Künneth Formulas of Hypergraphs

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Abstract. Hypergraphs are higher-dimensional generalizations of graphs. An (abstract) simplicial complex is a hypergraph such that all the faces of hyperedges are still hyperedges. In this paper, by using the embedded homology defined in (cf. [2]), as a generalization of the Künneth formula for the tensor product of chain complexes, we give a Künneth formula for the tensor product of graded submodules of chain complexes. Moreover, we study the Künneth formulas for join and product of hypergraphs successively.

1 Introduction

Hypergraph is an important model for complex networks, for example, the collaboration network. An edge in a graph consists of two vertices, while a hyperedge in a hypergraph allows multiple vertices (cf. [1]). In topology, a hypergraph can be obtained from a simplicial complex by deleting some non-maximal simplices (cf. [2, 15]). From this point of view, hypergraph is the complex of graph and simple complex, and it is the transition state from graph to simple complex. In this sense, hypergraph is the key hub to connect the simple complex in topology and graph in combinatorics, which is worth studying in theory and application.

Let $V_H$ be a totally-ordered finite set. Let $2^V$ denote the powerset of $V$. Let $\emptyset$ denote the empty set. A hypergraph is a pair $(V_H, H)$ where $H$ is a subset of $2^V \setminus \{\emptyset\}$ (cf. [1,15]). An element of $V_H$ is called a vertex. Let $k \geq 0$. We call a hyperedge $\sigma \in H$ consisting of $k + 1$ vertices a $k$-dimensional hyperedge, or a hyperedge of dimension $k$, and denote $\sigma^{(k)}$. For any hyperedges $\sigma, \tau \in H$, if $\sigma$ is a proper subset of $\tau$, then we write $\sigma < \tau$ or $\tau > \sigma$.

Throughout this paper, we assume that each vertex in $V_H$ appears in at least one hyperedge in $H$. Hence $V_H$ is the union $\bigcup_{\sigma \in H} \sigma$, and we simply denote a hypergraph $(V_H, H)$ as $H$.

Given two hypergraphs $H$ and $H'$, if each hyperedge of $H$ is also a hyperedge of $H'$, then we write $H \subseteq H'$ and say that $H$ can be embedded in $H'$.

An (abstract) simplicial complex is a hypergraph satisfying the following condition (cf. [13] p. 107), [16] Section 1.3): for any $\sigma \in H$ and any non-empty subset $\tau \subseteq \sigma$, $\tau$ must be a hyperedge in $H$. A hyperedge of a simplicial complex is called a simplex.

Let $\mathcal{H}$ be a hypergraph. We consider a single hyperedge $\sigma = \{v_0, v_1, \ldots, v_n\}$ of $\mathcal{H}$. The associated simplicial complex $\Delta \sigma$ of $\sigma$ is defined as the collection of all the nonempty subsets of $\sigma$

$$\Delta \sigma = \{\{v_{i_0}, v_{i_1}, \ldots, v_{i_k}\} \mid 0 \leq i_0 < i_1 < \cdots < i_k \leq n, 0 \leq k \leq n\}. \quad (1.1)$$

The associated simplicial complex $\Delta H$ of $\mathcal{H}$ is defined as the smallest simplicial complex that $\mathcal{H}$ can be embedded in (cf. [15]). Explicitly, we can write the set of simplices of $\Delta H$ as

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the union of the $\Delta\sigma$’s for all $\sigma \in \mathcal{H}$

$$\Delta \mathcal{H} = \{ \tau \in \Delta \sigma \mid \sigma \in \mathcal{H} \}.$$  \hspace{1cm} (1.2)

There are various (co)homology theories of hypergraphs. For example, A.D. Parks and S.L. Lipscomb [15], F.R.K. Chung and R.L. Graham [3], E. Emterand [7], J. Johnson [14], and Stephane Bressan, Jingyan Li, Shiquan Ren, Jie Wu [2] respectively studied this problem.

Throughout this paper, we let $R$ to be a principal ideal domain if there is no danger of confusion. Let $\mathcal{H}$ be a hypergraph and let $\mathcal{K}$ be a simplicial complex such that there is an injective map $i: \mathcal{H} \rightarrow \mathcal{K}$ sending each hyperedge of $\mathcal{H}$ to a simplex of $\mathcal{K}$. Let $C_\ast(\mathcal{H}; R)$ be the finitely generated free $R$-module with generators in $\mathcal{H}$. Then it is a graded sub-$R$-module of the chain complex $C_\ast(\mathcal{K}; R)$. In [2], it’s proved that both the infimum chain complex and the supremum chain complex of $C_\ast(\mathcal{H}; R)$ in $C_\ast(\mathcal{K}; R)$ do not depend on the choice of the simplicial complex $\mathcal{K}$ that $\mathcal{H}$ embedded in. Therefore, $\mathcal{K}$ can be taken as the associated simplicial complex of $\mathcal{H}$. By [2] Proposition 2.4, the homologies of these two chain complexes are isomorphic, which are defined as the embedded homology of $\mathcal{H}$ and denoted as $H_n(\mathcal{H})$. In particular, if the hypergraph is a simplicial complex, then the embedded homology coincides with the usual homology. Moreover, each morphism of hypergraphs from $\mathcal{H}$ to $\mathcal{H}'$ induces an homomorphism between the embedded homology $H_n(\mathcal{H}; R)$ and $H_n(\mathcal{H}'; R)$ ([2, Proposition 3.7]).

In this paper, base on the embedded homology defined in [2], we mainly study the Künneth formulas of hypergraphs. By generalising the classical Künneth formula in [13] and the Künneth formula for the path homology (with field coefficients) of digraphs in [9] and [11], we give a Künneth formula for the embedded Homology of the tensor product of graded submodules of chain complexes first, and then prove the Künneth formulas for the embedded homology (with $R$ coefficients) of join and product of hypergraphs respectively.

The framework of the paper is as follows. In Section 2 we define the tensor product for graded abelian subgroups of chain complexes, introduce the infimum chain complex and prove a Künneth formula for the Embedded Homology in Theorem 2.3. Furthermore, in Section 3 we introduce the join of hypergraphs and prove a Künneth formula for join of hypergraphs in Theorem 3.3. In Section 4 we review the Eilenberg-Zilber Theorem for simplicial sets in [5, 6, 12] and simplical sets representation of simplicial complexes in [10] firstly, then define the product of hypergraphs and prove a Künneth formula for the product of hypergraphs in Theorem 4.4. In Section 5 we discuss the relation between our Künneth formula for hypergraphs and the Künneth formula for digraphs proved in [9] Section 7 and [11].

2 A Künneth Formula for the Tensor Product of Graded Submodules of Chain Complexes

In this section, we define the tensor product for graded submodules of chain complexes. As a generalization of the Künneth formula for the tensor product of chain complexes, we prove a Künneth formula for the tensor product of graded submodules of chain complexes, in Theorem 2.3.
2.1 Tensor Product of Chain Complexes

We review the definition for the tensor product of chain complexes. The content of this subsection can be found in [13, Chapter 3, Section 3.B].

Let $C$ and $C'$ be chain complexes over $R$, 

$$C = \{C_n, \partial_n\}_{n \geq 0}, \quad C' = \{C'_n, \partial'_n\}_{n \geq 0}.$$ 

Their tensor product is a chain complex

$$C \otimes C' = \left\{ \bigoplus_{p+q=n, \ p,q \geq 0} C_p \otimes C'_q, \bigoplus_{p+q=n, \ p,q \geq 0} \partial_p \otimes \partial'_q \right\}_{n \geq 0}. \tag{2.1}$$

In (2.1), the tensor product of boundary maps is given by

$$(\partial_p \otimes \partial'_q)(u_p \otimes v_q) = (\partial_p u_p) \otimes v_q + (-1)^p u_p \otimes (\partial'_q v_q)$$

for any $u_p \in C_p$ and any $v_q \in C'_q$. For simplicity, we denote

$$(C \otimes C')_n = \bigoplus_{p+q=n, \ p,q \geq 0} C_p \otimes C'_q, \quad (\partial \otimes \partial')(n) = \bigoplus_{p+q=n, \ p,q \geq 0} \partial_p \otimes \partial'_q.$$ 

Then for any $u_p \in C_p$ and any $v_q \in C'_q$,

$$(\partial \otimes \partial')(n) \left( \sum_{p+q=n, \ p,q \geq 0} u_p \otimes v_q \right) = \sum_{p+q=n, \ p,q \geq 0} (\partial_p u_p) \otimes v_q + (-1)^p u_p \otimes (\partial'_q v_q). \tag{2.2}$$

It follows from (2.2) that

$$(\partial \otimes \partial')(n) : (C \otimes C')_n \rightarrow (C \otimes C')_{n-1}$$

is well-defined, and for any $n \geq 0$,

$$(\partial \otimes \partial')(n) \circ (\partial \otimes \partial')(n+1) = 0.$$ 

Hence (2.1) is a chain complex.

