The Bohm Interpretation of Quantum Cosmology

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I make a review on the applications of the Bohm-De Broglie interpretation of quantum mechanics to quantum cosmology. In the framework of minisuperspaces models, I show how quantum cosmological effects in Bohm’s view can avoid the initial singularity, isotropize the Universe, and even be a cause for the present observed acceleration of the Universe. In the general case, we enumerate the possible structures of quantum space and time.

1 INTRODUCTION

It is a great honour for me to write this contribution in memory of Prof. James T. Cushing. I met Prof. Cushing during a symposium in São Paulo about Bohm’s theory in 1999, where I could appreciate the clarity of his ideas, and his ability to find out the main point in a controversy. A gentleman and a deep thinker, capable to put in clear terms any difficult subject on his domain of interest.

One of the main topics of research of Prof. Cushing was the Bohm-De Broglie interpretation of quantum mechanics in its many aspects, and its comparison with other formulations of quantum mechanics. At the moment, there is no clear observation selecting one of these formulations. Hence, it is desirable to push these formulations to the frontiers of physics in order to discriminate them either by a possible feasible experiment or by a matter of principle (self consistency, resolution of fundamental issues, etc...).

One of these frontiers is cosmology, a unique domain in physics, where the totality of all physical systems, including spacetime itself and its geometry, is under investigation, composing a single system which cannot be manipulated or prepared, it can only be observed, and containing the observers...
themselves. The proposal of this contribution is to review the applications of the Bohm-De Broglie interpretation to quantum cosmology, see if it helps in the resolution of the traditional cosmological issues, and compare their results with the ones obtained in other formulations of quantum mechanics.

Two of the main questions one might ask in cosmology are:

1) Is the Universe eternal or it had a beginning, and in the last case, was this beginning given by an initial singularity?

2) Why the Universe we live in is remarkably homogeneous and isotropic, with very small deviations from this highly symmetric state?

The answer given by classical General Relativity (GR) to the first question, indicated by the singularity theorems [1], asserts that probably the Universe had a singular beginning. As singularities are out of the scope of any physical theory, this answer invalidates any description of the very beginning of the Universe in physical terms. One might think that GR and/or any other matter field theory must be changed under the extreme situations of very high energy density and curvature near the singularity, rendering the physical assumptions of the singularity theorems invalid near this point. One good point of view (which is not the only one) is to think that quantum gravitational effects become important under these extreme conditions. We should then construct a quantum theory of gravitation and apply it to cosmology. For instance, in the euclidean quantum gravity approach [2] to quantum cosmology, a second answer to the first question comes out: the Universe may have had a non-singular birth given by the beginning of time through a change of signature.

In the same way, the naive answer of GR and the standard cosmological scenario to the second question is not at all satisfactory: the reason for the Universe be highly homogeneous and isotropic is a matter of initial conditions. However, solutions of Einstein’s equations with this symmetry are of measure zero; so, why the Universe is not inhomogeneous and/or anisotropic? Inflation [3, 4] is an idea that tries to explain this fact. Nevertheless, in order for inflation to happen some special initial conditions are still necessary, although much more less stringent than in the case without inflation. Once again, quantum cosmology can help in this matter by providing the physical reasons for having the initial conditions for inflation.

Does it makes sense to quantize the whole Universe? Almost all physicists believe that quantum mechanics is a universal and fundamental theory, applicable to any physical system, from which classical physics can be recovered. The Universe is, of course, a valid physical system: there is a theory, Standard Cosmology, which is able to describe it in physical terms, and make predictions which can be confirmed or refuted by observations. In fact, the observations until now confirm the standard
cosmological scenario with a cosmological constant. Hence, supposing the universality of quantum mechanics, the Universe itself must be described by quantum theory, from which we could recover Standard Cosmology. However, the Copenhagen interpretation of quantum mechanics, which is the one taught in undergraduate courses and employed by the majority of physicists in all areas (specially the von Neumann’s approach), cannot be used in a Quantum Theory of Cosmology. This is because it imposes the existence of a classical domain. In von Neumann’s view, for instance, the necessity of a classical domain comes from the way it solves the measurement problem (see Ref. [8] for a good discussion). In an impulsive measurement of some observable, the wave function of the observed system plus macroscopic apparatus splits into many branches which almost do not overlap (in order to be a good measurement), each one containing the observed system in an eigenstate of the measured observable, and the pointer of the apparatus pointing to the respective eigenvalue. However, in the end of the measurement, we observe only one of these eigenvalues, and the measurement is robust in the sense that if we repeat it immediately after, we obtain the same result. So it seems that the wave function collapses, the other branches disappear. The Copenhagen interpretation assumes that this collapse is real. However, a real collapse cannot be described by the unitary Schrödinger evolution. Hence, the Copenhagen interpretation must assume that there is a fundamental process in a measurement which must occur outside the quantum world, in a classical domain. Of course, if we want to quantize the whole Universe, there is no place for a classical domain outside it, and the Copenhagen interpretation cannot be applied. Hence, if someone insists with the Copenhagen interpretation, she or he must assume that quantum theory is not universal, or at least try to improve it by means of further concepts. One possibility is by invoking the phenomenon of decoherence. In fact, the interaction of the observed quantum system with its environment yields an effective diagonalization of the reduced density matrix, obtained by tracing out the irrelevant degrees of freedom. Decoherence can explain why the splitting of the wave function is given in terms of the pointer basis states, and why we do not see superpositions of macroscopic objects. In this way, classical properties emerge from quantum theory without the need of being assumed. In the framework of quantum gravity, it can also explain how a classical background geometry can emerge in a quantum universe. In fact, it is the first quantity to become classical. However, decoherence is not yet a complete answer to the measurement problem. It does not explain the apparent collapse after the measurement is completed, or why all but one of the diagonal elements of the density matrix become null when the measurement is finished. The theory is unable to give an account of the existence of facts, their uniqueness as opposed to the multiplicity of possible phenomena. Further developments
are still in progress, like the consistent histories approach \[13\], which is however incomplete until now. The important role played by the observers in these descriptions is not yet explained \[14\], and still remains the problem on how to describe a quantum universe when the background geometry is not yet classical.

Nevertheless, there are some alternative solutions to this quantum cosmological dilemma which, together with decoherence, can solve the measurement problem maintaining the universality of quantum theory. One can say that the Schrödinger evolution is an approximation of a more fundamental non-linear theory which can accomplish the collapse \[15, 16\], or that the collapse is effective but not real, in the sense that the other branches disappear from the observer but do not disappear from existence. In this second category we can cite the Many-Worlds Interpretation \[17\] and the Bohm-de Broglie Interpretation \[18, 19, 20, 21\]. In the former, all the possibilities in the splitting are actually realized. In each branch there is an observer with the knowledge of the corresponding eigenvalue of this branch, but she or he is not aware of the other observers and the other possibilities because the branches do not interfere. In the latter, a point-particle in configuration space describing the observed system and apparatus is supposed to exist, independently on any observations. In the splitting, this point particle will enter into one of the branches (which one depends on the initial position of the point particle before the measurement, which is unknown), and the other branches will be empty. It can be shown \[19\] that the empty waves can neither interact with other particles, nor with the point particle containing the apparatus. Hence, no observer can be aware of the other branches which are empty. Again we have an effective but not real collapse (the empty waves continue to exist), but now with no multiplication of observers. Of course these interpretations can be used in quantum cosmology. Schrödinger evolution is always valid, and there is no need of a classical domain outside the observed system.

