Thermodynamics of Lemaître – Tolman – Bondi Model

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Here we consider our universe as inhomogeneous spherically symmetric Lemaître – Tolman – Bondi Model and analyze the thermodynamics of this model of the universe. The trapping horizon is calculated and is found to coincide with the apparent horizon. The Einstein field equations are shown to be equivalent with the unified first law of thermodynamics. Finally assuming the first law of thermodynamics validity of the generalized second law of thermodynamics is examined at the apparent horizon for the perfect fluid and at the event horizon for holographic dark energy.

Key words: Thermodynamics, Inhomogeneity, Tolman-Bondi model.

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I. INTRODUCTION

The discovery of Hawking radiation [1] completes the cyclic of identifying black hole (BH) as a thermodynamical object – the laws of BH physics and thermodynamical laws are equivalent. Since then there is a series of works [2] dealing with thermodynamical studies of the universe as thermodynamical system. Considering homogeneous and isotropic FRW model of the universe, most studies deal with validity of the generalized second law of thermodynamics (GSLT) starting from the first law when universe is bounded by the apparent horizon [3]. Considering various matter system and different gravity theories, it is generally found that there is a nice agreement of the thermodynamical laws with apparent horizon as the boundary. Also it is found that first law of thermodynamics and (modified) Einstein equations are equivalent at the apparent horizon. In contrast, there are few works [4] dealing with thermodynamics of the universe with event horizon as the boundary. Due to existence of the event horizon, the matter here is chosen as either quintessence or exotic in nature. Here validity of GSLT put some restrictions either on geometry or on the matter itself except when the matter is in the form of holographic dark energy (HDE) [5], no constraint is necessary.

In the present work, we consider our universe as in homogeneous Lemaître – Tolman – Bondi [LTB] Model. This simple inhomogeneous cosmological model agrees with current supernova and some other data [6]. Also very recently Clarkson and Marteens [7] give a justification for inhomogeneous model from the point view of perturbation analysis. The apparent horizon and the trapping horizon coincide for the model. We are able to show that Einstein field equations and unified first law are equivalent on the apparent horizon. Finally we determine the constrains to satisfy the GSLT on the apparent horizon for the perfect fluid and on the event horizon with matter as HDE.

II. BASIC EQUATIONS IN LTB MODEL:

The metric ansatz for inhomogeneous spherically symmetric LTB space time in a co-moving frame is given by

\[ dS^2 = -dt^2 + \frac{R^2}{1 + f(r)}dr^2 + R^2(d\theta^2 + \sin^2 \theta d\phi^2) \]  

(1)

where \( R = R(r, t) \) is the (area) radius of the spherical surface and \( f(r) > -1 \) is the curvature scalar (classifies the space-time as bounded, marginally bounded and unbounded depending on the range of its values which are

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respectively \( f(r) < 0, f(r) = 0, f(r) > 0 \). Let us suppose that the universe is filled with perfect fluid with energy momentum tensor

\[
T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} .
\] (2)

where \( \rho \) and \( p \) are respectively the matter density and pressure of the fluid and \( u^\mu \) is the fluid-four velocity of the fluid with normalization \( u_\mu u^\mu = -1 \). By introducing the mass function \( F(r, t) \) [8] (related to the mass contained within the co-moving radius \( r \))

\[
F(r, t) = R(\dot{R}^2 - f(r))
\] (3)

the Einstein equations can be written as

\[
8\pi G \rho = \frac{\dot{F}(r, t)}{R^2 \dot{R}}, \quad 8\pi G p = -\frac{\dot{F}(r, t)}{R^2 \dot{R}}
\] (4)

and the evolution equation for \( R \) is

\[
2R\ddot{R} + \dot{R}^2 + 8\pi G p R^2 = f(r)
\] (5)

The energy momentum conservation relation \( T^\nu_{\mu;\nu} = 0 \) gives

\[
\dot{\rho} + 3H(\rho + p) = 0 \quad \text{and} \quad p' = 0
\] (6)

where \( H = \frac{1}{3} \left( \frac{\ddot{R}}{R} + 2 \frac{\dot{R}}{R} \right) \), is the Hubble parameter.

