Cosmic Mach Number: A Sensitive Probe for the Growth of Structure

Yin-Zhe Ma\textsuperscript{1}, Jeremiah P. Ostriker\textsuperscript{2,1} and Gong-Bo Zhao\textsuperscript{3}

\textsuperscript{1}Kavli Institute for Cosmology and Institute of Astronomy, University of Cambridge, Madingley Road, Cambridge, CB3 0HA, UK
\textsuperscript{2}Department of Astrophysical Sciences, Princeton University, Princeton, NJ 08544, USA
\textsuperscript{3} Institute of Cosmology and Gravitation, University of Portsmouth, Dennis Sciama Building, Portsmouth, PO1 3FX, UK

We investigate the potential power of the Cosmic Mach Number (CMN), which is the ratio between the mean velocity and the velocity dispersion of galaxies as a function of cosmic scales, to constrain cosmologies. We first measure the CMN from 4 catalogs of galaxy peculiar velocity surveys at low redshift \((z \in [0.002, 0.03])\), and use them to contrast cosmological models. Overall, current data is consistent with the WMAP 7 A\(\Lambda\)CDM model. We find that the CMN is highly sensitive to the growth of structure on scales \(k \in [0.01, 0.1]\) h/Mpc in Fourier space. Therefore, modified gravity models, and models with massive neutrinos, in which the structure growth generically deviates from that of the A\(\Lambda\)CDM model in a scale-dependent way, can be well differentiated from the A\(\Lambda\)CDM model by using future CMN data.

Introduction— The Cosmic Mach Number (hereafter CMN) can provide a robust measure of the shape and growth rate of the peculiar velocity power spectrum of the galaxies in the universe. One considers a region of a given size \(r\) in the universe, and compares the bulk motion of the sphere with the random velocities within that region. The bulk motion provides a measurement of the forces on the region from irregularities external to it, so it measures the amplitude of perturbations on scales much larger than the region, whereas the random motions within the comoving region reflect gravitational perturbations on scales smaller than \(r\). Thus their ratio, \(M(r)\), depends on the shape of the perturbation spectrum while independent of its amplitude.

The concept was introduced by Ostriker and Suto in 1990 \cite{OS90} as a cosmological metric that would be relatively independent of the “bias” of the test particles being observed and also relatively independent of the then quite uncertain perturbation amplitude. They concluded that although the existing data were poor, they gave estimates of the CMN that appeared to be inconsistent with the then popular CDM model with \(\Omega_m = 1\) but seemed to prefer the open universe model instead. In a certain sense, the application of this test, correctly predicted the currently best validated cosmological models with a value of \(\Omega_m\) in the range of 0.2 – 0.3.

Subsequent to the original paper, Strauss et al (1993) \cite{S93} found again that the CDM models with \(\Omega_m = 1\) remained inconsistent with the better data they used, but some non-standard models passed the test (see also \cite{N01}). Nagamine et al (2001) \cite{N01} looked at \(\Lambda\)CDM models and found better agreement, but the then \(\Lambda\)CDM model with \(\Omega_m = 0.37\) again produced too high values of \(M\) over the range of \(3 - 40\) Mpc/h, whereas a model with \(\Omega_m \sim 0.2\) (actually closer to WMAP 7-yr constraint \cite{S93}) was more consistent with the observations. In addition, there are various other papers discussing the issues related to the CMN, such as non-linear clustering properties of dark matter on the CMN measurement \cite{MS11}, and constraints on the CMN from Sunyaev-Zeldovich effect \cite{S97}.

This history leads us to re-examine the issue in light of the much better knowledge now available from both the new peculiar velocity data and the range of models remaining plausible given the current cosmological constraints. In this paper, we will develop a new statistical tool to measure the CMN from the peculiar velocity surveys, and investigate the power of the CMN to distinguish various cosmological models, especially in the aspects of differentiating the \(\Lambda\)CDM model from variance with non-trivial growth function provided by Modified Gravity (hereafter MG) models, and from models with massive neutrinos.

