The usual Chern-Simons extension of Einstein gravity theory consists in adding a squared Riemann contribution to the Hilbert Lagrangian, which means that a square-curvature term is added to the linear-curvature leading term governing the dynamics of the gravitational field. However, in such a way the Lagrangian consists of two terms with a different number of curvatures, and therefore not homogeneous. To develop a homogeneous Chern-Simons correction to Einstein gravity we may, on the one hand, use the above-mentioned square-curvature contribution as the correction for the most general square-curvature Lagrangian, or on the other hand, find some linear-curvature correction to the Hilbert Lagrangian. In the first case, we will present the most general square-curvature leading term, which is in fact the already-known re-normalizable Stelle Lagrangian. In the second case, the topological current has to be an axial-vector built only in terms of gravitational degrees of freedom and with a unitary mass dimension, and we will display such an object. The comparison of the two theories will eventually be commented.

I. INTRODUCTION

Einstein gravity is perhaps one of the best theories ever to be built. Constructed starting from first principles of the most general validity, it has passed all the experimental tests performed in astrophysics and cosmology.

As a matter of fact, if dark matter is indeed, as it seems to be, some form of matter, and not a modification of the gravitational field, then there is not a single observation that Einsteinian gravitation would not fit. From a purely theoretical perspective, Einstein gravity has always been thought to be doomed by the necessity of singularity formation of singularity is no longer an unavoidable feature of Einstein gravity if it is complemented by torsion. The earliest attempt to include a Chern-Simons term into Einsteinian gravity is that of [2] (for a review see [3] and references therein), and it consists in adding one specific squared Riemann contribution to the Hilbert Lagrangian. The motivation for adding a term of type $R^{pqrs}R_{rquν}c_{pqc}$ is in analogy with the $F^{pq}F^{pc}Σ_{pq}$ found in electrodynamics, but while this can mathematically be done, nevertheless one cannot help but notice a certain lack of homogeneity. In fact, in electrodynamics the leading term has the structure $F^{ac}F_{ac}$ and therefore gravity should have a leading term $R^{pqrs}R_{pqrs}$ to obtain a full analogy. Such a square-curvature term, or a closely related one, is already studied by Sterle in [4, 5]. However the complementary way is to add $R^{pqrs}c_{pqc}$ to the term $R$ that constitutes the standard Hilbert Lagrangian. The problem with this term is that it is zero, and therefore a term that is a pseudo-scalar with 2 mass dimension must be found in alternative. This amounts to ask that such a topological correction be of the form $∇νKν$ with $Kν$ an axial-vector of unitary mass dimension. Or equivalently, $Kν$ must be an axial-vector built in terms of the connection. Although there seems to be no such a thing, recent studies are useful in providing this topological current as we are going to discuss in the present paper.

II. SPINOR FIELDS

As anticipated in the introduction, recent findings have enabled us to obtain a topological current $Kν$ constructed in terms of the connection alone. This can be done when Dirac spinorial matter fields are the space-time content. The Clifford matrices $γa$ are defined as $\{γaγb\}=2ηab[6]$ where $ηab$ is the Minkowskian matrix. So $\{γaγb\}=4σab$ defines the generators of the complex Lorentz algebra and the relationship $2iσ_{ab}=ε_{abcdπσ\sigma\sigma}$ defines $π$ (this matrix is usually denoted as a gamma matrix with an index five, but in space-time this index has no meaning, and so we use a notation with no index). By exponentiating $σ_{ab}$ we can compute the local complex Lorentz group $S$ and the spinor field $ψ$ is defined as what transforms according to $ψ→Sψ$ in general. With the Clifford matrices we define the adjoint spinor $ψ=γaψ†$ again in general. The set

$$\Sigma^{ab}=2\bar{ψ}\sigma^{ab}\piψ$$  \hspace{1cm} (1)

$$M^{ab}=2\bar{ψ}\sigma^{ab}\psi$$  \hspace{1cm} (2)

$$S^a=\bar{ψ}\gamma^a\psi$$  \hspace{1cm} (3)

$$U^a=\bar{ψ}\gamma^a\psi$$  \hspace{1cm} (4)

$$Θ=\bar{ψ}ψ$$  \hspace{1cm} (5)

$$Φ=\bar{ψ}ψ$$  \hspace{1cm} (6)

defines bi-linear spinor quantities, they are all real and such that $U^aU^a=S^aS^a=|Θ|^2+|Φ|^2$ and $U^aS^a=0$ hold.

These bi-lines can be used to perform the Lounesto classification [6, 7]: singular spinors are those for which $Θ=Φ\equiv0$ [8–14]; regular spinors are defined when either
$\Theta$ or $\Phi$ or both are not equal to zero identically. In this paper, we will be interested in regular spinors, and that is Dirac spinors, for which it is always possible to write

$$\psi = \phi e^{-i/2 \beta \pi S} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(7)
in chiral representation, for some complex Lorentz transformation $S$, with $\beta$ and $\phi$ called Yvon-Takahashi angle and module, and where the spinor is said to be in polar form [15]. The bi-linear spinor quantities reduce to

$$\Sigma^{ab} = 2\dot{\phi}^2 (\cos \beta u^a s^b - \sin \beta u^a s^b e^{ijk} s^i)$$

(8)

and

$$M^{ab} = 2\dot{\phi}^2 (\cos \beta u^a s^b e^{ijk} s^i + \sin \beta u^a s^b)$$

(9)

showing that they can always be written with the vectors

$$S^a = 2\dot{\phi}^2 s^a$$

(10)

$$U^a = 2\dot{\phi}^2 u^a$$

(11)

and the scalars

$$\Theta = 2\dot{\phi}^2 \sin \beta$$

(12)

$$\Phi = 2\dot{\phi}^2 \cos \beta$$

(13)

such that $u^a u^a = -s^a s^a = 1$ and $\epsilon_{abc} u^a = 0$ and which show that Yvon-Takahashi angle and module are the only 2 true degrees of freedom. The 8 real components of spinors are rearranged into the special configuration in which the 2 real scalar degrees of freedom, YT angle and module, become isolated from the 6 components that can always be transferred away, the spin and the velocity. We notice that the YT angle is a zero-dimension pseudo-scalar and therefore the module inherits the full 3/2-dimension that characterizes the spinor field. This is one most important remark for the following of the paper, as we shall see.

