Abstract

We present unfolded description of $AdS_4$ black hole with generic parameters of mass, NUT, magnetic and electric charges as well as two kinematical parameters one of which is angular momentum. A flow with respect to black hole parameters, that relates the obtained black hole unfolded system to the covariant constancy condition for an $AdS_4$ global symmetry parameter, is found. The proposed formulation gives rise to a coordinate-independent description of the black hole metric in $AdS_4$. The black hole charges are identified with flow evolution parameters while its kinematical constants are the first integrals of the black hole unfolded system expressed via invariants of the $AdS_4$ global symmetry parameter. It is shown how the proposed method reproduces various known forms of black hole metrics including the Carter and Kerr–Newman solutions. Free flow gauge parameters allow us to choose different metric representations such as Kerr–Schild, double Kerr–Schild or generalized Carter–Plebanski in the coordinate-independent way.
1 Introduction

In the paper [1] Kerr black hole in four-dimensional $AdS$ space-time was shown to admit unfolded formulation based on the Killing equation and the equation for the so-called Papapetrou field. As a starting point we used the following well known facts [2]

- Four-dimensional Einstein black holes are of Petrov D-type. In asymptotically flat space Riemann tensor is built of the derivatives of a Killing vector (Papapetrou field).
- Kerr–Schild Ansatz reduces nonlinear Einstein equations to linear Pauli–Fierz equations both on flat and on $AdS$ background.

The first property was generalized in [1] to the black hole on $AdS_4$ space within spinor approach leading to the description of $AdS_4$ Kerr black hole Weyl tensor in terms of $AdS_4$ Papapetrou field. Then we were able to show that $AdS_4$ black hole Kerr–Schild vector has background covariant nature and is built of $AdS_4$ Killing vector via certain coordinate-independent Killing projectors. This allowed us to describe Kerr black hole in $AdS_4$ covariant way via field redefinition of the $AdS_4$ global symmetry parameter covariant constancy condition. The important questions that have not been yet considered in [1] include

- As the unfolded formulation is by construction coordinate-free how does a particular choice of the $AdS_4$ global symmetry parameter affects the diffeomorphism-invariant properties of the resulting Kerr–Schild metric?
- How the approach of [1] can be generalized to a wider class of four-dimensional black holes to include electric charge, NUT parameter, etc.?

In this paper we answer these and some related questions. In our work we stick to the idea of [1] that $AdS_4$ global symmetry parameter covariant consistency equation

$$D_0 K_{AB} = 0, \quad D_0^2 = 0,$$  \hspace{1cm} (1.1)

where $A, B = 1, \ldots, 4$ are the $AdS_4$ spinor indices and $D_0$ is the $AdS_4$ covariant differential, admits a parametric deformation into a wider class of black holes. A novelty, however, is to rewrite (1.1) in terms of Killing vector and source-free Maxwell tensor field, rather than using Papapetrou field as in [1]. This redefinition turns out to be very convenient being particularly natural taking into account that all four-dimensional Einstein–Maxwell black holes have curvature tensor built of a sourceless Maxwell tensor. It follows that simple consistent deformation of (1.1) that preserves Killing and Maxwell properties of the system leads to Petrov D-type Weyl tensor with Ricci tensor given by Maxwell energy-momentum tensor and constant scalar...
curvature of $AdS_4$ space-time ($\Lambda = 3\lambda^2$). Defined this way unfolded system contains three real parameters $\mathcal{M} \in \mathbb{C}$ and $q \in \mathbb{R}$ instead of one real parameter of Kerr black hole mass of $\mathcal{I}$. We show that $\text{Re} \mathcal{M}$ and $\text{Im} \mathcal{M}$ correspond to black hole mass and NUT charge, respectively, while $q = 2(e^2 + g^2)$, where $e$ and $g$ are electric and magnetic charges, respectively.

In general, the obtained black hole unfolded system reproduces the so called Carter–Plebanski family of solutions, which in addition to aforementioned four real curvature parameters ($\text{Re} \mathcal{M}$, $\text{Im} \mathcal{M}$, $q$, $\lambda$) have two kinematic constants related to angular momentum $a$ and certain discrete parameter $\epsilon$ [3, 4, 5]. These kinematic parameters are shown to arise in the black hole unfolded system (BHUS) as two invariants of unfolded equations (first integrals).

To obtain explicit expressions for the metric resulting from our unfolded equations and to validate it is indeed of Carter–Plebanski family we use an efficient integrating flow method analogous to the one developed in [6, 7] for higher spin nonlinear equations. Applying the consistency requirement $[\partial_\chi, d] = 0$ to BHUS, where $\chi = (\mathcal{M}, q)$ are the deformation parameters and $d$ is space-time de Rham differential, we derive the first-order differential equations in the $\chi$ parameter space for all fields involved, i.e., vierbein, Killing vector, etc. The obtained flow equations can be easily integrated with the initial data $\mathcal{M} = 0, q = 0$ that correspond to pure $AdS_4$ vacuum leading to the coordinate-free description of generic Carter–Plebanski family of metrics.

The integrating flow reveals remarkable properties of the black hole parameters. In particular, the kinematic parameters turn out to be related to two invariants

$$C_2 = \frac{1}{4} K_{AB} K^{AB}, \quad C_4 = \frac{1}{4} \text{Tr}(K^4)$$

(1.2)

of the $AdS_4$ algebra $sp(4)$ which are modules that characterize the vacuum unfolded system. For example, a static black hole corresponds to

$$C_4 = C_2^2.$$  \hspace{1cm} (1.3)

One of the motivations for this work was to elaborate the unfolded approach to classical black holes appropriate to the analysis of black hole solutions in the spinor form of 4$d$ higher spin gauge theory in the form of [8, 9, 7] (see also [10, 11, 13, 12] for reviews of higher spin theory). In [11] we have shown that at least at the free field level the black hole solution admits a natural extension to higher spins. The natural question for the future study is whether those would receive corrections had

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1The term black hole that will be used throughout the paper is strictly speaking abuse of terminology as we do not restrict metric parameters to the domain that corresponds to the existence of horizons and absence of naked singularities.
the interaction switched on. To study this interesting question it is necessary to have black holes description in the spirit of higher spin unfolded formulation of [8, 9, 7]. In this paper we show that such a formulation is indeed available.

We believe, however, that the results of this paper may on their own right have useful applications in black hole physics. In particular, a wide class of black hole metrics formulated in coordinate-free form allows one to obtain straightforwardly their realization in any background coordinates. Moreover, by choosing appropriately free parameters in the integration flow one can reduce metric to either Kerr–Schild or double Kerr–Schild or “generalized” Carter–Plebanski form depending on the number of deformation parameters.

The rest of the paper is organized as follows. We start in Section 2 by summarizing main results obtained in this paper. In Section 3 we reformulate Einstein gravity using the Cartan formalism being most appropriate for our analysis. In Section 4 starting from the Killing equation in AdS\(_4\) we rewrite it in the unfolded form. Then we study its properties, particularly, find the first integrals, discrete symmetry, introduce certain Killing projectors giving rise to four Kerr–Schild vectors. In particular in Subsection 4.4 invariants of the AdS\(_4\) symmetry algebra \(sp(4)\) are discussed in connection with Killing symmetries of the system. Generic black hole unfolded system, obtained as a parametric deformation of initial AdS\(_4\) unfolded system, is presented in Section 5. We find that BHUS inherits most of the pre-deformed properties and symmetries. In particular, it expresses Weyl tensor in terms of Maxwell field making it manifestly of Petrov D-type. In Section 6 we apply the integrating flow technique to obtain the first order differential equations with respect to the deformation parameters that encode generic black hole solution. Integration of obtained flow-equations with AdS\(_4\) initial data is carried out in Section 7 giving rise to AdS\(_4\) covariant and coordinate-independent description of black hole metric. In Section 8 we use particular coordinate system for AdS\(_4\) space-time and its unfolded system. It allows us to reproduce in Subsection 9.1 canonical form of Carter–Plebanski metric and to identify BHUS modules with the physical black hole parameters. Section 10 contains summary and conclusions. The notation is summarized in Appendix A. The details on derivation of the integrating flow equations and their integration are given in Appendices B and C respectively. For the readers’ convenience the unfolded equations are rewritten in vector notation in Appendix D and some useful properties are presented in Appendix E. Finally, Plebanski–Demianski solution is commented in Appendix F.

2 Main results

The main result of our work is the unfolded formulation of generic AdS\(_4\) Einstein–Maxwell black hole solution. This formulation is coordinate-independent. Modules
of solutions include the real dynamical modules $M$, $N$ and $q$ which are, respectively, the black hole mass, NUT and a combination of electric and magnetic charges. The $o(3,2) \sim sp(4, \mathbb{R})$ transformations act on the modules of BH solution including three coordinates of the black hole position, three Lorentz boosts (i.e., velocities) and two angles of the rotation axis orientation. Two invariants of the $AdS_4$ transformations parameterize black hole kinematical parameters – its angular momentum per unit mass $a$ and Carter–Plebanski parameter $\epsilon$. The normalization of the $AdS_4$ invariants with $\epsilon = \pm 1$ or $0$ sets the scale for black hole curvature modules $M$, $N$ and $q$. The charge $q = 2(e^2 + g^2)$ arises as some inner $u(1)$-invariant, while the electro-magnetic duality mixes $M \leftrightarrow N$ and $e \leftrightarrow g$.

To reproduce generic black hole we use the idea of $[1]$, constructing black hole unfolded system as a deformation of $AdS_4$ global symmetry constancy equation

$$D_0 K_{AB} = 0 , \quad (2.1)$$

where $K_{AB}(x) = K_{BA}(x)$ is an $AdS_4$ symmetry parameter and $D_0$ is the $AdS_4$ co-variant differential (for notation see Appendix A). As explained in $[14]$ any solution of $(2.1)$ describes some symmetry of $AdS_4$. In particular, it gives rise to the corresponding Killing vector (see e.g. $[13]$).

Indeed, in two-component spinor notation $K_{AB}$ has the form

$$K_{AB} = \begin{pmatrix} \lambda^{-1} \kappa_{\alpha\beta} & V_{\alpha\beta} \\ V_{\beta\alpha} & \lambda^{-1} \bar{\kappa}_{\dot{\alpha}\dot{\beta}} \end{pmatrix} , \quad (2.2)$$

where $\lambda$ is the $AdS$ radius and $V_{\alpha\dot{\alpha}}$ is some vector. From $(2.2)$ it follows that $V_{\alpha\dot{\alpha}}$ satisfies

$$D V_{\alpha\dot{\alpha}} = \frac{1}{2} h_{\gamma\dot{\alpha}} \kappa_{\gamma\alpha} + \frac{1}{2} h_{\alpha\dot{\gamma}} \bar{\kappa}_{\dot{\alpha}\dot{\gamma}} , \quad (2.3)$$

where $D$ is the Lorentz derivative and $h_{\alpha\dot{\beta}}$ is the $AdS_4$ vierbein one-form. From $(2.3)$ it follows immediately that

$$D_{\alpha\dot{\alpha}} V_{\alpha\dot{\alpha}} = 0 , \quad (2.4)$$

or, equivalently, in tensor notation

$$D_i V_j + D_j V_i = 0 . \quad (2.5)$$

The equation $(2.5)$ means that $V_{\alpha\dot{\alpha}}$ is a Killing vector. $(2.1)$ does not impose any additional conditions being equivalent to $(2.5)$ along with the $AdS_4$ consistency $(1.1)$. The fields $\kappa_{\alpha\beta}$ and $\bar{\kappa}_{\dot{\alpha}\dot{\beta}}$ are (anti)self-dual parts of the Killing two-form$^2$$^2 \kappa_{ij} = D_i V_j$ (i, $j = 1, \ldots, 4$ are world indices).

$^2$One can easily check that it is a closed Killing–Yano tensor. Note that the reverse statement generally is not true, i.e., vector associated with closed Killing–Yano tensor is not necessary a Killing vector.
Note that the system (2.1) written down in components (2.2) provides the simplest example of unfolded equations consistent by virtue of zero-curvature condition (1.1) for AdS$_4$. In Section 5 we will show that a simple consistent deformation of (2.1) leads to certain Killing–Maxwell unfolded system that describes generic Carter–Plebanski metric. In terms out that it can be written down in terms of the fields of (2.2) in an AdS$_4$ covariant way.

To reproduce the metric explicitly we show that (2.1) generates four Kerr–Schild vectors built of components (2.2) in a coordinate-independent way. Two of them, $k^i$ and $n^i$, are real

\[ k_i k^i = n_i n^i = 0, \quad k^i D_i k_j = n^i D_i n_j = 0 \]  

and another two are complex-conjugated and orthogonal to $k^i$ and $n^i$

\[ l_i^+ l^{+i} = l_i^- l^{-i} = 0, \quad l^{+i} D_i l^+_j = l^{-i} D_i l^-_j = 0. \]

Their explicit realization in terms of AdS$_4$ fields $K_{AB}$ will be given in Section 4. To write down black hole metric in AdS$_4$ covariant and coordinate-free form we introduce the following Lorentz scalars

\[ \mathcal{G} = \frac{\lambda^2}{\sqrt{-x^2}}, \quad \bar{\mathcal{G}} = \frac{\lambda^2}{\sqrt{-\bar{x}^2}}, \]

where $\varepsilon_{\alpha\beta\gamma} = x^2 \varepsilon_{\alpha\gamma}$. This allows us to define “canonical scalars”

\[ 2r = \frac{1}{\mathcal{G}} + \frac{1}{\bar{\mathcal{G}}}, \quad 2iy = \frac{1}{\mathcal{G}} - \frac{1}{\bar{\mathcal{G}}}. \]

Using the unfolded analysis along with the integration flow method we show that the solution of the obtained first order flow equations describes generic Einstein–Maxwell black hole on AdS$_4$ space-time in coordinate-independent form

\[ ds^2 = ds_0^2 + \frac{2Mr - q}{r^2 + y^2} (\alpha_1(r) k_i dx^i + \alpha_2(r) n_i dx^i)^2 - \frac{2Ny + q}{r^2 + y^2} (\beta_1(y) l_i^+ dx^i + \beta_2(y) l_i^- dx^i)^2 \]

\[ + 4\alpha_1(r) \alpha_2(r) \frac{r^2 + y^2}{\Delta_r \Delta_y} (2Mr - \frac{q}{2}) dr^2 - 4\beta_1(y) \beta_2(y) \frac{r^2 + y^2}{\Delta_y \Delta_y} (2Ny + \frac{q}{2}) dy^2, \]

where $\alpha_1(r), \alpha_2(r)$ and $\beta_1(y), \beta_2(y)$ subjected to the constraints

\[ \alpha_1(r) + \alpha_2(r) = 1, \quad \beta_1(y) + \beta_2(y) = 1, \]

\[ \text{For notations used throughout this paper see Appendix A.} \]

\[ \text{The reason of this name is that in a certain reference frame, scalars } r \text{ and } y \text{ are equal to the canonical coordinates introduced by Carter in [I].} \]
are otherwise arbitrary, parameterizing some gauge ambiguity, \( ds_0^2 \) is the background \( AdS_4 \) metric, \( \hat{\Delta}_r \) and \( \hat{\Delta}_y \) are the following polynomials

\[
\hat{\Delta}_r = 2M r + r^2 (\lambda^2 r^2 + I_1) + \frac{1}{2} (-q + \frac{I_2}{2}),
\]

\[
\hat{\Delta}_y = 2N y + y^2 (\lambda^2 y^2 - I_1) + \frac{1}{2} (q + \frac{I_2}{2}),
\]

and

\[
\Delta_r = \hat{\Delta}_r \bigg|_{M,N,q=0} = r^2 (\lambda^2 r^2 + I_1) + \frac{1}{4} I_2,
\]

\[
\Delta_y = \hat{\Delta}_y \bigg|_{M,N,q=0} = y^2 (\lambda^2 y^2 - I_1) + \frac{1}{4} I_2,
\]

with \( I_1, I_2 \) being some first integrals of (2.1) related to the invariants (1.2) as follows

\[
C_2 = I_1, \quad C_4 = I_1^2 + \lambda^2 I_2.
\]

Note that, generally, the metric (2.10) is complex. Reality of the metric requires \( \beta_1 = \beta_2 = \frac{1}{2} \).

However, sometimes, it may be useful to consider complex metrics (for example, to reproduce the double Kerr–Schild form [5]).

