DYNAMICAL RELATIVISTIC LIQUID BODIES I: CONSTRAINT PROPAGATION

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Abstract. We introduce a new wave formulation for the relativistic Euler equations with vacuum boundary conditions that consists of a system of non-linear wave equations in divergence form with a combination of acoustic and Dirichlet boundary conditions. We show that solutions of our new wave formulation determine solutions of the relativistic Euler equations that satisfy the vacuum boundary conditions provided the initial data is chosen to make a specific set of constraints vanish on the initial hypersurface. Moreover, we prove that these constraints propagate. This article is the first step of a two step strategy to establish the local-in-time existence and uniqueness of solutions to the relativistic Euler equations representing dynamical liquid bodies in vacuum.

1. Introduction

Over the past two decades, a number of results that guarantee the local-in-time existence and uniqueness of solutions to the (non-relativistic) Euler equations that represent dynamical fluid bodies in vacuum have been established [7, 9, 10, 18, 25, 29, 30, 46, 47]. Recently, the first step towards extending these existence results to the relativistic setting have been taken. Specifically, a priori estimates for solutions to the relativistic Euler equations that satisfy the vacuum boundary conditions have been established for both liquids [41] and gases [21, 24]. Unlike most initial boundary value problems where well-known approximation schemes can be used to obtain local-in-time existence and uniqueness results from a priori estimates, it is highly non-trivial to obtain existence from a priori estimates for dynamical fluid bodies in vacuum, whether relativistic or not. The main reason for the difficulty is the presence of the free fluid-matter vacuum boundary, which make it necessary to exploit much of the structure of the Euler equations in order to derive a priori estimates. This makes the use of approximation methods problematic since any approximate equation would have to possess all of the essential structure of the Euler equations used to derive estimates, and to find such approximations has proved to be very difficult.

The a priori estimates from [41] were derived using a wave formulation of the Euler equations consisting of a fully non-linear system of wave equations in divergence form together with non-linear acoustic boundary conditions. This system of wave equations and acoustic boundary conditions were obtained by differentiating the Lagrangian representation of the Euler equations and vacuum boundary conditions in time and adding constraints that vanish identically on solutions. A priori estimates, without derivative loss, were then established using an existence and uniqueness theory that was developed for linear systems of wave equations with acoustic boundary conditions together with Sobolev-Moser type inequalities to handle the non-linear estimates. This approach to deriving a priori estimates suggests a two step strategy to obtain the local-in-time existence, without derivative loss, of solutions to the relativistic Euler equations that satisfy the vacuum boundary conditions. The first step is to show that the constraints used to derive the wave formulation propagate; that is, to show that if the constraints, when evaluated on a solution of the wave formulation, vanish on the initial hypersurface, then they must vanish identically everywhere on the world tube defined by support of the solution. The second step is to establish the local-in-time existence and uniqueness of solutions to the wave formulation, which would follow from a standard iteration argument using the linear theory and Sobolev-Moser inequalities developed in [41]. Such solutions would, by step one, then determine solutions of the relativistic Euler equation that satisfy the vacuum boundary conditions thereby establishing the local-in-time existence of solutions representing dynamical relativistic liquid bodies.

The main purpose of this article is to carry out the first step of the above strategy. For technical reasons, we do not use the wave formulation from [41], but instead, we consider a related version that differs by a choice of constraints. This new wave formulation involves an additional scalar field that solves a wave equation with Dirichlet boundary conditions. We will not address the second step in this article. It is carried out in the separate article [38], where we prove a local-in-time existence and uniqueness theorem that establishes the existence of solutions corresponding to dynamical relativistic liquid bodies.
1.1. Related and prior work. In the non-relativistic setting, a number of different approaches have been used to establish the local-in-time existence and uniqueness of solutions to the Euler equations that satisfy vacuum boundary conditions. Important early work was carried out by S. Wu, who, in the articles [16, 47], solved the water waves problem by establishing the local-in-time existence of solutions for an irrotational incompressible liquid in vacuum. This work improved on the earlier results [11, 37, 19], where existence for water waves was established under restrictions on the initial data. Wu’s results were later generalized, using a Nash-Moser scheme combined with extensions to earlier a priori estimates derived in [6], by H. Lindblad to allow for vorticity in [30]. This work was subsequently extended to compressible liquids in [29].

Due to the reliance on Nash-Moser, Lindblad’s existence results involve derivative loss. By using an approximation based on a parabolic regularization that reduces in the limit of vanishing viscosity to the Euler equations, the authors of [9] were able to establish, without derivative loss, a local-in-time existence result for incompressible fluid bodies, which they later generalized to compressible gaseous and liquid bodies in [10] and [7], respectively. Existence for compressible gaseous bodies was also established using a different approach in [25]. For other related results in the non-relativistic setting, which includes other approaches to a priori estimates and existence on small and large time scales, see the works [11, 2, 4, 8, 16, 20, 19, 22, 23, 31, 32, 33, 34, 36, 43, 48, 50] and references cited therein.

In the relativistic setting, much less is known. For gaseous relativistic bodies, the only existence result in the most physically interesting case where the square of the sound speed goes to zero like the distance to the boundary that we are aware of is [40], which is applicable to 2 spacetime dimensions. However, based on earlier work by Makino [35] in the non-relativistic setting, Rendall established the existence of solutions to the Einstein-Euler equations representing self-gravitating gaseous bodies that are undergoing collapse [42]. For relativistic liquids, a local-in-time existence result involving derivative loss has been established in [44] using a symmetric hyperbolic formulation in conjunction with a Nash-Moser scheme.

We note that the use of constraints to establish the existence of solutions has a long history in Mathematical Relativity with, perhaps, the most well known and important application being the proof of the existence and geometric uniqueness of solutions to the Einstein equations, which was first established in [12]. See also [8, 5, 15, 27, 28, 42] for related work when boundaries are present. We would also like to add that the work presented here and in [41] was inspired by the constraint propagation approach to the relativistic fluid body problem from [14].

1.2. Initial boundary value problem for relativistic liquid bodies. In order to define the initial boundary value problem (IBVP) for a relativistic fluid, we first need to introduce some geometric structure starting with a 4-dimensional manifold $\mathcal{M}$ equipped with a smooth Lorentzian metric

$$g = g_{\mu\nu}dx^\mu dx^\nu \quad (1.1)$$

of signature $(-, +, +, +)$. In the following, we let $\nabla_\mu$ denote the Levi-Civita connection of $g_{\mu\nu}$ and $\Omega_0 \subset \mathcal{M}$ be a bounded, connected spacelike hypersurface with smooth boundary $\partial \Omega_0$. The manifold $\Omega_0$ defines the initial hypersurface where we specify initial data for the fluid. The proper energy density $\rho$ of the fluid is initially non-zero on $\Omega_0$ and vanishes outside. The initial hypersurface $\Omega_0$ forms the “bottom” of the world tube $\Omega_T$ defined by the motion of the fluid body through spacetime, which is diffeomorphic to the cylinder $[0, T] \times \Omega_0$. We let $\Gamma_T$ denote the timelike boundary of $\Omega_T$, which is diffeomorphic to $[0, T] \times \partial \Omega_0$. By our conventions, $\Gamma_0 = \partial \Omega_0$.

The motion of the fluid body is governed by the relativistic Euler equations given by

$$\nabla_\mu T^{\mu\nu} = 0 \quad (1.2)$$

where

$$T^{\mu\nu} = (\rho + p)v^\mu v^\nu + pg^{\mu\nu}$$

is the stress energy tensor, $v^\mu$ is the fluid 4-velocity normalized by

$$g_{\mu\nu}v^\mu v^\nu = -1,$$

$^1$In this article, we, for simplicity, restrict our considerations to the physical spacetime dimension of $n = 4$. However, the results presented in this article are valid for all spacetime dimensions $n \geq 2$.

$^2$There is no need for $\Omega_0$ to be either connected or bounded; non-connected $\Omega_0$ correspond to multiple fluid bodies, while non-bounded components represent unbounded fluid bodies.

$^3$Here, by vanishing outside, we mean there exists a spacelike hypersurface $\Sigma \subset \mathcal{M}$ that properly contains $\Omega_0$ and $\rho$ vanishes in $\Sigma \setminus \Omega_0$.
\( p \) is the pressure, and \( \rho \), as above, is the proper energy density of the fluid. Projecting (1.2) into the subspaces parallel and orthogonal to \( v^\mu \) yields the following well-known form of the relativistic Euler equations

\[
\begin{align*}
v^\mu \nabla_\mu \rho + (\rho + p) \nabla_\mu v^\mu &= 0, \\
(\rho + p) v^\mu \nabla_\mu v^\nu + h^{\mu \nu} \nabla_\mu p &= 0,
\end{align*}
\]

where

\[
h_{\mu \nu} = g_{\mu \nu} + v_\mu v_\nu
\]
is the induced positive definite metric on the subspace orthogonal to \( v^\mu \).

In this article, we restrict our attention to fluids with a barotropic equation of state of the form

\[
\rho = \rho(p)
\]
where \( \rho \) satisfies

\[
\rho \in C^\infty((-\infty, \infty), [\rho_0, \rho_1]), \quad \rho(0) = \rho_0,
\]

and

\[
\frac{1}{s_1} \leq \rho'(p) \leq \frac{1}{s_0}, \quad -\infty < p < \infty,
\]

for some positive constants \( 0 < \rho_0 < \rho_1 \), and \( 0 < s_0 < s_1 < 1 \). Since the square of the sound speed is given by

\[
s^2 = \frac{1}{\rho'(p)},
\]

the assumption (1.7) implies that \( 0 < s_0^2 \leq s^2 \leq s_1^2 < 1 \), or in other words, that the sound speed is bounded away from zero and strictly less than the speed of light.

The boundary of the world-tube \( \Omega_T \), which separates the fluid body from the vacuum region, is defined by the vanishing of the pressure, i.e. \( p|_{\Gamma_T} = 0 \). By our assumption (1.6), this means that the proper energy density does not vanish at the boundary, and hence, there is a jump in the proper energy density across \( \Gamma_T \). Fluids of this type are referred to as \textit{liquids}. In addition to the vanishing of the pressure, the condition \( v|_{\Gamma_T} \in TT \Gamma_T \) must be satisfied to ensure that no fluid moves across \( \Gamma_T \). These two conditions form the vacuum boundary conditions satisfied by freely evolving fluid bodies. Collecting these boundary conditions together with the evolution equations (1.3)-(1.4), the complete Initial Boundary Value Problem (IBVP) for a relativistic liquid body is:

\[
\begin{align*}
v^\mu \nabla_\mu \rho + (\rho + p) \nabla_\mu v^\mu &= 0 \quad \text{in } \Omega_T, \\
(\rho + p) v^\mu \nabla_\mu v^\nu + h^{\mu \nu} \nabla_\mu p &= 0 \quad \text{in } \Omega_T, \\
p &= 0 \quad \text{in } \Gamma_T, \\
n_\nu v^\nu &= 0 \quad \text{in } \Gamma_T, \\
(\rho, v^\mu) &= (\tilde{\rho}, \tilde{v}^\mu) \quad \text{in } \Omega_0,
\end{align*}
\]

where \( (\tilde{\rho}, \tilde{v}^\mu) \) is the initial data, and \( n_\nu \) is the outward pointing unit conormal to \( \Gamma_T \).

1.3. Overview. We fix our notation and conventions used throughout this article in Section 2 and in Appendix A, where a number of definitions and formulas from differential geometry are collected. In Sections 3.1 and 3.3, we define the primary fields and constraints, respectively, that will be used in our wave formulation of the relativistic liquid body IBVP. The Eulerian representation of our wave formulation, which includes the freedom to add constraints, is introduced in Section 3.3. We then state and prove Theorem 3.1 in Section 4. Informally, this theorem guarantees the constraints, when evaluated on solutions of our wave formulation, vanish in \( \Omega_T \) provided they vanish initially on \( \Omega_0 \), i.e. they propagate, and moreover, that solutions to our wave formulation for which constraints vanish correspond to solutions of the relativistic liquid IBVP. We then, in Section 5, make a particular choice of the constraints that appear in the evolution equations and boundary conditions that define our wave formulation in order to bring the total system into form that is favorable for establishing the existence and uniqueness of solutions. In Section 6, we introduce Lagrangian coordinates and express system of equations and boundary conditions from Section 5 in these coordinates. In the final section, Section 7, we...
briefly discuss how the linear existence and uniqueness theory for systems of wave equations with acoustic boundary conditions from \([11]\) can be applied to the linearization of the Lagrangian representation of our wave formulation from Section \([8]\) and moreover, how this will be used in the article \([35]\) to establish the local-in-time existence and uniqueness of solutions to the relativistic liquid IBVP.

2. Preliminaries

In this section, we fix our notation and conventions that we will employ throughout this article; see also Appendix \([4]\) where we collect a number of definitions and formulas from differential geometry.

2.1. Indexing conventions. We will need to index various objects. The conventions that will employ are as follows:

| Alphabet | Examples | Index range | Index quantities |
|----------|----------|-------------|-----------------|
| Lowercase Greek | \(\mu, \nu, \gamma\) | \(0, 1, 2, 3\) | spacetime coordinate components |
| Uppercase Greek | \(\Lambda, \Sigma, \Omega\) | \(1, 2, 3\) | spatial coordinate components |
| Lowercase Latin | \(i,j,k\) | \(0, 1, 2, 3\) | spacetime frame components |
| Uppercase Latin | \(I,J,K\) | \(1, 2, 3\) | spatial frame components |
| Lowercase Calligraphic | \(i,j,k\) | \(0,1,2\) | spacetime boundary frame components |
| Uppercase Calligraphic | \(I,J,K\) | \(1,2\) | spatial boundary frame components |

2.2. Partial derivatives. We use

\[
\partial_\mu = \frac{\partial}{\partial x^\mu}
\]

to denote partial derivatives with respect to the coordinates \((x^\mu)\), and \(\partial\) to denote spacetime gradients so that \(\partial f = (\partial_\mu f)\) for scalar fields \(f = f(x^\mu)\). More generally, for \(k \in \mathbb{Z}_{\geq 0}\), we use \(\partial^k f = (\partial_\mu \partial_\nu \cdots \partial_\mu f)\) to denote the set of all partial derivatives of order \(k\), and similarly, \(\partial^k\) to denote the collection of derivatives of order \(k\) that are tangent to the boundary \(\Gamma_T\). We also let \(\partial^{[k]} f = \{ \partial f | 0 \leq j \leq k \}\) and \(\partial^{[k]} f = \{ \partial f | 0 \leq j \leq k \}\) denote the collection of partial derivatives of order less than or equal to \(k\).

2.3. Raising and lowering indices. We lower and raise spacetime coordinate indices without comment using the metric \(g_{\mu\nu}\) and its inverse \(g^{\mu\nu}\), respectively, while frame indices will be lowered and raised, again without comment, using the frame metric \(\gamma_{ij}\) and its inverse \(\gamma^{ij}\), respectively; see \([A,3]\) for a definition of the frame metric. We will have occasion to raise or lower indices using metrics other than \(g_{\mu\nu}\) or \(\gamma_{ij}\). In these situations, we will be explicit about this type of operation. For example, given a metric \(m^{\mu\nu}\) and a 1-form \(\lambda_\mu\), we would define the raised version using \(m\) explicitly by setting \(\lambda^\mu = m^{\mu\nu} \lambda_\nu\).

2.4. Norms. For a spacelike 1-form \(\lambda_\mu\), we define the spacetime norm \(|\lambda|_g\) by

\[
|\lambda|_g := \sqrt{g(\lambda, \lambda)} = \sqrt{g^{\mu\nu} \lambda_\mu \lambda_\nu},
\]

while if \(m = m^{\mu\nu}\) is a positive definite metric, then we define the \(m\)-norm of any 1-form \(\lambda_\mu\) by

\[
|\lambda|_m := \sqrt{m(\lambda, \lambda)} = \sqrt{m^{\mu\nu} \lambda_\mu \lambda_\nu}.
\]

Similar notation will also be used for inner products involving other objects carrying indices of some type; for example, we write \(|T|_m^2 = m_{\alpha\beta} m^{\mu\nu} T^\mu_\alpha T^\beta_\nu\) for a \((1, 1)\) tensor \(T^\mu_\alpha\), where \((m_{\alpha\beta}) := (m^{\alpha\beta})^{-1}\).

2.5. Constraint terms. To help encode the freedom to add constraints to evolution equations and boundary conditions, we reserve upper case Fraktur letters, e.g. \(\mathfrak{R}, \mathfrak{S}, \mathcal{Z}\), possibly endowed with spacetime indices, e.g. \(\mathfrak{R}^\nu\), to denote maps that depend linearly on a set of constraints \(\mathcal{Z}\). More precisely, if \(\mathcal{Z}\) is \(\mathbb{R}^N\)-valued, then

\[
\mathfrak{R}(\mathcal{Z}) = m \mathcal{Z}
\]

where \(m \in C^0(\overline{\Omega_T}, \mathbb{M}_{N \times N})\) if \(\mathfrak{R}(\mathcal{Z})\) is added to an evolution equation and \(m \in C^0(\overline{\Gamma_T}, \mathbb{M}_{N \times N})\) if \(\mathfrak{R}(\mathcal{Z})\) is added to a boundary condition.
3. The Eulerian wave formulation

3.1. Primary and auxiliary fields. The primary fields for our wave formulation consist of a scalar field $\zeta$ satisfying

$$\zeta > 0$$

and a future pointing, timelike 1-form $\dot{\theta}^0 = \dot{\theta}^0_\mu dx^\mu$. We use the primary fields to define a timelike 1-form, again future pointing, by

$$\theta^0 = \zeta \dot{\theta}^0,$$

which we complete to a coframe by introducing spacelike 1-forms $\theta^I = \theta^I_\mu dx^\mu$.

Along with these 1-forms, we introduce a collection of scalar fields $\sigma^{ikj}$. The set $\{\theta^I, \sigma^{ikj}\}$ defines the auxiliary fields that will evolve via simple transport equations. For latter use, we introduce a number of additional geometric fields beginning with the frame

$$e_j = e_j^\mu \partial_\mu \quad ((e^\mu_j) := (\theta^j_\mu)^{-1})$$

dual to $\theta^i$. Following our notation from Appendix A, we use $\gamma_{ij}$ and $\omega^{ikj}$ to denote the associated frame metric and connection coefficients, respectively; see the formulas (A.3) and (A.5). Finally, we define a future pointing, timelike vector field by

$$\xi^\mu = \frac{1}{\gamma_{00}} \theta_{00}.$$

3.2. Recovering $\rho$ and $v^\mu$. The fluid 4-velocity $v^\mu$ will be shown to be recoverable from the primary fields $\{\zeta, \dot{\theta}^0_\mu\}$ by normalizing the vector field $\xi^\mu$ to get

$$v^\mu := \frac{\xi^\mu}{\sqrt{-g(\xi, \xi)}} = -\frac{\dot{\theta}^0}{\sqrt{-g(\dot{\theta}^0, \dot{\theta}^0)}},$$

where in obtaining the second equality we used the fact that $\zeta > 0$ and $\gamma_{00} = g(\theta^0, \theta^0) < 0$. Recovering the proper energy density is more complicated. The first step is to define the pressure as a solution $p = p(\lambda)$ of the initial value problem

$$p'(\lambda) = \frac{1}{\lambda} \left( \rho(p(\lambda)) + p(\lambda) \right), \quad \lambda > 0,$$

$$p(\lambda_0) = p_0,$$

where $\lambda_0 > 0$ and $p_0 \geq 0$. To be definite, we set

$$p_0 = 0 \quad \text{and} \quad \lambda = 1.$$

From standard ODE theory, we see that $p = p(\lambda)$ is smooth for $\lambda > 0$, while from the IVP (3.5)-(3.6), it follows that $p$ is strictly increasing, which in turn, implies the invertibility of the map $\mathbb{R}_{\geq 1} \ni \lambda \mapsto p(\lambda) \in \mathbb{R}_{\geq 0}$. We will show that we can then use this map to recover the proper energy density from scalar field $\zeta$ by setting $\lambda = \zeta$ to give $p = p(\rho(\zeta))$.

To summarize, $\{\rho, v^\mu\}$ are determined from the primary fields $\{\zeta, \dot{\theta}^0_\mu\}$ via the formulas:

$$\rho = \rho(p(\zeta)),$$

$$v^\mu = -\frac{\theta^0}{\sqrt{-g(\dot{\theta}^0, \dot{\theta}^0)}},$$

3.3. Constraints. In this section, we define the constraints that will be essential for the definition of our wave formulation of the relativistic Euler equations with vacuum boundary conditions. We separate the constraints into bulk and boundary constraints, which are to be interpreted as being associated to $\Omega_T$ and $\Gamma_T$, respectively.
Bulk constraints:

\( a = \xi - e_0, \)  
\( b^J = \gamma^{0J}, \)  
\( c_{ij} = \sigma^i_j - \sigma^j_i, \)  
\( d^k_j = \sigma^k_0, \)  
\( e^K = d \theta^K + \frac{1}{2} \sigma^{K\ell} \theta^\ell \wedge \theta^J, \)  
\( F = d \theta^0 + \frac{1}{2} \sigma^I_j \theta^J \wedge \theta^I, \)  
\( g = \delta_g \left( \frac{1}{f(\xi)} \right), \)  
\( h = -\sqrt{-\det(\gamma_{ij})} - \frac{\xi}{f(\xi)}, \)  
\( j = g(\hat{\theta}^0, \hat{\theta}^0) + 1 \)

where \( f(\lambda) \) is defined by

\( f(\lambda) = -\lambda \exp \left( -\int_1^{\lambda} \frac{1}{\eta^2 s(\eta)} d\eta \right) \)

with the square of the sound speed given by \( s^2(\lambda) = (\rho'(p(\lambda)))^{-1}. \)

We collect together the following bulk constraints

\( \chi = (a^\mu, b^J, c_{ij}, d^k_j, e^K), \)

which, as we shall see in the proof of Theorem 4.1, satisfy simple transport equations. For latter use, we observe from (3.1), (3.3), (3.4), (3.10) and (A.3) that

\( v^\mu = \sqrt{-\gamma^{00}} \xi^\mu = \sqrt{-\gamma^{00}} e_0^\mu + \Re^\mu(a) = \sqrt{-\gamma^{00}} e_0^\mu + \Omega^\mu(a, \xi - 1), \)

where we have set

\( \hat{\gamma}^{00} = g(\hat{\theta}^0, \hat{\theta}^0) \)

and we are using \( \Re^\mu \) and \( \Omega^\mu \) to denote constraint terms in line with our notation set out in Section 2.5.

Boundary constraints:

\( \xi = \hat{\theta}^3 - n \)

where

\( \hat{\theta}^3 = \frac{\theta^3}{|\theta^3|} \)

and as in above, \( n \) is the outward pointing unit conormal to \( \Gamma_T. \)

3.4. **Eulerian IBVP.** The formulation (1.9)-(1.13) of the vacuum IBVP for the relativistic Euler equations is commonly referred to as the *Eulerian representation*. In this representation, the matter-vacuum boundary is free, or in other words, dynamical. This terminology is useful for distinguishing this form of the IBVP from the *Lagrangian representation* where the boundary is fixed. We will continue to use the Eulerian terminology for the wave formulation of the relativistic vacuum IBVP that we introduce in this section since the boundary is also free. Later, in Section 6, we will consider the Lagrangian representation where the boundary is fixed.

