Age-Limited Capacity of Massive MIMO

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Abstract—We investigate the age-limited capacity of the Gaussian many channel with total $N$ users, out of which a random subset of $K_u$ users are active in any transmission period, and a large-scale antenna array at the base station (BS). In an uplink scenario, where the transmission power is fixed among the users, we consider the setting in which both the number of users, $N$, and the number of antennas at the BS, $M$, are allowed to grow large at a fixed ratio $\zeta = M/N$. Assuming perfect channel state information (CSI) at the receiver, we derive the achievable bound under maximal ratio combining. As the number of active users, $K_u$, increases, the achievable spectral efficiency is found to increase monotonically to a limit $\log_2 \left(1 + \frac{M}{K_u}\right)$. Further extensions of the analysis to the zero-forcing receiver as well as imperfect CSI are provided, demonstrating the channel estimation penalty in terms of the mean squared error in estimation. Using the age of information (AoI) metric, first coined in [1], as our measure of data timeliness or freshness, we investigate the trade-offs between the AoI and spectral efficiency in the context of massive connectivity with large-scale receiving antenna arrays. As an extension of [2], based on our large system analysis, we provide an accurate characterization of the asymptotic (finite system size) spectral efficiency as a function of the number of antennas and the number of users, the attempt probability, and the AoI. It is found that while the spectral efficiency can be made large, the penalty is an increase in the minimumAoI obtainable. The proposed achievable bound is further compared against recent massive MIMO-based massive unsourced random access (URA) schemes.

Index Terms—Massive MIMO, Age of Information, Unsourced Random Access, Packet Error Probability, Spectral Efficiency

I. INTRODUCTION

A. Background and Motivation

MASSIVE random access in which a base station equipped with a large number of antennas is serving a large number of contending users has recently attracted considerable attention. This surge of interest is fuelled by the need to satisfy the soaring demand in wireless connectivity for many envisioned IoT applications such as massive machine-type communication (mMTC). Machine-type communication (MTC) has two distinct features [3] that make them drastically different from human-type communications (HTC) around which previous cellular systems have mainly evolved: i) machine-type devices (MTDs) require sporadic access to the network and ii) MTDs usually transmit small data payloads using short-packet signaling. The sporadic access leads to the overall mMTC traffic being generated by an unknown and random subset of active MTDs (at any given transmission instant or frame). This calls for the development of scalable random access protocols that are able to accommodate a massive number of MTDs. Short-packet transmissions, however, make the traditional grant-based access (with the associated scheduling overhead) fall short in terms of spectrum efficiency and latency, which are two key performance metrics in next-generation wireless networks. Hence, a number of grant-free random access schemes have been recently investigated within the specific context of massive connectivity (see [4], [5] and references therein). From the information-theoretic point of view, the problem of massive random access is not recent and dates back to the seminal work of Gallager in [6]. However, with an increasing number of possible applications the problem has reappeared in a new context [7], [8]. As opposed to classical treatments of the Gaussian multiple access channel in which the number of users stays fixed, in the new Gaussian channel formalism the number of users is allowed to grow with the blocklength [7] in a typical massive connectivity setup. Note that when a randomly varying subset of users (with different codebooks) are active over each transmission period, one is bound to sacrifice some of the spectral efficiency for user-identification [2], [7], [9]. However, when all the devices employ the same codebook (aka, unsourced access), the user-identification problem can be separated from the decoding problem as highlighted in [8]. In fact, by letting all the devices employ the same codebook, the system spectral efficiency depends on the number of active users only and not on total number of users, thereby making different multi-user decoders comparable against each other and to the random coding bound. In particular, it was shown in [8] that increasing the number of active users at a fixed per-user payload renders known solutions such as ALOHA far from the random coding achievability bound. The paradigm in [8] where all users share the same codebook with no need for user identification was later dubbed unsourced random access and now has a number of viable algorithmic solutions. However, most of the existing information-theoretic works on massive connectivity focus on the case of a single receive antenna at the BS. Yet, the idea of using a large-scale antenna array at the BS (i.e. massive MIMO) which was first pioneered in [10], has now become one of the main directions towards which the next-generation of cellular systems are projected to evolve.

From another perspective, in many real-time applications wherein the data is subject to abrupt variations, usefulness of the information when it arrives at the BS is directly related to its freshness. Due to infrequent access to the network, conventional performance metrics, such as delay fall short in characterizing the over-all freshness of the data [11]. In this respect, the AoI concept [11] was introduced to adequately characterize the freshness of the information at the receiver side. While many of the existing works on AoI focus primarily on grant-based access with AoI-constrained scheduling policies [12], [13], some have looked at uncoordinated transmission
schemes. Recently, a few information-theoretic works have investigated the trade-off between the AoI and achievable data rates [14], [15]. The performance of AoI has been investigated in Multiple-Input Multiple-Output (MIMO) systems [16]–[19]. In [16], the user scheduling problem has been investigated to minimize AoI in a multiuser MIMO status update system where multiple single-antenna devices send their information over a common wireless uplink channel to a multiple-antenna access point. In [18], a novel MIMO broadcast setting is studied to minimize the sum average AoI through precoding and transmission scheduling. In [17], the authors analyzed and optimized the performance of AoI in a grant-free random-access system with massive MIMO.

B. Contributions

The major contributions of this paper are summarized as follows.

• We derive a closed-form expression of the outage probability in the finite-user, finite-antenna regime and through use of the central-limit theorem (CLT) we express this outage probability in the asymptotic case where both the number of users and the number of antennas are allowed to grow large at a fixed ratio.

• Under the assumption of perfect CSI at the receiver, we derive an achievable bound using a maximal ratio combining (MRC) receiver. We demonstrate how this achievable bound scales with the number of users in the finite regime (e.g. in Theorem 1 in Section IV) and further elaborate on its behaviour in the limit (e.g. through Theorems 2 and 3 in Section IV).

• We show that fully uncoordinated non-orthogonal access can achieve minimum AoI as long as all the devices are active in each transmission period. Furthermore, our analysis reveals that with a large-scale antenna array at the BS both high spectral efficiency and low AoI can be achieved.

• We further extend the analysis to the case of imperfect CSI as well as the zero-forcing receiver. We derive the asymptotic as well as limiting spectral efficiency of both the MRC as well as the zero-forcing receiver when the estimation error is added to the noise contribution.

• Finally, using our bound, we gauge the performance of recent massive MIMO unsourced random access (URA) schemes.

The work that is most closely related to the results presented in this paper is reported in [2], where the authors considered a massive connectivity with massive MIMO system for uplink data communication. In their paper, they used the state evolution framework to obtain the limiting MSE of the approximate message passing (AMP) channel estimation/activity detection algorithm. They further calculated the achievable rate (interference limited capacity) with the MRC as well as the LMMSE receiver. The limitation of their approach is that it treats only the asymptotic convergence as both the number of antennas and users are infinite while the ratio of the number of antennas to the number of users stays finite. On the other hand, the outage probability formulation together with the approximation analysis presented in this work allows for the asymptotic spectral efficiency characterization of large, yet finite, systems. The non-asymptotic point of view, Theorem 1 in the manuscript, provides the spectral efficiency as well as a precise, $O(N^{-1.5})$, correction term.

C. Organization of the Paper and Notations

We structure the rest of this paper as follows. In Section II we introduce the system model. In Section III we derive the exact packet probability of error and also find its more insightful asymptotic approximation. In Section IV we state our main results on the trade-off between achievable spectral efficiency and the AoI. In Section V we extend the analysis to the case of imperfect CSI for the MRC and ZF receivers. These results are further corroborated by computer simulations in Section VI. Finally, we draw out some concluding remarks in Section VII and prove our various claims in the Appendices.

We also mention the common notations used in this paper. Lower- and upper-case bold fonts, $\mathbf{x}$ and $\mathbf{X}$, are used to denote vectors and matrices, respectively. $\mathbf{I}_M$ denotes the $M \times M$ identity matrix. The symbols $\|\cdot\|$ and $\|\cdot\|_2$ stand for the modulus and Euclidean norm, respectively. $\{\cdot\}^H$ stands for the Hermitian (transpose conjugate) operator. The shorthand notation $\mathbf{y} \sim CN(\mathbf{m}, \mathbf{R})$ means that the random vector $\mathbf{y}$ follows a complex circular Gaussian distribution with mean $\mathbf{m}$ and auto-covariance matrix $\mathbf{R}$. Likewise, $S \sim \Gamma(k, \theta)$ means that the random variable $S$ follows a Gamma distribution with shape parameter $k$ and scale parameter $\theta$. The statistical expectation is denoted as $\mathbb{E}\{\cdot\}$, and the notation $\triangleq$ is used for definitions.

II. SYSTEM MODEL, ASSUMPTIONS, AND METHODOLOGY OF ANALYSIS

A. System Model, Assumptions, and Definition of AoI

Consider a single-cell network consisting of $N$ single-antenna devices transmitting their status packets over an unreliable multiple-access channel to a BS with $M$ receive antenna elements. To aid synchronization, time is partitioned into slots of equal length $T$, which is the maximum amount of time for transmission and reception of a single information packet. This paper assumes sporadic device activity where at the start of every time slot user $i$ transmits is current status with probability $\tau_i$. We define $\{\varepsilon_i\}_{i=1}^N$ as the binary activity random variables which indicate whether user $i$ transmits its packet or remains idle in a given slot:

$$
\varepsilon_i = \begin{cases} 
1 & \text{if user } i \text{ transmits his packet,} \\
0 & \text{if user } i \text{ remains idle.}
\end{cases}
$$

(1)

Moreover, we assume $\{\varepsilon_i\}_{i=1}^N$ are independent in each time slot with marginal distributions $\text{Pr}(\varepsilon_i = 1) = \tau_i$. To maintain timely status updates, in every slot a new packet is generated by each user. In this static macro-cell environment, where the coherence time is on the order of hundreds of milliseconds and delay spread is on the order of microseconds [20], the
channel remains fairly constant and thus we assume a quasi-
static Rayleigh fading model for the duration of the slot in
which \( \mathbf{h}_i \sim \mathcal{CN}(0, \mathbf{I}_M) \) denotes the \( M \times 1 \) channel vector between the \( i \)'th user and the BS. The received signal at the BS at discrete time \( n \) can then be written as:

\[
y_n = \sum_{i=1}^{N} \mathbf{h}_i x_{i,n} + \mathbf{w}_n,
\]

where \( x_{i,n} \sim \mathcal{CN}(0, P_i) \) is the transmitted symbol, while \( \mathbf{w}_n \sim \mathcal{CN}(0, \sigma_w^2 \mathbf{I}_M) \) is the additive white Gaussian noise (AWGN) which is assumed spatially uncorrelated across all receive antennas. In the presence of \( K_a \) active users in a given transmission slot, the above formulation is a \( K_a \)-user single-input multiple-output (SIMO) fading Gaussian multiple access channel (GMAC).

