A note on $\sigma$-algebras on sets of affine and measurable maps to the unit interval

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Abstract
In The factorization of the Giry monad Kirk Sturtz considers two $\sigma$-algebras on convex spaces of functions to the unit interval. One of them is generated by the Boolean subobjects and the other is the $\sigma$-algebra induced by the evaluation maps. The author asserts that, under the assumptions given in the paper, the two $\sigma$-algebras coincide. We give examples contradicting this statement.

1 Introduction
The notation and terminology of this paper is mostly based on that of [1]; Cvx is the category of convex spaces with affine maps, Meas is the category of measurable spaces with measurable maps and Set denotes the category of sets.

For a convex space $A$, the binary operations defining its convex structure will be denoted by $+_r$, $r \in [0, 1]$. A subset $E \subseteq A$ is said to be convex if it is closed under all $+_r$ and it is said to be a Boolean subobject of $A$ if both $E$ and its complement $E^c$ are convex. It is immediate that the empty set, $A$ and also the inverse image of a convex set under an affine map are all convex, and similarly for Boolean subobjects.

Depending on the context, the unit interval $I$ will be viewed either as an object of Meas or Cvx. When considered as a measurable space, the Borel $\sigma$-algebra will be assumed. As a convex space, it is endowed with the natural convex structure given by

$$x +_r y = (1 - r)x + ry \quad \forall r \in [0, 1].$$

The Boolean subobjects of $I$ are exactly the empty set and the convex sets containing 0 or 1, i.e., the intervals $[0, x)$, $[0, x]$, $(x, 1]$ and $[x, 1]$ for all $x \in I$.

2 The measurable structure of a convex space
For a convex space $A$, the Boolean $\sigma$-algebra on $A$ is by definition the smallest $\sigma$-algebra containing all Boolean subobjects of $A$, and is denoted by $\Sigma_{bool}^A$. 
In particular, the Boolean σ-algebra on \( I \) coincides with its Borel σ-algebra. Note that for arbitrary convex spaces \( A \) and \( B \), every affine map \( A \to B \) is measurable w.r.t. the Boolean σ-algebras on \( A \) and \( B \).

For an arbitrary set \( A \), \( \text{Set}(A, \mathbb{I}) \) may be turned into a convex space by defining \( +_r \) pointwise, thus for \( f, g \in \text{Set}(A, \mathbb{I}) \) and \( a \in A \) we have

\[
(f +_r g)(a) = f(a) +_r g(a) \quad \forall r \in [0, 1].
\]

We are concerned with convex subsets \( A \subseteq \text{Set}(A, \mathbb{I}) \). In addition to the Boolean σ-algebra, \( A \) may be endowed with the evaluation σ-algebra, which is induced by the evaluation maps \( \text{ev}_a : A \to \mathbb{I}, a \in A \), where \( \text{ev}_a(f) = f(a) \). The evaluation σ-algebra will be denoted by \( \Sigma^A_{\text{ev}} \). Since the evaluations are affine, it follows that

\[
\Sigma^A_{\text{ev}} \subseteq \Sigma^A_{\text{bool}}.
\]

### 3 Overview of refuted statements

Let \( A \) be a convex space, and let \( \Sigma A = (A, \Sigma^A_{\text{bool}}) \). In [1] the author deals with the Boolean and evaluation σ-algebras on \( \text{Cvx}(A, \mathbb{I}) \) and \( \text{Meas}(\Sigma A, \mathbb{I}) \), and asserts:

- **Lemma 5.1.** For any convex space \( A 
  \)

\[
\Sigma_{\text{ev}}^{\text{Cvx}}(A, \mathbb{I}) = \Sigma_{\text{bool}}^{\text{Cvx}}(A, \mathbb{I}) \quad \text{(ks-5.1a)}
\]

\[
\Sigma_{\text{ev}}^{\text{Meas}}(\Sigma A, \mathbb{I}) = \Sigma_{\text{bool}}^{\text{Meas}}(\Sigma A, \mathbb{I}) \quad \text{(ks-5.1b)}
\]

- **Corollary 5.2.** For any affine map \( P : \text{Meas}(\Sigma A, \mathbb{I}) \to \mathbb{I} \n  \)

\[
P \text{ is measurable w.r.t. } \Sigma_{\text{ev}}^{\text{Meas}}(\Sigma A, \mathbb{I}) \quad \text{(ks-5.2)}
\]

- **Lemma 5.3.** For any measurable space \( X 
  \)

\[
\Sigma_{\text{ev}}^{\text{Meas}}(X, \mathbb{I}) = \Sigma_{\text{bool}}^{\text{Meas}}(X, \mathbb{I}) \quad \text{(ks-5.3)}
\]

We will show that taking the unit interval for \( A \) contradicts (ks-5.2), and hence (ks-5.1b), and thus, for \( X = \Sigma A \), disproves (ks-5.3) as well; it is, however, in accordance with (ks-5.1a). Next we show that the free convex space over an uncountable set contradicts (ks-5.1a) and (ks-5.1b).
4 Main results

Throughout this section, \( A \) is a convex space, and \( \mathcal{A} \) is a subset of \( \text{Set}(A, \mathbb{I}) \).

