Examples of Non-Associative Evolution Laws in Thermomechanical Framework for Elastoplastic Damaged Material

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Abstract. The goal of the paper is to outline the major steps of thermomechanical constitutive modelling with emphasis on incorporation of non-associative evolution laws. A brief description of the main assumptions and general equations are given. Then two examples are presented, both resulting in Beltrami-Michell failure condition, but differing in dissipation potential. The first example couples effects of plasticity and damage, while the second separates the phenomena. Some comments are given on basic technicalities in modelling.

1. Introduction

The objective of the work is showing examples of thermodynamically consistent elastoplastic damaged material models incorporating the non-associative evolution laws in order to outline the major steps necessary to achieve all the relations needed for a numerical solution of a problem for small deformations theory. The starting point is a proposal of two independent potentials: an energy potential and a dissipation function [2,8]. If the functions possess desired characteristics, the description of joined elasticity, plasticity and damage is complete based on only those two potentials.

In the classical approach to modelling of elastoplastic damaged material the constitutive relations are assumed *a priori* along with the yield and damage conditions and evolution laws [5]. Often, non-associative flow rules are necessary to correctly depict material behaviour, especially for quasi-brittle materials including some types of soil and concrete. The improper choice of those equations may cause the dissipation to be negative, which contradicts the second law of thermodynamics. Unfortunately, the positivity of the function can be checked *a posteriori*. This inconsistency is absent in the used approach [1,2,8].

The thermodynamically consistent model is by definition a model that allows to fulfil the first and the second laws of thermodynamics [8]. The modelling of such material is widely discussed in the literature[1-4,7-9]. It is based on postulating two potentials: an energy potential and a dissipation function. Depending on the choice of the arguments, the first can be Helmholtz free energy, Gibbs energy, internal energy or enthalpy. Various forms of those functions are well-recognized and similar for a broad class of engineering materials. On the contrary, the dissipation function strongly depends on the type of material and should be selected carefully to fit specific experimental data. The framework puts only two restrictions on those potentials: the restriction of convexity and the...
restriction of non-negativity \([2,5,8]\). On those basis, as two separate spaces, generalised stress tensors and dissipative stress tensors are derived \([1,2]\), tied by the orthogonality hypothesis. The distinction between those spaces makes it possible to incorporate the non-associative evolution laws in a way described below.

The theoretical foundation of the framework employed in this paper is well described, although the details of obtaining particular relationships remain unclear. There are publications showing examples of using the approach, see \([3]\), but the used models are quite complicated, taking into account singular plastic potentials, which combined with singularity of Legendre transformation itself, makes it difficult to understand the technique. In the present work simple functions are used to illustrate the framework. Furthermore, some remarks are made on the introduction of weak and strong coupling of elasticity, plasticity and damage.

2. Thermodynamically consistent model of an elastoplastic damaged material
Herein elastoplastic damaged material is considered in the frame of small deformation theory. The main assumption of thermodynamics is that the state of material is uniquely described when strain tensor \(\varepsilon\), temperature and suitable internal variables are known. In this case the variables are plastic strain tensor \(P\varepsilon\) and damage tensor \(M\). The temperature is constant during the process of loading \([8]\).

In order to fully describe the states of stress and strain in a material two potentials are required: an energy potential and a dissipation potential, both convex and non-negative. The fulfilment of Clausius-Duhem inequality \([8,9]\), i.e. the thermodynamic consistency is ensured by selecting a non-negative dissipation function. Next, the generalised stress tensors are derived from the energy potential and the dissipative stress tensors are obtained using the dissipation. The constitutive equations are then used and through Legendre transformation, the yield and/or damage condition is received. Subsequently, the laws for evolution of the internal variables are derived. Plastic/damage multiplier(s) can be calculated using consistency condition(s).

