Concentration of symplectic volumes on Poisson homogeneous spaces

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Abstract

For a compact Poisson-Lie group $K$, the homogeneous space $K/T$ carries a family of symplectic forms $\omega_s^\xi$, where $\xi \in t_+^*$, is in the positive Weyl chamber and $s \in \mathbb{R}$. The symplectic form $\omega^0_\xi$ is identified with the natural $K$-invariant symplectic form on the $K$ coadjoint orbit corresponding to $\xi$. The cohomology class of $\omega^s_\xi$ is independent of $s$ for a fixed value of $\xi$.

In this paper, we show that as $s \to -\infty$, the symplectic volume of $\omega^s_\xi$ concentrates in arbitrarily small neighborhoods of the smallest Schubert cell in $K/T \cong G/B$. This strengthens an earlier result of [10] and is a step towards a conjectured construction of global action-angle coordinates on $\text{Lie}(K)^* [4, Conjecture 1.1]$.

1 Introduction

Let $K$ be a compact connected Lie group with maximal torus $T$ and let $G = K^C$ denote its complexification. Let $t$ denote the Lie algebra of $T$. As our results concern the homogeneous space $K/T$, we may assume without loss of generality that $K$ is semisimple and simply connected.

The homogeneous space $K/T$ carries an interesting family of symplectic structures $\omega^s_\xi$ parameterized by $s \in \mathbb{R}$ and elements of a positive Weyl chamber, $\xi \in t_+^*$. Following [13], the Iwasawa decomposition $G = AN_-.K$ defines dual Poisson-Lie groups $(K, \pi_K)$ and $(AN_-, \pi_{AN_-})$. The symplectic leaves of $\pi_{AN_-}$ are the orbits of the so-called dressing action of $K$ on $AN_-$. Let $D_\xi \subset AN_-$ denote the dressing orbit through $\exp(\sqrt{-1}\xi)$, where $\xi \in t^*$ is identified with an element of $t$ via the Killing form. For all $s \neq 0$ and $\xi \in t_+^*$, fix the $K$-equivariant identification of $K/T$ with $D_\xi$ such that $eT \mapsto \xi$ and define

$$\pi^s_\xi := s\pi_{AN_-}|_{D_\xi}, \quad \omega^s_\xi := (\pi^s_\xi)^{-1}.$$  \hspace{1cm} (1)

For $s = 0$ and $\xi \in t_+^*$, fix the $K$-equivariant identification of $K/T$ with the coadjoint orbit $\Theta_\xi$ such that $eT \mapsto \xi$ and define $\omega^0_\xi$ to be the Kostant-Kirillov-Souriau symplectic form.

The family $\omega^s_\xi$ was studied in [1, 11] and has several nice properties. First, the action of $K$ on $K/T$ is Poisson: the action map $K \times K/T \to K/T$ is a Poisson map with respect to $s\pi_K$ and $\pi^s_\xi$ for all $s$ and $\xi$. In other words, $(K/T, \pi^s_\xi)$ is a Poisson homogeneous space for $(K, s\pi_K)$. Poisson homogeneous spaces for $(K, s\pi_K)$ were classified in [9]. Second, for a fixed value of $\xi$ the forms $\omega^s_\xi$ are isotopic for all $s \in \mathbb{R}$ [1]. It follows that for fixed $\xi$ and arbitrary $s$ the forms $\omega^s_\xi$ are cohomologous. In particular, their symplectic (Liouville) volumes are the same:

$$\text{Vol}(K/T, \omega^s_\xi) = \text{Vol}(K/T, \omega^0_\xi).$$  \hspace{1cm} (2)

Keywords: Poisson-Lie groups, homogeneous spaces, coadjoint orbits, symplectic geometry

1Note that for $s < 0$, $\omega^s_\xi$ is the symplectic structure on $K/T$ defined by $-s\pi_\lambda$, $\lambda = -s\sqrt{-1}\xi$, where $\pi_\lambda$ is the Poisson structure defined by Lu in [11, Notation 5.11].
Let $B \subset G$ be the positive Borel subgroup (corresponding to $t^*_+$. The flag variety $G/B$ is isomorphic to $K/T$ and admits a stratification into Schubert cells $BwB/B$, indexed by elements $w$ of the Weyl group. The smallest Schubert cell is the point $eB \in G/B$ and the biggest Schubert cell, $Bw_0B/B$, corresponding to the longest element $w_0 \in W$, is dense in $G/B$.

It follows from [11, Proposition 5.12] that the rescaled family of Poisson structures $s^{-1} \pi^s_\xi$ admits, for all $\xi$, a common limit $\pi^\infty$ when $s \rightarrow -\infty$. The Poisson structure $\pi^\infty$ coincides with the image of the standard Poisson structure $\pi_K$ under the projection map $K \rightarrow K/T$ and its symplectic leaves are exactly the Schubert cells. Theorem 2.2 in [10] implies the following:

**Theorem 1.1.** Let $\overline{U}$ be a compact subset of the big Schubert cell $Bw_0B/B$. Then, for any $\xi \in t^*_+$ and $\varepsilon > 0$, there exists $s_0 \in \mathbb{R}$ such that for $s \leq s_0$,

$$\text{Vol}(\overline{U}, \omega^s_\xi) < \varepsilon.$$ 

**Proof.** Fix $\xi \in t^*_+$ and identify $K/T$ with the dressing orbit $D_{s\xi}$ as above, equipped with $s\pi_{AN_-}$. Let $\text{pr}_A: G \rightarrow A$ denote projection with respect to the Iwasawa decomposition $G = AN_-K$. Identify $t \cong t^*$ via the Killing form. With these identifications,

$$\Psi_s: K/T \rightarrow t^*, \quad kT \mapsto \frac{1}{s\sqrt{-1}} \log \text{pr}_A(k \exp(s\sqrt{-1}\xi)),$$

is a moment map for the action of $T$ on $(K/T, \omega^s_\xi)$ by left multiplication, for all $s \neq 0$ [12, Theorem 4.13]. The $T$-fixed points, their weights, and their images under the moment map do not depend on $s$. Thus the Duistermaat-Heckman measure on the moment polytope defined by $\Psi_s$ is independent of $s$.

Fix a compact subset $\overline{U} \subset Bw_0B/B$. By [10, Theorem 2.2], there exists $r > 0$ such that

$$\|\log \text{pr}_A(k \exp(s\sqrt{-1}\xi)) - s w_0 \sqrt{-1}\xi\| < r$$

for all $\xi \in t^*_+$, $s < 0$, and $k \in \overline{U}$. The norm $\|\cdot\|$ is taken with respect to the Killing form. It follows that for fixed $\xi \in t^*_+$ and all $s < 0$,

$$\|\Psi_s(kT) - w_0\xi\| = \left\|\frac{1}{s\sqrt{-1}} \log \text{pr}_A(k \exp(s\sqrt{-1}\xi)) - w_0\xi\right\| < \frac{r}{|s|}$$

for all $k \in \overline{U}$. Since the Duistermaat-Heckman is independent of $s$, this implies that $\text{Vol}(\overline{U}, \omega^s_\xi) < \varepsilon$ for all $s < 0$ sufficiently large.

In other words, any compact subset of the big Schubert cell is depleted of symplectic volume as $s \rightarrow -\infty$. Since total volume is constant for fixed $\xi$, this implies that the volume concentrates in a small neighborhood of the other Schubert cells.

**Example 1.2.** As an illustration of this phenomenon, consider the example of $K = SU(2)$. Identify $t^* = \mathbb{R}$ and $\xi \in t^*_+ = \mathbb{R}_{>0}$. Let $(z, \varphi) \in (-1,1) \times (0, 2\pi)$ be cylindrical coordinates on the unit-sphere $S^2 \subset \mathbb{R}^3$ and fix the $K$-equivariant identification of $K/T$ with $S^2$ such that $eT$ is identified with the pole $z = 1$. The family of symplectic forms is

$$\omega^s_\xi = \begin{cases} \frac{\sinh(2s\xi)}{2s(\cosh(2s\xi) + z \sinh(2s\xi))} \, dz \wedge d\varphi, & s \neq 0; \\ \xi \, dz \wedge d\varphi, & s = 0. \end{cases}$$

One can derive this formula, for instance, from [11, Example 5.4]. Note that $\omega^0_\xi = \xi \, dz \wedge d\varphi$ are the rotation-invariant area forms on $S^2$. We leave it as an exercise to the reader to show that the cohomology class of $\omega^s_\xi$ is indeed independent of $s$ and that for $s \ll 0$ the volume concentrates near the pole $z = 1$, which was identified with the smallest Schubert cell, $eB$. 

