Perturbative Quantization of Gravity Theories

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We discuss relations between gravity and gauge theory tree amplitudes that follow from string theory. Together with $D$-dimensional unitarity, these relations can be used to perturbatively quantize gravity theories, i.e. they contain the necessary information for calculating complete gravity $S$-matrices to any loop orders. This leads to a practical method for computing non-trivial gravity $S$-matrix elements by relating them to much simpler gauge theory ones. We also describe arguments that $N = 8$ $D = 4$ supergravity is less divergent in the ultraviolet than previously thought.

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1 Introduction

In this talk we review work \[1,2,3,4,5\] that exploits perturbative relations between gravity and gauge theories. Although both theories have a local symmetry, their dynamical behaviors are quite different. Nevertheless, in the context of perturbation theory, it turns out that tree-level gravity amplitudes can, roughly speaking, be expressed as a sum of products of gauge theory amplitudes. These tree-level relations between gravity and gauge theory S-matrices are rather remarkable from a conventional Lagrangian or Hamiltonian point of view but can be most easily understood from the Kawai, Lewellen and Tye (KLT) \[6\] relations between open and closed string tree amplitudes. When combined with the \(D\)-dimensional unitarity methods described in refs. \[7,8\], it provides a new tool for investigating the ultra-violet behavior of quantum gravity. (The unitarity methods have also been applied to QCD loop computations of phenomenological interest and to supersymmetric gauge theory computations \[7,9,10\].)

Ultraviolet properties are a central issue for perturbative gravity. Although gravity is non-renormalizable by power counting, no divergence has, in fact, been established by a direct calculation for any four-dimensional supersymmetric theory of gravity. Explicit calculations have established that non-supersymmetric theories of gravity with matter generically diverge at one loop \[11,12,13\], and pure gravity diverges at two loops \[14\]. However, in any supergravity theory in \(D = 4\), supersymmetry Ward identities \[15\] forbid all possible one-loop \[16\] and two-loop \[17\] counterterms. Thus, at least a three-loop calculation is required to directly address the question of divergences in four-dimensional supergravity. There is a candidate counterterm at three loops for all supergravities including the maximally extended version (\(N = 8\)) \[18,19\]. However, no explicit three loop (super) gravity calculations have appeared. It is in principle possible that the coefficient of a potential counterterm can vanish, especially if the full symmetry of the theory is taken into account. Based on explicit calculation, we shall argue that this is indeed the case for the potential three-loop counterterm of \(N = 8\) supergravity.

With traditional perturbative approaches \[20\] to performing explicit calculations, as the number of loops increases the number of algebraic terms proliferates rapidly beyond the point where computations are practical. We will take a different approach, relying instead on a new formalism for perturbatively quantizing gravity.

2 Method for Investigating Perturbative Gravity

Our reformulation of quantum gravity is based on two ingredients:

1. The Kawai, Lewellen and Tye relations between closed and open string tree-level S-matrices \[6\].
2. The observation that the $D$-dimensional tree amplitudes contain all information necessary for building the complete perturbative $S$-matrix to any loop order.

2.1 The KLT tree-level relations.

In the field theory limit ($\alpha' \to 0$) the KLT relations for the four- and five-point amplitudes are

$$
M_{4}\text{tree}(1, 2, 3, 4) = -i s_{12} A_{4}\text{tree}(1, 2, 3, 4) A_{4}\text{tree}(1, 2, 4, 3),
$$

$$
M_{5}\text{tree}(1, 2, 3, 4, 5) = i s_{12} s_{34} A_{5}\text{tree}(1, 2, 3, 4, 5) A_{5}\text{tree}(2, 1, 4, 3) + i s_{13} s_{24} A_{5}\text{tree}(1, 3, 2, 4, 5) A_{5}\text{tree}(3, 1, 4, 2, 5),
$$