2.1. Energy potential
As mentioned, the energy potential is selected as one of the four: Helmholtz free energy, Gibbs free energy, internal energy or enthalpy. They depend on different arguments but are closely bound by Legendre transformations \([2,8]\). The choice of the potential is dictated by the type of formulation (stress-based or strain-based). In this paper Helmholtz free energy \(F(\varepsilon,\varepsilon_p, M)\) is used. It is frequently assumed that the energy is a function of the difference of the total strain tensor and the plastic strain tensor, i.e. the elastic strain tensor \(E\varepsilon\):

\[
F(\varepsilon, \varepsilon_p, M) = \tilde{F}(\varepsilon - \varepsilon_p, M) = \tilde{F}(\varepsilon, M).
\]

In a general case of anisotropic damage the form of Helmholtz energy is \([6]\):

\[
F(\varepsilon, \varepsilon_p, M) = \frac{1}{2} \tilde{\lambda} \text{tr}^2(\varepsilon - \varepsilon_p) + 2\tilde{\mu} \text{tr}(\varepsilon - \varepsilon_p)^2 + \frac{1}{2} \left[ (\eta \text{tr}^2(\varepsilon - \varepsilon_p) + \gamma \text{tr}(\varepsilon - \varepsilon_p)^2) \text{tr}M + \right. \\
+ \alpha \text{tr}(\varepsilon - \varepsilon_p) \text{tr}(\varepsilon - \varepsilon_p)^2 \text{tr}M + \chi \text{tr}(\varepsilon - \varepsilon_p)^3 \text{tr}M),
\]

where \(\text{tr}\) denotes the operation of trace, \(\tilde{\lambda}\) and \(\tilde{\mu}\) are Lame’s constants, \(\eta\), \(\gamma\), \(\alpha\) and \(\chi\) are other material parameters.

The generalised stress tensors are defined by the following formulas:

\[
\sigma = \frac{\partial F}{\partial \varepsilon}, \sigma_p = -\frac{\partial F}{\partial \varepsilon_p}, \sigma_M = -\frac{\partial F}{\partial M},
\]

(2)
where \( \sigma \) is the Cauchy stress tensor, \( \sigma_p \) is the generalised plastic stress tensor and \( \sigma_M \) is the generalised damage stress tensor.

2.2. Dissipation

The first and second law of thermodynamics take the form of the Clausius-Duhem inequality \[5,8\], stating that total dissipation in a process is non-negative. For the considered case the inequality is:

\[
D = D(\dot{\varepsilon}_p, \dot{\varepsilon}_p, \dot{\varepsilon}_M) = \sigma_p \cdot \dot{\varepsilon}_p + \sigma_M \cdot \dot{\varepsilon}_M \geq 0, \tag{3}
\]

The dissipation potential has to be non-negative and convex. Additionally, it should be homogenous of degree 1 with respect to the rates of change of the internal variables \[1,2,8,9\]. Then, the dissipative stress tensors are defined as follows:

\[
\overline{\sigma}_p = \frac{\partial D}{\partial \dot{\varepsilon}_p}, \quad \overline{\sigma}_M = \frac{\partial D}{\partial \dot{\varepsilon}_M}, \tag{4}
\]

Also, the following singular Legendre transformation occurs:

\[
D(\dot{\varepsilon}_p, \dot{\varepsilon}_p, \dot{\varepsilon}_M) + \lambda \overline{Y}(\overline{\sigma}_p, \overline{\sigma}_M) = \overline{\sigma}_p \cdot \dot{\varepsilon}_p + \overline{\sigma}_M \cdot \dot{\varepsilon}_M, \tag{5}
\]

connecting yield function \( \overline{Y} \) with the dissipation. As a result the yield/damage condition is given by:

\[
\overline{Y}(\overline{\sigma}_p, \overline{\sigma}_M) = 0. \tag{6}
\]

In case of uncoupled plasticity and damage there are two separate Legendre transformations concerning parts of the dissipation potential responsible for the phenomena and subsequently separate damage and yield conditions are obtained \[3\]. The details are shown in the second example.