As for the background, we have that using the metric we define the symmetric connection $A^\mu_{ab}$, and with it we define the spin connection $\Omega^a_\mu = \frac{1}{4} (\Omega_{ab}^\alpha \sigma^a_{ab} + i q A^\alpha_{ab})$ and

$$\Omega_\mu = \frac{1}{4} \Omega^a_\mu \sigma^a_{ab} + i q A^\mu_{ab}$$

(14)
in terms of the gauge potential $A_\mu$ and called spinorial connection. This is needed to write

$$\nabla_\mu \psi = \partial_\mu \psi + \Omega_\mu \phi$$

(15)
as spinorial covariant derivative. Then the commutator of spinorial covariant derivatives justifies the definitions

$$R^{ij}_j \mu = \partial_i \Omega_{j\mu} - \partial_j \Omega_{i\mu} + \Omega_{i\mu} \Omega_{j\nu} - \Omega_{j\mu} \Omega_{i\nu}$$

(16)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

(17)

that is the Riemann curvature and the Maxwell strength.

When the polar form is taken into account, and considering that we can formally write the expansion

$$S \partial_\mu S^{-1} = i \partial_\mu \alpha a + \frac{i}{2} \partial_\mu \theta_{ij} \sigma^{ij}$$

(18)

we can define

$$\partial_\mu \alpha - q A_\mu = P_\mu$$

(19)

$$\partial_\mu \theta_{ij} - \Omega_{ij\mu} = R_{ij\mu}$$

(20)
in which we used (14) and which can be demonstrated to be tensors and invariant under the gauge transformation simultaneously. With them we can eventually write

$$\nabla_\mu \psi = (-\frac{1}{2} \nabla_\mu \beta \pi + \nabla_\mu \ln \phi - i P_\mu \pi - \frac{1}{4} R_{ij\mu} \sigma^{ij} \psi)$$

(21)
as spinorial covariant derivative so that

$$\nabla_\mu s_i = R_{ji\mu} s^j$$

(22)

$$\nabla_\mu u_i = R_{ji\mu} u^j$$

(23)

are general geometric identities. Taking the commutator

$$q F_{\mu\nu} = -(\nabla_\mu P_{\nu} - \nabla_\nu P_{\mu})$$

(24)

$$R^i_{j\mu\nu} = - (\nabla_\mu R^i_{j\nu} - \nabla_\nu R^i_{j\mu} + R^i_{k\mu} R^k_{j\nu} - R^i_{k\nu} R^k_{j\mu})$$

(25)
in terms of the Riemann curvature and Maxwell strength, so that they encode electrodynamic and gravitational information. As we have been saying above, in writing the spinor field in its polar form, the spinor field is reconfigured so that its degrees of freedom are isolated from the components transferable into gauge and frames through the $\alpha$ phase and the $\theta_{ij}$ parameters in general. When the phase and parameters are added to gauge potential and spin connection, they do not alter their information, and thus (19, 20) have the information content of the gauge potential and the spin connection themselves, although in the combination non-covariant features fully cancel, so that (19, 20) are gauge invariant and Lorentz covariant, and for this reason they have been called gauge-invariant vector momentum and tensorial connection [16].

The Dirac spinor dynamics is given by the Lagrangian

$$\mathcal{L} = -i \overline{\gamma^\mu \nabla_\mu \psi} + m \overline{\psi} \psi$$

(26)
giving the Dirac spinor field equations

$$i \gamma^\mu \nabla_\mu \psi - m \psi = 0$$

(27)
as it is well known. By multiplying (27) by $\gamma^\alpha$ and $\gamma^\alpha \pi$ and by $\overline{\psi}$ and splitting real and imaginary parts gives

$$i (\psi \nabla^\alpha \psi - \overline{\psi} \overline{\nabla^\alpha \psi}) - \nabla^\alpha M^{\mu\alpha} - 2m U^{\alpha} = 0$$

(28)

$$\nabla_\rho \Phi - 2 (\overline{\psi} \sigma_{\rho\mu} \nabla_\mu \psi - \nabla_\mu \overline{\psi} \sigma_{\rho\mu} \psi) = 0$$

(29)

$$\nabla_\nu \Theta - 2i (\overline{\psi} \sigma_\nu \pi \nabla_\mu \psi - \nabla_\mu \overline{\psi} \sigma_\nu \pi \psi) + 2m S_\nu = 0$$

(30)

which are called Gordon decompositions and they have a great important in writing the Dirac spinor field equation in polar form while maintaining manifest covariance.