Using \( p_{e,i} \) to denote the slot-wise packet error probability (PEP) of the \( i \)'th user, the probability that the \( i \)'th user updates the BS with its current status is then given by:

\[
\gamma_i = \tau_i (1 - p_{e,i}).
\]

An example of the slotted system with \( N = 3 \) users with \( T = 1 \) is depicted in Fig. 1. Under perfect CSI at the receiver we use maximal-ratio combining (MRC) and assume that the blocklength is sufficiently large such that the capacity limit is approached within a packet, as justified in [21]. Consequently, \( p_{e,i} \) can be closely approximated by the following outage probability:

\[
1 - p_{e,i} = \Pr \left( \rho_i < \log_2 \left( 1 + \frac{\| \mathbf{h}_i \|^2 P_i}{\| \mathbf{w}_i \|^2 + \sum_{j \neq i} \| \mathbf{h}_j \|^2 x_j P_j} \right) \right),
\]

in which \( \rho_i \) [bits/channel-use] is the spectral efficiency of the \( i \)'th user. Recall that the ultimate goal of each node is to keep the BS updated with its most recent state. If the BS has node \( i \)'s state that was current at time \( t_0 \), the age of that user's state is defined by the random process \( \delta_i(t) = t - t_0 \). When node \( i \) attempts transmission and gets correctly decoded at the BS we call it an arrival. We denote \( i \)'th node's \( j \)th arrival epoch by \( t_{i,j} \).

Between two arrival epochs, the age grows as a staircase function of time and is reset to \( T \) when an arrival occurs, since this is the amount of time that it took for a packet to be transmitted. We denote by \( Z_{i,j} \) the inter-arrival time between the \( j \)th update and the \( (j+1) \)th update (i.e. \( Z_{i,j} = t_{i,j+1} - t_{i,j} \)). Assuming that the total number of users remains constant in each time slot, user \( i \) has a certain success probability, \( \gamma_i \), that it will update the BS with its state. After normalizing the slotted period to \( T = 1 \), it then follows that each \( Z_{i,j} \) is a geometric random variable with parameter \( \gamma_i \).

Similar to [22], we define the AoI, \( \Delta_i \), of each node \( i \) as:

\[
\Delta_i = \lim_{T \to \infty} \frac{1}{T} \int_0^T \delta_i(t) dt.
\]

For completeness, we show in Appendix A that the limit in (5) exists and that it converges with probability one (WP1) to

\[
\Delta_i = \frac{\mathbb{E}[Z_i^2]}{2\mathbb{E}[Z_i]} + \frac{1}{2},
\]

as was done in more general terms in [23]. Since the \( Z_i \)'s are geometric random variables, it follows that \( \mathbb{E}[Z_i] = \frac{1}{\gamma_i} \) and \( \mathbb{E}[Z_i^2] = \frac{3}{\gamma_i^2} - \frac{1}{\gamma_i} \), thereby leading to:

\[
\Delta_i = \gamma_i^{-1}.
\]

In the presence of \( N \) total users, we consider the network-wide average AoI, given by:

\[
\Delta = \frac{1}{N} \sum_{i=1}^{N} \Delta_i.
\]

B. Methodology of Analysis

Since the AoI solely depends on the parameter \( \tau_i \) and the outage probability \( p_{e,i} \), the main objective will be to find the outage probability. In the following, we will first do this in closed-form and thereafter, through use of the central limit theorem, we will find an approximation in the asymptotic regime. This later allows us to find an explicit relationship between the spectral efficiency, \( \rho \), the ratio \( M/N \), and the probability of error. To this end, we will illustrate exactly how the spectral efficiency scales in the finite-user, finite-antenna case (e.g., through Theorem 1 in Section IV). Thereafter, we will take the limit as the number of users and the number of antennas grow large and we will find a phase-transition where the AoI is minimized in one regime and grows unbounded in the other (as will be stated in Theorem 2 and Theorem 3 in Section IV).

III. DERIVATION OF PACKET ERROR PROBABILITY

A. Derivation of Exact PEP

Recall from (3) that in order for user \( i \) to successfully update the BS with its status in a given slot \( a \), it must attempt a transmission in that slot and \( b \) the transmitted packet must be decoded correctly. For ease of analysis, we consider a symmetric system wherein the \( N \) users have the same transmit

1 Notice that since the error probability of the \( i \)'th user \( p_{e,i} \) is averaged over the number of active users, the arrival process is still Bernoulli
Similarly, the second term in (13) is a sum of another gamma RV with scale parameter $\alpha_3 k$. Thus there are no other active users (i.e., $Z_k \equiv \tau$). Thus, the number of active users follows a gamma distribution with shape parameter $\kappa = \frac{M}{(k-1)\alpha_3}$ and scale parameter $\beta$. For notational compactness we define:

$$\alpha_\rho = \frac{1}{2\rho - 1},$$  \hspace{1cm} (10)

and we denote the inverse signal-to-noise ratio (SNR) as $\beta \triangleq \frac{\rho}{\rho - 1}$. Using these notations and further simplifying (9), we obtain:

$$1 - p_{e,i} = \Pr \left\{ \alpha_\rho \| \tilde{h}_i \|_2^2 - \sum_{j=1 \atop j \neq i}^N \| \tilde{h}_i \|_2^2 | \tilde{h}_j |^2 \geq 1 \right\}. \hspace{1cm} (11)$$

As was shown in [24], $\tilde{h}_i \sim \mathcal{CN}(0, 1)$ and they are mutually independent and also independent of $\| \tilde{h}_i \|_2^2$. In order to deal with the random sum in (11), we condition on the event of having $k$ other users being active with the $i$th user. Since the $\tilde{e}_j$’s are independent and identically distributed (i.i.d) Bernoulli random variables (RVs), the probability that $k$ users are active out of the remaining $N - 1$ users (i.e. after excluding user $i$) is the same as having $k$ successes in $N - 1$ Bernoulli trials. Thus, the number of active users follows a Binomial distribution with parameters $N - 1$ and $\tau$. In order to calculate $p_{e,i}$ for each $i$th user it is convenient to marginalize over the number of other active users, thereby leading to:

$$p_{e,i} = 1 - \sum_{k=0}^{N-1} \binom{N-1}{k} \tau^k (1-\tau)^{N-1-k} p_{i|k}, \hspace{1cm} (12)$$

where $p_{i|k}$ is defined as the conditional probability of successful decoding, conditioned on $k$ other users being also active. More specifically, we have:

$$p_{i|k} = \Pr \left\{ \alpha_\rho \| \tilde{h}_i \|_2^2 - \sum_{j=1}^k \| \tilde{h}_i \|_2^2 | \tilde{h}_j |^2 \geq 1 \right\}, \hspace{1cm} (13)$$

where $\| \tilde{h}_i \|_2^2$ follows a gamma distribution with shape parameter $M$ and scale parameter $1$ (i.e., $\| \tilde{h}_i \|_2^2 \sim \Gamma(M, 1)$). Similarly, the second term in (13) is a sum of $k$ complex normal RVs squared and hence follows a $\Gamma(k, 1)$ distribution. By defining $H \triangleq \alpha_\rho \| \tilde{h}_i \|_2^2$ and $X_k \triangleq \sum_{j=1}^k \| \tilde{h}_i \|_2^2$, we see that $p_{i|k}$ in (13) is the complementary distribution function of the RV $Z \triangleq H - X_k$, as a function of the inverse SNR. In the case there are no other active users (i.e. $k = 0$), $p_{i|k}$ is given by the complementary distribution of a gamma RV $\tilde{h}_i$. For $k > 0$, however, one can find the probability density function (pdf) of $Z$ through convolution, thereby leading to:

$$f_Z(z) = \begin{cases} \kappa \int_{-\infty}^z (-x)^{k-1}(z-x)^{M-1} e^{2\rho x} dx & \text{if } z < 0, \\ \kappa \int_{-\infty}^z (-x)^{k-1}(z-x)^{M-1} e^{2\rho x} dx & \text{if } z \geq 0, \end{cases} \hspace{1cm} (14)$$

where $\kappa = \frac{\alpha_\rho}{kM(2\rho - 1)}$. Since we are primarily interested in the probability that $Z$ is greater that $0$, we are only concerned with $f_Z(z)$ for non-negative values of $z$. By further manipulating the integral (14), it can be shown that the pdf for $z \geq 0$ can be written as:

$$f_Z(z) = \frac{z^{(M+k-2)}}{(M-1)\alpha_\rho 2^{M+1}} \exp \left( -\frac{z}{2} (2\rho - 2) \right) \tilde{W}_{M-k} \frac{1-M-k}{2\rho} (2\rho z) \hspace{1cm} (15)$$

where $\tilde{W}(.)$ denotes the Whittaker function. Averaging $p_{e,i}$ over the number of active users and incorporating everything together we can finally write $p_{e,i}$ as follows:

$$p_{e,i} = 1 - (1-\tau)^{N-1} (1 - \Pr \{ H \leq \beta \}) - \sum_{k=1}^{N-1} \binom{N-1}{k} \tau^k (1-\tau)^{N-1-k} \int_{-\infty}^\infty f_Z(z) dz. \hspace{1cm} (16)$$

B. Asymptotic Approximation of PEP

While (16) is an exact expression for PEP, it does not provide insights into the scaling law of error probability as the total number of users and the number of BS antenna branches both increase at a fixed ratio. In this Section, we derive an asymptotic approximation of PEP which becomes increasingly exact in the large system limit. More specifically, we will let the number of users $N$ and the number of antennas $M$ grow large, while keeping their ratio, $\zeta \triangleq \frac{M}{N}$, constant.

