Control in Probability for SDE Models of Growth Population

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Abstract
In this paper, we consider a (control) optimization problem, which involves a stochastic dynamic. The model proposes selecting the best control function that keeps bounded a stochastic process over an interval of time with a high probability level. Here, the stochastic process is governed by a stochastic differential equation affected by a stochastic process. This setting becomes a chance-constrained control optimization problem, where the constraint is given by the probability level of infinitely many random inequalities. Since such a model is challenging, we discretize the dynamic and restrict the space of control functions to piecewise mappings. On the one hand, it transforms the infinite-dimensional optimization problem into a finite-dimensional one. On the other hand, it allows us to provide the well-posedness of the problem and approximation. Finally, the results are illustrated with numerical results, where classical model for the growth of a population are considered.

Keywords Growth population models · Chance constrained optimization · Control in Probability

Mathematics Subject Classification 92-10 · 49N99 · 34H05 · 90C15 · 92B05

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1 Introduction

Models for the growth of a population have played an essential role in multiple fields long before the seminal work of T. Malthus (1798) in dynamical systems. Starting from simple Fibonacci sequence to Lotka-Volterra systems, epidemics SIR models, to more modern game theory works, we have advanced in the better understanding of the mechanisms of biological replication, ecological network stability properties and many other biological applications [3, 4]. For instance, SIR and all its extensions have been a critical element in predicting the behavior of the COVID-19 pandemic and have helped policymakers to understand the importance of data and modeling in the modern world [18]. Using classical dynamical systems and control theory has been a very prolific field in terms of not only theoretical but also applications (see e.g. [1, 12]). However, biological systems are intrinsically noisy, depending enormously on environmental conditions that cannot always be precisely determined. Even in controlled environments, models have to take into consideration small fluctuations in variables such as temperature or pressure [8] or directly consider an open system approach.

Stochastic programming emerges as a suitable tool for making decisions under noisy or random phenomena, which could affect the data and solution of the optimization problem. Consequently, the choice of the optimal point should be taken before the observation. It has been well-recognized as an important area in mathematical programming and operational research with plenty of real-world applications (see, e.g., [2, 13, 19]). The use of expectation and risk measures as objective functions or constraints provides machinery to model and handle the uncertainty in the decision-making process. Notably, in recent years, optimization problems under probabilistic constraints or chance-constrained optimization problems have captured the attention of several researchers, which has the intention of model problems where the uncertainty affects the restrictions of the problem. Probabilistic constraints offer a middle point between robustness and feasibility of the constraint (nonemptiness of the feasible set) because in this case, the requirements should be satisfied not for everyone, not for the average but for an event with high probability. Here, it is necessary to mention that during the last years, many authors have made an effort to incorporate probabilistic constraints in the area of control of systems of differential equations (see, e.g., [5–7]). It has motivated the study of new classes of probability functions in finite and infinite dimensional settings (see, e.g., [10, 23, 27] and the references therein).

In this paper, we are interested in studying a probabilistic approach to control a (stochastic) dynamical system. Instinctively, our desire is to find “the minimal cost” control $u$ which allows the dynamic $y_{u, \xi}(t)$ being lower than a quantity $\bar{b}$ for all $t \in [t_0, T]$. For instance, the dynamic $y_{u, \xi}$ might represent the biomass of a grasshopper living in a crop field or some parasite in a salmon farm context. Our desire is then to have levels of infection smaller than an upper regulatory safety bound $\bar{b}$. Since the intention is not to eliminate the host population if the threshold is attained, we only focus on having a good control (e.g., use of pesticides) that ensures that we do not cross the upper safety bound with high probability. The classical approach will consider the problem of always having (independently of the possible values of the stochastic process) $y_{u, \xi} < \bar{b}$, which turns out to be a very restrictive scenario for real-world applications. In our setting we relax this condition as follows.
Let \( y_{u,\xi}(t) = Y_{u,\xi}(t, \omega) \) represent the solution of a (random) dynamical system
\[
Y_{u,\xi}'(t) = \sigma(t, u(t), \xi_i(\omega)) f(t, Y_{u,\xi}(t), u(t)), \quad t \in [t_0, T], \quad Y_{u,\xi}(t_0) = y_0,
\] (1)
where \( \xi_i \) represents an “unknown” stochastic process and \( u : [t_0, T] \to \mathbb{R}^s \) is the control function. As a risk-averse formulation, our desire control problem corresponds to a chance-constrained control problem given in the form of
\[
\min \psi(u) \\
\text{s.t. } \varphi(u) \geq p, \\
u \in C.
\] (2)

where \( \psi \) is an objective function, \( C \) the set of admissible controls, and the probability function \( \varphi \) is given by
\[
\varphi(u) := \mathbb{P}\{Y_{u,\xi}(t) \leq \bar{b}, \text{ for all } t \in [t_0, T]\},
\] (3)
\( Y_{u,\xi}(\cdot) \) is the “unique” solution of the System (1), and \( p, \bar{b} \) are user-defined lower and upper bound safety levels. Nevertheless, the optimization problem (2) is far from a straightforward mathematical programming problem. For example, the first step in the well-posedness of (2) is the correct choice of the space of controls which should be large enough to have the existence of a solution to (2), but at the same time, the controls should have some regularity (continuity, differentiability, etc.) over the data to preserve the existence and uniqueness of (1).

In order to tackle Problem (2) we propose a semi-discretization of the dynamical system (1). Formally, we consider a partition of the interval \([t_0, T]\), let us say \( t_0 < t_1 < t_2 < \cdots < t_N = T \) and the space of controls \( u(\cdot) \) is considered as piecewise constant functions on \([t_i, t_{i+1}]\), and we identify this space of functions with \( \mathbb{R}^n = (\mathbb{R}^s)^N \) and we simply denote \( u = (u_1, \ldots, u_N) \) with each \( u_i \in \mathbb{R}^s \).

Therefore, given \( \omega \in \Omega \), we end up with a sequence of nested dynamical systems for \( i = 1, \ldots, N \):
\[
y_{i,\xi_i}'(t_i, u_i, \xi_i(t)) = \sigma_i(u_i, \xi_i(\omega)) f_i(y_{i,\xi_i}(t), u_i), \quad t \in [t_{i-1}, t_i], \quad (DS_i)
\]
\[
y_{i,\xi_i}(t_i) = y_{i-1,u_{i-1,\xi_{i-1}}}(t_{i-1}),
\]
where \( \sigma_i(\cdot, \cdot), f_i(\cdot, \cdot), u_i \) and \( \xi_i \) are approximations of \( \sigma(t, \cdot, \cdot), f(t, \cdot, \cdot), u(t) \) and \( \xi(t) \) over time interval \([t_{i-1}, t_i]\), respectively. Our desire is to study the following chance-constrained control problem
\[
\min \psi_N(u_1, \ldots, u_N) \\
\text{s.t. } \varphi(u_1, \ldots, u_N) \geq p, \\
(u_1, \ldots, u_N) \in C.
\] (4)
where \( \psi_N : \mathbb{R}^n \to \mathbb{R} \) is an objective function, \( C \subseteq \mathbb{R}^n \) is a nonempty closed set and the probability function \( \varphi : \mathbb{R}^n \to \mathbb{R} \) is given by
\[
\varphi(u) := \mathbb{P}\left( y_{u, \xi} (t) \leq \bar{b}, \text{ for all } t \in [t_0, T] \right),
\]
where \( y_{u, \xi} (\cdot) \) is formally defined by
\[
y_{u, \xi} (t) := y_{i, u_i, \xi_i} (t) \text{ if } t \in [t_{i-1}, t_i),
\]
with \( y_{i, u_i, \xi_i} (t) \) on \( [t_{i-1}, t_i] \) given by the “unique” solution of the dynamical system \( (DS_i) \) on the interval \( [t_{i-1}, t_i] \) with initial condition \( y_{i-1, u_{i-1}, \xi_{i-1}} (t_{i-1}) \). For simplicity of the calculations, we are assuming that we know exactly the initial condition value, meaning that \( y_0 \in \mathbb{R} \) is fixed and taken independently of \( u \) and \( \xi(t) \).

The rest of the paper is organized as follows: In Sect. 2 we review some definitions and notations. In Sect. 3 we start by reviewing some preliminary concepts on the well-posedness properties of the dynamical system considered, along with the conditions needed on the parameters for having consistency between \( (DS_i) \) and \( (1) \). We also finish that section with a revision on the most classical first order ODE models for population growth. In Sect. 4 we study the properties of the semi-probabilistic optimization problem, including geometrical properties, regularity and optimality conditions. Finally, in Sect. 5 we present a logistic growth model and by the use of the empirical mean over several simulations we approximate the probabilistic function \( \varphi \) finding how non trivial control functions that improve the full control system must be used in a classical approach.

### 2 Preliminary and Notation

In this section, we recall some notations and concepts from variational analysis and optimization. In the rest of the paper, we denote \( \mathbb{R}^n \) the \( n \)-dimensional Euclidean space equipped with the Euclidean norm, denoted by \( \| \cdot \|_n \). We omit the subindex \( n \) when there is no possible confusion. The zero vector in \( \mathbb{R}^n \) is denoted by \( 0_n \).