where the $M_n$’s are the amplitudes in a gravity theory stripped of couplings, the $A_n$’s are the color-ordered gauge theory sub-amplitudes also stripped of couplings and $s_{ij} \equiv (k_i + k_j)^2$. We suppress all $\varepsilon_j$ polarizations and $k_j$ momenta, but keep the ‘j’ labels to distinguish the external legs. Full gauge theory amplitudes are given in terms of the partial amplitudes $A_n$, via

$$
A_n^{\text{tree}}(1, 2, \ldots, n) = g^{(n-2)} \sum_{\sigma \in S_n/Z_n} \text{Tr} (T^{a_\sigma(1)} \cdots T^{a_\sigma(n)}) A_n^{\text{tree}}(\sigma(1), \ldots, \sigma(n)),
$$

where $S_n/Z_n$ is the set of all permutations, but with cyclic rotations removed, and $g$ is the gauge theory coupling constant. The $T^{a}$ are fundamental representation matrices for the Yang-Mills gauge group $SU(N_c)$, normalized so that $\text{Tr}(T^a T^b) = \delta^{ab}$. For states coupling with the strength of gravity, the full amplitudes including the gravitational coupling constant are,

$$
M_n^{\text{tree}}(1, \ldots, n) = \left(\frac{\kappa}{2}\right)^{(n-2)} M_n^{\text{tree}}(1, \ldots, n),
$$

where $\kappa^2 = 32\pi G_N$. The KLT equations generically hold for any closed string states, using their Fock space factorization into pairs of open string states.

Berends, Giele and Kuijf [21] exploited the KLT relations [1] and their $n$-point generalizations to obtain an infinite set of maximally helicity violating (MHV) graviton tree amplitudes, using the known MHV Yang-Mills amplitudes [22]. Cases of gauge theory coupled to gravity have recently been discussed in ref. [5]. Interestingly, the color charges associated with any gauge fields appearing in gravity theories are represented through the KLT equations as flavor charges carried either by scalars or fermions. For example, by applying the KLT equations the three-gluon one-graviton amplitude may be expressed as

$$
M_4^{\text{tree}}(1_g, 2_g, 3^+_g, 4^+_h) = -i g \frac{\kappa}{2} s_{12} A_4^{\text{tree}}(1_g, 2_g, 3^+_g, 4^+_h) A_4^{\text{tree}}(1_s, 2_s, 4^+_g, 3_s)
$$

$$
= \frac{\kappa}{2} \frac{\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 1 \rangle} \times \sqrt{2} f^{a_1 a_2 a_3} \frac{\langle 4 3 \rangle \langle 3 2 \rangle}{\langle 2 4 \rangle},
$$

2
where the ± superscripts denote the helicities and the subscripts \( h, g \) and \( s \) denote whether a given leg is a graviton, gluon or scalar. On the right-hand side of the equation, the group theory indices are flavor indices for the scalars. On the left-hand side they are reinterpreted as color indices for gluons. For simplicity, the amplitudes have been expressed in terms of \( D = 4 \) spinor inner products (see e.g. ref. \[23\]), although the factorization of the amplitude into purely gauge theory amplitudes holds in any dimension. The spinor inner products are denoted by \( \langle i j \rangle = \langle i^- | j^+ \rangle \) and \([i j] = \langle i^+ | j^- \rangle\), where \( |i^\pm\rangle \) are massless Weyl spinors of momentum \( k_i \), labeled with the sign of the helicity. They are antisymmetric, with norm \( | \langle i j \rangle | = | [i j] | = \sqrt{s_{ij}} \).

2.2 Cut Construction of Loop Amplitudes

We now outline the use of the KLT relations for computing multi-loop gravity amplitudes, starting from gauge theory amplitudes. Although the KLT equations hold only at the classical tree-level, \( D \)-dimensional unitarity considerations can be used to extend them to the quantum level. The application of \( D \)-dimensional unitarity has been extensively discussed for the case of gauge theory amplitudes \([7,8]\), so here we describe it only briefly.

