HARNACK INEQUALITY FOR THE FRACTIONAL NONLOCAL LINEARIZED MONGE–AMPÈRE EQUATION

DIEGO MALDONADO AND PABLO RAÚL STINGA

ABSTRACT. The fractional nonlocal linearized Monge–Ampère equation is introduced. A Harnack inequality for nonnegative solutions to the Poisson problem on Monge–Ampère sections is proved.

1. Introduction and main results

Throughout this paper we let \( \varphi \in C^3(\mathbb{R}^n) \) be a convex function with \( D^2 \varphi > 0 \) on \( \mathbb{R}^n \) and let \( \mu_\varphi \) denote its induced Monge–Ampère measure

\[ \mu_\varphi(x) := \det D^2 \varphi(x). \]

Associated to \( \varphi \) there are three, typically degenerate/singular, elliptic operators \( L_\varphi \), \( L_\varphi^c \), and \( L_\varphi^v \) defined as

\[ L_\varphi v := -\text{trace}((D^2 \varphi)^{-1} D^2 v), \]

\[ L_\varphi^c v := -\text{trace}(A_\varphi(x) D^2 v), \]

\[ L_\varphi^v v := -\text{div}(A_\varphi(x) \nabla v), \]

where \( A_\varphi(x) \) stands for the matrix of cofactors of \( D^2 \varphi(x) \), that is,

\[ A_\varphi(x) := \mu_\varphi(x)(D^2 \varphi(x))^{-1}. \]

From the fact that the columns of \( A_\varphi(x) \) are divergence-free, it follows that

\[ L_\varphi^c v = L_\varphi^v v = \mu_\varphi L_\varphi v. \]

The elliptic equation \(-L_\varphi^c u = f\) is the linearization of the Monge–Ampère equation \( \det D^2 u = f \) at the function \( \varphi \). The first identity in (1.1) implies that \( L_\varphi^c \) admits both nondivergence (trace) and divergence (variational) forms.

In their seminal works [4, 5], L. Caffarelli and C. Gutiérrez developed a real analysis associated to \( \varphi \) leading to their groundbreaking proof of a Harnack inequality for nonnegative solutions to \( L_\varphi^c u = 0 \). As a crucial feature of their approach stands the description of the intrinsic geometry to study the linearized Monge–Ampère equation. This geometry is given by the Monge–Ampère sections of \( \varphi \) defined as

\[ S_\varphi(x_0, R) := \{ x \in \mathbb{R}^n : \delta_\varphi(x_0, x) < R \}, \]

2010 Mathematics Subject Classification. Primary: 35R09, 35R11, 35J96. Secondary: 35B65, 35J15, 47D06.

Key words and phrases. Fractional linearized Monge–Ampère equation, Harnack inequality, language of semigroups.
where \( x_0 \in \mathbb{R}^n \) and \( R > 0 \) are called the center and the height of the section \( S_\varphi(x_0, R) \), respectively, and
\[
(1.3) \quad \delta_\varphi(x_0, x) := \varphi(x) - \varphi(x_0) - \langle \nabla \varphi(x_0), x - x_0 \rangle,
\]
where \( \langle \cdot, \cdot \rangle \) denotes the dot product in \( \mathbb{R}^n \).

Notice that the case of \( \varphi_2(x) := |x|^2/2 \) accounts for the Laplacian and the Euclidean balls, since \( L^{\varphi_2} = -\Delta \) and \( S_\varphi(x_0, R) = B(x_0, \sqrt{2R}) \) for every \( x_0 \in \mathbb{R}^n \) and \( R > 0 \).

The Caffarelli–Gutiérrez regularity theory for \( L^{\varphi} \) was originally motivated by its applications to fluid dynamics (see [5, Section 1] and references therein). Further applications have emerged, for instance, in relation to the affine Plateau problem in affine geometry and the prescribed affine mean curvature equation, as exposed in the work of N. Trudinger and X.-J Wang in [42, 43, 44], N. Q. Le in [24], and references therein.

After [4, 5], both the regularity theory and the associated real analysis for the linearized Monge–Ampère equation have seen further progress. The Caffarelli–Gutiérrez Harnack inequality has been proved to hold under minimal geometric conditions on \( \varphi \) in [31]. It has been later extended as to allow for lower-order terms in [30] and by N. Q. Le in [22]. Interior \( C^{1,\alpha}, C^{2,\alpha} \)- and \( W^{2,p} \)-estimates for solutions to \( L^{\varphi} u = f \) have been obtained by C. Gutiérrez and T. Nguyen in [17, 18] and C. Gutiérrez and F. Tournier in [19], respectively. Global (up to the boundary) \( C^{1,\alpha} \) and \( W^{2,p} \)-estimates have been proved by N. Q. Le and O. Savin in [27, 28] and N. Q. Le and T. Nguyen [25, 26], respectively. Estimates for Green’s functions on Monge–Ampère sections have been established in [32, 33] and by N. Q. Le in [23]. A Liouville-type theorem for entire solutions to \( L^{\varphi} u = 0 \) in \( \mathbb{R}^2 \) has been proved by O. Savin in [37]. Sobolev and Poincaré-type inequalities associated to \( L^{\varphi} \) have been proved in [31, 32] and by G. Tian and X.-J. Wang in [41].

In this paper we develop a nonlocal version of the linearized Monge–Ampère equation and establish a Harnack inequality on Monge–Ampère sections. More precisely, our purpose is to accomplish the following goals.

(a) To define the fractional powers \( L^s_\varphi \) and \( L^s_\varphi \) on arbitrary Monge–Ampère sections (within their natural nondivergence/divergence form contexts) and to prove existence and uniqueness of solutions to the nonlocal equations
\[
(1.4) \quad L^s_\varphi v = f \quad \text{and} \quad L^s_\varphi u = F, \quad \text{for } 0 < s < 1.
\]

(b) To show that the interplay between the (local) nondivergence and divergence structures in (1.1) persists on the (nonlocal) operators \( L^s_\varphi \) and \( L^s_\varphi \).

(c) To prove, under minimal geometric assumptions on \( \varphi \), a Harnack inequality for nonnegative solutions to (1.4) on the sections of \( \varphi \), showing in particular that the Monge–Ampère geometry carries over to our nonlocal equations.

The goals above could be regarded as a linear counterpart to the current efforts to provide a correct definition of a fractional nonlocal Monge–Ampère equation by L. Caffarelli and F. Charro [3] and L. Caffarelli and L. Silvestre [7].

Our results will hold true for every \( 0 < s < 1 \). Regarding (a), we should first observe that \( L_\varphi \) is an operator in nondivergence form. In Section 2, we show how to define the fractional powers \( L^s_\varphi \) on arbitrary Monge–Ampère sections via the semigroup
generated by $L_\varphi$. In Section 3 we illustrate the definition of $L^s_\varphi$ by computing an explicit example of its action on the Monge–Ampère quasi-distance. Using the corresponding eigenfunctions, in Section 4 we define the fractional powers $L^s u$. Then, in terms of existence and uniqueness of solutions to (1.4), we have the following result:

**Theorem 1.1.** Fix a section $S := S_\varphi(p_0, R)$.

(i) Given any $f \in C_0(S)$ there exists a unique solution $v \in \text{Dom}_S(L^s_\varphi)$ to

$$\begin{cases}
L^s_\varphi v = f, & \text{in } S, \\
v = 0, & \text{on } \partial S.
\end{cases}$$

(ii) Given any $F \in \text{Dom}_S(L^s_{\varphi}')$ there exists a unique solution $u \in \text{Dom}_S(L^s_\varphi)$ to

$$\begin{cases}
L^s_\varphi u = F, & \text{in } S, \\
u = 0, & \text{on } \partial S.
\end{cases}$$

Here $\text{Dom}_S(L^s_\varphi)$ and $\text{Dom}_S(L^s_{\varphi}')$ denote the domains of $L^s_\varphi$ and $L^s_{\varphi}'$, with respect to the section $S$, defined in (2.2)–(2.3) and (4.4); respectively. Parts (i) and (ii) of Theorem 1.1 are proved in Subsections 2.3 and 4.2, respectively.

As far as (b) is concerned, we show that the fractional powers $L^s_\varphi$ and $L^s_{\varphi}'$ do preserve the dual nondivergence/divergence nature of $L_\varphi$ and $L_{\varphi}'$ from (1.1). The following equality is proved in Section 5.

**Theorem 1.2.** Fix a section $S := S_\varphi(p_0, R)$. Then

$$L^s_\varphi v = L^s_{\varphi}' v,$$

for every $v \in \text{Dom}_S(L^s_\varphi)$.

Regarding (c), let us mention that by minimal geometric assumption on $\varphi$ we mean a doubling condition for $\mu_\varphi$ on the sections of $\varphi$ known as the $(\text{DC})_\varphi$-doubling condition. Namely, we write $\mu_\varphi \in (\text{DC})_\varphi$ if there exists a constant $C_d \geq 1$ such that

$$\mu_\varphi(S_\varphi(x, t)) \leq C_d \mu_\varphi(\frac{1}{2} S_\varphi(x, t)) \quad \forall x \in \mathbb{R}^n, \forall t > 0,$$

where, $\frac{1}{2} S_\varphi(x, t)$ denotes the $\frac{1}{2}$-contraction of $S_\varphi(x, t)$ with respect to its center of mass (see Section 6 for more about the $(\text{DC})_\varphi$-doubling condition). Throughout the article, a geometric constant will be a constant depending only on the $(\text{DC})_\varphi$-doubling constant in (1.5), dimension $n$, and $0 < s < 1$.

**Theorem 1.3.** Assume $\mu_\varphi \in (\text{DC})_\varphi$. There exist geometric constants $\kappa \in (0, 1)$ and $K_9, C_H > 1$ such that for every section $S_0 := S_\varphi(p_0, R_0)$, every $f \in C_0(S_0)$, every $v \in \text{Dom}_{S_0}(L^s_\varphi)$ solution to

$$\begin{cases}
L^s_\varphi v = f, & \text{in } S_0, \\
v \geq 0, & \text{in } S_0,
\end{cases}$$

and every section $S_\varphi(x_0, K_9 R) \subset S_0$, the following Harnack inequality holds true

$$\sup_{S_\varphi(x_0, \kappa R)} v \leq C_H \left( \inf_{S_\varphi(x_0, \kappa R)} v + R^s \| f \|_{L^\infty(S_\varphi(x_0, K_9 R))} \right).$$
Furthermore, there exist geometric constants $\varrho \in (0, 1)$ and $K_{10} > 0$ such that

\begin{equation}
|v(x_0) - v(x)| \leq K_{10} \delta_{\varphi}(x_0, x)^{\varrho} \left( \sup_{S_{\varphi}(x_0, K_9 R)} v + R^{\varrho} \|f\|_{L^\infty(S_{\varphi}(x_0, K_9 R))} \right),
\end{equation}

for every $x \in S_{\varphi}(x_0, R)$, where $\delta_{\varphi}$ denotes the intrinsic Monge–Ampère quasi-distance defined in (1.3). (For details on the geometric constants $\varrho$, $C_H$, $K_9$, and $K_{10}$ see the proof of Theorem 1.3 in Section 14.)

**Remark 1.4.** The Harnack inequality (1.7) makes a case for the central role of the Monge–Ampère geometry, based on the sections $S_{\varphi}$, also in the study of the nonlocal operators $L_{\varphi}$. In addition, from the definition of $\text{Dom}_{S_0}(L_{\varphi})$ in (2.3), we have $v \in W^{2,n}_{\text{loc}}(S_0)$ and, by the Sobolev embedding, $v \in C^\gamma_{\text{loc}}(S_0)$ for every $\gamma \in (0, 1)$, where $C^\gamma_{\text{loc}}(S_0)$ is the local $\gamma$-Hölder class with respect to the Euclidean distance. Now, inequality (1.8) says that $v \in C^\varrho_{\text{loc}, \delta_{\varphi}}(S_0)$ with respect to the intrinsic Monge–Ampère quasi-distance $\delta_{\varphi}$.

**Remark 1.5.** For the particular choice $\varphi_2(x) := |x|^2/2$, Theorem 1.3 complements, by also including a nonhomogeneous right hand side $f$, Theorem A from [40]. Indeed, such a result implies a Harnack inequality for nonnegative solutions to the fractional nonlocal equation $(-\Delta_D)^s v = 0$ in a ball $B \subset \subset B_1(0)$. Here $-\Delta_D$ stands for the Dirichlet Laplacian in the unit ball $B_1(0) \subset \mathbb{R}^n$.

An essential tool for the proof of our main results is the extension problem characterization of the fractional nonlocal operators $L_{\varphi}$ and $L_{\varphi}^{**}$. The celebrated extension problem for the fractional Laplacian on $\mathbb{R}^n$ was first explored from the PDE point of view in the pioneering work of L. Caffarelli and L. Silvestre [6]. This is a far reaching technique that allows to handle nonlocal problems for $(-\Delta)^s$ in a local way through a degenerate elliptic equation in $(n + 1)$-dimensions. Later on, the semigroup language approach and the extension problem for fractional powers of positive operators was developed in [38, 39]. In [40] a Harnack inequality for fractional nonlocal elliptic equations admitting variational form was proved. The most general extension problem so far has been obtained in [13]. It includes fractional powers of closed operators in Banach spaces allowing, in particular, to deal with nonvariational equations. Thus the results in [13] apply to our nondivergence form elliptic operator $L_{\varphi}$.

Theorems 1.1 and 1.2 will be consequences of the semigroup language approach, the localization provided by the extension problem of [13, 38, 39] and the variational structure of $L_{\varphi}$ given by (1.1).

The main steps in the proof of Theorem 1.3 are as follows. First, given $f \in C_0(\overline{S}_0)$ and the nonnegative solution $v \in \text{Dom}_{S_0}(L_{\varphi})$, we will establish the equivalence between the fractional nonlocal equation (1.6) and the local degenerate/singular extension problem in one more variable

\begin{equation}
\begin{cases}
-L_{\varphi} V + z^{2-1/s} V_{zz} = 0, & \text{for } x \in S_0, \ z > 0, \\
V(x, z) = 0, & \text{for } x \in \partial S_0, \ z \geq 0, \\
\lim_{z \to 0^+} V(x, z) = v(x), & \text{uniformly in } S_0, \\
-\lim_{z \to 0^+} V_z(x, z) = d_s f(x), & \text{uniformly in } S_0,
\end{cases}
\end{equation}
as shown in [13]. Here $d_s > 0$ is an explicit constant defined in (2.10). There is a unique nonnegative solution $V$, vanishing at infinity in the sense of (2.12), such that

$$V \in C^\infty((0, \infty); \text{Dom}_{\tilde{S}_0}(\mathcal{L}_\varphi)) \cap C^1([0, \infty); C_0(\overline{S_0}))\).$$

Second, by setting $\tilde{V} := V(x, |z|)$ for every $(x, z)$ in the cylinder $S_0 \times \mathbb{R}$ we have that $\tilde{V} \in C^2(\mathbb{R} \setminus \{0\}; \text{Dom}_{\tilde{S}_0}(\mathcal{L}_\varphi)) \cap \text{Lip}(\mathbb{R}; C_0(\overline{S_0}))$ solves

$$-\mathcal{L}_\varphi \tilde{V} + |z|^{2-1/s} \tilde{V}_{zz} = 0 \text{ pointwise in } S_0 \times (\mathbb{R} \setminus \{0\}).$$

Equation (1.10) can be recast as the linearized Monge–Ampère equation

$$\Phi(\tilde{V}) := -\text{trace}((D^2\Phi)^{-1}D^2\tilde{V}) = 0, \text{ pointwise in } S_0 \times (\mathbb{R} \setminus \{0\}),$$

where $\Phi \in C^1(\mathbb{R}^{n+1})$ is the strictly convex function (recall that $1/s > 1$) defined as

$$\Phi(x, z) := \varphi(x) + \frac{s^2}{(1-s)}|z|^{1/s}, \text{ for all } (x, z) \in \mathbb{R}^n \times \mathbb{R}.$$ 

In Section 9 we set $Q := S_0 \times \mathbb{R}$ and define a functional class $S(Q)$ that contains $\tilde{V}$. Then, Theorem 1.3 will result from (1.9) and the following Harnack inequality (see Section 9 for notation).

**Theorem 1.6.** Assume $\mu_\varphi \in (\text{DC})_\varphi$. Then, there exist geometric constants $\kappa \in (0, 1)$ and $K_7, \tilde{C}_H > 1$ such that for every nonnegative $W \in S(Q)$ solution to $L_\Phi(W) = 0$ pointwise in $Q^+$ and every section $S_\Phi(X_0, R)$ with

$$S_\Phi(X_0, K_7R) \subset \subset Q$$

the following Harnack inequality holds true

$$\sup_{S_\Phi(X_0, R)} W \leq \tilde{C}_H \left( \inf_{S_\Phi(X_0, R)} W + R^4W_{z,0}^+ (S_\Phi(X_0, K_7R)) \right).$$

Here $W_{z,0}^+ (S_\Phi(X_0, K_7R))$ stands for the $L^\infty$-norm of the normal derivative of $W$ on the intersection $S_\Phi(X_0, K_7R) \cap \{(x, z) \in \mathbb{R}^{n+1} : z = 0\}$, as defined in (9.2).

Consequently, there exist geometric constants $\varphi \in (0, 1)$ and $K_{11} > 0$ such that

$$|W(X_0) - W(X)| \leq K_{11} \delta_\Phi(X_0, X)\sup_{S_\Phi(X_0, R)} W + R^4W_{z,0}^+ (S_\Phi(X_0, K_4R)),$$

for every $X \in S_\Phi(X_0, R)$.

The major obstacles in the proof of Theorem 1.6 are the following:

(i) On the hyperplane $\{(x, z) \in \mathbb{R}^{n+1} : z = 0\}$, the matrix

$$D^2\Phi(x, z)^{-1} = \begin{pmatrix} D^2\varphi(x)^{-1} & 0 \\ 0 & |z|^{2-1/s} \end{pmatrix} \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$$

becomes degenerate (if $1/2 < s < 1$) or singular (if $0 < s < 1/2$). This degeneracy/singularity prevents the direct application (even under stronger assumptions than $\mu_\varphi \in (\text{DC})_\varphi$) of known Harnack inequalities for the linearized Monge–Ampère equation including [5, Theorem 5], [22, Theorem 1], [31, Theorem 1.4], and [30, Theorem 1], which require continuous second-order derivatives as well as positive-definite Hessians for the underlying convex function.
(ii) Furthermore, the fact that \( \det D^2 \Phi \) may vanish implies that the Caffarelli–Gutiérrez proof of the “passage to the double section” (that is, [5, Theorem 2], which is essential for the weak-Harnack inequality in [5, Theorem 4]) cannot be applied to (1.11).

(iii) Also, since \((D^2 \Phi)^{-1}\) becomes degenerate or singular, the solvability of the Dirichlet problem for \( L_\Phi \) in (1.10) cannot be taken for granted and then the argument from [30] used to prove the critical-density estimate (see [30, Theorem 2]) cannot be carried out for (1.11).

(iv) The function \( \tilde{V} \) does not belong apriori to \( W^{2,n+1}_{\text{loc}}(S_0 \times \mathbb{R}) \), which prevents its direct use in the Aleksandrov–Bakelman–Pucci maximum principle.

Notice that the hyperplane \( \{ (x, z) \in \mathbb{R}^{n+1} : z = 0 \} \) is central to our approach because it is precisely there where the connection between \( \tilde{V} \) and \( v \) takes place, see (1.9).

In addition to the obstacles above, we pursue (and prove) Theorem 1.6 under minimal geometric conditions, that is, under \( \mu_\varphi \in (\text{DC})_\varphi \) only. This is in contrast with [5].

In order to overcome those obstacles, in Section 10 we show that, despite the degeneracy or singularity of \((D^2 \Phi)^{-1}\) and under the hypothesis \( \mu_\varphi \in (\text{DC})_\varphi \) only, a critical-density estimate for \( \tilde{V} \) (Theorem 10.1 below) can be established, for some \( \varepsilon_0 \in (0, 1) \), in terms of the \( L^\infty(S_0) \)-norm of the normal derivative \( \lim_{z \to 0^+} \tilde{V}_z(x, z) = d_s f(x) \). That is, a suitable control on the normal derivative will allow for a critical-density estimate in the absence of the hypothesis of continuous second-order derivates and positive-definite Hessian. This represents a novelty in the study of the linearized Monge–Ampère equation.

With a critical-density estimate at hand, mean-value inequalities (see Theorems 11.2 and 11.3) can be obtained by fairly standard arguments, as described in Section 11.

The next step is to prove a weak-Harnack inequality for \( \tilde{V} \). As mentioned, the methods from [5, Sections 2 and 3] are not applicable. Instead, we rely on the variational side of the linearized Monge–Ampère equation.

The idea is the following. By (1.1), we have \( \mu_\varphi L_\varphi = L_\varphi \), and we know that \( V = V(x, z) \) comes from the extension problem (1.9) associated to \( L_\varphi \). Now, by developing the extension problem associated to the divergence-form operator \( L_\varphi \) from [8, 38, 39] (namely, (4.8) below) we find a solution \( U = U(x, y), x \in S_0, y \in \mathbb{R} \). Certainly, after contrasting the extension problems associated to \( L_\varphi \) and \( L_\varphi \), one cannot expect that \( U = V \). However, in Section 5 we prove that, modulo a change of variables (more precisely, a change in the extension variables \( y \) and \( z \)), the equality \( U = V \) does hold true. A key consequence of this equality is the energy estimate for \( V \) given by (5.8) in Remark 5.3. Although \( V \) solves the PDE in (1.9) which can also be written in divergence form, we were able to arrive at the energy estimate (5.8) only by going through \( U \).

