A CLASS OF SINGULAR INTEGRALS
ASSOCIATED WITH ZYGMUND DILATIONS

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Abstract. The main purpose of this paper is to study multi-parameter singular integral operators which commute with Zygmund dilations. We introduce a class of singular integral operators associated with Zygmund dilations and show the boundedness for these operators on $L^p$, $1 < p < \infty$, which covers those studied by Ricci–Stein [26] and Nagel–Wainger [24].

1. Introduction

Ricci–Stein [26] introduced multi-parameter singular integral operators and Fefferman–Pipher [14] considered specific singular integral operators associated with Zygmund dilations. The boundedness for these operators on $L^p$ and weighted $L^p_w$, $1 < p < \infty$, was obtained by Ricci–Stein [26] and Fefferman–Pipher [14], respectively. The main purpose of this paper is to introduce a class of singular integral operators associated with Zygmund dilations and show the boundedness for these operators on $L^p$, $1 < p < \infty$, which cover those studied by Ricci–Stein [26] and Nagel–Wainger [24].

We now set our work in context. In their well-known theory, Calderón and Zygmund [1] introduced certain convolution singular integral operators on $\mathbb{R}^n$ which generalize the Hilbert transform on $\mathbb{R}^1$. They proved that if $T(f) = \mathcal{K} * f$, where $\mathcal{K}$ is defined on $\mathbb{R}^n$ and satisfies the analogous estimates as $\frac{1}{x}$ does on $\mathbb{R}^1$, namely

$$|\mathcal{K}(x)| \leq \frac{C}{|x|^n},$$

$$|\nabla \mathcal{K}(x)| \leq \frac{C}{|x|^{n+1}},$$

and

$$\int_{a < |x| < b} \mathcal{K}(x) dx = 0 \quad \text{for all} \quad 0 < a < b,$$

then $T$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

The core of this theory is that the regularity and cancellation conditions are invariant with respect to the one-parameter family of dilations on $\mathbb{R}^n$ defined by $\delta(x_1, x_2, \cdots, x_n) = \cdots$
\((\delta x_1, \cdots, \delta x_n), \delta > 0\), in the sense that the kernel \(\delta^n K(\delta x)\) satisfies the same conditions with the same bound as \(K(x)\). Indeed, the classical singular integrals, maximal functions and multipliers are invariant with respect to such one-parameter dilations. The one-parameter theory is well understood up to now. On the other hand, the multiparameter theory of \(\mathbb{R}^n\) began with Zygmund’s study of the strong maximal function, which is defined by

\[
\mathcal{M}_n(f)(x) = \sup_{R \ni x} \frac{1}{|R|} \int_R |f(y)| dy,
\]

where \(R\) are the rectangles in \(\mathbb{R}^n\) with sides parallel to the axes, and then continued with Marcinkiewicz’s proof of his multiplier theorem. If we consider the family of product dilations defined by \(\delta(x_1, x_2, \cdots, x_n) = (\delta_1 x_1, \cdots, \delta_n x_n), \delta_i > 0, i = 1, \ldots, n\), then the strong maximal function and Marcinkiewicz’s multiplier are invariant under the product dilations. The multiparameter dilations are also associated with problems in the theory of differentiation of integrals. Jensen–Marcinkiewicz–Zygmund [17] proved that the strong maximal function in \(\mathbb{R}^n\) is bounded from the Orlicz space \(L(1 + (\log^+ L)^{n-1})\) to weak \(L^1\). Zygmund further conjectured that if the rectangles in \(\mathbb{R}^n\) had \(n\) side lengths which involve only \(k\) independent variables, then the resulting maximal operator should behave like \(\mathcal{M}_k\), the \(k\)-parameter strong maximal operator. More precisely, for \(1 \leq k \leq n\), and for positive functions \(\phi_1, \cdots, \phi_n\) as the side-lengths of the given collection of rectangles where the maximal function is defined, each one depending on parameters \(t_1 > 0, t_2 > 0, \cdots, t_k > 0\), assuming arbitrarily small values and increasing in each variable separately, then the resulting maximal function would be bounded from \(L(1 + (\log^+ L)^{k-1})\) to weak \(L^1\) according to Zygmund’s conjecture. Cordoba [7] showed that for the unit cube \(Q\) in \(\mathbb{R}^3\),

\[
|\{x \in Q : \mathcal{M}_3 f(x) > \lambda\}| \leq \frac{C}{\lambda} \|f\|_{L \log L(Q)},
\]

where \(\mathcal{M}_3 f\) denotes the maximal function on \(\mathbb{R}^3\) defined by

\[
\mathcal{M}_3 f(x) = \sup_{R \ni x} \frac{1}{|R|} \int_R |f(u)| du.
\]

The supremum above is taken over all rectangles with sides parallel to the axes and side lengths of the form \(s, t\), and \(\phi(s, t)\). Cordoba’s result was generalized to the case of \(\phi_1(s, t), \phi_2(s, t), \phi_3(s, t)\) by Soria [27] with some assumptions on \(\phi_1, \phi_2, \phi_3\). Moreover, Soria showed that Zygmund’s conjecture is not true even when \(\phi_1(s, t) = s, \phi_2(s, t) = s\phi(t), \phi_3(s, t) = s\psi(t)\), with \(\phi, \psi\) being positive and increasing functions.

In [15] Fefferman and Stein generalized the singular integral operator theory to the product space. They took the space \(\mathbb{R}^n \times \mathbb{R}^m\) along with the two-parameter family of dilations \((x, y) \mapsto (\delta_1 x, \delta_2 y), (x, y) \in \mathbb{R}^n \times \mathbb{R}^m, \delta_1, \delta_2 > 0\). Those operators considered in
generalize the double Hilbert transform on $\mathbb{R}^2$ given by $H(f) = f * \frac{1}{xy}$ and are of the form $T(f) = K * f$, where the kernel $K$ is characterized by the cancellation properties
\begin{align}
\int_{a<|x|<b} K(x, y) dx &= 0 \quad \text{for all } 0 < a < b \text{ and } y \in \mathbb{R}^m, \\
\int_{a<|y|<b} K(x, y) dy &= 0 \quad \text{for all } 0 < a < b \text{ and } x \in \mathbb{R}^n,
\end{align}
and the regularity conditions
\begin{equation}
|\partial^{\alpha}_{x} \partial^{\beta}_{y} K(x, y)| \leq C_{\alpha, \beta}|x|^{-n-|\alpha|}|y|^{-m-|\beta|}.
\end{equation}

Under the conditions (1.1) – (1.3), Fefferman and Stein proved the $L^p$, $1 < p < \infty$, boundedness of the product convolution operators $T(f) = K * f$. See [15] for more details. Note that the kernel $K$ satisfying the conditions (1.1) – (1.3) is invariant with respect to the product dilation in the sense that the kernel $\delta_1^a \delta_2^m K(\delta_1 x, \delta_2 y)$ satisfies conditions (1.1) – (1.3) with the same bound. For more discussions about the multiparameter product theory, see [2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 16, 18, 19, 20, 25, 28] and in particular the survey article of R. Fefferman [13] for development in this area. For singular integrals with flag kernels, see [21, 22, 23].

It has been widely considered that the next simplest multiparameter group of dilations after the product multiparameter dilations is the so-called the Zygmund dilation defined on $\mathbb{R}^3$ by $\rho_{s,t}(x_1, x_2, x_3) = (sx_1, tx_2, stx_3)$ for $s, t > 0$. Indeed, as far as $\mathcal{M}_3$ is concerned, E. M. Stein was the first to link the properties of maximal operators associated with Zygmund dilations to boundary value problems for Poisson integrals on symmetric spaces, such as Siegel’s upper half space. See the survey paper of R. Fefferman [11] on the future direction of research of multiparameter analysis on Zygmund dilations.

There are two operators intimately associated with Zygmund dilations. One is the maximal operator $M_3$ as mentioned above. Another is the singular integral operator $T_3$ introduced by Ricci and Stein [26], which commutes with this dilation. A special class of singular integral operators $T_3$ considered by Ricci and Stein is of the form $T_3 f = f * K$, where
\begin{equation}
K(x_1, x_2, x_3) = \sum_{k,j \in \mathbb{Z}} 2^{-2(k+j)} \phi^{k,j} \left( \frac{x_1}{2^k}, \frac{x_2}{2^j}, \frac{x_3}{2^{k+j}} \right)
\end{equation}
and the functions $\phi^{k,j}$ are supported in an unit cube in $\mathbb{R}^3$ with a certain amount of uniform smoothness and satisfy cancellation conditions
\begin{equation}
\int_{\mathbb{R}^2} \phi^{k,j}(x_1, x_2, x_3) dx_1 dx_2 = \int_{\mathbb{R}^2} \phi^{k,j}(x_1, x_2, x_3) dx_2 dx_3 = \int_{\mathbb{R}^2} \phi^{k,j}(x_1, x_2, x_3) dx_3 dx_1 = 0.
\end{equation}

It was shown in [26] that $T_3$ is bounded on $L^p(\mathbb{R}^3)$ for all $1 < p < \infty$. Particularly, as mentioned in [14], the above cancellation conditions are also necessary for the boundedness
of the above mentioned operators on $L^2(\mathbb{R}^3)$. It is easy to see that if the dyadic Zygmund dilation is given by $(\delta_{2^j,2^k}f)(x_1,x_2,x_3) = 2^{2(j+k)}f(2^jx_1,2^kx_2,2^{j+k}x_3)$, then we obtain that $(\delta_{2^j,2^k}T_1(f))(x_1,x_2,x_3) = T_1(\delta_{2^j,2^k}f)(x_1,x_2,x_3)$. This means that the operators studied by Ricci and Stein commute with Zygmund dilations of dyadic form. See [26] for more details.

R. Fefferman and Pipher [14] further showed that $T_1$ is bounded in $L^p_w$ spaces for $1 < p < \infty$ when the weights $w$'s satisfy an analogous condition of Muckenhoupt associated with Zygmund dilations. Related to the theory of operators like $M$ and $T_1$, several authors have considered singular integrals along surfaces in $\mathbb{R}^n$. See, for example, Nagel–Wainger [24].

To achieve our goal, the first aim of this paper is to develop a class of singular integral operators associated with Zygmund dilations, which covers those introduced in [26] and prove the $L^2$ boundedness. The second aim is to show the $L^p$, $1 < p < \infty$, boundedness for this class of singular integral operators.

Suppose that $K(x_1,x_2,x_3)$ is a function defined on $\mathbb{R}^3$ away from the union $\{0,x_2,x_3\} \cup \{x_1,0,x_3\} \cup \{x_1,x_2,0\}$ and all $\alpha, \beta$ and $\gamma$ are integers taking only values $0$ and $1$. We define

$$
\Delta_{x_1,h_1}\alpha K(x_1,x_2,x_3) = \alpha K(x_1+h_1,x_2,x_3)-K(x_1,x_2,x_3), \quad \alpha = 0 \text{ or } 1,
$$

$$
\Delta_{x_2,h_2}\beta K(x_1,x_2,x_3) = \beta K(x_1,x_2+h_2,x_3)-K(x_1,x_2,x_3), \quad \beta = 0 \text{ or } 1,
$$

and

$$
\Delta_{x_3,h_3}\gamma K(x_1,x_2,x_3) = \gamma K(x_1,x_2,x_3+h_3)-K(x_1,x_2,x_3), \quad \gamma = 0 \text{ or } 1.
$$

For simplicity, we denote $\Delta_{x_1,h_1} = \Delta_{x_1,h_1}^1$, $\Delta_{x_2,h_2} = \Delta_{x_2,h_2}^1$ and $\Delta_{x_3,h_3} = \Delta_{x_3,h_3}^1$.

The “regularity” conditions considered in this paper are characterized by

\[(R) \quad |\Delta_{x_1,h_1}^\alpha \Delta_{x_2,h_2}^\beta \Delta_{x_3,h_3}^\gamma K(x_1,x_2,x_3)| \leq \frac{C|h_1|^\alpha h_2|^\beta h_3|^\gamma}{|x_1|^\alpha |x_2|^\beta |x_3|^\gamma} \left( |x_1x_2| + \frac{|x_3|}{|x_2|} \right)^{\theta_2}
\]

for all $0 \leq \alpha \leq 1, 0 \leq \beta + \gamma \leq 1$ or $0 \leq \alpha + \gamma \leq 1, 0 \leq \beta \leq 1$, and $|x_1| \geq 2|h_1| > 0, |x_2| \geq 2|h_2| > 0, |x_3| \geq 2|h_3| > 0$, $h_1, h_2, h_3 \in \mathbb{R}$ and some $0 < \theta_1 \leq 1, 0 < \theta_2 < 1$.

Note that for any fixed non zero two variables, say, $x_1 \neq 0$ and $x_2 \neq 0$, $K(x_1,x_2,x_3)$ is an integrable function with respect to the variable $x_3$ and the resulting integral $\mathcal{K}(x_1,x_2) = \int_{\mathbb{R}} K(x_1,x_2,x_3)dx_3$, as a kernel on $\mathbb{R}^2$, satisfies the regularity conditions of the classical product kernel on $\mathbb{R}^2$ as studied by Fefferman and Stein in [15]. These facts, as mentioned above, can also be easily checked for singular integral operators studied by Ricci and Stein.
In this paper, we will consider three kinds of cancellation conditions. The first one is
given by

\[(C1.a) \quad \left| \int_{\delta_1 \leq |x_1| \leq r_1} \int_{\delta_2 \leq |x_2| \leq r_2} \int_{\delta_3 \leq |x_3| \leq r_3} \mathcal{K}(x_1, x_2, x_3)dx_1dx_2dx_3 \right| \leq C\]

uniformly for all \(\delta_1, \delta_2, \delta_3, r_1, r_2, r_3 > 0\);

\[(C1.b) \quad \left| \int_{\delta_1 \leq |x_1| \leq r_1} \int_{\delta_2 \leq |x_2| \leq r_2} \Delta_{x_3, \delta_3}^\gamma \mathcal{K}(x_1, x_2, x_3)dx_1dx_2 \right| \leq \frac{C|h_3|^\frac{\gamma}{\delta_3}}{|x_3|^\frac{\gamma}{\delta_3}+1}\]

uniformly for all \(\delta_1, \delta_2, r_1, r_2 > 0, |x_3| \geq 2|h_3| > 0\) and \(0 \leq \gamma \leq 1\);

\[(C1.c) \quad \left| \int_{\delta_2 \leq |x_2| \leq r_2} \int_{\delta_3 \leq |x_3| \leq r_3} \Delta_{x_1, \delta_1}^{\alpha} \mathcal{K}(x_1, x_2, x_3)dx_2dx_3 \right| \leq \frac{C|h_1|^{\alpha \delta_1}}{|x_1|^{\alpha \delta_1}+1}\]

uniformly for all \(\delta_2, \delta_3, r_2, r_3 > 0, |x_1| \geq 2|h_1| > 0\) and \(0 \leq \alpha \leq 1\);

\[(C1.d) \quad \left| \int_{\delta_1 \leq |x_1| \leq r_1} \int_{\delta_3 \leq |x_3| \leq r_3} \Delta_{x_2, \delta_2}^{\beta} \mathcal{K}(x_1, x_2, x_3)dx_3dx_1 \right| \leq \frac{C|h_2|^{\beta \delta_1}}{|x_2|^{\beta \delta_1}+1}\]

uniformly for all \(\delta_1, \delta_3, r_1, r_3 > 0, |x_2| \geq 2|h_2| > 0\) and \(0 \leq \beta \leq 1\).

The regularity conditions (R) and the cancellation conditions (C1.a) - (C1.d) imply the
following \(L^2\) boundedness.

**Theorem 1.1.** Suppose that \(\mathcal{K}\) is a function defined on \(\mathbb{R}^3\) and satisfies the conditions
(R) and (C1.a) - (C1.d). Set \(\mathcal{K}^N_\epsilon(x_1, x_2, x_3) = \mathcal{K}(x_1, x_2, x_3)\) if \(\epsilon \leq |x_1| \leq N_1, \epsilon \leq |x_2| \leq N_2\) and \(\epsilon \leq |x_3| \leq N_3\) and \(\mathcal{K}^N_\epsilon(x_1, x_2, x_3) = 0\) otherwise, where \(\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)\) and
\(N = (N_1, N_2, N_3)\) for all \(0 < \epsilon_1 \leq N_1 < \infty, \epsilon_2 \leq N_2 < \infty,\) and \(\epsilon_3 \leq N_3 < \infty\). Then, the
operator \(\mathcal{K}^N_\epsilon * f\) is bounded on \(L^2(\mathbb{R}^3)\) and moreover,

\[\|\mathcal{K}^N_\epsilon * f\|_{L^2(\mathbb{R}^3)} \leq A\|f\|_{L^2(\mathbb{R}^3)}\]

where the constant \(A\) depends only on the constant \(C\) but not on \(\epsilon_1, \epsilon_2, \epsilon_3, N_1, N_2\) and \(N_3\).

From Theorem [1.1] we will deduce the existence of the corresponding singular integrals
in the \(L^2\) norm as a limit of the truncated integrals.

**Corollary 1.2.** Suppose that \(\mathcal{K}\) is a function defined on \(\mathbb{R}^3\) and satisfies the conditions
(R), (C1.a) - (C1.d) and, in addition, the three integrals

\[\int_{|x_3| \leq 1} \int_{|x_2| \leq 1} \int_{|x_1| \leq 1} \mathcal{K}^N_\epsilon(x_1, x_2, x_3)dx_1dx_2dx_3,\]

\[\int_{|x_3| \leq 1} \int_{|x_2| \leq 1} \mathcal{K}^N_\epsilon(x_1, x_2, x_3)dx_2dx_3,\]

\[\int_{|x_3| \leq 1} \int_{|x_1| \leq 1} \mathcal{K}^N_\epsilon(x_1, x_2, x_3)dx_1dx_3.\]
converge almost everywhere as \( \epsilon_1, \epsilon_2, \epsilon_3 \to 0 \) and \( N_1, N_2, N_3 \to \infty \). Then the limit
\[
\lim_{N_1, N_2, N_3 \to \infty} K^N_{\epsilon} \ast f = K \ast f
\]
eexists in the \( L^2(\mathbb{R}^3) \) norm. Moreover,
\[
\|K \ast f\|_{L^2(\mathbb{R}^3)} \leq A\|f\|_{L^2(\mathbb{R}^3)}
\]
with the constant \( A \) depending only on the constant \( C \).

We remark in advance that the proof of Corollary 1.2 indeed implies that
\[
\lim_{N_1, N_2, N_3 \to \infty} K^N_{\epsilon} \ast f \text{ exists in the } L^p, 1 < p < \infty, \text{ norm for smooth functions } f \text{ having compact support.}
\]
This fact leads to the study of the \( L^p, p \neq 2 \), boundedness of the operator \( K \ast f \). For this purpose, we need the second kind of the cancellation conditions which are somewhat stronger than the first ones. They are given by

\[
\text{(C2.a)} \quad \left| \int_{\delta_1 \leq |x_3| \leq r_3} \int_{\delta_2 \leq |x_2| \leq r_2} \int_{\delta_1 \leq |x_1| \leq r_1} K(x_1, x_2, x_3) dx_1 dx_2 dx_3 \right| \leq C
\]
uniformly for all \( \delta_1, \delta_2, \delta_3, r_1, r_2, r_3 > 0; \)

\[
\text{(C2.b)} \quad \left| \int_{\delta_1 \leq |x_1| \leq r_1} \Delta_{x_2, h_2} \Delta_{x_3, h_3} \gamma \frac{K(x_1, x_2, x_3) dx_1 dx_2 dx_3}{x_2^{\theta_1+1} x_3^{\theta_1+1}} \right| \leq \frac{C|h_2|^\beta |h_3|^\gamma}{|x_3|^\beta} \left( \frac{1}{|x_1|^{\theta_1+1} x_3^{\theta_1+1}} \right)^{\beta_2} + \frac{1}{|x_3|^{\beta_1}}
\]
for all \( \delta_1, r_1 > 0, 0 \leq \beta + \gamma \leq 1, |x_2| \geq 2|h_2| > 0, |x_3| \geq 2|h_3| > 0; \)

\[
\text{(C2.c)} \quad \left| \int_{\delta_3 \leq |x_3| \leq r_3} \int_{\delta_2 \leq |x_2| \leq r_2} \int_{\delta_1 \leq |x_1| \leq r_1} \Delta_{x_1, h_1} \frac{K(x_1, x_2, x_3) dx_1 dx_2 dx_3}{x_2^{\theta_1+1} x_3^{\theta_1+1}} \right| \leq \frac{C|h_1|\alpha}{|x_1|^{\alpha \theta_1+1}}
\]
uniformly for all \( \delta_2, \delta_3, r_2, r_3 > 0, |x_1| \geq 2|h_1| > 0 \) and \( 0 \leq \alpha \leq 1 \). Or

\[
\text{(C2’.a)} \quad \left| \int_{\delta_3 \leq |x_3| \leq r_3} \int_{\delta_2 \leq |x_2| \leq r_2} \int_{\delta_1 \leq |x_1| \leq r_1} K(x_1, x_2, x_3) dx_1 dx_2 dx_3 \right| \leq C
\]
uniformly for all \( \delta_1, \delta_2, \delta_3, r_1, r_2, r_3 > 0; \)

\[
\text{(C2’.b)} \quad \left| \int_{\delta_2 \leq |x_2| \leq r_2} \Delta_{x_1, h_1} \Delta_{x_3, h_3} \frac{K(x_1, x_2, x_3) dx_2}{x_2^{\theta_1+1} x_3^{\theta_1+1}} \right| \leq \frac{C|h_1|\alpha}{|x_1|^{\alpha \theta_1+1}} \left( \frac{1}{|x_1|^{\theta_1+1} x_3^{\theta_1+1}} \right)^{\beta_2} + \frac{1}{|x_3|^{\beta_1}}
\]
for all \( \delta_2, r_2 > 0, 0 \leq \alpha + \gamma \leq 1, |x_1| \geq 2|h_1| > 0 \) and \( |x_3| \geq 2|h_3| > 0; \)

\[
\text{(C2’.c)} \quad \left| \int_{\delta_3 \leq |x_3| \leq r_3} \int_{\delta_1 \leq |x_1| \leq r_1} \Delta_{x_2, h_2} \frac{K(x_1, x_2, x_3) dx_3}{x_2^{\theta_1+1} x_3^{\theta_1+1}} \right| \leq \frac{C|h_2|^\beta}{|x_2|^{\beta \theta_1+1}}
\]
uniformly for all \( \delta_1, \delta_3, r_1, r_3 > 0, |x_2| \geq 2|h_2| > 0 \) and \( 0 \leq \beta \leq 1 \).

