BENDINGS BY FINITELY ADDITIVE TRANSVERSE COCYCLES

DRAGOMIR SARIĆ

Abstract. Let $S$ be any closed hyperbolic surface and let $\lambda$ be a maximal geodesic lamination on $S$. The amount of bending of an abstract pleated surface (homeomorphic to $S$) with the pleating locus $\lambda$ is completely determined by an $(\mathbb{R}/2\pi\mathbb{Z})$-valued transverse cocycle $\beta$ to the geodesic lamination $\lambda$. We give a sufficient condition on $\beta$ such that the corresponding pleating map $\tilde{f}_\beta : \mathbb{H}^2 \to \mathbb{H}^3$ induces a quasi-Fuchsian representation of the surface group $\pi_1(S)$. Our condition is genus independent.

1. Introduction

Let $S$ be a closed surface of genus at least two. The Teichmüller space $T(S)$ of the surface $S$ consists of all hyperbolic metrics on $S$ modulo isometries homotopic to the identity. Let $\lambda$ be a maximal geodesic lamination on $S$. Each component of the complement of $\lambda$, called a plaque, is an ideal hyperbolic triangle. Given a closed hyperbolic arc $k$ on a hyperbolic surface $S$ with endpoints in the plaques of $\lambda$, there is a well-defined real number associated to $k$ given by measuring the amount of the shearing of the hyperbolic metric on $S$ along $\lambda$ from one endpoint of $k$ to the other. The assignment of the real numbers to the geodesic arcs transverse to $\lambda$, called a transverse cocycle, is invariant under under homotopies relative $\lambda$ and it is a finitely additive signed measure on each arc (cf. [3]). Each transverse cocycle to $\lambda$ is induced by and induces a (Hölder) distribution on the space of Hölder continuous functions on the transverse intervals (cf. [2]). The Teichmüller space $T(S)$ is injectively mapped into the space $\mathcal{H}(\lambda; \mathbb{R})$ of all Hölder distributions on the transverse arcs to $\lambda$ and it is homeomorphic to an open cone $C(\lambda)$ of $\mathcal{H}(\lambda; \mathbb{R})$ (cf. Thurston [17] and Bonahon [3]). Various topological and analytic properties of the space $\mathcal{H}(\lambda, \mathbb{R})$ are given by Bonahon [3, 4, 5]. It is a compelling fact that Hölder distributions appear in the inner workings of the hyperbolic geometry.

An (abstract) pleated surface $S$ with the pleating locus $\lambda$ is a pleating map $\tilde{f} : \mathbb{H}^2 \to \mathbb{H}^3$ with the pleating locus $\lambda$ such that for each $\gamma \in \pi_1(S) < PSL_2(\mathbb{R})$ we have $\tilde{f} \circ \gamma \circ \tilde{f}^{-1} \in PSL_2(\mathbb{C})$ and that $\lambda$ is realized by $\tilde{f}$, where $\lambda$ is the lift of $\lambda$ (cf. [3]). A pleated surface with the pleating locus $\lambda$ induces a transverse cocycle to $\lambda$ which is valued in $\mathbb{C}/2\pi i\mathbb{Z}$, where the real part is given by the shearing of the path hyperbolic metric of the pleated surface $\tilde{f}(\mathbb{H}^2) \subset \mathbb{H}^3$ and the imaginary part by the amount of the bending along $\lambda$. A pleated surface with the pleating locus $\lambda$ induces a representation of the fundamental group $\pi_1(S)$ into $PSL_2(\mathbb{C})$ which realizes the lamination $\lambda$. Bonahon [3] proved that the space $\mathcal{R}(\lambda)$ of all representations of $\pi_1(S)$ which realize $\lambda$ is parametrized by the space $\mathcal{C}(S) \bigoplus i\mathcal{H}(\lambda, \mathbb{R}/2\pi\mathbb{Z})$ which

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is an open subset of the space of H"older distribution $\mathcal{H}(\lambda, \mathbb{C}/2\pi i \mathbb{Z})$. The Fuchsian locus corresponds to the space of real valued transverse cocycles $C(S)$ inside $\mathcal{H}(\lambda, \mathbb{C}/2\pi i \mathbb{Z})$. Since the quasiFuchsian space $Q(S)$ is an open subset of $R(\lambda)$, there is an open neighborhood of $C(S)$ inside $C(S) \oplus i\mathcal{H}(\lambda, \mathbb{R}/2\pi \mathbb{Z})$ such that the corresponding representations are quasiFuchsian. We consider the question of explicitly determining a neighborhood of $C(S)$ such that the corresponding representations are quasiFuchsian. This question is motivated by the recent proof of the well-known surface subgroup conjecture by Kahn and Markovic [10]. One of the main ingredients of the proof in [10] is a sufficient condition on the complex Fenchel-Nielsen coordinates of a pants decomposition $\mathcal{P}$ of $S$ which induces quasiFuchsian represen-itations. Assume that the real parts of the complex lengths of the geodesics of $\mathcal{P}$ are close to $R > 0$ and the real parts of the twists are close to 1 for a hyperbolic metric on $S$. A theorem from [10] states that there is a universal $\epsilon > 0$ such that when the imaginary parts of the complex lengths are less than $\epsilon$ and the imaginary parts of the complex twists are less than $\epsilon/R$ then the representation is quasiFuchsian when $R$ is large enough independently of the genus of $S$ (for another proof cf. [14]). When using Fenchel-Nielsen coordinates one is, by their definition, limited to giving conditions on finite laminations. Moreover the conditions on the real parts of the Fenchel-Nielsen coordinates impose conditions on the path hyperbolic metrics of the pleated surfaces. We extend the scope of the theorem on quasiFuchsian representations by allowing infinite geodesic laminations and arbitrary hyperbolic path metrics on the pleated surface.

Let $\alpha$ be a $\mathbb{C}/2\pi i \mathbb{Z}$-valued transverse cocycle to the maximal geodesic lamination $\lambda$ on the surface $S$ such that its real part is in $C(S)$. Give $S$ the hyperbolic metric corresponding to the real part of $\alpha$. Denote by $\beta$ the imaginary part of $\alpha$; $\beta$ is the bending cocycle which is $\mathbb{R}/2\pi \mathbb{Z}$-valued. In the special case when the bending cocycle $\beta$ is a countably additive measure $\mu$ on each arc $k$ transverse to $\lambda$ there is a well-understood, genus independent, condition that guarantees that the induced representation is quasiFuchsian. The Thurston norm $\|\mu\|$ of the transverse measure $\mu$ is defined to be the supremum of the deposited measure $|\mu(k)|$ over all unit length arcs $k$ transverse to $\lambda$. Then there is a universal $\epsilon > 0$ such that the bending map $f_\mu : \mathbb{H}^2 \to \mathbb{H}^3$ continuously extends to an injection $f_\mu : \partial_\infty \mathbb{H}^2 \to \partial_\infty \mathbb{H}^3$ thus inducing a quasiFuchsian representation when $\|\mu\| < \epsilon$ (cf. [9], [8], [13]). In fact, $f_\mu$ is a part of a holomorphic motion of $\partial_\infty \mathbb{H}^2$ and there is a quasiconformal extension of $f_\mu$ to $\partial_\infty \mathbb{H}^3$ whose quasiconformal constant converges to 1 as $\|\mu\| \to 0$. The above statement is independent of genus and it even holds on any hyperbolic surface including the unit disk (cf. [8] and [13]). Our main result is an analogous statement for arbitrary bending cocycle $\beta$ to $\lambda$.

In general, the restriction of a bending cocycle $\beta$ to an arc $k$ transverse to $\lambda$ is not a countably additive measure but rather a distribution supported on $k \cap \lambda$ which requires a subtle combination of analytic and geometric arguments (cf. [3], [2]). (One may think about the difference between working with absolutely and conditionally converging series of functions.) To illustrate the difficulty note that any (non-countably additive) finitely additive transverse cocycle $\beta$ satisfies

$$\sup_k |\beta(k)| = \infty$$

where the supremum is over all hyperbolic arcs $k$ of length 1 transverse to $\lambda$ unlike for the countably additive measures.
We fix a hyperbolic metric on $S$ and give a sufficient condition on the bending transverse cocycle $\beta$ to the maximal geodesic lamination $\lambda$ to guarantee the injectivity of the bending map $\hat{f}_\beta:\partial_s \mathbb{H}^2 \to \partial_s \mathbb{H}^3$. Let $\{k_1, \ldots, k_n\}$ be a family of closed geodesic arcs on $S$ with endpoints in the plaques of $\lambda$ such that each component of $\lambda \setminus \bigcup_{i=1}^n k_i$ is a finite arc, the arcs $\{k_1, \ldots, k_n\}$ intersect geodesics of $\lambda$ at angles between $\pi/4$ and $3\pi/4$.

(1) \[ \max_{1 \leq i \leq n} |k_i| \leq 1/20, \]

where $|k_i|$ is the length of $k_i$. In addition, collapsing each $k_i$ to a point (forming a vertex) and after identifying the homotopic (relative $k_i$’s) arcs of $\lambda \setminus \bigcup_{i=1}^n k_i$ (forming an edge) we obtain a “topological train track” where each vertex is either trivalent or bivalent. The long edges of a train track can be divided into shorter edges by introducing bivalent vertices which improve the estimates on the size of the region in $\mathcal{H}(\lambda, \mathbb{R}/2\pi\mathbb{Z})$ whose induced bending maps are injective.

If a set of arcs $\{k_1, \ldots, k_n\}$ satisfies the above properties then $\{k_1, \ldots, k_{n_1}\}$ is said to be geometric set of arcs. By the finite additivity, a transverse cocycle $\beta$ to $\lambda$ is completely determined by its values $\beta(k_i)$, for $i = 1, \ldots, n_1$, on a set of arcs as above (cf. [3]). We divide each arc $k_i$ (which makes a trivalent vertex) into two subarcs $k_i^1$ and $k_i^2$ with the division point in an interval of $k_i \setminus \lambda$ such that each family of homotopic arcs in $\lambda \setminus \bigcup_{i=1}^{n_1} k_i$ has all of its endpoints on $k_i$ in exactly one $k_i^1$ on the side of $k_i$ where two edges of the topological train track meet. For simplicity of notation, we write $\{k_i : i = 1, \ldots, n\}$ in place of $\{k_1, k_2^1 : 1 \leq i \leq n_1; j = 1, 2\}$. Let $e$ be an edge of the topological train track formed by the set of arcs $\{k_1, \ldots, k_n\}$, and let $k_{i(e)}$ and $k'_{i(e)}$ be the two arcs which form the vertices of $e$. Let $\text{diam}(k_{i(e)} \cup k'_{i(e)})$ be the diameter of the union of $k_{i(e)}$ and $k'_{i(e)}$. Define

(2) \[ m_0 = \max_e \text{diam}(k_{i(e)} \cup k'_{i(e)}). \]

Given an edge $e$, let $\text{dist}(k_{i(e)}, k'_{i(e)})$ be the length of the shortest geodesic arcs connecting $k_{i(e)}$ and $k'_{i(e)}$. Define

(3) \[ m_\ast = \min_e \text{dist}(k_{i(e)}, k'_{i(e)}). \]

Moreover, we define

\[ k_\ast = \max_{1 \leq i \leq n} |k_i| \]

and

\[ k_\ast = \min_{1 \leq i \leq n} |k_i|. \]

The above quantities $m_0$, $m_\ast$, $k_\ast$ and $k_\ast$ give the geometric information about the geometric set of arcs $\{k_i : 1 \leq i \leq n\}$ that we use in order to give a sufficient condition on the bending cocycles such that the bending map is injective.

