Derivative Formula and Gradient Estimates for Gruschin Type Semigroups

Feng-Yu Wang

School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China
and
Department of Mathematics, Swansea University, Singleton Park, SA2 8PP, UK
Email: wangfy@bnu.edu.cn; F.Y.Wang@swansea.ac.uk

April 23, 2012

Abstract

By solving a control problem and using Malliavin calculus, explicit derivative formula is derived for the semigroup $P_t$ generated by the Gruschin type operator on $\mathbb{R}^m \times \mathbb{R}^d$:

$$L(x,y) = \frac{1}{2} \left\{ \sum_{i=1}^{m} \partial_{x_i}^2 + \sum_{j,k=1}^{d} (\sigma(x)\sigma(x)^*)_{jk} \partial_{y_j} \partial_{y_k} \right\}, \quad (x,y) \in \mathbb{R}^m \times \mathbb{R}^d,$$

where $\sigma \in C^1(\mathbb{R}^m; \mathbb{R}^d \otimes \mathbb{R}^d)$ might be degenerate. In particular, if $\sigma(x)$ is comparable with $|x|^l I_{d \times d}$ for some $l \geq 1$ in the sense of [155], then for any $p > 1$ there exists a constant $C_p > 0$ such that

$$|\nabla P_t f(x,y)| \leq \frac{C_p(P_t|f|^p)^{1/p}(x,y)}{\sqrt{t} \wedge \sqrt{t(|x|^2 + t)^l}}, \quad t > 0, f \in B_b(\mathbb{R}^{m+d}), (x,y) \in \mathbb{R}^{m+d},$$

which implies a new Harnack type inequality for the semigroup. A more general model is also investigated.

AMS subject Classification: 60J75, 60J45.
Keywords: Gruschin semigroup, derivative formula, gradient estimate.

*Supported in part by NNSFC(11131003), SRFDP, the Laboratory of Mathematical and Complex Systems and the Fundamental Research Funds for the Central Universities.
1 Introduction

It is well-known that a hypoelliptic diffusion semigroup on $\mathbb{R}^d$ has a smooth transition density w.r.t. the Lebesgue measure (cf. [16]). An interesting research topic is then to derive explicit estimates on the derivatives of the diffusion semigroup. To this end, the derivative formula, which is called the Bismut formula or the Bismut-Elworthy-Li formula due to [9, 11], has become a powerful tool.

In the elliptic setting, the formula can be explicitly established by using the associated Bakry-Emery curvature tensor. But in the degenerate case the curvature is no-longer available and the existing formula established using the Malliavin covariance matrix is normally less explicit, see e.g. [1, Theorem 10] and [2, Theorem 3.2]. To establish explicit derivative formulae for hypoelliptic semigroups, one has to build and solve some control problems associated to the corresponding stochastic differential equations, see e.g. [13, 19, 20] for the study of generalized stochastic Hamiltonian systems, and see [2, Section 6] for some simple examples. See also [17] for the study of hypoelliptic Ornstein-Uhlenbeck semigroups.

Among Laplacian type hypoelliptic operators without drift term, two typical models are the Kohn-Laplacian on Heisenberg groups and the Gruschin operator on $\mathbb{R}^2$. In recent years, the gradient estimate and applications have been intensively investigated for the heat semigroup $P_t$ generated by the Kohn-Laplacian on finite- or infinite-dimensional Heisenberg groups, see [4, 8, 12, 14] and the references within. In particular, the gradient inequality

$$\Gamma_1(P_tf) \leq CP_t\Gamma_1(f), \quad t \geq 0, f \in C^1_b, t \geq 0$$

(1.1)

is confirmed in [12] for some constant $C > 0$, where $\Gamma_1$ is the associated square field. This gradient inequality has important applications, for instance, it implies the heat kernel Poincaré inequality and thus (cf. [4]),

$$\Gamma_1(P_tf) \leq \frac{c}{t}P_tf^2, \quad f \in C^1_b, t > 0$$

(1.2)

for some constant $c > 0$.

Accordingly, one may wish to prove (1.1) and (1.2) also for the semigroup generated by the Gruschin operator $\partial_x^2 + x^{2l}\partial_y^2$ on $\mathbb{R}^2$, where $l \in \mathbb{N}$. As pointed out to the author by the referee that when $l = 1$ these can be confirmed by using the known inequalities on the Heisenberg group and the submersion $\psi : (x, y, z) \mapsto (x, z + \frac{x^2}{2})$. Indeed, letting $\tilde{P}_t$ and $\tilde{\Gamma}_1$ be the semigroup and square field associated to the Kohn-Laplacian $\tilde{X}^2 + \tilde{Y}^2$ on $\mathbb{R}^2$, where $\tilde{X} := \partial_x - \frac{x}{2}\partial_x, \tilde{Y} := \partial_y + \frac{x}{2}\partial_z$, we have

$$\tilde{X}(f \circ \psi) = (\partial_x f) \circ \psi, \quad \tilde{Y}(f \circ \psi) = (x\partial_y f) \circ \psi, \quad f \in C^1(\mathbb{R}^2),$$

so that

$$\Gamma_1(P_tf) \circ \psi = \tilde{\Gamma}_1(\tilde{P}_t f \circ \psi), \quad (P_t \Gamma_1(f)) \circ \psi = \tilde{P}_t \tilde{\Gamma}_1(f \circ \psi)$$

hold. When $l \geq 2$, (1.1) is however not yet available. We also would like to mention that for $l = 1$, the generalized curvature-dimension condition introduced and applied in [5, 6, 7]...
holds, so that the corresponding results, in particular the gradient estimates and applications derived in [6], are valid. Even when \( l \geq 2 \), although their generalized curvature condition is no longer available, a more general version of curvature condition has been confirmed in [18], so that the \( L^2 \)-gradient estimate as in Corollary 1.2 below for \( p = 2 \) holds.

