Optimal Gathering over Meeting Nodes in Infinite Grid

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The gathering over meeting nodes problem requires the robots to gather at one of the predefined meeting nodes. This paper investigates the problem with respect to the objective function that minimizes the total number of moves made by all the robots. In other words, the sum of the distances traveled by all the robots is minimized while accomplishing the gathering task. The robots are deployed on the nodes of an anonymous two-dimensional infinite grid which has a subset of nodes marked as meeting nodes. The robots do not agree on a global coordinate system and operate under an asynchronous scheduler. A deterministic distributed algorithm has been proposed to solve the problem for all those solvable configurations, and the initial configurations for which the problem is unsolvable have been characterized. The proposed gathering algorithm is optimal with respect to the total number of moves performed by all the robots in order to finalize the gathering.

Keywords: Gathering, Meeting Nodes, Grid, Asynchronous, Look-Compute-Move cycle, Weber nodes

1. Introduction

One recent trend in robotics is using a group of small, inexpensive and mass-produced robots to perform complex tasks. The main focus of theoretical research in swarm robotics is to identify minimal sets of capabilities necessary to solve a particular problem. The gathering problem asks the mobile entities, which are initially
situated at distinct locations, to gather at a common location and remain there within a finite amount of time. In this paper, the robots are deployed on the nodes of an anonymous grid graph. Robots are assumed to be anonymous (no unique identifiers), autonomous (without central control), homogeneous (execute the same deterministic algorithm) and oblivious (no memory of past observations). They do not have any explicit means of communication, i.e., they cannot send any messages to other robots. They do not have any agreement on a global coordinate system or chirality. Each robot has its own local coordinate system with the robot’s current position as the origin. They are equipped with sensor capabilities in order to observe the positions of the other robots. No local memory is available on the nodes of the grid graph. The robots have unlimited visibility, i.e., they can perceive the entire graph. Robots operate in Look-Compute-Move (LCM) cycles. In the Look phase, a robot takes a snapshot of the entire configuration in its own local coordinate system. In the Compute phase, it decides either to stay idle or to move to one of its neighboring nodes. In the Move phase, it makes an instantaneous move to its computed destination. Based on the timing and activation of the robots, three types of schedulers are common in the literature. In the fully synchronous (FSYNC) setting, all the robots are activated simultaneously. The activation phase of all the robots can be divided into global rounds. In the semi-synchronous (SSYNC) setting, a subset of robots are activated simultaneously, i.e., not all the robots are necessarily activated in each round. FSYNC can be viewed as a special case of SSYNC. In the asynchronous (ASYNC) setting, there is no common notion of time. Moreover, the duration of the Look, Compute and Move phases is finite but unpredictable and is decided by the adversary for each robot. In this paper, we have considered the scheduler to be asynchronous. The scheduler is also assumed to be fair, i.e., each robot performs its LCM cycle within finite time and infinitely often. In the initial configuration, the robots are placed at the distinct nodes of the grid. The input graph also consists of some meeting nodes which are located on the distinct nodes of the grid graph. The meeting nodes are visible to the robots during the look phase, and they occupy distinct nodes of the grid. A robot can move to one of its adjacent nodes along the grid lines. The movement of the robots is assumed to be instantaneous, i.e., they can be seen only on the nodes of the input grid graph. They are equipped with global strong multiplicity detection capability, i.e., in the Look phase, they can count the exact number of robots occupying each node. In the global-weak version, a robot can detect whether a node is occupied by any robot multiplicity. Unlike in the global versions, the local versions refer to the ability of a robot to perceive the information about multiplicities concerning the node in which it resides. In this paper, the optimal gathering over meeting nodes problem has been studied in an infinite grid model. This is a variant of the gathering over meeting nodes in infinite grid problem, studied by Bhagat et al. [2,3]. This paper proposes a deterministic distributed algorithm for optimal gathering over meeting nodes problem
with \( n \geq 7 \) asynchronous robots in an infinite grid. The objective constraint is to minimize the total number of moves required by the robots in order to accomplish the gathering. In this paper, we have considered Weber nodes and observed that if the gathering node is a Weber node, the algorithm is optimal with respect to the total number of moves made by the robots. Moreover, the Weber node is not unique in general, even if the robots are non-collinear.

1.1. Motivation

Gathering over meeting nodes in infinite grids problem was studied by Bhagat et al. [2,3], where the robots are assumed to be deployed on the nodes of an infinite grid. The main aim of this paper is to study the problem under the optimization constraint that the sum of the distances traveled by the robots is minimized while accomplishing the gathering task. In order to complete the gathering task, the robots select a unique meeting node and move towards it in such a way that the sum of the lengths of the shortest paths from each robot to the selected meeting node is minimized. Since the robots are oblivious, the main challenge for designing a deterministic distributed algorithm lies in keeping the selected meeting node invariant while the robots move towards it. It is worth noting here that unlike in the continuous domain, where the robots are represented as points in \( \mathbb{R}^2 \), they are not allowed to perform infinitesimal movements with infinite precision if they are deployed on the nodes of a graph. This motivates us to consider the specified problem in a grid-based terrain where the robots are only allowed to move along the edges of the input grid graph.

1.2. Earlier works

In the continuous domain, the robots are represented as points in Euclidean plane \([1,9,15,16,22]\). Unlike in the graph model, the robots are placed on the nodes of an anonymous graph in general. In an anonymous graph, neither the nodes nor the edges of the graph are labeled, and no local memory is available on the nodes of the graph. The gathering problem in the discrete domain has been extensively studied in various topologies like rings \([21,12,18,19,20,13,14,24]\), finite and infinite grids \([11,23]\), bipartite graphs \([8]\), complete bipartite graphs \([8]\), trees \([11,24]\) and hypercubes \([5]\). Klasing et al. [21] studied the gathering problem in an anonymous ring using global weak-multiplicity detection capability. They proved that the gathering is impossible without the assumption of multiplicity detection capability. They proposed a deterministic distributed algorithm in the asynchronous model for gathering an odd number of robots. The algorithm also solves the gathering problem for an even number of robots when the initial configuration is asymmetric. D’Angelo et al. [12] studied the gathering problem in an anonymous ring, where the robots have global weak-multiplicity detection capability. They proposed a deterministic distributed
algorithm that solves the \textit{gathering} task for any initial configuration which is non-periodic and does not contain any edge-edge line of symmetry. Izumi et. al [18] studied the \textit{gathering} problem in an anonymous ring and proposed a deterministic algorithm using local weak-multiplicity detection capability. D’Angelo et al. [13] studied the \textit{gathering} problem on anonymous rings with 6 robots in the initial configuration. A unified strategy for all the \textit{gatherable} configurations has been provided in this paper. D’Angelo et al. [14] studied the \textit{exploration, graph searching} and \textit{gathering} problem in an anonymous ring where the initial configuration is aperiodic and asymmetric.

D’Angelo et al. [11], studied the \textit{gathering} problem on trees and finite grids. They showed that a configuration remains ungatherable if the configuration is periodic and the dimension of the finite grid consists of at least one even side. A configuration remains ungatherable even if it admits reflection symmetry with the reflection axis passing through the edges of the finite grid. The problem was solved for all the other remaining configurations without assuming any \textit{multiplicity detection capability}. Di Stefano et al. [24], studied the optimal \textit{gathering} of robots in arbitrary graphs. This paper also introduced the concept of \textit{Weber points} [10, 25] in graphs. A Weber point of a graph is a node of a graph that minimizes the sum of the distances from it to each robot. They proposed deterministic algorithms for the \textit{gathering} task on tree and ring topologies that always achieve \textit{optimal gathering} unless the initial configuration is ungatherable. In [23], the optimal \textit{gathering} problem in an infinite grid model was studied by Di Stefano et al. They proposed a deterministic distributed algorithm that minimizes the total distance traveled by all the robots. They proved that their assumed model represents the minimal setting to ensure \textit{optimal gathering}.

Cicerone et al. [8] studied the \textit{gathering} problem in arbitrary graphs and proposed a necessary and sufficient result for the feasibility of \textit{gathering} tasks in arbitrary graphs. They have also considered dense and symmetric graphs like complete and complete bipartite graphs. A deterministic algorithm was proposed that fully characterize the solvability of the \textit{gathering} task under the synchronous setting. Bose et al. [5] investigated the \textit{optimal gathering} problem in hypercubes, where the optimal criteria is to minimize the total distance traveled by each robot.

Fujina et al. [17] introduced the concept of \textit{fixed points} or \textit{landmarks} in the \textit{Euclidean plane}. In the \textit{landmarks covering problem}, the robots must reach a configuration, where at each \textit{landmark} point, there is precisely one robot. A distributed algorithm was proposed that assumes common orientation among the robots and minimizes the total number of moves traveled by all the robots. Cicerone et al. [6] studied the \textit{embedded pattern formation (EPF) problem} without assuming common chirality among the robots. The problem asks for a distributed algorithm that requires the robot to occupy all the \textit{fixed points} within a finite amount of time. A variant of the \textit{gathering} problem was studied by Cicerone et al. [7], where the \textit{gathering} is accomplished at one of the \textit{meeting points}. These are a finite set of points visible to all the robots during the \textit{Look} phase. They also studied the same
problem with respect to the two optimal criteria, one by minimizing the total number of moves traveled by all the robots and the other by minimizing the maximum distance traveled by a single robot. Bhagat et al. [23] studied the gathering over meeting nodes problem in an infinite grid. It was shown that even if the robots are endowed with multiplicity detection capability, some configurations remain un-gatherable. The $k$-circle formation problem was studied by Bhagat et al. [4] in the Euclidean plane under one axis agreement. For a given positive integer $k$, the $k$-circle formation problem asks a set of robots to form disjoint circles having $k$ robots on the boundary of the circles occupying distinct locations. The circles are centered at the set of fixed points. They have also shown that if the $k$-circle formation problem is deterministically solvable, then the $k$-EPF problem is also deterministically solvable.