Plugging the polar form in (26) gives

$$\mathcal{L} = -\dot{\phi}^2 [S^\mu (\nabla_\mu \beta + B_\mu) + 2u^\mu P_\mu - 2m \cos \beta]$$

(32)
while in (29, 30) it gives after some manipulation that
\[ B_{u} - 2P^{a}u_{[a}s_{[b]} + \nabla_{[a}s_{b]} + 2s_{m}m \cos \beta = 0 \]  
(33)
\[ R_{\mu} - 2P^{a}u^{a}\mu_{\rho \nu} + 2s_{m}m \sin \beta + \nabla_{\mu} \ln \phi^{2} = 0 \]  
(34)
with \( R_{\mu}{}^{a} = R_{\mu} \) and \( \frac{1}{2}R_{\mu}{}^{\rho \nu} = B_{\mu} \) which could be proven to be equivalent to polar form of Dirac equations, discussed thoroughly in [17]. The Dirac spinor field equations (27) consist of 8 real equations, which are as many as the 2 vectorial equations given by the (33, 34) above specifying all space-time derivatives of the two degrees of freedom given by YT angle and module. It is also important to notice that the YT angle, in its being the phase difference between chiral projections, must be expected to be present in the mass term, and it is as easy to see.

As known, Maxwell and Riemann tensors encode electrodynamic and gravitational information. As such, they act as filters keeping out information of gauge and frames. But the other hand, the gauge-invariant vector momentum and tensorial connection (19, 20) contain the information about electrodynamics and gravity but also additional information related to the gauge and the frames, and this last type of information is not necessarily trivial despite being described by covariant objects. To see what is the information related to the gauge and the frames one should consider the conditions given by
\[ \nabla_{\mu}P_{\gamma} - \nabla_{\gamma}P_{\mu} = 0 \]  
(35)
\[ \nabla_{\mu}R_{\beta}{}^{\gamma}_{\mu} - \nabla_{\gamma}R_{\beta}{}^{\mu}_{\mu} + R_{k}{}^{\mu}{}_{\gamma}R_{k}{}^{k}{}_{\mu} - R_{k}{}^{k}{}_{\mu}R_{\gamma}{}^{\mu} = 0 \]  
(36)
and find solutions for \( P_{\mu} \) and \( R_{\beta}{}^{\gamma}{}_{\mu} \) that are non-zero, as this would mean that they are non-trivial and since they are tensors then will remain such for all gauges and in all frames. However, they would contain no electrodynamic or gravitational information. The first instance is easy because (35) is solved for non-zero gauge-invariant vector momenta of the type \( F_{\mu} = \nabla_{\mu}P \) in general. As for the case of gravity things are more complicated because there does not appear to be a general solution. A special solution can however be found in spherical coordinates
\[ g_{\mu} = 1 \]  
(37)
\[ g_{\nu} = -1 \]  
(38)
\[ g_{\theta \theta} = -r^{2} \]  
(39)
\[ g_{\phi \phi} = -r^{2}\sin^{2}\theta \]  
(40)
with connection
\[ \Lambda_{\theta}^{\theta} = \frac{1}{r} \]  
(41)
\[ \Lambda_{\theta \theta} = 0 \]  
(42)
\[ \Lambda_{\phi}^{\phi} = 0 \]  
(43)
\[ \Lambda_{\phi \phi} = -r^{2}\sin^{2}\theta \]  
(44)
\[ \Lambda_{\phi \theta} = \cot\theta \]  
(45)
\[ \Lambda_{\phi \phi} = -\cos\theta \sin\theta \]  
(46)
by specifying to the case
\[ u_{\theta} = \cosh\alpha \]  
(47)
\[ u_{\phi} = r\sin\theta\sin\alpha \]  
(48)
\[ s_{\tau} = \cos\gamma \]  
(49)
\[ s_{\theta} = r\sin\gamma \]  
(50)
with \( \alpha = (r, \theta) \) and \( \gamma = (r, \theta) \) generic functions. Hence relations (22,23) can be solved for \( R_{ijk} \) giving
\[ R_{\tau\phi\tau} = R_{\tau\phi\phi} = R_{\tau\theta\tau} = 0 \]  
(51)
as well as
\[ r\sin\theta\theta_{\alpha} = R_{\tau\phi\theta} \]  
(52)
\[ r\sin\phi\tau_{\alpha} = R_{\tau\phi\tau} \]  
(53)
\[ -r(1 + \partial_{\theta}\gamma) = R_{\tau\phi\tau} \]  
(54)
\[ \partial_{\theta}\gamma = R_{\tau\tau\theta} \]  
(55)
linking the derivatives of the two above functions to four of the components of the \( R_{ijk} \) tensor and
\[ rR_{\theta\phi\tau} = R_{\tau\phi\tau} \tan\gamma \]  
(56)
\[ r\sin\theta R_{\tau\phi\tau} = (R_{\tau\phi\tau} - r^{2}\cos\theta\sin\theta) \tan\alpha \]  
(57)
\[ (R_{\phi\phi\tau} - r^{2}\sin\theta\cos\theta) \tan\gamma = r(R_{\tau\phi\tau} + r\sin\theta)^{2} \]  
(58)
\[ (R_{\tau\tau\tau} + r\sin\theta)^{2} \tan\alpha = r\sin\theta R_{\tau\tau\tau} \]  
(59)
as well as
\[ rR_{\tau\tau\theta} = R_{\tau\phi\theta} \tan\gamma \]  
(60)
\[ r\sin\theta R_{\tau\phi\theta} = R_{\tau\phi\phi} \tan\alpha \]  
(61)
\[ R_{\tau\phi\theta} \tan\gamma = rR_{\tau\tau\tau} \]  
(62)
\[ R_{\tau\tau\phi} \tan\alpha = r\sin\theta R_{\tau\tau\phi} \]  
(63)
and
\[ rR_{\tau\tau\tau} = R_{\tau\phi\theta} \tan\gamma \]  
(64)
\[ r\sin\theta R_{\tau\phi\theta} = R_{\tau\phi\phi} \tan\alpha \]  
(65)
\[ R_{\tau\phi\theta} \tan\gamma = rR_{\tau\tau\tau} \]  
(66)
\[ R_{\tau\tau\phi} \tan\alpha = r\sin\theta R_{\tau\tau\phi} \]  
(67)
with
\[ rR_{\tau\tau\theta} = R_{\tau\phi\theta} \tan\gamma \]  
(68)
\[ r\sin\theta R_{\tau\phi\theta} = R_{\tau\phi\phi} \tan\alpha \]  
(69)
\[ R_{\tau\phi\theta} \tan\gamma = rR_{\tau\tau\tau} \]  
(70)
\[ R_{\tau\tau\phi} \tan\alpha = r\sin\theta R_{\tau\tau\theta} \]  
(71)
grouped in four independent blocks each with four interlinked relations. Thus a working hypothesis might be to look for a solution in which we can set to zero some block while leaving different from zero others, such as
\[ R_{\tau\tau\tau} = R_{\tau\phi\tau} = R_{\phi\tau\phi} = 0 \]  
(72)
\[ R_{\tau\tau\theta} = R_{\tau\phi\theta} = R_{\phi\phi\phi} = 0 \]  
(73)
\[ R_{\tau\theta\phi} = R_{\tau\phi\phi} = 0 \]  
(74)
with
\[ R_{\tau\phi\phi} = -r^{2}\sin\theta \]  
(75)
\[ R_{\phi\phi\phi} = -r^{2}\cos\theta\sin\theta \]  
(76)
and
\[ R_{\mu
u} = -2\varepsilon \sinh \alpha \sin \gamma \] (77)
\[ R_{\nu\rho} = 2\varepsilon \sin \theta \cosh \alpha \sin \gamma \] (78)
\[ R_{\theta\mu} = 2\varepsilon \sin \rho \cosh \alpha \cos \gamma \] (79)
\[ R_{\rho\nu} = -2\varepsilon^2 \sin \theta \cosh \alpha \cos \gamma \] (80)