In this paper, we aim to establish the Bismut-type derivative formula and gradient estimates for the semigroup generated by the following Gruschin-type operators on \( \mathbb{R}^{m+d} \):

\[
L(x, y) = \frac{1}{2} \left\{ \sum_{i=1}^{m} \partial_{x_i}^2 + \sum_{j,k=1}^{d} (\sigma(x)\sigma(x)^*)_{jk} \partial_{y_j} \partial_{y_k} \right\}, \quad (x, y) \in \mathbb{R}^{m} \times \mathbb{R}^{d} = \mathbb{R}^{m+d},
\]

where \( \sigma \in C^1(\mathbb{R}^{m}, \mathbb{R}^{d} \otimes \mathbb{R}^{d}) \) might be degenerate. In this general case, it seems hard to adopt the above mentioned arguments developed for Heisenberg groups and subelliptic operators satisfying the generalized curvature. Our study is based on Malliavin calculus.

Let \( \Gamma_1 \) be the square field associated to \( L \). Then

\[
(1.3) \quad \Gamma_1(f)(x, y) = |\nabla f(\cdot,y)(x)|^2 + |\sigma(x)^*\nabla f(x,\cdot)(y)|^2, \quad (x, y) \in \mathbb{R}^{m+d}, f \in C^1(\mathbb{R}^{m+d}).
\]

We will use \( |\cdot| \) and \( \|\cdot\| \) to denote the Euclidean norm and the operator norm respectively.

To construct the associated diffusion process, we consider the stochastic differential equation on \( \mathbb{R}^{m+d} \):

\[
(1.4) \quad \begin{cases}
    dX_t = dB_t, \\
    dY_t = \sigma(X_t) d\tilde{B}_t,
\end{cases}
\]

where \((B_t, \tilde{B}_t)\) is a Brownian motion on \( \mathbb{R}^{m+d} \). It is easy to see that for any initial data the equation has a unique solution and the solution is non-explosive. Let \( \mathbb{E}^{x,y} \) stands for the expectation taken for the solution starting at \((x, y) \in \mathbb{R}^{m+d}\). We have

\[
P_t f(x, y) = \mathbb{E}^{x,y} f(X_t, Y_t), \quad f \in \mathcal{B}(\mathbb{R}^{m+d}), (x, y) \in \mathbb{R}^{m+d}, t \geq 0.
\]

To establish explicit derivative formula for \( P_t \), we need the following assumption.

(A) \quad For any \( T > 0 \) and \( x \in \mathbb{R}^{m} \), \( Q_T := \int_0^T \sigma(x + B_t)\sigma(x + B_t)^* dt \) is invertible such that

\[
\mathbb{E} \left\{ \|Q_T^{-1}\|^2 \int_0^T \left( \|\nabla \sigma(x + B_t)\|^4 + \|\sigma(x + B_t)\|^4 + 1 \right) dt \right\} < \infty.
\]

Obviously, \( Q_T \) is invertible if so is \( \sigma(x) \) for a.e. \( x \in \mathbb{R}^{m} \). According to the proof of Corollary 1.2 below, assumption (A) is ensured by (1.5) below.

**Theorem 1.1.** Assume (A). For any \( f \in C^1_b(\mathbb{R}^{m+d}) \) and \( v = (v_1, v_2) \in \mathbb{R}^{m+d} \),

\[
\nabla_v P_T f(x, y) = \mathbb{E}^{x, y} \left\{ f(X_T, Y_T) M_T \right\}, \quad (x, y) \in \mathbb{R}^{m+d}, T > 0
\]
\[ M_T = \frac{\langle v_1, B_T \rangle}{T} - \text{Tr} \left( Q_T^{-1} \int_0^T \frac{T-t}{T} \{ (\nabla_{v_1} \sigma^*) \sigma \} (x + B_t) dt \right) \]
\[ + \left\langle Q_T^{-1} \left\{ v_2 + \int_0^T \frac{T-t}{T} (\nabla_{v_1} \sigma)(x + B_t) d\tilde{B}_t \right\}, \int_0^T \sigma(x + B_t) d\tilde{B}_t \right\rangle, \]

where \( \nabla_v \) stands for the directional derivative along \( v \).

To derive explicit estimates, we assume that \( \sigma(x) \) is comparable with \( |x|^l I_{d \times d} \) in the sense of (1.5) below.

**Corollary 1.2.** Let \( l \in [1, \infty) \) and assume that

\[ \| \sigma(x) \| \geq a|x|^l, \quad \| \sigma(x) \| + \| \nabla \sigma(x) \| \cdot |x| \leq b|x|^l, \quad x \in \mathbb{R}^m \]

holds for some constants \( a, b > 0 \). Then for any \( p > 1 \) there exists a constant \( C_p > 0 \) such that for any \( v = (v_1, v_2) \in \mathbb{R}^{m+d} \),

\[ |\nabla_v P_T f(x, y)| \leq C_p (P_T |f|^p)^{1/p}(x, y) \left( \frac{|v_1|}{\sqrt{T}} + \frac{|v_2|}{\sqrt{T(|x|^2 + T^l)}} \right), \quad T > 0, (x, y) \in \mathbb{R}^{m+d}. \]

Consequently,

\[ \Gamma_1(P_T f) \leq \frac{C P_T f^2}{T}, \quad T > 0, f \in B_b(\mathbb{R}^{m+d}) \]

holds for some constant \( C > 0 \), where \( \Gamma_1 \) is given by (1.3).

Let \( P_t(z; \cdot) \) be the transition probability kernel of \( P_t \). It is easy to see that (1.6) implies

\[ \| P_t((x, y); \cdot) - P_t((x', y'); \cdot) \|_{\text{var}} \leq C \| f \|_{\infty} \frac{|x - x'|}{\sqrt{T}} + \frac{|y - y'|}{\sqrt{T^{3+l}}}, \quad T > 0, (x, y), (x', y') \in \mathbb{R}^{m+d} \]

for some constant \( C > 0 \), where \( \| \varphi \|_{\text{var}} := \sup \varphi(\cdot) - \inf \varphi(\cdot) \) is the total variational norm of a signed measure \( \varphi \). Consequently (cf. [15]), the Markov process has successful couplings. Moreover, according to the following result, (1.6) and (1.7) also imply Harnack type inequalities for \( P_T \).