2.2 Tensor Product of the Graded Submodules of Chain Complexes

We generalize the tensor product of chain complexes and define the tensor product for graded submodules of chain complexes. We introduce the infimum chain complex and prove Proposition 2.2.

For each $n \geq 0$, we consider sub-$R$-modules $D_n \subseteq C_n$ and $D'_n \subseteq C'_n$. Then we have graded sub-$R$-modules of the chain complexes

$$D = \{D_n\}_{n \geq 0}, \quad D' = \{D'_n\}_{n \geq 0}.$$ 

The tensor product of $D$ and $D'$ is defined as

$$D \otimes D' = \left\{ \bigoplus_{p+q=n, \ p,q \geq 0} D_p \otimes D'_q \right\}.$$ 

It is direct to verify that $D \otimes D'$ is a graded sub-$R$-modules of the chain complex $C \otimes C'$.

For simplicity, we denote

$$(D \otimes D')_n = \bigoplus_{p+q=n, \ p,q \geq 0} D_p \otimes D'_q.$$
Lemma 2.1. For any \( n \geq 0 \),
\[
(\partial \otimes \partial')_n(D \otimes D')_n = \sum_{p+q=n, \ p,q \geq 0} \partial_p D_p \otimes D'_q + D_p \otimes \partial'_q D'_q.
\]

Proof. By (2.2), the lemma follows from a calculation
\[
(\partial \otimes \partial')_n(D \otimes D')_n = (\partial \otimes \partial')_n\left( \bigoplus_{p+q=n, \ p,q \geq 0} D_p \otimes D'_q \right)
= \sum_{p+q=n, \ p,q \geq 0} (\partial \otimes \partial')_n(D_p \otimes D'_q)
= \sum_{p+q=n, \ p,q \geq 0} \partial_p D_p \otimes D'_q + D_p \otimes \partial'_q D'_q.
\]

By [2] Section 2, the infimum chain complex of \( D \) in \( C \) is defined as the largest sub-chain complex of \( C \) that is contained in \( D \) as graded submodules
\[
\text{Inf}_n(D,C) = D_n \cap \partial_n^{-1} D_{n-1}, \quad n \geq 0; \quad \tag{2.3}
\]
and the supremum chain complex of \( D \) in \( C \) is defined as the smallest sub-chain complex of \( C \) that contains \( D \) as graded submodules
\[
\text{Sup}_n(D,C) = D_n + \partial_{n+1} D_{n+1}, \quad n \geq 0. \quad \tag{2.4}
\]
By applying [2] Section 2] to the graded sub-\( R \)-module \( D \) in the chain complex \( C \), we have that the canonical inclusion of chain complexes \( \iota : \text{Inf}_*(D, C) \rightarrow \text{Sup}_*(D, C) \) induces an isomorphism of homologies
\[
\iota_* : H_n(\text{Inf}_*(D, C)) \xrightarrow{\cong} H_n(\text{Sup}_*(D, C)), \quad n \geq 0. \quad \tag{2.5}
\]
The \( n \)-th embedded homology group of \( D \) in \( C \), denoted as \( H_n(D,C) \), is defined as the homology group (2.5).

By substituting \( D \) with \( D' \) and \( D \otimes D' \), substituting \( C \) with \( C' \) and \( C \otimes C' \), and substituting \( \partial \) with \( \partial' \) and \( \partial \otimes \partial' \) in (2.3), (2.4) and (2.5) respectively, we can define \( \text{Inf}_n(D', C') \), \( \text{Inf}_n(D \otimes D', C \otimes C') \), \( \text{Sup}_n(D', C') \), \( \text{Inf}_n(D \otimes D', C \otimes C') \), \( \text{Sup}_n(D \otimes D', C \otimes C') \), and \( H_n(D \otimes D', C \otimes C') \).

Proposition 2.2. Let \( C, C' \) be chain complexes of finitely generated free \( R \)-modules, and let \( D, D' \) be graded sub-\( R \)-modules of \( C, C' \), respectively. Then for any \( n \geq 0 \), we have
\[
\text{Inf}_n(D \otimes D', C \otimes C') = (\text{Inf}_n(D, C) \otimes \text{Inf}_n(D', C')).n. \nonumber
\]

Proof. By substituting \( D \) and \( D' \) with \( D \cap \partial^{-1} D \) and \( D' \cap \partial'^{-1} D' \) in Lemma 2.1 respectively, it can be directly verified that
\[
(\partial \otimes \partial')_n((D \cap \partial^{-1} D) \otimes (D' \cap \partial'^{-1} D'))_n \subseteq (D \otimes D')_n. \nonumber
\]
Thus we have that
\[
\text{Inf}_n(D \otimes D', C \otimes C') \supseteq (\text{Inf}_n(D, C) \otimes \text{Inf}_n(D', C')).n. \nonumber
\]
On the other hand, for each element
\[
x_p \otimes y_q \in (D \otimes D')_n \cap (\partial \otimes \partial')_n^{-1}(D \otimes D')_{n-1}, \quad p + q = n, \quad p,q \geq 0,
\]
we have that
\[(\partial_p \otimes \partial'_q)(x_p \otimes y_q) = ((\partial_p x_p) \otimes y_q + (-1)^p x_p \otimes (\partial'_q y_q)) \in (D \otimes D')_{n-1}.
\]
By counting the degrees, we obtain that
\[(\partial_p x_p) \otimes y_q, x_p \otimes (\partial'_q y_q) \in (D \otimes D')_{n-1}.
\]
It follows that \(x_p \in D_p \cap \partial_p^{-1} D_{p-1}\) and \(y_q \in D'_q \cap \partial_q^{-1} D'_{q-1}\). Thus we have
\[(x_p \otimes y_q) \in ((D \cap \partial^{-1} D) \otimes (D' \cap \partial'^{-1} D'))_n. \tag{2.6}
\]
We assert that each element in \((D \otimes D')_n \cap (\partial \otimes \partial')^{-1}(D \otimes D')_{n-1}\) is of the form
\[
\sum_{i=1}^m x_{p_i} \otimes y_{q_i}, \quad (x_{p_i} \otimes y_{q_i}) \in (D \otimes D')_n \cap (\partial \otimes \partial')^{-1}(D \otimes D')_{n-1}, \quad p_i + q_i = n.
\]
Then it follows from (2.6) that
\[
\sum_{i=1}^m x_{p_i} \otimes y_{q_i} \in ((D \cap \partial^{-1} D) \otimes (D' \cap \partial'^{-1} D'))_n.
\]
Hence,
\[(D \otimes D')_n \cap (\partial \otimes \partial')^{-1}(D \otimes D')_{n-1} \subseteq ((D \cap \partial^{-1} D) \otimes (D' \cap \partial'^{-1} D'))_n.
\]
That is,
\[\text{Inf}_{\partial}(D \otimes D', C \otimes C') \subseteq (\text{Inf}_{\partial}(D, C) \otimes \text{Inf}_{\partial}(D', C'))_n.
\]
The proposition follows.

To prove the assertion, we assume that there is an element in
\[(D \otimes D')_n \cap (\partial \otimes \partial')^{-1}(D \otimes D')_{n-1}
\]
of the form
\[
\sum_{i=1}^m x_{p_i} \otimes y_{q_i}, \quad x_{p_i} \otimes y_{q_i} \notin (\partial \otimes \partial')^{-1}(D \otimes D')_{n-1}, \quad p_i + q_i = n.
\]
Then we have
\[
\partial x_{p_i} \otimes y_{q_i} + (-1)^{p_i} x_{p_i} \otimes \partial' y_{q_i} + \Phi \in (D \otimes D')_{n-1},
\]
\[
\Phi = \sum_{i=2}^m \partial x_{p_i} \otimes y_{q_i} + (-1)^{p_i} x_{p_i} \otimes \partial' y_{q_i}, \tag{2.7}
\]
and
\[
\partial x_{p_1} \otimes y_{q_1} + (-1)^{p_1} x_{p_1} \otimes \partial' y_{q_1} \notin (D \otimes D')_{n-1}.
\]
Firstly, if \(\partial x_{p_1} \otimes y_{q_1} \notin (D \otimes D')_{n-1}\), then we have \(\partial x_{p_1} \notin D_{p_1-1}\). To make the formula (2.7) hold, there must be a term \(-\partial x_{p_1} \otimes y_{q_1}\) in \(\Phi\). Without loss of generality, we set \(-\partial x_{p_1} \otimes y_{q_1}\) to be a term in \(\partial(x_{p_2} \otimes y_{q_2})\).

