Wormhole with Quantum Throat

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A wormhole with a quantum throat on the basis of an approximate model of the spacetime foam is presented. An effective spinor field is introduced for the description of the spacetime foam. The consequences of such model of the wormhole is preventing a “naked” singularity in the Reissner-Nordström solution with $|e|/m > 1$.

I. INTRODUCTION

By definition a wormhole (WH) is a bridge connecting two asymptotically flat regions. Usually such construction is a classical object and should satisfy to Einstein equations (see [1] for more detailed introduction to this field). The topology of the 4D WH is $R_1 \times R_2 \times S^2$, where $R_1$ is the time, $R_2$ is the radial coordinate and $S^2$ is the cross section of the WH. A space region near to the minimal cross section $S_{\text{min}}^2$ is called as a throat. The basic problem for existing of the WH is concentrated on the throat: in this region a matter violates the so-called null energy condition. In this paper we offer a model of the WH in which the throat is a set of quantum handles (wormholes) in the presence of the strong electric field. These quantum handles can be considered as quantum WH’s of a spacetime foam with separated mouthes (see Fig.1). We remind that the hypothesized spacetime foam [2], [3] is a set of quantum wormholes (handles) appearing in the spacetime on the Planck scale level ($l_{Pl} \approx 10^{-33}\text{cm}$). For the macroscopic observer these quantum fluctuations are smoothed and we have an ordinary smooth manifold with the metric submitting to Einstein equations.

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FIG. 1. The spacetime foam with separated mouthes of quantum handles. 1 are the quantum (virtual) handles (WHs); 2 are the mouthes of WHs.

The presence of the spacetime foam with virtual wormholes (VWH) which have separated mouthes gives a physical effect: VWH’s can entrap a part of the electric flux lines and in spite of the fact that at the infinity $|e|/m > 1$ we will have a non-singular spacetime ($e$ and $m$ are the charge and mass registered at the infinity).

The mechanism for creating the WH with quantum throat can be offered by the following way:

1. The VWH’s connect two regions with the strong electric field by such a way that the electric flux lines leak through VWH’s from one region to another one [4,5], see Fig. 2.

2. Each VWH is the solutions of the 5D Einstein equations with $G_{5t} \neq 0$.

3. As usually the $G_{5t}$ metric component can be considered as the 4D electric field which joins the electric flux lines of the above-mentioned two 4D regions.

4. For this model the whole spacetime is 5D one and outside of the throat the $G_{55}$ is non-dynamical variable and we have the 5D Kaluza-Klein theory in its initial interpretation ($G_{55} = 1$ and 5D gravity is equivalent to Einstein-Maxwell theory), inside of the throat the $G_{55}$ is dynamical variable and we have the ordinary Kaluza-Klein gravity with 4D metric + electromagnetic and scalar fields. Splitting off the 5D dimension takes place on the event horizon. The matching conditions of the 4D and 5D fields on the event horizon is discussed in Ref. [6]. As this composite WH is the classical solution of the 5D Kaluza-Klein equations a probability for the quantum creation of such VWH is not zero.

II. QUALITATIVE DESCRIPTION OF THE MODEL

The model of the VWH is presented on the Fig. 2.

FIG. 2. The model of the VWH in the presence of an external electric field. 1 is the force line of the external electric field; 2 is the 5D classical throat; 3 is the external spacetimes; 4 are the event horizons (mouthes of the VWH).
On the Fig. (3) are presented the two parts of the WH with the quantum throat.

**FIG. 3.** A is the part of the WH with force lines of the electric field incoming into mouthes of the VWH, respectively B is the part of the WH with force lines of the electric field outcoming from mouthes of the VWH. 1 are the mouthes of VWHs; 2 are the force lines of the electric field. Each mouth has a random velocity.

On the Fig.3A is presented the part of the WH with incoming force lines of the electric field and respectively on the Fig.3B the part with outcoming force lines. For the 4D observer each mouth is like to moving (+/-) electric charge. But this movement is stochastic one and as a consequence we have \(<\text{magnetic field} >= 0\) where \(\langle \rangle\) is the stochastical averaging.

In Ref. [7] is shown that such composite WH has a spin-like structure. The cause of this is following. The metric describing the 5D throat is

\[
ds^2 = \eta_{AB}\omega^A\omega^B = \\
-\frac{r_0^2}{\Delta(r)}(d\chi - \omega(r)dt)^2 + \Delta(r)dt^2 - dr^2 - a(r)\left(d\theta^2 + \sin\theta d\phi^2\right),
\]

\[
a = r_0^2 + r^2, \quad \Delta = \pm \frac{2r_0^2r^2 + r_0^4}{q \ r^2 - r_0^2}, \quad \omega = \pm \frac{4r_0^2}{q \ r^2 - r_0^2}.
\]

where \(\chi\) is the 5th extra coordinate; \(\eta_{AB} = (\pm, -, -, -, \mp), \ A, B = 0, 1, 2, 3, 5; \ r, \theta, \phi\) are the 3D polar coordinates; here \(r_0 > 0\) and \(q\) are some constants. We see that the signs of the \(\eta_{55}\) and \(\eta_{00}\) are not defined. We remark that this 5D metric is located behind the event horizon therefore the 4D observer is not able to determine the signs of the \(\eta_{55}\) and \(\eta_{00}\). Moreover this 5D metric (1) can fluctuate between these two possibilities. Hence the external 4D observer is forced to describe such composite WH by means of a spinor.

