On the $L_\infty$ formulation of Chern-Simons theories

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Abstract: $L_\infty$ algebras have been largely studied as algebraic frameworks in the formulation of gauge theories in which the gauge symmetries and the dynamics of the interacting theories are contained in a set of products acting on a graded vector space. On the other hand, FDAs are differential algebras that generalize Lie algebras by including higher-degree differential forms in their differential equations. In this article, we review the dual relation between FDAs and $L_\infty$ algebras. We study the formulation of standard Chern-Simons theories in terms of $L_\infty$ algebras and extend the results to FDA-based gauge theories. We focus on two cases, namely a flat (or zero-curvature) theory and a generalized Chern-Simons theory, both including high-degree differential forms as fundamental fields.

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1 Introduction

Using algebras is a way to mathematically express the symmetries of a physical system. When the physical system is described by physical theory, we have a set of symmetries that belong to the theory itself and these are usually such that they can be expressed as Lie algebras. One well-known exception to this is general relativity, which is globally invariant under diffeomorphisms. Such diffeomorphisms leave general relativity invariant but they do not constitute a Lie group. In the case of gauge theories, the gauge symmetry is usually encoded in the gauge algebra, a Lie algebra in most cases, defining a redundancy in the formulation, in general necessary for a covariant description. The information about physical interaction is contained in the equations of motion. In contrast, a formulation of gauge theories in terms of a single mathematical structure, known as \( L_\infty \) algebra, was studied on refs. [1, 2]. This means that the complete information of the theory, namely, its definition of gauge transformations (and a definition of covariant derivative through it), the gauge algebra and the dynamics, is included in an algebra that satisfies an enlarged version of the Jacobi identity, being the gauge algebra a certain subalgebra. Previous results by Barnich, Fulp, Lada and Stasheff already identify \( L_\infty \) algebras as structures that naturally appear in the study of Poisson brackets in field theory [3].

\( L_\infty \) algebras (also called strong homotopy algebras) are generalizations of Lie algebras in which the antisymmetric bilinear Lie product is replaced by a set of multilinear products satisfying a set of identities that generalize the standard Jacobi identity. In the simplest case, an \( L_\infty \) algebra is reduced to a Lie algebra. However, the presence of higher products allows the bilinear product not to satisfy the standard Jacobi identity, and therefore, a non-associative realization of the bilinear product is possible [4, 5]. In most cases, when using an \( L_\infty \) algebra to describe a gauge theory, the gauge subalgebra \( L_\infty^{\text{gauge}} \) is a Lie algebra, as expected, being the dynamics of the theory encoded into the remaining subspace. However, this is not always the case. Examples of this can be found in higher gauge theories [6]. Those are generalizations of the standard gauge theories that involve higher-degree tensors as fundamental fields, describing the dynamics of extended objects, such as string and branes, in a similar manner in which a standard gauge theory describes the dynamics of point particles. Further developments regarding the role of \( L_\infty \) algebras in string theory and supergravity can be found in refs. [7–12]. Moreover, for an extensive review on \( L_\infty \) algebras, its relation with the Batalin-Vilkovisky formalism and higher gauge theories, see refs. [13, 14]. As a consequence of including higher-degree tensors as gauge fields, a higher gauge theory can show an enlarged structure even in the gauge subalgebra when it is formulated as an \( L_\infty \) algebra. The simplest case of a higher gauge theory, known as \( p \)-form electrodynamics, was introduced in ref. [15] as a model in which the fundamental field is \( p \)-form evaluated on a Lie algebra. It was proved that the resulting theory is inconsistent in describing the parallel transport of extended objects because of its non-invariance under reparametrizations. That problem is immediately solved by removing the structure constants of the Lie algebra, making possible a gauge theory with \( p \)-forms for abelian groups.

Free differential algebras (FDAs), also known as Chevalley-Eilenberg algebras of Lie algebras, are differential algebras whose elements are dual to the Lie-valuated algebraic
elements and whose differential operator encodes the information of the Lie product. They were first introduced in the context of supergravity in ref. [16]. As happens with Lie algebras, FDAs can be gauged by considering non-vanishing field strengths (or curvatures), allowing the construction of invariant densities and giving rise to generalized Chern-Simons (CS) forms depending on higher-degree differential forms as gauge fields.

In standard gauge theory, CS (and transgression) forms are usually chosen as candidates to be Lagrangians, leading to background-free theories depending only on a one-form gauge field. Examples of these are the well-known 3D CS theories invariant theories under the Poincare and AdS Lie groups [17–19] and their generalizations to supersymmetric and higher-dimensional cases [20–23]. Moreover, CS theories that are invariant under FDAs have been studied in refs. [24–26] for particular cases in which the number of high-order gauge fields is truncated or in which the structure constants of the gauge algebra are restricted to particular cases.

As we will specify later, there is a dual relation between $L_\infty$ algebras and FDAs, being the first ones represented by a set of $n$-linear products acting on a graded vector space satisfying a generalized version of the Jacobi identity and the latter defined through a set of generalized Maurer-Cartan equations that extend those of Lie algebras. To extend a Lie algebra to a FDA is possible through the inclusion of new differential equations for higher-degree differential forms that cannot be trivially split into wedge products of one-forms (this would make the FDA equivalent to a Lie algebra) due to the presence of non-trivial cocycles, representatives of the Chevaley-Eilenberg cohomology classes of the original Lie algebra [27]. Interesting analyses on higher homotopy structures (in particular $L_\infty$ algebras) with a strong emphasis on their historical development and relation with differential algebras can be found in refs. [28] and [29].

This work aims to contribute to the study of the relation between $L_\infty$ algebras and classical field theories by providing explicit details of Zwiebach’s and Hohm’s formulation of field theories from ref. [1] for three particular cases. We start with standard CS theories and then focus in two theories based in a particular FDA, known as FDA1, which is obtained as the simplest algebraic extension of a Lie algebra by means of only one non-trivial cocycle [30–33]. The paper is organized as follows: in section 2, we briefly review $L_\infty$ algebras in the so-called $\ell$-picture and the general formulation of gauge theories in terms of $L_\infty$ algebras [1]. In section 3, we review FDAs, their gauging, the construction of transgression and CS forms from ref. [34] and the duality between FDAs and $L_\infty$ algebras. In section 4, we formulate standard CS theories in terms of $L_\infty$ algebras. In section 5, we write down the most simple FDA1-based theory in the $L_\infty$ formalism, namely, a flat theory whose dynamics is determined by the zero-curvature conditions. In section 6, we extend the results from section 4 and formulate the generalized CS theories for FDA1 in terms of $L_\infty$ algebras. There are also two appendices with details on the notation and further calculations.

2 $L_\infty$ algebras

$L_\infty$ algebras provide a framework to formulate gauge theories, containing in their mathematical structure not only the gauge symmetries but also the dynamics of the interacting
theory, being each one codified in the products of elements from specific subspaces. This section will shortly review the definition of $L_\infty$ algebras in the so-called $\ell$-picture and the procedure to write down the relevant information of an arbitrary gauge theory in terms of its products [1, 2].

2.1 Definition

An $L_\infty$ algebra is defined as a pair $(X, \{\ell_k\}_{k\in\mathbb{N}})$ where:

- $X$ is a graded vector space
  \[ X = \bigoplus_{n\in\mathbb{Z}} X_n. \] (2.1)
  Given an element $x \in X_n$ we say that $x$ has degree $n$, $\deg x = n$.

- $\{\ell_k\}_{k\in\mathbb{N}}$ is a set of $k$-linear products of degree $k - 2$ defined on $X$, i.e., given an arbitrary set of elements $x_1, \ldots, x_k \in X$
  \[ \deg \ell_k(x_1, \ldots, x_k) = k - 2 + \deg x_1 + \cdots + \deg x_k, \] (2.2)

- The products are graded symmetric
  \[ \ell_k(x_1, \ldots, x_k) = (-1)^{\sigma} \epsilon(\sigma, x) \ell_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}), \] (2.3)
  where $\epsilon(\sigma, x)$ is the Kozul sign defined by a graded symmetric product $x \wedge y = (-1)^{\deg x \deg y} y \wedge x$, which depends on the order of the elements $x_1, \ldots, x_k$ and the order of the permutation by means of the relation
  \[ x_1 \wedge \cdots \wedge x_k = \epsilon(\sigma, x) x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(k)}. \] (2.4)

- The products $\ell_k$ satisfy the so-called $L_\infty$ identities
  \[ \sum_{i+j=n+1} (-1)^{(i-1)} \sum_{\sigma \in U_n} (-1)^\sigma \epsilon(\sigma, x) \ell_j(x_{\sigma(1)}, \ldots, x_{\sigma(i)}, x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0, \] (2.5)
  with $n \geq 1$, and $U_n$ being the set of unshuffle permutations of $n$ elements, i.e., permutation whose arguments satisfy the following ordering relations
  \[ \sigma(1) < \cdots < \sigma(i), \] (2.6)
  \[ \sigma(i + 1) < \cdots < \sigma(n). \] (2.7)

2.2 $L_\infty$ formulation of gauge theories

Let us now focus on writing an arbitrary classical gauge theory in terms of an $L_\infty$ algebra (see [1]). We begin by introducing a gauge theory with fundamental field $A$ evaluated on a vector space $X_{-1}$. We also introduce gauge transformations through a gauge parameter $\varepsilon$ taking values on another vector space $X_0$. The dynamics is determined by the equation of motion $\mathcal{F} = 0$, where the off-shell functions $\mathcal{F}$ take values on a vector space $X_{-2}$. It is
possible to define an $L_\infty$ algebra on $X = X_0 \oplus X_{-1} \oplus X_{-2}$ such that the whole information of the gauge theory is codified on some non-vanishing $L_\infty$ products that satisfy the $L_\infty$ identities (2.5). It is necessary to include the information related to three aspects of the theory, namely, its definition of gauge transformations, the closed gauge algebra, and the equations of motion.

2.2.1 Gauge transformations

Given an arbitrary $L_\infty$ algebra, a set of gauge fields $A \in X_{-1}$ and a set of parameters $\varepsilon \in X_0$, the gauge variations are defined in terms of the $L_\infty$ products as follows

$$\delta_A \varepsilon = \sum_{n=0}^{\infty} \frac{(-1)^{n(n-1)}}{n!} \ell_{n+1} (\varepsilon, A^n).$$

(2.8)

There are also trivial gauge transformations, i.e., equations of motion symmetries. A particular case that will be important later and whose presence is due to the non-vanishing products of three or more elements is given by the following transformation

$$\delta_T \varepsilon_1, \varepsilon_2 A = \sum_{n=0}^{\infty} \frac{(-1)^{n(n-1)}}{n!} \ell_{n+3} (\varepsilon_1, \varepsilon_2, F, A^n).$$

(2.9)

This is a gauge transformation that vanishes on-shell and depends on two parameters. Notice that they do not appear in the case of Lie algebras because their products are bilinear.

2.2.2 Equations of motion

To define a gauge invariant action, it is necessary to introduce an inner product $\langle \cdot, \cdot \rangle_{L_\infty}$ on $X$, satisfying the following invariance conditions: given $n+1$ elements $x_0, \ldots, x_n \in X$,

$$\langle x_0, \ell_n (x_1, x_2, \ldots, x_n) \rangle_{L_\infty} = (-1)^{1+\deg x_0 \deg x_1} \langle x_1, \ell_n (x_0, x_2, \ldots, x_n) \rangle_{L_\infty},$$

(2.10)

$$\langle x_0, x_1 \rangle_{L_\infty} = (-1)^{\deg x_0 \deg x_1} \langle x_1, x_0 \rangle_{L_\infty}.$$  

(2.11)

The gauge transformations defined on eq. (2.8) leave the following action invariant

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n(n-1)}}{(n+1)!} \langle A, \ell_n (A^n) \rangle_{L_\infty}. $$

(2.12)

Taking the field variation of this action, one finds

$$\delta S = \langle \delta A, F \rangle_{L_\infty},$$

(2.13)

with

$$F = \sum_{n=1}^{\infty} \frac{(-1)^{n(n-1)}}{n!} \ell_n (A^n).$$

(2.14)

Assuming non-degeneracy of the inner product [1, 2], this leads to the equation of motion $F = 0$. 

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2.2.3 Gauge algebra

Given two parameters \( \varepsilon_1, \varepsilon_2 \in X_0 \), the commutator of its corresponding gauge transformations can be written in terms of two gauge transformations

\[
[\delta_{\varepsilon_2}, \delta_{\varepsilon_1}] A = \delta_{\varepsilon_3} A + \delta_{\varepsilon_1, \varepsilon_2} T A,
\]

(2.15)

where the parameter \( \varepsilon_3 \in X_0 \) is given by

\[
\varepsilon_3 = \sum_{n=0}^{\infty} \frac{(-1)^n n(n-1)}{2n!} \ell_{n+2} (\varepsilon_1, \varepsilon_2, A^n).
\]

(2.16)

The presence of a trivial gauge transformation on eq. (2.15) imposes some conditions on the dynamics. Starting from the definition of gauge transformations in an arbitrary theory, it is possible to find the allowed equation of motion by inspection of the algebraic element \( F \) on the trivial transformation. However, since every product on eq. (2.9) has three or more elements, this is not an issue when dealing with gauge symmetries described by bilinear products, such as in the case of Lie algebras.

From eq. (2.8) we can see that, given a gauge theory, the definition of gauge variations defines the products of the corresponding \( L_\infty \) algebra that involve elements on \( X_0 \) and \( X_-1 \).