Statistics of Cosmic Mach Number— In linear perturbation theory, the power spectrum of the velocity divergence \((\theta \equiv \nabla \cdot \mathbf{v})\) is related to the power spectrum of density fluctuations via \(P_\delta(k,z) = f^2(k,z)P(k,z)\), where \(f(k,z) \equiv -d\ln \delta/d\ln (1+z)\), and \(\delta\) is the density perturbation of matter. Since the data in our application are at very low-redshift, we assume that they have the same redshift \(z = 0\) throughout and drop the \(z\) dependence for brevity The mean square velocity dispersion \(\langle \sigma^2(r) \rangle\) and mean square bulk flow \(\langle V^2(r) \rangle\) in a window of size \(r\) can be calculated as \cite{OS90, N01}

\[
\langle V^2(r) \rangle = \frac{H_0^2}{2\pi^2} \int_0^\infty dk P_{\delta\delta}(k) W(kr),
\]

\[
\langle \sigma^2(r) \rangle = \frac{H_0^2}{2\pi^2} \int_0^\infty dk P_{\delta\delta}(k) [1 - W(kr)], \quad (1)
\]

where \(W(x) = [3(x - x \cos x)/x^3]^2\) is a top-hat window function. Note that \(W(x) \sim 1 (x \lesssim 1)\) and \(W\) drops to 0 quickly when \(x > 1\). Thus \(W\) effectively changes the...
We can see that the velocity dispersion \( \sigma \) can be easily determined by probes such as the Cosmic Data points are the data used in Ostriker and Suto 1990 [1]. The black line is the CMN prediction from WMAP 7-yr best-fit values [5]; the green dashed line is calculated by using (1990) ‘popular’ CDM cosmological parameters \( n_s = 0.96 \). The brown dashed and purple dot-dashed lines are the CMN from \( f(R) \) model with \( B_0 = 10^{-4} \) and from \( \Lambda \)CDM model with neutrino mass \( \sum m_\nu = 0.6 \text{eV} \).

Directly measuring \( \sigma(r) \) is simply \( \sigma^2(r) = \int dk P_{\theta \theta}(k) \) on large, and small scales. Using Eqs (2) and (3), one can reconstruct \( P_{\theta \theta}(k) \) up to an overall constant \( A \),

\[
P_{\theta \theta}(k = 1/r) = A \frac{dM(r)}{[1 + M^2(r)]^2} .
\]

Thus it basically measures the shape of \( P_{\theta \theta} \) by contrasting \( \int dk P_{\theta \theta}(k) \) on large, and small scales. Using Eqs (2) and (3), one can reconstruct \( P_{\theta \theta}(k) \) up to an overall constant \( A \),

\[
P_{\theta \theta}(k = 1/r) = A \frac{dM(r)}{[1 + M^2(r)]^2} .
\]

FIG. 1. (a): The 1-D posterior distribution of the CMN \( M(r) \) from 4 different catalogs. (b): The comparison between the CMN data ('3σ CL') and theoretical prediction: blue data points are the CMN data from posterior distributions shown in panel (a); red data points are the data used in Ostriker and Suto 1990 [1].

Then one can construct a joint likelihood function for \( u \) and \( M \) by contrasting \( S_n \) with the line-of-sight projection of the bulk flow \( \hat{r}_{n,i} u_i \). The uncertainty in \( (S_n - \hat{r}_{n,i} u_i) \) is simply \( \sigma_u^2 = \sigma_n^2 + \sigma_s^2 \), where \( \sigma_u = |u|/M \) is given by the definition of the CMN. Therefore, the likelihood function takes the form of,

\[
L(u,M) = \prod_{n=1}^{N} \frac{1}{1 + \frac{2}{(u^2 + (u|M|^2))^2}} \times \exp \left( -\frac{(S_n - \hat{r}_{n,i} u_i)^2}{2(\sigma_n^2 + (u|M|^2))^2} \right) .
\]

One can then marginalize over the 3D bulk flow vector \( u \) to obtain the distribution of \( M \) for each survey.

