MODULAR SYMMETRIES, THRESHOLD CORRECTIONS AND MODULI FOR $Z_2 \times Z_2$ ORBIFOLDS.

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ABSTRACT

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\( \mathbb{Z}_2 \times \mathbb{Z}_2 \) Coxeter orbifolds are constructed with the property that some twisted sectors have fixed planes for which the six-torus can not be decomposed into a direct sum \( T^2 \oplus T^4 \) with the fixed plane lying in \( T^2 \). The string loop threshold corrections to the gauge coupling constants are derived, and display symmetry groups for the \( T \) and \( U \) moduli that are subgroups of the full modular group \( \text{PSL}(2, \mathbb{Z}) \). The effective potential for duality invariant gaugino condensate in the presence of hidden sector matter is constructed and minimized for the values of the moduli. The effect of Wilson lines on the modular symmetries is also studied.
The spectrum of states for an orbifold [1, 2] is invariant under discrete transformations for the $T$ and $U$ moduli together with the winding numbers and momenta on the orbifold. These modular symmetries also appear as symmetries of the string loop threshold corrections which are important for the unification of gauge coupling constants [3-8]. Moreover, the form of the threshold corrections, which is dictated to a considerable extent by the modular symmetries, influences the form of the non-perturbative superpotential due to gaugino condensation in the hidden sector, and so the effective potential that determines the values of the $T$ and $U$ moduli [9-15].

In the absence of Wilson lines, provided all twisted sector fixed planes are such that the six-torus can be decomposed into a direct sum $T^2 \oplus T^4$ with the fixed plane lying in $T^2$, the group of modular symmetries of the threshold corrections is a product of $PSL(2, Z)$ factors one for each of the $T$ and $U$ moduli associated with the fixed planes. We shall refer to such orbifolds as decomposable orbifolds. However, for non-decomposable orbifolds, the group of modular symmetries is in general a product of congruence subgroups [17-19] of $PSL(2, Z)$. Wilson lines can also break the $PSL(2, Z)$ modular symmetries [20-23].

The modular symmetry groups for string loop threshold corrections have already been studied for non-decomposable $Z_N$ Coxeter orbifolds [17, 18, 19, 23]. It is our purpose here to extend the discussion to non-decomposable $Z_M \times Z_N$ Coxeter orbifolds. We shall find that $Z_2 \times Z_2$ provides the only examples.

A large class of orbifolds, the Coxeter orbifolds, can be obtained by taking the underlying lattice of the six-torus to be a direct sum of Lie group root lattices and constructing the generators of the point group from Coxeter elements, generalized Coxeter elements, or their powers for the various root lattices. A Coxeter element is the product of all the Weyl reflections for the root lattice. When the Dynkin diagram possesses an outer automorphism, we can also make a generalized Coxeter element by using those Weyl reflections associated with points in the Dynkin diagram that are not permuted by the outer automorphism together with one of
the permuted Weyl reflections and the outer automorphism itself [24].

To construct $\mathbb{Z}_M \times \mathbb{Z}_N$ Coxeter orbifolds, we need to choose Coxeter elements, generalized Coxeter elements, or their powers acting on the direct sum of root lattices, such that their eigenvalues give the correct action of the generators of $\mathbb{Z}_M$ and $\mathbb{Z}_N$ in the complex orthogonal space basis, and such that the generators of $\mathbb{Z}_M$ and $\mathbb{Z}_N$ commute. Using only Coxeter elements, we find no examples for non-decomposable $\mathbb{Z}_M \times \mathbb{Z}_N$ orbifolds. However, if we also deploy generalized Coxeter elements, there are examples in the case of $\mathbb{Z}_2 \times \mathbb{Z}_2$ point group.