With this insight on the variational side of \( V \), in Section 8 we develop a weak Poincaré inequality associated to \( \Phi \). One advantage of working in the variational context (which is based on energy estimates) is that the pointwise singularity of \( D^2 \Phi^{-1} \) will become inconsequential because \( |z|^{1/s-2} \in L^1_{\text{loc}}(\mathbb{R}) \) for every \( 0 < s < 1 \). In turn, in Section 12 we use the Poincaré inequality to prove a critical-density estimate for every \( \varepsilon \in (0, 1) \).
(see Theorem 12.2). In Section 13 we use the fact that the critical density can be taken small enough and combine it with a covering lemma to prove a weak-Harnack inequality for $\tilde{V}$ (see Theorem 13.2). Finally, in Section 14 we bring all the previous results together to prove Theorems 1.3 and 1.6.

We close this introduction by mentioning that no use of the normalization technique from [5, Section 1] or of the local Monge–Ampère-BMO space (which dominated the variational approach in [30, 31]) is made in this article.

2. Nondivergence Form: Fractional Powers $L^s_\varphi$ and Extension Problem

Fix any section $S := S_\varphi(x_0, R)$. On $S$ we consider the linearized Monge–Ampère equation with homogeneous Dirichlet boundary condition:

$$
\begin{align*}
L_\varphi v &\equiv -\operatorname{trace}((D^2\varphi)^{-1}D^2v) = f, \quad \text{in } S, \\
v & = 0, \quad \text{on } \partial S.
\end{align*}
$$

2.1. The semigroup generated by $L_\varphi$. Our first goal is to introduce the semigroup generated by $L_\varphi$. The matrix of coefficients $(D^2\varphi)^{-1}$ is symmetric and positive definite, with entries in $C^1(\overline{S})$. Since $\overline{S}$ is a compact set, such a matrix is uniformly elliptic on $S$. Notice that we use this fact without ever resorting to estimates depending on the size of the eigenvalues of $(D^2\varphi)(x)^{-1}$ for $x \in \overline{S}$. Let the domain of $L_\varphi$ be the space

$$
\operatorname{Dom}_S(L_\varphi) := \{ v \in C_0(\overline{S}) \cap W^{2,n}_{\text{loc}}(S) : L_\varphi v \in C(\overline{S}) \},
$$

where $C_0(\overline{S})$ is the Banach space $C_0(\overline{S}) := \{ v \in C(\overline{S}) : v = 0 \text{ on } \partial S \}$ endowed with the $L^\infty(S)$ norm. Here as usual $C(\overline{S})$ denotes the space of continuous functions on $\overline{S}$ under the $L^\infty(S)$ norm. It is clear that $\operatorname{Dom}_S(L_\varphi)$ depends on the section $S$. Also, let us define $\operatorname{Dom}_S(L^s_\varphi)$ as

$$
\operatorname{Dom}_S(L^s_\varphi) := \operatorname{Dom}_S(L_\varphi).
$$

It follows from [2, Theorem 4.1] that $(L_\varphi, \operatorname{Dom}_S(L_\varphi))$ generates a bounded holomorphic semigroup $\{e^{-tL_\varphi}\}_{t \geq 0}$ on $C(\overline{S})$. For convenience we recall the relevant definitions, see for example [1, 35, 45]. The family $\{e^{-tL_\varphi}\}_{t \geq 0}$ is a semigroup on $C(\overline{S})$ (see [35, Section 1.1]) if the following conditions hold:

(i) for each $t \geq 0$, $e^{-tL_\varphi}$ is a bounded linear operator from $C(\overline{S})$ into itself;

(ii) the semigroup property holds: for every $t_1, t_2 \geq 0$ and for any $v \in C(\overline{S}),$

$$
e^{-t_1L_\varphi}(e^{-t_2L_\varphi}v) = e^{-(t_1+t_2)L_\varphi}v;
$$

(iii) for every $v \in C(\overline{S})$ we have $e^{-0L_\varphi}v = v.$

The semigroup $e^{-tL_\varphi}$ is bounded holomorphic if the operator valued function $t \to e^{-tL_\varphi}$ from $[0, \infty)$ into the algebra of bounded linear operators on $C(\overline{S})$ has a holomorphic extension to an open sector of the complex plane contained in $\Re z > 0$, which is bounded on proper subsectors, see [1, p. 150]. It is shown in [2, Proposition 4.4] that $(L_\varphi, \operatorname{Dom}_S(L_\varphi))$ is dissipative, see also [1, Lemma 3.4.2]. Then, as a consequence of [1, Proposition 3.7.16], we obtain that the semigroup is a contraction. In other words, for every $v \in C(\overline{S})$,

$$
\|e^{-tL_\varphi}v\|_{L^\infty(S)} \leq \|v\|_{L^\infty(S)}, \quad \text{for all } t \geq 0.
$$
Therefore, for any $v \in \text{Dom}_S(L_{\varphi})$ the function $w(t, x) := e^{-tL_{\varphi}}v(x)$ is the unique solution to the parabolic equation
\[
\begin{cases}
\partial_t w = -L_{\varphi}w, & \text{for } t > 0, x \in S, \\
w(t, x) = 0, & \text{for } t \geq 0, x \in \partial S, \\
\lim_{t \to 0^+} w(t, x) = v(x), & \text{uniformly on } S.
\end{cases}
\]
see [35, Sections 1.1 and 1.2]. Observe that the semigroup $e^{-tL_{\varphi}}$, though a contraction, is not a $C_0$-semigroup on $C(S)$ because $\text{Dom}_S(L_{\varphi})$ is not dense in $C(S)$, see [35, Hille–Yosida Theorem in Section 1.3]. In particular, in order for $w(t, x)$ above to converge to the initial data $v(x)$ uniformly in $S$ we need to take $v \in \text{Dom}_S(L_{\varphi})$. As $e^{-tL_{\varphi}}$ is a bounded holomorphic semigroup, from [1, Theorem 3.7.19] we have that
\[
v \in C(S) \implies e^{-tL_{\varphi}}v \in \text{Dom}_S(L_{\varphi}), \text{ with sup } \|tL_{\varphi}e^{-tL_{\varphi}}v\|_{L^\infty(S)} < \infty.
\]
Finally, the following decay estimate holds: there are constants $M, \gamma > 0$ such that, for every $v \in \text{Dom}_S(L_{\varphi})$,
\[
\|e^{-tL_{\varphi}}v\|_{L^\infty(S)} \leq Me^{-\gamma t}\|v\|_{L^\infty(S)}, \quad \text{for all } t \geq 0,
\]
see [34, Theorem 1], also [2, Theorem 4.1, Proposition 4.7].

**Remark 2.1** (Positivity). It is important to notice that the semigroup $e^{-tL_{\varphi}}$ is positive on $\text{Dom}_S(L_{\varphi})$. Namely, if $v \in \text{Dom}(L_{\varphi})$ and $v \geq 0$ in $S$ then $e^{-tL_{\varphi}}v(x) \geq 0$, for every $x \in S$ and $t \geq 0$. Indeed, this follows from the well known weak minimum principle for parabolic equations in nondivergence form.

2.2. **The fractional nonlocal operator** $L_{\varphi}^s$. The semigroup generated by $L_{\varphi}$ allows us to define the fractional powers $L_{\varphi}^sv(x)$ as in [45, p. 260, (5)] and [13, Theorem 4.1].

**Definition 2.2.** Let $0 < s < 1$. The fractional operator $L_{\varphi}^sv(x)$ is defined for any $v \in \text{Dom}_S(L_{\varphi})$ and every $x \in S$ as
\[
L_{\varphi}^sv(x) = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-tL_{\varphi}}v(x) - v(x)) \frac{dt}{t^{1+s}}.
\]
The integral in (2.6) is absolutely convergent in the sense of Bochner. Indeed, the integrand $(e^{-tL_{\varphi}}v - v)t^{-(1+s)}$ is a function of $t \in (0, \infty)$ with values in $C_0(S)$ and, since $v \in \text{Dom}_S(L_{\varphi})$ and $e^{-tL_{\varphi}}$ is a contraction, we have (see [35, p.5, (2.5)–(2.6)]),
\[
\|e^{-tL_{\varphi}}v - v\|_{L^\infty(S)} \leq \int_0^t \|\partial_r e^{-rL_{\varphi}}v\|_{L^\infty(S)} \, dr = \int_0^t \|e^{-rL_{\varphi}}L_{\varphi}v\|_{L^\infty(S)} \, dr \leq \|L_{\varphi}v\|_{L^\infty(S)} t.
\]
On the other hand, by contractivity, $\|e^{-tL_{\varphi}}v - v\|_{L^\infty(S)} \leq 2\|v\|_{L^\infty(S)}$, for every $t \geq 0$. Therefore, for any $A > 0$,
\[
\int_0^\infty \|e^{-tL_{\varphi}}v - v\|_{L^\infty(S)} \frac{dt}{t^{1+s}} \leq \|L_{\varphi}v\|_{L^\infty(S)} \frac{dt}{t^s} + 2\|v\|_{L^\infty(S)} \int_A^\infty \frac{dt}{t^{1+s}} < \infty.
\]
In particular, the following fractional Sobolev-type interpolation inequality holds
\[
\|L^s_v\|_{L^\infty(S)} \leq \frac{sA^{1-s}}{\Gamma(2-s)}\|L^s_v\|_{L^\infty(S)} + \frac{2}{A^s\Gamma(1-s)}\|v\|_{L^\infty(S)}.
\]
for any \(v \in \text{Dom}_S(L^s_v), \ A > 0\) and \(0 < s < 1\), and, as a consequence,
\[
\|L^s_v\|_{L^\infty(S)} \leq \frac{2^{1-s}}{(2-s)\Gamma(2-s)}\|L^s_v\|_{L^\infty(S)}\|v\|_{L^\infty(S)}^{1-s}.
\]

We finally notice that, unlike the local differential operator \(L^s_v\) that in general has values in \(C(S)\), see (2.2), the fractional nonlocal operator \(L^s_v\) has range in \(C^0(S)\), namely,
\[
v \in \text{Dom}_S(L^s_v) \implies L^s_vv \in C^0(S).
\]

**Remark 2.3** (Maximum principle). The semigroup expression for \(L^s_v(x)\) in (2.6) yields the following maximum principle. Let \(v \geq 0\) and suppose that \(v(x_0) = 0\) for some \(x_0 \in S\). Then \(L^s_v(x_0) \leq 0\) in \(S\). Moreover, \(L^s_v(x_0) = 0\) if and only if \(v \equiv 0\) in \(S\). Indeed, since the semigroup is positive (see Remark 2.1) we have \(e^{-tL^s_v(x_0)} \geq 0\) and the first conclusion follows from (2.6) by noticing that \(\Gamma(-s) < 0\). Furthermore, \(\dot{L}^s_v(x_0) = 0\) if and only if \(e^{-tL^s_v(x_0)} = 0\) for all \(t > 0\), which in this case is equivalent to \(v \equiv 0\) in \(S\) by the usual weak maximum principle for parabolic equations.

2.3. **Proof of Theorem 1.1(i).** Given \(f \in C^0(S)\), it follows from general theory (see [1, Section 3.8] and [45, Chapter IX, Section 11]) that
\[
v(x) := L^{-s}_vf(x) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-tL^s_vf(x)} \frac{dt}{t^{1-s}}, \quad \forall x \in S,
\]
is a solution to \(L^s_v = f\) in \(S\) with \(v = 0\) on \(\partial S\). To prove uniqueness, assume that \(v \in \text{Dom}_S(L^s_v)\) is a non trivial solution to \(L^s_v = 0\) in \(S\). Let \(x_0 \in S\) be a point such that \(|v(x_0)| = \|v\|_{L^\infty(S)}\). By changing \(v\) by \(-v\) we can always assume that \(v(x_0) > 0\). Then, from (2.4) and (2.6) we conclude that \(L^s_v(x_0) \geq 0\). The case \(L^s_v(x_0) > 0\) is excluded by hypothesis. Thus, \(e^{-tL^s_v(x_0)} = v(x_0)\) for every \(t > 0\), contradicting (2.5). The proof of uniqueness is complete. \(\square\)

2.4. **The extension problem.** Using the language of semigroups we can characterize \(L^s_v\) with the extension problem established in [13, Theorem 1.1, Theorem 2.1]. Observe that the main results in [13] apply in principle to generators of bounded \(C_0\)-semigroups. Nevertheless, it is easy to follow the proofs there and conclude that we can extend them to our present case.

Let \(v \in \text{Dom}_S(L^s_v)\). For \(x \in S\) and \(z > 0\) we define
\[
V(x, z) := \frac{(s^2z^{1/s})^s}{\Gamma(s)} \int_0^\infty e^{-(s^2z^{1/s})/t} e^{-tL^s_v(x)} dt \frac{dt}{t^{1+s}}.
\]
Then \(V(\cdot, z) \in \text{Dom}_S(L^s_v)\) and, for each \(z > 0\),
\[
\|V(\cdot, z)\|_{L^\infty(S)} \leq \|v\|_{L^\infty(S)}.
\]
As a function of $z$, $V(x,z)$ is $C^\infty(0,\infty)$, for every $x \in S$. Furthermore, $V$ is a classical solution to the extension problem

$$
\begin{aligned}
-L_\varphi V + z^{2-1/s}V_{zz} &= 0, &\text{for } x \in S, \ z > 0, \\
V(x,z) &= 0, &\text{for } x \in \partial S, \ z \geq 0, \\
\lim_{z \to 0^+} V(x,z) &= v(x), &\text{uniformly in } S.
\end{aligned}
$$

(2.8)

Indeed, by usual elliptic regularity, $V \in C^{2,\alpha}_{\text{loc}}(S \times (0,\infty))$ for every $0 < \alpha < 1$. Moreover, $V(x,\cdot) \in C^1[0,\infty)$ and

$$
\lim_{z \to 0^+} V_z(x,z) = d_sL_\varphi^s v(x),
$$

uniformly in $S$, where

$$
d_s := \frac{s^{2s} \Gamma(1-s)}{\Gamma(1+s)} > 0.
$$

(2.9)

Hence, $V \in C^\infty((0,\infty); \text{Dom}_S(L_\varphi)) \cap C^1([0,\infty); C_0(\overline{S}))$ is a solution to the extension problem

$$
\begin{aligned}
-L_\varphi V + z^{2-1/s}V_{zz} &= 0, &\text{for } x \in S, \ z > 0, \\
V(x,z) &= 0, &\text{for } x \in \partial S, \ z \geq 0, \\
-\lim_{z \to 0^+} V_z(x,z) &= d_sL_\varphi^s v(x), &\text{uniformly in } S.
\end{aligned}
$$

(2.10)

For all the details see [13].

Remark 2.4 (Extension problem for negative powers). Given a function $f \in C_0(\overline{S})$, let $v \in \text{Dom}_S(L_\varphi)$ be the solution to $L_\varphi^sv = f$ in $S$. The solution $V$ in (2.7) can also be written as

$$
V(x,z) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-(s^2z^{1/s})/t} e^{-tL_\varphi} f(x) \frac{dt}{t^{1-s}}.
$$

Then, for every $x \in S$ we readily get

$$
V(x,0) = v(x) = L_\varphi^{-s} f(x).
$$

For the details see [13, Theorems 1.1, Theorem 2.1], also [8, 13, 38, 39].

Remark 2.5 (Uniqueness). By the weak maximum principle for elliptic equations, it is easy to see that there is at most one solution to the extension problem (2.8) such that

$$
\lim_{z \to \infty} \|V(\cdot, z)\|_{L^\infty(S)} = 0.
$$

(2.12)
Using the semigroup decay (2.5) it is readily checked that \( V(x,z) \) as defined in (2.7) satisfies (2.12), so this is indeed the unique solution.

### 3. An explicit example of \( L^s_\varphi v(x) \)

In this section we give two explicit examples on the action of \( L^s_\varphi \). Our examples are inspired by the identity

\[
L_\varphi(-\varphi) = \text{trace}((D^2\varphi)^{-1}D^2\varphi) = n.
\]

**Theorem 3.1.** Given an arbitrary section \( S := S_\varphi(x_0, R) \) introduce the function

\[
v_\varphi(x) := R - (\varphi(x) - \varphi(x_0) - \langle \nabla \varphi(x_0), x - x_0 \rangle)
\]

for \( x \in \overline{S} \).

Then, for every \( 0 < s < 1 \),

\[
L^s_\varphi(v_\varphi)(x) = n^s v_\varphi(x)^{1-s}, \quad \forall x \in S.
\]

**Proof.** Notice that \( v_\varphi \in C_0(S) \cap C^2(S) \subset \text{Dom}_S(L_\varphi) \) and that \( D^2v_\varphi = -D^2\varphi \) in \( S \). Also, from the definition of \( S_\varphi(x_0, R) \) in (1.2), we have \( v_\varphi > 0 \) in \( S \). In order to show (3.1) we first need to find the unique solution \( V \) to the extension equation

\[
\begin{cases}
-L_\varphi V + z^{2-1/s}V_{zz} = 0, & \text{for } x \in S, \ z > 0, \\
v(x, z) = 0, & \text{for } x \in \partial S, \ z \geq 0, \\
V(x, 0) = v_\varphi(x), & \text{for } x \in S,
\end{cases}
\]

that satisfies (2.12) in Remark 2.5. We do so by pursuing a solution \( V \) of the form

\[
V(x, z) = v_\varphi(x)g(z),
\]

where \( g : [0, \infty) \to \mathbb{R}^n \) has to be found. Notice that

\[
L_\varphi V = -\text{trace}((D^2\varphi(x))^{-1}D^2\varphi(x))g(z) = ng(z).
\]

Hence, by using the equation \(-L_\varphi V + z^{2-1/s}V_{zz} = 0\), the function \( g \geq 0 \) must be a solution to

\[
g'' + \left(\frac{-n}{v_\varphi(x)}\right) z^{1/s-2} g = 0, \quad g(0) = 1,
\]

that decays to zero as \( z \to \infty \). The equation (3.3) is a Bessel equation and in order to find its unique solution we follow the analysis in [39, Section 3.1], see also [29] for details about Bessel functions. To simplify the notation, fix \( x \in S \) and set

\[
\alpha := \frac{n}{v_\varphi(x)} \in (0, \infty).
\]

Thus, the general solution to (3.3) is

\[
g(z) = z^{1/2} \mathcal{Z}_\alpha(\pm i2\alpha^{1/2} z^{1/(2s)}),
\]

where \( \mathcal{Z}_\alpha(r) \) denotes a general cylinder function, see [29, p. 106]. By using the boundary condition \(|g(z)| \leq C\) as \( z \to \infty \) (see the asymptotic expansions of Bessel functions in [39, Section 3.1]) we obtain the modified Bessel function of the second kind \( \mathcal{K}_\alpha \):

\[
g(z) = C z^{1/2} \frac{2i^{-s-1}}{\pi} \mathcal{K}_\alpha(2\alpha^{1/2} z^{1/(2s)}),
\]
where $C$ is an arbitrary constant. Recall that $K_{\nu}(r) \sim 2^{\nu-1}\Gamma(\nu)r^{-\nu}$ as $r \to 0$. Hence,

$$g(z) \sim C \frac{i^{-s-1}\Gamma(s)}{\pi s^s \alpha^{s/2}}, \quad z \to 0,$$

and, in order to satisfy the initial condition $g(0) = 1$, we impose

$$C := \frac{\pi s^s \alpha^{s/2}}{i^{-s-1}\Gamma(s)}.$$

Therefore,\hspace{1cm}

$$(3.5) \quad g(z) = \frac{2^{1-s}}{\Gamma(s)} (2s \alpha^{1/2} z^{1/(2s)})^s K_s(2s \alpha^{1/2} z^{1/(2s)}),$$

is the unique bounded solution to (3.3). It is clear that $g \geq 0$. Moreover, from the asymptotic behavior $K_{\nu}(r) \sim \left(\frac{\pi}{(2r)}\right)^{1/2} e^{-r}$, as $r \to \infty$, it follows that $g(z) \to 0$ exponentially, as $z \to \infty$. In conclusion,

$$(3.6) \quad V(x, z) = v_\nu(x)g(z),$$

with $g(z)$ as in (3.5) and $\alpha$ as in (3.4), is the unique positive bounded solution to (3.2) that satisfies (2.12), see Remark 2.5. On the other hand, from (2.9) we know that if $V$ is the unique solution to the extension problem (3.2) then

$$(3.7) \quad -\frac{\Gamma(1+s)}{s^{2s}\Gamma(1-s)}V_z(x, 0) = L^s_\nu(v_\nu)(x) \quad \forall x \in S.$$

Again, to simplify the computation, let us put

$$\beta = \beta(z) := 2s \alpha^{1/2} z^{1/(2s)},$$

so that

$$g(z) = \frac{2^{1-s}}{\Gamma(s)} \beta^s K_s(\beta).$$

By using the chain rule, the properties of the derivatives of Bessel functions and the fact that $K_{-\nu}(r) = K_{\nu}(r)$, we can compute

$$\frac{dg}{dz}(z) = \frac{2^{1-s}}{\Gamma(s)} \frac{d}{d\beta}(\beta^s K_s(\beta)) \frac{d\beta}{dz}$$

$$= \frac{2^{1-s}}{\Gamma(s)} \beta^s K_{s-1}(\beta) \alpha^{1/2} z^{1/(2s)-1}$$

$$= -\frac{2s^s}{\Gamma(s)} \alpha^{(s+1)/2} z^{1/(2s)-1/2} K_{1-s}(2s \alpha^{1/2} z^{1/(2s)}).$$

Whence, from (3.6), (3.8) and the asymptotic behavior of $K_{\nu}(r)$ as $r \to 0$, we get

$$-\lim_{z \to 0^+} V_z(x, 0) = \frac{2s^s}{\Gamma(s)} v_\nu(x) \alpha^{(s+1)/2} \lim_{z \to 0^+} z^{1/(2s)-1/2} K_{1-s}(2s \alpha^{1/2} z^{1/(2s)})$$

$$= \frac{2s^s}{\Gamma(s)} v_\nu(x) \alpha^{(s+1)/2} \lim_{z \to 0^+} z^{1/(2s)-1/2} \frac{2^{-s}\Gamma(1-s)}{(2s \alpha^{1/2} z^{1/(2s)})^{1-s}}$$

$$= \frac{\Gamma(1-s)s^{2s}}{\Gamma(1+s)} v_\nu(x) \alpha^s.$$
Recalling the definition of $\alpha$ from (3.4) and the identity (3.7), we arrive at (3.1). \qed

As a consequence of Theorem 3.1 we obtain the following result on the fractional Dirichlet Laplacian.

