BPS state counting on singular varieties

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Abstract

We define new partition functions for theories with targets on toric singularities via products of old partition functions on crepant resolutions. We compute explicit examples and show that the new partition functions turn out to be homogeneous on MacMahon factors.

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1. Motivation for counting BPS states

BPS states are minimal energy states of supersymmetric field theories. These special states have had a crucial role in establishing various duality symmetries of superstring theory. One of the reasons for their pivotal role in studying dualities stems from the availability of information on exact masses and degeneracies of these states. The counting of BPS states is of great interest to string theory and supergravity. In certain instances, the counting of BPS states agrees with the counting of extremal black holes \cite{S1, IS}. In some cases, the string partition function matches with the black hole partition function, leading to a precise equivalence between the black hole entropy and the statistical entropy associated with an ensemble of BPS states \cite{S2}. Degeneracy of states is encoded in a partition function. Degeneracy of BPS D-branes in string theory depends on the background geometry. The spectrum of BPS D-branes changes across walls in the moduli space. As the moduli of the background are varied, the number of states can jump across walls of marginal stability. The walls thus partition the moduli space into chambers. In other words, across wall a BPS state may disappear, or ‘decay’, giving rise to a different spectrum of BPS states. The original BPS state is thus stable in a specific chamber, while the decay products are stable in another. Indeed, when D-branes are realized as BPS states, they are defined by the stable BPS states only. Characterizing the jumps of degeneracy of BPS states across walls in the moduli space, notwithstanding the continuity of appropriate
correlation functions, has been of immense interest recently [OSY, N, KS1, KS2, MMNS]. These studies unearthed a rich mathematical structure within the scope of topological string theories.

A class of BPS states in topological string theories is furnished by D-branes wrapping homology cycles of the target space. These D-branes as well as their bound states are described as objects in the derived category of coherent sheaves of the target space or objects in the Fukaya category, within the scope of the topological B and A models, respectively. On a Calabi–Yau target, the walls of marginal stability are detected from the alignment of charges of the D-branes in the spectrum. Across a wall, a D-brane decays into a finite or infinite collection of branes, with the charge of the parent brane aligning with the totality of charges of the products on the wall. The partition function of these branes can be calculated giving their degeneracies.

The partition function of the A-model generates the Gromov–Witten (GW) invariants of Calabi–Yau threefolds from the world-sheet perspective. From the target space perspective, it counts the Gopakumar–Vafa (GV) invariants. The GW invariants count holomorphic curves on the threefold, whereas the GV invariants count BPS states of spinning black holes in five dimensions obtained from M2-branes in M-theory on the Calabi–Yau threefold [AOVY]. Considering the topological A-model on the target $R^3 \times X \times S^1$, where $X$ denotes the Calabi–Yau space without 4-cycles and $S^1$ designates the compact Euclidean temporal direction, the partition function also counts the number of D0- and D2-brane bound states on a single D6-brane wrapped on $X$. M5-branes wrapping 4-cycles in $X$ may form bound states with M2-branes; these complications do not arise in the absence of 4-cycles in $X$ [AOVY]. From another point of view, the partition function of the A-model is also the generating function of the Donaldson–Thomas (DT) invariants in appropriate variables. Thus, the study of the degeneracy of states relates the GW, GV and DT invariants.

For a singular variety, for example an orbifold, the product of the partition functions for all its crepant resolutions may be considered. The homology groups of the crepant resolutions are isomorphic. For the BPS D-branes, the crepant resolutions correspond to different spectra of stable objects in different chambers with the isomorphism of homologies given by Seiberg duality. The product partition function then corresponds to a quiver variety, which is realized near the singularity or the orbifold point, possessing a derived equivalence with the crepant resolutions [Sz, N, Y]. However, different isomorphisms of homologies yield different partition functions. Here, we define a partition function for the generalized conifolds as the product of the crepant resolutions as above, but the isomorphism of the second homology groups is given by a direct identification of elements in terms of certain formal variables under a canonical ordering. In proving the main theorem on the homogeneity of the new partition function, we use a probabilistic argument which appears to relate the exponent of homogeneity to some kind of degeneracy of the singular variety. Finally, we discuss some combinatoric aspects of the T-dual type-IIA brane configurations with NS and NS′ branes corresponding to the crepant resolutions of $C_{m,n}$, which is related to the partition function of the quiver variety. We write down explicit formulas for the generalized conifold $C_{1,3}$ and compare the two partition functions.

2. New partition function via formal identification and main results

Let $X$ be a singular variety admitting a finite collection of crepant resolutions $X' \to X$ for an index $t \in T$, $|T| < \infty$. If a singular variety admits crepant resolutions each of which have trivial canonical bundle, then it will be called a singular Calabi–Yau variety. Let $X$ be a singular Calabi–Yau variety. Let us further assume that a partition function $Z_{old}(Y; Q, \ldots)$
is defined for a smooth Calabi–Yau space $Y$, where $Q = (Q_1, Q_2, \ldots)$ are formal variables corresponding to a basis of $H_2(Y; \mathbb{Z})$. Finally, we suppose that $H_2(X^s; \mathbb{Z}) \cong H_2(X^t; \mathbb{Z})$ for all $s, t \in T$.

We then define a new partition function as the product of partition functions of the resolutions,

$$Z_{\text{new}}(X; Q, \ldots) := \prod_{t \in T} Z_{\text{old}}(X^t; Q^t, \ldots).$$

The new partition function contains information about all crepant resolutions of $X$ and may thus be regarded as pertaining to the singular space $X$ itself. In the product, we do not include partial resolutions as they are contained in the full resolutions and their inclusion will but cause non-illuminating repetitions. This approach can be applied to various partition functions defined for Calabi–Yau spaces. In this paper, we restrict ourselves to partition functions of curve-counting type such as the GW and the DT partition functions.

The properties of the new partition function depend on the prescribed isomorphism of second homologies of the resolutions. Assuming a canonical ordering of elements of $H_2(X^t, \mathbb{Z})$, for all $t \in T$, we identify the formal variables $Q$ among all the resolutions giving the isomorphism of homologies by setting

$$Q^t_i = Q^t_i =: Q_i \quad \text{for all} \quad s, t \in T.$$  \hspace{1cm} (2.1)

The presence of 4-cycles in the resolutions complicates the ordering of second homologies. We shall restrict to varieties whose crepant resolutions do not possess homology 4-cycles.

By a singular Calabi–Yau threefold without contractible curves and/or compact 4-cycles, we refer to a singular Calabi–Yau variety admitting crepant resolutions; the latter containing no contractible curve and/or compact 4-cycle. We prove the following.

**Theorem.** Let $X$ be a singular toric Calabi–Yau threefold defined as a subset of $\mathbb{C}^4$ by $X = \mathbb{C}[x, y, z, w]/(xy - z^n w^m)$, where $m$ and $n$ are integers. Let $Z(Y; q, Q)$ be a partition function of curve-counting type (definition 5.6). Then, the total partition function,

$$Z_{\text{tot}}(X; q, Q) := \prod_{Y \rightarrow X} Z(Y; q, Q),$$

where the product ranges over all crepant resolutions of $X$, is homogeneous (definition 5.2) of degree

$$d = \frac{(m^2 - m + n^2 - n - 2mn)(m + n - 2)!}{m! n!}.$$  

In performing curve counting, the Calabi–Yau space is allowed to have contractible curves as well (corollary 5.9) in particular obtaining a counting of BPS states via the topological string partition function (corollary 5.11).