In the same way, from eqs. (2.9) and (2.16) we can see that the gauge-algebra-relations of a theory provide the information about the \( L_\infty \) products acting on at least two elements on \( X_0 \). Finally, by inspection of the equation of motion (2.13), we can see that the dynamics of the theory is contained in the products acting on elements of \( X_-1 \). This procedure was detailed explained and used to write down the \( L_\infty \) algebras that describe Yang-Mills and three-dimensional CS theories in ref. [1].

3 Free differential algebras

The Chevalley-Eilenberg cohomology algebras of Lie algebras can be formulated in terms of extended Maurer-Cartan equations that enlarge the mathematical structure described by the standard Maurer-Cartan equations for Lie algebras, including not only the usual left-invariant one-forms but higher-degree differential forms [16]. This allows to extend the gauge principle and study gauge invariant theories with higher-degree differential forms whose symmetry is described by enlarged algebraic structures. Let us consider an arbitrary manifold \( M \) and basis of differential forms \( \{ \Theta^A(p) \} \) defined on \( \Lambda^p (M) \). Each index \( A(p) \) runs over a different domain depending on the label \( p \), which also denotes the degree of the differential form \( \Theta^A(p) \). This allows to write a set of Maurer-Cartan equations that generalizes the standard Maurer-Cartan equations of Lie algebras, defining a mathematical structure known as FDA

\[
d\Theta^A(p) + \sum_{n=1}^{N+1} \frac{1}{n} C^A_{B_1(p_1)\cdots B_n(p_n)} \Theta^{B_1(p_1)} \wedge \cdots \wedge \Theta^{B_n(p_n)} = 0.
\]

(3.1)

The coefficients \( C^A_{B_1(p_1)\cdots B_n(p_n)} \) are called generalized structure constants (natural generalizations of the usual structure constants of Lie algebras) and have a graded symmetry
in their lower indices which is ruled by the (anti)symmetric wedge product between the differential forms in eq. (3.1). The nilpotent condition $d^2 \Theta^A(p) = 0$ leads to the following Jacobi identity [35]

$$\sum_{m,n=1}^{N+1} \frac{1}{m} C_{B_1(p_1)\cdots B_n(p_n)}^{A_1(q_1)\cdots A_m(q_m)} \Theta^{C_1(q_1)} \wedge \cdots \wedge \Theta^{C_m(q_m)} \wedge \Theta^{B_2(p_2)} \wedge \cdots \wedge \Theta^{B_n(p_n)} = 0.$$  

(3.2)

### 3.1 Dual formulation

Let us now consider a dual formulation for FDAs through the introduction of a set of products acting on a dual basis. We introduce a graded vector space $\tilde{X}$, a basis set $\{T_A(p)\}_{p=1}^N$ with $\deg_{\tilde{X}} \ T_A(p) = p$ and a set of $n$-linear products acting on $\tilde{X}$ ($n \geq 1$)

$$[T_{A_1(p_1)}, \ldots, T_{A_n(p_n)}]_n \in \tilde{X}. \quad (3.3)$$

The components $[T_{A_1(p_1)}, \ldots, T_{A_n(p_n)}]_n^{A(p)}$ are choosen to be proportional to the structure constants of a FDA

$$[T_{A_1(p_1)}, \ldots, T_{A_n(p_n)}]_n^{A(p)} = (n-1)! C_{A_1(p_1)\cdots A_n(p_n)}^A, \quad (3.4)$$

and therefore, they satisfy the same graded-symmetry relation

$$[T_{A_{\sigma(1)}(p_{\sigma(1)})}, \ldots, T_{A_{\sigma(n)}(p_{\sigma(n)})}]_n = \epsilon(\sigma, T) [T_{A_1(p_1)}, \ldots, T_{A_n(p_n)}]_n, \quad (3.5)$$

which, for $n = 2$, leads to the following rule

$$[T_{A(r)}, T_{B(s)}]_2 = (-1)^{rs} [T_{B(s)}, T_{A(r)}]. \quad (3.6)$$

$\quad$

From this definition it follows that the product $[T_{A_1(p_1)}, \ldots, T_{A_n(p_n)}]_n$ lies in the subspace $\tilde{X}_p$ with $p = p_1 + \cdots + p_n - 1$, otherwise, the corresponding structure constants vanish. Since the new products are proportional to the generalized structure constants of a FDA, it is possible to write down the generalized Jacobi in terms of them

$$\sum_{m,n=1}^{N+1} \frac{1}{m! (n-1)!} \left[ [T_{C_1(q_1)}, \ldots, T_{C_m(q_m)}]_m, T_{B_2(p_2)} \cdots, T_{B_n(p_n)}]_n^{A(p)} \Theta^{C_1(q_1)} \wedge \cdots \wedge \Theta^{C_m(q_m)} \wedge \Theta^{B_2(p_2)} \wedge \cdots \wedge \Theta^{B_n(p_n)} = 0, \quad (3.7)$$

where the sum runs over the combinations of $p_i$ and $q_i$ such that $q_1 + \cdots + q_m = p_1 + 1$ and $p_1 + \cdots + p_n = p + 1$. Since each element on eq. (3.7) is a power of order $m + n - 1$ in $\Theta$, we can separate them in different equations, each one with the same power. By renaming the indices and performing the sum over unsuffles, we can sum on $m! (n-1)!$ equivalent elements. We then remove the factorial factors and introduce a Kozul sign depending on
the order of the permutation and the degrees of the elements. The equation of power 
\( l = m + n - 1 \) in \( \Theta \) becomes

\[
\sum_{m+n=l-1}^{N+1} \sum_{\sigma \in U(l)} \epsilon(\sigma,T) \left[ \left[ T_{B_{\sigma(1)}}(q_{\sigma(1)}) \ldots T_{B_{\sigma(m)}}(q_{\sigma(m)}) \right]_{m} T_{B_{\sigma(m+1)}}(q_{\sigma(m+1)}) \ldots T_{B_{\sigma(l)}}(p_{\sigma(l)}) \right]_{n} = 0,
\]

(3.8)

where we have also removed the dependence on the differential forms.

We now define an \( L_\infty \) algebra as follows. We introduce a new \( \mathbb{Z} \)-graded vector space 
\( X = \oplus_n X_n \), with a basis \( \{ T_{A(p)} \}_{p=1}^N \) such that \( \deg X T_{A(p)} = p - 1 \), endowed with the following set of products

\[
\ell_n \left( T_{A_1(p_1)}, \ldots, T_{A_n(p_n)} \right) = (-1)^{(p_1-1)(n-1)+(p_2-1)(n-2)+\cdots+(p_{n-1}-1)} \left[ T_{A_1(p_1)}, \ldots, T_{A_n(p_n)} \right]_n.
\]

(3.9)

By replacing eq. (3.9) into eqs. (3.8) and (3.5) it is direct to prove that the \( L_\infty \) products satisfy graded symmetry relations and the \( L_\infty \) identities from eqs. (2.3) and (2.5), showing that FDAs are dual to \( L_\infty \) algebras. This could be anticipated by noting that eqs. (3.8) are the \( L_\infty \) identities in the so-called \( b \)-picture and eq. (3.9) is indeed the mapping between both equivalent formulations (see ref. [1] for details on the \( b \)-picture). Here we make explicit the relation between FDAs and \( L_\infty \) algebras in the \( \ell \)-picture because it will be necessary for the following sections. However, such relation can be more easily found in the \( b \)-picture. For extensive reviews on the relation between \( L_\infty \) algebras and graded differential algebras, see refs. [4, 5, 36].

As an example, let us consider a FDA carrying a one form \( \Theta^{A(1)} \) and a two-form \( \Theta^{A(2)} \).

Such FDA is described by the following Maurer-Cartan equations

\[
0 = d\Theta^{A(1)} + \frac{1}{2} C_{B(1)C(1)}^{A(1)} B^{(1)} \wedge \Theta^{C(1)},
\]

(3.10)

\[
0 = d\Theta^{A(2)} + \frac{1}{2} \left[ C_{B(2)C(1)}^{A(2)} B^{(2)} \wedge \Theta^{C(1)} + C_{B(1)C(2)}^{A(2)} B^{(1)} \wedge \Theta^{C(2)} \right] + \frac{1}{3} C_{B(1)C(1)D(1)}^{A(2)} B^{(1)} \wedge \Theta^{C(1)} \wedge \Theta^{D(1)},
\]

(3.11)

Notice that a term proportional to \( \Theta^{A(2)} \) is allowed in eq. (3.10). In this case, we have fixed the corresponding structure constant as \( C_{B(2)}^{A(1)} = 0 \). Thus eq. (3.10) defines a Lie algebra with the antisymmetric structure constants \( C_{B(1)C(1)}^{A(1)} \). In general, algebras that share this feature in their structure constants (i.e., \( C_{B(p+1)}^{A(p)} = 0 \)) are called minimal algebras. On the other hand, from eqs. (3.4) and (3.5), the structure constants \( C_{B(2)C(1)}^{(1)} \) are symmetric in the lower indices, allowing us to sum both terms in the right-hand side of (3.11) while the 3-cocycle \( C_{B(1)C(1)D(1)}^{A(2)} \) is completely antisymmetric. We now consider the two-form in the adjoint representation of the Lie algebra. This means that both indices \( A(1) \) and \( A(2) \) take the same values. We denote \( A(1) = A(2) = A \), making necessary to rename the FDA-potentials as \( \Theta^{A(1)} = \Theta_A^1 \) and \( \Theta^{A(2)} = \Theta_A^2 \). The structure constants of the FDA can
be simply written as

\begin{align}
C^{A(1)}_{B(1)C(1)} &= \left[ T_{B(1)}, T_{C(1)} \right]_{2}^{A(1)} = C^{A}_{BC}, \\
C^{A(2)}_{B(2)C(1)} &= \left[ T_{B(2)}, T_{C(1)} \right]_{2}^{A(2)} = C^{A}_{BC}, \\
C^{A(2)}_{C(1)B(2)} &= \left[ T_{C(1)}, T_{B(2)} \right]_{2}^{A(2)} = -C^{A}_{CB}, \\
C^{A(2)}_{B(1)C(1)D(1)} &= \frac{1}{2} \left[ T_{B(1)}, T_{C(1)}, T_{D(1)} \right]_{3}^{A(2)} = C^{A}_{B1C1D1},
\end{align}

where \( C^{A}_{BC} \) are the antisymmetric structure constants of the original Lie algebra. Notice that we are using an antisymmetric tensor \( C^{A}_{BC} \) to write down the components of a symmetric tensor \( C^{A(2)}_{B(2)C(1)} \). Thus, the Maurer-Cartan equations become

\begin{align}
0 &= d\Theta^{1} + \frac{1}{2} C^{A}_{BC} \Theta^{B} \wedge \Theta^{C}, \\
0 &= d\Theta^{2} + C^{A}_{BC} \Theta^{B} \wedge \Theta^{C} + \frac{1}{3} C^{A}_{BCD} \Theta^{B} \wedge \Theta^{C} \wedge \Theta^{D}.
\end{align}

We can write this FDA as an \( L_{\infty} \) algebra in the \( \ell \)-picture. We define the graded vector space \( X = X_{0} \oplus X_{1} \) with basis \( \{ T_{A}, \tilde{T}_{A} \} \) (with \( T_{A} \in X_{0} \) and \( \tilde{T}_{A} \in X_{1} \)) equipped with the following products

\begin{align}
\ell_{2} (T_{B}, T_{C}) &= C^{A}_{BC} T_{A}, \\
\ell_{2} (T_{B}, \tilde{T}_{C}) &= -C^{A}_{BC} \tilde{T}_{A}, \\
\ell_{3} (T_{B}, T_{C}, T_{D}) &= 2 C^{A}_{BCD} \tilde{T}_{A}, \\
\text{Others} &= 0.
\end{align}

Eqs. (3.18)–(3.20) carry the information of the structure constants from eqs. (3.12)–(3.15) in \( L_{\infty} \) formalism with \( T_{A(1)} = T_{A} \) and \( T_{A(2)} = \tilde{T}_{A} \). As we have seen in eq. (3.8), those products satisfy the \( L_{\infty} \) properties. This example explicitly shows how the original FDA-product \( \left[ \Theta^{(2)}, \Theta^{(1)} \right]^{A(2)}_{B(2)C(1)} = C^{A(2)}_{B(2)C(1)} \Theta^{B(2)} \wedge \Theta^{C(1)} \) is symmetric because of the symmetry of the structure constants in the lower indices \( C^{A(2)}_{B(2)C(1)} = C^{A(2)}_{C(1)B(2)} \). However, these constants are built with the antisymmetric structure constants of a Lie algebra. Moreover, we can also note that by imposing \( C^{A}_{BCD} = 0 \), the resulting \( L_{\infty} \) algebra becomes into a trivial enlargement of the Lie algebra on eqs. (3.18) and (3.19) due to the absence of a non-trivial cocycle providing additional structure.

