Silver block intersection graphs of Steiner 2-designs

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Abstract

For a block design $D$, a series of block intersection graphs $G_i$, or $i$-BIG($D$), $i = 0, \ldots, k$ is defined in which the vertices are the blocks of $D$, with two vertices adjacent if and only if the corresponding blocks intersect in exactly $i$ elements. A silver graph $G$ is defined with respect to a maximum independent set of $G$, called an $\alpha$-set. Let $G$ be an $r$-regular graph and $c$ be a proper $(r + 1)$-coloring of $G$. A vertex $x$ in $G$ is said to be rainbow with respect to $c$ if every color appears in the closed neighborhood $N[x] = N(x) \cup \{x\}$. Given an $\alpha$-set $I$ of $G$, a coloring $c$ is said to be silver with respect to $I$ if every $x \in I$ is rainbow with respect to $c$. We say $G$ is silver if it admits a silver coloring with respect to some $I$. Finding silver graphs is of interest, for a motivation and progress in silver graphs see [7] and [15]. We investigate conditions for 0-BIG($D$) and 1-BIG($D$) of Steiner 2-designs $D = S(2, k, v)$ to be silver.

keywords: Silver coloring, Block intersection graph, Steiner 2-design, and Steiner triple system

Subject class: 05C15, 05B05, 05B07, and 05C69

1 Introduction and preliminaries

We follow standard notations and concepts from design theory. For these, one may refer to, for example, [5] and [14].
A 2-\((v,k,\lambda)\) design \((2 < k < v)\) is a pair \((V,B)\) where \(V\) is a \(v\)-set and \(B\) is a collection of \(b\) \(k\)-subsets of \(V\) (blocks) such that any \(2\)-subset of \(V\) is contained in exactly \(\lambda\) blocks. A 2-\((v,k,1)\) design is called a Steiner 2-design and is denoted by \(S(2,k,v)\). An \(S(2,3,v)\) is a Steiner triple system or STS\((v)\). A design with \(b = v\) is a symmetric \((v,k,\lambda)\)-design. A symmetric \(S(2,k,v)\) is called a projective plane. If \(k\) is the size of the blocks then \(n := k - 1\) is called the order of the plane. This design is usually denoted by \(PG(2, n)\). A 2-\((n^2,n,1)\) design is called an affine plane. For such design we use the notation \(AG(2,n)\).

A partial parallel class is a set of blocks that contains no element of the design more than once. A parallel class (PC) or a resolution class in a design is a set of blocks that partition the set of elements \(V\). A near parallel class is a partial parallel class missing a single element. A resolvable balanced incomplete block design is a 2-\((v,k,\lambda)\) design whose blocks can be partitioned into parallel classes. The notation RBIBD\((v,k,\lambda)\) is commonly used. An affine plane of order \(n\) is an RBIBD\((n^2,n,1)\). A resolvable STS\((v)\) together with a resolution of its blocks is called a Kirkman triple system, KTS\((v)\).

Given a design \(D\), a series of block intersection graphs \(G_i\), or \(i\)-BIG, \(i = 0, \ldots, k\) can be defined in which the vertices are the blocks of \(D\), with two vertices are adjacent if and only if the corresponding blocks intersect in exactly \(i\) elements.

**Example 1** For STS\((7)\), 0-BIG is empty graph and 1-BIG is \(K_7\). For STS\((9)\), 0-BIG is disconnected and consists of four disjoint \(K_3\)’s and 1-BIG is \(K_{3,3,3,3}\).

The study of \(i\)-BIG\((D)\) is useful in characterizing block designs. Some researchers have studied properties of various kinds of block intersection graphs, see for example [1], [2], [4], [8], [9], [10], [16], and [17].

A graph of order \(v\) is strongly regular, denoted by \(SRG(v,k,\lambda,\mu)\), whenever it is not complete or edgeless and, (i) each vertex is adjacent to \(k\) vertices, (ii) for each pair of adjacent vertices there are \(\lambda\) vertices adjacent to both, (iii) for each pair of non-adjacent vertices there are \(\mu\) vertices adjacent to both.

**Remark 1** Let \(G_i\) be the \(i\)-block intersection graph of an \(S(2,k,v)\). Then for each \(i = 2,3, \ldots, k\), the graph \(G_i\) is empty. So we consider only \(G_0\) and \(G_1\). Graphs \(G_0\) and \(G_1\) are complements of each other. \(G_1\) is an \(SRG(b,k(r-1),r-2+(k-1)^2,k^2)\) and \(G_0\) is an \(SRG(b-b-k(r-1)-1,b-2k(r-1)+k^2-2,b-2kr+k^2+r-1)\) (see Chapter 21 of [13]).