3. The mathematics of curve counting

3.1. GW theory

**Definition 3.1.** By a curve, we mean a reduced scheme $C$ of pure dimension 1. The genus of $C$ is $g(C) := h^1(C; \mathcal{O}_C)$.

**Corollary 3.2.** A connected curve $C$ of genus-0 is a tree of rational curves.
Definition 3.3. An $n$-pointed curve $(C; P_1, \ldots, P_n)$ is called prestable if either every point of $C$ or a node singularity and the points $P_1, \ldots, P_n$ are smooth. A map $f : C \to X$ is called stable, if $(C; P_1, \ldots, P_n)$ is prestable and there are at least three marked or singular points on each contracted component.

Remark 3.4. Stability prohibits first-order infinitesimal deformations to the map $f$.

Let us denote by $\overline{M}_{g,n}(X, \beta)$, the collection of maps from stable, $n$-pointed curves of genus-$g$ into $X$ for which $[f(C)] = f_*(C) = \beta \in H_2(X; \mathbb{Z})$.

Behrend and Fantechi [BF1] showed that this has a coarse modulus (Deligne–Mumford) stack, Vistoli [V] studied the intersection theory on $\overline{M}_{g,n}(X, \beta)$ and constructed a perfect obstruction theory, and [BF1] also showed that there exists a virtual fundamental class of virtual dimension

$$vd = (1 - g)(\dim X - 3) - K_X(\beta) + n.$$ (We assume that $X$ does in fact have a canonical class $K_X \in H^2(X; \mathbb{Z})$, e.g. if $X$ is smooth.) Consequently, the dimension of the classes $[\overline{M}_{g,n}(X, \beta)]^{vir}$ is independent of $\beta$ when $K_X = 0$, that is, when $X$ is Calabi–Yau. Moreover, the unpointed moduli $\overline{M}_{0,0}(X, \beta)$ has virtual dimension zero for all $g$ if $\dim X = 3$, so on a three-dimensional Calabi–Yau, $\overline{M}_{0,0}(X, \beta)$ really ‘counts curves’.

Definition 3.5. Assume that $g(C) = 0$. Let $ev_i : \overline{M}_{0,n}(X, \beta) \to X$, $(f : (C; P_1, \ldots, P_n) \to X) \mapsto f(P_i)$ be the $i$th evaluation map. Assume that $\sum_{i=1}^n \deg(\gamma_i) = vd$ for some $\gamma_i \in H^*(\overline{M}_{0,n}(X, \beta))$. Then, the genus-$0$ GW invariants are

$$\langle \gamma_1, \ldots, \gamma_n \rangle_{\beta} := ev_1^*(\gamma_1) \cup \cdots \cup ev_n^*(\gamma_n) \cap [\overline{M}_{0,n}(X, \beta)]^{vir}.$$ For higher genera, the definition of the GW invariants requires the introduction of additional data called as descendent fields. Since we require only genus-$0$ for our purposes, we refer the interested reader to [MNOP2, section 2].

When $\dim X = 3$, $X$ is Calabi–Yau (i.e. $K_X = 0$), arbitrary genus-$g$ and $n = 0$, we have the unmarked GW invariants

$$N_{g,\beta}(X) := \int_{\overline{M}_{g,0}(X, \beta)} 1.$$ Example 3.6. If $X = \{pt\}$, then $\overline{M}_{g,0}(X, \beta) = \overline{M}_{g,n}$, the moduli of $n$-pointed curves.

Example 3.7. For $X = \mathbb{P}^1$, the genus-$0$ GW invariants are just the Hurwitz numbers.

The (unmarked) GW invariants are usually assembled into the unreduced and reduced generating functions, respectively,

$$F(X; u, v) = \sum_{\beta} \sum_{g \geq 0} N_{g,\beta}(X) u^{2g-2} v^\beta,$$

and

$$F'(X; u, v) = \sum_{\beta \neq 0} \sum_{g \geq 0} N_{g,\beta}(X) u^{2g-2} v^\beta,$$
where \( v = (v_1, \ldots, v_r) \) is an appropriate vector that can be paired with the \( r \) generators of \( H_2(X; \mathbb{Z}) \). The unreduced and reduced \textit{GW partition functions} are, respectively,

\[
Z_{GW}(X; u, v) = \exp F(X; u, v) = 1 + \sum_{\beta} Z_{GW}(X; u)_{\beta} v^\beta
\]

and

\[
Z'_{GW}(X; u, v) = \exp F'(X; u, v) = 1 + \sum_{\beta \neq 0} Z'_{GW}(X; u)_{\beta} v^\beta,
\]

where the last expressions define the homogeneous terms \( Z(X; u)_{\beta} \) and \( Z'(X; u)_{\beta} \) of degree \( \beta \).

### 3.2. DT theory

An \textit{ideal subsheaf} of \( O_X \) is a sheaf \( \mathcal{I} \) such that \( \mathcal{I}(U) \) is an ideal in \( O_X(U) \) for each open set \( U \subseteq X \). Alternatively, it is a torsion-free rank-1 sheaf with trivial determinant. It follows that \( \mathcal{I} \sim O_X \).

Thus, the evaluation map determines a quotient

\[
0 \rightarrow \mathcal{I} \rightarrow \mathcal{I} \sim O_X \rightarrow O_X/Io_X \rightarrow O_Y \\
(3.1)
\]

where \( Y \subseteq X \) is the support of the quotient and \( O_Y := (O_X/Io_X)|_{Y} \) is the structure sheaf of the corresponding subspace. Let \( [Y] \in H_2(X; \mathbb{Z}) \) denote the cycle class determined by the one-dimensional components of \( Y \) with their intrinsic multiplicities. We denote by

\[
I_n(X, \beta),
\]

the Hilbert scheme of ideal sheaves \( \mathcal{I} \subset O_X \) for which the quotient \( Y \) in (3.1) has dimension at most 1, \( \chi(O_Y) = n \) and \( [Y] = \beta \in H_2(X; \mathbb{Z}) \).

The work of DT was to show that \( I_n(X, \beta) \) has a canonical perfect obstruction theory (originally when \( X \) is smooth, projective and \( -K_X \) has non-zero sections) and a virtual fundamental class \( [I_n(X, \beta)]^{vir} \) of virtual dimension \( \int Y c_1(T_X) = -K_X(\beta) \). If \( X \) is a smooth, projective Calabi–Yau threefold, then the virtual dimension is zero, and we write

\[
\tilde{N}_{n, \beta}(X) := \int_{[I_n(X, \beta)]^{vir}} 1
\]

for the ‘number’ of such ideal sheaves. We assemble these numbers into a(unreduced) partition function,

\[
Z_{DT}(X; q, v) = \sum_{\beta} \sum_{n \in \mathbb{Z}} \tilde{N}_{n, \beta}(X) q^n v^\beta = \sum_{\beta} Z_{DT}(X; q)_{\beta} v^\beta,
\]

where again the last expression defines the unreduced terms of degree \( \beta \). The degree-0 term

\[
Z_{DT}(X; q)_0 = \sum_{n \geq 0} \tilde{N}_{n, \beta}(X) q^n
\]

is of special importance. We define the \textit{reduced DT partition function} as

\[
Z'_{DT}(X; q, v) = Z_{DT}(X; q, v)/Z_{DT}(X; q)_0 = 1 + \sum_{\beta \neq 0} Z'_{DT}(X; q)_{\beta} v^\beta,
\]

once again defining the reduced terms \( Z'_{DT}(X; q)_{\beta} \) of degree \( \beta \) implicitly.
3.3. The MNOP conjecture

For a smooth Calabi–Yau threefold \( X_{CY} \), the MNOP conjecture relates the reduced GW and DT partition functions,

\[
Z'_{GW}(X_{CY}; u, v) = Z'_{DT}(X_{CY}; -e^{iu}, v),
\]
signifying an equivalence between the GW and DT theories for Calabi–Yau threefolds. Proof of the MNOP relation was furnished for toric (hence non-compact) Calabi–Yau threefolds in [MNOP1, MNOP2] and for compact Calabi–Yau manifolds in [BF2, L].