We define the norm of $\beta$ for the family of arcs by

(4) \[ ||\beta||_{\max} = \max \{|\beta(k_i)| : 1 \leq i \leq n\}. \]

For a given $\delta > 0$, we introduce the $\delta$-variation of $\beta$ on $k_i$ as follows. A component of $k_i \setminus \lambda$ is called a gap of $k_i$. Let $\{d_l : l = 1, \ldots, l_i\}$ be finitely many gaps of $k_i$. Define $k_{d_l}$, for $l = 1, \ldots, l_i$, to be the subarc of $k_i$ whose initial point is the initial point of $k_i$ and whose endpoint is a point in $d_l$. Define

(5) \[ ||\beta||_{\text{var}, k_i} = \max_{1 \leq l \leq l_i} |\beta(k_{d_l})|, \]
where the set \( \{ d_l : l = 1, \ldots, l_i \} \) is chosen such that the length of \( k_i \setminus \bigcup_{l=1}^{d_l} d_l \) is less than \( \delta|k_i| \) (\( |k_i| \) denotes the length of \( k_i \)), and

\[
\|\beta\|_{\text{var}} = \max_{1 \leq i \leq n} \|\beta\|_{\text{var}, k_i}.
\]

We give a sufficient condition for the injectivity of the bending map corresponding to a transverse cocycle \( \beta \) in terms of a geometric set of arcs.

**Theorem 1.1.** There exist \( \epsilon > 0 \) and \( \delta > 0 \) such that for any closed hyperbolic surface \( S \) and a maximal geodesic lamination \( \lambda \) on \( S \) the following holds. Let \( \{k_1, \ldots, k_n\} \) be a geometric set of arcs for \( \lambda \) such that

\[
(7) \quad k^* < \frac{e^{-2\log \varepsilon} \tanh \frac{\delta}{2}}{8\pi}.
\]

If an \((\mathbb{R}/2\pi\mathbb{Z})\)-valued transverse cocycle \( \beta \) to \( \lambda \) satisfies

\[
(8) \quad \|\beta\|_{\text{max}} < \epsilon k^*,
\]

and

\[
(9) \quad \|\beta\|_{\text{var}} < \epsilon,
\]

then the developing map \( \tilde{f}_{\beta} : \mathbb{H}^2 \to \mathbb{H}^3 \) continuously extends to an injective map \( \tilde{f}_{\beta} : \partial_\infty \mathbb{H}^2 \to \partial_\infty \mathbb{H}^3 \).

**Remark 1.2.** The condition (8) is the standard condition that works for the transverse measures (cf. [3], [13]). The condition (9) is a new condition needed to control the geometry of the realization of the transverse cocycle \( \beta \) due to the fact that the variation of \( \beta \) is unbounded. Note that the condition (5) for a choice of a family of geometric arcs does not imply similar condition on an arbitrary arc of length at most 1. For this reason it is necessary that the arcs \( k_i \)'s are on a relatively large distance compared to their sizes which is made explicit by (7).

**Remark 1.3.** The size of \( m_0 \) and \( m^* \) depend on the lamination \( \lambda \). The constants \( \epsilon \) and \( \delta \) are computed in terms of \( m_0 \) and \( m^* \) in the proof of the theorem. If a geometric set of arcs \( \{k_1, \ldots, k_n\} \) does not satisfy (7) then we can divide each arc into several subarcs until the condition is satisfied. If \( \lambda \) contains short closed geodesics then \( m^* \) is small for any choice of a geometric set of arcs for \( \lambda \). A generic geodesic lamination \( \lambda \) contains no closed geodesics and the choice of a geometric set of arcs can be made such that \( m_0 \geq 1/5 \) and \( m^* \geq m_0/4 \) in which case we can choose \( k^* = 4.41719 \times 10^{-10} \) and \( \epsilon = \delta = 3.61749 \times 10^{-17} \). We give a table of values for \( \epsilon \) and \( \delta \) when \( m^* = m_0/4 \) for various values of \( m_0 \) (cf. Table 6). It seems that the optimal value is \( m_0 = 0.0238523 \) in which case \( \epsilon = \delta = 2.01795 \times 10^{-13} \) and \( k^* \leq 1.27126 \times 10^{-11} \).

Let \( \alpha \) be an \( \mathbb{R} \)-valued transverse cocycle to \( \lambda \) which is induced by the hyperbolic metric on \( S \) (cf [3]). For \( w \in \mathbb{C} \), define the transverse cocycle \( \alpha_w \) by

\[
\alpha_w(k) = (1 + w)\alpha(k) \mod (2\pi i\mathbb{Z})
\]

for each arc \( k \) transverse to \( \lambda \).

The developing shear-bend map \( \tilde{f}_w : \mathbb{H}^2 \to \mathbb{H}^3 \) (normalized to be the identity on a fixed plaque of \( \lambda \)) corresponding to the transverse cocycle \( \alpha_w \) induces a holomorphic family (in \( w \)) of representation of \( \pi_1(S) \) to \( \text{PSL}_2(\mathbb{C}) \) (cf. [3]). As a corollary to the above theorem, we obtain
Corollary 1.4. Let $\alpha$ be an $\mathbb{R}$-valued transverse cocycle to a geodesic lamination $\lambda$ corresponding to a hyperbolic metric on a closed surface $S$ and let $\tilde{f}_w$ be the shear-bend map for $\alpha_w$. Then there exists $\epsilon > 0$ such that the shear-bend map

$$\tilde{f}_w : \mathbb{H}^2 \to \mathbb{H}^3$$

extends by continuity to a holomorphic motion of $\partial_\infty \mathbb{H}^2$ in $\partial_\infty \mathbb{H}^3$ for the parameter $\{w \in \mathbb{C} : |w| < \epsilon\}$.

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2. Geodesic laminations

Let $S$ be a closed hyperbolic surface and $\lambda$ a maximal geodesic lamination on $S$. Each component of $S \setminus \lambda$ is an ideal hyperbolic triangle for the path metric of the complement called a plaque of $\lambda$. Let $\{k_1, \ldots, k_n\}$ be a collection of finite geodesic arcs on $S$ with endpoints in the plaques of $\lambda$ such that each geodesic of $\lambda$ is divided into finite length arcs by the set $\bigcup_{i=1}^n k_i$. The family of arcs $\lambda \setminus \bigcup_{i=1}^n k_i$ consists of finitely many homotopy classes relative $\{k_1, \ldots, k_n\}$ and we assume that after identifying all the arcs of the homotopy classes the obtained “topological train track” has the property that each vertex (corresponding to some $k_i$) is either trivalent or bivalent. The usual definition of train tracks does not allow bivalent vertices but we do allow them. The reason is that our train track holds geometric information.

We form a “metric train track” $\tau$ as follows. A gap of $k_i$ (with respect to $\lambda$) is a connected component of $k_i \setminus \lambda$. If $k_i$ corresponds to a trivalent vertex of the corresponding topological train track, we divide arc $k_i$ into two subarcs $k_i^1$ and $k_i^2$ with a division point in a gap of $k_i$ such that the endpoints of the arcs of $\lambda \setminus \bigcup_{i=1}^n k_i$ which belong to different homotopy classes lie in different subarcs. We connect the endpoints of $\{k_1, \ldots, k_n\}$ together with the points of the division of each $k_i$ (into $k_i^1$ and $k_i^2$) by geodesic arcs inside the plaques of $\lambda$ to obtain a finite collection of geodesic quadrilaterals whose two sides lie on $k_i$’s and the other two sides are obtained by connecting the chosen points on $k_i$’s inside the components of $S \setminus (\lambda \cup \bigcup_{i=1}^n k_i)$. We call these quadrilaterals, somewhat improperly, long rectangles. The sides of the rectangles which lie on $k_i$’s are said to be short and the other two sides are said to be long. The finite collection of long rectangles forms a (metric) train track $\tau$ on $S$ such that the long rectangles are the edges of $\tau$ and the switches of $\tau$ are the arcs $\{k_1, \ldots, k_n\}$, where we allow them to be either trivalent or bivalent. The geodesic lamination $\lambda$ is a subset of the interior of the train track $\tau$ and it is said that $\lambda$ is carried by $\tau$. The train track $\tau$ is homotopic to the topological train track. This kind of train tracks were introduced by Bonahon (cf. [2]).

Definition 2.1. Let $m_0$ be the maximum of the diameters of the long rectangles of $\tau$ and let $m_\ast$ be the shortest distance between two short sides of the long rectangles. Let $k_\ast$ be the minimum of the lengths of the short sides over all long rectangles of $\tau$ and let $k^\ast$ be the maximum of the lengths of the short sides over all long rectangles of $\tau$.

We impose two conditions on $\tau$. Namely we require that the angles at the vertices of each long rectangle lie in the interval $[\pi/4, 3\pi/4]$, and that $k^\ast \leq \frac{1}{20}$. A
(metric) train track which satisfies these conditions is said to be geometric and the corresponding collection of arcs \( \{k_1, k_2, \ldots, k_n\} \) is said to be geometric.

**Lemma 2.2.** Let \( R \) be a long rectangle of a geometric train track \( \tau \) with short sides \( k_1 \) and \( k_2 \). Then the distance \( d \) between the long sides of \( R \) satisfies

\[
d \geq \frac{1}{20e^{m_0}} \min\{|k_1|, |k_2|\},
\]

where \( |k_i| \) is the length of \( k_i \).

**Proof.** Let \( k_1 \) and \( k_2 \) be the short sides of \( R \), and let \( l_1 \) and \( l_2 \) be the long sides of \( R \). Denote by \( h_1 \) and \( h_2 \) orthogonal arcs from \( k_1 \cap l_2 \) and \( k_2 \cap l_1 \) onto \( l_1 \), respectively. The hyperbolic sine formula, the bounds on the angles at the vertices of \( R \) and the mean value theorem give

\[
|h_1| \cosh 1 \geq \sinh |h_1| \geq \frac{1}{\sqrt{2}} \sinh |k_i| \geq \frac{1}{\sqrt{2}} |k_i|
\]

which gives

\[
|h_i| \geq \frac{1}{\sqrt{2} \cosh 1} |k_i|.
\]

By possibly decreasing \( R \), we can assume that \( h_1 \) and \( h_2 \) have the same length \( |h_1| = |h_2| \geq \frac{1}{\sqrt{2} \cosh 1} \min\{|k_1|, |k_2|\} \). Let \( h \) be the arc which is orthogonal to both \( l_1 \) and \( l_2 \). An elementary hyperbolic geometry formula applied to the rectangle whose two sides are \( h_1 \) and \( h \) gives

\[
|h| \cosh 1 \geq \sinh |h| \geq \frac{\sinh \min\{|k_1|, |k_2|\}}{\sqrt{2} \cosh 1} \sqrt{\sinh^2 \frac{1}{2} \min\{|k_1|, |k_2|\} \cosh^2 \left(\min\{|k_1|, |k_2|\}\right) + 1}
\]

which in turn gives

\[
d = |h| \geq \frac{1}{2(\cosh^2 1) e^{m_0} + 1} \min\{|k_1|, |k_2|\}.
\]

\( \square \)

3. **Transverse cocycles for geodesic laminations**

**Definition 3.1.** A real-valued transverse cocycle \( \alpha \) for \( \lambda \) is an assignment of a finitely additive signed measure to each transverse arc \( k \) to \( \lambda \) which is invariant under homotopies relative \( \lambda \). Namely, \( \alpha \) assigns a real number \( \alpha(k) \) to each closed arc \( k \) transverse to \( \lambda \) whose endpoints are in \( S - \lambda \) such that if \( k' \) is an arc homotopic to \( k \) relative \( \lambda \) then \( \alpha(k) = \alpha(k') \). Moreover if \( k = k_1 \cup k_2 \), where \( k_1 \) and \( k_2 \) are transverse arcs to \( \lambda \) with disjoint interiors, then \( \alpha(k) = \alpha(k_1) + \alpha(k_2) \).