with \( \varepsilon \) being a generic constant, which can be interpreted as an integration constant since it comes from having set the Riemann curvature tensor to be zero identically [18].

Therefore, we have shown that it is indeed possible to have zero Riemann and Maxwell tensors, thus no gravity and electrodynamics, but still have non-vanishing covariant objects that as such contain information related only to frames and gauge, but which is non-trivial and cannot be removed with choices of frames and gauge. That this had to be the case is also clear from the fact that setting gauge-invariant vector momenta or tensorial connections to zero in general leads to unwanted consequences [19].

The tensorial connection is just the gravitational analogue of what the gauge-invariant vector momentum is in electrodynamics. However, the tensorial connection has a much richer structure since it can be decomposed into the vector trace \( R_\alpha \) and the axial-vector dual \( B_\alpha \) with a non-completely antisymmetric irreducible part accounting for the rest of the information on the degrees of freedom [20].

We notice that \( B_\alpha \) is an axial-vector that contains the same information of the connection and it also possesses the same mass dimension of the connection itself.

III. RE-NORMALIZABLE CHERN-SIMONS STELLE GRAVITY

In the introduction we have discussed that a first way to have a homogeneous Chern-Simons correction to Einstein gravity is to find the square-curvature leading term to be added to the already-mentioned square-curvature Chern-Simons type of gravitational topological current.

The most general square-curvature leading term is
\[ \mathcal{L} = X R^{\rho\sigma\tau\nu} R_{\rho\sigma\tau\nu} + Y R^{\rho\sigma} R_{\rho\sigma} + Z R^2 \] (81)
where \( R_{\rho\sigma\tau\nu} \) is the Riemann tensor, \( R^{\rho\sigma} = R^\alpha_{\rho\sigma\alpha} \) is the Ricci tensor and \( R = R^\mu_{\mu} \) is the Ricci scalar. Nonetheless, the Gauss-Bonnet identity tells that the square-Riemann term can always be written as a combination of the two square-Ricci terms up to a divergence. So there is no loss of generality in setting \( X \) to zero and considering
\[ \mathcal{L} = Y R^{\rho\sigma} R_{\rho\sigma} + Z R^2 \] (82)
as the square-curvature Lagrangian [4]. This Lagrangian is re-normalizable [5]. We call it Stelle Lagrangian.

To this we have to add the term
\[ \mathcal{L} = K \beta R^{\rho\sigma\tau\nu} R_{\rho\sigma\tau\nu} e_{pqac} \] (83)
with \( \beta \) a general pseudo-scalar [2]. The square-curvature term can be written as \( R^{\rho\sigma\tau\nu} R_{\rho\sigma\tau\nu} e_{pqac} = \nabla_{\mu} K^\mu \) as well known, but now with the tools developed in the previous section we can also see that the axial-vector is explicitly given by \( K^\mu = -4\varepsilon^{\rho\sigma\tau\nu}(R^\alpha_{\rho\sigma\nu} R^{\tau\nu}_{\alpha\rho} + \frac{1}{2} R^\tau_{\alpha\rho \nu} R^{\rho\nu}_{\alpha\tau}) \) in terms of the tensorial connection alone. And because the Yvon-Takabayashi angle is a pseudo-scalar of zero mass dimension, we will consider \( b \) to be \( \beta \) so that
\[ \mathcal{L} = K \beta R^{\rho\sigma\tau\nu} R_{\rho\sigma\tau\nu} e_{pqac} \] (84)
is a square-curvature of 4 mass dimension, and the topological term has the same properties of the leading term.