Given a set \( C \subseteq \mathbb{R}^n \), we say that \( C \) is convex if
\[
\lambda u_1 + (1 - \lambda) u_2 \in C \text{ for every } \lambda \in [0, 1] \text{ and all } u_1, u_2 \in C.
\]

Let us consider \( f : C \to \mathbb{R} \), defined on a convex set \( C \subseteq \mathbb{R}^n \). We say that \( f \) is convex on \( C \) (only convex if there is no confusion) provided that
\[
f(\lambda u_1 + (1 - \lambda) u_2) \leq \lambda f(u_1) + (1 - \lambda) f(u_2),
\]
for every \( \lambda \in [0, 1] \) and all \( u_1, u_2 \in C \). In addition if the last inequality holds strictly for all \( \lambda \in (0, 1) \) and \( u_1 \neq u_2 \), then the function \( f \) is said to be strictly convex on \( C \). The function \( f \) is said to be quasiconvex if for every \( \alpha \in \mathbb{R} \) the sublevel \( \{ x \in C : f(x) \leq \alpha \} \) is convex, and the function \( f \) is log-convex if \( \log \circ f \) is a convex
function. Finally, the function $f$ is said to be concave (quasiconcave or log-concave, respectively) if $-f$ is convex (quasiconvex or log-convex, respectively).

Given a point $u \in \mathbb{R}^n$, we say that $f$ is lower semicontinuous at $u$ if for any sequence $u_j \rightarrow u$, $\lim \inf_{j \rightarrow +\infty} f(u_j) \geq f(u)$. The function $f$ is upper semicontinuous at $u$ if $\lim \sup_{j \rightarrow +\infty} f(u_j) \leq f(u)$, for any sequence $u_j \rightarrow u$.

For a point $u \in \mathbb{R}^n$ and $\epsilon > 0$ the closed ball centered at $u$ with radius $\epsilon$ is denoted by

$$B_{\epsilon}(u) := \{w \in \mathbb{R}^n : \|w - u\| \leq \epsilon\}.$$

Now, we recall some notations of normal vectors to nonconvex sets. Given a closed set $C$ and a point $u \in C$, we define the regular normal cone by

$$\hat{N}(u, C) := \left\{ u^* \in \mathbb{R}^n : \lim \sup_{v \rightarrow C, \text{\tiny v} \rightarrow u} \langle u^*, v - u \rangle \|v - u\| \leq 0 \right\},$$

where $v \overset{C}{\rightarrow} u$ means $v \rightarrow u$ with $v \in C$. For convenience, we define $\hat{N}(u, C) = \emptyset$ for a point $u \notin C$. A much robust object is the basic normal cone, which is given as the Painlevé-Kuratowski upper limit of the regular vectors. Formally, given a point $u \in C$ we define the basic normal cone as

$$N(u, C) := \{ u^* \in \mathbb{R}^n : \exists u_n \rightarrow u \text{ and } u^*_n \in \hat{N}(u_n, C) \text{ such that } u^*_n \rightarrow u^* \}.$$

If $u \notin C$, we set $N(u, C) = \emptyset$. It is important to mentioning that when the set $C$ is closed and convex both notions, the regular and normal cone, coincide with the classical normal cone of convex analysis, that is,

$$\hat{N}(u, C) = N(u, C) = \{ u^* \in \mathbb{R}^n : \langle u^*, v - u \rangle \leq 0, \text{ for all } v \in C \}.$$

### 3 Well-Posedness of the Problem

In this section, we focus our attention in the study of the probability functions $\varphi$ defined in (4) by the piecewise solution (6) of the dynamical systems $(DS_i)$. To simplify the notation, we relabel and omit the index notation of the control parameter $u$ when no confusion is possible. We start by showing the well-posedness of Eqs. (1) and $(DS_i)$ and the consistency between the simplification and the general process as $N \rightarrow \infty$.

#### 3.1 Existence for the General Equation (1)

For a compact set $[t_0, T] \subset \mathbb{R}_+$, let $\mathcal{L}^1([t_0, T])$ be the space of real-valued stochastic processes $X(t)$ such that
\[ \mathbb{P} \left( \int_{t_0}^{T} |X(t)| \, dt < \infty \right) = 1. \]

**Lemma 1** Let \( \xi(t, \omega) \) be a separable measurable stochastic process with values in \( \mathbb{R}^m \). Define \( G : \mathbb{R}_+ \times \mathbb{R}^s \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R} \) as

\[ G(t, u, y, z) := \sigma(t, u, z) f(t, y, u), \quad (7) \]

and assume the following:

(a) There exists a stochastic process \( B(u(t), t, \omega) \in L^1([t_0, T]) \) such that for all \( x, y \in \mathbb{R} \):

\[ |G(t, u(t), x, \xi(t, \omega)) - G(t, u(t), y, \xi(t, \omega))| \leq B(u(t), t, \omega) |x - y|. \] (H1)

(b) For any positive constant \( R \), there exists a stochastic process \( M_R(t, \omega) \) such that

\[ |G(t, u(t), y, \xi(t, \omega))| \leq M_R(t, \omega) \in L^1([t_0, T]), \quad \forall |y| < R. \] (H2)

Then, the equation

\[ Y'(t) = \sigma(t, u(t), \xi(t, \omega)) f(t, Y(t), u(t)), \quad t \geq t_0, \quad Y(t_0) = Y(t_0, \omega) = y_0, \quad (8) \]

has a unique solution \( Y(\cdot) = Y(\cdot, \omega) \) which is almost surely absolutely continuous on \([t_0, T]\).

**Remark 1** This is a simple extension from [14, Theorem 1.5], so, the proof is actually built on it.

**Proof** Without loss of generality, assume that \( |y_0| < R \) for some positive constant \( R \). Let \( G(t, u, y, z) = \sigma(t, u, z) f(t, y, u) \). By applying the Picard’s method of successive approximations of (8) on \([t_0, T]\), we have:

\[ Y^{(k+1)}(t) = y_0 + \int_{t_0}^{t} G(s, u(s), Y^{(k)}(s), \xi(s, \omega)) \, ds, \quad Y^{(0)}(t) = y_0. \]

By (H1) and (H2) we obtain:

\[ |Y^{(1)}(t) - y_0| \leq \int_{t_0}^{t} M_R(s) \, ds \]
and
\[ |Y^{(k+1)}(t) - Y^{(k)}(t)| \leq \int_{t_0}^{t} B(u(s), s, \omega) |Y^{(k)}(s) - Y^{(k-1)}(s)| ds, \]
from which we conclude that
\[ |Y^{(k+1)}(t) - Y^{(k)}(t)| \leq \int_{t_0}^{t} M_R(s) ds \left( \int_{t_0}^{t} B(u(s), s, \omega) ds \right)^k. \]

Then, it follows that \( \lim_{k \to \infty} Y^{(k)}(t) \) exists and satisfies (8). By the usual argument from Grönwall’s inequality we obtain uniqueness, where the solution of (8),
\[ Y(t) = y_0 + \int_{t_0}^{t} \sigma(s, u(s), \xi(s, \omega)) f(s, Y(s), u(s)) ds, \]
is clearly absolutely continuous, and its derivative at \( t \) is given by the function \( \sigma(t, u(t), \xi(t, \omega)) f(t, Y(t), u(t)) \). Thus, the proof is finished.

**Remark 2** In practice, a more realistic assumption than the established by (H1) is the *locally Lipschitz* condition; that is, when (H1) is satisfied locally: whenever \( |x| \vee |y| < K \), for some \( K > 0 \), there exists a stochastic process \( B_K(u(t), t, \omega) \in L^1([t_0, T]) \) such that,
\[ |G(t, u(t), x, \xi(t, \omega)) - G(t, u(t), y, \xi(t, \omega))| \leq B_K(u(t), t, \omega)|x - y|. \]
However, this relaxation in the assumption does not guarantee the existence and uniqueness of a solution for all \( t \geq t_0 \), since it can explode in a finite time. To avoid it, in [14, Sect. 1.3], there are additional conditions in which the explosion in finite time is avoided. It basically consists of controlling the growth of a possible solution of (8), so that there is a stochastic \( \tau(K, \omega) \) such that
\[ \inf \{ t \geq t_0 : |Y(t, \omega)| \geq K \} \geq \tau(K, \omega), \]
where \( \tau(K, \omega) \uparrow \infty \) almost surely, when \( K \uparrow \infty \). To get this, it is enough to assume that, on any fixed time interval \([t_0, T]\), there exists a positive bounded stochastic process \( D(T, \omega) \) such that
\[ y G(t, u(t), y, z) \leq D(T, \omega)(1 + y^2) \]
(see also [15, Chapter 2, Theorem 3.5]). This condition will be fitted to the models that we can use in our context, where the solution of the process, with high probability, will be within some compact space.
3.2 Existence for the Discretization (DS$_i$)

The following lemma establishes the preliminary result for the well-posedness of the ordinary differential equation (DS$_i$) and some continuity properties on its parameters. Notice that in this subsection we work with given $z$ and not with the stochastic representation $\xi(t)$, therefore our results are modifications of classical tools in bifurcation theory.

**Lemma 2** For $i = 1, \ldots, N$ fixed, consider the differential equations

\[
\begin{align*}
y'_{u,z}(t) &= \sigma(u, z) f(y_{u,z}(t), u), \text{ for } t \in [t_{i-1}, t_i], \\
y_{u,z}(t_{i-1}) &= y_{i-1}(u, z),
\end{align*}
\]

(12)

with $(u, z)$ a vector of parameters in $\mathbb{R}^s \times \mathbb{R}^m$, and $y_{i-1}(u, z)$ in $\mathbb{R}_+$. Assume for each value of $u$ and $z$ that $\sigma(u, z) f(y, u)$ is Lipschitz continuous in $y$ with Lipschitz constant $L$ locally independent of $(u, z)$. For each fixed $(u, z)$ and $y_{i-1}(u, z)$ this problem has a unique solution in the interval $[t_{i-1}, t_i]$.