Case (i). \(x_{p_2} \otimes y_{q_2} = (-1)^{p_1} \partial x_{p_1} \otimes b\), where \(\partial' b = y_{q_1}\). Then we have
\[(\partial \otimes \partial')_n(x_{p_2} \otimes y_{q_2}) = -\partial x_{p_1} \otimes y_{q_1}.
\]
However, in this case, $x_{p_2} \otimes y_{q_2}$ is not a term in $\sum_{i=1}^m x_{p_i} \otimes y_{q_i}$, since $\partial x_{p_1} \notin D_{p_1-1}$. It’s a contradiction.

Case (ii). $x_{p_2} \otimes y_{q_2} = (-1)^{p_2+1}x_{p_2} \otimes y_{q_1}$, where $\partial x_{p_2} = (-1)^{p_2} \partial x_{p_1}$. Then we have

$$(\partial \otimes \partial')_n(x_{p_2} \otimes y_{q_2}) = -\partial x_{p_1} \otimes y_{q_1} - x_{p_2} \otimes \partial' y_{q_1}.$$ 

Let $a = x_{p_1} + (-1)^{p_2+1}x_{p_2}$. Then

$$a \otimes y_{q_1} = x_{p_1} \otimes y_{q_1} + x_{p_2} \otimes y_{q_2}.$$ 

Hence we can write

$$\sum_{i=1}^m x_{p_i} \otimes y_{q_i} = a \otimes y_{q_1} + \sum_{i=3}^m x_{p_i} \otimes y_{q_i}.$$ 

Secondly, if $\partial' y_{q_1} \in D'_{q_1-1}$, then $a \otimes y_1 \in (D \otimes D')_n \cap (\partial \otimes \partial')^{-1}_n(D \otimes D')_{n-1}$. If $\partial' y_{q_1} \notin D'_{q_1-1}$, similarly, we may set $x_{p_3} \otimes y_{q_3} = -a \otimes y_{q_2}$, where $\partial' y_{q_2} = \partial' y_{q_1}$. Let $b = y_{q_1} - y_{q_2}$. Then we can write

$$\sum_{i=1}^m x_{p_i} \otimes y_{q_i} = a \otimes b + \sum_{i=4}^m x_{p_i} \otimes y_{q_i}.$$ 

Note that $a \otimes b \in (D \otimes D')_n \cap (\partial \otimes \partial')^{-1}_n(D \otimes D')_{n-1}$. The above process leads to the assertion. \qed

### 2.3 A Künneth Formula for the Embedded Homology

By using the embedded homology, we prove a Künneth formula for the tensor product of graded submodules of chain complexes, in the next theorem.

**Theorem 2.3.** Let $R$ be a principal ideal domain. Let $C$ and $C'$ be chain complexes consisting of graded free $R$-modules. Let $D$ and $D'$ be graded sub-$R$-modules of $C$ and $C'$ respectively. Then for each $n \geq 0$, there is a short exact sequence

$$0 \longrightarrow (H_*(D, C) \otimes_R H_*(D', C'))_n \xrightarrow{\sim} H_n(D \otimes_R D', C \otimes_R C') \longrightarrow \bigoplus_i \text{Tor}_i(R, H_*(D), H_{n-i-1}(D', C')) \longrightarrow 0.$$ 

And this sequence splits.

**Proof.** By [13, Theorem 3B.5], we have a short exact sequence

$$0 \longrightarrow (H_*(\inf_*(D, C)) \otimes_R H_*(\inf_*(D', C')))_n \xrightarrow{\sim} H_n(\inf_*(D, C) \otimes_R \inf_*(D', C')) \longrightarrow \bigoplus_i \text{Tor}_i(R, H_*(\inf_*(D, C)), H_{n-i-1}(\inf_*(D', C'))) \longrightarrow 0.$$ 

And this sequence splits. By Proposition 2.2,

$$H_n(\inf_*(D, C) \otimes_R \inf_*(D', C')) = H_n(D \otimes_R D', C \otimes_R C').$$ 

The theorem follows from (2.8) and (2.9). \qed

Let $R$ be a field $\mathbb{F}$. The next corollary follows from Theorem 2.3.

**Corollary 2.4.** Suppose the chain complexes $C$ and $C'$ consist of graded vector spaces over a field $\mathbb{F}$. Let $D$ and $D'$ be graded vector subspaces of $C$ and $C'$ respectively. Then

$$H_*(D \otimes D', C \otimes C') \cong H_*(D, C) \otimes H_*(D', C').$$
3 Künneken Formula for Join of Hypergraphs

In this section, we define join of hypergraphs and prove the Künneken formula for the join of hypergraphs.

Let \( H \) and \( H' \) be two hypergraphs. By [4, Definition 4.8.], we define the join of \( H \) and \( H' \), denoted as \( H \ast H' \), as the hypergraph given by

1. the vertex set \( V(H \ast H') \) is the disjoint union of vertex sets \( V(H) \cup V(H') \);
2. whenever we have two hyperedges \( \sigma \in H \) and \( \sigma' \in H' \), we assign the disjoint union \( \sigma \uplus \sigma' \) as a hyperedge of \( H \ast H' \).

In particular, if both \( H \) and \( H' \) are simplicial complexes, then the join of \( H \) and \( H' \) is the classical join of simplicial complexes.

**Lemma 3.1.** Let \( H \) and \( H' \) be any hypergraphs. As simplicial complexes, we have
\[
\Delta(H \ast H') = (\Delta H) \ast (\Delta H').
\]

**Proof.** It suffices to prove that the set of maximal simplices of \( \Delta(H \ast H') \) is the same as the set of maximal simplices of \( (\Delta H) \ast (\Delta H') \). We notice that the set of maximal simplices of \( \Delta(H \ast H') \) is the same as the set of maximal hyperedges of \( H \ast H' \). By the item (2) in the definition of joins of hypergraphs, this set of maximal hyperedges is given by
\[
\{ \sigma \uplus \sigma' \mid \sigma \text{ is a maximal hyperedge of } H \text{ and } \sigma' \text{ is a maximal hyperedge of } H' \}. \tag{3.1}
\]

We notice that \( \sigma \) (resp. \( \sigma' \)) is a maximal hyperedge of \( H \) (resp. \( H' \)) if and only if it is a maximal simplex of \( \Delta H \) (resp. \( \Delta H' \)). Hence the set (3.1) can be written as
\[
\{ \sigma \uplus \sigma' \mid \sigma \text{ is a maximal simplex of } \Delta H \text{ and } \sigma' \text{ is a maximal simplex of } \Delta H' \}. \tag{3.2}
\]

By the definition of joins of simplicial complexes, we have that (3.2) gives the set of maximal simplices of \( (\Delta H) \ast (\Delta H') \). The lemma follows.

**Lemma 3.2.** Let \( H \) and \( H' \) be any hypergraphs. As chain complexes, we have
\[
\text{Inf}_n(H \ast H') \cong \left( \text{Inf}_n(H) \otimes \text{Inf}_n(H') \right)_{n-1}, \quad n \geq 0.
\]

**Proof.** By Lemma 3.1, we have that as chain complexes,
\[
C_n(\Delta(H \ast H'); R) = C_n((\Delta H) \ast (\Delta H'); R) \cong \bigoplus_{p + q = n - 1} C_p(\Delta H; R) \otimes_R C_q(\Delta H'; R)
\]
for any \( n \geq 0 \). As graded sub-\( R \)-modules of these chain complexes,
\[
R(H \ast H')_n \cong \bigoplus_{p + q = n - 1} R(H)_p \otimes_R R(H')_q \tag{3.3}
\]
for any \( n \geq 0 \). By taking the infimum chain complex for each graded \( R \)-module in (3.3), we have that as chain complexes,
\[
\text{Inf}_n(R(H \ast H')_\ast, C_\ast(\Delta(H \ast H'); R)) \cong \text{Inf}_{n-1}(R(H)_\ast \otimes_R R(H')_\ast, C_\ast(\Delta H; R) \otimes_R C_\ast(\Delta H'; R)) \cong \bigoplus_{p + q = n - 1} \text{Inf}_p(R(H)_\ast, C_\ast(\Delta H; R)) \otimes_R \text{Inf}_q(R(H')_\ast, C_\ast(\Delta H'; R)) \tag{3.4}
\]

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for any $n \geq 0$. The last isomorphism of chain complexes in (3.4) follows from Proposition 2.2. By the definition of the embedded homology for hypergraphs, (3.4) gives the lemma.

