Higher derivatives of the inverse tangent function 
and a summation formula involving binomial 
coefficients

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Abstract. In [1], O. Deiser and C. Lasser obtained an explicit 
formula for the $n$-th derivative of the inverse tangent function. We 
calculate this derivative by a different method based on Faà di 
Bruno’s formula. Comparing the two results leads to the following 
identity for binomial coefficients:

$$\sum_{i=m}^{[n/2]} \frac{(-1)^i}{4^i} \binom{i}{m} \binom{n-i}{i} = \frac{(-1)^m}{2^n} \binom{n+1}{2m+1},$$

where $n, m \in \mathbb{N}_0$ and $m \leq [n/2]$. As was pointed out to the author 
by C. Krattenthaler, this formula is a special case of Gauß’s formula 
for the hypergeometric function $_2F_1$.

1 Higher derivatives of arctan

The following explicit formula for the $n$-th derivative of the inverse tangent 
function $\arctan$ was proved by O. Deiser and C. Lasser in [1]:

$$\arctan^{(n)}(x) = (n-1)! \frac{q_{n-1}(x)}{(1+x^2)^n} \quad \forall x \in \mathbb{R}, \forall n \in \mathbb{N}, \quad (1.1)$$

where

$$q_n(x) := (-1)^n \sum_{k \text{ even}, 0 \leq k \leq n} \binom{n+1}{k+1} (-1)^{k/2} x^{n-k} \quad \forall x \in \mathbb{R}, \forall n \in \mathbb{N}_0 \quad (1.2)$$

(where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$).

Other representations of $\arctan^{(n)}$ can be found in [1, 4] and references 
therein. Here we want to obtain yet another explicit expression for $\arctan^{(n)}$ 
by using Faà di Bruno’s formula. Comparing our result to (1.1) then leads

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to a summation formula for binomial coefficients (which is a special case of Gauß’s formula for the hypergeometric function $_2F_1$, see section 2).

We start by recalling Faà di Bruno’s formula for the $n$-th derivative of the composition of two functions (see for instance the survey article [3] and references therein): Given two intervals $I, J \subseteq \mathbb{R}$ and two $n$-times differentiable functions $f : J \to \mathbb{R}$ and $g : I \to J$, we put $h := f \circ g$. Then $h$ is $n$-times differentiable and for every $x \in I$ we have

$$h^{(n)}(x) = \sum_{(l_1, \ldots, l_n) \in T_n} \frac{n!}{l_1! l_2! \cdots l_n!} f^{(l_1 + l_2 + \cdots + l_n)}(g(x)) \prod_{i=1}^n \left( \frac{g^{(i)}(x)}{i!} \right)^{l_i}, \quad (1.3)$$

where $T_n := \{(l_1, \ldots, l_n) \in \mathbb{N}_0^n : \sum_{i=1}^n il_i = n\}$.

There is also a slightly different version of Faà di Bruno’s formula based on Bell polynomials. After the first version of this preprint was published on arxiv.org, the author discovered the paper [4]. In this work one can also find an explicit formula for the higher derivatives of arctan, the proof of which is based on the Bell polynomial version of Faà di Bruno’s formula. But the formula for arctan$^{(n)}$ that we will obtain below (Proposition 1.2) is different from the one in [4].

Now, as a special case of (1.3) one gets the following result.

**Proposition 1.1.** Let $n \in \mathbb{N}$, $a > 0$ and let $f : (0, \infty) \to \mathbb{R}$ be an $n$-times differentiable function. We put $h(x) := f(a + x^2)$ for all $x \in \mathbb{R}$. Then $h$ is $n$-times differentiable and

$$h^{(n)}(x) = \sum_{k=0}^{[n/2]} \frac{n!}{k!(n-2k)!} (2x)^{n-2k} f^{(n-k)}(a + x^2) \quad \forall x \in \mathbb{R}. \quad (1.4)$$

**Proof.** We put $g(x) := a + x^2$ for $x \in \mathbb{R}$. Then $g'(x) = 2x$, $g''(x) = 2$ and $g^{(k)}(x) = 0$ for all $k \geq 3$ and all $x \in \mathbb{R}$.