### 3.2 Gauged FDAs

From now on, we will focus on the particular minimal FDA whose basis of differential forms \( A = (A^{A}, A^{i}) \) consists of multiplets of one-forms and \( p \)-forms, denoted by \( A^{A} \) and \( A^{i} \) and labeled by the algebraic indices \( A(1) = A \) and \( A(p) = i \) respectively. In such algebra, known as FDA1 and extensively studied in refs. [30–33], eq. (3.1) is reduced to
two Maurer-Cartan equations:

\[ \text{d}A^A + \frac{1}{2} C^A_{BC} A^B A^C = R^A = 0, \quad (3.22) \]
\[ \text{d}A^i + C^i_{Aij} A^j A^i + \frac{1}{(p + 1)!} C^i_{A_1 \cdots A_{p+1}} A^{A_1} \cdots A^{A_{p+1}} = R^i = 0. \quad (3.23) \]

An important feature of FDA1, shown by Castellani in ref. [33], is that its dual non-associative algebra contains the R-flux algebra of closed string theory [37]. Moreover, an example of FDA1 is the algebra of eleven-dimensional supergravity, which is obtained from the super Poincaré algebra as a FDA extension by including a three-form gauge field and a non-trivial four-cocycle [38, 39]. Although such FDA1 is obtained from a superalgebra, the procedure is analogous.

Let us now introduce gauge variations in terms of the standard and generalized covariant derivatives of a set of parameters. Let \( \varepsilon^A \) and \( \varepsilon^i \) be 0-form and a \((p - 1)\)-form gauge parameters respectively. The corresponding gauge variations are defined as follows

\[ \delta A^A = \text{d}\varepsilon^A + C^A_{BC} A^B \varepsilon^C, \quad (3.24) \]
\[ \delta A^i = \text{d}\varepsilon^i + C^i_{Aij} A^j \varepsilon^i - C^i_{Aij} \varepsilon^A A^j - \frac{1}{p!} C^i_{A_1 \cdots A_{p+1}} \varepsilon^{A_1} A^{A_2} \cdots A^{A_{p+1}}. \quad (3.25) \]

Eq. (3.24) corresponds to the usual gauge transformation of a one-form and eq. (3.25) extends the definition of the covariant derivative to a \((p - 1)\)-form involving the 0-form parameter and the new structure constants.

With those transformations defined, it is possible to write down a gauge invariant theory. The definition of transgression and CS forms for FDA1 can be found by studying the corresponding Chern-Weil theorem [34, 40, 41]. Let \( A = (A^A, A^i) \) be a set of gauge fields composed by a one-form and a \(p\)-form. Let \( R = (R^A, R^i) \) be its corresponding field strength whose components are the standard and generalized non-vanishing curvatures defined by the gauging of the Maurer-Cartan equations (3.22) and (3.23). By using combinations of the components of \( R \), it is possible to define an invariant \( q \)-form as follows

\[ \chi_q(A) = \sum_{m,n} g_{A_1 \cdots A_{m+1} i_1 \cdots i_n} R^{A_1} \cdots R^{A_m} R^{i_1} \cdots R^{i_n}, \quad (3.26) \]

where the sum runs over all the possible combinations such that \( 2m + (p + 1) n = q \). In order to \( \chi_q \) be gauge invariant, the coefficients \( g_{A_1 \cdots A_{m+1} i_1 \cdots i_n} \) must satisfy the following conditions

\[ \sum_{r=1}^{m} C^C_{A_0 A_r} g_{A_1 \cdots A_r C \cdots A_{m+1} i_1 \cdots i_n} + \sum_{s=1}^{n} C^k_{A_0 i_s} g_{A_1 \cdots A_{m+1} i_1 \cdots i_k \cdots i_n} = 0, \quad (3.27) \]
\[ \sum_{r=1}^{m+1} C^i_{A_r B_1 \cdots B_p} g_{A_1 \cdots A_r \cdots A_{m+1} i_1 \cdots i_n} = 0, \quad (3.28) \]
\[ \sum_{r=1}^{m+1} C^i_{A_r j} g_{A_1 \cdots A_r \cdots A_{m+1} i_1 \cdots i_n} = 0, \quad (3.29) \]

\(^1\)From now on, we omit the wedge product between differential forms.
where indices with hat denote the absence of such indices. Eqs. (3.27)–(3.29) are the corresponding generalization of the invariant tensor conditions for Lie algebras. The generalized invariant tensor conditions also make $\chi_q$ closed.

Let us introduce a second set of gauge fields and field strengths $\bar{A} = \left( \bar{A}^A, \bar{A}^i \right)$ and $\bar{R} = \left( \bar{R}^A, \bar{R}^i \right)$. It also is possible to write down an invariant $q$-form $\chi_q \left( \bar{A} \right)$ using only the components of $\bar{R}$ as building blocks. The difference between both invariant $q$-forms is given by the exterior derivative of a $(q-1)$-form, as follows

$$\chi_q(A) - \chi_q(\bar{A}) = dQ_{q-1}(A, \bar{A}). \quad (3.30)$$

The $(q-1)$-form $Q_{q-1}(A, \bar{A})$ is called transgression form and is explicitly given by

$$Q_{q-1}(A, \bar{A}) = \sum_{m,n} g_{A^1 \cdots A^m; i_1 \cdots i_n} \int_0^1 dt \left( m u^A R_i^A \cdots R_i^{A^m} R_i^{i_1} \cdots R_i^{i_n} + n R_i^{A^m} u^A R_i^{i_1} \cdots R_i^{i_n} \right), \quad (3.31)$$

where we introduce an homotopic gauge field $A_t = \bar{A} + tu = (\bar{A}^A, \bar{A}^i)$ with $u = A - \bar{A}$, and its corresponding field-strength $R_t = (R_t^A, R_t^i)$. By locally setting $\bar{A} = 0$, eq. (3.31) becomes a definition of $(q-1)$-dimensional CS form invariant under the transformations of FDA1. Moreover, by imposing $n = 0$, the generalized invariant tensor conditions (3.22) reproduce the standard one for Lie algebras. In the same way, eqs. (3.30) and (3.31) reproduce the standard Chern-Weil theorem and transgression forms for Lie algebras.

### 3.3 Dynamics

From now on, we will use a compact notation for FDA1-valued algebraic elements, which allows to write CS and transgression form in a free-index way. Since we deal with algebraic elements carrying components in both algebraic sectors (and therefore, with both algebraic indices $A$ and $i$), being each component a differential form of different degrees, it is helpful to denote their contraction with the invariant tensors in terms of brackets. By considering an arbitrary set of FDA1-valued elements $B_1, \ldots, B_{m+n}$, we denote

$$\langle B_1, \ldots, B_m; B_{m+1}, \ldots, B_{m+n} \rangle = g_{A^1 \cdots A^m; i_1 \cdots i_n} B_1^{A^1} \cdots B_m^{A^m} B_{m+1}^{i_1} \cdots B_{m+n}^{i_n},$$

where the semicolon separate the contraction corresponding to both algebraic sectors. This allows to write the gauge invariant $q$-form from eq. (3.26) in a more compact way

$$\chi_q(A) = \sum_{m,n} \langle R^m; R^n \rangle. \quad (3.32)$$

The details of this compact notation can be found in appendix A. Let us then consider a transgression action carrying one $p$-form extension defined on a $q-1$ dimensional manifold $M_{q-1}$

$$S_T = \int_{M_{q-1}} \sum_{m,n} \int_0^1 dt \left( m \langle u; R_i^{m-1}; R_i^n \rangle + n \langle R_i^{m}; u, R_i^{n-1} \rangle \right). \quad (3.33)$$
Notice that the Lagrangian form inside of the integral is the same that in the r.h.s. of eq. (3.31) in compact notation. By taking the total variation of \( S_T \) and integrating by parts we find

\[
\delta S_T = \int_{M_{q-1}} \sum_{m,n} \int_0^1 dt \left( m \left\langle \delta u, R_{t}^{m-1}; R_{t}^{n} \right\rangle + m (m-1) \left\langle \nabla_{t} u, \delta A_{t}, R_{t}^{m-2}; R_{t}^{n} \right\rangle \right) \\
+ mn \left\langle \nabla_{t} u, R_{t}^{m-1}; R_{t}^{n-1} \right\rangle + mn \left\langle \delta A_{t}, R_{t}^{m-1}; \nabla_{t} u, R_{t}^{n-1} \right\rangle + n \left\langle R_{t}^{m}; \delta u, R_{t}^{n-1} \right\rangle \\
- (-1)^n n (n-1) \left\langle R_{t}^{m}; \nabla_{t} u, \delta A_{t}, R_{t}^{n-2} \right\rangle \right) + \text{Boundary terms.} \tag{3.34}
\]

Using the definition of homotopic gauge fields and curvatures, we find the following relations

\[
\frac{dR_{t}}{dt} = \nabla_{t} u, \quad \frac{d\delta A_{t}}{dt} = \delta u. \tag{3.35}
\]

By plugging in eqs. (3.35) into eq. (3.34), integrating by parts with respect to the parameter \( t \) and neglecting the boundary terms, the variation of the action takes the form

\[
\delta S_T = \int_{M_{q-1}} \sum_{m,n} \left( m \left\langle \delta A, R_{t}^{m-1}; R_{t}^{n} \right\rangle + n \left\langle R_{t}^{m}; \delta A, R_{t}^{n-1} \right\rangle - m \left\langle \delta \tilde{A}, R_{t}^{m-1}; \tilde{R}_{t}^{n} \right\rangle - n \left\langle \tilde{R}_{t}^{m}; \delta \tilde{A}, \tilde{R}_{t}^{n-1} \right\rangle \right) \tag{3.36}
\]

Since the variations of the gauge fields \( A \) and \( \tilde{A} \) are independent, we obtain two equations of motion:

\[
\sum_{m,n} \left( m \left\langle \delta A, R_{t}^{m-1}; R_{t}^{n} \right\rangle + n \left\langle R_{t}^{m}; \delta A, R_{t}^{n-1} \right\rangle \right) = 0, \tag{3.37}
\]

\[
\sum_{m,n} \left( m \left\langle \delta \tilde{A}, \tilde{R}_{t}^{m-1}; \tilde{R}_{t}^{n} \right\rangle + n \left\langle \tilde{R}_{t}^{m}; \delta \tilde{A}, \tilde{R}_{t}^{n-1} \right\rangle \right) = 0. \tag{3.38}
\]

Those equations are reduced equations of motion for a standard transgression action if we impose \( n = 0 \). Moreover, by locally setting \( \tilde{A}^A = 0 \) and \( \tilde{A}^i = 0 \) we obtain the equations of motion for extended CS theory (or FDA1-CS theory) that can be separated again on its independent variations with respect to the one-form and \( p \)-form as follows

\[
\delta A^A : \sum_{m,n} m g_{A_{1}A_{2}...A_{m}} R^{A_{2}...A_{m}R^{A_{1}}...R^{n}} = 0, \tag{3.39}
\]

\[
\delta A^i : \sum_{m,n} n g_{A_{1}A_{1}...A_{n}} R^{A_{1}...A_{n}R^{i_{2}}...R^{n}} = 0. \tag{3.40}
\]

4 \( L_{\infty} \) formulation of CS theory

In this section we will follow the procedure introduced on ref. [1] to find the \( L_{\infty} \) structure of standard \( 2m-1 \) dimensional CS theory, whose action can be found from eq. (3.33) by setting \( \tilde{A} = 0 \) in absence of \( p \)-form gauge fields, i.e., \(^2\)

\[
S_{CS} = m \int_{M_{2m-1}} \int_0^1 dt \left\langle A, R_{t}^{m-1} \right\rangle_{\text{Lie}}, \tag{4.1}
\]

\(^2\)We include a label to denote the invariant tensor of the Lie algebra and distinguish it from the inner product of \( L_{\infty} \) algebras. We will later introduce a FDA1 invariant tensor that can be identified by a semicolon separating the algebraic sectors.
with \( A_t = tA \) and \( R_t = (t^2 - t) R \). By comparing with the general formulation of gauge theories in terms of \( L_\infty \) algebras, we will extract the relevant information of the theory contained in the gauge transformations, their closed gauge algebra, and the equations of motion and will write it in terms of algebraic products.

### 4.1 Gauge transformations

In standard CS theory, the fundamental field is the one-form gauge field \( A^A_\mu \) whose gauge variation is given by the Lie-covariant derivative of a 0-form gauge parameter \( \varepsilon^A \)

\[
\delta A^A_\mu = \partial_\mu \varepsilon^A + C^A_{BC} A^B_\mu \varepsilon^C. \tag{4.2}
\]

We identify the parameters \( \varepsilon^A \in X_0 \). In the \( L_\infty \) formulation of gauge theories, the gauge variation of \( A^A_\mu \) can be written in terms of the \( L_\infty \) products according to eq. (2.8). By direct inspection of eq. (4.2) we can remove every term in eq. (2.8) except by the ones that are powers of degree zero and one in the gauge field. This can be written in components as

\[
\delta A^A_\mu = [\ell_1 (\varepsilon)]^A_\mu + [\ell_2 (\varepsilon, A)]^A_\mu, \tag{4.3}
\]

which leads to the following information about the \( L_\infty \) products:

\[
[\ell_1 (\varepsilon)]^A_\mu = \partial_\mu \varepsilon^A, \tag{4.4}
\]

\[
[\ell_2 (\varepsilon, A)]^A_\mu = [A_\mu, \varepsilon]^A. \tag{4.5}
\]

Any other product involving one element on \( X_0 \) and elements from \( X_{-1} \) vanishes.