We will also need transverse cocycles which take values in \( \mathbb{R}/2\pi\mathbb{Z} \).

**Definition 3.2.** An \( \mathbb{R}/2\pi\mathbb{Z} \)-valued transverse cocycle \( \beta \) for \( \lambda \) is an assignment of \( \beta(k) \in \mathbb{R}/2\pi\mathbb{Z} \) to each transverse arc \( k \) to \( \lambda \) which is invariant under homotopy relative \( \lambda \) and which is finitely additive. For example, an \( \mathbb{R}/2\pi\mathbb{Z} \)-valued transverse cocycle \( \beta \) is obtained by taking a real-valued transverse cocycle \( \alpha \) and setting \( \beta(k) := \alpha(k) \mod (2\pi\mathbb{Z}) \).
A real-valued transverse cocycle for $\lambda$ induces a transverse Hölder distribution for $\lambda$ (cf. Bonahon [2]). However, this approach is not used in this paper and we refer the reader to [2] for more details.

Let $\alpha$ be either an $\mathbb{R}$-valued or an $\mathbb{R}/2\pi\mathbb{Z}$-valued transverse cocycle for $\lambda$. Given an edge $E \in \tau$, let $k_E$ be a geodesic arc which connects the two long boundary sides of $E$. Define $\alpha(E) := \alpha(k_E)$. Note that $\alpha(E)$ is independent of the choice of $k_E$ by the invariance of $\alpha$ under homotopy relative $\lambda$. The transverse cocycle $\alpha$ is completely determined by the values $\alpha(E), E \in \tau$ (cf. [2]).

4. The realizations of $\mathbb{R}$-valued and $\mathbb{R}/2\pi\mathbb{Z}$-valued transverse cocycles

We consider a hyperbolic surface $S$ and a maximal geodesic lamination $\lambda$ of $S$. Bonahon [3] defined an injective map from the Teichmüller space $T(S)$ of $S$ into the space $H(\lambda, \mathbb{R})$ of all $\mathbb{R}$-valued transverse cocycles for a fixed maximal geodesic lamination $\lambda$ which is a homeomorphism onto an open cone $C(\lambda)$ of $H(\lambda, \mathbb{R})$. Denote by $\sigma_0$ a fixed hyperbolic metric on $S$. Then $\sigma_0$ represents the base point in $T(S)$ and let $\alpha_0 \in H(\lambda, \mathbb{R})$ be the corresponding transverse cocycle. Then [3, Proposition 13] any other real-valued cocycle $\alpha \in H(\lambda, \mathbb{R})$ which is close enough to $\alpha_0$ is also in the image of $T(S)$ in $H(\lambda, \mathbb{R})$. Namely, when the difference $\alpha - \alpha_0$ is small in the sense that the norm $\|\alpha - \alpha_0\|$ is small, where

$$\|\alpha\| := \max_E |\alpha(E)|$$

and the maximum is over all edges $E$ of a train track $\tau$ that carries $\lambda$ then $\alpha$ determines a point in $T(S)$.

The proof of the above statement is given by constructing the realization of $\alpha_1$ starting from the realization of $\alpha_0$. We recall that $\lambda$ is a maximal geodesic lamination for the metric $\sigma_0$. We lift $\lambda$ to a geodesic lamination $\tilde{\lambda}$ of the universal covering $\mathbb{H}^2$. Components of $\mathbb{H}^2 \setminus \tilde{\lambda}$ are called plaques of $\tilde{\lambda}$ and they are lifts of plaques (i.e. connected components of $S \setminus \lambda$) of $\lambda$. Each plaque of $\tilde{\lambda}$ is an ideal hyperbolic triangle. Let $k$ be a geodesic arc in $\mathbb{H}^2$ which connects two plaques $P$ and $Q$ of $\tilde{\lambda}$. Let $\mathcal{P}_{P,Q}$ be the set of all plaques of $\tilde{\lambda}$ that separate $P$ and $Q$, and let $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ be a finite subset of $\mathcal{P}_{P,Q}$ such that the index increases from $P$ to $Q$. Let $g_i^P$ and $g_i^Q$ denote the geodesics on the boundary of the plaque $P_i$ that separate $P_i$ from $P$ and $Q$, respectively. Let $g_i^P$ be the geodesic on the boundary of $Q$ that separates $P$ and $Q$. Define $\alpha = \alpha_1 - \alpha_0$. Let $\alpha(P, P_i)$ denote the $\alpha$-mass of a geodesic arc with endpoints in $P$ and $P_i$; similar definition for $\alpha(P, Q)$. Define

$$\varphi_P = T_{g_i^P} T_{g_i^P}^{-\alpha(P, P_1)} T_{g_i^P}^{-\alpha(P, P_2)} \cdots T_{g_i^P}^{-\alpha(P, P_n)} T_{g_i^P}^{-\alpha(P, P)} T_{g_i^P}^{-\alpha(P, Q)}$$

where $T_{g_i^P}^{-\alpha(P, P_i)}$ is the hyperbolic translation with axis $g_i^P$ which is oriented to the left as seen from $P$ and with the translation length $\alpha(P, P_i)$. Similarly, $T_{g_i^Q}^{-\alpha(P, Q)}$ is the hyperbolic translation with axis $g_i^Q$, oriented to the left as seen from $P$, and with the translation length $-\alpha(P, P_i)$.

Then $\varphi_{P,Q} = \lim_{P \to \mathcal{P}_{P,Q}} \varphi_P$ and this defines the realization of the cocycle $\alpha$ (cf. [3]). Let

$$\psi_P = T_{g_i^P} T_{g_i^P}^{-\alpha(P, P_1)} T_{g_i^P}^{-\alpha(P, P_2)} \cdots T_{g_i^P}^{-\alpha(P, P_n)} T_{g_i^P}^{-\alpha(P, P)} T_{g_i^P}^{-\alpha(P, Q)}$$
and let \( \psi_{P,Q} = \lim_{p \to P_{P,Q}} \psi_p \). It follows that

\[
\varphi_{P,Q} = \psi_{P,Q} \circ T_{g_P^\beta}^{\alpha(P,Q)}.
\]

The quantity \( \psi_{P,Q} \) is the difference from \( T_{g_P^\beta}^{\alpha(P,Q)} \) of the realization \( \varphi_{P,Q} \) of \( \sigma \).

Bonahon [3] proved that for a fixed surface \( S \) and small \( ||\alpha|| = ||\alpha_1 - \alpha_0|| \) the difference always lies in a compact subset of \( PSL_2(\mathbb{R}) \). We obtain bounds on the difference independent of genus of the surface.

Our main interest are bending pleated surfaces. The bending of (abstract) pleated surfaces with the pleating locus \( \lambda \) is completely determined by an \( \mathbb{R}/2\pi\mathbb{Z} \)-valued transverse cocycles for the geodesic lamination \( \lambda \) on \( S \) and each \( \mathbb{R}/2\pi\mathbb{Z} \)-valued transverse cocycle to \( \lambda \) is realized by an abstract pleated surface with pleating locus \( \lambda \) (cf. [3]). Denote by \( \mathcal{H}(\lambda, \mathbb{R}/2\pi\mathbb{Z}) \) the space of all \( (\mathbb{R}/2\pi\mathbb{Z}) \)-valued transverse cocycles to the lamination \( \lambda \). The space of all abstract pleated surfaces with the pleating locus \( \lambda \) is parametrized by \( \mathcal{C}(\lambda) \oplus i\mathcal{H}(\lambda, \mathbb{R}/2\pi\mathbb{Z}) \), where \( \mathcal{C}(\lambda) \) is the open cone in \( \mathcal{H}(\lambda, \mathbb{R}) \) which parametrizes the Teichmüller space \( T(S) \).

Let \( \beta \in \mathcal{H}(\lambda, \mathbb{R}/2\pi\mathbb{Z}) \). The formula

\[
\varphi = R_{g_{P,P_1}^\beta}^{\beta(P,P_1)} R_{g_{P_2}^{\beta}}^{\beta(P,P_2)} R_{g_{P_3}^{\beta}}^{\beta(P,P_3)} \ldots R_{g_{P_n}^{\beta}}^{\beta(P,P_n)} R_{g_{P_0}^{\beta}}^{\beta(P,P_0)},
\]

where \( R_{g_{P,P_i}^\beta}^{\beta(P,P_i)} \) is the rotation with the axis \( g_i^\beta \) and with the angle \( \beta(P,P_i) \), defines the realization of bending on a finite subset \( \mathcal{P} = \{P_1, P_2, \ldots, P_n\} \) of \( \mathcal{P}_{P,Q} \), and \( \varphi_{P,Q} = \lim_{p \to P_{P,Q}} \varphi_p \) defines the realization of \( \beta \) (cf. [3]). Note that the realization exists for all \( \beta \in \mathcal{H}(\lambda, \mathbb{R}/2\pi\mathbb{Z}) \) because of the compactness of \( \mathbb{R}/2\pi\mathbb{R} \). The difference from \( R_{g_{P,Q}^\beta}^{\beta(P,Q)} \) of the realization of \( \beta \) is given by

\[
(10) \quad \psi_{P,Q} = \lim_{p \to P_{P,Q} \psi_p}
\]

where

\[
(11) \quad \psi_p = R_{g_{P,P_1}^\beta}^{\beta(P,P_1)} R_{g_{P_2}^{\beta}}^{\beta(P,P_2)} R_{g_{P_3}^{\beta}}^{\beta(P,P_3)} \ldots R_{g_{P_n}^{\beta}}^{\beta(P,P_n)} R_{g_{P_0}^{\beta}}^{\beta(P,P_0)}.
\]

Let \( \tilde{f} : \mathbb{H}^2 \to \mathbb{H}^3 \) be the bending map for the bending pleated surface defined by \( \beta \), where \( \mathbb{H}^2 \) is identified with the \((xz)\)-half-plane in the upper half-space \( \mathbb{H}^3 = \{(z,t) : z \in \mathbb{C}, t > 0\} \). Then \( \tilde{f} \) does not necessarily extend to an injective map from \( \partial \mathbb{H}^2 \) into \( \partial \mathbb{H}^3 \). We will give a sufficient condition on \( \beta \) such that this extension is injective and our condition will be genus independent.

5. THE NESTED CONES

Let \( g \subset \mathbb{H}^3 \) be a geodesic ray with initial point \( p_0 \), and let \( p \in g \) be a point.

For \( 0 < \theta < \pi \), the cone \( \mathcal{C}(p,g,\theta) \) with vertex \( p \), axis \( g \) and angle \( \theta \) is the set of all \( w \in \mathbb{H}^3 \) such that the angle at \( p \) between the positive direction of \( g \) and the geodesic ray from \( p \) through \( w \) is less than \( \theta \). A non-zero vector \( (p,v) \in T^1(\mathbb{H}^3) \) uniquely determines a geodesic ray \( g \) which starts at the basepoint \( p \) of \( v \) and which is tangent to \( v \). Then \( \mathcal{C}(p,v,\theta) \) is by the definition \( \mathcal{C}(p,g,\theta) \). The shadow of the cone \( \mathcal{C}(p,g,\theta) \) is the set \( \partial_\infty \mathcal{C}(p,g,\theta) \) of endpoints at \( \partial_\infty \mathbb{H}^3 \) of all geodesic rays starting at \( p \) and inside \( \mathcal{C}(p,g,\theta) \).

