Nonequilibrium dynamic transition in a kinetic Ising model driven by both deterministic modulation and correlated stochastic noises

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We report the nonequilibrium dynamical phase transition (NDPT) appearing in a kinetic Ising spin system (ISS) subject to the joint application of a deterministic external field and the stochastic mutually correlated noises simultaneously. A time-dependent Ginzburg-Landau stochastic differential equation, including an oscillating modulation and the correlated multiplicative and additive white noises, was addressed and the numerical solution to the relevant Fokker-Planck equation was presented on the basis of an average-period approach of driven field. The correlated white noises and the deterministic modulation induce a kind of dynamic symmetry-breaking order, analogous to the stochastic resonance in trend, in the kinetic ISS, and the reentrant transition has been observed between the dynamic disorder and order phases when the intensities of multiplicative and additive noises were changing. The dependencies of a dynamic order parameter $Q$ upon the intensities of additive noise $A$ and multiplicative noise $M$, the correlation $\lambda$ between two noises, and the amplitude of applied external field $h$ were investigated quantitatively and visualized vividly. A brief discussion was given to outline the underlying mechanism of the NDPT in a kinetic ISS driven by an external force and correlated noises.

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A. Introduction

Ising spin system (ISS) is an ideal theoretical model to describe a variety of physical phenomena of a uniaxial anisotropy system [1]. The nonequilibrium dynamic phase transition (NDPT) arises in a kinetic Ising spin system when a system is driven by both an external field and temperature simultaneously. Depending upon the combination of temperature and driving field, there exist in a kinetic ISS two sorts of dynamical phases, i.e. symmetry-breaking order phase and symmetric disorder counterpart, and the two sorts of dynamical phases transform into each other with the variation of driving field and temperature[2˜8]. The time-dependent Ginzburg-Landau (TDGL) equation offers, within an approximation of mean-field, a more tractable continuum solution and temperature[2˜8]. The time-dependent Ginzburg-Landau stochastic differential equation, including an oscillating modulation and the correlated multiplicative and additive white noises, was addressed and the numerical solution to the relevant Fokker-Planck equation was presented on the basis of an average-period approach of driven field. The correlated white noises and the deterministic modulation induce a kind of dynamic symmetry-breaking order, analogous to the stochastic resonance in trend, in the kinetic ISS, and the reentrant transition has been observed between the dynamic disorder and order phases when the intensities of multiplicative and additive noises were changing. The dependencies of a dynamic order parameter $Q$ upon the intensities of additive noise $A$ and multiplicative noise $M$, the correlation $\lambda$ between two noises, and the amplitude of applied external field $h$ were investigated quantitatively and visualized vividly. A brief discussion was given to outline the underlying mechanism of the NDPT in a kinetic ISS driven by an external force and correlated noises.

Introducing a stochastic fluctuation within a kinetic deterministic system is a topic of interest and it arouses an extensive concern of multidisciplinary researchers in recent years. A large quantity of evidence suggests that the noise plays an important and constructive role in a nonlinear system [9-17], such as the phase transitions induced by a noise, the stochastic resonance (SR), the complexity of biology and so on. Among these, the most challenging subject is associated with the NDPT of a kinetic ISS driven by a deterministic modulation, e.g. a sinusoidal oscillation, and a stochastic white noise simultaneously. Unfortunately, the studies on the NDPT in a kinetic ISS have so far focused simply on a single deterministic external field [2-7], a deterministic field jointly coupled with either a sole Gaussian white noise or pulse source [8,18-20]. Few studies have ever touched on the NDPT of a kinetic ISS including a correlated double noise source. Zaikin et al. investigated the double SR of bistable system driven by an external deterministic field with the addition of an uncorrelated double noise [13]. Denisov et al., however, studied the NDPT induced simply by a double multiplicative noise at a zero deterministic field [16]. Furthermore, Jia et al. reported the reentrant phenomenon appearing in a bistable system perturbed by correlated multiplicative noises at a zero deterministic field [14] as well as the SR due to the collaboration of both correlated noises and deterministic external field [15]. The above studies, although revealing from the different aspects some individual features of the NDPT and the SR of a bistable system in different parameter spaces, have not yet given a clear account for the NDPT of a kinetic ISS, especially the feature in dynamic responding when the correlation of a multiplicative and additive noises, the driving field and the system temperature vary accordingly.

