Generalized Euler, Smoluchowski and Schrödinger equations admitting self-similar solutions with a Tsallis invariant profile

Pierre-Henri Chavanis

1Laboratoire de Physique Théorique, Université de Toulouse, CNRS, UPS, France

The damped isothermal Euler equations, the Smoluchowski equation and the damped logarithmic Schrödinger equation with a harmonic potential admit stationary and self-similar solutions with a Gaussian profile. They satisfy an $H$-theorem for a free energy functional involving the von Weizsäcker functional and the Boltzmann functional. We derive generalized forms of these equations in order to obtain stationary and self-similar solutions with a Tsallis profile. In particular, we introduce a nonlinear Schrödinger equation involving a generalized kinetic term characterized by an index $q$ and a power-law nonlinearity characterized by an index $\gamma$. We derive an $H$-theorem satisfied by a generalized free energy functional involving a generalized von Weizsäcker functional (associated with $q$) and a Tsallis functional (associated with $\gamma$). This leads to a notion of generalized quantum mechanics and generalized thermodynamics. When $q = 2\gamma - 1$, our nonlinear Schrödinger equation admits an exact self-similar solution with a Tsallis invariant profile. Standard quantum mechanics (Schrödinger) and standard thermodynamics (Boltzmann) are recovered for $q = \gamma = 1$.

I. INTRODUCTION

In the standard theory of Brownian motion initiated by Einstein [1], the probabilistic evolution of a free particle (or an ensemble of noninteracting particles) in the overdamped limit is governed by the ordinary diffusion equation. When the particle is submitted to an external potential, one gets the Smoluchowski equation. These are both particular cases of the Fokker-Planck equation. These equations are associated with standard thermodynamics in the canonical ensemble based on the Boltzmann entropy and on the isothermal equation of state $P = \frac{1}{2}k_BT/m$. They satisfy an $H$-theorem associated with the Boltzmann free energy and relax towards the Boltzmann distribution at statistical equilibrium (when an external potential is present to counteract the effect of diffusion). The equilibrium state minimizes the Boltzmann free energy at fixed mass. On the other hand, the diffusion equation and the Smoluchowski equation with a harmonic potential admit a self-similar solution with a Gaussian invariant profile [1, 2, 7, 8].

In the last decades, several authors [9–54] have tried to generalize standard thermodynamics and Brownian theory (see, e.g., [42, 51, 53, 54] for reviews). For example, one can encounter situations in which the diffusion coefficient of the particles depends on their density as a power-law. In the absence of an external potential one gets the anomalous diffusion equation which was first introduced in the context of porous media [15]. When an external potential is present one gets the polytropic Smoluchowski equation [18]. These are both particular cases of nonlinear Fokker-Planck equations [42, 51, 53, 54]. These equations are associated with a notion of generalized thermodynamics in the canonical ensemble based on the Tsallis entropy [14] and on the polytropic equation of state $P = K\rho^{\gamma}$. They satisfy an $H$-theorem associated with the Tsallis free energy and relax towards the Tsallis distribution at statistical equilibrium (when an external potential is present to counteract the effect of diffusion). The equilibrium state minimizes the Tsallis free energy at fixed mass.$^1$ On the other hand, it has been shown that the anomalous diffusion equation and the polytropic Smoluchowski equation with a harmonic potential admit a self-similar solution with a Tsallis invariant profile [15, 18, 19, 47].

In this paper, we first show that these properties remain valid for the Euler and damped Euler equations with an isothermal or a polytropic equation of state. The Smoluchowski equation is recovered in the strong friction limit ($\xi \rightarrow +\infty$). These equations admit self-similar solutions with a Gaussian or a Tsallis invariant profile. Interestingly, the differential equation determining the evolution of the system’s size bears some analogies with the Friedmann equation in cosmology determining the evolution of the radius (scale factor) of the Universe. We then generalize these results to a quantum mechanics context related to generalized damped Gross-Pitaevskii (GP) equations [55]. We first consider the damped logarithmic GP equation. Using the Madelung transformation [56], we show that this

---

$^1$ Initially, nonlinear Fokker-Planck equations were introduced in relation to the Fermi-Dirac [17, 23], Bose-Einstein [17] and Tsallis [18, 19] entropies. This gave the impression that these entropies were special and that the generalized thermodynamical formalism was valid only for them. However, it was shown later by [22, 24, 26–29, 31, 32, 44, 49–52, 54] that a generalized thermodynamical formalism could be developed for an arbitrary form of entropy and for an arbitrary barotropic equation of state. In that case, the diffusion coefficient that appears in the corresponding Fokker-Planck equation is a nonlinear function of the density determined by the form of entropy or by the equation of state (see Refs. [51, 54]).
equation is equivalent to the quantum damped isothermal Euler equations. These equations admit a self-similar solution with a Gaussian invariant profile. This is because the standard quantum potential present in the quantum Euler equations has a structure compatible with the linear equation of state giving rise to the Gaussian solution (they “marry well”). We then consider the damped power-law GP equation. Using the Madelung transformation, we show that this equation is equivalent to the quantum damped polytropic Euler equations. In the Thomas-Fermi (TF) limit, where the quantum potential can be neglected, they reduce to the damped polytropic Euler equations and therefore admit a self-similar solution with a Tsallis invariant profile. However, when the standard quantum potential is taken into account, this property is lost because the structure of the standard quantum potential present in the quantum Euler equations is not compatible with the polytropic equation of state giving rise to the Tsallis solution (they do not “marry well”). We search and find a generalized form of quantum potential that allows us to restore the Tsallis self-similar solution. This leads to a generalized Schrödinger equation associated with a notion of generalized quantum (or wave) mechanics. More precisely, we introduce a generalized damped power-law GP equation involving a generalized kinetic term characterized by an index $q$ and a power-law nonlinearity characterized by an index $\gamma$. This equation is associated with a notion of generalized quantum mechanics (through the index $q$) and generalized Tsallis thermodynamics (through the index $\gamma$). Indeed, an $H$-theorem is satisfied by a generalized free energy functional involving a generalized von Weizsäcker functional (associated with $q$) and a Tsallis functional (associated with $\gamma$). The equilibrium state of this equation minimizes the generalized free energy functional under the normalization condition. Finally, this equation admits a self-similar solution with a Tsallis invariant profile provided that the condition $q = 2\gamma - 1$ is fulfilled. Standard quantum mechanics (Schrödinger) and standard thermodynamics (Boltzmann) are recovered for $q = \gamma = 1$.

In the main part of the paper, we explain our approach, present the basic equations under study, and exhibit their self-similar solutions. General results of our formalism and technical details are given in the Appendices. A detailed study of the self-similar solutions obtained in our paper is postponed to a forthcoming contribution.

II. SELF-SIMILAR SOLUTION OF THE EULER EQUATIONS

A. Euler equations

Let us consider the Euler equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$

(1)

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P - \nabla \Phi_{\text{ext}}$$

(2)

with a barotropic equation of state $P = P(\rho)$ and an external potential $\Phi_{\text{ext}}(\mathbf{r})$ in a space of dimension $d$. Eq. (1) is the equation of continuity which expresses the local conservation of mass. Eq. (2) is the momentum equation. In the following, we specifically consider a harmonic (quadratic) potential

$$\Phi_{\text{ext}} = \frac{1}{2} \omega_0^2 r^2$$

(3)

and a polytropic equation of state of the form

$$P = K \rho^\gamma,$$

(4)

where $\gamma$ is the polytropic index and $K$ is the polytropic constant (sometimes called the polytropic temperature $^4$). This includes, as a particular case, the isothermal equation of state

$$P = \frac{k_B T}{m}$$

(5)

\begin{enumerate}
\item In the absence of friction ($\xi = 0$) we get the quantum isothermal Euler equations and in the strong friction limit ($\xi \rightarrow +\infty$) we get the quantum Smoluchowski equation. This provides an interesting formal connection between quantum mechanics and Brownian theory.
\item Generalized Schrödinger equations have also been proposed recently by Nobre et al. However, their approach is substantially different from ours (see the Conclusion).
\item A physical justification of this terminology is given in Appendix A12.
\end{enumerate}
corresponding to \( \gamma = 1 \) and \( K = k_B T/m \). In most applications, we will assume that \( \omega_0^2 \), \( K \) and \( T \) are positive constants \((\geq 0)\).

Remark: We may also consider the logotropic equation of state \([47]\):

\[
P = A \ln \rho.
\]

Because of the identities \( A \nabla \ln \rho = (A/\rho) \nabla \rho \) and \( K \nabla \rho^\gamma = K \rho^{\gamma-1} \nabla \rho \), the logotropic equation of state can be viewed as a limit of the polytropic equation of state \( P = K \rho^\gamma \) when \( \gamma \to 0 \), \( K \to +\infty \) with \( K \gamma = A \ [47, 55] \).

### B. Self-similar solution

The Euler equations \([1-6]\) admit a self-similar solution of the form (see Appendix \([5]\))

\[
\rho(r, t) = \frac{M}{R(t)^d} f \left( \frac{r - \chi(t) r_0}{R(t)} \right), \quad u(r, t) = H(t) r + B(t) r_0
\]

with

\[
H = \frac{\dot{R}}{R}, \quad B = \dot{\chi} - H \chi,
\]

where \( R(t) \) is the typical size (radius) of the system and \( f(x) \) with \( x = [r - \chi(t) r_0]/R(t) \) is the invariant density profile. The density profile is spherically symmetric and contains all the mass \( \int \rho(r, t) \, dr = M \) so that \( \int f(x) \, dx = 1 \). The quantity \( \langle r \rangle(t) = \chi(t) r_0 \) represents the position of the center of the distribution (see Appendix \([59]\)). We take \( \chi(0) = 1 \) so that \( r_0 \) represents the initial position of the center of the distribution. The velocity field \( u(r, t) \) is an affine function of \( r \) with time-dependent factors \( H(t) \) and \( B(t) \). When \( B = 0 \), the velocity field is proportional to \( r \) with a proportionality factor \( H(t) = \dot{R}/R \). We note the formal analogy with the Hubble constant in cosmology, where \( R \) plays the role of the scale factor. The invariant density profile is given by the Tsallis distribution of index \( \gamma \), namely

\[
f(x) = \frac{1}{Z} \left[ 1 - (\gamma - 1) x^2 \right]^{1/(\gamma - 1)},
\]

where \([x]_+ = x \) if \( x \geq 0 \) and \([x]_+ = 0 \) if \( x \leq 0 \). When \( \gamma > 1 \), the density profile has a compact support, vanishing at \( x_{\text{max}} = 1/\sqrt{\gamma - 1} \). When \( \gamma < 1 \), the density extends to infinity and decreases at large distances as \( f(x) \sim x^{-2/(1-\gamma)} \). When \( \gamma \to 1 \), \( f(x) \) tends towards the Gaussian distribution (see below). The normalization constant \( Z \) is given by

\[
Z = \frac{\pi^{d/2} \Gamma \left( \frac{\gamma}{\gamma - 1} \right)}{(\gamma - 1)^{d/2} \Gamma \left( \frac{d}{\gamma - 1} \right)} \quad (\gamma \geq 1),
\]

\[
Z = \frac{\pi^{d/2} \Gamma \left( \frac{1}{1 - \gamma} - \frac{d}{2} \right)}{(1 - \gamma)^{d/2} \Gamma \left( \frac{1}{1 - \gamma} \right)} \quad \left( \frac{d - 2}{d} < \gamma \leq 1 \right).
\]

The distribution is normalizable provided that \( \gamma > (d - 2)/d \). For the logotropic equation of state \([6]\) in \( d = 1 \), the invariant profile \( f(x) \) is the Lorentzian corresponding to Eq. \([6]\) with \( \gamma = 0 \) \([47]\). The differential equation determining the evolution of \( \chi(t) \) is given by Eq. \([43, 59]\) and the differential equation determining the evolution of the radius \( \dot{R}(t) \) is given by

\[
\dot{R} + \omega_0^2 R = 2 \zeta^{-\gamma} K \gamma \frac{M^{\gamma - 1}}{R^{d \gamma - d + 1}}.
\]

For the logotropic equation of state \([6]\) in \( d = 1 \), the term on the right hand side of Eq. \([12]\) is a constant. In the particular case of an isothermal (linear) equation of state, the invariant profile is the Gaussian

\[
f(x) = \frac{1}{\pi^{d/2}} e^{-x^2}
\]

\[\text{(13)}\]
and the differential equation (12) determining the evolution of the radius reduces to

\[ \ddot{R} + \omega_0^2 R = \frac{2k_B T}{mR}. \]  

(14)

Eqs. (13) and (14) are limiting cases of Eqs. (9)-(12) for \( \gamma \to 1 \). In most applications, we will assume that \( R = 0 \) at \( t = 0 \) so that \( \rho(r,0) = M\delta(r-r_0) \).

Historical note: As far as we know, these self-similar solutions have not been given previously in the context of the Euler equations.

C. Steady state

A steady state of the Euler equations (1) and (2) satisfies the condition of hydrostatic equilibrium

\[ \nabla P + \rho \nabla \Phi_{\text{ext}} = 0. \]  

(15)

The time-independent distribution (7), corresponding to \( \chi(t) = 0, R(t) = R_e = \text{cst} \) and \( u = 0 \), when it exists, is a steady state of the Euler equations (1) and (2) with a harmonic potential. For the polytropic equation of state (4), assuming \( K > 0, \omega_0^2 > 0 \) and \( \gamma \geq \max\{0, (d-2)/d\} \), we obtain the Tsallis distribution

\[ \rho(r) = \frac{M}{R_e^d} \left[ 1 - (\gamma - 1)r^2/R_e^2 \right]^{\frac{1}{\gamma - 1}}, \]  

(16)

where

\[ R_e = \left( \frac{2K\gamma M^\gamma - 1}{Z^{\gamma - 1}\omega_0^2} \right)^{\frac{1}{2\gamma - 2(d-1)}} \]  

(17)

is the stationary solution of Eq. (12). For the isothermal equation of state (5), assuming \( T > 0 \) and \( \omega_0^2 > 0 \), we obtain the Boltzmann distribution

\[ \rho(r) = \frac{M}{\pi^{d/2} R_e^d} e^{-r^2/R_e^2}, \]  

(18)

where

\[ R_e = \left( \frac{2k_B T}{m\omega_0^2} \right)^{1/2} \]  

(19)

is the stationary solution of Eq. (14). By directly solving the differential equation (15) corresponding to the condition of hydrostatic equilibrium one can show that all the steady states of the Euler equations (1) and (2) are of that form (see Appendix A 8). The dynamical stability of a steady state of the Euler equations (1) and (2) can be determined by studying the dynamical stability of the steady state of the differential equations (12) and (14) [57]. It can be shown that the steady states defined by Eqs. (16)-(19) are stable.

III. SELF-SIMILAR SOLUTION OF THE DAMPED EULER EQUATIONS AND OF THE GENERALIZED SMOLUCHOWSKI EQUATION

A. Damped Euler equations and generalized Smoluchowski equation

In the recent years, generalized thermodynamics and nonlinear Fokker-Planck (NFP) equations have played an important role in physics (see the reviews [42, 51, 53, 54] and references therein). NFP equations can be introduced in the following heuristic manner which generalizes the arguments given by Einstein [1] in his seminal paper on Brownian theory. We start from the Euler equations (1) and (2) and introduce a linear friction force \( -\xi u \) in the momentum equation (2). This leads to the damped Euler equations

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0, \]  

(20)
\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P - \nabla \Phi_{\text{ext}} - \xi \mathbf{u}.
\]  
(21)

These hydrodynamic equations describe the motion of a fluid of particles (like Brownian particles) experiencing a friction with another fluid. In his theory of Brownian motion, Einstein considers an isothermal equation of state, the so-called osmotic pressure \( P = \rho k_B T/m \), in agreement with standard (Boltzmann) thermodynamics. Let us be more general and consider an arbitrary barotropic equation of state of the form \( P = P(\rho) \). In the strong friction limit \( \xi \to +\infty \), we can neglect the inertia of the particles, i.e., the left hand side of Eq. (21). In that case, the current of particles is given by

\[
\rho \mathbf{u} = -\frac{1}{\xi} (\nabla P + \rho \nabla \Phi_{\text{ext}}).
\]

Combining this relation with the continuity equation (20) we obtain the generalized Smoluchowski equation

\[
\xi \frac{\partial \rho}{\partial t} = \nabla \cdot (\nabla P + \rho \nabla \Phi_{\text{ext}}).
\]

For the isothermal equation of state (5), we recover the standard Smoluchowski (2) equation

\[
\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left( \frac{k_B T}{m} \nabla \rho + \rho \nabla \Phi_{\text{ext}} \right)
\]

which is a particular Fokker-Planck equation (3–5). For the polytropic equation of state (4), we obtain the polytropic Smoluchowski equation

\[
\xi \frac{\partial \rho}{\partial t} = \nabla \cdot (K \nabla \rho^\gamma + \rho \nabla \Phi_{\text{ext}}).
\]



### B. Generalized thermodynamics

Equations (20)–(26) are associated with a generalized thermodynamic formalism. In particular, they satisfy an \( H \)-theorem for the generalized free energy (see Refs. [51,52] and Appendix A)

\[
F = \int \rho \frac{u^2}{2} \, d\mathbf{r} + \int V(\rho) \, d\mathbf{r} + \int \rho \Phi_{\text{ext}} \, d\mathbf{r},
\]

where the potential \( V(\rho) \) is determined by the equation of state \( P(\rho) \) through the relation

\[
V(\rho) = \rho \int_0^\rho \frac{P(\rho')}{\rho'^2} \, d\rho'.
\]

5 From the generalized Kramers equation one can derive a hierarchy of hydrodynamic equations for the moments of the velocity distribution, but this hierarchy is not closed. It can be closed by making a local thermodynamic equilibrium (LTE) approximation leading to the damped Euler equations (20) and (21). However, it can be shown [51] that this LTE approximation is not rigorously justified except in the strong friction limit \( \xi \to +\infty \) in which the damped Euler equations (20) and (21) reduce to the generalized Smoluchowski equation (29).

6 We note that \( u(\rho) = V(\rho)/\rho \) represents the density of internal energy (\( U = \int V(\rho) \, d\mathbf{r} = \int \rho u \, d\mathbf{r} \)). It satisfies the first principle of thermodynamics \( du = -Pd(1/\rho) \), or equivalently, \( u'(\rho) = P(\rho)/\rho^2 \). The density of enthalpy is defined by \( h(\rho) = u(\rho) + P(\rho)/\rho \) (Euler relation) and it satisfies the Gibbs-Duhem relation \( dh = dP/\rho \), or equivalently, \( h'(\rho) = P'(\rho)/\rho \). We note that \( V'(\rho) = h(\rho) \). Defining \( C(\rho) = V(\rho)/T_{\text{eff}} \), where \( T_{\text{eff}} \) is an effective temperature, we can write the free energy (26) as \( F = E_* - T_{\text{eff}} S \) where \( S = -\int C(\rho) \, d\mathbf{r} \) is a generalized entropy and \( E_* = \Theta_* + W_{\text{ext}} \) is the sum of the macroscopic kinetic energy and the external energy (see Appendix A12). It can be shown [62] that this expression of the free energy arises naturally from ordinary thermodynamics in the canonical ensemble.
Inversely, the equation of state \( P(\rho) \) is related to the potential \( V(\rho) \) by

\[
P(\rho) = \rho V'(\rho) - V(\rho) = \rho^2 \left( \frac{V(\rho)}{\rho} \right)'.
\] (28)

An extremum of free energy at fixed mass determines a steady state of the (damped) Euler equations (20) and (21) satisfying the condition of hydrostatic equilibrium (15). A minimum of free energy is stable while a maximum or a saddle point is unstable. When \( \xi > 0 \) we can use the H-theorem (\( \dot{F} \leq 0 \)) to prove that the system relaxes towards a stable steady state (when it exists) for \( t \to +\infty \). The typical relaxation time is \( t_{\text{relax}} \sim \xi^{-1} \). When \( \xi = 0 \) the free energy is conserved (\( \dot{F} = 0 \)).

For the isothermal equation of state (5) the free energy can be written as \( F = E_* - TS_B \) where \( S_B \) is the Boltzmann entropy (A85). For the polytropic equation of state (4) the generalized free energy can be written as \( F = E_* - KS_\gamma \) where \( S_\gamma \) is the Tsallis entropy (A88) of index \( \gamma \). For the logotropic equation of state (6) the generalized free energy can be written as \( F = E_* - AS_L \) where \( S_L \) is the logarithmic entropy (A91). The extremization of \( F \) at fixed mass \( M \) leads to the Boltzmann and Tsallis distributions (18) and (16) respectively (see Refs. [51, 55] and Appendix A).

C. Isothermal case

The damped Euler equations (20) and (21) with the isothermal equation of state (5) and the harmonic potential (3) admit a self-similar solution of the form (7) with the Gaussian invariant profile (13) (see Appendix B). The differential equation determining the evolution of \( \chi(t) \) is given by Eq. (B39) and the differential equation determining the evolution of the radius \( R(t) \) is given by

\[
\ddot{R} + \xi \dot{R} + \omega_0^2 R = \frac{2k_B T}{mR}.
\] (29)

In the strong friction limit \( \xi \to +\infty \), corresponding to the Smoluchowski equation (24), the differential equation determining the evolution of \( \chi(t) \) is given by Eq. (B44) and the differential equation determining the evolution of the radius \( R(t) \) is given by

\[
\xi \dot{R} + \omega_0^2 R = \frac{2k_B T}{mR}.
\] (30)

When \( T > 0 \) and \( \omega_0^2 > 0 \), the steady states of the damped Euler equations and of the Smoluchowski equation are the same as those discussed in Sec. II C. They are given by Eqs. (18) and (19) and they are stable. When \( \xi > 0 \), the system relaxes towards these equilibrium states.

