Residuality of Families of $\mathcal{F}_\sigma$ Sets

Shingo SAITO

Abstract. We prove that two natural definitions of residuality of families of $\mathcal{F}_\sigma$ sets are equivalent. We make use of the Banach-Mazur game in the proof.

1 Introduction

Properties of a typical compact set in the Euclidean space are often discussed. Here we say that a property $P$ is fulfilled by a typical compact set if the set of all compact sets satisfying $P$ is residual in the space of all compact sets endowed with the Hausdorff metric. It is well-known that a typical compact set in the Euclidean space is Lebesgue null (see [3], for example). In this paper we consider what a typical $\mathcal{F}_\sigma$ set means, namely we define residuality of families of $\mathcal{F}_\sigma$ sets. To the best of the author’s knowledge, there has been no definition of such residuality.

We shall work in a compact, dense-in-itself metric space $(X, \rho)$ throughout this article. Without loss of generality, we may assume that $\rho(x, y) \leq 1$ for any $x, y \in X$. An $\mathcal{F}_\sigma$ set means an $\mathcal{F}_\sigma$ subset of $X$, and $\mathcal{F}_\sigma$ stands for the set of all $\mathcal{F}_\sigma$ sets. Let $\mathcal{K}$ denote the set of all compact (or equivalently closed) subsets of $X$. For $x \in X$ and $r > 0$, the closed ball of centre $x$ and radius $r$ is denoted by $\bar{B}(x, r)$. For $K \in \mathcal{K}$ and $r > 0$, we put $K[r] = \bigcup_{x \in K} \bar{B}(x, r)$. It is well-known that the Hausdorff metric $d$ makes $\mathcal{K}$ a compact metric space. Here we define $d(K, \emptyset) = 1$ for any nonempty set $K \in \mathcal{K}$. Then for $K, L \in \mathcal{K}$ and $r \in (0, 1)$, we have $d(K, L) \leq r$ if and only if $K \subset L[r]$ and $L \subset K[r]$, even when either $K$ or $L$ is empty.

Giving $\mathcal{F}_\sigma$ a topology would suffice to define residuality of families of $\mathcal{F}_\sigma$ sets, but no good topology on $\mathcal{F}_\sigma$ has been found so far. Bearing in mind that each $\mathcal{F}_\sigma$ set is the union of a sequence in $\mathcal{K}$, we look at the space of sequences in $\mathcal{K}$ instead. Here we might worry whether we should restrict ourselves only to increasing sequences, but our main theorem removes this concern. Let us proceed to rigorous definitions.

Convention 1.1. Every sequence begins with the term of subscript one and the set $\mathbb{N}$ of all positive integers does not contain zero.

The set of all sequences of sets in $\mathcal{K}$ is denoted by $\mathcal{K}^\mathbb{N}$ and endowed with the product topology. The closed subset $\mathcal{K}^\mathbb{N}_\prec$ of $\mathcal{K}^\mathbb{N}$ is defined as the set of all increasing sequences:

$$\mathcal{K}^\mathbb{N}_\prec = \{ (K_n) \in \mathcal{K}^\mathbb{N} \mid K_1 \subset K_2 \subset \cdots \}.$$
Definition 1.2. For a family $\mathcal{F}$ of $\mathcal{F}_\sigma$ sets, we put

$$K^N\mathcal{F} = \left\{(K_n) \in K^N \mid \bigcup_{n=1}^{\infty} K_n \in \mathcal{F}\right\}.$$ 

We say that $\mathcal{F}$ is $K^N$-residual if $K^N\mathcal{F}$ is residual in $K^N$ and that $\mathcal{F}$ is $K^N_F$-residual if $K^N\mathcal{F} \cap K^N_F$ is residual in $K^N_F$.

Our main theorem asserts that these two notions of residuality agree with each other:

**Main Theorem.** A family of $\mathcal{F}_\sigma$ sets is $K^N$-residual if and only if it is $K^N_F$-residual.

The equivalence seems to show the appropriateness of our definitions. Moreover our definitions match the properties of a typical compact set mentioned at the beginning. We prove a lemma before we state the precise relation.

**Lemma 1.3.** Let $Y$ be a second countable topological space and $Z$ a nonempty Baire space. Then a subset $A$ of $Y$ is residual if and only if $A \times Z$ is residual in $Y \times Z$.

