Finite Section Method in a Space with Two Norms

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Don Sarason, in memoriam.

Abstract. We compare the finite central truncations of a given matrix with respect to two non-equivalent Hilbert space norms. While the limit sets of the finite sections spectra are merely located via numerical range bounds, the weak ∗-limits of the counting measures of these spectra are proven in general to be gravi-equivalent with respect to the logarithmic potential in the complex plane. Classical methods of factorization of Volterra type or Wiener–Hopf type operators lead to a series of effective criteria of asymptotic equivalence, or uniform boundedness of the two sequences of truncations. Examples from function theory, integral equations and potential theory complement the theoretical results.

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1. Introduction

Weaker norm estimates often simplify the analysis of concrete operators, helping to locate spectra, obtain resolvent estimates, prove stability and so on. It is known for instance that the double layer potential, also known as the Poincaré-Neumann operator, has real spectrum on Lebesgue $L^2$-space of the respective boundary, but the explanation is far from obvious: this linear transform is self-adjoint only with respect to a weaker topology Hilbert space (a Sobolev space of negative fractional order). Moreover, the spectrum of the Neumann-Poincaré operator is highly sensitive on the choice of the underlying normed space, see for instance [1,14]. It is therefore natural to ask how the finite rank truncations (i.e. the Galerkin approximations) of a given operator behave when modifying the inner space structure. Such an inquiry was started in the note [14], but soon it become clear that a general framework is lurking...
in the background. The present article is a direct continuation of Kang and Putinar [14], building on some theoretical results outlined there.

Our first aim is to collect a series of general observations related to the comparison of finite central truncations of a prescribed infinite matrix, with respect to two non-equivalent Hilbert space norms. Norm relaxations in the spectral analysis on the Sobolev scale, $L^p$ scale or on Gelfand pairs are ubiquitous in applied mathematics, see for instance [12]. Added to these is the high degree of sophistication reached by the finite central truncation technique as part of Galerkin approximation or Krylov subspace method, see for instance [5,16,20,22]. In a totally different register, operator algebra experts have recognized in the finite central truncation scheme some familiar approximation schemes [2,4,6].

Independently, the theory of orthogonal polynomials is providing ample qualitative analysis of measure change consequences for the asymptotics of zeros or other relevant entities [7,17,19]. It is sufficient to recall that in the presence of a positive measure on the complex plane, admitting all moments, the finite central truncations of the multiplication $M_z$ by the complex variable are matrices whose spectra coincide with the zeros of the associated complex orthogonal polynomials, see for instance [16]. Thus a perturbation of the underlying measure will only change the orthogonal projections defining the finite central truncation, but not the multiplication operator $M_z$.

Our approach is rather elementary, exploiting solely classical works of the Ukrainian school of functional analysis, which in its turn has roots in the theory of Volterra or Wiener–Hopf type operators [3,9,10,15].

Quite specifically, we focus on the two sequences of compressions $T_n = P_n TP_n$ and $\tilde{T}_n = Q_n TQ_n$ of a given linear transformation $T$, where $P_n$ and $Q_n$ are projections onto the same finite dimensional subspace $H_n = P_n H$, but they are orthogonal with respect to two different norms. The projections $P_n$ are orthogonal and converge monotonically to the identity on a Hilbert space $H$, while $Q_n$ are orthogonal with respect to the inner product $\langle A, \cdot \rangle$ induced by a positive, in general non-invertible, linear operator on $H$. We compare the spectra of $T_n$ and $\tilde{T}_n$ regarded as linear transforms of $H_n$ and try to link the weak-* limits of the counting measures of these spectra, when $n$ tends to infinity. The last section incorporates some examples supporting this quest. We do not always expect simple answers, as the approximation theory of Toeplitz matrices amply testifies, even under a single inner product [5].

It turns out that bounds of the operator norm gap $\|T_n - \tilde{T}_n\|$ or the trace-class gap $|T_n - \tilde{T}_n|_1$ are at the key quantitative indicators to look for. Several competing factors contribute to the effective evaluation of these bounds: the adaptation of the chain of finite dimensional subspaces to the two norms, the matrix structure of the operator $T$ on the given chain of subspaces, or the intrinsic properties of $T$ such as compactness or quasi-diagonality. In complete analogy to the Cholesky factorization of Volterra type operators [8,10], a weaker norm induced by a positive operator of the form

$$A = (I + L)D(I + L^*)$$
is “universally good” for all matrices $T$. Above $D$ is block-diagonal with respect to the chain of subspaces $(H_n)$ while $L$ is strictly lower-triangular and compact. The departure from the standard theory is the non-invertibility of $D$. Several results of our work concur to the this conclusion. By “universally good” we mean the existence of a uniform bound for the operator norm gap $\|T_n - \tilde{T}_n\|$. If moreover,

$$\lim_{n} \frac{\dim H_n - \dim H_{n-1}}{\dim H_n} = 0,$$

and the matrix attached to $T$ has a Hessenberg structure, that is at most the first block sub-diagonal is non-zero, then the trace-class norm gap is bounded, and in this case any two weak-* limits of the counting measures of the spectra of $T_n$ and $\tilde{T}_n$ have the same logarithmic potential at infinity (sometimes called gravi-equivalent measures). As we will see, this scenario applies in particular to complex orthogonal polynomials. A special role is played by a non-invertible (usually compact) positive operator $A$ and the chain of subspaces spanned by its eigenvectors. These form a system of doubly orthogonal vectors with respect to the two norms, a concept well isolated and exploited in function theory and potential theory [11]. We analyze what kind of perturbations of such a doubly orthogonal system of vectors still produces a “universally good” chain of subspaces.

The general framework proposed in the subsequent sections provides only a basis for analyzing several specific situations. While our setting was motivated by the latter, in the present article we remain mostly at the abstract level, leaving for future works the return to, and enhancement of, the original sources.

2. Preliminaries

Let $H$ be a complex separable Hilbert space and $T \in \mathcal{L}(H)$ a linear bounded operator acting on $H$. The spectrum of $T$ is denoted $\sigma(T)$ and the numerical range of $T$ is $W(T) = \{\langle Tx, x \rangle, \|x\| = 1\}$. By a Theorem of Hausdorff and Toeplitz we know that $W(T)$ is a convex set. Moreover, it is not hard to prove that the closure of $W(T)$ contains $\sigma(T)$.

We endow $H$ with a weaker pre-hilbertian space norm:

$$(x, y) = \langle Ax, y \rangle, \quad x, y \in H,$$

where $A > 0$ is a positive, non-invertible bounded linear operator acting on $H$. Let $K$ denote the Hilbert space completion of $H$ with respect to the new norm. We have $H \subset K$, with dense range inclusion. Sometimes we will call $A$-norm, or $A$-convergence the respective entities induced by the norm of the Hilbert space $K$, denoting

$$\|x\|_A^2 = \langle Ax, x \rangle, \quad x \in H.$$

Let $H_n \subset H$ be an increasing sequence of finite-dimensional subspaces, whose union is dense in $H$. Unless otherwise stated we link the operator $T$ to the chain of subspaces $(H_n)$ by the assumption
This means that the block matrix decomposition of $T$ with respect to the orthogonal direct sum $H = H_0 \oplus (H_1 \ominus H_0) \oplus (H_2 \ominus H_1) \oplus \ldots$ has only the first sub-diagonal non-zero. This structure is known in numerical analysis as a block Hessenberg matrix. Remark that we do not assume $T$ to be bounded as a linear transformation from $K$ to $K$. However, $T$ can be regarded as a densely defined operator on $K$ and the graph of $T$ turns out to be closed in the $A$-norm.

We denote by $P_n$ the orthogonal projection of $H$ onto $H_n$, and by $Q_n$ the orthogonal projection of $K$ onto $H_n$. Note that $P_n \to I$ in the strong operator topology of $\mathcal{L}(H)$.

The two sets of projections satisfy:

$$P_nQ_n = Q_n, \quad Q_nP_n = P_n$$

when regarded as linear endomorphisms of $H$. The compression $A_n = P_nAP_n$ of the operator $A$ to the subspace $H_n$ is positive, hence invertible.

We recall first a few basic observations proved in the previous work [14]. The omitted proofs below are elementary and the reader may complete them as an exercise instead of consulting the details in [14].

**Lemma 2.1.** For every vector $x \in H$ one has

$$Q_nx = A_n^{-1}P_nAx.$$  \hspace{1cm} (2.1)

For a proof see Lemma 4.1 in [14].

We regard the finite central truncations $T_n = P_nTP_n$, versus $\tilde{T}_n = Q_nTQ_n$ as endomorphisms of the finite dimensional space $H_n$. Besides the set theoretic distance between spectra, we learned from probabilists or orthogonal polynomials experts to consider the finite atomic counting measures $\mu_n$, respectively $\tilde{\mu}_n$, defined on complex polynomials as:

$$\int p \, d\mu_n = \frac{1}{\dim H_n} \text{trace} \, p(T_n), \quad p \in \mathbb{C}[z],$$

and similarly

$$\int p \, d\tilde{\mu}_n = \frac{1}{\dim H_n} \text{trace} \, p(\tilde{T}_n), \quad p \in \mathbb{C}[z].$$

**Lemma 2.2.** Assume

$$\lim_{n \to \infty} \frac{1}{\dim H_n} \text{trace} |T_n - \tilde{T}_n| = 0,$$

and

$$\sup_n \|\tilde{T}_n\| < \infty.$$

Then

$$\lim_{n \to \infty} \left( \int pd\mu_n - \int pd\tilde{\mu}_n \right) = 0,$$

for every complex polynomial $p \in \mathbb{C}[z]$. 
Above, and throughout this article we denote $|X| = \sqrt{X^*X}$ the absolute value of a matrix $X$.

**Proof.** For square matrices $X, Y$ and $Z$, the inequalities

$$|\text{trace}(XYZ)| \leq \text{trace}|XYZ| \leq ||X||||Z||\text{trace}|Y|$$

hold true, see for instance [10]. The operator $T$ is bounded from start, while the compressed central truncations $\tilde{T}_n$ are uniformly bounded in norm. Let $M > 0$ be a uniform bound for both $||T_n||$ and $||\tilde{T}_n||$.

It is sufficient to prove the statement for a single monomial $p(z) = z^k$. Starting from the algebraic identity

$$T^k - \tilde{T}^k = T^{k-1}(T - \tilde{T}) + T^{k-2}(T - \tilde{T})\tilde{T} + \cdots + (T - \tilde{T})\tilde{T}^{k-1}$$

we infer from the above inequalities

$$\left|1\dim H_n \text{trace}(T^k - \tilde{T}^k)\right| \leq kM^{k-1}1\dim H_n \text{trace}|T_n - \tilde{T}_n|$$

Remarking that the support of the counting measure $\mu_n$ coincides with the spectrum of $T_n$ and in particular is contained in the numerical range $W(T)$, simply because by definition $W(T_n) \subset W(T)$.

