FURTHER PROPERTIES OF A FUNCTION OF OGG AND LIGOZAT

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Abstract. Certain identities of Ramanujan may be succinctly expressed in terms of the rational function $\tilde{g}_\chi = \tilde{f}_\chi - \frac{1}{\tilde{f}_\chi}$ on the modular curve $X_0(N)$, where $\tilde{f}_\chi = w_N f_\chi$ and $f_\chi$ is a certain modular unit on the Nebentypus cover $X_\chi(N)$ introduced by Ogg and Ligozat for prime $N \equiv 1 \pmod{4}$ and $w_N$ is the Fricke involution. These correspond to levels $N = 5, 13$, where the genus $g_N$ of $X_0(N)$ is zero. In this paper we study a slightly more general kind of relations for each $\tilde{g}_\chi$ such that $X_0(N)$ has genus $g_N = 1, 2$, and also for each $h_\chi = g_\chi + \tilde{g}_\chi$ such that the Atkin-Lehner quotient $X_0^+(N)$ has genus $g_N^+ = 1, 2$. It turns out that if $n$ is the degree of the field of definition $F$ of the non-trivial zeros of the latter, then the degree of the normal closure of $F$ over $\mathbb{Q}$ is the $n$-th solution of Singmaster’s Problem.

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1. INTRODUCTION

Let $X_0(N)$ be the usual compactification of the coarse moduli space $Y_0(N)$ of isomorphism classes of pairs $(E, E')$ of elliptic curves $E$ and $E'$ linked by a cyclic isogeny of degree $N$. Suppose $N$ is a prime number such that $N \equiv 1 \pmod{4}$ and let $X_\chi(N) \longrightarrow X_0(N)$ be the Nebentypus covering that corresponds to the kernel of the quadratic character $\chi$ on $(\mathbb{Z}/N\mathbb{Z})^\times$ as in (7.5.14) and (7.5.15) of Shimura’s book [15]. In Mazur’s paper [11, pp. 107, 108] it may be found a construction due to Ligozat of a modular unit $f_\chi$ on $X_\chi(N)$. Certain identities due to
Ramanujan may be succinctly expressed in terms of the rational function $\tilde{g}_\chi = \tilde{f}_\chi - \frac{1}{f_\chi}$ in $\mathbb{Q}(X_0(N))$ as $\tilde{g}_\chi(\tau) = P(t(\tau))$, where $t(\tau)$ is a certain modular unit on $X_0(N)$, $P(T)\mathbb{Z}[T]$ is a polynomial of degree 1, $\tilde{f}_\chi = w_N f_\chi$, and $w_N$ is the Fricke involution. (See Section 5.) In these cases the genus $g_N$ of $X_0(N)$ is $g_N = 0$. It turns out that for $X_0(N)$ of genus $g_N > 0$, identities of the form $\tilde{g}_\chi = P(t(\tau))$ with $P(T)\mathbb{Q}[T]$ arbitrary are unlikely. So the natural analogs of these identities for genus $g_N > 0$ are perhaps of the form $\tilde{g}_\chi = P(X,Y)$, for some “canonically defined” rational functions $X$ and $Y$ of $X_0(N)$. We suggest such identities for $\tilde{g}_\chi$ for $g_N = 1$ and 2, and also for the rational function $h_\chi = g_\chi + \tilde{g}_\chi \in \mathbb{Q}(X_0^+(N))$, where $g_\chi = f_\chi - \frac{1}{f_\chi}$ and $X_0^+(N) = X_0(N)/\{1, w_N\}$ is the Atkin-Lehner quotient of $X_0(N)$ defined by the involution $w_N$ and $X_0^+(N)$ has genus $g_N^+ = 1$ and 2. With the help of some Gröbner basis algorithms, the latter identities yield the field of definition of each of the zeros of $h_\chi$. These turn out to be either $\mathbb{Q}$ or a finite extension $F$ of the real quadratic field $\mathbb{Q}(\sqrt{N})$; the field extension $F/\mathbb{Q}$ has degree $n = \frac{1}{2}B_{2,\chi}$ if $N \equiv 1 \pmod{8}$ and $n = \frac{1}{2}B_{2,\chi} - 1$, if $N \equiv 5 \pmod{8}$, where $B_{m,\chi}$ is the $m$-th generalised Bernoulli number attached to the character $\chi$. Moreover, in each of these cases the Galois group $G = G(F^{\text{norm}}/\mathbb{Q})$ of the normal closure $F^{\text{norm}}$ of $F$ over the rationals $\mathbb{Q}$ is the wreath product $G = S_2 \wr C_2$ of the symmetric group $S_2$ of permutations of $\frac{n}{2}$ objects and the cyclic group of order two $C_2$. This means that in each of these cases the Galois group $G$ has order $\#(G) = 2(\frac{n}{2})!$, which turns out to be the $n$-th solution of Singmaster’s Problem [16].

