\[
f \in L^4(\mathbb{R}^3), \quad f + R_0(\lambda^2)Vf = 0 \implies f = 0.
\]

Let \( f \) be as on the left-hand side and set \( g = Vf \). Then \( g \in L^r \), where \( r = \frac{4p}{4 + p} < \frac{4}{3} \). By Lemma 6, \( f = -R_0(\lambda^2)g \) therefore belongs to \( L^q \cap L^4 \), where \( \frac{1}{q} - \frac{1}{4} = \frac{3 - 2p}{2 p} > 0 \).

This bootstrapping procedure can be repeated until it is shown that \( f \in L^{r'} \cap L^4 \). In fact, one can continue to the point where \( f \in L^\infty \), since \( R_0(\lambda^2) : L^{\frac{4}{2} - \varepsilon} \cap L^{\frac{4}{2} + \varepsilon} \to L^\infty \) is a bounded operator. What is important here is that \( f \) and \( g \) exist in spaces dual to each other.

Since \( V \) is real-valued, the duality pairing

\[
\langle f, g \rangle = \langle f, Vf \rangle = -\langle R_0(\lambda^2)g, g \rangle
\]

shows that \( \langle R_0(\lambda^2 \pm i0)g, g \rangle \) is real-valued. If \( \lambda^2 \not\in \mathbb{R} \), then the condition

\[
\Re \langle R_0(\lambda^2)g, g \rangle = \int_{\mathbb{R}^3} \left[ \frac{\Re(\lambda^2)}{(|\xi|^2 - \Re(\lambda^2))^2 + \Re(\lambda^2)^2} |\hat{g}(\xi)|^2 \right] d\xi = 0
\]

requires that \( \hat{g} = 0 \) almost everywhere.

On the boundary \( \lambda \in \mathbb{R} \), by the Stein-Tomas theorem

\[
\Re \langle R_0(\lambda \pm i0)^2g, g \rangle = \lim_{\varepsilon \to 0} \Re \langle R_0((\lambda + i\varepsilon)^2)g, g \rangle = c\lambda \int_{S^2} |\hat{g}(\lambda\omega)|^2 \sigma(d\omega)
\]

with some constant \( c \neq 0 \). Hence, \( \hat{g} = 0 \) on \( |\lambda|S^2 \) in the \( L^2 \) sense. Since \( g \in L^{r'}(\mathbb{R}^3) \), one concludes from Proposition 7 above that \( (1 + |x|)^{\frac{\delta - 4}{2}} R_0(\lambda^2 \pm i0)g \in L^2(\mathbb{R}^3) \) for some \( \delta > 0 \). Hence also \( (1 + |x|)^{\frac{\delta - 4}{2}} f \in L^2(\mathbb{R}^3) \) for some \( \delta > 0 \). Since \( (-\Delta + V - \lambda^2)f = 0 \) in the distributional sense, and one checks easily from Lemma 11 (remembering that \( f \in L^\infty \cap L^4 \)) that also \( f \in W^{1,2}_p(\mathbb{R}^3) \subset W^{1,2}_{\text{loc}}(\mathbb{R}^3) \), Theorem 2 implies that \( f = 0 \), as claimed. \( \square \)

The following two lemmas show that the inverses in the previous lemma have uniformly bounded norms.

**Lemma 10.** Let \( V \in L^p(\mathbb{R}^3), \frac{2}{3} \leq p \leq 2 \). The map \( \lambda \mapsto R_0(\lambda^2)V \) is continuous from the domain \( \mathbb{H} \setminus \{0\} \subset \mathbb{C} \) to the space of bounded operators on \( L^4(\mathbb{R}^3) \).
Then for any pair \( \lambda, \zeta \in \mathbb{H} \), \( |\lambda - \zeta| \leq \frac{1}{2\pi} \), we have

\[
|K(x)| \lesssim \begin{cases} 
|\lambda - \zeta|, & \text{if } |x| < |\lambda - \zeta|^{-1} \\
|x|^{-1}, & \text{if } |x| \geq |\lambda - \zeta|^{-1}
\end{cases}
\]

Thus \( \|(R_0(\lambda^2) - R_0(\zeta^2))Vf(x)\| \leq \begin{cases} 
|\lambda - \zeta||Vf|_1, & \text{if } |x| < |\lambda - \zeta|^{-1} \\
|x|^{-1}||Vf|_1, & \text{if } |x| \geq |\lambda - \zeta|^{-1}
\end{cases} \)

Approximate \( V \) by compactly supported \( \tilde{V} \in L^\infty \) so that \( \|V - \tilde{V}\|_p < \varepsilon \). By the above calculation, Lemma 5, and the simple identity

\[
\limsup_{\lambda \to \zeta} \|(R_0(\lambda^2) - R_0(\zeta^2))Vf\|_4 \leq |\lambda - \zeta|^{1/4}R(0)^{1/4}\|V\|_\infty\|f\|_4.
\]

Then for any \( \lambda, \zeta \in \mathbb{H} \), \( |\lambda - \zeta| \leq \frac{1}{2\pi} \), we have

\[
|K(x)| \lesssim \begin{cases} 
|\lambda - \zeta|, & \text{if } |x| < |\lambda - \zeta|^{-1} \\
|x|^{-1}, & \text{if } |x| \geq |\lambda - \zeta|^{-1}
\end{cases}
\]