In a graph \(G = (V,E)\) an independent set is a subset of vertices no two of which are adjacent. The independence number \(\alpha(G)\) is the cardinality of a largest set of independent vertices. We refer to any maximum independent set of a graph as an \(\alpha\)-set. Let \(c\) be a proper \((r+1)\)-coloring of an \(r\)-regular graph \(G\). A vertex \(x\) in \(G\) is said to be rainbow with respect to \(c\) if
every color appears in the closed neighborhood \( N[x] = N(x) \cup \{x\} \). Given an \( \alpha \)-set \( I \) of \( G \), the coloring \( c \) is said to be silver with respect to \( I \) if every \( x \in I \) is rainbow with respect to \( c \). We say \( G \) is silver if it admits a silver coloring with respect to some \( \alpha \)-set. If all vertices of \( G \) are rainbow, then \( c \) is called a totally silver coloring of \( G \) and \( G \) is said to be totally silver. Note that the definition of silver coloring depends on the chosen \( \alpha \)-set. For example in Figure 1 a graph \( G \) is shown which is silver when the \( \alpha \)-set (the bold vertices) is taken as in the left, but it does not have any silver coloring with the \( \alpha \)-set taken as on the right hand side.

![Figure 1: A silver coloring of a graph](image)

There are many different version of rainbow colorings in the literature, for example see \([3],[11],[12],[13]\). For a motivation and progress in silver graphs see \([7],[15]\). In fact silver graphs are closely related to a concept in graph coloring, called defining set. Let \( c \) be a proper \( k \)-coloring of a graph \( G \) and let \( S \subseteq V(G) \). If \( c \) is the only extension of \( c|_S \) to a proper \( k \)-coloring of \( G \), then \( S \) is called a defining set of \( c \). The minimum size of a defining set among all \( k \)-colorings of \( G \) is called a defining number and denoted by \( \text{def}(G,k) \). A more general survey of defining sets in combinatorics appears in \([6]\). Let \( G \) be an \( r \)-regular graph, then \( G \) is silver if and only if \( \text{def}(G,r+1) = |V(G)| - \alpha(G) \). In \([15]\) an open problem is raised:

**Question 1** Find classes of \( r \)-regular graphs \( G \), for which \( \text{def}(G,r+1) = |V(G)| - \alpha(G) \), i.e. determine classes of all silver graphs.

A silver cube is a silver graph \( G = K^d_n \), the Cartesian power of the complete graph \( K_n \). Silver cubes are generalizations of silver matrices, which are \( n \times n \) matrices where each symbol in \( \{1, 2, \ldots, 2n-1\} \) appears in either the \( i \)-th row or the \( i \)-th column of the matrix. In \([7]\) some algebraic constructions and a product construction of silver cubes are given. They show the relation of these cubes to codes over finite fields, dominating sets of a graph, Latin squares, and finite geometry. In particular the Hamming codes are used to produce a totally silver cube and the bound for the best binary codes is used to prove the non-existence of silver cubes for a large class of parameters with \( n = 2 \).

To study Question 1 here we consider \( i \)-BIGs of designs. First we give some examples of designs with silver \( i \)-BIGs.
Example 2 In any symmetric \((v, k, \lambda)\)-design \(\mathcal{D}\), every two distinct blocks have exactly \(\lambda\) elements in common, so for \(0 \leq i \leq k\), \(i \neq \lambda\), \(i\)-BIG(\(\mathcal{D}\)) is empty graph, and \(\lambda\)-BIG(\(\mathcal{D}\)) is complete graph. Hence all of these graphs are totally silver. Specifically for each \(k\) and \(0 \leq i \leq k + 1\), \(i\)-BIG(S(2, \(k + 1\), \(k^2 + k + 1\))) is totally silver.

If \(\mathcal{D}\) is an AG(2, \(n\)), then \(G_0 = 0\)-BIG(\(\mathcal{D}\)) consists of \((n + 1)\) disjoint \(K_n\)’s, so it is totally silver, and \(G_1 = 1\)-BIG(\(\mathcal{D}\)) is complete graph. Hence all of these graphs are totally silver. Specifically for each \(k\) and \(0 \leq i \leq k + 1\), \(i\)-BIG(S(2, \(k + 1\), \(k^2 + k + 1\))) is totally silver.