**Organisation of the paper.** In order to make the exposition as self contained as possible we include some standard results on modular curves and modular units tailored to our needs, in Section 2. We also include a variant of Ligozat’s construction based on an extension of a classical identity used by Gauß in his third proof of the Law of Quadratic Reciprocity, in Section 3. The main results are contained in Section 5, the $g_N = 1$ case (i.e. $N = 17$) and the $g_N = 2$ case (i.e. $N = 29, 37$), and in Section 6, the field of definition of the zeros of $h_\chi(\tau)$ for $g_N^+ = 1$ (i.e. $N = 37, 53, 61, 89, 101$), and for $g_N^+ = 2$ (i.e. $N = 73$). The paper concludes with a conjecture and an open problem, in Section 7.

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2. Background material

Let $X_1(N)$ be the usual compactification of the coarse moduli space $Y_1(N)$ of isomorphism classes of unordered pairs

$\{(E, P), (E, -P)\},$

where $E$ is an elliptic curve and $P$ a point of $E$. Let $\pi$ be the natural degeneracy morphism

$Y_1(N) \rightarrow Y_0(N)$

induced by the map

$\{(E, P), (E, -P)\} \mapsto (E, E/\langle P \rangle),$

where $\langle P \rangle$ denotes the group generated by the point $P$. Note that for each $r \in \mathbb{Z}$ such that $(r, N) = 1$ the map $(E, p) \mapsto (E_F, r \cdot P)$ induces an automorphism $\sigma_r$ of $X_1(N)$ over $X_0(N)$. Moreover, the map $r \mapsto \sigma_r$ induces an isomorphism from the multiplicative group $C_N = (\mathbb{Z}/N\mathbb{Z})^*/\{-1, 1\}$ onto the Galois group $G(X_1(N)^{an}/X_0(N)^{an})$ of $X_1(N)^{an}$ over $X_0(N)^{an}$) Recall that

$X_0(N)^{an} = \Gamma_0(N)\backslash H^*$,

and

$X_1(N)^{an} = \Gamma_1(N)\backslash H^*$,

where

$\Gamma_0(N) = \left\{ \mu = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) : \gamma \equiv 0 \pmod{N} \right\}.$

and

$\Gamma_1(N) = \left\{ \mu = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(N) : \delta \equiv 1 \pmod{N} \right\}.$

The map

$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \{\delta\}$

induces the above isomorphism from the set of right cosets $[\Gamma_0(N) : \Gamma_1(N)]$ onto the multiplicative group $C_N$, where

$\{\cdot\} : \mathbb{Z} \rightarrow \{0, 1, \ldots, \frac{N-1}{2}\}$

$a \mapsto \pm a \pmod{N},$
following the notation of Ogg [12] and Csirik’s thesis [4]. From now on we assume $N$ is a prime and such that $N \equiv 1 \pmod{4}$, so that $C_N$ is cyclic and the kernel $\Omega = \ker(\chi)$ of the quadratic character

$$\chi: C_N \longrightarrow \{-1, 1\},$$

$$n \mapsto \left(\frac{n}{N}\right),$$

is a (cyclic) subgroup of index 2. The Nebentypus curve $X_\chi(N)$ associated to the character $\chi$ is the intermediate covering $X_\chi(N) \rightarrow X_0(N)$ of the degeneracy morphism $\pi$ associated to the subgroup $\Omega$. In fact $X_\chi(N) = \Gamma_\chi(N)/\mathcal{H}^*$, where

$$\Gamma_\chi(N) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(N) : \chi(\delta) = 1 \right\}.$$

The curve $X_\chi(N)$ has 4 cups, namely the cusps $\infty_1$ and $\infty_2$, above the cusp $\infty$ and cusps $0_1$ and $0_2$, above the cusp $0$; the Galois group $G(X_\chi(N)/X_\chi(N))$ acts transitively on the set $\{\infty_1, \infty_2\}$ and on the set $\{0_1, 0_2\}$.

3. Variant of Ligozat’s construction

Recall that for odd primes $p$ and $q$ such that $p \neq q$ Gauß proved that

$$\left(\frac{q}{p}\right) = (-1)^{S(q,p)}$$

where $(\cdot)$ is the Legendre symbol,

$$S(q,p) = \sum_{r=1}^{\nu - 1} \left\lfloor \frac{r q}{p} \right\rfloor$$

and $[x]$ is the usual floor function. (Cf. p. 78 of Hardy and Wright [7].) This result may be generalised as follows.

Lemma 3.1. Suppose $n$ is an odd integer. If $\delta$ is any integer not divisible by $n$ then

$$\left(\frac{\delta}{n}\right) = (-1)^{S(\delta,n)}(-1)^{n^2/8\nu(\delta,n+1)}$$

where $(\cdot)$ is the Jacobi symbol.

Proof. On the one hand by the work of Jenkins [8] we know that $(\frac{\delta}{n}) = (-1)^{\nu(\delta,n)}$, where $\nu(\delta,n)$ is the number of elements of the set

$$\left\{ k \in \mathbb{Z}_{>0} : k < \frac{n}{2} < (k\delta \pmod{n}) \right\}.$$
On the other hand Lemma 3.1 of Zhi-Wei Sun [17] implies the congruence
\[ \nu \equiv S(\delta, n) + \frac{n^2 - 1}{8}(\delta + 1) \pmod{2} \]
and the lemma follows. \(\square\)

**Theorem 3.2** (Ogg, Ligozat). There is a holomorphic function on \( \mathcal{H} \) such that
\[ (3.1) \quad f_\chi(\mu \tau) = \chi(\delta) f_\chi(\tau)\chi(\delta), \]
where \( \mu = \left( \begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix} \right) \in \Gamma_0(N) \).