If furthermore $\sigma(u, z) f(y, u)$ is continuous, $|\sigma(u, z)|$ is bounded by some $C_\sigma > 0$ and $f(y, u)$ Lipschitz continuous with Lipschitz constant independent of $(u, z)$, then $y_{u,z}(t)$ is continuous in $(u, z)$ jointly. As a consequence, the set

\[ A_i := \{ z \in \mathbb{R}^m : y_{u,z}(t) \leq \alpha, \text{ for all } t \in [t_{i-1}, t_i] \}, \]

is closed.

**Proof** The existence and uniqueness of the solution $y_{u,z}$ is a direct consequence of the Picard Global Existence Theorem for Lipschitz right hand side. Take $(u_1, z_1)$ and $(u_2, z_2)$ two vector of parameters and call $y_1 = y_{u_1,z_1}$ and $y_2 = y_{u_2,z_2}$ the respective solutions to the equation (12). Notice that

\[
\begin{align*}
\frac{d}{dt} (y_1(t) - y_2(t))^2 &= 2(y_1(t) - y_2(t))(y_1(t) - y_2(t))' \\
&= 2(y_1(t) - y_2(t)) (\sigma(u_1, z_1) f(y_1(t), u_1) - \sigma(u_2, z_2) f(y_2(t), u_2)) \\
&= 2(y_1(t) - y_2(t)) \sigma(u_1, z_1) f(y_1(t), u_1) - f(y_2(t), u_2)) \\
&\quad + 2(y_1(t) - y_2(t)) \sigma(u_1, z_1) - \sigma(u_2, z_2)) f(y_2(t), u_2) \\
&\leq 2C_\sigma L |y_1(t) - y_2(t)| (|y_1(t) - y_2(t)| + |u_1 - u_2|) \\
&\quad + |y_1(t) - y_2(t)|^2 + |\sigma(u_1, z_1) - \sigma(u_2, z_2)|^2 |f(y_2(t), u_2)|^2 \\
&\leq (2C_\sigma L + 1) |y_1(t) - y_2(t)|^2 + C_\sigma L (|y_1(t) - y_2(t)|^2 + |u_1 - u_2|^2) \\
&\quad + |\sigma(u_1, z_1) - \sigma(u_2, z_2)|^2 |f(y_2(t), u_2)|^2 \\
&\leq (3C_\sigma L + 1) |y_1(t) - y_2(t)|^2 \\
&\quad + C_\sigma L |u_1 - u_2|^2 + |\sigma(u_1, z_1) - \sigma(u_2, z_2)|^2 |f(y_2(t), u_2)|^2,
\end{align*}
\]
which implies that
\[
\frac{d}{dt} \left( (y_1(t) - y_2(t))^2 e^{-(3\sigma_2 L + 1)t} \right) \leq \left[ C_{\sigma} L |u_1 - u_2|^2 + |\sigma(u_1, z_1) - \sigma(u_2, z_2)|^2 |f(y_2(t), u_2)|^2 \right] e^{-(3\sigma_2 L + 1)t}.
\]

Let \( c = 3\sigma_2 L + 1 \), then integrating from \( t_{i-1} \) to \( t \in (t_{i-1}, t_i] \) we find
\[
(y_1(t) - y_2(t))^2 \leq \left( y_0(u_1, z_1) - y_0(u_2, z_2) \right)^2 e^{-c(t_i - t_{i-1})} + \left[ c |u_1 - u_2|^2 + |\sigma(u_1, z_1) - \sigma(u_2, z_2)|^2 \max_{t \in [t_{i-1}, t_i]} |f(y_2(t), u_2)|^2 \right] \times (e^{c(t_i - t_{i-1})} - 1).
\]

The right-hand side goes to 0 as \((u_1, z_1)\) approaches \((u_2, z_2)\) therefore the continuity on \((u, z)\) follows. Moreover, the upperbond does not depend on \( t \) thus the continuity is actually uniform in time \( t \).

Take now a point \( z \in \mathbb{R}^m \) and a sequence \( \{z_k\} \) of values on the set \([z \in \mathbb{R}^m : y_{u, z}(t) \leq \alpha, \text{ for all } t \in [t_{i-1}, t_i]\} \), such that \( z_k \to z \). From the continuity of the map \((u, z) \mapsto y_{u, z}\) we now that for any \( t \in [t_{i-1}, t_i] \) it follows that
\[
y_{u, z_k}(t) \to y_{u, z}(t).
\]

Assume that there is some \( t \) such that \( y_{u, z} > \alpha \), then there is some \( k \) such that
\[
|y_{u, z_k}(t) - y_{u, z}(t)| < \frac{1}{3} (y_{u, z} - \alpha), \quad \Rightarrow \quad \frac{2}{3} y_{u, z} + \frac{\alpha}{3} < y_{u, z_k}(t)
\]
but since \( y_{u, z} > \alpha \), in particular
\[
\alpha < \frac{2}{3} y_{u, z} + \frac{\alpha}{3} < y_{u, z_k}(t),
\]
which is a contradiction with the definition of \( y_{u, z_k} \). In consequence, the set \([z \in \mathbb{R}^m : y_{u, z}(t) \leq \alpha, \text{ for all } t \in [t_{i-1}, t_i]\} \) is closed.

**Corollary 1** Under the same hypothesis of Lemma 2, for points \( u^N = (u_1, \ldots, u_N) \) in \( \mathbb{R}^{N \times s} \), and \( z^N = (z_1, \ldots, z_N) \) in \( \mathbb{R}^{m \times s} \), there exists a unique solution \( y_{u^N, z^N} = (y_{1, u_1, z_1}, \ldots, y_{N, u_N, z_N}) \) to
\[
y'_{i, u_i, z_i}(t) = \sigma_i(u_i, z_i) f_i(y_{i, u_i, z_i}(t), u_i), \quad t \in [t_{i-1}, t_i],
y_{i, u_i, z_i}(t_{i-1}) = y_{i-1, u_{i-1}, z_{i-1}}(t_{i-1}),
\]
for any \( y_0(u_0, z_0) = y_0 \) given. Moreover, the set
\[
\left\{ z^N \in \mathbb{R}^{N \times m} : y_{u^N, z^N}(t) \leq \alpha, \text{ for all } t \in [t_0, T] \right\}
\]
is closed.

**Proof** To define the solution to the problem we use an iterative strategy. Consider the initial sub-interval $[t_0, t_1]$ and the equation:

$$y_{1,u_1,z_1}'(t) = \sigma_1(u_1, z_1) f_1(y_{1,u_1,z_1}(t), u_1), \ t \in (t_0, t_1], \ y_{1,u_1,z_1}(t_0) = y_0.$$ 

Thanks to Lemma 2 the equation has a unique solution $y_1(\cdot) = y_1(\cdot, z_1)$ which is continuous in $t \in [t_0, t_1]$. Moreover, the integral form of the equations holds

$$y_{1,u_1,z_1}(t) = y_0 + \int_{t_0}^{t} \sigma_1(u_1, z_1) f_1(y_{1,u_1,z_1}(s), u_1) \, ds, \ t \in (t_0, t_1],$$

and the set

$$S_1 := \{(z_1, \ldots, z_N) \in \mathbb{R}^{N \times m} : y_{1,u_1,z}(t) \leq \alpha, \text{ for all } t \in [t_0, t_1]\}$$

is closed. Now, define

$$y_{2,z_2,u_2}^0 := y_{1,z_1,u_1}(t_1) = y_0 + \int_{t_0}^{t_1} \sigma_1(u_1, z_1) f_1(y_{1,u_1,z_1}(s), u_1) \, ds.$$ 

Analogously as was argued for the first sub-interval $[t_0, t_1]$, we have that the equation

$$y_{2,u_2,z_2}'(t) = \sigma_2(u_2, z_2) f_2(y_{2,u_2,z_2}(t), u_2), \ t \in (t_1, t_2], \ y_{2,u_2,z_2}(t_0) = y_{2,u_2,z_2}^0,$$

has a unique solution $y_{2,u_2,z_2}(\cdot)$ which is continuous for all $t \in [t_1, t_2]$. Moreover, it holds that

$$y_{2,u_2,z_2}(t) = y_{2,u_2,z_2}(t_1) + \int_{t_1}^{t} \sigma_2(u_2, z_2) f_2(y_{2,u_2,z_2}(s), u_2) \, ds$$

$$= y_0 + \int_{t_0}^{t_1} \sigma_1(u_1, z_1) f_1(y_{1,u_1,z_1}(s), u_1) \, ds$$

$$+ \int_{t_1}^{t} \sigma_2(u_2, z_2) f_2(y_{2,u_2,z_2}(s), u_2) \, ds$$

$$= y_0 + \int_{t_0}^{t} g_2(s) \, ds, \ t \in (t_0, t_2],$$
with

\[ g_2(t) = \sigma_1(u_1, z_1) f_1(y_1, u_1(t), u_1) \mathbb{1}_{[t_0, t_1]}(t) + \sigma_2(u_2, z_2) f_2(y_2, u_2, z_2(t), u_2) \mathbb{1}_{[t_1, t_2]}(t). \]

In consequence, the set

\[ S_2 := \{(z_1, \ldots, z_N) \in \mathbb{R}^{N \times m} : y_{1,u_1,z_1}(t) \mathbb{1}_{[t_0, t_1]}(t) + y_{2,u_2,z_2}(t) \mathbb{1}_{[t_1, t_2]}(t) \leq \alpha, \quad \forall t \in [t_0, t_2]\} \]

is also closed. Iterating the above construction we can conclude that the process

\[ y_{u,N,z,N}(t) = \sum_{i=1}^{N-1} y_{i,u_i,z_i}(t) \mathbb{1}_{[t_{i-1}, t_i]}(t) + y_{N,u_N,z_N}(t) \mathbb{1}_{[t_{N-1}, T]}(t), \]

is well-defined and absolutely continuous for all \( t \in [0, T] \). Finally, thanks to the definition of \( S_i \) we have that

\[ S_N := \{z^N \in \mathbb{R}^{N \times m} : y_{u,N,z,N}(t) \leq \alpha, \quad \text{for all } t \in [t_0, T]\} \]

is also closed.

\[ \square \]

### 3.3 Consistency Between the Approximate System (DS\(_i\)) and the General Formulation (1)

Next we will construct a (DS\(_i\)) approximation to the general equation (1), in which such an approximation converges in probability to (1) on any closed time interval, when the size of the subintervals of approximation converges to zero.

