INITIALLY REGULAR SEQUENCES AND DEPTHS OF IDEALS

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Abstract. For an arbitrary ideal $I$ in a polynomial ring $R$ and any term order, we define a new notion of initially regular sequences on $R/I$. These sequences share properties with regular sequences, and are relatively easy to construct using information obtained from the initial ideal of $I$. For any ideal $I$ we give a combinatorial description of elements that form an initially regular sequence on $R/I$, and identify situations where these sequences are also regular sequences. We use these results to obtain both a lower bound on depth $R/I$ and a concrete description of a sequence of linear polynomials that is either a regular sequence, or is sufficiently close to a regular sequence to realize the depth bound. We illustrate how polarization of the initial ideal can be used to strengthen the results, which is particularly useful when considering the depths of powers of an ideal. A general bound for the depths of powers of an ideal is given in the case of a squarefree monomial ideal $I$, which generalizes a known bound for depth $R/I$ in terms of the edgewise domination number of the corresponding hypergraph.

1. Introduction

A fundamental invariant in commutative algebra and algebraic geometry is the depth of a module. It appears naturally in the characterization of Cohen-Macaulay rings and modules or, more generally, in Serre’s criteria ($S_k$)'s (cf. [6, 42]). The notion of depth was initially introduced as a homological invariant (under the name of homological codimension — see [1]). Specifically, for a finitely generated module $M$ over a local (or graded) ring $R$ with a maximal (homogenous) ideal $\mathfrak{m}$, the depth of $M$ is

$$\text{depth } M := \min\{d \mid \text{Ext}_R^d(R/\mathfrak{m}, M) \neq 0\}.$$  

From duality theory, depth is also known to be closely related to local cohomology (cf. [19]). Particularly, depth $M = \min\{d \mid H^d_{\mathfrak{m}}(M) \neq 0\}$.

Our work is driven by the important fact that depth $M$ is measured by the maximum length of an $M$-regular sequence in $\mathfrak{m}$ (a sequence of elements $f_1, \ldots, f_d \in \mathfrak{m}$ is said to be an $M$-regular sequence if for each $i$, $f_i$ is a non-zerodivisor on $M/(f_1, \ldots, f_{i-1})M$). Making use of a regular element (or sequence) has been shown to be an essential tool in the proofs of many important results, especially when the technique involves taking hyperplane sections. Having a long regular sequence or, equivalently, knowing that the depth of a module is large is often of interest. In practice, however, finding a concrete description of regular sequences for specific examples is a difficult task.

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Our focus in this paper is on modules of the form \( R/I \), where \( R \) is a polynomial ring and \( I \subseteq R \) is an arbitrary ideal. We introduce a new notion, called an *initially regular sequence* on \( R/I \), whose concrete description is tractable and whose length gives an effective lower bound for the depth of \( R/I \). During the past two decades, many papers have appeared with various approaches to computing lower bounds for the depth, or equivalently upper bounds for the projective dimension, of \( R/I \) for a squarefree monomial ideal \( I \) (cf. \[9, 10, 11, 12, 15, 27, 28, 32, 33, 39, 40, 41\]). The general idea has been to associate to the ideal \( I \) a graph or hypergraph \( H \) and use *dominating* or *packing* invariants of \( H \) to bound the depth of \( R/I \). As a consequence of our work, we provide a new bound on depth of \( R/I \). When the initial ideal of \( I \) is the edge ideal of a graph, we provide examples that show our bound compares favorably with previously known bounds. In this setting, the main advantage of our work is the sequence produced, which is often a regular sequence. However, when the generators of the initial ideal of \( I \) have high degrees, our bound is frequently a substantial improvement over known bounds. In addition, we show that polarization can be combined with the techniques we use to form initially regular sequences to improve results.

**Definition 1.1.** Let \( R \) be a polynomial ring over a field, and fix a set of term orders \( >_1, \ldots, >_q \) on \( R \). Let \( I \subseteq R \) be a proper ideal. A sequence of nonconstant polynomials \( f_1, \ldots, f_q \) is said to be an *initially regular sequence* on \( R/I \) if for each \( i = 1, \ldots, q \), \( f_i \) is a regular element on \( R/I_i \), where \( I_i = \text{in}_{>_i}(I_{i-1}, f_{i-1}) \) (here, by convention, \( I_1 = \text{in}_{>_1}(I) \)).

In many cases, a single fixed term order will be used to create the initially regular sequence. If only one term order is specified, it will be understood that all term orders used are the same and that all initial ideals are formed with respect to the fixed term order.

**Example 1.2.** Let \( R = \mathbb{Q}[x_1, x_2, x_3, x_4, x_5] \). Let \( I = (x_1 x_2 x_3 + x_3 x_4, x_2 x_5 + x_1 x_2 x_4, x_3 x_5) \) be a polynomial ideal in \( R \). Using Macaulay 2 \[18\] we determine that depth \( R/I = 2 \). Notice that \( \text{in}(I) = (x_3 x_5, x_3 x_4^2, x_1 x_2 x_4, x_1 x_2 x_3) \) with respect to the graded reverse lexicographic order with \( x_1 > x_2 > x_5 > x_4 > x_3 \). Let \( f = x_1 + x_2 \), and \( g = x_5 + x_3 \). Then, \( f, g \) (and \( g, f \)) is an initially regular sequence on \( R/I \). In fact, \( f, g \) and \( g, f \) are also both regular sequences on \( R/I \).

Our first result, Theorem \[22\] shows that if \( f_1, \ldots, f_q \) form an initially regular sequence on \( R/I \) then depth \( R/I \geq q \). The remaining task is to explicitly construct initially regular sequences. By definition, \( f_1 \) will be a regular element on \( R/I_1 \), where \( I_1 = \text{in}(I) \) is a monomial ideal. To simplify notation, we will frequently assume this first initial ideal has been found and thus we will work with monomial ideals. Although taking repeated initial ideals appears to be rather cumbersome, we show in our next result that some basic linear sums will always form an initially regular sequence, giving both a combinatorial way to find a lower bound on the depth and a sequence of elements that in many cases is a regular sequence, and in others shares properties with one. Particularly, we prove the following theorem, where for a monomial ideal \( I \subseteq R \) and a variable \( x \) of \( R \), \( d_x(I) \) denotes the maximum power of \( x \) appearing in the minimal monomial generators of \( I \).

**Theorem 3.11.** Let \( I \) be an ideal in a polynomial ring \( R \) and \( >_1 \) a term order. Suppose that \( \{b_{i,j} \mid 1 \leq i \leq q, 0 \leq j \leq t_i \} \) are distinct variables of \( R \) such that:

1. \( d_{b_{i,j}}(\text{in}_{>_1}(I)) \leq 1 \), for all \( i \geq 1 \) and \( j \geq 1 \); and
Let \( f_i = \sum_{j=0}^{t_i} b_{i,j} \) for \( 1 \leq i \leq q \). Then \( f_1, \ldots, f_q \) is an initially regular sequence on \( R/I \) with respect to \( >_1 \) and any term orders \( >_2, \ldots, >_q \) for which \( b_{i,0} >_{i+1} b_{i,j} \) for all \( 1 \leq i \leq q-1 \) and all \( j \). In particular, \( \text{depth} R/I \geq q \).

The proof of Theorem 3.11 is an involved analysis of the structures of Gröbner bases when alternately finding initial ideals and regular elements. Particularly, we give a sufficient condition for a linear sum of variables to be regular with respect to a monomial ideal. Theorem 3.11 leads us to the following algorithm that exhibits an initially regular sequence with respect to any ideal in a polynomial ring.

**Algorithm 3.12.** Given an arbitrary ideal \( I \subseteq R = k[x_1, \ldots, x_n] \):

- Step 1: Choose a term order in \( R \) and compute \( J = \text{in}(I) \).
- Step 2: Let \( H = (V_H, E_H) \) be the underlying hypergraph associated to \( J \) (with degree at each vertex representing the highest power to which the vertex appears in the generators of \( J \)).
- Step 3: Set \( L = [ ] \) be an empty list and \( S = \emptyset \).
- Step 4: Pick a vertex \( b_0 \in V_H \setminus S \).
- Step 5: Let \( E = \{ e \in E_H \mid b_0 \in e \} \) be edges of \( H \) containing \( b_0 \). Select a set of vertices \( B = \{ b_1, \ldots, b_t \} \) such that \( b_i \notin S \) for all \( i \), \( B \cap e \neq \emptyset \) for all \( e \in E \), and for all \( i \), \( b_i \) has degree at most 1. (To optimize the process, select \( B \) to be minimal with respect to inclusion).
- Step 6: Let \( f = b_0 + \sum_{i=1}^{t} b_i \). Append \( f \) to \( L \) and add \( b_0, b_1, \ldots, b_t \) to \( S \).
- Step 7: Repeat Steps 4–6 until either \( S = V_H \) or for any \( b_0 \in V_H \setminus S \) there does not exist a set \( B \) satisfying the conditions above.
- Output: the list \( L \), which forms an initially regular sequence on \( R/I \) with respect to an appropriate term order.

This algorithm is best illustrated by an example.

**Example 1.3.** Let \( I = (x_1 x_2 + x_1 x_3, x_2 x_4 + x_2 x_3^2, x_3 x_4, x_4 x_5, x_5 x_1, x_1 x_6, x_5 x_7, x_7 x_8) \subset R = \mathbb{Q}[x_1, \ldots, x_8] \). Using Macaulay 2 [18], we see that \( \text{depth} R/I = 3 \). This value of depth \( R/I \) can also be obtained by Algorithm 3.12 as follows.

1. Choose the graded reverse lexicographic order in \( R \) with \( x_1 > \ldots > x_8 \). Then \( J = \text{in}(I) = (x_1 x_2, x_2 x_3, x_3 x_4, x_4 x_5, x_1 x_5, x_1 x_6, x_5 x_7, x_7 x_8) \). The graph \( H = H(J) \) is depicted below.

![Graph H(J)](image-url)
(2) Applying Algorithm 3.12 with any term order on $R$ in which $x_6 > x_1 > x_8 > x_7 > x_3 > x_2 > x_4 > x_5$, we get that
\[ x_6 + x_1, x_8 + x_7, x_3 + x_2 + x_4 \]
forms an initially regular sequence on $R/I$. Thus, depth $R/I \geq 3$ by Theorem 3.11.

In fact, $x_6 + x_1, x_8 + x_7, x_3 + x_2 + x_4$ is a regular sequence on $R/J$ by Corollary 2.7.

Computations show that $x_6 + x_1$, $x_8 + x_7$ is a regular sequence on $R/I$ but $x_3 + x_2 + x_4$ is not regular on $R/(I, x_6 + x_1, x_8 + x_7)$.

Note that although the definition of an initially regular sequence allows for the selection of a new term order at each step, Example 1.3 is representative of the general situation. In practice, when applying Theorem 3.11, we only use two term orders, one to compute the initial ideal of $I$ and a second to order the variables appearing in the initial ideal of $I$.

Although Algorithm 3.12 is particularly easy to visualize when $H$ is a graph — it is equivalent to packing stars in $H$ (a star in $H$ is a subgraph of $H$ consisting of a vertex $x$ and all its neighbors) — the algorithm has proven to be more effective when $H$ is a hypergraph, where there is a degree of freedom in choosing neighbor vertices in Step 5. The bound produced by Algorithm 3.12 can be viewed as a generalization of star packing to hypergraphs, where a star now consists of a center vertex and a set of neighbors that cover every edge containing the center.

In Example 1.3 the initially regular sequence found by Algorithm 3.12 is also a regular sequence on $R/\text{in}(I)$. We identify various situations where this is the case. We further discuss when regular sequences and initially regular sequences can be combined to give better estimates for the depth, and when Theorem 3.11 can be improved by allowing the reuse of vertices. For instance, we prove the following results.