The most general Lagrangian is given by
\[ \mathcal{L} = Y R^{\rho\sigma} R_{\rho\sigma} + Z R^2 + K \beta R^{\rho\sigma\tau\nu} R_{\rho\sigma\tau\nu} e_{pqac} \] (85)
which is in fact homogeneous and re-normalizable, and it is the Lagrangian of the gravitational sector. Then
\[ \mathcal{L} = Y R^{\rho\sigma} R_{\rho\sigma} + Z R^2 + K \beta R^{\rho\sigma\tau\nu} R_{\rho\sigma\tau\nu} e_{pqac} - \] (86)
\[ -i\gamma^\mu \nabla_\mu \psi + m\bar{\psi}\psi \]
or in polar form
\[ \mathcal{L} = Y R^{\rho\sigma} R_{\rho\sigma} + Z R^2 + K \beta R^{\rho\sigma\tau\nu} R_{\rho\sigma\tau\nu} e_{pqac} - \] (87)
\[ -\phi^2 [s^\mu (\nabla_\mu \beta + B_\mu) + 2a^\mu P_\mu - 2m \cos \beta] \]
is the Lagrangian of gravitational and material sectors. The variation of this Lagrangian is performed by employing the usual method, although here the condition of torsionlessness means that the variation of the connection is given in terms of the variation of the metric itself with the result that the gravitational field equations are

\[ Y \nabla^2 R_{\mu\nu} + \frac{1}{4}(4Z + Y) \nabla^2 R_{\mu\nu} - (2Z + Y) \nabla_\mu \nabla_\nu R + \] (88)
\[ +2Y R_{\mu\nu\rho\sigma} R^{\rho\sigma} - \frac{1}{2} Y R_{\alpha\rho\sigma} R^{\rho\sigma} g_{\mu\nu} + \] (89)
\[ +2Z R_{\mu\nu\rho\sigma} R^{\rho\sigma} g_{\mu\nu} + \] (90)
\[ +2K \nabla^\rho \nabla_\rho \beta R_{\mu\nu\rho\sigma} e_{\alpha\nu\rho\sigma} + \nabla^\rho \beta R_{\mu\nu\rho\sigma} e_{\alpha\nu\rho\sigma} \]
\[ = \frac{1}{\beta} (\bar{\psi} \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi) \]
while the variation with respect to the spinor field furnishes the material field equations
\[ i\gamma^\mu \nabla_\mu \psi - \frac{K}{2} R^{\rho\sigma\tau\nu} R_{\rho\sigma\tau\nu} e_{pqac} \phi^{-2} e^{i\beta} \bar{\psi} \psi - m\bar{\psi}\psi = 0 \]
or respectively in polar form
\[ Y \nabla^2 R_{\mu\nu} + \frac{1}{4}(4Z + Y) \nabla^2 R_{\mu\nu} - (2Z + Y) \nabla_\mu \nabla_\nu R + \] (91)
\[ +2Y R_{\mu\nu\rho\sigma} R^{\rho\sigma} - \frac{1}{2} Y R_{\alpha\rho\sigma} R^{\rho\sigma} g_{\mu\nu} + \] (92)
\[ +2Z R_{\mu\nu\rho\sigma} R^{\rho\sigma} g_{\mu\nu} + \] (93)
\[ +2K \nabla^\rho \nabla_\rho \beta R_{\mu\nu\rho\sigma} e_{\alpha\nu\rho\sigma} + \nabla^\rho \beta R_{\mu\nu\rho\sigma} e_{\alpha\nu\rho\sigma} \]
\[ = \frac{1}{\beta} (\bar{u}_i s_{\mu} s_{\nu} + u_i s_{\mu} + s_{\mu} s_{\nu} + \frac{3}{2} s_{\mu} s_{\nu}) - \] (94)
\[ -\frac{1}{2} R^{\rho\sigma\tau\nu} e_{\nu\rho\sigma\tau\nu} \]
and
\[ -2P_i u_j s_{\mu} + B_\mu + \nabla_\mu \beta + 2s_{\mu} m \cos \beta = 0 \]
\[ -2P^\mu s^\nu e_{\mu\nu\rho\sigma} + R_{\mu\nu} - s_{\mu} K R^{\rho\sigma\tau\nu} R_{\rho\sigma\tau\nu} e_{pqac} \phi^{-2} + \] (95)
\[ +\nabla_\mu \ln \phi^2 + 2s_{\mu} m \sin \beta = 0 \]
as the full set of gravity and matter field equations.

Taking the divergence of field equations (88) and using the Jacobi-Bianchi cyclic identities gives the constraints

$$-K\nabla_{\nu}\beta R^\rho_{\mu\sigma\tau} R^i_{\alpha\beta\gamma} \epsilon^{ijpq} =$$

$$= \frac{1}{4} \nabla^\nu (\overline{\psi} \partial_\nu \psi - \overline{\nabla}_\mu \psi \overline{\psi} + \overline{\psi} \nabla_\mu \psi - \overline{\nabla}_\mu \overline{\psi} \nabla_\mu \psi)$$

(93)

showing that the divergence of the symmetric energy density tensor is not zero. Nevertheless, computing it as

$$\frac{1}{4} \nabla^\nu \nabla_\mu \partial_\nu \psi - \overline{\nabla}_\mu \psi \overline{\psi} +$$

$$\overline{\psi} \nabla_\mu \psi - \overline{\nabla}_\mu \overline{\psi} \nabla_\mu \psi -$$

$$\nabla_\mu \nabla_\nu \overline{\psi} \nabla_\nu \psi$$

(94)

and repeatedly employing the Dirac equation one gets

$$\frac{1}{4} \nabla^\nu \nabla_\mu \partial_\nu \psi - \overline{\nabla}_\mu \psi \overline{\psi} +$$

$$\overline{\psi} \nabla_\mu \psi - \overline{\nabla}_\mu \overline{\psi} \nabla_\mu \psi -$$

$$\nabla_\mu \nabla_\nu \overline{\psi} \nabla_\nu \psi$$

(95)