1. Without external potential

In the absence of external potential, Eq. (24) reduces to the standard diffusion equation

\[
\frac{\partial \rho}{\partial t} = D \Delta \rho,
\] (31)

where the diffusion coefficient is given by the Einstein [1] relation

\[
D = \frac{k_B T}{\xi m}.
\] (32)

The standard diffusion equation has a Gaussian self-similar solution. The differential equation (B44) determining the evolution of \( \chi(t) \) reduces to Eq. (B10). Its solution is given by Eq. (B47). The differential equation (30) determining the evolution of the radius \( R(t) \) reduces to

\[
\dot{R} = \frac{2D}{R}.
\] (33)

Its solution is given by

\[
R(t) = (4Dt)^{1/2}.
\] (34)
Combining Eqs. (7), (13), (34) and (B47) we get
\[
\rho(r, t) = \frac{M}{(4\pi Dt)^{d/2}} e^{-\frac{(r-r_0)^2}{4Dt}}.
\] (35)

Using \(\langle r^2 \rangle = (d/2)R^2 + r_0^2\) obtained from Eqs. (B47), (B51) and (B52), and using Eq. (34), we find that
\[
\langle r^2 \rangle(t) = 2dDt + r_0^2.
\] (36)

This result can be obtained immediately from the virial theorem (A74). We also have \(\langle r(t) \rangle = r_0\) from Eqs. (B47) and (B50).

**Historical note:** The diffusion equation (31) was derived by Einstein [1] starting from a Markovian equation and expanding this equation in powers of the increment \(\Delta r\) in the position of the Brownian particle. He gave the solution of this equation [see Eq. (35)] and, from it, obtained the formula (36) giving the temporal evolution of the arithmetic mean of the squares of displacements of the Brownian particles. He also derived the Einstein relation (32) between the diffusion coefficient, the friction coefficient, and the temperature. Actually, the same relation was previously obtained by Sutherland [63] using essentially the same arguments. A similar relation was also derived by Smoluchowski [65]. On the other hand, Eq. (36) was derived by Langevin [66] from the Langevin equation.

2. With a harmonic potential

The Smoluchowski equation (24) with the harmonic external potential (3) can be written as
\[
\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left( k_B T \frac{m}{2} \nabla \rho + \rho \omega_0^2 r \right).
\] (37)

It has a Gaussian self-similar solution. The solution of the differential equation (B44) determining the evolution of \(\chi(t)\) is given by Eq. (B45). The solution of the differential equation (30) determining the evolution of the radius \(R(t)\) is given by
\[
R(t) = \left( \frac{2k_B T}{m \omega_0^2} \right)^{1/2} \left( 1 - e^{-2\omega_0^2 t/\xi} \right)^{1/2}.
\] (38)

For \(t \to +\infty\), we recover the equilibrium radius (19). Combining Eqs. (7), (13), (33) and (B45) we get
\[
\rho(r, t) = M \left[ \frac{m \omega_0^2}{2\pi k_B T (1 - e^{-2\omega_0^2 t/\xi})} \right]^{d/2} e^{-\frac{m \omega_0^2}{2k_B T} \left( r - e^{-\omega_0^2 t/\xi} r_0 \right)^2}.
\] (39)

For \(t \to +\infty\), this equation tends towards the stationary solution
\[
\rho(r) = M \left( \frac{m \omega_0^2}{2\pi k_B T} \right)^{d/2} e^{-\frac{m \omega_0^2 r^2}{2k_B T}.}
\] (40)

equivalent to Eq. (18). It corresponds to the Boltzmann distribution of statistical equilibrium for an ensemble of harmonic oscillators (comparing the steady state of the Smoluchowski equation with the Boltzmann distribution is the direct manner to establish the Einstein relation (32) [8]). Using Eq. (38) and \(\langle r^2 \rangle = (d/2)R^2 + \exp(-2\omega_0^2 t/\xi) r_0^2\) obtained from Eqs. (B45), (B51) and (B52), we find that
\[
\langle r^2 \rangle(t) = \frac{dk_B T}{m \omega_0^2} + \left( r_0^2 - \frac{dk_B T}{m \omega_0^2} \right) e^{-2\omega_0^2 t/\xi}.
\] (41)

7 This is the original method that led, later, to the Fokker-Planck equation [3–5] when one takes into account an additional deterministic drift term, like the effect of an external potential, in the evolution equation of the particle [8].

8 For that reason, Eq. (32) was originally called the Sutherland-Einstein relation (see Ref. [64], P. 569).
This result can be obtained immediately from the virial theorem \[11\]. We also have \((\mathbf{r}(t)) = e^{-\omega_0^2 t/\xi} \mathbf{r}_0\) from Eqs. \[14-15\] and \[15-20\]. For \(t \to +\infty\), we get \((\mathbf{r}^2)_\infty = d_k B T/m \omega_0^2\) and \((\mathbf{r})_\infty = 0\).

\*Historical note:* The Smoluchowski equation \[24\] was introduced by Smoluchowski in 1915 \[2\]. It can be viewed as a generalization of the diffusion equation considered by Einstein \[1\] when an external potential is acting on a Brownian particle \[9\]. This is also a particular form of the Fokker-Planck equation that was introduced later by Fokker \[3\] and Planck \[5\]. The solutions \[30\] and \[41\] were first obtained by Smoluchowski \[2, 71\]. The solution \[30\] was rederived later by Uhlenbeck and Ornstein \[7\] by different methods: (i) from the Langevin equation, (ii) from the Smoluchowski equation using Lord Rayleigh’s \[72\] method, and (iii) from the Smoluchowski equation using a trick due to Kramers (see Appendix II of \[7\]). They also derived Eq. \[41\] from the Langevin equation.\[10\]

3. Analogies with Lord Rayleigh’s kinetic theory

It is usually considered that the theory of Brownian motion started with Einstein’s seminal paper \[1\]. Actually, more than a decade before, Lord Rayleigh \[72\] studied the dynamics of massive particles bombarded by numerous small projectiles. Although he did not explicitly refer to Brownian motion, his paper may be considered as the first theory of Brownian motion with the important difference that Lord Rayleigh \[72\] considered the velocity distribution \(f(v, t)\) of homogeneously distributed particles while Einstein \[1\] considered the spatial density \(\rho(r, t)\) of Brownian particles in the strong friction limit \(\xi \to +\infty\) (or, equivalently, for large times \(t \gg \xi^{-1}\)). In his paper, Lord Rayleigh \[72\] derived a partial differential equation for the temporal evolution of the velocity distribution function \(f(v, t)\) of the particles which, in modern notations, may be written as

\[\frac{\partial f}{\partial t} = \xi \frac{\partial}{\partial v} \left( \frac{k_B T}{m} \frac{\partial f}{\partial v} + f v \right). \tag{42}\]

This equation includes a diffusion term in velocity space and a friction term. This is a particular form of the general Fokker-Planck equation \[4, 5\]. An equation related to Eq. \[42\] but including an advection term in phase space taking into account the presence of an external potential was introduced later by Klein \[74\], Kramers \[75\] and Chandrasekhar \[8\], and is usually called the Kramers equation. The Rayleigh equation \[42\] is formally equivalent to the Smoluchowski equation \[37\] with the harmonic external potential \[3\]. In this analogy, the velocity \(v\) plays the role of the position \(r\) and the linear friction the role of the linear harmonic force. In the absence of friction (or for sufficiently short times), Eq. \[42\] reduces to the diffusion equation in velocity space

\[\frac{\partial f}{\partial t} = D_v \frac{\partial^2 f}{\partial v^2}, \tag{43}\]

with a diffusion coefficient given by

\[D_v = \frac{\xi k_B T}{m}. \tag{44}\]

This can be viewed as the appropriate form of Einstein relation in the present context \[8\]. Lord Rayleigh \[72\] obtained the solution of Eq. \[43\], given by

\[f(v, t) = \frac{M}{V(4\pi D_v t)^{d/2}} e^{-\frac{(v-v_0)^2}{4 D_v t}}, \tag{45}\]

which is formally equivalent to Einstein’s solution \[65\]. He referred to it as Fourier’s solution. He gave the result \(\langle v^2 \rangle = 2D_v t + v_0^2\) which is formally equivalent to the Einstein result \[36\]. He also obtained the solution of Eq. \[12\], given by

\[f(v, t) = \frac{M}{V} \left[ \frac{m}{2\pi k_B T (1-e^{-2\xi t})} \right]^{d/2} e^{-\frac{m(v-v_0)^2}{2k_B T (1-e^{-2\xi t})}}, \tag{46}\]

---

\[9\] Actually, drift-diffusion equations similar to the Smoluchowski equation, and coupled to the Poisson equation by a mean field potential, were previously introduced by Nernst \[67, 68\] and Planck \[69\] in the context of electrolytes (see also Debye and Hückel \[71\]):

\[10\] Surprisingly, Langevin \[66\] is not quoted by Uhlenbeck and Ornstein \[7\] suggesting that his work was not well-known at that time (see in this respect the comment in the Introduction of \[73\]).
which is formally equivalent to the Smoluchowski (or Uhlenbeck-Ornstein) solution \([39]\). He pointed out that Eq. (46) relaxes towards the stationary solution

\[
f(v) = \frac{M}{V} \left( \frac{m}{2\pi k_B T} \right)^{d/2} e^{-\frac{mv^2}{2k_B T}},
\]

(47)
corresponding to Maxwell’s distribution in velocity space (comparing the steady state of the Rayleigh equation with the Maxwell-Boltzmann distribution is the direct manner to establish the Einstein relation \([44] \[8\]). This is the analogue of Eq. (40). He also pointed out the fact that the friction is necessary to avoid the divergence of the kinetic energy of the particles since \(\langle v^2 \rangle \sim 2dD_\ast t \to +\infty\) when the friction term is ignored. A similar argument was invoked later by Chandrasekhar \([76]\) in his famous paper on the \textit{Dynamical Friction}.\(^{11}\) When the friction is taken into account, one finds that

\[
\langle v^2 \rangle(t) = \frac{dk_B T}{m} + \left( v_0^2 - \frac{dk_B T}{m} \right) e^{-2\xi t}.
\]

(48)

which is the analogue of Eq. (41). We also have \(\langle v(t) \rangle = e^{-\xi t} v_0\). For \(t \to +\infty\), we get \(\langle v^2 \rangle_\infty = \frac{dk_B T}{m}\) and \(\langle v \rangle_\infty = 0\).

\section*{D. Polytropic case}

The damped Euler equations (20) and (21) with the polytropic equation of state \([41]\) and the harmonic potential \([43]\) admit a self-similar solution of the form \([8]\) with the Tsallis invariant profile (9) (see Appendix B). The differential equation determining the evolution of \(\chi(t)\) is given by Eq. (B39) and the differential equation determining the evolution of the radius \(R(t)\) is given by

\[
\ddot{R} + \xi \dot{R} + \omega_0^2 R = 2Z^{1-\gamma} K \frac{\gamma M^{\gamma-1}}{R^{d\gamma - d+1}}.
\]

(49)

In the strong friction limit \(\xi \to +\infty\), corresponding to the polytropic Smoluchowski equation (25), the differential equation determining the evolution of \(\chi(t)\) is given by Eq. (B44) and the differential equation determining the evolution of the radius \(R(t)\) is given by

\[
\xi \dot{R} + \omega_0^2 R = 2Z^{1-\gamma} K \frac{\gamma M^{\gamma-1}}{R^{d\gamma - d+1}}.
\]

(50)

When \(K > 0\), \(\omega_0^2 > 0\) and \(\gamma > \max\{0, (d - 2)/d\}\), the steady states of the damped Euler equations and of the polytropic Smoluchowski equation are the same as those discussed in Sec. II C. They are given by Eqs. (16) and (17) and they are stable. When \(\xi > 0\), the system relaxes towards these equilibrium states.

1. \textit{Without external potential}

In the absence of external potential, Eq. (25) reduces to the anomalous diffusion equation

\[
\frac{\partial \rho}{\partial t} = \frac{K}{\xi} \Delta \rho^\gamma
\]

(51)

originally introduced in the physics of porous media \([15]\). It has a Tsallis self-similar solution. The differential equation \([B41]\) determining the evolution of \(\chi(t)\) reduces to Eq. \([B46]\). Its solution is given by Eq. \([B47]\). The differential equation \([50]\) determining the evolution of the radius \(R(t)\) reduces to

\[
\xi \dot{R} = 2Z^{1-\gamma} K \frac{\gamma M^{\gamma-1}}{R^{d\gamma - d+1}}.
\]

(52)

\(^{11}\) Surprisingly, the paper of Lord Rayleigh \([72]\) is not mentioned at that occasion.
When $\gamma > \max\{0, (d - 2)/d\}$, the solution of Eq. (52) is given by

$$R = \left[2(d\gamma - d + 2)Z^{1-\gamma}K\gamma M^{\gamma-1}\right]^{1/(d\gamma-d+2)}.$$  \hfill (53)

When $\gamma > d(d+2)$ the variance $\langle r^2 \rangle$ is given by Eq. (B51) with Eqs. (B44), (B52), and (53). We note that the exponent in Eq. (53) is always positive since $\gamma > (d - 2)/d$. The dynamics is superdiffusive when $\gamma < 1$ and subdiffusive when $\gamma > 1$.

When $(d - 2)/d < \gamma < 0$ (this supposes $d < 2$), the solution of Eq. (52) is given by

$$R(t) = \left[2(d\gamma - d + 2)Z^{1-\gamma}K\gamma M^{\gamma-1}\right]^{1/(d\gamma-d+2)}.$$ \hfill (54)

We note that the radius $R(t)$ vanishes in a finite time $t_{\text{coll}}$ obtained by setting $t = 0$ in Eq. (54). Correspondingly, the density becomes infinite at $t_{\text{coll}}$. This corresponds to a finite time collapse. Note that the variance $\langle r^2 \rangle$ is not defined when $\gamma < 0$ (see Appendix B7).

Historical note: The similar solution of the anomalous diffusion equation (51), given by Eqs. (7), (11), (53), and (B47) was discovered by Barenblatt [15] in the context of porous media. It was later realized that the invariant profile is a Tsallis distribution (see Refs. [42, 53] and the discussion in Sec. VI.A. of [48]).

2. With a harmonic potential

The polytropic Smoluchowski equation [25] with the harmonic external potential [5] can be written as

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left(K \nabla \rho^\gamma + \rho \omega_0 \mathbf{r}\right).$$ \hfill (55)

It has a Tsallis self-similar solution. The solution of the differential equation (134) determining the evolution of $\chi(t)$ is given by Eq. (B45). When $\gamma > \max\{0, (d - 2)/d\}$, the solution of the differential equation (50) determining the evolution of the radius $R(t)$ is given by

$$R = \left(\frac{2Z^{1-\gamma}K\gamma M^{\gamma-1}}{\omega_0^2}\right)^{1/(d\gamma-d+2)} \left[1 - e^{-(d\gamma-d+2)\omega_0^2 t/\xi}\right]^{1/(d\gamma-d+2)}.$$ \hfill (56)

When $\gamma > d(d+2)$ the variance $\langle r^2 \rangle$ is given by Eq. (B51) with Eqs. (B44), (B52), and (53). For $t \to +\infty$, we recover the equilibrium state from Eqs. (7) and (17). When $(d - 2)/d < \gamma < 0$ (this supposes $d < 2$), the solution of Eq. (53) is given by

$$R(t) = \left(\frac{2Z^{1-\gamma}K\gamma M^{\gamma-1}}{\omega_0^2}\right)^{1/(d\gamma-d+2)} \left[e^{(d\gamma-d+2)\omega_0^2(t_{\text{coll}}-t)/\xi} - 1\right]^{1/(d\gamma-d+2)}.$$ \hfill (57)

We note that the radius $R(t)$ vanishes in a finite time $t_{\text{coll}}$ obtained by setting $t = 0$ in Eq. (57). Correspondingly, the density becomes infinite at $t_{\text{coll}}$. This corresponds to a finite time collapse. Note that the variance $\langle r^2 \rangle$ is not defined when $\gamma < 0$ (see Appendix B7).

Historical note: The Tsallis self-similar solution of the polytropic Smoluchowski equation with a harmonic potential was discovered by Plastino and Plastino [18]. However, they did not explicitly solve the differential equation for $R$. Tsallis and Bukman [19] solved the differential equation for $R$ in $d = 1$. The general solution is provided by Eqs. (50) and (B47) above. The Lorentzian self-similar solution of the logotropic Smoluchowski equation (corresponding to a polytrope $\gamma = 0$) has been obtained by Chavanis and Sire [47].

IV. SELF-SIMILAR SOLUTION OF THE GENERALIZED DAMPED GROSS-PITAEVSKI EQUATION AND OF THE GENERALIZED QUANTUM DAMPED EULER EQUATIONS

A. Generalized damped Gross-Pitaevskii equation and quantum damped Euler equations

We consider the generalized damped GP equation introduced in [55]:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \Phi_{\text{ext}} \psi + m \frac{dV}{d|\psi|^2} \psi - i\frac{\hbar}{2} \xi\left[\ln \left(\frac{\psi}{\psi^*}\right) - \left\langle \ln \left(\frac{\psi}{\psi^*}\right)\right\rangle\right] \psi,$$ \hfill (58)
where \( V(|\psi|^2) \) is the self-interaction potential of the bosons and \( \xi \) is the friction coefficient. This equation describes the evolution of the wavefunction \( \psi(\mathbf{r},t) \) of a dissipative BEC. For \( \xi = 0 \) and \( V(|\psi|^2) = (2\pi a_s \hbar^2/m^3)|\psi|^4 \), where \( a_s \) is the scattering length of the bosons, we recover the GP equation \( (58, 81) \) with a cubic nonlinearity. Performing the Madelung \( (56) \) transformation, we find (see \( (55) \) and Appendix A) that Eq. \( (58) \) is equivalent to the quantum damped Euler equations\(^{12}\)

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{59}
\]

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho} \nabla P - \nabla \Phi_{\text{ext}} - \frac{1}{m} \nabla Q - \xi \mathbf{u}, \tag{60}
\]

where

\[
Q = -\frac{\hbar^2}{2m} \Delta \sqrt{\rho} = -\frac{\hbar^2}{4m} \left[ \frac{\Delta \rho}{\rho} - \frac{1}{2} \frac{\nabla \rho}{\rho^2} \right] \tag{61}
\]

is the quantum potential taking into account the Heisenberg uncertainty principle. The barotropic equation of state \( P(\rho) \) is determined by the nonlinearity \( V(|\psi|^2) \) present in the GP equation \( (58) \) according to Eq. \( (28) \). Several examples of self-interaction potentials \( V(|\psi|^2) \), and the corresponding equations of state \( P(\rho) \), are given in \( (55) \). The specific cases of self-interaction potentials leading to isothermal and polytropic equations of state are discussed below.

In the absence of friction (\( \xi = 0 \)), Eqs. \( (59, 60) \) return the quantum Euler equations. In the strong friction limit (\( \xi \rightarrow +\infty \)) they return the generalized quantum Smoluchowski equation \( (62) \). Therefore, the generalized damped GP equation \( (58) \) provides an interesting formal connection between quantum mechanics and Brownian theory (see \( (55) \) and Appendix A) since it is equivalent to a nonlinear Schrödinger equation when \( \xi = 0 \) \( (82) \) and to a nonlinear Fokker-Planck equation when \( \xi \rightarrow +\infty \) \( (42, 51, 54, 53) \).

Equations \( (58, 61) \) are associated with a generalized thermodynamic formalism. In particular, they satisfy an \( H \)-theorem (see \( (55) \) and Appendix A) for the generalized free energy \( F \) obtained by adding to the free energy defined by Eq. \( (20) \) the von Weizsäcker \( (83) \) functional

\[
\Theta_Q = \frac{\hbar^2}{2m^2} \int (\nabla \sqrt{\rho})^2 \, dr = -\frac{\hbar^2}{2m^2} \int \sqrt{\rho} \Delta \sqrt{\rho} \, dr = \int \rho \frac{Q}{m} \, dr = \frac{\hbar^2}{8m^2} \int \frac{(\nabla \rho)^2}{\rho} \, dr. \tag{62}
\]

As already mentioned, for the isothermal equation of state \( (5) \) the generalized free energy is associated with the Boltzmann entropy and for the polytropic equation of state \( (4) \) the generalized free energy is associated with the Tsallis entropy (see \( (52) \) and Appendix A).

### B. Isothermal case

The damped logarithmic GP equation\(^{13}\)

\[
i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \Phi_{\text{ext}} \psi + 2k_B T \ln |\psi| \psi - i \hbar \frac{\xi}{2} \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \left\langle \ln \left( \frac{\psi}{\psi^*} \right) \right\rangle \right] \psi, \tag{63}
\]

corresponding to a self-interaction potential

\[
V(|\psi|^2) = \frac{k_B T}{m} |\psi|^2 \left( \ln |\psi|^2 - 1 \right), \tag{64}
\]

is equivalent to the quantum damped Euler equations \( (59) \) and \( (60) \) with the isothermal equation of state \( (5) \). For the harmonic external potential \( (4) \), these equations admit a Gaussian self-similar solution formed by Eqs. \( (7) \) and \( (1) \).

\(^{12}\) Since a BEC can be considered as a quantum fluid, these hydrodynamic equations have a truly physical nature. In particular, being equivalent to the (generalized) damped GP equation \( (58) \), there is no viscosity in the (generalized) quantum damped Euler equations \( (59) \) and \( (60) \). As a result, they describe a superfluid.