The analysis technique utilized in what follows capitalizes on the Berry-Esseen theorem. Proofs of the various claims introduced in this Section are detailed in Appendix [B]. Using symmetry arguments, it can be seen that $p_{e,i}$ does not depend on $i$ and after omitting that index it follows from (12) that:

$$1 - p_e = \sum_{k=0}^{N-1} \Pr \{ \tilde{e}_1 + \ldots + \tilde{e}_{N-1} = k \} G_{M,k}(\beta), \hspace{1cm} (17)$$

where $G_{M,k}(\beta) = \Pr \{ \sum_{m=1}^{2M} S_m + \sum_{l=1}^{2k} V_l \geq \beta \}$ in which $S_m \sim \Gamma(\frac{2}{\rho}, \alpha_\rho)$ and $V_l \sim \Gamma(\frac{1}{\rho}, 1)$. Using Berry-Esseen central limit theorem (BE-CLT), the inverse cumulative distribution function (CDF) of the sum of gamma RVs in (17) converges uniformly to the standard normal inverse CDF (see Lemma [I] in Appendix [B]), i.e.

$$\Pr \left\{ \sum_{m=1}^{2M} S_m + \sum_{l=1}^{2k} V_l \geq \beta \right\} = Q(w(k)) + O \left( \frac{1}{\sqrt{M+k}} \right), \hspace{1cm} (18)$$

where $w(k) = \frac{\beta - \alpha_\rho \rho M + k}{\sqrt{\alpha_\rho \rho M + k}}$ and $Q(.)$ is the standard Q-function, (i.e., the tail of the normal distribution):

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{+\infty} e^{-\frac{t^2}{2}} dt. \hspace{1cm} (19)$$
Now incorporating the result in (13) into (17) and then using the CLT on \( Pr(\xi_1 + \ldots + \xi_{N-1} = k) \), (17) can be re-written as (see Lemma 3 in Appendix B):

\[
p_e = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Q(w(s))e^{-\frac{s^2}{2}} ds + O\left(\frac{1}{\sqrt{N}}\right),
\]

where

\[
w(s) = \frac{\beta - \alpha_M s + s\sqrt{(N-1)(1-\tau) + (N-1)}}{\sqrt{\alpha_0^2 M + s\sqrt{(N-1)(1-\tau) + (N-1)}}}.
\]

For large \( N \), we approximate \( N - 1 \) by \( N \) (see Lemma 3 in Appendix B) thereby leading to:

\[
w(s) = \frac{\beta - \alpha_M s + s\sqrt{\tau(1-\tau) + \sqrt{N}\tau} + N\tau}{\sqrt{\alpha_0^2 M + s\sqrt{\tau(1-\tau) + \tau}}} + O\left(\frac{1}{\sqrt{N}}\right).
\]

Then by substituting \( \zeta = \frac{M}{N} \) and multiplying both the numerator and denominator by \( \frac{1}{\sqrt{N}} \) we obtain:

\[
w(s) = \frac{\beta - \alpha_M s + s\sqrt{\tau(1-\tau) + \sqrt{N}\tau}}{\sqrt{\alpha_0^2 \zeta + \frac{1}{\sqrt{N}} \sqrt{\tau(1-\tau) + \tau}}} + O\left(\frac{1}{\sqrt{N}}\right).
\]

We further neglect the terms which vanish for large \( N \) in (23), thereby leading to (see Lemma 4 in Appendix B):

\[
w(s) = \frac{\sqrt{\tau - \alpha_M s + s\sqrt{\tau(1-\tau) + \tau}}}{\sqrt{\alpha_0^2 \zeta + \tau}} + O\left(\frac{1}{\sqrt{N}}\right).
\]

We can also neglect the second term in the numerator of (24) that involves the integration variable \( s \). In fact, although \( s \) grows large inside the integral, the exponential makes the integrand function vanish for large-magnitude values of \( s \). Small values of \( s \), however, can also be neglected for large values of \( N \) (i.e., in the asymptotic regime). Finally, our approximation for \( w(s) \) makes it independent of \( s \) (see Lemma 5 in Appendix B):

\[
w = \frac{\sqrt{\tau - \alpha_M s}}{\sqrt{\alpha_0^2 \zeta + \tau}} + O(1).
\]

Consequently, one can take \( Q(w(s)) \) outside of the integral in (20) (see Lemma 6 in Appendix B):

\[
p_e = 1 - Q(w) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} ds + O\left(\frac{1}{\sqrt{N}}\right),
\]

which simplifies to:

\[
p_e = 1 - Q(w) + O\left(\frac{1}{\sqrt{N}}\right).
\]

IV. TRADE-OFF BETWEEN AOI AND SPECTRAL EFFICIENCY

In this Section, we characterize the trade-off between the achievable spectral efficiency and the AoI in multiuser systems with a large-scale antenna array at the BS. It is shown that as the number of users, \( N \), and the number of antennas, \( M \), increase while keeping their ratio constant (i.e. \( \zeta = \frac{M}{N} \)) the maximum achievable spectral efficiency approaches a well-characterized limit for any fixed AoI. The trade-off is manifested by making an observation that spectral efficiencies above the established limit can only be achieved by increasing the overall system AoI. To that end, we rewrite (8) more explicitly as a function of the system parameters (see Lemma 7 in Appendix B):

\[
\Delta(\zeta, N, \rho, \tau) = \frac{1}{\tau} \left(1 - Q\left(\sqrt{\frac{\alpha_0^2 (1-\zeta)\tau}{\sqrt{\alpha_0^2 \zeta + \tau}}}ight)\right) + O\left(\frac{1}{\sqrt{N}}\right),
\]

from which it follows, in the limit, that the minimum AoI for a given attempt probability \( \tau \) is given by

\[
\Delta_{\text{min}}(\tau) = \frac{1}{\tau}.
\]

We start by defining, for a given \( N, \tau, \) and \( \epsilon > 0 \), the set:

\[
\Psi_{\epsilon} \triangleq \{ \rho \in \mathbb{R}^+ \mid p_e < \epsilon \},
\]

as the set of all achievable spectral efficiencies for which the probability of error is less than \( \epsilon \). Note also that the condition \( p_e < \epsilon \) implies that \( \Delta(\zeta, N, \rho, \tau) < \frac{1}{\tau} \) thereby leading to the fundamental trade-offs in finite user case in the following theorem.

**Theorem 1.** For any \( 0 < \tau < 1, \zeta > 0, \) and \( \epsilon > 0 \), there exist \( N_0 \in \mathbb{N} \) such that for any \( N > N_0 \), the set \( \Psi_{\epsilon} \) is non-empty with a supremum

\[
\rho_N^* \triangleq \sup_{\rho} \Psi_{\epsilon} = \log_2 \left(1 + \frac{\zeta - \frac{Q^{-1}(\epsilon)^2}{N}}{\tau + \sqrt{\tau^2 + (\frac{1 - Q^{-1}(\epsilon)^2}{N})^2}} + \frac{1}{\sqrt{N}}\right).
\]

Proof. As shown in Appendix C, the condition that \( p_e < \epsilon \) leads to \( \alpha_\rho > \alpha_\rho^+ (\epsilon_1) \) where:

\[
\alpha_\rho^+ (\epsilon_1) = \frac{\tau + \sqrt{\tau^2 - \tau (1 - \frac{Q^{-1}(\epsilon_1)^2}{N}) (\frac{Q^{-1}(\epsilon_1)^2}{N})}}{\zeta - \frac{Q^{-1}(\epsilon_1)^2}{N}},
\]

and \( \epsilon_1 = \epsilon + O(1/\sqrt{N}) \). Due to the differentiability of \( Q^{-1} \), the error term can be taken out of \( Q^{-1} \) in (32) thereby leading to:

\[
\alpha_\rho^+ (\epsilon_1) = \frac{\tau + \sqrt{\tau^2 - \tau (1 - 1/2 + \epsilon + O(1/\sqrt{N}))}}{\zeta - 1/2 + \epsilon + O(1/\sqrt{N})},
\]

Recalling the definition of \( \alpha_\rho \), we see that \( \alpha_\rho > \alpha_\rho^+ (\epsilon_1) \) is equivalent to:

\[
\rho < \log_2 \left(1 + \frac{1}{\alpha_\rho^+ (\epsilon_1)}\right), \tag{34}
\]

One can also numerically optimize \( 25 \), ignoring the error term, and find the \( \tau \) that minimizes (28). Ignoring the error term will have minimal affect in this optimization as seen in the real-time simulation in Fig. 4.
Again, due to the differentiability of the logarithm and due to \( \epsilon \) being greater than \( \epsilon_0(N, \zeta) \) the error term can be taken out of the logarithm and we have:

\[
\rho < \log_2 \left( 1 + \frac{1}{\alpha_p^2(\epsilon)} \right) + \mathcal{O} \left( \frac{1}{N^{1.5}} \right). \tag{35}
\]

Therefore, \( \Psi_e \) can be re-written as:

\[
\Psi_e = \left\{ \rho \in \mathbb{R}^+ \mid \rho < \log_2 \left( 1 + \frac{1}{\alpha_p^2(\epsilon)} \right) + \mathcal{O} \left( \frac{1}{N^{1.5}} \right) \right\}. \tag{36}
\]

Note that the upper bound on \( \rho \) is always positive as we assume \( \epsilon > \epsilon_0(N, \zeta) \) and hence \( \Psi_e \) is non-empty and its supremum is given by \( \Psi_e \).

A special case of \textbf{Theorem 1} wherein the error probability vanishes in the limit is described in the following two theorems.

\textbf{Theorem 2.} (Achievability) For any \( 0 < \tau < 1 \) and \( \zeta > 0 \), we define the age-limited capacity as

\[
C_{\tau, \zeta} = \log_2 \left( 1 + \frac{\zeta}{\tau} \right), \tag{37}
\]

such that for any spectral efficiency, \( \rho < C_{\tau, \zeta} \), the error probability, \( p_e \to 0 \), and the AoI, \( \Delta(\zeta, N, \rho, \tau) \to \Delta_{\min}(\tau) \), as \( N \to \infty \).

\textbf{Proof.} Note that the second term in \( \frac{1}{\alpha_p^2(\epsilon)} \) goes to zero in the limit as \( N \to \infty \) and so the AoI is determined by the first term. Given the parameters \( \tau \) and \( \zeta \), we see that the AoI in \( \frac{1}{\alpha_p^2(\epsilon)} \) is monotonically increasing with \( p_e \). Therefore, as \( p_e \to 0 \) the AoI \( \Delta(\zeta, N, \rho, N) \to \Delta_{\min}(\tau) \). Now, fix \( \delta > 0 \) and \( \rho = C_{\tau, \zeta} + \delta \). The probability of error is determined by the Q-function or equivalently its argument. Hence, for a given \( N \) and a given value of

\[
\varphi(\rho) \equiv \frac{\alpha_p^2 - \tau}{\sqrt{\alpha_p^2 + \tau}}, \tag{38}
\]

the probability of error is well specified. Plugging \( \rho \) in \( \varphi(\rho) \), it follows that:

\[
\varphi(C_{\tau, \zeta} + \delta) = \frac{(\zeta + \tau)(1 - 2 - \delta)}{\sqrt{\zeta + \tau(2(C_{\tau, \zeta} + \delta) - 2)}} \tag{39}
\]

which is always positive. Therefore, as \( N \to \infty \) the argument inside the Q-function approaches \( +\infty \); hence \( p_e \to 0 \) and \( \Delta(\zeta, N, \rho, \tau) \to \Delta_{\min}(\tau) \).

\textbf{Theorem 3.} Given \( \tau, \zeta \) and any spectral efficiency \( \rho > C_{\tau, \zeta} \), the error probability, \( p_e \to 1 \), and the AoI \( \Delta(\zeta, N, \rho, \tau) \to \infty \) as, \( N \to \infty \).