2
Proof of Theorem 1.3. Let $U$ be an open neighborhood of the smallest Schubert cell $eB$. Then for any $\xi \in \mathfrak{t}^*_+$ and $\varepsilon > 0$, there exists $s_0 \in \mathbb{R}$ such that for $s \leq s_0$,

$$\text{Vol} \left( U, \omega^\varepsilon \right) > (1 - \varepsilon) \text{Vol} (K/T, \omega^\varepsilon) .$$

In other words, any compact subset of $G/B$ not containing $eB$ eventually gets depleted of symplectic volume as $s \to -\infty$.

The remainder of the paper is devoted to setting up the proof of Theorem 1.3, which is given below. Section 2 describes the dual Poisson-Lie group $(K^*, \pi_{K^*}) := (AN_-, \pi_{AN_-})$. There are two important maps defined for $s \neq 0$,

$$\mathcal{E}_s : \mathfrak{t}^* \to K^*$$

$$\mathcal{L}_s : \mathbb{R}^{r+m} \times T^m \to K^*$$

which are defined in Equations (5) and (9), respectively. Here $r = \dim (T)$, $2m = \dim (K/T)$, and $T^m$ is a compact torus of dimension $m$. The map $\mathcal{E}_s$ is a diffeomorphism. It is $K$-equivariant with respect to the coadjoint and dressing actions and has the property that $\mathcal{E}_s(\xi) = \exp(s \sqrt{-1} \xi)$ for all $\xi \in \mathfrak{t}^*$. The map $\mathcal{L}_s$ is a diffeomorphism onto its image and the image of $\mathcal{L}_s$ is an open dense subset of $K^*$ that is independent of $s$. The intersection $\mathcal{L}_s(\mathbb{R}^{r+m} \times T^m) \cap \mathcal{E}_s(\Theta_\xi)$ is an open dense subset of $\mathcal{E}_s(\Theta_\xi)$ for all $\xi \in \mathfrak{t}^*_+$. Moreover, all the maps in the following diagram are Poisson:

$$(\Theta_\xi, \pi^\varepsilon_\xi) \hookrightarrow (\mathfrak{t}^*, \pi^s) = \mathcal{E}_s(s\pi_{K^*}) \xrightarrow{\mathcal{E}_s} (K^*, s\pi_{K^*}) \xleftarrow{\mathcal{L}_s} (\mathbb{R}^{r+m} \times T^m, \mathcal{L}_s^*(s\pi_{K^*})). \quad (3)$$

There is a distinguished open subset $PT(K^*) \subset \mathbb{R}^{r+m} \times T^m$ called the partial tropicalization of $K^*$, introduced in [2], equipped with a constant Poisson structure $\pi_{PT}$. As $s \to -\infty$, the Poisson structure $\mathcal{L}_s^*(s\pi_{K^*})$ converges to $\pi_{PT}$ uniformly on certain subsets that exhaust $PT(K^*)$ (Section 2.3). Section 3 shows that the symplectic volume of the leaves of $\mathcal{L}_s^*(s\pi_{K^*})$ concentrates in $PT(K^*)$ as $s \to -\infty$ (Proposition 3.5). Section 4 contains the proof of Proposition 4.3, which says that, under the maps in (3), points of $PT(K^*)$ correspond to points near $\mathfrak{t}^*_+ \subset \mathfrak{t}^*$ when $s \ll 0$. This allows us to translate Proposition 3.5 into a statement about the symplectic volume of $(K/T, \omega^\varepsilon_\xi)$.

Proof of Theorem 1.3. Let $n_{s\xi} \subset \mathbb{R}^{r+m} \times T^m$ denote the preimage $(\mathcal{E}_s^{-1} \circ \mathcal{L}_s)^{-1}(\Theta_\xi)$, which is a symplectic leaf of $\mathcal{L}_s^*(s\pi_{K^*})$, and denote its symplectic form $\eta_{s\xi} = (\mathcal{E}_s^{-1} \circ \mathcal{L}_s)^*\omega^\varepsilon_\xi$. In Proposition 3.5, we prove that for all $\varepsilon > 0$, there is a compact subset $D_\varepsilon \subset PT(K^*)$ such that

$$\lim_{s \to -\infty} \text{Vol} \left( n_{s\xi} \cap D_\varepsilon, \eta_{s\xi} \right) \geq (1 - \varepsilon) \text{Vol}(n_{s\xi}, \eta_{s\xi}) = (1 - \varepsilon) \text{Vol} (K/T, \omega^\varepsilon_\xi) .$$

In Proposition 4.3, we show there exists $s_0 < 0$ such that for all $s \leq s_0$,

$$\mathcal{E}_s^{-1} \circ \mathcal{L}_s(n_{s\xi} \cap D_\varepsilon) \subseteq U .$$

Since $\mathcal{E}_s^{-1} \circ \mathcal{L}_s$ is a Poisson isomorphism, it preserves volumes of the symplectic leaves. Thus

$$\text{Vol} \left( U, \omega^\varepsilon_\xi \right) \geq \text{Vol} \left( \mathcal{E}_s^{-1} \circ \mathcal{L}_s(n_{s\xi} \cap D_\varepsilon), \omega^\varepsilon_\xi \right) = \text{Vol} \left( n_{s\xi} \cap D_\varepsilon, \eta_{s\xi} \right) .$$

Combining with the limit above completes the proof.
Figure 1: As $s \to -\infty$, volume of the symplectic leaves $n_{\xi} = (e_s^{-1} \circ \Sigma_s)^{-1}(O_{\xi})$ concentrates on subsets of $n_{\xi} \cap PT(K^*)$, illustrated in red. For $s$ sufficiently large, the image of the red subset is contained in an arbitrarily small neighborhood of $\xi$, illustrated in blue.

A motivation for our study is provided by the following idea. There exist Poisson isomorphisms between $\mathfrak{t}^*$ and $K^*$ called Ginzburg-Weinstein isomorphisms after the authors of [8]. Given a Ginzburg-Weinstein isomorphism $\gamma: \mathfrak{t}^* \to K^*$, its scaling $\gamma^*(x) := \gamma(sx)$ is a Poisson isomorphism with respect to $\pi_\mathfrak{t}^*$ and $s\pi_K^*$. Composing $\gamma^*$ with $\Sigma_{s}^{-1}$ defines coordinates on every regular coadjoint orbit which are almost global action-angle coordinates for $s \ll 0$. Conjecturally, the $s \to -\infty$ limit of this composition defines global action-angle coordinates on the regular coadjoint orbits. This has already been shown to be true for $K = U(n)$, where for a certain choice of Ginzburg-Weinstein diffeomorphism and cluster seed, the limit is the classical Gelfand-Zeitlin system [4].

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2 Background

Fix the following notation. Let $G$ be a connected simply-connected semisimple complex Lie group of rank $r$. Fix a compact real form $K \subset G$ and a Cartan subgroup $H \subset G$, and let $(\cdot)^*: G \to G$ be the anti-involution of $G$ under which elements $k \in K$ satisfy $k^{-1} = k^*$. Denote the Lie algebras of $G$, $K$, and $H$ by $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{h}$ respectively. Fix a choice of positive roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Denote the lattice of integral weights by $P$, and the semigroup of dominant integral weights by $P_\dagger$. We write $h \mapsto h^\mu \in \mathbb{C}^\times$ for the multiplicative character $H \to \mathbb{C}^\times$ determined by $\mu \in P$. Let $I = \{1, \ldots, r\}$ index the simple roots, $\alpha_i \in \mathfrak{h}^*$, the simple coroots, $\alpha_i^\vee \in \mathfrak{h}$, and the fundamental weights, $\omega_i$, which by definition satisfy $\omega_i(\alpha_j^\vee) = \delta_{ij}$. Denote the Weyl group of $G$ by $W$. Let $s_i \in W$ be the simple reflection generated by $\alpha_i$ and let $w_0$ be the longest element of $W$, with length denoted by $m$.