3.4.1. **Eulerian evolution equations.** Before stating the evolution equations, we first define a number of tensors that will be used repeatedly throughout this article starting with

\( \mathcal{W}^{\alpha\mu} = m^{\alpha\nu} \sigma^{\beta\nu}[\xi \nabla_{\beta} \hat{\theta}^0 + \sigma^I_0 \hat{\theta}^I \theta^J] \)

where \( m^{\alpha\nu} \) is a Riemannian metric and \( \sigma^{\alpha\beta} \) is a Lorentzian metric that are defined by

\( m^{\alpha\beta} = g^{\alpha\beta} + 2e^{\alpha\nu} e^{\beta\nu} \)

For solutions of our wave formulation that correspond to solutions of the relativistic Euler equations, the metric \( \sigma^{\alpha\beta} \) is conformal to the standard definition of the acoustic metric given by \( g^{\alpha\beta} + \left(1 - \frac{1}{2v^2}\right)v^\alpha v^\beta. \)
and

\[ a^{\alpha\beta} = -\frac{1}{f(\xi)} \left( g^{\alpha\beta} - \left( 1 - \frac{1}{s^2(\xi)} \right) \frac{\xi^\alpha \xi^\beta}{g(\xi, \xi)} \right), \]  

respectively. We use \( W^{\alpha\mu} \) to define

\[ \mathcal{E}^\mu = \nabla_\alpha W^{\alpha\mu} - \mathcal{H}^\mu \]  

where

\[
\mathcal{H}^\mu = \nabla_\alpha m^{\mu\nu} a^{\alpha\beta} \left[ \xi \nabla_\beta \hat{\theta}_\nu + \sigma_1^0 \theta_\beta^j \theta_\gamma^j \right] + m^{\mu\nu} \left[ -\nabla_\alpha \hat{\theta}_\gamma \right. \\
\left. \partial^\gamma \hat{\theta}_\nu + \sigma_1^0 \theta_\beta^j \theta_\gamma^j \right] \frac{\hat{\theta}_\nu}{g(\hat{\theta}^0, \theta^0)} + h_\nu^\gamma \left( \hat{\mathcal{H}}_\gamma - a^{\alpha\beta} \nabla_\xi \xi \nabla_\alpha \hat{\theta}_\gamma \right),
\]

\[
\hat{\mathcal{H}}_\nu = -\frac{1}{f} R^{\mu\nu} \theta_\mu^0 + \left( \delta_\alpha^\nu C^\omega_\alpha a^\lambda \beta - C^\beta_\alpha a^{\omega\alpha} \right) F_{\alpha\beta},
\]

and

\[ C^\lambda_\alpha = \frac{1}{2} a^{\lambda\gamma} \left[ \nabla_\omega a_\beta^\gamma + \nabla_\beta a_\gamma^\omega - \nabla_\gamma a_\beta^\omega \right], \quad \left( (a^{\alpha\beta})^{-1} \right).
\]

Here, we are using \( R_{\mu\nu} \) to denote the Ricci tensor of the metric \( g_{\mu\nu} \),

\[
F_{\alpha\beta} = \nabla_\alpha \theta_\beta^0 - \nabla_\beta \theta_\alpha^0 + \sigma_1^0 \theta_\beta^j \theta_\gamma^j
\]

to denote the coordinate components of the 2-form \( F \) defined by (3.15), i.e. \( F = \frac{1}{2} F_{\alpha\beta} \omega^\alpha \wedge \omega^\beta \), and \( h_{\alpha\beta} \) is as defined previously [1,3]. Using the above definitions, the evolution equations for our wave formulation are given by

\[
\nabla_\nu \mathcal{E}^\mu + \mathcal{R}^\mu \hat{\mathcal{E}}^{\hat{\mathcal{E}}}(\hat{\mathcal{E}}^{\hat{\mathcal{E}}} \hat{\mathcal{E}}) = 0 \quad \text{in} \ \Omega_T,
\]

\[
\nabla_\alpha (a^{\alpha\beta} \nabla_\beta \xi) = \mathcal{K} \quad \text{in} \ \Omega_T,
\]

\[
\mathcal{L}_\nu \theta_\mu^0 = 0 \quad \text{in} \ \Omega_T,
\]

\[
v(\sigma_1^k) = 0 \quad \text{in} \ \Omega_T,
\]

where

\[ \mathcal{K} = -\hat{\theta}_\nu \hat{\mathcal{H}}_\nu - \nabla_\nu \hat{\theta}_\gamma \left[ \xi \nabla_\beta \hat{\theta}_\nu + \sigma_1^0 \theta_\beta^j \theta_\gamma^j \right]. \]

**Remark 3.1.** The term \( \mathcal{R}^\mu \hat{\mathcal{E}}^{\hat{\mathcal{E}}} (\hat{\mathcal{E}}^{\hat{\mathcal{E}}} \hat{\mathcal{E}}) \) in (3.32) encodes the available freedom to add multiples of \( \mathcal{E}^\lambda \) and \( \chi \) and its first derivatives\(^6\) to the evolution equation \( \nabla_\nu \mathcal{E}^\mu = 0 \) in \( \Omega_T \). This freedom provides flexibility in choosing the form of the equations of motion, which we will take advantage of in Section 5 to bring the equations of motion into a form that is useful for establishing the local-in-time existence and uniqueness of solutions. We further note that it is clear from the equations of motion (3.34)-(3.35) that we are free to add terms proportional to \( L_{\nu} \theta_\mu^0 \) and \( v(\sigma_1^k) \) and their derivatives to evolution equations (3.32)-(3.33).

3.4.2. **Eulerian boundary conditions.** Before stating the boundary conditions for our wave formulation, we define

\[ \mathcal{B}^\mu = \theta_\alpha^3 W^{\alpha\mu} - \mathcal{L}^\mu \]

where

\[
\mathcal{L}^\mu = \left( \delta^\mu_\omega - \epsilon \Pi^\mu_\omega \right) \left( h^{\nu\alpha} s_{\alpha\beta} \hat{\gamma} \nabla_\gamma \hat{\theta}_\nu - h^{\nu\beta} \sqrt{-\hat{\gamma}} v^\alpha c_1^0 c_2^0 \nabla_\nu \nabla_\nu s_{\alpha\beta} \hat{\gamma} \right)
\]

\[
- \epsilon |N| \Pi^{\mu\nu} \frac{\nabla_\alpha \nabla_\nu \hat{\theta}_\nu}{|\nabla_\nu \hat{\theta}_\nu|_{\nu}} - \kappa v^\nu v^\nu \nabla_\nu \hat{\theta}_\nu
\]

\[ ^6\text{There is nothing stopping us from adding higher order derivatives of } \chi \text{ to the evolution equation. The only thing that would change in the analysis below is that the class of solutions that we are dealing with would have to have enough regularity for the derivatives of } \chi \text{ to make sense.} \]
with

\[ \nu_{\alpha\beta\gamma} = \sqrt{|g|} \epsilon_{\alpha\beta\gamma}, \]  
\[ N_\nu = -\frac{1}{\sqrt{-\gamma_{00}}} \nu_{\alpha\beta\gamma} \epsilon^\beta_1 e^\gamma_2, \]  
\[ \hat N_\nu = \frac{N_\nu}{|N|_g}, \]  
\[ p^{\mu\nu} = g^{\mu\nu} - \hat N^\mu \hat N^\nu, \]  
\[ \Pi^\mu_\nu = p^{\mu\nu} h^\lambda_\nu, \]  
\[ s_{\mu\nu} = \nu_{\mu\nu\lambda\omega} e^\lambda_1 e^\omega_2 \gamma^\nu - \nu_{\mu\nu\lambda\omega} e^\lambda_2 e^\omega_1 \gamma^\nu + \nu_{\mu\nu\lambda\omega} e^\lambda_1 e^\omega_2 \gamma^\nu, \]  
and \( \epsilon, \kappa \in \mathbb{R} \) constants to be fixed later. With \( B^\mu \) so defined, the boundary conditions are given by

\[ \nabla_v B^\mu + \Omega^\mu (\nabla^{|1|}_v h, \partial^{|1|}_v \chi, \partial^{|1|}_v \kappa, B) = 0 \quad \text{in } \Gamma_T, \]  
\[ \zeta = 1 \quad \text{in } \Gamma_T, \]  
\[ n_\nu v^\nu = 0 \quad \text{in } \Gamma_T. \]  

Remark 3.2. The term \( \Omega^\mu (\nabla^{|1|}_v h, \partial^{|1|}_v \chi, \partial^{|1|}_v \kappa, B) \) in (3.44) encodes the available freedom to add multiples of \( B^\lambda, \chi, \kappa \) and their indicated derivatives to the boundary conditions \( \nabla_v B^\mu = 0 \) in \( \Gamma_T \). This freedom provides flexibility in choosing the form of the boundary conditions, which we will take advantage of in Section 5 to bring the boundary conditions into a form that is useful for establishing the local-in-time existence and uniqueness of solutions. We also note that it is clear from the equations of motion (3.34) and (3.39) that the vector fields \( N^\nu \) and \( \xi^\nu \) are \( g \)-orthogonal, and hence satisfy

\[ h^\mu_\nu h^\lambda_\nu = h^\mu_\nu, \quad h^\mu_\nu \xi^\nu = 0, \]  

and

\[ p^\mu_\lambda p^\lambda_\nu = p^\mu_\nu, \quad p^\mu_\nu N^\nu = 0. \]  

Moreover, these projections commute, that is

\[ h^\mu_\nu p^\lambda_\nu = p^\mu_\nu h^\lambda_\nu, \]  

since it is clear from the definitions (3.3) and (3.39) that the vector fields \( N^\nu \) and \( \xi^\nu \) are \( g \)-orthogonal. It is worthwhile noting that we can raise the lower index of \( h^\mu_\nu \) using either \( g^{\nu\lambda} \) or \( m^{\nu\lambda} \) since \( h^\mu_\nu v^\lambda = 0 \), so that

\[ h^{\mu\nu} = m^{\mu\lambda} h^\nu_\lambda, \]  

where \( h^{\mu\nu} = g^{\nu\lambda} h^\lambda_\nu \) by our raising and lowering conventions. From the definition (3.42), it is then clear by (3.47) - (3.50) that \( \Pi^\mu_\nu \) defines a projection operator satisfying

\[ \Pi^\mu_\nu \Pi^\lambda_\nu = \delta^\mu_\nu, \quad \Pi^\mu_\nu N^\nu = 0, \quad \Pi^\mu_\nu \xi^\nu = 0 \]  

and

\[ \Pi^{\mu\nu} = m^{\mu\lambda} \Pi^\lambda_\nu. \]  

3.4.3. Projections. The operators \( h^\mu_\nu \) and \( p^\mu_\nu \) define projections into the subspaces \( g \)-orthogonal to \( \xi^\mu \) and \( N^\mu \), respectively, and hence satisfy

\[ h^\mu_\nu h^\lambda_\nu = h^\mu_\nu, \quad h^\mu_\nu \xi^\nu = 0, \]  

and

\[ p^\mu_\lambda p^\lambda_\nu = p^\mu_\nu, \quad p^\mu_\nu N^\nu = 0. \]  

We denote the complementary projection operator by

\[ \Pi^\mu_\nu = \delta^\mu_\nu - \Pi^\mu_\nu \]  

and set

\[ \Pi^{\mu\nu} = m^{\mu\lambda} \Pi^\lambda_\nu. \]  

\(^7\text{Again, there is nothing stopping us from adding higher order derivatives of } \chi \text{ and } \kappa \text{ to the boundary conditions.}\)
3.4.4. **Initial conditions.** In general, solutions of the IBVP problem consisting of the evolution equations (3.32)-(3.35) and boundary conditions (3.44)-(3.46) will not correspond to solutions of the relativistic Euler equations with vacuum boundary conditions given by (1.9)-(1.12). As we establish in Theorem 4.1 below, a solution \( \{ \zeta, \hat{\theta}_\mu^0, \theta^J_\mu, \sigma_{i,k} \} \) of (3.32)-(3.35) and (3.44)-(3.46) will only determine a solution, via the formulas (3.8)-(3.9), of (1.9)-(1.12) if the following conditions on the initial data are satisfied:

- \( a = 0 \) in \( \Omega_0 \), (3.55)
- \( b^J = 0 \) in \( \Omega_0 \), (3.56)
- \( c^k_j = 0 \) in \( \Omega_0 \), (3.57)
- \( d^k_j = 0 \) in \( \Omega_0 \), (3.58)
- \( e^K = 0 \) in \( \Omega_0 \), (3.59)
- \( \mathcal{F} = 0 \) in \( \Omega_0 \), (3.60)
- \( g = 0 \) in \( \Omega_0 \), (3.61)
- \( h = 0 \) in \( \Omega_0 \), (3.62)
- \( j = 0 \) in \( \Omega_0 \), (3.63)
- \( k = 0 \) in \( \Gamma_0 \), (3.64)
- \( d \mathcal{F} = 0 \) in \( \Omega_0 \), (3.65)
- \( \nabla_v g = 0 \) in \( \Omega_0 \), (3.66)
- \( \nabla_v h = 0 \) in \( \Omega_0 \), (3.67)
- \( \nabla_j = 0 \) in \( \Omega_0 \), (3.68)
- \( \mathcal{E}^\mu = 0 \) in \( \Omega_0 \) (3.69)

and

- \( \mathcal{B}^\mu = 0 \) in \( \Gamma_0 \). (3.70)

**Remark 3.3.** The constraints on the initial data (3.55)-(3.70) do not involve any constraints on the choice of initial data for the fluid, that is \( \rho \) and \( v^\mu \) or equivalently \( \hat{\theta}_\mu^0 \) and \( \zeta \), beyond the usual compatibility conditions that arise from the IBVP for the relativistic Euler equations with vacuum boundary conditions. The constraints that are unrelated to compatibility conditions for the physical fields involve the auxiliary fields, which are not physical and we are free to choose as we like; see Section 4.2 of [41] for details on how to choose the initial data for the auxiliary fields so that the above constraints are satisfied.

### 4. Constraint propagation

The precise relationship between solutions to the evolution equations and boundary conditions defined in the previous section and solutions to the relativistic Euler equations with vacuum boundary conditions is given in the following theorem. The main content of this theorem is that it guarantees that the constraints from Section 3.3 propagate for solutions of the evolution equations and boundary conditions defined in Sections 3.4.1 and 3.4.2 provided that the initial condition from Section 3.4.4 are satisfied.

**Theorem 4.1.** Suppose \( \epsilon \geq 0, \kappa \geq 0, \zeta \in C^2(\Omega_T), \hat{\theta}_\mu^0 \in C^3(\Omega_T), \theta^J_\mu \in C^2(\Omega_T), \sigma_{i,k} \in C^1(\Omega_T) \), there exists constants \( c_0, c_1 > 0 \) such that

- \( -g(\theta^0, \theta^0) \geq c_0 > 0 \) in \( \Omega_T \) (4.1)

and

- \( -i_{v^0} \nabla_v \theta^0 \geq c_1 > 0 \) in \( \Gamma_T \), (4.2)

and the quadruple \( \{ \zeta, \hat{\theta}_\mu^0, \theta^J_\mu, \sigma_{i,k} \} \) satisfies the initial conditions (3.55)-(3.70), the evolution equations (3.32)-(3.35) and the boundary conditions (3.44)-(3.46). Then the constraints (3.10)-(3.18) and (3.22) vanish in \( \Omega_T \) and \( \Gamma_T \), respectively, and the pair \( \{ \rho, v^\mu \} \) determined from \( \{ \hat{\theta}_\mu^0, \zeta \} \) via the formulas (3.8)-(3.9) satisfy the relativistic Euler equations with vacuum boundary conditions given by (1.9)-(1.12).
Proof. Propagation of $\theta^l$ in $\Omega_T$: From the definition of $\theta^l$, see (3.11), we compute
\[\begin{align*}
v(\theta^l) & \equiv L_v \left( \frac{1}{\sqrt{-\gamma^0}} i_v \theta^l \right) \\
& \equiv L_v \left( \frac{1}{\sqrt{-\gamma^0}} i_v \theta^l + L_v \left( \frac{1}{\sqrt{-\gamma^0}} \right) \gamma^0 \theta^l \right) \\
& \equiv L_v \left( \frac{1}{\sqrt{-\gamma^0}} \right) \sqrt{-\gamma^0} \theta^l, \tag{4.3}
\end{align*}\]
which holds in $\Omega_T$. The assumption (4.1) and the boundary condition (3.46) imply that $v$ is timelike and tangent to the boundary $\Gamma_T$, which in turn, implies that the set $\Omega_T$ is invariant under the flow of $v$.

From this fact, the transport equation (4.3) and the initial condition (3.56), it follows via the uniqueness of solutions to transport equations that
\[\theta^l = \gamma^0 \theta^l = 0 \quad \text{in} \quad \Omega_T. \tag{4.4}\]

Propagation of $a$ in $\Omega_T$: Since $i_\xi \theta^l = 0$ by (4.4) while $i_\xi \theta^0 = 1$ is a direct consequence of the definition (3.3), it follows immediately that
\[a = \xi - e_0 = 0 \quad \text{in} \quad \Omega_T. \tag{4.5}\]

Propagation of $c^k_{ij}$ and $d^k_j$ in $\Omega_T$: From the definitions (3.12)-(3.13) and the evolution equation (3.33), it is clear that the constraints $c^k_{ij}$ and $d^k_j$ satisfy the transport equations $v(c^k_{ij}) = 0$ and $v(d^k_j) = 0$ in $\Omega_T$. Since $c^k_{ij}$ and $d^k_j$ vanish on the initial hypersurface, see (3.37)-(3.38), we conclude, again by the uniqueness of solutions to transport equations, that
\[c^k_{ij} = \sigma^k_{ij} - \sigma^1_i \sigma^k_j = 0 \quad \text{and} \quad d^k_j = \sigma_0^k j = 0 \quad \text{in} \quad \Omega_T. \tag{4.6}\]

Propagation of $\epsilon^K$ in $\Omega_T$: From the definition (3.14) of $\epsilon^K$, we have that
\[L_v \epsilon^K = L_v d \theta^K + \frac{1}{2} \left( v(\sigma^K_j) \theta^i \wedge \theta^j + \sigma^K_J L_v \theta^i \wedge \sigma^K_J L_v \theta^j + \sigma^K_J \theta^i \wedge L_v \theta^j \right) \tag{4.7}\]
(by (A.7))
\[= L_v d \theta^K + \frac{1}{2} \left( \sigma^K_J L_v \theta^i \wedge \theta^j + \sigma^K_J \theta^i \wedge L_v \theta^j \right) \tag{4.8}\]
(by (3.35) and (4.6))
\[= d L_v \theta^K + \frac{1}{2} \left( \sigma^K_J L_v \theta^i \wedge \theta^j + \sigma^K_J \theta^i \wedge L_v \theta^j \right) \tag{4.9}\]
(by (A.5))
\[= 0 \tag{4.10}\]
in $\Omega_T$, and so
\[\epsilon^K = d \theta^K + \frac{1}{2} \sigma^K_J \theta^i \wedge \theta^j = 0 \quad \text{in} \quad \Omega_T\]
by the uniqueness of solutions to transport equations since $\epsilon^K$ vanishes on the initial hypersurface, see (3.59). We note, with the help of the Cartan structure equations (A.17), that (4.7) is equivalent to
\[\sigma^K_J = \omega^K_i \sigma^K_J - \omega^K_J \sigma^K_i \quad \text{in} \quad \Omega_T. \tag{4.11}\]

Propagation of $\xi$ in $\Gamma_T$: Letting $G_t(x^\lambda) = (\theta^\mu_t(x^\lambda))$ denote the flow of $v$ so that
\[\frac{d}{dt} G_t^\mu(x^\lambda) = v^\mu (G(x^\lambda)) \quad \text{and} \quad G_0^\mu(x^\lambda) = x^\mu, \]
we introduce Lagrangian coordinates ($\tilde{x}^\mu$) via
\[x^\mu = \phi^\mu(\tilde{x}) := G^\mu_{\tilde{x}}(0, \tilde{x}^\lambda), \quad \forall (\tilde{x}^0, \tilde{x}^\lambda) \in [0, T] \times \Omega_0. \tag{4.12}\]

Denoting the pull-back of $v$ via the map $\phi$ by $\bar{v} = \phi^* v$, we then have that
\[\bar{v} = \bar{\theta}_0 \quad \iff \quad \bar{v}^\mu = \delta^\mu_0. \tag{4.13}\]

We also observe that
\[\bar{v}^1(0, \tilde{x}^\lambda) = r(\tilde{x}^\lambda) \delta^0_{\mu} \quad \forall (\tilde{x}^\lambda) \in \partial \Omega_0 \tag{4.14}\]

\[0, T] \times \Omega_0 = \phi^{-1}(\Omega_T) \quad \text{and} \quad [0, T] \times \partial \Omega_0 = \phi^{-1}(\Gamma_T), \tag{4.15}\]
where $\tilde{x}^0$ defines a coordinate on the interval $[0, T]$ and the $(\tilde{x}^\lambda)$ define “spatial” coordinates on $\Omega_0$.

Without loss of generality, we may assume $\tilde{x}^3$ is a defining coordinate for the boundary $[0, T] \times \partial \Omega_0$ so that $d \tilde{x}^3$ is an outward pointing conormal on the boundary $[0, T] \times \partial \Omega_0$. Letting $\tilde{\theta}^3 = \phi^* \theta^3$ denote the pull-back of $\theta^3$ by $\phi$, we have from the choice of initial data, see (3.6-1), that
\[\tilde{\theta}^3_0(0, \tilde{x}^\lambda) = r(\tilde{x}^\lambda) \tilde{\theta}^3_0 \quad \forall (\tilde{x}^\lambda) \in \partial \Omega_0 \tag{4.16}\]
for some positive function \( r > 0 \). Pulling back the evolution equation \((3.34)\) for \( I = 3 \), we see from \((4.10)\) and \((4.9)\) that

\[
L_{\xi} \tilde{\theta}^3 = 0 \iff \tilde{\theta}_0 \tilde{\theta}^3 = 0,
\]

which implies by \((4.12)\) that \( \tilde{\theta}_\mu(x^0, x^A) = r(x^A) \delta^3_\mu \) for all \( (x^0, x^A) \in [0, T] \times \partial \Omega_0 \). But this shows that \( \tilde{\theta}^3 \) defines an outward pointing conormal to \( [0, T] \times \partial \Omega_0 \), and hence, that \( \theta^3 \) is an outward pointing conormal to \( \Gamma_T \) and

\[
\kappa = \tilde{\theta}^3 - n = 0 \quad \text{in } \Gamma_T \tag{4.13}
\]

by \((3.22)\).