In this contribution we will focus on the application of the Bohm-de Broglie interpretation to quantum cosmology \[22, 23, 24, 25\]. In this approach, the fundamental object of quantum gravity, the geometry of 3-dimensional spacelike hypersurfaces, is supposed to exist independently on any observation or measurement, as well as its canonical momentum, the extrinsic curvature of the spacelike hypersurfaces. Its evolution, labeled by some time parameter, is dictated by a quantum evolution that is different from the classical one due to the presence of a quantum potential which appears naturally from the Wheeler-DeWitt equation. This interpretation has been applied to many minisuperspace models \[22, 25, 26, 27, 28, 29, 30, 31\], obtained by the imposition of homogeneity of the spacelike hypersurfaces. The classical limit, the singularity problem, the cosmological constant problem, and
the time issue have been discussed. For instance, in some of these papers it was shown that in models involving scalar fields or radiation, which are good representatives of the matter content of the early universe, the singularity can be clearly avoided by quantum effects. In the Bohm-de Broglie interpretation description, the quantum potential becomes important near the singularity, yielding a repulsive quantum force counteracting the gravitational field, avoiding the singularity and yielding inflation. The classical limit (given by the limit where the quantum potential becomes negligible with respect to the classical energy) for large scale factors are usually attainable, but for some scalar field models it depends on the quantum state and initial conditions (see Ref.[33]). In fact it is possible to have small classical universes and large quantum ones [29]. For instance, in Ref.[30] we obtained models arising classically from a singularity, experiencing quantum effects in the middle of their expansion, and recovering their classical behaviour for large values of $\alpha$. These quantum effects may cause an acceleration of the expansion of these models as the one which is observed in our Universe. Such a possibility is explored in detail in Ref.[34] and shown to be a viable alternative explanation for the recent high redshift supernovae observations [35, 36]. About the time issue (different choices of time yielding different physical predictions [37, 38]), it was shown that for any choice of the lapse function the quantum evolution of the homogeneous hypersurfaces yield the same four-geometry [25]. In Ref.[31] we have shown how quantum cosmological effects in the Bohmian description may not only avoid the initial singularity but also can isotropize anisotropic models which classically never isotropizes.

Finally, we studied the bohmian view of quantum space and time in the full theory, where we are not restricted to homogeneous spacelike hypersurfaces [32]. The question is, given an initial hypersurface with consistent initial conditions, does the evolution of the initial three-geometry driven by the quantum bohmian dynamics yields the same four-geometry for any choice of the lapse and shift functions, and if it does, what kind of spacetime structure is formed? We know that this is true if the three-geometry is evolved by the dynamics of classical General Relativity (GR), yielding a non degenerate four geometry, but it can be false if the evolving dynamics is the quantum bohmian one. Our conclusion is that, in general, the quantum bohmian evolution of the three-geometries does not yield any non degenerate four-geometry at all. Only for very special quantum states a relevant quantum non degenerate four-geometry can be obtained, and it must be euclidean. In the general case, either the quantum bohmian evolution is consistent (still independent on the choice of the lapse and shift functions) but yielding a degenerate four-geometry, where special vector fields, the null eigenvectors of the four geometry, are present\(^1\). We arrive at these conclusions without assuming any

\(^1\)For instance, the four geometry of Newtonian spacetime is degenerate [39], and its single null eigenvector is the
regularization and factor ordering of the Wheeler-DeWitt equation. As we know, the Wheeler-DeWitt equation involves the application of the product of local operators on states at the same space point, which is ill defined [40]. Hence we need to regularize it in order to solve the factor ordering problem, and have a theory free of anomalies (for some proposals, see Refs [41, 42, 43]). Our conclusions are completely independent on these issues. Also, in the general case where there are degenerate four-geometries, we can obtain a picture of the quantum structure yielded by the bohmian dynamics, which is not a spacetime in the sense described above but something else, as the degenerate four-geometries compatible with the Carroll group [44].

This contribution is organized as follows: in the next section I present the Bohm-De Broglie interpretation of canonical quantum gravity and its restriction to the minisuperspace of homogeneous geometries. I show that, in this framework, the time issue can be avoided in the bohmian approach to quantum cosmology. In section III I study the singularity and isotropy problems in minisuperspace quantum cosmology. In section IV I treat the full superspace and the quantum picture of space and time coming from Bohm’s point of view. We end up with the conclusions.

2 THE BOHM-DE BROGLIE INTERPRETATION OF CANONICAL QUANTUM COSMOLOGY

Let me now apply the Bohm-de Broglie interpretation to canonical quantum cosmology. I will quantize General Relativity Theory (GR) where the matter content is constituted of a minimally coupled scalar field with arbitrary potential. All subsequent results remain essentially the same for any matter field which couples uniquely with the metric, not with their derivatives.

The classical hamiltonian of GR with a scalar field is given by:

\[ H = \int d^3x (N\mathcal{H} + N^j\mathcal{H}_j) \]  

where

\[ \mathcal{H} = \kappa G_{ijkl} \Pi^{ij} \Pi^{kl} + \frac{1}{2} h^{-1/2} \Pi_\phi^2 + h^{1/2} \left[ -\kappa^{-1} (R^{(3)} - 2\Lambda) + \frac{1}{2} h^{ij} \partial_i \phi \partial_j \phi + U(\phi) \right] \]  

(2)

\[ \mathcal{H}_j = -2D_i \Pi^i_j + \Pi_\phi \partial_j \phi. \]  

(3)

normal of the absolute hypersurfaces of simultaneity, the time. As we know, it does not form a single spacetime structure because it is broken in absolute space plus absolute time.
In these equations, \( h_{ij} \) is the metric of closed 3-dimensional spacelike hypersurfaces, and \( \Pi^{ij} \) is its canonical momentum given by

\[
\Pi^{ij} = -h^{1/2}(K^{ij} - h^{ij}K) = G^{ijkl}(\dot{h}_{kl} - D_kN_l - D_lN_k),
\]

where

\[
K_{ij} = -\frac{1}{2N}(\dot{h}_{ij} - D_iN_j - D_jN_i),
\]

is the extrinsic curvature of the hypersurfaces (indices are raised and lowered by the 3-metric \( h_{ij} \) and its inverse \( h^{ij} \)). The canonical momentum of the scalar field is

\[
\Pi_\phi = \frac{h^{1/2}}{N}\left(\dot{\phi} - N^i\partial_i\phi\right).
\]

The quantity \( R^{(3)} \) is the intrinsic curvature of the hypersurfaces and \( h \) is the determinant of \( h_{ij} \). The lapse function \( N \) and the shift function \( N_j \) are the Lagrange multipliers of the super-hamiltonian constraint \( \mathcal{H} \approx 0 \) and the super-momentum constraint \( \mathcal{H}^j \approx 0 \), respectively. They are present due to the invariance of GR under spacetime coordinate transformations. The quantities \( G_{ijkl} \) and its inverse \( G^{ijkl} \) (\( G^{ijkl}G_{ijab} = \delta_{ab} \)) are given by

\[
G^{ijkl} = \frac{1}{2}h^{-1/2}(h^{ik}h^{jl} + h^{il}h^{jk} - 2h^{ij}h^{kl}),
\]

\[
G_{ijkl} = \frac{1}{2}h^{1/2}(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}),
\]

which is called the DeWitt metric. The quantity \( D_i \) is the \( i \)-component of the covariant derivative operator on the hypersurface, and \( \kappa = 16\pi G/c^4 \).