In this paper we prove the following results: If an \(S(2, k, v)\) contains a parallel class, then a necessary condition for 1-BIG(S(2, \(k, v\))) to be silver is \(k^2 \mid v\). For each admissible \(v = 9m\) we construct a \(\mathcal{D}_1 = \text{KTS}(v)\), such that 1-BIG(\(\mathcal{D}_1\)) is silver. And in general for each \(k\) and \(v\) where an AG(2, \(k\)) and an RBIBD(v, 1) exist we construct a \(\mathcal{D}^\ast = \text{RBIBD}(kv, k, 1)\) such that 1-BIG(\(\mathcal{D}^\ast\)) is silver. Also a lower bound for \(\alpha(G_1)\) is given in order for 1-BIG(S(2, \(k, v\))) to be silver. For any admissible \(v\), the existence of a silver 1-BIG(S(2, \(k, v\))) which possesses a maximum possible independent set, i.e. of size \(\frac{v}{k}\) or \(\frac{v-1}{k}\), is settled. We prove that for \(v > k^3 - 2k^2 + 2k\) there is no silver 0-BIG(S(2, \(k, v\))). Also we settle the question of existence of silver 0-BIG(STS(\(v\))) for all admissible \(v\).

Since every vertex of \(i\)-BIG(\(\mathcal{D}\)) corresponds to a block of \(\mathcal{D}\), we will mostly refer to them as “blocks” rather than vertices. The following notation will be used in our discussion. Let \(G\) be a graph and \(I\) be an \(\alpha\)-set of \(G\). For each \(i = 1, \ldots, |I|\), we let

\[X_i := \{u | u \in V(G) \setminus I, \ u \text{ is adjacent to exactly } i \text{ vertices of } I\}\].

2 One block intersection graphs

The following is a necessary condition for 1-BIG(\(\mathcal{D}\)) of a Steiner system \(\mathcal{D} = S(2, k, v)\) with \(\alpha(G_1) = \frac{v}{k}\), to be silver.

Theorem 1 Let \(\mathcal{D}\) be an \(S(2, k, v)\), which has a parallel class, and let \(G_1\) be 1-BIG(\(\mathcal{D}\)). A necessary condition for \(G_1\) to be silver is \(k^2 \mid v\).

Proof. \(G_1\) is a \(\frac{k(v-k)}{(k-1)}\)-regular graph. Let \(I\) be an \(\alpha\)-set, and assume that \(G_1\) has a silver coloring with respect to \(I\) with \(C\) as the set of colors. We have \(|I| = \frac{v}{k}\), and \(|C| = \frac{k(v-k)}{k-1} + 1\). Since \(|C| > |I|\), a color like \(\iota\) exists that is not used in \(I\). The vertices of \(I\) are rainbow, and each vertex with color \(\iota\) from \(V(G_1) \setminus I\), must be adjacent to \(k\) distinct vertices of \(I\). Therefore \(|I|\) must be a multiple of \(k\), which implies \(k^2 \mid v\). \(\blacksquare\)
Example 3 There are 80 nonisomorphic STS(15)s, where 70 of them have parallel class (see [5], page 32). So by Theorem [1], none of those 70 has silver $G_1$.

By Theorem [1] if $v$ is not a multiple of 9, then no silver 1-BIG(KTS($v$)) exists. In the next lemma we show that for the case $9 \mid v$, when a KTS($v$) exists, i.e. $v = 18q + 9$, there exists a silver 1-BIG(KTS($v$)). This lemma is an illustration of a general structure which will be discussed in Theorem [2]

Lemma 1 If $v \equiv 3 \pmod{6}$, then a $K = KTS(3v)$ exists such that 1-BIG($K$) is silver.

Proof. Let $A = AG(2, 3) = STS(9)$ with $V(A) = \{(i, j) \mid 1 \leq i, j \leq 3\}$, and denote its parallel classes by:

- $\Theta_0$
  \begin{align*}
    \{(1, 1), (2, 1), (3, 1)\} \\
    \{(1, 2), (2, 2), (3, 2)\} \\
    \{(1, 3), (2, 3), (3, 3)\}
  \end{align*}

- $\Theta_1$
  \begin{align*}
    a_1 = \{(1, 1), (1, 2), (1, 3)\} \\
    a_2 = \{(2, 1), (2, 2), (2, 3)\} \\
    a_3 = \{(3, 1), (3, 2), (3, 3)\}
  \end{align*}

- $\Theta_2$
  \begin{align*}
    a_4 = \{(1, 1), (2, 2), (3, 3)\} \\
    a_5 = \{(1, 3), (2, 1), (3, 2)\} \\
    a_6 = \{(1, 2), (2, 3), (3, 1)\}
  \end{align*}

- $\Theta_3$
  \begin{align*}
    a_7 = \{(1, 1), (2, 3), (3, 2)\} \\
    a_8 = \{(1, 2), (2, 1), (3, 3)\} \\
    a_9 = \{(1, 3), (2, 2), (3, 1)\}
  \end{align*}

Consider a KTS($v$) $D = (V, B)$, $V = \{x_1, x_2, \ldots, x_v\}$ with parallel classes $\pi_1, \pi_2, \ldots, \pi_{\frac{v-1}{2}}$. Using its blocks we construct $K = (V^*, B^*)$, a KTS(3v) in the following manner.