Black hole Maxwell field \( F = dA \) is generated by a one-form potential that, up to a gauge freedom, can be chosen in the form

\[
A = \frac{r}{r^2 + y^2} k_i dx^i.
\]

The metric (2.10) is valid for any values of its parameters. However, in the case of zero NUT parameter \( N = 0 \) the flow integration can be performed differently giving rise to a simpler expression for the black hole metric. In particular we will show how the familiar Kerr–Schild form for Kerr–Newman black hole [15] can be obtained in arbitrary coordinates.

The solution (2.10) is characterized by two polynomial functions whose coefficients are determined by six arbitrary parameters. It belongs to Petrov D-type [16] class of solutions of Einstein–Maxwell equations including non-zero cosmological constant and electro-magnetic field such that the two degenerate principal null congruences of the Weyl tensor are aligned with the two principal null congruences of the Maxwell tensor. \( M \) plays the role of mass, \( N \) is a NUT charge, \( \lambda^2 \) is the cosmological term, \( I_2 \) is a rotational parameter \( a \), and \( I_1 \) is the Carter–Plebanski parameter which can be set
equal to 1, 0 or -1 by a rescaling transformation discussed below. It will be shown that 
$q = 2(e^2 + g^2)$, where $e$ and $g$ are electric and magnetic charges respectively. Note,
that the charges enter Equation (2.10) via $q$ combination and thus can not be distinguished 
unless some external charged fields introduced.

Particular solution types depend on the values of the curvature parameters — $M, N, q$ 
and $sp(4)$ invariants. Let us enlist the main cases:

- **Carter–Plebanski solution**

All of the six parameters are non-zero. The metric is given by Equation (2.10). It is easy to write 
it down in the well-known Carter–Plebanski form by setting $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \frac{1}{2}$ 
(see Subsection 9.1). The rotational parameter is $a^2 = I_2/4$, whereas the Carter– 
Plebanski parameter is $\epsilon = I_1$.

- **Double Kerr–Schild form of Carter–Plebanski**

The gauge choice $\alpha_1 = \beta_1 = 1, \alpha_2 = \beta_2 = 0$ leads to the so called double Kerr–Schild 
form of Equation (2.10)

$$ds^2 = ds^2_0 + 2\frac{r}{r^2 + y^2} (M - \frac{q}{4r}) k_i k_j dx^i dx^j - \frac{2y}{r^2 + y^2} (N + \frac{q}{4y}) l^+_i l^+_j dx^i dx^j$$

(2.19)

which is complex in Minkowski signature.

The following cases of zero NUT charge are important for physical applications:

- **N = 0, $C_2 = 1 + \lambda^2 a^2$, $C_4 = C^2_2 + 4\lambda^2 a^2$**

This case provides Kerr–Newman solution with $a$ being black hole angular momentum 
per unit mass. The metric can be written down in the Kerr–Schild form

$$ds^2 = ds^2_0 + \frac{2Mr - q}{r^2 + y^2} k_i k_j dx^i dx^j.$$ 

(2.20)

- **N = 0, $C_4 = C^2_2$ (equivalently, $K^C_A K_C^B = C_2 \delta^B_A$)**

The particular case of non-rotating solution results from the further degeneration 
$I_2 = 0, y = 0$. It describes Reissner–Nordström solution. Again, one can conveniently 
put the metric in Kerr–Schild form

$$ds^2 = ds^2_0 + \left(\frac{2M}{r} - \frac{q}{r^2}\right) k_i k_j dx^i dx^j.$$ 

(2.21)
Let us note, that all listed solutions are invariant under the rescaling

\[ K_{AB} \to \mu K_{AB}, \quad M \to \mu^3 M, \quad N \to \mu^3 N, \quad q^2 \to \mu^4 q^2 \]  

(2.22)

with real constant \( \mu \) that yields

\[ C_2 \to \mu^2 C_2, \quad C_4 \to \mu^4 C_4. \]  

(2.23)

This means that among two kinematical parameters of (2.10) one can be always chosen to be discrete 1, 0 or -1. Alternatively, using the scaling ambiguity (2.22) one can scale away the mass parameter \( M \) that will then be represented by the parameter \( \epsilon \).

In the following we will essentially use two-component spinor language which has great advantages in 4d description, so let us proceed to Cartan formalism of gravity.

3 Cartan formalism

In Riemannian approach to black hole in AdS4 gravity the metric and Maxwell gauge field verify Einstein–Maxwell equations

\[ R_{ij} = 3\lambda^2 g_{ij} + T_{ij}, \]  

(3.1)

\[ D_i F_{ij} = 0 \]  

(3.2)

with the energy-momentum tensor of the form

\[ T_{ij} = 4(e^2 + g^2) \left( F_{ik} F_k^j - \frac{1}{4} g_{ij} F_{kl} F^{kl} \right). \]  

(3.3)

Let us proceed to Cartan formulation of gravity. Let \( dx^m \Omega_m^{ab} \) be an antisymmetric Lorentz connection one-form and \( dx^m h_m^a \) be a vierbein one-form. These can be identified with the gauge fields of the AdS4 symmetry algebra \( o(3,2) \). The corresponding AdS4 curvatures \( R^{ab} = \frac{1}{2} R_{ij}^{ab} dx^i \wedge dx^j \) and \( R^a = \frac{1}{2} R_{ij}^a dx^i \wedge dx^j \) have the form

\[ R^{ab} = d\Omega^{ab} + \Omega^{ac} \wedge \Omega_c^b - \lambda^2 h^a \wedge h^b, \]  

(3.4)

\[ R^a = dh^a + \Omega^{ac} \wedge h_c, \]  

(3.5)

where \( a, b, c = 0, \ldots, 3 \) are Lorentz indices. Lorentz indices are raised and lowered with the flat metric \( \eta_{ab} = \text{diag}(1,-1,-1,-1) \). The zero-torsion condition \( R^a = 0 \) expresses algebraically the Lorentz connection \( \Omega \) via derivatives of \( h \). Then the \( \lambda \)-independent part of the curvature two-form (3.4) identifies with the Riemann tensor.
For the case of non-zero energy-momentum tensor it is convenient to decompose the curvature two-form into its traceless part associated with the Weyl tensor and tracefull one provided by $T_{ij}$

$$R_{ab} = \frac{1}{2} h^c \wedge h^d C_{cdab} + \frac{1}{2} (h_a T_b - h_b T_a),$$  \hspace{1cm} (3.6)

where $C_{abcd}$ is the Weyl tensor in the local frame, $C_{abcd} = -C_{bacd} = -C_{abdc} = C_{cdab}$ and $T_a = T_{ab} h^b$ is a one-form associated with the energy-momentum tensor. Equation (3.6) is equivalent to the metric form of Einstein equations (3.1) with the metric

$$g_{mn} = h^a_m h^b_n \eta_{ab}.$$  \hspace{1cm} (3.7)

Now we proceed to spinor reformulation of Einstein–Maxwell theory. Einstein equation (3.6) and torsion-free condition (3.5) can be rewritten in the spinor notation as follows

5. Lorentz connection one-forms $\Omega_{\alpha\alpha}$, $\bar{\Omega}_{\dot{\alpha}\dot{\alpha}}$ and vierbein one-form $h_{\alpha\dot{\alpha}}$ can be identified with the gauge fields of $sp(4) \sim o(3,2)$. It is easy to check that the equivalent spinor form of (3.3) is

$$T_{\alpha\beta} = -4(e^2 + g^2)F_{\alpha\beta} \bar{F}_{\alpha\beta}.$$  \hspace{1cm} (3.8)

It is obviously invariant under the electro-magnetic duality transformation

$$F_{\alpha\beta} \rightarrow e^{i\theta} F_{\alpha\beta}, \quad \bar{F}_{\dot{\alpha}\dot{\beta}} \rightarrow e^{-i\theta} \bar{F}_{\dot{\alpha}\dot{\beta}}.$$  \hspace{1cm} (3.9)

Then Einstein equations with cosmological constant acquire the form

$$R_{\alpha\beta} = d\Omega_{\alpha\beta} + \frac{1}{2} \Omega_{\alpha\gamma} \wedge \Omega_{\gamma\beta} = \frac{\lambda^2}{2} H_{\alpha\beta} + \frac{1}{8} H^{\gamma\gamma} C_{\gamma\gamma\alpha\beta} + \frac{e^2 + g^2}{2} H_{\gamma\gamma} F_{\gamma\gamma} F_{\alpha\beta}$$ \hspace{1cm} (3.10)

$$\bar{R}_{\dot{\alpha}\dot{\beta}} = d\Omega_{\dot{\alpha}\dot{\beta}} + \frac{1}{2} \Omega_{\dot{\alpha}\dot{\gamma}} \wedge \Omega_{\dot{\gamma}\dot{\beta}} = \frac{\lambda^2}{2} \bar{H}_{\dot{\alpha}\dot{\beta}} + \frac{1}{8} H^{\gamma\gamma} \bar{C}_{\gamma\gamma\dot{\alpha}\dot{\beta}} + \frac{e^2 + g^2}{2} H_{\gamma\gamma} F_{\gamma\gamma} \bar{F}_{\dot{\alpha}\dot{\beta}}$$ \hspace{1cm} (3.11)

$$\mathcal{R}_{\alpha\dot{\alpha}} = dh_{\alpha\dot{\alpha}} + \frac{1}{2} \Omega_{\alpha\gamma} \wedge h_{\gamma\dot{\alpha}} + \frac{1}{2} \Omega_{\dot{\alpha}\dot{\gamma}} \wedge h_{\alpha\gamma} = 0,$$ \hspace{1cm} (3.12)

where $R_{\alpha\beta}$ and $\bar{R}_{\dot{\alpha}\dot{\beta}}$ are the components of the Lorentz curvature two-form

$$D^2 \xi_{\alpha\dot{\alpha}} = \frac{1}{2} \mathcal{R}_{\alpha\beta} \bar{\xi}_{\beta\dot{\alpha}} + \frac{1}{2} \bar{\mathcal{R}}_{\dot{\alpha}\dot{\beta}} \xi_{\alpha\beta}$$ \hspace{1cm} (3.13)

and

$$H^{\alpha\dot{\alpha}} = h^{\alpha\dot{\alpha}} \wedge h^{\dot{\alpha}\dot{\alpha}}, \quad \bar{H}^{\dot{\alpha}\dot{\alpha}} = h^{\dot{\alpha}\dot{\alpha}} \wedge h^{\alpha\dot{\alpha}}.$$ \hspace{1cm} (3.14)

\(^5\)See Appendix A for notation.\(^6\)The symmetrization over denoted by the same letter spinor indices is implied.
4 \textbf{AdS}_4 \textbf{ unfolded system}

4.1 \textbf{Killing equations unfolded}

Let us formulate the unfolded system that describes AdS$_4$ geometry along with some its global symmetry. We start with an AdS$_4$ Killing vector $V^m$ and its covariant derivative

$$\kappa_{mn} = D_m V_n, \quad \kappa_{mn} = -\kappa_{nm}, \quad (4.1)$$

which will be referred to as the Killing two-form or Papapetrou field.

Since the AdS$_4$ Riemann curvature has vanishing Weyl tensor one can write down the following system

$$DV_{\alpha\dot{\alpha}} = \frac{1}{2} h^{\gamma\dot{\gamma}} \kappa_{\gamma\alpha}, \quad (4.2)$$

$$D\kappa_{\alpha\alpha} = \lambda^2 h^{\alpha\beta} V_{\alpha\beta}, \quad (4.3)$$

$$D\bar{\kappa}_{\dot{\alpha}\dot{\alpha}} = \lambda^2 h^{\gamma\dot{\alpha}} V_{\gamma\dot{\alpha}}, \quad (4.4)$$

which is consistent provided that

$$Dh_{\alpha\dot{\alpha}} = 0, \quad (4.5)$$

$$R_{\alpha\alpha} \equiv d\Omega_{\alpha\alpha} + \frac{1}{2} \Omega_{\alpha\beta} \wedge \Omega_{\beta\alpha} = \frac{\lambda^2}{2} h_{\alpha\dot{\alpha}} \wedge h_{\alpha\dot{\alpha}}, \quad (4.6)$$

$$\bar{R}_{\dot{\alpha}\dot{\alpha}} \equiv d\bar{\Omega}_{\dot{\alpha}\dot{\alpha}} + \frac{1}{2} \bar{\Omega}_{\dot{\alpha}\dot{\beta}} \wedge \bar{\Omega}_{\dot{\beta}\dot{\alpha}} = \frac{\lambda^2}{2} h_{\alpha\dot{\alpha}} \wedge h_{\alpha\dot{\alpha}}, \quad (4.7)$$

where $h_{\alpha\dot{\alpha}}$ is the AdS$_4$ vierbein, $\Omega_{\alpha\alpha}$ and $\bar{\Omega}_{\dot{\alpha}\dot{\alpha}}$ are components of Lorentz connection, $D$ is the background Lorentz covariant differential and $R_{\alpha\alpha}$, $\bar{R}_{\dot{\alpha}\dot{\alpha}}$ are the components of AdS$_4$ curvature two-form

$$D^2 \xi_{\alpha\dot{\alpha}} = \frac{1}{2} R_{\alpha\beta} \xi_{\beta\dot{\alpha}} + \frac{1}{2} \bar{R}_{\dot{\alpha}\dot{\beta}} \xi_{\alpha\beta}. \quad (4.8)$$

The equations (4.2)–(4.7) can be rewritten in the manifestly AdS$_4$ covariant form. Indeed, let $K_{AB}$ be the AdS$_4$ zero-form

$$K_{AB} = \begin{pmatrix} \lambda^{-1} \kappa_{\alpha\beta} & V_{\alpha\dot{\beta}} \\ V_{\beta\dot{\alpha}} & \lambda^{-1} \bar{\kappa}_{\dot{\alpha}\dot{\beta}} \end{pmatrix}, \quad (4.8)$$

and $\Omega_{AB}$ be the frame one-form

$$\Omega_{AB} = \begin{pmatrix} \Omega_{\alpha\beta} & -\lambda h_{\alpha\beta} \\ -\lambda h_{\beta\dot{\alpha}} & \bar{\Omega}_{\dot{\alpha}\dot{\beta}} \end{pmatrix}. \quad (4.9)$$
Then the manifestly $sp(4)$ covariant form of the system (4.2)–(4.7) reads as

$$D_0 K_{AB} = 0,$$

$$R_{0AB} \equiv d\Omega_{AB} + \frac{1}{2} \Omega_A^C \wedge \Omega_C B = 0,$$

where $D_0$ is the $AdS_4$ covariant differential. The first equation is the covariant constancy condition for global symmetry parameter, while the second one describes $AdS_4$.

Let us note, that the system (4.2)–(4.4) was used in [1] as a starting point in the construction of the Kerr black hole unfolded system. The deformation was performed in terms of Killing vector $V_{\alpha\dot{\alpha}}$ and Papapetrou field $\zeta_{\alpha\alpha}$, $\bar{\zeta}_{\alpha\dot{\alpha}}$. However, it turns out more convenient to rescale the Papapetrou field appropriately, rewriting the $AdS_4$ unfolded equations using the rescaled field. So, let us introduce self-dual Maxwell tensor $F_{\alpha\alpha}$ and its complex conjugate $\bar{F}_{\dot{\alpha}\dot{\alpha}}$ as

$$F_{\alpha\alpha} = -\lambda^{-2} \mathcal{G}^3 \zeta_{\alpha\alpha}, \quad \bar{F}_{\dot{\alpha}\dot{\alpha}} = -\lambda^{-2} \mathcal{G}^3 \bar{\zeta}_{\dot{\alpha}\dot{\alpha}},$$

where

$$\mathcal{G} = \frac{\lambda^2}{\sqrt{-\zeta^2}} = (-F^2)^{1/4}, \quad \bar{\mathcal{G}} = \frac{\lambda^2}{\sqrt{-\bar{\zeta}^2}} = (-\bar{F}^2)^{1/4}$$

and the roots on the right hand sides of (4.13) are chosen so as to have $\mathcal{G}$ and $\bar{\mathcal{G}}$ complex conjugated.