This allows us to do the following assumption:

**Assumption 1** The stochastical polarized spacetime foam approximately can be determined by some effective field: a spinor field \(\psi\).

In this case a density \(\epsilon\) of the VWH is

\[
\epsilon = |\psi|^2 = \bar{\psi}\psi
\]

where (\(\bar{\psi}\)) means the transposition and a density of an effective electric charge \(\rho\) (the mouthes of VWH entrapped the electric force lines) is
\[ \rho = e \tilde{\psi} \psi \]  

III. EXACT DESCRIPTION OF THE MODEL

As the hypothesized spacetime foam is a consequence of quantum gravity all our fields (metric \( g_{\mu \nu} \), electromagnetic filed \( A_\mu \) and spinor field \( \psi \)) in this model should be quantized fields. The interaction between these fields is very strong and we can not use the Feynmann diagram technique. For the quantization of this model we will use the Heisenberg quantization method which he applied for the non-linear spinor field [8]. The essence of this method consists in that the classical fields \( f(x^\mu) \) in field equations are exchanged on the field operators \( f(x^\mu) \rightarrow \hat{f}(x^\mu) \). In this case we have differential equations for operators. Certainly it is not clear what is a solution of such differential equations. In fact Heisenberg has shown that differential equations for the operator field is equivalent to some infinite set of differential equations for Green functions (for the small coupling constant this is Dayson-Schwinger equations system).

Following this way we write differential equations for the gravitational + electromagnetic fields in the presence of the spacetime foam (\( \psi \)) as follows

\[
\hat{R}_{\mu \nu} - \frac{1}{2} \hat{g}_{\mu \nu} \hat{R} = \hat{T}_{\mu \nu},
\]

\[
\left( i \gamma^\mu \partial_\mu + e \hat{A}_\mu - \frac{i}{4} \hat{\omega}_{\bar{a}b\mu} \gamma^{[\bar{a}} \gamma^{\bar{b}] \mu} - m \right) \hat{\psi} = 0,
\]

\[
D_\mu \hat{F}^{\mu \nu} = 4\pi e \left( \hat{\psi} \gamma^\mu \hat{\gamma}_\mu \hat{\psi} \right)
\]

where \( \hat{R}_{\mu \nu} \) is the operator of Ricci tensor; \( \hat{T}_{\mu \nu} \) is the operator of the sum of the energy-momentum tensor the spinor and Maxwell fields; \( \hat{g}_{\mu \nu} \) is the operator of the gravitational field; \( \hat{\gamma}^\mu = \hat{h}_a^\mu \gamma^a \) is the operator of the Dirac matrices; \( \hat{\gamma}^{\mu \nu} = (1/2)(\hat{\gamma}^\mu \hat{\gamma}^\nu + \hat{\gamma}^\nu \hat{\gamma}^\mu) \), \( \gamma^a \) means the antisymmetrization; \( \gamma^a (\bar{a} \text{ is the vier-bein index}) \) is the usual \( \gamma \)-matrices

\[
\gamma^\bar{0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}; \quad i = 1, 2, 3
\]

where \( \sigma^i \) are the Pauli matrices; \( \hat{h}_a^\mu \) is the vier-bein operator; Greek indexes are the spacetime indexes; Latin indexes with the bar are vier-bein indexes; \( \hat{F}_{\mu \nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu \) is the operator of the Maxwell tensor of the electromagnetic field; \( \hat{A}_\mu \) is the operator of the potential; \( m \) and \( e \) are some constants. Certainly this equation system is hopelessly complicated, and are impossible to find exact solutions.

We will consider equations for average values \( \langle g_{\mu \nu} \rangle, \langle A_\mu \rangle, \langle \psi \rangle \) and so on. In the first approximation we suppose that

\[
\langle \hat{f}(x) \rangle \approx f(\langle \hat{x} \rangle),
\]

where \( \hat{f} \) can be \( \hat{R}_{\mu \nu} \), \( \hat{\omega}_{\bar{a}b\mu} \) and so on; \( \hat{x} \) can be \( \hat{g}_{\mu \nu} \), \( \hat{\psi}, \hat{A}_\mu \) and so on. In this case we have the classical system of the Einstein-Dirac-Maxwell equations, \textit{i.e.} now the system (5)-(7) is without (\).

