The glass transition and the Coulomb gap in electron glasses

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We establish the connection between the presence of a glass phase and the appearance of a Coulomb gap in disordered materials with strongly interacting electrons. Treating multiparticle correlations in a systematic way, we show that in the case of strong disorder a continuous glass transition takes place whose Landau expansion is identical to that of the Sherrington-Kirkpatrick spin glass. We show that the marginal stability of the glass phase controls the physics of these systems: it results in slow dynamics and leads to the formation of a Coulomb gap.

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The relation between the slow dynamics of Coulomb glasses, disordered materials with strong electron-electron repulsions, and the appearance of a soft "Coulomb" gap in their density of states (DOS) has been a mystery for a long time. The strong effect of interactions on the DOS was first noticed by Pollak [1] and Srinivasan [2]. Efros and Shklovskii [3] predicted the Coulomb gap as resulting from the long-range Coulomb interactions between localized electrons in semiconductors, and leading to a crossover in the temperature dependence of the conductivity from Mott's law $\ln(\rho) \sim (T_M/T)^{1/4}$ to $\ln(\rho) \sim (T_{ES}/T)^{1/2}$ at low temperatures [4]. The Efros-Shklovskii law for the conductivity was verified in semiconductors, alloys and granular metals, and recently the gap itself was directly observed in semiconductors [5, 6]. The presence of disorder frustrates the Coulomb interactions and leads to glassy behavior in such materials, as predicted theoretically long ago [7]. The first evidence for glassy phenomena came from the slow relaxation of charge injected into compensated semiconductors [8]. Later, Ovadyahu's group established that the slow dynamics in Indium-oxides is indeed due to electron-electron interactions [9]. Very recently the same group has demonstrated memory and aging effects similar to those observed in spin glasses [10].

Despite the experimental progress a thorough understanding of the glass phase is still missing. The source of the difficulty is that in order to describe glassy phenomena one needs to take electron correlations into account, while the approach by Efros and Shklovskii is essentially a single particle theory. The necessity to include correlations has become clear from several recent numerical studies of the off-equilibrium dynamics of Coulomb glasses [11]. Further, an increasing number of experiments suggests that a quantitative description of the hopping conductivity should take multiparticle processes and correlations into account [12]. One also needs to go beyond a single particle theory in order to understand the relation between the Coulomb gap and the glass transition. The mean field solution of a model of uniformly interacting electrons in a disordered medium indicates that the glass transition and the formation of a pseudogap in the DOS are driven by the same mechanism, and a similar relation has been conjectured for the Coulomb glass [13].

The goal of this paper is to develop a formalism that accounts for the correlations between the electrons in a realistic model for Coulomb glasses in 3D. Our approach is based on the locator approximation developed for spin glasses in Refs. [14, 15, 16]. The idea is to map the original lattice problem into an effective single-site problem that encodes correlations by the distribution of a fluctuating local field, which gives exact results for infinite range models. In the limit of strong disorder, the Coulomb interactions are essentially unscreened, so that the effective number of neighbors is large and the locator approximation is parametrically well justified. In this regime, we find a replica symmetry breaking glass transition, which belongs to the same universality class as the transition in the Sherrington-Kirkpatrick (SK) spin glass [17]. Below the transition, the phase space divides into an exponential number of metastable states and ergodicity is broken. Like any generic glass, this electronic system freezes into one of many marginally stable states since the latter are the most abundant (the number of states increasing exponentially with decreasing stability). Marginally stable states are characterized by the presence of soft modes that lead to the slow relaxation dynamics observed in experiments. Above $T_c$, the DOS does not display any particular signature of the Coulomb interactions. We show that the Coulomb gap forms only below $T_c$ where it emerges as a direct consequence of marginal stability. Finally, we derive an asymptotic expression for the DOS at very low temperatures.

We consider the classical model [14] for strongly localized electrons occupying a fraction $0 < K < 1$ of a given set of impurity sites $i$,

$$H = \frac{1}{2} \sum_{i \neq j} n_i J_{ij} n_j + \sum_i n_i (\epsilon_i - \mu_K),$$

(1)

where $n_i \in \{0, 1\}$ is the occupation number of the site $i$. For simplicity, we take them to be arranged on a cubic lattice...
with lattice spacing $\ell \equiv 1$. The unscreened Coulomb interactions are described by $J_{ij} = 1/r_{ij}$ in units where $e^2/\ell \equiv 1$. The $\epsilon_i$’s denote random on-site energies, and $\mu_K$ is the chemical potential. We restrict ourselves to the case $K = 1/2$ where the particle-hole symmetry implies $\mu_{1/2} = 0$, and suggests to introduce spin variables $s_i = n_i - 1/2 = \pm 1/2$. Further, we assume a Gaussian distribution of width $W$ for the on-site energies $\epsilon_i$. Their randomness emulates the effect of the disorder in the site positions which is present in all physical electron glasses and generates rather large site-to-site fluctuations of the Coulomb potential. We focus on the limit of strong disorder, $W \gg 1$ and dimension $D = 3$. In this case screening is suppressed on short scales and the interactions remain long-range, which justifies the use of the locator approximation. We will see that at low temperatures the self-generated disorder outweighs $W$. We thus expect our results to be universal in that regime. Furthermore, this observation makes us believe that at low temperatures the locator approximation can be justified even in the case of weak disorder.

