Renormalisation Flow and Universality for Ultracold Fermionic Atoms

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A functional renormalisation group study for the BEC-BCS crossover for ultracold gases of fermionic atoms is presented. We discuss the fixed point which is at the origin of universality for broad Feshbach resonances. All macroscopic quantities depend only on one relevant parameter, the concentration $ak_F$, besides their dependence on the temperature in units of the Fermi energy. In particular, we compute the universal ratio between molecular and atomic scattering length in vacuum. We also present an estimate to which level of accuracy universality holds for gases of Li and K atoms.

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I. INTRODUCTION

The quantitatively precise understanding of the crossover from a Bose-Einstein condensate (BEC) to BCS superfluidity \cite{1} in gases of ultracold fermionic atoms \cite{2,3} is a theoretical challenge. If it can be met, the comparison with future experimental precision results could set a new milestone for the understanding of the transition from known microscopic laws to macroscopic observations at length scales several orders of magnitude larger than the characteristic atomic or molecular length scales. Furthermore, these systems could become a major testing ground for theoretical methods dealing with the fluctuation problem in complex many-body systems in a context where no small couplings are available. The theoretical progress could go far beyond the understanding of critical exponents, amplitudes and the equation of state near a second-order phase transition. Then, a whole range in temperature and coupling constants could become accessible to precise calculations and experimental tests \cite{4}.

The qualitative features of the BEC-BCS crossover through a Feshbach resonance can already be well reproduced by extensions of mean field theory which account for the contribution to the density from di-atom or molecule collective or bound states \cite{5}. Furthermore, in the limit of a narrow Feshbach resonance the crossover problem can be solved exactly \cite{6}, with the possibility of a perturbative expansion for a small Feshbach or Yukawa coupling. A systematic expansion beyond the case of small Yukawa couplings has been performed as a $1/N$-expansion \cite{7}. The crossover regime is also accessible to $\epsilon$-expansion techniques \cite{8}. At zero temperature \cite{9} as well as at finite temperature \cite{10}, numerical results based on various Monte-Carlo methods are available at the crossover. A unified picture of the whole phase diagram has arisen from functional field-theoretical techniques, in particular from self-consistent or $t$-matrix approaches \cite{11}, Dyson-Schwinger equations \cite{6,12}, 2PI methods \cite{13}, and the functional renormalisation group (RG) \cite{14}.

Recently, it has been advocated \cite{6} that a large universality holds for broad Feshbach resonances: all purely "macroscopic" quantities can be expressed in terms of only two parameters, namely the concentration $c = ak_F$ and the temperature in units of the Fermi energy, $T = T/\epsilon_F$. Here, $k_F$ and $\epsilon_F$ are defined by the density of atoms, $n = k_F^3/(3\pi^2)$, $\epsilon_F = k_F^2/(2M)$ and $a$ is the scattering length. Universality means that the thermodynamic quantities and the correlation functions can be computed independently of the particular realizations of microscopic physics, as for example the Feshbach resonances in $^6$Li or $^{40}$K. A special point in this large universality region is the location of the resonance, $c \to \infty$, where the scaling argument by Ho \cite{15} applies.

We emphasise that the universal description in terms of two parameters holds even in situations where the understanding of the microscopic physics may necessitate several other parameters. It is thus much more than the simple observation that the appropriate model involves only two parameters. The fact that two effective parameters are sufficient is related to the existence of a fixed point in the renormalisation group flow; for a given $T$, this fixed point has only one relevant direction, namely the concentration $c$. Then, the question of accuracy of a two-parameter description is related to the rate how fast this fixed point is approached by the flow towards the infrared. It depends on the scales involved and the physical questions under investigation. As in statistical physics, the deviations from universality can be related to the scaling dimensions of operators evaluated at infrared fixed points. We will address these questions quantitatively for Feshbach resonances in $^6$Li or $^{40}$K.

Even for broad Feshbach resonances not all quantities admit a universal description. An example is given by the number density of microscopic molecules which depends on further parameters \cite{3,12,16}.

The quantitative study of this universality has first been performed through the solution of Schwinger-Dyson equations, including the molecule fluctuations \cite{6,12,17}. Already in this work it has been argued that the key for a proper understanding of universality lies in the renormalisation flow and its (partial) fixed points. A similar point of view has been advocated by Nikolic and Sachdev \cite{7}. 

A first functional renormalisation group study evaluating the renormalisation flow for the whole phase diagram has been put forward in [14]. In the present paper, we perform a systematic study of the universality aspects of the renormalisation flow for various couplings thereby extending the results of [14] concerning universality. We relate the universality for broad Feshbach resonances to a non-perturbative fixed point which is infrared stable except for one relevant parameter corresponding to the concentration $c$.

Our approach is based on approximate solutions of exact functional renormalisation group equations [18], for reviews see [19, 20, 21], and for applications to non-relativistic fermions see [20]. These are derived by varying an effective “infrared” cutoff associated to a momentum scale $k$ (in units of $k_F$). The solution to the fluctuation problem corresponds then to the limit $k \to 0$. In this way, a continuous interpolation from the microscopic Hamiltonian or classical action to the macroscopic observables described by the effective action is achieved. In this paper, we will use a very simple cutoff associated to an additional negative chemical potential. This works well on the BEC side of the crossover and in the vacuum limit of vanishing density and temperature. Since the crucial characteristics of the fixed points relevant for universality can be found in the vacuum limit, this will be sufficient for our purpose. The investigation of the whole phase diagram in [14] has been performed with a different cutoff more amiable to this global task, see [21, 22].

The simple form of the cutoff allows for analytic solutions of the flow equations; these can be used for a direct check of the more general arguments, resulting from the investigation of fixed points and their stability. As a concrete example, we compute the universal ratio between the scattering length for molecules to the atomic scattering length. As the molecular binding energy or the scattering length depends on a magnetic field $B$. In Sect. XII we discuss the fixed-point behaviour for an additional parameter, namely the effective four-boson vertex $\lambda_\varphi$. This fixed point is responsible for the universal ratio between the scattering length for molecules and for atoms. We finally present a general discussion of universality for ultracold fermionic atoms in Sect. XII, and we estimate the deviations from the universal broad-resonance limit for Li and K in Sect. XIII. Sect. XIV contains our conclusions.

II. FUNCTIONAL INTEGRAL

We start from the partition function in presence of sources ($\int d\vec{x} = \int d^3\tilde{x}d\tau$)

$$Z_B[j_\phi] = \int D\hat{\psi}D\hat{\phi}\exp\left\{ -S[\hat{\psi},\hat{\phi}] \right\} \left( \int d\tilde{x} \left[ j_\phi(x)\hat{\phi}(x) + j_\phi(x)\hat{\phi}^*(x) \right] \right),$$

with

$$S = \int d\tilde{x} \left[ \hat{\phi}^\dagger \left( \hat{\partial}_\tau - \hat{\Delta} - \hat{\sigma} \right) \hat{\phi} + \frac{(\hat{\lambda}_\phi + \delta\hat{\lambda}_\phi)}{2} (\hat{\psi}^\dagger \hat{\psi})^2 \right. \\
\left. + \hat{\phi}^* \left( \hat{\partial}_\tau - \frac{\hat{\Delta}}{2} + \hat{\nu} + \delta\hat{\nu} - 2\hat{\sigma} \right) \hat{\phi} \right. \\
\left. - (\hat{h}_c + \delta\hat{h}_c) \left( \hat{\phi}^* \hat{\psi}_1 \hat{\psi}_2 - \hat{\phi} \hat{\psi}_1^\dagger \hat{\psi}_2^\dagger \right) \right].$$

Here, we have rescaled the fields and couplings, together with the space and time coordinates $\tilde{x} = k\tilde{x}$, $\tilde{\tau} = (k^2/2M)\tau$, with $M$ being the mass of the atoms. Our units are $\hbar = c = k_B = 1$. We use the Matsubara formalism with Euclidean time $\tau$ on a torus with a circumference given by the inverse temperature $T^{-1}$. The thermodynamic variables are $T = 2M/\tilde{k}^2$ and $\hat{\sigma} = 2M\sigma/k^2$, with $\sigma$ denoting the effective chemical potential.

The model parameters are:

1. the detuning of the magnetic field $B - B_0$ (with $\tilde{\mu} = 2\mu_B$ for $^6$Li and $\tilde{\mu} = 1.57\mu_B$ for $^{40}$K)

$$\hat{\nu} = \frac{2M}{\tilde{k}^2} \tilde{\mu}(B - B_0),$$

(2)
(2) the Feshbach or Yukawa coupling \( \hat{h}_\varphi \) which can be extracted from the properties of the quantum mechanical two-atom system as the molecular binding energy or the scattering cross section, 

(3) a point-like interaction for the fermionic atoms parameterised by \( \hat{\lambda}_\psi \). The shifts \( \delta \varphi, \delta \hat{\lambda}_\psi \) and \( \delta \hat{h}_\varphi \) are counter terms that are removed by the ultraviolet renormalisation. Details of the rescaling and the formulation can be found in [12, 14].

The scale \( \hat{k} \) (which sets the units) is arbitrary. (A typical value is \( \hat{k} = 1 \text{eV} \).) Under a rescaling of \( \hat{k} \to \hat{k}/\mu \) all quantities scale according to their canonical dimension, i.e.,

\[
\begin{align*}
\hat{x} &= \hat{k} x \to \hat{x}/\mu, \quad \hat{x} = (\hat{k}^2/(2M)) \tau \to \hat{x}/\mu^2, \\
\hat{T} &\to \mu^2 \hat{T}, \quad \hat{\sigma} \to \mu^2 \hat{\sigma}, \quad \hat{\nu} \to \mu^2 \hat{\nu}, \quad \hat{h}_\varphi \to \mu^{1/2} \hat{h}_\varphi, \\
\hat{\lambda}_\psi &\to \hat{\lambda}_\psi/\mu, \quad \hat{\psi} \to \mu^{3/2} \hat{\psi}, \quad \hat{\phi} \to \mu^{3/2} \hat{\phi}.
\end{align*}
\]

(3)

We observe that the canonical dimension of time is minus two and therefore the nonrelativistic Lagrangian has dimension five and not four, as for a relativistic quantum field theory. The total atom density \( n \) defines the Fermi momentum \( k_F \),

\[
n = \frac{k_F^3}{3\pi^2}.
\]

If one associates \( \hat{k} \) with \( k_F \) all quantities are expressed in units of the Fermi momentum [12].

From the partition function the effective action \( \Gamma \) obtains by the usual Legendre transform, \( \hat{\varphi} = \langle \phi \rangle \),

\[
\Gamma[\hat{\varphi}] = -\ln Z[j_\varphi [\hat{\varphi}]] + \int d\hat{x} (j_\varphi^* \hat{\varphi} + j_\varphi \hat{\varphi}^*). \tag{5}
\]

The order parameter \( \hat{\varphi}_0 \) for superfluidity corresponds to the minimum of \( \Gamma \) for \( j_\varphi = 0 \) and obeys the field equation

\[
\frac{\delta \Gamma}{\delta \hat{\varphi} |_{\hat{\varphi}_0} } = 0. \tag{6}
\]

The effective action generates the 1PI Green’s functions such that the propagators and transition amplitudes can be directly related to the functional derivatives of \( \Gamma \). The construction above is easily extended to an effective action that is also a functional of fermionic fields by introducing the appropriate fermionic sources, see below.

### III. Exact Functional Flow Equation and Truncation

The variation of \( \Gamma \) with the change of an effective infrared cutoff \( \hat{k} \) is given by an exact renormalisation group equation [18, 19, 20, 21]. For the present theory, this approach has been implemented in [14] within a study of the phase diagram; see also [26]. For RG studies of the purely bosonic system, see [27].