\textbf{Proof.} We proved this in a similar way as we did for \textbf{Theorem 2} In fact, we choose an arbitrary \( \delta > 0 \) and set \( \rho = C_{\tau, \zeta} + \delta \). Plugging the latter in \( \varphi(\rho) \) and simplifying we obtain:

\[
\varphi(C_{\tau, \zeta} + \delta) = -\frac{(\zeta + \tau)(2^{2} - 1)}{\sqrt{\zeta + \tau(2(C_{\tau, \zeta} + \delta) - 2)}}. \tag{40}
\]

Now, the argument inside the Q-function is negative for all values of \( N \) and approaches \( -\infty \) as \( N \to \infty \), from which it follows that \( p_e \to 1 \) and \( \Delta(\zeta, N, \rho, \tau) \to \infty \).

\textbf{Corollary 3.1.} For any \( \epsilon > 0 \), the age-limited capacity defined in \( \text{(37)} \) is given by

\[
C_{\tau, \zeta} = \lim_{N \to \infty} \rho_N^\epsilon. \tag{41}
\]

\textbf{Proof.} We see that as \( N \to \infty \) the threshold, \( \epsilon_0(N, \zeta) \) goes to zero as it decays with \( N \). Therefore, for any \( \epsilon > 0 \), \( \rho_N^\epsilon \) is well defined. As the function \( \log_2 x \) is continuous at \( x = 1 + \frac{\epsilon}{2} \), one can take the limit inside its argument in \( \rho_N^\epsilon \), from which the corollary follows.

\textbf{Remark 1.} It is interesting to observe the similarity between the age-limited capacity and the capacity of the AWGN channel. In the age-limited capacity, the ratio \( \frac{\delta}{\tau} \) plays the role of the SNR in the AWGN Capacity. In working with asymptotic scenarios where we have both a large number of users and a large number of antennas, the noise variance becomes negligible. In this asymptotic interference-limited scenario the decoding error probability is dominated by \( \tau \). It is insightful in this case to view \( \tau \) as the noise variance. Similarly, the ratio, \( \zeta \), of the number of antennas to the number of users plays the role of the transmit power.

\textbf{Remark 2.} Note also, that the age-limited capacity, \( C_{\tau, \zeta} = \log_2 \left( 1 + \frac{\zeta}{\tau} \right) \), is parameterized by \( \tau \) and \( \zeta \) and can be increased by decreasing the value of \( \tau \). Now, for any spectral efficiency below \( C_{\tau, \zeta} \) our analysis reveals that the age-limited capacity can be approached as \( N \to \infty \) in which case the minimum achievable AoI is given by \( \Delta_{\min}(\tau) \). Thus, while the aggregate spectral efficiency can be made large by decreasing \( \tau \), the price is an undesired increase in the Aoi. This should be expected on intuitive grounds since as \( \tau \) becomes small, the users that are lucky to transmit in a given slot can be easily separated in the spatial domain by making use of a large-scale antenna array at the BS.

\section{Analysis with Imperfect CSI}

We incorporate imperfect CSI into our analysis by using a channel estimator and writing the error in channel estimation as \( \mathcal{E} \triangleq \hat{H} - H \), where \( H \) is the channel estimate. We assume that \( \hat{H} \) and \( \mathcal{E} \) are independent and \( \mathcal{E} \sim \mathcal{CN}(0_{M}, \sigma_p^2I_M), \hat{H} \sim \mathcal{CN}(0_{M}, (1 - \sigma_p^2)I_M) \) where \( \ell \) are the \( \ell \)th columns of \( \mathcal{E} \) and \( \hat{H} \) respectively and \( \sigma_p^2 \) is the mean-squared error (MSE) in channel estimation. We further assume that the columns of \( \mathcal{E} \) and \( \hat{H} \) are independent. In the analysis we append the estimation error into the noise and interference terms. The MSE, \( \sigma_p^2 \), of the MMSE estimator could be obtained from the state evolution of the AMP algorithm [2].

\subsection{Maximal-Ratio Combining}

In the case of MRC our system model becomes:

\[
y_n = \sum_{i=1}^{N} \hat{h}_i \varepsilon_i x_{i,n} - \sum_{i=1}^{N} \mathcal{E}_i \varepsilon_i x_{i,n} + w_n. \tag{42}
\]

From this view, we can write the outage probability as:

\[
p_e, i = 1 - \Pr\{\rho_i < C_{\text{MRC}}\}. \tag{43}
\]
where
\[ C_{\text{MRC}} \triangleq \log_2 \left( 1 + \frac{\| \hat{h}_i \|^2 \sigma_n^2}{\sum_{j \neq i} \| \hat{h}_j \|^2 \gamma_j \sigma_j^2 + \sum_{j=1}^N \| h_j \|^2 \gamma_j \sigma_j^2} \right). \]
Conditioning on \( k \) out of the total \( N \) users being active and considering a symmetric system as before, i.e., \( P_i = P \forall i \), and simplifying we write the conditional outage probability, \( p_{e|k} \), as:
\[
1 - p_{e|k} = \Pr \left\{ \alpha \| \hat{h}_i \|^2 - \sum_{j=1}^k | \hat{h}_i^N \hat{h}_j |^2 - \sum_{j=k+1}^N | \hat{h}_i^N \hat{e}_j |^2 \geq \beta \right\},
\]
where \( \hat{h}_i = \frac{h_i}{\| h_i \|} \). To make this convenient for use of the CLT we write (43) as:
\[
1 - p_{e|k} = \Pr \left\{ \sum_{m=1}^{2M} S_m + \sum_{l=1}^{2k} V_l + \sum_{n=1}^{2(k+1)} T_n \geq \beta \right\},
\]
where \( S_m \sim \Gamma \left( \frac{1}{2}, \alpha \sigma_p (1 - \sigma_p^2) \right), \) \( V_l \sim \Gamma \left( \frac{1}{2}, (1 - \sigma_p^2) \right) \), and \( T_n \sim \Gamma \left( \frac{1}{2}, \sigma_p^2 \right) \). We now apply the CLT and use the same techniques utilized previously to get the asymptotic results. We will disregard the error terms in this analysis to avoid redundancy. Applying the CLT the conditional outage probability can be written as:
\[
p_{e|k} \approx 1 - Q \left( w(k) \right),
\]
where
\[
w(k) = \frac{\beta - \alpha \sigma_p (1 - \sigma_p^2) M + (1 - \sigma_p^2) k + (k + 1) \sigma_p^4}{\sqrt{\alpha \sigma_p^2 (1 - \sigma_p^2)^2 M + (1 - \sigma_p^2)^2 k + (k + 1) \sigma_p^4}}.
\]
Similar to the perfect CSI case, we make a normal approximation to the binomial distribution, \( k = s \sqrt{(N - 1)\tau (1 - \tau) + (N - 1)\tau} \), we approximate \( N - 1 \) by \( N \), and substitute \( \zeta \approx M \):
\[
w(s) = \frac{\sqrt{\zeta} (\tau - \alpha \sigma_p (1 - \sigma_p^2) \zeta + \sqrt{\zeta N + \sqrt{\zeta N (1 - \tau) N \tau}} + \sqrt{\zeta N (1 - \tau) N \tau}) + \alpha \sigma_p^2 (1 - \sigma_p^2)^2 \zeta + \tau ((1 + \sigma_p^2)^2 + \sigma_p^4)}{\sqrt{\zeta (1 - \sigma_p^2)^2 \zeta + \tau ((1 + \sigma_p^2)^2 + \sigma_p^4)}}.
\]
Next we divide both the numerator and denominator by \( \sqrt{N} \) and neglect terms that do not grow with \( N \):
\[
w(s) = \frac{\sqrt{N} (\tau - \alpha \sigma_p (1 - \sigma_p^2) \zeta)}{\sqrt{\alpha \sigma_p^2 (1 - \sigma_p^2)^2 \zeta + \tau ((1 + \sigma_p^2)^2 + \sigma_p^4)}}.
\]
As \( w(s) \) is now independent of \( s \), averaging over the binomial distribution will have no effect and thus we can write the total outage probability of the system as:
\[
p_e \approx Q \left( \frac{\sqrt{N} (\alpha \sigma_p (1 - \sigma_p^2) \zeta - \tau)}{\sqrt{\alpha \sigma_p^2 (1 - \sigma_p^2)^2 \zeta + \tau ((1 + \sigma_p^2)^2 + \sigma_p^4)}} \right).
\]
Similar to the perfect CSI case, we find the achievable rate for a given error probability of \( \epsilon \) given by:
\[
\rho_{\text{MRC}} \approx \log_2 \left( 1 + \frac{1}{\sqrt{\zeta \epsilon \sigma_p^2 \zeta \zeta + \tau (1 + \sigma_p^2)^2 + \tau \tau \sigma_p^4})}(1 + \sigma_p^2)^2 \zeta \zeta + \tau \sigma_p^2 \zeta \zeta + \tau \tau \sigma_p^4) \right),
\]
which in the limit leads to:
\[
\rho_{\text{MRC}} = \log_2 \left( 1 + \frac{\zeta \epsilon \sigma_p^2 \zeta \zeta + \tau (1 + \sigma_p^2)^2 + \tau \tau \sigma_p^4) \right).
\]

### B. Zero-Forcing

We first start by rewriting the system model given in (2) in a matrix-vector product form given by:
\[
y_n = H x_n + w_n,
\]
where \( H = [\varepsilon_1 h_1, \ldots, \varepsilon_N h_N] \) and \( x_n = [x_{1,n}, \ldots, x_{N,n}]^T \). Conditioned on \( k \) out of the \( N \) total users being active we can write ZF receiver as:
\[
W_{\text{ZF}}^H = \frac{1}{\sqrt{\rho}} (H^H H)^{-1} H^H,
\]
where now \( H \) is only composed of the channels of the \( k \) active transmitters. Applying this linear receiver to \( y_n \) we have:
\[
W_{\text{ZF}}^H y_n = s_n + v_n,
\]
where \( s_n \sim C(0, 1, \kappa_i) \) and \( v_n \) is coloured noise with \( \mathbb{E}[v_n v_n^H H] = \beta (H^H H)^{-1} \). In this case, the \( i \)th user will see a scalar channel and the maximum achievable rate over this channel will be given by \( \log_2 (1 + \text{SNR}^Z_{\text{ZF}}) \) where \( \text{SNR}^Z_{\text{ZF}} \) is given by:
\[
\text{SNR}^Z_{\text{ZF}} = \frac{\mathbb{E}[s_n^2] \beta (H^H H)^{-1}}{\mathbb{E}[v_n v_n^H H] H H^H} = \frac{1}{\beta (H^H H)^{-1}}.
\]
\( \text{SNR}^Z_{\text{ZF}} \) is found to follow a Chi-Squared distribution with \( 2(M - k + 1) \) degrees of freedom (DOF). For the case that no other users are active besides user \( i \), \( \text{SNR}^Z_{\text{ZF}} \) follows a Chi-Squared distribution with \( 2M \) DOF [20]. Therefore, averaging over all of the users we can write the error probability as:
\[
p_e = (1 - \tau)^{N - 1} \int_0^{\rho - 1} f_{\chi^2_{2M}} (\eta) d\eta
\]
\[
+ \sum_{k=1}^{N - 1} \int_0^{\rho - 1} f_{\chi^2_{2(M - k + 2)}} (\eta) d\eta,
\]
from which the AoI simply follows from (8).