**Theorem 2.5.** Let $I$ be a monomial ideal in a polynomial ring $R = k[x_1, \ldots, x_r]$. Fix a term order with $b_0 > b_1$. Let $f \in R' = k[x_1, \ldots, x_r]$ be a polynomial. Then $f$ is regular on $R/(I, b_0 + b_1)$ if and only if $f$ is initially regular on $R/(I, b_0 + b_1)$.

**Theorem 4.12.** Let $I$ be a monomial ideal in a polynomial ring $R$. Suppose that $b_0, \ldots, b_t$ are distinct variables in $R$ and $>$ is a fixed term order such that $b_0 > b_1 > \cdots > b_t$. Suppose that for some $q \leq t$, the sets \{b_0, b_1\}, \{b_1, b_2\}, \ldots, \{b_{q-2}, b_{q-1}\}, \{b_{q-1}, b_q, \ldots, b_t\}$ satisfy the conditions (1) and (2) of Theorem 3.11. Let $f_i = b_{i-1} + b_i$, $1 \leq i \leq q-1$, and $f_q = b_{q-1} + \ldots + b_t$. Then $f_1, \ldots, f_q$ is both a regular and an initially regular sequence on $R/I$.

In general, given an ideal $I \subseteq R$, it is not just the depth of $R/I$ that attracts significant attention; rather, it is the depth function depth $R/I^s$, for $s \in \mathbb{N}$, that is often of interest. A classical result of Burch that was later improved by Broadmann says that $\lim_{s \to \infty} \text{depth } R/I^s \leq \dim R - \ell(I)$, where $\ell(I)$ is the analytic spread of $I$ [3, 4]. Moreover, Eisenbud and Huneke [5, 7] showed that if, in addition, the associated graded ring, gr$_I(R)$, of $I$ is Cohen-Macaulay, then the above inequality becomes an equality. Therefore, one can say that the limiting behavior of the depth $R/I^s$ is quite well understood. It is then natural to consider the initial behavior of the depth function (cf. [3, 16, 21, 22, 24, 26, 29, 30, 34, 35, 36, 37, 43]).

Examples have been exhibited to show that the initial behavior of depth $R/I^s$ can be wild, see [3]. In fact, it was conjectured by Herzog and Hibi [24] that for any numerical function
The conclusion of Theorem 5.3 then follows by taking $H(1.1)$ dominated number of $G$.

Theorem 5.3. Let $I$ be a monomial ideal in $R = k[x_1, \ldots, x_n]$ and let $I^{pol} \subset R^{pol}$ be its polarization. Then the maximal length of an initially regular sequence on $R^{pol}/I^{pol}$ is at least $\sum_{i=1}^{n}(d_{x_i}(I) - 1)$, which is the number of polarizing variables.

In Example 4.17 we demonstrate how our results can be combined with polarization to obtain bounds on the depths of powers. This method, while proving to be interesting in specific examples, is difficult to use to produce general results unless one starts with a highly constrained class of ideals. To address this situation, we adapt a proof technique from [4] to generalize a bound for depth $R/I$, given by Dao and Schweig [10] in terms of the edgewise domination number. We provide a lower bound for the depth function $\text{depth} R/I^s$, $s \in \mathbb{N}$, when $I$ is a squarefree monomial ideal corresponding to a hyperforest.

Definition 1.4. Let $G = (V, E)$ be a simple hypergraph without any isolated vertices. A subset $F \subseteq E$ is called edgewise dominant if for every $v \in V$, there exists $e \in F$ and $u \in e$ such that $u$ and $v$ belong to an edge (i.e., $v$ is adjacent to a vertex contained in an edge of $F$). The edgewise domination number of $G$ is defined to be

$$\epsilon(G) = \min\{|F| \mid F \subseteq E \text{ is edgewise dominant}\}.$$  

It was proven in [10] Theorem 3.2] that $\text{depth} R/I(G) \geq \epsilon(G)$. We extend this result to higher powers of $I(G)$ when $G$ is a hyperforest (also referred to as a simplicial forest).

Theorem 5.3. Let $G$ be a hyperforest and let $I = I(G)$. Let $\epsilon(G)$ denote the edgewise domination number of $G$. Then for all $s \geq 1$,

$$\text{depth} R/I^s \geq \max\{\epsilon(G) - s + 1, 1\}.$$  

To prove Theorem 5.3, we write $I(G) = I(H) + I(T)$, where $H$ and $T$ are edge disjoint subhypergraphs of $G$, and establish a slightly more general statement that

$$(1.1) \quad \text{depth} R/[I(H) + I(T)^s] \geq \epsilon(G) - s + 1.$$  

The conclusion of Theorem 5.3 then follows by taking $H$ to be the empty hypergraph and $T = G$. The inequality (1.1) is proved by generalizing a nonstandard induction technique used in [4].

Throughout the paper, $R = k[x_1, \ldots, x_n]$ is a polynomial ring over an arbitrary field $k$. For a hypergraph $G = (V_G, E_G)$ over the vertex set $V_G = \{x_1, \ldots, x_n\}$, the edge ideal of $G$ is
defined to be
\[ I(G) = \left\langle \prod_{x \in e} x \mid e \in E_G \right\rangle \subseteq R. \]

This construction gives a one-to-one correspondence between squarefree monomial ideals in \( R = k[x_1, \ldots, x_n] \) and simple hypergraphs on the vertex set \( V = \{x_1, \ldots, x_n\} \). For a squarefree monomial ideal \( I \), we shall let \( H(I) \) denote the hypergraph corresponding to \( I \).

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2. Regular and initially regular sequences

In this section, we show that the length of an initially regular sequence on \( R/I \) gives a lower bound for depth \( R/I \), and discuss special situations where initially regular sequences are also regular sequences.

For convenience, we start by recalling the definition of an initially regular sequence from the introduction.

**Definition 2.1.** Let \( R \) be a polynomial ring over a field, and fix a set of term orders \( >_1, \ldots, >_q \) on \( R \). Let \( I \subseteq R \) be a proper ideal. A sequence of nonconstant polynomials \( f_1, \ldots, f_q \) is said to be an *initially regular sequence* on \( R/I \) if for each \( i = 1, \ldots, q \), \( f_i \) is a regular element on \( R/I_i \), where \( I_i = \in_{>_i}(I_{i-1}, f_{i-1}) \) (here, by convention, \( I_1 = \in_{>_1}(I) \)).

In general, there is no relationship between an element being regular and initially regular. Given an ideal in a polynomial ring, an element can be both regular and initially regular, either one without the other, or neither. However, our first result shows that initially regular sequences give a lower bound on the depth.

**Theorem 2.2.** Let \( I \) be an ideal in a polynomial ring \( R \). If \( f_1, \ldots, f_q \) form an initially regular sequence on \( R/I \) with respect to a sequence of term orders \( >_1, \ldots, >_q \), then
\[ \text{depth } R/I \geq q. \]

**Proof.** We proceed by induction on \( q \). If \( q = 1 \), then \( f_1 \) is regular on \( R/\in_{>_1}(I) \), and so by [23, Theorem 3.3.4], we have depth \( R/I \geq \text{depth } R/\in_{>_1}(I) \geq 1 \).

Suppose that \( q \geq 2 \). Then \( f_2, \ldots, f_q \) is an initially regular sequence on \( R/(\in_{>_1}(I), f_1) \) and by induction, depth \( R/(\in_{>_1}(I), f_1) \geq q - 1 \). Thus, by [23, Theorem 3.3.4] again, we have depth \( R/I \geq \text{depth } R/\in_{>_1}(I) = \text{depth } R/(\in_{>_1}(I), f_1) + 1 \geq q \). \( \square \)

In light of [23, Theorem 3.3.4], which shows depth \( R/I \geq \text{depth } R/\in(I) \) for whichever term order is selected for the first step, we will often simplify statements by assuming step
one has been completed. That is, when convenient, we can assume that we are starting with a monomial ideal.

In practice it is often the case that the initially regular sequence that we construct is also a regular sequence. We will show instances where the two notions are equivalent. To do so we first examine the initial ideal of \((I, b_0 + b_1)\), where \(b_0, b_1\) are distinct variables in a polynomial ring \(R\) and \(I\) is a monomial ideal in \(R\). Note that in our examination, we describe a set of monomials that generate \(\text{in}(I, b_0 + b_1)\). Although the set need not be minimal, it is convenient to describe the set in terms of the minimal generators of \(I\).

To state the result we introduce a notation for the degree of a variable in a monomial. For a monomial \(N\) and a variable \(x\), define \(d_x(N) = \max\{t \mid x^t \text{ divides } N\}\).

**Lemma 2.3.** Let \(I\) be a monomial ideal in a polynomial ring \(R\). Suppose that \(b_0, b_1\) are distinct variables of \(R\). Fix a term order with \(b_0 > b_1\). Then

\[\text{in}(I, b_0 + b_1) = \left\{b_0, b_1^{d_{b_0}(M)} \frac{M_j}{b_0^{d_{b_0}(M_j)}} \mid M_j \text{ a minimal monomial generator of } I\right\}.\]

**Proof.** Let \(R = k[x_1, \ldots, x_r, b_0, b_1]\), \(R_1 = k[x_1, \ldots, x_r]\), and \(R_2 = k[b_0, b_1]\). We may write \(I = (M_1, \ldots, M_r, M_{r+1}, \ldots, M_p)\), where \(b_0 \mid M_j\) for \(1 \leq j \leq r\), and \(b_0 \nmid M_j\) for \(r+1 \leq j \leq p\). Observe that \(b_0 \in \text{in}(I, b_0 + b_1)\) since \(b_0\) is the leading term of \(b_0 + b_1\).

In order to compute a Gröbner basis for \((I, b_0 + b_1)\) we must consider the reductions of all possible \(S\)-resultants. Notice that the \(S\)-resultant of two monomials is 0. Thus, it initially suffices to consider all possible \(S\)-resultants involving \(f_1 = b_0 + b_1\). Let \(M\) be a monomial. Then \(S(M, f_1)\) is again monomial. In fact,

\[S(M, f_1) = \frac{\text{lcm}(M, b_0)}{b_0}b_0 + b_1 - \frac{\text{lcm}(M, b_0)}{M} = \frac{\text{lcm}(M, b_0)}{b_0}b_1.\]

For \(r_1 + 1 \leq j \leq p\), we have

\[S(M_j, f_1) = \frac{\text{lcm}(M_j, b_0)}{b_0}b_1 = M_j b_1,\]

since \(b_0 \nmid M_j\). In this case, \(S(M_j, f_1)\) reduces to 0 modulo the generators of \(I\).

For \(1 \leq j \leq r_1\), we have

\[S(M_j, f_1) = \frac{\text{lcm}(M_j, b_0)}{b_0}b_1 = \frac{M_j b_1}{b_0}.\]

If \(b_0\) does not divide \(\frac{M_j b_1}{b_0}\), then

\[\frac{M_j b_1}{b_0} = b_1^{d_{b_0}(M_j)} \frac{M_j}{b_0^{d_{b_0}(M_j)}}.\]

If \(b_0\) divides \(\frac{M_j b_1}{b_0}\) then we reduce by \(b_0 + b_1\) to get

\[\frac{M_j b_1}{b_0} - \left(\frac{M_j b_1}{b_0^2}\right) = \frac{-M_j b_1^2}{b_0^2}.\]

Iterating the reduction step of the algorithm, we continue to reduce until the result is no longer divisible by \(b_0\). That is, until we reach \(b_1^{d_{b_0}(M_j)} \frac{M_j}{b_0^{d_{b_0}(M_j)}}\). The assertion follows. \(\square\)
If \(a, b\) is an initially regular sequence on \(R/I\) and \(f\) is an initially regular element on \(R/(I, a, b)\), then \(a, b, f\) need not be an initially regular sequence on \(R/I\) since in general \(\in(I, a, b) \neq \in(in(I, a), b)\) even in the case where only one term order is used. The situation is more clear when \(a\) and \(b\) are sums of variables, some of which are distinct. In this case, Lemma 2.3 can be applied to obtain the following result.