We have provided an insight into the influence of correlated additive and multiplicative noises on the growth rate of a tumor using the Logistic model in a previous paper [21]. Here we present some latest results of our study on the NDPT of a kinetic ISS. We visualize in a clear and vivid way the NDPT occurring in the multi-parameter space consisting of the correlation coefficient ($\lambda$) between additive and multiplicative noises as well as their intensities ($A$, $M$), and the amplitude of deterministic driving field ($h_0$). This paper is organized with five sections, including the current introductory section. In sec. 2, we give a brief description of a kinetic ISS and the
basic feature of the NDPT, the stochastic TDGL equation with a correlated multiplicative and additive Gaussian noise source as well as the algorithm for solving numerically the relevant Fokker-Planck equation. The computational results and the relevant analysis, especially the mechanism underlying the peculiar reentrant NDPT in a kinetic ISS, are presented in detail in sec 3 and 4, respectively. The characteristics of the NDPT in a kinetic ISS are summarized in sec 5.

B. Nonequilibrium dynamical phase transition and theoretical model

Considering the kinetic Ising model of N interacting spins driven by an external deterministic field, one can express its Hamiltonian as

$$\dot{H} = -\frac{J}{N} \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j - h(t) \sum_i \vec{S}_i$$

(1)

where J is the exchange-coupling constant and spin S = ±1. Symbol i and \{ij\} stand for a single spin-site and a spin-pair of the nearest neighbor, respectively. The time variation of driving field h(t) is written in a simple cosinoidal form. The stochastic TDGL equation of Eq. (1) within the approximation of mean-field is described by

$$\frac{\partial m}{\partial t} = -\Gamma \frac{\delta H(m)}{\delta m} + m \xi(t) + \rho(t)$$

(2-1)

H(m) is the usual m^4 Hamiltonian for the instant order-parameter m(t) and consists of the following terms:

$$H(m) = \int d^4r \left\{ \frac{r_0}{2} m^2 + \frac{u_0}{4} m^4 - m \cdot h_0 \cos(\omega t) \right\}$$

(2-2)

Γ and u_0 are two phenomenological coefficients. The reduced temperature r_0 is defined as ~T-T_C0; T and T_C0 are the temperature of system and static critical temperature, respectively; h_0 and \omega denote the amplitude and frequency of a deterministic driving field; t is the evolution time of the system; \xi(t) and \rho(t) represent the multiplicative and additive Gaussian noises which satisfy the zero-mean and autocorrelation, expressed as

$$\langle \xi(t) \rangle = 0, \langle \xi(t) \xi(t') \rangle = 2M \delta(t - t')$$

(3-1)

$$\langle \rho(t) \rangle = 0, \langle \rho(t) \rho(t') \rangle = 2A \delta(t - t')$$

(3-2)

$$\langle \xi(t) \rho(t') \rangle = 2 \lambda \sqrt{MA} \delta(t - t')$$

(3-3)

Here M, A and \lambda in eqs.(3) symbolize the intensities of mutually correlated multiplicative and additive noises, and the correlation intensity between them, respectively.

To quantify the process of dynamic phase transition, the dynamic order parameter Q is defined as

$$Q = \frac{\omega}{2\pi} \int m(t) dt$$

(4)

Actually, dynamic order-parameter Q is the period-averaged of the instant order-parameter m over evolution time t. Q=0 and Q≠0 correspond to a symmetric dynamic disorder phase and a symmetry-breaking dynamic order phase, respectively. And they were also referred to as the symmetry-restoring oscillation (SRO) and symmetry-breaking oscillation (SBO) [8]. The purpose of the current paper is to investigate quantitatively the dependence of Q parameter of a kinetic ISS upon M, A, \lambda and h_0 combined.

The relevant Fokker-Planck (F-P) equation of eq.(2) is given as with the initial and boundary conditions specified

$$\frac{\partial p(m,t)}{\partial t} = -2 \frac{\partial [C(m,t)p(m,t)]}{\partial m} + \frac{\partial^2}{\partial m^2}[B(m)p(m,t)]$$

(5-1)

$$C(m,t) = (1 + M)m^2 - m^4 + h_0 \cos(\omega t), \text{(5-2(1))}$$

$$B(m) = Mm^2 + 2\lambda \sqrt{MA} m + A \text{ \ (5-2(2))}$$

p(m,0) = \delta(m - 0), p(-\infty,t) = p(\infty,t) = 0 \text{ \ (5-3)}$$

Apparently, partial differential eq. (5-1), owing to its explicit time-dependence, is non-autonomous in nature and it is feasible to work out the numerical solution of F-P eq. (5-1), i.e. probability distribution of m against time t, p(m,t).

