Long-time behavior for a hydrodynamic model on nematic liquid crystal flows with asymptotic stabilizing boundary condition and external force

Maurizio Grasselli * and Hao Wu†

February 22, 2012

Abstract

In this paper, we consider a simplified Ericksen–Leslie model for the nematic liquid crystal flow. The evolution system consists of the Navier–Stokes equations coupled with a convective Ginzburg–Landau type equation for the averaged molecular orientation. We suppose that the Navier–Stokes equations are characterized by a no-slip boundary condition and a time-dependent external force \( g(t) \), while the equation for the molecular director is subject to a time-dependent Dirichlet boundary condition \( h(t) \). We show that, in 2D, each global weak solution converges to a single stationary state when \( h(t) \) and \( g(t) \) converge to a time-independent boundary datum \( h_\infty \) and \( 0 \), respectively. Estimates on the convergence rate are also obtained. In the 3D case, we prove that global weak solutions are eventually strong so that results similar to the 2D case can be proven. We also show the existence of global strong solutions, provided that either the viscosity is large enough or the initial datum is close to a given equilibrium.

Keywords: Nematic liquid crystal flow, non-autonomous Navier–Stokes equations, time-dependent Dirichlet boundary condition, long-time behavior, Lojasiewicz–Simon inequality.

AMS Subject Classification: 35B40, 35Q35, 76A15, 76D05.

1 Introduction

We consider the following hydrodynamical model for the flow of nematic liquid crystals

\[
\begin{align*}
\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} - \nu \Delta \mathbf{v} + \nabla \pi &= -\lambda \nabla \cdot (\nabla \mathbf{d} \otimes \nabla \mathbf{d}) + g(t), \\
\nabla \cdot \mathbf{v} &= 0, \\
\mathbf{d}_t + \mathbf{v} \cdot \nabla \mathbf{d} &= \eta (\Delta \mathbf{d} - f(\mathbf{d})),
\end{align*}
\]

in \( \Omega \times \mathbb{R}^+ \), where \( \Omega \subset \mathbb{R}^n \) \( (n = 2, 3) \) is a bounded domain with sufficiently smooth boundary \( \Gamma \), \( \mathbf{v} = (v_1, \ldots, v_n)^{tr} \) is the velocity field of the flow and \( \mathbf{d} = (d_1, \ldots, d_n)^{tr} \) represents the averaged macroscopic/continuum molecular orientations in \( \mathbb{R}^n \) \( (n = 2, 3) \). \( \pi \) is a scalar function representing the pressure (including both the hydrostatic and the induced elastic part from the orientation field). The external volume force is represented by \( g \). The positive constants \( \nu, \lambda \) and \( \eta \) stand for

*Dipartimento di Matematica, Politecnico di Milano, Milano 20133, Italy, maurizio.grasselli@polimi.it
†School of Mathematical Sciences and Shanghai Key Laboratory for Contemporary Applied Mathematics, Fudan University, Shanghai 200433, China, haowu@fudan.edu.cn
viscosity, the competition between kinetic energy and potential energy, and macroscopic elastic relaxation time (Deborah number) for the molecular orientation field. \( \nabla d \otimes \nabla d \) denotes the \( n \times n \) matrix whose \((i,j)\)-th entry is given by \( \nabla_i d \cdot \nabla_j d \), for \( 1 \leq i, j \leq n \). We assume that \( f(d) = \nabla_d F(d) \) for some smooth bounded function \( F : \mathbb{R}^n \to \mathbb{R} \). In particular, one uses the Ginzburg–Landau approximation \( f(d) = \frac{1}{\epsilon^2}(|d|^2 - 1)d \) to relax the nonlinear constraint \(|d| = 1\) on molecule length (cf. [17,18]).

System (1.1)–(1.3) was firstly proposed in [16] as a simplified approximate system of the original Ericksen–Leslie model for the nematic liquid crystal flows (cf. [5,15]). Well-posedness of the autonomous version of system (1.1)–(1.3) (namely, with \( g = 0 \), no-slip boundary condition for \( v \), and time-independent Dirichlet boundary condition for \( d \)) has been analyzed in [18] (see also [6,12,19] and, for different boundary conditions, [22]). For numerical approximation we refer to [21,23,24]. Problem (1.1)–(1.3) has also been investigated on a Riemannian manifold in [26], where the existence of a global attractor in the 2D case was proven. As far as the long-time behavior of the single trajectory is concerned, in [18], a natural question on the uniqueness of asymptotic limit for global solutions (to the autonomous system) was raised. This question was answered in [31], where it is proven that each trajectory converges to a single steady state (cf. [25] for some generalization). The proof is based on a suitable Lojasiewicz–Simon type inequality (see [27], cf. also [10] and references cited therein).

The technically more challenging case of time-dependent Dirichlet boundary conditions for \( d \) has been recently analyzed in [13,14,18]. For instance, under proper assumptions on the time-dependent boundary condition and assuming that \( g = 0 \), the existence of global weak solution, the existence of global regular solution for viscosity coefficient big enough, and the weak/strong uniqueness were obtained in [13]. Regularity criteria for solutions in the 3D case can be found in [14]. Besides, the presence of a time-dependent external force is allowed in [13] and existence of global and exponential attractors is proven in the 2D case. In this paper, we want to extend the results of [31] to the non-autonomous case treated in [1]. Thus we consider system (1.1)–(1.3) subject to the boundary conditions

\[
\begin{align*}
\mathbf{v}(x,t) &= 0, \quad \mathbf{d}(x,t) = \mathbf{h}(x,t), \quad (x,t) \in \Gamma \times \mathbb{R}^+, \\
|\mathbf{v}|_{t=0} &= \mathbf{v}_0(x) \quad \text{with} \quad \nabla \cdot \mathbf{v}_0 = 0, \quad |\mathbf{d}|_{t=0} = \mathbf{d}_0(x), \quad x \in \Omega.
\end{align*}
\]

In the 2D case, we prove that each weak/strong solution converges to a single stationary state when \( h(t) \) and \( g(t) \) converge to a time-independent boundary datum \( h_\infty \) and \( 0 \), respectively. In the 3D case, we first show the eventual regularity of global weak solutions, and the existence of global strong solutions provided that either the viscosity is large enough or the initial datum is close to a given equilibrium. Then an analogous result on the long-time behavior as in 2D is also obtained. In both cases, we provide an estimate on the convergence rate.

Before ending this section, we state some key ingredients of the present paper. System (1.1)–(1.3) is non-autonomous due to the time-dependent boundary data \( h \) and external force \( g \). This brings some additional difficulties into our subsequent proofs. First, in order to obtain the energy inequalities that play crucial roles in the proof of well-posedness as well as in the long-time behavior of global solutions (cf. Lemmas 2.2, 2.5, 2.6, 5.1), we have to introduce proper lifting functions (cf. (2.7) and (2.21) below). The idea was first used in [31], but the lifting function introduced in this paper is different from the one in [3]. This is due to the
fact that we need some specific energy inequalities which not only yield uniform estimates of
the solutions, but also provide estimates of the convergence rate (cf. Section 4). The second
issue regards the application of the Lojasiewicz–Simon approach (cf. [27]) which has been shown
to be very useful in the study of long-time behavior of global solutions to nonlinear evolution
equations (cf., for instance, [9, 11, 13, 30, 32] and references therein). In particular, convergent
results related to various evolution equations with asymptotically autonomous source terms were
established, e.g., in [27, 10]. However, our current case is more complicated than the previous
cases, because the Lojasiewicz–Simon inequality involves $d$ which is subject to a time-dependent
boundary datum. To overcome this difficulty, we derive an extended Lojasiewicz–Simon type
inequality for vector functions with arbitrary nonhomogeneous Dirichlet boundary data, which
is associated with the lifted energy (cf. Corollary 3.1). This generalizes the results in [10, 31] and
may have its own interest. Third, in the 3D case, we also apply the Lojasiewicz–Simon approach
to prove the existence of global strong solutions provided that the initial datum is close to a local
minimizer of the elastic energy and the non-autonomous terms are properly small perturbations
of their asymptotic limits (cf. Section 5). Then we further discuss the stability of these energy
minimizers. This extends the previous results in [18, 31] for the autonomous system, where the
initial datum was required to be sufficiently close to a global energy minimizer.

The remaining part of the paper is organized as follows. The next section is devoted to
report some existence and uniqueness results and basic a priori estimates for the solution. The
extended Lojasiewicz–Simon inequality we need is discussed in Section 3. In Section 4 we show
the convergence of each global weak/strong solution to a single steady state and provide uniform
estimates on the convergence rate in 2D. Results in 3D are presented in Section 5. In particular,
we study the eventual regularity of global weak solutions as well as the well-posedness when the
initial data are close to local minimizers of the elastic energy. Long-time convergence of global
solutions and stability of such minimizers are also proved. In the final Section 6, some useful
properties of the lifting functions are reported.

2 Preliminaries: well-posedness and a priori estimates

Without loss of generality, from now on we set $\lambda = \eta = 1$. Let us introduce the function spaces
we shall work with. As usual, $L^p(\Omega)$ and $W^{k,p}(\Omega)$ stand for the Lebesgue and the Sobolev spaces
of real valued functions, with the convention that $H^k(\Omega) = W^{k,2}(\Omega)$. The spaces of vector-valued
functions are denoted by bold letters, correspondingly. Without any further specification, $\| \cdot \|$ stands for the norm in $L^2(\Omega)$ or $L^2(\Omega)$. This norm is induced by the scalar inner product
$(u, v) = \int_\Omega u v dx$, where for vector valued functions the product $uv$ is replaced by the Euclidean
inner product $u \cdot v$. We set, as usual,

$$H = \nabla L^2(\Omega), \quad V = \nabla H^1_0(\Omega), \quad \text{where} \quad V = \{ v \in C_0^{\infty}(\Omega, \mathbb{R}^n) : \nabla \cdot v = 0 \}.$$ 

For any Banach space $B$, we denote its dual space by $B^*$. In particular, we denote the dual
space of $H^1_0(\Omega)$ by $H^{-1}(\Omega)$.

In the following text, we will use the regularity result for Stokes problem (see, e.g., [29])

**Lemma 2.1.** For the Stokes operator $S : D(S) = V \cap H^2(\Omega) \to H$ defined by

$$Su = -\nabla u + \nabla \pi \in H, \quad \forall u \in D(S),$$

3
it holds
\[ \|u\|_{H^2} + \|\pi\|_{H^2} \leq C\|Su\|, \quad \forall u \in D(S), \]
for some positive constant \(C\) only depending on \(\Omega\) and \(n\).

We begin to report the existence of a weak solution (see [1, Corollary 1.1]).

**Proposition 2.1.** For any given \(T > 0\), assume
\[
\begin{align*}
g &\in L^2(0,T;V^*), \\
h &\in L^2(0,T;H^2(\Gamma)), \\
h_t &\in L^2(0,T;H^{-\frac{1}{2}}(\Gamma)) \\
|h|_{R^n} \leq 1, &\quad \text{a.e. on } \Gamma \times [0,T], \quad (2.4) \\
d_0|_{\Gamma} = h|_{t=0}. &\quad (2.5)
\end{align*}
\]
Then for any \((v_0,d_0) \in H \times H^1(\Omega)\) with \(|d_0|_{R^n} \leq 1\) almost everywhere in \(\Omega\), problem (1.1)–(1.5) admits a weak solution \((v,d)\) such that
\[
\begin{align*}
v &\in L^\infty(0,T;H) \cap L^2(0,T;V), \\
d &\in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)), \\
|d|_{R^n} \leq 1, &\quad \text{a.e. on } \Omega \times [0,T]. \quad (2.6)
\end{align*}
\]

In the case \(n = 2\), we also have the following results on uniqueness.

**Proposition 2.2.** (cf. [1, Theorem 1.4]) Let \(n = 2\) and let the assumptions of Proposition 2.1 hold. Then the weak solution \((v,d)\) to problem (1.1)–(1.5) given by Proposition 2.1 is unique. Moreover, we have \((v,d) \in C([0,T];H \times H^1(\Omega)).\)

In order to obtain proper energy inequalities, we recall that suitable lifting functions were introduced in [3,4] to overcome the technical difficulties related to the time-dependent boundary datum for \(d\). The first lifting function \(d_E = d_E(x,t)\) is of elliptic type (cf. [4]):
\[
\begin{align*}
-\Delta d_E & = 0, \quad \text{in } \Omega \times \mathbb{R}^+, \\
d_E & = h, \quad \text{on } \Gamma \times \mathbb{R}^+.
\end{align*}
\]
(2.7)

In particular, we define the lifting function \(d_{E0}\) for the initial datum:
\[
\begin{align*}
-\Delta d_{E0} & = 0, \quad \text{in } \Omega, \\
d_{E0} & = d_0, \quad \text{on } \Gamma.
\end{align*}
\]
(2.8)

Set now
\[
\hat{d} = d - d_E.
\]
(2.9)

Then system (1.1)–(1.5) can be rewritten into the following form:
\[
\begin{align*}
v_t + v \cdot \nabla v - \nu \Delta v + \nabla \pi & = -\Delta \hat{d} \cdot \nabla d + g(t), \quad (2.10) \\
\nabla \cdot v & = 0, \quad (2.11) \\
\hat{d}_t + v \cdot \nabla d & = \Delta \hat{d} - f(d) - \partial_t d_E(t), \quad (2.12)
\end{align*}
\]
with homogeneous Dirichlet boundary conditions and initial conditions
\[ v = 0, \quad \hat{d} = 0, \quad \text{on } \Gamma \times \mathbb{R}^+, \quad (2.13) \]
\[ v|_{t=0} = v_0, \quad \hat{d}|_{t=0} = d_0 - d_{EO}, \quad \text{in } \Omega. \quad (2.14) \]

Note that we have used the identity \( \nabla \cdot (\nabla d \odot \nabla d) = \frac{1}{2} \nabla ((|\nabla d|^2) + \Delta d \cdot \nabla d) \) to absorb the gradient term into pressure (cf. [18]).

Let us introduce the lifted energy
\[
\hat{E}(t) = \frac{1}{2} \|v(t)\|^2 + \frac{1}{2} \|\nabla \hat{d}(t)\|^2 + \int_\Omega F(d(t))dx, \quad t \geq 0. \quad (2.15)
\]

Then we can derive the basic energy inequality for system \((1.1)-(1.5)\).

**Lemma 2.2.** Let the assumptions of Proposition [2.7] be satisfied for all \( T > 0 \). Then, any weak solution which is smooth enough satisfies the following inequality for \( t \geq 0 \)
\[
\frac{d}{dt} \hat{E}(t) + \nu \|\nabla v\|^2 + \frac{1}{2} \|\Delta \hat{d} - f(d)\|^2 \leq \frac{1}{2} \|\partial_t d_E\|^2 + C\|\partial_t d_E\| + C\|\mathbf{g}\|_{V^*}^2 := r(t), \quad (2.16)
\]
where \( C \) is a positive constant independent of \( v \) and \( d \).

**Proof.** Multiplying \((2.10)\) and \((2.12)\) by \( v \) and \(-\Delta \hat{d} + f(d)\), respectively, integrating over \( \Omega \) and adding the results together, we get
\[
\frac{d}{dt} \left( \frac{1}{2} \|v\|^2 + \frac{1}{2} \|\nabla \hat{d}\|^2 + \int_\Omega F(d)dx \right) + \nu \|\nabla v\|^2 + \|\Delta \hat{d} - f(d)\|^2 = (\partial_t d_E, \Delta \hat{d}) + (\mathbf{g}, \mathbf{v}). \quad (2.17)
\]

In above, we have used the facts \((v \cdot \nabla v, v) = (\nabla P, v) = (v \cdot \nabla d, f(d)) = 0\) due to the impressibility condition \(\nabla \cdot v = 0\). By the Poincaré inequality \(\|v\| \leq C_P\|\nabla v\|\) and \((2.6)\), the right-hand side of \((2.17)\) can be estimated as follows
\[
\begin{align*}
|(&\partial_t d_E, \Delta \hat{d}) + (\mathbf{g}, \mathbf{v})| \\
\leq & |(&\partial_t d_E, \Delta \hat{d} - f(d))| + |(&\partial_t d_E, f(d))| + |(\mathbf{g}, \mathbf{v})| \\
\leq & \|\Delta \hat{d} - f(d)\|\|\partial_t d_E\| + \|f(d)\|\|\partial_t d_E\| + \|v\|\|\mathbf{g}\|_{V^*} \\
\leq & \frac{\nu}{2} \|\nabla v\|^2 + \frac{1}{2} \|\Delta \hat{d} - f(d)\|^2 + \frac{1}{2} \|\partial_t d_E\|^2 + C\|\partial_t d_E\| + C\|\mathbf{g}\|_{V^*}^2.
\end{align*}
\]

The proof is complete. \(\square\)

**Remark 2.1.** We fix the calculations in [4] Lemma 2 in which the term \((\partial_t d_E, \Delta \hat{d})\) is missing. Though it does not affect the proof of existence, it does have influence on the long-time behavior of global solutions (especially the convergence rate).

Let us now introduce the following (Banach) spaces of translation bounded functions
\[
L^q_{tb}(0, +\infty; X) := \left\{ h \in L^q_{loc}([0, +\infty); X) : \|h\|^q_{L^q_{tb}(0, +\infty; X)} := \sup_{t \geq 0} \int_0^{t+1} \|h(\tau)(\cdot)\|_X^q d\tau < +\infty \right\},
\]
where \( X \) is a (real) Banach space and \( q \in [1, +\infty) \) is given.