The relationships revealing evolution of internal variables are as follows:

\[
\dot{\varepsilon}_p = \lambda \frac{\partial \overline{Y}}{\partial \overline{\sigma}_p}, \quad \dot{\varepsilon}_M = \lambda \frac{\partial \overline{Y}}{\partial \overline{\sigma}_M}, \tag{7}
\]

where \( \lambda \) is a multiplier obtained from consistency equation, i.e. \( \overline{Y}(\overline{\sigma}_p, \overline{\sigma}_M) = 0 \). Again, if the model is uncoupled, there are two multipliers determined by separate consistency conditions.

2.3. Orthogonality hypothesis

Combination of definitions (3) and (4), using Euler theorem for homogenous functions results in the following:

\[
(\overline{\sigma}_p - \sigma_p) \cdot \dot{\varepsilon}_p + (\overline{\sigma}_M - \sigma_M) \cdot \dot{\varepsilon}_M = 0. \tag{8}
\]

Relationship (8) is always true if generalised and dissipative stress tensors are equal, i.e.:

\[
\overline{\sigma}_p = \sigma_p \quad \text{and} \quad \overline{\sigma}_M = \sigma_M. \tag{9}
\]
which is called the orthogonality hypothesis [2,3]. Equation (9) allows to connect yield/damage condition (6) and evolution equations (7) with \( \sigma \) or \( \varepsilon \).

### 3. Coupled elastoplastic-damage model based on Beltrami-Michell yield condition

In this section a coupled elastoplastic-damage model with Beltrami-Michell condition is presented. Two potentials: Helmholtz free energy and dissipation are proposed resulting in one damage/yield condition.

The Helmholtz free energy is assumed as:

\[
F(\varepsilon, \varepsilon_p, d) = (1 - d) \left( \frac{1}{2} K \text{tr}^2 (\varepsilon - \varepsilon_p) + G \text{tr} (\varepsilon - \varepsilon_p)^2 \right), \tag{10}
\]

where \( \varepsilon \) and \( \varepsilon_p \) are deviatoric parts of the total strain tensor and the plastic strain tensor, respectively. \( 0 \leq d \leq 1 \) is a damage parameter, a scalar internal variable interpreted as a ratio of the actual cross-section area (decreased during the loading process) to the initial area [5, 6]. Isotropic damage is considered, i.e. \( \mathbf{M} = d \mathbf{I} \), \( \mathbf{I} \) being second order unit tensor, compare equation (1). \( K \) is the bulk modulus and \( G \) is the shear modulus.

The dissipation is of the following form, compare [4,7]:

\[
D(\dot{\varepsilon}_p, \dot{d}) = \sqrt{\kappa' (\sigma_p) \left[ R_p^2 \left( \frac{A^2}{\alpha} p^2 + \frac{B^2}{\beta} q^2 \right) + \left( \frac{R_d}{k(\varepsilon, \varepsilon_p, d)} \right)^2 \right]}, \tag{11}
\]

where \( p \) and \( q \) are the rates of change of the isotropic and deviatoric parts of the plastic strain tensor respectively, i.e. \( p = (\text{tr} \dot{\varepsilon}_p) / \sqrt{3}, \ q = \sqrt{\text{tr} \dot{\varepsilon}_p^2} \). \( A, B, \alpha \) and \( \beta \) are material parameters and \( \alpha, \beta \in (0,1) \). \( R_p, R_d > 1 \) are constants measuring the coupling between damage and plasticity, connected by the relation [3]:

\[
\frac{1}{R_p^2} + \frac{1}{R_d^2} = 1, \tag{12}
\]

The coupling between damage and plasticity is introduced by the use of the square root in the dissipation potential, but also through the employment of two functions, namely \( k \) and \( \kappa' \). The first is of the form:

\[
k(\varepsilon, \varepsilon_p, d) = \frac{(1 - d)^2}{F(\varepsilon, \varepsilon_p, d)} \sqrt{\frac{3K^2}{A^2} \text{tr}^2 (\varepsilon - \varepsilon_p) + \frac{4G^2}{B^2} \text{tr} (\varepsilon - \varepsilon_p)^2}. \tag{13}
\]