For \( d > 0 \), let \( p_d \in g \) be the point on \( g \) which is on the distance \( d \) from \( p_0 = p \). Let \( \eta > 0 \) be the maximal angle such that \( \mathcal{C}(p_d,g,\eta) \subset \mathcal{C}(p_0,g,\theta) \). Then \( \eta = \eta(d,\theta) \)
is a continuous function of $d$ and $\theta$. For a fixed $0 < \theta < \pi$, we have $\eta(d, \theta) > \theta$ and $\eta(d, \theta) \to \theta$ as $d \to 0$. These properties are elementary.

We use the quaternions to represent the upper half-space model $\mathbb{H}^3 = \{z + tj : z \in \mathbb{C}, t > 0\}$ of the hyperbolic three-space (cf. Beardon [1]). The space of isometries of $\mathbb{H}^3$ is identified with $PSL_2(\mathbb{C})$ which is equipped with the norm

$$\|A\| = \max\{|a| + |b|, |c| + |d|\}$$

where $A(z) = \frac{az + b}{cz + d} \in PSL_2(\mathbb{C})$ and $ad - bc = 1$. The Poincaré extension to $\mathbb{H}^3$ of the action of $A \in PSL_2(\mathbb{C})$ on $\hat{\mathbb{C}}$ is computed in [1] to be

$$A(z + tj) = \frac{(az + b)(cz + d) + a\bar{c}t^2 + tj}{|cz + d|^2 + |c|^2h^2}.$$

An isometry of $\mathbb{H}^3$ which is close to the identity moves points on a bounded distance from $j \in \mathbb{H}^3$ by a small amount and the tangent vectors are rotated by a small angle with respect to the Euclidean parallel transport in $\mathbb{R}^3$. We give a quantitative statement for the above including the situation when the points are on the unbounded distances from $j \in \mathbb{H}^3$.

Given $P = z + tj \in \mathbb{H}^3$, we define

$$ht(P) = t$$

and

$$Z(P) = z.$$

Let $(P, v), (Q, w) \in T\mathbb{H}^3$ be two points in the tangent space $T\mathbb{H}^3$ of $\mathbb{H}^3$, where $P, Q \in \mathbb{H}^3$ and $v, w \in T\mathbb{H}^3$. Define the distance on $T\mathbb{H}^3$ by

$$D_{T\mathbb{H}^3}((P, v), (Q, w)) = \max\{|ht(P) - ht(Q)|, |Z(P) - Z(Q)|, |\angle(v, w)|\}$$

where $\angle(v, w)$ is the angle between the vectors $v$ and the Euclidean parallel translate of $w$ at the point $P$.

**Lemma 5.1.** Let $m_0 > 0$ and $C > 0$ be fixed. Define $\eta'(m_0, C) = \min\{\frac{1}{10(C + 1)m_0}, 1\}$. If $0 \leq \eta < \eta'$, $0 < m \leq m_0$, $A \in PSL_2(\mathbb{C})$ with

$$\|A - Id\| < \eta m$$

and $(P, v) \in T\mathbb{H}^3$ such that

$$D_{T\mathbb{H}^3}((P, v), (e^{-m}j, -j)) < C\eta m$$

then

$$D_{T\mathbb{H}^3}(A(P, v), (e^{-m}j, -j)) < C_1\eta m$$

where $C_1 = 2\pi(60C + 9)$.

**Proof.** Let $ht(P) = h$ and $Z(P) = z$. Let $P_1 = A(P)$, and $ht(P_1) = h_1$ and $Z(P_1) = z_1$. For $A \in PSL_2(\mathbb{C})$, the Poincaré extension (cf. [1]) of $A$ is given by the formula

$$A(P) = \frac{(az + b)(cz + d) + a\bar{c}h^2}{|cz + d|^2 + |c|^2h^2} + \frac{h}{|cz + d|^2 + |c|^2h^2}. j.$$

By the assumption, $|Z(P) - Z(e^{-m}j)| = |Z(P)| = |z| < C\eta m$ and $|ht(P) - ht(e^{-m}j)| = |h - e^{-m}| < C\eta m$. We set

$$\eta'(m_0, C) = \min\{\frac{1}{10(C + 1)m_0}, 1\}.$$
If $\eta \leq \eta'(m_0, C)$ then $C\eta m \leq \frac{1}{10}$ and $\eta m \leq \frac{1}{10}$. Moreover, $\|A - Id\| < \eta m$ implies that $|b|, |c|, |a - 1|, |d - 1| < \eta m$. The above inequalities together with some elementary computations give
\[
|ht(P_1) - e^{-m}| \leq (2C + 1)\eta m
\]
and
\[
|Z(P_1)| \leq 4(C + 1)\eta m.
\]
Let $v = v_1 + v_2i - j$ and $w = A'(P)v = w_1 + w_2i + w_3j$. The assumption $|\angle(v, -j)| < C\eta m$ implies that $|v_1|, |v_2| < \frac{\pi}{2}C\eta m$. Note that
\[
w_l = \frac{\partial A_l}{\partial x}v_1 + \frac{\partial A_l}{\partial y}v_2 + \frac{\partial A_l}{\partial h}v_3
\]
for $l = 1, 2, 3$, where $A = A_1 + A_2i + A_3j$. Some more elementary (and long) computations give
\[
|\frac{\partial A_1}{\partial x}|, |\frac{\partial A_1}{\partial y}|, |\frac{\partial A_2}{\partial x}|, |\frac{\partial A_2}{\partial y}| \leq 15
\]
at the point $P$,
\[
|\frac{\partial A_1}{\partial h}|, |\frac{\partial A_2}{\partial h}| \leq 9\eta m
\]
also at the point $P$,
\[
|\frac{\partial A_3}{\partial x}|, |\frac{\partial A_3}{\partial y}| \leq 12\eta m
\]
and
\[
|\frac{\partial A_3}{\partial h}| \geq 1 - 6\eta m
\]
at the point $P$. The above estimate provide
\[
|w_1|, |w_2| \leq (60C + 9)\eta m
\]
and
\[
|w_3| \geq 1 - (4C + 6)\eta m.
\]
Then
\[
|\angle(w, -j)| \leq \frac{\pi(60C + 9)}{1 - (4C + 6)\eta m} \eta m.
\]
Since $\eta \leq \frac{1}{(4C + 6)m_0}$ we obtain
\[
|\angle(w, -j)| \leq 2\pi(60C + 9)\eta m.
\]
(12)

**Lemma 5.2.** Fix $m_0 > 0$ and $C > 0$. Let $\eta''(m_0, C) = \min\{\frac{e^{-m_0}}{2(C + 1)m_0}, \frac{e^{-m_0}}{66\pi(60C + 9)}\}$. Then for each $0 \leq \eta < \eta''(m_0, C)$, for each $0 < m \leq m_0$ and for each $A \in PSL_2(\mathbb{C})$ with
\[
\|A - Id\| < \eta m
\]
we have
\[
\partial_{\infty}C(j, -j, \frac{\pi}{2}) \supset \partial_{\infty}C(A(P), A'(P)v, \frac{\pi}{2}),
\]
where $(P, v) \in T\mathbb{H}^3$ is such that
\[
D_{T\mathbb{H}^3}((P, v), (e^{-m}j, -j)) < C\eta m.
\]
Proof. Recall that \( h_1 = h (P_1) \) and \( z_1 = Z(P_1) \), where \( e^{-m} - (2C + 1)\eta m \leq h_1 \leq e^{-m} + (2C + 1)\eta m \) and \( |z_1| \leq 4(C + 1)\eta m \) by the proof of the above. Let \( y = |z_1| \).

Note that \( z_1 \) is the foot in \( \mathbb{C} \) of the vertical line through \( P_1 \) perpendicular to \( \mathbb{C} \).

Let \( x \in \mathbb{C} \) be the center of the euclidean hemisphere in \( \mathbb{H}^3 \) which passes through \( P_1 \) and is tangent to the unit radius euclidean hemisphere centered at \( 0 \in \mathbb{C} \). Let \( \varphi \) be the angle at \( P_1 \) between the radius of the hemisphere centered at \( x \) and the vertical line through \( P_1 \) (cf. Figure 1).

From Figure 1, we have

\[
(1 - x)^2 = (x - y)^2 + h_1^2 \leq x^2 + h_1^2
\]

which implies

\[
x \geq \frac{1 - h_1^2}{2} \geq \frac{1 - h_1}{2} \geq \frac{1 - e^{-m} - (2C + 1)\eta m}{2} \geq \frac{1 - (2C + 1)\eta m}{2}.
\]

If \( \eta \) satisfies

\[
\eta \leq \frac{1}{2(2C + 1)},
\]

then

\[
x \geq \frac{1}{4} m.
\]

This gives

\[
\tan \varphi = \frac{x - y}{h_1} \geq \frac{\frac{1}{2}m - 4(C + 1)\eta m}{e^{-m} + (2C + 1)\eta m} \geq \frac{1}{16} m
\]

when \( \eta \leq \frac{1}{32m_0(C + 1)} \).

Let \( \varphi_0 \) be the angle at \( P_1 \) in Figure 1 when \( m = m_0 \). Then we have

\[
\varphi \geq \frac{\varphi_0}{\tan \varphi_0} \tan \varphi \geq \frac{\varphi_0}{\tan \varphi_0} \frac{1}{16} m.
\]

We need an upper bound on \( \frac{\tan \varphi_0}{\varphi_0} \). Let \( x_0 \) and \( y_0 \) be the values of \( x \) and \( y \) when \( m = m_0 \). Similar to the above, we have

\[
(1 - x_0)^2 = (x_0 - y_0)^2 + h^2 \geq (x_0 - y_0)^2
\]
Lemma 5.3. Let \( \varphi \) be the positive \( z \)-axis in \( \mathbb{H}^3 \) and let \( h \) be the geodesic in the \( xz \)-plane that intersects \( g \) at the point \( e^{-m}j \in \mathbb{H}^3 \) subtending the angle \( \theta \). Then for any \( m \) with \( 0 < m < m_0 \), any \( \theta \) with \( 0 < \theta < \frac{\pi}{16} \), and any \( (P, v) \in T\mathbb{H}^3 \) with

\[
D_{\mathbb{H}^3}((P, v), (e^{-m}j, -j)) = \delta < \frac{1}{4}
\]

we have

\[
D_{\mathbb{H}^3}(R^\varphi_h(P, v), (e^{-m}j, -j)) \leq 20\delta + 40\sqrt{2}e^{m_0}\theta
\]

for any \( \varphi \in \mathbb{R} \), where \( R^\varphi_h \) is the rotation of \( \mathbb{H}^3 \) with the angle \( \varphi \) around the axis \( h \).

