Perturbative 3-manifold invariants by cut-and-paste topology

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We give a purely topological definition of the perturbative quantum invariants of links and 3-manifolds that were originally defined by Kontsevich [20] and that are associated with Chern-Simons field theory. Our definition is as close as possible to one given by Kontsevich. We will also establish some basic properties of these invariants, in particular that they are universally finite type in the expected way. The purpose of this article is to give a purely topological definition of the perturbative quantum invariants of links and 3-manifolds (an association with Chern-Simons field theory. Our definition is as close as possible to one given by Kontsevich [20] and that are associated with Chern-Simons field theory. We will also establish some basic properties of these invariants, in particular that they are universally finite type in the expected way. The main difference between our treatment and that of Kontsevich is that we will use cohomology rather than differential forms and the pairing between homology and cohomology rather than integration. We will define the perturbative invariants of degree \(n\) of a given closed, framed rational homology 3-sphere \(M\) as the degree of a generalized Gauss map

\[ \Phi : X_n \to Y_n, \]

where \(X_n\) and \(Y_n\) are modified configuration spaces that are constructed from \(M\) using cut-and-paste topology. Otherwise we will follow Kontsevich closely, since his definition is terse but essentially rigorous.

The purely topological approach was first considered by Bott and Taubes [8], and later by the second author [30], and others [2, 26]. The definition given there generalized the Gauss map

\[ \Phi : K_1 \times K_2 \to S^2, \]

whose degree is the linking number between two knots \(K_1\) and \(K_2\) in \(\mathbb{R}^3\), to other maps whose degrees give all of the Vassiliev invariants of knots and links. Our maps can also be defined for links in 3-manifolds, a generalization which we will discuss later.

More precisely, we will construct an invariant

\[ \Phi : (C_n, D) \to (P \times 3^n, Q), \]

\[ \Phi^* : H^{6n}(P \times 3^n, Q) \to H^{6n}(C_n, D; Q), \]

where \(6n\) is the degree of the top non-vanishing rational cohomology (or homology) of certain spaces \(C_n\) and \(P \times 3^n\) that depend on \(M\), and \(Q\) and \(D\) are certain degenerate loci associated with the infinite asymptote in an asymptotically flat model of \(M\). The space \(P\) has a generating class

\[ \alpha \in H^2(P; \mathbb{Q}) \]

called a propagator. The space \(C_n\) is defined using the combinatorics of Jacobi diagrams. The space \(V_n^\ast\) of primitive weight systems of degree \(n\) embeds in the homology space \(H_{6n}(C_n; \mathbb{Q})\). If \(w \in V_n^\ast\) is a weight system, let \(\mu_w\) be a corresponding cycle. Then we can define an invariant

\[ I_w(M) = \langle w, \Phi^*(\alpha^{\otimes 3n}) \rangle = \langle \Phi_\ast(w), \alpha^{\otimes 3n} \rangle \]

depending on a weight system. Dually, we can define a universal invariant (in a sense given below) as an element \(I_n(M) \in V_n^\ast\).

**Theorem 1.** The invariant \(I_n(M)\) of framed rational homology spheres is additive under connected sums:

\[ I_n(M_1 \# M_2) = I_n(M_1) + I_n(M_2). \]

In particular, \(I_n(S^3) = 0\) (if the modified tangent bundle \(T'S\) as defined in Section 43 is given the canonical framing).

**Theorem 2.** The invariant \(I_n(M)\) is a finite-type invariant of degree \(n\) in both the algebraically split and Torelli senses for framed rational homology spheres \(M\), and it is universal for integer homology spheres.

The phrase “finite-type invariant” merits some explanation. In general, suppose that \(\mathcal{M}\) is some set of topological objects with the structure of a cubical complex \(C\). Certain pairs of elements are connected by edges, certain pairs of pairs form squares, and so on. Then a function \(I\) on \(\mathcal{M}\) (a topological invariant) taking values in an abelian group extends to \(C\) by taking alternating sums, or repeated finite differences. For example, if \(M_0\), \(M_1\), \(M_2\), and \(M_{12}\) form a square \(C\), then we can define

\[ I''(C) = I(M_0) - I(M_1) - I(M_2) + I(M_{12}). \]
In this general context an invariant \( I \) is \textit{finite-type} of order \( n \) if the \( n+1 \)st order finite difference \( I^{(n+1)} \) vanishes. Another view is to interpret \( S \) as a symbol for a formal linear combination

\[
S = M_0 - M_1 - M_2 + M_{12}
\]

and then extend \( I \) linearly. In this interpretation, we define \( \mathcal{M}_n \) to be the span (in the space of rational linear combinations) of all \( n \)-cells of \( C \).

As a motivating example, let \( \mathcal{M}_n \) be the set of \( n \)-dimensional parallelepipeds on a vector space (i.e., a collection of \( n \) vectors together with a base point). Then the functions on the vector space that satisfy the above definition of finite-type of degree \( n \) are the polynomials of degree \( n \).

In our case, \( \mathcal{M} \) is the set of homeomorphism types of oriented rational homology 3-spheres, and the cubes in \( C \) are defined in one of two ways: The vertices may be connected by surgery on sublinks of an algebraically split link, or by Torelli surgery on subsets of a collection of disjoint handlebodies. Here an algebraically split link is a framed link whose linking matrix is the identity; at the end of Section 6.4 we will also consider a rational generalization. A \textit{Torelli surgery} is the operation of removing a handlebody from \( M \) and gluing it back after applying an element of the Torelli group to the boundary. (The Torelli group of a surface is the subgroup of the mapping class group that acts trivially on homology.) Algebraically split surgery was defined by Ohtsuki \cite{25}, while Torelli surgery generalizes both blink surgery as defined by Garoufalidis and Levine \cite{15} and clasper surgery as defined by Habiro \cite{17}. Garoufalidis and Levine \cite{15} showed that these two notions of finite type are equivalent to each other for integer homology spheres. Moreover, they showed that there is a surjection

\[
\kappa : V_n \to \mathcal{M}_{kn} / \mathcal{M}_{kn+1},
\]

where \( k = 3 \) in the algebraically split case and \( k = 2 \) in the Torelli case. A finite-type invariant is \textit{universal} if its finite difference of order \( kn \) is a right inverse

\[
I^{(kn)} : \mathcal{M}_{kn} \to V_n,
\]

thereby showing that the map \( \kappa \) is an isomorphism. We will argue universality directly in both cases. (Note that for unframed 3-manifolds, \( \mathcal{M}_{kn+j} = \mathcal{M}_{kn+j+1} \) when \( k \) does not divide \( j \). The framed theory is the same except that \( \mathcal{M}_1 \) is 1-dimensional and detects change of framing.)

Finally in Section 6.5 we will prove the following theorem.

**Theorem 3.** There is an invariant \( \delta_n(M) \in V_n \) of homology 3-spheres \( M \) decorated with a framing or bundled bordism such that the difference

\[
\tilde{I}_n(M) = I_n(M) - \delta_n(M)
\]

is independent of the decoration. Moreover, the framing correction \( \delta_n(M) \) is finite type of degree 1 in the Torelli and algebraically split senses.

In particular, the unframed invariant \( \tilde{I}_n(M) \) is also universal.

It is also known \cite{15} that surgery on boundary links (links whose components admit disjoint Seifert surfaces) again gives an equivalent finite-type theory for integer homology spheres. Thus Theorem 2 has the following corollary.

**Corollary 4.** The invariant \( I_n(M) \) is universally finite type of order \( n \) for boundary link surgery in the class of integer homology spheres.

We do not yet have a direct proof of Corollary 4. We also have the following closely related conjectures.

**Conjecture 5.** The unframed invariant

\[
\omega(M) = \sum_n m^n \tilde{I}_n(M),
\]

where \( m = |H_1(M; \mathbb{Z})| \), equals the surgery-defined invariant of Le, Murakami, and Ohtsuki \cite{23}.

Conjecture 3 asserts that \( \tilde{I}_n \) satisfies the Le-Murakami-Ohtsuki surgery formula. At the moment, we can only compute appropriate finite differences to find the highest order term, analogous to the leading coefficient of a polynomial. Since both invariants are universal, Conjecture 5 holds to highest order.

**Conjecture 6.** The framing correction \( \delta_n(M) \) vanishes for \( n > 1 \).

By a remark in Section 5.1.2, Conjecture 6 holds for \( n \) even.

### 1.1 Related work and further directions

These definitions and results generalize to arbitrary rational homology spheres and to links in rational homology spheres. One interesting variant that we have not analyzed is the definition of Axelrod and Singer \cite{3, 4}, further developed by Bott and Cattaneo \cite{9, 10}. The main difference between that definition and the one due to Kontsevich (and ours) is that Kontsevich punctures the 3-manifold \( M \) so that the space of pairs of distinct points in \( M \) (the building block of the space \( P \) above) is a homology 2-sphere, while Axelrod and Singer “smear out” the puncture using a volume form. These variations were considered in more detail by Cattaneo \cite{8}. In this article we use a compact version of Kontsevich’s space, denoted \( C_{c, \infty}(M) \); without puncturing it would be just \( C_c(M) \). Algebraically, we need to know that

\[
H^2(C_{c, \infty}(M); \mathbb{Q}) \cong \mathbb{Q}.
\]

Following Axelrod and Singer, one could, without puncturing \( M \), choose a propagator

\[
\alpha \in Z^2(C_c(M); \mathbb{Q})
\]
such that the coboundary satisfies
\[ \delta \alpha = \mu \otimes 1 + 1 \otimes \mu, \]
where \( \mu \) is a cocycle in \( \mathbb{Z}^3(M) \).

Taubes \[27, 28\] defines and studies an invariant that is very close to the invariant \( I_1(M) \) that we define, using the canonical framing of a rational homology 3-sphere. He finds that his quantity is invariant under spin cobordism, implying that it is trivial for integer homology 3-spheres. On the other hand, Theorems 3 and 4 imply that our invariant is the Casson invariant. (A standard relation for the Casson invariant \[1\] implies that it is finite type of degree 3 in the algebraically split sense; on the other hand the space of invariants of this degree is 1-dimensional \[15\].) We have no explanation for the discrepancy, but we plan to consider the invariant \( I_1(M) \) in more detail in a future article \[13\].

Two other generalizations that can be considered are invariants of graphs in 3-manifolds, and invariants associated to other flat connections \[3\]. We will analyze these in future work. Among other things, there should be a general relation between flat bundles and links in 3-manifolds on the one hand and finite covers and branched covers on the other hand \[13\].

Kontsevich has discussed yet other generalizations. There should be corresponding invariants for a higher-dimensional smooth, framed manifold \( M \) which produce certain characteristic classes of an \( M \)-bundle over another topological space \[22\]. Our analysis may extend to these invariants. (Although the methods are still combinatorial, they do use the tangent bundle, so it’s not clear if the invariants would descend to PL invariants.) More recently \[27\], he explained that all perturbative invariants are examples of homotopy functors from a certain category of coordinate patches in \( M \).