1.3. Our contributions

This paper proposes a deterministic distributed algorithm for optimal gathering over meeting nodes problem, where the initial configurations comprise at least seven robots. The robots are deployed on the nodes of an infinite grid. The optimization criterion considered in this paper is the minimization of the total number of moves made by the robots to finalize the gathering. In this paper, a meeting node that minimizes the sum of the distances from all the robots is defined as a Weber node. Di Stefano et al. [24] proved that to ensure gathering by minimizing the total number of moves, the robots must gather at one of the Weber points. In our restricted gathering model, the robots must gather at one of the Weber nodes to ensure gathering with a minimum number of moves. It has been shown that there exist some configurations where gathering over Weber nodes cannot be ensured, even if the robots are endowed with the multiplicity detection capability. This includes the following collection of configurations:

1. Configurations admitting a single line of symmetry without any robots or Weber nodes on the reflection axis.
2. Configurations admitting rotational symmetry without a robot or a meeting node on the center of rotation.

In this paper, the assumption on the multiplicity detection refers to the global strong multiplicity detection capability. It has been shown that, without such an assumption, there are configurations where gathering cannot be accomplished as soon as a multiplicity is created. However, there are initial configurations where gathering can be ensured over a meeting node, but not on the set of Weber nodes. This includes the configuration admitting a single line of symmetry without any robots or Weber nodes on the reflection axis, but at least one meeting node exists on the reflection axis. In that case, the feasibility of gathering over meeting nodes has been studied.
1.4. Outline

The following section describes the robot model, and the notations used in the paper. Section 2 provides the formal definition of the problem and the impossibility results for the solvability of the gathering task. Section 4 proposes a deterministic distributed algorithm to solve the optimal gathering over meeting nodes problem. Section 5 describes the correctness of the proposed algorithm. Section 6 discusses the optimal gathering for the configurations where gathering over a meeting node can be ensured, but cannot be ensured over a Weber node. Finally, in Section 7 we conclude the paper with some discussion for future research.

2. Optimal Gathering over Meeting Nodes

The optimal gathering over meeting nodes problem has been considered in an infinite grid graph. The objective is to minimize the total distance traveled by all the robots. In order to ensure optimal gathering over a meeting node, the robots must finalize gathering over a meeting node that minimizes the total distance traveled by all the robots, i.e., each robot must gather at one of the Weber nodes.

2.1. Terminology

In this subsection, some terminologies and definitions have been proposed.

- **System Configuration**: Let $P = (\mathbb{Z}, E')$ denote the infinite path graph, where the vertex set corresponds to the set of integers $\mathbb{Z}$ and the edge set is defined by the ordered pair $E' = \{(i, i+1) | i \in \mathbb{Z}\}$. The input grid graph is defined as the Cartesian Product of the graph $P \times P$. Let $V$ and $E$ denote the set of nodes and edges of the input grid graph, respectively. $d(u, v)$ denote the lengths of the shortest paths between the nodes $u$ and $v$. Let $M = \{m_1, m_2, \ldots, m_s\}$ denote the finite set of meeting nodes located on the nodes of the grid graph. $R(t) = \{r_1(t), r_2(t), \ldots, r_n(t)\}$ denote the finite set of robots deploying on the nodes of the grid at any time $t$ respectively. $C(t) = (R(t), M)$ represents the system configuration at time $t$.

- **Symmetry**: An automorphism of a graph $G = (V, E)$ is a bijective map $\phi : V \rightarrow V$ such that $u$ and $v$ are adjacent if and only if $\phi(u)$ and $\phi(v)$ are adjacent. $f_t : V \rightarrow \{0, 1\}$ at any time $t \geq 0$ is defined as follows:

$$f_t(v) = \begin{cases} 0 & \text{if } v \text{ is not a meeting node} \\ 1 & \text{if } v \text{ is a meeting node} \end{cases}$$

Let $\lambda_t : V \rightarrow N$ be a function denoting the number of robots on each node $v \in V$ at any time $t \geq 0$. An automorphism of a configuration $(C(t), f_t, \lambda_t)$ is an automorphism $\phi$ of the input grid graph such that $f_t(v) = f_t(\phi(v))$ and $\lambda_t(v) = \lambda_t(\phi(v))$ for all $v \in V$. The grid graph is embedded in the Cartesian plane. As a result, a grid can admit only three types of automorphisms:
translation, reflection and rotation, and compositions of them. Since the number of robots and meeting node is finite, a translational automorphism is not possible. An axis of reflection defines a reflection automorphism, while the center of rotation and the angle of rotation determine a rotational automorphism. If the configuration admits reflectional symmetry, the axis of reflection can be horizontal, vertical, or diagonal. The axis of symmetry can pass through the nodes or edges of the graph. In the case of rotational symmetry, the angle of rotation can be $90^\circ$ or $180^\circ$. The center of rotation can be a node, a center of an edge, or the center of a unit square.

![Diagram](image)

**Fig. 1.** (a) In this figure, cross represents meeting node and black circles represent robot positions. $m_2$ is the unique Weber node. (b) Robots $r_2$ and $r_3$ move towards $m_2$ and create a multiplicity $\mu_2$. Similarly, $r_4$, $r_5$ and $r_6$ move towards $m_2$ and creates a multiplicity $\mu_1$. $m_2$ remains the unique Weber node but robots will not be able to compute it correctly if they do not have global strong multiplicity detection capability.

- **Weber node:** Since in the initial configuration the robots are deployed at the distinct nodes of the grid graph, $\lambda_t(v) \leq 1$, $\forall v \in V$. In the final configuration, all the robots are on a single meeting node $m \in M$. For a configuration to be final, there must exist a $m \in M$ such that $\lambda_t(m) = n$ and $\lambda_t(v) = 0$ for each $v \in V \setminus \{m\}$. The *consistency* of a node $m$ at any time $t$ is defined as $c_t(m) = \sum_{v \in V} d(v, m) \lambda_t(v)$. A node $m$ is defined as a Weber node if it minimizes the value $c_t(m)$. In other words, a Weber node $m$ is defined as the meeting node which minimizes the sum of the distances from all the robots to itself. The Weber node may not be unique in general. Let $W(t)$ denote the set of all the Weber nodes at some time $t$. A deterministic distributed algorithm that gathers all the robots in a Weber node via the shortest paths will be optimal with respect to the total number of moves made by the robots.

The robots are equipped with global strong multiplicity detection capability, i.e., they can detect the exact number of robots occupying any node. Without this assumption, the Weber nodes cannot be detected correctly by the robots as soon as a multiplicity is created. As a result, the total number of moves made by the robots to accomplish the gathering might not be optimized. For example, consider the configuration in Figure 1(a). If the
robots compute the *Weber node* in this configuration, a unique *Weber node* would be computed. Due to the robot’s movement, if the configuration in Figure 1(b) is reached, then without the *global strong multiplicity detection capability*, the robots will not be able to compute the unique *Weber node* correctly. This example shows that without assuming *strong multiplicity detection capability*, the gathering over *Weber nodes* is no longer possible as soon as a multiplicity is created.

![Diagram](image_url)

*Fig. 2. ABCD denotes the minimum enclosing rectangle MER. The lexicographic minimum string in this figure is $s_{DC} = (1, 2, 6, 8, 16, 18, 22)$. Here $\alpha_{DC} = ((0, 0), (1, 0), (1, 0), (0, 1), (0, 0), (0, 1), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (0, 0), (1, 0), (0, 1), (0, 0), (1, 0), (0, 0), (0, 0), (0, 0), (1, 0), (1, 0), (0, 1), (0, 0), (1, 0), (0, 0), (0, 0), (0, 0), (0, 0), (1, 0), (0, 0), (1, 0), (0, 0), (0, 0), (0, 0), (0, 1))*."

- **Minimum Enclosing Rectangle**: Let $MER = ABCD$ denote the minimum enclosing rectangle of $R \cup M$. Let the dimension of $MER$ be $p \times q$, where $|AB| = p$ and $|AD| = q$. The side length $|AB|$ of $MER$ is defined as the number of grid edges on it. Similarly, the other side lengths are defined. For a corner $A$, consider the two possible strings, which are denoted by $s_{AD}$ and $s_{AB}$. The string $s_{AD}$ is defined as follows: Starting from the corner $A$, scan the entire grid by proceeding along the respective direction and sequentially considering all the grid lines parallel to $AD$. While scanning the grid, let $(m_1, m_2, \ldots, m_s)$ denote the list of *meeting nodes*, in the order of their appearance. Let $d_i$ denote the number of hops from the corner $A$ to $m_i$, if measured along the direction parallel to $s_{AD}$. The string $s_{AD}$ is defined by $s_{AD} = (d_1, d_2, \ldots, d_s)$ (Figure 2). The string $s_{AB}$ is defined similarly. Consider the two possible strings for each corner. Thus, there are a total of eight strings of distances of length $s$ that are obtained by traversing $MER$. If $MER$ is a non-square rectangle with $p > q$, the string $s_{AD}$ or $s_{AB}$ is associated for the corner $A$ in the direction of the smallest side ($AD$ in this case). If the *meeting nodes* are asymmetric, there exists a unique string, which is lexicographically minimum among all the possible strings ($s_{DC}$ in Figure 2). Otherwise, if any two possible strings are equal, then the *meeting nodes* are symmetric. The corner associated with the minimum lexicographic string is defined as a *leading corner*, and the string associated with the *leading corner* is defined as the *string direction* for the respective
corner. If MER is a square, consider the two strings associated with a corner. The string which is lexicographically smaller between the two strings is selected as the string direction for the respective corner. The two strings for a respective corner are equal if the meeting nodes are symmetric with respect to the diagonal passing through that corner of MER. If all the robots and meeting nodes lie on a single line, then MER is a p \times 1 rectangle with A = D and B = C, and length of AD and BC is 1. Note that, in this case, s_{AD} and s_{DA} refer to the same string. The meeting nodes are symmetric when the strings s_{AD} and s_{DA} are equal.

- Potential Weber nodes: In general, the Weber nodes in an infinite grid is not unique. If it is possible to gather at one of the Weber nodes, then all the robots must decide to agree on a common Weber node for gathering. Depending on the symmetricity of the set M, the number of leading corners is 1, 2 or 4 respectively. Consider the Weber nodes that represents the last Weber nodes in the string directions associated to the leading corners. Note that there exists at most four such Weber nodes. Let W_p(t) denote the set of such Weber nodes at time t \geq 0. The set W_p(t) is defined as the set of Potential Weber nodes.