**Propagation of \( k \) in \( \Gamma_T \):** By definition \((3.22)\), \( \theta^3_\mu \) is the inverse of \( e^\mu_j \), and consequently,

\[
\theta^3_\mu = \frac{\text{cof}(e^\mu_j)}{\det(e)} = -\frac{1}{\det(e)} e_{\mu \alpha \beta \gamma} e^\alpha_1 e^\beta_2 e^\gamma_2, \quad (e = (e^\mu_j)),
\]

by cofactor expansion, which is equivalent to

\[
\sqrt{|g|} \det(e) \theta^3_\mu = -\nu_{\mu \alpha \beta \gamma} e^\alpha_0 e^\beta_1 e^\gamma_2 \tag{4.14}
\]

by \((3.38)\). Since \( \xi = e_0 \) by \((4.5)\), we can, using the boundary condition \((3.35)\), write \((4.14)\) as

\[
\sqrt{|g|} \det(e) \theta^3_\mu = N_\mu \quad \text{in } \Gamma_T,
\]

where \( N_\mu \) is defined by \((3.39)\). But, we also have that

\[
\theta^3_\mu = \frac{1}{\det(e)} (\det(e) \theta^3_\mu) = \sqrt{|g|} \sqrt{-\det(\gamma^{kl})} \det(e) \theta^3_\mu = \sqrt{|g|} \left( -\frac{\zeta \sqrt{-\gamma^{00}}}{f} \right) \det(e) \theta^3_\mu, \tag{4.15}
\]

where in deriving the last equality we used \((3.17)\). Combining \((4.14)\) and \((4.15)\) yields

\[
\theta^3_\mu = \frac{\zeta \sqrt{-\gamma^{00}}}{f} \nu_{\mu \alpha \beta \gamma} e^\alpha_0 e^\beta_1 e^\gamma_2 - \sqrt{|g|} \sqrt{-\gamma^{00}} k \det(e) \theta^3_\mu. \tag{4.16}
\]

Next, from \((3.19)\) and \((3.45)\), we have that

\[
f = f(\xi) = -1 \quad \text{in } \Gamma_T. \tag{4.17}
\]

Using this together with \((4.5)\), we find, after a short calculation, that

\[
\nabla_\xi \left( \frac{\zeta \sqrt{-\gamma^{00}}}{f} \nu_{\mu \alpha \beta \gamma} e^\alpha_0 e^\beta_1 e^\gamma_2 \right) = -\nu_{\mu \alpha \beta \gamma} \left( e^\beta_2 \nabla_\xi (\sqrt{-\gamma^{00}} \xi^\alpha) + \sqrt{-\gamma^{00}} \xi^\alpha e^\beta_2 \nabla_\xi \xi^\beta + \sqrt{-\gamma^{00}} \xi^\alpha e^\beta_2 \nabla_\xi \xi^\gamma \right)
\]

\[
- \sqrt{-\gamma^{00}} \xi^\alpha e^\beta_2 \nabla_\xi \nu_{\mu \alpha \beta \gamma} - \nu_{\mu \alpha \beta \gamma} \sqrt{-\gamma^{00}} \left( e^\alpha_0 e^\beta_1 [e_0, e_1]^\beta + e^\alpha_0 e^\beta_1 [e_0, e_2]^\gamma \right)
\]

in \( \Gamma_T \). Recalling that \( v^\alpha = \sqrt{-\gamma^{00}} \xi^\alpha = -(-g(\tilde{\theta}^0, \tilde{\theta}^0))^{-1/2} \tilde{\theta}^0_{\alpha} \) by \((3.3)-(3.4)\), we see, with the help of \((3.1)\), \((3.3)\) and the boundary condition \((3.35)\), that applying \( \nabla_\xi \) to \( \sqrt{-\gamma^{00}} \xi^\alpha \) yields

\[
\nabla_\xi (\sqrt{-\gamma^{00}} \xi^\alpha) = -\frac{1}{\sqrt{-\gamma^{00}}} h^{\alpha \beta \gamma} \nabla_\xi \tilde{\theta}^\beta, \tag{4.18}
\]

where \( h^{\alpha \beta \gamma} \) is as defined previously by \((1.5)\). Then since

\[
\nu_{\mu \alpha \beta \gamma} \left( \sqrt{-\gamma^{00}} \xi^\alpha e^\beta_2 \nabla_\xi \xi^\beta + \sqrt{-\gamma^{00}} \xi^\alpha e^\beta_2 \nabla_\xi \xi^\gamma \right) = \nu_{\mu \alpha \beta \gamma} \left( \xi^\alpha e_2 \nabla_\xi (\sqrt{-\gamma^{00}} \xi^\beta) + \xi^\alpha e_1 \nabla_\xi (\sqrt{-\gamma^{00}} \xi^\gamma) \right)
\]

by the anti-symmetry in the indices of the volume form \( \nu_{\mu \alpha \beta \gamma} \), we can use \((4.20)\) to write \((4.19)\) as

\[
\nabla_\xi \left( \frac{\zeta \sqrt{-\gamma^{00}}}{f} \nu_{\mu \alpha \beta \gamma} e^\alpha_0 e^\beta_1 e^\gamma_2 \right) = -\frac{1}{\sqrt{-\gamma^{00}}} s_{\mu \beta \gamma} h^{\beta \nu} \nabla_\xi \tilde{\theta}^\nu - \sqrt{-\gamma^{00}} \xi^\alpha e^\beta_2 \nabla_\xi \nu_{\mu \alpha \beta \gamma}
\]

\[
- \nu_{\mu \alpha \beta \gamma} \sqrt{-\gamma^{00}} \left( e^\alpha_0 e^\beta_2 [e_0, e_1]^\beta + e^\alpha_0 e^\beta_2 [e_0, e_2]^\gamma \right) \quad \text{in } \Gamma_T, \tag{4.21}
\]

\[\text{More explicitly, it follows from } (4.14) \text{ and the formulas } (5.1), (5.3), (5.10), (5.21) \text{ and } (5.59) \text{ that}
\]

\[
\sqrt{|g|} \det(e) \theta^3_\mu = N_\mu + \Omega_\mu (\alpha, \xi - 1),
\]

which, in particular, implies \((4.15)\) when \( \alpha = 0 \) and \( \xi = 1 \).
where $s_{\mu\gamma}$ is as defined previously by (3.13). From the identity $[e_0, e_j]^\beta = -d\theta^k(e_0, e_j)e_\xi^\beta$, it then follows, with the help of the definitions (3.13-3.15), that
\[
[e_0, e_j]^\beta = -\mathcal{F}(e_0, e_j)e_\xi^\beta + e^K(e_0, e_j)e^K_\xi + \frac{1}{2}c_0K_\xi e^K_\xi,
\] (4.22)
which, in turn, implies that
\[
[e_0, e_j]^\beta = \mathcal{F}(e_0, e_j)e_0^\beta
\] (4.23)
via (4.13)-(4.17). Substituting (4.23) into (4.21) gives
\[
\nabla_\xi\left(\sqrt{-g}f e_0\right) = -\frac{1}{\gamma_{00}}s_{\mu\beta\gamma}h^{\beta\nu}\nabla_\gamma\theta^0_\mu - \sqrt{-g}e_{\xi e_0}e_2\nabla_\xi\nu_{\mu\alpha\beta\gamma} \quad \text{in } \Gamma_T,
\] (4.24)
while applying $\nabla_\xi$ to (4.17), yields, with the help of (4.24),
\[
-\gamma_{00}\nabla_\xi\theta^0_\mu = s_{\mu\beta\gamma}h^{\beta\nu}\nabla_\gamma\theta^0_\mu - (-\gamma_{00})\frac{1}{2}\xi^\alpha e_{\xi e_0}e_2\nabla_\xi\nu_{\mu\alpha\beta\gamma} - \gamma_{00}(\nabla_\xi\tilde{\theta}^3_\mu + \tilde{h}\nabla_\xi\tilde{\theta}^3_\mu) \quad \text{in } \Gamma_T
\] (4.25)
where
\[
\tilde{h} = -\sqrt{|g|}\sqrt{-g}00\det(e)|\theta^3| g h.
\] (4.26)

To continue, we compute
\[
\theta^3_\alpha g^{\alpha\beta}\nabla_\nu\theta^0_\beta = \theta^3_\alpha g^{\alpha\beta}(\nabla_\nu\theta^0_\beta - \nabla_\nu\theta^0_\beta + \nabla_\nu\theta^0_\beta)
\]
\[
= \theta^3_\alpha g^{\alpha\beta} F_{\beta\nu} - \sigma^3_\alpha j_{\beta\nu}\theta^0_\beta - \nabla_\nu\theta^3_\alpha g^{\alpha\beta}\theta^0_\beta - g^\alpha\beta\theta^0_\nu \nabla_\alpha\theta^0_\beta
\] (by 3.15 and 4.11)
\[
= \theta^3_\alpha g^{\alpha\beta} F_{\beta\nu} - \sigma^3_\alpha j_{\beta\nu}\theta^0_\beta - g^\alpha\beta\theta^0_\nu \nabla_\alpha\theta^0_\beta - g^\alpha\beta\theta^0_\nu \nabla_\alpha\theta^0_\beta + \sigma^3_\alpha j_{\beta\nu}\theta^0_\beta g^\alpha\beta\theta^0_\beta
\] (by 4.14)
\[
= \theta^3_\alpha g^{\alpha\beta} F_{\beta\nu} - \theta^3_\alpha g^{\alpha\beta} \sigma^3_\alpha j_{\beta\nu}\theta^0_\beta - \sigma^3_\alpha j_{\beta\nu}\theta^0_\beta g^\alpha\beta\theta^0_\beta
\] (4.27)
where in deriving the last equality we used (4.4), (4.6) and (4.7). Rearranging (4.27), we find, by (3.1), (5.3) and (4.26), that
\[
\theta^3_\alpha g^{\alpha\beta} F_{\beta\nu} = \theta^3_\alpha g^{\alpha\beta} (\nabla_\nu\theta^0_\beta + \nabla_\nu\theta^0_\beta + \nabla_\nu\theta^0_\beta) + (-\gamma_{00})\frac{1}{2}\xi^\alpha e_{\xi e_0}e_2\nabla_\xi\nu_{\mu\alpha\beta\gamma} + \gamma_{00}(\nabla_\xi\tilde{\theta}^3_\mu + \tilde{h}\nabla_\xi\tilde{\theta}^3_\mu) \quad \text{in } \Gamma_T.
\] (4.29)

Contracting the above expression with $\theta^3_\nu = \theta^3_\mu g^{\mu\nu}$, we find, via the anti-symmetry of $F_{\alpha\beta}$ and (4.1), that
\[
\theta^3_\mu (\nabla_\nu\theta^0_\mu + \nabla_\nu\theta^0_\mu + \nabla_\nu\theta^0_\mu) + (-\gamma_{00})\frac{1}{2}\xi^\alpha e_{\xi e_0}e_2\nabla_\xi\nu_{\mu\alpha\beta\gamma} + \gamma_{00}(\nabla_\xi\tilde{\theta}^3_\mu + \tilde{h}\nabla_\xi\tilde{\theta}^3_\mu) \quad \text{in } \Gamma_T.
\] (4.30)

But, by (3.39), (4.17) and (4.18), we observe that
\[
\theta^3_\mu = \sqrt{-\gamma_{00}}N_\mu - \sqrt{|g|}\sqrt{-\gamma_{00}}\det(e)\theta^3_\mu
\] (4.29)
which we note, using the definition (3.56) and (4.21), can be used to write (4.28) as
\[
\theta^3_\mu B^\mu = -\gamma_{00}\theta^3_\mu g e_\xi\tilde{h} + a\tilde{h} \quad \text{in } \Gamma_T
\] (4.30)
for some function $a \in C^0(\Gamma_T)$.

Next, we observe, by (3.45), (4.4), (4.3), (4.6), (4.7), (4.13) and (4.26), that the boundary condition (3.44) can be expressed as
\[
\nabla_\xi B^\mu = \sum_{\ell=0}^1 b^\mu_\ell \nabla_\xi h + c^\mu_\ell B^\nu \quad \text{in } \Gamma_T
\] (4.31)
for some collection of functions $b^\mu_\ell \in C^0(\Gamma_T)$, $\ell = 0, 1$, and $c^\mu_\ell \in C^0(\Gamma_T)$. Contracting (4.31) with $\theta^3_\mu$, we see, with the help of (4.30), that $\tilde{h}$ satisfies
\[
\nabla_\xi^2 h = \sum_{\ell=0}^1 d_\ell \nabla_\xi^2 h + e_\mu B^\mu \quad \text{in } \Gamma_T
\] (4.32)
for some collection of functions $d_\ell \in C^0(\Gamma_T)$, $\ell = 0, 1$, and $e_\mu \in C^0(\Gamma_T)$. From the choice of initial data, see (3.62), (3.67) and (3.70), it then follows immediately from the transport equations (4.31) and (4.32) that
\[
\tilde{h} = 0 \quad \text{in } \Gamma_T
\] (4.33)
and

\[ \mathcal{B}^\mu = 0 \quad \text{in } \Gamma_T. \] (4.34)

Propagation of \( j \) in \( \Omega_T \): Contracting (4.34) with \( \hat{\theta}_\nu^0 \) shows, with the help of (3.45), (4.31) and (4.33), that the constraint \( j \) satisfies the boundary condition

\[ |\theta|^2 n_\alpha a^{\alpha\beta} \nabla_\beta j = \kappa g(\hat{\theta}^0, \hat{\theta}^0) \nabla_\xi j \quad \text{in } \Gamma_T, \] (4.35)

where we note that

\[ \kappa g(\hat{\theta}^0, \hat{\theta}^0) \leq 0 \] (4.36)

since \( \kappa \geq 0 \) and \( \hat{\theta}^0 \) is timelike.

To derive an evolution equation for \( j \), we first observe from (4.4)–(4.7) and (3.32) that \( \mathcal{E}^\mu \) satisfies \( \nabla_\nu \mathcal{E}^\mu + 2 \mathcal{R}^\mu (\mathcal{E}) = 0 \) in \( \Omega_T \). From this equation and the initial conditions (3.69), we then deduce that \( \mathcal{E}^\mu = 0 \) in \( \Omega_T \), and hence, by (3.27), that

\[ \nabla_\alpha (a^{\alpha\beta} [\nabla_\beta \theta^0_\nu + \sigma_\nu^0 j \theta^0_\gamma \theta^0_\delta]) = h^0_\alpha (\hat{H}_\mu - a^{\alpha\beta} \nabla_\alpha \nabla_\beta \theta^0_\mu) - \nabla_\alpha \hat{\theta}^0_\mu a^{\alpha\beta} [\nabla_\beta \theta^0_\nu + \sigma_\nu^0 j \theta^0_\gamma \theta^0_\delta] \frac{\hat{\theta}^0_\nu}{g(\hat{\theta}^0, \hat{\theta}^0)} \]

in \( \Omega_T \). Contracting this equation with \( \hat{\theta}^0_\nu \) while using (4.34) shows that

\[ \nabla_\alpha (a^{\alpha\beta} \nabla_\beta j) = 0 \quad \text{in } \Omega_T. \] (4.37)

But solutions to the IBVP defined by (3.69), (3.68), (4.33) and (4.37) are unique by (4.30) and Theorem 2.2 of [20], and therefore, we conclude that

\[ j = g(\hat{\theta}^0, \hat{\theta}^0) + 1 = 0 \quad \text{in } \Omega_T. \] (4.38)

Propagation of \( g \) in \( \Omega_T \): Noting that \( h^0_\nu \nabla_\nu \hat{\theta}^0_\mu = \nabla_\nu \hat{\theta}^0_\mu \) follows directly from (4.5) and (4.38), we can, with the help of (3.33) and (4.38), write (4.37) as

\[ \nabla_\alpha (a^{\alpha\beta} [\nabla_\beta \theta^0_\nu + \sigma_\nu^0 j \theta^0_\gamma \theta^0_\delta]) = \mathcal{H}_\mu \quad \text{in } \Omega_T. \] (4.39)

Next, observing from (3.1) and (4.3) that (4.38) implies

\[ \zeta = \sqrt{-\gamma^0} \quad \text{in } \Omega_T, \] (4.40)

we calculate

\[ \nabla_\nu \left( \frac{1}{f} \hat{\theta}^0_\mu \right) = - \frac{f'}{f} \nabla_\nu \zeta \hat{\theta}^0_\mu + \frac{1}{f} \nabla_\nu \hat{\theta}^0_\mu \]

\[ = - \frac{1}{\zeta f} \left( 1 - \frac{1}{s^2} \right) \nabla_\nu \theta^0_\mu + \frac{1}{f} \delta^\alpha_\mu \nabla_\nu \hat{\theta}^0_\alpha \quad \text{(by (3.19))} \]

\[ = - \frac{1}{f} \left( 1 - \frac{1}{s^2} \right) \theta^0_\mu \theta^0_\beta \nabla_\nu \theta^0_\alpha + \frac{1}{f} \delta^\alpha_\mu \nabla_\nu \hat{\theta}^0_\alpha \quad \text{(by (4.30))} \]

\[ = - a^{\mu\nu} \nabla_\nu \theta^0_\alpha \] (4.41)

where \( a^{\mu\nu} \) is the acoustic metric defined by (3.20). From this, we see that

\[ \nabla_\nu \delta_j \left( \frac{1}{f} \hat{\theta}^0_\mu \right) = - \nabla_\nu \nabla_\mu \left( \frac{1}{f} \hat{\theta}^0_\mu \right) \]

\[ = - \nabla_\nu \nabla_\mu \left( \frac{1}{f} \hat{\theta}^0_\mu \right) + \frac{1}{f} R_{\nu\mu \lambda} \theta^0_\lambda \quad \text{(by (A.19))} \]

\[ = \nabla_\mu (a^{\mu\alpha} \nabla_\nu \hat{\theta}^0_\alpha) + \frac{1}{f} R_{\nu \mu \lambda} \theta^0_\lambda \quad \text{(by (4.11))} \]

\[ = \nabla_\mu (a^{\mu\alpha} \nabla_\nu \theta^0_\alpha + a^{\mu\alpha} [\nabla_\nu \theta^0_\alpha - \nabla_\alpha \theta^0_\nu]) + \frac{1}{f} R_{\nu \mu \lambda} \theta^0_\lambda, \]

which, we observe, using (3.31) and (4.3), can be written as

\[ \nabla_\nu \delta_j \left( \frac{1}{f} \hat{\theta}^0_\mu \right) = \nabla_\alpha (a^{\alpha\beta} [\nabla_\beta \theta^0_\nu + \sigma_\nu^0 j \theta^0_\gamma \theta^0_\delta]) + \frac{1}{f} R_{\nu \mu \lambda} \theta^0_\lambda + \nabla_\alpha (a^{\alpha\beta} \mathcal{F}_{\nu\beta}) \quad \text{in } \Omega_T. \] (4.42)
Using (A.20) to express the last term in (4.42) as
\[
\nabla_a (a^{\alpha \beta} F_{\nu \beta}) = - \nabla_a (a^{\alpha \beta} F_{\beta \nu}) + \left[ C^\beta_{\omega \lambda} a^{\omega \alpha} - \hat{a}^\beta_{\nu} C^\omega_{\omega \lambda} a^{\lambda \alpha} \right] F_{\alpha \beta},
\]
(4.43) where \( C^\beta_{\omega \lambda} \) is defined by (3.30) and \( \nabla \) is the Levi-Civita connection of the acoustic metric \( a_{\alpha \beta} \), we see from (3.39), (4.42) and (4.43) that
\[
d g - \delta_a F = 0 \quad \text{in } \Omega_T.
\]
(4.44) Applying the codifferential \( \delta_a \) to this expression, we see, with the help of (A.15), that \( g \) satisfies the wave equation
\[
d g - \delta_a F = 0 \quad \text{in } \Omega_T.
\]
(4.45)

To complete the proof that the constraint \( g \) propagates, we need to show that \( g \) satisfies an appropriate boundary condition. We determine the boundary condition by first noting, see (A.12), that \( g \) is \( g \)-orthogonal to the co-frame fields \( \theta^I \). Since \( \theta^0 \) is orthogonal to the \( \theta^I \) by (4.4), \( \theta^0 \) must be proportional to \(*_g (\theta^1 \land \theta^2 \land \theta^3) \), and so
\[
q *_g \theta^0 = \theta^1 \land \theta^2 \land \theta^3 \quad \text{in } \Omega_T
\]
(4.46) for some function \( q \). To find \( q \), we wedge the above expression with \( \theta^0 \) to get \( q \theta^0 \lor *_g \theta^0 = \theta^0 \land \theta^1 \land \theta^2 \land \theta^3 \) and then use the formulas (A.11) and (A.10) to obtain \( q = \sqrt{-\det(\gamma_{\alpha \beta})} / \gamma_{00} \). Substituting this into (4.46) yields \( *_g \left( \frac{\sqrt{-\det(\gamma_{\alpha \beta})}}{\gamma_{00}} \theta^0 \right) = \theta^1 \land \theta^2 \land \theta^3 \) which, after taking the exterior derivative, gives
\[
d *_g \left( \frac{\sqrt{-\det(\gamma_{\alpha \beta})}}{\gamma_{00}} \theta^0 \right) = \partial_1 \theta^1 \land \partial_2 \theta^2 \land \partial_3 \theta^3 - \partial_1 \theta^2 \land \partial_1 \theta^3 + \partial_1 \theta^2 \land \partial_2 \theta^3 + \partial_2 \theta^1 \land \partial_2 \theta^3 + \partial_2 \theta^1 \land \partial_3 \theta^3 + \partial_3 \theta^1 \land \partial_2 \theta^2 + \partial_1 \theta^3 \land \partial_2 \theta^2 + \partial_1 \theta^3 \land \partial_3 \theta^2 + \partial_2 \theta^3 \land \partial_1 \theta^2 + \partial_3 \theta^2 \land \partial_1 \theta^3 + \partial_3 \theta^1 \land \partial_2 \theta^1 + \partial_3 \theta^2 \land \partial_2 \theta^1 + \partial_3 \theta^3 \land \partial_1 \theta^2 + \partial_3 \theta^3 \land \partial_2 \theta^1
\]
(4.47)

where in deriving the last two equalities we used the relations (4.6) and (4.7) along with the fact that \( \theta^I \land \partial_1 \theta^J \land \partial_1 \theta^K \land \partial_1 \theta^L = 0 \) for any choice of \( I, J, K, L \in \{1, 2, 3\} \). But this implies \( \delta_3 \left( \frac{\sqrt{-\det(\gamma_{\alpha \beta})}}{\gamma_{00}} \theta^0 \right) = 0 \) in \( \Omega_T \), which, in turn, implies that
\[
g = -\delta_3 \left( \frac{h}{\sqrt{-\gamma_{00}}} \theta^0 \right) = -\sqrt{-\gamma_{00}} \nabla \xi h - \delta_3 \left( \frac{\theta^0}{\sqrt{-\gamma_{00}}} \right) h \quad \text{in } \Omega_T
\]
(4.48)

by (3.3), (3.10), (3.17) and (4.41). Evaluating (4.44) on the boundary \( \Gamma_T \) yields the Dirichlet boundary condition
\[
g = 0 \quad \text{in } \Gamma_T
\]
by (4.16) and (4.18). From the trivial initial data (3.61) and (3.63), and the uniqueness of solutions to the wave equation (4.45) with Dirichlet boundary conditions, we deduce that
\[
g = \delta_3 \left( \frac{1}{f} \theta^0 \right) = 0 \quad \text{in } \Omega_T.
\]
(4.48)

Propagation of \( h \) in \( \Omega_T \): Since \( -\sqrt{-\gamma_{00}} \nabla \xi \delta_3 \left( \frac{\theta^0}{\sqrt{-\gamma_{00}}} \right) h = 0 \) in \( \Omega_T \) by (4.48) and (4.49) and \( h \) vanishes initially, see (3.62), it follows immediately from the uniqueness of solutions to transport equations that
\[
h = -\frac{\sqrt{-\det(\gamma_{\alpha \beta})}}{\sqrt{-\gamma_{00}}} - \frac{\zeta}{f(\xi)} = 0 \quad \text{in } \Omega_T.
\]
(4.49)

Propagation of \( F \) in \( \Omega_T \): First, from (4.44) and (4.48), we get
\[
d_\nu F = 0 \quad \text{in } \Omega_T,
\]
(4.50) while we see that
\[
L_\nu d F = \frac{1}{2} d \left( L_{\nu} (\sigma^0_{ij}) \theta^I \land \theta^J + \sigma^0_{ij} L_\nu \theta^I \land \theta^J + \sigma^0_{ij} \theta^I \land \theta^J \land L_\nu \theta^J \right) = 0 \quad \text{in } \Omega_T
\]
(4.51) follows from (3.15), (3.14), (3.55), (A.5) and (A.7). Since \( d F \) vanishes initially, see (3.65), it follows immediately from (4.51) that
\[
d F = 0 \quad \text{in } \Omega_T.
\]
(4.52)

Together (4.50) and (4.52) show that \( F \) satisfies Maxwell’s equations in \( \Omega_T \).
In order to verify that the constraint $F$ propagates, we need to show that $F$ satisfies an appropriate boundary condition. To this show, we begin by noting that the equality

$$N_\nu = \theta^\gamma_\nu \quad \text{in } \Gamma_T$$

(4.53)
is an immediate consequence of the relations (4.29), (4.33) and (4.40), and the boundary conditions (4.45)-(4.46).