The set of elements of $K$ is $V^* = \{1, 2, 3\} \times V$, and the blocks are introduced in the following 4 types of parallel classes, $\Omega_{0, \beta}$, $\Omega_{1, \beta}$, $\Omega_{2, \beta}$ and $\Omega_{3, \beta}$.

- $\Omega_{0, \beta} : \\{(1, x_i), (2, x_i), (3, x_i)\} \mid 1 \leq i \leq v\}.$

We denote every block of $D$ by $\{x_i, x_j, x_k\}$, where $i < j < k$. In the following a label $(m, \beta)$ for each block is its color, the block with label $(m, \beta)$ is obtained by using the block $a_m$ of $A$.

- $\Omega_{1, \beta} : \\{(1, x_i), (1, x_j), (1, x_k)\}^{(1, \beta)}$, \{(2, x_i), (2, x_j), (2, x_k)\}^{(2, \beta)}$, \{(3, x_i), (3, x_j), (3, x_k)\}^{(3, \beta)} | \\{x_i, x_j, x_k\} \in \pi_\beta\)$, for $1 \leq \beta \leq \frac{v-1}{2}$,

- $\Omega_{2, \beta} : \\{(1, x_i), (2, x_j), (3, x_k)\}^{(4, \beta)}$, \{(1, x_k), (2, x_i), (3, x_j)\}^{(5, \beta)}$, \{(1, x_j), (2, x_k), (3, x_i)\}^{(6, \beta)}) | \\{x_i, x_j, x_k\} \in \pi_\beta\)$, for $1 \leq \beta \leq \frac{v-1}{2}$. 

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\[ \cdot \Omega_{3,\beta} : \{ (1, x_i), (2, x_k), (3, x_j) \}_{(7, \beta)}, \{ (1, x_j), (2, x_i), (3, x_k) \}_{(8, \beta)}, \{ (1, x_k), (2, x_j), (3, x_i) \}_{(9, \beta)}, \{ x_i, x_j, x_k \} \in \pi_{\beta} \}, \text{ for } 1 \leq \beta \leq \frac{v - 1}{2}. \]

Figures 2 and 3 demonstrate the 4 types of blocks.

We note that there is only one parallel class in \( \Omega_{0,\beta} \), but there are \( \frac{v - 1}{2} \) parallel classes in each of other types, so we have \( 3 \frac{v - 1}{2} \) parallel classes and each class has \( v \) blocks.

Clearly, \( K \) is a KTS(3\( v \)). The number of colors needed in a silver coloring of 1-BIG(\( K \)) is equal to \( \frac{9v - 7}{2} \). We color 0 the vertices corresponding to the blocks in \( \Omega_{0,\beta} \) class. The label of each block in other classes, which is shown as its index, is the color of its corresponding vertex in 1-BIG(\( K \)): \( (m, \beta), 1 \leq m \leq 9, 1 \leq \beta \leq \frac{v - 1}{2} \). It is easy to check that this is a proper coloring and all vertices in \( \Omega_{0,\beta} \) class, i.e. the \( \alpha \)-set, are rainbow.

Next theorem is a generalization of the construction introduced in Lemma 1.

**Theorem 2** Assume there exist an affine plane \( A = AG(2, k) \), and a resolvable balanced incomplete block design \( D = RBIBD(v, k, 1) \). Then there exists a \( D^* = RBIBD(kv, k, 1) \) where 1-BIG(\( D^* \)) is silver.

**Proof.** Let \( V(A) = \{(i, j) \mid 1 \leq i, j \leq k\} \) and denote its parallel classes by \( \Theta_0, \Theta_1, \ldots, \Theta_k \). Specifically we let

\[ \Theta_0 = \{(1, j), (2, j), \ldots, (k, j)\} \mid j = 1, 2, \ldots, k \}. \]
Also we let $V(D) = \{x_1, x_2, \ldots, x_v\}$ with parallel classes $\pi_1, \pi_2, \ldots, \pi_{v-1}$.

For each block $b = \{x_{s_1}, x_{s_2}, \ldots, x_{s_k}\}$ of $D$ we consider an ordering on $b$ such that

$$x_{s_i} < x_{s_j} \iff s_i < s_j,$$

and define a function:

$$\Psi_b : V(A) \rightarrow \{1, 2, \ldots, k\} \times \{x_{s_1}, x_{s_2}, \ldots, x_{s_k}\}$$

$$\Psi_b(i, j) = (i, x_{s_j}).$$

We extend $\Psi_b$ for each block $a$ of $A$ as $\Psi_b(a) = \{\Psi_b(i, j) \mid (i, j) \in a\}$.