Let $S := \{(l_1, \ldots, l_n) \in T_n : l_i = 0$ for $i = 3, \ldots, n\}$. Since $h = f \circ g$ it follows from Faà di Bruno’s formula that

$$h^{(n)}(x) = \sum_{(l_1, \ldots, l_n) \in S} \frac{n!}{l_1! l_2!} f^{(l_1)}(a + x^2)(2x)^{l_1} \quad \forall x \in \mathbb{R},$$

which can be rewritten as (1.4). \[\square\]

Alternatively, one can also prove this statement directly by induction, without using Faà di Bruno’s formula (see the Appendix).

Proposition 1.1 now allows us to obtain the following formula for the higher derivatives of the inverse tangent function.
1. Higher derivatives of arctan

Proposition 1.2. For every \( n \in \mathbb{N}_0 \) and every \( x \in \mathbb{R} \) one has

\[
\arctan^{(n+1)}(x) = \frac{n!2^n(-1)^n}{(1+x^2)^{n+1}} \sum_{m=0}^{\lfloor n/2 \rfloor} a_{mn} x^{n-2m},
\]

where

\[
a_{mn} := \sum_{k=m}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{4^k} \binom{k}{m} \binom{n-k}{k} \quad \forall m = 0, \ldots, \lfloor n/2 \rfloor.
\]

Proof. Let \( h(x) := 1/(1 + x^2) = \arctan'(x) \) for all \( x \in \mathbb{R} \) and \( f(y) := 1/y \) for all \( y \in \mathbb{R} \setminus \{0\} \). Then \( f^{(k)}(y) = k!(-1)^k/y^{k+1} \) holds for every \( k \in \mathbb{N}_0 \). Since \( h(x) = f(1 + x^2) \) it follows from Proposition 1.1 that

\[
\arctan^{(n+1)}(x) = h^{(n)}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!(-1)^{n-k}(n-k)!}{k!(n-2k)!(1+x^2)^{n+1-k}} (2x)^{n-2k}
\]

\[
= \frac{n!2^n(-1)^n}{(1+x^2)^{n+1}} \sum_{k=0}^{\lfloor n/2 \rfloor} (1+x^2)^k \frac{(-1)^k(n-k)!}{k!(n-2k)!} \frac{x^{n-2k}}{4^k}
\]

\[
= \frac{n!2^n(-1)^n}{(1+x^2)^{n+1}} \sum_{k=0}^{\lfloor n/2 \rfloor} (1+x^2)^k \frac{(-1)^k(n-k)}{k} \frac{x^{n-2k}}{4^k}.
\]

Using the binomial theorem, we obtain

\[
\arctan^{(n+1)}(x) = \frac{n!2^n(-1)^n}{(1+x^2)^{n+1}} \sum_{k=0}^{\lfloor n/2 \rfloor} k \sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{n-k}{k} \frac{x^{n-2k+2i}}{4^k}
\]

\[
= \frac{n!2^n(-1)^n}{(1+x^2)^{n+1}} \sum_{(k,i) \in A} (-1)^i \binom{k}{i} \binom{n-k}{k} \frac{x^{n-2k+2i}}{4^k},
\]

where \( A := \{(k,i) : k \in \{0, \ldots, \lfloor n/2 \rfloor\}, i \in \{0, \ldots, k\}\}. \)

Let \( A_m := \{(k,i) \in A : k-i = m\} \) for all \( m = 0, \ldots, \lfloor n/2 \rfloor \). It follows that

\[
\arctan^{(n+1)}(x) = \frac{n!2^n(-1)^n}{(1+x^2)^{n+1}} \sum_{m=0}^{\lfloor n/2 \rfloor} x^{n-2m} \sum_{(k,i) \in A_m} (-1)^i \binom{k}{i} \binom{n-k}{k}.
\]

But

\[
\sum_{(k,i) \in A_m} \frac{(-1)^i}{4^k} \binom{k}{i} \binom{n-k}{k} = \sum_{k=m}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{4^k} \binom{k}{k-m} \binom{n-k}{k}
\]

\[
= \sum_{k=m}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{4^k} \binom{k}{m} \binom{n-k}{k} = a_{mn} \quad \forall m = 0, \ldots, \lfloor n/2 \rfloor,
\]

which completes the proof. \( \square \)
2 A summation formula involving binomial coefficients

Now we can compare our result Proposition 1.2 with the simpler formula (1.1) that was found in [1] and obtain the following summation formula for binomial coefficients.