According to the definition of Q parameter in eq. (4) as well as the numerical solution of p(m,t) above, we can figure out finally the dynamic order parameter Q of a kinetic ISS using

$$Q_{F-P} = \left| \frac{\int_{-\infty}^{\infty} m(t) p(m,t) dt}{\int_{-\infty}^{\infty} p(m,t) dt} \right|$$

(6)

Fig. 1a shows an asymptotically steady and symmetric p(m,t) at a zero driving field and the zero correlation of two sorts noises. Time-dependent oscillations
of \( p(m,t) \) are displayed in Fig.1b (nonzero driving field but zero correlation of noises ) and 1c (nonzero driving field and nonzero correlation). The symmetry of \( p(m,t) \) is destroyed by the mutually correlated noises and no asymptotically steady solution is available when a time-dependent driving field is involved.

C. Results and analysis

Fig. 2 displays the dependence of the reduced Q parameter upon both M and A under the condition of a different correlation intensity \( \lambda \). The dynamic order parameter \( Q \), due to the cooperation of the driving field with the correlated noises, attains a distinctive resonant peak within a certain range of M and A, quite similar to the characteristic of a common stochastic resonance when the ratio of signal versus noise is evaluated. A larger correlation intensity leads to a higher Q peak as shown in Fig. 2. The existence of a peaking Q parameter indicates the enhancement of dynamic ordering in a kinetic ISS if the deterministic driving force could synchronize with the stochastic force. Judging from the viewpoint of the NDPT, the resonance-like dependence of Q parameter is a kind of reentrant phenomenon that occur usually in a peculiar noise circumstance. Both additive and multiplicative noises give rise to a reentrant trend of the dynamic order parameter but, according to Fig. 2, the multiplicative noise has a more evident effect than the additive one does.

Fig. 3 exhibits two four-dimensional panoramic plots of the sliced contour-lines of dynamic order Q within the parameter space of A, M, \( \lambda \) and A, M, \( h_0 \). It has been displayed vividly that a correlated noise source induces the enhancement of dynamic ordering as well as the NDPT of reentrant characteristic. In contrast to noise intensity, the amplitude of a deterministic driving field simply results in a monotonic trend of Q parameter.

D. Discussion

When an interacting collaboratively many-body system, such as the ISS in the present paper, is driven in time by an external field, the system cannot respond instantaneously to the perturbation due to the relaxational delay of the system itself [2], and therefore there come along a series of intriguing nonequilibrium phenomena, such as dynamic response, dynamic hysteresis and dynamic phase transition. There exist two competing time scales in a kinetic spin system, namely the oscillation period of an external driving field \( \tau_\omega = 2\pi/\omega \) and the intrinsic relaxation time of a spin system \( \tau_s \). The latter tends to diverge because of the critical slowing-down when temperature approaches to a critical point. The NDPT takes place only when the two competing time scales, \( \tau_\omega \) and \( \tau_s \), match each other someway in quantity [3,4]. The involvement of a noise source introduces the third competing time scale \( \tau_K \). As for an activated Brownian particle jumping between the double potential wells, the noise-induced hopping between the local equilibrium states with the Kramers rate \( \gamma_K \) as given as [9],

\[
\gamma_K = \gamma_0 \exp\left(-\frac{\Delta V}{D}\right)
\]