To analyze our system asymptotically under ZF we write our system model as:
\[
y_n = \sqrt{P(1 - \sigma_p^2)} H s_n - \sqrt{P} \varepsilon s_n + w_n.
\]
Conditioning on \( k \) out of the \( N \) users being active the ZF filter can be written as:
\[
W_{\text{ZF}}^H = \frac{1}{\sqrt{P(1 - \sigma_p^2)}} (H^H H)^{-1} H^H,
\]
where now $\hat{H}$ is only composed of the channels of the $k$ active transmitters. Applying this linear receiver to $y_n$ we have:

$$W_{ZF}^H y_n = s_n + v_n,$$

where $s_n \sim \mathcal{CN}(0,k)$ and $v_n$ is coloured noise with the estimation error term, i.e.:

$$v_n = -\sqrt{P}W_{ZF}^H \mathcal{E} s_n + W_{ZF}^H w_n.$$

In this case, the $i^{th}$ user will see a scalar channel and the maximum achievable rate over this channel will be given by $\log_2(1 + \text{SNR}_i^{ZF})$ where $\text{SNR}_i^{ZF}$ is given by:

$$\text{SNR}_i^{ZF} = \frac{E[s_n s_n^H]_{i,i}}{E[v_n v_n^H]_{i,i}} = \frac{1}{E[v_n v_n^H]_{i,i}}.$$

It is easy to show that:

$$E[v_n v_n^H] = \left(\frac{k\sigma^2 + \beta}{1 - \sigma_p^2}\right) (\hat{H})^{-1}(\hat{H}).$$

Therefore, we can now write the conditional outage probability as:

$$p_{c|k} = \Pr \left\{ \frac{1}{(\hat{H})^{-1}(\hat{H})_{i,i}} \leq \frac{1}{\alpha_q} \right\}.$$

For the asymptotic analysis we let $M$ and $N$ grow large while holding their ratio, $\zeta = M/N$, constant. In the case of ZF we additionally have the condition that $\zeta \geq 1$. We can see the first term in (53) goes to 0 as $N$ grows large and we focus on the second term. We note that the RV:

$$Z = \frac{\chi^2(M-k+1) - (M-k+1)/\left(\frac{k\sigma^2 + \beta}{1 - \sigma_p^2}\right)}{\sqrt{M-k+1}/\left(\frac{k\sigma^2 + \beta}{1 - \sigma_p^2}\right)},$$

tend to a normal distributions by the central limit theorem as the number of DOF grow large. Therefore we can rewrite the second integral in (54) as:

$$\int_0^{2^\eta-1} f_{\chi^2(M-k+1)}(\eta) d\eta \rightarrow \int_{-\sqrt{M-k+1}}^{\lambda_{M,k}} f_Z(z) dz,$$

where:

$$\lambda_{M,k} = \frac{k\sigma^2 + \beta}{\alpha_q(1 - \sigma_p^2)\sqrt{M-k+1}}.$$

We can write the integral in (63) as a difference of two Q-functions given by the upper and lower bounds of the integral, i.e.:

$$\int_{-\sqrt{M-k+1}}^{\lambda_{M,k}} f_Z(z) dz = Q\left(\sqrt{M-k+1}\right) - Q(\lambda_{M,k}).$$

As we let $M$ grow large for a fixed $k$ the term on the left goes to 1. Making the substitution $M = \xi N$, approximating $N - 1$ by $N$, and approximating the binomial distribution using the CLT through $k = s\sqrt{(N-1)\tau(1-\tau)} + (N-1)\tau$, we can write the total probability error as:

$$p_{e,i} = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Q(w(s)) e^{-\frac{s^2}{2}} ds,$$

where $w(s)$ is given by:

$$w(s) = \frac{\left((-k+1)\beta - 1\right) - (s(\sqrt{s+1} + N\tau) - \left((-k+1)\beta - 1\right))}{\sqrt{N - (s(\sqrt{s+1} + N\tau) - \left((-k+1)\beta - 1\right) + 1}}.$$

Dividing the numerator and denominator of $w(s)$ by $\sqrt{N}$ and neglecting terms that do not grow with $N$ we get:

$$w(s) = \sqrt{N} \left(\tau - \frac{(\zeta - \zeta)\sqrt{N}}{\sqrt{\tau}}\right).$$

Finally, since this term does not depend on $s$ we take it out of the integral and write the asymptotic packet error probability as:

$$p_{e,i} = Q\left(\sqrt{N} \left(\zeta - \frac{(\zeta - \zeta)\sqrt{N}}{\sqrt{\tau}}\right)\right) .$$

Similarly we find the achievable rate for a given error probability of $\epsilon$:

$$\rho^{ZF} = \log_2 \left(1 + \frac{1 - \sigma_p^2}{\sigma_p^2} \left(\frac{\zeta - 1 - (\zeta - \zeta)\sqrt{N}}{\sqrt{\tau}}\right)\right) ,$$

which in the limit leads to:

$$\rho^{ZF} = \log_2 \left(1 + \frac{1 - \sigma_p^2}{\sigma_p^2} \left(\frac{\zeta - 1}{\sqrt{\tau}}\right)\right).$$

VI. NUMERICAL RESULTS AND DISCUSSION

We now illustrate the results found in the previous sections and further compare recent URA schemes against our bound.

A. Trade-Off Between AoI and Spectral Efficiency

In order to determine the direct relationship between the spectral efficiency and the AoI, we start by re-writing (28) as a function of the spectral efficiency for a given probability of error, $\rho = \epsilon$. In this case, the AoI reduces simply to:

$$\Delta(\zeta, N, \rho, \tau) = \frac{1}{\tau(1-\epsilon)} + O\left(\frac{1}{\sqrt{N}}\right),$$

where $\tau_c$ is found by solving for $\tau$ in the Q-function in (28) and is given by

$$\tau_c = \alpha_q \zeta + \frac{Q^{-1}(\epsilon)\alpha_q}{2N} - \frac{1}{\sqrt{2N}}\left(\alpha_q \zeta + \frac{Q^{-1}(\epsilon)\alpha_q}{2N} \right)^2.$$

The identity in (22) is valid for $\rho \geq \rho_{min}(\epsilon)$ where $\rho_{min}(\epsilon)$ is given in (31) evaluated at $\tau = 1$ (this is seen when solving for $\tau$ from the Q-function and observing where the solution is valid). In Fig. 2 we plot (71), ignoring the error terms, for $p_c = 10^{-3}$, $\zeta = 0.3$, and $N = \{100, 500, 1000\}$. We also plot the case of infinite number of users with $p_c = 0$ whose curve is obtained by taking $N \rightarrow \infty$ in (22), i.e.:

$$\lim_{N \rightarrow \infty} \tau_c = \alpha_q \zeta.$$
Each spectral efficiency on the $N = \infty$ curve is the age-limited capacity for a given set of $\tau$ and $\zeta$. In both the finite- and infinite-number-of-users scenarios, the AoI is minimized by setting $\tau$ to 1, as seen in (71). In the finite case, however, the minimum AoI is limited by the probability of error and is given by:

$$
\Delta_{\epsilon} = \frac{1}{1 - \epsilon} + O\left(\frac{1}{\sqrt{N}}\right).
$$

On the other hand, in the infinite-number-of-users regime the probability of error is driven to zero and the AoI takes the minimum value possible, $\Delta = 1$. This is clearly seen in the zoomed portion of Fig. 2. We also add a simulation for $N = 100$. In our simulation we choose a large number of time slots, $10^5$, and simulate the network AoI. Indeed, we can see from Fig. 2 that the error term in our approximations is small.

### B. Performance under Imperfect CSI

We compare the achievable rates for MRC and ZF for a given MSE in channel estimation in Fig. 3. Indeed as expected the ZF receiver performs better given that the channel estimate is good. However, the MRC receiver is more robust and is not as sensitive to imperfect channel estimates. Additionally, the constraint $\zeta \leq 1$ for ZF heavily limits its use in mMTC where the total number of users is much larger than the number of BS antennas. Similar results follow for both AoI and Outage performance.

### C. Application to Unsourced Random Access (URA)

The URA paradigm, initially analyzed in [8], in which the base station is tasked with providing multiple access to a large number of uncoordinated users, has attracted considerable attention. In [8], the random coding achievable bound was derived and compared against popular multiple-access schemes. A number of algorithmic solutions for URA were proposed in [25]–[28]. However, all of the above theoretical and algorithmic works focused on the case of a single receive antenna at the BS. To date, the only two algorithmic solutions for the URA paradigm that have so far investigated the

- use of the massive MIMO technology are [29], [30] whose performances have not been gauged against any achievability bound. We will refer to these two massive MIMO-based URA schemes in [29], [30] as “clustering-based” and “covariance-based”, respectively.

In Fig. 4 we use computer simulations to compare both schemes to the new achievable bound established in Theorem 1 as well as to the exact expression established in (16). For both the schemes, we fix the bandwidth to $W = 10$ MHz and the noise variance to $\sigma^2 = 10^{-19.9} \times W$ [Watts] and then calculate the required transmit power that yields the SNR, $\beta^{-1} = 30$ dB. For the clustering-based URA scheme, we use a Gaussian prior for HyGAMP-based compressed sensing (CS) and communicate $B = 102$ information bits per user/packet over $L = 6$ slots. For the covariance-based scheme, we fix the number of information bits per user/packet to $B = 104$ bits which are communicated over $L = 17$ slots. The parity bit allocation for the outer tree code is set to $p = [0, 8, 8, \ldots, 14]$. 
We also use $J = 14$ coded bits per slot which leads to the total rate of the outer code $R_{out} = 0.437$. For both the schemes, we simulate 3 data points with $K_a = [50, 75, 100]$ active users and $M = [30, 45, 60]$ antennas at the base station in which the achievable spectral efficiency in (37) is given by

$$\log_2 \left( 1 + \frac{P_L}{K_a} \right).$$

Note that even though the achievable spectral efficiency does not depend on the total number of users, as is the case in [8], the AoI does. In fact, as the total number of users increases, the AoI grows unbounded for any fixed number of active users.