A more exotic possible generalization would be to pass from three real dimensions to three complex dimensions. It is possible that the holomorphic cohomology of a Calabi-Yau 3-fold has all of the necessary properties to generalize the definition of the invariant \( I_n \).

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2 HOMOLOGICAL CONVENTIONS

None of the ideas in this article depend in any fundamental way on the model of homology used: De Rham, singular, simplicial/cellular, Čech, etc. For concreteness it is convenient to use simplicial homology with coefficients in \( \mathbb{Q} \) and with unspecified triangulations. We will henceforth omit the coefficients. (Note that most of the constructions would work identically with arbitrary coefficients.)

Recall that the cup product in simplicial homology depends on an ordering of the vertices, and that it is not graded commutative on the level of chains. This deficiency can be ameliorated when working over \( \mathbb{Q} \) or any other coefficient ring that contains \( \mathbb{Q} \). Namely, we can average over the \((a + b + 1)!\) orderings of the vertices of each \( a + b \)-simplex when taking the cup product of an \( a \)-cochain and a \( b \)-cochain. If such a cochain is a cocycle, then it can be represented by a differential form which is constant on each simplex, and a cup product is then identically equal to the corresponding wedge product. In this sense, simplicial cohomology is a kind of “mock De Rham cohomology”.

The degenerate locus \( Q \) is constructed as a semi-algebraic set rather than with a cut-and-paste method. Hence it does not a priori have simplicial homology. A foundational result of Hironaka states that semi-algebraic sets can be ambiently triangulated, and the simplices of such a triangulation can be straightened \[18, 19\].

We will need the following extension lemma, which is elementary in the setting of simplicial cohomology.

Lemma 7. If \( K \) is a subcomplex of a simplicial complex \( L \), and if a cohomology class \( \alpha \in H^*(K) \) extends to a class \( \beta \in H^*(L) \), then any simplicial cocycle in \( K \) representing \( \alpha \) extends to a cocycle in \( L \) representing \( \beta \).

3 JACOBI DIAGRAMS

In this section we review the definition of different kinds of Jacobian diagrams, which are also variously called chord diagrams, Chinese characters, Chinese character diagrams, and Feynman diagrams. Technically we will need this formalism only much later (in Lemma 8 and Section 5), but we present it here as a fundamental preliminary.

3.1 Parity functors

Let \( P \) be the category of two-element sets in which morphisms are bijections; it has a natural tensor product operation if you view it as the category of affine spaces over the multiplicative group \( \{1, -1\} \). (More concretely: The identity map from any object of \( P \) to itself is called 1 and the other map is called \(-1\). If \( A = \{a, b\} \) and \( X = \{x, y\} \) are in \( P \), then \( A \otimes X \) has the two elements \( \{(a, x), (b, y)\} \) and \( \{(b, x), (a, y)\} \).) A parity functor is a functor from some other category with invertible morphisms to \( P \). For example, let \( A(S) \) be the set of sign-orderings of a finite set \( S \), i.e., the set of linear orderings quotiented by the action of the alternating group \( \text{Alt}(S) \). \( A \) and the orientation functor for finite-dimensional vector spaces are the two most commonly used non-trivial parity functors. (Arguably the trivial functor 1, a special case of which is defined below, is even more commonly used.)

Let \( G \) be the category of connected, finite graphs \( \Gamma \) (multiple edges and loops are allowed) in which the morphisms are
graph isomorphisms. One can consider the following parity functors on $\mathcal{G}$:

$1(\Gamma)$ is the trivial functor that takes every graph to $\{1, -1\}$.
$A(\Gamma)$ is the set of sign-orderings of the edges of $\Gamma$.
$B(\Gamma)$ is the set of sign-orderings of the odd-valence vertices.
$C(\Gamma)$ is the set of sign-orderings of the even-valence vertices.
$D(\Gamma)$ is the set of orientations of all edges, quotiented by the operation of negating any two.
$E(\Gamma)$ is the set of sign-orderings of the flags (pairs consisting of an edge and one of its vertices).
$F(\Gamma)$ is the set of sign-orderings of the edges incident to each vertex, up to negating any two sign-orderings.
$G(\Gamma)$ is the set of sign-orderings of all vertices, equivalently the set of orientations of the vector space of simplicial 0-chains $Z_0(\Gamma; \mathbb{R})$.
$H(\Gamma)$ is the set of orientations of the vector space of 1-chains $Z_1(\Gamma; \mathbb{R})$.
$I(\Gamma)$ is the set of orientations of $H_1(\Gamma; \mathbb{R})$.

These functors, modulo isomorphism of functors (via natural transformations), generate an abelian group with exponent 2 (since $X \otimes X \cong 1$ for any parity functor $X$) with the relations:

$E \cong D \quad H \cong A \otimes E$
$F \cong B \otimes E \quad I \cong G \otimes H$
$G \cong B \otimes C$

For example, the functors $D$ and $E$ are isomorphic as follows: An orientation of an edge $e$ of a graph $\Gamma$ can be expressed as an ordering of the two flags that include $e$. Listings the edges $e_1, e_2, \ldots, e_n$ in any order, we get an ordering of the flags $f_{1,1}, f_{1,2}, f_{2,1}, f_{2,2}, f_{3,1}, f_{3,2}, \ldots, f_{n,2}$, where the flags of the edge $e_i$ are ordered $(f_{i,1}, f_{i,2})$. The sign of this ordering of the flags does not depend on the ordering of the edges, establishing a canonical isomorphism $D(\Gamma) \cong E(\Gamma)$. We leave the other relations as an exercise.

Each parity functor determines a homomorphism

$\text{Aut}(\Gamma) \rightarrow \{1, -1\}$.

There are choices for $\Gamma$ for which $A$, $B$, $C$, and $D$ induce independent homomorphisms, for example the one in Figure 1.

Thus, no further relations are possible. The parity functors listed above can be expressed in terms of the four generators according to Table 1.

| A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|
| A |   |   |   |   |   |   |   |   |
| B |   |   |   |   |   |   |   |   |
| C |   |   |   |   |   |   |   |   |
| D |   |   |   |   |   |   |   |   |

TABLE 1: Nine parity functors in terms of four generators.

On the subcategory of $\mathcal{G}$ of odd-valence graphs, $C \cong 1$, but $A$, $B$, and $D$ remain independent.

We define a Lie orientation of $\Gamma$ to be an element of $(D \otimes G)(\Gamma)$. This parity functor is naturally associated to invariants and characteristic classes of odd-dimensional manifolds. In the association between graph homology and the twisted equivariant homology of “outer space” $[1, 20]$, the isomorphic parity functor $A \otimes I$ appears. The parity functor $F \otimes C$ is also isomorphic; Bar-Natan $[3]$ defines Lie orientations in the odd-valence case using just $F$. Note that the parity functor $A$ leads to the other kind of graph homology; it corresponds to the untwisted equivariant homology of outer space and to configuration spaces on even-dimensional manifolds.

### 3.2 Diagrams and relations

A closed Jacobi diagram is a Lie-oriented graph $\Gamma$ with trivalent vertices. (A non-closed Jacobi diagram may also have univalent vertices.) A closed diagram has $2n$ trivalent vertices if and only if it has $n + 1$ loops, where the loop number is just the first Betti number of the diagram. The Vassiliev space $V_n$ is the vector space over $\mathbb{Q}$ of isomorphism classes of connected Jacobi diagrams with $n + 1$ loops, modulo the Jacobi relation (also called the $IHX$ relation):

\[
\begin{array}{c}
\text{Figure}
\end{array}
\]

This is a linear relation among any three graphs that are the same except at the indicated subgraphs. The edges incident to each vertex are cyclically ordered clockwise in the diagram.
This is an $F$-orientation (the \textit{blackboard} orientation), which by previous considerations is equivalent to a Lie orientation for trivalent graphs.

We will also consider dual vectors $w \in V_n^*$, which are called \textit{primitive weight systems}.

\textit{Remark.} The IHX relation is compatible with many kinds of decorations on Jacobi diagrams. The edges may be ordered; the homology or the fundamental group may have distinguished elements or other decorations; there may be univalent vertices which may or may not be labelled; and the diagram may be attached to a link or a graph. These decorations are important for generalizations of the invariant $I_n(M)$ and for analyzing Vassiliev spaces, but in this article we only need the simplest of all Vassiliev spaces.

\textit{Example.} The spaces $V_n$ are 1-dimensional for $n = 1, 2$. For $n = 1$ there is only one diagram, the theta graph:

\begin{center}
\includegraphics[width=0.1\textwidth]{theta_graph.png}
\end{center}

For $n = 2$ there are two, a double theta and a tetrahedron, and the former is twice the latter:

\begin{center}
\includegraphics[width=0.15\textwidth]{theta_tetrahedron.png}
\end{center}

As above, we assume the blackboard orientation in this equation.

4 \ \textit{CONFIGURATION SPACES}

In this section we will define a certain compactification of the configuration space of maps from the vertices of a graph $\Gamma$ to a manifold $M$ such that vertices connected by an edge are distinct. The idea is to blow up diagonals corresponding to the edges in the space of all maps $M^\Gamma$. This is more complicated than one might expect, since these diagonals are not mutually transverse. We will rely on a general construction for resolving non-transverse blowups of this type.

4.1 Blowups: The balls, beams, and plates construction

In this section we will discuss blowing up a manifold $M$ along a general type of closed subset $X$ called a Whitney-stratified space \cite{16, 23}. By virtue of its Whitney-stratification, $X$ decomposes into a locally finite, partially ordered set of smoothly embedded manifolds,

\[ X = \bigcup_{i \in S} X_i. \]

The decomposition and the partial ordering are compatible according to the condition that

\[ i \prec j \iff X_i \subset \overline{X}_j \iff X_i \cap \overline{X}_j \neq \emptyset \]

for $i \neq j$. In our case, we additionally require that $X$ is locally smoothly equivalent to a cone over another Whitney-stratified space; i.e., for each $p \in X$ there is a tangent cone $T_pX$. We call such a Whitney-stratified space \textit{cone-like}; one which is not conelike can have cusps and other singularities in which strata kiss.

We will need a generalization of this definition which we call a \textit{Whitney-stratified immersion}. As before, $X$ decomposes into smoothly embedded manifolds, and we assume that

\[ i \prec j \iff X_i \subset \overline{X}_j. \]

But the third condition, that $X_i$ and $\overline{X}_j$ are disjoint if $i$ and $j$ are incomparable, is replaced by two weaker conditions:

1. Each $\overline{X}_i$ is a union of strata.

2. If $i_1, \ldots, i_n$ are an anti-chain, then the corresponding strata $X_{i_1}, \ldots, X_{i_n}$ are mutually transverse.

Here an \textit{anti-chain} is a pairwise incomparable set.

If $M$ is a manifold with a cone-like, Whitney-stratified immersion $X$, there is a way to blow up $M$ along $X$. It is convenient (but not strictly necessary) to give $M$ a Riemannian metric. The blowup $B_X(M)$ is formed by successively blowing up $X_i$ as $i$ increases. This means that we replace each $p \in \overline{X}_i$ by the set of rays in $T_p(M)$ which are normal to $T_p(X_i)$; here $\overline{X}_i$ is the closure of $X_i$ in the partially blown up model of $M$. If some strata in an anti-chain intersect transversely, then their blowups commute, so they can be performed in either order.