We would like to point out that the condition (C2.b) implies (C1.b) and (C1.d) while (C2’.b) implies (C1.b) and (C1.c), and all the above regularity and cancellation conditions are invariant with respect to the Zygmund dilation in the sense that the kernel
$$\delta_1^2 \delta_2^2 K(\delta_1 x_1, \delta_2 x_2, \delta_1 \delta_2 x_3)$$ satisfies the same conditions with the exactly same bounds as $K(x_1, x_2, x_3)$.

The $L^p$ estimate then is given by the following

**Theorem 1.3.** Suppose that $K$ is a function defined on $\mathbb{R}^3$ and satisfies the conditions (R) and (C2.a) – (C2.c) (or (R), (C2’).a – (C2’.c)) and in addition the three integrals

$$\int_{|x_3| \leq 1} \int_{|x_2| \leq 1} \int_{|x_1| \leq 1} K^N(x_1, x_2, x_3) dx_1 dx_2 dx_3,$$

$$\int_{|x_3| \leq 1} \int_{|x_2| \leq 1} K^N(x_1, x_2, x_3) dx_2 dx_3,$$

$$\int_{|x_3| \leq 1} \int_{|x_1| \leq 1} K^N(x_1, x_2, x_3) dx_1 dx_3$$

converge almost everywhere as $\epsilon_1, \epsilon_2, \epsilon_3 \to 0$ and $N_1, N_2, N_3 \to \infty$. Then the operator

$$K * f := \lim_{N_1, N_2, N_3 \to \infty} K^N * f$$

defined initially on $L^2 \cap L^p, 1 < p < \infty$, extends to a bounded operator on $L^p(\mathbb{R}^3)$; moreover,

$$\|K * f\|_{L^p(\mathbb{R}^3)} \leq A\|f\|_{L^p(\mathbb{R}^3)}$$

with the constant $A$ depending only on the constant $C$.

In many applications, singular integral operators are of the form $K * f$ where $K$ is a distribution that equals a function $K$ on $\mathbb{R}^3$ away from the union $\{0, x_2, x_3\} \cup \{x_1, 0, x_3\} \cup \{x_1, x_2, 0\}$ and satisfy certain regularity and cancellation conditions. For this purpose, we begin with recalling the bump functions introduced by Stein in [28]. A normalized bump function (n.b.f.) is a smooth function $\phi$ supported on the unit ball and is bounded by a fixed constant together with its gradient. The third kind of the cancellation conditions considered in this paper is characterized by

(C3.a) \[ \int \int \int K(x_1, x_2, x_3) \phi(R_1 x_1, R_2 x_2, R_1 R_2 x_3) dx_1 dx_2 dx_3 \leq C \]

for every n.b.f. $\phi$ on $\mathbb{R}^3$ and all $R_1, R_2 > 0$;

(C3.b) \[ \int \Delta_2^{\beta} \Delta_3^{\gamma} K(x_1, x_2, x_3) \phi(R x_1) dx_1 \leq \frac{C h_2^{\beta |h_3| \gamma_1}}{|x_2|^{\beta_1} |x_3|^{\gamma_1 + 1} (|x_2|^{\beta_1} + |x_3|^{\gamma_1 + 1})^{\gamma_2}} \]

for all $0 \leq \beta + \gamma \leq 1$, every n.b.f. $\phi$ on $\mathbb{R}$, $|x_2| \geq 2|h_2| > 0$, $|x_3| \geq 2|h_3| > 0$ and all $R > 0$;

(C3.c) \[ \int \Delta_1^{n_1} K(x_1, x_2, x_3) \phi(R_1 x_2, R_2 x_3) dx_2 dx_3 \leq \frac{C h_1^{\alpha_1}}{|x_1|^{\alpha_1 + 1}} \]

for all $0 \leq \alpha + 1 \leq 1$, every n.b.f. $\phi$ on $\mathbb{R}$, $|x_2| \geq 2|h_2| > 0$, $|x_3| \geq 2|h_3| > 0$ and all $R > 0$.\]
for all $0 \leq \alpha \leq 1$, every n.b.f. $\phi$ on $\mathbb{R}^2$, $|x_1| \geq 2|h_1| > 0$ and all $R_1, R_2 > 0$. Or

\[(C3'.a)\quad \iint K(x_1, x_2, x_3) \phi(R_1 x_1, R_2 x_2, R_1 R_2 x_3) dx_1 dx_2 dx_3 \leq C\]

for every n.b.f. $\phi$ on $\mathbb{R}^3$ and all $R_1, R_2 > 0$;

\[(C3'.b)\quad \int \Delta^\alpha_{x_1, h_1} \Delta^\gamma_{x_1, h_3} K(x_1, x_2, x_3) \phi(R x_2) dx_2 \leq \frac{C|h_1|^{|\alpha|}|h_3|^{|\gamma|}}{|x_1|^{|\alpha|+1}|x_3|^{|\gamma|+1}(|R x_2| + |\frac{x_1}{R x_3}|)^{\theta^2}}\]

for all $0 \leq \alpha + \gamma \leq 1$, every n.b.f. $\phi$ on $\mathbb{R}$, $|x_1| \geq 2|h_1| > 0, |x_3| \geq 2|h_3| > 0$ and all $R > 0$;

\[(C3'.c)\quad \iint \Delta^\beta_{x_2, h_2} K(x_1, x_2, x_3) \phi(R_1 x_1, R_2 x_3) dx_1 dx_3 \leq \frac{C|h_2|^{|\beta|}}{|x_2|^{|\beta|+1}}\]

for all $0 \leq \beta \leq 1$, every n.b.f. $\phi$ on $\mathbb{R}^2$, $|x_2| \geq 2|h_2| > 0$ and all $R_1, R_2 > 0$.

**Theorem 1.4.** (a) Suppose that $K$ is a distribution that equals a function on $\mathbb{R}^3$ away from the union $\{0, x_2, x_3\} \cup \{x_1, 0, x_3\} \cup \{x_1, x_2, 0\}$ and satisfies the conditions (R) and (C3.a) – (C3.c) (or (R), (C3'.a) – (C3'.c)). Then, the operator $K * f$ is bounded on $L^p(\mathbb{R}^3)$, $1 < p < \infty$; moreover,

$$\|K * f\|_{L^p(\mathbb{R}^3)} \leq A\|f\|_{L^p(\mathbb{R}^3)}$$

with the constant $A$ depending only on the constant $C$.

(b) Suppose that $K$ is a distribution that equals a function on $\mathbb{R}^3$ away from the union $\{0, x_2, x_3\} \cup \{x_1, 0, x_3\} \cup \{x_1, x_2, 0\}$ and satisfies the conditions (R) and (C2.a) – (C2.c) (or (R), (C2'.a) – (C2'.c)). Then, the operator $K * f$ is bounded on $L^p(\mathbb{R}^3)$, $1 < p < \infty$, and,

$$\|K * f\|_{L^p(\mathbb{R}^3)} \leq A\|f\|_{L^p(\mathbb{R}^3)}$$

with the constant $A$ depending only on the constant $C$.

**Remark 1.5.** We would like to point out that all regularity and cancellation conditions given above are invariant with respect to Zygmund dilations. Moreover, the operators studied by Ricci and Stein, as mentioned before, satisfy all above regularity and cancellation conditions. So our results provide another proof of the boundedness for operators in [26] on $L^p, 1 < p < \infty$. And the boundedness results in this paper can be extended to higher dimensions. The consideration of regularity and cancellation conditions in this paper leads naturally to the study of non-convolution singular integral operators which are associated with Zygmund dilations. We will discuss all these topics in the forthcoming works.

In the next section, we will show the $L^2$ boundedness for singular integral operators associated with Zygmund dilations, namely Theorem 1.1 and Corollary 1.2. The proofs of $L^p$ boundedness, Theorems 1.3 and 1.4 will be given in section 3. In the last section, some
examples and applications of singular integral operators in our class will be discussed. In particular, we show that the kernels of singular integral operators $T_j$ in the special class studied by Ricci and Stein satisfy the conditions $(R)$ and $(C2.a) - (C2.c)$ (or $(R)$, $(C2'.a) - (C2'.c)$), and $(R)$, $(C3.a) - (C3.c)$ (or $(R)$, $(C3'.a) - (C3'.c)$). We also show that the operator considered by Nagel and Wainger $[24]$, where only the $L^2$ boundedness is proved, belongs to our class, and therefore, as a consequence of Theorem $1.3$ is bounded on $L^p$, $1 < p < \infty$.

2. $L^2$ BOUNDEDNESS

The main task of this section is to provide proofs of Theorem $1.1$ and Corollary $1.2$. Before proving Theorem $1.1$ we first show the following simple result which will be used frequently below.

**Lemma 2.1.** For any $f(x) \in L^1_{loc}(\mathbb{R})$ and $N > 8$, we have

$$\left| \int_{8 \leq |x| \leq N} f(x)e^{-ix}dx \right| \leq \frac{1}{2} \int_{E_N} |f(x)|dx + \frac{1}{2} \int_{8 \leq |x| \leq N} |f(x) - f(x + \pi)|dx,$$

where $E_N = \{x : 4 \leq |x| \leq 12\} \cup \{x : N - \pi \leq |x| \leq N + \pi\}$.

**Proof.** We write

$$\int_{8 \leq |x| \leq N} f(x)e^{-ix}dx = \int_{8 \leq |x+\pi| \leq N} f(x + \pi)e^{-i(x+\pi)}dx = -\int_{8 \leq |x+\pi| \leq N} f(x + \pi)e^{-ix}dx.$$

Therefore,

$$\left| \int_{8 \leq |x| \leq N} f(x)e^{-ix}dx \right| = \frac{1}{2} \left| \int_{8 \leq |x| \leq N} f(x)e^{-ix}dx - \int_{8 \leq |x+\pi| \leq N} f(x + \pi)e^{-ix}dx \right|$$

$$\leq \frac{1}{2} \left| \int_{8 \leq |x| \leq N} (f(x) - f(x + \pi))e^{-ix}dx \right|$$

$$+ \frac{1}{2} \int_{\{x : 8 \leq |x| \leq N\} \cap \{x : 8 \leq |x+\pi| \leq N\}} f(x + \pi)e^{-ix}dx$$

$$+ \frac{1}{2} \int_{\{x : 8 \leq |x+\pi| \leq N\} \cap \{x : 8 \leq |x| \leq N\}} f(x + \pi)e^{-ix}dx$$

$$= \frac{1}{2} \left| \int_{8 \leq |x| \leq N} (f(x) - f(x + \pi))e^{-ix}dx \right|$$

$$+ \frac{1}{2} \int_{\{x : 8 \leq |x-\pi| \leq N\} \cap \{x : 8 \leq |x| \leq N\}} f(x)e^{-ix}dx$$

$$+ \frac{1}{2} \int_{\{x : 8 \leq |x| \leq N\} \cap \{x : 8 \leq |x-\pi| \leq N\}} f(x)e^{-ix}dx.$$
More precisely, we write

\[ \frac{1}{2} \int_{|x| \leq N} |f(x) - f(x + \pi)| \, dx + \frac{1}{2} \int_{E_N} |f(x)| \, dx \]

and Lemma 2.1 follows.

We now prove Theorem 1.1.

**Proof of Theorem 1.1.** By the Plancherel theorem, the \( L^2 \) boundedness of \( \mathcal{K}_\epsilon^N \ast f \) is equivalent to \( |\mathcal{K}_\epsilon^N(\chi, \eta, \xi)| \leq A \), where \( \mathcal{K}_\epsilon^N \) is the Fourier transform of \( \mathcal{K}_\epsilon^N \), \( A \) is the constant depending only on the constant \( C \) but not on \( \epsilon = (\epsilon_1, \epsilon_2, \epsilon_3) \), and \( N = (N_1, N_2, N_3) \). To obtain such an estimate, we may assume that \( \chi \) and \( \eta \) are both positive. Note that

\[
\mathcal{K}_\epsilon^N(\chi, \eta, \xi) = \int_{x_3 \leq |x_3| \leq N_3} \int_{x_2 \leq |x_2| \leq N_1} \int_{x_1 \leq |x_1| \leq N_1} \mathcal{K}(x_1, x_2, x_3) e^{-ix_1\chi} e^{-ix_2\eta} e^{-ix_3\xi} \, dx_1 dx_2 dx_3
\]

\[
\mathcal{K}(x_1, x_2, x_3) = \frac{1}{\pi^2} \frac{\mathcal{K}(\frac{x_1}{\chi}, \frac{x_2}{\eta}, \frac{x_3}{\xi})}{\chi^2 \eta^2} e^{-ix_1\chi} e^{-ix_2\eta} e^{-ix_3\xi/(\chi \eta)} \, dx_1 dx_2 dx_3.
\]

As remarked above, the assumptions on \( \mathcal{K} \) are invariant in the sense that \( \delta_1 \delta_2 \mathcal{K}(\delta_1 x_1, \delta_2 x_2, \delta_3 x_3) \) satisfies the same assumptions as \( \mathcal{K} \) with the same constant \( C \), independent of \( \delta_1, \delta_2 > 0 \). Thus \( \frac{1}{\chi^2 \eta^2} \mathcal{K}(\frac{x_1}{\chi}, \frac{x_2}{\eta}, \frac{x_3}{\xi}) \) satisfies all conditions (R) and (C1.a) - (C1.d) with the same bounds uniformly for \( \chi, \eta \). Therefore, it suffices to show that \( \mathcal{K}_\epsilon^N(1, 1, \xi) \) is a bounded function uniformly for \( 0 < \epsilon_1, \epsilon_2, \epsilon_3, N_1, N_2, N_3 < \infty \). To do this, for simplicity, we set \( \epsilon_4 = \epsilon_3 |\xi| \) and \( N_4 = N_3 |\xi| \). Without loss of generality, we may assume that \( \epsilon_1, \epsilon_2, \epsilon_4 \leq 8 \leq N_1, N_2, N_4 \) since all other cases can be written as a finite linear combination of these cases and can be handled similarly.

The bound of \( \mathcal{K}_\epsilon^N(1, 1, \xi) \) follows from the regularity and cancellation conditions on \( \mathcal{K} \). More precisely, we write

\[
\mathcal{K}_\epsilon^N(1, 1, \xi) = \int_{x_3 \leq |x_3| \leq N_3} \int_{x_2 \leq |x_2| \leq N_2} \int_{x_1 \leq |x_1| \leq N_1} \mathcal{K}(x_1, x_2, x_3) e^{-ix_1\chi} e^{-ix_2\eta} e^{-ix_3\xi} \, dx_1 dx_2 dx_3
\]

\[
= \int_{x_3 \leq |x_3| \leq N_4} \int_{x_2 \leq |x_2| \leq N_2} \int_{x_1 \leq |x_1| \leq N_1} \frac{1}{\xi} \mathcal{K}(x_1, x_2, x_3) e^{-ix_1\chi} e^{-ix_2\eta} e^{-ix_3\xi/(\chi \eta)} \, dx_1 dx_2 dx_3
\]

\[= I + II,
\]

where \( I \) is the result of integrating over the set \( \{ 8 \leq |x_1| \leq N_1, \epsilon_2 \leq |x_2| \leq N, \epsilon_4 \leq |x_3| \leq N_4 \} \) and \( II \) over the set \( \{ \epsilon_1 \leq |x_1| < 8, \epsilon_2 \leq |x_2| \leq N_1, \epsilon_4 \leq |x_3| \leq N_4 \} \).

For term \( I \), using Lemma 2.1 with \( f(x_1) = \int_{x_2 \leq |x_2| \leq N_2} \int_{x_3 \leq |x_3| \leq N_4} \frac{1}{\xi} \mathcal{K}(x_1, x_2, x_3) e^{-ix_2\eta} e^{-ix_3\xi/(\chi \eta)} \, dx_3 dx_2 \), we obtain

\[|I| \leq \int_{E_{N_1}} \left| \int_{x_2 \leq |x_2| \leq N_2} \int_{x_3 \leq |x_3| \leq N_4} \frac{1}{\xi} \mathcal{K}(x_1, x_2, x_3) e^{-ix_2\eta} e^{-ix_3\xi/(\chi \eta)} \, dx_3 dx_2 \right| \, dx_1\]
\[ + \int_{8 \leq |x_1| \leq N_4} \int_{\epsilon_2 \leq |x_2| \leq |x_3| \leq \epsilon_4} \int_{\epsilon_4 \leq |x_4| \leq N_4} \Delta_{x_1, \pi} \left( \frac{1}{\xi} K(x_1, x_2, \frac{x_3}{\xi}) \right) e^{-ix_2} e^{-ix_3} dx_3 dx_2 \, dx_1 \]

\[ := I_1 + I_2. \]

Then

\[ I_1 \leq \int_{E_{N_1}} \int_{8 \leq |x_2| \leq N_2} \int_{\epsilon_4 \leq |x_3| \leq N_4} \frac{1}{\xi} K(x_1, x_2, \frac{x_3}{\xi}) e^{-ix_2} e^{-ix_3} dx_3 dx_2 \, dx_1 \]

\[ + \int_{E_{N_1}} \int_{\epsilon_2 \leq |x_2| \leq 8} \int_{\epsilon_4 \leq |x_3| \leq N_4} \frac{1}{\xi} K(x_1, x_2, \frac{x_3}{\xi}) e^{-ix_2} e^{-ix_3} dx_3 dx_2 \, dx_1 \]

\[ := I_{1,1} + I_{1,2}. \]

To estimate term $I_{1,1}$, using Lemma 2.1 with $f(x_2) = \int_{\epsilon_4 \leq |x_3| \leq N_4} \frac{1}{\xi} K(x_1, x_2, \frac{x_3}{\xi}) e^{-ix_3} dx_3$ we get

\[ |I_{1,1}| \lesssim \int_{E_{N_1}} \int_{E_{N_2}} \int_{\epsilon_4 \leq |x_3| \leq N_4} \frac{1}{\xi} K(x_1, x_2, \frac{x_3}{\xi}) e^{-ix_3} dx_3 \, dx_2 \, dx_1 \]

\[ + \int_{E_{N_1}} \int_{E_{N_2}} \int_{\epsilon_4 \leq |x_3| \leq N_4} \Delta_{x_2, \pi} \left( \frac{1}{\xi} K(x_1, x_2, \frac{x_3}{\xi}) \right) e^{-ix_3} dx_3 \, dx_2 \, dx_1 \]

\[ \lesssim 1, \]

where we use the condition (R) above on $K$ with $\alpha = \beta = \gamma = 0$ and $\alpha = 0, \beta = 1, \gamma = 0$, respectively.