In the plots of Fig. 4, apart from the gap between the newly established bound and the existing algorithmic solutions, it is seen that the achievable spectral efficiency of both schemes decreases as the number of active users increases. Actually, it should be possible to rigorously prove this limitation for any CS-based decoding scheme. Roughly speaking, as the number of active users grows, increasing the per-user spectral efficiency requires one to decrease the blocklength thereby rendering the CS-based support recovery task more challenging. In fact, the fundamental limitation of support recovery requires (see Chapter 7 of [31]) the blocklength $L = O(K_a \log(\frac{K_a^2}{N^2}))$ to scale a little faster than the number of active users with a single antenna at the BS. On the contrary, in [30], [32], it was shown that the covariance-based URA scheme with a large-scale antenna array at the BS can recover the support perfectly as long as $K_a \log^2 \left( \frac{2^J/L}{N^2} \right) = O(L^2)$ and $K_a^2 = o(1)$. However, in this case, the achievable spectral efficiency goes to infinity and the achievable performance with respect to Theorem 1 has to be investigated carefully. In particular, a sharper characterization of the $\frac{K_a^2}{M^2} = o(1)$ term is required.

VII. CONCLUSION

We have established achievability and converse results in the $K_a$-user GMAC with a large-scale antenna array at the BS. We have defined the age-limited capacity as the maximum spectral efficiency achieving such that the AoI is finite in asymptotic system limits. In this case, we have shown that in order to minimize the system AoI all devices must be active in every transmission period. This is also the case in finite system sizes in which the AoI is, however, limited by the probability of error. We have used our bound to compare the two recent massive MIMO URA algorithms, thereby revealing a huge gap between their performance and the overall spectral efficiency that can be potentially achieved in practice. In future work, considering the overloaded system $\hat{\zeta}/\tau < 1$, it is desirable to do some scheduling in order to control the inter-user interference. One approach could be to subdivide each slot into $J$ scheduling intervals. Then the asymptotic joint optimization of the spectral efficiency as well as the AoI could be performed over the number of scheduling intervals.

APPENDIX

A. Derivation of the AoI

Recall from (5) the definition of AoI. The latter is a limit of a time-average of the age sample function as the time horizon grows large. Instead of computing the integral in (5) directly, we can express it as a function of the inter-update times, as shown pictorially in Fig. 5.

The area below the sample function shown in Fig. 5 consists of a rectangular base of width $t'$ and unit height, jagged triangular structures, and boundary pieces (both above the base rectangle). We denote the area of the triangular pieces by $Q_l$ and the first and last boundary pieces by $\tilde{Q}_F$ and $\tilde{Q}_L$, respectively. Now (5) can be re-written as:

$$\int_{0}^{t'} \delta_i(t)dt = t' + \tilde{Q}_F + \tilde{Q}_L + \sum_{k=1}^{N(t')-1} Q_k,$$

where $N(t')$ denotes the number of arrivals by time $t'$. The $Q_i$’s can be written in terms of the inter-update intervals as:

$$Q_l = \frac{Z_i(Z_i - 1)}{2}.$$

Dividing the right-hand side of (75) by $t'$ and taking the limit (as $t'$ goes to $+\infty$) we have:

$$\Delta_i = \lim_{t' \to \infty} \left( 1 + \frac{\tilde{Q}_F + \tilde{Q}_L}{t'} + \frac{\sum_{k=1}^{N(t')-1} Z_{ik}(Z_{ik} - 1)}{2t'} \right).$$

The first term can be taken out of the limit and the second term goes to 0 WP1. We re-write the last term in (77) as follows:

$$\lim_{t' \to \infty} \left( \frac{N(t') - 1}{2t'} \right) \sum_{k=1}^{N(t')-1} Z_{ik}(Z_{ik} - 1).$$

As the Bernoulli process is a renewal process, we have $\frac{N(t')}{t'} \to 1$ WP1. Also, from the strong law of large numbers (SLLN) the sum in the limit converges to $\mathbb{E}[Z_i^2] - \mathbb{E}[Z_i]$ WP1. Therefore, we have:

$$\Delta_i = \frac{\mathbb{E}[Z_i^2]}{2\mathbb{E}[Z_i]} + \frac{1}{2}.$$

B. Proofs of the Various Approximations

Lemma 1. The inverse CDF of the sum of random variables, $\sum_{m=1}^{2M} S_m + \sum_{k=1}^{2k} V_i$, converges uniformly to the standard normal inverse CDF, where the convergence rate is bounded above by $O\left( \frac{1}{\sqrt{M+k}} \right)$.

$$\Pr \left\{ \sum_{m=1}^{2M} S_m + \sum_{k=1}^{2k} V_i \geq \beta \right\} = Q(w(k)) + O\left( \frac{1}{\sqrt{M+k}} \right),$$

(80)
in which \( w(k) = \frac{3-\alpha}{\sqrt{\sigma^2_k + k}} \).

**Proof.** Let \( B_{2M+2k} \) be the following normalized RV:

\[
B_{2M+2k} \triangleq \frac{1}{\sqrt{\sigma^2_k + k}} \left[ \sum_{m=1}^{2M} S_m + \sum_{l=1}^{2k} V_l - (M\alpha - k) \right],
\]

whose inverse CDF is denoted as \( F_{2M+2k}(x) \). From the Berry-Esseen inequality for non identically distributed random variables, the difference between \( F_{2M+2k}(x) \) and the standard normal inverse CDF is bounded uniformly, i.e.

\[
\sup_{x \in \mathbb{R}} |F_{2M+2k}(x) - Q(x)| \leq K \phi_1(2M + 2k) \phi_2(2M + 2k),
\]

where \( K \) is a constant, and

\[
\phi_1(2M + 2k) = \left( \sum_{m=1}^{2M} \mathbb{E}[|S_m - \mathbb{E}[S_m]|^2] + \sum_{m=1}^{2k} \mathbb{E}[|V_l - \mathbb{E}[V_l]|^2] \right)^{\frac{1}{2}},
\]

\[
\phi_2(2M + 2k) = \left( \sum_{m=1}^{2M} \mathbb{E}[|S_m - \mathbb{E}[S_m]|^3] + \sum_{m=1}^{2k} \mathbb{E}[|V_l - \mathbb{E}[V_l]|^3] \right)^{\frac{1}{2}}.
\]

Since \( S_m \sim \Gamma(\frac{3}{2}, \alpha_p) \) and \( V_l \sim \Gamma(\frac{1}{2}, 1) \), it can be verified by computing the fourth central moment that the second- and third-order central moments are finite. Let \( \mathbb{E}[|S_m - \mathbb{E}[S_m]|^2] = \frac{\alpha_p}{2} > 0 \), \( \mathbb{E}[|V_l - \mathbb{E}[V_l]|^2] = \frac{\alpha_p}{2} > 0 \), \( \mathbb{E}[|S_m - \mathbb{E}[S_m]|^3] = \frac{\alpha_p}{2} > 0 \), and \( \mathbb{E}[|V_l - \mathbb{E}[V_l]|^3] = \frac{\alpha_p}{2} > 0 \). Using these notations, we rewrite (81) as follows:

\[
|F_{2M+2k}(x) - Q(x)| \leq \frac{C_1 M + C_k}{(C_3 M + C_k)^{\frac{3}{2}}} \quad \forall x.
\]

Since \( C_1 M + C_k < \max\{C_1, C_2\}(M + k) \) and \( C_3 M + C_k > \min\{C_3, C_4\}(M + k) \) we obtain:

\[
|F_{2M+2k}(x) - Q(x)| \leq \frac{\max\{C_1, C_2\}(M + k)}{\min\{C_3, C_4\}(M + k)^{\frac{3}{2}}} = \frac{C}{\sqrt{M + k}} \quad \forall x,
\]

where \( C = \frac{\max\{C_1, C_2\}}{\min\{C_3, C_4\}} \). This implies:

\[
F_{2M+2k}(x) - Q(x) = \mathcal{O}\left(\frac{1}{\sqrt{M + k}}\right) \quad \forall x.
\]

This can be used to show that:

\[
\Pr\left\{ \sum_{m=1}^{2M} S_m + \sum_{l=1}^{2k} V_l \geq \beta \right\} = F_{2M+2k}(w(k)) = Q(w(k)) + \mathcal{O}\left(\frac{1}{\sqrt{M + k}}\right).
\]

**Lemma 2.** The probability of error in (17) is given by:

\[
p_e = 1 - \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} Q(w(s)) ds + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right).
\]

**Proof.** We begin by noting that the expression in (17) is the expected value of (18) with respect to a Binomial distribution with parameters \( N - 1 \) and \( \tau \). Here, we show that this expected value can be instead taken with respect to a Gaussian distribution with \( \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) \) error term. To start with, since the Binomial distribution does not admit a density function it is more convenient to use Stieltjes integrals over a compact interval before going to infinity. First, we show that \( \mathcal{O}\left(\frac{1}{\sqrt{M + k}}\right) = \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) \) by noticing that:

\[
\mathcal{O}\left(\frac{1}{\sqrt{M + k}}\right) < \frac{C}{\sqrt{N + k}} = \frac{C}{\sqrt{N(\zeta + \delta)}} = \tilde{C}(\delta),
\]

where \( 0 < \delta = \frac{k}{N} < 1 \). We also write the error resulting from integrating with respect to different CDFs as follows:

\[
\left| \int_{-x}^{x} \left( Q(w(k)) + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) \right) dF(k) - \int_{-x}^{x} \left( Q(w(k)) + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) \right) d\tilde{F}(k) \right|,
\]

where \( F(k) \) is the CDF of a Binomial RV with mean \( (N - 1)\tau \) and variance \( (N - 1)\tau(1 - \tau) \) and \( \tilde{F}(k) \) is the CDF of the approximating Gaussian distribution with the same mean and variance. Since both \( F(k) \) and \( \tilde{F}(k) \) are non-decreasing on any compact interval \([-x, x]\), the difference \( F(k) - \tilde{F}(k) \) is of bounded variation, and hence the Stieltjes integral with respect to \( F(k) - \tilde{F}(k) \) is defined for any continuous function. Since \( Q(w(k)) \) is continuous at \( w(k) \), we can re-write (89) as follows:

\[
\left| \int_{-x}^{x} \left( Q(w(k)) + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) \right) d\left(F(k) - \tilde{F}(k)\right) \right| = \left| M_1(x) - M_2(x) - M_3(x) \right|,
\]

with

\[
M_1(x) = \left( Q(x) + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) \right) \left( F(x) - \tilde{F}(-x) \right),
\]

\[
M_2(x) = \left( Q(-x) + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) \right) \left( F(-x) - \tilde{F}(-x) \right),
\]

\[
M_3(x) = \int_{-x}^{x} \left( F(k) - \tilde{F}(k) \right) dQ(w(k)).
\]