Consider a partition of the interval \([t_0, T]\), \(t_0 < t_1 < \ldots < t_N = T\), such that \(t_i - t_{i-1} \leq \Delta_N\), where \(\lim_{N \to \infty} \Delta_N = 0\). Define

\[ \sigma^N(t, u(t), z) := \sum_{i=1}^{N} \sigma(t_{i-1}, u(t_{i-1}), z) \mathbb{1}_{(t_{i-1}, t_i]}(t), \]

\[ f^N(t, Y(t), u(t)) := \sum_{i=1}^{N} f(t_{i-1}, Y(t_{i-1}), u(t_{i-1})) \mathbb{1}_{(t_{i-1}, t_i]}(t) \]

and

\[ \xi^N(t, \omega) := \sum_{i=1}^{N} \xi(t_{i-1}, \omega) \mathbb{1}_{(t_{i-1}, t_i]}(t). \]

Consider \( G(t, u, y, z) \) defined in (7), and additionally assume that:

[88] Springer
(a) For all \((u, y, z)\) and \((u', y', z')\) in \(\mathbb{R}^s \times \mathbb{R} \times \mathbb{R}^m\), there exists a constant \(C > 0\) such that

\[
\left| G(t, u, y, z) - G(t, u', y', z') \right|^2 \leq C(\|u - u'\|_x^2 + |y - y'|^2 + \|z - z'\|_m^2),
\]

and for any \(t, t' \in [t_0, T]\), there exists a positive constant \(K\) such that \(\|u(t) - u(t')\|^2 \leq K |t - t'|^2\).

(b) For all \(t \in [t_0, T]\) we have that \(\mathbb{E}\|\xi(t, \omega)\|^2_m < \infty\), and for any \(t \geq t' \geq t_0\),

\[
\mathbb{E}\|\xi(t, \omega) - \xi(t', \omega)\|^2_m \leq \gamma(t - t'),
\]

where \(\gamma(t)\) is a continuous function such that \(\gamma(0) = 0\).

**Proposition 1** Under hypotheses (H1), (H2), (H3) and (H4), let \(Y^N\) be the process defined by unique solution of

\[
Y^N(t) = y_0 + \int_{t_0}^t \sigma^N(s, u(s), \xi^N(s, \omega)) f^N(s, Y^N(s), u(s)) ds,
\]

and \(Y\) the solution of (8). Then, \(\sup_{t \in [t_0, T]} \mathbb{E} \left| Y^N(t) - Y(t) \right|^2 \to 0\), as \(\Delta_N \to 0\).

**Remark 3** Notice that an approximation system \((DS_i)\) arises by defining the mappings \(\sigma_i(\cdot, \cdot) = \sigma(t_{i-1}, \cdot, \cdot), f_i(\cdot, \cdot) = f(t_{i-1}, \cdot, \cdot), u_i = u(t_{i-1})\) and \(\xi_i = \xi(t_{i-1})\), so that the solution \(Y^N(t)\) is the corresponding solution of \((DS_i)\).

**Proof** Let \(G^N(t, u, y, z) = \sigma^N(t, u, z) f^N(t, y, u)\). We have that on \([t_0, T]\),

\[
\left| Y(t) - Y^N(t) \right|^2 \leq (T - t_0) \int_{t_0}^t \left| G^N(s, u(s), Y^N(s), \xi^N(s, \omega)) - G(s, u(s), Y(s), \xi(s, \omega)) \right|^2 ds
\]

\[
\leq C(T - t_0) \int_{t_0}^t (K \Delta_N^2 + \left| Y^N(s) - Y(s) \right|^2 + \|\xi(s, \omega) - \xi^N(s, \omega)\|^2_m) ds.
\]

Now, by taking expectation in the above expression we obtain:

\[
\mathbb{E} \left| Y^N(t) - Y(t) \right|^2 \leq C(T - t_0)[K(T - t_0)\Delta_N^2 + (T - t_0)\gamma(\Delta_N) + \int_{t_0}^t \mathbb{E} \left| Y^N(s) - Y(s) \right|^2 ds],
\]

\(\Box\) Springer
where the result is obtained by using Grönwall’s inequality. Thus, the proof is finished.

\[ \square \]

**Remark 4** Similarly as mentioned in Remark 2, it is enough that the Lipschitz conditions in \((H3)\) can be satisfied locally, as long as the existence and uniqueness of a solution of the original equation for all \(t \geq t_0\) is guaranteed, since approximation will occur in any ball of \(\mathbb{R}\). Typical stochastic noises that satisfy \((H4)\) are the standard Brownian motion in \(\mathbb{R}^m\), \(W(t)\), and the \(\mathbb{R}^m\)-dimensional compensated Poisson process \(\mathcal{M}(t) = \mathcal{N}(t) - \lambda t \mathbf{1}\), where \(\mathcal{N}(t)\) is an \(\mathbb{R}^m\)-dimensional of independent and identically distributed Poisson processes with instantaneous rate \(\lambda > 0\), and \(\mathbf{1}\) is the \(m\)-vector of 1’s. They are square-integrable martingales whose associated increasing process are proportional to \(t\), in which \((H4)\) holds.

Let us notice that thanks to Lemma 2 the probability function \((5)\) is well-defined. Therefore, the feasible set in Problem 4 is given by a so-called probabilistic/robust (probust) constraint, which has been studied in recent articles (see, e.g., \([9, 23, 27]\)). Such a representation and a posterior study is quiet complex because, on the one hand, by the possibly infinite number of inequalities, and on the other hand, it is not clear how the variational and geometrical behavior of the control parameter \(u\) and the random inflow \(\xi\) in the resulting dynamic \(y_{u, \xi}\) are. In the following result, we show that the optimization problem \((4)\) has a suitable representation as a joint chance-constrained optimization problems.

**Theorem 1** Let \(u \in \mathbb{R}^n\), and consider \(y_{u, \xi}\) as the unique solution of the System \((6)\) with initial condition \(y_0(u) \leq \bar{b}\) and parameter \(u\). Suppose that there exist \(a_1 < a_2^i\) for \(i = 1, \ldots, N\) such that

(a) \(\bar{b} < a_2^i\) for all \(i = 1, \ldots, N\)

(b) \(y_{u, \xi}(\omega)(t) \in (a_1^i, a_2^i)\) for all \(t \in [t_{i-1}, t_i]\) and all \(\omega \in \Omega\).

(c) \(f_i(s, u) > 0\) for all \(s \in (a_1^i, a_2^i)\).

Then, the probability function \((5)\) associated to the solution of \((6)\) can be rewritten as

\[
\varphi(u) = \mathbb{P}\left( \sigma_i(u, \xi_i(\omega)) \leq F_{\bar{b}}^i(u, y_{u, \xi}(t_{i-1})) \text{ for all } i = 1, \ldots, N \right). \tag{13}
\]

where the function \(F_{\bar{b}}^i : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}\) is defined by

\[
F_{\bar{b}}^i(u, a) := \frac{1}{t_i - t_{i-1}} \int_{a}^{\bar{b}} ds \frac{f_i(s, u)}{f_i(s, u) / F_{\bar{b}}^i(u, a)}. \tag{14}
\]

**Proof** Let us consider the sets for \(k = 1, \ldots, N\)

\[
A_k := \{ \omega \in \Omega : y_{u, \xi}(t) \leq \bar{b}, \text{ for all } t \in [t_0, t_k] \}
\]

\[
B_k := \{ \omega \in \Omega : \sigma(u, \xi_i(\omega)) \leq F_{\bar{b}}^i(u) \text{ for all } i = 1, \ldots, k \}
\]
and $A_0 = B_0 = \Omega$. Let us show by induction that $A_k = B_k$ for all $i = 0, \ldots, N$. Suppose that $A_k = B_k$. Since $f_i(s, u)$ is strictly positive on $(a_1^k, a_2^k)$ and $y_{u, \xi}(t) \in (a_1^k, a_2^k)$ for all $i = 1, \ldots, N$ we have that $G(w) := \frac{1}{t_{k+1}-t_k} \int_{y_{u, \xi}(t_k)}^{w} \frac{ds}{f_k(s, u)}$ is well-defined and non-decreasing. Now, consider $\omega \in \Omega$, and the system of inequalities

$$y_{u, \xi}(t) \leq \bar{b}, \text{ for all } t \in [t_k, t_{k+1}].$$

(15)

On the one hand, if $\sigma_i(u, \xi_i(\omega)) \leq 0$, we have that $y_{u, \xi}(t)$ is non-increasing because $y_{u, \xi}(t) \in (a_1^k, a_2^k)$ and $f_k(s, u)$ is positive on $(a_1^k, a_2^k)$. The system of inequalities (15) reduces to $y_{u, \xi}(t_k) \leq \bar{b}$, which holds by our inductive hypothesis, and consequently in this case $\omega \in A_{k+1}$ if and only if $\omega \in B_{k+1}$. On the other hand, suppose that $\sigma(u, \xi_k(\omega)) > 0$, then $y_{u, \xi}(t)$ is non-decreasing over $[t_k, t_{k+1}]$, so the system of inequalities (15) reduces to $y_{u, \xi}(t_{k+1}) \leq \bar{b}$. Now, applying the function $G$ to the last inequality, we get that $G(y_{u, \xi}(t_{k+1})) \leq G(\bar{b})$. Now, using separation of variables in (DS$_i$) we get that $G(y_{u, \xi}(t_{k+1})) = \sigma(u, \xi_{k+1}(\omega))$. Consequently, the system of inequalities (15) reduces to $\sigma(u, \xi_k(\omega)) \leq G(\bar{b}) = F_{\bar{b}}^{k+1}(u, y_{u, \xi}(t_k))$, which show in this case that also $\omega \in A_{k+1}$ if and only if $\omega \in B_{k+1}$, and that proofs the equality of the sets.

