Recurrence time in the quantum dynamics of the 1D Bose gas

Eriko Kaminishi, Jun Sato†, and Tetsuo Deguchi†
Department of Physics, Graduate School of Science, the University of Tokyo,
7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan
† Department of Physics, Graduate School of Humanities and Sciences,
Ochanomizu University, 2-1-1 Ohtsuka, Bunkyo-ku, Tokyo 112-8610, Japan
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Recurrence time is successfully evaluated for some initially localized quantum states in the onedimensional (1D) Bose gas with repulsive short range interactions. We suggest that the recurrence time is given typically by several minutes or seconds in experiments of cold atoms trapped in one dimension. It is much shorter than the estimated recurrence time of a generic quantum many-body system, which is usually as long as the age of the universe. We show numerically how the recurrence time depends on the interaction strength. In the free-bosonic and the free-fermionic regimes we evaluate the recurrence time rigorously and show that it is proportional to the square of the number of particles. For instance, the result is exact in the impenetrable 1D Bose gas.

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INTRODUCTION

Recurrence is one of the central concepts not only in classical mechanics but also in quantum statistical mechanics [1]. Due to recent experiments of cold atomic systems confined in one dimension [2], it has become quite attractive to study recurrence phenomena in isolated quantum systems theoretically. The cold atomic experiments have created a huge motivation for studying fundamental aspects of quantum statistical mechanics: Equilibration and relaxation of isolated quantum many-body systems [3] and ergodic theorems [4] in quantum statistical mechanics from the viewpoint of typicality [9–12]. Furthermore, the dynamics of isolated quantum many-body systems in one dimension is extensively studied by both experiments and theories [4,6,13–17].

Oscillating dynamical behavior was observed in experiments of cold atomic gases in one dimension [14]. It is considered that the system in the experiment is close to the integrable system of the one-dimensional (1D) interacting bosons with delta-function potentials, which we call the 1D Bose gas. The finding triggered further theoretical studies on the dynamics of isolated quantum many-body systems. It seems, however, that studies on quantum recurrent phenomena are rather rare. Here we remark that the approach to equilibrium in quantum many-body systems has been studied in a spin model with long-range interaction [18,20]. Through quantum analogues of Poincare’s recurrence theorem, it is argued that quantum statistical systems are almost periodic [1,21,22]. It is also demonstrated that quantum systems with time-periodic Hamiltonians are almost periodic [23]. However, the recurrence time of a generic quantum system is usually very long. It takes an extremely long period of time, which may be as long as the age of the universe, for an isolated quantum system with incommensurable energy levels to have a recurrent phenomenon [25]. Thus, although the time evolution of any given isolated quantum system is recurrent, it is very rare to observe a recurrent phenomenon in it practically, unless we choose the system or the initial state quite properly.

In this Letter, we present concrete examples of recurrence for some quantum states in the 1D Bose gas, and we evaluate the recurrence time for them. Let us call the squared amplitude between an initial state and the time-evolved state the fidelity. We observe that periodic patterns appear in the fidelity as a function of time for several different values of the interaction strength. We thus evaluate the recurrence time by the shortest period of time among the observed periodic patterns. Here, the system size is small and the number of particles $N$ is given by $N = 8, 20,$ and 30, typically. As an initial state we employ the quantum state given by the superposition of all the type-II excitations [26]. We remark that the quantum state has the same density profile as a dark soliton [27]. Furthermore, in the cases of infinite and zero interaction strength, i.e., in the free-fermionic and free-bosonic regimes, respectively, we can rigorously calculate the recurrence time for various quantum states. For instance, we derive the recurrence time for the quantum state associated with a dark soliton which is constructed from the type-II excitations. Moreover, we suggest that the recurrence phenomena in the Letter can be observed in experiments of cold atomic systems. We show that the recurrence time in the 1D Bose gas is given typically by the order of minutes or seconds in cold atomic gases trapped in one dimension [1]. It is much shorter than the age of the universe.