Now we construct a design $D^* = (V^*, B^*)$, as in the following:

$V^* = \{1, 2, \ldots, k\} \times V(D)$.

$B^* = \{\Psi_b(a) \mid b \text{ and } a \text{ are blocks of } D \text{ and } A, \text{ respectively}\}$.

See Figure 4.

$D^*$ is an RBIBD with the following parallel classes:

$$\Omega_{\alpha, \beta} = \{\Psi_b(a) \mid a \in \Theta_\alpha, b \in \pi_\beta\}, \text{ for each } 0 \leq \alpha \leq k \text{ and } 1 \leq \beta \leq \frac{v-1}{k-1}.$$

Note that:

$$\Omega_{0,1} = \Omega_{0,2} = \cdots = \Omega_{0, \frac{v-1}{k-1}} = \big\{(1, x_s), (2, x_s), \ldots, (k, x_s)\big\} \mid s = 1, 2, \ldots, v\big\}.$$

We show that $1$-BIG($D^*$) is silver with respect to the $\alpha$-set

$$I^* = \{\Psi_b(a) \mid a \in \Theta_0 \text{ and } b \text{ is a block of } D\}$$

$$= \big\{(1, x_s), (2, x_s), \ldots, (k, x_s)\big\} \mid s = 1, 2, \ldots, v\big\}.$$
by the following coloring:

\[
c : \mathcal{B}^* \rightarrow \{0\} \cup \{(a, \beta) \mid a \text{ is a block of } \mathcal{A} \setminus \Theta_0 \text{ and } 1 \leq \beta \leq \frac{v-1}{k-1}\}
\]

\[
\Psi_b(a) \mapsto \begin{cases} 
0 & \text{if } a \in \Theta_0, \\
(a, \beta) & \text{if } a \notin \Theta_0 \text{, and } b \in \pi_\beta.
\end{cases}
\]

We show that \(c\) is a proper coloring and any vertex \(b^* \in I^*\) is rainbow. Note that all the vertices of \(I^*\) have color 0. Let \(\Psi_{b_1}(a_1)\) and \(\Psi_{b_2}(a_2)\) be two blocks of \(\mathcal{D}^*\) with the same color \((a, \beta)\). Then we have \(b_1, b_2 \in \pi_\beta\). Therefore \(b_1\) and \(b_2\) are disjoint blocks of \(\mathcal{D}\), so \(\Psi_{b_1}(a_1)\) and \(\Psi_{b_2}(a_2)\) are disjoint. Thus \(c\) is proper.

To show silverness, for a fixed \(s\) let \(b^*_s = \{(1, x_s), (2, x_s), \ldots, (k, x_s)\}\) be a block of \(I^*\). By definition, for any given nonzero color like \((a, \beta)\) we have \(a \notin \Theta_0\), and there exists a unique block \(b\) of \(\pi_\beta\) which contains \(x_s\) and the color of \(\Psi_b(a)\) is \((a, \beta)\). Since in \(\mathcal{A}\), the block \(a\) intersects each block of \(\Theta_0\), thus by definition of \(\mathcal{B}^*\), \(\Psi_b(a)\) intersects \(b^*_s\) in \(\mathcal{D}^*\), so the color \((a, \beta)\) appears in the neighborhood of \(b^*_s\).

In the next theorem for any \(\mathcal{D} = S(2, k, v)\), we show a lower bound for \(\alpha(G_1)\), in order \(G_1 = 1-\text{BIG}(\mathcal{D})\) to be silver.

**Theorem 3** Let \(\mathcal{D}\) be an \(S(2, k, v)\), and \(G_1 = 1-\text{BIG}(\mathcal{D})\). If \(\alpha(G_1) > k\lfloor \frac{v(u-1)}{k^2v-2k^3+k^2-k} \rfloor\), then \(G_1\) is not silver.

**Proof.** \(G_1\) is a \(\frac{k(u-k)}{(k-1)}\)-regular graph with \(\frac{v(u-1)}{k(k-1)}\) vertices. Let \(I\) be an \(\alpha\)-set, and assume that \(G_1\) has a silver coloring with respect to \(I\) with \(C\) as the set of colors, \(|C| = \frac{k(u-k)}{k-1} + 1\).