**Proof.** Following Kubert and Lang [9] we define the \textit{Siegel function}
\[ g_a(\tau) = t_a \begin{pmatrix} \tau \\ 1 \end{pmatrix} \Delta_{\frac{1}{12}}(\tau) \]
where \( g_a \) is the Klein form associated to a non-zero element \( a = (a_1, a_2) \)
of \( \left( \frac{1}{N}\mathbb{Z}/\mathbb{Z}\right) \times \left( \frac{1}{N}\mathbb{Z}/\mathbb{Z}\right) \), and \( \Delta_{\frac{1}{12}}(\tau) \) is the principal part of the 12-th root of the cusp form
\[ \Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \quad (q = e^{2\pi i \tau}) \]
of weight 12. It is well-known that
\[ g_a(\mu \tau) = \xi(\mu) g_{a\mu}(\tau) \]
where \( \xi(\mu) \) is a the 12-th root of unity such that \( \xi \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) = i \) and \( \xi \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right) = \zeta_{12} \). (See Serre’s book [14].) Now for each integer \( r \) such that \( \gcd(r, N) = 1 \) we define
\[ f_r(\tau) = g(0, \frac{r}{N})(\tau) \]
So the function of Ogg and Ligozat (cf. Mazur [11], p. 107) may be defined by the product
\[ f_\chi(\tau) = \prod_{r=1}^{N-1} f_r(\tau)^{\chi(r)}. \]

By the work of Kubert and Lang [10] we know that
\[ f_r(\mu \tau) = \xi(\mu)e^{\pi i (-\frac{r^2}{N} \gamma \delta + \frac{r \gamma}{N} + \frac{r \delta}{N})} f_{(r\delta)}(\tau) \]
So we have
\[ f_\chi(\mu \tau) = \prod_{r=1}^{N-1} f_r(\mu \tau)^{\chi(r)} = e^{\pi i (S_1 + S_2 + S_3)} f_\chi(\tau)^{\chi(\delta)}, \]
where
\[ S_1 = -\frac{\gamma \delta}{N^2} \sum_{r=1}^{N-1} \chi(r)r^2 \equiv \delta \gamma \cdot N^2 - 1 \cdot 24 \equiv \delta \gamma \cdot \frac{N^2 - 1}{8} \pmod{2}, \]
\[ S_2 = \frac{\gamma}{N} \sum_{r=1}^{N-1} \chi(r)r \equiv \frac{\gamma}{N} \cdot \frac{N^2 - 1}{8} \equiv \gamma \cdot \frac{N^2 - 1}{8} \quad \text{(mod 2)}, \]

\[ S_3 = \sum_{r=1}^{N-1} \chi(r) \left[ \frac{r}{N} \delta \right] \equiv (\delta + 1) \cdot \frac{N^2 - 1}{8} + k_{N,\delta} \quad \text{(mod 2)}, \]

and \( k_{\delta,N} = 1 \), if \( (\frac{\delta}{N}) = -1 \) and \( k_{\delta,N} = 0 \) otherwise. (The latter congruence follows from Lemma 3.1.) Therefore

\[ f_{\chi}(\mu \tau) = \chi(\delta)(-1)(\delta + 1)(\gamma + 1) \frac{N^2 - 1}{8} f_{\chi}(\tau) \chi(\delta). \]

The equation \( \alpha \delta - \beta \gamma = 1 \) implies that \( \gamma \) and \( \delta \) may not have the same parity. So the theorem follows. \( \square \)

4. Infinite products and Fourier expansions

The generalised Bernoulli numbers \( B_{n,\chi} \) attached to the non-trivial primitive character \( \chi \) of conductor \( N \) may be defined by the formal power series

\[ \sum_{r=1}^{N} \chi(r) \frac{Xe^{rX}}{e^{NX} - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{X^n}{n!}. \]

Lemma 4.1. The quotient of polynomials

\[ \Psi(X) = \prod_{r=1}^{N-1} (1 - X\zeta_N^r)^{\chi(r)} \]

has a formal power series expansion of the form

\[ \Psi(X) = 1 - \sqrt{N}X + \cdots \in k[[X]], \]

where \( k = \mathbb{Q}(\sqrt{N}) \).

Proof. Let \( H \) denote the (unique) subgroup of index 2 of the Galois group \( G(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \) and note that our assumption \( N \equiv 1 \) (mod 4) implies that the fixed field of \( H \) is the real quadratic field \( k \). Now let \( \sigma \) be an element of \( H \) and let \( \rho \) be the element of \( (\mathbb{Z}/N\mathbb{Z})^\times \) with the property \( \sigma(\zeta_N) = \zeta_N^\rho \). Clearly

\[ \sigma(\Psi(X)) = \sigma \left( \prod_{r=1}^{N-1} (1 - X\zeta_N^r)^{\chi(r)} \right) = \prod_{r=1}^{N-1} (1 - X\zeta_N^{\rho r})^{\chi(r)} = \Psi(X). \]