In general, let \( E \) be a connected differential manifold and let \( \Gamma_1 \) be a square field of type

\[ \Gamma_1(f) = \sum_{i=1}^l (X_i f)^2 \]

for some continuous vector fields \( \{X_i\}_{i=1}^d \). For any vector \( v \in T_x E \), the intrinsic norm of \( v \) induced by \( \Gamma_1 \) is

\[ |v|_{\Gamma_1} = \sup \{|vf|(x) : \Gamma_1(f)(x) \leq 1\}. \]
For any $C^1$-curve $\gamma : [0, 1] \to E$, the length of $\gamma$ induced by $\Gamma_1$ is

$$\ell(\gamma) = \int_0^1 |\dot{\gamma}_s|_{\Gamma_1} ds.$$ 

Finally, for any $z, z' \in E$, the intrinsic distance between them induced by $\Gamma_1$ is

$$\rho(z, z') = \inf \{ \ell(\gamma) : \gamma \text{ is a } C^1\text{-curve linking } z \text{ and } z' \}.$$ 

It is well known that $\rho$ is finite if $\{X_i\}_{i=1}^d$ are smooth vector fields satisfying Hörmander’s condition. An alternative way to define $\rho$ is to use the subunit curve. Recall that a $C^1$-curve $\gamma : [0, T] \to E$ is called subunit w.r.t. $\Gamma_1$ if $|\frac{d}{dt} f(\gamma_t)| \leq \sqrt{\Gamma_1(f)(\gamma_t)}$, $t \in [0, T]$. Then

$$\rho(z, z') = \inf \{ T > 0 : \text{there exists a subunit curve } \gamma : [0, T] \to M, \gamma_0 = z, \gamma_T = z' \}.$$ 

**Proposition 1.3.** Let $\Gamma_1$ and $\rho$ be fixed as above on a connected differential manifold $E$ such that $\rho$ is finite. Let $P$ be a (sub-)Markov operator on $\mathcal{B}_b(E)$, the set of all bounded measurable functions on $E$. Then for any constant $C > 0$,

$$(1.8) \quad \Gamma_1(Pf) \leq C^2 Pf^2, \quad f \in C_b^1(E)$$

is equivalent to the Harnack type inequality

$$(1.9) \quad Pf(z') \leq Pf(z) + C \rho(z, z') \sqrt{Pf^2(z')}, \quad z, z' \in E, f \geq 0, f \in \mathcal{B}_b(E).$$

A simple application of (1.9) is the following Harnack inequality for the transition kernel $P(z, \cdot)$ of $P$: taking $f = 1_A$ in (1.9) for measurable set $A$, we obtain

$$P(z, \cdot) \leq P(z', \cdot) + C \rho(z, z') \sqrt{P(z', \cdot)}, \quad z, z' \in E.$$ 

We will prove Theorem 1.1 in Section 2 and prove Corollary 1.2 and Proposition 1.3 in Section 3. Finally, in section 4 we extend Theorem 1.1 to a more general model.

## 2 Proof of Theorem 1.1

To establish the derivative formula, we first briefly recall the integration by parts formula for the Brownian motion. Let $T > 0$ be fixed and let

$$\mathbb{H} = \left\{ h \in C([0, T]; \mathbb{R}^{m+d}) : h(0) = 0, \|h\|_{\mathbb{H}}^2 := \int_0^T |h'(t)|^2 dt < \infty \right\}$$

be the Cameron-Martin space. Let $\mu$ be the distribution of $(B_t, \tilde{B}_t)_{t \in [0, T]}$, which is a probability measure (i.e. Wiener measure) on the path space $W = C([0, T]; \mathbb{R}^{m+d})$. A function $F \in L^2(W; \mu)$ is called differentiable if for any $h \in \mathbb{H}$, the directional derivative

$$D_h F := \lim_{\varepsilon \to 0} \frac{F(\cdot + \varepsilon h) - F(\cdot)}{\varepsilon}$$

is well defined for $F(\cdot)$ almost surely.
exists in $L^2(W; \mu)$. We write $F \in \mathcal{D}(D)$ if moreover

$$H \ni h \mapsto D_h F \in L^2(W; \mu)$$

is a bounded linear operator. In this case the Malliavin gradient $DF$ is defined as the unique element in $L^2(W \to H; \mu)$ such that $\langle DF, h \rangle_H = D_h F$ for $h \in H$. It is well known that $(D, \mathcal{D}(D))$ is a closed operator in $L^2(W; \mu)$, whose adjoint operator $(\delta, \mathcal{D}(\delta))$ is called the divergence operator. That is,

$$\int_W D_h F d\mu = \int_W F \delta(h) d\mu, \quad F \in \mathcal{D}(D), h \in \mathcal{D}(\delta). \quad (2.1)$$

**Theorem 2.1.** For fixed $T > 0$ and $v = (v_1, v_2) \in \mathbb{R}^{m+d}$, let $h_1 \in C^1([0, T]; \mathbb{R}^m)$ with $h_1(0) = 0$ and $h_1(T) = v_1$. If there exists a process $\{h_2(t)\}_{t \in [0, T]}$ on $\mathbb{R}^d$ such that $h_2(0) = 0$, and $h := (h_1, h_2) \in \mathcal{D}(\delta)$ satisfying

$$\int_0^T \sigma(X_t)h_2'(t)dt + \int_0^T (\nabla h_1(t) - v_1 \sigma)(X_t)d\tilde{B}_t = v_2, \quad (2.2)$$

then

$$\nabla_v P_T f = \mathbb{E}\{f(X_T, Y_T)\delta(h)\}, \quad f \in C^1_b(\mathbb{R}^2).$$

