Abstract

This is the first paper of a series of our works on the self-similar orbit-averaged Fokker-Planck (OAFP) equation and shows its accurate pre-collapse solution. At the late stage of relaxation evolution of dense star clusters, standard stellar dynamics predicts that the clusters may evolve in a self-similar fashion forming a collapsing core. However, the corresponding mathematical model, the self-similar OAFP equation for distribution function of stars in isotropic star clusters, has never been solved on the whole energy domain ($-1 < E < 0$). The existing works based on kinds of finite difference methods provide solutions only on the truncated domain $-1 < E < -0.2$. To broaden the range of the truncated domain, the present work resorts to a (highly accurate- and efficient-) Gauss-Chebyshev pseudo-spectral method. We provide a Chebyshev spectral solution, whose number of significant figures is four, on the whole domain. Also, The solution can reduce to a semi-analytical form whose degrees of polynomials are only eighteen holding three significant figures. We also provide the new eigenvalues; $c_1 = 9.0925 \times 10^{-4}$, $c_2 = 1.1118 \times 10^{-3}$, $c_3 = 7.1975 \times 10^{-2}$ and $c_4 = 3.303 \times 10^{-2}$, corresponding to the core collapse rate $\xi = 3.64 \times 10^{-1}$, scaled escape energy $\chi_{esc} = 13.881$ and power-law exponent $\alpha = 2.2305$. Since the solution on the whole domain is unstable against change in degree of Chebyshev polynomials, we show spectral solutions on truncated domains ($-1 < E < E_{\text{max}}$, where $-0.35 < E_{\text{max}} < -0.03$) to explain how to handle the instability. By reformulating the OAFP equation in several ways, we improve the accuracy of the spectral solution and reproduce an existing self-similar solution; we consider existing solutions have only one significant figure at most.

Keywords: dense star cluster; core collapse; self-similar evolution; orbit-averaged Fokker-Planck model; isotropic; numerical; pseudo-spectral method; Gauss- Chebyshev polynomial

1. Introduction

The present paper is the first paper for a series of our works on the solution of self-similar orbit-averaged Fokker-Planck (ss-OAFP) equation and shows an accurate Gauss-Chebyshev spectral solution for pre-collapse stage. In the second and third papers, we discuss the physical properties of the ss-OAFP model focusing on the negative specific heat of the core and application to the observed structural profiles of globular clusters with resolved cores in Milky Way.

The relaxation evolution of core-collapsing dense star clusters (e.g. globular clusters) can not result in a state of thermal equilibrium of stars due to the ‘negative’ specific heat; as relaxation processes mostly in the core cause stars and kinetic energy to flow from the core to the halo, the core heats up and halo cools down. Once the core density reaches so high that the cluster undergoes the gravothermal instability (Antonov, 1985), it begins to show a self-similar density profile in the core and inner halo (Lynden-Bell and Eggleton, 1980). Without existence of primordial binary stars or formation and growth of binary stars, the most probable distribution of stars, still in sense...
of increment of Boltzmann entropy for distribution function (DF) of stars, is a core-collapsed (infinite-density) profile that can be achieved during a finite time duration at the late stage of the self-similar evolution (Hénon, 1961; Cohn, 1980). While the core-collapsing self-similar profile of stars is just a mathematical idealization, it has been one’s concern (e.g. Baumgardt et al., 2003; Szell et al., 2005; Pavlic and Subi, 2018) since it may provide, in addition to conceptual understandings of the late stage of relaxation evolution, the (asymptotic value of) physical parameters to characterize the evolution; the core collapse rate $\dot{\epsilon}$, the power-law index $\alpha$ in spatial density profile and scaled escape energy $\chi_{esc}$ in energy distribution function (Heggie and Stevenson, 1983). In the rest of the present section, we explain OAFP equation (Section 1.1), its self-similar form (Section 1.2) and numerical difficulty in integration of the ss-OAFP equation (Section 1.3).

1.1. Orbit-averaged Fokker-Planck (OAFP) equation

The ideal model of a dense star cluster would be a collection of $N$ equal-mass stars that is isotropic in velocity space and spherical in configuration space; the model can provide a good qualitative understandings of relaxation evolution (Cohn, 1980; Takahashi, 1995).

Due to the nature of the long-range interacting stars, the DF of stars may be considered to reach a state of quasi-stationary equilibrium (Virial-equilibrium) through the rapid fluctuation in m.f. potential (’violent relaxation’, Jeans, 1902). Due to the nature of the long-range interacting stars, the DF of stars may be considered to reach a state of quasi-stationary equilibrium (Virial-equilibrium) through the rapid fluctuation in m.f. potential (’violent relaxation’, phase- or chaotic- mixing, ... ). Hence, by assuming that the m.f. potential is regular, the strong Jeans theorem (e.g. Binney and Tremaine, 2011) may allow one to simplify the phase-space probability DF at time $t$ as $f(r, v, t) \approx f(\epsilon)$ for the isotropic system in which the energy of star per unit mass is as follows $\epsilon = \psi(r, t) + \frac{v^2}{2}$ where $r$ and $v$ are stellar position and velocity.

This collisionless dynamical-evolution scenario breaks down on relaxation time scales ($t_{rel} \sim N t_{dyn}/ \ln N$) due to the effect of finiteness of total number $N$ of stars; the ‘smooth’ orbits of stars are gradually changed due to (stochastic) irregular forces via many-body Newtonian interaction and the system could reach various quasi-stationary states. In this sense, the explicit time-dependence of DF may be retrieved ($f(r, v, t) \approx f(\epsilon, t)$) and the m.f. potential is to be determined by Poisson equation

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r} \frac{\partial \psi}{\partial r} = \rho \left[ \psi(r, t) \right] \equiv 16\pi^2 G m \int_{\psi(r, t)}^{0} f(\epsilon, t) \sqrt{2\epsilon - 2\psi(r, t)} d\epsilon.' \tag{1.1}$$

Stellar dynamists have conventionally modeled the effect of many-body interaction, in first-order approximation of $1/N$ as (a cumulative) weak two-body encounter with a homogeneous background approximation. The corresponding time-evolution model of DF is a (nonlinear) Fokker-Planck equation averaged over the radial period between the apocenter and pericenter of the orbits, which is known as the orbit-averaged Fokker-Planck (OAFP) equation (e.g. Hénon, 1961; Spitzer, 1988)

$$\frac{\partial f(\epsilon, t)}{\partial t} + \frac{\partial q(\epsilon, t)}{\partial \epsilon} + \frac{\partial q(\epsilon, t)}{\partial t} \frac{\partial f(\epsilon, t)}{\partial \epsilon} = \Gamma \frac{\partial}{\partial \epsilon} \left\{ f(\epsilon, t) \left[ f(\epsilon, t)q(\epsilon, t) - j(\epsilon, t) \right] + \frac{\partial f(\epsilon, t)}{\partial \epsilon} \left[ i(\epsilon, t) + q(\epsilon, t)g(\epsilon, t) \right] \right\}, \tag{1.2a}$$

$$\Gamma \equiv (4\pi Gm)^2 \ln N \tag{1.2b}$$

where $G$ is the gravitational constant, and $m$ the stellar mass. The $q$-integral (the integral associated with the radial action) reads

$$q(\epsilon, t) = \frac{1}{3} \int_{0}^{\chi_{esc}(\epsilon, t)} \left[ 2\epsilon - 2\psi(r', t) \right]^{3/2} r^2 dr', \tag{1.3}$$

1More realistic star clusters must be modeled as anisotropic systems in velocity space based on statistical and dynamical principles (Polyachenko and Shukhman, 1982; Luciani and Pellat, 1987), numerical results (Cohn, 1979; Takahashi, 1995; Giersz and Spurzem, 1994; Baumgardt et al, 2002) and observation (Meylan, 1987; Meylan and Heggie, 1997).

2See Gilbert, 1968 [19, 2018] for more statistically-exact treatment of $1/N$-expansion in N-body Liouville equation which includes the effects of the inhomogeneity in encounter, gravitational polarization, statistical acceleration and/or strong encounter.
where $r_{\text{max}}(\epsilon, t) = \psi^{-1}(\epsilon)$. The integrals associated with dynamical friction and energy diffusion read

\[ i(\epsilon, t) \equiv \int_{-\infty}^{\epsilon} f(\epsilon', t) q(\epsilon', t) \, d\epsilon', \quad (1.4a) \]
\[ j(\epsilon, t) \equiv \int_{-\infty}^{\epsilon} \frac{\partial f(\epsilon', t)}{\partial \epsilon'} q(\epsilon', t) \, d\epsilon', \quad (1.4b) \]
\[ g(\epsilon, t) \equiv \int_{\epsilon}^{0} f(\epsilon', t) \, d\epsilon'. \quad (1.4c) \]

where $\psi(0) = -1$ is assumed.

1.2. Self-similar OAFP equation

The OAFP system (i.e. the system of OAFP equation (1.2a) and Poisson equation (1.1)) predicts that at the early stage of relaxation evolution the DF of stars may be characterized by a lowered-Maxwellian while at the late stage the cluster may undergo a self-similar evolution (Cohn, 1980). To reflect the self-similar evolution of a core-collapsing isotropic cluster, the following self-similar variables are employed in equations (1.2a) and (1.1) for independent variables concerned

\[ E = \epsilon/E_c(t), \quad (1.5a) \]
\[ R = r/r_c(t), \quad (1.5b) \]

and for dependent variables concerned

\[ F(E) = f(\epsilon, t)/f_c(t), \quad (1.6a) \]
\[ Q(E) = q(\epsilon, t)/q_c(t), \quad (1.6b) \]
\[ \Psi(R) = \psi(r, t)/\psi_c(t), \quad (1.6c) \]
\[ I(E) = i(\epsilon, t)/i_c(t), \quad (1.6d) \]
\[ J(E) = j(\epsilon, t)/j_c(t), \quad (1.6e) \]
\[ G(E) = g(\epsilon, t)/g_c(t). \quad (1.6f) \]

where suffix $c$ means that the variables depend only on time $t$. Following [Heggie and Stevenson, 1988], one can obtain the ss-OAFP (self-similar Orbit-Averaged Fokker-Planck) system; a system of four ordinary differential equations (4ODEs)

\[ [I(E) + G(E)Q(E)] \frac{dF}{dE} = c_1 Q(E) F(E) + \frac{2c_1 - 3c_2}{4} J(E) - F(E) [F(E)Q(E) - J(E)], \quad (1.7a) \]
\[ \frac{dG}{dE} = -F(E), \quad (1.7b) \]
\[ \frac{dI}{dE} = Q(E) F(E), \quad (1.7c) \]
\[ \frac{dJ}{dE} = Q(E) \frac{dF}{dE}. \quad (1.7d) \]

the $Q$-integral

\[ Q(E) = \frac{1}{3} \int_{0}^{R_{\text{max}}(E)} [2E - 2\Psi(R')]^{3/2} R^2 \, dR', \quad (R_{\text{max}}(E) = \psi^{-1}(E)), \quad (1.8) \]

and Poisson equation

\[ \frac{\partial^2 \Psi}{\partial R^2} + \frac{2}{R} \frac{\partial \Psi}{\partial R} = D |\Psi(R)| \equiv \int_{\psi(R)}^{0} F(E') \sqrt{2E' - 2\Psi(R')} \, dE'. \quad (1.9) \]
The self-similar parameters read
\[
\begin{align*}
c_1 &= \frac{1}{\Gamma(f(t))} \frac{d}{dt} f(t), \\
c_2 &= \frac{1}{\Gamma(f(t))} \frac{d}{dt} E(t),
\end{align*}
\]
and the corresponding physical parameters concerned are
\[
\begin{align*}
\alpha &= \frac{2(3 + 2\beta)}{2\beta + 1}, \\
\xi &= \frac{c_1 + c_2}{0.167 \sqrt{\alpha}}, \\
\chi_{\text{esc}} &= \frac{F_{\text{BC}}(F_{\text{BC}} - c_1)}{c_3}.
\end{align*}
\]

where \(c_3(\equiv G(E = -1))\) is the third eigenvalue and the value \(F_{\text{BC}}\) is a boundary value to be assigned.\(^3\) The new eigenvalue \(\beta(\equiv c_1/c_2)\) characterizes the power-law profile of stars in the halo for each of dependent variables in the following boundary conditions for the 4ODEs (1.12a)-(1.12d)
\[
\begin{align*}
F(E \to 0) &= c_4(\beta + 1)(-E)^\beta, & F(E = -1) &= F_{\text{BC}}, \\
G(E \to 0) &= c_3(-E)^{\beta+1}, & G(E = -1) &= c_1, \\
I(E \to 0) &= c_4 \frac{4(\beta + 1)}{2\beta - 7} (-E)^{\beta+1} Q(E \to 0), & I(E = -1) &= 0, \\
J(E \to 0) &= -c_4 \frac{4(\beta + 1)}{2\beta - 3} (-E)^\beta Q(E \to 0), & J(E = -1) &= 0,
\end{align*}
\]

where \(\sigma = -3(2\beta - 1)/4\) and \(c_4\) is the fourth eigenvalue. The boundary condition for Poisson equation is
\[
\frac{d\Psi(R = 0)}{dR} = 0, \quad \Psi(R = 0) = -1.
\]

1.3. Numerical problems in integration of ss-OAFP equation and spectral methods

Mathematical execution of the ss-OAFP system, solving a system of equations (1.7)-(1.9) for the set of dependent variables \(\{F, G, I, J, \Phi\}\) and four eigenvalues \(\{c_1(\equiv \beta), c_2, c_3, c_4\}\) based on the boundary conditions (1.12)-(1.13), is supposed to be a simple task compared to more exact models (e.g. time-dependent OAFP model and N-body direct simulations), but it was studied only in a few works (Heggie and Stevenson 1988; Takahashi and Inagaki 1992; Takahashi 1993) in which clear difficulties in numerical integration of the ss-OAFP system were reported. Although Heggie and Stevenson (1988); Takahashi (1993) found their self-similar solutions, their works are not complete due to the following reasons. First, the domains of their solutions are truncated in energy space, which means the solutions may depend on the extrapolation of power-law profile; they did not discuss the relationship between their solutions and a solution obtained on the whole domain. Second, the the value of scaled escape energy \(\chi_{\text{esc}}\) in Heggie and Stevenson 1988 is reported to be \(\chi_{\text{esc}} = 13.85\) while this value is not compatible with a result of Cohm 1980 in which, at the same epoch of the energy (13.85), the central density reaches only \(10^{20}\) times higher than the initial density; if the value 13.85 is correct, the Cohm 1980’s time-evolution model is supposed to reach an infinite density; one has yet to discuss which of their works is a more accurate result. Third, Takahashi and Inagaki 1992; Takahashi 1993 tried to reproduce the result of Heggie and Stevenson 1988 based on a variational principle though it was not a plentiful work specifies the value of \(F_{\text{BC}}\) when it is necessary.

\(^3\)Although the boundary condition for the DF in Heggie and Stevenson 1988 Takahashi and Inagaki 1992 was set to \(F(-1) = 1\), the present work specifies the value of \(F_{\text{BC}}\) when it is necessary.
result; as reported in both works, the Newton iteration method did not well converge unless the initial guess for solution is very close to the ‘true’ solution.

In the present work, we employ a Gauss-Chebyshev pseudo-spectral method to overcome the numerical difficulties associated with the ss-OAFP model and to obtain a solution on the whole domain. Spectral methods are very accurate and efficient compared to finite difference (deferred correction) methods, also they can provide a closed form of solution different from finite element methods. Especially, Chebyshev spectral method has the advantages over other spectral methods in the sense that the explicit expression of Chebyshev nodes, numerical differentiation and integrals are known and that its numerical stability and efficiency have been extensively studied (e.g. Boyd, 2001). The ss-OAFP system is associated with infinite-domain problems through Poisson equation; the infinite domain problems have been a matter of concern in applied-mathematics and computational-physics community as an end-point singularity problem last decades, especially which was discussed for Lane-Emden equations and the variants in astrophysical context (e.g. Parand and Shahini, 2010; Caruntu and Bota, 2013; Ito et al., 2018). The present work also aims at extending the numerical scheme developed in (Ito et al., 2018) to the ss-OAFP system.

The present paper is organized as follows. Section 2 explains the transformation of functions and change of variables for the ss-OAFP system. Section 3 explains Chebyshev spectral method. Sections 4 and 5 show the results obtained on whole- and truncated-domains respectively. Section 6 reproduces the Heggie-Stevenson solution. Section 7 makes a conclusion.

2. Mathematical formulation

The domains of 4ODEs (1.7a) - (1.7d) and $Q$-integral (equation (1.2)) are a finite ($E \in [-1,0)$) while the domain of Poisson equation (1.9) is semi-infinite ($R \in [0,\infty)$). To employ the Chebyshev spectral method throughout the present work, in Section 2.1, we convert the domain of the latter to the same domain as the former employing an inverse function of the m.f. potential $\Psi(R)$, following the inverse-function method employed in (Ito et al., 2018). Also, since all the dependent variables have power-law profiles forming large-scale gaps between terms in the 4ODEs and Poisson equation, we regularize the variables by the factor $(-E)\beta$, $DF(E)$ and/or the integral $Q(E)$ in Section 2.2. Also since the truncation of the domain is essential following (Heggie and Stevenson, 1988; Takahashi, 1993), Section 2.3 provides the explicit expression of the $Q$- and $D$- integrals on the whole- and truncated-domains.