So \( \Psi(X) \) is a quotient of two polynomials defined over \( \mathbb{Q}(\sqrt{N}) \), and thus the formal power series of \( \Psi(X) \) has coefficients in \( \mathbb{Q}(\sqrt{N}) \). Finally note that

\[ (1 - X\zeta_N^r)^{\chi(r)} = \begin{cases} 1 + \zeta_N^rX + \zeta_N^{2r}X^2 + \ldots, & \text{if } \chi(r) = -1, \\ 1 - \zeta_N^rX, & \text{if } \chi(r) = +1. \end{cases} \]
So clearly
\[ \Psi(X) = 1 - \left( \sum_{r=1}^{N-1} \chi(r) \zeta^r \right) X + \cdots = 1 - \sqrt{N}X + \ldots \]
where the last equality follows from a well-known result of Gauß and the fact that \( N \equiv 1 \pmod{4} \). \( \Box \)

As before put \( \check{f}_\chi(\tau) = w_N f_\chi(\tau) = f_\chi(w_N(\tau)) \), where \( w_N \) is the Fricke involution \( w_N(\tau) = -\frac{1}{N\tau} \).

**Theorem 4.2.** Assume for simplicity that \( N > 5 \). We have the following \( q \)-expansions.

**A:** If \( k = \mathbb{Q}(\sqrt{N}) \) and \( h(N) \) is the class number of \( k \), then
\[ f_\chi(\tau) = u^{h(N)} (1 - \sqrt{N}q + \ldots) \in k[[q]], \]
where \( u \) is a fundamental unit of \( k \) and \( h(N) \) is the class number of \( k \).

**B:** If as above \( B_{2,\chi} \) denotes the second Bernoulli number attached to \( \chi \) then
\[ \check{f}_\chi(\tau) = q^{\frac{1}{2}B_{2,\chi}} \prod_{n=1}^\infty (1 - q^n)^{\chi(n)} \in \mathbb{Z}[[q]] \]

**Proof.** By well-known results on Siegel functions (see Kubert and Lang [10]) and Lemma 4.1 we have
\[ f_\chi(\tau) = \prod_{r=1}^{N-1} \left( \frac{-q^{-\frac{1}{2}\zeta^{-r}_N}}{2\pi i} (1 - \zeta_N^r)^{\chi(r)} \prod_{n=1}^\infty (1 - q^n\zeta_N^r)(1 - q^n\zeta_N^{-r}) \right)^{\chi(r)} = \lambda \prod_{r=1}^{N-1} \zeta_{2N}^{-\chi(r)r} (1 - \zeta_N^r)^{\chi(r)} = \lambda \prod_{r=1}^{N-1} \Psi(q^n), \]
where
\[ \lambda = \prod_{r=1}^{N-1} \zeta_{2N}^{-\chi(r)r} (1 - \zeta_N^r)^{\chi(r)}. \]
So it remains to show that \( \lambda = u^{h(N)} \). Note
\[ \lambda^\kappa = \left( \prod_{r=1}^{N-1} (\zeta_N^{-\frac{r}{2}} - \zeta_N^{\frac{r}{2}})^{\chi(r)} \right)^\kappa = (-1)^{\frac{N-1}{2}} \lambda = \lambda, \]
where \( \kappa \) denotes complex conjugation. So \( \lambda \) is real. Moreover
\[ \lambda^2 = \lambda\lambda^\kappa = \prod_{r=1}^{N-1} (1 - \zeta_N^r)^{\chi(r)} \prod_{r=1}^{N-1} (1 - \zeta_N^{-r})^{\chi(-r)} = \prod_{r=1}^{N-1} (1 - \zeta_N^r)^{\chi(r)}. \]
By the work of Tate [18] (cf. Darmon [5]) we know
\[ \prod_{r=1}^{N-1} (1 - \zeta_N^r)^{\chi(r)} = u^{2h(N)}, \]
where \( u \) is a fundamental unit of the real quadratic field \( k = \mathbb{Q}(\sqrt{N}) \). Therefore \( \lambda = u^{\pm h(N)} \) and (A) follows. Now consider \( N \equiv 1 \pmod{4} \) implies
\[ N \sum_{r=1}^{N-1} \chi(r)B_2 \left( \frac{r}{N} \right) = \frac{1}{2}B_{\chi,2}, \]
where \( B_2(X) \) is the second Bernoulli polynomial introduced earlier. Moreover,
\[ f_\chi(\tau) = \prod_{r=1}^{N-1} \left( \frac{-1}{2\pi} q^{\frac{1}{2}B_2(\frac{r}{N})} (1 - q^r) \prod_{n=1}^{\infty} (1 - q^{Nn+r})(1 - q^{Nn-r}) \right)^{\chi(r)} = q^{\frac{1}{2}B_{2,\chi}} \prod_{r=1}^{N-1} (1 - q^r) \prod_{n=1}^{\infty} (1 - q^{Nn+r}) \chi(r)(1 - q^{Nn-r}) \chi(-r) = q^{\frac{1}{2}B_{2,\chi}} \prod_{n=1}^{\infty} (1 - q^n) \chi(n), \]
and (B) follows. \( \square \)

From Theorem 4.2 and Theorem 3.2 it follows that \( f_\chi \) defines a rational function on \( X_0(N) \), provided that \( \frac{1}{2}B_{n,\chi} \) is integral. In fact \( \frac{1}{2}B_{n,\chi} \) is integral for all primes \( N \) except when \( N = 5 \), where \( \frac{1}{2}B_{n,\chi} = \frac{1}{5} \).