For a function \( f \) and a set \( u \), \( f\mid_u \), \( f^\leftarrow(u) \) and \( f^\rightarrow(u) \) denote the restriction, image and inverse image, respectively.

Let us define \( \mathcal{E} \subseteq \mathcal{P}(\mathcal{A}) \) by
\[
\mathcal{E} = \{ E \subseteq A : (\exists u \in [A]^{\leq \omega}) (\forall f \in E)(\forall g \in \mathcal{A})(f\mid_u = g\mid_u \implies g \in E) \},
\]
where \( [A]^{\leq \omega} \) denotes the set of countable subsets of \( A \). Then \( \mathcal{E} \) has the following properties:

1. \( \emptyset \in \mathcal{E}, \ A \in \mathcal{E} \);
2. \( \mathcal{E} \) is closed under countable unions and intersections;
3. for \( K \subseteq \mathbb{I} \) and \( a \in A \), \( ev_a^\leftarrow(K) \in \mathcal{E} \).

Proof. (1) is obvious. Regarding (2), let \( C \subseteq \mathcal{E} \) be countable. For each \( E \in C \) fix \( u_E \) such that
\[
(\forall f \in E)(\forall g \in \mathcal{A})(f\mid_{u_E} = g\mid_{u_E} \implies g \in E)
\]
and employ \( u = \bigcup\{u_E : E \in C\} \), which is countable, to show that both \( \bigcup C \) and \( \bigcap C \) belong to \( \mathcal{E} \). The proof of (3) is obvious on setting \( u = \{a\} \).

Now it is immediate that \( \mathcal{S} = \{ E \subseteq A : E \in \mathcal{E} \lor E^c \in \mathcal{E} \} \) is a \( \sigma \)-algebra and from \( (ev_a^\leftarrow(K))^c = ev_a^\rightarrow(K^c) \) it follows that \( ev_a^\leftarrow(K) \in \mathcal{S} \). We conclude that \( \Sigma^A_{ev} \subseteq \mathcal{S} \), which brings us to

Lemma 1. For every convex space \( A \), and every \( \mathcal{A} \subseteq \text{Set}(A, \mathbb{I}) \):
\[
\Sigma^A_{ev} \subseteq \mathcal{E}.
\]

We have already mentioned that \( \Sigma^\mathbb{I} = \mathbb{I} \). From the following two lemmas it follows that \( A = \mathbb{I} \) provides a counterexample to (ks-5.1b), though it is in accordance with (ks-5.1a).

Lemma 2. Let \( P : \text{Meas}(\mathbb{I}, \mathbb{I}) \to \mathbb{I} \) map every function of \( \text{Meas}(\mathbb{I}, \mathbb{I}) \) to its Lebesgue integral. Then \( P \) is affine but not measurable with respect to the evaluation \( \sigma \)-algebra on \( \text{Meas}(\mathbb{I}, \mathbb{I}) \).

Proof. We apply Lemma 1 for \( A = \mathbb{I} \) and \( \mathcal{A} = \text{Meas}(\mathbb{I}, \mathbb{I}) \) to show that \( P^{-1}(0) \notin \Sigma^\text{Meas}(\mathbb{I}, \mathbb{I})_{ev} \), from which the conclusion of the lemma follows. For any countable subset \( u \subseteq \mathbb{I} \), consider the zero constant function \( \emptyset \in P^{-1}(0) \), and the characteristic function \( \chi_u^\varnothing \in \text{Meas}(\mathbb{I}, \mathbb{I}) \). Then \( \emptyset\mid_u = \chi_u^\| \mid_u \), but \( P(\chi_u^\varnothing) = 1 \), thus \( \chi_u^\varnothing \notin P^{-1}(0) \). Therefore, \( P^{-1}(0) \notin \mathcal{E} \).
Lemma 3. The Boolean and evaluation $\sigma$-algebras coincide for $\text{Cvx}(I, I)$.

Proof. The affine map $\Phi : \text{Cvx}(I, I) \to I \times I$ given by

$$\Phi(f) = (f(0), f(1))$$

is an isomorphism of convex spaces. Under this isomorphism, the evaluations $ev_0$ and $ev_1$ correspond to the projections of the product, and it is straightforward that $\Sigma^{\times I}_I$ coincides with the usual Borel $\sigma$-algebra on $I \times I$. Thus, given a Boolean subobject $E$ of $I \times I$, our business is to show that $E$ is a Borel set.

Recall that for every pair of elements $p, q \in I \times I$ there is a unique affine map $\pi_{p,q} : I \to I \times I$ such that $\pi_{p,q}(0) = p$ and $\pi_{p,q}(1) = q$, which is given explicitly by

$$\pi_{p,q}(r) = p + r q.$$

From the Closed map lemma it follows that $\pi_{p,q}$ is a closed map, and from this it is clear that the image of every convex subset of $I$ under $\pi_{p,q}$ is a Borel subset of $I \times I$.

