FOLKLORE ON VECTOR-VALUED HOLOMORPHIC FUNCTIONS IN SEVERAL VARIABLES

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Abstract. In the present paper we give some explicit proofs for folklore theorems on holomorphic functions in several variables with values in a locally complete locally convex Hausdorff space $E$ over $\mathbb{C}$. Most of the literature on vector-valued holomorphic functions is either devoted to the case of one variable or to infinitely many variables whereas the case of (finitely many) several variables is only touched or is subject to stronger restrictions on the completeness of $E$ like sequential completeness. The main tool we use is Cauchy's integral formula for derivatives for an $E$-valued holomorphic function in several variables which we derive via Pettis-integration. This allows us to generalise the known integral formula, where usually a Riemann-integral is used, from sequentially complete $E$ to locally complete $E$. Among the classical theorems for holomorphic functions in several variables with values in a locally complete space $E$ we prove are the identity theorem, Liouville's theorem, Riemann's removable singularities theorem and the density of the polynomials in the $E$-valued polydisc algebra.

1. Introduction

This paper is not meant as a survey article but only to ease our mind when it comes to vector-valued holomorphic functions in several variables, i.e. holomorphic functions $f: \Omega \to E$ from an open set $\Omega \subset \mathbb{C}^d$ to a complex locally convex Hausdorff space $E$, by giving some proofs that are missing in the literature or only touched with reference to the case of one variable.

There is a lot of work available on $\mathbb{C}$-valued holomorphic functions in several variables, like the books by Gunning and Rossi [22], Hörmander [23], Jarnicki and Pflug [24] and Krautz [29]. But when it comes to vector-valued holomorphic functions then most of the work is either restricted to the case of one variable, i.e. $d = 1$, or directly jumps to infinitely many variables, i.e. $\Omega$ is an open subset of a complex infinite dimensional locally convex Hausdorff space $F$. Holomorphy of vector-valued functions in infinitely many variables is discussed for instance by Mujica in [34], where $F$ and $E$ are Banach spaces, and by Dineen in [12], [13] for general locally convex spaces $F$ and $E$. These references also contain results on finitely many variables ($F = \mathbb{C}^d$) but the emphasis is on infinitely many variables.

Banach-valued holomorphic functions in one variable are handled and characterised by Dunford [14, Theorem 76, p. 354] and more recently by Arendt and Nikolski [1], [2]. Holomorphic functions in one variable with values in a locally convex Hausdorff space $E$ are considered in [27, Satz 10.11, p. 241] by Kaballo if $E$ is quasi-complete, in [21] by Grothendieck if $E$ has the convex compactness property (cf. [24, 16.7.2 Theorem, p. 362-363]), in [3] by Bogdanowicz if $E$ is sequentially complete, in [19], [20] by Grosse-Erdmann if $E$ is locally complete and several equivalent conditions describing holomorphy are given. In particular, in all these cases

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holomorphy coincides with weak holomorphy which means that \( f: \Omega \to E \) is holomorphic if and only if the \( \mathbb{C} \)-valued functions \( e' \circ f \) are holomorphic for each \( e' \in E' \) where \( E' \) is the dual space of \( E \). Moreover, the interesting question is treated under which conditions one can replace \( E' \) by a separating subspace \( G \subset E' \) and still can conclude holomorphy from the holomorphy of \( e' \circ f \) for each \( e' \in G \). More general, the extension problem for \( E \)-valued holomorphic functions which have weak holomorphic extensions is studied, in one variable by Grosse-Erdmann in [20], in several variables by Bonet, Frerick and Jordá in [3], [17] and Vitali’s and Harnack’s type results are derived in [20] if \( E \) is locally complete. Further results on vector-valued holomorphic functions in several variables may be found in [1] by Bochnak and Siciak where \( E \) is sequentially complete and in a survey by Barletta and Dragomir [1], extended in [2], but here \( E \) is often restricted to have the convex compactness property or even to be a Fréchet space.