The result $B_X(M)$ is a smooth manifold with right-angled corners: a manifold locally diffeomorphic to a closed cube. It has a codimension 1 face $F_i$ for each $i$, and the interior of $F_i$ blows down to the open stratum $X_i$. Lower-dimensional faces correspond to flags (ordered chains) of strata. This is topologically and combinatorially equivalent to the complement of an open regular neighborhood of $X$. The latter is also called the “balls, beams, and plates” construction when it appears in geometric topology.

\textit{Example.} Let $M$ be a square and let $X$ be a fish on a line, as in Figure 2. Note that there would be a geometric pathology at the univalent points if we tried to blow up all of $X$ in one go. These pathologies become extreme in high dimensions.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{blowup.png}
\caption{Iterated blowup of a square at a fish on a line.}
\end{figure}
Remark. The construction is actually more general in several important cases. First, instead of a Whitney-stratified immersion in a manifold, we could consider an immersion of one cone-like space in another one. Even if the target space is a manifold, this allows the blowup of each point in $X$ to be a transversely immersed submanifold. Second, in our blowups we quotient by multiplication by scalars in $\mathbb{R}^+$. We could use instead quotients by scalar multiplication by $\mathbb{R}^*$ or $\mathbb{C}^*$, provided that the tangent cone at each point in $X$ is invariant under this larger group of homotheties. The iterated $C^*$ blowups of complex configuration spaces is called the Fulton-Macpherson compactification [4].

4.2 Geometry of blowups

If $p \in F_i$ blows down to $q$ for $q \in X_i$, then $p$ can be thought of as a point “infinitely close” to $q$. More formally, it is an element of the quotient $(T_p(M) - T_q(X_i))/\mathbb{R}^+$, where $\mathbb{R}^+$ acts by positive rescaling in the directions normal to $T_q(X_i)$. We can and will use the vector space structure of $T_p(M)$ to describe $p$. If $p$ lies in a corner of $B_X(M)$, for example in the intersection of $F_i$ and $F_j$ for $i < j$, then it can be understood as infinitely close to both $X_i$ and $X_j$, but infinitely closer to $X_j$ than to $X_i$.

In the main construction we will label part of the blowup locus as being “at infinity” and give $T_p(M)$ an inverted linear structure at points $p$ in this locus. In the simplest example, $M$ has a marked point $\infty$. Define $M_{\text{fin}}$, the finite part of $M$, as

$$M_{\text{fin}} = M \setminus \{\infty\}.$$  

If $M$ has a Riemannian metric, then we can give $M_{\text{fin}}$ an asymptotically flat Riemannian metric by inverting the exponential map from the point $\infty$. If we add a sphere at infinity to $M_{\text{fin}}$ in the usual way by adding endpoints to infinite rays, the result $M_{\text{fin}}'$ is combinatorially equivalent to the blowup $B_\infty(M)$ of $M$ at $\{\infty\}$.

Although it is more complicated to describe, in the general case any subcomplex $Y \subset X$ can be considered the finite locus, and the blowup of $M$ along $Y$ can be given an inverted geometry. The idea is to invert the exponential map normal to each stratum $Y_i$.

These geometries will be described more explicitly in the case of interest in the next section.

4.3 Blowups for configuration spaces

Let $M$ be a $d$-dimensional manifold and let $\Gamma$ be a connected graph with $n$ vertices. The graph $\Gamma$ may have self-loops and multiple edges, but these do not affect the construction in this section. Let the symbol $\Gamma$ also denote the vertex set of $\Gamma$, so that

$$M^\Gamma = \{f : \Gamma \to M\}$$

is equivalent to a Cartesian product $M^{\times n}$. Points in $M^\Gamma$, and in other spaces that we will form from it, are called configurations. If $\Gamma'$ is a subgraph of $\Gamma$, let $\Delta_{\Gamma'}$ denote the diagonal in $M^{\Gamma'}$ in which all vertices of $\Gamma'$ are sent to the same point.

In order to define a Gauss map, we need to blow up the diagonal $\Delta_{\infty}$, for every edge $e \subset \Gamma$. Such a diagonal is called principal, and blowing it up produces a codimension 1 face $F_e$ called a principal face. The principal diagonals are not mutually transverse, so we must blow up other diagonals first. Specifically, we blow up $\Delta_{\Gamma'}$ for every vertex-2-connected subgraph $\Gamma' \subset \Gamma$. (A cut vertex of a connected graph is a vertex whose removal disconnects the graph. A graph is vertex-2-connected if it is connected and has no cut vertices. Note that a single edge is vertex-2-connected.) These will be used as the strata in Section 4.1. Unless $\Gamma'$ is a single edge, the diagonal $\Delta_{\Gamma'}$ is called hidden and blowing it up produces a codimension 1 face $F_{\Gamma'}$ called a hidden face. If $\Gamma$ itself is 2-connected, then $F_\Gamma$ is called the anomalous face. We denote the result $C_T(M)$, the (compactified) $\Gamma$-configuration space of $M$. Its construction is valid modulo the following lemma:

**Lemma 8.** The system of diagonals $\{\Delta_{\Gamma'}\}$ corresponding to 2-connected subgraphs forms a conelike, Whitney-stratified, self-transverse immersion in $M^\Gamma$.

**Proof.** (Sketch) Checking that the diagonals are Whitney-stratified and cone-like is complicated but routine; the more significant issue is self-transversality.

We describe the minimal strata containing a configuration $c \subset M^{\Gamma'}$. Let $\Gamma_1, \ldots, \Gamma_k$ be the connected components of the inverse images in $c$ of points in $M$, not including components which consist of a single point. If $c$ lies on some of the diagonals, then this list of subgraphs is non-empty. A subgraph $\Gamma_i$ might not be 2-connected. Rather, it is has tree-like structure consisting of maximal 2-connected subgraphs which share cut vertices. We call such a structure a cactus and the 2-connected subgraphs lobes. A diagonal $\Delta_{\Gamma'}$ is a minimal stratum containing $c$ if and only if $\Gamma'$ is a maximal 2-connected subgraph of some $\Gamma_i$. Checking that these strata are mutually transverse is again complicated but routine.

Each codimension 1 face $F_{\Gamma'}$ has a geometric structure that we will use to describe certain gluings. The face $F_{\Gamma'}$ fibers over a smaller configuration space $C_{T'\Gamma'}(M)$, where $\Gamma'/\Gamma'$ is the graph $\Gamma$ with $\Gamma'$ contracted to a vertex $p \in M$. Let $f_{\Gamma',p}$ be a fiber where no other point of $\Gamma'$ is close to $p$. Then this fiber is just $C_{T'}(T_pM)/\Theta$, where $\Theta = \Theta(T_pM)$ is the $d+1$-dimensional Lie group of translation and homothety (scalar multiplication by $\mathbb{R}^+$) in the tangent space $T_pM$. Later we will need the quotient

$$C_{T'}(V) = C_{T'}(V)/\Theta(V)$$

for an arbitrary $d$-dimensional vector space $V$. If $\Gamma' = e$ is an edge, each $f_{\Gamma',p}$ is diffeomorphic to $S^{d-1}$. The reader can check that $C_T(M)$ is $dn$-dimensional and that $dn - 1$ is the total dimension of the fiber structure of each codimension 1 face.

The general corner (i.e., face with codimension $\geq 2$) of $C_T(M)$ is given by a list of codimension 1 faces that meet

6
it, or equivalently a list of edges and 2-connected subgraphs. In general corners in the balls, beams, and plates construction can either come from transverse intersections or from flags of strata; corners of both types are illustrated in Figure 2. In our case corners that are purely of the first type have the same combinatorics as the corresponding transverse intersections as described in the proof of Lemma 3. As a configuration approaches the corner, the graph $\Gamma$ develops one or more cactus structures whose lobes are 2-connected subgraphs; the vertices in each node converge and the nodes converge together. Corners that are purely of the second type consist of nested 2-connected subgraphs

$$\Gamma_1 \supset \Gamma_2 \supset \ldots \supset \Gamma_k,$$

where possibly the innermost graph $\Gamma_k$ is a single edge. In this case the vertices of each $\Gamma_i$ draw together as a configuration approaches the corner, but those of $\Gamma_{i+1}$ draw together at a faster rate than those of $\Gamma_i$. Most corners are of mixed type: Each is given by a cactus forest of 2-connected subgraphs, and there may be more cactus forests nested in the cactus lobes.

Example. Figure 3 shows an example of a configuration $c$ in which nine vertices of $\Gamma$ have converged. Because these vertices form a double square and a triangle connected by an edge, $c$ lies on a principal face and two hidden faces. In addition two of the vertices in the double square have converged more quickly to each other than to the other four vertices in their cluster, which means that $c$ lies on a second principal face as well. Thus, the face has codimension 4. (Note that some of the gluings described in Section 5.1 reduce the dimensions of some of the faces and corners.)

These blowups together with the diagonal blowups yield the space $C_{\Gamma, \infty}(M)$.

Example. Suppose for simplicity that we only blew up $M^\Gamma$ for one such subgraph $\Gamma'$ and that we did not blow up $\Delta_{\Gamma''}$ for any subgraph $\Gamma''$ of $\Gamma'$. Let $R_{\Gamma'}$ be the set of endpoints of rays in $M^\Gamma_{\text{fin}}$, which is asymptotically flat just as $M_{\text{fin}}$ is. Then the blown up locus is $R_{\Gamma'} \times M^\Gamma_{\text{fin}}$. In a configuration in the blown up locus, the vertices of $\Gamma'$ lie at an astronomical scale compared to $M_{\text{fin}}$. We retain the relative distances between these points and $M_{\text{fin}}$ and the angles, but not the scale of these distances relative to the internal geometry of $M_{\text{fin}}$. See Figure 4.

Finally, to match the inverted geometry at $\infty$, we will use a modified tangent bundle $T' M$. This is the unique bundle on $M$ whose sections pull back to asymptotically constant sections on $TM_{\text{fin}}$; alternatively, it is the push forward of the bundle $T(M \# B^3)$ to $M$, mapping the boundary sphere of $B^3$ to a point and identifying the fibers of $TB^3$ using its natural trivialization (from the inclusion $B^3 \subset \mathbb{R}$). If $M$ is an orientable 3-manifold, $T'M$ is isomorphic to $TM$ since they are both trivial, but the isomorphism is not canonical. (Note that $T'S^d$ is always trivial, while $T'S^n$ is non-trivial for even $d$.)

5 EXISTENCE OF THE INVARIANT

5.1 The gluings

In this section we will construct a grand configuration space $C_n$ using all decorated, closed Jacobi diagrams $\Gamma$ with $n + 1$ loops. Multiple edges are allowed, but self-loops are not. The way that we decorate the diagrams is that we explicitly orient every edge and we fully order the edges. We will need the Lie orientation of $\Gamma$ only later, in Lemma 8. The vertices of $\Gamma$ are not ordered. Let $J_n$ be the set of such diagrams up to isomorphism.