It is nontrivial to evaluate recurrence time in an interacting quantum system. We recall that a recurrent phenomenon occurs very rarely in a measurable period of time, although the time evolution of an isolated quantum system is recurrent for any number $N$ of particles. Fortunately at some values of interaction strength, periodic structures appear in the time evolution of the fidelity
in the 1D Bose gas, and we can determine the recurrence time by the shortest period of time of the observed periodic patterns. We remark that quantum collapse and revival in a one-atom maser were experimentally demonstrated, which had been studied in the Jaynes-Cummings model \[28\]. However, it has not been observed that recurrence of a quantum state occurs in a quantum many-body system.

For the free-fermionic and free-bosonic regimes we derive the recurrence time rigorously in the 1D Bose gas and show that it is proportional to the square of the number of particles, \(N^2\). For some quantum systems it is shown that relaxation times diverge exponentially with respect to \(N\) \[18, 20\]. Since recurrence times are much longer than relaxation times, we expect that the recurrence time also increases exponentially with respect to \(N\) for a generic quantum many-body system.

The time evolution of “a quantum dark-soliton state” of the 1D Bose gas was recently obtained numerically and exactly by the Bethe ansatz method \[17, 27\]. We suggest that the quantum state corresponds to the quantum state of a dark soliton realized in the experiments of cold atomic systems trapped in one dimension. Here the periodic boundary conditions of the system size \(L\) are assumed on the wavefunctions. Hereafter, we consider the repulsive interaction: \(c > 0\). We define interaction parameter \(\gamma\) by \(\gamma := c/u\), where \(u = N/L\) is the particle density. The bulk and static properties of the LL model are characterized by parameter \(\gamma\). We employ the system of units with \(2m = \hbar = 1\), where \(m\) is the particle mass. The unit of time in our numerical calculation is proportional to \(L^{-2}\).

In the LL model, the Bethe ansatz offers an exact eigenstate with an exact energy eigenvalue for a given set of quasi-momenta \(k_1, k_2, \ldots, k_N\) satisfying the Bethe equations for \(j = 1, 2, \ldots, N\).

\[
k_jL = 2\pi I_j - 2 \sum_{\ell \neq j}^{N} \arctan \left( \frac{k_j - k_\ell}{c} \right).
\]

Here \(I_j\)’s are integers for odd \(N\) and half-odd integers for even \(N\). We call them the Bethe quantum numbers. The total momentum \(P\) and energy eigenvalue \(E\) are written in terms of the quasi-momenta as \(P = \sum_{j=1}^{N} k_j = \frac{2\pi}{L} \sum_{j=1}^{N} I_j\), \(E = \sum_{j=1}^{N} k_j^2\).

Superposing Lieb’s type II excitations \[20\], i.e. one-hole excitations, we construct a quantum state with an initially localized density profile, which coincides with the amplitude profile of a dark-soliton solution of the Gross-Pitaevskii equation \[27\]. We remark that translational symmetry is broken in the quantum state. In the type II branch, for each integer \(p\) in the set \(\{0, 1, \ldots, N - 1\}\), we consider momentum \(P = 2\pi p/L\) and denote by \(|P, N\rangle\) the normalized Bethe eigenstate of \(N\) particles with total momentum \(P\). The Bethe quantum numbers of \(|P, N\rangle\) are given by \(I_j = -(N + 1)/2 + j\) for integers \(j\) with \(1 \leq j \leq N - p\) and \(I_j = -(N + 1)/2 + j + 1\) for \(j\) with \(N - p + 1 \leq j \leq N\). For each integer \(q\) satisfying \(0 \leq q \leq N - 1\) we define the coordinate state \(|X\rangle\) of \(X = qL/N\) by the discrete Fourier transformation:

\[
|X\rangle := \frac{1}{\sqrt{N}} \sum_{p=0}^{N-1} \exp(-2\pi i pq/N) |P, N\rangle. \tag{3}
\]

**TIME EVOLUTION OF FIDELITY**

**Definition of fidelity**

Let us define symbol \(|X(t)\rangle\) by \(|X(t)\rangle := \exp(-i\mathcal{H}t)|X\rangle\). Taking advantage of quantum integrability, we numerically obtain all the energy eigenvalues \(E_p\) of \(|P\rangle\)’s in the type II branch and follow the time evolution for quite a long time. We define the fidelity by the overlap between the initial state \(|X(0)\rangle\) and the time-evolved state \(|X(t)\rangle\) at time \(t\) as

\[
F(t) := |\langle X(t)|X(0)\rangle|^2 = \frac{1}{N^2} \sum_{p=0}^{N-1} |\exp(-iE_p t)|^2. \tag{4}
\]

Here, \(E_p\) is the energy of the one-hole excited state with momentum \(P = 2\pi p/L\).