We shall illustrate features of the new partition function using the DT partition function, for which toric computational techniques have been developed by [LLLZ]. We consider a special class of threefolds admitting crepant resolutions without compact 4-cycles, obtained as orbifolds of the conifold or their partial resolutions [MP, U, vU, N].

4. Generalized conifolds

Given a pair of non-negative integers \( m, n \), we consider the toric varieties \( C_{m,n} := \{(x, y, z, w)|xy - zmw^n = 0\} \subset \mathbb{C}^4 = \text{Spec} \mathbb{C}[x, y, z, w] \).

We suppose \( n \geq m \) without any loss of generality. Two cases arise:

(i) \( n > m = 0 \). Then, \( C_{0,n} \) are quotients of \( \mathbb{C}^3 \) by \( \mathbb{Z}/n\mathbb{Z} \) acting on a two-dimensional subspace \( \mathbb{C}^2 \) as \((a, b, c) \mapsto (\varepsilon a, \varepsilon^{-1}b, c)\), with \( \varepsilon^n = 1 \). These spaces have one-dimensional singularities, as \( C_{0,n} \cong K_n \times \mathbb{C} \), where \( K_n = \{(x, y, z)|xy - z^n = 0\} \subset \mathbb{C}^3 \) is the Kleinian surface, with a singular point at the origin.

(ii) \( n \geq m \geq 1 \). The space \( C_{1,1} = \{(x, y, z, w)|xy - zw = 0\} \subset \mathbb{C}^4 \) is the conifold. All other \( C_{m,n} \) are obtained either as quotients of the conifold, if \( n = m \), or through their partial resolutions, otherwise.

Let us briefly describe \( C_{m,n} \), referred to as a \textit{generalized conifold} in the sequel, as a toric variety. The toric fan of \( C_{m,n} \) is generated by a three-dimensional cone \( \sigma \) with ray generators \( v_i, i = 1, 2, 3, 4 \), which are vectors in a lattice \( N \) of rank 3 in \( \mathbb{R}^3 \) given by the columns of the matrix

\[
\begin{pmatrix}
v_1 & v_2 & v_3 & v_4 \\
0 & 0 & m & n \\
0 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 \\
\end{pmatrix},
\]

all of which lie on the height-one \( z \)-plane along the perimeter of a trapezoid, thereby rendering the canonical divisor trivial. The inward-pointing normals to the facets subtended by a pair of these vectors given by their cross products, namely \( n_i = v_{i+1} \times v_i \) in cyclic order, define the semigroup \( S_{\sigma} = \sigma^\vee \cap M \), with \( M \) being the dual lattice of \( N \). The dual cone is

\[
\sigma^\vee = \{n \in \mathbb{R}^3 | (n, v) \geq 0, \forall v \in \sigma \}.
\]

The various vectors and the cone are depicted in figure 1(left). Then, \( S_{\sigma} \) is generated by the four columns of the following matrix:

\[
T = \begin{pmatrix}
n_1 & n_2 & n_3 & n_4 \\
1 & 0 & -1 & 0 \\
0 & -1 & m-n & 1 \\
0 & 1 & n & 0 \\
\end{pmatrix},
\]

6
which provides the toric data. The relation among these four three-dimensional vectors is given through the kernel of $T$:

$$\ker T = (1, -n, 1, -m)^T.$$  (4.4)

Hence, the toric variety $C_{m,n}$ is given by the equation

$$x_1x_3 - x_4^m x_2^n = 0.$$  (4.5)

Since all the ray generators $v_i$ lie in the height one $z$-plane, it suffices, especially for the purpose of exhibiting triangulations considered below, to draw the intersection of the cone $\sigma$ with this plane. We shall henceforth refer to the trapezoidal polygon on this plane formed by the vertices $(0, 0, 0)$, $(0, 1, 0)$, $(m, 1, 0)$ and $(n, 0, 0)$, illustrated below, as the toric data for the variety $C_{m,n}$.

In general, blowing up the singular locus of a generalized conifold results in a non-Calabi–Yau variety. This can be seen by constructing the star subdivision of the singular subcone. The new ray generator does not lie on the $z = 1$ hyperplane. However, small resolutions are crepant and therefore result in a smooth Calabi–Yau variety. We obtain these resolutions by triangulating the cone $\sigma$, as shown in figure 1(right). This is equivalent to constructing a lattice triangulation of the trapezoid:

Internal edges in the triangulation of the strip correspond to two-dimensional cones in the toric fan of the resolved threefold; they describe the irreducible components of the exceptional curve. The absence of lattice points in the interior of the cone signifies that the resolution does not contain compact 4-cycles. Its second homology is thus generated by the components of the exceptional curve. Each prime component of the exceptional set is a smooth rational curve.
Figure 2. The point $(1, 1)$ is contained in four triangles. A pair of neighbouring triangles intersect at an interior edge containing the point, while non-neighbouring triangles intersect at the point only.

We shall consider all possible crepant resolutions of $C_{m,n}$, which correspond to all maximal lattice triangulations of the strip (i.e. triangulations in which each triangle has area $\frac{1}{2}$). We shall abuse notation to use $C_{m,n}$ to refer to the strip as well as to the variety which it defines.

Let us first collect some combinatorial properties of these triangulations.

**Proposition 4.1.**

1. Each triangulation of the polygon $C_{m,n}$ has $N_F = m + n$ triangles and $N_E = m + n - 1$ interior edges.
2. There are $N_\triangle = \binom{m+n}{m}$ triangulations of $C_{m,n}$.
3. The Euler characteristic of any crepant resolution of $C_{m,n}$ is $m + n$.

**Proof.** The area of each regular triangle in a tessellation of the polygon is $\frac{1}{2}$, as mentioned above. The area of the trapezoid is $\frac{(m+n)}{2}$. Hence the number of triangles in each triangulation is $N_F = m + n$.

Since every interior edge of a triangulation emanates from a point in the upper row (also ends on a point in the lower), it suffices to count the number of lines emanating from the points in the upper row. Considering a triangulation, let $N_i$ denote the number of triangles containing the point $(i, 1)$ in the upper row; $0 \leq i \leq m$. These triangles have a totality of $\binom{N_i}{2}$ pairwise intersections of which $\binom{N_i - 1}{1}$ intersections are at the point alone, while the rest, $\binom{N_i - 1}{1} = N_i - 1$ intersections, are along an interior edge, containing the point (cf figure 2). Hence the total number of interior edges is $N_E = \sum_{i=0}^{m} (N_i - 1) = \sum_{i=0}^{m} N_i - (m + 1)$. On the other hand, since the point $(i, 1)$ is shared by $N_i$ triangles and there is a single triangle containing two consecutive points in the upper row, summing $N_i$ over all the points in the upper row counts the number of triangles with $m$ triangles counted twice, ergo $\sum_{i=0}^{m} N_i = N_E + m$. From these two expressions and the expression for $N_F$ obtained above, we have $N_E = m + n - 1$. This proves statement (1).