**Note:** From this point until we establish the propagation of the constraint $F$, we will raise and lower spacetime indices with the acoustic metric $a_{\mu\nu}$. After that, we will return to our standard convention of raising and lowering spacetime indices with the metric $g_{\mu\nu}$.

From (4.27), (4.32), and (4.53), it is then clear that $\Pi^\alpha_\nu$, see (4.42), projects onto the $a$-orthogonal subspace to the span of the vector fields $n^\mu$ and $v^\mu$. Next, we observe from (4.43), (4.26) and (4.33) that (4.25) can be written as

$$-g^{\alpha\beta}\theta^\gamma_\mu \nabla_\gamma \theta^\beta_\mu = s_{\mu\gamma}h^{\beta\nu} \nabla_\gamma \theta^\beta_\nu - \sqrt{-\gamma_0}e_\alpha \epsilon^{\gamma}_{\alpha\beta} \nabla \xi_{\nu \mu \alpha \beta \gamma} \quad \text{in } \Gamma_T.$$

From this result and (4.27), it then follows, with the help of (4.13) and (4.34), that

$$\Pi^\mu_{\alpha\nu} a^{\alpha\beta} F_{\beta\nu} = -\epsilon \Pi^\mu_{\alpha\nu} \frac{\nabla_\xi \theta^0_\mu + \nabla_\xi \theta^0_\nu}{|\nabla_\xi \theta^0_\mu|_{m} \nabla_\xi \theta^0_\nu} \quad \text{in } \Gamma_T.$$  

(4.54)

Furthermore, we observe, using (4.14)-(4.15) and (4.16)-(4.18), that $\nabla_\xi \theta^0_\nu$ is given by

$$\nabla_\xi \theta^0_\nu = Y_K \theta^K_\nu - \omega^0_3 \theta^3_\nu - \frac{1}{\epsilon} e_\nu (\gamma_0) \theta^K_\nu,$$

(4.55)

where we have set

$$Y_K = F(e_0, e_K).$$

(4.56)

Since the frame components $e_\mu e^\mu$ satisfy $n_\mu e_\mu = 0$ in $\Gamma_T$ by (4.13), the vector fields $e_\mu$ must be tangent to $\Gamma_T$, and consequently, $e_\nu (\gamma_0) = 0$ in $\Gamma_T$ by (4.3), (4.10) and the boundary condition (3.40). Substituting this into (4.55) yields

$$\nabla_\xi \theta^0_\nu = Y_K \theta^K_\nu - \omega^0_3 \theta^3_\nu \quad \text{in } \Gamma_T,$$

(4.57)

which, in turn, implies that

$$-\frac{\nabla_\xi \theta^0_\nu - Y_K \theta^K_\nu}{|\nabla_\xi \theta^0_\nu - Y_K \theta^K_\nu|_{m}_{\nu}} = \theta^3_\nu \quad \text{in } \Gamma_T.$$  

(4.58)

since $\omega^0_3 = -i_\gamma \nabla_\gamma \omega^0_3 > 0$ in $\Gamma_T$ by assumption. Adding $|\nabla_\xi \theta^0_\mu|_m^{-1} \nabla_\xi \theta^0_\nu$ to both sides of (4.58), we see that

$$\frac{\nabla_\xi \theta^0_\mu}{|\nabla_\xi \theta^0_\mu|_{m}} + \theta^3_\mu = \frac{\nabla_\xi \theta^0_\nu}{|\nabla_\xi \theta^0_\nu|_{m}} - \frac{\nabla_\xi \theta^0_\nu - Y_K \theta^K_\nu}{|\nabla_\xi \theta^0_\nu - Y_K \theta^K_\nu|_{m}} \quad \text{in } \Gamma_T.$$  

(4.59)

Applying $\nabla_\xi$ to this expression, we find after a short calculation that

$$\nabla_\xi \left( \frac{\nabla_\xi \theta^0_\mu}{|\nabla_\xi \theta^0_\mu|_{m}} + \theta^3_\mu \right) = \frac{1}{\epsilon} a \xi (Y_K) \theta^K_\mu + b^K_\nu Y_K \quad \text{in } \Gamma_T,$$

for some functions $\{a, c^K_\nu\} \subset C^0(\Gamma_T)$ with $a > 0$ in $\Gamma_T$. Using this result and (4.57), it is not difficult to see, with the help of (4.33), that we can write (4.54) as

$$\Pi^\mu_{\nu} \hat{n}^3 F_{\beta\nu} = -\Pi^\mu_{\nu} (a \xi (Y_K) \theta^K_\nu + b^K_\nu Y_K) \quad \text{in } \Gamma_T,$$

(4.59)

for an appropriate choice of functions $\{b^K_\nu\} \subset C^0(\Gamma_T)$, where we have defined

$$\hat{n}^\mu = \frac{n^\mu}{|n|_a}.$$

We then use (4.59) to express $\hat{n}^\mu F_{\mu} a^{\alpha\beta} F_{\nu\beta} \xi^\nu$ on the boundary $\Gamma_T$ as follows:

$$\hat{n}^\mu F_{\mu} a^{\alpha\beta} F_{\nu\beta} \xi^\nu = \hat{n}^\mu F_{\mu} \left( \Pi^\alpha_{\beta} + \hat{n}^3 \hat{n}^3 - \nu^\alpha \nu^\beta \right) F_{\nu\beta} \xi^\nu = \hat{n}^\mu F_{\mu} \Pi^\alpha_{\beta} F_{\nu\beta} \xi^\nu = -\Pi^\alpha_{\beta} (a \xi (Y_K) \theta^K_\beta + b^K_\nu Y_K) F_{\nu\beta} \xi^\nu \quad \text{by (4.50)}$$

$$= -\Pi^\alpha_{\beta} (a \xi (Y_K) \theta^K_\beta + b^K_\nu Y_K) F_{\nu\beta} \xi^\nu \quad \text{by (4.5) and } F_{\alpha\beta} = -F_{\beta\alpha},$$

$$= -\Pi^\alpha_{\beta} (a \xi (Y_K) \theta^K_\beta + b^K_\nu Y_K) Y_{\gamma} \theta^\gamma_{\beta} \quad \text{since } \Pi^\alpha_{\beta} \theta^3_{\beta} = 0.$$  

(4.60)
We now claim that (4.60) defines the required boundary condition needed to show that the constraint \( \mathcal{F} \) propagates. To see this, we introduce a time function \( \tau \) that foliates \( \Omega_T \) and choose a future pointing timelike vector field \( \tau^\nu \) such that \( \tau^\nu \partial_\tau = 1 \) in \( \overline{\Omega_T} \) and \( \tau^\nu \hat{n}_\nu = 0 \) in \( \Gamma_T \). We further define the level sets
\[
\Omega(t) = \tau^{-1}(t) \cap \Omega_T \cong \{ t \} \times \Omega_0 \quad \text{and} \quad \Gamma(t) = \tau^{-1}(t) \cap \Gamma_T \cong \{ t \} \times \partial \Omega_0,
\]
and we let
\[
\Omega_t = \bigcup_{0 \leq t \leq T} \Omega(t) \cong [0, T] \times \Omega_0 \quad \text{and} \quad \Gamma_t = \bigcup_{0 \leq t \leq T} \Gamma(t) \cong [0, T] \times \partial \Omega_0.
\]
Since \( \mathcal{F} \) satisfies the Maxwell’s equations in \( \Omega_T \) by (4.51)-(4.52), we obtain the integral identity
\[
\int_{\Gamma(t)} \tilde{\tau}^\nu T_{\nu\mu} \xi^\mu + 2 \int_{\Gamma_T} \left[ \Pi^{JK} Y_{\nu J}(\xi_{\nu K}) + \Lambda^{JK} Y_{\nu J} Y_{\nu K} \right] = \int_{\Gamma_0} \tilde{\tau}^\nu T_{\nu\mu} \xi^\mu - \frac{1}{2} \int_{\Omega_T} T^{\mu\nu} L_{\xi} a_{\mu\nu} \quad (4.61)
\]
directly from Lemma B.1 and (4.60), where \( \tilde{\tau}^\mu = (\sigma(\tau, \tau))^{-1/2} \tau^\mu \),
\[
T_{\mu\nu} = \mathcal{F}_{\mu\nu} \mathcal{F}_{\nu\alpha} - \frac{1}{2} a_{\mu\nu} \mathcal{F}_{\alpha\beta} \mathcal{F}^{\alpha\beta}
\]
is the stress energy tensor of \( \mathcal{F} \), and we have set
\[
\Pi^{JK} = \theta^J_\alpha \theta^K_\beta \quad \text{and} \quad \Lambda^{JK} = \Pi^{\alpha\beta} \theta^K_\alpha \theta^J_\beta.
\]
It is important note that the matrix the 2-by-2 symmetric matrix \( \Pi^{JK} = \theta^J_\alpha \Pi^{\alpha\beta} \theta^K_\beta \) is positive definite.
Using the Divergence Theorem, we can express the boundary integral in (4.61) as
\[
\int_{\Omega(t)} \Pi^{JK} Y_{\nu J}(\xi_{\nu K}) + \Lambda^{JK} Y_{\nu J} Y_{\nu K} = \int_{\Gamma(t)} (\tilde{\tau}^\nu \xi^\mu) \Pi^{JK} Y_{\nu J} Y_{\nu K} - \int_{\Gamma_0} (\tilde{\tau}^\nu \xi^\mu) \Pi^{JK} Y_{\nu J} Y_{\nu K} - \int_{\Gamma_T} (\xi(\Pi^{JK})) + \text{div}(\xi) \Pi^{JK} - \Lambda^{JK} Y_{\nu J} Y_{\nu K} \quad (4.62)
\]
Noting that \( \tilde{\tau}^\nu \xi^\mu > 0 \), by virtue of \( \tilde{\tau}^\nu \) and \( \xi^\mu \) both being future pointing timelike vector fields, and \( |\mathcal{F}|_m^2 \leq |\tilde{\tau}^\nu F_{\nu\mu} \xi^\mu|^2 \) by Lemma B.2 it is clear that the energy estimate
\[
\int_{\Omega(t)} |\mathcal{F}|^2_m + \int_{\Gamma(t)} |Y|^2_\Pi = \int_{\Omega_0} |\mathcal{F}|^2_m + \int_{\Gamma_0} |Y|^2_\Pi + \int_{\Gamma_t} |\mathcal{F}|^2_m + \int_{\Gamma_t} |Y|^2_\Pi, \quad 0 \leq t \leq T,
\]
is a direct consequence of (4.61)-(4.62). But this implies
\[
\int_{\Omega(t)} |\mathcal{F}|^2_m + \int_{\Gamma(t)} |Y|^2_\Pi = \int_{\Omega_0} |\mathcal{F}|^2_m + \int_{\Gamma_0} |Y|^2_\Pi, \quad 0 \leq t \leq T,
\]
by Gronwall’s inequality, and so we conclude that
\[
\mathcal{F} = d \theta^0 + \frac{1}{2} \sigma^I_0 \theta^I \wedge \theta^J = 0 \quad \text{in} \ \Omega_T, \quad (4.63)
\]
since \( \mathcal{F} \) and \( Y \) vanish at \( t = 0 \) in \( \Omega_0 \) and \( \Gamma_0 \), respectively, by (4.50), (4.56). For use below, we note, with the help of the Cartan structure equations (A.17) and (4.60), that (4.63) is equivalent to
\[
\sigma^I_0 = \omega^0_{ij} - \omega^0_{ji} \quad \text{in} \ \Omega_T. \quad (4.64)
\]

Solution of the relativistic Euler equations: With the proof of the propagation of constraints complete, we now turn to showing that the frame \( \theta^I_\mu \) determines a solution of the relativistic Euler equations. To start, we use (4.41) and (4.40) to express (4.49) as
\[
\left( \frac{2}{\gamma_{00} - \gamma_{ij}} \right) = \det(\gamma_{ij}) \quad \text{in} \ \Omega_T. \quad (4.65)
\]
where \( s^2 = s^2(\gamma_{00} - \gamma_{ij}) \), while
\[
\omega^0_{0j} + \omega^0_{0j} = 0 \quad \text{in} \ \Omega_T \quad (4.66)
\]
follows from applying \( e_0 \) to (4.4). We also note from (4.4), (4.18) and (4.64) that
\[
\omega^0_{ij} - \omega^0_{ij} = 0 \quad \text{in} \ \Omega_T. \quad (4.67)
\]
From \((4.1)\) and \((4.65)-(4.67)\), it is then clear that the equations
\[
\left(3 + \frac{1}{s^2}\right) \frac{1}{\gamma_{00}} - 3\gamma^{00} \omega_{000} - \gamma_{IJ}\omega_{I,J0} - 2\gamma^{0J}(\omega_{J00} + \omega_{0,J0}) = 0,
\]
\[
2\gamma^{00}\omega_{000} + \gamma_{IJ}(\omega_{J00} + \omega_{0,J0}) = 0,
\]
hold in \(\Omega_T\). Setting
\[
A^{ijk} = \left(3 + \frac{1}{s^2}\right) \frac{\delta_i^j\delta_j^k - \delta_i^k}{\gamma_{00}} + \delta_0^i\delta_j^k + \delta_0^j\delta_i^k + \delta_0^i\delta_0^k,
\]
a short calculation shows that the above equations can be written as
\[
A^{ijk}\omega_{k0} = 0 \quad \text{in} \quad \Omega_T,
\]
which, in turn, are easily seen to be equivalent to
\[
A_{\mu\nu}\gamma_{\gamma}w^\mu = 0 \quad \text{in} \quad \Omega_T,
\]
where
\[
w^\mu := e^\mu_0 = \frac{1}{\gamma^{00}}g^{\mu\nu}\theta^0_\nu
\]
and
\[
A_{\mu\nu} = \left(3 + \frac{1}{s^2}\right) \frac{w^\mu w^\nu}{g(w, w)} - \frac{\gamma_{00}}{\sqrt{g}} \omega^\gamma + \frac{\delta^\gamma_\mu}{\gamma} w_\mu + \frac{\delta_\mu}{\gamma} w_\nu + w^\gamma g^{\mu\nu}.
\]

**Remark 4.2.** Since the solutions \(\{\zeta, \theta^0_\mu, \theta^I_\sigma, \sigma^K_i\}\) from Theorem \(4.1\) satisfy \(\gamma^{0J} = 0\) and \(\omega_0^k = \omega_0^k\) in \(\Omega_T\) and \(\gamma^{00} = -1\) on \(\Gamma_T\) due to the propagation of the constraints, we observe that
\[
-\frac{1}{2}\gamma^{00}\nabla_n\gamma_00 \geq c_1 > 0 \quad \text{in} \quad \Gamma_T,
\]
is fulfilled. By Remark 2.1.(iii) of \(11\), this inequality is equivalent to the Taylor sign condition, which is the condition that the pressure \(p\) of the fluid solution \(\{\rho, v^\mu\}\) determined by \(\{\zeta, \theta^0_\mu, \theta^I_\sigma, \sigma^K_i\}\) satisfies
\[
-\nabla_n p \geq c_p > 0 \quad \text{in} \quad \Gamma_T
\]
for some positive constant \(c_p\).

**Remark 4.3.** From the proof of Theorem \(4.1\) it is clear the propagation of the constraints \(\{a, b^J, e^K_i, c^K_i\}\) depends only on the evolution equations \((3.34)\) and \((3.35)\) for \(\theta^I_\sigma\) and \(\sigma^K_i\), respectively, and the boundary condition \((3.46)\).

5. **Choice of constraints**

With the goal of establishing the local-in-time existence of solutions to guide us, we will, in this section, make particular choices for the constraints that appear in the evolution equations \((3.32)\) and boundary conditions \((3.41)\). The reason for these particular choices will be discussed in more detail in the following sections. Before proceeding, we introduce some notation. Given a vector field \(A^\mu\), we define the projections
\[
A^\mu_\parallel := \Pi^\mu_\omega A^\omega \quad \text{and} \quad A^\mu_\perp := \tilde{\Pi}^\mu_\omega A^\omega.
\]
Furthermore, given a tensor field \(B^{\mu\nu}\), we define vector fields \(B^{\mu\nu}\) by
\[
B^{\mu\nu} := \theta^I_\mu B^{I\nu},
\]
which, using our notation above, we can write the projected components as
\[
B^{\mu\nu}_\parallel = \theta^I_\mu \Pi^\nu_\omega B^{I\omega} \quad \text{and} \quad B^{\mu\nu}_\perp = \theta^I_\mu \tilde{\Pi}^\nu_\omega B^{I\omega}.
Lemma 5.1. Suppose $\epsilon \in (0,1)$. Then there exist remainder terms $\mathcal{R}^\mu$ and $\Omega^\mu$ such that
\[
\left( \frac{1}{1-\epsilon} \Pi_\mu^\epsilon + \tilde{\Pi}_\mu^\epsilon \right) \left( \nabla_v \mathcal{E}^\tau + \mathcal{R}^\tau (\partial^{[1]} \chi, \mathcal{E}) \right) = \nabla_\alpha \left( A^{\alpha \beta \mu \nu} \nabla_\beta \nabla_\mu \delta_0^\nu + X^{\alpha \mu} \right) - F^\mu \tag{5.1}
\]
and
\[
\left( \frac{1}{1-\epsilon} \Pi_\mu^\epsilon + \tilde{\Pi}_\mu^\epsilon \right) \left( \nabla_v \mathcal{B}^\tau + \Omega^\tau (\tilde{h}, \partial^{[1]} \chi, B, L_v \theta^3, \zeta - 1) \right) = \theta_\alpha^3 \left( A^{\alpha \beta \mu \nu} \nabla_\beta \nabla_\mu \delta_0^\nu + X^{\alpha \mu} \right)
- \frac{1}{\sqrt{\gamma}} \left( S^{\mu \nu \gamma} \nabla_\gamma \nabla_\nu \delta_0^\mu \right)
- \frac{\epsilon}{1-\epsilon} |N|^\beta \Pi_\nu \nu \left( \Pi_\nu \nabla_v \nabla_\nu \delta_0^\mu \right)
+ P^\mu \nu \left( \nabla_v \nabla_\nu \delta_0^\mu \right) + G^\mu \tag{5.2}
\]
where
\[
A^{\alpha \beta \mu \nu} = \zeta \left( \frac{1}{1-\epsilon} \Pi_\mu^\epsilon + \tilde{\Pi}_\mu^\epsilon \right) \alpha^{\alpha \beta},
\]
\[
X^{\alpha \mu} = \left( \frac{1}{1-\epsilon} \Pi_\mu^\epsilon + \tilde{\Pi}_\mu^\epsilon \right) \left[ \zeta m^{\mu \nu \lambda} a^{\alpha \beta} (v^\lambda R_{\beta \gamma \lambda} \theta_0^\alpha - \nabla_{\gamma} \nabla_{\lambda} \delta_0^\nu) + \nabla_v (\zeta m^{\mu \nu \lambda} a^{\alpha \beta} \nabla_\lambda \delta_0^\nu)
+ \nabla_v \left( \zeta m^{\mu \nu \lambda} a^{\alpha \beta} \sigma_f^0 \theta_0^j \theta_0^j \right) + e_i^\alpha \nabla_\lambda \delta_0^\nu \lambda \delta_0^\mu - e_i^\alpha \nabla_{\lambda \nu} \lambda \delta_0^\nu \right] \tag{5.3}
\]
\[
F^\mu = \nabla_\alpha \left( \frac{1}{1-\epsilon} \Pi_\mu^\epsilon + \tilde{\Pi}_\mu^\epsilon \right) \left( \nabla_v \mathcal{W}^{\alpha \nu \tau} + e_i^\alpha \nabla_\nu \theta_0^i \mathcal{W}^{\omega \mu} - e_i^\alpha \nabla_\omega \mathcal{W}^{\nu \mu} \right)
+ \left( \frac{1}{1-\epsilon} \Pi_\mu^\epsilon + \tilde{\Pi}_\mu^\epsilon \right) \left[ \nabla_\alpha e_i^\alpha \nabla_v \mathcal{W}^{\alpha \nu \tau} - \nabla_\nu \mathcal{W}^{\alpha \nu \tau} \right] + \nabla_v \left( \nabla_v \mathcal{W}^{\alpha \nu \tau} \right)
- \nabla_v (\nabla_v \mathcal{W}^{\nu \mu} \mathcal{W}_{\mu \nu}) - \nabla_v (\mathcal{W}_{\mu \nu} \mathcal{W}^{\nu \mu})
- \nabla_v (\mathcal{W}^{\nu \mu} \mathcal{W}_{\mu \nu}) + \nabla_v \mathcal{W}^{\nu \mu} \mathcal{W}^{\nu \mu} - \nabla_v \mathcal{W}^{\mu \nu} \mathcal{W}^{\mu \nu}
+ \nabla_v \mathcal{W}^{\mu \nu} \mathcal{W}^{\nu \mu} - \nabla_v \mathcal{W}^{\nu \mu} \mathcal{W}^{\mu \nu} + \nabla_v \mathcal{W}^{\nu \mu} \mathcal{W}^{\mu \nu} \right] \tag{5.4}
\]
\[
S^{\mu \nu \gamma} = h^{\mu \alpha \beta} s_{\alpha \beta \gamma} h^{\gamma \delta},
\]
\[
P^{\mu \nu} = \frac{1}{\sqrt{-g_{\mu \nu}}} \left( \frac{\epsilon}{1-\epsilon} \Pi_\mu^\epsilon + \tilde{\Pi}_\mu^\epsilon \right) \left( -v (|N|^\beta \Pi_\alpha^\lambda) + |N|^\beta \tilde{\Pi}_\alpha^\lambda \Pi_\beta \right) \tag{5.5}
\]
\[
G^\mu = S^{\mu \nu \gamma} (v^\sigma R_{\gamma \nu \lambda} \theta_0^\alpha - \nabla_v \delta_0^\alpha \nabla_\nu \delta_0^\mu + \frac{1}{\sqrt{-g_{\gamma \nu}}} \nabla_\nu \left( h^{\mu \alpha \beta \nu \lambda} (\sqrt{-g_{\gamma \nu}}) v_\alpha \theta_0^\beta \theta_0^\lambda \delta_0^\nu - \nu_{\alpha \beta \lambda \nu} \nu^\nu \delta_0^\gamma \right)
+ \nu_{\alpha \beta \lambda \nu} \nu^\nu \delta_0^\gamma \right) \left( \nabla_\nu \delta_0^\mu - \frac{\kappa}{\sqrt{-g_{\gamma \nu}}} \nabla_\nu \left( \sqrt{-g_{\gamma \nu}} h^{\mu \alpha \beta \nu \lambda} \nabla_\lambda \delta_0^\nu \right) \right)
\]
\[
- \frac{1}{\sqrt{-g_{\gamma \nu}}} \nabla_\nu \left( \nabla_v \mathcal{W}^{\nu \mu} \mathcal{W}^{\mu \nu} \right)
+ \frac{1}{\sqrt{-g_{\gamma \nu}}} \nabla_\nu \left( \nabla_v \mathcal{W}^{\nu \mu} \mathcal{W}^{\mu \nu} \right)
- \frac{1}{\sqrt{-g_{\gamma \nu}}} \nabla_\nu \left( \nabla_v \mathcal{W}^{\nu \mu} \mathcal{W}^{\mu \nu} \right)
\]
\[
p^{\mu \nu} = -v^\alpha \nu^\nu + e_i^\alpha \nabla_\nu \theta_0^i \tag{5.6}
\]
\[
l^{\mu} = \left( \delta_\alpha^\mu - \epsilon_\mu^\epsilon \right) \left( \frac{h^{\alpha \beta \gamma} s_{\alpha \beta \gamma} h^{\beta \lambda} \nabla_\nu \theta_0^\lambda - \nu_{\alpha \beta \lambda \nu} \nu^\nu \delta_0^\gamma \right)
- \kappa \nu^\mu \nu^\nu \nabla_v \theta_0^\nu \tag{5.7}
\]
and all other quantities are as previously defined.