Let $M$ be a closed 3-manifold with a marked point $\infty$, and assume a framing of the modified tangent bundle $T'M$. This is equivalent to an asymptotically constant framing of $M_{\text{fin}}$ and is not much different from a framing of $M$.

![Figure 4: A configuration at the astronomical scale.](image-url)
We start with a dismembered version of the Gauss map $\Phi$. Let $C_n(M)$ be the disjoint union of all $C_{\Gamma,\infty}(M)$:

$$C_n(M) = \coprod_{\Gamma \in J_n} C_{\Gamma,\infty}(M)$$

and let

$$P(M) = C_{e,\infty}(M)$$

where the graph $e$ is an edge. The map

$$\Phi : C_n(M) \to P(M)^{\times 3n}$$

is defined in the $k$th factor by erasing the vertices of $\Gamma$ other than the two in the $k$th edge. Thus the configuration space $P(M)$ is a kind of topological propagator. It has the desired homology $H^2(P(M)) \cong \mathbb{Q}$, but $C_n(M)$ has no degree $6n$ (or top) homology because each component has faces. So we will glue the faces of $C_n(M)$ to each other, or otherwise collapse, or relativize them, and correspondingly modify $P(M)$ as necessary without destroying its homology. This process is equivalent to Kontsevich’s arguments that certain improper integrals vanish or cancel. The result will be a commutative diagram:

$$\begin{array}{ccc}
C_n(M) & \xrightarrow{\Phi} & P(M)^{\times 3n} \\
\downarrow & & \downarrow \\
\overline{C}_n(M) & \xrightarrow{\Phi} & \overline{P}(M)^{\times 3n}
\end{array}$$

As desired, the Vassiliev space $V_n$ will appear as a quotient of the top cohomology of the glued configuration space $\overline{C}_n(M)$ computed relative to the the degenerate locus $D$.

The gluings are as follows:

![Figure 5: Identifying six principal faces.](image)

**Principal faces:** For each simple edge $e$ we glue together the principal faces $F_e$ of the configuration spaces $C_{\Gamma,\infty}(M)$ for six different graphs $\Gamma$. The graphs are paired by reversing the orientation of $e$, and the three pairs differ by the Jacobi relation with $e$ in the middle, as in Figure 3. The ordering of the edges changes only implicitly, by virtue of the fact that the edges are re-connected. (This rule implies that six distinct faces are always glued.)

**Hidden faces:** Recall that the hidden faces correspond to 2-connected subgraphs $\Gamma'$; a double edge is counted as a hidden face rather than a principal one. Suppose that a pair of edges $e_1$ and $e_2$ in $\Gamma'$ separates it into two subgraphs $\Psi_1$ and $\Psi_2$. Then we can glue $F_{\Gamma'}$ to another face in which $e_1$ and $e_2$ are switched and their orientations are reversed. Since all points of $\Gamma'$ lie in some tangent space $T_pM$, modulo the homothety group, we can describe this operation explicitly in terms of linear algebra in $T_pM$. We leave $\Psi_2$ fixed and send every vertex $q \in \Psi_1$ to $q - e_1 - e_2$, where $e_1$ and $e_2$ also denote vectors corresponding to the edges $e_1$ and $e_2$ point from $\Psi_2$ to $\Psi_1$. Note that this involution changes the extra decoration on $\Gamma$, namely the ordering and orientation of its edges, but not the underlying graph.

We need to know that there is at least one involution, since $\Gamma'$ is not all of $\Gamma$, it has a vertex $q$ connected to $\Gamma \setminus \Gamma'$. Since $\Gamma'$ is 2-connected, $q$ has valence 2 in $\Gamma'$. We let $\Psi_1$ be $q$ and call the neighboring vertices $p_1$ and $p_2$, so that $e_1 = (p_1, q)$ and $e_2 = (p_2, q)$. (If $\Gamma'$ is a double edge, then $p_1 = p_2$.) Figure 6 illustrates the hidden face involution in this case.

![Figure 6: The involution for a hidden face.](image)

**The anomalous face:** This face is a compactification of a bundle with fiber $c_1(T(M_\infty))$ over $M_\text{fin}$. We identify all fibers with each other using the framing of $M_\text{fin}$. We perform the same operation in the topological propagator $P(M)$.

The unique face corresponding to $\Theta$, the unique Jacobi diagram with two vertices, is treated as anomalous rather than principal or hidden.
Infinite faces: First, the topological propagator $P(M)$ has two semi-infinite faces with one vertex at infinity and the other not, and it has a totally infinite face with both vertices at infinity. A configuration in any of these faces determines an element of $S^2$ by taking the unit vector from vertex 1 to vertex 2. We identify all three faces with standard $S^2$ using this correspondence; this $S^2$ is necessarily identified with the remnant of the anomalous face of $P(M)$. Denote the result $\overline{\Gamma}(M)$.

The infinite faces of the domain $C_n(M)$, including the totally infinite face, form the degenerate locus $D$. The degenerate locus

$$Q = \bigcup_A Q_A \times \overline{\Gamma}(M)^{3n-\lvert A \rvert}$$

is the union of pieces, one for each set $A$ of the edges numbered from 1 to $3n$. Given $A = \{a_1, \ldots, a_k\}$, the locus $Q_A$ consists of those elements

$$(v_{a_1}, \ldots, v_{a_k}) \in (S^2)^{\times A}$$

such that the unit vectors $\{v_{a_i}\}$ can be realized as the directions of the edges of some graph with $k$ edges which has been linearly mapped into $\mathbb{R}^3$. The graph is required to have no vertices of valence 1 and at most one of valence 2, although multiple edges are allowed.

The result is a glued configuration space $\overline{C_n}(M)$ and a glued topological propagator $\overline{P}(M)$.

5.1.1 The transitive closure of the gluings

To describe the transitive closure of the gluings of the principal and hidden faces, we begin with the geometry of the configuration space after only the principal faces are glued. The principal edges of a configuration $c$ form a forest of trees in the graph $\Gamma$. (Because of the hidden blowups, these edges cannot form closed loops.) By virtue of the principal blowups, which may be performed simultaneously after all hidden blowups, each of these edges has a well-defined direction but not a length, not even a relative length when compared with any other edge of $\Gamma$. The principal gluings then identify $c$ with all other configurations in which each tree of principal edges is replaced by some other tree with the edges pointing in the same directions. In the glued space the trees lose their identity. The data that remains is a graph $\overline{\Gamma}$ in which each tree of principal edges in $\Gamma$ is contracted to a point; a vertex in $\overline{\Gamma}$ at the point $p \in M$ of valence $n > 3$ is also assigned a list of $n - 3$ unit tangent vectors in $T_p M$.

Some of the corners that are glued to each other do not have the same dimension, because the reconnection in Figure 5 can change whether or not a subgraph is 2-connected. Section 4.3 describes how a corner before gluing is determined by nested cactus structures in the diagram $\Gamma$, and that the codimension of the corner equals the total number of lobes of the cacti. After the gluings of the principal faces, a general corner is described by the contracted graph $\overline{\Gamma}$ together with nested cacti in $\Gamma$. Each lobe of each cactus now corresponds to a hidden face and cannot be a single edge. The total codimension of the corner is then the total number of principal edges in $\Gamma$ plus the total number of cactus lobes in $\overline{\Gamma}$. For example, before the principal gluings the configuration on the left in Figure 6 lies on a codimension 4 face. The reconnection in Figure 5 glues it to the codimension 5 face in Figure 6.

Assuming that the principal gluings are completed, we describe the hidden face involutions. The description rests on two facts. First, following the description in the previous paragraph, a hidden face involution always glues together faces of the same dimension. If two edges $e_1$ and $e_2$ of a configuration of some face separate the contracted subgraph $\Gamma'$, then the inverse image of a cactus lobe in $\overline{\Gamma}$ contains either both or neither. This is true even if one or both of $e_1$ or $e_2$ is principal. Second, the involutions for any given $\Gamma$ generate a finite group, because they are given by permuting and reversing edges. (In fact it is a product of symmetric groups, each acting on an equivalence class of edges, where two edges are equivalent if they separate $\Gamma'$.)

5.1.2 Remarks on the construction

The entire construction has a folded version in which $P(M)$ is defined as the space of unordered pairs of points, the Cartesian product $\overline{P}(M)^{3n}$ is replaced by the symmetric power $S^{3n}(\overline{P}(M))$, and $\mathbb{R}^+$-blowups are replaced by $\mathbb{R}^+$-blowups throughout. The propagator space $P(M)$ becomes a homology $\mathbb{R}P^2$ rather than a homology sphere, and its relevant second cohomology group has coefficients in the twisted flat line bundle over $P(M)$ or $\overline{P}(M)$. The edges of $\Gamma$ are no longer explicitly oriented, nor are the edges ordered. This version is formally cleaner, but it is harder to visualize. It essentially hides signs and denominators in homological algebra rather than removing them.

Many hidden faces admit more symmetries than those generated by the given involutions. Namely for each subgraph $\Gamma'' \subseteq \Gamma'$ connected to $\Gamma'$ at two vertices $p_1$ and $p_2$, we can reverse every edge in $\Gamma''$ and switch $p_1$ and $p_2$ relative to
\( \Gamma'' \setminus \Gamma' \). Also we can reverse every edge of \( \Gamma' \). These involutions generate a larger gluing group. The key property of any such gluing group is that half of its elements negate the map \( f \) in Lemma 3. Reversing \( \Gamma'' \) has this effect if and only if \( \Gamma'' \) has an odd number of edges plus vertices; reversing all of \( \Gamma' \) does if and only if it has an even number of edges plus vertices. Thus reversing a single edge does not negate \( f \), but if \( n \) is even, reversing all of \( \Gamma \) in the anomalous face does \([3, 6]\). This removes the need to collapse the anomalous face using the framing; in physics terminology, the anomaly cancels. The other involutions available also eliminate the anomaly if \( \Gamma \) isn’t edge-3-connected.

Kontsevich \([20]\) blows up every diagonal of \( M \times 2n \) to form an analogue of \( \hat{C}_r(M) \) that does not depend on \( \Gamma \). While his convention may have a rigorous analytic interpretation, it does not work well in topologically, because a hidden face involution can send some vertices of \( \Gamma' \) on top of others. In other works \([11, 23, 30]\) (and in the first version of this work), the diagonal \( \Delta_{\Gamma'} \) is blown up for every connected subgraph \( \Gamma' \subseteq \Gamma \). Then the hidden face involutions are more complicated and also inconsistent at corners. The usual remedy is to observe that corners are irrelevant to degree calculations; only codimension 1 faces are important. This is equivalent to relegate all of the corners and their images to the relative locus. We feel that it is more natural to only blow up \( \Delta_{\Gamma''} \) when \( \Gamma'' \) is 2-connected.