**Proof.** From [14] it is easy to see that the derivative process $(\nabla_v X_t, \nabla_v Y_t)_{t \geq 0}$ solve the equation

$$\begin{cases}
    d\nabla_v X_t = 0, & \nabla_v X_0 = v_1, \\
    d\nabla_v Y_t = (\nabla \nabla_v X_t \sigma)(X_t)d\tilde{B}_t, & \nabla_v Y_0 = v_2.
\end{cases}$$

So,

$$\begin{align*}
    \nabla_v X_t &= v_1, \\
    \nabla_v Y_t &= v_2 + \int_0^t (\nabla \nabla_v X_s \sigma)(X_s) d\tilde{B}_s. \quad (2.3)
\end{align*}$$

Next, for $h$ given in the theorem, we have

$$\begin{align*}
    dD_h X_t &= h_1'(t)dt, & D_h X_0 &= 0, \\
    dD_h Y_t &= \sigma(X_t)h_2'(t)dt + (\nabla D_h X_t \sigma)(X_t)d\tilde{B}_t, & D_h Y_0 &= 0.
\end{align*}$$

Thus,

$$\begin{align*}
    D_h X_t &= h_1(t), \\
    D_h Y_t &= \int_0^t \sigma(X_s)h_2'(s)ds + \int_0^t (\nabla h_1(s) \sigma)(X_s)d\tilde{B}_s.
\end{align*}$$

Since $h_1(T) = v_1$, combining this with (2.2) and (2.3) we obtain

$$(\nabla_v X_T, \nabla_v Y_T) = (D_h X_T, D_h Y_T).$$
Therefore, for any $f \in C^1_b(\mathbb{R}^2)$, it follows from (2.1) that
\[
\nabla_v P_T f = \mathbb{E}\langle \nabla f(X_T, Y_T), (\nabla_v X_T, \nabla_v Y_T) \rangle = \mathbb{E}\langle \nabla f(X_T, Y_T), (D_h X_T, D_h Y_T) \rangle = \mathbb{E}D_h \{f(X_T, Y_T)\} = \mathbb{E}\{f(X_T, Y_T)\delta(h)\}.
\]

\[\Box\]

To prove Theorem 1.1 the key point is to solve the control problem (2.2). To this end, we will need the following fundamental lemma.

**Lemma 2.2.** Let $\rho_t$ be a predictable process on $\mathbb{R}^d$ with $\mathbb{E}\int_0^T |\rho_t|^q < \infty$ for some $q \geq 2$. Then
\[
\mathbb{E}\left| \int_0^T \langle \rho_t, d\tilde{B}_t \rangle \right|^q \leq \left\{ \frac{q(q-1)}{2} \right\}^{q/2} \left( \int_0^T (\mathbb{E}|\rho_t|^q)^{2/q} dt \right)^{q/2} \leq \left\{ \frac{q(q-1)}{2} \right\}^{q/2} T^{(q-2)/2} \int_0^T \mathbb{E}|\rho_t|^q dt.
\]

**Proof.** It suffices to prove the first inequality since the second follows immediately from Jensen’s inequality. Let $N_t = \int_t^\infty \langle \rho_s, d\tilde{B}_s \rangle$, $t \geq 0$. Then $d\langle N \rangle_t = |\rho_t|^2 dt$ and
\[
dN_t^2 = 2N_t dN_t + |\rho_t|^2 dt.
\]
Noting that $|N_t|^q = (N_t^2)^{q/2}$, by Itô’s formula we obtain
\[
d|N_t|^q = \frac{q}{2} (N_t^2)^{(q-2)/2} dN_t^2 + \frac{q(q-2)}{2} (N_t^2)^{(q-4)/2} N_t^2 |\rho_t|^2 dt
= qN_t |N_t|^{q-2} dN_t + \frac{q(q-1)}{2} |N_t|^{q-2} |\rho_t|^2 dt.
\]
Therefore,
\[
\mathbb{E}|N_T|^q = \frac{q(q-1)}{2} \int_0^T \mathbb{E}\{|N_t|^{q-2} |\rho_t|^2\} dt
\leq \frac{q(q-1)}{2} \int_0^T (\mathbb{E}|N_t|^q)^{(q-2)/q} (\mathbb{E}|\rho_t|^q)^{2/q} dt
\leq \frac{q(q-1)}{2} (\mathbb{E}|N_T|^q)^{(q-2)/q} \int_0^T (\mathbb{E}|\rho_t|^q)^{2/q} dt.
\]
Up to an approximation argument we may assume that $\mathbb{E}|N_T|^q < \infty$, so that this implies
\[
\mathbb{E}|N_T|^q \leq \left\{ \frac{q(q-1)}{2} \right\}^{q/2} \left( \int_0^T (\mathbb{E}|\rho_t|^q)^{2/q} dt \right)^{q/2}.
\]
\[\Box\]
Proof of Theorem 1.1. We assume that $(X_0, Y_0) = (x, y)$ and simply denote $\mathbb{E}^{x,y}$ by $\mathbb{E}$. Let

$$h_1(t) = \frac{tv_1}{T}, \quad t \in [0, T]$$

and

$$h_2(t) = \left( \int_0^t \sigma(X_s)^* ds \right) Q_T^{-1} \left( v_2 + \int_0^T \frac{T-s}{T} (\nabla v_1 \sigma)(X_s) d\tilde{B}_s \right), \quad t \in [0, T].$$