In this special case we will find it convenient to switch to the notation
\[ f_\chi(\tau) = \left( \prod_{r=1}^{N-1} f_r(\tau)^{\chi(r)} \right)^5. \]

5. On some identities of Ramanujan

Suppose \( N \equiv 1 \pmod{4} \) prime. Let \( \tilde{g}_\chi = \tilde{f}_\chi - \frac{1}{f_\chi} \). It follows from Theorem 3.2 that \( \tilde{g}_\chi \) defines a rational function on \( X_0(N) \). Also let
\[ t(\tau) = q^{\frac{1}{mN}} \prod_{n > 0 \atop n \equiv 0 \pmod{N}} (1 - q^n)^{\frac{24}{m}}, \]
where \( m = \gcd(N-1, 12) \). It is well-known that \( t(\tau) \) defines a rational function on the modular curve \( X_0(N) \). Certain identities due to Ramanujan may be succinctly expressed in terms of \( \tilde{g}_\chi \) and \( t(\tau) \). Indeed, suppose \( N = 5 \) and consider Entry 11(iii) of Chapter 19 of Berndt’s book [1] may be expressed as
\[ ((11 + t(\tau))^2 + 1)^{\frac{1}{2}} - (11 + t(\tau)) = \tilde{f}_\chi(\tau), \]
and thus
\[ \tilde{f}_x^2(\tau) + (11 + t(\tau))\tilde{f}_x(\tau) - 1 = 0. \]

In other words
\[ (5.1) \quad \tilde{g}_x(\tau) = 11 + t(\tau). \]

Now suppose \( N = 13 \). Then \( \mu_2\mu_3\mu_4 = \tilde{f}_x \) and \( \mu_1\mu_5\mu_6 = \frac{1}{\tilde{f}_x} \), where
\[
\begin{align*}
\mu_1 &= q^{-\frac{7}{13}} \frac{f(-q^4, -q^9)}{f(-q^2, -q^{11})}, \\
\mu_2 &= q^{-\frac{6}{13}} \frac{f(-q^6, -q^7)}{f(-q^3, -q^{10})}, \\
\mu_3 &= q^{-\frac{5}{13}} \frac{f(-q^2, -q^{11})}{f(-q, -q^{12})}, \\
\mu_4 &= q^{-\frac{2}{13}} \frac{f(-q^5, -q^8)}{f(-q^4, -q^9)}, \\
\mu_5 &= q^{\frac{3}{13}} \frac{f(-q^3, -q^{10})}{f(-q^5, -q^8)}, \\
\mu_6 &= q^{\frac{15}{13}} \frac{f(-q, -q^{12})}{f(-q^6, -q^7)},
\end{align*}
\]

and \( f(a, b) \) is Ramanujan’s two-variable theta function
\[ f(a, b) = \prod_{n=0}^{\infty} (1 + a^{n+1}b^n)(1 + a^nb^{n+1})(1 - a^{n+1}b^{n+1}). \]

So Entry 8 of Chapter 20 of Berndt’s book [1]
\[ t + 3 = \mu_2\mu_3\mu_4 - \mu_1\mu_5\mu_6 \]

yields
\[ \tilde{f}_x^2(\tau) + (t(\tau) + 3)\tilde{f}_x(\tau) - 1 = 0. \]

Hence
\[ (5.2) \quad \tilde{g}_x(\tau) = t(\tau) + 3. \]

From the point of view of the function theory of the curve \( X_0(N) \) Equation 5.1 and Equation 5.2 are essentially obvious. Indeed, consider the well-known fact that the divisor of poles of \( t(\tau) \) is concentrated at the cusp \( \infty \) with degree \( v_N = \frac{1-N}{m} \), and that the divisor of poles of \( g_x(\tau) \) is concentrated also at the cusp \( \infty \), but with degree \( v_x = \frac{1}{2}B_{2x} \). Assume \( N = 5, 13 \). Thus both, the modular unit \( t(\tau) \) and the rational function \( g_x(\tau) \) have valence one. So there must be a polynomial \( P(T) \in \mathbb{Z}[T] \) of degree one such that \( g_x(T) = P(t(\tau)) \), which may be explicitly obtained using the principal part of the \( q \)-expansion of each of these functions.

An equation of the form \( g_x(T) = P(t(\tau)) \), for some polynomial \( P(T) \in \mathbb{Q}[T] \), for prime level \( N \equiv 1 \pmod{4} \) greater than 13 seems unlikely. Indeed, experimental evidence suggests that for such levels

\footnote{The function \( f_x(\tau) \) also appears in connection with the Rogers-Ramanujan’s continued fractions, e.g. \( 1 + \frac{e^{-2\pi}}{1 + \frac{e^{-4\pi}}{1 + \frac{e^{-6\pi}}{1 + \ldots}}} = \left( \sqrt{\frac{3+\sqrt{5}}{2}} - \frac{1+\sqrt{5}}{2} \right) e^{\frac{2\pi}{\sqrt{5}}} \) in §19.15 (p. 294-295) of Hardy and Wright [7].}
By subtracting from \( g_\chi(\tau) \) the relevant powers of \( t(\tau) \), for each level \( N \) in the above table it is easy to see that a relation of the form \( g_\chi(\tau) = P(t(\tau)) \) is not possible.