In the case of long-range interactions, diagrammatic expansions can be efficiently resummed since the large number of effective neighbors allows one to approximate the self-energy by an average local term, see Fig. 1. This observation further suggests to replace the interactions of a given spin with its environment by an effective local field described by couplings of the spin to its own replicas. This reduces the model to a single-site problem, translating the complexity of the environment into a non-trivial replica structure of the one-site Hamiltonian [13, 16].

We note that this procedure resembles the way in which the SK-model is transformed into a one-site problem. Indeed, the locator approximation applied to the SK-model sums all tree-like diagrams with doubled interaction lines which becomes exact in the large $N$ limit.

To make contact between the Hamiltonians (1) and (2), we require that they both yield the same single-site spin correlation functions,

$$\langle s_i s_j \rangle = \sum_{\{s_a\}} s_i s_j e^{-\frac{1}{2} \sum_{\alpha, \gamma} s_{\alpha} B_{\alpha \gamma} s_{\gamma}} = \left[ \frac{1}{\tilde{B} - \Sigma} \right]_{ab}$$

We have averaged over the random energies $\epsilon_i$, and as usual, the number of replicas $n$ is implicitly assumed to tend to zero in the end of calculations. $\mathcal{I}$ denotes a $n \times n$ block matrix with all entries equal to 1, and we have defined $\tilde{B} = B - \beta^2 W^2 \mathcal{I}$. In Eqs. (3) we approximated the full propagator for either model as a simple geometric series with a local self-energy insertion $\Sigma_{ab} \delta_{ij}$, as motivated above. Since the mapping is to preserve correlations, the self-energy has to be the same for both models [16]. From (2), we obtain the free energy

$$n \beta F(B) = -\ln \left[ \sum_{\{s_a\}} e^{-\frac{1}{2} \sum_{\alpha, \gamma} s_{\alpha} \tilde{B}_{\alpha \gamma} s_{\gamma}} \right] + \frac{U(B)}{2}$$

where $U(B)$ has to be determined such that the saddle point equations with respect to $B$ yield back Eqs. (3). Up to a function of temperature, we find

$$U(B) = tr \left[ \ln(\tilde{B} - \Sigma) \right] - \frac{1}{V} Tr \ln \left( \beta J - \beta^2 W^2 \mathcal{I} - \Sigma \right)$$

FIG. 1: For long-range interactions the self-energy $\Sigma$ can be approximated by a local operator. The full propagator is obtained as a simple geometric series.
where \(tr\) denotes the trace in replica space. We emphasize that in this expression the self-energy \(\Sigma\) has to be considered as an implicit function of \(B\) as defined through Eq. (3). In the following we shall need spatial traces like 
\[
g_n(x) = V^{-1} tr(1/(\beta J_x + x))
\]
which we evaluate in Fourier space 
\[
g_n(x) = \frac{1}{(2\pi)^d} \int \frac{d^d k}{2\pi^d} \frac{1}{(\beta J_k + x)}
\]
with \(J_k = 4\pi/k^2\) at small \(k\). We assume some cut-off procedure that regularizes the small scale physics so that \(\int d k = 1\) and \(\int d^2 k = 0\). For \(x \gg \beta\) we obtain 
\[
g_n(x) \approx x^{-n}[1 + C_n(\beta/x)^{3/2}]
\]
where \(C_1 = 2/\sqrt{\pi}\) and \(C_2 = 5/\sqrt{\pi}\).

Let us first discuss the replica symmetric (RS) solution of Eqs. (3) for which we assume \(\Sigma_{ab} = -\Sigma_0 \delta_{ab} + \Sigma_I I\), and 
\[
B_{ab} = - B_0 \delta_{ab} + B_2 I.
\]
For \(W \gg 1\), we find \(\Sigma_0 \approx \sqrt{2W}\beta\), suggesting the interpretation of \(\Sigma_0^{-1}\) as the fraction of thermally active sites (for \(T_c < T \ll W\)). The distribution of local fields obtained from this RS solution matches remarkably well numerical simulation data for finite temperatures and \(W \lesssim 1\), even though the locator approximation is difficult to justify in this parameter regime [18]. A depletion of sites in small fields is found due to strong correlations in this “Coulomb plasma”. However, a closer analysis reveals that there is no true pseudogap on the replica symmetric level, the depletion disappearing completely for strong disorder. This is also reflected in the charge susceptibility \(\chi_{RS} = \beta(1/4 - \langle s_a s_b \rangle_{a \neq b}) \approx \beta/\Sigma_0\) which tends to a finite constant \((\sim 1/W)\) within this solution. The genuine Coulomb gap is formed only when the replica symmetry is broken. For \(W \gg 1\), the RS solution indeed exhibits an instability when the condition

\[
\int_{-\infty}^{\infty} dy \frac{e^{-y^2/2(W^2 + B_2/\beta^2)}}{[2\cosh(\beta y/2)]^4} \frac{1}{g_1^{-2}(\Sigma_0) - g_2^{-1}(\Sigma_0)^{-1}} \approx \pi \beta \Sigma_0^{-1/2},
\]

is met, from which we extract the critical temperature \(T_c \approx W^{-1/2}/[6(2/\pi)^{1/4}] \ll 1 \ll W\). We emphasize that the difference \(g_1^{-2}(\Sigma_0) - g_2^{-1}(\Sigma_0)\) is controlled by the contribution from large scales, \(1/k \sim \sqrt{W}\), which justifies our assumption of a large number of effective neighbors.