**Corollary 3.2.** For every $0 < s < 1$ we have

$$(-\Delta_D)^s(1 - |\cdot|^2)(x) = (2n)^s(1 - |x|^2)^{1-s}, \quad \forall x \in B_1(0),$$

where $(-\Delta_D)^s$ is the fractional Dirichlet Laplacian in the unit ball $B_1(0) \subset \mathbb{R}^n$.

**Proof.** Use Theorem 3.1 with $\varphi(x) \equiv \varphi_2(x) := |x|^2$ for every $x \in \mathbb{R}^n$ and notice that $B_1(0) = S_{\varphi_2}(0, 1)$ and $D^2\varphi_2 = 2I$, which gives $L_{\varphi_2} = -\frac{1}{2}\Delta$. \qed

### 4. Divergence form: fractional powers $L^s_{\varphi}$ and extension problem

This section is devoted to the definition of the fractional powers of the divergence form operator $L_{\varphi}$ subject to the homogeneous Dirichlet boundary condition. As in Section 2, fix any section $S := S_\varphi(x_0, R)$. For $f \in L^2(S, d\mu_\varphi)$, consider the following Dirichlet problem for $L_{\varphi}$:

$$\begin{cases}
L_{\varphi}u \equiv -\text{div}(A_{\varphi}(x)\nabla u) = \mu_\varphi f, & \text{in } S, \\
u = 0, & \text{on } \partial S.
\end{cases}$$

Observe that the right hand side $f$ in (4.1) appears multiplied by the Monge–Ampère measure $\mu_\varphi$. This can always be assumed by considering $f/\mu_\varphi$.

#### 4.1. The fractional nonlocal operator $L^s_{\varphi}$

The fractional powers $L^s_{\varphi}$, $0 < s < 1$, will be defined by using the Dirichlet eigenfunctions and eigenvalues along the lines of [8, 38, 39, 40].

Let $W^{1,2}_{0,\varphi}(S)$ denote the completion of $C_c^1(S)$ with respect to the norm

$$\|u\|_{W^{1,2}_{0,\varphi}(S)}^2 := \|u\|_{L^2(S, d\mu_\varphi)}^2 + \|\nabla^\varphi u\|_{L^2(S, d\mu_\varphi)}^2.$$  

Here $\nabla^\varphi$ stands for the Monge–Ampère gradient, which is defined as

$$\nabla^\varphi u := (D^2\varphi)^{-1/2}\nabla u.$$

By the Sobolev inequality for the Monge–Ampère quasi-metric structure, see [32, Theorem 1], an equivalent norm in $W^{1,2}_{0,\varphi}(S)$ is $\|\nabla^\varphi u\|_{L^2(S, d\mu_\varphi)}$.

A weak solution $u$ to (4.1) is a function $u \in W^{1,2}_{0,\varphi}(S)$ such that

$$\int_S \langle \nabla^\varphi u, \nabla^\varphi h \rangle \ d\mu_\varphi = \int_S fh \ d\mu_\varphi, \quad \text{for every } h \in W^{1,2}_{0,\varphi}(S).$$

Notice that the matrix of coefficients $A_{\varphi}(x)$ is symmetric and uniformly elliptic in the compact set $S$ and that $L^2(S, d\mu_\varphi)$ is isometrically embedded in $L^2(S, dx)$. Therefore, by standard techniques (see for example [10, Chapter 6] or [14, Section 8.12]), there exist a sequence of eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \nearrow \infty$ and a corresponding family of eigenfunctions $\{e_k\}_{k \geq 1} \subset W^{1,2}_{0,\varphi}(S)$ such that

$$\begin{cases}
L_{\varphi}e_k = \mu_\varphi \lambda_k e_k, & \text{in } S, \\
e_k = 0, & \text{on } \partial S.
\end{cases}$$

(4.2)
in the weak sense. In other words, for every \( h \in W^{1,2}_0(S) \) and every \( k \in \mathbb{N} \),

\[
(4.3) \quad \int_S \langle \nabla^\varphi e_k, \nabla^\varphi h \rangle \, d\mu_\varphi = \lambda_k \int_S e_k h \, d\mu_\varphi.
\]

Moreover, \( \{e_k\}_{k \in \mathbb{N}} \) forms an orthonormal basis of \( L^2(S, d\mu_\varphi) \).

For \( s \geq 0 \), we consider the Hilbert space

\[
(4.4) \quad \mathcal{H}_\varphi^s(S) := \text{Dom}_S(\mathcal{L}_\varphi^s) := \left\{ u = \sum_{k=1}^\infty u_k e_k \in L^2(S, d\mu_\varphi) : \sum_{k=1}^\infty \lambda_k^s u_k^2 < \infty \right\},
\]

endowed with the inner product

\[
\langle u, h \rangle_{\mathcal{H}_\varphi^s(S)} := \sum_{k=1}^\infty \lambda_k^s u_k h_k, \quad \text{for } u, h \in \mathcal{H}_\varphi^s(S),
\]

where \( h = \sum_{k=1}^\infty h_k e_k \). Observe that \( \mathcal{H}_\varphi^0(S) = L^2(S, d\mu_\varphi) \). From (4.3) it is readily verified that \( \mathcal{H}_\varphi^1(S) = W^{1,2}_0(S) \) as Hilbert spaces and

\[
(4.5) \quad \int_S \langle \nabla^\varphi u, \nabla^\varphi h \rangle \, d\mu_\varphi = \sum_{k=1}^\infty \lambda_k^s u_k h_k, \quad \text{for any } u, h \in \mathcal{H}_\varphi^1(S).
\]

We read the right-hand side in (4.5) as the definition of \( \mathcal{L}_\varphi^s u \) for \( u \in \mathcal{H}_\varphi^1(S) \). We are now in position to define the fractional power \( \mathcal{L}_\varphi^s \) in \( \mathcal{H}_\varphi^s(S) \).

**Definition 4.1.** Let \( 0 < s < 1 \). The fractional operator \( \mathcal{L}_\varphi^s u \) is defined for any \( u \in \mathcal{H}_\varphi^s(S) \) as the unique element \( \mathcal{L}_\varphi^s u \) in the dual space \( \mathcal{H}_\varphi^s(S)' \) acting as

\[
(4.6) \quad (\mathcal{L}_\varphi^s u)(h) = \sum_{k=1}^\infty \lambda_k^s u_k h_k, \quad \text{for every } h = \sum_{k=1}^\infty h_k e_k \in \mathcal{H}_\varphi^s(S).
\]

4.2. **Proof of Theorem 1.1(ii).** Given \( F = \sum_{k=1}^\infty F_k e_k \in \mathcal{H}_\varphi^s(S)' \), the unique solution \( u \in \mathcal{H}_\varphi^s(S) \) is given by \( u = \sum_{k=1}^\infty \lambda_k^{-s} F_k e_k \in \mathcal{H}_\varphi^s(S) \). In particular, if \( F \in L^2(S, d\mu_\varphi) \) then \( u \in \mathcal{H}_\varphi^{2s}(S) \).

4.3. **The extension problem.** Let us introduce

\[
a := 1 - 2s \in (-1, 1),
\]

and, for \( x \in S \),

\[
(4.7) \quad B_\varphi(x) := \begin{pmatrix} A_\varphi(x) & 0 \\ 0 & \mu_\varphi(x) \end{pmatrix} \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}.
\]

By reasoning as in [38, 39], see also [8, 13], the extension problem characterization of (4.6) can now be obtained as follows. Let \( u \in \mathcal{H}_\varphi^s(S) \). We say that a function \( U = U(x, y) \), defined for \( x \in S \) and \( y \geq 0 \), is a weak solution to the extension problem

\[
(4.8) \quad \begin{cases}
\text{div}_{x,y}(y^{-s} B_\varphi(x) \nabla_{x,y} U) = 0, & \text{for } x \in S, \ y > 0, \\
U(x, y) = 0, & \text{for } x \in \partial S, \ y \geq 0, \\
U(x, 0) = u(x), & \text{for } x \in S,
\end{cases}
\]
if $U$, $\nabla^\varphi U$ and $U_y$ belong to the weighted space $L^2(S \times (0, \infty), y^a d\mu_\varphi dy)$, $U = 0$ on $\partial S \times (0, \infty)$ in the sense of traces, $U(x, y) \to u(x)$ as $y \to 0^+$ in $L^2(S, d\mu_\varphi)$, and for every test function $W = W(x, y)$ such that $W(x, 0) = 0$ in $S$, we have

$$
\int_0^\infty \int_S y^a \langle B_\varphi(x) \nabla_{x,y} U, \nabla_{x,y} W \rangle \, dx \, dy = \int_0^\infty \int_S y^a \langle \nabla^\varphi U, \nabla^\varphi W \rangle \, d\mu_\varphi \, dy + \int_0^\infty \int_S y^a U_y W_y \, d\mu_\varphi \, dy = 0.
$$

By proceeding as in [38, Section 3.3.1] or [39, Section 3.1], see also [8, Section 2.3], the unique weak solution $U$ that weakly vanishes as $y \to \infty$ can be written using the Fourier coefficients $u_k$ of $u$ and the eigenfunctions $e_k$ as

$$
U(x, y) = \sum_{k=1}^\infty c_k(y) u_k e_k(x),
$$

where the coefficients $c_k(y)$ are given by

$$
c_k(y) = \frac{2^{1-s}}{\Gamma(s)} (\lambda_k^{1/2} y)^s K_s(\lambda_k^{1/2} y), \quad \text{for } y > 0 \text{ and } k \geq 1.
$$

Here $K_s$ is the modified Bessel function of the second kind and parameter $s$. Moreover,

$$
\lim_{y \to 0^+} y^a U_y = c_s L^s_\varphi u, \quad \text{in } H^s_\varphi(S),
$$

where

$$
c_s := \frac{\Gamma(1-s)}{4^{s-1/2}\Gamma(s)}.
$$

That is, if $W(x, y)$ is a test function then (4.9) reads

$$
\int_0^\infty \int_S y^a \langle \nabla^\varphi U, \nabla^\varphi W \rangle \, d\mu_\varphi \, dy + \int_0^\infty \int_S y^a U_y W_y \, d\mu_\varphi \, dy = c_s (L^s_\varphi u)(W(\cdot, 0)).
$$

By using $U$ as a test function we obtain the energy identity

$$
\int_0^\infty \int_S y^a |\nabla^\varphi U|^2 \, d\mu_\varphi \, dy + \int_0^\infty \int_S y^a |U_y|^2 \, d\mu_\varphi \, dy = c_s \int_S |L^{s/2}_\varphi u|^2 \, d\mu_\varphi,
$$

where we used that $L^{s/2}_\varphi u \in L^2(S, d\mu_\varphi)$.

**Remark 4.2.** As in [8] we could have used the semigroup generated by $L^s_\varphi$ to obtain an equivalent expression for $L^s_\varphi u$ which explicitly shows that the fractional operator $L^s_\varphi$ is a nonlocal integro-differential operator in divergence form. It is also possible to write the solution $U$ above by using the semigroup generated by $L_\varphi$ (see [8, 38, 39]). Instead, by means of a change of variables, in Section 5 we will directly relate the solution $U$ of the extension problem in divergence form (4.8) to the solution of the extension problem in nondivergence form (2.8).
5. NONDIVergence Form MEETS DIVergence Form: PROOF OF THEOREm 1.2

In this section we establish the connection between the nondivergence form and divergence form extension problems (2.8) and (4.8), which will ultimately lead us to the proof of Theorem 1.2. Let us start with the following proposition.

**Proposition 5.1.** For every section \( S := S_\varphi(x_0, R) \) the following inclusion holds true

\[
\text{Dom}_S(L_\varphi^s) \subset \text{Dom}_S(\mathcal{L}_\varphi^s).
\]

**Proof.** Recall from Definition 2.2 that \( \text{Dom}_S(L_\varphi^s) = \text{Dom}_S(L_\varphi) \) as previously defined in (2.2). On the other hand, \( \text{Dom}_S(\mathcal{L}_\varphi^s) = \mathcal{H}_\varphi^s(S) \), see (4.4). Let us remark that, since the eigenvalues \( \{\lambda_k\}_{k \geq 1} \) from (4.2) increase towards \( +\infty \), we have \( \lambda_k^{\varphi} < \lambda_k \) for all sufficiently large \( k \), and then \( \mathcal{H}_\varphi^1(S) \subset \mathcal{H}_\varphi^s(S) \). Also, the fact that the Hessian \( D^2 \varphi \) is a positive definite matrix with entries in \( \mathbb{C} \) on \( [14, p. 241] \) and notice that the extension equation (2.8), \( L(5.6) \), Also, from (1.1), \( L \) by the space with respect to Lebesgue measure.

**5.1. Proof of Theorem 1.2.** The connection between the fractional powers \( L_\varphi^s \) and \( \mathcal{L}_\varphi^s \) will materialize through the change of variables

\[
z = (y/(2s))^{2s}, \quad z > 0 \quad \leftrightarrow \quad y = (2s)z^{1/(2s)}, \quad y > 0,
\]

see [6]. Define

\[
U(x, y) := V(x, z),
\]

for \( x \in S \) and \( y > 0 \), where \( V \) is the unique solution to the extension equation satisfying (2.12), see Section 2. Then,

\[
U_y = V_z z_y = (y/(2s))^{2s-1}V_z,
\]

and

\[
U_{yy} = (y/(2s))^{4s-2}V_{zz} + \frac{2s-1}{2s}(y/(2s))^{2s-2}V_z.
\]

Therefore,

\[
\frac{a_y}{y}U_y + U_{yy} = (y/(2s))^{4s-2}V_{zz} = z^{2-1/s}V_{zz}.
\]

Also, from (1.1),

\[
L^2 \varphi V = \mathcal{L}_\varphi V = \mathcal{L}_\varphi U, \quad \text{in } S \times (0, \infty),
\]

where in the second identity we noticed that \( \mathcal{L}_\varphi \) acts only in the variable \( x \in S \) for each fixed \( z > 0 \) and \( y > 0 \). Therefore, from (5.6) and (5.5), since \( V \) is the solution to the extension equation (2.8),

\[
0 = -L^2 \varphi V + \mu_\varphi z^{2-1/s}V_{zz} = -\mathcal{L}_\varphi U + \mu_\varphi \left( \frac{a_y}{y}U_y + U_{yy} \right)
\]
\[ y^a \operatorname{div}_{x,y}(y^a B_\varphi(x) \nabla_{x,y} U), \]

where \( B_\varphi(x) \) is as in (4.7). Therefore, \( U \) defined by (5.3) is a solution to (4.8) with \( U(x,0) = V(x,0) = v(x) \) for \( v \in \text{Dom}_S(L_\varphi) \). Moreover, by (2.12), we see that \( U(\cdot, y) \to 0 \), as \( y \to \infty \), weakly in \( L^2(S,d\mu_\varphi) \). Hence \( U \) in (5.3) is the unique solution to (4.8).

From (4.9), for every \( x \in S \),
\[ -\lim_{y \to 0^+} y^a U_y(x,y) = c_s L_\varphi^* v(x). \]

On the other hand, it is readily verified that
\[ -y^{1-2s} U_y = -\frac{1}{(2s)^{2s-1}} V_z. \]

Thus,
\[ c_s L_\varphi^* v(x) = -\lim_{y \to 0^+} y^a U_y(x,y) = -\frac{1}{(2s)^{2s-1}} \lim_{z \to 0^+} V_z(x,z) = \frac{d_s}{(2s)^{2s-1}} L_\varphi^* v(x). \]

Hence, as \( c_s = d_s/(2s)^{2s-1} \), for \( v \in \text{Dom}_S(L_\varphi) \) we get
\[ L_\varphi^* v = L_\varphi^* v. \]

Therefore, \( L_\varphi^* v(x) = L_\varphi^* v(x) \) and both the divergence and nondivergence structures occur simultaneously. \( \square \)

**Remark 5.2.** Notice that, in order to keep the equality in (5.7) consistent with (1.1), the righthand sides of \( L_\varphi^* v \) and \( L_\varphi^* v \) in (1.4) must differ by \( \mu_\varphi \), as stemming from the equations (2.1) and (4.1).

**Remark 5.3.** The following finite-energy estimate will play a key role in Section 9:
\[ \int_S \int_0^\infty |\nabla^a V(x,z)|^2 z^{1/s-2} \, dz \, d\mu_\varphi(x) + \int_S \int_0^\infty V_z(x,z)^2 \, dz \, d\mu_\varphi(x) < \infty. \]

We prove (5.8) by using the change of variables (5.2). Indeed,
\[
\int_S \int_0^\infty |\nabla^a V(x,z)|^2 z^{1/s-2} \, dz \, d\mu_\varphi(x) \\
= \int_S \int_0^\infty |\nabla^a U(x,y)|^2 \left( \frac{y}{2s} \right)^{2s(1/s-2)} \left( \frac{y}{2s} \right)^{2s-1} \, dy \, d\mu_\varphi(x) \\
= (2s)^{2s-1} \int_S \int_0^\infty |\nabla^a U(x,y)|^2 y^a \, dy \, d\mu_\varphi(x).
\]

On the other hand, from (5.4),
\[
\int_S \int_0^\infty V_z(x,z)^2 \, dz \, d\mu_\varphi(x) = \int_S \int_0^\infty U_y(x,y)^2 \left( \frac{y}{2s} \right)^{1-2s} \, dy \, d\mu_\varphi(x) \\
= (2s)^{2s-1} \int_S \int_0^\infty U_y(x,y)^2 y^a \, dy \, d\mu_\varphi(x).
\]

Therefore, by the energy identity (4.10),
\[
\int_S \int_0^\infty |\nabla^a V(x,z)|^2 z^{1/s-2} \, dz \, d\mu_\varphi(x) + \int_S \int_0^\infty V_z(x,z)^2 \, dz \, d\mu_\varphi(x)
\]
\[= (2s)^{2s-1} \left[ \int_S \int_0^\infty |\nabla^r U(x, y)|^2 y^s dy \, d\mu_\phi(x) + \int_S \int_0^\infty U(x, y)^2 y^s dy \, d\mu_\phi(x) \right] \]
\[= (2s)^{2s-1} c_s \int_S |\mathcal{L}_\phi^2 u(x)|^2 \, d\mu_\phi(x) < \infty. \quad \square \]

6. Notation and Monge–Ampère Background

Throughout the article, the function \( \phi \) will denote a generic convex function which is used as a placeholder for the functions \( \varphi, \Phi, \) and \( h_s \) to be introduced in Section 7. Let also \( N \) denote a generic dimension that will take the values \( n, n+1, \) or \( 1. \)

For a strictly convex function \( \phi \in C^1(\mathbb{R}^N) \) (strictly convex in the sense that its graph contains no line segments), its associated Monge–Ampère measure \( \mu_\phi \) acts on a Borel set \( E \subset \mathbb{R}^N \) as
\[
\mu_\phi(E) := |\nabla \phi(E)|,
\]
where \( |F| \) denotes the Lebesgue measure of a subset \( F \subset \mathbb{R}^N \). Given \( x \in \mathbb{R}^N \) and \( R > 0 \), its Monge–Ampère section \( S_\phi(x, R) \) is defined as the open, convex set
\[
S_\phi(x, R) := \{ y \in \mathbb{R}^N : \delta_\phi(x, y) < R \}
\]
where
\[
(6.1) \quad \delta_\phi(x, y) := \phi(y) - \phi(x) - \langle \nabla \phi(x), y-x \rangle \quad \forall x, y \in \mathbb{R}^N.
\]
We write \( \mu_\phi \in (DC)_\phi \) if there exists a constant \( C_d \geq 1 \) such that
\[
(6.2) \quad \mu_\phi(S_\phi(x, t)) \leq C_d \mu_\phi(\frac{1}{2} S_\phi(x, t)) \quad \forall x \in \mathbb{R}^N, \forall t > 0,
\]
where, for a convex set \( S \), \( \frac{1}{2} S \) denotes its \( \frac{1}{2} \)-contraction with respect to its center of mass (with the computation of the center of mass based on the Lebesgue measure). The condition \( \mu_\phi \in (DC)_\phi \) is equivalent to the structure of space of homogeneous type for the triple \( (\mathbb{R}^N, \mu_\phi, \delta_\phi) \) (see [31] and references therein), which we understand as the minimal structure to carry out real analysis. It is in this sense that we refer to \( \mu_\phi \in (DC)_\phi \) as a minimal geometric condition.