A color like \(\iota\) exists that is used in the coloring of at most \(\left\lfloor \frac{|V(G_1)|}{|C|} \right\rfloor = \left\lfloor \frac{v(u-1)}{k^2v-2k^3+k^2-k} \right\rfloor\) vertices of \(G_1\). For a set \(X \subseteq V(G_1)\) we denote the set of vertices with color \(\iota\) in \(X\) by \(X(\iota)\). By counting the number of appearances of color \(\iota\) in \(I\) and in the neighborhood of \(I\) we obtain,

\[
\alpha(G_1) = |I(\iota)| + |X_1(\iota)| + 2|X_2(\iota)| + \cdots + k|X_k(\iota)| \\
\leq k(|I(\iota)| + |X_1(\iota)| + |X_2(\iota)| + \cdots + |X_k(\iota)|) \\
\leq k\lfloor \frac{v(u-1)}{k^2v-2k^3+k^2-k} \rfloor \\
< \alpha(G_1).
\]

A contradiction.

**Example 4** It is easy to check that for any of two \(\text{STS}(13)s\), \(\alpha(G_1) = 4\). For 80 nonisomorphic \(\text{STS}(15)s\), we have \(\alpha(G_1) = 4\) or 5 (see [5], page 32). Also there are 18 nonisomorphic \(S(2, 4, 25)\) (see [5], page 34), by a computer search they have \(\alpha(G_1) = 5\) or 6. So by Theorem 3 none of them has a silver \(G_1\).
Remark 2 Let $G_1$ be the 1-block intersection graph of an $S(2, k, v)$ with a parallel class. Then $\alpha(G_1) = \frac{v}{k}$, and all the elements of $V$ appear in the blocks corresponding to each $\alpha$-set. Let $I$ be an $\alpha$-set for $G_1$, therefore any vertex of $V(G_1) \setminus I$ is adjacent to $k$ vertices of $I$. Thus $|X_1| = |X_2| = \cdots = |X_{k-1}| = 0$, $|X_k| = \frac{v(v-k)}{k(k-1)}$.

If an $S(2, k, v)$ has a near parallel class, then $\alpha(G_1) = \frac{v-1}{k}$, and each $\alpha$-set contains all the elements of $V$ except one. Hence in this case any vertex of $V(G_1) \setminus I$ is adjacent to either $(k-1)$ or $k$ vertices of $I$, and $|X_1| = |X_2| = \cdots = |X_{k-2}| = 0$; $|X_{k-1}| = \frac{v-1}{k-1}$, $|X_k| = \frac{(v-1)(v-2k+1)}{k(k-1)}$.

Theorem 4 Let $\mathcal{D}$ be an $S(2, k, v)$, with a near parallel class. Then $G_1 = 1$-BIG($\mathcal{D}$) is not silver.

Proof. Let $I$ be an $\alpha$-set for $G_1$. Assume that $G_1$ has a silver coloring with respect to $I$ and $C$ is the set of colors. $G_1$ is $\frac{k(v-k)}{k-1}$-regular, $|C| = \frac{k(v-k)}{k-1} + 1$ and $|I| = \frac{v-1}{k}$. By Remark 2 $|X_{k-1}| = \frac{v-1}{k-1}$ and $|X_k| = \frac{(v-1)(v-2k+1)}{k(k-1)}$. Since $|C| > |I \cup X_{k-1}|$, a color like $\iota$ exists that is used only in the coloring of vertices of $X_k$. The vertices of $I$ are rainbow, so each of the vertices of $X_k$ that have color $\iota$, must be adjacent to $k$ different vertices of $I$. Thus $|I| \geq k$ a multiple of $k$, say $|I| = mk$.

Since $|X_{k-1}| = \frac{v-1}{k-1} > |I|$, a color like $\iota'$ exists that is used in the coloring of vertices of $X_{k-1}$ but is not used in $I$. The induced subgraph on $X_{k-1}$ is a clique, so $\iota'$ appears only in one vertex of $X_{k-1}$ and it has $(k-1)$ neighbors in $I$. Thus $|I| - k + 1$ vertices of $I$, each must have a neighbor in $X_k$ with color $\iota'$. Again vertices from $X_k$ that have color $\iota'$, each must be adjacent to $k$ different vertices of $I$. Therefore $|I| - k + 1 = (m-1)k + 1$ is also a multiple of $k$. This is impossible.

Example 5 The 1-block intersection graph of any Hanani triple system (see [5], page 67 for the definition) is not silver.

Note that by Theorems 1, 2, 5 and 4 for any admissible $v$ the problem of existence of a silver 1-BIG($S(2, k, v)$) which possesses maximum possible independent set is settled.

3 Zero block intersection graphs

In this section we discuss 0-block intersection graphs of $S(2, k, v)$. 