The main purpose of the present paper is to derive some equivalent characterisations of holomorphic functions in several variables with values in a locally complete space \( E \) (see Corollary 3.19, Theorem 3.20, Corollary 3.22) with explicit proofs avoiding the usual ‘like in the case of one variable’ (see e.g. the four-line Section 4.1, p. 409). Of course, the short reference to the case of one variable is often due to constraints, like page limits or the perception as folklore since it is known to everyone from the field how to transfer the results from one variable to several variables but never written down not least because of the low chance to get it published. This is the reason why we wrote this down so that we have a reference with explicit proofs, not more, not less. Anyway, our main tool to obtain the equivalent characterisations of holomorphic functions in several variables with values in a locally complete space \( E \) is Cauchy’s integral formula for derivatives which we obtain via Pettis-integration (Theorem 3.12). To the best of our knowledge Cauchy’s integral formula for derivatives for holomorphic functions with values in a locally complete space \( E \) is not contained in the literature. Usually, Riemann-integration is used instead of Pettis-integration and \( E \) has to be sequentially complete or the derivatives have to be considered in the completion of \( E \). On the way to our main Theorem 3.20 we derive Fubini’s theorem (Theorem 3.27) and Leibniz’ rule for differentiation under the integral sign (Lemma 3.11) for holomorphic functions with values in a locally complete space. We use our main theorem to prove some classical theorems like the identity theorem (Theorem 3.24), Liouville’s theorem (Theorem 3.25), Riemann’s removable singularities theorem (Theorem 3.26) and the density of the polynomials in the \( E \)-valued polydisc algebra (Corollary 3.27).

2. Notation and Preliminaries

We equip the spaces \( \mathbb{R}^d \) and \( \mathbb{C}^d \), \( d \in \mathbb{N} \), with the usual Euclidean norm \( |\cdot| \). Moreover, we denote by \( B_r(x) := \{ w \in \mathbb{R}^d \mid |w - x| < r \} \) the ball around \( x \in \mathbb{R}^d \) with radius \( r > 0 \) and use the same notation when \( \mathbb{R}^d \) is replaced by \( \mathbb{C}^d \). Furthermore, for a subset \( M \) of a topological space \( X \) we denote by \( \overline{M} \) the closure of \( M \) in \( X \). For a subset \( M \) of a topological vector space \( X \), we write \( \text{aconv}(M) \) for the closure of the absolutely convex hull \( \text{aconv}(M) \) of \( M \) in \( X \).

By \( E \) we always denote a non-trivial locally convex Hausdorff space (lcHs) over the field \( K = \mathbb{R} \) or \( \mathbb{C} \) equipped with a directed fundamental system of seminorms \( (p_\alpha)_{\alpha \in \mathcal{A}} \). If \( E = K \), then we set \( (p_\alpha)_{\alpha \in \mathcal{A}} := \{ |\cdot| \} \). Further, we write \( \hat{E} \) for the completion of \( E \) and for a disk \( D \subset F \), i.e. a bounded, absolutely convex set, we write \( E_D := \bigcup_{n \in \mathbb{N}} nD \) which becomes a normed vector space if it is equipped with gauge functional of \( D \) as a norm (see [24], p. 151). The space \( E \) is called locally complete if \( E_D \) is a Banach space for every closed disk \( D \subset F \) (see [24], 10.2.1 Proposition, p. 197). In particular, every sequentially complete space is locally complete.
complete and this implication is strict. Further, we recall the following definitions from [38, p. 259] and [39, 9-2-8 Definition, p. 134]. A locally convex Hausdorff space is said to have the \textit{metric} convex compactness property (\textit{metric} ccp) if the closure of the absolutely convex hull of every \textit{metrisable} compact set is compact. Equivalently this definition can be phrased with the convex hull instead of the absolutely convex hull. Every locally convex Hausdorff space with ccp has metric ccp, every quasi-complete locally convex Hausdorff space has ccp, every sequentially complete locally convex Hausdorff space has metric ccp and every locally convex Hausdorff space with metric ccp is locally complete and all these implications are strict (see [38, p. 3-4] and the references therein). For more details on the theory of locally convex spaces see [16, 24] or [38].

For \( k \in \mathbb{N}_{0, \infty} \) we denote by \( \mathcal{C}^k(\Omega, E) \) the space of \( k \)-times continuously partially differentiable functions on an open set \( \Omega \subset \mathbb{R}^d \) with values in a locally convex Hausdorff space \( E \). We say that a function \( f: \Omega \to E \) is weakly \( \mathcal{C}^k \) if \( e' \circ f \in \mathcal{C}^k(\Omega) := \mathcal{C}^k(\Omega, \mathbb{K}) \) for each \( e' \in E' \). By \( L(F, E) \) we denote the space of continuous linear operators from \( F \) to \( E \) where \( F \) and \( E \) are locally convex Hausdorff spaces.

