Quickest Detection of Markov Networks

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Abstract
Detecting correlation structures in large networks arises in many domains. Such detection problems are often studied independently of the underlying data acquisition process, rendering settings in which data acquisition policies (e.g., the sample-size) are pre-specified. Motivated by the advantages of data-adaptive sampling, this paper treats the inherently coupled problems of data acquisition and decision-making for correlation detection. Specifically, this paper considers a Markov network of agents generating random variables, and designs a sequential strategy for discerning the correlation model governing the Markov network. By abstracting the Markov network as an undirected graph, designing the quickest sampling strategy becomes equivalent to sequentially and data-adaptively identifying and sampling a sequence of vertices in the graph. Designing such coupled sensing and inference mechanisms is closely related to the notion of controlled sensing. The paper establishes that the known controlled sending approaches, inspired by Chernoff’s rule for controlled sensing, are not optimal in the context of the problem studied, and devises alternative information measures based on which optimal sensing and inference rules are characterized. Performance analyses and numerical evaluations demonstrate the gains of the proposed sequential approach over the existing fixed-sample size and sequential approaches.

1 Introduction
Driven by advances in information sensing and acquisition, many application domains have evolved towards interconnected networks of information sources in which large-scale and complex data is constantly generated and processed for various inferential and decision-making purposes. Induced by their physical couplings, such information sources generate data streams that often bear strong correlation structures. The couplings and the associated correlation model can represent, for instance, the adjacency of modules in physical networks, the interaction of members in social networks, or the electrical connectivity of buses in power grids. Determining the correlation structure governing the data generated by such networks is significantly necessary for designing various inference rules.

In this paper the focus is placed on detecting correlation structures of Markov networks 1. Such networks are effective in modeling the interactions among network constituents, and are widely used for modeling networks in a wide range of domains (e.g., physics, artificial intelligence, social science) 2 5. The

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objective of this paper is to design an optimal sequential and data-adaptive strategy for collecting data from
a network of agents in order to determine the underlying correlation structure in the data generated by a
Markov network. Correlation detection has immediate applications in a number of domains, e.g., biometric
authentication \[6\], blind source separation \[7,8\], sensor networks \[9\], and neural coding \[10\].
Correlation detection can be applied for deciding between correlation and independence, which is referred
to as testing against independence, or more generally for deciding in favor of one correlation model against
a group of models. This can be abstracted by a hypothesis testing where the collected data samples bear
certain correlation structures under different hypotheses. There exist studies in which the decision-making
and data acquisition processes are decoupled and the focus is placed on forming the most reliable detection
decisions based on a given set of data samples (e.g., \[8–15\]). These approaches lack efficiency when facing large
networks and high dimensional data, in which data acquisition incurs substantial communication, sensing,
and decision delay costs. Driven by controlling such costs, it is imperative to determine the fundamentally
minimum number of measurements required for forming decisions with desired reliability, and to characterize
the attendant sampling strategies. Such sampling strategies over networks are specified by the number
of measurements to be collected, as well as the order in which they are collected. When the order is
pre-specified, determining the optimal sampling strategy reduces to minimizing the (average) number of
measurements. This can be effectively facilitated via sequential hypothesis testing, which is well-investigated.
In sequential hypothesis testing the measurements are collected sequentially according to a pre-specified order,
and the sampling strategy dynamically decides whether to take more measurements, or to terminate the
process and form a decision \[16–19\]. However, incorporating dynamic decisions about the order of sampling,
especially in networked data, is less-investigated. Forming such dynamic decisions that pertain to data
acquisition naturally arises in a broad range of applications such as sensor management \[20\], inspection and
classification \[21\], medical diagnosis \[22\], cognitive science \[23\], generalized binary search \[24\], and channel
coding with feedback \[25\], to name a few. One directly applicable approach to treat such coupled sampling
and decision-making process is controlled sensing, originally developed by Chernoff for binary composite
hypothesis testing through incorporating a controlled information gathering process that dynamically decides
about taking one of a finite number of possible actions at each time \[26\]. Under the assumption of uniformly
distinguishable hypotheses and having independent control actions, Chernoff’s rule decides in favor of the
action with the best immediate return according to proper information measures, and achieves optimal
performance in the asymptote of diminishing rate of erroneous decisions. Chernoff’s rule, specifically, at each
time identifies the most likely true hypothesis based on the collected data, and takes the action that reinforces
the decision. Extensions of the Chernoff’s rule to various settings are studied in \[27–30\]. Specifically, studies
in \[27\] and \[28\] investigate the extension of Chernoff rule to accommodate infinite number of available actions
and infinite number of hypothesis, and \[29\] and \[30\] provide alternative rules that are empirically shown to
outperform the Chernoff rule in the non-asymptotic regimes. Recent advances on controlled sensing that
are relevant to the scope of this paper include \[31–34\]. In \[31\] the Chernoff rule is modified to relax the
assumption that the hypotheses should be uniformly distinguishable in the multi-hypothesis setting. In
this modified rule, a randomization policy is introduced into the selection rule such that at certain time
instants it ignores the Chernoff rule and randomly selects one action according to a uniform distribution. This rule is shown to admit the same asymptotic performance as the Chernoff rule. The results are extended to the setting in which the available data belongs to a discrete alphabet and follows a stationary Markov model [32]. An application of the Chernoff rule to anomaly detection in a dataset is investigated in [33], where it is shown that when facing a finite number of sequences consisting of an anomalous one, the Chernoff rule is asymptotically optimal even without assuming that the hypotheses are distinguishable, or exerting randomized actions. The study in [34] imposes a cost on switching among different actions and offers a modification of the Chernoff rule, which randomly decides between repeating the previous action, and a new action based on the Chernoff rule, and it achieves the same asymptotic optimality property as the Chernoff rule. Similarly, the Chernoff rule is also applied to sparse signal recovery [35], sequential estimation [36], and classification problems [37,38] often resulting in great performance gains.

Besides the Chernoff rule and its variations, other efforts have also been made to devise alternative strategies admitting certain optimality guarantees. In pioneering studies, [39] and [40] offer a strategy that initially takes a number of measurements according to a pre-designated rule in order to identify the true hypothesis, after which it selects the action that maximizes the information under the identified hypothesis. The study in [41] proposes a heuristic strategy and characterizes the deviation of its average delay from the optimal rule. Recently, more studies have investigated the Bayesian setting [42–46]. The study in [42] considers a sequential multi-hypothesis testing problem with multiple control action for which the optimal strategy is the solution to a dynamic programming that is computationally intractable. Hence, it designs two heuristic policies and investigates their non-asymptotic and asymptotic performances. For the same problem, performance bounds and the gains of sequential sampling and optimal data-adaptive selection rules are analyzed in the asymptote of large cost of erroneous decisions [43]. The study in [44] restricts the measurements to be generated by the exponential family distributions, and shows that the dimension of the sufficient statistic space is less than both the number of parameters governing the exponential family and the number of hypotheses. Hence, the exactly optimal policy can be characterized with only moderate computational complexity. Other heuristic approaches for anomaly detection are also investigated in [45] and [46] which select the action with the minimum immediate effect on the total Bayesian cost, and are shown to achieve the same optimality guarantees suggested by Chernoff [26].

Despite their discrepancies in settings and approaches, all the aforementioned studies on controlled sensing assume that the available actions are independent, or they follow a first-order stationary Markov process. This is in contrast to the setting of this paper, in which the correlation structure in the generated data under one hypothesis or both induces co-dependence among all control actions. In this paper, we devise a sequential sampling strategy for detecting Markov networks, in which the correlation model plays a significant role in forming the sampling decisions. Specifically, the devised selection rule, unlike the Chernoff rule, incorporates the correlation structure into the decision-making via accounting for the impact of each action on the future ones and selecting the one with the largest expected information under the most likely true hypothesis. The associated optimality guarantees are established, and the specific results for the special case of Gaussian distributions are characterized. The gains of the proposed selection rule are also delineated analytically and
The contributions of this paper are threefold. First we tackle the correlation detection problem posed in [9] in a sequential data-adaptive setting. The study in [9] considers this problem in the fixed-sample-size setting and characterizes the error exponent of the Neyman-Pearson test for Gauss-Markov random fields with nearest neighbor dependency graphs. While the Chernoff rule [26] is the widely used approach for designing sequential sampling strategies, we provide a counter examples which shows that in the presence of the correlation structure the Chernoff rule and its modified version [31] are not necessarily optimal, and propose a novel approach which incorporates the correlation structure into the decisions. Secondly, since the proposed approach becomes computationally complex for large-scale networks, we show that the Markov properties of the network can be leveraged to reduce the computational cost. Specifically, instead of considering the correlation structure among all the nodes (actions), we show that it suffices to consider the correlation among the neighbor nodes in the dependency graph. Next, we provide the optimality properties of the proposed approach and for some special networks we quantify its gains over the Chernoff rule.

2 Data Model and Problem Formulation

2.1 Markov Random Field

A Markov random field (MRF) is a graphical model that encodes certain dependency structures among a collection of random variables. Specifically, corresponding to an undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ with nodes $\mathcal{V} = \{1, 2, \ldots, n\}$ and edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, the ordered set of random variables $\mathcal{X} = \{X_1, \ldots, X_n\}$ form a Markov random field with dependency graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ if the global Markov property stated below holds. Throughout the paper, for any given set $A \subseteq \{1, \ldots, n\}$ we define $X_A = \{X_i : i \in A\}$.

**Definition 1. Global Markov property:** Any two disjoint subsets of random variables are conditionally independent given a separating set, i.e.,

$$X_A \perp \perp X_B \mid X_C,$$

where $C$ separates $A$ and $B$ such that every path between a node in $A$ and a node in $B$ passes through at least one node in $C$.

One immediate result of the global Markov property is that

$$(i, j) \notin \mathcal{E} \iff X_i \perp \perp X_j \mid X_{\mathcal{V} \setminus \{i, j\}}. \tag{1}$$

2.2 Data Model

Consider a network with $n \in \mathbb{N}$ nodes, in which each node $i \in \{1, \ldots, n\}$ can generate a random variable denoted by $X_i \in \mathbb{R}$. The generated random variables $\{X_1, \ldots, X_n\}$ form an MRF corresponding to a
dependency graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$. The true correlation model of $\{X_1, \ldots, X_n\}$ is unknown, and it is assumed to obey one of the two possible models, identifying which can be formalized as the binary hypothesis test:

$$
\begin{align*}
H_0 & : (X_1, \ldots, X_n) \sim F_0 \\
H_1 & : (X_1, \ldots, X_n) \sim F_1
\end{align*}
$$

(2)

where $F_0$ and $F_1$ denote the joint cumulative distribution functions (cdfs) of $\{X_1, \ldots, X_n\}$ under the two models. Figure 1 depicts such a dichotomy correlation model. For convenience in notations, we assume that the distributions of the random variables under each hypothesis $\ell \in \{0, 1\}$ are absolutely continuous with respect to a common distribution and have well-defined probability density functions (pdfs). For every non-empty set $A \subseteq \mathcal{V}$, we denote the joint pdf of $X_A$ under $H_\ell$ by $f_\ell(\cdot; A)$. We also define $T \in \{H_0, H_1\}$ as the true hypothesis and denote the prior probability that hypothesis $H_\ell$ is true by $\epsilon_\ell$, i.e.,

$$
\epsilon_\ell \triangleq \mathbb{P}(T = H_\ell) \quad \text{for } \ell \in \{0, 1\},
$$

(3)

where $\epsilon_0 + \epsilon_1 = 1$.

### 2.3 Sampling Model

We consider a fully sequential data acquisition mechanism, in which we select one node at-a-time and collect one measurement from the selected node. The goal is to identify an optimal sequence of nodes, such that with the minimum number of measurements the true model $T \in \{H_0, H_1\}$ can be determined. Measurements are collected sequentially such that at any time $t$ and based on the information accumulated up to that time, the sampling procedure takes one of the following actions.

- **A$_1$) Exploration:** Due to lack of sufficient confidence, making any decision is deferred and one more measurement is taken from another node in the network. Under this action, the node to be selected should also be specified.

- **A$_2$) Detection:** Data collection process is terminated and a reliable decision about the true model of the network is formed. Under this action, the stopping time and the final decision rule upon stopping should be specified.
The sampling process can be expressed uniquely by its stopping time, final decision, and the data-adaptive node selection rule. In order to formalize the decision-making (detection) action, we define $\tau_n \in \mathbb{N}$ as the stochastic stopping time, at which the sampling process is terminated and a decision is formed, and define $\delta_n \in \{0, 1\}$ as the decision rule at the stopping time, where $\delta_n = \ell$ indicates a decision in favor of $H_\ell$, for $\ell \in \{0, 1\}$. Furthermore, in order to characterize the information-gathering process (exploration), we define the node selection function $\psi_n : \{1, \ldots, \tau\} \rightarrow \mathcal{V}$, where $\psi_n(t)$ returns the index of the node observed at time $t$. Accordingly, we define $\psi_n^t$ as the ordered set of nodes observed up to time $t$, i.e.,

$$\psi_n^t \triangleq \{\psi_n(1), \ldots, \psi_n(t)\}.$$  

(4)

We also define $\varphi_n^t$ as the set of nodes that are unobserved up to time $t$ and can be observed at $t$, i.e.,

$$\varphi_n^t \triangleq \mathcal{V} \setminus \{\psi_n(1), \ldots, \psi_n(t - 1)\}.$$  

(5)

Accordingly, we define the tuple $\Phi_n \triangleq (\tau_n, \delta_n, \psi_n^\tau)$ to uniquely describe the sampling strategy. We also remark that the subscript $n$ is included in all decision rules to signify the effect of network size. Furthermore, we denote the measurement taken from node $\psi_n(t)$ at time $t$ by $Y_t \triangleq X_{\psi_n(t)}$, and denote the sequence of measurements accumulated up to time $t$ by $^tY \triangleq (Y_1, \ldots, Y_t)$.