The unitarity cuts of a loop amplitude can be expressed in terms of amplitudes containing fewer loops. For example, the two-particle cut of a one-loop four-point amplitude in the channel carrying momentum \( k_1 + k_2 \), as shown in fig. 1, can be expressed as the cut of,

\[
\sum_{\text{states}} \int \frac{d^D L_1}{(4\pi)^D} \frac{i}{L_1^2} M_{4\text{tree}}(-L_1,1,2,L_3) \frac{i}{L_3^2} M_{4\text{tree}}(-L_3,3,4,L_1) \bigg|_{\text{cut}},
\]

where \( L_3 = L_1 - k_1 - k_2 \), and the sum runs over all states crossing the cut. We label \( D \)-dimensional momenta with capital letters and four-dimensional ones with lower case. We apply the on-shell conditions \( L_1^2 = L_3^2 = 0 \) to the amplitudes appearing in the cut even though the loop momentum is unrestricted; only functions with a cut in the given channel under consideration are reliably computed in this way.

Complete amplitudes are found by combining all cuts into a single function with the correct cuts in all channels. If one works with an arbitrary dimension \( D \) in

![Figure 1: The two-particle cut at one loop in the channel carrying momentum \( k_1 + k_2 \).](image)
eq. (2), and takes care to keep the full analytic behavior as a function of $D$, then the results will be free of subtraction ambiguities that are commonly present in cutting methods [23,7,8]. (The regularization scheme dependence remains, of course.) An advantage of the cutting approach is that the gauge-invariant amplitudes on either side of the cut may be simplified before attempting to evaluate the cut integral [8].

3 Recycling Gauge Theory Into Gravity Loop amplitudes

As a relatively simple example, consider the one-loop amplitude with four identical helicity external gravitons and a scalar in the loop [2,3]. The cut in the $s_{12}$ channel is

$$\int \frac{d^D L_1}{(2\pi)^D} \frac{i}{L_1^2} M^\text{tree}_4(-L^s_1, 1^+_h, 2^+_g, L^s_3) \frac{i}{L_3^2} M^\text{tree}_4(-L^s_3, 3^+_h, 4^+_g, L^s_1) \bigg|_{\text{cut}},$$

where the superscript $s$ indicates that the cut lines are scalars and the subscript $h$ indicates that the external particles are gravitons. Using the KLT expressions [11] we may replace the gravity tree amplitudes appearing in the cuts with products of gauge theory amplitudes. The required gauge theory tree amplitudes, with two external scalar legs and two gluons, are relatively simple to obtain using Feynman diagrams and are,

$$A^\text{tree}_4(-L^s_1, 1^+_h, 2^+_g, L^s_3) = i \frac{\mu^2 [12]}{(12)} \left[ \frac{1}{(\ell_1 - k_1)^2 - \mu^2} + \frac{1}{(\ell_1 - k_2)^2 - \mu^2} \right],$$

$$A^\text{tree}_4(-L^s_1, 1^+_g, L^s_3, 2^+_g) = -i \frac{\mu^2 [12]}{(12)} \left[ \frac{1}{(\ell_1 - k_1)^2 - \mu^2} + \frac{1}{(\ell_1 - k_2)^2 - \mu^2} \right],$$

where $L_1 = \ell_1 + \mu$, where the subscript $g$ means the lines are gluons. The gluon momenta are four-dimensional, but the scalar momenta are allowed to have a $(-2\epsilon)$-dimensional component $\vec{\mu}$, with $\vec{\mu} \cdot \vec{\mu} = \mu^2 > 0$. The overall factor of $\mu^2$ appearing in these tree amplitudes means that they vanish in the four-dimensional limit, in accord with a supersymmetry Ward identity [15]. In the KLT relation [11], one of the propagators cancels, leaving

$$M^\text{tree}_4(-L^s_1, 1^+_h, 2^+_g, L^s_3) = -i \left( \frac{\mu^2 [12]}{\langle 12 \rangle} \right)^2 \left[ \frac{1}{(\ell_1 - k_1)^2 - \mu^2} + \frac{1}{(\ell_1 - k_2)^2 - \mu^2} \right].$$

By symmetry, the tree amplitudes appearing in any of the other cuts are the same up to relabelings. We then inserting these trees, with appropriate leg labels, into the cut (3).