For the purpose of the present work, we extend the truncation used in [14] as well as using a particular version of the flow equation where the cutoff acts like a shift in the respective chemical potentials for \( \psi \) and \( \varphi \), as also done in [28]. To that end, we introduce an infrared-regularised partition function \( Z_k \) by adding a cutoff term to the action in Eq. (1),

\[
S[\hat{\psi}, \hat{\phi}] \to S[\hat{\psi}, \hat{\phi}] + \Delta S_k[\hat{\psi}, \hat{\phi}], \tag{7}
\]

where the cutoff term \( \Delta S_k \) is chosen as

\[
\Delta S_k[\hat{\psi}, \hat{\phi}] = \int d\hat{x} (R_k^{(F)} \hat{\psi}^* \hat{\psi} + R_k^{(B)} \hat{\phi}^* \hat{\phi}), \tag{8}
\]

with

\[
R_k^{(F)} = Z_\psi(k)k^2, \quad R_k^{(B)} = 2Z_\varphi(k)k^2,
\]

\[
\frac{\partial R_k^{(F)}}{\partial k^2} = Q_\psi Z_\varphi, \quad \frac{\partial R_k^{(B)}}{\partial k^2} = 2Q_\varphi Z_\varphi,
\]

\[
Q_\psi,\varphi = 1 - \eta_\psi,\varphi/2, \quad \eta_\psi,\varphi = -k \frac{\partial}{\partial k} \ln Z_\psi,\varphi. \tag{9}
\]

The \( k \)-dependent wave function renormalisation \( Z_\psi,\varphi \) will be determined below, and we note that \( k \) is measured in units of the fixed scale \( \hat{k} \). Introducing the effective average action \( \Gamma_k \) in terms of a modified Legendre transform,

\[
\Gamma_k[\hat{\psi}, \hat{\varphi}] = -\ln Z_k[j_\varphi, j_\psi] - \Delta S_k[\hat{\psi}, \hat{\varphi}] + \int d\hat{x} (j_\varphi^* \hat{\varphi} + j_\varphi \hat{\varphi}^* + j_\psi^* \hat{\psi} - \hat{\psi}^t j_\psi), \tag{10}
\]

the exact functional renormalisation group equation (flow equation) can straightforwardly be derived,

\[
\frac{\partial \Gamma_k}{\partial k^2} = \int d\hat{x} (Z_\varphi Q_\psi \hat{\psi}^* \hat{\varphi} \hat{\psi} c + 2Z_\varphi Q_\varphi \hat{\phi}^* \hat{\phi} \hat{\psi} c) = \begin{cases} 
\text{Tr} \left\{ -Z_\psi Q_\psi (\Gamma_k^{(2)} + R_k)^{-1} \hat{\psi} \hat{\psi} + 2Z_\varphi Q_\varphi (\Gamma_k^{(2)} + R_k)^{-1} \hat{\phi} \hat{\phi} \right\} & , \\
\end{cases} \tag{11}
\]

Here, we have expressed the propagators of the fermionic and the bosonic fields by the corresponding components of the inverse of the matrix of second functional derivatives of \( \Gamma_k \). The trace \( \text{Tr} \) contains an integration over \( \hat{x} \) or a corresponding momentum integration as well as a trace over all internal indices. Both \( \Gamma_k \) and \( \Gamma_k^{(2)} \) are functionals of arbitrary fields \( \hat{\varphi} \) and \( \hat{\psi} \) which are kept fixed for the \( k \) derivative in Eq. (11).

The flow equation (11) is a nonlinear functional differential equation and we can only hope to find approximate solutions by suitable truncations of the most general form.
functions in order:

\[ \Gamma_k = \int d\tilde{x} \left[ Z_\psi \hat{\psi}^\dagger \left( \partial_\tau - A_\psi \hat{\Delta} - \hat{\sigma} \right) \hat{\psi} 
+ Z_\varphi \hat{\varphi}^\dagger \left( \partial_\tau - A_\varphi \hat{\Delta} \right) \hat{\varphi} + u(\varphi, \sigma) 
- \hat{h}_\varphi \left( \varphi^* \hat{\psi}_1 \hat{\psi}_2 - \varphi \hat{\psi}_1^* \hat{\psi}_2^* \right) + \frac{\lambda_\psi}{2} (\hat{\psi}^\dagger \hat{\psi})^2 \right] \]

\[ = \int d\tilde{x} \left[ \psi^\dagger \left( \partial_\tau - A_\psi \hat{\Delta} - \hat{\sigma} \right) \psi 
+ \varphi^* \left( \partial_\tau - A_\varphi \hat{\Delta} \right) \varphi + u(\varphi, \sigma) 
- \hat{h}_\varphi \left( \varphi^* \psi_1 \psi_2^* - \varphi^* \psi_1^* \psi_2 \right) + \frac{\lambda_\varphi}{2} (\varphi^* \varphi)^2 \right]. \tag{12} \]

In the second equation, we have used renormalised fields

\[ \psi = Z_{\psi}^{1/2} \hat{\psi}, \quad \varphi = Z_{\varphi}^{1/2} \hat{\varphi} \tag{13} \]

and renormalised couplings

\[ h_\varphi = \hat{h}_\varphi Z_{\psi}^{-1} Z_{\varphi}^{-1/2}, \quad \lambda_\psi = \hat{\lambda}_\psi Z_{\psi}^{-2}. \tag{14} \]

The ansatz (12) generalises that of (11) by the four-fermion interaction with coupling \( \lambda_\psi \). This term is particularly important in the limit of broad Feshbach resonances as will be discussed later.

The global symmetry of \( U(1) \) phase rotations implies that the effective potential \( u \) can only depend on \( \rho = \varphi^* \varphi \). If the ground state corresponds to a homogeneous \( \varphi_0 \) the pressure and density can be computed from the properties of the effective potential

\[ \frac{\partial u}{\partial \varphi} \bigg|_{\varphi_0} = 0, \quad u_0(\sigma) = u(\varphi_0, \sigma), \]

\[ p = -\frac{\kappa^2}{2M} u_0, \quad n = -\frac{\kappa^3}{2} \frac{\partial u_0}{\partial \sigma}. \tag{15} \]

At this point, some comments concerning the wave function renormalisations are in order:

(i) Analytic continuation to “real time” and a Fourier transform to (real) frequency space results in \( \partial_\tau \rightarrow -\omega \). We define \( Z_{\psi,\varphi} \) by the coefficient of the term linear in \( \omega \) in the full inverse propagator \( \tilde{P}_{\psi,\varphi} \) given by the terms quadratic in \( \hat{\psi} \) or \( \hat{\varphi} \) in \( \Gamma_k \). More precisely, we choose in Fourier space

\[ Z_{\psi,\varphi} = \left. \frac{-\partial \tilde{P}_{\psi,\varphi}}{\partial \omega} \right|_{\hat{q}=0, \omega=0}. \tag{16} \]

(ii) With this definition of \( Z_{\psi,\varphi} \), the renormalised fields \( \hat{\psi}, \hat{\varphi} \) have a unit residuum for the pole in the propagator for \( \hat{q} \rightarrow 0 \) if the “on shell value” of \( \omega \) vanishes for \( \hat{q} = 0 \).

(iii) We use the same \( Z_{\psi,\varphi} \) in the definition of the cutoff (9) as in Eq. (12).

(iv) The fields \( \hat{\psi} \) and \( \hat{\varphi} \) describe “microscopic” or “bare” atoms and molecules while the renormalised field \( \varphi \) describes dressed molecules. The wave function renormalisation \( Z_{\varphi} \) accounts for a description of dressed molecules as a mixing of microscopic molecules and di-atom states [3, 7, 12].

IV. INITIAL CONDITIONS

Inserting the truncation (12) into the flow equation (11), and taking appropriate functional derivatives, leads to a coupled set of differential equations for the couplings \( Z_\psi, A_\psi, Z_\varphi, A_\varphi, h_\varphi, \lambda_\psi \) as well as the effective potential \( u(\varphi) \). The task is to follow the flow of these couplings as the cutoff scale \( k \) is changed. The initial values for large \( k \) are taken as

\[ Z_\varphi = Z_\psi = 1, \quad A_\varphi = \frac{1}{2}, \quad A_\psi = 1, \]

\[ u = m_\varphi^2 \rho, \]

\[ m_\varphi^2 = \nu + 2\delta - 2\delta. \tag{17} \]

Here \( \nu \) is related to the detuning of the magnetic field

\[ \nu = \frac{2M}{k^2} \bar{\mu}(B - B_0) \tag{18} \]

and we have to fix \( \delta \nu \) (i.e., the renormalised counterpart of \( \delta \nu \) in Eq. (11)) such that the Feshbach resonance in vacuum occurs for \( \nu = 0 \). The initial values of \( \lambda_\psi \) and \( h_\varphi \) are chosen such that the molecular binding energy and scattering of two atoms in vacuum is correctly described.

A realistic model contains an effective ultraviolet cutoff \( \Lambda \) which is given roughly by the inverse of the range of the van der Waals force. For \( k \ll \Lambda/k \) we can take the limit \( \Lambda \rightarrow \infty \) since all momentum integrals are ultraviolet finite. All ultraviolet “divergencies” are already absorbed in the computation of \( \Gamma_k \) [12, 13]. For \( ^6\text{Li} \) or \( ^{40}\text{K} \) one has \( \Lambda/k \) of order \( 10^2 \) to \( 10^3 \).

V. RUNNING COUPLINGS AND ANOMALOUS DIMENSIONS

The flow equations for the various couplings and the densities derived from (11) have a simple interpretation as renormalisation group improved one-loop equations [18] with full propagators and dressed vertices, but are exact. The renormalisation constants \( Z_\psi, Z_\varphi \) are related to the dependence of the unrenormalised inverse propagators (for \( \hat{\varphi} \) and \( \hat{\psi} \)) on the (Minkowski) frequency, i.e., the coefficients of the terms linear in \( \omega(\pm \partial/\partial \tau) \) in \( \Gamma^2 \). The second derivative of Eq. (11) yields exact flow equations for the inverse propagator [18, 19, 20, 21] and therefore for \( Z_\psi, Z_\varphi \). We define

\[ \eta_{\psi,\varphi} = \eta_{\psi,\varphi}/2k^2 = -\partial \ln Z_{\psi,\varphi}/\partial k^2 \tag{19} \]
and obtain in our approximation
\[ \hat{\eta}_\varphi = -\frac{h^2 Q_\varphi}{2} \frac{\partial}{\partial k^2} I^{(F)}_\varphi + \hat{\eta}_\varphi^{(B)}. \] (20)

Here the fermion loop integral
\[ I^{(F)}_\varphi = \frac{1}{8T^2} \int \frac{d^3 \hat{q}}{(2\pi)^3} \gamma_\varphi^{-3} \left[ \tanh \gamma_\varphi - \gamma_\varphi \cosh^{-2} \gamma_\varphi \right] \] (21)
involves
\[ \gamma_\varphi = \frac{1}{2T} \sqrt{(A_\varphi q^2 + k^2 - \hat{\sigma})^2 + h^2 k^2}, \]
\[ \gamma = \frac{1}{2T} (A_\varphi q^2 + k^2 - \hat{\sigma}). \] (22)

In our truncation, the boson fluctuations contribute to \( \hat{\eta}_\varphi \) only in the superfluid phase \( \langle \hat{\eta}_\varphi^{(B)}(\rho = 0) \rangle \) [14]. Crucial quantities for the investigation of this problem are the running of the Yukawa coupling \( h_\varphi \),
\[ \frac{\partial}{\partial k^2} h^2_\varphi = (\hat{\eta}_\varphi + 2\hat{\eta}_\varphi)h^2_\varphi + 2h^2_\varphi \lambda_\psi Q_\psi J^{(F)}_\varphi, \] (23)
with
\[ J^{(F)}_\varphi = \frac{1}{16T^2} \int \frac{d^3 \hat{q}}{(2\pi)^3} \gamma_\varphi^{-5} \left[ (3\gamma^2 - \gamma_\varphi^2) \tanh \gamma_\varphi \right. \]
\[ + \left. \gamma_\varphi \cosh^{-2} \gamma_\varphi \left( -3\gamma_\varphi + \gamma_\varphi^2 \right) 
+ 2\gamma_\varphi (\gamma_\varphi^2 - \gamma_\varphi^2) \tanh \gamma_\varphi \right], \] (24)
and the point-like four-fermion interaction
\[ \frac{\partial}{\partial k^2} \lambda_\psi = 2\hat{\eta}_\varphi \lambda_\psi + \lambda^2_\psi Q_\psi (I^{(F)}_\varphi + I_{\lambda_\psi}), \] (25)
\[ I_{\lambda_\psi} = -\frac{1}{4T^2} \int \frac{d^3 \hat{q}}{(2\pi)^3} \gamma_\varphi^{-1} \tanh \gamma_\varphi \cosh^{-2} \gamma_\varphi. \]

For both quantities, we neglect the contribution from the molecule fluctuations. This is a valid approximation for the parameter ranges for which we give quantitative results. In other ranges, the molecule fluctuations give more dominant contributions, see [14] and the discussion below. For \( \rho = 0 \), we note \( J^{(F)}_\varphi = I^{(F)}_\varphi \).

### VI. FLOW OF THE EFFECTIVE POTENTIAL

Keeping now \( \rho \) (and not \( \hat{\rho} \)) fixed, the flow of the effective potential obeys,
\[ \frac{\partial}{\partial k^2} u(\rho) = \zeta_F(\rho) + \zeta_B(\rho) + \hat{\eta}_\varphi \rho \alpha \varphi(\rho), \] (26)
with
\[ \zeta_F(\rho) = -Q_\varphi \int \frac{d^3 \hat{q}}{(2\pi)^3} \left( \frac{\gamma_\varphi}{\gamma_\varphi} \tanh \gamma_\varphi - 1 \right), \]
\[ \zeta_B(\rho) = Q_\varphi \int \frac{d^3 \hat{q}}{(2\pi)^3} \left( \frac{\alpha + \kappa_2}{\alpha_\varphi} \coth \alpha_\varphi - 1 \right). \] (27)

Here, we define
\[ \alpha_\varphi = \frac{1}{2T} \left[ (A_\varphi q^2 + k^2 + u') \right], \]
\[ \alpha = \frac{A_\varphi q^2 + k^2 + u}{2T}, \]
\[ \kappa_2 = \frac{\rho \alpha u''}{2T}, \kappa_3 = \frac{\rho^2 u'''}{2T}. \] (28)

and the primes denote derivatives with respect to \( \rho \). The contribution of the boson fluctuations is computed in a basis where the complex field \( \varphi \) is written in terms of two real fields with inverse bosonic propagator
\[ \mathcal{P}_\varphi = \begin{pmatrix} A_\varphi q^2 + k^2 + u' + 2\rho u'' & -2\pi n \hat{T} \ A_\varphi q^2 + k^2 + u' \end{pmatrix} \] (29)

Then, Eq. (11) evaluated for constant \( \varphi \) becomes
\[ \zeta_B(\rho) = \hat{T} Q_\varphi \pi \int \frac{d^3 \hat{q}}{(2\pi)^3} \left( \mathcal{P}_\varphi^{-1} - \frac{1}{A_\varphi q^2 + k^2} \right) \] (30)
in accordance with [29]. The subtraction of a \( \rho \)-independent part renders the momentum integral ultraviolet finite. For extensive numerical studies one would rather rely on cutoff choices as in [14] which do not require any ultraviolet subtraction, and minimise the numerical costs. However, this is of no importance for the quantities computed in this paper – the dependence of \( A_\varphi \) on \( \hat{\sigma} \) and \( \hat{T} \) is negligible, and simplicity of the approach is the more important property.