### 4.2 Gauge algebra

As second step, we need to ensure the closure of the gauge subalgebra \( L_\infty^{\text{gauge}} \). The commutator between two gauge transformations can be written in terms of a third gauge transformation without introducing on-shell symmetries

\[
(\delta_2 \delta_1 - \delta_1 \delta_2) A^A_\mu = \partial_\mu \varepsilon^A + C^A_{DB} A^B_\mu \varepsilon^C. \tag{4.6}
\]

The components of the new gauge parameter \( \varepsilon_3 \) are given by the Lie product between the original parameters, i.e., \( \varepsilon^A_3 = C^A_{BC} \varepsilon^B \varepsilon^C \). On the other hand, from eqs. (2.15) and (2.16) we get

\[
(\delta_2 \delta_1 - \delta_1 \delta_2) A^A_\mu = [\ell_1 (\ell_2 (\varepsilon_1, \varepsilon_2))]^A_\mu + [\ell_1 (\ell_3 (\varepsilon_1, \varepsilon_2, A))]^A_\mu + [\ell_2 (\ell_2 (\varepsilon_1, \varepsilon_2, A))]^A_\mu. \tag{4.7}
\]

We can find other products that, for consistency, must be non-vanishing. Since \( \ell_2 (\varepsilon_1, \varepsilon_2), \ell_3 (\varepsilon_1, \varepsilon_2, A) \in X_0 \) we get

\[
[\ell_1 (\ell_2 (\varepsilon_1, \varepsilon_2))]^A_\mu = \partial_\mu [\ell_2 (\varepsilon_1, \varepsilon_2)]^A, \tag{4.8}
\]

\[
[\ell_1 (\ell_3 (\varepsilon_1, \varepsilon_2, A))]^A_\mu = \partial_\mu [\ell_3 (\varepsilon_1, \varepsilon_2, A)]^A. \tag{4.9}
\]

and therefore, by comparing eqs. (4.6) and (4.7), we obtain the following products between elements in \( X_0 \)

\[
[\ell_2 (\varepsilon_1, \varepsilon_2)]^A = C^A_{BC} \varepsilon^B \varepsilon^C, \quad [\ell_3 (\varepsilon_1, \varepsilon_2, A)]^A = 0. \tag{4.10}
\]
In summary, at this point, the information is codified into the following $L_\infty$ products

$$[\ell_1 (\varepsilon)]^A_\mu = \partial_\mu \varepsilon^A, \quad [\ell_2 (\varepsilon_1, \varepsilon_2)]^A = [\varepsilon_2, \varepsilon_1]^A.$$

Gauge transformations

Any other product involving elements on $X_0$ and $X_{-1}$ vanishes. Those products define an $L_\infty$ algebra and describe a consistent gauge theory if we include the information concerning the dynamics.

### 4.3 Equations of motion

Starting from the $2m - 1$ dimensional CS action (4.1), one finds the following variation

$$\delta S_{CS} = \int_{M_{2m-1}} \left\langle \delta A, R^{m-1} \right\rangle_{\text{Lie}}. \quad (4.12)$$

By expanding the power of the curvature, it is possible to write the variation of $S_{CS}$ as

$$\delta S_{CS} = \int_{M_{2m-1}} \frac{1}{2k} \left( m - 1 \right) \left( m - k \right) \left\langle \delta A, (dA)^{m-k-1}, \frac{1}{2k} [A, A]^k \right\rangle_{\text{Lie}}, \quad (4.13)$$

or, explicitly writing algebraic indices and component of the differential forms:

$$\delta S_{CS} = \int dx^{2m-1} \frac{1}{2k} \left( m - 1 \right) \left( m - k \right) \varepsilon_{\mu_1 \cdots \mu_{2m-1}} g_{A_{B_1 \cdots B_{m-1}}} [\delta A_{\mu_1}, (dA)^{m-k-1}]_{\text{Lie}} [A, A]^k_{\text{Lie}}, \quad (4.14)$$

where $\varepsilon_{\mu_1 \cdots \mu_{2m-1}}$ is the Levi-Civita pseudotensor and $g_{A_{B_1 \cdots B_m}}$ are the components of the invariant tensors of the Lie algebra. Note that the invariant tensor is given by the trace over the Lie algebra’s basis and it can be understood as a multilinear product. On the other hand, the inner product of the $L_\infty$ algebra is bilinear. By comparing eqs. (2.13) and (4.14) we can identify the inner product of the $L_\infty$ algebra in terms of the invariant tensor of the Lie algebra. Given two algebraic elements $x \in X_{-1}$ and $y \in X_{-2}$ evaluated on the Lie algebra, we identify

$$\langle x, y \rangle_{L_\infty} = \int dx^{2m-1} \eta^{\mu \nu} \langle x_\mu, y_\nu \rangle_{\text{Lie}}. \quad (4.15)$$

Then, the variation of the action can be written in terms of $F$ as follows:

$$\delta S_{CS} = \left\langle \delta A, F \right\rangle_{L_\infty}$$

$$= \int dx^{2m-1} \eta^{\mu \nu} \langle \delta A_\mu, F_\nu \rangle_{\text{Lie}}. \quad (4.16)$$

In this case, the algebraic index is the Lie algebra index. Therefore we can write

$$\langle \delta A, F \rangle_{L_\infty} = \int dx^{2m-1} \eta^{\mu \nu} g_{A_{B_1 \cdots B_m}} A^A F^B, \quad (4.17)$$
where \( g_{AB} \) is the Cartan-Killing metric of the Lie algebra. By comparing eqs. (4.14) and (4.17) we obtain an explicit expression for \( F \), namely

\[
F^A_{\nu} = \sum_{k=0}^{m-1} \frac{1}{2^k} \binom{m-1}{k} \varepsilon^\nu_{\mu_1\ldots\mu_{2m-2}} g^{A}_{B_1\ldots B_{m-1}} \partial_{\mu_1} A_{\mu_2} \ldots \partial_{\mu_{2m-2k-3}} A_{\mu_{2m-2k-2}} B_{m-k}^{m-1} \times [A_{\mu_{2m-2k-1}}, A_{\mu_{2m-2k}}] B_{m-k} \ldots [A_{\mu_{2m-3}}, A_{\mu_{2m-2}}] B_{m-1}.
\]

(4.18)

On the other hand, the algebraic element \( F \), given in general by eq. (2.14), can be written for the case as

\[
F^A = \sum_{l=1}^{\infty} \frac{(-1)^{\frac{(l-1)}{2}}}{l!} \left[ \ell_1 \left( A' \right) \right]_l^A.
\]

(4.19)

We can now extract the corresponding information about the \( L_\infty \) products. From eqs. (4.18) and (4.19) we have

\[
\sum_{l=1}^{\infty} \frac{(-1)^{\frac{(l-1)}{2}}}{l!} \left[ \ell_1 \left( A' \right) \right]_l^A = \sum_{k=0}^{m-1} \frac{1}{2^k} \binom{m-1}{k} \varepsilon^\nu_{\mu_1\ldots\mu_{2m-2}} g^{A}_{B_1\ldots B_{m-1}} \partial_{\mu_1} A_{\mu_2} \ldots \partial_{\mu_{2m-2k-3}} A_{\mu_{2m-2k-2}} B_{m-k}^{m-1} \times [A_{\mu_{2m-2k-1}}, A_{\mu_{2m-2k}}] B_{m-k} \ldots [A_{\mu_{2m-3}}, A_{\mu_{2m-2}}] B_{m-1}.
\]

(4.20)

We compare then the terms of equal powers of \( A \). The \( k \)-th element in the sum of the right-hand-side has degree \( m+k-1 \) in \( A \). There is only one element of such degree. We can match one element on the left-hand side with one element on the right side. Therefore, given a fixed value of \( m \), we have many values for \( k (k = 0, \ldots, m-1) \), and given a value for \( k \), \( l \) is completely determined by

\[
l = m + k - 1.
\]

(4.21)

In other words, for a fixed value of \( m \), we have a \( 2m-1 \) dimensional gauge theory whose dynamics is described by a set of non-vanishing products \( \ell_j \) with \( l = m-1, \ldots, 2m-2 \) acting on \( X_{-1} \). In general, the \( l \)-th product is given by

\[
\left[ \ell_1 \left( A' \right) \right]_l^A = (-1)^{\frac{(l-1)}{2}} \frac{1}{2^{l-m+1}} \frac{l! (m-1)!}{(2m-l-2)! (l-m-1)!} \varepsilon^\nu_{\mu_1\ldots\mu_{2m-2}} g^{A}_{B_1\ldots B_{m-1}} \partial_{\mu_1} A_{\mu_2} \ldots \partial_{\mu_{4l-2l-5}} A_{\mu_{4l-2l-4}} B_{2m-l}^{m-1} \times [A_{\mu_{4l-2l-3}}, A_{\mu_{4l-2l-2}}] B_{2m-l-1} \ldots [A_{\mu_{2m-3}}, A_{\mu_{2m-2}}] B_{m-1}.
\]

(4.22)

In summary, we can write the \( 2m+1 \) dimensional CS theory (for convenience, we change \( m \rightarrow m+1 \)) as an \( L_\infty \) algebra defined by a vector space \( X_0 \oplus X_{-1} \oplus X_{-2} \) endowed with the following products

\[
[\ell_1 (\varepsilon)]_\mu^A = \partial_\mu \varepsilon^A,
\]

(4.23)

\[
[\ell_2 (\varepsilon, A)]_\mu^A = [A_\mu, \varepsilon]^A,
\]

(4.24)

\[
[\ell_2 (\varepsilon_1, \varepsilon_2)]_\mu^A = [\varepsilon_2, \varepsilon_1]^A,
\]

(4.25)

\[
[\ell_2 (\varepsilon, E)]_\mu^A = [E_\mu, \varepsilon]^A,
\]

(4.26)
\[ [\ell_l(A_1,\ldots,A_l)]^A_\mu = \frac{1}{2^{l-m}} \frac{(-1)^{N(l-1)}}{(l-m)!} \varepsilon_\mu^{\mu_1\cdots\mu_2m} g^A_{B_1\cdots B_{m-l}} \partial_{\mu_1} (A_{(1)})^B_{\mu_2} \cdots \partial_{\mu_{l-1}} (A_{2m-l})^B_{\mu_{2m-l}} \times [(A_{2m-l+1})_{\mu_{2m-2l+1}} g^{A_1}_{B_1\cdots B_{m-l}} \partial_{\varepsilon} (A_{(1)})^B_{\mu_2} \cdots \partial_{\mu_{l-1}} (A_{2m-l})^B_{\mu_{2m-l}} \varepsilon_A] ]_{m}, \tag{4.27} \]

with \( \varepsilon, \varepsilon_1, \varepsilon_2 \in X_0, A, A_1, \ldots, A_l \in X_{-1}, E \in X_{-2} \) and \( l = m, \ldots, 2m \). Note that eq. (4.27) is obtained directly from eq. (4.22) by considering arbitrary elements on \( X_{-1} \) instead of the same gauge field \( l \) times. The corresponding normalized symmetrization denoted with braces is included in order to ensure that the symmetry rule of the \( L_\infty \) products holds (this is \( A_1 \cdots A_l = \frac{1}{l!} \sum_{\sigma \in S_l} A_{\sigma(1)} \cdots A_{\sigma(l)} \), with \( S_l \) the group of permutations of \( l \) elements). Moreover, the product involving elements on space \( X_{-2} \) in eq. (4.27) is obtained by consistency with the \( L_\infty \) identities (2.5). The calculation can be found in appendix B.1.

As an example, let us consider the three-dimensional CS theory. This case is obtained by setting \( m = 1 \). In such case, there are only two products \( \ell_l \) \((l = 1, 2)\) in the dynamical sector

\[ [\ell_1 (A_1)]^A_\mu = \varepsilon^{\mu_1\mu_2} \partial_{\mu_1} (A_1)^A_{\mu_2}, \tag{4.28} \]
\[ [\ell_2 (A_1, A_2)]^A_\mu = -\varepsilon^{\mu_1\mu_2} [(A_1)^{A_1}_{\mu_1}, (A_2)^{A_2}_{\mu_2}]^A. \tag{4.29} \]

The corresponding \( L_\infty \) algebra is given by eqs. (4.23)–(4.26) together with eqs. (4.28) and (4.29). This reproduces the algebraic formulation of 3D CS theory from ref. [1].