**Lemma 2.3.** Assume that $\lim sup ||T_n - \tilde{T}_n|| = r$. Then the support of any weak-* limit point of the measures $\tilde{\mu}_n$ is contained the closed $r$ neighborhood of $W(T)$.

**Proof.** Any point belonging to the closed support a weak-* limit of the measures $\tilde{\mu}_n$ is a limit of a sub-sequence of points belonging to the respective supports.

Let $\lambda_n \in W(\tilde{T}_n)$. That is, there is a vector $x \in H_n$, of unit length, so that $\lambda_n = \langle \tilde{T}_nx, x \rangle$. Then

$$|\langle T_nx, x \rangle - \langle \tilde{T}_nx, x \rangle| \leq ||T_n - \tilde{T}_n||,$$

whence

$$\text{dist}(\lambda_n, W(T)) \leq ||T_n - \tilde{T}_n||.$$

Assume that $\lambda = \lim_k \lambda_n(k)$, where $n(k)$ is a subsequence converging to infinity. Then

$$\text{dist}(\lambda, W(T)) \leq \lim sup ||T_n - \tilde{T}_n||.$$  \hfill \Box

A weaker assumption, leading to the same conclusion of the Lemma is

$$\lim sup w(T_n - T) \leq r,$$

where $w(A) = \sup\{||\lambda||, \lambda \in W(A)\}$ denotes the numerical radius of an operator. Obviously one can refine the conclusion of the above lemma by working with the upper limit set of the sequences of spectra $\sigma(T_n)$, invoking the theory of psudospectra [20]. A notable particular case is described below.

**Corollary 2.4.** Assume that operator $T$ is normal and $\lim ||T_n - \tilde{T}_n|| = 0$. Then the support of any weak-* limit point of the measures $\tilde{\mu}_n$ is contained in the convex hull of the spectrum of $T$. 
Proof. The numerical range of a normal operator coincides with the convex hull of the spectrum. □

A direct consequence of Lemma 2.2 is the fact that any weak-∗limit points of the measures $\mu_n$, respectively $\tilde{\mu}_n$, have the same complex moments. First remark that the sequences of probability measures $(\mu_n)$ and $(\tilde{\mu}_n)$ have support in the closed disk $\overline{D}(0,M)$, therefore they are relatively compact in the weak-* topology. Thus they possess (possibly non-unique) limit points, say $\mu$, respectively $\tilde{\mu}$. We infer from Lemma 2.2 that

$$\int z^k d\mu = \int z^k d\tilde{\mu}, \quad k \geq 0.$$  

Since the supports of the two measures $\mu$ and $\tilde{\mu}$ are bounded, a Neumann series expansion leads to the conclusion

$$\int \frac{d\mu(\zeta)}{\zeta - z} = \int \frac{d\tilde{\mu}(\zeta)}{\zeta - z}, \quad |z| > M.$$  

In other terms, the two measures have the same Cauchy transform in a neighborhood of the point at infinity. One step further, one can integrate the latter identity to

$$\int \log |\zeta - z| d\mu(\zeta) = \int \log |\zeta - z| d\tilde{\mu}(\zeta), \quad |z| > M. \quad (2.2)$$  

Note that he constant of integration disappears because the two measure have the same mass. Two measure satisfying identity (2.2) are called gravi-equivalent, that is the logarithmic potentials produced by them (far away from the source) are identical. By no means two gravi-equivalent measures are equal, think for instance to Gauss mean value theorem for harmonic functions: a point mass at $z = a$ and a uniform mass supported by a disk centered at $z = a$ will have the same logarithmic potential at infinity.

3. Asymptotic Equivalence of Finite Central Truncations

This section is devoted to a study of the distance between the finite central truncations $T_n$ and $\tilde{T}_n$, within the framework and notation introduced in the previous section. The starting point is the following simple identity stated and proved in [14] as Lemma 4.2.

**Lemma 3.1.** Let $T \in \mathcal{L}(H)$ be an operator satisfying $TH_n \subset H_{n+1}$ for all $n \geq 0$. Then

$$\tilde{T}_n - T_n = A_n^{-1} P_n A(P_{n+1} - P_n) TP_n, \quad n \geq 0.$$  

If the operator $T$ is not necessarily adapted by the assumption $T(H_n) \subset H_{n+1}, \quad n \geq 0$, to the chain of subspaces $(H_n)$ we obtain a similar formula:

$$\tilde{T}_n - T_n = A_n^{-1} P_n A(I - P_n) TP_n, \quad n \geq 0. \quad (3.1)$$  

The key residual term $X_n = A_n^{-1} P_n A(P_{n+1} - P_n)$ naturally appears in the Cholesky type decomposition of the one-step extension of the matrix $A_n$, to $A_{n+1}$. We reproduce from [14] this observation in some detail. The
matrix $A_{n+1}$ has the following block structure factorization with respect to
the orthogonal decomposition $H_{n+1} = H_n \oplus (H_{n+1} \ominus H_n)$:

$$A_{n+1} = \begin{pmatrix} A_n & P_nA(P_{n+1} - P_n) \\ (P_{n+1} - P_n)AP_n & (P_{n+1} - P_n)A(P_{n+1} - P_n) \end{pmatrix}$$

$$= \begin{pmatrix} I & 0 \\ X_n^* & I \end{pmatrix} \begin{pmatrix} A_n & 0 \\ 0 & D_{n+1} \end{pmatrix} \begin{pmatrix} I \\ X_n \end{pmatrix},$$

where $D_{n+1} = (P_{n+1} - P_n)A(P_{n+1} - P_n) - X_n^*A_nX_n$.

On the abstract side, we isolate below a basic matrix identity necessary
for our study. Assume that the original Hilbert space is decomposed into two
orthogonal subspaces:

$$H = H_0 \oplus H_1,$$

with orthogonal projections $P_0$ and $P_1$, respectively. The weaker norm is
induced by a positive and bounded operator $A$ of the form

$$A = LDL^*,$$

where $L^*$ leaves invariant $H_0$ and $D > 0$ leaves invariant both $H_0$ and $H_1$.
Note that $D$ may not be invertible. In matrix form:

$$A = \begin{pmatrix} L_{00} & 0 \\ L_{10} & L_{11} \end{pmatrix} \begin{pmatrix} D_0 & 0 \\ 0 & D_1 \end{pmatrix} \begin{pmatrix} L_{00}^* & L_{10}^* \\ 0 & L_{11}^* \end{pmatrix}.$$ 

**Lemma 3.2.** Assume, with the above conventions, that $L_{00}$ is invertible. Then
the projection $Q_0$ in the $A$-norm of $H$ onto $H_0$ is

$$Q_0 = (L_{00}^*)^{-1}P_0L^*.$$ (3.2)

**Proof.** We check separately the formula for elements of $H_0$ and $H_1$. If $x \in H_0$, then

$$(L_{00}^*)^{-1}P_0L^*x = (L_{00}^*)^{-1}P_0L_{00}x = (L_{00}^*)^{-1}L_{00}x = x.$$

For $y \in H_1$ and $x \in H_0$ we find

$$\langle A(y - (L_{00}^*)^{-1}P_0L^*y), x \rangle = \langle P_0A(y - (L_{00}^*)^{-1}P_0L^*y), x \rangle$$

$$= \langle P_0LDL^*(y - (L_{00}^*)^{-1}P_0L^*y), x \rangle$$

$$= \langle L_{00}D_0P_0L^*(y - (L_{00}^*)^{-1}P_0L^*y), x \rangle$$

$$= \langle L_{00}D_0P_0L^*y - L_{00}D_0P_0L^*P_0(L_{00}^*)^{-1}P_0L^*y, x \rangle$$

$$= \langle L_{00}D_0P_0L^*y - L_{00}D_0L_{00}(L_{00}^*)^{-1}P_0L^*y, x \rangle$$

$$= 0. \quad \square$$

A first main result derived from these simple observations follows.

**Theorem 3.3.** Let $H$ be a separable Hilbert space endowed with a weaker
norm and let $(H_n)_{n=0}^\infty$ be an increasing chain of finite dimensional subspaces
converging to $H$. Assume that

$$\lim_n \frac{\dim H_n - \dim H_{n-1}}{\dim H_n} = 0.$$ (3.3)
Let \( T \in \mathcal{L}(H) \) satisfy \( T(H_n) \subset H_{n+1}, \ n \geq 0. \) If the finite central truncations \((T_n)\) and \((\tilde{T}_n)\) associated to the chain of subspaces \((H_n)\) with respect to the original, respectively, weaker norm remain close in norm:

\[
\sup_n \|T_n - \tilde{T}_n\| < \infty, \tag{3.4}
\]

then any weak-* cluster measures \( \mu, \tilde{\mu} \) of the counting measures of the spectra of \( T_n, \) respectively \( \tilde{T}_n, \) have the same logarithmic potentials at infinity.

In practice the spaces \( H_n \) are formed by polynomials of degree less than or equal to \( n \) on a support of \( d \) variables, which is not constrained by an algebraic dependence. Then condition (3.3) is satisfied. The equality of logarithmic potentials in a neighborhood of infinity propagates by analytic continuation to the unbounded connected component of \( C \setminus (\text{supp}(\mu) \cup \text{supp}(\tilde{\mu})) \).

Proof. We simply remark that \( \text{rank}(T_n - \tilde{T}_n) \leq \dim H_{n+1} - \dim H_n, \) whence

\[
\text{trace}|T_n - \tilde{T}_n| \leq (\dim H_{n+1} - \dim H_n)\|T_n - \tilde{T}_n\|, \ n \geq 0.
\]

On the other hand, \( \sup_n \|\tilde{T}_n\| < \infty \) by the first assumption in the statement. We infer that for every polynomial \( p \in \mathbb{C}[z], \)

\[
\lim_n \frac{1}{\dim H_n} \text{trace}(p(T_n) - p(\tilde{T}_n)) = 0
\]

hence any limiting measures \( \mu, \) respectively \( \tilde{\mu}, \) are gravi-equivalent. \( \square \)

Next we focus on conditions assuring \( \lim_{n \to \infty} \|X_n TP_n\| = 0. \) One obvious instance being the block-diagonal structure of the operator \( A, \) with respect to the chain of subspaces \( H_n: \)

\[
A = \text{diag} (D_0, D_1, D_2, \ldots). \tag{3.5}
\]

A second sufficient condition is

\[
\lim_{n \to \infty} \|A_n^{-1} P_n A(P_{n+1} - P_n)\| = 0 \tag{3.6}
\]

and a third

\[
\sup_n \|A_n^{-1} P_n A(P_{n+1} - P_n)\| < \infty \quad \text{and} \quad \lim_{n \to \infty} \|(I - P_n) TP_n\| = 0. \tag{3.7}
\]

We prove below that the latter two sufficient conditions for the asymptotic equivalence of the two sequences of finite central truncations are not affected by a structured perturbation of the operator \( A. \) By a strictly lower triangular operator with respect to the chain of subspaces \( (H_n)_{n=0}^{\infty} \) we mean an element \( L \in \mathcal{L}(H) \) satisfying

\[
P_n L = P_n L P_{n-1}, \ n \geq 1,
\]

or more intuitively and equivalently

\[
L^* P_n = P_{n-1} L^* P_n, \ n \geq 1,
\]
which in turn means \( L^*(H_n) \subset H_{n-1} \).