Finally, (13) follows from the equality $A_N = B_N$, which ends the proof. □

Remark 5 It is important to notice that the above result shows that the computation of the probability function $\varphi$, defined in (5), which is given by a system of infinitely many inequalities, reduces to a finite number of inequalities (classical joint chance constrained optimization problem). Therefore, numerical computation of such a probability only requires the estimation of the dynamic at points $t_1, t_2, \ldots, t_N$ instead of the complete trajectory $y_{u, \xi}$ over $[t_0, T]$.

In the rest of this section we review classical growth models, which fits with the assumption of Theorem 1. For simplicity of the explanation, we will assume in the following examples that the control affects only the rate growth function $\sigma$.

3.3.1 Exponential Growth Model

One of the most well known models for population dynamics is the exponential growth model proposed by Robert Malthus. Assuming no limitation in population growth, the exponential model for a population reads:

$$y'_{u, \xi}(t) = \sigma(u, \xi)y_{u, \xi}(t), \quad y(t_0) = y_0.$$

(16)

The parameter $\sigma$ is a continuously differentiable function, let us say, $\sigma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ of our control $u \in \mathbb{R}^n$ and an $m$-dimensional random vector $\xi = (\xi_1, \ldots, \xi_N)$. Assume that the growth rate function $\sigma$ is nonnegative and bounded by some $\sigma_{max}$. The explicit solution to this model is a simple exponential function:

$$y_{u, \xi}(t) = y_0 \exp(\sigma(u, \xi)(t - t_0)), \quad y(t_0) = y_0.$$

(17)
Take any $\bar{b} \in \mathbb{R}_+$ and consider the first interval $[t_0, t_1]$. We have then
\[ a_1^1 := 0 \leq y_{1, u_1, \xi_1}(t) \leq \bar{b} \exp(\sigma_{\text{max}}(t_1 - t_0)) =: a_2^1, \]
and $f_1(s, u) = s > 0$ for all $s \in (a_1^1, a_2^1)$. In the second interval, since
\[ y_{2, u_2, \xi_2}(t_1) = y_{1, u_1, \xi_1}(t_1) \leq a_2^1, \]
it holds that
\[ 0 < y_{2, u_2, \xi_2}(t) \leq a_2^1 \exp(\sigma_{\text{max}}(t_2 - t_1)) =: a_2^2, \quad \forall t \in [t_1, t_2]. \]
Iterating the argument for next time intervals, we find the existence of the family of upper bounds \( \{ a_i^2 \}_{i=1}^N \) and by consequence the exponential model fits the hypotheses of Theorem 1. We remark that in the exponential model, all bounds depends explicitly on the initial value of $y_0$ (or an upper bound for that quantity) and on the size of the time interval $(T - t_0)$.

### 3.3.2 Logistic Model with Allee Effect

A much more elaborated version of the logistic model describes the situations in which the sparsity of individuals leads to a reduced survival of the offspring. In biological terms, the Allee effect is related to the extinction of the population due to critically low levels of individuals. The new dynamics is given by the equation
\[
y_{u, \xi}'(t) = \sigma(u, \xi) y_{u, \xi} \left( \frac{y_{u, \xi}(t)}{r} - 1 \right) \left( 1 - \frac{y_{u, \xi}(t)}{k} \right), \quad y(t_0) = y_0, \tag{18}
\]
with $0 < r < k$ the size of the Allee effect.

If the initial condition fits $\bar{b} \geq y_0 > r$, once again the respective solution $y_{u, \xi}$ is strictly increasing going asymptotically to $k$. Similarly, if $0 < y_0 < r$ the respective solution $y_{u, \xi}$ is strictly decreasing going asymptotically to $0$. In both cases Theorem 1 applies by adapting the arguments presented to the exponential and $\theta$-logistic models.

### 4 Semi-probabilistic Optimization Problem

Thanks to Theorem 1, the probability function (5) can be computed using only a finite number of random inequalities. Consequently, our control problem (4) can be seen as a regular chance-constrained optimization problem. Nevertheless, the dependence of each $F_i^{t_b}$ defined in (14), on the random parameter $\xi$ is not explicitly. It makes it problematic, and it is not clear how to get benefits from such a formulation, and therefore how to explore further properties of the probability function (5). For that reason, it is convenient to avoid the random dependency of $F_i^{t_b}$ inside of the probability function replacing the stochastic function $F_i^{t_b}(u, y_{u, \xi}(t_{i-1}))$ for some deterministic
version $\hat{F}_b^i(u)$, which depends only on the control parameter $u$, and at the same time split the constraint $\varphi(u) \geq p$ into several constraints. Formally, we consider the following optimization problem

$$
\min \psi_N(u_1, \ldots, u_N) \\
\varphi_i(u) \geq p_i \text{ for all } i = 1, \ldots, N \\
\quad u \in C,
$$

(19)

where $C \subseteq \mathbb{R}^n$ is a nonempty closed set and the probability functions $\varphi_i : \mathbb{R}^n \to \mathbb{R}$ are given by

$$
\varphi_i(u) := \mathbb{P} \left( \sigma_i(u, \xi_i(\omega)) \leq \hat{F}_b^i(u) \right).
$$

(20)

In order to fix ideas, the reader can think in the deterministic function

$$
\hat{F}_b^i(u) := F_b^i(u, \hat{y}_u(t_{i-1})),
$$

where $F_b^i$ defined in (14) and $\hat{y}_u(t_{i-1})$ represents the average of the dynamic solutions defined on (6). Other possible choice for $\hat{F}_b^i(u)$ is the expected value of $F_b^i(u, y_u, \xi(t_{i-1}))$. Both of them can be computed numerically before the computation of each probability functions $\varphi_i$. It is worth mentioning that the probability functions $\varphi_i$, given in (20), can be seen as a (disjoint) semi-probabilistic version of (13), so (under suitable assumptions) the optimization problem (19) can be seen as a (disjoint) semi-probabilistic version of (5).

In the rest of the section we will assume that the random vectors $\xi_i$ has continuous density with respect to the Lebesgue measure. It will allow us to study variational and geometric properties of the optimization problem (19).

### 4.1 Topological and Geometric Properties of Semi-probabilistic Model

**Proposition 2** Let $i \in \{1, \ldots, N\}$ be a fixed index, and suppose that the functions $\sigma_i$ and $\hat{F}_b^i$, given in (20), are continuous functions and that $\xi_i$ posses density with respect to the $m$-dimensional Lebesgue measure. Then, the probability function $\varphi_i$ defined in (20) is upper semicontinuous. Moreover, $\varphi_i$ is continuous at a point $\bar{u}$ provided that

$$
\left\{ z \in \mathbb{R}^m : \sigma_i(\bar{u}, z) = \hat{F}_b^i(\bar{u}) \right\}
$$

(21)

has null-measure (with respect to the $m$-dimensional Lebesgue measure).

**Proof** Consider $u_j \to \bar{u}$. Now, let us define the sequence of functions

$$
f_j(z) := 1_{A_j}(z), \quad f(z) := 1_A(z),
$$

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where \( A_j := \{ z \in \mathbb{R}^m : \sigma_i(u_j, z) \leq \hat{F}_b^i(u_j) \} \) and \( A := \{ z \in \mathbb{R}^m : \sigma_i(\bar{u}, z) \leq \hat{F}_b^i(\bar{u}) \} \). Let us verify that

\[
\limsup f_j(z) \leq f(z), \quad \text{for all } z \in \mathbb{R}^m.
\] (22)

Indeed, fix \( z \in \mathbb{R}^m \). On the one hand if \( \sigma_i(\bar{u}, z) \leq \hat{F}_b^i(\bar{u}) \), then (22) holds trivially. On the other hand suppose that \( \sigma_i(\bar{u}, z) > \hat{F}_b^i(\bar{u}) \), so by continuity of the functions \( \sigma_i(\cdot, z) \) and \( \hat{F}_b^i \) at \( \bar{u} \), we have that the inequality \( \sigma_i(u_j, z) > \hat{F}_b^i(u_j) \) must holds for large enough \( j \), hence (22) does holds as an equality.

Then, by Fatou’s lemma and Theorem 1 we get that

\[
\varphi_i(\bar{u}) = \mathbb{E}_{\xi_b}(f) \geq \limsup_{j \to \infty} \mathbb{E}_{\xi_b}(f_j) = \limsup_{j \to \infty} \varphi(u_j),
\]

which shows the upper-semicontinuity.

Now, if the set in (21) has null measure, then using similar arguments we can show that (22) holds as an equality for almost all \( z \in \mathbb{R}^m \), then using Lebesgue’s convergence theorem we get that

\[
\varphi_i(\bar{u}) = \mathbb{E}_{\xi_b}(f) = \lim_{j \to \infty} \mathbb{E}_{\xi_b}(f_j) = \lim_{j \to \infty} \varphi(u_j)
\]

which shows the continuity of \( \varphi \) at \( \bar{u} \).