- Key corner: Consider all the leading corners of MER and the strings s_i associated with each leading corner i. Assume that there exists at least two leading corners. Without loss of generality, assume that A and D are the leading corners and the strings parallel to AD and DA are the string directions associated with the leading corners. The string \alpha_{AD} is defined as follows: Starting from the corner A, scan the grid along the string direction of A, i.e., along AD and associate the pair (f_t(v), \lambda_t(v)) to each node v (Figure 2). The string \alpha_{DA} is defined similarly. Consider the strings \alpha_{AD} and \alpha_{DA}. If C(t) is asymmetric, there always exists a unique string which is lexicographically smaller than \alpha_{DA}, then the corner A is defined as the key corner. If C(t) is symmetric, there may exist more than one key corner. Similarly, the strings \beta_i, for each non-leading corner i is defined.

In [24], it was proved that a Weber point remains invariant under the movement of a robot towards itself. In our restricted gathering model, where gathering can be finalized only on meeting nodes, we have the following lemma.

**Lemma 1.** Let m be a Weber node in a given configuration C(t). Suppose C(t') denotes the configuration after a single robot or a robot multiplicity moves towards the Weber node m. Then the following results hold.

1. m \in W(t')
2. W(t') \subseteq W(t)

**Proof.**
(1) By definition we have, $c_t(m) = \sum_{v \in V} d(v, m) \lambda_t(v)$ and $W(t) = \{ m | \min_{m \in M} c_t(m) \}$. Suppose $r(t) = a$ and $r(t') = b$, i.e., $r$ has moved from the vertex $a$ to $b$ along a shortest path towards $m$ in the time interval $[t, t']$. After the movement of the robot $r$, $\lambda_t(a)$ and $\lambda_t(b)$ becomes $\lambda_t(a) - 1$ and $\lambda_t(b) + 1$, respectively. Since $b$ lies on the shortest path from $r$ to $m$ and $d(a, b) = 1$, $c_t'(m)$ becomes $c_t(m) + d(b, m) - d(a, m)$, which is again equivalent to $c_t(m) - 1$. Therefore, $\min_{m \in M} c_t'(m) = \min_{m \in M} c_t(m) - 1$ and hence $m \in W(t')$. Similarly, if a robot multiplicity moves from some vertex $a$ to $b$ at time $t'$ via any shortest path towards $m$, then after the movement of the robot multiplicity, $\lambda_t(a)$ and $\lambda_t(b)$ becomes $\lambda_t(a) - j$ and $\lambda_t(b) + j$, respectively, where $j \geq 2$ denotes the number of robots that move from node $a$ to $b$. This implies that, $c_t'(m) = c_t(m) - j$ and hence, $\min_{m \in M} c_t'(m) = \min_{m \in M} c_t(m) - j$. Therefore, the Weber nodes in the new configuration $C(t')$ are the Weber nodes of $C(t)$ which are on some shortest path from $r$ to $m$. Hence $m \in W(t')$.

(2) Assume that $m \in W(t')$. This implies that $m$ minimizes the value $c_t'(m)$. The first part of the proof implies that $\min_{m \in M} c_{t'}(m) = \min_{m \in M} c_t(m) - j$, where $j \geq 1$ denotes the number of robots that move from node $a$ to $b$. In other words, no node can become a Weber node if it was not before the move. Therefore, $m$ must belong to $W(t)$ and hence $W(t') \subseteq W(t)$. 

![Fig. 3. Multiple Weber nodes in a configuration.](image)

This lemma proves that the Weber node remains invariant under the movement of robots towards itself via a shortest path. In Figure 3, the configuration admits rotational symmetry. The Weber node is not unique. There are three Weber nodes $m_3, m_4$ and $m_5$ in the configuration.

**Observation 1.** Let $C(0)$ be any initial configuration that admits rotational symmetry. Assume that the center of the rotational symmetry $c$ contains a meeting node $m$. Then $m$ is a Weber node.
3. Problem Definition and Impossibility Results

In this section, we have formally defined the problem. A partitioning of the initial configurations has also been provided in this section.

3.1. Problem Definition

Let $C(t) = (R(t), M)$ be a given configuration. The goal of the optimal gathering over meeting nodes problem is to finalize the gathering at one of the Weber nodes of $C(0)$. We have proposed a deterministic distributed algorithm that ensures gathering over a Weber node, where the initial configuration consists of at least seven robots. If $|W(t)| = 1$, then all the robots finalize the gathering at the unique Weber node. Otherwise, all the robots must agree on a common Weber node and finalize the gathering.

3.2. Partitioning of the Initial Configurations

All the initial configurations can be partitioned into the following disjoint classes.

1. $I_1$: Any configuration for which $|W(t)| = 1$ (Figure 4(a)).
2. $I_2$: Any configuration for which $M$ is asymmetric and $|W(t)| \geq 2$ (Figure 4(b)).
3. $I_3$: Any configuration for which $M$ admits a unique line of symmetry $l$ and $|W(t)| \geq 2$. This can be further partitioned into:
   (a) $I_3^1$: $C(t)$ is asymmetric (Figure 5(a)).
   (b) $I_3^2$: $C(t)$ is symmetric with respect to the line $l$. This can be further partitioned into: $I_3^{21}$: there exists at least one Weber node on $l$. $I_3^{22}$: there exists at least one robot position on $l$ but no Weber nodes on $l$ (Figure 5(b)), $I_3^{23}$: there does not exist any Weber node or robot position on $l$, but there may exist a meeting node on $l$ (Figure 7(a)) and $I_3^{24}$: there does not exist any meeting node or robot position on $l$.

Fig. 4. (a) $I_1$ configuration. $m_3$ is the unique Weber node. (b) $I_2$ configuration. $m_2$ and $m_3$ are the Weber nodes. $A$ is the leading corner.
Fig. 5. (a) $I_3^3$ configuration. $A$ is the key corner, $m_1$ and $m_4$ are the Weber nodes. (b) $I_3^{b1}$ configuration, $m_2$ and $m_3$ are the Weber nodes. (c) $I_3^{b2}$ configuration, $m_3$ is a meeting node on $l$, but not a Weber node on $l$. $r_1$ and $r_4$ are the robot positions on $l$.

Fig. 6. (a) $I_4^4$ configuration, $m_2$ and $m_3$ are the Weber nodes. (b) $I_3^{b1}$ configuration, $m_3$, $m_4$ and $m_5$ are the Weber nodes. (c) $I_3^{b2}$ configuration, $m_1$, $m_2$, $m_3$ and $m_4$ are the Weber nodes. Robot $r_9$ on the center of rotation.

Fig. 7. (a) $I_3^{b3}$ configuration. It contains a meeting node $m_1$ on $l$ but it is not a Weber node. $m_3$ and $m_4$ are the Weber nodes. (b) $I_3^{b3}$ configuration without robots or meeting nodes on $c$.

(4) $I_4$ — Any configuration for which $M$ admits rotational symmetry with center of rotation $c$ and $|W(t)| \geq 2$. This can be further partitioned into:

(a) $T_4^1 - C(t)$ is asymmetric (Figure 6(a)).
(b) $I_{4}^{1} = C(t)$ is symmetric with respect to rotational symmetry. This can be further partitioned into: $I_{4}^{1} -$ there exists a meeting node on $c$ (Figure 6(b)), $I_{4}^{2} -$ there exists a robot position on $c$ (Figure 6(c)) and $I_{4}^{3} -$ there does not exist any meeting node or robot positions on $c$ (Figure 6(b)).

We assume that if the meeting nodes are symmetric with respect to a single line of symmetry, then $l$ is the line of symmetry. Similarly, if the meeting nodes are symmetric with respect to rotational symmetry, then $c$ is the center of rotational symmetry. Since the partitioning of the initial configurations depends only on the position of meeting nodes which are fixed nodes, all the robots can determine the class of configuration in which it belongs without any conflict. Let $I$ denote the set of all initial configurations.

**Lemma 2.** If the initial configuration $C(0) \in I_{4}^{2} \cup I_{4}^{3}$, then the gathering over Weber nodes problem cannot be solved.

The proof of the above lemma can be observed as a corollary to Theorem 1, proved in Bhagat et al. [23]. In [23], it was proved that $I_{4}^{4}$ is ungatherable. Let $\mathcal{U}$ denote the set of all configurations for which gathering over a Weber node cannot be ensured. According to Lemma 2, this includes all the configurations,

1. admitting a single line of symmetry $l$, and $l \cap (R \cup W(t)) = \phi$,
2. admitting rotational symmetry with center $c$ and $\{c\} \cap (R \cup M) = \phi$.

Note that according to Observation 1 if $c$ is a meeting node on $c$, then it must be a Weber node.

4. Algorithm

4.1. Overview of the Algorithm

In this subsection, a deterministic distributed algorithm has been proposed to solve the optimal gathering problem by gathering each robot at one of the Weber nodes. The proposed algorithm works for all the configurations $C(t) \in I \setminus (\mathcal{U} \cup I_{4}^{4})$ consisting of at least seven robots. The main strategy of the algorithm is to select a Weber node among all the possible Potential Weber nodes and allow the robots to move towards the selected Weber node. The proposed algorithm mainly consists of the following phases: Guard Selection, Target Weber Node Selection, Leading Robot Selection, Symmetry Breaking, Creating Multiplicity on Target Weber Node and Finalisation of Gathering. In the Target Weber Node Selection phase, the Potential Weber node for optimal gathering is selected. The Weber node selected for gathering is defined as the target Weber node. A set of robots denoted as guards are selected in the Guard Selection phase. Guards are selected in order to ensure that the initial MER remains invariant. In the Leading Robot Selection phase, a robot is selected as a leading robot and placed. A unique robot is selected and allowed to
move towards an adjacent node in the Symmetry Breaking phase. This movement of
the robot transforms a symmetric configuration into an asymmetric configuration.
All the non-guard robots move towards the target Weber node, thus creating a mul-
tiplicity on it in the Creating Multiplicity on Target Weber Node phase. Finally, all
the guards move towards the uniquely identifiable (robots have global strong mul-
tiplicity detection capability) target Weber node in the Finalisation of Gathering
phase and finalizes the gathering.