The following statement was stated and proved in [14] as Theorem 4.3.

**Theorem 3.4.** Let \( (x, y) = \langle Ax, y \rangle \) be a second, weaker inner product structure on a Hilbert space \( H \), implemented by the positive operator \( A \in \mathcal{L}(H) \). Denote by \( P_n \) the orthogonal projections on \( H_n \), and set \( A_n = P_n A P_n \).

Consider a strictly lower triangular operator \( L \in \mathcal{L}(H) \), and the multiplicative perturbation \( B = (I + L)A(I + L^*) \).

(a) If the operator \( L \) is compact and

\[
\lim_{n \to \infty} \| A_n^{-1} P_n A(P_{n+1} - P_n) \| = 0 \tag{3.8}
\]

then

\[
\lim_{n \to \infty} \| B_n^{-1} P_n B(P_{n+1} - P_n) \| = 0. \tag{3.9}
\]

(b) If \( I + L \) is invertible and

\[
\sup_n \| A_n^{-1} P_n A(P_{n+1} - P_n) \| < \infty. \tag{3.10}
\]

then

\[
\sup_n \| B_n^{-1} P_n B(P_{n+1} - P_n) \| < \infty. \tag{3.11}
\]

On the theoretical side, we can now construct a large class of examples of finite central truncations with the same operator norm asymptotics.

**Corollary 3.5.** Let \( T \in \mathcal{L}(H) \) be a linear bounded operator possessing a block Hessenberg matrix with respect to the complete and increasing chain of finite dimensional subspaces \((H_n)_{n=0}^{\infty}\). Let \( D \in \mathcal{L}(H) \) be a positive block-diagonal operator and let \( L \in \mathcal{L}(H) \) be any compact, strictly lower triangular operator, both with respect to the same chain \((H_n)_{n=0}^{\infty}\). Define a pre-hilbert space structure on \( H \) by

\[
(x, y) = \langle (I + L)D(I + L^*)x, y \rangle, \quad x, y \in H.
\]

Then the finite central truncations \( T_n \) and \( \tilde{T}_n \) defined by orthogonal projections onto \( H_n \) in the two norms satisfy

\[
\lim_n \| T_n - \tilde{T}_n \| = 0.
\]

In view of Theorem 3.3 one can relax the structure of the weak norm, with control of the logarithmic potential of the limit of counting measures, provided the dimensions in the chain \((H_n)\) have a moderate growth. Specifically, we state mutatis mutandis the following observation.

**Corollary 3.6.** Let \( T \in \mathcal{L}(H) \) be a linear bounded operator possessing a block Hessenberg matrix with respect to the complete and increasing chain of finite dimensional subspaces \((H_n)_{n=0}^{\infty}\). Let \( D \in \mathcal{L}(H) \) be a positive block-diagonal operator and let \( L \in \mathcal{L}(H) \) be any bounded strictly lower triangular operator, both with respect to the same chain \((H_n)_{n=0}^{\infty}\). Define a pre-hilbert space structure on \( H \) by

\[
(x, y) = \langle (I + L)D(I + L^*)x, y \rangle, \quad x, y \in H.
\]
Then the finite central truncations $T_n$ and $\tilde{T}_n$ defined by orthogonal projections onto $H_n$ in the two norms satisfy
\[
\sup_n \|T_n - \tilde{T}_n\| < \infty.
\]

In the two corollaries above only the case of non-invertible diagonal $D$ is of interest. In most examples $D$ is also a compact operator.

The classical factorization theory of integral operators [9,10] offers a second, useful criteria of producing admissible weak norm, as above, via small additive perturbations of a diagonal operator.

**Theorem 3.7.** Let $(H_n)_{n=0}^{\infty}$ be a chain of finite dimensional spaces in a Hilbert space $H$ with $\dim H_n = n, n \geq 0$, and let $D > 0$ be a bounded, positive, block-diagonal operator with respect to the chain $(H_n)$.

Assume that $F \in L(H)$ is a Hilbert–Schmidt operator on $H$, which is $D$-symmetric (i.e. $DF = F^*D$). If $D + DF > 0$, then there is a bounded diagonal operator $D'$ and a strictly lower triangular operator $L \in L(H)$ with the property
\[
D + DF = (I + L)D'(I + L^*).
\]

**Proof.** According to Krein’s boundedness criterion [15], the operator $F$ is bounded in the $D$-norm, and has the same spectrum with respect to $H$ and the $D$-norm. In particular, the spectrum of $F$ is real, and $-1$ is not an eigenvalue, otherwise $D(I + F)$ would not be positive. By the invariance of the Fredholm index under compact perturbations, $I + F$ is an invertible operator. According to [8] Theorem XXII.4.2, or see [10] Chapter IV for an earlier version, a factorization of the type
\[
I + F = (I + L)E(I + U)
\]
exists, where $E$ is a diagonal bounded operator with respect to the chain $(H_n)$, $L$ is a strictly lower triangular, Hilbert–Schmidt operator on $H$ and $U$ is a strictly upper triangular, Hilbert–Schmidt operator on $H$. Note that $I + L$ and $I + U$ are also invertible, by the same index invariance. Then $I + F^* = (I + U^*)E^*(I + L^*)$ is a similar factorization, and they are related by the intertwining formula
\[
D(I + F) = (I + F^*)D,
\]
that is
\[
D(I + L)E(I + U) = (I + U^*)E^*(I + L^*)D.
\]

Therefore
\[
(I + U^*)^{-1}D(I + L)E = E^*(I + L^*)D(I + U)^{-1}.
\]

But the left hand side of the latter identity is lower triangular, while the righthand side is upper triangular, hence both are diagonal. That is
\[
(I + U^*)^{-1}D(I + L)E = DE = E^*D = E^*(I + L^*)D(I + U)^{-1}.
\]

This proves that $E = E^*$ and
\[
D(I + L) = (I + U^*)D.
\]
In conclusion
\[ D(I + F) = D(I + L)E(I + U) = (I + U^*)DE(I + U) \]
\[ \square \]

Since the operator \( F \) in the statement of the Theorem is bounded in the \( D \)-norm, there exists a linear bounded operator \( S \in \mathcal{L}(H) \), so that
\[ \sqrt{DF} = S\sqrt{D}. \]

Then
\[ DF = \sqrt{DS} \sqrt{D} \]
is self-adjoint, and, according to Krein’s Theorem [15], the spectrum of \( S \) coincides with the spectrum of \( F \), with respect to either space. The perturbation in the statement of the theorem becomes
\[ D + DF = D + \sqrt{DS} \sqrt{D}, \]
with \( S \) a Hilbert–Schmidt operator on \( H \).

4. Perturbations of Orthonormal Bases

A natural question to investigate in the context in the present article is the gap between two orthonormal bases in the original, respectively the weaker inner product space. As a byproduct we will isolate a couple of sufficient conditions on the positive operator \( A \) defining the weak topology to be “universally good” for our asymptotic analysis, assuring that condition (3.6) is fulfilled.

**Proposition 4.1.** Let \( (H_n)_{n=0}^{\infty} \) be an ascending chain of subspaces of a complex Hilbert space \( H \), satisfying \( \text{dim } H_n = n \) for all \( n \geq 0 \). Let \( P_n \) denote the orthogonal projection on \( H_n \), \( n \geq 0 \), respectively, and denote by \( (\phi_n)_{n=0}^{\infty} \) the associated orthonormal basis: \( H_n = \text{span}\{\phi_0, \phi_1, \ldots, \phi_{n-1}\} \).

Assume \( A \in \mathcal{L}(H) \) is a positive, bounded linear operator with matrix entries \( a_{k\ell} = \langle A\phi_k, \phi_\ell \rangle \), \( k, \ell \geq 0 \). If
\[ \sup_{k<\ell} a_{k\ell} a_{kk} < \infty \quad (4.1) \]
and
\[ \sum_{k=0}^{\infty} \sum_{\ell=k+1}^{\infty} \frac{|a_{k\ell}|^2}{a_{kk} a_{\ell\ell}} < \infty \quad (4.2) \]
then the central truncations \( A_n = P_n A P_n \) satisfy
\[ \lim_n A_n^{-1} P_n A (P_{n+1} - P_n) = 0. \]

**Proof.** Since \( a \) is a positive operator, the diagonal entries \( a_{nn} \) are all positive. Denote by \( D = \text{diag}(\sqrt{a_{00}}, \sqrt{a_{11}}, \sqrt{a_{22}}, \ldots) \) the square root of the diagonal of \( A \). We can factor
\[ A = D(I + S)D \]
with the matrix $S$ of entries
\[ s_{k\ell} = \frac{a_{k\ell}}{\sqrt{a_{kk}a_{\ell\ell}}}, k \neq \ell, \]
and zero on the diagonal.

Assumption (4.2) is equivalent to $S$ being Hilbert–Schmidt. According to the main factorization theorem (see for instance [10] Chapter IV) there exists a Hilbert–Schmidt operator $L$ which is strictly lower triangular with respect to the chain $(H_n)$, with the property
\[ I + S = (I + L)E(I + L^*), \]
where $E$ is a diagonal operator. In view of Theorem 3.4 we deduce:
\[ \lim_n \| (I_n + S_n)^{-1}P_n(I + S)(P_{n+1} - P_n) \| = 0. \]

To infer a similar conclusion for the operator $A$ we start by the identities:
\[ A_n = D_n(I_n + S_n)D_n, \]
and
\[ D(P_{n+1} - P_n) = \sqrt{a_{nn}}(P_{n+1} - P_n), n \geq 0. \]

Whence
\[
A_n^{-1}P_n A(P_{n+1} - P_n) = D_n^{-1}(I_n + S_n)^{-1}D_n(P(I + S)(P_{n+1} - P_n)\sqrt{a_{nn}}
= \text{diag}\left(\frac{a_{nn}}{a_{00}}, \ldots, \frac{a_{nn}}{a_{(n-1)(n-1)}}\right)^{1/2}(I_n + S_n)^{-1}P_n(I + S)
\times (P_{n+1} - P_n).
\]
The boundedness assumption in the statement takes over and completes the proof. \[ \square \]

The second condition in the statement of the above Proposition simply states that the matrix formed by the cosine values with respect to the $A$-norm
\[ \cos(\phi_k, \phi_\ell)_A = \frac{\langle A\phi_k, \phi_\ell \rangle}{\| \phi_k \|_A\| \phi_\ell \|_A}, \quad k, \ell \geq 0, \]
is a Hilbert–Schmidt perturbation of the identity matrix. In other terms, the vectors $\phi_k/\| \phi_k \|_A$ form a Bari basis of the weaker norm Hilbert space $K$. A Bari basis, in the terminology of Krein, is a Riesz basis which is quadratically close to an orthonormal basis, see [3,9].