For \( X_0(N) \) of genus \( g_N = 1 \) it is not difficult compute explicit expressions of the form \( g_\chi(\tau) = P(X(\tau), Y(\tau)) \), where \( P(X,Y) \in \mathbb{Q}[X,Y] \), and \( X \) and \( Y \) are given generators of the function field of \( X_0(N) \) over \( \mathbb{Q} \). Indeed, let \( X \) and \( Y \) be the coordinate functions of the modular parametrisation of \( X_0(N) \), regarded as an elliptic curve. Then by computing a few terms of the \( q \)-expansion of \( g_\chi(\tau) \) (see Theorem 4.2) it is easy to find an integer \( a(r,s) \), and positive integers \( r \) and \( s \) such that the rational function \( g_\chi(\tau) - a(r,s)X^rY^s \) has a pole at \( \infty \) of degree less than that of \( g_\chi(\tau) \) at \( \infty \). Proceeding recursively, we may arrive at a function with no poles that vanishes at \( \infty \), thus the zero function on \( X_0(N) \). This yields the desired expression \( g_\chi = P(X,Y) \).

Example 5.1. Suppose \( N = 17 \). Using Cremona’s Tables [3] we may see that the curve \( X_0(N) \) of genus \( g_N = 1 \) is isomorphic to curve 17A1. Thus it has a global minimal Weierstraß model

\[
Y^2 + XY + Y = X^3 - X^2 - X - 14,
\]

and a modular parametrisation

\[
X(\tau) = \frac{1}{q^2} + \frac{1}{q} + 1 + O(q),
\]

\[
Y(\tau) = -\frac{1}{q^3} - \frac{2}{q^2} - \frac{2}{q} - 3 + O(q).
\]

Since \( \tilde{g}_\chi \) has exactly one pole, and \( \tilde{g}_\chi = -\frac{1}{q^2} - \frac{1}{q} - 2 + O(q) \), it follows that

\[
\tilde{g}_\chi(\tau) = -(X(\tau) + 1).
\]

For \( X_0(N) \) of genus \( g_N = 2 \) the situation is nearly as simple.

Example 5.2. Suppose \( N = 29 \). The modular curve \( X_0(N) \) is an hyperelliptic curve of genus \( g_N = 2 \) with a model

\[
Y^2 = X^6 + 2X^5 - 17X^4 - 66X^3 - 83X^2 - 32X - 4,
\]

where \( X = \sqrt{2} \cdot \frac{f^{s-1}}{f^{s-1}} \) and \( Y = q \left( \frac{d^s}{dq^s}X \right) \frac{1}{f^{s-1}} \), and \( f \) is the (unique) newform of level \( N \) weight 2 and trivial character. With the help of
the fact that $w_N X = X$ and $w_N Y = -X$ it may be found that

$$\tilde{g}_\chi(\tau) = \frac{1}{2}(Y + X^3 + X^2 - 9X) + 7.$$  

**Example 5.3.** Suppose $N = 37$. It is well-known that the curve $X_0(N)$ has genus $g_N = 2$ and a model

$$Y^2 = X^6 + 14X^5 + 35X^4 + 48X^3 + 35X^2 + 14X + 1$$

where $X = \frac{f_2 + f_1}{f_2 - f_1}$ and $Y = 2q \left( \frac{d}{dq} X \right) \frac{1}{f_1 - f_2}$, and $f_1$ (resp. $f_2$) is the rational newform for $\Gamma_0(N)$ such that the Fourier coefficient $a_{f_1}(N) = -1$ (resp. $a_{f_2}(N) = 1$). Then $\tilde{g}_\chi(\tau)$ is $-\frac{1}{2}$ times

$$X^5 + 16X^4 + 67X^3 + YX^2 + 87X^2 + 9YX + 62X + 11Y + 13$$

**6. THE ZEROS OF $h_\chi$ FOR SMALL $N$**

As above fix a prime $N \equiv 1 \pmod{4}$. Define

$$h_\chi = g_\chi + \tilde{g}_\chi = f_\chi - \frac{1}{f_\chi} + \tilde{f}_\chi - \frac{1}{\tilde{f}_\chi}.$$  

A consequence of Theorem 3.2 is that $h_\chi$ may be regarded as a rational function on $X^+_0(N)$. Using again basic results from the Theory of Algebraic Curves it is an easy matter to obtain equations of the form $h_\chi(\tau) = P(X(\tau), (X(\tau))$, where $P(X, Y) \in \mathbb{Q}[X, Y]$, and $X$ and $Y$ are generators of the function field of $X^+_0(N)$ over $\mathbb{Q}$. The polynomial $P(X, Y)$ may be obtained in a completely analogous way as we did for the rational function $\tilde{g}_\chi(\tau)$ on $X_0(N)$, but now using the formula