(6)

The information accumulated sequentially can be abstracted by the filtration $\{\mathcal{F}_t : t = 1, 2, \ldots\}$, where

$$\mathcal{F}_t \triangleq \sigma(Y_1, \ldots, Y_t).$$  

(7)

Both the stopping time and selection rule are $\mathcal{F}_t$-measurable functions. Finally, we define two information measures that are instrumental to formalizing and analyzing various decision rules throughout the paper. Specifically, for any given $\psi_n^t$ and $A \subseteq \{1, \ldots, n\} \setminus \psi_n^t$ we define

$$J_0(A, \psi_n^t) \triangleq D_{KL}(f_0(X_A; A \mid \mathcal{F}_t) \parallel f_1(X_A; A \mid \mathcal{F}_t)),$$

(8)

and

$$J_1(A, \psi_n^t) \triangleq D_{KL}(f_1(X_A; A \mid \mathcal{F}_t) \parallel f_0(X_A; A \mid \mathcal{F}_t)).$$  

(9)

where $D_{KL}(f \parallel g)$ denotes the Kullback-Leibler (KL) divergence from a statistical model with pdf $g$ to a model with pdf $f$.

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1Throughout the paper, depending on the context of the measure defined, the random variables $Y_t$ and $X_{\psi_n(t)}$ are used interchangeably.
2.4 Problem Formulation

The coupled information-gathering strategy and decision-making processes are uniquely specified by the triplet $\Phi_n = (\tau_n, \delta_n, \psi^\tau_n)$. Designing the optimal sampling strategy for achieving the quickest reliable decision involves resolving the tension between the \textit{quality} and \textit{agility} of the process as two opposing measures. The agility of the process is captured by the average delay in reaching a decision, i.e., $\mathbb{E}\{\tau\}$, and the decision quality is captured by the frequency of erroneous decisions. Such error rates include the probability of false alarm (erroneously deciding in favor of $H_1$), i.e.,

$$P^\text{fa}_n \triangleq \mathbb{P}(\delta_n = 1 | T = H_0) ,$$

and the probability of missed-detection (erroneously missing $H_1$), i.e.,

$$P^\text{md}_n \triangleq \mathbb{P}(\delta_n = 0 | T = H_1) .$$

There exists an inherent tension between the accuracy and the agility of the process, since improving one penalizes the other one. Hence, to formalize the quickest reliable decision, we control the quality of the decision and minimize the average number of measurements over all possible combinations of $\Phi_n = (\tau_n, \delta_n, \psi^\tau_n)$. Hence, the optimal sampling strategy of interest is the solution to

$$\inf_{\Phi_n} \mathbb{E}\{\tau_n\} \quad \text{s.t.} \quad P^\text{fa}_n \leq \alpha_n , \quad P^\text{md}_n \leq \beta_n ,$$

where $\alpha_n \in (0, 1)$ and $\beta_n \in (0, 1)$ control the error probability terms $P^\text{fa}_n$ and $P^\text{md}_n$, respectively. The values of $\alpha_n$ and $\beta_n$ for $\ell \in \{0, 1\}$ are selected such that problem (12) is feasible almost surely. The conditions under which (12) is feasible are analyzed in Section 4.1. Furthermore, we define

$$\bar{\alpha} \triangleq -\lim_{n \to \infty} \frac{1}{n} \log \alpha_n , \quad \text{and} \quad \bar{\beta} \triangleq -\lim_{n \to \infty} \frac{1}{n} \log \beta_n .$$

3 Data-Adaptive Sampling

In this section, we offer a coupled sampling and decision-making strategy $\Phi_n = (\tau_n, \delta_n, \psi^\tau_n)$ as a solution to (12) and analyze its optimality properties in Sec. 4.

3.1 Stopping Time and Decision Rules

The detection action consists of determining the stopping time of the sampling process and the final decision rule. To proceed, let us define

$$A_t \triangleq \frac{\log f_1(\mathcal{F}_t)}{f_0(\mathcal{F}_t)}$$

\footnote{For simplicity in notations, throughout the rest of the paper, we omit the subscript $n$ in terms $\psi^\ell_n$ and $\psi_n(t)$.}
as the log-likelihood ratio (LLR) of the measurements collected up to time $t$. It can be readily verified that

$$
\Lambda_{t+1} = \Lambda_t + \log \frac{f_1(Y_{t+1}; \psi(t+1) | F_t)}{f_0(Y_{t+1}; \psi(t+1) | F_t)}.
$$

(15)

To determine the stopping time of the sampling process, we define

$$
\gamma^L_n \triangleq \log \beta_n, \quad \text{and} \quad \gamma^U_n \triangleq -\log \alpha_n,
$$

(16)

and specify the stopping time through the following sequential likelihood ratio test (LRT):

$$
\tau_n \triangleq \inf \{ t : \Lambda_t \notin (\gamma^L_n, \gamma^U_n) \text{ or } t = n \}.
$$

(17)

When $\Lambda_t \notin (\gamma^L_n, \gamma^U_n)$, the thresholds specified in (16) guarantee that the error probability constraints in (12) are satisfied and the problem is feasible. Furthermore, at the stopping time we make a decision about the model according to

$$
\delta_n = \begin{cases} 
0, & \text{if } \Lambda_t < 0 \\
1, & \text{if } \Lambda_t \geq 0
\end{cases}.
$$

(18)

Based on (17) and (18), the sampling process resumes as long as $\Lambda_t \in (\gamma^L_n, \gamma^U_n)$ and terminates as $\Lambda_t$ falls outside this band or $t = n$. Furthermore, if $\Lambda_t$ exits this interval from the upper threshold $\gamma^U_n$ the set \{X$_1$, ..., X$_n$\} is deemed to form a Markov network with cdf $F_1$, and if it falls below the lower threshold $\gamma^L_n$ we make a decision in favor of $F_0$. It is noteworthy that when problem (12) is feasible almost surely, then

$$
P(\Lambda_t \in (\gamma^L_n, \gamma^U_n)) = 0.
$$

(19)

The following theorem establishes that the error probability constraints in (12) are satisfied when deploying the LRT described above.

**Theorem 1.** When problem (12) is feasible, for any data adaptive sampling strategy with the stopping time and decision rule specified in (17) and (18), respectively, and the thresholds given in (16) we have $P_{fa}^n \leq \alpha_n$ and $P_{md}^n \leq \beta_n$.

**Proof:** See Appendix A.

Decision rules in (17) and (18) specify the decisions at the stopping time. Prior to that, we need to dynamically characterize $\psi_n(t)$. In other words, based on (17), as long as the LLR value varies between the two thresholds, we need to take at least one more measurement from the network. In the next subsection we characterize the selection function $\psi_n(t)$ which identifies the node to be observed at each time.

### 3.2 Node Selection Rules

Prior to the stopping time $\tau_n$, at any time $t$ the sampling process should dynamically identify one unobserved node that provides the most relevant information about the true hypothesis, and take measurement from this selected node. Specifically, at time $t$, and based on the information accumulated up to time $t - 1$, the sampling process decides about $\psi(t) \in \varphi^t_n$. 

In this subsection we provide two approaches for the dynamic selection function $\psi(t)$. The first approach is based on the Chernoff rule, which is used widely for relevant problems in controlled sensing and admits optimality properties under certain conditions. Once we establish the formulations and results for the Chernoff rule, we point to its shortcomings for the problem at hand, and offer an alternative rule, which circumvents the shortcomings of the Chernoff rule. The Chernoff rule is provided to furnish some of the elements for designing the optimal approach, and also serves as a strong baseline for assessing the performance of the proposed rule.

### 3.2.1 Chernoff Rule

In the context of the problem studied in this paper, the Chernoff rule first forms the maximum likelihood (ML) decision about the true model of the data $T \in \{H_0, H_1\}$. By denoting the ML decision about the true hypothesis at time $t$ by $\delta_{\text{ML}}(t)$ we have

$$
\delta_{\text{ML}}(t) \doteq \begin{cases} H_0, & \text{if } \Lambda_t < 0 \\ H_1, & \text{if } \Lambda_t \geq 0 \end{cases}.
$$

(20)

Next, based on this decision Chernoff’s rule at any time $t$ selects and samples the node whose measurement is expected to maximize the reliability the decision about the model in the at time $(t+1)$. We define $\psi_{\text{ch}}(t)$ as the node selected by Chernoff’s rule at time $t$, and accordingly define the ordered set $\psi_{\text{ch}}^t = \{\psi_{\text{ch}}(1), \ldots, \psi_{\text{ch}}(t)\}$.

To formalize the Chernoff’s rule in the context of the hypothesis testing problem considered in this paper, and in order to quantify the information gained from each measurement, we define the following two measures:

$$
D_{i0}^t(t) \doteq J_0(\{i\}, \psi_{\text{ch}}^{t-1})
$$

and

$$
D_{i1}^t(t) \doteq J_1(\{i\}, \psi_{\text{ch}}^{t-1}).
$$

(21)

(22)

where $J_0$ and $J_1$ are defined in (8)-(9). Measure $D_{i\ell}^t(t)$ quantifies the information gained by observing node $i$ at time $t$ when the true hypothesis is $H_{\ell}$. These measures can be also represented by

$$
D_{i0}^t(t) = \mathbb{E}_0 \left\{ \log \frac{f_0(Y_i; \{i\} | \mathcal{F}_{t-1})}{f_1(Y_i; \{i\} | \mathcal{F}_{t-1})} \right\},
$$

and

$$
D_{i1}^t(t) = \mathbb{E}_1 \left\{ \log \frac{f_1(Y_i; \{i\} | \mathcal{F}_{t-1})}{f_0(Y_i; \{i\} | \mathcal{F}_{t-1})} \right\}.
$$

(23)

(24)

The Chernoff rule selects the node that maximizes the distance between $f_{\ell}$ and its alternative when the ML decision is in favor of $H_{\ell}$. Therefore, according to the Chernoff rule we obtain the following node selection function:

$$
\psi_{\text{ch}}(t) \doteq \begin{cases} \arg \max_{i \in \psi_{\text{ch}}^{t-1}} D_{i0}^t(t), & \text{if } \delta_{\text{ML}}(t-1) = H_0 \\ \arg \max_{i \in \psi_{\text{ch}}^{t-1}} D_{i1}^t(t), & \text{if } \delta_{\text{ML}}(t-1) = H_1 \end{cases}.
$$

(25)

To avoid any ambiguities, whenever $\arg \max_{i \in \psi_{\text{ch}}^{t-1}} D_{i\ell}^t(t)$ is not unique (for instance, at the beginning of the sampling process), we select one node randomly according to a uniform distribution. The Chernoff rule
minimizes the average delay in the asymptote of small rate of erroneous decisions, if all the selection actions are independent \[26,31\], which in the context of this paper translates to testing for two distributions without any correlation structures. In this paper, however, the available actions, i.e., selecting unobserved nodes, are co-dependent due to the correlation structure of the underlying MRF. Therefore, the Chernoff rule, which ignores such correlation, fails to exploit it. Specifically, by selecting the best immediate action, the Chernoff rule ignores the perspective of the decisions and the impact of the current decision on the future ones. We provide an example in Section 5.2 for which we show that the Chernoff rule is not asymptotically optimal anymore. Our analyses show that incorporating the impact of the decisions on future actions improves the agility of the process significantly. This, in turn, brings about computational complexities, which we will show that can be reduced considerably by leveraging the MRF structure. Another disadvantage of the Chernoff rule, in the context of the problem analyzed in this paper, is when the Markov network is comprised of multiple disconnected subgraphs, in which case the sampling strategy is trapped in one subgraph until it exhausts all the nodes in that subgraph before moving to another one. This limits the flexibility of the sampling strategy for freely navigating the entire graph. Another shortcoming of the Chernoff rule which penalizes the quickness significantly is when the highly correlated nodes (random variables) are concentrated in a cluster with a size considerably smaller than that of the graph, i.e., \(n\). In such cases, our proposed selection rule approaches the cluster more rapidly, compared with the Chernoff rule.