From the basic energy inequality \((2.16)\), through a suitable Galerkin approximation scheme, one can derive uniform-in-time estimates for any weak solution (the proof is a minor modification of [1] Lemma 1.2, Remark 1.1]).
Lemma 2.3. Let the assumptions of Proposition 2.1 hold for all $T > 0$. In addition, suppose that
\begin{align}
g & \in L^2(0, +\infty; V^*), \\
h & \in L^2_0(0, +\infty; H^\frac{1}{2}(\Gamma)), \\
h_t & \in L^2(0, +\infty; H^{-\frac{1}{2}}(\Gamma)) \cap L^1(0, +\infty; H^{-\frac{1}{2}}(\Gamma)).
\end{align}
Then a weak solution $(v, d)$ to problem (1.1)–(1.5) given by Proposition 2.1 is a global solution on $[0, +\infty)$ and fulfills the following uniform bounds
\begin{align}
\|v(t)\| & \leq C, \quad \|d(t)\|_{H^1} \leq C, \quad \forall \ t \geq 0, \\
\int_0^t (\nu \|\nabla v(\tau)\|^2 + \|\Delta d - f(d)(\tau)\|^2) d\tau & \leq C, \quad \forall \ t \geq 0.
\end{align}
Here $C$ is a positive constant depending on $\|v_0\|$, $\|d_0\|_{H^1}$, $\|g\|_{L^2(0, +\infty; V^*)}$, $\|h\|_{L^2_0(0, +\infty; H^\frac{1}{2}(\Gamma))}$, $\|h_t\|_{L^2(0, +\infty; H^{-\frac{1}{2}}(\Gamma))}$ and $\|h_t\|_{L^1(0, +\infty; H^{-\frac{1}{2}}(\Gamma))}$.

Next, we introduce the lifting function $d_P = d_P(x, t)$ of parabolic type, which satisfies
\begin{equation}
\begin{cases}
\partial_t d_P - \Delta d_P = 0, & \text{in } \Omega \times \mathbb{R}^+, \\
d_P = h, & \text{on } \Gamma \times \mathbb{R}^+, \\
d_P(0) = d_{E0}, & \text{in } \Omega.
\end{cases}
\end{equation}

The motivation of introducing the parabolic lifting function $d_P$ is that we now have, by definition, $\Delta(d - d_P) - f(d)|_\Gamma = 0$. This fact is crucial when we use integration by parts to derive some higher-order differential inequalities of system (1.1)–(1.5) (cf. [18]). We note that $d_P$ in (2.21) is different from the one introduced in [11] as they have different initial values. Both choices are valid for the proof of existence result, but the current definition of $d_P$ is necessary for the study of long-time behavior. Denote
\begin{equation}
\tilde{d} = d - d_P.
\end{equation}
System (1.1)–(1.5) can now be rewritten into the following form:
\begin{align}
v_t + v \cdot \nabla v - \nu \Delta v + \nabla P & = -\Delta d \cdot \nabla d + g(t), \\
\nabla \cdot v & = 0, \\
\tilde{d}_t + v \cdot \nabla \tilde{d} & = \Delta \tilde{d} - f(d),
\end{align}
with homogeneous Dirichlet boundary conditions and initial conditions
\begin{align}
v & = 0, \quad \tilde{d} = 0, & \text{on } \Gamma \times \mathbb{R}^+, \\
v|_{t=0} = v_0, \quad \tilde{d}|_{t=0} = d_0 - d_{E0}, & \text{in } \Omega.
\end{align}
In the sequel, we shall frequently use the following lemma (cf. [11])

Lemma 2.4. The following equivalence between norms hold
\begin{align}
\|v\|_{H^1} & \approx \|\nabla v\|, \quad \|\tilde{d}\|_{H^1} \approx \|\nabla \tilde{d}\|, \quad \text{in } H^1_0(\Omega), \\
\|v\|_{H^2} & \approx \|\Delta v\|, \quad \|\tilde{d}\|_{H^2} \approx \|\Delta \tilde{d}\|, \quad \text{in } H^1_0(\Omega) \cap H^2(\Omega),
\end{align}
\[ \|\tilde{d}\|_{H^1} \approx \|\nabla(\Delta \tilde{d})\| + \|\Delta \tilde{d}\|, \text{ in } H^1_0(\Omega) \cap H^2(\Omega). \]

If \(d\) and \(d_P\) are functions which are smooth enough and \(|d|_{\mathbb{R}^n} \leq 1, |d_P|_{\mathbb{R}^n} \leq 1\), then we have

\[
\begin{align*}
\|\Delta d\| & \leq \|\Delta d_P\| + \|\Delta \tilde{d} - f(d)\| + C, \\
\|\nabla \Delta d\| & \leq \|\nabla \Delta d_P\| + \|\nabla(\Delta \tilde{d} - f(d))\| + C\|\nabla d\|.
\end{align*}
\]

where \(C\) is a positive constant independent of \(d\) and \(d_P\).

Let us introduce the quantity

\[ A_P(t) = \|\nabla v(t)\|^2 + \|\Delta \tilde{d}(t) - f(d(t))\|^2, \quad t \geq 0. \]

**Lemma 2.5.** Let \(n = 2\) and let the assumptions of Lemma 2.3 hold. If the weak solution \((v, d)\) is smooth enough then it satisfies the inequality

\[ \frac{d}{dt}A_P(t) \leq C(A_P^2(t) + A_P(t) + R_1(t)), \quad \tag{2.27} \]

where

\[ R_1(t) = \|\partial_1 d_P(t)\|^4 + \|\partial_2 d_P(t)\|^2 + \|\nabla \Delta d_P(t)\|^2 + \|g(t)\|^2. \quad \tag{2.28} \]

Here \(C\) is a positive constant depending on \(\nu, \|v_0\|, \|d_0\|_{H^1}, \|g\|_{L^2(0, +\infty; \mathbb{R}^n)}, \|h_1\|_{L^2(0, +\infty; \mathbb{H}^{1/2}(\Gamma))}, \|h_r\|_{L^2(0, +\infty; \mathbb{H}^{-1/2}(\Gamma))}\) and \(\|h\|_{L^2(0, +\infty; \mathbb{H}^{-1}(\Gamma))}\).

**Proof.** Taking the time derivative of \(A_P(t)\), we obtain by a direct calculation that

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt}A_P(t) + \nu \|sv\|^2 + \|\nabla(\Delta \tilde{d} - f(d))\|^2
& = -(sv, \nabla v) + (sv, g) - (sv, \Delta d \cdot \nabla d) - (\nabla (v \cdot \nabla d), \nabla (\Delta \tilde{d} - f(d))) \\
& \quad - (f'(d)d, \Delta \tilde{d} - f(d)) \\
& : = \sum_{j=1}^5 I_j. \quad \tag{2.29}
\end{align*}
\]

To get this identity we have used the fact that \(\Delta \tilde{d} - f(d)\rvert_\Gamma = 0\) as well as \((sv, v_i) = (\Delta v, v_i)\).

It is not difficult to see that

\[
\begin{align*}
|I_1| & \leq \|sv\|\|v\|_{L^4}\|\nabla v\|_{L^4} \leq C\|sv\|((\nabla v)^{1/2}\|v\|^{1/2})(\|\Delta v\|^2\|\nabla v\|^{1/2}) \\
& \leq C\|sv\|^2\|\nabla v\| \leq \epsilon\|sv\|^2 + C\|\nabla v\|^2, \\
|I_2| & \leq \epsilon\|sv\|^2 + C\|g\|^2.
\end{align*}
\]

For \(I_3\), we have

\[
|I_3| = |(sv, (\Delta \tilde{d} - f(d)) \cdot \nabla d) + (sv, f(d) \cdot \nabla d) + (sv, \partial_1 d_P \cdot \nabla d)| \\
\leq \|sv\|\|\nabla d\|_{L^4}\|\Delta \tilde{d} - f(d)\|_{L^4} + \|sv\|\|\nabla d\|_{L^\infty}\|\partial_1 d_P\| \\
\leq \epsilon\|sv\|^2 + C\|\nabla d\|_{L^4}^2\|\Delta \tilde{d} - f(d)\|_{L^4}^2 + C\|\nabla d\|_{L^\infty}^2\|\partial_1 d_P\|^2.
\]

On account of Lemma 2.3, we infer from the Sobolev embedding theorem that

\[
\|\nabla d\|_{L^4}^2 \leq C\|\Delta d\|\|\nabla d\|^2 + C\|\nabla d\|^2 \leq C\|\Delta \tilde{d}\| + C\|\partial_1 d_P\| + C
\]

7
Recalling (2.24), we have
\[ \|\nabla d\|^2 \lesssim \|\nabla \Delta \tilde{d} - f(d)\| + \|\Delta \tilde{d} - f(d)\| + C, \]
\[ \|\nabla d\|_{L^\infty}^2 \lesssim C\|\nabla \Delta \tilde{d}\|\|\nabla d\| + C\|\nabla d\|^2 \]
\[ \lesssim C\|\nabla \Delta d_P\| + C(1 + \|\nabla (\Delta \tilde{d} - f(d))\|), \]
\[ \|\Delta \tilde{d} - f(d)\|_{L^4}^2 \lesssim C\|\nabla (\Delta \tilde{d} - f(d))\|\|\Delta \tilde{d} - f(d)\|. \]

Using the above estimates, we obtain the estimates for $I_3$ and $I_4$:
\[ |I_3| \leq \varepsilon\|S v\|^2 + C\|\nabla (\Delta \tilde{d} - f(d))\|\|\Delta \tilde{d} - f(d)\| + \|\partial_t d_P\| + 1 + C\|\partial_t d_P\|^2 + \|\nabla \Delta d_P\| + 1, \]
\[ |I_4| \leq \varepsilon\|\nabla (\Delta \tilde{d} - f(d))\|^2 + C\|\nabla v\|^2 \|\nabla d\|_{L^4} \|\nabla d\|_{L^4} + \|v\|_{L^\infty} \|d\|_{H^2} \]
\[ \leq \varepsilon\|\nabla (\Delta \tilde{d} - f(d))\|^2 + C\|\nabla v\|^2 \|\nabla d\|_{L^4}^2 + C\|v\|^2 \|\Delta d\|^2 + 1, \]
\[ \leq \varepsilon\|\nabla (\Delta \tilde{d} - f(d))\|^2 + C\|\nabla v\|^2 \|\nabla (\Delta \tilde{d} - f(d))\| + \|\partial_t d_P\| + 1 + C\|\nabla d\|^2 + \|\partial_t d_P\|^2 + 1, \]
\[ \leq \varepsilon\|S v\|^2 + \varepsilon\|\nabla (\Delta \tilde{d} - f(d))\|^2 + C\|\nabla v\|^4 + C\|\Delta \tilde{d} - f(d))\|^2 \]
\[ + C\|\nabla v\|^2 + C\|\Delta \tilde{d} - f(d))\|^2 + \|\partial_t d_P\|^2. \]

We now observe that
\[ I_5 = -(f'(d)\partial_t \tilde{d}, \Delta \tilde{d} - f(d)) = (f'(d)\partial_t d_P, \Delta \tilde{d} - f(d)) := I_{5a} + I_{5b}. \]

Recalling (2.24), we have
\[ |I_{5a}| = |(f'(d)(v \cdot \nabla) d, \Delta \tilde{d} - f(d)) - (f'(d)(\Delta \tilde{d} - f(d)), \Delta \tilde{d} - f(d))| \]
\[ \leq \|f'(d)\|_{L^\infty} \|v\|_{L^4} \|\nabla d\|_{L^4} \|\Delta \tilde{d} - f(d)\| + \|\Delta \tilde{d} - f(d)\|^2 \]
\[ \leq C\|\nabla v\|^2 \|\nabla d\|_{L^4}^2 + C\|\Delta \tilde{d} - f(d)\|^2 \]
\[ \leq C\|\nabla v\|^2 + C\|\Delta \tilde{d} - f(d)\|^2 + C\|\partial_t d_P\|^2, \]
\[ |I_{5b}| \leq \|f'(d)\|_{L^\infty} \|\partial_t d_P\| \|\Delta \tilde{d} - f(d)\| \leq C\|\Delta \tilde{d} - f(d)\|^2 + C\|\partial_t d_P\|^2. \]

Finally, collecting the above estimates and taking $\varepsilon$ sufficiently small, we deduce that
\[ \frac{d}{dt} A_P(t) + (\nu\|S v\|^2 + \|\nabla (\Delta \tilde{d} - f(d))\|^2) \]
\[ \leq C(A_P^2(t) + A_P(t) + C\|\partial_t d_P\|^2 \|\partial_t d_P\|^2 + \|\nabla d_P\| + 1) + \|g\|^2, \]
which easily implies the inequality (2.27).

Taking advantage of Lemmas 2.5, 6.1 and 6.2, one can deduce the following results on the regularity of weak solutions as well as the existence of strong solutions to system (1.1)–(1.3) in 2D.

**Theorem 2.1.** Let $n = 2$ and let the assumptions of Proposition 2.1 hold for all $T > 0$. In addition, suppose that
\[ g \in L^2(0, +\infty; H), \]
\[ g \in L^2(0, +\infty; H), \]
Lemma 2.6.\] where \( C \) is a positive constant depending on \( \nu \), \( \|v_0\|, \|d_0\|_{H^1}, \|g\|_{L^2(0, \infty; H)}, \|h\|_{L^2_{th}(0, \infty; H^{\frac{5}{2}}(\Gamma))}, \|h_t\|_{L^2(0, \infty; H^{\frac{7}{2}}(\Gamma))}, \|h_t\|_{L^1(0, \infty; H^{-\frac{1}{2}}(\Gamma))}\).

(ii) If \((v_0, d_0) \in V \times H^\frac{7}{2}(\Omega)\), then problem \((1.1) - (1.5)\) admits a unique global strong solution \((v, d)\) satisfying
\[
\|v(t)\|_V \leq C, \quad \|d(t)\|_{H^2} \leq C, \quad \forall t \geq 0, \quad \int_0^t (\|v(\tau)\|^2_{H^2} + \|d(\tau)\|^2_{H^1}) d\tau \leq CT, \quad t \in [0, T],
\]
where \( C \) is a positive constant depending on \( \nu \), \( \|v_0\|, \|d_0\|_{H^2}, \|g\|_{L^2(0, \infty; H)}, \|h\|_{L^2_{th}(0, \infty; H^{\frac{5}{2}}(\Gamma))}, \|h_t\|_{L^2(0, \infty; H^{\frac{7}{2}}(\Gamma))}, \|h_t\|_{L^1(0, \infty; H^{-\frac{1}{2}}(\Gamma))}\).

Remark 2.2. Lemma 2.3 and Theorem 2.1 still hold when \( g \) and \( h_t \) are translation bounded with respect to time (see \[8\]).

Next, we consider the 3D case. Instead of Lemma 2.5, we have the following higher-order energy inequality

**Lemma 2.6.** Let \( n = 3 \) and let the assumptions of Lemma 2.3 hold. If a weak solution \((v, d)\) is smooth enough then it satisfies the following inequality
\[
\frac{d}{dt} \tilde{A}_P(t) + \left( \nu - c_1 \tilde{A}_P(t) \right) \|Sv\|^2 + \left( 1 - \frac{c_2}{\nu} \tilde{A}_P(t) \right) \|\nabla(\Delta \tilde{d} - f(d))\|^2 \leq C(1 + \nu^{-2}) (A_P(t) + R_2(t)), \quad t \geq 0,
\]
where \( \tilde{A}_P(t) = A_P(t) + 1 \) and
\[
R_2(t) = \|\partial_t d_P(t)\|^2 + \|\partial_t d_P(t)\|^6 + \|\nabla \partial_t d_P(t)\|^2 + \|g(t)\|^2.
\]
Here \( c_1, c_2 \) are positive constants that may depend on \( \|v_0\|, \|d_0\|_{H^1} \) and on \( \|g\|_{L^2(0, \infty; V^\ast)} \), \( \|h\|_{L^2_{th}(0, \infty; H^{\frac{2}{3}}(\Gamma))}, \|h_t\|_{L^2(0, \infty; H^{-\frac{1}{3}}(\Gamma))}, \|h_t\|_{L^1(0, \infty; H^{-\frac{1}{2}}(\Gamma))} \), but they are independent of \( \nu \).