The above expression was chosen to obtain a desired expression for the yield/damage condition, that is Beltrami-Michell condition. The argument of the second function is the generalised plastic stress tensor:

\[
\kappa' (\sigma_p) = 1 - \frac{1}{R_p^2} \left( \frac{1 - \alpha}{A^2} \sigma_p^2 + \frac{1 - \beta}{B^2} \sigma_p^2 \right), \tag{14}
\]
where $\xi_p = (\text{tr} \sigma_p) / \sqrt{3}$ and $r_p = \sqrt{\text{tr} s_p^2}$ with $s_p = \sigma_p - \left(\xi_p / \sqrt{3}\right) I$. The choice of the arguments of $\kappa$ allows to introduce the non-associative evolution laws by a careful derivation described by (7), shown in the subsequent part of the paper. The selection of the independent variables is arbitrary with the exception of the dissipative stress tensors. Proposed dissipation (11) is always non-negative and convex for $p$, $q$ and $d$.

The next step, after proposing the potentials, is determining the generalised stress tensors using equation (2):

$$\sigma = \frac{\partial F}{\partial \varepsilon} = (1-d) \left( K \text{tr} (\varepsilon - \varepsilon_p) I + 2G (\varepsilon - \varepsilon_p) \right), \quad \sigma_p = -\frac{\partial F}{\partial \varepsilon_p} = \sigma,$$

$$\sigma_g = -\frac{\partial F}{\partial d} = \frac{1}{2} K \text{tr}^2 (\varepsilon - \varepsilon_p) + G \text{tr} (\varepsilon - \varepsilon_p)^2. \quad (15)$$

The nominal stress tensor $\sigma$ is connected to both total and plastic strain tensors. The variable conjugated with the damage parameter is the elastic strain energy for an undamaged material. Equations (15) result in the following relations describing volumetric and distortional deformation:

$$\text{tr} (\varepsilon - \varepsilon_p) = \frac{\xi}{\sqrt{3}(1-d)K} = \frac{\xi_p}{\sqrt{3}(1-d)K}, \quad \varepsilon - \varepsilon_p = \frac{1}{2(1-d)G} s = \frac{1}{2(1-d)G} s_p. \quad (16)$$

Now, the dissipative stress tensors have to be determined employing equations (4):

$$\bar{\sigma}_p = \frac{\partial D}{\partial \bar{\varepsilon}_p} = \frac{\kappa R_p^2}{D} \left( A^2 p + B^2 \bar{\varepsilon}_p \right), \quad \bar{\sigma}_g = \frac{\partial D}{\partial \bar{d}} = \frac{\kappa^2 R_p^2}{D k^2 d}. \quad (17)$$

If not necessary, the arguments of the functions will be omitted for the sake of clarity.

The resultant failure condition is obtained by adding up scalar products of the isotropic and the deviatoric parts of $\bar{\sigma}_p$ and squared $\bar{\sigma}_d$:

$$\frac{\alpha}{\kappa R_p^2 A} \left( \frac{1}{3} \text{tr} (\bar{\sigma}_p) I \right) \left( \frac{1}{3} \text{tr} (\bar{\sigma}_p) I \right) + \frac{\beta}{\kappa R_p^2 B^2} S_p \cdot S_p + \frac{k^2}{\kappa R_d} \bar{\sigma}_d^2 = 1. \quad (18)$$

Substituting $\xi_p = (\text{tr} \sigma_p) / \sqrt{3}$ and $r_p = \sqrt{\text{tr} s_p^2}$, $\bar{S}_p$ being the deviator: $\bar{S}_p = \bar{\sigma}_p - \left(\xi / \sqrt{3}\right) I$, the following failure condition is received:

$$\gamma (\bar{\sigma}_p, \bar{\sigma}_d) = \frac{1}{R_p^2} \left( \frac{\alpha}{A^2} \xi_p + \frac{\beta}{B^2} r_p \right)^2 + \frac{k^2}{R_d} \bar{\sigma}_d^2 - \kappa = 0. \quad (19)$$