3.2.2 Correlation-Based Selection Rule

We start by introducing new information measures that dynamically incorporate the impact of the decision at any given time on all possible future ones. Specifically, these measures are devised to facilitate selecting the nodes, the measurements of which maximizes the combination of immediate information and future expected information. To this end, at time \(t\) and for each node \(i \in \varphi_n\) we define the set \(\mathcal{R}_i^t\) as the set of all subsets of \(\varphi_n\) that contain \(i\), i.e.,

\[
\mathcal{R}_i^t \triangleq \{ S : S \subseteq \varphi_n \text{ and } i \in S \}.
\] (26)

Corresponding to the measurements collected from the nodes in the set \(S \in \mathcal{R}_i^t\) under \(H_0\) and \(H_1\) we define the following information measures:

\[
M_{0i}^t(t, S) \triangleq J_0(S, \psi^{t-1}),
\] (27)

and

\[
M_{1i}^t(t, S) \triangleq J_1(S, \psi^{t-1}).
\] (28)

These information measures can be equivalently expressed as

\[
M_{0i}^t(t, S_i^t) = \mathbb{E}_0 \left\{ \frac{f_0(X_{S_i^t}; S_{i^t}^t | F_{t-1})}{f_1(X_{S_i^t}; S_{i^t}^t | F_{t-1})} \right\},
\] (29)

and

\[
M_{1i}^t(t, S_i^t) = \mathbb{E}_1 \left\{ \frac{f_1(X_{S_i^t}; S_{i^t}^t | F_{t-1})}{f_0(X_{S_i^t}; S_{i^t}^t | F_{t-1})} \right\}.
\] (30)

The terms \(M_{0i}^t(t, S_i^t)\) capture the information content of \(|S_i^t|\) measurements. Hence, the normalized terms \(\frac{M_{0i}^t(t, S)}{|S|}\) account for the average information content per measurement. Based on these two normalized
measures, an optimal action is to select the node that maximizes the average information over all possible future decisions. Therefore, the node selection function is the solution of the following optimization problem over all combinations of the unobserved nodes:

$$
\psi(t) = \begin{cases} 
\arg \max_{i \in \varphi_n^t} \max_{S \in \mathcal{R}_{i}} \frac{M^i_0(t, S)}{|S|}, & \text{if } \delta_{\text{ML}}(t-1) = H_0 \\
\arg \max_{i \in \varphi_n^t} \max_{S \in \mathcal{R}_{i}} \frac{M^i_1(t, S)}{|S|}, & \text{if } \delta_{\text{ML}}(t-1) = H_1 
\end{cases}
$$

In this selection rule, an ML decision about the true hypothesis is formed based on the collected data, and the node that maximizes the average information over all future possible sequences of measurements is selected. We note that the sets $S$ are selected such that they contain node $i$, which is a candidate to be observed at time $t$, and possibly additional nodes that will be observed in the future. Mimicking this distinction, for $S \in \mathcal{R}_{i}$, the information measure $M^i_{\ell}(t, S)$ can be decomposed according to

$$
M^i_{\ell}(t, S) = J_\ell(\{i\}, \psi_{t-1}) + J_\ell(S \setminus \{i\}, \psi_{t-1}), \quad \text{for } \ell \in \{0, 1\},
$$

where $J_\ell(S \setminus \{i\}, \psi_{t-1})$ is given by

$$
J_0(S \setminus \{i\}, \psi_{t-1}) = \mathbb{E}_0 \left\{ \log \frac{f_0(X_{S \setminus \{i\}}; S \setminus \{i\} \mid \mathcal{F}_{t-1}, X_i; \{i\})}{f_1(X_{S \setminus \{i\}}; S \setminus \{i\} \mid \mathcal{F}_{t-1}, X_i; \{i\})} \right\},
$$

and

$$
J_1(S \setminus \{i\}, \psi_{t-1}) = \mathbb{E}_1 \left\{ \log \frac{f_1(X_{S \setminus \{i\}}; S \setminus \{i\} \mid \mathcal{F}_{t-1}, X_i; \{i\})}{f_0(X_{S \setminus \{i\}}; S \setminus \{i\} \mid \mathcal{F}_{t-1}, X_i; \{i\})} \right\}.
$$

In the expansion in (32), $J_\ell(\{i\}, \psi_{t-1})$, defined in (8) and (9), is the information gained by observing node $i$ at time $t$, and $J_\ell(S \setminus \{i\}, \psi_{t-1})$ is the expected information gained from future measurements from the nodes contained in $S \setminus \{i\}$ when $\psi(t) = i$. This second term constitutes the key distinction of the proposed rule compared to the Chernoff rule, which accounts for incorporating every possible future action. Finding the optimal node $i$ and set $S$ in (31) involves an exhaustive search over all the remaining nodes, which can become computationally prohibitive. However, by leveraging the Markov properties of an MRF, and considering an acyclic dependency graph and identical marginal distributions for each node under both hypotheses, the exhaustive search for an optimal $S \in \mathcal{R}_{i}$ can be simplified significantly. Specifically, for each node $i$, the optimal set $S$ is restricted to only contain the neighbors of $i$ that are not observed prior to time $t$, i.e., $S \subseteq \mathcal{L}_{i}$ where

$$
\mathcal{L}_{i} \triangleq \{i\} \cup \{N_i \cap \varphi_n^t\},
$$

in which $N_i \triangleq \{j \in \mathcal{V} : j \neq i, (i, j) \in \mathcal{E}\}$ is defined as the set of neighbors of node $i$. This indicates that for determining the node to select at each time in an acyclic MRF, it is sufficient to consider a significantly shorter future sampling path for each node. The cardinality of the set of subsets of $\mathcal{L}_{i}$ is significantly smaller than that of $\varphi_n^t$, which translates to a substantial reduction in the complexity of characterizing the optimal selection functions. This observation is formalized in the following theorem.
Algorithm 1: Correlation-based quickest detection strategy

1. set $t = 0$, $\varphi^1_n = \{1, \ldots, n\}$, $\Lambda_0 = 0$, $\gamma^L_n = \log \beta_n$, and $\gamma^U_n = |\log \alpha_n|
2. $t \leftarrow t + 1$
3. for $i \in \varphi^i_n$
4. $\mathcal{L}_i^t \leftarrow \mathcal{N}_i \cap \varphi^i_n$
5. for any $S \subseteq \mathcal{L}_i^t$
6. compute $M^0_i(t, S)$ and $M^1_i(t, S)$ according to (27)–(28)
7. end for
8. end for
9. find $\psi(t)$ based on (37)
10. $\varphi^{t+1}_n \leftarrow \varphi^{t+1}_n \setminus \psi(t)$
11. compute $\Lambda_t$ according to (15)
12. if $\gamma^L_n < \Lambda_t < \gamma^U_n$ and $t < n$
13. go to step 2
14. else if $\Lambda_t < 0$
15. set $\delta_n = 0$ and $\tau_n = t$
16. else
17. set $\delta_n = 1$ and $\tau_n = t$
18. end if

Theorem 2. Under an acyclic dependency graph and identical marginal distributions for each node under both hypotheses, i.e., $f_0(x; \{i\}) = f_1(x; \{i\})$, at each time $t$ and for $\ell \in \{0, 1\}$ we have

\[
\arg \max_{i \in \varphi^i_n} \max_{S \subseteq \mathcal{R}_i^1} \frac{M^\ell_i(t, S)}{|S|} = \arg \max_{i \in \varphi^i_n} \max_{S \subseteq \mathcal{L}_i^t} \frac{M^\ell_i(t, S)}{|S|}.
\]

(36)

Proof: See Appendix [3].

Based on this theorem, the selection function given in (31) simplifies to

\[
\psi(t) = \begin{cases} 
\arg \max_{i \in \varphi^i_n} \max_{S \subseteq \mathcal{R}_i^1} \frac{M^0_i(t, S)}{|S|}, & \text{if } \delta_{\mathrm{ML}}(t - 1) = H_0 \\
\arg \max_{i \in \varphi^i_n} \max_{S \subseteq \mathcal{L}_i^t} \frac{M^1_i(t, S)}{|S|}, & \text{if } \delta_{\mathrm{ML}}(t - 1) = H_1
\end{cases}
\]

(37)

Based on the results of Theorem 1 and Theorem 2, Algorithm 1 provides the detailed steps for detecting a Markov network with a certain correlation structure. The performance of this algorithm is analyzed in the next section.
4 Performance Analysis

4.1 Feasibility Analysis

In this subsection we examine problem (12) in the non-asymptotic regime, i.e., finite $n$, and characterize its feasibility conditions. We aim to determine the conditions on distributions $F_0$ and $F_1$, and the error probability parameters $\alpha_n$ and $\beta_n$, under which the proposed sampling strategy terminates prior to exhausting all the nodes with probability 1. To this end, we use Bhattacharyya distance (BD) as a measure of similarity of two distributions. Specifically, corresponding to two distributions with pdfs $f$ and $g$ we define

$$B(f, g) \triangleq \int f^\frac{1}{2}(y)g^\frac{1}{2}(y) \, dy .$$  \hfill (38)

BD can be used to determine the relative closeness of the two distributions as well as the degree of their separability. By applying the Cauchy-Schwartz inequality it can be readily verified that $B(f, g) \leq 1$ where the equality holds if and only if $f \equiv g$. We adopt the BD for the joint pdfs $f_0(\cdot; V)$ and $f_1(\cdot; V)$, and apply the chain rule to obtain a tighter upper bound on the BD between the two joint distributions. Specifically, corresponding to any ordered sampling sequence $\psi_n = \{\psi(1), \ldots, \psi(n)\}$, we define

$$B(f_0, f_1; \psi_n) \triangleq \int f_0^\frac{1}{2}(y_1^n; \psi^n) f_1^\frac{1}{2}(y_1^n; \psi^n) \, dy^n$$
$$= \int \prod_{t=1}^{n} f_0^\frac{1}{2}(y_t; \psi(t) \mid F_{t-1}) \prod_{t=1}^{n} f_1^\frac{1}{2}(y_t; \psi(t) \mid F_{t-1}) \, dy^n ,$$  \hfill (40)

from which by successively applying the Cauchy-Schwartz inequality we obtain

$$B(f_0, f_1; \psi_n) \leq \prod_{t=1}^{n} \rho(\psi^n, t) ,$$ \hfill (41)

where we have defined

$$\rho(\psi^n, t) \triangleq \int f_0^\frac{1}{2}(y_t; \psi(t) \mid F_{t-1}) f_1^\frac{1}{2}(y_t; \psi(t) \mid F_{t-1}) \, dy_t$$ \hfill (42)

We note that the terms $\rho(\psi^n, t)$ vary for different $\psi^n$. Also, for any sampling sequence $\psi^n$, it might happen that $\rho(\psi^n, t) = 1$ for certain values of $t$, which refers to the node in the $t^{th}$ order of sampling sequence. We define the set $\mathcal{P}_n$ to include all the indices $t$ corresponding to which $\rho(\psi^n, t) < 1$, i.e.,

$$\mathcal{P}_n \triangleq \{t : \rho(\psi^n, t) < 1\} .$$ \hfill (43)

Furthermore, we define $\lambda_n \in (0, 1)$ as the fraction of the indices $t$ corresponding to which $\rho(\psi^n, t) < 1$, i.e.,

$$\lambda_n \triangleq \frac{|\mathcal{P}_n|}{n} .$$ \hfill (44)

Finally, we define

$$\rho_{\text{max}}(\psi^n) \triangleq \max_{t \in \mathcal{P}_n} \rho(\psi^n, t) ,$$ \hfill (45)
based on which from (41) we have
\[ B(f_0, f_1; \psi^n) \leq [\rho_{\max}(\psi^n)]^n \lambda_n. \] (46)

Next, by defining
\[ \kappa_n \triangleq \sup_{\psi^n} - \frac{1}{n} \log \left( B(f_0, f_1; \psi^n) \right), \] (47)
the following theorem establishes a sufficient condition for the feasibility of problem (12) when relying on the detection and selection rules defined in (16)-(18), and (37).

**Theorem 3.** When adopting the detection and selection rules defined in (16)-(18), and (37), respectively, for the problem formulated in (12) we have
\[ P(\tau \leq n) \geq 1 - \left( \frac{\epsilon_0}{\sqrt{\beta_0}} \exp \left\{ -n \left( \kappa_n - \frac{\beta}{2} \right) \right\} + \frac{\epsilon_1}{\sqrt{\alpha_0}} \exp \left\{ -n \left( \kappa_n - \frac{\alpha}{2} \right) \right\} \right), \] (48)
where we have defined \( \alpha_0 \triangleq \alpha_n \exp \{ n\bar{\alpha} \} \) and \( \beta_0 \triangleq \beta_n \exp \{ n\bar{\beta} \} \).

**Proof:** See Appendix C. 

This theorem provides probabilistic guarantees for the feasibility of the proposed test. It can be readily verified that when the error exponents \( \bar{\alpha} \) and \( \bar{\beta} \) are sufficiently small, increasing the size of the network increases the probability of stopping and making a reliable decision prior to exhausting all the nodes in the network. The observation is formalized in the next corollary.