Let $T$ be the maximal torus of $K$ which has Lie algebra $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$. Let $a = \sqrt{-1}t$ and denote the corresponding subgroup of $G$ by $A$. Corresponding to the choice of positive roots, we have opposite maximal unipotent subgroups $N_{\pm}$ with Lie algebras $n_{\pm}$ as well as opposite Borel subgroups $B_{\pm} = HN_{\pm}$ with Lie algebras $\mathfrak{h} \oplus n_{\pm}$. Fix a set of Chevalley generators $F_i \in n_-$, $\alpha_i^\vee \in \mathfrak{h}$, $E_i \in n$, $i \in I$. Recall the Iwasawa decompositions $G = AN_+K$ and $\mathfrak{g} = n_- \oplus a \oplus \mathfrak{t}$.

Fix an invariant non-degenerate bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$. The isomorphism $\mathfrak{t} \cong \mathfrak{t}^*$ determined by $(\cdot, \cdot)$ embeds $\mathfrak{t}^+ \subseteq \mathfrak{t}^*$, as the image of $\mathfrak{t}$. Let $t^*_+ \subseteq \mathfrak{t}^*$ be the open cone such that $\sqrt{-1}t^*_+ \subseteq \mathfrak{h}^*$ is the interior of the real cone spanned by $P_\dagger$. We refer to both $t^*_+$ and $\sqrt{-1}t^*_+$ as the positive Weyl chamber.
2.1 Dressing orbits and compact Poisson-Lie groups

Recall that a Poisson-Lie group \((K, \pi)\) is a Lie group \(K\) equipped with a Poisson structure \(\pi\) such that the multiplication map \(K \times K \to K\) is Poisson (with respect to the product Poisson structure on \(K \times K\)). For example, the canonical Lie-Poisson structure \(\pi^*\) on the dual \(\mathfrak{t}^*\) of a Lie algebra \(\mathfrak{t}\) is linear, so \((\mathfrak{t}^*, \pi^*)\) is a Poisson-Lie group with respect to vector addition.

For \(G\) as above, both \(K\) and \(AN_\cdot\) have natural Poisson-Lie group structures defined as follows (see [13] for details). Let \(\Im(\cdot, \cdot)\) be the imaginary part of the fixed \(G\)-invariant non-degenerate bilinear form \(\langle \cdot, \cdot \rangle\). Then \(\mathfrak{t} \oplus \mathfrak{n}_-\) are isotropic subspaces with respect to \(2\Im(\cdot, \cdot)\), and \(2\Im(\cdot, \cdot)\) defines an isomorphism \(\mathfrak{n}_- \cong \mathfrak{t}^*\). This identification endows \(\mathfrak{t}\) and \(\mathfrak{t}^*\) with the structure of dual Lie bialgebras. Since \(K\) and \(AN_\cdot\) are simply connected, the Lie bialgebra structures on \(\mathfrak{t}\) and \(\mathfrak{t}^*\) integrate to define Poisson-Lie group structures \(\pi_K\) on \(K\) and \(\pi_{K^*}\) on \(AN_\cdot\), respectively. These Poisson-Lie group structures are dual, since they arise by integrating dual Lie bialgebras, thus one denotes \(K^* = AN_\cdot\), and refers to \((K^*, \pi_{K^*})\) as the dual Poisson-Lie group of \((K, \pi_K)\).

Both \(\mathfrak{t}^*\) and \(K^*\) have naturally defined \(K\) actions. The coadjoint action of \(K\) on \(\mathfrak{t}^*\) is defined in terms of the adjoint action by the equation

\[
\langle \text{Ad}^*_k \xi, x \rangle = \langle \xi, \text{Ad}_{k^{-1}} x \rangle, \quad k \in K, \xi \in \mathfrak{t}^*, \text{ and } x \in \mathfrak{t}.
\]

The coadjoint action preserves \(\pi_{\mathfrak{t}^*}\), and the symplectic leaves of \(\pi_{\mathfrak{t}^*}\) are the coadjoint orbits. The dressing action of \(K\) on \(K^*\) is defined by re-factorizing \(kb \in G\) according to the Iwasawa decomposition. If

\[
kb = b'k' \in AN_\cdot K, \quad k, k' \in K, \quad b, b' \in K^*,
\]

then the dressing action of \(k\) on \(b\) is defined as \(kb = b'\). The symplectic leaves of \(\pi_{K^*}\) are the dressing orbits. In other words, they are the joint level sets of the Casimir functions [13],

\[
C_i(b)^2 := \text{Tr} \left( \rho^\psi \left( bb^* \right) \right), \quad b \in K^*,
\]

where \(\rho^\psi\) is the fundamental irreducible \(G\)-representation with highest weight \(\omega_i \in P_\cdot\). The map \(\varphi: b \mapsto bb^*\) is a diffeomorphism of \(K^*\) onto the set \(S = \{ g \in G \mid g^* = g \}\).

There is a family of diffeomorphisms \(\xi_s: \mathfrak{t}^* \to K^*\) parameterized by \(s \neq 0\) [7]. Let \(\psi: \mathfrak{t}^* \to \mathfrak{t}\) be the \(K\)-equivariant isomorphism given by the fixed bilinear form on \(\mathfrak{g}\). Then, define

\[
\xi_s: \mathfrak{t}^* \xrightarrow{\psi} \mathfrak{t} \xrightarrow{\exp(2s\sqrt{-1}\psi)} S \xrightarrow{\varphi^{-1}} K^* = AN_\cdot.
\]

The map \(\xi_s\) is equivariant with respect to the coadjoint and dressing actions of \(K\). Let \(\Theta_\xi\) be the coadjoint orbit through \(\xi \in \mathfrak{t}_\cdot^*\). Denote by \(D_\xi\) the dressing orbit through \(\xi_s(\xi) = \exp \left( s\sqrt{-1}\psi(\xi) \right)\). Since \(\xi_s\) is \(K\)-equivariant, \(\xi_s(\Theta_\xi) = D_\xi\).

2.2 Cluster coordinates on double Bruhat cells

The double Bruhat cell determined by a pair of elements \(u, v \in W\), is the intersection

\[
G^{u,v} := BuB \cap B_- vB_- \subset G.
\]

In particular, we will consider \(G^{u_0v_0} = Bu_0B \cap B_-\) which is an open dense subset of \(B_-\).

Let \(G_0 = N_- H N\) be the open dense subset of elements in \(G\) that admit a Gaussian decomposition. For a dominant weight \(\mu \in P_+\), the principal minor \(\Delta_{\mu, \mu}\) is a regular function \(G \to \mathbb{C}\) uniquely determined by its value on \(G_0\):

\[
\Delta_{\mu, \mu}(n-hn) = h^\mu, \text{ for any } n_- \in N_-, h \in H, n \in N.
\]
For any two weights $\gamma$ and $\delta$ of the form $\gamma = w\mu$, $\delta = v\mu$, for some $w, v \in W$, the **generalized minor** $\Delta_{w\mu, v\mu}$ is the regular function on $G$ given by

$$\Delta_{\gamma, \delta}(g) = \Delta_{w\mu, v\mu}(g) = \Delta_{\mu, \mu}(\overline{w^{-1}g\overline{v}}), \text{ for } g \in G,$$

where $\overline{w}$ is a specific lift of $w \in W$ to $G$ as in [6, Equation 1.5].

Fix a reduced word $i = (i_1, \ldots, i_m)$, $i_j \in I$, for $w_0 = s_{i_1} \cdots s_{i_m}$. Let $R = R^- \cup R^+$, where $R^- = [-r, -1]$ and $R^+ = [1, m]$. For $1 < k < m$, let $v_k = s_{i_m} \cdots s_{i_{k+1}}$ and let $v_m = e$. For $k \in R^-$, let $i_k = -k$ and $v_k = w_0$. Consider the functions

$$\Delta_k := \Delta_{v_k \omega_k, \omega_k}, \quad k \in R.$$ 

The functions $\Delta_k$ form a seed for the upper cluster algebra structure on $\mathbb{C}[G^{w_0, e}]$ described in [5].