4
For our model we use the following ansatz: for the spherically symmetric metric

\[ ds^2 = e^{2\nu(r)} \Delta(r) dt^2 - \frac{dr^2}{\Delta(r)} - r^2 \left( d\theta^2 + \sin^2 d\varphi^2 \right), \]

(10)

for the electromagnetic potential

\[ A_\mu = (-\phi, 0, 0, 0), \]

(11)

for the spinor field

\[ \tilde{\psi} = e^{-i\omega t} \frac{e^{-\nu/2}}{r^{1/4}} \left( f, 0, i g \cos \theta, i g \sin \theta e^{i\varphi} \right). \]

(12)

The following is very important for us: the ansatz (12) for the spinor field \( \psi \) has the \( T_{t\varphi} \) component of the energy-momentum tensor and the \( J_{\varphi} = 4\pi e(\tilde{\psi}\gamma^\varphi\psi) \) component of the current. Let we remind that \( \psi \) determines the stochastic gas of the VWH's which can not have a preferred direction in the spacetime. This means that after substitution expression (10)-(12) into field equations they should be averaged by the spin direction of the ansatz (12). After this averaging we have \( T_{t\varphi} = 0 \) and \( J_{\varphi} = 0 \) and we have the following equations system describing our spherically symmetric spacetime

\[ f'\sqrt{\Delta} = \frac{f}{r} - g \left( (\omega - e\phi) \frac{e^{-\nu}}{\sqrt{\Delta}} + m \right), \]

(13)

\[ g'\sqrt{\Delta} = f \left( (\omega - e\phi) \frac{e^{-\nu}}{\sqrt{\Delta}} - m \right) - \frac{g}{r}, \]

(14)

\[ r\Delta' = 1 - \Delta - \kappa \frac{e^{-2\nu}}{\Delta} (\omega - e\phi) \left( f^2 + g^2 \right) - r^2 e^{-2\nu} \phi'^2, \]

(15)

\[ r\Delta\nu' = \kappa \frac{e^{-2\nu}}{\Delta} (\omega - e\phi) \left( f^2 + g^2 \right) - \kappa \frac{e^{-\nu}}{\sqrt{\Delta}} fg - \frac{\kappa}{2} m \frac{e^{-\nu}}{\sqrt{\Delta}} \left( f^2 - g^2 \right), \]

(16)

\[ r^2 \Delta\phi'' = -8\pi e \left( f^2 + g^2 \right) - (2r\Delta - r^2 \Delta\nu') \phi', \]

(17)

where \( \kappa \) is some constant. This equations system was investigated in [9] and result is the following. A particle-like solution exists which has the following expansions near \( r = 0 \)

\[ f(r) = f_1 r + O(r^2), \quad g(r) = O(r^2), \]

(18)

\[ \Delta(r) = 1 + O(r^2), \quad \nu(r) = O(r^2), \quad \phi(r) = O(r^2) \]

(19)

and the following asymptotical behaviour

\[ \Delta(r) \approx 1 - \frac{2m_\infty}{r} + \frac{(2e_\infty)^2}{r^2}, \quad \nu(r) \approx \text{const}, \]

(20)

\[ \phi(r) \approx \frac{2e_\infty}{r}, \]

(21)

\[ f \approx f_0 e^{-\alpha r}, \quad g \approx g_0 e^{-\alpha r}, \quad \frac{f_0}{g_0} = \sqrt{\frac{m_\infty + \omega}{m_\infty - \omega}}, \quad \alpha^2 = m_\infty^2 - \omega^2. \]

(22)

1 another words the averaging \( \langle \rangle \) in the expression (9) is not only quantum but stochastical, too.
where \( m_\infty \) is the mass for the observer at infinity and \( 2e_\infty \) is the charge of this solution. Our interpretation of this solution is presented on the Fig.(4).

\[ e/m > 1 \quad e/m > 1 \quad e/m < 1 \quad e/m < 1 \]

**FIG. 4.** 1 are the quantum (virtual) WHs, 2 are two solutions with \(|e_\infty|/m_\infty|>1\). Such object can be named as the wormhole with quantum throat.

The solution exists for \((|e_\infty|/m_\infty|>1\) and \((|e_\infty|/m_\infty|<1\) but for us is essential the first case with \((|e_\infty|/m_\infty|>1\). In this case the classical Einstein-Maxwell theory leads to the “naked” singularity. The presence of the spacetime foam drastically changes this result: the appearance of the VWH’s can prevent the formation of the “naked” singularity in the Reissner-Nordström solution with \(|e|/m > 1\).

**IV. CONCLUSIONS**

Thus our model is based on the following assumptions:

- in basic the quantum (virtual) WHs of the spacetime foam have the 5D throat.
- each quantum WH has a spin-like structure,
- the spacetime foam effectively can be described with the help of a spinor field.

As the consequence we have the result that the strong electric field separates the VWH’s of the spacetime foam by such a way that they can prevent a singularity in the Reissner-Nordström solution (with \(|e|/m > 1\) on account of the formation of VWHs.

In the spirit of the Einstein idea that the right-hand of gravitational equations should be zero we can note that this model of the WH with quantum throat is the vacuum model since the gauge fields can be considered as components of the metric in some multidimensional Kaluza-Klein gravity.

\[ ^2 \text{it depends on the mass} \ m \]
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