**Corollary 1.** Under the condition
\[ \kappa_n > \frac{1}{2} \max\{\bar{\alpha}, \bar{\beta}\}, \] (49)
the problem in (12) is feasible almost surely in the asymptote of large networks.

### 4.2 Asymptotic Analysis

In the previous subsection we characterized the feasibility conditions of problem (12). In this subsection, we assume that the problem is feasible and analyze the asymptotic performance of the proposed selection rule as the size of the network \( n \) grows, and compare it with that of the Chernoff rule. We note that the proposed selection rule capitalizes on the discrepancies in the information measures corresponding to selecting different nodes. In general, a wider range of information measures leads to more effectively distinguishing the most informative nodes to sample, which in turn reduces the average delay for reaching a sufficiently confident decision. In order to quantify the performance gain of the optimal selection rule over that of the Chernoff rule, we consider different form of variations in the information level of different nodes. For this purpose, corresponding to any subsequence of nodes \( \{\psi(t)\}_{t=1}^m \) and sequence of observations from these nodes \( \{Y_t\}_{t=1}^m \), we define \( I_0(\psi^m) \) and \( I_1(\psi^m) \) as the convergence limits of the normalized log-likelihood ratios (LLRs) as follows
\[
\text{nLLR}_0(Y^m, \psi^m) \triangleq \frac{1}{m} \log \frac{f_0(F_m)}{f_1(F_m)} \rightarrow I_0(\psi^m), \quad \text{under } H_0, \quad (50)
\]
and
\[
\text{nLLR}_1(Y^m, \psi^m) \triangleq \frac{1}{m} \log \frac{f_1(F_m)}{f_0(F_m)} \rightarrow I_1(\psi^m), \quad \text{under } H_1, \quad (51)
\]
when the convergence is complete when \( n \to \infty \) and \( m \to \infty \). In this case, LLRs can be considered random walks with variable step sizes, and limits in (50) and (51) imply that the fluctuations in the average step sizes are diminishing as \( n \) grows. The limits in (50) and (51) converge completely if and only if \( \mathbb{E}_t\{T_\ell(h)\} < \infty \) for all \( h > 0 \), where
\[
T_\ell(h) \triangleq \sup \{ t : |\text{nLLR}_\ell(Y^m, \psi^m) - I_\ell(\psi^m)| \geq h \} . \quad (52)
\]
When the correlation structure of the network is homogeneous, measure \( I_\ell(\psi^m) \) is the same for any subsequence \( \psi^m \), denoted by \( \bar{I}_\ell \), and therefore, observing any subsequence of nodes provides the same amount of information on average. In contrast, when the correlation structure is heterogeneous, measures \( I_0(\psi^m) \) and \( I_1(\psi^m) \) vary for different subsequences. In this case we define
\[
I^*_0 \triangleq \sup_{\psi^m \subseteq \mathcal{V}} I_0(\psi^m), \quad \text{and} \quad I^*_1 \triangleq \sup_{\psi^m \subseteq \mathcal{V}} I_1(\psi^m) . \quad (53)
\]
Based on the measures defined in (50), (51), and (53), in this subsection we analyze the performance in the asymptote of growing \( n \). For this purpose, we first focus on a homogeneous setting, and establish the optimality of the final detection rule characterized in (16)–(18). Next, we compare the gain of the proposed sequential approach to that of the non-sequential approach analyzed in [9]. Finally, we provide the optimality guarantee of the proposed strategy for the heterogeneous settings.

First, for a homogeneous correlation model we characterize the performance of any sequential sampling strategy. The following theorem provides the performance bounds of any feasible solution to problem (12) for a network with a homogeneous correlation structure.

**Theorem 4.** Under the assumptions in (50) and (51), in the asymptote of large \( n \), any feasible solution to problem (12) satisfies
\[
\frac{\mathbb{E}_0\{\tau\}}{n} \geq \frac{\bar{\beta}}{I_0}(1 + o(1)) , \quad (54)
\]
and
\[
\frac{\mathbb{E}_1\{\tau\}}{n} \geq \frac{\bar{\alpha}}{I_1}(1 + o(1)) . \quad (55)
\]

**Proof:** See Appendix [D].

Next, we show that any selection rule combined with the likelihood ratio test given in (16)–(18) achieves asymptotic optimality under this setting.
Theorem 5. Under the assumptions in (50) and (51), a sequential likelihood ratio test with the detection rule specified in (16)–(18), and any node selection rule is an optimal solution to problem (12) in the asymptote of large \( n \), i.e.,

\[
\frac{\mathbb{E}_0\{\tau_n\}}{n} \leq \frac{\bar{\beta}}{I_0} (1 + o(1)) ,
\]

and

\[
\frac{\mathbb{E}_1\{\tau_n\}}{n} \leq \frac{\bar{\alpha}}{I_1} (1 + o(1)) ,
\]

where \( \tau_n \) is the stopping time of the sequential likelihood ratio test.

Proof: See Appendix E.

Next we characterize the gain obtained from data-adaptive stopping time. To this end, we compare the performance of sequential sampling procedures with that of the fixed sample-size setting in terms of their associated error exponents, when the detection decision is made according to the Neyman-Pearson (NP) rule investigated in [9]. For this purpose, we define

\[
\bar{P}_{fa}^n \triangleq P(\delta_n = 1 | T = H_0) ,
\]

and

\[
\bar{P}_{md}^n \triangleq P(\delta_n = 0 | T = H_1) ,
\]

as the frequencies of erroneous decisions by the NP test based on \( n \) samples, and also define

\[
E_{fa}^f \triangleq -\lim_{n \to \infty} \frac{1}{n} \log \bar{P}_{fa}^n ,
\]

and

\[
E_{md}^f \triangleq -\lim_{n \to \infty} \frac{1}{n} \log \bar{P}_{md}^n ,
\]

as the associated error exponents. Similarly, we define

\[
E_{fa}^s \triangleq -\lim_{m \to \infty} \frac{1}{m} \log P_{fa}^n(m) ,
\]

and

\[
E_{md}^s \triangleq -\lim_{m \to \infty} \frac{1}{m} \log P_{md}^n(m) ,
\]

as the error exponents of sequential sampling, where \( P_{fa}^n(m) \) and \( P_{md}^n(m) \) are the error probabilities of sequential sampling when the average number of measurements is \( m \). The connections between the error exponents of the NP test and sequential sampling strategies are established in the following theorem.

Theorem 6 (Gain of adaptivity). The error exponents of the proposed sequential approaches is related to that of the NP test through

\[
E_{md}^s = I_0 \quad \text{and} \quad E_{fa}^s = I_1 ,
\]

\[
E_{md}^f = I_0 \quad \text{and} \quad E_{fa}^f = 0 .
\]

Proof: See Appendix F.

Theorem 6 quantifies the gain of the sequential approaches over the fixed sample-size setting, but does not distinguish the performance gap between different sequential approaches. Hence, in order to quantify the
gains obtained by dynamic node selection rules as well as the performance gap between the Chernoff rule and
the proposed sampling strategy, we prove the optimality properties of the proposed sampling strategy for the
class of networks studied in this paper for which the Chernoff rule is not necessarily optimal. An example
of such networks is provided in Section 5.2. The following theorem establishes the asymptotic optimality of
the proposed selection rule given in (37).

**Theorem 7 (Optimality).** The sampling strategy with the detection action given in (16)–(18), and the
selection function given in (37), is an optimal solution to problem (12) in the asymptote of large $n$, i.e., for
any feasible solution we have

$$
\frac{E_0\{\tau\}}{n} \geq \frac{\beta}{I_0}(1 + o(1)),
$$

(66)

and

$$
\frac{E_1\{\tau\}}{n} \geq \frac{\alpha}{I_1}(1 + o(1)),
$$

(67)

and for the proposed strategy we have

$$
\frac{E_0\{\tau_n\}}{n} \leq \frac{\beta}{I_0}(1 + o(1)),
$$

(68)

and

$$
\frac{E_1\{\tau_n\}}{n} \leq \frac{\alpha}{I_1}(1 + o(1)).
$$

(69)

**Proof:** See Appendix G. 

The proposed selection rule, under each hypothesis $H_\ell$, identifies the subsequence of nodes that achieves
the largest values for $I_\ell$ in (50) and (51) and, since the average delay is inversely proportional to $I_\ell$, it also
minimizes the average delay.

It is noteworthy that the Chernoff rule, which is widely used for controlled sensing, is known to achieve
asymptotic optimality under the assumption of independent actions [26, 31]. In the next subsection we
provide an example for which the actions are dependent due to the correlation structure of the network, and
show that a selection function based on the Chernoff rule is not optimal for such a network.

## 5 Special Cases

In this section, we consider a few special cases, for each of which we present more specialized results. First,
we consider the setting in which both distributions are Gaussian and characterize measures defined for
designing the sampling strategy in terms of the covariance matrices of the distributions. Next, we consider
a line graph with Gaussian data model that also serves as a counter example establishing that the Chernoff
rule is not asymptotically optimal for carrying out the detection decisions in the MRFs considered in this
paper. Finally, we consider detecting whether a given MRF contains a cluster of nodes whose data form
a given correlation model. In all the special cases we quantify the performance gap between the proposed
sampling strategy and the Chernoff rule.
5.1 Gauss-Markov Random Field

Consider a network in which the random variables $X = \{X_1, \ldots, X_n\}$ form a Gauss-Markov random field (GMRF) with dependency graph $G(V, E)$. Specifically, these random variables have a multivariate Gaussian distribution, and for every pair $u, v \in V$ if $(u, v) \notin E$ random variables $X_u$ and $X_v$ are conditionally independent given $X \setminus \{X_u, X_v\}$. A GMRF with covariance matrix $\Sigma$ is non-degenerate if $\Sigma$ is positive-definite, in which case, the potential matrix associated with the GMRF is denoted by $J \triangleq \Sigma^{-1}$. The non-zero elements of the potential matrix are in one-to-one correspondence with the edges of the dependency graph in the sense that

$$J_{uv} = 0 \iff (u, v) \notin E. \quad (70)$$

In a GMRF, the properties of the network are strongly influenced by the structure of the underlying dependency graph. GMRFs with acyclic dependency represent an important class of GMRFs in which there exists at most one path between any pair of nodes, and consequently, the cross-covariance value between any two non-neighbor nodes in the graph is related to the cross-covariance values of the nodes connecting them. Specifically, corresponding to any two edges $(i, j) \in E$ and $(i, k) \in E$, which share node $i \in \mathcal{N}$, we have

$$\Sigma_{jk} = \Sigma_{ji} \Sigma_{ii}^{-1} \Sigma_{ik}, \quad \text{for all } \{j, k\} \subseteq \mathcal{N}_i. \quad (71)$$

In a GMRF with an acyclic graph, the elements and the determinant of the potential matrix can be expressed explicitly in terms of the elements of the covariance matrix.

**Theorem 8** ([9], Theorem 1). For a GMRF with an acyclic dependency graph $G = (V, E)$ and covariance matrix $\Sigma$, the elements of the potential matrix $J$ are given by

$$J_{ii} = \frac{1}{\Sigma_{ii}} \left( 1 + \sum_{j \in \mathcal{N}_i} \frac{\Sigma_{ij}^2}{\Sigma_{ii} \Sigma_{jj} - \Sigma_{ij}^2} \right), \quad \forall i \in \{1, \ldots, n\}, \quad (72)$$

and

$$J_{ij} = \begin{cases} -\Sigma_{ij} & \text{if } (i, j) \in E \\ \frac{-\Sigma_{ii} \Sigma_{jj} - \Sigma_{ij}^2}{\Sigma_{ii} \Sigma_{jj} - \Sigma_{ij}^2} & \text{if } (i, j) \notin E \\ 0 & \text{if } (i, j) \notin E \end{cases}. \quad (73)$$

Furthermore, the determinant of the potential matrix is also given by

$$|J| = \prod_{i \in \mathcal{V}} \Sigma_{ii}^{\text{deg}(i)-1} \prod_{(i,j) \in \mathcal{E}} |\Sigma_{ii} \Sigma_{jj} - \Sigma_{ij}^2|^{-\frac{1}{2}}, \quad (74)$$

where $\text{deg}(i) \triangleq |\mathcal{N}_i|$ counts the number of neighbors of node $i$.

Based on these characteristics of the potential matrix, we express the joint pdf of the random variables forming a GMRF in terms of the cross-covariance terms associated with the dependency graph, which in turn are leveraged to obtain closed-form expressions for the information measures defined in (23)-(24) and (27)-(28), and the selection rules characterized in Section 3. To this end, we first assume that corresponding
to a given acyclic graph $G(V, E)$, the random variables generated by the entire network under the null and alternative hypothesis are distributed according to $\mathcal{N}(\mathbf{0}, \mathbf{I}_n)$ and $\mathcal{N}(0, \mathbf{\Sigma})$, respectively, where $\mathbf{\Sigma}$ is the positive-definite covariance matrix of the random variables generated by the network under $H_1$. We denote the $(i, j)^{th}$ entry of $\mathbf{\Sigma}$ by $\sigma_{ij}$, and assume that $\sigma_{ii} = 1$ for $i \in \{1, \ldots, n\}$ in order to ensure that the marginal distributions generated by each node under both hypotheses are identical.