Proof. We begin establishing (5.2) by computing
\[
\nabla_v B^\mu = \nabla_v B_{\perp}^\mu + L_v B_{\parallel}^\mu + \nabla_v v^\mu
\]
\[
= \nabla_v \mathcal{W}_{\perp}^{\mu \nu} + L_v \mathcal{W}_{\parallel}^{\mu \nu} - \nabla_v \mathcal{L}_{\perp}^\mu - L_v \mathcal{L}_{\parallel}^\mu + \Omega^\mu (B^\lambda)
\]
\[
= \theta_\alpha^3 \left( e_i^\alpha \left[ \nabla_v \mathcal{W}_{\perp}^{\mu \nu} + L_v \mathcal{W}_{\parallel}^{\mu \nu} \right] - \nabla_v \mathcal{L}_{\perp}^\mu - L_v \mathcal{L}_{\parallel}^\mu + \Omega^\mu (B^\lambda) \right),
\]
where here and below, in line with our conventions, we use $\Omega$ and $\mathcal{R}$ to denote generic remainder terms that may change from line to line. Observing that
\[
e_i^\alpha \left[ \nabla_v \mathcal{W}_{\perp}^{\mu \nu} + L_v \mathcal{W}_{\parallel}^{\mu \nu} \right] = e_i^\alpha \nabla_v \mathcal{W}_{\perp}^{\mu \nu} - e_i^\alpha \nabla_v \mathcal{W}_{\parallel}^{\mu \nu},
\]
(5.10)
we can write (5.11) as
\[\nabla_v B^\mu + \Omega^\mu (\mathcal{B}) = \theta^0_\alpha (\nabla_v W^\alpha^\mu + e_i^\alpha \nabla_v \theta_i^\mu) - \nabla_v L^\mu - L_v L^\mu. \quad (5.13)\]
From the definitions (3.37) and (5.10), we further observe that the last two terms of the above expression may be expressed as
\[\nabla_v L^\mu + L_v L^\mu = \nabla_v L^\mu + L_v L^\mu - \nabla_v (\epsilon |N|_g \Pi^\alpha^\mu \nabla_v \nabla_\nu \theta^0_\nu / |\nabla_v \theta^0_\nu|_m) = \nabla_v L^\mu - \nabla_v L^\mu - \epsilon |N|_g \Pi^\alpha^\mu \nabla_v \nabla_\nu \theta^0_\nu / |\nabla_v \theta^0_\nu|_m. \quad (5.14)\]
Noticing that
\[N_\alpha \nabla_v (\Pi^\alpha^\mu) = \frac{|N|^2}{|\theta^3|_g} L_v (\Pi^\alpha^\mu) + \Omega^\alpha_\alpha (a, \zeta - 1) \quad \text{(by (4.15))}\]
\[= \frac{|N|^2}{|\theta^3|_g} L_v (\theta^0_\alpha \Pi^\alpha^\mu) + \Omega^\alpha_\alpha (L_v \theta^3, a, \zeta - 1)\]
\[= \Omega^\alpha_\alpha (L_v \theta^3, \partial^3 \alpha_\lambda, \zeta - 1), \quad \text{(by (4.15) and (3.51))}\]
holds for any one form \(\lambda_\mu\), we can rewrite (5.14) as
\[\nabla_v L^\mu + L_v L^\mu = \nabla_v L^\mu - \nabla_v L^\mu - \epsilon |N|_g \Pi^\alpha^\mu \nabla_v \nabla_\nu \theta^0_\nu / |\nabla_v \theta^0_\nu|_m + \Omega^\mu (L_v \theta^3, \partial^3 \alpha_\lambda, \zeta - 1). \quad \text{(5.15)}\]
Combining this with (5.10) and recalling the definitions (3.21) and (3.17) of \(\hat{\gamma}^0\) and \(\hat{\mu}\), respectively, gives
\[\nabla_v B^\mu + \Omega^\mu (\hat{\mathcal{B}}, \partial^3 \alpha_\lambda, \zeta - 1) = \theta^3 (\nabla_v W^\alpha^\mu + e_i^\alpha \nabla_v \theta_i^\mu) - e_i^\alpha \nabla_v \theta^0_\nu / |\nabla_v \theta^0_\nu|_m\]
\[- \frac{1}{\sqrt{-\gamma}} \left[ \nabla_v L^\mu - \nabla_v L^\mu - \epsilon |N|_g \Pi^\alpha^\mu \nabla_v \nabla_\nu \theta^0_\nu / |\nabla_v \theta^0_\nu|_m \right].\]
Using the commutator formula (A.19) and the definition (3.24) of \(W^\alpha^\mu\), we further observe that \(\nabla_v W^\alpha^\mu\) is given by
\[\nabla_v W^\alpha^\mu = \zeta m^\alpha^\mu \alpha^\beta \nabla_\beta \theta^0_\nu + \nabla_v \theta^3 \theta^3 \nabla_\nu \theta^0_\nu - \nabla_v \theta^3 \nabla_\nu \theta^0_\nu + \nabla_v (\zeta m^\alpha^\mu \alpha^\beta \sigma^I_\nu \theta^I_0$.\]
Next, we notice that
\[p^\mu^\nu = e_i^\mu \theta_i^\mu (\gamma^\beta \theta^0_\nu + \hat{\delta}^\beta \theta^0_\nu) e_i^\nu + e_i^\mu \theta_i^\mu (\theta^0_\nu + \hat{\theta}^0_\nu) e_i^\nu\]
\[= e_i^\mu \left( \gamma^\beta + \frac{\gamma^\beta \gamma^\gamma}{\gamma^3} \right) e_i^\nu + \Omega^\mu^\nu (a, \zeta - 1), \quad (5.17)\]
where in deriving the last equality we used (4.15). But
\[(\gamma^\beta) = \left( \begin{array}{cc} \gamma^0 & 0 \\ 0 & \gamma^3 \end{array} \right) \quad (5.18)\]
with \(\hat{\gamma}^0\) given by (5.10). The importance of (5.18) is that it allows us to express projected derivatives \(p^\mu^\nu \partial^\nu\) on the boundary \(\Gamma_T\) explicitly in terms of the vector fields \(v^\mu\) and \(e_\beta^\mu\), which will be shown to be tangent to the boundary \(\Gamma_T\) in Section 6.
Substituting (5.19) into (5.15) and multiplying the resulting expression by \( \frac{1}{1-\epsilon} \Pi_\mu^\alpha + \tilde{\Pi}_\mu^\alpha \), we find, with the help of (3.42) and (3.51), that

\[
\left( \frac{1}{1-\epsilon} \Pi_\mu^\alpha + \tilde{\Pi}_\mu^\alpha \right) (\nabla_v B^\mu + \Omega^\mu(\delta, B, \partial^{[1]} \chi, L_v \theta^2, \zeta - 1)) = \left( \frac{1}{1-\epsilon} \Pi_\mu^\alpha + \tilde{\Pi}_\mu^\alpha \right) \left( \theta_\alpha^\mu (\nabla_v \mathcal{W}^\alpha)^\mu \right.
\]
\[
+ e_i^\alpha \nabla_v e_i^\mu \mathcal{W}^\alpha_{\mu} - e_i^\alpha \nabla \mathcal{W}_v^\mu \nu^\alpha \right) - \frac{1}{1-\epsilon} \Pi_\mu^\alpha + \tilde{\Pi}_\mu^\alpha \right) \left[ \nabla_v \ell^\mu - h_{\gamma \omega} \tilde{\mathcal{W}}^\omega \nabla_\chi \lambda^\mu \right.
\]
\[
- \left. p^\mu \nu \left( \epsilon |N| \Pi^{\alpha \lambda} \Pi^{\chi} \nabla_v \nabla_\nu \hat{\theta}_v^\alpha |N| \Pi^{\alpha \lambda} \Pi^{\chi} \nabla_v \nabla_\nu \hat{\theta}_v^\alpha \right) \right].
\]

(5.20)

To proceed, we examine the term \( \nabla_v \ell^\mu \) in more detail by first noting that

\[
s_{\alpha \beta}^\gamma \theta_i^\mu = \left( \nu_{\alpha \beta \lambda} e_i^\lambda \right) \sqrt{\gamma_{00}} \frac{\epsilon_i}{2} - \nu_{\alpha \beta \lambda} v^\lambda \epsilon_i^\delta_i \left( 1 + \nu_{\mu \lambda \omega} v^\lambda \epsilon_i^\delta_i \right) + \Omega_{\alpha \beta}^\gamma (A, \zeta - 1)
\]

(5.21)

and

\[
s_{\alpha \beta}^\gamma \theta_i^\mu = s_{\alpha \beta}^\gamma + \Omega_{\alpha \beta}^\gamma (a)
\]

(5.22)

follow directly from (3.20), (3.43) and \( \theta_i^\mu \epsilon_i^\delta_i = \delta_i^\gamma \). We also observe that

\[
[v, e_i] = \left[ \sqrt{-\gamma_{00}} e_0, e_i \right] + \mathcal{R}(\partial^{[1]} a, \partial^{[1]} \theta^i)
\]

(by (3.30)-(3.31), (3.40)-(3.41))

\[
= \frac{1}{\sqrt{-\gamma_{00}}} \left[ \frac{1}{2} \epsilon_i (\gamma_{00}) e_0 - \gamma_{00} e_i e_0, e_i \right] + \mathcal{R}(\partial^{[1]} a, \partial^{[1]} \theta^i)
\]

\[
= \frac{1}{\sqrt{-\gamma_{00}}} \left[ g(\theta^0, \nabla_v \theta^0) + \gamma_{00} F(v, e_i) \right] e_i + \mathcal{R}(\partial^{[1]} a, \partial^{[1]} \theta^i, \epsilon^K, \epsilon^p, r^q)
\]

(by (4.22))

\[
= \frac{1}{\sqrt{-\gamma_{00}}} \left[ g(\theta^0, \nabla_v \theta^0) + \gamma_{00} (i e_i, \nabla_v \theta^0 - i e_i \nabla_v \theta^0) \right] e_i + \mathcal{R}(\partial^{[1]} a, \partial^{[1]} \theta^i, \epsilon^K, \epsilon^p, r^q)
\]

\[
= - i e_i \nabla_v \theta^0 v + \mathcal{R}(\partial^{[1]} \chi),
\]

(5.23)

which allows us, with the help of (3.20), to write \( \nabla_{[v, e_i]} \hat{\theta}_v^\alpha \) as

\[
\nabla_{[v, e_i]} \hat{\theta}_v^\alpha = - \frac{1}{\sqrt{-\gamma_{00}}} e_i^\lambda \nabla_v \hat{\theta}_v^\alpha \nabla_v \hat{\theta}_v^\alpha + \Omega_{\alpha \beta}^\gamma (A, \zeta - 1, \nabla_v \zeta).
\]

(5.24)

Differentiating \( h^\omega_{\alpha \beta \gamma} h^\delta \nu \nabla_v \gamma \hat{\theta}_v^\gamma \), we obtain

\[
\nabla_v \left( h^\omega_{\alpha \beta \gamma} h^\delta \nu \nabla_v \gamma \hat{\theta}_v^\gamma \right) = h^\omega_{\alpha \beta \gamma} h^\delta \nu \theta_i^i \nabla_v \nabla_v \hat{\theta}_v^\mu + \nabla_v \left( h^\omega_{\alpha \beta \gamma} h^\delta \nu \theta_i^i \nabla_v \nabla_v \hat{\theta}_v^\mu + \Omega^\omega (\partial^{[1]} a) \right)
\]

(by (5.22))

\[
= h^\omega_{\alpha \beta \gamma} h^\delta \nu \theta_i^i \nabla_v \nabla_v \hat{\theta}_v^\mu + \nabla_v \left( h^\omega_{[v, e_i]} \hat{\theta}_v^\mu + e_i \nabla_v \theta^0 + \nabla_v \left( h^\omega_{\alpha \beta \gamma} h^\delta \nu \theta_i^i \nabla_v \theta^0 + \Omega^\omega (\partial^{[1]} a) \right) \right)
\]

\[
+ \nabla_v \left( h^\omega_{\alpha \beta \gamma} h^\delta \nu \theta_i^i \nabla_v \nabla_v \hat{\theta}_v^\mu + \Omega^\omega (\partial^{[1]} a) \right),
\]

(5.25)

where in deriving the last equality, we used the commutator formula (A.19). It is then easy to verify using (5.10), (5.16), (5.21), (5.22), (5.23), (5.25) and the identity

\[
\left( \frac{1}{1-\epsilon} \Pi_\mu^\alpha + \tilde{\Pi}_\mu^\alpha \right) (\delta^\lambda - c \Pi^\lambda) = \left( \frac{1}{1-\epsilon} \Pi_\mu^\alpha + \tilde{\Pi}_\mu^\alpha \right) ((1 - \epsilon) \Pi^\lambda + \tilde{\Pi}^\lambda) (5.26)
\]

that (5.20) is equivalent to (3.24).
Turning to the bulk relation, we observe that

\[
\nabla_v \mathcal{E}^\mu = \nabla_v \nabla_\alpha (e^\alpha_{\nu} W^{\nu \mu}) - \nabla_v \mathcal{H}^\mu
\]

\[
= \nabla_v (\nabla_{e_i} W_{\nu}^{\mu} + \nabla_{e_i} W_{\nu}^{\mu} + \nabla_\alpha e^\alpha_{\nu} W^{\nu \mu}) - \nabla_v \mathcal{H}^\mu,
\]

\[
= \nabla_v \nabla_\alpha (e^\alpha_{\nu} W^{\nu \mu} + \nabla_\alpha L_{e_i} W_{\nu}^{\mu} + \nabla_v (W_{\mu} W_{\nu}^{\mu} + \nabla_\alpha e^\alpha_{\nu} W^{\nu \mu})) - \nabla_v \mathcal{H}^\mu,
\]

\[
= \nabla_v \nabla_\alpha (e^\alpha_{\nu} W^{\nu \mu} + L_{e_i} W_{\nu}^{\mu} + \nabla_\alpha (W_{\mu} W_{\nu}^{\mu} + \nabla_\alpha e^\alpha_{\nu} W^{\nu \mu})) - \nabla_v \mathcal{H}^\mu,
\]

\[
= \nabla_v (e^\alpha_{\nu} W^{\nu \mu} + L_{e_i} W_{\nu}^{\mu} + \nabla_\alpha (W_{\mu} W_{\nu}^{\mu} + \nabla_\alpha e^\alpha_{\nu} W^{\nu \mu})) - \nabla_v \mathcal{H}^\mu
\]

\[
+ \nabla_\alpha (\nabla_\mu W_{\nu}^{\mu} + L_{e_i} W_{\nu}^{\mu} + \nabla_\alpha (W_{\mu} W_{\nu}^{\mu} + \nabla_\alpha e^\alpha_{\nu} W^{\nu \mu})) - \nabla_v \mathcal{H}^\mu,
\]

\[
= \nabla_\alpha (\nabla_v W_{\nu}^{\mu} + L_{e_i} W_{\nu}^{\mu} + \nabla_\alpha (W_{\mu} W_{\nu}^{\mu} + \nabla_\alpha e^\alpha_{\nu} W^{\nu \mu})) - \nabla_v \mathcal{H}^\mu
\]

\[
+ \nabla_\alpha (\nabla_\mu W_{\nu}^{\mu} + L_{e_i} W_{\nu}^{\mu} + \nabla_\alpha (W_{\mu} W_{\nu}^{\mu} + \nabla_\alpha e^\alpha_{\nu} W^{\nu \mu})) - \nabla_v \mathcal{H}^\mu,
\]

\[
= \nabla_\alpha (\nabla_v W_{\nu}^{\mu} + L_{e_i} W_{\nu}^{\mu} + \nabla_\alpha (W_{\mu} W_{\nu}^{\mu} + \nabla_\alpha e^\alpha_{\nu} W^{\nu \mu})) - \nabla_v \mathcal{H}^\mu
\]

\[
+ \nabla_\alpha (\nabla_v W_{\nu}^{\mu} + L_{e_i} W_{\nu}^{\mu} + \nabla_\alpha (W_{\mu} W_{\nu}^{\mu} + \nabla_\alpha e^\alpha_{\nu} W^{\nu \mu})) - \nabla_v \mathcal{H}^\mu
\]

(5.27)

For the wave formulation to be useful for establishing the existence of solutions, the only derivative of \(W^{\nu \mu}\) that can appear outside the principal term \(\nabla_\alpha (\nabla_v W^{\nu \mu} + e^\alpha_e W^{\nu \mu} - e^\alpha_e W_{\nu}^{\mu})\) is the “time” derivative \(\nabla_v W^{\nu \mu}\). We therefore must further decompose the terms \(\nabla_\alpha W^{\nu \mu}, L_{e_i} W^{\nu \mu}\) and \(\nabla_L e_i W^{\nu \mu}\) in (5.27). We begin with \(L_{e_i} W^{\nu \mu}\), which we can write as follows:

\[
L_{e_i} W^{\nu \mu} = \nabla_\alpha W^{\nu \mu} - \nabla_{W_{\mu}^{\nu}} e^\alpha_{\nu}
\]

\[
= \nabla_\alpha (e^\alpha_{\nu} W^{\nu \mu}) - \nabla_\alpha e^\alpha_{\nu} W^{\mu \nu} - \nabla_{W_{\mu}^{\nu}} e^\alpha_{\nu}
\]

\[
= \nabla_\alpha (e^\alpha_{\nu} W^{\nu \mu}) - \nabla_\alpha e^\alpha_{\nu} W^{\mu \nu} - \nabla_{W_{\mu}^{\nu}} e^\alpha_{\nu}
\]

\[
= \nabla_\alpha (e^\alpha_{\nu} W^{\nu \mu}) - \nabla_\alpha e^\alpha_{\nu} W^{\mu \nu} - \nabla_{W_{\mu}^{\nu}} e^\alpha_{\nu}
\]

\[
= \nabla_\alpha (e^\alpha_{\nu} W^{\nu \mu}) - \nabla_\alpha e^\alpha_{\nu} W^{\mu \nu} - \nabla_{W_{\mu}^{\nu}} e^\alpha_{\nu}
\]

\[
= \nabla_\alpha (\nabla_{e_i} W^{\mu \nu} + \nabla_{e_i} W_{\nu}^{\mu} + \nabla_{e_i} (W_{\mu} W_{\nu}^{\mu} + \nabla_{e_i} e^\alpha_{\nu} W^{\nu \mu})) - \nabla_v \mathcal{H}^\mu.
\]

(5.28)

Substituting (5.28) and (5.23) into (5.27) gives

\[
\nabla_v \mathcal{E}^\mu + \mathcal{R}^\mu (\vartheta^{\mu} \chi, \mathcal{E}) = \nabla_\alpha (\nabla_v W^{\nu \mu} + e^\alpha_{\nu} \nabla_v (e^\alpha_{\nu} W^{\mu \nu}) - e^\alpha_{\nu} W_{\nu}^{\mu}) - \nabla_\alpha e^\alpha_{\nu} (\nabla_v W^{\nu \mu} - \nabla_{W_{\mu}^{\nu}} e^\alpha_{\nu})
\]

\[
- \nabla_\alpha e^\alpha_{\nu} (\nabla_{e_i} W^{\mu \nu} + \nabla_{e_i} W^{\nu \mu} + \nabla_{e_i} (W_{\mu} W_{\nu}^{\mu} + \nabla_{e_i} e^\alpha_{\nu} W^{\nu \mu})) - \nabla_\alpha e^\alpha_{\nu} (\nabla_v W^{\nu \mu} - \nabla_{W_{\mu}^{\nu}} e^\alpha_{\nu})
\]

\[
+ \nabla_\alpha (\nabla_{e_i} W^{\mu \nu} + \nabla_{e_i} W^{\nu \mu} + \nabla_{e_i} (W_{\mu} W_{\nu}^{\mu} + \nabla_{e_i} e^\alpha_{\nu} W^{\nu \mu})) - \nabla_\alpha e^\alpha_{\nu} (\nabla_v W^{\nu \mu} - \nabla_{W_{\mu}^{\nu}} e^\alpha_{\nu}) - \nabla_v \mathcal{H}^\mu.
\]

which is easily seen to be equivalent to (5.2).

The following corollary is then an immediate consequence of the above lemma and Theorem 5.1

**Corollary 5.2.** Suppose \(e \in [0, 1), \kappa \geq 0, \zeta \in C^2(\Omega_T), \vartheta^{\mu}_0 \in C^3(\Omega_T), \vartheta^{\mu}_1 \in C^3(\Omega_T), \sigma^{\nu \mu} \in C^1(\Omega_T), \) there exists constants \(c_0, c_1 > 0\) such that

\[
- g(\vartheta^{\mu}_0, \vartheta^{\mu}_0) \geq c_0 > 0 \text{ in } \Omega_T \quad \text{and} \quad - i_{e_3} \nabla_{e_3} \vartheta^{\mu}_0 \geq c_1 > 0 \text{ in } \Gamma_T,
\]

(5.29)
and the quadruple \( \{ \zeta, \hat{\theta}_\mu, \theta^{J}_\mu, \sigma^{k}_{i,j} \} \) satisfies the initial conditions \((3.55)\)-(3.70) and
\[
\nabla_\alpha (A^{\alpha \beta \mu \nu} \nabla_\beta \nabla_v \hat{\theta}_\nu + X^{\alpha \mu}) = F^\mu \quad \text{in } \Omega_T, \quad (5.30)
\]
\[
\theta^{J}_\alpha (A^{\alpha \beta \mu \nu} \nabla_\beta \nabla_v \hat{\theta}_\nu + X^{\alpha \mu}) = \frac{1}{\sqrt{|\gamma|}} \left[ S^{\mu \nu \gamma} \nabla_\gamma \nabla_v \hat{\theta}_\nu - \frac{\kappa}{1-\epsilon} |N|_\beta \Pi^{\alpha \mu \nu} \left( \Pi^\lambda v_\lambda \nabla_\nu \hat{\theta}_\nu \right) \right] + P^{\mu \nu} \left( \frac{\nabla_\nu \nabla_v \hat{\theta}_\nu}{|\nabla_v \hat{\theta}_\nu|} \right) - \frac{\kappa}{\sqrt{-\gamma_{00}}} v^\mu v^\nu \nabla_\nu \nabla_v \hat{\theta}_\nu + G^\mu \quad \text{in } \Gamma_T, \quad (5.31)
\]
\[
n_\nu v^\nu = 0 \quad \text{in } \Gamma_T, \quad (5.32)
\]
\[
\nabla_\alpha (a^{\alpha \beta} \nabla_\beta \zeta) = K
\]
\[
\zeta = 1 \quad \text{in } \Omega_T, \quad (5.33)
\]
\[
L_\nu \theta^I = 0 \quad \text{in } \Gamma_T, \quad (5.34)
\]
\[
v(\sigma^{k}_{i,j}) = 0 \quad \text{in } \Omega_T, \quad (5.35)
\]
\[
Q^{\mu_1 \cdots \mu_r}_{\nu_1 \cdots \nu_s} \text{ is treated as an unknown and (6.3) is viewed as an evolution equation for } \theta_{\mu_0}. \]

Then the constraints \((3.10)-(3.18)\) and \((3.22)\) vanish in \( \Omega_T \) and \( \Gamma_T \), respectively, and the pair \( \{ \rho, \nu^\mu \} \) determined from \( \{ \theta^{J}_\mu, \zeta \} \) via the formulas \((3.8)-(3.9)\) satisfy the relativistic Euler equations with vacuum boundary conditions given by \((1.9)-(1.13)\).