Evolution laws (7) become:

$$\dot{\varepsilon}_p = \frac{\lambda}{R_p^2} \left( \frac{2\alpha}{A^2} \xi_p + \frac{2\beta}{B^2} r_p \right), \quad \dot{d} = \frac{\lambda}{R_d^2} \left( \frac{2\alpha}{A^2} \xi_p + \frac{2\beta}{B^2} r_p \right). \quad (20)$$
It needs to be emphasized that \( \kappa \) as a function of the generalized stress tensor is not subjected to differentiation, which, in turn, allows to obtain evolution laws associated in the space of dissipative stresses but non-associated in the generalized stress space. Only after those steps, the orthogonality hypothesis (9) can be used in the following form [2,9]:

\[
\overline{\sigma}_p = \sigma_p \quad \text{and} \quad \overline{\sigma}_d = \sigma_d,
\]

(21)

At this point, failure condition (19) and evolution laws (20) can be expressed via the generalized plastic stress tensor and the damage parameter. Using the definitions of \( \kappa \) and \( \kappa \), i.e. equations (13) and (14), it is obtained:

\[
\begin{align*}
Y^*(\sigma_p, \sigma_d) &= \frac{1}{R_p^2} \left( \frac{\varepsilon_p^2}{A^2} + \frac{\kappa^2}{B^2} \right) + \frac{k^2}{R_p^2} \sigma_d^2 - 1 = 0, \\
\dot{\varepsilon}_p &= \lambda \left( \frac{2\alpha}{\sqrt{3}A^2} \varepsilon^2 + \frac{2\beta}{B^2} s_p \right), \\
\dot{d} &= \frac{2\lambda k^2}{R_d^2} \sigma_d.
\end{align*}
\]

(22)

(23)

Condition (22) allows for function \( \kappa \) to be always non-negative, which ensures the non-negativeness of the dissipation potential, hence fulfilment of the second law of thermodynamics. Equations (23) state that as \( R_p \to 1 \), \( 1/R_d^2 \to 0 \) and \( d \to 0 \), thus the behaviour of the material becomes purely elastoplastic. On the contrary, if \( R_p \to 1 \) then \( 1/R_d^2 \to 0 \) and \( \varepsilon_p \to 0 \), after reaching a certain level of stress indicated by equation (22), the deterioration of the elastic properties begins with no plastic flow. Parameters \( \alpha \) and \( \beta \) describe the degree of departure of the direction of plastic flow from gradient of the yield surface. If \( \alpha = \beta = 1 \) then \( \kappa(\sigma) = 1 \) and the evolution law for \( \varepsilon_p \) is associated with failure condition (22). \( \alpha < 1 \) is responsible for the deviation of the isotropic part of \( \varepsilon_p \) from the gradient of the failure surface, while \( \beta < 1 \) accounts for modification of the deviatory part’s direction. Regardless, the evolution law of the damage parameter is always associative.

It is convenient to express the failure condition and evolution laws via the nominal stress tensor and the damage parameter using equations (15) as follows:

\[
\begin{align*}
Y(\sigma) &= \frac{\varepsilon_p^2}{A^2} + \frac{\kappa^2}{B^2} - 1 = 0, \\
\dot{\varepsilon}_p &= \lambda \left( \frac{2\alpha}{\sqrt{3}A^2} \varepsilon^2 + \frac{2\beta}{B^2} s_p \right), \\
\dot{d} &= \frac{2\lambda k^2}{R_d^2} \sigma_d,
\end{align*}
\]

(24)

(25)

Multiplier \( \lambda \) can be found using persistency condition:

\[
\dot{Y}(\overline{\sigma}_p, \overline{\sigma}_M) = 0 \quad \text{or (equivalently)} \quad \dot{Y}(\sigma) = 0,
\]

(26)