Thus \( \|(R_0(\lambda^2) - R_0(\zeta^2))Vf\|_4 \leq |\lambda - \zeta|^{1/4}R(0)^{1/4}\|V\|_\infty\|f\|_4. \)

By the resolvent identity, for any \( \lambda \in \mathbb{C} \) such that \( \|R_0(\lambda^2)^{-1}\|_4 < \frac{1}{2} \), we have

\[
(R_0(\lambda^2) - R_0(\zeta^2)V = R_0(\lambda^2)(V - \tilde{V}) + (R_0(\lambda^2) - R_0(\zeta^2))\tilde{V} - R_0(\zeta^2)(V - \tilde{V}),
\]

we see that \( \limsup_{\zeta \to \lambda} \|(R_0(\lambda^2) - R_0(\zeta^2)V\|_4 \leq |\lambda|^{1/4}R(0)^{1/4}\|V\|_\infty\|f\|_4. \)

\[
\sup_{\|\mathcal{R}(\lambda)\| \geq \lambda_0} \left\|(I + R_0(\lambda^2)^{-1})\right\|_4 < \infty.
\]

Proof. In view of Lemma 5, there is some finite \( \lambda_1 \in \mathbb{R} \) so that \( \|R_0(\lambda^2)V\|_4 < \frac{1}{2} \) provided \( |\lambda| > \lambda_1 \). It therefore suffices to prove (12) on the compact set \( \{ \lambda \in \mathbb{C} : \lambda_0 \leq |\lambda| \leq \lambda_1, |\mathcal{R}(\lambda)| \geq \lambda_0 \} \). The previous two lemmas, however, show that \( (I + R_0(\lambda^2)^{-1}) \) is a continuous function of \( \lambda \) on this set, hence it is uniformly bounded from above.

It is now a simple matter to prove Theorem 1.

Proof of Theorem 1. By the resolvent identity, for any \( \varepsilon \neq 0 \),

\[
R_V(\lambda^2 + i\varepsilon) = R_0(\lambda^2 + i\varepsilon) - R_0(\lambda^2 + i\varepsilon)VR_V(\lambda^2 + i\varepsilon).
\]

By Lemma 11, one therefore has

\[
R_V(\lambda^2 + i\varepsilon) = (I + R_0(\lambda^2 + i\varepsilon)V)^{-1}R_0(\lambda^2 + i\varepsilon)
\]

and the right-hand side is uniformly bounded for \( \lambda \geq \lambda_0 \geq 0 \) as well as \( 0 < \varepsilon \leq 1 \) in the \( L^4 \) operator norm. In fact, the last factor contributes a decaying factor of \( \lambda^{-\frac{1}{2}} \) as \( L^4 \) operator norm in view of Lemma 4.

Proof of Proposition 4. There is only one point in the argument where the condition \( V \in L^\frac{3}{2}(\mathbb{R}^3) \) is used, namely the step in Lemma 3 where we wish to make use of Theorem 2. It otherwise suffices to assume that \( V \in L^p(\mathbb{R}^3), \frac{3}{2} < p < 2 \).

For the second claim, one observes the following consequence of Lemma 5. If \( V \in L^p, \frac{3}{2} < p \leq 2, \) and \( r > 4 \), then \( R_0(\lambda^2 \pm 10)V : L^4 \cap L^r \to L^4 \cap L^s, \) where \( \frac{1}{s} = \max\left(\frac{1}{r} + \frac{1}{p} - \frac{2}{3}, 0\right) \). The same is true for any \( V \in L^q, p \leq q \leq 2 \). This allows the bootstrapping procedure on \( \tilde{V} \) to continue normally, and furthermore \( V = V\tilde{f} \) is still an element of \( L^\frac{3}{2} - \varepsilon \), as desired. Therefore, the only matter of concern is whether the conclusion of Theorem 2 will hold for such a potential \( V \).
4 Further estimates on the free resolvent

Returning to Proposition 7, we note that a sharper estimate can be made at the endpoint \( p = 1 \).

**Proposition 12.** Let \( f \) be a function in \( L^1(\mathbb{R}^3) \) such that \( \hat{f} = 0 \) on the unit sphere \( S^2 \). Then

\[
\sup_{\varepsilon > 0} \| R_0(1 \pm i \varepsilon)f \|_2 \leq \frac{1}{\sqrt{8\pi}} \| f \|_1
\]

**Proof.** Define the trace function

\[
G(\lambda) = \lambda^{-2} \| \hat{f}|_{S^2} \|_2^2 = 4\pi \int_{\mathbb{R}^6} f(x) \frac{\sin(\lambda|x-y|)}{\lambda|x-y|} \bar{\hat{f}}(y) \, dx \, dy
\]

By inspection,

\[
G(\lambda) = 2\pi \int_{\mathbb{R}^3} \int_{\mathbb{R}^6} f(x) \bar{\hat{f}}(y) \chi_{|x-y|}(\tau) e^{i\lambda \tau} \, d\tau \, dx \, dy
\]

where \( \chi_{|x-y|} \) denotes the characteristic function of the interval \( \{|\tau| \leq |x-y|\} \). The integrand on the right-hand side is in \( L^1(\mathbb{R}^7) \), so Fubini’s Theorem implies that \( G \) is the inverse Fourier transform of an \( L^1 \) function.