**Proof.** It suffices to show that a subset $A$ of $Y$ is meagre if and only if $A \times Z$ is meagre in $Y \times Z$.

Suppose that $A$ is meagre. Then there exist nowhere dense sets $A_1, A_2, \ldots$ such that $A = \bigcup_{n=1}^{\infty} A_n$. It is easy to see that $A_n \times Z$ is nowhere dense in $Y \times Z$ for every $n \in \mathbb{N}$. Thus $A \times Z = \bigcup_{n=1}^{\infty} (A_n \times Z)$ is meagre.

Conversely suppose that $A \times Z$ is meagre. Then the Kuratowski-Ulam theorem shows that for every $z$ in a residual set in $Z$, the set $\{ y \in Y \mid (y, z) \in A \times Z \} = A$ is meagre. Therefore $A$ is meagre since $Z$ is a nonempty Baire space.

**Remark 1.4.** We shall use this lemma for $Y = \mathcal{K}$ and $Z = \mathcal{K}^N$ in the next proposition. In this situation, the ‘if’ part can be replaced by the following lemma, which is Lemma 4.25 of [2] by Phelps:

Let $M$ be a complete metric space, $Y$ a Hausdorff space and $f: M \rightarrow Y$ a continuous open surjective mapping. If $G$ is the intersection of countably many dense open subsets of $M$, then its image $f(G)$ is residual in $Y$.

Indeed it suffices to substitute $\mathcal{K} \times \mathcal{K}^N$ for $M$, $\mathcal{K}$ for $Y$, and the first projection for $f$. In order to prove this lemma, Phelps used the Banach-Mazur game, which we shall look at from the next section onwards.

**Proposition 1.5.** Let $\mathcal{I}$ be a $\sigma$-ideal on $X$. Then $\mathcal{I} \cap \mathcal{K}$ is residual in $\mathcal{K}$ if and only if $\mathcal{I} \cap \mathcal{F}_\sigma$ is $\mathcal{K}^N$-residual.
Residuality of Families of $\mathcal{F}_\sigma$ Sets

Proof. Since

$$\left\{ (K_n) \in \mathcal{K}^\mathbb{N} \mid \bigcup_{n=1}^\infty K_n \in \mathcal{I} \right\} = \left\{ (K_n) \in \mathcal{K}^\mathbb{N} \mid K_n \in \mathcal{I} \text{ for every } n \in \mathbb{N} \right\} = \bigcap_{n=1}^\infty \left( \mathcal{K} \times \cdots \times \mathcal{K} \times (\mathcal{I} \cap \mathcal{K}) \times \mathcal{K} \times \mathcal{K} \times \cdots \right),$$

we see that $\mathcal{I} \cap \mathcal{F}_\sigma$ is $\mathcal{K}^\mathbb{N}$-residual if and only if $(\mathcal{I} \cap \mathcal{K}) \times \mathcal{K} \times \mathcal{K} \times \cdots$ is residual in $\mathcal{K}^\mathbb{N}$. Lemma 1.3 shows that this is equivalent to the condition that $\mathcal{I} \cap \mathcal{K}$ is residual in $\mathcal{K}$. 

This proposition shows, for example, that a typical $\mathcal{F}_\sigma$ subset of the interval $[0, 1]$ is null.

Acknowledgements. The author expresses his deep gratitude to his supervisor Professor David Preiss for invaluable suggestions and a lot of encouragement. In addition he is grateful to Mr Tim Edwards and Mr Hiroki Kondo for their careful reading of the manuscript. He also acknowledges the financial support by a scholarship from Heiwa Nakajima Foundation and by the Overseas Research Students Award Scheme.

2 Banach-Mazur games

It is known that we can grasp residuality in terms of the Banach-Mazur game.

Definition 2.1. Let $Y$ be a topological space, $S$ a subset of $Y$, and $\mathcal{A}$ a family of subsets of $Y$. Suppose that every set in $\mathcal{A}$ has nonempty interior and that every nonempty open subset of $Y$ contains a set in $\mathcal{A}$. The $(Y, S, \mathcal{A})$-Banach-Mazur game is described as follows. Two players, called Player I and Player II, alternately choose a set in $\mathcal{A}$ with the restriction that they must choose a subset of the set chosen in the previous turn. Player II will win if the intersection of all the sets chosen by the players is contained in $S$; otherwise Player I will win.