**Proposition 2.1.** Let \( n, m \in \mathbb{N}_0 \) such that \( m \leq \lfloor n/2 \rfloor \). Then we have

\[
\sum_{i=m}^{\lfloor n/2 \rfloor} \frac{(-1)^i}{4^i} \binom{n-i}{i} = \frac{(-1)^m}{2^m} \binom{n+1}{2m+1}. \tag{2.1}
\]

**Proof.** From Proposition 1.2 and the result of [1] (formulas (1.1) and (1.2)) it follows that

\[
2^n \sum_{m=0}^{\lfloor n/2 \rfloor} a_{mn} x^{n-2m} = \sum_{m=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2m+1} (-1)^m x^{n-2m}
\]

holds for all \( x \in \mathbb{R} \). Hence

\[
\frac{(-1)^m}{2^m} \binom{n+1}{2m+1} = a_{mn} = \sum_{i=m}^{\lfloor n/2 \rfloor} \frac{(-1)^i}{4^i} \binom{i}{m} \binom{n-i}{i}
\]

for every \( m \in \{0, \ldots, \lfloor n/2 \rfloor \} \). \( \square \)

After the first version of this preprint was published on arxiv.org, the author was informed by Christian Krattenthaler that this formula is a special case of Gauß’s formula for the hypergeometric function \( _2F_1 \). Here is a sketch of the argument: \( _2F_1 \) is defined by

\[
_2F_1(a, b; c; z) := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{z^k}{k!},
\]

where \((q)_k\) is the Pochhammer symbol, i.e. \((q)_k := q(q+1)\ldots(q+k-1)\) for \(k \geq 1\) and \((q)_0 := 1\).

Gauß’s formula reads

\[
_2F_1(a, b; c; 1) = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)},
\]

where \(\Gamma\) denotes the Gamma function (information on \(\Gamma\) and \( _2F_1 \) can be found, for instance, in Chapter 1 resp. 2 of [2]).

The sum in (2.1) can be expressed via \( _2F_1 \) as

\[
\sum_{i=m}^{\lfloor n/2 \rfloor} \frac{(-1)^i}{4^i} \binom{i}{m} \binom{n-i}{i} = \frac{(-1)^m}{m!4^m} \binom{n-2m+1}{m}. 
\]
Applying Gauß’s formula then leads to the result of Proposition 2.1.

A consequence of (2.1) is the following formula (which is probably also known, but the author was unable to find a reference).

**Corollary 2.2.** For all \( n \in \mathbb{N}_0 \) we have

\[
\sum_{i=0}^{n} \frac{(-1)^i}{4^i} \binom{2n+1-i}{i} = \begin{cases} 
0 & \text{if } n \text{ is odd,} \\
\frac{4}{n+1} & \text{if } n \text{ is even.}
\end{cases}
\]

*Proof.* For each \( n \in \mathbb{N}_0 \) we put

\[
C_n := \sum_{i=0}^{n} \frac{(-1)^i}{4^i} \binom{2n-i}{i}.
\]

By Proposition 2.1 (with \( m = 0 \) and \( 2n \) instead of \( n \)) we have

\[
C_n = \frac{4}{2n+1}.
\]

It follows that

\[
C_{n+1} - C_n = \frac{2}{4^{n+1}}.
\]

On the other hand, we have

\[
C_{n+1} - C_n = \sum_{i=1}^{n+1} \frac{(-1)^i}{4^i} \binom{2n+2-i}{i} - \sum_{i=0}^{n} \frac{(-1)^i}{4^{i+1}} \binom{2n-i}{i}.
\]

For \( i \in \{1, \ldots, n\} \) we have

\[
\binom{2n+2-i}{i} + \binom{2n+1-i}{i-1} = \frac{(2n+2-i)! + (2n+1-i)!}{i!(2n+2-2i)! + (i-1)!(2n+2-2i)!} = \frac{(2n+1-i)!(2n+2)}{i!(2n+2-2i)!} = \frac{2n+2}{2n+2-2i} \binom{2n+1-i}{i}.
\]