Using the Plancherel identity (in 3 dimensions), and noting that \( G \) is an even function,

\[
\| R_0(1 \pm i \varepsilon)f \|_2^2 = \frac{1}{2(2\pi)^3} \int_{-\infty}^{\infty} \lambda^2 \frac{\lambda^2}{|\lambda^2 - (1 + i \varepsilon)|^2} G(\lambda) \, d\lambda
\]

For any \( \varepsilon > 0 \), the multiplier \( M_\varepsilon(\lambda) = \frac{\lambda^2}{|\lambda^2 - (1 + i \varepsilon)|^2} \) is integrable, hence it has Fourier transform \( \hat{M}_\varepsilon \in L^\infty(d\tau) \). By Parseval’s formula, this time in one dimension,

\[
\| R_0(1 \pm i \varepsilon)f \|_2^2 = \frac{1}{2(2\pi)^4} \int_{\mathbb{R}} \hat{G}(\tau) \hat{M}_\varepsilon(-\tau) \, d\tau
\]

An explicit formula for \( \hat{M}_\varepsilon(\tau) \) can be obtained via residue integrals:

\[
\hat{M}_\varepsilon(\tau) = \frac{\pi}{2\varepsilon}(\sqrt{1 + i \varepsilon} e^{i|\tau|\sqrt{1+i\varepsilon}} + \sqrt{1 - i \varepsilon} e^{-i|\tau|\sqrt{1-i\varepsilon}})
\]

This, along with (15), can be immediately substituted back into equation (17).

\[
\| R_0(1 \pm i \varepsilon)f \|_2^2 = \frac{1}{8\pi \varepsilon} \int_{\mathbb{R}^6} \int_{0}^{\tau|\tau|} f(x) \bar{\hat{f}}(y) \left( e^{i|x-y|\sqrt{1+i\varepsilon}} + e^{-i|x-y|\sqrt{1-i\varepsilon}} \right) \, d\tau \, dx \, dy
\]

Boundedness of \( \hat{M}_\varepsilon \) enables us to continue applying Fubini’s theorem to the multiple integral. We have also simplified the expression by noting that \( M_\varepsilon \) is an even function. Recall definition (14) and
subtract \( \frac{1}{16\pi^2\varepsilon} G(1) \) from both sides of the equation.

\[
\| R_0(1 \pm i\varepsilon) f \|_2^2 - \frac{1}{16\pi^2\varepsilon} G(1)
= \frac{1}{8\pi i\varepsilon} \int_{\mathbb{R}^6} \frac{f(x)\overline{f(y)}}{|x-y|} \left( (e^{i|x-y|\sqrt{1+i\varepsilon}} - e^{-i|x-y|\sqrt{1-i\varepsilon}}) - (e^{-i|x-y|\sqrt{1+i\varepsilon}} - e^{i|x-y|\sqrt{1-i\varepsilon}}) \right) dx \, dy
= \frac{1}{8\pi i\varepsilon} \int_{\mathbb{R}^6} \frac{f(x)\overline{f(y)}}{|x-y|} K(|x-y|) \, dx \, dy
\]

where \(|K(|x-y|)| \leq \varepsilon|x-y|\). This leads to the conclusion

\[
\| R_0(1 \pm i\varepsilon) f \|_2^2 - \frac{1}{16\pi^2\varepsilon} G(1) \leq \frac{1}{8\pi} \| f \|_1^2.
\]

If \( f \) satisfies the hypothesis \( \hat{f}|_{S^2} = 0 \), then \( G(1) = 0 \).

**Corollary 13.** Let \( f \) be a function in \( L^1(\mathbb{R}^3) \) such that \( \hat{f} = 0 \) on the unit sphere \( S^2 \). Then

\[
\| R_0(1 \pm i0) f \|_2 \leq \frac{1}{\sqrt{8\pi}} \| f \|_1
\]

**Proof.** This follows immediately from (19) and monotone convergence.

The condition \( \hat{f} = 0 \) is crucial in Proposition 7. Indeed, recall that for \( f \in L^p(\mathbb{R}^3) \) real-valued with \( 1 \leq p \leq \frac{4}{3} \), one has

\[
\exists \| R_0(1 + i0) f = c (\hat{\sigma}_{S^2} \ast f)
\]

for some constant \( c \). This follows by writing \( R_0(1 + i\varepsilon) \) as a sum of its real and imaginary parts, as well as from the fact that the operation of restriction \( f \mapsto \hat{f}(r\cdot) \) is continuous in \( r > 0 \) as a map \( L^p(\mathbb{R}^3) \to L^2(S^2) \). However, it is clear that for any \( \delta > 0 \)

\[
\| (1 + |x|^\delta \hat{\sigma}_{S^2} \ast f) \|_2 = \infty
\]
even for smooth bump-functions \( f \) since the function inside the norm decays like \( (1 + |x|)^{\delta - \frac{3}{2}} \) which just fails to be \( L^2(\mathbb{R}^3) \). The following simple lemma shows, on the other hand, that \( \delta < 0 \) does lead to a finite norm in (21).