4.2. Half-planes and Quadrants
Assume that the initial configuration $C(0)$ is asymmetric. First, consider the case
when the locations of the meeting nodes are symmetric with respect to a single
line of symmetry $l$. The line $l$ divides MER into two half-planes. The half-planes
defined in this section are open half-planes, i.e., excluding the nodes on $l$. If the
meeting nodes are symmetric with respect to rotational symmetry and $c$ is the
center of rotation, then consider the lines $l$ and $l'$ which pass through $c$. These
lines are perpendicular to each other and divide the MER into four quadrants. The
quadrants defined in this section are open quadrants, i.e., the quadrants exclude
the nodes belonging to the lines $l$ and $l'$. A configuration is said to be balanced if
the following conditions hold:

1. $C(0) \in I_3$ and the half-planes delimited by $l$ contain an equal number of
robots.
2. $C(0) \in I_4$. Assume that there exist at least two quadrants that contain
the maximum number of Potential Weber nodes. Suppose more than one
quadrant contains either the maximum or the minimum number of robots
among all the specified quadrants. In that case, the configuration is said to
be balanced.

If the initial configuration is not balanced, then it is an unbalanced configuration.
An initial configuration $C(0)$ satisfies the following conditions:

- $C_1$: there exists a unique half-plane or quadrant that contains the maximum
number of Potential Weber nodes.
- $C_2$: there exists multiple half-planes or quadrants that contain the max-
imum number of Potential Weber nodes. Any configuration $C(0)$ satisfying
condition $C_2$ is said to satisfy $C_{21}$, if $C(0)$ is balanced. Otherwise, it satisfies
$C_{22}$, if the initial configuration is unbalanced.
- $C_3$: there does not exist any Potential Weber node on the half-planes or on
the quadrants.
Fig. 8. (a) \( l \) denotes the single line of symmetry. \( m_3 \) and \( m_4 \) are the Weber nodes. \( \mathcal{H}^+ \) is defined as the half-plane with the maximum number of robots. (b) \( A \) and \( D \) are the leading corners. \( A \) is the key corner. \( m_2 \) and \( m_4 \) are the Weber nodes. \( \mathcal{H}^+ \) is defined as the half-plane not containing the key corner \( A \).

| Demarcation of the half-planes for fixing the target | \( \mathcal{H}^+ \) |
|-----------------------------------------------------|------------------|
| Initial Configuration \( C(0) \) | The unique half-plane containing the Potential Weber nodes |
| satisfy \( C_1 \) | The unique half-plane not containing the key corner |
| satisfy \( C_{21} \land l \) is a horizontal or vertical line of symmetry | The half-plane which lies in the direction of \( AD \), if \( \alpha_{AD} \) is lexicographically larger than \( \alpha_{AB} \) |
| satisfy \( C_{21} \land l \) is a diagonal line of symmetry \( \land \exists \) a unique leading corner | The half-plane containing the corners \( A \) and \( D \), if \( \alpha_{AD} \) is lexicographically larger than \( \alpha_{CD} \) |
| satisfy \( C_{21} \land l \) is a diagonal line of symmetry \( \land \exists \) two leading corners | The unique half-plane with the maximum number of robots |

Table 1. Demarcation of the half-planes

4.2.1. Demarcation of the Half-planes for fixing the target

Assume that the meeting nodes are symmetric with respect to a single line of symmetry \( l \). Note that \( |W_p(t)| \leq 2 \). Further, assume that \( |W_p(t)| = 2 \) and \( C(0) \) does not satisfy \( C_3 \). This implies that there exists at least one Potential Weber node located on the half-planes. Note that, if \( l \) is a diagonal line of symmetry, then there may exist one or two leading corners. If there exists a unique leading corner, then without loss of generality, let \( A \) be the leading corner. Otherwise, if there exist two leading corners, then assume that \( A \) and \( C \) are the leading corners and the string directions associated to the corners \( A \) and \( C \) are along the sides \( AD \) and \( CD \), respectively. \( \mathcal{H}^+ \) is defined according to Table 1. The other half-plane delimited by \( l \)
Fig. 9. (a) $m_1$ and $m_2$ are the Weber nodes. $\mathcal{H}^{++}$ denotes the unique quadrant with the maximum number of robots. (b) $m_1$ and $m_2$ are the Weber nodes. The quadrants containing the corners $C$ and $D$, contain the maximum number of robots. $D$ is the key corner. $\mathcal{H}^{++}$ denotes the quadrant with the maximum number of robots and not containing the unique key corner.

| Demarcation of the quadrants for fixing the target | $\mathcal{H}^{++}$ |
|--------------------------------------------------|---------------------|
| Initial Configuration $C(0)$ | The unique quadrant containing the maximum number of Potential Weber nodes |
| satisfy $C_1$ | |
| satisfy $C_{21}$ \& the angle of rotation is $180^\circ$ $\wedge$ $\exists$ at least one quadrant that contains the Potential Weber nodes as well as the leading corners | The unique quadrant containing the leading corner with which the largest lexicographic string $\alpha_i$ is associated, and that contains the maximum number of robots |
| satisfy $C_{21}$ \& the angle of rotation is $180^\circ$ $\wedge$ the quadrants that contains the Potential Weber nodes, does not contain the leading corners | The unique quadrant containing the non-leading corner with which the largest lexicographic string $\beta_i$ is associated, and that contains the maximum number of robots |
| satisfy $C_{21}$ \& the angle of rotation is $90^\circ$ | The quadrant containing the corner with which the largest lexicographic string $\alpha_i$ is associated, and that contains the maximum number of robots |
| satisfy $C_{22}$ | The unique quadrant with the maximum number of robots |
| satisfy $C_3$ \& unbalanced | The unique quadrant containing the minimum number of robots |
| satisfy $C_3$ \& balanced | The unique quadrant that contains the smallest lexicographic string $\alpha_i$, associated with the leading corner and containing the minimum number of robots |

Table 2. Demarcation of the quadrants.

is defined as $\mathcal{H}^-$ (Figure 8 (a) and 8 (b)).
4.2.2. Demarcation of Quadrants for fixing the target

First, consider the case when the meeting nodes are symmetric with respect to rotational symmetry without multiple lines of symmetry and \( W_p(t) \geq 2 \). The quadrant \( H^{++} \) is defined according to Table 2. The other quadrants are defined as follows.

- \( H^{+} \): The quadrant adjacent to \( H^{++} \) with respect to the line \( l \).
- \( H^{-} \): The quadrant adjacent to \( H^{++} \) with respect to the line \( l' \).
- \( H^{-} \): The quadrant which is non-adjacent to \( H^{++} \) (Figure 9(a) and 9(b)).

If \( MER \) is a square, and the configuration admits multiple lines of symmetry, there can be at most four lines of symmetry. If there are more than two lines of symmetry, the two lines that are perpendicular to each other and do not pass through any corner of \( MER \) are selected and considered as \( l \) and \( l' \). Consider the quadrants delimited by the lines \( l \) and \( l' \). The quadrants are defined similarly, as in the case when \( MER \) admits rotational symmetry without multiple lines of symmetry.

4.3. Phases of the Algorithm

The proposed algorithm mainly consists of the following phases.

4.3.1. Guard Selection

In this phase, a set of robots is selected as guards in order to keep the initial \( MER \) invariant. If there does not exist any meeting nodes on a side of the boundary of \( MER \), then there must exist at least one robot on that particular side of the boundary. Guards are selected in such a way that they remain uniquely identifiable. If a side of the boundary of \( MER \) contains at least one meeting node, then a guard robot is not required for that particular side of the boundary. Therefore, consider
the case when the boundary of MER does not contain any meeting nodes. Consider the robots which are on the boundary of the MER. First, assume that \( C(t) \) is asymmetric. Let \( G \) denote the set of guards. Let \( G_C \) denote the set of guard corner and is defined as follows.

- The unique leading corner, if the meeting nodes are asymmetric.
- The leading corner contained in \( H^+ \), if the meeting nodes are symmetric with respect to a horizontal or vertical single line of symmetry \( l \). The unique key corner contained in \( H^+ \), if the meeting nodes are symmetric with respect to a diagonal line of symmetry.
- The leading corner contained in \( H^{++} \), if the meeting nodes are symmetric with respect to rotational symmetry.

The robot positions on the sides adjacent to the unique guard corner and are closest to the guard corner are considered as guards. Similarly, the robots that are farthest from the guard corner measured along the string direction and lying on the sides non-adjacent to the guard corner are also considered as guards. Note that, in each case, there are exactly four guard robot positions that are selected in this phase (Figure 10(a)).

If \( C(t) \) is symmetric with respect to a unique line of symmetry \( l \) and \( l \) is a horizontal or vertical line of symmetry, there are exactly two leading corners. Consider the robot positions on the sides adjacent to the leading corners and which are closest to the leading corners. These two robots and their symmetric images are selected as guards. The robots which are farthest from the leading corners and lying on the side which are non-adjacent to the leading corners are also selected as guards. Hence, there are exactly six guard robots that are selected when \( C(t) \) is symmetric with respect to \( l \) (Figure 10(b)). Otherwise, if \( l \) is a diagonal line of symmetry and there exists a unique leading corner, then the robots positions on the sides adjacent to the leading corner and are closest to the leading corner are selected as guards. The robot positions on the sides non-adjacent to the leading corner and farthest from the leading corner are also selected as guards. Note that they are symmetric images of each other. If there are two leading corners, the robots which are closest to the leading corners and lying on the sides adjacent to the leading corners are selected as guards. Note that, if \( C(t) \) is symmetric with respect to rotational symmetry, then since the center of rotational symmetry is also the center of fixed meeting nodes and gathering is finalized in the center, the Guard Selection phase is not executed in this case.