In the symmetric phase (SYM), one has \( \rho_0 = 0, u'(0) = m_\varphi, u''(0) = \lambda_\varphi \), whereas in the presence of spontaneous symmetry breaking (SSB) we use \( \rho_0 > 0, u'(\rho_0) = 0, u''(\rho_0) = \lambda_\varphi \). Eq. (25) yields in the symmetric phase
\[ \frac{\partial}{\partial k^2} m_\varphi^2 = h^2 Q_\varphi I^{(F)}_\varphi (\rho = 0) - Q_\varphi I^{(B)}_\varphi (\rho = 0) + \hat{\eta}_\varphi m_\varphi^2. \] (31)

Here, we define the boson loop integral
\[ I^{(B)}_\varphi = \frac{\lambda_\varphi}{2T} \int \frac{d^3 \hat{q}}{(2\pi)^3} \]
\[ \times \left\{ \left[ (2\alpha + \kappa_2) u'' + \alpha \rho u''' \right] \frac{\alpha + \kappa_2}{\alpha_\varphi^2 \sinh^2 \alpha_\varphi} \right. \]
\[ \left. + \left[ \kappa_2 (\kappa_2 - \alpha) - \alpha \kappa_3 \right] u'' \coth \alpha_\varphi \frac{1}{\alpha_\varphi^2} \right\}, \]
such that
\[ I^{(B)}_\varphi (\rho = 0) = \frac{\lambda_\varphi}{T} \int \frac{d^3 \hat{q}}{(2\pi)^3} \sinh^{-2} \alpha. \] (33)

The flow starts with a positive \( m_\varphi^2 \) for large \( k \) and is therefore in the symmetric regime. As \( k \) decreases, \( m_\varphi^2 \)
decreases according to Eq. 31, with \( \hat{\eta}_\varphi \) being positive. At high temperature, \( m^2_\varphi \) stays positive for all \( k \) and the system is in the symmetric phase and ungapped for all momenta. At temperatures below a pseudo-critical temperature \( T_p \), \( m^2_\varphi \) hits zero at some critical \( k_c \). For \( k < k_c \), the flow first continues in the broken-symmetry regime with non-zero \( \rho_0 \). Whereas fermionic fluctuations tend to increase \( \rho_0 \), molecule fluctuations cause its depletion. For temperatures inbetween the critical temperature and pseudo-critical temperature, \( T_c < T < T_p \), molecule fluctuations eventually win out over the fermionic fluctuations and \( \rho_0 \) vanishes again at smaller \( k \); here, the system is in the symmetric phase. Identifying the \( k \) dependence of the fermionic 2-point function with the momentum dependence, the non-zero value of \( \rho_0 \) for finite \( k \) can be associated with a pseudo-gap [30]. Below the pseudo-critical temperature, \( T < T_c \), molecule fluctuations cause its depletion. Whereas fermionic fluctuations tend to increase \( \rho_0 \), molecule fluctuations cause its depletion. For \( \rho_0 \) non-zero, i.e., for \( k < k_c \) and below the pseudo-critical temperature \( T_p \), we find the coupled system of flow equations for the symmetric phase

\[ \frac{\partial h^2}{\partial K} = \frac{h^4}{32\pi} K^{-2} + \frac{h^2_\varphi \lambda_\varphi}{4\pi}, \]

\[ \frac{\partial \lambda_\varphi}{\partial K} = \frac{\lambda^2_\varphi}{8\pi}, \]  

(38)

For \( \rho = 0 \), \( T = 0 \), \( \sigma < 0 \), one also finds

\[ I^{(F)}_\varphi = \frac{1}{16\pi} A^{-3/2}_\varphi K^{-1}, \quad I_{\lambda_\varphi} = 0, \]  

(36)

and, in our approximation, the contributions of the boson loops to \( \hat{\eta}_\varphi \), \( \partial h^2_\varphi / \partial k^2 \) and \( \partial \lambda_\varphi / \partial k^2 \) vanish for \( m^2_\varphi > 0 \).

At \( \hat{T} = 0 \) and \( \sigma < 0 \), we can use \( Z_\varphi = A_\varphi = 1, Q_\varphi = 1 \). This is an exact result [7] as long as all propagators in the relevant diagrams have simple poles in the imaginary \( q_0 \) plane. With

\[ \hat{\eta}_\varphi = \frac{h^2_\varphi}{64\pi} K^{-3}, \]  

(37)

we find the coupled system of flow equations for the symmetric phase

\[ \frac{\partial h^2}{\partial K} = \frac{h^4}{32\pi} K^{-2} + \frac{h^2_\varphi \lambda_\varphi}{4\pi}, \]

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\[ \frac{\partial h^2}{\partial K} = \frac{h^4}{32\pi} K^{-2} + \frac{h^2_\varphi \lambda_\varphi}{4\pi}, \]

\[ \frac{\partial \lambda_\varphi}{\partial K} = \frac{\lambda^2_\varphi}{8\pi}, \]  

(38)

The solution of the last equation,

\[ \lambda^{-1}_\varphi = \lambda^{-1}_{\varphi, \text{in}} + \frac{1}{8\pi} (K_{\text{in}} - K) \]

\[ = \lambda^{-1}_{\varphi, 0} - \frac{K}{8\pi}, \]  

(39)

renders \( \lambda_\varphi \) almost independent of \( K \) if \( K \ll K_{\text{in}} + 8\pi/\lambda_{\varphi, \text{in}} \). Here, we assume implicitly that \( \lambda_{\varphi, \text{in}} \) is not too much negative such that \( \lambda_\varphi \) remains finite in the whole \( k \) range of interest. For positive \( \lambda_\varphi \), the self-consistency of the flow requires an upper bound \( \lambda_{\varphi, 0} (8\pi) < 1/K_{\text{in}} \).

In contrast, for \( \lambda_\varphi = 0 \) the solution for \( h^2_\varphi \),

\[ h^{-2}_\varphi = h^{-2}_{\varphi, \text{in}} + \frac{1}{32\pi} (K^{-1} - K_{\text{in}}^{-1}) \]  

(40)

is dominated by small \( K \).

For \( \lambda_\varphi \neq 0 \), the flow of \( h_\varphi \) is modified, however, without changing the characteristic behaviour for \( K \rightarrow 0 \). Indeed, in the limit \( K \rightarrow 0 \) the term \( \sim \lambda_\varphi \) becomes subdominant for the evolution of \( h^2_\varphi \). The flow for the ratio \( h^2_\varphi / K \) reaches a fixed point \( 32\pi \). We can explicitly solve the system [33] by consecutive integrations. One obtains
for \( \hat{h}_\varphi^2 = Z_\varphi h_\varphi^2 \)

\[
\dot{\hat{h}}_\varphi^2 = \hat{h}_{\varphi,\text{in}}^2 \exp \left\{ -\frac{1}{4\pi} \int dK \lambda_\varphi(x) K_{\text{in}} \right\}
= \hat{h}_{\varphi,\text{in}}^2 (1 - \hat{c}_0 K_{\text{in}})^2 (1 - \hat{c}_0 K)^{-2}
= \hat{h}_{\varphi,0}^2 (1 - \hat{c}_0 K)^{-2}.
\]

This yields for the wave function renormalisation

\[
Z_\varphi = 1 + \frac{1}{32\pi} \int K_{\text{in}}^{-1} d(x^{-1}) \dot{\hat{h}}_\varphi^2 (x^{-1})
= 1 + \frac{\hat{h}_{\varphi,0}^2}{32\pi} \left[ \frac{1}{K} \left( 1 - \frac{\hat{c}_0}{1/K - \hat{c}_0} \right) \right. \\
\left. + 2\hat{c}_0 \ln \left( \frac{1}{K} - \hat{c}_0 \right) - (K \to K_{\text{in}}) \right],
\]

with

\[
\hat{c}_0 = \frac{\lambda_{\psi,0}}{8\pi}, \quad \frac{1}{K} > \hat{c}_0. \tag{43}
\]

For \( K \to 0 \), the ratio \( \hat{h}_{\varphi}^2 / \hat{h}_{\varphi,\text{in}}^2 \) obviously reaches a constant which depends on \( \lambda_\psi, K_{\text{in}} \), whereas \( Z_\varphi \) diverges \( \approx \hat{h}_{\varphi,0}^2 / (32\pi K) \), as is consistent with the fixed-point behaviour \( \hat{h}_{\varphi}^2 \approx 32\pi \). Neglecting \( \hat{c}_0 \) (as always appropriate for small enough \( K \)) and assuming a broad resonance \( \hat{h}_{\varphi,0} \gg 32\pi \), we arrive at the simple relation

\[
Z_\varphi \approx \frac{\hat{h}_{\varphi,0}^2}{32\pi K}. \tag{44}
\]

In this limit, the factor \( Q_\varphi \) appearing in the flow equations is given by

\[
Q_\varphi = 1 - \frac{k^2}{2(k^2 - \hat{c}_0)} \tag{45}
\]
and approaches \( 1/2 \) for \( k^2 \gg -\hat{c}_0 \). We note that in this range the bosonic cutoff function \( R_k(B) = 2Z_\varphi k^2 \) is effectively linear in \( k \) and not quadratic.

We finally investigate the flow equation for \( m_\varphi^2 \) in the symmetric phase. As we will see below, the boson loops (the last contribution in Eq. (35)) vanish in our truncation. This yields

\[
\frac{\partial m_\varphi^2}{\partial K} = \frac{h_\varphi^2}{8\pi} + \frac{h_\varphi^2 m_\varphi^2}{32\pi} K^{-2}, \tag{46}
\]

or

\[
\frac{\partial m_\varphi}{\partial K} = \frac{\partial}{\partial K} (Z_\varphi m_\varphi^2) = \frac{\dot{\hat{h}}_\varphi^2}{8\pi}. \tag{47}
\]

The solution reads

\[
\dot{m}_\varphi^2 = \frac{2M\hat{\mu}(B - B_0)}{k^2} - 2\sigma + \delta\nu \tag{48}
\]

\[
- \frac{\hat{h}_{\varphi,\text{in}}^2 (1 - \hat{c}_0 K_{\text{in}})^2}{8\pi c_0} \left( \frac{1}{1 - \hat{c}_0 K_{\text{in}}} - \frac{1}{1 - \hat{c}_0 K} \right).
\]

The counterterm \( \delta\nu \) is determined from the condition \( \dot{m}_\varphi^2 (B = B_0, \sigma = 0, k = 0) = 0 \), such that (up to minor corrections)

\[
\dot{m}_\varphi^2 = \frac{2M\hat{\mu}(B - B_0)}{k^2} + \frac{\hat{h}_{\varphi,0}^2}{8\pi} K - 2\sigma. \tag{49}
\]

In particular, for \( k = 0 \) and \( \sigma < 0 \) one has \( K = \sqrt{-\sigma} \), and Eq. (49) yields the expression for \( \dot{m}_\varphi^2 \) in the presence of all fluctuations

\[
\dot{m}_\varphi^2 = \frac{2M\hat{\mu}(B - B_0)}{k^2} + \frac{\hat{h}_{\varphi,0}^2 \sqrt{-\sigma}}{8\pi - \lambda_{\psi,0} \sqrt{-\sigma}} - 2\sigma. \tag{50}
\]

At this point, we have the explicit solution of the flow equation for \( T = 0 \). In the present simple truncation, we expect that the description of the system is qualitatively correct. Higher quantitative precision will require a more extended truncation; however, this is not the main emphasis of the present work which rather concentrates on the structural properties related to the fixed points.

VIII. FLOW EQUATIONS AND FIXED POINTS

The explicit solution of the preceding section is indeed very useful for verifying and explicitly demonstrating the general fixed-point properties. The following features of the flow equations hold in a much wider context of different microscopic actions and different cutoffs. The overall pattern of the flow is governed by the existence of fixed points. Some of these fixed points may correspond to particularly simple situations, being less relevant for the physics under discussion. By contrast, the stability or instabilities of small deviations from the various fixed points are much more important, as they determine the topology of the flow in the space of coupling constants, as demonstrated in Fig. 1. If the system is in the vicinity of any of the different fixed points the number of effective couplings needed for a description of the macrophysics (beyond \( T \)) corresponds to the number of relevant directions.