which can eventually be easily turned into

$$\frac{1}{4} \nabla^\nu \nabla_\mu \partial_\nu \psi - \overline{\nabla}_\mu \psi \overline{\psi} +$$

$$\overline{\psi} \nabla_\mu \psi - \overline{\nabla}_\mu \overline{\psi} \nabla_\mu \psi -$$

$$\nabla_\mu \nabla_\nu \overline{\psi} \nabla_\nu \psi$$

(96)

as clear. Putting together the last three expressions gives

$$\frac{1}{4} \nabla^\nu (\overline{\psi} \partial_\nu \psi - \overline{\nabla}_\mu \psi \overline{\psi} + \overline{\psi} \nabla_\mu \psi - \overline{\nabla}_\mu \overline{\psi} \nabla_\mu \psi) =$$

$$= -K R^\rho_{\mu\sigma\tau} R^i_{\alpha\beta\gamma} \epsilon^{ijpq} \nabla_\nu \beta$$

(97)

to be compared against (93) above. In doing so, we finally see that there is no constraint developed in the end.

This is important since in [2] the authors state that the theory must be restricted to have $R^\rho_{\mu\sigma\tau} R^i_{\alpha\beta\gamma} \epsilon^{ijpq} \equiv 0$ in order for the divergencelessness of the symmetric energy density tensor to hold. Such a restriction is not necessary as the divergencelessness of the symmetric energy density tensor does not need to be implemented in the first place due to the fact that the $eta R^\rho_{\mu\sigma\tau} R^i_{\alpha\beta\gamma} \epsilon^{ijpq}$ is a potential energy of interaction between matter and the space-time.

Lack of conservation of energy simply means that there is an energy flux from/to matter, which does not have to be zero so long as it is exactly compensated by the energy flux to/from the space-time, and here we proved it is.

We will see however that this is not always true.

IV. LEAST-DERIVATIVE CHERN-SIMONS HILBERT GRAVITY

In the introduction, we have discussed about the necessity to have an object $K_\alpha$ being an axial-vector and built in terms of the connection. And in the previous section, we have seen that the axial-vector of the tensorial connection has such features. As such $B_\alpha$ seems the perfect candidate for the topological current we are seeking.

To the least-order derivative the leading term is

$$\mathcal{L} = R$$

(98)

which is the least-order derivative Lagrangian. Obviously this is the very well known usual Hilbert Lagrangian.

The least-order derivative topological term can now be added straightforwardly as it has to be of the form

$$\mathcal{L} = k\beta \nabla_\mu B^\mu$$

(99)

with $b$ generic pseudo-scalar. However, if we want to keep homogeneity, and since the gravitational Lagrangian has 2 mass dimension, then also this term must have 2 mass dimension, and the only way we have to do this is to take $b$ to be a pseudo-scalar of zero mass dimension, which can only be the Yvon-Takahayashi angle, so that

$$\mathcal{L} = k\beta \nabla_\mu B^\mu$$

(100)

is the only option for the least-order topological term.

Altogether we have

$$\mathcal{L} = R + k\beta \nabla_\mu B^\mu$$

(101)

which is homogeneous and least-order derivative, as the Lagrangian for gravity. Therefore we have that

$$\mathcal{L} = R + k\beta \nabla_\mu B^\mu - \frac{i}{2} \overline{\psi} \gamma^\mu \nabla_\mu \psi + m \overline{\psi} \psi$$

(102)

or in polar form

$$\mathcal{L} = R + k\beta \nabla_\mu B^\mu - \phi^2 [\delta^\mu_\nu (\nabla_\mu \beta + B_\mu) +$$

$$+ 2m^2 \nabla_\mu \psi - 2m \nabla_\mu \phi \phi_{\phi^2 \psi} - m \overline{\psi} \psi)$$

(103)

for the gravitational and the material sectors together.

The variation of this Lagrangian then gives

$$R^\sigma_{\mu\nu} - \frac{1}{2} g^\sigma_{\nu} R + \frac{1}{2} \beta \nabla_\mu B^\mu g^\sigma_{\nu} +$$

$$+ \frac{1}{2} \nabla_\mu \beta (\epsilon_{\mu\nu\rho\sigma} R^\rho_{\nu\sigma} + \epsilon_{\mu\nu\rho\sigma} R^\rho_{\nu\sigma} - 2B^\rho g^\sigma_{\nu})$$

(104)

and

$$\gamma^\mu \nabla_\mu \psi - \frac{1}{2} \nabla_\nu \beta \phi_{\phi^2 \psi} - m \overline{\psi} \psi = 0$$

(105)

or respectively in polar form

$$R^\sigma_{\mu\nu} - \frac{1}{2} g^\sigma_{\nu} R + \frac{1}{2} \beta \nabla_\mu B^\mu g^\sigma_{\nu} +$$

$$+ \frac{1}{2} \nabla_\mu \beta (\epsilon_{\mu\nu\rho\sigma} R^\rho_{\nu\sigma} + \epsilon_{\mu\nu\rho\sigma} R^\rho_{\nu\sigma} - 2B^\rho g^\sigma_{\nu})$$

(106)

and

$$-2P^\mu u_\mu s_{\beta} + B_\mu + \nabla_\mu \beta + 2s_{\mu} m \cos \beta = 0$$

(107)

$$-2P^\mu u^\mu s_{\beta} + R_\mu - s_{\mu} \nabla_\mu \Phi_{\phi^2 \psi} +$$

$$+ \nabla_\mu \ln \phi^2 + 2s_{\mu} m \sin \beta = 0$$

(108)
as the full set of gravity and matter field equations.