Remark 2.2. The assumptions on $\mathcal{A}$ ensure that the players can continue to take sets.

Fact 2.3. The $(Y, S, \mathcal{A})$-Banach-Mazur game has a winning strategy for Player II if and only if $S$ is residual in $Y$.

For the proof of this fact, we refer the reader to Theorem 1 in [1].

In order to prove our main theorem, we look at the following Banach-Mazur games:

Definition 2.4. Let $\mathcal{F}$ be a family of $\mathcal{F}_\sigma$ sets.

Let $\mathcal{B}$ denote the family of all sets of the form

$$\bar{B}((K_n), a, r) = \left\{ (A_n) \in \mathcal{K}^\mathbb{N} \mid d(A_n, K_n) \leq r \text{ for } n = 1, \ldots, a \right\},$$

where $a$ is a positive integer, $(K_n)$ is a sequence in $\mathcal{K}^\mathbb{N}$ such that $K_1, \ldots, K_a$ are pairwise disjoint finite sets, and $r$ is a positive real number less than 1 such that any two distinct

$$\text{3}$$
points in $\bigcup_{j=1}^{a} K_j$ have distance at least $3r$. The $(K^N, K^N_B, B)$-Banach-Mazur game is called the $(K^N, F)-BM$ game for ease of notation.

Let $B_\sigma$ denote the family of all sets of the form

$$B_\sigma((L_n), b, s) = \{ (A_n) \in K^N_\sigma \mid d(A_n, L_n) \leq s \text{ for } n = 1, \ldots, b \},$$

where $b$ is a positive integer, $(L_n)$ is a sequence in $K^N_\sigma$ such that $L_1, \ldots, L_b$ are finite, and $s$ is a positive real number less than 1 such that any two distinct points in $L_b$ have distance at least $3s$. The $(K^N_\sigma, K^N_B \cap K^N_\sigma, B_\sigma)$-Banach-Mazur game is called the $(K^N_\sigma, F)$-BM game.

**Remark 2.5.** Notice that the families $B$ and $B_\sigma$ satisfy the assumptions in Definition 2.1 since $X$ is dense-in-itself.

**Convention 2.6.** Whenever we write $B((K_n), a, r)$ or $B((L_n), b, s)$, we assume that $(K_n)$, $a$, $r$; $(L_n)$, $b$, $s$ satisfy the conditions in Definition 2.4.

**Remark 2.7.** A trivial observation shows that $B((K_n), a, r) \subset B((K_n'), a', r')$ implies $a \geq a'$ and $r \leq r'$ and that $B_\sigma((L_n), b, s) \subset B_\sigma((L_n'), b', s')$ implies $b \geq b'$ and $s \leq s'$.

Fact 2.8 enables us to translate our main theorem into the following:

**Theorem 2.8.** For a family $F$ of $F_\sigma$ sets, the $(K^N_\sigma, F)$-BM game has a winning strategy for Player II if and only if the $(K^N_\sigma, F)$-BM game does.

## 3 Proof of our main theorem

In this section we shall prove Theorem 2.8, which, as we have already mentioned, implies our main theorem. Hereafter we fix a family $F$ of $F_\sigma$ sets and call the Banach-Mazur games without referring to $F$.

### 3.1 Outline of the proof

This subsection is devoted to the outline of the proof that $K^N_\sigma$-residuality implies $K^N$-residuality, or equivalently, that if the $K^N_\sigma$-BM game has a winning strategy for Player II then so does the $K^N$-BM game. Figure 1 illustrates this, and Figure 2 allows us to guess easily the outline of the proof of the other implication.