If \( \phi : \mathbb{R}^N \to \mathbb{R} \) is twice differentiable at point \( x_0 \in \mathbb{R}^N \) with \( \mu_\phi(x_0) = \det D^2 \phi(x_0) > 0 \), then the matrix of cofactors of \( D^2 \phi \) at \( x_0 \) is given by
\[A_\phi(x_0) := D^2 \phi(x_0)^{-1} \mu_\phi(x_0).\]
If \( \phi \) is three times differentiable at a point \( x_0 \in \mathbb{R}^N \) with \( \mu_\phi(x_0) > 0 \), then \( A_\phi(x_0) \) has divergence-free columns (see, for instance, [10, p. 462]). Then, given \( h : \mathbb{R}^N \to \mathbb{R} \) twice differentiable at \( x_0 \) we have
\[
\text{div}(A_\phi(x_0) \nabla h(x_0)) = \text{trace}(A_\phi(x_0) D^2 h(x_0)).
\]

6.1. Convex conjugates. Suppose that \( \phi \in C^2(\mathbb{R}^N) \) with \( D^2 \phi > 0 \). If \( \mu_\phi \in (DC)_\phi \) then \( \nabla \phi : \mathbb{R}^N \to \mathbb{R}^N \) is a continuously differentiable homeomorphism and the convex conjugate of \( \phi \), denoted by \( \psi : \mathbb{R}^N \to \mathbb{R} \), satisfies
\[
(6.3) \quad \nabla \psi(\nabla \phi) = \nabla \phi(\nabla \psi) = id : \mathbb{R}^N \to \mathbb{R}^N.
\]
In particular, we have
\[
(6.4) \quad \psi \in C^2(\mathbb{R}^N) \quad \text{and} \quad D^2 \psi > 0.
\]
In addition, $\mu_\phi \in (DC)_\phi$ implies $\mu_\psi \in (DC)_\psi$ (with doubling constants depending on the ones for $\mu_\phi$ and dimension $N$) and the Monge–Ampère sections of $\phi$ and $\psi$ are related as follows
\begin{equation}
S_\phi(x, \kappa_1 R) \subset \nabla \psi(S_\psi(\nabla \phi(x), R)) \subset S_\phi(x, K_1 R) \quad \forall x \in \mathbb{R}^N, R > 0,
\end{equation}
for constants $0 < \kappa_1 < 1 < K_1 < \infty$ depending only on $C_d$ in (6.2) and dimension $N$. See [12, Section 5] for these and related results.

6.2. Poincaré inequalities. Associated to a strictly convex $\phi \in C^2(\mathbb{R}^N)$ with $D^2 \phi > 0$ and $\mu_\phi \in (DC)_\phi$, Poincaré inequalities with respect to its Monge–Ampère sections and gradient $\nabla \phi := (D^2 \phi)^{-1/2} \nabla$ have been proved in [31, Theorem 1.3]. Namely, there exists a constant $C_P > 0$, depending only on the (DC) constant $N$, such that for every section $S_\phi := S_\phi(x_0, R)$ and every $u \in C^1(S_\phi)$ we have
\begin{equation}
\frac{1}{|S_\phi|} \int_{S_\phi} |u(x) - u_{S_\phi}| \, dx \leq C_P R^2 \left( \frac{1}{|S_\phi|} \int_{S_\phi} |\nabla^\phi u(x)|^2 \, dx \right)^{\frac{1}{2}},
\end{equation}
where $u_{S_\phi} := \frac{1}{|S_\phi|} \int_{S_\phi} u(x) \, dx$ and $|\nabla^\phi u(x)|^2 = \langle D^2 \phi(x)^{-1} \nabla u(x), \nabla u(x) \rangle$.

From now on, for a Borel measure $\mu$, which will be either a Monge–Ampère measure $\mu_\phi$ or the Lebesgue measure, and a measurable set $E \subset \mathbb{R}^N$ we put
$$\int_E f(x) \, d\mu(x) := \frac{1}{\mu(E)} \int_E f(x) \, d\mu(x).$$

6.3. The $L^2(S, d\mu_\phi)$-energy of the quasi-distance $\delta_\phi$. Here we record the following consequence of Lemma 3.1 from [33]: given any strictly convex function $\phi \in C^3(\mathbb{R}^N)$ (no doubling assumptions required) and any section $S := S_\phi(x_0, R)$, we have
\begin{equation}
\int_S \langle D^2 \phi(x)^{-1} (\nabla \phi(x) - \nabla \phi(x_0)), \nabla \phi(x) - \nabla \phi(x_0) \rangle \, d\mu_\phi(x) \leq n R \mu_\phi(S_\phi(x_0, R)).
\end{equation}
By recalling the definition of $\delta_\phi$ from (6.1), for each fixed $x_0 \in \mathbb{R}^N$, we have
$$\nabla \delta_\phi(x_0, x) = \nabla \phi(x) - \nabla \phi(x_0),$$
and then
$$|\nabla^\phi \delta_\phi(x_0, x)|^2 = \langle D^2 \phi(x)^{-1} (\nabla \phi(x) - \nabla \phi(x_0)), \nabla \phi(x) - \nabla \phi(x_0) \rangle,$$
which makes (6.7) an estimate on the $L^2(S, d\mu_\phi)$-energy, with respect to the Monge–Ampère gradient $\nabla^\phi$, of the mapping $x \mapsto \delta_\phi(x_0, x)$.

7. The function $\Phi$

Henceforth, fix $\varphi \in C^3(\mathbb{R}^n)$ with $D^2 \varphi > 0$ in $\mathbb{R}^n$ and $\mu_\varphi \in (DC)_\varphi$. Given $0 < s < 1$ introduce
$$h_s(z) := \frac{s^2}{(1 - s)} |z|^{1/s}, \quad \forall z \in \mathbb{R},$$
and set
\begin{equation}
\Phi(x,z) := \varphi(x) + h_s(z), \quad \forall (x,z) \in \mathbb{R}^n \times \mathbb{R}.
\end{equation}

Observe that both \( h_s \) and \( \Phi \) are strictly convex (in the sense that their graphs do not contain line segments), continuously differentiable functions. From (7.1), for every \((x,z) \in \mathbb{R}^n \times (\mathbb{R} \setminus \{0\})\) we have
\begin{equation}
\mu_\varphi(x,z) = \mu_\varphi(x) h_s(z) = \mu_\varphi(x) |z|^{1/s-2}.
\end{equation}

7.1. The function \( \Phi \) and the (DC) doubling property. By defining (6.1) for \( \Phi \) as in (7.1), for \( X = (x,z), X_0 = (x_0,z_0) \in \mathbb{R}^{n+1} \), we have
\begin{equation}
\begin{align*}
\delta_\Phi(x_0, X) &= \delta_\varphi(x_0, x) + \delta_{h_s}(z_0, z) \\
&= \varphi(x) - \varphi(x_0) - (\nabla \varphi(x_0), x - x_0) + h_s(z) - h_s(z_0) - h_s(z)(z - z_0).
\end{align*}
\end{equation}

Notice that, since \( 0 < s < 1 \), the function \( h_s^p(z) = |z|^{1/s-2} \) is a Muckenhoupt \( A_p \)-weight on the real line for some \( 1 < p < \infty \) if and only if \( 1/s < p + 1 \) and \( h_s^p \in A_1 \) if and only if \( 1/s \leq 2 \) (see Example 9.1.7 on [15, p. 286]). Hence, \( h_s^p \in A_{\infty} \) for every \( 0 < s < 1 \), which makes it a doubling weight on the real line (equivalently, \( h_s \in (DC)_{h_s} \), since in dimension 1 the (DC) doubling property coincides with the usual doubling property) whose doubling constant depends only on \( s \). In particular, there exists \( K_s \geq 1 \), depending only on \( 0 < s < 1 \), such that
\begin{equation}
\delta_{h_s}(z,z') \leq K_s (\min \{ \delta_{h_s}(z,z''), \delta_{h_s}(z'', z) \} + \min \{ \delta_{h_s}(z',z''), \delta_{h_s}(z'', z') \})
\end{equation}
for every \( z, z', z'' \in \mathbb{R} \).

Now, since \( \mu_\varphi \in (DC)_\varphi \) and \( h_s \in (DC)_{h_s} \), from [11, Lemma 6] it follows that \( \Phi \), being the tensor sum of \( \varphi \) and \( h_s \), satisfies \( \Phi \in (DC)_\Phi \) with constants depending only on the (DC) constants for \( \mu_\varphi \), dimension \( n \), and \( s \). In addition, the condition \( \mu_\Phi \in (DC)_\Phi \) is quantitatively equivalent to the existence of \( K \geq 1 \) such that
\begin{equation}
\delta_\Phi(X,Y) \leq K (\min \{ \delta_\Phi(Z,X), \delta_\Phi(X,Z) \} + \min \{ \delta_\Phi(Y,Z), \delta_\Phi(Y,Z) \})
\end{equation}
for every \( X,Y,Z \in \mathbb{R}^{n+1} \).

By [11, Lemma 6] the sections of \( \Phi \) are related to the ones of \( \varphi \) and \( h_s \) by
\begin{equation}
S_\Phi((x_0,z_0), R) \subset S_\varphi(x_0, R) \times S_{h_s}(z_0, R) \subset S_\Phi((x_0,z_0), 2R)
\end{equation}
for every \((x_0,z_0) \in \mathbb{R}^n \times \mathbb{R} \) and \( R > 0 \).

We recall that constants depending only on the (DC) constants for \( \mu_\varphi \) in (6.2), \( 0 < s < 1 \) and dimension \( n \) will be called geometric constants.

By [16, Corollary 3.3.2], the condition \( \mu_\Phi \in (DC)_\Phi \) implies the following doubling property for \( \mu_\Phi \): there exists a geometric constant \( K_d > 1 \) such that
\begin{equation}
\mu_\Phi(S_\Phi(X,2R)) \leq K_d \mu_\Phi(S_\Phi(X,R)) \quad \forall X \in \mathbb{R}^{n+1}, R > 0.
\end{equation}

Iterations of (7.7) yield
\begin{equation}
\mu_\Phi(S_\Phi(X,R)) \leq K_d \left( \frac{R}{r} \right)^\nu \mu_\Phi(S_\Phi(X,r)) \quad \forall X \in \mathbb{R}^{n+1}, 0 < r < R,
\end{equation}
where \( \nu := \log_2 K_d \). Also, by \( \mu_\Phi \in (DC)_\phi \) (as well as the hypotheses \( \Phi \in C^1(\mathbb{R}^{n+1}) \) and its strict convexity), there exists a geometric constant \( K_3 > 1 \) such that for every section \( S := S_\Phi(X,R) \) we have
\[
\mu_\Phi(S) |S| \leq K_3 R^{n+1},
\]
see for instance [12, Theorem 1].

7.2. The matrix of cofactors of \( D^2 \Phi \). From (7.1), for every \( (x,z) \in \mathbb{R}^n \times (\mathbb{R} \setminus \{0\}) \)
\[
D^2 \Phi(x,z) = \begin{pmatrix} D^2 \varphi(x)^{-1} & 0 \\ 0 & |z|^{2-1/s} \end{pmatrix} \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1},
\]
where we have excluded the value \( z = 0 \) to avoid the singularities of \( |z|^{1/s-2} \) or \( |z|^{2-1/s} \).

From (7.2) and (7.10), for every \( (x,z) \in \mathbb{R}^n \times (\mathbb{R} \setminus \{0\}) \) the matrix of cofactors of \( D^2 \Phi(x,z) \) equals
\[
A_\Phi(x,z) := \begin{pmatrix} A_\varphi(x)|z|^{1/s-2} & 0 \\ 0 & \mu_\varphi(x) \end{pmatrix} \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}.
\]

Important features are that \( A_\Phi(x,z) \in L_{loc}^1(\mathbb{R}^n \times \mathbb{R}) \) and \( A_\Phi(x,z) \) is differentiable for (Lebesgue) a.e. \( (x,z) \in \mathbb{R}^n \times \mathbb{R} \). Notice that the first \( n \) columns of \( A_\Phi \) are differentiable with respect to \( x \) and the last column is differentiable with respect to \( z \). Also, since the columns of \( A_\varphi \) are divergence free, so are the columns of \( A_\Phi \).

If \( H : \mathbb{R}^{n+1} \rightarrow \mathbb{R} \) is differentiable at a point \( X = (x,z) \in \mathbb{R}^n \times (\mathbb{R} \setminus \{0\}) \), the Monge–Ampère gradient of \( H \) at \( X \) is then given by
\[
\nabla^\Phi H(X) = D^2 \Phi(X)^{-1/2} \nabla H(X) = (D^2 \varphi(x)^{-1/2} \nabla_x H(x,z), |z|^{1-\frac{1}{s}} H_z(x,z)) \in \mathbb{R}^{n+1},
\]
which implies
\[
|\nabla^\Phi H(X)|^2 = |D^2 \varphi(x)^{-1} \nabla_x H(x,z) \nabla_x H(x,z)| + |z|^{2-1/s} H_z(x,z)^2
= |\nabla \varphi H(x,z)|^2 + |z|^{2-1/s} H_z(x,z)^2.
\]

7.3. The \( L^2(S_\Phi,d\mu_\Phi) \)-energy of the quasi-distance \( \delta_\Phi \). For \( \Phi \) as in (7.1), we will next prove the following counterpart to (6.7): For every section \( S_\Phi(X_0,R) \) it holds true that
\[
\int_{S_\Phi(X_0,R)} \langle A_\Phi(X) (\nabla \Phi(X) - \nabla \Phi(X_0)), \nabla \Phi(X) - \nabla \Phi(X_0) \rangle \, dX
\leq (n+2) K_d R \mu_\Phi(S_\Phi(X_0,R)).
\]
Notice that \( \Phi \notin C^3(\mathbb{R}^{n+1}) \), so we cannot directly apply (6.7) with \( \phi = \Phi \). Instead, we will use the tensorial nature of \( \Phi \). Given a section \( S_\Phi := S_\Phi(X_0,R) \), with \( X_0 = (x_0,z_0) \in \mathbb{R}^{n+1} \), by means of (7.11) and (7.3) we can write
\[
\int_{S_\Phi} \langle A_\Phi(X) (\nabla \Phi(X) - \nabla \Phi(X_0)), \nabla \Phi(X) - \nabla \Phi(X_0) \rangle \, dX
\]
\[
= \int_{S_\Phi} |\nabla^\Phi \delta_\Phi(X_0,X)|^2 \, d\mu_\Phi(X)
\]
\[
= \int_{S_\Phi} \left( (D^2 \varphi(x)^{-1} \nabla \delta_\Phi(X_0,X), \nabla \delta_\Phi(X_0,X)) + |z|^{2-1/s} \frac{\partial \delta_\Phi}{\partial z}(X_0,X)^2 \right) \, d\mu_\Phi(X)
\]
From the first inclusion in (7.6) we have

\[ \int_{S_h} (|\nabla \delta \varphi(x_0, x)|^2 + |z|^{2-1/s}(h'_s(z) - h'_s(z_0))^2) \, d\mu_\varphi(X). \]

Now, from the first inclusion in (7.6) and (6.7) (used with \( \phi = \varphi \), since \( \varphi \in C^3(\mathbb{R}^n) \)),

\[ \int_{S_h} |\nabla \delta \varphi(x_0, x)|^2 \, d\mu_\varphi(X) \leq \int_{S_{\varphi(x_0,R)}} |\nabla \delta \varphi(x_0, x)|^2 \, d\mu_\varphi(x) \times \int_{S_{h_s(z_0,R)}} h'_s(z) \, dz \]

\[ \leq nR \mu_\varphi(S_{\varphi(x_0,R)}) \mu_{h_s}(S_{h_s(z_0,R)}) = nR \mu_\varphi(S_{\varphi(x_0,R)} \times S_{h_s(z_0,R)}) \]

\[ \leq nR \mu_\varphi(S_{\Phi(X_0,2R)}) \leq nK_d R \mu_\varphi(S_{\Phi(X_0, R)}), \]

where for the last two inequalities above we used the second inclusion in (7.6) and the doubling property (7.7). On the other hand, by (7.2),

\[ \int_{S_h} |z|^{2-1/s}(h'_s(z) - h'_s(z_0))^2 \, d\mu_\varphi(X) = \int_{S_h} (h'_s(z) - h'_s(z_0))^2 \, d\mu_\varphi(x) \, dz. \]

Let us write the one-dimensional section \( S_{h_s(z_0,R)} \) as \( S_{h_s(z_0,R)} = (z_\ell, z_r) \), where \( z_\ell, z_r \in \mathbb{R} \) satisfy

\[ h_s(z_\ell) - h_s(z_0) - h'_s(z_0)(z_\ell - z_0) = h_s(z_r) - h_s(z_0) - h'_s(z_0)(z_r - z_0) = R. \]

From the first inclusion in (7.6) we have

\[ \int_{S_h} (h'_s(z) - h'_s(z_0))^2 \, d\mu_\varphi(x) \, dz \leq \mu_\varphi(S_{\varphi(x_0,R)}) \int_{z_\ell}^{z_r} (h'_s(z) - h'_s(z_0))^2 \, dz. \]

As \( h'_s \) is increasing,

\[ \int_{z_\ell}^{z_r} (h'_s(z) - h'_s(z_0))^2 \, dz \leq (h'_s(z_r) - h'_s(z_\ell)) \int_{z_\ell}^{z_r} (h'_s(z) - h'_s(z_0))^2 \, dz \]

\[ = \mu_{h_s}(z_\ell, z_r) \int_{z_\ell}^{z_r} |h'_s(z) - h'_s(z_0)| \, dz. \]

At this point we split the last integral above as

\[ \int_{z_\ell}^{z_r} |h'_s(z) - h'_s(z_0)| \, dz = \int_{z_\ell}^{z_0} (h'_s(z_0) - h'_s(z)) \, dz + \int_{z_0}^{z_r} (h'_s(z) - h'_s(z_0)) \, dz \]

\[ = (h'_s(z_0)(z_0 - z_\ell) - h_s(z_0) + h_s(z_\ell)) + (h_s(z_r) - h_s(z_0) - h'_s(z_0)(z_r - z_0)) = 2R, \]

where the last equality is due to (7.13). Therefore,

\[ \int_{S_h} |z|^{2-1/s}(h'_s(z) - h'_s(z_0))^2 \, d\mu_\varphi(X) \leq \mu_\varphi(S_{\varphi(x_0,R)}) \mu_{h_s}(z_\ell, z_r) \int_{z_\ell}^{z_r} |h'_s(z) - h'_s(z_0)| \, dz = \mu_\varphi(S_{\varphi(x_0,R)}) \mu_{h_s}(S_{h_s(z_0,R)}) 2R \]

\[ \leq 2R \mu_\varphi(S_{\Phi(X_0,2R)}) \leq 2K_d R \mu_\varphi(S_{\Phi(X_0, R)}), \]

and (7.12) follows. \( \square \)
8. The function $\Phi$ and a weak Poincaré inequality

Our goal in this section is to prove a version of the Poincaré inequality (6.6) with Lebesgue measure being replaced by the Monge–Ampère measure $\mu_\Phi$. We will reason along the lines of [32, Section 4], where the change in the opposite direction (i.e. from Monge–Ampère to Lebesgue measure) was made by means of convex conjugation. In the case of $\Phi$, however, an approximation argument will be used to circumvent the fact that $\Phi \notin C^2(\mathbb{R}^{n+1})$.

**Theorem 8.1.** Let $\Phi$ be as in (7.1). Then there exist geometric constants $K_2 > 1$ and $K_P > 0$, such that for every section $S_\Phi := S_\Phi(X_0, R)$ and $G \in C(S_\Phi(X_0, K_2 R))$ with $\nabla^2 G \in L^2(S_\Phi(X_0, K_2 R), d\mu_\Phi)$ we have

$$
\int_{S_\Phi} |G(X) - G_{S_\Phi}| d\mu_\Phi(X) \leq K_P R^{\frac{1}{2}} \left( \int_{S_\Phi(X_0, K_2 R)} |\nabla^2 G(X)|^2 d\mu_\Phi(X) \right)^{\frac{1}{2}},
$$

where

$$G_{S_\Phi} := \int_{S_\Phi} G(X) d\mu_\Phi(X).$$

**Proof.** Let $\eta \in C^1(\mathbb{R})$ be nonnegative and compactly supported in $[-1, 1]$ with $\int_{\mathbb{R}} \eta = 1$. For $\varepsilon > 0$ set $\eta_\varepsilon(z) := \frac{1}{\varepsilon} \eta(\frac{z}{\varepsilon})$ and define

$$h_{s,\varepsilon}(z) := h_s \ast \eta_\varepsilon(z) \quad \forall z \in \mathbb{R}$$

and

$$\Phi_\varepsilon(x, z) := \varphi(x) + h_{s,\varepsilon}(z) \quad \forall (x, z) \in \mathbb{R}^n \times \mathbb{R}.$$ 

Then, for every $\varepsilon > 0$, we have that $\Phi_\varepsilon \in C^2(\mathbb{R}^{n+1})$ with $D^2 \Phi_\varepsilon > 0$. In addition, $h_{s,\varepsilon}''$ converges to $h_s''$ in $L^1_{\text{loc}}(\mathbb{R})$ and, consequently, $\mu_{\Phi_\varepsilon}$ converges to $\mu_\Phi$ in $L^1_{\text{loc}}(\mathbb{R}^{n+1})$ as $\varepsilon$ tends to 0. Also, $\nabla \Phi_\varepsilon$ converges to $\nabla \Phi$ uniformly on compact sets of $\mathbb{R}^{n+1}$.