Then it is easy to see that (2.2) holds. To see that $h := (h_1, h_2) \in \mathcal{D}(\delta)$ and to calculate $\delta(h)$, let

$$g_i = \left\langle e_i, Q_T^{-1} \left( v_2 + \int_0^T \frac{T-s}{T} (\nabla v_1 \sigma)(X_s) d\tilde{B}_s \right) \right\rangle,$$

$$\tilde{h}_i(t) = \int_0^t \sigma(X_s)^* e_i ds, \quad i = 1, \ldots, d,$$

where $\{e_i\}_{i=1}^d$ is the canonical ONB on $\mathbb{R}^d$. We have

$$h(t) = (h_1(t), 0) + \sum_{i=1}^d g_i(0, \tilde{h}_i(t)).$$

It is easy to see that $h_1$ and $\tilde{h}_i$ are adapted and

$$\delta((h_1, 0)) = \int_0^T \langle h'_i(t), dB_i \rangle = \frac{\langle v_1, B_T \rangle}{T},$$

$$\delta((0, \tilde{h}_i)) = \int_0^T \langle \tilde{h}'_i(t), d\tilde{B}_i \rangle = \int_0^T \langle \sigma(X_t)^* e_i, d\tilde{B}_i \rangle.$$
for some constants $c, c' > 0$. So, (A) implies $g_i \delta((0, \tilde{h}_i)) \in L^2(\mathbb{P})$ for $i = 1, \cdots d$. Hence, if for any $i \in \{1, \cdots d\}$ one has $D(0, \tilde{h}_i)g_i \in L^2(\mathbb{P})$, then $h \in \mathcal{D}(\delta)$ and by (2.6) and (2.7),

$$(2.8) \quad \delta(h) = \frac{\langle v_1, B_T \rangle}{T} + \sum_{i=1}^{d} \left\{ g_i \int_{0}^{T} \langle \sigma(X_t)^* e_i, d\tilde{B}_t \rangle - D(0, \tilde{h}_i)g_i \right\}.$$  

Noting that $X_t = x + B_t$ is independent of $\tilde{B}$, it is easy to see that $D(0, \tilde{h}_i)g_i = \langle e_i, Q^{-1} \int_{0}^{T} T - t \langle \nabla_{v_1} \sigma(X_t) \tilde{h}_i'(t) dt \rangle$  

$= \langle e_i, Q^{-1} \int_{0}^{T} T - t \{ (\nabla_{v_1} \sigma)^* \} (X_t)e_i dt \rangle$,

which is in $L^2(\mathbb{P})$ according to (A). Combining this with (2.8) and noting that $X_t = x + B_t$, we conclude that $h \in \mathcal{D}(\delta)$ and $\delta(h) = M_T$. Then the proof is finished by Theorem 2.1.  

3 Proofs of Corollary 1.2 and Proposition 1.3

To verify (A) for $\sigma$ given in Corollary 1.2, we first present the following lemma.

**Lemma 3.1.** For any $n \in [1, \infty)$ and $\alpha > 0$, there exists a constant $c > 0$ such that

$$(3.1) \quad \mathbb{E}^{x,y} \left( \int_{0}^{T} |X_t|^{2n} dt \right)^{-\alpha} \leq \frac{c}{T^{\alpha}(|x|^2 + T)^{\alpha n}}, \quad T > 0, (x, y) \in \mathbb{R}^2.$$  

**Proof.** We shall simply denote $\mathbb{E}^{x,y}$ by $\mathbb{E}$. Since $X_t = x + B_t$, for any $\lambda > 0$ we have (see e.g. [10, page 142])

$$\mathbb{E}e^{-\lambda \int_{0}^{T} |X_t|^2 dt} = \prod_{i=1}^{m} \mathbb{E} e^{-\lambda \int_{0}^{T} (x_i + B_t^{(i)})^2 dt} \leq \frac{\exp[-\frac{x^2}{2} \tanh(\sqrt{2}T)]}{\{\coth(\sqrt{2}T)\}^{m/2}}$$

$$\leq 2^{m/2} \exp \left[ -(\frac{T \sqrt{\lambda}}{\sqrt{2}} - \frac{x^2 \sqrt{\lambda}}{2 \sqrt{2}} \{ (\sqrt{2T} \lambda) \wedge 1 \}) \right]$$

$$\leq 2^{m/2} \exp \left[ -\frac{(x^2 + T) \sqrt{\lambda}}{2 \sqrt{2}} \{ (\sqrt{2T} \lambda) \wedge 1 \} \right]$$

$$\leq 2^{m/2} \exp \left[ -\frac{(x^2 + T) \sqrt{\lambda}}{2 \sqrt{2}} \right] + 2^{m/2} \exp \left[ -\frac{(x^2 + T) \lambda T}{\sqrt{2}} \right].$$

This implies that for any $r > 0$,  

$$\mathbb{E}^{x,y} \left( \int_{0}^{T} |X_t|^{2n} dt \right)^{-\alpha} \leq \frac{c}{T^{\alpha}(|x|^2 + T)^{\alpha n}}.$$ 

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\[ \mathbb{E} \exp \left[ -\lambda \int_0^T |X_t|^{2n} \, dt \right] = \mathbb{E} \exp \left[ -\int_0^T \left( \lambda^{1/n} |X_t|^2 \right)^n \, dt \right] \]
\[ \leq \mathbb{E} \exp \left[ \int_0^T \left( \frac{n - 1}{n(n-1)} \right) r^{n/(n-1)} - r \lambda^{1/n} |X_t|^2 \right] \, dt \]
\[ \leq 2^{n/2} \exp \left[ \frac{T(n - 1)}{n(n-1)} r^{n/(n-1)} - \frac{2}{2^{1/2} T(n-1)/2n} \right] \left( \exp \left[ -\left( \frac{x^2 + T}{2^{1/2}} \right) \lambda^{1/(2n)} \sqrt{r} \right] + \exp \left[ -\left( \frac{x^2 + T}{2^{1/2}} \right) \lambda^{1/n} \right] \right) \].