Proof. Let \( b < 0 \) and \( a > 0 \) be the endpoints of \( h \). Since the angle between \( g \) and \( h \) is \( \theta \), it follows \( a = -b \tan^2 \theta \). Moreover \( h \) intersects \( g \) in the geodesic arc \([j, e^{-m_0}]\) which gives \( e^{-m_0} \leq \sqrt{-2ab} \leq 1 \). Consequently

\[
\frac{\sqrt{2}e^{-m_0}}{\theta} \leq |b| \leq \frac{\pi}{\sqrt{2}} \frac{1}{\theta}
\]

and

\[
\frac{e^{-2m_0}}{\sqrt{2}\pi} \theta \leq a \leq \frac{e^{m_0}}{2\sqrt{2}} \theta.
\]

Note that

\[
R^\varphi_h(z) = \frac{a - e^{i\varphi}b}{e^{i\varphi/2}(a-b)} z + \frac{ab(e^{i\varphi} - 1)}{e^{i\varphi/2}(a-b)} + \frac{e^{i\varphi}a - b}{e^{i\varphi/2}(a-b)}
\]
and let \( R^c_h(z + tj) \) be the extension of \( R^c_h \) to \( \mathbb{H}^3 \). Then

\[
Z(R^c_h(z + tj)) = \frac{1}{\cosh^2(tj + \frac{1}{2}e^{-tj})} \left( \frac{1}{\cosh^2(tj + \frac{1}{2}e^{-tj})} z + \frac{1}{\cosh^2(tj + \frac{1}{2}e^{-tj})} \right) + \frac{1}{\cosh^2(tj + \frac{1}{2}e^{-tj})} \left( \frac{1}{\cosh^2(tj + \frac{1}{2}e^{-tj})} z + \frac{1}{\cosh^2(tj + \frac{1}{2}e^{-tj})} \right)
\]

and

\[
ht(R^0_h(P, v)) = \frac{t}{\cosh^2(\frac{1}{2}e^{-t}) + \frac{1}{\cosh^2(\frac{1}{2}e^{-t})} \left( \frac{1}{\cosh^2(\frac{1}{2}e^{-t})} z + \frac{1}{\cosh^2(\frac{1}{2}e^{-t})} \right)^2}.
\]

The bounds on \( \delta \) and \( \theta \) give that \( a < \frac{1}{3}, |z| < \frac{1}{3} \) and \( |b| > 4 \). Similar to the proof of Lemma 5.4, we obtain the desired estimates. \( \square \)

**Lemma 5.4.** Let \( A \) be a rotation in \( \mathbb{H}^3 \) with fixed points \( a, b \in \mathbb{R} \) that rotates by an angle \( \epsilon \). If the geodesic with endpoints \( a \) and \( b \) intersects the ball of radius \( m_0 > 0 \) centered at \( j \in \mathbb{H}^3 \), then

\[
\|A - Id\| \leq (1 + e^{2m_0}) |\epsilon|/2.
\]

**Proof.** Note that

\[
A(z) = \frac{a - be^{i\epsilon}}{(a - b)e^{i\epsilon}} z + \frac{ab(e^{i\epsilon} - 1)}{(a - b)e^{i\epsilon}}.
\]

Assume that \( |a| \leq |b| \). Since \( l_{a,b} \) intersects the ball of radius \( m_0 \) centered at \( j \in \mathbb{H}^3 \), it follows that \( |a| \leq e^{m_0} \) and \( |a - b| \geq 2e^{m_0} \). Then

\[
\left| \frac{a - be^{i\epsilon}}{(a - b)e^{i\epsilon}} - 1 \right| = \frac{|1 - e^{i\epsilon}| \cdot |a + be^{i\epsilon}|}{|a - b|} \leq (1 + \frac{|a|}{|a - b|} \cdot \epsilon) \leq (1 + \frac{2|a|}{|a - b|} \cdot \epsilon) \leq (1 + e^{2m_0}) \cdot \epsilon/2.
\]

Further, we claim that

\[
\frac{ab(e^{i\epsilon} - 1)}{a - b} |\epsilon| \leq e^{2m_0} \cdot \epsilon.
\]

To see this, note that \( j \in \mathbb{H}^3 \) is on the distance at most \( m_0 \) from the geodesic \( l_{a,b} \). Then [1 formula (7.20.4)] gives

\[
\sinh m_0 \geq \frac{\cosh^2 m_0 + ab}{(|b - a|) \cosh m_0}
\]

which implies the above. In a similar fashion, we obtain

\[
\left| \frac{ae^{i\epsilon} - b}{(a - b)e^{i\epsilon}} - 1 \right| \leq (1 + e^{2m_0}) \cdot \epsilon/2
\]

and

\[
\left| \frac{1 - e^{i\epsilon}}{(a - b)e^{i\epsilon}} \right| \leq e^{m_0} \epsilon.
\]

\( \square \)

The following lemma is well-known [3] and we estimate the constant involved.
Lemma 5.5. Let $g_1$ and $g_2$ be geodesics in $\mathbb{H}^2 \subset \mathbb{H}^3$ that have a common endpoint and that intersect the ball $B_{m_0}(i)$ of radius $m_0 > 0$ centered at $i \in \mathbb{H}^2$. Let $s$ be a geodesic arc that connects $g_1$ and $g_2$ inside $B_{m_0}(i)$. Then

$$||R_{g_1}^\epsilon \circ R_{g_2}^{-\epsilon} - Id|| \leq 2e^{m_0}(1 + e^{m_0})|s| \cdot |\epsilon|,$$

where $|s|$ is the hyperbolic length of the geodesic arc $s$.

Proof. Let $R_i$ be a rotation around $i \in \mathbb{H}^2$. Then $R_i(z) = \frac{az + b}{cz + d}$ with $a^2 + b^2 = 1$, $a, b \in \mathbb{R}$. Then $||R_i|| = |a| + |b| \leq 2$.

Let $m \in \mathbb{R} \cup \{\infty\}$ be the common endpoint of the geodesics $g_1$ and $g_2$. If the rotation $R_i$ is chosen such that $R_i(x) = \infty$, then $R_i \circ R_{g_j}^\epsilon \circ R_i^{-1} = R_{g_j}^\epsilon$ for $j = 1, 2$ where $g_j'$ also intersects the ball around $i \in \mathbb{H}^2$ of radius $m_0$ and $R_{g_j}^\epsilon$ is the rotation around the axis $g_j'$ by the angle $\epsilon$. Note that

$$||R_{g_1}^\epsilon \circ R_{g_2}^{-\epsilon} - Id|| \leq ||R_{g_1}^{-1}|| \cdot ||R_{g_2}^\epsilon \circ R_{g_2}^{-\epsilon} - Id|| \cdot ||R_i|| = 2||R_{g_1}^\epsilon \circ R_{g_2}^{-\epsilon} - Id||.$$

Let $a$ and $b$ be the endpoints of $g_1'$ and $g_2'$, respectively. A short computation gives

$$||R_{g_1}^\epsilon \circ R_{g_2}^{-\epsilon} - Id|| = 2|a - b| \cdot |\sin \frac{\epsilon}{2}| \leq |a - b| \cdot |\epsilon|.$$

Let $P_1 \in g_1'$ and $P_2 \in g_2'$ be the endpoints of $s$. Without loss of generality, assume that $ht(P_1) \geq ht(P_2)$. Let $l'$ be the arc issued from $P_2$ that is orthogonal to $g_1'$. Let $x = ht(P_2)$. A direct computations gives

$$l' = \log \left[ \frac{|a - b|}{x} + \sqrt{1 + \left( \frac{|a - b|}{x} \right)^2} \right]$$

Note that $h = \frac{|a - b|}{x}$ is the length of the horocyclic arc centered at $i$ between $g_1'$ and $g_2'$ at the height $x$. Let $h_0$ is the maximum of the lengths of horocyclic arcs with the center at $\infty$ and inside the ball of radius $m_0$ centered at $i \in \mathbb{H}^2$. Then we obtain

$$l' \geq \log(1 + h) \geq \frac{1}{1 + h_0}h$$

which implies

$$|a - b| \leq x(1 + h_0)|l'| \leq e^{m_0}(1 + e^{m_0})|l'|$$

and lemma follows.

6. Injectivity on the boundary

We fix an embedding of $\mathbb{H}^2$ in $\mathbb{H}^3$ defined by the mapping $z = x + yi \mapsto x + yj$ for $y > 0$ and $x \in \mathbb{R}$. We show that the bending map of a pleated surface realizing a transverse cocycle $\beta \in \mathcal{H}(\lambda, \mathbb{R}/2\pi\mathbb{Z})$ induces an injective map from $\partial_\infty \mathbb{H}^2$ into $\partial_\infty \mathbb{H}^3$ under geometric conditions on the bending transverse cocycle $\beta$ given in Theorem 1.3.

Proof of Theorem 1.3. Let $\{k_1, \ldots, k_n\}$ be a set of arcs obtained by subdividing a geometric set of arcs for the geodesic lamination $\lambda$ as in §2. Let $\tau$ be the corresponding geometric train track. Let $\lambda$ and $\tilde{\tau}$ be lifts to $\mathbb{H}^2$ of the geodesic lamination $\lambda$ and the geometric train track $\tau$. For simplicity, we denote by $k_j$ any lift of an arc $k_j$ and by $E$ any lift of an edge $E$.

Let $g$ be an arbitrary geodesic in $\mathbb{H}^2$. Our goal is to show that the endpoints of $g$ in $\partial_\infty \mathbb{H}^2$ are mapped to distinct points in $\partial_\infty \mathbb{H}^3$ under the bending map $\tilde{f}$.
corresponding to the transverse cocycle $\beta$. If $g$ is contained in $\tilde{\tau}$ then it coincides with a geodesic of $\tilde{\lambda}$. Since the bending map $\tilde{f}$ sends a geodesic in $\tilde{\lambda}$ to a geodesic in $\partial_{\infty}\mathbb{H}^3$, it follows that the endpoints of $g$ are mapped to distinct points.

The main case to consider is when $g$ intersects $\mathbb{H}^2 - \tilde{\tau}$. Let $p \in g \cap (\mathbb{H}^2 - \tilde{\tau})$. Then $p$ divides the geodesic $g$ into two geodesic rays $g_1$ and $g_2$. We consider the geodesic ray $g_1$ and similar conclusions hold for $g_2$.

Divide the geodesic ray $g_1$ into subarcs using the points of intersections of $g_1$ with the boundary sides of the edges of $\tilde{\tau}$ (i.e., boundary sides of rectangles) as follows. The point $p$ is the initial point of $g_1$. The first point $a_1$ of the intersection of $g_1$ with $\tilde{\tau}$ is at a long side of an edge $E$ of $\tilde{\tau}$. We consider the next point $p_1$ of the intersection of $g_1$ with a boundary side of an edge $E$ of $\tilde{\tau}$. If $p_1$ is on the long side of $E$ then we set $b_1 = p_1$ and $[a_1, b_1]$ is the first subarc in the division of $g_1$. If $p_1$ is on the short side of the edge $E$ then we consider the next point $q_1$ of the intersection of $g_1$ with boundary sides of the edges of $\tilde{\tau}$. If $q_1$ is on a long side of an edge then we set $b_1 = q_1$. If $q_1$ is on a short side and the next point of the intersection $r_1$ is on long side, then we set $b_1 = r_1$. Note that the arc $[a_1, b_1]$ intersects interiors of at most three edges of $\tilde{\tau}$.

Assume that we have defined first $n$ arcs $\{[a_1, b_1], \ldots, [a_n, b_n]\}$ and we proceed to define $(n + 1)$-st arc. If $b_n$ is on the boundary of two edges of $\tilde{\tau}$, then we set $a_{n+1} = b_n$; otherwise we let $a_{n+1}$ to be the first intersection point of $g_1$ with the boundary sides of the edges of $\tilde{\tau}$ that comes after $b_n$. Then $b_{n+1}$ is chosen in a same fashion as $b_1$ above. We continue this process indefinitely. Thus we obtain a family of arcs $\{[a_n, b_n]\}_{n \in \mathbb{N}}$. If $a_n$ does not belong to a plaque of $\tilde{\lambda}$ then we replace it with a nearby point on $g_1$ which belongs to a plaque and call the new point $a_n$ again. Do the same for $b_n$. This situation occurs when $a_n$ or $b_n$ belong to the intersections of a short side of an edge of $\tilde{\tau}$ and the geodesic lamination $\lambda$. The complement in $g_1$ of the union of arcs $[a_n, b_n]$ does not intersect $\tilde{\lambda}$.