To handle term $I_{1,2}$, we write

\[ I_{1,2} \leq \int_{E_{N_1}} \int_{\epsilon_2 \leq |x_2| \leq 8} \int_{8 \leq |x_3| \leq N_4} \frac{1}{\xi} K(x_1, x_2, \frac{x_3}{\xi}) e^{-ix_2} e^{-ix_3} dx_3 dx_2 \, dx_1 \]

\[ + \int_{E_{N_1}} \int_{\epsilon_2 \leq |x_2| \leq 8} \int_{\epsilon_4 \leq |x_3| \leq 8} \frac{1}{\xi} K(x_1, x_2, \frac{x_3}{\xi}) e^{-ix_2} e^{-ix_3} dx_3 dx_2 \, dx_1 \]

\[ := I_{1,2,1} + I_{1,2,2}. \]

By Lemma 2.1 with $f(x_2) = \int_{\epsilon_2 \leq |x_2| \leq 8} \frac{1}{\xi} K(x_1, x_2, \frac{x_3}{\xi}) e^{-ix_2} dx_2$, we get

\[ |I_{1,2,1}| \lesssim \int_{E_{N_1}} \int_{E_{N_2}} \int_{\epsilon_2 \leq |x_2| \leq 8} \frac{1}{\xi} K(x_1, x_2, \frac{x_3}{\xi}) e^{-ix_2} dx_2 \, dx_3 \, dx_1 \]

\[ + \int_{E_{N_1}} \int_{8 \leq |x_3| \leq N_4} \int_{\epsilon_2 \leq |x_2| \leq 8} \Delta_{x_3, \pi} \left( \frac{1}{\xi} K(x_1, x_2, \frac{x_3}{\xi}) \right) e^{-ix_2} dx_2 \, dx_3 \, dx_1 \]
Then \((C1.c)\) with

\[
\|I\| \lesssim \frac{1}{|x_1|x_2x_3^3}\left(\frac{|x_2x_3^2|}{x_3} + \left|\frac{x_3}{x_1x_2^2}\right|\right)\theta_2 dx_2dx_3dx_1
\]

\[
+ \int_{E_{N_1}} \frac{1}{|x_1|x_2x_3^3}\left(\frac{|x_2x_3^2|}{x_3} + \left|\frac{x_3}{x_1x_2^2}\right|\right)\theta_2 dx_2dx_3dx_1
\]

\[
\lesssim 1,
\]

where we use the regularity condition (R) above with \(\alpha = \beta = \gamma = 0\) and \(\alpha = \beta = 0, \gamma = 1\), respectively.

To estimate \(I_{1,2,1}\), we note that

\[
I_{1,2,1} \lesssim \int_{E_{N_1}} \left|\int_{e_2 \leq |x_2| \leq 8} \int_{e_4 \leq |x_3| \leq 8} \frac{1}{\xi} K(x_1, x_2, x_3) e^{-ix_2} e^{-ix_3} - 1\right| dx_3dx_2 dx_1
\]

\[
+ \int_{E_{N_1}} \left|\int_{e_2 \leq |x_2| \leq 8} \int_{e_4 \leq |x_3| \leq 8} \frac{1}{\xi} K(x_1, x_2, x_3) dx_3dx_2\right| dx_1
\]

\[
\lesssim \int_{E_{N_1}} \left|\int_{e_2 \leq |x_2| \leq 8} \int_{e_4 \leq |x_3| \leq 8} \left|\frac{x_2}{|x_2|} + \frac{x_3}{|x_3|}\right| \theta_2 dx_3dx_2 dx_1 + \int_{E_{N_1}} \frac{1}{|x_1|} dx_1
\]

\[
\lesssim 1,
\]

where we use the condition (R) with \(\alpha = \beta = \gamma = 0\), and the cancellation condition (C1.c) with \(\alpha = 0\).

Next we consider \(I_2\). Set \(I_{2,1}\) and \(I_{2,2}\) to be

\[
I_{2,1} = \int_{8 \leq |x_1| \leq N_1} \int_{8 \leq |x_2| \leq 8} \int_{e_4 \leq |x_3| \leq N_4} \Delta_{x_1,\pi} \left(\frac{1}{\xi} K(x_1, x_2, x_3)\right) e^{-ix_2} e^{-ix_3} dx_3dx_2 dx_1
\]

and

\[
I_{2,2} = \int_{8 \leq |x_1| \leq N_1} \int_{e_2 \leq |x_2| \leq 8} \int_{e_4 \leq |x_3| \leq N_4} \Delta_{x_1,\pi} \left(\frac{1}{\xi} K(x_1, x_2, x_3)\right) e^{-ix_2} e^{-ix_3} dx_3dx_2 dx_1.
\]

Then \(I_2 \lesssim |I_{2,1}| + |I_{2,2}|\). Similarly, applying Lemma 21 with

\[
f(x_2) = \int_{e_4 \leq |x_3| \leq N_4} \Delta_{x_1,\pi} \left(\frac{1}{\xi} K(x_1, x_2, x_3)\right) e^{-ix_3} dx_3,
\]

we obtain

\[
|I_{2,1}| \lesssim \int_{8 \leq |x_1| \leq N_1} \int_{e_4 \leq |x_3| \leq N_4} \Delta_{x_1,\pi} \left(\frac{1}{\xi} K(x_1, x_2, x_3)\right) e^{-ix_3} dx_3 dx_2 dx_1
\]

\[
+ \int_{8 \leq |x_1| \leq N_1} \int_{8 \leq |x_2| \leq 8} \int_{e_4 \leq |x_3| \leq N_4} \Delta_{x_2,\pi} \Delta_{x_1,\pi} \left(\frac{1}{\xi} K(x_1, x_2, x_3)\right) e^{-ix_3} dx_3 dx_2 dx_1
\]

\[
\approx \int_{8 \leq |x_1| \leq N_1} \int_{e_4 \leq |x_3| \leq N_4} \left|\frac{x_1}{|x_1|} + \frac{x_2}{|x_2|}\right| \theta_2 dx_3dx_2 dx_1
\]
where we use the condition (R) above with $\alpha = 1, \beta = \gamma = 0$ and $\alpha = \beta = 1, \gamma = 0$, respectively.

For term $I_{2,2}$, note that

\[
I_{2,2} \leq \int_{8 \leq |x_1| \leq N_1} \int_{x_2 \leq |x_2| \leq 8} \int_{8 \leq |x_3| \leq N_4} \frac{1}{|x_1|^{1+\theta_1}|x_2|^{1+\theta_1}|x_3|\left(\left|\frac{x_1 x_2 \xi}{x_3}\right| + \left|\frac{x_3}{x_1 x_2 \xi}\right|\right)^{\theta_2}} dx_3 dx_2 dx_1
\]

\[
= I_{2,2,1} + I_{2,2,2}.
\]

By Lemma 2.1 with $f(x_3) = \int_{x_2 \leq |x_2| \leq 8} \Delta_{x_1, \pi} \left(\frac{1}{\xi} K(x_1, x_2, \frac{x_3}{\xi})\right)e^{-ix_2} dx_2$, we have

\[
|I_{2,2,1}| \leq \int_{8 \leq |x_1| \leq N_1} \int_{E_{N_4}} \int_{x_2 \leq |x_2| \leq 8} \Delta_{x_1, \pi} \left(\frac{1}{\xi} K(x_1, x_2, \frac{x_3}{\xi})\right)e^{-ix_2} dx_2 dx_3 dx_1
\]

\[
+ \int_{8 \leq |x_1| \leq N_1} \int_{x_2 \leq |x_2| \leq 8} \int_{8 \leq |x_3| \leq N_4} \Delta_{x_3, \pi} \Delta_{x_1, \pi} \left(\frac{1}{\xi} K(x_1, x_2, \frac{x_3}{\xi})\right)e^{-ix_2} dx_2 dx_3 dx_1
\]

\[
\lesssim 1,
\]

where we use the conditions (R) above with $\alpha = 1, \beta = \gamma = 0$ and $\alpha = \gamma = 1, \beta = 0$, respectively.

The estimate for term $I_{2,2,2}$ follows from a similar way as term $I_{1,2,2}$. Indeed,

\[
I_{2,2,2} = \int_{8 \leq |x_1| \leq N_1} \int_{x_2 \leq |x_2| \leq 8} \int_{x_1 \leq |x_3| \leq 8} \Delta_{x_1, \pi} \left(\frac{1}{\xi} K(x_1, x_2, \frac{x_3}{\xi})\right)(e^{-ix_2} e^{-ix_3} - 1) dx_3 dx_2 dx_1
\]

\[
+ \int_{8 \leq |x_1| \leq N_1} \int_{x_2 \leq |x_2| \leq 8} \int_{8 \leq |x_3| \leq N_4} \Delta_{x_1, \pi} \left(\frac{1}{\xi} K(x_1, x_2, \frac{x_3}{\xi})\right) dx_3 dx_2 dx_1
\]

\[
\lesssim 1,
\]
where we use the condition (R) with $\alpha = 1, \beta = \gamma = 0$ and the condition (C1,c), respectively.

Now we turn to the estimate for term II. We first write

$$II = \int_{\epsilon_4 \leq |x_3| \leq N_4} \int_{\epsilon_2 \leq |x_2| \leq N_2} \int_{\epsilon_1 \leq |x_1| \leq 8} \frac{1}{\xi} K(x_1, x_2, \frac{x_3}{\xi})(e^{-ix_1} - 1)e^{-ix_2}e^{-ix_3}dx_1dx_2dx_3$$

$$+ \int_{\epsilon_4 \leq |x_3| \leq N_4} \int_{\epsilon_2 \leq |x_2| \leq N_2} \int_{\epsilon_1 \leq |x_1| \leq 8} \frac{1}{\xi} K(x_1, x_2, \frac{x_3}{\xi})e^{-ix_2}e^{-ix_3}dx_1dx_2dx_3$$

$$:= II_1 + II_2.$$

We further write

$$|II_1| = \int_{\epsilon_4 \leq |x_3| \leq N_4} \int_{\epsilon_2 \leq |x_2| \leq N_2} \int_{\epsilon_1 \leq |x_1| \leq 8} \frac{1}{\xi} K(x_1, x_2, \frac{x_3}{\xi})(e^{-ix_1} - 1)e^{-ix_2}e^{-ix_3}dx_1dx_2dx_3$$

$$+ \int_{\epsilon_4 \leq |x_3| \leq N_4} \int_{\epsilon_2 \leq |x_2| \leq N_2} \int_{\epsilon_1 \leq |x_1| \leq 8} \frac{1}{\xi} K(x_1, x_2, \frac{x_3}{\xi})e^{-ix_2}e^{-ix_3}dx_1dx_2dx_3$$

$$:= II_{1,1} + II_{1,2}.$$ 

For term $II_{1,1}$, using Lemma 2.1 with $f(x_2) = \int_{\epsilon_4 \leq |x_3| \leq N_4} \int_{\epsilon_1 \leq |x_1| \leq 8} \frac{1}{\xi} K(x_1, x_2, \frac{x_3}{\xi})(e^{-ix_1} - 1)e^{-ix_3}dx_1dx_3$, we obtain

$$|II_{1,1}| \lesssim \int_{E_{N_2}} \int_{\epsilon_4 \leq |x_3| \leq N_4} \int_{\epsilon_1 \leq |x_1| \leq 8} \frac{1}{\xi} K(x_1, x_2, \frac{x_3}{\xi})(e^{-ix_1} - 1)e^{-ix_3}dx_1dx_3 dx_2$$

$$+ \int_{\epsilon_4 \leq |x_3| \leq N_4} \int_{\epsilon_1 \leq |x_1| \leq 8} \frac{1}{\xi} K(x_1, x_2, \frac{x_3}{\xi})e^{-ix_3}dx_1dx_3 dx_2$$

$$\lesssim \int_{E_{N_2}} \int_{\epsilon_4 \leq |x_3| \leq N_4} \int_{\epsilon_1 \leq |x_1| \leq 8} \frac{1}{\xi} K(x_1, x_2, \frac{x_3}{\xi})e^{-ix_3}dx_1dx_2dx_3$$

where we use the condition (R) above for $\alpha = \beta = \gamma = 0$ and $\beta = 1, \alpha = \gamma = 0$, respectively.

Similarly,

$$II_{1,2} = \int_{\epsilon_4 \leq |x_3| \leq N_4} \int_{\epsilon_2 \leq |x_2| \leq 8} \int_{\epsilon_1 \leq |x_1| \leq 8} \frac{1}{\xi} K(x_1, x_2, \frac{x_3}{\xi})(e^{-ix_1} - 1)e^{-ix_2}e^{-ix_3}dx_1dx_2dx_3$$

$$+ \int_{\epsilon_4 \leq |x_3| \leq 8} \int_{\epsilon_2 \leq |x_2| \leq 8} \int_{\epsilon_1 \leq |x_1| \leq 8} \frac{1}{\xi} K(x_1, x_2, \frac{x_3}{\xi})e^{-ix_2}e^{-ix_3}dx_1dx_2dx_3$$

$$:= II_{1,2,1} + II_{1,2,2}.$$ 

The bounds of $II_{1,2,1}$ and $II_{1,2,2}$, we follow from the similar estimates as terms $I_{2,2,1}$ and $I_{2,2,2}$, respectively.
Finally, we estimate term $II_2$. Denote $II_2 = II_{2,1} + II_{2,2}$, where

$$II_{2,1} = \int_{\epsilon_4 \leq |x_4| \leq N_4} \int_{8 \leq |x_2| \leq N_2} \int_{\epsilon_1 \leq |x_1| \leq 8} \frac{1}{\xi} \mathcal{K}(x_1, x_2, \frac{x_3}{\xi}) e^{-ix_2} e^{-ix_3} dx_1 dx_2 dx_3$$

and

$$II_{2,2} = \int_{\epsilon_4 \leq |x_4| \leq N_4} \int_{\epsilon_2 \leq |x_2| \leq 8} \int_{\epsilon_1 \leq |x_1| \leq 8} \frac{1}{\xi} \mathcal{K}(x_1, x_2, \frac{x_3}{\xi}) e^{-ix_2} e^{-ix_3} dx_1 dx_2 dx_3.$$

Note that

$$|II_{2,1}| = \int_{8 \leq |x_2| \leq N_2} \int_{\epsilon_1 \leq |x_1| \leq 8} \int_{\epsilon_1 \leq |x_1| \leq 8} \frac{1}{\xi} \mathcal{K}(x_1, x_2, \frac{x_3}{\xi}) e^{-ix_2} e^{-ix_3} dx_1 dx_2 dx_3$$

Applying Lemma 2.1 with $f(x_2) = \int_{\epsilon_1 \leq |x_1| \leq 8} \frac{1}{\xi} \mathcal{K}(x_1, x_2, \frac{x_3}{\xi}) e^{-ix_3} dx_1 dx_3$ first, then $f(x_3) = \int_{\epsilon_1 \leq |x_1| \leq 8} \frac{1}{\xi} \mathcal{K}(x_1, x_2, \frac{x_3}{\xi}) dx_1$, and combining with the condition (R), we obtain

$$|II_{2,1,1}| \approx \int_{E_{N_2}} \int_{8 \leq |x_3| \leq N_4} \int_{\epsilon_1 \leq |x_1| \leq 8} \frac{1}{\xi} \mathcal{K}(x_1, x_2, \frac{x_3}{\xi}) e^{-ix_3} dx_1 dx_3 dx_2$$

To estimate $II_{2,1,2}$, inserting $e^{-ix_3} = [e^{-ix_3} - 1] + 1$ and then applying Lemma 2.1, we get

$$II_{2,1,2} \approx \int_{8 \leq |x_2| \leq N_2} \int_{\epsilon_1 \leq |x_1| \leq 8} \int_{\epsilon_4 \leq |x_4| \leq 8} \frac{1}{\xi} \mathcal{K}(x_1, x_2, \frac{x_3}{\xi}) e^{-ix_2} (e^{-ix_3} - 1) dx_3 dx_1 dx_2$$

and

$$II_{2,2,1} = \int_{8 \leq |x_2| \leq N_2} \int_{\epsilon_1 \leq |x_1| \leq 8} \int_{\epsilon_4 \leq |x_4| \leq 8} \frac{1}{\xi} \mathcal{K}(x_1, x_2, \frac{x_3}{\xi}) e^{-ix_2} dx_3 dx_1 dx_2$$

and

$$II_{2,2,2} = \int_{8 \leq |x_2| \leq N_2} \int_{\epsilon_1 \leq |x_1| \leq 8} \int_{\epsilon_4 \leq |x_4| \leq 8} \frac{1}{\xi} \mathcal{K}(x_1, x_2, \frac{x_3}{\xi}) dx_3 dx_1 dx_2.$$
\[ + \int_{8 \leq |x_3| \leq N_2} \int_{\epsilon_1 \leq |x_1| \leq 8} \int_{\epsilon_4 \leq |x_4| \leq 8} \left| \Delta_{x_2, \pi} \left( \frac{1}{\xi} \mathcal{K}(x_1, x_2, \frac{x_3}{\xi}) \right) \right| \left| x_3 \right| dx_3 dx_4 dx_2 \]

\[ + \int_{E_{N_2}} \int_{\epsilon_1 \leq |x_1| \leq 8} \int_{\epsilon_4 \leq |x_4| \leq 8} \frac{1}{\xi} \mathcal{K}(x_1, x_2, \frac{x_3}{\xi}) dx_3 dx_1 \right| \left| x_2 \right| dx_2 \]

\[ + \int_{8 \leq |x_2| \leq N_2} \int_{\epsilon_1 \leq |x_1| \leq 8} \int_{\epsilon_4 \leq |x_4| \leq 8} \Delta_{x_2, \pi} \left( \frac{1}{\xi} \mathcal{K}(x_1, x_2, \frac{x_3}{\xi}) \right) dx_3 dx_1 \right| \left| x_2 \right| dx_2. \]

The required bound then follows from the conditions (R) for the first two integrals while the condition (C1.d) for the last two integrals.

For \( II_{2,2} \), splitting the set \( \{ \epsilon_4 \leq |x_4| \leq N_4 \} \) into two parts \( \{ \epsilon_4 \leq |x_3| \leq 8 \} \) and \( \{ 8 \leq |x_3| \leq N_4 \} \), and inserting \( e^{-ix_2}e^{-ix_3} = (e^{-ix_2} - 1)(e^{-ix_3} - 1) + (e^{-ix_2} - 1) + (e^{-ix_3} - 1) + 1 \) for the integral over the first set and \( e^{-ix_2}e^{-ix_3} = (e^{-ix_2} - 1)e^{-ix_3} + e^{-ix_3} \) for the integral over the second set, we obtain

\[ II_{2,2} \lesssim \int_{\epsilon_4 \leq |x_4| \leq 8} \int_{\epsilon_2 \leq |x_2| \leq 8} \int_{\epsilon_1 \leq |x_1| \leq 8} \frac{1}{\xi} \mathcal{K}(x_1, x_2, \frac{x_3}{\xi}) \left( (e^{-ix_2} - 1)(e^{-ix_3} - 1) \right) dx_1 dx_2 dx_3 \]

\[ + \int_{\epsilon_4 \leq |x_4| \leq 8} \int_{\epsilon_2 \leq |x_2| \leq 8} \int_{\epsilon_1 \leq |x_1| \leq 8} \frac{1}{\xi} \mathcal{K}(x_1, x_2, \frac{x_3}{\xi}) \left( e^{-ix_2} - 1 \right) dx_1 dx_2 dx_3 \]

\[ + \int_{\epsilon_4 \leq |x_4| \leq 8} \int_{\epsilon_2 \leq |x_2| \leq 8} \int_{\epsilon_1 \leq |x_1| \leq 8} \frac{1}{\xi} \mathcal{K}(x_1, x_2, \frac{x_3}{\xi}) \left( e^{-ix_3} - 1 \right) dx_1 dx_2 dx_3 \]

\[ + \int_{\epsilon_4 \leq |x_4| \leq 8} \int_{\epsilon_2 \leq |x_2| \leq 8} \int_{\epsilon_1 \leq |x_1| \leq 8} \frac{1}{\xi} \mathcal{K}(x_1, x_2, \frac{x_3}{\xi}) \left( e^{-ix_2} \right) dx_1 dx_2 dx_3 \]

\[ + \int_{8 \leq |x_3| \leq N_4} \int_{\epsilon_2 \leq |x_2| \leq 8} \int_{\epsilon_1 \leq |x_1| \leq 8} \frac{1}{\xi} \mathcal{K}(x_1, x_2, \frac{x_3}{\xi}) \left( e^{-ix_2} - 1 \right) e^{-ix_3} dx_1 dx_2 dx_3 \]

\[ + \int_{8 \leq |x_3| \leq N_4} \int_{\epsilon_2 \leq |x_2| \leq 8} \int_{\epsilon_1 \leq |x_1| \leq 8} \frac{1}{\xi} \mathcal{K}(x_1, x_2, \frac{x_3}{\xi}) \left( e^{-ix_3} \right) e^{-ix_3} dx_1 dx_2 dx_3. \]

The first four items follow from the conditions (R),(C1.d),(C1.b), and (C1.a), respectively. To estimate the fifth and sixth terms, we apply Lemma 2.11 to get

\[ \left| \int_{8 \leq |x_3| \leq N_4} \int_{\epsilon_2 \leq |x_2| \leq 8} \int_{\epsilon_1 \leq |x_1| \leq 8} \frac{1}{\xi} \mathcal{K}(x_1, x_2, \frac{x_3}{\xi}) \left( e^{-ix_2} - 1 \right) e^{-ix_3} dx_1 dx_2 dx_3 \right| \]

\[ \lesssim \int_{E_{N_4}} \int_{\epsilon_2 \leq |x_2| \leq 8} \int_{\epsilon_1 \leq |x_1| \leq 8} \frac{1}{\xi} \mathcal{K}(x_1, x_2, \frac{x_3}{\xi}) \left( e^{-ix_2} - 1 \right) dx_1 dx_2 \right| \left| x_3 \right| dx_3 \]

\[ + \int_{8 \leq |x_3| \leq N_4} \int_{\epsilon_2 \leq |x_2| \leq 8} \int_{\epsilon_1 \leq |x_1| \leq 8} \Delta_{x_3, \pi} \left( \frac{1}{\xi} \mathcal{K}(x_1, x_2, \frac{x_3}{\xi}) \right) \left( e^{-ix_2} - 1 \right) dx_1 dx_2 \right| \left| x_3 \right| dx_3 \]

\[ + \int_{E_{N_4}} \int_{\epsilon_2 \leq |x_2| \leq 8} \int_{\epsilon_1 \leq |x_1| \leq 8} \frac{1}{\xi} \mathcal{K}(x_1, x_2, \frac{x_3}{\xi}) dx_1 dx_2 \right| \left| x_3 \right| dx_3 \]

This completes the proof of the estimate for \( II_{2,2} \).
where we use the condition (R) for the first two terms and (C1.b) for the last two terms above. Thus these estimates yield the bound of $II_{2,2}$ and hence the required bound for term $II$. The $L^2$ boundedness of $K^N_{\epsilon} \ast f$ follows.