Taking \( r = T^{-(n-1)/n} \) we obtain
\[ \mathbb{E} \exp \left[ -\lambda \int_0^T |X_t|^{2n} \, dt \right] \leq c_1 \left( \exp \left[ -\left( \frac{x^2 + T}{2^{1/2} T(n-1)/2n} \right) \lambda^{1/(2n)} \sqrt{T} \right] + \exp \left[ -\left( \frac{x^2 + T}{2^{1/2} T(n-1)/2n} \right) \lambda T^{1/n} \right] \right) \]
for some constant \( c_1 > 0 \). Noting that
\[ \int_0^\infty \lambda^{\alpha-1} e^{-\theta \lambda^{1/n}} \, d\lambda = \frac{l}{\theta \alpha l} \int_0^\infty e^{-s \lambda^{1/n}} \, ds = \frac{l \Gamma(\alpha l)}{\theta \alpha l} \]
holds for all \( l \geq 1 \) and \( \theta, \alpha > 0 \), we conclude that
\[ \mathbb{E} \left( \int_0^T |X_t|^{2n} \, dt \right)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty \lambda^{\alpha-1} \mathbb{E} \exp \left[ -\lambda \int_0^T |X_t|^{2n} \, dt \right] \, d\lambda \]
\[ \leq c_1 \int_0^\infty \lambda^{\alpha-1} \left\{ \exp \left[ -\left( \frac{|x|^2 + T}{2^{1/2} T(n-1)/2n} \right) \lambda^{1/(2n)} \sqrt{T} \right] + \exp \left[ -\left( \frac{|x|^2 + T}{2^{1/2} T(n-1)/2n} \right) \lambda T^{1/n} \right] \right\} \, d\lambda \]
\[ \leq \frac{c_2 T^{\alpha(n-1)}}{(|x|^2 + T)^{2an}} + \frac{c_3}{(|x|^2 + T)^{anT^{\alpha}}} \leq \frac{c}{(|x|^2 + T)^{anT^{\alpha}}} \]
holds for some constants \( c_2, c_3 \) and \( c \). □

**Proof of Corollary** By Jensen’s inequality, it suffices to prove for \( p \in (1, 2] \) so that
\[ q := \frac{p}{p-1} \geq 2. \] In fact, once (1.6) holds for \( p = 2 \), it also holds for \( p > 2 \) with \( C_p = C_2 \) since in this case \( (P_T f^2)^{1/2} \leq (P_T f^p)^{1/p} \).

It is easy to see that (1.5) implies
\[ Q_T \geq \left( a^2 \int_0^T |X_t|^{2l} \, dt \right) I_{d \times d}, \]
and hence,
\[ (3.1) \quad \|Q_T^{-1}\| \leq \frac{1}{a^2 \int_0^T |X_t|^{2l} \, dt}. \]

Since \( \{X_t\}_{t \in [0,T]} \) is measurable w.r.t. \( \mathcal{G} \) and due to (1.5)
\[ \|\{(\nabla_{v_i} \sigma^*) (X_t)\} \leq b^2 |X_t|^{2l-1}, \]
we obtain
\[
\mathbb{E}\left(\tfrac{\langle v_1, B_T \rangle}{T} - \text{Tr}\left( Q_T^{-1} \int_0^T \frac{T-t}{T} \{ (\nabla v_t \sigma) \sigma^* \} (X_t) dt \right) \right)^q \left\| \mathcal{C} \right\|
\]
\[
= \left| \frac{\langle v_1, B_T \rangle}{T} - \text{Tr}\left( Q_T^{-1} \int_0^T \frac{T-t}{T} \{ (\nabla v_t \sigma) \sigma^* \} (X_t) dt \right) \right|^q
\]
\[
\leq c_1 |v_1|^q \left( \frac{|B_T|^q}{T^q} + \frac{T^{q-1} \int_0^T |X_t|^{(2l-1)q} dt}{(\int_0^T |X_t|^{2q} dt)^q} \right)
\]  
for some constant $c_1 > 0$. Moreover, since $\tilde{B}_t$ is independent of $\mathcal{C}$, due to (3.1) and Lemma 2.2 there exist constants $c_2, c_3 > 0$ such that
\[
\mathbb{E}\left( \|Q_T^{-1}\|^q \left| \left\langle v_2, \int_0^T \sigma(X_t) d\tilde{B}_t \right\rangle \right| \left\| \mathcal{C} \right\| \right) \leq \frac{c_2 |v_2|^q T^{q-1} \int_0^T |X_t|^{(2l-1)q} dt}{(\int_0^T |X_t|^{2q} dt)^q}
\]
and
\[
\mathbb{E}\left( \|Q_T^{-1}\|^q \left| \left\langle v_2, \int_0^T \sigma(X_t) d\tilde{B}_t \right\rangle \right| \left\| \mathcal{C} \right\| \right) \leq \frac{c_2 |v_2|^q T^{q-1} \int_0^T |X_t|^{(2l-1)q} dt}{(\int_0^T |X_t|^{2q} dt)^q}
\]
\[
\mathbb{E}\left( \|Q_T^{-1}\|^q \left| \left\langle v_2, \int_0^T \sigma(X_t) d\tilde{B}_t \right\rangle \right| \left\| \mathcal{C} \right\| \right) \leq \frac{c_3 |v_1|^q T^{q-1} \int_0^T |X_t|^{(2l-1)q} dt}{(\int_0^T |X_t|^{2q} dt)^q}
\]
hold. Combining these with (3.2) we obtain
\[
\mathbb{E}|M_T|^q = \mathbb{E}\left\{ \mathbb{E}( |M_T|^q | \mathcal{C} ) \right\}
\]
\[
\leq c_3 \left\{ \frac{|v_1|^q}{T^{q/2}} + \frac{|v_2|^q T^{q-1} \int_0^T |X_t|^{(2l-1)q} dt}{(\int_0^T |X_t|^{2q} dt)^q} \right\}
\]
By Lemma 3.1 and noting that $X_t = x + B_t$, we conclude that for any $\beta \geq 1,$
\[
\mathbb{E}\left( \int_0^T |X_t|^\beta dt \right)^{q/2} \leq \left\{ \mathbb{E}\left( \int_0^T |X_t|^{2q} dt \right)^2 \right\}^{1/2} \left\{ \mathbb{E}\left( \int_0^T |X_t|^{2q} dt \right)^{-2q} \right\}^{1/2}
\]
\[
\leq c_3 (T \mathbb{E}\int_0^T |X_t|^{2\beta} dt)^{1/2} \leq c_4 T(|x|^2 + T)^{\beta/2} \leq \frac{c_4}{T^{q-1}(|x|^2 + T)^{(q-1)/2}}
\]
holds for some constants $c_3, c_4 > 0$. Substituting this into (3.3) we arrive at
\[
(\mathbb{E}|M_T|^q)^{1/q} \leq c_5 \left\{ \frac{|v_1|}{T^{q/2}} + \frac{|v_2|}{T^{q/2}(|x|^2 + T)^{q/2}} \right\}^{1/q}.
\]
for some constant $c_5 > 0$. Therefore, (1.6) follows since according to Theorem 1.1