The matrix of cofactors of $D^2 \Phi_\varepsilon(x, z)$ is given by

$$A_{\Phi_\varepsilon}(x, z) := \begin{pmatrix} A_{\varphi}(x) & h_{s,\varepsilon}''(z) \\ 0 & \mu_\varepsilon(x) \end{pmatrix} \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}.$$ 

Now, a simple computation shows that, for each $\varepsilon > 0$, the measure $h_{s,\varepsilon}''$ is a doubling measure on the real line with doubling constant smaller than or equal to the doubling constant for $h_s''$, which, in turn, depends only on $s$. Indeed, let $C_s \geq 1$ denote the doubling constant for $h_s''$ as measure on the real line. Given $c \in \mathbb{R}$ and $r > 0$, let $I_c := [c - r, c + r]$ and $2I_c := [c - 2r, c + 2r]$. Then,

$$\int_{2I_c} h_{s,\varepsilon}''(z) dz = \int_{2I_c} \int_{\mathbb{R}} \eta_\varepsilon(y) h_{s,\varepsilon}''(z - y) dy dz = \int_{\mathbb{R}} \int_{2I_c} \eta_\varepsilon(y) h_{s,\varepsilon}''(z - y) dz dy.$$ 

For each $y \in \mathbb{R}$, by changing variables $w := y - z$ we get $z \in 2I_c$ if and only if $w \in 2I_{c-y}$. Hence, using that $h_{s,\varepsilon}''$ is even and doubling with constant $C_s$,

$$\int_{\mathbb{R}} \int_{2I_{c-y}} \eta_\varepsilon(y) h_{s,\varepsilon}''(w) dw = C_s \int_{I_{c-y}} \int_{\mathbb{R}} \eta_\varepsilon(y) h_{s,\varepsilon}''(y - z) dy dz = C_s \int_{I_c} h_{s,\varepsilon}''(z) dz.$$
Thus,
\[ \int_{2I_\varepsilon} h_{\varepsilon, z}(z) \, dz \leq C_\varepsilon \int_{I_\varepsilon} h_{\varepsilon, z}(z) \, dz, \]
uniformly in \( \varepsilon > 0 \). By [11, Lemma 6], it follows that the (DC) constants for \( \Phi_\varepsilon \) are controlled by the one for \( \Phi \) uniformly in \( \varepsilon > 0 \).

Next, let \( \Psi_\varepsilon \) denote the convex conjugate of \( \Phi_\varepsilon \). By (6.4) we have \( \Psi_\varepsilon \in C^2(\mathbb{R}^{n+1}) \) with \( D^2\Psi_\varepsilon > 0 \) for every \( \varepsilon > 0 \).

Fix \( Y_0 \in \mathbb{R}^{n+1}, R > 0 \), and set \( X_0 := \nabla \Phi_\varepsilon(Y_0) \). Let \( \kappa_1, K_1 \) be the geometric constants from (6.5) applied to \( \Phi_\varepsilon \) and define \( S_{\Phi_\varepsilon} := S_{\Phi_\varepsilon}(Y_0, R/k_1) \) and \( K_2 := K_1/k_1 \). Hence, the inclusions (6.5) yield
\[ S_{\Phi_\varepsilon}(X_0, R) \subset S_{\Psi_\varepsilon} := \nabla \Psi_\varepsilon(S_{\Phi_\varepsilon}) \subset S_{\Phi_\varepsilon}(X_0, K_2 R). \]
Assume first that \( G \in C^1(S_{\Phi_\varepsilon}(X_0, K_2 R)) \) and define \( H \in C^1(S_{\Psi_\varepsilon}) \) as
\[ H(Y) := G(\nabla \Psi_\varepsilon(Y)) \quad \forall Y \in S_{\Psi_\varepsilon}, \]
which, by putting \( Y := \nabla \Phi_\varepsilon(X) \), yields
\[ \nabla \Psi_\varepsilon H(Y) = D^2 \Psi_\varepsilon(Y)^{-1/2} \nabla H(Y) = D^2 \Psi_\varepsilon(Y)^{-1/2} D^2 \Psi_\varepsilon(Y) \nabla G(X) = D^2 \Phi_\varepsilon(X)^{-1/2} \nabla G(X) = \nabla \Phi_\varepsilon G(X). \]
Now, by the Poincaré inequality (6.6) applied to \( \Psi_\varepsilon \) on the section \( S_{\Psi_\varepsilon} \) and \( H \in C^1(S_{\Psi_\varepsilon}) \), we get
\[ \left( \int_{S_{\Psi_\varepsilon}} |H(Y) - H_{S_{\Psi_\varepsilon}}| \, dY \right)^{1/2} \leq C_P R \left( \int_{S_{\Psi_\varepsilon}} |\nabla \Psi_\varepsilon H(Y)|^2 \, dY \right)^{1/2}, \]
with \( C_P > 0 \) a geometric constant, and by changing variables \( Y := \nabla \Phi_\varepsilon(X) \) in (8.3) and using that the identity (6.3) gives \( dY = \mu_{\Phi_\varepsilon}(X) \, dX \) and
\[ |\nabla \Psi_\varepsilon H(Y)|^2 \, dY = |\nabla \Phi_\varepsilon G(X)|^2 \mu_{\Phi_\varepsilon}(X) \, dX = \langle A_{\Phi_\varepsilon}(X) \nabla G(X), \nabla G(X) \rangle \, dX. \]
we obtain
\[ \left( \int_{S_{\Psi_\varepsilon}} |G(X) - G_{S_{\Psi_\varepsilon}}| \mu_{\Phi_\varepsilon}(X) \, dX \right)^{1/2} \leq \left( \frac{C_P}{\mu_{\Phi_\varepsilon}(S_{\Psi_\varepsilon})} \int_{S_{\Psi_\varepsilon}} |\nabla \Psi_\varepsilon G(X)|^2 \mu_{\Phi_\varepsilon}(X) \, dX \right)^{1/2}, \]
where we have put
\[ G_{S_{\Psi_\varepsilon}} := \frac{1}{\mu_{\Phi_\varepsilon}(S_{\Psi_\varepsilon})} \int_{S_{\Psi_\varepsilon}} G(X) \mu_{\Phi_\varepsilon}(X) \, dX \]
and used that \( |S_{\Psi_\varepsilon}| = |\nabla \Phi_\varepsilon(S_{\Psi_\varepsilon})| = \mu_{\Phi_\varepsilon}(S_{\Psi_\varepsilon}) \), due to the definition of \( S_{\Psi_\varepsilon} := \nabla \Psi(S_{\Phi_\varepsilon}) \) in (8.2). Next, the inclusions (8.2) and the doubling property (7.8) give
\[ \frac{1}{K_d K^2_\mu_{\Phi_\varepsilon}(S_{\Phi_\varepsilon}(X_0, R))} \leq \frac{1}{\mu_{\Phi_\varepsilon}(S_{\Phi_\varepsilon}(X_0, K_2 R))} \leq \frac{1}{\mu_{\Phi_\varepsilon}(S_{\Psi_\varepsilon})} \leq \frac{1}{K_d K^2_\mu_{\Phi_\varepsilon}(S_{\Phi_\varepsilon}(X_0, R))} \]
Consequently, the inclusions (8.2), the inequality (8.4), and the fact that \( A_{\Phi_\varepsilon} \) converges to \( A_\Phi \) in \( L^1_{loc}(\mathbb{R}^{n+1}) \), imply (8.1) with \( K_P := 2C_P(K_d K_2)^{3/2} \) in the case \( G \in \).
we have 

\( C^1(S_\phi(X_0, K_2R)) \). Then, (8.1) in the case of \( \nabla^s \phi G \in L^2(S_\phi(X_0, K_2R), d\mu_\phi) \) follows by approximation in \( L^2(S_\phi(X_0, K_2R), d\mu_\phi) \). Just notice that smooth functions are dense in \( L^2(\Omega, w(X)dX) \) for every open, bounded, convex subset \( \Omega \subset \mathbb{R}^{n+1} \) and every \( w \in A_\infty(\Omega) \) and that on the compact set \( \overline{\Omega} \) we have \( \mu_\phi(x, z) \sim |z|^{1/s-2} \in A_\infty(\Omega) \). □

Next we record a simple consequence of the weak Poincaré's inequality known as “Fabes lemma”.

**Corollary 8.2.** Let \( \Phi \) be as in (7.1) and let \( K_P > 0 \) and \( K_2 > 1 \) be the geometric constants from Theorem 8.1. Then, for every \( \varepsilon \in (0, 1) \), every section \( S := S_\phi(X_0, R) \) and every \( G \in C(S_\phi(X_0, K_2R)) \) with \( \nabla^s \phi G \in L^2(S_\phi(X_0, K_2R), d\mu_\phi) \) and

\[
\mu_\phi(\{X \in S : G(X) = 0\}) \geq \varepsilon \mu_\phi(S),
\]

we have that

\[
\left(1 + \frac{1}{\varepsilon}\right) K_P \left( \int_{S_\phi(X_0, K_2R)} |\nabla^s \phi G(X)|^2 d\mu_\phi(X) \right)^{1/2} \leq \int_S |G(X)| d\mu_\phi(X) \leq \left(1 + \frac{1}{\varepsilon}\right) \int_S |G - G_S| d\mu_\phi(X).
\]

**Proof.** Setting \( E := \{X \in S : G(X) = 0\} \) and \( G_E := \int_E G \ d\mu_\phi = 0 \), for \( X \in S \) we have

\[
|G(X)| \leq |G(X) - G_S| + |G_S - G_E| \leq |G(X) - G_S| + \left(\frac{\mu_\phi(S)}{\mu_\phi(E)}\right) \int_S |G - G_S| d\mu_\phi
\]

so that

\[
\int_S |G(X)| d\mu_\phi(x) \leq \left(1 + \frac{1}{\varepsilon}\right) \int_S |G(X) - G_S| d\mu_\phi(X)
\]

and then (8.5) follows from the weak-Poincaré’s inequality (8.1). □

**9. The class \( S(Q) \)**

From here on we fix an arbitrary \( p_0 \in \mathbb{R}^n \) and \( R_0 > 0 \) and define the cylinder

\[ Q := S_\phi(p_0, R_0) \times \mathbb{R} \subset \mathbb{R}^{n+1}. \]

Also, let us put

\[
Z^+ := \{(x, z) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n, z > 0\}, \\
Z^- := \{(x, z) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n, z < 0\}, \\
Z_0 := \{(x, 0) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n\},
\]

and define

\[
Q^+ := Q \cap Z^+ \quad \text{and} \quad Q^- := Q \cap Z^-.
\]

Henceforth, all of our sub- and super-solutions will belong to a class \( S(Q) \) modeled after some key properties of \( \overline{V}(x, z) := V(x, |z|) \) where \( V \), as in (2.7), is the solution to the extension problem (2.11) in \( S_0 := S_\phi(p_0, R_0) \) for a nonnegative \( v \in \text{Dom}_{S_0}(L_\varphi) \).
9.1. The definition of $S(Q)$. We write $F \in S(Q)$ if the following conditions hold.

(i) $F(x, z) = F(x, -z)$ for every $(x, z) \in Q$;
(ii) $F \in C(Q) \cap C^2(Q \setminus Z_0)$, in particular, $\nabla F$, and then $\nabla^\phi F$, exist a.e. in $Q$;
(iii) for every section $S_\phi(X_0, R)$ with $S_\phi(X_0, 2R) \subset\subset Q$ we have
   
   $$\nabla^\phi F \in L^2(S_\phi(X_0, 2R), d\mu_\phi);$$

(iv) for every $x \in S_\phi(p_0, R_0)$ the limit
   
   $$F_z(x, 0^+) := \lim_{z \to 0^+} F_z(x, z)$$

   exists and the function $x \mapsto F_z(x, 0^+)$ is continuous in $S_\phi(p_0, R_0)$. Notice that if
   the limit in (9.1) exists, then we have
   
   $$F_z(x, 0^+) = \lim_{z \to 0^+} \frac{F(x, z) - F(x, 0)}{z}.$$  

9.2. The definition of $F_{z, 0^+}$. Given $F \in S(Q)$ and an open subset $\Omega \subset\subset Q$, define

$$F_{z, 0^+}(\Omega) := \left\{ \begin{array}{ll}
\sup \{ |F_z(x, 0^+)| : (x, 0) \in \Omega \cap Z_0 \} & \text{if } \Omega \cap Z_0 \neq \emptyset, \\ 
0 & \text{if } \Omega \cap Z_0 = \emptyset.
\end{array} \right.$$  

The continuity of the function $x \mapsto F_z(x, 0^+)$ in $S_\phi(p_0, R_0)$ ensures that $F_{z, 0^+}(\Omega) < \infty$ for every open subset $\Omega \subset\subset Q$.

9.3. The fact that $\tilde{V} \in S(Q)$. Let $V$ be the solution to the extension problem (2.11) in the section $S_\phi(p_0, R_0)$ and set $\tilde{V}(x, z) := V(x, |z|)$. Then $\tilde{V}$ satisfies that

$$\tilde{V} \in C(Q) \cap C^2(Q \setminus Z_0), \quad \tilde{V} \text{ is even with respect to } z,$$

and for every $x \in S_\phi(p_0, R_0), \tilde{V}_z(x, 0^+) := \lim_{z \to 0^+} \tilde{V}_z(x, z) = \lim_{z \to 0^+} V_z(x, z) = -d_s L^s_\phi v(x),$

with $L^s_\phi v \in C_0(S_\phi(p_0, R_0))$ for every $v \in \text{Dom}_S(L^s_\phi)$. Also, given $X_0 = (x_0, z_0) \in \mathbb{R}^{n+1}$ and $R > 0$ such that $S_\phi(X_0, 2R) \subset\subset Q$, the inclusions (7.6) yield

$$S_\phi(X_0, R) \subset S_\phi(x_0, R) \times S_{h_0}(z_0, R) \subset S_\phi(X_0, 2R) \subset\subset Q.$$

Now, by recalling (7.11), to get an expression for $|\nabla^\phi \tilde{V}(X)|^2$, and by the energy estimate (5.8) from Remark 5.3, we obtain

$$\int_{S_\phi(X_0, R)} |\nabla^\phi \tilde{V}(X)|^2 d\mu_\phi(X) \leq \int_{Q} |\nabla^\phi \tilde{V}(X)|^2 d\mu_\phi(X)$$

$$= \int_{S_\phi(p_0, R)} \int_{-\infty}^{\infty} \left( |\nabla^\phi \tilde{V}(x, z)|^2 + |z|^{2-1/s} \tilde{V}_z(x, z)^2 \right) |z|^{1/s-2} dz d\mu_\phi(x)$$

$$= 2 \int_{S_\phi(p_0, R)} \int_{0}^{\infty} |\nabla^\phi \tilde{V}(x, z)|^2 z^{1/s-2} dz d\mu_\phi(x) + \int_{S_\phi(p_0, R)} \int_{0}^{\infty} \tilde{V}_z(x, z)^2 dz d\mu_\phi(x) < \infty.$$
10. The critical-density estimate

For $s \in (0, 1)$ introduce

$$M_s := \frac{2^{2+s}(1-s)^s}{s^2} + 8 \quad \text{and} \quad \beta_s := 4K_s(1 + 8K_s),$$

where $K_s \geq 1$ is the quasi-triangle constant in (7.4), which depends only on $s$. The main result in this section is the following critical-density estimate.

**Theorem 10.1.** Let $\Phi$ be as in (7.1). There exist geometric constants $\theta_0, \varepsilon_0 \in (0, 1)$ such that for every section $S_R := S_\Phi(X_0, R)$ with $S_{\beta_s R} := S_\Phi(X_0, \beta_s R) \subset Q$ and every $W \in \mathcal{S}(Q)$ with $L_\Phi W = -\text{trace} \left( (D^2 \Phi)^{-1} D^2 W \right) \geq 0$ pointwise in $Q^+$, the inequalities

$$W_{z,0}^+(S_{\beta_s R})R^s \leq \theta_0$$

and

$$\inf_{S_R} W \leq 1$$

imply

$$\mu_\Phi \left( \{ X \in S_{\beta_s R} : W(X) < M_s \} \right) \geq \varepsilon_0 \mu_\Phi (S_{\beta_s R}).$$

As mentioned in the introduction, due to the degeneracy of $D^2 \Phi$ on the hyperplane $Z_0$, one cannot directly resort to the proof of Theorem 1 in [5] or the one for Theorem 2 in [30]. In order to address the degeneracy or singularity of $D^2 \Phi$ on the hyperplane $Z_0$, the proof of Theorem 10.1 will be broken down into three cases according to the position of the sections of $\Phi$ with respect to $Z_0$. These three cases are illustrated in Figure 1.

In the case of sections intersecting $Z_0$, the $L^\infty$-norm of normal derivatives, in the sense of (9.1), of super-solutions will play a key role in addressing the degeneracy or singularity of $L_\Phi$.

![Figure 1](image-url)  

Figure 1. The three cases for a section of $\Phi$ (within $Q$) to be considered in the proof of Theorem 10.1. When the section intersects $Z_0$, the control on the size of the normal derivative of a solution, as defined by (9.2), will counteract the degeneracy or singularity of $L_\Phi$ at $Z_0$. 


Let us start by stating a version of the Aleksandrov–Bakelman–Pucci maximum principle on which the cases are built.

10.1. The ABP maximum principle. Recall that given a domain \( \Omega \subset \mathbb{R}^N \) and \( u : \Omega \to \mathbb{R} \), the normal mapping of \( u \) at \( x \in \Omega \), denoted by \( \partial u(x) \), is defined as the set

\[
\partial u(x) := \{ p \in \mathbb{R}^N : u(y) \geq u(x) + \langle p, y - x \rangle \; \forall y \in \Omega \}
\]

and, given \( E \subset \Omega \),

\[
\partial u(E) := \bigcup_{x \in E} \partial u(x).
\]

Also, let \( \Gamma_u \) denote the convex envelope of \( u \) in \( \Omega \) and let \( \mathcal{C}_u := \{ x \in \Omega : u(x) = \Gamma_u(x) \} \) denote the contact set of \( u \) and \( \Gamma_u \) in \( \Omega \). In particular (see [16, pp.13-16]),

\[
(10.2) \quad \partial \Gamma_u(\mathcal{C}_u) = -\partial(-u)(\mathcal{C}_u).
\]

The next lemma allows to replace \( \text{diam}(\Omega) \) in [16, Theorem 1.4.5] with \( |\Omega|^{1/N} \) when \( \Omega \) is bounded and convex (its proof is a combination of the one for Theorem 1.4.5 from [16, p.15] and the normalization technique in [5, Section 1]).

**Lemma 10.2.** Let \( \Omega \subset \mathbb{R}^N \) be open, convex, bounded and let \( U \in C(\overline{\Omega}) \) satisfy \( U \leq 0 \) on \( \partial \Omega \). Let \( \Gamma_{-U} \) and \( \mathcal{C}_{-U} \) denote the convex envelope of \( -U \) in \( \Omega \) and the contact set of \( -U \) with \( \Gamma_{-U} \) in \( \Omega \), respectively. Then, there exists a dimensional constant \( C_N > 0 \) such that

\[
\max_{\Omega} U \leq C_N |\Omega|^{1/N} |\partial(\Gamma_{-U})(\mathcal{C}_{-U})|^{1/N}.
\]

We will be using the following consequence of Lemma 10.2.

**Corollary 10.3.** Let \( \Omega \subset \mathbb{R}^N \) be open, convex, and bounded. Suppose that \( H \in C(\overline{\Omega}) \) satisfies the following conditions

(i) \( H \geq 0 \) on \( \partial \Omega \),

(ii) there is an open set \( \Omega' \subset \Omega \) such that \( H \in C^2(\Omega') \) and \( \mathcal{C}_{-H^-} \subset \Omega' \), where \( H^- := \max\{0, -H\} \).