**Lemma 14.** For any \( R \geq 1 \) one has

\[
\left\| \chi_{|x|<R}|\hat{\sigma}_{S^2} \ast f| \right\|_2 \lesssim \sqrt{R} \| f \|_{\frac{4}{3}}
\]

for all \( f \in L_{\frac{4}{3}}(\mathbb{R}^3) \).

**Proof.** Let \( \phi \) be a smooth cut-off function with \( \hat{\phi} \) compactly supported. Then by Plancherel, and Cauchy-Schwartz,

\[
\left\| \chi_{|x|<R}|\hat{\sigma}_{S^2} \ast f| \right\|_2^2 = R^6 \int_{\mathbb{R}^3} \left| \int_{S^2} \hat{\chi}(R(\xi - \eta))\hat{f}(\eta) \sigma_{S^2}(d\eta) \right|^2 \, d\xi
\lesssim R^6 \int_{\mathbb{R}^3} \int_{S^2} |\hat{\chi}(R(\xi - \eta'))| \, d\eta' \int_{S^2} |\hat{\chi}(R(\xi - \eta))| |\hat{f}(\eta)|^2 \, \sigma_{S^2}(d\eta) \, d\xi
\lesssim R \| \hat{f} \|_{L^2(S^2)}^2 \lesssim R \| f \|_{\frac{4}{3}}^2,
\]
as claimed.
The previous lemma suggests that one should also have the bound
\[(22) \sup_{\varepsilon > 0} \left\| \chi_{|x|<R} R_0(1 \pm i\varepsilon) f \right\|_2 \leq \sqrt{R} \left\| f \right\|_4^{1/3}.\]

While this bound remains open\(^1\), it is easy to show that
\[(23) \sup_{\varepsilon > 0} \left\| \chi_{|x|<R} R_0(1 \pm i\varepsilon) f \right\|_2 \lesssim R^{3/4} \left\| f \right\|_4.\]

Indeed, denoting the operator on the left-hand side by \(T\) for a fixed \(\varepsilon > 0\), observe that by Lemma 4
\[T^* T = R_0 (1 - i\varepsilon) \chi_{|x|<R} R_0 (1 + i\varepsilon)\]
satisfies \(\|T^* T\|_{1/3 \rightarrow 4} \lesssim \|\chi_{|x|<R}\|_2 \lesssim R^{3/4}\), which is the same as (23). One would of course expect that the Knapp example determines the power of \(R\) in (22). Note that in the case of a Knapp example of dimensions \(\delta \times \sqrt{\delta} \times \sqrt{\delta}\) where \(\delta = R^{-1}\), one does not encounter the \(L^2\) norm of \(\|\chi_{|x|<R}\|_2\) in the previous \(T^* T\) argument, but rather the \(L^2\) norm of a \(R \times \sqrt{R} \times \sqrt{R}\)-tube, which gives the conjectured \(R\). Conversely, in what follows we show how to approach (22) by a decomposition into Knapp examples. This leads to an improvement over the simple bound (23) by \(\frac{1}{8}\). Such arguments originate in the analysis of Bochner Riesz multipliers as well as the restriction theory of the Fourier transform. We will use a square function bound from Bourgain [Bou1]. For the convenience of the reader, we reproduce the details.

The following lemma is a discrete version of the Stein-Tomas theorem, see Lemma 6.2 in [Bou1]. It can also be proved by expanding the \(L^4\)-norm on the left-hand side explicitly and then using the usual geometric arguments based on counting overlap. However, the following approach does not depend on any special arithmetic properties of the Stein-Tomas exponent and therefore generalizes to other dimensions as well.

**Lemma 15.** Let \(R \geq 1\) and let \(\{\xi_\alpha\}_\alpha \subset S^2\) be a collection of \(R^{-1/2}\)-separated points and \(\{a_\alpha\}_\alpha\) a sequence of arbitrary numbers. Then
\[(24) \left\| \sum_\alpha e^{ix \cdot \xi_\alpha} a_\alpha \right\|_{L^4(Q)} \lesssim R^{1/2} \left( \sum_\alpha |a_\alpha|^2 \right)^{1/2}\]
for any cube \(Q\) of side-length \(R^{1/2}\).

**Proof.** Fix \(R \geq 1\), some cube \(Q\) and a smooth cut-off function \(\chi_Q\) adapted to \(Q\). We assume that \(\text{supp}(\hat{\chi}_Q)\) is contained inside a \(CR^{-\frac{1}{2}}\)-cube with some big constant \(C\). Then
\[
\left\| \sum_\alpha e^{ix \cdot \xi_\alpha} a_\alpha \right\|_{L^4(Q)} \lesssim \left\| \left[ \sum_\alpha a_\alpha \hat{\chi}_Q (\cdot - \xi_\alpha) \right]^\vee \right\|_4
\lesssim \int_{1-R^{1/2}}^{1+R^{-1/2}} \left\| \int_{S^2} e^{i \omega \cdot x} \sum_\alpha a_\alpha \hat{\chi}_Q (r \omega - \xi_\alpha) \sigma (d\omega) \right\|_{L^4_x} dr
\lesssim R^{-1/2} R^{3/2} \left( \sum_\alpha R^{-1} |a_\alpha|^2 \right)^{1/2} = R^{1/2} \left( \sum_\alpha |a_\alpha|^2 \right)^{1/2},
\]
as claimed.\(\square\)

