Finite-Time Convergence Rates of Nonlinear Two-Time-Scale Stochastic Approximation under Markovian Noise

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Abstract

We study the so-called two-time-scale stochastic approximation, a simulation-based approach for finding the roots of two coupled nonlinear operators. Our focus is to characterize its finite-time performance in a Markov setting, which often arises in stochastic control and reinforcement learning problems. In particular, we consider the scenario where the data in the method are generated by Markov processes, therefore, they are dependent. Such dependent data result to biased observations of the underlying operators. Under some fairly standard assumptions on the operators and the Markov processes, we provide a formula that characterizes the convergence rate of the mean square errors generated by the method to zero. Our result shows that the method achieves a convergence in expectation at a rate $O(1/k^{2/3})$, where $k$ is the number of iterations. Our analysis is mainly motivated by the classic singular perturbation theory for studying the asymptotic convergence of two-time-scale systems, that is, we consider a Lyapunov function that carefully characterizes the coupling between the two iterates. In addition, we utilize the geometric mixing time of the underlying Markov process to handle the bias and dependence in the data. Our theoretical result complements for the existing literature, where the rate of nonlinear two-time-scale stochastic approximation under Markovian noise is unknown.

1 Nonlinear two-time-scale SA

Stochastic approximation (SA), introduced by [1], is a simulation-based approach for finding the root (or fixed point) of some unknown operator $F$ represented by the form of an expectation, i.e., $F(x) = \mathbb{E}_\pi[F(x, \xi)]$, where $\xi$ is some random variable with a distribution $\pi$. Specifically, this method seeks a point $x^*$ such that $F(x^*) = 0$ based on the noisy observations $F(x; \xi)$. The iterate $x$ is iteratively updated by moving along the direction of $F(x; \xi)$ scaled by some step size. Through a careful choice of this step size, the “noise” induced by the random samples $\xi$ can be averaged out across iterations, and the algorithm converges to $x^*$. SA has found broad applications in many areas including statistics, stochastic optimization, machine learning, and reinforcement learning [2, 3, 4].

In this paper, we consider the two-time-scale SA, a generalized variant of the classic SA, which is used to find the root of a system of two coupled nonlinear equations. Given two unknown operators $F : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $G : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ represented by

$$F(x, y) = \mathbb{E}_\pi[F(x, y; \xi)] \quad \text{and} \quad G(x, y) = \mathbb{E}_\pi[G(x, y; \xi)],$$

we seek to find $x^*$ and $y^*$ such that

$$\begin{cases}
F(x^*, y^*) = 0 \\
G(x^*, y^*) = 0.
\end{cases} \quad (1)$$

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Since $F$ and $G$ are unknown, we assume that there is a stochastic oracle that outputs noisy values of $F(x, y)$ and $G(x, y)$ for a given pair $(x, y)$, i.e., we only have access to $F(x, y; \xi)$ and $G(x, y; \xi)$. Using this stochastic oracle, we study the two-time-scale nonlinear SA for solving problem (1), which iteratively updates the iterates $x_k$ and $y_k$, the estimates of $x^*$ and $y^*$, respectively, for any $k \geq 0$ as

$$
\begin{align*}
x_{k+1} &= x_k - \alpha_k F(x_k, y_k; \xi_k) \\
y_{k+1} &= y_k - \beta_k G(x_k, y_k; \xi_k),
\end{align*}
$$

where $x_0$ and $y_0$ are arbitrarily initial conditions and $\{\xi_k\}$ is a sequence of random variables. We consider the case where $\{\xi_k\}$ is a Markov chain, whose stationary distribution is $\pi$. Thus, $\xi_k$ are dependent and the observations are biased, i.e.,

$$
\mathbb{E}_{\xi_k}[F(x; y; \xi_k)] \neq F(x, y) \quad \text{and} \quad \mathbb{E}_{\xi_k}[G(x; y; \xi_k)] \neq G(x, y).
$$

In (2), $\alpha_k$ and $\beta_k$ are two nonnegative step sizes chosen such that $\beta_k \ll \alpha_k$, i.e., the second iterate is updated using step sizes that are very small as compared to the ones used to update the first iterate. Thus, the update of $x_k$ is referred to as the “fast-time scale” while the update of $y_k$ is called the “slow-time scale”. The time-scale difference here is loosely defined as the ratio between the two step sizes, i.e., $\beta_k / \alpha_k$. In addition, the update of the fast iterate depends on the slow iterate and vice versa, that is, they are coupled to each other. To handle this coupling, the two step sizes have to be properly chosen to guarantee the convergence of the method. Indeed, an important problem in this area is to select the two step sizes so that the two iterates converge as fast as possible. Our main focus is, therefore, to derive the finite-time convergence of (2) in solving (1) under some proper choice of these two step sizes and to understand their impact on the performance of the nonlinear two-time-scale SA under Markovian randomness caused by the Markov process $\{\xi_k\}$.

### 1.1 Motivating applications

Nonlinear two-time-scale SA, Eq. (2), has found numerous applications in many areas including stochastic optimization [5, 6], distributed control over cluster networks [7], distributed optimization under communication constraints [8, 9], and reinforcement learning [10, 11]. The Markov setting we consider in this paper can be found in the applications where the generated data are dependent and evolve through time, for example, they are sampled from some dynamical systems. Notable examples include robust estimation [12], stochastic control/reinforcement learning [10, 11], Markov chain Monte Carlo methods [13], and (distributed) incremental stochastic optimization [14, 15]. We provide below two such motivating applications.

One concrete example is to model different variants of the well-known stochastic gradient descent (SGD) where the data is generated by a Markov process such as in robust estimation [12]. In this problem, we assume that the data is generated by an autoregressive process, that is, the data points $\xi_k = (\xi^1_k, \xi^2_k) \in \mathbb{R}^d \times \mathbb{R}$ is generated as follows

$$
\xi^1_k = A \xi^1_{k-1} + e_1 W_k, \quad \xi^2_k = \langle x, \xi^1_k \rangle + V_k,
$$

where $e_1$ is the first basis vector, and $W_k$ and $V_k$ are sampled i.i.d from the standard normal distribution $N(0, 1)$. In addition, $A \in \mathbb{R}^{d \times d}$ is a subdiagonal matrix, where $A_{i,i-1}$ is drawn uniformly from $[.8, .99]$. Obviously, since $\{W_k\}$ and $\{V_k\}$ are i.i.d $\{\xi_k\}$ is a Markov chain. The objective of robust identification is to estimate the system parameter $x$ from these Markov samples. In robust identification problems, we want to find $x$ that optimizes

$$
\min_{x \in \mathbb{R}^d} f(x) = \mathbb{E}_\pi [F(x; \xi)],
$$

where $\mathbb{E}_\pi$ denotes the expectation with respect to the stationary distribution $\pi$. For such problems, it has been shown that $\alpha_k = \beta_k \ll 1$ and $\alpha_k \beta_k / \alpha_k \approx \ln 2$ ensures the fast convergence of the two-time-scale SA.
where $F$ is some loss function, e.g., $F(x; \xi) = (\langle x, \xi^1 \rangle - \xi^2)^2$. For solving this problem, we consider SGD with the Polyak-Ruppert averaging, where an additional averaging iterate is used to improve the performance of the classic SGD [16, 17]

$$
y_{k+1} = y_k - \beta_k \nabla f(y_k, \xi_k),$$

$$
x_{k+1} = \frac{1}{k+1} \sum_{t=0}^{k} y_k = x_k + \frac{1}{k+1} (y_k - x_k),$$

which is a special form of (2). Other applications of SGD under Markov samples can be found in incremental optimization [14, 15], where the iterates are updated based on a finite Markov chain. In general, it has been shown in [18] that using Markov samples also help to improve the performance of SGD as compared to the case of i.i.d samples. In this case, SGD with Markov samples, while using less data and computation, converges faster than the i.i.d counterpart.

As another example, two-time-scale SA has been used extensively to model reinforcement learning methods, for example, gradient temporal difference (TD) learning and actor-critic methods [19, 11, 20, 21, 22] and the references therein. In reinforcement learning, problems are often modeled as Markov decision processes, therefore, Markov samples are a natural setting. To be specific, we consider the gradient TD learning for solving the policy evaluation problem under nonlinear function approximations studied in [19], which can be viewed as a variant of (2). In this problem, we want to estimate the cumulative rewards $V$ of a stationary policy using function approximations $V_y$, that is, our goal is to find $y$ so that $V_y$ is as close as possible to the true value $V$. Here, $V_y$ can be represented by a neural network where $y$ is the weight vector of the network. Let $\zeta$ be the environmental state, whose transition is governed by a Markov process. In addition, let $\gamma$ be the discount factor, $\phi(\zeta) = \nabla V_y(\zeta)$ be the feature vector, and $r$ be the reward returned by the environment. Given a sequence of samples $\{\zeta_k, r_k\}$, one version of GTD is

$$
x_{k+1} = x_k + \alpha_k (\delta_k - \phi(\zeta_k)^T x_k) \phi(\zeta_k)

y_{k+1} = y_k + \beta_k \left[ (\phi(\zeta_k) - \gamma \phi(\zeta_{k+1})) \phi(\zeta_k)^T x_k - h_k \right],$$

where $\delta_k$ and $h_k$ are defined as

$$
\delta_k = r_k + \gamma V_{y_k}(\zeta_{k+1}) - V_{y_k}(\zeta_k),

h_k = (\delta_k - \phi(\zeta_k)^T x_k) \nabla^2 V_{y_k}(\zeta_k) x_k,

$$

which is clearly a variant of (2) under some proper choice of $F$ and $G$. It has been observed that gradient TD is more stable and performs better compared to the single-time-scale counterpart (TD learning) under off-policy learning and nonlinear function approximations [19].

### 1.2 Main contributions

The focus of this paper is to derive the finite-time performance of the nonlinear two-time-scale SA under Markov randomness. In particular, under some proper choice of step sizes $\alpha_k$ and $\beta_k$, we show that the method achieves a convergence in expectation at a rate $O(\log(k)/k^{2/3})$, where $k$ is the number of iterations. Our convergence rate is similar to the ones in the i.i.d setting except for the log factor, which captures the mixing rate of the Markov chain. Our analysis is mainly motivated by the classic singular perturbation theory for studying the asymptotic convergence of two-time-scale systems, that is, we consider a Lyapunov function that carefully characterizes the coupling between the two iterates. In addition, we utilize the geometric mixing time of the underlying Markov process to handle the bias and dependence in the data.
1.3 Related works

Given the broad applications of SA in many areas, its convergence properties have received much interests for years. In particular, the asymptotic convergence of SA, including its two-time-scale variant, can be established by using the (almost) Martingale convergence theorem when the noise are i.i.d or the ordinary differential equation (ODE) method for more general noise settings; see for example [2, 23]. Under the right conditions both of these methods show that the noise effects eventually average out and the SA iterate asymptotically converges to the desired solutions.

The convergence rate of the single-time-scale SA has been studied extensively for years under different settings due to its broad applications in machine learning and stochastic optimization. The asymptotic rate of this method can be studied by using the Central Limit Theorem (CLT), but requiring substantially stronger assumptions [2, 24, 23]. On the other hand, the finite-time bounds of SA has been studied under both i.i.d and Markov settings; see for example [25, 26, 27, 26, 28, 29, 30, 31] and the references therein. We also note that there are also different work on studying the finite-time performance of SA in the context of SGD both in the i.i.d and Markovian noise models; see for example [32, 18, 33, 34] and the references therein.