Suppose that Player I chose $B((K_n^{(1)}), a^{(1)}, r^{(1)})$ in the first turn. Player II transfers it to a certain set, say $B_\sigma((\tilde{K}_n^{(1)}), \tilde{a}^{(1)}, \tilde{r}^{(1)})$, in the $K^N_\sigma$-BM game. Then the winning strategy in the $K^N_\sigma$-BM game tells Player II to take a set $B_\sigma((L_n^{(1)}), b^{(1)}, s^{(1)})$. Player II transfers it to a set $B(\tilde{L}_n^{(1)}, \tilde{b}^{(1)}, \tilde{s}^{(1)})$, which will be the real reply in the $K^N$-BM game. In a similar way, after Player I replies $B((K_n^{(2)}), a^{(2)}, r^{(2)})$, Player II obtains $B_\sigma((\tilde{K}_n^{(2)}), \tilde{a}^{(2)}, \tilde{r}^{(2)}), \tilde{B}_\sigma((\tilde{L}_n^{(2)}), \tilde{b}^{(2)}, \tilde{s}^{(2)}), \text{ and } B((\tilde{L}_n^{(2)}), \tilde{b}^{(2)}, \tilde{s}^{(2)})$. Player II continues this strategy.

Since $K^N$ and $K^N_\sigma$ are compact, the intersections of the closed sets chosen by the players are nonempty. By modifying the winning strategy for the $K^N_\sigma$-BM game, we may assume that...
Residuality of Families of $\mathcal{F}_\sigma$ Sets

Figure 1: Outline of the proof that $\mathcal{K}_N^{\mathcal{N}}$-residuality implies $\mathcal{K}_N^{\mathcal{N}}$-residuality

Figure 2: Outline of the proof of that $\mathcal{K}_N^{\mathcal{N}}$-residuality implies $\mathcal{K}_N^{\mathcal{N}}$-residuality
lim_{m \to \infty} s^{(m)} = 0$, so that the intersection in this game is a singleton. Furthermore, since the transfers are executed so that $\tilde{s}^{(m)} \leq s^{(m)}$ holds for every $m \in \mathbb{N}$ as will be stated below, the intersection in the $\mathcal{K}^{\mathbb{N}}$-BM game is also a singleton.

We write
\[ \bigcap_{m=1}^{\infty} B((K_n^{(m)}), a^{(m)}, r^{(m)}) = \bigcap_{m=1}^{\infty} B((\tilde{L}_n^{(m)}), \tilde{b}^{(m)}, \tilde{s}^{(m)}) = \{(P_n)\} \quad \text{and} \quad \bigcap_{m=1}^{\infty} B_{\mathcal{N}}((\tilde{K}_n^{(m)}), \tilde{a}^{(m)}, \tilde{r}^{(m)}) = \bigcap_{m=1}^{\infty} B_{\mathcal{N}}((L_n^{(m)}), b^{(m)}, s^{(m)}) = \{(Q_n)\}. \]

Notice that
\[ \lim_{m \to \infty} (K_n^{(m)}) = \lim_{m \to \infty} (\tilde{L}_n^{(m)}) = (P_n) \quad \text{and} \quad \lim_{m \to \infty} (\tilde{K}_n^{(m)}) = \lim_{m \to \infty} (L_n^{(m)}) = (Q_n). \]

Since Player II follows the winning strategy in the $\mathcal{K}^{\mathbb{N}}$-BM game, we have $(Q_n) \in \mathcal{K}^{\mathbb{N}}_\mathcal{F} \cap \mathcal{K}^{\mathbb{N}}_{\mathcal{J}}$, or equivalently $\bigcup_{n=1}^{\infty} Q_n \in \mathcal{F}$. Thus all we have to show is that $(P_n) \in \mathcal{K}^{\mathbb{N}}_\mathcal{F}$, and to this aim it suffices to prove that $\bigcup_{n=1}^{\infty} P_n = \bigcup_{n=1}^{\infty} Q_n$.

### 3.2 Details of the transfers

#### 3.2.1 Conditions and definitions

A stage consists of two moves (one in the $\mathcal{K}^{\mathbb{N}}$-BM game and one in the $\mathcal{K}^{\mathbb{N}}_{\mathcal{J}}$-BM game) which lie at the same height in Figures 1 and 2. When we describe the situation at a fixed stage, we omit the integer $m$ indicating the stage unless ambiguity may be caused: for example, we write $K_n$ in place of $K_n^{(m)}$. This is not only for simple notation; we try to offer explanation of the transfers which will go in the proofs of both implications, and this omission solves the problem that when we describe the stage having, say, $K_n^{(m)}$, the previous stage can have $L_n^{(m-1)}$ or $L_n^{(m)}$ depending on which implication we look at.