$$h_\chi(\tau) = -T_{k/\mathbb{Q}}(u^{h(N)}) - \frac{1}{f_\chi} + O(q),$$

where $u$ is a fundamental unit of $k$, and $h(N)$ is the class number of $k$, and $T_{k/\mathbb{Q}}$ is the trace from the real quadratic field $k = \mathbb{Q}(\sqrt{N})$ to the field of the rational numbers $\mathbb{Q}$. (See Theorem 4.2.) Once we know $P(X, Y)$ we may (at least in principle) obtain the field of definition of each of the zeros of $h_\chi(\tau)$ by decomposing into irreducible factors over $\mathbb{Q}$ a generator $g_X$ of the kernel of the natural map

$$\mathbb{Q}[X, Y]/(W, P) \longrightarrow \mathbb{Q}[X, Y]/(Y)$$

and a generator $g_Y$ of the kernel of the natural map

$$\mathbb{Q}[X, Y]/(W, P) \longrightarrow \mathbb{Q}[X, Y]/(X),$$

provided $X^+_0(N)$ is embeddable in the projective plane $\mathbb{P}^2$ and that we have an explicit equation $W(X, Y) = 0$ for $X^+_0(N)$.

**Example 6.1.** Let $E$ be elliptic curve $37A1$. It is the elliptic curve of conductor $N = 37$ with a minus sign in the functional equation of its
A-function. (See Cremona’s Tables [3].) So $X_0^+(N)$ may be identified with $E$ via the modular parametrisation $\tau \mapsto (X(\tau), Y(\tau))$, where

$$
X(\tau) = \frac{1}{q^2} + \frac{2}{q} + 5 + O(q)
$$

$$
Y(\tau) = -\frac{1}{q^3} - \frac{3}{q^2} - \frac{9}{q} - 21 + O(q)
$$

and $q = e^{2\pi i \tau}$. Now consider that the modular unit $\tilde{f}_\chi$ has divisor

$$(\tilde{f}_\chi) = \frac{1}{2} B_{\chi,2}(\infty_1 - \infty_2) = 5(\infty_1 - \infty_2)$$

Thus the poles of the function $h_\chi(\tau)$ are concentrated at the only cusp of $X_0^+(N)$, with multiplicity 5. So the Riemann-Roch theorem implies that $h_\chi$ may be expressed as a (unique) linear combination of the 5 functions $1, X, Y, X^2$ and $XY$. So a simple exercise in linear algebra involving the principal parts of the $q$-expansion

$$
h_\chi(\tau) = -\frac{1}{q^5} - \frac{1}{q^4} - \frac{1}{q^2} - \frac{2}{q} - 12 + O(q),
$$

(see Theorem 4.2) and of the $q$-expansions of the above 5 functions yield

$$
h_\chi = 1 + 6X + 4Y - 4X^2 - XY.
$$

Now with the help of Grayson’s MACAULAY2 [6], the latter equation together with the Weierstraß equation of $E$:

$$
Y^2 + Y = X^3 - X,
$$

imply that the $X$-coordinates of the zeros of $h_\chi$ are exactly the roots of

$$
g_X(T) = T^5 - 24T^4 + 67T^3 - 42T^2 - 5T + 3 \in \mathbb{Q}[T].
$$

The polynomial $p_x(T)$ factors into irreducibles (over $\mathbb{Q}$) as

$$
g_X(T) = (T - 1)(T^4 - 23T^3 + 44T^2 + 2T - 3).
$$

Similarly, the $Y$-coordinates of the zeros of $h_\chi$ are exactly the roots of

$$
g_Y(T) = T^5 + 95T^4 - 86T^3 - 279T^2 - 72T + 27,
$$

which factors into irreducibles (over $\mathbb{Q}$) as

$$
g_Y(T) = (T + 1)(T^4 + 94T^3 - 180T^2 - 99T + 27).
$$

The computations for $X_0^+(N)$ of genus $g_N^+ = 2$ are nearly as simple as for $g_N^+ = 1$, so we do not discuss the details of these computations here any further. For the $N = 53, 61, 89,$ and $101$ (i.e. the rest of the $g_N^+ = 1$ cases) and for $N = 73$ (the only $g_N^+ = 2$ case under our hypothesis) we found exactly one nontrivial Galois orbit of roots of $h_\chi(\tau)$. The minimal polynomial $p_N(X) \in \mathbb{Z}[T]$ (normalised so that its

\footnote{See Zagier’s paper [19] for an excellent account on the construction of this modular parametrisation.}
coefficients are coprime) of a generator of the field of definition of these points is given as follows.

\[ p_{53}(T) = T^6 - 20T^5 + 95T^4 - 156T^3 + 145T^2 - 174T - 44. \]

\[ p_{61}(T) = T^{10} + 72T^9 + 1000T^8 - 2327T^7 - 2810T^6 + 3994T^5 + 1905T^4 - 2283T^3 - 221T^2 + 404T - 60. \]

\[ p_{69}(T) = T^{26} - 22T^{25} + 108T^{24} + 489T^{23} - 3164T^{22} - 9330T^{21} + 30025T^{20} + 120140T^{19} - 4651T^{18} - 483581T^{17} - 576269T^{16} + 246025T^{15} + 882959T^{14} + 596485T^{13} + 263186T^{12} - 289362T^{11} - 263968T^{10} - 43576T^9 - 51782T^8 - 44804T^7 + 3680T^5 + 7476T^4 - 6260T^3 - 4128T^2 + 3680T - 352. \]