To count the number of triangulations, let us note that all of the $N_E$ interior edges start from one of the $m$ points in the upper row, which can happen in $\binom{N_E}{m}$ ways. Also, all of these lines end on one of the $n$ points in the bottom row, which can happen in $\binom{N_E}{n}$ ways. Adding, we have the number of triangulations $N_\triangle = \binom{m+n-1}{m} + \binom{m+n-1}{n} = \binom{m+n}{m}$, proving statement (2).

Finally, for any crepant resolution the Euler characteristic is $\chi(X) = h^0(X; \mathbb{Z}) + h^2(X; \mathbb{Z})$ in the absence of higher dimensional homology cycles. Moreover, for the cases at hand, the 2-cycles are given by the interior edges, so that $h^2(X; \mathbb{Z}) = N_E$, while $h^0(X; \mathbb{Z}) = 1$. Statement (3) follows.

**Aliter.** We can count the number $N_F$ in another way by observing that each triangle in a triangulation has a unique horizontal side, which is either at the top or at the bottom of the strip, corresponding to vertical coordinate 1 or 0, respectively. We shall refer to this side as the base of the triangle. Since there are $m$ segments on the top line and $n$ on the bottom, each
of which is the base of one and only one triangle, the number of triangles in a triangulation is $N_F = m + n$.

### 4.1. Enumerating triangulations

In the following, we require a means to enumerate triangulations and label its triangles and edges. There is a natural ordering of triangles in a triangulation ‘from left to right’. We start with the unique triangle $t_1$ having the line $(0, 0)–(0, 1)$ as its side and move towards the right across the unique other non-horizontal edge to arrive at the next triangle $t_2$. Continuing and labelling triangles on the way seriatim, we finally arrive at the unique triangle $t_{m+n}$ which has the line $(m, 1)–(n, 0)$ as its side. From the expression of $N_\triangle$ obtained above, it is clear that specifying the $m$ triangles based on the top line, or, alternatively, the $n$ triangles based on the bottom line, fixes a triangulation. However, since we have assumed $m \leq n$, the first choice is more economic and we shall adhere to it. Hence, we denote each triangulation of $C_{m,n}$ by a subset $T \subset \{1, 2, \ldots, N_F\}$ with length $|T| = m$, where the base of the triangle $t_k, k \in T$ is at the upper line of the strip and $\{t_1, \ldots, t_{m+n}\}$ denotes the set of all triangles. These are illustrated in the following.

**Example 4.2.** Let $m = 2$ and $n = 4$. Here are some of the triangulations of $C_{2,4}$ given by subsets of length 2 of $\{1, \ldots, 6\}$.

- $T = \{1, 2\}$:
  ![Diagram](Diagram1)

- $T = \{1, 3\}$:
  ![Diagram](Diagram2)

- $T = \{3, 6\}$:
  ![Diagram](Diagram3)

Interior edges are labelled using intersection of adjacent triangles. We define the $i$th edge $e_i$ as

$$e_i := t_i \cap t_{i+1}, \quad i = 1, \ldots, N_F.$$  

In a given triangulation $T \subset \{1, \ldots, N_F\}$, there are two possibilities for each edge $e_i$, namely it is either the intersection of two triangles $t_i$ and $t_{i+1}$ both having bases on the same horizontal line (top or bottom) of the strip or they have bases on different lines. In the former case either $i, i+1 \in T$ or $i, i+1 \notin T$, we say that $e_i$ is of type ‘+’ and draw the edge as a dashed line in the toric diagram. These correspond to $O(-2, 0)$ curves. In the latter case either $i \in T, i+1 \notin T$ or $i \notin T, i+1 \in T$, $e_i$ is said to be of type ‘−’ and we depict it by a solid line. These correspond to $O(-1, -1)$ curves. We let $\tau(e_i) = \pm 1$ according to whether $e_i$ is of type ‘+’ or ‘−’. This furnishes a canonical scheme for ordering and characterizing the edges, which correspond to bases of the second co-homology group of crepant resolutions.

### 4.2. Computing triangulations

In working with triangulations, implementation of the above scheme in computer programs is useful. Let us briefly discuss some aspects. The triangulation was carried out using the software
TOPCOM [TOP]. The function points2allfinetriangs. triangulates a strip using triangles of equal, minimal area producing a list of all possible triangulations.

In TOPCOM, points in a point set are given in homogeneous coordinates, so for our purposes the vertex \((i, j)\) corresponds to the point \([i, j, 1]\). We label the \(m + n + 2\) vertices sequentially, assigning the range \(0, \ldots , m\) to the vertices \(v_0 := [0, 0, 1], v_1 := [1, 0, 1], \ldots, v_m := [m, 0, 1]\), and the range \(m + 1, \ldots, m + n + 1\) to \(v_{m+1} := [0, 1, 1], v_{m+2} := [1, 1, 1], \ldots, v_{m+n+1} := [n, 1, 1]\). The output of TOPCOM consists of lists of triplets \((v_a, v_b, v_c)\) of vertices giving the triangulation of the strip. The internal edges in a triangulation are extracted from this list.

Their types are determined as follows. The natural ordering ‘from left to right’ of the non-horizontal edges is precisely the lexicographic ordering of either the top or the bottom vertices \((i, j)\). When the edges are ordered in this fashion, the \(k\)th edge, corresponding to the vertex \((i_k, j_k)\), is of type ‘+’ if \(j_{k-1} = j_k = j_{k+1}\) and \(i_{k-1} + 1 = i_k = i_{k+1} - 1\). Otherwise, it is of type ‘−’.

From this data, we can construct the partition function of any particular resolution of \(C_{m,n}\) corresponding to a specific triangulation.

5. Curve counting on singular varieties

For any complex threefold \((X, O_X)\), the Hilbert scheme of ideal sheaves \(\mathcal{I} \subset O_X\) with fixed Euler characteristic \(\chi(\mathcal{I}) = k\) and support \([\text{supp}(\mathcal{I})] = \beta \in H_2(X, \mathbb{Z})\), written \(I_k(X, \beta)\), has a perfect obstruction theory of virtual dimension \(\int_\beta c_1(T_X) = -K_X(\beta)\), see [DT]. When \(K_X = 0\), the numbers

\[
N_{k,\beta}(X) := \int_{[I_k(X,\beta)]^{vir}} 1
\]

are the DT invariants of \(X\). Let \(Q = (Q_1, \ldots, Q_h)\), \(h = \dim H_2(X, \mathbb{Z})\), be a set of symbols corresponding to generators of \(H_2(X, \mathbb{Z})\). The DT invariants are collected into the DT partition function:

\[
Z(X; q, Q) := \sum_{k=0}^{\infty} \sum_{\beta \in H_2(X, \mathbb{Z})} N_{k,\beta}(X) \ q^k \ Q^\beta,
\]

where \(Q^\beta = Q_1^{\beta_1} \cdots Q_h^{\beta_h}\). We single out the degree-0 contributions,

\[
Z_0(X; q) := \sum_{k=0}^{\infty} N_{k,0}(X) \ q^k,
\]

and we define the reduced DT partition function as

\[
Z'(X; q, Q) := Z(X; q, Q) / Z_0(X; q).
\]