Unlike the single-time-scale SA, the convergence rates of the two-time-scale SA are less understood due to the complicated interactions between the two step sizes and the iterates. Specifically, the rates of the two-time-scale SA has been studied mostly for the linear settings in both i.i.d and Markovian settings, i.e, when $F$ and $G$ are linear functions w.r.t their variables; see for example in [35, 36, 37, 38, 39, 40, 41, 37]. For the nonlinear settings, we are only aware of the work in [42, 43], which considers the convergence rates of the nonlinear two-time-scale SA in (2) under i.i.d settings. In particular, under the stability condition (Assumption 1 in [42], $\lim_{k \to \infty} (x_k, y_k) = (x^\star, y^\star)$) and when $F$ and $G$ can be locally approximated by linear functions in a neighborhood of $(x^\star, y^\star)$, a convergence rate of (2) in distribution is provided in [42]. This work also shows that the rates of the fast-time and slow-time scales are asymptotically decoupled under proper choice of step sizes, which agrees with the previous observations of linear two-time-scale SA; see for example [35]. On the other hand, the work in [43] studies the finite-time bound of (2) under different assumptions on the operators $F$ and $G$ as compared to the ones considered in [42]. The setting considered in this paper is similar to the ones studied in [43], explained in detail in Section 2. However, unlike the work in [42] and [43], we study the finite-time performance of (2) under Markov randomness, where we consider different techniques as compared to the ones in [42, 43] due to the dependence and bias of the observations in our updates. More details are discussed in the next section.

2 Main Results

In this section, we present in detail the main results of this paper, that is, we provide a finite-time analysis for the convergence rates of (2) in mean square errors. Under some certain conditions explained below, we show that the mean square errors converge to zero at a rate

$$
\mathbb{E} \left[ \|y_k - y^\star\|^2 \right] + \frac{1}{\alpha_k} \mathbb{E} \left[ \|x_k - x^\star\|^2 \right] \leq \mathcal{O} \left( \frac{1}{(k + 1)^2} + \frac{\log(k + 1)}{(k + 1)^{2/3}} \right),
$$

where the choice of $\beta_k \ll \alpha_k$ will be discussed explicitly in the next section. We note that this convergence rate is the same as the one in the i.i.d settings [43], except for the log factor that captures the mixing time of the underlying Markov chain $\{\xi_k\}$. To derive our theoretical result, in the next two subsections we present main technical assumptions and preliminaries used in our analysis.
2.1 Main Assumptions

We first present the main technical assumptions used to derive our finite-time convergence results. First, we discuss the assumptions on the operators $F$ and $G$, which are motivated by the ones required to establish the stability of the corresponding deterministic two-time-scale differential equations of (2) in [44]. For an ease of exposition, we assume here that $(x^*, y^*) = (0, 0)$. Since $\beta_k \ll \alpha_k$ the update of $x_k$ is referred to as the “fast-time” scale while $y_k$ is updated at a “slow-time” scale. The time-scale difference between these two updates is loosely defined by the ratio $\beta_k/\alpha_k \ll 1$, which is equivalent to the one in [44]. To further present our motivation we consider the case of constant step sizes, i.e., $\alpha_k = \alpha$ and $\beta_k = \beta \ll \alpha$ for some proper chosen constants $\alpha, \beta$. Under appropriate choice of step sizes and proper conditions on the noise sequence $\xi_k$, the ODE method shows that the asymptotic convergence of the iterates in (2) is equivalent to the stability the following differential equations [2]

$$
\dot{x} = \frac{dx}{dt} = -F(x(t), y(t))
$$

$$
\dot{y} = \frac{dy}{dt} = -\frac{\beta}{\alpha}G(x(t), y(t)).
$$

First, since $\beta/\alpha \ll 1$, $y(t)$ is updated much slower than $x(t)$, therefore, one can view $y(t)$ being static in $\dot{x}$. By fixing $y(t) = y$ we have

$$
\frac{dx}{dt} = -F(x(t), y).
$$

Thus, to study the stability of $x(t)$ one needs at least to guarantee that this ODE equation has a solution for any given $y$. In this case, the equilibrium of (5) is a function of $y$, i.e., there exists some operator $H$ such that $F(H(y), y) = 0$. A standard condition to guarantee the existence of an equilibrium of (5) is that the operators $H$ and $F$ are Lipschitz continuous as stated in the following assumption.

**Assumption 1.** Given $y \in \mathbb{R}^d$ there exists an operator $H : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $x = H(y)$ is the unique solution of

$$
F(H(y), y) = 0,
$$

where $H$ and $F$ are Lipschitz continuous with constant $L_H$ and $L_F$, respectively, i.e., $\forall x_1, x_2, y_1, y_2 \in \mathbb{R}^d$

$$
\|H(y_1) - H(y_2)\| \leq L_H \|y_1 - y_2\|, \quad (6)
$$

$$
\|F(x_1, y_1) - F(x_2, y_2)\| \leq L_F (\|x_1 - x_2\| + \|y_1 - y_2\|). \quad (7)
$$

**Remark 1.** In the case of linear two-time-scale SA, i.e., $F$ and $G$ are linear

$$
F(x, y) = A_{11}x + A_{12}y,
$$

$$
G(x, y) = A_{21}x + A_{22}y,
$$

where $A_{11}$ is negative definite (but not necessarily symmetric) [35, 40]. We then have $H(y) = -A_{11}^{-1}A_{12}y$ is a linear operator.

Second, for the global asymptotic convergence of $x(t)$ to the equilibrium of (5) it is necessary that this equilibrium is unique. This condition is guaranteed if $F$ is strong monotone.

**Assumption 2.** $F$ is strongly monotone w.r.t $x$ when $y$ is fixed, i.e., there exists a constant $\mu_F > 0$

$$
\langle x - z, F(x, y) - F(z, y) \rangle \geq \mu_F \|x - z\|^2. \quad (8)
$$
These two assumptions are also considered under different variants in the context of both linear and nonlinear two-time-scale SA studied in [35, 36, 38, 39, 40, 41, 42]. Similarly, once \( x(t) \) converges to \( H(y(t)) \) the convergence of \( y(t) \) can be shown through studying the stability of the following differential equation

\[
\frac{dy}{dt} = -\frac{\beta}{\alpha}G(H(y(t)), y(t)).
\tag{9}
\]

We again require two similar assumptions to guarantee the existence and uniqueness of the solution of (9).

**Assumption 3.** The operator \( G(\cdot, \cdot) \) is Lipschitz continuous with constant \( L_G \), i.e., \( \forall x_1, x_2, y_1, y_2 \in \mathbb{R}^d \),

\[
\|G(x_1, y_1) - G(x_2, y_2)\| \leq L_G (\|x_1 - x_2\| + \|y_1 - y_2\|).
\tag{10}
\]

Moreover, \( G \) is 1-point strongly monotone w.r.t \( y^* \), i.e., there exists a constant \( \mu_G > 0 \) such that for all \( y \in \mathbb{R}^d \)

\[
\langle y - y^*, G(H(y), y) \rangle \geq \mu_G \|y - y^*\|^2.
\tag{11}
\]

Assumptions 1–3 are used in [44, Chapter 7] to study the globally asymptotic stability of (4). Our focus is on the finite-time convergence of the stochastic variant (2) of (4). We, therefore, require the following assumption on the Lipschitz continuity of \( F \) and \( G \).

**Assumption 4.** Given any \( \xi \), the operators \( F(\cdot, \cdot; \xi) \) and \( G(\cdot, \cdot; \xi) \) are Lipschitz continuous with constant \( L_F \) and \( L_G \), respectively. That is, for any \( x_1, x_2, y_1, y_2 \in \mathbb{R}^d \) we have

\[
\|F(x_1, y_1; \xi) - F(x_2, y_2; \xi)\| \leq L_F (\|x_1 - x_2\| + \|y_1 - y_2\|) \quad \text{a.s.,}
\]

\[
\|G(x_1, y_1; \xi) - G(x_2, y_2; \xi)\| \leq L_G (\|x_1 - x_2\| + \|y_1 - y_2\|) \quad \text{a.s.}
\tag{12}
\]

Note that assumption 4 is weaker than the boundedness condition on \( F \) and \( G \). Under this assumption, the iterates \( \{x_k, y_k\} \) can be potentially unbounded.

Finally, we present the assumption on the noise model, which basically states that the Markov chain \( \{\xi_k\} \) has geometric mixing time. In particular, we denote by \( \tau(\alpha) \) the mixing time of \( \{\xi_k\} \) associated with a positive constant \( \alpha \), which basically tells us how long the Markov chain gets close to its stationary distribution \( \pi \) [45]. The following assumption formally states the condition of \( \tau(\alpha) \).

**Assumption 5.** The sequence \( \{\xi_k\} \) is a Markov chain with a compact state space \( \Xi \) and has stationary distribution \( \pi \). For all \( x, y \in \mathbb{R}^d \) and \( \xi \in \Xi \) and a given \( \alpha > 0 \) we have \( \forall k \geq \tau(\alpha) \)

\[
\|\mathbb{E}[F(x, y; \xi_k)] - F(x, y) \mid \xi_0 = \xi\| \leq \alpha
\]

\[
\|\mathbb{E}[G(x, y; \xi_k)] - G(x, y) \mid \xi_0 = \xi\| \leq \alpha.
\tag{13}
\]

Moreover, \( \{\xi_k\} \) has a geometric mixing time, i.e., there exists a positive constant \( C \) such that

\[
\tau(\alpha) = C \log \left( \frac{1}{\alpha} \right).
\tag{14}
\]

Assumption 5 basically sates that given \( \alpha > 0 \) there exists \( C > 0 \) s.t. \( \tau(\alpha) = C \log(1/\alpha) \) and

\[
\|P^k(\xi_0, \cdot) - \pi\|_{TV} \leq \alpha, \quad \forall k \geq \tau(\alpha), \forall \xi_0 \in \Xi,
\tag{15}
\]
where \( \| \cdot \|_{TV} \) is the total variance distance and \( \mathbb{P}^k(\xi_0, \xi) \) is the probability that \( \xi_k = \xi \) when we start from \( \xi_0 \) [45]. This assumption holds in various applications, e.g., in incremental optimization [14, 15], where the iterates are updated based on a finite Markov chain, and in reinforcement learning problems with a finite number states [46]. Assumption 5 is used in the existing literature to study the finite-time performance of SGD and SA under Markov randomness; see [18, 29, 41] and the references therein. Finally, since \( \Xi \) is compact the Lipschitz continuity of \( F \) and \( G \) also gives the following result.

**Lemma 1.** Let \( B \) be a constant defined as

\[
B = \max \{ \max_{\xi \in \Xi} \{ \| F(0, 0, \xi) \|, \| G(0, 0, \xi) \| \}, \| F(0, 0) \|, \| G(0, 0) \|, L_F, L_G, L_H \}.
\]

Then for all \( x, y \) we have

\[
\max \{ \| F(x, y, \xi) \|, \| F(x, y) \| \} \leq B(\| x \| + \| y \| + 1) \quad a.s.,
\]

\[
\max \{ \| G(x, y, \xi) \|, \| G(x, y) \| \} \leq B(\| x \| + \| y \| + 1) \quad a.s.
\]

### 2.2 Preliminaries

We now present some preliminaries, which will be useful to derive our main result studied in the next subsection. For an ease of exposition, we present the analysis of these results in Section 3.