Then,

\[
\max_{\Omega} H^- \leq C_N |\Omega|^{1/N} \left( \int_{\mathcal{C}_{-H^-}} |\det D^2 H(X)| \, dX \right)^{1/N}.
\]

**Proof.** Lemma 10.2 applied to \( H^- \) yields

\[
(10.3) \quad \max_{\Omega} H^- \leq C_N |\Omega|^{1/N} |\partial(\Gamma_{-H^-})(\mathcal{C}_{-H^-})|^{1/N}.
\]

By (10.2) with \( u = H^- \),

\[
(10.4) \quad \partial(\Gamma_{-H^-})(\mathcal{C}_{-H^-}) = -\partial(-H^-)(\mathcal{C}_{-H^-}).
\]

Now, for \( X \in \mathcal{C}_{-H^-} \) the fact that \( H^- = 0 \) on \( \partial \Omega \) yields \( -H^-(X) = H(X) \leq 0 \) (with \( H(X) < 0 \) unless \( H \equiv 0 \)). Let us see that this implies the inclusion

\[
(10.5) \quad \partial(-H^-)(\mathcal{C}_{-H^-}) \subset \partial H(\mathcal{C}_{-H^-}).
\]

Indeed, given \( X \in \mathcal{C}_{-H^-} \) and \( P \in \partial(-H^-)(X) \) the definition of normal mapping gives

\[
(10.6) \quad -H^-(Y) \geq -H^-(X) + \langle P, Y - X \rangle \quad \forall Y \in \Omega,
\]
and since \(-H^\pm(X) = H(X)\) and \(-H^\pm(Y) \leq H(Y)\) for every \(Y \in \Omega\), the inequality (10.6) implies that \(P \in \partial H(X)\), thus proving (10.5). By combining (10.4) and (10.5), we have

\[
\partial(\Gamma_{-H^-})(C_{-H^-}) \subset -\partial H(C_{-H^-}),
\]

so that, from (10.3),

\[
\max_{\Omega} H^- \leq C_N |\Omega|^{\frac{1}{n}} |\partial H(C_{-H^-})|^\frac{1}{n}.
\]

Finally, by the assumptions \(C_{-H^-} \subset \Omega'\) and \(H \in C^2(\Omega')\), the inequality

\[
|\partial H(C_{-H^-})| \leq \int_{C_{-H^-}} |\det D^2 H(X)| \, dX
\]

follows from the usual formula of change of variables and the proof is complete. \(\square\)

10.2. The case of sections of \(\Phi\) away from \(Z_0\).

**Theorem 10.4.** Let \(\Phi\) be as in (7.1). Let \(X_0 \in \mathbb{R}^n \times \mathbb{R}\) and \(R > 0\) such that \(S_\Phi(X_0, 2R) \subset \subset Q\). Set \(S_R := S_\Phi(X_0, R)\), \(S_{2R} := S_\Phi(X_0, 2R)\) and suppose that \(S_{2R} \cap Z_0 = \emptyset\). Then, there exists a geometric constant \(\varepsilon_1 \in (0, 1)\) such that for every \(W \in \mathcal{S}(Q)\) with \(L_\Phi W = -\text{trace}((D^2\Phi)^{-1} D^2 W) \geq 0\) and \(W \geq 0\) in \(S_{2R}\), the inequality

(10.7)

\[
\inf_{S_R} W \leq 1
\]

implies

(10.8)

\[
\mu_\Phi(\{X \in S_{2R} : W(X) < 4\}) \geq \varepsilon_1 \mu_\Phi(S_{2R}).
\]

**Proof.** Introduce the auxiliary function

\[
H(X) := W(X) + 4 \left( \frac{\delta_\Phi(X_0, X)}{2R} - 1 \right) \quad \forall X \in \mathbb{R}^{n+1},
\]

where the function

\[
X \mapsto \delta_\Phi(X_0, X) := \Phi(X) - \Phi(X_0) - \langle \nabla \Phi(X_0), X - X_0 \rangle,
\]

is convex with \(D^2 \delta_\Phi(X_0, \cdot) = D^2 \Phi\). By (10.7), there is \(X_1 \in S_\Phi(X_0, R)\) such that \(W(X_1) < 1\), and then

\[
H(X_1) = W(X_1) + 4 \left( \frac{\delta_\Phi(X_0, X_1)}{2R} - 1 \right) < 1 + 4 \left( \frac{1}{2} - 1 \right) = -1.
\]

Now, the ABP maximum principle in Corollary 10.3 applied to the function \(H^\pm\) on the convex set \(\Omega = \Omega' = S_{2R}\) (notice that \(H = W \geq 0\) on \(\partial S_{2R}\)) yields

\[
1 < H^\pm(X_1)^{n+1} \leq C_n |S_{2R}| \int_{C_{-H^-}} |\det D^2 H(X)| \, dX.
\]

In particular, on the contact set \(C_{-H^-}\) we have that \(H\) is negative, and then

\[
C_{-H^-} \subset \{X \in S_{2R} : H(X) < 0\} = \{X \in S_{2R} : W(X) < 4(1 - \delta_\Phi(X_0, X)/2R)\}\]

\[
\subset \{X \in S_{2R} : W(X) < 4\}.
\]
Also, $D^2H \geq 0$ on $C_{-H}$ so that, recalling that $A_\Phi := (\det D^2\Phi)D^2\Phi^{-1}$ is positive in $S_{2R}$, on $C_{-H}$ we have

$$0 \leq \det D^2H = \det(A_\Phi D^2H)(\det A_\Phi)^{-1} \leq \left[\frac{\text{trace}(A_\Phi D^2H)}{n + 1}\right]^{n+1}(\det D^2\Phi)^{-n}$$

$$= \left[\frac{\text{trace}(A_\Phi D^2W) + \frac{2}{R}\text{trace}(A_\Phi D^2\Phi)}{n + 1}\right]^{n+1}(\det D^2\Phi)^{-n}$$

$$\leq \left(\frac{2\det D^2\Phi}{R}\right)^{n+1}(\det D^2\Phi)^{-n} = \left(\frac{2}{R}\right)^{n+1}\det D^2\Phi.$$

Therefore,

$$1 \leq C_n|S_{2R}|\left(\frac{2}{R}\right)^{n+1}\int_{C_{-H}}\det D^2\Phi(X)\,dX$$

$$\leq C_n|S_{2R}|\left(\frac{2}{R}\right)^{n+1}\mu_\Phi(\{X \in S_{2R} : W(X) < 4\}).$$

Then, by recalling the definition of $K_3$ in (7.9),

$$1 \leq \frac{C_n4^{n+1}K_3}{\mu_\Phi(S_{2R})}\mu_\Phi(\{X \in S_{2R} : W(X) < 4\})$$

and (10.8) follows with $\varepsilon_1 := (C_n4^{n+1}K_3)^{-1}$. \qed

10.3. The case of sections of $\Phi$ centered at $Z_0$. For each $s \in (0,1)$, set

$$q_s := \frac{2^s(1-s)^s}{s^{2s}}.$$

**Theorem 10.5.** Let $\Phi$ be as in (7.1). Let $x_0 \in \mathbb{R}^n$ and $R > 0$ such that $S_{\Phi}(x_0,0), 2R) \subset Q$. Put $S_R := S_{\Phi}(x_0,0), R)$ and $S_{2R} := S_{\Phi}(x_0,0), 2R)$. Suppose that $W \in S(Q)$ satisfies $L_{\Phi}W = -\text{trace}((D^2\Phi)^{-1}D^2W) \geq 0$ and $W \geq 0$ pointwise in $S_{2R} \setminus Z_0$.

Then, there exist geometric constants $\theta_2, \varepsilon_2 \in (0,1)$ such that the inequalities

$$W_{z,0} + (2R)R^s \leq \theta_2 \tag{10.9}$$

and

$$\inf_{S_R} W \leq 1 \tag{10.10}$$

imply

$$\mu_\Phi(\{X \in S_{2R} : W(X) < 4q_s + 8\}) \geq \varepsilon_2\mu_\Phi(S_{2R}). \tag{10.11}$$

**Proof.** Notice that the expression for $\delta_\Phi$ in (7.3) and the fact that $h_s(0) = h'_s(0) = 0$ give

$$\delta_\Phi((x_0,0), X) = \varphi(x) - \varphi(x_0) - \langle \nabla \varphi(x_0), x - x_0 \rangle + h_s(z) \quad \forall X = (x, z) \in \mathbb{R}^{n+1},$$

which makes all sections centered at $Z_0$ symmetric with respect with $z$. Let

$$Q_s := 4q_s W_{z,0} + (2R)R^s + 8. \tag{10.12}$$
For $X = (x, z) \in S_{2R}$, introduce the auxiliary function $H \in C(S_{2R})$ as

$$H(X) := W(X) + Q_s \left( \frac{\delta \Phi((x_0, 0), X)}{2R} - 1 \right) - W_{z, 0^+}(2R)|z| - \frac{|z|}{q_s R^s} + W_{z, 0^+}(2R)q_s R^s + 1$$

which makes $H$ symmetric in $z$ as well (that is, $H(x, z) = H(x, -z)$ for every $(x, z) \in S_{2R}$). Now, by the inclusions (7.6),

$$S_{2R} \subset S_{\varphi}(x_0, 2R) \times S_{h_s}(0, 2R)$$

so that $X = (x, z) \in S_{2R}$ implies $z \in S_{h_s}(0, 2R)$ and then $h_s(z) \leq 2R$ (because $h_s(0) = h'_s(0) = 0$). Consequently, from the definition of $h_s(z)$, $X = (x, z) \in S_{2R}$ implies

$$|z| \leq 2^s (1 - s)^s R^s =: q_s R^s. \quad (10.13)$$

In particular, for $X = (x, z) \in \partial S_{2R}$ (where it holds that $\delta \Phi((x_0, 0), X) = 2R$) and using that $W \geq 0$ and

$$-W_{z, 0^+}(2R)|z| + W_{z, 0^+}(2R)q_s R^s \geq 0 \quad \text{and} \quad -\frac{|z|}{q_s R^s} + 1 \geq 0, \quad (10.14)$$

we have

$$H(X) = W(X) - W_{z, 0^+}(2R)|z| - \frac{|z|}{q_s R^s} + W_{z, 0^+}(2R)q_s R^s + 1 \geq 0.$$

On the other hand, for every $(x, 0) \in S_{2R} \cap Z_0$ (and using $h'_s(0) = 0$ again),

$$\frac{\partial H}{\partial z^+}(x, 0) = \frac{\partial W}{\partial z^+}(x, 0) - W_{z, 0^+}(2R) - \frac{1}{q_s R^s} \leq -\frac{1}{q_s R^s} < 0.$$

As a consequence, the convex envelope of $-H^-$ in $S_{2R}$ cannot touch $-H^-$ on $Z_0$; moreover, the contact set $C_{-H^-}$ lies at a positive distance from $Z_0$. Therefore, $C_{-H^-} \subset S_{2R} \setminus Z_0$. By Corollary 10.3 applied to $H$ with $\Omega' = S_{2R} \setminus Z_0$ and $\Omega = S_{2R}$ (notice that $H \in C^2(S_{2R} \setminus Z_0)$) we obtain

$$\left( \max_{S_{2R}} H^- \right)^{n+1} \leq C_n |S_{2R}| \int_{C_{-H^-}} |\det D^2 H(X)| dX. \quad (10.15)$$

For $X$ in the contact set $C_{-H^-} \subset S_{2R} \setminus Z_0$ we have

$$|\det D^2 H(X)| = \mu_\Phi(X) \det D^2 \Phi(X)^{-1} |\det D^2 H(X)| \leq \mu_\Phi(X) \left( \frac{\text{trace}(D^2 \Phi(X)^{-1} D^2 H(X))}{n + 1} \right)^{n+1}.$$

In $S_{2R} \setminus Z_0$ we have $D^2 H = D^2 W + \frac{Q_s}{2R} D^2 \Phi$ and $-L_\Phi(W) = \text{trace}(D^2 \Phi(X)^{-1} D^2 W(X)) \leq 0$. So that for $X \in C_{-H^-} \subset S_{2R} \setminus Z_0$,

$$|\det D^2 H(X)| \leq \left( \frac{Q_\Phi}{2R} \right)^{n+1} \mu_\Phi(X),$$
which combined with (10.15) gives

\[
(10.16) \quad \left( \max_{S_{2R}} H^{-} \right)^{n+1} \leq C_n \left( \frac{Q_s}{2R} \right)^{n+1} |S_{2R}| \mu_\Phi(C_{-H^{-}}).
\]

Now, by (10.10), there exists \( X_1 = (x_1, z_1) \in S_R \) such that \( W(X_1) \leq 1 \) and then

\[
H(X_1) = W(X_1) + Q_s \left( \frac{\delta_\Phi((x_0, 0), X_1)}{2R} - 1 \right) - W_{z,0+}(2R)|z_1| - \frac{|z_1|}{q_s R^s} + W_{z,0+}(2R)q_s R^s + 1
\]

\[
< 1 - \frac{Q_s}{2} - W_{z,0+}(2R)|z_1| - \frac{|z_1|}{q_s R^s} + W_{z,0+}(2R)q_s R^s + 1
\]

\[
< 3 + 2q_s W_{z,0+}(2R)R^s - \frac{Q_s}{2},
\]

where we have used that, from (10.13),

\[
0 \leq 1 - \frac{|z_1|}{q_s R^s} \leq 1 + \frac{|z_1|}{q_s R^s} \leq 2
\]

and

\[
0 \leq W_{z,0+}(2R)(q_s R^s - |z_1|) \leq W_{z,0+}(2R)(q_s R^s + |z_1|) \leq 2q_s W_{z,0+}(2R)R^s.
\]

By the definition of \( Q_s \) in (10.12) we get \( 3 + 2q_s W_{z,0+}(2R)R^s - \frac{Q_s}{2} = -1 \) and then \( H^{-}(X_1) > 1 \).

Then, from (10.16), the fact that \( H^{-}(X_1) > 1 \) and (7.9), we deduce

\[
(10.17) \quad \mu_\Phi(S_{2R}) \leq C_n K_3 Q_s^{n+1} \mu_\Phi(C_{-H^{-}}).
\]

Next, on the contact set we have \( H \leq 0 \), so that by using (10.9) we get (recalling (10.14))

\[
C_{-H^{-}} \subset \{ X \in S_{2R} : H(X) \leq 0 \} \subset \{ X \in S_{2R} : W(X) \leq Q_s \}
\]

\[
\subset \{ X \in S_{2R} : W(X) \leq 4q_s + 8 \}.
\]

On the other hand,

\[
Q_s^{n+1} = (4q_s W_{z,0+}(2R)R^s + 8)^{n+1} \leq (4q_s \theta_2 + 8)^{n+1} \leq (8q_s \theta_2)^{n+1} + 16^{n+1},
\]

so that (10.17) implies

\[
\mu_\Phi(S_{2R}) \leq C_n K_3 (8q_s \theta_2)^{n+1} \mu_\Phi(S_{2R}) + C_n K_3 16^{n+1} \mu_\Phi(C_{-H^{-}})
\]

and by choosing \( \theta_2 \in (0, 1) \) such that \( C_n K_3 (8q_s \theta_2)^{n+1} \leq 1/2 \), the inequality (10.11) follows with

\[
\varepsilon_2 := \frac{1}{2C_n K_3 16^{n+1}}.
\]

\( \square \)
10.4. The case of sections of $\Phi$ intersecting $Z_0$. When a section $S_\Phi(X_0, R)$, not necessarily centered somewhere at $\mathbb{R}^n \times Z_0$, satisfies $\overline{S_\Phi(X_0, R)} \cap Z_0 \neq \emptyset$, the first step will be to relate it to a section centered at $\mathbb{R}^n \times Z_0$ of comparable height. More precisely, we have

**Lemma 10.6.** Given a section $S_\Phi(X_0, R)$ centered at $X_0 = (x_0, z_0)$ with $\overline{S_\Phi(X_0, R)} \cap Z_0 \neq \emptyset$, there exists $R_r \in (R, 2K_sR)$ such that

$$S_\Phi(X_0, R) \subset S_\Phi((x_0, 0), 2R_r) \subset S_\Phi((x_0, 0), 4R_r) \subset S_\Phi(X_0, \beta_s R/2),$$

where $K_s \geq 1$, depending only on $s$, is the quasi-triangle constant for $h_s$ in (7.4) and $\beta_s$, also depending only on $s$, is as in (10.1).

**Proof.** Given $S_\Phi(X_0, R)$ with $\overline{S_\Phi(X_0, R)} \cap Z_0 \neq \emptyset$, the first inclusion in (7.6) gives

$$S_\Phi(X_0, R) \subset S_\Phi(x_0, R) \times S_{h_s}(z_0, R).$$

The fact that $\overline{S_\Phi(X_0, R)} \cap Z_0 \neq \emptyset$ implies that the closed interval $S_{h_s}(z_0, R) \subset \mathbb{R}$ contains 0, that is, $S_{h_s}(z_0, R) = (z_l, z_r)$ with $z_l \leq 0 \leq z_r$. Without loss of generality, let us assume that $z_r \geq |z_l|$. By putting $R_r := h_s(z_r)$ we have $(-z_r, z_r) = S_{h_s}(0, R_r)$ (see Figure 2) and then

$$S_{h_s}(z_0, R) = (z_l, z_r) \subset (-z_r, z_r) = S_{h_s}(0, R_r).$$

![Figure 2](image-url)

**Figure 2.** On the inclusion $S_{h_s}(z_0, R) = (z_l, z_r) \subset (-z_r, z_r) = S_{h_s}(0, R_r)$.

Also, from the quasi-triangle inequality for $h_s$ in (7.4) along with $\delta_{h_s}(z_0, z_r) = R$ and $\delta_{h_s}(z_0, 0) \leq R$, we get

(10.18) \[ R_r = \delta_{h_s}(0, z_r) \leq K_s(\delta_{h_s}(z_0, z_r) + \delta_{h_s}(z_0, 0)) \leq 2K_sR, \]

and, using that $R < R_r$,

$$S_\Phi(X_0, R) \subset S_\Phi(x_0, R) \times S_{h_s}(z_0, R) \subset S_\Phi(x_0, R_r) \times S_{h_s}(0, R_r) \subset S_\Phi((x_0, 0), 2R_r).$$

Next, we claim that

(10.19) \[ S_{h_s}(0, 4R_r) \subset S_{h_s}(z_0, K_s(1 + 8K_s)R). \]
Indeed, given \( z^\prime \in S_{h_\ast}(0,4R_r) \) and using the triangle inequality (7.9) along with \( \delta_{h_\ast}(z_0,0) \leq R \) and (10.18),
\[
\delta_{h_\ast}(z_0,z^\prime) \leq K_\ast(\delta_{h_\ast}(z_0,0) + \delta_{h_\ast}(0,z^\prime)) < K_\ast(R + 4R_r) \leq K_\ast(1 + 8K_\ast)R,
\]
which proves (10.19). Consequently,
\[
S_\Phi((x_0,0),4R_r) \subset S_\Phi((x_0,0),4R_r) \times S_{h_\ast}(0,4R_r) \\
\subset S_\Phi(x_0,K_\ast(1 + 8K_\ast)R) \times S_{h_\ast}(z_0,K_\ast(1 + 8K_\ast)R) \subset S_\Phi(x_0,2K_\ast(1 + 8K_\ast)R),
\]
and the lemma is proved.

**Theorem 10.7.** Let \( \Phi \) be as in (7.1) and \( \beta_s \) be as in (10.1) (which is the same constant as in Lemma 10.6). Let \( X_0 \in \mathbb{R}^{n+1} \) and \( R > 0 \) such that \( S_\Phi(X_0,\beta_sR) \subset \subset Q \). Put \( S_R := S_\Phi(X_0,R) \), \( S_{\beta_s,R} := S_\Phi((x_0,0),\beta_sR) \), and suppose that \( S_{\Phi}(X_0,2\hat{R}) \cap Z_0 \neq \emptyset \).

Then, there exist geometric constants \( \theta_3, \varepsilon_3 \in (0,1) \) such that for every \( W \in S(Q) \) with \( L_\Phi(W) = -\text{trace} \left( (D^2\Phi)^{-1}D^2W \right) \geq 0 \) and \( W \geq 0 \) in \( S_{\beta_s,R} \setminus Z_0 \) the inequalities
\[
W_{z,0^+}(\beta_sR) < \varepsilon_3
\]
and
\[
\inf_{S_R} W \leq 1
\]
implies
\[
\mu_\Phi(\{X \in S_{\beta_s,R} : W(X) < 4q_s + 8}\}) \geq \varepsilon_3 \mu_\Phi(S_{\beta_s,R}).
\]

**Proof.** Let us take \( \theta_3 := (8K_\ast)^s\theta_2 \) with \( \theta_2 \in (0,1) \) as in Theorem 10.5. By Lemma 10.6 used with \( 2\hat{R} \) and setting \( \hat{R} := (2R_r) \in (2R,4K_\ast R) \), we have
\[
S_\Phi((x_0,0),2\hat{R}) \subset S_\Phi((x_0,0),2\hat{R}) \subset S_\Phi((x_0,0),4\hat{R}) \subset S_\Phi(x_0,\beta_sR),
\]
since, by definition, \( \beta_s := 4K_\ast(1 + 8K_\ast) \). In particular, by hypothesis (10.20) and the definition of \( \theta_3 \),
\[
W_{z,0^+}(4\hat{R})(2\hat{R})^s \leq W_{z,0^+}(\beta_sR)(8K_\ast R)^s \leq (8K_\ast)^s\theta_3 = \theta_2.
\]
Now, inequality (10.21) and the first inclusion in (10.23) imply that \( \inf_{S_\Phi((x_0,0),2\hat{R})} W \leq 1 \), which combined with (10.24) allows us to use Theorem 10.5 with the sections \( S_\Phi((x_0,0),2\hat{R}) \) and \( S_\Phi((x_0,0),4\hat{R}) \) to obtain
\[
\mu_\Phi(\{X \in S_\Phi((x_0,0),4\hat{R}) : W(X) < 4q_s + 8\}) \geq \varepsilon_2 \mu_\Phi(S_\Phi((x_0,0),4\hat{R})).
\]

By the doubling property (7.8) and the first inclusion in (10.23) we get
\[
\mu_\Phi(S_\Phi(x_0,\beta_sR)) \leq K_d\beta_s^\nu_\ast \mu_\Phi(S_\Phi(x_0,R)) \leq K_d\beta_s^\nu_\ast \mu_\Phi(S_\Phi((x_0,0),4\hat{R})),
\]
and then (10.22) follows, with \( \varepsilon_3 := \varepsilon_2(K_d\beta_s^\nu_\ast)^{-1} \), from (10.25), (10.26), and the last inclusion in (10.23). \( \square \)

10.5. **Proof of Theorem 10.1.** The proof follows, with the geometric constants \( \theta_0 := \theta_3 \) and \( \varepsilon_0 := \varepsilon_3 \), from Theorems 10.4 and 10.7, and the fact that \( \beta_s > 2 \) and \( M_\ast > 4 \). \( \square \)
11. Local boundedness

Let us define
\[ \beta_K := \max\{K, \beta_s\}, \]
where \( K \) is as in (7.5) and \( \beta_s \) is as in (10.1). With the critical-density estimate from Theorem 10.1 at hand, we can deduce the following local-boundedness results.