**Lemma 2.4.** Let \(I\) be a monomial ideal in a polynomial ring \(R\). Let \(\{x_i, y_i\}_{i=1}^\ell\) be pairs of variables, and \(>\) be a term order such that \(x_i > y_i\) for all \(i\) and \(x_i > x_j\) for all \(i < j\). Then

\[
\in(I, x_1 + y_1, \ldots, x_\ell + y_\ell) = \in(\in(\ldots, \in(I, x_1 + y_1), x_2 + y_2), \ldots), x_\ell + y_\ell).
\]

**Proof.** By iterated use of Lemma 2.3, the right hand side can be generated by \(x_1, \ldots, x_\ell\) and the monomials obtained from the monomial generators of \(I\) by successively replacing \(x_1\) by \(y_1\), then \(x_2\) by \(y_2\), . . . , and eventually \(x_\ell\) by \(y_\ell\). Notice that once \(x_i\) is replaced by \(y_i\) in a monomial generator of \(I\), \(x_i\) will not reappear in the generating set through subsequent replacements since by the term order \(x_i \neq y_j\) for any \(i < j\).

On the other hand, the left hand side is generated by the leading terms of a Gröbner basis of \((I, x_1 + y_1, \ldots, x_\ell + y_\ell)\). Observe that the \(S\)-resultant of \(x_i + y_i\) and \(x_j + y_j\) reduces to 0 modulo \(\{x_1 + y_1, \ldots, x_\ell + y_\ell\}\). As in Lemma 2.3, the reduction of an \(S\)-resultant of a monomial with \(x_i + y_i\) is formed by successively replacing \(x_i\) by \(y_i\) until \(x_i\) no longer divides the resulting monomial. If this resulting monomial is divisible by \(x_j\) for some \(j\), the reduction process continues, eventually yielding a monomial where \(x_j\) has been replaced by \(y_j\) for all \(j\). Hence, the left hand side can also be generated by the set consisting of \(x_1, \ldots, x_\ell\) and the monomials obtained from monomial generators of \(I\) by successively replacing \(x_i\) by \(y_i\), for \(i = 1, \ldots, \ell\).

Our next result establishes an instance where the notion of initially regular is equivalent to being regular.

**Theorem 2.5.** Let \(I\) be a monomial ideal in a polynomial ring \(R = k[x_1, \ldots, x_r, b_0, b_1]\). Fix a term order with \(b_0 > b_1\). Let \(f \in R' = k[x_1, \ldots, x_r, b_1]\) be a polynomial. Then \(f\) is regular on \(R/(I, b_0 + b_1)\) if and only if \(f\) is initially regular on \(R/(I, b_0 + b_1)\).

**Proof.** Since \(I\) is a monomial ideal, write \(I = (M_1, \ldots, M_\ell, M_{\ell+1}, \ldots M_p)\), where \(M_j\) is divisible by \(b_0\) or \(b_1\) if and only if \(1 \leq j \leq \ell\). For simplicity of notation, for a monomial \(M\), set

\[
\bar{M} = b_1^{d_0(M)} b_0^{d_1(M)}.
\]

By Lemma 2.3 we have

\[
\in(I, b_0 + b_1) = (b_0, M_1, \ldots, M_\ell, M_{\ell+1}, \ldots, M_p).
\]

Define \(\phi : R \to R'\) to be the ring homomorphism that sends \(b_0\) to \(-b_1\) and identifies all other variables in \(R\). It is easy to see that \(\phi\) is onto and its kernel is \((b_0 + b_1)\). Set

\[
I' = \in(I, b_0 + b_1) \setminus \{b_0\} = (\bar{M}_1, \ldots, \bar{M}_\ell, \bar{M}_{\ell+1}, \ldots, M_p).
\]

Define \(\bar{\phi} : R \to R'/I'\) by \(\bar{\phi}(a) = \phi(a) + I'\) for \(a \in R\). Notice that \(\bar{\phi}\) is an onto homomorphism, and since \(\phi(I) \subseteq I'\), \((I, b_0 + b_1) \subseteq \ker(\bar{\phi})\). To see that this is an equality, consider an arbitrary
monomial $M$ of $R$ such that $\phi(M) \in I'$. If $M$ is not divisible by $b_0$ or $b_1$, then $\phi(M) = M$ and $M \in \langle M_{\ell+1}, \ldots, M_{\ell} \rangle \subseteq I$. If $b_0$ or $b_1$ divides $M$, then $b_1$ divides $\phi(M)$ and thus $\phi(M)$ is divisible by $\hat{M}_j$ for some $1 \leq j \leq \ell$. By the definition of $\phi$, this implies that there is a monomial $N = b_0^{t_0} b_1^{t_1} \hat{M}_j$ that divides $M$ for some $t_0 + t_1 = t = d_{b_1}(\hat{M}_j)$. Set $M'_j = \hat{M}_j$. Notice that $M_j = b_0^{s_0} b_1^{s_1} M'_j$ for some $s_0, s_1 \geq 0$ with $s_0 + s_1 = t$.

Next we claim that $N \in (I, b_0 + b_1)$. If $t_0 = s_0$, then $t_1 = s_1$ and $N = M_j \in I \subseteq (I, b_0 + b_1)$. If $s_0 \geq 1$, then $\frac{M_{b_0}}{b_0} = (b_0 + b_1) \frac{M_{b_0}}{b_0} - M_j \in (I, b_0 + b_1)$. Iterating the process shows that $N \in (I, b_0 + b_1)$ for all $t_0 \leq s_0$. Similarly, if $s_1 \geq 1$, then $\frac{M_{b_1}}{b_1} = (b_0 + b_1) \frac{M_{b_1}}{b_1} - M_j \in (I, b_0 + b_1)$. Again, iterating the process shows that $N \in (I, b_0 + b_1)$ for all $t_1 \leq s_1$. Since $t_0 + t_1 = s_0 + s_1$, this shows $N \in (I, b_0 + b_1)$ for all such $N$. Thus, $\ker(\phi) = (I, b_0 + b_1)$ and so

$$R/(I, b_0 + b_1) \cong R'/I'.$$

Notice that $R/\text{in}(I, b_0 + b_1) \cong R/(I', b_0) \cong R'/I' \cong R/(I, b_0 + b_1)$ and therefore, since $\phi(f) = f$, then $f$ is regular on $R/(I, b_0 + b_1)$ if and only if $f$ is regular on $R/\text{in}(I, b_0 + b_1)$. 

**Remark 2.6.** In the setting of Theorem 2.5 suppose $f \in R'' = k[x_1, \ldots, x_r]$. Then the roles of $b_0$ and $b_1$ can be reversed. As a result, $f$ will be regular on $R/(I, b_0 + b_1)$ if and only if it is initially regular on $R/(I, b_0 + b_1)$ for any term order. When this is the case, we will omit the term order but assume one has been fixed.

We now obtain the following corollary.

**Corollary 2.7.** Let $I$ be a monomial ideal in a polynomial ring $R$. Let $\{x_i, y_i\}_{i=1}^{q-1}$ be pairs of variables and let $> b$ be a term order such that $x_i > y_i$ for all $i$ and $x_i > x_j$ for all $i < j$. Let $c_1, \ldots, c_r$ be distinct variables disjoint from $\bigcup_{i=1}^{q-1}\{x_i, y_i\}$ (with the only exception possibly at $c_1 = y_{q-1}$). Suppose that $c_1 > c_i$ for all $2 \leq i \leq r$. Set $f_i = x_i + y_i$ for all $1 \leq i \leq q - 1$ and let $f_q = c_1 + \ldots + c_r$. Then, $f_1, \ldots, f_q$ is a regular sequence on $R/I$ if and only if $f_1, \ldots, f_q$ is an initially regular sequence on $R/I$.

**Proof.** The result follows by induction, using Theorem 2.5 and Lemma 2.4. \qed

We shall apply Corollary 2.7 later on, in Section 4 to construct examples of initially regular sequences which are also regular sequences. Note that the condition that the ideal $I$ is monomial in Corollary 2.7 is necessary as can be seen in Example 1.3. However, at times these sequences can also be regular sequences on $R/I$, where $I$ is not necessarily monomial. For instance, in Example 1.3 a variation of the original sequence, namely $x_6 + x_1, x_8 + x_7, x_4 + x_5 + x_3$, is both a regular and an initially regular sequence on $R/I$.

### 3. Constructing Initially Regular Sequences

This section is devoted to the task of deriving an algorithm to construct initially regular sequences. Recall that Theorem 2.3 gives a lower bound for the depth of $R/I$ provided that initially regular sequences on $R/I$ can be found.

Since initially regular sequences require that an element is regular on an initial ideal at each step, it is helpful to understand the structure of the Gröbner basis at each step in
the construction. For background information on Gröbner bases or Buchberger’s algorithm, see [2]. For simplicity and convenience of notation, in the remainder of this section, unless otherwise specified, we shall assume the following set-up:

**Set-up 3.1.** Let \( R = k[x_1, \ldots, x_r, b_0, \ldots, b_t] \) be a polynomial ring over a field and let \( I \) be a monomial ideal in \( R \). Suppose that \( R \) has a fixed term order, and set \( R_1 = k[x_1, \ldots, x_r] \) and \( R_2 = k[b_0, \ldots, b_t] \).

The following notion allows us to focus on ideals and Gröbner bases of a particular form, which is an essential part of our construction of initially regular sequences.

**Definition 3.2.** A polynomial \( f \in R \) is called \((R_1, R_2)\)-factorable if \( f = Mg \in R \), where \( M \in R_1 \) is a monomial and \( g \in R_2 \) is a polynomial. An ideal \( I \) that admits a minimal set of generators \( \{f_1, \ldots, f_p\} \) in which \( f_i \) is \((R_1, R_2)\)-factorable for all \( i \) is called an \((R_1, R_2)\)-factorable ideal. A Gröbner basis whose elements are \((R_1, R_2)\)-factorable is called an \((R_1, R_2)\)-factorable Gröbner basis.

Note that all monomial ideals are \((R_1, R_2)\)-factorable, where 1 is considered a monomial in \( R_1 \) when necessary. The next two lemmas show that the two key steps of Buchberger’s algorithm, forming \( S \)-resultants and the reduction process, preserve \((R_1, R_2)\)-factorability. Recall that for a polynomial \( g \) in \( R \) under a fixed term order, \( \text{in}(g) \) represents the leading term of \( g \).

**Lemma 3.3.** Let \( f, g \in R \) be \((R_1, R_2)\)-factorable polynomials. Then \( S(f, g) \) is also \((R_1, R_2)\)-factorable.