To study the performance of SA one can analyze the convergence rate of the following mean square error

\[
\mathbb{E}[\| x_k - x^* \|^2 + \| y_k - y^* \|^2] \quad \text{to zero.}
\]

However, this mean square error does not explicitly characterize the coupling between the two iterates. We, therefore, consider a different notion of mean square error, which will help us to facilitate our development. In particular, under Assumption 1 and by (1) we have \( x^* = H(y^*) \) and

\[
F(H(y^*), y^*) = 0 \quad \text{and} \quad G(H(y^*), y^*) = 0.
\]

The coupling between \( x \) and \( y \) is represented through \( H \), motivated us to consider the following two residual variables

\[
\hat{x}_k = x_k - H(y_k)
\]

\[
\hat{y}_k = y_k - y^*.
\]

Obviously, if \( \hat{y}_k \) and \( \hat{x}_k \) go to zero, \( (x_k, y_k) \to (x^*, y^*) \). Thus, to establish the convergence of \( (x_k, y_k) \) to \( (x^*, y^*) \) one can instead study the convergence of \( (\hat{x}_k, \hat{y}_k) \) to zero. The rest of this paper is to focus on deriving the convergence rates of these variables to zero.

For our analysis, we consider nonincreasing and nonnegative time-varying sequences of step sizes \( \{\alpha_k, \beta_k\} \) satisfying

\[
\sum_{k=0}^{\infty} \alpha_k = \sum_{k=0}^{\infty} \beta_k = \infty \quad \text{and} \quad \sum_{k=0}^{\infty} \left( \alpha_k^2 + \beta_k^2 + \frac{\beta_k^2}{\alpha_k} \right) < \infty,
\]

which also implies that \( \beta_k \ll \alpha_k \) and \( \lim_{k \to \infty} \alpha_k = \lim_{k \to \infty} \beta_k = 0 \). Since \( \alpha_k \) decreases to zero and \( \tau(\alpha_k) = \log(1/\alpha_k) \) there exists a positive integer \( K^* \) s.t.

\[
\alpha_k; \tau(\alpha_k) \triangleq \sum_{t=k-\tau(\alpha_k)}^{k} \alpha_t \leq \tau(\alpha_k) \alpha_{k-\tau(\alpha_k)} \leq \min \left\{ \frac{\log(2)}{2B}, \alpha_0 \right\}, \quad \forall k \geq K^*.
\]
For convenience, we denote by
\[
\psi_k = F(x_k, y_k; \xi_k) - F(x_k, y_k), \\
\zeta_k = G(x_k, y_k; \xi_k) - G(x_k, y_k),
\]
so (2) can be rewritten as
\[
x_{k+1} = x_k - \alpha_k(F(x_k, y_k) + \psi_k) \\
y_{k+1} = y_k - \beta_k(G(x_k, y_k) + \zeta_k),
\]
(22)

Note that \(\{\psi_k, \zeta_k\}\) are Markovian, therefore, they are dependent and have mean different to zero. We denote by \(Q_k\) the filtration contains all the history generated by the algorithms up to time \(k\), i.e.,
\[
Q_k = \{x_0, y_0, \xi_0, \psi_1, \psi_2, \ldots, \xi_{k-1}\}.
\]
Finally, for an ease of exposition we define
\[
z = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \hat{z} = \begin{bmatrix} x - H(y) \\ y - y^* \end{bmatrix} = \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}.
\]
(23)

The main challenges in our analysis are two fold: 1) the coupling between the fast and slow iterates and 2) the dependence and bias in the observations of \(F\) and \(G\) due to the Markov model. We handle the first challenge by introducing a proper weighted Lyapunov function that combines the norms of \(\hat{x}_k\) and \(\hat{y}_k\), which we will discuss in the next section. On the other hand, we utilize the geometric mixing time to handle the Markovian noise, which is used in the following three lemmas to characterize the sizes of the two residual variables.

**Lemma 2.** Suppose that Assumptions 1–5 hold. Then we have for all \(k \geq K^*\)
\[
E[\|\hat{x}_{k+1}\|^2] \leq (1 - \mu_F \alpha_k) E[\|\hat{x}_k\|^2] + 32(1 + B)^6 \left( \frac{5 \beta_k^2}{\mu_F \alpha_k} + \beta_k^2 + \alpha_k \tau(\alpha_k) \alpha_k \right) E[\|\hat{z}_k\|^2] \\
\quad + 32(1 + B)^6 (\|y^*\| + \|H(0)\| + 1)^2 \left( \frac{5 \beta_k^2}{\mu_F \alpha_k} + \beta_k^2 + \alpha_k \tau(\alpha_k) \right),
\]
(24)

**Lemma 3.** Suppose that Assumptions 1–5 hold. Then we have for all \(k \geq K^*\)
\[
E[\|\hat{y}_{k+1}\|^2] \leq (1 - \mu_G \beta_k) E[\|\hat{y}_k\|^2] + 18(1 + B)^4 (\alpha_k \beta_k + 10B \alpha_k \tau(\alpha_k) \beta_k + 3 \beta_k^2) E[\|\hat{z}_k\|^2] \\
\quad + 24(1 + B)^4 (\|y^*\| + \|H(0)\| + 1)^2 (\beta_k^2 + 7B \alpha_k \tau(\alpha_k) \beta_k) + \frac{B^2}{\mu_G} \beta_k E[\|\hat{x}_k\|^2].
\]
(25)

**Lemma 4.** Suppose that Assumptions 1–5 hold. Let \(D_1\) and \(D_2\) be defined as
\[
D_1 = \sum_{k=0}^{\infty} \frac{\beta_k^2}{\alpha_k} + \frac{\beta_k^2}{\alpha_k \tau(\alpha_k)} < \infty, \\
D_2 = 160(1 + B)^6 (\|y^*\| + \|H(0)\| + 1)^2.
\]
(26)

Then we obtain for all \(k \geq K^*\)
\[
E[\|\hat{z}_k\|^2] \leq D \triangleq E[\|\hat{z}_0\|^2] e^{160D_1(B+1)^6} + D_1 D_2 e^{32D_1(B+1)^6}.
\]
(27)
2.3 Convergence Rates

In this section, we present the main result of this paper, which is the convergence rate of (2). To do it, we introduce the following candidate of Lyapunov function, which takes into account the time-scale difference between these two residual variables

$$V(\hat{x}_k, \hat{y}_k) = \mathbb{E}[\|\hat{y}_k\|^2] + \frac{2B^2}{\mu_F \mu_G} \frac{\beta_k}{\alpha_k} \mathbb{E}[\|\hat{x}_k\|^2],$$

(28)

where $\frac{2B^2}{\mu_F \mu_G} \frac{\beta_k}{\alpha_k}$ is to characterize the time-scale difference between the two residuals. Our main result, which is the finite-time bound of the rates of the residual variables to zero in expectation, is formally stated in the following theorem.

**Theorem 1.** Suppose that Assumptions 1–5 hold. Let \{x_k, y_k\} be generated by (2) with \(x_0\) and \(y_0\) initialized arbitrarily. Let \(\alpha_k, \beta_k\) be two sequence of nonnegative and nonincreasing step sizes satisfying

$$\alpha_k = \frac{\alpha_0}{(k + 1)^{2/3}}, \beta_k = \frac{\beta_0}{k + 1}, \quad \frac{\beta_0}{\alpha_0} \leq \max \left\{ \frac{\mu_F}{2 \mu_G}, \frac{\mu_F \mu_G}{B^2} \right\}, \quad \beta_0 \geq \frac{1}{\mu_G}. \quad (29)$$

Moreover, let \(C\) be defined in (14), and \(D_2, D\) be given in Lemma 4. Then we have for all \(k \geq K^*\)

$$V_{k+1} \leq \frac{(K^*)^2 V_{K^*}}{(k + 1)^2} + \frac{5D_2 + 64D(1 + B)^8}{2\mu_F \mu_G} \left( \frac{5\beta_0^3 + 2\mu_F \beta_0 \alpha_0^3}{\mu_F \alpha_0^2} \right) \frac{1}{(k + 1)^{2/3}} + \frac{4C \beta_0 \alpha_0 \log((k + 1)/\alpha_0)}{(k + 1)^{2/3}}. \quad (30)$$

**Proof:** For convenience, we denote by \(\omega_k\)

$$\omega = \frac{2B^2}{\mu_F \mu_G} \frac{\beta_k}{\alpha_k}.$$

Since \(\beta_k/\alpha_k\) is nonincreasing and less than 1, multiplying both sides of (24) by \(\omega_k\) we have

$$\omega_{k+1} \mathbb{E}[\|\hat{x}_{k+1}\|^2] \leq \omega_k (1 - \mu_F \alpha_k) \mathbb{E}[\|\hat{x}_k\|^2] + \frac{64B^2(1 + B)^6}{\mu_F \mu_G} \left( \frac{5\beta_k^3}{\mu_F \alpha_k^2} + \frac{\beta_k^2}{\mu_F \alpha_k^2} + \alpha_{k; \tau(\alpha_k)} \beta_k \right) \mathbb{E}[\|\hat{x}_k\|^2]$$

$$+ \frac{64B^2(1 + B)^6}{\mu_F \mu_G} \left( \|y^*\| + \|H(0)\| + 1 \right)^2 \left( \frac{5\beta_k^3}{\mu_F \alpha_k^2} + \frac{\beta_k^2}{\mu_F \alpha_k^2} + \alpha_{k; \tau(\alpha_k)} \beta_k \right)$$

$$\leq (1 - \mu_G \beta_k) \omega_k \mathbb{E}[\|\hat{x}_k\|^2] + (\mu_G \beta_k - \mu_F \alpha_k) \omega_k \mathbb{E}[\|\hat{x}_k\|^2]$$

$$+ \frac{64B^2(1 + B)^6}{\mu_F \mu_G} \left( \frac{5\beta_k^3}{\mu_F \alpha_k^2} + \frac{\beta_k^2}{\mu_F \alpha_k^2} + \alpha_{k; \tau(\alpha_k)} \beta_k \right) \mathbb{E}[\|\hat{x}_k\|^2]$$

$$+ \frac{64B^2(1 + B)^6}{\mu_F \mu_G} \left( \|y^*\| + \|H(0)\| + 1 \right)^2 \left( \frac{5\beta_k^3}{\mu_F \alpha_k^2} + \frac{\beta_k^2}{\mu_F \alpha_k^2} + \alpha_{k; \tau(\alpha_k)} \beta_k \right).$$
which when adding to (25) and using (28) we obtain