The next theorem follows from Theorem 2.3 and Lemma 3.2.

**Theorem 3.3.** Let $\mathcal{H}$ and $\mathcal{H}'$ be hypergraphs. Let $R$ be a principal ideal domain. Then for each $n \geq 0$, there is a short exact sequence

$$0 \rightarrow (H_*(\mathcal{H}; R) \otimes_R H_*(\mathcal{H}'; R))_n \xrightarrow{\phi} H_{n+1}(\mathcal{H} \ast \mathcal{H}'; R) \rightarrow \bigoplus_i \text{Tor}_i(R(H_*(\mathcal{H}); R), H_{n-i-1}(\mathcal{H}'; R)) \rightarrow 0.$$

And this sequence splits.

**Proof.** By Theorem 2.3 we have a short exact sequence

$$0 \rightarrow \left( H_*(\Inf_*(\mathcal{H})) \otimes_R H_*(\Inf_*(\mathcal{H}')) \right)_n \xrightarrow{\phi} H_n(\Inf_*(\mathcal{H}) \otimes \Inf_*(\mathcal{H}')) \rightarrow \bigoplus_i \text{Tor}_i(R(H_*(\Inf_*(\mathcal{H})); R), H_{n-i-1}(\Inf_*(\mathcal{H}')))) \rightarrow 0. \tag{3.5}$$

And this sequence splits. By Lemma 3.2 we have a commutative diagram

$$\begin{array}{ccc}
(\Inf_*(\mathcal{H}) \otimes \Inf_*(\mathcal{H}'))_{n+1} & \xrightarrow{\phi} & \Inf_{n+2}(\mathcal{H} \ast \mathcal{H}') \\
\downarrow \partial_{n+1} & & \downarrow \partial_{n+2} \\
\left( \Inf_*(\mathcal{H}) \otimes \Inf_*(\mathcal{H}') \right)_n & \xrightarrow{\phi} & \Inf_{n+1}(\mathcal{H} \ast \mathcal{H}') \\
\downarrow \partial_n & & \downarrow \partial_{n+1} \\
\left( \Inf_*(\mathcal{H}) \otimes \Inf_*(\mathcal{H}') \right)_{n-1} & \xrightarrow{\phi} & \Inf_n(\mathcal{H} \ast \mathcal{H}').
\end{array}$$

Hence

$$H_n(\Inf_*(\mathcal{H}) \otimes \Inf_*(\mathcal{H}')) \cong H_{n+1}(\Inf_*(\mathcal{H} \ast \mathcal{H}')). \tag{3.6}$$

By substituting (3.6) into the short exact sequence (3.5), we obtain the splitting short exact sequence stated in the theorem.

The next corollary follows from Theorem 3.3 or alternatively from Corollary 2.4 and Lemma 3.2.

**Corollary 3.4.** Let $\mathcal{H}$ and $\mathcal{H}'$ be hypergraphs. Let $\mathbb{F}$ be a field. Then for each $n \geq 0$, we have the isomorphism of $R$-modules

$$H_{n+1}(\mathcal{H} \ast \mathcal{H}'; \mathbb{F}) \cong (H_*(\mathcal{H}; \mathbb{F}) \otimes_{\mathbb{F}} H_*(\mathcal{H}'; \mathbb{F}))_n.$$

**4 Kühneth Formula for Product of Hypergraphs**

In this section, we review the definition of Eilenberg-Zilber map and the cross product of simplicial complexes. Moreover, we define the product of hypergraphs and give the Kühneth-type formula in Theorem 4.4.
4.1 The Eilenberg-Zilber Theorem for Simplicial Sets

We review the Eilenberg-Zilber Theorem for simplicial sets. The content of this subsection is proved by S. Eilenberg and S. MacLane in [5, 6], and is summarized in [12].

A simplicial set is a sequence of sets \( X = \{X_n\}_{n \geq 0} \) with faces \( d_i : X_n \rightarrow X_{n-1} \) and degeneracies \( s_i : X_n \rightarrow X_{n+1} \), \( 0 \leq i \leq n \), such that

1. \( d_j d_i = d_{i-1} d_j \) for \( j < i \);
2. \( s_j s_i = s_{i+1} s_j \) for \( j \leq i \);
3. \( d_j s_i = s_{i+1} d_j \) for \( j < i \);
4. \( d_j s_i = s_i d_{j-1} \) for \( j > i + 1 \);
5. \( d_j s_i = id \) for \( j = i, i + 1 \).

For any \( x \in X \), if \( x = s_i y \) for some \( y \in X \) and some degeneracy \( s_i \), then \( x \) is called degenerate.

Let \( X = \{X_n\}_{n \geq 0} \) be a simplicial set. Let \( R \) be a principal ideal domain. For each \( n \geq 0 \), let \( C_n(X; R) \) be the free \( R \)-module generated by the elements of \( X_n \). Let boundary maps \( \partial_n : C_n(X; R) \rightarrow C_{n-1}(X, R) \) be \( \partial_n = \sum_{i=0}^{n} (-1)^i d_i \). Then we have a chain complex

\[
C_*(X; R) = \{ C_n(X; R), \partial_n \}_{n \geq 0}.
\]

Let \( s(C_*(X; R)) \) be the graded \( R \)-module generated by all the degenerate elements of \( X \). Then \( s(C_*(X; R)) \) is a sub-chain complex of \( C_*(X; R) \). Thus we have the quotient chain complex

\[
C^N_*(X; R) = \{ C_n(X; R)/s(C_{n-1}(X; R)), \partial_n \}_{n \geq 0}.
\]  (4.1)

We call (4.1) the normalized chain complex associated to \( X \).

Given two simplicial sets \( X = \{X_n\}_{n \geq 0} \) and \( Y = \{Y_n\}_{n \geq 0} \), their Cartesian product is a simplicial set \( X \times Y = \{(X_n, Y_n)\}_{n \geq 0} \), with faces \((d^X_i, d^Y_i) : (X_n, Y_n) \rightarrow (X_{n-1}, Y_{n-1})\) and degeneracies \((s^X_i, s^Y_i) : (X_n, Y_n) \rightarrow (X_{n+1}, Y_{n+1})\).

Let \( X \) and \( Y \) be two simplicial sets. An Eilenberg-Zilber contraction from \( C^N_*(X \times Y; R) \) to \( C^N_*(X; R) \otimes_R C^N_*(Y; R) \) is defined by a triple \( (\nu, \mu, \phi) \) where the projection (called the Alexander-Whitney operator)

\[
\nu : C^N_*(X \times Y; R) \rightarrow C^N_*(X; R) \otimes_R C^N_*(Y; R)
\]

and the inclusion (called the Eilenberg-MacLane operator)

\[
\mu : C^N_*(X; R) \otimes_R C^N_*(Y; R) \rightarrow C^N_*(X \times Y; R)
\]

are chain maps, and

\[
\phi : C^N_*(X \times Y; R) \rightarrow C^N_{*+1}(X \times Y; R)
\]

(called the Shih operator) is a homomorphism of \( R \)-modules raising degree by 1, with the followings satisfied

1. \( \nu \mu = id; \)
(2) \( \phi \partial + \partial \phi + \mu \nu = \text{id} \);

(3) \( \phi \mu = 0, \ \nu \phi = 0, \ \phi \phi = 0 \).