**Lemma 11.1.** Let \( \Phi \) be as in (7.1). There exist geometric constants \( N_1, N_2, N_3 > 0 \) such that for every \( X_0 \in \mathbb{R}^{n+1} \) and \( R > 0 \) with \( S_\Phi(X_0, \beta_K R) \subset \subset Q \) and every \( W \in \mathcal{S}(Q) \) with \( L_\Phi W = -\text{trace} ( (D^2 \Phi)^{-1} D^2 W) \leq 0 \) and \( W \geq 0 \) pointwise in \( Q^+ \) the inequalities
\[ R^s W_{z,0^+}(S_\Phi(X_0, \beta_s R)) \leq N_1 \quad \text{and} \quad \int_{S_\Phi(X_0,2KR)} W \, d\mu_\Phi \leq N_2 \]

imply
\[ \sup_{S_\Phi(X_0, R/2)} W \leq N_3. \]

**Theorem 11.2.** Let \( \Phi \) be as in (7.1). There exist geometric constants \( N_4, N_5 > 0 \) such that for every \( X_0 \in \mathbb{R}^{n+1} \) and \( R > 0 \) with \( S_\Phi(X_0, \beta_K R) \subset \subset Q \) and every \( W \in \mathcal{S}(Q) \) with \( L_\Phi W = -\text{trace} ( (D^2 \Phi)^{-1} D^2 W) \leq 0 \) and \( W \geq 0 \) pointwise in \( Q^+ \) we have
\[ \sup_{S_\Phi(X_0, R/2)} W \leq N_4 \int_{S_\Phi(X_0,2KR)} W \, d\mu_\Phi + N_5 R^s W_{z,0^+}(S_\Phi(X_0, \beta_s R)). \]

**Theorem 11.3.** Let \( \Phi \) be as in (7.1). Then there are geometric constants \( \kappa \in (0,1) \) and \( K_4 \geq K \) such that for every \( q > 0 \) there exist constants \( C_{1,q}, C_{2,q} > 0 \), depending only on geometric constants and \( q \), such that for every section \( S_\Phi(X_0, K_4 R) \subset \subset Q \) and every \( W \in \mathcal{S}(Q) \) with \( L_\Phi W = -\text{trace} ( (D^2 \Phi)^{-1} D^2 W) \leq 0 \) and \( W \geq 0 \) pointwise in \( Q^+ \), we have
\[ \sup_{S_\Phi(X_0, \kappa R)} W \leq C_{1,q} \left( \int_{S_\Phi(X_0, R)} W^q \, d\mu_\Phi \right)^{1/q} + C_{2,q} R^s W_{z,0^+}(S_\Phi(X_0, K_4 R)). \]

About the proofs. The proof of Lemma 11.1 follows as the proof of Lemma 6 in [30]. Essentially, the only modification is to have the expression
\[ \frac{t}{\mu_\varphi(S_\varphi(z,t))^{1/n}} \| f \|_{L^\infty(S_\varphi(z,2KR), \, d\mu_\varphi)} \]

(in the proof of [30, Lemma 6]) replaced with \( W_{z,0^+}(S_{\beta,s} R) \) (and, of course, replacing the function \( \varphi \) with \( \Phi \)). A key point in the proof is that the sub-solution be locally bounded, which is the case in Lemma 11.1 since \( W \in \mathcal{S}(Q) \).

Similarly, Theorem 11.2 follows from Lemma 11.1 as in the proof of Theorem 3 on [30, p.2004] and Theorem 11.3 follows from Theorem 11.2 as in the proof of Theorem 7 on [30, pp.2005-8].
12. AN ARBITRARILY SENSITIVE CRITICAL-DENSITY ESTIMATE

By relying on the divergence-form side of the linearized Monge–Ampère operator, in this section we extend Theorem 10.1 by proving that every $\varepsilon \in (0,1)$ can work as a critical-density parameter (see Theorem 12.2 and Corollary 12.3 below).

All a.e. statements are referred to Lebesgue measure, which is equivalent to a.e. with respect to $\mu_\Phi$ due the hypothesis $\varphi \in C^3(\mathbb{R}^n)$ with $D^2 \varphi > 0$ in $\mathbb{R}^n$ and the fact that $\mu_{\Phi}(z) = |z|^{1/s-2}$.

**Lemma 12.1.** Let $\Phi$ be as in (7.1). Fix a section $S_\Phi(X_0, 2R)$ and suppose that $H \in C(S_\Phi(X_0, 2R))$ satisfies the following conditions:

(i) $D^2 H$ exists a.e. in $S_\Phi(X_0, 2R)$,
(ii) $L_\Phi(H) = -\text{trace}((D^2 \Phi)^{-1} D^2 H) \geq 0$ a.e. in $S_\Phi(X_0, 2R)$,
(iii) there exists $\tau > 0$ such that $H(X) \geq \tau$ for every $X \in S_\Phi(X_0, 2R)$,
(iv) $\nabla^\Phi H \in L^2(S_\Phi(X_0, 2R), d\mu_\Phi)$, that is,

$$\int_{S_\Phi(X_0, 2R)} (\nabla_\Phi H, \nabla H) dX < \infty.$$

Then,

$$\int_{S_\Phi(X_0, R)} |\nabla^\Phi \log H|^2 d\mu_\Phi \leq \frac{32(n + 2)K^2_d}{R},$$

where $K_d > 1$ is the doubling constant from (7.7).

**Proof.** Multiply the inequality $L_\Phi(H) \geq 0$ by $\mu_\Phi$ to obtain, a.e. in $S_\Phi(X_0, 2R),

(12.3) \quad 0 \geq \mu_\Phi \text{trace}((D^2 \Phi)^{-1} D^2 H) = \text{trace}(A_\Phi D^2 H) = \text{div}(A_\Phi \nabla H).

For $\gamma \in C^1(\mathbb{R})$, supported in $[-2, 2]$, with $\gamma \equiv 1$ on $[0,1]$ and $\|\gamma\|_{L^\infty(\mathbb{R})} \leq 2$ define

$$\zeta(X) := \gamma \left( \frac{\delta_\Phi(X_0, X)}{R} \right) \quad \forall X \in \mathbb{R}^n.$$

Now multiply (12.3) by $\zeta^2/H$ and integrate by parts on $S_\Phi(X_0, 2R)$ to obtain

$$\int_{S_\Phi(X_0, 2R)} \frac{\zeta^2}{H^2} (A_\Phi \nabla H, \nabla H) dX \leq \int_{S_\Phi(X_0, 2R)} \frac{2\zeta}{H} (A_\Phi \nabla H, \nabla \zeta) dX.$$

From Young’s inequality,

$$\int_{S_\Phi(X_0, 2R)} \frac{2\zeta}{H} (A_\Phi \nabla H, \nabla \zeta) dX \leq \frac{1}{2} \int_{S_\Phi(X_0, 2R)} \frac{\zeta^2}{H^2} (A_\Phi \nabla H, \nabla H) dX + \int_{S_\Phi(X_0, 2R)} (A_\Phi \nabla \zeta, \nabla \zeta) dX.$$

Hence,

$$\int_{S_\Phi(X_0, 2R)} \frac{\zeta^2}{H^2} (A_\Phi \nabla H, \nabla H) dX \leq 4 \int_{S_\Phi(X_0, 2R)} (A_\Phi \nabla \zeta, \nabla \zeta) dX,$$
where we have used that \( \zeta^2 \leq 1, 0 < \tau \leq H \), and (12.1) to guarantee that
\[
\int_{S_\Phi(X_0,2R)} \frac{\zeta^2}{H^2} \langle A_\Phi \nabla H, \nabla H \rangle dX < \infty.
\]
On the other hand, since
\[
\nabla \zeta(X) = \frac{1}{R} \gamma' \left( \frac{\delta_\Phi(X_0,X)}{R} \right) (\nabla \Phi(X) - \nabla \Phi(X_0))
\]
we get
\[
\int_{S_\Phi(X_0,2R)} \langle A_\Phi \nabla \zeta, \nabla \zeta \rangle dX \leq \frac{4}{R^2} \int_{S_\Phi(X_0,2R)} \langle A_\Phi(X) (\nabla \Phi(X) - \nabla \Phi(X_0)), (\nabla \Phi(X) - \nabla \Phi(X_0)) \rangle dX \leq \frac{8(n+2)K_d}{R} \mu_\Phi(S_\Phi(X_0,2R)),
\]
where for the last inequality we used the energy estimate (7.12). Therefore,
\[
\int_{S_\Phi(X_0,R)} |\nabla^\Phi \log H|^2 d\mu_\Phi = \int_{S_\Phi(X_0,R)} \frac{1}{H^2} \langle A_\Phi \nabla H, \nabla H \rangle dX \leq \int_{S_\Phi(X_0,2R)} \frac{\zeta^2}{H^2} \langle A_\Phi \nabla H, \nabla H \rangle dX \leq \frac{32(n+2)K_d}{R} \mu_\Phi(S_\Phi(X_0,2R)),
\]
and (12.2) follows from the doubling property (7.7).

\[\square\]

**Theorem 12.2.** Let \( \Phi \) be as in (7.1). Fix \( W \in S(Q) \) with \( W \geq 0 \) and \( L_\Phi W = -\operatorname{trace}((D^2 \Phi)^{-1} D^2 W) \geq 0 \) in \( Q^+ \). Fix \( X_0 \in \mathbb{R}^{n+1} \) and \( R > 0 \) with \( S_\Phi(X_0,K_4R) \subset Q \) and put \( S := S_\Phi(X_0,R) \).

Then, given \( \varepsilon, \tau \in (0,1) \), the inequalities
(12.4) \( W_{z,0+}(S_{K_4R}) R^s \leq \tau \)
and
(12.5) \( \mu_\Phi(\{ X \in S : W(X) \geq 1 - \tau \}) \geq \varepsilon \mu_\Phi(S) \)

imply that
(12.6) \( \inf_{S_\Phi(X_0,\kappa R)} W + \tau \geq e^{-C_0(\varepsilon)} \),

where \( \kappa \in (0,1) \) and \( K_4 > 1 \) are the geometric constants from Theorem 11.3 and

(12.7) \( C_0(\varepsilon) := C_{1,1} K_P K_d \left( 1 + \frac{1}{\varepsilon} \right) \sqrt{\frac{32(n+2)}{K_2}} + C_{2,1}, \)

where \( K_P, K_2 > 0 \) are the geometric constants from the weak Poincaré inequality (8.1) and \( C_{1,1}, C_{2,1} > 0 \) are the geometric constants from Theorem 11.3 corresponding to \( q = 1 \).
Consequently, and the fact that \((iii)\) follows from the definition of \(S\).

Let us see that and introduce \((12.9)\)

\[\ell\] for \(\{\ell_{\tau,\epsilon}\}_{\epsilon \in (0,1)}\) be a smooth approximation of \(\ell\) such that \((12.9)\)

\[\ell_{\tau,\epsilon}'' \geq 0, \quad \ell_{\tau,\epsilon}' \leq 0, \quad \|\ell_{\tau,\epsilon}'\|_{L^{\infty}(0,\infty)} \leq 1/\tau, \quad \forall \epsilon \in (0,1)\]

and introduce

\[G^{(c)}(X) := \ell_{\tau,\epsilon}(W(X) + \tau) \quad \forall X \in Q.\]

Let us see that \(G^{(c)} \in S(Q)\) for every \(\epsilon \in (0,1)\) by checking the conditions (i)-(iv) from the definition of \(S(Q)\) in Section 9. Conditions (i) and (ii) are immediate. Condition (iii) follows from

\[\nabla \Phi G^{(c)} = \ell_{\tau,\epsilon}'(W(X) + \tau)\nabla \Phi(W(X)) \quad \forall X \in Q\]

and the fact that \(\|\ell_{\tau,\epsilon}'\|_{L^{\infty}(0,\infty)} \leq 1/\tau, \) uniformly in \(\epsilon \in (0,1)\). Similarly, condition (iv), follows from

\[G_{z}^{(c)} = \ell_{\tau,\epsilon}'(W(X) + \tau)W_{z}(X) \quad \forall X \in Q.\]

Consequently, \(G^{(c)} \in S(Q)\) for every \(\epsilon \in (0,1)\).

On the other hand,

\[D^{2}G^{(c)} = \ell_{\tau,\epsilon}'(W + \tau)(\nabla W \otimes \nabla W) + \ell_{\tau,\epsilon}'(W + \tau)D^{2}W \quad \text{in } Q^{+}\]

and then, always in \(Q^{+}\),

\[-L_{\Phi}(G^{(c)}) = \text{trace}((D^{2}\Phi)^{-1}D^{2}G^{(c)})
\begin{align*}
&= \ell_{\tau,\epsilon}''(W + \tau)(D^{2}\Phi)^{-1}\nabla W, \nabla W + \ell_{\tau,\epsilon}'(W + \tau)\text{trace}((D^{2}\Phi)^{-1}D^{2}W) \\
&\geq \ell_{\tau,\epsilon}'(W + \tau)\text{trace}((D^{2}\Phi)^{-1}D^{2}W) \geq 0.
\end{align*}\]

That is, \(G^{(c)} \in S(Q)\) satisfies \(L_{\Phi}(G^{(c)}) \leq 0\) and \(G^{(c)} \geq 0\) in \(Q^{+}\). By Theorem 11.3 applied to \(G^{(c)}\) with \(q = 1\) we have

\[\sup_{S_{\Phi}(X_{0,\kappa}R)} G^{(c)} \leq C_{1,1} \int_{S_{\Phi}(X_{0,R})} G^{(c)} d\mu_{\Phi} + C_{2,1}G_{z,0}^{(c)}(S_{K_{4}R})R^{s}.\]

And, by \((12.9)\) and \((12.4)\),

\[G_{z,0}^{(c)}(S_{K_{4}R})R^{s} \leq \frac{1}{\tau}W_{z,0}(S_{K_{4}R})R^{s} \leq 1, \quad \forall \epsilon \in (0,1),\]

and then, by taking limits as \(\epsilon \to 0\) in \((12.10)\), it follows that

\[\sup_{S_{\Phi}(X_{0,\kappa}R)} G \leq C_{1,1} \int_{S_{\Phi}(X_{0,R})} G d\mu_{\Phi} + C_{2,1}.\]
Now, since
\[ \{ X \in S : W(X) \geq 1 - \tau \} = \{ X \in S : W(X) + \tau \geq 1 \} = \{ X \in S : G(X) = 0 \}, \]
the hypothesis (12.5) says that \( \mu_\Phi(\{ X \in S : G(X) = 0 \}) \geq \varepsilon \mu_\Phi(S) \) and, by Corollary 8.2,
\[
\int_{S_\Phi(X_0, R)} G \, d\mu_\Phi \leq K_P (1 + \frac{1}{\varepsilon}) R^{1/2} \left( \int_{S_\Phi(X_0, K_2 R)} |\nabla^\Phi G|^2 \, d\mu_\Phi \right)^{1/2}.
\]

At this point we use Lemma 12.1 with \( H := W + \tau \) in the section \( S_\Phi(X_0, K_2 R) \) to get
\[
\int_{S_\Phi(X_0, R)} G \, d\mu_\Phi \leq K_P K_d \left( 1 + \frac{1}{\varepsilon} \right) \sqrt{\frac{32(n + 2)}{K_2}}
\]
and then
\[
(12.11) \quad \sup_{S_\Phi(X_0, R)} G \leq C_{1,1} K_P K_d \left( 1 + \frac{1}{\varepsilon} \right) \sqrt{\frac{32(n + 2)}{K_2}} + C_{2,1} =: C_0(\varepsilon).
\]

By the definition of \( G \) in (12.8), we have
\[
(12.12) \quad \sup_{S_\Phi(X_0, R)} G = \ell_\tau \left( \inf_{S_\Phi(X_0, R)} (1 - \tau)^{\frac{1}{\tau}} \right)
\]
and (12.6) follows from (12.11) and (12.12). \( \square \)

**Corollary 12.3.** Let \( \Phi \) be as in (7.1). Fix \( W \in S(Q) \) with \( W \geq 0 \) and \( L_\Phi W = -\text{trace}((D^2\Phi)^{-1} D^2 W) \geq 0 \) in \( Q^+ \). Fix \( X_0 \in \mathbb{R}^{n+1} \) and \( R > 0 \) with \( S_\Phi(X_0, K_4 R) \subset Q \) and put \( S := S_\Phi(X_0, R) \). Then, for every \( \varepsilon \in (0, 1) \) there exists \( \tau \in (0, 1) \), depending only on \( \varepsilon \) and geometric constants, such that the inequalities
\[
W_{\varepsilon,0}(S_\Phi(X_0, K_4 R)) \leq \frac{\tau}{1 - \tau}
\]
and
\[
(12.13) \quad \mu_\Phi(\{ X \in S : W(X) \geq 1 \}) \geq \varepsilon \mu_\Phi(S)
\]
imply that
\[
\inf_{S_\Phi(X_0, R)} W \geq \frac{\tau}{1 - \tau},
\]
where \( \kappa \in (0, 1) \) is the geometric constant from Theorem 11.3.

**Proof.** Given \( \varepsilon \in (0, 1) \) let \( \tau \in (0, 1) \) be defined by
\[
(12.14) \quad 2\tau := e^{-C_0(\varepsilon)}
\]
and apply Theorem 12.2 to \( W_{\tau} := (1 - \tau)W \). \( \square \)
13. The weak-Harnack inequality

In this section we prove a weak-Harnack inequality for nonnegative super-solutions in $S(Q)$ (see Theorem 13.2). The proof relies on establishing a “non-homogeneous” version of Theorem 7.1 in [20]. Towards that end, we next adapt a result known as the “crawling ink spots lemma” to the elliptic Monge–Ampère context. The “crawling ink spots lemma” has been developed by Krylov-Safonov in [21, Section 2]; the lemmas in [21, Section 2] correspond to the parabolic case; see [36, Lemma 1.1] for the elliptic case in $\mathbb{R}^n$, and [20, Lemma 7.2], for instance, for a version in doubling metric spaces).

Lemma 13.1. Let $\Phi$ be as in (7.1). Fix any $K_0 > K(2K + 1)$, where $K \geq 1$ is the quasi-triangle constant from (7.5). Given a section $S := S_\Phi(X_0, R)$, a measurable subset $E \subset S$, and $\delta \in (0, 1)$ define the open set

$$E_\delta := \bigcup_{0 < \rho < K_0 R} \{ S_\Phi(X, \rho) \cap S : X \in S \text{ and } \mu_\Phi(E \cap S_\Phi(X, \rho)) > \delta \mu_\Phi(S_\Phi(X, \rho)) \}.$$

Then either $E_\delta = S$ or

$$\mu_\Phi(E) \leq \delta K_d K_0^{\nu} \mu_\Phi(E_\delta),$$

where $\nu \geq 1$ is as in (7.8).

Proof. The proof follows as the one for [20, Lemma 7.2] by means of Vitali’s covering lemma for Monge–Ampère sections. In turn, Vitali’s covering lemma for Monge–Ampère sections follows as in the proof of Theorem 1.2 in [9, p.69] for general spaces of homogeneous type. In the Monge–Ampère case, the dilation constant in Vitali’s lemma can be any $K_0$ with $K_0 > 2K^2 + K$. □

Theorem 13.2. Let $\Phi$ be as in (7.1). There exist geometric constants $\sigma \in (0, 1)$ and $K_6, K_7 > 1$ such that for every $W \in S(Q)$ with $L_\phi W = -\text{trace} ((D^2\Phi)^{-1} D^2 W) \geq 0$ and $W \geq 0$ in $Q^+$ and every $(X_0, R) \in \mathbb{R}^{n+1} \times (0, \infty)$ with $S_\Phi(X_0, K_7 R) \subset \subset Q$, we have

$$\left( \int_{S_\Phi(X_0, R)} W(X)^{\sigma} \mu_\Phi(X) \right)^{\frac{1}{\sigma}} \leq K_6 \left( \inf_{S_\Phi(X_0, \kappa R)} W + R^{\kappa} W_{z,0+}(S_\Phi(X_0, K_7 R)) \right),$$

where $\kappa \in (0, 1)$ is the geometric constant from Theorem 11.3.

Proof. Let us start by defining $K_7 > 1$ as

$$K_7 := K(K_4 K_0 + \kappa),$$

with $K_4 \geq K$ being the geometric constant from Theorem 11.3. Let us fix $\delta \in (0, 1)$ such that

$$\delta_0 := \delta K_d K_0^{\nu} < 1,$$

where $K_d K_0^{\nu}$ is the product of geometric constants from (13.1) in Lemma 13.1, and for $\delta_0$ as in (13.4) choose $\varepsilon \in (0, 1)$ such that

$$\varepsilon := \frac{\delta K_\nu}{K_d} < \delta_0.$$
With this choice of $\varepsilon$, and recalling the definition of $C_0(\varepsilon)$ in (12.7), define $\tau \in (0, 1)$ by means of (12.14) and put $\lambda := \tau/(1 - \tau) \in (0, 1)$, which makes $\tau$, $\delta_0$, and $\lambda$ geometric constants.