\section{5 \( L_\infty \) formulation of FDA1-based theories}

From now on, we will focus on gauge theories whose symmetries are not described by Lie algebras but FDAs. The procedure to obtain the \( L_\infty \) formulation of extended gauge theory is analogous to the one shown in the previous case. However, the gauge subalgebra \( L_\infty^{\text{gauge}} \) will no longer be a Lie algebra and it cannot be trivially reduced into one by writing the higher-degree forms in terms of one-forms as long as the cocycle with which the algebra is defined is non-trivial. As we have seen, the gauge transformations are given in eqs. (3.24) and (3.25) by the standard and extended versions covariant derivatives of a 0-form and a \((p-1)\)-form gauge parameters. Since the gauge variation of the one-form \( A^A \) is the same as with Lie algebras, the \( L_\infty \) products obtained from its definition and the corresponding gauge algebra are the same as in the previous section (see eq. (4.11)). Thus, we will focus on obtaining the products corresponding to the extended variations. As before, we identify the gauge parameters and fields with the spaces \( X_0 \) and \( X_{-1} \) respectively

\[ \varepsilon = (\varepsilon^A, \varepsilon^i) \in X_0, \tag{5.1} \]
\[ A = (A^A, A^i) \in X_{-1}. \tag{5.2} \]

\subsection{5.1 Gauge transformations}

The algebraic elements \( \delta A \) carry both algebraic indices, each one being a one-form and a \( p \)-form respectively. Therefore, the gauge variation must be separated into its components
\[ \delta A^\mu_i = \delta A_{\mu_1 \cdots \mu_p}^i \] and \[ \delta A_{\mu_1 \cdots \mu_p}^i. \] From eq. (3.25) we can see that the expression for \[ \delta A_{\mu_1 \cdots \mu_p}^i \] in terms of \( L_\infty \) products is truncated, resulting only in those terms that are powers of degree zero, one and \( p \) in the gauge fields

\[
\delta A_{\mu_1 \cdots \mu_p}^i = \left[ \ell_1 (\varepsilon) \right]_{\mu_1 \cdots \mu_p}^i + \left[ \ell_2 (\varepsilon, A) \right]_{\mu_1 \cdots \mu_p}^i + \frac{(-1)^{p(p-1)}}{p!} \left[ \ell_{p+1} (\varepsilon, A, \ldots, A) \right]_{\mu_1 \cdots \mu_p}^i. \tag{5.3}
\]

Here we must point out an important difference with the previous section. Each algebraic element on \( X \) carries two differential forms of different degrees. Since the products between elements on \( X \) are also elements on \( X \), they can be decomposed into its \( A \)- and \( i \)-components, being those also differential forms of different degrees. This is why some \( L_\infty \) products carry a different number of coordinate indices depending on which algebraic sector and subspace of \( X \) lie. For simplicity, we introduce differential form products as follows

\[
[\ell_r (x_1, \ldots, x_r)] = \frac{1}{s!} [\ell_r (x_1, \ldots, x_r)]_{\mu_1 \cdots \mu_s} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_s}, \tag{5.4}
\]

with \( x_1, \ldots, x_n \in X \) and \( \ell_r (x_1, \ldots, x_r) \) some non-vanishing \( r \)-linear product carrying \( s \) antisymmetric space-time indices. This allows to write the product between a large number of elements without overloading of indices. One might then think that this is an \( L_\infty \) product between differential forms. However, this is not the case; the product is still between algebraic elements that carry differential forms of different degrees and the result is being written in terms of a basis for differential forms. From now on, we will write down the \( L_\infty \) products in terms of differential forms. In the case of the gauge variation of \( A \) in the extended sector of the algebra, this becomes

\[
\delta A^i = \left[ \ell_1 (\varepsilon) \right]^i + \left[ \ell_2 (\varepsilon, A) \right]^i + \frac{(-1)^{p(p-1)}}{p!} \left[ \ell_{p+1} (\varepsilon, A, \ldots, A) \right]^i, \tag{5.5}
\]

which leads to the following information about the \( L_\infty \) products

\[
[\ell_1 (\varepsilon)]^i = d\varepsilon^i, \tag{5.6}
\]
\[
[\ell_2 (\varepsilon, A)]^i = [A, \varepsilon]^i - [\varepsilon, A]^i, \tag{5.7}
\]
\[
[\ell_{p+1} (\varepsilon, A^p)]^i = (-1)^{1+\frac{p(p-1)}{2}} [\varepsilon, A^p]^i. \tag{5.8}
\]

Every other product involving one element on \( X_0 \) and elements from \( X_{-1} \) vanishes. The next step is to obtain the information concerning the gauge algebra.

### 5.2 Gauge algebra

As second step, we need to ensure the closure of the commutator of two gauge transformations. By applying two consecutive transformations with parameters \( \varepsilon_1 = (\varepsilon_1^A, \varepsilon_1^i) \) and
\[ \varepsilon_2 = (\varepsilon_2^A, \varepsilon_2^i) \] and taking the \( i \)-component, we find

\[
(\delta_2 \delta_1 - \delta_1 \delta_2) A^i = [d \varepsilon_2, \varepsilon_1] - [\varepsilon_1, d \varepsilon_2] + [\varepsilon_1, \varepsilon_2] + d [\varepsilon_1, \varepsilon_2] + [\varepsilon_2, d \varepsilon_1] + [\varepsilon_1, \varepsilon_2] + [\varepsilon_2, [A, \varepsilon_1]] - [\varepsilon_1, [A, \varepsilon_2]] + \frac{1}{p!} \varepsilon_2^1 \varepsilon_1^i + \frac{1}{(p-1)!} [\varepsilon_1, \varepsilon_2, A^p-1] + \frac{1}{(p-1)!} [\varepsilon_2, [A, \varepsilon_1], A^p-1] i. \quad (5.9)
\]

On the other hand, from the generalized Jacobi identity, it is possible to prove the following relations

\[
[\varepsilon_1, [\varepsilon_2, A]] = [\varepsilon_2, [\varepsilon_1, A]] = [\varepsilon_1, \varepsilon_2, A] = 0, \quad (5.10)
\]

\[
[\varepsilon_1, [A, \varepsilon_2]] + [[A, \varepsilon_1], \varepsilon_2] - [A, [\varepsilon_1, \varepsilon_2]] = 0. \quad (5.11)
\]

Using eqs. (5.10) and (5.11), the relation \( dA^A = R^A - \frac{1}{2} [A, A]^A \) and integrating by parts, eq. (5.9) becomes

\[
(\delta_2 \delta_1 - \delta_1 \delta_2) A^i = \delta_3 A^i - \frac{1}{(p-2)!} [\varepsilon_2, \varepsilon_1, R, A^{p-2}] i, \quad (5.12)
\]

where we introduce a third composite parameter \( \varepsilon_3 = (\varepsilon_3^A, \varepsilon_3^i) \) whose components depends on the original parameters and gauge fields as follows

\[
\varepsilon_3^A = [\varepsilon_2, \varepsilon_1]^A, \quad (5.13)
\]

\[
\varepsilon_3^i = [\varepsilon_2, \varepsilon_1]^i - [\varepsilon_1, \varepsilon_2]^i + \frac{1}{(p-1)!} [\varepsilon_2, \varepsilon_1, A^{p-1}] i. \quad (5.14)
\]

Let us consider again two gauge transformations including both parameters \( \varepsilon^A \) and \( \varepsilon^i \). In the \( \ell \)-picture, the commutator between two gauge transformations is given by eq. (2.15) as the sum of a trivial and non-trivial transformation. In order to extract the relevant information, we compare eqs. (2.15) and (5.12) to truncate the expansion in terms of \( L_\infty \) products. Then we can write \( \delta_\varepsilon A \) in terms of a small set of products and split the sum in a more convenient way, namely

\[
\delta_\varepsilon A = \ell_1 (\varepsilon_3) + \ell_2 (\varepsilon_3, A) + \frac{1}{p!} (-1)^{\frac{p(p-1)}{2}} \ell_{p+1} (\varepsilon_3, A^p)
\]

\[
= [\delta_\varepsilon A]_0 + [\delta_\varepsilon A]_1 + [\delta_\varepsilon A]_{p-1} + [\delta_\varepsilon A]_p, \quad (5.15)
\]

where we denote \([\delta_\varepsilon A]_k\) to the sum of terms on \( \delta_\varepsilon A \) with power \( k \) in \( A \). By plugging in
eq. (5.13) and (5.14) into eq. (5.15) we obtain an explicit expression for each term:

\[
\begin{align*}
[\delta_{\epsilon_3} A]_0 &= \ell_1 (\ell_2 (\epsilon_1, \epsilon_2)), \\
[\delta_{\epsilon_3} A]_1 &= \ell_1 (\ell_3 (\epsilon_1, \epsilon_2, A)) + \ell_2 (\ell_2 (\epsilon_1, \epsilon_2), A), \\
[\delta_{\epsilon_3} A]_{p-1} &= \left(\frac{(-1)^{(p-1)(p-2)}}{(p-1)!}\right) \ell_1 (\ell_{p+1} (\epsilon_1, \epsilon_2, A^{p-1})) \\
&\quad + \left(\frac{(-1)^{(p-2)(n-3)}}{(p-2)!}\right) \ell_2 (\ell_p (\epsilon_1, \epsilon_2, A^{p-2}), A), \\
[\delta_{\epsilon_3} A]_p &= \left(\frac{(-1)^{(p-1)(p-2)}}{p!}\right) \ell_1 (\ell_{p+2} (\epsilon_1, \epsilon_2, A^p)) + \left(\frac{(-1)^{(p-1)(p-2)}}{(p-1)!}\right) \ell_2 (\ell_{p+1} (\epsilon_1, \epsilon_2, A^{p-1}), A) \\
&\quad + \left(\frac{(-1)^{(p-1)(p-2)}}{p}\right) \ell_{p+1} (\ell_2 (\epsilon_1, \epsilon_2), A^p).
\end{align*}
\]  

From eqs. (5.13) and (5.14) it follows that the variation with respect to \(\epsilon_3\) can be written in terms of \(\epsilon_1\) and \(\epsilon_2\) as

\[
\delta_3 A^i = d \left\{ [\epsilon_2, \epsilon_1]^i - [\epsilon_1, \epsilon_2]^i + \frac{1}{(p-1)!} [\epsilon_2, \epsilon_1, A^{p-1}]^i \right\} + [A, [\epsilon_2, \epsilon_1] - [\epsilon_1, \epsilon_2] + \frac{1}{(p-1)!} [\epsilon_2, \epsilon_1, A^{p-1}]]^i - [[\epsilon_2, \epsilon_1], A]^i - \frac{1}{p!} [[\epsilon_2, \epsilon_1], A^p]^i.
\]  

We now compare eq. (5.20) with eqs. (5.16)–(5.19) to obtain four relations.

**First relation (power 0 in A).**

\[
[\ell_1 (\ell_2 (\epsilon_1, \epsilon_2))]^i = d \left\{ [\epsilon_2, \epsilon_1]^i - [\epsilon_1, \epsilon_2]^i \right\}.
\]  

**Second relation (power 1 in A).**

\[
[\ell_1 (\ell_3 (\epsilon_1, \epsilon_2, A)) + \ell_2 (\ell_2 (\epsilon_1, \epsilon_2), A)]^i = [A, ([\epsilon_2, \epsilon_1] - [\epsilon_1, \epsilon_2])^i - [[\epsilon_2, \epsilon_1], A]^i].
\]  

**Third relation (power \(p-1\) in A).**

\[
(-1)^{(p-1)(p-2)} \left[ \ell_1 (\ell_{p+1} (\epsilon_1, \epsilon_2, A^{p-1})) \right]^i + (-1)^{(p-2)(n-3)} (p-1)! \left[ \ell_2 (\ell_p (\epsilon_1, \epsilon_2, A^{p-2}), A) \right]^i = d [\epsilon_2, \epsilon_1, A^{p-1}]^i.
\]  

**Fourth relation (power \(p\) in A).**

\[
[\ell_1 (\ell_{p+2} (\epsilon_1, \epsilon_2, A^p))]^i + p \left[ \ell_2 (\ell_{p+1} (\epsilon_1, \epsilon_2, A^{p-1}), A) \right]^i + [\ell_{p+1} (\ell_2 (\epsilon_1, \epsilon_2), A^p)]^i = (-1)^{(p-1)(p-2)} \left[ A, [\epsilon_2, \epsilon_1, A^{p-1}] \right]^i - (-1)^{(p-1)(p-2)} [[\epsilon_2, \epsilon_1], A^p]^i.
\]  

The products \(\ell_2 (\epsilon_1, \epsilon_2), \ell_3 (\epsilon_1, \epsilon_2, A), \ell_p (\epsilon_1, \epsilon_2, A^{p-2}), \ell_{p+1} (\epsilon_1, \epsilon_2, A^{p-1})\) and \(\ell_{p+2} (\epsilon_1, \epsilon_2, A^p)\) lie in \(X_0\), and we can therefore use the information obtained from the
definition of gauge transformations contained in eqs. (5.6)–(5.7) into eqs. (5.21)–(5.24) to obtain explicit expressions for them. Moreover, we have to compare the trivial transformation in the right-hand side from eq. (2.15) with the last term on eq. (5.12). This allows to write a product involving an element \( R \in X_{-2} \).

\[
(-1)^{(p-2)(p-3)/2} \ell_{p+1} \left( \varepsilon_1, \varepsilon_2, R, A^{p-2} \right) = -\frac{1}{(p-2)!} \left[ \varepsilon_2, \varepsilon_1, R, A^{p-2} \right]^i. \tag{5.25}
\]

**Summary.** At this point, we have found the relevant information contained in the gauge transformations and the closure rule of the gauge algebra. That information is codified into the following products:

\[
\begin{align*}
[\ell_1 (\varepsilon_1)]^A &= d\varepsilon_1^A, \\
[\ell_1 (\varepsilon_1)]^i &= d\varepsilon_1^i, \\
[\ell_2 (\varepsilon_1, A_1)]^A &= [A_1, \varepsilon_1]^A, \\
[\ell_2 (\varepsilon_1, A_1)]^i &= [A_1, \varepsilon_1]^i - [\varepsilon_1, A_1]^i, \\
[\ell_{p+1} (\varepsilon_1, A_1, \ldots, A_p)]^i &= (-1)^{1+\frac{p(p+1)}{2}} [\varepsilon, A_1, \ldots, A_p]^i, \\
[\ell_2 (\varepsilon_1, \varepsilon_2)]^A &= [\varepsilon_2, \varepsilon_1]^A, \\
[\ell_2 (\varepsilon_1, \varepsilon_2)]^i &= [\varepsilon_2, \varepsilon_1]^i - [\varepsilon_1, \varepsilon_2]^i, \\
[\ell_{p+1} (\varepsilon_1, \varepsilon_2, A_1, \ldots, A_{p-1})]^i &= (-1)^{\frac{(p-1)(p-2)}{2}} [\varepsilon_2, \varepsilon_1, A_1, \ldots, A_{p-1}]^i, \\
[\ell_{p+1} (\varepsilon_1, \varepsilon_2, E, A_1, \ldots, A_{p-2})]^i &= (-1)^{\frac{(p-2)(p-3)}{2}} [\varepsilon_1, \varepsilon_2, E, A_1, \ldots, A_{p-2}]^i,
\end{align*}
\]

where \( \varepsilon_1, \varepsilon_2 \in X_0, A_1, \ldots, A_p \in X_{-1} \) and \( E \in X_{-2} \). Any other product involving elements on both subspaces \( X_0 \) and \( X_{-1} \) vanishes. These products will define an \( L_\infty \) algebra and describe a consistent gauge theory if we include the information coming from the equations of motion. Note that different theories with the same gauge symmetry will share the previously found products. Starting from this point, we will consider two separate cases, being the first one a flat FDA1 theory, in which the dynamics is governed by the Maurer-Cartan equations, i.e., the zero-curvature conditions. The second case to analyze will be the \( q \)-dimensional CS gauge theory invariant under FDA1.