The instability (6) signals a continuous glass transition with full replica symmetry breaking. We may analyze it further by expanding the free energy with respect to the replicon mode \(\delta B\) (with \(\delta B_{\alpha \alpha} = 0\) and \(\delta B I = 0\)) around the RS solution,

\[
n\beta \delta F = \frac{c_1}{W^{3/2}} \left[ tr(-\tau \delta B^2 + c_2 \delta B^3) + c_3 \sum_{\alpha, \gamma} \delta B^3_{\alpha \gamma} \right]
\]

where \(\tau = 1 - T/T_c\). This shows that the glass transition in Coulomb glasses belongs to the same universality class as the one in the SK-model. Hence, many results about the critical behavior known for infinite range spin glasses [19] should be directly applicable to the present case. This might be interesting in particular for the aging and memory effects observed in experiments [10], even though the dynamics of spin glasses obey slightly different rules than in Coulomb glasses. Vice versa, electron glasses present an appealing testing ground for many theoretical ideas developed in the context of the SK-model.

We now turn to a more detailed analysis of the physics far below \(T_c\). Since we expect that \(\Sigma_0 \approx \beta \chi^{-1} \gg \beta\) we may expand the free energy (6) in \(\beta J/\Sigma\). Using Eqs. (3) we eliminate \(\Sigma\) and obtain \(U(B) = -tr(B^3)/12\pi^2 \beta^3\), resembling the SK-model where \(U(B) \sim -tr(B^3)/\beta^2\). The exponent reflects the spatial dimension \(D = 3\) and is responsible for the shape of the pseudogap \((\rho(E) \sim E^{-D-1})\). In order to derive this result, it is more convenient to keep the self-energy in the formalism. Let us suppose that the replica symmetry is broken at the level of \(K\) steps. We represent the Parisi matrices as \(\Sigma = -\Sigma_0 + \sum_{k=1}^K \Sigma_k R_{mk}\), where \(R_{mk}\) consist of blocks of size \(m_k\) on the diagonal with all entries equal to 1. Let us focus on the set \(C\) of the \(m_1\) spins corresponding to one of the innermost blocks. These spins experience an effective field \(y\) created by all other spins. We describe its thermal fluctuations by a distribution \(P(y)\), which in the RS case was a simple Gaussian (see Eq. (6)). In the case of continuous replica symmetry breaking, \(P(y)\) can in principle be obtained by integration of Parisi’s differential equation using the methods of Refs. [20]. Here, we will only exploit the fact that the Coulomb glass is in a marginally stable state (the Hessian \(\partial^2 F/\partial B^2\) has a vanishing eigenvalue in the replicon mode \(\delta B\) characterized by \(\delta B_{\alpha \alpha} = 0\) and \(\delta B R_{mk} = 0\) for all \(k\)), which imposes the constraint

\[
\int_{-\infty}^{\infty} dy P(y) \frac{1}{[2 \cosh(\beta y/2)]^4} \frac{1}{g_1^{-2}(\Sigma_0) - g_2^{-1}(\Sigma_0)^{-1}} = 0.
\]

Further, the innermost component of Eqs. (3) reads

\[
\chi \equiv \beta \left[ \frac{1}{4} - \langle s_a s_b \rangle_{a \neq b \in C} \right] = \beta g_1(\Sigma_0)
\]
hopping conductivity, tunneling experiments \([5]\) probe the system on very short time scales, sampling the distribution \(P(y)/T\) to the thermodynamic field relaxation of the environment, the gap is much narrower.

Expanding \(g_n\) for \(\Sigma_0/\beta \gg 1\), one can see that at low temperatures these two equations only admit a solution if \(\Sigma_0 \sim \beta^3\) and \(P(y)\) takes the scaling form

\[
T^{-2}P(y \equiv zT) \to p(z) \quad (T \to 0)
\]

with \(p(z) \sim z^2\) for \(z \gg 1\). This implies that the susceptibility obeys the scaling \(\chi \sim T^2\), and the (static) screening length diverges at low temperatures as \(l_{sc} = (4\pi\chi)^{-1/2} \sim T^{-1}\). Note that \(\chi\) measures the charge response to a local potential change when the particles on other sites are allowed to readjust to the induced charge. Thus, it is associated with the thermodynamic local fields \(y_i\) defined by \(\langle s_i \rangle = m_i = \tanh(\beta y_i/2)/2\). While we expect \(\chi\) to control the hopping conductivity, tunneling experiments \([5]\) probe the system on very short time scales, sampling the distribution \(\tilde{P}(h)\) of instantaneous local fields \(h_i = \sum_j J_{ij}s_j\). The thermal average of these local fields, \(\langle h_i \rangle = \sum_j J_{ij}m_j\), is related to the thermodynamic field \(y_i\) via a Thouless-Anderson-Palmer (TAP) equation \([21]\), \(\langle h_i \rangle = y_i + \langle s_i \rangle h_O\), where the Onsager term