For any smooth, toric threefold \(X\), we have \(K_X(0) = 0\) and so we can define the degree-0 partition function \(Z_0(X; q)\). It is known [MNOP1] that

\[
Z_0(X; -q) = M(1, q)^{\chi(T_X \otimes K_X)},
\]

and in particular if \(X\) is Calabi–Yau, then

\[
Z_0(X; -q) = M(1, q)^{\chi(X)},
\]

4 We are grateful to Jesus Martinez-Garcia for writing the program.
where \( \chi(X) \) denotes the Euler characteristic of \( X \) and
\[
M(x, q) := \prod_{k=1}^{\infty} \frac{1}{(1 - xq^k)^k} = \exp \sum_{j=1}^{\infty} \frac{i}{j} x^j q^{ij}.
\]
denotes the (generalized) MacMahon function. The nexus between the partition function and the MacMahon function originates from the fact that the MacMahon function counts box partitions, and degree-0 toric ideal sheaves are given by monomial ideals, which can indeed be arranged like ‘boxes stacked into a corner’.

5.1. DT invariants of generalized conifolds

If \( X \) is a crepant resolution of \( C_{m,n} \), then it is a smooth, toric Calabi–Yau threefold. The DT partition function can be computed combinatorially by the topological vertex method (see [LLLZ, IK]). We shall always take the curves corresponding to the interior edges \( e_i \) as our preferred basis for \( H_2(X; \mathbb{Z}) \), that is,
\[
\beta = \sum_{i=1}^{N_E} \beta[e_i] \in H_2(X; \mathbb{Z}),
\]
where \( N_E = m + n - 1 \), by proposition 4.1. Furthermore, we have \( \chi(X) = m + n \).

We need to establish some terminology to describe \( Z'(X; q, Q) \). A set \( P = \{i, i+1, \ldots, j\} \) is called an edge path if \( 1 \leq i \leq j \leq N_E \). It is to be thought of as a sequence of consecutive interior edges of the triangulation \( T \) of \( C_{m,n} \) corresponding to the resolution \( X \). An edge path \( P \) has length \( |P| := j - i + 1 \) and connects the triangles \( t_i \) and \( t_{i+1} \). In a triangulation, there are \( m + n - 1 \) edge paths of length 1, \( m + n - 2 \) of length 2, and so forth, and 1 of length \( m + n - 1 \), so in total there are \( \binom{m+n}{2} \) edge paths. An edge path is literally a path along the compact edges of the dual tropical curve of the triangulation \( T \).

If \( P = \{i, i+1, \ldots, j\} \) is an edge path, we write \( Q_P = Q_{ij} = Q_i \cdots Q_j \), so for example \( Q_{22} = Q_2 \) and \( Q_{35} = Q_3 Q_4 Q_5 \). We define
\[
f(P, q, Q) = M(Q_P, q)^{\tau(e_i) \tau(e_{i+1})^{-\tau(e_j)}}.
\]
Thus, \( f(P, q, Q) \) is either the MacMahon function or its reciprocal, depending upon whether \( P \) contains an even or an odd number of edges of type ‘−’. The whole partition function of \( X \) is the product of such terms over all edge paths, that is,
\[
Z'(X; q, Q) = \prod_{P} f(P, q, Q) = \prod_{1 \leq i < j \leq N_E} \prod_{|P|}^{\infty} (1 - Q_P q^{|P|})^{-k \tau(e_{i+1})^{-\tau(e_j)}}.
\]
(5.2)
Since this partition function is determined entirely by the triangulation, i.e. by the subset \( T \subset \{1, 2, \ldots, N_E\} \), \( |T| = m \), alluded to above, we write \( Z'_T(C_{m,n}; q, Q^T) \) for the partition function, where we abbreviate \( Q^T = (Q^T_1, \ldots, Q^T_{N_E}) \). We now consider the collection of all possible triangulations of \( C_{m,n} \).

**Definition 5.1.** We define the total partition function:
\[
Z'_{\text{tot}}(C_{m,n}; q, Q) := \prod_{T \subset \{1, 2, \ldots, N_E\}} Z'_T(C_{m,n}; q, Q).
\]
Let us consider the following ad hoc definition.
Definition 5.2. A partition function \( Z(q, Q) \) of variables \( Q = (Q_1, Q_2, \ldots) \) is called homogeneous if
\[
Z(q, Q) = \left( \prod M(\prod_{i \in A} Q_i, q) \right)^d,
\]
where the first product is over an arbitrary finite collection of index set \( A \subset \{1, 2, \ldots\} \). The exponent \( d \) is called the degree of \( Z \).

Example 5.3. Let us consider \( m = 1, n = 1 \), in which case the strip is a single square admitting two triangulations, namely

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{triangle1} \\
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{triangle2} \\
\end{array}
\]
yielding the partition function \( Z'_\text{tot}(C_{1,1}; -q, Q) = M(Q_1, q)^{-2} \), which is homogeneous with degree \(-2\). Triangulations on smaller strips can be extended to triangulations of bigger strips. To illustrate this, let us consider the following two ways to pass from a triangulation of \( C_{m,n} \) to a triangulation of \( C_{m,n+1} \). In the first case, the rightmost edge of \( C_{m,n} \) turns into an internal edge of \( C_{m,n+1} \) of ‘+’ type, as

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{triangle3} \\
\end{array}
\quad \rightarrow \quad
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{triangle4} \\
\end{array}
\]
The exponent of \( M(Q_{1,m+n-1}, q) \) coming from this triangulation of \( C_{m,n} \) is the same as the exponent of \( M(Q_{1,m+n}, q) \) for the corresponding triangulation of \( C_{m,n+1} \). Hence, there is a correspondence between such kinds of triangulations of the two strips, maintaining equality of exponents of the MacMahon factors. In the second case, the rightmost edge of \( C_{m,n} \) turns into an internal edge of \( C_{m,n+1} \) of ‘−’ type, as

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{triangle5} \\
\end{array}
\quad \rightarrow \quad
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{triangle6} \\
\end{array}
\]
Now in the triangulation on the right-hand side, we have MacMahon factors as \( M(Q_1, q)^{-1} \) and \( M(Q_1Q_2, q)^{-1} \), which appear to give rise to different exponents. However, since every parallelogram has two diagonals, there is another triangulation obtained by flopping the diagonal on rightmost parallelogram of the previous figure, and we obtain an extra triangulation of \( C_{m,n+1} \) (this one not coming from a triangulation of \( C_{m,n} \)) as

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{triangle7} \\
\end{array}
\]
that contributes factors of \( M(Q_1, q)^{-1} \) and \( M(Q_1Q_2, q)^{-1} \), cancelling out the seemingly unbalanced contributions from the previous one.

In general, we have the following.

Proposition 5.4. For \( 0 < m \leq n \), \( Z'_\text{tot}(C_{m,n}; -q, Q) \) is homogeneous of degree \( d \), where
\[
d = \frac{(m^2 - m + n^2 - n - 2mn)(m + n - 2)!}{m!n!},
\]
(5.3)
namely
\[
Z'_\text{tot}(C_{m,n}; -q, Q) = \prod_{1 \leq i < j \leq m+n-1} M(Q_{ij}, q)^{d},
\]
**Proof.** We first present a purely combinatorial proof. The proposition consists of two separate parts, and so does the proof. The first statement is that each MacMahon factor $M(Q_{ij}, q)$ appears with the same power in the total partition function.