Then (4.2)–(4.4) can be rewritten as

$$DV_{\alpha\dot{\alpha}} = \frac{1}{2} \rho h_{\gamma}^\alpha F_{\gamma\alpha} + \frac{1}{2} \bar{\rho} h_{\alpha}^{\dot{\gamma}} \bar{F}_{\dot{\alpha}\dot{\gamma}}$$

$$DF_{\alpha\alpha} = -\frac{3}{2\mathcal{G}} h^{\beta\gamma} V_{\gamma}^\beta F_{(\beta\beta} F_{\alpha)\alpha}$$

$$D\bar{F}_{\dot{\alpha}\dot{\alpha}} = -\frac{3}{2\bar{\mathcal{G}}} h^{\beta\dot{\gamma}} V_{\dot{\gamma}}^\beta \bar{F}_{(\beta\beta} \bar{F}_{\dot{\alpha}\dot{\alpha})}.$$

with

$$\rho = -\lambda^2 \mathcal{G}^{-3}, \quad \bar{\rho} = -\lambda^2 \bar{\mathcal{G}}^{-3}.$$

In what follows this Killing–Maxwell system along with the $AdS_4$ curvature equations (4.5)–(4.7) will be referred to as $AdS_4$ unfolded system. Note, that so defined field strength (4.12) is well defined in the flat limit $\lambda \to 0$.

The important property of (4.14)–(4.16) is that $F_{\alpha\alpha}$ and $\bar{F}_{\dot{\alpha}\dot{\alpha}}$ satisfy source-free Maxwell equations and Bianchi identities

$$D_{\gamma\dot{\alpha}} F_{\alpha\gamma} = 0, \quad D_{\alpha\dot{\gamma}} \bar{F}_{\dot{\alpha}\dot{\gamma}} = 0.$$
Using (4.15) and (4.16) one obtains useful equations for $G$ and $\bar{G}$

\[ dG = -\frac{1}{2} h^{\alpha\dot{\alpha}} V_\alpha F_{\dot{\alpha}}, \quad d\bar{G} = -\frac{1}{2} h^{\alpha\dot{\alpha}} V_\alpha \bar{F}_{\dot{\alpha}}. \]  

(4.19)

Unfolded equations (4.14)–(4.16) have a number of remarkable properties. In particular, the system can be shown to possess Killing–Yano tensor and another Killing vector built of $V_{\alpha\dot{\alpha}}$ and $F_{\alpha\dot{\alpha}}$, $\bar{F}_{\alpha\dot{\alpha}}$. These properties are summarized in Appendix E.

An important property related to the description of the kinematical parameters of 4d black holes is that the system (4.14)–(4.16) possesses two first integrals

\[ I_1 = V^2 - \frac{\lambda^2}{2} \left( \frac{1}{G^2} + \frac{1}{\bar{G}^2} \right), \]

\[ I_2 = \frac{1}{G^3\bar{G}^3} V^{\alpha\dot{\alpha}} V^{\dot{\alpha}\beta} F_{\alpha\beta} - V^2 \left( \frac{1}{G^2} + \frac{1}{\bar{G}^2} \right) + \frac{\lambda^2}{4} \left( \frac{1}{G^2} - \frac{1}{\bar{G}^2} \right)^2, \]

(4.20)

(4.21)

where $V^2 = \frac{1}{2} V_{\alpha\dot{\alpha}} V^{\alpha\dot{\alpha}}$. Using (4.14)–(4.16) one can straightforwardly check that $dI_1 = 0$ and $dI_2 = 0$. Obviously enough, these conserved quantities are related to two invariants of $sp(4)$ algebra as we will see more explicitly in Subsection 4.4.

Finally, $AdS_4$ unfolded system is invariant under the following transformation

\[ \tau_\mu : (V_{\alpha\dot{\alpha}}, F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}}) \rightarrow (\mu V_{\alpha\dot{\alpha}}, \frac{1}{\mu} F_{\alpha\beta}, \frac{1}{\mu} \bar{F}_{\dot{\alpha}\dot{\beta}}), \]

(4.22)

where $\mu$ is a real parameter. Yet another symmetry of the system is the parity transformation

\[ \pi : (V_{\alpha\dot{\alpha}}, h_{\alpha\dot{\alpha}}) \rightarrow (-V_{\alpha\dot{\alpha}}, -h_{\alpha\dot{\alpha}}). \]

(4.23)

### 4.2 Killing Projectors

As explained in [1], the key element of the black hole unfolding, that eventually gives rise to (2.10), is the construction of Kerr–Schild vectors out of the $AdS_4$ global symmetry parameter. The proposed procedure is essentially four-dimensional being based on certain projectors we are in a position to define, namely, we split the spinor space into two orthogonal sectors using the projectors constructed from the Maxwell field. In what follows they will be referred to as Killing projectors.

Let two pairs of mutually conjugated projectors $\Pi_{\alpha\dot{\beta}}^\pm$ and $\Pi_{\dot{\alpha}\beta}^\pm$ have the form

\[ \Pi_{\alpha\dot{\beta}}^\pm = \frac{1}{2} (\epsilon_{\alpha\dot{\beta}} \pm \frac{1}{G^2} F_{\alpha\beta}), \quad \Pi_{\dot{\alpha}\beta}^\pm = \frac{1}{2} (\epsilon_{\dot{\alpha}\beta} \pm \frac{1}{\bar{G}^2} \bar{F}_{\dot{\alpha}\dot{\beta}}), \]

(4.24)

so that

\[ \Pi_{\alpha\dot{\beta}}^+ + \Pi_{\alpha\dot{\beta}}^- = \epsilon_{\alpha\dot{\beta}}, \quad \Pi_{\dot{\alpha}\beta}^+ + \Pi_{\dot{\alpha}\beta}^- = \epsilon_{\dot{\alpha}\beta}, \]

(4.25)
\[ \Pi^\pm_\alpha \Pi^\pm_\beta \gamma = \Pi^\pm_\alpha \gamma, \quad \Pi^\pm_\alpha \beta \Pi^\pm_\beta \gamma = 0, \quad \Pi^\pm_\alpha \beta \Pi^\pm_\beta \gamma = \Pi^\pm_\alpha \gamma, \quad \Pi^\pm_\alpha \beta \Pi^\pm_\beta \gamma = 0. \quad (4.26) \]

From the definition (4.24) it follows
\[ \Pi^\pm_\alpha \beta = -\Pi^\mp_\beta \alpha, \quad \bar{\Pi}^\pm_\alpha \beta = -\bar{\Pi}^\mp_\beta \alpha. \quad (4.27) \]

From (4.24), (4.15) and (4.19) one finds the following differential properties
\[ D\Pi^\pm_\alpha \beta = \pm G^2 (\Pi^\pm_\alpha \beta + \Pi^\pm_\alpha \gamma \Pi^\pm_\beta \gamma) h^\gamma \dot{\gamma} V^\gamma \dot{\gamma}, \quad (4.28) \]
\[ D\bar{\Pi}^\pm_\alpha \beta = \pm \bar{G}^2 (\bar{\Pi}^\pm_\alpha \beta + \bar{\Pi}^\pm_\alpha \gamma \bar{\Pi}^\pm_\beta \gamma) h^\gamma \dot{\gamma} V^\gamma \dot{\gamma}. \quad (4.29) \]

Hereinafter we will focus on the holomorphic (i.e., undotted) sector of the system. All relations in the antiholomorphic sector result by conjugation.

The projectors (4.24) split the two-dimensional (anti)holomorphic spinor space into the direct sum of two one-dimensional subspaces. For any \( \psi_\alpha \) we set
\[ \psi^\pm_\alpha = \Pi^\pm_\alpha \beta \psi_\beta, \quad \psi^+_\alpha + \psi^-_\alpha = \psi_\alpha, \quad (4.30) \]
so that \( \Pi^\mp_\alpha \beta \psi^\pm_\beta = 0 \). This allows us to build light-like vectors with the aid of the projectors. Indeed, consider an arbitrary vector \( V^\pm_\alpha \dot{\alpha} \). Using (4.24) define \( V^\pm_\alpha \dot{\alpha} \) and \( V^\pm_\mp_\alpha \dot{\alpha} \) as
\[ V^\pm_\alpha \dot{\alpha} = \Pi^\pm_\alpha \beta \Pi^\pm_\beta \beta V^\beta \dot{\beta}, \quad V^+_\alpha \dot{\alpha} = \Pi^+_\alpha \beta \Pi^+_\beta \beta V^\beta \dot{\beta}, \quad V^-_\alpha \dot{\alpha} = \Pi^-_\alpha \beta \Pi^+_\beta \beta V^\beta \dot{\beta}. \quad (4.31) \]

Since the projectors have rank one, they can be expressed via a pair of some basis spinors \( (\xi_\alpha, \eta_\alpha) \) as follows
\[ \Pi^+_\alpha \beta = \frac{\eta_\alpha \xi_\beta}{\eta_\gamma \xi_\gamma}, \quad \Pi^-_\alpha \beta = \frac{\xi_\alpha \eta_\beta}{\xi_\gamma \eta_\gamma}. \quad (4.32) \]

Obviously, \( V^\pm_\alpha \dot{\alpha} V^\pm_\alpha \dot{\alpha} = 0 \) and \( V^\pm_\alpha \dot{\alpha} V^\pm_\alpha \dot{\alpha} = 0 \). Then \( V^\pm_\alpha \dot{\alpha} \) and \( V^\pm_\mp_\alpha \dot{\alpha} \) can be cast into the form
\[ V^-_\alpha \dot{\alpha} = \xi_\alpha \bar{\xi}_\dot{\alpha}, \quad V^+_\alpha \dot{\alpha} = \eta_\alpha \bar{\eta}_\dot{\alpha}, \quad V^+_\alpha \dot{\alpha} = q \eta_\alpha \xi_\dot{\alpha}, \quad V^-_\alpha \dot{\alpha} = \bar{q} \xi_\alpha \eta_\dot{\alpha}, \quad (4.33) \]
where \( q(x) \) and \( \bar{q}(x) \) are some complex functions. As a consequence of (4.32), we also have
\[ F_{\alpha \dot{\alpha}} = 2 G^2 \frac{\xi_\alpha \eta_\dot{\alpha}}{\eta_\gamma \xi_\gamma}. \quad (4.34) \]
Now, from (4.33) it is obvious that
\[
V_{\alpha\beta}^+ V_{\beta\alpha}^+ = V_{\alpha\alpha}^+ V_{\beta\beta}^+ , \quad V_{\alpha\beta}^- V_{\beta\alpha}^- = -\frac{(V^- + V^+)}{(V^- - V^+)} V_{\alpha\alpha}^- V_{\beta\beta}^+ , \tag{4.35}
\]
where
\[
(AB) = A_{\alpha\dot{\alpha}} B^{\alpha\dot{\alpha}} .
\]
Also note that
\[
V^2 = (V^+ V^-) + (V^+ V^-) = (1 - q\bar{q})(V^+ V^-) . \tag{4.36}
\]
It is worth to note that according to Papapetrou [19] any stationary axisymmetric solution of empty-space Einstein’s equations have a discrete symmetry upon simultaneous inversion of the angular and time Killing vectors. Boyer and Lindquist [20] have written a special transformation which casts the empty-space Kerr metric into a form manifestly invariant under such an inversion. In the \(AdS\) unfolded system, this symmetry is \(\tau_{-1} \) (4.22) that interchanges the projectors
\[
\tau_{-1} : \quad \Pi_{\alpha\beta}^\pm \rightarrow \Pi_{\alpha\beta}^{\mp} , \quad \bar{\Pi}_{\alpha\beta}^\pm \rightarrow \bar{\Pi}_{\alpha\beta}^{\mp} . \tag{4.37}
\]
### 4.3 Kerr–Schild null-vector basis

Now we are in a position to introduce the complete set of null-vectors (complex null tetrad)
\[
k_{\alpha\dot{\alpha}} = \frac{2}{(V^+ V^-)} V_{\alpha\dot{\alpha}}^- , \quad n_{\alpha\dot{\alpha}} = \frac{2}{(V^+ V^-)} V_{\alpha\dot{\alpha}}^+ , \tag{4.38}
\]
\[
l_{\alpha\dot{\alpha}}^{++} = \frac{2}{(V^+ V^+ - V^-)} V_{\alpha\dot{\alpha}}^{++} , \quad l_{\alpha\dot{\alpha}}^{--} = \frac{2}{(V^- V^- - V^+)} V_{\alpha\dot{\alpha}}^{--} . \tag{4.39}
\]
Note, that \(k_{\alpha\dot{\alpha}}\) and \(n_{\alpha\dot{\alpha}}\) are real vectors, whereas \(l_{\alpha\dot{\alpha}}^{++}\) and \(l_{\alpha\dot{\alpha}}^{--}\) are mutually conjugated
\[
l_{\alpha\dot{\alpha}}^{++} = \bar{l}_{\alpha\dot{\alpha}}^{--} .
\]
The discrete symmetry \(\tau_{-1} \) (4.37) interchanges the null-vectors
\[
\tau_{-1} : \quad k_{\alpha\dot{\alpha}} \rightarrow n_{\alpha\dot{\alpha}} , \tag{4.40}
\]
\[
\tau_{-1} : \quad l_{\alpha\dot{\alpha}}^{++} \rightarrow l_{\alpha\dot{\alpha}}^{--} . \tag{4.41}
\]
Schematically, in terms of spinors (4.32), one can think of these null vectors as
\[
k_{\alpha\dot{\alpha}} \sim \xi_{\alpha} \bar{\xi}_{\dot{\alpha}} , \quad n_{\alpha\dot{\alpha}} \sim \eta_{\alpha} \bar{\eta}_{\dot{\alpha}} , \quad l_{\alpha\dot{\alpha}}^{++} \sim \eta_{\alpha} \bar{\xi}_{\dot{\alpha}} , \quad l_{\alpha\dot{\alpha}}^{--} \sim \xi_{\alpha} \bar{\eta}_{\dot{\alpha}} . \tag{4.42}
\]
It is convenient to arrange this set of null-vectors into the array

\[ e_{I,\alpha \dot{\alpha}} = \left(k_{\alpha \dot{\alpha}}, n_{\alpha \dot{\alpha}}, l_{\alpha \dot{\alpha}}^+, l_{\alpha \dot{\alpha}}^-\right) \tag{4.43} \]

with the evident properties

\[ e_{I,\alpha \dot{\alpha}} e_{I,\alpha \dot{\alpha}} = 0, \quad \frac{1}{2} e_{I,\alpha \dot{\alpha}} V^{\alpha \dot{\alpha}} = 1, \tag{4.44} \]

where \( I = 1, \ldots, 4 \) (no summation over \( I \)). Obviously,

\[ (e_1 e_2) = \frac{4}{(V^+ V^-)}, \quad (e_3 e_4) = \frac{4}{(V^+ V^-)}. \tag{4.45} \]

From (4.14)–(4.16) it follows that

\[
De_{I,\alpha \dot{\alpha}} = (-1)^{\sigma I} \frac{G}{4} \left( \rho \bar{G} e_{I,\alpha \dot{\alpha}} e_{I,\beta \dot{\beta}} + e_{I,\beta \dot{\beta}} V_\alpha \bar{V}_\gamma e_{I,\gamma \dot{\gamma}} \right) h^{\beta \dot{\beta}} \\
+ (-1)^{\sigma I} \frac{G}{4} \left( \bar{\rho} \bar{G} e_{I,\alpha \dot{\alpha}} e_{I,\beta \dot{\beta}} + e_{I,\alpha \dot{\alpha}} V_\beta \bar{V}_\gamma e_{I,\gamma \dot{\gamma}} \right) h^{\beta \dot{\beta}}, \tag{4.46}
\]

where \( \sigma_I \) counts the number of \( \eta_\alpha \) in \( e_{I,\alpha \dot{\alpha}} \) (4.43), i.e., \( \sigma_I = (0, 1, 1, 0) \) and \( \bar{\sigma}_I \) counts the number of \( \bar{\eta}_\dot{\alpha} \), i.e., \( \bar{\sigma}_I = (0, 1, 0, 1) \).

A simple consequence of (4.46) and (4.42) is that \( e_{I,\alpha \dot{\alpha}} \) obey the geodesity condition

\[ e_{I,\alpha \dot{\alpha}} D_{\alpha \dot{\alpha}} e_{I,\beta \dot{\beta}} = 0. \tag{4.47} \]

(No summation over \( I \)). In other words, all null-vectors (4.38) and (4.39) are Kerr–Schild, that is light-like and each satisfying (4.47). In addition, \( e_{I,\alpha \dot{\alpha}} \) are eigenvectors of the Maxwell tensors \( F_{\alpha \beta} \), \( F_{\alpha \dot{\beta}} \) as follows from (4.38), (4.39) and (4.33)

\[
F_{\alpha \beta} e_{I,\beta \dot{\beta}} = (-1)^{\sigma I} \frac{G}{4} e_{I,\alpha \dot{\alpha}}, \tag{4.48}
\]

\[
F_{\alpha \dot{\beta}} e_{I,\alpha \dot{\alpha}} = (-1)^{\bar{\sigma}_I} \bar{G} e_{I,\alpha \dot{\alpha}}. \tag{4.49}
\]

For the tensor version of these and related formulae we refer the reader to Appendix D.