Let us now consider the set $\pi_{p,q}^-(E)$. As it is a Boolean subobject of $I$, its image under $\pi_{p,q}$ is a Borel set; we will denote this image by $E_{p,q}^-$. In the sequel, we will employ the following fact:

$$E_{p,q}^- \subseteq E \quad \& \quad \pi_{p,q}^-(I) \setminus E_{p,q}^- \subseteq E^c.$$

If $p \in E$, then $0 \in \pi_{p,q}^-(E)$; the supremum of $\pi_{p,q}^-(E)$ will be called the dividing point determined by $\pi_{p,q}$. The dividing point determined by $\pi_{p,q}$, denoted for the moment by $u$, is characterized by the following property:

$$\pi_{p,q}^-(0, u) \subseteq E \quad \& \quad \pi_{p,q}^-(u, 1) \subseteq E^c.$$

Now that we are through the preliminaries, we will distinguish four cases, depending on how many corners of $I \times I$ belong to $E$ and $E^c$.

1. If $E$ does not contain any corner, then all of them are contained in $E^c$, and we have $E^c = I \times I$, so that $E = \emptyset$, hence $E$ is a Borel set.

2. Suppose $E$ contains one corner and $E^c$ contains the remaining three corners. We proceed for $(0, 0) \in E$, the other cases are analogous. Let $u$ and $v$ be the dividing points determined by $\pi_{(0,0),(1,0)}$ and $\pi_{(0,0),(1,0)}$, respectively. Then it is readily checked that

$$E = \{(x, y) \in I \times I : uy + vx < uv\} \cup E_{(0,v), (u,0)},$$

thus $E$ is a Borel set.
(3) Suppose $E$ contains two neighboring corners and $E^c$ contains the other two corners. We proceed for $(0,0), (1,0) \in E$, the other cases are analogous. Let $u$ and $v$ be the dividing points determined by $\pi(0,0), (0,1)$ and $\pi(1,0), (1,1)$, respectively. Then we have

$$E = \{(x,y) \in \mathbb{I} \times \mathbb{I} : y < (1-x)u + xv\} \cup E_{(0,u),(1,v)},$$

thus $E$ is a Borel set.

(4) Suppose $E$ contains two opposite corners. Then it must also contain at least one of the remaining corners, because otherwise the point $(\frac{1}{2}, \frac{1}{2})$ would belong to both $E$ and $E^c$, a contradiction. But then we deal with a complementary case to (1) or (2).

The next result builds on a free convex space $A$ over an (arbitrary) uncountable set $M$. Recall that $A$ is the set of maps $\alpha \in \text{Set}(M, \mathbb{I})$ with $\alpha^{-1}(0)$ cofinite in $M$ and $\sum_{m \in M} \alpha(m) = 1$. $A$ is endowed with the pointwise convex structure. As a consequence of the forthcoming lemma we have

1. $\Sigma^\text{Cvx}(A,\mathbb{I}) \subseteq \Sigma^\text{bool}(A,\mathbb{I})$, which contradicts (ks-5.1a);
2. $\Sigma^\text{Meas}(A,\mathbb{I}) \subseteq \Sigma^\text{bool}(A,\mathbb{I})$, which contradicts (ks-5.1b).

**Lemma 4.** Let $\emptyset$ be the zero constant function $A \rightarrow \mathbb{I}$. For every convex subset $A \subseteq \text{Set}(A,\mathbb{I})$ that includes $\text{Cvx}(A,\mathbb{I})$ we have

$$\{\emptyset\} \in \Sigma^A \setminus \Sigma^A_{\text{ev}}.$$

**Proof.** Convexity of $\{\emptyset\}$ and its complement is facile, we focus on $\{\emptyset\} \notin \Sigma^A_{\text{ev}}$. For a countable subset $u \subseteq A$ let $M_0 = \bigcap \{\alpha^{-1}(0) : \alpha \in u\}$, and define $g : A \rightarrow \mathbb{I}$ by

$$g(\alpha) = \sum_{m \in M_0} \alpha(m).$$

Then $g$ is affine, thus $g \in A$. For $\alpha \in u$ we have $g(\alpha) = 0$, so that $g|_u = \emptyset|_u$. However, $g \neq \emptyset$. To see this, observe that $M_0$ is non-empty as it is a countable intersection of cofinite subsets of $M$, which has been chosen to be uncountable. For an arbitrary $m_0 \in M_0$ we take the unique $\alpha_0 \in A$ with $\alpha_0(m_0) = 1$, hence $g(\alpha_0) = 1$. This proves $g \notin \{\emptyset\}$ and, as $u$ was arbitrary, we conclude that $\{\emptyset\} \notin \mathcal{E}$. Now apply Lemma 1 to show $\{\emptyset\} \notin \Sigma^A_{\text{ev}}$. ■

**References**

[1] Kirk Sturtz, *The factorization of the Giry monad* (2018), available at arXiv:1707.00488v2[math.CT].