The generators $\theta$ and $\omega$ of $\mathbb{Z}_2 \times \mathbb{Z}_2$ have the action on the space basis

$$\theta = (-1, 1, -1), \quad \omega = (1, -1, -1). \quad (1)$$

A non-decomposable example, in the sense discussed earlier, is obtained using the lattice $SU(4) \times SO(4) \times SU(2)$. Then, in terms of Coxeter elements, the action on the $SU(4)$, $SO(4)$ and $SU(2)$ sub-lattices is given by $\left(\mathcal{C}^2(SU(4)), \mathcal{C}(SO(4)), 1\right)$ and $\left(\mathcal{C}^3(SU(4)^{[2]}), I, -1\right)$ for $\theta$ and $\omega$ respectively, where $\mathcal{C}$ denotes the Coxeter element, and $\mathcal{C}(SU(4)^{[2]})$ is the generalized Coxeter element for the $SU(4)$ lattice. Other examples are obtained by replacing $SO(4)$ by $SO(5)$, $G_2$ or $SU(3)$ and $\mathcal{C}(SO(4))$ by $\mathcal{C}^2(SO(5))$, $\mathcal{C}^3(G_2)$ or $\mathcal{C}^3(SU(3)^{[2]})$. However, all of these have same action on the 2-dimensional sub-lattices as $\mathcal{C}(SO(4))$ and consequently there is no relevant distinction between these possibilities for our purposes. We shall focus on the $SO(4)$ case in what follows.

The action of $\theta$ and $\omega$ on the basis vectors $e_1, \cdots, e_6$ of the lattice is then

$$\theta : e_i \rightarrow \Theta_{ij} e_j, \quad \Theta = \text{diag} \left( \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}, -I_2, 1 \right)$$

$$\omega : e_i \rightarrow \Omega_{ij} e_j, \quad \Omega = \text{diag} \left( \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, I_2, -1 \right). \quad (2)$$

where $I_2$ is the $2 \times 2$ identity matrix, and the eigenvalues of $\mathcal{C}^2(SU(4))$ are
(-1, -1, 1). It is therefore clear that the fixed plane in the \( \theta \) twisted sector lies partly in the \( SU(4) \) sub-lattice and partly in the \( SU(2) \) sub-lattice so that the \( T^2 \oplus T^4 \) decomposition does not occur.

The moduli that occur in the string loop threshold corrections to the gauge coupling constants are those associated with fixed planes in some twisted sectors [4] (the \( N = 2 \) moduli.) In the present case, the moduli \((T_1, U_1), (T_2, U_2)\) and \((T_3, U_3)\) are associated with fixed planes in the \( \omega, \theta \) and \( \theta \omega \) twisted sectors, respectively. The \( \omega \) twisted sector fixed plane lies in the \( SO(4) \) sub-lattice, Consequently, the \( T_1 \) and \( U_1 \) dependent threshold corrections have the standard [4] full modular symmetry. For the \( \theta \) twisted sector, the boundary conditions requires the winding number \( w \) and momentum \( p \) on the orbifold to satisfy

\[
Qw = w, \quad Q^*p = p, \quad (3)
\]

where \( Q \equiv \Theta^t \). Thus, for this sector

\[
w = \begin{pmatrix} n_1 \\ 0 \\ n_1 \\ 0 \\ 0 \\ n_6 \end{pmatrix}, \quad p = \begin{pmatrix} m_1 \\ -m_1 \\ m_1 \\ 0 \\ 0 \\ m_6 \end{pmatrix}, \quad (4)
\]

where \( m_1, m_6, n_1 \) and \( n_6 \) are arbitrary integers. For the \( \theta \omega \) twisted sector, the boundary conditions requires

\[
Q\bar{Q}w = w, \quad Q^*\bar{Q}^*p = p, \quad (5)
\]

\* \( Q^* = Q^{(-1)} \)
\[ \bar{Q} \equiv \Omega^t, \] and the corresponding windings and momenta are

\[
\mathbf{w} = \begin{pmatrix}
    n_1' \\
    n_2' \\
    n_2' - n_1' \\
    0 \\
    0 \\
    0
\end{pmatrix}, \quad \mathbf{p} = \begin{pmatrix}
    m_1' \\
    m_2' \\
    -m_1' \\
    0 \\
    0 \\
    0
\end{pmatrix}, \tag{6}
\]

where \( m_1', m_2', n_1' \) and \( n_2' \) are arbitrary integers.