**Proof.** By assumption, there exist monomials \( M, N \in R_1 \) and polynomials \( f', g' \in R_2 \) with \( f = Mf' \) and \( g = Ng' \). Write \( f' = f_1 + \tilde{f} \) and \( g' = g_1 + \tilde{g}, \) where \( f_1 \) and \( g_1 \) are the leading terms of \( f' \) and \( g' \) respectively under the fixed term order. By the definition of an \( S \)-resultant, and using the fact that \( M \) and \( N \) are monomials in \( R_1 \), we have

\[
S(Mf', Ng') = \frac{\text{lcm}(Mf_1, Ng_1)}{Mf_1}Mf' - \frac{\text{lcm}(Mf_1, Ng_1)}{Ng_1}Ng' \\
= \frac{\text{lcm}(M, N)\text{lcm}(f_1, g_1)}{f_1}f' - \frac{\text{lcm}(M, N)\text{lcm}(f_1, g_1)}{g_1}g' \\
= \text{lcm}(M, N)S(f', g').
\]

**Lemma 3.4.** Let \( f, g \in R \) be \((R_1, R_2)\)-factorable polynomials. When \( f \) is reduced modulo \( g \) in Buchberger’s algorithm, then the remainder will also be \((R_1, R_2)\)-factorable. Moreover, the \( R_1 \)-monomial term in the \((R_1, R_2)\)-factorization of \( f \) is the same as the \( R_1 \)-monomial term of the remainder.

**Proof.** Suppose that \( f = Mf', g = Ng' \), where \( M, N \in R_1 \) are monomials, and \( f', g' \in R_2 \) are polynomials. Write \( f = \sum_{i=1}^{t_1} f_i = M \sum_{i=1}^{t_1} f'_i \) and \( g = \sum_{j=1}^{t_2} g_j = N \sum_{j=1}^{t_2} g'_j \) where the \( f_i, g_j \) are the monomial terms of \( f \) and \( g \), respectively.
Observe that $f$ can be reduced modulo $g$ in the Buchberger’s algorithm (see for example [2]) if the leading term $g_1 = \text{in}(g)$ divides a monomial term $f_i$. That is, $f_i = \alpha g_i$ for some monomial $\alpha \in R$. Write $f' = f_i' + \tilde{f}$ and $g' = g_i' + \tilde{g}$ for some polynomials $\tilde{f}, \tilde{g} \in R_2$.

Let $h = f - \alpha g$. Write $\alpha = \alpha_1 \alpha_2$, where $\alpha_1 \in R_1$ and $\alpha_2 \in R_2$. Since $M f_i' = \alpha N g_i'$, it follows that $\alpha_1 N = M$, and $\alpha_2 g_i' = f_i''$. Therefore,

$$h = f - \alpha g = M f' - \alpha N g' = M f_i' + M \tilde{f} - \alpha N g_i' - \alpha N \tilde{g} = M \tilde{f} - \alpha_1 N \alpha_2 \tilde{g} = M (\tilde{f} - \alpha_2 \tilde{g}).$$

The conclusion now follows since the remainder of $f$ modulo $g$ is obtained by repeating this process until no monomial term of $f$ is divisible by $\text{in}(g)$.

Using Lemmas \ref{lem:3.3} and \ref{lem:3.4}, we show that Buchberger’s algorithm preserves $(R_1, R_2)$-factorability.

**Proposition 3.5.** Let $I = (f_1, \ldots, f_p)$ be an $(R_1, R_2)$-factorable ideal in $R$ such that for each $i = 1, \ldots, p$, $f_i = M_i g_i$, where $M_i \in R_1$, $g_i \in R_2$, and $M_i$ are monomials. Then, there exists an $(R_1, R_2)$-factorable Gröbner basis of $I$ in which every element is of the form $f = Mg$, where $M \in R_1$, $g \in R_2$, and $M = \text{lcm}(M_{i_1}, \ldots, M_{i_\ell})$, for some $1 \leq i_1, \ldots, i_\ell \leq p$. Furthermore, the unique reduced Gröbner basis of $I$ is also $(R_1, R_2)$-factorable and consists of elements of this form.

**Proof.** We follow Buchberger’s algorithm to produce a Gröbner basis for $I$. Set $G_1 = \{f_1, \ldots, f_p\}$. For $i \neq j$ form the S-resultant $S = S(f_i, f_j)$. By Lemma \ref{lem:3.3} $S(f_i, f_j)$ has the desired form with $S = S(f_i, f_j) = \text{lcm}(M_i, M_j)S(g_i, g_j)$. If $\text{in}(f_k)$ divides $\text{in}(S)$ for any $k$, reduce $S$ modulo $f_k$. Note that by Lemma \ref{lem:3.4} the reduction has the desired form and the monomial term of the reduction remains $\text{lcm}(M_i, M_j)$. Repeat this process until $S = \sum_{k=1}^{p} \alpha_k f_k + f_{p+1}$, where $\text{in}(f_k)$ does not divide $\text{in}(f_{p+1})$ for all $k$. That is, $f_{p+1}$ is the remainder when $S$ is reduced modulo $G_1$. If $f_{p+1} \neq 0$, add it to $G_1$. Thus, the new set $G_1$ again consists entirely of $(R_1, R_2)$-factorable elements. Repeating this process produces a Gröbner basis $G_1 = \{f_1, \ldots, f_p, f_{p+1}, \ldots, f_n\}$, where every element has the desired form.

To produce the (unique) reduced Gröbner basis, the elements of $G_1$ need to be further reduced so that for each $1 \leq i \leq n$, $\text{in}(f_i)$ does not divide any monomial term of $f_j$ for $i \neq j$. Again by Lemma \ref{lem:3.4}, passing to the reduced Gröbner basis preserves $(R_1, R_2)$-factorability and the form of the $R_i$-monomial terms.

A closer examination of the proof in Proposition \ref{prop:3.5} shows that if $I$ is an $(R_1, R_2)$-factorable ideal, the maximum degree of a variable $x_i$ that divides one of the generators of $I$ will not increase when passing to an $(R_1, R_2)$-factorable Gröbner basis. In particular, if the monomial terms $M_i \in R_1$ associated to the original generating set of $I$ are squarefree, then so are the $R_1$ monomial terms of the Gröbner basis. Recall that for a monomial ideal $I = (f_1, \ldots, f_p)$ set $d_x(I) = \max\{d_x(f_i) \mid 1 \leq i \leq p\}$. Notice that this is well defined as the set of minimal monomial generators of $I$ is unique.

**Corollary 3.6.** Let $I = (f_1, \ldots, f_p)$ be an $(R_1, R_2)$-factorable ideal with $f_i = M_i g_i$. Then $\max \{d_x(M_i)\} \geq d_x(\text{in}(I))$ for every variable $x$ in $R_1$. 


Proof. Let $G = \{h_1, \ldots, h_m\}$ be the reduced Gröbner basis for $\text{in}(I)$ and notice that by Proposition 3.5, $G$ is an $(R_1, R_2)$-factorable Gröbner basis. Let $h_i = N_i h'_i$ for $i = 1, \ldots, m$ with $N_i$ monomials in $R_1$ and $h'_i \in R_2$.

By Proposition 3.5, each $N_i$ is the least common multiple of some of the $M_1, \ldots, M_p$. Thus, the assertion follows by observing that

\[
d_x(\text{lcm}(M_1, \ldots, M_p)) = \max\{d_x(M_i) \mid 1 \leq j \leq \ell\}
\leq \max\{d_x(M_i) \mid 1 \leq i \leq p\}.
\]

\[\square\]

Additional control over the maximal degree of a variable will be needed in special cases once we begin to form the initially regular sequences. The following lemma provides such control. Recall that we are still in the setting of Set-up 3.1

Lemma 3.7. Let $I$ be a monomial ideal in $R$. Let $J = (I, b_0 + b_1 + \ldots + b_t)$. Then $d_{x_i}(I) \geq d_{x_i}(\text{in}(J))$ for all $1 \leq i \leq r$. Furthermore, suppose that $>$ is a monomial order such that $b_0 > b_j$ for all $j = 1, \ldots, t$. Then $d_{b_0}(\text{in}(J)) = 1$.

Proof. Since all monomial ideals are $(R_1, R_2)$-factorable, let $I = (f_1, \ldots, f_p)$, where $f_i = M_i g_i$ with $M_i \in R_1$, $g_i \in R_2$, and $M_i, g_i$ monomials. Setting $M_{p+1} = 1$ we have $f_{p+1} = b_0 + b_1 + \ldots + b_t = M_{p+1} f_{p+1}$ and thus $J$ is an $(R_1, R_2)$-factorable ideal. Thus, by Proposition 3.5, there exists an $(R_1, R_2)$-factorable Gröbner basis of $J$. The first statement follows from Corollary 3.6 after noting that for every variable $x_i$,

\[
d_{x_i}(I) = \max\{d_{x_i}(f_i) \mid 1 \leq \ell \leq p + 1\} = \max\{d_{x_i}(M_i) \mid 1 \leq \ell \leq p + 1\}.
\]

The second statement is obvious since the leading term of $b_0 + \ldots + b_t$ is $b_0$.

\[\square\]

Lemma 3.8. Let $I$ be a monomial ideal in a polynomial ring $R$. Suppose that $h = \sum_{i=0}^{t} b_i$ is a sum of distinct variables in $R$ and suppose that $d_{b_i}(I) \leq 1$ for all $i \geq 1$. Then $I : h$ is a monomial ideal.

Proof. Suppose that $f \in R$ and $fh \in I$. Since $I$ is monomial and $h$ is homogeneous, we may assume that $f$ is a homogeneous polynomial. Let $f = \sum_{i=1}^{\ell} f_i = \sum_{i=1}^{\ell} c_i f'_i$, where $f'_1, \ldots, f'_\ell$ are distinct monomials of the same degree and $f_i = c_i f'_i$ with $c_i \in k$ for all $i$. If $f_i(b_0 + \ldots + b_t) \in I$ for some $1 \leq i \leq \ell$ and $\ell \geq 2$, then we may replace $f$ by $f - f_i$. Thus, we can assume that either $\ell = 1$ or $\ell \geq 2$ and $f_i h \notin I$ for all $i = 1, \ldots, \ell$. It suffices to show that $\ell = 1$.

Suppose that $\ell \geq 2$ and for every $1 \leq i \leq \ell$ we have $f_i h = f_i (b_0 + \ldots + b_t) \notin I$. Since $I$ is a monomial ideal, this implies that for each $1 \leq i \leq \ell$, we have $f_i b_j \notin I$ for some $0 \leq j \leq t$. Observe that among those pairs $(i, j)$ such that $f_i b_j \notin I$, there must be such a pair in which $j \neq 0$. Indeed, if that is not the case then we would have $f_i b_j \in I$ for all $i$ and $j > 0$ and hence $f_i b_0 \notin I$ for all $i$. This would imply that $f b_0 = f (b_0 + \ldots + b_t) - f (b_1 + \ldots + b_t) \in I$, a contradiction.

Now, among all pairs $(i, j)$ (with $j > 0$) such that $f_i b_j \notin I$, let $(\alpha, \beta)$ be such a pair so that $d_{b_\beta}(f_\alpha)$ is maximal possible.
Since \( f_\alpha b_\beta \not\in I \), then as \( f(b_0 + \ldots + b_t) \in I \), there must exist \((\gamma_i, \delta_i)\), for \( 1 \leq i \leq v \), with \( \delta_i \neq \beta \) such that \( f_\alpha b_\beta + \sum_{i=1}^v f_\gamma_i b_\delta_i = 0 \). That is, \( c_\alpha f'_\alpha b_\beta + \sum_{i=1}^v c_\gamma_i f'_\gamma_i b_\delta_i = 0 \), where \( c_\alpha + \sum_{i=1}^v c_\gamma_i = 0 \) and \( f'_\alpha b_\beta = f'_\gamma_i b_\delta_i \) for each \( i \). In particular, set \( f_\gamma = f_\gamma_1 \) and \( b_\delta = b_\delta_1 \). Then \( f_\gamma b_\delta \not\in I \).