### 6. Lagrangian Coordinates

In order for the wave formulation of the relativistic Euler equations given by \((5.30)-(5.36)\) to be useful for either establishing the local-in-time existence of solutions or for constructing numerical solutions, the dynamical matter-vacuum boundary must be fixed. We achieve this through the use of Lagrangian coordinates
\[
\phi : [0,T] \times \Omega_0 \longrightarrow \Omega_T : \bar{x}^\lambda \longmapsto (\phi^\mu (\bar{x}^\lambda))
\]
that were defined previously by \((1.9)\). In the following, we use
\[
J^\mu_\nu = \bar{\partial}_\nu \phi^\mu \quad \text{(6.1)}
\]
and
\[
(J^\mu_\nu)^{-1} = \bar{J}^\mu_\nu \quad \text{(6.2)}
\]
to denote the Jacobian matrix and its inverse, respectively.

**Notation 6.1.** For scalars fields \( f \) defined on \( \Omega_T \), we employ the notation
\[
\bar{f} = f \circ \phi
\]
to denote the pullback of \( f \) by \( \phi \) to \( [0,T] \times \Omega_0 \). More generally, we use this notation also to denote the pullback tensor field components \( Q^{\mu_1 \cdots \mu_r}_{\nu_1 \cdots \nu_s} \) treated as scalar fields defined on \( \Omega_T \), that is
\[
\bar{Q}^{\mu_1 \cdots \mu_r}_{\nu_1 \cdots \nu_s} = Q^{\mu_1 \cdots \mu_r}_{\nu_1 \cdots \nu_s} \circ \phi.
\]
Using this notation, we can then write the geometric pullback of a tensor field \( Q^{\mu_1 \cdots \mu_r}_{\nu_1 \cdots \nu_s} \) by \( \phi \) as
\[
\bar{Q}^{\mu_1 \cdots \mu_r}_{\nu_1 \cdots \nu_s} := (\phi^* Q)^{\mu_1 \cdots \mu_r}_{\nu_1 \cdots \nu_s} = \bar{J}^{\mu_1}_{\alpha_1} \cdots \bar{J}^{\mu_r}_{\alpha_r} \frac{\partial^{\alpha_1}}{\partial x^{\nu_1}} \cdots \frac{\partial^{\alpha_r}}{\partial x^{\nu_s}} \bar{Q}^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s}.
\]

Since the Lagrangian coordinates are defined via the flow of the fluid velocity \( v \), it follows that the components of the pullback \( \bar{v} = \phi^* v \) are given by
\[
\bar{v}^\mu = \delta^\mu_0. \quad \text{(6.2)}
\]
Substituting this into the transformation law \( J^\mu_0 \bar{v}^\nu = J^\mu_\nu \) shows that \( \phi^\mu \) satisfies
\[
\bar{\partial}_0 \phi^\mu = v^\mu \quad \text{(6.3)}
\]
where we note, see \((3.21)\), that
\[
\bar{z}^{00} = g^{\mu \nu} \phi^\mu \phi^\nu.
\]
In the Lagrangian representation, the map \( \phi = (\phi^\mu) \) is treated as an unknown and \((6.3)\) is viewed as an evolution equation for \( \phi \).
Pulling back the evolution equations (5.35) and (5.36) using the map $\phi$, we see, with the help of formulas (6.2) and (A.9), and the naturalness property $\phi^* L_v = L_v\phi^*$ of Lie derivatives, that
\[
\tilde{\partial}_\nu \tilde{\theta}_\mu = 0 \quad \text{and} \quad \tilde{\partial}_\nu \tilde{\sigma}^k_{\ j} = 0 \quad \text{in} \quad [0, T] \times \Omega_0.
\] (6.4)
By (4.4), it is clear that $\phi$ satisfies $\phi(\Omega_0) = \Omega_0$ from which it follows that
\[
J^\mu_\nu(0, \tilde{x}^\Sigma) = v^\mu(0, \tilde{x}^\Sigma)\theta^\nu_\sigma + \delta^\nu_\sigma \delta^\Lambda_\sigma, \quad \forall (\tilde{x}^\Sigma) \in \Omega_0,
\]
by (6.3) while
\[
v^\mu(0, \tilde{x}^\Sigma)\theta^\nu_\sigma(0, \tilde{x}^\Sigma) = 0, \quad \forall (\tilde{x}^\Sigma) \in \Omega_0,
\]
follows from our choice of initial data, see (3.11) and (3.56). The above two results together with (6.3), (6.4) and the transformation law $\tilde{\theta}^\mu_\nu = J^\mu_\nu \theta^\nu_\sigma$ yield the explicit representations
\[
\tilde{\theta}^\mu_\nu(\tilde{x}^0, \tilde{x}^\Sigma) = \delta^\Lambda_\nu \theta^\mu_\Lambda(0, \tilde{x}^\Sigma) \quad \text{and} \quad \tilde{\sigma}^k_{\ j}(\tilde{x}^0, \tilde{x}^\Sigma) = \sigma^k_{\ j}(0, \tilde{x}^\Sigma), \quad \forall (\tilde{x}^0, \tilde{x}^\Sigma) \in [0, T] \times \Omega_0,
\] (6.5)
of the unique solution to the evolution equations (5.35) and (5.36). We also recall, see (4.12), that
\[
\tilde{\theta}^3 = \tilde{\theta}^3_{\mu} d\tilde{x}^\mu = \theta^3_{\mu}(0, \tilde{x}^\Sigma) d\tilde{x}^\Lambda
\]
defines an outward pointing conormal to the boundary $[0, T] \times \partial \Omega_0$ by our choice of initial data.

Remark 6.2.

(i) Since (6.5) represents the unique solution of the evolution equations (5.35) and (5.36) given the particular choice of initial data, we consider them solved and remove these equations from our system of equations. Furthermore, from the transformation law $\tilde{\theta}^\mu_\nu = J^\mu_\nu \theta^\nu_\sigma$ and (6.5), it is clear that we can express $\tilde{\theta}^\mu_\nu$ as
\[
\tilde{\theta}^\mu_\nu(\tilde{x}^0, \tilde{x}^\Sigma) = J^\mu_\nu(\tilde{x}^0, \tilde{x}^\Sigma) \theta^\nu_\Lambda(0, \tilde{x}^\Sigma), \quad \forall (\tilde{x}^0, \tilde{x}^\Sigma) \in [0, T] \times \Omega_0,
\] (6.7)
which, in particular, shows that the components $\tilde{\theta}^\mu_\nu(\tilde{x}^0, \tilde{x}^\Sigma)$ are determined completely in terms of the initial data $\theta^\nu_\Lambda(0, \tilde{x}^\Sigma)$ and the derivatives of $\theta^\nu_\sigma$, since by definition $(J^\mu_\nu) = (\tilde{\partial}_\nu \phi^\mu)^{-1}$.

(ii) The boundary condition (5.32) is automatically satisfied since $\tilde{\theta}^3$ defines an outward pointing conormal to the boundary $[0, T] \times \partial \Omega_0$ and $\tilde{v}^\mu \tilde{\theta}^3 = 0$ follows immediately from (6.2) and (6.6). We, therefore, consider the boundary condition (5.32) as satisfied and do not consider it further.

By definition, the frame field components $\tilde{e}^\nu_\Lambda$ are given by
\[
(\tilde{e}^\nu_\Lambda) = (\tilde{\theta}^\nu_\Lambda)^{-1} = \left( \begin{array}{c} \tilde{\theta}^0_\Lambda - \tilde{\theta}^0_{\mu} \tilde{\theta}^\mu_\Lambda \tilde{\theta}^K_0 \\
- \tilde{\theta}^0_\Lambda \tilde{\theta}^\mu_{\mu K} \\
\tilde{\theta}^0_\Lambda - \tilde{\theta}^0_{\mu} \tilde{\theta}^\mu_\Lambda \tilde{\theta}^K_0 \\
\end{array} \right),
\]
where $(\tilde{\theta}^\nu_\Lambda)^{-1}$, which by (6.5), reduces to
\[
(\tilde{e}^\nu_\Lambda) = (\tilde{\theta}^\nu_\Lambda)^{-1} = \left( \begin{array}{c} \frac{1}{\tilde{\theta}^0_\Lambda} - \tilde{\theta}^0_{\mu} \tilde{\theta}^\mu_\Lambda \\
0 \\
\tilde{\theta}^0_\Lambda - \tilde{\theta}^0_{\mu} \tilde{\theta}^\mu_\Lambda \\
\end{array} \right),
\] (6.8)
Since $\tilde{e}^\nu_\Lambda = 0$ by duality, and $\tilde{\theta}^3$ is conormal to $[0, T] \times \partial \Omega_0$, we have that $T([0, T] \times \partial \Omega_0) = \text{Span} \{ \tilde{e}_k = \tilde{e}^\nu_\Lambda \tilde{\theta}^\nu_\Lambda \}$, which, by (6.8), implies that $\tilde{\partial}_\nu$ and the vector fields $\tilde{Z}_K$ defined by
\[
\tilde{Z}_K(\tilde{x}^0, \tilde{x}^\Sigma) = \tilde{\theta}^\nu_{\Lambda K}(0, \tilde{x}^\Sigma) \tilde{\theta}^\Lambda_\nu
\] (6.9)
span the tangent space to the boundary $[0, T] \times \partial \Omega_0$, that is
\[
T([0, T] \times \partial \Omega_0) = \text{Span} \{ \tilde{\partial}_\nu, \tilde{Z}_K \}.
\] (6.10)
Appealing to transformation law $\tilde{\theta}^\mu_\nu = J^\mu_\nu \theta^\nu_\sigma$, we see from (6.8) and (6.9) that
\[
(\tilde{e}^\nu_\Lambda) = \left( \begin{array}{c} \tilde{\theta}^\mu_\nu \tilde{Z}_K(\phi^\nu) \\
\tilde{\theta}^\mu_\nu \tilde{\partial}_\nu \phi^\nu \\
\tilde{\theta}^\mu_\nu \tilde{Z}_K(\phi^\nu) \\
\end{array} \right),
\]
while
\[
\tilde{\theta}^\mu_\nu = \tilde{\partial}_\nu \phi^\nu = \frac{\tilde{g}^\mu_{\nu} \tilde{\theta}^\nu_0}{\sqrt{-\tilde{g}}}
\] (6.11)
follows from (6.3). Combining these two results yields
\[ \bar{e}_K = -\frac{1}{\sqrt{-\frac{1}{2} g_{00} \bar{Z}_K(\phi^\mu)}} \partial_0 + \bar{Z}_K, \] (6.12)
which, when used in conjunction with the transformation law \( e^\mu_K = J^\mu_\nu \bar{e}_\nu^K \), gives
\[ e^\mu_K = \bar{e}_K(\phi^\mu) = -\frac{1}{\sqrt{-\frac{1}{2} g_{00} \bar{Z}_K(\phi^\mu)}} \partial_0 \phi^\mu + \bar{Z}_K(\phi^\mu). \] (6.13)

Since
\[ \bar{g}^0_\mu = \hat{\bar{g}}^0_\mu \text{ in } [0, T] \times \partial \Omega_0 \] (6.14)
by (3.1) and the boundary condition (5.34), we find, after restricting (6.12) and (6.13) to the boundary, that
\[ \bar{e}_K = \frac{1}{\sqrt{-\frac{1}{2} \bar{g}_{00}}} \partial_0 + \bar{Z}_K \] (6.15)
and
\[ e^\mu_K = \frac{1}{\sqrt{-\frac{1}{2} \bar{g}_{00}}} \partial_0 \phi^\mu + \bar{Z}_K(\phi^\mu) \] (6.16)
in \([0, T] \times \partial \Omega_0\).

Next, we consider the determinant \( |\gamma| \) of the frame metric evaluated in the Lagrangian coordinates. By definition \( |\gamma| = -\text{det}(g(e_i, e_j)) \), and so pulling this back by \( \phi \) gives
\[ |\gamma| = -\text{det}(e^\mu_\nu \partial_\nu \phi^\mu) = |g| \text{det}(J)^2 \text{det}(\bar{e}), \]
where in deriving the second equality we have used the transformation law \( J^\mu_\nu \bar{e}_\nu^K = e^\mu_K \). Using this formula in conjunction with (6.8), (6.11) and (6.14), we find that
\[ \frac{1}{\sqrt{|\gamma|}} = \frac{\text{det}(\bar{g})}{\text{det}(J) \sqrt{|g|}} \text{ in } [0, T] \times \partial \Omega_0, \] (6.17)
where the coframe components \( \bar{g}^\mu_\nu \) are given by (6.5).

Letting \( \eta = \eta_{\mu \nu} dx^\mu dx^\nu \), where \( (\eta_{\mu \nu}) = \text{diag}(-1, 1, 1, 1) \), denote the Minkowski metric, we recall the following transformation formula for the divergence of a vector field \( Y = Y^\mu \partial_\mu \):
\[ |\eta|^\frac{1}{2} \partial_\mu (|\eta| \frac{1}{2} Y^\mu) = |\eta|^\frac{1}{2} \partial_\mu (|\eta| \frac{1}{2} Y^\mu). \]
Since \( |\eta| = -\text{det}(\eta_{\mu \nu}) = 1 \), \( |\eta| = -\text{det}(J^\mu_\nu \eta_{\alpha \beta} J^\beta_\nu) = \text{det}(J)^2 \), and \( Y^\mu = \bar{Y}^\mu \), the above formula can be written as
\[ \text{det}(J)^{-1} \partial_\mu (\text{det}(J) \bar{Y}^\mu) = \partial_\mu Y^\mu. \]
Using this along with the transformation law \( \partial_\mu = \bar{J}^\mu_\nu \partial_\nu \) for partial derivatives and the formula (6.17), it is not difficult to see that the remaining equations (5.30)-(5.31) and (5.33)-(5.34), when expressed in Lagrangian coordinates, become:
\[ \bar{\partial}_\alpha (\omega^{\alpha \beta \mu} \partial_\beta \psi_\nu + \Phi^{\alpha \mu}) = \Phi^{\mu} \] in \([0, T] \times \Omega_0, \) (6.18)
\[ \bar{\partial}_\alpha (\omega^{\alpha \beta \mu} \partial_\beta \psi_\nu + \Phi^{\alpha \mu}) = \Phi^{\mu} \gamma_\mu \partial_0 \psi_\nu + \Phi^{\mu} \partial_0 (\Pi^\lambda_\nu \Psi_\lambda) + \Phi^{\mu} \Psi_\nu + \Phi^{\mu} \] in \([0, T] \times \partial \Omega_0, \) (6.19)
\[ \bar{\partial}_\alpha (\mathcal{M}^{\alpha \beta} \partial_\beta \zeta) = \mathcal{K} \] in \([0, T] \times \Omega_0, \) (6.20)
\[ \zeta = 1 \] in \([0, T] \times \partial \Omega_0, \) (6.21)
\[ \partial_0 \phi^\mu = \frac{1}{\sqrt{-\frac{1}{2} g_{00}}} g^{\mu \nu} \bar{g}^0_\nu \] in \([0, T] \times \Omega_0, \) (6.22)
\[ \partial_0 \bar{\partial}_0 = \psi_\nu + \psi^\lambda \Gamma_\nu^\lambda \bar{\partial}_0 \] in \([0, T] \times \Omega_0, \) (6.23)
\[ \partial_0 \psi_\nu = \bar{\psi}_m \Psi_\nu + \psi^\lambda \Gamma_\nu^\lambda \psi_\gamma \] in \([0, T] \times \Omega_0, \) (6.24)
where we have defined

$$\mathcal{J}^{\alpha\beta\mu\nu} = \det(J) \sqrt{|g|} \tilde{J}_\sigma \tilde{A}^{\alpha\gamma\mu\nu} \tilde{J}_\gamma,$$

$$\mathcal{X}^{\alpha\mu} = \det(J) \sqrt{|g|} (\tilde{X}^{\alpha\mu} - \tilde{A}^{\alpha\gamma\mu\nu} \tilde{J}_\gamma),$$

$$\mathcal{F}^\mu = \det(J) \sqrt{|g|} \left( \tilde{F}^\mu - \Gamma^\mu_{\alpha\lambda} \left( \tilde{J}_\beta (\partial_\gamma \widetilde{\phi}) - \Gamma^\gamma_{\beta\lambda} + \tilde{A}^{\alpha\beta\gamma} \right) \right),$$

$$\mathcal{F}^{\mu\nu\gamma} = \det(\tilde{\theta}^\lambda) \sqrt{-\frac{\tilde{\Lambda}}{\tilde{\Lambda}}} \tilde{S}^{\mu\nu\lambda} \tilde{J}_\lambda,$$

$$\mathcal{Q}^{\mu\nu} = -\frac{\epsilon}{1 - \epsilon} \det(\tilde{\theta}^\lambda) \sqrt{-\frac{\tilde{\Lambda}}{\tilde{\Lambda}}} \tilde{N}_{\alpha\beta} \tilde{P}^{\mu\nu},$$

$$\mathcal{R}^{\mu\nu} = \det(\tilde{\theta}^\lambda) \left( \sqrt{-\frac{\tilde{\Lambda}}{\tilde{\Lambda}}} \tilde{D}^{\mu\nu} - \tilde{S}^{\mu\nu\gamma} \tilde{S}_{\gamma\alpha} \tilde{\phi} \right),$$

$$\mathcal{M}^{\alpha\beta} = \det(J) \sqrt{|g|} \tilde{J}_\sigma \tilde{A}^{\alpha\gamma} \tilde{J}_\gamma,$$

$$\mathcal{K} = \det(J) \sqrt{|g|} \tilde{K}.$$

and we are using the notation $|g| = -\det(\tilde{g}_{\alpha\beta})$ for the Lagrangian representation of the metric determinant and $|X|^2 = \tilde{g}^{\alpha\beta} X_\alpha X_\beta$, with $X_\mu$ spacelike, and $|Y|^2 = \tilde{m}^{\alpha\beta} Y_\alpha Y_\beta$ for the Lagrangian representations of the inner-products.

The systems (6.18)-(6.24) along with the relations (6.34) defines the Lagrangian representation of the system (5.30)-(5.36). The unknowns to solve for are $\{\phi^\mu, \zeta, \tilde{\theta}^\lambda, \psi_\nu, \Psi_\nu\}$. All other quantities in the system can be computed in terms of these fields and the time-independent frame fields $\tilde{\theta}^\lambda$ and functions $\tilde{g}_{\alpha\beta}$ defined by (6.34). In particular, the Lagrangian representation of the metric and metric related quantities, i.e. Christoffel symbols and curvature, are to be viewed as functions of $\phi^\mu$. To see this, we observe that if $g_{\alpha\beta}(x^\lambda)$ are the components of the Lorentzian metric, in the Eulerian representation, with associated Christoffel symbols $\Gamma_{\alpha\beta\gamma}^\lambda(x^\lambda)$ and Curvature tensor $R_{\alpha\beta\gamma\lambda}(x^\lambda)$, then in the Lagrangian representation, we have that

$$g_{\alpha\beta}(x^\mu) = g_{\alpha\beta}(\phi^\lambda(\tilde{x}^\mu)), \quad \Gamma_{\alpha\beta\gamma}^\lambda(\tilde{x}^\mu) = \Gamma_{\alpha\beta\gamma}^\lambda(\phi^\lambda(\tilde{x}^\mu)) \quad \text{and} \quad R_{\alpha\beta\gamma\lambda}(\tilde{x}^\mu) = R_{\alpha\beta\gamma\lambda}(\phi^\lambda(\tilde{x}^\mu)).$$

Moreover, the Lagrangian representation of the components of any covariant derivative can be reduced to partial derivatives with respect to the Lagrangian coordinates using computations like the following:

$$\nabla_\gamma Y_\nu(\tilde{x}^\mu) = \tilde{J}^\mu_\gamma(\tilde{x}^\mu) \tilde{\partial}_\gamma Y_\nu(\tilde{x}^\mu) - \Gamma_{\nu\mu}^\lambda(\tilde{x}^\mu) Y_\lambda(\tilde{x}^\mu).$$

Remark 6.3.

(i) The evolution equations (6.23) and (6.24) for $\tilde{\theta}^\lambda$ and $\psi_\nu$ are easily seen to be equivalent to

$$\nabla_\nu \psi_\nu = \nabla_\nu \tilde{\theta}^\lambda \quad \text{and} \quad \Psi_\nu = \nabla_\nu \tilde{\theta}^\lambda \quad \text{for independent variables},$$

they are, by the choice of evolution equations, just labels for the derivatives $\nabla_\nu \tilde{\theta}^\lambda$ and $\nabla_\nu \tilde{\theta}^\lambda$, respectively. We take this point of view because it turns out to be convenient for establishing the local-in-time existence of solutions via energy methods; see the following section for further discussion.

(ii) It is an immediate consequence of the above remark and Corollary 5.2 that solutions of $\{\phi^\mu, \zeta, \tilde{\theta}^\lambda, \psi_\nu, \Psi_\nu\}$ of (6.18)-(6.24) for which the initial conditions (5.55)-(5.70) and the Lagrangian version of (6.29) are satisfied determine solutions of the relativistic Euler equations with vacuum boundary conditions.

(iii) From the formulas (3.43), (5.6) and (5.23) that define $\mathcal{F}^{\mu\nu\gamma}$ and the fact that the vector fields

$\{\tilde{v} = \tilde{\partial}_0, \tilde{e}_\nu = \tilde{e}_\nu^\lambda \tilde{\partial}_\lambda\}$ 1-form $\tilde{\theta}^\lambda = \tilde{\theta}^\lambda_\mu \tilde{e}_\mu$ are tangent, see (6.10) and (6.13), and normal to the boundary $[0, T] \times \partial \Omega_0$, respectively, it follows that $\mathcal{F}^{\mu\nu\gamma}$ satisfies the following two crucial properties:

$$\mathcal{F}^{\mu\nu\gamma} \tilde{\theta}^\lambda_\gamma = 0 \quad \text{in} \quad [0, T] \times \partial \Omega_0$$

and

$$\mathcal{F}^{\mu\nu} = -\mathcal{F}^{\nu\mu} \quad \text{in} \quad [0, T] \times \overline{\Omega}_0.$$

(6.34)}
Note that (6.34) is equivalent to the statement that the operator $\mathcal{F}^{\mu\nu}\bar{\partial}_{\gamma}$ involves only derivatives that are tangential to the boundary $[0, T] \times \partial \Omega_0$.