\(^{13}\) See Ref. \( (84) \) for a derivation of this equation from the theory of scale relativity.
The differential equation determining the evolution of \( \chi(t) \) is given by Eq. (13) and the differential equation determining the evolution of the radius \( R(t) \) is given by

\[ \dot{R} + \xi \dot{R} + \omega_0^2 R = \frac{2k_B T}{mR} + \frac{h^2}{m^2 R^3}. \] (65)

For \( \xi = T = 0 \), it has the general analytical solution

\[ R^2(t) = \frac{1}{\omega_0^2} \left[ a \cos(2\omega_0 t) + b \sin(2\omega_0 t) + E \right] \quad \text{with} \quad a^2 + b^2 = E^2 - \left( \frac{h \omega_0}{m} \right)^2, \] (66)

corresponding to a quantum harmonic oscillator (the constants \( a \) and \( b \) are determined by the initial condition) [57]. If we consider a free quantum particle \( (\omega_0 = 0) \), we obtain

\[ R^2(t) = R_0^2 + \frac{\hbar^2 t^2}{m^2 R_0^4}, \quad H(t) = \frac{\hbar^2 t}{m^2 R_0^4} + \frac{\hbar^2}{m^2 R_0^4}, \] (67)

where we have assumed \( \dot{R}(0) = 0 \). On the other hand, for \( T = 0 \) and \( \xi \to +\infty \), Eq. (65) has the general analytic solution

\[ R^4(t) = \frac{\hbar^2}{m^2 \omega_0^2} \left[ 1 - \left( 1 - \frac{m^2 \omega_0^2}{\hbar^2} R_0^2 \right) e^{-2\omega_0 t/\xi} \right]. \] (68)

For \( \omega_0 = 0 \), we obtain

\[ R^4(t) = R_0^4 + \frac{4\hbar^2 t}{\xi m^2}, \quad H(t) = \frac{\hbar^2}{R_0^4 + \frac{4\hbar^2 t}{\xi m^2}}. \] (69)

**Remark:** It is remarkable that the quantum damped Euler equations [59] and [60] with the isothermal equation of state [5] still admit a Gaussian self-similar solution despite the presence of the quantum force. This is because, when acted on the Gaussian distribution formed by Eqs. (7) and (13), the quantum force \(-(1/m)\nabla Q\) and the isothermal pressure force \(-(k_B T/m)\nabla \ln \rho\) are both proportional to \( x \) with a prefactor depending on time but independent of \( x \) (see Appendices B and C). Therefore, the quantum force and the isothermal equation of state “marry well”. We may consider this striking feature as a coincidence. Inversely, we can regard this coincidence as being fundamental. This may reveal some connection between standard quantum mechanics (through the quantum potential \( Q \)) and standard thermodynamics (through the isothermal equation of state \( P = \rho k_B T/m \)). In other words, the standard quantum potential may be linked to the isothermal equation of state. Assuming that this idea is correct, we can now look for the generalized quantum potential that is linked to the polytropic equation of state (see below). This will lead to a notion of generalized quantum mechanics (through the generalized quantum potential \( Q_{\gamma} \)) associated with Tsallis generalized thermodynamics (through the polytropic equation of state \( P = K \rho^\gamma \)). Determining whether this idea is physically relevant is beyond the scope of this paper. In any case, it allows us to obtain a generalized Schrödinger (or GP) equation that is of interest at a formal level.

**C. Polytropic case**

The damped power-law GP equation

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m\Phi_{\text{ext}} \psi + \frac{K \gamma m}{\gamma - 1} |\psi|^{2(\gamma - 1)} \psi - \frac{\hbar}{2} \xi \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \ln \left( \frac{\psi^*}{\psi} \right) \right] \psi, \] (70)

corresponding to a self-interaction potential

\[ V(|\psi|^2) = \frac{K}{\gamma - 1} |\psi|^{2\gamma}, \] (71)

is equivalent to the quantum damped Euler equations [59] and [60] with the polytropic equation of state [41]. In the absence of quantum force \( (\hbar = 0) \), and for the harmonic potential [59], we have seen that the damped Euler equations admit a Tsallis self-similar solution. This is no more true for the quantum damped Euler equations \( (\hbar \neq 0) \). We
suggest therefore to look for a generalized form of quantum potential that enables us to maintain a Tsallis self-similar solution. We find that a suitable form of generalized quantum potential is given by (see Appendix C1)

\[ Q_g = -\frac{\hbar^2}{2m} \Delta[(\rho/\rho_0)^{\gamma-1/2}] = -\frac{\hbar^2}{4m} (2\gamma - 1) \left[ \frac{\Delta(\rho/\rho_0)}{(\rho/\rho_0)^{3-2\gamma}} - \frac{1}{2} (3 - 2\gamma) \left( \nabla(\rho/\rho_0) \right)^2 \right], \]  

(72)

where \( \rho_0 \) is a constant with the dimension of a density that has been introduced for dimensionality reasons. For \( \gamma = 1 \), we recover the standard quantum potential \[ 61 \]. The generalized GP equation leading, through the Madelung transformation, to the generalized quantum damped Euler equations

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \]  

(73)

\[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P - \nabla \Phi_{\text{ext}} - \frac{1}{m} \nabla Q_g - \xi \mathbf{u}, \]  

(74)

with the generalized quantum potential \[ 72 \] and the polytropic equation of state \[ 41 \] is (see Appendices A and C)

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + \frac{\hbar^2}{2m} |\psi|^2 \psi - \frac{\hbar^2}{2m} \frac{\Delta}{(|\psi|^2/\rho_0)^{3-2\gamma}} \psi \]  

\[ + \frac{K\gamma m}{\gamma - 1} |\psi|^{2(\gamma-1)} \psi + m \Phi_{\text{ext}} \psi - i\hbar \xi \left[ \ln \left( \frac{\psi}{\psi_0} \right) - \left( \frac{\psi}{\psi_0} \right) \right]. \]  

(75)

For the harmonic external potential \[ 53 \], these equations admit a Tsallis self-similar solution formed by Eqs. \[ 7 \] and \[ 9 \] (see Appendix B). The differential equation determining the evolution of \( \chi(t) \) is given by Eq. \[ B39 \] and the differential equation determining the evolution of the radius \( R(t) \) is given by

\[ \ddot{R} + \xi \dot{R} + \omega_0^2 R = 2 Z^{1-\gamma} K \gamma \frac{M^{\gamma-1}}{R_c^{\gamma-3d+1}} + \frac{h^2}{m^2 \rho_0}^{2(1-\gamma)} \frac{M^{2(\gamma-1)}}{R_c^{2d(\gamma-1)+1}} \frac{2\gamma - 1}{2\gamma} \frac{1}{Z^{2(\gamma-1)}} [1 + d(\gamma - 1)]. \]

(76)

### D. Steady states

A steady state of the generalized quantum damped Euler equations \[ 73 \] and \[ 74 \] satisfies the condition of generalized quantum hydrostatic equilibrium

\[ -\frac{1}{\rho} \nabla P - \nabla \Phi_{\text{ext}} - \frac{1}{m} \nabla Q_g = 0. \]  

(77)

The time-independent distribution \[ 7 \], corresponding to \( \chi(t) = 0, R(t) = R_c = \text{cst} \) and \( \mathbf{u} = 0 \), when it exists, is a steady state of the generalized quantum damped Euler equations \[ 73 \] and \[ 74 \] with a harmonic potential. For an isothermal equation of state this equilibrium distribution is Gaussian and for a polytropic equation of state it is given by a Tsallis profile. The equilibrium radius of the system is determined by the equation

\[ \omega_0^2 R_c = 2 Z^{1-\gamma} K \gamma \frac{M^{\gamma-1}}{R_c^{\gamma-3d+1}} + \frac{h^2}{m^2 \rho_0}^{2(1-\gamma)} \frac{M^{2(\gamma-1)}}{R_c^{2d(\gamma-1)+1}} \frac{2\gamma - 1}{2\gamma} \frac{1}{Z^{2(\gamma-1)}} [1 + d(\gamma - 1)]. \]  

(78)

When the equilibrium state is unique and stable, and when \( \xi > 0 \), the system relaxes towards this steady state. However, the solution described previously is just a particular solution of Eqs. \[ 73 \] and \[ 74 \]. This may not be the only solution. The differential equation \[ 77 \] corresponding to the condition of generalized quantum hydrostatic equilibrium may have other solutions that are not given by the Boltzmann or by the Tsallis distribution.

### V. ANALOGY WITH COSMOLOGY

Let us consider the Euler equations \[ 11 \] and \[ 12 \] with the harmonic potential \[ 53 \] and the polytropic equation of state \[ 41 \] in \( d = 3 \) dimensions. The differential equation \[ 122 \] governing the evolution of the radius \( R(t) \) of the system can be written as

\[ \dot{R} = \frac{\kappa}{R^{3\gamma-2}} - \omega_0^2 R, \]

(79)
where we have introduced the notation \( \kappa = 2Z^{1-\gamma}K\gamma M^{\gamma-1} \). This equation is formally identical to the fundamental equation of dynamics (Newton’s equation) for a fictive particle of unit mass in a one dimensional potential \( V(R) \). Indeed, it can be rewritten as

\[
\dot{R} = \frac{dV}{dR},
\]  

with

\[
V(R) = \frac{\kappa}{3(\gamma - 1)R^{3(\gamma - 1)}} + \frac{1}{2}\omega_0^2 R^2 \quad (\gamma \neq 1),
\]  

\[
V(R) = -\kappa \ln R + \frac{1}{2}\omega_0^2 R^2 \quad (\gamma = 1).
\]

The first integral of motion is

\[
E = \frac{1}{2} \left( \frac{dR}{dt} \right)^2 + V(R),
\]

where \( E \) is a constant representing the energy of the fictive particle. The evolution of the radius \( R(t) \) of the system is therefore given by the integral

\[
t = \pm \int_{R_0}^{R(t)} \frac{dR}{\sqrt{2(E - V(R))}},
\]

where \( R_0 \) is the value of the radius at \( t = 0 \) and the sign in front of the integral is + when \( dR/dt > 0 \) and − when \( dR/dt < 0 \). Substituting the expression of the potential from Eqs. (81) and (82) into Eq. (84), we obtain

\[
t = \pm \int_{R_0}^{R(t)} \frac{dR}{\sqrt{2E - \frac{2\kappa}{3(\gamma - 1)R^{3(\gamma - 1)}} - \frac{\omega_0^2}{2} R^2}} \quad (\gamma \neq 1),
\]

\[
t = \pm \int_{R_0}^{R(t)} \frac{dR}{\sqrt{2E + 2\kappa \ln R - \frac{\omega_0^2}{2} R^2}} \quad (\gamma = 1).
\]

The first integral of motion (83) can be rewritten as

\[
H^2 = \left( \frac{\dot{R}}{R} \right)^2 = \frac{2E}{R^2} - \frac{2V(R)}{R^2}.
\]

Using the expressions (81) and (82) of the potential, we get

\[
H^2 = \left( \frac{\dot{R}}{R} \right)^2 = \frac{2E}{R^2} - \frac{2\kappa}{3(\gamma - 1)R^{3(\gamma - 1)}} - \frac{\omega_0^2}{2} \quad (\gamma \neq 1),
\]

\[
H^2 = \left( \frac{\dot{R}}{R} \right)^2 = \frac{2E}{R^2} + \frac{2\kappa \ln R}{R^2} - \frac{\omega_0^2}{2} \quad (\gamma = 1).
\]

Equation (87) is similar to the Friedmann equation in cosmology\(^{14}\)

\[
H^2 = \left( \frac{\dot{R}}{R} \right)^2 = -\frac{k c^2}{3c^2} + \frac{8\pi G}{3c^2} \varepsilon + \frac{\Lambda}{3},
\]

\(^{14}\) The basic equations of cosmology needed in the present section as well as a short account of the early development of modern cosmology can be found in Ref. [85].
which determines the evolution of a universe with curvature $k$, energy density $\epsilon$ and cosmological constant $\Lambda$. In this analogy, $R$ plays the role of the scale factor, $H = R/R$ plays the role of the Hubble parameter, $-2E$ plays the role of the curvature constant $kek^2$, and $-2V(R)/R^2$ plays the role of the energy density term $8\pi G \epsilon/3c^2$. We can therefore draw certain analogies between the evolution of the radius of our system and the evolution of the scale factor in the Friedmann-Lemaître-Robertson-Walker (FLRW) universe.

First of all, comparing Eqs. (88) and (89) with Eq. (10), we note that $-3\omega_0^2$ plays the role of the cosmological constant $\Lambda$. A repulsive harmonic potential $\omega_0^2 = -\Omega_0^2 < 0$ corresponds to a positive cosmological constant $\Lambda = 3\Omega_0^2 > 0$ (de Sitter universe) while an attractive harmonic potential $\omega_0^2 > 0$ corresponds to a negative cosmological constant $\Lambda = -3\omega_0^2 < 0$ (anti de Sitter universe).

On the other hand, in a universe filled with a fluid described by a linear equation of state $P_{\text{cosmo}} = \alpha \epsilon$, the energy density is related to the scale factor by $\epsilon \propto R^{-3(1+\alpha)}$. For $\gamma \neq 1$ the potential term in Eq. (88) is analogous to an energy density $\epsilon \propto R^{-3(\gamma-1)}$ in cosmology. This corresponds to an equation of state parameter $\alpha = \gamma - 4/3$. Note that in our system the “energy density” may be positive or negative (depending on the signs of $\kappa$ and $\gamma - 1$) while in cosmology the energy density is generally positive. We also note that the energy term $\propto R^{-2}$ in Eqs. (88) and (89), the curvature term in Eq. (90), and the thermal term $\propto \ln R/R^2$ in Eq. (89) have the same effect (up to a logarithmic correction in the latter case) as a fluid described by an equation of state $P_{\text{cosmo}} = -\epsilon/3$ (gas of cosmic strings) for which $\epsilon \propto 1/R^2$.

If we take $E = 0$ and $\gamma \neq 1$, our system behaves, in the cosmological analogy, as a fluid with an equation of state $P_{\text{cosmo}} = (\gamma - 4/3)\epsilon$ in a flat universe ($k = 0$) with a cosmological constant $\Lambda$ (dark energy). Interestingly, the index $\gamma = 4/3$, implying $P_{\text{cosmo}} = 0$ and $\epsilon \propto R^{-3}$, leads to the standard $\Lambda$CDM model (we need, however, to assume $K < 0$ in order to have a positive energy density). Let us now consider an arbitrary index $\gamma \neq 1$ and assume that $\omega_0 = 0$ (we also restrict ourselves to the case where the system is expanding). Using the cosmological analogy, we have the following results:

(i) For $\gamma > 1$, corresponding to $\alpha > -1/3$, the expansion is decelerating. For $\gamma < 1$, corresponding to $\alpha < -1/3$, the expansion is accelerating.

(ii) For $\gamma > 1/3$, corresponding to $\alpha > -1$, the evolution of our system is similar to the evolution of a normal universe in which the energy density decreases as the scale factor increases. For $\gamma < 1/3$, corresponding to $\alpha < -1$, the evolution of our system is similar to the evolution of a phantom universe in which the energy density increases as the scale factor increases. For $\gamma = 1/3$, corresponding to $\alpha = -1$, i.e. $P_{\text{cosmo}} = -\epsilon$, the energy density is constant and it plays the same role as the cosmological constant ($\Lambda$).

VI. CONCLUSION

In this paper, we have introduced a nonlinear wave equation (A20) that generalizes the standard Schrödinger equation. The Schrödinger equation is recovered for $q = 1$. We have also introduced a nonlinear wave equation (A11) that generalizes the GP equation. The GP equation is recovered for $q = 1$, $\xi = 0$, and for a cubic nonlinearity. By making the Madelung transformation, we have shown that the generalized damped GP equation (A11) is equivalent to the generalized quantum damped Euler equations (A13) and (A19). They include a generalized quantum potential (A11), a pressure force determined by the self-interaction potential $V(|\psi|^2)$, and a friction force. For conservative systems ($\xi = 0$), we obtain the generalized quantum Euler equations (A21) and (A22). In the strong friction limit $\xi \to +\infty$, we obtain the generalized quantum Smoluchowski equation (A24). In the TF limit ($h = 0$), where the quantum potential can be neglected, we recover the Euler equations (1) and (2) when $\xi = 0$, and the generalized Smoluchowski equation (23) when $\xi \to +\infty$.

We have shown that these equations are associated with a notion of generalized quantum mechanics, generalized Brownian theory and generalized thermodynamics. They satisfy an $H$-theorem for a generalized form of free energy functional. The generalized quantum potential (A11) is associated with the generalized Von Weizäcker functional (A11)-(A14). On the other hand, a logarithmic potential in the generalized damped GP equation (A11) leads to an isothermal equation of state (5) associated with the Boltzmann entropy (A85) while a power-law potential in the generalized damped GP equation (A11) leads to a polytropic equation of state (6) associated with the Tsallis entropy (A88).

---

15 We also note from Eq. (60) that the standard quantum potential term is analogous to an energy density $\epsilon \propto -1/R^4$ in cosmology. This corresponds to $\alpha = 1/3$ like for the standard radiation. However, the energy density is negative.
When considering the harmonic external potential \( V \) and the isothermal equation of state \( T \), these equations admit stationary and self-similar solutions with an invariant profile that has the form of a Gaussian distribution. When considering the harmonic external potential \( V \) and the polytropic equation of state \( P \), these equations admit stationary and self-similar solutions with an invariant profile that has the form of a Tsallis distribution. This is true provided that the exponent \( q \) of the generalized quantum potential \( A_{11} \) and the exponent \( \gamma \) of the polytropic equation of state \( A_4 \) are related by

$$ q = 2\gamma - 1. \quad (91) $$

This equation provides a relation between generalized quantum mechanics \( (q) \) and generalized thermodynamics \( (\gamma) \). For \( q = \gamma = 1 \), we recover standard quantum mechanics and standard thermodynamics.

Several researchers, including de Broglie \([86]\), have tried to obtain nonlinear wave equations generalizing the (linear) Schrödinger equation. They considered that the Schrödinger equation is not the most fundamental equation of quantum mechanics and that it may be an approximation of a more fundamental nonlinear wave equation. The Schrödinger equation is the simplest (because it is linear) wave equation that returns the relation \( E = \frac{p^2}{2m} \) between the energy and the impulse of a free classical particle if we consider plane waves of the form \( e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \) with the correspondances \( E = \hbar \omega \) (Planck-Einstein) and \( \mathbf{p} = \hbar \mathbf{k} \) (de Broglie). But this is not the only equation enjoying this property. We can construct nonlinear wave equations that preserve these relations. One example has been proposed recently by Nobre et al. \([58, 60]\). Their nonlinear wave equation admits q-plane waves of the form \( e_q^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \) that preserve the above relations. Another example is provided by our nonlinear wave equation \( A_{10} \) that preserves the above relations for standard plane waves of the form \( e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \) (see the end of Appendix \( A_1 \)). In addition, it admits wave packet solutions which, instead of having a Gaussian profile, have a q-Gaussian profile (see Appendix \( B_3 \) and the end of Appendix \( B_1 \)). This is interesting because one never exactly measures pure Gaussians, but rather q-Gaussians \([77]\). Therefore, one cannot reject the possibility that quantum mechanics may be described by a nonlinear wave equation of the form of Eq. \( A_{20} \).

Several researchers \([54] \) have also tried to generalize Brownian theory and obtain nonlinear Fokker-Planck equations generalizing the (linear) Fokker-Planck equation associated with standard (Boltzmann) thermodynamics. However, until now, no connection was made between generalized Schrödinger equations and generalized Fokker-Planck equations. Our approach, initiated in \([55]\), is a first step in that direction. An interest of our formalism is to connect (generalized) quantum mechanics, (generalized) Brownian motion, and (generalized) thermodynamics. Indeed, the (generalized) damped GP equation \( A_{11} \) contains the (generalized) Schrödinger equation of quantum mechanics and the (generalized) Smoluchowski equation of Brownian theory as special cases. The (generalized) Schrödinger equation is obtained in the no friction limit \((\xi = 0)\) while the (generalized) Smoluchowski equation is obtained in the strong friction limit \( \xi \to +\infty \).

The starting point of our generalization of quantum mechanics was based on the following remark. The logarithmic Schrödinger equation \( B_{30} \) admits self-similar solutions with a Gaussian profile like in standard (Boltzmann) thermodynamics. This is because the Laplacian \( \Delta \) in quantum mechanics “marries well” with the logarithmic term \( 2k_B T \ln |\psi| \) associated with the isothermal equation of state \( B_{25} \) of standard thermodynamics. We then looked for a generalization of quantum mechanics such that the generalized Laplacian \( \Delta_q \) (see Eq. \( A_{31} \)) in generalized quantum mechanics “marries well” with the power-law term \( K|\psi|^{2(\gamma - 1)} \) associated with the polytropic equation of state \( A_4 \) of generalized thermodynamics. This leads to the generalized power-law Schrödinger equation \( C_{25} \) that admits self-similar solutions with a Tsallis profile like in generalized (Tsallis) thermodynamics.