The classical 4-metric

\[
ds^2 = -(N^2 - N^iN_i)dt^2 + 2N_i dx^i dt + h_{ij}dx^idx^j
\]

and the scalar field which are solutions of the Einstein’s equations can be obtained from the Hamilton’s equations of motion

\[
\dot{h}_{ij} = \{h_{ij}, H\},
\]

\[
\dot{\Pi}^{ij} = \{\Pi^{ij}, H\},
\]

\[
\dot{\phi} = \{\phi, H\},
\]
\[ \Pi_\phi = \{ \Pi_\phi, H \}, \]  

for some choice of \( N \) and \( N^i \), and if we impose initial conditions compatible with the constraints

\[ \mathcal{H} \approx 0, \]  

\[ \mathcal{H}_i \approx 0. \]  

It is a feature of the hamiltonian of GR that the 4-metrics constructed in this way, with the same initial conditions, describe the same four-geometry for any choice of \( N \) and \( N^i \).

The algebra of the constraints close in the following form (we follow the notation of Ref. [45]):

\[ \{ \mathcal{H}(x), \mathcal{H}(x') \} = \mathcal{H}^i(x)\partial_i \delta^3(x, x') - \mathcal{H}^i(x')\partial_i \delta^3(x', x) \]  

\[ \{ \mathcal{H}_i(x), \mathcal{H}(x') \} = \mathcal{H}(x)\partial_i \delta^3(x, x') \]  

\[ \{ \mathcal{H}_i(x), \mathcal{H}_j(x') \} = \mathcal{H}_i(x)\partial_j \delta^3(x, x') + \mathcal{H}_j(x')\partial_i \delta^3(x, x') \]  

To quantize this constrained system, we follow the Dirac quantization procedure. The constraints become conditions imposed on the possible states of the quantum system, yielding the following quantum equations:

\[ \hat{\mathcal{H}}_i | \Psi > = 0 \]  

\[ \hat{\mathcal{H}} | \Psi > = 0 \]

In the metric and field representation, the first equation is

\[ -2h_{ii} D_j \frac{\delta \Psi(h_{ij}, \phi)}{\delta h_{ij}} + \frac{\delta \Psi(h_{ij}, \phi)}{\delta \phi} \partial_i \phi = 0, \]  

which implies that the wave functional \( \Psi \) is an invariant under space coordinate transformations.

The second equation is the Wheeler-DeWitt equation [46, 47]. Writing it unregulated in the coordinate representation we get

\[ \left\{ -h^2 \left[ \kappa G_{ijkl} \frac{\delta}{\delta h_{ij}} \frac{\delta}{\delta h_{kl}} + \frac{1}{2} h^{-1/2} \frac{\delta^2}{\delta \phi^2} \right] + V \right\} \Psi(h_{ij}, \phi) = 0, \]  

where \( V \) is the classical potential given by

\[ V = h^{1/2} \left[ -\kappa^{-1} (R_{(3)} - 2\Lambda) + \frac{1}{2} h^{ij} \partial_i \phi \partial_j \phi + U(\phi) \right]. \]
This equation involves products of local operators at the same space point, hence it must be regularized. After doing this, one should find a factor ordering which makes the theory free of anomalies, in the sense that the commutator of the operator version of the constraints close in the same way as their respective classical Poisson brackets. Hence, Eq. (20) is only a formal one which must be worked out.

Let us now see what is the Bohm-de Broglie interpretation of the solutions of Eqs. (19) and (20) in the metric and field representation. First we write the wave functional in polar form \( \Psi = A \exp(iS/\hbar) \), where \( A \) and \( S \) are functionals of \( h_{ij} \) and \( \phi \). Substituting it in Eq. (19), we get two equations saying that \( A \) and \( S \) are invariant under general space coordinate transformations:

\[
-2h_{li}D_j\frac{\delta S(h_{ij},\phi)}{\delta h_{lj}} + \frac{\delta S(h_{ij},\phi)}{\delta \phi}\partial_i\phi = 0, \tag{22}
\]

\[
-2h_{li}D_j\frac{\delta A(h_{ij},\phi)}{\delta h_{lj}} + \frac{\delta A(h_{ij},\phi)}{\delta \phi}\partial_i\phi = 0. \tag{23}
\]

The two equations we obtain for \( A \) and \( S \) when we substitute \( \Psi = A \exp(iS/\hbar) \) into Eq. (20) will of course depend on the factor ordering we choose. However, in any case, one of the equations will have the form

\[
\kappa G_{ijkl} \frac{\delta S}{\delta h_{ij}} \frac{\delta S}{\delta h_{kl}} + \frac{1}{2} \hbar^{-1/2} \left( \frac{\delta S}{\delta \phi} \right)^2 + V + Q = 0, \tag{24}
\]

where \( V \) is the classical potential given in Eq. (21). Contrary to the other terms in Eq. (24), which are already well defined, the precise form of \( Q \) depends on the regularization and factor ordering which are prescribed for the Wheeler-DeWitt equation. In the unregulated form given in Eq. (20), \( Q \) is

\[
Q = -\hbar^2 \frac{1}{A} \left( \kappa G_{ijkl} \frac{\delta^2 A}{\delta h_{ij} \delta h_{kl}} + \frac{\hbar^{-1/2}}{2} \frac{\delta^2 A}{\delta \phi^2} \right). \tag{25}
\]

Also, the other equation besides (24) in this case is

\[
\kappa G_{ijkl} \frac{\delta}{\delta h_{ij}} \left( A^2 \frac{\delta S}{\delta h_{kl}} \right) + \frac{1}{2} \hbar^{-1/2} \frac{\delta}{\delta \phi} \left( A^2 \frac{\delta S}{\delta \phi} \right) = 0. \tag{26}
\]

Let me now implement the Bohm-de Broglie interpretation for canonical quantum gravity. First of all we note that Eqs. (22) and (24), which are always valid irrespective of any factor ordering of the Wheeler-DeWitt equation, are like the Hamilton-Jacobi equations for GR, supplemented by an extra term \( Q \) in the case of Eq. (24), which we will call the quantum potential. By analogy with the cases of non-relativistic particle and quantum field theory in flat spacetime, we will postulate that the 3-metric of spacelike hypersurfaces, the scalar field, and their canonical momenta always exist, independent on
any observation, and that the evolution of the 3-metric and scalar field can be obtained from the
guidance relations

\[ \Pi^{ij} = \frac{\delta S(h_{ab}, \phi)}{\delta h_{ij}}, \]

(27)

\[ \Pi_\phi = \frac{\delta S(h_{ij}, \phi)}{\delta \phi}, \]

(28)

with \( \Pi^{ij} \) and \( \Pi_\phi \) given by Eqs. (4) and (6), respectively. Like before, these are first order differential
equations which can be integrated to yield the 3-metric and scalar field for all values of the \( t \) parameter.
These solutions depend on the initial values of the 3-metric and scalar field at some initial hypersurface.
The evolution of these fields will of course be different from the classical one due to the presence of
the quantum potential term \( Q \) in Eq. (24). The classical limit is once more conceptually very simple:
it is given by the limit where the quantum potential \( Q \) becomes negligible with respect to the classical
energy. The only difference from the cases of the non-relativistic particle and quantum field theory
in flat spacetime is the fact that Eq. (24) for canonical quantum gravity cannot be interpreted as a
continuity equation for a probability density \( A^2 \) because of the hyperbolic nature of the DeWitt metric
\( G_{ijkl} \). However, even without a notion of probability, which in this case would mean the probability
density distribution for initial values of the 3-metric and scalar field in an initial hypersurface, we can
extract a lot of information from Eq. (24) whatever is the quantum potential \( Q \), as we will see. After
we get these results, we will return to this probability issue in the last section.