where \( \gamma_0 \) is a prefactor, \( \Delta V \) and D are the height of a potential barrier and a noise intensity, respectively. Obviously, \( \tau_K = 1/\gamma_K \) and therefore \( \tau_K \) is adjustable through the noise intensity D. The match between \( \tau_\omega \) and \( \tau_K \) brings about the common SR [9], whereas the match between \( \tau_s \) and \( \tau_K \) causes a noise-induced nonequilibrium phase transition. It is the match of three competing time scales, \( \tau_\omega, \tau_s \) and \( \tau_K \), that gives rise to the nonequilibrium dynamic phase transition which we address in this paper. For the sake of simplicity, we assume a fixed amplitude and frequency of the driving field and a constant reduced temperature. Hence, \( \tau_\omega \) and \( \tau_s \) are invariable in discussion below. Set \( \omega = 2 \) then \( \tau_\omega = \pi \). According to either the theory of para- and ferro-magnetic resonance [22] or the linear responding theory [23], one can work out an analogic temperature dependence of \( \tau_s \) out of either theory above. The \( \tau_s \) we figure out by means of the linear responding theory [23] is between 2.5~6 within a wide range of temperature. We plot in Fig. 4 the variation of \( \tau_K \) versus the additive noise intensity A using a reduced equation 7, \( \tau_K = 1/(\gamma_K/\gamma_0) \). For the TDGL model given by eq. (2-2), \( \Delta V \) = 1/4. It is easily observed from Fig. 4 that if \( A < 0.25 \), the \( \tau_K \) is nearly divergent, and when \( 0.25 < A < 1 \), \( \tau_K \) is between 1.5~10 which matches with \( \tau_\omega \) and \( \tau_s \) in an order of magnitude. When \( A > 1 \), \( \tau_K = 1 \). The trend of \( \tau_K \) against the additive noise intensity A in Fig. 4 complies excellently with the Q A trend demonstrated in Fig. 2. The optimal additive noise \( A_P \) at which the dynamic order parameter Q has a peak value is around 0.25~0.5. The effect of a multiplicative noise, similar to the trend of an additive noise as mentioned above, is more noticeable in that the hopping of a Brownian particle agitated by a multiplicative noise is even modulated by the instant order parameter \( m(t) \) itself. Because of \( |m(t)| \leq 1 \) as well as the \( m(t) \) declines with the rise of temperature, the multiplicative noise must satisfy the requirement of \( MP > A_P \) so as to achieve the same effect that an additive noise has. Fig. 3 shows that the optimal multiplicative noise \( M_P \) is about 3, greater than its counterpart \( A_P \). Considering that the modulation the order parameter m impacts on the multiplicative noise intensity M, one can understand why the dynamic order parameter Q relies upon the M more drastically than the A when \( M > M_P \). And that is
the reason why the range of reentrance induced by a multiplicative noise is much narrower than that caused by an additive noise, as typically shown in Figs. 2 and 3. The noise-induced dynamic ordering and the NDPT come into being only when $\tau_K$ complies someway with $\tau_a$ and $\tau_s$ in quantity. In a word, the matches among the three competing time scales, what we deal with and present in this paper, are the fundamental condition for the occurrence of a dynamic ordering, and hence the nonequilibrium dynamic phase transition. Within a multi-parameter space consisting of the amplitudes of a deterministic external field and a correlated noise source simultaneously.

E. Conclusions

The kinetic Ising spin system, when driven by a deterministic external field and a stochastic source of correlated additive and multiplicative noise simultaneously, holds a characteristic of stochastic resonance and gives rise to the nonequilibrium dynamic phase transition. Within a multi-parameter space consisting of the amplitudes of a deterministic external field, the intensities of additive and multiplicative noises, the relevant correlation intensity between the two sorts of noises together, the dynamic order parameter take an apparent reentrant trend against noise intensity, indicating the development and the fading away of dynamic ordering due to the variation from a cooperative match to a mismatch among the deterministic driving field, stochastic source and the intrinsic relaxation of a spin system itself. A multiplicative noise has a more obvious effect on inducing the dynamic order in a kinetic Ising spin system than an additive noise does.

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G. References

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H. Figures
FIG. 1: (a) $A=M=0.5$, $h_0=0.0$, $\lambda=0.0$

FIG. 2: (b) $A=M=0.5$, $h_0=0.2$, $\lambda=0.0$

FIG. 3: (c) $A=M=0.5$, $h_0=0.2$, $\lambda=0.5$
FIG. 4: The distribution function of probability $p(m,t)$ versus instant order parameter $m$ and evolution time $t$ at zero driving field and zero correlation of noise (a), nonzero driving field and zero correlation of noise (b), and nonzero driving field and nonzero correlation of noise (c).

FIG. 5: figure2: (a)
FIG. 6: figure 2: (b)

FIG. 7: figure 2: (c)
FIG. 8: figure 2: (d) Fig2. The dependence of dynamic order $Q$ upon additive noise intensity $A$ and multiplicative noise $M$ at different correlation intensities of noises ($\lambda=0.2$, $\lambda=0.5$).

FIG. 9: figure 3: (a)
FIG. 10: figure 3: (b) Fig. 3. The sliced contour of dynamic order parameter Q within the parameter space of A, M and $\lambda$ (above), of A, M, and h0 (below).

FIG. 11: Figure 4. Reduced $\tau_K$ versus additive noise intensity A.
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