It is in general very difficult to find exact analytical solutions of nonlinear partial differential equations, especially for \( d \neq 1 \) (for \( d = 1 \) several nonlinear partial differential equations admit soliton solutions). Therefore, our work may be of interest to mathematicians. We have introduced a nonlinear wave equations \( A_{11} \) with a generalized Laplacian \( q \) and a power-law nonlinearity \( (\gamma) \) that admits exact analytical self-similar solutions in the form of wave packets with a Tsallis profile (in certain limits they reduce to soliton-like solutions that we called \( \gamma \)-gaussians). These solutions are valid not only for a free particle but also for a particle in a harmonic potential. These solutions are not valid for more complicated potentials but it is probably possible to make perturbative expansions about the harmonic potential.

For conservative systems \((\xi = 0)\) we have shown that the radius of the system \( R(t) \) in the self-similar solution satisfies a differential equation that shares some analogies with the Friedmann equations in cosmology describing the evolution of the radius (scale factor) of the Universe. This analogy will be developed in a forthcoming contribution \([57]\) where analytical solutions of the equations derived in the present paper are given.

At the present stage of the development of the theory all our results are formal. Still, there are “exact” in the sense that there is no approximation of any sort in our approach once the basic equations are assumed. Therefore, the equations presented in this paper have nice mathematical properties. It will be important in future works to determine if our formalism can have physical applications. We have already proposed to apply these generalized wave equations to the physics of dark matter halos made of Bose-Einstein condensates (BECs) \([88]\). In that case, they
must be coupled to the gravity through the Poisson equation, leading to the (generalized) Gross-Pitaevskii-Poisson (GPP) equations [53, 89]. In the strong friction limit, and in the TF approximation, we recover the (generalized) Smoluchowski-Poisson equations studied in Ref. [58] (and references therein). Another line of research would consist in extending these ideas to the context of general relativity in relation to the Klein-Gordon-Einstein (KGE) equations. Some work in that direction has already been performed in Ref. [91]. On the other hand, in Ref. [92], we have given an illustration of the self-similar solution of the classical Euler equations presented in Sec. III in relation to the explosion of a star (supernova). These are different examples where the equations of the present paper, and their self-similar solutions, may find physical and astrophysical applications. It is likely that many other applications will be discovered in the future.

Appendix A: Basic properties of the generalized damped GP equation

1. Generalized damped GP equation

We consider a generalized damped GP equation of the form

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + \frac{\hbar^2}{2m} \Delta \left\| \psi \right\| \psi - \frac{\hbar^2}{2m} \Delta \left[ \left( \frac{\left\| \psi \right\|}{\left\| \psi_0 \right\|} \right)^q \right] \psi + m \left[ \frac{dV}{d\left\| \psi \right\|^2} + \Phi + \Phi_{\text{ext}} \right] \psi - \frac{\hbar}{2} \xi \left( \ln \left( \frac{\psi}{\psi^*} \right) - \left\langle \ln \left( \frac{\psi}{\psi^*} \right) \right\rangle \right) \psi \]  

(A1)

describing a dissipative BEC with a mass density \( \rho(r, t) = |\psi|^2(r, t) \), where \( \psi(r, t) \) is the wave function of the condensate.\(^{16} \) In this equation \( V(|\psi|^2) \) is the self-interaction potential of the bosons, \( \Phi(r, t) \) is the mean field potential created by the bosons, \( \Phi_{\text{ext}}(r) \) is a fixed external potential, and \( \xi \) is the friction coefficient.\(^{18} \) The brackets denote a space average over the entire domain: \( \left\langle X \right\rangle = \frac{1}{M} \int \rho X \, dr \). The mean field potential can be written as

\[ \Phi(r, t) = \int u(|r - r'|)|\psi|^2(r', t) \, dr', \]  

(A2)

where \( u(|r - r'|) \) is a binary potential of interaction between the bosons. The mean field approximation is valid for long-range potentials of interaction, such as the gravitational potential \( u = -G/|r - r'| \), in a proper thermodynamic limit where the number of particles \( N \to +\infty \). Self-gravitating BECs have been proposed as a model of dark matter in cosmology (see Refs. [53, 89] and references therein). Equation (A1) involves a nonlinear Laplacian operator

\[ \Delta_q \psi = \Delta \psi - \frac{\Delta \left\| \psi \right\|}{\left\| \psi \right\|} \psi + \Delta \left[ \left( \frac{\left\| \psi \right\|}{\left\| \psi_0 \right\|} \right)^q \right] \psi, \]  

(A3)

where \( \psi_0 \) is a constant with the dimension of a wave function introduced for dimensional reasons. For \( q = 1 \), \( \Delta_q \) reduces to the ordinary Laplacian operator \( \Delta \) and Eq. (A1) reduces to the generalized damped GP equation (GPP) studied in [53] in the context of standard quantum mechanics. We refer to this paper for a detailed discussion of this equation and for its physical interpretation. The aim of this Appendix is to summarize in a synthetic manner the main results of this study and to generalize them in the case where the linear Laplacian operator \( \Delta \) is replaced by the nonlinear Laplacian operator \( \Delta_q \). This leads to a notion of generalized quantum mechanics (see Appendix C). Introducing the function

\[ h(|\psi|^2) = \frac{dV}{d|\psi|^2}, \quad \text{i.e.} \quad h(\rho) = V'(\rho), \]  

(A4)

the generalized damped GP equation (A1) can be rewritten as

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + \frac{\hbar^2}{2m} \Delta \left\| \psi \right\| \psi - \frac{\hbar^2}{2m} \Delta \left[ \left( \frac{\left\| \psi \right\|}{\left\| \psi_0 \right\|} \right)^q \right] \psi + \frac{m}{2} \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \left\langle \ln \left( \frac{\psi}{\psi^*} \right) \right\rangle \right] \psi. \]  

(A5)

---

\(^{16} \) We may also regard this nonlinear Schrödinger equation as describing the evolution of the wavefunction of a single quantum particle. However, in that case, the interpretation of the different terms demands a more delicate discussion that is left for a future contribution.

\(^{17} \) The (generalized) GP equation can be derived from the (generalized) Klein-Gordon equation in the nonrelativistic limit \( c \to +\infty \). In that case, \( V(|\psi|^2) \) is the potential that appears in the KG equation.

\(^{18} \) We may also consider a time dependent external potential \( \Phi_{\text{ext}}(r, t) \). It could account for a stochastic forcing as in Ref. [92]. Such a term is often written under the form \( \Phi_{\text{ext}}(r, t) = A(t) \cdot r \) where \( A(t) \) is a random force.\(^{93} \)
It can be shown (see below) that the generalized damped GP equation conserves the mass $M = \int \rho \, d\mathbf{r}$.

**Remark:** If we consider a free particle ($V = \Phi = \Phi_{\text{ext}} = \xi = 0$), we obtain the generalized Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + \frac{\hbar^2}{2m} \frac{\Delta |\psi|}{|\psi|} \psi - \frac{\hbar^2}{2m} \Delta \left[ |\psi|/|\psi_0|\right] \psi.$$ \quad (A6)

Looking for a solution in the form of a plane wave $\psi(\mathbf{r}, t) \sim e^{i (\mathbf{k} \cdot \mathbf{r} - \omega t)}$ we obtain the dispersion relation $\omega = \hbar k^2 / 2m$. Using the Planck-Einstein relation $E = \hbar \omega$ and the de Broglie relation $p = \hbar \mathbf{k}$, we recover the relation $E = p^2 / 2m$ between the energy and the impulse of a classical free particle. Therefore, our generalized Schrödinger equation is consistent with the wave-particle duality which is at the basis of quantum mechanics (see, e.g., the introduction of [90]).

### 2. Madelung transformation

We use the Madelung [56] transformation to rewrite the generalized damped GP equation (A5) under the form of hydrodynamic equations.\(^{19}\) We write the wave function as

$$\psi(\mathbf{r}, t) = \sqrt{\rho(\mathbf{r}, t)} e^{i S(\mathbf{r}, t)/\hbar},$$ \quad (A7)

where the phase $S(\mathbf{r}, t) = -i(\hbar/2) \ln(\psi/\psi^*)$ represents the action. Following Madelung, we introduce the velocity field

$$\mathbf{u} = \frac{\nabla S}{m} = -i \frac{\hbar}{2m} \frac{\psi^* \nabla \psi - \psi \nabla \psi^*}{|\psi|^2}.$$ \quad (A8)

Since the velocity is potential, the flow is irrotational: $\nabla \times \mathbf{u} = 0$. Substituting Eq. (A7) into Eq. (A5) and separating real and imaginary parts, we obtain

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$ \quad (A9)

$$\frac{\partial S}{\partial t} + \frac{1}{2m} (\nabla S)^2 + m [(\rho + \Phi + \Phi_{\text{ext}}) + Q_g + \xi (S - \langle S \rangle)] = 0,$$ \quad (A10)

where

$$Q_g = -\frac{\hbar^2}{2m} \frac{\Delta [(\rho/\rho_0)^{q/2}]}{(\rho/\rho_0)^{q/2}} = \frac{\hbar^2}{4m} q \left[ \frac{\Delta (\rho/\rho_0)}{(\rho/\rho_0)^{2-q}} - \frac{1}{2} (2-q) \frac{\nabla (\rho/\rho_0)^2}{(\rho/\rho_0)^{3-q}} \right]$$ \quad (A11)

is the generalized quantum potential (see Appendix C1) and $\rho_0 = \psi_0^2$ is a constant with the dimensions of a mass density. When $q = 1$ we recover the standard quantum potential from Eq. (61). Eq. (A9) is similar to the equation of continuity in hydrodynamics. It accounts for the local conservation of mass. Eq. (A10) has a form similar to the classical Hamilton-Jacobi equation with an additional generalized quantum potential and a source of dissipation. It can also be interpreted as a generalized Bernoulli equation for a potential flow. Taking the gradient of Eq. (A10), and using the well-known identity of vector analysis ($\mathbf{u} \cdot \nabla \mathbf{u} = \nabla (\mathbf{u}^2/2) - \mathbf{u} \times (\nabla \times \mathbf{u})$) which reduces to ($\mathbf{u} \cdot \nabla \mathbf{u} = \nabla (\mathbf{u}^2/2)$) for an irrotational flow, we obtain an equation similar to the Euler equation with a linear friction and a generalized quantum force

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla h - \nabla \Phi - \nabla \Phi_{\text{ext}} - \frac{1}{m} \nabla Q_g - \xi \mathbf{u}.$$ \quad (A12)

\(^{19}\) As discussed in [55, 94], this hydrodynamic representation has a clear physical meaning in the case where the generalized damped GP equation (A5) describes a BEC. Its interpretation is less clear (and was strongly criticized by the founders of quantum mechanics like Pauli) when the nonlinear Schrödinger equation (A5) describes a single quantum particle. In that respect, we recall that the Madelung transformation (also introduced independently by de Broglie [63, 66] in his pilot wave theory of relativistic particles) was rediscovered by Bohm [67, 68] who gave it an interpretation in terms of particle trajectories (see Appendix B9). For that reason the quantum potential is sometime called the Bohm potential although it was found previously by Madelung.
We can also write Eq. (A12) under the form
\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{1}{\rho} \nabla P - \nabla \Phi - \nabla \Phi_{\text{ext}} - \frac{1}{m} \nabla Q_g - \xi \mathbf{u},
\] (A13)
where \( P(\mathbf{r}, t) \) is the pressure. Since \( h(\mathbf{r}, t) = h[\rho(\mathbf{r}, t)] \), the pressure \( P(\mathbf{r}, t) = P[\rho(\mathbf{r}, t)] \) is a function of the density, i.e., the flow is barotropic. The equation of state \( P(\rho) \) is determined by the potential \( h(\rho) \) through the relation
\[
h'(\rho) = \frac{P'(\rho)}{\rho}.
\] (A14)
This equation, which can be viewed as the Gibbs-Duhem relation \( dh = dP/\rho \), shows that the effective potential \( h \) appearing in the generalized damped GP equation (A5) can be interpreted as an enthalpy in the hydrodynamic equations. We shall see later (in Appendix A8) that \( h \) is one component of the chemical potential. Equation (A14) can be integrated into
\[
P(\rho) = \rho h(\rho) - V(\rho) = \rho V'(\rho) - V(\rho) = \rho^2 \left[ \frac{V(\rho)}{\rho} \right]'.
\] (A15)
This determines the equation of state \( P(\rho) \) as a function of the self-interaction potential \( V(\rho) \). The speed of sound is
\[
c^2 = P'(\rho) = \rho V''(\rho).
\] (A16)
In conclusion, the generalized damped GP equation (A5) is equivalent to the hydrodynamic equations
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,
\] (A18)
\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P - \nabla \Phi - \nabla \Phi_{\text{ext}} - \frac{1}{m} \nabla Q_g - \xi \mathbf{u}.
\] (A19)
For the harmonic potential defined by Eq. (3), we have \( \nabla \Phi_{\text{ext}} = \omega_0^2 \mathbf{r} \). Using the continuity equation (A18), the Euler equation (A19) can be rewritten as
\[
\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla(\rho \mathbf{u} \otimes \mathbf{u}) = -\nabla P - \rho \nabla \Phi - \rho \nabla \Phi_{\text{ext}} - \frac{\rho}{m} \nabla Q_g - \xi \rho \mathbf{u}.
\] (A20)
We shall refer to these equations as the generalized quantum damped Euler equations. We note that the hydrodynamic equations (A18) and (A19) do not involve viscous terms so they describe a superfluid. When the generalized quantum potential can be neglected, we recover the classical damped Euler equations. For dissipationless systems (\( \xi = 0 \)), Eqs. (A18) and (A19) reduce to the generalized quantum Euler equations
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,
\] (A21)
\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P - \nabla \Phi - \nabla \Phi_{\text{ext}} - \frac{1}{m} \nabla Q_g.
\] (A22)
When the generalized quantum potential can be neglected, we recover the classical Euler equations. On the other hand, in the overdamped limit \( \xi \to +\infty \), we can formally neglect the inertia of the particles in Eq. (A19) so that
\[
\xi \mathbf{u} \simeq -\frac{1}{\rho} \nabla P - \nabla \Phi - \nabla \Phi_{\text{ext}} - \frac{1}{m} \nabla Q_g.
\] (A23)
Substituting this relation into the continuity equation (A18), we obtain the generalized quantum Smoluchowski equation
\[ \xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left( \nabla P + \rho \nabla \Phi + \rho \nabla \Phi_{\text{ext}} + \frac{\rho}{m} \nabla Q_\rho \right). \] (A24)

When the generalized quantum potential can be neglected, we recover the generalized classical Smoluchowski equation [51]. Finally, if we neglect the advection term \( \nabla (\rho \textbf{u} \otimes \textbf{u}) \) in Eq. (A20), but retain the term \( \partial (\rho \textbf{u}) / \partial t \), and combine the resulting equation with the continuity equation (A18), we obtain the generalized quantum telegraph equation
\[ \frac{\partial^2 \rho}{\partial t^2} + \xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left( \nabla P + \rho \nabla \Phi + \rho \nabla \Phi_{\text{ext}} + \frac{\rho}{m} \nabla Q_\rho \right). \] (A25)

It can be seen as a generalization of the quantum Smoluchowski equation (A24) taking memory effects into account. As a result, the generalized damped GP equation (A5) contains many important equations of physics as particular cases. In particular, it provides an interesting formal connection between (generalized) quantum mechanics and (generalized) Brownian theory [55] since it is equivalent to a nonlinear Schrödinger equation when \( \xi = 0 \) and to a nonlinear Fokker-Planck equation when \( \xi \to +\infty \) [42, 51, 53, 54].

Remark: When \( V = \Phi = \xi = 0 \), the generalized damped GP equation (A1) reduces to a generalized Schrödinger equation of the form
\[ i \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + \frac{\hbar^2}{2m} \left( |\psi| \frac{\Delta |\psi|}{|\psi|} - \frac{2}{q} |\psi| \frac{\Delta |\psi|}{|\psi|} \right) + m \Phi_{\text{ext}} \psi + \frac{1}{m} \nabla Q_\rho \psi. \] (A26)

If we ignore the third term in the right hand side of Eq. (A26), we get the classical wave equation (see Appendix L.3. of [84])
\[ i \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \Phi_{\text{ext}} \psi. \] (A27)

This equation is equivalent, through the Madelung transformation, to the continuity equation (A18) and to the classical (i.e. without quantum potential) Euler equation
\[ \frac{\partial \textbf{u}}{\partial t} + (\textbf{u} \cdot \nabla) \textbf{u} = -\nabla \Phi_{\text{ext}}. \] (A28)

As a result, the generalized Schrödinger equation (A26) is equivalent, through the Madelung transformation, to the continuity equation (A18) and to the Euler equation with the generalized quantum potential
\[ \frac{\partial \textbf{u}}{\partial t} + (\textbf{u} \cdot \nabla) \textbf{u} = -\nabla \Phi_{\text{ext}} - \frac{1}{m} \nabla Q_\rho. \] (A29)

This is basically how we have originally obtained Eq. (A26): we have first removed the contribution of the standard quantum potential, then added the contribution of the generalized quantum potential. As another remark, we note that when \( q \to 1 \) the generalized Schrödinger equation (A26) reduces to
\[ i \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi - (q-1) \frac{\hbar^2}{2m} \left( |\psi| \frac{\Delta |\psi|}{|\psi|} \right) + \ln \left( \frac{|\psi|}{\psi_0} \right) \frac{\Delta |\psi|}{|\psi|} \psi + m \Phi_{\text{ext}} \psi. \] (A30)

This provides the first correction to the Schrödinger equation.

3. Generalized quantum force

The generalized quantum force can be written as
\[ \frac{1}{m} \partial_i Q_\rho = \frac{1}{\rho} \partial_j P_{ij}^g, \] (A31)

where \( P_{ij}^g \) is a generalized quantum pressure tensor defined by (see Appendix C.4)
\[ P_{ij}^g = \frac{\hbar^2}{4m^2} q^1 \rho_0^{1-q} \left[ \frac{1}{\rho^{2-q}} \partial_i \rho \partial_j \rho - A \frac{1}{\rho^{1-q}} \delta_{ij} \Delta \rho - B \frac{1}{\rho^{1-q}} \partial_i \rho \partial_j \rho + C \frac{1}{\rho^{2-q}} (\nabla \rho)^2 \delta_{ij} \right]. \] (A32)
\[ A = \frac{1 - k(2 - q)}{q - 1}, \quad B = \frac{k - 1}{q - 1}, \quad C = \frac{k}{2}(3 - q), \quad (A33) \]

where \( k \) is an arbitrary parameter. When \( q = 1 \) we recover the standard quantum pressure tensor \( P_{ij} \) given by Eq. (C54). We note that the contracted quantum pressure tensor can be written as

\[ P_{ii}^q = \frac{\hbar^2}{4m^2q} \frac{1}{\mathcal{R}^q} \left[ (1 + dC) \frac{(\nabla \rho)^2}{\rho^{2-q}} - (B + dA) \frac{\Delta \rho}{\rho^{1-q}} \right], \quad (A34) \]

where a summation is implied over repeated indices.

4. Generalized time-independent GP equation

If we consider a wave function of the form

\[ \psi(r, t) = \phi(r) e^{-iEt/\hbar}, \quad (A35) \]

where \( \phi(r) = \sqrt{\rho(r)} \) is real, and substitute Eq. (A35) into the generalized damped GP equation (A33), we obtain the generalized time-independent GP equation

\[ -\frac{\hbar^2}{2m} \frac{\Delta \left[ (\phi/\psi_0)^q \right]}{(\phi/\psi_0)^{2-q}} \phi = m[\Phi + \hbar(\phi^2) + \Phi_{\text{ext}}] \phi = E\phi. \quad (A36) \]

Equation (A36) defines a nonlinear eigenvalue problem for the wave function \( \phi(r) \) where the eigenvalue \( E \) is the energy (we call it the eigenenergy). Dividing Eq. (A36) by \( \phi(r) \) and using \( \rho = \phi^2 \) and Eq. (A11), we get

\[ m\Phi + m\hbar(\rho) + m\Phi_{\text{ext}} + Q_g = E. \quad (A37) \]

This relation can be directly derived from the generalized quantum damped Hamilton-Jacobi equation (A10) by setting \( S = -Et \).

5. Generalized quantum hydrostatic equilibrium

The generalized time-independent GP equation (A37) can also be obtained from the generalized quantum damped Euler equation (A13) which is equivalent to the generalized damped GP equation (A33). The equilibrium state of the generalized quantum damped Euler equation, obtained by taking \( \partial_t = 0 \) and \( \mathbf{u} = \mathbf{0} \), satisfies

\[ \nabla P + \rho \nabla \Phi + \rho \nabla \Phi_{\text{ext}} + \frac{\rho}{m} \nabla Q_g = 0. \quad (A38) \]

This equation generalizes the usual condition of hydrostatic equilibrium by incorporating the contribution of the generalized quantum potential. Equation (A38) describes the balance between the pressure force due to the self-interaction of the bosons, the mean field force, the external force, and the generalized quantum force. This equation is equivalent to Eq. (A37). Indeed, integrating Eq. (A38) using Eq. (A14), we obtain Eq. (A37) where the eigenenergy \( E \) appears as a constant of integration.

6. Generalized free energy

The generalized free energy associated with the generalized damped GP equation (A1), or equivalently with the generalized quantum damped Euler equations (A18) and (A19), can be written as

\[ F = \Theta_c + \Theta_Q^q + U + W + W_{\text{ext}}, \quad (A39) \]

where the different functionals are defined below using Madelung’s hydrodynamic variables (see [55] for a more detailed discussion).