Taking the divergence of field equations (104) and using the Jacobi-Bianchi cyclic identities yields that

\[
k \nabla_i \left[ -\beta \nabla_i B^\mu g^{\sigma \alpha} + \frac{1}{2} \nabla_\mu \beta \varepsilon^{\mu \sigma \alpha \gamma} R^\gamma_{\alpha \eta} + \varepsilon^{\mu \sigma \alpha \gamma} R^\gamma_{\alpha \eta} - 2 B^\mu g^{\sigma \alpha} \right] = \\
\frac{1}{2} k \nabla_\nu \left( \overline{\psi} \gamma^\nu \psi - \nabla^\nu \overline{\psi} \gamma^\nu \psi + \overline{\psi} \gamma^\nu \nabla^\nu \psi - \nabla^\nu \gamma^\nu \psi \right) \]

(109)

showing that the divergence of the symmetric energy density tensor is not zero, similarly as before. However, now by following the same strategy we would obtain that

\[
k \nabla_\nu \left( \overline{\psi} \gamma^\nu \psi - \nabla^\nu \overline{\psi} \gamma^\nu \psi + \overline{\psi} \gamma^\nu \nabla^\nu \psi - \nabla^\nu \gamma^\nu \psi \right) = \\
- \frac{1}{2} k \nabla \cdot \nabla^\sigma \beta \]

(110)

to be compared against (109) above. In doing so, we get

\[
\nabla_\mu \nabla_\nu \varepsilon^{\mu \sigma \alpha \gamma} R^\gamma_{\alpha \eta} - 2 \nabla_\mu \nabla_\sigma B^\alpha + \\
\nabla_\mu \varepsilon^{\mu \sigma \alpha \gamma} \nabla_\nu R^\gamma_{\alpha \eta} + \nabla_\mu \beta \varepsilon^{\mu \sigma \alpha \gamma} \nabla_\nu R^\gamma_{\alpha \eta} - \\
- 2 \nabla_\mu \nabla_\sigma B^\alpha - 2 \beta \nabla_\nu \nabla_\mu B^\alpha = 0
\]

(111)

with the restriction still holding. Because of the different dependence on the differential structure of the YT angle, there is no way to have this solved in general. Therefore, in the present case we remain with a true restriction that has to be imposed on the structure of the background.

The reason for this occurrence is that for determining the conservation laws we have to use field equations which for the tensorial connection \( R_{ij\alpha} \) do not exist [20]. In fact, not only the \( R_{ij\alpha} \) have no dynamical equations coupling them to sources, but they also have non-local properties and in particular, they do not need to vanish at infinity.

The field equations (104) can now be studied to see the effects on the background. Effects on the curvature have to be expected, but now effects on the topological sector should be expected as well. To screen the curvature, and to simplify the problem we consider that, in order to study the topological features at infinity, we are allowed to take into account the macroscopic approximation.

In this case, all internal structures are negligible and with them also the spin contributions. We can thus take the momentum to be \( P_\sigma = m \cos \beta u_\sigma \) [20], so to write

\[
k \left( \nabla_\mu \beta \varepsilon^{\mu \sigma \alpha \gamma} R^\gamma_{\alpha \eta} + \varepsilon^{\mu \sigma \alpha \gamma} R^\gamma_{\alpha \eta} \right) - \\
\nabla_\mu \left( \beta B^\alpha g^{\sigma \alpha} \right) \approx 2 \phi^2 m \cos \beta u^\alpha u^\sigma
\]

(121)

so that in the co-moving frame only the time-time component would account for a coupling to the material distribution. This will be eventually given by

\[
- \nabla_\mu \left( k \beta B^\alpha \right) \approx 2 \phi^2 m \cos \beta
\]

(122)

where on the right-hand side we have the mass density.

Integrating over the whole volume and considering the normalization for the matter distribution we may write

\[
\int_{\partial V} \beta B_\mu dS^\mu \approx -m/k \neq 0
\]

(123)

where \( dS^\mu \) is the element of surface \( S = \partial V \) constituting the border of the volume \( V \) of integration. This condition clearly shows that the vector \( \beta B^\alpha \) can never go to zero.

Therefore, the tensorial connection, as well as the YT angle, display some topological features at infinity.

V. FINAL COMPARISON

We have studied the re-normalizable Chern-Simons extension of Stelle gravity, described by the Lagrangian

\[
\mathcal{L} = Y R^{\sigma\nu} R_{\sigma\nu} + Z R^2 + K \beta R \varepsilon^{\rho\sigma\nu\tau} R_{\rho\sigma} \varepsilon_{\tau\mu\nu\rho}\]

(124)

and the least-order derivative Chern-Simons extension of the Hilbert gravity, described by the Lagrangian

\[
\mathcal{L} = R + k \beta \nabla_\mu B^\mu
\]

(125)

and it is now time for a comparison of the two.

In analogy with one another, the two corrections consist of an interaction between the Yvon-Takabayashi angle and the background, and therefore we should expect an energy exchange between these two. So such an energy exchange would entail the failure of the conservation law for the energy density of matter allowing only the conservation law for the energy density of the system of matter

\[
-2 P_\nu u_\nu + B_\mu + \nabla_\nu \beta + 2 s_\mu m \cos \beta = 0
\]

(119)

\[
-2 P_\nu u_\nu s_\alpha + R_\mu - s_\mu k \nabla_\nu B \phi^{-2} + \\
\nabla_\mu \ln \phi^2 + 2 s_\mu m \sin \beta = 0
\]

(120)

with (112-113), (114-117) to be plugged in.