In order to describe the effect of the sequential sampling process on different measures that we use, sequentially we construct the sequence of graphs $\{G_t(V_t, E_t) : t \in \{1, \ldots, \tau\}\}$ such that the graph $G_t(V_t, E_t)$ at time $t$ is adapted to the nodes observed up to time $t$. Specifically, we set $V_t = \psi^t$, and for each pair of nodes $i, j \in V_t$ we include an edge $(i, j) \in E_t$ if and only if either $(i, j) \in E$, or there exists a path between nodes $i$ and $j$ in the original graph $G(V, E)$ such that none of the nodes on this path has been observed up to time $t$ (except for $i$ and $j$). Figure 2 depicts a toy example on the evolution of $G_t(V_t, E_t)$ over time for $t \in \{1, 2, 3\}$ corresponding to an underlying graph $G(V, E)$. Furthermore, for any $(i, j) \in E_t$ we define

$$\text{LLR}(i, j) = \frac{1}{2} \left[ \log \frac{1}{1 - \sigma_{ij}^2} - \frac{\sigma_{ij}^2}{1 - \sigma_{ij}^2} (X_i^2 + X_j^2) + \frac{2\sigma_{ij}}{1 - \sigma_{ij}^2} X_i X_j \right]. \quad (75)$$

Under these definitions and by assuming that $G_t(V_t, E_t)$ remains acyclic at time $t$, for the LLR of the measurements up to time $t$ defined in (14) we have

$$\Lambda_t = \sum_{i \in V_t} \sum_{j \in \mathcal{N}_i^t} \text{LLR}(i, j), \quad (76)$$

where $X_i$ is the measurement taken from node $i$ and $\mathcal{N}_i^t \triangleq \{ j \in V_t : (i, j) \in E_t \}$. Next, by invoking the GMRF structure, the information measures defined for the Chernoff rule in (23) and (24) for any $i \in \varphi_i$ can be further simplified and expressed in terms of the correlation coefficients. Specifically, corresponding to the Chernoff rule and its associated sampling sequence $\psi_{\text{ch}}$, we have

$$D_0(t) = \frac{1}{2} \sum_{j \in \mathcal{N}_i^t} \left[ \log (1 - \sigma_{ij}^2) + \frac{\sigma_{ij}^2}{1 - \sigma_{ij}^2} (X_j^2 + 1) \right], \quad (77)$$

and

$$D_1(t) = \frac{1}{2} \sum_{j \in \mathcal{N}_i^t} \left[ \log \frac{1}{1 - \sigma_{ij}^2} + \frac{\sigma_{ij}^2}{1 - \sigma_{ij}^2} (X_j^2 - 1) \right]. \quad (78)$$

Fig. 2: Toy example for the evolution of $G_t(V_t, E_t)$ over time for $\psi^3 = \{1, 4, 3\}$. 
Furthermore, by defining
\[ \Delta_i^t \triangleq \{(j,k) : j,k \in \mathcal{N}_i^t\} , \] (79)
from (3) and (9) for the node selection sequence designed by the proposed rule we have
\[ J_0(\{i\}, \psi^{t-1}) = \frac{1}{2} \sum_{j \in \mathcal{N}_i^t} \log(1 - \sigma_{ij}^2) + \frac{1}{2} \sum_{(j,k) \in \Delta_i^t} \text{LLR}(j,k) , \] (80)
and
\[ J_1(\{i\}, \psi^{t-1}) = \frac{1}{2} \sum_{j \in \mathcal{N}_i^t} \log \frac{1}{1 - \sigma_{ij}^2} - \frac{1}{2} \sum_{(j,k) \in \Delta_i^t} \log \frac{1}{1 - \sigma_{jk}^2} \]
\[ + \frac{1}{2} \left[ \sum_{j \in \mathcal{N}_i^t} \frac{\sigma_{ij}^2}{1 - \sigma_{ij}^2} (X_j^2 - 1) + \sum_{(j,k) \in \Delta_i^t} \text{LLR}(j,k) \right] \times \frac{\prod_{j \in \mathcal{N}_i^t} (1 - \sigma_{ij}^2)}{\prod_{(j,k) \in \Delta_i^t} (1 - \sigma_{jk}^2)} . \] (82)
Similarly, by leveraging the result of Theorem 2, for any \( S \in \mathcal{L}_i^t \) we find
\[ J_0(S \backslash \{i\}, \psi^{t-1}) = \frac{1}{2} \sum_{j \in S \backslash \{i\}} \left[ \log(1 - \sigma_{ij}^2) + \frac{2\sigma_{ij}^2}{1 - \sigma_{ij}^2} \right] , \] (83)
and
\[ J_1(S \backslash \{i\}, \psi^{t-1}) = \frac{1}{2} \sum_{j \in S \backslash \{i\}} \left[ \log \frac{1}{1 - \sigma_{ij}^2} \right] . \] (84)
Subsequently, based on (32), the closed-form expression for \( M_i^t(t, S) \) is obtained from
\[ M_i^t(t, S) = J_t(\{i\}, \psi^{t-1}) + J_t(S \backslash \{i\}, \psi^{t-1}) , \quad \text{for } \ell \in \{0, 1\} . \] (85)
These closed-form expressions of the information measures in terms of the covariance matrix entries and the dependency graph structure substantially reduces the computational complexities involved in calculating these measures from the expected values in (23), (24), (29), and (30).

### 5.2 Counter Example for the Optimality of Chernoff rule

Building on the results for the GMRF, in this subsection we provide an example of a heterogeneous network for which the Chernoff rule is not asymptotically optimal, and quantify the gap between its performance and our proposed rule. For this purpose, we consider a setting in which the random variables \( X_V = \{X_i : i \in V\} \) are independent under \( H_0 \), while under \( H_1 \) they form a GMRF with covariance matrix \( \Sigma \). As depicted in Fig. 3, the dependency graph of the GMRF consists of two disjoint line graphs corresponding to the nodes in sets \( A \) and \( B = V \backslash A \). By denoting the covariance matrix of the random variables generated by sets \( A \) and \( B \) by \( \Sigma^A \) and \( \Sigma^B \), respectively, we assume that for any \( (i,j) \in \mathcal{E} \) we have
\[ |\Sigma^A_{ij}| > a , \quad \text{and} \quad |\Sigma^B_{ij}| < b , \] (86)
where \( a > b \). This means that the random variables generated by the nodes in set \( A \) have stronger correlation than the ones generated by the nodes in set \( B \). For such a network, the performance gap between the proposed rule and the Chernoff rule is established in terms of \( a \) and \( b \) in the following theorem.
Theorem 9. Consider an independence versus GMRF test, where GMRF consists of two disjoint line graphs corresponding to the sets of nodes in $A$ and $B$. If the correlation coefficient values between the neighbors in set $A$ are greater than $a$, while in set $B$ they are less than $b$ and $|A| = p = o(n)$, then as $n$ grows for $\ell \in \{0, 1\}$

$$
\frac{E_\ell \{\tau_c\}}{E_\ell \{\tau_n\}} = \frac{I^A_\ell}{I^B_\ell} \geq \left(\frac{a}{b}\right)^2 ,
$$

where we have defined

$$
I^A_\ell \triangleq \sup_{\psi^m \subseteq A} I_\ell (\psi^m) \quad \text{and} \quad I^B_\ell \triangleq \sup_{\psi^m \subseteq B} I_\ell (\psi^m) ,
$$

and $\tau_c$ and $\tau_n$ are the stopping times of the strategies based on the Chernoff rule and the proposed selection rule, respectively.

Proof: See Appendix [H].

This theorem establishes that the Chernoff rule is not necessarily an asymptotically optimal sampling strategy when the actions are correlated.

5.3 Cluster Detection

In this subsection, we analyze cases in which the two statistical models under $H_0$ and $H_1$ are all similar except for a small cluster of nodes that exhibit two different correlation models. Specifically, first we consider a model in which there exists a subset of nodes $B \subseteq \{1, \ldots, n\}$ such that random variables $X_B \triangleq \{X_i : i \in B\}$ are statistically independent under both models $H_0$ and $H_1$. This indicates that the correlation models under $H_0$ and $H_1$ differ only in their distributions over the random variables from nodes $A \triangleq \{1, \ldots, n\} \setminus B$, as depicted in Fig. 4. Also, we assume that the random variables $X_A \triangleq \{X_i : i \in A\}$ form a homogeneous correlation structure, i.e., for $\ell \in \{0, 1\}$

$$
\forall \psi^m \subseteq A : \quad I_\ell (\psi^m) = \bar{I}^A_\ell ,
$$

which means that observing any subsequence of the nodes in set $A$, on average provides the same amount of information. Clearly, for any set of nodes $\psi^m \subseteq B$, we have

$$
\forall \psi^m \subseteq B : \quad I_\ell (\psi^m) = 0 .
$$
In this setting we show that there is a constant gap between the expected stopping times of the proposed rule and the Chernoff rule. This gap stems from the fact that our proposed approach directly starts from sampling the nodes in $A$, and does not waste any sampling time by taking samples from set $B$. However, the Chernoff rule, on average, takes a number of samples from $B$ before getting its sampling focused on $A$. The gap between the stopping times is formulated in the next theorem.

**Theorem 10.** In a network of size $n$, when there exists a subset of nodes $A$ with size $p$ forming an MRF with a connected graph, while the rest of the network generate independent measurements, we have

$$
\mathbb{E}_\ell \{\tau_c\} - \mathbb{E}_\ell \{\tau_n\} = O\left(\frac{n}{p}\right), \quad \text{for } \ell \in \{0, 1\},
$$

(91)

where $\tau_c$ and $\tau_n$ are the stopping times of the strategies based on the Chernoff rule and the proposed selection rule, respectively.

**Proof:** See Appendix [I].

This theorem establishes the zero order asymptotic gain of the proposed strategy over the Chernoff rule in a special setting. Note that as $p$ (the size of $A$) becomes smaller, which leads to more similar and less distinguishable models under $H_0$ and $H_1$, the performance gap increases according to $n/p$.

Next, we further generalize the above setting to one in which besides $X_A$, random variables $X_B$ also form a homogeneous correlation structure (not independent anymore) with a connected dependency graph, i.e., for $\ell \in \{0, 1\}$

$$
\forall \psi^m \subseteq B : \ I_\ell(\psi^m) = \bar{I}_\ell^B.
$$

(92)

This setting is depicted in Fig. [5]. If for set $A$ we have $|A| = o(n)$, then the Chernoff rule starts the sampling process from set $B$ almost surely and it remains in set $B$ until it exhausts all the nodes of $B$, while the proposed rule always identifies the most informative nodes to take the measurements. The following theorem characterizes the performance gap between the Chernoff and the proposed rule in this setting.
Fig. 5: Independence versus a MRF consisting of two clusters.

**Theorem 11.** Consider a network of size $n$ partitioned into sets $A$ and $B$ that satisfy (89) and (92), respectively. In the asymptote of large $n$, if the dependency graph of the nodes in both $A$ and $B$ are connected and $|A| = o(n)$, then

$$\frac{E_0\{\tau_c\}}{E_0\{\tau_n\}} = \max\{\bar{I}_A^0, \bar{I}_B^0\}, \quad \text{and} \quad \frac{E_1\{\tau_c\}}{E_1\{\tau_n\}} = \max\{\bar{I}_A^1, \bar{I}_B^1\}. \quad (93)$$

**Proof:** When $p = o(n)$ the Chernoff rule starts the sampling process from set $B$ with probability 1. By invoking the results of Theorem 6 and Theorem 7 we can conclude the proof.

According to the theorem above, when the size of $A$ is sufficiently small such that most of the time the Chernoff rule starts the sampling process from set $B$, the Chernoff loses its first-order asymptotic optimality property, as shown in the counter example in Section 5.2.

The settings discussed in this subsection highlight the advantages of the proposed selection rule by quantifying two main gains: the gain of selecting the best node at the beginning of the sampling process, and the gain obtained from freely navigating throughout the entire network by jumping over subgraphs in order to find the most informative nodes. Although these settings are special cases, the gains of the proposed rule for a general network is a combination of these two gains.

### 6 Simulation Results

In this section, we evaluate the performance of the proposed sampling strategy by comparing it with the existing approaches through simulations. For this purpose, we use the NP test as the fixed sample-size approach, and for the sequential sampling we consider random (non-adaptive) sampling order and the Chernoff rule. We consider zero-mean Gaussian distributions for data, and test covariance matrix under $H_1$ versus $I_n$ under $H_0$. We also set $\epsilon_0 = \epsilon_1 = 0.5$.