**Proof.** We estimate the right-hand side of (2.29) by using the 3D version of Sobolev embedding theorems. We have
\[
|I_1| \leq \|Sv\| \|\nabla v\|_{L^2(\Omega)} \|\Delta v\|_{L^2(\Omega)} \leq C \|Sv\| \|\nabla v\| (\|\Delta v\|_{H^{\frac{1}{2}}} \|\nabla v\|_{H^{\frac{1}{2}}}) \leq C \|Sv\|_{H^{\frac{5}{2}}} \|\nabla v\|_{H^{\frac{5}{2}}} \leq \frac{1}{2} \|\nabla v\|_{H^{\frac{5}{2}}} \|Sv\|^2 + C \|\nabla v\|^2,
\]
\[
|I_2| \leq \frac{\nu}{8} \|Sv\|^2 + \frac{2}{\nu} \|g\|^2.
\]
Recalling that \(\|d\|_{H^1} \leq C\) (cf. Lemma 2.34), from the Sobolev embedding theorems as well as Agmon’s inequality in dimension three, we infer

\[
\begin{align*}
\|\nabla d\|_{L^3} & \leq C\|\Delta d\|^{\frac{1}{2}}\|\nabla d\|^{\frac{1}{2}} + C\|\nabla d\| \leq C(\|\Delta \tilde{d} - f(d)\| + \|\partial_t d_P\|)^{\frac{1}{2}} + C, \\
\|\nabla d\|_{L^6} & \leq C\|\Delta d\| + C\|\nabla d\| \leq C\|\Delta \tilde{d} - f(d)\| + C\|\partial_t d_P\| + C, \\
\|\nabla d\|_{L^\infty} & \leq C\|\nabla d\|_{H^1} \leq C(\|\nabla \Delta d\|^{\frac{1}{2}}\|\Delta d\|^{\frac{1}{2}} + \|\Delta d\| + 1) \\
& \leq C(\|\nabla (\Delta \tilde{d} - f(d))\|^{\frac{1}{2}}\|\Delta \tilde{d} - f(d)\|^{\frac{1}{2}} + \|\Delta \tilde{d} - f(d)\|^{\frac{1}{2}}\|\nabla \Delta d_P\|^{\frac{1}{2}} \\
& + \|\Delta \tilde{d} - f(d)\|^{\frac{1}{2}}\|\partial_t d_P\|^{\frac{1}{2}} + \|\nabla \nabla d_P\|^{\frac{1}{2}}\|\Delta d_P\|^{\frac{1}{2}} + \|\nabla (\Delta \tilde{d} - f(d))\|^{\frac{1}{2}} \\
& + \|\nabla \Delta d_P\|^{\frac{1}{2}} + \|\Delta \tilde{d} - f(d)\| + \|\Delta d_P\| + 1), \\
\|\Delta \tilde{d} - f(d)\|_{L^3} & \leq C\|\nabla (\Delta \tilde{d} - f(d))\|^{\frac{1}{2}}\|\Delta \tilde{d} - f(d)\|^{\frac{1}{2}}.
\end{align*}
\]

Thus we have

\[
\begin{align*}
|I_3| & \leq \|S\|\|\nabla d\|_{L^6}\|\Delta \tilde{d} - f(d)\|_{L^3} + \|S\|\|\nabla d\|_{L^\infty}\|\partial_t d_P\| \\
& \leq C\|S\|\|\Delta \tilde{d} - f(d)\| + \|\partial_t d_P\| + 1)(\|\nabla (\Delta \tilde{d} - f(d))\|^{\frac{1}{2}}\|\Delta \tilde{d} - f(d)\|^{\frac{1}{2}} \\
& + C\|S\|\|\nabla d_P\|\|\nabla \Delta d_P\|^{\frac{1}{2}}\|\nabla \partial_t d_P\| + \|\Delta \tilde{d} - f(d)\|^{\frac{1}{2}}\|\nabla \partial_t d_P\|^{\frac{1}{2}} \\
& + \|\nabla \partial_t d_P\|^{\frac{1}{2}} + \|\Delta \tilde{d} - f(d)\| + \|\partial_t d_P\| + 1) \\
& \leq \left(\frac{\nu}{8} + \frac{1}{2}\|\Delta \tilde{d} - f(d)\|^2\right)\|S\|^2 + \frac{1}{8}\|\nabla (\Delta \tilde{d} - f(d))\|^2 \\
& + C(1 + \nu^{-2})(\|\Delta \tilde{d} - f(d)\|^2 + \|\partial_t d_P\|^6 + \|\partial_t d_P\|^2 + \|\nabla \Delta d_P\|^2),
\end{align*}
\]

\[
\begin{align*}
|I_4| & \leq \|\nabla (\Delta \tilde{d} - f(d))\|\|\nabla d\|_{L^6}\|\nabla d\|_{L^6} + \|\nabla d\|_{L^\infty}\|d\|_{H^2} \\
& \leq C\|\nabla (\Delta \tilde{d} - f(d))\|\|\nabla d\|^{\frac{1}{2}}\|\Delta \tilde{d} - f(d)\| + \|\partial_t d_P\| + 1) \\
& \leq \left(\frac{\nu}{8} + \frac{1}{2}\|\nabla v\|^2\right)\|S\|^2 + \left(\frac{1}{8} + \frac{1}{2\nu^2}\|\Delta \tilde{d} - f(d)\|^2\right)\|\nabla (\Delta \tilde{d} - f(d))\|^2 \\
& + C\|\Delta \tilde{d} - f(d)\|^2 + C(1 + \nu^{-1})\|\nabla v\|^2 + C\|\partial_t d_P\|^4,
\end{align*}
\]

\[
\begin{align*}
|I_{5a}| & \leq \|f'(d)\|_{L^\infty}\|\nabla d\|_{L^6}\|\Delta \tilde{d} - f(d)\| + \|\Delta \tilde{d} - f(d)\| \\
& \leq C\|\nabla v\|^2\|\nabla d\|^{\frac{1}{2}} + C\|\Delta \tilde{d} - f(d)\|^2 \\
& \leq C\|\Delta v\|\|\nabla \Delta \tilde{d} - f(d)\| + \|\partial_t d_P\| + 1) + C\|\Delta \tilde{d} - f(d)\|^2 \\
& \leq \frac{\nu}{8}\|\nabla v\|^2 + C(1 + \nu^{-1})\|\nabla v\|^2 + \|\Delta \tilde{d} - f(d)\|^2 + \|\partial_t d_P\|^2).
\end{align*}
\]

We observe that \(I_{5b}\) can be estimated as in 2.31. Then, collecting all the estimates of \(I_j\), we have

\[
\begin{align*}
\frac{d}{dt} A_P(t) + \left(\nu - \|\nabla v\|^2 \|\nabla v\|^2\right)\|S\|^2 \\
+ \left(1 - \frac{1}{\nu^2}\|\Delta \tilde{d} - f(d)\|^2\right)\|\nabla (\Delta \tilde{d} - f(d))\|^2 \\
\leq C(1 + \nu^{-2})(A_P(t) + \|\partial_t d_P\|^2 + \|\partial_t d_P\|^6 + \|\nabla \Delta d_P\|^2 + \|\nabla \Delta d_P\|^2).
\end{align*}
\]
As a result, there exist constants $c_1, c_2 > 0$ independent of $\nu$ such that the following inequality holds

\[
\frac{d}{dt} A_P(t) + (\nu - c_1 \tilde{A}_P(t)) \| Sv \|^2 + \left(1 - \frac{c_2}{\nu^2} \tilde{A}_P(t) \right) \| \nabla (\Delta \tilde{d} - f(d)) \|^2 \\
\leq C (1 + \nu^{-2}) (A_P(t) + \| \partial_t d \|^2 + \| \partial_t d_P \|^2 + \| \nabla \Delta d_P \|^2 + \| g \|^2),
\]

which implies (2.37). \hfill \Box

On account of Lemma 2.6, one can deduce that system (2.32)–(2.34) and (2.31) are satisfied. For any $(v_0, d_0) \in V \times H^2(\Omega)$ satisfying (2.29) and $|d_0|_{\mathbb{R}^3} \leq 1$, there exists a $\nu_0 > 0$, depending on $\| (v_0, d_0) \|_{V \times H^2}$ and $\| g \|_{L^2(0, +\infty; H^l)}, \| h_0 \|_{L^2(0, +\infty; H^h_2(\Gamma))}, \| h_1 \|_{L^2(0, +\infty; H^h(\Gamma))}$, such that, for any $\nu \geq \nu_0$, problem (2.32)–(2.34) admits a global strong solution $(v, d)$ which satisfies the same uniform estimates as in the 2D case (cf. (2.35) and (2.36)).

Remark 2.3. When $n = 3$, the weak-strong uniqueness result obtained in [4, Theorem 7] still holds in our case. Thus, the global strong solution $(v, d)$ obtained in Theorem 2.2 is unique.

### 3 Extended Łojasiewicz–Simon type inequality

For all $d \in \mathcal{N} := \{ \phi \in H^1(\Omega) : \phi|_\Gamma = h_\infty \}$, where $h_\infty \in H^\frac{1}{2}(\Gamma)$ is given, we consider the functional

\[
E(d) = \frac{1}{2} \| \nabla d \|^2 + \int_\Omega F(d) dx.
\]

It is straightforward to verify that

Lemma 3.1. If $\psi \in H^1(\Omega)$ is a weak solution to the elliptic problem

\[
\begin{aligned}
-\Delta \phi + f(\phi) &= 0, \\
\phi|_\Gamma &= h_\infty,
\end{aligned}
\]

then $\psi$ is a critical point of the functional $E(d)$ in $\mathcal{N}$. Conversely, if $\psi$ is a critical point of the functional $E(d)$ in $\mathcal{N}$, then $\psi$ is a weak solution to problem (3.2).

Remark 3.1. If $h_\infty$ is more regular, then $\psi$ is more regular. For instance, if $h_\infty \in H^2(\Gamma)$, then $\psi \in H^2(\Omega)$.

Lemma 3.2. Suppose that $\psi$ is a critical point of $E(d)$ in $\mathcal{N}$. Then there exist constants $\beta_1 > 0$, $\theta \in (0, \frac{1}{2})$ depending on $\psi$ such that, for any $w \in \mathcal{N}$ that satisfies $\| w - \psi \|_{H^1} < \beta_1$, there holds

\[
\| - \Delta w + f(w) \|_{H^{-1}} \geq |E(w) - E(\psi)|^{1-\theta}.
\]

Remark 3.2. The above lemma can be viewed as an extended version of Simon’s result [27] for scalar function under $L^2$-norm. We can refer to [10, Chapter 2, Theorem 5.2], in which the vector case subject to homogeneous Dirichlet boundary condition was considered. We observe that the result can be easily proved by modifying the argument in [10] using a simple transformation (cf. also [31, Remark 2.1]).
The Lojasiewicz–Simon type inequality \[\text{(3.3)}\] only applies to proper perturbations of the critical point of energy \(E\) in the set \(\mathcal{N}\) and it is not enough for our evolutionary problem \(\text{(1.1)}–\text{(1.5)}\), whose boundary datum is time-dependent (not necessary in \(\mathcal{N}\)). In order to overcome this difficulty, we prove the following extended result that also involves the perturbation of boundary:

**Theorem 3.1.** Suppose that \(\psi\) is a critical point of \(E(d)\) in \(\mathcal{N}\). Then there exists a constant \(\beta \in (0,1)\) depending on \(\psi\) such that, for any \(d \in H^1(\Omega)\) satisfying \(\|d - \psi\|_{H^1} < \beta\), there holds

\[
C \left(\|d\|_{H^\frac{1}{2}(\Gamma)} - h_\infty \|h\|_{H^\frac{1}{2}(\Gamma)} \right) + \| - \Delta d + f(d)\|_{H^{-1}} \geq |E(d) - E(\psi)|^{1-\theta}, \tag{3.4}
\]

where \(\theta \in (0, \frac{1}{2})\) is the same constant as in Lemma \(\text{3.2}\), while \(C\) is a positive constant depending on \(\psi\).

**Proof.** For any \(d \in H^1(\Omega)\), we have that \(\Delta d \in H^{-1}(\Omega)\). Then we consider the elliptic boundary value problem

\[
\begin{align*}
\Delta w &= \Delta d, \\
\mid w\mid = h_\infty.
\end{align*}
\tag{3.5}
\]

It easily follows from the elliptic regularity theory (cf. e.g., \[28\] Proposition 5.1.7) that

\[
\|w - d\|_{H^1} \leq C\|d\|_{H^\frac{1}{2}(\Gamma)} - h_\infty \|h\|_{H^\frac{1}{2}(\Gamma)},
\tag{3.6}
\]

which implies

\[
\|w - \psi\|_{H^1} \leq \|w - d\|_{H^1} + \|d - \psi\|_{H^1} \leq C\|d\|_{H^\frac{1}{2}(\Gamma)} - h_\infty \|h\|_{H^\frac{1}{2}(\Gamma)} + \|d - \psi\|_{H^1}
\leq C\|d - \psi\|_{H^1}.
\tag{3.7}
\]

Let \(\beta_1\) be the constant in Lemma \(\text{3.2}\). We infer from the above inequality that if \(\beta \in (0,1)\) is chosen sufficiently small, then we have \(\|w - \psi\|_{H^1} < \beta_1\). As a consequence of Lemma \(\text{3.2}\), we have

\[
\| - \Delta w + f(w)\|_{H^{-1}} \geq |E(w) - E(\psi)|^{1-\theta}. \tag{3.8}
\]

On the other hand, by the definition of \(w\), we can see that

\[
|E(w) - E(\psi)|^{1-\theta} \leq \| - \Delta w + f(w)\|_{H^{-1}} \leq \| - \Delta d + f(d)\|_{H^{-1}} + C\|f(d) - f(w)\|_{L^2(\Omega)}
\leq \| - \Delta d + f(d)\|_{H^{-1}} + C\|d - w\|_{H^1}
\leq \| - \Delta d + f(d)\|_{H^{-1}} + C\|d\|_{H^\frac{1}{2}(\Gamma)} - h_\infty \|h\|_{H^\frac{1}{2}(\Gamma)}. \tag{3.9}
\]

We deduce from \(\theta \in (0, \frac{1}{2})\) that

\[
|E(d) - E(\psi)|^{1-\theta} \leq |E(w) - E(\psi)|^{1-\theta} + |E(d) - E(w)|^{1-\theta}, \tag{3.10}
\]

and

\[
|E(d) - E(w)|^{1-\theta} \leq \left(\frac{1}{2}\right)^{1-\theta} \left|\|d\|_{H^1}^2 - \|w\|_{H^1}^2\right|^{1-\theta} + \left|\int \Omega (F'(d) - F'(w))dx\right|^{1-\theta}
\leq C(\|d\|_{H^1}, \|w\|_{H^1})\|d - w\|_{H^1}^{1-\theta} \leq C\|d\|_{H^\frac{1}{2}(\Gamma)} - h_\infty \|h\|_{H^\frac{1}{2}(\Gamma)}, \tag{3.11}
\]

where in \(\text{3.11}\) we use the facts that \(\|d\|_{H^1} \leq \|\psi\|_{H^1} + \beta\) and \(\|w\|_{H^1} \leq \|\psi\|_{H^1} + \beta_1\).

Hence, combining \(\text{3.9}–\text{3.11}\), we deduce \(\text{3.4}\). □
Since the basic energy inequality (2.16) (cf. Lemma 2.2) is only valid for the lifted energy \( \hat{E} \) (2.15), in order to apply the Łojasiewicz–Simon approach to our problem, we need to consider the following auxiliary functional corresponding to energy \( E \) (cf. (3.11)):

\[
\hat{E}(d) = \frac{1}{2} \| \nabla \hat{d} \|^2 + \int_{\Omega} F(d) dx, \quad \forall \ d \in H^1(\Omega),
\]

where

\[
\hat{d} = d - d_E,
\]

and \( d_E \) is the elliptic lifting function satisfying the following elliptic problem (cf. (2.7))

\[
\begin{aligned}
-\Delta d_E &= 0, \quad x \in \Omega, \\
\psi_E &= d|_{\Gamma}, \quad x \in \Gamma.
\end{aligned}
\]

Then we have

**Corollary 3.1.** Suppose that \( \psi \) is a critical point of \( E(d) \) in \( N \). Then there exist constants \( \beta \in (0, 1) \) and \( \theta \in (0, \frac{1}{2}) \) depending on \( \psi \) such that, for any \( d \in H^1(\Omega) \) satisfying \( \| d - \psi \|_{H^1} < \beta \), there holds

\[
C \| d|_{\Gamma} - h_{\infty} \|_{H^2(\Gamma)}^{1-\theta} + \| -\Delta \hat{d} + f(d) \|_{H^{-1}} \geq | \hat{E}(d) - \hat{E}(\psi) |^{1-\theta},
\]

where \( C \) is a positive constant depending on \( \psi \) and \( h_{\infty} \).

**Proof.** From the definition of \( \hat{E}(d) \), we set, for \( \psi \in N \),

\[
\hat{E}(\psi) = \frac{1}{2} \| \nabla \hat{\psi} \|^2 + \int_{\Omega} F(\psi) dx,
\]

where \( \hat{\psi} = \psi - \psi_E \) and \( \psi_E \) satisfies

\[
\begin{aligned}
-\Delta \psi_E &= 0, \quad x \in \Omega, \\
\psi_E &= h_{\infty}, \quad x \in \Gamma.
\end{aligned}
\]

A direct calculation yields that

\[
\hat{E}(d) = E(d) + \frac{1}{2} \| \nabla d_E \|^2 - \int_{\Omega} \nabla d : \nabla d_E dx,
\]

\[
\hat{E}(\psi) = E(\psi) + \frac{1}{2} \| \nabla \psi_E \|^2 - \int_{\Omega} \nabla \psi : \nabla \psi_E dx,
\]

where we used the notation \( A : B = \sum_{i,j=1}^n A_{ij} B_{ij} \). Theorem 3.1 implies that there exist constants \( \beta \in (0, 1) \) and \( \theta \in (0, \frac{1}{2}) \), such that for any \( d \in H^1(\Omega) \) satisfying \( \| d - \psi \|_{H^1} < \beta \), 3.1 holds. Next, we proceed to estimate the quantity \( | \hat{E}(d) - \hat{E}(\psi) |^{1-\theta} \)

\[
\begin{align*}
| \hat{E}(d) - \hat{E}(\psi) |^{1-\theta} \\
&\leq | E(d) - E(\psi) |^{1-\theta} + \left( \frac{1}{2} \right)^{1-\theta} \left| \int_{\Omega} (\nabla (d - \psi_E) : \nabla (d - \psi_E) dx \right|^{1-\theta} \\
&\quad + \left| \int_{\Omega} (\nabla d : \nabla d_E - \nabla \psi : \nabla \psi_E) dx \right|^{1-\theta} \\
&:= J_1 + J_2 + J_3.
\end{align*}
\]
The estimate for $J_1$ follows from (3.4). Since $\|d - \psi\|_{H^1} < \beta < 1$, then $\|d\|_{H^1} \leq \|\psi\|_{H^1} + 1$. For $J_2$, we infer from the elliptic estimate (cf. [28 Proposition 5.1.7]) that

\[
J_2 \leq C\|\nabla (d_E - \psi_E)\|^{1-\theta}\|\nabla (d_E + \psi_E)\|^{1-\theta}
\leq C\|d\|_\Gamma - h_\infty\|^{1-\theta}\|H^{1/2}_\Gamma\| + \|H^{1/2}_\Gamma\|^{1-\theta}
\leq C\|d\|_\Gamma - h_\infty\|^{1-\theta}\|H^{1/2}_\Gamma\|^{1-\theta}
\leq C\|d\|_\Gamma - h_\infty\|^{1-\theta}\|H^{1/2}_\Gamma\|.
\]