Being an upper cluster algebra implies that any $f \in \mathbb{C}[G^{w_0, e}]$ is a Laurent polynomial in the functions $\Delta_k$. The functions $\Delta_k$ then determine an open embedding

$$\sigma(i) : (\mathbb{C}^\times)^{m+r} \to G^{w_0, e},$$

which is a (birational) inverse to

$$G^{w_0, e} \to \mathbb{C}^{m+r}; \quad g \mapsto (\Delta_{-r}(g), \ldots, \Delta_m(g)).$$

Note that there is no term $\Delta_k$ with index $k = 0$.

We conclude this section by recalling how generalized minors appear in matrix entries of representations of $G$. A dominant integral weight $\mu \in P_+$ can be written uniquely as

$$\mu = \sum_{i \in I} c_i(\mu)\omega_i, \quad c_i(\mu) \in \mathbb{Z}_{\geq 0}.$$ 

Then the function $\Delta_{w_0\mu, \mu}$ can be written as

$$\Delta_{w_0\mu, \mu} = \prod_{i \in I} \Delta_{w_0\mu, \omega_i}^{c_i(\mu)}.$$ (7)

One can check that

$$h \cdot \Delta_{w_0\mu, \mu} \cdot h' = h^{-w_0\mu}h'\Delta_{w_0\mu, \mu}, \quad E_i \cdot \Delta_{w_0\mu, \mu} = \Delta_{w_0\mu, \mu} \cdot E_i = 0 \text{ for } i \in I,$$

where $h, h' \in H$, and $G$ acts on $\mathbb{C}[G]$ in the standard way

$$(g \cdot f)(x) = f(g^{-1}xh), \quad g, h, x \in G, \quad f \in \mathbb{C}[G].$$

For a sequence of indices $j = (j_1, \ldots, j_n)$ in $I$, write $F_j = F_{j_1}F_{j_2} \cdots F_{j_n} \in U(\mathfrak{g})$. Recall that the functions $F_j \cdot \Delta_{w_0\mu, \mu} \cdot F_k$ arise from representations of $G$ as follows. Let $(V, \rho : G \to \text{GL}(V))$ be the irreducible $G$-module with highest weight $\mu$. Let $v_1, \ldots, v_n$ be a weight basis of $V$, where $H$ acts on $v_j$ with weight $\text{wt}(v_j) \in \mathfrak{h}^*$, and assume $\text{wt}(v_1) = \mu$ and $\text{wt}(v_n) = w_0\mu$. Let $\rho_{j,k}(g)$ be the $(j, k)$-entry of the matrix for $\rho(g)$ with respect to the basis $\{v_j\}$. Then $\rho_{n,1} = c \Delta_{w_0\mu, \mu}$, for some $c \in \mathbb{C}^\times$. We may choose the weight basis such that $c = 1$. Each $\rho_{j,k}$ is a linear combination of terms of the form $F_j \cdot \Delta_{w_0\mu, \mu} \cdot F_k$, where $j$ and $k$ are such that

$$h \cdot (F_j \cdot \Delta_{w_0\mu, \mu} \cdot F_k) \cdot h' = h^{-\text{wt}(v_j)}(h')^{\text{wt}(v_k)}(F_j \cdot \Delta_{w_0\mu, \mu} \cdot F_k)$$ (8)

for all $h, h' \in H$.
2.3 The partial tropicalization and its symplectic leaves

Recall from Section 2.1 that $K^* = AN_\omega$. Let $S = \{k \in R \mid v_k \omega_{i_k} \neq \omega_{i_k}\}$. Then $|R \setminus S| = r$, and $\Delta_k(K^*) \subset \mathbb{R}_+$ if and only if $k \in R \setminus S$. The collection of functions

$$\{\Delta_k \mid k \in R\} \cup \{\Delta_k \mid k \in S\}$$

define a real coordinate system on an open dense subset of $K^*$. Equip $\mathbb{R}^{r+m} \times \mathbb{T}^m$ with coordinates $(\lambda_R, \varphi_S)$, where $\lambda_R = (\lambda_k)_{k \in R}$ and $\varphi_S = (\varphi_k)_{k \in S}$.

There is a Poisson manifold $(PT(K^*), \pi_{PT})$, called the partial tropicalization of $K^*$, which was introduced in [2]. As a manifold, $PT(K^*)$ is defined as

$$PT(K^*) := C \times \mathbb{T}^m \subset \mathbb{R}^{r+m} \times \mathbb{T}^m,$$

where $C$ is an open convex polyhedral cone of dimension $r + m$ defined by inequalities described in [6] and [2, Theorem 6.24]. The definition of $C$ depends on the choice of reduced word $i$ fixed in Section 2.2. More precisely, $C$ is the set of points $x \in \mathbb{R}^{m+r}$ satisfying an inequality $\Phi^i(x) > 0$, where $\Phi^i : \mathbb{R}^{m+r} \rightarrow \mathbb{R}$ is a certain piecewise-linear function called the tropical Berenstein-Kazhdan potential.

The Poisson structure $\pi_{PT}$ is constant in the coordinates $(\lambda_R, \varphi_S)$. The symplectic leaves of $PT(K^*)$ are the joint level sets of the coordinates $\lambda_{R^-} = (\lambda_{r}, \ldots, \lambda_{-1})$ [3, Theorem 6.5].

There is a correspondence between symplectic leaves of $PT(K^*)$ and regular coadjoint orbits of $K$, which we now describe. To each $\xi \in \mathfrak{t}_\omega^*$, we associate $\lambda_{R^-} \in \mathbb{R}^r$ with coordinates

$$\lambda_{-i} = (w_0\omega_i, \sqrt{-1}\xi) \text{ for } i = -r, \ldots, -1.$$ 

Denote the symplectic leaf of $PT(K^*)$ that is the fiber of $\lambda_{R^-}$ by $P_\xi$. The corresponding coadjoint orbit is $\Theta_\xi$. For each fixed value of $s \neq 0$, the leaf $P_\xi$ also corresponds to the dressing orbit $D_{\lambda_\xi}$, defined in Section 2.1.

Each symplectic leaf $P_\xi \subset PT(K^*)$ inherits a symplectic form from $\pi_{PT}$ denoted by $\nu_\xi$.

**Theorem 2.1.** [3, Theorem 6.11] The symplectic volume of $(P_\xi, \nu_\xi)$ equals the symplectic volume of the coadjoint orbit $\Theta_\xi \subset \mathfrak{t}^*$ equipped with the Kirillov-Kostant-Souriau symplectic form:

$$\text{Vol}(P_\xi, \nu_\xi) = \text{Vol}(\Theta_\xi, \omega_\xi).$$

**Remark 2.2.** Although [3, Theorem 6.11] is only stated for leaves parameterized by regular dominant integral weights, the theorem here follows by scaling and continuity.

In order to compare the Poisson structures of $PT(K^*)$ and $K^*$, we define the detropicalization map $\Sigma_s : PT(K^*) \rightarrow K^*$ as follows. For $s < 0$, let

$$\Sigma_s(\lambda_R, \varphi_S) = \sigma(1) \left( e^{s\lambda_{-r}}(-1)^{s-r}, \ldots, e^{s\lambda_{m}}(-1)^{s-m} \right),$$

where we understand $\varphi_k = 0$ on the right hand side if $k \notin S$. Denote $b_s = \Sigma_s(\lambda_R, \varphi_S)$.

**Remark 2.3 (Conventions).** We follow the conventions of [3, 6] for (partial) tropicalization, which are opposite to those of [2]. We consider $K^* \subset B_\omega$, as in [3], rather than $K^* \subset B$, as in [2], and take the limit $s \rightarrow -\infty$. This accounts for the minus signs in (9).