The string loop threshold corrections \( \Delta_a \) may now be calculated from partition functions for \( \mathcal{N} = 2 \) twisted sectors \((h, g)\) of the orbifold, where \( h \) and \( g \) refer to twists in the \( \sigma \) and \( t \) directions, respectively, with both \( h \) and \( g \) leaving fixed the same complex plane. It is convenient [16] to perform the calculation in terms of a subset of \( \mathcal{N} = 2 \) twisted sectors \((h_0, g_0)\), referred to as the fundamental elements, then,

\[
\Delta_a = \sum_{(h_0, g_0)} b_a^{(h_0, g_0)} \int_{\tilde{\mathcal{F}}} \frac{d^2\tau}{\tau_2} \mathcal{Z}_{(h_0, g_0)}(\tau, \bar{\tau}) - b_a^{\mathcal{N}=2} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2}, \tag{7}
\]

where \( \mathcal{Z}_{(h, g)} \) denotes the moduli dependent parts of the partition functions, \( b_a^{(h, g)} \) is the contribution of the \((h, g)\) sector to the one loop renormalization group equation coefficients, \( b_a^{\mathcal{N}=2} \) is the contribution of all \( \mathcal{N} = 2 \) twisted sectors, \( \mathcal{F} \) is the fundamental region for the world sheet modular group \( PSL(2, \mathbb{Z}) \), and \( \tilde{\mathcal{F}} \) is the fundamental region for the world sheet modular symmetry group of \( \mathcal{Z}_{(h_0, g_0)} \). Here, the single twisted sector \((h_0, g_0)\) replaces a set of twisted sectors that can be obtained from it by applying \( SL(2, \mathbb{Z}) \) transformations which generate \( \tilde{\mathcal{F}} \) from \( \mathcal{F} \). In the present case, the fundamental elements for the non-decomposable fixed planes are \((I, \theta)\) and \((I, \theta\omega)\). The partition functions \( \mathcal{Z}_{(I, \theta)} \) \( \mathcal{Z}_{(I, \theta\omega)} \) may now be derived using methods for non-decomposable orbifolds discussed elsewhere [17-19] with the
\[ Z(I, \theta) = \sum_{m_1, m_6 \atop n_1, n_6} e^{2\pi i \tau (2m_1 n_1 + m_6 n_6)} e^{\frac{\pi \tau_2}{Im T_1 Im U_2}} |T_2 U_2 n_6 + T_2 n_1 - 2U_2 m_1 + m_6|^2 \]

\[ Z(I, \theta \omega) = \sum_{m'_1, m'_2 \atop n'_1, n'_2} e^{2\pi i \tau (2m'_1 n'_1 + m'_2 n'_2)} e^{\frac{\pi \tau_2}{Im T_3 Im U_3}} |T_3 U_3 n'_2 + T_3 n'_1 - 2U_3 m'_1 + m'_2|^2, \]

where \( m'_2 = m'_1 - m'_2 \). After Poisson resummation of the partition functions and \( \tau \) integration \([4, 17, 19]\) we find

\[ \Delta_a = -b_a(I, \theta) \log \left( k \, Im T_2 |\eta(T_2)|^4 Im U_2 |\eta(2U_2)|^4 \right) - b_a(I, \theta) \log \left( k \, Im T_2 |\eta(\frac{T_2}{2})|^{4 Im U_2 |\eta(U_2)|^4} \right) - b_a(I, \theta \omega) \log \left( k \, Im T_3 |\eta(T_3)|^4 Im U_3 |\eta(2U_3)|^4 \right) - b_a(I, \theta \omega) \log \left( k \, Im T_3 |\eta(\frac{T_3}{2})|^{4 Im U_3 |\eta(U_3)|^4} \right) - b_a(I, \omega) \log \left( k \, Im T_1 |\eta(T_1)|^4 Im U_3 |\eta(U_1)|^4 \right) \]

where \( \eta \) is the Dedekind function, and \( k = \frac{8\pi}{3\sqrt{3}} e^{(1 - \gamma_E)} \), where \( \gamma_E \) the Euler-Mascheroni constant. The threshold correction \( \Delta_a \) is invariant under the target space modular symmetry group