### 5.3 Flat FDA1 theory

In this case, we consider zero-curvatures. This is not necessarily equivalent to a flat spacetime. The corresponding field equations are immediately written as follows

\[
F^A = dA^A + \frac{1}{2} C_{BC}^A A^B A^C, \tag{5.28}
\]

\[
F^i = dA^i + C_{Aj} A^A A^j + \frac{1}{(p+1)!} C_{A_1 \ldots A_{p+1}} A^{A_1} \ldots A^{A_{p+1}}. \tag{5.29}
\]

By comparing eqs. (5.28) and (5.29) with the general equation of motion of eq. (2.14) we can see that the expansion in terms of \( L_\infty \) products gets truncated in different ways for
both algebraic sectors

\[
\mathcal{F}^A = \left[\ell_1 (A)^A - \frac{1}{2} \left[\ell_2 \left( A^2 \right) \right]^A \right],
\]

\[
\mathcal{F}^i = \left[\ell_1 (A)^i - \frac{1}{2} \left[\ell_2 \left( A^2 \right) \right]^i + \frac{(-1)^{\frac{p(p+1)}{2}}}{(p+1)!} \left[\ell_{p+1} \left( A^{p+1} \right) \right]^i \right].
\]

We can now obtain the information of the \(L_\infty\) products acting on \(X_{-1}\). For the Lie sector we get

\[
\left[\ell_1 (A)^i \right]^i = dA^i,
\]

\[
\left[\ell_2 \left( A^2 \right)^i \right]^i = -C_{BC}^A A^B A^C,
\]

while for the extended sector we obtain one extra non-vanishing product, carrying the information of the cocycle

\[
\left[\ell_1 (A)^i \right]^i = dA^i,
\]

\[
\left[\ell_2 \left( A^2 \right)^i \right]^i = -2C_{ij} A^i A^j,
\]

\[
\left[\ell_{p+1} \left( A^{p+1} \right)^i \right]^i = (-1)^{\frac{p(p+1)}{2}} C_{A_1 \ldots A_{p+1}} A^A_{p+1} \ldots A^A_{p+1}.
\]

The dynamical sector for this theory is then summarized by the following products

\[
\begin{align*}
\left[\ell_1 (A)^A \right]^A &= dA^A, \\
\left[\ell_2 \left( A^2 \right)^A \right]^A &= -C_{BC}^A A^B A^C, \\
\left[\ell_2 \left( A^2 \right)^A \right]^A &= -C_{ij} A_i^A A_j^A, \\
\left[\ell_{p+1} \left( A^{p+1} \right)^A \right]^A &= (-1)^{\frac{p(p+1)}{2}} C_{A_1 \ldots A_{p+1}} A^A_{p+1} \ldots A^A_{p+1}, \\
\left[\ell_2 \left( \varepsilon, E \right)^A \right]^A &= [E, \varepsilon]^A, \\
\left[\ell_2 \left( \varepsilon, E \right)^i \right]^i &= [E, \varepsilon]^i - [\varepsilon, E]^i, \\
\left[\ell_{p+1} \left( \varepsilon, E, A_1, \ldots, A_{p+1} \right)^i \right]^i &= (-1)^{1+\frac{(p-1)(p-2)}{2}} [\varepsilon, E, A_1, \ldots, A_{p+1}]^i,
\end{align*}
\]

where \(\varepsilon \in X_0\), \(A, A_1, \ldots, A_{p+1} \in X_{-1}\) and \(E \in X_{-2}\). As before, the consistency products involve elements on \(X_{-2}\) and are obtained by plugging in the already found products into the \(L_\infty\) identities. An explicit calculation can be found in appendix B.2. Eqs. (5.26), (5.27), (5.38) and (5.37) define a consistent \(L_\infty\) algebra encoding the information of the flat FDA1 gauge theory. Note that this is consistent with the trivial gauge transformation found in the gauge algebra from eq. (5.12). The trivial gauge transformations vanish on-shell, ensuring the closure of the gauge subalgebra \(L^\text{gauge}_\infty\) and thus the whole \(L_\infty\) algebra.

6 \(L_\infty\) formulation of FDA1-CS theory

Let us now consider extended CS theories introduced in section 3.3. The corresponding action principle is directly obtained from eq. (3.33) by locally setting \(\hat{A} = 0\), and consequently
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Getting an expression for the Lie sector of the equations of motion. From (6.3) and (2.14) we consider first the Lie sector of the equations of motion. Let us introduce the algebraic elements $u = \left( u^A_{\mu_1 \cdots \mu_q} \right) \in X_{-1}$ and $v = (v^A_{\mu_1 \cdots \mu_{q-2}}, v^B_{\mu_1 \cdots \mu_{q-1}}) \in X_{-2}$. Notice that, in this case we define the indices for the elements on $X_{-2}$ in a different way that allows to easily write the equations of motion in terms of them. We also define the inner product between elements on these subspaces as

$$\langle u, v \rangle_{L_{\infty}} = \int dx^{2m-1} \epsilon^{\mu_1 \cdots \mu_{q-1}} \left( g_{AB} u^A_{\mu_1} v^B_{\mu_2 \cdots \mu_{q-2}} + \epsilon^{\mu_1 \cdots \mu_{q-1}} g_{ij} u^A_{\mu_1 \cdots \mu_p} v^j_{\mu_{p+1} \cdots \mu_{q-1}} \right),$$  \hspace{1cm} (6.1)

where $g_{AB}$ and $g_{ij}$ are the components of the rank-2 invariant tensor defined on eqs. (3.27)–(3.29). Components with mixed indices as $g_{A_i}$ are also allowed but they are not necessary in the definition of the inner product. As mentioned, by setting $\bar{A} = 0$ in eq. (3.34) we can write the variation of the action principle in terms of the inner product of the $L_{\infty}$ algebra as

$$\delta S = \langle \delta A, F \rangle_{L_{\infty}} = \int dx^{2m-1} \epsilon^{\mu_1 \cdots \mu_{q-1}} \left( \frac{1}{(q-2)!} g_{AB} \delta A^A_{\mu_1} F^B_{\mu_2 \cdots \mu_{q-2}} + \frac{1}{p!(q-p-1)!} \epsilon^{\mu_1 \cdots \mu_{q-1}} g_{ij} \delta A^A_{\mu_1 \cdots \mu_p} F^i_{\mu_{p+1} \cdots \mu_{q-1}} \right),$$ \hspace{1cm} (6.2)

where the components of the algebraic element $F = (F^A_{\mu_1 \cdots \mu_{q-3}}, F^i_{\mu_1 \cdots \mu_{q-p-1}}) \in X_{-2}$ can be written in terms of differential forms as follows

$$F^A = \sum_{m,n} mg^{A}_{A_1 A_2 \cdots A_{m+i} \cdots i_n} R^{A_2} \cdots R^{A_m} R^{i_1} \cdots R^{i_n},$$ \hspace{1cm} (6.3)

$$F^i = \sum_{m,n} ng^{i}_{i_1 i_2 \cdots i_n} R^{A_1} \cdots R^{A_m} R^{i_2} \cdots R^{i_n}.$$ \hspace{1cm} (6.4)

The indices $A$ and $i$ were raised in the original expressions (3.39) and (3.40) using the rank-2 invariant tensors $g^{AB}$ in the case of the Lie sector and $g^{ij}$ for the extended one (see appendix A). This allows to get an explicit expression for $F^A$ in terms of the algebraic products.

### 6.1 Lie sector

Let us consider first the Lie sector of the equations of motion. From (6.3) and (2.14) we get an expression for $F^A$, namely

$$F^A = \sum_{l=1}^{\infty} \frac{(-1)^{\frac{l(l+1)}{2}}}{l!} \left[ \ell_l \left( A^i \right) \right]^A = \sum_{m,n} mg^{A}_{A_1 \cdots A_{m+i} \cdots i_n} R^{A_2} \cdots R^{A_m} R^{i_1} \cdots R^{i_n}. \hspace{1cm} (6.5)$$
Moreover, by replacing the definition of curvatures (3.22) and (3.23) into (6.3) we can explicitly write $\mathcal{F}^A$ in terms of the gauge fields and their derivatives as

$$
\mathcal{F}^A = \sum_{m,n} \sum_{k=0}^{m-1} \sum_{r+s+t=n} \frac{1}{2^k \,(p+1)! \,k! \,(m-k-1)! \,r! \,s! \,t!} 
	imes g^{A_1 \ldots A_{m-1} i_1 \ldots i_n} \,dA A_1 \ldots dA^{m-k-1} [A, A]^{A_{m-k} \ldots [A, A]^{A_{m-1}}}
	imes dA^{i_1} \ldots dA^{i_r} [A, A]^{i_{r+1}} \ldots [A, A]^{i_{r+s}} [A^{p+1}]^{i_{r+s+1}} \ldots [A^{p+1}]^{i_n}. 
\tag{6.6}
$$

In order to isolate the contributions to $\mathcal{F}^A$ corresponding to different $L_\infty$ products, we will separate the terms of the sum in eq. (6.6) that are powers of the same degree in the gauge fields, i.e.,

$$
\mathcal{F}^A = \sum_{l=1}^{\infty} \left[ \mathcal{F}^A \right]_l, 
\tag{6.7}
$$

with

$$
\left[ \mathcal{F}^A \right]_l = \frac{(-1)^{\frac{l(l-1)}{2}}}{l!} \left[ \ell_l \left( A^l \right) \right]^A. 
\tag{6.8}
$$

Each term on the sum on the right-hand side of eq. (6.6) is a power of degree $m + n + k + s + pt - 1$ in $A$. Thus, we can say that $\left[ \mathcal{F}^A \right]_l$ is equal to the sum of those terms verifying $m + n + k + s + pt - 1 = l$. This allows us to write

$$
\left[ \mathcal{F}^A \right]_l = \sum_{m,n,r+s+t=n} \frac{1}{2^{k_{st} \,(p+1)! \,k_{st}! \,(m-k_{st}-1)! \,r! \,s! \,t!} 
	imes g^{A_1 \ldots A_{m-1} i_1 \ldots i_n} \,dA A_1 \ldots dA^{m-k_{st}-1} [A, A]^{A_{m-k_{st}} \ldots [A, A]^{A_{m-1}}}
	imes dA^{i_1} \ldots dA^{i_r} [A, A]^{i_{r+1}} \ldots [A, A]^{i_{r+s}} [A^{p+1}]^{i_{r+s+1}} \ldots [A^{p+1}]^{i_n}, 
\tag{6.9}
$$

where $k_{st} = l + 1 - m - n - s - pt$. By comparing eqs. (6.8) and (6.9) we obtain the non-vanishing $L_\infty$ products that describe the dynamics of the theory for the Lie sector of the algebra

$$
\left[ \ell_l \left( A^l \right) \right]^A = \frac{(-1)^{\frac{l(l-1)}{2}}}{l!} \sum_{m,n,r+s+t=n} \frac{1}{2^{k_{st} \,(p+1)! \,k_{st}! \,(m-k_{st}-1)! \,r! \,s! \,t!} 
	imes g^{A_1 \ldots A_{m-1} i_1 \ldots i_n} \,dA A_1 \ldots dA^{m-k_{st}-1} [A, A]^{A_{m-k_{st}} \ldots [A, A]^{A_{m-1}}}
	imes dA^{i_1} \ldots dA^{i_r} [A, A]^{i_{r+1}} \ldots [A, A]^{i_{r+s}} [A^{p+1}]^{i_{r+s+1}} \ldots [A^{p+1}]^{i_n}. 
\tag{6.10}
$$

### 6.2 Extended sector

For the equation of motion corresponding to the extended sector, we get a similar expression. From eq. (6.4) and the definition of curvatures, we write the equation of motion in terms of the fields and their derivatives

$$
\mathcal{F}^A = \sum_{m,n,k=0}^{m} \sum_{r+s+t=n-1} \frac{1}{2^k \,(p+1)! \,k! \,(m-k-1)! \,r! \,s! \,t!} g^{A_1 \ldots \ldots A_{m}} dA^{A_1} \ldots dA^{A_{m-k}} [A, A]^{A_{m-k+1}} \ldots [A, A]^{A_{m}}
\times dA^{i_1} \ldots dA^{i_r} [A, A]^{i_{r+1}} \ldots [A, A]^{i_{r+s}} [A^{p+1}]^{i_{r+s+1}} \ldots [A^{p+1}]^{i_n}. 
\tag{6.11}
$$
We now need to extract the part of the sum that has the same order on $A$. Each term in the sum is a power of degree $m + n + k + s + pt - 1$ in $A$. The terms on $F^i$ that are powers of degree $l$ are then given by

$$
[f^i_l] = \sum_{m,n,r,s,t=0} \frac{1}{2^{k_{st}}(p+1)^t} \frac{m!}{(m-k_{st})!} \frac{n!}{n!} \times dA^1 \cdots dA^{m-k_{st}} [A,A]^{A_{m-k_{st}+1}} \cdots [A,A]^{A_{m}} \times dA^{i_1} \cdots dA^{i_{r+s}} [A,A]^{[r+s]} [A^p+1]^{i_{r+s+1}} \cdots [A^p+1]^{i_{n-1}},
$$

(6.12)

where $k_{st} = l + 1 - m - n - s - pt$. Since there are no symmetry or antisymmetry rules for the different kind of indices on $g_{A_1 \cdots A_{m_1} \cdots i_n} = g_{i_1 \cdots i_n A_1 \cdots A_m}$.