The first term is the classical kinetic energy

\[ \Theta_c = \int \rho \frac{u^2}{2} \, d\mathbf{r}. \quad (A40) \]
The second term is the generalized quantum kinetic energy. It is given by the equivalent expressions (see Appendix C2)

\[ \Theta_Q^g = \frac{\hbar^2}{2m^2 q} \int \left\{ \nabla \left[ \frac{(\rho/\rho_0)^{q/2}}{(\rho/\rho_0)^{2}} \right] \right\}^2 \, dr, \]  

(A41)

\[ \Theta_Q^g = \frac{\hbar^2}{8m^2 \rho_0 q} \int \frac{\nabla (\rho/\rho_0)^{2}}{(\rho/\rho_0)^{2}} \, dr, \]  

(A42)

\[ \Theta_Q^g = -\frac{\hbar^2}{2m^2 \rho_0 q} \int \left( \frac{\rho}{\rho_0} \right)^{q/2} \Delta \left[ \left( \frac{\rho}{\rho_0} \right)^{q/2} \right] \, dr, \]  

(A43)

\[ \Theta_Q^g = \frac{1}{q} \int \rho \frac{Q_g}{m} \, dr. \]  

(A44)

According to Eq. (A44) the generalized quantum kinetic energy can also be interpreted as a potential energy associated with the generalized quantum potential \( Q_g \) (this analogy is not obvious, however, since \( Q_g \) is a function of the density, not an external potential). The functional (A42) is a generalization of the von Weizsäcker functional \([83]\). It is related to a generalization of the Fisher \([99]\) entropy by Eq. (C31). The functional (A43) is a generalization of the Madelung functional \([56, 100]\). For \( q = 1 \) we recover the standard expressions given in \([55]\):

\[ \Theta_Q = \frac{\hbar^2}{2m^2} \int (\nabla \sqrt{\rho})^2 \, dr = \frac{\hbar^2}{8m^2} \int \left( \frac{\nabla \rho}{\rho} \right)^2 \, dr = \frac{\hbar^2}{2m^2} \int \sqrt{\rho} \Delta \sqrt{\rho} \, dr = \int \rho \frac{Q}{m} \, dr \quad (q = 1). \]  

(A45)

The third term is the internal energy

\[ U = \int \rho \int^\rho \frac{P(\rho')}{\rho^2} \, d\rho' \, dr = \int [\rho h(\rho) - P(\rho)] \, d\rho = \int V(\rho) \, dr. \]  

(A46)

The fourth term is the mean field energy

\[ W = \frac{1}{2} \int \rho \Phi \, dr. \]  

(A47)

The fifth term is the external potential energy

\[ W_{ext} = \int \rho \Phi_{ext} \, dr. \]  

(A48)

For the harmonic potential \([59]\), one has

\[ W_{ext} = \frac{1}{2} \omega_0^2 I, \quad \text{where} \quad I = \int \rho r^2 \, dr \]  

(A49)

is the moment of inertia. We note that \( I = M \langle r^2 \rangle \) where \( \langle r^2 \rangle \) measures the dispersion of the particles (or the size of the BEC).

Regrouping these results, the free energy associated with the generalized damped GP equation (A1), or equivalently with the generalized quantum damped Euler equations (A18) and (A19), can be explicitly written as

\[ F = \int \rho \frac{u^2}{2} \, dr + \frac{1}{qm} \int \rho Q_g \, dr + \int V(\rho) \, dr + \frac{1}{2} \int \rho \Phi \, dr + \int \rho \Phi_{ext} \, dr. \]  

(A50)

On the other hand, the free energy associated with the generalized quantum Smoluchowski equation (A24) is given by

\[ F = \Theta_Q^g + U + W + W_{ext} = \frac{1}{qm} \int \rho Q_g \, dr + \int V(\rho) \, dr + \frac{1}{2} \int \rho \Phi \, dr + \int \rho \Phi_{ext} \, dr \]  

(A51)

since the classical kinetic energy \( \Theta_c \), which is of order \( O(\xi^{-2}) \), can be neglected in the overdamped limit \( \xi \to +\infty \).
Remark: the functionals $\Theta_c, U, W$ and $W_{\text{ext}}$ are the same as in [55]. Their first and second order variations are given in Appendix C of [55]. The functional $\Theta^g_Q$ is a generalization of the standard von Weizsäcker functional $\Theta_Q$. However, its first variations satisfy the fundamental relation

$$\delta \Theta^g_Q = \int \frac{Q_g}{m} \delta \rho \, dr,$$

as in the standard case [55]. Actually, we determined the expression of $\Theta^g_Q$ precisely in order to satisfy this relation (see Appendix C 2).

7. $H$-theorem

The time derivative of the generalized free energy (A50) satisfies the identity

$$\dot{F} = -\xi \int \rho u^2 \, dr = -2\xi \Theta_c.$$ (A53)

When $\dot{F} = 0$, we get $u = 0$ leading to the condition of generalized quantum hydrostatic equilibrium (A38). For dissipationless systems ($\xi = 0$), Eq. (A53) shows that the generalized GP equation, or the generalized quantum Euler equations, conserve the free energy ($\dot{F} = 0$). For dissipative systems ($\xi > 0$), Eq. (A53) shows that the generalized damped GP equation, or the generalized quantum damped Euler equations, decrease the free energy ($\dot{F} \leq 0$). This provides a form of $H$-theorem. In the strong friction limit $\xi \to +\infty$, the time derivative of the generalized free energy (A51) associated with the generalized quantum Smoluchowski equation satisfies the $H$-theorem

$$\dot{F} = -\frac{1}{\xi} \int \rho \left( \nabla P + \rho \nabla \Phi + \rho \nabla \Phi_{\text{ext}} + \frac{\rho}{m} \nabla Q_g \right)^2 \, dr \leq 0.$$ (A54)

When $\dot{F} = 0$, the term in parenthesis vanishes leading to the condition of generalized quantum hydrostatic equilibrium (A38). A more detailed discussion of these results is given in Sec. 3.3 of [55].

8. Equilibrium state

According to the results of the preceding section, a stable equilibrium state of the generalized (damped) GP equation, or of the generalized quantum (damped) Euler equations, is the solution of the minimization problem

$$F(M) = \min_{\mu, M} \{F[\rho, u] \mid M \text{ fixed} \}.$$ (A55)

An extremum of free energy at fixed mass is an equilibrium state of the generalized (damped) GP equation, or of the generalized quantum (damped) Euler equations. It is determined by the variational principle

$$\delta F - \frac{\mu}{m} \delta M = 0,$$ (A56)

where $\mu$ is a Lagrange multiplier (chemical potential) taking into account the mass constraint. This variational problem gives $u = 0$ (the equilibrium state is static) and

$$m \Phi + m \Phi_{\text{ext}} + mh(\rho) + Q_g = \mu.$$ (A57)

This is the generalization of the Gibbs condition of constancy of chemical potential (see, e.g., Landau and Lifshitz [101]). Indeed, if we define the out-of-equilibrium chemical potential by

$$\frac{\mu(r, t)}{m} = \frac{\delta F}{\delta \rho} = \Phi + \Phi_{\text{ext}} + h(\rho) + \frac{1}{m} Q_g,$$ (A58)

we find that, at equilibrium, $\mu(r, t) = \mu$ is a constant. Taking the gradient of Eq. (A57), and using Eq. (A14), we recover the condition of generalized quantum hydrostatic equilibrium (A38). Equation (A57) is also equivalent to the generalized time-independent GP equation (A37) provided that we make the identification

$$\mu = E.$$ (A59)
This shows that the Lagrange multiplier $\mu$ (chemical potential) in the variational problem (A55) can be identified with the eigenenergy $E$. Inversely, the eigenenergy $E$ may be interpreted as a chemical potential. Multiplying Eq. (A57) by $\rho/m$, integrating over the whole domain and using Eq. (A59) we get

$$NE = q\Theta_Q^2 + \int \rho h(\rho) \, dr + 2W + W_{\text{ext}}. \tag{A60}$$

Finally, according to Eq. (A57), an equilibrium state of the generalized (damped) GP equation, or of the generalized quantum (damped) Euler equations, satisfies the relation

$$\rho = h^{-1} \left( \frac{\mu}{m} - \frac{Q_g}{m} - \Phi - \Phi_{\text{ext}} \right). \tag{A61}$$

When $Q_g = \Phi = 0$, this equation directly determines the equilibrium distribution $\rho(r)$. More generally, Eq. (A61) is a differential, or an integrodifferential, equation for $\rho(r)$. An equilibrium state which just extremizes the free energy at fixed mass may, or may not, be stable. Considering the second order variations of the free energy, we find that the equilibrium state is stable if, and only if,

$$\delta^2 F = \frac{1}{2} \int h'(\rho)(\delta \rho)^2 \, dr + \frac{1}{2} \int \frac{\hbar^2}{8m^2} \rho_1^{-q} \rho_2 \left[ (2-q) \left( \frac{\Delta \rho}{\rho} - \frac{1}{2} (3-q) \frac{(\nabla \rho)^2}{\rho^2} \right) (\delta \rho)^2 + (\nabla \delta \rho)^2 \right] \, dr > 0 \tag{A62}$$

for all perturbations that conserve mass: $\int \delta \rho \, dr = 0$. This inequality can also be written as

$$\delta^2 F = \frac{1}{2} \int h'(\rho)(\delta \rho)^2 \, dr + \frac{1}{2} \int \delta \rho \Phi \, dr + \frac{\hbar^2}{8m^2} \rho_1^{-q} \int \left[ \nabla \left( \frac{\delta \rho}{\rho^{1-q/2}} \right) \right]^2 \, dr + \frac{\hbar^2}{8m^2} \rho_1^{-q} \int (2-q) \int \frac{\Delta (\rho^{1/2})}{\rho^{2-q/2}} (\delta \rho)^2 \, dr > 0. \tag{A63}$$

9. Functional derivatives

The results of Sec. 3.6 of [55] concerning the functional derivatives of $F$ remain valid in the present context provided that we make the substitution $Q \rightarrow Q_g$.

10. Virial theorem

The virial of the generalized quantum force is defined by

$$(W_{ii}^Q)_g = - \int \frac{\rho}{m} r \cdot \nabla Q_g \, dr. \tag{A64}$$

Using Eq. (A31) we get

$$(W_{ii}^Q)_g = \int P_{ii}^g \, dr, \tag{A65}$$

where $P_{ii}^g$ is given by Eq. (A34). It is important to note that $(W_{ii}^Q)_g \neq 2\Theta_Q^g$ when $q \neq 1$ contrary to the standard quantum mechanics case ($q = 1$) where $W_{ii}^Q = 2\Theta_Q$. This is because $\Theta_e + \Theta_Q^g$ does not correspond to the standard kinetic energy $\Theta = \frac{\hbar^2}{2m} \int |\nabla \psi|^2 \, dr$ when we make the Madelung transformation. Indeed, the generalized kinetic energy $\Theta_g = \Theta_e + \Theta_Q^g$ is given by

$$\Theta_g = \frac{\hbar^2}{2m^2} \int |\nabla \psi|^2 \, dr + \frac{\hbar^2}{8m^2} \psi_0^2 \left[ \frac{\nabla (|\psi|^2 / \psi_0^2)}{|\psi|^2 - \psi_0^2} \right]^2 \, dr - \frac{\hbar^2}{8m^2} \int \frac{[\nabla (|\psi|^2)]^2}{|\psi|^2} \, dr. \tag{A66}$$

Apart from this difference, the results of Sec. 4 and Appendix G of [55] remain valid. The scalar virial theorem associated with the generalized damped GP equation, or with the generalized quantum damped Euler equations, writes

$$\frac{1}{2} \dot{I} + \frac{1}{2} \epsilon \dot{I} + \omega^2 I = 2\Theta_e + \int P_{ii}^g + d \int P \, dr + W_{ii}, \tag{A67}$$
where $W_{ii}$ is the virial of the mean field force and we have considered the case of an external harmonic potential. In the strong friction limit $\xi \to +\infty$, the generalized quantum damped Euler equations reduce to the generalized quantum Smoluchowski equation and the scalar virial theorem becomes

$$\frac{1}{2} \dot{I} + \omega_0^2 I = \int P_{ii}^g + d \int P \, dr + W_{ii}. \tag{A68}$$

In all cases, the equilibrium scalar virial theorem writes

$$\int P_{ii}^g + d \int P \, dr + W_{ii} - \omega_0^2 I = 0. \tag{A69}$$

For classical systems ($\hbar = 0$) described by the damped Euler equations with an isothermal equation of state $P = \rho k_B T/m$, and in the absence of mean field force ($\Phi = 0$), the scalar virial theorem takes the form

$$\frac{1}{2} \dot{I} + \frac{1}{2} \xi \dot{I} + \omega_0^2 I = 2\Theta_e + dN k_B T. \tag{A70}$$

In the strong friction limit $\xi \to +\infty$, the damped Euler equations reduce to the Smoluchowski equation and the scalar virial theorem becomes

$$\frac{1}{2} \dot{I} + \omega_0^2 I = dN k_B T. \tag{A71}$$

At equilibrium, we get

$$dN k_B T - \omega_0^2 I = 0. \tag{A72}$$

We note that Eq. (A71) is a closed differential equation for $I$. Solving this equation and using $I = M \langle r^2 \rangle$ we directly obtain Eq. (41). In the absence of external potential ($\omega_0 = 0$), the foregoing equations reduce to

$$\frac{1}{2} \dot{I} + \frac{1}{2} \xi \dot{I} = 2\Theta_e + dN k_B T \tag{A73}$$

and

$$\frac{1}{2} \xi \dot{I} = dN k_B T. \tag{A74}$$

The solution of Eq. (A74) directly leads to Eq. (36).

11. Conservation of linear and angular momentum

Using hydrodynamic variables, the linear momentum of the system is given by

$$P = -i \frac{\hbar}{m} \int \psi^* \nabla \psi \, dr \Rightarrow P = \int \rho u \, dr. \tag{A75}$$

Taking its time derivative and using Eq. (A20), we obtain

$$\frac{dP}{dt} = -\xi P \quad \text{i.e.} \quad P(t) = P_0 e^{-\xi t}. \tag{A76}$$

To get this result, we have used the identity $\int \rho \nabla \Phi \, dr = 0$ (easily obtained from Eq. (A2) by interchanging the dummy variables $r$ and $r'$) and the identity $\int (\rho/m) \nabla Q_g \, dr = \int \partial_j P_{ij}^g \, dr = 0$ obtained from Eq. (A31). The linear momentum is conserved when $\xi = 0$.

Using hydrodynamic variables, the angular momentum of the system is given by

$$L = -i \frac{\hbar}{m} \int \psi^* (r \times \nabla) \psi \, dr \Rightarrow L = \int \rho r \times u \, dr. \tag{A77}$$

Taking its time derivative and using Eq. (A20), we obtain

$$\frac{dL}{dt} = -\xi L \Rightarrow L(t) = L_0 e^{-\xi t}. \tag{A78}$$
To get this result, we have used the virial identity (see Appendix G of [55])

\[ \int x_i \frac{\partial}{\partial t}(pu_{ij}) \, dx = \int pu_{i,j} \, dx + \delta_{ij} \int P \, dx + W_{ij} + W_{ij}^{ext} + (W_{ij}^Q)_g - \xi \int \rho x_{i,j} \, dx. \]  
(A79)

Since all the tensors on the right hand side except the last one are symmetric, we get

\[ \frac{d}{dt} \int \rho (x_{i,j} - x_{j,i}) \, dx = -\xi \int \rho (x_{i,j} - x_{j,i}) \, dx, \]  
(A80)

leading to Eq. (A78). The angular momentum is conserved when \( \xi = 0 \).

12. Generalized thermodynamics and Tsallis entropy

In Ref. [55] we have shown that the damped GP equation (68) is associated with a generalized thermodynamical formalism. The generalized entropy \( S \) that appears in this formalism is determined by the nonlinearity \( V(|\psi|^2) \) in the damped GP equation (58) or by the equation of state \( P(\rho) \) in the quantum damped Euler equation (60). In the strong friction limit \( \xi \to +\infty \), and when quantum effects are neglected (TF approximation), the damped GP equation (58) becomes equivalent to the generalized Smoluchowski equation (23). In that case, we recover the generalized thermodynamical formalism associated with NFP equations considered in Ref. [51]. The generalized thermodynamical formalism of Ref. [55] remains valid for the generalized damped GP equation (A1). The only modification to make with respect to Ref. [55] is to replace the von Weizsäcker functional (A45) by the generalized von Weizsäcker functional (A42) in the expression of the free energy \( F \).

The generalized free energy (A39) can be written as

\[ F = E_* + U = E_*$ \ - \ T_{eff} S, \]  
(A81)

where

\[ E_* = \Theta_c + \Theta_Q^2 + W + W_{ext}, \]  
(A82)

is the energy that includes the classical kinetic energy \( \Theta_c \), the generalized quantum kinetic energy \( \Theta_Q^2 \), the mean field energy \( W \), and the external potential energy \( W_{ext} \). It can be written explicitly as

\[ E_* = \int \rho \frac{\mu^2}{2} \, dx + \frac{1}{qm} \int \rho Q_g \, dx + \frac{1}{2} \int \rho \Phi \, dx + \int \rho \Phi_{ext} \, dx. \]  
(A83)

On the other hand, we have written the internal energy as \( U = -T_{eff} S \), where \( T_{eff} \) is an effective temperature and \( S = -\int C(\rho) \, d\rho \) is a generalized entropy determined by the potential \( V(\rho) \) according to the relation \( C(\rho) = V(\rho)/T_{eff} \). We note that \( h(\rho) = T_{eff} C'(\rho) \) so that the equilibrium state determined by Eq. (A61) can be rewritten as

\[ \rho = (C')^{-1} \left[ \frac{1}{T_{eff}} \left( \frac{\mu}{m} - \frac{Q_g}{m} - \Phi - \Phi_{ext} \right) \right]. \]  
(A84)

Several examples of generalized entropies associated with the generalized damped GP equation (68) have been given in [55]. We consider below two of them.

The generalized logarithmic damped GP equation (83) is equivalent to the generalized quantum damped Euler equations (72) and (60) with the isothermal equation of state (55). In that case \( h(\rho) = (k_B T/m) \ln \rho \) and \( V(\rho) = (k_B T/m) \rho (\ln \rho - 1) \). The associated entropy is the Boltzmann entropy

\[ S_B = -k_B \int \rho (\ln \rho - 1) \, d\rho \]  
(A85)

and the free energy (A39) can be written as

\[ F_B = E_* - T S_B. \]  
(A86)

The equilibrium state (A34) is the Boltzmann distribution

\[ \rho = e^{-(m \Phi + m \Phi_{ext} + Q_g - \mu)/k_B T}. \]  
(A87)
The generalized power-law damped GP equation \((70)\) is equivalent to the generalized quantum damped Euler equations \((59)\) and \((60)\) with the polytropic equation of state \((4)\). In that case \(h(\rho) = [K(\gamma/(\gamma-1)]\rho^{\gamma-1}\) and \(V(\rho) = [K(\gamma/(\gamma-1)]\rho^\gamma\). The associated entropy is the Tsallis entropy\(^{20}\)

\[
S_\gamma = -\frac{1}{\gamma-1} \int (\rho^\gamma - \rho) \, dr
\]  

(A88)

and the free energy \((A89)\) can be written as

\[
F_\gamma = E_\star - KS_\gamma.
\]  

(A89)

We note that the polytropic constant \(K\) plays the role of a generalized temperature: \(T_\text{eff} = K\). The equilibrium state \((A83)\) is the Tsallis distribution

\[
\rho = \left(\frac{\gamma-1}{\gamma}\right)^{\frac{1}{\gamma-1}} \left[\frac{1}{K} \left(\frac{\mu - \Phi - \Phi_{\text{ext}} - Q_\rho}{m}\right)\right]^{\frac{1}{\gamma-1}}.
\]  

(A90)

For \(\gamma \to 1\), the Tsallis entropy, the Tsallis free energy and the Tsallis distribution \((A88)-(A90)\) return the Boltzmann entropy, the Boltzmann free energy and the Boltzmann distribution \((A84)-(A87)\) (see \([55]\) for more details).

Remark: The generalized free energy associated with the logotropic equation of state \((6)\) can be written as

\[
F = E_\star - A S_L, \quad S_L = \int \ln \rho \, dr
\]  

(A91)

is the logarithmic entropy \([47]\). The corresponding equilibrium distribution is the Lorentzian. The logarithmic entropy, the logarithmic free energy and the Lorentzian distribution can be obtained from the Tsallis entropy, the Tsallis free energy and the Tsallis distribution \((A88)-(A90)\) in the limit \(\gamma \to 0\), \(K \to +\infty\) with \(K\gamma = A\) (see \([47, 52]\) for more details).