Further the LTB line element can also be written as

\[
ds^2 = h_{ab} dx^a dx^b + R^2 d\Omega_2^2
\] (7)

where

\[
d\Omega_2^2 = d\theta^2 + \sin^2 \theta d\phi^2 \text{ is the metric on unit two sphere}
\]

and

\[
h_{ab} = \text{diag} \left(-1, \frac{R^2}{1 + f(r)} \right), \quad (a, b = 0, 1 \text{ with } x^0 = t, x^1 = r)
\]

is the metric on the 2D hyper surface normal to the 2-sphere. We now introduce two null vectors \( \partial_+ \) and \( \partial_- \) normal to the 2-sphere (i.e. on the 2D hyper surface) as

\[
\partial_+ = -\sqrt{2} \left( \partial_t - \frac{\sqrt{1 + f(r)}}{R'} \partial_r \right) \quad \text{and} \quad \partial_- = -\sqrt{2} \left( \partial_t + \frac{\sqrt{1 + f(r)}}{R'} \partial_r \right)
\] (8)

The dynamical apparent horizon which is essentially the marginally trapped surface with vanishing expansion is a spherical surface of radius \( R = R_A \) satisfying [9]

\[
h^{ab} = \partial_a R \partial_b R = 0
\]

i.e.,

\[
R_A = F(r, t) \quad \text{and} \quad \dot{R}_A^2 = 1 + f(r)
\] (9)

A trapping horizon \((R_T)\) is defined as a hyper surface foliated by marginal spheres and is characterized by [9]

\[
\partial_+ R_T = 0
\]

i.e.,

\[
\dot{R}_T = \sqrt{1 + f(r)}
\] (10)

Thus trapping horizon coincides with the apparent horizon and the result is in agreement with Lemma (1) in ref.[9].
III. THERMODYNAMICS ON THE APPARENT HORIZON:

We start this section with the following theorems:

**Theorem 1.** The unified first law is equivalent to the Einstein field equations at any spherical surface.

**Proof:** The unified first law states that

\[ dE = A\psi + WdV \]  

(11)

where,

\[ E = \frac{R}{2G} (1 - h^{ab}\partial_a R \partial_b R), \]

is the Misner-sharp mass.

\[ \Psi = \psi_a dx^a, \quad A = 4\pi R^2, \quad V = \frac{4}{3}\pi R^3, \quad \text{areal volume} \]

\[ \psi_a = T^b_a \partial_b R + W \partial_a R, \]

is the energy flux (or momentum density) and

\[ W = -\frac{1}{2} \text{Trace}(T), \]

is the work function or energy density. Note that here trace is referred to the two-dimensional space normal to the spheres of symmetry.

For the present LTB model,

\[ E = \frac{R}{2G} (\dot{R}^2 - f(r)) \]  

(12)

So

\[ dE = \frac{1}{2G} [\dot{R} (\dot{R}^2 + 2R\ddot{R} - f(r)) dt + (R' (\dot{R}^2 - f(r)) + R(2\dot{R}\ddot{R} - f'(r))) dr] \]  

(13)

\[ W = \frac{1}{2}(\rho - p) \]

\[ \psi_0 = -\frac{1}{2}(\rho + p)\dot{R}, \quad \psi_1 = \frac{1}{2}(\rho + p)R' \]

(14)

Therefore,

\[ \Psi = -\frac{1}{2}(\rho + p)(\dot{R}dt - R'dr) \]

Thus

\[ A\psi + WdV = 4\pi R^2 \left[ \rho R' dr - p\dot{R}dt \right] \]  

(15)

Hence equating (13) and (15) according to the unified first law (11) we have

\[ \dot{R}^2 + 2R\ddot{R} - f(r) = -8\pi G\rho R^2 \]  

(16)
and

\[( R'( \dot{R}^2 - f(r)) + R(2\ddot{R} \dot{R}' - f'(r))) = 8\pi GR'^2 \rho \]  \hspace{1cm} (17)

We see that eq. (16) is nothing but the evolution equation (5). Also equation (17) can be written (after some simplification)

\[ 8\pi G \rho = \frac{d}{dr} \left\{ R'^2 - f(r)R \right\} \frac{R'}{R'^2} \]

The other equation in eq. (4) for \( p \) can be obtained by differentiating equation (3) and using the evolution equation (5). Therefore, we write

Unified First Law of Thermodynamics ⇔ Einstein Equations.