\(^1\)Note added in proof: This conjecture is solved in a forthcoming paper by Ionescu and the second author [IonSch], which also contains other improvements on the results obtained here.
In what follows, we associate with every $R \geq 1$ a decomposition $\{S^{(R)}_{\alpha}\}_{\alpha}$ of the shell $S_R := \{\xi \in \mathbb{R}^3 : |\xi| - 1 < R^{-1}\}$ into Knapp caps $S^{(R)}_{\alpha}$ of size about $R^{-\frac{1}{2}} \times R^{-\frac{1}{2}} \times R^{-1}$. We furthermore assume bounded overlap, i.e., that

$$\sup_{R \geq 1} \left\| \sum_{\alpha} \chi_{S^{(R)}_{\alpha}} \right\|_{\infty} < \infty.$$ 

**Lemma 16.** Let $g \in L^4(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ be so that $\hat{g} \subset S_R$. Write $g = \sum_{\alpha} g_{\alpha}$ where each $\hat{g}_{\alpha}$ is supported in $S^{(R)}_{\alpha}$. Then

$$\|g\|_4 \lesssim R^{\frac{1}{2}} \left\| \left( \sum_{\alpha} |g_{\alpha}|^2 \right)^{\frac{1}{2}} \right\|_4.$$ 

Dually, one has for any $f \in L^4_{\pi}(\mathbb{R}^3)$ that

$$\left\| \left( \sum_{\alpha} |f_{\alpha}|^2 \right)^{\frac{1}{2}} \right\|_{\frac{4}{\pi}} \lesssim R^\frac{1}{2} \|f\|_{\frac{4}{\pi}}$$

where $\sum_{\alpha} f_{\alpha} = \hat{f} \chi_{S_R}$ and each $f_{\alpha}$ is Fourier supported in $S^{(R)}_{\alpha}$.

**Proof.** Firstly, fix $R \geq 1$, a cube $Q$ of size $\sqrt{R}$, the cap decomposition $\{S^{(R)}_{\alpha}\}_{\alpha}$, as well as some $\xi_{\alpha} \in S^{(R)}_{\alpha}$ for each $\alpha$. Secondly, fix some smooth cut-off $\chi_0$ so that $\chi_0(\sqrt{R} y)$ is adapted to the cube $Q_0 := [-c_0 \sqrt{R}, c_0 \sqrt{R}]^3$ where $c_0 > 0$ is small. We require that $\chi_0(\sqrt{R}(\xi - \xi_{\alpha})) = 1$ for all $\xi \in S^{(R)}_{\alpha}$ and each $\alpha$. Set

$$\mu(\xi) := \chi_0(\sqrt{R} \xi) \left( \int_{Q_0} e^{ix \cdot \xi} dy \right)^{-1}$$

and $\mu_{\alpha}(\xi) = \mu(\xi - \xi_{\alpha})$ for every $\alpha$. Note that this is again a smooth cut-off function adapted to a ball of size $R^{-\frac{1}{2}}$ (together with the natural derivative bounds with constants uniform in $R$). In particular, $\sup_{\alpha} \|\hat{\mu}_{\alpha}\|_1 = \|\hat{\mu}\|_1 < \infty$ uniformly in $R$. Then

$$\|g\|_{L^4(Q)}^4 = \left\| \sum_{\alpha} \int e^{ix \cdot \xi} \hat{g}_{\alpha}(\xi) d\xi \right\|_{L^4(Q)}^4 \leq \int_Q \int_{Q_0} \left\| \sum_{\alpha} e^{-iy \cdot \xi_{\alpha}} \int e^{i(x+y) \cdot \xi} \hat{g}_{\alpha}(\xi) \mu_{\alpha}(\xi) d\xi \right\|_{L^4(Q)}^4 dy \, dx$$

(27)

$$\lesssim \int_{2Q} \int_{Q_0} \left\| \sum_{\alpha} e^{-iy \cdot \xi_{\alpha}} \int e^{ix \cdot \xi} \hat{g}_{\alpha}(\xi) \mu_{\alpha}(\xi) d\xi \right\|_{L^4(Q)}^4 dy \, dx$$

(28)

$$\lesssim \int_{2Q} \int_{Q_0} \left\| \sum_{\alpha} \left( \int e^{ix \cdot \xi} \hat{g}_{\alpha}(\xi) \mu_{\alpha}(\xi) d\xi \right)^2 \right\|_{L^4(Q)}^2 dx$$

(29)

$$\lesssim R^2 \left\| \left( \sum_{\alpha} |h_{\alpha}|^2 \right)^{\frac{1}{2}} \right\|_{L^4(2Q)}^4$$

(30)

where $h_{\alpha} = g_{\alpha} * k_{\alpha}$ and $\hat{\mu}_{\alpha} = \mu_{\alpha}$. Here (27) follows from Jensen's inequality as well as the definition of $\mu_{\alpha}$, (28) follows by changing variables (and $2Q$ is the cube with the same center but twice the size),
whereas (29) is a consequence of Lemma 15. Summing (30) over a partition \( \{Q\} \) of \( \mathbb{R}^3 \) consisting of congruent cubes one obtains that

\[
\|g\|_4 \lesssim R^{1/2} \left( \sum_{\alpha} |h_\alpha|^2 \right)^{1/2} \lesssim R^{1/2} \left( \sum_{\alpha} |g_\alpha|^2 \sup |k_\alpha| \right)^{1/2} \lesssim R^{1/2} \left( \sum_{\alpha} |g_\alpha|^2 \right)^{1/2},
\]

since \( \|\tilde{\mu}\|_1 \lesssim 1 \). This proves (25) and (26) follows by duality. Indeed,

\[
\sup_{\|g\|_4 \leq CR^{1/2}} |\langle f, g \rangle| \lesssim R^{1/2} \|f\|_4.
\]

In line (31), the supremum is taken over all \( g \) as in (25), whereas in (32), we drop the condition \( \text{supp}(\hat{g}) \subset S_R \).