Observe further that \( d_{b_\beta}(f_\gamma) = d_{b_\beta}(f_\gamma b_\delta) = d_{b_\beta}(f_\alpha) = d_{b_\beta}(f_\gamma) + 1 \). Thus, by the maximality of \( d_{b_\beta}(f_\alpha) \), we must have \( f_\gamma b_\beta \in I \). However, \( d_{b_\beta}(f_\gamma b_\beta) = d_{b_\beta}(f_\gamma) + 1 \geq 2 \), and since \( d_{b_\beta}(I) \leq 1 \) we must have that \( f_\gamma \in I \), which is a contradiction. \( \square \)

We are now ready to begin creating an initially regular sequence on \( R/I \). Note that there are other types of regular elements that we will explore later on.

**Lemma 3.9.** Suppose \( I \) is a monomial ideal in a polynomial ring \( R \). Let \( b_0, b_1, \ldots, b_t \) be distinct variables of \( R \) such that:

(a) \( d_{b_i}(I) \leq 1 \), for all \( i \geq 1 \); and
(b) if \( M \) is a monomial generator of \( I \) and \( b_0 \) divides \( M \), then there exists an \( i \geq 1 \) such that \( b_i \) divides \( M \).

Then \( b_0 + b_1 + \ldots + b_t \) is a regular element on \( R/I \).

**Proof.** Suppose that \( f \in R \) and \( f(b_0 + \ldots + b_t) \in I \). We shall show that \( f \in I \). Since \( I \) is a monomial ideal, we may assume that \( f \) is a monomial by Lemma 3.8. Then \( f(b_0 + \ldots + b_t) \in I \) implies that \( fb_j \in I \) for every \( j \geq 0 \). Since \( fb_0 \in I \), we can write \( fb_0 = Mg \) for a minimal generator \( M \) of \( I \). It follows that either \( b_0 | g \) in which case \( f \in I \), or \( b_0 \not| M \). If \( b_0 \not| M \), condition (b) implies that \( b_i \) divides \( M \) for some \( i \geq 1 \). This, in particular, shows that \( b_i \not| f \). Now, consider \( fb_i \in I \). Noting that \( d_{b_i}(fb_i) \geq 2 \), condition (a) then implies that \( f \in I \). \( \square \)

**Remark 3.10.** Notice that condition (a) in Lemma 3.9 requires that all the \( b_i \) have degree at most one for all \( i \geq 1 \). This allows us to consider polynomials where \( b_0 \) has a higher degree. A careful examination of the proofs of Lemmas 3.8 and 3.9 shows that the condition \( d_{b_i}(I) \leq 1 \) for all \( i \geq 1 \) can be relaxed to instead require that the degree of \( b_i \) in all monomial generators of \( I \) is either 0 or a fixed constant.

We are ready to state our primary theorem, which provides a process for creating an initially regular sequence for any ideal in a polynomial ring. Note that in this first version, we are giving the basics. There are special cases where we can fine-tune the process to achieve improved lower bounds on the depth. These cases will be discussed in detail in Section 4.

**Theorem 3.11.** Let \( I \) be an ideal in a polynomial ring \( R \) and \( >_1 \) a term order. Suppose that \( \{b_{i,j} \mid 1 \leq i \leq q, 0 \leq j \leq t_i \} \) are distinct variables of \( R \) such that:

1. \( d_{b_{i,j}}(\text{in}_{>_1}(I)) \leq 1 \), for all \( i \geq 1 \) and \( j \geq 1 \); and
2. for each \( i = 1, \ldots, q \), if \( M \) is a monomial generator of \( \text{in}_{>_1}(I) \) and \( b_{i,0} \) divides \( M \), then there exists a \( j \geq 1 \) such that \( b_{i,j} \) divides \( M \).

Let \( f_i = \sum_{j=0}^{t_i} b_{i,j} \), for \( 1 \leq i \leq q \). Then \( f_1, \ldots, f_q \) is an initially regular sequence on \( R/I \) with respect to \( >_1 \) and any term orders \( >_2, \ldots, >_q \) for which \( b_{i,0} >_{i+1} b_{i,j} \) for all \( 1 \leq i \leq q - 1 \) and all \( j \). In particular, \( \text{depth } R/I \geq q \).
Remark 3.13. There are freedoms of choice in Algorithm 3.12 which, in practice, can be utilized to give us a sharper bound for the depth of $R/I$. Specifically,

1. the initial term order in $R$ can be chosen so that $J = \text{in}(I)$ and $H = H(J)$ are combinatorially easy to visualize, as was done in Example 1.3 and
2. (in high generating degrees) the variables $b_i$ in Step 5 can be chosen appropriately so that the iterated process can be done as many times as possible.
In condition (1) of Theorem 3.11 we require the degree of all the variables we use to build the initially regular sequences to be one, with the exception of the degrees of the variables $b_{i,0}$. The following example illustrates this.

**Example 3.14.** Let $R = \mathbb{Q}[a, b, c, d]$ and let $I = (a^2 b, abcd, c^2 d)$ be the ideal corresponding to the hypergraph $G$ depicted below, where the degrees of the vertices $a$ and $c$ indicate the maximal power to which they appears in a minimal generator of $I$.

![Diagram showing hypergraph with vertices a, b, c, d and edges a(2)-b-c(2)-d](image)

Notice that by Theorem 3.11 and Corollary 2.7 we have $a + b, c + d$ is both a regular and an initially regular sequence on $R/I$ with respect to any term order such that $a > b$ and $c > d$. Using Macaulay 2 [18] we can confirm that depth $R/I = 2$.

In the following examples we apply Algorithm 3.12 to obtain initially regular sequences and bounds on the depth of $R/I$. We also explain how the bound obtained compares to known bounds.

**Example 3.15.** Let $I = (x_1 x_2, x_2 x_3, x_1 x_3, x_2 x_4, x_4 x_5, x_3 x_6, x_6 x_7) \subseteq R = \mathbb{Q}[x_1, \ldots, x_7]$ be the edge ideal of the graph $G$ depicted below.

![Diagram showing graph with vertices x1, x2, x3, x4, x5, x6, x7](image)

Previously known bounds from [9] give depth $R/I \geq \max\{\varepsilon(G), \tau(G)\} = 3$. Our Theorem 3.11 confirms that depth $R/I \geq 3$ in this example. Notice that $x_5 + x_4, x_7 + x_6, x_1 + x_2 + x_3$ is an initially regular sequence on $R/I$, where $x_1 > x_2 > x_3 > x_5 > x_4 > x_7 > x_6$. Computations in Macaulay 2 [18] indeed verify that depth $R/I = 3$. It is also worth noting that $x_5 + x_4, x_7 + x_6, x_1 + x_2 + x_3$ is a regular sequence on $R/I$ as well by Corollary 2.7.

The next example shows that in the case of hypergraphs a careful selection of the vertex sets used can result in a significant improvement from known results.

**Example 3.16.** Let $I = (x_1 x_2 x_3, x_2 x_3 x_4, x_2 x_5 x_6, x_3 x_7 x_8, x_4 x_9 x_{10}) \subseteq R = \mathbb{Q}[x_1, \ldots, x_{10}]$ be the edge ideal of the hypergraph $G$ depicted below. The previously known bound of [10] Theorem 3.2] shows depth $R/I \geq 1$. The bounds from this section ensure depth $R/I \geq 4$. Notice that $x_1 + x_2, x_5 + x_6, x_7 + x_8, x_9 + x_{10}$ is both a regular and an initially regular sequence on $R/I$, where $x_1 > x_2 > x_5 > x_6 > x_7 > x_8 > x_3 > x_9 > x_{10} > x_4$ by Theorem 3.11 and Corollary 2.7.
Using Macaulay 2 [18] we have that depth $R/I = 6$. In the next section, we show that our results can be further refined to improve accuracy. For example, using Theorem 4.12 we will see that $x_1 + x_2, x_5 + x_6, x_7 + x_8, x_8 + x_3, x_9 + x_{10}, x_{10} + x_4$ is both a regular and an initially regular sequence on $R/I$ (relative to the order above) and thus achieving the actual bound for depth $R/I$.

4. Extensions of initially regular sequences

In this section, we discuss some extensions of Theorem 3.11, where initially regular sequences and regular sequences can be combined to get longer initially regular sequences, and where the reuse of variables in the algorithm is possible.

We begin by showing that under suitable assumptions initially regular sequences remain initially regular after enlarging the ideal appropriately.

**Proposition 4.1.** Let $I$ be a monomial ideal in a polynomial ring $R$. Let $B = \{b_{i,j} \mid 1 \leq i \leq q, 0 \leq j \leq t_i\}$ be distinct variables in $R$ satisfying the conditions of Theorem 3.11. Let $f_i = \sum_{j=0}^{t_i} b_{i,j}$, for $i = 1, \ldots, q$. Let $Y = \{y_1, \ldots, y_r\}$ be a collection of variables in $R$ that is disjoint from $B$, and let $h_1, \ldots, h_\ell \in k[y_1, \ldots, y_r] \subseteq R$. Then $f_1, \ldots, f_q$ is an initially regular sequence on $R/(I, h_1, \ldots, h_\ell)$.

**Proof.** We prove the statement by applying Theorem 3.11 to the ideal $H = \text{in}(I, h_1, \ldots, h_\ell)$. Let $K = (I, h_1, \ldots, h_\ell)$ and notice that $K$ is an $(R_1, R_2)$-factorable ideal, where $R_1 = k[x_1, \ldots, x_u]$, $R_2 = k[y_1, \ldots, y_r]$, and $B \subseteq \{x_1, \ldots, x_u\}$. Then $d_{b_{i,j}}(H) \leq d_{b_{i,j}}(I) \leq 1$ for all $i \geq 1$ and $j \geq 1$ since $B \cap Y = \emptyset$, by Corollary 3.6. Hence, condition (1) of Theorem 3.11 is satisfied.

To see that condition (2) is satisfied, let $N$ be a monomial generator of $H$ such that $b_{i,0} \mid N$ for some $i$. Then $N = \text{lcm}(M_{i_1}, \ldots, M_{i_v})g$, where $\{M_{i_1}, \ldots, M_{i_v}\}$ is a subcollection of $R_1$-factors of the minimal generators of $K$ (as in Proposition 3.5) and $g \in R_2$. Since $b_{i,0} \not\in \{y_1, \ldots, y_r\}$, we have that $b_{i,0} \mid \text{lcm}(M_{i_1}, \ldots, M_{i_v})$. Moreover, since $h_i \in R_2$, then $b_{i,0} \mid M_{i_v}$ for some $M_{i_v}$ that is a factor of a generator of $I$. By Proposition 3.5 we may assume that $N_{i_v} = M_{i_v}g_{i_v}$ is the corresponding monomial generator of $I$ with $g_{i_v} \in R_2$. Hence, by the construction of the initially regular sequence $f_1, \ldots, f_q$, there must exist $j > 0$ such that $b_{i,j} \mid N_{i_v}$. Therefore, $b_{i,j} \mid M_{i_v}$ since $B \cap Y = \emptyset$. Hence, condition (2) of Theorem 3.11 is satisfied, and the conclusion now follows. □
If in addition to the assumptions of Proposition 4.1 we assume that the sequence $h_1, \ldots, h_\ell$ is a regular sequence on $R/I$ then we can get a better bound on the depth.

**Corollary 4.2.** Let $I$, $B$, and $Y$ be as in Proposition 4.1. Suppose further that $h_1, \ldots, h_\ell$ is a regular sequence on $R/I$. Then depth $R/I \geq \ell + q$.

**Proof.** By Theorem 2.2 and Proposition 4.1 depth $R/(I, h_1, \ldots, h_\ell) \geq q$. Moreover, since $h_1, \ldots, h_\ell$ is a regular sequence on $R/I$, then depth $R/I = \text{depth } R/(I, h_1, \ldots, h_\ell) + \ell$. □

Our next goal is to construct sequences that are both regular and initially regular. Our construction is inspired by the notion of leaves in graphs. We say that a variable $x$ is a leaf in a monomial ideal $I$ if there exists a unique monomial generator $M \in I$ such that $x \mid M$.