Appendix B: Self-similar solution

1. Generalized damped GP equation

We consider the generalized damped power-law GP equation

\[
\frac{i\hbar}{\partial t} \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + \frac{\hbar^2}{2m} \frac{\Delta |\psi|^2}{|\psi|^2} \psi - \frac{\hbar^2}{2m} \frac{\Delta (|\psi|^2/\psi_0)^{q}}{(|\psi|^2/\psi_0)^{2-q}} \psi + \frac{K m \gamma}{\gamma - 1} |\psi|^{2(\gamma-1)} \psi + m \Phi_{\text{ext}} \psi - i \frac{\hbar}{2} |v| \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \left\langle \ln \left( \frac{\psi}{\psi^*} \right) \right\rangle \right] \psi
\]  

(B1)

with the harmonic potential \([13]\). This equation is associated with generalized quantum mechanics (characterized by the parameter \(q\)) and generalized Tsallis thermodynamics (characterized by the parameter \(\gamma\)). We show below that Eq. \((B1)\) admits a self-similar solution with a Tsallis invariant profile when the parameters \(q\) and \(\gamma\) satisfy the relation

\[
q = 2\gamma - 1.
\]  

(B2)

In that case, Eq. \((B1)\) can be rewritten as

\[
\frac{i\hbar}{\partial t} \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + \frac{\hbar^2}{2m} \frac{\Delta |\psi|^2}{|\psi|^2} \psi - \frac{\hbar^2}{2m} \frac{\Delta (|\psi|^2/\psi_0)^{2\gamma-1}}{(|\psi|^2/\psi_0)^{3-2\gamma}} \psi + \frac{K m \gamma}{\gamma - 1} |\psi|^{2(\gamma-1)} \psi + m \Phi_{\text{ext}} \psi - i \frac{\hbar}{2} |v| \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \left\langle \ln \left( \frac{\psi}{\psi^*} \right) \right\rangle \right] \psi.
\]  

(B3)

\(^{20}\) A power-law entropic functional similar to Eq. \((A88)\), but written in phase space as \(S = -\int f^q \, dv\), was introduced by Ipser \([103]\) in a dynamical (Vlasov) context, in relation to Eddington’s stellar polytropes \([102]\). Ipser also introduced functionals of the form \(S = -\int C(f) \, dv\) (where \(C(f)\) is convex) that we now call generalized entropic functionals \([51, 52]\). Actually, Ipser’s polytropic functional \(S = -\int \rho^\gamma \, dr\) corresponds to the first term of Eq. \((A88)\). In principle the second term of Eq. \((A88)\), which is proportional to the mass \(M\), is a constant that can be omitted in the entropy. However, an interest of the expression \((A88)\) given by Tsallis \([14]\) is that it reduces to the Boltzmann entropy for \(\gamma \to 1\) thanks to L’Hospital’s rule (see Refs. \([14, 104]\) for more details and for a thorough discussion between dynamical and thermodynamical stability).
For $q = \gamma = 1$, we recover the damped logarithmic GP equation
\[
i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + 2k_B T \ln |\psi| \psi + m\Phi_{\text{ext}} \psi - i\frac{\hbar}{2}\xi \left[ \ln \left( \frac{\psi}{\psi^*} \right) - \langle \ln \left( \frac{\psi}{\psi^*} \rangle \right) \right] \psi. \tag{B4}
\]
This equation is associated with standard quantum mechanics ($q = 1$) and standard thermodynamics ($\gamma = 1$). We show below that this equation admits a self-similar solution with a Gaussian invariant profile.

**Remark:** In order to recover formally the logarithmic GP equation from the power-law GP equation when $\gamma \to 1$, we should write the power-law nonlinearity as $\frac{K m}{\gamma - 1} (|\psi|^2 (\gamma - 1) - 1)$. However, the constant can be absorbed in the potential energy so, for economy, we do not write it explicitly.

### 2. Generalized quantum damped Euler equations

Using the Madelung transformation (see Appendix A), the generalized damped power-law GP equation (B3) is equivalent to the generalized quantum damped polytropic Euler equations
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{B5}
\]
\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P - \nabla \Phi_{\text{ext}} - \frac{1}{m} \nabla Q_g - \xi \mathbf{u}, \tag{B6}
\]
involving the harmonic potential (3), the polytropic equation of state (4) and the generalized quantum potential (72).

For $\gamma = 1$, we recover the standard quantum damped isothermal Euler equations involving the harmonic potential (3), the isothermal equation of state (5) and the standard quantum potential (61).

### 3. Scaling ansatz

We look for a self-similar solution of Eqs. (B5) and (B6) of the form
\[
\rho(r, t) = \frac{M}{R(t)^d} f\left( \frac{r - \chi(t)r_0}{R(t)} \right), \quad \mathbf{u}(r, t) = H(t) \mathbf{r} + B(t) r_0. \tag{B7}
\]
We have assumed that the velocity field is an affine function of $\mathbf{r}$. Defining
\[
x = \frac{r - \chi(t)r_0}{R(t)}, \quad (\mathbf{r} = R(t)x + \chi(t)r_0), \tag{B8}
\]
we can rewrite Eq. (B7) as
\[
\rho(r, t) = \frac{M}{R(t)^d} f(x), \quad \mathbf{u}(r, t) = H(t)R(t)x + [H(t)\chi(t) + B(t)]r_0. \tag{B9}
\]
In the foregoing equations $R(t)$ is the typical size (radius) of the BEC at time $t$, $(\mathbf{r})(t) = \chi(t)r_0$ represents the position of the center of the BEC, and $f(x)$ is the invariant density profile. We assume that the self-similar density profile contains all the mass ($M = \int \rho(r, t) d\mathbf{r}$) so that $\int f(x) d\mathbf{x} = 1$.

The continuity equation (B5) can be rewritten as
\[
\frac{\partial \ln \rho}{\partial t} + \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \ln \rho = 0. \tag{B10}
\]
From Eq. (B7), we obtain
\[
\frac{\partial \ln \rho}{\partial t} = -\left( \frac{\dot{R}}{R} \mathbf{x} + \frac{\dot{\chi}}{R} r_0 \right) \cdot \nabla \ln f - \dot{R} \frac{\dot{R}}{R} \nabla \ln f, \quad \nabla \ln \rho = \frac{1}{R} \nabla \chi \ln f, \quad \nabla \cdot \mathbf{u} = dH. \tag{B11}
\]
Substituting the foregoing relations into Eq. (B10), we get
\[
\left( H - \frac{\dot{R}}{R} \right) (d + x \cdot \nabla_x \ln f) + \frac{r_0}{R} \cdot \nabla_x \ln f (H \chi + B - \dot{\chi}) = 0. \tag{B12}
\]
This equation must be satisfied for all \( x \). This implies
\[
H(t) = \frac{\dot{R}}{R} \quad \text{and} \quad B(t) = \dot{\chi} - H \chi. \tag{B13}
\]
We note the formal analogy between the first term and the Hubble constant in cosmology. Using Eq. (B13) we can rewrite Eq. (B7) as
\[
u(r, t) = H(t)(r - \chi(t)r_0) + \dot{\chi}r_0. \tag{B14}
\]
Similarly, Eq. (B9) can be rewritten as
\[
u(r, t) = \dot{R}x + \dot{\chi}r_0. \tag{B15}
\]
Comparing Eqs. (B8) and (B15) we note that
\[
\frac{\partial \nu}{\partial t} = \left( \frac{\partial}{\partial t} \right)_x \frac{\partial \nu}{\partial x} \tag{B16}
\]
According to Eq. (B13), we have
\[
\dot{H} + H^2 = \frac{\dot{R}}{R}, \quad \dot{B} + HB = \dot{\chi} - (H + H^2)\chi. \tag{B17}
\]
Combining Eqs. (B16) and (B17), we obtain
\[
\frac{D\nu}{Dt} = \frac{\partial \nu}{\partial t} + (u \cdot \nabla)u = \dot{R}x + \dot{\chi}r_0. \tag{B18}
\]
Comparing Eqs. (B8) and (B18) we note that \( \frac{D\nu}{Dt} = (\partial^2 r / \partial t^2)_x \) where the time derivative is taken at fixed \( x \).

For the polytropic equation of state \( (4) \), the pressure term in the right hand side of the generalized quantum damped Euler equation (B6) is given by
\[
- \frac{1}{\rho} \nabla P = -K \gamma \rho^{-2} \nabla \rho. \tag{B19}
\]
With the scaling ansatz from Eq. (B7), we obtain
\[
- \frac{1}{\rho} \nabla P = -K \gamma \frac{M^{\gamma-1}}{R^{d\gamma-d+1}} f^{\gamma-2} \nabla_x f. \tag{B20}
\]
Assuming that \( f \) depends only on \( x = |x| \), we get
\[
- \frac{1}{\rho} \nabla P = -K \gamma \frac{M^{\gamma-1}}{R^{d\gamma-d+1}} f^{\gamma-2} f'(x) \frac{1}{x}. \tag{B21}
\]
On the other hand, we have
\[
- \nabla \Phi_{ext} = -\omega_0^2 r = -\omega_0^2 (R \dot{x} + \dot{\chi} r_0) \tag{B22}
\]
and
\[
\xi u = -\xi \dot{R}x - \xi \dot{\chi} r_0. \tag{B23}
\]
The generalized quantum force \(-(1/m) \nabla Q_g\) will be considered later (see Appendix B5). For the moment we do not take it into account. Substituting Eqs. (B18), (B21), (B22) and (B23) into the generalized quantum damped polytropic Euler equation (B6), we get
\[
\ddot{\chi} + \xi \dot{\chi} + \omega_0^2 \chi = 0 \tag{B24}
\]
and
\[ \ddot{R} + \xi \dot{R} + \omega_0^2 R = -K\gamma \frac{M^{\gamma-1}}{R^{d\gamma-d+1}} f^{\gamma-2} \frac{f'(x)}{x}. \] (B25)

The first equation determines the evolution of the center \( \langle r \rangle = \chi(t)r_0 \) of the BEC. We see that it follows the classical equation of motion of a damped particle in a harmonic potential. The general solution the differential equation (B24) is given in Appendix B 6. Below, we focus on Eq. (B25). The variables of position and time separate provided that
\[ f^{\gamma-2} \frac{df}{dx} + 2Ax = 0 \] (B26)
and
\[ \ddot{R} + \xi \dot{R} + \omega_0^2 R = 2AK\gamma \frac{M^{\gamma-1}}{R^{d\gamma-d+1}}, \] (B27)
where \( A \) is a constant (the factor 2 has been introduced for convenience). These differential equations determine the invariant density profile \( f(x) \) of the BEC and the evolution of its radius \( R(t) \).

Remark: We can add in the external potential a term of the form \( \Phi_{\text{ext}} = -A \cdot \mathbf{r} \) corresponding to a constant force. The self-similar solution remains valid provided that \( x_0 \) lies in the direction of \( A \). In that case, Eq. (B24) is replaced by
\[ \ddot{\chi} + \xi \dot{\chi} + \omega_0^2 \chi = \frac{A}{x_0}. \] (B26)
The other equations are not altered. We also note that the self-similar solution remains valid when \( \omega_0(t) \) and \( A(t) \) depend on time.

4. Invariant density profile and system’s size

The differential equation (B26) determining the invariant density profile of the system\(^{21} \) can be integrated into
\[ f(x) = \left[ C - (\gamma - 1)Ax^2 \right]^{1/(-\gamma-1)}, \] (B28)
where \([x]_+ = x \) if \( x \geq 0 \) and \([x]_+ = 0 \) if \( x \leq 0 \). Therefore, the invariant profile is given by a Tsallis distribution of index \( \gamma \). We can take \( C = A \) without loss of generality. Denoting this constant by \( Z^{1-\gamma} \), we get
\[ f(x) = \frac{1}{Z} \left[ 1 - (\gamma - 1)x^2 \right]_+^{1/(-\gamma-1)}, \] (B29)
where \( Z \) is determined by the normalization condition \( \int f(x) \, dx = 1 \). This yields
\[ Z = \int_0^{x_{\text{max}}} \left[ 1 - (\gamma - 1)x^2 \right]_+^{1/(-\gamma-1)} S_d x^{d-1} \, dx, \] (B30)
where \( x_{\text{max}} = 1/\sqrt{\gamma-1} \) if \( \gamma \geq 1 \) and \( x_{\text{max}} = +\infty \) if \( (d-2)/d < \gamma \leq 1 \) (the distribution is not normalizable when \( \gamma \leq (d-2)/d \)). We recall that \( S_d = 2\pi^{d/2}/\Gamma(d/2) \) is the surface of the unit sphere in \( d \) dimensions. The integrals can be expressed in terms of Gamma functions leading to Eqs. (10) and (11). On the other hand, the differential equation (B27) determining the evolution of the system’s radius becomes
\[ \ddot{R} + \xi \dot{R} + \omega_0^2 R = 2Z^{1-\gamma}K\gamma \frac{M^{\gamma-1}}{R^{d\gamma-d+1}}. \] (B31)

For \( \gamma = 1 \), corresponding to the isothermal equation of state \( \mathbf{1} \), the differential equation (B26) reduces to
\[ \frac{df}{dx} + 2Ax f(x) = 0. \] (B32)
We can take \( A = 1 \) without loss of generality. In that case, Eq. (B32) leads to the Gaussian invariant profile
\[ f(x) = \frac{1}{\pi^{d/2}} e^{-x^2}, \] (B33)

\(^{21} \) This differential equation was enlightened in Sec. VI.A. of \( \mathbf{48} \).
where we have used the normalization condition $\int f(x) \, dx = 1$. We can check that the Gaussian distribution \[B33\] is the limiting form of the Tsallis distribution \[B29\] for $\gamma \to 1$. On the other hand, the differential equation \[B27\] determining the evolution of the system’s radius reduces to

$$\ddot{R} + \xi \dot{R} + \omega_0^2 R = \frac{2k_B T}{mR}.$$ \(B34\)

This is the limiting form of the differential equation \[B31\] for $\gamma \to 1$.

5. Accounting for the quantum potential

When $q = 2\gamma - 1$, as assumed at the beginning of this Appendix, the generalized quantum potential is given by Eq. \[B2\]. When $\rho(r,t)$ is given by Eq. \[B7\] where $f(x)$ is the Tsallis distribution \[B29\], we have the relation (see Appendix \[C1\])

$$-\frac{1}{m} \nabla Q_g = \frac{\hbar^2}{m^2} \rho_0^{2(1-\gamma)} \frac{M^{2(\gamma-1)}}{R^{2d(\gamma-1)+3}} \frac{2\gamma - 1}{Z^{2(\gamma-1)}} [1 + d(\gamma - 1)] x.$$ \(B35\)

Since the generalized quantum force \[B35\] is proportional to $x$ with a proportionality constant depending only on $t$, and not on $x$, we conclude that the generalized quantum damped polytropic Euler equations \[B5\] and \[B6\] admit a Tsallis self-similar solution. The differential equation determining the evolution of the radius $R(t)$ is given by

$$\ddot{R} + \xi \dot{R} + \omega_0^2 R = 2Z^{1-\gamma} K \gamma \frac{M^{\gamma-1}}{R^{d(\gamma-1)+1}} + \frac{\hbar^2}{m^2} \rho_0^{2(1-\gamma)} \frac{M^{2(\gamma-1)}}{R^{2d(\gamma-1)+3}} \frac{2\gamma - 1}{Z^{2(\gamma-1)}} [1 + d(\gamma - 1)].$$ \(B36\)

For $q = \gamma = 1$, the standard quantum potential is given by Eq. \[B1\]. When $\rho(r,t)$ is given by Eq. \[B7\] where $f(x)$ is the Boltzmann distribution \[B33\], we have the relation (see Appendix \[C1\])

$$-\frac{1}{m} \nabla Q = \frac{\hbar^2}{m^2} \frac{1}{R^3} x.$$ \(B37\)

Using the same argument as before, we conclude that the standard quantum damped isothermal Euler equations \[B5\] and \[B6\] admit a Boltzmann self-similar solution. The differential equation determining the evolution of the radius $R(t)$ is given by

$$\ddot{R} + \xi \dot{R} + \omega_0^2 R = \frac{2k_B T}{mR} + \frac{\hbar^2}{m^2 R^3}.$$ \(B38\)

This is the limiting form of the differential equation \[B36\] for $\gamma \to 1$.

6. The function $\chi(t)$

The differential equation determining the evolution of $\chi(t)$ is given by

$$\ddot{\chi} + \xi \dot{\chi} + \omega_0^2 \chi = 0.$$ \(B39\)

This is the classical equation of motion of a damped particle submitted to a one dimensional harmonic potential. The solution of this equation is well-known. We have to distinguish three cases:

(i) When $\xi > 2\omega_0$,

$$\chi(t) = e^{-\xi t/2} \left( A e^{\frac{\xi}{2} \sqrt{\xi^2 - 4\omega_0^2} t} + B e^{-\frac{\xi}{2} \sqrt{\xi^2 - 4\omega_0^2} t} \right).$$ \(B40\)

(ii) When $\xi < 2\omega_0$,

$$\chi(t) = e^{-\xi t/2} \left[ A \cos \left( \frac{1}{2} \sqrt{4\omega_0^2 - \xi^2} t \right) + B \sin \left( \frac{1}{2} \sqrt{4\omega_0^2 - \xi^2} t \right) \right].$$ \(B41\)

(iii) When $\xi = 2\omega_0$,

$$\chi(t) = e^{-\xi t/2} \left( At + B \right).$$ \(B42\)
In these expressions, $A$ and $B$ are two constants of integration determined by the initial conditions. In particular, for $\xi = \omega_0 = 0$ we have

$$\chi(t) = \frac{\omega}{r_0} t + 1. \quad (B43)$$

In the strong friction limit $\xi \to +\infty$, the differential equation determining the evolution of $\chi(t)$ reduces to

$$\dot{\chi} + \frac{\omega_0^2}{\xi}\chi = 0. \quad (B44)$$

Its solution is

$$\chi(t) = e^{-\frac{\omega_0^2}{\xi} t}, \quad (B45)$$

where we have assumed that the density profile is initially centered on $r_0$ so that $\chi(0) = 1$. In the absence of external force ($\omega_0 = 0$), Eq. (B44) reduces to

$$\dot{\chi} = 0 \quad (B46)$$

and its solution is

$$\chi(t) = 1. \quad (B47)$$

7. Moments of the distribution

We define the first and second moments of the distribution $\rho(r,t)$ by

$$\langle r \rangle = \frac{1}{M} \int \rho r \, dr, \quad (B48)$$

$$\langle r^2 \rangle = \frac{1}{M} \int \rho r^2 \, dr = \frac{I}{M}, \quad (B49)$$

where $I$ is the moment of inertia (A49). Using Eqs. (B7) and (B8), and recalling that $f(x)$ is spherically symmetric, we get

$$\langle r \rangle = \chi(t)r_0, \quad \langle v \rangle = \frac{d\langle r \rangle}{dt} = \dot{\chi}r_0, \quad (B50)$$

and

$$\langle r^2 \rangle = R^2(t)\langle x^2 \rangle + \chi^2(t)r_0^2, \quad \langle (r - \chi(t)r_0)^2 \rangle = R^2(t)\langle x^2 \rangle. \quad (B51)$$

For the Gaussian distribution (13), we have

$$\langle x^2 \rangle = \frac{d}{2}. \quad (B52)$$

For the Tsallis distribution (9), we have

$$\langle x^2 \rangle = \frac{d}{d(\gamma - 1) + 2\gamma}. \quad (B53)$$

The variance exists provided that $\gamma > d/(d + 2)$. In particular, it is not defined when $\gamma < 0$. 
8. The wavefunction

From Eqs. (A8) and (B15) we find that the action (phase) is given by

\[ S(r, t) = \frac{1}{2} m\dot{R} r^2 + m R \chi \dot{r}_0 \cdot \mathbf{x} + S_0(t), \]  

(B54)

where \( S_0(t) \) is a “constant” of integration that can depend on time. Using Eqs. (A7), (B7) and (B54), the wavefunction can be written as

\[ \psi(r, t) = \frac{\sqrt{M}}{R(t)^{d/2}} e^{-\frac{(r - \chi(t)r_0)^2}{2R(t)^2}} e^{i \frac{\Phi}{2}} \left[ \frac{1}{R(t)} (r - \chi(t)r_0)^2 + \dot{\chi} r_0 (r - \chi(t)r_0) + \frac{S_0(t)}{m} \right]. \]  

(B55)

It can either represent the condensate wavefunction of a BEC or the wave packet of a single quantum particle (see footnote 16). Depending on the interpretation \( r(t) = \chi(t)r_0 \) represents either the position of the center of the BEC or the position of the center of the wavepacket associated with the quantum particle, and \( v(t) = \dot{\chi}(t)r_0 \) represents its velocity (see Appendix B.7). According to Eq. (B34), these quantities follow the classical equations of motion of a damped particle in a harmonic potential. This is a particular case of the Ehrenfest theorem (see Appendix C of [84]). On the other hand, \( R(t) \) is a measure of the size of the BEC or a measure of the width of the wavepacket (see Appendix B.7). Stationary solutions can be interpreted as the equilibrium state of the BEC or the localization of the wave packet of a quantum particle.