(3.18)

Recalling the function $w$ introduced in (3.3), we estimate $J_3$ as follows

\[
J_3 = \left| \int_{\Omega} [\nabla (d - w) : \nabla d_E + \nabla w : \nabla (d_E - \psi_E) + \nabla (w - \psi) : \nabla \psi_E] \, dx \right|^{1-\theta}
\leq \left| \int_{\Omega} [\nabla (d - w) : \nabla d_E] \, dx \right|^{1-\theta} + \left| \int_{\Omega} [\nabla w : \nabla (d_E - \psi_E)] \, dx \right|^{1-\theta}
+ \left| \int_{\Omega} [\nabla (w - \psi) : \nabla \psi_E] \, dx \right|^{1-\theta}
:= J_{3a} + J_{3b} + J_{3c}.
\]

(3.19)

Using (3.6) and (3.7) and the fact $\|d - \psi\|_{H^1} < \beta$, we observe that

\[
J_{3a} \leq \|\nabla (d - w)\|^{1-\theta}\|\nabla d_E\|^{1-\theta} \leq C\|d\|_\Gamma - h_\infty\|^{1-\theta}\|H^{1/2}_\Gamma\|^{1-\theta}
\leq C\|d\|_\Gamma - h_\infty\|^{1-\theta}\|H^{1/2}_\Gamma\| \leq C\|d\|_\Gamma - h_\infty\|^{1-\theta}\|H^{1/2}_\Gamma\|.
\]

(3.20)

\[
J_{3b} \leq \|\nabla w\|^{1-\theta}\|\nabla (d_E - \psi_E)\|^{1-\theta} \leq C (\|\psi\|_{H^1} + C\beta)^{1-\theta}\|d\|_\Gamma - h_\infty\|^{1-\theta}\|H^{1/2}_\Gamma\|
\leq C\|d\|_\Gamma - h_\infty\|^{1-\theta}\|H^{1/2}_\Gamma\|.
\]

(3.21)

For $J_{3c}$, using integration by parts and noticing that $\Delta \psi_E = 0$, $(w - \psi)_\Gamma = 0$, we obtain

\[
\int_{\Omega} [\nabla (w - \psi) : \nabla \psi_E] \, dx = -\int_{\Omega} (w - \psi) \cdot \Delta \psi_E \, dx + \int_{\Gamma} (w - \psi) \cdot \partial_n \psi_E dS = 0,
\]

(3.22)

where $n$ is the unit outer normal to the boundary $\Gamma$. Thus (3.22) implies that

\[
J_{3c} = 0.
\]

(3.23)

Finally, since $1-\theta \in (0, 1)$, we have $\|d\|_\Gamma - h_\infty\|^{1-\theta}\|H^{1/2}_\Gamma\| \leq C\|d\|_\Gamma - h_\infty\|^{1-\theta}\|H^{1/2}_\Gamma\|$. Summing up, we can conclude from (3.4), (3.17)–(3.23), and $\hat{\Delta} d = \Delta d$ that (3.14) holds. The proof is complete.  

\[\square\]

Remark 3.3. If $\theta \in (0, \frac{1}{2})$ is such that (3.14) holds, then, for all $\theta' \in (0, \theta)$ and any $d \in H^1(\Omega)$ satisfying $\|d - \psi\|_{H^1} < \beta$, we still have

\[
C \left( \|d\|_\Gamma - h_\infty\|^{1-\theta'}\|H^{1/2}_\Gamma\| + \|\Delta d + f(d)\|_{H^{-1}} \right) \geq |\hat{E}(d) - \hat{E}(\psi)|^{1-\theta'},
\]

(3.24)
where \( C \) is a (properly adjusted) positive constant depending on \( \psi \) and \( h_\infty \). To see this, we first notice that, since \( 2 > \frac{1-\theta'}{\theta} > 1 \), for any \( a, b \geq 0 \), it holds \((a + b)^{\frac{1-\theta'}{\theta}} \leq 2 (a^{\frac{1-\theta'}{\theta}} + b^{\frac{1-\theta'}{\theta}})\). Then it follows from (3.14) that

\[
|\tilde{E}(d) - \tilde{E}(\psi)|^{1-\theta'} = \left( |\tilde{E}(d) - \tilde{E}(\psi)|^{1-\theta} \right)^{\frac{1-\theta'}{1-\theta}} \\
\leq C \left( \|d|_\Gamma - h_\infty \|_{H^1(\Gamma)}^{1-\theta} + \| - \Delta \tilde{d} + f(d)\|_{H^{-1}} \right)^{\frac{1-\theta'}{1-\theta}} \\
\leq C \left( \|d|_\Gamma - h_\infty \|_{H^1(\Gamma)}^{1-\theta} + \| - \Delta \tilde{d} + f(d)\|_{H^{-1}} \right).
\]

4 Long-time behavior in 2D

In this section, we focus on the case \( n = 2 \). In order to study the long-time behavior of global solutions to problem (1.1)–(1.5), we need some decay conditions on the time-dependent external force \( g \) and boundary data \( h \), namely,

(H1) \( \int_t^{+\infty} \|h_t(\tau)\|_{H^1(\Gamma)}^2 d\tau \leq C (1 + t)^{-1-\gamma} \);  

(H2) \( \int_t^{+\infty} \|h_t(\tau)\|_{H^2(\Gamma)}^2 d\tau \leq C (1 + t)^{-1-\gamma} \);  

(H3) \( \int_t^{+\infty} \|g(\tau)\|^2 d\tau \leq C (1 + t)^{-1-\gamma} \);  

(H4) \( \|g(t)\|^2 \leq C (1 + t)^{-2-\gamma} \);  

(H5) \( \|h_t(t)\|_{L^2(\Gamma)} \leq C (1 + t)^{-1-\gamma} \);

for all \( t \geq 0 \). Here \( C \) and \( \gamma \) are given positive constants. We also note that (H4) entails (H3).

Since in the 2D case weak solutions become strong for positive times (cf. Theorem 2.1), we can confine ourselves to consider strong solutions. We recall that, for any given global strong solution \( (v, d) \), we have the uniform estimate (2.35). It follows that the \( \omega \)-limit set of the corresponding initial datum \( (v_0, d_0) \) is non-empty. Namely, for any unbounded increasing sequence \( \{t_n\}_{n=1}^{\infty} \), there are functions \( v_\infty \in V \) and \( d_\infty \in H^2(\Omega) \) such that, up to a subsequence \( \{t_j\}_{j=1}^{\infty} \subset \{t_n\}_{n=1}^{\infty} \), we have

\[
\lim_{j \to +\infty} \|v(t_j) - v_\infty\| = 0, \quad \lim_{j \to +\infty} \|d(t_j) - d_\infty\|_{H^1} = 0. \quad (4.1)
\]

Next, we characterize the structure of the \( \omega \)-limit set. In order to do that, we first recall a technical lemma (see [32, Lemma 6.2.1])

**Lemma 4.1.** Let \( T \) be given with \( 0 < T \leq +\infty \). Suppose that \( y \) and \( h \) are nonnegative continuous functions defined on \([0, T]\) and satisfy the following conditions:

\[
\frac{dy}{dt} \leq c_1 y^2 + c_2 + h, \quad \int_0^T y(t) dt \leq c_3, \quad \int_0^T h(t) dt \leq c_4, \quad \text{where } c_i \ (i = 1, 2, 3, 4) \text{ are given nonnegative constants.}
\]

Then for any \( \rho \in (0, T) \), the following estimates holds:

\[
y(t + \rho) \leq \left( \frac{c_1}{\rho} + c_2 \rho + c_4 \right) e^{c_1 c_3}, \quad \text{for all } t \in [0, T - \rho].
\]

Furthermore, if \( T = +\infty \), then \( \lim_{t \to +\infty} y(t) = 0 \).
Proposition 4.1. Let the assumptions of Theorem 2.1 hold. Then the \( \omega \)-limit set \( \omega(v_0, d_0) \) is a subset of
\[
S = \{(0, u) : u \in \mathcal{N} \cap H^2(\Omega) \text{ such that } -\Delta u + f(u) = 0, \text{ in } \Omega\}.
\]
Moreover, we have
\[
\lim_{t \to +\infty} \| v(t) \|_V = 0, \quad \text{(4.2)}
\]
\[
\lim_{t \to +\infty} \| -\Delta d(t) + f(d(t)) \| = 0. \quad \text{(4.3)}
\]

Proof. It follows from Lemma 2.2 that \( \int_0^{+\infty} \| \nabla v(t) \|^2 + \| \Delta \hat{d}(t) - f(d(t)) \|^2 \, dt < +\infty \), which together with the definition of \( A_P \) and (6.3) yields
\[
\int_0^{+\infty} A_P(t) \, dt \leq \int_0^{+\infty} (\| \nabla v(t) \|^2 + 2(\| \Delta \hat{d}(t) - f(d(t)) \|^2 + 2\| \partial_t d_P(t) \|^2) \, dt < +\infty. \quad \text{(4.4)}
\]
Using Lemma 2.7 and Lemma 4.1 we can see that
\[
\lim_{t \to +\infty} A_P(t) = 0,
\]
which implies \( \lim_{t \to +\infty} \| \nabla v(t) \| = 0 \). Hence, for any \( (v_\infty, d_\infty) \in \omega(v_0, d_0) \), we have \( v_\infty = 0 \).

On the other hand, by definition of \( A_P \), (4.4) also yields that
\[
\lim_{t \to +\infty} \| -\Delta \hat{d}(t) + f(d(t)) \| = 0. \quad \text{(4.5)}
\]
From Lemma 6.2 we have \( \lim_{t \to +\infty} \| \partial_t d_P(t) \| = 0 \) (cf. (6.5)). As a result, it follows from the inequality
\[
0 \leq \| -\Delta d(t) + f(d(t)) \| \leq \| -\Delta \hat{d}(t) + f(d(t)) \| + \| \partial_t d_P(t) \|, \quad t \geq 0 \quad \text{(4.6)}
\]
that (4.3) holds. Concerning the limit function \( d_\infty \), we infer from (2.35) that \( d_\infty \in H^2(\Omega) \) and (4.1) holds. We now check the boundary condition for \( d_\infty \). Since \( h_t \in L^1(0, +\infty; H^{-\frac{1}{2}}(\Gamma)) \), \( h(t) \)
strongly converges to a certain function \( h_\infty \in H^{-\frac{1}{2}}(\Gamma) \) as time goes to infinity with a controlled rate, namely,
\[
\| h(t) - h_\infty \|_{H^{-\frac{1}{2}}(\Gamma)} \leq \int_t^{+\infty} \| h(\tau) \|_{H^{-\frac{1}{2}}(\Gamma)} \, d\tau \to 0, \quad \text{as } t \to +\infty. \quad \text{(4.7)}
\]
On the other hand, we infer from (2.33) and (2.34) that \( h \in L^\infty(0, +\infty; H^{\frac{1}{2}}(\Gamma)) \). Consequently, \( h_\infty \in H^{\frac{1}{2}}(\Gamma) \) and \( h \) weakly-* converges to \( h_\infty \) in \( L^\infty(0, +\infty; H^{\frac{1}{2}}(\Gamma)) \). By interpolation, we have
\[
\lim_{t \to +\infty} \| h(t) - h_\infty \|_{L^2(\Gamma)} = 0. \quad \text{Thus, from the asymptotic behavior of the boundary datum } h, \text{ we have for any } j \in \mathbb{N},
\]
\[
0 \leq \| d_\infty |r - h_\infty \|_{L^2(\Gamma)} \leq \| d_\infty |r - h(t_j) \|_{L^2(\Gamma)} + \| h(t_j) - h_\infty \|_{L^2(\Gamma)} \leq C \| d_\infty - d(t_j) \|_{H^1} + \| h(t_j) - h_\infty \|_{L^2(\Gamma)}.
\]
Hence, letting \( j \to +\infty \) in the above inequality, we deduce from (4.1) and (4.7) that \( d_\infty |r = h_\infty \).
For any \( z \in H^1_0(\Omega) \) and \( j \in \mathbb{N} \), we have
\[
\int_{\Omega} (-\Delta d_\infty + f(d_\infty)) \cdot z \, dx
\]
Summing up, we can see that
\[ \text{given in Corollary 3.1 (depending on Theorem 4.1. Proposition 4.2.}
\]
\[
\text{Let the assumptions of Theorem 2.1 hold. Then the lifted energy functional}
\]
\[ E(\tau) \text{ defined by (2.15) is constant on the } \omega \text{-limit set } \omega(v_0, d_0). \text{ Namely, there exists a constant } \hat{E}_\infty \text{ such that } \hat{E}(d_\infty) = \hat{E}_\infty, \text{ for all } (0, d_\infty) \text{ with } d_\infty \in N \cap H^2(\Omega). \text{ Moreover, we have}
\]
\[
\lim_{t \to +\infty} \hat{E}(t) = \hat{E}_\infty. \quad (4.8)
\]
\[
\text{Proof. From the previous argument, we know that for arbitrary } (0, d_1^{(1)}), (0, d_2^{(2)}) \in \omega(v_0, d_0) \text{ there exist unbounded increasing sequences } \{t_j^{(1)}\}_{j=1}^\infty \text{ and } \{t_j^{(2)}\}_{j=1}^\infty \text{ such that (4.1) holds. As a result, we have}
\]
\[
\lim_{j \to +\infty} \hat{E}(t_j^{(1)}) = \hat{E}(d_1^{(1)}), \quad \lim_{j \to +\infty} \hat{E}(t_j^{(2)}) = \hat{E}(d_2^{(2)}).
\]
\[
\text{On the other hand, it follows from the basic energy inequality (2.16) that for any } t' > t'' > 0,
\]
\[
|\hat{E}(t') - \hat{E}(t'')| \leq \int_{t'}^{t''} r(t) dt \to 0, \quad \text{as } t', t'' \to +\infty.
\]
\[
\text{Then by}
\]
\[
|\hat{E}(d_1^{(1)}) - \hat{E}(d_2^{(2)})| \leq |\hat{E}(t_j^{(1)}) - \hat{E}(t_j^{(2)})| + |\hat{E}(d_1^{(1)}) - \hat{E}(t_j^{(1)})| + |\hat{E}(t_j^{(2)}) - \hat{E}(d_2^{(2)})|,
\]
\[
\text{letting } j \to +\infty, \text{ we can see that } \hat{E}(d_1^{(1)}) = \hat{E}(d_2^{(2)}). \text{ Namely, } \hat{E} \text{ is a constant (denoted by } \hat{E}_\infty) \text{ on the } \omega \text{-limit set } \omega(v_0, d_0). \text{ Moreover, for any } t > 0 \text{ there exist } t_j < t_{j+1} \text{ such that } t \in [t_j, t_{j+1}]\text{ and } |\hat{E}(t) - \hat{E}_\infty| \leq |\hat{E}(t) - \hat{E}(t_j)| + |\hat{E}(t_j) - \hat{E}_\infty| \text{ which yields (4.8).} \quad \square
\]
\[
\text{4.1 Convergence to equilibrium}
\]
\[
\text{Theorem 4.1. Let the assumptions of Theorem 2.1 hold. If, in addition, we assume (H1)–(H3), then any strong solution } (v(t), d(t)) \text{ convergence to an equilibrium } (0, d_\infty) \text{ strongly in } V \times H^2(\Omega) \text{ as } t \to +\infty.
\]
\[
\text{Proof. On account of (4.1) we only need to prove that } d(t) \text{ converges to } d_\infty \text{ as } t \to +\infty \text{ given by (4.1). Below we adapt the idea in [27] to achieve our goal. Indeed, observe that we can find an integer } j_0 \text{ such that for all } j \geq j_0, \|d(t_j) - d_\infty\|_{H^1} < \beta, \text{ where } \beta \in (0, 1) \text{ is the constant given in Corollary 3.1 (depending on } d_\infty). \text{ Consequently, we define}
\]
\[
s(t_j) = \sup\{\tau \geq t_j : ||d(\tau) - d_\infty||_{H^1} < \beta\}.
\]
Since \( d \in C([0, +\infty); \mathbf{H}^1(\Omega)) \), we can see that \( s(t_j) > t_j \) for any \( j \geq j_0 \). By Lemma 2.2 and Proposition 4.2, we have

\[
|\hat{E}(t) - \hat{E}(d_\infty)| \geq \frac{1}{4} \min\{\nu, 1\} \int_t^{+\infty} D^2(\tau)d\tau, \\
\]

where

\[
D(t) = \|\nabla v(t)\| + \|\Delta \hat{d}(t) - f(d(t))\|, \\
\]

and \( r \) is defined in (2.10) such that, thanks to (H1)–(H3), we have

\[
\int_t^{+\infty} r(\tau)d\tau \leq C(1 + t)^{-1-\gamma}, \quad \forall \ t \geq 0. \\
\]

Let the constant \( \theta \) be as in Corollary 3.1 (depending on \( d_\infty \)). Using Remark 3.3 we can choose \( \theta' \in (0, \theta] \) such that \( \theta' \) also satisfies

\[
0 < \theta' < \frac{\gamma}{2(1 + \gamma)}. \quad (4.9) \\
\]

If \( \theta \) itself satisfies (4.9), we just take \( \theta' = \theta \). For any fixed \( t_j \) with \( j \geq j_0 \), we introduce the sets

\[
K_j = [t_j, s(t_j)), \quad K_j^{(1)} = \{ t \in K : D(t) > (1 + t)^{-(1-\theta')(1+\gamma)} \}, \quad K_j^{(2)} = K \setminus K_j^{(1)}. \\
\]

Consider the following functional on \( K_j \)

\[
\Phi(t) = \hat{E}(t) - \hat{E}(d_\infty) + 2 \int_t^{s(t_j)} r(\tau)d\tau, \quad \forall \ t \in K_j. \\
\]

It easily follows that

\[
\lim_{j \to +\infty} \Phi(t_j) = 0. \quad (4.10) \\
\]