Lemma 3.1 guarantees than that the norm gap between the finite central truncations $T_n$ and $\tilde{T}_n$ of any linear bounded operator $T \in \mathcal{L}(H)$ tends to zero. Since $\dim H_n = n + 1$, $n \geq 0$, in the conditions of Proposition 4.1, Theorem 3.3 applies. We state explicitly only one possible consequence of these observations.

**Corollary 4.2.** Assume, in the conditions of Proposition 4.1, that $T = T^*$ is a bounded self-adjoint operator on $H$. Then the counting measures $\tilde{\mu}_n$ of the spectra of the finite central truncations of $T$ in the $A$-norm converge in the weak-* topology along subsequences to positive measures supported on the smallest closed interval of the real line containing the spectrum of $T$. 

Proof. Let $\mu_n$ denote the counting measures of the spectra of the finite central truncations $T_n$. Since $T$ is a self-adjoint operator, so are all truncations $T_n$, $n \geq 0$, and hence the measures $\mu_n$ are supported by the numerical range $W(T)$ which is a closed interval of the real axis. Their weak-* cluster points are measures $\mu$ supported by $W(T)$. Corollary 2.4 implies that the support of any weak-* limit point $\tilde{\mu}$ of the measures $\tilde{\mu}_n$ is contained in $W(T)$. □

The theory of orthogonal polynomials provides examples of self-adjoint operators $T$ and chains of projections $(P_n)$ with the property that the counting measures $\mu_n$ of the spectra of the finite central truncations $T_n = P_n TP_n$ do not have a unique limit point in the weak-* topology. Specifically, see irregular measures on the real line in the sense of Stahl and Totik [19] for the case of $T = M_x$ the multiplication operator with the real variable and $H_n$ the space generated by all polynomials of degree less than $n$.

5. Krylov Subspaces

An important scenario for the phenomena described in the previous sections is offered by the cyclic subspaces of a fixed operator $T \in \mathcal{L}(H)$. Specifically, we fix a non-zero vector $\xi \in H$ and define the finite dimensional subspaces

$$H_n = \text{span}\{\xi, T\xi, \ldots, T^{n-1}\xi\}.$$  

We assume that $\dim H_n = n$, $n \geq 0$, that is there is no degeneracy in the $T$-cyclic subspaces defined by the vector $\xi$. We can also assume without loss of generality that the union of spaces $H_n$ is dense in $H$. The second, weaker inner product structure is induced by a positive, non-invertible operator $A$.

In this case the “orthogonal polynomials” associated to the chain of subspaces $(H_n)$ and the two norms play a central role:

$$\phi_n \in H_{n+1} \ominus H_n, \quad \|\phi_n\| = 1, \quad n \geq 0,$$

$$\psi_n \in H_{n+1} \ominus A H_n, \quad (\psi_n, \psi_n) = (A\psi_n, \psi_n) = 1, \quad n \geq 0.$$

Indeed, we can write

$$\phi_n = \kappa_n T^n \xi + \kappa_{n-1} T^{n-1} \xi + \cdots,$$

$$\psi_n = \gamma_n T^n \xi + \gamma_{n-1} T^{n-1} \xi + \cdots.$$

The choice of positive leading coefficients

$$\frac{1}{\kappa_n} = \inf_{h \in H_n} ||T^n \xi - h||,$$

$$\frac{1}{\gamma_n} = \inf_{h \in H_n} ||\sqrt{A}(T^n \xi - h)||$$

leaves no room for ambiguity.

The orthogonal projections introduced in the previous section are:

$$P_{n+1} = \phi_n \langle \cdot, \phi_n \rangle + \phi_{n-1} \langle \cdot, \phi_{n-1} \rangle + \cdots + \phi_0 \langle \cdot, \phi_0 \rangle,$$

respectively

$$Q_{n+1} = \psi_n \langle \cdot, \psi_n \rangle + \psi_{n-1} \langle \cdot, \psi_{n-1} \rangle + \cdots + \psi_0 \langle \cdot, \psi_0 \rangle.$$
We translate the computations of the general framework in these terms.

**Proposition 5.1.** The difference between the two finite central truncations of the operator $T$ along the Krylov subspaces $\text{span}\{\xi, T\xi, \ldots, T^{n-1}\xi\}$ is of rank one and has the expression

$$
\hat{T}_n - T_n = \frac{\kappa_{n-1}}{\kappa_n} \left( \phi_n - \frac{\kappa_n}{\gamma_n} \psi_n \right) \langle \cdot, \phi_{n-1} \rangle.
$$

*Proof.* Start with the rank one projection

$$
P_{n+1} - P_n = \phi_n \langle \cdot, \phi_n \rangle
$$

and consider a vector $h \in H_n$:

$$
h = h_{n-1} \phi_{n-1} + h_{n-2} \phi_{n-2} + \cdots,
$$

where $h_k = \langle h, \phi_k \rangle$. Then

$$
(P_{n+1} - P_n)Th = h_{n-1}(P_{n+1} - P_n)T\phi_{n-1} = h_{n-1}\phi_n \langle T\phi_{n-1}, \phi_n \rangle
$$

$$
= h_{n-1}\phi_n \langle \kappa_{n-1}T^n\xi + \cdots, \phi_n \rangle = h_{n-1}\phi_n \frac{\kappa_{n-1}}{\kappa_n},
$$

due to the orthogonality property of $\phi_n$.

Further on, let $u = A_n^{-1}P_nA\phi_n$, that is $u \in H_{n-1}$ and

$$
Au - A\phi_n \perp H_{n-1},
$$

which implies $\phi_n - u = \lambda \psi_n$. The comparison of the leading coefficients yields $\lambda \gamma_n = \kappa_n$, whence

$$
A_n^{-1}P_nA\phi_n = \phi_n - \frac{\kappa_n}{\gamma_n} \psi_n.
$$

In conclusion Lemma 3.1 yields

$$
A_n^{-1}P_nA(P_{n+1} - P_n)TP_n = \frac{\kappa_{n-1}}{\kappa_n} \left( \phi_n - \frac{\kappa_n}{\gamma_n} \psi_n \right) \langle \cdot, \phi_{n-1} \rangle. \quad \Box
$$

**Corollary 5.2.** In the conditions of Proposition 5.1,

$$
\text{trace}\left| \hat{T}_n - T_n \right| = \| \hat{T}_n - T_n \| = \frac{\kappa_{n-1}}{\kappa_n} \sqrt{\frac{\kappa_n^2}{\gamma_n^2} \| \psi_n \|^2 - 1}.
$$

*Proof.* Indeed,

$$
\left\langle \phi_n, \frac{\kappa_n}{\gamma_n} \psi_n \right\rangle = 1,
$$

therefore Lemma 3.1 implies

$$
\left\| \phi_n - \frac{\kappa_n}{\gamma_n} \psi_n \right\|^2 = \frac{\kappa_n^2}{\gamma_n^2} \| \psi_n \|^2 - 1.
$$

Second, the norm and trace norm coincide on any rank one operator. \quad \Box
The “orthogonal polynomials” point of view allows a simple interpretation of the LDU decomposition of the operator $A$. To this aim, we write

$$\phi_n = c_{n,n} \psi_n + c_{n,n-1} \psi_{n-1} + \cdots + c_{n,0} \psi_0, \quad n \geq 0,$$

so that

$$\langle A\phi_n, \phi_k \rangle = \sum_{j \leq n} \sum_{\ell \leq k} \langle A\psi_j, \psi_\ell \rangle c_{n,j} c_{k,\ell} = \sum_{j \leq \min(n,k)} c_{n,j} \overline{c}_{k,j}.$$ 

Thus the lower triangular matrix $C = (c_{n,j})$ satisfies, at all stages $n \geq 0$:

$$A_n = P_n A P_n = P_n C P_n C^* P_n = P_n C C^* P_n.$$ 

That is $C$ represents in the orthonormal basis $(\phi_n)$ a linear bounded operator, also denoted by $C$. Let $C_n = P_n C P_n$. The diagonal elements are identifiable from the linear decomposition above:

$$c_{n,n} = \frac{\kappa_n}{\gamma_n}.$$ 

Also,

$$\langle A\phi_n, \phi_n \rangle = c_{n,n}^2 + |c_{n,n-1}|^2 + \cdots + |c_{n,0}|^2,$$

and

$$c_{n,n-1} \psi_{n-1} + \cdots + c_{n,0} \psi_0 = \phi_n - c_{n,n} \psi_n = A_n^{-1} P_n A \phi_n.$$ 