We use four different catalogs of galaxy peculiar velocity surveys, namely, SBF [8], ENEAR [9], Type Ia Supernovae (SN) [10], and SFI++ [11], to constrain the CMN by using Eq. (5). To calculate the characteristic depth \( \tau \) of each catalog, we use error-weighted depth as

\[
\tau = \sum_n w_n n / \sum_n w_n , \quad w_n = 1/(\sigma_n^2 + \sigma_s^2) .
\]

We marginalize over the ‘bulk flow’ velocity \( u \) in the 4-D parameter space and obtain the 1-D posterior distribution of \( M \) as shown in panel (a) of Fig. 1. In the panel (a), one can see that the distribution of the CMN is very Gaussian, and the width depends primarily on the number of data entries in each catalog. In addition, since each catalog probes the CMN on various depths, different catalogs form a complimentary set of tests of cosmic structures.

In panel (b) of Fig. 1, we put together the old (1990) and current CMN data with various predictions computed from theoretic models: The [Blue] triangle data points are the current CMN values computed from likelihood (5) by using the four-catalogs, which provide more and deeper data than those used by Ostriker and Suto [1] in 1990 ([Red] square data).
The ΛCDM model with WMAP 7-yr best-fit parameters ($\Omega_m = 0.271$, $h = 0.704$, $\Omega_b = 0.967$, $\Omega_B = 0.0455$, [Black] solid line) is mildly consistent with the current CMN data at 3 σ CL. In comparison, we overplot the theoretical $M(r)$ ([Green] dashed line) by using 1990’s ‘popular’ CDM parameters. One can immediately understand the reason why Refs. [1, 2] claimed that there was a strong conflict between the data ([Red] points) and popular CDM model ([Green] dashed line). There was in fact no inconsistency between data and a model with massive neutrino (dot-dashed line) models. [c]: Same comparison as (b) but for growth function $f$. We summarize the experimental conditions for future experiment that can sharpen the CMN test: (1) there should be considerably more galaxy samples ($\gtrsim 10^5$) on scales [10, 150] Mpc/h; (2) the smaller the measurement error $\sigma_{\eta}$ is, and the larger sky area it covers, the better it can reduce the overall error of $M$. One should also notice that, when the CMN data is used to constrain cosmology, the cosmic variance (CV) due to the limited volume of the survey should be taken into account. The cosmic variance $\sigma_{CV}(r)$ can be estimated as $\left[ \frac{\sigma_{CV}(r)}{A(r)} \right]^2 = \frac{1}{4} \left( \frac{\sigma_A(r)}{A(r)} \right)^2 + \left( \frac{\sigma_B(r)}{B(r)} \right)^2$, where $A$ and $B$ stand for $(V^2(r))$ and $(\sigma^2(r))$ respectively, as defined in Eq (1), and $\sigma_A(r)$ and $\sigma_B(r)$ can be calculated as, $\sigma_A(r) = \frac{\sigma^2_{\theta \theta}}{2\pi^2} \int \frac{1}{\sqrt{k^2 - \Delta k^2}} P_{\theta \theta}(k) W(kr) dk$, $\sigma_B(r) = \frac{\sigma^2_{\phi \phi}}{2\pi^2} \int \frac{1}{\sqrt{k^2 - \Delta k^2}} P_{\theta \theta}(k) [1 - W(kr)] dk$, where $\Delta k$ and $V$ are the width of the $k$ bin and the volume of the survey respectively, and $W$ is the same window function as in Eq (1). We found that for the current velocity surveys of SBF, ENEAR, SN and SFI+++, CV is 21, 19, 5 and 10% of the measurement error. For the 6df survey, CV is smaller than 5% of the statistical errors in all the bins.