2.1. Inverse form of Poisson equation

Using the inverse mapping $R = \Psi$ through the local theorem

$$\frac{d\Psi}{dR} = \frac{1}{\frac{d\Psi}{dE}}, \quad \frac{d^2\Psi}{dR^2} = -\frac{d^2R}{d\Psi^2} \left( \frac{1}{\frac{d\Psi}{dE}} \right)^3,$$

we reduced Poisson equation (1.9) to

$$R(\Psi)\frac{d^2R}{d\Psi^2} = -2 \left( \frac{dR}{d\Psi} \right)^2 + R(\Psi) \left( \frac{dR}{d\Psi} \right)^3 D(\Psi) = 0.$$  \hspace{1cm} (2.2)

The asymptotic approximation of the inverse form of Poisson equation (2.2) near $\Psi = -1$ (corresponding to the boundary condition (1.13) at $R = 0$) reads

$$R(\Psi \to -1) = (1 + \Psi)^{1/2}.$$  \hspace{1cm} (2.3)

Also, the asymptotic approximation of the dependent variable $R$ near $\Psi = 0$ is

$$R(\Psi \to 0) \propto (-\Psi)^\nu, \quad \nu = \frac{-2\beta + 1}{4}.$$  \hspace{1cm} (2.4)
2.2. Regularization of ss-OAFP system

We introduced the following independent variables \( x \) and \( y \) to employ Chebyshev polynomial (which is defined on \((-1, 1)\) to be explained in Section 3):

\[
x \equiv 2(-E)^{1/L} - 1, \quad y \equiv 2(-\Psi)^{1/L} - 1,
\]

where \( L \) is a numerical parameter introduced to deal with a certain kind of end-point singularities of Chebyshev polynomials [Ito et al., 2018]. Making use of the known asymptotic approximation of dependent variables (i.e. equations (1.12), (2.3) and (2.4)), we regularized the dependent variables as follows

\[
\begin{align*}
v_R(y) &\equiv \ln \left[ R(y) \left( \frac{1-y}{2} \right)^{-1/2} \left( \frac{1+y}{2} \right)^{-L_r} \right], \\
v_S(y) &\equiv \ln \left[ -S(y) \left( \frac{1-y}{2} \right)^{1/2} \left( \frac{1+y}{2} \right)^{-L_r} \right], \\
v_G(x) &\equiv Q(x) \left( \frac{1+x}{2} \right)^{1/3}, \\
v_F(x) &\equiv \ln \left[ F(x) \left( \frac{1+x}{2} \right)^{\beta L_r} \right], \\
v_G(x) &\equiv \frac{G(x)}{F(x)}, \\
v_I(x) &\equiv \frac{I(x)}{F(x)Q(x)}, \\
v_J(x) &\equiv \frac{(2\beta - 3)J(x)}{4\beta F(x)Q(x)},
\end{align*}
\]

where the following new dependent variable was introduced for convenience

\[
S(y) \equiv 2 \frac{dR}{dy}.
\]

The regularized variables provide more straightforward boundary conditions to understand the relation between the conditions and eigenvalues, compared to the original ones (equations (1.12a)-1.12d));

\[
\begin{align*}
v_F(x \to -1) &= \ln c_4^*, & v_F(x = 1) &= \ln F_{BC}, \\
v_I(x \to -1) &= 0, & v_I(x = 1) &= 0, \\
v_G(x \to -1) &= 0, & v_G(x = 1) &= c_3, \\
v_J(x \to -1) &= -1, & v_J(x = 1) &= 0. \\
\end{align*}
\]

where \( c_4^* \) is a newly-introduced eigenvalue for convenience and the relation of the eigenvalue \( c_4^* \) with \( c_4 \) in Heggie and Stevenson, 1988’s work is

\[
c_4^* \equiv c_4(\beta + 1).
\]

Since all the 4ODEs (1.12a)-1.12d) are first order in differentiation, the eigenvalues (end-point values at \( x = -1 \) \( c_4^* \) and \( c_3 \)) would be directly associated with the boundary conditions at the opposite ends \( v_F(x = 1) \) and \( v_G(x \to -1) = 0 \) while the eigenvalues \( c_1 \) and \( c_2 \) would be determined by the boundary conditions for \( v_I(x) \) and \( v_J(x) \).

The inverse form of Poisson equation (2.2) reduces to a system of the following two ODEs

\[
\begin{align*}
2(1 - y)(1 + y) \frac{dR}{dy} - 2yL(1 - y) = 0, \\
2(1 - y)(1 + y) \frac{dS}{dy} - 4e^{\sigma y}(-c_6y^2(1 - y) - 4L^2e^{\psi y} + 2L(1 - y)(1 + y) + 4e^{\sigma y}(-c_6y^2 - 4L^2e^{\psi y}) = 0,
\end{align*}
\]

(2.10a, b)
where the regularized density \( v_D(y) \) is
\[
v_D(y) \equiv D(y) \left( \frac{1 + y}{2} \right)^{-(\beta + 3/2)L} = \frac{L}{2} \left( \frac{1 + y}{2} \right)^{L(\beta + 3/2)} \int_{x}^{y} A_L(y, x') e^{\beta x'} (x') \sqrt{\frac{1 + x'}{2}} \, dx',
\]
where the factor \( A_L(x, x') \) is
\[
A_L(x, x') = \sqrt{\frac{1 + x}{2}} - \sqrt{\frac{1 + x'}{2}} \left( \frac{x - x'}{2} \right)^{-1/2}.
\]
We did not need to employ any boundary conditions for Poisson equations (2.10a)-(2.10b) since the equations are completely regularized at each end point of the domains of \( v_D(x) \) and \( v_E(x) \); in other words; the equations themselves include their boundary conditions, which appears after the limits of \( x \to \pm \) are taken at equation level.

We regularized the integral \( Q(x) \) (equation (1.2)) as follows
\[
[Q(x)]^3 = \frac{L}{6 \sqrt{\pi}} \left( \frac{1 + x}{2} \right)^{\nu L} \int_{x}^{1} A_L(y, x) e^{\beta (y - x)} \sqrt{\frac{1 + y}{2}} \, dy'.
\]

4ODEs (1.7a) - (1.7d) reduce to
\[
\frac{2}{L} \left( \frac{1 + x}{2} \right)^{\nu L} \frac{d v_D}{dx} [v_f(x) + v_E(x)] + \left( \frac{1 + x}{2} \right)^{\nu L} \beta \left[ v_f(x) + v_E(x) \right] + \left( \frac{1 + x}{2} \right)^{2 L} (4 \nu \sigma \nu L) = 0,
\]
\[
\frac{1 + x}{L} \frac{d v_E(x)}{dx} + v_E(x) \left( \frac{1 + x}{2} \right)^{\nu L} \beta \left[ v_f(x) \right] = 0,
\]
\[
\frac{1 + x}{L} \frac{d v_f(x)}{dx} = \left( \frac{1 + x}{2} \right)^{\nu L} \beta \left[ v_f(x) \right] = 0.
\]

2.3. The integral formulations on the whole- and truncated- domains

When we solved the ss-OAFP system on the whole-domains \( x, y \in (-1, 1) \), we numerically integrated the following integrals \( v_D(y) \) (equation (2.11)) and \( Q(x) \) (equation (2.13))
\[
v_D(y) \equiv \frac{L}{2 \sqrt{\pi}} \left( \frac{1 + y}{2} \right)^{\nu L} \int_{x}^{1} A_L \left[ y, \left( \frac{1 + y}{2} \right)^{\nu L} \beta \left( 1 + x \right), 1 \right] e^{\beta (x - y) / (1 + x)} \sqrt{\frac{1 + x}{2}} \, dx',
\]
\[
[Q(x)]^3 = \frac{L}{24} \left( \frac{1 + x}{2} \right)^{3 \nu L} \int_{x}^{1} A_L \left[ 1 - \left( \frac{1 - y}{2} \right)^{\nu L} \beta \left( 1 + x \right), 1 \right] e^{3 \nu (1 - y) / (1 + x)} \sqrt{\frac{1 + x}{2}} \, dx'.
\]

On one hand, given that we solved the ss-OAFP system on the truncated domains \( x, y \in (x_{min}, 1) \) where \( -1 < x_{min} < 1 \), we introduced the following new independent variables
\[
z \equiv -2 \frac{1 - x}{1 - x_{min}} + 1, \quad w \equiv -2 \frac{1 - y}{1 - x_{min}} + 1.
\]

We had to differently treat the integrals \( v_D(z) \) and \( v_D(w) \) given that we truncated the domain; the latter requires \( v_f(x) \) to be extrapolated on \( -1 < x < x_{min} \). The contribution of the function \( v_f(x) \) defined on the domain \( x_{min} < x < 1 \) to the integral \( v_D(w) \) is as follows
\[
v_{D_{\text{main}}}(w) \equiv \frac{1}{2 \sqrt{2}} \left( \frac{1 - x_{min}}{2} \right)^{3/2} \left[ 1 + \frac{2 + x_{min}(1 - w)}{2(1 + w)} \right]^{3/2} \int_{z}^{1} e^{3 \nu (1 - y - z) / (1 + x)} \sqrt{\frac{1 + x}{2}} \, dz',
\]
where $L = 1$ is assumed for simplicity. On the domain $-1 < x < x_{\text{min}}$, we extrapolated the function $v_F(x)$ as follows

\[
v_F^{(\text{ex})}(x) = \begin{cases} 
\ln\left(c_4 + \frac{2c_3}{x_{\text{min}} - x_{\text{min}}} \frac{d}{dx}\ln(1 + x_{\text{min}})\right) (x - x_{\text{min}}), & \text{(smooth at } x = x_{\text{min}}), \\
\ln[c_4^+], & \text{(non-smooth at } x = x_{\text{min}} \text{ or } c \to \infty),
\end{cases}
\]

(2.18)

where $c$ and $d$ are numerical parameters. The contribution of the function $v_F^{(\text{ex})}(x)$ to the integral $v_D(w)$ is as follows

\[
v_D^{(\text{ex})}(w) = \frac{1}{2} \int_{-1}^{1} \left( \frac{1 + x_{\text{min}}}{1 + y} \right)^{\beta/3} e^{i\beta y} \left( \frac{1 + x_{\text{min}}}{z} \right)^{\beta} \sqrt{1 + y} \left( 1 + x_{\text{min}} \right) \left( 1 - y \right)^{\beta/2} \left( \frac{1 + z''}{2} \right)^{\beta} dz''.
\]

(2.19)

where $y = \frac{1}{2} \left[ 1 + x_{\text{min}} + z(1 - x_{\text{min}}) \right]$. Hence, the integral $v_D(w)$ is composed of the total of contributions from the extrapolated DF $v_F^{(\text{ex})}(x)$ on $-1 < x < x_{\text{min}}$ and DF $v_F(x)$ on $x_{\text{min}} < x < 1$ as follows

\[
v_D(w) = v_D^{(\text{min})}(w) + v_D^{(\text{ex})}(w).
\]

(2.20)

We employed the following explicit formula to numerically integrate the integral $v_Q(z)$

\[
[v_Q(z)]^3 = \frac{1}{24} \left( \frac{1 - x}{2} \right)^{3/2} \left( \frac{1 + x}{2} \right)^{3/2} \int_{-1}^{1} e^{3y} \left( \frac{1 + z''(1 + w'')}{2} \right)^{3} \left( 1 + \frac{(1 - z)(1 + w')(1 - x_{\text{min}})}{4(1 + x)} \right)^{3y} dw' .
\]

(2.21)

where $L = 1$ is assumed and $x = \frac{1}{2} \left[ 1 + x_{\text{min}} + z(1 - x_{\text{min}}) \right]$.

3. Gauss-Chebyshev spectral method and numerical treatments of ss-OAFP system

Sections 3.1 and 3.2 explain Gauss-Chebyshev pseudo-spectral method and numerical treatment of the ss-OAFP system respectively.

3.1. The Gauss-Chebyshev pseudo-spectral method

Chebyshev polynomials of the first kind are defined on domain $x \in [-1, 1]$ as (e.g. Boyd, 2001; Mason and Handscomb, 2002)

\[
T_n(x) = \cos\left( n \cos^{-1}(x) \right), \quad (n = 0, 1, 2, \cdots, N)
\]

(3.1)

Due to the singularities in the 4ODES (2.14a)-(2.14b) and Poisson equation (2.10) at the endpoints $x = \pm 1$, we must solve the equations as an open-interval problem $x \in (-1, 1)$. Hence, the discretized domain of the polynomials at the Gauss-Chebyshev points is

\[
x_k = \cos(t_k) \equiv \cos\left( \frac{2k - 1}{2N} \pi \right), \quad (k = 1, 2, 3 \cdots N)
\]

(3.2)

The discrete Gauss-Chebyshev polynomials $T_n(x_k)$ of the first kind satisfy the orthogonality condition (e.g. Mason and Handscomb, 2002)

\[
\sum_{j=1}^{N} T_n(x_j) T_m(x_j) = \begin{cases} 
0, & (1 < n \neq m < N) \\
N, & (n = m = 0) \\
\frac{N}{2}, & (0 < n = j \leq N)
\end{cases}
\]

(3.3)

\text{The Poisson equation is regular singular at both of end points of } x \text{ and } D^- \text{ and } Q^-\text{integral also have a singular property as } x \to -1. \text{ In this sense, to handle the singularities, in the present work we employ Gauss-Chebyshev nodes by considering the domain to be an open interval (e.g. Bhrawy and Alofi, 2013; Boyd, 2013).}
Hence, the discrete Gauss-Chebyshev polynomial expansion of any function \( h(x) \) and its derivative are

\[
h(x_j) = \sum_{n=1}^{N} a_n T_{n-1}(x_j), \quad \frac{dh(x_j)}{dx} = \sum_{n=1}^{N} a_n n \sin(n \cos^{-1} x_j) \sin\left(\cos^{-1} x_j\right),
\]

and the Chebyshev-Gauss expansion can be inverted to

\[
a_1 = \frac{1}{n} \sum_{j=1}^{n} T_0(x_j) h(x_j), \quad a_n = \frac{2}{n} \sum_{j=1}^{n} T_{n-1}(x_j) h(x_j), \quad (2 \leq n \leq N)
\]

3.2. Numerical treatments of the ss-OAFP equation

In a similar way to Heggie and Stevenson (1988)’s work, we had to carry out many numerical arrangements. First, Newton iteration method for the whole-domain formulation did not work at all. Hence we truncated the domain of \( v_Q \) and differentiation \( \frac{d}{dx} \) in the 4ODEs employing equation (3.14). Then, this arrangement provided spectral solutions on \( x_{\text{min}} \approx -0.96 < x < 1 \). Also, truncated-domain formulation did not work, hence, we regularized \( v_Q(x) \) and \( v_\beta(x) \) by the factor \((1 + x)/2\) so that \( \lim_{x \to -x_{\text{min}}} 2v_Q(x)/(1 + x) = 4/(2a - 7) \) and \( \lim_{x \to -x_{\text{min}}} 2v_\beta(x)/(1 + x) = 1/(a + 1) \). This arrangement provided solutions on \( x_{\text{min}} \approx -0.2 < x < 1 \). To broaden the range of the effective interval \( (x_{\text{min}} < x < 1) \), following Heggie and Stevenson (1988), we shortened the Newton steps in the iteration process though, it did not work.

To overcome the difficulty in convergence of Newton method, we fixed the eigenvalue \( \beta \) to a certain value during iteration process. For the fixed \( \beta \)-value, once we found a solution at a specific \( x_{\text{min}} \), we chose a new \( \beta \) that is close to the old \( \beta \). Then, we found a new solution for the new \( \beta \) using Newton iteration method. We repeated this process until \( v_Q(x = 1) \) reached its minimum. Then, at a new \( x_{\text{min}} \) that is very close to the old \( x_{\text{min}} \) with new \( \beta \) that is very close to old \( \beta \) we repeated the whole process above. As a result, \( x_{\text{min}} \) reached \(-1\) for the whole-domain formulation while \( x_{\text{min}} \) reached \(-0.96\) for the domain-truncated formulation.