By comparing with the general expression for $F$ in eq. (2.14), we get an expression for the products between gauge fields

$$
[f^i_l (A^l)] = (-1)^{\frac{l(l-1)}{2}} \sum \sum \frac{l!}{2^{k_{st}}(p+1)^t} \frac{m!}{(m-k_{st})!} \frac{n!}{n!} \times g^{ij}_{i_1 \cdots i_n A_1 \cdots A_m} dA^{A_{m-k_{st}}} [A,A]^{A_{m-k_{st}+1}} \cdots [A,A]^{A_{m}} dA^{A_{1}} \cdots dA^{A_{r+s}} [A,A]^{[r+s]} [A^p+1]^{i_{r+s}} \cdots [A^p+1]^{i_{n-1}}.
$$

(6.13)

Eqs. (6.10) and (6.13) describe the complete dynamical sector of the theory. As in the previous cases, for the products above to verify the $L_\infty$ identities, it is necessary to include some consistency products involving elements on $X_{-2}$. Such products are given by

$$
\begin{align*}
[f^i_2(\varepsilon,E)]^A &= C_{BC}^A E^B \varepsilon^C - g^{AB} g_{ij} C_{Bk}^i E^j \varepsilon^k, \\
[f^i_{p+1}(\varepsilon,E,A^{p-1})]^A &= (-1)^{\frac{p(p+1)}{2}} g^{AB} g_{ij} C_{B_1 \cdots B_{p+1}}^i E^{B_1} \cdots A^{B_{p+1}} E^j, \\
[f^i_2(\varepsilon,E)]^i &= g^{ij} g_{kl} C_{Bj}^i \varepsilon^B E^l,
\end{align*}
$$

(6.14)

with $\varepsilon \in X_0, A \in X_{-1}$ and $E \in X_{-2}$. An explicit calculation for the consistency products can be found in appendix B.3.

In summary, the complete set of products that define a consistent $L_\infty$ algebra for FDA1-CS theory is given by eqs. (5.26) and (5.27), describing the gauge symmetries, together with eqs. (6.10), (6.13) and (6.14) describing the dynamical sector. For simplicity, we do not explicitly write the product between arbitrary elements on $X_{-1}$ as we did for the standard CS theory. Such expression can be obtained from eqs. (6.10) and (6.13) by considering different algebraic elements in the argument of the products and including the corresponding symmetrization. Note that, as it happens in the standard CS theory, there is a different number of non-vanishing products in the dynamical sector, depending on the dimensionality of the theory. Let us recall that the original CS action is $q - 1$ dimensional and the values of $m$ and $n$ are the integer and non-negative solutions to the equation $2m + (p + 1)n = q$. This implies that the possible values of $l$ change case by case and they are restricted between $l_{\min} = n$ and $l_{\max} = 2m - 2$ for eq. (6.10) (i.e., in the Lie-sector) and $l_{\min} = n - 1$ and $l_{\max} = 2m$ for eq. (6.13) (i.e., in the extended sector). This completes the
algebraic products of the $L_\infty$ formulation of FDA1-CS theory. Contrary to the standard case, the gauge subalgebra is not a Lie algebra but also an $L_\infty$ algebra.

6.3 Five-dimensional theory

As an example, let us consider a five-dimensional CS theory for a FDA1 with $p = 3$ in absence of a cocycle. The corresponding FDA is given by the following Maurer-Cartan equations

$$dA^A + \frac{1}{2} C^A_{BC} A^B A^C = R^A = 0,$$

$$dA^i + C^i_A A^A A^j = R^i = 0.$$  \hfill (6.15) \hfill (6.16)

This allows to formulate a gauge theory with non-trivial coupling between a one-form and a three-form whose gauge invariant action is given by

$$S_5[A] = \int_M \int_0^1 dt \left( 3g_{A_1 A_2 A_3} A^{A_1} R_{A_2} A^{A_1} + g_{A_1 i_1} A^{A_1} R_{i_1} + g_{A_1 i_1} R_{A_1} A^{i_1} \right),$$  \hfill (6.17)

where $A_t = tA$, and being $R_t$ its corresponding field strength. An example of this action for a particular bosonic FDA can be found in ref. [34].

The $L_\infty$ products containing the information of the symmetries and interacting theory are then given by

$$[\ell_1(\varepsilon)]^A_\mu = \partial_\mu \varepsilon^A,$$

$$[\ell_1(\varepsilon)]^i_\mu = \partial_\mu \varepsilon^i,$$

$$[\ell_2(\varepsilon,A)]^A_\mu = C^A_{BC} A^B A^C,$$

$$[\ell_2(\varepsilon,A)]^i_\mu = C^i_A A^A A^j - \varepsilon^k A^j_{\mu \nu \rho \sigma} \varepsilon^A A^A A^j,$$

Gauge transformations \hfill Gauge algebra \hfill (6.18)

\begin{align*}
[\ell_4(\varepsilon_1,\varepsilon_2)]^A_{\mu \nu \rho \sigma} &= 3 \times 3! \times 4! g^{A B C D} g^B C^A_{\mu \rho \sigma} \varepsilon_1 \varepsilon_2 \varepsilon_3 [A(1)]_{\mu \rho \sigma} [B(2)]_{\nu \sigma} [C(3)]_{\rho \sigma} \varepsilon_4, \\
[\ell_3(\varepsilon_1,\varepsilon_2)]^A_{\nu \rho \sigma} &= -3 \times 3! \times 4! g^{A B C D} g^B C^A_{\nu \rho \sigma} \varepsilon_1 \varepsilon_2 [A(1)]_{\mu \rho \sigma} \varepsilon_4, \\
[\ell_2(\varepsilon_1,A)]^A_{\mu \nu \rho \sigma} &= -3 \times 3! \times 4! g^{A B C D} g^B C^A_{\mu \nu \rho \sigma} \varepsilon_1 \varepsilon_2 \varepsilon_3 [A(1)]_{\nu \sigma} [B(2)]_{\rho \sigma} \varepsilon_4, \\
[\ell_2(\varepsilon_1,A)]^i_{\mu \nu \rho \sigma} &= -2 g^{A B C D} g^B C^A_{\mu \nu \rho \sigma} \varepsilon_1 \varepsilon_2 \varepsilon_3 [A(1)]_{\mu \nu \sigma} [B(2)]_{\rho \sigma} \varepsilon_4, \\
[\ell_2(\varepsilon,E)]^A_{\mu \nu \rho \sigma} &= C^A_{B C} E^B_{\mu \nu \rho \sigma} \varepsilon^C - 3 g^{A B C} g_{D E} C^i_D E^j_{\mu \nu \rho \sigma} \varepsilon^k, \\
[\ell_2(\varepsilon,E)]^i_{\mu \nu \rho \sigma} &= 2 g^{A B C} g_{D E} C^i_D E^j_{\mu \nu \rho \sigma}. \quad \hfill \text{Consistency products} \hfill (6.19)
\end{align*}

A particular case of this theory is the so-called BF theory, in which only the third term in the integral at the right side of eq. (6.17) is not vanishing.
7 Concluding remarks

We have formulated the standard CS theory and two cases of FDA-based theories in terms of \( L_\infty \) algebras. In the standard CS case, we wrote down the products between algebraic elements. Such products satisfy the graded symmetry rule from eq. (2.3). In later cases, for FDA-based theories, the corresponding products were written in terms of differential forms for simplicity. It is important to note that the algebraic products satisfy the mentioned graded symmetry from eq. (2.3) if we write them in terms of the gauge fields without using differential forms. By writing the products in terms of differential forms, we add a new algebraic structure, and therefore all the products found starting from section 5 satisfy a modified symmetry rule. The same difference is present in the formulation of standard gauge theories, where the Lie product is antisymmetric but, when dealing with differential forms, it becomes symmetric or antisymmetric depending on their differential degrees. The reason for this choice in the latter cases is that the algebraic products take on a particularly simple form due to the natural presence of higher-degree forms as gauge fields in FDA-based theories. As we can see in eq. (5.4), the \( L_\infty \) products, satisfying the graded-symmetry rule (2.3) can be easily found by removing the dependence on the basis of differential forms and including the corresponding antisymmetrization. This leads to larger expressions that we explicitly write for the five-dimensional case in the example at the end of section 6.

A FDA-based gauge theory can naturally involve higher-degree differential forms as gauge fields due to the presence of higher-degree products in the dynamical sector. However, an important issue must be noticed. Higher-degree components of the field strength do not transform covariantly. This was pointed out as a general feature of \( L_\infty \) gauge theories involving higher-degree forms in ref. [42]. As a consequence, the gauge algebra is closed only on-shell, as shown in eq. (2.15) due to the presence of a trivial gauge transformation that vanishes by imposing the equations of motion. The definition of gauge transformations imposes then a constraint on the action. In the case of FDA1, the gauge algebra closes on-shell only if the equations of motions imply that the two-form curvature \( R^A \) vanishes (there are no constraints on the extended curvature \( R^i \)). Otherwise, such gauge transformations do not close, and therefore, the \( L_\infty \) algebra that describes the whole theory has a missing definition in its gauge subalgebra. The generalized CS action remains invariant under the FDA1 gauge transformations but it is not an entirely well-defined gauge theory. The \( L_\infty \) formulation of FDA1-CS theory is therefore valid, only for cases in which the equations of motion are not inconsistent with the closure of the gauge algebra. Simple examples of this can be found by choosing symmetries described by trivial extensions of Lie algebras, i.e., when the extended Maurer-Cartan equation does not include a non-trivial cocycle. Moreover, if the theory is three-dimensional or the FDA invariant tensors do not present mixed indices corresponding to both algebraic sectors (as \( g_{Ai} \)), a non-trivial FDA with closed gauge algebra is also possible. In contrast, the gauge algebra always close on-shell for the flat FDA theory (this is also true for any other theory verifying \( R^A = 0 \) on-shell). Since the trivial gauge transformations on eq. (2.9) appear when dealing with products of three or more elements, this is not an issue for algebras with bilinear products either, as is the case of Lie algebras or in the five-dimensional example of section 6.
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A Notation

In this appendix, we will introduce a compact notation for FDAs, similar to the one usually used for brackets and invariant tensors of Lie algebras. Let \( x \in \bar{X} \) be an algebraic element for certain FDA as it was introduced in section 3. This means that we consider a collection of differential forms

\[
x = (x^{A(1)}, \ldots, x^{A(N)}),
\]

(A.1)
each one belonging to a certain subspace of the graded space \( \bar{X} \) (\( \deg \bar{X} x^{A(p)} = p \)). We introduce a FDA-degree that will be useful to identify the differential-form degree of each component. Such degree corresponds to the differential form degree of its first component, i.e., if \( x^{A(1)} \) is a \( r \)-form, we say that \( \deg_{\text{FDA}} x = r \). Therefore, we can say that each component \( x^{A(p)} \) is given by a \( (r+p-1) \)-form. The FDA-degree is particularly useful in the case of the FDA from eqs. (3.22) and 3.23) where \( X \) is reduced to two subspaces and \( x \) to the following pair

\[
x = (x^{A(1)}, x^{A(p)}) \quad \text{r-form} \quad (r+p-1)-\text{form}.
\]

(A.2)

For convenience, will denote \( x^{A(p)} \) as an \( \bar{r} \)-form with \( \bar{r} = r + p - 1 \).