The Casimir functions for $K^*$ are related to the coordinates $\lambda_R, \varphi_S$ by the detropicalization map via $r$ equations (one for each Casimir function):

$$C_i(b_s) = \text{Tr}(\rho^{\omega_i}(b_s b_s^*)^2) = \sum_j \rho^{\omega_i}_{j,i}(b_s b_s^*) = \sum_{j,k} |\rho^{\omega_i}_{j,k}(b_s)|^2$$

$$= \sum_{j,k} \left| \sum_{i,j} c_{i,j}(F_j \Delta_{w_0i,\omega_i} F_j)(b_s) \right|^2$$

$$= |\Delta_{w_0i,\omega_i}(b_s)|^2 \left( 1 + \sum_{j,k} \left| \sum_{i,j} c_{i,j} \frac{(F_j \Delta_{w_0i,\omega_i} F_j)(b_s)}{\Delta_{w_0i,\omega_i}(b_s)} \right|^2 \right).$$

(10)
Since \( b_s = \Sigma_s(\lambda, \varphi_S) \), the last line on the right side can be rewritten as a Laurent polynomial in the functions \( e^{s\lambda_k - \sqrt{-1}pk} \). The term \( |\Delta_{w_0\omega_i}(b_s)|^2 = e^{2s\lambda_i} \) dominates the expression for \( s \ll 0 \), and the exponents in the other terms are controlled by their distance from the boundary of \( C \), as follows.

Recall that \( C \) is the set of points \( x \in \mathbb{R}^{m+r} \) satisfying the inequality \( \Phi^i(x) > 0 \). For \( \delta > 0 \), let \( C^\delta \subset C \) be the set of points \( x \in \mathbb{R}^{m+r} \) which satisfy the inequality \( \Phi^i > \delta \). Then,

**Proposition 2.4.** \([2, \text{Theorem 4.13 and Lemma 6.17}]\) For \((\lambda, \varphi) \in C^\delta \times \mathbb{T}^m\), each term

\[
\left| \sum_{i,j} c_{ij} \frac{(F_i \Delta_{w_0\omega_i}(F_j))(b_s)}{\Delta_{w_0\omega_i}(b_s)} \right| = O(e^{sd}).
\]

Here and throughout, a function \( f(s) \) is in \( O(g(s)) \), \( g(s) \geq 0 \), if there exists \( c > 0 \) such that

\[-cg(s) \leq f(s) \leq cg(s).\]

As a direct consequence of Proposition 2.4 and Equations \((10)\), we have:

**Corollary 2.5.** \([3, \text{Remark 6.6}]\) For all \( \xi \in t^*_+ \) and \((\lambda, \varphi) \in P_\xi \), and for each \( i = 1, \ldots, r \),

\[
\lim_{s \to -\infty} \frac{1}{s} \log \circ C_{\xi} \circ \Sigma_s(\lambda, \varphi_S) = \lambda_{-i} = (w_0\omega_i, \sqrt{-1}\xi).
\]

**Remark 2.6.** Corollary 2.5 says that points \( \Sigma_s(P_\xi) \) in the image of a tropical leaf under the detropicalization map approach the corresponding scaled dressing orbit \( D_{s\xi} \) in the limit \( s \to -\infty \). It is useful to note that points in \( \Sigma_s(P_\xi) \) will concentrate near a certain region of \( D_{s\xi} \), not the entire orbit: there are points in the preimages of the scaled dressing orbits \( \Sigma_s^{-1}(D_{s\xi}) \) that remain far away from \( PT(K^*) \), even as \( s \to -\infty \) (see Figure 2).

### 3 Symplectic volumes of the leaves of \( \pi_s \)

In this section we study volumes of the symplectic leaves of the Poisson bivector

\[
\pi_s := (\Sigma_s)^*(s\pi_{K^*}).
\]

Note that the pullback of a bivector under a diffeomorphism is by definition the pushforward under the inverse diffeomorphism. The symplectic leaves in question are submanifolds of \( \mathbb{R}^{r+m} \times \mathbb{T}^m \). Roughly, for \( s \ll 0 \) each of these leaves has a piece which lies inside \( PT(K^*) = C \times \mathbb{T}^m \), close to the corresponding leaf of \( \pi_{PT} \) (Section 3.1). For \( s \ll 0 \), the volume of the symplectic leaves concentrate there (Proposition 3.5). This is illustrated in Figure 2.

Let us first establish some notation. Each symplectic leaf of \( \pi_s \) is the preimage under \( \Sigma_s \) of a dressing orbit. We denote the leaf and its symplectic form by

\[
\eta_{s\xi} := \Sigma_s^{-1}(D_{s\xi}), \quad \eta_{s\xi} := (\pi_s)^{-1}.
\]

There is a corresponding symplectic leaf \( P_\xi \) of \( PT(K^*) \) equipped with \( \nu_\xi \), as described in Section 2.3. Recall, for \( \xi \in t^*_+ \),

\[
P_\xi := \{(\lambda, \varphi) \in PT(K^*) \mid \lambda_{-i} = (w_0\omega_i, \sqrt{-1}\xi), i = -r, \ldots, -1 \},
\]

which is a product of an open polytope (a fiber in \( C \) of projection to the first \( r \) coordinates) times a torus.

We will often reference the open subset \( P^\delta := P_\xi \cap (C^\delta \times \mathbb{T}^m) \) and its closure \( \overline{P}_{\xi}^\delta \).
Figure 2: Volume of the symplectic leaves $N_{s\xi}$ of $\pi_s$ concentrates on the part of $N_{s\xi}$ that is close to the corresponding tropical leaf, $P_\xi$.

3.1 The implicit function theorem argument

Consider the map
\[ F_{s\xi} = (f_{r-1}, \ldots, f_{-1}): \mathbb{R}^r \times \mathbb{R}^m \times \mathbb{T}^m \to \mathbb{R}^r \]
with coordinates $f_{-i}$ defined by composing the detropicalization map (9) with the Casimir functions (4) on $K^*$,
\[ f_{-i}(\lambda_R, \varphi_S) = \frac{1}{s} \log \circ C_i \circ L_s(\lambda_R, \varphi_S). \]

The fiber $F_{s\xi}^{-1}(\lambda_R)$ is the symplectic leaf $N_{s\xi}$. The following lemma will allow us to apply the implicit function theorem at certain points in $N_{s\xi}$.

**Lemma 3.1.** For all $(\lambda_R, \varphi_S) \in C_\delta \times \mathbb{T}^m$, the derivatives
\[ D_{\lambda_R} F_{s\xi} = I_r + O(e^{2s\delta}); \]
\[ D_{\lambda_R} F_{s\xi} = O(e^{2s\delta}); \]
\[ D_{\varphi_S} F_{s\xi} = O(e^{2s\delta}). \]

(Here $I_r$ is the $r \times r$ identity matrix and $O(e^{s\delta})$ denotes a matrix of the appropriate dimensions whose entries are $O(e^{2s\delta})$ as functions of $s$.)

**Proof.** By the formula for $f_{-i}$, Equations (10), and the comment directly following Equations (10),
\[ e^{2sf_{-i}(\lambda_R, \varphi_S)} = e^{2s\lambda_{-i}} \left( 1 + \sum_{j,k} c_{j,k} e^{2sL_{j,k}(\lambda_R, \varphi_S)} \right). \]

for $-i = -r, \ldots, -1$, constants $c_{j,k}$, and some linear combinations $L_{j,k}(\lambda_R, \varphi_S)$. Differentiating these equations gives
\[ \frac{\partial f_{-i}}{\partial \lambda_k} = e^{2s(\lambda_{-i} - f_{-i}(\lambda_R, \varphi_S))} \left( \delta_{-i,k} + \sum_{j,k} \left( \frac{\partial L_{j,k}}{\partial \lambda_k} + \delta_{-i,k} \right) c_{j,k} e^{2sL_{j,k}(\lambda_R, \varphi_S)} \right); \]
\[ \frac{\partial f_{-i}}{\partial \varphi_k} = e^{2s(\lambda_{-i} - f_{-i}(\lambda_R, \varphi_S))} \sum_{j,k} \frac{\partial L_{j,k}}{\partial \varphi_k} c_{j,k} e^{2sL_{j,k}(\lambda_R, \varphi_S)}. \]