Explicitly, the maps \( \nu, \mu \) and \( \phi \) are given as follows:

1. For any \( x_m \in X_m \) and any \( y_m \in Y_m \),

\[
\nu(x_m \times y_m) = \sum_{i=0}^{m} d_{i+1} \cdots d_{m} x_m \otimes d_0 \cdots d_{i-1} y_m;
\]

2. For any \( x_p \otimes y_q \in C_p(X; R) \otimes C_q(Y; R) \),

\[
\mu(x_p \otimes y_q) = \sum_{(\alpha, \beta) \in \{(p, q)\text{-shuffles}\}} (-1)^{\text{sig}(\alpha, \beta)} (s_{\beta_q} \cdots s_{\beta_1} x_p, s_{\alpha_p} \cdots s_{\alpha_1} y_q)
\]

where \( (p, q) \)-shuffle \((\alpha, \beta)\) is a partition of the set \( \{0, 1, \ldots, p+q-1\} \) into two disjoint subsets \( \alpha_1 < \cdots < \alpha_p \) and \( \beta_1 < \cdots < \beta_q \), and \( \text{sig}(\alpha, \beta) = \sum_{i=1}^{p} (\alpha_i - i + 1) \);

3. For any \( x_m \in X_m \) and any \( y_m \in Y_m \),

\[
\phi(x_m, y_m) = \sum (-1)^{\bar{m}+\text{sig}(\alpha, \beta)+1} (s_{\beta_q}+m \cdots s_{\beta_0} \cdots s_{\alpha_m-1} d_{m-q+1} \cdots d_m x_m, s_{\alpha_p+1+m} \cdots s_{\alpha_0+m} d_{m-q} \cdots d_0 y_m)
\]

where \( \bar{m} = m - p - q, \ \text{sig}(\alpha, \beta) = \sum_{i=1}^{p+1} (\alpha_i - i + 1) \) and the sum is taken over all the indices \( 0 \leq q \leq m - 1, 0 \leq p \leq m - q - 1 \) and \( (\alpha, \beta) \in \{(p+1, q)\text{-shuffles}\} \).

The next theorem follows from the Eilenberg-Zilber contraction.

**Theorem 4.1** (Eilenberg-Zilber Theorem). (cf. [5][6][12]) Let \( X \) and \( Y \) be simplicial sets. Then we have induced isomorphisms in homology

\[
H_*(C_*(X \times Y; R)) \xrightarrow{\nu \otimes \mu} H_*(C_*(X; R) \otimes_R C_*(Y; R)).
\]

\[\square\]

### 4.2 Cross Product of Simplicial Complexes

We review the definition of the cross product of simplicial complexes. The content of this subsection can be found in [13 Chapter 3, Section 3.B].

Firstly, we review that simplicial complexes can be represented by simplicial sets (cf. [10 Chapter 2]). Given an (abstract) simplicial complex \( K \), let \( X_n(K) \), \( n \geq 0 \), be the set of the sequences \( (v_0, v_1, \ldots, v_n) \) where

1. \( v_0, v_1, \ldots, v_n \) are vertices of \( K \) with \( v_0 \preceq v_1 \preceq \ldots \preceq v_n \);

2. after deleting the repeated vertices, \( \{v_0, v_1, \ldots, v_n\} \) gives a simplex of \( K \).

For \( 0 \leq i \leq n \), we define the faces \( d_i : X_n(K) \longrightarrow X_{n-1}(K) \) by sending \((v_0, v_1, \ldots, v_n)\) to \((v_0, v_1, \ldots, v_i, \ldots, v_n)\); and define the degeneracies \( s_i : X_n(K) \longrightarrow X_{n+1}(K) \) by sending \((v_0, v_1, \ldots, v_n)\) to \((v_0, v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_n)\). Then equipped with the \( d_i \)'s and \( s_i \)'s, we have a simplicial set

\[
X(K) = \{X_n(K)\}_{n \geq 0}.
\]
Let $R$ be a principal ideal domain. For each $n \geq 0$, let $C_n(K; R)$ be the $R$-module generated by the $n$-simplices of $K$. Then $C_n^X(X(K); R)$ is the same as $C_n(K; R)$. Hence (4.1) gives the chain complex $\{C_n(K; R), \partial_n\}_{n \geq 0}$.

Secondly, we define the cross product of two simplicial complexes. Let $K$ and $K'$ be two (abstract) simplicial complexes with boundary maps $\partial$ and $\partial'$ respectively. Take two simplices $\sigma^{(p)} \in K$ and $\tau^{(q)} \in K'$, where $p, q \geq 0$. We label the vertices of $\sigma$ and $\tau$ respectively as

$$\sigma = \{v_0, v_1, \ldots, v_p\}, \quad \tau = \{w_0, w_1, \ldots, w_q\}.$$ 

We regard the pairs $(i, j)$ with $0 \leq i \leq p$ and $0 \leq j \leq q$ as the vertices of an $p \times q$ rectangular grid in the plane.

Let $\gamma$ be a path formed by a sequence of $p$ horizontal edges and $q$ vertical edges in the grid, starting at $(0, 0)$ and ending at $(p, q)$, always moving either to the right or upward. To such a path $\gamma$, we associate a simplex

$$\eta(\gamma) = \{(i_k, j_k) \mid (i_k, j_k) \text{ is the } k\text{-th vertex of the path } \gamma, 0 \leq k \leq m + n\}.$$ 

Let $\sigma$ run over all simplices of $K$, $\tau$ run over all simplices of $K'$, and $\gamma$ run over all possible paths. We obtain a simplicial complex $K \times K'$. Let $R$ be a principal ideal domain. We define a simplicial cross product

$$\mu : C_p(K; R) \otimes_R C_q(K'; R) \longrightarrow C_{p+q}(K \times K'; R)$$  \hspace{1cm} \text{(4.3)}$$

by the rule

$$\mu(\sigma^{(p)} \otimes \tau^{(q)}) = \sum \gamma (-1)^{|\gamma|} \eta(\gamma).$$  \hspace{1cm} \text{(4.4)}$$

Here $|\gamma|$ is the number of squares in the grid lying below the path $\gamma$. We use $\partial^X$ to denote the boundary maps of $K \times K'$. The boundary maps satisfy

$$\partial^X_{p+q}(\mu(\sigma^{(p)} \otimes \tau^{(q)})) = \mu((\partial_p \sigma^{(p)} \otimes \tau^{(q)}) + (-1)^p \mu(\sigma^{(p)} \otimes (\partial_q \tau^{(q)}))).$$  \hspace{1cm} \text{(4.5)}$$

Hence $\mu$ gives a chain map by (4.3).

Finally, for any simplicial complexes $K$ and $K'$, we observe

$$X(K \times K') = X(K) \times X(K').$$  \hspace{1cm} \text{(4.6)}$$

Here in (4.6), the $\times$ on the left-hand side is the product of simplicial complexes, while the $\times$ on the right-hand side is the Cartesian product of simplicial sets. Hence the cross product $\mu$ given by (4.3) is essentially the Eilenberg-MacLane operator on the simplicial set level. By Theorem (4.1), $\mu$ induces an isomorphism in homology

$$\mu_* : H_\ast(C_\ast(K; R) \otimes_R C_\ast(K'; R)) \longrightarrow H_\ast(C_\ast(K \times K'; R)).$$  \hspace{1cm} \text{(4.7)}$$

### 4.3 The product of hypergraphs

We define the product of hypergraphs and prove the Künneth formula for product of hypergraphs by the method of quasi-isomorphism.
**Definition 4.1.** Let $\mathcal{H}, \mathcal{H}'$ be hypergraphs, then the *product of hypergraphs* $\mathcal{H} \boxtimes \mathcal{H}'$ consists of the elements of the following form

$$\omega = \{(v_0, w_{\beta(0)}), (v_{\alpha(1)}, w_{\beta(1)}) \cdots, (v_{\alpha(p+q)}, w_{\beta(p+q)})\}$$

for all $\sigma = \{v_0, \cdots, v_p\} \in \mathcal{H}, \tau = \{w_0, \cdots, w_q\} \in \mathcal{H}'$, where for each $i$ either

$$\begin{cases} 
\alpha(i + 1) = \alpha(i) \\
\beta(i + 1) = \beta(i) + 1
\end{cases}$$

or

$$\begin{cases} 
\alpha(i + 1) = \alpha(i) + 1 \\
\beta(i + 1) = \beta(i)
\end{cases}$$

Note that if $\mathcal{H}, \mathcal{H}'$ are simplicial complexes, the *product of hypergraphs* is not always a simplicial complex, but its associated simplicial complex coincides with the product of simplicial complexes. Precisely, we have $\Delta(\mathcal{H} \boxtimes \mathcal{H}') = \mathcal{H} \times \mathcal{H}'$. More generally, we obtain

**Proposition 4.2.** $\Delta(\mathcal{H} \boxtimes \mathcal{H}') = \Delta \mathcal{H} \times \Delta \mathcal{H}'$.

**Proof.** Note that $\mathcal{H} \boxtimes \mathcal{H}' \subseteq \Delta \mathcal{H} \times \Delta \mathcal{H}'$. Since $\Delta \mathcal{H} \times \Delta \mathcal{H}'$ is a simplicial complex, by definition, we have $\Delta(\mathcal{H} \boxtimes \mathcal{H}') \subseteq \Delta \mathcal{H} \times \Delta \mathcal{H}'$.