For the Gaussian self-similar solution, using Eq. (B3), we get

\[ \psi(r, t) = \frac{\sqrt{M}}{R(t)^{d/2}} e^{-\frac{(r - \chi(t)r_0)^2}{2R(t)^2}} e^{i \frac{\Phi}{2}} \left[ \frac{1}{R(t)} (r - \chi(t)r_0)^2 + \dot{\chi} r_0 (r - \chi(t)r_0) + \frac{S_0(t)}{m} \right]. \]  

(B56)

For a free quantum particle with \( \omega_0 = \xi = T = 0 \), \( R(t) \) is given by Eq. (67), \( \chi(t) \) is given by Eq. (B47), and the wavefunction (B56) takes the form

\[ \psi(r, t) = \frac{\sqrt{M}}{\pi R_0^2 \left( 1 + \frac{\hbar^2 \omega_0^2}{m R_0^2} \right)^{d/4}} e^{-\frac{(r - \chi(t)r_0)^2}{2R_0^2}} e^{i \frac{\Phi}{2}} \left[ \frac{1}{R_0} (r - \chi(t)r_0)^2 + \dot{\chi} r_0 (r - \chi(t)r_0) + \frac{S_0(t)}{m} \right]. \]  

(B57)

with \( p = mv \) and \( E = \frac{p^2}{2m} \). For \( t \to +\infty \), the radius \( R(t) \to +\infty \) [see Eq. (67)] because of the spreading of the wave packet. In that case, Eq. (B57) reduces to a pure plane wave \( \psi(r, t) \propto e^{i \frac{\Phi}{2}} (p \cdot r - Et) \) and the quantum particle is completely delocalized. On the other hand, when the differential equation (B38) determining the evolution of the radius \( R(t) \) has a stable stationary solution, \( R(t) = R_0 \) [22] the logarithmic GP equation (63) admits a solitonic solution of the form

\[ \psi(r, t) = \frac{\sqrt{M}}{R_0^{d/2}} e^{-\frac{(r - \chi(t)r_0)^2}{2R_0^2}} e^{i \frac{\Phi}{2}} \left[ \dot{\chi} r_0 (r - \chi(t)r_0) + \frac{S_0(t)}{m} \right]. \]  

(B58)

In particular, when \( \xi = \omega_0 = 0 \), the foregoing equation reduces to

\[ \psi(r, t) = \frac{\sqrt{M}}{R_0^{d/2}} e^{-\frac{(r - \chi(t)r_0)^2}{2R_0^2}} e^{i \frac{\Phi}{2}} (p \cdot r - Et). \]  

(B59)

This solution is called a gausson [105]. It corresponds to a uniformly moving Gaussian wave packet modulated by the de Broglie plane wave \( (p = h\mathbf{k}, E = \hbar \omega) \). The gausson arises from the invariance of the logarithmic GP equation under Galilean transformation. In this manner, we can use the static solution of the logarithmic GP equation to generate

\[ \text{In that case we have to assume that } T < 0 \text{ (see Sec. 6.2 of [84]).} \]
uniformly moving solutions characterized by the velocity $\mathbf{v}$ and the initial position $\mathbf{r}_0$. The particle is localized in a region of size [see Eq. (B38)]

$$R_e = \frac{\hbar}{\sqrt{2mk_BT}}$$  \hspace{1cm} (B60)

which has the form of a de Broglie length (with a negative temperature). If, for curiosity, we compare $R_e$ with the classical radius of the electron $r_e = e^2/m_e c^2 = 2.82 \times 10^{-15} \text{ m}$, we obtain a temperature scale $k_BT = m_e e^4 \hbar^2/2e^4 = m_e c^2/2\alpha^2 = 4.80 \text{ GeV}$, where $\alpha = e^2/\hbar c \approx 1/137$ is the fine-structure constant.

For the Tsallis self-similar solution, using Eq. (9), we get

$$\psi(r, t) = \frac{\sqrt{M}}{Z^{1/2}R(t)^{d/2}} \left[ 1 - (\gamma - 1) \left( \frac{\mathbf{r} - \chi(t)\mathbf{r}_0}{R(t)} \right)^{2\gamma/(\gamma - 1)} \right] e^{i \frac{\hbar}{\sqrt{2m^2c^4}} \left[ \frac{1}{2}(r-\chi(t)r_0)^2 + \chi r_0 \cdot \mathbf{r} - \gamma^2 r_0 \cdot \mathbf{r}_0 - \frac{\hbar^2}{2m^2c^4} \right]}. \hspace{1cm} (B61)$$

When the differential equation (B36) determining the evolution of the radius $R(t)$ has a stable stationary solution, $R(t) = R_e$, the generalized power-law GP equation (75) admits a solitonic solution of the form

$$\psi(r, t) = \frac{\sqrt{M}}{Z^{1/2}R_e^{d/2}} \left[ 1 - (\gamma - 1) \left( \frac{\mathbf{r} - \chi(t)\mathbf{r}_0}{R_e} \right)^{2\gamma/(\gamma - 1)} \right] e^{i \frac{\hbar}{\sqrt{2m^2c^4}} \left[ \frac{1}{2}(r-\chi(t)r_0)^2 + \chi r_0 \cdot \mathbf{r} - \gamma^2 r_0 \cdot \mathbf{r}_0 - \frac{\hbar^2}{2m^2c^4} \right]}. \hspace{1cm} (B62)$$

In particular, when $\xi = \omega_0 = 0$, the foregoing equation reduces to

$$\psi(r, t) = \frac{\sqrt{M}}{Z^{1/2}R_e^{d/2}} \left[ 1 - (\gamma - 1) \left( \frac{\mathbf{r} - \mathbf{r}_0 - \mathbf{v} t}{R_e} \right)^{2\gamma/(\gamma - 1)} \right] e^{i \frac{\hbar}{\sqrt{2m^2c^4}} \left[ \frac{1}{2}(r-\mathbf{r}_0 - \mathbf{v} t)^2 + \mathbf{r}_0 \cdot \mathbf{v} \right]}. \hspace{1cm} (B63)$$

This solution could be called a $\gamma$-gaussian. It is localized in a region of size [see Eq. (B36)]

$$R_e = \left\{ \frac{\hbar^2}{2m^2c^4} \left( \frac{M}{\gamma} \right)^{1/2} \frac{2^{(1-\gamma)} M^{\gamma - 1} 2\gamma - 1}{\gamma \left[ 1 + d(\gamma - 1) \right]} \right\}^{1/2 + d(\gamma - 1)}. \hspace{1cm} (B64)$$

**Remark:** The self-similar solution obtained in the previous subsections can also be obtained by substituting the Ansatz (B55) for the wavefunction into the generalized damped power-law GP equation (B1) and separating real and imaginary parts. However, the algebra is more complicated (and more obscure) than using the hydrodynamic representation of the GP equation.

9. **Bohm’s Lagrangian point of view**

The fluid equations (B55) and (B60) arising from the Madelung transformation have been written in the Eulerian point of view, i.e., the velocity field $\mathbf{u}(\mathbf{r}, t)$ is calculated at a fixed position $\mathbf{r}$. Alternatively, following Bohm [97, 98], we can adopt a Lagrangian point of view and consider the motion of “fluid particles”, which form an ensemble of “Bohmian particles”, with position $\mathbf{r}_B(t)$. Their equation of motion is obtained by writing

$$\frac{d\mathbf{r}_B}{dt} = \mathbf{u}(\mathbf{r}_B(t), t). \hspace{1cm} (B65)$$

A given Bohmian particle can be labeled by its initial position $\mathbf{r}_{B,0}$. Now, for the self-similar solution considered in this Appendix, the velocity field $\mathbf{u}(\mathbf{r}, t)$ is given by Eq. (B15). Using Eq. (B3) it can be rewritten as

$$\mathbf{u}(\mathbf{r}, t) = \frac{\dot{R}}{R} (\mathbf{r} - \langle \mathbf{r} \rangle) + \langle \dot{\mathbf{r}} \rangle, \hspace{1cm} (B66)$$

where we recall that $\langle \mathbf{r} \rangle = \chi(t)\mathbf{r}_0$ denotes the position of the center of the wavepacket, which may also be interpreted as the position of a “classical particle” following the Newtonian equation of motion (B39) (see Appendix B3). Combining Eqs. (B65) and (B66), we obtain

$$\frac{d}{dt} (\mathbf{r}_B - \langle \mathbf{r} \rangle) = \frac{\dot{R}}{R} (\mathbf{r}_B - \langle \mathbf{r} \rangle), \hspace{1cm} (B67)$$
which is immediately integrated into
\[
(r_B - \langle r \rangle)(t) = \frac{R(t)}{R_0} (r_B - \langle r \rangle_0). \tag{B68}
\]

Interestingly, this equation gives the evolution of the separation between a “Bohmian particle” with position \(r_B(t)\) and the “classical particle” (or the center of the wave packet) with position \(\langle r \rangle(t)\). We note that the Bohmian particle which coincides with the classical particle initially \((r_B,0) = (\langle r \rangle_0)\), coincides with it for all times \((r_B(t) = \langle r \rangle(t))\).

As a simple example, let us consider the case of a free quantum particle with \(\omega_0 = T = 0\) described by the standard Schrödinger equation. In the nondissipative limit \(\xi = 0\), using Eqs. (B17), (B14) and Eq. (B43), we obtain
\[
u_B(t) = (r_B,0) - \langle r \rangle_0 \sqrt{1 + \frac{\hbar^2 t^2}{m^2 R_0^2}} + v. \tag{B71}
\]

Solving Eq. (B65) with this velocity field, we find that
\[
r_B(t) - \langle r \rangle_0 = (r_B,0) - \langle r \rangle_0 \left( 1 + \frac{4\hbar^2 t}{\xi m^2 R_0^2} \right)^{1/4}, \tag{B73}
\]
\[
v_B(t) = (r_B,0) - \langle r \rangle_0 \left( 1 + \frac{4\hbar^2 t}{\xi m^2 R_0^2} \right)^{-3/4} \frac{\hbar^2}{\xi m^2 R_0^2}. \tag{B74}
\]

The interpretation is essentially the same as given previously, except that the wavepacket does not move \((v = 0)\) in the present case. The extension of the preceding results to the generalized Schrödinger equation associated with Tsallis distributions will be given in a future contribution [57].

10. Another self-similar solution

Let us look for a self-similar solution of Eqs. (B55) and (B60) of the form of Eq. (B7) without assuming the relation (B2), i.e., using the general expression (A11) of the quantum potential. Substituting Eq. (B7) into Eq. (A11) and

23 For a pure plane wave, the Bohmian particles all have the same velocity \(v_B = u = \nabla S/m = h\hbar/km = p/m = v\). On the other hand, the wave moves with a phase velocity \(v_\phi = \omega/k = E/p = p/2m\). They differ by a factor 2.
Interestingly, this is the polytropic index of a nonrelativistic Fermi gas at the radius \( q \).

The variables of position and time separate provided that

\[
\frac{1}{m} \nabla Q_g = \frac{\hbar^2}{2m^2 \rho_0^{-1}} \frac{1}{R^3} \left( \frac{M}{R^d} \right)^{q-1} \frac{1}{x \, dx} \left[ \frac{\Delta_x f(x)^{q/2}}{f(x)^{1-q/2}} \right] x. \tag{B75}
\]

Substituting Eqs. \((B18), (B21), (B22), (B23)\) and \((B75)\) into the generalized quantum damped polytropic Euler equation \((B6)\), we get Eq. \((B24)\) and

\[
\tilde{R} + \xi \dot{R} + \omega_0^2 R = -K \gamma \frac{M^{q-1}}{R^{d+q-1}} \frac{f''(x)}{f(x)} \frac{f'(x)}{x} + \frac{\hbar^2}{2m^2 \rho_0^{-1}} \frac{1}{R^3} \left( \frac{M}{R^d} \right)^{q-1} \frac{1}{x \, dx} \left[ \frac{\Delta_x f(x)^{q/2}}{f(x)^{1-q/2}} \right]. \tag{B76}
\]

In general, the variables of position and time do not separate so that Eqs. \((B35)\) and \((B6)\) do not systematically admit a self-similar solution. An exception, as we have seen, is when \( q = 2 \gamma - 1 \) because, in that case, the terms depending on \( x \) in Eq. \((B76)\) become constant for the Tsallis distribution \((B29)\). Indeed, one has (see Appendix C1):

\[
f(x)^{q/2} = 2 Z^1 \gamma, \quad \frac{1}{x \, dx} \left[ \frac{\Delta_x f(x)^{q-1/2}}{f(x)^{3/2-q}} \right] = \frac{4}{Z^2(\gamma-1)} \left( \gamma - \frac{1}{2} \right) \left[ d(\gamma - 1) + 1 \right]. \tag{B77}
\]

In that case, we recover Eq. \((B26)\). Another exception is when the two terms on the right hand side of Eq. \((B76)\) have the same time dependence. This imposes \( d \gamma - d + 1 = 3 + d(q - 1) \), i.e.,

\[
\gamma = q + \frac{2}{d}. \tag{B78}
\]

For \( q = 1 \) this condition corresponds to \( \gamma = 1 + 2/d \).\(^{24}\) In that case, the variables of position and time separate provided that

\[
f(x)^{2/d+q-2} \frac{df}{dx} = -K \frac{1}{q+2} \frac{1}{M^{2/d} \rho_0^{-1/2}} \frac{h^2}{2m^2} \frac{1}{x \, dx} \left[ \frac{\Delta_x f(x)^{q/2}}{f(x)^{1-q/2}} \right] + 2Ax = 0 \tag{B79}
\]

and

\[
\tilde{R} + \xi \dot{R} + \omega_0^2 R = 2AK \left( q + \frac{2}{d} \right) \frac{M^{2/d}}{R^3} \left( \frac{M}{R^d} \right)^{q-1}, \tag{B80}
\]

where \( A \) is a constant. These differential equations determine the invariant density profile \( f(x) \) and the evolution of the radius \( R(t) \) of the system. For \( q = 1 \) they reduce to

\[
f(x)^{2/d-1} \frac{df}{dx} = -K \frac{1}{1 + \frac{2}{d}} \frac{1}{M^{2/d} \rho_0^{-1/2}} \frac{h^2}{2m^2} \frac{d}{dx} \left[ \frac{\Delta_x \sqrt{f(x)}}{\sqrt{f(x)}} \right] + 2Ax = 0 \tag{B81}
\]

and

\[
\tilde{R} + \xi \dot{R} + \omega_0^2 R = 2AK \left( 1 + \frac{2}{d} \right) \frac{M^{2/d}}{R^3}. \tag{B82}
\]

The invariant density profile \( f(x) \) is different from the Gaussian or the Tsallis profiles considered previously.\(^{25}\)

Remark: In the case where \( K = 0 \), Eq. \((B76)\) reduces to

\[
\tilde{R} + \xi \dot{R} + \omega_0^2 R = \frac{h^2}{2m^2 \rho_0^{-1/2}} \frac{1}{R^3} \left( \frac{M}{R^d} \right)^{q-1} \frac{1}{x \, dx} \left[ \frac{\Delta_x f(x)^{q/2}}{f(x)^{1-q/2}} \right]. \tag{B83}
\]

The variables of position and time separate provided that

\[
\frac{1}{x \, dx} \left[ \frac{\Delta_x f(x)^{q/2}}{f(x)^{1-q/2}} \right] = \frac{2q}{Z^{q-1}} \left( \frac{d}{2} (q - 1) + 1 \right) \tag{B84}
\]

\(^{24}\) Interestingly, this is the polytropic index of a nonrelativistic Fermi gas at \( T = 0 \) in dimension \( d \).\(^{106}\)

\(^{25}\) Such profiles would be a particular solution of Eqs. \((B29)\) and \((B31)\) provided that \( q = 2 \gamma - 1 \) and \( \gamma = q + 2/d \), leading to \( \gamma = (d - 2)/d \). But, in that case, the profile is not normalizable.
and
\[ \ddot{R} + \xi \dot{R} + \omega_0^2 R = \frac{\hbar^2}{m^2 R^2} \frac{1}{\rho_0^2} R^3 \left( \frac{M}{Rd} \right)^{q-1} \frac{1}{Z^{2q-1}} q \left[ \frac{d}{2} (q-1) + 1 \right], \]  
\hspace{2cm} (B85)

where \( Z \) is a constant (the right hand side of Eq. (B84) has been written under that form for commodity). A particular solution of Eq. (B84) is the Tsallis distribution (B29) with index \( \gamma = (q+1)/2 \) [see Eq. (B77)], reducing to the Gaussian for \( q = 1 \). However, the Tsallis distribution may not be the only solution of Eq. (B84).

11. Eigenenergy

The eigenenergy \( E \) of the generalized time-independent GP equation can be obtained by applying Eq. (A37) at \( r = 0 \). This yields
\[ E = m\hbar \rho(0) + Q_g(0). \]  
\hspace{2cm} (B86)

In the isothermal case [see Eq. (64)], we have
\[ V(\rho) = \frac{k_B T}{m} \rho (\ln \rho - 1), \quad h(\rho) = V'(\rho) = \frac{k_B T}{m} \ln \rho. \]  
\hspace{2cm} (B87)

In the polytropic case [see Eq. (71)], we have
\[ V(\rho) = \frac{K}{\gamma - 1} \rho^\gamma, \quad h(\rho) = V'(\rho) = \frac{K\gamma}{\gamma - 1} \rho^{\gamma-1}. \]  
\hspace{2cm} (B88)

Let us first consider, for simplicity, the case \( T = K = 0 \) for which Eq. (B86) reduces to
\[ E = Q_g(0). \]  
\hspace{2cm} (B89)

For the standard Schrödinger equation \((q = 1)\), the equilibrium density is Gaussian. Using Eqs. (B38), (B89) and (C5), we recover the standard results
\[ R_e = \left( \frac{\hbar}{m\omega_0} \right)^{1/2}, \quad E = \frac{d}{2} \hbar \omega_0 \]  
\hspace{2cm} (B90)

of the quantum harmonic oscillator’s ground state. For the generalized Schrödinger equation \((q = 2\gamma - 1)\), the equilibrium density is a Tsallis distribution. Using Eqs. (B38), (B89) and (C16), we find that
\[ R_e = \left\{ \frac{\hbar^2}{m^2 \omega_0^2} \rho_0^{2(1-\gamma)} M^{2(\gamma-1)} \frac{2\gamma - 1}{Z^{2(\gamma-1)}(1 + d(\gamma - 1))} \right\}^{1/[2d(\gamma-1)+4]}, \quad E = \frac{d}{2} m\omega_0^2 R_e^2 \frac{1}{1 + d(\gamma - 1)}. \]  
\hspace{2cm} (B91)

For \( \gamma = 1 \), we recover Eq. (B90). The general case \( (T \neq 0 \text{ or } K \neq 0) \) can be treated similarly by substituting the expression (B87) or (B88) of the enthalpy in the eigenenergy (B86) and using \( \rho(0) = (M/R_d) f(0) \) with \( f(0) = 1/\pi^{d/2} \) for the Gaussian distribution and \( f(0) = 1/Z \) for the Tsallis distribution.

Appendix C: Generalized quantum mechanics

In this Appendix, we expose the ideas leading to the generalized quantum damped Euler equations (A18) and (A19) which are equivalent to the generalized damped GP equation (A1). Our aim is to motivate the introduction of the generalized quantum potential (A11) associated with the nonlinear Laplacian operator (A3).

1. Generalized quantum potential

Let us first consider the standard quantum potential
\[ Q = -\frac{\hbar^2}{2m} \Delta \sqrt{\rho} = -\frac{\hbar^2}{4m} \left[ \frac{\Delta \rho}{\rho} - \frac{1}{2} \left( \nabla \rho \right)^2 \right]. \]  
\hspace{2cm} (C1)
Applying the Laplacian operator to the Gaussian distribution \((\text{13})\), we find that

\[
\Delta x \sqrt{f} = \sqrt{A}(x^2 - d)e^{-x^2/2}.
\]

(C2)

Therefore, the ratio

\[
\frac{\Delta x \sqrt{f}}{\sqrt{f}} = x^2 - d
\]

(C3)

is a quadratic function of \(x\). As a result, the self-similar Gaussian profile defined by Eqs. \((\text{7})\) and \((\text{13})\) satisfies

\[
\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} = 1 \frac{R^2}{(r^2 R^2 - d)}.
\]

(C4)

Substituting this result into Eq. \((\text{C1})\), we obtain

\[
Q = -\frac{\hbar^2}{2m R^2} \left( \frac{r^2}{R^2} - d \right),
\]

(C5)

implying

\[
-\frac{1}{m} \nabla Q = \frac{\hbar^2}{m^2 R^3} x.
\]

(C6)

Therefore, for a Gaussian self-similar distribution, the standard quantum force \(-(1/m)\nabla Q\) is proportional to \(x\) with a prefactor depending on time but not on \(x\). According to the calculations of Appendix \((\text{B})\) this implies that the standard damped logarithmic GP equation \((\text{63})\), or the standard quantum damped isothermal Euler equations \((\text{5}), (\text{59})\) and \((\text{60})\), admit an exact self-similar solution with a Gaussian invariant profile.

We now generalize this procedure to the case of polytropes. We want to find a generalized form of quantum potential such that the generalized damped power-law GP equation \((\text{75})\), or the generalized quantum damped polytropic Euler equations \((\text{4}), (\text{73})\) and \((\text{74})\), admit an exact self-similar solution with a Tsallis invariant profile. Applying the Laplacian operator to a power \(\alpha\) of the Tsallis distribution \((\text{9})\) we find that

\[
\Delta x (f^\alpha) = \frac{2\alpha}{Z_\alpha} \left[ 1 - (\gamma - 1)x^2 \right]^{\frac{\alpha - 2(\gamma - 1)}{1 - \gamma}} \left[ (2\alpha + (d - 2)(\gamma - 1)x^2 - d \right].
\]

(C7)

Therefore, the ratio

\[
\frac{\Delta x (f^\alpha)}{f^\alpha - 2(\gamma - 1) = \frac{2\alpha}{Z_\alpha} \left[ (2\alpha + (d - 2)(\gamma - 1)x^2 - d \right]
\]

(C8)

is a quadratic function of \(x\). As a result, the self-similar Tsallis profile defined by Eqs. \((\text{7})\) and \((\text{9})\) satisfies

\[
\frac{\Delta (\rho^\alpha)}{\rho^\alpha - 2(\gamma - 1) = \frac{M^{2(\gamma - 1)}}{R^{2d(\gamma - 1)+2}} \frac{2\alpha}{Z^{2(\gamma - 1)}} \left[ (2\alpha + (d - 2)(\gamma - 1)x^2 - d \right].
\]

(C9)

This suggests introducing the generalized quantum potential

\[
Q_g = -\frac{\hbar^2}{2m} \frac{\Delta [(\rho/\rho_0)^\alpha]}{(\rho/\rho_0)^\alpha - 2(\gamma - 1)},
\]

(C10)

which reduces to the standard quantum potential \((\text{C1})\) for \(\gamma = 1\) and \(\alpha = 1/2\). Substituting Eq. \((\text{C9})\) into Eq. \((\text{C10})\), we obtain

\[
Q_g = -\frac{\hbar^2}{2m \rho_0^{2(1-\gamma)}} \frac{M^{2(\gamma - 1)}}{R^{2d(\gamma - 1)+2}} \frac{2\alpha}{Z^{2(\gamma - 1)}} \left[ (2\alpha + (d - 2)(\gamma - 1)x^2 - d \right]
\]

(C11)

implying

\[
-\frac{1}{m} \nabla Q_g = \frac{\hbar^2}{m^2 \rho_0^{2(1-\gamma)}} \frac{M^{2(\gamma - 1)}}{R^{2d(\gamma - 1)+3}} \frac{2\alpha}{Z^{2(\gamma - 1)}} \left[ (2\alpha + (d - 2)(\gamma - 1)\right] x.
\]

(C12)
Therefore, for a Tsallis self-similar distribution, the generalized quantum force \(-\frac{1}{m}\nabla Q_g\) is proportional to \(x\) with a prefactor depending on time but not on \(x\). According to the calculations of Appendix [C], this implies that the generalized damped power-law GP equation (75), or the generalized quantum damped polytropic Euler equations (4), (26) and (44), admit an exact self-similar solution with a Tsallis invariant profile.