Now, by the triangle inequality, it follows from (91) that:

\[
\left| \int_{-x}^{x} \left( Q(w(k)) + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) \right) d\left(F(k) - \tilde{F}(k)\right) \right| \leq \left| M_1(x) \right| + \left| M_2(x) \right| + \left| M_3(x) \right|.
\]

Then, since \( 0 < Q(x) < 1 \), (95) can be further bounded by:

\[
K_1 \left| F(x) - \tilde{F}(x) \right| + \sum_{k \in [-x, x]} \left| F(k) - \tilde{F}(k) \right| V_{x}^{\tau}(Q),
\]

where \( V_{x}^{\tau}(Q) < \infty \) is the total variation of the Q-function over \([-x, x]\). Recall, also that \( F(x) \) represents the CDF of a large sum of \( N - 1 \) binary random variables. Consequently,
upon appropriate normalization and by applying the Berry-
Eessen inequality for i.i.d random variables we see that (96) is
less than:

$$ \frac{A_1}{\sqrt{N}} + \frac{A_2}{\sqrt{N}} + \frac{A_3}{\sqrt{N}} = O \left( \frac{1}{\sqrt{N}} \right), \tag{97} $$

for some positive constants $A_1$, $A_2$ and $A_3$. Now, coming back
to (89) we see that:

$$ \int_{-\infty}^{x} \left( Q(w(k)) + O \left( \frac{1}{\sqrt{N}} \right) \right) dF(k) = \int_{-x}^{\infty} \left( Q(w(k)) + O \left( \frac{1}{\sqrt{N}} \right) \right) dF(k) + O \left( \frac{1}{\sqrt{N}} \right), \tag{98} $$

where the $O \left( \frac{1}{\sqrt{N}} \right)$ on the left- and right-hand sides can be
safely dropped since both CDFs are bounded, thereby yielding:

$$ \int_{-x}^{\infty} Q(w(k))dF(k) = \int_{-x}^{\infty} Q(w(k))d\bar{F}(k) + O \left( \frac{1}{\sqrt{N}} \right). \tag{99} $$

The right-hand side integral in (99) is with respect to a
Gaussian CDF and can now be explicitly written in terms of
a density by making the substitution $s = \frac{k}{\sqrt{N(1-\tau)}}$.

$$ \int_{-x}^{\infty} Q(w(k))dF(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Q(w(s))e^{-\frac{s^2}{2}}ds. \tag{100} $$

Since the approximation is uniform in $x$ and the integrals are
convergent, we can go to the limit in (99) and obtain:

$$ \int_{-\infty}^{\infty} Q(w(k))dF(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Q(w(s))e^{-\frac{s^2}{2}}ds + O \left( \frac{1}{\sqrt{N}} \right). \tag{101} $$

By further noticing that the left-hand side of (98) is nothing
but the expression of $1 - p_e$, we finally obtain:

$$ p_e = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} Q(w(s))e^{-\frac{s^2}{2}}ds + O \left( \frac{1}{\sqrt{N}} \right). \tag{102} $$

Lemma 3. $w(s)$ in (21) can be approximated as follows,

$$ w(s) = \beta - \alpha_0 M + s\sqrt{N\tau(1-\tau)} + N\tau \sqrt{\alpha_0^2 M + s\sqrt{N\tau(1-\tau)} + N\tau} + O \left( \frac{1}{\sqrt{N}} \right). \tag{103} $$

Proof. We denote the first term in (105) by $w_a(s)$ and start
by re-writing (21) as:

$$ w(s) = \beta - \alpha_0 M + s\sqrt{N\tau(1-\tau)} + N\tau \sqrt{\alpha_0^2 M + s\sqrt{N\tau(1-\tau)} + N\tau} + \frac{s\sqrt{\tau(1-\tau)(\sqrt{N-1}-\sqrt{N}) - \tau}}{\sqrt{\alpha_0^2 M + s\sqrt{(N-1)\tau(1-\tau)} + (N-1)\tau}}. \tag{104} $$

Dividing and multiplying the first term in (104) by the denominator of the first term in (103) yields:

$$ w(s) = w_a(s) \frac{\sqrt{\alpha_0^2 M + s\sqrt{N\tau(1-\tau)} + N\tau}}{\sqrt{\alpha_0^2 M + s\sqrt{(N-1)\tau(1-\tau)} + (N-1)\tau} + \frac{s\sqrt{\tau(1-\tau)(\sqrt{N-1}-\sqrt{N}) - \tau}}{\sqrt{\alpha_0^2 M + s\sqrt{(N-1)\tau(1-\tau)} + (N-1)\tau}}}. \tag{105} $$

Recall that $M$ grows proportionally to $N$ with proportionality
coefficient $\zeta$ (i.e. $M = \zeta N$). Therefore, after multiplying and
dividing the second term in (103) by $\frac{1}{\sqrt{N}}$, one can see it is
$O \left( \frac{1}{\sqrt{N}} \right)$. Using this fact and simplifying the first term, (105)
can be written as:

$$ w(s) = w_a(s) \left( 1 + \frac{\sqrt{\tau(1-\tau)(\sqrt{N-1}-\sqrt{N}) + \tau}}{\alpha_0^2 \zeta + \sqrt{\tau(1-\tau) + N\tau}} \right)^{\frac{1}{2}} + O \left( \frac{1}{\sqrt{N}} \right). \tag{106} $$

Taking a first order Taylor expansion of the coefficient of
$w_a(s)$ in (106) we obtain the desired result.

Lemma 4. By neglecting the terms $\frac{\beta}{\sqrt{N}}$ and $\frac{\alpha_0}{\sqrt{N}} \sqrt{\tau(1-\tau)}$
in (23) we obtain:

$$ w(s) = \beta - \alpha_0 M \left( \sqrt{\frac{\tau(1-\tau)}{\alpha_0^2 \zeta + \sqrt{\tau(1-\tau) + N\tau}}} + O \left( \frac{1}{\sqrt{N}} \right) \right). \tag{107} $$

Proof. We denote the first term in (107) by $w_a'(s)$. Factoring
out $\frac{\beta}{\sqrt{N}}$ from the first term in (23) and resorting to some
simplifications, (23) is re-written as follows:

$$ w(s) = \frac{\sqrt{N\tau(1-\tau) + \tau}}{\sqrt{\alpha_0^2 \zeta + \sqrt{\tau(1-\tau) + N\tau}}} + O \left( \frac{1}{\sqrt{N}} \right). \tag{108} $$

Then, multiplying and dividing the first term in (108) by
$\sqrt{\alpha_0^2 \zeta + \tau}$ leads to:

$$ w(s) = w_a'(s) \frac{\sqrt{\alpha_0^2 \zeta + \sqrt{\tau(1-\tau) + N\tau}}}{\sqrt{\alpha_0^2 \zeta + \sqrt{\tau(1-\tau) + N\tau}}} + O \left( \frac{1}{\sqrt{N}} \right), \tag{109} $$

which is equivalent to:

$$ w(s) = w_a'(s) \left( 1 + \frac{s\sqrt{\tau(1-\tau)}}{(\sqrt{\alpha_0^2 \zeta + \sqrt{\tau(1-\tau) + N\tau}}} \right)^{-\frac{1}{2}} + O \left( \frac{1}{\sqrt{N}} \right). \tag{110} $$

Taking a first order Taylor expansion of the coefficient of
$w_a'(s)$ in (110) we obtain the desired result.

Lemma 5. By ignoring the term $s\sqrt{\tau(1-\tau)}$ in (24), we obtain:

$$ w = \frac{\sqrt{N\tau + \tau}}{\sqrt{\alpha_0^2 \zeta + \sqrt{\tau(1-\tau)}}}, $$

Proof. Since the term $\frac{\sqrt{\tau(1-\tau)}}{\sqrt{\alpha_0^2 \zeta + \sqrt{\tau(1-\tau)}}}$ is independent of $N$ we make a
constant error in neglecting it from (24).
Lemma 6. Taking \(Q(w(s))\) outside of the integral in (20) leads to
\[
p_c = 1 - Q(w) + O\left(\frac{1}{\sqrt{N}}\right),
\]
(112)

Proof. We first rewrite \(w(s)\), given in (24), as follows:
\[
w(s) = \sqrt{N} \left(\tau - \alpha_p \zeta\right) + \sqrt{s} \frac{\tau(1 - \tau)}{\alpha_p^2 s + \tau} + O\left(\frac{1}{\sqrt{N}}\right),
\]
(113)
and denote the first term by \(w\). We denote the error probability in (20) by \(p_t\) and the one in (26) by \(p_a\). The absolute error between \(p_t\) and \(p_a\) is then given by:
\[
|p_t - p_a| = \left|\frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} (Q(w) - Q(w(s))) e^{-\frac{s^2}{2}} ds\right) + O\left(\frac{1}{\sqrt{N}}\right)\right|.
\]
(114)

Using the definition of the Q-function in (19) we can rewrite (114) as follows:
\[
|p_t - p_a| = \left|\frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} w^{w(s)} e^{-\frac{s^2}{2}} ds\right) + O\left(\frac{1}{\sqrt{N}}\right)\right|.
\]
(115)

Since \(e^{-\frac{s^2}{2}} \leq 1\ \forall t\) we can replace the inner integrand function by 1 thereby leading to:
\[
|p_t - p_a| \leq \left|\frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} (w(s) - w) e^{-\frac{s^2}{2}} ds + O\left(\frac{1}{\sqrt{N}}\right)\right)\right|.
\]
\[
= \left|\frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} s \sqrt{\frac{\tau(1 - \tau)}{\alpha_p^2 s + \tau}} + O\left(\frac{1}{\sqrt{N}}\right)\right) e^{-\frac{s^2}{2}} ds + O\left(\frac{1}{\sqrt{N}}\right)\right|.
\]
\[
= \left|\frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} \frac{\tau(1 - \tau)}{\alpha_p^2 s + \tau} + \int_{-\infty}^{\infty} \frac{s}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} ds + O\left(\frac{1}{\sqrt{N}}\right)\right)\right|.
\]
(116)

The remaining integral in (116) is nothing but the expected value of a zero-mean Gaussian RV and therefore we have:
\[
|p_t - p_a| \leq \frac{K}{\sqrt{N}},
\]
(117)
for some positive constant \(K\). Since the absolute error is \(O\left(\frac{1}{\sqrt{N}}\right)\) we obtain the desired result.