The transfers are executed so that the following conditions, written as $(\ast)$ afterwards, are fulfilled:

1. $\tilde{a} \geq a$, $\tilde{b} \geq b$, $\tilde{r} \leq r/2$, and $\tilde{s} \leq s/2$;
2. $\bigcup_{j=1}^{n} K_j \subset \tilde{K}_n$ for $n = 1, \ldots, a$, and $\bigcup_{j=1}^{n} L_j \subset L_n$ for $n = 1, \ldots, b$;
3. $\bigcup_{n=1}^{\infty} K_n = \tilde{K}_{\tilde{a}}$ and $\bigcup_{n=1}^{\infty} L_n = L_{\tilde{b}}$.

For $x \in \bigcup_{n=1}^{\infty} K_n = \tilde{K}_{\tilde{a}}$, its affiliation $(n_1, n_2)$ is the pair of the integer $n_1 \in \{1, \ldots, a\}$ with $x \in K_{n_1}$, called the first affiliation of $x$, and the least integer $n_2 \in \{1, \ldots, \tilde{a}\}$ with $x \in \tilde{K}_{n_2}$, called the second affiliation of $x$. We give a similar definition for the points in $\bigcup_{n=1}^{\infty} L_n = L_{\tilde{b}}$: for $x \in \bigcup_{n=1}^{\infty} L_n = L_{\tilde{b}}$, its affiliation $(n_1, n_2)$ is the pair of the integer $n_1 \in \{1, \ldots, \tilde{b}\}$ with $x \in \tilde{L}_{n_1}$, called the first affiliation of $x$, and the least integer $n_2 \in \{1, \ldots, \tilde{b}\}$ with $x \in L_{n_2}$, called the second affiliation of $x$. Strictly speaking, we should specify the stage at which the
affiliations are defined, because, for instance, it may be that \( L^{(m)} \cap L^{(m')} \neq \emptyset \) for distinct \( m \) and \( m' \). However, since we can easily guess the stage from the context, we choose not to specify it in order to avoid complexity.

**Remark 3.1.** Condition (2) in (*) is equivalent to the condition that the first affiliation is always greater than or equal to the second affiliation.

Let us look at \( \bar{B}((K_n), a, r) \in B \) and \( \bar{B}((\tilde{K}_n), \tilde{a}, \tilde{r}) \in B \) at any stage except the first one. We have \( \bar{B}((\tilde{L}_n), \tilde{b}, \tilde{s}) \in B \) and \( \bar{B}((L_n), b, s) \in B \) at the previous stage. Since \( \bar{B}((K_n), a, r) \subset \bar{B}((\tilde{L}_n), \tilde{b}, \tilde{s}) \), for each \( x \in \bigcup_{n=1}^{\tilde{b}} \tilde{L}_n \) there exists a unique \( y \in \bigcup_{n=1}^{\tilde{b}} \tilde{L}_n = L_b \) satisfying \( \rho(x, y) \leq \tilde{s} \), where uniqueness follows from the assumption that any two distinct points in \( \bigcup_{n=1}^{\tilde{b}} \tilde{L}_n \) have distance at least \( 3\tilde{s} \). This \( y \) is called the parent of \( x \). Observe that if \( x \in K_n \) then \( y \in \tilde{L}_n \). We give a similar definition also when we look at \( \bar{B}((L_n), b, s) \in B \) and \( \bar{B}((\tilde{L}_n), \tilde{b}, \tilde{s}) \in B \): the parent of \( x \in L_{\tilde{a}} \) is the unique \( y \in \bigcup_{n=1}^{\tilde{b}} K_n = \tilde{K}_{\tilde{a}} \) satisfying \( \rho(x, y) \leq \tilde{r} \).

### 3.2.2 Transfers from the \( \mathcal{K}^N \)-BM game to the \( \mathcal{K}^N_{\tilde{a}} \)-BM game

Given a move \( \bar{B}((K_n), a, r) \in B \), we shall construct its transfer \( \bar{B}((\tilde{K}_n), \tilde{a}, \tilde{r}) \in B \). If it is the first move of Player I, then we put \( \tilde{a} = a \), \( \tilde{r} = r/2 \), and \( \tilde{K}_n = \bigcup_{j=1}^{n} K_j \) for every \( n \in \mathbb{N} \), and we can easily see that the conditions (*) are fulfilled. So suppose otherwise. Then we already know \( \bar{B}((L_n), b, s) \in B \) and its transfer \( \bar{B}((\tilde{L}_n), \tilde{b}, \tilde{s}) \in B \), and we have \( \bar{B}((K_n), a, r) \subset \bar{B}((\tilde{L}_n), \tilde{b}, \tilde{s}) \).