We have to show that each MacMahon factor $M(Q_{ij}, q)$ appears with the same power in the total partition function and compute the value of this exponent. The problem is entirely combinatorial. In terms of finite sets, it takes the following form. Let us simply write $N$ for the finite set $\{1, 2, \ldots, N\}$. For any subset $T \subseteq N$ and any fixed, ordered subset $S = \{s_1, \ldots, s_k\} \subseteq N$, we define the characteristic sequence

$$\chi_T(S) := (\chi_T(s_1), \ldots, \chi_T(s_k)),$$

where $\chi_T : N \rightarrow \{0, 1\} = \mathbb{Z}/2\mathbb{Z}$ is the characteristic function of $T$. (It will be opportune to think of the two-element set as the additive group of order 2.)

In our application, we shall take $S$ to be a ‘contiguous’ subset of the form $\{i, i+1, \ldots, j\}$ corresponding to some edge path. For such a subset, we define the difference sequence as

$$\Delta_T(S) := (\chi_T(s_1) - \chi_T(s_2), \chi_T(s_2) - \chi_T(s_3), \ldots, \chi_T(s_{k-1}) - \chi_T(s_k)),$$

and we define the $T$-signature of $S$ as

$$\sigma_T(S) := \prod_{b \in \Delta_T(S)} (-1)^b \in [+1, -1].$$

(Since we are only interested in the $T$-signature, we may consider the elements of $\Delta_T(S)$ to take values in $\mathbb{Z}/2\mathbb{Z}$ and identify $+1$ and $-1$.) Finally, the exponent of $M(Q_{ij}, q)$ in the total partition function of $C_{m,n}$ is the $m$-signature of the set $S = \{i, i+1, \ldots, j\}$, defined as

$$\sigma(S) = \sum_{T \subseteq N : |T|=m} \sigma_T(S),$$

where $N = m + n$.

So much for the setup. The first observation is that any action $\pi \in \Sigma_N$ that preserves the contiguous ordering of the elements of $S$ does not alter the value of the total signature: $\sigma(\pi S) = \sigma(S)$. Therefore, we may assume without loss of generality that $S = \{1, 2, \ldots, k\}$.

Next, any subset $T \subseteq N$ with $|T| = m$ is of the form $T = U \cup T'$, where $U \subseteq \{1, 2, \ldots, k\}$ with $|U| = i$ and $T' \subseteq \{k+1, k+2, \ldots, N\}$ with $|T'| = m-i$ for $i = 0, 1, \ldots, k$. Now observe that all we need to compute the $m$-signature is $\Delta_U(S)$, or rather $\sigma_U(S) = \sigma_T(S)$. Since there are $\binom{N}{m}$ subsets in total, we have

$$\sigma(S) = \left|\{T : \sigma_T(S) = +1\}\right| - \left|\{T : \sigma_T(S) = -1\}\right| = \binom{N}{m} - 2\left|\{T : \sigma_T(S) = -1\}\right|.$$

The combinatorics of this are easily determined: subsets $T = U \cup T'$ for which $\sigma_U(S) = -1$ are those for which $\Delta_U(S)$ has an odd number of 1s, and there are $2\binom{k-2}{i-1}$ of those, where $i = |U|$. Summing over all $i$, we find

$$\sigma(S) = \binom{N}{m} - 4 \sum_{i=1}^{m-1} \binom{k-2}{i-1} \binom{N-k}{m-i}.$$ 

The last factor accounts for all the possible subsets $T'$. The sum evaluates to $\binom{N-2}{m-1}$ leading to

$$\sigma(S) = \binom{N}{m} - 4 \binom{N-2}{m-1} = \frac{(N^2 - N + 4m^2 - 4mn)(N-2)!}{m!(N-m)!}.$$ 

This is true for any contiguous set $S = \{i, i+1, \ldots, j\}$, and the result follows by substituting $N = m + n$.

The second statement is the value of the exponent. Since the exponent is the same for each factor $M(Q_{ij}, q)$ by the first part, we may compute it by just computing the exponent of
triangulations, the contribution to the partition function comes with the exponent connecting two triangles. If it connects two triangles both having bases on the top line, then the contribution to the partition function comes with a negative exponent. The number of even odd $1$'s is thus the sum of the number of triangulations of $C_{m-2,n}$ and $C_{m,n-2}$, and the number of odd $1$'s is twice the number of triangulations of $C_{m-1,n-1}$.

**Aliter.** We present another proof using probabilities. As discussed before, the interior edges $e_i, 1, 2, \ldots, N_E$ are numbered from left to right in a unique fashion. An edge path connects two triangles both having bases on the bottom line is $ pt^i$, as there are $\binom{m+n}{m}$ such paths. Similarly, the probability that an edge path connects two triangles both having bases on the top line is $ pb^i$, as there are $\binom{m+n}{n}$ such paths. The path edges and their exponent in the partition function are also indicated.

$M(Q_1, q)$, i.e. the factor corresponding to the edge path $t_1$. Each triangulation $T$ contributes either an exponent $+1$ or $-1$. The exponent is $+1$ if $1, 2 \in T$ or $1, 2 \notin T$, and it is $-1$ if $1 \in T, 2 \notin T$ or $1 \notin T, 2 \in T$. The number of $+1$'s is thus the sum of the number of triangulations of $C_{m-2,n}$ and $C_{m,n-2}$, and the number of $-1$'s is twice the number of triangulations of $C_{m-1,n-1}$.

For a triangulation, given an edge path $Q_{ij}$, the probability that the triangle $t_i$ has its base on the top line is $m/(m+n)$, as there are $m$ triangles with bases on the top line in any triangulation and there are $m+n$ triangles in total. Then, the probability that the triangle $t_{j+1}$ also has its base on the top line is $(m-1)/(m+n-1)$. The probability that the edge path connects two triangles both having bases on the top line is thus $p_t = m(m-1)/(m+n)(m+n-1)$. Similarly, the probability that an edge path connects two triangles both having bases on the bottom line is $p_b = n(n-1)/(m+n)(m+n-1)$. The probability that an edge path connects triangles having bases on different lines is then $1 - p_t - p_b$. Hence, considering all the $N_\Delta$ triangulations, the contribution to the partition function comes with the exponent

$$d = \left( p_t + p_b - (1 - p_t - p_b) \right) N_\Delta$$

$$= \left( \frac{2m}{(m+n)(m+n-1)} + \frac{2n}{(m+n)(m+n-1)} - 1 \right) \frac{(m+n)!}{m!n!}$$

$$= 2 \left( \frac{m+n-2}{m-2} \right) + 2 \left( \frac{m+n-2}{n-2} \right) - \left( \frac{m+n}{n} \right)$$

$$= \frac{(m^2 - m + n^2 - n - 2mn)(m+n-2)!}{m!n!}.$$

(5.4)
While the integrality of the exponent $d$ is obvious from its definition, we made it conspicuous by writing it as a combination of binomial coefficients in the third line. 