For each $\omega \in \Delta \mathcal{H} \boxtimes \Delta \mathcal{H}'$, we assume that

$$\omega = \{(v_{\alpha(0)}, w_{\beta(0)}), (v_{\alpha(1)}, w_{\beta(1)}) \cdots, (v_{\alpha(p+q)}, w_{\beta(p+q)})\},$$

where $\sigma = \{v_0, \cdots, v_p\} \in \Delta \mathcal{H}, \tau = \{w_0, \cdots, w_q\} \in \Delta \mathcal{H}'$. Thus there exist two elements $\tilde{\sigma} \in \mathcal{H}$ and $\tilde{\tau} \in \mathcal{H}'$ such that

$$\sigma \subseteq \tilde{\sigma}, \quad \tau \subseteq \tilde{\tau}.$$

We may write

$$\tilde{\sigma} = \{v_0, \cdots, v_m\}, \quad \tilde{\tau} = \{w_0, \cdots, w_n\}$$

and choose $i_0, \cdots, i_p$ and $j_0, \cdots, j_q$ corresponding to the order of elements in $\tilde{\sigma}$ and $\tilde{\tau}$. For $\tilde{\sigma} \in \mathcal{H}$ and $\tilde{\tau} \in \mathcal{H}'$, we obtain an element $\tilde{\omega} \in \mathcal{H} \boxtimes \mathcal{H}'$ as follows.

$$\tilde{\omega} = \begin{pmatrix}
(v_0, w_0), & (v_1, w_0), & \cdots & (v_{\alpha(0)}, w_0), \\
(v_{\alpha(0)}, w_1), & (v_{\alpha(0)}, w_2), & \cdots & (v_{\alpha(0)}, w_{\beta(0)}), \\
(v_{\alpha(0)}, w_{\beta(0)+1}), & (v_{\alpha(0)+1}, w_{\beta(0)+1}), & \cdots & (v_{\alpha(1)}, w_{\beta(0)+1}), \\
& \vdots & & \vdots \\
(v_{\alpha(p+q)}, w_{\beta(p+q)+1}), & (v_{\alpha(p+q)+1}, w_{\beta(p+q)+1}), & \cdots & (v_{\alpha(p+q)+1}, w_{\beta(p+q)+1}), \\
(v_m, w_{\beta(p+q)+1}), & (v_m, w_{\beta(p+q)+2}), & \cdots & (v_m, w_{\beta(p+q)+2})
\end{pmatrix}$$

It follows that $\omega \subseteq \tilde{\omega}$. Thus we have $\omega \in \Delta(\mathcal{H} \boxtimes \mathcal{H}')$. It shows that $\Delta \mathcal{H} \boxtimes \Delta \mathcal{H}' \subseteq \Delta(\mathcal{H} \boxtimes \mathcal{H}')$. Thus we have $\Delta \mathcal{H} \times \Delta \mathcal{H}' = \Delta(\Delta \mathcal{H} \boxtimes \Delta \mathcal{H}') \subseteq \Delta(\mathcal{H} \boxtimes \mathcal{H}')$. The desired result follows. 

Next, we will illustrate various product definitions of hypergraph and its associated simplicial complex through specific example.

\[12\]
Example 4.1. Let $\mathcal{H} = \{v_0, v_0v_1\}$, $\mathcal{H}' = \{w_1, w_0w_1\}$, then we have

$$
\begin{align*}
\Delta \mathcal{H} &= \{v_0, v_1, v_0v_1\}, \\
\Delta \mathcal{H}' &= \{w_0, w_1, w_0w_1\} \\
\mathcal{H} \otimes \mathcal{H}' &= \\{(v_0, w_0), (v_0, w_1), (v_0, v_1), (v_1, w_0), (v_0, w_0), (v_1, w_1)\}, \\
\Delta \mathcal{H} \otimes \Delta \mathcal{H}' &= \\{(v_0, v_0), (v_0, v_1), (v_0, w_1), (v_1, w_0), (v_1, v_1), (v_1, w_1)\}, \\
\Delta \mathcal{H} \times \Delta \mathcal{H}' &= \Delta \mathcal{H} \otimes \Delta \mathcal{H}' \cup \{(v_0, w_0), (v_1, w_1)\}.
\end{align*}
$$

Proposition 4.3. Let $R$ be a principal ideal domain, and let $\mathcal{H}, \mathcal{H}'$ be hypergraphs. Then there is a quasi-isomorphism

$$
\inf(C_\ast(\mathcal{H}; R) \otimes C_\ast(\mathcal{H}'; R), C_\ast(\Delta \mathcal{H}; R) \otimes C_\ast(\Delta \mathcal{H}'; R)) \rightarrow \inf(C_\ast(\mathcal{H} \boxtimes \mathcal{H}'; R), C_\ast(\Delta \mathcal{H} \times \Delta \mathcal{H}'; R))
$$

Here, $C_\ast(\mathcal{H}; R)$ denotes finitely generated free $R$-module with generators in $\mathcal{H}$.

Proof. Recall that there are two morphisms of chain complexes

$$
\begin{align*}
\nu : C_\ast(\Delta \mathcal{H} \times \Delta \mathcal{H}'; R) &\rightarrow C_\ast(\Delta \mathcal{H}; R) \otimes C_\ast(\Delta \mathcal{H}'; R), \\
\mu : C_\ast(\Delta \mathcal{H}; R) \otimes C_\ast(\Delta \mathcal{H}'; R) &\rightarrow C_\ast(\Delta \mathcal{H} \times \Delta \mathcal{H}'; R).
\end{align*}
$$

Moreover,

$$
\nu \circ \mu = \id, \quad \mu \circ \nu \simeq \id.
$$

By the definition of Eilenberg-Zilber map, we have that

$$
\mu(\sigma \otimes \tau) = \sum_{\omega \in \mathcal{H} \boxtimes \mathcal{H}'} (-1)^{|\omega|} \omega \in C_\ast(\mathcal{H} \boxtimes \mathcal{H}'; R), \quad \sigma \otimes \tau \in C_\ast(\mathcal{H}; R) \otimes C_\ast(\mathcal{H}'; R),
$$

where $|\omega|$ is determined by $\omega$. It follows that

$$
\mu(C_\ast(\mathcal{H}; R) \otimes C_\ast(\mathcal{H}'; R)) \subseteq C_\ast(\mathcal{H} \boxtimes \mathcal{H}'; R).
$$

On the other hand, by straightforward calculation, we have

$$
\nu(C_\ast(\mathcal{H} \boxtimes \mathcal{H}'; R)) \subseteq C_\ast(\mathcal{H}; R) \otimes C_\ast(\mathcal{H}'; R).
$$

Thus we obtain

$$
\mu(\inf(C_\ast(\mathcal{H}; R) \otimes C_\ast(\mathcal{H}'; R), C_\ast(\Delta \mathcal{H}; R) \otimes C_\ast(\Delta \mathcal{H}'; R))) \subseteq \inf(C_\ast(\mathcal{H} \boxtimes \mathcal{H}'; R), C_\ast(\Delta \mathcal{H} \times \Delta \mathcal{H}'; R)),
$$

and

$$
\nu(\inf(C_\ast(\mathcal{H} \boxtimes \mathcal{H}'; R), C_\ast(\Delta \mathcal{H} \times \Delta \mathcal{H}'; R))) \subseteq \inf(C_\ast(\mathcal{H}; R) \otimes C_\ast(\mathcal{H}'; R), C_\ast(\Delta \mathcal{H}; R) \otimes C_\ast(\Delta \mathcal{H}'; R)).
$$

In addition, we have

$$
\nu \circ \mu|_{\inf(C_\ast(\mathcal{H}; R) \otimes C_\ast(\mathcal{H}'; R), C_\ast(\Delta \mathcal{H}; R) \otimes C_\ast(\Delta \mathcal{H}'; R))} = \id.
$$

It follows that

$$
H(\nu|_{\inf(C_\ast(\mathcal{H}; R) \otimes C_\ast(\mathcal{H}'; R), C_\ast(\Delta \mathcal{H}; R) \otimes C_\ast(\Delta \mathcal{H}'; R))})H(\mu|_{\inf(C_\ast(\mathcal{H}; R) \otimes C_\ast(\mathcal{H}'; R), C_\ast(\Delta \mathcal{H}; R) \otimes C_\ast(\Delta \mathcal{H}'; R))}) = \id,
$$

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and $H(\nu|_{\text{Inf}_n(C_\ast(\mathcal{H} \otimes \mathcal{H}'; R), C_\ast(\Delta \mathcal{H} \times \Delta \mathcal{H}' ; R))})$ is surjective.