As a consequence of (4.14)–(4.16) and (4.19) the following properties can be verified

\[
D_{\alpha \dot{\alpha}} e_{I,\alpha \dot{\alpha}} = -2((-1)^{\sigma I} G + (-1)^{\sigma I} \bar{G}), \quad e_{I,\alpha \dot{\alpha}} D_{\alpha \dot{\alpha}} G = 2(-1)^{\sigma I} G^2, \tag{4.50}
\]

\[
D_{\alpha \dot{\alpha}}((-1)^{\sigma I} G + (-1)^{\sigma I} \bar{G}) e_{I,\alpha \dot{\alpha}} = -4 \bar{G} G, \quad D_{\alpha \dot{\alpha}}(G \bar{G} e_{I,\alpha \dot{\alpha}}) = 0, \tag{4.51}
\]

\[
e_{I,\alpha \dot{\alpha}} D_{\alpha \dot{\alpha}} e_{I,\gamma \dot{\gamma}} = (-1)^{\sigma I} \bar{G} e_{I,\alpha \dot{\alpha}} V_{\gamma \dot{\gamma}} e_{I,\alpha \dot{\alpha}}. \tag{4.52}
\]
Using (4.19) and (4.46) one can make sure that each Kerr–Schild vector (4.43) generates the Maxwell field (4.12) via

\[ F_{\alpha\alpha} = \frac{1}{2} D_{\alpha} \dot{\epsilon}_{\alpha} \left( (\mathcal{G} + (-1)^{\sigma I + \sigma I}) \epsilon \dot{I}_I \dot{\epsilon}_I, \dot{\epsilon}_I \dot{\epsilon}_I \right), \]  

(4.53)

\[ F_{\dot{\alpha}\dot{\alpha}} = \frac{1}{2} D_{\dot{\alpha}} \dot{\epsilon}_{\dot{\alpha}} \left( (\mathcal{G} + (-1)^{\sigma I + \sigma I}) \epsilon \dot{I}_I \dot{\epsilon}_I, \dot{\epsilon}_I \dot{\epsilon}_I \right). \]  

(4.54)

Now let us give the explicit expressions for \((V^+V^-)\) and \((V^+V^-^+)\) which will be useful in what follows. Using (4.20) and (4.21) and making change of variables (2.9) we obtain

\[ (V^+V^-) = \frac{\Delta_r}{r^2 + y^2}, \]  

(4.55)

\[ (V^+V^-^+) = -\frac{\Delta_y}{r^2 + y^2}, \]  

(4.56)

with

\[ \Delta_r = r^2(\lambda^2 r^2 + I_1) + \frac{I_2}{4}, \]  

(4.57)

\[ \Delta_y = y^2(\lambda^2 y^2 - I_1) + \frac{I_2}{4}. \]  

(4.58)

The important remark is that this way we define the so called Carter canonical coordinates \(r\) and \(y\) (see [33] for more detail) which naturally arise in our approach, being related to the Maxwell field.

Let us introduce one-forms \(\mathcal{E}_I\) corresponding to the null-vectors (4.43) which will play an important role in metric construction

\[ \mathcal{E}_I = \frac{1}{2} \epsilon_{I,\alpha\dot{\alpha}} h^{\alpha\dot{\alpha}} = (K, N, L^{+}, L^{-}) \]  

(4.59)

Using (4.53) and (4.54) we observe that the vector-poten\(A_{1,2} = \frac{r}{r^2 + y^2} \mathcal{E}_{1,2}\)s

\[ A_{1,2} = \frac{r}{r^2 + y^2} \mathcal{E}_{1,2} \]  

(4.60)

generate the same Maxwell tensor field \(F = dA_{1,2}\).

The second pair of vector-poten\(A_{3,4} = \frac{y}{r^2 + y^2} \mathcal{E}_{3,4}\)s

\[ A_{3,4} = \frac{y}{r^2 + y^2} \mathcal{E}_{3,4} \]  

(4.61)
gives the Hodge dual field strength $\ast F = dA_{3,4}$.

One can check that
\[ K - N = \frac{2(r^2 + y^2)}{\Delta_r} \, dr, \quad L^{+-} - L^{-+} = \frac{2(r^2 + y^2)}{i\Delta_y} \, dy. \] (4.62)

From here it is obvious that the one-form potentials $A_{1,2}$ (4.60) ($A_{3,4}$ (4.61)) belong to the same gauge class and generate the same Maxwell field $F_{ij} (\ast F_{ij})$.

### 4.4 $AdS_4$ invariants

To reveal the algebraic nature of the first integrals (4.20) it is instructive to use the $AdS_4$ covariant form (4.10–4.11) of the unfolded system (4.2)–(4.7).

Consider the $AdS_4$ invariants constructed out of $K_{AB}$. Calculation of the square of $K_{AB}$ yields
\[ K_{AC} K^C_B = \begin{pmatrix} (V^2 + \lambda^2 \kappa^2) \epsilon_{\alpha\beta} & \lambda^{-1}(\kappa_{\alpha\gamma} V_{\gamma\beta} - \kappa_{\gamma\beta} V_{\alpha\gamma}) \\ -\lambda^{-1}(\kappa_{\gamma\alpha} V_{\gamma\delta} - \kappa_{\delta\alpha} V_{\gamma\beta}) & (V^2 + \lambda^{-2} \kappa^2) \epsilon_{\alpha\beta} \end{pmatrix}. \] (4.63)

$AdS_4$ indices are raised and lowered with the aid of canonical $sp(4)$-form (see Appendix A). All higher powers of $K_{AB}$ have the same structure with the scalar coefficients changed. The two independent $sp(4)$ invariants are
\[ C_2 = \frac{1}{4} K_{AB} K^{AB} = I_1, \] (4.64)
\[ C_4 = \frac{1}{4} \text{Tr}(K^4) = I_1^2 + \lambda^2 I_2, \] (4.65)

where $I_{1,2}$ are defined in (4.20) and (4.21). Note that all odd invariants are zero $\frac{1}{4} \text{Tr}(K^n) = 0$, for odd $n$. All higher even invariants are expressed in terms of $C_{2,4}$ (equivalently, $I_{1,2}$) in the agreement with the fact that the algebra $sp(4)$ has rank two.

Few comments are now in order. First of all, as follows from (4.22), $\tau_\mu$ symmetry makes it possible to set one of the $AdS_4$ invariants to 1, 0 or -1. As we will seen it gives a black hole two kinematic parameters one of which can be always taken discrete by diffeomorphism. Another observation is that the Kerr–Schild vectors $l_{\alpha\dot{\alpha}}^\pm$ and $l_{\alpha\dot{\alpha}}^-\dot{\gamma}$ may not exist for some values of $AdS_4$ invariants. Indeed, consider the case with $K^C_A K^B_C \sim \delta_A^B$ where
\[ C_4 = C_2^2. \] (4.66)

It is easy to see, that in this case
\[ \kappa_{\alpha\gamma} V_{\gamma\beta} = \bar{\kappa}_{\gamma\dot{\beta}} V_{\alpha\dot{\gamma}}. \] (4.67)
Direct consequence of (4.67) is
\[ \kappa^2 = \bar{\kappa}^2, \quad G = \bar{G}. \] (4.68)

From (4.63) and (4.67) it follows that
\[ K_{AC} K^{CB} = C_2 \begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon_{\alpha\beta} \end{pmatrix}. \] (4.69)

Using the definition (4.31) and (4.32) we have
\[ V_\alpha = V_\alpha^+ + V_\alpha^- + V_{\alpha\bar{\alpha}}^+ + V_{\alpha\bar{\alpha}}^- \]. (4.70)

Substituting (4.70) into (4.67) and using (4.33), (4.34) we find
\[ V_{\alpha\bar{\alpha}}^+ = V_{\alpha\bar{\alpha}}^- = 0. \] (4.71)

Then from (4.39) it follows that
\[ l_{\alpha\bar{\alpha}}^+ \to \infty, \quad l_{\alpha\bar{\alpha}}^- \to \infty. \] (4.72)

Moreover, taking into account (4.56) we obtain \( I_2 = 0 \). As we will see later this case provides a black hole with vanishing rotation parameter and only one non-zero invariant \( C = I_1 \), while \( I_2 = 0 \).

## 5 Black hole unfolded system

The equations (4.14)–(4.16) admit a natural deformation of the AdS\(_4\) unfolded system, that preserves its Killing and Maxwell properties, i.e., we require the deformed unfolded system to be built of Killing vector and source-free Maxwell tensor. As we will see this deformation describes generic AdS\(_4\) black hole.

Let us relax the equation (4.17) for \( \rho \) in (4.14) by allowing it to be an arbitrary function of \( G \) and \( \bar{G} \)
\[ \rho = \rho(G, \bar{G}). \] (5.1)

In this case the consistency condition for the system (4.14)–(4.16) turns out to be very restrictive. Solving Bianchi identities for (4.14)–(4.16) and taking (5.1) into account we find the following most general solution for \( \rho \)
\[ \rho = \mathcal{M} - \lambda^2 G^{-3} - q \bar{G}, \] (5.2)

where \( \mathcal{M} \) and \( q \) are, respectively, arbitrary complex and real parameters.
As a result the complete consistent deformed unfolded equations read

\[ \mathcal{D} \nu_{\alpha \beta} = \frac{1}{2} \rho h^{\gamma \delta} F_{\gamma \alpha} + \frac{1}{2} \bar{\rho} h_{\alpha}^\gamma \bar{F}_{\alpha \gamma}, \]  
(5.3)

\[ \mathcal{D} F_{\alpha \alpha} = -\frac{3}{2G} h^{\beta \gamma} \mathcal{V}_{\gamma}^{\beta} F_{\beta \beta F_{\alpha \alpha}}, \]  
(5.4)

\[ \mathcal{D} \bar{F}_{\alpha \alpha} = -\frac{3}{2\bar{G}} h^{\beta \gamma} \mathcal{V}_{\gamma}^{\beta} \bar{F}_{\beta \beta \bar{F}_{\alpha \alpha}}, \]  
(5.5)

with the following curvature two-forms

\[ \mathcal{R}_{\alpha \alpha} = \frac{\lambda^2}{2} H_{\alpha \alpha} - \frac{3(M - q \bar{G})}{4G} H^{\beta \beta} F_{\beta \beta F_{\alpha \alpha}} + \frac{q}{4} H^{\beta \beta} \bar{F}_{\beta \beta F_{\alpha \alpha}}, \]  
(5.6)

\[ \bar{R}_{\alpha \alpha} = \frac{\lambda^2}{2} \bar{H}_{\alpha \alpha} - \frac{3(M - q \bar{G})}{4\bar{G}} \bar{H}^{\beta \beta} \bar{F}_{\beta \beta \bar{F}_{\alpha \alpha}} + \frac{q}{4} \bar{H}^{\beta \beta} F_{\beta \beta F_{\alpha \alpha}}, \]  
(5.7)

\[ \mathcal{D} h_{\alpha \alpha} = 0, \]  
(5.8)

and

\[ \rho = M - \lambda^2 \bar{G}^{-3} - q \bar{G}, \quad \bar{\rho} = \bar{M} - \lambda^2 \bar{G}^{-3} - q \bar{G}, \]  
(5.9)

\[ G = (-F^2)^{1/4}, \quad \bar{G} = (-\bar{F}^2)^{1/4}, \]  
(5.10)

where \( H_{\alpha \alpha} \) and \( \bar{H}_{\alpha \alpha} \) are defined by (3.14), \( \mathcal{R}_{\alpha \alpha} \) and \( \bar{R}_{\alpha \alpha} \) are the curvatures (3.10) and (3.11). For \( G \) and \( \bar{G} \) one finds the same consequence as (4.19)

\[ dG = -\frac{1}{2} h^{\alpha \beta} \mathcal{V}_{\alpha \beta} F_{\alpha \alpha}, \quad d\bar{G} = -\frac{1}{2} h^{\alpha \beta} \mathcal{V}_{\alpha \beta} \bar{F}_{\alpha \alpha}. \]  
(5.11)

The last term in (5.6) and (5.7) has the form of energy-momentum tensor for Maxwell field and is invariant under \( U(1) \) rotations (3.9). Then the integration constant \( q \) is interpreted as the sum of squares of the electric and magnetic charges and can be written as \( q = 2(e^2 + g^2) \).

We call the system (5.3)–(5.10) black hole unfolded system (BHUS). (Note, that the BHUS of [1] is a particular case of (5.3)–(5.10) with \( M = \bar{M} \) and \( q = 0 \).) The Weyl tensor it yields is of Petrov D-type. Comparing (5.6) and (5.7) with (3.10) and (3.11) we find out that the deformation (5.1) of \( AdS_4 \) algebra leads to vacuum Maxwell-Einstein equations, with the Maxwell tensor \( F_{\alpha \alpha} \) and Weyl tensor given by

\[ C_{\alpha \beta \alpha \beta} = -\frac{6(M - q \bar{G})}{\bar{G}} F_{\alpha \alpha} F_{\alpha \alpha}. \]  
(5.12)

It follows from (4.22) and (4.23) that it is \( \tau_\mu \)–invariant in accordance with [21] where it was shown that the Papapetrou’s result concerning additional discrete symmetry
can be generalized to non-empty spaces with the matter tensor invariant under simultaneous inversion of the time and axial angle and that this holds automatically in the case of source-free electromagnetic field.

By analogy with the $AdS_4$ case, using (5.3)–(5.5) one can straightforwardly check that the following expressions conserve in BHUS

\[ I_1 = \mathcal{V}^2 - \mathcal{M} \mathcal{G} - \overline{\mathcal{M}} \overline{\mathcal{G}} - \frac{\lambda^2}{2} \left( \frac{1}{\overline{\mathcal{G}}^2} + \frac{1}{\mathcal{G}^2} \right) + q \mathcal{G} \overline{\mathcal{G}}, \quad (5.13) \]

\[ I_2 = \frac{1}{\mathcal{G}^3 \overline{\mathcal{G}}^3} \mathcal{V}^{\dot{\alpha} \dot{\beta}} \mathcal{V}^{\alpha \beta} F_{\alpha \alpha} \overline{F}_{\dot{\alpha} \dot{\beta}} - 2 \mathcal{M} \mathcal{G} - I_1 \left( \frac{1}{\overline{\mathcal{G}}^2} + \frac{1}{\mathcal{G}^2} \right) - \frac{\lambda^2}{4} \left( \frac{1}{\overline{\mathcal{G}}^2} + \frac{1}{\mathcal{G}^2} \right) - \frac{3\lambda^2}{2\mathcal{G}^2 \overline{\mathcal{G}}^2}, \quad (5.14) \]

\[ dI_1 = dI_2 = 0. \]

In other words $I_1$ and $I_2$ are the first integrals in the unfolded system.

Remarkably, all the differential and algebraic properties of BHUS literally coincide with those of the vacuum $AdS_4$ system of Section 4 with (4.20), (4.21) replaced by (5.13), (5.14) and (4.57), (4.58) by

\[ \hat{\Delta}_r = 2\mathcal{M} r + r^2 (\lambda^2 r^2 + I_1) + \frac{1}{2} (-q + \frac{I_2}{2}), \quad (5.15) \]

\[ \hat{\Delta}_y = 2\mathcal{N} y + y^2 (\lambda^2 y^2 - I_1) + \frac{1}{2} (q + \frac{I_2}{2}), \quad (5.16) \]

where

\[ \mathcal{M} = \frac{1}{2} (\mathcal{M} + \overline{\mathcal{M}}), \quad \mathcal{N} = \frac{1}{2i} (\mathcal{M} - \overline{\mathcal{M}}). \quad (5.17) \]

In Subsection 9.1 we will see that $\mathcal{M}$ and $\mathcal{N}$ can be interpreted as mass and NUT-parameter of a black hole. Analogously, one can build Kerr–Schild vectors in the system using the same projector procedure. The differential properties of these Kerr–Schild vectors are given by (4.46)–(4.49) with the function $\rho$ of the form (5.9). Using (5.13) and (5.14) and making the change of variables (2.9) we obtain

\[ (\mathcal{V}^+ \mathcal{V}^-) = \frac{\hat{\Delta}_r}{r^2 + y^2}, \quad (5.18) \]

\[ (\mathcal{V}^+ \mathcal{V}^{++}) = \frac{\hat{\Delta}_y}{r^2 + y^2}, \quad (5.19) \]

Vector potentials for Maxwell tensor and its Hodge dual again read as

\[ A_{1,2} = \frac{r}{r^2 + y^2} \hat{E}_{1,2}, \quad A_{3,4} = \frac{y}{r^2 + y^2} \hat{E}_{3,4}, \quad (5.20) \]
where

\[ \dot{e}_I = \frac{1}{2} \hat{e}_{I,\alpha} h^{\alpha\dot{a}} = (\dot{K}, \dot{N}, \dot{L}^+, \dot{L}^-). \]  

(5.21)

Here

\[ \dot{e}_{I,\alpha} = (\dot{k}_{\alpha\dot{a}}, \dot{n}_{\alpha\dot{a}}, \dot{l}_{\alpha\dot{a}}^+, \dot{l}_{\alpha\dot{a}}^-) \]  

(5.22)

are defined the same way as in (4.43). Recall, that (un)hatted quantities are associated with the (un)deformed unfolded system.