The powers of \( R \) in (21), as well as in (25) and (26) are optimal. In the case of (24), this can be seen by taking all \( a_\alpha = 1 \) and similarly for the square function.

Recall that Agmon’s limiting absorption principle states that

\[
\sup_{\varepsilon > 0} \left\| (1 + |x|)^{-1/2} R_0 (1 \pm i\varepsilon) f \right\|_2 \lesssim \left\| (1 + |x|)^{1/2} f \right\|_2.
\]

This in particular implies that

\[
\sup_{\varepsilon > 0} \left\| \chi_{|x| < R_0} R_0 (1 \pm i\varepsilon) \chi_{|x| < R_0} f \right\|_2 \lesssim R^{1/2} \|f\|_2.
\]

The following lemma is an improved version of (34), see the decay in \(|v|\). The proof is self-contained. In particular, it does not rely on (33).

**Lemma 17.** For any \( v \in \mathbb{R}^3 \) one has

\[
\sup_{\varepsilon > 0} \left\| \chi_{|x| < R} R_0 (1 \pm i\varepsilon) \chi_{|x-v| < R} f \right\|_2 \lesssim R (1 + |v|/R)^{-1} \|f\|_2.
\]

for all \( R \geq 1 \). The constants are uniform in \( R \) and \( v \).

**Proof.** This is basically a simple consequence of Hörmander’s variable coefficient Plancherel theorem, see [Hör], and especially Wolff’s notes [Wol], page 55. We start with the case \(|v| \gg R\). Consider the operator

\[
T_{R,v} f(x) = \int_{\mathbb{R}^3} \psi(x) e^{iR|x-y|} |x-y| \psi(y-v/R) f(y) dy
\]

where \( \psi \) is a smooth bump function at zero. This operator is of the form

\[
T_{R,v} f(x) = \int_{\mathbb{R}^3} e^{iR \Phi(x,y)} a_v(x,y) f(y) dy
\]
where \( \Phi \) is smooth on the support of \( a_v(x, y) \), and \( \text{rank}[\partial_{xy}^2 \Phi(x, y)] = 2 \). Moreover, \( a_v(x, y) \) is smooth, the size of its support is uniformly bounded in \( v \), and \( \|\partial^3 a_v\|_\infty \lesssim (1 + |v|/R)^{-1} \). Hence, Hörmander’s variable coefficient Plancherel theorem implies that

\[
\|T_{R,v}\|_{2 \to 2} \lesssim R^{-1}(1 + |v|/R)^{-1}.
\]

One now checks that (35) follows from this by means of a change of variables.

If \( |v| \lesssim R \), one can argue similarly, but needs to introduce a Whitney decomposition away from the singularity \( x = y \). Firstly, note that with \( \psi \) as above,

\[
\left\| \int_{\mathbb{R}^3} \psi(x) e^{iR|x-y|/|x-y|} \chi_{[|x-y| \leq R^{-1/2}]} \psi(y) f(y) \, dy \right\|_2 \lesssim \int_{[|x| < R^{-1/2}]} \|f\|_2 \lesssim R^{-1} \|f\|_2.
\]

Next, let \( \rho \in C_0^\infty(\mathbb{R}) \) be a smooth function so that \( \rho(t) = 1 \) if \( 1 < t < 2 \) and \( \rho(t) = 0 \) if \( t > 2 \) or \( t < \frac{1}{2} \). Then we claim that

\[
\left\| \int_{\mathbb{R}^3} \psi(x) e^{iR|x-y|/|x-y|} \rho(2^{-j}|x-y|) \psi(y) f(y) \, dy \right\|_2 \lesssim 2^j R^{-1} \|f\|_2
\]

for all \( R^{-\frac{1}{2}} < 2^j \leq 1 \). To this end introduce a further decomposition \( 1 = \sum j \omega(2^{-j}(x - x_j)) \) where the sum runs over a lattice of points \( \{x_j\}_j \) in \( \mathbb{R}^3 \) that are \( 2^j \)-spaced, and \( \omega \) is some smooth cut-off which is adapted to the unit cube. Exploiting orthogonality, (36) follows from the following estimate

\[
\left\| \int_{\mathbb{R}^3} \omega(2^{-j}(x - x_j)) e^{iR|x-y|/|x-y|} \rho(2^{-j}|x-y|) \omega(2^{-j}(y - x_k)) f(y) \, dy \right\|_2 \lesssim 2^j R^{-1} \|f\|_2.
\]

This, however, is again reduced the Hörmander’s bound by means of an obvious rescaling. \( \square \)

We are now ready to formulate our estimate which lies between the conjecture (22) and the simple bound (25).