Also, since the eigenvalue \( \beta \) was fixed during the iteration process, we speeded up the numerical integration of the integrals \( v_Q(x) \) and \( v_\beta(x) \) by applying the Fejér’s first rule quadrature to the integrals before the iteration process starts. For example, we discretized \( v_\beta(y) \) as follows

\[
v_\beta(y_j) = \sum_{n=1}^{N} f_{\text{linear}}^{\beta} \mathcal{D}_n(y_j)
\]

where \( \{f_{\text{linear}}^{\beta}\} \) is the Chebyshev coefficients of function \( v_\beta(x) \exp[v_T(x)] \). One can obtain \( \{f_{\text{linear}}^{\beta}\} \) from \( v_T(x) \) by using equations (3.4) and (3.5). The matrix \( \mathcal{D}_n(y_j) \) is a preset matrix to be integrated before the Newton-iteration (loop) process begins and explicitly reads

\[
\mathcal{D}_n(y_j) = \frac{L}{2 \sqrt{2}} \left(\frac{1 + y}{2}\right)^{\frac{N-1}{2}} \times \int_{-1}^{1} \Lambda_n \left[ y_j, \frac{(y_j + 1)(x' + 1)}{2} - 1 \right] T_n \left[ \frac{(y + 1)(x' + 1)}{2} - 1 \right] \sqrt{1 - x'}^i (1 + x')^{i(\beta+1)-1} dx'.
\]

We also prepared a similar preset matrix for \( v_Q(x) \). As a result, the two preset matrices made the iteration process 10 ~ 100 times more efficient than the original iteration process in which we implemented the Fejér’s first rule quadrature for each iteration.

\footnote{For example, to find the whole-domain solution, the change \( \delta \beta \) in \( \beta \) was 0.03 from \( x_{\text{min}} = -0.94 \) to \(-0.96\), \( \delta \beta = 0.001 \) from \( x_{\text{min}} = -0.9994 \) to \(-0.9996\), and \( \delta \beta = 0.000001 \) from \( x_{\text{min}} = -0.999994 \) to \(-0.999995\).}

\footnote{Using a 2.4 GHz CPU processor, the resulting CPU time for \( 10^8 \) iterations was \(~\) a few min for \( N = 70 \), which was needed to find solutions near \( x_{\text{min}} = -1\).}
4. Self-similar solution on the whole domain

We provide the whole-domain solution, its semi-analytical form and eigenvalues (Section 4.1) as the main result. Section 4.2 details the asymptotic approximation of the solution and the characteristics of the Chebyshev coefficients. Section 4.3 discusses the numerical stability of the solution and reports that the solution is unstable under change of degree $\mathcal{N}$.

4.1. Numerical results (main results of the present chapter)

We report (i) a whole-domain solution, (ii) eigenvalues and (iii) semi-analytical expression of the solution as the main result of the present chapter.

4.1.1. (i) Whole-domain solution

We found the whole-domain solution compatible to the HS solution. Figures 1(a) and 2(a) depict $DF(E)$ and m.f. potential $\Phi(R)$ obtained from the whole-domain spectral solution. In the figures, the HS solution is also depicted. The spectral- and HS- solutions are visually almost identical on the figures. For the whole-domain solution, the optimal values of numerical parameters are $\mathcal{N} = 70$, $F_{\text{BC}} = 1$, $L = 1$. The optimal eigenvalue of $\beta$ is

$$\beta_0 \equiv 8.1783711596581. \quad (4.1)$$

We chose the value of $\beta_0$ so that $\eta_1(x)$ reached its minimum value ($\sim 10^{-12}$). In order to make Newton iteration method work, we needed to correctly specify at least eight significant figures of $\beta_0$ (Appendix B.1). Also, degree $\mathcal{N} = 70$ is the minimum value among $70 \leq \mathcal{N} \leq 400$ for which Newton iteration method worked (Section 4.3). Figures 1(b) and 2(b) show the magnified figures for the solutions. The spectral solutions slightly deviate from the HS solution around $E = -0.3$

![Graphs](image.png)

Figure 1: (a) Distribution function $F(E)$ of stars on the whole domain and (b) its magnified graph on $-1 \leq E < -0.25$. ($\mathcal{N} = 70$, $F_{\text{BC}} = 1$, $L = 1$.)
4.1.2. (ii) Eigenvalues and physical parameters

The eigenvalues we found are the same as those of HS’s work up to 1 significant figure. Table 1 lists the eigenvalues obtained from the spectral solution. Our eigenvalues \( c_1, c_2 \) and \( c_3 \) are the same as two significant figures of the HS’s values while \( c_4 \) is only one significant figure with relative error of 6.7%. On one hand, the physical parameters \( \alpha, \chi_{\text{esc}} \) and \( \xi \) are the same as the HS’s values up to three significant figures. The present value of \( \chi_{\text{esc}} \) is greater than the HS’s value 13.85. This is consistent with the result of [Cohn, 1980] that predicted \( \chi_{\text{esc}} \approx 13.9 \) at which a complete core-collapse (an infinite central density) occurs.

In the rest of sections, we call the following eigenvalues and \( \beta_0 \) the reference eigenvalues for comparison with other solutions

\begin{align*}
  c_{10} &= 9.09254120455 \times 10^{-4}, \\
  c_{40}^* &= 3.03155222 \times 10^{-1}.
\end{align*}

The reference eigenvalues were obtained from the whole-domain solution when \( \beta = \beta_0, \ n = 70, \ L = 1 \) and \( F_{\text{BC}} = 1 \).

4.1.3. (iii) Semi-analytical expression of the whole-domain solution

We report the semi-analytical solution of the ss-OAFP system for any application. Since spectral-method studies generally provide a solution of equation concerned with low degrees of polynomials\footnote{Spectral methods can provide a ‘semi-analytical’ solution in sense that the solution can be expanded in terms of a few to tens of degrees of polynomials or base functions (e.g. Legendre polynomials, Geggenbauer polynomials, Hermite functions,...) for application. We show solutions based on different formulations of the ss-OAFP system in Sections 5 and Appendix B.3, hence we could construct variants of the semi-analytic solutions in the present work. However, they do not have an outstanding property. For example, the corresponding semi-analytical solutions on the truncated domain (Section 5) and contracted domain (Appendix B.3) need only 12 – 13 degrees to achieve a relative error of \( 10^{-4} \), but they are not be practical since they depend on parameters \( L \) and \( x_{\text{min}} \). Also, the degrees of the exponential of the regularized solution \( \exp[c_f(x)] \) still needs 16.}, Table 2 lists ‘semi-analytical’ forms of \( F(E), Q(E) \) and \( R(\Phi) \). The degrees of polynomials are at most eighteen and \% error is 0.1\% compared to the whole-domain solution with degree \( n = 70 \).

4.2. The detail analyses regarding the whole-domain solution and its asymptotic feature

The present section details the mathematical- and numerical-characteristics of the whole-domain solution. We discuss (i) the Chebyshev coefficients of the regularized solution, (ii) regularized solution and (iii) detail structure of \( \nu_3 \).
Table 1: Comparison of the present eigenvalues and physical parameters with the results of 'HS' (Heggie and Stevenson, 1988) and 'T' (Takahashi, 1993). The relative error between Heggie-Stevenson(HS)'s eigenvalues and the present ones are also shown. The present eigenvalues are based on the results for various combinations of numerical parameters ($13 < H < 560$, $10^{-4} < F_{BC} < 10^{4}$ and $L = 1/2, 3/4, 1$), different formulations (Sections 5 and 6) and stability analyses (Appendix B).

Table 2: Semi-analytical forms of the Chebyshev spectral solutions $F(E)$, $R(E)$ and $Q(E)$. The relative error of the semi-analytic form from the whole-domain solution is order of $10^{-4}$.
4.2.1. Chebyshev coefficients

The Chebyshev coefficients of the regularized solutions are depicted in Figure 3 in which the coefficients are divided by their own first \((n = 1)\) coefficients. The minimum absolute values of all the coefficients reach \(\sim 10^{-12}\) around at \(n = 70\). This implies that possible relative error of the spectral solutions is \(\sim 10^{-10}\%\) at best. The coefficients show geometric convergences; \(|F_n/F_1| \sim |G_n/G_1| \sim |I_n/I_1| \sim |J_n/J_1| \sim \exp(-0.3n)\) and \(|R_n/R_1| \sim |Q_n/Q_1| \sim \exp(-0.4n)\).

![Figure 3: Absolute values of the normalized Chebyshev spectral coefficients for the regularized solutions. \((I = 70, F_{\text{BC}} = 1\) and \(L = 1)\). The coefficients are divided by their own first coefficients.](image)

4.2.2. Regularized solution and its asymptotic approximation

To discuss the fine difference between the spectral- and HS- solutions, Figure 4 compares the corresponding regularized solutions. One can find a discrepancy between the two solutions as \(E \to 0\) for \(v_f(E), v_h(\Phi)\) and \(v_Q(E)\). The figure indicates that the HS solutions were obtained outside the domain on which the present solutions asymptotically behave as constant functions. This implies that the actual number of significant figures of the HS solution may not be more than one. This matter is discussed in detail in Section 6.3 (and Appendix F).
Since the asymptotic approximations of $F(E)$, $G(E)$, $I(E)$ and $J(E)$ as $E \to 0$ read

\begin{align}
F_{\text{asy}}(E) &\equiv \ln[c_4^*](-E)^\beta, \\
G_{\text{asy}}(E) &\equiv \frac{1}{a+1}(-E)^{\beta+1}, \\
I_{\text{asy}}(E) &\equiv \frac{4}{2\beta-7}(-E)^{\beta+1}Q(E), \\
J_{\text{asy}}(E) &\equiv -\frac{4\beta}{2\beta-3}(-E)^\beta Q(E),
\end{align}

one can compute the relative errors of the spectral solutions from $F_{\text{asy}}(E)$, $G_{\text{asy}}(E)$, $I_{\text{asy}}(E)$ and $J_{\text{asy}}(E)$ (Figure 5). The figure also depicts the corresponding errors for the HS solutions. Both our and HS- solutions show that $F_{\text{asy}}(E)$, $G_{\text{asy}}(E)$ and $J_{\text{asy}}(E)$ can well approximate $F(E)$, $G(E)$ and $J(E)$ near $E = -0.6$ or $E = -0.7$ and the relative errors between them are order of $10^{-3}$. Since the value of $c_4$ is relatively different in between the present- and HS- works, one finds a discrepancy between the works for $|1 - F(E)/F_{\text{asy}}(E)|$. 

![Figure 4: Regularized spectral solutions. (N = 70, $F_{\text{BC}} = 1$ and $L = 1.$)](image-url)
order of machine precision (See Appendix D for detail).

Figure 5: Relative error of the whole-domain spectral solutions from the asymptotic approximations ($n = 70, F_{\text{gc}} = 1, L = 1$). The circles are the corresponding relative errors of the HS solutions. Heggie and Stevenson (1988) listed the numerical values of their solution up to second decimal places, meaning the present solution can compare to their solutions up to $5 \times 10^{-3}$ at best.

4.2.3. Detail structure of $v_J$

Since the higher order of asymptotic approximation for $v_J(x)$ is analytically tractable, we discuss the feature qualitatively and quantitatively. First, we can qualitatively find a consistency of the spectral whole-domain solution by examining the asymptotic approximation of the solution $J(E)$ that can be explicitly found from one of the four ODEs (Appendix A).

$$| 1 - J(E \to 0)/J_{\text{asy}}(E) | = -c_4^* \frac{(2\beta + 7)(6\beta - 3)}{c_1 (2\beta - 7)(2\beta - 3)(\beta + 1)} (-E)^{\beta} \equiv C_\beta (c_1, c_4^*) (-E)^{\beta}. \quad (4.4)$$

This $(-E)^{\beta}$-dependence is numerically reproduced in Figure 6(a). Figure 6(b) depicts the characteristics of $| 1 - J(E)/J_{\text{asy}}(E) | \sim (-E)^{-d}$ that it is still approximately constant on the interval $-4 \times 10^{-2} \lesssim E \lesssim -1 \times 10^{-1}$.

To quantitatively see the consistency of the spectral solution, we numerically calculated the values of $\beta$ and $C_\beta (c_1, c_4^*)$ of equation (4.2.3). Figure 7(a) shows the relative error between $\beta$, and the logarithmic derivative of $| 1 - J(E)/J_{\text{asy}}(E) |$ and Figure 7(b) depicts the error between $| 1 - J(E)/J_{\text{asy}}(E) | \sim (-E)^{-d}$ and $C_\beta (c_1, c_4^*)$. One can find the former is correct at order of $4.3 \times 10^{-6}$ at best and the latter $2.0 \times 10^{-7}$. The logarithmic derivative and asymptotic approximation lose their accuracies at energies greater than $E = -0.07 \sim -0.05$. This is since the expression in equation (4.2.3) is correct under the limited condition that the factor $(0.5 + 0.5x^d)_{\text{deg}}$ in equation (2.14a) does not reach order of machine precision (See Appendix D for detail).
Figure 6: (a) Characteristics of $|1 - J(E)/J_{asy}(E)|$ and (b) characteristics of $|1 - J(E)/J_{asy}(E)|(-E)^{-\beta}$ ($N = 70$, $F_{BC} = 1$, $L = 1$). The circles are the corresponding characteristics of Heggie and Stevenson’s work.

Figure 7: (a) Logarithmic derivative of $|1 - J(E)/J_{asy}(E)|$ with respect to $E$ and (b) characteristics of $|1 - J(E)/J_{asy}(E)|(-E)^{-\beta}$ ($N = 70$, $F_{BC} = 1$, $L = 1$).
4.3. Instability of the whole-domain solution

The present section explains the numerical instability of the whole-domain solution. As explained in detail in Appendix B the whole-domain solution is stable under change of some numerical parameters. For a broad range of $\beta, F_{BC}, L$ and the total number of nodes for the Fejér’s rule quadrature, the eigenvalues $c_1$ and $c_1^*$ can preserves seven- and five- significant figures compared to $c_{10}$ and $c_{10}^*$. On one hand, the whole-domain solution is unstable to change in $N$. The Newton iteration method well worked only for $70 \leq N \lesssim 400$. It did not work at all for $N$ less than 70 while it still worked for $N > 400$ but high $N$ increased the condition number and $v_I(x = 1)$, costing an unfeasible CPU time.

Figure 8(a) shows that $v_I(x = 1)$ increases with $N$ but the rate of change becomes calm for higher $N$ and Figure 8(b) depicts the condition number of the Jacobian matrix for the 4ODEs and $v_O$-integral and it monotonically increases with $N$. Both $v_I(x = 1)$ and the condition number reach their lowest values when $N = 70$. Hence, we compared the spectral solution for $N = 70$ to the solutions with different degrees $N$ (Figure 8(c)). The figure shows that the solutions are stable up to order of $10^{-4}$. Also, Figure 9 shows the power-law profiles under the asymptotic approximations (that appear if the domain of Figure 5 is extended to $E \to 0$) lose their characteristics as $N$ increases. In the figure, the power-law profiles for $N = 400$ and $N = 70$ are shown.
Figure 8: (a) Relative error between $c_4$ and $c_4^*$ and the value of $v_1(x = 1)$, (b) Condition number of the Jacobian matrix for the 4ODEs and $Q$-integral. The condition number was computed when $(a_q)^{old} - (a_q)^{new}$ reach order of $10^{-13}$ in Newton iteration process and (c) Relative error between the whole-domain solutions with different $N$ and the solution with $N = 70$. ($L = 1$ and $F_{bc} = 1$)
5. Self-similar solutions on truncated domain

The present section provides spectral solutions on several truncated domains and show an optimal truncated-domain solution is identical to the reference solution up to order of $10^{-9}$ on certain truncated domains. Since the spectral solution on the whole domain is unstable to change in degree $N$ and also since the present work relies on a collocation method, it is imperative for us to construct a spectral solution whose accuracy improves with increasing $N$. To find such a solution, we truncated the domain of the ss-OAFP system. According to Section 4, we extrapolated $v_f(x)$ using $v_f^{\text{asy}}(x)$ (equation (2.18)) on $-0.2 \lesssim E \lesssim 0$, on which the regularized solutions behave like constant functions of $E$. This means one may expect to obtain several kinds of solutions for different maximum energy $E_{\text{max}}$. The following lists kinds of solutions based on the absolute value of each term in equation (2.14a) (Refer to Appendix
for the details of the classification.)

(i) \( E_{\text{max}} \lesssim -0.25 \) (Incorrect solution)
Solutions and eigenvalues significantly differ from the existing solutions.

(ii) \(-0.25 \lesssim E_{\text{max}} \lesssim -0.05\) (Stable solution)
Chebyshev coefficients are relatively stable against change in degree \( N \).

(iii) \(-0.05 \lesssim E_{\text{max}} \lesssim -0.005\) (Semi-stable solution)
Chebyshev coefficients are stable up to a certain degree \( N \).

(iv) \(-0.005 \lesssim E_{\text{max}}\) (unstable solution)
Chebyshev coefficients are unstable against change in degree \( N \).

The goal of the present section is, based on four cases (i) - (iv), to show some optimal truncated-domain solutions.
First, Section 5.1 explains the condition to obtain a truncated solution by examining cases (i) and (ii). Sections 5.2 and 5.3 discusses cases (ii) and (iii) to find an optimal truncated-domain solution and compare. Especially, Section 5.2 shows solutions on truncated domains with optimal values of \( \beta \). Section 5.3 discusses the difference between the solutions obtained on whole- and truncated-domain for fixed \( \beta = \beta_o \).

For comparison in the rest of sections, we call the whole-domain solution with \( N = 70, L = 1, F_{\text{BC}} \) and \( \beta = \beta_o \) (shown in Section 4.1) the reference solution. The solution is labeled with subscript symbol ‘\( o \)‘, such as \( F_o(E), \Phi_o(R_v), v_{F_o}, v_{G_o} \) ... and so on.