First we note that, whatever is the form of the quantum potential \( Q \), it must be a scalar density of
weight one. This comes from the Hamilton-Jacobi equation (24). From this equation we can express
\( Q \) as

\[ Q = -\kappa G_{ijkl} \frac{\delta S}{\delta h_{ij}} \frac{\delta S}{\delta h_{kl}} - \frac{1}{2} h^{-1/2} \left( \frac{\delta S}{\delta \phi} \right)^2 - V. \]

(29)

As \( S \) is an invariant (see Eq. (22)), then \( \delta S/\delta h_{ij} \) and \( \delta S/\delta \phi \) must be a second rank tensor density
and a scalar density, both of weight one, respectively. When their products are contracted with \( G_{ijkl} \)
and multiplied by \( h^{-1/2} \), respectively, they form a scalar density of weight one. As \( V \) is also a scalar
density of weight one, then \( Q \) must also be. Furthermore, \( Q \) must depend only on \( h_{ij} \) and \( \phi \) because it
comes from the wave functional which depends only on these variables. Of course it can be non-local,
i.e., depending on integrals of the fields over the whole space, but it cannot depend on the momenta.

A minisuperspace is the set of all spacelike geometries where all but a set of the \( h_{ij}^{(n)}(t) \) and the
corresponding \( \Pi_{ij}^{(n)}(t) \) are put identically to zero.

Evidently, this procedure violate the uncertainty principle. However, we expect that the quanti-
zation of these minisuperspace models retains many of the qualitative features of the full quantum

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theory, which are easier to study in this simplified model. For more details on minisuperspace models, see Refs. [48, 49, 50].

In the case of a minisuperspace of homogeneous models, the supermomentum constraint $H^i$ is identically zero, and the shift function $N_i$ can be set to zero in equation (1) without losing any of the Einstein’s equations. The hamiltonian (1) is reduced to:

$$H_{GR} = N(t)\mathcal{H}(p^\alpha(t), q_\alpha(t)),$$

where $p^\alpha(t)$ and $q_\alpha(t)$ represent the homogeneous degrees of freedom coming from $\Pi^j(x, t)$ and $h_{ij}(x, t)$. Equations (24-28) become:

$$\frac{1}{2} f_{\alpha\beta}(q_\mu) \frac{\partial S}{\partial q_\alpha} \frac{\partial S}{\partial q_\beta} + U(q_\mu) + Q(q_\mu) = 0,$$

$$Q(q_\mu) = -\frac{1}{N} f_{\alpha\beta} \frac{\partial^2 R}{\partial q_\alpha \partial q_\beta},$$

$$p^\alpha = \frac{\partial S}{\partial q_\alpha} = f^{\alpha\beta} \frac{1}{N} \frac{\partial q_\beta}{\partial t} = 0,$$

where $f_{\alpha\beta}(q_\mu)$ and $U(q_\mu)$ are the minisuperspace particularizations of $G_{ijkl}$ and $-h^{1/2}R^{(3)}(h_{ij})$, respectively.

Equation (33) is invariant under time reparametrization. Hence, even at the quantum level, different choices of $N(t)$ yield the same spacetime geometry for a given non-classical solution $q_\alpha(x, t)$.

3 AVOIDANCE OF SINGULARITIES AND QUANTUM ISOTROPIZATION OF THE UNIVERSE

One of the fluids which may represent the matter content of the very early Universe is a massless free scalar field, which is equivalent to stiff matter [51] ($p = \rho$, sound velocity equal to the speed of light). As for this type of matter content in a universe which is spatially homogeneous and isotropic (an excellent approximation for the very early Universe), $\rho \propto a^{-6}(t)$, where $a(t)$ is the scale factor of the homogeneous and isotropic hypersurfaces, in the very early Universe, where $a(t)$ approaches zero, this term dominates over radiation and dust, whose energy densities depends on $a(t)$ as $a^{-4}(t)$ and $a^{-3}(t)$, respectively. We will concentrate on this model now. If this scalar field is not present, radiation will be the dominant term in the early Universe. This case is studied in detail in Ref. [28].
Let us take the lagrangian
\[ L = \sqrt{-g} \left( R - \frac{1}{2} \phi \rho \phi^\rho \right) , \]  
(34)
where \( R \) is the Ricci scalar of the metric \( g_{\mu \nu} \) with determinant \( g \), and \( \phi \) is the scalar field. The gravitational part of the minisuperspace model is given by the homogeneous and anisotropic Bianchi I line element
\[ ds^2 = -N^2(t) dt^2 + \exp[2\beta_0(t) + 2\beta_+ (t) + 2\sqrt{3}\beta_- (t)] \ dx^2 + \]
\[ \exp[2\beta_0(t) + 2\beta_+ (t) - 2\sqrt{3}\beta_- (t)] \ dy^2 + \exp[2\beta_0(t) - 4\beta_+ (t)] \ dz^2. \]  
(35)
This line element will be isotropic if and only if \( \beta_+ (t) \) and \( \beta_- (t) \) are constants [1]. Other authors have studied Bianchi IX models adopting other interpretations of quantum cosmology [52, 53, 54, 55, 56].

Inserting Equation (35) into the action \( S = \int L \ d^4 x \), supposing that the scalar field \( \phi \) depends only on time, discarding surface terms, and performing a Legendre transformation, we obtain the following minisuperspace classical Hamiltonian
\[ H = \frac{N}{24 \exp (3\beta_0)} \left( -p_0^2 + p_+^2 + p_-^2 + p^2 \right), \]  
(36)
where \( (p_0, p_+, p_-, p_\phi) \) are canonically conjugate to \( (\beta_0, \beta_+, \beta_-, \phi) \), respectively, and we made the trivial redefinition \( \phi \to \sqrt{C_w/6} \ \phi \).