Proof of Corollary L.3. It suffices to show that $K^N_{\epsilon} \ast f$ converges in $L^2(\mathbb{R}^3)$, as $\epsilon_1, \epsilon_2, \epsilon_3 \to 0$ and $N_1, N_2, N_3 \to \infty$, for a dense subset of $L^2(\mathbb{R}^3)$. For this purpose, we consider smooth functions $f$ having compact support. We may assume that $\epsilon_1, \epsilon_2, \epsilon_3 < 1$ and $N_1, N_2, N_3 > 1$.

We write $\iiint_{\mathbb{R}^3} K^N_{\epsilon}(u)f(x-u)\,du$ as a sum of eight terms; that is, the integrals over the sets (i) $|u_1| \leq 1, |u_2| \leq 1, |u_3| \leq 1$; (ii) $|u_1| \leq 1, |u_2| \leq 1, |u_3| > 1$; (iii) $|u_1| \leq 1, |u_2| > 1, |u_3| \leq 1$; (iv) $|u_1| \leq 1, |u_2| > 1, |u_3| > 1$; (v) $|u_1| > 1, |u_2| \leq 1, |u_3| \leq 1$; (vi) $|u_1| > 1, |u_2| \leq 1, |u_3| > 1$; (vii) $|u_1| > 1, |u_2| > 1, |u_3| \leq 1$; (viii) $|u_1| > 1, |u_2| > 1, |u_3| > 1$.

Inserting

$$f(x-u) = [f(x_1 - u_1, x_2 - u_2, x_3 - u_3) - f(x_1 - u_1, x_2 - u_2, x_3 - u_3)]$$

$$+ [f(x_1, x_2, x_3) - f(x_1, x_2, x_3)]$$

$$+ [f(x_1, x_2 - u_2, x_3 - u_3) - f(x_1, x_2 - u_2, x_3)]$$

$$+ [f(x_1, x_2 - u_2, x_3) - f(x_1, x_2, x_3)]$$

$$+ [f(x_1, x_2 - u_2, x_3) - f(x_1, x_2, x_3)]$$

$$+ [f(x_1, x_2, x_3) - f(x_1, x_2, x_3)]$$

into the first term

$$\int_{|u_1| \leq 1} \int_{|u_2| \leq 1} \int_{|u_3| \leq 1} K(u_1, u_2, u_3)f(x_1 - u_1, x_2 - u_2, x_3 - u_3)\,du_1\,du_2\,du_3$$

yields five integrals. In view of the conditions of $f$ and the condition (R) on $K$, the first integral is dominated by

$$F_1(x_1)F_2(x_2)F_3(x_3) \int_{|u_1| \leq 1} \int_{|u_2| \leq 1} \int_{|u_3| \leq 1} \frac{1}{|u_1||u_2||u_3|}(\frac{|u_1u_2|}{u_3} + \frac{u_3}{u_1u_2})^{-\theta_2} \times |u_1||u_2| + |u_3|\,du_1\,du_2\,du_3,$n

where $F_1(x_1), F_2(x_2)$ and $F_3(x_3)$ are bounded functions with bounded supports. Thus, as $\epsilon_1, \epsilon_2, \epsilon_3 \to 0$, the limit of the first integral exists for each $x_1, x_2, x_3$ and, moreover, is dominated by a fixed bounded function with compact support. Therefore, the first integral converges in $L^2$ as $\epsilon_1, \epsilon_2, \epsilon_3 \to 0$. The third integral can be handled by the same way. To see the second integral, by the condition (C1.c) and the assumption on $K$ we observe that the limit $\int_{|u_2| \leq 1} \int_{|u_3| \leq 1} K(u_1, u_2, u_3)\,du_2\,du_3$ exists as $\epsilon_2, \epsilon_3 \to 0$, and is dominated by $C|u_1|^{-1}$. This fact together with the smoothness condition on $f$ implies the
second integral converges in $L^2$ as $\epsilon_1, \epsilon_2, \epsilon_3 \to 0$ and the limit is dominated by a bounded function with compact support. Similarly, the required results for the fourth and the last integrals follow from the conditions (C1.d) and (C1.a), respectively, together with the assumptions on $K$.

Note that in fact $K(u)$ is integrable over the sets (ii) $|u_1| \leq 1, |u_2| \leq 1, |u_3| \geq 1$ and (vii) $|u_1| \geq 1, |u_2| \geq 1, |u_3| \leq 1$. Thus we have all the required results over these two sets.

Observe that

\[
\int \int \int_{|u_3| \geq 1} |K(u)f(x-u)|du \lesssim \int \int \int_{|u_3| \geq 1} \frac{1}{|u_1||u_2||u_3|} \left( \left| \frac{u_1 u_2}{u_3} \right| + \left| \frac{u_3}{u_1 u_2} \right| \right)^{-\theta_2} \\
\times \frac{1}{(1 + |x_1 - u_1|)^2(1 + |x_2 - u_2|)^2(1 + |x_3 - u_3|)^2} \, du,
\]

which belongs to $L^2(\mathbb{R}^3)$. This implies the required results over the corresponding sets (iv), (vi) and (viii).

To handle the integral over the set (iii) $|u_1| \leq 1, |u_2| \geq 1, |u_3| \leq 1$, inserting

\[
f(x_1 - u_1, x_2 - u_2, x_3 - u_3) = [f(x_1 - u_1, x_2 - u_2, x_3 - u_3) - f(x_1, x_2 - u_2, x_3 - u_3)] \\
+ [f(x_1, x_2 - u_2, x_3 - u_3) - f(x_1, x_2 - u_2, x_3)] \\
+ f(x_1, x_2 - u_2, x_3)
\]

yields three integrals over the set (iii). The first two integrals, by the condition (R) and the smoothness of $f$, are dominated by

\[
F_1(x_1)F_3(x_3) \int_{|u_1| \leq 1} \int_{|u_2| \geq 1} \int_{|u_3| \leq 1} \frac{1}{|u_1||u_2||u_3|} \left( \left| \frac{u_1 u_2}{u_3} \right| + \left| \frac{u_3}{u_1 u_2} \right| \right)^{-\theta_2} \frac{1}{(1 + |x_2 - u_2|)^2} \, du_1 \, du_2 \, du_3
\]

and

\[
F_1(x_1)F_3(x_3) \int_{|u_1| \leq 1} \int_{|u_2| \geq 1} \int_{|u_3| \leq 1} \frac{1}{|u_1||u_2||u_3|} \left( \left| \frac{u_1 u_2}{u_3} \right| + \left| \frac{u_3}{u_1 u_2} \right| \right)^{-\theta_2} \frac{1}{(1 + |x_2 - u_2|)^2} \, du_1 \, du_2 \, du_3,
\]

where $F_1(x_1)$ and $F_3(x_3)$ are bounded functions with bounded supports. Thus, we obtain a domination, independent of $\epsilon_1, \epsilon_3$ and $N_2$, by a function which belongs to $L^2(\mathbb{R}^3)$, so the limits as $\epsilon_1, \epsilon_3 \to 0$ and $N_2 \to \infty$ exist. Condition (C1.d) with $\beta = 0$ yields that the last integral is bounded by

\[
F_1(x_1)F_3(x_3) \int_{|u_2| \geq 1} \frac{1}{|u_2|(1 + |x_2 - u_2|)^2} \, du_2,
\]

which belongs to $L^2(\mathbb{R}^3)$ and the limit as $\epsilon_1, \epsilon_3 \to 0$ and $N_2 \to \infty$ exists.

Finally, for the integral over the set (v) $|u_1| \geq 1, |u_2| \leq 1, |u_3| \leq 1$, by inserting

\[
f(x_1 - u_1, x_2 - u_2, x_3 - u_3) = [f(x_1 - u_1, x_2 - u_2, x_3 - u_3) - f(x_1 - u_1, x_2 - u_2, x_3)] \\
+ [f(x_1 - u_1, x_2 - u_2, x_3) - f(x_1 - u_1, x_2, x_3)] \\
+ f(x_1 - u_1, x_2, x_3)
\]
and then applying the condition (R) for the first two integrals and (C1.c) with \( \alpha = 0 \) on the last integral, this integral is dominated by

\[
F_2(x_2)F_3(x_3) \int_{|u_1| \geq 1} \frac{1}{|u_1| (1 + |x_1 - u_1|)^2} du_1,
\]

where \( F_2(x_2) \) and \( F_3(x_3) \) are bounded functions with bounded supports. The existence of the limit is concluded. The \( L^2 \) boundedness of \( K \ast f \) then follows from Theorem 1.1. \( \square \)

**Remark 2.2.** As mentioned early in section 1, we have incidentally shown that \( \mathcal{K}_{\epsilon}^N \ast f \) converges in \( L^p \) norm and almost everywhere as \( \epsilon_1, \epsilon_2, \epsilon_3 \to 0 \) and \( N_1, N_2, N_3 \to \infty \) whenever \( f \) is a smooth function with compact support. We also point out that the condition (C1.b) is not used in the proof of Corollary 1.2.

### 3. \( L^p \) estimates for \( 1 < p < \infty \)

In this section, we will prove Theorem 1.3 and Theorem 1.4. The main tools to show the \( L^p, 1 < p < \infty \), estimates are

- the \( L^2 \) boundedness of \( K \ast f \);
- the Littlewood–Paley theory associated with Zygmund dilation;
- the almost orthogonality argument.

We first recall the Littlewood–Paley theory. As mentioned in section 1, to handle the \( L^p, 1 < p < \infty \), boundedness of operators, one only needs the continuous Littlewood-Paley square function. To do this, let \( \mathcal{S}(\mathbb{R}^i) \) denote the Schwartz class in \( \mathbb{R}^i, i = 1, 2, 3 \). We construct a function defined on \( \mathbb{R}^3 \) by

\[
\phi(x_1, x_2, x_3) = \phi^{(1)}(x)\phi^{(2)}(x_2, x_3),
\]

where \( \phi^{(1)} \in \mathcal{S}(\mathbb{R}), \phi^{(2)} \in \mathcal{S}(\mathbb{R}^2) \) with the supports contained in the unit ball centered at the origin in \( \mathbb{R}^3 \), and satisfy

\[
\sum_{j \in \mathbb{Z}} |\hat{\phi}^{(1)}(2^j \xi_1)|^2 = 1 \quad \text{for all } \xi_1 \in \mathbb{R} \setminus \{0\},
\]

\[
\sum_{k \in \mathbb{Z}} |\hat{\phi}^{(2)}(2^k \xi_2, 2^k \xi_3)|^2 = 1 \quad \text{for all } (\xi_2, \xi_3) \in \mathbb{R}^2 \setminus \{(0, 0)\},
\]

and the moment conditions

\[
\int_{\mathbb{R}} x_1^\alpha \phi^{(1)}(x_1) dx_1 = \int_{\mathbb{R}^2} x_2^\beta x_3^\gamma \phi^{(2)}(x_2, x_3) dx_2 dx_3 = 0 \quad \text{for } 0 \leq \alpha, \beta, \gamma \leq 10.
\]
For \( f \in L^p \), \( 1 < p < \infty \), the continuous Littlewood–Paley square function \( g^\epsilon_j(f) \) of \( f \) associated with the Zygmund dilation is defined by

\[
g^\epsilon_j(f)(x) = \left\{ \sum_{j,k \in \mathbb{Z}} |(\phi_{j,k} * f)(x)|^2 \right\}^{\frac{1}{2}},
\]

where

\[
(3.5) \quad \phi_{j,k}(x_1, x_2, x_3) := 2^{-2(j+k)}\phi^{(1)}(2^{-j}x_1)\phi^{(2)}(2^{-k}x_2, 2^{-(j+k)}x_3).
\]

By taking the Fourier transform, it is easy to see that the following Calderón’s identity

\[
(3.6) \quad f(x) = \sum_{j,k \in \mathbb{Z}} (\phi_{j,k} * \phi_{j,k} * f)(x)
\]

holds on \( L^2(\mathbb{R}^3) \). Using the \( L^p \) boundedness of operators for \( 1 < p < \infty \) in [26], as mentioned in section 1, we have

\[
\left\| \sum_{(j,k) \in F} \epsilon(j,k)(\phi_{j,k} * f) \right\|_p \leq C\| f \|_p
\]

for every sequence \( \epsilon(j,k) \), taking the values 1 and \(-1\), where \( F \) is any finite subset of \( j, k \in \mathbb{Z} \). By Khinchin’s well-known inequality,

\[
\| g^\epsilon_j(f) \|_p \leq C_2\| f \|_p \quad \text{for} \quad 1 < p < \infty.
\]

This estimate together with Calderón’s identity on \( L^2 \) allows us to obtain the \( L^p \) estimates of \( g^\epsilon_j \) for \( 1 < p < \infty \). Namely, there exist constants \( C_1 \) and \( C_2 \) such that, for \( 1 < p < \infty \),

\[
(3.7) \quad C_1\| f \|_p \leq \| g^\epsilon_j(f) \|_p \leq C_2\| f \|_p.
\]

Now we turn to the proof of Theorem 1.3. First note that \( \mathcal{K} \) satisfies the conditions (C1.a) – (C1.d) since (C2.b) implies (C1.b) and (C1.d), as mentioned in section 1. Therefore, by Corollary 1.2, the operator \( \mathcal{K} * f = \lim_{N_1,N_2,N_3 \to \infty} \mathcal{K}_\epsilon^N * f \) is bounded on \( L^2(\mathbb{R}^3) \). To obtain the \( L^p \) boundedness of \( \mathcal{K} * f \), it suffices to show this for all \( f \in L^2 \cap L^p \) since the subspace \( L^2 \cap L^p \) is dense in \( L^p \), \( 1 < p < \infty \). By the \( L^p \) estimates of the Littlewood–Paley square function given in (3.7), the \( L^p \) boundedness of \( \mathcal{K} * f \) will follow from the estimate

\[
(3.8) \quad \| g^\epsilon_j(\mathcal{K} * f) \|_p \lesssim \| f \|_p.
\]

To prove (3.8) for all \( f \in L^2 \cap L^p \), using the fact that \( \mathcal{K} * f \) is bounded on \( L^2 \), as mentioned above, and Calderón’s identity on \( L^2 \) given in (3.6), we write

\[
\phi_{j,k} * (\mathcal{K} * f)(x_1, x_2, x_3) = \sum_{j',k' \in \mathbb{Z}} (\phi_{j,k} * \mathcal{K} * \phi_{j',k'} * \phi_{j',k'} * f)(x_1, x_2, x_3).
\]

The proof of Theorem 1.3 now follows from the following almost orthogonality argument.
Lemma 3.2. follows from the following lemma.

Proof. For simplicity, let
\[
\max(\lambda, \theta, \theta_2) = \min(\frac{1}{2} \min(\theta_1, \theta_2), \lambda)
\]
where we use Fefferman–Stein’s vector-valued maximal inequality and the Littlewood–Paley square function estimate for \( L^p \), \( 1 < p < \infty \), in the last two inequalities, respectively.

Assuming Proposition 3.1 for the moment, we then observe that
\[
|\phi_{j,k} * \mathcal{K} * \phi_{j',k'}(x_1, x_2, x_3)| \leq C 2^{-|j-j'|} 2^{-|k-k'|} \frac{2^{-j|j'|}}{(1 + 2^{-j|j'|}|x_1|)^{1+\lambda}} \times \frac{2^{-j|j'|}}{(1 + 2^{-j|j'|}|x_2|)^{1+\lambda}} \frac{2^{-j|j'|}}{(1 + 2^{-j|j'|}|x_3|)^{1+\lambda}},
\]
where the constant \( C \) depends only on \( \lambda \) and \( \mathcal{K} * f \) is defined for \( f \in L^2 \) as in Corollary 1.2 and \( j \vee j' \) means \( \max(j, j') \).

Proposition 3.1. Suppose that \( \phi_{j,k} \) is defined as in (3.5) and \( \mathcal{K} \) is a function on \( \mathbb{R}^3 \) satisfying the conditions (R) and (C2.a) – (C2.c). Then, for \( \lambda = \frac{1}{2} \min(\theta_1, \theta_2) \),
\[
|\phi_{j,k} * \phi_{j',k'}(x_1, x_2, x_3)| \leq 2^{-|j-j'|} 2^{-|k-k'|} \left( \frac{2^{-j|j'|}}{(1 + 2^{-j|j'|}|x_1|)^{1+\lambda}} \times \frac{2^{-j|j'|}}{(1 + 2^{-j|j'|}|x_2|)^{1+\lambda}} \frac{2^{-j|j'|}}{(1 + 2^{-j|j'|}|x_3|)^{1+\lambda}} \right),
\]
where \( \mathcal{M}_s \) is the strong maximal function on \( \mathbb{R}^3 \). Hölder’s inequality implies
\[
\|g_{j}(\mathcal{K} * f)\|_p = \left\| \sum_{j,k} |\phi_{j,k} * \mathcal{K} * f|^2 \right\|_p \leq C \left\| \sum_{j',k'} |\mathcal{M}_s(\phi_{j',k'} * f)|^2 \right\|_p \\
\leq C \left\| \sum_{j',k'} |\phi_{j',k'} * f|^2 \right\|_p \leq C \|f\|_p,
\]
where we use Fefferman–Stein’s vector-valued maximal inequality and the Littlewood–Paley square function estimate for \( L^p \), \( 1 < p < \infty \), in the last two inequalities, respectively.

To finish the proof of Theorem 1.3, we only need to show Proposition 3.1 whose proof follows from the following lemma.

Lemma 3.2. Suppose that \( \phi^{(1)} \) and \( \phi^{(2)} \) satisfy the conditions (3.1) – (3.2) and \( \mathcal{K} \) is a function on \( \mathbb{R}^3 \) satisfying the conditions (R) and (C2.a) – (C2.c). Then, for \( \lambda = \frac{1}{2} \min(\theta_1, \theta_2) \),
\[
|\mathcal{K} * (\phi^{(1)} \otimes \phi^{(2)})(x_1, x_2, x_3)| \leq \frac{C_\lambda}{(1 + |x_1|)^{1+\lambda}(1 + |x_2|)^{1+\lambda}(1 + |x_3|)^{1+\lambda}},
\]
where \( C_\lambda \) is the constant depending only on \( \lambda \).

Proof. For simplicity, let \( S = \lim_{N_1, N_2, N_3 \to \infty} \int_{c_2 \leq |x_2 - u_2| \leq N_1} \int_{c_3 \leq |x_3 - u_3| \leq N_1} \mathcal{K}(x_1 - u_1, x_2 - u_2, x_3 - u_3) \phi^{(1)}(u_1) \phi^{(2)}(u_2, u_3) du_3 du_2 du_1 \). We consider the following eight cases.