3.1. Uniaxial loading test

Now, the case of uniaxial tension test is considered in order to show the influence of parameters on the results – the components of the plastic strain tensor and the damage parameter. It is assumed that:
Parameters $A$ and $B$ are obtained from failure condition (24) assuming that the failure surface passes through points representing the uniaxial tension test and the pure shear test. The result is:

$$A = \frac{\sigma_0 \sigma_v}{\sqrt{3} \sigma_v^2 - \sigma_0^2} \quad \text{and} \quad B = \sqrt{2} \sigma_v \quad \text{for} \quad \sqrt{3} \sigma_v > \sigma_0,$$

where $\sigma_0$ is the plastic limit for the uniaxial test and $\sigma_v$ marks the initiation of plastic flow and damage for the pure shear test. Hardening is omitted for the purpose of clarity.

The yield/damage condition is reduced to the following:

$$Y(\sigma) = \frac{\sigma^2}{\sigma_0} - 1 = 0,$$

and evolution laws (25) become:

$$\dot{\varepsilon}_p = \frac{2\sigma_0}{3R_p} \left( \frac{\alpha}{A^2} + \frac{2\beta}{B^2} \right) \lambda, \quad \dot{\varepsilon}_{PB} = \frac{2\sigma_0}{3R_p} \left( \frac{\alpha}{A^2} - \frac{\beta}{B^2} \right) \lambda, \quad \dot{d} = \frac{2E}{\sigma_0^2 R_d} (1 - d)^2 \lambda,$$  

Consistency condition (26) for the case states that:

$$\sigma = 0 \Rightarrow (1 - d)E(\dot{\varepsilon}_p - \dot{\varepsilon}_P) - dE(\varepsilon - \varepsilon_P) = 0,$$

where $E$ is the Young modulus.

Elimination of $\lambda$ by dividing (30)$_1$ and (30)$_3$ and using (31) results in the following set of differential equations:

$$\frac{\dot{d}(t)}{1-d(t)} = \gamma \dot{\varepsilon}_p(t), \quad \text{where} \quad \gamma = \text{const} = \frac{6E}{\sigma_0^2 \left( \frac{\alpha}{A^2} + \frac{2\beta}{B^2} \right) R_p^2} \quad \text{and}$$

$$\left[ \gamma \{1 - d(t)\} \varepsilon(t) + 1 \right] \dot{\varepsilon}_p(t) - \gamma \{1 - d(t)\} \varepsilon_p(t) \dot{\varepsilon}_p(t) - \dot{\varepsilon}(t) = 0,$$

which is complemented by the initial conditions: $\varepsilon_p(t_0) = 0$ and $d(t_0) = 0$. $t_0$ is an instant of the outset of plasticity and damage. The straining program is assumed as $\varepsilon(t) = \alpha t$. Then, the lateral component of the plastic strain tensor is calculated using equations (30) with $\varepsilon_{PB}(t_0) = 0$. As a result the following functions are found:
\[ \varepsilon_p(t) = \frac{a(t-t_0)}{1+\gamma at}, \quad \varepsilon_{vb}(t) = \frac{\alpha - \beta}{\alpha + \frac{2\beta}{A}} \varepsilon_p(t), \quad d(t) = \frac{a \gamma (t-t_0)}{1+\gamma at}. \] (33)

Considering the process of loading for linear input strain, the plastic strain responses are also linear with respect to time and the damage parameter grows nonlinearly with the horizontal asymptote of value 1 (fully damaged material).

Diagrams are obtained for the following values: \( \sigma_0 = 300 \text{ MPa} \), \( \sigma_v = 200 \text{ MPa} \), \( E = 200 \text{ GPa} \) and the rate of change of the longitudinal strain component is \( a = 0.001/\text{s}^2 \). It is calculated: \( A = 346.4 \text{ MPa} \), \( B = 282.8 \text{ MPa} \) and \( t_0 = 1.5 \text{s} \).