Let $D = \text{diag}(\frac{c_{n,n}^2}{\gamma_n})$, so that there exists a strictly lower diagonal matrix $L_n$ with the property

$$C_n = (I + L_n) \sqrt{D_n}, \quad n \geq 0.$$ 

Specifically

$$L_{nj} = \frac{c_{nj}}{c_{jj}}, \quad j < n.$$ 

Note that one can recover the entries of the diagonal matrix $D$ via determinantal formulae:

$$\det A_{n+1} = \det D_{n+1} = c_{00} c_{11} \cdots c_{nn} = \left[ \frac{\kappa_0 \kappa_1 \cdots \kappa_n}{\gamma_0 \gamma_1 \cdots \gamma_n} \right]^2,$$

and

$$\kappa_n^2 = \frac{\gamma_n^2}{\gamma_n^2} = \frac{\det A_{n+1}}{\det A_n}.$$ 

In this way we recover the LDU decomposition

$$A_n = C_n C_n^* = (I + L_n) D_n (I + L_n^*)$$

According to Corollary 3.5, a sufficient condition for the asymptotic equivalence of the two sequences of finite central truncations is the finiteness of the Hilbert–Schmidt norm of the infinite matrix $L$. We state this observation as a partial conclusion of the computations above.
Proposition 5.3. Let $T \in \mathcal{L}(H)$ be a linear bounded operator with cyclic vector $\xi$ and let $A$ be a positive, non-invertible linear operator on $H$. The ascending chain of Krylov subspaces of $T$ carries the orthonormal basis $(\phi_n)_{n=0}^\infty$ and the $A$-orthonormal basis $(\psi_n)_{n=0}^\infty$. If the transition matrix $(c_{nj})$ between the two bases (5.1) satisfies

$$
\sum_{n=1}^\infty \sum_{j=0}^{n-1} \left| \frac{c_{nj}}{c_{jj}} \right|^2 < \infty, \quad (5.5)
$$

then the finite central truncations of $T$ are asymptotically equivalent

$$
\lim_{n \to \infty} \| T_n - \tilde{T}_n \| = 0.
$$

Similarly, if we start with the inverse of the transition matrix $B = (b_{nk})$:

$$
\psi_n = b_{nn} \phi_n + b_{n,n-1} \phi_{n-1} + \cdots + b_{n0} \phi_0 \quad (5.6)
$$

we end up with a factorization

$$
B_n A_n B_n^* = I, \quad n \geq 0.
$$

Since $B_n = P_n B P_n$ is a lower triangular matrix, or directly, we infer

$$
B_n C_n = B_n (I + L_n) \sqrt{D_n} = I,
$$

where $L_n$ is the matrix defined by (5.2). Whence

$$
B_n = \sqrt{D_n}^{-1} (I + L_n)^{-1} = \sqrt{D_n}^{-1} (I + M_n)
$$

where $M_n$ is another strictly lower triangular matrix defined by the above identity. Allowing $n$ to tend to infinity, note that the full matrix $L$ is Hilbert–Schmidt if and only if $M$ is:

$$
M_n = (I + L_n)^{-1} - I = (I + L_n)^{-1} L_n
$$

plus the convergence of $(I + L_n)^{-1}$ in norm to $(I + L)^{-1}$, and vice-versa. Remark that

$$
b_{nn} = \gamma_n = \kappa_n^{-1}.
$$

We conclude that the condition in the statement of the above Proposition is equivalent to

$$
\sum_{n} \sum_{j<n} \left| \frac{b_{nj}}{b_{nn}} \right|^2 < \infty.
$$

Along the same lines, we remark that

$$
A_n^{-1} P_n A \phi_n = \phi_n - \frac{\psi_n}{b_{nn}} = - \sum_{j<n} \frac{b_{nj}}{b_{nn}} \phi_j,
$$

and, for every $n \geq 0$,

$$
\| A_n^{-1} P_n A \phi_n \|^2 = \sum_{j<n} \left| \frac{b_{nj}}{b_{nn}} \right|^2.
$$
Corollary 5.4. Let \((b_{nj})\) denote the base change matrix (5.6) of the A-orthonormal basis \((\psi_n)\) into the orthonormal basis \((\phi_n)\).

If

\[
\lim_n \sum_{j<n} \left| b_{nj} b_{nn} \right|^2 = 0,
\]

then

\[
\lim_{n \to \infty} \| T_n - \tilde{T}_n \| = 0.
\]

Proof. To complete the proof we only need to remark that the extra factor \(\kappa_n^{-1}\) in the difference \(T_n - \tilde{T}_n\) is uniformly bounded:

\[
\frac{\kappa_n^{-1}}{\kappa_n} = \langle T \phi_{n-1}, \phi_n \rangle \leq \|T\|. \quad \square
\]

Next we relax the decay assumption (5.7) and state a similar result to Proposition 5.3, with a weaker conclusion.

Proposition 5.5. Let \(T \in \mathcal{L}(H)\) be a linear bounded operator with cyclic vector \(\xi\) and let \(A\) be a positive, non-invertible linear operator on \(H\). The ascending chain of Krylov subspaces of \(T\) carries the orthonormal basis \((\phi_n)_{n=0}^{\infty}\) and the A-orthonormal basis \((\psi_n)_{n=0}^{\infty}\). If the transition matrix \((c_{nj})\) between the two bases (5.1) satisfies

\[
\sup_j \left( \sum_{n>j} |c_{nj}| \right) \sup_n \left( \sum_{j<n} |c_{nj}| \right) < 1,
\]

then the two sequences of finite central truncations of \(T\) remain at finite distance

\[
\sup_n \| T_n - \tilde{T}_n \| < \infty.
\]

Proof. In order to use Theorem 3.4 we have to check that the operator \(I + L\) is invertible on \(\ell^2(\mathbb{N})\), where \(L\) has matrix entries \(L_{nj} = \frac{c_{nj}}{c_{jj}}\) for \(j < n\) and zero otherwise. We use Schur’s boundedness criterion:

\[
\|Lx\|^2 = \sum_n \left| \sum_{j<n} L_{nj} x_j \right|^2 \leq \sum_n \left( \sum_{j<n} |L_{jn}| \right) \left( \sum_{j<n} |L_{jn}|^2 \right) \leq M \sum_{j<n} |L_{jn}|^2 \leq MN \|x\|^2,
\]

where \(M = \sup_n \left( \sum_{j<n} |c_{nj}| \right)\) and \(N = \sup_j \left( \sum_{n>j} |c_{nj}| \right)\). \quad \square

6. Examples

We collect in this final section several relevant examples for the theoretical results proved in the main body of the article. A common theme in all specific situations we develop below is the high benefits of possessing the doubly orthogonal system of vectors, with respect to the pair of sesquilinear forms.
which define the finite central truncations we are interested in. A second major theme is that relatively small perturbations of the weaker norm will not affect the asymptotics of the counting spectrum measures of the finite central truncations of a prescribed operator.

We start by recalling a couple of classical, apparently trivial examples, which nevertheless lie at the foundation of understanding finite central truncations of Toeplitz matrices, respectively differential operators.

6.1. Hardy Space Operators

Let $H^2(\mathbb{D})$ denote the Hardy space of the disk centered at $z = 0$ and of radius $r > 0$. That is the space of Fourier series with non-negative coefficients

$$f(re^{i\theta}) = \sum_{k=0}^{\infty} c_k r^k e^{ik\theta},$$

which are square summable:

$$\|f\|_r^2 = \sum_{k=0}^{\infty} |c_k|^2 r^{2k} < \infty.$$ 

Then $f(z) = \sum_{k=0}^{\infty} c_k z^k$ is an analytic function in the open disk $r\mathbb{D}$, with non-tangential boundary values $f(re^{i\theta})$.

It is relevant for our study to remark that the monomials $z^k, k \geq 0,$ are simultaneously orthogonal on all disks $r\mathbb{D}$. In addition, the norms $\|f\|_r^2$ are non-decreasing as functions of $r > 0$. Let $H_n = \text{span}\{1, z, \ldots, z^{n-1}\}$ denote the subspace of polynomials of degree less than $n$.

Fix a value $0 < r < 1$ and consider the Hilbert space $H^2(\mathbb{D})$ and the weaker norm $\|\cdot\|_r$ on it, giving rise by completion to the Hilbert space $H^2(r\mathbb{D})$. The positive, compact and diagonal operator

$$D = \text{diag}(1, r^2, r^4, r^6, \ldots)$$

links, in the spirit of the present note, the two norms:

$$\langle Df, f \rangle = \langle f, f \rangle_r.$$ 

Let $T : H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D})$ denote an arbitrary linear and bounded operator, and let $T_n, \tilde{T}_n$ denote its finite central truncations with respect to the chain $(H_n)$, in the norms $\|\cdot\|_1$, respectively $\|\cdot\|_r$. Lemma 3.1 implies then $T_n = \tilde{T}_n$. This shows that the asymptotic values of the spectra of the truncations $T_n$ have very little to do with the spectrum of $T$. Indeed, consider an analytic Toeplitz operator, that is the multiplier $T = M_F$ with a bounded analytic function, defined in a neighborhood of the closed unit disk. Then the operator

$$T : H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D})$$

is subnormal and has spectrum equal to $F(\mathbb{D})$, while

$$T : H^2(r\mathbb{D}) \rightarrow H^2(r\mathbb{D})$$

has spectrum equal to $F(r\mathbb{D})$. 

Originally, this invariance phenomenon was discovered on Toeplitz matrices by Schmidt and Spitzer [18], and led to a fascinating, yet far from being complete, analysis of the respective eigenvalue distribution, see [5].

Remark that instead of a second Hardy space $H^2(rD)$ we can take any Hilbert space of analytic functions on the disk $rD$ which admits the monomials as an orthogonal basis. As for instance a Bergman type space with a rotationally invariant weight. Next we exploit in the same context the Hilbert–Schmidt perturbation observation contained in Theorem 3.7.

Let $L(z, w)$ be a positive semi-definite function, analytic in $z, |z| < \frac{1}{r}$ and anti-analytic in $w, |w| < \frac{1}{r}$, continuous on the closed bidisk $\frac{1}{r}D \times \frac{1}{r}D$. We define the positive semi-definite, Hilbert–Schmidt operator $S \in L(H^2(D))$ by

$$(Sf)(z) = \int_T L(z, w) f(w) \frac{dw}{iw}, \quad f \in H^2(D).$$

With the choice of the diagonal operator $D$ as before, we want to assure the continuity of $F = \sqrt{D^{-1}} S \sqrt{D}$, that is of the integral operator

$$(Ff)(z) = \int_T L\left(\frac{z}{r}, \frac{w}{r}\right) f(rw) \frac{dw}{iw} = \int_T L\left(\frac{z}{r}, \frac{w}{r}\right) f(w) \frac{dw}{iu}.$$

But $F$ is Hilbert–Schmidt on the Hardy space due to the continuity assumption on the kernel $L$.

Let $(Af)(z) = f(rz) + (Sf)(z), \ f \in H^2(D)$, and consider the weaker norm $\|f\|_A^2 = \langle Af, f \rangle$. Theorem 3.4 implies then that for an arbitrary operator $T \in L(H^2(D))$ the finite central truncations with respect to the chain of polynomial spaces $(H_n)$ and the norms $\| \cdot \|, \| \cdot \|_A$ are asymptotically equivalent. Note that the norm $\| \cdot \|_A$ is equivalent to $\| \cdot \|_r$, but the monomials are no longer orthogonal in the inner product induced by the operator $A$.