**Remark 4.3.** By employing a change of variables if needed (see [38, Lemmas 3.3 and 3.5]), the depth of a monomial ideal is unchanged if we assume that $d_x(I) = 1$ for any leaf $x$ of $I$.

**Lemma 4.4.** Let $I$ be a monomial ideal in a polynomial ring $R$. Suppose that $x$ and $y$ are two leaves in $I$ with $M_1, M_2$ the unique monomial generators in $I$ such that $x \mid M_1$ and $y \mid M_2$. Suppose there exist monomials $z, w \in R$ such that $x \nmid z$, $z \mid M_1$, $y \nmid w, w \mid M_2$, $\gcd(z, w) = 1$, and $zw \in I$. Then $x + y$ is a regular element on $R/I$.

**Proof.** First notice that if $M_1 = M_2$, then $x_1 + y_1$ is a regular element on $R/I$, by Lemma 3.9 and Remark 3.10. Hence, we may assume that $M_1 \neq M_2$.

Suppose that $g(x + y) \in I$, for some $g \in R$. Then we may assume that $g$ is a monomial by Lemma 3.8 and Remark 3.10. Since $I$ is a monomial ideal, then $gx \in I$ and $gy \in I$. Thus, if $g \notin I$, then $M_1 \mid gx$, since $x$ appears only in $M_1$, and similarly $M_2 \mid gy$. Thus, $gx = xzM'_1$ for some monomial $M'_1$ and therefore $g = zM'_1$. Similarly, $g = wM'_2$, for some monomial $M'_2$. Hence, $zw \mid g$ and therefore $g \in I$, since $\gcd(z, w) = 1$ and $zw \in I$. □

Notice that if $I$ is the edge ideal of a graph, the condition $\gcd(z, w) = 1$ in Lemma 4.4 means that the two leaves we are considering are distance three apart as long as $M_1 \neq M_2$. Moreover, the result does not hold in general if the distance is not three as can be seen in the next example.

**Example 4.5.** Let $I = (x_1x_2, x_2x_3, x_2x_4, x_4x_5, x_5x_6, x_5x_7) \subseteq R = \mathbb{Q}[x_1, \ldots, x_7]$ be the edge ideal of the graph $G$ depicted below.

![Graph](image)

Notice that $x_1, x_3, x_6$, and $x_7$ are all leaves in $I$, no two of which are distance three apart. It can be checked that no sum of any two of these leaves is a regular element.

**Remark 4.6.** Under the assumptions of Lemma 4.4 we may assume as in the proof that $M_1 \neq M_2$. Moreover, we may assume that $zw$ is a minimal generator of $I$. Indeed, since $zw \in I$, then $zw = MN$, where $M$ is a monomial generator of $I$ and $N \in R$ is another monomial.
Let $z = z'z''$ and $w = w'w''$, with $z' \mid M$, $w' \mid M$, and $\gcd(z'', M) = \gcd(w'', M) = 1$. Since $\gcd(z, w) = 1$, then $\gcd(z', w') = 1$, and therefore $M = z'w'$.

Finally, since $x \nmid z$, then $x \nmid z'$ and similarly, $y \nmid w'$. Also, since $z \mid M_1$, then $z' \mid M_1$ and similarly, $w' \mid M_2$. Therefore, we may replace $z$ and $w$ by $z'$ and $w'$, respectively and assume that $zw$ is indeed a minimal monomial generator of $I$.

**Definition 4.7.** An ordered pair of leaves $x, y$ of a monomial ideal $I$ which satisfies the conditions of Lemma 4.4 with $M_1 \neq M_2$ is called a leaf pair. We will say that two leaf pairs $x, y$ and $a, b$ are disjoint if $\{x, y\} \cap \{a, b\} = \emptyset$.

We are now ready to show that disjoint leaf pairs can be used to form an initially regular sequence. Using Theorem 2.5, we see that the sequence is also a regular sequence.

**Theorem 4.8.** Let $I$ be a monomial ideal in a polynomial ring $R$. Let $\{x_i, y_i\}_{i=1}^\ell$ be a set of disjoint leaf pairs with respect to $I$. Then $x_1 + y_1, \ldots, x_\ell + y_\ell$ forms both a regular sequence and an initially regular sequence on $R/I$ with respect to any term order such that $x_i > y_i$ for all $i$.

**Proof.** We start with the case where $\ell = 2$. For ease of notation, let $x, y$ and $a, b$ denote the given two leaf pairs (with $x > y$ and $a > b$). Let $M_1, M_2, N_1,$ and $N_2$ be the monomial generators of $I$ that are divisible by $x, y, a,$ and $b$, respectively.

By the definition of a leaf pair and Lemma 4.4, $x+y$ is regular on $R/I$. Since $I$ is monomial, $x + y$ is also initially regular. It remains to show that $a + b$ is regular on $R/(I, x+y)$ and on $R/\ln(I, x+y)$. By Theorem 2.5 it is enough to show that $a + b$ is regular on $R/\ln(I, x+y)$. As in Theorem 2.5, $R/\ln(I, x+y) \cong R'/I'$, where $R = k[x, y, a, b, x_5, \ldots, x_n]$, $R' = k[y, a, b, x_5, \ldots, x_n]$ and $I' = \langle \ln(I, x+y) \rangle = \langle I \setminus \{M_1 \cup \{M_2\}\} \rangle$, where $M_1 = \frac{y^{d_x(a)\ell + d_y(b)\ell}}{x_{d_x(a)\ell}}M_1$. The isomorphism is induced by the map $\phi : R \to R'$ that sends $x$ to $-y$ and fixes all other variables. In light of Lemma 4.4, it suffices to show that $\{\phi(a), \phi(b)\}$ is a leaf pair with respect to the ideal $I'$ in $R'$.

By the definition of a leaf pair, $a \mid N_1$, $b \mid N_2$, and $N_1 \neq N_2$. In addition, since $b$ is a leaf, then $b \nmid N_1$. Since $\phi(b) = b$, we have $\phi(b) \mid \phi(N_2)$ but $\phi(b) \nmid \phi(N_1)$, so $\phi(N_1) \neq \phi(N_2)$. Also, there exist monomials $\alpha \mid N_1$ and $\beta \mid N_2$ with $a \nmid \alpha$ and $b \nmid \beta$ with $\alpha\beta \in I$ and $\gcd(\alpha, \beta) = 1$. Notice that since $\alpha \mid N_1$ and $\beta \mid N_2$, we have $\phi(\alpha) \mid \phi(N_1)$ and $\phi(\beta) \mid \phi(N_2)$. Since $\alpha\beta \in I$ and, as in the proof of Theorem 2.5, $\phi(I) \subseteq I'$, we have $\phi(\alpha)\phi(\beta) \in I'$. Since $a \neq x$, we have $\phi(a) = a$. Observe that if $\phi(a) \mid \phi(\alpha)$ then we must have $\phi(\alpha) \neq \alpha$, so $a \nmid \frac{y^{d_x(a)\ell}}{x_{d_x(a)\ell}}\alpha$. This implies that $a \nmid \alpha$, since $a \neq y$, which is a contradiction. Thus, $\phi(a) \nmid \phi(\alpha)$. Similarly, $\phi(b) \nmid \phi(\beta)$.

It remains to show that $\gcd(\phi(\alpha), \phi(\beta)) = 1$. If $x \nmid \alpha$ and $x \nmid \beta$, then $\phi(\alpha) = \alpha$ and $\phi(\beta) = \beta$ and therefore $\gcd(\phi(\alpha), \phi(\beta)) = 1$. Otherwise, since $\gcd(\alpha, \beta) = 1$, $x$ can divide at most one of $\alpha$ and $\beta$. Without loss of generality, suppose that $x \mid \alpha$. Then, $\phi(\alpha) = \frac{y^{d_x(a)\ell}}{x_{d_x(a)\ell}}\alpha$ and $\phi(\beta) = \beta$. By Remark 4.6, we may assume that $\alpha\beta$ is a minimal generator of $I$, and therefore at most one of $x$ or $y$ divides $\alpha\beta$. Since we have assumed $x \mid \alpha$, it follows that $x, y \nmid \beta$. Therefore, $\gcd(\phi(\alpha), \phi(\beta)) = \gcd\left(\frac{y^{d_x(a)\ell}}{x_{d_x(a)\ell}}\alpha, \beta\right) = \gcd(-y^{d_x(a)\ell}\alpha, \beta) = 1$. 

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To see the general case, for any \( \ell \geq 2 \), note that \( \{\phi(x_i), \phi(y_i)\}_{i=2}^{\ell} \) is a set of disjoint leaf pairs with respect to \( I' \subseteq R' \), and so the conclusion follows by induction.

Our next result shows that the initially regular sequences formed by Theorem 3.11 can be combined with leaf pairs to create longer initially regular sequences, and thus improve the depth bound.

**Corollary 4.9.** Let \( I \) be a monomial ideal in a polynomial ring \( R \). Suppose that \( B = \{b_{i,j} | 1 \leq i \leq q, 0 \leq j \leq t_i\} \) are distinct variables in \( R \) satisfying the conditions of Theorem 3.11, and let \( f_i = \sum_{j=0}^{t_i} b_{i,j} \) for \( i = 1, \ldots, q \). Suppose that there exist a sequence of disjoint pairs of leaves \( \{x_k, y_k\}_{k=1}^{\ell} \) in \( I \) such that each pair \( x_k, y_k \) satisfies the conditions of Lemma 4.4. Assume further that \( B \cap \{x_1, y_1, \ldots, x_\ell, y_\ell\} = \emptyset \). Then \( x_1 + y_1, \ldots, x_\ell + y_\ell, f_1, f_2, \ldots, f_q \) is an initially regular sequence on \( R/I \) with respect to any term order such that \( x_k > y_k \) for all \( k \leq \ell \) and \( b_{i,0} > b_{i,j} \) for \( i < q \) and \( j \leq t_i \). Particularly, \( \text{depth} R/I \geq \ell + q \).

**Proof.** By Theorem 4.8, \( x_1 + y_1, \ldots, x_\ell + y_\ell \) forms an initially regular sequence on \( R/I \). By Proposition 4.1, \( f_1, \ldots, f_q \) is an initially regular sequence on \( R/(I, x_1 + y_1, \ldots, x_\ell + y_\ell) \). Now, by Lemma 2.4, we have
\[
\text{in}(I, x_1 + y_1, \ldots, x_\ell + y_\ell) = \text{in}(\ldots \text{in}(I, x_1 + y_1), x_2 + y_2, \ldots), x_\ell + y_\ell).
\]
Thus, we can concatenate \( x_1+y_1, \ldots, x_\ell+y_\ell \) and \( f_1, \ldots, f_q \) to get an initially regular sequence on \( R/I \). The last claim follows from Theorem 2.2. \( \square \)

The following examples illustrate Corollary 4.9 in the special case of edge ideals of graphs. For graphs, our bounds are similar to known bounds, but our results give a regular sequence or an approximation of one that achieves the bound. For hypergraphs in general, our results are significantly better than known results, and still produce a regular sequence or an approximation of one.