Next, we have

\[
\frac{d}{dt}(|\Phi(t)|^{\theta'} \text{sgn}\Phi(t)) = \theta'|\Phi(t)|^{\theta'-1} \frac{d}{dt}\Phi(t) \leq -\frac{\theta'}{4} \min\{\nu, 1\}|\Phi(t)|^{\theta'-1} D^2(t) \leq 0, \quad (4.11) \\
\]

which implies that the functional \( |\Phi(t)|^{\theta'} \text{sgn}\Phi(t) \) is decreasing on \( K_j \). Keeping in mind that \( \theta' \leq \theta \) and \( 2(1-\theta') > 1 \), we can apply Corollary 3.1 (cf. also Remark 3.3) to obtain that

\[
|\Phi(t)|^{1-\theta'} \leq |\hat{E}(t) - \hat{E}(d_\infty)|^{1-\theta'} + C \left( \int_t^{+\infty} r(\tau)d\tau \right)^{1-\theta'} \leq \left( \frac{1}{2} \right)^{2(1-\theta')} \|v\|^{2(1-\theta')} + C\|h(t) - h_\infty\|_{\mathbf{H}^2(\Gamma)}^{1-\theta'} + C\| - \Delta \hat{d} + f(d)\|_{\mathbf{H}^{-1}} + C \left( \int_t^{+\infty} r(\tau)d\tau \right)^{1-\theta'} \leq C \|\nabla v\| + C\| - \Delta \hat{d} + f(d)\| + C \left( \int_t^{+\infty} \|h(t)\|_{\mathbf{H}^2(\Gamma)}d\tau \right)^{1-\theta'} + C \left( \int_t^{+\infty} r(\tau)d\tau \right)^{1-\theta'} \leq C \|\nabla v\| + C\| - \Delta \hat{d} + f(d)\| + C(1 + t)^{-(1-\theta')(1+\gamma)}. \quad (4.12) \\
\]

Thus, on \( K_j^{(1)} \), we have

\[
|\Phi(t)|^{1-\theta'} \leq CD(t), \\
\]
which together with (4.11) yields that on $K_j^{(1)}$,
\[ -\frac{d}{dt}(|\Phi(t)|^{\theta} \text{sgn}\Phi(t)) \geq C\mathcal{D}(t). \] (4.13)

As a consequence, we have
\[ \int_{K_j^{(1)}} \mathcal{D}(t) dt \leq -C \int_{K_j} \frac{d}{dt}(|\Phi(t)|^{\theta} \text{sgn}\Phi(t)) dt \leq C(|\Phi(t_j)|^{\theta'} + |\Phi(s(t_j))|^{\theta'}) < +\infty, \] (4.14)

where $\Phi(s(t_j)) = 0$ if $s(t_j) = +\infty$. On the other hand, on $K_j^{(2)}$, we have
\[ \int_{K_j^{(2)}} \mathcal{D}(t) dt \leq C \int_{t_j}^{\infty} (1 + t)^{-(1-\theta')(1+\gamma)} dt = \frac{C}{-\gamma \theta' - \theta' + \gamma} (1 + t_j)^{\gamma \theta' + \theta' - \gamma}. \] (4.15)

Here, we notice that $\gamma \theta' + \theta' - \gamma < 0$ due to (4.9). Then (4.14) and (4.15) imply that
\[ \int_{K_j} \mathcal{D}(t) dt = \int_{K_j^{(1)}} \mathcal{D}(t) dt + \int_{K_j^{(2)}} \mathcal{D}(t) dt < +\infty, \]
for any $j$. On the other hand, it follows from (2.35) and (2.12) that
\[ \|d_i(t)\| \leq \|v \cdot \nabla d\| + \|\Delta \mathbf{d} - f(d)\| \leq \|v\|_{L^4} \|\nabla d\|_{L^4} + \|\nabla d\| \leq C\mathcal{D}(t). \] (4.16)

As a consequence,
\[ \int_{K_j} \|d_i(t)\| dt \leq C(|\Phi(t_j)|^{\theta'} + |\Phi(s(t_j))|^{\theta'}) + C(1 + t_j)^{\gamma \theta' + \theta' - \gamma}. \] (4.17)

**Proposition 4.3.** Let the assumptions of Theorem 2.1 hold. Then there exists an integer $j_1 \geq j_0$ such that $s(t_{j_1}) = +\infty$. Thus
\[ \|d(t) - d_\infty\|_{H^1} < \beta, \quad \forall t \geq t_{j_1}. \]

**Proof.** The conclusion follows from a contradiction argument (cf. [13]). Suppose that for any $j \geq j_0$ we have $s(t_{j}) < +\infty$. Then, by definition, we have
\[ \|d(s(t)) - d_\infty\|_{H^1} = \beta > 0. \] (4.18)

Besides, it follows from (4.1), (4.10) and (1.17) that
\[ \|d(s(t_j)) - d_\infty\| \leq \|d(s(t_j)) - d(t_j)\| + \|d(t_j) - d_\infty\| \leq \int_{t_j}^{s(t_j)} \|d_i(t)\| dt + \|d(t_j) - d_\infty\| \to 0, \quad \text{as} \; j \to +\infty. \]

Using uniform estimate (2.35) and interpolation inequality, we obtain
\[ \|d(s(t_j)) - d_\infty\|^2_{H^1} \leq \|d(s(t_j)) - d_\infty\|^2_{H^2} \|d(s(t_j)) - d_\infty\| \to 0, \quad \text{as} \; j \to +\infty, \]

which leads a contradiction with (4.18). The proof is complete.
Due to Proposition 4.3, we have $s(t_{j_1}) = +\infty$ for some $j_1 \geq j_0$. Arguing as above, we can prove
\[ \int_{t_{j_1}}^{+\infty} \|d_i(t)\| dt < +\infty. \]
Thus $d(t)$ converges in $L^2$ and recalling (4.1), by compactness we conclude that
\[ \lim_{t \to +\infty} \|d(t) - d_\infty\|_{H^1} = 0. \] (4.19)
Finally, observe that
\[ \|\Delta d(t) - \Delta d_\infty\| = \|\Delta d(t) + f(d(t))\| + \|f(d(t)) - f(d_\infty)\| \leq \|\Delta d(t) + f(d(t))\| + C\|d(t) - d_\infty\|. \] (4.20)
Then (4.3) and (4.19) entail that
\[ \lim_{t \to +\infty} \|d(t) - d_\infty\|_{H^2} = 0 \]
and this finishes the proof. \qed

4.2 Convergence rate

Theorem 4.2. Let the assumptions of Theorem 2.1 hold. If, in addition, we assume (H1)–(H2) and (H4)–(H5), then we have
\[ \|v(t)\| + \|d(t) - d_\infty\|_{H^1} \leq C(1 + t)^{-\theta'} t \geq 0. \]
Moreover, if (H2) and (H5) are replaced by, respectively,
(H6) $\|h_i(t)\|_{H^2(\Gamma)} \leq C(1 + t)^{-1-\gamma}$;
(H7) $\|h(t) - h_\infty\|_{H^2(\Gamma)} \leq C(1 + t)^{-1-\gamma}$;
the following higher-order estimate holds
\[ \|v(t)\|_{V} + \|d(t) - d_\infty\|_{H^2} \leq C(1 + t)^{-\theta' t^{1-2\theta}}, \quad t \geq 0. \]

Proof. The proof consists of several steps.

Step 1. $L^2$-estimate of $d - d_\infty$. This follows from an argument devised in [7]. For the readers’ convenience, we sketch the proof here. From the previous argument, we only have to work on the time interval $[t_{j_1}, +\infty)$. Denote
\[ \Phi(t) = \hat{E}(t) - \hat{E}(d_\infty) + 2 \int_{t}^{+\infty} r(\tau)d\tau. \]
Since
\[ \frac{d}{dt}\Phi(t) \leq -\theta' \min\{\nu, 1\} D^2(t) - r(t) \leq 0, \]
and $\lim_{t \to +\infty} \Phi(t) = 0$, we know that $\Phi(t)$ is decreasing and $\Phi(t) \geq 0$ for $t \geq t_{j_1}$.
First, if the boundary datum $h$ and the external force $g$ become time-independent in finite time, i.e., there exists time $T_0$ such that for $t \geq T_0$, $h = h_\infty$ and $g = 0$. Then the problem
Therefore, we infer

Due to (4.9), it follows that $t^* = 0$, for all $t \geq t^*$ and this is a contradiction since $r(t)$ cannot identically vanish from any finite time on. Therefore, we can suppose

$$\Phi(t) > 0, \quad \forall \ t \geq t_{j1}.$$  

If the open set $K_{j1}^{(1)}$ is bounded, then there exists $t^* \geq t_{j1}$ such that $[t^*, +\infty) \subset K_{j1}^{(2)}$. As a result, $D(t) \leq (1 + t)^{-1-(\theta') (1+\gamma)}$ and by (4.16), we have

$$\|d(t) - d_{\infty}\| \leq \int_t^{+\infty} \|d_\tau\|d\tau \leq \frac{C}{-\gamma \theta' - \theta' + \gamma} (1 + t)^{\gamma \theta' + \theta' - \gamma}, \quad \forall \ t \geq t^*.$$  

Next, we treat the case when the open set $K_{j1}^{(1)}$ is unbounded. There exists a countable family of disjoint open sets $(a_n, b_n)$ such that $K_{j1}^{(1)} = \bigcup_{n=1}^{\infty} (a_n, b_n)$. On $K_{j1}^{(1)}$, recalling (4.12), we can see that on any $(a_n, b_n) \subset K_{j1}^{(1)}$, it holds

$$\frac{d}{dt}\Phi(t) + C\Phi^{2(1-\theta')}(t) \leq 0.$$  

As a result, for any $t \in (a_n, b_n)$,

$$\Phi(t) \leq \left[\Phi(a_{n^*})^{2\theta'-1} + C(1 - 2\theta')(t - a_{n^*})\right]^{-\frac{1}{1-2\theta'}}, \quad (4.21)$$  

where by the definition of $K_{j1}^{(1)}$ and (4.12) we have

$$\Phi(a_n) \leq CD(a_n)^{-\frac{1}{1-\theta'}} + C(1 + a_{n^{*}})^{-1+\gamma} = C(1 + a_n)^{-1-\gamma}.$$  

Using the fact $(1 + \gamma)(1 - 2\theta') > 1$ (cf. (4.30)), we can take $n^* \in \mathbb{N}$ sufficiently large such that

$$\Phi(a_{n^*})^{2\theta'-1} - C(1 - 2\theta')a_{n^*} \geq a_{n^*}^{(1+\gamma)(1 - 2\theta')} = C(1 - 2\theta')a_{n^*} \geq 1. \quad (4.22)$$  

Therefore, we infer

$$\Phi(t) \leq C(1 + t)^{-\frac{1}{1-2\theta'}}, \quad \forall \ t \in (a_{n^*}, \infty) \cap K_{j1}^{(1)}.$$  

Similar to (4.13), we have (since $\Phi(t) > 0$)

$$-\frac{d}{dt}\Phi(t)^{\theta'} \geq CD(t), \quad \forall \ t \in (a_{n^*}, \infty) \cap K_{j1}^{(1)}.$$  

Due to (4.9), it follows that $-\gamma \theta' - \theta' + \gamma \geq \frac{\theta'}{1-2\theta'}$. Now for any $t > a_{n^*}$, we can conclude that

$$\|d(t) - d_{\infty}\| \leq C \int_{(t, \infty) \cap K_{j1}^{(1)}} D(\tau) d\tau + C \int_t^{+\infty} (1 + \tau)^{-(1-\theta')(1+\gamma)} d\tau \leq C\Phi(t)^{\theta'} + C(1 + t)^{\gamma \theta' + \theta' - \gamma} \leq C(1 + t)^{-\frac{\theta'}{1-2\theta'}}.$$  

21
Using (2.35), after properly adjusting the constant $C$, we have
\[ \|d(t) - d_{\infty}\| \leq C(1 + t)^{-\frac{\nu}{1 - 2p}}, \quad \forall \ t \geq 0. \quad (4.23) \]

**Step 2.** $H \times H^1$-estimate. It easily from the basic energy inequality (2.16) that
\[ \frac{d}{dt} y(t) + \frac{\nu}{2} \|\nabla v\|^2 + \frac{1}{2} \|\Delta \tilde{d} - f(d)\|^2 \leq r(t), \quad (4.24) \]
where
\[ y(t) = \frac{1}{2} \|v(t)\|^2 + \frac{1}{2} \|\nabla (d(t) - \tilde{d}_{\infty})\|^2 + \int_{\Omega} [F(d(t)) - F(d_{\infty})] d\tau. \]

As in [31], using (2.35), we can show that
\[ \left\| \int_{\Omega} [F(d(t)) - F(d_{\infty}) - f(d_{\infty})(d(t) - d_{\infty})] d\tau \right\| \leq C \|d(t) - d_{\infty}\|^2. \]

Keeping in mind the definition of lifting functions, we have $\tilde{d} - \tilde{d}_{\infty} |_{\Gamma} = 0$ so that
\[ \|\nabla (\tilde{d} - \tilde{d}_{\infty})\| \leq C \|\Delta (\tilde{d} - \tilde{d}_{\infty})\| \leq \|\Delta \tilde{d} + f(d)\| + C \|f(d) - f(d_{\infty})\| \leq C \|\Delta \tilde{d} + f(d)\| + C \|\nabla (d(t) - d_{\infty})\|, \]
\[ \|\nabla (d - d_{\infty})\| \leq \|\nabla (d)\| + \|\nabla (d_{\infty})\| \leq C \|\Delta \tilde{d} + f(d)\| + C \|\nabla (d(t) - d_{\infty})\| + C \int_{t}^{+\infty} \|h_{t}(\tau)\|_{H^{\frac{1}{2}}(\Gamma)} d\tau. \]

Thus it follows that
\[ y(t) \geq \frac{1}{2} \|v(t)\|^2 + \frac{1}{2} \|\nabla (d - d_{\infty})\|^2 - C \|d(t) - d_{\infty}\|^2 - C \left( \int_{t}^{+\infty} \|h_{t}(\tau)\|_{H^{\frac{1}{2}}(\Gamma)} d\tau \right)^2 \quad (4.25) \]
\[ y(t) \leq C \|\nabla v\|^2 + C \|\Delta \tilde{d} + f(d)\|^2 + C \|\nabla (d(t) - d_{\infty})\|^2. \quad (4.26) \]

Condition (4.20) implies that $\frac{2\nu}{1 - 2p} < \gamma$. Then we deduce from (4.24), (4.23), (H4)–(H5) and Lemma (6.1) that
\[ \frac{d}{dt} y(t) + \alpha y(t) \leq C(r(t) + \|d(t) - d_{\infty}\|^2) \leq C(1 + t)^{-\frac{2\nu}{1 - 2p}}, \quad (4.27) \]
where $\alpha > 0$ is sufficiently small. The above inequality implies that
\[ y(t) \leq C(1 + t)^{-\frac{2\nu}{1 - 2p}}, \quad \forall \ t \geq 0. \quad (4.28) \]

Combining it with (4.25) and recalling (H1), we get
\[ \|v(t)\|^2 + \|d(t) - d_{\infty}\|^2_{H^1} \leq Cy(t) + C \|d(t) - d_{\infty}\|^2 + C \left( \int_{t}^{+\infty} \|h_{t}(\tau)\|_{H^{\frac{1}{2}}(\Gamma)} d\tau \right)^2 \]
\[ \leq C(1 + t)^{-\frac{2\nu}{1 - 2p}}, \quad \forall \ t \geq 0. \quad (4.29) \]

**Step 3.** $V \times H^2$-estimate. Taking advantage of the stronger assumptions (H6)–(H7) and (4.27), we now get a higher-order estimate. Observe first that
\[ \|\Delta \tilde{d} + f(d)\| \leq \|\Delta \tilde{d} + f(d)\| + \|\Delta d_{\rho}\| = \|\Delta \tilde{d} + f(d)\| + \|\tilde{\partial} d_{\rho}\|, \]

22
then we have
\[ y(t) \leq C\|\nabla v\|^2 + C - \Delta \tilde{d} + f(d)\|^2 + C\|\partial_1 d_P\|^2 + C\|d(t) - d_\infty\|^2. \]