First, we consider a nearest neighbor dependency graph for uniformly distributed nodes in a two-dimensional space, for which the cross covariance value between two nearest neighbors is a function of their distance. We denote the distance between nodes $i$ and $j$ by $R_{ij}$ and set the correlation coefficient between nodes $i$ and $j$ to $\Sigma_{ij} = Me^{-aR_{ij}}$, where $a, M \in \mathbb{R}_+$. Under $H_0$ we set $M = 0$, which corresponds
to independent measurements. Under $H_1$ as $M$ increases the KL divergence between the distributions corresponding to $f_0$ and $f_1$ grows. In Fig. 6 we set $M = 0.1$, $a = 0$, and $\beta_n = 0.1$ and compare the performance of different approaches. To this end, for different values of $n$ we find the the false alarm probability for NP test, based on which we design the sequential sampling strategy for the Cherenoff and proposed selection rules and find the average delay. It is observed that the proposed sampling procedure outperforms both the NP test and the Chernoff rule in terms of the reliability-agility trade-off.

In order to compare the performance of different selection rules for different levels of correlation strength, Fig. 7 compares the average delays incurred by the proposed approach, the Chernoff rule and a random selection rule for different values of $M$ when $n = 1000$, $\alpha_n = \beta_n = 0.2$, and $a = 1$. It is observed that both the Chernoff rule and the proposed approach outperform the random selection rule and as the KL divergence grows by increasing $M$ the improvement is more significant.
We also compare the performance of the proposed strategy with that of the Chernoff rule and the random selection rule in a heterogeneous network. For this purpose, we generate a subgraph with three nodes and two edges, in which the cross-covariance values between the neighbors are 0.5 and 0.1. We use 500 copies of this subgraph as the building block for a network consisting of 1500 nodes. For such a network, the optimal rule is to select the nodes with larger cross-covariance values. Figure 8 demonstrates the average delay before reaching a confident decision for different target accuracies and the selection rules when $\alpha_n = \beta_n$. By comparing Fig. 8 and Fig. 6 it is observed that in heterogeneous networks the proposed strategy improves significantly compared to the Chernoff rule. The reason is the larger discrepancy in the amount of information gained from different nodes. Furthermore, in Fig. 9 the error exponents are compared where it is observed that, the proposed strategy has an error exponent twice as large as that of the Chernoff rule and both of them outperform the strategy based on a random selection of nodes.
In order to verify the results of Theorem 10, we consider a network with \( n = 30000 \) nodes, in which only a subset \( A \) consisting of \( p \) nodes generate correlated random variables under one of the two hypotheses, while the random variables generated by the rest of the nodes are independent under both hypotheses. Figure 10 demonstrates the average delay of the Chernoff rule in taking its first measurement from set \( A \). The upper bound \( \frac{a}{p} \) and lower bound \( \frac{a + b}{2p} \) obtained in Theorem 10 are also shown for comparison. It is observed that the delay difference is always between the obtained bounds, which confirms that it is \( O\left(\frac{n}{p}\right) \).

Finally, we consider a network with 10000 nodes, from which 50 nodes, denoted by set \( A \), are strongly correlated, i.e., the cross-covariance values between the neighbors in set \( A \), denoted by \( \Sigma^A_{ij} \), are greater than a constant \( a \in (0.3, 0.6) \), while the rest of the nodes, denoted by set \( B \), also form a connected graph with cross-covariance values \( \Sigma^B_{ij} \) less than a constant \( b = 0.2 \). In Fig. 11, the ratio between the average delay of the proposed sampling strategy is compared with the lower bound \( (\frac{a}{p})^2 \) obtained in Theorem 9 for different values of \( a \). We also include the ratio between the average delays for the setting in which the cross-covariance values in sets \( A \) and \( B \) are equal to \( a \) and \( b \), respectively, for which it is observed that the lower bound is tighter.

7 Conclusion

We have considered the quickest detection of a correlation structure in a Markov network, with the objective of determining the true model governing the measurements generated by different nodes in the network. After discussing the widely used Chernoff rule and its shortcomings, we have designed a sequential and data-adaptive sampling strategy in order to determine the true correlation structure with the fewest average number of measurements while, in parallel, the final decision is controlled to meet a target reliability. The proposed sampling strategy, which judiciously incorporates the correlation structure of the network into its decision rules, involves dynamically deciding whether to terminate the sampling process, or to continue collecting further evidence, and prior to terminating the process which node to observe at each time. We
have established the optimality properties of the proposed sampling strategy, and leveraged the Markov properties of the network to reduce the computational complexities involved in the implementation of the proposed approach. We have provided an example for which the Chernoff rule is not optimal. Finally, we have quantified the advantages of the proposed rule over the Chernoff rule for some special cases.

Notations throughout the Proofs

For the convenience in notations, throughout the proofs we drop the necessary subscript $n$ in all the measures that are defined in such ways that depend on $n$. This includes the majority of the metrics defined in Section 2 and Section 3.

A Proof of Theorem 1

The feasibility assumption guarantees that $\Lambda_{\tau} \notin (\gamma^L, \gamma^U)$. Therefore, for $P_{fa}$ we have

$$P_{fa} = P(\delta = 1 \mid T = H_0)$$

$$= \sum_{k=1}^{n} P(\delta = 1, \tau = k \mid T = H_0)$$

$$= P(0 \leq \Lambda_{\tau} < \gamma^U, \tau = n \mid T = H_0) + \sum_{k=1}^{n} P(\Lambda_{k} \geq \gamma^U, \tau = k \mid T = H_0)$$

$$= \sum_{k=1}^{n} P(\Lambda_{k} \geq \gamma^U, \tau = k \mid T = H_0)$$

$$= \sum_{k=1}^{n} \int_{(\Lambda_{k} \geq \gamma^U, \tau = k)} f_0(Y^k; \psi^k) dY^k$$

$$\leq \sum_{k=1}^{n} \int_{(\Lambda_{k} \geq \gamma^U, \tau = k)} e^{-\gamma^U} f_1(Y^k; \psi^k) dY^k$$
\[= \alpha_n \sum_{k=1}^{n} \int_{(\Lambda_k \geq \gamma^U, \tau = k)} f_1(Y^k; \psi^k) dY^k \tag{100}\]

\[= \alpha_n \sum_{k=1}^{n} P(\delta = 1, \tau = k | T = H_1) \tag{101}\]

\[= \alpha_n \cdot P(\delta = 1 | T = H_1) \tag{102}\]

\[\leq \alpha_n, \tag{103}\]

where (97) is due to the feasibility assumption and (99) holds since at the stopping time we have \( \Lambda_\tau \geq \gamma^U \).

By following the same line of argument for \( P^{md} \) we obtain

\[P^{md} \leq e^{\gamma^L} \cdot P(\delta = 0 | T = H_0) \tag{104}\]

\[= \beta_n \cdot P(\delta = 0 | T = H_0) \tag{105}\]

\[\leq \beta_n, \tag{106}\]

which concludes the proof.

B Proof of Theorem 2

We show that any node \( k \not\in L^i_t \) falls into one of the following two categories:

1. Its measurement is independent of the measurement we want to take from node \( i \) at time \( t \), i.e., the data from node \( i \) has no impact on the information that will be acquired by measuring node \( k \) in the future.

2. Despite its dependence on the measurement we may take from node \( i \), the amount of its information is less than the average information of the best subset of \( L^i_t \), i.e.,

\[\max_{S \subseteq L^i_t} \frac{M^i_j(t, S \cup \{k\})}{|S| + 1} \leq \max_{S \subseteq L^i_t} \frac{M^i_j(t, S)}{|S|}. \tag{107}\]

It is clear that for both categories, inclusion of node \( k \) in set \( S \) reduces metric \( M^i_j(t, S) \).

First, suppose that \( k \) and \( i \) belong to two disconnected subgraphs. Hence, random variables \( X_i \) and \( X_k \) are unconditionally independent and the selection of node \( i \) has no impact on the selection of node \( k \) in the future, which means that \( k \) belongs to the first category. Next, consider the case in which nodes \( i \) and \( k \) belong to the same subgraph. Since the graph is acyclic, there exists a unique path between these two nodes, because otherwise, two distinct paths between two nodes create a cycle in the graph. Therefore, there exists a unique node \( j \in N_i \) on the path between nodes \( i \) and \( k \). If node \( j \) belongs to the set \( S^i_t \) that maximizes the right hand side of (107), due to the global Markov property and the fact that node \( j \) separates nodes \( i \) and \( k \), node \( k \) will be conditionally independent of \( i \) given \( j \) (category 1).

Now we only need to show that when node \( j \) does not belong to the set \( S \subseteq L^i_t \) that maximizes the right hand side of (107), inclusion of node \( k \) will reduce the average information. To this end, we show that the
information gained from observing node \( j \) is greater than that of node \( k \), and as a result, since \( j \) does not belong to the set that maximizes the normalized \( M'_f(t, S) \), node \( k \) is not in that set, too. For convenience in notations in the remainder of the proof, for the marginal pdf or joint pdf of any combinations of \( X_i \), \( X_j \) and \( X_k \), we remove the index of the node, i.e., the joint pdf of \( X_i \), \( X_j \) and \( X_k \) under \( H_0 \) is denoted by \( f_{\ell}(X_i, X_j, X_k) \) instead of \( f_{\ell}(X_i, X_j, X_k; \{i, j, k\}) \).

To show that the information of the measurement from node \( j \) is greater than that of node \( k \), we first consider the case that under \( H_0 \) all the nodes are independent. We expand

\[
D_{KL}(f_1(X_i, X_j, X_k) \parallel f_1(X_i)f_1(X_j, X_k))
\]

in two ways and compare the results. Hence, we have

\[
D_{KL}(f_1(X_i, X_j, X_k) \parallel f_1(X_i)f_1(X_j, X_k)) = D_{KL}(f_1(X_i, X_j) \parallel f_1(X_i)f_1(X_j)) \quad \quad (109)
\]

\[
= D_{KL}(f_1(X_i, X_j)f_1(X_k|X_j) \parallel f_1(X_i)f_1(X_j)f_1(X_k|X_j)) \quad \quad (110)
\]

\[
= D_{KL}(f_1(X_i, X_j) \parallel f_1(X_i)f_1(X_j)) \quad \quad (111)
\]

\[
= D_{KL}(f_1(X_i, X_j) \parallel f_0(X_i)f_0(X_j)) . \quad \quad (112)
\]

On the other hand

\[
D_{KL}(f_1(X_i, X_j, X_k) \parallel f_1(X_i)f_1(X_j, X_k)) = D_{KL}(f_1(X_i, X_k)f_1(X_j|X_i, X_k) \parallel f_1(X_i)f_1(X_k)f_1(X_j|X_k)) \quad \quad (113)
\]

\[
= D_{KL}(f_1(X_i, X_k) \parallel f_1(X_i)f_1(X_k)) \quad \quad (114)
\]

\[
= D_{KL}(f_1(X_i, X_k) \parallel f_1(X_i)f_1(X_k) + D_{KL}(f_1(X_j|X_i, X_k) \parallel f_1(X_j|X_k)) \quad \quad (115)
\]

\[
\geq D_{KL}(f_1(X_i, X_k) \parallel f_1(X_i)f_1(X_k)) \quad \quad (117)
\]

\[
= D_{KL}(f_1(X_i, X_k) \parallel f_0(X_i)f_0(X_k)) , \quad \quad (118)
\]

where the inequality holds due to the non-negativity of KL divergence. Since the left hand side of (109) and (113) are the same, we have

\[
D_{KL}(f_1(X_i, X_j) \parallel f_0(X_i)f_0(X_j)) \geq \quad \quad (119)
\]

\[
D_{KL}(f_1(X_i, X_k) \parallel f_0(X_i)f_0(X_k)) . \quad \quad (120)
\]

By following the same line of argument for \( D_{KL}(f_1(X_i)f_1(X_j, X_k) \parallel f_1(X_i, X_j, X_k)) \) we obtain

\[
D_{KL}(f_0(X_i)f_0(X_j) \parallel f_1(X_i, X_j)) \geq \quad \quad (121)
\]

\[
D_{KL}(f_0(X_i)f_0(X_k) \parallel f_1(X_i, X_k)) . \quad \quad (122)
\]

From (119)–(122) we can conclude that for independence versus an acyclic MRF setting, the divergence of two distributions is maximal between neighboring nodes, which concludes the proof.