**Proposition 18.** Let \( f \in L^\frac{4}{3}(\mathbb{R}^3) \). Then there is the bound

\[
\sup_{\varepsilon > 0} \left\| \chi_{[|x| < R]} R_0(1 \pm i\varepsilon) f \right\|_2 \lesssim R^{\frac{k}{2}} \|f\|_{\frac{4}{3}}.
\]

for all \( R \geq 1 \).

**Proof.** If \( \text{supp}(\hat{f}) \subset \{ \xi \in \mathbb{R}^3 : ||\xi| - 1| > \frac{1}{2} \} \), then one has

\[
\sup_{\varepsilon > 0} \left\| \chi_{[|x| < R]} R_0(1 \pm i\varepsilon) f \right\|_2 \lesssim \|(-\Delta + 1)^{-1} f\|_2 \lesssim \|f\|_{\frac{4}{3}}.
\]

Hence we can assume that \( \text{supp}(\hat{f}) \subset \{ \xi \in \mathbb{R}^3 : ||\xi| - 1| \leq \frac{1}{2} \} \) and we write \( f = \sum f_k \), where \( \text{supp}(\hat{f}_k) \subset \{ \xi \in \mathbb{R}^3 : ||\xi| - 1| \geq 2^k R^{-1} \} \) for \( k \geq 1 \) and

\[
\text{supp}(\hat{f}_0) \subset \{ \xi \in \mathbb{R}^3 : ||\xi| - 1| \leq R^{-1} \}.
\]
Now $R_0(1 + i\varepsilon)f_0 = R_0(1 + i\varepsilon) \sum \alpha f_\alpha$ with $\sum \alpha f_\alpha$ as in Lemma 16. Consider any of the pieces $f_\alpha$. Then one can write $f_\alpha = \sum_j f_{\alpha,j}$ where each $f_{\alpha,j}$ lives on a tube of dimensions $R \times \sqrt{R} \times \sqrt{R}$ and the tubes corresponding to different $j$'s are disjoint. Moreover, the bound

$$\|f_{\alpha,j}\|_\infty \lesssim \|f_{\alpha,j}\|_p R^{-\frac{2}{p}}$$

holds for every $1 \leq p \leq \infty$. Therefore,

$$\|f_\alpha\|^2 \approx \sum_j \|f_{\alpha,j}\|^2 \lesssim \sum_j \|f_{\alpha,j}\|^2 R^2$$

$$\lesssim \sum_j \|f_{\alpha,j}\|^2 \frac{1}{R} R^{-1} \lesssim \left( \sum_j \|f_{\alpha,j}\|^\frac{3}{4} \right)^\frac{4}{3} R^{-1}$$

(38)

$$\lesssim \|f_\alpha\|^2 \frac{1}{4} R^{-1}.$$ We will apply this bound not to $f_\alpha$, but to $\psi((\cdot - v)/R)f_\alpha$ for a smooth cut-off $\psi$ and arbitrary $v$. This is justified since the Fourier support of $\psi((\cdot - v)/R)f_\alpha$ still lies in the cap $S_{\alpha}^{(R)}$. In combination with (35), (38) yields

$$\sup_{\varepsilon > 0} \|\chi_{|x| < R} R_0(1 + i\varepsilon)f_\alpha\|_2 \approx \sum_{\alpha \in \mathbb{R}^3} \sup_{\varepsilon > 0} \|\chi_{|x| < R} R_0(1 + i\varepsilon)\chi_{|x-v| < R} f_\alpha\|_2$$

$$\lesssim R \sum_{\varepsilon \in \mathbb{R}^3} (1 + |v|/R)^{-1} \|\psi((\cdot - v)/R)f_\alpha\|_2$$

$$\lesssim R \sum_{\varepsilon \in \mathbb{R}^3} (1 + |v|/R)^{-1} R^{-\frac{1}{2}} \|\psi((\cdot - v)/R)f_\alpha\|_{\frac{3}{4}}$$

(39)

$$\lesssim R^\frac{3}{4} \left( \sum_{j \in \mathbb{Z}^3} (1 + |j|)^{-4} \right)^\frac{1}{2} \left( \sum_{j \in \mathbb{Z}^3} \|\psi((\cdot - Rj)/R)f_\alpha\|_{\frac{3}{4}} \right)^\frac{4}{3} \lesssim R^{\frac{3}{4}} \|f_\alpha\|_{\frac{3}{4}}.$$ It remains to sum up (39) exploiting orthogonality as provided by Lemma 16.

$$\sup_{\varepsilon > 0} \|\chi_{|x| < R} R_0(1 + i\varepsilon)f_\alpha\|_2^2 \approx \sum_\alpha \sup_{\varepsilon > 0} \left\|\psi\left(\frac{\cdot}{R}\right) R_0(1 + i\varepsilon)f_\alpha\right\|_2^2$$

$$\lesssim R \sum_\alpha \|f_\alpha\|_{\frac{3}{4}}^2 \lesssim R^{\frac{3}{4}} \|f\|_{\frac{3}{4}}^2.$$ The final bound uses (26). It is easier to deal with $f_k$ where $k \geq 1$. Fix such a $k$ and let $f_k = \sum_\beta f_\beta$ be the corresponding decomposition into Knapp caps. Then, by (38),