We now introduce a product between algebraic elements in terms of the brackets from eq. (3.4), which in this particular FDA case is reduced to a bilinear bracket and a \( (p+1) \)-linear one. Let us consider a set of algebraic elements \( B_1, \ldots, B_{p+1} \) on FDA1, each one of FDA-degree \( b_1, \ldots, b_{p+1} \). In terms of the structure constants of the FDA1, we define:

1. A bilinear product \([B_1, B_2]\) such that

\[
[B_1, B_2]^A = C^A_{BC} B_1^B B_2^C,
\]

(A.3)

\[
[B_1, B_2]^i = C^i_{i'j} B_1^j B_2^j.
\]

(A.4)

2. A \((p+1)\)-linear product \([B_1, \ldots, B_{p+1}]\) such that

\[
[B_1, \ldots, B_{p+1}]^A = 0,
\]

(A.5)

\[
[B_1, \ldots, B_{p+1}]^i = C^i_{A_1 \cdots A_{p+1}} B_1^{A_1} \cdots B_{p+1}^{A_{p+1}}.
\]

(A.6)

Let us now consider a two sets of algebraic elements \( B_1, \ldots, B_m \) and \( E_1, \ldots, E_n \), each one of FDA-degree \( b_1, \ldots, b_m \) and \( e_1, \ldots, e_n \) respectively. We introduce a compact notation for the contraction of their algebraic components with the FDA1 invariant tensor

\[
\langle B_1, \ldots, B_m; E_1, \ldots, E_n \rangle = g_{A_1 \cdots A_{m1} \cdots i_n} B_1^{A_1} \cdots B_m^{A_m} E_1^{e_1} \cdots E_n^{e_n}.
\]

(A.7)
This bracket is equivalent to the symmetrized trace notation for Lie algebras for $n = 0$. In general, it separates the algebraic sectors before and after the semicolon, being the first ones evaluated in the Lie sector and the latter in the extended sector. In this notation, the following properties are fulfilled

\[
\langle \ldots, B_r, B_{r+1}, \ldots; E_1, \ldots, E_n \rangle = (-1)^{b_r b_{r+1}} \langle \ldots, B_{r+1}, B_r, \ldots; E_1, \ldots, E_n \rangle, \quad (A.8)
\]

\[
\langle B_1, \ldots, B_m; E_s, E_{s+1}, \ldots \rangle = (-1)^{\bar{e}_s \bar{e}_{s+1} + \ell+1} \langle B_1, \ldots, B_m; \ldots, E_{s+1}, E_s, \ldots \rangle, \quad (A.9)
\]

The invariant properties of $g_{A_1 \cdots A_m i_1 \cdots i_n}$ provide us with a notion of covariance and contravariance. Given an algebraic element $B = (B^A, B^i)$ with FDA-degree $b$, we define the contravariant duals of its components as $B^A = g^{AB} B^B$ and $B^i = g^{ij} B^j$. Note that, although the components with mixed indices $g_{Ai}$ are in general non-vanishing, we do not include them into the definition in order not to change the differential form degree of the components of $B$. In this way, $B^A$ and $B^i$ are still a $b$-form and $\bar{b}$-form respectively. In the same way, we define the inverse components $g_{AB}$ and $g^{ij}$ through the following relations

\[
g_{AB} = g_{AC} g_{BD} g^{CD}, \quad g^{ij} = g^{ik} g^{jl} g^{kl}. \quad (A.10)
\]

In the standard case, $g_{AB}$ is reduced to the Cartan-Killing metric for Lie algebras. Note that this is not a rigorous definition of a generalized Cartan-Killing metric for FDA but a notation that will be useful when writing products without introducing ambiguity.

### B Consistency products on $X_{-2}$

In this appendix, we will obtain the missing products in the $L_\infty$ formulation of CS theory and FDA-based gauge theories. Such products do not contain information coming directly from the gauge transformations, gauge algebra, or equations of motion but must be non-vanishing for consistency. They act on $X_{-2}$ and can be found by replacing the already found products into the $L_\infty$ identities.

A simple procedure to find them is to use the definition eq. (2.14) and take its gauge variation:

\[
\delta F = \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^{\frac{k(k-1)}{2} + \bar{r} (r-1)}}{k! \bar{r}!} \ell_k \left( \ell_{r+1} (\varepsilon, A^r), A^{n-1} \right). \quad (B.1)
\]

Using the $L_\infty$ identities (2.5) this can be written as

\[
\delta F = \sum_{k=0}^{\infty} \frac{(-1)^{\frac{k(k-1)}{2}}}{k!} \ell_{k+2} \left( \varepsilon, F, A^k \right). \quad (B.2)
\]

We can now directly compare this expression with the variation of $F$ obtained case by case that contains the information about the products of $X_{-1}$. Let us treat this separately.

#### B.1 Products in CS theory

In this case, the algebraic element $F \in X_{-2}$ is given by

\[
F^A_\nu = \varepsilon^\mu_1 \cdots \mu_{2m-2} g^{A B_1 \cdots B_{m-1}} R^{B_1}_{\mu_1 \mu_2} \cdots R^{B_{m-1}}_{\mu_{2m-3} \mu_{2m-2}}. \quad (B.3)
\]
By plugging in the well-known relation $\delta R_{\mu\nu}^A = C_{BC}^A R_{\mu B}^B \epsilon^C$ we find

$$\delta F_{\nu}^A = (m - 1) \epsilon_{\nu}{}^{\mu_1 \cdots \mu_{2m-2}} g_{B_{1} \cdots B_{m-1}} C_{BC}^A R_{B_1}^B \epsilon_{\mu_2 \mu_4} \cdots R_{\mu_{2m-3} \mu_{2m-2}}^{B_m-1}.$$ (B.4)

Using the definition of invariant tensor for Lie algebras (such definition can be obtained by setting $n = 0$ on eq. (3.27)), we find

$$\delta F_{\nu}^A = \epsilon_{\mu_1 \cdots \mu_{2m-2}} g_{B_{1} \cdots B_{m-1}} C_{BC}^A R_{B_1}^B \epsilon_{\mu_2 \mu_4} \cdots R_{\mu_{2m-3} \mu_{2m-2}}^{B_m-1}.$$ (B.5)

This shows that, in the case of the standard CS theory, the equation-of-motion term $F_{\nu}^A$ inherits the transformation law of the 2-form curvature. By directly comparing eqs. (B.2) and (B.5) we obtain the following product involving elements on $X_0$ and $X_{-2}$:

$$[\ell_2 (\epsilon, F)]^A = C_{BC}^A F_{B}^C \epsilon^C.$$ (B.6)

### B.2 Products in FDA-flat theory

In this case, the equations of motions are equivalent to the Maurer-Cartan equations for FDA1, i.e., $\left( F^A, F^i \right) = \left( R^A, R^i \right)$. The gauge variation of $R$ under a transformation with parameter $\epsilon = (\epsilon^A, \epsilon^i)$ is then given by

$$\delta R^A = C_{BC}^A R^B \epsilon^C,$$ (B.7)

$$\delta R^i = C_{Aj}^i R^A \epsilon^j - C_{Aj}^i \epsilon^A R^j - \frac{1}{(p - 1)!} C_{A_1 \cdots A_{p+1}}^i \epsilon^{A_1} R^{A_2} A^{A_3} \cdots A^{A_{p+1}}.$$ (B.8)

By comparing eqs. (B.7), (B.8) and (B.2) we immediately find the following brackets (as mentioned in section 5, we write the products in terms of differential forms for FDA-based theories)

$$[\ell_2 (\epsilon, F)]^A = C_{BC}^A F_{B}^C \epsilon^C,$$ (B.9)

$$[\ell_2 (\epsilon, F)]^i = C_{Aj}^i \left( F^A \epsilon^j - \epsilon^A F^j \right),$$ (B.10)

$$\left[ \ell_{p+1} (\epsilon, F, A^{p-1}) \right]^i = (-1)^{\frac{p-1}{2}} \frac{(p-1)(p-2)}{2} C_{A_1 \cdots A_{p+1}}^i \epsilon^{A_1} F^{A_2} A^{A_3} \cdots A^{A_{p+1}}.$$ (B.11)

### B.3 Products in FDA1-CS theory

Unlike the previous ones, the algebraic element $F$ does not have the same transformation law as the field strength $R$, leading to more complicated expressions for the consistency products.
\textbf{B.3.1 Lie sector}

From the variation of the curvatures on eqs. (B.7), (B.8) and the definition on eq. (6.3) we obtain the gauge variation of $\mathcal{F}^A$:

\begin{equation}
\delta \mathcal{F}^A = \sum_{m,n} g A_{\alpha} \partial_m \bar{A}_{\alpha,m} \left( (m-1) C_{BC}^A R^B \varepsilon^C R^A_1 \ldots R^A_m R^i_1 \ldots R^i_n \\
+ n R^A_2 \ldots R^A_m C_{AJ}^{A_j} R^A_3 \varepsilon^j R^2 \ldots R^i_n - n R^A_2 \ldots R^A_m C_{AJ}^{A_j} R^i_1 \ldots R^i_n \\
- \frac{n}{(p-1)!} R^A_2 \ldots R^A_m h_{\alpha_A \alpha_{i-1}} \varepsilon^A R^A_2 A^A_3 \ldots A^A_{p+1} R^i_2 \ldots R^i_n \right). \tag{B.12}
\end{equation}

Using the invariant tensor conditions (3.27)–(3.29), it is possible to prove the following relations:

\begin{align}
0 &= g A_{\alpha} \partial_m \bar{A}_{\alpha,m} \left( (m-1) C_{BC}^A R^B \varepsilon^C R^A_3 \ldots R^A_m R^i_1 \ldots R^i_n - n R^A_2 \ldots R^A_m C_{Bj}^{A_j} R^j R^i_2 \ldots R^i_n \right) \\
&- g A_{\alpha} \partial_m \bar{A}_{\alpha,m} C_{Bj}^{A_j} R^j R^i_2 \ldots R^i_n, \tag{B.13}
\end{align}

\begin{align}
0 &= m g A_{\alpha} \partial_m \bar{A}_{\alpha,m} f A_{A_1} R^A_2 \ldots R^A_m C_{Bj}^{A_j} B_1 \ldots B_{p+1} R^1 \varepsilon^2 B A^3 \ldots A^B_{p+1} R^2 \ldots R^i_n \\
&+ g A_{\alpha} \partial_m \bar{A}_{\alpha,m} C_{Bj}^{A_j} R^j R^i_2 \ldots R^i_n, \\
0 &= m g A_{\alpha} \partial_m \bar{A}_{\alpha,m} f A_{A_1} R^A_2 \ldots R^A_m C_{Bj}^{A_j} R^j R^i_2 \ldots R^i_n \\
&+ g A_{\alpha} \partial_m \bar{A}_{\alpha,m} C_{Bj}^{A_j} R^j R^i_2 \ldots R^i_n. \tag{B.14}
\end{align}

We replace eqs. (B.13)–(B.15) into eq. (B.12) and write the variation of $\mathcal{F}^A$ in terms of $\mathcal{F}^A$ and $\mathcal{F}^i$:

\begin{equation}
\delta \mathcal{F}^A = C_{BC}^A \mathcal{F}^B \varepsilon^C - g A B g_{ik} C_{Bj}^{A_i} \mathcal{F}^j \varepsilon^i - \frac{1}{(p-1)!} g A B g_{ik} C_{Bj}^{A_i} \varepsilon^2 B A^3 \ldots A^B_{p+1} \mathcal{F}^i. \tag{B.16}
\end{equation}

By comparing eqs. (B.16) and (B.2) we obtain the following products:

\begin{align}
\left[ f_2 \left( \varepsilon, \mathcal{F}, A^N \right) \right]^{A} &= C_{BC}^A \mathcal{F}^B \varepsilon^C - g A B C_{Bj}^{A_i} \mathcal{F}^j \varepsilon^i, \tag{B.17}
\left[ f_{p+1} \left( \varepsilon, \mathcal{F}, A^{p-1} \right) \right]^{A} &= (-1)^{1 + \frac{p-1}{2}} C_{AB_2 \ldots B_{p+1}} \varepsilon^2 B A^3 \ldots A^B_{p+1} \mathcal{F}^i. \tag{B.18}
\end{align}

\textbf{B.3.2 Extended sector}

We now compute the variation of $\mathcal{F}^i$:

\begin{equation}
\delta \mathcal{F}^i = \sum_{m,n} n g_{ij} f A_{\alpha} \partial_m \bar{A}_{\alpha,m} \left( m C_{BC}^A R^B \varepsilon^C R^A_2 \ldots R^A_m R^i_1 \ldots R^i_n \right) \\
+ \left( n - 1 \right) R^A_1 \ldots R^A_m C_{Bj}^{A_j} R^j R^i_2 \ldots R^i_n - \left( n - 1 \right) R^A_1 \ldots R^A_m C_{Bj}^{A_j} R^i_1 \ldots R^i_n \\
- \frac{\left( n - 1 \right)}{(p-1)!} R^A_1 \ldots R^A_m C_{Bj}^{A_j} R^i_2 \ldots R^i_n. \tag{B.19}
\end{equation}

Using again the invariant tensor conditions (3.27)–(3.29) we prove the following relations:

\begin{align}
0 &= g A_{\alpha} \partial_m \bar{A}_{\alpha,m} \left( m C_{BC}^A R^B \varepsilon^C R^A_2 \ldots R^A_m f^i R^i_2 \ldots R^i_n - \left( n - 1 \right) R^A_1 \ldots R^A_m f^i C_{Bj}^{A_j} R^j R^i_2 \ldots R^i_n \right) \\
&- g A_{\alpha} \partial_m \bar{A}_{\alpha,m} C_{Bj}^{A_j} R^j R^i_2 \ldots R^i_n, \tag{B.20}
\end{align}

\begin{align}
0 &= g A_{\alpha} \partial_m \bar{A}_{\alpha,m} f A_{A_1} R^A_2 \ldots R^A_m C_{Bj}^{A_j} B_1 \ldots B_{p+1} \varepsilon^1 B R^2 A^3 \ldots A^B_{p+1} R^3 \ldots R^i_n, \tag{B.21}
0 &= g A_{\alpha} \partial_m \bar{A}_{\alpha,m} f A_{A_1} R^A_2 \ldots R^A_m C_{Bj}^{A_j} R^j R^i_2 \ldots R^i_n. \tag{B.22}
\end{align}
By replacing eqs. (B.20)–(B.22) into eq. (B.19) we obtain that
\[ \delta F^i = g^{ij} g^{kl} C^k_B B^j \epsilon B^l. \]  
(B.23)

Finally, by comparing eq. (B.23) with the general expression in eq. (B.2), we obtain one consistency product for the extended sector
\[ [\ell_2 (\varepsilon, F)]^i = g^{ij} g^{kl} C^k_B B^j \epsilon B^l. \]  
(B.24)

This completes the obtention of the $L_\infty$ products acting on $X_{-2}$.

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