5.1. Stable solutions on truncated domains with \(-0.35 < E_{\text{max}} < -0.05\)

While Newton iteration method itself worked quite well for \( E_{\text{max}} < -0.1 \), the spectral solutions on the truncated domains have a transition point around at \( E_{\text{soln}} = -0.225 \) that separates the solutions into incorrect and stable solutions. To see this, the present section shows truncated-domain solutions obtained near \( E_{\text{soln}} \) for \( L = 1, F_{\text{BC}} = 1 \) and \( \beta = 8.1783^8 \). Figure 10(a) shows the calculated eigenvalues \( | 1 - c_1/c_{10} |, | 1 - c_4/c_{40} | \) and \( | v(x = 1) | \) for \(-0.35 < E_{\text{max}} < -0.1 \) and Figure 10(b) the condition number of the Jacobian matrix of the 4ODEs and \( Q \)-integral. All the values depict significant changes around at \( E_{\text{soln}} \). Heggie and Stevenson (1988) reported this transition as a difficulty in convergence of Newton method. Since the eigenvalues for \( E_{\text{soln}} > E_{\text{max}} \) significantly deviate from the both existing and our eigenvalues, they should be considered as incorrect solutions.

\(^8\)The value of \( \beta \) is set to five digits, meaning if one applies the same accuracy relation discussed in Appendix [i] to this case, the relative error of solutions would be \(| 1 - c_4/c_{40} | \sim 10^{-1} \).
Spectral solutions for $-0.225 \leq E_{\text{max}} < -0.05$ are relatively stable to change in $N$. Especially at $E_{\text{max}} = -0.225$ the solution is the most stable. For $E_{\text{max}} = -0.225$ and $\beta = 8.17837$, Figure 11(a) shows the characteristics $|1 - c_1/c_{10}|$, $|1 - c_4^*/c_{40}^*|$ and $|v_\ell(x = 1)|$ against degree $N$. The Newton iteration worked well even for $N = 360$ and the accuracy improves with increasing $N$ in sense the eigenvalues approach the reference eigenvalues. Also, higher $N$ provides smaller absolute values of Chebyshev coefficients. Figure 11(b) shows the coefficients for $N = 50$ and $N = 360$.

The reason why the coefficients do not decay rapidly with high index $n$ would be that the rapid decay was hindered by the discontinuous behavior of the $Q$-integral (Appendix E.1). The cause of the discontinuity is that one could not correctly specify the value of $\beta$ with high accuracy or even an accurate solution does not exist when $E_{\text{max}}$ is not close to zero. In fact, the discontinuous behavior disappears for semi-stable solutions with $E_{\text{max}} \approx 0.05$ (Section 5.2).
5.3.1. (i) Semi-stable solutions with \( \beta = \beta_0 \)

Semi-stable solutions with \( \beta = \beta_0 \) approach the reference solution up to certain degrees of polynomial and they lose accuracy beyond the degrees. Figure 12 shows the characteristics of \( \psi_2(x = 1) \) against \( N \) and the relative accuracy.

\( ^9 \) The condition number of the Jacobian is order of \( 10^7 \) in Newton iteration process for the whole-domain solution. This means one can obtain approximately five significant-figure solution since the minimum of the ‘practical’ machine precision is order of \( 10^{-12} \) (Appendix B). Considering that the gap in accuracy is order of \( 10^{-3} \) between \( c_1 \) (or \( \beta \)) and \( c_4 \), the ten maximum significant figures are a reasonable choice for \( \beta \).
Table 3: Numerical results for the truncated-domain formulation at different $E_{\text{max}}$ ($L = 1$ and $F_{\text{BC}} = 1$). The minimum values of $|1 - c_i/c_{\text{lo}}|$ and $v_i(x = 1)$ are underlined to highlight the accuracy for each $H$. The results do not include some data in which the value of $v_i(x = 1)$ are greater than the order of $10^{-7}$ for convenience. The Newton iteration method was very hard to work for $H < 25$ and the degrees beyond the maximum values of $H$ for each $E_{\text{max}}$; those condition provided the change $|\alpha^{\text{new}} - \alpha^{\text{old}}| \lesssim 10^{-9}$ while the data in the table are the results when it reached order of $10^{-12}$.

| $E_{\text{max}}$ | $H$ | Eigenvalue $\beta$ | $|1 - c_i/c_{\text{lo}}|$ | $|1 - c_i^*/c_{\text{lo}}^*|$ | $v_i(x = 1)$ |
|-----------------|-----|--------------------|-----------------|-----------------|----------------|
| $-0.03$         | 55  | 8.178371160        | $6.5 \times 10^{-10}$ | $2.6 \times 10^{-6}$ | $1.6 \times 10^{-8}$ |
|                 | 50  | 8.178371160        | $7.1 \times 10^{-10}$ | $2.9 \times 10^{-6}$ | $1.7 \times 10^{-8}$ |
|                 | 40  | 8.178371165        | $2.4 \times 10^{-9}$  | $6.9 \times 10^{-7}$ | $4.3 \times 10^{-9}$ |
|                 | 30  | 8.178370376        | $1.3 \times 10^{-7}$  | $9.8 \times 10^{-6}$ | $8.1 \times 10^{-10}$ |
|                 | 25  | 8.1783436          | $8.1 \times 10^{-7}$  | $1.3 \times 10^{-4}$ | $1.1 \times 10^{-9}$ |

| $E_{\text{max}}$ | $H$ | Eigenvalue $\beta$ | $|1 - c_i/c_{\text{lo}}|$ | $|1 - c_i^*/c_{\text{lo}}^*|$ | $v_i(x = 1)$ |
|-----------------|-----|--------------------|-----------------|-----------------|----------------|
| $-0.04$         | 55  | 8.178371160        | $5.1 \times 10^{-10}$ | $2.1 \times 10^{-6}$ | $1.3 \times 10^{-8}$ |
|                 | 50  | 8.178371160        | $1.9 \times 10^{-10}$ | $8.3 \times 10^{-7}$ | $5.1 \times 10^{-9}$ |
|                 | 40  | 8.178371170        | $1.4 \times 10^{-10}$ | $1.7 \times 10^{-7}$ | $4.0 \times 10^{-9}$ |
|                 | 30  | 8.17837159         | $1.8 \times 10^{-7}$  | $4.7 \times 10^{-6}$ | $6.7 \times 10^{-9}$ |
|                 | 25  | 8.1783585          | $1.7 \times 10^{-6}$  | $6.8 \times 10^{-5}$ | $5.2 \times 10^{-9}$ |

| $E_{\text{max}}$ | $H$ | Eigenvalue $\beta$ | $|1 - c_i/c_{\text{lo}}|$ | $|1 - c_i^*/c_{\text{lo}}^*|$ | $v_i(x = 1)$ |
|-----------------|-----|--------------------|-----------------|-----------------|----------------|
| $-0.05$         | 55  | 8.178371160        | $2.8 \times 10^{-9}$  | $1.0 \times 10^{-5}$ | $6.8 \times 10^{-8}$ |
|                 | 50  | 8.178371158        | $5.5 \times 10^{-9}$  | $9.7 \times 10^{-7}$ | $3.0 \times 10^{-9}$ |
|                 | 40  | 8.1783723          | $2.1 \times 10^{-8}$  | $3.7 \times 10^{-6}$ | $3.1 \times 10^{-8}$ |
|                 | 30  | 8.178373           | $1.5 \times 10^{-6}$  | $1.9 \times 10^{-5}$ | $4.0 \times 10^{-9}$ |

| $E_{\text{max}}$ | $H$ | Eigenvalue $\beta$ | $|1 - c_i/c_{\text{lo}}|$ | $|1 - c_i^*/c_{\text{lo}}^*|$ | $v_i(x = 1)$ |
|-----------------|-----|--------------------|-----------------|-----------------|----------------|
| $-0.06$         | 40  | 8.178371160        | $8.3 \times 10^{-10}$ | $8.7 \times 10^{-7}$ | $2.2 \times 10^{-9}$ |
|                 | 30  | 8.17837225         | $1.5 \times 10^{-7}$  | $5.0 \times 10^{-6}$ | $8.1 \times 10^{-9}$ |
|                 | 25  | 8.1783813          | $2.4 \times 10^{-7}$  | $1.6 \times 10^{-5}$ | $2.3 \times 10^{-9}$ |

| $E_{\text{max}}$ | $H$ | Eigenvalue $\beta$ | $|1 - c_i/c_{\text{lo}}|$ | $|1 - c_i^*/c_{\text{lo}}^*|$ | $v_i(x = 1)$ |
|-----------------|-----|--------------------|-----------------|-----------------|----------------|
| $-0.07$         | 35  | 8.178371159        | $8.6 \times 10^{-9}$  | $2.3 \times 10^{-6}$ | $1.4 \times 10^{-9}$ |
|                 | 30  | 8.1783717          | $2.2 \times 10^{-7}$  | $1.9 \times 10^{-5}$ | $9.7 \times 10^{-8}$ |
|                 | 25  | 8.178382           | $1.1 \times 10^{-6}$  | $5.0 \times 10^{-5}$ | $9.8 \times 10^{-8}$ |

| $E_{\text{max}}$ | $H$ | Eigenvalue $\beta$ | $|1 - c_i/c_{\text{lo}}|$ | $|1 - c_i^*/c_{\text{lo}}^*|$ | $v_i(x = 1)$ |
|-----------------|-----|--------------------|-----------------|-----------------|----------------|
| $-0.08$         | 25  | 8.1783768          | $2.8 \times 10^{-6}$  | $4.7 \times 10^{-5}$ | $7.9 \times 10^{-9}$ |
errors of \( c_1 \) and \( c_4^* \) from the reference values. For \( E_{\text{max}} > -0.07 \), \(| v_I(x = 1) |, | 1 - c_1/c_{10} | \) and \(| 1 - c_4^*/c_{40}^* | \) show an ideal characteristics under change in \( N \). The values decrease with increasing \( N \) and can reach very small (\( \approx 10^{-9} \sim 10^{-13} \)) at certain degrees. Beyond the degrees, the Newton iteration method, however, did not work or the spectral solutions significantly loses their accuracies. On one hand, for \( E_{\text{max}} < -0.07 \) the characteristics of the \( N \)-dependence are less ideal. \(| v_I(x = 1) |, | 1 - c_1/c_{10} | \) and \(| 1 - c_4^*/c_{40}^* | \) stall with increasing \( N \) while the minimum values still can be found at relatively-low degrees (\( N = 27 \sim 35 \)). This would be since the optimal value of \( \beta \) is not close to \( \beta_0 \) as found in Table 3 near \( E_{\text{max}} = -0.08 \) the optimal value may be larger than \( \beta_0 \).
Figure 12: Relative errors of $c_1$ and $c_4^*$ from the reference eigenvalues and characteristics of $e_\nu(x = 1)$ against $n$ for the truncated-domain solutions with $\beta = \beta_o$. ($-0.1 \leq E_{\text{max}} \leq -0.03$, $L = 1$ and $F_{\text{BC}} = 1$.)
5.3.2. (ii) Truncated domains and numerical instability against change in degree $N$

We consider the numerical stability that occurred to the semi-stable solutions and reference solution originates from the property that ss-OAFP system may not have a solution when the terms of the system reach order of machine precision at equation level and beyond the accuracy. In Figure 12, the minimum of $| v_I(x = 1) |$ occurs at degrees $N_{\text{best}} = [27, 25, 35, 35, 47, 57, 65, 65]$ for $E_{\text{max}} = [-0.10, -0.09, -0.08, -0.07, -0.06, -0.05, -0.04, -0.03]$. To consider why the truncated-domain solutions lose their accuracy beyond $N_{\text{best}}$, Figure 13 depicts the $E_{\text{max}}$-dependence of $| v_I(x = 1) |$, $| 1 - c_4/c_{10}^* |$ and $| 1 - c_4^*/c_{10}^* |$ obtained at each $N_{\text{best}}$. $| 1 - c_4^*/c_{10}^* |$ decreases in a power-law-like fashion with increasing $E_{\text{max}}$. One may understand this characteristics by introducing a power-law profile $c_{10}(-E_{\text{max}}^\beta)/c_{40}^*$. This profile originates from the power-law dependence of the last term in equation 2.14a (See Appendix D). In Figure 13, the decrease of $c_{10}(-E_{\text{max}}^\beta)/c_{40}^*$ is similar to that of $| 1 - c_4^*/c_{10}^* |$. On one hand, $| 1 - c_1/c_{10}^* |$ and $| v_I(x = 1) |$ stops decreasing at $E_{\text{max}}$ larger than $\approx -0.05$. This may be understood as a limit of double precision. In addition to $c_{10}(-E_{\text{max}}^\beta)/c_{40}^*$ characterizing the accuracy of $c_{10}^*$ (correspondingly the solution), the infinity norm of $| [F_n]^\text{old} - [F_n]^\text{new} |$ for Newton method reaches order of $10^{-13}$ at best (Appendix D). Under these circumstances, $c_{10}(-E_{\text{max}}^\beta)/c_{40}^*$ reaches order of $10^{-13}$ at $E_{\text{max}} \approx -0.0523$ that is the maximum value of $E_{\text{max}}$ to preserve numerical accuracy. This result implies that, for the truncated-domain solutions at $E_{\text{max}} \lesssim -0.05$, machine precision is not enough precise to obtain more accurate solution. Hence, a semi-stable solution is stable up to a certain degree. On one hand, in case of the reference solution, the solution is not truncated on large $E$, meaning the power-law boundary conditions in the ss-OAFP system can be satisfied only when they reach order of machine precision. Hence, unlike the semi-stable solution, the reference solution does not improve accuracy for low degrees of polynomials and it only loses its accuracy with increasing degree. One can find more detail discussion for machine precision and the convergence of Newton iteration method in Appendix D.

![](image.png)

Figure 13: Relative error of $c_1$ and $c_4^*$ from the reference values and characteristics of $| v_I(x = 1) |$ that are obtained at each $N_{\text{best}}$. The guideline $c_{10}(-E_{\text{max}}^\beta)/c_{40}^*$ is shown for comparison. ($L = 1, F_{\text{inc}} = 1$ and $\beta = \beta_c$.)

5.3.3. (iii) an optimal semi-stable solution

Lastly, we propose an optimal truncated-domain solution to compare to the reference solution. For $E_{\text{max}} = -0.03$ in Figure 12, $| 1 - c_4^*/c_{10}^* |$ reaches order of $10^{-9}$. This is the same order as that of $| 1 - c_4^*/c_{40}^* |$ computed against different $\beta$ in Section B.1. Also, $v_I(x = 1)$ for $H = 65$ is one of the least values $6.1 \times 10^{-12}$ among that for semi-stable truncated solutions. Hence er calculated the relative errors between the solution with $H = 65$ and solutions with different $H$ for $E_{\text{max}} = -0.03$ (Figure 14(a)). We obtained the ideal tendency that as $H$ increases, the solutions gradually converge to the solution with $H = 65$. Hence the truncated-domain solution with $E_{\text{max}} = -0.03, H = 65$ and
\( \beta = \beta_o \) is the optimal truncated-domain solution in the present work. Figure 14(b) shows the relative error between this optimal solution and the reference solution at the points that are less associated with the Gauss-Chebyshev nodes. Our optimal truncated-domain solution validates the reference solution and the largest relative error between them is order of \( 10^{-9} \) at the prescribed points.

\[
| 1 - F(E)^{(N=65)}/F(E) | \quad | 1 - F(E)^{(N=65)}/F_o(E) |
\]

Figure 14: (a) Relative errors of solutions with different \( N \) from the solution with \( E_{\text{max}} = -0.03 \). (\( L = 1 \) and \( F_{\text{BC}} = 1 \)) (b) Relative error between the reference solution \( F_o(E) \) and the optimal truncated-domain solution. (\( L = 1, F_{\text{BC}} = 1 \) and \( \beta = \beta_o \)).

6. Discussion: Modifying the mathematical formulation of the ss-OAFP system to reproduce the HS solution

The present section discusses how to improve the whole-domain solution and reproduce the solution of \( \text{Heggie and Stevenson, 1988} \) using the spectral method to discuss the accuracy of the reference solution. In Section 5.4, we mentioned that the stable truncated-domain solution on \(-0.35 < E < -0.1\) is closer to the reference solution rather than the HS solution that was obtained on almost the same domain. Our goal of the present section is to show that the discrepancy between our and the HS solution originate from the difference in mathematical formulation of the ss-OAFP system between the two works. In order to explain the discrepancy and also see the consistency of our result compared to the HS solution, we discuss several classes of ss-OAFP solutions by modifying the regularized independent variables. We found that only modification of \( v_J(x), v_R(x) \) and \( v_F(x) \) provides significant change in ss-OAFP solution while that of the rest of the regularized function did not change the solution. Sections 6.1 and 6.2 detail the effect of modifying \( v_J(x) \) and \( v_R(x) \) to improve the asymptotic behavior of \( v_J(x) \) and to discuss the effect of discontinuity in \( v_R(x) \). Based on the modification of \( v_R(x) \), Section 6.3 reproduces the HS solution with limited degrees and shows that the formulation can provide both the HS- and reference- solutions only by controlling \( x_{\text{min}} \) or \( E_{\text{max}} \). For brevity, further detail discussion on reproducing the HS solution is included into Appendix F in which we discuss how to take off the limitation. This can be done by modifying \( v_R(x) \) and \( v_F(x) \).