**Remark 5.5.** The case $n > m = 0$ is excluded from the first proof of the proposition, since $C_{0,n}$ only admits one unique triangulation, and all interior edges are of type ‘+’. Writing $X$ for the resolution, we have

$$Z'(X; -q, Q) = \prod_{1 \leq i < j \leq n-1} M(Q_{ij}, q) \quad \text{and} \quad Z(X; -q, Q) = M(1, q)^n Z'(X; -q, Q).$$

We have indeed $d = 1$ in equation 5.3 whenever $m = 0$.

The second proof, on the other hand, only excludes the case $m = 0, n = 1$, for not having any interior edge. It is more general in this sense.

**Definition 5.6.** A partition function for a Calabi–Yau manifold $Y$ is of curve-counting type if it can be expressed in terms of the DT partition function up to a factor depending only on the Euler characteristic of $Y$.

We have thus proved the following:

**Theorem 5.7.** Let $X$ be a toric singular Calabi–Yau threefold without contractible curves or compact 4-cycles. Let $Z(Y; q, Q)$ be any partition function of curve-counting type. Then, the total partition function for $X$ is given by

$$Z_{\text{tot}}(X; q, Q) := \prod_Y Z(Y; q, Q),$$

where the product ranges over all crepant resolutions of $X$, is homogeneous, and its degree is given by proposition 5.4.

For a general singular toric Calabi–Yau threefold $X$ without compact 4-cycles, we can use this theorem to factor the partition function into homogeneous factors. The toric diagram $\Delta_1$ of $X$ is a strip of shape $C_{m,n}$ with an arbitrary number of internal edges filled in, for example,

```
0 1 2 3 4
0 1 2 3 4
```

Let us partition the integers $m, n$ according to the already filled-in interior edges, that is,

$$(m, n) = \sum_{k=1}^{P} (m_k, n_k) = (m_1 + m_2 + \cdots + m_p, n_1 + n_2 + \cdots + n_p).$$

In the example above, we have $(m, n) = (3, 4)$, and the single interior edge corresponds to the partition $(3, 4) = (2 + 1, 1 + 3)$. It is clear that the number of maximal triangulations of this shape is

$$\prod_{k=1}^{P} \binom{m_k + n_k}{n_k},$$

where each factor counts the number of triangulations of the embedded subdiagram $C_{m_k,n_k} =: C_k$. If we restrict our attention to some fixed subdiagram $C_k$, then the entire collection of triangulations of $\Delta$ contains many triangulations with the same restriction to $C_k$. It is clear
that for any fixed triangulation of $C_k$, there are $b_k$ triangulations of $\Delta$ that restrict to the given triangulation, where
\[
b_k = \prod_{j \neq k} \binom{m_j + n_j}{n_j}.
\]
We extend definition 5.1 in a straightforward manner to the following.

**Definition 5.8.** If $X$ is a singular Calabi–Yau threefold without compact 4-cycles such that the convex hull of its toric diagram is $C_{m,n}$ (that is, there exists a birational map $X \to C_{m,n}$), we define the total partition function to be
\[
Z'_\text{tot}(X; -q, Q) := \prod_T Z'_T(C_{m,n}, -q, Q).
\]
Here, the term in the product of the right-hand side is the same as in definition 5.1, except that the product is taken only over those triangulations $T$ which correspond to resolutions of $X$.

Now theorem 5.7 implies the following.

**Corollary 5.9.** If $X$ is a singular Calabi–Yau threefold without compact 4-cycles and $(m, n)$, $P$ and $b_k$ are as above, then the total partition function of $X$ factors is as follows:
\[
Z'_\text{tot}(X; -q, Q) = Z''(-q, Q) \prod_{k=1}^P Z'_\text{tot}(C_{m_k, n_k}, -q, Q)^{b_k}.
\]
The factors in the product on the right are homogeneous as per theorem 5.7, and the function $Z''$ only contains factors $M(Q_{ij}, q)$ for which the edge path corresponding to $Q_{ij}$ crosses one of the interior edges of the toric diagram of $X$.

**Example 5.10.** In the above example with $(m, n) = (3, 4) = (2 + 1, 1 + 3)$, the two homogeneous factors are $Z'_\text{tot}(C_{1,2}; -q, Q)^3$ and $Z'_\text{tot}(C_{3,1}; -q, Q)^2$, and the inhomogeneous factor contains only terms $M(Q_{ij}, q)$ with $i \leq 3 \leq j$, because the third edge is already fixed in the diagram.

### 5.2. BPS counting and relation to black holes

Here is one application to BPS state counting. The topological string partition function of $X$ is
\[
Z_{\text{top}}(X; q, Q) = M(1, q)^{\chi(X)/2} Z'(X; -q, Q),
\]
so it is a partition function of curve-counting type.

**Corollary 5.11.** Writing $X_T$ for the resolution of $C_{m,n}$ corresponding to the triangulation $T$, we have
\[
\prod_T Z_{\text{top}}(X_T; q, Q) = M(1, q)^{\chi(X)/2} \prod_{1 \leq i \leq j \leq m+n-1} M(Q_{ij}, q)^{\frac{m^2+n^2+2mn}{2m+n} \binom{m+n}{m}}.
\]

**Proof.** This follows immediately from the fact that $\chi(X_T) = m+n$ for all $T$ and that there are $\binom{m+n}{m}$ triangulations. \(\square\)
6. Partition function via change of variables

It has been mentioned earlier that the product of partition functions corresponding to different triangulations depends upon the explicit isomorphism between homologies of crepant resolutions. For purposes of comparison, let us briefly discuss the product of partition functions in the case when the map between the homologies of crepant resolutions in different chambers in the moduli space is given by Seiberg duality [Sz, N, Y]. We shall consider the combinatorial aspects of the partition function in terms of the dual type-IIA picture, given by a gauge theory of NS five-branes with D4-branes stretched between them, interpreted as fractional branes. Depending on the spatial directions occupied by the NS branes in the target space, two types of branes, referred to as NS and NS’ branes, are considered. The arrangement of the two types of NS-branes on a circle corresponds to the triangulations of the trapezoidal strip [U]. The field theory of such configuration of branes is well developed [U, MP, vU]. We shall not discuss the field theory here but focus only on certain combinatorial aspects of arrangement of branes.

The T-dual type-IIA theory on \( C_{m,n} \) has \( m \) NS branes and \( n \) NS’ branes. For any triangulation of the strip, an NS brane corresponds to a regular triangle based on the top line and we denote it by a dark circle in figure 3. An NS’ brane, on the other hand, corresponds to a triangle based on the bottom line and will be denoted by a white circle. The D4-branes stretched between these are denoted by a line, which also serves to designate the relative separation between the NS-branes, given by the period of the B-field.

Considering an arrangement of NS-branes, a pair of branes linked by a line corresponds to an edge path and contributes a factor to \( Z'(X; -q, Q) \) in (5.2). According to the combinatorial rule laid out earlier, the index of the factor is positive if the branes are of the same type, that is, the edge path connects either an NS–NS or an NS’–NS’ pair and negative otherwise. Indeed, a curve connecting two adjacent cones in the toric diagram is \( O(-2) \) if the branes in the cones are of the same type and is \( O(-1, -1) \) otherwise. Thus, in particular, NS and NS’ branes are exchanged under a flop, as in \( Q_1 \) to \( R_1 \) in figure 3. For example, \( Q_1 \) in figure 3 contributes \( \prod_{k=1}^{\infty} (1 - q^k Q_1) \) to the partition function, as it connects branes of different types. On the other hand, \( Q_{13} = Q_1 Q_3 \) contributes a factor of \( \prod_{k=1}^{\infty} (1 - q^k Q_{13}) \) as it connects branes of the same type.