\[
h_O = \beta \int \frac{J_k^2}{\beta J_k + \Sigma_0} \approx 2 \sqrt{\pi \beta / \Sigma_0} \approx 2 \sqrt{\pi \chi}
\]

accounts for the extra polarizations induced by the presence of the charge \(\langle s_i \rangle\). For the consistency with the locator approximation, we have retained only terms corresponding to a local self-energy. The deviation of the local field \(h_i\) from its mean \(\langle h_i \rangle\) is essentially a Gaussian variable with width \(h_O\). More precisely, the relation

\[
\tilde{P}(h) = \int dy P(y) \frac{\cosh(\beta h/2)}{\cosh(\beta y/2)} \frac{e^{-\beta(h-y)^2/2h_O}}{\sqrt{2\pi h_O} e^{-\beta h_O/2}}
\]

holds \([22]\), which generalizes a known result for the SK-model \([23]\). The tunneling density of states at zero bias then follows from \(\nu_0 = \beta \int dh \tilde{P}(h)[2 \cosh(\beta h/2)]^{-2}\). Eq. \([12]\) implies that \(\tilde{P}(h)\) obeys a scaling analogous to Eq. \([10]\), and hence \(\nu_0 \sim T^2\). Generally, in order to make quantitative predictions, one needs to know the functional form of the field distributions. It turns out, however, that certain parameters are not very sensitive to their details. It is convenient to assume a simple form \(P(\langle h \rangle) = \alpha(\langle h \rangle)^2 + \gamma T^2\) for the distribution of average fields, obtain \(P(y)\) via the TAP-equations and solve Eqs. \([8,9]\). This yields \(\chi, \nu_0\) and \(\alpha\) as slowly varying functions of \(\gamma\) \([24]\): \(\alpha \approx 0.204 - 0.0067\gamma, \chi \approx (22.27 - 0.81\gamma) T^2, \nu_0 \approx (2.178 - 0.0087\gamma) T^2\). The tunneling DOS \(\nu_0\) is roughly an order of magnitude smaller than the full susceptibility \(\chi\), as is also evident from the typical distributions shown in Fig. \(2\). This agrees well with the experimental observation \([12]\) that the susceptibilities governing tunneling and hopping transport differ significantly. The value of \(\alpha\) should be compared to the Efros-Shklovskii prediction \(\alpha_{ES} = 3/\pi \approx 0.95\) \([8]\) which is larger than our estimate because their self-consistency argument imposes stability only with respect to single electron hops. By contrast, our estimate includes multiparticle constraints that decrease \(\alpha\) below \(\alpha_{ES}\) in agreement with large-scale numerical simulations \([25]\).

In conclusion, we have developed the locator approximation for Coulomb glasses, allowing us to include multiparticle correlations. We have used this formalism to provide evidence for a continuous glass transition below which the
Coulomb glass gets stuck in a marginally stable state, resulting in subexponential relaxation dynamics and giving rise to the Coulomb gap. A priori, the locator approximation is justified for large disorder. However, as long as crystallization is prevented, a structural glass transition will provide sufficient self-generated disorder, so that we expect our results to hold at low temperatures even in the case of weak external disorder. We verified \cite{22} that in this limit the local observables are still determined by the contribution from large scales and reveal the Efros-Shklovskii gap. Further, we found that the locator approximation gives a significant decrease of the DOS with temperature already above $T_c$, in agreement with numerics. Moreover, it predicts a discontinuous glass transition at a scale of $T_c \approx 0.030$ which depends, however, on the details of the cutoff at small scales. The validity of this prediction remains thus unclear.

The locator approximation not only provides new insight into classical Coulomb glasses, but also allows for quantitatively new predictions that go beyond the single particle theory. It thus sets the stage for further theoretical developments to understand the puzzles of correlated transport and glassy relaxation of these systems. For instance, it allows one to study the collective modes of the electrons which induce fluctuations in the local electric fields \cite{11} and thus enhance the probability for resonant tunneling. Experiments indicate that such a mechanism might provide an alternative to phonon assisted tunneling, in particular at low temperatures when phonons freeze out \cite{26}. Finally, extensions of the formalism to include quantum effects may be envisioned.

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APPENDIX: JUSTIFICATION OF THE LOCATOR APPROXIMATION

In order to justify the locator approximation we calculate the leading terms ignored by the locator approximation and show that their effect on the physical results is small in the limit \( W \gg 1 \). To calculate the higher order terms we need a diagram technique. This is not convenient in the spin representation of the original problem

\[
H[\{s_i\}] = \frac{1}{2} \sum_{i \neq j} s_i s_j + \sum_i s_i \epsilon_i,
\]

but turns out to be rather simple in terms of the fields \( \phi_i \) conjugated to the spin variables \( s_i \) that appear after we perform a Hubbard-Stratonovich transformation:

\[
Z = \sum_{\{s_i\}} e^{-\beta H[\{s_i\}]} = \int \prod_i d\phi_i e^{\frac{i}{\beta} \sum_{i,j} \phi_i (\beta \mathcal{J})_{ij} \phi_j + \sum_i (\beta \epsilon_i + i \lambda_i) s_i}.
\]

The field \( i \beta^{-1} \phi_i \) can be thought of as the mean Coulomb potential at site \( i \) created by all other electrons \( 2 \).