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Notation 1 Let $x$ be a given element of $S(2, k, v)$, and denote by $T(x)$ the set of \( \frac{v-1}{k-1} \) blocks containing $x$.

It is trivial that $T(x)$ is an independent set for $G_0$, thus $\alpha(G_0) \geq \frac{v-1}{k-1}$.

Lemma 2 Let $\mathcal{D}$ be an $S(2, k, v)$, and $G_0 = 0$-BIG($\mathcal{D}$). If $v > k^3 - 2k^2 + 2k$ then any maximum independent set of $G_0$ is of the form $T(x)$, therefore $\alpha(G_0) = \frac{v-1}{k-1}$.

Proof. Let $I$ be an $\alpha$-set of $G_0$. Suppose $I$ is not of the form $T(x)$. There exists an element $x_0$ of $\mathcal{D}$ which appears in at least two blocks of $I$. Let $I_1 = \{B_1, B_2, \ldots, B_p\} = \{B \mid B \in I \cap T(x_0)\}$, and $I \setminus I_1 = \{B_{p+1}, B_{p+2}, \ldots, B_{p+q}\}$. Since $\lambda = 1$, for $1 \leq i < j \leq p$, $(B_i \setminus \{x_0\}) \cap (B_j \setminus \{x_0\}) = \emptyset$. Every two blocks in $I$ have one intersection. So, for each block $B \in I \setminus I_1$ we have $B \cap B_i = \{a_i\}$, $i = 1, 2, \ldots, p$. So $p \leq |B| = k$.

Now suppose $B_1, B_2 \in I_1$. There exist exactly $(k-1)^2$ pairs $\{x, y\}$ where $x \in B_1 \setminus \{x_0\}$ and $y \in B_2 \setminus \{x_0\}$, and each of these pairs appears at most in one of the blocks of $I \setminus I_1$. Thus $q \leq (k-1)^2$.

So $|I| = p + q \leq k + (k-1)^2$. But since $v > k^3 - 2k^2 + 2k$, for each $x$ we have $|T(x)| = \frac{v-1}{k-1} > k + (k-1)^2 \geq |I|$. Hence the statement follows.

Theorem 5 Let $\mathcal{D}$ be an $S(2, k, v)$. For $v > k^3 - 2k^2 + 2k$, $G_0 = 0$-BIG($\mathcal{D}$) is not silver.

Proof. $G_0$ is a $\frac{v^2+k^3-v(k^2+1)-k^2+k}{k(k-1)}$-regular graph (Remark 1). Let $I$ be any $\alpha$-set for $G_0$. By Lemma 2 $I = T(x)$ and $|I| = \alpha(G_0) = \frac{v-1}{k-1}$. Since each block out of $I$ intersects exactly $k$ blocks of $I$, each vertex of $V(G_0) \setminus I$ is adjacent to $\frac{v-1}{k-1} - k = \frac{v-1-k^2+k}{k(k-1)}$ vertices of $I$. Then $V(G_0) = I \cup X_{\frac{v-1}{k-1}-k^2+k}$ and $|X_{\frac{v-1}{k-1}-k^2+k}| = \frac{(v-1)(v-k)}{k(k-1)}$.

To the contrary, $G_0$ has a silver coloring with respect to $I$. Let $C$ be the set of colors, $|C| = \frac{v^2-k^2+3}{k(k-1)}$. Since $|C| > \frac{1}{k-1}$, a color like $\iota$ exists that is not used in the coloring of $I$. The vertices of $I$ are rainbow, and the vertices from $X_{\frac{v-1}{k-1}-k^2+k}$ that have color $\iota$, each must be adjacent to $\frac{v-1-k^2+k}{k(k-1)}$ different vertices of $I$. Therefore $|I|$ must be divisible by $\frac{v-1-k^2+k}{k-1}$, then $(v-k^2+k-1) | (v-1)$ which is impossible, since $v > k^3 - 2k^2 + 2k$. Therefore graph $G_0$ is not silver with respect to any $\alpha$-set.

3.1 0-BIG for Steiner triple systems

Both 0-BIG($\text{STS}(v)$) for $v = 7$ and $v = 9$, by Example 2, are totally silver.

Theorem 6 For any admissible $v > 9$, $G_0 = 0$-BIG($\text{STS}(v)$) is not silver.
Proof. For \( v > 15 \), it follows by Theorem 5.