It follows that

\[
\frac{1}{2n+2} \left( C_{n+1} - C_n \right) = \frac{1}{2n+2} \left( \frac{2n+1-i}{4^{n+1}} \right) + \sum_{i=1}^{n} \frac{(-1)^i}{4^i} \binom{2n+2-i}{i} + \frac{(-1)^{n+1}2}{4^{n+1}}.
\]
Together with (2.2) this implies
\[
\frac{(-1)^{n+1}}{4^{n+1}(n+1)} + \sum_{i=0}^{n} \frac{(-1)^i}{4^i(2n + 2 - 2i)} \binom{2n+1-i}{i} = \frac{1}{4^{n+1}(n+1)}
\]
and thus
\[
\sum_{i=0}^{n} \frac{(-1)^i}{4^i(n+1-i)} \binom{2n+1-i}{i} = \frac{2(1 - (-1)^{n+1})}{4^{n+1}(n+1)} = \begin{cases} 0 & \text{if } n \text{ is odd}, \\ \frac{4-n}{n+1} & \text{if } n \text{ is even}. \end{cases}
\]

\[\square\]

3 Appendix

Here we want to give a direct proof of Proposition 1.1 via induction (without using Faà di Bruno’s formula). We recall the Proposition’s statement:

If \( n \in \mathbb{N}, a > 0, f : (0, \infty) \to \mathbb{R} \) is an \( n \)-times differentiable function and \( h(x) := f(a + x^2) \) for all \( x \in \mathbb{R} \), then \( h \) is \( n \)-times differentiable and

\[
h^{(n)}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{k!(n-2k)!}(2x)^{n-2k} f^{(n-k)}(a + x^2) \quad \forall x \in \mathbb{R}. \quad (3.1)
\]

Proof. For \( n = 1 \) this follows immediately from the chain-rule.

Now suppose that the statement is true for some \( n \in \mathbb{N} \) and assume that \( f \) is even \((n+1)\)-times differentiable. Differentiating (3.1) gives

\[
h^{(n+1)}(x) = A(x) + B(x),
\]

where

\[
A(x) := \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{2^{n-2k}n!}{k!(n-2k)!} (n-2k)x^{n-2k-1} f^{(n-k)}(a + x^2),
\]

\[
B(x) := \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{k!(n-2k)!} (2x)^{n+1-2k} f^{(n+1-k)}(a + x^2).
\]

If \( n \) is even, say \( n = 2l \), we get

\[
A(x) = \sum_{k=1}^{l} \frac{2^{n-2k+2}n!}{(k-1)!(n-2k+1)!} x^{n+1-2k} f^{(n+1-k)}(a + x^2)
\]

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and hence
\[ h^{(n+1)}(x) - (2x)^{n+1} f^{(n+1)}(a + x^2) = \sum_{k=1}^{l} (2x)^{n+1-2k} f^{(n+1-k)}(a + x^2) \left( \frac{2}{(k-1)!(n-2k+1)!} + \frac{1}{k!(n-2k)!} \right) \]
\[ = \sum_{k=1}^{l} \frac{(n+1)!}{k!(n+1-2k)!} (2x)^{n+1-2k} f^{(n+1-k)}(a + x^2). \]

This implies
\[ h^{(n+1)}(x) = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \frac{(n+1)!}{k!(n+1-2k)!} (2x)^{n+1-2k} f^{(n+1-k)}(a + x^2). \]

If \( n \) is odd (\( n = 2l + 1 \)), then we have
\[ A(x) = \sum_{k=1}^{l} \frac{2^{n-2k+2}n!}{(k-1)!(n-2k+1)!} (2x)^{n+1-2k} f^{(n+1-k)}(a + x^2) + \frac{2n!}{l!} f^{(l+1)}(a + x^2) \]
and by a similar calculation as in the even case we obtain
\[ h^{(n+1)}(x) - (2x)^{n+1} f^{(n+1)}(a + x^2) \]
\[ = \sum_{k=1}^{l} \frac{(n+1)!}{k!(n+1-2k)!} (2x)^{n+1-2k} f^{(n+1-k)}(a + x^2) + \frac{2n!}{l!} f^{(l+1)}(a + x^2) \]
\[ = \sum_{k=1}^{l+1} \frac{(n+1)!}{k!(n+1-2k)!} (2x)^{n+1-2k} f^{(n+1-k)}(a + x^2). \]

Thus we have again
\[ h^{(n+1)}(x) = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \frac{(n+1)!}{k!(n+1-2k)!} (2x)^{n+1-2k} f^{(n+1-k)}(a + x^2). \]

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