$$|
abla_s P_T f(x, y)| = |\mathbb{E}\{f(X_T, Y_T) M_T\}| \leq (P_T|f|^p)^{1/p} (\mathbb{E}|M_T|^q)^{1/q}.$$ 

Proof of Proposition 1.3 (1.8) ⇒ (1.9). By the monotone class theorem, it suffices to prove (1.9) for $f \in C_b(E)$. For $z, z' \in E$, let $\rho = \rho(z, z')$. Up to an approximation argument we assume that $\rho$ is reached by a subunit curve $\gamma : [0, \rho] \to E$ with $\gamma_0 = z, \gamma_\rho = z'$. Then, due to (1.8), for any positive $f \in C_b(E)$ we have

$$\frac{d}{ds} P\left(\frac{f}{1 + r \rho f}\right)(\gamma_s) \leq -P\left(\frac{rf^2}{(1 + rsf)^2}\right)(\gamma_s) + \sqrt{\Gamma_1\left(\frac{f}{1 + rsf}\right)(\gamma_s)}$$

$$\leq -rP\left(\frac{f^2}{(1 + rsf)^2}\right)(\gamma_s) + C \sqrt{P\left(\frac{f}{1 + rsf}\right)^2(\gamma_s)}$$

$$\leq \frac{C^2}{4r}.$$ 

Integrating over $[0, \rho]$ w.r.t. $ds$ we obtain

$$P\left(\frac{f}{1 + r \rho f}\right)(z') \leq Pf(z) + \frac{C^2}{4r}.$$ 

Combining this with the fact that

$$\frac{f}{1 + r \rho f} = f - \frac{r \rho f^2}{1 + r \rho f} \geq f - r \rho f^2,$$

we obtain

$$Pf(z') \leq Pf(z) + \frac{C^2 \rho}{4r} + r \rho Pf^2(z').$$ 

Minimizing the right-hand side in $r > 0$ we prove (1.9).

(1.9) ⇒ (1.8). By (1.9), we have

$$|Pf(x) - Pf(z')| \leq C \rho(x, y) \|f\|_\infty, \quad f \in C_b(M).$$ 

So, $Pf$ is $\rho$-Lipschitz continuous for any $f \in \mathcal{B}_b(E)$. Let $z \in E$ and $\gamma : [0, 1] \to M$ be $C^1$-curve such that $\gamma_0 = z, \rho(\gamma_0, \gamma_s) = s$ and

$$\frac{d}{ds} Pf(\gamma_s)|_{s=0} = \sqrt{\Gamma(Pf)(z)}.$$ 

Then it follows from (1.9) that

$$\sqrt{\Gamma(Pf)(z)} = \lim_{s \to 0} \frac{Pf(\gamma_s) - Pf(\gamma_0)}{s} \leq C \lim_{s \to 0} \sqrt{Pf^2(\gamma_s)} = C \sqrt{Pf^2(z)}.$$ 

Therefore, (1.8) holds.
4 An extension

Consider the following SDE on $\mathbb{R}^{m+d}$:

\begin{equation}
\begin{cases}
\text{d}X_t = \sigma_1(X_t)\text{d}B_t + b_1(X_t)\text{d}t, \\
\text{d}Y_t = \sigma_2(X_t)\text{d}\tilde{B}_t + b_2(X_t)\text{d}t,
\end{cases}
\end{equation}

where $(B_t, \tilde{B}_t)$ is a Brownian motion on $\mathbb{R}^{m+d}$, $\sigma_1 \in C^1_b(\mathbb{R}^m; \mathbb{R}^m \otimes \mathbb{R}^m)$ is invertible with $\|\sigma_1^{-1}\| \leq c$ for some constant $c > 0$, $\sigma_2 \in C^1(\mathbb{R}^m; \mathbb{R}^d \otimes \mathbb{R}^d)$ might be degenerate, $b_1 \in C^1_b(\mathbb{R}^m; \mathbb{R}^m)$ and $b_2 \in C^1(\mathbb{R}^m; \mathbb{R}^d)$. It is easy to see that for any initial data the solution exists uniquely and is non-explosive. Let $P_t$ be the associated Markov semigroup. To establish the derivative formula, let $v = (v_1, v_2) \in \mathbb{R}^{m+d}$ and $T > 0$ be fixed, and let $\xi_t$ solve the following SDE on $\mathbb{R}^m$:

\begin{equation}
\text{d}\xi_t = (\nabla_{\xi_t}\sigma_1)(X_t)\text{d}B_t + \left\{ (\nabla_{\xi_t}b_1)(X_t) - \frac{\xi_t}{T-t} \right\}\text{d}t, \quad \xi_0 = 0.
\end{equation}