Another approach to relativizing the semi-infinite faces, which may be what Kontsevich had in mind, is power counting. If \( \alpha \in H^2(\overline{P}(M)) \) (defined in Section 5.3) is a Hodge form, it vanishes as \( L^{-2} \) on a length scale \( L \) in the asymptotic part of \( \overline{P}(M) \). At the same time the available volume for a single vertex grows as \( L^3 \). The product is a negative power of \( L \) for semi-infinite faces of \( C_T(M) \), which means that these faces are irrelevant in the degree formula for \( \Phi \). It may be possible to phrase this argument in terms of spectral sequences of filtrations, since \( \overline{P}(M) \times 3n \) can be filtered according to how many coordinates lie in \( S^2 \subset \overline{P}(M) \), while \( \overline{C}_n(M) \) can be filtered according to how many vertices are at infinity.

The power counting argument does not work for the totally infinite face. In this case an alternative is to cap \( F_{T, \infty}(M) \) with \( C_T(S^3) \), since the geometry of the face does not depend on the manifold \( M \).

### 5.2 The bordism variant of framings

Instead of collapsing the anomalous face of \( C_T(M) \), we can instead cap it using a bordism of \( M \). Although a special case of this formulation is entirely equivalent to the framing approach, it will be more convenient for the constructions in Section 5.4.

More precisely, let \( W \) be a 4-manifold bounded by \( M \) and let \( E \) be a 3-plane bundle that restricts to \( T'M \) on \( M \). Then we can cap the anomalous face \( F_T(M) \) with a certain configuration bundle \( c_T(E) \) over \( W \) for all graphs \( \Gamma \). The fiber over \( p \in W \) of this bundle is the configuration space \( c_T(E_p) \) of the fiber \( E_p \). This configuration bundle has its own principal and hidden faces, which are glued in the same way as faces of \( C_{T, \infty}(M) \) to form a bundle \( c_\alpha(E) \). All of the other faces of \( C_{T, \infty}(M) \) are also glued the same way as before. We denote the resulting glued space \( \hat{C}_n(M) \).

Likewise we can cap the diagonal face of \( P(M) \) with \( c_\alpha(E) \), which is just the unit sphere bundle \( SE \). We can also refrain from collapsing the semi-infinite faces or the totally face of \( P(M) \). Call the result \( \hat{P}(M) \). This propagator space may have some spurious second cohomology coming from the homology of \( W \), but there is a unique second cohomology class in \( SE \) which can be represented by a cocycle which is antisymmetric under the antipodal map on fibers. (The antipodal map on the fibers, which extends to the map switching the two factors of \( C_2(M) \), splits the (rational) cohomology into the odd and even subspace. All the cohomology classes from \( W \) are even, by definition.) This class extends to the propagator class \( \alpha \in \hat{P}(M) \). The Gauss map \( \Phi \) is defined as before.

If \( T'M \) has a framing that extends to \( E \), then there is a quotient map

\[
\pi : \hat{C}_n(M) \to \overline{C}_n(M)
\]

given by collapsing \( W \) to a single point (and \( E \) to a single fiber). The map \( \pi \) induces an isomorphism of the top homology of the configuration spaces, and the analogous map on propagators takes \( \alpha \) to \( \alpha \). (This property can be used as the definition of \( \alpha \in \hat{P}(M) \).) The map \( \pi \) then forms a commutative square with \( \Phi \):

\[
\begin{array}{ccc}
\hat{C}_n(M) & \xrightarrow{\Phi} & \hat{P}(M) \\
\downarrow{\pi} & & \downarrow{\pi} \\
\overline{C}_n(M) & \xrightarrow{\Phi} & \overline{P}(M)
\end{array}
\]

This square and the isomorphism properties of \( \pi \) demonstrate that, if the bundle \( E \) matches the framing of \( T'M \), \( \hat{C}_n(M) \) produces the same invariant \( I_n(M) \) as \( \overline{C}_n(M) \).

Indeed, the bordism \( W \) need not be a manifold, but only a homology manifold. (A homology \( n \)-manifold for us is a simplicial complex such that the link of each vertex is a homology \( n-1 \)-manifold and a homology \( n-1 \)-sphere.) In particular, if \( W \) is the cone over \( M \), a bundle over \( W \) extending \( T'M \) is equivalent to a framing of \( T'M \).

### 5.3 Cohomology

**Lemma 9.** The top cohomology of the glued configuration space \( \overline{C}_n(M) \) is independent of \( M \) and has a surjection onto the Vassiliev space \( V_n \):

\[
H^{6n}(\overline{C}_n(M), D) \to V_n.
\]

**Proof.** Let \( X_n \) be the union of all faces of \( C_n \) (both finite and infinite) and let \( \overline{X}_n \) be its image in \( \overline{C}_n \). Consider the cohomology exact sequence of the triple \( D \subset \overline{X}_n \subset \overline{C}_n \):

\[
H^{6n-1}(\overline{X}_n, D) \to H^{6n}(\overline{C}_n, \overline{X}_n) \to H^{6n}(\overline{C}_n, D) \to 0.
\]
On the other hand,
\[ H^{6n}(\overline{C}_n, D \cup \overline{X}_n) \cong H^{6n}(C_n, X_n) \cong \bigoplus_{\Gamma \in J_n} \mathbb{Q}\Gamma \]

since \( D \cup \overline{X}_n \) cuts \( \overline{C}_n \) into the pieces of \( C_n \), and on each of these we have a unique top cohomology class, the fundamental class. Thus, \( H^{6n}(\overline{C}_n(M), D) \) is a space of graphs modulo some relations. These graphs are not quite Lie-oriented as in the definition of \( V_n \), since the edges are labelled and each edge is oriented. (A Lie-oriented graph has a global choice of orientations up to sign.) But there is a forgetful map \( f \) from \( J_n \) to Lie-oriented graphs.

To prove the lemma, we only need to check that the relations given by \( H^{6n-1}(\overline{X}_n, D) \) become trivial or the Jacobi relation under \( f \). The space \( H^{6n-1}(\overline{X}_n, D) \) might be rather complicated, but by the same exact sequence it is generated by one cohomology class for each face (of \( C_n \)). By cases:

**Principal Faces:** The sum of the six graphs in Figure 3 descends to the Jacobi relation.

**Hidden faces:** We glue together several different graphs which become identical (up to sign) under \( f \). Each involution defined in Sections 5.1.2 negates \( f(\Gamma) \). For example, if the graph \( \Psi_2 \) is the single point \( q \), then
\[ q \mapsto p_1 + p_2 - q \]
is orientation-reversing, and two edges are reversed. Thus half of the elements of the group generated by these involutions negates \( f(\Gamma) \), so the total sum vanishes.

**Infinite faces:** Since cohomology is computed relative to these faces, they impose no relation.

**Anomalous face:** Since we reduce the dimension of this face, it imposes no relation.

**Remark.** It may appear as if we are discarding information present in the rest of \( H^{6n}(\overline{C}(n)(M), D) \) by relying on Lemma 3. However, the arguments of Section 5.1.2 imply that any invariant determined by the action of \( \Phi \) on \( H^{6n}(\overline{C}(n)(M), D) \) is finite type. Since \( I_n \) is universal among finite type invariants by Theorem 2, it determines all other such invariants. In the minimal construction mentioned in Section 5.1.2, \( H^{6n} \) is isomorphic to \( V_n \); the spurious cohomology is absent.

**Lemma 10.** If \( M \) is a rational homology sphere, then the second cohomology \( H^2(\overline{P}(M)) \) of the glued topological propagator is generated by the fundamental class \( \alpha \) of the standard sphere \( S^2 \subset \overline{P}(M) \). Moreover, there is a well-defined cohomology class
\[ \alpha^{\otimes 3n} \in H^{6n}(\overline{P}(M)^{\times 3n}, Q). \]

**Proof.** The existence of \( \alpha \) originates with the geometry of the configuration space \( C_{c,\infty}(M) \). This is a manifold with corners whose interior is \( M^{n^2+2} \setminus \Delta \), the space of pairs of distinct points in \( M_{6n} \). If \( M_{6n} = \mathbb{R}^3 \), it is clearly homotopy equivalent to \( S^2 \). In the general case it has the same homology by a Mayer-Vietoris argument. Each of the gluings used to make \( \overline{P}(M) \) from \( P(M) \) is chosen to preserve the second cohomology, although higher cohomology may also appear.

The class \( \alpha^{\otimes 3n} \) clearly exists in the absolute cohomology \( H^{6n}(\overline{P}(M)^{\times 3n}) \); the question is whether it exists uniquely in cohomology relative to \( Q \). Observe first that if \( |A| = k \), then \( Q_A \subset (S^2)^k \) has codimension at least 3. Each allowed graph \( \Gamma' \) on \( A \) with \( k \) edges (of which there are finitely many) has at most \( (2k + 1)/3 \) vertices. Thus there are at most \( 2k + 1 \) degrees of freedom in embedding \( \Gamma' \) in \( \mathbb{R}^3 \). In addition, 4 of these degrees of freedom are absorbed by invariance under the homothety group \( \text{Th}(\mathbb{R}^3) \), so \( Q_A \) has dimension at most \( 2k - 3 \).

Choose a point
\[ p = (p_1, \ldots, p_{3n}) \in (S^2)^{3n} \setminus Q. \]
For each \( i \), choose a cocycle \( \alpha_i \in Z^2(\overline{P}(M)) \) that represents the class \( \alpha \) and that is localized at \( p_i \) (or for concreteness, a small simplex containing \( p_i \)) in the standard sphere \( S^2 \subset \overline{P}(M) \). Recall that the space of relative cocycles \( Z^{6n}(\overline{P}(M)^{\times 3n}, Q) \) is a subspace of the space of absolute cocycles \( Z^{6n}(\overline{P}(M)) \). The cocycle
\[ \alpha_p = \alpha_1 \otimes \ldots \otimes \alpha_{3n} \]
exists as a relative cocycle because it avoids \( Q \). It represents a non-trivial cohomology class because relativization can only diminish the space of boundaries. Thus \( \alpha^{\otimes 3n} \) exists in relative cohomology.

To show uniqueness, suppose that \( a \subset S^2 \) is an arc connecting \( p_1 \) with some point \( p'_1 \) and which is disjoint from \( Q \):
\[ a \times (p_2, p_3, \ldots, p_{3n}) \subset (S^2)^{3n} \setminus Q. \]
If \( a_1' \) represents \( a \) and is localized at \( p'_1 \), then there is a 1-cochain \( \beta \) localized along \( a \) which is a homology between \( a_1 \) and \( a_1' \):
\[ \delta \beta = a_1' - a_1. \]
In this case
\[ \beta \otimes \alpha_2 \otimes \alpha_3 \otimes \ldots \otimes \alpha_{3n} \]
is a homology between \( a_p \) and \( a_{p'} \), where
\[ p' = (p_1', p_2, p_3, \ldots, p_{3n}). \]
Since \( Q \) has codimension 3 in \( (S^2)^{3n} \), any two points in its complement can be connected by a sequence of moves of this type. Hence \( \alpha^{\otimes 3n} \) is unique. \( \square \)

Having defined all elements of the map 3 and equation 3 (taking \( C_n = \overline{C}_n(M) \) and \( P = \overline{P}(M) \)), the definition of the invariant \( I_n(M) \) for framed, rational homology spheres is complete.
6 PROPERTIES

6.1 Connected sums

In this subsection we prove Theorem 1. Although the conclusion is in the spirit of properties of surgery, the argument has more in common with the definition of $I_w(M)$. Suppose that $M = M_1 \# M_2$ is a rational homology sphere. We may realize $M_{\text{fin}}$ by patching very small copies of $(M_1)_{\text{fin}}$ and $(M_2)_{\text{fin}}$ into a flat $\mathbb{R}^3$, as in Figure 8. In fact, $(M_1)_{\text{fin}}$ and $(M_2)_{\text{fin}}$ can be infinitely small. More precisely, we blow up $\mathbb{R}^3$ (with the trivial framing) at $(0,0,0)$ and $(1,0,0)$ and we glue the spheres at infinity of $(M_1)_{\infty}$ and $(M_2)_{\infty}$ to the blown up points. In addition to the astronomical scale used to compactify $M_{\text{fin}}$, this geometry gives it an intermediate scale, which we called the planetary scale. On the planetary scale, $(M_1)_{\text{fin}}$ and $(M_2)_{\text{fin}}$ are reduced to points but are a unit distance from each other. Call the set of points in this region $\mathcal{P}(M_1, M_2)$.