After combining all three cuts into a single function that has the correct cuts in
all channels one obtains the one-loop graviton amplitude with a scalar in the loop,

\[ M_{4\text{-loop}}^{1}(1_{h}^{+}, 2_{h}^{+}, 3_{h}^{+}, 4_{h}^{+}) = 2 \frac{[1 2]^{2} [3 4]^{2}}{(1 2)^{2} (3 4)^{2}} \left( \mathcal{I}_{4\text{-loop}}^{1}[\mu^{8}](s, t) + \mathcal{I}_{4\text{-loop}}^{1}[\mu^{8}](s, u) + \mathcal{I}_{4\text{-loop}}^{1}[\mu^{8}](t, u) \right), \]

where \( s = s_{12}, \ t = s_{14}, \ u = s_{13} \) are the usual Mandelstam variables and

\[ \mathcal{I}_{4\text{-loop}}^{1}[\mathcal{P}](s, t) = \int \frac{d^{D}L}{(2\pi)^{D}} \frac{\mathcal{P}}{L^{2}(L - k_{1})^{2}(L - k_{1} - k_{2})^{2}(L + k_{4})^{2}} \]

is the scalar box integral depicted in fig. 2 with the external legs arranged in the order 1234. In eq. (4) the numerator \( \mathcal{P} \) is \( \mu^{8} \). The two other scalar integrals that appear correspond to the two other distinct orderings of the four external legs. The spinor factor \( [1 2]^{2} [3 4]^{2} / (\langle 1 2 \rangle^{2} \langle 3 4 \rangle^{2}) \) in eq. (4) is actually completely symmetric, although not manifestly so. By rewriting this factor and extracting the leading \( \mathcal{O}(\epsilon^{0}) \) contribution from the integral, the final one-loop \( D = 4 \) result after reinserting the gravitational coupling is

\[ M_{4\text{-loop}}^{1}(1_{h}^{+}, 2_{h}^{+}, 3_{h}^{+}, 4_{h}^{+}) = -i \frac{\kappa^{2}}{(4\pi)^{2}} \left( \frac{2}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 1 \rangle} \right)^{2} \left( s^{2} + t^{2} + u^{2} \right) \frac{1}{120}, \]

in agreement with a previous calculation \[25\].

4 **Maximal Supergravity**

Maximal \( N = 8 \) supergravity can be expected to be the least divergent of the four-dimensional supergravity theories due to its high degree of symmetry. Moreover, from a technical viewpoint maximally supersymmetric \( N = 8 \) amplitudes are by far the easiest to deal with in our formalism because of spectacular supersymmetric cancellations. For these reasons it is logical to re-investigate the divergence properties of this theory first \[1\]. It should be possible to apply similar methods to theories with less supersymmetry.

\[ \text{Figure 2: The one loop box integral.} \]
4.1 Cut Construction

Again we obtain supergravity amplitudes by recycling gauge theory calculations. For the $N = 8$ case, we factorize each of the 256 states of the multiplet into a tensor product of $N = 4$ super-Yang-Mills states. The key equation for obtaining the two-particle cuts is,

$$\sum_{N=8 \text{ states}} M^\text{tree}_4(-L_1,1,2,L_3) \times M^\text{tree}_4(-L_3,3,4,L_1) = s^2 \sum_{N=4 \text{ states}} A^\text{tree}_4(-L_1,1,2,L_3) \times A^\text{tree}_4(-L_3,3,4,L_1) \times \sum_{N=4 \text{ states}} A^\text{tree}_4(L_3,1,2,-L_1) \times A^\text{tree}_4(L_1,3,4,-L_3),$$