\[
V_{k+1} \leq (1 - \mu_G \beta_k) V_k + \left(\mu_G \beta_k - \mu_F \alpha_k + \frac{B^2 \beta_k}{\mu_G} \omega_k\right) \omega_k \mathbb{E}[\|\hat{x}_k\|^2] \\
+ \frac{64 B^2 (1 + B)^6}{\mu_F \mu_G} \left(\frac{5 \beta_k^3}{\mu_F \alpha_k^2} + \beta_k^2 + \alpha_k;\tau(\alpha_k) \beta_k\right) \mathbb{E}[\|\hat{z}_k\|^2] \\
+ \frac{64 B^2 (1 + B)^6}{\mu_F \mu_G} (\|y^*\| + \|H(0)\| + 1)^2 \left(\frac{5 \beta_k^3}{\mu_F \alpha_k^2} + \beta_k^2 + \alpha_k;\tau(\alpha_k) \beta_k\right) \\
+ 18(1 + B)^4 (\alpha_k \beta_k + 10 B \alpha_k;\tau(\alpha_k) \beta_k + 3 \beta_k^2) \mathbb{E}[\|\hat{z}_k\|^2] \\
+ 24(1 + B)^4 (\|y^*\| + \|H(0)\| + 1)^2 (\beta_k^2 + 7B \alpha_k;\tau(\alpha_k) \beta_k) \\
\leq (1 - \mu_G \beta_k) V_k + \frac{64 (1 + B)^8}{\mu_F \mu_G} \left(\frac{5 \beta_k^3}{\mu_F \alpha_k^2} + 2 \beta_k^2 + 4 \alpha_k;\tau(\alpha_k) \beta_k\right) \mathbb{E}[\|\hat{z}_k\|^2] \\
+ \frac{64 (1 + B)^8}{\mu_F \mu_G} (\|y^*\| + \|H(0)\| + 1)^2 \left(\frac{5 \beta_k^3}{\mu_F \alpha_k^2} + 2 \beta_k^2 + 4 \alpha_k;\tau(\alpha_k) \beta_k\right) \\
\overset{26}{=}(1 - \mu_G \beta_k) V_k + \frac{64 (1 + B)^8}{\mu_F \mu_G} \left(\frac{5 \beta_k^3}{\mu_F \alpha_k^2} + 2 \beta_k^2 + 4 \alpha_k;\tau(\alpha_k) \beta_k\right) \mathbb{E}[\|\hat{z}_k\|^2] \\
+ \frac{5 D_2}{2 \mu_F \mu_G} \left(\frac{5 \beta_k^3}{\mu_F \alpha_k^2} + 2 \beta_k^2 + 4 \alpha_k;\tau(\alpha_k) \beta_k\right),
\] (31)

where the second inequality we use (29) to have

\[
\mu_G \beta_k - \mu_F \alpha_k + \frac{B^2 \beta_k}{\mu_G} \omega_k = \mu_G \beta_k - \mu_F \alpha_k + \frac{\mu_F \alpha_k}{2} \leq 0.
\]

First, since \( \beta_k = \beta_0/(k+1) \) and \( \beta_0 \geq 2/\mu_G \) we have

\[
(k + 1)^2 (1 - \mu_G \beta_k) \leq (k + 1)^2 \left(1 - \frac{k - 1}{k + 1}\right) \leq k^2.
\]

Multiplying both sides of (31) by \((k + 1)^2\) and using the preceding relation and (27) we obtain

\[
(k + 1)^2 V_{k+1} \leq k^2 V_k + \frac{5 D_2 + 64 D (1 + B)^8}{2 \mu_F \mu_G} \left(\frac{5 \beta_k^3}{\mu_F \alpha_k^2} + 2 \beta_k \alpha_k + 4 \beta_k \alpha_k;\tau(\alpha_k)\right) (k + 1)^2 \\
\leq k^2 V_k + \frac{5 D_2 + 64 D (1 + B)^8}{2 \mu_F \mu_G} \left(\frac{5 \beta_0^3 (k + 1)^{1/3}}{\mu_F \alpha_0^2} + 2 \beta_0 \alpha_0 (k + 1)^{1/3} + \frac{4 \beta_0 \alpha_0 \tau(\alpha_k) (k + 1)}{(k + 1 - \tau(\alpha_k))^{2/3}}\right). \tag{32}
\]

By using (20) we have for all \( k \geq K^* \)

\[
\frac{(k + 1)}{(k + 1 - \tau(\alpha_k))} \leq 2(k + 1)^{1/3}.
\]

Thus, we obtain from (32) for all \( k \geq K^* \)

\[
(k + 1)^2 V_{k+1} \leq k^2 V_k + \frac{5 D_2 + 64 D (1 + B)^8}{2 \mu_F \mu_G} \left(\frac{5 \beta_0^3 (k + 1)^{1/3}}{\mu_F \alpha_0^2} + 2 \beta_0 \alpha_0 (k + 1)^{1/3} + 8 \beta_0 \alpha_0 \tau(\alpha_k) (k + 1)^{1/3}\right) \\
\leq (K^*)^2 V_{K^*} + \frac{5 D_2 + 64 D (1 + B)^8}{2 \mu_F \mu_G} \sum_{t=K^*}^k \left(\frac{5 \beta_0^3 (t + 1)^{1/3}}{\mu_F \alpha_0^2} + 2 \beta_0 \alpha_0 (t + 1)^{1/3} + 8 \beta_0 \alpha_0 \tau(\alpha_k) (k + 1)^{1/3}\right) \\
\leq (K^*)^2 V_{K^*} + \frac{5 D_2 + 64 D (1 + B)^8}{2 \mu_F \mu_G} \left(\frac{5 \beta_0^3 (k + 1)^{4/3}}{\mu_F \alpha_0^2} + 2 \beta_0 \alpha_0 (k + 1)^{4/3} + 6 \beta_0 \alpha_0 \tau(\alpha_k) (k + 1)^{4/3}\right),
\]
where we use the integral test to have
\[
\sum_{t=K^*}^{k} (t + 1)^{1/3} \leq 1 + \int_{t=0}^{k} (t + 1)^{1/3} dt \leq \frac{1}{4} + \frac{3}{4}(k + 1)^{4/3},
\]
\[
\sum_{t=K^*}^{k} \tau(\alpha_t)(t + 1)^{1/3} \leq \int_{t=0}^{k} \tau(\alpha_t)(t + 1)^{1/3} dt \leq \frac{3}{4} \tau(\alpha_k)(k + 1)^{4/3}.
\]
Diving both sides of the equation above by (k+1) and using \(\tau(\alpha_k) = \frac{2C}{3} \log((k + 1)/\alpha_0)\) yields (30). □

3 Proof of Main Lemmas

In this section we present the analysis of the results presented in Lemmas 2–4. We require the following technical lemmas, whose proofs are presented in the appendix.

**Lemma 5.** Suppose that Assumptions 1–5 hold. Then for all \(k \geq K^*\) we have
\[
\mathbb{E}[-\hat{x}_k^T \psi_k] \leq 9(1 + B)^4 \alpha_k \mathbb{E}[\|\hat{z}_k\|^2] + 180B^2(1 + B)^3 \alpha_{k;T(\alpha_k)} \mathbb{E}[\|\hat{z}_k\|^2]
+ 156B(1 + B)^4 \alpha_{k;T(\alpha_k)}(\|y^*\| + \|H(0)\| + 1)^2.
\]

**Lemma 6.** Suppose that Assumptions 1–5 hold. Then for all \(k \geq K^*\) we have
\[
\mathbb{E}[-\hat{y}_k^T \zeta_k] \leq 9(1 + B)^4 \alpha_k \mathbb{E}[\|\hat{z}_k\|^2] + 180B^2(1 + B)^3 \alpha_{k;T(\alpha_k)} \mathbb{E}[\|\hat{z}_k\|^2]
+ 156B(1 + B)^4 \alpha_{k;T(\alpha_k)}(\|y^*\| + \|H(0)\| + 1)^2.
\]

**Lemma 7.** Suppose that Assumptions 1–5 hold. Then for all \(k \geq K^*\) we have
\[
\|\hat{z}_k - \hat{z}_{k-\tau(\alpha_k)}\| \leq 4B\alpha_{k;T(\alpha_k)}(\|\hat{z}_{k-\tau(\alpha_k)}\| + 1).
\]
\[
\|\hat{z}_k - \hat{z}_{k-\tau(\alpha_k)}\| \leq 12B\alpha_{k;T(\alpha_k)}(\|\hat{z}_k\| + 1).
\]

**Lemma 8.** Suppose that Assumptions 1–5 hold. Then for all \(k \geq K^*\) we have
\[
\|\hat{z}_k - \hat{z}_{k-\tau(\alpha_k)}\| \leq 4B(1 + B)^2 \alpha_{k;T(\alpha_k)}(\|\hat{z}_{k-\tau(\alpha_k)}\| + \|y^*\| + \|H(0)\| + 1).
\]
\[
\|\hat{z}_k - \hat{z}_{k-\tau(\alpha_k)}\| \leq 12B(1 + B)^2 \alpha_{k;T(\alpha_k)}(\|\hat{z}_k\| + \|y^*\| + \|H(0)\| + 1).
\]
\[
\|\hat{z}_k - \hat{z}_{k-\tau(\alpha_k)}\|^2 \leq 288B^2(1 + B)^4 \alpha_{k;T(\alpha_k)}(\|\hat{z}_k\|^2 + (\|y^*\| + \|H(0)\| + 1)^2).
\]

**Lemma 9.** Let Assumptions 1–10 hold. Then we have
\[
\|\psi_k\| \leq 2B(1 + B)(\|\hat{z}_k\| + \|y^*\| + \|H(0)\| + 1),
\]
\[
\|\zeta_k\| \leq 2B(1 + B)(\|\hat{z}_k\| + \|y^*\| + \|H(0)\| + 1).
\]

3.1 Proof of Lemma 2

**Proof.** Recall that \(\hat{x}_k = x_k - H(y_k)\). By (18) and (22) we have
\[
\|\hat{x}_{k+1}\|^2 = \|x_{k+1} - H(y_{k+1})\|^2 = \|x_k - \alpha_k (F(x_k, y_k) + \psi_k) - H(y_{k+1})\|^2
= \|\hat{x}_k - \alpha_k F(x_k, y_k) + (H(y_k) - H(y_{k+1})) - \alpha_k \psi_k\|^2
= \|\hat{x}_k - \alpha_k F(x_k, y_k)\|^2 + \|H(y_k) - H(y_{k+1}) - \alpha_k \psi_k\|^2
+ 2(\hat{x}_k - \alpha_k F(x_k, y_k))^T (H(y_k) - H(y_{k+1})) - 2\alpha_k \psi_k^T (\hat{x}_k - \alpha_k F(x_k, y_k)).
\]
We next analyze each term on the right-hand side of (41). First, since $F(H(y_k), y_k) = 0$ (by Assumption 1) and by (8) and (16) we consider

\[
(\hat{x}_k - \alpha_k F(x_k, y_k))^2 = (x_k - H(y_k)) - \alpha_k F(x_k, y_k))^2 \\
= \|\hat{x}_k\|^2 - 2\alpha_k (x_k - H(y_k))^T F(x_k, y_k) + \alpha_k^2 \|F(x_k, y_k)\|^2 \\
= \|\hat{x}_k\|^2 - 2\alpha_k (x_k - H(y_k))^T (F(x_k, y_k) - F(H(y_k), y_k)) + \alpha_k^2 \|F(x_k, y_k) - F(H(y_k), y_k)\|^2 \\
\leq (8) \|\hat{x}_k\|^2 - 2\mu_F \alpha_k \|x_k - H(y_k)\|^2 + L_F^2 \alpha_k^2 \|x_k - H(y_k)\|^2 \\
\leq (1 - 2\mu_F \alpha_k + B^2 \alpha_k^2) \|\hat{x}_k\|^2, \tag{42}
\]

where in the first inequality we also use (7). Second, by (6), (10), (16), and $G(H(y^*), y^*) = 0$ we consider

\[
\|G(x_k, y_k)\|^2 \leq (\|G(x_k, y_k) - G(H(y_k), y_k)\| + \|G(H(y_k), y_k) - G(H(y^*), y^*)\|)^2 \\
\leq (B \|\hat{x}_k\| + B (\|H(y_k) - H(y^*)\| + \|\hat{y}_k\|))^2 \leq (B \|\hat{x}_k\| + B(B + 1) \|\hat{y}_k\|)^2 \\
\leq 2B^2 \|\hat{x}_k\|^2 + 2B^2 (B + 1)^2 \|\hat{y}_k\|^2. \tag{43}
\]