A different example on Hardy space, this time exploiting Theorem 3.3, can be constructed as follows. For a function $f \in H^2(D)$ we denote by $\hat{f}(n)$ its Fourier transform, that is $(\hat{f}(n))$ are the coefficients of the Taylor expansion of $f$ at $z = 0$:

$$f(z) = \hat{f}(0) + \hat{f}(1)z + \hat{f}(2)z^2 + \cdots.$$

Consider a bounded analytic function in the disk $h \in H^\infty(D)$ subject to the normalization $\|h\|_\infty = \sup_{|z| < 1} |h(z)| < 1$. We define the Toeplitz operator

$$(Rf)(z) = f(z) + \frac{1}{2\pi} \int_T \frac{\overline{h(\zeta)f(\zeta)}}{\zeta - z} \frac{d\zeta}{i\zeta}, \quad f \in H^2, \ |z| < 1.$$

The matrix associated to $R$ in the orthonormal basis $(z^n)$ is of the form $I + L^*$, where $L$ is strictly lower-triangular and strictly contractive $\|L\| < 1$. Let $(\gamma_n)$ denote an arbitrary sequence of positive numbers, converging to zero. We define the weaker norm on Hardy space by

$$\langle Af, f \rangle = \sum_{n=0}^{\infty} \gamma_n |\hat{Rf}(n)|^2, \quad f \in H^2.$$
Then Theorem 3.3 asserts that any Hessenberg matrix

$$T = \begin{pmatrix} t_{00} & t_{01} & t_{02} & t_{03} & \ldots \\ t_{10} & t_{11} & t_{12} & t_{13} & \ldots \\ 0 & t_{21} & t_{22} & t_{23} & \ldots \\ 0 & 0 & t_{32} & t_{33} & \ldots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

bounded as an operator in the $\ell^2$-norm, has the central finite truncations $(T_n)$ and $(\tilde{T}_n)$ at uniform distance:

$$\sup_n \|T_n - \tilde{T}_n\| < \infty.$$ 

Moreover, if the sub diagonal of $T$ converges to zero: $\lim_n t_{n+1,n} = 0$, then

$$\lim_n \|T_n - \tilde{T}_n\| = 0.$$ 

In both cases, the counting measures of the spectra of $T_n$, respectively $\tilde{T}_n$ will have limit measures with the same logarithmic potential at infinity.

### 6.2. Weighted $\ell^2$-spaces

We start with a positive, bounded weight $w : \mathbb{Z}^n \rightarrow \mathbb{R}$. The norm

$$\|f\|_w^2 = \sum_{\alpha \in \mathbb{Z}^n} w(\alpha) |f_\alpha|^2, \quad f = (f_\alpha),$$

is continuous on $\ell^2(\mathbb{Z}^n)$, and in general weaker than the standard norm

$$\|f\|^2 = \sum_{\alpha \in \mathbb{Z}^n} |f_\alpha|^2, \quad f = (f_\alpha).$$

Denoting by $(e_\alpha)_{\alpha \in \mathbb{Z}^n}$ the standard orthonormal basis on $\ell^2(\mathbb{Z}^n)$, we find that the norm $\| \cdot \|_w$ is implemented by the diagonal operator $D = \text{diag}(\sqrt{w(\alpha)})$. Given an order on $\mathbb{Z}^n$, for instance derived from the lexicographical order of the positive orthant, we define the finite dimensional subspaces

$$H_\beta = \text{span}\{e_\alpha, \alpha < \beta\}.$$ 

Given a linear bounded operator $T$ on $\ell^2(\mathbb{Z}^n)$, we discover the rather tautological fact that the finite central truncations of $T$ with respect to the chain of subspaces $H_\alpha, \alpha \in \mathbb{Z}^n$, in the two norms $\| \cdot \|_w$ are identical. This observation can be proved directly, as $f_\alpha = \frac{e_\alpha}{\sqrt{w(\alpha)}}$ is the orthonormal system of vectors in the weaker norm:

$$(x, f_\alpha)f_\alpha = \frac{(D x, e_\alpha)e_\alpha}{w(\alpha)} = \frac{x, De_\alpha}{w(\alpha)} = \frac{e_\alpha}{\sqrt{w(\alpha)}}.$$ 

As an application we consider the chain of Sobolev spaces on the torus $\mathbb{T}^n$. The Fourier system $\exp(\alpha \cdot), \alpha \in \mathbb{Z}^n$ is orthogonal on every Sobolev space. Let $P(x, D)$ denote a pseudo-differential operator of order zero, on $\mathbb{T}^n$. According to the observation above, the finite central truncations of $P(x, D)$ with respect to the Fourier orthogonal system is independent of the
order of the Sobolev space norm in which is computed. Specifically, the linear bounded operators
\[ P(x, D) : H^s(\mathbb{T}^n) \rightarrow H^s(\mathbb{T}^n) \]
have the same finite central truncations along the chain of Fourier modes of increasing higher frequency, independent of the order \( s \in \mathbb{R} \).

### 6.3. Jacobi Matrices

Let \( \mu \) be a positive measure supported on a bounded interval \([0, a]\) of the real axis. The operator of multiplication by the variable \( A = M_x \) is positive, bounded and injective as soon as \( \mu \) does not have a point mass at zero. We will take \( H = L^2(\mu) \) and for the \( A \)-norm we consider the space \( K = L^2(x\mu) \).

The filtration given by polynomials and their degrees
\[ H_n = \{ p \in \mathbb{C}[z]; \ \deg p \leq n - 1 \}, \ n \geq 0, \]
produces systems of orthonormal polynomials \( (\phi_n) \subset L^2(\mu) \) and \( (\psi_n) \subset L^2(x\mu) \). The positive operator \( A = M_x \) of multiplication by the variable has a tri-diagonal matrix representation with respect to the basis of \( \mu \)-orthogonal polynomials \( (\phi_j) \):

\[ x\phi_j(x) = b_{j+1}\phi_{j+1}(x) + a_j\phi_j(x) + b_j\phi_{j-1}(x), \ j \geq 0. \]

Note that in this case the orthogonal projections \( P_n \) onto the subspaces \( H_n \) in the chain satisfy
\[ P_nA(I - P_n) = P_nA(P_{n+1} - P_n), \ n \geq 0. \]

Let \( T \in \mathcal{L}(L^2(\mu)) \) be any linear bounded operator. In view of the proof of Lemma 3.1 we infer
\[ T_n - \tilde{T}_n = A^{-1}_n P_nA(I - P_n)TP_n = A^{-1}_n P_nA(P_{n+1} - P_n)TP_n \]
hence
\[ \lim_n \|T_n - \tilde{T}_n\| = 0 \]
as soon as
\[ \lim_n \|A^{-1}_n P_nA(P_{n+1} - P_n)\| = 0. \]

Summing-up we have obtained the following result.

**Proposition 6.1.** Let
\[ A = \begin{pmatrix} b_0 & a_1 & 0 & 0 & \ldots \\ \bar{a}_1 & b_1 & a_2 & 0 & \ldots \\ 0 & \bar{a}_2 & b_2 & a_3 & \ldots \\ 0 & 0 & \bar{a}_3 & b_3 & \ldots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \]
be a positive Jacobi matrix subject to the conditions
\[ b_{n+1} \leq b_n, \ n \geq 0 \]
and
\[
\sum_{k=1}^{\infty} \frac{|a_k|^2}{b_{k-1}b_k} < \infty.
\]

Then any linear bounded operator \( T : \ell^2(N) \to \ell^2(N) \) has asymptotically equivalent finite central truncations with respect to the main diagonal, in the \( \ell^2 \), respectively \( A \)-norms.

The most interesting case is a compact and positive Jacobi matrix \( A \), and this is directly linked to Stieltjes famous memoir devoted to continued fractions and their convergence domains. Under these assumptions the original measure \( \mu \) is discrete, with only \( x = 0 \) as accumulation point of its point masses \( \lambda_k > 0 \). Specifically
\[
\| f \|^2 = \int |f|^2 d\mu = \sum_{k=1}^{\infty} \gamma_k |f(\lambda_k)|^2,
\]

is the original norm, while the weaker one is
\[
\| f \|^2_A = \sum_{k=1}^{\infty} \gamma_k \lambda_k^2 |f(\lambda_k)|^2.
\]

The weights \( (\gamma_k) \) above are positive and constrained by the two assumptions in the Proposition. For details, examples and ample references on compact Jacobi matrices see [21].

### 6.4. Integral Kernels. I (Discrete Chains)

Let \( \mu \) be a positive measure with compact support in \( \mathbb{C} \). We consider the Lebesgue space \( L^2(\mu) \) and the closure of complex polynomials \( H = P^2(\mu) \) in its norm. The chain of subspaces defined by the degree filtration
\[
H_n = \{ p \in \mathbb{C}[z]; \deg p < n \}, \ n \geq 1,
\]
is dense in \( H \), and we assume it is non-stationary, that is the measure \( \mu \) is not supported by a finite set.

Let \( K(z, w) = \sum_{j=0}^{\infty} P_j(z)\overline{P_j(w)} \) be a positive semi-definite kernel, obtained as an infinite sum of non-negative rank-one kernels with polynomial components \( P_j \). We assume \( \deg P_j = j \) for all \( j \geq 0 \) and denote
\[
P_j(z) = \delta_j z^j + \text{lower order terms}.
\]
The choice \( \delta_j > 0 \) for the leading coefficient does not affect the kernel \( K(z, w) \).

We impose the Hilbert–Schmidt condition
\[
\sum_{j=0}^{\infty} \| P_j \|^2 < \infty.
\]

In this way the integral operator
\[
( Af)(z) = \int K(z, w)f(z) d\mu(z), \quad f \in H,
\]
is compact and non-negative. We also assume that the system \( (P_j) \) is complete in \( H \), that is the range of \( A \) is dense in \( H \), hence its kernel is trivial.
We denote as before by \((\phi_n)_{n=0}^{\infty}\) the sequence of orthonormal polynomials in the metric of \(H\). The leading coefficient of every \(\phi_n\) is chosen to be positive:

\[
\phi_n(z) = \kappa_n z^n + \text{lower order terms.}
\]

The matrix (still denoted by \(A\)) associated to the operator \(A\) in this basis is

\[
A_{j\ell} = \langle A\phi_j, \phi_{\ell} \rangle = \sum_{k \geq j} \langle \phi_j, P_k \rangle \langle P_k, \phi_{\ell} \rangle.
\]

The diagonal entries are

\[
a_{jj} = \sum_{k \geq j} |\langle \phi_j, P_k \rangle|^2 \geq |\langle \phi_j, P_j \rangle|^2 = \frac{\delta_j^2}{\kappa_j^2}.
\]

The upper triangular matrix

\[
B = (b_{jk}), \quad b_{jk} = \langle \phi_j, P_k \rangle,
\]

satisfies \(A = BB^*\) and its diagonal

\[
D = \text{diag} \left( \frac{\delta_j}{\kappa_j} \right),
\]

has positive entries.