One may note that in ref [10], Cai et al stated that unified first law is an identity concerning the (0, 0) component of the Einstein equation. But here we have shown that unified first law is more general — it is equivalent to the Einstein field equations at any spherical surface of symmetry.

**Theorem II.** The validity of Clausius relation at any spherical surface of symmetry depends on the choice of the tangent vector.

**Proof:** The Clausius relation states that

\[ \langle A\psi, z \rangle = \frac{\kappa}{8\pi G} \langle dA, z \rangle \]  \hspace{1cm} (18)

where \( \kappa \) is the surface gravity and \( z \) is any vector tangential to the spherical surface of symmetry. Let us choose

\[ z = z^+ \partial_+ + z^- \partial_- \]

\[ = z_1 \partial_t + z_2 \frac{\sqrt{1 + f}}{R'} \partial_r \]  \hspace{1cm} (19)

where \( z_1, z_2 \) (i.e., \( z^+, z^- \)) are constant parameters. By definition, the surface gravity is given by

\[ \kappa = \frac{1}{2\sqrt{-h}} \partial_a \left( \sqrt{-h} h^{ab} \partial_b R \right), \]

which for the present model has the expression

\[ \kappa = \frac{\sqrt{1 + f}}{2R'} \left[ R'' - \frac{\left( R' \dot{R} + \ddot{R} \dot{R}' \right)}{\sqrt{1 + f}} \right] \]  \hspace{1cm} (20)

Now, \( \langle A\psi, z \rangle = \frac{\kappa}{8\pi G} \langle dA, z \rangle \) gives on simplification,

\[ \frac{z_1}{z_2} = \frac{\sqrt{1 + f}}{\dot{R}} \left[ \frac{2\pi R (\rho + p) - \frac{\kappa}{G}}{2\pi R (\rho + p) + \frac{\kappa}{G}} \right] \]  \hspace{1cm} (21)

This clearly shows that the ratio \( \frac{z_1}{z_2} \) will have different values at different spherical surface. Therefore, validity of the Clausius relation depends on the choice of the approximate tangent vector and is in agreement with lemma III of ref [9].

Now assuming the Clausius relation at the (apparent or event) horizon we examine the validity of the GSLT. Let \( S_H \) and \( S_I \) be the entropy of the horizon and the matter bounded by the horizon. Then GSLT states that

\[ \frac{\partial}{\partial t} (S_H + S_I) \geq 0 \]  \hspace{1cm} (22)

Since we are considering equilibrium thermodynamics so temperature of the inside matter distribution can be considered as that of the horizon \( (T_H) \).
Now to find the variation of entropy at the horizon we start with unified first law

$$dE = A\Psi + W dV$$

using equation (14) and (15) we write

$$dE = -4\pi r^2 (\rho + p) \dot{R} dt + 4\pi R^2 \rho R' dR \tag{23}$$

Hence from the Clausius relation at a horizon

$$T_H dS_H = \delta Q = -dE = 4\pi R_H^2 (\rho + p) \dot{R}_H dt$$

i.e.,

$$\frac{dS_H}{dt} = \frac{4\pi R_H^2 \dot{R}_H (\rho + p)}{T_H} \tag{24}$$

where $S_H$ and $T_H$ are the entropy and temperature at the horizon.