Put \( \tilde{a} = a \) and \( \tilde{r} = \min\{s - \tilde{s}, r/2\} \), and define \( \tilde{K}_n = \bigcup_{j=1}^{\tilde{b}} K_j \) for \( n > \tilde{b} \). We define \( \tilde{K}_n \) for \( n \leq \tilde{b} \) by declaring that the second affiliation of each \( x \in \bigcup_{n=1}^{\tilde{b}} K_n \) is the same as that of the parent of \( x \).

**Claim.** We have \( d(\tilde{K}_n, L_n) \leq \tilde{s} \) for \( n = 1, \ldots, b \).

**Proof.** Fix such an integer \( n \).

Let \( x \in \tilde{K}_n \) and denote its affiliation by \( (n_1, n_2) \). Then the parent \( y \) of \( x \) has affiliation \( (n_1, n_2) \) and so belongs to \( L_{n_2} \). It follows from \( y \in L_{n_2} \subset L_n \) and \( \rho(x, y) \leq \tilde{s} \) that \( x \in L_n[\tilde{s}] \).

Conversely let \( y \in L_n \) and denote its affiliation by \( (n_1, n_2) \). Then there exists a point \( x \in K_{n_1} \) with \( \rho(x, y) \leq \tilde{s} \) because \( d(K_{n_1}, \tilde{L}_{n_1}) \leq \tilde{s} \). Since \( y \) is the parent of \( x \), the affiliation of \( x \) is \( (n_1, n_2) \). Therefore \( x \in \tilde{K}_{n_2} \subset \tilde{K}_n \) and so \( y \in \tilde{K}_n[\tilde{s}] \).

We may deduce from this claim that \( \bar{B}((\tilde{K}_n), \tilde{a}, \tilde{r}) \subset \bar{B}((L_n), b, s) \) using the triangle inequality and \( \tilde{r} + \tilde{s} \leq s \). Therefore \( \bar{B}((\tilde{K}_n), \tilde{a}, \tilde{r}) \) is a valid reply in the \( \mathcal{K}^N_{\tilde{a}} \)-BM game. It is easy to see that the conditions (*) are fulfilled.

### 3.2.3 Transfers from the \( \mathcal{K}^N_{\tilde{a}} \)-BM game to the \( \mathcal{K}^N \)-BM game

Given a move \( \bar{B}((L_n), b, s) \in B \), we shall construct its transfer \( \bar{B}((\tilde{L}_n), \tilde{b}, \tilde{s}) \in B \). If it is the first move of Player I, then we put \( \tilde{b} = b \), \( \tilde{s} = s/2 \), \( \tilde{L}_1 = L_1 \), and \( \tilde{L}_n = L_n \setminus L_{n-1} \) for
every \( n \geq 2 \). We can easily see that the conditions (*) are fulfilled in this case. So suppose otherwise. Then we already know \( B((K_n), a, r) \in \mathcal{B} \) and its transfer \( \tilde{B}((\tilde{K}_n), \tilde{a}, \tilde{r}) \in \mathcal{B} \), and we have \( \tilde{B}(\tilde{K}_n, b, s) \subset \tilde{B}(\tilde{K}_n, \tilde{a}, \tilde{r}) \).

Put \( \bar{b} = b + 1 \) and \( \bar{s} = \min\{r - \bar{r}, s/2\} \), and define \( \bar{L}_n = L_{n-1} \) for \( n > \bar{b} \). We define \( \bar{L}_n \) for \( n \leq \bar{b} \) by determining the first affiliation of each point in \( L_b \) as follows. Let \( x \in L_b \) and denote its second affiliation by \( n_2 \). If \( n_2 > \bar{a} \), then the first affiliation of \( x \) is \( n_2 \). Suppose \( n_2 \leq \bar{a} \), and let \( y \in \bar{K}_{n_2} \) denote the parent of \( x \). If the second affiliation of \( y \) is \( n_2 \), then the first affiliation of \( x \) is the same as that of \( y \); otherwise the first affiliation of \( x \) is \( \bar{b} \).

\[ \text{Claim. We have } d(\bar{L}_n, K_n) \leq \tilde{r} \text{ for } n = 1, \ldots, a. \]

\[ \text{Proof. Fix an integer } n. \]

Let \( x \in \bar{L}_n \) and denote its parent by \( y \). Then it follows that \( x \) and \( y \) have the same affiliation, and so \( y \in K_n \). Hence we may infer from \( \rho(x, y) \leq \tilde{r} \) that \( x \in K_n[\tilde{r}] \).