Case 1. \( |x_1| \geq 3, |x_2| \geq 3, |x_3| \geq 3 \). For this case, we use the cancellation conditions in (3.4) to write
\[
S = \lim_{N_1, N_2, N_3 \to \infty} \int_{c_1 \leq |x_1 - u_1| \leq N_1} \int_{c_2 \leq |x_2 - u_2| \leq N_1} \int_{c_3 \leq |x_3 - u_3| \leq N_1} \left[ \mathcal{K}(x_1 - u_1, x_2 - u_2, x_3 - u_3) - \mathcal{K}(x_1, x_2 - u_2, x_3 - u_3) - \mathcal{K}(x_1 - u_1, x_2, x_3) + \mathcal{K}(x_1, x_2, x_3) \right] \\
\times \phi^{(1)}(u_1) \phi^{(2)}(u_2, u_3) du_3 du_2 du_1.
\]
Note that \( \mathcal{K}(x_1 - u_1, x_2 - u_2, x_3 - u_3) - \mathcal{K}(x_1, x_2 - u_2, x_3 - u_3) - (\mathcal{K}(x_1 - u_1, x_2, x_3) - \mathcal{K}(x_1, x_2, x_3)) = \Delta_{x_2, -u_2} \Delta_{x_1, -u_1} \mathcal{K}(x_1, x_2, x_3 - u_3) + \Delta_{x_3, -u_3} \Delta_{x_1, -u_1} \mathcal{K}(x_1, x_2, x_3). \) Thus, by the condition (R) with \( \alpha = \beta = 1, \gamma = 0 \) and \( \alpha = \gamma = 1, \beta = 0, \) respectively,

\[
|S| \lesssim \int_{|u_1| \leq 1} \int_{|u_2| \leq 1} \int_{|u_3| \leq 1} \left| \frac{|u_1|^\theta_1 (|u_2|^\theta_1 + |u_3|^\theta_1)}{|x_1|^{1+\theta_1}|x_2 + u_2||x_3 - u_3|} \right| \left( \frac{|x_1 x_2|}{x_3} + \left| \frac{x_3}{x_1 x_2} \right| \right)^{-\lambda} \, du_3 du_2 du_1 \lesssim \frac{1}{(1 + |x_1|)^{1+\lambda}(1 + |x_2|)^{1+\lambda}(1 + |x_3|)^{1+\lambda}}.
\]

Case 2. \(|x_1| \geq 3, |x_2| \geq 3, |x_3| < 3. \) By the cancellation condition of \( \phi^{(1)} \),

\[
S = \lim_{N_1, N_2, N_3 \to \infty} \int_{|u_1| \leq |x_1| - u_1 \leq N_1} \int_{|u_2| \leq |x_2| - u_2 \leq N_2} \int_{|u_3| \leq |x_3| - u_3 \leq N_3} \Delta_{x_1, -u_1} \mathcal{K}(x_1, x_2 - u_2, x_3 - u_3) \times \phi^{(1)}(u_1) \phi^{(2)}(u_2, u_3) \, du_3 du_2 du_1.
\]

Therefore, by the condition (R) with \( \alpha = 1 \) and \( \beta = \gamma = 0, \) we obtain

\[
|S| \lesssim \int_{|u_1| \leq 1} \int_{|u_2| \leq 1} \int_{|u_3| \leq 1} \frac{|u_1|^\theta_1}{|x_1|^{1+\theta_1}|x_2 - u_2||x_3 - u_3|} \times \left( \frac{|x_1 (x_2 - u_2)|}{x_3 - u_3} + \left| \frac{x_3 - u_3}{x_1 (x_2 - u_2)} \right| \right)^{-\lambda} \, du_3 du_2 du_1 \lesssim \frac{1}{(1 + |x_1|)^{1+\lambda}(1 + |x_2|)^{1+\lambda}(1 + |x_3|)^{1+\lambda}}.
\]

Case 3. \(|x_1| \geq 3, |x_2| < 3, |x_3| \geq 3. \) The same expression for \( S \) as in case 2 yields

\[
|S| \lesssim \int_{|u_1| \leq 1} \int_{|u_2| \leq 1} \int_{|u_3| \leq 1} \frac{|u_1|^\theta_1}{|x_1|^{1+\theta_1}|x_2 - u_2||x_3|} \left| \frac{x_3}{x_1 (x_2 - u_2)} \right|^{-\lambda} \, du_3 du_2 du_1 \lesssim \frac{1}{(1 + |x_1|)^{1+\lambda}(1 + |x_2|)^{1+\lambda}(1 + |x_3|)^{1+\lambda}}.
\]

Case 4. \(|x_1| \geq 3, |x_2| < 3, |x_3| < 3. \) Using the cancellation condition of \( \phi^{(1)} \), we write

\[
S = \lim_{N_1, N_2, N_3 \to \infty} \int_{|u_1| \leq |x_1| - u_1 \leq N_1} \int_{|u_2| \leq |x_2| - u_2 \leq N_2} \int_{|u_3| \leq |x_3| - u_3 \leq N_3} \left[ \mathcal{K}(x_1 - u_1, u_2, u_3) - \mathcal{K}(x_1, u_2, u_3) \right] \times \phi^{(1)}(u_1) \phi^{(2)}(x_2 - u_2, x_3 - u_3) - \phi^{(2)}(x_2, x_3) \, du_3 du_2 du_1
\]

\[
+ \lim_{N_1, N_2, N_3 \to \infty} \int_{|u_1| \leq |x_1| - u_1 \leq N_1} \int_{|u_2| \leq |x_2| - u_2 \leq N_2} \int_{|u_3| \leq |x_3| - u_3 \leq N_3} \left[ \mathcal{K}(x_1 - u_1, u_2, u_3) - \mathcal{K}(x_1, u_2, u_3) \right] \times \phi^{(1)}(u_1) \phi^{(2)}(x_2, x_3) \, du_3 du_2 du_1.
\]
By the condition (R) with $\alpha = 1$, $\beta = \gamma = 0$ for the first integral, and the cancellation condition (C2.c) with $\alpha = 1$ for the second integral,

$$|S| \lesssim \int_{|u_1| \leq 1} \int_{|u_2| \leq 1} \int_{|u_3| \leq 1} \frac{|u_1|^\theta_1}{|u_1||x_1|^{1+\theta_1}} |u_3|^{-\theta_2} (|u_2| + |u_3|) du_3 du_2 du_1$$

$$+ \int_{|u_1| \leq 1} \frac{|u_1|^\theta_1}{|x_1|^{1+\theta_1}} du_1$$

$$\lesssim \frac{1}{|x_1|^{1+\theta_1}}$$

$$\lesssim \frac{1}{(1 + |x_1|)^{1+\lambda}(1 + |x_2|)^{1+\lambda}(1 + |x_3|)^{1+\lambda}}.$$ 

Case 5. $|x_1| < 3, |x_2| \geq 3, |x_3| \geq 3$. Similar to case 4, using the cancellation condition of $\phi^{(2)}$, we write

$$S = \lim_{N_1, N_2, N_3 \to \infty} \int_{|x_1| \leq 1} \int_{|x_2| \leq 1} \int_{|x_3| \leq 1} \left[ K(u_1, x_2 - u_2, x_3 - u_3) - K(u_1, x_2, x_3) \right] \times \left( \phi^{(1)}(x_1 - u_1) - \phi^{(1)}(x_1) \right) \phi^{(2)}(u_2, u_3) du_3 du_2 du_1$$

$$+ \lim_{N_1, N_2, N_3 \to \infty} \int_{|x_1| \leq 1} \int_{|x_2| \leq 1} \int_{|x_3| \leq 1} \left[ K(u_1, x_2 - u_2, x_3 - u_3) - K(u_1, x_2, x_3) \right] \times \phi^{(1)}(x_1) \phi^{(2)}(u_2, u_3) du_3 du_2 du_1.$$

Note that $K(u_1, x_2 - u_2, x_3 - u_3) - K(u_1, x_2, x_3) = \Delta_{x_2, u_2} K(u_1, x_2, x_3) + \Delta_{x_3, u_3} K(u_1, x_2, x_3)$ and $K(u_1, x_2 - u_2, x_3 - u_3) - K(u_1, x_2, x_3) = \Delta_{x_2, u_2} K(u_1, x_2, x_3) + \Delta_{x_3, u_3} K(u_1, x_2, x_3)$.

Thus, using the condition (R) on $K$, the smoothness of $\phi^{(1)}$ for the first integral, and the cancellation conditions (C2.b) with $\beta = 1, \gamma = 0$ and $\beta = 0, \gamma = 1$, respectively, for the second integral, and applying the dominated convergence theorem, we obtain

$$|S| \lesssim \int_{|u_1| \leq 1} \int_{|u_2| \leq 1} \int_{|u_3| \leq 1} \left( \frac{|u_2|^\theta_1}{|u_1||x_1|^{1+\theta_1}} + \frac{|u_3|^\theta_1}{|u_1||x_2|^{1+\theta_1}} \right) \times \left( \frac{|u_1 x_2|}{x_3} + \frac{|x_3|}{u_1 x_2} \right)^{-\lambda} |u_1| du_3 du_2 du_1$$

$$+ \int_{|u_2| \leq 1} \int_{|u_3| \leq 1} \left( \frac{|u_2|^\theta_1}{|x_2|^{1+\theta_1}} + \frac{|u_3|^\theta_1}{|x_2|^{1+\theta_1}} \right) \left( \frac{4 x_2}{x_3} + \frac{|x_3|}{4 x_2} \right)^{-\lambda} du_3 du_2$$

$$\lesssim \frac{1}{(1 + |x_1|)^{1+\lambda}(1 + |x_2|)^{1+\lambda}(1 + |x_3|)^{1+\lambda}}.$$
Case 6. \(|x_1| < 3, |x_2| \geq 3, |x_3| < 3\). Note that

\[
S = \lim_{N_1,N_2,N_3 \to \infty} \int_{\epsilon_1 \leq |u_1| \leq 4} \int_{\epsilon_2 \leq |x_2-u_2| \leq N_2} \int_{\epsilon_3 \leq |x_3-u_3| \leq N_3} \mathcal{K}(u_1, x_2 - u_2, x_3 - u_3) \\
\times (\phi^{(1)}(x_1 - u_1) - \phi^{(1)}(x_1)) \phi^{(2)}(u_2, u_3) du_2 du_3 du_1 \\
+ \lim_{N_1,N_2,N_3 \to \infty} \int_{\epsilon_1 \leq |u_1| \leq 4} \int_{\epsilon_2 \leq |x_2-u_2| \leq N_2} \int_{\epsilon_3 \leq |x_3-u_3| \leq N_3} \mathcal{K}(u_1, x_2 - u_2, x_3 - u_3) \\
\times \phi^{(1)}(x_1) \phi^{(2)}(u_2, u_3) du_2 du_3 du_1.
\]

By the condition (R) with \(\alpha = \beta = \gamma = 0\) and the smoothness condition of \(\phi^{(1)}\) on the first integral, the condition (C2.b) with \(\beta = \gamma = 0\) for the second integral, and the dominated convergent theorem,

\[
|S| \lesssim \int_{|u_1| \leq 4} \int_{|u_2| \leq 1} \int_{|u_3| \leq 1} \frac{1}{|u_1||x_2||x_3 - u_3| \left( \frac{|x_1 x_2|}{x_3 - u_3} + \frac{x_3 - u_3}{u_1 x_2} \right)^\lambda} |u_1| du_2 du_3 du_1 \\
+ \int_{|u_2| \leq 1} \int_{|u_3| \leq 1} \frac{1}{|x_2||x_3 - u_3| \left( \frac{4|x_2|}{x_3 - u_3} + \frac{x_3 - u_3}{4|x_2|} \right)^\lambda} du_2 du_3 du_1 \\
\lesssim \frac{1}{|x_2|^{1+\lambda}} \\
\lesssim \frac{1}{(1 + |x_2|)^{1+\lambda}(1 + |x_3|)^{1+\lambda}}.
\]

Case 7. \(|x_1| < 3, |x_2| < 3, |x_3| \geq 3\). The required estimate follows directly from the condition (R):

\[
|S| \lesssim \int_{|u_1| \leq 1} \int_{|u_2| \leq 1} \int_{|u_3| \leq 1} \frac{1}{|x_1 - u_1||x_2 - u_2||x_3| \left( x_3 \right)^\lambda (x_1 - u_1)(x_2 - u_2)} du_2 du_3 du_1 \\
\lesssim \frac{1}{|x_3|^{1+\lambda}} \\
\lesssim \frac{1}{(1 + |x_1|)^{1+\lambda}(1 + |x_2|)^{1+\lambda}(1 + |x_3|)^{1+\lambda}}.
\]

Case 8. \(|x_1| < 3, |x_2| < 3, |x_3| < 3\). Inserting

\[
\phi^{(1)}(x_1 - u_1) \phi^{(2)}(x_2 - u_2, x_3 - u_3) = [\phi^{(1)}(x_1 - u_1) - \phi^{(1)}(x_1)] [\phi^{(2)}(x_2 - u_2, x_3 - u_3) - \phi^{(2)}(x_2, x_3)] \\
+ \phi^{(1)}(x_1)[\phi^{(2)}(x_2 - u_2, x_3 - u_3) - \phi^{(2)}(x_2, x_3)] \\
+ [\phi^{(1)}(x_1 - u_1) - \phi^{(1)}(x_1)] \phi^{(2)}(x_2, x_3) + \phi^{(1)}(x_1) \phi^{(2)}(x_2, x_3),
\]

we write

\[
S = \lim_{N_1,N_2,N_3 \to \infty} \int_{\epsilon_1 \leq |u_1| \leq 4} \int_{\epsilon_2 \leq |u_2| \leq 4} \int_{\epsilon_3 \leq |u_3| \leq 4} \mathcal{K}(u_1, u_2, u_3)
\]
Suppose that Lemma 3.3.

and hence Theorem 1.3 is proved.

as four integrals. Using the condition (R) with \( \alpha = \beta = \gamma = 0 \), the smoothness condition of \( \phi^{(1)} \) for the first integral, the cancellation conditions (C2.b), (C2.c), (C2.a) for the last three integrals, and the dominated convergent theorem, we obtain

\[
|S| \lesssim \int_{|u_1| \leq 4} \int_{|u_2| \leq 4} \int_{|u_3| \leq 4} \frac{1}{|u_1||u_2||u_3|} \left( \frac{|u_1u_2|}{u_3} + \frac{|u_3|}{|u_1u_2|} \right)^{-\theta_2} |u_1|(|u_2| + |u_3|)du_3du_2du_1
\]

\[
+ \int_{|u_2| \leq 4} \int_{|u_3| \leq 4} \frac{1}{|u_2||u_3|} \left( \frac{4u_2}{u_3} + \frac{u_3}{4u_2} \right)^{-\theta_2} (|u_2| + |u_3|)du_3du_2
\]

\[
+ \int_{|u_1| \leq 4} \frac{1}{|u_1|} |u_1|du_1 + 1
\]

\[
\lesssim 1
\]

\[
\lesssim \frac{1}{(1 + |x_1|)^{1+\lambda}(1 + |x_2|)^{1+\lambda}(1 + |x_3|)^{1+\lambda}}.
\]

The proof of Lemma 3.2 is completed.

Recall that \( \phi_{j,k}(u_1, u_2, u_3) = 2^{-2(j+k)}\phi^{(1)}(2^{-j}u_1)\phi^{(2)}(2^{-k}u_2, 2^{-(j+k)}u_3) \), and the assumptions on \( \mathcal{K} \) are invariant with respect to Zygmund dilation. By Lemma 3.2 we have the following estimate

\[
|\langle \mathcal{K} * \phi_{j,k} \rangle(x_1, x_2, x_3)| \leq C \frac{2^{-j}}{(1 + 2^{-j}|x_1|)^{1+\lambda}} \frac{2^{-k}}{(1 + 2^{-k}|x_2|)^{1+\lambda}} \frac{2^{-(j+k)}}{(1 + 2^{-(j+k)}|x_3|)^{1+\lambda}}.
\]

Now the proof of Proposition 3.1 follows from the above estimate with replacing \( \phi_{j,k} \) by \( \phi_{j,k} * \phi_{j',k'} \). Note that, by Lemma 3.3 given below, \( \phi_{j,k} * \phi_{j',k'} \) satisfies the same properties as \( \phi_{j\vee j', k\vee k'} \) but with the bound \( C2^{-|j-j'|}2^{-|k-k'|} \). Thus, the proof of Proposition 3.1 follows and hence Theorem 1.3 is proved.

The following lemma is an almost orthogonal estimate.

Lemma 3.3. Suppose that \( \phi_{j,k} \) is defined as in (3.3). Then

\[
|\phi_{j,k} * \phi_{j',k'}(x)| \leq 2^{-|j-j'|L2^{-|k-k'|L}} \frac{2^M(j\vee j')}{(2^{j\vee j'} + |x_1|)^{1+M}} \frac{2^M}{2^M(k\vee k')} \frac{2^M(k\vee k')}{2^M(k\vee k') + |x_2| + 2^{-j^*}|x_3|^2 + M}
\]

for any fixed \( L, M > 0 \), where \( x = (x_1, x_2, x_3) \in \mathbb{R}^3, j^* = j \) if \( k \geq k' \) and \( j^* = j' \) if \( k < k' \).
Proof. Let \( \phi^{(2)}_{j,k}(x_2, x_3) = 2^{-j}2^{-2k}\phi^{(2)}(2^{-k}x_2, 2^{-k-j}x_3) \). Note that \( \phi^{(1)}_j(x_1) = 2^{-j}\phi^{(1)}(2^{-j}x_1) \). Then \( \phi_{j,k}(x_1, x_2, x_3) = \phi^{(1)}_j(x_1)\phi^{(2)}_{j,k}(x_2, x_3) \), and hence (3.9) follows if we prove the following estimates:

\[
|\phi^{(1)}_j * \phi^{(1)}_j(x_1)| \lesssim 2^{-ij-j'\left|L_1\right|} \frac{2^{M(j'j')}}{(2^{j'j'} + |x_1|)^{1+M}}
\]

and

\[
|\phi^{(2)}_{j,k} * \phi^{(2)}_{j',k'}(x_2, x_3)| \lesssim 2^{(ij-j'((L_2+M+2)2^{-|k-k'|L_2}) - 2^{-|k-k'|L_2})} \frac{2^{M(kk')}}{2^{j'}(2^{k'k'} + |x_2| + 2^{-j'}|x_3|)^{2+M}}
\]

for any fixed \( L_1, L_2, M > 0 \).

Inequality (3.10) is the classical almost orthogonality estimate and thus it suffices to show (3.11).

By symmetry, we can only consider the case when \( k > k' \). Applying the cancellation conditions on \( \phi^{(2)}_{j,k} \) and the smoothness conditions on \( \phi^{(2)}_{j,k} \), we write

\[
\left| \int_{\mathbb{R}^2} \phi^{(2)}_{j,k}(x_2 - u_2, x_3 - u_3)\phi^{(2)}_{j',k'}(u_2, u_3)du_2du_3 \right|
\]

\[
= \left| \int_{\mathbb{R}^2} [\phi^{(2)}_{j,k}(x_2 - u_2, x_3 - u_3) - P_{L-1}[\phi^{(2)}_{j,k}](x_2, x_3)]\phi^{(2)}_{j',k'}(u_2, u_3)du_2du_3 \right|
\]

\[
\lesssim \int_{\mathbb{R}^2} \left( \frac{|u_2|}{2^k} + \frac{|u_3|}{2^{j+k}} \right) \frac{2^{2-2k-j}}{(1 + 2^{-k}|\xi_2| + 2^{-j-k}|\xi_3|)^{M_1}}
\]

\[
\times \frac{2^{-2k' - j'}}{(1 + 2^{-k'}|u_2| + 2^{-j'-k'}|u_3|)^{M_2}}du_2du_3
\]

for some \( (\xi_2, \xi_3) \) on the segment joining \((x_2 - u_2, x_3 - u_3)\) to \((x_2, x_3)\), where \( P_{L-1}[\phi^{(2)}_{j,k}](x_2, x_3) \) denotes the Taylor polynomial of order \( L - 1 \) of \( \phi^{(2)}_{j,k} \) at \((x_2, x_3)\).