We consider a sequence of nested cones $C(a_n, g_1, \frac{\pi}{2}) \supset C(b_n, g_1, \frac{\pi}{2})$ for $n \in \mathbb{N}$. Normalize the bending map $\tilde{f}$ to be the identity at $a_n$. It is enough to prove that

$$\partial_{\infty}\tilde{f}(C(b_n, g_1, \frac{\pi}{2})) \subset \partial_{\infty}C(a_n, g_1, \frac{\pi}{2}).$$

Because this property is geometric, it is independent of the normalization of the bending map and it can be repeated along the sequence of arcs $\{[a_n, b_n]\}$. Thus we obtain that the sequence of the images under the bending map of the shadows of the cones is nested and in particular, the image under $\tilde{f}$ of the endpoint of $g_1$ is contained in $\partial_{\infty}C(p, g_1, \frac{\pi}{2})$. Similarly, the image under $\tilde{f}$ of the endpoint of $g_2$ is contained in $\partial_{\infty}C(p, g_2, \frac{\pi}{2})$. Since $\partial_{\infty}C(p, g_1, \frac{\pi}{2}) \cap \partial_{\infty}C(p, g_2, \frac{\pi}{2}) = \emptyset$, the bending map $\tilde{f}$ sends the endpoints of $g$ into distinct points of $\partial_{\infty}\mathbb{H}^3$.

To finish the proof, it remains to show that

$$(19) \quad \partial_{\infty}\tilde{f}(C(b_n, g_1, \frac{\pi}{2})) \subset \partial_{\infty}C(a_n, g_1, \frac{\pi}{2})$$

where the bending map $\tilde{f} : \mathbb{H}^2 \to \mathbb{H}^3$ is normalized to be the identity at the point $a_n$. The rest of the proof is divided into cases depending on the combinatorics of the intersection of $[a_n, b_n]$ with the edges of $\tilde{\tau}$. 

Let \( m_0 \) be the maximum of the diameters of the edges of the train track \( \tilde{\tau} \). Then each \([a_n, b_n]\) has length less than or equal to \(3m_0\) since it intersects at most three edges of \( \tilde{\tau} \).

**Case 1.** Assume that \([a_n, b_n]\) intersects interior of a single edge \( E \) of \( \tilde{\tau} \) and that it connects the long sides of \( E \). Let \( P \) and \( Q \) be the plaques that contain \( a_n \) and \( b_n \), respectively. We normalize such that \( a_n = i \in \mathbb{H}^2 \). Let \( k_j \) be a short side of \( E \) (which means that it is a lift to \( \mathbb{H}^2 \) of an arc in \([k_1, \ldots, k_n]\) denoted by \( k_j \) again, cf. §2). Then \(|k_j| \leq k^* \leq \frac{\pi}{20} \). Note that \( k_j \) is contained in the ball of radius \( m_0 \) centered at \( a_n = i \in \mathbb{H}^2 \). Recall that

\[
\psi_{P,Q} = \lim_{P_i \to P, Q} \psi_{i}
\]

where \( \mathcal{P}_i = \{P_1, P_2, \ldots, P_l\} \) is a set of plaques between \( P \) and \( Q \) in that order, and

\[
\psi_{i} = R_{g_{P_i}^P}^{\beta(P, P_i)} \circ R_{g_{Q}^P}^{\beta(P, P_i)} \circ \cdots \circ R_{g_{P_1}^P}^{\beta(P, P_i)} \circ R_{g_{Q}^P}^{\beta(P, P_i)} = \prod_{i=1}^{l} R_{g_{P_i}^P}^{\beta(P, P_i)} \circ R_{g_{Q}^P}^{\beta(P, P_i)}
\]

with \( g_{P_i}^P \) being the geodesic on the boundary of \( P_i \) which separates \( P_i \) from \( P \); similar for \( g_{Q}^P \); and \( R_g^{a} \) being a rotation with the axis \( g \) and the rotation angle \( a \).

Since points of \( k_j \) are on the distance at most \( m_0 \) from \( i \in \mathbb{H}^2 \), it follows by Lemma 5.5 that

\[
\| R_{g_{P_i}^P}^{\beta(P, P_i)} \circ R_{g_{Q}^P}^{\beta(P, P_i)} - Id \| \leq 2em_0(1 + e^{m_0})|\beta(P, P_i)| \cdot |d_i|
\]

where \(|d_i|\) is the length of the gap \( d_i = k_j \cap P_i \) and \( \beta(P, P_i) \) is taken to be in the interval \((-\pi, \pi]\).

We estimate the norm of \( \prod_{i=1}^{l} R_{g_{P_i}^P}^{\beta(P, P_i)} \circ R_{g_{Q}^P}^{\beta(P, P_i)} \) for arbitrary \( l \). By (20), we have that

\[
\| \prod_{i=1}^{l} R_{g_{P_i}^P}^{\beta(P, P_i)} \circ R_{g_{Q}^P}^{\beta(P, P_i)} \| \leq \prod_{i=1}^{l} \| R_{g_{P_i}^P}^{\beta(P, P_i)} \circ R_{g_{Q}^P}^{\beta(P, P_i)} \| \leq \prod_{d} (1 + 2\pi e^{m_0}(1 + e^{m_0})|d|)
\]

where the last product is over all gaps \( d \) of \( k_j \) with respect to \( \tilde{\lambda} \). Then

\[
\log \prod_{i=1}^{l} \| R_{g_{P_i}^P}^{\beta(P, P_i)} \circ R_{g_{Q}^P}^{\beta(P, P_i)} \| \leq \sum_{d} \log(1 + 2\pi e^{m_0}(1 + e^{m_0})|d|) \leq \sum_{d} 2\pi e^{m_0}(1 + e^{m_0})|d| \leq 2\pi e^{m_0}(1 + e^{m_0})|k_j|.
\]

Since \(|k_j| \leq \frac{\pi}{20} \), we have

\[
\| \prod_{i=1}^{l} R_{g_{P_i}^P}^{\beta(P, P_i)} \circ R_{g_{Q}^P}^{\beta(P, P_i)} \| \leq e^{2\pi e^{m_0}(1 + e^{m_0})|k_j|} \leq C(m_0)
\]

where

\[
C(m_0) = e^{2m_0}.
\]

This implies

\[
\| \psi_{P,Q} - Id \| \leq C(m_0) \sum_{d} \| R_{g_{P_d}^P}^{\beta(P, P_d)} \circ R_{g_{Q}^P}^{\beta(P, P_d)} - Id \|
\]
where the sum is over all gaps $d$ of $k_j$ (i.e. components of $k_j \setminus \lambda$) except the two components which contain the endpoints of $k_j$: $P_d$ is the plaque which contains $d$.

We divide $\sum ||R^{\beta(P,P_d)}_{g_d} \circ R^{-\beta(P,P_d)}_{g_d} - Id||$ into two sums as follows. The first sum $\sum'$ is over finitely many gaps $\{d_l : l = 1, \ldots, n_j\}$ of $k_j$ used in the definition of $||\beta||_{\text{var},k_j}$ and the second sum $\sum''$ is over the remaining gaps of $k_j$. We have that each term of $\sum'$ corresponding to a gap $d$ of $k_j$ is bounded from above by $2e^{m_0}(1 + e^{m_0})|d| \cdot ||\beta||_{\text{var}}$ by (20). Thus $\sum' \leq 2e^{m_0}(1 + e^{m_0})|k_j| \cdot ||\beta||_{\text{var}}$. The term of the second sum $\sum''$ which corresponds to a gap $d$ of $k_j$ is bounded by $2\pi e^{m_0}(1 + e^{m_0})|d|$ by (20). Since the sum of the lengths of all $d \in \sum''$ is less than $\delta|k_j|$, we obtain $\sum'' \leq 2\pi e^{m_0}(1 + e^{m_0})\delta|k_j|$. Thus taking the two estimates together, we have

$$\|\psi_{P,Q} - Id\| \leq C'(m_0)(||\beta||_{\text{var}} + \delta)|k_j|$$

where

$$C'(m_0) = 2\pi e^{m_0}(1 + e^{m_0})e^{2m_0}.$$

By Lemma 5.4 we get that $||R^{\beta(P,P_d)}_{g_d} \circ R^{-\beta(P,P_d)}_{g_d} - Id|| \leq (1 + e^{2m_0})|\beta(k_j)|/2 \leq \frac{1 + e^{2m_0}}{2}||\beta||_{\max}$ by the assumption and by $\beta(P,Q) = \beta(k_j)$. Thus $\|R^{\beta(P,P_d)}_{g_d} \circ R^{-\beta(P,P_d)}_{g_d} - Id\| \leq \frac{3 + e^{2m_0}}{2}$ if we restrict to $\beta$ with $||\beta||_{\max} \leq 1$. Consequently, we obtain

$$\|\varphi_{P,Q} - Id\| \leq C'(m_0)\frac{3 + e^{2m_0}}{2}(||\beta||_{\text{var}} + \delta)|k_j| + \frac{1 + e^{2m_0}}{2}||\beta||_{\max} \leq C''(m_0)(||\beta||_{\text{var}} + \delta)|k_j| + \frac{1 + e^{2m_0}}{2}||\beta||_{\max} \leq C''(m_0)(||\beta||_{\text{var}} + \delta)|k_j| + \frac{1 + e^{2m_0}}{2}||\beta||_{\max}$$

where

$$C''(m_0) = C'(m_0)\frac{3 + e^{2m_0}}{2} = \pi(1 + e^{m_0})(3 + e^{2m_0})e^{m_0}e^{2m_0}.$$

The inequality (22) holds for both short sides $k_1$ and $k_2$ of the edge $E$. Without loss of generality we assume that $|k_1| = \min\{|k_1|, |k_2|\}$. Lemma 2.2 implies that $|[a_n, b_n]| \geq \frac{1}{20e^{m_0}}|k_1|$, where $|[a_n, b_n]|$ is the length of the hyperbolic arc $[a_n, b_n]$. Let $\eta''(m_0, 0)$ be the constant from Lemma 5.4 for the given $m_0$ and $C = 0$. Lemma 5.4 implies that the desired nesting of the cones at the right side of (22) is less than $\eta''(m_0, 0)|[a_n, b_n]|$. To achieve this, it is enough to set

$$\epsilon = \frac{1}{2} \min\left\{ \frac{\eta''(m_0, 0)}{60e^{m_0}C''(m_0)}, \frac{2\eta''(m_0, 0)}{3(1 + e^{2m_0})} \right\} = \frac{\eta''(m_0, 0)}{120e^{m_0}C''(m_0)}$$

and

$$\delta = \frac{\eta''(m_0, 0)}{120e^{m_0}C''(m_0)}$$

for $\eta''(m_0, 0)$ given by Lemma 5.4. The nesting of the cones at $a_n$ and $b_n$ is guaranteed by Lemma 5.4 because $|[a_n, b_n]| \leq m_0$.