A particularity of our system concerns a certain redundancy in the description: a pointlike interaction can be described either by the four-fermion coupling \( \lambda_\psi \) or by a limiting behaviour of the scalar exchange. Indeed, for \( \hat{h}_{\varphi,\text{in}} \to \infty \) and fixed \( \hat{h}_{\varphi,\text{in}}^2/\dot{m}_\varphi^2 \), the molecule exchange interaction becomes effectively point-like. Therefore, we may define an effective point-like coupling

\[
\lambda_\psi, \text{eff} = \lambda_\psi - \frac{\hat{h}_{\varphi,0}^2}{m_\varphi^2}, \tag{51}
\]

which describes the interaction in the zero-momentum limit. Combining the flow equations (38), (49) for \( \lambda_\psi, \hat{h}_\varphi^2, \dot{m}_\varphi^2 \), one obtains

\[
\frac{\partial \lambda_\psi, \text{eff}}{\partial K} = \frac{\lambda_\psi, \text{eff}^2}{8\pi}. \tag{52}
\]
This is the same flow equation as for $\lambda_\psi$ (38).

Next, we may consider the renormalised couplings in units of $\hbar k$ instead of $\hbar \bar{\kappa}$. This will reveal the relevant fixed points for the flow more clearly. We define, according to the scaling dimensions of Eq. (3),

\[
\lambda_\psi = \lambda_\psi k, \quad \hat{\lambda}_\psi,_{\text{eff}} = \lambda_\psi,_{\text{eff}} k,
\]

\[
h_\varphi = \frac{\hbar^2}{\sqrt{k}}, \quad \tilde{m}_\varphi^2 = m_\varphi^2 / k^2,
\]

\[
t = \ln \frac{k}{k_\text{in}},
\]

and obtain the dimensionless flow equations

\[
\partial_t \hat{\lambda}_\psi = \hat{\lambda}_\psi + \frac{\hat{\lambda}_\psi^2 y}{8\pi}, \quad \partial_t \hat{\lambda}_\psi,_{\text{eff}} = \hat{\lambda}_\psi,_{\text{eff}} + \frac{\hat{\lambda}_\psi,_{\text{eff}} y}{8\pi},
\]

\[
\partial_t \tilde{h}_\varphi^2 = -\tilde{h}_\varphi^2 + \frac{\tilde{h}_\varphi^2 y^3}{32\pi} + \frac{\tilde{h}_\varphi^2 \lambda_\psi y}{4\pi},
\]

\[
\partial_t \tilde{m}_\varphi^2 = -2\tilde{m}_\varphi^2 + \frac{\tilde{h}_\varphi^2 y^3}{8\pi} + \frac{\tilde{h}_\varphi^2 \tilde{m}_\varphi^2 y^3}{32\pi},
\]

with

\[
y = \left(\frac{k^2}{k^2 - \sigma}\right)^{1/2}.
\]

Let us first consider $\hat{\sigma} = 0$, or, more generally, $k^2 \gg -\hat{\sigma}$, such that $y = 1$. In this case, we observe two fixed points for $\hat{\lambda}_\psi$, either $\hat{\lambda}_\psi = 0$ or $\hat{\lambda}_\psi = -8\pi$. The corresponding fixed points for $\tilde{h}_\varphi^2$ are

\[(A): \quad \hat{\lambda}_\psi = 0 \quad , \quad \tilde{h}_\varphi^2 = 32\pi \quad , \quad \tilde{m}_\varphi^2 = 4\]

\[(B): \quad \hat{\lambda}_\psi = -8\pi \quad , \quad \tilde{h}_\varphi^2 = 96\pi \quad , \quad \tilde{m}_\varphi^2 = -12.\]  

(We will discuss later the fixed points with $\tilde{h}_\varphi^2 = 0 \quad , \quad \tilde{m}_\varphi^2 = 0$.) The fixed point (A) is infrared stable for $\tilde{h}_\varphi^2$ and $\tilde{h}_\varphi^2$ – both couplings run towards their fixed-point values as $k$ is lowered, as is visualised in Fig. 1. In contrast, $\tilde{m}_\varphi^2$ is infrared unstable – the detuning $B - B_0$ corresponds to a relevant perturbation of the fixed point. With $\tilde{h}_\varphi^2$ at the fixed-point value, $\tilde{m}_\varphi^2$ deviates from its fixed point with an anomalous dimension

\[
\tilde{m}_\varphi^2 = 4 + \delta \tilde{m}_\varphi^2 \frac{k_{\text{in}}}{k}.
\]

Comparing this with the explicit solution (49) for $\lambda_\psi = 0$, $\hat{\sigma} = 0$, $\hat{h}_\varphi = \hat{h}_\varphi,_{\text{in}}$, namely

\[
\tilde{m}_\varphi^2 = \frac{\tilde{h}_\varphi^2}{8\pi} + \frac{2M \bar{\mu} (B - B_0)}{Z_\varphi k^2 k^2}
\]

with Eq. (42),

\[
Z_\varphi \approx \frac{\tilde{h}_\varphi^2,_{\text{in}}}{32\pi k}.
\]

We conclude that broad Feshbach resonances (large enough Yukawa couplings) can be characterised by the fixed point (A), with $\tilde{h}_\varphi$ and $\hat{\lambda}_\psi$ being irrelevant and $\tilde{m}_\varphi^2 \sim B - B_0$ the relevant coupling.

For the second fixed point (B), $\hat{\lambda}_\psi$ becomes a relevant parameter. For $\tilde{h}_\varphi^2$, the fixed point (60) with $\tilde{h}_\varphi^2 = 96\pi$ remains infrared attractive, but the fixed point for $\tilde{m}_\varphi^2$ occurs for negative $\tilde{m}_\varphi^2$ where our computation in the symmetric phase is no longer valid.

Finally, we turn to the fixed points with $\tilde{h}_\varphi^2 = 0$. In this case, $\tilde{h}_\varphi^2$ is always a relevant coupling and increases as $k$ is lowered. The fixed point

\[(C): \quad \hat{\lambda}_\psi = 0 \quad , \quad \tilde{h}_\varphi^2 = 0 \quad , \quad \tilde{m}_\varphi^2 = 0\]  

is infrared attractive for $\hat{\lambda}_\psi$. For this case of a narrow Feshbach resonance, the crossover problem can be solved exactly (6), and perturbation theory around the exact solution becomes valid for small $\tilde{m}_\varphi^2$. The coupling $\tilde{m}_\varphi^2$ is a second relevant parameter around this fixed point. As one increases $\tilde{h}_\varphi^2,\,_{\text{in}}$, a crossover to the fixed point (A) occurs (6). The “narrow-resonance fixed point” (C) describes the limit of a combined model with free fermions and free bosons, sharing the same chemical potential.

We emphasise that, for small $\tilde{h}_\varphi$, the equivalent purely fermionic description typically has a nonlocal interaction.

For the fourth fixed point of our system,

\[(D): \quad \hat{\lambda}_\psi = -8\pi \quad , \quad \tilde{h}_\varphi^2 = 0 \quad , \quad \tilde{m}_\varphi^2 = 0,\]

all three parameters are relevant. This point corresponds again to strong attractive interactions between...
the fermionic atoms. The flow away from this fixed point for non-vanishing $h_{\varphi}^2/\tilde{m}_{\varphi}^2$ can be characterised by the flow of the ratio $(y = 1)$

$$\partial_t \left( \frac{\tilde{h}_{\varphi}^2}{\tilde{m}_{\varphi}^2} \right) = \left( 1 + \frac{\lambda_{\psi}}{4\pi} \right) \frac{\tilde{h}_{\varphi}^2}{\tilde{m}_{\varphi}^2} - \frac{1}{8\pi} \left( \frac{\tilde{h}_{\varphi}^2}{\tilde{m}_{\varphi}^2} \right)^2$$

(63)

which does not have a fixed point for positive $\tilde{h}_{\varphi}^2, \tilde{m}_{\varphi}^2$. Formally, the flow runs for negative $\tilde{m}_{\varphi}^2$ towards the fixed point (B) with $\tilde{h}_{\varphi}/\tilde{m}_{\varphi} = -8\pi$. Actually, it may be possible to consider $\lambda_{\psi}$ as a redundant parameter using partial bosonisation and rebosonisation during the flow [31], see also [21]. Then, the fixed points with $\tilde{\lambda}_{\psi} = -8\pi$ may again be associated with broad Feshbach resonances. We show the four fixed points A, B, C, D and the infrared flow of the couplings $\tilde{h}_{\varphi}^2$ and $\tilde{\lambda}_{\psi}$ in Fig. [1].

We should emphasise that the inclusion of the omitted contributions from boson fluctuations to the flow of $\tilde{\lambda}_{\psi}$ could result in corrections $(y = 1)$,

$$\partial_t \tilde{\lambda}_{\psi} = \tilde{\lambda}_{\psi} + \frac{\tilde{h}_{\varphi}^2}{8\pi} + \frac{c_1}{32\pi} \tilde{h}_{\varphi}^4 + \frac{c_2}{32\pi} \tilde{\lambda}_{\psi} \tilde{h}_{\varphi}^2,$$

(64)

such that the flow of $\tilde{\lambda}_{\psi}$ remains no longer independent of $\tilde{h}_{\varphi}^2$. This will change the precise location of fixed points (A) and (B). We expect that the qualitative characteristics of the fixed point (A) remain unchanged, whereas the fate of the fixed point (B) is less clear. We may consider the flow of the ratio $\tilde{\lambda}_{\psi}/\tilde{h}_{\varphi}^2$,

$$\partial_t \left( \frac{\tilde{\lambda}_{\psi}}{\tilde{h}_{\varphi}^2} \right) = 2 \left( \frac{\tilde{\lambda}_{\psi}}{\tilde{h}_{\varphi}^2} \right)$$

(65)

$$+ \frac{\tilde{h}_{\varphi}^2}{32\pi} \left[ c_1 + (c_2 - 1) \frac{\tilde{\lambda}_{\psi}}{\tilde{h}_{\varphi}^2} - 4 \left( \frac{\tilde{\lambda}_{\psi}}{\tilde{h}_{\varphi}^2} \right)^2 \right].$$

For any possible fixed point, one has

$$\tilde{h}_{\varphi}^2 = 32\pi \left( 1 - \frac{\tilde{\lambda}_{\psi}}{4\pi} \right) \Rightarrow \frac{\tilde{h}_{\varphi}^2}{32\pi} = \left( 1 + 8 \frac{\tilde{\lambda}_{\psi}}{\tilde{h}_{\varphi}^2} \right)^{-1},$$

(66)

and therefore obtains the fixed point condition for $Q = \tilde{\lambda}_{\psi}/\tilde{h}_{\varphi}^2$ as

$$c_1 + (c_2 + 1)Q + 12Q^2 = 0.$$  

(67)

In general, this quadratic equation has two distinct solutions, where the larger value of $Q$ is infrared stable and corresponds to (A), while the smaller value of $Q$ is unstable and corresponds to (B).

We finally include the effect of a nonzero negative chemical potential $\tilde{\sigma}$. As soon as $k^2$ becomes smaller than $\tilde{\sigma}$, the parameter $y$ rapidly goes to zero. Then, only the first term in the flow equations [31] matters. In this range of $k$, the couplings $\tilde{\lambda}_{\psi}, \tilde{h}_{\varphi}^2, \tilde{m}_{\varphi}^2$ remain constant. A negative $\tilde{\sigma}$ acts as an additional infrared cutoff, such that the flow is effectively stopped for $k^2 < -\tilde{\sigma}$. This yields a simple rough picture: the couplings $\tilde{\lambda}_{\psi}, \tilde{h}_{\varphi}^2, \tilde{m}_{\varphi}^2$ flow according to Eq. [67] for $k^2 > -\tilde{\sigma}$ until the flow stops when $k^2 < -\tilde{\sigma}$.

The fixed points are relevant not only for $T = 0$. We demonstrate the influence of the fixed point (A) on the flow of the Yukawa coupling $h_{\varphi}$ for $T = 0.5$ and $c = 1$ in Figs. [2, 3]. In Fig. [2] we observe that the renormalised Yukawa coupling $h_{\varphi}$ at small $k$ becomes almost independent of its initial value $h_{\varphi,0}$ if the latter is large. Figure [3] demonstrates the influence of the background scattering length $a_{bg}$ for a fixed value of $a$, again for $T = 0.5$ and $c = 1$.

FIG. 2: Sensitivity of the renormalised Yukawa coupling $h_{\varphi}$ to the two-body Yukawa coupling $h_{\varphi,0}$ at fixed $\tilde{\lambda}_{\psi,0}$: (a) UV and (b) IR flow of the Yukawa coupling. We use $k = k_F = 1$ eV, $a_{bg,0} = \tilde{\lambda}_{\psi,0}/(8\pi) = 0.38$ as appropriate for $^6$Li and $\tilde{T} = 0.5$, $c^1 = 1$. The different curves correspond to the two-body value of the Yukawa coupling $h_{\varphi,0}^2 = 3.72 \cdot (10^3, 10^4, 10^5)$ (from top to bottom), where the last value corresponds to $^6$Li while the first one is comparable to that of $^{40}$K. Universality with respect to the value of the two-body Yukawa coupling is very strong even for a comparatively large (fixed) $\tilde{\lambda}_{\psi}$. 