It leaves us to show $H(\nu|_{\text{Inf}_n(C_\ast(\mathcal{H} \otimes \mathcal{H}'; R), C_\ast(\Delta \mathcal{H} \times \Delta \mathcal{H}' ; R))})$ is injective. If $\omega \in \mathcal{H} \otimes \mathcal{H}'$ is a cycle, then $\nu(\omega) = 0$. Since $\nu$ is a quasi-isomorphism, by Proposition 4.2 we obtain that $\omega$ is a boundary in $C_\ast(\Delta \mathcal{H} \times \Delta \mathcal{H}' ; R)$. To show $H(\nu|_{\text{Inf}_n(C_\ast(\mathcal{H} \otimes \mathcal{H}'; R), C_\ast(\Delta \mathcal{H} \times \Delta \mathcal{H}' ; R))})$ is injective, it suffices to show $\omega$ is also a boundary in $C_\ast(\mathcal{H} \otimes \mathcal{H}' ; R)$.

Firstly, we consider the cycle of the form

$$\omega = \{(v_0, v_0), (v_1, v_0), \ldots, (v_p, v_0)\} \in \mathcal{H} \otimes \mathcal{H}'.$$

(4.8)

Then $\nu(\omega) = \{v_0, v_1, \ldots, v_p\} \otimes \{v_0\}$. If $\nu(\omega)$ is boundary in $C_\ast(\mathcal{H}; R) \otimes C_\ast(\mathcal{H}'; R)$, then $\{v_0, v_1, \ldots, v_p\}$ is a boundary in $C_\ast(\mathcal{H}; R)$. It implies that $\omega$ is a boundary in $C_\ast(\mathcal{H} \otimes \mathcal{H}'; R)$.

Secondly, we may assume that $p, q \geq 1$. Note that the boundary in $C_\ast(\Delta \mathcal{H} \times \Delta \mathcal{H}' ; R)$ is generated by $\partial \omega$ for $\omega \in \Delta \mathcal{H} \times \Delta \mathcal{H}'$.

Case (i). $\omega \in \Delta \mathcal{H} \times \Delta \mathcal{H}'$. Then we have

$$\omega = \{(v_0, w_0), (v_{\alpha(1)}, w_{\beta(1)}), \ldots, (v_{\alpha(p+q)}, w_{\beta(p+q)})\}$$

for $\sigma = \{v_0, \ldots, v_p\} \in \Delta \mathcal{H}$, $\tau = \{w_0, \ldots, w_q\} \in \Delta \mathcal{H}'$. If $\partial \omega \in C_\ast(\mathcal{H} \otimes \mathcal{H}'; R)$, then each term of $\partial \omega$ is of the form

$$\{(v_0, w_0), \ldots, (v_{\alpha(i)}, w_{\beta(i)}), \ldots, (v_{\alpha(p+q)}, w_{\beta(p+q)})\} \in \mathcal{H} \otimes \mathcal{H}'.$$

Since $p, q \geq 1$, by Pigeonhole Principle, we have $\sigma \in \mathcal{H}$, $\tau \in \mathcal{H}'$. It follows that $\omega \in \mathcal{H} \otimes \mathcal{H}'$.

Case (ii). $\omega \in \Delta \mathcal{H} \times \Delta \mathcal{H}'$ and $\omega \notin \Delta \mathcal{H} \otimes \Delta \mathcal{H}'$. Then we can write

$$\omega = \{(v_0, v_0), \ldots, (v_i, v_i), (v_{i+1}, w_{i+1}), \ldots, (v_n, w_n)\}$$

such that $v_{i+1} \neq v_i, w_{i+1} \neq w_i$.

When $n \geq 2$, at least one of the terms

$$\{(v_0, w_0), \ldots, (v_i, v_i), (v_{i+1}, w_{i+1}), \ldots, (v_n, w_n)\},$$

$$\{(v_0, w_0), \ldots, (v_i, v_i), (v_{i+1}, w_{i+1}), \ldots, (v_n, w_n)\}$$

is not in $\Delta \mathcal{H} \otimes \Delta \mathcal{H}'$. Thus $\partial \omega \notin C_\ast(\Delta \mathcal{H} \otimes \Delta \mathcal{H}' ; R)$. Especially, we have $\partial \omega \notin C_\ast(\mathcal{H} \otimes \mathcal{H}' ; R)$.

When $n = 1$, it follows that

$$\partial \omega = \{(v_1, w_1)\} - \{(v_0, w_0)\}.$$

If $\partial \omega \in C_\ast(\mathcal{H} \otimes \mathcal{H}' ; R)$ is a cycle, then $\{(v_1, w_1)\}, \{(v_0, w_0)\} \in \mathcal{H} \otimes \mathcal{H}'$ are cycles in $C_\ast(\mathcal{H} \otimes \mathcal{H}' ; R)$. It reduces to the form (4.8).

Finally, from the above discussion, if $\partial \omega \in \text{Inf}_n(C_\ast(\mathcal{H} \otimes \mathcal{H}' ; R), C_\ast(\Delta \mathcal{H} \times \Delta \mathcal{H}' ; R))$ is a cycle in $\text{Inf}_n(C_\ast(\mathcal{H} \otimes \mathcal{H}' ; R), C_\ast(\Delta \mathcal{H} \times \Delta \mathcal{H}' ; R))$ for $\omega \in \Delta \mathcal{H} \times \Delta \mathcal{H}'$, then we have $\omega \in \mathcal{H} \otimes \mathcal{H}'$.

Thus $\partial \omega$ is also a boundary in $\text{Inf}_n(C_\ast(\mathcal{H} \otimes \mathcal{H}' ; R), C_\ast(\Delta \mathcal{H} \times \Delta \mathcal{H}' ; R))$. It implies that

$$H(\nu|_{\text{Inf}_n(C_\ast(\mathcal{H} \otimes \mathcal{H}' ; R), C_\ast(\Delta \mathcal{H} \times \Delta \mathcal{H}' ; R))})$$

is injective.

\begin{tikzpicture}
\end{tikzpicture}

Remark 4.1. (1). In the proof of Proposition 4.3

$$H(\nu|_{\text{Inf}_n(C_\ast(\mathcal{H} \otimes \mathcal{H}' ; R), C_\ast(\Delta \mathcal{H} \times \Delta \mathcal{H}' ; R))})$$

is injective.
and

\[ H(\mu|_{\text{Inf}_n(C_*(H;R) \otimes C_*(H';R), C_*(\Delta H;R) \otimes C_*(\Delta H';R))}) \]

represent the morphisms of homologies induced by

\[ \nu|_{\text{Inf}_n(C_*(H \boxtimes H';R), C_*(\Delta H \times \Delta H';R))} \]

and

\[ \mu|_{\text{Inf}_n(C_*(H;R) \otimes C_*(H';R), C_*(\Delta H;R) \otimes C_*(\Delta H';R))} \]

respectively.

(2). Quasi-isomorphism is a weaker condition than chain homotopy and the special chain homotopy (the contraction of chain complex) in the Eilenberg-Zilber Theorem. Proposition 4.3 can be seen as a generalization of Theorem 4.1 on hypergraphs.