Since $\nabla\sigma$ and $\nabla b_1$ are bounded, the equation has a unique solution up to time $T$. It is easy to see from the Itô formula that

\begin{align*}
\text{d}\left\{ \frac{|\xi_t|^2}{T-t} \right\} &= 2\left\langle \frac{\xi_t}{T-t}, (\nabla_{\xi_t}\sigma)(X_t)\text{d}B_t \right\rangle + \left( \frac{\|\nabla_{\xi_t}b_1(X_t)\|^2}{T-t} + \frac{2\langle \xi_t, (\nabla_{\xi_t}b_1)(X_t) \rangle}{T-t} - \frac{|\xi_t|^2}{(T-t)^2} \right)\text{d}t \\
&\leq 2\left\langle \frac{\xi_t}{T-t}, (\nabla_{\xi_t}\sigma)(X_t)\text{d}B_t \right\rangle + \left( C\frac{|\xi_t|^2}{T-t} - \frac{|\xi_t|^2}{(T-t)^2} \right)\text{d}t, \quad t \in [0, T)
\end{align*}

holds for some constant $C > 0$. This implies that for $t \in [0, T)$,

\begin{equation}
\mathbb{E}|\xi_t|^2 \leq (T-t)e^{Ct}, \quad \mathbb{E} \int_0^T \frac{|\xi_t|^2}{(T-t)^2}\text{d}t < \infty,
\end{equation}

Consequently, we may set $\xi_T = 0$ so that $\xi_t$ solves (4.2) for $t \in [0, T]$. Moreover, for any $n \geq 1$ we have

\begin{equation}
\text{d}|\xi_t|^{2n} \leq 2n|\xi_t|^{2(n-1)}\left\langle \xi_t, (\nabla_{\xi_t}\sigma)(X_t)\text{d}B_t \right\rangle + c(n)|\xi_t|^{2n}\text{d}t
\end{equation}

for some constant $c(n) > 0$. Therefore,

\begin{equation}
\sup_{t \in [0, T]} \mathbb{E}|\xi_t|^{2n} < \infty, \quad n \geq 1.
\end{equation}

We are now able to state the derivative formula for $P_t$ as follows.

**Theorem 4.1.** Let $Q_T = \int_0^T \sigma_2(X_t)\sigma_2(X_t)^*\text{d}t$ be invertible such that

\begin{equation}
\mathbb{E}^{\varepsilon, y}\left( \|Q_T^{-1}\|^2 \int_0^T \left\{ \|\sigma_2(X_t)\|^4 + \|\nabla\sigma_2(X_t)\|^4 + \|\nabla b_2(X_t)\|^4 + 1 \right\}\text{d}t \right) < \infty.
\end{equation}
Then
\[ \nabla_v P_T f(x, y) = \mathbb{E}^x \{ f(X_T, Y_T) M_T \} \]
holds for \( f \in C^1_b(\mathbb{R}^{m+d}) \) and
\[
M_T = \int_0^T \left( \frac{\sigma_1(X_s)^{-1}\xi_t}{T-t} dB_t \right) - \text{Tr} \left( Q_T^{-1} \int_0^T \frac{T-t}{T} \{ (\nabla \xi, \sigma_2) \{X_t\} dt \} \right) + \left( Q_T^{-1} \left\{ v_2 + \int_0^T \frac{T-t}{T} (\nabla \xi, \sigma_2) (X_t) d\tilde{B}_t + \int_0^T (\nabla \xi, b_2) (X_t) \right\}, \int_0^T \sigma_2(X_t) d\tilde{B}_t \right). \]

Proof. Let \( h = (h_1, h_2) \), where
\[
h_1(t) = \int_0^t \frac{\sigma_1(X_s)^{-1}\xi_s}{T-s} ds, \ t \in [0, T],
\]
\[
h_2(t) = \left( \int_0^t \sigma_2(X_s)^* ds \right) Q_T^{-1} \left( v_2 + \int_0^T (\nabla \xi, \sigma_2) (X_t) d\tilde{B}_t + \int_0^T (\nabla \xi, b_2) (X_t) dt \right).
\]
As in the proof of Theorem 1.1, it is easy to see from (4.3), (4.4), (4.5) and \( \| \sigma^{-1}_i \| \leq c \) that \( h \in \mathcal{D}(\delta) \) with \( \delta(h) = M_T \). Therefore, it remains to verify that \( (\nabla_v X_t, \nabla_v Y_t) = (D_h X_t, D_h Y_t) \).

It is easy to see that both \( \nabla_v X_t \) and \( D_h X_t + \xi_t \) solve the equation
\[ dV_t = (\nabla_v \sigma_1(X_t) dB_t + (\nabla_v b_1)(X_t) dt, \ t \in [0, T], V_0 = v_1. \]

By the uniqueness of the solution we have \( \nabla_v X_t = D_h X_t + \xi_t \) for \( t \in [0, T] \). Since \( \xi_T = 0 \), this implies that \( \nabla_v X_T = D_h X_T \). Moreover, we have
\[
\begin{cases}
    d\nabla_v Y_t = (\nabla_{\nabla_v X_t} \sigma_2)(X_t) d\tilde{B}_t + (\nabla_{\nabla_v X_t} b_2)(X_t) dt, & \nabla_v Y_0 = v_2, \\
    dD_h Y_t = (\nabla_{D_h X_t} \sigma_2)(X_t) d\tilde{B}_t + \sigma_2(X_t) h'_2(t) dt + (\nabla_{D_h X_t} b_2)(X_t) dt, & D_h Y_0 = 0.
\end{cases}
\]

Combining this with the definition of \( h_2 \) and \( D_h X_t = \nabla_v X_t - \xi_t \), we obtain
\[
D_h Y_T = \nabla_v Y_T - v_2 - \int_0^T (\nabla \xi, \sigma_2) (X_t) d\tilde{B}_t + \int_0^T \sigma_2(X_t) h'_2(t) dt - \int_0^T (\nabla \xi, b_2)(X_t) dt = \nabla_v Y_T.
\]

Therefore, the proof is finished. \( \square \)

Acknowledgement. The author would like to thank the referee for careful reading and useful comments.

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