Using (6), (2), (16), and (43) we consider the second term on the right-hand side of (41)

\[
\|H(y_k) - H(y_{k+1}) - \alpha_k \psi_k\|^2 \leq 2\|H(y_k) - H(y_{k+1})\|^2 + 2\alpha_k^2 \|\psi_k\|^2 \\
\leq 2B^2 \|y_{k+1} - y_k\|^2 + 2\alpha_k^2 \|\psi_k\|^2 \leq 2B^2 \beta_k^2 \|G(x_k, y_k) + \zeta_k\|^2 + 2\alpha_k^2 \|\psi_k\|^2 \\
\leq 4B^2 \beta_k^2 \|G(x_k, y_k)\|^2 + 4B^2 \beta_k^2 \|\zeta_k\|^2 + 2\alpha_k^2 \|\psi_k\|^2 \\
\leq 8B^4 \beta_k^2 \|\hat{x}_k\|^2 + 8B^4 (B + 1)^2 \beta_k^2 \|\hat{y}_k\|^2 + 4B^2 \beta_k^2 \|\zeta_k\|^2 + 2\alpha_k^2 \|\psi_k\|^2. \tag{44}
\]

Third, applying the same line of analysis as in (43) we obtain

\[
\|G(x_k, y_k)\| \leq (\|G(x_k, y_k) - G(H(y_k), y_k)\| + \|G(H(y_k), y_k) - G(H(y^*), y^*)\|) \\
\leq (B \|\hat{x}_k\| + B (\|H(y_k) - H(y^*)\| + \|\hat{y}_k\|)) \leq B \|\hat{x}_k\| + B(B + 1) \|\hat{y}_k\|. 
\]

Using the preceding relation, (6), and $F(H(y_k), y_k) = 0$ we consider

\[
(\hat{x}_k - \alpha_k F(x_k, y_k))^T (H(y_k) - H(y_{k+1})) \leq (\|\hat{x}_k\| + \alpha_k \|F(x_k, y_k)\|) \|H(y_k) - H(y_{k+1})\| \\
\leq B \beta_k (\|\hat{x}_k\| + \alpha_k \|F(x_k, y_k)\|) \|y_{k+1} - y_k\| \\
= B \beta_k (\|\hat{x}_k\| + \alpha_k \|F(x_k, y_k) - F(H(y_k), y_k)\|) \|G(x_k, y_k) + \zeta_k\| \\
\leq B \beta_k (\|\hat{x}_k\| + B \alpha_k \|\hat{x}_k\|)(\|G(x_k, y_k)\| + \|\zeta_k\|) \\
\leq B(1 + B \alpha_k \beta_k) \|\hat{x}_k\| (B \|\hat{x}_k\| + B(B + 1) \|\hat{y}_k\| + \|\zeta_k\|),
\]

which by using the relation $2ab \leq a^2/\eta + \eta b^2$ for any $\eta > 0$ yields

\[
(\hat{x}_k - \alpha_k F(x_k, y_k))^T (H(y_k) - H(y_{k+1})) \\
\leq \frac{\mu_F}{2} \alpha_k \|\hat{x}_k\|^2 + \frac{2B^2 (1 + B \alpha_k)^2 \beta_k^2}{\mu_F \alpha_k} (B \|\hat{x}_k\| + B(B + 1) \|\hat{y}_k\| + \|\zeta_k\|)^2 \\
\leq \frac{\mu_F}{2} \alpha_k \|\hat{x}_k\|^2 + \frac{4B^2 (1 + B \alpha_k)^2 \beta_k^2}{\mu_F \alpha_k} (B \|\hat{x}_k\| + B(B + 1) \|\hat{y}_k\|)^2 + \frac{4B^2 (1 + B \alpha_k)^2 \beta_k^2}{\mu_F \alpha_k} \|\zeta_k\|^2 \\
\leq \frac{\mu_F}{2} \alpha_k \|\hat{x}_k\|^2 + \frac{16B^4 (B + 1)^2 \beta_k^2}{\mu_F \alpha_k} \|\hat{x}_k\|^2 + \frac{16B^2 \beta_k^2}{\mu_F \alpha_k} \|\zeta_k\|^2, \tag{45}
\]
where in the last inequality we use (20) to have $1 + B\alpha_k \leq 2$. Finally, using (16) and $F(H(y_k), y_k) = 0$ we obtain

$$
2\alpha_k^2 \psi_k^T \psi F(x_k, y_k) \leq \alpha_k^2 \|\psi_k\|^2 + \alpha_k^2 \|F(x_k, y_k)\|^2 = \alpha_k^2 \|\psi_k\|^2 + \alpha_k^2 \|F(x_k, y_k) - F(H(y_k), y_k)\|^2 \\
\leq \alpha_k^2 \|\psi_k\|^2 + B^2 \alpha_k^2 \|\hat{x}_k\|^2.
$$

(46)

Taking the expectation on (41) and using the preceding relations (42)–(46) yields

$$
\mathbb{E}[\|\hat{x}_{k+1}\|^2] \leq (1 - 2\mu_F \alpha_k + \frac{B^2 \alpha_k^2}{\mu_F \alpha_k})\mathbb{E}[\|\hat{x}_k\|^2] - 2\alpha_k \mathbb{E}[\psi_k^T \hat{x}_k] + \alpha_k^2 \mathbb{E}[\|\psi_k\|^2] + B^2 \alpha_k^2 \mathbb{E}[\|\hat{x}_k\|^2] + 8B^4 \beta_k^2 \mathbb{E}[\|\hat{y}_k\|^2] + 8B^4 \beta_k^2 \mathbb{E}[\|\hat{y}_k\|^2] + 4B^2 \beta_k^2 \mathbb{E}[\|\zeta_k\|^2] + 2\alpha_k^2 \mathbb{E}[\|\psi_k\|^2] \\
+ \frac{\mu_F}{2} \alpha_k \mathbb{E}[\|\hat{x}_k\|^2] + \frac{16B^4 (B + 1)^2 \beta_k^2}{\mu_F \alpha_k} \mathbb{E}[\|\hat{z}_k\|^2] + \frac{16B^2 \beta_k^2}{\mu_F \alpha_k} \mathbb{E}[\|\zeta_k\|^2] \\
\leq (1 - \mu_F \alpha_k)\mathbb{E}[\|\hat{x}_k\|^2] - 2\alpha_k \mathbb{E}[\psi_k^T \hat{x}_k] + 3\alpha_k^2 \mathbb{E}[\|\psi_k\|^2] + \left(\frac{16B^2 \beta_k^2}{\mu_F \alpha_k} + 4B^2 \beta_k^2\right) \mathbb{E}[\|\zeta_k\|^2] \\
+ B^2 \left(\alpha_k^2 + 8B^2 \beta_k^2\right) \mathbb{E}[\|\hat{x}_k\|^2] + 8B^4 (B + 1)^2 \beta_k^2 \mathbb{E}[\|\hat{y}_k\|^2] \\
+ \frac{16B^4 (B + 1)^2 \beta_k^2}{\mu_F \alpha_k} \mathbb{E}[\|\hat{z}_k\|^2].
$$

(47)

By (40) we have

$$
\|\psi_k\|^2 \leq 8B^2 (1 + B)^2 \|\hat{z}_k\|^2 + 8B^2 (1 + B)^2 (\|y^*\| + \|H(0)\| + 1)^2, \\
\|\zeta_k\|^2 \leq 8B^2 (1 + B)^2 \|\hat{z}_k\|^2 + 8B^2 (1 + B)^2 (\|y^*\| + \|H(0)\| + 1)^2,
$$

which by using (33) and $\alpha_k \leq \alpha_{k,\tau(\alpha_k)}$ gives

$$
-2\alpha_k \mathbb{E}[\psi_k^T \hat{x}_k] + 3\alpha_k^2 \mathbb{E}[\|\psi_k\|^2] + \left(\frac{16B^2 \beta_k^2}{\mu_F \alpha_k} + 4B^2 \beta_k^2\right) \mathbb{E}[\|\zeta_k\|^2] \\
\leq 18(1 + B)^4 \alpha_k^2 \|\hat{z}_k\|^2 + 360B^2 (1 + B)^3 \alpha_{k,\tau(\alpha_k)} \alpha_k \|\hat{z}_k\|^2 \\
+ 312B(1 + B)^4 (\|y^*\| + \|H(0)\| + 1)^2 \alpha_{k,\tau(\alpha_k)} \alpha_k \\
+ 8B^2 (1 + B)^2 \left(\frac{16B^2 \beta_k^2}{\mu_F \alpha_k} + 4B^2 \beta_k^2 + 3\alpha_k^2\right) \|\hat{z}_k\|^2 \\
+ 8B^2 (1 + B)^2 (\|y^*\| + \|H(0)\| + 1)^2 \left(\frac{4B^2 (1 + B\alpha_k)^2 \beta_k^2}{\mu_F \alpha_k} + 4B^2 \beta_k^2 + 3\alpha_k^2\right) \\
\leq 8(1 + B)^4 \left(\frac{16B^2 \beta_k^2}{\mu_F \alpha_k} + 4B^2 \beta_k^2 + (45B + 6) \alpha_{k,\tau(\alpha_k)} \alpha_k\right) \|\hat{z}_k\|^2 \\
+ 8(1 + B)^4 (\|y^*\| + \|H(0)\| + 1)^2 \left(\frac{16B^2 \beta_k^2}{\mu_F \alpha_k} + 4B^2 \beta_k^2 + (39B + 3) \alpha_k \alpha_{k,\tau(\alpha_k)}\right).
$$
Substituting the preceding relation into (47) gives
\[
\mathbb{E}[\|\hat{x}_{k+1}\|^2] \leq (1 - \mu_F \alpha_k)\mathbb{E}[\|\hat{x}_k\|^2] \\
+ 8(1 + B)^4 \left( \frac{16B^2 \beta_k^2}{\mu_F \alpha_k} + 4B^2 \beta_k^2 + (45B + 6)\alpha_k;\tau(\alpha_k) \alpha_k \right) \|\hat{z}_k\|^2 \\
+ 8(1 + B)^4 (\|y^*\| + \|H(0)\| + 1)^2 \left( \frac{16B^2 \beta_k^2}{\mu_F \alpha_k} + 4B^2 \beta_k^2 + (39B + 3)\alpha_k;\tau(\alpha_k) \alpha_k \right) \\
+ B^2 \left( \alpha_k^2 + 8B^2 \beta_k^2 \right) \mathbb{E}[\|\hat{x}_k\|^2] + 8B^4 (B + 1)^2 \beta_k^2 \mathbb{E}[\|\hat{y}_k\|^2] \\
+ \frac{16B^4 (B + 1)^2 \beta_k^2}{\mu_F \alpha_k} \mathbb{E}[\|\hat{z}_k\|^2] \\
\leq (1 - \mu_F \alpha_k)\mathbb{E}[\|\hat{x}_k\|^2] \\
+ 8(1 + B)^4 \left( \frac{18B^2 \beta_k^2}{\mu_F \alpha_k} + 4B^2 \beta_k^2 + (45B + 6)\alpha_k;\tau(\alpha_k) \alpha_k \right) \|\hat{z}_k\|^2 \\
+ 8(1 + B)^4 (\|y^*\| + \|H(0)\| + 1)^2 \left( \frac{16B^2 \beta_k^2}{\mu_F \alpha_k} + 4B^2 \beta_k^2 + (39B + 3)\alpha_k;\tau(\alpha_k) \alpha_k \right) \\
+ B^2 \alpha_k^2 \mathbb{E}[\|\hat{z}_k\|^2] + 8B^4 (B + 1)^2 \beta_k^2 \mathbb{E}[\|\hat{z}_k\|^2] \\
\leq (1 - \mu_F \alpha_k)\mathbb{E}[\|\hat{x}_k\|^2] + 32(1 + B)^6 \left( \frac{5\beta_k^2}{\mu_F \alpha_k} + \beta_k^2 + \alpha_k;\tau(\alpha_k) \alpha_k \right) \|\hat{z}_k\|^2 \\
+ 32(1 + B)^6 (\|y^*\| + \|H(0)\| + 1)^2 \left( \frac{5\beta_k^2}{\mu_F \alpha_k} + \beta_k^2 + \alpha_k;\tau(\alpha_k) \alpha_k \right),
\]
which concludes our proof. \(\square\)