Here $\delta_{-i,k}$ is the Kronecker-delta function. By Proposition 2.4, for $(\lambda_R, \varphi_S) \in C_\delta \times \mathbb{T}^m$,
\[ e^{2s(\lambda_{-i} - f_{-i}(\lambda_R, \varphi_S))} = 1 + O(e^{2s\delta}); \]
\[ e^{2sL_{j,k}(\lambda_R, \varphi_S)} = O(e^{2s\delta}), \]
which completes the proof. \(\diamondsuit\)
Fix an arbitrary element \( p = (\lambda_R^-, \lambda_R^+, \varphi_S) \in \mathcal{P}_\xi \) and consider the subspace
\[
S_p := \mathbb{R}^r \times \{ \lambda_R^+ \} \times \{ \varphi_S \} \subseteq \mathbb{R}^r \times \mathbb{R}^m \times T^m.
\]
By an intermediate value theorem argument, we can show that for \( s \ll 0 \), \( \mathcal{N}_{s\xi} \) intersects \( S_p \) near \( p \):

Lemma 3.2. For all \( \xi \in t^*_+ \) and for all \( \delta, \upsilon > 0 \) sufficiently small, there exists \( s_0 < 0 \) such that for all \( s \leqslant s_0 \) and \( p \in \mathcal{P}_\xi^\delta \), the intersection \( S_p \cap \mathcal{N}_{s\xi} \cap B_\upsilon(\mathcal{P}_\xi) \) is non-empty (see Figure 3).

\[
\text{Figure 3: The intersection described in Lemma 3.2. The intersection of } \mathcal{N}_{s\xi} \text{ with the shaded region is locally the graph of a function defined on } \mathcal{P}_\xi^\delta \text{ (Proposition 3.3). In the figure, } \mathcal{P}_\xi^\delta \text{ is the thick part of } \mathcal{P}_\xi.
\]

Proof. Consider the equivalent problem of showing there is a \( s_0 \) such that for all \( s \leqslant s_0 \) and \( p \in \mathcal{P}_\xi^\delta \), the submanifold \( L_s(\mathcal{S}_p \cap B_\upsilon(\mathcal{P}_\xi)) \) intersects the dressing orbit \( \mathcal{D}_{s\xi} \). Since dressing orbits are joint level sets of the Casimir functions \( C_i \), showing this intersection is non-empty is equivalent to showing that \( \lambda_R^- \) is contained in the image of \( \mathcal{S}_p \cap B_\upsilon(\mathcal{P}_\xi) \) under the map \( F_{s\xi} \) defined in Equations (11) and (12).

Fix \( \delta > 0 \) (small enough that \( \mathcal{P}_\xi^\delta \) is nonempty). By Corollary 2.5, for \( \epsilon > 0 \) sufficiently small,
\[
\lim_{s \to -\infty} f_{-i}(\lambda_{-r}, \ldots, \lambda_{-r} \pm \epsilon, \ldots, \lambda_{-1}, \lambda_R^+, \varphi_S) = \lambda_{-i} \pm \epsilon.
\]
Thus, for all \( p \in \mathcal{P}_\xi^\delta \), there is a \( s_p \) such that for \( s \leqslant s_p \), the map \( F_{s\xi} \) satisfies the assumptions of the Poincaré-Miranda Theorem on the box
\[
[\lambda_{-r} - \epsilon, \lambda_{-r} + \epsilon] \times \cdots \times [\lambda_{-1} - \epsilon, \lambda_{-1} + \epsilon] \times \{ \lambda_R^+ \} \times \{ \varphi_S \} \subseteq S_p.
\]
Take \( \epsilon > 0 \) sufficiently small so that the box is contained in \( \mathcal{S}_p \cap B_\upsilon(\mathcal{P}_\xi) \) and, without loss of generality (making \( \upsilon \) smaller if necessary), assume that \( \mathcal{S}_p \cap B_\upsilon(\mathcal{P}_\xi) \subset C^{\delta/2} \times T^m \) for all \( p \in \mathcal{P}_\xi^\delta \). It follows by the Poincaré-Miranda theorem that \( \lambda_R^- \) is contained in the image of the box under the map \( F_{s\xi} \) for \( s \leqslant s_p \).

By transversality of the intersection of \( \mathcal{S}_p \) and \( \mathcal{N}_{s\xi} \) at points in \( C^{\delta/2} \times T^m \), for \( s \) less than some \( s' \) (Lemma 3.1), each \( p \in \mathcal{P}_\xi^\delta \) has a neighborhood \( U_p \) such that for \( p' \in U_p \) and \( s \leqslant s_p \), the intersection \( \mathcal{S}_{p'} \cap \mathcal{N}_{s\xi} \cap B_\upsilon(\mathcal{P}_\xi) \) is non-empty. Passing to a finite subcover \( U_{p_k}, k = 1, \ldots, n \) and letting \( s_0 = \min \{ s', s_{p_k} \} \) completes the proof.

Define
\[
\mathcal{U}_{\xi, \delta} := \bigcup_{p \in \mathcal{P}_\xi^\delta} S_p.
\]
From this point forward, take \( v > 0 \) sufficiently small so that \( U_{\xi, \delta} \cap B_v(\partial\xi) \subset C^{\delta/2} \times T^m \). The region \( U_{\xi, \delta} \cap B_v(\partial\xi) \) is shaded blue in Figure 3.

**Proposition 3.3.** For all \( \delta > 0 \) and \( s \leq s_0 \) as in Lemma 3.2, the intersection \( N_{s\xi} \cap U_{\xi, \delta} \cap B_v(\partial\xi) \) is locally the graph of a function

\[
g_s : \mathcal{P}_\xi^\delta \to \mathbb{R}^r.
\]

**Proof.** Combine Lemmas 3.1, 3.2, and the implicit function theorem. \( \diamond \)

### 3.2 Comparing symplectic volumes on the leaves of \( \pi_s \)

In this subsection, we compare the symplectic volumes of \( (\mathcal{P}_\xi, \nu_\xi) \) and \( (N_{s\xi}, \eta_{s\xi}) \). By Proposition 3.3, the intersection of \( N_{s\xi} \) with \( U_{\xi, \delta} \cap B_v(\partial\xi) \) is locally the graph of a function \( g_s \), i.e. locally there is a diffeomorphism

\[
G_s : \mathcal{P}_\xi^\delta \to N_{s\xi}, \quad (\lambda_R, \varphi_S) \mapsto (g_s(\lambda_{R+}, \varphi_S), \lambda_{R+}, \varphi_S)
\]

**Lemma 3.4.** For \( s \leq s_0 \) as in Lemma 3.2, at points in \( N_{s\xi} \cap U_{\xi, \delta} \cap B_v(\partial\xi) \subset C^{\delta/2} \times T^m \),

\[
(G_s)_* \nu_\xi = \eta_{s\xi} + O(e^{s\delta})
\]

(here \( O(e^{s\delta}) \) denotes a 2-form whose coefficients in coordinates \( (\lambda_R, \varphi_S) \) are \( O(e^{s\delta}) \) as functions of \( s \)).

**Proof.** Fix \( p = (\lambda_R, \varphi_S) \in \mathcal{P}_\xi^\delta \). By the implicit function theorem, for all \( (X, Y) \in T_p \mathcal{P}_\xi^\delta = \mathbb{R}^m \times \mathbb{R}^m \),

\[
D_p G_s(X, Y) = \left( -(D_{\lambda_R} F_{\xi})^{-1}(D_{\lambda_R} F_{\xi} X + D_{\varphi_S} F_{\xi} Y), X, Y \right)
\]

The constant bivector \( \pi_{PT} \) has the form

\[
\pi_{PT} = \sum_k X_k \wedge Y_k
\]

for some \( X_k, Y_k \in T_p \mathcal{P}_\xi^\delta \). By Lemma 3.1 and the formula for \( D_p G_s \) above, we find \( (G_s)_* \pi_{PT} = \pi_{PT} + O(e^{s\delta}) \), where \( O(e^{s\delta}) \) denotes a bivector whose coefficients in coordinates \( (\lambda_R, \varphi_S) \) are \( O(e^{s\delta}) \) as functions of \( s \). The 2-form \( (G_s)_* \nu_\xi = ((G_s)_* \pi_{PT})^{-1} = \pi_{PT}^{-1} + O(e^{s\delta}). \)

On the other hand, by the proof of [2, Theorem 6.18], at \( G_s(p) \in C^{\delta/2} \times T^m \),

\[
\eta_{s\xi} = (\pi_s)^{-1} = \left( \pi_{PT} + O(e^{s\delta}) \right)^{-1} = \pi_{PT}^{-1} + O(e^{s\delta}). \diamond
\]

Finally, we show that for \( s \ll 0 \), the symplectic volume of \( N_{s\xi} \) is concentrated on the piece that lies in \( C^{\delta/2} \times T^m \).