We use the planetary scale to compactify $P(M)$ slightly differently. If $(p, q) \in P(M)$ and at least one of $p$ and $q$ is in $\mathcal{P}(M_1, M_2)$, or if one is in $(M_1)_{\text{fin}}$ and the other is in $(M_2)_{\text{fin}}$, we glue $(p, q)$ to the point in $S^2$ given by the direction from $p$ to $q$.

![Figure 8: The planetary scale for $M_1 \# M_2$.](image)

We may slightly enlarge the degenerate locus $Q$ without changing $H^\infty(P(M) \times S^1, Q)$. In the definition of $Q$ in Section 4, we allow graphs in $\mathbb{R}^2$ with at most two vertices of valence 2 rather than at most one, and we allow the graph consisting of a single edge from $(0,0,0)$ to $(1,0,0)$ (or vice versa). The resulting locus $Q'$ has codimension 2 rather than codimension 3, but 2 is still enough for the arguments of Lemmas 2 and 10.

If a configuration in $\overline{C}_n(M)$ has any points at the planetary scale as in Figure 8, or if it has some points in $(M_1)_{\text{fin}}$ and others in $(M_2)_{\text{fin}}$, then the map $\Phi$ sends it to the locus $Q'$. The only other possibilities are that all vertices are in $(M_1)_{\text{fin}}$, or that all vertices are in $(M_2)_{\text{fin}}$. This realizes the cocycle $\alpha_{\text{fin}}$ as the sum of cocycles on $\overline{C}_n(M_1)$ and $\overline{C}_n(M_2)$, which establishes the identity

$$I_n(M) = I_n(M_1) + I_n(M_2).$$

Since these propagators are defined on configuration spaces for different manifolds, we will dismember the configuration spaces so that some of the pieces are the same, and then calculate with the propagators on these common pieces.

We begin by more precisely defining the cubical complex $C$ mentioned in Section 5 in the algebraically split and Torelli cases.

We will consider a knot $K$ in a 3-manifold $M$ to be a closed solid torus that does not contain the marked point $\infty$, and a link $L$ to be the union of finitely many disjoint knots $\{K_1, \ldots, K_k\}$. For each such link $L$ we will consider the $2^k$ sublinks of the form

$$L_I = \bigcup_{i \in I} K_i$$

where $I \subseteq [k] = \{1, \ldots, k\}$ is a set of indices. For each such $L_I$ we will let $M_I$ be the result of $+1$ surgery on each component of $L_I$. Recall that a link $L$ in an integer homology sphere is algebraically split if the linking number between each pair of components vanishes. In the case we interpret the pair $(M, L)$ as an element of $\mathcal{M}$ given by the alternating sum

$$\langle M, L \rangle = \sum_{I \subseteq [k]} (-1)^{|I|} M_I.$$  

Thus if $f(M)$ is an invariant, then $f^{(k)}(M, L)$, the $k$th algebraically split finite difference of $f$, is defined by the same sum.

We use the same conventions for Torelli surgery. As mentioned in the introduction, a Torelli surgery on an integer homology 3-sphere $M$ consists of removing a handlebody $H$ and gluing back a handlebody $H'$ that differs by a surface automorphism which acts trivially on $H^1(\partial H)$ (an element of the Torelli group of $\partial H$). The locus $T$ of a Torelli surgery is the union of finitely many disjoint handlebodies $\{H_1, \ldots, H_k\}$ (a multi-handlebody in $M$), where each $H_i$ is decorated with an element of the Torelli group of $\partial H_i$. For each multi-handlebody $T$ we will consider the sub-multi-handlebodies $T_I$ for each $I \subseteq [k]$ and we let $M_I$ be the result of surgery on $T_I$. We let

$$\langle M, U \rangle = \sum_{I \subseteq [k]} (-1)^{|I|} M_I$$

and if $f(M)$ is an invariant, we let $f^{(k)}(M, T)$ be the $k$th Torelli finite difference of $f$.

Consider a surgery (either algebraically split or Torelli) on a manifold $M$ in which a submanifold $N$ is replaced by some other submanifold $N'$. If $M$ is framed, we will assume that $N'$ has a framing which agrees with the framing of $N$ at the boundary. Likewise if $M$ has a bundle bordism $(W, E)$, then we will assume a cobordism $W'$ between $N$ and $N'$ to attach to $W$. If $W$ has a bundle $E$ extending the modified tangent bundle $T'M$, we can extend it to $W'$. The choices for this extra data will not matter, as long as we always make the same choice for a surgery component $N$ which is shared by many
multi-component surgeries. Note that because of spin obstructions, a framing does not extend across an algebraically split surgery if any of the surgery slopes are odd. But bundle bordisms always extend.

6.3 Dismemberment and bubble wrap

The best way to understand dismemberment of a manifold \( M \) is as a kind of blowing up. If \( S \) is a surface in \( M \), we can blow up \( M \) along \( S \), which amounts to cutting \( M \) along \( S \), to make a manifold \( B \). We can also add configurations in \( M^\Gamma \) that meet \( S \) to the blowup loci used to construct \( C_{T,\infty}(M) \). Call the resulting configuration space \( C_{T,\infty}(B) \). There is a blow-down map

\[
\pi : C_{T,\infty}(B) \to C_{T,\infty}(M).
\]

For example, suppose that \( M \) consists of two manifolds \( M_1 \) and \( M_2 \) identified along a connected surface \( S \). (It is immaterial here which of \( M_1 \) and \( M_2 \) has the point \( \infty \), as long as it is not on \( S \) itself.) Then the blowup \( Z \) is

\[
B = M_1 \amalg M_2.
\]

If \( e \) is an edge, then \( C_e(B) \) has four components, defined by which of the vertices of the edge are in \( M_1 \) and which are in \( M_2 \). The four components are homeomorphic to \( C_e(M_1) \), \( C_e(M_2) \), \( M_1 \times M_2 \), and \( M_2 \times M_1 \). Their geometry is slightly different, because if \( p, q, r \in M \) are coincident on \( S \), the point \( (p, q) \) is blown up to record the direction from \( p \) to \( q \) and the ratio of the distance from \( p \) to \( S \) to the distance from \( q \) to \( S \). Nonetheless by abuse of notation we will refer to the components as \( C_e(M_1) \), \( M_1 \times M_2 \), etc.

In the definition of \( C_n(M) \), the gluings of the hidden faces and the anomalous face are difficult to reconcile with blowing up along a surface \( S \). However, the anomalous face poses no problem if we cap it using a bundle bordism \( (W, E) \), since we can then extend \( S \) to a hypersurface \( T \) in \( W \) and blow that up too. Thus the topological propagator \( \hat{P}(M) \) can be dismembered to make \( \hat{P}(B) \). Instead of dismembering \( \hat{C}_n(M) \), we will pull back propagators defined on it to the pieces \( C_{T,\infty}(M) \) and \( c_1(E) \), which we will then dismember.

In comparing propagators, we only need to compare the first algebraically split discrete derivative. Let \( M \) be an integer homology sphere and let \( K = K_1 \subset M \) be a knot. Let \( M_1 \) be the result of replacing \( K \) by \( K' \) in \( M \), where \( K' \) and \( K \) differ by a +1 Dehn twist.

**Lemma 11.** If two integer homology spheres \( M \) and \( M_1 \) differ by +1 surgery on a knot \( K \), and if \( \alpha \in H^2(\hat{P}(M)) \) and \( \alpha_1 \in H^2(\hat{P}(M_1)) \) are cohomological propagators, then \( \alpha_1 - \alpha \) is homologous to \( \beta \otimes \beta_1 \) on \( \hat{P}(M \setminus K) \), where \( \beta \) is a 1-cocycle dual to a Seifert surface of \( K \) (a Seifert cocycle).

**Proof.** We can measure \( \alpha_1 - \alpha \) by pairing it with 2-cycles in \( \hat{P}(M \setminus K) \). There are several kinds of these, but the only kind that can have non-zero pairing is represented by a torus \( J_1 \times J_2 \), where \( J_1 \) and \( J_2 \) are two disjoint knots in \( M \setminus K \). In this case \( \alpha \) measures their linking number in \( M \):

\[
\langle J_1 \times J_2, \alpha \rangle = \text{lk}_M(J_1, J_2)
\]

Likewise \( \alpha' \) measures their linking number in \( M_K \). The difference is the product of linking numbers with \( K \):

\[
\text{lk}_{M_1}(J_1, J_2) - \text{lk}_M(J_1, J_2) = \text{lk}_{M_1}(J_1, K)\text{lk}_M(J_2, K)
\]

This is easy to see when \( K \) is an unknot, since surgery on \( K \) has the effect of twisting \( J_1 \) and \( J_2 \) about each other without changing \( M \), as in Figure 9. Since \( \alpha_1 - \alpha \) pairs with homology classes in the same way as \( \beta_1 \otimes \beta_1 \), the two cocycles are homologous.

The significance of Lemma 11 is that by Lemma 7, we can define \( \alpha_1 \) to be an extension of \( \alpha \) adjusted by \( \beta_1 \):

\[
\alpha_1 \overset{\text{def}}{=} \alpha + \beta_1 \otimes \beta_1
\]

on \( \hat{P}(M \setminus K) \). Note also that we can assume that the support of \( \beta_1 \) is a neighborhood of any desired Seifert surface \( S \) of \( K \).