We can write this Hamiltonian in a compact form by defining \( y^\mu = (\beta_0, \beta_+, \beta_-, \phi) \) and their canonical momenta \( p_\mu = (p_0, p_+, p_-, p_\phi) \), obtaining
\[ H = \frac{N}{24 \exp (3\beta_0)} \eta^{\mu \nu} p_\mu p_\nu, \]  
(37)
where \( \eta^{\mu \nu} \) is the Minkowski metric with signature \((-+++\)). The equations of motion are the constraint equation obtained by varying the Hamiltonian with respect to the lapse function \( N \)
\[ \mathcal{H} = \eta^{\mu \nu} p_\mu p_\nu = 0, \]  
(38)
and the Hamilton’s equations
\[ \dot{y}^\mu = \frac{\partial \mathcal{H}}{\partial p_\mu} = \frac{N}{12 \exp (3\beta_0)} \eta^{\mu \nu} p_\nu, \]  
(39)
\[ \dot{p}_\mu = -\frac{\partial \mathcal{H}}{\partial y^\mu} = 0. \]  
(40)
The solution to these equations in the gauge $N = 12 \exp(3y_0)$ is

$$y^\mu = \eta^{\mu\nu} p_\nu t + C^\mu,$$  \hspace{1cm} (41)

where the momenta $p_\nu$ are constants due to the equations of motion and the $C^\mu$ are integration constants. We can see that the only way to obtain isotropy in these solutions is by making $p_1 = p_+ = 0$ and $p_2 = p_- = 0$, which yields solutions that are always isotropic, the usual Friedmann-Robertson-Walker (FRW) solutions with a scalar field. Hence, there is no anisotropic solution in this model which can classically becomes isotropic during the course of its evolution. Once anisotropic, always anisotropic. If we suppress the $\phi$ degree of freedom, the unique isotropic solution is flat space-time because in this case the constraint (38) enforces $p_0 = 0$.

To discuss the appearance of singularities, we need the Weyl square tensor $W^2 \equiv W^{\alpha\beta\mu\nu} W_{\alpha\beta\mu\nu}$. It reads

$$W^2 = \frac{1}{432} e^{-12\beta_0} (2p_0 p_+^3 - 6p_0 p_+^2 p_+ + p_+^4 + 2p_+^2 p_- + p_+^4 + p_0^2 p_+^2 + p_0^2 p_-^2).$$ \hspace{1cm} (42)

Hence, the Weyl square tensor is proportional to $\exp(-12\beta_0)$ because the $p$’s are constants (see Equations (40)) and the singularity is at $t = -\infty$. The classical singularity can be avoided only if we set $p_0 = 0$. But then, due to Equation (38), we would also have $p_i = 0$, which corresponds to the trivial case of flat space-time. Therefore, the unique classical solution which is non-singular is the trivial flat space-time solution.

The Dirac quantization procedure yields the Wheeler-DeWitt equation through the imposition of the condition

$$\hat{H} \Psi = 0,$$ \hspace{1cm} (43)

on the quantum states, with $\hat{H}$ defined as in Equation (38) (we are assuming the covariant factor ordering) using the substitutions

$$p_\mu \rightarrow -i \frac{\partial}{\partial y^\mu}. \hspace{1cm} (44)$$

Equation (43) reads

$$\eta^{\mu\nu} \frac{\partial^2}{\partial y^\mu y^\nu} \Psi(y^\mu) = 0.$$ \hspace{1cm} (45)

For the minisuperspace we are investigating, the guidance relations in the gauge $N = 12 \exp(3y_0)$ are (see Equations (39))

$$p_\mu = \frac{\partial S}{\partial y^\mu} = \eta_{\mu\nu} y^\nu,$$ \hspace{1cm} (46)
where \( S \) is the phase of the wave function.

I will investigate spherical-wave solutions of Equation (45). They read

\[
\Psi_1 = \frac{1}{y} \left[ f(y^0 + y) + g(y^0 - y) \right],
\]

where \( y \equiv \sqrt{\sum_{i=1}^{3} (y_i^2)} \).

One particular example is the Gaussian superposition of plane wave solutions of Equation (45),

\[
\Psi_2(y^\nu) = \int \{ F(\vec{k}) \exp[i(|\vec{k}|y^0 + \vec{k} \cdot \vec{y})] + G(\vec{k}) \exp[i(|\vec{k}|y^0 - \vec{k} \cdot \vec{y})]\} d^3 k ,
\]

where \( \vec{k} \equiv (k_1, k_2, k_3), \vec{y} \equiv (y^1, y^2, y^3), |\vec{k}| \equiv \sqrt{\sum_{i=1}^{3} (k_i^2)}, \) with \( F(\vec{k}) \) and \( G(\vec{k}) \) given by

\[
F(\vec{k}) = G(\vec{k}) = \exp \left[ -\frac{(|\vec{k}| - d)^2}{\sigma^2} \right].
\]

After performing the integration in Equation (48) using spherical coordinates we obtain

\[
\Psi_2(y^0, y) = \frac{i\pi^{3/2}}{y} \left\{ \left[ 2d\sigma + i(y^0 - y)\sigma^3 \right] \exp \left[ -\left(\frac{y^0 - y}{\sigma^2}\right)^2 \right] \exp[i(y^0 - y)] \right. \\
\left. \left[ 1 + \Phi \left( \frac{d}{\sigma} + i(y^0 - y)\frac{\sigma}{2} \right) \right] \right.
\left. + \left[ -2d\sigma + i(y^0 + y)\sigma^3 \right] \exp \left[ -\left(\frac{y^0 + y}{\sigma^2}\right)^2 \right] \exp[i(y^0 + y)] \right. \\
\left. \left[ 1 + \Phi \left( \frac{d}{\sigma} + i(y^0 + y)\frac{\sigma}{2} \right) \right] \right\},
\]

where \( \Phi(x) \equiv \left( \frac{2}{\sqrt{\pi}} \right) \int_{0}^{x} \exp(-t^2) dt \) is the probability integral. The wave function \( \Psi_4 \) is a spherical solution with the form of Equation (47) with \( g = -f \). In order to simplify \( \Psi_4 \), we will take the limit \( \sigma^2 >> d \) and \( (y^0 \pm y)\sigma >> 1 \) in Equation (50) yielding

\[
f(z) \approx -\frac{16\pi d}{\sigma^2 z^3} + i2\pi \left( \frac{2}{z^2} + \sigma^2 \right).
\]

Let us study the spherical wave solutions of Equation (45). The guidance relations (46) are

\[
p_0 = \partial_0 S = \text{Im} \left( \frac{\partial_0 \Psi_1}{\Psi_1} \right) = -y^0,
\]

\[
p_i = \partial_i S = \text{Im} \left( \frac{\partial_i \Psi_1}{\Psi_1} \right) = y^i,
\]
where \( S \) is the phase of the wave function. In terms of \( f \) and \( g \) the above equations read

\[
y^0 = -\text{Im} \left( \frac{f'(y^0 + y) + g'(y^0 - y)}{f(y^0 + y) + g(y^0 - y)} \right),
\]

\[
y^i = \frac{y^i}{y} \text{Im} \left( \frac{f'(y^0 + y) - g'(y^0 - y)}{f(y^0 + y) + g(y^0 - y)} \right),
\]

where the prime means derivative with respect to the argument of the functions \( f \) and \( g \), and \( \text{Im}(z) \) is the imaginary part of the complex number \( z \).