A sensitive test of growth of structure—Since the CMN measures the shape of the peculiar velocity power spectrum $P_{\theta \theta}$ by design, it is sensitive to any distortion of $P_{\theta \theta}$. In the ΛCDM model, the growth is scale-independent. However, in the modified gravity models, and models with massive neutrinos, the growth is generally scale-dependent, thus $P_{\theta \theta}$ for these models is a distorted version comparing with the ΛCDM model, making the CMN an excellent tool to distinguish these models from ΛCDM.

Let us consider the $f(R)$ model and ΛCDM model with massive neutrinos as examples. In a viable $f(R)$ model, where the Einstein-Hilbert action is extended to be a general function of the Ricci scalar $R$, the effective type galaxies, which is roughly 10% of the total sample, and they are located at $z \lesssim 0.05$, corresponding to the depth $r \lesssim 150$ Mpc/h. We further assume that the measurement error for line-of-sight velocity is around 20%, which is a typical error for the fundamental plane distance measurement. We divide these data into different redshift bins, and in each shell $(r, r + dr)$, we calculate the Fisher Matrix value for the CMN $F_{MM}$ from Eq (5) which leads to the forecasted error of the CMN as

$$\sigma_M(r) \simeq \sqrt{\frac{M(r)}{2N(r)}} \left[ 1 + \frac{\sigma_B^2}{(u(r)/M(r))^2} \right],$$

where $N(r)$ is the number of data points in the shell $(r, r + dr)$, and $u(r)$ is the average bulk flow magnitude on depth $r$. We plot these forecast data in the panel (a) of Fig. 2. Comparing with panel (b) of Fig. 1, We find the full range of the CMN data on scales [10, 150] Mpc/h from 6df can improve the constraint on the variation of the scale-dependent growth factor significantly. We summarize the experimental conditions for future experiment that can sharpen the CMN test: (1) there should be considerably more galaxy samples ($\gtrsim 10^5$) on scales [10, 150] Mpc/h; (2) the smaller the measurement error $\sigma_{\eta}$ is, and the larger sky area it covers, the better it can reduce the overall error of $M$. One should also notice that, when the CMN data is used to constrain cosmology, the cosmic variance (CV) due to the limited volume of the survey should be taken into account. The cosmic variance $\sigma_{CV}(r)$ can be estimated as $\left[ \frac{\sigma_{CV}(r)}{A(r)} \right]^2 = \frac{1}{4} \left( \frac{\sigma_A(r)}{A(r)} \right)^2 + \left( \frac{\sigma_B(r)}{B(r)} \right)^2$, where $A$ and $B$ stand for $(V^2(r))$ and $(\sigma^2(r))$ respectively, as defined in Eq (1), and $\sigma_A(r)$ and $\sigma_B(r)$ can be calculated as, $\sigma_A(r) = \frac{\sigma^2_{\theta \theta}}{2\pi^2} \int \frac{1}{\sqrt{k^2 - \Delta k^2}} P_{\theta \theta}(k) W(kr) dk$, $\sigma_B(r) = \frac{\sigma^2_{\phi \phi}}{2\pi^2} \int \frac{1}{\sqrt{k^2 - \Delta k^2}} P_{\theta \theta}(k) [1 - W(kr)] dk$, where $\Delta k$ and $V$ are the width of the $k$ bin and the volume of the survey respectively, and $W$ is the same window function as in Eq (1). We found that for the current velocity surveys of SBF, ENEAR, SN and SFI+++, CV is 21, 19, 5 and 10% of the measurement error. For the 6df survey, CV is smaller than 5% of the statistical errors in all the bins.