The importance of (6.34) and (6.35) is that they allow us to write the evolution equation (6.18) and boundary condition (6.19) as

$$
\bar{\partial}_{\alpha}(\mathcal{F}^{\alpha\beta\mu} \bar{\partial}_{\beta} \psi + \mathcal{X}^{\alpha}) = \mathcal{F}^{\alpha} + \bar{\partial}_{\alpha}\left(\frac{2}{|\beta|^2} \mathcal{F}^{\mu \nu [\alpha} m_{\beta]} \lambda \bar{\theta}_{\alpha}^{3}\right) \bar{\partial}_{\beta} \psi \quad \text{in } [0, T] \times \Omega_0,
$$

(6.36)

$$
\bar{\theta}_{\alpha}^{3}(\mathcal{F}^{\alpha\beta\mu} \bar{\partial}_{\beta} \psi + \mathcal{X}^{\alpha}) = \mathcal{Q}^{\mu \nu \alpha} \bar{\partial}_{\alpha} \left(\Pi_{\lambda} \Psi_{\lambda}\right) + \mathcal{P}^{\mu \nu} \Psi_{\nu} + \mathcal{G}^{\mu} \quad \text{in } [0, T] \times \partial \Omega_0,
$$

(6.37)

where

$$
\mathcal{F}^{\alpha\beta\mu} = \mathcal{F}^{\alpha\beta\mu} + \frac{2}{|\beta|^2} \mathcal{F}^{\mu \nu [\alpha} m_{\beta]} \lambda \bar{\theta}_{\alpha}^{3}.
$$

(6.38)

This form of the equations is crucial for establishing energy estimates, see [41]. However, this form is not enough by itself to derive energy estimates, the following coercive type of elliptic estimate must also hold: there exists positive constants $c_0, c_1$ such that

$$
\langle \bar{\partial}_{\lambda} u_{\mu}, \mathcal{F}^{\lambda \Sigma \mu} (\bar{x}^{0}) \bar{\partial}_{\Sigma} u_{\nu}\rangle_{L^2(\Omega_0)} \geq c_0 \|u\|_{H^1(\Omega_0)}^2 - c_1 \|u\|_{L^2(\Omega_0)}^2
$$

(6.39)

for all $\bar{x}^0 \in [0, T]$ and $u = (u_{\mu}) \in H^1(\Omega_0)$. That $\mathcal{F}^{\alpha\beta\mu}$, defined by (6.38), satisfies this coercive estimate for $\epsilon \in (0, 1)$ follows from a slight variation of Lemma 8.3 from [41].

(iv) From the formulas (3.26), (3.29), (3.32), (6.34), (6.35), (6.37), (5.6)–(5.10), (6.1), (6.28)–(6.31), (6.2)–(6.3), and (6.19)–(6.16), it is not difficult to verify, with the help of the evolution equations (6.22) that the boundary the coefficients and source terms $\{\mathcal{F}^{\mu \nu}, \mathcal{Q}^{\mu \nu}, \mathcal{P}^{\mu \nu}, \bar{\theta}_{\alpha}^{3} \Pi_{\lambda} \Psi_{\lambda}, \mathcal{G}^{\mu}\}$ that appear on the right hand side of the boundary condition (6.19) involve only the fields $\{\bar{\theta}_{\alpha}^{1} \phi_{\mu}, \bar{\theta}_{\alpha}^{1} \phi_{\mu}\}$, where in line with our previous notation, we are using $\bar{\theta}$ to denote derivatives tangential to the boundary $[0, T] \times \partial \Omega_0$.

7. Remarks on local-in-time existence and uniqueness

In this section, we briefly remark on the properties of the system (6.18)–(6.24) that will allow us to use the existence results and energy estimates for systems of wave equations with variable coefficients from the articles [26, 41] to prove the local-in-time existence and uniqueness of solutions to (6.18)–(6.24); the complete existence proof will be presented in the separate article [38]. The first step in the existence proof from [38] uses a contraction mapping argument that effectively reduces the problem of establishing local-in-time existence of solutions to the non-linear system to that of establishing the existence of solutions to the linear system with variable coefficients that arises from “freezing” the non-principal coefficients of (6.18)–(6.24). Since this step is standard, we will not discuss it further here.

This leaves us to discuss the existence problem for solutions to the corresponding variable coefficient linear system. With this in mind, we recall that linear wave equations with variable coefficients of the form that come from freezing the coefficients of (6.20) and enforcing a Dirichlet boundary condition, see (6.21), have already been thoroughly analyzed in [26]. We are therefore left to consider the remaining evolution equations from the system (6.18)–(6.24). Freezing the coefficients of (6.18)–(6.19) and (6.22)–(6.24) yields the variable coefficient linear system

$$
\bar{\partial}_{\alpha}(\mathcal{F}^{\alpha\beta\mu} \bar{\partial}_{\beta} \psi + \mathcal{X}^{\alpha}) = \mathcal{F} \quad \text{in } [0, T] \times \Omega_0,
$$

(7.1)

$$
\bar{\theta}_{\alpha}^{3}(\mathcal{F}^{\alpha\beta\mu} \bar{\partial}_{\beta} \psi + \mathcal{X}^{\alpha}) = \mathcal{Q}^{\gamma \mu \alpha} \bar{\partial}_{\alpha} \left(\Pi_{\lambda} \Psi_{\lambda}\right) + \mathcal{P}^{\mu \nu} \Psi_{\nu} + \mathcal{G}^{\mu} \quad \text{in } [0, T] \times \partial \Omega_0,
$$

(7.2)

$$
\bar{\partial}_{\alpha} \psi + \alpha \bar{\theta}_{\alpha}^{0} = 0 \quad \text{in } [0, T] \times \Omega_0,
$$

(7.3)

$$
\bar{\partial}_{\alpha} \psi = \psi + \beta \bar{\theta}_{\alpha}^{0} \quad \text{in } [0, T] \times \Omega_0,
$$

(7.4)

$$
\bar{\partial}_{\alpha} \psi = \lambda \Psi + \beta \psi \quad \text{in } [0, T] \times \Omega_0.
$$

(7.5)
where we are now employing matrix notation\(^{10}\) and have set
\[
\gamma^\alpha{}_{\beta\mu} = (\gamma^\alpha{}_{\beta\mu}), \quad \varphi = (\varphi^{\mu\nu}), \quad \alpha = (\alpha^{\mu\nu}), \quad \beta = (\beta^{\mu\nu}),
\]
\[
\mathcal{D} = (\mathcal{D}^{\mu\nu}), \quad \Pi = (\Pi^{\mu}), \quad \mathcal{F} = (\mathcal{F}^{\alpha}), \quad \mathcal{G} = (\mathcal{G}^{\mu}).
\]
Thus, the acoustic boundary conditions that was developed in Section 7 of \([41]\). However, at the moment, the system
\[
\text{is computed in terms of the frozen fields, which we will also denote by}
\]
\[
\hat{\Theta}, \hat{\varphi}, \hat{\phi}, \psi, \hat{\psi}.
\]
Here, \{\phi, \psi, \hat{\phi}^0, \Psi\} denote the unknown variable fields to be determined while the coefficients
\[
\{\lambda, \mathcal{D}^\alpha{}_{\beta}, \mathcal{F}^\gamma, \mathcal{D}, \varphi, \alpha, \beta, \Pi, \mathcal{F}, \mathcal{G}, \mathcal{D}\}
\]
are computed in terms of the frozen fields, which we will also denote by \{\hat{\phi}, \hat{\psi}, \hat{\phi}^0, \hat{\Psi}\}. This should lead to no confusion since the interpretation of the fields \{\phi, \psi, \hat{\phi}^0, \hat{\Psi}\} as being variable or frozen will be clear from context, that is whether or not they appear in one of the coefficients \(\{7.6\}\). Additionally, we assume that \(\lambda\) satisfies\(^{11}\)
\[
\lambda \geq c_\lambda > 0 \quad \text{in} \quad [0, T] \times \Omega_0
\]
for some positive constant \(c_\lambda\) and we assume, as previously, that \(\epsilon \in [0, 1)\) and \(\kappa \geq 0\).

7.1. **The modified system.** Our strategy to obtain the existences of solutions with energy estimates for systems of the form \(\{7.1\}-\{7.3\}\) is to use the linear existence theory for systems of wave equations with acoustic boundary conditions that was developed in Section 7 of \([41]\). However, at the moment, the system \(\{7.1\}-\{7.3\}\) is not in the required form to apply this theory. So instead, we will consider a modified system and obtain solutions to that system first, and subsequently show these solutions determine solutions to the original system \(\{7.1\}-\{7.3\}\). As will be clear from Proposition 7.4 below, the modified system is obtained by performing an orthogonal transformation \(\Psi \mapsto E \Psi\) followed by differentiating the evolution equation \(\{7.1\}\) and boundary conditions \(\{7.2\}\) in time, i.e. with respect to \(\xi^0\).

In order to define the modified system, we first introduced an orthogonal transformation \(E\) via the following orthonormal change of basis. Letting
\[
\Xi = (\Xi^\nu) \quad \text{and} \quad \hat{\Xi} = (\hat{\Xi}^\nu)
\]
denote frozen versions of the Lagrangian representation of the vector fields \(v^\nu\) and \(\hat{V}^\nu\), respectively, it follows from the definitions \(\{3.4\}, \{3.25\}, \{3.40\}\) that
\[
\mathcal{M}(v, \nu) = \mathcal{M}(\hat{V}, \hat{\nu}) = 1 \quad \text{and} \quad \mathcal{M}(\hat{V}, \nu) = 0
\]
where \(\mathcal{M}_{\alpha\beta}\) is the inverse of the frozen version of the Lagrangian representation of the positive definite metric \(\mathcal{M}^\alpha{}_{\beta}\) defined by \(\{3.25\}\) and we are using the notation \(\mathcal{M}(X, Y) = \mathcal{M}_{\alpha\beta}X^\alpha Y^\beta\). Setting
\[
E^\mu_0 = \tilde{v}^\mu \quad \text{and} \quad E^\mu_3 = \tilde{\Xi}^\mu,
\]
we can use the Gram-Schmidt algorithm to complete \(\{E^\mu_0, E^\mu_3\}\) to an orthonormal basis \(\{E^\mu_0, E^\mu_1, E^\mu_2, E^\mu_3\}\) with respect to \(\mathcal{M}\), that is
\[
\mathcal{M}(E_i, E_j) = \mathcal{M}_{\alpha\beta}E^\alpha_i E^\beta_j = \delta_{ij}.
\]
Next, we let \(\{\Theta^i_\mu, \Theta^1_\nu, \Theta^2_\mu, \Theta^3_\mu\}\) denote the dual basis, which satisfies
\[
\Theta^i_\mu E^\mu_i = \delta^i_j
\]
and
\[
\mathcal{M}(\Theta^i, \Theta^j) = \mathcal{M}_{\alpha\beta} \Theta^i_\alpha \Theta^j_\beta = \delta^{ij}.
\]
\(^{10}\)Here, we are using the following conventions for matrix notation: if \(L = (L^{\mu\nu})\), \(M = (M^\mu_\nu)\) and \(N = (N^\nu_\mu)\) are matrices and \(Y = (Y^\mu_\nu)\) and \(X = (X^\nu)\) are vectors, then the various matrix products are defined as follows:
\[
LY = (L^{\mu\nu}Y_\nu), \quad MY = (M^\mu_\nu Y_\nu), \quad LM = (L^{\mu\nu}M^\nu_\lambda) \quad \text{and} \quad MN = (M^\mu_\nu N^\nu_\mu).
\]
With these conventions, we have that
\[
M^{\mu\nu} L = (M^\mu_\nu L^{\nu\rho}) \quad \text{and} \quad M^{\mu\nu} X = (M^\mu_\nu X^\rho) \quad \text{and} \quad M^{\mu\nu} N^{\nu\rho} = (M^\mu_\nu N^\nu_\rho).
\]
\(^{11}\)The assumption \(\lambda = |\psi|^m > 0\) is equivalent to \(|\nabla_x \hat{\phi}^0|^m > 0\). As discussed in Remark 4.2 of \([41]\), \(|\nabla_x \hat{\phi}^0|^m\) is automatically bounded away from 0 on the vacuum boundary \([0, T] \times \partial \Omega_0\) due to the Taylor sign condition being satisfied. Thus, \(|\nabla_x \hat{\phi}^0|^m\) is bounded away from zero in a neighborhood of \([0, T] \times \partial \Omega_0\). Away from the boundary, we can, via the finite propagation speed, obtain solutions to the relativistic Euler equations using the theory of symmetric hyperbolic systems, and so, it is enough to consider the problem in a neighborhood of the boundary. Consequently, we lose no generality by assuming \(|\nabla_x \hat{\phi}^0|^m > 0\) on \([0, T] \times \partial \Omega_0\).
We then define the matrices
\[ E = (E_i^µ) \quad \text{and} \quad \Theta = (\Theta_µ^i), \]
which, by (7.9), are inverses of each other, that is
\[ \Theta E = 1, \quad (7.11) \]
where 1 is the 4 × 4 identity matrix.

Remark 7.1. It follows from the Gram-Schmidt process that \( E \) and \( \Theta \) can be viewed as matrix valued maps that depend analytically on the vectors \( \mathbf{v} \) and \( \mathbf{N} \). This allows us to write
\[ E = E(\mathbf{v}, \mathbf{N}) \quad \text{and} \quad \Theta = \Theta(\mathbf{v}, \mathbf{N}) \]
for some maps \( E(X, Y) \) and \( \Theta(X, Y) \) that are analytic in their arguments.

In the following, we will use the definitions:
\[ m = (m_αβ), \quad \Pi = (Π_µ^ν), \quad \tilde{Π} = (\tilde{Π}_µ^ν), \quad p = (p_µ^ν) \quad \text{and} \quad \mathbf{v} \otimes \mathbf{v} = (v_µ v_ν). \]

We further note that
\[ (Π_µ^ν) = m Π_µ^ν \quad \text{and} \quad (\tilde{Π}_µ^ν) = m \tilde{Π}_µ^ν \quad (7.12) \]
by (3.52) and (3.54).

Lemma 7.2. The following relations hold:
\[ \Theta^ν m Θ = 1, \quad (7.13) \]
\[ E Π Θ = P, \quad (7.14) \]
\[ E \tilde{Π} Θ = \tilde{P}, \quad (7.15) \]
\[ E p Π Θ = P_0, \quad (7.16) \]
and
\[ \Theta^ν v \otimes \mathbf{v} Θ = P_0 \quad (7.17) \]
where
\[ P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad P_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. (7.18) \]

Proof. First, we note that (7.13) is easily seen to be equivalent to (7.10). Next, from the definitions (3.4), (3.25) and (3.40), it is clear that
\[ \mathbf{v}_µ = -m_µν \mathbf{v}^ν \quad \text{and} \quad \tilde{N}_µ = m_µν \tilde{N}^ν, \]
from which the relations
\[ \Theta^0_µ = -\mathbf{v}_µ \quad \text{and} \quad \Theta^3_µ = \tilde{N}_µ \quad (7.19) \]
follow by (7.11)-(7.9). Using (7.8) and (7.19), we see from (3.41), (3.42) and (3.52) that we can write the projection operators \( p_µ^ν, \Pi_µ^ν \) and \( \tilde{Π}_µ^ν \) as
\[ p_µ^ν = δ_µ^ν - E_0^ν Θ_µ^0; \quad (7.20) \]
\[ \Pi_µ^ν = δ_µ^ν - E_0^ν Θ_µ^0 - E_0^3 Θ_µ^3 \]
and
\[ \tilde{Π}_µ^ν = E_0^ν Θ_µ^0 + E_0^3 Θ_µ^3, \quad (7.21) \]
respectively. We then get
\[ E \Pi Θ = (E_i^µ) \Pi_µ^ν Θ_ν^i = (δ_µ^i - δ_0^i δ_0^0 - δ_3^i δ_3^0) \]
from (7.21), which establishes (7.14). Formulas (7.15) and (7.16) follow from (7.20) and (7.22) via a similar calculation. To complete the proof, we observe that
\[ \Theta^ν v \otimes \mathbf{v} Θ = (Θ_µ^ν v_µ Θ_ν^i) = (Θ_µ^ν E_0^ν Θ_µ^0) = (δ_0^ν δ_0^0), \]
which establishes the final formula (7.17). \( \square \)
Lemma 7.3. The following relations hold:
\[ \Theta^\alpha {}_r \Phi^r = q \mathbb{P}, \]  
\[ \Theta^\alpha {}_r \Phi^r = \frac{\epsilon}{1 - \epsilon} \Phi \mathbb{P} + \epsilon \Phi_0 \mathbb{P} + \kappa \Phi_0, \]  
(7.23)  
(7.24)
where \( q, \phi \) and \( \Phi \) are computed in terms of the frozen fields by the formulas
\[ q = -\frac{\epsilon}{1 - \epsilon} \text{det}(\hat{\beta}_j^\gamma) \sqrt{\hat{\Sigma}_\gamma^0 g_{\mu\nu}}, \]  
\[ \phi = -\text{det}(\hat{\beta}_j^\gamma) \psi|_\Sigma, \]  
(7.25)  
(7.26)
and
\[ \Phi^\alpha = \text{det}(\hat{\beta}_j^\gamma) \left( \Theta^\alpha_v \left[ -\hat{\partial}_h (|N| \tilde{\Pi}^\mu_{\nu\gamma}) + |N| g_{\mu\nu} \tilde{\gamma} \tilde{\gamma} \tilde{\partial}_\alpha \tilde{\Pi}^\mu_{\nu\gamma} \right] \Theta^\nu_\delta \right), \]  
(7.27)
respectively.

Proof. From the definition (6.29) of \( \mathbb{Q}^{\mu\nu} \) and (7.12), we see that
\[ \mathbb{Q} = q \mathbb{P} \mathbb{P}, \]  
where \( q \) is defined as above by (7.25). Multiplying this expression of the left and right by \( \Theta^\alpha \) and \( \Theta \), respectively, we find, with the help of (7.11) and (7.13)-(7.14), that
\[ \Theta^\alpha \mathbb{Q} = q \Theta^\alpha \mathbb{Q} \mathbb{Q} = q \mathbb{P}, \]  
(7.28)
which establishes the first formula (7.23). To establish the second formula (7.24), we first observe from (7.25), (7.26) and (7.11) that
\[ \Phi = \left( \frac{\epsilon}{1 - \epsilon} \mathbb{P}^\alpha \mathbb{P} + \epsilon \Phi_0 \mathbb{P} + \kappa \Phi_0 \right) \otimes \psi, \]  
(7.29)
Multiplying this expression of the left and right by \( \Theta^\alpha \) and \( \Theta \), respectively, we get
\[ \Theta^\alpha \Phi \Theta = \frac{\epsilon}{1 - \epsilon} \Theta^\alpha \mathbb{P}^\alpha \mathbb{P} + \epsilon \Phi_0 \mathbb{P} + \kappa \Phi_0 \]  
(7.30)
where in deriving this expression we used (7.14) and (7.16) and (7.17), and the proof is complete. \( \square \)

With the preliminaries out of the way, we are now ready to define the modified system as follows:
\[ \tilde{\partial}_\beta (\Theta^\alpha \mathbb{Q}^\alpha \tilde{\Theta} \tilde{\Theta}^\gamma \tilde{\Psi} + \mathbb{Q}^\alpha) = \mathcal{H} \]  
\[ \tilde{\partial}_0 \phi = \mu \tilde{\theta}^0 \]  
\[ \tilde{\partial}_0 \tilde{\theta}^0 = \psi + \beta \tilde{\theta}^0 \]  
\[ \tilde{\partial}_0 \psi = \lambda \tilde{\theta}^0 + \beta \psi \]  
(7.32)
where \( \{ \mu, \beta, \psi, \tilde{\psi} \} \) are the unknowns and
\[ \mathcal{H} = \tilde{\partial}_0 \left( \Theta^\alpha \mathcal{F} + \tilde{\partial}_0 \Theta^\alpha \tilde{\Theta} \tilde{\Psi} + \Phi \right) \]  
\[ \mathcal{R} = \tilde{\partial}_0 \Phi \mathbb{P} + \epsilon \Phi_0 \mathbb{P} + \kappa \Phi \mathbb{P} \]  
\[ \mathcal{I} = \Theta^\alpha \mathcal{F} = \tilde{\partial}_0 \left( \tilde{\partial}_0 \Phi \mathbb{P} + \Theta^\alpha \mathcal{P} \right) \tilde{\Psi} + \tilde{\partial}_0 (\Theta^\alpha \mathcal{I} \tilde{\Psi}), \]  
(7.33)  
(7.34)  
(7.35)  
(7.36)
with \( q, \phi \) and \( \Phi \), as defined above by (7.25), (7.26) and (7.27), respectively.

The relationship between solutions of (7.28)-(7.32) and (7.33)-(7.35) is clarified in the following proposition. In particular, we give the precise conditions on the initial data for solutions of (7.28)-(7.32) that are needed to generate solutions to (7.33)-(7.35).
Proposition 7.4. Suppose \( \{ \phi, \tilde{\psi}, \tilde{\Psi} \} \) is a classical solution to (7.28)-(7.32) and let \( \Psi = \Theta \tilde{\Psi} \). Then
\[
\mathcal{I} = \Theta^{tr} \left[ \tilde{\alpha} \left( \mathcal{A}^{\alpha\beta} \tilde{\partial}_\beta \psi + \mathcal{X}^{\alpha} \right) \right] - \mathcal{F}
\]
and
\[
\mathcal{J} = \Theta^{tr} \left[ \tilde{\beta} \left( \mathcal{A}^{\alpha\beta} \tilde{\partial}_\beta \psi + \mathcal{X}^{\alpha} \right) \right] - \mathcal{F}^\gamma \tilde{\partial}_\gamma \psi - 2\tilde{\partial}_0 \left( \Pi \Psi \right) - \mathcal{R} \Psi - \mathcal{Y}
\]
satisfy \( \tilde{\partial}_0 \mathcal{I} = 0 \) and \( \tilde{\partial}_0 \mathcal{J} = 0 \) in \( [0,T] \times \Omega_0 \) and \( [0,T] \times \partial \Omega_0 \), respectively. In particular, if \( \{ \phi, \tilde{\psi}, \tilde{\Psi} \} \) satisfies \( \mathcal{J}|_{x^0=0} = 0 \) and \( \mathcal{F}|_{x^0=0} \) initially, then \( \mathcal{I} \) and \( \mathcal{J} \) vanish everywhere in \( [0,T] \times \Omega_0 \) and \( [0,T] \times \partial \Omega_0 \), respectively, and \( \{ \phi, \tilde{\psi}, \tilde{\Psi}, \Psi = \Theta \tilde{\Psi} \} \) defines a solution of (7.11)-(7.35).

Proof. Writing \( \mathcal{I} \) as
\[
\mathcal{I} = \tilde{\partial}_\alpha \left( \Theta^{tr} \mathcal{A}^{\alpha\beta} \tilde{\partial}_\beta \psi + \Theta^{tr} \mathcal{X}^{\alpha} \right) - \left( \mathcal{F} + \tilde{\partial}_\alpha \Theta^{tr} \mathcal{A}^{\alpha\beta} \tilde{\partial}_\beta \psi + \mathcal{X}^{\alpha} \right)
\]
and then differentiating with respect to \( x^0 \) gives
\[
\tilde{\partial}_0 \mathcal{I} = \tilde{\partial}_\alpha \left( \Theta^{tr} \mathcal{A}^{\alpha\beta} \tilde{\partial}_\beta \psi + \partial_0 \left( \Theta^{tr} \mathcal{A}^{\alpha\beta} \right) \tilde{\partial}_\beta \psi + \partial_0 \left( \Theta^{tr} \mathcal{X}^{\alpha} \right) \right) - \tilde{\partial}_\alpha \left( \mathcal{F} + \partial_0 \Theta^{tr} \mathcal{A}^{\alpha\beta} \tilde{\partial}_\beta \psi + \mathcal{X}^{\alpha} \right).
\]
Using the evolution equation (7.32) to replace \( \tilde{\partial}_0 \psi \) with \( \lambda \Theta \tilde{\Psi} + \beta \psi \) in (7.37), we see, with the help of the identity
\[
E \tilde{\partial}_\beta \Theta = - \tilde{\partial}_\beta E \Theta
\]
that follows from differentiating (7.11), that we can write (7.37) as
\[
\tilde{\partial}_0 \mathcal{I} = \tilde{\partial}_\alpha \left( \Theta^{tr} \mathcal{A}^{\alpha\beta} \tilde{\partial}_\beta \psi + \mathcal{X}^{\alpha} \right) - \mathcal{H},
\]
where \( \mathcal{X}^{\alpha} \) and \( \mathcal{H} \) are as defined above by (7.33) and (7.34), respectively. But the right hand side of this expression vanishes by virtue of the evolution equation (7.28) showing that \( \tilde{\partial}_0 \mathcal{I} = 0 \) in \( [0,T] \times \Omega_0 \).