From Equations (54) and (55) we obtain that

\[
\frac{dy^i}{dy^0} = \frac{y^i}{y^0},
\]

which implies that \( y^i(t) = c^i_j y^j(t) \), with no sum in \( j \), where the \( c^i_j \) are real constants, \( c^1_j = 1/c^0_j \) and \( c^1_1 = c^2_2 = c^3_3 = 1 \). Hence, apart some positive multiplicative constant, knowing about one of the \( y^i \) means knowing about all \( y^i \). Consequently, we can reduce the four equations (54) and (55) to a planar system by writing \( y = C|y^3| \), with \( C > 1 \), and working only with \( y^0 \) and \( y^3 \), say. The planar system now reads

\[
y^0 = -\text{Im} \left( \frac{f'(y^0 + C|y^3|)}{f(y^0 + C|y^3|) + g(y^0 - C|y^3|)} \right),
\]

\[
y^3 = \text{sign}(y^3) \frac{1}{C} \text{Im} \left( \frac{f'(y^0 + C|y^3|) - g'(y^0 - C|y^3|)}{f(y^0 + C|y^3|) + g(y^0 - C|y^3|)} \right).
\]

Note that if \( f = g \), \( y^3 \) stabilizes at \( y^3 = 0 \) because \( \dot{y}^3 \) as well as all other time derivatives of \( y^3 \) are zero at this line. As \( y^i(t) = c^i_j y^j(t) \), all \( y^i(t) \) become zero, and the cosmological model isotropizes forever once \( y^3 \) reaches this line. Of course one can find solutions where \( y^3 \) never reaches this line, but in this case there must be some region where \( \dot{y}^3 = 0 \), which implies \( \dot{y}^i = 0 \), and this is an isotropic region. Consequently, quantum anisotropic cosmological models with \( f = g \) always have an isotropic phase, which can become permanent in many cases.

As a concrete example, let us take the Gaussian \( \Psi_2 \) given in Equation (50). It is a spherical wave solution of the Wheeler-DeWitt equation (45) with \( f = -g \), and hence it does not necessarily have isotropic phases as described above for the case \( f = g \). The most interesting case happens when \( d \) is negative, as it is shown in Figure 1 for \( d/\sigma^2 = -10^{-4} \) and \( C = 2 \). Realistic cosmological models without singularities (in fact, periodic Universes, as one can see in the closed lines depicted in Figure 1) are obtained, with expanding phases (increasing \( \beta_0 \)) which are isotropic [remember that
lines parallel to the $\beta_0$ axes ($\phi \approx \text{const.}$) are also lines with $\beta_\pm \approx \text{const.}$ (isotropic phases), which can be made arbitrarily large in the region $|\phi| >> |\beta_0|$. Hence, what was classically forbidden (a nonempty, nonsingular anisotropic model which becomes isotropic in the course of its evolution) is possible within the bohmian quantum dynamics described above.

\[ \dot{\beta}_0 = \frac{2d}{\sigma^2} (3\beta_0^4 + 6C^2\beta_0^2\phi^2 - C^4\phi^4)}{\beta_0^2(\beta_0^2 - C^2\phi^2)^2 + \frac{16d^2}{\sigma^2}(3\beta_0^2 + C^2\phi^2)^2}, \]  
\[ \dot{\phi} = \frac{16d\beta_0^3\phi}{\beta_0^2(\beta_0^2 - C^2\phi^2)^2 + \frac{16d^2}{\sigma^2}(3\beta_0^2 + C^2\phi^2)^2}, \]  
where we have reset $y^0 = \beta_0$ and $y^3 = \phi$. This approximation is not reliable in the lines $\beta_0 = \pm C\phi$.

As one can see immediately from these equations, $\beta_\pm = \text{const.}$ whenever $|\phi| >> |\beta_0|$, and the sign of $d$ defines the trajectories direction.

**Figure 1**: Field plot of the system of planar equations (57-58) for $\sigma = 100$, $d = -1$ and $C = 2$, which uses the Bohm-de Broglie interpretation with the wave function $\Psi_2$, Equation (50). Each arrow of the vector field is shaded according to its true length, black representing short vectors and white, long ones. The dark shade of gray shows the regions where the derivative of the vector field points clockwise (the light shade of gray means the opposite) and this shading allows to see the regions with arbitrarily long periods of isotropic evolution. The trajectories are the black or white curves with direction arrows.

In order to get some analytical insight over Figure 1, we present the planar system obtained from the guidance relations corresponding to the wave function $\Psi_2$, Equation (50) in the approximation (51):

\[ \dot{\beta}_0 = \frac{2d}{\sigma^2} (3\beta_0^4 + 6C^2\beta_0^2\phi^2 - C^4\phi^4)}{\beta_0^2(\beta_0^2 - C^2\phi^2)^2 + \frac{16d^2}{\sigma^2}(3\beta_0^2 + C^2\phi^2)^2}, \]  
\[ \dot{\phi} = \frac{16d\beta_0^3\phi}{\beta_0^2(\beta_0^2 - C^2\phi^2)^2 + \frac{16d^2}{\sigma^2}(3\beta_0^2 + C^2\phi^2)^2}, \]  
where we have reset $y^0 = \beta_0$ and $y^3 = \phi$. This approximation is not reliable in the lines $\beta_0 = \pm C\phi$. As one can see immediately from these equations, $\beta_\pm = \text{const.}$ whenever $|\phi| >> |\beta_0|$, and the sign of $d$ defines the trajectories direction.
In this section I make general study of the quantum bohmian trajectories in full superspace. I will investigate the following important problem. From the guidance relations (27) and (28) we obtain the following first order partial differential equations:

\[ \dot{h}_{ij} = 2NG_{ijkl} \frac{\delta S}{\delta h_{kl}} + D_iN_j + D_jN_i \]  

and

\[ \dot{\phi} = Nh^{-1/2} \frac{\delta S}{\delta \phi} + N^i \partial_i \phi. \]

The question is, given some initial 3-metric and scalar field, what kind of structure do we obtain when we integrate this equations in the parameter \( t \)? Does this structure form a 4-dimensional geometry with a scalar field for any choice of the lapse and shift functions? Note that if the functional \( S \) were a solution of the classical Hamilton-Jacobi equation, which does not contain the quantum potential term, then the answer would be in the affirmative because we would be in the scope of GR. But \( S \) is a solution of the modified Hamilton-Jacobi equation (24), and we cannot guarantee that this will continue to be true. We may obtain a complete different structure due to the quantum effects driven by the quantum potential term in Eq. (24). To answer this question we will move from this Hamilton-Jacobi picture of quantum geometrodynamics to a hamiltonian picture. This is because many strong results concerning geometrodynamics were obtained in this later picture [45, 58]. We will construct a hamiltonian formalism which is consistent with the guidance relations (27) and (28). It yields the bohmian trajectories (61) and (62) if the guidance relations are satisfied initially. Once we have this hamiltonian, we can use well known results in the literature to obtain strong results about the Bohm-de Broglie view of quantum geometrodynamics.

Examining Eqs. (22) and (24), we can easily guess that the hamiltonian which generates the bohmian trajectories, once the guidance relations (27) and (28) are satisfied initially, should be given by:

\[ H_Q = \int d^3x \left[ N(\mathcal{H} + Q) + N^i \mathcal{H}_i \right] \]

where we define

\[ \mathcal{H}_Q \equiv \mathcal{H} + Q. \]

The quantities \( \mathcal{H} \) and \( \mathcal{H}_i \) are the usual GR super-hamiltonian and super-momentum constraints given by Eqs. (2) and (3). In fact, the guidance relations (27) and (28) are consistent with the constraints
\( H_Q \approx 0 \) and \( H_i \approx 0 \) because \( S \) satisfies (22) and (24). Furthermore, they are conserved by the hamiltonian evolution given by (63). Then we can show that indeed Eqs. (61,62) can be obtained from \( H_Q \) with the guidance relations (27) and (28) viewed as additional constraints. For details, see Ref. [32].