3.2 Proof of Lemma 3

Proof. Using (2) we consider
\[
\hat{y}_{k+1} = y_{k+1} - y^* = \hat{y}_k - \beta_k G(x_k, y_k) - \beta_k \zeta_k \\
= \hat{y}_k - \beta_k G(H(y_k), y_k) - \beta_k \zeta_k + \beta_k (G(H(y_k), y_k) - G(x_k, y_k)),
\]
which implies that
\[
\|\hat{y}_{k+1}\|^2 = \|\hat{y}_k - \beta_k G(H(y_k), y_k)\|^2 - 2\beta_k^2 \|G(H(y_k), y_k)\|^2 - 2\beta_k \hat{y}_k^T \zeta_k + 2\beta_k^2 G(H(y_k), y_k)^T \zeta_k \\
+ \|\beta_k (G(H(y_k), y_k) - G(x_k, y_k)) - \beta_k \zeta_k\|^2.
\] (48)

We next analyze each term on the right-hand side of (48). First, using \(G(H(y^*), y^*) = 0\), (11), (6), and (10) we consider
\[
\|\hat{y} - \beta_k G(H(y_k), y_k)\|^2 - 2\beta_k^2 \|G(H(y_k), y_k)\|^2 \leq \|\hat{y}_k\|^2 - 2\beta_k \hat{y}_k^T G(H(y_k), y_k) \\
\leq (11) \|\hat{y}_k\|^2 - 2\mu_G \beta_k \|\hat{y}_k\|^2 = (1 - 2\mu_G \beta_k) \|\hat{y}_k\|^2.
\] (49)

Second, using (10), (17), and \(G(H(y^*), y^*) = 0\) we consider
\[
2\beta_k^2 G(H(y_k), y_k)^T \zeta_k \leq \beta_k^2 \|G(H(y_k), y_k)\|^2 + \beta_k^2 \|\zeta_k\|^2 \\
= \beta_k^2 \|G(H(y_k), y_k) - G(H(y^*), y^*)\|^2 + \beta_k^2 \|\zeta_k\|^2 \\
\leq B^2 \beta_k^2 \|H(y_k) - H(y^*)\| + \|y_k - y^*\|^2 + \beta_k^2 \|\zeta_k\|^2 \\
\leq B^2 (B + 1)^2 \beta_k^2 \|\hat{y}_k\|^2 + \beta_k^2 \|\zeta_k\|^2.
\] (50)
Third, using (10), (16), and the relation $2ab \leq a^2/\eta + \eta b^2$ for all $\eta > 0$, we obtain

$$2\beta_k \hat{y}_k^T (G(H(y_k), y_k) - G(x_k, y_k)) \leq 2B \beta_k ||\hat{y}_k|| ||\hat{x}_k|| \leq \mu_G \beta_k ||\hat{y}_k||^2 + \frac{B^2}{\mu_G} \beta_k ||\hat{x}_k||^2,$$  

(51)

Next, using (10), (17), and $G(H(y^*), y^*) = 0$ we have

$$2\beta_k^2 G(H(y_k), y_k)^T G(x_k, y_k) = 2\beta_k^2 \left( G(H(y_k), y_k) - G(H(y^*), y^*) \right)^T \left( G(x_k, y_k) - G(H(y^*), y^*) \right) 
\leq 2B^2 \beta_k^2 \left( ||H(y_k) - H(y^*)|| + ||y_k - y^*|| \right) ||x_k - H(y^*)|| 
\leq 2B^2 (B + 1) \beta_k ||\hat{y}_k|| \left( ||x_k - H(y_k)|| + ||H(y_k) - H(y^*)|| \right) 
\leq 2B^3 (B + 1) \beta_k ||\hat{y}_k||^2 + 2B^2 (B + 1) \beta_k^2 ||\hat{x}_k|| ||\hat{x}_k|| 
\leq 3B^3 (B + 1) \beta_k ||\hat{y}_k||^2 + B(B + 1) \beta_k^2 ||\hat{x}_k||^2,$$  

(52)

where the last inequality is due to the relation $2ab \leq a^2/\eta + \eta b^2$ for all $\eta > 0$. Finally, using (10) and (16) we have

$$||\beta_k (G(H(y_k), y_k) - G(x_k, y_k)) - \beta_k \zeta_k||^2 \leq 2\beta_k^2 ||G(H(y_k), y_k) - G(x_k, y_k)||^2 + 2\beta_k^2 ||\zeta_k||^2 
\leq 2B^2 \beta_k^2 ||\hat{x}_k||^2 + 2\beta_k^2 ||\zeta_k||^2.$$  

(53)

Taking the expectation of (48) and using (49)–(53) yields

$$\mathbb{E}[||\hat{y}_{k+1}||^2] \leq (1 - 2\mu_G \beta_k) \mathbb{E}[||\hat{y}_k||^2] - 2\beta_k \mathbb{E}[||\hat{y}_k\zeta_k||] + B^2 (B + 1)^2 \beta_k^2 \mathbb{E}[||\hat{y}_k||^2] + \beta_k^2 \mathbb{E}[||\zeta_k||^2] 
+ \mu_G \beta_k \mathbb{E}[||\hat{y}_k||^2] + \frac{B^2}{\mu_G} \beta_k \mathbb{E}[||\hat{x}_k||^2] + 3B^3 (B + 1) \beta_k^2 \mathbb{E}[||\hat{y}_k||^2] + B(B + 1) \beta_k^2 \mathbb{E}[||\hat{x}_k||^2] 
+ 2B^2 \beta_k^2 \mathbb{E}[||\hat{x}_k||^2] + 2\beta_k^2 \mathbb{E}[||\zeta_k||^2] 
\leq (1 - \mu_G \beta_k) \mathbb{E}[||\hat{y}_k||^2] - 2\beta_k \mathbb{E}[||\hat{y}_k\zeta_k||] + 3B^2 \beta_k^2 \mathbb{E}[||\hat{y}_k||^2] 
+ \frac{B^2}{\mu_G} \beta_k \mathbb{E}[||\hat{x}_k||^2] + 3(B + 1)^2 \beta_k^2 \mathbb{E}[||\hat{x}_k||^2] + 4(B + 1)^2 \beta_k^2 \mathbb{E}[||\hat{y}_k||^2].$$  

(54)

By using (34) and (40) we consider

$$-2\beta_k \mathbb{E}[[\hat{y}_k^T \zeta_k] + 3\beta_k^2 \mathbb{E}[||\zeta_k||^2] 
\leq 18(1 + B)^4 \alpha_k \beta_k \mathbb{E}[||\hat{z}_k||^2] + 180B^2 (1 + B)^3 \alpha_k \beta_k \mathbb{E}[||\hat{z}_k||^2] 
+ 156B(1 + B)^4 (||y^*|| + ||H(0)|| + 1)^2 \alpha_k \beta_k \mathbb{E}[||\hat{z}_k||^2] 
+ 24B^2 (1 + B)^2 \beta_k^2 \mathbb{E}[||\hat{z}_k||^2] + 24B^2 (1 + B)^2 (||y^*|| + ||H(0)|| + 1)^2 \beta_k^2 
\leq 18(1 + B)^4 (\alpha_k \beta_k + 10B \alpha_k \beta_k + \beta_k^2) \mathbb{E}[||\hat{z}_k||^2] 
+ 24(1 + B)^4 (||y^*|| + ||H(0)|| + 1)^2 (\beta_k^2 + 7B \alpha_k \beta_k \beta_k).$$

Substituting the preceding relation into (54) yields

$$\mathbb{E}[||\hat{y}_{k+1}||^2] \leq (1 - \mu_G \beta_k) \mathbb{E}[||\hat{y}_k||^2] + 18(1 + B)^4 (\alpha_k \beta_k + 10B \alpha_k \beta_k + \beta_k^2) \mathbb{E}[||\hat{z}_k||^2] 
+ 24(1 + B)^4 (||y^*|| + ||H(0)|| + 1)^2 (\beta_k^2 + 7B \alpha_k \beta_k \beta_k) 
+ \frac{B^2}{\mu_G} \beta_k \mathbb{E}[||\hat{x}_k||^2] + 3(B + 1)^2 \beta_k^2 \mathbb{E}[||\hat{x}_k||^2] + 4(B + 1)^2 \beta_k^2 \mathbb{E}[||\hat{y}_k||^2] 
\leq (1 - \mu_G \beta_k) \mathbb{E}[||\hat{y}_k||^2] + 18(1 + B)^4 (\alpha_k \beta_k + 10B \alpha_k \beta_k + \beta_k^2) \mathbb{E}[||\hat{z}_k||^2] 
+ 24(1 + B)^4 (||y^*|| + ||H(0)|| + 1)^2 (\beta_k^2 + 7B \alpha_k \beta_k \beta_k) + \frac{B^2}{\mu_G} \beta_k \mathbb{E}[||\hat{x}_k||^2],$$

which concludes our proof.
3.3 Proof of Lemma 4

Proof. Adding (24) to (25) and using \( \beta_k \leq \alpha_k \leq \alpha_{k;\tau(\alpha_k)} \) we obtain