The following relations remain true

\[ \dot{K} - \dot{N} = \frac{2(r^2 + y^2)}{\Delta r} dr, \quad \dot{L}^+ - \dot{L}^- = \frac{2(r^2 + y^2)}{i\Delta y} dy. \]  

(5.23)

Again, \( \dot{K} \) and \( \dot{N} \) are real one-forms, while \( \dot{L}^+ \) and \( \dot{L}^- \) are complex conjugated. Finally, \( A_{1,2} \) generate the same Maxwell tensor \( F_{ij} \) and \( A_{3,4} \) generate \( \ast F_{ij} \).

The free and deformed unfolded systems are similar in many respects. In particular, they have the same number of first integrals, Kerr–Schild vectors, both have source-free Maxwell and closed Killing–Yano tensors. At \( \mathcal{M} = 0, q = 0 \) the two systems just coincide. All this suggests that there should be some integrating flow with respect to the parameters \( \mathcal{M} \) and \( q \) that maps one system to another. The existence of such an integrating flow is also natural in the context of expected relationship of the proposed construction with yet unknown black hole solution of the nonlinear higher spin gauge theory. Indeed, the integrating flow approach we are about to explore is to large extend analogous to the integrating flow in higher spin gauge theory [7] that maps solutions on nonlinear higher spin equations to those of free higher spin equations.

\section{Integrating flow}

Now we are in a position to construct the integrating flows with respect to \( \mathcal{M} \) and \( q \). The benefit of using the integrating flows which are first order differential equations with respect to the modules of the black hole solution is that by solving these equations it is easy to reconstruct the black hole solutions in terms of the initial data that describe the vacuum \( AdS_4 \) geometry. In other words, the idea is to obtain complicated black hole solutions of Einstein theory as solutions of simple and easily integrable flow equations whose form is fixed by the formal consistency conditions

\[
\begin{bmatrix}
\frac{\partial}{\partial \mathcal{M}}, & \frac{\partial}{\partial x^m}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial}{\partial \mathcal{M}}, & \frac{\partial}{\partial x^m}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial}{\partial \mathcal{M}}, & \frac{\partial}{\partial q}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial}{\partial \mathcal{M}}, & \frac{\partial}{\partial q}
\end{bmatrix} = 0
\]  

(6.1)

with equations (5.3)–(5.5), (5.8).
We require the Maxwell tensor to be constant along the flows and hence so are $G$ and $\bar{G}$ (equivalently, $r$ and $y$)

$$\frac{\partial}{\partial M} F_{\alpha\alpha} = \frac{\partial}{\partial \bar{M}} F_{\alpha\alpha} = \frac{\partial}{\partial q} F_{\alpha\alpha} = 0.$$ (6.2)

Although the requirement (6.2) is not necessary, it drastically simplifies the analysis. Also it is natural in a sense that known examples of black holes curvature tensors can be reduced to the form that agrees with (6.2). Indeed, the condition (6.2) turns out to be consistent with (5.3)–(5.5). Note, that the integrating flow with respect to $M$ can be obtained by complex conjugation of the $M$-flow. Leaving the detail of derivation for Appendix C, the final result for the integrating flows is

$$\partial_M V_{\alpha\dot{\alpha}} = \sum_{I=1}^{4} \phi_I \hat{e}_{I,\alpha\dot{\alpha}} , \quad \partial_M h_{\alpha\dot{\alpha}} = \sum_{I=1}^{4} \phi_I \hat{e}_{I,\alpha\dot{\alpha}} \hat{E}_I$$ (6.3)

and

$$\partial_q V_{\alpha\dot{\alpha}} = \sum_{I=1}^{4} \psi_I \hat{e}_{I,\alpha\dot{\alpha}} , \quad \partial_q h_{\alpha\dot{\alpha}} = \sum_{I=1}^{4} \psi_I \hat{e}_{I,\alpha\dot{\alpha}} \hat{E}_I ,$$ (6.4)

For the generic case with arbitrary complex $M$ the functions $\phi_I$ and $\psi_I$ read

$$\phi_1 = \frac{G + \bar{G}}{4} \alpha_1(r), \quad \phi_2 = \frac{G + \bar{G}}{4} \alpha_2(r), \quad \phi_3 = \frac{G - \bar{G}}{4} \beta_1(y), \quad \phi_4 = \frac{G - \bar{G}}{4} \beta_2(y)$$ (6.5)

and

$$\psi_1 = -\frac{G \bar{G}}{4} \theta_1(r), \quad \psi_2 = -\frac{G \bar{G}}{4} \theta_2(r), \quad \psi_3 = -\frac{G \bar{G}}{4} \vartheta_1(y), \quad \psi_4 = -\frac{G \bar{G}}{4} \vartheta_2(y) ,$$ (6.6)

where functions $\alpha, \beta, \theta, \vartheta$ satisfy the constraints

$$\alpha_1(r) + \alpha_2(r) = \beta_1(y) + \beta_2(y) = 1 ,$$ (6.9)

$$\theta_1(r) + \theta_2(r) = 1 - \frac{\gamma}{2} , \quad \vartheta_1(y) + \vartheta_2(y) = 1 + \frac{\gamma}{2}$$ (6.10)

and

$$\alpha_1(r) \theta_2(r) = \alpha_2(r) \theta_1(r) , \quad \beta_1(y) \vartheta_2(y) = \beta_2(y) \vartheta_1(y) ,$$ (6.11)

\footnote{Strictly speaking it is true given the reality conditions \((6.12)\) imposed.}
where \( \gamma \) is an arbitrary real constant. In addition, we assume that \( \alpha, \beta, \theta, \vartheta, \gamma \) are \( M \)- and \( q \)-independent. The reality condition for the gravitational fields requires in addition
\[
\beta_1(y) = \beta_2(y) = \frac{1}{2}, \quad \vartheta_1(y) = \vartheta_2(y) = \frac{1}{2} + \frac{\gamma}{4}.
\] (6.12)
Note that sometimes it is useful to deal with complex metric (particularly, with its double Kerr–Schild form).

The flows (6.3), (6.4) also imply
\[
\partial_M I_1 = \partial_M I_2 = 0, \quad \partial_q I_1 = 0, \quad \partial_q I_2 = \gamma.
\] (6.13)

Let us emphasize, that the arbitrary functions \( \alpha_{1,2}(r), \beta_{1,2}(y) \) and \( \gamma \) represent the pure gauge ambiguity. These arise as integration constants in the integrating flow equations (see Appendix B) and restricted only by the reality condition (6.12) if necessary. This gauge ambiguity encodes in a rather nontrivial way the ambiguity in the choice of one or another coordinate system, giving the integrating flow approach the wide area of applicability.

Note, that the case of \( M = \overline{M} \) and \( q = 0 \) was considered in [1] within the same unfolded approach. However, the simple Kerr–Schild shift used there to map BHUS into the free AdS\(_4\) system does not work when \( M \neq \pm \overline{M} \). The reason is that the Kerr–Schild shift used in [1] is not compatible with the reality of metric in this case. The answer in terms of the integrating flow reduces to different constraint conditions for flow functions (see Appendix B).

In the case \( M = \overline{M} \), we perform different flow integration, than that of general complex \( M \), resulting in notable simplification of (6.3), (6.4). In this case we fix the gauge freedom as follows
\[
\phi_1 = \frac{1}{2}(G + \bar{G}) , \quad \phi_2 = \phi_3 = \phi_4 = 0 ,
\] (6.14)
\[
\psi_1 = \frac{1}{2}G\bar{G} , \quad \psi_2 = \psi_3 = \psi_4 = 0 .
\] (6.15)
This in turn implies
\[
\partial_M I_1 = \partial_M I_2 = 0 , \quad \partial_q I_1 = 0 , \quad \partial_q I_2 = -2 .
\] (6.16)

Note, that the difference between the cases with \( M = \pm \overline{M} \) and \( M \neq \pm \overline{M} \) arises as a consequence of the possibility to replace the two flows with respect to \( M \) and \( \overline{M} \) in the latter case by a single flow with respect to \( M \) in the former. (For more detail see Appendix B.)
Let us stress, that the condition (6.2) has been extensively used in the derivation of (6.3), (6.4). The obtained integrating flows can be explicitly integrated, giving the AdS4 covariant and coordinate-free description of a black hole metric as we demonstrate in the next section.

The following comment is now in order. In the derivation of the integrating flow equations we have fixed the gauge freedom. In principle, this could have been done in variety of ways keeping one or another amount of arbitrary gauge parameters. Our strategy was to leave those as few as possible, though still enough to encompass the most important representations, such as Kerr–Schild, double Kerr–Schild and the generalized Carter–Plebanski. In principle one can think of further generalization of the form of the integrating flows to describe even more general forms of the black hole solutions.

7 Flow integration with AdS4 initial data

7.1 Solution for Kerr–Schild vectors

To restore BHUS frame fields from integrating flow equations it is convenient to start with Kerr–Schild vectors. Let us start with the generic case of arbitrary complex $\mathcal{M}$.

It is convenient to restrict the integrating flow gauge parameters (6.9)–(6.11) as follows

$$
\alpha_1(r) = \theta_1(r), \quad \alpha_2(r) = \theta_2(r), \quad \beta_1(y) = \vartheta_1(y), \quad \beta_2(y) = \vartheta_2(y), \quad \gamma = 0. \quad (7.1)
$$

Note that this gauge choice is compatible with the reality condition (6.12) only if $\beta_1 = \beta_2 = 1/2$. As already mentioned, the other choices may still be useful to incorporate the double Kerr–Schild form of the complex black hole metric.

As explained in Appendix C, the flow equations (6.3), (6.4) for the Kerr–Schild vectors give

$$
\hat{k}_{\alpha\hat{\alpha}} = k_{1,\alpha\hat{\alpha}} \left( \frac{\Delta_r}{\Delta_r} \right)^{\alpha_2}, \quad \hat{n}_{\alpha\hat{\alpha}} = n_{\alpha\hat{\alpha}} \left( \frac{\Delta_r}{\Delta_r} \right)^{\alpha_1},
$$

$$
\hat{l}_{\alpha\hat{\alpha}}^+ = l_{\alpha\hat{\alpha}}^+ \left( \frac{\Delta_y}{\Delta_y} \right)^{\beta_2}, \quad \hat{l}_{\alpha\hat{\alpha}}^- = l_{\alpha\hat{\alpha}}^- \left( \frac{\Delta_y}{\Delta_y} \right)^{\beta_1},
$$

where the unhatted quantities correspond to the initial data of the AdS4 unfolded system (4.14)–(4.16)

$$
\Delta_{r,y} = \hat{\Delta}_{r,y} \bigg|_{\mathcal{M},\bar{\mathcal{M}},q=0}, \quad e_{I,\alpha\hat{\alpha}} = \hat{e}_{I,\alpha\hat{\alpha}} \bigg|_{\mathcal{M},\bar{\mathcal{M}},q=0}. 
$$
Analogously, the integration of the Kerr–Schild one-forms $\hat{E}_I$ (5.21) gives

$$\hat{K} = K - \alpha_2(r) \frac{\Delta r}{\Delta r} (K - N), \quad \hat{N} = N - \alpha_1(r) \frac{\Delta r}{\Delta r} (N - K),$$

(7.5)

$$\hat{L}^{+-} = L^{+-} - \beta_2(y) \frac{\Delta y}{\Delta y} (L^{+-} - L^{+-}), \quad \hat{L}^{-+} = L^{-+} - \beta_1(y) \frac{\Delta y}{\Delta y} (L^{-+} - L^{-+}).$$

(7.6)

Note, that

$$\alpha_1 K + \alpha_2 N = \alpha_1 \hat{K} + \alpha_2 \hat{N}, \quad (7.7)$$

$$\beta_1 L^{+-} + \beta_2 L^{-+} = \beta_1 \hat{L}^{+-} + \beta_2 \hat{L}^{-+}. \quad (7.8)$$

Note also that $\hat{L}^{+-}$ and $\hat{L}^{-+}$ are complex conjugated only if $\beta_1 = \beta_2 = \frac{1}{2}$. The first integrals for that particular gauge choice (7.1) coincide with those of $AdS_4$

$$\mathcal{I}_1 = I_1, \quad \mathcal{I}_2 = I_2. \quad (7.9)$$

Now consider the special case of $\mathcal{M} = \overline{\mathcal{M}}$. Using the simplest gauge choice for the flow coefficients (6.14), (6.15) the integration of the vierbein field gives the Kerr–Schild form

$$\hat{k}_{\alpha\dot{\alpha}} = k_{\alpha\dot{\alpha}}, \quad \hat{K} = K, \quad (7.10)$$

$$h_{\alpha\dot{\alpha}} = h_{\alpha\dot{\alpha}} + \frac{1}{2} \left( M(\mathcal{G} + \mathcal{G}) - q\mathcal{G}\mathcal{G} \right) k_{\alpha\dot{\alpha}} K \quad (7.11)$$

with

$$\mathcal{I}_1 = I_1, \quad \mathcal{I}_2 = I_2 - 2q. \quad (7.12)$$

Let us stress again that for $C_4 = C_2^2$ the Kerr–Schild vectors $l_{\alpha\dot{\alpha}}^{++}$ and $l_{\alpha\dot{\alpha}}^{-+}$ are ill-defined already in the vacuum $AdS_4$ geometry and hence the vectors $\hat{l}_{\alpha\dot{\alpha}}^{++}$ and $\hat{l}_{\alpha\dot{\alpha}}^{-+}$ can not be expressed in terms of their $AdS_4$ counterparts.

The following comment is now in order. The $AdS_4$ unfolded system (4.14)–(4.16) admits well defined flat limit $\lambda \to 0$. Therefore one can integrate BHUS with Minkowski space-time set of initial data by introducing an additional flow with respect to cosmological constant $\lambda^2$. In this case BHUS first integrals would be related to invariants of Poincaré algebra and the solution would be written in Minkowski covariant way. We, however, prefer to work in $AdS_4$ covariant way rather than in Poincaré.

### 7.2 $AdS_4$ covariant form of a black hole metric

The obtained Kerr–Schild vectors and one-forms (7.2), (7.3), (7.5), (7.6) allow us to reconstruct black hole vierbein and metric. Let us consider the generic case of
arbitrary complex \( \mathcal{M} \). To reproduce the metric we use the following identity

\[
\hat{h}_{\alpha\dot{\alpha}} = \frac{2}{(kn)} \left( \hat{k}_{\alpha\dot{a}} \hat{N} + \hat{n}_{a\dot{a}} \hat{K} \right) + \frac{2}{(l\dot{+}\dot{l}+)} \left( \hat{\dot{i}}_{\alpha\dot{a}} \hat{\dot{L}} + \hat{i}_{a\dot{a}} \hat{\dot{L}}+ \right),
\]

(7.13)

which arises as a consequence of the completeness relation for two-component spinors

\[
\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} = \frac{1}{(kn)} \left( \hat{k}_{\alpha\dot{a}} \hat{\dot{n}}_{\dot{\beta}\dot{\beta}} + \hat{n}_{a\dot{a}} \hat{\dot{k}}_{\dot{\beta}\dot{\beta}} \right) + \frac{1}{(l\dot{+}\dot{l}+)} \left( \hat{i}_{\alpha\dot{a}} \hat{\dot{i}}_{\dot{\beta}\dot{\beta}} + \hat{i}_{a\dot{a}} \hat{\dot{i}}_{\dot{\beta}\dot{\beta}} \right),
\]

(7.14)

which can be straightforwardly checked using (4.33). Substituting (7.2), (7.3), (7.5), (7.6) into (7.13) we restore vierbein in a coordinate-independent way.