In order to get rid of the disorder and to restore translational invariance, we apply the replica trick and calculate \( \langle Z^n \rangle_e \), where \( \langle \rangle_e \) denotes the average over the random energies \( \epsilon_i \). We assume the latter to be independent local variables with distribution \( P(\epsilon) \). The replicated partition function, summed over spin degrees of freedom, reads

\[
\langle Z^n \rangle_e = \left\langle \int \prod_{i,a} d\phi_i^n \exp \left[ -\frac{1}{2} \sum_{i,j,a} \phi_i^n (\beta \mathcal{J})_{ij} \phi_j^n - \frac{1}{2} \frac{1}{\beta W} \sum_{i,a} (\phi_i^n)^2 - \frac{1}{2} \sum_{i,a,b} \phi_i^n \phi_j^n \phi_i^n \phi_j^n \right] \right\rangle_e.
\]

We expand the local terms \( \ln[ \cosh(\beta \epsilon_i + i g \phi_i^n / 2) ] \) with respect to \( \phi_i^n \) and evaluate the disorder average by a cumulant expansion,

\[
\langle Z^n \rangle_e = \int \prod_{i,a} d\phi_i^n \left\langle \exp \left[ -\frac{1}{2} \sum_{i,j,a} \phi_i^n (\beta \mathcal{J})_{ij} \phi_j^n - \frac{1}{2} \frac{1}{\beta W} \sum_{i,a} (\phi_i^n)^2 - \frac{1}{2} \sum_{i,a,b} \phi_i^n \phi_j^n \phi_i^n \phi_j^n \right] \right\rangle_e
\]

where \( \frac{1}{\beta W}, \zeta, \eta, \lambda \) denote vertex coefficients:

\[
\frac{1}{\beta W} = \left\langle \frac{1}{2 \cosh(\beta \epsilon/2)^2} \right\rangle_e \approx \frac{1}{\sqrt{2\pi} \beta W},
\]

\[
\zeta = \left\langle \tanh^2(\beta \epsilon/2)^2 \right\rangle_e \approx \frac{1}{4} (1 - \frac{1}{\sqrt{2\pi} \beta W}),
\]

\[
\eta = \left\langle \frac{d^2}{d(\beta \epsilon)^2} \frac{1}{2 \cosh(\beta \epsilon/2)^2} \right\rangle_e \approx \frac{1}{\sqrt{2\pi} (\beta W)^3},
\]

\[
\lambda = \left\langle \frac{1}{2 \cosh(\beta \epsilon/2)^2} \right\rangle_e - \left\langle \frac{1}{2 \cosh(\beta \epsilon/2)^2} \right\rangle_e \approx \frac{1}{6 \sqrt{2\pi} \beta W}.
\]

Here we assumed a symmetric distribution of \( \epsilon_i \) and dropped the zeroth order term in \( \mathcal{I} \) since it is irrelevant in the replica limit \( n \to 0 \). The last equality in each equation gives the leading order in \( 1/\beta W \) for Gaussian disorder. Note that typically vertex coefficients scale as \( \sim 1/\beta W \), except for the coefficients \( c_m \) corresponding to the replica conserving vertices \( (\phi_i^n)^m \). The latter are suppressed by additional powers of \( \beta W \), \( c_m \sim 1/((\beta W)^{m-1}) \) because they correspond to disorder averages of exact derivatives of \( 1/\cosh^2(\beta \epsilon/2) \). To establish the connection with the locator approximation, notice that in the leading order in \( 1/\beta W \) the quantity \( \tilde{W} \) coincides with the diagonal part of the self energy \( \Sigma_0 \) of the replica symmetric theory.

A systematic diagrammatic expansion is now obtained by taking \( \left[ G_0^{-1} \right]_{ij} = \left\{ (\beta \mathcal{J})_{ij}^{-1} + \delta_{ij}/\beta W \right\} \delta_{ab} + \zeta \delta_{ij} \mathcal{I}^{ab} \) as bare propagator and treating all other terms in the cumulant expansion as interactions. In Fourier space, we have

\[
G_0^{ab}(k) = g_0(k) \delta^{ab} + g_1 \mathcal{I}^{ab},
\]
To estimate this effect we replace the screening part of the bare propagator, electric field and enhances the interactions. In this regime, the renormalizations might become more important.

\[
g_0(k) = \frac{\beta}{2\pi + 1/W},
g_1(k) = -\zeta g_0^2(k).
\]

The leading non-local contribution, \(\Sigma^{(1)}\), to the replica diagonal part of the self-energy is a tripled propagator line, connecting two replica-conserving 4-vertices. Its \(k\)-dependent part evaluates to

\[
\Sigma_{ab}^{(1)}(k) - \Sigma_{ab}^{(1)}(0) = \delta_{ab}\eta^2 \int g_0^3(r)(\exp(ikr) - 1)d^3r
\]

\[
= \eta^2 \cdot 4\pi\beta^3\delta_{ab}\left\{1 - \frac{3}{r_0k} \arctan\left(\frac{r_0k}{3}\right) - \frac{1}{2} \ln\left[1 + \left(\frac{r_0k}{3}\right)^2\right]\right\},
\]

where \(r_0 = (\tilde{W}/4\pi)^{1/2}\). The local part is of the order of

\[
\Sigma_{ab}^{(1)}(0) \approx \delta_{ab}\eta^2 \cdot 4\pi\beta^3\ln(r_0) \sim \delta_{ab}\frac{\ln(\tilde{W})}{\beta^3W^6}.
\]