By the triangle inequality,

\[
|x_2| \leq |\xi_2| + |x_2 - \xi_2| \leq |\xi_2| + |u_2|,
\]

\[
|x_3| \leq |\xi_3| + |x_3 - \xi_3| \leq |\xi_3| + |u_3|.
\]

From (3.13) and the fact that \( k > k' \),

\[
2^{-k}|x_2| \leq 2^{-k}|\xi_2| + 2^{-k}|u_2| \leq 2^{-k}|\xi_2| + 2^{-k'}|u_2|.
\]

Using (3.14) and \( k > k' \) again, we get

\[
2^{-j-k}|x_3| \leq 2^{-j-k}|\xi_3| + 2^{-j-k}|u_3| \leq 2^{-j-k}|\xi_3| + 2^{j-j'}2^{-j'-k'}|u_3|
\]

\[
\leq 2^{j-j'}(2^{-j-k}|\xi_3| + 2^{-j'-k'}|u_3|).
\]

Putting (3.15) and (3.16) together gives

\[
1 + 2^{-k}|x_2| + 2^{-j-k}|x_3| \leq 2^{j-j'}(1 + 2^{-k}|\xi_2| + 2^{-k'}|u_2| + 2^{-j-k}|\xi_3| + 2^{-j'-k'}|u_3|)
\]

\[
\leq 2^{j-j'}(1 + 2^{-k}|\xi_2| + 2^{-j-k}|\xi_3|)(1 + 2^{-k'}|u_2| + 2^{-j'-k'}|u_3|).
This is equivalent to
\[
\frac{1}{1 + 2^{-k}|\xi_2| + 2^{-j-k}|\xi_3|} \leq 2^{j-j'}\frac{1 + 2^{-k'}|u_2| + 2^{-j'-k'}|u_3|}{1 + 2^{-k}|x_2| + 2^{-j-k}|x_3|}.
\]
We also have
\[
\left(\frac{|u_2|}{2^k} + \frac{|u_3|}{2^j+k}\right)^L \leq 2^{[|j-j'|+(k'-k)]L} \left(\frac{|u_2|}{2^{k'}} + \frac{|u_3|}{2^{j'+k'}}\right)^L.
\]
We insert these estimates to the last integral in (3.12) and use the fact that \(M_2 > M_1 + L + 2\) to get
\[
\left|\int_{\mathbb{R}^2} \phi_{j,k}^{(2)}(x_2 - u_2, x_3 - u_3)\phi_{j',k'}^{(2)}(u_2, u_3)du_2du_3\right|
\leq 2^{(k'-k)L}2^{(j-j')(L+M_1)}2^{-2j-1} \frac{2^{2k-j}}{(1 + 2^{-k}|x_2| + 2^{-j-k}|x_3|)^{M_1}}
\times \int_{\mathbb{R}^2} (1 + 2^{-k'}|u_2| + 2^{-j'-k'}|u_3|)^{M_2-M_1-L}du_2du_3
\leq 2^{(k'-k)L}2^{(j-j')(L+M_1)}2^{-2k-j} \frac{2^{2k-j}}{(1 + 2^{-k}|x_2| + 2^{-j-k}|x_3|)^{M_1}},
\]
which gives (3.11) with \(L = L_2, M_1 = M + 2\). This concludes the proof of Lemma 3.3. \(\square\)

We now turn to the proof of Theorem 1.4. To prove part (a), we first show the \(L^2\) boundedness of \(K * f\). This is similar to the proof of Theorem 1.1. We only outline the proof as follows.

By the Plancherel theorem, the \(L^2\) boundedness of \(K * f\) is equivalent to \(|\hat{K}(\chi, \eta, \xi)| \leq A\), where \(\hat{K}\) is the Fourier transform of \(K\) in the sense of distributions and \(A\) is the constant depending only on the constant \(C\).

Let \(\zeta_1(x_1)\) be a smooth function on \(\mathbb{R}\) with \(\zeta_1(x_1) = 1\) if \(|x_1| \leq 8\) and \(\zeta_1(x_1) = 0\) if \(|x_1| \geq 16\), and let \(\zeta_2 = 1 - \zeta_1\). For simplicity, we denote by \(\tilde{K}(x_1, x_2, x_3) = \frac{1}{\chi \eta \xi} \hat{K}(\frac{x_1}{\chi}, \frac{x_2}{\eta}, \frac{x_3}{\xi})\). We write
\[
\tilde{K}(\chi, \eta, \xi) = \int \int \int \tilde{K}(x_1, x_2, x_3)\zeta_2(x_1)e^{-ix_1}e^{-ix_2}e^{-ix_3}dx_1dx_2dx_3
\]
\[
+ \int \int \int \tilde{K}(x_1, x_2, x_3)\zeta_1(x_1)e^{-ix_1}e^{-ix_2}e^{-ix_3}dx_1dx_2dx_3
\]
\[
:=I + II.
\]
To estimate \(I\), we write
\[
|I| = \frac{1}{2} \int \int \int \Delta_{x_1, \eta} \left(\tilde{K}(x_1, x_2, x_3)\zeta_2(x_1)\right)e^{-ix_1}e^{-ix_2}e^{-ix_3}dx_1dx_2dx_3
\]
\[
\lesssim \int \int \int \Delta_{x_1, \eta} \left(\tilde{K}(x_1, x_2, x_3)\zeta_2(x_1)\right)e^{-ix_1}\zeta_2(x_2)e^{-ix_2}e^{-ix_3}dx_1dx_2dx_3
\]
To estimate the term II, note that

\[
\Delta_{x_1 \pi} \left( \mathcal{K}(x_1, x_2, x_3) \xi_2(x_1) \right) e^{-ix_1} \xi_1(x_2) e^{-ix_2} e^{-ix_3} dx_1 dx_2 dx_3
\]

\[= I_1 + I_2.\]

Note that

\[
I_1 \lesssim \left\{ \Delta_{x_2 \pi} \left( \Delta_{x_1 \pi} \left( \mathcal{K}(x_1, x_2, x_3) \xi_2(x_1) \right) \xi_2(x_2) \right) e^{-ix_1} e^{-ix_2} e^{-ix_3} dx_1 dx_2 dx_3 \right\}
\[
\lesssim \int \int \int_{|x_2| \geq 8, |x_1| \geq 8} \frac{1}{|x_1|^{1+\theta_1}|x_2|^{1+\theta_1}|x_3|} \left( \frac{|x_1x_2\xi_2|}{x_3\eta} + \frac{|x_3\eta|}{x_1x_2\xi_2} \right)^{-\theta_2} dx_1 dx_2 dx_3
\]

\[\lesssim 1.\]

For term \(I_2\), note that

\[
|I_2| \leq \left\{ \Delta_{x_1 \pi} \left( \mathcal{K}(x_1, x_2, x_3) \xi_2(x_1) \right) e^{-ix_1} \xi_1(x_2) e^{-ix_2} \xi_2(x_3) e^{-ix_3} dx_1 dx_2 dx_3 \right\}
\[
+ \left\{ \Delta_{x_1 \pi} \left( \mathcal{K}(x_1, x_2, x_3) \xi_2(x_1) \right) e^{-ix_1} \xi_1(x_2) e^{-ix_2} \xi_1(x_3) e^{-ix_3} dx_1 dx_2 dx_3 \right\}
\]

\[:= I_{2,1} + I_{2,2}.\]

Thus,

\[
I_{2,1} = \frac{1}{2} \left\{ \Delta_{x_1 \pi} \left( \mathcal{K}(x_1, x_2, x_3) \xi_2(x_1) \right) e^{-ix_1} \xi_1(x_2) e^{-ix_2} e^{-ix_3} dx_1 dx_2 dx_3 \right\}
\[
\lesssim \int \int \int_{|x_2| \geq 8, |x_1| \geq 8} \frac{1}{|x_1|^{1+\theta_1}|x_2|^{1+\theta_1}|x_3|} \left( \frac{|x_1x_2\xi_2|}{x_3\eta} + \frac{|x_3\eta|}{x_1x_2\xi_2} \right)^{-\theta_2} dx_1 dx_2 dx_3
\]

\[\lesssim 1.\]

To estimate \(I_{2,2}\), we write

\[
I_{2,2} = \left\{ \Delta_{x_1 \pi} \left( \mathcal{K}(x_1, x_2, x_3) \xi_2(x_1) \right) e^{-ix_1} \xi_1(x_2) \xi_1(x_3) \left( e^{-ix_2} e^{-ix_3} - 1 \right) dx_1 dx_2 dx_3 \right\}
\[
+ \left\{ \Delta_{x_1 \pi} \left( \mathcal{K}(x_1, x_2, x_3) \xi_2(x_1) \right) e^{-ix_1} \xi_1(x_2) \xi_1(x_3) dx_1 dx_2 dx_3 \right\}
\]

Inserting \(|e^{-ix_2} e^{-ix_3} - 1| \leq |x_2| + |x_3|\) into the first integral together with the condition (R) and using the cancellation condition (C3.c) for the second integral, we get

\[
I_{2,2} \lesssim \int \int \int_{|x_2| \leq 16, |x_1| \leq 16} \frac{1}{|x_1|^{1+\theta_1}|x_2|^{1+\theta_1}|x_3|} \left( \frac{|x_1x_2\xi_2|}{x_3\eta} + \frac{|x_3\eta|}{x_1x_2\xi_2} \right)^{-\theta_2} (|x_2| + |x_3|) dx_2 dx_3
\]

\[+ \int |x_1|^{-1-\theta_1} dx_1,\]

which is dominated by a constant. Altogether, we obtain the required bound for term \(I\).

Now we estimate term \(II\). We first write

\[
II = \left\{ \mathcal{K}(x_1, x_2, x_3) \xi_1(x_1) (e^{-ix_1} - 1) e^{-ix_2} e^{-ix_3} dx_1 dx_2 dx_3 \right\}
\]
\[ + \iint K(x_1, x_2, x_3)\zeta_1(x_1)e^{-ix_2e^{-ix_3}}dx_1dx_2dx_3 \]
\[ := II_1 + II_2. \]

We further write
\[ II_1 = \iint K(x_1, x_2, x_3)\zeta_1(x_1)(e^{-ix_1} - 1)\zeta_2(x_2)e^{-ix_2}e^{-ix_3}dx_1dx_2dx_3 \]
\[ + \iint K(x_1, x_2, x_3)\zeta_1(x_1)(e^{-ix_1} - 1)\zeta_1(x_2)e^{-ix_2}e^{-ix_3}dx_1dx_2dx_3 \]
\[ := II_{1,1} + II_{1,2}. \]

For term \( II_{1,1} \), we have
\[ |II_{1,1}| \leq \frac{1}{2} \iint \Delta_{x_2,p}(K(x_1, x_2, x_3)\zeta_2(x_3))\zeta_1(x_1)(e^{-ix_1} - 1)e^{-ix_2}e^{-ix_3}dx_1dx_2dx_3. \]

Then the required bound follows from the fact that \( |e^{-ix_1} - 1| \leq |x_1| \) and the condition (R).

Similarly, we write
\[ II_{1,2} = \iint K(x_1, x_2, x_3)\zeta_1(x_1)(e^{-ix_1} - 1)\zeta_1(x_2)e^{-ix_2}\zeta_1(x_3)e^{-ix_3}dx_1dx_2dx_3 \]
\[ + \iint K(x_1, x_2, x_3)\zeta_1(x_1)(e^{-ix_1} - 1)\zeta_2(x_2)e^{-ix_2}\zeta_1(x_3)e^{-ix_3}dx_1dx_2dx_3 \]
\[ := II_{1,2,1} + II_{1,2,2}. \]

Since
\[ |II_{1,2,1}| \leq \frac{1}{2} \iint \Delta_{x_3,p}(K(x_1, x_2, x_3)\zeta_2(x_3))\zeta_1(x_1)(e^{-ix_1} - 1)\zeta_1(x_2)e^{-ix_2}e^{-ix_3}dx_1dx_2dx_3. \]

The required bound for \( II_{1,2,1} \) is concluded by the fact that \( |e^{-ix_1} - 1| \leq |x_1| \) and the condition (R). To estimate term \( II_{1,2,2} \), we write
\[ II_{1,2,2} = \iint K(x_1, x_2, x_3)\zeta_1(x_1)(e^{-ix_1} - 1)\zeta_1(x_2)\zeta_1(x_3)(e^{-ix_2}e^{-ix_3} - 1)dx_1dx_2dx_3 \]
\[ + \iint K(x_1, x_2, x_3)\zeta_1(x_1)(e^{-ix_1} - 1)\zeta_1(x_2)\zeta_1(x_3)dx_1dx_2dx_3. \]

Using the facts that \( |e^{-ix_1} - 1| \leq |x_1| \) and \( |e^{-ix_2}e^{-ix_3} - 1| \leq |x_2| + |x_3| \), the condition (R) for the first integral, and the condition (C3c) for the second integral, we obtain the desired bound for \( II_{1,2,2} \).

Finally, we estimate term \( II_2 \). Denote \( II_2 = II_{2,1} + II_{2,2} \), where \( II_{2,1} \) and \( II_{2,2} \) are given by \( \iint K(x_1, x_2, x_3)\zeta_1(x_1)\zeta_2(x_2)e^{-ix_2}e^{-ix_3}dx_1dx_2dx_3 \) and \( \iint K(x_1, x_2, x_3)\zeta_1(x_1)\zeta_1(x_2)e^{-ix_2}e^{-ix_3}dx_1dx_2dx_3 \), respectively. Then
\[ |II_{2,1}| \leq \frac{1}{2} \iint \Delta_{x_2,p}(K(x_1, x_2, x_3)\zeta_2(x_2))\zeta_1(x_1)e^{-ix_2}e^{-ix_3}dx_1dx_2dx_3 \leq 1. \]
For $II_{2,2}$, we insert

$$
\zeta_1(x_1)\zeta_1(x_2)e^{-ix_2}e^{-ix_3} = \zeta_1(x_1)\zeta_1(x_2)e^{-ix_2}\zeta_2(x_3)e^{-ix_3}
+ \zeta_1(x_1)\zeta_1(x_2)\zeta_1(x_3)(e^{-ix_2}e^{-ix_3} - 1) + \zeta_1(x_1)\zeta_1(x_2)\zeta_1(x_3)
$$

into

$$
\tilde{K}(x_1, x_2, x_3)\zeta_1(x_1)\zeta_1(x_2)e^{-ix_2}e^{-ix_3}dx_1dx_2dx_3
$$

and apply condition (C1.b), (C1.b), and (C1.a). Thus these estimates yield the bound of $II_{2,2}$ and hence the required bound for term $II$. The $L^2$ boundedness of $K*f$ follows.

Next, to show the $L^p$ boundedness of the operator $K*f$, similar to the proof of Theorem 3.3, it suffices to prove the following lemma.

**Lemma 3.4.** Suppose that $\phi^{(1)}$ and $\phi^{(2)}$ satisfy the conditions (3.1) – (3.3) and $K$ is a distribution defined on $\mathbb{R}^3$ satisfying conditions (R) and (C3.a) – (C3.c). Then, for $\lambda = \frac{1}{2}\min(\theta_1, \theta_2),

$$
|K*(\phi^{(1)} \otimes \phi^{(2)})(x_1, x_2, x_3)| \leq \frac{C_\lambda}{(1 + |x_1|)^{1+\lambda}(1 + |x_2|)^{1+\lambda}(1 + |x_3|)^{1+\lambda}},
$$

where $C_\lambda$ is the constant depending only on $\lambda$.

**Proof.** The proof the Lemma 3.4 is similar to the proof of Lemma 3.2. For simplicity, let $S = K*(\phi^{(1)} \otimes \phi^{(2)})(x_1, x_2, x_3)$. We consider the following eight cases.

Case 1. $|x_1| \geq 3, |x_2| \geq 3, |x_3| \geq 3$. For this case, we use (3.4) to write

$$
S = \int_{|u_1| \leq 1} \int_{|u_2| \leq 1} \int_{|u_3| \leq 1} \frac{|u_1|^{\theta_1}|u_2|^{\theta_1}}{|x_1|^{1+\theta_1}|x_2|^{1+\theta_1}|x_3|^{1+\theta_1}} \left( |x_1x_2| + |x_3| \right)^{-\lambda} du_3du_2du_1.
$$

Note that $K(x_1 - u_1, x_2 - u_2, x_3 - u_3) - K(x_1, x_2 - u_2, x_3 - u_3) - (K(x_1 - u_1, x_2, x_3) - K(x_1, x_2, x_3)).$ Thus, by the condition (R) with $\alpha = \beta = 1, \gamma = 0$ and $\alpha = \gamma = 1, \beta = 0$, respectively,

$$
|S| \lesssim \left( \int_{|u_1| \leq 1} \int_{|u_2| \leq 1} \int_{|u_3| \leq 1} \frac{|u_1|^{\theta_1}|u_2|^{\theta_1}}{|x_1|^{1+\theta_1}|x_2|^{1+\theta_1}|x_3|^{1+\theta_1}} \left( |x_1x_2| + |x_3| \right)^{-\lambda} du_3du_2du_1
+ \int_{|u_1| \leq 1} \int_{|u_2| \leq 1} \int_{|u_3| \leq 1} \frac{|u_1|^{\theta_1}|u_3|^{\theta_1}}{|x_1|^{1+\theta_1}|x_2|^{1+\theta_1}|x_3|^{1+\theta_1}} \left( |x_1x_2| + |x_3| \right)^{-\lambda} du_3du_2du_1
\right.
\left. \leq \frac{1}{(1 + |x_1|)^{1+\lambda}(1 + |x_2|)^{1+\lambda}(1 + |x_3|)^{1+\lambda}}.
\right.
$$

Case 2. $|x_1| \geq 3, |x_2| \geq 3, |x_3| < 3$. By the cancellation condition of $\phi^{(1)},$

$$
S = \int_{|u_1| \leq 1} \int_{|u_2| \leq 1} \int_{|u_3| \leq 1} \frac{|u_1|^{\theta_1}|u_2|^{\theta_1}}{|x_1|^{1+\theta_1}|x_2|^{1+\theta_1}|x_3|^{1+\theta_1}} \left( |x_1x_2| + |x_3| \right)^{-\lambda} du_3du_2du_1.
$$
\[ x \phi^{(1)}(u_1) \phi^{(2)}(u_2, u_3) du_3 du_2 du_1. \]

Therefore, using the condition (R) with \( \alpha = 1 \) and \( \beta = \gamma = 0 \), we obtain

\[
|S| \lesssim \int_{|u_1| \leq 1} \int_{|u_2| \leq 1} \int_{|u_3| \leq 1} \frac{|u_1|^\theta_1}{|x_1|^{1+\theta_1}} \frac{|x_1 x_2|}{x_3 - u_3} \frac{|x_1 x_2|}{x_3 - u_3}^{-\lambda} du_3 du_2 du_1 \\
\lesssim \frac{1}{(1 + |x_1|)^{1+\lambda}(1 + |x_2|)^{1+\lambda}(1 + |x_3|)^{1+\lambda}}.
\]

Case 3. \( |x_1| \geq 3, |x_2| < 3, |x_3| \geq 3. \) The same expression for \( S \) as in Case 2 yields

\[
|S| \lesssim \int_{|u_1| \leq 1} \int_{|u_2| \leq 1} \int_{|u_3| \leq 1} \frac{|u_1|^\theta_1}{|x_1|^{1+\theta_1}} \frac{|x_2 - u_2|}{x_3 - u_3} \frac{|x_3|}{x_2 - u_2} \frac{|x_1|}{x_3 - u_3}^{-\lambda} du_3 du_2 du_1 \\
\lesssim \frac{1}{(1 + |x_1|)^{1+\lambda}(1 + |x_2|)^{1+\lambda}(1 + |x_3|)^{1+\lambda}}.
\]

Before handling the other cases, we introduce a bump function \( \tilde{\phi} \) on \( \mathbb{R} \), with \( \tilde{\phi}(x_1) = 1 \) if \( |x_1| \leq 1/2 \) and \( \tilde{\phi}(x_1) = 0 \) if \( |x_1| \geq 1 \).