It follows from (22.27) and (4.27) that
\[ \frac{d}{dt} z(t) + \alpha_2 z(t) \leq C(1 + t)^{-\frac{2\theta'}{1 - 2\theta'}} + C(R_1(t) + \|\partial_1 d_P(t)\|^2), \tag{4.30} \]
where
\[ z(t) = y(t) + \alpha_1 A_P(t), \tag{4.31} \]
and \( \alpha_1 \) and \( \alpha_2 \) are sufficiently small positive constants. From the definition of \( R_1 \), (6.6) and the fact \( \frac{2\theta'}{1 - 2\theta'} < 2 + 2\gamma \) (cf. (4.9)), we have
\[ R_1(t) + \|\partial_1 d_P(t)\|^2 \leq C(1 + t)^{-\frac{2\theta'}{1 - 2\theta'}} + C\|\nabla d_P(t)\|^2. \tag{4.32} \]
Hence from (4.30) we infer that
\[
\begin{align*}
  z(t) &\leq z(0)e^{-\alpha_2 t} + C e^{-\alpha_2 t} \int_0^t e^{\alpha_2 \tau} \left[ C(1 + \tau)^{-\frac{2\theta'}{1 - 2\theta'}} + \|\nabla d_P(\tau)\|^2 \right] d\tau \\
  &\leq C e^{-\alpha_2 t} + e^{-\alpha_2 t} \int_0^t e^{\alpha_2 \tau} \left[ C(1 + \tau)^{-\frac{2\theta'}{1 - 2\theta'}} + \|\nabla d_P(\tau)\|^2 \right] d\tau \\
  &\quad + e^{-\alpha_2 t} \int_t^{t+\frac{\theta}{2}} e^{\alpha_2 \tau} \left[ C(1 + \tau)^{-\frac{2\theta'}{1 - 2\theta'}} + \|\nabla d_P(\tau)\|^2 \right] d\tau \\
  &= C e^{-\alpha_2 t} + Z_1(t) + Z_2(t). \tag{4.33}
\end{align*}
\]
It follows from (6.4) and (H6) that
\[
\begin{align*}
  Z_1(t) &\leq C e^{-\alpha_2 t} \int_0^{t+\frac{\theta}{2}} \left[ C(1 + \tau)^{-\frac{2\theta'}{1 - 2\theta'}} + \|\nabla d_P(\tau)\|^2 \right] d\tau \\
  &\leq C e^{-\alpha_2 t} \left( t + \int_0^{t+\frac{\theta}{2}} (1 + \tau)^{-2 - 2\gamma} d\tau \right) \leq C(1 + t)^{-\frac{2\theta'}{1 - 2\theta'}}. \tag{4.34}
\end{align*}
\]
Next, by (6.7) and the fact \( \frac{2\theta'}{1 - 2\theta'} < 1 + 2\gamma \), we deduce that
\[
\begin{align*}
  Z_2(t) &\leq C e^{-\alpha_2 t} \left( 1 + \frac{t}{2} \right)^{-\frac{2\theta'}{1 - 2\theta'}} \int_\frac{t}{2}^t e^{\alpha_2 \tau} d\tau + C \int_\frac{t}{2}^t \|\nabla d_P(\tau)\|^2 d\tau \\
  &\leq C(1 + t)^{-\frac{2\theta'}{1 - 2\theta'}} + C(1 + t)^{-1 - 2\gamma} \leq C(1 + t)^{-\frac{2\theta'}{1 - 2\theta'}}. \tag{4.35}
\end{align*}
\]
As a result, we obtain that
\[ z(t) \leq C(1 + t)^{-\frac{2\theta'}{1 - 2\theta'}}, \quad \forall \ t \geq 0. \tag{4.36} \]
In particular, we have
\[ A_P(t) \leq C(1 + t)^{-\frac{2\theta'}{1 - 2\theta'}}, \quad t \geq 0, \tag{4.37} \]
which together with (4.20) and (4.29) yields the following estimate
\[ \|v(t)\|^2_V + \|\Delta d(t) - \Delta d_\infty\|^2 \leq C(1 + t)^{-\frac{2\theta'}{1 - 2\theta'}}, \quad t \geq 0. \]
Finally, using a standard elliptic estimate, we obtain (cf. (H7))
\[ \|d(t) - d_\infty\|_{H^2} \leq C\|\Delta d(t) - \Delta d_\infty\| + C\|h(t) - h_\infty\|_{H^2} \leq C(1 + t)^{-\frac{\theta'}{1 - 2\theta'}}, \]
for all \( t \geq 0 \) and this finishes the proof.
5 Long-time behavior in 3D

As in the classical Navier-Stokes case (see [14]), we can prove the eventual regularity of any global weak solution. Thus the convergence results can also be extended to the 3D case. Indeed, comparing with Lemma 2.6 we derive first an alternative higher-order energy inequality.

**Lemma 5.1.** Let the assumptions of Proposition 2.7 hold for all $T > 0$. Suppose, in addition, that (2.22)–(2.31) are satisfied. If a weak solution $(\mathbf{v}, \mathbf{d})$ is smooth enough then it fulfills the following inequality

$$
\frac{d}{dt} A_P(t) + \nu \|\mathbf{Sv}\|^2 + \|\nabla(\Delta \mathbf{d} - \mathbf{f}(\mathbf{d}))\|^2 \leq C_*(A_P^3(t) + A_P(t) + R_3(t)),
$$

where

$$
R_3(t) = \|\partial_t \mathbf{d}_P(t)\|^2 + \|\partial_t \mathbf{d}_P(t)\|^2 + \|\nabla \Delta \mathbf{d}_P(t)\|^2 + \|g(t)\|^2,
$$

for all $t \geq 0$. Here $C_*$ is a positive constant that may depend on $\nu$, $\|\mathbf{v}_0\|$, $\|\mathbf{d}_0\|_{H^1}$, $\|\mathbf{g}\|_{L^2(0, +\infty; \mathbb{V}^\perp)}$, $\|h\|_{L^2(0, +\infty; H^2(\Gamma))}$, $\|h_t\|_{L^2(0, +\infty; \dot{H}^{-\frac{2}{3}}(\Gamma))}$, $\|h_t\|_{L^1(0, +\infty; \dot{H}^{-\frac{2}{3}}(\Gamma))}$.

**Proof.** We reconsider the estimates in the proof of Lemma 2.6. Recalling (2.29) and (2.30), thanks to the Young inequality, it is not difficult to obtain that

$$
\begin{align*}
|I_1| &\leq \|\mathbf{Sv}\|_L^2 \|\nabla \mathbf{v}\|_L^3 \leq C\|\mathbf{Sv}\|_L^2 \|\nabla \mathbf{v}\|_L^2 \leq \varepsilon \|\mathbf{Sv}\|^2 + C \|\nabla \mathbf{v}\|^6, \\
|I_2| &\leq \varepsilon \|\mathbf{Sv}\|^2 + C \|g\|^2, \\
|I_3| &\leq \varepsilon \|\mathbf{Sv}\|^2 + \varepsilon \|\nabla(\Delta \mathbf{d} - \mathbf{f}(\mathbf{d}))\|^2 + C\|\Delta \mathbf{d} - \mathbf{f}(\mathbf{d})\|^6 + C\|\Delta \mathbf{d} - \mathbf{f}(\mathbf{d})\|^2 \\
&\quad + C\|\partial_t \mathbf{d}_P\|^2 + C\|\partial_t \mathbf{d}_P\|^2 + C \|\nabla \Delta \mathbf{d}_P\|^2, \\
|I_4| &\leq \varepsilon \|\mathbf{Sv}\|^2 + \varepsilon \|\nabla(\Delta \mathbf{d} - \mathbf{f}(\mathbf{d}))\|^2 + C\|\Delta \mathbf{d} - \mathbf{f}(\mathbf{d})\|^6 + C \|\nabla \mathbf{v}\|^6 + C \|\nabla \mathbf{v}\|^2 + C \|\partial_t \mathbf{d}_P\|^2, \\
|I_{5a}| &\leq \varepsilon \|\mathbf{Sv}\|^2 + C\|\Delta \mathbf{d} - \mathbf{f}(\mathbf{d})\|^2 + C \|\nabla \mathbf{v}\|^2 + C \|\partial_t \mathbf{d}_P\|^2.
\end{align*}
$$

In addition, $I_{5b}$ can be exactly estimated as (2.31). Collecting all the estimates, and taking $\varepsilon$ to be sufficiently small, we obtain our conclusion (5.1).

Then we prove the following sufficient condition.

**Proposition 5.1.** Suppose that the assumptions of Proposition 2.7 and (2.32)–(2.34) are satisfied. In addition, assume that $(\mathbf{v}_0, \mathbf{d}_0) \in \mathbb{V} \times H^2(\Omega)$. If there exists a sufficiently small $\varepsilon_0 \in (0, 1]$ such that

$$
\int_0^{+\infty} (\nu \|\mathbf{v}(t)\|^2 + \|\Delta \mathbf{d}(t) - \mathbf{f}(\mathbf{d}(t))\|^2)dt \leq \varepsilon_0.
$$

then problem (1.1)–(1.3) admits a unique global strong solution $(\mathbf{v}, \mathbf{d})$ in $\Omega \times (0, +\infty)$, provided that $\|h_t\|_{L^2(0, +\infty; H^2(\Gamma))}$ is small enough.

**Proof.** For simplicity, we give a formal proof. To make it rigorous we should work within a proper approximation scheme (see, for instance, [14]). Let $L_i > 0$ ($i = 1, 2, 3, 4, 5$) be the constants such that

$$
\begin{align*}
\|\mathbf{v}_0\| + \|\mathbf{d}_0\|_{H^1} &\leq L_1, \\
\|h_t\|_{L^2(0, +\infty; H^2(\Gamma))} &\leq L_2, \\
\|h_t\|_{L^2(0, +\infty; H^2(\Gamma))} &\leq L_3.
\end{align*}
$$

It is easy to see that

$$
\frac{d}{dt} A_P(t) + \nu \|\mathbf{Sv}\|^2 + \|\nabla(\Delta \mathbf{d} - \mathbf{f}(\mathbf{d}))\|^2 \leq C_* (A_P^3(t) + A_P(t) + R_3(t)).
$$

where

$$
R_3(t) = \|\partial_t \mathbf{d}_P(t)\|^2 + \|\partial_t \mathbf{d}_P(t)\|^2 + \|\nabla \Delta \mathbf{d}_P(t)\|^2 + \|g(t)\|^2,
$$

for all $t \geq 0$. Here $C_*$ is a positive constant that may depend on $\nu$, $\|\mathbf{v}_0\|$, $\|\mathbf{d}_0\|_{H^1}$, $\|\mathbf{g}\|_{L^2(0, +\infty; \mathbb{V}^\perp)}$, $\|h\|_{L^2(0, +\infty; H^2(\Gamma))}$, $\|h_t\|_{L^2(0, +\infty; \dot{H}^{-\frac{2}{3}}(\Gamma))}$, $\|h_t\|_{L^1(0, +\infty; \dot{H}^{-\frac{2}{3}}(\Gamma))}$.
\[ \|h_t\|_{L^1(0, +\infty; H^{-\frac{1}{2}}(\Gamma))} \leq L_4, \]
\[ \|g\|_{L^2(0, +\infty; V^*)} \leq L_5. \]

It follows from the basic energy inequality (2.16) that
\[ \hat{\mathcal{E}}(t) + \frac{1}{2} \int_0^t \left( \nu \|\nabla v\|^2 + \|\Delta \hat{d} - f(d)\|^2 \right) dt \leq \hat{\mathcal{E}}(0) + \int_0^{+\infty} r(t)dt, \quad \forall t \geq 0. \]  

Then, by definition of \( \hat{\mathcal{E}} \) and Lemma 6.1, we have
\[ \|v(t)\| + \|d(t)\|_{H^1} \leq C_1, \quad \forall t \geq 0, \]  
\[ \int_0^{+\infty} \left( \nu \|\nabla v\|^2 + \|\Delta \hat{d} - f(d)\|^2 \right) dt \leq C_2, \]
where the constants \( C_1, C_2 \) depend on \( L_1, ..., L_5 \) and \( \Omega \).

Let \( K > 0 \) be such that
\[ \nu \|\nabla v_0\|^2 + \|\Delta \hat{d}(0) - f(d_0)\|^2 \leq K. \]  

Keeping Lemma 5.1 in mind and arguing as in [18], we consider the following Cauchy problem
\[ \frac{d}{dt} Y(t) = C_s(Y(t)^3 + Y(t)) + C_s R_3(t), \quad Y(0) = \max \left\{ 1, \nu^{-1} \right\} K \geq A_P(0). \]

We denote by \( I = [0, T_{max}) \) the (right) maximal interval for the existence of a (nonnegative) solution \( Y(t) \) so that \( \lim_{t \to T_{max}} Y(t) = +\infty \). It easily follows from (5.1) and the comparison principle that \( 0 \leq A_P(t) \leq Y(t) \), for any \( t \in I \). Consequently, \( A_P(t) \) is finite on \( I \). We deduce from Lemma 6.2 that
\[ \int_0^{+\infty} R_3(t) dt \leq C_3, \]
where \( C_3 \) is a constant depending on \( \Omega, \|g\|_{L^2(0, +\infty; H)} \) and \( L_2 \). Besides, we note that \( T_{max} \) is determined by \( Y(0), C_s \) and \( C_3 \) such that \( T_{max} = T_{max}(Y(0), C_s, C_3) \) is increasing when \( Y(0) \geq 0 \) is decreasing. Taking \( t_0 = \frac{1}{2} T_{max} > 0 \), then it follows that \( Y(t) \) (as well as \( A_P(t) \)) is uniformly bounded on \([0, t_0]\). This easily implies the local existence of a unique strong solution to problem (1.1)–(1.2) (at least) on \([0, t_0]\) (actually on \([0, T_{max})\), but we lose uniform estimates on such maximal interval).

By Lemma 6.2 (cf. (6.15)), we have
\[ \sup_{t \geq 0} \|\Delta (d_P(t) - d(t))\|^2 \leq c \|h_t\|^2_{L^2(0, +\infty; H^\frac{1}{2}(\Gamma))}, \]
where \( c \) is a constant that depends only on \( \Omega \). Set now
\[ \bar{\varepsilon}_0 = \min \left\{ 1, \frac{t_0 K}{8} \right\}, \quad L_6 = \min \left\{ 1, L_2^2, \frac{K}{4c} \right\}. \]

From the assumption, there exists a small constant \( \varepsilon_0 \leq \bar{\varepsilon}_0 \) such that (5.3) is satisfied. Therefore, we can find \( t_* \in \left[ \frac{L_6}{2}, t_0 \right] \) such that
\[ \nu \|\nabla v(t_*)\|^2 + \|\Delta \hat{d}(t_*) - f(d(t_*))\|^2 \leq 2\bar{\varepsilon}_0 t_0^{-1}. \]
where for the second inequality we have used Lemma 6.2(i) and the fact that 

\[ \nu \left\| \Delta \tilde{v}(t) - f(\tilde{d}(t)) \right\| \leq 4\varepsilon_0 t_0^{-1} + 2cL6 \leq K \]

Taking \( t_* \) as the initial time for equation (6.13), we infer from the above argument that \( A_P(t) \) is uniformly bounded at least on \( [0, \frac{2\varepsilon_0}{K}] \subset [0, t_* + t_0] \). Moreover, its bound only depends on \( \Omega, \nu, L_1, ..., L_5, C_* \) and \( t_0 \). Then by an iterative argument we can show that \( A_P(t) \) is uniformly bounded for all \( t \geq 0 \) and this enable us to extend the local strong solution to the whole time interval \( [0, +\infty) \). The proof is complete. \( \square \)

A consequence of the above proposition is the eventual regularity of global weak solutions.

**Theorem 5.1.** Suppose that the assumptions of Proposition 2.1 and (2.32), (2.34) are satisfied. Let \((v, d)\) be a global weak solution to (1.1)–(1.3). Then there exists a large time \( T^* \in (0, +\infty) \) such that \((v, d)\) is a strong solution on \((T^*, +\infty)\).

**Proof.** Let \( L_1, L_2, L_3, L_4, L_5 > 0 \) be the constants as in the proof of Proposition 5.2. For a weak solution \((v, d)\), we still have the uniform estimates (5.10) and (5.11). Considering the ODE problem (5.13), we fix the constants \( \varepsilon_0, L_6, t_0 \). Taking \( \varepsilon_0 = \varepsilon_0 \), we observe that there must exist a sufficiently large \( T_1 > 0 \) such that

\[
\int_{T_1}^{+\infty} \left( \nu \left\| \nabla v \right\|^2 + \left\| \Delta \tilde{d} - f(d) \right\|^2 \right) dt \leq \varepsilon_0, \quad (5.16)
\]

\[
\left\| \Delta d_P(t) - \Delta d_E(t) \right\| \leq L_6, \quad \forall t \geq [T_1, +\infty), \quad (5.17)
\]

where for the second inequality we have used Lemma 6.2(i) and the fact that \( \partial_t d_P(t) = \Delta d_P(t) - \Delta d_E(t) \). Also, (5.13) implies that there is \( T^* \in [T_1, T_1 + 2t_0] \) such that

\[
\nu \left\| \nabla v(T_*) \right\|^2 + \left\| \Delta \tilde{d}(T_*) - f(d(T_*)) \right\|^2 \leq \frac{\varepsilon_0}{t_0}, \quad (5.18)
\]

As a result,

\[
\nu \left\| \nabla v(T_*) \right\|^2 + \left\| \Delta \tilde{d}(T_*) - f(d(T_*)) \right\|^2 \leq \frac{2\varepsilon_0}{t_0} + 2cL6 \leq K.
\]

Taking \( T^* \) as the initial time, then we can apply Proposition 5.1 to conclude that problem (1.1)–(1.3) admits a unique global strong solution \((\tilde{v}, \tilde{d})\). By the weak/strong uniqueness result (Theorem 7), we see that \((v, d)\) coincide with \((\tilde{v}, \tilde{d})\) on \([T^*, +\infty)\). The proof is complete. \( \square \)
Thanks to the eventual regularity we can argue as in the previous section to prove

**Theorem 5.2.** Suppose that the assumptions of Theorem 5.1 hold. Then any global weak solution given by Proposition 2.1 converges in $V \times H^2(\Omega)$ to a single equilibrium $(0, d_{\infty})$ with estimates on the convergence rate similar to the 2D case, provided that $g$ and $h$ fulfill the corresponding hypotheses (H1)–(H7) as in Theorems 4.1 and 4.2.

**Remark 5.1.** We recall that there exists a (unique) global strong solution when the viscosity is large enough (cf. Theorem 2.2). Consequently, due to Lemma 2.6, all the results proven in Section 4 (i.e., Theorem 4.1 and Theorem 4.2) still hold with the same assumptions on the data. The related proofs just require some minor modifications.

The existence of a global strong solution is also ensured (with no restrictions on viscosity) when the initial data are close to a given equilibrium and the time dependent boundary data satisfies suitable bounds. First, recall that the basic energy inequality (2.16) implies (cf. (5.9))

$$\int_0^t (\nu \|\nabla v(t)\|^2 + \|\Delta \hat{d}(t) - f(d(t))\|^2)dt \leq 2(\hat{E}(0) - \hat{E}(t)) + 2 \int_0^{+\infty} r(t)dt,$$

and

$$\int_0^{+\infty} r(t)dt \leq C_r \left(\|h_t\|_{L^2(0, +\infty; H^{-1/2}(\Gamma))} + \|h_t\|_{L^1(0, +\infty; H^{-1/2}(\Gamma))} + \|g\|_{L^2(0, +\infty; V^*)}\right), \quad (5.19)$$

where $C_r$ is a universal constant. Then we can easily deduce from Proposition 5.1 that if the lifted energy stays sufficiently close to its initial state, then system (1.1)–(1.5) admits a unique global strong solution (cf. [18] for the autonomous case).