4.3.2. Target Weber Node Selection

In this phase, the Weber node for gathering is selected. The target meeting node must remain invariant during the execution of the algorithm. Depending on the class of configuration to which \( C(t) \) belongs, the target Weber node is selected according to Table 3. The pseudo-code corresponding to this phase is given in Algorithm 1.
Target Weber Node Selection

| Configuration \( C(t) \) | Target Weber Node |
|--------------------------|-------------------|
| Admitting a unique Weber node | The unique Weber node |
| Admitting a unique Potential Weber node | The unique Potential Weber node |
| \( \mathcal{I}_3 \land \) there exists a Weber node on \( l \) | The northernmost Weber node on \( l \) |
| \( \mathcal{I}_3^p \land \) there does not exist any Weber node on \( l \land |W_p(t)| = 2 \) \( \land l \) is a horizontal or vertical line of symmetry | The Potential Weber node in \( \mathcal{H}^+ \). Ties are broken by considering the Potential Weber node which appears last in the string direction associated to the leading corner in \( \mathcal{H}^+ \) |
| \( \mathcal{I}_3^p \land \) there does not exist any Weber node on \( l \land |W_p(t)| = 2 \) \( \land l \) is a diagonal line of symmetry \( \land \) there exists a unique leading corner | The Potential Weber node in \( \mathcal{H}^+ \) which appears last in the string direction associated to the unique leading corner |
| \( \mathcal{I}_4 \land \) there exists a Weber node on \( c \) | The Weber node on \( c \) |
| \( \mathcal{I}_4 \land \) there exists a Weber node on \( c \land |W_p(t)| \geq 2 \) \( \land \) there does not exist any Weber node on the quadrants | The Potential Weber node which is closest from the unique key corner in the string direction and lying on either \( l \) or \( l' \) |
| \( \mathcal{I}_4^p \land \) there does not exist a Weber node on \( c \land |W_p(t)| \geq 2 \) \( \land \) there exists a Weber node on the quadrants | The Potential Weber node in \( \mathcal{H}^{++} \) which is farthest from the leading corner contained in \( \mathcal{H}^{++} \) in the string direction |

Table 3. Target Weber Node selection.

We have the following observation.

**Observation 2.** If the meeting nodes are symmetric with respect to a unique line of symmetry \( L \), and there exists at least one meeting node on \( L \), then the meeting nodes on \( L \) are orderable.

The northernmost meeting node on \( l \) is defined as the meeting node on \( l \) which is farthest from the leading corner(s). Similarly, the northernmost robot on \( l \) is defined.

4.3.3. Leading Robot Selection

If the initial configuration is balanced and asymmetric, a robot \( r \) is selected as a leading robot in the Leading Robot Selection phase (Figure 11). The leading robot moves towards the half-plane or quadrant containing the target Weber node \( m \). While \( r \)
Algorithm 1: Target Weber Node Selection()

Input: $C(t) = (R(t), M)$

1. if $C(t) \in I_1$ then
2. Select the unique Weber node $m$;
3. else if $C(t) \in I_2$ then
4. Select the unique Potential Weber node $m$;
5. else if $C(t) \in I_3$ then
6. if $C(t)$ is asymmetric and $l \cap W(t) \neq \phi$ then
7. Select the northernmost Weber node on $l$;
8. else if $C(t)$ is asymmetric and $(l \cap W(t) = \phi \land |W_p(t)| = 1)$ then
9. Select the unique Potential Weber node $m$;
10. else if $C(t)$ is asymmetric and $(l \cap W(t) = \phi \land |W_p(t)| = 2)$ then
11. if there exists a unique leading corner in $H^+$ then
12. Select the Potential Weber node in $H^+$ which appears last in the string direction associated to the unique leading corner;
13. else if there exist two leading corner in $H^+$ then
14. Select the Potential Weber node that appears last in the string direction associated to the key corner in $H^+$;
15. else if $C(t)$ is symmetric with respect to the line $l$ then
16. Select the northernmost Weber node on $l$;
17. else if $C(t) \in I_4$ then
18. if $C(t)$ is asymmetric and $\{c\} \cap W(t) \neq \phi$ then
19. Select the Weber node on $c$;
20. else if $C(t)$ is asymmetric and $(\{c\} \cap W(t) = \phi \land |W_p(t)| = 1)$ then
21. Select the unique Potential Weber node $m$;
22. else if $C(t)$ is asymmetric and $(\{c\} \cap W(t) = \phi \land |W_p(t)| \geq 2)$ then
23. if All the Potential Weber nodes lie either on line $l$ or $l'$ then
24. Select the Potential Weber node which is farthest from the unique key corner in the string direction and lying on either $l$ or $l'$;
25. else if there exist a Potential Weber node lying on the quadrants then
26. Select the Potential Weber node in $H^{++}$ which is farthest from the leading corner contained in $H^{++}$ in the string direction;
27. else if $C(t)$ is symmetric with a meeting node on the center of rotation $c$ then
28. Select the meeting node on $c$;

reaches the half-plane or the quadrant containing $m$, the configuration transforms into an unbalanced configuration, and the asymmetry of the configuration remains invariant. Since the initial configuration is balanced, assume that $C(t) \in I_3 \cup I_4$. Further, assume that the initial configuration does not satisfy the condition $C_3$.

Depending on the class of configuration to which $C(t)$ belongs, the leading robot is selected according to Table 4. In case, the configuration is in $I_3$ and there exists a robot on $l$ (resp. $l'$), the leading robot first move along the line $l$ (resp. $l'$) and when it becomes collinear with $m$, it starts moving along $l'$ (resp. $l$).
Fig. 11. Balanced $I_a^3$ configuration. $m_2$ and $m_4$ are the Weber nodes lying on different half-planes. $m_2$ is selected as the target Weber node. Robot $r_2$ is selected as the leading robot.

| Configuration $C(t)$ | Leading Robot |
|---------------------|----------------|
| $I_a^3 \wedge$ there exists a robot on $l$ | The northernmost robot on $l$ |
| $I_a^3 \wedge$ there does not exist any robot on $l$ | The robot closest to $l$ and lying on $H^-$. Ties are broken by considering the robot on $H^-$ which is farthest from the leading corner contained in $H^-$ in the string direction |
| $I_a^4 \wedge$ there exists a robot either on $l$ or $l'$ | The robot closest to the target Weber node and lying on $l$ or $l'$. Ties are broken by considering the robot either on $l$ or $l'$ which is closest from the leading corner contained in $H^{++}$ in the string direction |
| $I_a^4 \wedge$ there does not exist any robot on $l$ and $l' \wedge$ there exists a non-guard robot in a quadrant adjacent to $H^{++}$ | The robot lying on a quadrant adjacent to $H^{++}$ and closest to the target Weber node. Ties are broken by considering the robot, which is closest from the leading corner contained in $H^{++}$ in the string direction |
| $I_a^4 \wedge$ there does not exist any robot on $l$ and $l' \wedge$ there does not exist any non-guard robot in the quadrants adjacent to $H^{++}$ | The robot lying on the quadrant non-adjacent to $H^{++}$ and closest to the target Weber node. Ties are broken by considering the robot, which is closest from the leading corner contained in $H^{++}$ in the string direction |

Table 4. Leading Robot Selection.

4.3.4. Symmetry Breaking

In this phase, all the symmetric configurations that can be transformed into asymmetric configurations are considered. A unique robot is identified that allows the transformation. We have the following cases.
(1) $C(t) \in I_{3}^{2}$. In this class of configurations, at least one robot exists on $l$. Let $r$ be the northernmost robot on $l$. $r$ moves towards an adjacent node that does not belong to $l$, and the configuration becomes asymmetric.

(2) $C(t) \in I_{4}^{2}$. In this class of configurations, there exists a robot (say $r$) on $c$. The robot $r$ moves towards an adjacent node. If the configuration admits rotational symmetry with multiple lines of symmetry and there is a robot $r$ at the center, $r$ moves towards an adjacent node. This movement creates a unique line of symmetry $l'$. However, the new position of $r$ might have a multiplicity. If that happens to be the northernmost robot on $l'$, moving robots from there might still result in a configuration with a line of symmetry. Even so, the unique line of symmetry $l'$ would still contain at least one robot position without multiplicity, and the number of robot positions on $l'$ will be strictly less than the number of robots on the line of symmetry in the original configuration. Thus, the repeated movement of the robot on $l'$ guarantees to transform the configuration into an asymmetric configuration.

4.3.5. Creating Multiplicity on Target Weber Node

The target Weber node $m$ is selected in the Target Weber Node Selection phase. Since there is a unique target Weber node $m$, all the non-guard robots move towards $m$ in the Creating Multiplicity on Target Weber Node phase. Note that, since the guards do not move during this phase, the MER remains invariant. As a result, $m$ remains invariant. Eventually, a robot multiplicity is created on $m$, while the non-guards moves towards it. Depending on the class of configuration to which $C(t)$ belongs, the following cases are to be considered.

(1) $C(t) \in I_{1}$: All the robots moves towards the unique Weber node $m$.

(2) $C(t) \in I_{2}$: All the non-guards move towards the unique target meeting node $m$.

(3) $C(t) \in I_{3}$: If $C(t) \in I_{3}^{3}$ and there exists a Weber node on $l$, each non-guards move towards the target meeting node $m$.

Next, consider the case when $C(t) \in I_{3}^{2}$ and there does not exist any Weber node on $l$. A leading robot in the Leading Robot Selection phase transforms a balanced configuration into an unbalanced configuration. All the non-guard robots from $H^{-}$ move towards $m$. This movement is required in order to ensure that $H^{+}$ remains invariant. While such robots reach $H^{+}$, all the non-guard robots in $H^{+}$ move towards $m$, thus creating a multiplicity on $m$.

Finally, if $C(t) \in I_{3}^{2}$, each non-guard which is closest to $m$, moves towards $m$ either synchronously or there may be a possible pending move due to the asynchronous behavior of the scheduler. Ties are broken by considering the closest robots which are farthest from the leading corners in their respective
string directions.

(4) $C(t) \in I_4$: First consider the case when the target Weber node $m$ is the center of rotational symmetry. Each robot moves towards $m$.

Next, consider the case when $C(t) \in I_4^a$ and the target Weber node $m$ is on $\mathcal{H}^{++}$. After the Leading Robot Selection phase, each non-guard robot in the quadrants different from $\mathcal{H}^{++}$ as well as on the lines $l$ or $l'$ moves towards the target Weber node $m$ in $\mathcal{H}^{++}$. First, the non-guards in the quadrants adjacent to $\mathcal{H}^{++}$ and the robots on $l$ or $l'$ moves towards $\mathcal{H}^{++}$. While they reach $\mathcal{H}^{++}$, the non-guards in the quadrants non-adjacent to $\mathcal{H}^{++}$ moves towards $\mathcal{H}^{++}$. This movement is required in order to ensure that $\mathcal{H}^{++}$ remains invariant. While such non-guards reach $\mathcal{H}^{++}$, each non-guards in $\mathcal{H}^{++}$ moves towards $m$, thus creating a robot multiplicity on $m$. Otherwise, consider the case when $m$ is on either on $l$ or $l'$. First all the non-guards in $\mathcal{H}^{++}$ moves towards $m$. While they reach $l$ or $l'$, the other non-guards moves towards $m$. This movement is required in order to ensure that $\mathcal{H}^{++}$ remains invariant. Finally, a robot multiplicity is created at $m$.