**Example 4.10.** Let \( R = \mathbb{Q}[a, b, c, d, e, f] \), let \( I = (ab, bc, be, de, ef) \) be the edge ideal of the graph \( G \) depicted below, and fix a term order with \( a > b > c > d > e > f \).

```
          a
          |
          b
    ---
          |
          c
        ---
        d
          |
          e
    ---
          f
```

Notice that \( a + b, d + e \) is an initially regular sequence on \( R/I \) by Theorem 3.11. Also, \( c, f \) is a leaf pair in the sense of Definition 4.7. Hence, by Corollaries 2.7 and 4.9, we have that \( c + f, a + b, d + e \) is both a regular and an initially regular sequence on \( R/I \). Therefore, \( \text{depth} R/I \geq 3 \). By [9, Corollary 4.2], \( \text{depth} R/I \geq \max\{\epsilon(G), \tau(G)\} = 2 \). However, by [31, Theorem 1.1] we have that \( \text{depth} R/I = 6 - \text{bight} I = 3 \). In this example, Corollary 4.9 provides an optimal construction in the sense that it produces a maximal regular and initially regular sequence on \( R/I \).

Note that in Example 4.10 we have a tree and, in this case, \( \text{depth} R/I \) is determined by [31, Theorem 1.1]. The next example is a slight modification of the previous one.
Example 4.11. Let $R = \mathbb{Q}[a, b, c, d, e, f, g]$, let $I = (ab, ae, bc, be, de, ef, bg)$ be the edge ideal of the graph $G$ depicted below, and fix a term order with $a > b > c > d > e > f > g$.

![Graph Diagram]

By [9, Corollary 4.2], depth $R/I \geq \max\{e(G), \tau(G)\} = 2$. Also the diameter of this graph is 3 and hence $R/I \geq 2$ by [10, Theorem 3.1]. Using Macaulay 2 [18] we have that depth $R/I = 3$. Notice that $d + g, c + f, a + b + e$ is both a regular sequence and an initially regular sequence on $R/I$ by Corollary 2.7, Theorem 3.11 and Corollary 4.9. Our results, again, provide a sharp bound for depth as well as a sequence that realizes the depth.

Next we exhibit a special situation where a variable can be reused in the creation of initially regular sequences.

Theorem 4.12. Let $I$ be a monomial ideal in a polynomial ring $R$. Suppose that $b_0, \ldots, b_t$ are distinct variables in $R$ and $>$ is a fixed term order such that $b_0 > b_1 > \cdots > b_t$. Suppose that for some $q \leq t$, the sets $\{b_0, b_1\}, \{b_1, b_2\}, \ldots, \{b_{q-2}, b_{q-1}\}, \{b_{q-1}, b_q, \ldots, b_t\}$ satisfy the conditions (1) and (2) of Theorem 3.11. Let $f_i = b_{i-1} + b_i$, $1 \leq i \leq q - 1$, and $f_q = b_{q-1} + \cdots + b_t$. Then $f_1, \ldots, f_q$ is both a regular and an initially regular sequence on $R/I$.

Proof. By Corollary 2.7, it suffices to show that $f_1, f_2, \ldots, f_q$ is an initially regular sequence on $R/I$. We will proceed by induction. When $q = 1$ the result follows from Theorem 3.11.

By induction, it suffices to show that $\{b_1, b_2\}, \ldots, \{b_{q-1}, \ldots, b_t\}$ satisfy the conditions (1) and (2) of Theorem 3.11 applied to $H = \text{in}(I, f_1) = \text{in}(I, b_0 + b_1)$. By Corollary 3.6, $d_0(H) \leq 1$ for $i \geq 2$, so condition (1) of Theorem 3.11 holds for the sets $\{b_1, b_2\}, \ldots, \{b_{q-1}, \ldots, b_t\}$ relative to $H$.

Let $I = (M_1, \ldots, M_p)$, where $M_1, \ldots, M_p$ is a minimal set of monomial generators of $I$ such that $b_0 \mid M_i$ if and only if $1 \leq i \leq \ell$. By Lemma 2.3, $H = \left(b_0, \widehat{M}_1, \ldots, \widehat{M}_\ell, M_{\ell+1}, \ldots, M_p\right)$, where $\widehat{M}_i = b_1^{d_0(M_i)} \frac{M}{b_0^{d_0(M_i)}}$.

By the definition of $\widehat{M}_i$, for $j \neq 0, 1, b_j \mid M$ if an only if $b_j \mid \widehat{M}$. Thus, condition (2) of Theorem 3.11 on $\{b_2, b_3\}, \ldots, \{b_{q-1}, \ldots, b_t\}$ follows from the original hypotheses. If $b_1 \mid M_i$, for any $i$, by hypothesis, $b_2 \mid M_i$. If $b_1 \mid \widehat{M}_i$ for some $i$, then by definition $b_0 \mid M_i$ and again by hypothesis $b_1 \mid M_i$. It follows that $b_2$ divides both $M_i$ and $\widehat{M}_i$ and so condition (2) of Theorem 3.11 holds for $\{b_1, b_2\}$. The result now follows by induction.

It is interesting to note that in the situation of Theorem 4.12 the given set of generators for $H$ is a minimal generating set. Before proving the final result of the section, we give a series of examples. In the first example, as an immediate application of Theorem 4.12, we obtain a sharp bound for the depth of a tetrahedron. It is worth noting that none of
the previously known combinatorial bounds were able to capture the exact value for this example.

**Example 4.13.** Let \( R = \mathbb{Q}[a, b, c, d] \) and let \( I = (abcd) \) be the edge ideal corresponding to the hypergraph \( G \) of a tetrahedron depicted below.

![Tetrahedron Diagram]

It is easy to see that depth \( R/I = 3 \). However, the known combinatorial bound of \cite{[10]} Theorem 3.2 gives at most depth \( R/I \geq 1 \). It follows immediately by Theorem 4.12 that \( a + b, b + c, c + d \) is both a regular and an initially regular sequence on \( R/I \) with respect to a term order with \( a > b > c > d \), and that depth \( R/I \geq 3 \).

In the next example we consider the case of the edge ideal of an octagon. It is worth noting here that we can exhibit a regular sequence that accurately computes the depth, however the fact that the last term of the sequence is regular on the appropriate module does not follow from any of our results. Therefore, there are other regular and initially regular sequences that one can compute and more work can be done in the direction of fully understanding how to construct such sequences.

**Example 4.14.** Let \( I = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_6x_7, x_7x_8, x_1x_8) \subseteq R = \mathbb{Q}[x_1, \ldots, x_8] \) be the edge ideal of the graph \( G \) of the octagon depicted below.

![Octagon Diagram]

First we note that depth \( R/I = 3 \). Using Theorem 3.11 we can only create a maximal initially regular sequence of length two on \( R/I \). For example, let \( f = x_2 + x_1 + x_3 \) and \( g = x_5 + x_6 + x_4 \) and notice that \( f, g \) is an initially regular sequence on \( R/I \) with respect to any term order such that \( x_2 > x_1 > x_3 > x_5 > x_6 > x_4 \). Moreover, \( f, g \) is a regular sequence on \( R/I \) as can be verified by Macaulay 2 \cite{[15]}.

In search for a third element to complete our regular sequence we note that the only variables that were not used are \( x_7, x_8 \). But neither \( x_7 + x_8 + x_6 \) nor \( x_8 + x_7 + x_1 \) are regular on \( R/(I, f, g) \) or initially regular on \( R/(\text{in}(I, f), g) \). However, using Macaulay 2 \cite{[15]} for instance we can see that \( h = x_7 + x_8 + x_6 + x_1 \) is regular on \( R/(I, f, g) \). Moreover, \( f, g, h \) is both a regular and an initially regular sequence on \( R/I \) with respect to the any term order such that \( x_7 > x_2 > x_1 > x_3 > x_5 > x_6 > x_4 > x_8 > x_6 \).

In the next example, we shall see that when there is a freedom of choice in Algorithm 3.12 our bound on the depth can at times be made to be the actual value.
**Example 4.15.** Let \( R = \mathbb{Q}[a, b, c, d, e, f, g, h] \) and let \( I = (abc, acd, bcd, de, efgh) \) be the edge ideal of the following hypergraph.

![Hypergraph Diagram]

Note that \( f + h, h + g, g + e, a + c, c + b + d \) is both a regular and an initially regular sequence on \( R/I \) by Theorems 3.11 and 4.12 with respect to any order such that \( f > h > g > e, a > c > b, \) and \( c > d. \) Hence, depth \( R/I \geq 5 \) and computations on Macaulay 2 [18] show that this is actually an equality.

The final result of this section shows that the method of creating initially regular sequences produces a bound that can be effectively combined with the use of polarization when bounding the depths of non-squarefree monomial ideals. That is, the bound produced will be sufficiently large to at least recover the number of polarizing variables. Note that the prior known depth bound for general hypergraphs, \( \varepsilon \), is not generally effective when combined with this technique due to the nature of polarization. By definition, hyperedges of the polarization that contain polarizing variables will also contain the corresponding original variables, creating a situation where it is relatively easy for a few edges to dominate many others.

**Theorem 4.16.** Let \( I \) be a monomial ideal in \( R = k[x_1, \ldots, x_n] \) and let \( I^{\text{pol}} \subset R^{\text{pol}} \) be its polarization. Then the maximal length of an initially regular sequence on \( R^{\text{pol}}/I^{\text{pol}} \) is at least \( \sum_{i=1}^{n} (d_{x_i}(I) - 1) \), which is the number of polarizing variables.

**Proof.** Set \( d_i = d_{x_i}(I) \). Then \( x_i \) is polarized by variables \( x_{i,1}, x_{i,2}, x_{i,3}, \ldots, x_{i,d_i} \). Set \( x_{i,1} = x_i \) for ease of notation. Let \( R^{\text{pol}} = R[x_{i,1}, \ldots, x_{i,d_i} | 1 \leq i \leq n] \) and let \( I^{\text{pol}} \) denote the polarization of \( I \) in \( R^{\text{pol}} \). Then by the definition of polarization, for any \( 1 \leq i \leq n, d_{x_{i,j}}(I^{\text{pol}}) = 1 \) for all \( 1 \leq j \leq d_i \) and if \( x_{i,j} \) divides a monomial generator \( M \) of \( I^{\text{pol}} \), then \( x_{i,k} \) divides \( M \) for all \( 1 \leq k \leq j \). In particular, the sets \( \{x_{i,j}, x_{i,j-1}\} \) satisfy the conditions of Theorem 4.12 for \( 2 \leq j \leq d_i \). By Theorem 4.12, the elements

\[
x_{i,d_i} + x_{i,d_i-1}, x_{i,d_i-1} + x_{i,d_i-2}, \ldots, x_{i,2} + x_{i,1}
\]

form an initially regular sequence on \( R^{\text{pol}}/I^{\text{pol}} \) with respect to an appropriate term order. Using Lemma 2.4 and the same arguments as in Proposition 4.11 we can concatenate these sequences to see that

\[
x_{1,d_1} + x_{1,d_1-1}, x_{1,d_1-1} + x_{1,d_1-2}, \ldots, x_{1,2} + x_{1,1}, x_{2,d_2} + x_{2,d_2-1}, \ldots, x_{n,2} + x_{n,1}
\]

form an initially regular sequence on \( R^{\text{pol}}/I^{\text{pol}} \) with respect to an appropriate term order. \( \square \)

Application of Theorem 4.16 illustrates the power of the choices made when forming initially regular sequences. The goal is to produce the longest possible initially regular sequence by judicious choice of elements satisfying the hypotheses of Theorem 3.11 and
its extensions. When this maximal length is greater than the minimum guaranteed by Theorem 4.16, a positive lower bound for the depth of the original monomial ideal results.

**Example 4.17.** Let $R = \mathbb{Q}[a, b, c]$ and let $I = (ab, bc)$ be the edge ideal of the graph $G$ of a path of length 2 depicted below.

\[ a \quad b \quad c \]

Consider the ideal $H = I^2 = (a^2b^2, ab^2c, b^2c^2)$. Notice that since $d_a(I) = d_b(I) = d_c(I) = 2$ we may not use any of our previous results to obtain any regular or initially regular elements on $R/I$.