We will adapt this scenario to Proposition 4.1 by assuming that there exists a positive constant \(C\) with the property

\[
C \frac{\kappa_k}{\kappa_\ell} \leq \frac{\delta_k}{\delta_\ell}, \quad \ell > k.
\]

Next we factor formally

\[
B = D(I + U)
\]

where the matrix \(U\) is strictly upper-triangular with entries

\[
u_{jk} = \frac{\kappa_j \langle \phi_j, P_k \rangle}{\delta_j}.
\]

A second condition we impose is that \(U\) is Hilbert–Schmidt:

\[
\sum_{j=0}^{\infty} \sum_{k=j+1}^{\infty} \frac{\kappa_j^2 |\langle \phi_j, P_k \rangle|^2}{\delta_j^2} < \infty.
\]

Hence we can factor the operator \(A\) as

\[
A = D(I + U)(I + U^*)D,
\]

with \((I + U)(I + U^*) = I + S\) and \(S\) Hilbert–Schmidt. Note that the product above is in the wrong order when compared to the standard \(LDU\) factorization.

We state the following conclusion derived from Proposition 4.1.
Proposition 6.2. A Hilbert–Schmidt integral operator \( A : L^2(\mu) \rightarrow L^2(\mu) \) with kernel \( K(z, w) = \sum_k P_k(z)\overline{P_k(w)} \), where \( \deg P_k = k, \ k \geq 0 \), subject to conditions (6.1) and (6.2) is “universally good”, that is every linear bounded operator \( T : P^2(\mu) \rightarrow P^2(\mu) \) subject to the Hessenberg matrix condition
\[
\deg(Tp) \leq \deg(p) + 1, \ \ p \in \mathbb{C}[z],
\]
has asymptotically equivalent finite central truncations in the original norm and with respect to the \( A \)-norm.

6.5. Integral Kernels. II (Continuous Chains)

From the extensive literature on Volterra type operators we extract an illustrative case. The ground Hilbert space \( H \) is \( L^2[0, \infty) \) with respect to Lebesgue measure. For every \( t \geq 0 \) the space of functions in \( H \) with support contained in \([0, t)\) is a closed subspace called \( H_t \). The orthogonal projection onto \( H_t \) is denoted \( P_t \). Let \( \gamma \in L^1[0, \infty) \) and consider the associated Wiener–Hopf operator:
\[
(Lf)(x) = \int_0^x \gamma(x-t)f(t)dt, \ \ f \in H.
\]

We know from Gelfand’s theory that \( I + L \) is invertible on \( H \) if and only if its symbol does not vanish:
\[
1 + \int_0^\infty e^{i\xi t} \gamma(t)dt \neq 0, \ \xi \in \mathbb{R}, \quad (6.3)
\]
see for instance Theorem XXX.2.6 in [8]. The triangular nature of \( L \) is also reflected by \( P_t L(I - P_t) = 0, \ t \geq 0 \). Let \( \delta \in L^\infty[0, \infty) \) be a monotonically decreasing function with positive values, and define the multiplication operator
\[
(Df)(x) = \delta(x)^2 f(x), \ \ f \in H.
\]

Note that for every fixed \( t > 0 \), the multiplicator \( f \mapsto Df \) is positive and bounded from below, hence invertible on \( L^2[0, t] \).

Under these conditions the linear operator \( A = (I + L)D(I + L^*) \) is bounded and positive, but possibly not invertible on the whole space. The \( A \)-norm is given in closed form by the expression:
\[
\|f\|^2_A = \int_0^\infty |\delta(x)f(x) + \delta(x)\int_0^\infty \gamma(t-x)f(t)dt|^2dx.
\]

An adaptation of the proof of Theorem 3.4 shows that the finite central truncations \( T_s \), respectively \( \tilde{T}_s \) of any linear bounded operator \( T \in \mathcal{L}(L^2[0, \infty)) \) with respect to the chain \( (H_t) \) in the original, respectively \( A \)-norm, satisfy
\[
\sup_{s>0} \|T_s - \tilde{T}_s\| < \infty.
\]

We leave the details to the interested reader.
6.6. Doubly Orthogonal Systems

An immediate consequence of our first computations is that a doubly orthogonal basis in the original metric and the weaker one will give identical finite central truncations for any operator. Under such a scenario the positive operator $A \in \mathcal{L}(H)$ implementing the weak norm is diagonalized by this system of vectors. Obviously not every positive operator has pure point spectrum, hence can be diagonalized. Two relevant functional models for such a scenario are briefly recorded below.

6.6.1. Natural Hilbert Spaces of Potential Theory. A classical framework involving simultaneous spectral analysis in two different Hilbert spaces is offered by potential theory. The note [14] based on Krein’s foundational work [15] is a recent illustration for this theme. Rather than discussing spectral permanence in the presence of two norms, we indicate some possible applications of the present article to the rapidly growing field of spectral analysis of layer potentials, see [1].

Let $\Gamma$ be a closed surface in $\mathbb{R}^d, d \geq 2$, sufficiently regular (for instance of Lipschitz type). Two Hilbert spaces stand out in the study of Dirichlet’s problem with data on $\Gamma$ via layer potentials. First is the Lebesgue space $L^2(\Gamma, d\sigma)$ with respect to area measure on $\Gamma$, and second is the energy space $\mathcal{H}$ consisting of pairs $h = (h_i, h_e)$ of harmonic functions on $\Omega_i$ (the interior of $\Gamma$), respectively $\Omega_e$ (the exterior of $\Gamma$) possessing finite energy

$$[h]^2 = \int_{\Omega_i} |\nabla (h_i)|^2 dx + \int_{\Omega_e} |\nabla (h_e)|^2 dx < \infty.$$  

We denote by $-E(x)$ the fundamental solution of Laplace operator: $\Delta E = -\delta$. The two norms are related by the single layer potential

$$S_f(z) = \int_{\Gamma} E(z - y)f(y)d\sigma(y), \quad z \notin \Gamma, \quad f \in L^2(\Gamma),$$

and by the operator obtained by passing to boundary values on $\Gamma$:

$$(Sf)(x) = \int_{\Gamma} E(x - y)f(y)d\sigma(y), \quad x \in \Gamma, \quad f \in L^2(\Gamma).$$

It turns out that the singularity appearing in the latter integral operator is removable and the result, modulo a scaling and exclusion of constant functions in dimension $d = 2$, is a positive compact operator $S : L^2(\Gamma) \rightarrow L^2(\Gamma)$. The identity connecting the two norms is very simple

$$[S_f]^2 = \langle S_f, f \rangle, \quad f \in L^2(\Gamma).$$

In this way the continuous inclusion $L^2(\Gamma) \subset \mathcal{H}$ fits into the framework developed in the present article.

But it is the double layer potential

$$D_f(z) = \int_{\Gamma} \frac{\partial}{\partial n_y} E(z - y)f(y)d\sigma(y)$$

and its boundary induced operator historically responsible for the solvability of the Dirichlet problem. The latter is a compact perturbation of the identity
only for smooth $\Gamma$, its essential spectrum being in the general case highly relevant for an array of applications, see for details [1].

Since Galerkin approximation is ubiquitous in all numerical experiments, a conclusion of formula (3.1) is in order.

**Lemma 6.3.** Let $T$ be a linear bounded operator acting on $L^2(\Gamma, d\sigma)$, where $\Gamma$ is a closed, Lipschitz surface in Euclidean space. The finite central truncations of $T$ along the system of eigenvectors $(\phi_j)_{j=0}^{\infty}$ of the single layer potential operator attached to $\Gamma$ coincide in the $L^2$, respectively energy space norms.

**Proposition 6.4.** Let $(\phi_j)$ denote the doubly orthogonal system of functions in the spaces $L^2(\Gamma)$ and $H$. Assume that $K : H \rightarrow L^2(\Gamma)$ is a compact operator with the property that $I + K : L^2(\Gamma) \rightarrow L^2(\Gamma)$ is invertible.

The two sequences of finite central truncations of a linear bounded operator $T \in L(L^2)$, with respect to the chain of finite dimensional subspaces generated by the vectors $(I + K)\phi_j$, $j \geq 0$, satisfy:

$$\lim_n \|T_n - \tilde{T}_n\|_{L(L^2, H)} = 0.$$  

**Proof.** The operator $I + K$ is also invertible from $H$ to $H$, as it is Fredholm of zero index and possesses dense range. Note also that the restriction of $K$ to $L^2(\Gamma)$ is a compact operator from $L^2(\Gamma)$ to itself. For consistency, we return to the standard notation of this article and denote the weaker norm by

$$\langle x, y \rangle_H = \langle Ax, y \rangle_{2,\Gamma}, \quad x, y \in L^2(\Gamma).$$

Denote $R = I + K$ and let $H_n$ denote the linear span of the vectors $\phi_0, \phi_1, \ldots, \phi_{n-1}$. We compute in closed form the orthogonal projection $\hat{P}_n$ of $L^2$ onto the finite dimensional subspace $\hat{H}_n = RH_n$:

$$\hat{P}_n = RP_n(P_n R^* R P_n)^{-1} P_n R^*.$$  

Indeed, $\hat{P}_n = \hat{P}_n^2 = \hat{P}_n^3$ and the range of $\hat{P}_n$ is $\hat{H}_n$.