**Periodicity of the fidelity**

We now show periodic patterns in the graph of fidelity as a function of time. In Fig. 1 periodic patterns in the time evolution of the fidelity are shown in the weak coupling case \((\gamma = 1.0 \times 10^{-4})\), upper panel) and in the strong coupling case \((\gamma = 100,\) lower panel).

Let us consider the case of intermediate values of interaction parameter \(\gamma\), which is not very small or very large. If interaction parameter \(\gamma\) is very large (very small), the recurrence time coincides with that of the infinite coupling case (the zero coupling case). When interaction parameter \(\gamma\) increases from zero to a finite nonzero value
which is not very small, the recurrence time enhances abruptly at some value of \( \gamma \). In Table I it increases when interaction parameter increases from \( \gamma = 0.001 \) to \( 0.01 \) for \( N = 5, 8, 20 \).

We observe numerically that the recurrence time tends to decrease as interaction parameter \( \gamma \) increases when \( \gamma \) is small such as \( \gamma < 0.1 \) in Table I. Then, for some values of \( \gamma \) such as \( \gamma = 10 \), we do not determine recurrence time because any periodic structure does not appear.

At some intermediate values of \( \gamma \) such as \( \gamma = 0.05, 0.1, 1, \) and \( 10 \) for \( N = 15 \), there is no periodic structure in the fidelity as a function of time as shown in Fig. 3. For the values of \( \gamma \), the fidelity does not return to or become close to 1.0, so that we do not determine the recurrence time.

TABLE I: Recurrence time of the quantum state associated with a dark soliton, \( |X(t)\rangle \). Here \( \gamma \geq 0.1 \) and \( N/L = 1 \).

| \( N \) | \( \gamma \) | 0.1 | 1 | 10 | 100 | 200 | 300 | 1000 |
|---|---|---|---|---|---|---|---|---|
| 5 | 3.97787 787.82 | 397.88 | 262.627 | 198.955 | 163.104 |
| 8 | 10.1818 1268.17 | 646.793 | 432.936 | 331.041 | 269.89 |
| 20 | 63.5964 3406.82 | N/A | N/A | N/A | N/A |

For \( \gamma > 10 \), the recurrence time increases as interaction parameter \( \gamma \) increases when \( \gamma \) is large such as \( \gamma = 100 \). In Figs. 2 and 3 it is illustrated how the fidelity depends on interaction parameter \( \gamma \) for \( N = 8 \) and \( N = 15 \). We confirm in Figs. 2 and 3 that the time evolution of the fidelity becomes more complex as the number of particles increases from \( N = 8 \) to \( N = 15 \).

In the intermediate regimes of interaction parameter \( \gamma \), where \( \gamma \) is large but not very large, the fidelity does not return to 1.0. However, we numerically evaluate the recurrence time by taking advantage of the fact that the fidelity has periodic patterns as a function of time at some values of interaction parameter \( \gamma \). For example, in the lower panel of Fig. 1, for the first recurrence at \( t = 1559.59 \), the value of the fidelity is given by 0.995, where interaction parameter is given by \( \gamma = 100 \) and the number of particles \( N = 20 \).

At some large value of interaction parameter \( \gamma \) the recurrence time suddenly decreases to the value of the infinite interaction parameter case (i.e., \( \gamma = \infty \)). In Table II it increases when interaction parameter increases from \( \gamma = 300 \) to 1000 for \( N = 5, 8, 20 \).

Weak or strong coupling cases

The strong coupling \((\gamma \to \infty)\) and the weak coupling \((\gamma \to 0)\) cases correspond to the free-fermionic and the free-bosonic regimes, respectively. We first numerically calculated the recurrence time, and then arrived at rigorous derivation. We calculated the recurrence time for small numbers of particles such as \( N = 5 \sim 30 \) for the free-bosonic regime, and \( N = 5 \sim 150 \) for the free-fermionic regime.