**Corollary 2** Let \( C \subseteq \mathbb{R}^n \) be a closed subset, and suppose that for each \( i = 1, \ldots, N \) the functions \( \sigma_i \) and \( \hat{F}_b^i \), given in (20), are continuous functions and that \( \xi_i \) posses density with respect to the \( m \)-dimensional Lebesgue measure. Then the set \( \{ x \in C : \varphi_i(x) \geq p_i, \text{ for all } i = 1, \ldots, N \} \) is closed.

**Proof** Since, the assumptions of Proposition 2 hold we have that each function \( \varphi_i \) is upper-semicontinuous, which shows that the set \( \{ x \in C : \varphi_i(x) \geq p_i, \text{ for all } i = 1, \ldots, N \} \) is necessarily closed.

Finally, let us shows sufficient conditions to ensure the convexity of the feasible set of the optimization problem (19). We recall that a random vector has quasiconcave density provided that its density with respect to the Lebesgue is quasiconcave.

**Proposition 3** Let \( C \subseteq \mathbb{R}^n \) be a closed and convex and suppose that the functions \( \sigma_i \) and \( \hat{F}_b^i \), given in (20), are continuous functions. In addition suppose that \( \sigma_i \) is quasiconvex, \( \hat{F}_b^i \) is quasiconcave and assume that \( \xi_i \) has quasiconcave density with respect to the \( m \)-dimensional Lebesgue measure. Then, the set

\[
\{ u \in C : \varphi_i(u) \geq p_i \text{ for all } i = 1, \ldots, N \}
\]

is convex for all \( p_i \in (0, 1] \).
Proof Let us consider the set \( C_0 := \{ u \in C : \phi_i(u) \geq p_i \text{ for all } i = 1, \ldots, N \} \) and \( C_i := \{ u : \phi_i(u) \geq p_i \} \) for all \( i = 1, \ldots, N \). By [19, Theorem 4.39] we have that the function \( \phi_i \) is quasi-concave, and consequently the set \( C_i \) is convex for all \( i = 1, \ldots, N \). Then, the set \( C_0 \) is convex being the intersection of the convex sets \( C, C_1, \ldots, C_N \).

4.2 Existence and Uniqueness of Solution

Now, we can provide necessary conditions to the existence and uniqueness of the solution to the semi-probabilistic optimization problem (19). The first result of this subsection shows the general existence of a solution, and the second one under convexity shows that it must be unique.

Theorem 2 Let \( C \subseteq \mathbb{R}^n \) be a nonempty closed set, let us suppose that for each \( i = 1, \ldots, N \) the functions \( \sigma_i \) and \( \hat{F}_b^i \), given in (20), are continuous. In addition, assume that one of the following conditions holds:

(a) The objective function \( \psi_N \) of optimization problem (19) is coercive, i.e., \( \psi_N(u) \rightarrow +\infty \) as \( \|u\| \rightarrow +\infty \).
(b) The set \( C \) is bounded.
(c) There exists \( \alpha \in \mathbb{R} \) such that the set
\[
\{ u \in C : \phi_i(u) \geq p \text{ for all } i = 1, \ldots, N \text{ and } \psi_N(u) \leq \alpha \}
\]
is nonempty and bounded.

Then, the optimization problem (19) has a solution provided that it is feasible.

Proof It is easy to see that 2 and 2 follows from 2. Moreover, by Corollary 2 and under 2 we have that the optimization problem (19) reduces to a minimization of a lower semicontinuous function over a compact set, which by classical arguments has at least one solution (see, e.g., [17, Theorem 1.9] for more details).

Corollary 3 Under the assumptions of Proposition 3 suppose that the objective function \( \psi_N \) of optimization problem (19) is coercive and strictly convex on \( \{ u \in C : \phi_i(u) \geq p_i \text{ for all } i = 1, \ldots, N \} \). Then, optimization problem (19) is convex and has a unique solution provided that it is feasible.

Proof First, we have by Theorem 2 that the optimization problem (19) has a solution. Moreover, under the assumption of Proposition 3 the set \( \{ u \in C : \phi_i(u) \geq p_i \text{ for all } i = 1, \ldots, N \} \) is convex, so the optimization problem is convex. Finally, the solution of this optimization problem is unique due to the fact that \( \psi_N \) is strictly convex over the feasible set (see, e.g., [17, Theorem 2.6]).

4.3 Differentiability of \( \phi \)

Now, we turn into the study of the differentiability of the probability function (20). In that case, we establish results for differentiability and formulae for gradients of
the probability functions $\varphi_i$ using of the so-called spherical-like radial decomposition described in [25]. It is worth mentioning that such representation also can be used for computing the values of the probability function $\varphi_i$, we refer to [11, 22, 24, 26–28] for more results in same research line.

It will be convenient to adopt the following notation. Given a random vector $\xi : \Omega \to \mathbb{R}^m$, we denote its density with respect to the $m$-dimensional Lebesgue measure by $f_{\xi}$ (when this exists). The following definition corresponds to a growth condition used to control the gradients of the data function to compute the derivatives of probability functions.

**Definition 1** ($\eta_{\xi}$-growth condition) Consider a random vector $\xi : \Omega \to \mathbb{R}^m$ and a function $\eta_{\xi} : \mathbb{R}^m \to [0, +\infty)$ such that

$$\lim_{\|z\| \to +\infty} \|z\|^m f_{\xi}(z) \eta_{\xi}(z) = 0.$$ 

We say that the mapping $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ satisfies the $\eta_{\xi}$-growth condition at $\bar{u}$ if for some $k, l, \epsilon > 0$

$$\|\nabla_u \sigma_i(u, z)\| \leq k \eta_{\xi}(z)$$

for all $u \in B_{\epsilon}(\bar{u})$ and for all $\|z\| \geq l$.

**Remark 6** (Remark on $\eta_{\xi}$-growth condition) It is worth mentioning that for most used density distribution the choice of the function $\eta_{\xi}$ simplifies in above definition. For instance, for the Gaussian distribution it can be taken $\eta_{\xi}(z) := \exp(\|z\|)$ (see, e.g., [20]). More general for the class of elliptically symmetric distribution with generator function $\kappa$ (see, [22, Definition 2.3] for its formal definition) the function $\eta_{\xi}$ can be chosen by $\eta_{\xi}(z) := \nu(\|z\|/\|L\|)$, where $\nu$ is any real-valued function satisfying $\lim_{r \to \infty} r^m \nu(r) \kappa(r^2) = 0$ (see, e.g., [22, Definition 4.14]).

Now, given a nonsingular matrix $L$, and a reference point $\hat{z}$ we define the density-like function $\theta : \mathbb{R}_+ \times \mathbb{S}^{m-1} \to \mathbb{R}$ given by

$$\theta_{\xi}(r, v) := \frac{2\pi^{\frac{m}{2}} |\det(L)|}{\Gamma\left(\frac{m}{2}\right)} r^{m-1} f_{\xi}(rLv + \hat{z}).$$

(23)

where, $\mathbb{S}^{m-1} := \{z \in \mathbb{R}^m : \|z\| = 1\}$ is the $m$-dimensional unit sphere, $\det(L)$ refers to the determinant of the matrix $L$, and $\Gamma$ is the Gamma function.

To rewrite the probability function $\varphi_i$, given in (5), let us consider the uniform probability measure $\mu : \mathcal{B}(\mathbb{S}^{m-1}) \to [0, 1]$, defined by

$$\mu(A) = \frac{m \Gamma\left(\frac{m}{2}\right)}{2\pi^{\frac{m}{2}}} \cdot \lambda_m \left(\{z \in \mathbb{R}^m : \exists r \in [0, 1], \exists v \in A \ s.t. \ z = rv\}\right).$$

(24)

where $\mathcal{B}(\mathbb{S}^{m-1})$ denotes the Borel $\sigma$-algebra over $\mathbb{S}^{m-1}$.

The following results gives criteria for differentiability of the probability functions $\varphi_i$ defined in (20). The first one use the geometrical assumption that $\sigma_i$ is convex with
respect to the second argument. Similarly, the second result shows differentiability but under the assumption of concavity in the second variable instead of convexity. Both results follows from the gradient formulae found in [25] (see, also, [20, 21, 23, 27, 28]).

Theorem 3 Let \( i \in \{1, \ldots, N\} \) be a fixed index, and suppose that the function \( \sigma_i \), given in (20), is continuously differentiable and convex with respect to the second variable. In addition, assume that at our point of interest \( \bar{u} \) the following assumptions hold:

(a) There exists \( \hat{z} \in \mathbb{R}^m \) such that \( \sigma_i(\bar{u}, \hat{z}) < \hat{F}_b^i(\bar{u}) \),
(b) the function \( \sigma_i \) satisfies the \( \eta_{\xi_i} \)-growth condition at \( \bar{u} \), for some function \( \eta_{\xi_i} \).
(c) The function \( \hat{F}_b^i \), given in (20), is continuously differentiable at \( \bar{u} \).