Since $K$ is compact, $\lim_n (P_n K - K P_n) = 0$ in the operator norm, hence

$$\lim_n \|\hat{P}_n - P_n\| = \lim_n \|R P_n(P_n R^* R P_n)^{-1} P_n R^* - P_n R(R^* R)^{-1} R^* P_n\| = \lim_n \|P_n(R(R^* R)^{-1} R^* - I) P_n\| = 0.$$  

We infer that for any bounded linear operator $T \in L(L^2)$ one has

$$\lim_n \|P_n T P_n - \hat{P}_n T \hat{P}_n\| = 0.$$  

On the other hand, according to the Lemma, the orthogonal projections $Q_n$ onto $H_n$ in the $H_n$-norm coincide with $P_n$. By repeating the argument above, and using the fact that $K \in L(H)$ is compact, we find that the orthogonal projections $\hat{Q}_n$ onto $RH_n$ satisfy in the $H$ metric:

$$\lim_n \|\hat{Q}_n - Q_n\|_{H} = 0$$  

and a fortiori

$$\lim_n \|\hat{Q}_n - Q_n\|_{L(L^2, H)} = 0. \quad (6.4)$$
For a given linear bounded transform \( T \in \mathcal{L}(L^2) \) we note the estimates:
\[
\| (\hat{P}_n - \hat{Q}_n) T \|_{\mathcal{L}(L^2, H)} \leq \| A \|^{1/2} \| (\hat{P}_n - P_n) T \|_{\mathcal{L}(L^2)}
+ \| (Q_n - \hat{Q}_n) T \|_{\mathcal{L}(L^2, H)},
\]
whence
\[
\lim_n \| (\hat{P}_n - \hat{Q}_n) T \|_{\mathcal{L}(L^2, H)} = 0,
\]
as desired. \(\square\)

We would like to offer a second proof of the asymptotic equivalence (6.4), and for this we will enter into the structure of the compact operator \( K : \mathfrak{H} \rightarrow L^2(\Gamma) \). More precisely, the boundedness of \( K \) implies
\[
K = C \sqrt{A},
\]
where \( C \in \mathcal{L}(L^2) \), while the compactness of \( K \) implies that \( C \) is compact as an endomorphism of \( L^2 \). Let us denote \( K^\sharp \) the adjoint with respect to the inner product of \( \mathfrak{H} \). The identity
\[
\langle AKx, y \rangle = \langle Ax, K^\sharp y \rangle
\]
implies
\[
\sqrt{A} K^\sharp = C^* A.
\]

We claim that
\[
\lim_n \| \sqrt{A}(\hat{Q}_n - Q_n) \|_{\mathcal{L}(L^2)} = 0
\]
which is the same as (6.4). Indeed, notice first that the chain of projections \( P_n = Q_n \) commute with the positive operator \( A \):
\[
P_n A = AP_n = P_n A P_n.
\]
The projection \( \hat{Q}_n \) has the closed form:
\[
\hat{Q}_n = R P_n [P_n R^\sharp R P_n]^{-1} P_n R^\sharp,
\]
or
\[
\hat{Q}_n = (I + C \sqrt{A}) P_n [P_n(I + K^\sharp)(I + C \sqrt{A}) P_n]^{-1} P_n(I + K^\sharp).
\]
Remark that
\[
\sqrt{A}(I + C \sqrt{A}) = (I + \sqrt{A}C) \sqrt{A},
\]
and
\[
\sqrt{A} P_n(I + K^\sharp) = P_n \sqrt{A}(I + K^\sharp) = P_n(I + C^* \sqrt{A}) \sqrt{A}.
\]
Consequently
\[
\sqrt{A} \hat{Q}_n = (I + \sqrt{A}C) P_n \sqrt{A} [\sqrt{A} P_n(I + K^\sharp)(I + C \sqrt{A}) P_n]^{-1} \sqrt{A} P_n(I + K^\sharp)
= (I + \sqrt{A}C) P_n [I + C^* \sqrt{A}(I + \sqrt{A}C) P_n \sqrt{A}]^{-1} P_n(I + C^* \sqrt{A}) \sqrt{A}
= (I + \sqrt{A}C) P_n [(I + C^* \sqrt{A})(I + \sqrt{A}C) P_n]^{-1} P_n(I + C^* \sqrt{A}) \sqrt{A}.
\]
Finally the compactness of \( C \) takes over and we find
\[
\lim_n \| \sqrt{A}(\hat{Q}_n - P_n) \|_{\mathcal{L}(L^2)} = 0.
\]
With the aid of the eigenfunctions $\phi_j$ of the single layer potential operator $S$ and its spectrum

$$S\phi_j = \lambda_j \phi_j, \quad j \geq 0,$$

one can illustrate the corollary above by an explicit matrix condition. Quite specifically, assume for all $j \geq 0$ that $\|\phi_j\|_{2,\Gamma} = 1$, so that $\left( \frac{\phi_j}{\sqrt{\lambda_j}} \right)$ is an orthonormal basis of the energy space $\mathcal{H}$. Let

$$a_{jk} = \langle K\phi_j, \phi_k \rangle, \quad j, k \geq 0,$$

denote the matrix entries of a linear transformation $K$. If

$$\sum_{j,k=0}^{\infty} \frac{|a_{jk}|^2}{\lambda_j} < \infty$$

then the operator $K$ is Hilbert–Schmidt as a linear transformation from $\mathcal{H}$ to $L^2$.

A classical example of doubly orthogonal system in $L^2(\Gamma)$ and energy space $\mathcal{H}$ is offered by the unit sphere $\Gamma = S^{d-1}$ in $\mathbb{R}^d, d \geq 3$. Indeed, all spherical harmonics $h$ diagonalize the single layer potential operator:

$$Sh = \frac{h}{2\deg h + d - 2}.$$ 

Of course the multiplicity of the eigenvalue $\frac{1}{2\deg h + d - 2}$ depends on the number of linearly independent spherical harmonic polynomials of a prescribed degree.

6.6.2. Analytic Extension by Series Expansion. A similar scenario for doubly orthogonal systems (this time of analytic functions) is offered by the so-called embedding or restriction operators. While this is a rich area of continuous research, closely related to sampling and interpolation of analytic functions, we choose a simple, yet representative, example.

Let $\Omega$ be an open set in the complex plane and let $\mu$ denote a positive measure compactly supported on $\Omega$, so that the restriction map from Bergman’s space of square integral analytic function with respect to area measure

$$R : L^2_a(\Omega) \longrightarrow L^2(\mu), \quad Rf = f|_{\text{supp}\mu},$$

is well defined and continuous. Specifically that means that there is a positive constant $C$ with the property

$$\|f\|_{2,\mu} \leq C\|f\|_{2,\Omega}, \quad f \in L^2_a(\Omega).$$

Moreover, Montel’s Theorem implies that the restriction operator $R$ is compact and moreover the eigenvalues of the modulus $R^* R$, denoted $\lambda_n$, decay exponentially, see [13] for the precise statement.

Assume the support of the measure $\mu$ infinite, so that $R$ is also an injective map and let $f_n \in L^2_a(\Omega)$ denote the orthonormal basis of Bergman space formed by the eigenvectors of the positive and compact operator $A = R^* R$: 
In other words
\[
\int_{\Omega} f_n g dA = \frac{1}{\lambda_n} \int f_n g d\mu, \quad n \geq 0, \quad g \in L^2(\Omega).
\]

In particular we infer that \((f_n / \sqrt{\lambda_n})\) is an orthonormal basis of a closed subspace of \(L^2(\mu)\). We adopt the ad-hoc and ambiguous notations \(L^2_a(\mu)\) rather than \(L^2_a(\mu, \Omega)\), for this subspace of \(L^2(\mu)\), the closure of the range of \(R\). One immediate and remarkable consequence of these elementary observations is the characterization of all elements of \(L^2(\mu)\) which analytically extend to \(\Omega\) and are square summable there: namely these are functions \(h \in L^2(\mu)\) satisfying

\[
h = \sum_{n=0}^{\infty} c_n \frac{f_n}{\sqrt{\lambda_n}},
\]

subject to the additional decay condition for the coefficients:

\[
\sum_{n=0}^{\infty} \frac{|c_n|^2}{\lambda_n} < \infty.
\]

This global analytic extension phenomenon, by means of more sophisticated sums than power series has far reaching consequences, see for instance [11]. For the topics of the present article two conclusions are in order.

To relate the above construct to our general setting, we deal in this case with the Bergman space \(L^2(\Omega)\) and the weaker norm of \(L^2_a(\mu)\) induced by the positive and compact operator \(A = R^* R\). Let \(T \in \mathcal{L}(L^2(\Omega))\) be an arbitrary linear and bounded transformation of Bergman space. We can regard \(T : \mathcal{D} \subset L^2_a(\mu) \longrightarrow L^2_a(\mu)\) as a densely defined operator, with closed graph, in the weaker norm. The first observation is that the double orthogonal system of functions \((f_n)\) in \(L^2_a(\Omega)\), respectively \(L^2_a(\mu)\) gives rise to identical finite central truncations of \(T\) with respect to the two norms. Exactly as the basis of monomials behave on concentric disks for Toeplitz or more general operators.

Second, we isolate a class of perturbations of the doubly orthogonal system of functions \((f_n)\) which do not alter too much the finite central truncations of a given operator. The proof of Proposition 6.4 applies line by line with the following result.

**Proposition 6.5.** Let \(\Omega\) be a planar domain and \(\mu\) a positive measure compactly supported by an infinite subset of \(\Omega\). Denote by \((f_n)\) the system of doubly orthogonal functions, in Bergman space \(L^2_a(\Omega)\) and \(L^2(\mu)\). Let \(K \in \mathcal{L}(L^2(\mu), L^2_a(\Omega))\) be a compact operator so that \(I + K\) is invertible on Bergman space. Denote \(g_n = f_n + Kf_n, \ n \geq 0\). For every linear bounded transformation \(T \in \mathcal{L}(L^2_a(\Omega))\) the finite central truncations \(T_n, \widetilde{T}_n\) along the subspaces generated by \(\{g_0, g_1, ..., g_{n-1}\}, \ n \geq 1\), in the Bergman space norm and respectively the weak norm \(L^2(\mu)\), satisfy:
\[ \lim_n \|T_n - \hat{T}_n\|_{\mathcal{L}(L^2(\mu), L^2_a(\Omega))} = 0. \]

The conclusion of the proposition becomes effective for a linear operator \( T \) which is bounded from \( L^2(\mu) \) to \( L^2_a(\Omega) \). Indeed, in this case there exists a positive constant \( M \) with the property:

\[ \|Tf\|_{2,\Omega} \leq M\|f\|_{2,\mu}, \quad f \in L^2(\mu). \]

Within the notation of the proof of Proposition 6.4:

\[ \|\sqrt{A}(\hat{Q}_n - \hat{P}_n)Tg\|_{2,\Omega} \leq M\|\sqrt{A}(\hat{Q}_n - \hat{P}_n)\|_{\mathcal{L}(L^2_\mu(\Omega))}\|\sqrt{A}g\|_{2,\Omega}, \quad g \in \hat{H}_n, \]

hence

\[ \lim_n \|T_n - \hat{T}_n\|_{\mathcal{L}(L^2(\mu))} = 0, \]

with the counting measure asymptotics consequences we outlined at the beginning of this article. Of course this is no surprise as the above “regularizing” assumption turns \( T \) into a compact operator on \( L^2(\mu) \). However, the proof of Proposition 6.4 uses only the assumption \( AP_n = P_nA = P_nAP_n, \quad n \geq 1 \), and this scenario covers way more general situations. We do not expand here the details.

A classical example of double orthogonality is provided by an ellipse and the measure \( \sqrt{1 - x^2}dx \) supported on the interval between the two foci, located at \( \pm 1 \). In this situation Chebyshev polynomials of the second kind \( (U_n)_{n=0}^{\infty} \) are doubly orthogonal, see for instance [11]. Note that all confocal ellipses possess the same system of simultaneous orthogonal polynomials. A converse was proved by Szegö and Walsh, see [11].

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