The next case is algebraically split surgery with two components. Let \( L = \{K_1, K_2\} \) be a link in \( M \). Then the each of the four topological propagators \( \hat{P}(M) \), \( \hat{P}(M_1) \), \( \hat{P}(M_2) \), and \( \hat{P}(M_{1,2}) \) dismember into nine pieces. The dismemberment of \( \hat{P}(M) \) looks like this:

| \( \hat{P}(K_1) \) | \( K_1 \times M \setminus L \) | \( K_1 \times K_2 \) |
| \( M \setminus L \times K_1 \) | \( \hat{P}(M \setminus L) \) | \( M \setminus L 	imes K_2 \) |
| \( K_2 \times K_1 \) | \( K_2 \times M \setminus L \) | \( \hat{P}(K_2) \) |

Here we have circled \( \hat{P}(M \setminus K_1) \) and \( \hat{P}(M \setminus K_2) \). Choosing Seifert surfaces \( S_1 \) and \( S_2 \) and Seifert cocycles \( \beta_1 \) and \( \beta_2 \), we define \( \alpha_1 \) and \( \alpha_2 \) by equation (6) and the extension principle.
We assume that $S_1$ is disjoint from $K_2$ and vice versa. Finally, $\tilde{P}(M_{K_1,K_2})$ dismembers as follows:

\[
\begin{array}{c}
\tilde{P}(K_1') \\
M \setminus L 	imes K_1' \\
K_1' \times K_1' \\
K_1' \times K_1' \times M \setminus L \\
M \setminus L 	imes K_1' \\
K_1' \times K_1' \times K_1' \times K_1' \\
K_1' \times K_1' \times K_1' \times K_1' \times K_1' \\
\end{array}
\]

In this diagram the northwest square is shared with $\tilde{P}(M_1)$, while the southeast square is shared with $\tilde{P}(M_2)$. By the boundary-disjointness of the Seifert surfaces, if we define

\[
\alpha_{K_1,K_2} = \alpha + \beta_1 \otimes \beta_1 + \beta_2 \otimes \beta_2
\]
on $\tilde{P}(M \setminus L)$, we can extend it by $\alpha_3$ and $\alpha_2$ on the rest of the shared pieces. This leaves the two remaining pieces $K_1' \times K_2'$ and $K_2' \times K_2'$. We claim that $\alpha_{1,2}$ automatically extends to these pieces, because they can cannot create any second homology. In other words, the inclusion

\[
\tilde{P}(M_{1,2}) \setminus (K_1' \times K_2' \cup K_2' \times K_1') \subset \tilde{P}(M_{1,2})
\]
is an isomorphism on $H^2$. This may be seen by a general position argument, where we abbreviate the inclusion as just $X \subset Y$: Since $K_1' \times K_2'$ and $K_2' \times K_1'$ are thickened 2-tori in the interior of $Y$, a 6-manifold with boundary, any 2-cycle in $Y$ used to measure 2-cocycles can be perturbed to lie in $X$. Furthermore, if a 2-cycle bounds a 3-chain in $Y$, the 3-chain can be perturbed to lie in $X$ as well.

Finally in the general case, let $L = \{K_1, K_2, \ldots, K_k\}$ be a $k$-component link in $M$. Given an arbitrary propagator $\alpha$ on $\tilde{P}(M)$, we choose 1-cocycles $\beta_1, \ldots, \beta_k$ and construct propagators $\alpha_i$ and $\alpha_{i,j}$ as above. If $I$ has at least three elements, then the dismemberment $\tilde{P}(B_I)$ of $\tilde{P}(M_I)$ consists entirely of shared pieces. We define

\[
\alpha_I = \alpha + \sum_{i \in I} \beta_i \otimes \beta_i
\]
on $\tilde{P}(M \setminus L)$. We extend $\alpha_I$ to each of the other shared pieces by reusing either $\alpha_i$ or $\alpha_{i,j}$. The conclusion is the following technical lemma:

**Lemma 12.** Let $L \subset M$ be a link with $k$ components. For each component $K_i$, let $\beta_i$ be a Seifert cocycle. Then for each $I \subseteq [k]$, there is a homological propagator $\alpha_I$ on the manifold $M_I$ such that

\[
\alpha_I = \alpha + \sum_{i \in I} \beta_i \otimes \beta_i
\]
on $\tilde{P}(M \setminus L)$, and otherwise $\alpha_I$ and $\alpha_{I'}$ agree on each component shared by the dismemberments $\tilde{P}(B_I)$ and $\tilde{P}(B_{I'})$ of $\tilde{P}(M_I)$ and $\tilde{P}(M_{I'})$ along $\partial L$.

Finally let $M_{\text{dis}}$ be the union of the dismemberments $B_I$ of all $M_I$. Likewise let $P(M_{\text{dis}})$, $C_{\Gamma,\infty}(M_{\text{dis}})$, and $c_{\Gamma}(E_{\text{dis}})$ be the union, respectively, of all dismemberments of topological propagators, configuration spaces, and bundles associated to each $M_I$. We extend each $\alpha_I$ by 0 to define it on all of $P(M_{\text{dis}})$.

For Torelli surgery we will use a dual construction called *bubble wrap* in which we glue configuration spaces together instead of dismembering them. More precisely, if $T \subset M$ is a multi-handlebody with $k$ components $H_1, \ldots, H_k$, and if $H'_1, \ldots, H'_k$, the bubble-wrap model $M_{\text{bub}}$ of the pair $(M, T)$ is given by gluing in both $H_i$ and $H'_i$ to $M \setminus T$ for each $i$. The topological propagator $\tilde{P}(M_{\text{bub}})$ and the configuration space $\tilde{C}_{\Gamma}(M_{\text{bub}})$ are likewise formed from $M_I$ by gluing together $\tilde{P}(M_I)$ and $\tilde{C}_{\Gamma}(M_I)$, ranging over all $I \subseteq [k]$, wherever these spaces agree. In the Torelli analogue of Lemma 11, $\alpha_I - \alpha$ is null-homologous where it is defined. Consequently the above reasoning allows us to choose the $\alpha_I$ to agree on their common domains, which leads to the following conclusion.

**Lemma 13.** Let $T \subset M$ be a multi-handlebody with $k$ components. Then the cohomological propagators $\alpha_I$, ranging over all $I \subseteq [k]$, form a cocycle $\alpha \in \tilde{P}(M_{\text{bub}})$.

Since in the bubble wrap model there is only one cocycle, we will instead add and subtract cycles. For this purpose, given a weight system $w$, we define $\mu_{w,I}$ as a cycle on $C_{\Gamma,\infty}(M_{\text{bub}})$ by extending $\mu_w$, which exists on $C_{\Gamma,\infty}(M_I)$, by 0. Dually, all $\mu_{w,I}$ exist as chains on their common domain on $C_{\Gamma,\infty}(M_{\text{dis}})$ and on $c_{\Gamma}(E_{\text{dis}})$, although they are no longer cycles because of dismemberment and because we have suppressed gluing. They form a chain $\mu_w$.

### 6.4 The invariants are finite type

#### 6.4.1 Torelli type

We first discuss the Torelli case since it is a bit simpler than the algebraically split case. In light of Lemma 11, we cannot take a tensor power of a cohomological propagator $\alpha \in H^2(\tilde{P}(M))$ at the cochain level; instead we use a tensor product

\[
\alpha_1 \otimes \alpha_2 \otimes \ldots \otimes \alpha_{3n}.
\]

Nonetheless the arguments of Section 6.3 apply to each $\alpha_i$ separately. For brevity we let $\gamma$ be its pull-back under $\Phi^*$ to $\tilde{C}_{\Gamma,\infty}(M_{\text{bub}})$.

The constructions of Section 6.3 leave us with a cocycle $\alpha$ on $\tilde{P}(M_{\text{bub}})$ as well as a family of cycles $\mu_{w,I}$ on $\tilde{C}_{\Gamma,\infty}(M_{\text{bub}})$, and we wish to compute the alternating sum of pairings

\[
I_w^{(k)}(M, T) = \sum_{I \subseteq [k]} (-1)^{|I|} \langle \mu_{w,I}, \gamma \rangle.
\]

Observe that the cycles $\mu_I$ form a parallelepiped in the vector space of all cycles on $\tilde{C}_{\Gamma,\infty}(M_{\text{bub}})$. In other words, there is a
cycle-valued, affine-linear functional \( \mu_w(t) \), where \( t \in \mathbb{R}^k \) is a vector of parameters, such that

\[
\mu_{w,I} = \mu_w(t_I),
\]

where \( (t_I)_i = 1 \) for \( i \in I \) and 0 for \( i \not\in I \). Let

\[
I_w(t) = (\mu_w(t), \gamma).
\]

Then \( I^{(k)}_w(M, T) \) is a finite difference

\[
I^{(k)}_w(M, T) = \Delta_1 \Delta_2 \ldots \Delta_k I_w(t)|_0, \tag{7}
\]

where by definition

\[
\Delta_i I = I(t_i) - I(t_i + 1).
\]

Also let

\[
\nu = \Delta_1 \Delta_2 \ldots \Delta_k \mu_w(t)|_0
\]

be the finite difference as the cycle level; then

\[
I^{(k)}_w(M, T) = (\nu_w, \gamma) \tag{8}
\]

Equation (7) passes from formal finite differences of 3-manifold invariants to traditional finite differences of polynomials. It follows that \( I^{(k)}_w(M) \) vanishes when \( k > 2n \), because \( I_w(t) \) is a polynomial of degree \( 2n \) in \( t \). Indeed, the cycle-valued finite difference \( \nu \) vanishes identically when \( k > 2n \).

A more precise calculation gives us the borderline finite difference \( I^{(2n)}_w(M, T) \). Observe that if a configuration \( f : \Gamma \rightarrow M_{\text{bub}} \) is disjoint from a bubble

\[
B_i = H_i \cup H'_i
\]

of the Torelli surgery then at this point \( \mu_w(t) \) is independent of \( t_i \); consequently \( \nu \) vanishes here. Since there are as many bubbles as vertices, \( \Gamma \) must have exactly one vertex in each bubble in the non-vanishing part of the pairing. Moreover the bubbles are 3-manifolds; on their product, the cycle \( \nu_w \) is just the fundamental homology class times the weight \( w(\Gamma) \). So we may write the pairing (8) as

\[
\sum_{\Gamma} w(\Gamma) \sum_{f : \Gamma \rightarrow [2n]} ([B_1 \times B_2 \times \ldots \times B_{2n}], \gamma). \tag{9}
\]

Given that in this sum each edge of \( \Gamma \) connects two distinct bubble \( B_i \) and \( B_j \), the corresponding factor of the cohomological propagator \( \gamma \) measures the linking between 1-cycles in the handlebody \( H_i \) (or \( H'_i \)) and 1-cycles in the handlebody \( H_j \) (or \( H'_j \)). This linking is the same before and after Torelli surgery, and the inclusion \( H_i \subset B_i \) is an isomorphism in first homology.

In conclusion the pairing (9) becomes a contraction of tensors: A vertex in the bubble \( B_i \) is replaced by the trilinear form

\[
\tau_i : H^1(B_i)^{\times 3} \rightarrow \mathbb{Q}
\]

given by the triple cup product, an edge connecting \( B_i \) to \( B_j \) is replaced by the pairing

\[
\lambda_{i,j} : H_1(B_i) \times H_1(B_j) \rightarrow \mathbb{Q}
\]

given by linking in any \( M_i \), and when an edge is incident to a vertex, the tensors are contracted. Figure 10 gives an example of such a replacement using arrow notation for tensor contractions [22]. These tensor expressions are summed over Jacobi diagrams \( \Gamma \) with vertices decorated by bubbles. Finally there is a factor of \( 2^{2n}(3n)! \) arising from orderings and orientations of the edges of \( \Gamma \), which are now vestigial. This leads to the desired value for \( I^{(2n)}(M, T) \) (implicit in work of Garoufalidis and Levine [13]).