Case 4. \( |x_1| \geq 3, |x_2| < 3, |x_3| < 3. \) Using the cancellation condition of \( \phi^{(1)} \), we write

\[
\mathcal{K} \ast (\phi^{(1)} \otimes \phi^{(2)})(x_1, x_2, x_3) = \iiint (\mathcal{K}(x_1 - u_1, u_2, u_3) - \mathcal{K}(x, u_2, u_3)) \phi^{(1)}(u_1) \\
\times (\phi^{(2)}(x_2 - u_2, x_3 - u_3) - \phi^{(2)}(x, x_3)) \tilde{\phi}(\frac{u_2}{10}) \tilde{\phi}(\frac{u_3}{10}) du_3 du_2 du_1 \\
+ \iiint (\mathcal{K}(x_1 - u_1, u_2, u_3) - \mathcal{K}(x_1, u_2, u_3)) \phi^{(1)}(u_1) \\
\times \phi^{(2)}(x_2, x_3) \tilde{\phi}(\frac{u_2}{10}) \tilde{\phi}(\frac{u_3}{10}) du_3 du_2 du_1.
\]

Hence, by the condition (R) with \( \alpha = 1, \beta = \gamma = 0 \) for the first integral and the cancellation condition (C3.3) with \( \alpha = 1 \) for the second integral,

\[
|S| \lesssim \int_{|u_1| \leq 1} \int_{|u_2| \leq 10} \int_{|u_3| \leq 10} \frac{|u_1|^\theta_1}{|x_1|^{1+\theta_1}} \frac{|x_1 u_2|}{u_3} \left( \frac{|x_1 u_2|}{u_3} \right) \frac{|u_3|}{x_1 u_2}^{-\lambda} du_3 du_2 du_1 \\
+ \int_{|u_1| \leq 1} \frac{|u_1|^\theta_1}{|x_1|^{1+\theta_1}} du_1 \\
\lesssim \frac{1}{(1 + |x_1|)^{1+\lambda}(1 + |x_2|)^{1+\lambda}(1 + |x_3|)^{1+\lambda}}.
\]

Case 5. \( |x_1| < 3, |x_2| \geq 3, |x_3| \geq 3. \) Similar to Case 4, using the cancellation condition of \( \phi^{(2)} \), we write

\[
S = \iiint [\mathcal{K}(u_1, x_2 - u_2, x_3 - u_3) - \mathcal{K}(u_1, x_2, x_3)]
\]
\[
\times (\phi^{(1)}(x_1 - u_1) - \phi^{(1)}(x_1)) \phi^{(2)}(u_2, u_3)\tilde{\phi}(\frac{u_1}{10}) du_3 du_2 du_1 \\
+ \iiint [\mathcal{K}(u_1, x_2 - u_2, x_3 - u_3) - \mathcal{K}(u_1, x_2, x_3)] \phi^{(1)}(x_1) \phi^{(2)}(u_2, u_3)\tilde{\phi}(\frac{u_1}{10}) du_3 du_2 du_1.
\]

Note that \(\mathcal{K}(u_1, x_2 - u_2, x_3 - u_3) - \mathcal{K}(u_1, x_2, x_3) = \Delta_{x_2-u_2}\mathcal{K}(u_1, x_2, x_3) + \Delta_{x_3-u_3}\mathcal{K}(u_1, x_2, x_3).\) Using condition (R) on \(\mathcal{K},\) the smoothness of \(\phi^{(1)}\) for the first integral, the cancellation conditions (C3.b) with \(\beta = 1, \gamma = 0\) and \(\beta = 0, \gamma = 1\) for the second integral, and applying the dominated convergence theorem, we obtain

\[
|S| \lesssim \int_{|u_1| \leq 10} \int_{|u_2| \leq 1} \int_{|u_3| \leq 1} \left( \frac{|u_2|^\theta_1}{|u_1||x_2|^{1+\theta_1}|x_3|} + \frac{|u_3|^\theta_1}{|u_1||x_2||x_3|^{1+\theta_1}} \right) \\
\times \left( \left| \frac{u_1 x_2}{x_3} \right| + \left| \frac{x_3}{u_1 x_2} \right| \right)^{-\lambda} |u_1| du_3 du_2 du_1 \\
+ \int_{|u_2| \leq 1} \int_{|u_3| \leq 1} \left( \frac{|u_2|^\theta_1}{|x_2|^{1+\theta_1}|x_3|} + \frac{|u_3|^\theta_1}{|x_2||x_3|^{1+\theta_1}} \right) \left( \frac{4x_2}{x_3} \right) \\
+ \left( \left| \frac{x_3}{u_1 x_2} \right| \right)^{-\lambda} du_3 du_2 \\
\lesssim \frac{1}{(1 + |x_1|)^{1+\lambda}(1 + |x_2|)^{1+\lambda}(1 + |x_3|)^{1+\lambda}}.
\]

Case 6. \(|x_1| < 3, |x_2| \geq 3, |x_3| < 3.\) Note that

\[
S = \iiint \mathcal{K}(u_1, x_2 - u_2, x_3 - u_3)(\phi^{(1)}(x_1 - u_1) - \phi^{(1)}(x_1)) \phi^{(2)}(u_2, u_3)\tilde{\phi}(\frac{u_1}{10}) du_3 du_2 du_1 \\
+ \iiint \mathcal{K}(u_1, x_2 - u_2, x_3 - u_3)\phi^{(1)}(x_1) \phi^{(2)}(u_2, u_3)\tilde{\phi}(\frac{u_1}{10}) du_3 du_2 du_1.
\]

By the condition (R) with \(\alpha = \beta = \gamma = 0\) and the smoothness condition of \(\phi^{(1)}\) on the first integral, and the condition (C3.b) with \(\beta = \gamma = 0\) for the second integral,

\[
|S| \lesssim \int_{|u_1| \leq 10} \int_{|u_2| \leq 1} \int_{|u_3| \leq 1} \frac{1}{|u_1||x_2||x_3-u_3|} \left( \left| \frac{u_1 x_2}{x_3 - u_3} \right| + \left| \frac{x_3 - u_3}{u_1 x_2} \right| \right)^{-\lambda} |u_1| du_3 du_2 du_1 \\
+ \int_{|u_2| \leq 1} \int_{|u_3| \leq 1} \frac{1}{|x_2||x_3-u_3|} \left( \left| \frac{x_2}{x_3 - u_3} \right| + \left| \frac{x_3 - u_3}{x_2} \right| \right)^{-\lambda} du_3 du_2 \\
\lesssim \frac{1}{|x_2|^{1+\lambda}} \\
\lesssim \frac{1}{(1 + |x_1|)^{1+\lambda}(1 + |x_2|)^{1+\lambda}(1 + |x_3|)^{1+\lambda}}.
\]

Case 7. \(|x_1| < 3, |x_2| < 3, |x_3| \geq 3.\) The required estimate follows directly from the condition (R):

\[
|S| \lesssim \int_{|u_1| \leq 1} \int_{|u_2| \leq 1} \int_{|u_3| \leq 1} \frac{1}{|x_1-u_1||x_2-u_2||x_3|} \left( \left| \frac{x_3}{u_1 x_2} \right| \right)^{-\lambda} du_3 du_2 du_1 \\
\lesssim \frac{1}{|x_3|^{1+\lambda}}
\]
for the last three integrals, we obtain
\[ \phi(x_1 - u_1, x_2 - u_2, x_3 - u_3) = [\phi(x_1 - u_1) - \phi(x_1)] [\phi(x_2 - u_2, x_3 - u_3) - \phi(x_2, x_3)] + \phi(x_1) [\phi(x_2 - u_2, x_3 - u_3) - \phi(x_2, x_3)] + [\phi(x_1 - u_1) - \phi(x_1)] \phi(x_2, x_3) + \phi(x_1) \phi(x_2, x_3), \]
we write
\[ S = \iiint \mathcal{K}(u_1, u_2, u_3) \times \left\{ [\phi(x_1 - u_1) - \phi(x_1)] [\phi(x_2 - u_2, x_3 - u_3) - \phi(x_2, x_3)] + \phi(x_1) [\phi(x_2 - u_2, x_3 - u_3) - \phi(x_2, x_3)] + [\phi(x_1 - u_1) - \phi(x_1)] \phi(x_2, x_3) + \phi(x_1) \phi(x_2, x_3) \right\} \tilde{\phi}(u_1) \tilde{\phi}(u_2) \tilde{\phi}(u_3) du_3 du_2 du_1 \]
as four integrals. Using the condition (R) with \( \alpha = \beta = \gamma = 0 \) and the smoothness condition of \( \phi^{(1)} \) for the first integral, the cancellation conditions (C3.b), (C3.c) and (C3.a) for the last three integrals, we obtain
\[
|S| \lesssim \int_{|u_1| \leq 4} \int_{|u_2| \leq 4} \int_{|u_3| \leq 4} \frac{1}{|u_1| |u_2| |u_3|} \left( \frac{u_1 u_2}{u_3} + \frac{|u_3|}{u_1 u_2} \right) \frac{u_2}{u_3} (\frac{4 u_2}{u_3} + \frac{u_3}{4 u_2})^{-\theta_2} (|u_2| + |u_3|) du_3 du_2 du_1
\]
\[
\lesssim \int_{|u_1| \leq 4} \int_{|u_2| \leq 4} \int_{|u_3| \leq 4} \frac{1}{|u_1| |u_2| |u_3|} \left( \frac{u_1 u_2}{u_3} + \frac{|u_3|}{u_1 u_2} \right) \frac{u_2}{u_3} (\frac{4 u_2}{u_3} + \frac{u_3}{4 u_2})^{-\theta_2} (|u_2| + |u_3|) du_3 du_2 du_1
\]
\[
\lesssim 1
\]
\[
\lesssim \frac{1}{(1 + |x_1|)^{1+\lambda} (1 + |x_2|)^{1+\lambda} (1 + |x_3|)^{1+\lambda}}.
\]
This completes the proof of Lemma 3.4. \( \square \)

The proof of part (b) of Theorem 1.4 follows from part (a). Indeed, the conditions (R) and (C2.a) – (C2.c) imply the conditions (C3.a) – (C3.c). To see this, inserting
\[
\tilde{\phi}(x_1, x_2, x_3) = \left[ (\tilde{\phi}(x_1, x_2, x_3) - \tilde{\phi}(0, x_2, x_3)) - (\tilde{\phi}(x_1, 0, 0) - \tilde{\phi}(0, 0, 0)) \right] + (\tilde{\phi}(x_1, 0, 0) - \tilde{\phi}(0, 0, 0)) + (\tilde{\phi}(0, x_2, x_3) - \tilde{\phi}(0, x_2, x_3)) + \tilde{\phi}(0, x_2, 0) - \tilde{\phi}(0, 0, 0) + \tilde{\phi}(0, 0, 0)
\]
into \( \iiint \mathcal{K}(x_1, x_2, x_3) \tilde{\phi}(x_1, x_2, x_3) dx_1 dx_2 dx_3 \), we obtain that
\[
\iiint \mathcal{K}(x_1, x_2, x_3) \tilde{\phi}(R_1 x_1, R_2 x_2, R_1 R_2 x_3) dx_1 dx_2 dx_3
\]
\[
\lim_{\epsilon_1, \epsilon_2, \epsilon_3 \to 0} \int_{E(\epsilon, R_1, R_2)} \mathcal{K}(x_1, x_2, x_3) \phi(R_1 x_1, R_2 x_2, R_1 R_2 x_3) \, dx_1 \, dx_2 \, dx_3.
\]

Let \( E(\epsilon, R_1, R_2) = \{ x \in \mathbb{R}^3 : \epsilon_1 \leq |x_1| \leq \frac{1}{R_1}, \epsilon_2 \leq |x_2| \leq \frac{1}{R_2}, \epsilon_3 \leq |x_3| \leq \frac{1}{R_1 R_2} \} \). Then

\[
\int_{E(\epsilon, R_1, R_2)} \mathcal{K}(x_1, x_2, x_3) \phi(R_1 x_1, R_2 x_2, R_1 R_2 x_3) \, dx_1 \, dx_2 \, dx_3
\]

\[
\leq \int_{E(\epsilon, R_1, R_2)} \mathcal{K}(x_1, x_2, x_3) \left\{ \phi(R_1 x_1, R_2 x_2, R_1 R_2 x_3) - \phi(0, R_2 x_2, R_1 R_2 x_3) \right\} \, dx_1 \, dx_2 \, dx_3
\]

\[
+ \int_{E(\epsilon, R_1, R_2)} \mathcal{K}(x_1, x_2, x_3) (\phi(R_1 x_1, 0, 0) - \phi(0, 0, 0)) \, dx_1 \, dx_2 \, dx_3
\]

\[
+ \int_{E(\epsilon, R_1, R_2)} \mathcal{K}(x_1, x_2, x_3) (\phi(0, R_2 x_2, R_1 R_2 x_3) - \phi(0, R_2 x_2, 0)) \, dx_1 \, dx_2 \, dx_3
\]

\[
+ \int_{E(\epsilon, R_1, R_2)} \mathcal{K}(x_1, x_2, x_3) (\phi(0, R_2 x_2, 0) - \phi(0, 0, 0)) \, dx_1 \, dx_2 \, dx_3
\]

\[
\leq 1,
\]

where we apply conditions (R) and (C2.c) for the first and second term, respectively, (C2.b) for the third and fourth term, and (C2.a) for the last term above. Hence \( \mathcal{K} \) satisfies (C3.a).

Similarly, for any \( 0 \leq \beta + \gamma \leq 1 \), n.b.f. \( \tilde{\phi} \) on \( \mathbb{R} \) and \( R > 0 \), we can write

\[
\left| \Delta_{x_2, h_2}^\beta \Delta_{x_3, h_3}^\gamma \mathcal{K}(x_1, x_2, x_3) \phi(R x_1) \, dx_1 \right|
\]
\[ \lim_{\varepsilon \to 0} \left| \int_{|x_1| \leq \frac{1}{\varepsilon}} \Delta_{x_2, h_2}^\beta \Delta_{x_3, h_3}^\gamma \mathcal{K}(x_1, x_2, x_3) \tilde{\phi}(Rx_1) dx_1 \right| \]

and

\[ \left| \int_{|x_1| \leq \frac{1}{\varepsilon}} \Delta_{x_2, h_2}^\beta \Delta_{x_3, h_3}^\gamma \mathcal{K}(x_1, x_2, x_3) \tilde{\phi}(Rx_1) dx_1 \right| \]

\[ \leq \left| \int_{|x_1| \leq \frac{1}{\varepsilon}} \Delta_{x_2, h_2}^\beta \Delta_{x_3, h_3}^\gamma \mathcal{K}(x_1, x_2, x_3)(\tilde{\phi}(Rx_1) - \tilde{\phi}(0)) dx_1 \right| \]

\[ + \left| \int_{|x_1| \leq \frac{1}{\varepsilon}} \Delta_{x_2, h_2}^\beta \Delta_{x_3, h_3}^\gamma \mathcal{K}(x_1, x_2, x_3) \tilde{\phi}(0) dx_1 \right| \]

\[ \leq \int_{|x_1| \leq \frac{1}{\varepsilon}} \frac{|h_2|^{|\beta_1|} |h_3|^{|\gamma_1|}}{|x_1|^{|\beta_1|+1} |x_2| |x_3|^{|\gamma_1|+1}} \left( \left| \frac{x_1 x_2}{x_3} \right| + \left| \frac{x_3}{x_1 x_2} \right| \right)^{-\theta_2} |Rx_1| dx_1 \]

Taking \( \varepsilon \to 0 \), then (C3.b) is obtained.

Finally we verify (C3.c). For any \( 0 \leq \alpha \leq 1 \), n.b.f. \( \tilde{\phi} \) on \( \mathbb{R}^2 \) and \( R_1, R_2 > 0 \), we write

\[ \left| \iint \Delta_{x_1, h_1}^\alpha \mathcal{K}(x_1, x_2, x_3) \tilde{\phi}(R_1 x_2, R_2 x_3) dx_2 dx_3 \right| \]

\[ = \lim_{\varepsilon_1, \varepsilon_2 \to 0} \left| \int_{\varepsilon_2 \leq |x_3| \leq \frac{1}{\varepsilon_2}} \int_{\varepsilon_1 \leq |x_2| \leq \frac{1}{\varepsilon_1}} \Delta_{x_1, h_1}^\alpha \mathcal{K}(x_1, x_2, x_3) \tilde{\phi}(R_1 x_2, R_2 x_3) dx_2 dx_3 \right| \]

and

\[ \left| \int_{|x_2| \leq |x_3| \leq \frac{1}{\varepsilon_2}} \int_{|x_1| \leq |x_2| \leq \frac{1}{\varepsilon_1}} \Delta_{x_1, h_1}^\alpha \mathcal{K}(x_1, x_2, x_3) \tilde{\phi}(R_1 x_2, R_2 x_3) dx_2 dx_3 \right| \]

\[ \leq \left| \int_{|x_2| \leq |x_3| \leq \frac{1}{\varepsilon_2}} \int_{|x_1| \leq |x_2| \leq \frac{1}{\varepsilon_1}} \Delta_{x_1, h_1}^\alpha \mathcal{K}(x_1, x_2, x_3)(\tilde{\phi}(R_1 x_2, R_2 x_3) - \tilde{\phi}(0)) dx_2 dx_3 \right| \]

\[ + \left| \int_{|x_2| \leq |x_3| \leq \frac{1}{\varepsilon_2}} \int_{|x_1| \leq |x_2| \leq \frac{1}{\varepsilon_1}} \Delta_{x_1, h_1}^\alpha \mathcal{K}(x_1, x_2, x_3) \tilde{\phi}(0) dx_2 dx_3 \right| \]

\[ \leq \int_{|x_3| \leq \frac{1}{\varepsilon_2}} \int_{|x_2| \leq \frac{1}{\varepsilon_1}} \frac{|h_1|^{|\alpha_1|}}{|x_1|^{|\alpha_1|+1} |x_2| |x_3|} \left( \left| \frac{x_1 x_2}{x_3} \right| + \left| \frac{x_3}{x_1 x_2} \right| \right)^{-\theta_2} \left( |R_1 x_2| + |R_2 x_3| \right) dx_2 dx_3 \]

\[ + \frac{|h_1|^{|\alpha_1|}}{|x_1|^{|\alpha_1|+1}} \]
Thus (C3.c) is obtained. This completes the proof of part (b), and hence Theorem 1.4 is concluded.

4. Examples and applications

As mentioned in section 1, the original motivation for this paper is to introduce a class of singular integral operators which cover those studied by Ricci and Stein in [26]. Now in this section we show that a special class of singular integrals studied by Ricci and Stein in [26]. Indeed, for $(x_1, x_2, x_3) \in \mathbb{R}^3$, it was proved in [26] that $K(x_1, x_2, x_3) = \sum_{j,k} 2^{2j+2k} \phi^{(1)}(2^j x_1) \phi^{(2)}(2^k x_2, 2^{j+k} x_3)$, where $\phi^{(1)}$ and $\phi^{(2)}$ are defined as in (3.1), is a distribution kernel on $\mathbb{R}^3$. The following result shows that this kernel satisfies the regularity condition (R) and the cancellation conditions (C2.a) – (C2.c).

**Theorem 4.1.** Suppose that $\phi^{(1)}$ and $\phi^{(2)}$ are defined as in (3.1) and

$$K(x_1, x_2, x_3) = \sum_{j,k} 2^{2j+2k} \phi^{(1)}(2^j x_1) \phi^{(2)}(2^k x_2, 2^{j+k} x_3).$$

Then

$$|\partial_{x_1}^{\alpha} \partial_{x_2}^{\beta} \partial_{x_3}^{\gamma} K(x_1, x_2, x_3)| \leq \frac{C_{\alpha,\beta,\gamma,\theta_2}}{|x_1|^\alpha |x_2|^\beta |x_3|^\gamma + 1} \left( \frac{|x_1 x_2|}{x_3} + \frac{|x_3|}{|x_1 x_2|} \right)^{-\theta_2}$$

for all $\alpha, \beta, \gamma \geq 0$ and $0 < \theta_2 < 1$;

$$\int_{\delta_1 \leq |x_1| \leq r_1} \int_{\delta_2 \leq |x_2| \leq r_2} \int_{\delta_3 \leq |x_3| \leq r_3} K(x_1, x_2, x_3) dx_1 dx_2 dx_3 \leq C$$

uniformly for all $\delta_1, \delta_2, \delta_3, r_1, r_2, r_3 > 0$;

$$\int_{\delta \leq |x_1| \leq r} \partial_{x_2}^{\beta} \partial_{x_3}^{\gamma} K(x_1, x_2, x_3) dx_1 \leq \frac{C_{\beta,\gamma,\theta_2}}{|x_1|^\beta |x_3|^\gamma + 1} \left( \frac{1}{|x_2| |x_3|} + \frac{1}{|x_2| |x_3|} \right)^{\theta_2},$$

for all $\delta, r > 0$, $\beta, \gamma \geq 0$ and $0 < \theta_2 < 1$;

$$\int_{\delta_1 \leq |x_2| \leq r_1} \int_{\delta_2 \leq |x_3| \leq r_2} \partial_{x_1}^{\alpha} K(x_1, x_2, x_3) dx_2 dx_3 \leq \frac{C_{\alpha}}{|x_1|^{\alpha+1}}$$

uniformly for all $\delta_1, \delta_2, r_1, r_2 > 0$ and $\alpha \geq 0$.

To show Theorem 4.1 we need the following simple lemmas.