6.1. Modification of function \( v_J(x) \) and its asymptotic behavior

The present section shows that one can improve the reference solution by modifying the formulation for the regularized function \( v_J(x) \). Even after all the independent variables of the ss-OAFP system are completely regularized (so that the variables reaches certain constant values at the end points of the domain), the terms of the regularized ss-OAFP system significant change at equation level. All the terms in the 4ODEs (equations (2.14a) and (2.14d))
change like at least \( \sim (0.5 + 0.5x)^{\beta} \) as \( x \to -1 \). Section 4 showed the consequence of the large-scale gap in Figure 7 in which the accuracy in the logarithmic derivative and higher order of \( \sim (0.5 + 0.5x)^{\beta} \) in \( v_f \) are divergent as \( x \to -1 \). One can improve the result (or weaken the divergence) by modifying \( v_f(x) \) as follows

\[
v_f^{(m)}(x) \equiv (v_f(x) + 1)^{\left(\frac{1 + x}{2}\right)^{-bL}},
\]

where \( b \) is a real number. In equation (2.14a) the highest orders of \( (0.5 + 0.5x)^{\beta} \) is the term \( \frac{d\psi(x)}{dx} \left(\frac{1 + x}{2}\right)^{\beta-b} \), hence the function \( v_f^{(m)}(x) \) can reduce it to \( \frac{d\psi(x)}{dx} \left(\frac{1 + x}{2}\right)^{\beta-b} \). We solved the ss-OAFP system again following the procedure of Section 3.2, but this time with \( a = b \) in which the accuracy in the logarithmic derivative and higher order of \( \sim (0.5 + 0.5x)^{\beta} \) in \( v_f \) were obtained for \( \beta \) at most order of \( \sim (0.5 + 0.5x)^{\beta} \) in \( v_f \).

We found solutions for \( 1 \leq b \leq 6 \) on both whole- and truncated-domains and \( b = 6 \) provided the best result in accuracy. The results are quite well; the modification of \( v_f \) improved the asymptotic behaviors of the logarithmic derivative of \( v_f(x) \) and the approximation \( C_\beta(c_1, c_2) \) as \( x \to -1 \) on the whole domain (Figure 15). Also, the eigenvalues that were obtained for \( b = 6 \) on the truncated- and whole- domains are almost identical to the reference eigenvalues (Table 4). The relative error between the reference solution and the truncated solution for \( b = 6 \) and \( E_{\text{max}} = -0.025 \) is at most order of \( \sim 3 \times 10^{-8} \) (Figure 16). This result infers that one can obtain a suitable solution that is less divergent in higher order of \( \sim (0.5 + 0.5x)^{\beta} \) as \( x \to -1 \) by correctly regularizing function \( J(E) \).

Figure 15: (a) The logarithmic derivative of \( \mid 1 - J(E)/J_{\text{asy}}(E) \mid \) with respect to \( E \) and (b) \( \mid 1 - J(E)/J_{\text{asy}}(E) \mid (\mid E \mid^{\beta})^{-b} \) on the whole domain for \( b = 6 \). (\( H = 150, F_{\text{BC}} = 1, L = 1 \)). The solution with \( b = 0 \) corresponds to the reference solution.
Table 4: Numerical results for the integration of the ss-OAF P system for \( b = 6 \). \( (L = 1 \) and \( F_{BC} = 1 \)). The eigenvalues are compared to the reference eigenvalues.

| domain          | \( N \) | optimal \( \beta \) | \( |1 - c_1/c_{10}| \) | \( |1 - c'_1/c'_{10}| \) | \( v_f(x = 1) \) | condition number |
|-----------------|--------|----------------------|----------------------|----------------------|----------------|-----------------|
| whole           | 150    | \( \beta_0 \)       | \( 2.4 \times 10^{-11} \) | \( 7.7 \times 10^{-8} \) | \( 1.6 \times 10^{-9} \) | \( 8.8 \times 10^7 \) |
| truncated \( (x_{min} = -0.95) \) | 55     | \( \beta_0 + 3.0 \times 10^{-12} \) | \( 1.6 \times 10^{-12} \) | \( 2.6 \times 10^{-8} \) | \( 3.0 \times 10^{-12} \) | \( 1.2 \times 10^8 \) |

Figure 16: Relative error of the truncated-domain solution with \( x_{min} = -0.95 \), \( b = 6 \) and \( N = 55 \) from the reference solution. \( (F_{BC} = 1 \) and \( L = 1 \)).

6.2. Modification of function \( v_f(x) \) and its discontinuous behavior

The asymptotic behavior of \( v_f(x) \) as \( x \to -1 \) is important to see the effect of discontinuity in the ss-OAFP solutions and the discontinuity clearly appears in the solutions that are obtained without the assumption that \( v_f(x) \) is regular at \( x = -1 \) (this assumption is made implicitly in Section 2 by regularizing \( R \) with \( (1 - E) \)). We show this by modifying the formulation for \( v_f(x) \) as follows

\[
v_f^{(m)}(x) = [v_f(x)]^2 \left(1 - \frac{x}{2}\right).
\] (6.2)

The square of \( v_f(x) \) can avoid the endpoint singularity at the branch point \( x = 1 \). Again we solved the ss-OAFP system but this time with \( v_f^{(m)}(x) \) (without including \( v_f(x) \) and \( v_f^{(m)}(x) \) in the ss-OAFP system.) following the procedure of Section 3.2.

We found solutions for \( v_f(x) \) and \( v_f^{(m)}(x) \) with \( N = 540 \) that well explains the feature of discontinuity in the ss-OAFP system. Figures 17 shows the maximum relative errors of \( v_f^{(m)}(x) \) and \( v_f(x) \) are order of \( 2 \times 10^{-5} \) and \( 5 \times 10^{-5} \) from their reference solutions. The order of errors well reflects the relative error of the solutions from their asymptotic approximations (Figure 18). Figure 19 depicts the Chebyshev coefficients of \( v_f(x) \) and \( v_f^{(m)}(x) \). Slow decay appears in both coefficients for \( v_f(x) \) and \( v_f^{(m)}(x) \). The former apparently flattens (more exactly, decays like \( 7 \times 10^{-8} n^{-0.1} \)) and the latter decays like \( 1.5 \times 10^{-9} n^{-1} \). It is not easy to find the cause of the flattening and slow decay due to the mathematically complex structure of the ss-OAFP system. Yet, the asymptotic behavior \( a_n \sim 1/n \ (n \to \infty) \) has
approximately the same decay rate as Chebyshev coefficients for discontinuous functions (Boyd, 2001; Xiang, 2013). Hence, Appendices E.1 and E.2 show the numerical results that were obtained by calculating the $Q$ integral for a fixed discontinuous $v_R$ and also by solving the Poisson equation for a fixed discontinuous $v_D$. The former provides a slow decay of Chebyshev coefficients like $1/n$ or much slower (Figure E.29) and the latter a flattening of Chebyshev coefficients for large $n$ (Figure E.30). These unique behaviors occurred only when the point of discontinuity was very close to endpoints of the domain (See Appendices E.1 and E.2 for detail).

### 6.3. Reproducing the HS- Solution and eigenvalues with limited degrees

The present section reproduces the HS solution with only low degrees ($N < 20$) of polynomials by modifying the formulation for $v_R$. According to (Heggie and Stevenson, 1988), the numerical values of their solutions are “thought to be accurate about three significant figures”.

Due to these ambiguous expressions and lack of detail description for their error analysis in their work, the present section aims to reproduce at least two significant figures of the HS’s solutions and eigenvalues. We show the results obtained by reformulating the ss-OAFP system based on $v_R^{(m)}$ (explained in Section 5.3) and by using the numerical procedure of Section 3.2. However, the results reproduced only either of the HS’s solution and eigenvalues for a certain $N$, not both of them. To understand the reproduced solutions, the present section examines two kinds of solutions. 

(i) The first kind of solution reproduces the HS solution but the eigenvalues are the same as only two significant figures of the HS eigenvalues. 

(ii) The second kind reproduces the HS’s eigenvalues but the solution is the same as only two significant figures of the HS solution. 

For comparison, the HS solution is labeled hereafter by subscript ‘HS’, such as $F_{HS}$ for stellar DF.

#### 6.3.1. (i) Reproducing the same solution as HS’s work

We found spectral solutions with low degrees ($N = 13 \sim 19$) that are the same as the numerical values of $\ln[F_{HS}(E)]$ reported in (Heggie and Stevenson, 1988), however the eigenvalues are different from the HS eigenvalues (Table 5). The eigenvalues are stable up to two significant figures $\beta = 8.2$, $c_1 = 9.1$ and $c_4 = 3.5$ and the physical parameters up to three significant figures $\chi_{esc} = 13.8$ and $\alpha = 2.23$. The measures of accuracy hold approximately the same order for different $E_{max}$ and $N$, that is, $F_R \approx 10^{-4}$ and $v_f(x = 1) \approx 10^{-4} \sim 10^{-5}$.

Available degrees $N$ that can reproduce the HS solution are limited. Figure 20 shows the relative error of the spectral solution with $E_{max} = -0.275$ from the HS solution for different $N$. Since the HS’s work reported their solution up to second decimal places, the values of $0.005/\ln[F_{HS}]$ are also shown in the figure as reference. The spectral solution reproduced the HS solution for $N = 15$ and $N = 17$, that is, in the figure all the relative errors are below $0.005/\ln[F_{HS}]$. Yet, beyond $N = 17$, the spectral solution deviates from the HS solution and the eigenvalues rather approach the reference eigenvalues.
Figure 18: (a) Comparison of asymptotic behaviors between $v_R^{(m)}$ and $v_{Ro}$; the relative errors from the corresponding end values $v_R^{(m)}(x = -0.1)$ and $v_{Ro}(x = -1)$ are shown. (b) Comparison of asymptotic behaviors between $v_{Fo}$ and $v_F$; the relative errors from the corresponding end values $v_{Fo}(x = 0.1)$ and $v_F(x = -1)$ are shown.

Figure 19: (a) Absolute value of Chebyshev coefficients for $v_R^{(m)}$ with a dashed guideline for measure of slow decay. (b) Absolute value of the Chebyshev coefficients for $v_F$ with a dashed guideline for measure of slow decay.
Table 5: Eigenvalues obtained when the spectral solution reproduced the HS solution. Heggie and Stevenson (1988) reported the numerical values of their solution on $-1 \leq E \leq -0.317$. They mentioned that their Newton iteration method worked up to $E_{\text{max}} \approx -0.223$ and it could work beyond -0.223.

![Figure 20](image-url): Relative error between the spectral solution $\ln[F]$ with $E_{\text{max}} = -0.275$ and HS solution $\ln[F_{\text{HS}}]$ for different degrees $N$.
Chebyshev polynomials is 70 and the minimum of the normalized Chebyshev coefficients. The equation, however, has never been solved with an agreeable accuracy and the existing solutions were domain-truncated, whose domain is $-\beta < x < -0.317$. They mentioned that their Newton iteration method worked up to $E_{\text{max}} = -0.223$ and it could work beyond -0.223.

6.3.2. (ii) Finding solution whose eigenvalues are the same as the HS eigenvalue

We found spectral solutions whose eigenvalues are the same as the HS's eigenvalues ($c_1 = 9.10$ and $c_4 = 3.52$) with $N = 15$ near $E_{\text{max}} = -0.225$. For the solutions, Table 6 shows $\beta$, $\chi_{\text{esc}}$ and measures of accuracy ($F_n$ and $v_f(x = 1)$). The measures of accuracy are approximately the same order as the reproduced HS solution (shown in Table 5), $F_n \approx v_f(x = 1) \sim 10^{-4}$. Interestingly, for $E_{\text{max}} = -0.225$, $\chi_{\text{esc}}$ reaches the HS’ value ($\pm 13.85$). The numerical values of $\ln[F]$ reproduced 2 ~ 4 significant figures of $\ln[F_{\text{HS}}]$. The relative error between $\ln[F]$ and $\ln[F_{\text{HS}}]$ is at most order of $1 \times 10^{-3}$ for $E \geq -0.9$ (Figure 21). This result would infer that the spectral solution reproduced “about three significant figures” of the HS solution for the same eigenvalues.

6.3.3. (iii) Successfully reproducing HS solution and accuracy of the reference solution

We briefly explain the condition to obtain the both reference- and HS- solution on truncated domains only based on a single mathematical formulation of the ss-OAFP model. For brevity the detail discussion is made in Appendix D and we explain only the results. The most important result in Appendix D is that one can find the HS solution if the absolute value of the coefficients $|I_n|$ for $v_f(x)$ reach approximately $10^{-4} \sim 10^{-3}$ for $E_{\text{max}} \approx -0.25$ and also the reference solution if they reach order of $10^{-6} \sim 10^{-7}$ for $E_{\text{max}} \approx -0.05$ (Figure 24). We believe the reason why we could not find out the condition in the present section is that the decay rate of the Chebyshev coefficients is too rapid and provided only limited degrees to obtain the HS solution for $E_{\text{max}} \approx -0.25$. Hence, in the Appendix D we intentionally included the effect of the singular and non-regular properties into dependent variables by reformulating the variables $v_R$ (with a discontinuity) and $v_f$ (with a logarithmic dependence).

We believe our numerical accuracy of the reference solution is at least four significant figures based on the detail analyses done for various formulations in the present section, Sections B and C and Appendixes B and D. What we made efforts in the majority of the present work is to find a truncated-domain solution which is close to HS-solution for small $E_{\text{max}}$ but still close to the reference solution for small $E_{\text{max}}$ based on a single formulation. Among the variant formulations, the $v_f^{(m)}$-formulation of the present section not only reproduced the both HS- and reference- solutions but also provided the smallest relative error of $4 \times 10^{-5}$ from the reference solution (See Figure 17). This number corresponds with the relative error of $c_4$ from the reference eigenvalue. Hence, in Table 7 the value of $c_4$ is described with four significant figures and the rest of eigenvalues are with five (since $c_1$, $c_2$ and $c_3$ are more stable than $c_4$ for any formulations.)

| $E_{\text{max}}$ | $\beta$ | $\chi_{\text{esc}}$ | $F_n$ | $v_f(x = 1)$ |
|-----------------|---------|---------------------|-------|--------------|
| $-0.240$        | 8.17310 | 13.840              | $4.1 \times 10^{-4}$ | $2.1 \times 10^{-4}$ |
| $-0.225$        | 8.17460 | 13.845              | $4.5 \times 10^{-4}$ | $2.3 \times 10^{-4}$ |
| $-0.215$        | 8.17516 | 13.847              | $4.7 \times 10^{-4}$ | $2.4 \times 10^{-4}$ |
| $-0.200$        | 8.17526 | 13.850              | $4.6 \times 10^{-4}$ | $2.3 \times 10^{-4}$ |

Table 6: Eigenvalues of the reproduced HS solution with eigenvalues $c_1 = 9.10 \times 10^{-4}$ and $c_4 = 3.52 \times 10^{-2}$ for $N = 15$. [Heggie and Stevenson, 1988] reported the numerical values of their solution on $-1 \lesssim E \leq -0.317$. They mentioned that their Newton iteration method worked up to $E_{\text{max}} = -0.223$ and it could work beyond -0.223.

7. Conclusion

The self-similar OAFP equation to model core-collapsing star clusters is important in the sense that it provides a conceptual understanding of the late stage of the relaxation evolution of isotropic-spherical dense star clusters and useful physical parameters. The equation, however, has never been solved with an agreeable accuracy and the existing solutions were domain-truncated, whose domain is $-0.2 < E < 1$.

In the present paper, we employed a Gauss-Chebyshev pseudo-spectral method to find an accurate self-similar solution on the whole domain ($-1 < E < 0$). In Section 4 we provided the whole-domain solution whose degrees of Chebyshev polynomials is 70 and the minimum of the normalized Chebyshev coefficients for all of normalized independent variables reach order of $10^{-12}$. We obtained the corresponding eigenvalues more mathematically satisfactory
compared to the existing works. Also, we provided a semi-analytical form of the whole-domain solution whose degree of polynomials is at most 13. The eigenvalues results in the following physical parameters; the power-law index $\alpha$ is 2.230, the collapse rate $\xi = 3.64 \times 10^{-3}$ and the scaled escape energy $\chi_{esc} = 13.89$.

Since the whole-domain solution depends on the degree $N$ of polynomials in an undesirable way, in Section 5 we aimed at finding truncated-domain solutions that are less sensitive to the degree $N$. We obtained truncated-domain solutions stable up to 8 significant figures for $-0.08 \leq E_{\text{max}} < -0.04$ and the degree of the polynomials needs only $N \approx 25 \sim 55$. To find an optimal truncated-domain solution that is close to the whole-domain solution, we obtained the truncated-domain solutions with $\beta = \beta_0$ for $-0.05 \leq E_{\text{max}} < -0.02$ and those solutions are very stable up to their specific degrees of polynomials. At point $E_{\text{max}} = -0.03$, the truncated-domain solution has the same order of accuracy in $c_4^*$ as the whole-domain solution. Hence, the truncated-domain solution with $N = 65$ at $E_{\text{min}} = -0.03$ was compared to the whole-domain solution with $N = 70$; the relative deviation between the solutions are approximately $10^{-9}$ at certain energy-domain points.