where we have suppressed the particle labels. The external labels are those for any particles in the supermultiplet, while the sum on the left-hand side runs over all states in the $N = 8$ super-multiplet. On the right-hand side the two sums run over the states of the $N = 4$ super-Yang-Mills multiplet: a gluon, four Weyl fermions and six real scalars. Given the corresponding $N = 4$ Yang-Mills two-particle sewing equation [9],

$$\sum_{N=4 \text{ states}} A^\text{tree}_4(-L_1,1,2,L_3) \times A^\text{tree}_4(-L_3,3,4,L_1) = -ist A^\text{tree}_4(1,2,3,4) \frac{1}{(L_1 - k_1)^2} \frac{1}{(L_3 - k_3)^2},$$

it is a simple matter to evaluate eq. (7), yielding

$$\sum_{N=8 \text{ states}} M^\text{tree}_4(-L_1,1,2,L_3) \times M^\text{tree}_4(-L_3,3,4,L_1) = istuM^\text{tree}_4(1,2,3,4) \left[ \frac{1}{(L_1 - k_1)^2} + \frac{1}{(L_1 - k_2)^2} \right] \left[ \frac{1}{(L_3 - k_3)^2} + \frac{1}{(L_3 - k_4)^2} \right].$$

The sewing equations for the $t$ and $u$ channels are similar.

A remarkable feature of the cutting equation (8) is that the external-state dependence of the right-hand side is entirely contained in the tree amplitude $M^\text{tree}_4$. This fact allows us to iterate the two-particle cut algebra to all loop orders! Although this is not sufficient to determine the complete multi-loop four-point amplitudes, it does provide a wealth of information.

Applying eq. (8) at one loop to each of the three channels yields the one-loop four
graviton amplitude of $N = 8$ supergravity,

$$\mathcal{M}_4^{1\text{-loop}}(1, 2, 3, 4) = -i\left(\frac{K}{2}\right)^4 stuM_4^{\text{tree}}(1, 2, 3, 4)$$

$$\times \left(\mathcal{I}_4^{1\text{-loop}}(s,t) + \mathcal{I}_4^{1\text{-loop}}(s,u) + \mathcal{I}_4^{1\text{-loop}}(t,u)\right),$$

in agreement with previous results \cite{26}. We have reinserted the gravitational coupling $\kappa$ in this expression. The scalar integrals are defined in eq. (5) with $\mathcal{P} = 1$.

At two loops, the two-particle cuts are given by a simple iteration of the one-loop calculation. The three-particle cuts can be obtained by recycling the corresponding cuts for the case of $N = 4$ super-Yang-Mills. It turns out that the three-particle cuts introduce no other functions than those already detected in the two-particle cuts. Combining all the cuts into a single function yields the $N = 8$ supergravity two-loop amplitude \cite{1}.

$$\mathcal{M}_4^{2\text{-loop}}(1, 2, 3, 4) = \left(\frac{K}{2}\right)^6 stuM_4^{\text{tree}}(1, 2, 3, 4)$$

$$\times \left(s^2 \mathcal{I}_4^{2\text{-loop},P}(s,t) + s^2 \mathcal{I}_4^{2\text{-loop},P}(s,u)ight.$$  

$$+ s^2 \mathcal{I}_4^{2\text{-loop},NP}(s,t) + s^2 \mathcal{I}_4^{2\text{-loop},NP}(s,u) + \text{cyclic}\right),$$

where ‘+ cyclic’ instructs one to add the two cyclic permutations of legs (2,3,4), and $\mathcal{I}_4^{2\text{-loop},P/\text{NP}}$ are depicted in fig. 3.

We comment that using the two-loop amplitude \cite{3}, Green, Kwon and Vanhove \cite{27} provided an explicit demonstration of the non-trivial M theory duality between $D = 11$ supergravity and type II string theory.