The metric is

\[
ds^2 = \frac{1}{2} \hat{h}_{\alpha\dot{a}} \hat{h}_{\dot{\alpha}\dot{a}} dx^i dx^j.
\]

(7.15)

Substituting (7.13) we obtain coordinate-independent representation for the metric in the form

\[
ds^2 = \frac{\hat{\Delta}_r}{r^2 + y^2} \hat{K} \hat{N} - \frac{\hat{\Delta}_y}{r^2 + y^2} \hat{\dot{L}}\dot{\dot{L}} + \hat{\dot{L}}\hat{\dot{L}} - ,
\]

(7.16)

where \( \hat{\Delta}_r \) and \( \hat{\Delta}_y \) are defined in (5.15) and (5.16), respectively. Now with the help of (7.5) and (7.6) one can rewrite the metric in terms of AdS_4 fields (4.14)–(4.16)

\[
ds^2 = ds_0^2 + \frac{2Mr - q/2}{r^2 + y^2} \left( \alpha_1(r)K + \alpha_2(r)N \right)^2 - \frac{2Ny + q/2}{r^2 + y^2} \left( \beta_1(y)L^- + \beta_2(y)L^+ \right)^2
\]

\[+ 4\alpha_1(r)\alpha_2(r) \frac{r^2 + y^2}{\Delta_r \hat{\Delta}_r} \left( 2Mr - q/2 \right) dr^2 - 4\beta_1(y)\beta_2(y) \frac{r^2 + y^2}{\Delta_y \hat{\Delta}_y} \left( 2Ny + q/2 \right) dy^2,
\]

(7.17)

where \( \alpha_1(r) + \alpha_2(r) = \beta_1(y) + \beta_2(y) = 1 \) and \( ds_0^2 \) is the background AdS_4 metric which can be represented in the form analogous to (7.16)

\[
ds_0^2 = \frac{\Delta_r}{r^2 + y^2} K \hat{N} - \frac{\Delta_y}{r^2 + y^2} L^+ \hat{L}^-.
\]

(7.18)

The reality condition for (7.17) requires

\[
\beta_1 = \beta_2 = \frac{1}{2}.
\]

(7.19)

The case of complex metric with, say, \( \alpha_1 = \beta_1 = 1, \alpha_2 = \beta_2 = 0 \) yields the double Kerr–Schild form of (7.17)

\[
ds^2 = ds_0^2 + \frac{2r}{r^2 + y^2} \left( M - \frac{e^2 + g^2}{2r} \right) KK - \frac{2y}{r^2 + y^2} \left( N + \frac{e^2 + g^2}{2y} \right) L^- \hat{L}^+,
\]

(7.20)
which might be useful as it satisfies both linearized and nonlinear Einstein–Maxwell equations.\(^{10}\)

To reveal physical meaning of the metric (7.17) let us first recall its parameter space. There are three parameters \(M, N, q\) and \(\lambda\) that cannot be redefined by diffeomorphisms as they enter the Riemann curvature tensor in (5.6), (5.7). Then, there are two parameters associated with the first integrals (7.9) and expressed via \(AdS_4\) invariants by (4.64), (4.65). These are so called kinematical parameters one of which can be always chosen to be -1, 0 or 1. Indeed, the obtained integrating flow transforms BHUS into \(AdS_4\) global symmetry parameter equation (4.10), (4.11) which is invariant under rescaling (2.22). This allows us to set, say, \(C_2 = \pm 1\) or 0.

As a result, the diffeomorphism invariant black hole parameter space consists of three curvature parameters \(M, N, q\) (and \(\lambda\)) and two kinematical ones, discrete \(C_2\) and continuous \(C_4\).

Let us show that (7.17) is nothing but a coordinate-independent realization of the \(AdS_4\)-Kerr–Newman-Taub-NUT solution originally discovered by Carter [4] and Plebanski [5]. To this end we choose certain two-parametric coordinate realization of \(AdS_4\) space that covers whole range of values of invariants for a particular \(AdS_4\) Killing vector and calculate the resulting metric (7.17).

## 8 Coordinate realization of \(AdS_4\) background

Following [41] it is convenient to specify \(AdS_4\) metric in certain two-parametric form. Using the coordinate system \(\{\tau, \psi, r, y\}\) \(AdS_4\) metric can be written down in the form

\[
\begin{align*}
\bar{ds}^2 = & \frac{\Delta_r}{r^2 + y^2} (d\tau + y^2 d\psi)^2 - \frac{\Delta_y}{r^2 + y^2} (d\tau - r^2 d\psi)^2 - \frac{r^2 + y^2}{\Delta_r} dr^2 - \frac{r^2 + y^2}{\Delta_y} dy^2, \\
\end{align*}
\]

where

\[
\begin{align*}
\Delta_r = r^2 (\lambda^2 r^2 + \epsilon) + a^2, \quad \Delta_y = y^2 (\lambda^2 y^2 - \epsilon) + a^2.
\end{align*}
\]

The parameters \(a\) and \(\epsilon\) that enter the \(AdS_4\) metric (8.1) as pure gauge arbitrary constants\(^{11}\) become the Carter–Plebanski black hole kinematic parameters upon \(AdS_4\) deformation. The metric verifies \(AdS_4\) Einstein equations and provides

\[
R_{ij} = 3 \lambda^2 g_{ij}.
\]

Now, the \(\frac{\partial}{\partial t}\) Killing vector

\[
V^i = \{1, 0, 0, 0\}
\]

\(^{10}\)Note, that (7.20) becomes real upon Wick rotation to (2,2) signature.

\(^{11}\)\(a^2\) can be chosen to be negative as well.
renders via (4.14) the source-free Maxwell two-form
\[
F = \frac{1}{(r^2 + y^2)^2} ((d\tau + y^2\psi) \wedge (r^2 - y^2)dr + 2(d\tau - r^2\psi) \wedge rydy), \quad (8.5)
\]
that can be generated by the vector potential one-form
\[
F = dA, \quad A = \frac{r}{r^2 + y^2}(d\tau + y^2\psi). \quad (8.6)
\]
Maxwell tensor (8.5) along with the Killing vector (8.4) fulfill \(AdS_4\) unfolded equations (4.14)--(4.16). Coordinates \(r\) and \(y\) can be checked to coincide with the canonical coordinates (2.9).

Using (4.38) and (4.39) we find Kerr–Schild one-forms in the specified coordinates
\[
K = d\tau + y^2d\psi + \frac{r^2 + y^2}{\Delta_r}dr, \quad N = d\tau + y^2d\psi - \frac{r^2 + y^2}{\Delta_r}dr, \quad (8.7)
\]
\[
L^+ = d\tau - r^2d\psi + \frac{r^2 + y^2}{i\Delta_y}dy, \quad L^- = d\tau - r^2d\psi - \frac{r^2 + y^2}{i\Delta_y}dy. \quad (8.8)
\]
Note, that having expressions for background one-forms (8.7) and (8.8) it is easy to calculate coordinate form of Killing–Yano and closed Killing–Yano tensors (E.2) and (E.5), respectively
\[
Y = ydr \wedge (dt + y^2d\psi) + rdy \wedge (dt - r^2d\psi), \quad (8.9)
\]
\[
*Y = rdr \wedge (dt + y^2d\psi) - ydy \wedge (dt - r^2d\psi). \quad (8.10)
\]
The first integrals for the \(AdS_4\) unfolded system (4.20), (4.21) amount to
\[
I_1 = \epsilon, \quad I_2 = 4a^2 \quad (8.11)
\]
The obtained \(AdS_4\) formulae lead to the Carter–Plebanski metric.

9 Particular solutions

9.1 Carter–Plebanski solution
Consider the real case (7.19) of (7.17) and let us fix the gauge functions as
\[
\alpha_1(r) = \alpha_2(r) = \frac{1}{2} \quad (9.1)
\]
Substituting (8.1) and (8.7), (8.8) into (7.17) we obtain
\[
ds^2 = \frac{\hat{\Delta}_r}{r^2 + y^2} (d\tau + y^2 d\psi)^2 - \frac{\hat{\Delta}_y}{r^2 + y^2} (d\tau - r^2 d\psi)^2 - \frac{r^2 + y^2}{\hat{\Delta}_r} dr^2 - \frac{r^2 + y^2}{\hat{\Delta}_y} dy^2 , \quad (9.2)
\]
where
\[
\hat{\Delta}_r = 2M r - e^2 - g^2 - r^2 (\lambda^2 r^2 + \epsilon) + a^2 , \quad (9.3)
\]
\[
\hat{\Delta}_y = 2N y + e^2 + g^2 + y^2 (\lambda^2 y^2 - \epsilon) + a^2 . \quad (9.4)
\]
(Recall, that \( q = 2(e^2 + g^2) \).) The metric (9.2) is the well-known Carter–Plebanski solution of vacuum Einstein–Maxwell equations that describes Petrov D-type metric characterized by the mass \( M \), NUT-parameter \( N \), electric and magnetic charges \( e \) and \( g \). It possesses two first integrals (5.13), (5.14) \( a^2 \) and \( \epsilon \) one of which can be chosen to be 1, 0 or -1. On account of (7.9) and (4.64), (4.65) these kinematical parameters are related to the \( AdS_4 \) invariants
\[
\epsilon = C_2 , \quad 4\lambda^2 a^2 = C_4 - C_2^2 . \quad (9.5)
\]
Since the coordinate realization of the \( AdS_4 \) metric (8.1) and Killing vector (8.4) covers the whole range of \( AdS_4 \) invariants (8.11) it is shown that BHUS with generic parameters describes Carter–Plebanski solution (9.2). The parametric form (7.17) allows one to choose its different representations. In particular, the double Kerr–Schild form (7.20) is constructed from mutually orthogonal background null-vectors \( k_{\alpha\dot{\alpha}} , l^{+\alpha}_{\alpha\dot{\alpha}} \) and depends on the deformation parameters linearly. By an appropriate change of variables that changes the metric signature one can obtain usual real form of the metric in Minkowski signature [22].

### 9.2 Kerr–Newman solution

For physical applications it is instructive to consider the case with vanishing NUT parameter \( N = 0 \) that corresponds to real \( \mathcal{M} \). This case was considered in [1] for \( q = 0 \) (Kerr metric) within the unfolded dynamics approach. The case of \( q \not= 0 \) corresponds to Kerr–Newman solution that describes rotating and electro-magnetically charged black hole (provided its charge and rotation parameter are such, that the metric is free of naked singularities). To describe this case one can simply set \( N = 0 \) in (7.17). However as we have already seen, the integrating flow with \( \mathcal{M} = \overline{\mathcal{M}} \) admits a different integration (7.11) giving rise to the simpler Kerr–Schild form of a metric
\[
ds^2 = ds_0^2 + \frac{2Mr - q}{r^2 + y^2} KK . \quad (9.6)
\]
To associate $AdS_4$ invariants with the black hole rotational parameter one can use the coordinate realization of [23] for $AdS_4$ metric and certain $AdS_4$ Killing vector (see [1] for detail) to find

$$C_2 = 1 + \lambda^2 a^2, \quad C_4 = C_2^2 + 4\lambda^2 a^2.$$  \hspace{1cm} (9.7)

The important particular case of static solution (charged Schwarzschild black hole) arises if $a = 0$, or equivalently $C_4 = C_2^2 \neq 0$.

10 Summary and discussion

To conclude let us summarize the results obtained in this paper. It is shown that a wide class of black hole metrics (Carter–Plebanski) admits simple unfolded description in terms of Killing and source-free Maxwell fields. The system is obtained as a parametric deformation of $AdS_4$ global symmetry equation. Two deformation parameters $M \in \mathbb{C}$ and $q \in \mathbb{R}$ are associated with black hole mass $M = \text{Re} M$, NUT charge $N = \text{Im} M$ and electro-magnetic charges $q = 2(e^2 + g^2)$. Black hole kinematic characteristics related to the angular momentum $a$ and the discrete parameter $\epsilon$ are expressed via two first integrals of the unfolded system.

It is shown that $AdS_4$ global symmetry equation and BHUS are related by the integrating flows, that describe an evolution with respect to black hole charges. The integrating flows allowed us to describe the black hole vierbein, metric, Killing fields, and other characteristics in the $AdS_4$ covariant and coordinate-independent form. This was done by the straightforward integration of the first order flow evolution equations with the initial data corresponding to the $AdS_4$ vacuum space. One of the consequences of this procedure is that black hole kinematic parameters acquired invariant interpretation in terms of two $AdS_4$ invariants. Let us stress, that the flow correspondence between vacuum and black hole systems is similar to the one studied in the nonlinear higher spin gauge theory [6, 7].

We believe that the obtained results can have various useful applications. One of the most striking features of the obtained description is that it does not refer to any particular coordinate system. Many important algebraic objects resided in black holes such as Killing–Yano tensors, Kerr–Schild vectors acquire simple and natural origin in the proposed formulation.

One of the most straightforward applications could be the study of fluctuations of different types of fields in the black hole geometry in the unfolded dynamics approach. Fortunately, the unfolded formulation of free fields of various types is available in the literature (see e.g. [8] for massless fields in $AdS_4$, [24] for the case of scalar field of any mass in arbitrary dimension and [7, 13] for more references).
Another intriguing development is to study a possible generalization of the obtained results to the full nonlinear higher spin gauge theory that rests on the unfolded formulation. Hopefully, the obtained results will allow us to challenge this problem at least perturbatively.

An interesting direction for the future study is to explore the higher-dimensional generalization. It is well-known that black holes in higher dimensions have reacher properties than their four-dimensional counterparts. In particular, in $d > 4$ there are black holes with non-spherical (ring) horizon topology [23]. Besides, the curvature tensor of higher-dimensional black holes is not necessarily of generalized D-type [20] (e.g., in the case of black rings). Even though the approach demonstrated in this paper is essentially four-dimensional, we hope it can be extended to higher dimensions. Let us note in this respect that the analysis of [27], where the hidden symmetries of higher-dimensional black holes were discovered, suggests that the unfolded description of these black holes is likely to be based on the differential forms of higher ranks.

An alternative possibility for a higher-dimensional generalization of the obtained unfolded system is the analysis of the black hole-like solutions in $Sp(2M)$ spaces with matrix coordinates [28, 29, 30, 31]. Since these models provide a higher-dimensional generalization of the spinor approach used in this paper, one can speculate that such an extension may be even simpler than in the usual tensorial setup.

The last but not the least is to understand better the origin of the integrating flow, which very likely is a manifestation of some hidden higher dimensional and/or integrable structure in the system.

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Appendix

A Notations

Capital Latin letters $A, B, \ldots$ label $Sp(4)$ vector (i.e., 4d Majorana spinor) indices, $A, B = 1, \ldots 4$. Indices $i, j, \cdot \cdot \cdot = 1, \ldots 4$ are world (base), while $a, b, \cdot \cdot \cdot = 1, \ldots 4$ are fiber ones. Capital Latin indices from the middle of alphabet $I, J = 1, \ldots , 4$ are used
to enumerate basis null-vectors in integrating flow decomposition. To distinguish between $AdS_4$ background and black hole quantities the latter are endowed with hats.

The analysis in four dimensions considerably simplifies in spinor notation. Vector notation is translated to the spinor one and vice versa with the help of $\sigma$-matrices ($\sigma^0_{\dot{\alpha} \dot{\alpha}}$ is the unit matrix and $\sigma^{1,2,3}_{\alpha \dot{\alpha}}$ are Pauli matrices) that obey the condition

$$\eta_{ab} \sigma^a_{\alpha \dot{\alpha}} \sigma^b_{\beta \dot{\beta}} = 2 \varepsilon_{\alpha \beta} \varepsilon_{\dot{\alpha} \dot{\beta}}, \quad (A.1)$$

where $\alpha$, and $\dot{\alpha} = 1, 2$ are mapped to each other by complex conjugation $\alpha \leftrightarrow \dot{\alpha}$. For a Lorentz vector $V_a$ we have

$$V_{\alpha \dot{\alpha}} = (\sigma^a_{\alpha \dot{\alpha}} V_a, \quad V_a = \frac{1}{2} (\sigma_a)^{\alpha \dot{\alpha}} V_{\alpha \dot{\alpha}}. \quad (A.2)$$

Spinor indices are raised and lowered by the $sp(2)$ antisymmetric tensors $\varepsilon_{\alpha \beta}$ and $\varepsilon_{\dot{\alpha} \dot{\beta}}$

$$\xi_\alpha = \varepsilon^\beta \varepsilon_{\beta \alpha}, \quad \xi^\alpha = \varepsilon_{\alpha \beta} \xi_\beta, \quad \bar{\xi}_{\dot{\alpha}} = \varepsilon_{\dot{\beta} \dot{\alpha}} \bar{\xi}^{\dot{\beta}}, \quad \bar{\xi}^{\dot{\alpha}} = \varepsilon_{\dot{\alpha} \dot{\beta}} \bar{\xi}^{\dot{\beta}}, \quad (A.3)$$

where $\varepsilon_{12} = \varepsilon^{12} = 1, \varepsilon_{\alpha \beta} = -\varepsilon_{\beta \alpha}, \varepsilon_{\dot{\alpha} \dot{\beta}} = -\varepsilon^{\dot{\beta} \dot{\alpha}}$.