Note that if the configuration is asymmetric or symmetric with respect to rotational symmetry, there may exist at most four robot positions that are not on $m$ during this phase. Otherwise, if the configuration is symmetric with respect to a horizontal or vertical line of symmetry, there may exist at most six robot positions that are not on $m$.

4.3.6. Finalization of Gathering

Let $m$ be the target Weber node, where a robot multiplicity is created during the Creating Multiplicity on Target Weber Node phase. The following are the cases in which the robots will identify that the Finalization of Gathering is in progress:

1. The configuration has at most four robot positions that are not on $m$ containing robot multiplicity. Moreover, each side of the MER contains at most one robot position.

2. The configuration has exactly six robot positions that are not on $m$ containing robot multiplicity. Moreover, there exist exactly two sides of the MER that contain two robot positions and are symmetric images of each other.

In this phase, all the guards move towards $m$. During their movement, they do not create any multiplicity on a Weber node other than $m$. In order to ensure this, all the guards first move along the boundary of MER, and when it becomes collinear with $m$, it starts moving towards $m$. A guard robot moves by minimizing the Manhattan distance between $m$ and itself. This implies that during their movement, no other multiplicity would be created on any other Weber node and gathering would be finalized on $m$. 
4.4. Optimal Gathering()

Our main algorithm Optimal Gathering() considers the following cases. If \( C(t) \in I_1 \), then each robot finalizes the gathering on the unique Weber node. Consider the case when the meeting nodes are asymmetric. There exists a unique Potential Weber node. The guards are selected in the Guard Selection phase. Each non-guard moves towards the unique Potential Weber node, creating a multiplicity on it. Finally, the guards move towards the multiplicity and finalizes the gathering on it.

Next, consider the case when the configuration is balanced and asymmetric. A leading robot is selected in the Leading Robot Selection phase, which transforms the configuration into an unbalanced configuration. The guards are selected in the Guard Selection phase. In the Creating Multiplicity on the Target Weber node phase, each non-guard moves towards the target Weber node, selected in the Target Weber Node Selection phase. Finally, the guards move towards the multiplicity and finalizes the gathering.

If \( C(t) \) is symmetric and there exists a Weber node on \( l \cup \{c\} \), the gathering is finalized on the target Weber node \( m \) selected in the Target Weber node Selection phase. Otherwise, if the configuration is symmetric and there exists a robot on either \( l \) or \( c \), then in the Symmetry Breaking phase, the configuration is transformed into an asymmetric configuration. Note that, in case \( C(t) \in I_3^b \), there may exist exactly six robots that are selected as guards. In case \( n = 7 \), there must exist at least one robot position on \( l \). The northernmost robot on \( l \) moves towards an adjacent node away from \( l \), if there does not exist any Weber nodes on \( l \). Hence, the configuration becomes asymmetric, and the algorithm proceeds similarly, as in the asymmetric case for \( n = 7 \). Otherwise, if there exists at least one Weber node on \( l \), the northernmost Weber node \( m \) on \( l \) is selected as the target Weber node. The closest robot on \( l \) and the northernmost in case of a tie move towards \( m \). While the robot moves towards \( m \), it remains invariant. After the robot reaches \( m \), \( m \) is uniquely identifiable and the gathering is finalized in the Finalization of Gathering phase.

5. Correctness

In this section, we describe the correctness of our proposed algorithm. Lemmas 3 and 4 proves that the leading robot remains invariant during the movement towards its destination.

**Lemma 3.** If \( C(t) \in I_3^a \), then in the Leading Robot Selection phase, the leading robot remains the unique robot while it moves towards the half-plane \( \mathcal{H}^+ \).

**Proof.** Let \( C(t) \) be any balanced configuration that belongs to \( I_3^a \). Since the configuration is balanced and asymmetric, the number of robots in the two half-planes delimited by \( l \) are equal and there exists a unique key corner. If there exists at
least one robot position on \( l \), then the northernmost robot on \( l \) is the leading robot. The northernmost robot moves towards an adjacent node away from \( l \), and the configuration becomes unbalanced. Consider the case, when there does not exist any robot position on \( l \). Without loss of generality, assume that \( l \) is a vertical line of symmetry. Let \( r \) be the leading robot in \( \mathcal{H}^- \) selected in the Leading Robot Selection phase. Without loss of generality, let \( A \) be the unique key corner and \( \alpha_{AD} = a_1, a_2, \ldots, a_{pq} \) is the unique smallest lexicographic string associated to the corner \( A \). Similarly, let \( B \) be the other leading corner and \( \alpha_{BC} = b_1, b_2, \ldots, b_{pq} \) be the string associated to \( B \). Let \( u_i \) and \( v_i \) denote the nodes, which the positions \( a_i \) and \( b_i \) represents in \( \alpha_{AD} \) and \( \alpha_{BC} \), respectively. Since the meeting nodes are symmetric, \( f_i(u_i) = f_i(v_i) \), for each \( i = 1, 2, \ldots, pq \). As \( \alpha_{AD} = a_1, a_2, \ldots, a_{pq} \) is the unique smallest lexicographic string among the \( \alpha'_s \), there must exist a position \( k' \) such that \( \lambda_i(u_{k'}) = 0 < \lambda_i(v_{k'}) = 1 \). Without loss of generality, let \( i \) be the position of the leading robot in \( \alpha_{AD} \). Let \( k \) be the first position, where \( \lambda_i(u_k) \) and \( \lambda_i(v_k) \) differs. Note that, \( \lambda_i(u_k)=0 \) and \( \lambda_i(v_k) = 1 \). We have to prove that after the movement of the leading robot, \( \alpha_{AD} <_i \alpha_{BC} \), where \( ' <' \) denote the relation that \( \alpha_{AD} \) is lexicographically smaller than \( \alpha_{BC} \). Assume that at time \( t' \), the leading robot moves towards an adjacent node. Depending on the possible values of \( i \) and \( k \) in \( \alpha_{AD} \), the following cases are considered.

**Case 1.** The position of \( i \) is less than \( k \) in \( \alpha_{AD} \). While the leading robot moves towards \( l \), \( \lambda_i(u_i) \) becomes 0, but \( \lambda_i(v_i) \) equals 1. Hence, after the movement of the leading robot towards an adjacent node, \( \alpha_{AD} <_i \alpha_{BC} \).

**Case 2.** The position of \( i \) is equal to \( k \) in \( \alpha_{AD} \). Since each robot is deployed at the distinct nodes of the grid in the initial configuration, this case is not possible.

**Case 3.** The position of \( i \) is greater than \( k \) in \( \alpha_{AD} \). While the leading robot moves towards \( l \), the position \( k \) remains invariant. Hence, after the movement of the leading robot towards an adjacent node, \( \alpha_{AD} <_i \alpha_{BC} \).

Note that, after a single movement of the leading robot towards \( l \), it becomes the unique robot that is eligible to move towards \( \mathcal{H}^+ \). Since \( \alpha_{AD} \) remains the unique lexicographically smallest string at \( t' \), \( \mathcal{H}^+ \) remains invariant. Clearly, after a finite number of movements towards \( l \), \( \mathcal{H}^+ \) remains invariant, and ultimately, the configuration becomes unbalanced. The proof is similar when the meeting nodes admits a horizontal or a diagonal line of symmetry.

**Lemma 4.** If \( C(t) \in T^4_4 \), then in the Leading Robot Selection phase, the leading robot remains the unique robot while it moves towards the target Weber node.

**Proof.** Let \( C(t) \) be any balanced configuration that belongs to \( T^4_4 \). Since the configuration is balanced and asymmetric, there exist at least two quadrants that contain the maximum number of Potential Weber nodes with the maximum number of robots on such quadrants. We have to prove that while the leading robot moves
towards the target Weber node, the quadrant $\mathcal{H}^{++}$ remains invariant. First, consider the case when the leading robot $r$ is on either $l$ or $l'$. Note that in this case, $r$ may be one or more than one node away from $\mathcal{H}^{++}$. There is nothing to prove when $r$ is one node away from $\mathcal{H}^{++}$. In this case, a move of $r$ transforms the configuration into an unbalanced configuration. Therefore, consider the case when $r$ is more than one node away from $\mathcal{H}^{++}$. Without loss of generality, let $r$ be on $l$. Let $MER = ABCD$ be such that the corner $C$ is the corner diagonally opposite to $A$ and the corners $A$ and $B$ are separated by line $l$. Similarly, $A$ and $D$ are the corners separated by line $l'$. $\mathcal{H}^{++}$ is the quadrant containing $A$. Let $\alpha_{AD} = a_{1}, a_{2}, \ldots , a_{pq}$ and $\alpha_{BC} = b_{1}, b_{2}, \ldots , b_{pq}$ be the strings associated to the corners $A$ and $B$. While $r$ moves along $l$, we have to prove that that $\alpha_{AD}$ remains lexicographic larger than $\alpha_{BC}$. It is noteworthy that $\alpha_{AD}$ is lexicographic larger than $\alpha_{CB}$ and $\alpha_{AD}$ while $r$ moves. Note that, we have consider the case when the string directions are along the width of the rectangle. Let $i$ be the position of leading robot in $\alpha_{AD}$ and $\alpha_{BC}$. Let $u_{i}$ and $v_{i}$ denote the nodes, which the positions $a_{i}$ and $b_{i}$ represents in $\alpha_{AD}$ and $\alpha_{BC}$, respectively. Since the meeting nodes are symmetric, $f_{l}(u_{i}) = f_{l}(v_{i})$, for each $i = 1, 2, \ldots , pq$. After a movement of the leading robot along the line $l$, note that $\mathcal{H}^{++}$ remains invariant. After a finite number of movements, the robot $r$ becomes one node away from $\mathcal{H}^{++}$, and the proof proceeds similarly as before. Next, consider the case when the leading robot is on a quadrant adjacent to $\mathcal{H}^{++}$. Without loss of generality, assume that the leading robot is on $\mathcal{H}^{+-}$. While the leading robot moves, it can be observed that $\alpha_{AD}$ is lexicographically larger than $\alpha_{CB}$ and $\alpha_{DA}$. We have to prove that $\alpha_{AD}$ remains lexicographic larger than $\alpha_{BC}$ while the leading robot moves. Let $i$ and $j$ be the position of the leading corner in $\alpha_{AD}$ and $\alpha_{BC}$, respectively. Note that $i < j$, as the leading robot is selected on $\mathcal{H}^{+-}$. Let $k$ be the first position for which $b_{k} < a_{k}$. We have the following cases.