**Lemma 4.2.** Suppose $a, b, c > 0$ with $b > a$ and $r_1, r_2, r_3 > 0$. Then, for all $0 < \varepsilon < 1$,

$$\sum_j 2^{ja} \frac{1}{(1+2^jr_1)^b} \frac{1}{(r_2 + 2^jr_3)^c} \leq C \varepsilon r_1^{-a} r_2^{-c} \left( 1 + \frac{r_3}{r_1 r_2} \right)^{-(a\wedge c)+1-\varepsilon}.$$
Proof. We first write
\[
\sum_j 2^{ja} \frac{1}{(1 + 2^j r_1)^b} \frac{1}{(r_2 + 2^j r_3)^c} = \sum_j 2^{ja} \frac{1}{(1 + 2^j r_1)^b} \frac{1}{(1 + 2^j r_2)^c} r_2^c
\]
\[
\lesssim \sum_{j: 2^j > r_1^{-1}, 2^j > r_2 r_3^{-1}} 2^{ja} \frac{1}{(2^j r_1)^b} \frac{1}{(2^j r_2)^c} r_2^c + \sum_{j: 2^j > r_1^{-1}, 2^j \leq r_2 r_3^{-1}} 2^{ja} \frac{1}{(2^j r_1)^b} r_2^c
\]
\[
I + II + III + IV.
\]

For term I, we observe that
\[
I \lesssim \left(1 + \frac{r_3}{r_1 r_2}\right)^{a-b-c} r_1^{-a} r_2^{-c} \left(\frac{r_3}{r_1 r_2}\right)^{b-a} \lesssim r_1^{-a} r_2^{-c} \left(1 + \frac{r_3}{r_1 r_2}\right)^{-c}.
\]

For II, since \(r_3 < r_1 r_2\),
\[
II \lesssim r_1^{-a} r_2^{-c} \lesssim r_1^{-a} r_2^{-c} \left(1 + \frac{r_3}{r_1 r_2}\right)^{-c}.
\]

For III, note that \(r_3 > r_1 r_2\). We consider three cases. In the first case where \(a > c\), we obtain
\[
III \lesssim r_1^{-a} r_2^{-c} \left(\frac{r_3}{r_1 r_2}\right)^{-c} \lesssim r_1^{-a} r_2^{-c} \left(1 + \frac{r_3}{r_1 r_2}\right)^{-c}.
\]

If \(a < c\), then
\[
III \lesssim r_1^{-a} r_2^{-c} \left(\frac{r_3}{r_1 r_2}\right)^{-a} \lesssim r_1^{-a} r_2^{-c} \left(1 + \frac{r_3}{r_1 r_2}\right)^{-a}.
\]

When \(a = c\), we have
\[
III \lesssim r_3^{-c} \log \left(\frac{r_3}{r_1 r_2}\right) \lesssim r_1^{-a} r_2^{-c} \left(1 + \frac{r_3}{r_1 r_2}\right)^{-a+1-\theta_2}.
\]

Finally, for term IV, we have
\[
IV \lesssim \left(\frac{r_1 r_2}{r_3 + r_1 r_2}\right)^{a} r_1^{-a} r_2^{-c} = r_1^{-a} r_2^{-c} \left(1 + \frac{r_3}{r_1 r_2}\right)^{-a}.
\]

These estimates yield the required bound and Lemma 4.2 is proved.

Lemma 4.3. For any \(N > 0\), \(r > 0\) and \(k \in \mathbb{Z}\), we have
\[
\int_{\{x_1 \in \mathbb{R}: |x_1| \leq r\}} \frac{1}{(1 + 2^k |x_1|)^N} dx_1 \lesssim \frac{r}{1 + 2^k r}
\]
and
\[
\int_{\{x_1 \in \mathbb{R}: |x_1| > r\}} \frac{1}{(1 + 2^k |x_1|)^N} dx_1 \lesssim \frac{2^{-k}}{(1 + 2^k r)^{N-1}}.
\]
Proof. We consider two cases. For the case \( r \leq 2^{-k} \), we clearly have \( \int_{|x_1| \leq r} \frac{1}{(1 + 2^k |x_1|)^N} dx_1 \lesssim r \). The second inequality follows from

\[
\int_{R} \frac{1}{(1 + 2^k |x_1|)^N} dx_1 = 2^{-k} \int_{R} \frac{1}{(1 + |x_1|)^N} dx_1 \lesssim 2^{-k}.
\]

If \( r > 2^{-k} \), then the first inequality follows again from (4.5) while the second follows from

\[
\int_{|x_1| > r} \frac{1}{(1 + 2^k |x_1|)^N} dx_1 \lesssim \frac{2^{-k}}{(1 + 2^k r)^{N-1}}.
\]

The proof of Lemma 4.3 is finished.

We now return to show Theorem 4.1.

Proof of Theorem 4.1. We prove the regularity estimate (4.1) first. By the definition of \( K \) and the conditions on \( \phi^{(1)} \) and \( \phi^{(2)} \), we have

\[
|\partial_x^\alpha \partial_{x_2}^\beta \partial_{x_3}^\gamma K(x_1, x_2, x_3)| \lesssim \sum_{j,k} 2^{2j+2k+\beta+\gamma + k(\beta+\gamma)} (1 + 2^j |x_1|)^{3+\alpha+\gamma} (1 + 2^k |x_2| + 2^{j+k} |x_3|)^{3}.
\]

Note that

\[
\sum_k \frac{2^{2k+k(\beta+\gamma)}}{(1 + 2^k |x_2| + 2^{j+k} |x_3|)^3} = \sum_{k: 2^k \leq (|x_2|+2^k |x_3|)^{-1}} \frac{2^{2k+k(\beta+\gamma)}}{(1 + 2^k |x_2| + 2^{j+k} |x_3|)^3} + \sum_{k: 2^k > (|x_2|+2^k |x_3|)^{-1}} \frac{2^{2k+k(\beta+\gamma)}}{(1 + 2^k |x_2| + 2^{j+k} |x_3|)^3} \lesssim \frac{1}{(|x_2| + 2^j |x_3|)^{2+\beta+\gamma}}.
\]

Inserting this estimate into the above inequality, we obtain

\[
|\partial_x^\alpha \partial_{x_2}^\beta \partial_{x_3}^\gamma K(x_1, x_2, x_3)| \lesssim \sum_j \frac{2^{2j+\beta+\gamma + k(\beta+\gamma)}}{(1 + 2^j |x_1|)^{3+\alpha+\gamma} (|x_2| + 2^j |x_3|)^{2+\beta+\gamma}} \lesssim \frac{1}{|x_1|^a |x_2|^b |x_3|^c (1 + \frac{|x_3|}{x_1 x_2})^{(a \wedge b) + c + r_1 r_2 r_3}},
\]

where we apply Lemma 4.2 with \( a = 2 + \alpha + \gamma, b = 3 + \alpha + \gamma, c = 2 + \beta + \gamma, r_1 = |x_1|, r_2 = |x_2| \) and \( r_3 = |x_3| \) in the last inequality. This implies the required estimate.

We now show the cancellation conditions (4.2) – (4.4). To verify (4.3), we observe that

\[
\left| \int_{\delta \leq |x_1| \leq r} \partial_{x_2} \partial_{x_3} \partial_{x_3}^\alpha K(x_1, x_2, x_3) dx_1 \right| \lesssim \sum_{j,k} 2^{2j+2k+\beta+\gamma + k(\beta+\gamma)} \int_{\delta \leq |x_1| \leq r} \phi^{(1)}(2^j x_1) \phi^{(2)}(2^k x_2, 2^{j+k} x_3) dx_1.
\]
Note that, for all \( N \geq 2 \),
\[
\int_{|x_1| \leq r} |\phi^{(1)}(2^j x_1)| dx_1 \lesssim r,
\]
and, by the vanishing condition of \( \phi^{(1)} \),
\[
\int_{|x_1| \leq r} |\phi^{(1)}(2^j x_1)| dx_1 = \int_{|x_1| > r} |\phi^{(1)}(2^j x_1)| dx_1 \leq C_N \int_{|x_1| > r} \frac{1}{(2^j |x_1|)^N} dx_1 \leq C_N 2^{-j N} r^{-N}.
\]
Therefore,
\[
\int_{|x_1| \leq r} |\phi^{(1)}(2^j x_1)| dx_1 \leq C_N \frac{r}{(1 + 2^j r)^N},
\]
which implies
\[
\left| \int_{\delta \leq |x_1| \leq r} \partial_x^\beta \partial_{x_2} \partial_{x_3} K(x_1, x_2, x_3) dx_1 \right| \lesssim \sum_{j,k} \frac{2^{2j+2k+j(\beta+\gamma)}}{(1 + 2^j r)^{3+\gamma}} \left( \frac{2^j}{1 + 2^j \delta} \right)^{\beta+2} \left( \frac{1}{1 + 2^j \delta} \right)^{\gamma+1}.
\]
Summing over \( k \) first yields that the two summations above are dominated by
\[
\sum_j \left( \frac{r}{(1 + 2^j r)^{3+\gamma}} + \frac{\delta}{(1 + 2^j \delta)^{3+\gamma}} \right) \left( \frac{2^j}{1 + 2^j \delta} \right)^{\beta+2} \left( \frac{1}{1 + 2^j \delta} \right)^{\gamma+1}.
\]
Applying Lemma 4.2 with \( a = 2+\gamma, b = 3+\gamma, c = 2+\gamma+\beta, r_1 = r \) or \( \delta, r_2 = |x_2|, r_3 = |x_3| \), we obtain
\[
\left| \int_{\delta \leq |x_1| \leq r} \partial_x^\beta \partial_{x_2} \partial_{x_3} K(x_1, x_2, x_3) dx_1 \right| \lesssim \frac{1}{r^{\gamma+1} |x_2|^{\beta+\gamma+2} \left( 1 + \frac{\beta}{\alpha x_2} \right) \gamma+1+\theta_2} + \frac{1}{\delta^{\gamma+1} |x_2|^{\beta+\gamma+2} \left( 1 + \frac{\beta}{\alpha x_2} \right) \gamma+1+\theta_2},
\]
which implies the desired cancellation condition (4.3).

To show the cancellation condition (4.4), we start with
\[
\left| \int_{\delta_1 \leq |x_2| \leq r_1} \int_{\delta_2 \leq |x_3| \leq r_2} \partial_{x_1}^\alpha K(x_1, x_2, x_3) dx_2 dx_3 \right| \lesssim \sum_{j,k} 2^{2j+2k+j\alpha} \left| \int_{|x_2| \leq r_1} \int_{|x_3| \leq r_2} \left( \partial_{x_1}^\alpha \phi^{(1)}(2^j x_1) \phi^{(2)}(2^k x_2, 2^{j+k} x_3) \right) dx_2 dx_3 \right|
\]
\[
+ \sum_{j,k} 2^{2j+2k+j\alpha} \left| \int_{|x_2| \leq \delta_1} \int_{|x_3| \leq \delta_2} \left( \partial_{x_1}^\alpha \phi^{(1)}(2^j x_1) \phi^{(2)}(2^k x_2, 2^{j+k} x_3) \right) dx_2 dx_3 \right|.
\]
By the vanishing condition of $\phi^{(2)}$, 
\[
\int_{|x_2| \leq r_1} \int_{|x_3| \leq r_2} \phi^{(2)}(2^k x_2, 2^j \cdot k x_3) dx_2 dx_3 = \int_{|x_2| > r_1} \int_{|x_3| \leq r_2} \phi^{(2)}(2^k x_2, 2^j \cdot k x_3) dx_2 dx_3 \\
+ \int_{|x_2| \leq r_1} \int_{|x_3| > r_2} \phi^{(2)}(2^k x_2, 2^j \cdot k x_3) dx_2 dx_3 \\
+ \int_{|x_2| > r_1} \int_{|x_3| > r_2} \phi^{(2)}(2^k x_2, 2^j \cdot k x_3) dx_2 dx_3.
\]
Applying the size condition of $\phi^{(2)}$ and Lemma 4.3, we obtain
\[
\left| \int_{|x_2| \leq r_1} \int_{|x_3| \leq r_2} \phi^{(2)}(2^k x_2, 2^j \cdot k x_3) dx_2 dx_3 \right| \lesssim \frac{2^{-k} r_2}{r_1} \frac{2^{-j-k}}{(1 + 2^k r_1)^3 (1 + 2^j \cdot k r_2)^3},
\]
On other hand, the size condition on $\phi^{(2)}$ yields
\[
\left| \int_{|x_2| \leq r_1} \int_{|x_3| \leq r_2} \phi^{(2)}(2^k x_2, 2^j \cdot k x_3) dx_2 dx_3 \right| \leq \frac{r_1}{1 + 2^k r_1} \frac{r_2}{1 + 2^k r_2}.
\]
Therefore,
\[
\sum_{j,k} 2^{2j+2k+j} \alpha \left| \int_{|x_2| \leq r_1} \int_{|x_3| \leq r_2} \phi^{(2)}(2^k x_2, 2^j \cdot k x_3) dx_2 dx_3 \right| \lesssim \sum_{j,k} 2^{2j+2k+j} \alpha \min \left\{ \left( \frac{2^{-k} r_2}{(1 + 2^k r_1)^3 (1 + 2^j \cdot k r_2)^3} \right) + \frac{2^{-j-k}}{r_1 (1 + 2^k r_1) (1 + 2^j \cdot k r_2)^3} \right\},
\]
Summing over $k$ first and considering the four cases: (i) $2^k \leq r_1^{-1}$ and $2^k \leq 2^{-j} r_2^{-1}$; (ii) $2^k \leq r_1^{-1}$ and $2^k > 2^{-j} r_2^{-1}$; (iii) $2^k > r_1^{-1}$ and $2^k \leq 2^{-j} r_2^{-1}$; (iv) $2^k > r_1^{-1}$ and $2^k > 2^{-j} r_2^{-1}$, we obtain that the last summation above is dominated by
\[
\sum_j \frac{2^{2j+2k+j} \alpha}{(1 + 2^j \cdot k |x_1|)^{3+\alpha}} 2^{-j},
\]
which yields the cancellation condition (4.4).
Finally the cancellation (4.2) follows directly from the following estimates.
\[
\left| \int_{\delta_1 \leq |x_1| \leq r_1} \int_{\delta_2 \leq |x_2| \leq r_2} \int_{\delta_3 \leq |x_3| \leq r_3} K(x_1, x_2, x_3) dx_1 dx_2 dx_3 \right| \lesssim \sum_{j,k} 2^{2j+2k} \alpha \left| \int_{\delta_1 \leq |x_1| \leq r_1} \phi^{(1)}(2^j x_1) dx_1 \right|
\times \left| \int_{\delta_2 \leq |x_2| \leq r_2} \int_{\delta_3 \leq |x_3| \leq r_3} \phi^{(2)}(2^k x_2, 2^j \cdot k x_3) dx_2 dx_3 \right|,
\]
The proof of Theorem 4.1 is complete. □

As mentioned in section 1, a special class of singular integral operators $T_3$ considered by Ricci and Stein [26] is of the form $T_3f = f * K$, where

$$K(x_1, x_2, x_3) = \sum_{k,j \in \mathbb{Z}} 2^{2(k+j)} \phi \left( 2^j x_1, 2^k x_2, 2^j + k x_3 \right)$$

and the function $\phi$ is supported in a unit cube in $\mathbb{R}^3$ and satisfies a certain amount of uniform smoothness with cancellation conditions

$$\int_{\mathbb{R}^2} \phi(x_1, x_2, x_3) dx_1 dx_2 = \int_{\mathbb{R}^2} \phi(x_1, x_2, x_3) dx_2 dx_3 = \int_{\mathbb{R}^2} \phi(x_1, x_2, x_3) dx_3 dx_1 = 0.$$

Fefferman and Pipher [14] showed that the above cancellation conditions are necessary for the $L^2$ boundedness for singular integral $T_3$. Moreover, if $\phi$ satisfies the above cancellation conditions, then $\phi$ can be decomposed by $\phi = \phi_1 + \phi_2$, where $\phi_1$ and $\phi_2$ have the following cancellation conditions

$$\int_{\mathbb{R}} \phi_1(x_1, x_2, x_3) dx_1 = \int_{\mathbb{R}^2} \phi_1(x_1, x_2, x_3) dx_2 dx_3 = 0$$

and

$$\int_{\mathbb{R}} \phi_2(x_1, x_2, x_3) dx_1 = \int_{\mathbb{R}^2} \phi_2(x_1, x_2, x_3) dx_2 dx_3 = 0.$$

This means that the operator $T_3$ studied by Ricci and Stein can be decomposed as $T_3 = T_3^1 + T_3^2$, where the kernels of $T_3^1$ and $T_3^2$ are given, respectively, by

$$K_1(x_1, x_2, x_3) = \sum_{k,j \in \mathbb{Z}} 2^{2(k+j)} \phi_1 \left( 2^j x_1, 2^k x_2, 2^j + k x_3 \right)$$

and

$$K_2(x_1, x_2, x_3) = \sum_{k,j \in \mathbb{Z}} 2^{2(k+j)} \phi_2 \left( 2^j x_1, 2^k x_2, 2^j + k x_3 \right).$$

Theorem 4.1 shows that the kernel $K_1$ satisfies the regularity (R) and cancellation conditions (C2.a) – (C2.c) while the kernel $K_2$ satisfies the regularity (R) and cancellation conditions (C2’.a) – (C2’.c). Therefore, these operators $K_1$ and $K_2$ belong to our class.

**Remark 4.4.** Actually, based on the proof of Theorem 4.1 we note that the kernel

$$K(x_1, x_2, x_3) = \sum_{j,k \in \mathbb{Z}} 2^{2j+2k} \phi^{(1)}(2^j x_1) \phi^{(2)}(2^k x_2, 2^j + k x_3)$$

as in Theorem 4.1 satisfies the following stronger conditions
(i) \[ |\partial_{x_1}^\alpha \partial_{x_2}^\beta \partial_{x_3}^\gamma \mathcal{K}(x_1, x_2, x_3)| \leq \frac{C_{\alpha,\beta,\gamma,\theta_2}}{|x_1|^{\alpha+\gamma+2} |x_2|^{\beta+\gamma+2} (1 + \frac{x_1}{|x_1|})^{(\alpha+\beta)+\gamma+1+\theta_2}}; \]

(ii) \[ \int_{\delta_1 \leq |x_1| \leq r_1} \int_{\delta_2 \leq |x_2| \leq r_2} \int_{\delta_3 \leq |x_3| \leq r_3} \mathcal{K}(x_1, x_2, x_3)dx_1dx_2dx_3 \leq C \]

uniformly for all \( \delta_1, \delta_2, r_1, r_2, r_3 > 0; \)

(iii) \[ \int_{\delta \leq |x_1| \leq r} |\partial_{x_2}^\beta \partial_{x_3}^\gamma \mathcal{K}(x_1, x_2, x_3)|dx_1 \]

\[ \leq C_{\beta,\gamma,\theta_2} \left( \frac{1}{r^{\gamma+1} |x_2|^{\beta+\gamma+2} (1 + \frac{x_1}{r})^{\gamma+1+\theta_2}} + \frac{1}{\delta^{\gamma+1} |x_2|^{\beta+\gamma+2} (1 + \frac{x_1}{\delta})^{\gamma+1+\theta_2}} \right) \]

for all \( \delta, r > 0, \beta, \gamma \geq 0 \) and \( 0 < \theta_2 < 1; \)

(iv) \[ \int_{\delta_1 \leq |x_2| \leq r_1} \int_{\delta_2 \leq |x_3| \leq r_2} |\partial_{x_1}^\alpha \mathcal{K}(x_1, x_2, x_3)|dx_2dx_3 \leq \frac{C_{\alpha}}{|x_1|^{\alpha+1}} \]

uniformly for all \( \delta_1, \delta_2, r_1, r_2 > 0 \) and \( \alpha \geq 0. \)

In [24], Nagel and Wainger considered the \( L^2 \) boundedness of certain singular integral operators on \( \mathbb{R}^n \) whose kernels have appropriate homogeneities with respect to a multi-parameter group of dilations, generated by a finite number of diagonal matrices. In particular, they considered the following two-parameter dilation group

\[ (4.6) \]

\[ \delta(s, t)(x_1, x_2, x_3) = (sx_1, tx_2, s^\alpha t^\beta x_3) \]

acting on \( \mathbb{R}^3 \) for \( s, t, \alpha, \beta > 0. \) They defined a singular kernel \( \mathcal{K} \) by

\[ \mathcal{K}(x_1, x_2, x_3) = \text{sgn}(x_1x_2) \left\{ \frac{|x_1|^{\alpha-1} |x_2|^{\beta-1}}{|x_1|^{2\alpha} |x_2|^{2\beta} + x_3^2} \right\} \]

and proved that convolution with \( \mathcal{K} \) is bounded on \( L^2(\mathbb{R}^3). \)

It is easy to see that when \( \alpha = \beta = 1, \mathcal{K}(x_1, x_2, x_3) \) satisfies all conditions in Corollary 1.2 and Theorem 1.3. Therefore, by Theorem 1.3, the convolution singular integral operator \( \mathcal{K} * f \) with \( \alpha = \beta = 1 \) is also bounded on \( L^p(\mathbb{R}^3) \) for \( 1 < p < \infty, \) where \( \mathcal{K} * f \) is defined by the limit of \( \mathcal{K}_c^N * f \) in the \( L^p, 1 < p < \infty, \) norm. It is worthwhile to point out that the theory we are developing here can be easily generalized to the “anisotropic” case (adapted to \( \delta(s, t) \) in (4.6)). The details are left to the interested reader.

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