If \( v \leq 15 \), then suppose \( I \) is an \( \alpha \)-set of \( G_0 \), and \( I \) is not of the form \( T(x) \). Then it is easy to check that, each element of \( \text{STS}(v) \) appears at most in 3 blocks of \( I \). If it has 3 blocks containing an element \( x \), then such a set has at most 7 blocks, and they are contained in \( I_1 \), where:

\[
I_1 = \{ \{x, a, b\}, \{x, c, d\}, \{x, e, f\}, \{a, c, f\}, \{a, d, e\}, \{b, c, e\}, \{b, d, f\} \} \approx \text{STS}(7).
\]

Now we discuss possible cases.

\( v = 15 \):

For \( v = 15 \) an \( \alpha \)-set, \( I \), may be of the form \( T(x) \) or it may come from a subsystem \( \text{STS}(7) \), in either case \( \alpha(G_0) = 7 \). From 80 non-isomorphic \( \text{STS}(15) \)s, 23 of them have a subsystem \( \text{STS}(7) \) (5, page 32). It is straightforward to check that in all of \( \text{STS}(15) \)s for any \( \alpha \)-set \( I \), each block out of \( I \) has intersection with exactly three blocks of \( I \). So each vertex in \( V(G_0) \setminus I \) is adjacent to exactly four vertices of \( I \). In any silver coloring with \( C \) as the set of colors of \( G_0 \), we have \( |C| = 17 > 7 = |I| \). So there exists a color \( \iota \) which is not used in \( I \). Every vertex with the color \( \iota \) has exactly 4 neighbors in \( I \), therefore 7 must be a multiple of 4. So \( G_0 \) does not have a silver coloring.

\( v = 13 \):

For \( v = 13 \) there are two non-isomorphic \( \text{STS}(13) \)s. No \( \text{STS}(13) \) has a subsystem of \( \text{STS}(7) \), even no \( \text{STS}(13) \) has 6 blocks of an \( \text{STS}(7) \). So, in \( G_0 \) for both of them, the sets of the form \( T(x) \), are the only \( \alpha \)-sets and \( \alpha(G_0) = 6 \). Suppose \( I \) is any \( \alpha \)-set.

First, we show that it is always possible to find three vertices in \( I \) with no common neighbor:

- One of two \( \text{STS}(13) \)s, Type 1, has a cyclic automorphism, and we can construct its blocks on \( \{1, 2, \ldots, 13\} \) by the following base blocks:

\[
\{1, 2, 5\}, \quad \{1, 3, 8\} \mod 13.
\]

If \( I = T(1) \), then \( B_1 = \{1, 2, 5\}, B_2 = \{1, 3, 8\} \), and \( B_3 = \{1, 10, 11\} \) do not have common neighbor. Let \( x \neq 1 \) be a given element of \( \text{STS}(v) \), and \( I = T(x) \). Three vertices of \( I \), \( B'_1, B'_2, B'_3 \) are obtained by adding \( (x - 1) \) to all members of blocks \( B_1, B_2, B_3 \), do not have common neighbor.

- The other \( \text{STS}(13) \) is non-cyclic and we can construct its blocks from Type 1 by
replacing four blocks of trade $T_1$ with four blocks of trade $T_2$ as follows:

\[
\begin{array}{ccc}
  1 & 2 & 5 \\
  1 & 3 & 8 \\
 10 & 2 & 8 \\
10 & 3 & 5 \\
\end{array}
\]

\[
\begin{array}{ccc}
  1 & 2 & 8 \\
  1 & 3 & 5 \\
 10 & 2 & 5 \\
10 & 3 & 8 \\
\end{array}
\]

Let $I = T(x)$ for some $x$. If $x$ is an element of $T_2$, i.e. $x \in \{1, 2, 3, 5, 8, 10\}$, then there are two blocks say $B_1$ and $B_2$ of $T_2$ which contain $x$. There exists one element $y$, such that $y \in T_2$ but $y \notin B_1 \cup B_2$. We consider $B_3$, the block containing $x$ and $y$. Then these three blocks do not have common neighbor. If $x$ is not in $T_2$, then we consider several cases for $I = T(x)$, and show that there exist three vertices of $I$, which do not have common neighbor.

Now, assume for some STS(13), $G_0 = 0$-BIG(STS(13)) is silver with respect to some $\alpha$-set $I = T(x) = \{B_1, B_2, B_3, B_4, B_5, B_6\}$. The color of all neighbors of $B_i$, $i = 1, \ldots, 6$, must be distinct. Assume $\{B_1, B_2, B_3\} \subset I$ do not have common neighbor. Let $N(B_i)$ be the set of neighbors of $B_i$. $G_0 = SRG(26, 10, 3, 4)$, so $|N(B_1) \cap N(B_2)| + |N(B_2) \cap N(B_3)| + |N(B_1) \cap N(B_3)| = 12$. Thus the color of these vertices must be distinct, while we have only 11 colors. Therefore $G_0$ does not have a silver coloring.

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