Conversely let \( y \in K_n \) and denote its second affiliation by \( n_2 \). Then there exists a point \( x \in L_{n_2} \) with \( \rho(x, y) \leq \tilde{r} \) because \( d(\bar{K}_{n_2}, L_{n_2}) \leq \tilde{r} \). Since \( y \) is the parent of \( x \) and has the same second affiliation as \( x \), the first affiliation of \( x \) is \( n \). Therefore \( y \in \bar{L}_n[\tilde{r}] \). \]

We may deduce from the claim that \( \bar{B}((\bar{L}_n), \bar{b}, \bar{s}) \subset \bar{B}((K_n), a, r) \) using the triangle inequality and \( \bar{r} + \bar{s} \leq r \). Therefore \( \bar{B}((\bar{L}_n), \bar{b}, \bar{s}) \) is a valid reply in the \( K^\infty \)-BM game. It is easy to see that the conditions (*) are fulfilled.

### 3.3 Proof of \( \bigcup_{n=1}^{\infty} P_n = \bigcup_{n=1}^{\infty} Q_n \)

We shall prove that \( \bigcup_{n=1}^{\infty} P_n = \bigcup_{n=1}^{\infty} Q_n \), which will complete the proof of Theorem 2.8 and hence of our main theorem. Recall that \( (K_n^{(m)}) \) and \( (\tilde{K}_n^{(m)}) \) converge to \( (P_n) \) and \( (Q_n) \) respectively as \( m \) tends to infinity. In other words we have \( \lim_{m \to \infty} K_n^{(m)} = P_n \) and \( \lim_{m \to \infty} \tilde{K}_n^{(m)} = Q_n \) for every \( n \in \mathbb{N} \).

In order to prove \( \bigcup_{n=1}^{\infty} P_n \subset \bigcup_{n=1}^{\infty} Q_n \), it is enough to show that \( \bigcup_{j=1}^{n} P_j \subset Q_n \) for every \( n \in \mathbb{N} \). The set \( \{(A, B) \in K^2 \mid A \subset B \} \) is closed in \( K^2 \) and contains \( \bigcup_{j=1}^{n} K_j^{(m)} \) for all \( m \in \mathbb{N} \). Since \( \bigcup_{j=1}^{n} K_j^{(m)} \to \bigcup_{j=1}^{n} P_j, Q_n \) as \( m \) tends to infinity, which follows from the continuity of the map \( A_1, \ldots, A_n \to \bigcup_{j=1}^{n} A_j \) from \( K^n \) to \( K \), we obtain \( \bigcup_{j=1}^{n} P_j \subset Q_n \).

Now we shall prove \( \bigcup_{n=1}^{\infty} Q_n \subset \bigcup_{n=1}^{\infty} P_n \). Let \( x \in \bigcup_{n=1}^{\infty} Q_n \), and denote by \( n \) the least positive integer with \( x \in Q_n \). Since it is easy to observe that \( K_1^{(m)} = \tilde{K}_1^{(m)} \) for every \( m \in \mathbb{N} \), which implies \( P_1 = Q_1 \), we may assume that \( n \geq 2 \). Because \( Q_{n-1} \) is closed and \( x \notin Q_{n-1} \), there exists a positive real number \( r \) less than 1 satisfying \( \bar{B}(x, 4r) \cap Q_{n-1} = \emptyset \), that is, \( x \notin Q_{n-1}[4r] \). Fix a positive integer \( m_0 \) such that \( \tilde{a}^{(m)} \geq n, \tilde{r}^{(m)} \leq r \), and \( d(\tilde{K}_n^{(m)}, Q_{n-1}) \leq r \) for every \( m \geq m_0 \). Observe that \( x \notin \tilde{K}_n^{(m)}[3r] \) for every \( m \geq m_0 \).