Lorentz irreducible spinor decompositions of the Maxwell and Weyl tensors $F_{ab}$ and $C_{abcd}$ read, respectively, as

$$F_{\alpha \beta \dot{\alpha} \dot{\beta}} = \varepsilon_{\alpha \beta} F_{\dot{\alpha} \dot{\beta}} + \varepsilon_{\dot{\alpha} \dot{\beta}} F_{\alpha \beta}, \quad C_{\alpha \beta \dot{\alpha} \dot{\beta} \gamma \delta \dot{\gamma} \dot{\delta}} = \varepsilon_{\alpha \beta} \varepsilon_{\gamma \delta} C_{\dot{\alpha} \dot{\beta} \dot{\gamma} \dot{\delta}} + \varepsilon_{\dot{\alpha} \dot{\beta}} \varepsilon_{\gamma \delta} C_{\alpha \beta \gamma \delta}, \quad (A.4)$$

where the symmetrization over spinor indices denoted by the same letter is implied. $F_{\alpha \beta}, C_{\alpha \beta \gamma \delta}$ and their conjugated are totally symmetric multispinors.

Hodge duality for two-forms is translated as follows. By definition, $P_{ij}$ and $*P_{ij}$ are related as

$$*P_{ij} = \frac{\sqrt{-g}}{2} \varepsilon_{ijkl} P^{kl}, \quad (A.5)$$

where $g$ is the determinant of the metric and $\varepsilon_{ijkl}$ is Levi-Civita symbol ($\varepsilon_{0123} = -\varepsilon^{0123} = 1$). Then for spinor components we have

$$*P_{\alpha \alpha} = -i P_{\alpha \alpha}, \quad *\bar{P}_{\dot{\alpha} \dot{\alpha}} = i \bar{P}_{\dot{\alpha} \dot{\alpha}}. \quad (A.6)$$

In vector notation $P_{\alpha \alpha}, \bar{P}_{\dot{\alpha} \dot{\alpha}}$ correspond to the (anti)self-dual parts $P^\pm_{ij}$ of antisymmetric tensor $P_{ij}$ defined by

$$P^\pm_{ij} = \frac{1}{2} (P_{ij} \pm i *P_{ij}). \quad (A.7)$$

$AdS_4$ spinor indices $A, B, \ldots$ unify left and right Weyl spinor indices $A = (\alpha, \dot{\alpha})$. These are raised and lowered by the canonical $sp(4)$ form

$$\varepsilon_{AB} = \begin{pmatrix} \varepsilon_{\alpha \beta} & 0 \\ 0 & \varepsilon_{\dot{\alpha} \dot{\beta}} \end{pmatrix}. \quad (A.8)$$
Sketch of derivation of integrating flows

Let us sketch the idea of derivation of the equations (6.3) and (6.4). Since the method is similar for the flows with respect to $\mathcal{M}$ and $\overline{\mathcal{M}}$ we confine ourselves to $\partial\mathcal{M}$-flow with complex $\mathcal{M}$.

Consider the most general decomposition of $\partial\mathcal{M}v_{\alpha\dot{\alpha}}$ and $\partial\mathcal{M}h_{a\dot{a}}$ in the basis of Kerr–Schild vectors (5.22)

$$
\partial\mathcal{M}v_{\alpha\dot{\alpha}} = 4 \sum_{I=1}^{4} t_I e_{I,a\dot{a}}, \quad \partial\mathcal{M}h_{a\dot{a}} = 4 \sum_{I,J=1}^{4} \Phi_{IJ} \hat{e}_{I,a\dot{a}} \hat{e}_{J,\gamma\dot{\gamma}} h^{\gamma\dot{\gamma}}
$$

with some set of functions $t_I(x,\mathcal{M},\ldots)$ and $\Phi_{IJ}(x,\mathcal{M},\ldots)$. Applying $[\partial\mathcal{M},d]$ to (4.19) and using $\partial\mathcal{M}G = \partial\mathcal{M}\overline{G} = 0$ (see (6.2)) we obtain

$$
2 \sum_{I} (-1)^{\sigma_I + \sigma_J} \Phi_{IJ} = t_J,
$$

$$
2 \sum_{I} (-1)^{\sigma_I + \bar{\sigma}_J} \Phi_{IJ} = \bar{t}_J.
$$

Some components of $\Phi_{IJ}$ can be eliminated using the gauge freedom of Cartan equations. Indeed, the gauge transformation of vierbein is

$$
\delta h_{a\dot{a}} = D\xi_{a\dot{a}} + \xi^{\dot{\alpha}} \bar{h}_{\dot{\alpha}a} + \bar{\xi}_{\dot{\alpha}} h_{a\dot{\alpha}},
$$

where $\xi_{a\dot{a}}, \xi^{\dot{\alpha}}$ are arbitrary gauge parameters. Using (B.1) to extract pure gauge part in

$$
\delta \mathcal{M} h_{a\dot{a}} = 4 \sum_{I,J=1}^{4} \Phi_{IJ} \hat{e}_{I,a\dot{a}} \hat{E}_J \delta \mathcal{M},
$$

one can see that six parameters $\xi_{aa}$ and $\bar{\xi}_{a\dot{a}}$ make it possible to eliminate the antisymmetric part $\Phi_{[IJ]}$. Choose $\xi_{a\dot{a}}$-gauge parameter in the form

$$
\xi_{a\dot{a}} = \sum_{I=1}^{4} S_I \hat{e}_{I,a\dot{a}} \delta \mathcal{M},
$$

where $S_I(\mathcal{G},\overline{\mathcal{G}},\mathcal{M},\ldots)$ is some set of functions. Using (5.3)–(5.5) one obtains

$$
D\xi_{a\dot{a}} = \sum_{I,J=1}^{4} (B_{[IJ]} + B_{(IJ)}) \hat{e}_{I,a\dot{a}} \hat{E}_J,
$$

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where $B_{(IJ)}$ and $B_{[IJ]}$ are symmetric and antisymmetric. Finally, making use of the
gauge parameters $\xi_{\alpha\dot{\alpha}}$ and $\bar{\xi}_{\dot{\alpha}\alpha}$ one eliminates the
antisymmetric part $\Phi_{[IJ]}$. The symmetric part $\Phi_{(IJ)}$ is restricted by \[B.2\], \[B.3\] to the form

$$
\Phi_{(IJ)} = \begin{pmatrix}
\Phi_{11} & X & Z & Z \\
X & \Phi_{22} & Z & Z \\
Z & Z & \Phi_{33} & Y \\
Z & Z & Y & \Phi_{44}
\end{pmatrix}.
$$ \[B.8\]

Its off-diagonal part is parameterized by three parameters $X, Y, Z$. The leftover gauge
freedom in $\xi_{\alpha\dot{\alpha}}$ allows us to set $X, Y, Z$ to zero. At this stage, one gauge parameter
in $\xi_{\alpha\dot{\alpha}}$ remains free. An important observation is the following. The gauge fixing
that makes $\Phi_{IJ}$ diagonal turns out to impose in addition to \[6.2\] $\mathcal{M}$-independence
condition on Maxwell two-form

$$
\partial_M F = 0.
$$ \[B.9\]

Thus, the gauge fixing leads to the following structure functions

$$
\Phi_{IJ} = \frac{1}{2} \delta_{IJ} \phi_J, \quad t_I = \phi_I.
$$ \[B.10\]

The condition $\left[ \partial_M, d \right] = 0$ applied to \[5.3\]–\[5.5\] after somewhat annoying but straightforward calculation using the relations \[5.3\]–\[5.5\] and \[4.46\] gives the following simple compatibility conditions

$$
\frac{\partial \phi_1}{\partial y} + (\mathcal{G} - \mathcal{\bar{G}}) \phi_1 = 0, \quad \frac{\partial \phi_3}{\partial r} + (\mathcal{G} + \mathcal{\bar{G}}) \phi_3 = 0,
$$ \[B.11\]

$$
\frac{\partial \phi_2}{\partial y} + (\mathcal{G} - \mathcal{\bar{G}}) \phi_2 = 0, \quad \frac{\partial \phi_4}{\partial r} + (\mathcal{G} + \mathcal{\bar{G}}) \phi_4 = 0
$$ \[B.12\]

along with the following constraints

$$
\phi_1 + \phi_2 + \phi_3 + \phi_4 = \frac{1}{2} \mathcal{G} + \frac{1}{2} \partial_M I_1,
$$ \[B.13\]

$$
\phi_1 + \phi_2 - \phi_3 - \phi_4 = \frac{1}{2} \mathcal{\bar{G}} + \frac{\mathcal{G}^2 + \mathcal{\bar{G}}^2}{4 \mathcal{G} \mathcal{\bar{G}}} \partial_M I_1 + \frac{\mathcal{G} \mathcal{\bar{G}}}{4} \partial_M I_2,
$$ \[B.14\]

which can be equivalently rewritten as

$$
\phi_1 + \phi_2 = \frac{\mathcal{G} + \mathcal{\bar{G}}}{4} (1 + r \partial_M I_1 + \frac{1}{4r} \partial_M I_2),
$$ \[B.15\]

$$
\phi_3 + \phi_4 = \frac{\mathcal{G} - \mathcal{\bar{G}}}{4} (1 - iy \partial_M I_1 + \frac{i}{4y} \partial_M I_2).
$$ \[B.16\]
Let us note, that the conditions (B.13), (B.14) arise if $\partial M$-flow is applied to (5.13) and (5.14), respectively.

Solutions for $\phi_I$ exist for any values of $I_1$ and $I_2$. However, the case with $\partial M I_1, 2 \neq 0$, although being consistent with (B.11), (B.12) turns out to be incompatible with the reality of the metric and even in the complex case it does not seem to lead to any simplification. Therefore, we demand

$$\partial_M I_1 = 0, \quad \partial_M I_2 = 0.$$  \hspace{1cm} (B.17)

Performing straightforward integration of (B.11) and (B.12) we obtain (6.5), (6.6) and (6.9) with $\alpha_{1,2}(r)$ and $\beta_{1,2}(y)$ arising as the integration parameters.

The analysis for $\partial q$-flow is analogous leading to the same differential equations (B.11), (B.12) for $\psi_I$-functions along with the constraints

$$\psi_1 + \psi_2 + \psi_3 + \psi_4 = -\frac{G\bar{G}}{2} + \frac{1}{2} \partial_q I_1$$ \hspace{1cm} (B.18)

$$\psi_1 + \psi_2 - \psi_3 - \psi_4 = \frac{G^2 + \bar{G}^2}{4G\bar{G}} \partial_q I_1 + \frac{G\bar{G}}{4} \partial_q I_2$$ \hspace{1cm} (B.19)

or, equivalently,

$$\psi_1 + \psi_2 = -\frac{G\bar{G}}{4}(1 - 2r^2 \partial_q I_1 - \frac{1}{2} \partial_q I_2).$$ \hspace{1cm} (B.20)

$$\psi_3 + \psi_4 = -\frac{G\bar{G}}{4}(1 - 2y^2 \partial_q I_1 + \frac{1}{2} \partial_q I_2).$$ \hspace{1cm} (B.21)

It is convenient to set $\partial_q I_1 = 0$ reproducing (6.7), (6.8) and (6.10), where $\gamma = \partial_q I_2$. Finally, the condition $[\partial_M, \partial_q] = 0$ leads to (6.11). The reason to keep $\partial_q I_2 \neq 0$ is that it allows to reduce black hole metric to convenient Kerr–Schild form when $\mathcal{M} = \overline{\mathcal{M}}$.

Finally, the case with real $\mathcal{M} = \overline{\mathcal{M}}$ can be considered separately as it admits some simplification of the metric. When $\mathcal{M}$ is real one is left with the only mass flow instead of two for complex $\mathcal{M}$. This case provides the same system of differential equations for the structure functions (B.11), (B.12). Analogously, to have the complete set of consistency conditions for $\phi_I$ one acts with $\partial_M$ on the first integrals (5.13), (5.14). Note, that in this case one should set $\mathcal{M} = \overline{\mathcal{M}}$ on the right hand sides of (5.13), (5.14) prior their differentiation. Pretty much as in the complex case, it is convenient to demand (B.17). As a result, in addition to (B.11), (B.12) one gets the following constraints

$$\phi_1 + \phi_2 + \phi_3 + \phi_4 = \frac{1}{2}(G + \bar{G}), \quad \phi_1 + \phi_2 - \phi_3 - \phi_4 = \frac{1}{2}(G + \bar{G}).$$ \hspace{1cm} (B.22)
General solution of (B.11), (B.12), (B.22) is
\[ \phi_1 = \frac{G + \tilde{G}}{2} \alpha_1(r), \quad \phi_2 = \frac{G + \tilde{G}}{2} \alpha_2(r), \] (B.23)
\[ \phi_3 = \frac{G - \tilde{G}}{2} \beta_1(y), \quad \phi_4 = \frac{G - \tilde{G}}{2} \beta_2(y), \] (B.24)
where
\[ \alpha_1(r) + \alpha_2(r) = 1, \quad \beta_1(y) + \beta_2(y) = 0. \] (B.25)
Note, that (B.23) differs from (6.5) by 2 factor and the constraints for \( \beta_{1,2} \) differ from that in (6.9). This integration allows one to fix gauge parameters \( \alpha_2 = \beta_1 = \beta_2 = 0 \) to get (6.14).

Let us note, that for each of the integrating flows \( \partial_\chi = (\partial_M, \partial_M', \partial_q) \) the integrability condition
\[ [\partial_\chi, d] = 0 \] (B.26)
provides the same differential equations for the flow structure functions (B.11), (B.12). These equations are not sufficient for the consistency (B.26). The rest of the conditions, such as (B.13), (B.14) result from the requirement that (5.13), (5.14) are constant in BHUS. Together with (B.11) and (B.12) they satisfy (B.26). Having solved (B.26) one is left with
\[ [\partial_\chi, \partial_\chi'] = 0 \] (B.27)
which is straightforward to analyze.

C   Integration of Kerr–Schild vectors

Consider the general case of complex \( \mathcal{M} \). To restore Kerr–Schild vectors in terms of their \( AdS_4 \) counterparts (7.2), (7.3) via the flow integration we differentiate (B.22) along the flows and use their definition to obtain
\[ \partial_M \hat{k}_{a\dot{a}} = -\frac{\alpha_2}{\Delta_r} \hat{k}_{a\dot{a}}, \quad \partial_M \hat{j}^{+\cdot} = -\frac{\beta_2 y}{i \Delta_y} \hat{j}^{+\cdot}, \] (C.1)
\[ \partial_M \hat{n}_{a\dot{a}} = -\frac{\alpha_1}{\Delta_r} \hat{n}_{a\dot{a}}, \quad \partial_M \hat{i}^{-\cdot} = -\frac{\beta_1 y}{i \Delta_y} \hat{i}^{-\cdot}. \] (C.2)
Analogously, applying \( \partial_q \)–flow we obtain
\[ \partial_q \hat{k}_{a\dot{a}} = \frac{\theta_2}{2 \Delta_r} \hat{k}_{a\dot{a}}, \quad \partial_q \hat{j}^{+\cdot} = -\frac{\theta_2 y}{2 \Delta_y} \hat{j}^{+\cdot}, \] (C.3)
\[ \partial_q \hat{n}_{a\dot{a}} = \frac{\theta_1}{2 \Delta_r} \hat{n}_{a\dot{a}}, \quad \partial_q \hat{i}^{-\cdot} = -\frac{\theta_1 y}{2 \Delta_y} \hat{i}^{-\cdot}. \] (C.4)
Recall, that the parameters $\alpha_{1,2}(r), \beta_{1,2}(y), \theta_{1,2}(r), \vartheta_{1,2}(y)$ arise as the integration constants in (B.11), (B.12). Integration with the constraint (7.1) gives (7.2), (7.3).