**Proposition 3.5.** For \( \xi, \delta, \nu, \) and \( s \leq s_0 \) as in Lemma 3.2, the symplectic volume of \( N_{s\xi} \cap U_{\xi, \delta} \cap B_v(\partial\xi) \subset C^{\delta/2} \times T^m \) satisfies the inequalities

\[
\text{Vol}(N_{s\xi}, \eta_{s\xi}) \geq \text{Vol}(N_{s\xi} \cap U_{\xi, \delta} \cap B_v(\partial\xi), \eta_{s\xi}) \geq \text{Vol}(N_{s\xi}, \eta_{s\xi}) - \text{Vol}(P_{\xi} \setminus \mathcal{P}_\xi^\delta, \nu_\xi) + O(e^{s\delta}).
\]

Note that \( \text{Vol}(P_{\xi} \setminus \mathcal{P}_\xi^\delta, \nu_\xi) \to 0 \) as \( \delta \to 0 \).

**Remark 3.6.** In the proof of Theorem 1.3, we choose \( \delta, \nu > 0 \) sufficiently small and let \( D_x \) be the closure of \( U_{\xi, \delta} \cap B_v(\partial\xi) \subset C^{\delta/2} \times T^m \).
Proof. The first inequality follows since volume is monotonic. By Proposition 3.3 and Lemma 3.4, 
\[ N_\delta \cap U_{\xi,\delta} \cap B_v(P_\xi) \] is locally the image of a diffeomorphism \( G_\delta \) with domain in \( P^\delta_\xi \) and \( (G_\delta)_* \nu_\xi = \eta_\delta + O(e^{\delta}) \), so 
\[ \text{Vol}(N_\delta \cap U_{\xi,\delta} \cap B_v(P_\xi),\eta_\delta) \geq \text{Vol}(P^\delta_\xi,\nu_\xi) + O(e^{\delta}). \]

By definition of \( P^\delta_\xi = P_\xi \cap (C^\delta \times T^m) \), 
\[ \text{Vol}(P^\delta_\xi,\nu_\xi) = \text{Vol}(P_\xi,\nu_\xi) - \text{Vol}(P_\xi \setminus P^\delta_\xi,\nu_\xi). \]

Finally, by Theorem 2.1, 
\[ \text{Vol}(P_\xi,\nu_\xi) - \text{Vol}(P_\xi \setminus P^\delta_\xi,\nu_\xi) + O(e^{\delta/2}) = \text{Vol}(N_\delta,\eta_\delta) - \text{Vol}(P_\xi \setminus P^\delta_\xi,\nu_\xi) + O(e^{\delta}). \]

\[ \square \]

4 Preimages of points in \( PT(K^*) \)

The goal of this section is to show that for a fixed value of \( \xi \in t^*_+ \) and \( s \ll 0 \), if \( \mathcal{E}_s(\text{Ad}_k^* \xi) \in \mathcal{L}_s(PT(K^*)) \), then \( \text{Ad}_k^* \xi \) must be close to \( \xi \) in the coadjoint orbit \( O_\xi \).

Fix a faithful irreducible representation \( (\rho,V) \) of \( G \). Let \( n = \dim(V) \), and fix a Hermitian inner product on \( V \) which is preserved by \( \rho(K) \). For the representation \( V \), fix a unitary weight basis \( v_1, \ldots, v_n \).

Consider the wedge product \( \langle \rho^l, \wedge^k V \rangle \) of the representation \( (\rho,V) \). Note that \( \wedge^k V \) has basis 
\[ \{ v_1 := v_1 \wedge \cdots \wedge v_i \mid I = (i_1, \ldots, i_l) \text{ and } i_1 < \cdots < i_l \}. \]

We can reorder the unitary weight basis \( \{ v_l \} \) so that, for all \( l \in [n] \), the vector \( v_{[l]} = v_1 \wedge \cdots \wedge v_l \) is a minimal weight vector of \( \wedge^l V \). For \( I, J \subset [n] \) with \( |I| = |J| = l \), denote by \( \Delta_{I,J} \) the \( l \times l \) minor of elements of \( GL(V) \) in the basis \( v_i \), with rows \( I \) and columns \( J \). Define the map 
\[ p_{\xi}^*: PT(K^*) \to t^*_+; \quad x \in P_\xi \mapsto \xi. \]

Lemma 4.1. Let \( l \in [n] \), and let \( J \subset [n] \) with \( |J| = l \) and \( [l] \not= J \). For all \( \delta > 0 \) and \( s < 0 \), define 
\[ U_s = \{ k \in K \mid \mathcal{E}_s(\text{Ad}_k^* \xi) \in \mathcal{L}_s(p) \text{ for some } p \in C^\delta \times T^m, \xi \in p_{\xi}^*(C^\delta \times T^m) \}. \]

Then there exists \( a > 0 \) such that for all \( k \in U_s \), 
\[ |\Delta_{[l],J}(\rho(k))| \leq ae^{\delta}, \]

in the unitary weight basis \( \{ v_i \} \).

Proof. Let \( \text{wt}(v_{[l]}) = w_0 \zeta \), where \( \zeta \in P_+ \) is a dominant weight, and consider the irreducible sub-representation \( G \cdot v_{[l]} \) of \( \wedge^l V \) which is generated by \( v_{[l]} \). Then in this subrepresentation, \( v_{[l]} \) will be of lowest weight. Let \( L \) denote the index of the highest weight vector of this subrepresentation. It follows that \( \text{wt}(v_{[l]}) = \zeta \). Write the matrix entries of \( \rho^l(g) \) in the basis \( \{ v_1 \} \) as \( \rho^l_{1,1}(g) \). Note that 
\[ \rho^l_{1,1}(g) = \Delta_{[l],1}(\rho(g)). \]

Because \( v_{[l]} \) is of lowest weight in the subrepresentation \( G \cdot v_{[l]} \), we have 
\[ \rho^l(g)v_{[l]} = \sum_{w_0 \zeta < \text{wt}(v_{[l]})} \rho^l_{1,1}(g)v_{1}, \]

where the sum on the right hand side is over weight vectors \( v_{1} \) such that \( w_0 \zeta - \text{wt}(v_{1}) \) is a negative weight or \( J = [l] \). In other words, \( \rho^l_{1,1}(g) = 0 \) unless \( w_0 \zeta < \text{wt}(v_{1}) \) or \( J = [l] \).

Using the definition of the dressing action and the fact that the map \( \mathcal{E}_s \) is \( K \)-equivariant, we have 
\[ k \cdot (\mathcal{E}_s(\xi))^2 \cdot k^* = \mathcal{E}_s(\text{Ad}_k^* \xi) \cdot \mathcal{E}_s(\text{Ad}_k^* \xi)^* \cdot k^* . \]
Rewrite (15) as

\[ k \cdot d_s^2 \cdot k^* = b_s \cdot b_s^* \]  \hspace{1cm} (16)

where \( d_s = \exp\left(s\sqrt{-1}\psi(\xi)\right) \) and \( b_s = \Sigma_s(p) \).