FIG. 3: IR flow of the Yukawa coupling $h_{\varphi}$ for $T = 0.5$ and $c = 1$.
For an estimate of the condensate fraction \( \Omega_C = \hat{n}_C/\hat{n} \), we use (up to small corrections involving \( K_{in} \))

\[
\frac{\partial \hat{n}_C^2}{\partial \hat{\sigma}} = -2Z_\varphi \left[ 1 + O \left( \hat{c}_0 K \ln \frac{1}{K} \right) \right],
\]

(71)

where we have used (111) and (110). For small \( K \), the term \(-2Z_\varphi \) dominates and we conclude that the condensate fraction is given by the renormalised order parameter \( \rho_0 \),

\[
\Omega_C = \frac{2\rho_0}{\hat{n}}.
\]

(72)

The above formula is one viable definition of the condensate fraction. In general, it still needs to be related to the definition of a condensate fraction as measured in a particular experiment.

The density receives contributions from atoms and molecules with different momenta. In order to describe each momentum mode accurately, we first compute the change of the density with \( k \),

\[
\frac{d}{dk^2} \hat{n} = - \frac{d}{dk^2} \frac{\partial u_0}{\partial \sigma} = - \frac{\partial}{\partial k^2} \frac{\partial u_0}{\partial \sigma} \bigg|_{\rho_0} - \frac{\partial^2 u}{\partial \rho \partial \sigma} \bigg|_{\rho_0} \frac{d\rho_0}{dk^2},
\]

(73)

and then integrate from \( k_{in} \) to \( k = 0 \). The r.h.s. will be dominated by momenta \( q^2 \approx k^2 \). The first term in Eq. (73) is given by the \( \hat{\sigma} \) derivative of Eq. (27). The different contributions have an approximate interpretation in terms of unbound atoms, molecules and the condensate density

\[
\frac{d\hat{n}_F}{dk^2} = - \frac{\partial}{\partial k^2} \zeta_F(\rho_0), \quad 2\frac{d\hat{n}_M}{dk^2} = - \frac{\partial}{\partial \rho} \zeta_B(\rho_0),
\]

\[
\frac{d\hat{n}_C}{dk^2} = - \frac{\partial^2 u}{\partial \rho \partial \sigma} \bigg|_{\rho_0} \left( \frac{d\rho_0}{dk^2} + \hat{n}_C \right).
\]

(74)

For example, for \( \rho_0 = 0 \) one rediscovers the integral over the usual Fermi distribution (\( Q_\varphi = 1 \))

\[
\frac{d\hat{n}_F}{dk^2} = \int \frac{d^3 \hat{q}}{(2\pi)^3} \exp \left\{ \left( A_\varphi \hat{q}^2 + k^2 - \hat{\sigma} \right)/T \right\} + 1.
\]

(75)

In (73) we have traded the \( \hat{\sigma} \) derivative for a \( k^2 \) derivative, since \( \gamma \) depends only on the combination \( k^2 - \hat{\sigma} \). As long as \( A_\varphi \) depends only on \( K^2 = k^2 - \hat{\sigma} \), the \( k^2 \) derivative also acts on \( A_\varphi \), and Eq. (75) can be integrated,

\[
\hat{n}_F = \int \frac{d^3 \hat{q}}{(2\pi)^3} \frac{2}{\exp \left\{ \left( A_\varphi \hat{q}^2 - \hat{\sigma} \right)/T \right\} + 1} + \hat{n}_F(k_{in}) - \zeta_F(\rho = 0, k_{in}) = 0. \tag{76}
\]

We may evaluate the initial value \( \hat{n}_F(k_{in}) \) in perturbation theory, finding both quantities being exponentially suppressed, \( \hat{n}_F(k_{in}) = \zeta_F(\rho = 0, k_{in}) \approx 0 \). In our approximation, (\( A_\varphi = 1 \)) \( \hat{n}_F \) is simply the density of a free gas of fermionic atoms as long as \( \rho_0 = 0 \). In the presence

**FIG. 3:** Sensitivity of the renormalised Yukawa coupling \( h_\varphi \) to the two-body background scattering length \( \lambda_\varphi \) for fixed two-body Yukawa coupling \( h_{\varphi,0} = 3.72 \cdot 10^6, k = k_\varphi = 1 \text{eV} \), as appropriate for \( ^6\text{Li} \): (a) UV and (b) IR flow of the Yukawa coupling. We plot curves for \( a_{bg} \hat{k}_n = \lambda_\varphi/(8\pi) = -0.38/(10^{-3}, 10^{-2}, 10^{-1}, 1) \) (the last value corresponds to \( ^6\text{Li} \)) and \( T = 0.5, c^{-1} = 1 \). The uppermost line in (b) has the largest \( a_{bg} \) and starts in (a) as the lowest line.

**IX. DENSITY AND CONDENSATE FRACTION**

Let us next compute the density

\[
\hat{n} = \frac{n}{k^3} = - \frac{\partial u_0}{\partial \sigma},
\]

(68)

where the \( \hat{\sigma} \) derivative will be taken at fixed \( \rho \) and \( k \). This is possible, since for the minimum \( d\rho_0/\partial \sigma = \partial u/\partial \sigma_{\rho_0} + (\partial u/\partial \varphi)(\partial \varphi_0/\partial \sigma) = \partial u/\partial \sigma_{\rho_0} \). Before doing a more detailed calculation, we approximate the potential by

\[
u_s(\varphi, \hat{\sigma}) = \nu_s(\hat{\sigma}) \hat{\rho} + \frac{1}{2} \lambda_\varphi \hat{\rho}^2
\]

(69)

with \( \lambda_\varphi \) independent of \( \hat{\sigma} \). The density then receives contributions from unbound atoms and molecules (\( \hat{n}_F + 2\hat{n}_M \)) and atoms in the condensate (\( \hat{n}_C \)) for \( \rho_0 \neq 0 \),

\[
\hat{n}_F + 2\hat{n}_M = - \frac{\partial \hat{\nu}}{\partial \hat{\sigma}}, \quad \hat{n}_C = - \frac{\partial \hat{\nu}_M}{\partial \hat{\sigma}} \hat{\rho}_0.
\]

(70)
of a condensate ($\rho_0 \neq 0$), we typically find a flow where $\rho_0(k) = 0$ for $k > k_c$, such that the flow of $\hat{n}_F(k)$ remains unchanged in this range. On the other hand, the contribution from modes with $q^2 < k_c^2$ will be suppressed by the presence of a gap $\Delta = \hbar \sigma \sqrt{\rho_0}$ in the propagator. We emphasise that the derivative $\partial / \partial k^2$ in Eq. (75) does not act on $\rho_0$, since the $\sigma$ derivative in Eq. (74) is taken at fixed $\rho$. Therefore the flow for $\hat{n}_F$ has to be integrated numerically if $\rho_0 \neq 0$.

For the computation of the molecule density $\hat{n}_M$, we need $\partial \zeta_B / \partial \sigma$. Let us concentrate on $\rho_0 = 0$ and neglect the $\sigma$ dependence of $Q_\varphi$ and $A_\varphi$, such that

$$\frac{\partial \zeta_B}{\partial \sigma} = 2Q_\varphi \int \frac{d^3 \hat{q}}{(2\pi)^3} \frac{\partial}{\partial \sigma} \left( \exp \left\{ (A_\varphi \hat{q}^2 + 2k^2 + m^2_\varphi) / |T| \right\} - 1 \right)^{-1}$$

$$= Q_\varphi \frac{m^2_\varphi}{\partial k^2} \int \frac{d^3 \hat{q}}{(2\pi)^3} \times \left( \exp \left\{ (A_\varphi \hat{q}^2 + 2k^2 + m^2_\varphi) / |T| \right\} - 1 \right)^{-1}.$$  

(77)

We next use Eq. (74)

$$\frac{dm^2_\varphi}{d\sigma} \approx -2 - m^2_\varphi \frac{\partial}{\partial \sigma} \ln Z_\varphi.$$  

(78)

The second term can be approximated for small $K$ by

$$m^2_\varphi \frac{\partial}{\partial \sigma} \ln Z_\varphi \approx \frac{m^2_\varphi}{2(k^2 - \sigma)}.$$  

(79)

If we neglect this term and set $Q_\varphi = 1$ we find the intuitive formula

$$\hat{n}_M = \int \frac{d^3 \hat{q}}{(2\pi)^3} \frac{1}{\exp \left\{ (A_\varphi \hat{q}^2 + m^2_\varphi) / |T| \right\} - 1}.$$  

(80)

This is the expression for a free boson gas. What is crucial, however, is the appearance of the inverse propagator for the renormalised bosonic field $\varphi$ instead of the microscopic or “bare” molecule field $\hat{\varphi}$. The propagators for the renormalised and bare fields are related by a relative factor $Z_\varphi$, such that we find for the density of bare molecules

$$\hat{n}_{M, \text{bare}} = Z_\varphi^{-1} \hat{n}_M.$$  

(81)

As a consequence, one may have a substantial molecule fraction $2\hat{n}_M / \hat{n}$ even if the density of bare molecules is tiny. These features reproduce the results from a solution of the Schwinger-Dyson equations [8] where a bare molecule density in accordance with the experimental result of Partridge et al. [32] has been found. It is one of the important advantages of our functional renormalisation group approach that it accounts for the distinction between renormalised and bare molecules in a very direct and straightforward manner [14]. Within a Hamiltonian formulation, this issue is related to the mixing between open-channel and closed-channel atoms, and its correct treatment is crucial for a quantitatively reliable description of the crossover. We emphasise that the renormalised molecule field plays an important role even on the BCS side of the crossover. The composite bosons are crucial effective degrees freedom, even though the microscopic theory can be very well approximated by a point-like interaction between fermionic atoms without any reference to molecules. On the BCS side, the renormalised field $\varphi$ describes Cooper pairs. Nevertheless, one never needs these physical interpretations explicitly, since the density only involves the $\sigma$ dependence of the potential at its minimum, which is a well-defined quantity.

We finally turn to the condensate density. If we can approximate the $\sigma$ dependence of $(\mu - \bar{u})$ by a term $-2\sigma \rho$, as suggested by Eq. (78), we infer

$$\frac{d\hat{n}_C}{dk^2} = 2 \frac{d\rho_0}{dk^2} + 2\hat{n}_C \rho_0 = 2Z_\varphi \frac{d}{dk^2} \frac{\rho_0}{Z_\varphi}.$$  

(82)

(This approximation needs to hold only in the range of $k$ where $\rho_0$ differs from zero.) The dominant flow of $\hat{n}_C$ typically arises from a region where the $k$ dependence of $Z_\varphi$ is already sub-leading, resulting in $\hat{n}_C \approx 2\rho_0$, as found above. In practice, all these various approximations of our analytical discussion need not to be made, since it is sufficient to follow the flow of $\hat{\varphi}(k)$ numerically, starting from an initial value $\hat{n}(k_{\text{in}}) \approx 0$ and extracting the physical density for $k = 0$.

X. VACUUM AND TWO-ATOM SCATTERING

It is one of the advantages of our method that it can access simultaneously the many-body physics of a gas in thermal equilibrium and the two-body physics of atom scattering and molecular binding. Indeed, the two-body physics describes excitations above the vacuum. In turn, the vacuum and the properties of its excitations obtain in our formalism simply by taking the limit of vanishing density and temperature. (More precisely, the limit should be taken such that the ratio $T / \hat{n}^{2/3}$ is large enough that no condensate occurs.) The scattering cross section between two atoms can then be directly inferred from the Yukawa coupling and the propagator of the molecule field in the vacuum.

For $T = 0$, the condition of zero density requires $\hat{\sigma} \leq 0$ in order to ensure $\hat{n}_F = 0$, cf. Eq. (70). On the other hand, we infer from Eq. (80) that $m^2_\varphi \geq 0$ is needed for $\hat{n}_M = 0$, whereas a vanishing condensate density requires $\rho_0 = 0$. The vacuum is the state that is reached as density and temperature approach zero from above. It therefore corresponds to the boundary of the region where $\hat{\sigma} \leq 0, m^2_\varphi \geq 0$. There are two branches of vacuum states (for a more detailed discussion, see [15]). The first has a negative $\hat{\sigma} = \hat{\sigma}_A < 0$ and $m^2_\varphi = 0$. In this case, the single-atom excitations have a gap $-\hat{\sigma}_A > 0$ which corresponds
to half the binding energy of the stable molecules,

$$\epsilon_M = \frac{k^2 \delta_{\Lambda}}{M} , \quad \epsilon_M = 2\delta_{\Lambda}. \quad (83)$$

We identify this state with the “molecule phase” where stable molecules exist in the vacuum. This phase is realized for \( B < B_0 \), with \( \epsilon_M \) or \( \delta_{\Lambda} \) being a function of \( B \) that vanishes for \( B \to B_0 \). The other branch corresponds to \( \delta = 0, \, m^2_{\varphi} > 0 \). This “atom phase” of the vacuum is realized for \( B > B_0 \), with \( m^2_{\varphi} \) a function of \( B \) vanishing for \( B \to B_0 \). In the atom phase the “binding energy” \( \epsilon_M = \frac{k^2 m^2_{\varphi}}{(2M)} \) is positive and the “molecules” correspond to unstable resonances. We observe a continuous phase transition between the molecule and atom phase for \( m^2_{\varphi} = 0, \, \delta = 0 \), corresponding to the location of the Feshbach resonance at \( B = B_0 \). This fixes \( \delta \nu \) in the initial value of \( m^2_{\varphi} \) by the requirement that for \( \nu = 0 \) the mass term \( m^2_{\varphi}(\delta) \) vanishes for \( \delta = 0 \) if \( T = 0 \).