Also, in Section 6 by modifying the regularized independent variable $v_J$, we improved the divergent asymptotic behavior in differentiations of the whole-domain and truncated-domain solutions. Also, the new regularization of $v_R$ and $v_F$ helped us to reproduce the Heggie-Stevenson’s solution around at $E_{\text{max}} = -0.225$ while it still can be the same as the whole domain solution around at $E_{\text{max}} = -0.05$ with accuracy of order of $10^{-5}$. We consider that one can find the Heggie-Stevenson solution as a result of low accuracy for small $E_{\text{max}}$ and the actual number of their significant figures is one.

We will discuss the physical properties and application of the ss-OAFP model in the follow-up papers (the second paper for its thermodynamic property and the third for application to globular clusters in Milky Way) though, we are planning to extend our numerical code to the post-core-collapse solution in future work. The present model can be meaningful only to the clusters that (i) have already reached in complete-core-collapsed state (if possible) and (ii) are approximately undergoing core collapse. Our numerical code can extend to those models such as the ss-OAFP model (Heggie and Stevenson, 1988) and the self-similar conductive gaseous model (Goodman, 1984) to a FP model.
Appendix A  The asymptotic approximations of function in the 4ODEs

We detail the asymptotic approximations of the regularized functions \( v_f(x) \) and \( v_J(x) \) (Appendix A.1) and \( v_J(x) + 1 \) (Appendix A.2).

A.1 The asymptotic approximation of the functions \( v_f(x) \) and \( v_J(x) \)

The function \( v_f(x) \) is important to determine the eigenvalue \( \beta \) and its asymptotic approximation is related to the boundary condition of the ss-OAFP system. Equation (2.14b) for \( v_f(x) \) does not include \( c_1 \) at first-order differential equation level and even the asymptotic approximation in first-order differentiation do not include \( c_1 \) around endpoints

\[
v_f(x \to -1) = \frac{4}{2\beta - 7} \left( \frac{x + 1}{2} \right)^L + \cdots, \quad (A.1)
\]

\[
v_f(x \to 1) = \frac{L}{4} \left[ 1 - \left( \frac{1 + x}{2} \right)^L \right] + \cdots. \quad (A.2)
\]

On one hand, the eigenvalue \( c_1 \) is associated with \( v_J(x) \) since equation (2.14d) for \( v_J(x) \) includes \( c_1 \) in its asymptotic approximation

\[
v_J(x \to -1) = -\left( \frac{x + 1}{2} \right)^L + \cdots, \quad (A.3)
\]

\[
v_J(x \to 1) = v_f(x \to 1) \frac{\beta}{2} = \frac{1}{2F_{bc}} \frac{2\beta - 3}{4\beta} \left( \frac{F_{bc} - c_1}{c_3} \right) \left[ 1 - \left( \frac{1 + x}{2} \right)^L \right] + \cdots. \quad (A.4)
\]

The relation between the eigenvalues and boundary conditions can be confirmed by fixing the value of \( \beta \) during iteration process and by seeing how the value of \( v_J(x) \) reaches the expected boundary numerical value, i.e. 0, for different values of \( \beta \) (See Appendix B.1).

A.2 The asymptotic approximation of the factor \([v_f(x) + 1]\)

Careful readers would have realized that 4ODEs (2.14a) - (2.14d) do not apparently include an equation to describe the asymptotic approximation of \( v_f(x) \) in a limit of \( x \to -1 \) while they include the corresponding approximations of \( v_I, v_J \) and \( v_G \). To see this, take a limit of \( x \to -1 \) in equation (2.14a), it would be obviously seen that the factor \([1 + v_f(x)]\) is proportional to \((1/2 + x/2)^{\beta}\). Hence, one may introduce a new dependent variable

\[
\eta_J(x) \equiv \frac{1 + v_f(x)}{(1/2)^{\beta L}}. \quad (A.5)
\]

By the new variable, Equations (2.14a) and (2.14d) can be rewritten as

\[
\left[ \frac{1 + x}{L} \frac{dv_J}{dx} + \left( \frac{1 + x}{2} \right)^{\beta L} \right] [v_f(x) + v_G(x)] + \left( \frac{1 + x}{2} \right)^L \frac{4\beta}{2\beta - 3} \left( v_f(x) \left( \frac{1 + x}{2} \right)^{\beta L} \right) - 1 + \beta c_2 e^{-v_f(x) v_J(x)} \left( \frac{1 + x}{2} \right)^L = 0,
\]

\[
\frac{1 + x}{L} v_G(x) \frac{dv_J}{dx} + \eta_J(x) \left( \frac{1 + x}{2} \right)^{\beta L} \left( 3 \frac{2\beta + 1}{4} v_G(x) + \frac{1 + x}{2L} v_G(x) \frac{dv_E}{dx} + 3 \frac{dv_G}{dx} \right) - \frac{1 + x}{2L} v_G(x) \frac{dv_E}{dx} \frac{3(2\beta - 1)}{2} - 6 \frac{dv_G}{dx} L \frac{dv_J}{dx} = 0. \quad (A.6a)
\]
Taking a limit of \( x \to -1 \) in equation (A.6a) provides the asymptotic approximation \( v_I(x \to -1) \):

\[
v_I(x \to -1) = -\frac{c_4^*}{c_2} \frac{(2\beta + 7)(6\beta - 3)}{\beta (2\beta - 7)(2\beta - 3)\beta + 1}.
\] (A.7)

It is to be noted that the expression for the asymptotic approximation (equation (A.7)) is correct in a limited sense that it is correct under a numerical limit in double precision; as explained in Appendix D, one may assume that the first term in equation (A.6a) is zero when it reaches order of machine precision; strictly speaking the first term should not vanish.

**Appendix B Stability analyses of the whole-domain solution**

The present Appendix shows the numerical stability of the reference solution (the whole-domain solution). We detail the dependence of the solution on eigenvalue \( \beta \) (Appendix B.1), the nodes of Fejér’s quadrature (Appendix B.2), the boundary condition for \( v_I(x) \) (Appendix B.3), and the numerical parameter \( L \) (Appendix B.4).

**B.1 Stability of the whole-domain solution against change in the eigenvalue \( \beta \)**

Throughout the present work the boundary value \( v_I(x = 1) \) is important since it determines the eigenvalue \( \beta = c_1/c_2 \) and the present Appendix shows its stability. Figure B.22 shows the \( \beta \)-dependence of the values of \( v_I(x = 1) \), \( |1 - c_1/c_{1o}| \) and \( |1 - c_4^*/c_{4o}^*| \). We solved the ss-OAFP system for different \( \beta \)-value with \( \pm 10^{-8} \) from the reference value \( \beta_0 \). The figure shows that all the values consistently converge to their reference values, \( c_1 \approx c_{1o} \), and \( c_4^* \approx c_{4o}^* \) when \( \beta \) reaches \( \beta_0 \). One can find the following approximate relationship in the order of value

\[
|1 - c_1/c_{1o}| \sim |1 - \beta/\beta_0| \sim \frac{v_I(x = 1)}{10^2} \sim \frac{|1 - c_4^*/c_{4o}^*|}{10^4}.
\] (B.1)

The relation implies that one would need 5 \( \sim \) 6 significant figures for the value of \( \beta \) and \( c_1 \) to determine one significant figure for of \( c_4^* \). It is also to be noted that all of the values are almost symmetric about \( \beta_0 = 8.1783711596581 \).

The Newton iteration did not work when the value of \( \beta \) deviate from the reference value \( \beta_0 \) by \( 1.3 \times 10^{-6} \% \) in the lower limit while the iteration was given up at the relative deviation of \( 2.5 \times 10^{-7} \% \) in the upper limit due to an expensive CPU cost. Hence, the minimum condition to let the iteration process work for the whole-domain solution is that one needs to specify the value of \( \beta \) with eight or nine significant figures \( (8.17837105 \leq \beta \lesssim 8.17837119) \) in the present approach.

![Figure B.22: Values of \( v_I(x = 1) \), \( |1 - c_1/c_{1o}| \) and \( |1 - c_4^*/c_{4o}^*| \) against change \( \Delta \beta \) in eigenvalue \( \beta \) around the reference value \( \beta_0 \) \( (\Delta \beta = \beta - \beta_0) \). Numerical parameters are \( II = 70, L = 1 \) and \( F_{BC} = 1 \).](image)

\(^{10}\)Over one million iterations were needed when the eigenvalue \( \beta \) deviates more than \( 1 \times 10^{-7} \% \) above the reference value \( \beta_0 \)
B.2 Stability of the whole-domain solution against change in the number of nodes in Fejér’s first-rule quadrature

Figure B.23 shows the dependence of the eigenvalues $c_1$ and $c_4$ and boundary value $v_f(x = 1)$ on the number of nodes in Fejér’s first-rule quadrature. The total number of nodes are chosen between $150$ to $10^4$ for fixed $\beta = \beta_0$ and $n = 70$; the Newton iteration did not work for the nodes less than $150$. The eigenvalues get stable for the nodes over $\sim 580$ points if the $c_1$ and $c_4$ are compared to the reference eigenvalues $c_{10}$ and $c_{40}^*$.

Also, the boundary value $v_f(x = 1)$ shows a good resemblance in qualitative behavior to the eigenvalues. The infinity norm of $\{a_{p}^{\text{old}}\} - \{a_{p}^{\text{new}}\}$ in Newton iteration process converges to values between $8 \times 10^{-14}$ and $2 \times 10^{-12}$ regardless of the number of the quadrature nodes.

![Graph showing dependence of eigenvalues and boundary value on quadrature nodes]

Figure B.23: Dependence of the eigenvalues $c_1$ and $c_4$ and boundary value $v_f(x = 1)$ on the node number of Fejér’s first-rule quadrature. The eigenvalues are compared to their reference eigenvalues $c_{10}$ and $c_{40}^*$ and the following numerical parameters are employed; $n = 70$, $L = 1$ and $F_{BC} = 1$.

B.3 Stability of the whole-domain solution against change in the boundary value $F_{BC}$

While the boundary condition $F(E = 1) = 1$ was employed in (Heggie and Stevenson, 1988; Takahashi, 1993), there is no specific reason to choose the value 1 unless one needs to change the central density. Hence, we employed different boundary values of $F(E = 1)$ to see the consistency of the eigenvalues. The left panel in Figure B.24 shows the values of $c_1$ and $c_4$ against the different values of $F_{BC}$ between 0.0001 and 10000. We found that the eigenvalues are proportional to the value $F_{BC}$ while the eigenvalues $\beta$ and $c_3$ are the same in their eight significant figures. Also, as the value of $F_{BC}$ increases, the same characteristics of data similar to the eigenvalues was found in the condition number of the Jacobian matrix for the 4ODEs and condition number reached $\sim 10^{12}$. Due to the linear relation between the eigenvalues and the boundary value, we divided the eigenvalues by $F_{BC}$ and compared to the reference values $c_{10}$ and $c_{40}^*$ acquired at $F_{BC} = 1$. We confirmed the eigenvalues ($c_1$ and $c_4^*$) are proportional to the value $F_{BC}$ with accuracy of $\sim 10^{-8}$ for $c_4$ and $\sim 10^{-13}$ for $c_4^*$ while the high condition number did not interfere the accuracies (Figure B.24 right panel).
have an numerically intrinsic property against change in $F$ where condition number does not change significantly against change in $F$ system is proportional to $F$ taken. ($F_{BC} = 1$, $L = 1$ and $N = 70$).

To avoid the significant change in the condition number for high values of $F_{BC}$, we regularized the ss-OAFP system by dividing the function $F(E)$ by $F_{BC}$. This regularization corresponds with that only the density $D(E)$ in the system is proportional to $F_{BC}$. We again solved the regularized ss-OAFP system for different $F_{BC}$. As expected, the condition number does not change significantly against change in $F_{BC}$ (Figure B.25). Also, the eigenvalues are stable against change in $F_{BC}$: $\Theta(10^{-14}) < |1 - c_1/F_{BC}/c_{1o}| < \Theta(10^{-13})$ and $\Theta(10^{-10}) < |1 - c_4/F_{BC}/c_{4o}| < \Theta(10^{-8})$.

In conclusion, (i) the eigenvalues are less sensitive to high condition number (ii) the eigenvalues $\beta$ (or $\alpha$) and $c_3$ have an numerically intrinsic property against change in $F_{BC}$ while $c_1$, $c_2$ and $c_4$ are extrinsic:

$$c_1(F_{BC}) \propto \left( c_{1o} + \Theta \left( 10^{-13} \right) \right) F_{BC}, \quad c_2(F_{BC}) \propto \left( c_{2o} + \Theta \left( 10^{-8} \right) \right) F_{BC}, \quad \beta(F_{BC}) = \beta_o + \Theta(10^{-8}), \quad (B.2)$$

where $-10^{-4} < F_{BC} < 10^{4}$.

Figure B.24: (Left panel) Dependence of the eigenvalues $c_1$ and $c_4$ on the boundary condition $F(x = 1)$ compared to the condition number of the Jacobian matrix for the 4ODEs. (Right panel) Relative deviation of the regularized eigenvalues $c_1/F_{BC}$ and $c_4/F_{BC}$ from the reference eigenvalues $c_{1o}$ and $c_{4o}$ taken. ($F_{BC} = 1$, $L = 1$ and $N = 70$).

Figure B.25: (Left panel) Condition number of the Jacobian matrix for the 4ODEs regularized through $F(E)/F_{BC}$. (Right panel) Relative deviation of the regularized eigenvalues $c_1/F_{BC}$ and $c_4/F_{BC}$ from the reference eigenvalues $c_{1o}$ and $c_{4o}$ for the 4ODEs regularized through $F(E)/F_{BC}$. ($F_{BC} = 1$, $L = 1$ and $N = 70$).
contracted-domain formulation provides slow decay of Chebyshev coefficients due to endpoint singularity. Newton iteration converged only when the Chebyshev coefficients reach as low as of the order of \(10^{-9}\). We found spectral solutions of the ss-OAFP system with the mapping parameters as low as of the order of \(10^{-10}\). Solutions with low accuracy cannot provide the reference solution.

### Stability of the whole-domain solution against change in the parameter \(L\)

The parameter-\(L\)-dependence of the solutions provides an understanding of the property of the ss-OAFP equation. We found spectral solutions of the ss-OAFP system with the mapping parameters \(L = 1/2\) and \(L = 3/4\) (Table B.4) while the values greater than \(L\) were hard to work with. In this sense, we call a solution with \(L < 1\) the ‘contracted-domain’ solution of the ss-OAFP system. The contracted-domain solutions provide some advantages over the reference solution; they are still whole-domain solutions (since they are not truncated) while need less degrees of polynomials and significant digits of eigenvalue \(\beta\) compared to the reference solution with \(L = 1\). The convergence rate of Chebyshev coefficients for ‘large \(n\)’ is apparently characterized by \(a_n \propto \left(\frac{1}{n}\right)^{1+2L}\) due to the end-point singularity \((1 \pm x)^2\) at branch points \(x \pm 1\). The characteristics of the low convergence rate for the function \(v_F(x)\) clearly appears when the degree \(N\) is greater than 65 and 75 for \(L = 0.75\) and \(L = 0.5\) respectively. The Newton iteration converged only when the Chebyshev coefficients reach as low as of the order of \(10^{-9}\) for \(L = 3/4\) and \(10^{-6}\) for \(L = 1/2\). Recalling the Newton iteration worked only when the Chebyshev coefficients of the whole-domain solution with \(L = 1\) reach order of \(10^{-12}\) (Table 3), we infer the technical rule for the relationship between the coefficients and iteration method that Newton iteration method could work when the minimum value of Chebyshev coefficients reaches as low as of the order of \(10^{-12L}\).

| \(N\)  | Eigenvalue \(\beta\) | \(1 - c_1/c_{1n}\) | \(1 - c_2/c_{2n}\) | \(v_F(x = 1)\) | \(N\)  | Eigenvalue \(\beta\) | \(1 - c_1/c_{1n}\) | \(1 - c_2/c_{2n}\) | \(v_F(x = 1)\) |
|-------|---------------------|---------------------|---------------------|-----------------|-------|---------------------|---------------------|---------------------|------------------|
| 60    | 8.178371160         | \(2.1 \times 10^{-10}\) | \(6.6 \times 10^{-7}\) | \(9.4 \times 10^{-9}\) | 35    | 8.17837104         | \(4.9 \times 10^{-8}\) | \(2.3 \times 10^{-6}\) | \(2.1 \times 10^{-10}\) |
| 55    | 8.178371160         | \(2.1 \times 10^{-10}\) | \(1.3 \times 10^{-6}\) | \(1.7 \times 10^{-9}\) | 50    | 8.178371160         | \(9.8 \times 10^{-6}\) | \(1.3 \times 10^{-9}\) | 

Table B.7: Numerical results for the contracted-domain formulation with \(L = 1\) and \(F_{BC} = 1\).