Case 1. $i < j < k$. Note that $u_{i-1}$ cannot be a robot position, otherwise $r$ would not be selected as a leading robot. After a move of $r$, $u_{i-1}$ is a robot position but $v_{i-1}$ cannot be a robot position.

Case 2. $i < j = k$. Since each robot is deployed at the distinct nodes of the grid in the initial configuration, this case is not possible.

Case 3. $i < k < j$. After a move of $r$, $u_{i-1}$ is a robot position, but $v_{i-1}$ cannot be a robot position, as $k$ is the first position where $a_{k} > b_{k}$.

Case 4. $i = k < j$. The proof is similar to the previous case.

Case 5. $k < i < j$. After a move of $r$, it may be the case that $k = i - 1$ in $\alpha_{AD}$. In that case, $\lambda_{l}(u_{i-1}) = 2$, but $\lambda_{l}(v_{i-1}) = 0$. Otherwise, the proof is similar as $a_{k} > b_{k}$.

The proof is similar when the string directions are along the lengths of $MER$. Next, consider the case when the leading robot $r$ is selected on a quadrant non-adjacent
to $\mathcal{H}^{++}$. Without loss of generality, we assume that $r$ first starts moving towards $l'$. While the leading robot moves, it can be observed that $\alpha_{AD}$ is lexicographically larger than $\alpha_{CB}$ and $\alpha_{DA}$. We have to prove that $\alpha_{AD}$ remains lexicographic larger than $\alpha_{BC}$ while $r$ moves. Let $i$ and $j$ be the position of the leading corner in $\alpha_{AD}$ and $\alpha_{BC}$, respectively. Note that $i > j$, as the leading robot is selected on $\mathcal{H}^{--}$. We have the following cases.

**Case 1.** $i > j > k$. Note that $u_{i-1}$ cannot be a robot position, otherwise $r$ would not be selected as a leading robot. After a move of $r$, $a_{i-1} \geq b_{i-1}$, depending on whether there exists a robot position on $b_{i-1}$ or not.

**Case 2.** $i > j = k$. Since each robot is deployed at the distinct nodes of the grid in the initial configuration, this case is not possible.

**Case 3.** $i = k > j$. After a move of $r$, $a_{i-1} \geq b_{i-1}$, depending on whether there exists a robot position on $b_{i-1}$ or not.

**Case 4.** $k > i > j$. Note that $a_{i-1}$ cannot be a robot position before the move, otherwise $r$ would not be selected as a robot position. After a move of $r$, $a_{i-1}$ is a robot position, but $b_{i-1}$ cannot be a robot position as $k$ is the first position where $a_k$ and $b_k$ differs.

**Case 5.** $j < k < i$. After a move of $r$, it may be the case that $k = i - 1$ in $\alpha_{AD}$. In that case, $\lambda_t(u_{i-1}) = 2$, but $\lambda_t(v_{i-1}) = 0$. Otherwise, the proof is similar as $a_k \geq b_k$.

From all the above cases, the leading robot remains invariant while it moves towards its destination. \(\square\)

The next three lemmas prove that the target Weber node remains invariant in the Creating Multiplicity on the Target Weber node phase.

**Lemma 5.** If $C(t) \in \mathcal{L}_2 \cup \mathcal{L}_3 \cup \mathcal{T}_4$, then the target Weber node remains invariant in the Creating Multiplicity on Target Weber Node phase.

**Proof.** In the Creating Multiplicity on Target Weber Node phase, all the non-guard robot move towards the target Weber node. According to Lemma[1] the Weber node remains invariant under the movement of robots towards itself. The MER remains invariant unless the guard moves. The following cases are to be considered.

**Case 1.** $C(t) \in \mathcal{L}_2$: Since the meeting nodes are asymmetric, there exists a unique Potential Weber node. The unique Potential Weber node is selected as the target Weber node. The unique Potential Weber node of the configuration is defined with respect to the position of the leading corner. The leading corner remains invariant unless the MER changes. As the guards do not move in the Creating Multiplicity on Target Weber Node phase, the MER remains invariant. Hence, the target Weber node remains invariant.
Case 2. \( C(t) \in I_3^b \): The northernmost Weber node on \( l \) is selected as the target Weber node. Since the northernmost agreement depends on the position of the leading corner(s), the agreement remains invariant unless the MER changes. As the guards do not move in the Creating Multiplicity on Target Weber Node phase, the MER remains invariant. Hence, target Weber node remains invariant.

Case 3. \( C(t) \in I_4^b \): The center of rotational symmetry \( c \) is the target Weber node. Since \( c \) is also the center of rotational symmetry for the meeting nodes also, the target Weber node remains invariant.

Lemma 6. If \( C(t) \in I_3^a \), then the target Weber node remains invariant in the Creating Multiplicity on Target Weber Node phase.

Proof. The meeting nodes are symmetric with respect to a single line of symmetry \( l \). In the Creating Multiplicity on Target Weber Node phase, all the non-guard robot move towards the target Weber node. The MER remains invariant unless the guard moves. The following cases are to be considered.

Case 1. There exists at least one Weber node on \( l \). The northernmost Weber node on \( l \) is selected as the target Weber node. Since the northernmost agreement depends on the position of the leading corner(s), and the leading corner(s) remains invariant unless the MER changes, the agreement remains invariant. Hence, the target Weber node remains invariant.

Case 2. \( C(t) \) satisfy \( C_1 \). In this case, we have to prove that \( H^+ \) remains invariant in the Creating Multiplicity on Target Weber Node phase. Note that, in this phase, all the non-guards move towards the target Weber node. According to Lemma 1, the Weber nodes remains invariant while the robots move towards it. As MER remains invariant unless the guard robot moves, the leading corner(s) remains invariant. Since, the Potential Weber nodes are defined with respect to the positions of the leading corner(s), \( H^+ \) remains invariant. Hence, the target Weber node remains invariant.

Case 3. \( C(t) \) satisfy \( C_2 \). The following subcases are to be considered.

Subcase 1. \( C(t) \) satisfy \( C_{22} \). \( H^+ \) is the half-plane that contains the maximum number of robots. All the non-guard robots in \( H^- \) move towards the target Weber node in \( H^+ \). During this movement of the robots, \( H^+ \) still contains the maximum number of robots. Hence, the target Weber node remains invariant.

Subcase 2. \( C(t) \) satisfy \( C_{21} \). The leading robot in \( H^- \) move towards the target Weber node in \( H^+ \) resulting in transforming the configuration into an unbalanced configuration. While the leading robot moves towards \( H^+ \), the unique lexicographic smallest string \( \alpha_i \) remains invariant according to Lemma 3. As the key corner remains invariant, \( H^+ \) remains invariant. The moment the leading robot reaches \( l \), the
configuration becomes unbalanced. The rest of the proof follows from the previous case.

Lemma 7. If $C(t) \in T_a^4$, then the target Weber node remains invariant in the
Creating Multiplicity on Target Weber Node phase.

Proof. Since $C(t) \in T_a^4$, the meeting nodes are symmetric with respect to rotational
symmetry. Let $c$ be the center of the rotational symmetry for $M$. According to
Lemma 1, the Weber nodes remain invariant while all the robots move towards it.

Case 1. There exists a Weber node on $c$. It is selected as the target Weber node.
Since $c$ is the center of rotational symmetry for the fixed meeting nodes, the target
Weber node remains invariant while the robots move towards it.

Case 2. $C(0)$ satisfy $C_1$. The target Weber node is selected in $H^{++}$ as the Weber
node which is farthest from the leading corner contained in $H^{++}$ in the string
direction. We have to prove that $H^{++}$ remains invariant while the robots move
towards the target Weber node. Note that since the guards do not move during this
phase, the MER remains invariant. As a result, the Potential Weber nodes and
$H^{++}$ remains invariant. Hence, the target Weber node remains invariant.

Case 3. $C(0)$ satisfy $C_2$. We have to prove that $H^{++}$ remains invariant while the
robots move towards the target Weber node. Considering such quadrants that con-
tain the maximum number of Potential Weber nodes, the target Weber node is
selected as the Potential Weber node in $H^{++}$. Ties are broken by considering the
Weber node in $H^{++}$ which is farthest from the leading corner in $H^{++}$ in the string
direction. If $C(0)$ satisfy $C_{22}$, first, all the non-guard robots in the quadrants ad-
just adjacent to $H^{++}$ and on $l \cup l'$ move towards the target Weber node in $H^{++}$. Finally,
the other non-guard robots move towards $m$. Since $H^{++}$ is the unique quadrant
that contains the maximum number of robot positions, it still contains the maximum
number of robots while all such robots reach $H^{++}$. Hence, the target Weber
node remains invariant. Otherwise, if $C(0)$ satisfy $C_{21}$, there exists more than one
quadrant that contains the maximum number of robot positions. Considering such
quadrants and the corners contained in those quadrants. $H^{++}$ is the quadrant con-
taining the largest lexicographic string among those $\alpha_i$ that are associated with
the leading corners contained in such quadrants. A leading robot is selected in the
Leading Robot Selection phase and is allowed to move towards the target Weber
node in $H^{++}$. While the leading robots moves towards the target Weber node in $H^{++}$, $H^{++}$ remains invariant according to Lemma 4. As a result, the configuration
becomes unbalanced. The rest of the proof follows similarly, as in the unbalanced
case.

Case 4. $C(0)$ satisfy $C_3$. The target Weber node is selected on either $l$ or $l'$. Note
that, in this case, if the configuration is unbalanced, $H^{++}$ is the quadrant that
contains the minimum number of robots. Otherwise, if the configuration is balanced, then $H^+$ is the quadrant containing the smallest lexicographic string among all those $\alpha_i's$. In both the cases, all the non-guard robots on $l \cup l'$, and the non-guard robots on $H^+$, move towards the target Weber node $m$. After such robots reach $H^+$, $H^+$ remains the unique quadrant with the minimum number of robots. As a result, $H^+$ remains invariant. Hence, the target Weber node remains invariant.