We will use the method of polarization to obtain a bound on the depth of $R/I^2$. Let $a_1, b_1, c_1$ be polarizing variables for $a, b,$ and $c$, respectively. Then

$$H_{\text{pol}} = (aa_1bb_1, abb_1c, bb_1cc_1) \subseteq R[a_1, b_1, c_1] = R_{\text{pol}}.$$ 

By Theorem 3.11, Proposition 4.1 and Theorem 4.12, we have that $c_1 + c, a_1 + a, a + b, b + b_1$ is both a regular and an initially regular sequence on $R_{\text{pol}}/H_{\text{pol}}$ with respect to a term order such that $c_1 > c, a_1 > a > b > b_1$. Hence, depth $R_{\text{pol}}/H_{\text{pol}} \geq 4$ and therefore, depth $R/I^2 \geq 4 - 3 = 1$, by [23, Corollary 1.6.3]. Finally, we can verify that depth $R/I^2 = 1$, using Macaulay 2 [18]. Notice that $\epsilon(H_{\text{pol}}) = 1$, so the prior known depth bound for general hypergraphs yields depth $R_{\text{pol}}/H_{\text{pol}} \geq 1$, which is not large enough to account for the three polarizing variables.

This last example shows how our results on initially regular sequences and the technique of polarization can lead to estimates on the depth of higher powers of monomial ideals. However, the bounds obtained are highly dependent on the structure of the original monomial ideal. Thus, this technique is currently most useful when applied to a specific monomial ideal. To obtain depth bounds on a class of monomial ideals, we will apply an alternate method. The next section handles estimates of the depths of higher powers in the special case of simplicial forests.

5. Depths of powers of squarefree monomial ideals

In this section, we adapt a technique introduced in [4] to give a general lower bound for the depth function of a squarefree monomial ideal when the underlying hypergraph is a hyperforest (also known as a simplicial forest). Simplicial forests were defined by Faridi in [14], where it was shown that the edge ideals of these hypergraphs are always sequentially Cohen-Macaulay. They have also been used in the study of standard graded (symbolic) Rees algebras of squarefree monomial ideals [25]. We first recall the definition of a simplicial forest (or a hyperforest for short).

**Definition 5.1.** Let $G = (V, E)$ be a simple hypergraph.

1. An edge $e \in E$ is called a leaf if either $e$ is the only edge in $G$ or there exists $e \neq g \in E$ such that for any $e \neq h \in E$, $e \cap h \subseteq e \cap g$.
2. A leaf $e$ in $G$ is called a good leaf if the set $\{e \cap h \mid h \in E\}$ is totally ordered with respect to inclusion.
(3) $G$ is called a simplicial forest (or simply, a hyperforest) if every subhypergraph of $G$ contains a leaf. A simplicial tree (or simply, a hypertree) is a connected hyperforest.

It follows from [25, Corollary 3.4] that every hyperforest contains good leaves.

**Example 5.2.** For the hypergraphs depicted below, the first one is not a hypertree while the second one is, see also [14, Examples 1.4, 3.6].

![Diagram showing two hypergraphs](image)

A bound for the depth function of a squarefree monomial ideal based on the edgewise domination number was introduced in [10]. Recall that for a hypergraph $G = (V, E)$ without isolated vertices, a subset $F \subseteq E$ is called edgewise dominant if every vertex $v \in V$ is adjacent to a vertex contained in an edge of $F$, and the edgewise domination number of $G$ is defined to be

$$\epsilon(G) = \min \{|F| \mid F \subseteq E \text{ is edgewise dominant} \}.$$  

We use this bound to obtain a lower bound for the depths of powers of the edge ideal of a simplicial forest in our final theorem. For simplicity of notation, we write $V_H$ and $E_H$ to denote the vertex and edge sets of a hypergraph $H$.

**Theorem 5.3.** Let $G$ be a hyperforest and let $I = I(G)$. Then for all $s \geq 1$,

$$\text{depth } R/I^s \geq \max \{\epsilon(G) - s + 1, 1\}.$$  

**Proof.** It follows from [25, Corollary 3.3] (see also [17]) that the symbolic Rees algebra of $I$ is standard graded. That is, $I^{(s)} = I^s$ for all $s \geq 1$. In particular, this implies that $I^s$ has no embedded primes for all $s \geq 1$. Thus, depth $R/I^s \geq 1$ for all $s \geq 1$.

It remains to show that depth $R/I^s \geq \epsilon(G) - s + 1$. Indeed, this statement and, hence, Theorem 5.3 follows from the following slightly more general result.

**Proposition 5.4.** Let $G$ be a hyperforest. Let $H$ and $T$ be subhypergraphs of $G$ such that $E_H \cup E_T = E_G$ and $E_H \cap E_T = \emptyset$.

Then we have

$$\text{depth } R/[I(H) + I(T)^s] \geq \max \{\epsilon(G) - s + 1, 0\}.$$  

**Proof.** It suffices to show that depth $R/[I(H) + I(T)^s] \geq \epsilon(G) - s + 1$. We shall use induction on $|E_T|$ and $s$. If $|E_T| = 0$ then the statement follows from [9, Theorem 3.2]. If $s = 1$ then the statement also follows from [9, Theorem 3.2]. Suppose that $|E_T| \geq 1$ and $s \geq 2$.

Let $e$ be a good leaf of $T$. By the proof of [8, Theorem 5.1], we have $I(T)^s : e = I(T)^{s-1}$. This implies that

$$(I(H) + I(T)^s) : e = (I(H) : e) + I(T)^{s-1}.$$
Moreover,
\[ I(H) + I(T)^s + (e) = I(H + e) + I(T \setminus e)^s. \]
Thus, we have the exact sequence
\[ 0 \to R/[I(H) : e] + I(T)^{s-1}] \to R/[I(H) + I(T)^s] \to R/[I(H + e) + I(T \setminus e)^s] \to 0 \]
which, in turns, gives
\[(5.1) \quad \text{depth } R/[I(H) + I(T)^s] \geq \min \{ \text{depth } R/[I(H) : e] + I(T)^{s-1}], \text{depth } R/[I(H + e) + I(T \setminus e)^s] \}. \]

Observe that \( G = (H + e) + (T \setminus e) \) and \( E_{H+e} \cap E_{T\setminus e} = \emptyset \). Thus, by induction on \(|E_T|\), we have
\[ \text{depth } R/[I(H + e) + I(T \setminus e)^s] \geq \epsilon(G) - s + 1. \]
On the other hand, let \( Z = \{ z \in V_H | \exists h \in E_H \text{ such that } \{ z \} = h \setminus e \} \). Let \( H' \) be the hypergraph obtained from \( I(H) : e \) by deleting any isolated vertices of \( H \) and the vertices in \( Z \). Let \( T' \) be the hypergraph obtained from \( T \) by deleting any vertices in \( V_T \cap Z \).

Let \( G' = H' + T' \), let \( R' = k[V_H \cup V_T] \), and let \( \text{is}(H) \) be the collection of vertices in \( H \) which do not belong to \( Z \) nor any edges of \( H' \). Then
\[ I(H) : e = I(H') + (z | z \in Z). \]
Therefore, it follows from induction on \( s \) that
\[
\text{depth } R/[I(H) : e] + I(T)^{s-1}] = \text{depth } R/[I(H') + I(T')^{s-1} + (z | z \in Z)]
\[= \text{depth } R'/[I(H') + I(T')^{s-1}] + |\text{is}(H)| \]
\[\geq \epsilon(G') - (s - 1) + 1 + |\text{is}(H)| \]
\[= (\epsilon(G') + 1 + |\text{is}(H)|) - s + 1. \]

Now, let \( F' \subseteq E_{G'} \) be an edgewise dominant set in \( G' \). By the construction of \( H' \), for each \( f' \in F' \cap E_{H'} \), there is an edge \( f \in E_H \) such that \( f' = f \setminus e \). Let \( F \) be the set obtained from \( F' \) by replacing each \( f' \in F' \cap E_{H'} \) by such \( f \). Observe that for any vertex \( v \in V_G \), either \( v \in \text{is}(H) \), or \( v \in Z \), or \( v \in V_{G'} \). If \( v \in Z \) then \( v \) is dominated by \( e \). If \( v \in V_{G'} \) then \( v \) is dominated by some edge in \( F' \). Thus, \( F \cup \{ e \} \) together with one edge for each vertex in \( \text{is}(H) \) will form an edgewise dominant set in \( G \). This implies that
\[ \epsilon(G') + 1 + |\text{is}(H)| \geq \epsilon(G). \]
Therefore,
\[ \text{depth } R/[I(H) : e] + I(T)^{s-1}] \geq \epsilon(G) - s + 1. \]
Hence, by \((5.1)\), we eventually have
\[ \text{depth } R/[I(H) + I(T)^s] \geq \epsilon(G) - s + 1. \]

\[ \square \]

For a random hypertree \( G \), computations indicate that the depth function \( \text{depth } R/I(G)^s \) decreases incrementally as \( s \) increases as predicted by Theorem \((5.3)\). However, \( \epsilon(G) \) is often not the right place to start. On the other hand, for hypertrees \( G \) for which \( \epsilon(G) = \text{depth } R/I(G) \), the depth function \( \text{depth } R/I(G)^s \) usually does not decrease incrementally as \( s \) increases. These statements are illustrated by the following pair of examples.
Example 5.5. Let $I = (x_1x_2, x_2x_3, x_3x_4, x_3x_5, x_3x_6, x_6x_7, x_6x_8, x_8x_9, x_8x_{10}, x_8x_{11}, x_8x_{12}) \subseteq R = \mathbb{Q}[x_1, \ldots, x_{12}]$ be the edge ideal of the graph $G$ depicted below.

![Graph Diagram]

Computation in Macaulay 2 [18] shows that the depth function of $I$ is 4, 3, 2, 1, 1, \ldots. Thus, Theorem 5.3 predicts correctly how the depth function behaves. However, in this example, $\epsilon(G) = 2$ does not give the right value for depth $R/I$. Note that our bound from Theorem 3.11 gives depth $R/I \geq 4$.

Example 5.6. Let $I = (x_1x_2, x_1x_3, x_1x_4, x_4x_5, x_5x_6, x_6x_7, x_6x_8, x_8x_9, x_8x_{10}, x_8x_{11}, x_8x_{12}) \subseteq R = \mathbb{Q}[x_1, \ldots, x_{12}]$ be the edge ideal of the graph $G$ depicted below.

![Graph Diagram]

Then $\epsilon(G) = 3$. Computation in Macaulay 2 [18] shows that the depth function of $I$ is 3, 3, 3, 1, 1, \ldots. The bound in Theorem 5.3 gives the depth function of $I$ to be at least 3, 2, 1, 1, 1, \ldots. In this example, while $\epsilon(G)$ gives the right value for depth $R/I$, Theorem 5.3 does not predict correctly how the depth function of $I$ behaves.

Examples 5.5 and 5.6 show that to get a sharp bound for the depth function of random hypertrees, we may want to start with invariants other than $\epsilon(G)$ which give better bounds for depth $R/I(G)$ in general.

Remark 5.7. The proof of Proposition 5.4 shows that we can replace $\epsilon(G)$ by any invariant $a(G)$, for which depth $R/I(G) \geq a(G)$ and

$$a(G') + 1 + |\text{is}(H)| \geq a(G).$$

It would be interesting to know whether the length of an initially regular sequence with respect to $I(G)$, or improved bounds for depth $R/I(G)$ obtained in Section 4 could be used to get better bounds for the depth function than those given in Theorem 5.3. For instance, in Example 5.5 one bound from Theorem 3.11 gives depth $R/I(G) \geq 4$, which is the right place to start.

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