---

1\(^1\)Choosing high numbers of \(L\) (e.g. \(L = 1.5\) and \(L = 2\)) resulted in much more difficulty in Newton interaction convergence. We had to shorten the Newton steps from 1 to a fraction less than 0.01. On one hand, low numbers of \(L\) less than 1/2 did not work; this is perhaps because contracted-domain formulation provides slow decay of Chebyshev coefficients, accordingly low accuracy of the solutions. As discussed in Section 6.3, solutions with low accuracy cannot provide the reference solution.

1\(^2\)This slow convergence does not originate from the branch point. This is since the regularized function \(v_F\) behaves like \(c_1(1 + b(0.5 + 0.5x)^2)\) where \(b\) is constant. In fact, as we increased the digits of \(\beta\), coefficients decay rapidly and reach order of \(10^{-11}\) a the maximum degree \((N = 65)\) on table B.36.
the extrapolated DF at and $10^{-c}$. To understand the structure, one must refer to the values of the infinity norms of the derivatives (Table C.8). In the table, the set of parameters $(c, d) = (1, 10)$ provided the best accuracy in the sense that $\nu_f(x) = 1$ is $7.3 \times 10^{-12}$. The combination $(c, d) = (\infty, NA)$ means the extrapolated function is constant.

| $c$ | $d$ | $|1-c_1/c_\text{ex}|$ | $|1-c_2/c_\text{ex}|$ | $\nu_f(x=1)$ |
|-----|-----|-----------------|-----------------|-------------|
| $\infty$ | RA | $1.2 \times 10^{-11}$ | $5.3 \times 10^{-10}$ | $1.0 \times 10^{-11}$ |
| 10 | 1 | $1.2 \times 10^{-11}$ | $5.3 \times 10^{-10}$ | $1.0 \times 10^{-11}$ |
| 5 | 1 | $7.7 \times 10^{-10}$ | $5.3 \times 10^{-10}$ | $1.0 \times 10^{-11}$ |
| 1 | 1 | $1.0 \times 10^{-9}$ | $4.5 \times 10^{-10}$ | $9.8 \times 10^{-12}$ |
| 0.1 | 1 | $1.6 \times 10^{-10}$ | $9.6 \times 10^{-10}$ | $1.2 \times 10^{-11}$ |
| 0.01 | 1 | $1.4 \times 10^{-11}$ | $9.1 \times 10^{-10}$ | $1.1 \times 10^{-11}$ |

Table C.8: Numerical results for different extrapolated DF ($L = 1$ and $F_{BC} = 1$). The eigenvalues are compared to $c_{\text{ex}}$ and $c_{\text{ex}}^* \equiv 3.03155223 \times 10^{-1} (4 c_{\text{ex}}^* + 1 \times 10^{-1})$ obtained for $(c, d) = (1, 10)$ and the value of $\nu_f(x=1)$ is $7.3 \times 10^{-12}$. The combination $(c, d) = (\infty, NA)$ means the extrapolated function is constant.

Figure B.26: (Left panel) Chebyshev coefficients of $\nu_f(x)$ for $L = 0.75$ in the following cases (a) $N = 60$ and $\beta = 8.178371160$ and (ii) $N = 65$ and $\beta = 8.178371275$. The iteration method for the latter did not work satisfactorily stalling around $|1-c_1/c_\text{ex}| = 6.0 \times 10^{-3}$ and $\nu_f = 0.4 \times 10^{-2}$. Yet, it is shown here for comparison. (Right panel) Chebyshev coefficients of $\nu_f(x)$ for $L = 0.5$ in the following cases (a) $N = 35$ and $\beta = 8.178371160$ and (ii) $N = 50$ and $\beta = 8.1783712$.

Appendix C Stability of the truncated-domain solution against change in extrapolated DF

We found that the accuracy of the truncated-domain solutions is little sensitive to the expression of the extrapolated DF (equation (2.18)). We compared the effects of change in the extrapolated DF on the eigenvalues (Table C.8). In the table, the set of parameters $(c, d) = (1, 10)$ provided the best accuracy in the sense that $\nu_f(x)$ reaches a minimum of $7.3 \times 10^{-12}$ among the chosen parameters, hence we compared the rest of the eigenvalues to the eigenvalues obtained for $(c, d) = (1, 10)$.

For combinations of different sets of parameters among $0.01 < c < 10$ and $0.1 < d < 10$, the relative deviation of eigenvalues are of the order of $10^{-13}$ in $c_1$ compared to its reference value and $10^{-9}$ in $c_2$ at most holding small values of $\nu_f(x = 1) \approx 1 \times 10^{-11}$. Even the effect of discontinuity in derivative of the extrapolated DF at $E = E_{\text{min}}$ ($c \to \infty$) is not significant compared to the effect of large value of $c (= 5, 10)$.

Appendix D Why is Newton iteration method hard to work for the ss-OAFP system?

The difficulty in numerical integration of the ss-OAFP system may originate from the complicated mathematical structure of the 4ODEs (2.14a) - (2.14d). To understand the structure, one must refer to the values of the infinity norms of the difference between ‘new’ and ‘old’ Chebyshev coefficients associated with 4ODE in the process...
of Newton iteration method; we found the following values were universally output for all the truncated-domain-, whole-domain- and contracted-domain-formulations

\[
\left\| \left[F_n\right]_{\text{new}} - \left[F_n\right]_{\text{old}} \right\|_\infty \approx \Theta \left(10^{-13}\right),  
\left\| [G_n]_{\text{new}} - [G_n]_{\text{old}} \right\|_\infty \sim \left\| [J_n]_{\text{new}} - [J_n]_{\text{old}} \right\|_\infty \approx \Theta \left(10^{-16}\right) \approx \text{eps}.  
\left\| c_1 - c_1^\text{old} \right\|_\infty \approx \Theta \left(10^{-16}\right),  
\left\| c_4 - c_4^\text{old} \right\|_\infty \approx \Theta \left(10^{-13}\right),
\]

where \(\text{eps}\) means the machine precision of MATLAB \((\approx 2.2 \times 10^{-16})\). Only the norms for \([F_n]\) and \(c_4^\text{old}\) are approximately \(10^3\) higher than the others, implying that equation (2.14a) associated with \(v_f\) may have a mathematically internal conflict. Equation (2.14a) has the following mathematical structure

\[
4L \left( \frac{1 + x}{2} \right)^{\beta + 1} \frac{d}{dx}(\eta(x;v_G,v_I)) + \left( \frac{1 + x}{2} \right)^\beta \mu(x;v_G,v_I) + c_1 e^{-\tau_f(x)(v_f + 1)} = 0,
\]

where \(\eta(x;v_G,v_I)\) and \(\mu(x;v_G,v_I)\) are functions of \(v_G(x)\) and \(v_f(x)\) and their absolute values are order of unity on the whole domain. We explain possible relationships of the Newton’s method with the mathematical structures focusing on problems in equation (D.4) at limits of \(x \to 1\) (Section D.1) and \(x \to -1\) (Section D.2) for \(v_f\), in the derivative of \(v_f\) (Section D.3). Also, we show equation (D.4) is important in integration of the ODE at equation level (Section D.4) and explain some other numerical difficulties in integrating the ss-OAFP system (Section D.5).

### D.1 A problem in solving equation for \(v_f\) at limit of \(x \to 1\)

A problem in solving equation (D.4) is that the factor \(c_1\) forms a gap between terms at equation level. First, take a limit of \(x \to 1\) in equation (D.4)

\[
4c_3 \frac{d}{dx}(x \to 1) + \beta c_3 - 1 + \frac{c_1}{F_{BC}} = 0.
\]

where \(L = 1\) is chosen for simplicity. In equation (D.5), since \(\beta c_3\) is approximately unity \((\approx 0.59)\), the largest gap is the difference between the third- and fourth- terms that is order of \(10^{-3}\) regardless of the value of \(F_{BC}\) (since \(c_1\) is proportional to \(F_{BC}\) as explained in Appendix B.3). Hence, the equation can turn into an overdetermined problem at equation level greater than order of \(10^{-3}\), which would be one of the reasons why the Newton method is hard to work. Also, the smallness of the gap could explain the large value of the norms for \(c_4\) and \([F_n]\) (equation (D.1)); the boundary value \(F_{BC}(= \ln(\tau_f(x = 1)))\) is effective only up to 13 digits in the sense that it consistently determine the differentiation \(\frac{d}{dx}\); digits more than 13 would be counted as rounding error due to the gap in \(1 - c_1/F_{BC}\). Due to this mathematical structure, we call order of \(10^{-13}\) ‘practical’ machine precision as comparison to Matlab machine precision \(\approx 2.2 \times 10^{-16}\).

### D.2 A problem in solving equation for \(v_f\) at limit of \(x \to -1\)

Another problem in solving equation (D.4) is that the factors \(\left(\frac{1 + x}{2}\right)^\beta\), \(\frac{d}{dx}(x \to -1)\), and \((v_f + 1)\) form power law profiles as \(x \to -1\). Take a limit of \(x \to -1\) in equation (D.4)

\[
\frac{6(2\beta - 1)}{(2\beta - 7)(\beta + 1)} \left( \frac{1 + x}{2} \right)^{\beta + 1} \frac{d}{dx}(x \to -1) + \left( \frac{1 + x}{2} \right)^\beta \frac{4\beta^2 - 4\beta + 37}{(2\beta - 7)(\beta + 1)(2\beta - 3)} + \frac{c_1}{c_4} (v_f(x \to -1) + 1) = 0.
\]

where the second line represents the power-law dependence of each term; the differentiation \(\frac{d}{dx}(x \to -1)\) behaves like \(\left(\frac{1 + x}{2}\right)^{\beta - 1}\) according to the result of Section D.2 and \((v_f(x \to -1) + 1)\) is explicitly proportional to \(\left(\frac{1 + x}{2}\right)\) as explained in Section A.2. The first term in equation (D.6) describes the ‘time-evolution’ equation with respect to \(x\) in the sense that the equation is first order in differentiation or an initial value problem. Hence, one may consider the
first term in equation (D.6) is important to determine the interval on which one can solve the equation satisfactorily beginning from \( x = 1 \). The factor \( (\frac{1}{2})^{2p} \), of course, does not contribute to the numerical integration of equation (D.6) if it reaches order of machine precision \( \sim 10^{-16} \). Hence, by equating the first term to machine precision of Matlab

\[
\frac{6(2^{b-1})}{(2p-q+1)} \left( \frac{1}{2} \right)^{q+1} = 2.2 \times 10^{-16},
\]

where \( \frac{1}{c_1} = \frac{dE(x)}{dx}(x = -1) \), we can estimate the lower limit of the interval is \( x_{\text{min}} \approx -0.82 \) (or \( E_{\text{const}} \approx -0.09 \)). This discussion implies that one can not effectively determine the value of \( c_4^* \) at equation level with a numerical accuracy greater than \( 10^{-9} (\approx (0.5 + 0.5x_{\text{const}}^0)) \) (Since \( c_4^* \) is related to the third term in equation (D.6)). This order of values well reflects the result in Figure B.22 in which \( c_4^* \) is stable up to order of \( 10^{-9} \) against change in \( \beta \). Also, it may explain the reason that the relative error of the optimal truncated solution to the reference solution is at most \( 10^{-9} \) as shown in Figure 14.

### D.3 Absolute values of terms in equation for \( \nu_F \) and classification of truncated-domain solutions

The present Appendix compares the orders of values of terms in equation (2.14a) to detail the mathematical structures and explains the classification of the truncated-domain solutions. Figure (D.27) depicts the absolute values of the first through third terms in equation (2.14a) together with relative deviation \( | 1 - \nu_F(x)/\ln[c_j^*] | \) and practical machine precision (~ \( 10^{-15} \)). Also, the sum of the three terms is depicted. The absolute value of the first term reaches the total of the three terms approximately at \( E \approx -0.05 \) while the second and third terms reach it at \( E \approx -0.005 \). Since we expect that we can satisfactorily solve equation (2.14b) at \( E < -0.05 \), we name the solutions that we can obtain on interval \( -1 < E < -0.05 \) as the 'stable solution'. This well reflects the result for the reference solution in Figure 7 in which the asymptotic behavior in differentiation of \( \nu_F(x) \) loses accuracy at \( E > -0.05 \). Also, the truncated-domain solution holds accuracy beyond \( E = -0.05 \) as shown in Figure 13. On one hand, we call solutions obtained for \( -0.05 < E_{\text{max}} < -0.005 \) as 'semi-stable' solutions. This is since the second and third terms can determine the value of \( \nu_F \) in place of the derivative of \( \nu_F \), which result in that the accuracy of \( c_4 \) does not change with increasing \( E_{\text{max}} \). The practical machine precision well describes the hold in accuracy of \( c_4 \). Lastly, beyond \( E_{\text{max}} > -0.005 \) there does not exist a meaningful term with over machine precision, hence we can not solve the equation. We believe the cause of the numerical instability is determined with the relationship between \( \nu_F \) and \( \nu_Q \) as shown in Appendix E.3; one can not integrate the \( Q \)-integral for \( E_{\text{max}} > -0.005 \). We call the solutions that we obtain for \( E_{\text{max}} > -0.005 \) as the 'unstable' solution.

We can show Figure (D.27) has two more important characteristics of the solution focusing on the second and third terms. First, we can obtain the solution that is close to the reference solution and HS solution only for \( E_{\text{max}} < -0.25 \) (Section 5.1). This nature comes out when the second and third terms reach the same order of value and cancel out each other. Around at \( E = -0.25 \), the absolute values of terms are order of \( 10^{-5} \). This reflects the order of value to which \( c_4 \) is stable against change in \( \beta \) (Figure B.1). Also, this infers that, in order to make Newton’s method work, one must prepare an accurate 'initial guess' for solution whose accuracy is order of \( 10^{-5} \) to effectively determine the first digit of the eigenvalue \( c_4 \), which would make the Newton method hard to work. Another important characteristics is that the value of \( | 1 - \nu_F(x)/\ln[c_j^*] | \) multiplied by the maximum value of the third term is greater than that of the first term. This means the reason why \( \nu_F \) can behave like a constant function as \( E \to 0 \) is not because the first term reaches machine precision and lose its significance. This property is important to secure the consistency of our solution.
Appendices D.1, D.2, and D.3 only focused on equation (2.14a) among the 4ODEs; to emphasize the importance of the equation we compare the equation to the rest of the equations. To analyze the mathematical structures of the 4ODEs, we rewrite the 4ODE with new functions for convenience

\[ O_1(E) \equiv 0, \quad O_2(E) \equiv 0, \quad O_3(E) \equiv 0, \quad O_4(E) \equiv 0, \] (D.7)

where \( O_1(x) \) through \( O_4(x) \) are functions on the left hand sides of equations (2.14a)-(2.14d). In Figure D.28 (Top Panel) we depict the absolute values of \( O_1 \) through \( O_4 \) at Gauss-Chebyshev nodes on the whole domain. In the figure only \( O_1 \) is regularized by dividing \( O_1 \) by \( c_1 \). All the functions \( O_1(x) \) through \( O_4(x) \) lose accuracy on the unstable region increasing their absolute values as \( x \to -1 \). One can see the absolute values of \( O_3(x) \) and \( O_4(x) \) are very alike around the semi-stable region, which well describes the fact that \( O_1 \) and \( O_4 \) ’switch’ their roles; they determine \( v_J \) and \( v_F \) as \( x \to -1 \) while \( v_F \) and \( v_J \) as \( x \to +1 \), as explained in Appendix A.2. Since the absolute values of the functions in Figure D.28 are not regularized consistently to compare their absolute values, Figure D.28 (Bottom panel) shows the regularized functions \( O_1(x) - O_4(x) \); we regularized the absolute values of \( O_1(x) - O_4(x) \) by dividing each function by the term whose values is the largest in the corresponding equation in a limit of \( E \to -1 \). As expected, \( O_2(x) \), \( O_3(x) \) and \( O_4(x) \) stalls near the machine precision except for the unstable region. On one hand, \( O_1(x) \) significantly loses accuracy as \( E \) approaches 0 and it reflects well the relation with \( | 1 - v_F(x)/\ln[c_1^*] | \) in a similar way to Figure 7. In Figure D.28 (Bottom Panel) \( | 1 - v_F(x)/\ln[c_1^*] | \) is also regularized by the same term for equation (2.14a). This result signifies the dominant effect of equation (2.14a) to determine the accuracy of the 4ODE as \( x \to -1 \).
Figure D.28: Values of the regularized functions $O_1$ through $O_4$ in 4ODEs (2.14a)-(2.14d) at Gauss-Chebyshev nodes. (Top) only $O_1$ is divided by $c_1$ (Bottom panel) All the functions are normalized so that the largest value of terms in each equation approaches unity as $E \to -1$. The horizontal lines represent limits of precision. On the bottom panel, $|1 - v_f(x)/\ln[c_1^*]|$ is further regularized by the first term of equation (2.14a).

D.5 Some problems in numerical integration of ss-OAFP system

Lastly, we summarize the three more difficulties that we faced in numerical integration of the ss-OAFP system.