Let us apply the representation \( \rho^\delta \) to both sides of (16), and consider the \([\ell],[l]\)-entry of these matrices. Using the fact that \( \{y_k\} \) is a unitary basis for \( \wedge^2 V \), matrix multiplication and (14) gives us:

\[
\sum_{u_0\zeta < \text{wt}(v_\ell)} |\rho^\delta_{[\ell],[l]}(k^*)|^2 \cdot |\rho^\delta_{[\ell],[l]}(d_s)|^2 = \sum_{u_0\zeta < \text{wt}(v_\ell)} |\rho^\delta_{[\ell],[l]}(b_s^*)|^2. \hspace{1cm} (17)
\]

Since \( \rho^\delta(k) \cdot \rho^\delta(k^*) = \rho^\delta(kk^*) = 1 \), we have

\[
\sum_{u_0\zeta < \text{wt}(v_\ell)} |\rho^\delta_{[\ell],[l]}(k^*)|^2 = 1. \hspace{1cm} (18)
\]

Rewrite (18) as

\[ |\rho^\delta_{[\ell],[l]}(k^*)|^2 = 1 - \sum_{u_0\zeta < \text{wt}(v_\ell)} |\rho^\delta_{[\ell],[l]}(k^*)|^2 \]

and plug it into (17). After rearranging, we get

\[
|\rho^\delta_{[\ell],[l]}(d_s)|^2 = \sum_{u_0\zeta < \text{wt}(v_\ell)} |\rho^\delta_{[\ell],[l]}(b_s^*)|^2 + \sum_{u_0\zeta < \text{wt}(v_\ell)} |\rho^\delta_{[\ell],[l]}(k^*)|^2 \cdot \left( |\rho^\delta_{[\ell],[l]}(d_s)|^2 - |\rho^\delta_{[\ell],[l]}(d_s)|^2 \right). \hspace{1cm} (19)
\]

Since \( u_0\zeta < \text{wt}(v_\ell) \) and the terms \( |\rho^\delta_{[\ell],[l]}(d_s)|^2 - |\rho^\delta_{[\ell],[l]}(d_s)|^2 \) are positive, by discarding terms on the right hand side of (19), one has for any \( J \) with \( u_0\zeta < \text{wt}(v_\ell) \),

\[ |\rho^\delta_{[\ell],[l]}(d_s)|^2 > |\rho^\delta_{[\ell],[l]}(b_s^*)|^2 + |\rho^\delta_{[\ell],[l]}(k^*)|^2 \cdot \left( |\rho^\delta_{[\ell],[l]}(d_s)|^2 - |\rho^\delta_{[\ell],[l]}(d_s)|^2 \right). \]

Hence

\[
|\rho^\delta_{[\ell],[l]}(k^*)|^2 < \frac{|\rho^\delta_{[\ell],[l]}(d_s)|^2 - |\rho^\delta_{[\ell],[l]}(b_s^*)|^2}{|\rho^\delta_{[\ell],[l]}(d_s)|^2 - |\rho^\delta_{[\ell],[l]}(d_s)|^2} = \frac{1 - |\rho^\delta_{[\ell],[l]}(b_s^*)|^2}{1 - |\rho^\delta_{[\ell],[l]}(d_s)|^2}. \hspace{1cm} (20)
\]

From Proposition 2.4, because \( p \in C^\delta \times \mathbb{T}^m \), we have

\[ C_s(b_s) = |\Delta_{w_{\gamma_j},\omega_i}(b_s)| \left( 1 + O(e^{2s\delta}) \right). \]

On the other hand, from (15), for \( s < 0 \),

\[ C_s(b_s)^2 = \text{Tr}(\rho^\omega(d_s^2)) = \sum_j c_j e^{2s(\gamma_j,\sqrt{-1}\xi)} = e^{2s(w_{\gamma_j}\omega_i,\sqrt{-1}\xi)} \left( 1 + O(e^{2s\delta}) \right). \]

Here, the weights \( \gamma_j \) are those which appear in the representation \( \rho^\omega \), and \( c_j = 1 \) when \( \gamma_j \) is the extremal weight \( w_{\gamma_j}\omega_i \). The last equality holds because, by assumption, \( \xi \in \text{pr}_{\omega}(C^\delta \times \mathbb{T}^m) \), which in turn guarantees that \( (\alpha_i,\sqrt{-1}\xi) > \delta \) for all \( i \in I \).

Combining the previous two equations, since \( e^{s(w_{\gamma_j}\omega_i,\sqrt{-1}\xi)} = \Delta_{w_{\gamma_j}\omega_i,w_{\gamma_j}\omega_i}(d_s) \), we have

\[ \left| \frac{\Delta_{w_{\gamma_j}\omega_i,w_{\gamma_j}\omega_i}(b_s)}{\Delta_{w_{\gamma_j}\omega_i,w_{\gamma_j}\omega_i}(d_s)} \right|^2 - 1 = O(e^{2s\delta}), \hspace{1cm} \text{for all } i \in I. \]
For $\zeta \in P_+$, by using (7) we get

$$\left| \frac{\Delta_{w_0\zeta,\zeta}(b_s)}{\Delta_{w_0\zeta,\zeta}(d_s)} \right|^2 - 1 = O(e^{2\delta}),$$

for $s \ll 0$. By the discussion at the end of Section 2, we know

$$\rho_{[l],[l]}^l = c\Delta_{w_0\zeta,\zeta} \quad \text{and} \quad \rho_{[l],[L]}^l = c\Delta_{w_0\zeta,\zeta}$$

for some $c \in \mathbb{C}^\times$. By (21) and (20), we find $|\Delta_{[l],J}(\rho(k))| = |\Delta_{J,[l]}(\rho(k^s))| = O(e^{s\delta})$. \hfill $\Diamond$

**Lemma 4.2.** Let $g: (-\infty, 0) \to U(n)$ be an element of $U(n)$ depending on a parameter $s$. Assume there exists $\delta > 0$ such that

$$|\Delta_{[l],J}(g(s))| = O(e^{s\delta}) \quad \text{for all } l \in [n] \text{ and all } J \subset [n] \text{ with } |J| = l \text{ and } [l] \neq J.$$

Then, the matrix entries satisfy $|g_{i,j}(s)| = O(e^{s\delta})$ for all $i \neq j$.

**Proof.** We proceed by induction on $i$. When $i = 1$, we have $|g_{1,j}| = O(e^{s\delta})$ for $j \neq 1$. Assume the statement is known for $1, \ldots, i - 1$. By induction hypothesis and the fact that $g$ is unitary, we have

$$1 - |g_{i,j}| = O(e^{s\delta}) \text{ for } j < i.$$

By taking inner product of the $i^{th}$ row with the previous rows and again using the fact that $g$ is unitary, we have $|g_{i,j}| = O(e^{s\delta})$ for $j < i$. For $j > i$, consider the minor $\Delta_{[i],J}(g)$, where $J = \{1, \ldots, i-1, j\}$. By assumption, $|\Delta_{[i],J}(g)| = O(e^{s\delta})$. Expanding this minor along the $j^{th}$ column and applying the induction hypothesis, we have that $|g_{i,j}| = O(e^{s\delta})$. \hfill $\Diamond$

Recall that $N_{s\xi}$ is the preimage $(E_s^{-1} \circ \mathcal{S}_s)^{-1}(\Theta_s)$.

**Proposition 4.3.** For all $\xi \in \ell_*^+$, if $U \subset \Theta_s$ is an open subset with $\xi \in U$, then for all sufficiently small $\delta > 0$, there exists $s_0 \in \mathbb{R}$ so that, for all $s \leq s_0$,

$$E_s^{-1} \circ \mathcal{S}_s \left( N_{s\xi} \cap (C^\delta \times T^m) \right) \subseteq U.$$

**Proof.** Fix $\xi \in \ell_*^+$, $U \subset \Theta_s$ open with $\xi \in U$, and $\delta > 0$ sufficiently small so that $\xi \in pr_{t_*}(C^\delta \times T^m)$. Observe that for all $s < 0$,

$$U'_s = \{ k \in K \mid \mathcal{S}_s(Ad_k^* \xi) \in \mathcal{S}_s(N_{s\xi} \cap (C^\delta \times T^m)) \} \subseteq U_s.$$

By Lemma 4.1, there exists $a > 0$ such that for all $k \in U'_s$,

$$|\Delta_{[l],J}(\rho(k))| \leq ae^{s\delta}.$$

By Lemma 4.2 and since $\rho$ faithful, there exists $s_0 < 0$ such that for all $s \leq s_0$,

$$E_s^{-1} \circ \mathcal{S}_s \left( N_{s\xi} \cap (C^\delta \times T^m) \right) \subseteq U.$$ \hfill $\Diamond$
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