The vacuum state can be used to fix the parameters of our model by direct comparison to experimentally measured binding energies and cross sections. Let us first consider the molecular binding energy \( \epsilon_M(B) \) that can be computed from Eq. (50) by requiring \( m^2_{\varphi} = 0 \), i.e.,

$$\epsilon_M(B) = \frac{\delta_{\Lambda}(B) \hat{k}^2}{M} = \frac{\hat{\mu}(B - B_0)}{16\pi\sqrt{M}} \sqrt{\epsilon_M} \left( 1 - \frac{\lambda_{\varphi,0} \sqrt{M |\epsilon_M|}}{8\pi\hbar k} \right)^{-1}. \quad (84)$$

In the vicinity of the Feshbach resonance (small \(|\epsilon_M|\)) the binding energy depends quadratically on \( B \)

$$\epsilon_M = -\left( \frac{16\pi}{\hat{\mu}(B - B_0)} \right)^2 \hat{k}^2 \hat{h}^2_{\varphi,0}. \quad (85)$$

Using

$$a_0 = \frac{\lambda_{\varphi,0}}{8\pi\hbar} , \quad \hat{h}^2_{\varphi,0} = \frac{\hat{k}^2_{\varphi,0}}{4M^2} , \quad \hat{a} = -\frac{\hat{h}^2_{\varphi,0} \hat{\mu}}{4\pi \left( \hat{\mu}(B - B_0) - \epsilon_M \right)} + a_0$$

Eq. (84) reads

$$\sqrt{M |\epsilon_M|} = \frac{1}{\hat{a}}. \quad (87)$$

The scattering length for the fermionic atoms is determined by the total cross section in the zero-momentum limit, \( \sigma = 4\pi a^2 \), such that

$$a = \frac{M}{4\pi} \lambda_{\varphi,eff} = \frac{\lambda_{\varphi,eff}}{8\pi\hbar k} \quad (88)$$

More details are described in the appendix. For the atom phase \( (\epsilon_M > 0) \), one has \( \hat{h}^2_{\varphi,0} \)

$$\lambda_{\varphi,eff} = \lambda_{\varphi,0} - \frac{\hat{h}^2_{\varphi,0}}{m^2_{\varphi}(\delta = 0)} = \lambda_{\varphi,0} + \lambda_{\varphi,R}. \quad (89)$$

For the molecule phase, the effective interaction depends on the energy \( \omega \) of the exchanged molecule

$$\lambda_{\varphi,R}(\omega = 0, \bar{p} = 0) = -\frac{\hbar^2(\delta_{\Lambda})}{2Z_{\varphi}(\delta_{\Lambda}) \delta_{\Lambda}}, \quad \lambda_{\varphi,R}(\omega = -2\delta_{\Lambda}, \bar{p} = 0) = -\frac{\hbar^2(\delta_{\Lambda})}{4Z_{\varphi}(\delta_{\Lambda}) \delta_{\Lambda}}. \quad (90)$$

We find for the atom phase

$$a = a_0 - \frac{\hbar^2_{\varphi,0} M}{4\pi \hat{\mu}(B - B_0)} = a_0 + a_{res}. \quad (91)$$

which agrees with \( \bar{a} \) for \( \epsilon_M = 0 \), Eq. (80) therefore relates the binding energy in the molecule phase to the scattering length in the atom phase.

The value of \( \hat{h}_{\varphi,0} \) can now be extracted from the resonant part, \( a_{res} \), whereas \( \lambda_{\varphi,0} \) follows from the \( B \)-independent “background scattering length” \( a_0 = a_{bg} \), which has been measured as \( a_{bg}^{(Li)} = -1420a_B \), \( a_{bg}^{(K)} = 174a_B \) (the Bohr radius), or, expressed in units \( h = c = k_B = 1 \) (\( a_B = 2.6817 \cdot 10^{-4}c\text{V}^{-1} \)), \( a_{bg}^{(Li)} = -0.38e\text{V}^{-1} \), \( a_{bg}^{(K)} = 4.67 \cdot 10^{-2}c\text{V}^{-1} \). For \( \hbar = 1e\text{V} \) corresponding to a density \( n = 4.4 \cdot 10^{17}c\text{m}^{-3} \), we find the dimensionless expressions for \( ^6\text{Li} \) and \( ^{40}\text{K} \):

$$\hat{h}_{\varphi,0}^{(Li)} = 3.72 \cdot 10^5 , \quad \hat{h}_{\varphi,0}^{(K)} = 6.1 \cdot 10^3, \quad (92)$$

$$\lambda_{\varphi,0}^{(Li)} = -9.55 , \quad \lambda_{\varphi,0}^{(K)} = 1.17. \quad (93)$$

From these values, one can compute the initial values \( \lambda_{\varphi,in}, \hat{h}_{\varphi,in}^{(Li)} \). For \( ^6\text{Li} \), we use the values

$$k_{in} = 10^3, \quad \hat{h}_{\varphi,in}^{(Li)} = 2.56, \quad \lambda_{\varphi,in}^{(Li)} = -2.51 \cdot 10^{-2}. \quad (93)$$

whereas for \( ^{40}\text{K} \) we take

$$k_{in} = \sqrt{300}, \quad \hat{h}_{\varphi,in}^{(K)} = 1.67 \cdot 10^5, \quad \lambda_{\varphi,in}^{(K)} = 6.14. \quad (94)$$

For \( ^{40}\text{K} \), we observe large values of \( \hat{h}_{\varphi,in}^{(K)} \) and \( \lambda_{\varphi,in}^{(K)} \). Similar values for \( ^6\text{Li} \) are observed in the corresponding \( K \) range, e.g. \( \hat{h}_{\varphi,in}^{(K)}(K^2 = 300) = 6490, \lambda_{\varphi,in}^{(K)}(K^2 = 300) = 2024 \). The broad Feshbach resonances indeed describe strongly coupled systems. In our approximation, the solutions of the flow equations for \( \hat{h}_{\varphi,\lambda_{\varphi}} \) and \( Z_{\varphi} \) do not depend on \( m^2_{\varphi} \) provided \( m^2_{\varphi} \) remains positive during the flow.

### XI. SCATTERING LENGTH FOR MOLECULES

So far, we have only discussed the contributions of the molecule fluctuations to \( \partial\sigma/\partial\delta \) in order to extract the molecule density \( \hat{n}_M \). In this section, we discuss their influence on the effective potential for \( T = 0 \) more systematically. In particular, we will extract the scattering length for molecule-molecule scattering in vacuum.
from \( \lambda(s) = \partial^2 u/\partial \rho^2 |_{\rho_0} \). From Eq. (26), we infer for the derivative of the potential (primes denote derivatives

\[
\frac{\partial}{\partial k^2} u''(B) = -\frac{1}{4\pi^2} \int_0^\infty dx \, \frac{x^2}{(A_x x^2 + 2k^2 + u')^{1/2}} \left[ \rho'' \frac{1}{A_x x^2 + 2k^2 + u'} - \frac{3}{2} \rho'' + 2\rho'' u'' \right]
\]

(95)

This contribution vanishes for \( \rho = 0 \), and we find in the symmetric phase no influence on the running of \( m^2 \).

The bosonic contribution to the running of \( \lambda(s) = u''(0) \),

\[
\frac{\partial}{\partial k^2} \lambda(B) = \frac{\lambda^2}{2\pi^2} \int_0^\infty dx \, \frac{x^2}{(A_x x^2 + 2k^2 + m^2)^2}
\]

(96)

combines with the fermionic contribution and the anomalous dimension

\[
\frac{\partial}{\partial k^2} \lambda = \frac{h^2_\psi}{32\pi^2} (k^2 - \sigma)^{-3/2} + \frac{\partial}{\partial k^2} \lambda_{(F)} + \frac{\partial}{\partial k^2} \lambda(B),
\]

\[
\frac{\partial}{\partial k^2} \lambda_{(F)} = \frac{-3h_\psi^4}{8\pi^2} \int_0^\infty dx \, \frac{x^2}{(x^2 + k^2 - \sigma)^4}
\]

(97)

\[
= -\frac{3h_\psi^4}{256\pi} (k^2 - \sigma)^{-5/2}.
\]

Similar to the flow equation for \( \lambda(s) \), the equation for the gradient coefficient \( A(s) \) decouples from eqs. (81)-(83) in the symmetric phase at \( T = 0 \) and reads

\[
\frac{\partial}{\partial k^2} A = \frac{h_\psi^2}{64\pi} (k^2 - \sigma)^{-3/2} \left( A - \frac{1}{2} \right).
\]

(98)

There are no boson contributions in this regime. The equation is solved by \( A = 1/2 \) for our initial condition \( A_{\infty,\infty} = 1/2 \) (the initial condition is actually irrelevant, since \( A = 1/2 \) is a stable fixed point in Eq. (83)).

It is interesting to compare the scattering length for molecules \( a_M = \lambda(s)/M/(2\pi) \) with the fermionic scattering length \( a \) (83). For this purpose, we are interested in the flow of the ratio \( R_a = a_M/a = 2\lambda(s)/\lambda_{(s)} \). For the sake of a simple analytic discussion, we take \( A(s) = 1/2, \lambda(s,0) = 0 \) and consider the broad-resonance limit. Let us first consider

\[
\tilde{R} = \frac{8\lambda(s)}{h_\psi^2} (k^2 - \sigma).
\]

(99)

which approaches \( R_a \) in the infrared limit \( k \to 0 \), \( \tilde{R}(k \to 0) = R_a \). With \( K^2 = k^2 - \sigma \), we find the flow equation

\[
\frac{K^2}{\partial k^2} \tilde{R} = \tilde{R} - \frac{8K^4}{h_\psi^2} \frac{\partial h_\psi^2}{K^2} + \frac{8K^4}{h_\psi^2} \frac{\partial \lambda(s)}{K^2}
\]

(100)

\[
= \tilde{R} + \frac{4K^4}{32\pi^2} \left( 1 - \frac{3}{2} \tilde{R} - 3 \sqrt{k^2 - \sigma} \frac{K^4}{k^2 + m^2/2} \right) \tilde{R}.
\]

(101)

If we neglect the contribution from the molecule fluctuations (the last term in Eq. (101)) we find an infrared stable fixed point at \( \tilde{R} = 2 \). For this fixed point, one obtains for \( k \to 0 \), \( \sigma \to \sigma_A \):

\[
\lambda(s) = -\frac{h_\psi^2}{4\sigma_A} = -\frac{h_\psi^2}{2M(s)}.
\]

(102)

to be compared with \( \lambda_{(s)} = -h_\psi^2/(2\epsilon_M) \), as appropriate for the molecule phase. For the atom scattering in the molecule phase, the propagator of the exchanged molecule has to be evaluated for nonzero \( \omega = -\epsilon_M/2 \), cf. Eq. (60). We discuss this issue in the appendix. We conclude that this fixed point corresponds to the mean-field result \( a_M = 2a, R_a = 2 \), see also [14].

The bosonic fluctuations will lower the infrared value of \( R \). Using the fixed point for \( h_\psi^2 \) and the expression (81) for the binding energy in Eq. (49), one finds

\[
k^2 + \frac{m^2}{2} = 3k^2 - \tilde{\epsilon}_M - \sqrt{\epsilon_M/\sqrt{2k^2 - \tilde{\epsilon}_M}}.
\]

(103)

At the resonance point \( \tilde{\epsilon}_M = 0 \), the flow equation for \( \tilde{R} \) becomes

\[
\frac{k^2}{\partial k^2} \tilde{R} = -3 + \frac{3}{2} \tilde{R} + \frac{1}{\sqrt{3}} \tilde{R}^2.
\]

(104)

The infrared-stable fixed point now occurs for

\[
\tilde{R}_* = \frac{\sqrt{3}}{2} \left( \frac{9}{4} + \frac{12}{\sqrt{3}} \right) \approx 1.325.
\]

(105)
Precisely on the Feshbach resonance ($B = B_0$), the asymptotic behaviour obeys the scaling form

$$\lambda_\varphi = \frac{\hbar^2 R_*}{8k^2} = \frac{4\pi R_*}{k}. \quad (106)$$

For $B < B_0$, the mass term $m^2_\varphi$ is smaller as compared to the critical value for $B = B_0$. For a given $K$ the expression $\sqrt{2k^2 + m^2_\varphi} = \sqrt{2k^2 + 2\pi + m^2_\varphi}$ is therefore smaller and the term $\sim R^2$ in the flow Eq. is enhanced. As a consequence, $\lambda_\varphi$ turns out smaller than the critical value $[21, 22]$. lead to RG flows in the present truncation with optimised cutoffs $k_{\text{tr}}$.