\[ \Gamma = [SL(2, Z)]_{T_1} \times [SL(2, Z)]_{U_1} \times [\Gamma^0(2)]_{T_2} \times [\Gamma_0(2)]_{U_2} \times [\Gamma^0(2)]_{T_3} \times [\Gamma_0(2)]_{T_3} \]

where \( \Gamma_0(n) \) and \( \Gamma^0(n) \) are congruence subgroups of \( SL(2, Z) \) transformations defined by

\[ T \rightarrow \frac{aT + b}{cT + d}, \quad \text{with} \ c = 0 \ (\text{mod} \ n) \ \text{and} \ b = 0 \ (\text{mod} \ n), \]

respectively.
The string loop threshold corrections may now be used to construct the effective potential due to duality invariant gaugino condensates [9, 16]. In general, an effective potential with a realistic minimum for the dilaton expectation value, and so realistic values for the gauge coupling constants, is not obtained in the absence of hidden matter sector. When the hidden sector matter is present, the requirement that the effective superpotential should have the correct modular weight can prevent the occurrence of such a realistic minimum when there are some $\mathcal{N} = 1$ moduli in the theory [15, 16], i.e., when there are some complex planes which are rotated by the action of the point group in all twisted sectors. This difficulty does not occur here because all moduli are $\mathcal{N} = 2$ moduli.

A simple model for the hidden sector matter [15] is to have two factors in the hidden sector gauge group (two gaugino condensate) with the matter charged under these two factors coupled to singlet scalars $A_1$ and $A_2$ with self couplings $A_3^1$ and $A_3^2$. In such a model, the perturbative superpotential $W_p$ is given by

$$W_p = MA_1 \sum_\alpha Q_\alpha \bar{Q}_\alpha + MA_2 \sum_\beta R_\beta \bar{R}_\beta + A_3^1 + A_3^2$$  \hspace{1cm} (12)$$

where $Q_\alpha$ and $R_\beta$ are the matter fields coupled to the two factors of the gauge group, with $M$ is the string scale. The renormalization group equations for the hidden sector gauge coupling constants $g_a(\mu)$, $a = 1, 2$, at scale $\mu$, are

$$g_a^{-2}(\mu) = g_a^{-2}(M) + \frac{(b_a)_0}{16\pi^2} \log \frac{(M_a)_I^2}{\mu^2} + \frac{(b_a)}{16\pi^2} \log \frac{(M_a)^2}{(M_a)_I^2} + \frac{\Delta_a}{16\pi^2}$$  \hspace{1cm} (13)$$

where $(b_a)_0$ is the renormalization group coefficient for the pure gauge case, $b_a$ is the complete renormalization group coefficient, $(M_a)_I$ are the intermediate scales

$$(M_a)_I = M < A_a >, \hspace{1cm} a = 1, 2$$  \hspace{1cm} (14)$$

In general, when the modular group symmetries are congruence subgroups of
\( SL(2, Z) \) the string loop threshold corrections have the form [17, 19]

\[
\Delta_a = - \sum_i (b'_a - \delta_{GS}^i) \left( \log \left( \frac{T_i + \bar{T}_i}{l_{im}} \right) + \sum_m \frac{c_{im}}{2} \log \left( \eta(T_i) \frac{l_{im}}{l_{in}} \right)^4 \right)
- \sum_i (\tilde{b}'_a - \tilde{\delta}_{GS}^i) \left( \log \left( \frac{U_i + \bar{U}_i}{l_{in}} \right) + \sum_n \frac{\tilde{c}_{in}}{2} \log \left( \eta(U_i) \frac{l_{in}}{l_{in}} \right)^4 \right)
\]

(15)

where \( b'_a, \delta_{GS}^i, \tilde{b}'_a \) and \( \tilde{\delta}_{GS}^i \) are the usual [6, 7] duality anomaly and Green-Schwarz coefficients for the \( T \) and \( U \) moduli, \( l_{im} \) and \( \bar{l}_{in} \) are rational numbers, and the integer-valued coefficients \( c_{in} \) and \( c_{im} \) satisfy

\[
\sum_n c_{in} = \sum_m c_{im} = 2.
\]

(16)

(This ensures that the corresponding non-perturbative superpotential has the correct modular weights). In (15), and what follows we have replaced the original \( T_i \) and \( U_i \) by \(-iT_i\) and \(-iU_i\) to conform with the notations usually employed in the discussion of effective potentials.