Set \( k_0 = \lceil 1/r \rceil \). For each \( k \geq k_0 \), choose \( m_k \geq m_0 \) satisfying \( d(\tilde{K}_n^{(m)}, Q_n) \leq 1/k \) for every \( m \geq m_k \), and for each \( m \geq m_k \) take \( y_{km} \in \tilde{K}_n^{(m)} \) with \( \rho(x, y_{km}) \leq 1/k \) and let \( z_{km} \in \tilde{K}_n^{(m)} \) denote the unique point satisfying \( \rho(y_{km}, z_{km}) \leq \tilde{r}^{(m)} \).
Claim. The two points $y_{km}$ and $z_{km}$ have the same affiliation.

Proof. By an ancestor of $y_{km}$ we mean a point that can be written as ‘the parent of . . . the parent of $y_{km}$.’ Observe that $z_{km}$ is an ancestor of $y_{km}$. Indeed if we denote by $z_{km}'$ the ancestor of $y_{km}$ in $K_n^{(mo)}$, then

$$\rho(y_{km}, z_{km}') < \tilde{r}^{(mo)} + \frac{\tilde{r}^{(mo)}}{2} + \frac{\tilde{r}^{(mo)}}{2^2} + \cdots = 2\tilde{r}^{(mo)}$$

and so $\rho(z_{km}, z_{km}') < 3\tilde{r}^{(mo)}$, which implies $z_{km} = z_{km}'$.

In order to prove our claim, it suffices to prove that the second affiliation of the ancestor $w \in K_n^{(m')}$ of $y_{km}$ is $n$ for any $m' \in \{m_0, \ldots, m\}$. We can see $\rho(w, y_{km}) \leq 2\tilde{r}^{(m')} \leq 2r$ by the same reasoning as above. Therefore we have

$$\rho(w, x) \leq \rho(w, y_{km}) + \rho(y_{km}, x) \leq 2r + \frac{1}{k} \leq 3r.$$ 

Thus the second affiliation of $w$ cannot be less than $n$ because $x \notin K_{n-1}^{(m')}[3r]$.  

Note that all $z_{km}$ belong to the single finite set $K_n^{(mo)}$. We can choose $z_k \in K_n^{(mo)}$ for $k \geq k_0$ inductively so that the set

$$\{ m \geq m_k \mid z_{km} = z_{k_0}, \ldots, z_{km} = z_k \}$$

is infinite for any $k \geq k_0$. Then we take $z \in K_n^{(mo)}$ for which $\{ k \geq k_0 \mid z_k = z \}$ is infinite, and put $\{ k \geq k_0 \mid z_k = z \} = \{k_1, k_2, \ldots\}$, where $k_1 < k_2 < \cdots$. Since the set

$$\{ m \geq m_{k_j} \mid z_{kjm} = \cdots = z_{kjm} = z \}$$

is infinite for every $j \in \mathbb{N}$, we may construct a strictly increasing sequence $m'_1, m'_2, \ldots$ of positive integers satisfying $z_{k_1m'_1} = \cdots = z_{k_jm'_j} = z$.

Let $l$ denote the first affiliation of $z$. Then the foregoing claim shows that whenever $i \leq j$, the first affiliation of $y_{km_j'}$ is $l$, which implies that $x \in K_i^{(m_j')}[1/k_i]$. For any $i \in \mathbb{N}$, since $d(K_j^{(m_j')}, P_i) \leq 1/k_i$ for sufficiently large $j$, we have $x \in P_i[2/k_i]$. Hence $x \in \bigcap_{i=1}^{\infty} P_i[2/k_i] = P_i$. This completes the proof.

References

[1] John C. Oxtoby, *The Banach-Mazur Game and Banach Category Theorem*, Contributions to the Theory of Games, vol. 3, Annals of Mathematics Studies 39 (1957), 159–163, Princeton University Press.

[2] Robert R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, second edition, Lecture Notes in Mathematics 1364 (1993), Springer-Verlag.

[3] Tudor Zamfirescu, *How Many Sets Are Porous?*, Proceedings of the American Mathematical Society 100 (1987), vol. 2, 383–387.