**Proposition 5.2.** Assume (2.32)–(2.34) and (2.4) hold. Moreover, suppose that $(v_0, d_0) \in V \times H^2(\Omega)$ satisfying (2.5) and $|d_0|_{\mathbb{R}^3} \leq 1$. If there exists a sufficiently small $\varepsilon_0 \in (0, 1]$ such that

$$\hat{E}(t) \geq \hat{E}(0) - \varepsilon_0, \quad \forall \ t \geq 0, \quad (5.20)$$

where $\hat{E}$ is the lifted energy defined by (2.15), then problem (1.1)–(1.5) admits a unique global strong solution $(v, d)$ in $\Omega \times (0, +\infty)$, provided that $\|h_t\|_{L^2(0, +\infty; H^{-1/2}(\Gamma))}$, $\|h_t\|_{L^1(0, +\infty; H^{-1/2}(\Gamma))}$ and $\|g\|_{L^2(0, +\infty; V^*)}$ are small enough.

We can prove the global existence of a strong solution that originates near a local minimizer of the lifted energy. For this purpose, we assume that for all $t \geq 0$ (comparing with assumptions (H1), (H4), (H5))

(H1') $\int_t^{+\infty} \|h_t(\tau)\|_{H^{-1/2}(\Gamma)}d\tau \leq M_1(1 + t)^{-1-\gamma};$

(H4') $\|g(t)\|^2 \leq M_2(1 + t)^{-2-\gamma};$

(H5') $\|h_t(t)\|_{L^2(\Gamma)} \leq M_3(1 + t)^{-1-\gamma}.$

Here $M_j, \ j = 1, 2, 3$ and $\gamma$ are positive constants. $\gamma$ characterizes the decay rate of non-autonomous terms, while $M_j$ control their magnitude.
Theorem 5.3. Assume \((2.32) - (2.34)\) and \((2.3)\) hold. Moreover, suppose that \((v_0, d_0) \in V \times H^2(\Omega)\) satisfying \((2.3)\) and \(|d_0|_{\mathbb{R}^3} \leq 1\). Denote by \(d^*_E\) the unique solution to
\[
\begin{cases}
-\Delta d^*_E = 0, & x \in \Omega, \\
d^*_E = h, & x \in \Gamma,
\end{cases}
\]
and set
\[
\mathcal{E}(d) = \frac{1}{2} \| \nabla (d - d^*_E) \|^2 + \int_{\Omega} F(d)dx, \quad \forall d \in \mathcal{N}.
\]
Let \(d^* \in \mathcal{N} \cap H^2(\Omega)\) be a local minimizer of \(\mathcal{E}(d)\) in the sense that \(\mathcal{E}(d) \geq \mathcal{E}(d^*)\) for all \(d \in \mathcal{N}\) satisfying \(\|d - d^*\|_{H^1} < \delta\). Suppose also that the initial data \(v_0\) and \(d_0\) satisfy
\[
\|v_0\| \leq 1, \quad \|d_0 - d^*\|_{H^2} \leq 1.
\]
There exist positive constants \(\sigma_1, \sigma_2, M_1, M_2, M_3, L_0\), which may depend on the system coefficients, on \(\Omega\) and on \(d^*\), such that if the initial data \((v_0, d_0)\) and \(h\) also fulfill
\[
\|v_0\| \leq \sigma_1, \quad \|d_0 - d^*\|_{H^1} \leq \sigma_2, \quad \|h\|_{L^2(0, \infty; H^2(\Gamma))} \leq L_0,
\]
and \((H1'), (H4'), (H5')\) hold with such \(M_j, \; j = 1, 2, 3, \; \text{and} \; \gamma > 1\), then problem \((1.1) - (1.5)\) admits a unique global strong solution \((v, d)\).

Proof. Without loss of generality, we assume \(\delta \in (0, 1]\). In the subsequent proof, \(C_i\) \((i \in \mathbb{N})\) stand for a positive constant which only depends on \(\Omega, \nu, \gamma\) and \(d^*\). Since \((5.22)\) holds, it is not difficult to see that the constants \(L_1\) and \(K\) in \((5.32)\) and \((5.12)\) depend on \(d^*\) only. We just take \(L_2 = L_3 = L_4 = L_5 = 1\) in \((5.6)\) for the sake of simplicity. Then we have the uniform estimate (cf. \((5.10)\))
\[
\|v(t)\| + \|\dot{d}(t)\|_{H^1} \leq C_1, \quad t \geq 0.
\]
Arguing as in the proof of Proposition 5.1 we find that problem \((1.1) - (1.5)\) admits a unique strong solution (at least) on \([0, t_0]\), whose \(V \times H^2\) norm is uniformly bounded on \([0, t_0]\):
\[
\|v(t)\|_V + \|\dot{d}(t)\|_{H^2} \leq C_3, \quad t \in [0, t_0].
\]
Besides, we can also fix the constants \(\bar{\varepsilon}_0\) and \(L_6\) (see \((5.13)\)). Here, we just take \(\varepsilon_0 = \bar{\varepsilon}_0\) and \(L_0 = L_6\). It follows from \((5.19)\) that
\[
\int_0^{+\infty} r(t)dt \leq C_r C_s (M_1 + M_2 + M_3^2) \leq \frac{\varepsilon_0}{4},
\]
provided that \(M_1, M_2, M_3 > 0\) are assumed to be properly small and satisfying
\[
M_1 + M_2 + M_3^2 \leq \frac{\varepsilon_0}{4C_r C_s},
\]
where \(C_s\) is a universal constant due to the Sobolev embedding. Hence, according to Propositions 5.1 and 5.2 in order to prove the existence of global strong solution, we only have to verify that
\[
\hat{E}(t) - \hat{E}(0) \geq -\frac{\varepsilon_0}{2}, \quad \forall t \geq 0.
\]
First, we notice that (recalling \((2.7), (2.8)\) and \((3.12)\))
\[
\hat{E}(0) - \hat{E}(t) \leq \frac{1}{2} \|v_0\|^2 + \hat{E}(d_0) - \hat{E}(d(t))
\]

\begin{equation}
\frac{1}{2} \|v_0\|^2 + \frac{1}{2} \|\nabla (d_0 - d_{E0})\|^2 - \frac{1}{2} \|\nabla (d(t) - d_E)\|^2 + \int_\Omega F(d_0) - F(d(t))dx \\
\leq \frac{1}{2} \|v_0\|^2 + C_4(\|d_0 - d(t)\|_{H^1} + \|d_{E0} - d_E\|_{H^1}).
\tag{5.25}
\end{equation}

On the other hand, thanks to standard elliptic estimates, we have
\begin{align*}
\|d_{E0} - d_E\|_{H^1} &\leq c\|d_0|_\Gamma - h(t)\|_{H^{\frac{3}{2}}(\Gamma)} \\
&\leq c\|d_0|_\Gamma - h_\infty\|_{H^{\frac{3}{2}}(\Gamma)} + c\|h_\infty - h(t)\|_{H^{\frac{3}{2}}(\Gamma)} \\
&\leq c\|d_0 - d^*\|_{H^1} + c \int_\Gamma \|h(t)\|_{H^{\frac{3}{2}}(\Gamma)} d\Gamma \\
&\leq c\sigma_2 + cM_1, \quad \forall \ t \geq 0.
\tag{5.26}
\end{align*}

Let
\[\sigma_1 \leq \min \left\{1, \frac{\sqrt{\varepsilon_0}}{2}\right\}, \quad \sigma_2 \leq \frac{\varepsilon_0}{8C_4} \min \left\{1, e^{-1}\right\}, \quad M_1 \leq \min \left\{1, \frac{\varepsilon_0}{8C_4}\right\}.
\]

Due to (5.25) and (5.26), in order to prove (5.24), we only have to verify
\begin{equation}
\|d_0 - d(t)\|_{H^1} \leq \frac{\varepsilon_0}{8C_4}, \quad \forall \ t \geq 0.
\tag{5.27}
\end{equation}

Since \(d^* \in \mathcal{N} \cap H^2(\Omega)\) is the local minimizer of \(\mathcal{E}\), it is easily to verify that \(d^*\) satisfies (5.2) and thus is the critical point of \(E\). As a consequence, Corollary 3.1 holds for \(d^*\) with constants \(\theta, \beta\) determined by \(d^*\). By (4.9), \(\varepsilon'\) can be determined by \(\theta\) and \(\gamma\). In addition, we further choose \(\varepsilon'\) smaller if necessary such that (recall that \(\gamma > 1\))
\begin{equation}
\varepsilon' \leq \frac{\gamma - 1}{2\gamma}.
\tag{5.28}
\end{equation}

Let us define
\[\varpi = \min \left\{\frac{\beta}{2}, \frac{\delta}{2}, \frac{\varepsilon_0}{10C_4}\right\},
\]
and set
\[\bar{t}_0 = \sup \{t \in [0, t_0], \|d(t) - d^*\|_{H^1} < \varpi, \forall \ s \in [0, t]\}.
\]

If we assume
\begin{equation}
\sigma_2 \leq \frac{1}{4} \varpi,
\tag{5.29}
\end{equation}
then by the continuity of \(d(t)\) in \(H^1(\Omega)\), we have \(\bar{t}_0 > 0\). Next, we shall prove that \(\bar{t}_0 > t_0\) by contradiction. We introduce the auxiliary functional
\[\Psi_1(t) = \tilde{E}(t) - \tilde{E}(d^*) + 2 \int_t^{+\infty} r(\tau)d\tau,
\]
and the function
\[\bar{d}(t) = d(t) - d_E + d_E^*.
\]

It easily follows that
\begin{align*}
\Psi_1(t) &\geq \tilde{E}(d(t)) - \tilde{E}(d^*) = \tilde{E}(d(t)) - \mathcal{E}(d(t)) + \mathcal{E}(d(t)) - \tilde{E}(d^*) \\
&= \int_\Omega F(d(t)) - F(d(t))dx + \mathcal{E}(d(t)) - \tilde{E}(d^*).
\tag{5.30}
\end{align*}
By definition, \( \bar{d}(t) \in N \). Moreover, on \([0, \bar{t}_0]\),

\[
\| \bar{d}(t) - \bar{d}' \|_{H^1} \leq \| \bar{d}(t) - d^* \|_{H^1} + \| d_E - d_E^* \|_{H^1}
\]

\[
\leq \varepsilon + c \| h(t) - h \|_{H^1(\Omega)} \leq \frac{\delta}{2} + c \int_{\tau}^{+\infty} \| h_t(\tau) \|_{H_{\frac{1}{2}}(\Omega)} \, d\tau
\]

\[
\leq \frac{\delta}{2} + cM_1.
\]

Taking

\[
M_1 = \min \left\{ 1, \frac{\delta}{4c} \right\},
\]

then we have \( \| \bar{d}(t) - d^* \|_{H^1} \leq \delta \). Since \( d^* \) is a local minimizer of \( \mathcal{E} \), we see that

\[
\mathcal{E}(\bar{d}(t)) - \bar{E}(d^*) = \mathcal{E}(\bar{d}(t)) - \mathcal{E}(d^*) \geq 0, \quad t \in [0, \bar{t}_0],
\]

On the other hand, since \( |\bar{d}(t)|_{\mathbb{R}^3} \leq 1 \) and \( |\bar{d}(t)|_{\mathbb{R}^3} \leq 3 \) (this is due to the maximum principle), we infer from the standard elliptic estimate and \((H5')\) that

\[
\left| \int_{\Omega} F(d(t)) - F(\bar{d}(t)) \, dx \right| \leq C_3 \| d_E - d_E^* \| \leq C_5 c \int_{\tau}^{+\infty} \| h_t(\tau) \|_{L^2(\Gamma)} \, d\tau
\]

\[
\leq C_5 c M_3 \eta^{-1}(1 + t)^{-\gamma}.
\]

Let us introduce now two further functions

\[
z(t) = (C_5 c + 1)M_3 \eta^{-1}(1 + t)^{-\gamma}, \quad \Psi(t) = \Psi_1(t) + z(t).
\]

We deduce from (5.31)-(5.33) that

\[
\Psi(t) \geq M_3 \eta^{-1}(1 + t)^{-\gamma} > 0, \quad t \in [0, \bar{t}_0],
\]

and by the basic energy inequality (2.16)

\[
\frac{d}{dt} \Psi(t) = \frac{d}{dt} \bar{E}(t) - 2 \bar{r}(t) - (C_5 c + 1)M_3 (1 + t)^{-1-\gamma}
\]

\[
\leq \left( - \frac{1}{4} \min \{ \nu, 1 \} D^2(t) - (C_5 c + 1)M_3 (1 + t)^{-1-\gamma} \right)
\]

\[
\leq - C_6 \left( D(t) + M_3 \eta^2 (1 + t)^{-\frac{1+\gamma}{2}} \right)^2,
\]

where \( D(t) = \| \nabla v(t) \| + \| \Delta \bar{d}(t) - f(d(t)) \| \). Arguing as to get (4.12), using Remark 3.3 and assumptions \((H1')\), \((H4')\), we deduce

\[
\Psi(t)^{1-\theta} \leq C_7 \left( D(t) + (M_1 + M_2)(1 + t)^{1-\gamma} + M_3 \eta^{-\theta}(1 + t)^{-1-\gamma} \right).
\]

Assuming

\[
M_1 \leq \frac{1}{2}M_3^{\frac{1}{2}}, \quad M_2 \leq \frac{1}{2}M_3^{\frac{1}{2}}, \quad M_3 \leq 1,
\]

we can see that

\[
\Psi(t)^{1-\theta} \leq C_7 \left( D(t) + 2M_3^{\frac{1}{2}}(1 + t)^{1-\gamma} \right).
\]
As a result, we find

\[-\frac{d}{dt}\Psi(t)^{\theta'} = -\theta'\Psi(t)^{\theta'-1}\frac{d}{dt}\Psi(t) \geq \frac{C_6}{C_7} \left( D(t) + M_3^\frac{1}{4} (1 + t)^{-\frac{1+\gamma}{4}} \right)^2 \]

which implies

\[\geq C_6 \left( D(t) + M_3^\frac{1}{4} (1 + t)^{-\frac{1+\gamma}{4}} \right), \quad (5.35)\]

where we have used the fact that \(\frac{1+\gamma}{2} \leq (1-\theta')\gamma\) (cf. (5.28)). It follows from (5.34), (5.35), assumptions (H1'), (H4'), (H5') and the definition of \(\Psi\) that

\[\int_0^t \|d_i(t)\|dt \leq C_9\Psi(0)^{\theta'} \]

\[\leq C_{10} \left( \|v_0\|^2 + \|d_0 - d^*\|_{H^1} + \|d_{E0} - d^*_E\|_{H^1} + \int_0^{+\infty} r(t)dt + z(0) \right)^{\theta'} \]

\[\leq C_{11} \left( \|v_0\|^2 + \|d_0 - d^*\|_{H^1} + M_3^\frac{1}{4} \right)^{\theta'}. \quad (5.36)\]

By (5.23), (5.36) and an interpolation inequality, we get

\[\|d(\bar{t}_0) - d^*\|_{H^1} \leq \|d(\bar{t}_0) - d_0\|_{H^1} + \|d_0 - d^*\|_{H^1} \]

\[\leq C_{12} \left( \|d(\bar{t}_0)\|_{H^2} + \|d_0\|_{H^2} \right)^\frac{1}{2} \|d(\bar{t}_0) - d_0\|^{\frac{1}{2}} + \|d_0 - d^*\|_{H^1} \]

\[\leq C_{13} \left( \|v_0\|^{\theta'} + \|d_0 - d^*\|_{H^1}^{\theta'} + M_3^{\theta'} \right) + \|d_0 - d^*\|_{H^1}. \quad (5.37)\]

Taking now

\[\sigma_1 \leq \min \left\{ 1, \frac{\sqrt{\sigma_2}}{2}, \left( \frac{\omega}{6C_{13}} \right)^\frac{1}{\theta'} \right\}, \quad \sigma_2 \leq \min \left\{ 1, \frac{1}{4} \omega, \left( \frac{\omega}{6C_{13}} \right)^\frac{1}{\theta'} \right\}, \]

\[M_3 \leq \min \left\{ 1, \left( \frac{\omega}{6C_{13}} \right)^\frac{1}{\theta'} \right\}, \]

we infer from (5.37) that

\[\|d(\bar{t}_0) - d^*\|_{H^1} \leq \frac{3}{4} \omega < \omega. \]

This leads to a contradiction with the definition of \(\bar{t}_0\). As a result, we have \(\bar{t}_0 > t_0\), and

\[\|d_0 - d(t)\|_{H^1} \leq \|d_0 - d^*\|_{H^1} + \|d^* - d(t)\|_{H^1} \leq \sigma_2 + \omega \leq \frac{5}{4} \omega \leq \frac{\varepsilon_0}{8C_4}, \quad \forall \ t \in [0, t_0]. \quad (5.38)\]

Thus, (5.24) holds on \([0, t_0]\), which implies

\[\int_0^{t_0} (\nu\|\nabla v(t)\|^2 + \|\Delta d(t) - f(d(t))\|^2)dt \leq \varepsilon_0. \]

As in Proposition 5.1 there exists \(t_\ast \in [\frac{t_0}{2}, t_0]\) such that

\[\nu\|\nabla v(t_\ast)\|^2 + \|\Delta d(t_\ast) - f(d(t_\ast))\|^2 \leq 2\varepsilon_0 t_0^{-1}, \]

31
and again we have $A_P(t_\ast) \leq \max\{1, \nu^{-1}\} K$. Taking $t_\ast$ as the initial time for the Cauchy problem (5.13), we can extend the (unique) strong solution to $[0, \frac{3}{2} t_0]$ and its $V \times H^2$-norm is uniformly bounded by the same constant $C_3$ as on $[0, t_0]$. Repeating the above argument in $[0, \frac{3}{2} t_0]$, we can verify that (5.24) still holds. By iteration we can show that (5.24) holds for all $t \geq 0$. Hence, our conclusion follows from Proposition 5.24.