Next, we consider the setting in which \( X_j \) is a neighbor of both \( X_i \) and \( X_k \) under both hypotheses, i.e., for \( \ell \in \{0, 1\} \) we have

\[
f_{\ell}(X_i, X_j, X_k) = f_{\ell}(X_i, X_j)f_{\ell}(X_k|X_j) . \quad \quad (123)
\]
Under this setting, we aim to show the same results similar to the independence versus MRF setting. To this end, we compute $D_{\text{KL}}(f_1(X_i, X_j, X_k) \parallel f_0(X_i, X_j, X_k))$ in two different ways as follows:

$$D_{\text{KL}}(f_1(X_i, X_j, X_k) \parallel f_0(X_i, X_j, X_k)) = D_{\text{KL}}(f_1(X_i, X_j) \parallel f_0(X_i, X_j))$$  \hspace{1cm} (124)
$$+ D_{\text{KL}}(f_1(X_k|X_j) \parallel f_0(X_k|X_j))$$  \hspace{1cm} (125)
$$= D_{\text{KL}}(f_1(X_i, X_j) \parallel f_0(X_i, X_j))$$  \hspace{1cm} (126)
$$+ D_{\text{KL}}(f_1(X_j, X_k) \parallel f_0(X_j, X_k))$$  \hspace{1cm} (127)

and

$$D_{\text{KL}}(f_1(X_i, X_j, X_k) \parallel f_0(X_i, X_j, X_k)) = D_{\text{KL}}(f_1(X_i, X_k) \parallel f_0(X_i, X_k))$$  \hspace{1cm} (128)
$$+ D_{\text{KL}}(f_1(X_j|X_i, X_k) \parallel f_0(X_j|X_i, X_k))$$  \hspace{1cm} (129)

where \((126)\) holds since $f_0(X_j) = f_1(X_j)$, and consequently, $D_{\text{KL}}(f_1(X_j) \parallel f_0(X_j)) = 0$. Now, if we show that

$$D_{\text{KL}}(f_1(X_j|X_i, X_k) \parallel f_0(X_j|X_i, X_k)) \geq D_{\text{KL}}(f_1(X_j|X_k) \parallel f_0(X_j|X_k))$$  \hspace{1cm} (130)

from \((124)-(128)\) we can conclude that

$$D_{\text{KL}}(f_1(X_i, X_k) \parallel f_0(X_i, X_k)) \leq D_{\text{KL}}(f_1(X_i, X_j) \parallel f_0(X_i, X_j))$$  \hspace{1cm} (131)

which indicates that KL divergence between $f_0$ and $f_1$ for two neighbor nodes is greater than the non-neighbor ones. This means that by looking at the neighbors of $i$ we can achieve the maximum metric value $M_i$. To prove \((130)\) we can leverage the convexity of the KL divergence in its first and second argument, and the fact that

$$f_{\ell}(X_j|X_k) = \int f_{\ell}(X_j|X_i, X_k)f_{\ell}(X_i|X_k) \, dX_i .$$  \hspace{1cm} (132)

C Proof of Theorem 3

We first find an upper bound on the probability that the number of samples required for satisfying the reliability constraints is greater than $n$, i.e.,

$$P(\tau > n) = P(\Lambda_t \in (\gamma^L, \gamma^U) \text{ for } t \in \{1, 2, \ldots, n\})$$  \hspace{1cm} (133)
$$\leq P(\Lambda_n \in (\gamma^L, \gamma^U))$$  \hspace{1cm} (134)
$$= \epsilon_0 P_0(\Lambda_n \in (\gamma^L, \gamma^U)) + \epsilon_1 P_1(\Lambda_n \in (\gamma^L, \gamma^U)).$$  \hspace{1cm} (135)

Now, for the first probability term on the right hand side we have

$$P_0(\Lambda_n \in (\gamma^L, \gamma^U)) \leq P_0(\Lambda_n > \gamma^L)$$  \hspace{1cm} (136)
$$= P_0(\sqrt{\exp\{\Lambda_n\}} > \sqrt{\exp\{\gamma^L\}})$$  \hspace{1cm} (137)
$$\leq \frac{E_0\{\sqrt{\exp\{\Lambda_n\}}\}}{\sqrt{\exp\{\gamma^L\}}}.$$  \hspace{1cm} (138)
where (137) holds since exponential function is monotonically increasing, (138) is due to the Markov inequality, and (139) is obtained by replacing $\kappa_n$ and $\gamma^L$ with their values. By following the same procedure we have

$$P_1(\Lambda_n \in (\gamma^L, \gamma^U)) \leq P_1(\Lambda_n < \gamma^U)$$

(140)

$$= P_1(\sqrt{\exp(-\Lambda_n)} > \sqrt{\exp(-\gamma^U)})$$

(141)

$$\leq \sqrt{\exp(\gamma^U)} E_1\{\sqrt{\exp(-\Lambda_n)}\}$$

(142)

$$\leq \frac{\exp(-n(\kappa_n - \bar{\beta}^2))}{\sqrt{\alpha_0}}.$$  

(143)

which combined with (135) and (139) concludes the proof.

**D Proof of Theorem 4**

We provide the proof for (55) and the same line of argument proves (54). In order to prove (55) we show that for any feasible solution to (12) and all $0 < \rho < 1$

$$\text{and } P_1\left(\frac{T}{n} > \rho \frac{\alpha}{I_1}\right) = 1.$$  

(144)

Then, by applying the Markov inequality, we obtain

$$E_1\left\{\frac{T}{n} \right\} \geq \rho \cdot P_1\left(\frac{T}{n} > \rho \frac{\alpha}{I_1}\right) \geq 1,$$  

(145)

which concludes (54). To prove (144) let us define the event

$$A(i, L) \triangleq \{ \delta = i, \tau \leq L \}.$$  

(146)

Then, for any $0 < L < n$ and $F > 0$ we have

$$P_{fa} = P_0(\delta = 1)$$  

(147)

$$= E_0\{I(\delta = 1)\}$$  

(148)

$$= E_1\{I(\delta = 1) \exp(-\Lambda_r)\}$$  

(149)

$$\geq E_1\{I(\mathcal{A}(1, L), \Lambda_r < F) \exp(-\Lambda_r)\}$$  

(150)

$$\geq e^{-F} P_1(\mathcal{A}(1, L), \Lambda_r < F)$$  

(151)

$$\geq e^{-F} P_1\left(\mathcal{A}(1, L), \sup_{t<L} \Lambda_t < F\right)$$  

(152)

$$= e^{-F}\left( P_1(\mathcal{A}(1, L)) - P_1\left(\sup_{t<L} \Lambda_t \geq F\right)\right)$$  

(153)

$$= e^{-F}\left(P_1(\delta = 1) - P_1(\tau > L) - P_1\left(\sup_{t<L} \Lambda_t \geq F\right)\right),$$  

(154)
where (153) and (154) hold due to the properties of set difference operation. Now we have

\[ P_1(\tau > L) \geq P_1(\delta = 1) - e^F P_0(\delta = 1) - P_1(\sup_{t < L} \Lambda_t \geq F) \]

\[ = 1 - \beta_n - e^F \alpha_n - P_1(\sup_{t < L} \Lambda_t \geq F) . \]

(155)

(156)

Since (155) holds for any \( F > 0 \), we set \( F = cL\bar{I}_1 \) for some \( c > 1 \). Then for any \( 1 < K < L \) we obtain

\[ P_1(\sup_{t < L} \Lambda_t \geq cL\bar{I}_1) \leq P_1(\sup_{t < K} \Lambda_t + \sup_{K < t < L} \Lambda_t \geq (c - 1)\bar{I}_1) \]

\[ \leq P_1(\sup_{t < K} \Lambda_t + \sup_{K < t < L} (\frac{L}{t} \Lambda_t) - L\bar{I}_1 \geq (c - 1)\bar{I}_1) \]

\[ \leq P_1(\sup_{t < K} \Lambda_t + \sup_{K < t < L} (\frac{\Lambda_t}{t} - \bar{I}_1) \geq (c - 1)\bar{I}_1) \]

\[ \leq P_1(\frac{1}{L} \sup_{t < K} \Lambda_t + \sup_{t > K} |\frac{\Lambda_t}{t} - \bar{I}_1| \geq (c - 1)\bar{I}_1) . \]

(157)

(158)

(159)

(160)

(161)

(162)

According to (51), for any \( \epsilon > 0 \) there exist a \( \hat{K}(\epsilon) \) such that

\[ P_1\left(\left|\frac{\Lambda_t}{t} - \bar{I}_1\right| \leq \epsilon \right) = 1 , \quad \forall t > \hat{K}(\epsilon) . \]

(163)

Hence, we have

\[ P_1(\sup_{t < L} \Lambda_t \geq cL\bar{I}_1) \leq P_1(\frac{1}{L} \sup_{t < \hat{K}(\epsilon)} \Lambda_t \geq (c - 1)\bar{I}_1 - \epsilon) . \]

(164)

Since, this is true for any \( L < n \) and \( c > 1 \), we assume the case that \( n, L \to \infty \) and \( c > 1 + \frac{\rho}{\bar{I}_1} \). In this setting the right hand side of (164) approaches zero which indicates that for every \( c > 1 \)

\[ \lim_{L \to \infty} P_1(\sup_{t < L} \Lambda_t \geq cL\bar{I}_1) = 0 . \]

(165)

Next, by setting

\[ L = \rho \frac{|\log \alpha_n|}{\bar{I}_1} , \]

(166)

for any \( 0 < \rho < \frac{1}{\epsilon} \), from (155) we obtain

\[ P_1\left(\tau > \rho \frac{|\log \alpha_n|}{\bar{I}_1}\right) \geq 1 - \beta_n - \alpha_n^{1 - \rho c} - P_1(\sup_{t < L} \Lambda_t \geq cL\bar{I}_1) . \]

(167)

(168)

Now, by combining (165) and (168), and for the setting in which \( \alpha_n \) and \( \beta_n \) approach zero we obtain

\[ P_1\left(\tau > \rho \frac{|\log \alpha_n|}{\bar{I}_1}\right) = P_1\left(\tau > \rho \frac{\bar{I}_1}{\bar{I}_1}\right) = 1 . \]

(169)

Since (169) holds regardless of the sampling procedure and stopping rule and only depends on the error performance of the strategy, it is valid for any strategy with the decision quality constraints in (12).
E Proof of Theorem 5

We show that the sequential strategies of this paper achieve the lower bounds on delay given in (54) and (55). To this end, first we leverage the properties of complete convergence in (50) and (51) which yield
\[
E_\ell \{ T_\ell(h) \} < \infty, \quad \forall h > 0, \quad \ell \in \{0, 1\}.
\]  
(170)

According to the definition of the stopping time in (17),
\[
\Lambda_{\tau-1} < \gamma^U.
\]
(171)

Also, from the definition of \(T_1(h)\) in (52) when \(\tau > T_1(h) + 1\) we have
\[
\Lambda_{\tau-1} > (\tau - 1)(1 - h)I_1.
\]
(172)

By combining inequalities in (171) and (172) we obtain
\[
\tau < 1 + \frac{\gamma^U}{I_1(1 - h)} + \frac{\gamma^U}{I_1(1 - h)} + \frac{\gamma^U}{I_1(1 - h)} + T_1(h)
\]
(173)

\[
\leq 1 + \frac{\gamma^U}{I_1(1 - h)} + T_1(h).
\]
(174)

\[
\leq 1 + \frac{\gamma^U}{I_1(1 - h)} + T_1(h).
\]
(175)

Then, by taking the expectation and using (170) we can conclude that
\[
E_1 \{ \tau \} \leq \frac{\gamma^U}{I_1(1 + o(1))},
\]
(176)

and by replacing \(\gamma^U\) from (16) we have
\[
\frac{E_1 \{ \tau \}}{n} \leq \frac{\bar{\alpha}}{I_1(1 + o(1))}.
\]
(177)

By following the same line of argument for \(\Lambda_{\tau-1} > \gamma^L\) we can derive
\[
\frac{E_0 \{ \tau \}}{n} \leq \frac{\bar{\beta}}{I_0(1 + o(1))},
\]
(178)

which concludes the proof.