**Theorem 4.4** (Künneth Formula for Hypergraphs). *Let \( R \) be a principle ideal domain, and let \( \mathcal{H}, \mathcal{H}' \) be hypergraphs. Then there is a natural exact sequence*

\[
0 \to \bigoplus_{p+q=n} H_p(H) \otimes H_q(H') \to H_n(H \boxtimes H') \to \bigoplus_{p+q=n} \text{Tor}_1^R(H_p(H), H_{q-1}(H')) \to 0.
\]

**Proof.** By [13, Theorem 3B.5], there is an exact sequence

\[
0 \to \bigoplus_{p+q=n} H_p(H) \otimes H_q(H') \to H_n(\text{Inf}_*(H) \otimes \text{Inf}_*(H'))
\]

\[
\to \bigoplus_{p+q=n} \text{Tor}_1^R(H_p(H), H_{q-1}(H')) \to 0.
\]

By Proposition 2.2, we have

\[
\text{Inf}_*(H) \otimes \text{Inf}_*(H') = \text{Inf}_n(C_*(H; R) \otimes C_*(H'; R), C_*(\Delta H; R) \otimes C_*(\Delta H'; R)).
\]

By Proposition 4.3, we obtain

\[
H(\text{Inf}_*(H) \otimes \text{Inf}_*(H')) \cong H(\text{Inf}_n(C_*(H \boxtimes H'; R), C_*(\Delta H \times \Delta H'; R))).
\]

Note that \( \Delta H \times \Delta H' = \Delta (H \boxtimes H') \), we have

\[
H(\text{Inf}_*(H) \otimes \text{Inf}_*(H')) \cong H(H \boxtimes H').
\]

This implies our result. \( \square \)

**Corollary 4.5.** *Let \( \mathcal{H} \) and \( \mathcal{H}' \) be hypergraphs. Let \( \mathbb{F} \) be a field. Then for each \( n \geq 0 \), we have the isomorphism of \( R \)-modules*

\[
H_n(H \boxtimes H'; \mathbb{F}) \cong (H_*(H; \mathbb{F}) \otimes_{\mathbb{F}} H_*(H'; \mathbb{F}))_n.
\]

The next example illustrates the above result.
Example 4.2. Let \( \mathcal{H} = \{v_0, v_0v_1, v_1v_2, v_0v_2\} \), \( \mathcal{H}' = \{w_0, w_1, w_0w_1\} \), and \( \mathbb{F} \) is a field. Then we have that

\[
\mathcal{H} \boxtimes \mathcal{H}' = \begin{cases}
\{(v_0, w_0)\}, & \{(v_0, w_1)\}, \\
\{(v_0, w_0), (v_0, w_1)\}, & \{(v_0, w_0), (v_0, w_1)\}, \\
\{(v_1, w_0), (v_0, w_1)\}, & \{(v_1, w_1), (v_2, w_1)\}, \\
\{(v_1, w_1), (v_2, w_1)\}, & \{(v_1, w_1), (v_2, w_1)\}, \\
\{(v_0, w_0), (v_1, w_1)\}, & \{(v_0, w_0), (v_1, w_1)\}, \\
\{(v_0, w_0), (v_1, w_1), (v_2, w_1)\}, & \{(v_0, w_0), (v_1, w_1), (v_2, w_1)\}
\end{cases}
\]

According to the definition of the embedded homology and Proposition 2.3 in [2], we know that

\[
H_n(\mathcal{H}; \mathbb{F}) = H_n(\text{Inf}_*(\mathcal{H})) = \text{Ker}(\partial_n |_{C_n(\mathcal{H}; \mathbb{F})})/(C_n(\mathcal{H}; \mathbb{F}) \cap \partial_{n+1}(C_{n+1}(\mathcal{H}; \mathbb{F})))
\]

\[
H_0(\mathcal{H} \boxtimes \mathcal{H}'; \mathbb{F}) = \text{Ker}(\partial_0 |_{\langle\langle(v_0, w_0), (v_0, w_1)\rangle\rangle})/\langle\langle(v_0, w_1)\rangle\rangle - \langle\langle(v_0, w_0)\rangle\rangle
\]

\[
= \mathbb{F}
\]

\[
H_0(\mathcal{H}; \mathbb{F}) = \text{Ker}(\partial_0 |_{\langle\langle v_1 \rangle\rangle})/\langle\langle v_1 \rangle\rangle - \langle\langle v_0 \rangle\rangle \cap \langle\langle v_0 \rangle\rangle
\]

\[
= \mathbb{F}
\]

\[
H_0(\mathcal{H}'; \mathbb{F}) = \text{Ker}(\partial_0 |_{\langle\langle w_0 \rangle\rangle})/\langle\langle w_1 \rangle\rangle - \langle\langle v_0 \rangle\rangle
\]

\[
= \mathbb{F}
\]

\[
H_p(\mathcal{H} \boxtimes \mathcal{H}'; \mathbb{F}) = 0, H_p(\mathcal{H}; \mathbb{F}) = H_p(\mathcal{H}'; \mathbb{F}) = 0 \text{ for any } p \geq 1.
\]

Thus

\[
H_n(\mathcal{H} \boxtimes \mathcal{H}'; \mathbb{F}) \cong (H_*(\mathcal{H}; \mathbb{F}) \otimes_{\mathbb{F}} H_*(\mathcal{H}'; \mathbb{F}))_n.
\]

5 Further Discussions

In this section, we discuss briefly that our proof for the Künneth formula for hypergraphs is applicable to give an alternative proof for the Künneth formula for the path homology of digraphs.

A digraph is a pair \((V, E)\) where \(V\) is the vertex set and \(E\) is a subset of \(V \times V\). For any \((a, b) \in E\), we write \(a \to b\) and call it a directed edge. In [8, 11], the professors A. Grigor’yan, Y. Lin, Y. Muranov, V. Vershikin and S. T. Yau defined the path complex for a digraph where the allowed paths go along the arrows of the directed edges; and with the help of path complex, the path homology for a digraph is defined and studied. The Künneth formula for the path homology (with field coefficients) of digraphs is proved in [9, Section 7] and [11].

By applying Theorem 2.3 and the Eilenberg-Zilber Theorem of simplicial sets, we can generalize the Künneth formula for the path homology with field coefficients and obtain a Künneth formula for the path homology with coefficients in a general principal ideal domain \(R\). We regard the set of allowed paths in a digraph as a graded subset of certain simplicial set, and regard the path complex as a graded abelian subgroup of certain chain complex. By a similar argument of Theorem 1.3 we would obtain an alternative proof for the Künneth
formula for the path homology, and generalize the coefficients from a field to general principal ideal domains.

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References

[1] C. Berge, *Graphs and hypergraphs*. North-Holland Mathematical Library, Amsterdam, 1973.

[2] S. Bressan, J. Li, S. Ren and J. Wu, *The embedded homology of hypergraphs and applications*. Asian J. Math. 23 (3) (2019), 479-500.

[3] F.R.K. Chung and R.L. Graham, *Cohomological aspects of hypergraphs*. Trans. Amer. Math. Soc. 334 (1) (1992), 365-388.

[4] A. M. Duval and V. Reiner, *Shifted simplicial complexes are Laplacian integral*. Trans. Amer. Math. Soc. 354 (11) (2002), 4313-4344.

[5] S. Eilenberg and S. MacLane, *On the groups H(π, n), I*. Ann. Math. 58 (1953), 55-106.

[6] S. Eilenberg and S. MacLane, *On the groups H(π, n), II*. Ann. Math. 60 (1954), 49-139.

[7] E. Entander, *Betti numbers of hypergraphs*. Commun. Algebra 37 (5), (2009), 1545-1571.

[8] A. Grigor’yan, Y. Lin, Y. Muranov and S.T. Yau, *Path complexes and their homologies*, preprint, 2015. https://www.math.uni-bielefeld.de/~grigor/dnote.pdf. to appear in Int. J. Math.

[9] A. Grigor’yan, Y. Lin, Y. Muranov and S.T. Yau, *Homologies of path complexes and digraphs*. arXiv (2012), http://arxiv.org/abs/1207.2834

[10] A. Grigor’yan, Y. Muranov, V. Vershinin and S.T. Yau, *path homology theory of multigraphs and quivers*. Forum Math. 30 (5) (2018), 1319-1337.

[11] A. Grigor’yan, Y. Muranov and S.T. Yau, *Homologies of digraphs and Künneth formulas*. Commun. Anal. Geom. 25 (5) (2017), 969-1018.

[12] R. González-Díaz and P. Real, *a combinatorial method for computing Steenrod squares*. Journal of Pure and Applied Algebra 139 (1999), 89-108.

[13] A. Hatcher, *Algebraic topology*. Cambridge University Press, Cambridge, 2001.

[14] J. Johnson, *Hyper-networks of complex systems*, in *Complex Sciences*, Series Lecture Notes of the Institute for Computer Sciences, Social Informatics and Telecommunications Engineering 4, (2009), 364-375.
[15] A.D. Parks and S.L. Lipscomb, *Homology and hypergraph acyclicity: a combinatorial invariant for hypergraphs*. Naval Surface Warfare Center, 1991.

[16] J. Wu, *Simplicial objects and homotopy groups*, in *Braids. Introductory Lectures on Braids, Configurations and their Applications*. World Scientific, Hackensack, 2010, 31-181.

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