### 4.2 Divergence Properties of $N = 8$ Supergravity

Though a momentum cutoff scheme leads to a one-loop divergence for $N = 1$, $D = 11$ supergravity, in dimensional regularization there are no one-loop divergences

![Figure 3: The planar (a) and non-planar (b) scalar integrals, $\mathcal{I}_4^{2\text{-loop},P}(s,t)$ and $\mathcal{I}_4^{2\text{-loop},NP}(s,t)$, appearing in the two-loop $N = 8$ amplitudes. Each internal line represents a scalar propagator.](image-url)
in $D = 11$, so the first potential divergence in this theory is at two loops. Dimensional regularization is a rather convenient way to extract divergence properties as an analytic function of dimension, allowing us to directly relate properties of the $N = 1$, $D = 11$ supergravity to $N = 8$ $D = 4$ supergravity. (Some care is needed, however, to preserve supersymmetry [28].)

Since the two-loop $N = 8$ supergravity amplitude [9] has been expressed in terms of scalar $\phi^3$ loop momentum integrals, it is straightforward to extract the divergence properties. The scalar integrals diverge only for $D \geq 7$; hence the two-loop $N = 8$ amplitude is manifestly finite in $D = 5$ and 6, contrary to earlier expectations based on superspace power-counting arguments [19]. The discrepancy between the above explicit results and the earlier superspace power counting arguments is due to a previously unaccounted higher dimensional gauge symmetry. Once this symmetry is accounted for, superspace power counting gives the same degree of divergence as the explicit calculation [29].

The manifest $D$-independence of the cutting algebra allowed us to extend the calculation to $D = 11$, though there is no corresponding $D = 11$ super-Yang-Mills theory. The result [9] then explicitly demonstrates that $N = 1$ $D = 11$ supergravity diverges even when using dimensional regularization. The $D = 11$ two-loop divergence may be extracted from the amplitude in eq. (9) yielding [1] a non-vanishing counterterm. Further work on the structure of the $D = 11$ counterterm has been carried out in refs. [30].

Since the two-particle cut sewing equation iterates to all loop orders, one can compute all contributions which can be assembled solely from two-particle cuts [1]. Counting powers of loop momenta in these contributions suggests the simple finiteness formula,

$$L < \frac{10}{(D - 2)}, \quad \text{(with } L > 1\text{),}$$

where $L$ is the number of loops. This formula indicates that $N = 8$ supergravity is finite in some other cases where the previous superspace bounds suggest divergences [19], e.g. $D = 4$, $L = 3$. The first $D = 4$ counterterm detected via the two-particle cuts of four-point amplitudes occurs at five, not three loops. Further evidence that the finiteness formula is correct stems from the maximally helicity violating contributions to $m$-particle cuts, in which the same supersymmetry cancellations occur as for the two-particle cuts [1]. Moreover, a recent superspace power counting analysis taking the appropriate symmetries into account confirms the finiteness bound [29]. Further work would, however, be required to prove that there are no additional hidden cancellations which could improve the finiteness condition beyond eq. (10). Interestingly, there has been a suggestion by Chalmers that dualities might accomplish this [31].
5 Concluding Comments

There are also a number of other interesting open questions. For example, the methods described here have been used to investigate only maximal supergravity. It would be interesting to systematically re-examine the divergence structure of non-maximal theories. (Some interesting recent work on this may be found in ref. [32].) Using the methods described in this talk it might, for example, be possible to systematically determine finiteness conditions order-by-order in the loop expansion. A direct derivation of the Kawai-Lewellen-Tye decomposition of gravity amplitudes in terms of gauge theory ones starting from the Einstein-Hilbert Lagrangian perhaps might lead to a useful reformulation of gravity. Some initial steps to gain an understanding of the Kawai-Lewellen-Tye relations, starting from the Lagrangian was presented in ref. [4]. (See also ref. [33].) Connected with this is the question of whether the heuristic notion that gravity is the square of gauge theory can be given meaning outside of perturbation theory. In particular, an intriguing question is whether it is possible to relate more general solutions of the classical equations of motion for gravity to those for gauge theory.

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