On figure 1 the processes of loading complemented by unloading program for various values of \( R_p \) and \( \alpha = \beta = 1 \) (associative evolution laws) are shown to illustrate the possibility of controlling the coupling between plasticity and damage. For \( R_p \to 1 \) the behaviour of the material is purely elastoplastic. As the value of \( R_p \) increases, the damage growth starts influencing the results, causing degradation of the elastic properties seen as a drop in value of the effective Young modulus (the decrease of the slopes of the unloading paths). The limit case is \( R_p \to \infty \) when plasticity is absent and the decline of the slope causes the material to return to the starting point after every cycle of loading.

To capture the influence of parameter \( \alpha \), diagrams are presented for fixed \( \beta = 1 \) and \( R_p = R_v = \sqrt{2} \). As shown on figure 2, its increase causes the increase in the longitudinal strain component and the drop in value of the lateral strain component. The impact of \( \alpha \) on the damage parameter is negligible. Figure 3 presents the components of plastic strain and the damage parameter for different values of \( \beta \) for \( \alpha = 1 \) and \( R_p = R_v = \sqrt{2} \). Again, the influence of change of \( \beta \) on the damage parameter is slight. However, considerably greater differences are obtained for the plastic strain components when modifying the values of \( \beta \) than for altering \( \alpha \). Also, there exists a limit value \( \beta_0 \) that is associated with null lateral plastic strain. Above \( \beta_0 \), the lateral plastic strain component is positive for tension (although total lateral strain, i.e. \( \varepsilon_b - \varepsilon_{vb} \), remains negative).

![Figure 1](image1.png)  
**Figure 1** Loading and unloading process for uniaxial loading for various values of \( R_p \).
4. Uncoupled elastoplastic-damage Beltrami-Michell model

In the previous section a fully coupled material model was presented, while in this section the failure conditions and the evolution laws are derived for uncoupled damage and plasticity phenomena. The lack of interaction is accomplished through the dissipation potential. It is a sum of two separate parts – the first dependent only on the degree of damage and the second connected to plastic flow [7]. However, as the Helmholtz energy is described by equation (10), depending on the total strain tensor, the plastic strain tensor and the damage parameter, therefore still indirect coupling between the phenomena exists.

In the case of uncoupled model, the dissipation potential is assumed as follows:

\[ D(\mathbf{\epsilon}_p, \dot{d}) = D_p(\mathbf{\epsilon}_p) + D_d(\dot{d}) = \kappa(\sigma_p) \left( \frac{A^2}{\alpha^2} p^2 + \frac{B^2}{\beta^2} q^2 \right) + h(d) \dot{d}, \]  \hspace{1cm} (34)

where \( h(d) \) is a positive increasing function of the damage parameter and:

\[ \kappa(\sigma_p) = 1 - \left( \frac{1 - \alpha}{A^2} \xi_p^2 + \frac{1 - \beta}{B^2} \eta_p^2 \right), \]  \hspace{1cm} (35)

so \( D \) is convex and non-negative.

Equations (15) and (16) hold, while the dissipative stresses are the following:
As a consequence of the assumed split of the dissipation potential, two separate conditions for damage and yielding are received. The manner of obtaining the conditions is similar as in section 3. Adding up scalar products of the isotropic and the deviatoric parts of the dissipative plastic stress tensor results in:

\[
\frac{\alpha}{\kappa A^2} \left( \frac{1}{3} \text{tr} (\mathbf{\bar{\sigma}}_p) \mathbf{I} \right) + \frac{\beta}{\kappa B^2} \mathbf{\bar{\xi}}_p \cdot \mathbf{\bar{\xi}}_p = 1 = 0 \quad \text{and} \quad \mathbf{\bar{\sigma}}_d - h(d) = 0. \tag{36}
\]

Substituting \( \bar{\xi} \) and \( \bar{\tau} \), the following yield and damage conditions are obtained:

\[
\bar{Y}_p (\mathbf{\bar{\sigma}}_p) = \frac{\alpha}{A^2} \bar{\xi}^2 + \frac{\beta}{B^2} \bar{\tau}^2 - \kappa = 0 \quad \text{and} \quad \bar{Y}_d (\mathbf{\bar{\sigma}}_d) = \mathbf{\bar{\sigma}}_d - h(d) = 0. \tag{37}
\]