(i) The effect of discontinuity in solutions was an issue for truncated-domain formulation; see some discussion in Appendix E which would have made harder guessing a ‘good’ initial solutions in Newton iteration process. (ii) We also employed the Radau-Chebyshev spectral method and boundary condition $v_f(x = 1) = 0$ so that we can determine a spectral solution when the value $v_f(x = -1)$ is minimized by changing the value of $\beta$, but such solution included very strong discontinuous property in both of whole- and truncated-domain solutions. This could be due to the gap $|1 - c_1/c_{1\text{ex}}|/v_f(x = 1) \approx 10^{-2}$ that prevents us from imposing the boundary condition $v_f(x = 1) = 0$. If one would like to determine 15 significant digits for $v_f(x = 1)$, one must find 17 significant digits of eigenvalue $c_1$, which is beyond the limit of double-precision. (iii) The Newton iteration was hard to work for truncated-domain solutions for $-0.1 < E_{\text{max}} < -0.4$. This would simply reflect the fact that an extrapolation of DF by the power-law profile on the domain is not a proper treatment.
Figure E.29: Chebyshev Coefficients of $Q$-integral for a discontinuous test function $v_R = 0.1\Theta(0.50 + 0.5|x - x_{\text{trans}}|) + 1$. Recall $E = -(0.5 + 0.5x)^2$, here $L = 1$.

**Appendix E  Solving part of the ss-OAFP system with a fixed independent variable**

The present Appendix shows the results of numerical integration of part of the ss-OAFP system that we solved with some fixed independent variables (without self-consistently solving the entire system). Appendices E.1 and E.2 show the effect of discontinuities in independent variable on the convergence rate of Chebyshev coefficients for integration of the Poisson equation and $Q$-integral respectively. The results possibly explain the slow convergence rate of the truncated-domain solutions. Also, Appendix E.3 shows that the numerical instability (reported in Section 4.3) does not occur for integration of 4ODE with a fixed $\{Q_n\}$. This infers that the instability may originate from the relation between the 4ODE and $Q$-integral rather than 4ODE itself.

**E.1  $Q$-integral with fixed discontinuous $v_R$**

In the present work, all the spectral solutions that we obtained with truncated-domain formulations include a certain flattening in Chebyshev coefficients as index $n$ becomes large. To find a possible cause of the flattening, we calculated the Chebyshev coefficients of the $Q$-integral for the following discontinuous test function for $v_R$

$$v_R^{(\text{tes})} = 0.1 \Theta\left(\frac{1 + x - x_{\text{trans}}}{2}\right) + 1,$$

(E.1)

where $x_{\text{trans}}$ is a small positive number and $\Theta(\cdot)$ the Heaviside function. When the point of discontinuity is relatively close to order of unity, say $x_{\text{trans}} = 0.1$, the Chebyshev coefficients for $Q$-integral slowly decay like $\sim 1/n^2$ for large $n$ (Left panel, Figure E.29) in a similar way to Chebyshev coefficients for discontinuous functions (e.g. [Boyd, 2001; Xiang, 2013]). However, once the discontinuity point more closely approaches the end point of the domain such as $x_{\text{trans}} = 0.001$ (Right panel, Figure E.29), the coefficients show a flattening with large $n$. Since for $x_{\text{trans}} = 0.001$ the majority of domain is covered by a constant function, one can find a rapid decay for small $n$. One can also observe for very large $n$ that the coefficients reach the same order of value regardless of the value of $x_{\text{trans}}$.

**E.2  Poisson equation with fixed discontinuous $v_D$**

In Section 6.2 the modification of function from $v_R$ to $v_R^{(m)}$ changes the numerical result significantly; especially, a slow decay of the Chebyshev coefficients are observed. This also may be associated with the effect of
discontinuous behavior of independent variable, this time, \( v_D \) on \( v_R^{(m)} \) in Poisson equation. The following test function is employed

\[
v_D^{(\text{tes})} = v_D(x) \left( A_{\text{trans}} \Theta \left[ \frac{1 + x - 0.001}{2} \right] + 1 \right),
\]

(E.2)

where \( A_{\text{trans}} \) is a small positive number and \( v_D(x) \) is the whole-domain solution with \( N = 70 \) (obtained in Section 4). The Poisson equation was solved with the fixed \( v_D^{(\text{tes})} \) and different \( A_{\text{trans}} \). When the value of \( A_{\text{trans}} \) is very small such as 0.00001, the solutions \( \sqrt{v_R^{(m)}} \) and \( \exp(v_R) \sqrt{0.5 + 0.5x} \) (that are supposed to be the same if the Poisson equation is successfully integrated) are compared (Right panel, Figure E.30). The difference appears only at order of \( 10^{-4} \). On one hand, when \( A_{\text{trans}} \) is close to unity such as 0.1, not only the difference appears in the value of coefficients at order of 0.1 but also \( v_R \) showed a slower decay compared to \( v_R^{(m)} \) (Left Panel, Figure E.30).

Figure E.30: Chebyshev Coefficients of \( \sqrt{v_R^{(m)}} \) and \( \exp(v_R) \sqrt{0.5 + 0.5x} \) for discontinuous test function \( v_D^{(\text{tes})} \). The dashed guidelines are added only for measure of slow decays.

E.3 ss-OAFP equation with Fixed \( \{Q_n\} \)

To test if the origin of numerical instability in integration of the ss-OAFP system is only from the large change in value of independent variables due to the factor \( (-E)^\delta \) in 4ODE (1.7a) - (1.7d), the present section shows a result of solving the 4ODE and \( Q \)-integral for fixed coefficients \( \{Q_n\} \). As test coefficients the whole-domain solution with \( N = 70 \) (depicted in Figure 3) is employed with \( \beta = \beta_o \). For the fixed \( Q_n \), the Chebyshev coefficients of the spectral solution show very stable behavior (Figure E.31); coefficients \( \{F_n\} \), \( \{G_n\} \), \( \{I_n\} \) and \( \{J_n\} \) reach order of \( 10^{-15} \) around at \( n = 90 \) and show a flattening even up to \( n = 1000 \) due to the rounding error. Also, the values of \( c_1 \) and \( c_4 \) at different degrees \( N = 10, 20, \ldots, 900 \) are compared to the corresponding values at \( N = 1000 \) in Figure E.32. The deviations in
$c_1$ and the value of $v_j(x = -1)$ reach almost machine precision around at $N = 400$ while the deviation in $c_4$ reaches order of $10^{-13}$ at $N = 1000$. The deviation in $c_1$ flattens with order of $10^{-13}$ which appears on degrees $70 \leq N \leq 300$; this would reflect the fact that the minimum absolute value of test coefficients $\{Q_n\}$ is order of $10^{-13}$ and to gain more accurate solution, perhaps, one needs more Chebyshev nodes near the endpoints.

The result of the present Appendix is important to consider the cause of the numerical instability. The difference between the 4ODE with fixed $\{Q_n\}$ and those with unfixed may appear in equation (A.6b). In the equation as $x \to -1$ the differentiations of $v_F$, $v_Q$ and $v_J$ becomes significant compared to the rest of factors and terms. For fixed $v_Q$, one can determine $v_F$ in the equation while $v_J$ is determined from equation (2.14a). In case of non-fixed $v_Q$, as one can see the form of the $Q$-integral, the value of the integral is undetermined beyond $E \approx -0.06$ at which $(-E)^\sigma$ reaches machine precision. This infers $v_Q$ must be also further determined as $E \to 0$ with an extra equation. Hence, for non-fixed $v_Q$ equation (A.6b) becomes an underdetermined problem at $E < -0.06$. 


Figure E.31: Absolute values of Chebyshev coefficients for regularized functions for which ss-OAFP system (only 4ODE and Q-integral) was solved with fixed $\{g_{n}\}$ on the whole domain ($\Omega = 1000$, $L = 1$, $F_{BC} = 1$).
Appendix F Relation between the reference- and Heggie-Stevenson solutions

The reproduced HS solution in Section 6.3 is not satisfactory due to the limited degrees \( N \), hence the present Appendix tests variants of modified independent variables aiming to detail distinct conditions to systematically find the reference solution and the HS solution even for high degree of polynomials. We employed formulations similar to the formulation of (Heggie and Stevenson, 1988). Appendix F.1 shows the results from the ss-OAFP system we solved for modified variables \( v_R, v_I, v_G, \) and \( v_J \) and Appendix F.2 for \( v_J \). The latter provided the reference- and HS-solution with reasonable accuracy but still available degrees are limited in the same way as in Section 6.3. This encouraged us to apply to the ss-OAFP system combinations of modified variables employed in Appendix F.2 and Section 6.3 (Appendix F.3). This treatment reproduces the HS- and reference- solutions even for high (~ 200) degree of polynomials.

F.1 Modifying variables \( v_R, v_I, v_G, \) and \( v_J \)

We first examined formulations similar to (Heggie and Stevenson, 1988)’s formulation but they are not useful, rather it increases the condition number for the 4ODEs and \( Q \)-integral. First, we introduce the following modified independent variables

\[
\begin{align*}
    v_R^{(m2)}(x) &\equiv (0.5 + 0.5x)^2 v_R(x), \\
    v_I^{(m)}(x) &\equiv (0.5 + 0.5x)^b v_I(x), \quad (b \geq 0) \\
    v_G^{(m)}(x) &\equiv (0.5 + 0.5x)^b v_G(x), \\
    v_J^{(m2)}(x) &\equiv (0.5 + 0.5x)^b v_J(x).
\end{align*}
\]

If one applies all the modified functions to the ss-OAFP model, the new system is very similar to the HS’s formulation. We found the whole- and truncated- domain solutions for the ss-OAFP system with the modified functions using the
procedure of Section 3.2, after we tested many different combinations of the modified variables. The first three modified independent variables \((v_R^{(m)}, v_{l1}^{(m)})\) did not change the results almost at all from the reference solution. On one hand, the fourth modification \((v_{G}^{(m)})\) provided very high condition numbers. For \(b = 1\), we found a spectral solution on whole domain and it is almost identical to the reference solution, while the condition number was high \(\sim 10^{11}\). For \(b > 1\), the Newton method was hard to work due to higher condition numbers on whole domain. On one hand, we found solutions on truncated domain with \(b = \beta\) near \(E_{\text{max}} = -0.225\). These numerical parameters are close to those used in HS’s work. However, the condition number is still high \((\sim 10^{11})\). Solutions with high condition numbers (close to a reciprocal of machine precision) are generally less trustful. Also, as done in [Heggie and Stevenson, 1988], we had to shorten the Newton step (whose default is unity) to less than 0.1 to find those solutions using Newton iteration method, which costed an unfeasible CPU time.

### F.2 Modifying variables \(v_F\)

We employed the following modification that provided a sensible condition to find the both HS- and reference-solutions only by controlling \(E_{\text{max}}\)

\[
v_F^{(m)} = \ln \left( \exp[v_F] \left[ \frac{1 + x^j}{2} \right]^n \right).
\]

(F.2)

We solved the ss-OAFP system for \(v_F^{(m)}\) and unmodified variables \(v_S, v_G, v_l, v_j, v_R\) using the procedure of Section 3.2. In a similar way to the modification \(v_{R}^{(m)}\) (Sections 6.2 and 6.3), the spectral solution based on \(v_F^{(m)}\)-formulation is close to the HS solution for large \(E_{\text{max}}\) while it also can be close to the reference solution for small \(E_{\text{max}}\) (See Table 6.9 in which \((v_F^{(m)}, v_R)\) is the corresponding formula). Due to the logarithmic endpoint singularity of \(v_F^{(m)}\), the degree \(l\) can reach high number \((\sim 500)\) and the Chebyshev coefficients \([F_n^{(m)}]\) for \(v_F^{(m)}\) show slow decays for both large- and small- \(E_{\text{max}}\). A more distinct slow decays appears in Chebyshev coefficients \([I_n]\) for \(v_F\) especially when \(E_{\text{max}}\) is large (Figure F.34). Interestingly, the value of \(v_F(x = 1)\) is still order of \(10^{-4}\) for large \(E_{\text{max}}\) and the same order as the value given by the modification \(v_{R}^{(m)}\) in Sections 6.2 and 6.3. This infers that the HS solution may be obtained when a numerical scheme has a low accuracy and \(E_{\text{max}}\) is small \(\approx -0.225\). This condition could occur to solutions obtained by the spectral method when Chebyshev coefficients decays slowly by intentionally including the non-analytic and non-regular properties in the solutions. The modification \(v_F^{(m)}\) provides the HS solution only for small \(l\), hence one may further be able to find the HS solution with larger \(l\) by controlling the singularities.

### F.3 Modified variables \((v_R^{(m)}, v_F^{(m)})\)

Double modification \((v_R^{(m)}, v_F^{(m)})\) provides a proper feature of ss-OAFP solutions in the sense that one can obtain the reference- and HS- solutions for high degrees \((l \gtrsim 200)\). The results of Appendix F.2 showed that slowing the rapid decay is also a key to find the both HS- and reference solutions based on a single formulation. Hence, we combined the two formulations of Appendix F.2 and Section 6.3 that provided consistent characteristics of solutions. As expected, we found the HS- and reference solution only by controlling the value of \(E_{\text{max}}\) based on double modification \((v_R^{(m)}, v_F^{(m)})\). This double modification provided the spectral coefficients of the solutions that can reach high degree, such as \(l = 200\) for \(E_{\text{max}} = -0.225\), while it also provided a spectral solution close to the reference solution for \(E_{\text{max}} = -0.04\) and \(l = 80\) (Table 6.9). One may conclude that the HS solutions can be found around \(E_{\text{max}}(\approx -0.225)(\approx 0.05)\) with low accuracy \((v_F(x = -1) = O(10^{-4}))\) while the reference solution can be found for large \(E_{\text{max}}(\approx -0.05)\) with high accuracy (at least \(v_F(x = -1) = O(10^{-6})\)). Also, the Chebyshev coefficients \([F_n]\) and \([I_n]\) showed a distinctive difference between the two solutions. \([F_n]\) decay in different fashions depending on the combination of modifications for \(v_F\) and \(v_R\) (Figure F.35) while the absolute values of \([I_n]\) stall approximately to \(10^{-4} \sim 10^{-5}\) for \(E_{\text{max}} \approx -0.25\) and to \(10^{-6} \sim 10^{-7}\) for \(E_{\text{max}} \approx -0.05\) (Figure F.34). The latter would well reflect the fact that \([I_n]\) is directly associated with \(v_F\) and also, we believe, the determination of \(\beta\).

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Table F.9: Eigenvalues and $\nu_I(x = 1)$ for combinations of $\nu_F^{(m)}$ and $\nu_R^{(m)}$. The upper three rows are the data that reproduced the HS eigenvalues while the lower three rows are the data that reproduced three significant figures of the reference eigenvalues. In the modified ss-OAFP systems, the variables $\nu_S, \nu_Q, \nu_G, \nu_I$ and $\nu_J$ are not modified.

| function | $N$ | $E_{\text{max}}$ | $\beta$ | $c_1$ | $c_4$ | $\nu_I(x = 1)$ |
|----------|-----|-----------------|--------|-------|------|----------------|
| $(\nu_F^{(m)}, \nu_R)$ | 15 | $-0.24$ | 8.181 | $9.1014 \times 10^{-4}$ | $3.516 \times 10^{-1}$ | $3.3 \times 10^{-4}$ |
| $(\nu_F, \nu_R^{(m)})$ | 15 | $-0.24$ | 8.1731 | $9.1023 \times 10^{-4}$ | $3.524 \times 10^{-1}$ | $2.1 \times 10^{-4}$ |
| $(\nu_F^{(m)}, \nu_R^{(m)})$ | 200 | $-0.225$ | 8.175860 | $9.1018 \times 10^{-4}$ | $3.523 \times 10^{-1}$ | $2.9 \times 10^{-4}$ |
| $(\nu_F^{(m)}, \nu_R)$ | 70 | $-0.00525$ | 8.1783712 | $9.0925 \times 10^{-4}$ | $3.304 \times 10^{-1}$ | $2.2 \times 10^{-6}$ |
| $(\nu_F, \nu_R^{(m)})$ | 200 | $-0.04$ | 8.178371129 | $9.0925 \times 10^{-4}$ | $3.301 \times 10^{-1}$ | $4.9 \times 10^{-7}$ |
| $(\nu_F^{(m)}, \nu_R^{(m)})$ | 80 | $-0.04$ | 8.1783683 | $9.0926 \times 10^{-4}$ | $3.301 \times 10^{-1}$ | $8.9 \times 10^{-7}$ |

Figure F.33: Absolute values of Chebyshev coefficients $|F_n^{(m)}|$ for $\nu_F^{(m)}$. In the modified ss-OAFP system, $\nu_S, \nu_Q, \nu_G, \nu_I$ and $\nu_J$ are not modified. The maximum values $E_{\text{max}}$ of the truncated domain are also depicted in the figure.
Figure F.34: Absolute values of Chebyshev coefficients $|I_n|$ for $n$. In the modified ss-OAFP system, $v_5, v_7, v_9, v_1, v_I$ are not modified. The maximum values $E_{\text{max}}$ of the truncated domain are also depicted in the figure.

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