In order to determine the time variation of the matter entropy. We start with the Gibb’s equation [11]

$$T_H dS_I = dE_I + pdV \tag{25}$$

where $V = \frac{4}{3}\pi R_H^3$, $E_I = \rho V$ and $(\rho, p)$ are the matter density and pressure of the fluid bounded by the horizon. Now using the energy conservation equation (6) we have from Gibb’s equation (after some simplification)

$$T_H \frac{dS_I}{dt} = 4\pi R_H^2 (\rho + p) \frac{dR_h}{dt} - 4\pi R_H^2 H (\rho + p) \tag{26}$$

Hence combining (24) and (26) we have,

$$T_H \frac{d}{dt}(S_H + S_I) = \frac{4\pi R_H^3}{3} (\rho + p) \left\{ \frac{4\dot{R}_H}{R_H} - \frac{\dot{R}'_H}{R_H^2} \right\} \tag{27}$$

Now we shall examine the validity of GSLT both at the apparent horizon and at the event horizon.

**Case-I : Universe filled with perfect fluid and bounded by the apparent horizon.**

The apparent horizon for the LTB model is characterized by

$$\dot{R}^2 = 1 + f(r)$$

or equivalently,

$$R = F.$$

So eq. (27) now becomes

$$T_A \frac{d}{dt}(S_A + S_I) = \frac{4\pi F^3 \sqrt{1 + f(r)}}{3F'(r, t)} \left[ \frac{4F'}{F} - \frac{f'(r)}{2(1 + f(r))} \right] \tag{28}$$

**IV. CASE-II : UNIVERSE FILLED WITH HDE AND BOUNDED BY THE EVENT HORIZON**

As geometrically event horizon ($R_E$) can not be evaluated for LTB model so we try to evaluate $R_E$ from physical consideration. It is HDE in which energy density $(\rho_D)$ can be written as [12]

$$\rho_D = \frac{3\dot{r}^2}{R_E^2} \tag{29}$$
where, $c$ is any free dimensionless parameter estimated by observational data [13]. However, in the present work we have taken $c$ to be arbitrary. Now from the energy conservation relation (6) we obtain

$$
\dot{R}_E = \frac{3}{2} H R_E (1 + \omega_D)
$$

(30)

where $\omega_D = \frac{p_{DE}}{\rho_{DE}}$ is the equation of state parameter for the HDE. Then from eq (27)

$$
T_E \frac{d}{dt} (S_E + S_I) = \frac{6\pi c^2 R_E^2 (1 + \omega_D)^2 H}{R_E'} \left[ 3 \frac{R_E}{R_E'} \frac{H'}{H} - \frac{\omega_D}{1 + \omega_D} \right]
$$

(31)

Now we shall analyze the above results for the validity of GSLT (i.e., inequality (22)).

**Apparent Horizon**

**(a) Quintessence Era ($\rho + p > 0$)**

In an expanding universe both $\dot{R}$ and $R'$ are positive. So from the expression of $\rho$ (in eq (4)) we see that $F'$ must be positive. But $\dot{F}$ is negative or positive depending on whether $p$ is positive or negative. Also from equation (3) $F$ should be positive. Hence from equation (27) GSLT will be valid if

\( (i) f'(r) \leq 0 \)

or

\( f'(r) > 0 \) and \( \frac{\partial}{\partial r} \left( \ln \frac{F^4}{\sqrt{1 + f}} \right) > 0 \)

**(b) Phantom Era ($\rho + p < 0$)**

In this case GLST will be valid if

\( f'(r) > 0 \) and \( \frac{\partial}{\partial r} \left( \ln \frac{F^4}{\sqrt{1 + f}} \right) < 0 \)

Therefore, validity of GSLT at the apparent horizon depends on the arbitrary integration functions appear in the Einstein field equations. Finally, one may note that here we have not used any explicit expression for entropy and temperature at the horizon.

**Event Horizon**

From eq.(31) we see that validity of GSLT does not depend explicitly on whether the HDE satisfies the weak energy condition or not. Essentially GSLT will be satisfied at the event horizon if both $R_E$ and $\frac{R_E^2}{H(1 + \omega_D)}$ are simultaneously increasing (or decreasing) function of $r$.

For future work, it will be interesting to examine whether Bekenstein entropy and Hawking temperature formulae hold at the apparent horizon for the present inhomogeneous LTB model. Also it will be nice to make an attempt for determining event horizon in this model.

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