\[
\mathbb{E}[\|\hat{z}_{k+1}\|^2] \leq (1 - \mu_F \alpha_k) \mathbb{E}[\|\hat{x}_k\|^2] + 32(1 + B)^6 \left( \frac{5\beta_k^2}{\mu_F \alpha_k} + \beta_k^2 + \alpha_{k;\tau(\alpha_k)} \right) \|\hat{z}_k\|^2 \\
+ 32(1 + B)^6 (\|y^*\| + \|H(0)\| + 1)^2 \left( \frac{5\beta_k^2}{\mu_F \alpha_k} + \beta_k^2 + \alpha_{k;\tau(\alpha_k)} \right) \\
+ (1 - \mu_G \beta_k) \mathbb{E}[\|\hat{y}_k\|^2] + 18(1 + B)^4 (\alpha_k \beta_k + 10B \alpha_{k;\tau(\alpha_k)} \beta_k + 3\beta_k^2) \mathbb{E}[\|\hat{z}_k\|^2] \\
+ 24(1 + B)^4 (\|y^*\| + \|H(0)\| + 1)^2 (\beta_k^2 + 7B \alpha_{k;\tau(\alpha_k)} \beta_k) + \frac{B^2}{\mu_G} \beta_k \mathbb{E}[\|\hat{x}_k\|^2] \\
\leq \mathbb{E}[\|\hat{z}_k\|^2] + 160(1 + B)^6 \left( \frac{\beta_k^2}{\mu_F \alpha_k} + \beta_k^2 + \alpha_{k;\tau(\alpha_k)} \right) \|\hat{z}_k\|^2 \\
+ 160(1 + B)^6 (\|y^*\| + \|H(0)\| + 1)^2 \left( \frac{\beta_k^2}{\mu_F \alpha_k} + \beta_k^2 + \alpha_{k;\tau(\alpha_k)} \right),
\]

where the last inequality we use (29) to have

\[-\mu_F \alpha_k + \frac{B^2}{\mu_G} \beta_k \leq 0.
\]

Let \( w_k \) satisfy \( w_0 = 1 \) and

\[
w_k = \prod_{t=0}^{k} \left( 1 + 160(B + 1)^6 \left( \frac{\beta_t^2}{\mu_F \alpha_t} + \beta_t^2 + \alpha_{t;\tau(\alpha_t)} \right) \right),
\]

(56)

On the one hand, using \( (1 + x) \leq e^x \) for all \( x \geq 0 \) and (26) we have

\[
w_k \leq e^{160(B+1)^6 \sum_{t=0}^{k} \left( \frac{\beta_t^2}{\mu_F \alpha_t} + \beta_t^2 + \alpha_{t;\tau(\alpha_t)} \right)} \leq e^{160D_1 (B+1)^6}.
\]

(57)

On the other hand, using \( 1 + x \geq e^{-x} \) for all \( x \geq 0 \) and (26) we obtain

\[
w_k \geq e^{160(B+1)^6 \sum_{t=0}^{k} \left( \frac{\beta_t^2}{\mu_F \alpha_t} + \beta_t^2 + \alpha_{t;\tau(\alpha_t)} \right)} \leq e^{-160D_1 (B+1)^6}.
\]

(58)

Thus, dividing both sides of (55) by \( w_{k+1} \) and using (58) and (26) give

\[
\frac{\mathbb{E}[\|\hat{z}_{k+1}\|^2]}{w_{k+1}} \leq \frac{\mathbb{E}[\|\hat{z}_k\|^2]}{w_k} + \frac{D_2}{w_{k+1}} \left( \frac{\beta_k^2}{\mu_F \alpha_k} + \beta_k^2 + 2\alpha_{k;\tau(\alpha_k)} \right) \\
\leq \mathbb{E}[\|\hat{z}_0\|^2] + D_2 e^{160D_1 (B+1)^6 \sum_{t=0}^{k} \left( \frac{\beta_t^2}{\mu_F \alpha_t} + \beta_t^2 + 2\alpha_{t;\tau(\alpha_t)} \right)} \\
\leq \mathbb{E}[\|\hat{z}_0\|^2] + D_1 D_2 e^{160D_1 (B+1)^6},
\]

which by using Eq. (58) immediately gives Eq. (27).
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A  Proof of Lemma 5

Proof. We consider

$$\begin{align*}
-\dot{x}_k^T \psi_k &= -\dot{x}_k^T (\psi_k - \psi_{k-\tau(\alpha_k)}) - \dot{x}_k^T \psi_{k-\tau(\alpha_k)} \\
&= -\dot{x}_k^T (\psi_k - \psi_{k-\tau(\alpha_k)}) - (\dot{x}_k - \dot{x}_{k-\tau(\alpha_k)})^T \psi_{k-\tau(\alpha_k)} - \dot{x}_{k-\tau(\alpha_k)}^T \psi_{k-\tau(\alpha_k)}.
\end{align*}$$

(59)

We next analyze each term on the right-hand side of (59). First, using (14) and (21) we have \(\forall k \geq K^*\)

$$\begin{align*}
\mathbb{E}[-\dot{x}_k^T \psi_{k-\tau(\alpha_k)} &\mid Q_{k-\tau(\alpha_k)}] = -\dot{x}_k^T (\psi_k - \psi_{k-\tau(\alpha_k)}) \\
&\leq \|\dot{x}_{k-\tau(\alpha_k)}\| \mathbb{E}[\|\psi_{k-\tau(\alpha_k)}\| \mid Q_{k-\tau(\alpha_k)}]\leq \|\dot{x}_{k-\tau(\alpha_k)}\| \|\psi_{k-\tau(\alpha_k)}\| \\
&\leq \alpha_k \|\hat{z}_k\| + \alpha_k \|\hat{z}_k - \hat{x}_{k-\tau(\alpha_k)}\| \\
&\leq \alpha_k \|\hat{z}_k\|^2 + \alpha_k \|\hat{z}_k - \hat{x}_{k-\tau(\alpha_k)}\|^2 + 2\alpha_k \\
&\leq 9(1 + B)^4 \alpha_k \|\hat{z}_k\|^2 + 10(1 + B)^4 (\|y^*\| + \|H(0)\| + 1)^2 \alpha_k,
\end{align*}$$

(60)

where the last inequality we use (39) and (20) (i.e., \(2B\alpha_{k;\tau(\alpha_k)} \leq \log(2) \leq 1/3\)) to have for all \(k \geq K^*\)

$$\begin{align*}
\|\hat{z}_k - \hat{x}_{k-\tau(\alpha_k)}\|^2 &\leq 288B^2 (1 + B)^4 \alpha_{k;\tau(\alpha_k)}^2 (\|\hat{z}_k\|^2 + (\|y^*\| + \|H(0)\| + 1)^2) \\
&\leq 8(1 + B)^4 (\|\hat{z}_k\|^2 + (\|y^*\| + \|H(0)\| + 1)^2).
\end{align*}$$

Second, using (7), (4), and (16) we consider

$$\begin{align*}
-\dot{x}_k^T (\psi_k - \psi_{k-\tau(\alpha_k)}) &\leq \|\dot{x}_k\| \|\psi_k - \psi_{k-\tau(\alpha_k)}\| \\
&\leq 2B \|\hat{z}_k\| \|\hat{z}_k - \hat{x}_{k-\tau(\alpha_k)}\| \leq 24B^2 \alpha_{k;\tau(\alpha_k)} \|\hat{z}_k\| (\|\hat{z}_k\| + 1) \\
&\leq 24B^2 (1 + B) \alpha_{k;\tau(\alpha_k)} \|\hat{z}_k\|^2 + 24B^2 (1 + B) \alpha_{k;\tau(\alpha_k)} \|\hat{z}_k\| (\|y^*\| + \|H(0)\| + 1) \\
&\leq 36B^2 (1 + B) \alpha_{k;\tau(\alpha_k)} \|\hat{z}_k\|^2 + 12B^2 (1 + B) \alpha_{k;\tau(\alpha_k)} (\|y^*\| + \|H(0)\| + 1)^2,
\end{align*}$$

(61)

where the last inequality we use the Cauchy-Schwarz inequality. Third, by using (36) and (20) we have \(\forall k \geq K^*\)

$$\begin{align*}
\|\hat{z}_{k-\tau(\alpha_k)}\| &\leq \|\hat{z}_k - \hat{x}_{k-\tau(\alpha_k)}\| + \|\hat{z}_k\| \overset{(36)}{\leq} 12B \alpha_{k;\tau(\alpha_k)} (\|\hat{z}_k\| + 1) + \|\hat{z}_k\| \overset{(20)}{\leq} 3 \|\hat{z}_k\| + 2.
\end{align*}$$

Moreover, by using (17) and (21) we have

$$\|\psi_{k-\tau(\alpha_k)}\| = \|F(x_{k-\tau(\alpha_k)}; y_{k-\tau(\alpha_k)}; \xi_{k-\tau(\alpha_k)}) - F(x_{k-\tau(\alpha_k)}; y_{k-\tau(\alpha_k)})\| \leq 2B (\|\hat{z}_{k-\tau(\alpha_k)}\| + 1)$$

Using the preceding two relations, and (38), we have

$$\begin{align*}
- (\dot{x}_k - \dot{x}_{k-\tau(\alpha_k)})^T \psi_{k-\tau(\alpha_k)} &\leq \|\dot{x}_k - \dot{x}_{k-\tau(\alpha_k)}\| \|\psi_{k-\tau(\alpha_k)}\| \\
&\overset{(17)}{\leq} 2B \|\dot{x}_k - \dot{x}_{k-\tau(\alpha_k)}\| (\|\hat{z}_{k-\tau(\alpha_k)}\| + 1) \leq 6B \|\dot{x}_k - \dot{x}_{k-\tau(\alpha_k)}\| (\|\hat{z}_k\| + 1) \\
&\overset{(67)}{\leq} 6B (1 + B) \|\dot{x}_k - \dot{x}_{k-\tau(\alpha_k)}\| (\|\hat{z}_k\| + \|y^*\| + \|H(0)\| + 1) \\
&\overset{(38)}{\leq} 72B^2 (1 + B)^3 \alpha_{k;\tau(\alpha_k)} (\|\hat{z}_k\| + \|y^*\| + \|H(0)\| + 1)^2 \\
&\leq 144B^2 (1 + B)^3 \alpha_{k;\tau(\alpha_k)} (\|y^*\| + \|H(0)\| + 1)^2.
\end{align*}$$

(62)
Thus, taking the expectation on both sides of (59) and using (60)–(62) we obtain (33), i.e.,

\[
\mathbb{E}[-\hat{z}_k^T \psi_k] = 9(1 + B)^4 \alpha_k \mathbb{E}[||\hat{z}_k||^2] + 10(1 + B)^4 (||y^*|| + ||H(0)|| + 1)^2 \alpha_k \\
+ 36B^2(1 + B)\alpha_k;\tau(\alpha_k)\mathbb{E}[||\hat{z}_k||^2] + 12B^2(1 + B)\alpha_k;\tau(\alpha_k)(||y^*|| + ||H(0)|| + 1)^2 \\
+ 144B^2(1 + B)^3 \alpha_k;\tau(\alpha_k)\mathbb{E}[||\hat{z}_k||^2] + 144B^2(1 + B)^3 \alpha_k;\tau(\alpha_k)(||y^*|| + ||H(0)|| + 1)^2 \\
\leq 9(1 + B)^4 \alpha_k \mathbb{E}[||\hat{z}_k||^2] + 180B^2(1 + B)^3 \alpha_k;\tau(\alpha_k)\mathbb{E}[||\hat{z}_k||^2] \\
+ 156B(1 + B)^4 \alpha_k;\tau(\alpha_k)(||y^*|| + ||H(0)|| + 1)^2,
\]

where in the last inequality we use \( \alpha_k \leq \alpha_k;\tau(\alpha_k) \).