We have a hamiltonian, \( H_Q \), which generates the bohmian trajectories once the guidance relations (27) and (28) are imposed initially. In the following, we can investigate if the evolution of the fields driven by \( H_Q \) forms a four-geometry like in classical geometrodynamics. First we recall a result obtained by Claudio Teitelboim [58]. In this paper, he shows that if the 3-geometries and field configurations defined on hypersurfaces are evolved by some hamiltonian with the form

\[
\bar{H} = \int d^3x \left( N\bar{H} + N^i\bar{H}_i \right),
\]

and if this evolution can be viewed as the “motion” of a 3-dimensional cut in a 4-dimensional spacetime (the 3-geometries can be embedded in a four-geometry), then the constraints \( \bar{H} \approx 0 \) and \( \bar{H}_i \approx 0 \) must satisfy the following algebra

\[
\{ \bar{H}(x), \bar{H}(x') \} = -\epsilon[\bar{H}_i(x)\partial_i\delta^3(x',x)] - \bar{H}_i(x')\partial_i\delta^3(x,x') \quad (66)
\]

\[
\{ \bar{H}_i(x), \bar{H}_j(x') \} = \bar{H}_i(x)\partial_j\delta^3(x,x') - \bar{H}_j(x')\partial_i\delta^3(x,x') \quad (67)
\]

\[
\{ \bar{H}_i(x), \bar{H}_j(x') \} = \bar{H}_i(x)\partial_j\delta^3(x,x') - \bar{H}_j(x')\partial_i\delta^3(x,x') \quad (68)
\]

The constant \( \epsilon \) in (66) can be \( \pm 1 \) depending if the four-geometry in which the 3-geometries are embedded is euclidean (\( \epsilon = 1 \)) or hyperbolic (\( \epsilon = -1 \)). These are the conditions for the existence of spacetime.

The above algebra is the same as the algebra (16) of GR if we choose \( \epsilon = -1 \). But the hamiltonian (63) is different from the hamiltonian of GR only by the presence of the quantum potential term \( Q \) in \( H_Q \). The Poisson bracket \( \{ H_i(x), H_j(x') \} \) satisfies Eq. (68) because the \( H_i \) of \( H_Q \) defined in Eq. (63) is the same as in GR. Also \( \{ H_i(x), H_Q(x') \} \) satisfies Eq. (67) because \( H_i \) is the generator of spatial coordinate transformations, and as \( H_Q \) is a scalar density of weight one (remember that \( Q \) must be a scalar density of weight one), then it must satisfies this Poisson bracket relation with \( H_i \). What remains to be verified is if the Poisson bracket \( \{ H_Q(x), H_Q(x') \} \) closes as in Eq. (66). We now recall the result of Ref. [45]. There it is shown that a general super-hamiltonian \( \bar{H} \) which satisfies Eq. (66), is a scalar density of weight one, whose geometrical degrees of freedom are given only by the three-metric \( h_{ij} \) and its canonical momentum, and contains only even powers and no non-local term.
in the momenta (together with the other requirements, these last two conditions are also satisfied by \( \mathcal{H}_Q \) because it is quadratic in the momenta and the quantum potential does not contain any non-local term on the momenta), then \( \bar{\mathcal{H}} \) must have the following form:

\[
\bar{\mathcal{H}} = \kappa G_{ijkl}\Pi^{ij}\Pi^{kl} + \frac{1}{2}h^{-1/2}\pi^2 + V_G,
\]

where

\[
V_G \equiv -\epsilon h^{1/2}\left[ -\kappa^{-1}(R^{(3)} - 2\Lambda) + \frac{1}{2}h^{ij}\partial_i\phi\partial_j\phi + \bar{U}(\phi) \right].
\]

With this result we can now establish three possible scenarios for the Bohm-de Broglie quantum geometrodynamics, depending on the form of the quantum potential:

### 4.1 Quantum geometrodynamics evolution is consistent and forms a non degenerate four-geometry

In this case, the Poisson bracket \( \{\mathcal{H}_Q(x), \mathcal{H}_Q(x')\} \) must satisfy Eq. (66). Then \( Q \) must be such that

\[
V + Q = V_G
\]

with \( V \) given by (21) yielding:

\[
Q = -h^{1/2}\left[ (\epsilon + 1)\left( -\kappa^{-1}R^{(3)} + \frac{1}{2}h^{ij}\partial_i\phi\partial_j\phi \right) + \frac{2}{\kappa}(\epsilon\Lambda + \Lambda) + \epsilon\bar{U}(\phi) + U(\phi) \right].
\]

Then we have two possibilities:

#### 4.1.1 The spacetime is hyperbolic (\( \epsilon = -1 \))

In this case \( Q \) is

\[
Q = -h^{1/2}\left[ \frac{2}{\kappa}(-\Lambda + \Lambda) - \bar{U}(\phi) + U(\phi) \right].
\]

Hence \( Q \) is like a classical potential. Its effect is to renormalize the cosmological constant and the classical scalar field potential, nothing more. The quantum geometrodynamics is indistinguishable from the classical one. It is not necessary to require the classical limit \( Q = 0 \) because \( V_G = V + Q \) already may describe the classical universe we live in.

#### 4.1.2 The spacetime is euclidean (\( \epsilon = 1 \))

In this case \( Q \) is

\[
Q = -h^{1/2}\left[ 2\left( -\kappa^{-1}R^{(3)} + \frac{1}{2}h^{ij}\partial_i\phi\partial_j\phi \right) + \frac{2}{\kappa}(\Lambda + \Lambda) + \bar{U}(\phi) + U(\phi) \right].
\]
Now $Q$ not only renormalize the cosmological constant and the classical scalar field potential but also change the signature of spacetime. The total potential $V_G = V + Q$ may describe some era of the early universe when it had euclidean signature, but not the present era, when it is hyperbolic. The transition between these two phases must happen in a hypersurface where $Q = 0$, which is the classical limit.

We can conclude from these considerations that if a quantum spacetime exists with different features from the classical observed one, then it must be euclidean. In other words, the sole relevant quantum effect which maintains the non-degenerate nature of the four-geometry of spacetime is its change of signature to a euclidean one. The other quantum effects are either irrelevant or break completely the spacetime structure. This result points in the direction of Ref. [2].

4.2 Quantum geometrodynamics evolution is consistent but does not form a non-degenerate four-geometry

In this case, the Poisson bracket $\{H_Q(x), H_Q(x')\}$ does not satisfy Eq. (66) but is weakly zero in some other way. Let us examine some examples.