Then, the probability function \( \varphi_i \), defined in (20), is continuously differentiable at \( \bar{u} \). Furthermore, given any nonsingular matrix \( L \), the following formula holds for all \( u \) in a suitable neighbourhood of \( \bar{u} \)

\[
\nabla \varphi_i(u) = \int_{v \in D_i(u)} \frac{\theta_{\xi_i}(\rho_i(u, v), v)}{\left( \nabla_z \sigma_i(\bar{u}, \rho_i(u, v) L v + \hat{z}), L v \right)} \left( \nabla_u \sigma_i(x, \rho_i(u, v) L v + \hat{z}) - \nabla \hat{F}_b^i(u) \right) d\mu(v), \tag{25}
\]

where \( \theta \) is defined in (23), \( D_i(u) := \{ v \in S^{m-1} : \text{There exists } r > 0 \text{ s.t. } \sigma_i(u, r L v + \hat{z}) > \hat{F}_b^i(u) \} \), and the radial function \( \rho_i \) is given by

\[
\rho_i(u, v) := \sup\{ r > 0 : \sigma_i(u, r L v + \hat{z}) \leq \hat{F}_b^i(u) \}.
\]

Proof We notice that the probability functions defined in (20) can be rewrite as

\[
\varphi_i(u) = \mathbb{P} \left( g_i(u, \xi_i) \leq 0 \right),
\]

where \( g_i(u, z) := \sigma_i(u, z) - \hat{F}_b^i(u) \). Then, applying [25, Corollary 2], we get the desire conclusion of the result. \( \square \)

Remark 7 It is important to mention that in many application the choice of the reference point \( \hat{z} \) is given by the precise media. Particularly, in the case of Gaussian distribution, that is, \( \xi \sim \mathcal{N}(m, \Sigma) \), the violation of the inequality \( \sigma(\bar{u}, m) < F_b(\bar{u}) \), implies that the probability of the even is less than \( 1/2 \) (see [20, Proposition 3.11]), which for many application means that \( \bar{u} \) is not a feasible point of a chance-contrained optimization, where the requirement is allays with probability level \( p \) close to 1, for instance \( p \) grater than 0.9.

Furthermore, the non singular matrix \( L \) and the reference point \( \hat{z} \) have been used for simplifications and numerical computations of the aforementioned formulae. Particularly, when \( \xi \sim \mathcal{N}(m, \Sigma) \), the reference point \( \hat{z} \) is the media \( m \), and the non singular matrix \( L \) corresponds to the Choleski decomposition of \( \Sigma \), that is, \( \Sigma = L^T L \). We refer to [11, 20–22] and the references therein for more precise results.

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In the setting of Theorem 3 let us suppose that the function \( \sigma_i \) is concave with respect to the second variable instead of being convex. Then, the probability functions \( \varphi_i \), defined in (20), is continuously differentiable at \( \bar{u} \). Furthermore, given any nonsingular matrix \( L \), the following formula holds for all \( u \) in a suitable neighborhood of \( \bar{u} \)

\[
\nabla \varphi_i(u) = \int_{v \in E_i(u)} \frac{\theta_{\hat{\varepsilon}_i}(\hat{\rho}_i(u, v), v)}{\langle \nabla_z \sigma_i(u, \hat{\rho}_i(u, v) L v + \hat{z}) - \nabla \hat{F}_b^i(u), d
\mu(v)\rangle}
\]

where \( \theta \) is defined in (23), \( E_i(u) := \{ v \in S^{m-1} : \text{There exists } r > 0 \text{ s.t. } \sigma_i(u, rL v + \hat{z}) < \hat{F}_b^i(u) \} \), and the radial function \( \rho_i \) is given by

\[
\hat{\rho}_i(u, v) := \sup\{ r > 0 : \sigma_i(u, rL v + \hat{z}) \geq \hat{F}_b^i(u) \}. \]

**Proof** Let us notice that the probability function \( \varphi_i(u) = 1 - \phi_i(u) \), where

\[
\phi_i(u) := \mathbb{P}(\hat{g}_i(u, \xi_i) \leq 0) \text{ and } \hat{g}_i(u, z) := \hat{F}_b^i(u) - \sigma_i(u, z). \]

Then, we can apply [25, Corollary 2] to the probability function \( \phi_i \), and consequently we get the corresponding differentiability result to the probability function \( \varphi_i \) and its corresponding formula (26) (see also [25, Corollary 5]). \( \square \)

4.4 Optimality Conditions

In this section, we study optimality conditions for the control problem (19). The first result corresponds to necessary conditions to optimality, and the second one, under suitable convexity assumptions, corresponds to a sufficient condition to optimality.

Given a point \( \bar{u} \) we consider the following qualification condition

\[
\text{co}\{ \nabla \varphi_i(\bar{u}) : i = 1, \ldots, N \} \cap N_C(\bar{u}) = \emptyset, \]

where \( \text{co}(A) \) denotes the convex hull of a set \( A \).

**Theorem 5** Let us suppose that \( \bar{u} \) is a local solution of the optimization problem (19). Assume that the objective function \( \psi_N \) and \( \varphi_i \) are continuously differentiable at \( \bar{u} \) (for instance under the assumptions of Theorem 3 or 4). Then, we have that the following optimality condition holds: There exist multipliers \( \lambda_i \leq 0 \) for all \( i = 1, \ldots, N \) such that \( \lambda_i(\varphi_i(\bar{u}) - p_i) = 0 \) for all \( i = 1, \ldots, N \) and

\[
0_n \in \nabla \psi_N(\bar{u}) + \sum_{i=1}^{N} \lambda_i \nabla \varphi_i(\bar{u}) + N_C(\bar{u}), \]

provided that the qualification condition (27) holds at \( \bar{u} \).
Proof Let us notice that the optimization problem (19) can be written as

$$\min \psi_N(u)$$
$$p_i - \psi_i(u) \leq 0 \text{ for all } i = 1, \ldots, N$$
$$u \in C,$$

Therefore, by [16, Theorem 5.21], there are multipliers $\eta_i \geq 0$ such that $\sum_{i=0}^{N} \eta_i = 1$, $\eta_i (\psi_i(\bar{u}) - p_i) = 0$ for all $i = 1, \ldots, N$ and

$$0_n \in \eta_i \nabla \psi_N(\bar{u}) - \sum_{i=1}^{N} \eta_i \nabla \psi_i(\bar{u}) + NC(\bar{u}). \tag{29}$$

Now, if $\eta_0 = 0$, we have that (29) implies that $0_n \in -\sum_{i=1}^{N} \eta_i \nabla \psi_i(\bar{u}) + NC(\bar{u})$, which contradicts (27), so $\eta_0 > 0$. Now, dividing (29) by $\eta_0$ and defining $\lambda_i = -\eta_i/\eta_0$ we get that (28) holds.

The final result of this section shows that the condition (28) is also sufficient for a point to be a minimum of (19) under suited geometric assumptions.

Theorem 6 Under assumption of Theorem 5 let us suppose that $\psi_N$ and $C$ are convex, the functions $\sigma_i$ and $\hat{F}^i_{\bar{b}}$, given in (5), are quasiconvex and quasiconcave, respectively and that $\xi_i$ has log-concave density with respect to the $m$-dimensional Lebesgue measure. Furthermore, assume that all the prescribed probability level $p_i > 0$. Then, the fulfillment of (28) by a feasible point of problem (19) implies that this point is a global minimum of the (19).

Proof Let us consider the functions $\phi_i(u) := -\log \phi_i(u) + \log(p_i)$. By [19, Theorem 4.39] the function $\phi_i$ is convex over the set feasible set of the problem (19). Due to (28), there are multipliers $\lambda_i \leq 0$ and $v^* \in NC(\bar{u})$ such that $\lambda_i (\phi_i(\bar{u}) - p_i) = 0$ for all $i = 1, \ldots, N$, and $0_n = \nabla \psi(\bar{u}) - \sum_{i=1}^{N} \lambda_i \nabla \phi_i(\bar{u}) + v^*$. Now, let us consider a feasible point $u$ of problem (19). By convexity of $\phi_i$ we have that for all $i = 1, \ldots, N$

$$\langle \nabla \phi_i(\bar{u}), u - \bar{u} \rangle \leq -\log \phi_i(u) + \log \phi_i(\bar{u}).$$

and by convexity of $\psi_N$ and $C$ we have $\langle \nabla \psi_N(\bar{u}), u - \bar{u} \rangle \leq \psi_N(u) - \psi_N(\bar{u})$, and $\langle v^*, u - \bar{u} \rangle \leq 0$, respectively. Consequently

$$\psi_N(\bar{u}) \leq \langle -\nabla \psi_N(\bar{u}), u - \bar{u} \rangle + \psi_N(u)$$
$$\leq - \sum_{i=1}^{N} \lambda_i \langle \nabla \phi_i(\bar{u}), u - \bar{u} \rangle + \langle v^*, u - \bar{u} \rangle + \psi_N(u)$$
$$\leq \sum_{i=1}^{N} \lambda_i \log(\phi_i(\bar{u})) \langle \nabla \phi_i(\bar{u}), u - \bar{u} \rangle + \psi_N(u)$$

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\[ \sum_{i=1}^{N} \lambda_i \log(\varphi_i(\bar{u})) \left[ -\log \varphi_i(u) + \log \varphi_i(\bar{u}) \right] + \psi_N(u) \leq 0 \]

which shows the optimality of \( \bar{u} \).