For $t > 0$ and $i \in \mathbb{N}_0$ set

$$A_{t,i} := \{ X \in S_\Phi(X_0, \kappa R) : W(X) \geq t\lambda^i \}$$

and let $j = j(t) \in \mathbb{N}$ satisfy

$$\delta_0 \leq \frac{\mu_\Phi(A_{t,0})}{\mu_\Phi(S_\Phi(X_0, R))} \leq \delta_0^{-1},$$

which yields

$$\left( \frac{\mu_\Phi(A_{t,0})}{\mu_\Phi(S_\Phi(X_0, R))} \right)^{\gamma} \leq \lambda^{j-1}$$

where

$$\gamma := \frac{\log \lambda}{\log \delta_0}$$

is a geometric constant. We will show that

$$t\lambda^{j-1} \leq K_8 \left( \inf_{S_\Phi(X_0, \kappa R)} W + R^*W_{z,0^+}(S_\Phi(X_0, K_7 R)) \right),$$

where $K_8 > 1$ is the geometric constant defined as

$$K_8 := K_0^1 \lambda^{-1}.$$ 

Indeed, given $i \in \{1, \ldots, j\}$ we consider two cases: when

$$K_0^1 R^*W_{z,0^+}(S_\Phi(X_0, K_7 R)) > t\lambda^i$$

and when

$$K_0^1 R^*W_{z,0^+}(S_\Phi(X_0, K_7 R)) \leq t\lambda^i.$$ 

If (13.10) holds true for some $i \in \{1, \ldots, j\}$, then (13.8) is immediate from the definition of $K_8$ in (13.9) and the fact that $\lambda \in (0, 1)$ and $i \in \{1, \ldots, j\}$ imply $\lambda^i \geq \lambda^j$.

Suppose then that (13.11) holds true for every $i \in \{1, \ldots, j\}$. We will prove (13.8) by repeatedly applying Corollary 12.3 and Lemma 13.1.

If for some $X \in S_\Phi(X_0, \kappa R)$ and $\rho \in (0, \kappa K_0 R)$ we have

$$\mu_\Phi(A_{t,i-1} \cap S_\Phi(X, \rho)) > \delta \mu_\Phi(S_\Phi(X, \rho)),$$

(with $\delta$ as in (13.4)) by the doubling property (7.8) (and recalling that $\kappa \in (0, 1)$) it follows that

$$\mu_\Phi \left( \{ Y \in S_\Phi(X, \rho/\kappa) : \frac{W(Y)}{t\lambda^{i-1}} \geq 1 \} \right) \geq \mu_\Phi(A_{t,i-1} \cap S_\Phi(X, \rho)) > \delta \mu_\Phi(S_\Phi(X, \rho))$$

$$\geq \frac{\delta \kappa^\nu}{K_d} \mu_\Phi(S_\Phi(X, \rho/\kappa)) = \varepsilon \mu_\Phi(S_\Phi(X, \rho/\kappa)),$$

where for the last equality we used the definition of $\varepsilon$ in (13.5). Next, let us see that $\rho \in (0, \kappa K_0 R)$ and $X \in S_\Phi(X_0, \kappa R)$ imply the inclusion

$$S_\Phi(X, K_4 \rho/\kappa) \subset S_\Phi(X_0, K_7 R).$$
Indeed, given \( Y \in S_\Phi(X, K_4 \rho/\kappa) \), by the \( K \)-quasi-triangle inequality (7.5)

\[
\delta_\Phi(X_0, Y) \leq K (\delta_\Phi(X, Y) + \delta_\Phi(X_0, X)) < K \left( \frac{K_4 \rho}{\kappa} + \kappa R \right) \leq K (K_4 K_0 R + \kappa R) = K_7 R,
\]

where for the last equality we used the definition of \( K_7 \) in (13.3). Now, from the fact that \( \rho/\kappa < K_9 R \), the inclusion (13.12), and the hypothesis (13.11), we get

\[
(\rho/\kappa)^s W_{z,0+}(S_\Phi(X, K_4 \rho/\kappa)) \leq K_6^s R^s W_{z,0+}(S_\Phi(X_0, K_7 R)) \leq t \lambda^j = t \lambda^{i-1} \frac{\tau}{1 - \tau}.
\]

Then, Corollary 12.3 applied to \( \frac{W(X)}{\lambda^j} \) on the section \( S_\Phi(X, \rho/\kappa) \) with \( \varepsilon \) as in (13.5) yields

\[
\inf_{S_\Phi(X, \rho)} W \geq t \lambda^j
\]

and, consequently, \( S_\Phi(X_0, \kappa R) \cap S_\Phi(X, \rho) \subseteq A_{t, i} \). By Lemma 13.1 applied to the section \( S_\Phi(X_0, \kappa R) \) and the set \( E := A_{t, i-1} \) it follows that either \( A_{t, i-1} = S_\Phi(X_0, \kappa R) \) or

\[
(13.13) \quad \frac{1}{\delta_0} \mu_\Phi(A_{t, i-1}) \leq \mu_\Phi(E_\delta) \leq \mu_\Phi(A_{t, i}),
\]

where \( \delta_0 \in (0, 1) \) is as in (13.4). Now, if \( A_{t, i-1} = S_\Phi(X_0, \kappa R) \) for some \( i \in \{1, \ldots, j\} \), then (due to the inclusion \( A_{t, i-1} \supset A_{t, j-1} \)) we have \( A_{t, j-1} = S_\Phi(X_0, \kappa R) \), which means

\[
\inf_{S_\Phi(X_0, \kappa R)} V \geq t \lambda^{j-1},
\]

and the inequality (13.8) follows. Hence, we can assume that (13.13) holds for every \( i \in \{1, \ldots, j\} \) and then write

\[
\mu_\Phi(A_{t, j-1}) \geq \frac{1}{\delta_0} \mu_\Phi(A_{t, j-2}) \geq \cdots \geq \frac{\mu_\Phi(A_{t, 0})}{\delta_0^{j-1}} \geq \delta_0 \mu_\Phi(S_\Phi(X_0, \kappa R)),
\]

where for the last inequality we used the definition of \( j \) in (13.6). In particular,

\[
\mu_\Phi(\{X \in S_\Phi(X_0, \kappa R) : W(X) \geq t \lambda^{j-1}\}) = \mu_\Phi(A_{t, j-1}) \geq \delta_0 \mu_\Phi(S_\Phi(X_0, \kappa R)) > \varepsilon \mu_\Phi(S_\Phi(X_0, \kappa R))
\]

and, using that \( \kappa < K_0 \) and \( \kappa K_4 < K_7 \), by (13.11) applied with \( i = j \), we obtain

\[
(\kappa R)^s W_{z,0+}(S_\Phi(X_0, \kappa K_4 R)) \leq K_6^s R^s W_{z,0+}(S_\Phi(X_0, K_7 R)) < t \lambda^j = t \lambda^{j-1} \frac{\tau}{1 - \tau},
\]

so that Corollary 12.3 applied to \( \frac{W(X)}{\lambda^j} \) on the section \( S_\Phi(X_0, \kappa R) \) with \( \varepsilon \) as in (13.5) yields

\[
\inf_{S_\Phi(X_0, \kappa R)} W \geq t \lambda^j
\]

and (13.8) follows. Now, by setting \( \xi := K_8 \inf_{S_\Phi(X_0, \kappa R)} W + R^s W_{z,0+}(S_\Phi(X_0, K_7 R)) \), from (13.7) and (13.8) we obtain

\[
\mu_\Phi(\{X \in S_\Phi(X_0, \kappa R) : W(X) \geq t\}) = \mu_\Phi(A_{t, 0}) \leq \left( \frac{\xi}{t} \right)^{\frac{1}{\gamma}} \forall t > 0
\]
and then, for any $\sigma \in (0, 1/\gamma)$,
\[
\int_{S_\Phi(X_0, rR)} W(X)^\sigma \, d\mu_\Phi(X) \\
\leq \sigma \int_0^t t^{\sigma - 1} \, dt + \frac{\sigma}{\mu_\Phi(S_\Phi(X_0, \kappa R))} \int_t^\infty t^{\sigma - 1} \mu_\Phi(A_t, 0) \, dt \leq \left( \frac{\gamma}{1 - \sigma \gamma} + 1 \right) \xi^\sigma,
\]
which proves (13.2) with $K_6 := \left( \frac{\gamma}{1 - \sigma \gamma} + 1 \right)^{1/\sigma} K_8$. \qed

14. Proofs of Theorems 1.3 and 1.6

We are now position to prove Theorems 1.3 and 1.6. Let us start with

Proof of Theorem 1.6. By Theorem 13.2 applied to $W \in S(Q)$, we get
\[
\left( \int_{S_\Phi(X_0, rR)} W(X)^\sigma \, d\mu_\Phi(X) \right)^{\frac{1}{\sigma}} \leq K_6 \left( \inf_{S_\Phi(X_0, rR)} W + R^4 W_{z, 0+}(S_\Phi(X_0, K_7 R)) \right).
\]

Now, by Theorem 11.3 applied to $W$ with $q = \sigma$,
\[
\sup_{S_\Phi(X_0, rR)} W \leq C_{1, \sigma} \left( \int_{S_\Phi(X_0, rR)} W(X)^\sigma \, d\mu_\Phi(X) \right)^{\frac{1}{\sigma}} + C_{2, \sigma} R^4 W_{z, 0+}(S_\Phi(X_0, K_4 R)),
\]
and notice that, from (13.3), we have $K_4 \leq K_7$. Hence, the Harnack inequality (1.12) follows with $C_H := C_{1, \sigma} K_6 + C_{2, \sigma}$.

Next, for $0 < r < R$, consider the functions
\[
M(r) := \sup_{S_\Phi(X_0, r)} W \quad \text{and} \quad m(r) := \inf_{S_\Phi(X_0, r)} W.
\]
A standard argument (see for instance [14, Section 8.9]) implies the existence of geometric constants $\varrho \in (0, 1)$ and $K_{11} > 0$ such that
\[
(14.1) \quad M(r) - m(r) \leq K_{11} r^\varrho \left( \sup_{S_\Phi(X_0, R)} W + R^4 W_{z, 0+}(S_\Phi(X_0, K_4 R)) \right), \quad \forall r \in (0, R),
\]
which, in turn, implies (1.13). \qed

Proof of Theorem 1.3. Given a section $S_0 := S_\varphi(p_0, R_0)$, $f \in C_0(S_0)$, and a nonnegative $v \in \text{Dom}_{S_0}(L_\varphi)$ solution to $L_\varphi^* v = f$ in $S_0$, let $V$ be the solution to the extension problem (2.8). In particular, we have $V \in C_0(Q)$ and $\lim_{z \to 0^+} V(x, z) = v(x)$ uniformly in $S_0$.

Now, let us set $K_9 := 2K_7$ so that given a section $S := S_\varphi(x_0, R)$ with $S_\varphi(x_0, K_9 R) \subset S_0$, by the first inclusion in (7.6) we get
\[
S_\Phi((x_0, 0), 2K_7 R) \subset S_\varphi(x_0, 2K_7 R) \times S_{h_+(0, 2K_7 R)} = S_\varphi(x_0, K_9 R) \times S_{h_+}(0, K_9 R) \subset S_0 \times \mathbb{R} = Q,
\]
and then the Harnack inequality (1.12) for \( \tilde{V} \) on the section \( S_\Phi((x_0,0),2R) \) gives

\[
\sup_{S_\Phi((x_0,0),2\kappa R)} \tilde{V} \leq \tilde{C}_H \left( \inf_{S_\Phi((x_0,0),2\kappa R)} \tilde{V} + R^* \tilde{V}_{z,0+}(S_\Phi((x_0,0),2K_7R)) \right).
\]

On the other hand, the second inclusion in (7.6) yields

\[
S_\Phi(x_0,\kappa R) \times \{0\} \subset S_\Phi(x_0,\kappa R) \times S_{h_\nu}(0,\kappa R) \subset S_\Phi((x_0,0),2\kappa R)
\]

and, since \( \tilde{V}(x,0) = V(x,0) = v(x) \) for every \( x \in S_0 \), it follows that

\[
\sup_{S_\Phi((x_0,\kappa R)} v = \sup_{S_\Phi((x_0,\kappa R)} \tilde{V}(\cdot,0) \leq \sup_{S_\Phi((x_0,0),2\kappa R)} \tilde{V}
\]

as well as

\[
\inf_{S_\Phi((x_0,0),2\kappa R)} \tilde{V} \leq \inf_{S_\Phi((x_0,0),\kappa R)} \tilde{V}(\cdot,0) = \inf_{S_\Phi((x_0,\kappa R)} v,
\]

which, along with (14.2) implies

\[
\sup_{S_\Phi((x_0,\kappa R)} v \leq \tilde{C}_H \left( \inf_{S_\Phi((x_0,0),\kappa R)} v + R^* \tilde{V}_{z,0+}(S_\Phi((x_0,0),2K_7R)) \right).
\]

But, by the inclusion \( S_\Phi((x_0,0),2K_7R) \subset S_\Phi(x_0,2K_7R) \times S_{h_\nu}(0,2K_7R) \), we have

\[
S_\Phi((x_0,0),2K_7R) \cap Z_0 \subset S_\Phi(x_0,2K_7R) \times \{0\},
\]

so that from the definition of \( \tilde{V}_{z,0+}(S_\Phi((x_0,0),2K_7R)) \) in (9.2) and the limit in (2.9), we get

\[
\tilde{V}_{z,0+}(S_\Phi((x_0,0),2K_7R)) \leq d_\nu \|L^* v\|_{L^\infty(S_\Phi(x_0,2K_7R))}.
\]

Then, the Harnack inequality (1.7) follows from (14.3) and (14.4) with \( K_\mu := 2K_7 \) and \( C_H := d_\nu \tilde{C}_H \).

Also by a restriction argument, the Monge–Ampère Hölder estimate (1.8) follows from (1.13) applied to \( \tilde{V} \) and then restricting to \( Z_0 \). Notice that in principle we cannot prove (1.8) directly from the Harnack inequality (1.7), as we did to go from (1.12) to (1.13) via (14.1). This is due to the fact that if \( v \in \text{Dom}_{S}(L^*_\nu) \) and \( C \in \mathbb{R} \setminus \{0\} \), then it is not true that \( v - C \in \text{Dom}_{S}(L^*_\nu) \) because \( v - C \) does not vanish on \( \partial S \).

\[ \square \]

**Acknowledgements**

The authors would like to thank the referee for suggestions that helped to improve the presentation of the paper. The first author was supported by NSF under grant DMS 1361754. The second author was partially supported by grant MTM2015-66157-C2-1-P (MINECO/FEDER) from Government of Spain.

**References**

[1] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems*, Monographs in Mathematics 96, Birkhäuser, Basel, 2001.

[2] W. Arendt and R. M. Schätzle, Semigroups generated by elliptic operators in nondivergence form on \( C_0(\Omega) \), *Ann. Sc. Norm. Super. Pisa. Cl. Sci. (5) XIII* (2014), 1–18.

[3] L. A. Caffarelli and F. Charro, On a fractional Monge–Ampère operator, *Ann. PDE* 1 (2015), 1–47.
[4] L. A. Caffarelli and C. Gutiérrez, Real analysis related to the Monge–Ampère equation, *Trans. Amer. Math. Soc.* **348** (1996), 1075–1092.

[5] L. A. Caffarelli and C. Gutiérrez, Properties of the solutions of the linearized Monge–Ampère equation, *Amer. J. Math.* **119** (1997), 423–465.

[6] L. A. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, *Comm. Partial Differential Equations* **32** (2007), 1245–1260.

[7] L. A. Caffarelli and L. Silvestre, A nonlocal Monge–Ampère equation, *Comm. Anal. Geom.* **24** (2016), 307–335.

[8] L. A. Caffarelli and P. R. Stinga, Fractional elliptic equations, Caccioppoli estimates and regularity, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **33** (2016), 767–807.

[9] R. R. Coifman and G. Weiss, *Analyse Harmonique Non-commutative Sur Certains Espaces Homogènes*, Lecture Notes in Mathematics 242, Springer, 1971.

[10] L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics **19**, Second Edition, American Mathematical Society, Providence, RI, 2010.

[11] L. Forzani and D. Maldonado, A mean-value inequality for nonnegative solutions to the linearized Monge–Ampère equation, *Potential Anal.* **30** (2009), 251–270.

[12] L. Forzani and D. Maldonado, Properties of the solutions to the Monge–Ampère equation, *Nonlinear Anal.* **57** (2004), 815–829.

[13] J. E. Galé, P. J. Miana and P. R. Stinga, Extension problem and fractional operators: semigroups and wave equations, *J. Evol. Equ.* **13** (2013), 343–368.

[14] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[15] L. Grafakos, *Modern Fourier Analysis*, Graduate Texts in Mathematics **250**, Second Edition, Springer-Verlag, 2009.

[16] C. Gutiérrez, *The Monge–Ampère Equation*, Progress in Nonlinear Differential Equations and Their Applications **44**, Birkhäuser, 2001.

[17] C. Gutiérrez and T. Nguyen, Interior gradient estimates for solutions to the linearized Monge–Ampère equation, *Adv. Math.* **228** (2011), 2034–2070.

[18] C. Gutiérrez and T. Nguyen, Interior second derivative estimates for solutions to the linearized Monge–Ampère equation, *Trans. Amer. Math. Soc.* **367** (2015), 4537–4568.

[19] C. Gutiérrez and F. Tournier, $W^{2,p}$-estimates for the linearized Monge–Ampère equation, *Trans. Amer. Math. Soc.* **358** (2006), 4843–4872.

[20] J. Kinnunen and N. Shanmugalingam, Regularity of quasi-minimizers on metric spaces, *Manuscripta Math.* **105** (2001), 401–423.

[21] N. V. Krylov and M. V. Safonov, A certain property of solutions of parabolic equations with measurable coefficients, *USSR Izvestija* **6** (1981), 151–164.

[22] N. Q. Le, On the Harnack inequality for degenerate and singular elliptic equations with unbounded lower order terms via sliding paraboloids, *Commun. Contemp. Math.*, to appear.

[23] N. Q. Le, Remarks on the Green’s function of the linearized Monge–Ampère operator, *Manuscripta Math.* **149** (2016), 45–62.

[24] N. Q. Le, $W^{2,p}$ solution to the second boundary value problem of the prescribed affine mean curvature and Abreu’s equations, *J. Differential Equations* **260** (2016), 4285–4300.

[25] N. Q. Le and T. Nguyen, Geometric properties of boundary sections of solutions to the Monge–Ampère equation and applications, *J. Funct. Anal.* **264** (2013), 337–361.

[26] N. Q. Le and T. Nguyen, Global $W^{2,p}$ estimates for solutions to the linearized Monge–Ampère equations, *Math. Ann.* **358** (2014), 629–700.

[27] N. Q. Le and O. Savin, Boundary regularity for solutions to the linearized Monge–Ampère equations, *Arch. Ration. Mech. Anal.* **210** (2013), 813–836.

[28] N. Q. Le and O. Savin, On boundary Hölder gradient estimates for solutions to the linearized Monge–Ampère equations, *Proc. Amer. Math. Soc.* **143** (2015), 1605–1615.

[29] N. N. Lebedev, *Special Functions and Their Applications*, revised edition, translated from the Russian and edited by Richard A. Silverman, Dover, New York, 1972.
[30] D. Maldonado, Harnack’s inequality for solutions to the linearized Monge–Ampère operator with lower-order terms, *J. Differential Equations* **256** (2014), 1987–2022.

[31] D. Maldonado, On the $W^{2,1+\varepsilon}$-estimates for the Monge–Ampère equation and related real analysis, *Calc. Var. Partial Differential Equations* **50** (2014), 94–114.

[32] D. Maldonado, The Monge–Ampère quasi-metric structure admits a Sobolev inequality, *Math. Res. Lett.* **20** (2013), 527–536.

[33] D. Maldonado, $W^{1,1+\varepsilon}$-estimates for Green’s functions of the linearized Monge–Ampère operator, *Manuscripta Math.* **152** (3) (2017), 539–554.

[34] J. K. Oddson, On the rate of decay of solutions of parabolic differential equations, *Pacific J. Math.* **29** (1969), 389–396.

[35] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences **44**, Springer-Verlag, New York, 1983.

[36] M. V. Safonov, Harnack inequality for elliptic equations and the Hölder property of their solutions, *J. Soviet Math.* **21** (1983), 851–863.

[37] O. Savin, A Liouville theorem for solutions to the linearized Monge–Ampère equation, *Discrete Contin. Dyn. Syst.* **28** (2010), 865–873.

[38] P. R. Stinga, Fractional powers of second order partial differential operators: extension problem and regularity theory, PhD Thesis, Universidad Autónoma de Madrid, 2010.

[39] P. R. Stinga and J. L. Torrea, Extension problem and Harnack’s inequality for some fractional operators, *Comm. Partial Differential Equations* **35** (2010), 2092–2122.

[40] P. R. Stinga and C. Zhang, Harnack’s inequality for fractional nonlocal equations, *Discrete Cont. Dyn. Syst.* **33** (2013), 3153–3170.

[41] G. Tian and X.-J. Wang, A class of Sobolev type inequalities, *Methods Appl. Anal.* **15** (2008), 263–276.

[42] N. Trudinger and X.-J Wang, Affine complete locally convex hypersurfaces, *Invent. Math.** **150** (2002), 45–60.

[43] N. Trudinger and X.-J Wang, The affine Plateau problem, *J. Amer. Math. Soc.* **18** (2005), 253–289.

[44] N. Trudinger and X.-J. Wang, The Bernstein problem for affine maximal hypersurfaces, *Invent. Math.** **140** (2000), 399–422.

[45] K. Yosida, *Functional Analysis*, Reprint of the sixth (1980) edition, Classics in Mathematics, Springer-Verlag, Berlin, 1995.

Department of Mathematics, Kansas State University, 138 Cardwell Hall, Manhattan, KS 66506, USA.

E-mail address: dmaldona@math.ksu.edu

Department of Mathematics, Iowa State University, 396 Carver Hall, Ames, IA 50011, USA.

E-mail address: stinga@iastate.edu