A sensitive test of growth of structure—Since the CMN measures the shape of the peculiar velocity power spectrum $P_{\theta \theta}$ by design, it is sensitive to any distortion of $P_{\theta \theta}$. In the ΛCDM model, the growth is scale-independent. However, in the modified gravity models, and models with massive neutrinos, the growth is generally scale-dependent, thus $P_{\theta \theta}$ for these models is a distorted version comparing with the ΛCDM model, making the CMN an excellent tool to distinguish these models from ΛCDM.
value of Newton’s constant $G_{\text{eff}}$ has both time and scale dependence, namely, $G_{\text{eff}} = \mu(a, k)G$, where $\mu(a, k) = (1 + \frac{1}{2} \lambda \xi k^2 a^4)/(1 + \lambda \xi k^2 a^4)$, and $G$ is the Newton’s constant in general relativity (GR) \[14, 15\]. The only free parameter is $\lambda^2$, which quantifies the Compton wavelength of the scalar field $f_R \equiv df/dR$ and characterizes the scale-dependent growth rate. It is more convenient to redefine a dimensionless $B_0$ which is $B_0 = 2H_0^2 \lambda^2 / c^2$, and $B_0 = 0$ for GR \[15\].

In panels (b) and (c) of Fig. 2, we show the relative difference in $P_{\theta\theta}$ and $f$ with respect to that in the $\Lambda$CDM model for a $f(R)$ model with $B_0 = 10^{-4}$ (dashed lines). The growth rate for $f(R)$ model is enhanced on scales of $k \gtrsim 0.01 \text{ h/Mpc}$, thus the integration of $P_{\theta\theta}(k)$ cumulates the deviation in $k$–space and exhibits the substantial difference of CMN between $\Lambda$CDM and $f(R)$ model. As a result shown in panel (a) of Fig. 2, a 0.01% of $B_0$ in $f(R)$ model can produce a 20% suppression in the CMN, which is potentially observable by 6dF.

Similarly, the CMN is sensitive to the neutrino mass. When neutrinos became non-relativistic and the Universe was deeply in the matter dominated era, neutrino thermal velocities damped out the perturbation under the characteristic scale $k_{nr} \simeq 0.018\Omega_m^{1/2}(\sum \nu /1\text{eV})\text{h/Mpc}$, suppressing the power spectrum on small scales. On scales greater than $k_{nr}$, neutrinos affect the overall expansion of the Universe and therefore shift the peak of power spectrum to larger scales. In panels (b) and (c) of Fig. 2, we can see that neutrino with mass of 0.6eV can suppress the power spectrum on scales of $k \gtrsim 0.01\text{h/Mpc}$ significantly, which exactly falls in the detection window of the CMN. Therefore, the cumulative ‘integral’ of $P_{\theta\theta}(k)$ in CMN can manifest the neutrino free-streaming effect by enhancing its value on all scales. Comparing with panel (b) of Fig. 1 and panel (a) of Fig. 2, we find that future CMN data is potentially able to distinguish the nonzero neutrino mass.

Note that the possible degeneracy among cosmological parameters might dilute the constraints. Therefore we employ a Fisher matrix forecast \[16\] to study the power of CMN on cosmological parameter constraints. The Fisher matrix for CMN is,

$$F_{\mu\nu} = \sum_{i,j} \frac{\partial M(r_i)}{\partial p_\mu} \text{Cov}^{-1}[M(r_i), M(r_j)] \frac{\partial M(r_j)}{\partial p_\nu}$$ \quad (7)

where $p$ denotes the cosmological parameter, and Cov is the data covariance matrix for CMN. We include the cosmic variance of CMN in our analysis. We make a forecast for two models: $A = \{B_0, CP\}$, $B = \{\Sigma m_\nu, CP\}$ where CP is a set of basic cosmological parameters: $CP = \{\Omega_m, \sigma_8, H_0, n_s\}$. We choose the best-fit model for WMAP7 as a fiducial model \[5\], namely, $CP = \{0.266, 0.801, 71, 0.93\}$, and choose $B_0 = 10^{-5}$ and $\Sigma m_\nu = 0.05 \text{ eV}$ as fiducial values for models A and B respectively. We also marginalize over the nuisance parameter $C$, whose fiducial value was evaluated using the $\Lambda$CDM model. We also combine the Fisher matrices for the current data of CMB (WMAP \[5\]) and SNe (UNION2 \[17\]) to improve the constraint (see details for CMB and SNe Fisher matrices in \[14, 18\]). The result is shown in Fig 3. As we can see, the constraints on $B_0$ and neutrino mass are largely improved when the CMN data from 6dF is combined. Quantitatively, the 95% CL. upper limits on $B_0$ and $\Sigma m_\nu$ are $B_0 < 0.4$ \[15\] and $\Sigma m_\nu < 1.3 \text{ eV}$ \[5\] using WMAP7 combined with UNION2 data, and the constraints can be tightened to $B_0 < 5 \times 10^{-5}$ and $\Sigma m_\nu < 0.65 \text{ eV}$ when the CMN data from 6dF is included.