The gauge kinetic function may be read off from (13) and (15), and the corresponding two-gaugino condensate non-perturbative superpotential \( W_{np} \) takes the form

\[
W_{np} = \sum_{a=1,2} d_a e^{\frac{24\pi^2 S}{(b_a)_0}} A^\mu_a H_a(T_i, U_i),
\]

(17)

where \( S \) is the dilaton field,

\[
\mu_a = \frac{3 \left( (b_a)_0 - b_a \right)}{(b_a)_0}, \quad d_a = \frac{b_a}{96\pi^2 g_a}
\]

(18)

and

\[
H_a(T_i, U_i) = \prod_{i,m} \left( \eta \left( \frac{T_i}{l_{im}} \right) \right)^{3c_{im}(b'_a - \delta_{GS}^i)} \prod_{i,m} \left( \eta \left( \frac{U_i}{l_{in}} \right) \right)^{3\tilde{c}_{in}(\tilde{b}'_a - \tilde{\delta}_{GS}^i)}
\]

(19)
The complete superpotential is

\[ W = W_{np} + W_p, \]  

where

\[ \delta^i_{GS} = \tilde{\delta}^i_{GS} = 0, \]  

for all \( i \)

as is true for the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) orbifold. After ‘integrating out’ the singlet scalars \( A_1 \) and \( A_2 \), we obtain for the effective potential

\[
(S + S) \prod_i (T_i + \bar{T}_i)(U_i + \bar{U}_i)V_{eff} = |W_{eff} - (S + S) \frac{\partial W_{eff}}{\partial S}|^2 + \\
\sum_i \left\{|W_{eff} - (T_i + \bar{T}_i) \frac{\partial W_{eff}}{\partial T_i}|^2 + |W_{eff} - (U_i + \bar{U}_i) \frac{\partial W_{eff}}{\partial U_i}|^2\right\} - 3|W_{eff}|^2,
\]

with

\[ W_{eff} = \Omega(S)\Psi(T_i, U_i) \]

where

\[
\Omega(S) = - \sum_a \frac{b_a}{(b_a - (b_a)_0)} \left( - \frac{\mu_a d_a}{3} \right) \frac{(b_a)_0}{b_a} e^{24\pi^2 S},
\]

\[
\Psi(T_i, U_i) = \prod_{i,m} \left( \eta(T_i) \eta(U_i) \right)^{-c_{im}} \prod_{i,n} \left( \eta(U_i) \eta(2U_i) \right)^{-\tilde{c}_{in}}
\]

In the present case, (9) implies that

\[
\Psi(T_i, U_i) = \left( \eta(T_1)\eta(U_1) \right)^{-2} \prod_{j=2,3} \left( \eta(T_j)\eta(T_j)\eta(U_j)\eta(2U_j) \right)^{-1}
\]

Equation (22) has the usual extremum for \( S \) at

\[
\Omega(S) - (S + S) \frac{d\Omega}{dS} = 0,
\]

and it is known [15] that this can give a realistic minimum for \( S \) with \( S_R \approx 2 \) for many possible \( SU(N_1) \times SU(N_2) \) hidden sector gauge group with hidden sector...
matter in fundamental $SU(N_1)$ and $SU(N_2)$ representations and their conjugates. At the minimum for $S$,

$$|\Omega(S)|^{-2}(S + \bar{S}) \prod_i (T_i + \bar{T}_i)(U_i + \bar{U}_i)V_{eff} =$$

$$\sum_i \left\{ |\Psi - (T_i + \bar{T}_i) \frac{\partial \Psi}{\partial T_i}|^2 + |\Psi - (U_i + \bar{U}_i) \frac{\partial \Psi}{\partial U_i}|^2 \right\} - 3|\Psi|^2$$