We note that the generalized quantum potential (C10) depends on two parameters \(\alpha\) and \(\gamma\). We shall impose the relation

\[
\alpha = \gamma - \frac{1}{2},
\]

(C13)

This relation allows us to write the functional associated with the generalized quantum potential (C10) as a generalized von Weizsäcker functional (see Appendix C2). With the relation (C13), the generalized quantum potential (C10) takes the form

\[
Q_g = -\frac{\hbar^2}{2m}\Delta[(\rho/\rho_0)^{\gamma-1/2}].
\]

(C14)

It can also be written as

\[
Q_g = -\frac{\hbar^2}{4m}(2\gamma - 1)\left[\frac{\Delta(\rho/\rho_0)}{(\rho/\rho_0)^{3-2\gamma}} - \frac{1}{2}(3 - 2\gamma)(\nabla(\rho/\rho_0))^2(\rho/\rho_0)^{4-2\gamma}\right].
\]

(C15)

Setting \(q = 2\gamma - 1\), we obtain Eq. (A11). For \(\gamma = 1\) we recover the standard quantum potential (C1). On the other hand, with the relation (C13), Eqs. (C11) and (C12) reduce to

\[
Q_g = -\frac{\hbar^2}{2m}\rho_0^{2(1-\gamma)}\frac{M^2(\gamma-1)}{R^{2d(\gamma-1)+3}}\left[2\gamma - 1 \right] (2\gamma - 1 + (d - 2)(\gamma - 1))\frac{r^2}{R^2} - d\),
\]

(C16)

\[
-\frac{1}{m}\nabla Q_g = \frac{\hbar^2}{m^2}\rho_0^{2(1-\gamma)}\frac{M^2(\gamma-1)}{R^{2d(\gamma-1)+3}}\left[2\gamma - 1 \right] (1 + d(\gamma - 1))x.
\]

(C17)

For \(\gamma = 1\), we recover the results from Eqs. (C5) and (C6).

2. Generalized von Weizsäcker functional

We want to find a functional \(\Theta_Q^g\) such that

\[
\delta\Theta_Q^g = \int \frac{Q_g}{m} \delta \rho \, dr,
\]

(C18)

where \(Q_g\) is the generalized quantum potential defined by Eq. (C10). The identity (C18) is necessary to obtain the equilibrium condition (A17) from the extremization of the generalized free energy functional (A39). In this manner, a stationary solution of the generalized damped GP equation (A1) is guaranteed to be an extremum of free energy at fixed mass and the \(H\)-theorem from Appendix A7 is satisfied. In standard quantum mechanics, the functional \(\Theta_Q\) associated with the standard quantum potential (C1) is the von Weizsäcker functional (A45). The object of this Appendix is to find the proper generalization of this functional in relation to the generalized quantum potential (C10).

Let us consider a functional of the form

\[
\Theta_Q^g = C \int [\nabla(\rho^g)]^2 \, dr = \alpha^2 C \int \frac{(\nabla \rho)^2}{\rho^{2(1-\alpha)}} \, dr,
\]

(C19)

where \(C\) is a constant. Its first variations are

\[
\delta\Theta_Q^g = -2\alpha C \int \frac{\Delta(\rho^g)}{\rho^{1-\alpha}} \delta \rho \, dr.
\]

(C20)

\(^{26}\) It corresponds to the quantum kinetic energy term \(\Theta_Q = (\hbar^2/8m^2) \int [|\nabla \rho|^2/\rho] \, dr\) when we perform the Madelung transformation in the total kinetic energy \(\Theta = \langle \psi | H | \psi \rangle = (\hbar^2/2m^2) \int |\nabla \psi|^2 \, dx\) of a quantum particle, the other term being the classical kinetic energy \(\Theta_c = (1/2) \int \rho u^2 \, dx\). We thus have \(\Theta = \Theta_c + \Theta_Q\) (see [52] for more details).
Considering expression (C10) of the generalized quantum potential, we obtain Eq. (C18) provided that \(1 - \alpha = \alpha - 2(\gamma - 1)\), i.e., \(\alpha = \gamma - 1/2\). This is the relation announced in the preceding section [see Eq. (C13)]. We have therefore found a functional \(\Theta_Q^g\) such that Eq. (C18) is satisfied when \(Q_g\) is the generalized quantum potential defined by Eq. (C14). Using relation (C18), we can rewrite Eq. (C19) as

\[
\Theta_Q^g = C \int \left[\nabla (\rho^{\gamma - 1/2})\right]^2 dx = C \int \frac{(\nabla \rho)^2}{\rho^{3 - 2\gamma}} dx \tag{C21}
\]

and Eq. (C20) as

\[
\delta \Theta_Q^g = -(2\gamma - 1)C \int \frac{\Delta (\rho^{\gamma - 1/2})}{\rho^{3/2 - \gamma}} \delta \rho dx. \tag{C22}
\]

Considering Eqs. (C14) and (C22), we find that relation (C18) is exactly satisfied by taking

\[
C = \frac{\hbar^2}{2m^2} \frac{1}{2\gamma - 1} \frac{1}{\rho_0^{2(\gamma - 1)}}. \tag{C23}
\]

Therefore, the final expression of the generalized von Weizsäcker functional is

\[
\Theta_Q^g = \frac{\hbar^2}{2m^2} \frac{\rho_0}{2\gamma - 1} \int \left\{\nabla [((\rho/\rho_0)^{\gamma - 1/2})^2] \right\}^2 dx. \tag{C24}
\]

It can be written under the equivalent forms

\[
\Theta_Q^g = -\frac{\hbar^2}{2m^2} \frac{\rho_0}{2\gamma - 1} \int \left(\frac{\rho}{\rho_0}\right)^{\gamma - 1/2} \Delta \left[\left(\frac{\rho}{\rho_0}\right)^{\gamma - 1/2}\right] dx, \tag{C25}
\]

\[
\Theta_Q^g = \frac{1}{2\gamma - 1} \int \frac{Q}{m} dx, \tag{C26}
\]

\[
\Theta_Q^g = \frac{\hbar^2}{8m^2} \rho_0 (2\gamma - 1) \int \frac{[\nabla (\rho/\rho_0)]^2}{(\rho/\rho_0)^{3 - 2\gamma}} dx. \tag{C27}
\]

Setting \(q = 2\gamma - 1\), we obtain Eqs. (A44)-(A47).

### 3. Generalized Fisher functional

Let us first consider the standard diffusion equation

\[
\frac{\partial \rho}{\partial t} = D \Delta \rho. \tag{C28}
\]

If we compute the rate of change of the Boltzmann entropy (A85) and use Eq. (C28), we obtain

\[
\dot{S}_B = k_B D S_F, \tag{C29}
\]

where

\[
S_F = \frac{1}{m} \int \frac{(\nabla \rho)^2}{\rho} dx \tag{C30}
\]

is the Fisher entropy (99). We note that the Fisher entropy is related to the von Weizsäcker functional (A45) by

\[
\Theta_Q = \frac{\hbar^2}{8m} S_F. \tag{C31}
\]
Therefore, we can write

$$\dot{S}_B = k_B D \frac{8m}{\hbar^2} \Theta_Q.$$  \hfill (C32)

This equation relates the Boltzmann entropy to the von Weizsäcker functional. It provides an intriguing relation between standard thermodynamics (Boltzmann) and standard quantum mechanics (Schrödinger).

In connection to the generalized von Weizsäcker functional (C27), we introduce a generalized Fisher entropy so as to preserve the relation (C31). Writing

$$\Theta_Q^g = \frac{\hbar^2}{8m} S_F^g,$$  \hfill (C33)

we obtain a generalized Fisher entropy of the form

$$S_F^g = \frac{\rho_0}{m} (2\gamma - 1) \int \frac{[\nabla (\rho/\rho_0)]^2}{(\rho/\rho_0)^{3-2\gamma}} \,dr.$$  \hfill (C34)

It reduces to Eq. (C30) when $\gamma = 1$. Let us now consider the anomalous diffusion equation

$$\xi \partial \rho \partial t = K \Delta \rho.$$  \hfill (C35)

If we compute the rate of change of the Tsallis entropy (A88) and use Eqs. (C34) and (C35) we obtain

$$\dot{S}_\gamma = \frac{Km}{\xi} \frac{\gamma^2}{2\gamma - 1} \rho_0^{2(\gamma - 1)} S_F^g.$$  \hfill (C36)

Using Eq. (C33), we can write

$$\dot{S}_\gamma = \frac{Km}{\xi} \frac{\gamma^2}{2\gamma - 1} \rho_0^{2(\gamma - 1)} \frac{8m}{\hbar^2} \Theta_Q^g.$$  \hfill (C37)

This equation relates the Tsallis entropy to the generalized von Weizsäcker functional. It provides an intriguing relation between generalized thermodynamics (Tsallis) and the form of generalized quantum mechanics introduced in this paper.

4. Generalized quantum pressure tensor

In standard quantum mechanics, the quantum force can be written as the gradient of a quantum pressure tensor $P_{ij}$ \hfill [53]. Indeed, we have

$$(F_Q)_i = -\frac{1}{m} \partial_i Q = -\frac{1}{\rho} \partial_j P_{ij}.$$  \hfill (C38)

with

$$P_{ij}^{(1)} = -\frac{\hbar^2}{4m^2} \rho \partial_i \partial_j \ln \rho = \frac{\hbar^2}{4m^2} \left( \frac{1}{\rho} \partial_i \rho \partial_j \rho - \partial_i \partial_j \rho \right) \quad \text{or} \quad P_{ij}^{(2)} = \frac{\hbar^2}{4m^2} \left( \frac{1}{\rho} \partial_i \rho \partial_j \rho - \delta_{ij} \Delta \rho \right).$$  \hfill (C39)

This tensor is manifestly symmetric: $P_{ij} = P_{ji}$. The tensors defined by Eq. (C39) are related to each other by

$$P_{ij}^{(1)} = P_{ij}^{(2)} + \frac{\hbar^2}{4m^2} (\delta_{ij} \Delta \rho - \partial_i \partial_j \rho).$$  \hfill (C40)

They differ by a tensor $\chi_{ij} = \delta_{ij} \Delta \rho - \partial_i \partial_j \rho$ satisfying $\partial_i \chi_{ij} = 0$.

We show below that Eq. (C38) remains valid for the generalized quantum force, i.e., we show that the generalized quantum force can be written as

$$(F_Q^g)_i = -\frac{1}{m} \partial_i Q_g = -\frac{1}{\rho} \partial_j P_{ij}^g.$$  \hfill (C41)
where $P^g_{ij}$ is a generalized quantum potential tensor. To determine $P^g_{ij}$, we first note that the generalized quantum potential (C15) is proportional to

$$Q^g = \frac{\Delta \rho}{\rho^{3-2\gamma}} - \frac{1}{2} (3 - 2\gamma) (\nabla \rho)^2 \rho^{3-2\gamma}$$

so that its gradient is proportional to

$$-\partial_i Q^g = (3 - 2\gamma) \frac{1}{\rho^{3-2\gamma}} \partial_i \partial_j \rho - (3 - 2\gamma) (2 - \gamma) \frac{1}{\rho^{3-2\gamma}} \partial_i (\nabla \rho)^2 \rho^{3-2\gamma}$$

We want to find a tensor $P^g_{ij}$ such that

$$-k \partial_i Q^g = \frac{1}{\rho} \partial_j P^g_{ij},$$

where $k$ is a constant. To find the most general form of $P^g_{ij}$, we consider an expression of the form

$$P^g_{ij} = \frac{1}{\rho^{3-2\gamma}} \partial_i \partial_j \rho - A \frac{1}{\rho^{3-2\gamma}} \delta_{ij} \Delta \rho - B \frac{1}{\rho^{3-2\gamma}} \partial_{ij} \rho + C \frac{1}{\rho^{3-2\gamma}} (\nabla \rho)^2 \delta_{ij},$$

where $A$, $B$ and $C$ are some constants. We have

$$\frac{1}{\rho} \partial_j P^g_{ij} = \frac{1}{\rho^{4-2\gamma}} \partial_i \partial_j \rho \partial_j \rho \partial_j \rho [1 + 2A(1 - \gamma)] + \frac{1}{\rho^{4-2\gamma}} \partial_{ij} \rho \partial_j \rho [1 + 2B(1 - \gamma) + 2C]$$

$$- \frac{1}{\rho^{5-2\gamma}} \partial_i (\nabla \rho)^2 (3 - 2\gamma) (1 + C) - \frac{1}{\rho^{3-2\gamma}} \partial_i \partial_j \rho (A + B).$$

Substituting Eqs. (C43) and (C46) into Eq. (C44), we obtain the system of equations

$$k(3 - 2\gamma) = 1 + 2A(1 - \gamma),$$

$$k(3 - 2\gamma) = 1 + 2B(1 - \gamma) + 2C,$$

$$k(3 - 2\gamma)(2 - \gamma) = (3 - 2\gamma)(1 + C),$$

$$k = A + B.$$ 

Coming back to the original variables, we find that the generalized quantum potential (C14) is related to $Q^g$ by

$$Q^g = -\frac{\hbar^2}{4m} (2\gamma - 1) \rho_0^{2-2\gamma} Q^g.$$

According to Eqs. (C43), (C44) and (C51), the generalized pressure tensor is given by

$$P^g_{ij} = \frac{\hbar^2}{4m} (2\gamma - 1) \frac{1}{k} \rho_0^{2-2\gamma} P^g_{ij},$$

where $P^g_{ij}$ is given by Eq. (C45) with the coefficients $A$, $B$, $C$ and $k$ determined by Eqs. (C47)-(C50).

For $\gamma = 1$, Eqs. (C44) and (C50) reduce to

$$k = 1, \quad C = 0, \quad A + B = 1.$$ 

We note that there is a freedom since $A$ and $B$ are not individually determined. Only their sum is fixed to unity. As a result, the general expression of the standard quantum pressure tensor is

$$P_{ij} = \frac{\hbar^2}{4m} \left[ \frac{1}{\rho} \partial_i \partial_j \rho - A \delta_{ij} \Delta \rho - (1 - A) \partial_i \rho \right].$$
It returns the usual expressions $P^{(1)}_{ij}$ and $P^{(2)}_{ij}$ given by Eq. [C59] when we take $A = 0$ or $A = 1$ respectively. We can also take $A = B = 1/2$ leading to

$$P_{ij} = \frac{\hbar^2}{4m^2} \left[ \frac{1}{\rho} \partial_i \rho \partial_j \rho - \frac{1}{2} \delta_{ij} \Delta \rho - \frac{1}{2} \partial_i \rho \right]. \quad \text{(C55)}$$

For $\gamma \neq 1$, the solution of Eqs. [C47]-[C50] is

$$A = \frac{1 - k(3 - 2\gamma)}{2(\gamma - 1)}, \quad B = \frac{k - 1}{2(\gamma - 1)}, \quad C = k(2 - \gamma) - 1. \quad \text{(C56)}$$

Therefore, the generalized pressure tensor is given by

$$P_{ij} = \frac{\hbar^2}{4m^2} (2\gamma - 1 - k)\rho_0^{3-2\gamma} \left[ \frac{1}{\rho^{3-2\gamma}} \partial_i \rho \partial_j \rho - \frac{1}{\rho^{2-2\gamma}} \delta_{ij} \Delta \rho - \frac{1}{\rho^{3-2\gamma}} \partial_i \rho + \frac{1}{\rho^{3-2\gamma}} (\nabla \rho)^2 \delta_{ij} + (1 - \gamma) \frac{1}{\rho^{3-2\gamma}} (\nabla \rho)^2 \delta_{ij} \right]. \quad \text{(C57)}$$

We note that $C \neq 0$ in general. This brings a new term in the generalized quantum pressure tensor that is absent in the expression of the standard quantum pressure tensor [C54] when $\gamma = 1$. On the other hand, we are free to take $k$ as we please provided that $k = 1$ when $\gamma = 1$. Let us consider particular cases.

(i) For $k = 1$ we get $A = 1$, $B = 0$ and $C = 1 - \gamma$ yielding

$$P_{ij} = \frac{\hbar^2}{4m^2} (2\gamma - 1)\rho_0^{3-2\gamma} \left[ \frac{1}{\rho^{3-2\gamma}} \partial_i \rho \partial_j \rho - \frac{1}{\rho^{2-2\gamma}} \delta_{ij} \Delta \rho + (1 - \gamma) \frac{1}{\rho^{3-2\gamma}} (\nabla \rho)^2 \delta_{ij} \right]. \quad \text{(C58)}$$

When $\gamma = 1$ this returns the expression $P^{(2)}_{ij}$ from Eq. [C39].

(ii) For $k = 1/(3 - 2\gamma)$ we get $A = 0$, $B = 1/(3 - 2\gamma)$ and $C = (\gamma - 1)/(3 - 2\gamma)$ yielding

$$P_{ij} = \frac{\hbar^2}{4m^2} \frac{2\gamma - 1}{3 - 2\gamma} \rho_0^{3-2\gamma} \left[ \frac{1}{\rho^{3-2\gamma}} \partial_i \rho \partial_j \rho - \frac{1}{\rho^{2-2\gamma}} \delta_{ij} \Delta \rho + \frac{\gamma - 1}{3 - 2\gamma} \frac{1}{\rho^{3-2\gamma}} (\nabla \rho)^2 \delta_{ij} \right]. \quad \text{(C59)}$$

When $\gamma = 1$ this returns the expression $P^{(1)}_{ij}$ from Eq. [C39].

(iii) For $k = 1/(2 - \gamma)$ we get $A = B = 1/[2(2 - \gamma)]$ and $C = 0$ yielding

$$P_{ij} = \frac{\hbar^2}{4m^2} \frac{2\gamma - 1}{2 - \gamma} \rho_0^{3-2\gamma} \left[ \frac{1}{\rho^{3-2\gamma}} \partial_i \rho \partial_j \rho - \frac{1}{2(2 - \gamma)} \frac{1}{\rho^{2-2\gamma}} \delta_{ij} \Delta \rho - \frac{1}{2(2 - \gamma)} \frac{1}{\rho^{2-2\gamma}} \partial_i \rho \right]. \quad \text{(C60)}$$

When $\gamma = 1$ this returns the expression from Eq. [C55].
[84] P.H. Chavanis, Eur. Phys. J. Plus 132, 286 (2017)
[85] P.H. Chavanis, Eur. Phys. J. Plus 129, 38 (2014)
[86] L. de Broglie, Nonlinear Wave Mechanics (Elsevier, Amsterdam, 1960)
[87] C. Vignat, A. Plastino, Physica A 388, 601 (2009)
[88] L. Hui, J. Ostriker, S. Tremaine, E. Witten, Phys. Rev. D 95, 043541 (2017)
[89] P.H. Chavanis, Phys. Rev. D 84, 043531 (2011)
[90] P.H. Chavanis, T. Matos, Eur. Phys. J. Plus 132, 30 (2017)
[91] P.H. Chavanis, B. Denet, M. Le Berre and Y. Pomeau, (to be published)
[92] P.H. Chavanis, Phys. Rev. D 84, 063518 (2011)
[93] M.D. Kostin, J. Chem. Phys. 57, 3589 (1972)
[94] L. de Broglie, J. Physique 8, 225 (1927)
[95] L. de Broglie, Compt. Rend. Acad. Sci. Paris 185, 380 (1927)
[96] L. de Broglie, Compt. Rend. Acad. Sci. Paris 185, 1118 (1927)
[97] D. Bohm, Phys. Rev. 85, 166 (1952)
[98] D. Bohm, Phys. Rev. 85, 180 (1952)
[99] R.A. Fisher, Proc. Cambridge Philos. Soc. 22, 700 (1925)
[100] E. Madelung, Naturwiss. 14, 1004 (1926)
[101] L.D. Landau, E.M. Lifshitz, Statistical Physics (Pergamon, 1959)
[102] A.S. Eddington, Monthly Not. Roy. Astron. Soc. 76, 572 (1916)
[103] J.R. Ipser, Astrophys. J. 193, 463 (1974)
[104] A. Campa, P.H. Chavanis, J. Stat. Mech. 06, 06001 (2010)
[105] I. Białynicki-Birula, J. Mycielski, Ann. Phys. 100, 62 (1976)
[106] P.H. Chavanis, Phys. Rev. D 76, 023004 (2007)