It is instructive to study the flow in the range where $k^2 \ll -\epsilon_M/2$, $K \approx \sqrt{-\epsilon_M/2 + k^2/\sqrt{-2\epsilon_M}}$ and therefore

$$k^2 + \frac{m^2_\varphi}{2} \approx 2k^2. \quad (107)$$

The flow equation for $\hat{R}$ reads

$$k^2 \frac{\partial}{\partial k^2} \hat{R} = \left\{ \frac{3}{2} \hat{R} - 2 + \frac{\sqrt{2k^2 - \epsilon_M}}{2k} \hat{R}^2 \right\} \frac{k^2}{k^2 - \epsilon_M/2}. \quad (108)$$

For $k \to 0$, the second term dominates,

$$\frac{\partial \hat{R}}{\partial k} = \frac{\sqrt{2\hat{R}^2}}{\sqrt{k^2 - \epsilon_M/2}}, \quad (109)$$

and the running stops for $k \to 0$, resulting in

$$\hat{R}^{-1}(k = 0) \approx \hat{R}^{-1}(k_{\text{tr}}) + \sqrt{2}\text{Arsinh}\left(\frac{\sqrt{2k_{\text{tr}}}}{\sqrt{-\epsilon_M}}\right). \quad (110)$$

Here, $k_{\text{tr}}$ is a typical value for the transition from the scaling behaviour $[114]$ for $k^2 \gg -\epsilon_M/2$ to the boson dominated running $[109]$ for $k^2 \ll -\epsilon_M/2$. We may give a rough estimate using $k_{\text{tr}} = \sqrt{-\epsilon_M/2}$ and $\hat{R}(k_{\text{tr}}) = \hat{R}_*$,

$$\hat{R}(0) = \frac{\hat{R}_*}{1 + \sqrt{2}\text{Arsinh}(1)\hat{R}_*} \approx 0.5. \quad (111)$$

The value for a numerical solution turns out somewhat higher and we find for $k \to 0$

$$\hat{R}(k \to 0) = \frac{a_M}{a} = 0.81. \quad (112)$$

This is mainly due to the presence of fermion fluctuations which enhance $\hat{R}$.

Following the flow of $\lambda_\varphi$ in a simple truncation thus predicts $a_M/a \approx 0.81$, in qualitative agreement with rather refined quantum mechanical computations $[33]$, numerical simulations $[3]$, and sophisticated diagrammatic approaches $[34]$, $a_M/a \approx 0.6$. The resummation obtained from our flow equation is equivalent to the one performed in $[33]$, yielding $a_M/a \approx 0.78 \pm 4$. Functional RG flows in the present truncation with optimised cutoffs $[21, 22]$ lead to $a_M/a \approx 0.71$ $[14]$. It should be possible to further improve the quantitative accuracy by extending the truncation, for example by including an interaction of the type $\psi^\dagger \psi^\dagger \varphi$ corresponding to atom-molecule scattering. Already at this stage, it is clear that the final value of $R_*\lambda$ results from an interesting interplay between the fermionic and bosonic fluctuations. In a model describing only interacting bosonic particles (atoms or molecules), the flow of $\lambda_\varphi$ can be taken from Eq. $[106]$ for $m^2_\varphi = 0$, $A_\varphi = \frac{1}{2}$

$$\frac{\partial \lambda_\varphi}{\partial k} = \frac{1}{2\pi} \lambda_\varphi^2. \quad (113)$$

(For an application to bosonic atoms, one should recall that our normalisation of the particles corresponds to atom number 2 and mass $2M$. The bosonic fluctuation effects have the tendency to drive $\lambda_\varphi$ towards zero, according to the solution of Eq. $[113]$ for $k \to 0$,

$$\lambda_\varphi = \frac{\lambda_{\varphi, \text{in}}}{1 + \lambda_{\varphi, \text{in}} k_{\text{in}}/2\pi}. \quad (114)$$

This solution reflects the infrared-stable fixed point for $\lambda_{\varphi} = \lambda_{\varphi} k$,

$$\left(\lambda_{\varphi}\right)_* = 0, \quad (115)$$

resulting from

$$\lambda_{\varphi} = \lambda_{\varphi} + \frac{\lambda_{\varphi}^2}{2\pi}. \quad (116)$$

On the other hand, the fermion fluctuations have the opposite tendency: they generate a non-vanishing $\lambda_\varphi$ even for an initial value $\lambda_{\varphi, \text{in}} = 0$ and slow down the decrease of $\lambda_\varphi$ as compared to Eq. $[113]$ in the range of small $k$. The resulting $R_\lambda$ results from a type of balance between the opposite driving forces. If the scaling behaviour $[104]$ is approximately reached during the flow, the final value of $R_\lambda$ is a universal ratio which does not depend on the microscopic details. This is a direct consequence of the “loss of microscopic memory” for the fixed point $[103]$.

XII. UNIVERSALITY

The essential ingredient for the universality of the BEC-BCS crossover for a broad Feshbach resonance is the fixed point in the renormalisation flow for large $\hbar^2 \varphi$ (fixed point (A) in sect. [1X]. This fixed point is infrared stable for all couplings except one relevant parameter corresponding to the detuning $B - B_0$. For $T = 0$, this fixed point always dominates the flow at $B = B_0$ as long as effects from a nonzero density can be neglected. However, for any $k_F \neq 0$ the flow will finally deviate from the scaling form due to the occurrence of a condensate $\rho_0 \neq 0$. Typically, this happens once $k$ reaches the gap for single fermionic atoms, $k \approx \Delta = h_\varphi / \sqrt{\rho_0}$. (If we use $k = k_F$ the
relevant scale is \( k = 16 \Omega_c/(3 \pi) \). For \( k \ll \Delta \), the contribution from fermionic fluctuations to the flow gets suppressed. Also the contribution from the “radial mode” of the bosonic fluctuations will be suppressed due to a mass-like term \( m_\sigma^2 \approx 2 \lambda_\sigma \rho_0 \). Only the Goldstone mode corresponding to the phase of \( \varphi \) will have un-suppressed fluctuations.

For nonzero density it is convenient to express all quantities in units of the Fermi momentum \( k_F \) and the Fermi energy \( \varepsilon_F \), i.e., to set \( k = k_F \). As compared to the scaling form at zero density the flow equations are now modified by two effects. They concern the influence of \( \rho_0 \neq 0 \) and the nonzero value for \( \hat{\sigma} = \hat{\sigma}_s \) which corresponds to \( B = B_0 \). (Typically, this value \( \hat{\sigma}_s \) is positive, as opposed to \( \hat{\sigma}_s = 0 \) for the zero density case.) Nevertheless, all these modifications only concern the flow for small \( k \). If the flow for large \( k \) has already approached the fixed point close enough no memory is left from the microscopic details. We can immediately conclude that all physical quantities are universal for \( B = B_0 \).

The same type of arguments applies for nonzero temperature. For \( T \neq 0 \), a new effective infrared cutoff is introduced for the fermionic fluctuations. Also the contributions from the bosonic fluctuations get modified. Again, this only concerns the flow for \( k \ll \pi T \). The loss of microscopic memory due to the attraction of the flow for large \( k \) towards the fixed point remains effective.

In a similar way, we can consider the flow for \( B \neq B_0 \). Away from the exact location of the resonance, the observable quantities will now depend on the relevant parameter. The latter can be identified with the concentration \( c = \alpha k_F \). Still, the deviation from the scaling flow only concerns small \( k \), namely the range when the first term in Eq. (49) becomes important. This happens for

\[
k \approx \frac{16 \pi M \mu |B - B_0|}{\hbar^2 \varepsilon_F (k_F)^2} = \frac{1}{c}.
\]

If the flow for large \( k \) has been attracted close enough to the fixed point all physical quantities at nonzero density are universal functions of two parameters, namely \( c \) and \( \hat{T} = T/\varepsilon_F \). These functions describe all broad resonances.

**XIII. DEVIATIONS FROM UNIVERSALITY**

The functional flow equations also allow for simple quantitative estimates for the deviations from universality for a given atom gas. Typically, these corrections are power suppressed \( \sim (\Lambda_{IR}/\Lambda_{UV})^p \). The microscopic scale \( \Lambda_{UV} \) roughly denotes the inverse of the characteristic range of van der Waals forces and we may associate \( \Lambda_{UV} \approx 1/(1000 \lambda_\varphi) \) with \( \lambda_\varphi \) the “radius” of the atoms. More precisely, \( \Lambda_{UV} \) corresponds to the scale at which the attraction to the fixed point sets in. In some cases, this may be substantially below the van der Waals scale, as for the case of \( ^{40} \)K, as we will argue below. The relevant infrared scale \( \Lambda_{IR} \) depends on the density, \( \Lambda_{IR} = L(c, \hat{T}) k_F \).

In view of the discussion in the preceding section, we approximate \( L \) by the highest value of \( k \) where the deviation from the scaling form of the flow equations sets in, i.e.,

\[
L = \max\{|c^{-1}|, \pi \hat{T}, \sqrt{|\hat{\sigma}|}, \hat{\Delta}\}.
\]

We note that \( \hat{\sigma} \) and \( \hat{\Delta} = h \varphi \sqrt{\rho_0} \) depend on \( c \) and \( \hat{T} \).

The power \( p \) of the suppression factor reflects the “strength of attraction” of the fixed point. More precisely, we may consider the flow of various dimensionless couplings \( \alpha_i \). If we denote their values at the fixed point by \( \hat{\alpha}_i \), the flow of small deviations \( \delta \alpha_i = \alpha_i - \hat{\alpha}_i \) is governed by the “stability matrix” \( A_{ij} \)

\[
k \frac{\partial}{\partial k} \delta \alpha_i = A_{ij} \delta \alpha_j.
\]

The relevant parameter corresponds to the negative eigenvalue of \( A \). All other eigenvalues of \( A \) are positive, and \( p \) is given by the smallest positive eigenvalue of \( A \).

Typical couplings \( \alpha_i \) are \( (\hat{\lambda}_\psi, \hat{h}_\lambda^2, \hat{m}_\sigma^2, \hat{R}) \). For these couplings, the stability matrices reads for fixed point \( \Lambda \):

\[
A = \begin{pmatrix}
1, & 0, & 0, & 0 \\
8, & 1, & 0, & 0 \\
0, & \frac{1}{4\pi}, & -1, & 0 \\
0, & e_1, & e_2, & e_3
\end{pmatrix}
\]

with \( e_1 = (\hat{R}_s - 6 + 2 \hat{R}_s^2)/(32 \pi), e_2 = -\hat{R}_s^2/(6 \sqrt{3}), e_3 = 3 + 4 \hat{R}_s^2/3 \). The eigenvalues are \( (1, 1 - 1, e_3) \) and we conclude \( p = 1 \). This situation is not expected to change if more couplings like \( A_\sigma, A_\psi \) or the \( \rho_0^2 \) term in the potential are included.

We may use our findings for a rough estimate for the range in \( c^{-1} \) and \( \hat{T} \) for which deviations from universality are smaller than \( 1\% \). From our previous considerations this holds for

\[
|c^{-1}| \lesssim \Lambda_{UV}/100 k_F, \quad \sqrt{\hat{T}} \lesssim \Lambda_{UV}/300 k_F.
\]

As a condition for \( \Lambda_{UV} \), we require that all couplings are in a wider sense in the “vicinity” of the fixed point at the scale \( k_{UV} = \Lambda_{UV}/k_F \). Of course, one necessary condition is \( \Lambda_{UV} < \Lambda \). Also, the flow equations must be meaningful for \( k < k_{UV} \). While this poses no restriction on Li where \( \lambda_{\psi, 0} < 0 \), the case of K with positive \( \lambda_{\psi, 0} \) requires \( k_{UV} \delta_0 < 1 \) or \( k_{UV} \lesssim 20 \). If we would try to extrapolate the flow to even higher \( k \) the value of \( \hat{h}_\lambda^2(k) \) would diverge as can be seen from Eq. (111). As a further condition for the flow to be governed by the universal behaviour of the fixed point we require \( Z_\varphi - 1 \gtrsim 1 \). This tells us that the notion of a di-atom state \( \varphi \) starts to be independent of the detailed microscopic properties. For Li and K, this happens for scales only somewhat below \( k_{UV} \).