As we will show below, the glass physics is dominated by long scales, \(k \lesssim 1/r_0\). For these momenta one can approximate the momentum dependent part of the self-energy by

\[
\Sigma_{ab}^{(1)}(k) - \Sigma_{ab}^{(1)}(0) \approx \frac{4\pi}{54}\eta^2\beta^3(r_0k)^2 + O\left((r_0k)^4\right).
\]

which results in an insignificant renormalization of the Coulomb interaction at temperatures of the order of the glass transition (\(T_c \sim W^{1/2}\)). In this regime we can thus safely approximate the self-energy by its local part \(\Sigma^{(1)}(0)\).

At very low temperatures the density of states at low energies is suppressed, this reduces the screening of the electric field and enhances the interactions. In this regime, the renormalizations might become more important. To estimate this effect we replace the screening part of the bare propagator, \(1/W = \beta/[2\cosh(\beta\epsilon/2)^{-2}]\) by the average thermodynamic susceptibility \(\chi = \beta \int dyP(y)[2\cosh(\beta y/2)^{-2} - 1/\beta^2] \approx 1/\beta^2\). This gives rise to the bare propagator \(g_0(k) = \beta/(k^2/4\pi + \chi)\) with the longer screening length \(r_0 \to (4\pi\chi)^{-1/2} \sim \beta\). Further, as a consequence of the broken replica symmetry, the replica mixing vertices might also contribute to the renormalization of the replica diagonal part of the Greens function. At low temperatures the vertex coefficients generically scale as \(1/\beta^3\). This might increase the importance of the non-local corrections and even make them marginally relevant below some temperature \(T^*\). By continuity we must have \(T^* \ll T_c\). We do not know any physical argument that supports the appearance of the second temperature (energy) scale and it seems unlikely to us that it happens. However, in order to check that non-local contributions remain parametrically small at all temperatures one needs to calculate them against the background of the full replica symmetry breaking solution.

Incorporating the local term \(\delta_{ij}\delta^{ab}/W\) of the bare propagator into the self-energy \(\Sigma_\phi\) we get the full Greens function

\[
\langle \phi^a_i \phi^b_j \rangle_c = (G^{-1})_{ij}^{ab} = \left[\frac{1}{(\beta\mathcal{J})^{-1} - \Sigma_{ij}}\right]_{ij}^{ab}.
\]

Before discussing the physical interpretation of this quantity in more detail we mention that the spin-spin correlation function follows easily via partial integration in \(\mathcal{H}\) with respect to the conjugated fields,

\[
\langle s^a_i s^b_j \rangle_c = \langle \beta\mathcal{J} \rangle_{ij}^{-1} - \sum_{i,m} \langle \beta\mathcal{J} \rangle_{ii}^{-1} \langle \phi^a_i \phi^b_m \rangle_c \langle \beta\mathcal{J} \rangle_{mj}^{-1} = \left[\frac{1}{\beta\mathcal{J} - \Sigma}\right]_{ij}^{ab},
\]

where \(\Sigma = \Sigma^{-1}\).

The locality of the self-energy suggests the mapping to an effective single site model, for which the same self-energy is assumed. For self-consistency we then have to require that the average local spin-spin correlators be the same.

\[
\langle s^a_i s^b_j \rangle_c = \left[\frac{1}{B - \Sigma}\right]_{ij}^{ab} = \frac{1}{V}\text{Tr} \left[\langle s^a_i s^b_j \rangle_c\right] = \frac{1}{V}\text{Tr} \left[\frac{1}{\beta\mathcal{J} - \Sigma}\right]_{ij}^{ab}.
\]

The equation \((14)\) for spin-spin correlator is a central element in the mapping to a single site-model. Physically, this expression and the locator approximation in general, is based on the assumption that a typical spin interacts
with many thermally active neighbors. The calculations above show that this assumption is satisfied by a typical spin that contributes to the density of states and other thermodynamic properties. This, however, does not exclude the presence of other types of low-energy excitations. For instance, an occupied and a nearby empty site may form a strongly coupled dipole, in which the electron can hop to the empty site at very low energy expense. Such pairs of sites are strongly coupled, and are not described by the locator approximation. The small effect of non-local corrections evaluated here physically mean that such dipoles renormalize weakly the dielectric susceptibility without interfering strongly in thermodynamics of carrier sites. The effect of these dipoles on other properties remain an open question.