F Proof of Theorem 6

The error exponents of NP test are studied in [47] where it is shown that when \(P_{fa}\) is fixed, which is equivalent to an error exponent of 0, the error exponent of missed-detection is the convergence limit of \(n\text{LLR}_0(Y^n; \psi^n)\) as \(n\) grows under the assumption that \(\{Y_1, \ldots, Y_n\}\) are drawn from distribution \(f_0\). This is equivalent to the definition of \(\bar{I}_0\) provided in (50). Hence, for the NP test we have \(E_{md}^E = \bar{I}_0\) and \(E_{fa}^E = 0\). For the sequential sampling setting, based on the analysis of average delay in Theorem 5 we have
\[
E_0 \{ \tau \} = \frac{\log P_{md}}{I_0} (1 + o(1))
\]
(179)

and
\[
E_1 \{ \tau \} = \frac{\log P_{fa}}{I_1} (1 + o(1)).
\]
(180)

which immediately gives the error exponents characterized in (64).
G Proof of Theorem 7

We start the proof by finding the minimum number of measurements that any sampling strategy requires to achieve the target accuracy when $n$ grows to infinity. To this end, we show that

$$ E_0(\tau) \geq \frac{|\log \beta_n|}{I_0} \label{eq:G7_1} $$

and

$$ E_1(\tau) \geq \frac{|\log \alpha_n|}{I_1} \label{eq:G7_2} . $$

But these two statements are the similar to (54) and (55) for different values of KL divergences, and the proof follows the same line of argument. We show that for the proposed selection rule we have

$$ E_0(\tau) \leq \frac{|\log \beta_n|}{I_0} (1 + o(1)) \label{eq:G7_3} $$

and

$$ E_1(\tau) \leq \frac{|\log \alpha_n|}{I_1} (1 + o(1)) \label{eq:G7_4} . $$

For this purpose, we define $\hat{\tau}$ as the first time instant after which the ML decision about the hypothesis is always true, i.e., for $t \geq \hat{\tau}$

$$ \begin{cases} 
  \Lambda_t < 0 & \text{if } T = H_0 \\
  \Lambda_t \geq 0 & \text{if } T = H_1 
\end{cases} \label{eq:G7_5} $$

We show that $\hat{\tau}$ is finite, and more precisely, we have

$$ P(\hat{\tau} > t) \leq B_1 e^{-c_1 t} \label{eq:G7_6} . $$

To proceed, we note that

$$ P_1(\delta_{ML}(t) = H_0) = P_1(\Lambda_t < 0) \label{eq:G7_7} . $$

For any $s \leq 0$ we have

$$ E_1(\exp\{s \Lambda_t\} \mathbb{1}_{(\Lambda_t < 0)}) = P_1(\Lambda_t < 0) \cdot E_1(\exp\{s \Lambda_t\} \mathbb{1}_{(\Lambda_t < 0)}) \label{eq:G7_8} $$

$$ \leq E_1(\exp\{s \Lambda_t\}) \label{eq:G7_9} . $$

and since $E_1(\exp\{s \Lambda_t\} | \mathbb{1}_{(\Lambda_t < 0)}) \geq 1$ for any $s \leq 0$, it yields

$$ P_1(\Lambda_t < 0) \leq E_1(\exp\{s \Lambda_t\}) \label{eq:G7_10} . $$

The right hand side of (190) can be rewritten by using the towering property of expectation as

$$ E_1(\exp\{s \Lambda_t\}) = E_1(\exp\{s \Lambda_{t-1}\} E_1\left\{ \left( \frac{f_1(Y_t; \psi(t)|F_{t-1})}{f_0(Y_t; \psi(t)|F_{t-1})} \right)^s \left| Y^{t-1} \right. \right\} ) \label{eq:G7_11} . $$

Now we consider the inner expectation and define

$$ \xi_t(s) \equiv E_1\left\{ \left( \frac{f_1(Y_t; \psi(t)|F_{t-1})}{f_0(Y_t; \psi(t)|F_{t-1})} \right)^s \left| Y^{t-1} \right. \right\} \label{eq:G7_12} , $$

34
which is a convex function of \( s \) and is equal to 1 for \( s = 0, -1 \). There are two possible cases for \( \xi_t(s) \) when \(-1 < s < 0\). In the first case, it is constant and \( \xi_t(s) = 1, \forall s \in (-1, 0) \), which occurs only if the likelihood ratio inside the expectation is equal to 1, i.e., the measurement taken at time \( t \) has the same distribution under both hypotheses. Since the marginal distributions under both hypotheses are identical, this can happen when the selection function jumps to a new subgraph that has not been observed yet, which cannot occur all the time since when it jumps to a new subgraph we expect the selection rule to take at least two samples from the new subgraph due to the structure of average information measures, set \( S \subseteq L_i \), and the selection rule in (37).

In the second case, \( \xi_t(s) < 1, \forall s \in (-1, 0) \). It means that there exists a constant \( \hat{c}_1 > 0 \) such that for some \( 0 < s^* < 1 \) we have

\[
\xi_t(s^*) \leq e^{-\hat{c}_1} < 1. \tag{193}
\]

By successive application of this approach and accounting for case 1 we obtain

\[
\mathbb{P}_1(\Lambda_t < 0) \leq \mathbb{E}_1 \left\{ \exp\left(s^* \Lambda_1\right) \right\} \leq e^{-\bar{c}_1 t}, \tag{194}
\]

for some \( \bar{c}_1 > 0 \) which concludes the proof for (186) under hypothesis \( H_1 \). By following the same line of argument for \( \mathbb{P}_0(\Lambda_t > 0) \) we can show that \( \mathbb{P}_0(\Lambda_t > 0) \leq \exp\{\bar{c}_1 t\} \) for some \( \bar{c}_1 > 0 \). Then, defining \( c_1 \triangleq \min\{\hat{c}_1, \bar{c}_1\} \) concludes the proof for (186).

Next, we prove the asymptotic optimality property under \( H_1 \), given in (184), and the optimality under \( H_0 \) follows the same line of argument. To prove (184), we show that

\[
\mathbb{P}_1(\Lambda_t < -\log \alpha_n) \leq B_2 e^{-c_2 t} \text{ for } t \geq \frac{-\log \alpha_n}{I_1^* (1 + o(1))}. \tag{195}
\]

By denoting the optimal selection rule by \( \psi_{\text{opt}} \) we have

\[
\Lambda_t = \left[ \Lambda_t - \mathbb{E}_1 \left\{ \log \frac{f_1(F_t)}{f_0(F_t)} \right\} \right] \\
+ \left[ \mathbb{E}_1 \left\{ \log \frac{f_1(F_t)}{f_0(F_t)} \right\} - \mathbb{E}_1 \left\{ \log \frac{f_1(Y^t; \psi_{\text{opt}}^t)}{f_0(Y^t; \psi_{\text{opt}}^t)} \right\} \right] \\
+ \mathbb{E}_1 \left\{ \log \frac{f_1(Y^t; \psi_{\text{opt}}^t)}{f_0(Y^t; \psi_{\text{opt}}^t)} \right\}. \tag{196}
\]

We note that the last term in (196) is equal to \( I_1^* t \) for sufficiently large \( t \). For the first bracket in (196) we have

\[
\Lambda_t - \mathbb{E}_1 \left\{ \log \frac{f_1(F_t)}{f_0(F_t)} \right\} \tag{197}
\]

\[
= \Lambda_{t-1} - \mathbb{E}_1 \left\{ \log \frac{f_1(F_{t-1})}{f_0(F_{t-1})} \right\} \tag{198}
\]

\[
+ \log \frac{f_1(Y; \psi(t)|F_{t-1})}{f_0(Y; \psi(t)|F_{t-1})} - \mathbb{E}_1 \left\{ \log \frac{f_1(Y; \psi(t)|F_{t-1})}{f_0(Y; \psi(t)|F_{t-1})} \right\}. \tag{199}
\]
Since $E_1\{\ell_t\} = 0$, for any $\eta_1 > 0$ we have $E_1\{\ell_t + \eta_1\} > 0$. Now consider the moment generating function of $\ell_t + \eta_1$ for any $s < 0$. Since for $s = 0$ the value of the moment generating function is 1 and

$$\frac{d}{ds} E_1\{\exp\{s(\ell_t + \eta_1)\}\} \bigg|_{s=0} = \eta_1 > 0,$$

we can conclude that there exist some $\hat{s} < 0$ and $\hat{c}_2 > 0$ such that

$$E_1\{\exp\{\hat{s}(\ell_t + \eta_1)\}\} \leq e^{-\hat{c}_2} < 1.$$  

Successively applying this technique yields

$$E_1\left\{ \exp \left\{ \hat{s} \left( \Lambda_t - E_1\left\{ \log \frac{f_1(F_t)}{f_0(F_t)} \right\} + \eta_1 t \right) \right\} \right\} \leq e^{-\hat{c}_2 t},$$

Next by using the same line of thought as in (190) we obtain

$$P_1 \left( \Lambda_t - E_1\left\{ \log \frac{f_1(F_t)}{f_0(F_t)} \right\} < -\eta_1 t \right) \leq e^{-\hat{c}_2 t}.$$  

For the second bracket in (196), according to the definition of $\hat{t}$ we have

$$E_1\left\{ \log \frac{f_1(Y_t, \psi(t)|F_{t-1})}{f_0(Y_t, \psi(t)|F_{t-1})} \right\} = E_1\left\{ \log \frac{f_1(Y_t; \psi_{opt}(t)|Y_{t-1}; \psi_{opt}(t))}{f_0(Y_t; \psi_{opt}(t)|Y_{t-1}; \psi_{opt}(t))} \right\} \text{ for } t > \hat{t}.$$  

Therefore, we can conclude that

$$\left| E_1\left\{ \log \frac{f_1(F_t)}{f_0(F_t)} \right\} - E_1\left\{ \log \frac{f_1(Y_t; \psi_{opt}(t))}{f_0(Y_t; \psi_{opt}(t))} \right\} \right| \leq \hat{c} \Delta \text{ for some } \Delta > 0,$$

and since $\hat{t}$ is exponentially bounded according to (186), for some $\hat{c} > 0$ we have

$$P_1 \left( E_1\left\{ \log \frac{f_1(F_t)}{f_0(F_t)} \right\} - E_1\left\{ \log \frac{f_1(Y_t; \psi_{opt})}{f_0(Y_t; \psi_{opt})} \right\} < -\eta_1 t \right) \leq e^{-\hat{c}_2 t} < 1.$$  

Finally, by defining $\eta_3 \triangleq \eta_1 + \eta_2$ and combining (196), (205), and (208) we obtain

$$P_1 \left( \Lambda_t < (I_1^* - \eta_3) t \right) \leq B_2 e^{-\hat{c}_2 t},$$

which concludes the proof.

**H Proof of Theorem 9**

For a GMRF with a line dependency graph, when $\sigma_{ij} = \sigma$ for all the neighbor nodes, the closed-form expression of $\bar{I}_0$ and $\bar{I}_1$ are calculated from the law of large numbers according to

$$\bar{I}_0 = \log(1 - \sigma^2) + \frac{2\sigma^2}{1 - \sigma^2},$$

and

$$\bar{I}_1 = \log \frac{1}{1 - \sigma^2}.$$
By applying this identity for sets $A$ and $B$ separately, and since $I_0$ and $I_1$ are monotonically increasing function of $|\sigma|$ we have

\[ \frac{I_0^A}{I_0^B} \geq \frac{\log(1 - a^2) + \frac{2a^2}{1 - a^2}}{\log(1 - b^2) + \frac{2b^2}{1 - b^2}} \geq -a^2 - \frac{2a^2}{1 - b^2} - \frac{b^2}{3} - \frac{2b^2}{1 - b^2} - \frac{b^2}{3} = a^2 + \frac{2}{3} a^4 + \frac{5}{3} a^6 + o(a^6) \]

\[ = \frac{1}{b^2}(1 + \frac{1}{2} a^2 + \frac{1}{4} a^4 + o(a^4)) \geq a^2 \cdot b^2 \cdot \frac{1}{b^2} . \]

For the expected delays under $H_1$ we have

\[ \frac{I_1^A}{I_1^B} \geq \frac{-\log(1 - a^2)}{-\log(1 - b^2)} \geq a^2 \cdot b^2 \cdot \frac{1}{b^2} . \]

When $|A| = o(n)$ the Chernoff rule starts the sampling process from set $B$ with probability 1 and since the graph is connected stays in set $B$ until it exhaust all its nodes. By invoking the results of Theorem 5, we can conclude that the expected delay of the Chernoff rule under $H_1$ is inversely proportional to $I_1^B$. Furthermore, from Theorem 7 the expected delay of the proposed strategy under $H_1$ is inversely proportional to $I_1^A$, which concludes the proof.

I Proof of Theorem 10

We denote the delay before taking one sample from set $A$ by $\tau_d$ and prove this theorem by showing that both upper bound and lower bound of $\mathbb{E}\{\tau_d\}$ are $O\left(\frac{n}{p}\right)$. For the upper bound we note that the case in which we can take more than one measurement from each node incurs more delay for taking the first measurement from set $A$. By considering this setting, we have

\[ \mathbb{E}\{\tau_d\} \leq \sum_{k=1}^{n-p+1} \frac{k(n-p)^{k-1} p}{n^k} = \frac{p}{n} \sum_{k=1}^{n-p+1} k(1 - \frac{p}{n})^{k-1} \leq O(n/p) . \]
For the lower bound, we have

\[ E\{\tau_d\} = \sum_{k=1}^{n-p+1} k \cdot \mathbb{P}(\tau_d = k) \]  
\[ = \sum_{k=1}^{n-p+1} k \cdot \left(\frac{(n-p)!}{k!(n-k)!}\right)^p \]  
\[ \geq \sum_{k=1}^{\lfloor \frac{n-p}{2} \rfloor} k \cdot \left(\frac{(n-p)!}{k!(n-k)!}\right)^p \]  
\[ \geq \sum_{k=1}^{\lfloor \frac{n-p}{2} \rfloor} k \cdot \left(\frac{(n-p)^{k-1}}{(\frac{n}{2})^k}\right) \]  
\[ \geq \frac{2p}{n+p} \sum_{k=1}^{\lfloor \frac{n-p}{2} \rfloor} k \left(\frac{n-p}{n+p}\right)^{k-1} \]  
\[ = \frac{2p}{n+p} \sum_{k=1}^{\lfloor \frac{n-p}{2} \rfloor} k \left(1 - \frac{2p}{n+p}\right)^{k-1} \]  
\[ = O\left(\frac{n}{p}\right), \]  

where (227) holds since for \( k \leq \frac{n-p}{2} \) we always have \( \frac{n-p}{2} \) nodes outside set \( A \) and \( \frac{n+p}{2} \) nodes from the entire network to take sample from. Hence, that both upper bound and lower bound are \( O\left(\frac{n}{p}\right) \).

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