\( \square \)

### B Proof of Lemma 6

**Proof.** We consider

\[
-\hat{y}_k^T \zeta_k = -\hat{y}_k^T (\zeta_k - \zeta_k;\tau(\alpha_k)) - \hat{y}_k^T \zeta_k;\tau(\alpha_k) \\
= -\hat{y}_k^T (\zeta_k - \zeta_k;\tau(\alpha_k)) - (\hat{y}_k - \hat{y}_k;\tau(\alpha_k))^T \zeta_k;\tau(\alpha_k) - \hat{y}_k;\tau(\alpha_k) \zeta_k;\tau(\alpha_k).
\]

We next analyze each term on the right-hand side of (59). First, using (13) and (38) we have \( \forall k \geq K^* \)

\[
\mathbb{E}[-\hat{y}_k^T \zeta_k;\tau(\alpha_k) | Q_k;\tau(\alpha_k)] = -\hat{y}_k^T \zeta_k;\tau(\alpha_k) \mathbb{E}[\zeta_k;\tau(\alpha_k) | Q_k;\tau(\alpha_k)] \\
\leq ||\hat{y}_k;\tau(\alpha_k)|| ||\mathbb{E}[\zeta_k;\tau(\alpha_k) | Q_k;\tau(\alpha_k)]|| \leq \alpha_k ||\hat{y}_k;\tau(\alpha_k)|| \\
\leq \alpha_k ||\hat{z}_k|| + \alpha_k ||\hat{z}_k - \hat{z}_k;\tau(\alpha_k)|| \leq \alpha_k ||\hat{z}_k||^2 + \alpha_k ||\hat{z}_k - \hat{z}_k;\tau(\alpha_k)||^2 + 2\alpha_k \\
\leq 9(1 + B)^4 \alpha_k ||\hat{z}_k||^2 + 10(1 + B)^4 (||y^*|| + ||H(0)|| + 1)^2 \alpha_k,
\]

(64)

where the last inequality we use (39) and (20) (i.e., \( 2B\alpha_k;\tau(\alpha_k) \leq \log(2) \leq 1/3 \)) to have for all \( k \geq K^* \)

\[
||\hat{z}_k - \hat{z}_k;\tau(\alpha_k)||^2 \leq 288B^2(1 + B)^4 \alpha_k;\tau(\alpha_k)(||\hat{z}_k||^2 + (||y^*|| + ||H(0)|| + 1)^2) \\
\leq 8(1 + B)^4 (||\hat{z}_k||^2 + (||y^*|| + ||H(0)|| + 1)^2).
\]

Second, similar to (61) we obtain

\[
-\hat{y}_k^T (\zeta_k - \zeta_k;\tau(\alpha_k)) \leq ||\hat{y}_k|| ||\zeta_k - \zeta_k;\tau(\alpha_k)|| \leq 2B ||\hat{y}_k|| ||\hat{z}_k - \hat{z}_k;\tau(\alpha_k)|| \leq 2B ||\hat{z}_k|| ||\hat{z}_k - \hat{z}_k;\tau(\alpha_k)|| \\
\leq 36B^2(1 + B)\alpha_k;\tau(\alpha_k) ||\hat{z}_k||^2 + 12B^2(1 + B)\alpha_k;\tau(\alpha_k)(||y^*|| + ||H(0)|| + 1)^2.
\]

(65)

Third, using the same line of analysis as in (62) we have

\[
-(\hat{y}_k - \hat{y}_k;\tau(\alpha_k))^T \zeta_k;\tau(\alpha_k) \leq ||\hat{y}_k - \hat{y}_k;\tau(\alpha_k)|| ||\zeta_k;\tau(\alpha_k)|| \leq 2B ||\hat{y}_k - \hat{y}_k;\tau(\alpha_k)|| (||z_k - \tau(\alpha_k)|| + 1) \\
\leq 144B^2(1 + B)^3 \alpha_k;\tau(\alpha_k) ||\hat{z}_k||^2 + 144B^2(1 + B)^3 \alpha_k;\tau(\alpha_k)(||y^*|| + ||H(0)|| + 1)^2.
\]

(66)

Thus, taking the expectation on both sides of (63) and using (64)–(66) we obtain (34).

\( \square \)

### C Proof of Lemma 7

**Proof.** By (2) and since \( \beta_k \leq \alpha_k \) we have for all \( k \geq K^* \)

\[
||\hat{z}_{k+1}|| \leq ||\hat{z}_k|| + \alpha_k \left( ||F(x_k, y_k; \xi_k)|| + \frac{\beta_k}{\alpha_k} ||G(x_k, y_k; \xi_k)|| \right) \leq (1 + 2B\alpha_k)||\hat{z}_k|| + 2B\alpha_k,
\]

(17)
which by using the relation $1 + x \leq \exp(x)$ for all $x \geq 0$ yields for all $k \geq K^*$ and $t \in [k - \tau(\alpha_k), k]$

\[
\|z_{t+1}\| \leq \prod_{\ell=k-\tau(\alpha_k)}^{t} (1 + 2B\alpha_\ell)\|z_{k-\tau(\alpha_k)}\| + 2B \sum_{\ell=k-\tau(\alpha_k)}^{t} \alpha_\ell \prod_{u=\ell+1}^{t} (1 + 2B\alpha_u)
\]

\[
\leq \|z_{k-\tau(\alpha_k)}\| \exp (2B \sum_{\ell=k-\tau(\alpha_k)}^{t} \alpha_\ell) + 2B \sum_{\ell=k-\tau(\alpha_k)}^{t} \alpha_\ell \exp (2B \sum_{u=\ell+1}^{t} \alpha_u)
\]

\[
\leq 2\|z_{k-\tau(\alpha_k)}\| + 4B\alpha_{t;\tau(\alpha_k)},
\]

where the last inequality is due to

\[
\sum_{t=k-\tau(\alpha_k)}^{k} \alpha_t \leq \tau(\alpha_k)\alpha_{k-\tau(\alpha_k)} \leq \frac{\log(2)}{2B}, \quad \forall k \geq K^*.
\]

Using the relation above we obtain (35), i.e.,

\[
\|z_k - z_{k-\tau(\alpha_k)}\| \leq \sum_{t=k-\tau(\alpha_k)}^{k-1} \|z_{t+1} - z_t\| \leq \sum_{t=k-\tau(\alpha_k)}^{k-1} 2B\alpha_t(\|z_t\| + 1)
\]

\[
\leq \sum_{t=k-\tau(\alpha_k)}^{k-1} 2B\alpha_t(2\|z_{k-\tau(\alpha_k)}\| + 4B\alpha_{t;\tau(\alpha_k)} + 1)
\]

\[
\leq 4B\alpha_{k;\tau(\alpha_k)}\|z_{k-\tau(\alpha_k)}\| + 4B\alpha_{k;\tau(\alpha_k)};
\]

where in the last inequality we use $4B\alpha_{k;\tau(\alpha_k)} \leq 2\log(2) \leq 1$ for all $k \geq K^*$. Finally, using the triangle inequality the preceding relation yields

\[
\|z_k - z_{k-\tau(\alpha_k)}\| \leq 4B\alpha_{k;\tau(\alpha_k)}(\|z_k - z_{k-\tau(\alpha_k)}\| + \|z_k\|) + 4B\alpha_{k;\tau(\alpha_k)};
\]

which by rearranging both sides and using $4B\alpha_{k;\tau(\alpha_k)} \leq 2\log(2) \leq 2/3$ gives (36). \qed

**D Proof of Lemma 8**

**Proof.** By (18) and (6) we have

\[
\|\hat{x}_k\| = \|x_k - H(y_k)\| \geq \|x_k\| - L\|y_k\| - \|H(0)\|,
\]

\[
\|\hat{y}_k\| = \|y_k - y^*\| \geq \|y_k\| - \|y^*\|,
\]

which by (16) implies that

\[
\|y_k\| \leq \|\hat{y}_k\| + \|y^*\|,
\]

\[
\|x_k\| \leq \|\hat{x}_k\| + B\|\hat{y}_k\| + \|H(0)\| + L\|y^*\|.
\]

Thus, using $z = [x, y]^T$ and $\hat{z} = [\hat{x}, \hat{y}]^T$ we obtain $\forall k \geq 0$

\[
\|z_k\| \leq (1 + B)\|\hat{z}_k\| + (1 + B)(\|y^*\| + \|H(0)\|).
\]

(67)
We now use (16), (18) and (35) to obtain (37), i.e., \( \forall k \geq K^* \)

\[
\| \hat{z}_k - \hat{z}_{k-\tau(\alpha_k)} \| = \| \hat{x}_k - \hat{x}_{k-\tau(\alpha_k)} \| + \| \hat{y}_k - \hat{y}_{k-\tau(\alpha_k)} \|
\]
\[
= \| x_k - x_{k-\tau(\alpha_k)} - H(y_k) + H(y_{k-\tau(\alpha_k)}) \| + \| y_k - y_{k-\tau(\alpha_k)} \|
\]
\[
\leq \| x_k - x_{k-\tau(\alpha_k)} \| + (B + 1) \| y_k - y_{k-\tau(\alpha_k)} \| \leq (1 + B) \| z_k - z_{k-\tau(\alpha_k)} \|
\]  
\[ (68) \]
\[
\leq 4B(1 + B)\alpha_{k;\tau(\alpha_k)}(\| z_k - z_{k-\tau(\alpha_k)} \| + 1)
\]
\[
\leq 4B(1 + B)^2\alpha_{k;\tau(\alpha_k)}(\| \hat{z}_k - \hat{z}_{k-\tau(\alpha_k)} \| + (\| y^* \| + \| H(0) \| + 1)).
\]

Next, by (68) and (36) we achieve (38), i.e., \( \forall k \geq K^* \)

\[
\| \hat{z}_k - \hat{z}_{k-\tau(\alpha_k)} \| \leq 12B(1 + L)\alpha_{k;\tau(\alpha_k)}(\| \hat{z}_k \| + 1)
\]
\[
\leq 12B(1 + B)^2\alpha_{k;\tau(\alpha_k)}\| \hat{z}_k \| + 12B(1 + B)^2\alpha_{k;\tau(\alpha_k)}(\| y^* \| + \| H(0) \| + 1).
\]

Finally, using the relation \( (a + b)^2 \leq 2a^2 + 2b^2 \) and the preceding equation immediately gives (39). \( \square \)

D.1 Proof of Lemma 9

Proof. By (16), (21), (17), and (67) we have

\[
\| \psi_k \| \leq 2B(\| z_k \| + 1) \leq 2B(1 + B)\| \hat{z}_k \| + 2B(1 + B)(\| y^* \| + \| H(0) \| + 1).
\]

The proof of \( \| \zeta_k \| \) can be shown in a similar step. \( \square \)