Case 2. Assume that $[a_n, b_n]$ enters the edge $E_1$ through a long side, then enters the edge $E_2$ through a short side in common with $E_2$ and exists $E_2$ through a long side of $E_2$. Note that the set of geodesics of $\lambda$ which intersect the arc $[a_n, b_n]$ is either the set of geodesics which traverses the edge $E_1$ or the set of geodesics which traverses the edge $E_2$. For definiteness, assume that we are in the former case. Let $k_1$ be the short side of $E_1$ that intersects the boundary of $E_2$ and let $c_n = [a_n, b_n] \cap k_1$. Normalize such that $a_n = i \in \mathbb{H}^2$. Let $k_1^i$ and $k_2^i$ be the two
arcs obtained by dividing $k_1$ with the point $c_n$ such that the arcs $[a_n, c_n]$ and $k_1^1$ have endpoints on the same long side $l_1$ of $E_1$, and that the arcs $[c_n, b_n]$ and $k_2^1$ have endpoints on the same long side $l_2$ of $E_2$. Let $h_1$ and $h_2$ be two arcs from $c_n$ orthogonal to $l_1$ and $l_2$, respectively.

It is immediate that $|[a_n, c_n]| \geq |h_1|$ and $|[c_n, b_n]| \geq |h_2|$. The hyperbolic sine rule and the fact that $|k_1| \leq \frac{1}{20}$ give $|k_1^1| \leq \cosh \frac{1}{20}|h_1| \leq \cosh \frac{1}{20}|[a_n, c_n]|$ and $|k_2^1| \leq \cosh \frac{1}{20}|h_2| \leq \cosh \frac{1}{20}|[c_n, b_n]|$. We obtain $|k_1| \leq \cosh \frac{1}{20}|a_n, b_n|$.

Similar to Case 1, we get

$$
\|\varphi_{P,Q} - Id\| \leq C''(2m_0)(\|\beta\|_{\text{var}} + \delta)\|k_1\| + \frac{1 + \epsilon^{4m_0}}{2}\|\beta\|_{\text{max}}
$$

where we use $2m_0$ instead of $m_0$ because the diameter of $E_1 \cup E_2$ is $2m_0$.

The nesting of the cones at $a_n$ and $b_n$ follows as in Case 1 with the constants $\epsilon = \delta = \frac{\eta''(2m_0, 0)}{120e^{2m_0}C''(2m_0)}$ for $\eta''(2m_0, 0)$ given by Lemma 5.2. The later case is dealt with in the same fashion with the same constants.

Case 3. Assume that $[a_n, b_n]$ enters a short side of an edge $E_1$, then it enters a short side of an edge $E_2$ which is in common with $E_1$ and it exists a long side of $E_2$. See Figure 2 and Figure 3 for different possibilities of the relative positions of $E_1$, $E_2$ and $g_1$. Let $P$ and $Q$ be the plaques that contain $a_n$ and $b_n$, respectively.

For the position in Figure 2 we argue as follows. Let $E_1^1$ be the incoming edge which meets $E_2$ at the same short side as $E_1$. Let $Q'$ be the plaque that separates the geodesics of $\tilde{\lambda}$ that traverse $E_1$ from the geodesics of $\tilde{\lambda}$ that traverse $E_1^1$. Then we have

$$
\varphi_{P,Q} = \varphi_{P,Q'} \circ \varphi_{Q',Q}.
$$

Note that $\beta(Q', Q) = \beta(E_1^1)$ which implies that $|\beta(Q', Q)| \leq \|\beta\|_{\text{max}}$. 

\[\text{Figure 2.}\]
By reasoning as in Case 1, we obtain

\[
\| \varphi_{Q',Q} - \text{Id} \| \leq C''(2m_0)(\| \beta \|_{\text{vars}} + \delta)|k_1| + \frac{1 + e^{4m_0}}{2}\| \beta \|_{\text{max}}
\]

where \(k_1\) is the short side of \(E_1^{1}\) intersecting \(E_2\). Recall that we normalized such that \(a_n = j\) and \(b_n = e^{-m}j\) for \(0 \leq m \leq 2m_0\). Since \([a_n, b_n]\) connects the short sides of the edge \(E_1\), it follows that \(||a_n, b_n|| \geq m_* > 0\).

We apply Lemma 5.1 to \((e^{-m}j, -j) \in T\mathbb{H}^3\) with the constants \(2m_0, C = 0\) and \(m \geq m_* > 0\). We get

\[
D_{T\mathbb{H}^3}(\varphi_{Q',Q}(e^{-m}j, -j), (e^{-m}j, -j)) \leq 18\pi[C''(2m_0)(\| \beta \|_{\text{vars}} + \delta)|k_1| + \frac{1 + e^{4m_0}}{2}\| \beta \|_{\text{max}}] < \eta'(2m_0, 0)
\]

(23)

whenever

\[
\frac{C''(2m_0)(\| \beta \|_{\text{vars}} + \delta)|k_1| + \frac{1 + e^{4m_0}}{2}\| \beta \|_{\text{max}}}{m_*} < \frac{\eta'(2m_0, 0)}{18\pi}.
\]

(24)

Recall that

\[
\varphi_{P,Q'} = \psi_{P,Q'} \circ R^\beta_{\beta(P,Q')}\]

where, in general, \(\beta(P, Q') \neq \beta(E_1)\) since \(a_n\) is on a short side of \(E_1\). Let \(c_n\) be the point of the intersection between \([a_n, b_n]\) and the common boundary \(k_1\) of \(E_1\) and \(E_2\). It follows that the arc \([a_n, c_n]\) and the subarc of any geodesic of \(\hat{\beta}\) that traverses \(E_1\) are remaining close for the length \(m_*\). This implies that they intersect at small angles. We give a numerical statement.

**Lemma 6.1.** Let \(E\) be an edge of a train track such that the shortest geodesic connecting the short sides has length at least \(l > 0\) and the maximum of the lengths of the short sides is at most \(x\). Let \(a_1\) and \(a_2\) be geodesic arcs in \(E\) connecting the short sides intersecting at an angle \(\phi\). Then

\[
\phi \leq \frac{\pi}{2}(\coth \frac{l}{2})x.
\]

**Proof.** Let \(A = a_1 \cap a_2\). Then \(A\) divides \(a_1\) into two sub arcs \(a'_1\) and \(a'_2\). Without loss of generality, we can assume that the length of \(a'_1\) is at least \(\frac{1}{2}l\). Then the length of the subarc \(a'_2\) of the arc \(a_2\) which connects the short side of \(\hat{E}\) which contains an endpoint of \(a'_1\) to the point \(A\) is at least \(\frac{1}{2}l - x\). Let \(h\) be the geodesic arc issued from the endpoint of \(a'_1\) on a short side of \(E\) orthogonal to \(a'_2\). The length of \(h\) is at most \(x\). We obtained a right angled triangle with one angle \(\varphi\) whose opposite side has length at most \(x\), and the side opposite to the right angle has the length at most \(\frac{1}{2}l\). The hyperbolic sine rule gives

\[
\sin \phi = \frac{\sinh x}{\sinh \frac{l}{2}}
\]

which implies

\[
\phi \leq \frac{\pi}{2}(\coth \frac{l}{2})x.
\]

\(\square\)
We apply $R_{g_{g_Q}^P}^{P(Q')}$ to $\varphi_{Q',Q}(e^{-m_j},-j)$. By Lemma 6.1 the angle of intersection $\phi$ between $g_{g_Q}^P$ and $g_1$ satisfies

$$\phi \leq \frac{\pi}{2}(\coth \frac{m}{2})k^*.$$  

By Lemma 5.3 we have

$$D_{\mathcal{H}}([R_{g_{g_Q}^P}^{P(Q')} \circ \varphi_{Q',Q}](e^{-m_j},-j),(e^{-m_j},-j)) \leq$$

$$20D_{\mathcal{H}}(\varphi_{Q',Q}(e^{-m_j},-j),(e^{-m_j},-j)) + 40\sqrt{2}\epsilon^{2m_0} \phi$$

when $D_{\mathcal{H}}(\varphi_{Q',Q}(e^{-m_j},-j),(e^{-m_j},-j)) < \frac{1}{4}$ and $\phi < \frac{e^{-2m_0}}{16}$. The former condition is satisfied because $D_{\mathcal{H}}(\varphi_{Q',Q}(e^{-m_j},-j),(e^{-m_j},-j)) < \eta'(2m,0) \leq \frac{1}{4}$. To achieve the later condition we require that

$$\frac{\pi}{2}(\coth \frac{m}{2})k^* < \frac{e^{-2m_0}}{16}$$

which implies that

$$k^* < \frac{e^{-2m_0} \tanh \frac{m}{2}}{8\pi}.$$  

By (23), (26) and (20) we have

$$D_{\mathcal{H}}([R_{g_{g_Q}^P}^{P(Q')} \circ \varphi_{Q',Q}](e^{-m_j},-j),(e^{-m_j},-j)) \leq$$

$$\frac{560\pi}{m_\ast} C''(2m_0)(\|\beta\|_{\text{vars}} + \delta)|k_1| + \frac{1 + e^{4m_0}}{2}\|\beta\|_{\text{max}}m_\ast +$$

$$+ 20\sqrt{2}\epsilon^{2m_0} \coth \frac{m}{2}k^* m_\ast$$

Let $\eta''(2m_0,1)$ be the constant from Lemma 5.2 If

$$\delta, \|\beta\|_{\text{vars}} \leq \frac{m_\ast}{4 \cdot 560\pi |k_1| C''(2m_0)} \eta''(2m_0,1),$$

$$\|\beta\|_{\text{max}} \leq \frac{m_\ast}{4 \cdot 560\pi 3e^{4m_0}} \eta''(2m_0,1)$$

and

$$k^* \leq \frac{m_\ast}{20\sqrt{2}\pi e^{2m_0} \coth \frac{m}{2}} \eta''(2m_0,1),$$

then

$$D_{\mathcal{H}}([R_{g_{g_Q}^P}^{P(Q')} \circ \varphi_{Q',Q}](e^{-m_j},-j),(e^{-m_j},-j)) \leq \eta''(2m_0,1)m_\ast.$$  

By Case 1, we immediately obtain the estimate

$$\|\psi_{P,Q'} - Id\| \leq C''(2m_0)(\|\beta\|_{\text{vars}} + \delta)|k_1|.$$  

It follows that

$$\|\psi_{P,Q'} - Id\| \leq \eta''(2m_0,1)$$

if

$$\delta, \|\beta\|_{\text{vars}} \leq \frac{m_\ast}{2C''(2m_0)|k_1|}$$

which is satisfied because $C''(2m_0) \leq C''(2m_0)$ and by (28). Therefore, Lemma 5.2 and $|k_j| \leq 1/20$ implies the nesting of the cones if

$$\epsilon = \frac{m_\ast \eta''(2m_0,1)}{130\pi C''(2m_0)} \leq \min\left\{ \frac{m_\ast \eta''(2m_0,1)}{4 \cdot 560\pi C''(2m_0)|k_1|}, \frac{m_\ast \eta''(2m_0,1)}{4 \cdot 560\pi \frac{1+e^{4m_0}}{2}} \right\}.$$
and
\[ \delta \leq \frac{m_* \eta''(2m_0,1)}{130 \pi C''(2m_0)} \]
and (30) holds.

We consider the positions in Figure 3. The top left position in Figure 3 is a subcase of the position in Figure 3 where we set \( \varphi_{Q'Q} = 1d \) and the nesting follows for the same choices of \( \epsilon, \delta \) and \( k^* \). The top right position is exactly dealt as with the top left position. The bottom position in Figure 3 is exactly equal to the position in Figure 2 and the nesting is achieved by choosing the same constants.