$$\|f_\beta\|_2 \lesssim \|f_\beta\|_{\frac{3}{4}} (2^k R^{-1})^{\frac{1}{2}}$$

and instead of (39) one now has

$$\sup_{\varepsilon > 0} \|\chi_{|x| < R} R_0(1 + i\varepsilon)f_\beta\|_2 \approx (2^k R^{-1})^{-1} \|f_\beta\|_2 \lesssim (2^k R)^\frac{1}{2} \|f_\beta\|_{\frac{3}{4}}.$$
Invoking (26) finally allows one to conclude that
\[
\sup_{\varepsilon > 0} \left\| \chi_{|x| < R_0} R_0 (1 \pm i \varepsilon) f_k \right\|_2 \lesssim (2^{-k} R) \frac{2}{3} \| f \|_{\frac{4}{3}}.
\]
Summing over \( k \) finishes the proof. \( \square \)

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**References**

[Agu] Agmon, S. *Spectral properties of Schrödinger operators and scattering theory.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 2 (1975), no. 2, 151–218.

[BenKla] Ben-Artzi, M., Klainerman, S., *Decay and regularity for the Schrödinger equation.* J. Anal. Math. 58 (1992), 25–37.

[Bou1] Bourgain, J. *Besicovitch type maximal operators and applications to Fourier analysis.* Geom. Funct. Anal. 1 (1991), no. 2, 147–187.

[ConSau1] Constantin, P., Saut, J.-C. *Local smoothing properties of dispersive equations.* J. Amer. Math. Soc. 1 (1988), no. 2, 413–439.

[ConSau2] Constantin, P., Saut, J.-C. *Local smoothing properties of Schrödinger equations.* Indiana Univ. Math. J. 38 (1989), no. 3, 791–810

[CyFrKiSi] Cycon, H. L., Froese, R. G., Kirsch, W., Simon, B. *Schrödinger operators with application to quantum mechanics and global geometry.* Texts and Monographs in Physics. Springer Study Edition. Springer-Verlag, Berlin, 1987.

[Doi] Doi, S.-I. *Smoothing effects of Schrödinger evolution groups on Riemannian manifolds.* Duke Math. J. 82 (1996), no. 3, 679–706

[Hör] Hörmander, L. *Oscillatory integrals and multipliers on FLp.* Ark. Mat. 11, 1–11 (1973).

[IonJer] Ionescu, A., Jerison, D. *On the absence of positive eigenvalues of Schrödinger operators with rough potentials.* To appear in GAFA, preprint 2002.

[IonSch] Ionescu, A., Schlag, W. *Agmon-Kato-Kuroda theorems for a general class of potentials.* preprint, 2004.

[Kat] Kato, T. *Wave operators and similarity for some non-selfadjoint operators.* Math. Ann. 162 (1965/1966), 258–279.

[KatYaj] Kato, T., Yajima, K. *Some examples of smooth operators and the associated smoothing effect.* Rev. Math. Phys. 1 (1989), no. 4, 481–496.

[KeelTao] Keel, M., Tao, T. *Endpoint Strichartz estimates.* Amer. J. Math. 120 (1998), no. 5, 955–980.
[KenRuiSog] Kenig, C. E., Ruiz, A., Sogge, C. D. *Uniform Sobolev inequalities and unique continuation for second-order constant coefficient differential operators*. Duke Math. J. 55 (1987), 329–347.

[KoTa] Koch, H., Tataru, D., *Sharp counterexamples in unique continuation for second order elliptic equations*. J. Reine Angew. Math. 542 (2002), 133–146.

[KoTa2] Koch, H., Tataru, D., personal communication and forthcoming.

[LieLos] Lieb, E., Loss, M. *Analysis*. Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001.

[ReeSim] Reed, M., Simon, B. *Methods of modern mathematical physics. IV. Analysis of operators*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.

[RodSch] Rodnianski, I., Schlag, W. *Time decay for solutions of Schrödinger equations with rough and time-dependent potentials*. to appear in Invent. Math.

[RuiVeg] Ruiz, A., Vega, L., *On local regularity of Schrödinger equations*. Intl. Math. Res. Not. (1993), 13–27.

[Sim] Simon, B. *Best constants in some operator smoothness estimates*. J. Funct. Anal. 107 (1992), no. 1, 66–71.

[Sjo] Sjölin, P. *Regularity of solutions to Schrödinger equations*. Duke Math. J. 55 (1987), 699–715

[Ste] Stein, E. *Beijing lectures in harmonic analysis*. Proceedings of a summer symposium on analysis in China held at Beijing University, Beijing, September 1984. Edited by E. M. Stein. Annals of Mathematics Studies, 112. Princeton University Press, Princeton, NJ, 1986.

[Veg] Vega, L. *Schrödinger equations: Pointwise convergence to the initial data*. Proc. Amer. Math. Soc. 102 (1988), 874–878

[Wol] Wolff, Thomas H. *Lectures on harmonic analysis*. With a foreword by Charles Fefferman and preface by Izabella Laba. Edited by Laba and Carol Shubin. University Lecture Series, 29. American Mathematical Society, Providence, RI, 2003.

Division of Astronomy, Mathematics, and Physics, 253-37 Caltech, Pasadena, CA 91125, U.S.A.

email: mikeg@caltech.edu, schlag@caltech.edu