The yield functions are bound with the suitable parts of the dissipation potential by the transformations:

\[
D_p (\mathbf{\bar{\epsilon}}_p) + \lambda_p \bar{Y}_p (\mathbf{\bar{\sigma}}_p) = \mathbf{\bar{\sigma}}_p \cdot \mathbf{\bar{\epsilon}}_p \quad \text{and} \quad D_d (\mathbf{\bar{\delta}}) + \lambda_d \bar{Y}_d (\mathbf{\bar{\sigma}}_d) = \mathbf{\bar{\sigma}}_d \mathbf{\bar{\delta}}. \tag{38}
\]

The evolution laws are as follows:

\[
\dot{\mathbf{\bar{\epsilon}}}_p = \lambda_p \frac{\partial \bar{Y}_p}{\partial \mathbf{\bar{\sigma}}_p} = 2 \lambda_p \left( \frac{\alpha}{\sqrt{3} A^2} \bar{\xi} \mathbf{I} + \frac{\beta}{B^2} \mathbf{\bar{\xi}}_p \right), \quad \dot{\mathbf{\bar{\delta}}} = \lambda_d \frac{\partial \bar{Y}_d}{\partial \mathbf{\bar{\sigma}}_d} = \lambda_d. \tag{39}
\]

Introducing orthogonality hypothesis (21), the yield and damage conditions and the evolution laws expressed via the nominal stress tensor and the damage parameter become:

\[
Y_p (\mathbf{\sigma}) = \frac{\xi^2}{A^2} + \frac{\tau^2}{B^2} - 1 = 0 \quad \text{and} \quad Y_d (\mathbf{\sigma}, d) = \frac{1}{(1-d)^2} \left( \frac{\xi^2}{6K} + \frac{\tau^2}{4G} \right) - h(d) = 0. \tag{40}
\]

\[
\dot{\mathbf{\bar{\epsilon}}}_p = 2 \lambda_p \left( \frac{\alpha}{\sqrt{3} A^2} \xi \mathbf{I} + \frac{\beta}{B^2} \mathbf{\bar{s}} \right), \quad \dot{\mathbf{\bar{\delta}}} = \lambda_d. \tag{41}
\]

The relations given above are complemented by the following consistency equations:

\[
\dot{Y}_p (\mathbf{\sigma}) = 0 \quad \text{and} \quad \dot{Y}_d (\mathbf{\sigma}, d) = 0. \tag{42}
\]

5. Summary
The considered framework is a powerful tool allowing to ensure the fulfilment of the laws of thermodynamics. The general governing equations and basic assumptions are clear and well described in the technical literature [2,3,8,9]. The great advantage of the framework is that it is sufficient to deliver two potentials, non-negative and convex, in order to determine all the other relationships.
The coupling between plasticity and damage is governed by choosing an appropriate form of the dissipation potential. Naturally, if the potential is a sum of two functions, the first of which involving only damage internal variable(s) and the second dependent on the plastic internal variable(s) exclusively, then the model is called uncoupled. In fact, an indirect connection between plasticity and damage exists because of the universally accepted forms of the energy potentials. Coupling between the phenomena can be included in the form of the dissipation potential, using any non-negative, convex and homogenous function of the rates of change of internal variables for example the square root, or, not straightforwardly, in the definition of material parameters that change along with the evolution of damage or plastic strain.

As shown above, incorporating the non-associative evolution laws is achieved by a proper choice of arguments of the functions involved, followed by a careful differentiation. Evolution laws expressed via the dissipative stresses are associated with the adequate failure condition. Then, the orthogonality principle allows to change the considered stress space to this connected to the generalized stress tensors, which in turn generates the non-associative evolution laws. This feature is especially desired when modelling quasi-brittle materials, in particular concrete. The models with arbitrary chosen evolution laws require an additive calculation check after every simulation to prove the thermodynamic consistency. The considered approach removes this inconvenience and as such is worth of special attention.

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