A reasonable estimate at this stage may be \( k_{UV}^{(Li)} = 800, k_{UV}^{(K)} = 15 \). One percent agreement with the universal behaviour would then hold for

\[
\text{Li: } |c^{-1}| \lesssim 8, \quad \sqrt{\hat{T}} \lesssim 3, \\
\text{K: } |c^{-1}| \lesssim 0.15, \quad \sqrt{\hat{T}} \lesssim 0.05,
\]
which corresponds to

\[ \text{Li : } |ak_F| \gtrsim 0.13, \quad T/T_F \lesssim 9, \]
\[ \text{K : } |ak_F| \gtrsim 6.7, \quad T/T_F \lesssim 0.0025. \] (123)

Already at this stage, we conclude that the universal behaviour for K covers only a much smaller range in c and \( T \) as compared to Li. In particular, experiments with Li are indeed performed within the universal regime which extends far off resonance and to temperatures well above the quantum degenerate regime \( T/T_F \approx 1 \), while for K it will be hard to reach the truly universal domain, since the lowest available temperatures range down to \( T/T_F \approx 0.04 \), and the magnetic field tuning is too low to resolve a regime where \( |ak_F| \gtrsim 6.7 \).

At first sight, the limitation to a rather small value \( k_{\text{UV}}^{(K)} = 15 \) for \( 40K \) may seem to be of a technical nature, enforced by the breakdown of the flow for \( \lambda_{\psi} \) for \( k > k_{\text{UV}}^{(K)} \). However, this technical shortcoming most likely reveals the existence of some additional physical scale close to \( k_{\text{UV}}^{(K)} \), as, for example, associated to a further nearby resonance state not accounted for in our simple model. Indeed, if our model (without modifications and additional effective degrees of freedom) would be valid for \( k \gg k_{\text{UV}}^{(K)} \), one could infer an effective upper bound for \( \lambda_{\psi} \) from its flow. Starting at some \( k_0 \) with an arbitrarily large positive \( \lambda_{\psi,\text{in}} \), the value of \( \lambda_{\psi}(k) \) would be renormalised to a finite value, bounded by \( \lambda_{\psi}(K = 0) < 8\pi/K_{\text{in}} \) (cf. Eq. (39)). This would lead to a contradiction with the observed value unless \( K_{\text{in}} \) is sufficiently low. The observed value of \( \lambda_{\psi,0} \) therefore implies the existence of a scale near \( k_{\text{UV}}^{(K)} \) where our simple description needs to be extended. This issue is very similar to the “triviality bound” in the standard model for the electroweak interactions in particle physics.

As further concern, one may ask if \( \tilde{h}_{\varphi}^2(k_{\text{UV}}) \) is already close enough to the fixed point value \( 32\pi \) and if \( \lambda_{\psi}(k_{\text{UV}}) \) is close enough to zero. This is an issue, since we know of the existence of a different fixed point for narrow Feshbach resonances at \( \tilde{h}_{\varphi}^2 = \lambda_{\psi} = 0 \). At the scale \( k_{\text{UV}} \), the flow should not be in the vicinity of this “narrow-resonance fixed point” anymore, but at least in the crossover region towards the “broad-resonance fixed point”. If we evaluate the couplings at the scales \( k_{\text{UV}}^{(\text{Li})} = 800, k_{\text{UV}}^{(K)} = 15 \) we obtain for \( c^{-1} = 0 \)

\[ \text{Li : } \tilde{h}_{\varphi}^2(k_{\text{UV}}) = 5.00 \cdot 10^{-3}, \quad \lambda_{\psi}(k_{\text{UV}}) = -25.0, \]
\[ m_{\varphi}^2(k_{\text{UV}}) = 6.07 \cdot 10^{-2}; \]
\[ \text{K : } \tilde{h}_{\varphi}^2(k_{\text{UV}}) = 4.53 \cdot 10^3, \quad \lambda_{\psi}(k_{\text{UV}}) = 58.8, \]
\[ m_{\varphi}^2(k_{\text{UV}}) = 54.0. \] (124)

For K, the relevance of the broad-resonance fixed point seems plausible, but the small value of \( \tilde{h}_{\varphi}^2(k_{\text{UV}}) \) for Li may shed doubts. However, one should keep in mind that for a strong enough \( \lambda_{\psi,\text{eff}} \) the distribution between \( \lambda_{\psi} \) and \( -\tilde{h}_{\varphi}^2/m_{\varphi}^2 \) concerns mainly the dependence of the scattering length on the magnetic field \( B \) and not so much the physics for a given \( a \) or \( c \). Indeed, we could absorb \( \lambda_{\psi} \) by partial bosonisation (Hubbard-Stratonovich transformation) in favour of a change of \( h_{\varphi}^2 \) and \( m_{\varphi}^2 \). Up to terms \( \sim q^4 \), which are sub-leading for the low-momentum physics, a model with given \( \lambda_{\psi}, h_{\varphi}^2, m_{\varphi}^2 \) is equivalent to a model with \( \lambda_{\psi}' = 0 \), but new values \( h_{\varphi}'^2 \) and \( m_{\varphi}'^2 \) related to the original parameters by

\[ h_{\varphi}'^2 = h_{\varphi}^2 - 2\lambda_{\psi} m_{\varphi}^2 \frac{\lambda_{\psi}^2 m_{\varphi}^4}{h_{\varphi}^2}, \quad m_{\varphi}'^2 = m_{\varphi}^2 - \lambda_{\psi} m_{\varphi}^4 \frac{h_{\varphi}^2}{h_{\varphi}^2}. \] (125)

The value of the new Yukawa coupling for Li, \( \tilde{h}_{\varphi}^2(k_{\text{UV}}) = 4.65 \cdot 10^2 \), is much larger and seems acceptably close to the broad resonance fixed point. Using partial bosonisation and the technique of rebosonisation during the flow, it may actually be possible to treat \( \lambda_{\psi} \) as a redundant parameter, thus enlarging the “space of attraction” of the fixed point (A). In this context, we note that we should include the contribution of the bosonic fluctuations to the running of \( \lambda_{\psi} \) for our truncation. This will shift the precise location of the fixed points. In a language with rebosonisation where \( \lambda_{\psi} \) remains zero, these additional contributions will be shifted into the flow of \( m_{\varphi}^2 \) and \( h_{\varphi}^2 \).

Our estimate for the deviations from universality concerns only the overall suppression factor. Detailed proportionality coefficients \( c_{O} \) for the deviation of a given observable \( O \) from the universal strong-resonance limit, \( \delta O = c_{O}\Lambda_{IR}/\Lambda_{UV} \), depend on the particular observable. The deviations from universality should therefore be checked by explicit solutions of the flow equation for Li and K.

**XIV. CONCLUSIONS**

In this paper, we have performed a functional renormalisation group study for ultracold gases of fermionic atoms. We have concentrated on four parameters: the Yukawa coupling of the molecules to atoms \( h_{\varphi} \), the offset between molecular and open channel energy levels \( m_{\varphi}^2 \) which is related to the detuning, the background atom-atom-interaction \( \lambda_{\psi} \) and the molecule-molecule interaction \( \lambda_{\varphi} \). This system exhibits a fixed point for the physics for a given \( c_{O} \), with the galaxy of the microphysical details except for one relevant parameter. This parameter can be associated with the inverse concentration \( c_{O} = (ak_F)^{-1} \).

Whenever for a given system the trajectories of the flow in coupling-constant space approach this fixed point close enough, the microscopic physics looses the memory of the microphysical details except for one parameter, namely \( c_{O} \). In consequence, for a certain parameter in \( c_{O} \) around 0 and for a certain range in temperature \( T \) all dimensionless macroscopic quantities can only depend on \( c \) and \( T \). Here dimensionless quantities are typically
obtained by multiplication with appropriate powers of $k_F$ or $\epsilon_F$.

The macroscopic quantities include all thermodynamic variables and, more generally, all quantities that can be expressed in terms of $n$-point functions for renormalised fields at low momenta. In particular, the correlation functions for atoms and “molecules” as well as their interactions depend only on $c$ and $T$, independently of the concrete microscopic realisation of a broad Feshbach resonance. Here “molecules” refers to bosonic quasi particles as collective di-atom states which may be quite different from the notion of microscopic or “bare” molecules. In a certain sense this situation has an analogue in the universal critical behaviour near a second order phase transition. In contrast to this, however, the universal description includes now a temperature range between $T = 0$ and substantially above the critical temperature, and the universal quantities depend on an additional relevant parameter $c^{-1}$. We may also compare to a quantum phase transition at $T = 0$ which is realized in function of a parameter analogous to $c^{-1}$. In our case, universality is extended to nonzero temperature as well.

The “broad resonance fixed point” is not the only fixed point of the system. Another “narrow resonance fixed point” at $\tilde{h}_\psi^2 = 0$, $\lambda_\psi = 0$ allows for an exact solution of the crossover problem for narrow Feshbach resonances and a perturbative expansion around this solution. The narrow-resonance fixed point has two relevant parameters $c^{-1}$ and $\tilde{h}_\psi^2$. A typical flow away from this fixed point at small enough $c^{-1}$ and $\lambda_\psi$ describes a crossover towards the broad resonance fixed point.

For a given physical system characterised by some microscopic Hamiltonian, it is important to determine how close it is to the universal behaviour of the broad-resonance limit. We have performed here a first estimate for the range in $c^{-1}$ and $T$ for which universality holds within one percent accuracy for the experimentally studied Feshbach resonances in $^6$Li and $^{40}$K. It will be an experimental challenge to verify of falsify these predictions of universality.

**APPENDIX: SCATTERING LENGTH AND CROSS SECTION**

In this appendix, we collect useful formulae for the scattering of molecules in vacuum. In quantum field theory, the scattering cross section for identical nonrelativistic bosons is given by

$$\sigma_B = \frac{1}{2} \int d\Omega \frac{d\sigma}{d\Omega}. \tag{A.1}$$

The factor $1/2$ is a convention, motivated by integration over half the spatial angle in order to avoid double counting of the identical particles. The differential cross section is related to the scattering amplitude by

$$\frac{d\sigma}{d\Omega} = \frac{M_F^2|A_B|^2}{16\pi^2}, \tag{A.2}$$

such that, in case of isotropic (e.g. s-wave) scattering

$$\sigma_B = \frac{M_F^2|A_B|^2}{8\pi}. \tag{A.3}$$

Next, we relate the above results to nonrelativistic quantum mechanics: The scattering length $a$ is a quantity which is directly meaningful in nonrelativistic quantum mechanics only. It measures the strength of the $1/r$ decay of the scattered wave function at low energies as $a/r$. Equivalently, $a$ is defined by the $l = 0$ phase shift for the scattered wave function. This definition leads to the relation between scattering length and cross section (identical bosons)

$$\sigma_B = 8\pi a_B^2. \tag{A.4}$$

We can use Eqs. (A.3), (A.3) in order to relate the scattering amplitude to the scattering length or rather to define this relation,

$$|A_B| = \frac{8\pi a_B}{M_B} = \frac{2\lambda_\psi}{Z_\psi^2}. \tag{A.5}$$

The scattering amplitude is given by the effective four-boson vertex, as obtained by functional derivatives of the effective action, $Z_\psi^2 A_B = \Gamma^{(4)} = 2\lambda_\psi$. In the limit of a point-like interaction, $\lambda_\psi$ is a constant. (Note that the notion of scattering amplitude is used to describe different quantities in quantum mechanics and QFT.)

For fermions, the definitions are similar,

$$\sigma_F = \int d\Omega \frac{d\sigma}{d\Omega}. \tag{A.6}$$

Now, the integration covers the full space angle, since the fermions can be distinguished by their spin. The differential cross section is related to the scattering amplitude by

$$\frac{d\sigma}{d\Omega} = \frac{M_F^2|A_F|^2}{16\pi^2}, \tag{A.7}$$

such that for isotropic scattering

$$\sigma_F = \frac{M_F^2|A_F|^2}{4\pi}. \tag{A.8}$$

For distinguishable fermions, one has in quantum mechanics

$$\sigma = 4\pi a_F^2, \tag{A.9}$$

such that the scattering amplitude and scattering length are related by

$$|A_F| = \frac{4\pi a_F}{M_F} = \lambda_\psi^{\text{eff}}(\omega, \vec{q}). \tag{A.10}$$

The scattering amplitude $A_F$ is given by the tree level graph for the molecule exchange fermions plus a contribution from the fermionic 1PI vertex $\lambda_\psi$. The energy-
and momentum-dependent resonant four-fermion vertex generated by the molecule exchange reads

\[ \tilde{\lambda}_{\psi,\text{eff}}(\omega, \vec{q}) - \tilde{\lambda}_{\psi,0} = -\frac{\tilde{h}_\omega^2(\sigma_A)}{P_\psi(\omega, \vec{q})} \]  

(A.11)

with

\[ \Delta P_\psi(\omega, \vec{q}) = \frac{\tilde{h}_\omega^2(\sigma_A)M^{3/2}}{4\pi} \sqrt{-\omega - 2\sigma_A + \frac{q^2}{4M}}. \]  

(A.12)

Using the definition of the renormalised Yukawa coupling \( \tilde{h}_\psi = \tilde{h}_\omega^2 / Z_\psi \), we may express

\[ \lambda_{\psi,\text{eff}}(-2\sigma_A, 0) - \lambda_{\psi,0} = \frac{8\pi}{\sqrt{-2\sigma_A}} = \frac{h_\omega^2}{4\sigma_A} - \frac{h_\omega^2}{2\sigma_m}. \]  

(A.17)

In the last two expressions, we have used the leading terms in Eqs. (11), (12) for \( K = \sqrt{-\sigma_A} \rightarrow 0 \).

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