To obtain (7.5), (7.6) we contract the second equation of (6.3) with all Kerr–Schild vectors (5.22). This gives

$$
\partial_M \hat{K} = \frac{\alpha_2 r}{\Delta_r} (\hat{N} - \hat{K}), \quad \partial_M \hat{L}^+ = \frac{\beta_2 y}{i \Delta_y} (\hat{L}^- + \hat{L}^+), \quad (C.5)
$$

$$
\partial_M \hat{N} = \frac{\alpha_1 r}{\Delta_r} (\hat{K} - \hat{N}), \quad \partial_M \hat{L}^- = \frac{\beta_1 y}{i \Delta_y} (\hat{L}^+ - \hat{L}^-). \quad (C.6)
$$

The analysis for $\partial_q$–flow is analogous and the integration at the condition (7.1) gives (7.5), (7.6).

## D Vector form of AdS\(_4\) unfolded system

AdS\(_4\) unfolded system in vector notation reads as

$$
D_i V_j = \kappa_{ij}, \quad (D.1)
$$

$$
D_k \kappa_{ij} = \lambda^2 (g_{kj} V_i - g_{ki} V_j). \quad (D.2)
$$

Let us introduce antisymmetric tensor $F_{ij} = F_{ij}^+ + F_{ij}^-$ by

$$
\kappa_{ij}^+ = \rho F_{ij}^+, \quad \kappa_{ij}^- = \bar{\rho} F_{ij}^-, \quad (D.3)
$$

where

$$
\rho = -\lambda^2 G^{-3}, \quad \bar{\rho} = -\lambda^2 \bar{G}^{-3}, \quad (D.4)
$$

and

$$
F_{ij}^+ F^{+ij} = -G^4, \quad F_{ij}^- F^{-ij} = -\bar{G}^4, \quad (D.5)
$$

where $\pm$ denote (anti)self-dual parts of the corresponding two-forms. One can easily check that $F_{ij}$ fulfills Maxwell equation and Bianchi identities

$$
D_k F^k_i = 0, \quad D_{[i} F_{jk]} = 0. \quad (D.6)
$$

Then the system (D.1), (D.2) can be equivalently rewritten as

$$
D_i V_j = \rho F_{ij}^+ + \bar{\rho} F_{ij}^-, \quad (D.7)
$$

$$
D_k F_{ij}^+ = V^p C_{pki}, \quad (D.8)
$$

$$
D_k F_{ij}^- = V^p C_{pki}, \quad (D.9)
$$

where $C_{ijkl}^\pm$ are the following quadratic combinations of $F_{ij}^\pm$

$$
C_{ijkl}^\pm = -G^{-1} \left( 2 F_{ij}^+ F_{kl}^+ + F_{ik}^+ F_{jl}^+ - F_{ij} F_{jk}^+ + \frac{G^4}{4} (g_{ik} g_{jl} - g_{ij} g_{jk}) \right), \quad (D.10)
$$

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\[
C^-_{ijkl} = -\mathcal{G}^{-1}\left(2F^-_{ij}F^-_{kl} + F^-_{ik}F^-_{jl} - F^-_{il}F^-_{jk} + \frac{\mathcal{G}^4}{4}(g_{ik}g_{jl} - g_{il}g_{jk})\right).
\]

(D.11)

Note that \(C^\pm_{ijkl}\) describe the (anti)self-dual parts of the black hole Weyl tensor.

The Killing projectors in vector indices are

\[
\Pi^\pm_{ij} = \frac{1}{2}g_{ij} \pm \mathcal{G}^{-2}F^+_{ij},
\]

(D.12)

\[
\bar{\Pi}^\pm_{ij} = \frac{1}{2}g_{ij} \pm \mathcal{G}^{-2}F^-_{ij}.
\]

(D.13)

They possess the following obvious properties

\[
\Pi^\pm_i \Pi^\pm_j = \Pi^\pm_k, \quad \Pi^\pm_i \bar{\Pi}^\pm_j = 0,
\]

(D.14)

and

\[
\Pi^\pm_i \Pi^\pm_j = \Pi^\pm_k, \quad \Pi^\pm_i \bar{\Pi}^\pm_j = 0.
\]

(D.15)

In spinor notation these projectors yield (4.24).

The Mutually orthogonal null-vectors (4.38)–(4.39) are the following projections of a Killing vector

\[
k_i = \frac{1}{(V+V^-)}V^-_i, \quad n_i = \frac{1}{(V+V^-)}V^+_i,
\]

(D.16)

\[
l^{+-}_i = \frac{1}{(V^-+V^-)}V^{+-}_i, \quad l^-_i = \frac{1}{(V^-+V^-)}V^{-+}_i,
\]

(D.17)

where \((AB) = A_iB^i\) and

\[
V^-_i = \Pi^-_i \bar{\Pi}^-_{jk}V^k, \quad V^+_i = \Pi^+_i \bar{\Pi}^+_{jk}V^k,
\]

(D.18)

\[
V^{+-}_i = \Pi^+_i \bar{\Pi}^-_{jk}V^k, \quad V^{-+}_i = \Pi^-_i \bar{\Pi}^+_{jk}V^k.
\]

(D.19)

Let us also note that

\[
\Pi^+_i \Pi^-_j = \Pi^-_i \Pi^+_j.
\]

(D.20)

The vectors (D.16), (D.17) define the four geodesic congruences

\[
k^jD_jk_i = 0, \quad n^jD_jn_i = 0, \quad l^{+-}D_jl^{+-} = 0, \quad l^{-+}D_jl^{-+} = 0.
\]

(D.21)

Now one can check that consistency of the Killing-Maxwell system (D.7)–(D.8) demands the function \(\rho\) to be of the form (5.2). The Riemann tensor reads as

\[
R_{ijkl} = \lambda^2(g_{ik}g_{jl} - g_{il}g_{jk}) + 2(e^2 + g^2)(g_{ik}T_{jl} + g_{jl}T_{ik} - g_{il}T_{jk} - g_{jk}T_{il})
+ 6(\mathcal{M} - 2(e^2 + g^2)\mathcal{G})C^+_ijkl + 6(\mathcal{M} - 2(e^2 + g^2)\mathcal{G})C^-_{ijkl},
\]

(D.22)
where the energy-momentum tensor $T_{ij}$ has the following simple form

$$T_{ij} = 2F^{+}_{ik}F_{j}^{-k} = 2F^{-}_{ik}F_{j}^{+k}. \quad (D.23)$$

Rewriting (4.48) and (4.49) in vector notation yields

$$F_{ij}k^j = \frac{G^2 + \bar{G}^2}{2}k_i, \quad F_{ij}n^j = \frac{G^2 + \bar{G}^2}{2}n_i, \quad (D.24)$$

$$F_{ij}l^{++} = \frac{G^2 - \bar{G}^2}{2}l_i^{++}, \quad F_{ij}l^{+-} = -\frac{G^2 - \bar{G}^2}{2}l_i^{-+} \quad (D.25)$$

and

$$*F_{ij}k^j = \frac{i(G^2 - \bar{G}^2)k_i}{2}, \quad *F_{ij}n^j = \frac{i(G^2 - \bar{G}^2)n_i}{2}, \quad (D.26)$$

$$*F_{ij}l^{++} = -\frac{i(G^2 + \bar{G}^2)}{2}l_i^{++}, \quad *F_{ij}l^{+-} = \frac{i(G^2 + \bar{G}^2)}{2}l_i^{-+}. \quad (D.27)$$

E Some useful unfolded system properties

Considered unfolded systems have a number of important properties, such as the existence of Killing–Yano tensors and additional Killing vector. Let us show this in some detail for the case of $AdS_4$.

Using (4.3) and (4.12) we find

$$D\left(\frac{1}{G^3} F_{\dot{\alpha} \dot{\alpha}}\right) = -h_{\alpha} \dot{\alpha} V_{\alpha \dot{\alpha}}. \quad (E.1)$$

Therefore, the Maxwell tensor generates Killing–Yano tensor

$$Y_{\alpha \alpha} = \frac{i}{G^3} F_{\alpha \alpha}, \quad \dot{Y}_{\dot{\alpha} \dot{\alpha}} = -\frac{i}{G^3} \bar{F}_{\dot{\alpha} \dot{\alpha}}. \quad (E.2)$$

Indeed, (E.2) gives

$$D_{\alpha \dot{\alpha}} Y_{\alpha \alpha} = 0, \quad D_{\beta \dot{\beta}} Y_{\beta \alpha} + D_{\alpha \dot{\beta}} \bar{Y}_{\dot{\beta} \dot{\dot{\alpha}}} = 0, \quad (E.3)$$

which is equivalent to the Killing–Yano equation in vector indices [32]

$$D_{(k} Y_{m)n} = 0, \quad Y_{mn} = -Y_{nm}. \quad (E.4)$$

The next observation is that the Hodge dual tensor (see Appendix A) $*Y_{ij}$ that has the following irreducible components

$$*Y_{\alpha \alpha} = \frac{1}{G^3} F_{\alpha \alpha}, \quad *\bar{Y}_{\dot{\alpha} \dot{\alpha}} = \frac{1}{G^3} \bar{F}_{\dot{\alpha} \dot{\alpha}}. \quad (E.5)$$
is the closed Killing–Yano tensor since it fulfils the equation \[ d * Y = 0, \text{i.e.,} \]
\[ \partial_i * Y_{jk} = 0, \quad *Y_{mn} = - *Y_{nm}. \]  
(E.6)
where brackets denote antisymmetrization over indices. It is obvious that
\[ * Y_{mn} = - \frac{1}{\lambda^2} \kappa_{mn}, \]  
(E.7)
where \( \kappa_{mn} \) is given by (4.1).

One can see that it is possible to express Killing vector \( V^i \) as
\[ V^i = \frac{1}{3} D_j * Y^{ji}. \]  
(E.8)
Note that another Killing vector can be constructed from \( V^i \) by means of the second-rank Killing tensor \( K_{ij} \) generated by the Killing–Yano tensor \( Y_{ij} \) (see [18])
\[ \phi_i = K_{ij} V^j, \quad K_{ij} = Y_{ik} Y^{kj}. \]  
(E.9)
In spinor notation this gives the following relation between Killing vectors
\[ \phi_{\dot{a} \dot{a}} = \frac{1}{4 G^3 \bar{G}^3} F_{\alpha \beta} \bar{F}_{\dot{\alpha} \dot{\beta}} V_{\beta \dot{\beta}} - \frac{1}{4} \left( \frac{1}{G^2} + \frac{1}{\bar{G}^2} \right) V_{\dot{a} \dot{a}}. \]  
(E.10)
One can make sure that it solves the Killing equation
\[ D \phi_{\dot{a} \dot{a}} = \frac{1}{2} h_{\beta \dot{\beta}} \varphi_{\dot{a} \dot{a}} + \frac{1}{2} h^{\alpha \dot{\alpha}} \varphi_{\alpha \alpha}, \]  
(E.11)
where \( \varphi_{\alpha \alpha}, \bar{\varphi}_{\dot{a} \dot{a}} \) are the (anti)self-dual components of the Killing two-form
\[ \varphi_{\alpha \alpha} = - \frac{1}{2 G^3} \left( I_1 + \frac{\lambda^2}{G^2} \right) F_{\alpha \alpha} - \frac{1}{2 \bar{G}^3} V_{\dot{a} \dot{a}} V_{\dot{a} \dot{a}} \bar{F}_{\dot{a} \dot{a}}. \]  
(E.12)
Note, that when \( C_4 = C_2^2 \) this Killing vanishes \( \phi_{\dot{a} \dot{a}} = 0 \) and \( \varphi_{\alpha \alpha} = 0. \)

As a result the global symmetry parameter \( K_{AB} \) (4.13) generates another global symmetry parameter \( \tilde{K}_{AB} \) in the following way
\[ \tilde{K}_{AB} = \begin{pmatrix} \varphi_{\alpha \beta} & \lambda \phi_{\alpha \beta} \\ \lambda \phi_{\beta \alpha} & \bar{\varphi}_{\dot{\alpha} \dot{\beta}} \end{pmatrix}, \quad D_0 \tilde{K}_{AB} = 0. \]  
(E.13)
Its \( sp(4) \) invariants read
\[ C_2 = \tilde{K}_{AB} \tilde{K}^{AB} = I_1 I_2, \]  
(E.14)
\[ C_4 = \text{Tr}(\tilde{K}^4) = \frac{I_2^2}{4} (I_1^2 + \lambda^2 I_2). \]  
(E.15)

\(^{12}\)Note that covariant differential acts on forms as ordinary de Rham differential.
Remarkably, the existence of the Killing–Yano tensor and additional Killing vector also takes place in the black hole unfolded system (5.3)–(5.8). One can show that the formulae (E.1)–(E.10) are valid in BHUS upon redefinition

\[ D \rightarrow D, \quad V_{\alpha\dot{\alpha}} \rightarrow V_{\alpha\dot{\alpha}}. \]  

(E.16)

F  Comment on Plebanski–Demianski solution

The considered unfolded system (5.3)–(5.5) has been shown to describe generic Carter–Plebanski family of metrics. This family can be obtained by some limiting procedure from the so called Plebanski–Demianski metric [34], which is believed to be the most general D-type solution of vacuum Einstein–Maxwell equations with aligned principle directions. Although it has the same number of parameters as the Carter–Plebanski solution all of them are continuous unlike those in Carter–Plebanski case with one being discrete. Physical meaning of this additional continuous parameter according to [34] is acceleration. The metric generalizes Carter–Plebanski solution reproducing the latter in non-accelerated limit. In this Appendix we demonstrate that the constructed black hole unfolded system does not contain Plebanski–Demianski metric.

Consider Plebanski–Demianski class of solutions of the Einstein–Maxwell equations with electric and magnetic charges and the cosmological constant. Up to notation, its original form reads as

\[ g = \frac{1}{(1 - qp)^2} \left( \frac{Q(q)(d\tau - p^2d\psi)^2}{q^2 + p^2} - \frac{P(p)(d\tau + q^2d\psi)^2}{q^2 + p^2} - \frac{q^2 + p^2}{Q(q)} dq^2 - \frac{q^2 + p^2}{P(p)} dp^2 \right), \]  

(F.1)

where

\[ Q(q) = (\lambda^2/2 + \gamma + e^2) - 2mq + \varepsilon q^2 - 2nq^3 - (\lambda^2/2 - \gamma + g^2)q^4, \]  

(F.2)

\[ P(p) = (\lambda^2/2 + \gamma - g^2) + 2np - \varepsilon p^2 + 2mp^3 - (\lambda^2/2 - \gamma - e^2)p^4. \]  

(F.3)

Choosing the vector potential in the form

\[ A = \frac{q}{q^2 + p^2} d\tau - \frac{qp^2}{q^2 + p^2} d\psi \]  

(F.4)

one can check that the following equations hold

\[ R_{ij} = 3\lambda^2 g_{ij} - 2(e^2 + g^2)(F_{ik}F_j^k - \frac{1}{4}g_{ij}F^{kl}F_{kl}), \]  

(F.5)

where \( F_{ij} \) is the Maxwell tensor \( F = dA \)

\[ F = \frac{1}{(q^2 + p^2)^2} \left[ (d\tau - p^2d\psi) \wedge (q^2 - p^2)dq + 2(d\tau + q^2d\psi) \wedge pqdp \right]. \]  

(F.6)
The Weyl tensor of (F.1) is built of (F.6) according to (5.12) thus having a chance to be described by (5.3)–(5.5) with some Killing vector \( V^i \). Let us show that this is not the case for any real \( V^i \).

Assuming, that Plebanski–Demianski metric can be described by BHUS we use its Maxwell tensor (F.6) to express corresponding Killing vector \( V^i \) via (4.19) in two different ways

\[
(a) \quad V^i = \frac{1}{2\mathcal{G}^4} h_{\alpha\dot{\alpha}}^i F^{\alpha\gamma} \partial_\gamma \dot{\alpha} \mathcal{G}, \\
(b) \quad V^i = \frac{1}{2\mathcal{G}^4} h^{\alpha\dot{\alpha}} F^{\alpha\dot{\alpha}} \partial_\gamma \dot{\alpha} \mathcal{G}.
\]

(F.7) (F.8)

By construction, in the Carter–Plebanski case these two formulae give the same result with real \( V^i \). For (F.6) this is not the case and we obtain two complex-conjugated Killing vectors

\[
V^i_{(a)} = \{1, -i, 0, 0\}, \quad V^i_{(b)} = \{1, i, 0, 0\}.
\]

(F.9)

Thus, Plebanski–Demianski metric (F.1) is governed by a complex Killing vector or, equivalently, two real ones. Its unfolded system requires some modification to include complex Killing vector and will be considered elsewhere.

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