Fluctuating correlations and criticality

The replica diagonal part of the Green function (21) is simply the disorder average (or equivalently, the average over sites at fixed distance) of the connected correlation function. Assuming a local self-energy, we explicitly obtain

\[
\langle \phi_i^a \phi_j^a \rangle_c = \langle \phi_i^a \phi_j^a \rangle_c = \beta e^{-r/r_0}
\]

where the overbar denotes the disorder average, and \( r_0 = \sqrt{\Sigma_0/(4\pi\beta)} \) (\( = (W/4\pi)^{1/2} \) in the limit of large \( W \) and at temperatures above \( T_c \)). This expression may be misleading since it seems to suggest that potentials are screened on average on a distance \( r_0 \). However, we have to bear in mind that (21) describes only the disorder averaged correlations. To study the fluctuations around this average, we have to calculate \( \langle \phi_i^a \phi_j^a \rangle - \langle \phi_i^a \phi_j^a \rangle_c \). Within the replica formalism the first term can be calculated as

\[
\langle \phi_i^a \phi_j^a \rangle_c^2 = \left\langle \left( \phi_i^a \phi_j^a - \phi_i^b \phi_j^b \right) \right\rangle,
\]

(22)
a, b, c and d being all distinct replicas that see the same disorder configuration. (In the glass phase one has to choose all indices in the same innermost block, which imposes that the two replicas belong to the same metastable state.)

This four point function can be evaluated by the diagram technique, the leading diagrams being shown in Fig. 3. The correlation function (22) is given by the coefficient, \( C \), of \( \delta_{ae}\delta_{bd} \) of the full expression for \( \langle \phi_i^a \phi_j^a \phi_i^c \phi_j^d \rangle \). It can easily be calculated by resumming a geometric series, which yields

\[
C(r) = \int d^3r e^{ikr} \langle \phi(0)\phi(r) \rangle^2 = \Sigma^{(2)}(k) \left[ 1 + \lambda \Sigma^{(2)}(k) + \left( \lambda \Sigma^{(2)}(k) \right)^2 + \ldots \right] = \frac{1}{1/\Sigma^{(2)}(k) - \lambda}.
\]

(23)

Here, \( \Sigma^{(2)}(k) \) denotes the insertion of a doubled propagator line,

\[
\Sigma^{(2)}(k) = \int g_0^2(r) e^{ikr} d^3r = 2\pi r_0 \left( 1 - \frac{(r_0 k)^2}{12} + O((r_0 k)^4) \right).
\]

Using the expressions at high temperature in the large \( W \)-limit, one easily verifies that at the critical temperature \( T_c = W^{-1/2}/[6(2\pi)^{1/4}] \) (cf. Eq. 3), the \( k \)-independent terms in (24) cancel, leading to a power law decay of the potential correlations,

\[
\langle (\beta^{-1}\phi(0))(\beta^{-1}\phi(r)) \rangle^2_{T=T_c} \sim \frac{1}{W^2} \frac{\beta_+}{r}.
\]

(24)

FIG. 3: Leading diagrams for the computation of the four point function \( \langle \phi_i^a \phi_j^a \rangle^2 \).
We separated a prefactor $1/\tilde{W}^2$ which has the interpretation of the probability that both sites at 0 and $r$ are thermally active.

Note that the critical temperature is exactly the same as we found by mapping to a single-site problem. In the original lattice, the replicon instability receives a natural interpretation: The correlations in the Coulomb potential created by the electron configuration become critical at $T_c$ where they only decay with a power law. Below $T_c$, the phase space splits into an exponential number of metastable states (ergodic components) each of which is characterized by the finite expectation values $\psi_i$ of the conjugated potential fields $\psi_i = i\beta^{-1}\langle \phi_i \rangle$. From Eq. (14) we see that within such a metastable state we should consider $\epsilon_i + \psi_i$ as the effective field on the site $i$ to be used to calculate the vertex coefficients. Anticipating an Efros-Shklovskii-type distribution for those effective fields then suggests that the vertex coefficients averaged over sites, scale as $1/\beta^3$ as mentioned above. In particular, the mass term of the propagator will be replaced by the average susceptibility corresponding to the distribution of effective fields $\epsilon_i + \psi_i$.

At this point the diagram technique in the original lattice becomes very complicated since translational invariance is spontaneously broken by the emergence of spontaneous expectation values. The advantage of the mapping to a single site problem is now obvious: The replica structure implicitly takes the statistics of a spin’s environment into account, and allows, e.g., for a self-consistent determination of the distribution of local fields.

**Spin-spin correlations and inhomogeneous charge response**

The criticality of the correlations of potential fields translates directly to the criticality of spins. The fact that the glass phase is marginally stable has the simple interpretation that these correlations remain long range (power law) throughout the low temperature phase. This self-organized criticality has an important consequence: One can show that by imposing the spin on site $i$ to take a definite value one induces a polarization at site $j$ which is proportional to $\langle s_i s_j \rangle_c$. The above results imply that this response of the system in the glass phase decays only as a power law with distance, i.e., screening is almost absent. Furthermore, we note that the polarization induced on different sites do not have a definite sign. It should be clear from these considerations that typical spins interact with a large number of effective neighbors, which in turn self-consistently confirms the initial assumption suggesting the locator approximation.