### 6.4.2 Algebraically split surgery

In the algebraically split case, there is one chain \( \mu_w \) on all of \( C_{T, \infty}(M_{\text{dis}}) \) and \( C_T(E_{\text{dis}}) \), but there are \( 2^k \) cocycles \( \alpha_t \). These also form a parallelepiped in the space of all cocycles on \( \tilde{P}(M_{\text{dis}}) \), which is also encoded by an affine-linear function \( \alpha(t) \), with \( t \in \mathbb{R}^k \), such that

\[
\alpha_t = \alpha(t_I)
\]

for all \( I \subseteq [k] \). By Lemma 13, the function \( \alpha(t) \) has the explicit form

\[
\alpha(t) = \alpha + \sum_i t_i \beta_i \otimes \beta_i \tag{10}
\]

on the link complement \( M \setminus \Lambda \); slightly more generally, the formula also shows the dependence of \( \alpha(t) \) on \( t_i \) everywhere outside of the component \( K_i \). We correspondingly let

\[
\gamma(t) = \Phi^*(\alpha_1(t) \otimes \alpha_2(t) \otimes \ldots \otimes \alpha_k(t)
\]

in keeping with Lemma 14 and we define

\[
\kappa = \Delta_1 \Delta_2 \ldots \Delta_k \gamma(t)|_0.
\]
We would like to compute
\[ I^{(k)}(M, L) = \sum_{\Gamma} (\mu_w, \kappa). \] (11)

Two properties of this finite difference can be argued relatively easily. If \( k > 3n \), then \( \kappa \) vanishes identically, because \( \gamma(t) \) is a polynomial of degree \( 3n \) in \( t \). If \( k \leq 2 \), then \( \kappa \) vanishes on \( C_{1, \infty}^s(E_{\text{dis}}) \), because on each component of \( E_{\text{dis}} \), \( \gamma(t) \) is either proportional to a single \( t_i \) (if the component bounds the knot \( K_i \)) or it is constant (if the component is shared for all surgeries).

As with Torelli surgery, the marginal case \( k = 3n \) simplifies because \( \kappa \) is non-zero only when \( \Gamma \) is distributed among all components of the surgery. The following lemma expresses this principle of resource exhaustion.

**Lemma 14.** If \( k = 3n \), then \( \gamma(t) \) is independent of some \( t_i \) at a configuration \( f : \Gamma \to M_{\text{dis}} \) unless each point in \( f(\Gamma) \) lies at a triple intersection of Seifert surfaces of the link \( L \) in \( M \).

**Proof.** Say that a vertex of \( \Gamma \) provides a dollar to the component \( K_i \) if it lies in the knot \( K_i \), and it provides 50 cents if it lies in the Seifert surface \( S_i \). By equation (10), each component \( K_i \), of which there are \( 3n \) dollars, needs a dollar in order for \( \kappa(t) \) to depend on \( t_i \) at the configuration \( f \). Each vertex, of which there are \( 2n \), can provide at most \( $1.50 \), and only by lying at the intersection of three Seifert surfaces. The components need \( 3n \) dollars, which is the most that the vertices can provide. Therefore the vertices lie on the Seifert surfaces.

Having established that the finite difference \( \kappa \) is supported in the link complement \( M \setminus L \), we can compute \( I^{(3n)}(M, L) \) using the relative cohomology ring \( H^*(M, L) \). Equation (10) implies that
\[ \kappa = \Phi^*(\prod_i \beta_i \otimes \beta_i). \]

This cocycle blows down from the configuration space \( C^s_{1, \infty}(M \setminus L) \) to the Cartesian product \( (M \setminus L)^3 \). After blowing down, the chain \( \mu_w \) is now proportional to the fundamental class:
\[ \mu_w = w(\Gamma)(M \setminus L)^3. \]

The upshot is that the pairing (11) evaluates to another numerical formula with the geometry of \( \Gamma \): the total weight of all diagrams \( \Gamma \) decorated with a bijection with the link components. Here the weight of any single diagram is the product of the weights of its vertices. If a vertex has incoming edges \( i, j, \) and \( k \), its weight is the triple linking number of the knots \( K_i, K_j, \) and \( K_k \). This is again the desired answer (15).

**Remark.** Blowing down from the configuration space to the Cartesian product is one solution to a geometric difficulty in the computation of \( I^{(k)}(M, L) \): Two vertices of \( \Gamma \) might want to lie at the same triple intersection of Seifert surfaces in \( M \setminus L \), but it is then difficult to see the behavior of the propagator between them. In differential terms, the operation of blowing down says that the diagonal singularities of the propagators cancel when we take suitable finite differences. Another approach is to choose two Seifert surfaces \( S_i \) and \( S_i' \) for each link component \( K_i \), so that Lemma (12) becomes
\[ \alpha_{\Gamma} = \alpha + \sum_i \beta_i \otimes \beta_i'. \]

If all of the Seifert surfaces are in general position, then the triple points on \( S_i \) and on \( S_i' \) will be disjoint, and the computation of \( I^{(3n)}(M, L) \) reduces to counting transverse intersections of manifolds far away from the blowup loci.

For rational homology spheres there is an interesting generalization of algebraically split surgery: the framing of each link component \( K_i \) can be a non-zero rational number \( p_i/q_i \). In this case Lemma (12) becomes
\[ \alpha_{\Gamma} = \alpha + \sum_i \frac{q_i}{p_i} \beta_i \otimes \beta_i. \]

It follows that the marginal finite difference \( I^{(3n)}(M, L) \) is multilinear in the reciprocals of the framings.

### 6.5 An unframed invariant

The proof of Theorem 3 rests on three constructions.

First, let \( W \) be a closed homology 4-manifold with a 3-plane bundle \( E \). Following Section 5.2, the sphere bundle \( c_n(E) \) has a canonical cohomology class \( \alpha \) which is antisymmetric with respect to fiberwise inversion, and there is a bundle \( c_n(E) \) of total configuration spaces of the fibers. As usual, the pull-back
\[ \Phi^*(\alpha \otimes 3n) \in H^{6n}(c_n(E)) \]
maps to an element in the Jacobi diagram space \( V_n \), yielding a universal invariant \( I_n(E) \).

The class \( \alpha \) is not only canonical, but *functorial* with respect to pull-backs of bundles. The rest of the construction is fiberwise and therefore also functorial. On the other hand, since \( E \) is a real 3-plane bundle, its only rational characteristic number is its Pontryagin number \( p_1(E) \) [24]. Consequently
\[ I_n(E) = r_n p_1(E) \]
for some universal vector \( r_n \in V_n \).

Second, if \( E \) is an oriented 4-plane bundle over some space, it has two associated 3-plane bundles \( F_k' = \Lambda^2 \oplus (F_k \otimes \Lambda^2) \) whose fibers are the spaces of self-dual and anti-self-dual antisymmetric 2-tensors. If \( W \) is an orientable Riemannian 4-manifold with boundary \( M \), the bundles \( T_k' W \) both canonically restrict to \( T M \). Also \( W \) has a modified tangent bundle \( T' W \) that extends \( T' M \), and correspondingly \( T_k' W \) extend \( T_k' M \).

If \( F \) is any oriented 4-plane bundle over it, then the average of the Pontryagin numbers \( p_1(F_k') \) is the Pontryagin number \( p_1(F) \). If \( F = T W \), then the Hirzebruch signature theorem says that the Pontryagin number is thrice the signature \( \sigma(W) \),
If we perform surgery on a knot $p$ if the intersection form of $W$ has signature $(a, b)$ \cite[Th. 19.4]{Fulton-MacPherson}. Algebraically,

$$p_1(TW) = \frac{1}{2}(p_1(T_+W) + p_1(T_-W)) = 3\sigma(W). \tag{12}$$

If $W = W_1 \cup W_2$ is the union of two 4-manifolds which share boundary $M$ with a marked point $\infty$, then it has a modified tangent bundle $T'W$ extending $T'W_1$ and $T'W_2$. The Euler number of $T'W$ differs by 2 from that of $TW$, but the Pontryagin number is the same, so $T'W$ satisfies equation (12) as well. (Since $TW$ and $T'W$ differ only in the neighborhood of $\infty$ and only in a canonical way, this fact can be verified with a single example: $T'S^4$ is trivial while $\chi(TS^4) = 2$ and $p_1(TS^4) = 0$.)

Third, if $M$ is a homology 3-sphere decorated with a bundle bordism $E$ over a homology 4-manifold $W$. Let $W_1$ be a smooth 4-manifold with boundary $M$ and signature 0, and let $E_1$ be the bundles formed by extending $E$ by $T^*_1W_1$. Then we define

$$\delta_n(M) = \frac{r_n}{2}(p_1(E^+) + p_1(E^-)).$$

By equation (12) this quantity does not depend on $W_1$. (If we replace $W_1$ by $W_2$, their union has signature 0 because $M$ is a rational homology sphere; consequently the Pontryagin number, which determines the change in $\delta_n(M)$, is 0 as well.)

Also the difference

$$\tilde{I}_n(M) = I_n(M) - \delta_n(M)$$

is independent of $E$ by the definition of $r_n$.

It remains to show that $\delta_n(M)$ is finite type of degree 1. The argument is clearer if we restrict to certain specific bundle bordisms on $M$ and its relatives obtained by surgery. Namely we choose a 4-manifold $W$ with boundary $M$ and we decorate $M$ with the formal average of the bundles $\Lambda^3_2(T^*W)$. In this case the framing correction is given by

$$\delta_n(M) = 3r_n\sigma(W).$$

If we perform surgery on a knot $K \subset M$ or a Torelli surgery on a handlebody $H \subset M$, we extend $W$ arbitrarily. In this case the intersection form of $W$ changes by taking direct sums with matrices that depend only on the surgery. Since the signature of a form is linear under direct sums, it is finite type of degree 1, as desired. The argument that it is finite type for general decorations of $M$ is similar.

To conclude this section, we compute the first framing correction coefficient $r_1$. The invariant $I_1(M)$ lies in the 1-dimensional vector space $V_1$ generated by a theta graph; we choose a basis such that

$$\langle \alpha^{\pm 3}/6, [\tilde{C}_2(M)] \rangle.$$

The simplest twisted $S^2$-bundle on $S^4$ has Pontryagin number $p_1 = 4$ and total space $\mathbb{C}P^3$; one model of it is the sequence

$$S^7 \to \mathbb{C}P^3 \to \mathbb{H}P^1 = S^4$$
given by quotienting $S^7$ by complex and quaternionic multiplication. By the ring structure of $H^*(\mathbb{C}P^3)$ and the fact that $\alpha$ generates it,

$$\langle \alpha^{\pm 3}/6, [\mathbb{C}P^3] \rangle = 1/6.$$

Thus $r_1 = 1/24$.

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