**Proof** Let us consider the functions \( \phi_i(u) := -\log \varphi_i(u) + \log(p_i) \). By [19, Theorem 4.39] the function \( \phi_i \) is convex over the set feasible set of the problem (19). Due to (28), there are multipliers \( \lambda_i \leq 0 \) and \( v^* \in N_C(\bar{u}) \) such that \( \lambda_i(\varphi_i(\bar{u}) - p_i) = 0 \) for all \( i = 1, \ldots, N \), and \( 0_n = \nabla \psi(\bar{u}) - \sum_{i=1}^{N} \lambda_i \nabla \varphi_i(\bar{u}) + v^* \). Now, let us consider a feasible point \( u \) of problem (19). By convexity of \( \phi_i \) we have that for all \( i = 1, \ldots, N \)

\[ \langle \nabla \phi_i(\bar{u}), u - \bar{u} \rangle \leq -\log \varphi_i(u) + \log \varphi_i(\bar{u}) \]

and by convexity of \( \psi_N \) and \( C \) we have \( \langle \nabla \psi_N(\bar{u}), u - \bar{u} \rangle \leq \psi_N(u) - \psi_N(\bar{u}) \), and \( \langle v^*, u - \bar{u} \rangle \leq 0 \), respectively. Consequently

\[ \psi_N(\bar{u}) \leq \langle \nabla \psi_N(\bar{u}), u - \bar{u} \rangle + \psi_N(u) \]

\[ \leq - \sum_{i=1}^{N} \lambda_i \langle \nabla \varphi_i(\bar{u}), u - \bar{u} \rangle + \langle v^*, u - \bar{u} \rangle + \psi_N(u) \]

\[ \leq \sum_{i=1}^{N} \lambda_i \log(\varphi_i(\bar{u})) \langle \nabla \phi_i(\bar{u}), u - \bar{u} \rangle + \psi_N(u) \]

\[ \leq \sum_{i=1}^{N} \lambda_i \log(\varphi_i(\bar{u})) \left[ -\log \varphi_i(u) + \log \varphi_i(\bar{u}) \right] + \psi_N(u) \]

\[ \leq \sum_{i=1}^{N} \log(\varphi_i(\bar{u})) \lambda_i \left[ -\log(p_i) + \log \varphi_i(\bar{u}) \right] + \psi_N(u) = \psi_N(u), \]

which shows the optimality of \( \bar{u} \).

**5 Numerical Example**

In this last section we approximate an optimal control in a particular population model. To that aim we use as stochastic process \( \xi(t) = \alpha W(t) \), with \( 0 < \alpha < 1 \) and \( W(t) \) a one dimensional standard Brownian motion. For the population dynamics, we consider the logistic model \( f(s) = s(1 - s/k) \) with fixed \( k \), and the following family of growth rate functions:

\[ \sigma(t, u(t), \xi(t)) = \sigma_+ + (\sigma_+ - \sigma_-)\phi(a \xi(t) - b u(t)), \]
Fig. 1 Numerical approximation of the logistic growth model without control \( u^N = (0, \ldots, 0) \) (upper panel) and fully controlled \( u^N = (1, \ldots, 1) \) (lower panel). In dashed bold line we plot the empirical mean over \( M = 3 \times 10^4 \) simulations. The carrying capacity is \( k = 100[\text{kt}] \) (maximal biomass supported by the environment), and the upper bound safety level or limit biomass used for defining the probability function \( \bar{b} = 80[\text{kt}] \). The growth factor function and its corresponding regular cut-off function are \( \sigma(u, z) = 10^{-10} + \phi(0.5z - 2u) \) and \( \phi(s) = (1 + \tanh(s))/2 \), and the diffusion parameter on \( \xi(t) \) is \( \alpha = 0.5 \).

with \( \sigma_- \) a minimal positive growth value and \( \phi_+ \) the maximal allowed growth rate. We take \( a \) and \( b \) positive constants and \( \phi \) an increasing \( C^\infty(\mathbb{R}) \) function, for instance \( \phi(s) = \frac{1 + \tanh(s)}{2} \). This choice of growth rate functions is such that

\[
0 < \sigma_- := \lim_{s \to -\infty} \sigma(s) \leq \sigma_+ := \lim_{s \to +\infty} \sigma(s) < \infty,
\]

and \( \sup_{s \in \mathbb{R}} |\sigma'(s)| < +\infty \).

For simplicity of the calculations, we are assuming that we know exactly the initial condition values, meaning that \( y_0 \in \mathbb{R} \). To avoid confusions, we denote \( u^N = (u_1, \ldots, u_N) \) and \( \xi^N = (\xi_1, \ldots, \xi_N) \) for a given discretization of the stochastic process \( \xi \).

In Fig. 1 we show the numerical approximation \( y_{u^N, \xi^N} \) for the system with no control \( u^N = (0, \ldots, 0) \) and with full control \( u^N = (1, \ldots, 1) \) for \( N = 4 \). In all simulations performed in the present section we use \( y_0 = 10 \), independently of the values of \( u \) and \( z \).
Table 1  Probability and cost function values obtained for the logistic model in the discrete set of control scenario \( u^N \in \{0, 1\}^N \).

| Control scenario \( u^N \) | Increasing (in time) control | Decreasing (in time) control |
|-----------------------------|-----------------------------|-----------------------------|
|                             | Probability \( \varphi \)   | Cost \( \psi \)             | Probability \( \varphi \)   | Cost \( \psi \)             |
| \((1, 1, 1, 1)\)            | 0.9748                      | 36.1720                     | 0.9753                      | 0.8507                      |
| \((1, 1, 1, 0)\)            | 0.7754                      | 16.0864                     | 0.7754                      | 0.8009                      |
| \((1, 1, 0, 1)\)            | 0.7818                      | 26.6842                     | 0.7821                      | 0.7453                      |
| \((1, 1, 0, 0)\)            | 0.4563                      | 6.5987                      | 0.4567                      | 0.6955                      |
| \((1, 0, 1, 1)\)            | 0.8083                      | 31.6903                     | 0.8074                      | 0.6276                      |
| \((1, 0, 1, 0)\)            | 0.4485                      | 11.6047                     | 0.4462                      | 0.5778                      |
| \((1, 0, 0, 1)\)            | 0.4478                      | 22.2025                     | 0.4481                      | 0.5222                      |
| \((1, 0, 0, 0)\)            | 0.2924                      | 2.1170                      | 0.2925                      | 0.4724                      |
| \((0, 1, 1, 1)\)            | 0.8383                      | 34.0551                     | 0.8383                      | 0.3783                      |
| \((0, 1, 1, 0)\)            | 0.4402                      | 13.9694                     | 0.4399                      | 0.3285                      |
| \((0, 1, 0, 1)\)            | 0.4345                      | 24.5672                     | 0.4321                      | 0.2729                      |
| \((0, 1, 0, 0)\)            | 0.2612                      | 4.4817                      | 0.2623                      | 0.2231                      |
| \((0, 0, 1, 1)\)            | 0.4256                      | 29.5733                     | 0.4252                      | 0.1552                      |
| \((0, 0, 1, 0)\)            | 0.2436                      | 9.4877                      | 0.2441                      | 0.1054                      |
| \((0, 0, 0, 1)\)            | 0.2427                      | 20.0855                     | 0.2429                      | 0.0498                      |
| \((0, 0, 0, 0)\)            | 0.1655                      | 0                          | 0.1651                      | 0                          |

Parameters are shown in Fig. 1. The lower bound safety level \( p \) is 0.75. In the case \( c > 0 \) (third column) the best result is obtained for the control \((1, 1, 1, 0)\) meaning that the system reduced the growth rate for times close to the initial time \( t_0 = 0 \), which is in concordance to the fact that as \( t_i \) increases, the effect of \( u_i = 1 \) in the cost function also does. In the opposite case \( c < 0 \) (fourth column), the best result is obtained for the control \((0, 1, 1, 1)\) meaning that the system reduced the growth rate for times away from \( t_0 = 0 \).

Regarding the costs and possible control functions, we consider the following time dependant expression:

\[
\psi_N(u^N) = \sum_{i=1}^{N} u_i^2 e^{c(t_i - t_0)}, \quad u^N = (u_1, u_2, \ldots, u_N),
\]

we vary the values of \( c \) to capture the effect on the optimal \( u^N \) when the future costs \( u_{i+1} \) are more/less important than current ones. In this setting, we look for a solution of the chance-constrained control problem (4) with probability level \( p = 0.75 \).

Moreover, for numerical purposes, we consider first the finite set of control functions:

\[
u^N = (u_1, \ldots, u_N), \quad u_i \in \{0, 1\},
\]

In Table 1, we show the results for both models for \( N = 4 \) and \( M = 30000 \) (see the caption of Fig. 1 for parameter values) by approximating the probability by the empirical rate positive/total. In this case, the total number of possible control functions...
is $2^4 = 16$ thus we show all the costs $\psi(u^N)$ for $c = 1/5$ and $c = -1/5$. We see that the uncontrolled system has a small probability of having $y(T) \leq \bar{b}$. When $c > 0$, activating the coordinate $u_i$ for further times contribute more to $\psi_N$ than initial times. In this scenario, we find that the optimal control is $(1, 1, 1, 0)$. Naturally, if $c < 0$ then previous argument flips and the optimal control is now $(0, 1, 1, 1)$.

In a second stage, we consider the hyper-cube $[0, 1]^N$ as possible control functions, i.e.:

$$u^N = (u_1, \ldots, u_N), \quad 0 \leq u_i \leq 1, \ i = 1, \ldots, N.$$  

We performed a simple gradient search with adaptive step increment, starting with different controls. When $c = -1/5$, we use increasing initial control and we find a local minimum at $u^N = (0.4019, 0.4744, 0.6242, 0.6684)$ with $\varphi(u^N) = 0.75133$ and $\psi_N(u^N) = 0.1898$. If $c = 1/5$, we use a decreasing initial control to find a local minimum at $u^N = (0.7542, 0.5918, 0.6145, 0.3383)$, $\varphi(u^N) = 0.75$ and $\psi_N(u^N) = 8.6558$. We see that in both cases, the result outperforms the discrete scenario.

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**Declarations**

**Conflict of interest**  The authors declare that they have no conflict of interest.

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