\[ p_{89}(T) = T^{26} - 22T^{25} + 108T^{24} + 489T^{23} - 3164T^{22} - 9330T^{21} + 30025T^{20} + 120140T^{19} - 4651T^{18} - 483581T^{17} - 576269T^{16} + 246025T^{15} + 882959T^{14} + 596485T^{13} + 263186T^{12} - 289362T^{11} - 263968T^{10} - 43576T^9 - 51782T^8 - 44804T^7 + 3680T^5 + 7476T^4 - 6260T^3 - 4128T^2 + 3680T - 352. \]

\[ p_{101}(T) = T^{18} - 19T^{17} + 135T^{16} - 434T^{15} + 548T^{14} + 145T^{13} - 1028T^{12} + 1631T^{11} - 2464T^{10} + 1016T^9 + 4005T^8 - 6040T^7 - 1811T^6 + 5457T^5 - 55T^4 - 3417T^3 - 195T^2 + 1694T + 676. \]

\[ p_{73}(T) = 8T^{22} - 84T^{21} - 874T^{20} + 142T^{19} + 15945T^{18} + 26187T^{17} - 98676T^{16} - 300010T^{15} + 117375T^{14} + 1211979T^{13} - 802441T^{12} + 1804645T^{11} - 23443277T^{10} + 714633T^9 + 1965510T^8 - 93748T^7 - 882954T^6 + 62476T^5 + 225574T^4 + 47106T^3 - 22652T^2 + 9508T - 984. \]

Note that the degree of each of the above polynomials is one less than the degree of the divisor of zeros of \( h_\chi(\tau) \) for \( N = 37, 53, 61, \) and 101. These are exactly the prime levels such that \( N \equiv 5 \pmod{8} \) considered just above. This is explained by the existence of a “trivial” rational zero of \( h_\chi(\tau) \).

**Lemma 6.2.** If \( N \equiv 5 \pmod{8} \) then \( h_\chi(\tau) \) vanishes at the common image in \( X_0^+(N) \) of the two Heegner points of discriminant \( D = -4 \) on the curve \( X_0(N) \).

**Proof.** The congruence \( N \equiv 5 \pmod{8} \) implies that the non-trivial element of the Galois group \( G(X_\chi(N)/X_0(N)) \) may be represented by the matrix

\[ \mu = \begin{pmatrix} \rho & 1 \\ -\rho^2 + 1 & -\rho \end{pmatrix} \in \Gamma_0(N), \]

where \( \rho \) is any integer such that \( \rho^2 \equiv -1 \pmod{N} \). Note that \( \mu \) is an elliptic matrix of order 2 that fixes the Heegner points \( \tau_{-4, \pm r} = -\frac{1}{r \pm r} \in \mathcal{H} \), of discriminant \( D = -4 \) of \( X_0(N) \). Equation 3.1 yields
\( f_\chi(\tau-4,r)^2 = -1 \). But \( f_\chi(\tau)^\kappa = f_\chi(\tau^\kappa) \), where \( \kappa \) denotes complex conjugation. Thus \( f(\tau-4,r) = i \) implies
\[
\check{g}_\chi(\tau-4,-r) = \check{f}_\chi(\tau-4,-r) - \frac{1}{\check{f}_\chi(\tau-4,-r)} = -2i.
\]
Similarly, if \( \check{f}(\tau-4,r) = -i \) then
\[
\check{g}_\chi(\tau-4,-r) = \check{f}_\chi(\tau-4,-r) - \frac{1}{\check{f}_\chi(\tau-4,-r)} = 2i.
\]
But \( w_N(\tau-4,r) = \tau-4,-r \). Therefore \( \check{g}_\chi(\tau-4,\pm r) = -g_\chi(\tau-4,\pm r) \) and the lemma follows. \( \Box \)

7. Final remarks

Some computations we performed on the above polynomials \( p_N(T) \) are consistent with the following.

**Conjecture 7.1.** The field of definition each of the zeros of \( h_\chi \) is either \( \mathbb{Q} \) or an extension \( F \) of the real quadratic field \( k = \mathbb{Q}(\sqrt{N}) \) of degree \( n = \frac{1}{2}B_{2,\chi} \), if \( N \equiv 1 \pmod{8} \) and \( n = \frac{1}{2}B_{2,\chi} - 1 \), if \( N \equiv 5 \pmod{8} \), where \( B_{m,\chi} \) is the \( m \)-th generalised Bernoulli number attached to the character \( \chi \).

In other words we conjecture that the Galois group is in a sense as “large as possible”. Numerical evidence suggests that the above conjecture is true also for the genus \( g_N^+ = 0 \) case, i.e. for \( N = 17, 29 \), and \( N = 41 \). It is hoped that a more extensive numerical evidence from higher genus examples, e.g. \( g_N^+ = 3 \) may perhaps shed some light on the nature of phenomena.

Finally, perhaps it is worth investigating if this potential connexion with Singmaster’s Problem [16] might suggest a yet to be discovered, interesting combinatorial aspect of the Galois group \( G = G(F^{nrm}/\mathbb{Q}) \) of the normal closure \( F^{nrm} \) of \( F \) over the rationals \( \mathbb{Q} \).

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