Finally, we can conclude with the following local stability result:

**Theorem 5.4.** Let the assumptions of Theorem 5.3 hold. Then any global strong solution given by Theorem 5.3 converges in $V \times H^2(\Omega)$ to a single equilibrium $(0, d_\infty)$ with $d_\infty \in N \cap H^2(\Omega)$ such that $\mathcal{E}(d_\infty) = \mathcal{E}(d^*)$. In addition, convergence rate estimates similar to the 2D case hold provided that $g$ and $h$ fulfill the corresponding hypotheses (i.e., assumptions (H1), (H4), (H5) are replaced by (H1'), (H4'), (H5'), respectively). Actually, $d^*$ is (locally) Lyapunov stable, and in particular, if $d^*$ is an isolated local minimizer of $\mathcal{E}$, then it is (locally) asymptotically stable.

**Proof.** Arguing as in Section 4 we find $\lim_{t \to +\infty} (\|v(t)\|_V + \|d(t) - d_\infty\|_H^2) = 0$ for some $d_\infty \in N \cap H^2(\Omega)$. Then the estimate on the convergence rates can be obtained following the proof of Theorem 4.2. Recalling the proof of Theorem 5.3 we actually showed that, for all $t \geq 0$, $\|d(t) - d^*\|_{H^1} \leq \omega \leq \frac{1}{2} \beta$, which implies that (let $t_0$ be large)

$$\|d_\infty - d^*\|_{H^1} \leq \|d(t) - d_\infty\|_{H^1} + \|d(t) - d^*\|_{H^1} < 2 \omega.$$  

Thus, taking $d = d_\infty$ and $\psi = d^*$ in Corollary 5.1 we see that $|\mathcal{E}(d_\infty) - \mathcal{E}(d^*)|^{1-\theta} \leq \delta$ for some $\delta > 0$. Therefore, $d_\infty$ is also a minimizer of $\mathcal{E}$, because from the definition of $\omega$, $\|d_\infty - d^*\|_{H^1} \leq \delta$. Actually, the proof of Theorem 5.3 implies that $d^*$ is (locally) Lyapunov stable. Moreover, it is easy to see that if $d^*$ is an isolated local minimizer then $d_\infty = d^*$.

6 Appendix

We report some properties of the lifting functions $d_E$ and $d_P$ (cf. (2.4) and (2.24)) that have been used in the previous sections. Below we denote by $c$ a generic positive constant which depends on $n$ and $\Omega$ at most.

**Lemma 6.1.** For any $t \geq 0$, and $k = 0, 1, 2, \ldots$, $j = 0, 1$, we have

(i) $\|\partial^j t d_E(t)\|_{H^k} \leq c \|\partial^j t h(t)\|_{H^{k-\frac{j}{2}}(\Gamma)}$;

(ii) $\|d_E(t) - d_\ast\|_{H^k} \leq c \|h(t) - h_\ast\|_{H^{k-\frac{j}{2}}(\Gamma)}$, where $d_\ast$ is the unique solution to

$$\begin{cases} -\Delta d_\ast = 0, & x \in \Omega, \\
                 d_\ast = h_\ast, & x \in \Gamma. \end{cases}$$

(iii) $\|d_E(t) - d_\ast\|_{H^k} \leq c \|h(t) - h_\ast\|_{H^{k-\frac{j}{2}}(\Gamma)}$.

**Proof.** The conclusion follows from the classical elliptic regularity theory (cf., e.g., [20, 25]).

**Lemma 6.2.** Let $d_0 \in H^2(\Omega)$ with $|d_0|_{\mathbb{R}^n} \leq 1$. Suppose that $h$ satisfy (2.4)–(2.8) and $h_\tau \in L^2_{\text{loc}}([0, +\infty); H^2(\Gamma))$. Then, for any $t > 0$, the following estimates hold

$$\|d_P(t) - d_E(t)\|_{H^1}^2 \leq ce^{-t} \int_0^t e^{\tau} \|h_\tau(\tau)\|_{H^{1-\frac{j}{2}}(\Gamma)}^2 d\tau,$$

(6.2)
\[ \|\partial_t d_P(t)\|^2 + \|d_P(t) - d_E(t)\|^2 \leq c \int_0^t \|h_t(\tau)\|^2_{H^\frac{1}{2}(\Gamma)} \, d\tau, \quad (6.3) \]
\[ \int_0^t \|\nabla \Delta d_P\|^2 \, d\tau \leq c \int_0^t \|h_t(\tau)\|^2_{H^\frac{1}{2}(\Gamma)} \, d\tau. \quad (6.4) \]

In addition, we have

(i) if \( h_t \in L^2(0, +\infty; H^\frac{1}{2}(\Gamma)) \) then
\[ \lim_{t \to +\infty} \|\partial_t d_P(t)\| = 0, \quad (6.5) \]

(ii) if \( h_t \) satisfies (H6) then, for all \( t \geq 0 \),
\[ \|\partial_t d_P(t)\|^2 + \|d_P(t) - d_E(t)\|^2 \leq c(1 + t)^{-2-2\gamma}, \quad (6.6) \]
\[ \int_0^t \|\nabla \Delta d_P(\tau)\|^2 \, d\tau \leq c(1 + t)^{-1-2\gamma}. \quad (6.7) \]

**Proof.** It follows from (2.7) and (2.21) that
\[ \begin{cases} -\Delta (d_P - d_E) = -\partial_t d_P, & \text{in } \Omega \times \mathbb{R}^+, \\ d_P - d_E = 0, & \text{on } \Gamma \times \mathbb{R}^+, \end{cases} \quad (6.8) \]
and
\[ \begin{cases} \partial_t (d_P - d_E) - \Delta (d_P - d_E) = -\partial_t d_E, & \text{in } \Omega \times \mathbb{R}^+, \\ d_P - d_E = 0, & \text{on } \Gamma \times \mathbb{R}^+, \\ d_P - d_E|_{t=0} = 0, & \text{in } \Omega. \end{cases} \quad (6.9) \]

Multiplying the first equation in (6.9) by \( (d_P - d_E) - \Delta (d_P - d_E) \), integrating by parts and using the Poincaré inequality, we obtain
\[ \begin{aligned} &\frac{1}{2} \frac{d}{dt} (\|d_P - d_E\|^2 + \|\nabla (d_P - d_E)\|^2 + \|\Delta (d_P - d_E)\|^2) \\
&\leq (\|d_P - d_E\| + \|\Delta (d_P - d_E)\|) \|\partial_t d_E\| \\
&\leq (C_P \|\nabla (d_P - d_E)\| + \|\Delta (d_P - d_E)\|) \|\partial_t d_E\| \\
&\leq \frac{1}{2} \|\nabla (d_P - d_E)\|^2 + \frac{1}{2} \|\Delta (d_P - d_E)\|^2 + \left( \frac{1}{2} C_P^2 + \frac{1}{2} \right) \|\partial_t d_E\|^2, \quad (6.10) \end{aligned} \]

which, together with Lemma 6.1 implies
\[ \|d_P(t) - d_E(t)\|^2_{H^1} \leq ce^{-c_1 t} \int_0^t e^{c_1 \tau} \|\partial_t d_E(\tau)\|^2 \, d\tau \leq ce^{-c_1 t} \int_0^t e^{c_1 \tau} \|h_t(\tau)\|^2_{H^{-\frac{1}{2}}(\Gamma)} \, d\tau, \quad (6.11) \]

that is, (6.2).

Applying now the Laplacian to the first equation in (6.9), we get
\[ \begin{cases} \partial_t \Delta (d_P - d_E) - \Delta^2 (d_P - d_E) = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ \Delta (d_P - d_E) = h_t, & \text{on } \Gamma \times \mathbb{R}^+, \\ \Delta (d_P - d_E)|_{t=0} = 0, & \text{in } \Omega. \end{cases} \quad (6.12) \]

Multiplying the first equation of (6.12) by \( \Delta (d_P - d_E) \) and integrating by parts, we get
\[ \frac{1}{2} \frac{d}{dt} \|\Delta (d_P - d_E)\|^2 + \|\nabla \Delta (d_P - d_E)\|^2 \]

33
\[ \|\partial h \Delta (d_P - d_E)\|_{H^{-\frac{1}{2}}(\Gamma)} \leq c \|\Delta (d_P - d_E)\|_{H^1} \|h_t\|_{H^\frac{1}{2}(\Gamma)} \]
\[ \leq \frac{1}{2} (\|\nabla \Delta (d_P - d_E)\|_{H^2}^2 + \|\Delta (d_P - d_E)\|_{L^2}^2) + c \|h_t\|_{H^\frac{1}{2}(\Gamma)}^2. \] (6.13)

Hence, from (6.10) and (6.13) we infer
\[ \frac{d}{dt} \|d_P(t) - d_E(t)\|_{H^2}^2 + c_2 (\|d_P(t) - d_E(t)\|_{H^2}^2 + \|\nabla \Delta (d_P - d_E)\|_{L^2}^2) \leq c \|h_t\|_{H^\frac{1}{2}(\Gamma)}^2, \] (6.14)
which entails (6.4) and
\[ \|d_P(t) - d_E(t)\|_{H^2}^2 \leq c \int_0^t \|h_t(\tau)\|_{H^\frac{1}{2}(\Gamma)}^2 d\tau. \] (6.15)

Thus (6.3) follows from (6.13) and the fact \( \|\partial_t d_P(t)\| = \|\Delta d_P(t)\| \).

Now if \( h_t \in L^2(0, +\infty; H^\frac{1}{2}(\Gamma)) \), we infer from (6.10) that
\[ \int_0^{+\infty} \|\Delta (d_P(t) - d_E(t))\|_{L^2}^2 dt \leq c \int_0^{+\infty} \|\partial_t d_E(t)\|_{H^2}^2 dt \leq c \int_0^{+\infty} \|h_t(t)\|_{H^\frac{1}{2}(\Gamma)}^2 dt < +\infty. \] (6.16)
Then it follows from (6.13), (6.16) and Lemma 4.1 that \( \lim_{t \to +\infty} \|\Delta (d_P(t) - d_E(t))\|_{L^2}^2 = 0 \), which implies (6.5).

Furthermore, if (H6) holds, then (6.14) implies that (cf., e.g., [30])
\[ \|d_P(t) - d_E(t)\|_{H^2}^2 \leq c(1 + t)^{-2-2\gamma}, \quad \forall t \geq 0. \]

Using (6.14) once more, we have
\[ \int_0^t \|\nabla d_P(\tau)\|_{H^2}^2 d\tau = \int_0^t \|\nabla \Delta (d_P - d_E)(\tau)\|_{L^2}^2 d\tau \]
\[ \leq \frac{1}{2} \left( \left\| \frac{t}{2} - \frac{t}{2} \right\|_{H^2}^2 + c \int_0^t \|h_t(\tau)\|_{H^\frac{1}{2}(\Gamma)}^2 d\tau \right) \]
\[ \leq \frac{1}{2} \left( 1 + \frac{t}{2} \right)^{-2-2\gamma} + \frac{c}{1 + 2\gamma} \left( 1 + \frac{t}{2} \right)^{-1-2\gamma} \leq c(1 + t)^{-1-2\gamma}, \quad \forall t \geq 0, \]
and this gives (6.7). The proof is complete. \( \square \)

Acknowledgments. This work originated from a visit of the first author to the Fudan University whose hospitality is gratefully acknowledged. The first author has also been partially supported by the Italian MIUR-PRIN Research Project 2008 Transizioni di fase, isteresi e scale multiple. The second author was partially supported by NSF of China 11001058, SRFDP and the Fundamental Research Funds for the Central Universities.

References

[1] S. Bosia, Well-posedness and long term behavior of a simplified Ericksen–Leslie non-autonomous system for nematic liquid crystal flow, Comm. Pure Appl. Anal., 11 (2012), 407–441.

[2] R. Chill and M.A. Jendoubi, Convergence to steady states in asymptotically autonomous semilinear evolution equations, Nonlinear Anal., 53 (2003), 1017–1039.
[3] B. Climent-Ezquerra, F. Guillén-González and M.A. Rojas-Medar, Reproductivity for a nematic liquid crystal model, Z. Angew. Math. Phys., 576 (2006), 984–998.

[4] B. Climent-Ezquerra, F. Guillén-González and M. Jesus Moreno-Iraberte, Regularity and time-periodicity for a nematic liquid crystal model, Nonlinear Anal., 71 (2009), 539–549.

[5] J.L. Ericksen, Conservation laws for liquid crystals, Trans. Soc. Rheol., 5 (1961), 22–34.

[6] J. Fan and T. Ozawa, Regularity criteria for a simplified Ericksen–Leslie system modeling the flow of liquid crystals, Discrete Contin. Dyn. Syst., 25 (2009), 859–867.

[7] M. Grasselli, H. Petzeltová and G. Schimperna, Convergence to stationary solutions for a parabolic–hyperbolic phase–field system, Commun. Pure Appl. Anal., 5 (2006), 827–838.

[8] F. Guillén-González, M.A. Rodríguez-Bellido and M.A. Rojas-Medar, Sufficient conditions for regularity and uniqueness of a 3D nematic liquid crystal model, Math. Nachr., 282 (2009), 846–867.

[9] A. Haraux and M.A. Jendoubi, Decay estimates to equilibrium for some evolution equations with an analytic nonlinearity, Asymptot. Anal., 26 (2001), 21–36.

[10] S.-Z. Huang, Gradient inequalities, with applications to asymptotic behavior and stability of gradient-like systems, Mathematical Surveys and Monographs, 126, AMS, Providence, RI, 2006.

[11] S.-Z. Huang and P. Takác, Convergence in gradient-like systems which are asymptotically autonomous and analytic, Nonlinear Anal., 46 (2001), 675–698.

[12] X.-P. Hu and D.-H. Wang, Global solution to the three-dimensional incompressible flow of liquid crystals, Commun. Math. Phys., 296 (2010), 861–880.

[13] M.A. Jendoubi, A simple unified approach to some convergence theorem of L. Simon, J. Funct. Anal., 153 (1998), 187–202.

[14] J. Leray, Sur le mouvement d’un liquide visqueux emplissant l’espace, Acta Math., 63 (1934), 193–248.

[15] F.M. Leslie, Theory of flow phenomena in liquid crystals, in Advances in Liquid Crystals, 4, 1–81, Academic Press, New York, 1979.

[16] F.-H. Lin, Nonlinear theory of defects in nematic liquid crystals: Phase transitions and flow phenomena, Comm. Pure Appl. Math., 42 (1989), 789–814.

[17] F.-H. Lin, J.-Y. Lin and C.-Y. Wang, Liquid crystal flows in two dimensions, Arch. Ration. Mech. Anal., 197 (2010), 297–336.

[18] F.-H. Lin and C. Liu, Nonparabolic dissipative system modeling the flow of liquid crystals, Comm. Pure Appl. Math., 48 (1995), 501–537.

[19] F.-H. Lin and C. Liu, Partial regularities of the nonlinear dissipative systems modeling the flow of liquid crystals, Discrete Contin. Dyn. Syst., 2 (1996), 1–23.

[20] J.-L. Lions and E. Magenes, Nonhomogeneous boundary value problems and applications, 1, Springer-Verlag, New York, 1972.

[21] P. Lin and C. Liu, Simulations of singularity dynamics in liquid crystal flows: a C^0 finite element approach, J. Comput. Phys., 215 (2006), 348—362.
[22] C. Liu and J. Shen, On liquid crystal flows with free–slip boundary conditions, Discrete Contin. Dyn. Syst., 7 (2001), 307–318.

[23] C. Liu and N.J. Walkington, Approximation of liquid crystal flows, SIAM J. Numerical Analysis, 37 (2000), 725–741.

[24] C. Liu and N.J. Walkington, Mixed methods for the approximation of liquid crystal flows, Math. Model. Numer. Anal., 36 (2002), 205–222.

[25] H. Petzeltová, E. Rocca and G. Schimperna, On the long-time behavior of some mathematical models for nematic liquid crystals, Calc. Var. Partial Differential Equations, (2012), online first, DOI: 10.1007/s00526-012-0496-1.

[26] S. Shkoller, Well-posedness and global attractors for liquid crystals on Riemannian manifolds, Comm. Partial Differential Equations, 27 (2001), 1103–1137.

[27] L. Simon, Asymptotics for a class of nonlinear evolution equation with applications to geometric problems, Ann. Math. (2), 118 (1983), 525–571.

[28] M. Taylor, Partial differential equations, Vol. I, Applied Math. Sciences, 115, Springer-Verlag, New York, 1996.

[29] R. Temam, Navier–Stokes equations and nonlinear functional analysis, Second edition, CBMS-NSF Reg. Conf. Ser. Appl. Math., 66, SIAM, Philadelphia, PA, 1995.

[30] H. Wu, M. Grasselli and S. Zheng, Convergence to equilibrium for a parabolic–hyperbolic phase–field system with Neumann boundary conditions, Math. Models Methods Appl. Sci., 17 (2007), 1–29.

[31] H. Wu, Long–time behavior for nonlinear hydrodynamic system modeling the nematic liquid crystal flows, Discrete Contin. Dyn. Syst., 26 (2010), 379–396.

[32] S. Zheng, Nonlinear evolution equations, Pitman series Monographs and Survey in Pure and Applied Mathematics, 133, Chapman & Hall/CRC, Boca Raton, Florida, 2004.