4.2.1 Real solutions of the Wheeler-DeWitt equation

For real solutions of the Wheeler-DeWitt equation, which is a real equation, the phase $S$ is null. Then, from Eq. (24), we can see that $Q = -V$. Hence, the quantum super-hamiltonian (64) will contain only the kinetic term, yielding

$$\{H_Q(x), H_Q(x')\} = 0.$$  \hspace{1cm} (74)

This is a strong equality. This case is connected with the strong gravity limit of GR [59, 60, 62]. If we take the limit of big gravitational constant $G$ (or small speed of light $c$, where we arrive at the Carroll group [44]), then the potential in the super-hamiltonian constraint of GR can be neglected and we arrive at a super-hamiltonian containing only the kinetic term. The Bohm-de Broglie interpretation is telling us that any real solution of the Wheeler-DeWitt equation yields a quantum geometrodynamics satisfying precisely this strong gravity limit. The classical limit $Q = 0$ in this case implies also that $V = 0$. It should be interesting to investigate further the structure we obtain here.
4.2.2 Non-local quantum potentials

Any non-local quantum potential breaks spacetime. As an example take a quantum potential of the form

\[ Q = \gamma V, \tag{75} \]

where \( \gamma \) is a function of the functional \( S \) (here comes the non-locality). Calculating \( \{ H_Q(x), H_Q(x') \} \), we obtain (see Ref. [32]),

\[
\{ H_Q(x), H_Q(x') \} = (1 + \gamma) [\mathcal{H}^i(x) \partial_i \delta^3(x, x') - \mathcal{H}^i(x') \partial_i \delta^3(x', x)]
\]

\[
- \frac{d\gamma}{dS} V(x') [2\mathcal{H}_Q(x) - 2\kappa G_{klij} (x) \Pi^{ij} (x) \Phi^{kl} (x) - h^{-\frac{1}{2}} \Phi(x') \Phi(x)]
\]

\[
+ \frac{d\gamma}{dS} V(x) [2\mathcal{H}_Q(x') - 2\kappa G_{klij} (x') \Pi^{ij} (x') \Phi^{kl} (x') - h^{-\frac{1}{2}} \Phi(x') \Phi(x')]
\]

\[
\approx 0 \tag{76}
\]

The rhs in the last expression is weakly zero because it is a combination of the constraints and the guidance relations. Note that it was very important to use the guidance relations to close the algebra. It means that the hamiltonian evolution with the quantum potential (75) is consistent only when restricted to the bohmian trajectories. For other trajectories, it is inconsistent.

In the examples above, we have explicitly obtained the “structure constants” of the algebra that characterizes the “pre-four-geometry” generated by \( H_Q \) i.e., the foam-like structure pointed long time ago in early works of J. A. Wheeler [46] [63].

Finally, there are no inconsistent bohmian trajectories [61].

5 CONCLUSION

The Bohm-de Broglie interpretation of canonical quantum cosmology yields a quantum geometrodynamic picture where the bohmian quantum evolution of three-geometries may form, depending on the wave functional, a consistent non degenerate four geometry which must be euclidean (but only for a very special local form of the quantum potential), and a consistent but degenerate four-geometry indicating the presence of special vector fields and the breaking of the spacetime structure as a single entity (in a wider class of possibilities). Hence, in general, and always when the quantum potential is non-local, spacetime is broken. The three-geometries evolved under the influence of a quantum potential do not in general stick together to form a non degenerate four-geometry, a single spacetime
with the causal structure of relativity. This is not surprising, as it was anticipated long ago [63].

Among the consistent bohmian evolutions, the more general structures that are formed are degenerate four-geometries with alternative causal structures. We obtained these results taking a minimally coupled scalar field as the matter source of gravitation, but it can be generalized to any matter source with non-derivative couplings with the metric, like Yang-Mills fields.

As shown in the previous section, a non-degenerate four-geometry can be attained only if the quantum potential have the specific form (71). In this case, the sole relevant quantum effect will be a change of signature of spacetime, something pointing towards Hawking’s ideas.

In the case of consistent quantum geometrodynamical evolution but with degenerate four-geometry, we have shown that any real solution of the Wheeler-DeWitt equation yields a structure which is the idealization of the strong gravity limit of GR. This type of geometry, which is degenerate, has already been studied [62]. Due to the generality of this picture (it is valid for any real solution of the Wheeler-DeWitt equation, which is a real equation), it deserves further attention. It may well be that these degenerate four-metrics were the correct quantum geometrodynamical description of the young universe. It would be also interesting to investigate if these structures have a classical limit yielding the usual four-geometry of classical cosmology.

As the Bohm-de Broglie interpretation is a more detailed description of quantum phenomena, in this framework we can investigate further what kind of structure is formed in quantum geometrodynamics by using the Poisson bracket relation (66), and the guidance relations (61) and (62). By assuming the existence of 3-geometries, field configurations, and their momenta, independently on any observations, the Bohm-de Broglie interpretation allows us to use classical tools, like the hamiltonian formalism, to understand the structure of quantum geometry. The Bohm-de Broglie interpretation yields a lot of information about quantum geometrodynamics which cannot be obtained from the many-worlds interpretation. If this information is useful, I do not know. However, we cannot answer this question precisely if we do not investigate further, and the tools are at our disposal.

We would like to remark that all these results were obtained without assuming any particular factor ordering and regularization of the Wheeler-DeWitt equation. Also, we did not use any probabilistic interpretation of the solutions of the Wheeler-DeWitt equation. Hence, it is a quite general result. However, we would like to make some comments about the probability issue in quantum cosmology. The Wheeler-DeWitt equation when applied to a closed universe does not yield a probabilistic interpretation for their solutions because of its hyperbolic nature. However, it has been suggested many times [26, 64, 65, 66, 67] that at the semiclassical level we can construct a probability measure.
with the solutions of the Wheeler-DeWitt equation. Hence, for interpretations where probabilities are essential, the problem of finding a Hilbert space for the solutions of the Wheeler-DeWitt equation becomes crucial if someone wants to get some information above the semiclassical level. For instance, at the minisuperspace level, we have obtained a lot of information concerning the isotropization of the Universe and avoidance of singularities in sections III and IV from quantum cosmological wave equations which have not a well defined notion of probability. Of course, probabilities are also useful in the Bohm-de Broglie interpretation. When we integrate the guidance relations (61) and (62), the initial conditions are arbitrary, and it should be nice to have some probability distribution on them. However, as we have seen along this contribution, we can extract a lot of information from the full quantum gravity level using the Bohm-de Broglie interpretation, without appealing to any probabilistic notion. In this interpretation, probabilities are not essential. Hence, we can take the Wheeler-DeWitt equation as it is, without imposing any probabilistic interpretation at the most fundamental level, but still obtaining information using the Bohm-de Broglie interpretation, and then recover probabilities when we reach the semiclassical level.

It would also be important to investigate the Bohm-de Broglie interpretation for other quantum gravitational systems, like black holes. Attempts in this direction have been made, but within spherical symmetry in empty space [68], where we have only a finite number of degrees of freedom. It should be interesting to investigate more general models. These cases are, however, qualitatively different from quantum closed cosmological models. There is no problem in thinking of observers outside an ensemble of black holes. It is quantum mechanics of an open system, with less conceptual problems of interpretation.

The conclusions of this paper are of course limited by many strong assumptions we have tacitly made, as supposing that a continuous three-geometry exists at the quantum level (quantum effects could also destroy it), or the validity of quantization of standard GR, forgetting other developments like string theory. However, even if this approach is not the appropriate one, it is nice to see how far we can go with the Bohm-de Broglie interpretation, even in such incomplete stage of canonical quantum gravity. It seems that the Bohm-de Broglie interpretation may at least be regarded as a good alternative view to be used in quantum cosmology, as it will prove harder, or even impossible, to reach the detailed conclusions of this contribution using other interpretations. Furthermore, if the finer view of the Bohm-de Broglie interpretation of quantum cosmology can yield useful information in the form of observational effects, (as in the possibility raised in Ref. [34] saying that the present acceleration of the Universe may be a quantum cosmological effect obtained from bohmian quantum
dynamics) then we will have means to decide between interpretations, something that will be very important not only for quantum cosmology, but for quantum theory itself.

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