Lemma 7. The \(O\left(\frac{1}{\sqrt{N}}\right)\) term in (27) can be taken out of the denominator in the AoI expression and the AoI expression becomes
\[
\Delta(\zeta, N, \rho, \tau) = \frac{1}{\tau} \left(1 - Q\left(\frac{\sqrt{N}(\alpha_p \zeta - \tau)}{\sqrt{\alpha_p^2 s + \tau}}\right)\right) + O\left(\frac{1}{\sqrt{N}}\right).
\]
(118)

Proof. We denote the first term in (118) by \(\Delta_a\) and define \(\Delta_t\) as:
\[
\Delta_t = \frac{1}{\tau} \left(1 - Q\left(\frac{\sqrt{N}(\alpha_p \zeta - \tau)}{\sqrt{\alpha_p^2 s + \tau}}\right)\right) + O\left(\frac{1}{\sqrt{N}}\right).
\]
(119)
The absolute error between \(\Delta_t\) and \(\Delta_a\) is given by,
\[
|\Delta_t - \Delta_a| = \left|\frac{1}{\tau} \left(1 - Q\left(\frac{\sqrt{N}(\alpha_p \zeta - \tau)}{\sqrt{\alpha_p^2 s + \tau}}\right)\right) - \left(1 - Q\left(\frac{\sqrt{N}(\alpha_p \zeta - \tau)}{\sqrt{\alpha_p^2 s + \tau}}\right)\right)\right|.
\]
Combining the two terms and resorting to some simplifications we obtain:
\[
|\Delta_t - \Delta_a| = O\left(\frac{1}{\sqrt{N}}\right).
\]
(120)
The fact that the absolute error between \(\Delta_t\) and \(\Delta_a\) is \(O\left(\frac{1}{\sqrt{N}}\right)\) implies the desired result in (118).

C. Finding the Spectral Efficiency from the Packet Error Probability

From (27), we see that the condition \(p_c < \epsilon\) leads to:
\[
Q\left(\frac{\sqrt{N}(\alpha_p \zeta - \tau)}{\sqrt{\alpha_p^2 s + \tau}}\right) < \epsilon_1,
\]
(121)
where \(\epsilon_1 = \epsilon + O\left(\frac{1}{\sqrt{N}}\right)\). Inverting the Q-function and re-arranging the terms yields:
\[
\zeta \left(\frac{\zeta - Q^{-1}(\epsilon_1)^2}{N}\right) \alpha_p^2 - (2\zeta)\alpha_p + \tau \left(\tau - Q^{-1}(\epsilon_1)^2\right) > 0.
\]
(122)
The left-hand side of (121) equation is a quadratic form of \(\alpha_p\) whose zeroes are given by:
\[
\alpha_p^+ = \frac{1 + \sqrt{1 - \frac{\tau Q^{-1}(\epsilon_1)^2}{N}}}{\tau - \frac{\tau Q^{-1}(\epsilon_1)^2}{N}},
\]
(123)
and
\[
\alpha_p^- = \frac{1 - \sqrt{1 - \frac{\tau Q^{-1}(\epsilon_1)^2}{N}}}{\tau - \frac{\tau Q^{-1}(\epsilon_1)^2}{N}},
\]
(124)
Recall that \(\alpha_p = \frac{1}{\sqrt{\eta}}\) and since \(\rho \geq 0\) then \(\alpha_p\) is non-negative.

Case 1: \(\zeta > Q^{-1}(\epsilon_1)^2\)

In the case that \(\zeta > Q^{-1}(\epsilon_1)^2\), the parabola in (122) is convex with real roots and, therefore, the inequality in (122) is satisfied whenever \(\alpha_p < \alpha_p^-\) or \(\alpha_p > \alpha_p^+\). Note that \(\alpha_p^+\) is always positive and any \(\alpha_p > \alpha_p^+\) is a valid solution. However, if \(\tau \leq \frac{Q^{-1}(\epsilon_1)^2}{N}\), \(\alpha_p^-\) will be non-positive and solutions \(\alpha_p < \alpha_p^-\) are invalid as \(\alpha_p^\) must be non-negative. Yet, even in the case \(\tau > \frac{Q^{-1}(\epsilon_1)^2}{N}\), \(\alpha_p < \alpha_p^-\) leads to incorrect solutions for small \(\epsilon_1\). Re-writing (121), we have:
\[
\frac{\sqrt{N}(\alpha_p \zeta - \tau)}{\sqrt{\alpha_p^2 s + \tau}} > Q^{-1}(\epsilon_1).
\]
(125)
Now if \(\epsilon_1\) is small, i.e. smaller than \(\frac{1}{2}\), then \(Q^{-1}(\epsilon_1) > 0\) and hence the term on the left-hand side of (125) should be
positive as well. In this case, $\alpha_\rho$ should be greater than $\frac{1}{N}$ and re-writing $\alpha_\rho^-$ as:

$$\alpha_\rho^- = \frac{\tau}{\zeta} \left(1 - \frac{1 - (1 - \frac{Q^{-1}(\epsilon_1)^2}{N \zeta})(1 - \frac{Q^{-1}(\epsilon_1)^2}{N \tau})}{1 - \frac{Q^{-1}(\epsilon_1)^2}{N \zeta}}\right),$$

we see that this is equivalent to the term inside the brackets in (126) being greater than 1. In that case, we have:

$$1 - \frac{1 - (1 - \frac{Q^{-1}(\epsilon_1)^2}{N \zeta})(1 - \frac{Q^{-1}(\epsilon_1)^2}{N \tau})}{1 - \frac{Q^{-1}(\epsilon_1)^2}{N \zeta}} > 1 - \frac{Q^{-1}(\epsilon_1)^2}{\zeta N}.$$  

(127)

This implies:

$$\left(\frac{Q^{-1}(\epsilon_1)^2}{N \zeta}\right)^2 > \left(\frac{Q^{-1}(\epsilon_1)^2}{N \zeta}\right)\left(1 - \frac{Q^{-1}(\epsilon_1)^2}{N \tau}\right) + \frac{Q^{-1}(\epsilon_1)^2}{N \tau}.$$  

(128)

Dividing both sides of (128) by $\left(\frac{Q^{-1}(\epsilon_1)^2}{N \zeta}\right)$ and rearranging the terms, we obtain:

$$\left(\frac{Q^{-1}(\epsilon_1)^2}{N \zeta}\right)\left(1 + \frac{\zeta}{\tau}\right) > \left(1 + \frac{\zeta}{\tau}\right),$$

from which it follows that:

$$\zeta < \frac{Q^{-1}(\epsilon_1)^2}{N}.$$  

(130)

As we are assuming $\zeta > \frac{Q^{-1}(\epsilon_1)^2}{N}$ we have a contradiction and thus $\alpha_\rho < \alpha_\rho^-$ is an invalid solution.

Case 2: $\zeta < \frac{Q^{-1}(\epsilon_1)^2}{N}$

In this case, the quadratic form involved in (122) is concave. Its zeroes are still given by (124) and (124) and solutions to (122) are $\alpha_\rho^+ < \alpha_\rho < \alpha_\rho^-$. In this case, $\alpha_\rho^+$ is always negative and only when $\tau > \frac{Q^{-1}(\epsilon_1)^2}{N}$, $\alpha_\rho^-$ is positive. Since $\alpha_\rho$ must always be non-negative, valid solutions to (122) are $0 < \alpha_\rho < \alpha_\rho^-$ under the condition that $\tau > \frac{Q^{-1}(\epsilon_1)^2}{N}$. Yet, even we will show that this leads to invalid solutions. If $\epsilon_1$ is small, i.e. less than $\frac{1}{N}$, then $Q^{-1}(\epsilon_1)$ is positive and so $\alpha_\rho^+ - \tau$ must be greater than 0 or we have a contradiction. Assuming that $\alpha_\rho^+ - \tau$ is positive, it follows that:

$$\alpha_\rho^+ - \tau > 0,$$  

(131)

which is equivalent to:

$$\frac{\tau}{\zeta} \left(1 + \frac{(\frac{Q^{-1}(\epsilon_1)^2}{N \zeta} - 1)(1 - \frac{Q^{-1}(\epsilon_1)^2}{N \tau})}{\frac{Q^{-1}(\epsilon_1)^2}{N \zeta} - 1}\right) > \frac{\tau}{\zeta},$$

where in (132) we multiplied the numerator and denominator of $\alpha_\rho^-$ given in (124) by $\frac{1}{N \zeta}$. Simplifying (132) further leads to:

$$\zeta > \frac{Q^{-1}(\epsilon_1)^2}{N}.$$  

(133)
[23] B. Li, R. Li, and A. Eryilmaz, “Throughput-optimal scheduling design with regular service guarantees in wireless networks,” *IEEE/ACM Transactions on Networking*, vol. 23, no. 5, pp. 1542–1552, 2014.

[24] A. Shah and A. M. Haimovich, “Performance analysis of maximal ratio combining and comparison with optimum combining for mobile radio communications with cochannel interference,” *IEEE Transactions on Vehicular Technology*, vol. 49, no. 4, pp. 1454–1463, 2000.

[25] V. K. Amalladinne, A. Vem, D. K. Soma, K. R. Narayanan, and J.-F. Chamberland, “A coupled compressive sensing scheme for unsourced multiple access,” in *2018 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, 2018, pp. 6628–6632.

[26] R. Calderbank and A. Thompson, “Chirrap: a practical algorithm for unsourced multiple access,” *Information and Inference: A Journal of the IMA*, vol. 9, no. 4, pp. 875–897, 2020.

[27] A. Pradhan, V. Amalladinne, A. Vem, K. R. Narayanan, and J.-F. Chamberland, “A joint graph based coding scheme for the unsourced random access gaussian channel,” in *2019 IEEE Global Communications Conference (GLOBECOM)*. IEEE, 2019, pp. 1–6.

[28] A. K. Pradhan, V. K. Amalladinne, K. R. Narayanan, and J.-F. Chamberland, “Polar coding and random spreading for unsourced multiple access,” in *ICC 2020-2020 IEEE International Conference on Communications (ICC)*. IEEE, 2020, pp. 1–6.

[29] V. Shtianov, F. Bellili, A. Mezghani, and E. Hossain, “Massive unsourced random access based on uncoupled compressive sensing: Another blessing of massive mimo,” *IEEE Journal on Selected Areas in Communications*, vol. 39, no. 3, pp. 820–834, 2020.

[30] A. Fengler, G. Caire, P. Jung, and S. Haghighatshoar, “Massive mimo unsourced random access,” *arXiv preprint arXiv:1901.00828*, 2019.

[31] M. J. Wainwright, *High-dimensional statistics: A non-asymptotic viewpoint*. Cambridge University Press, 2019, vol. 48.

[32] A. Fengler, S. Haghighatshoar, P. Jung, and G. Caire, “Non-bayesian activity detection, large-scale fading coefficient estimation, and unsourced random access with a massive mimo receiver,” *IEEE Transactions on Information Theory*, vol. 67, no. 5, pp. 2925–2951, 2021.

[33] R. G. Gallager, *Stochastic Processes: Theory for Applications*. Cambridge University Press, 2013.