Conclusion—We provide a statistical tool to measure the CMN, and further demonstrate that it is a sensitive probe of the structure growth. By design, the CMN is immune to the uncertainty in the overall amplitude of the density perturbation, and to linear galaxy bias. Also it is highly sensitive to any scale-dependent distortion of $P_{\theta\theta}$ since any small distortion can be amplified via the integral effect. Therefore it is an excellent tool to test alternative theories of gravity, and models with non-zero neutrino mass.

We first perform a likelihood analysis of the CMN from the current peculiar velocity field data, and further confront the WMAP7 $\Lambda$CDM model and 1990s popular CDM model with the CMN data from current and old observations. We confirm that the $\Lambda$CDM model with WMAP 7-yr parameters is consistent with current CMN data at 3σ CL. level. Based on our forecast for 6dF, we find that the CMN can improve the constraints on the modified gravity parameter $B_0$ by 4 orders of magnitude, and it can also tighten the present constraints on the neutrino mass. The CMN information from future surveys, such as the Square Kilometre Array (SKA) \[19\], will be more powerful to constrain cosmologies, especially for modified gravity models, and models with massive neutrinos.

Acknowledgement—We would like to thank A. Challinor, G. Efstathiou, K. Koyama and K. Masters for
helpful discussions. GBZ is supported by STFC grant ST/H002774/1.

[1] J. P. Ostriker and Y. Suto, ApJ, 348 (1990) 378-382.
[2] M. A. Strauss, R. Cen and J. P. Ostriker, ApJ, 408 (1993) 389-402.
[3] Y. Suto, N. Gouda and N. Sugiyama, ApJ, 74 (1990) 665-674; Y. Suto, R. Cen and J. P. Ostriker, ApJ, 395 (1992) 1-20.
[4] K. Nagamine, J. P. Ostriker and R. Cen, ApJ, 553 (2001) 513-527.
[5] E. Komatsu et al., ApJ, 192 (2011) 18.
[6] M. Gramann, N. A. Bahcall, R. Cen and J. R. Gott, ApJ, 441 (1995) 449-457.
[7] F. Atrio-Barandela, A. Kashlinsky and J. P. Mucket, ApJ, 601 (2004) 111-114.
[8] J. L. Tonry et al., ApJ, 546 (2001) 681.
[9] L. N. da Costa et al., Astron. J, 120 (2000) 95; M. Bernardi, et al., Astron. J, 123 (2002) 2990.; G. Wenger et al., Astron. J, 126 (2003) 2268.
[10] J. L. Tonry, et al., Astrophys.J., 594 (2003) 1.
[11] C. M. Springob, et al., ApJ Supp, 172 (2007) 599.
[12] E. Jennings, C. Baugh and S. Pascoli, MNRAS 410 (2011) 2081-2094.
[13] D. H. Jones et al., MNRAS 399 (2009) 683-698.
[14] G. B. Zhao, L. Pogosian, A. Silvestri, and J. Zylberberg, Phys. Rev. D 79 (2009) 083513.
[15] T. Giannantonio, M. Martinelli, A. Silvestri, A. Melchiorri, JCAP, 04 (2010) 030.
[16] M. Tegmark, A. Taylor, A. Heavens, Astrophys. J. 480, 22 (1997).
[17] R. Amanullah et al., ApJ, 716 (2010) 712-738.
[18] L. Pogosian, et al. Phys. Rev. D72 (2005) 103519.
[19] http://www.skatelescope.org