These field equations should now be studied in specific situations. However, finding solutions is not an easy task, and to simplify the problem we consider that, in order to study the topological features at infinity, we are allowed to take into account the macroscopic approximation.
and space-time taken together. In [2] a constraint on the structure of the space-time was implemented in order to save the conservation of energy for matter while here we have demonstrated this is not necessary so long as we can prove that there is conservation of the energy for the full matter-gravity system. And we did in fact prove that it is the case, for the re-normalizable Chern-Simons extension of Stelle gravity. Nevertheless we also proved that it is not the case for the least-order derivative Chern-Simons extension of the Hilbert gravity since we got (111) which is not verified identically. This accounts for what we see as the main difference between these two extensions.

We believe that the reason for this difference is to be sought in the fact that in the re-normalizable instance the higher-order derivative structure of the correction allows it to be written entirely in terms of the curvature tensor while in the least-order derivative instance the correction is not written in terms of the curvature but in terms of the tensorial connection. The difference between the curvature and tensorial connection is that while the former contains only information about gravity, the latter contains information about both gravity and inertial contributions. The first case is entirely physical, the correction is fully dynamically determined and hence the energy of the total system is conserved. The second case is physical only in the gravitational information, so the correction is dynamically determined only for gravitation and thus the energy non-conservation of matter is compensated by the energy non-conservation of gravity, but there the inertial information is not physical, it does not have a field equation and it is only by having it constrained that the full conservation of energy is ensured. We have observed that even when the flat space-time has a non-trivial tensorial connection the restriction (111) may still be not verified.

The fact that the tensorial connection may contain information that does not necessarily vanish at infinity but the gravitational field must always vanish at the boundary of the space-time has a straightforward consequence, in view of the comparison of the two theories. And that is, there is more information about the large scale structure of the space-time in the least-derivative CS extension of the Hilbert Lagrangian than in the re-normalizable CS extension of the Stelle Lagrangian. This should not be so surprising, since re-normalizable theories are those valid also in the ultraviolet and hence at small scales.

VI. CONCLUSION

In this paper, we have discussed in what way it is possible to exploit the polar form of spinor fields, and hence the tensorial connection, to construct the re-normalizable higher-derivative CS extension of square-curvature Stelle Lagrangian and the least-derivative CS extension of the linear-curvature Hilbert Lagrangian. Then, we discussed the analogies and the differences of the two theories.

So far as we are aware, these are the only two theories in which both the leading terms and their corrections are homogeneous in the number of curvatures that are used to build the dynamics of the gravitational field.

Combining the two and allowing for the cosmological constant gives the re-normalizable topological completion of the most general space-time dynamics.

[1] L.Fabbri, “Singularity-free spinors in gravity with propagating torsion”, Mod.Phys.Lett. A32, 1750221 (2017).
[2] R.Jackiw, S.Y.Pi, “Chern-Simons modification of general relativity”, Phys.Rev. D68, 104012 (2003).
[3] S.Alexander, N.Yunes, “Chern-Simons Modified General Relativity”, Phys.Rept. 480, 1 (2009).
[4] K.S.Stelle, “Classical Gravity with Higher Derivatives”, Gen.Rel.Grav. 9, 353 (1978).
[5] K.S.Stelle, “Renormalization of Higher Derivative Quantum Gravity”, Phys.Rev. D 16, 953 (1977).
[6] P.Lounesto, Clifford Algebras and Spinors (Cambridge University Press, 2001).
[7] R.T.Cavalcanti, “Classification of Singular Spinor Fields and Other Mass Dimension One Fermions”, Int.J.Mod.Phys. D23, 1444002 (2014).
[8] J.M.Hoff da Silva, R.T.Cavalcanti, “Revealing how different spinors can be: the Lounesto spinor classification”, Mod.Phys.Lett. A32, 1750032 (2017).
[9] J.M.Hoff da Silva, R.da Rocha, “Unfolding Physics from the Algebraic Classification of Spinor Fields”, Phys. Lett. B718, 1519 (2013).
[10] R.Ablamowicz, I.Gonçalves, R.da Rocha, “Bilinear Covariants and Spinor Fields Duality in Quantum Clifford Algebras”, J. Math. Phys. 55, 103501 (2014).
[11] W.A.Rodrigues, R.da Rocha, J.Vaz, “Hidden consequence of active local Lorentz invariance”, Int.J.Geom.Meth.Mod.Phys. 2, 305 (2005).
[12] J.M.Hoff da Silva, R.da Rocha, “From Dirac Action to ELKO Action”, Int. J. Mod. Phys. A24, 3227 (2009).
[13] R.da Rocha, J.M.Hoff da Silva, “ELKO, flagpole and flag-dipole spinor fields, and the instanton Hopf fibration”, Adv. Appl. Clifford Algebras 20, 847 (2010).
[14] R.da Rocha,L.Fabbri,J.M.Hoff da Silva,R.T.Cavalcanti, J.A.Silva-Neto, “Flag-Dipole Spinor Fields in ESK Gravities”, J. Math. Phys. 54, 102505 (2013).
[15] L.Fabbri, “A generally-relativistic gauge classification of the Dirac fields”, Int.J.Geom.Meth.Mod.Phys. 13, 1650078 (2016).
[16] L.Fabbri, “Covariant inertial forces for spinors”, Eur.Phys.J. C78, 783 (2018).
[17] L.Fabbri, “Torsion Gravity for Dirac Fields”, Int.J.Geom.Meth.Mod.Phys. 14, 1750037 (2017).
[18] L.Fabbri, “Polar solutions with tensorial connection of the spinor equation”, Eur.Phys.J. C79, 188 (2019).
[19] L.Fabbri, “General Dynamics of Spinors”, Adv. Appl. Clifford Algebras 27, 2901 (2017).
[20] L.Fabbri, “Spinors in Polar Form”, arXiv:2003.10825