**Case 4.** Assume that \([a_n, b_n]\) enters \( E \) on a short edge and that it exists \( E \) on the opposite short edge. The argument in this case is contained in the second part of Case 3 and the bounds are the same.

**Case 5.** Assume that \([a_n, b_n]\) enters an edge \( E_1 \) of \( \tilde{\tau} \) through a long side, enters another edge \( E_2 \) through a short side in common with \( E_1 \) and then enters an edge \( E_3 \) through a short side in common with \( E_2 \), and then exists \( E_3 \) through a long side. Let \( P \) and \( Q \) be the plaques of \( \tilde{\tau} \) which contain \( a_n \) and \( b_n \), respectively. Note that the arc \([a_n, b_n]\) has length at least \( m_* \) because it traverses the edge \( E_2 \). Moreover, since the arc \([a_n, b_n]\) connects two long sides of different edges of \( \tilde{\tau} \) it follows that the set of geodesics of \( \tilde{\tau} \) that intersect \([a_n, b_n]\) is disjoint union of at most three sets of geodesics each of them traversing an edge of \( \tilde{\tau} \). The situation in Figure 4 illustrates the case when this union consists of the geodesics traversing the edge above \( E_2 \), the edge \( E_2 \) and the edge below \( E_2 \). Note that the short sides of these three geodesics are on the distance at most \( 3m_0 \) from \( a_n = i \). Other
Figure 4.

possibilities can be easily checked by drawing pictures. It always happen that the set of geodesics of \( \tilde{\lambda} \) intersecting \([a_n, b_n]\) is the disjoint union of at most three sets of geodesics traversing three edges of \( \tilde{\tau} \) whose short sides are on the distance at most \( 3m_0 \) from \( a_n \). Therefore \( \varphi_{P,Q} \) is the composition of at most three Möbius maps \( \varphi_{E_i}' \), for \( i = 1, 2, 3 \), each corresponding to an edge \( E_i' \) of \( \tilde{\tau} \).

We use the argument from Case 1 to estimate \( \|\varphi_{E_i}' - I d\| \). Namely, it is enough to replace \( m_0 \) with \( 3m_0 \) to obtain

\[
\|\varphi_{E_i}' - I d\| \leq C'(3m_0)(\|\beta\|_{\text{var}} + \delta)|k_i'| + \frac{1 + \epsilon^{6m_0}}{2}\|\beta\|_{\text{max}}.
\]

Consequently, we have

\[
\|\varphi_{P,Q} - I d\| \leq 3(1 + C'(3m_0)(\|\beta\|_{\text{var}} + \delta)|k_i'| + \frac{1 + \epsilon^{6m_0}}{2}\|\beta\|_{\text{max}}) \times (C'(3m_0)(\|\beta\|_{\text{var}} + \delta)|k_i'| + \frac{1 + \epsilon^{6m_0}}{2}\|\beta\|_{\text{max}}).
\]

Assume that \( \delta, \|\beta\|_{\text{var}} \leq \frac{1}{2} \) and \( \|\beta\|_{\text{max}} \leq 1 \). Then \( 1 + C'(3m_0)(\|\beta\|_{\text{var}} + \delta)|k_i'| + \frac{1 + \epsilon^{6m_0}}{2}\|\beta\|_{\text{max}} \leq 1 + C'(3m_0) + \frac{1 + \epsilon^{6m_0}}{2} \) because \( |k_i'| \leq \frac{1}{20} \). We choose

\[
\epsilon = \delta = \frac{m_0 \eta''(3m_0, 0)}{18C'(3m_0)(1 + C'(3m_0) + \frac{1 + \epsilon^{6m_0}}{2})^2}.
\]

We established that the cones are nested along the sequence \{\([a_n, b_n]\)\}_{n \in \mathbb{N}}. Thus the bending map \( \tilde{f} \) is injective on \( \partial_{\infty} \mathbb{H}^2 \) as claimed. We choose \( \epsilon \) and \( \delta \) to be the minimum over all cases and the nesting is guaranteed always. This ends the proof of Theorem 1.1.

The size of \( \epsilon, \delta \) and \( k^* \) depends on the above constants \( m_0 \) and \( m_* \) (cf. Table 6). The minimum \( m_* \) of the distances between short sides of the edges of \( \tilde{\tau} \) can be arbitrary small. In fact, when there are short closed geodesics contained in the geodesic lamination \( \lambda \) then the train track \( \tau \) cannot be modified such that \( m_* \) is
bigger than a universal positive constant. This fact forces us to include $m_*$ as a part of the geometric information for the geodesic lamination $\lambda$.

If $\lambda$ does not contain closed geodesics then there exists a choice of a geometric train track $\tau$ which carries $\lambda$ such that $m_* \geq 1/20$ and $m_0 = 1/5$. In this case we explicitly compute the constants in Theorem 1.1 to be

$$k^* = 4.41719 \times 10^{-10}$$

and

$$\epsilon = \delta = 3.61749 \times 10^{-17}.$$

### 7. Holomorphic motions and shear-bend cocycle

Let $K \subset \hat{\mathbb{C}}$ and let $D = \{ w \in \mathbb{C} : |w| < 1 \}$. A holomorphic motion of a set $K$ is a map

$$f : K \times D \to \hat{\mathbb{C}}$$

such that

$$f(z, w) : K \to \hat{\mathbb{C}}$$

is injective for each $w \in D$, $f(z, 0) = z$ for all $z \in K$, and

$$f(z, \cdot) : D \to \hat{\mathbb{C}}$$

is holomorphic in $w \in D$ for each $z \in K$ (see [12]). The variable $w \in D$ is called the parameter of the holomorphic motion of $K$. It is also possible to define holomorphic motions over simply connected regions of $\mathbb{C}$ when we specify the point where the motion is the identity.

The lambda lemma states that a holomorphic motion of $K$ extends to a holomorphic motion of the closure $\bar{K}$ of $K$ (see [12]). Slodkowski [15] proved that a holomorphic motion of a closed set $K$ which contains at least three points extends to a holomorphic motion of $\hat{\mathbb{C}}$. In fact, if a holomorphic motion of $K$ is invariant
under a subgroup $G$ of $PSL_2(\mathbb{C})$ then the extension of the holomorphic motion can be chosen to be $G$-equivariant on $\bar{C}$.

A shear-bend transverse cocycle $\beta$ for a geodesic lamination $\lambda$ on a closed hyperbolic surface $S$ assigns to each arc $k$ transverse to $\lambda$ (with endpoints of $k$ in the plaques of $\lambda$) a number $\beta(k) \in \mathbb{C}/2\pi i \mathbb{Z}$ such that if $k = k_1 \cup k_2$ and $k_1, k_2$ have disjoint interiors then $\beta(k) = \beta(k_1) + \beta(k_2)$. Denote by $H(\lambda, \mathbb{C}/2\pi i \mathbb{Z})$ the space of all shear-bend transverse cocycles for $\lambda$. Bonahon $[3]$ proved that the space of all representations of the fundamental group $\pi_1(S)$ of $S$ in $PSL_2(\mathbb{C})$ which realize $\lambda$ is homeomorphic to an open subset of $H(\lambda, \mathbb{C}/2\pi i \mathbb{Z})$, where the real part is restricted to belong to the image of $T(S)$ in $H(\lambda, \mathbb{R})$ and there is no restrictions on the imaginary part.

Let $\alpha \in H(\lambda, \mathbb{R})$ be in the image of the Teichmüller space $T(S)$. For $w = u + iv \in \mathbb{C}$, we define $\beta_w(k) = (w\alpha(k)) (\bmod 2\pi i \mathbb{Z})$ for each arc $k$ transverse to $\lambda$. Then $\beta_w \in H(\lambda, \mathbb{C}/2\pi i \mathbb{Z})$. Let $f_w : \mathbb{H}^2 \to \mathbb{H}^3$ be the shear-bend map corresponding to $\beta_w$ as in $[3]$.

**Theorem 7.1.** Let $\alpha \in H(\lambda, \mathbb{R})$ be in the image of $T(S)$ and let $f_{(1+w)}$ be the shear-bend map for $\beta_{(1+w)} \in H(\lambda, \mathbb{C}/2\pi i \mathbb{Z})$. Then there exists $r > 0$ such that the shear-bend map

$$f_{(1+w)} : \mathbb{H}^2 \to \mathbb{H}^3$$

extends by continuity to a holomorphic motion of $\partial_\infty \mathbb{H}^2$ in $\partial_\infty \mathbb{H}^3$ for the parameter $\{w \in \mathbb{C} : |w| < r\}$.

**Proof.** For $w = u + iv$, consider the hyperbolic surface $S_{1+w}$ obtained by shearing along the real part of $\beta_{1+w}$ which is $(1+u)\alpha \in H(\lambda, \mathbb{R})$. By $[3]$, the image of $T(S)$ is a cone in $H(\lambda, \mathbb{R})$ and therefore $S_{1+w}$ exists for $r$ small enough. Note that $S_1$ is the original hyperbolic surface $S$.

Let $\tau$ be the train track that carries $\lambda$ used in the proof of Theorem $[2]$. We choose $\tau$ such that $k^* = \frac{1}{2} \cdot \frac{e^{-2\pi u} \tanh \frac{m_s}{\pi}}{m_s}$. For $|u|$ small enough, the endpoints of the switches of $\tau$ under the shear map $f_{(1+w)}$ are close to the switches of $\tau$. By connecting the switches with geodesics for the hyperbolic metric of $(1+u)\alpha$ we construct a train track $\tau_{1+u}$ which is homotopic to $\tau$. For $|u|$ small enough, we have the constants $m_0(\tau_{1+u}), m_s(\tau_{1+u})$ and $k^*(\tau_{1+u})$ are as close as we need to the original constants $m_0, m_s$ and $k^*$ of the train track $\tau = \tau_1$. The constants $k^*(\tau_{1+u})$ and $\epsilon(\tau_{1+u}) = \delta(\tau_{1+u})$ from the proof of Theorem $[1]$ depend continuously on $m_0(\tau_{1+u}), m_s(\tau_{1+u})$ and $k^*(\tau_{1+u})$. Thus they depend continuously on $u$ and are bounded away from 0 for $u$ small enough. Then the proof of Theorem $[1]$ applies to each $\beta_{1+w}$ to obtain an injective map $\partial_\infty \mathbb{H}^2$ with $|Im(w)|$ obtained from Theorem $[2]$. It is clear that when $w = 0$, we have $f_{(1+0)} = id$.

Finally, we fix $z \in \partial_\infty \mathbb{H}^2$ and consider $w \mapsto f_{(1+w)}(z)$. Bonahon $[3]$ proved that the shear-bend map is holomorphic in the transverse cocycle when restricted to a single plaque of $\lambda'$. Since the endpoints of the plaques are dense in $\partial_\infty \mathbb{H}$, it follows that $w \mapsto f_{(1+w)}(z)$ is holomorphic in $w$ for $z$ in a dense subset of $\partial_\infty \mathbb{H}$. By the lambda lemma, this is enough to claim that $f_{(1+w)}$ extends to a holomorphic motion of $\partial_\infty \mathbb{H}^2$ for $w$ in the described neighborhood of $0 \in \mathbb{C}$. \[\square\]
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Department of Mathematics, Queens College of CUNY, 65-30 Kissena Blvd., Flushing, NY 11367

E-mail address: Dragomir.Saric@qc.cuny.edu

Mathematics PhD. Program, The CUNY Graduate Center, 365 Fifth Avenue, New York, NY 10016-4309