(27)

Because $\delta_G^i$ and $\tilde{\delta}_G^i$ are zero, there are extrema for $T_i$ and $U_i$ at the self-dual (or fixed) points of the modular symmetry group (10),

$$T_1 = 1 \text{ or } e^{\frac{i\pi}{6}}, \quad U_1 = 1 \text{ or } e^{\frac{i\pi}{6}}, \quad T_2 = T_3 = 1 - i, \quad U_2 = U_3 = \frac{(1 + i)}{2}. \quad (28)$$

These extrema are saddle points and maxima for the fixed points in $(T_1, U_1)$, and maxima at the remaining fixed points listed in eqn. (28).

A numerical minimization of $V_{eff}$ shows that the minima for $T_i$ and $U_i$ occur at

$$T_1 = U_1 = 1.26, \quad T_2 = 1.41,$$

$$U_2 = 0.71 + i, \quad T_3 = 1.41, \quad U_3 = 0.71 + i. \quad (29)$$

This provides an elegant example of an orbifold for which there is an anisotropic solution for the moduli.

To date, it is not known how to calculate explicit string loop threshold corrections in the presence of Wilson lines. However, it is possible to identify modular symmetries that those threshold corrections will possess, and a set of conditions to determine these modular symmetries has been written down elsewhere [22, 23]. For the $Z_2 \times Z_2$ case, the inequivalent Wilson lines $\tilde{a}_{bI}$, $b = 1, \cdots, 6$, $I = 1, \cdots, 16$, satisfy

$$\tilde{a}_{3I} = \tilde{a}_{1I}, \quad 2\tilde{a}_{bI} \in \Lambda_{E_8 \times E_8} \quad (30)$$

The matrices $A^t$ of refs [22, 23] can be constructed from $\tilde{a}_{bI}$,

$$A^t_{bB} = \tilde{a}_{bI} E^I_B \quad (31)$$

where $E^I_B$ is a vielbein for the $E_8$ lattice, and the matrices $K_1$, $K_2$ and $K_3^t$ of ref
are given by

\[ K_1 = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3^t = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \]  

(32)

in the \( \theta \) sector and

\[ K_1 = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3^t = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \]  

(33)

in the \( \theta \omega \) sector. The modular symmetry groups for \( T_1 \) and \( U_1 \) may now be deduced from (48)-(55) of ref [22], or (5.15) and (5.16) of ref [23]. The modular symmetry groups for \( T_2, U_2, T_3 \) and \( U_3 \) may be deduced from (6.13) and (6.14) of ref [23]. A simple example is obtained by taking

\[ A^t = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \]  

(34)

in which case the modular symmetry groups associated with the various \( T \) and \( U \) moduli are as follows. Define the subgroup \( \tilde{\Gamma}_0(2) \) of the congruence subgroup \( \Gamma_0(2) \) as in eqn.(11), but with the additional constraint \( b = 0 \) (mod2). Then for
$T_1, U_1$ we obtain $[\text{PSL}(2, Z)]_{T_1}, [\text{PSL}(2, Z)]_{U_1}$, for $T_2, U_2, [\tilde{\Gamma}_0(2)]_{T_2}, [\tilde{\Gamma}_0(2)]_{U_2}$ and finally for $T_3, U_3$ we find the subgroups $[\tilde{\Gamma}_0(2)]_{T_3}, [\Gamma_0(4)]_{U_3}$.

In conclusion, we have shown that $\mathbb{Z}_2 \times \mathbb{Z}_2$ is the only $\mathbb{Z}_M \times \mathbb{Z}_N$ Coxeter orbifold with a choice of lattice for which the modular symmetry groups, in the absence of Wilson lines, are congruence subgroups of $\text{SL}(2, Z)$ rather than the full modular group. The explicit string loop threshold corrections displaying these modular symmetries have been calculated and used to derive the values of the $T$ and $U$ moduli by minimizing duality invariant effective potential. Finally, we have discussed possible modular symmetries of threshold corrections for $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds with Wilson lines background.

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