The recurrence time has even-odd dependence with respect to the number of particles \( N \). We can prove that recurrence time in the free-fermionic regime with odd \( N \) is equal to that in the free-bosonic regime, while recurrence time in the free-fermionic regime with even \( N \) is half of that with odd \( N \). We confirm the even-odd dependence of the recurrence times in Fig. 4. We plot the fidelity as a function of time in Fig. 3 with particle density \( N/L = 1 \) in the strong coupling case of \( \gamma = 10^{10} \) and in the weak coupling case of \( \gamma = 1.0 \times 10^{-9} \). We observe that the recurrence time in the free-fermionic regime with even \( N \) is half of the recurrence time in the free-bosonic regime.
FIG. 2: The dependence of fidelity $F(t) = |\langle X(t)|X(0) \rangle|^2$ on interaction parameter $\gamma$. Here, the number of particles is given by $N = 8$ and the particle density $N/L = 1$.

Rigorous derivation of recurrence time

We evaluate rigorously the recurrence time in the free-fermionic and the free-bosonic regimes, where $\gamma = \infty$ and $\gamma = 0$, respectively. Let us express the difference between a one-hole excitation energy $E_p$ and the ground state energy $E_g$ in terms of $e_p$ as follows.

$$E_p - E_g = \left(\frac{2\pi}{L}\right)^2 e_p. \quad (5)$$

We remark that $e_p$ is given by an integer. We have $e_p = \{(N + 1)/2\}^2 - \{(N + 1)/2 - p\}^2$ in the free-fermionic regime, and $e_p = p$ in the free-bosonic regime. Here we remark that in the free-fermionic regime we have a particle at $I_N = (N + 1)/2$ and a hole at $I_{N-p} = (N + 1)/2 - p$; in the free bosonic regime we have a particle at $I = p$ and $N - 1$ particles at $I = 0$.

Putting Eq.(5) into Eq.(4) we express the fidelity at recurrence time $T$ as follows.

$$F(T) = \frac{1}{N^2} \left| \sum_{p=0}^{N-1} \exp \{i \left(\frac{2\pi}{L}\right)^2 e_p T\} \right|^2. \quad (6)$$

From the condition: $F(T) = 1$ we obtain the recurrence time

$$T = \frac{L^2}{2\pi G}. \quad (7)$$

Here, $G$ is the greatest common divisor among the integers in the set $\{e_1, e_2, ..., e_{N-1}\}$. The expression (7) of recurrence time is exact. When $N/L = 1$ we have $T = N^2/(2\pi G)$.

For various other quantum states, the recurrence time is given by the same formula (7). For instance, it is also valid for the state given by the sum over all two-hole excitations from the ground state.

When the fidelity returns to 1 at time $t = T$, the state returns to the initial one except for a relative phase factor, and all the physical quantities take the same values as in the initial state.
We suggest that the quantum state, \( |X(t)\rangle \), which is given by the sum over all one-hole excitations from the ground state, can be realized in experiments of cold atomic systems. There are several experimental methods such as the phase imprinting method to construct a dark soliton in cold atomic systems [5]. We expect that by the method we can realize the state \( |X(t)\rangle \) in cold atomic gases trapped in one dimension.

**CONCLUSIONS**

In conclusion, we have observed that periodic patterns appear in the fidelity as a function of time in the quantum many-body system of the 1D Bose gas for the state \( |X(t)\rangle \) which is given by the sum over all one-hole excitations. We suggest that it is associated with dark solitons. We have evaluated the recurrence time through the periodic structures in the time evolution of the fidelity for small numbers of particles such as \( N = 5, 8, 20 \). The recurrence time thus depends on the number of particles \( N \) and interaction parameter \( \gamma \). It is not a bulk and static quantity. Furthermore, in the free-bosonic and the free-fermionic regimes we can calculate the recurrence time rigorously for some states, and have shown that the recurrence time of the state \( |X(t)\rangle \) is proportional to the square of the number of particles, \( N^2 \). Moreover, we suggest that the recurrence time of the quantum state can be observed in experiments. We suggest that the recurrence time of the 1D Bose gas is given by the order of minutes or seconds in cold atomic experiments confined in one dimension. Finally, we suggest that the recurrent phenomena in the Letter show the diversity in the dynamics of isolated quantum systems.

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