**APPENDIX: THE REPLICON INSTABILITY**

Within the effective single-site model, the glass transition (6) and the marginal stability condition (8) both derive from the vanishing of the eigenvalue of the Hessian $\partial^2 F/\partial B^2$ in the replicon mode. The variation of the spin-part in (4) is standard and yields

$$\delta^2 \beta F_{\text{spin}} = -tr(\delta B^2) \int_{-\infty}^{\infty} dy P(y) \frac{1}{[2 \cosh(\beta y/2)]^4}. \quad (25)$$

When varying $U(B)$, (5), we need to take into account that $\Sigma$ is a function of $B$. The selfconsistency condition

$$\frac{1}{B - \Sigma} = g_1(-\Sigma) \quad (26)$$

imposes, upon variation in the replicon mode, that $(\delta B - \delta \Sigma)g_2^2(\Sigma_0) = -g_2(\Sigma_0)\delta \Sigma$. The variation of $U(B)$ thus evaluates to

$$\delta^2 U(B) = -g_1^2(\Sigma_0)tr(\delta B - \delta \Sigma)^2 + g_2(\Sigma_0)tr(\delta \Sigma)^2 = \frac{tr(\delta B^2)}{g_1^2(\Sigma_0) - g_2^{-1}(\Sigma_0)}, \quad (27)$$

which together with (25) leads to (6) and (8).

**APPENDIX: GENERALIZED ONSAGER TERM**

We have shown above that the spin $s_0$ at site 0 polarizes its environment in a large spatial region. As is well-known from the TAP-approach to spin glasses, in order to obtain the thermodynamic field $y_0 = 2\beta^{-1}\tanh^{-1}(2m_0)$,
the back reaction of this polarization on the spin itself has to be subtracted from the thermally averaged field
\[ \langle h_0 \rangle = - \sum_{j \neq 0} J_{0j} m_j, \]
\[ y_0 = \langle h_0 \rangle_{s=s_0} - s_0 h_O, \quad (28) \]
where \( h_O \) is the famous Onsager back reaction. The usual term \( h_O = \sum_j J_{0j}^2 \chi_j \), familiar from spin glasses, has to be generalized to the case of Coulomb glasses where this expression is clearly divergent. Indeed, to obtain a finite response we have to sum up all higher order polarizations,
\[ h_O = \sum_{j_1} J_{ij_1} \chi_{j_1} J_{j_1 i} - \sum_{j_1, j_2} J_{ij_1} \chi_{j_1} J_{j_1 j_2} \chi_{j_2} J_{j_2 i} + \ldots, \quad (29) \]
their alternating sign reflecting the antiferromagnetic nature of the Coulomb interactions [2]. Approximating the on-site susceptibilities \( \chi_j \) by their average \( \chi \) (which is justified since we average over a very large number of spins), we may perform the sum
\[ h_O = \text{Tr} \frac{J^2}{\chi^{-1} + \Sigma} \approx \beta \text{Tr} \left( \frac{J^2}{\beta J + \Sigma_0} \right) \approx 2\pi \sqrt{\Sigma_0} \sim T. \quad (30) \]
The last approximations are valid at low temperatures.

The distribution of instantaneous fields can be obtained from
\[ P(h, s) = \frac{1}{V} \sum_i \int \frac{d\lambda}{2\pi} e^{i\lambda(h - i\Sigma_j J_{ij} s_j) \delta_{si}} \approx \frac{1}{V} \sum_i \int \frac{d\lambda}{2\pi} e^{i\lambda(h - i\Sigma_j s_j) \delta_{si}} \approx 2\pi e^{i\lambda h - i\lambda \sum_j J_{ij} s_j} e^{-\lambda^2/2\Sigma_j J_{ij} s_j \delta_{si}}. \quad (31) \]
We have only retained the first two cumulants, to be consistent with in the locator approximation. From the generalized TAP-equations [28] we may identify the first cumulant as
\[ \langle h_i \rangle_{s_i = s} = y_i + s h_O. \quad (32) \]
The second cumulant is almost insensitive to the value of the spin at site \( i \), and evaluates to
\[ \sum_{j,k} J_{ij} \langle s_j s_k \rangle_c J_{ki} = \left[ \frac{1}{\beta J + \Sigma_0} \right]_{ii} \approx \text{Tr} \left[ \frac{J^2}{\beta J + \Sigma_0} \right] = \beta^{-1} h_O. \quad (33) \]
in the locator approximation.

To carry out the site average, we have to weight the pairs \( (y, s) \), according to their joint probability density \( P(y) \exp(\beta y s)/(2 \cosh(\beta y/2)), \)
\[ P(h, s) = \int dy P(y) \left[ \frac{1}{2 \cosh(\beta y/2)} \int \frac{d\lambda}{2\pi} e^{i\lambda y} e^{i\lambda(h - y - sH(\beta))} e^{-\lambda^2/2H(\beta)/2} \right]. \quad (34) \]
Performing the \( \lambda \)-integral we find
\[ P(h, s) = \int dy P(y) \frac{e^{i\beta y}}{2 \cosh(\beta y/2)} \exp[-\beta(h - y - s h_O)^2/2 h_O] / \left[ 2\pi h_O/\beta \right]^{1/2}. \quad (35) \]
and summing over \( s \) we find the local field distribution
\[ P(h) = \int dy P(y) \frac{\cosh(\beta h/2)}{\cosh(\beta y/2)} \frac{\exp[-\beta(h - y)^2/2 h_O]}{2\pi h_O/\beta} \exp(\beta h_O/8). \quad (36) \]