Next, differentiating \( \mathcal{J} \) with respect to \( x^0 \), we find, with the help of Lemmas (7.2) and (7.3) and the time independence of \( \tilde{\partial}_0 \), i.e., \( \tilde{\partial}_0 \tilde{\partial}_0 = 0 \), that
\[
\tilde{\partial}_0 \mathcal{J} = \tilde{\partial}_\alpha \left( \Theta^{tr} \mathcal{A}^{\alpha\beta} \tilde{\partial}_\beta \partial_0 \psi + \partial_0 \left( \Theta^{tr} \mathcal{A}^{\alpha\beta} \right) \tilde{\partial}_\beta \psi + \partial_0 \left( \Theta^{tr} \mathcal{X}^{\alpha} \right) \right) - \tilde{\partial}_\alpha \left( \mathcal{F} + \partial_0 \Theta^{tr} \mathcal{A}^{\alpha\beta} \tilde{\partial}_\beta \psi + \mathcal{X}^{\alpha} \right).
\]
Using again the evolution equation (7.32) to replace \( \tilde{\partial}_0 \psi \) with \( \lambda \Theta \tilde{\Psi} + \beta \psi \) in (7.39), a straightforward calculation using Lemma (7.3) and the identity (7.38) shows that
\[
\tilde{\partial}_0 \mathcal{J} = \tilde{\partial}_\alpha \left( \Theta^{tr} \mathcal{A}^{\alpha\beta} \tilde{\partial}_\beta \left( \Theta \tilde{\Psi} + \mathcal{X}^{\alpha} \right) \right) - \Theta^{tr} \tilde{\partial}_\gamma \mathcal{R} \tilde{\partial}_\gamma \psi - \mathcal{R} \tilde{\partial}_0 \psi - \mathcal{J},
\]
which in turn, implies via the evolution equation (7.29) that \( \tilde{\partial}_0 \mathcal{J} = 0 \) in \( [0,T] \times \partial \Omega_0 \). \( \square \)

In light of the above proposition, establishing the existence of solutions to the original linear system (7.1)-(7.5) now becomes the problem of establishing the existence of solutions to the modified system (7.28)-(7.32). Although we will not consider this in detail here, we remark that the existence of solutions to (7.28)-(7.32) is a direct consequence of the following: (a) Theorem 7.16 from [41], (b) the coercive estimates (6.39), which as discussed above follow from a slight modification of Lemma 8.3 from [41], and (c) the following lemma, which guarantees that the matrices \( q \mathcal{P} \) and \( \mathcal{R} \) satisfy a condition needed to apply Theorem 7.16 from [41].

Lemma 7.5. Suppose \( \epsilon \in (0,1), s \in \mathbb{R}, \) and \( a \leq -c_a < 0, p \leq -c_p < 0, \) and \( |\tilde{\partial}_0 q| + \| q \mathcal{E} \tilde{\partial}_0 \Theta \|_{op} + \| \mathcal{P} \|_{op} \leq c_s \) in \( [0,T] \times \partial \Omega_0 \) for some positive constants \( c_a, c_p \) and \( c_s \), and let
\[
\kappa = \frac{1}{2c_p}, \quad \text{and} \quad C = \left( |s| + \frac{a + \epsilon}{2} \right) \frac{c_s}{c_a}.
\]
Then
\[
\mathcal{R} + s \tilde{\partial}_0 (q \mathcal{P}) + Cq \mathcal{P} \leq 0 \quad \text{in} \quad [0,T] \times \partial \Omega_0.
\]

\footnote{Here, we are using the following matrix notation: given vectors \( X, Y \in \mathbb{R}^4 \), we let \( \langle X | Y \rangle = X^{tr} Y \) and \( |X| = \sqrt{\langle X | X \rangle} \) denote the Euclidean inner-product and norm, respectively. We also use the notation \( \| A \|_{op} = \sup_{X \in \mathbb{R}^4, Y \neq 0} \frac{\| AX \|}{\| Y \|} \) for the operator norm of a matrix \( A \in \mathbb{M}_{4 \times 4} \), and given two matrices \( A, B \in \mathbb{M}_{4 \times 4} \), we write \( A \leq B \) if and only if \( \langle X | AX \rangle \leq \langle X | BX \rangle \) for all \( X \in \mathbb{R}^4 \).}
Proof. From the definition \((7.35)\) of \(\mathcal{R}\) and the assumptions \(q \leq -c_q < 0\) and \(p \leq -c_p < 0\), we see for \(X \in \mathbb{R}^4\), \({C, \kappa}\) \(\subset \mathbb{R}_{\geq 0}\), \(\epsilon \in (0, 1)\) and \(s \in \mathbb{R}\) that

\[
(X|\mathcal{R}|X) + s(X|\tilde{\partial}_0(q\mathcal{P})X) + C(X|q\mathcal{P}X) = (Cq + (s + 1)\tilde{\partial}_0q)|\mathcal{P}X|^2 + (\mathcal{P}X|qE\tilde{\partial}_0\Theta\mathcal{P}X)
+ \frac{\epsilon}{1 - \epsilon}(\mathcal{P}X|\mathcal{P}_s\mathcal{P}X) + \epsilon(\mathcal{P}_0X|\mathcal{P}_s\mathcal{P}X) + \kappa|\mathcal{P}_0X|^2
\]

from which the inequality

\[
(X|\mathcal{R}|X) + s(X|\tilde{\partial}_0(q\mathcal{P})X) + C(X|q\mathcal{P}X) \leq \left(-Cq + \left(|s| + 2 + \frac{\epsilon}{1 - \epsilon}\right)c_s\right)|\mathcal{P}X|^2 + c_s|\mathcal{P}_0X|^2 + \left(-\kappa c_p + \frac{1}{2}\right)|\mathcal{P}_0X|^2
\]

follows from the Cauchy-Schwarz inequality and the inequalities \(q \leq -c_q < 0\), \(p \leq -c_p < 0\) and \(\tilde{\partial}_0q + |qE\tilde{\partial}_0\Theta|_{op} + |\mathcal{P}_s|_{op} \leq c_s\). Choosing \(\kappa = \frac{1}{\epsilon c_p}\) and \(C = \left(|s| + \frac{2}{\epsilon} + \frac{\epsilon}{1 - \epsilon}\right)\frac{\epsilon}{c_q}\) then yields the desired inequality \((X|\mathcal{R}|X) + s(X|\tilde{\partial}_0(q\mathcal{P})X) + C(X|q\mathcal{P}X) \leq 0\). \(\square\)

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**Appendix A. Differential geometry formulas**

In this appendix, we collect together some useful formulas from differential geometry that will be used throughout this article. In the following, we let

\[g = g_{\mu\nu}dx^\mu dx^\nu\]

denote a smooth Lorentzian metric on a four dimensional manifold \(M\), and we use \(\nabla\) to denote the Levi-Civita connection of this metric. We also use the indexing conventions from Section 2.1. Given a local frame

\[e_j = e^\mu_j \partial_\mu\]

on \(M\), we denote the dual frame by

\[\theta^j = \theta^j_\mu dx^\mu \quad ((\theta^j_\mu) := (e^\mu_j)^{-1}),\]

and use the notation

\[\gamma_{ij} = g(e_i, e_j) = g_{\mu\nu}e^\mu_i e^\nu_j \quad \text{and} \quad \gamma^{ij} = g(\theta^i, \theta^j) = g^{\mu\nu}\theta^\mu_i \theta^\nu_j\]

for the frame metric and its inverse, respectively.

**A.1. Lie and exterior derivatives**. Given vector fields \(X, Y, a\) scalar field \(f\), a \(q\)-form \(\alpha\), and a \(p\)-form \(\beta\), the following identities hold:

\[\mathcal{L}_X \alpha = i_X d\alpha + d i_X \alpha,\]  
\[d\mathcal{L}_X \alpha = L_X d\alpha,\]  
\[i_Y L_X \alpha = L_X i_Y \alpha + i_{[Y,X]} \alpha,\]  
\[L_X (\alpha \wedge \beta) = L_X \alpha \wedge \beta + \alpha \wedge L_X \beta\]

and

\[\mathcal{L}_f X = f L_X \alpha + d f \wedge i_X \alpha.\]

Expressing \(\alpha\) locally as

\[\alpha = \frac{1}{q!} \alpha_{\mu_1\mu_2...\mu_q} dx^{\mu_1} \wedge dx^{\mu_2} \wedge ... \wedge dx^{\mu_q},\]

the exterior derivative of \(\alpha\) can be computed using the formula

\[(d \alpha)_{\mu_1...\mu_q+1} = (q + 1) \partial_{[\mu_1} \alpha_{\mu_2...\mu_{q+1}]},\]

Furthermore, given the local expression

\[X = X^\mu \partial_\mu,\]
the Lie derivative $L_X$ of $\alpha$ can be computed using
\[(L_X \alpha)_{\mu_1 \mu_2 \ldots \mu_q} = X^\nu \partial_\nu \alpha_{\mu_1 \mu_2 \ldots \mu_q} + \alpha_{\nu \mu_2 \ldots \mu_q} \partial_\mu X^\nu + \alpha_{\mu_1 \nu \mu_3 \ldots \mu_q} \partial_{\mu_2} X^\nu + \cdots + \alpha_{\mu_1 \mu_2 \ldots \mu_q \nu} \partial_{\nu} X^\nu. \quad (A.9)\]
For functions, we employ the alternate notation
\[X(f) = L_X(f) = X^\mu \partial_\mu f\]
for the Lie derivative, and more generally, we use this notation locally on coordinate components of tensors, e.g. $X(Y^\nu) = X^\mu \partial_\mu Y^\nu$.

A.2. **Volume form.** We use $\nu$ to denote the volume form of the metric $g$, which is given locally by
\[\nu = \frac{1}{4!} \nu_{\mu\alpha\beta\gamma} dx^\mu \wedge dx^\alpha \wedge dx^\beta \wedge dx^\gamma,\]
where the components are computed using
\[\nu_{\mu\alpha\beta\gamma} = \sqrt{|g|} \epsilon_{\mu\alpha\beta\gamma}.\]
Here, $\epsilon_{\mu\alpha\beta\gamma}$ denotes the completely anti-symmetric symbol and we employ the standard notation
\[|g| = -\det(g_{\mu\nu})\]
for the negative of the determinant of the metric $g_{\mu\nu}$. The volume form is also given locally in terms of the coframe by
\[\nu = \sqrt{|\gamma|} \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3, \quad (A.10)\]
where
\[|\gamma| = -\det(\gamma_{kl}).\]

A.3. **Hodge star operator.** The Hodge star operator $^*_g$ associated to $g$ satisfies
\[\alpha \wedge ^*_g \beta = g(\alpha, \beta) \nu \quad (A.11)\]
for any 1-forms $\alpha$, $\beta$, where $\nu$, as above, is the volume form of $g$. From (A.10), (A.11) and the identity
\[^*_g \ast_g \alpha = (-1)^{(d-q)+1} \alpha\]
for $q$-forms, we obtain
\[g(\theta^i, (\theta^1 \wedge \theta^2 \wedge \theta^3)) \nu_g = \theta^i \wedge \theta^1 \wedge \theta^2 \wedge \theta^3, \]
from which it follows that
\[g(\theta^i, (\theta^1 \wedge \theta^2 \wedge \theta^3)) = 0, \quad I \in \{1, 2, 3\}. \quad (A.12)\]

A.4. **Codifferential.** The codifferential $\delta_g$ associated to $g$ is given by the formula
\[\delta_g \alpha = (-1)^{(d-q-1)} \ast g \, d \ast_g \alpha \quad (A.13)\]
when acting on $q$-forms, and it can be computed locally via the formula
\[(\delta_g \alpha)_{\mu_2 \ldots \mu_q} = -\nabla^\mu \alpha_{\mu_1 \mu_2 \ldots \mu_q} = -\frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} g^{\mu\nu} \alpha_{\mu_1 \mu_2 \ldots \mu_q}). \quad (A.14)\]
The codifferential satisfies the identity
\[\delta_g^2 = 0. \quad (A.15)\]

A.5. **Cartan structure equations.** The connection coefficients $\omega^k_{\mu j}$ of the metric $g$ with respect to the frame $\{e_i\}$ are defined by
\[\nabla_{e_i} e_j = \omega^k_{\mu j} e_k, \]
or equivalently
\[\nabla_{e_i} \theta^k = -\omega^k_{\mu j} \theta^j. \quad (A.16)\]
Defining the connection one forms $\omega^k_{\mu j}$ by
\[\omega^k_{\mu j} = \omega^k_{\mu j} \theta^i, \]
the **Cartan structure equations** are given by
\[d \theta^i = -\omega^i_{\mu j} \wedge \theta^j, \quad (A.17)\]
\[d \gamma_{ij} = \omega_{ij} + \omega_{ji}, \quad (A.18)\]
where
\[\omega_{ij} = \gamma_{ijk} \omega^k_{\mu j}.\]
A.6. Curvature. We use
\[ \nabla_\mu \nabla_\nu \lambda_\gamma - \nabla_\nu \nabla_\mu \lambda_\gamma = R_{\mu \nu \gamma}^\sigma \lambda_\sigma, \]
(A.19)
to define the curvature tensor \( R_{\mu \nu \gamma}^\sigma \) of the metric \( g \), and we define the Ricci tensor \( R_{\mu \gamma} \) by
\[ R_{\mu \gamma} = R_{\mu \nu \gamma}^\nu. \]

A.7. Covariant derivatives and changes of metrics. Letting \( \bar{\nabla} \) denote the Levi-Civita connection of another Lorentian metric \( g \) needed in the proof of Theorem 4.1.

\[ \hat{\nabla} = \hat{\nabla}_{\mu \nu} dx^\mu dx^\nu \]
on \( M \), the covariant derivatives with respect to the metrics \( g \) and \( \hat{g} \) are related via the formula
\[ \bar{\nabla}_\gamma T^\mu_{\nu_1 \ldots \nu_s} = \nabla_\gamma T^\mu_{\nu_1 \ldots \nu_s} + C^\mu_{\lambda \nu_1 \ldots \nu_s} + \ldots + C^\mu_{\nu_1 \ldots \nu_s - 1 \lambda} - C^\lambda_{\nu_1 \ldots \nu_s} T^\mu_{\lambda \nu_1 \ldots \nu_s - 1}, \]
(A.20)
where
\[ C_{\alpha \beta} = \frac{1}{2} g^{\lambda \gamma} (\nabla_\alpha \hat{g}_{\beta \gamma} + \nabla_\beta \hat{g}_{\alpha \gamma} - \nabla_\gamma \hat{g}_{\alpha \beta}). \]

Appendix B. Maxwell’s equations

As in the introduction, let \( \Omega_0 \subset M \) be a bounded, connected spacelike hypersurface with smooth boundary \( \partial \Omega_0 \), and \( \Omega_T \) be a timelike cylinder diffeomorphic to \( [0, T] \times \Omega_0 \). We use \( \Gamma_T \) to denote the timelike boundary of \( \Omega_T \), which is diffeomorphic to \( [0, T] \times \partial \Omega_0 \), and note that \( \Gamma_0 = \partial \Omega_0 \). We denote the outward unit conormal to \( \Gamma_T \) by \( n = n_\nu dx^\nu \), which we arbitrarily extend to all of \( M \), and we let \( \Omega^T \cong \{ T \} \times \Omega_0 \) and \( \Gamma^T = \partial \Omega^T \cong \{ T \} \times \partial \Omega_0 \) denote the “top” of the spacetime cylinder and its boundary, respectively. We further assume that \( \tau = \tau^\alpha \partial_\alpha \) and \( \xi = \xi^\mu \partial_\mu \) are timelike, future pointing \( C^1 \) vector fields on \( \Omega_T^T \) that satisfy \( \tau(x), \xi(x) \in T_x \Gamma_T \) for all \( x \in \Gamma_T \), \( \tau \) is normal to \( \Omega_0 \) and \( \Omega^T \), and \( \tau \) is unit length.

Maxwell’s equations on the world tube \( \Omega_T \) are given by
\[ \begin{align*}
\delta_\xi F &= 0 \quad \text{in } \Omega_T, \\
\epsilon_\nu = 0 &= 0 \quad \text{in } \Omega_T, 
\end{align*} \]
(B.1)
where \( F = \frac{1}{2} F_{\mu \nu} dx^\mu \wedge dx^\nu \) is the electromagnetic tensor. We recall that the stress-energy tensor \( T^{\mu \nu} \) of the electromagnetic field is defined by
\[ T^{\mu \nu} = 2 F^{\mu \alpha} F^{\nu \alpha} - \frac{1}{2} g^{\mu \nu} F_{\alpha \beta} F^{\alpha \beta}. \]
(B.3)
For solutions to Maxwell’s equations, \( T^{\mu \nu} \) satisfies
\[ \nabla_\mu T^{\mu \nu} = 0 \quad \text{in } \Omega_T. \]
(B.4)
Integrating this expression over \( \Omega_T \) leads to the following well-known integral relation, which will be needed in the proof of Theorem 4.1.

Lemma B.1. Suppose \( F \in C^1(\Omega_T) \) solves (B.1)-(B.2). Then
\[ \int_{\Omega_T} \tau^{\mu \nu} F_{\mu \nu} = \int_{\Omega_0} \tau^{\mu \nu} F_{\mu \nu} \xi_\mu + 2 \int_{\Gamma_T} n_\nu F^{\mu \alpha} F^{\nu \alpha} \xi_\mu - \frac{1}{2} \int_{\Omega_T} T^{\mu \nu} L_\xi g_{\mu \nu}. \]
Proof. Since any solution \( F \in C^1(\Omega_T) \) of (B.1)-(B.2) satisfies (B.3) in \( \Omega_T \), we have that
\[ \nabla_\mu (T^{\mu \nu} \xi_\nu) = T^{\mu \nu} \nabla_\mu \xi_\nu = \frac{1}{2} T^{\mu \nu} L_\xi g_{\mu \nu} \quad \text{in } \Omega_T. \]
Integrating this expression over \( \Omega_T \), we find using the Divergence Theorem that
\[ \int_{\Omega_T} \tau^{\mu \nu} F_{\mu \nu} = \int_{\Omega_0} \tau^{\mu \nu} F_{\mu \nu} \xi_\mu + 2 \int_{\Gamma_T} n_\nu F^{\mu \alpha} F^{\nu \alpha} \xi_\mu - \frac{1}{2} \int_{\Omega_T} T^{\mu \nu} L_\xi g_{\mu \nu}, \]
where in deriving this we have used the fact that \( n_\mu \xi_\mu = 0 \) in \( \Gamma_T \).

In the proof of Theorem 4.1, we will also need the inequality from the following lemma, which is used in literature to show that the electromagnetic stress-energy tensor satisfies the Dominant Energy Condition. Before stating the lemma, we first denote the unit-normalized version of \( \xi_\mu \) by \( v_\mu = (-g(\xi, \xi))^{-1/2} \xi_\mu \) and we let \( h_{\mu \nu} = g_{\mu \nu} + v_\nu v_\mu \) denote the induced positive definite metric on the subspace \( g \)-orthogonal to \( v_\mu \). We also define a positive definite metric by \( m_{\mu \nu} = g_{\mu \nu} + 2v_\mu v_\nu \).
Lemma B.2. There exists a constant $c > 0$, independent of $F$, such that

$$v_\mu T^{\mu\nu} \tau_\nu \geq c |F^2_m| \text{ in } \Omega_T.$$  

Proof. Starting from the standard decomposition, see [17] Ch. 13,

$$F_{\mu\nu} = 2v_\mu E_\nu - v_{\mu\alpha\beta\gamma} v^\alpha B^{\beta\gamma},$$  \hspace{1cm} (B.5)

of the electromagnetic tensor in terms of the electric and magnetic fields relative to $v^\mu$, which are defined by

$$E_\mu = F_{\mu\nu} v^\nu \text{ and } B_\mu = \frac{1}{2} v_{\mu\alpha\beta\gamma} v^\alpha F^{\beta\gamma},$$  \hspace{1cm} (B.6)

respectively, a straightforward calculation, see [17] Ch. 15, shows that $F^{\mu\nu} F_{\mu\nu} = 2(B_\mu B^\mu - E_\mu E^\mu)$. Using

$$E_\mu v^\mu = B_\mu v^\mu = 0,$$  \hspace{1cm} (B.7)

we can then write $F^{\mu\nu} F_{\mu\nu}$ as $F^{\mu\nu} F_{\mu\nu} = 2(|B|^2_m - |E|^2_m)$. From this and (B.7), it is then not difficult to verify that

$$|F|^2_m = m^{\alpha\beta} m^{\mu\nu} F_{\alpha\mu} F_{\beta\nu} = 2(|B|^2_m + |E|^2_m).$$  \hspace{1cm} (B.8)

Next, we recall that the energy density relative to $v^\mu$ is given by, see [17] Ch. 15, the formula

$$v_\mu T^{\mu\nu} v_\nu = |E|^2_m + |B|^2_m.$$  \hspace{1cm} (B.9)

Using this, we compute

$$v_\mu T^{\mu\nu} \tau_\nu = (-\tau_\lambda v^\lambda) v_\mu T^{\mu\nu} \tau_\nu + v_\mu T^{\mu\nu} h^\gamma_\nu \tau_\gamma = (-\tau_\lambda v^\lambda)(|E|^2_m + |B|^2_m) + 2E^\alpha F^{\alpha\beta} h^\beta_\nu \tau_\lambda \qquad \text{(by (B.3), (B.6) & (B.9))}$$

$$= (-\tau_\lambda v^\lambda)(|E|^2_m + |B|^2_m) - 2E^\alpha B^\nu \nu_{\mu\alpha} F^{\beta\gamma} h^\beta_\nu h^\gamma_\lambda \tau_\lambda,$$  \hspace{1cm} (B.10)

where $\nu_{\mu\alpha} = \nu_{\mu\alpha\beta\gamma} v^\beta$ and in deriving the last equality we used (B.5). This result together with the inequality

$$|E^\mu B^\nu \nu_{\mu\alpha} h^\alpha_\beta \tau_\alpha| \leq |E|^2_m |B| |h| \tau_h = |E|^2_m |B|^2_m |\tau|_h \leq \frac{1}{2} (|E|^2_m + |B|^2_m) |\tau|_h$$

implies

$$v_\mu T^{\mu\nu} \tau_\nu \geq \left(-\tau_\lambda v^\lambda \right)(|E|^2_m + |B|^2_m) + \frac{(-\tau_\lambda v^\lambda - |\tau|_h)(|E|^2_m + |B|^2_m)}{2}.$$  \hspace{1cm} (B.11)

Furthermore, since $v^\mu$ and $\tau^\mu$ are both future pointing and timelike by assumption, we have that $v_\mu \tau^\mu > 0$ and $\tau_\mu \tau^\mu = |\tau|^2_h - (-\tau_\lambda v^\lambda)^2 < 0$, from which it follows that $(-\tau_\lambda v^\lambda - |\tau|_h) \geq c > 0$ in $\Omega_T$ for some positive constant $c$. The proof now follows from this inequality and (B.11). \hfill \Box

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