The next two lemmas proves that any initial configuration $C(0) \in I \setminus U$, would never reach a configuration $C(t) \in U$, at any point of time $t > 0$ during the execution of the algorithm $Optimal Gathering()$.

Lemma 8. Given $C(0) \in I_3$ and $t > 0$ be an arbitrary instant of time at which at least one robot has completed its LCM cycle. If $C(0) \notin I_3^{b3}$, then during the execution of the algorithm $Optimal Gathering()$, $C(t) \notin I_3^{b3}$.

Proof. According to Lemma 1, the Weber nodes remains invariant while the robots move towards it. Since the meeting nodes admits a single line of symmetry $l$ and there does not exist any Weber node on $l$, assume that $C(0) \notin I_3^a \cup I_3^{b2}$. The following cases are to be considered.

Case 1. $C(0) \in I_3^a$. Note that there does not exist any Weber node on $l$, otherwise according to Lemma 1, $C(t) \notin I_3^{b3}$. Depending on the position of potential Weber nodes, the following subcases may arise.

Subcase 1. $C(0)$ satisfy $C_1$. All the non-guard robots in $H^- \cup l$ move towards the target Weber node in $H^+$. So, at any arbitrary instant of time $t > 0$, $C(t)$ remains asymmetric and hence $C(t) \notin I_3^{b3}$.

Subcase 2. $C(0)$ satisfy $C_2$. If the configuration is unbalanced, all the robots in $H^- \cup l$ moves towards the target Weber node in $H^+$. As a result, the configuration remains unbalanced and hence asymmetric. Otherwise, if the configuration is balanced, the leading robot moves towards the target Weber node at some time $t' > 0$. According to Lemma 3, the configuration remains asymmetric during its movement and ultimately, the configuration becomes unbalanced. Proceeding similarly, as in the unbalanced case, at any arbitrary instant of time $t > 0$, $C(t)$ remains asymmetric and hence $C(t) \notin I_3^{b3}$, where $t \geq t'$.

Case 2. $C(0) \in I_3^{b2}$. Assume that at time $t' > 0$, the northernmost robot on $l$ moves towards an adjacent node away from $l$, which transforms the configuration into an unbalanced asymmetric configuration. The rest of the proof follows from the previous case. Hence, $C(t) \notin I_3^{b3}$, where $t' \geq t$.

Lemma 9. Given $C(0) \in I_4$ and $t > 0$ be an arbitrary instant of time at which at least one robot has completed its LCM cycle. If $C(0) \notin I_4^{b3}$ then during the execution of the algorithm $Optimal Gathering()$, $C(t) \notin I_4^{b3}$.
Proof. According to Lemma [1], the Weber nodes remains invariant while the robots move towards it. Since, the meeting nodes admit rotational symmetry and there does not exist any Weber node on c, assume that \( C(0) \in I_a^4 \cup I_a^{12} \). The following cases are to be considered.

Case 1. \( C(0) \in I_a^4 \). If there exists a Weber node on c, then all the robots move towards it and finalize the gathering. According to Lemma [1], since the Weber node remains invariant while all the robots move towards it, \( C(t) \notin I_{b3}^4 \). Consider the case when there does not exist any Weber node on c. The following subcases may arise.

Subcase 1. \( C(0) \) satisfy \( C_1 \). All the non-guard robots from the other quadrants as well as on \( l \) or \( l' \) move towards the target Weber node in \( H^{++} \). So at any arbitrary instant of time \( t > 0 \), \( C(t) \) remains asymmetric and hence \( C(t) \notin I_{b3}^4 \).

Subcase 2. \( C(0) \) satisfy \( C_2 \). If the configuration is unbalanced, all the robots in the quadrants different from \( H^{++} \) as well as the robots on \( l \) or \( l' \) move towards the target Weber node in \( H^{++} \) in the Creating Multiplicity on Target Weber Node phase. While such a robot reaches \( H^{++} \), the configuration remains unbalanced and hence asymmetric. If the configuration is balanced, a leading robot is selected in the Leading Robot Selection phase. According to Lemma [4], the configuration remains asymmetric during the movement of the leading robot towards the target Weber node at some time \( t' > 0 \). While the leading robot reaches \( H^{++} \), the configuration becomes unbalanced and remains asymmetric. So, at any arbitrary instant of time \( t > 0 \), \( C(t) \notin I_{b3}^4 \), where \( t \geq t' \).

Subcase 3. \( C(0) \) satisfy \( C_3 \). All the robots in \( H^{++} \), move towards the target Weber node. After all the robots in \( H^{++} \), reaches the target Weber node \( m \), all the non-guard robots from the other quadrants as well as on \( l \) or \( l' \) move towards \( m \), thus creating a multiplicity on \( m \). During this robot movement, \( C(t) \) remains asymmetric and hence \( C(t) \notin I_{b3}^4 \).

Case 2. \( C(0) \in I_{b2}^4 \). Assume that at time \( t' > 0 \), the robot on c move towards one of the adjacent nodes which transforms the configuration into a configuration which may be asymmetric or admits a single line of symmetry. Proceeding similarly as in the case of \( C(0) \in I_a^4 \cup I_a^{12} \), at any arbitrary instant of time \( t > 0 \), \( C(t) \) remains asymmetric and hence \( C(t) \notin I_{b3}^4 \), where \( t \geq t' \).

Theorem 10. If the initial configuration belongs to the set \( I \setminus U \), then algorithm Optimal Gathering() ensures gathering over Weber nodes.

Proof. Assume that \( C(0) \in I \setminus U \). If \( C(t) \) is not a final configuration for some \( t \geq 0 \), each active robot executes algorithm Optimal Gathering(). According to the Lemmas [8] and [9], any initial configuration \( C(0) \in I \setminus U \), would never reach a configuration \( C(t) \in U \), at any point of time \( t > 0 \) during the execution of the algorithm Optimal Gathering(). The following cases are to be considered.
Case 1. There exists a unique Weber node. All the robots move towards the unique Weber node and finalize the gathering.

Case 2. There exists more than one Weber node. The target Weber node is selected in Target Weber node Selection phase. According to the Lemmas 3, 4, and 5, the target Weber node remains invariant during the execution of the algorithm Optimal Gathering(). If $C(0)$ is a balanced configuration, then a leading robot is selected in Leading Robot Selection phase. Lemmas 3 and 4 ensure that the leading robot remains invariant during its movement. Without loss of generality, assume that $m$ is the target Weber node. Assume that, at any point of time $t$, there exists at which at least one robot $r$ that has completed its LCM cycle. If $r$ is a non-guard robot, then it must have moved at least one unit distance towards $m$ at time $t' > t$. Since, each non-guard robot move towards $m$ via a shortest path in the Creating Multiplicity on Target Weber node phase, this implies that eventually at time $t'' > t'$, there exist a robot multiplicity on $m$. Finally, in the Finalization of Gathering phase, since the robots have global strong-multiplicity detection capability, all the guard robots move towards $m$ and finalizes the gathering without creating any other multiplicity on a meeting node. Since each robot finalizes the gathering, by moving towards $m$ via a shortest path, gathering over Weber nodes is ensured.

6. Optimal Gathering for $C(t) \in U$

We have proposed a deterministic distributed algorithm that ensures gathering over a Weber node for any initial configuration $C(0) \in I \setminus U$. Let $U' \subset U$ denote the set of all the initial configurations which admit a unique line of symmetry $l$ and no Weber nodes or robot positions exist on $l$. However, there exists at least one meeting node on $l$. The set $U'$ includes the initial configurations for which gathering is feasible on a meeting node. Note that, if $C(t) \in U \setminus U'$, then it is ungatherable. To ensure gathering deterministically, the target point must lie on $l$. At this point of time, one optimal feasible solution for a configuration $C(0) \in U'$ would be to finalize the gathering at a meeting node $m \in l$ at which the total number of moves is minimized. Ties may be broken by considering the northernmost such meeting node. Another very important
assumption that is not highlighted much in the literature is that initially, all the robots are static. The correctness of our proposed algorithm fails to hold when the optimal target point is dynamically selected. As a consequence, termination may not be guaranteed with optimal number of moves. For example, we consider one possible execution for an initial configuration \( C(0) = (\{r_1, r_2\}, \{m_1, m_2, m_3, m_4\}) \) in figure 12(a). At \( t = 0 \), \( m_3 \) and \( m_4 \) are the Weber nodes. Between \( m_1 \) and \( m_2 \), the number of total moves will be minimized if the robots gather at \( m_1 \). While \( r_1 \) and \( r_2 \) start moving towards \( m_1 \), there may be a pending move due to the asynchronous behavior of the scheduler. Consider the case when \( r_2 \) has completed its LCM cycle while \( r_1 \)’s move is pending. At \( t = t_1 > 0 \), \( m_3 \) becomes the unique Weber node (figure 12(b)). At \( t_2 > t_1 \), assume that \( r_1 \) has reached \( m_1 \) and \( r_2 \) has moved by one hop distance towards \( m_3 \). At \( t_2 \), \( m_1 \) becomes the unique Weber node (figure 12(c)). Next, the gathering would be finalized eventually at \( m_1 \). Initially, the minimum number of moves required to finalize the gathering is 8 (figure 12(a)). The number of moves required to finalize the gathering in this execution is 10. It is not guaranteed that the minimum number of moves required to finalize the gathering in the initial configuration is achievable.

7. Conclusion
In this paper, the optimal gathering over meeting nodes problem has been investigated over an infinite grid. The objective function is to minimize the total distance traveled by all the robots. We have characterized all the configurations for which gathering over a Weber node cannot be ensured. For the remaining configurations, a deterministic distributed algorithm has been proposed that solves the gathering over Weber nodes for at least seven robots. One future direction of work would be to consider the min-max gathering over meeting nodes problem, where the objective function is to minimize the maximum distance traveled by a robot. Since there remain some initial symmetric configurations, for which gathering over Weber nodes cannot be ensured, it would be interesting to consider randomized algorithms for those configurations. Another direction for future interest would be to consider multiplicities in the initial configuration.

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