In this dual theory, each triangulation of the strip corresponds to a ‘phase’ of the field theory described by a quiver gauge theory with a superpotential. Different phases correspond to different paths to approach the singularity from the asymptotic large-volume region.

We consider products over all crepant resolutions, that is, phases, again and we still assume all relevant resolutions to have isomorphic second homologies. However, instead of formally identifying the elements of \( H_2 \), we change coordinates to write each element of \( H_2(X', Z') \) in terms of a fixed basis \( Q' \). We set

\[
Z_{\text{new}}^u(X; Q, \ldots) := \prod_{r \in T} Z_{\text{old}}(X^r; Q'(Q'), \ldots).
\]

Example 6.1. For the conifold, we have two crepant resolutions

\[
Q \quad \text{and} \quad R,
\]

with respective bases for the second homologies denoted as \( \{Q_1\} \) and \( \{R_1\} \). Since they are related by a flop, the change of coordinates reads \( R_1 = Q_1^{-1} \). We obtain the partition function [Sz],

\[
Z_{\text{tot}}^u(C_{1,1}; -q, Q) = M(Q, q) M(Q_1^{-1}, q),
\]
to be contrasted with the partition function $Z_{\text{tot}}$ with degree $d = -2$ obtained earlier in example 5.3.

**Example 6.2.** The following are the four triangulations corresponding to the crepant resolutions of the generalized conifold $C_{1,3} := \{(x, y, z, w)|xy - zw^3 = 0\}$.

These four resolutions of $xy - zw^3 = 0$ are obtained from each other by a series of flops,

\[ Q \xrightarrow{\text{flop } Q} R \xrightarrow{\text{flop } R} S \xrightarrow{\text{flop } S} T. \]

Let us recall from example 6.1 that under a flop of a $(-1, -1)$-line the formal variable changes from $Q$ to $Q^{-1}$. Thus, the formal variables of the different triangulations are identified as

\begin{align*}
R_1 &= Q_1^{-1}, & S_1 &= R_1, & T_1 &= S_1, \\
R_2 &= Q_2, & S_2 &= R_2^{-1}, & T_2 &= S_2, \\
R_3 &= Q_3, & S_3 &= R_3, & T_3 &= S_3^{-1},
\end{align*}

specifying the isomorphism of second homologies.

Partition functions for the four triangulations are then written down using (5.2) as

\begin{align*}
Z_{\text{op}}(q, Q) &= M(1, q)^2 \prod_{k=1}^{\infty} (1 - Q_1 q^k)^{-1}(1 - Q_2 q^k)^{-1}(1 - Q_3 q^k)^{-1}(1 - Q_4 q^k)^{-1}(1 - Q_5 q^k)^{-1} \\
Z_{\text{op}}(q, R) &= M(1, q)^2 \prod_{k=1}^{\infty} (1 - R_1 q^k)^{-1}(1 - R_2 q^k)^{-1}(1 - R_3 q^k)^{-1}(1 - R_4 q^k)^{-1}(1 - R_5 q^k)^{-1} \\
Z_{\text{op}}(q, S) &= M(1, q)^2 \prod_{k=1}^{\infty} (1 - S_1 q^k)^{-1}(1 - S_2 q^k)^{-1}(1 - S_3 q^k)^{-1}(1 - S_4 q^k)^{-1}(1 - S_5 q^k)^{-1} \\
Z_{\text{op}}(q, T) &= M(1, q)^2 \prod_{k=1}^{\infty} (1 - T_1 q^k)^{-1}(1 - T_2 q^k)^{-1}(1 - T_3 q^k)^{-1}(1 - T_4 q^k)^{-1}(1 - T_5 q^k)^{-1}.
\end{align*}

Expressing them all in terms of the $Q$ variables, we obtain

\begin{align*}
Z_{\text{op}}(q, Q) &= M(1, q)^2 \prod_{k=1}^{\infty} (1 - Q_1 q^k)^{-1}(1 - Q_2 q^k)^{-1}(1 - Q_3 q^k)^{-1}(1 - Q_4 q^k)^{-1}(1 - Q_5 q^k)^{-1} \\
Z_{\text{op}}(q, R) &= M(1, q)^2 \prod_{k=1}^{\infty} (1 - Q_1 q^k)^{-1}(1 - Q_2 q^k)^{-1}(1 - Q_3 q^k)^{-1}(1 - Q_4 q^k)^{-1}(1 - Q_5 q^k)^{-1} \\
Z_{\text{op}}(q, S) &= M(1, q)^2 \prod_{k=1}^{\infty} (1 - Q_1 q^k)^{-1}(1 - Q_2 q^k)^{-1}(1 - Q_3 q^k)^{-1}(1 - Q_4 q^k)^{-1}(1 - Q_5 q^k)^{-1} \\
Z_{\text{op}}(q, T) &= M(1, q)^2 \prod_{k=1}^{\infty} (1 - Q_1 q^k)^{-1}(1 - Q_2 q^k)^{-1}(1 - Q_3 q^k)^{-1}(1 - Q_4 q^k)^{-1}(1 - Q_5 q^k)^{-1}.
\end{align*}

Taking the product to assemble the full partition function of the singularity $C_{1,3}$, after cancellations, we are left with

\[ Z_{\text{tot}}^{(2)}(C_{1,3}; q, Q) = M(1, q)^{8} \prod_{k=1}^{\infty} \left( \frac{(1 - Q_1 q^k)^{-1}(1 - Q_2 q^k)^{-1}(1 - Q_3 q^k)^{-1}(1 - Q_4 q^k)^{-1}(1 - Q_5 q^k)^{-1}}{(1 - Q_1 q^k)^{-1}(1 - Q_2 q^k)^{-1}(1 - Q_3 q^k)^{-1}(1 - Q_4 q^k)^{-1}(1 - Q_5 q^k)^{-1}} \right)^4. \]
which can be rewritten, using expression (5.1) for the generalized MacMahon function, as
\[
Z_{\text{tot}}(C_{1,3}; q, Q) = M(1, q) \frac{M(Q_1^{-1}, q)M(Q_1^{-1}Q_2, q)M(Q_1^{-1}Q_2Q_3, q)}{M(Q_1, q)M(Q_1Q_2, q)M(Q_1Q_2Q_3, q)}
\times \frac{M(Q_2, q)M(Q_1^{-1}Q_2Q_3, q)M(Q_1^{-1}Q_2Q_3^{-1}Q_1, q)}{M(Q_1^{-1}, q)M(Q_2^{-1}, q)M(Q_1^{-1}Q_2^{-1}Q_1^{-1}Q_3, q)}.
\]

This expression corresponds to the partition function of a quiver variety that enjoys a derived equivalence with the crepant resolutions \([N, Y]\). The partition function \(Z_{\text{tot}}\) for this case is of vanishing degree, by (5.3).

To summarize, we defined a partition function for a generalized conifold through the product of partition functions of all its crepant resolutions. The second homologies of the resolutions are identified through a canonical ordering of elements, facilitated by the absence of homology 4-cycles in the resolutions. We proved that the new partition function is homogeneous with respect to MacMahon factors. This has been contrasted with the same product of partition functions with the relation between the elements of the second homology group given by the Seiberg duality.

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