If \( E = \mathbb{K} \), we just write \( F' := L(\mathbb{F}, \mathbb{K}) \) for the dual space. We write \( L_{\varepsilon}(F, E) \) for the space \( L(F, E) \) equipped with the locally convex topology of uniform convergence on compact subsets of \( F \) if \( t = c \), on the absolutely convex, compact subsets of \( F \) if \( t = b \) and on the bounded subsets of \( F \) if \( t = b \). The so-called \( \varepsilon \)-product of Schwartz is defined by

\[
F \in E := L_{\varepsilon}(F'_{\kappa}, E)
\]

where \( L(F'_{\kappa}, E) \) is equipped with the topology of uniform convergence on the equicontinuous subsets of \( F' \) (see e.g. [37, Chap. 1, §1, Définition, p. 18]). For more information on the theory of \( \varepsilon \)-products see [24] and [27].

3. Holomorphic functions in several variables

3.1. Definition ((weakly, separately, Gâteaux-) differentiable, holomorphic). Let \( E \) be an lcHs over \( \mathbb{K} \), let \( \Omega \subset \mathbb{K}^d \) be open and \( f: \Omega \to E \).

- a) \( f \) is called \textit{differentiable} (on \( \Omega \)) if for every \( z \in \Omega \) there is a \( \mathbb{K} \)-linear map \( df(z) := dfz: \mathbb{K}^d \to \hat{E} \) such that

\[
\lim_{w \to z, w \neq z} \frac{f(w) - f(z) - dfz(w - z)}{|w - z|} = 0 \quad \text{in} \ \hat{E}
\]

and the map \( df(z): \Omega \to \hat{E} \) is continuous for every \( v \in \mathbb{K}^d \).

- b) \( f \) is called the \textit{Gâteaux-differentiable} (on \( \Omega \)) if

\[
Df(z)[v] := D_G f(z)[v] := \lim_{h \to 0} \frac{f(z + hv) - f(z)}{h} \quad \text{exists in} \ \hat{E}
\]

for every \( z \in \Omega \) and \( v \in \mathbb{K}^d \).

- c) If \( v = e_j \) is the \( j \)-th unit tim unit vector for \( 1 \leq j \leq d \) and \( z \in \Omega \), we write

\[
(\partial^c_k)^Ef(z) := (\partial z_j)^E f(z) := D_G f(z)[e_j]
\]

if \( D_G f(z)[e_j] \) exists in \( E \). Especially, we use \( f'(z) := (\partial^c_k)^E f(z) \) if \( d = 1 \).

- d) For \( z = (z_1, \ldots, z_d) \in \Omega \) we define the continuous function

\[
\pi_{z,j}: \mathbb{K} \to \mathbb{K}^d, \quad \pi_{z,j}(w) := (z_1, \ldots, z_{j-1}, w, z_{j+1}, \ldots, z_d).
\]

\( f \) is called \textit{separately differentiable} (on \( \Omega \)) if \( f \) is a differentiable function in each variable, i.e. \( f \circ \pi_{z,j}: \mathbb{K}^1 \to E \) is differentiable for every \( z \in \Omega \) and \( 1 \leq j \leq d \).

- e) \( f \) is called \textit{weakly (separately, Gâteaux-) differentiable} (on \( \Omega \)) if \( e' \circ f: \Omega \to \mathbb{K} \) is (separately, Gâteaux-) differentiable for every \( e' \in E' \).
f) If $\mathbb{K} = \mathbb{C}$, we say holomorphic or complex differentiable instead of differentiable on the open set $\Omega$ and, if $\mathbb{K} = \mathbb{R}$, we sometimes say real differentiable.

3.2. Remark. Let $E$ be an lcHs over $\mathbb{K}$, $\Omega \subset \mathbb{K}^d$ open and $f: \Omega \to E$.

a) If $f$ is differentiable, then $df: \Omega \times \mathbb{K}^d \to \overline{E}$ is continuous.

b) If $f$ is differentiable, then $f: \Omega \to E$ is continuous.

c) If $f$ is differentiable, then $f$ is Gâteaux- and separately differentiable and

$$df(z)[v] = Df(z)[v] = \sum_{j=1}^{d} (\partial^j_E f)(z)v_j, \quad z \in \Omega, \ v = (v_1, \ldots, v_d) \in \mathbb{K}^d.$$

d) If $f$ is (separately, Gâteaux-) differentiable, then $f$ is weakly (separately, Gâteaux-) differentiable.

e) If $(\partial^j_E f)(z) \in E$ for some $1 \leq j \leq d$ and $z \in \Omega$, then

$$(\partial^j_E f)(z) = (\partial^j_E f)(z).$$

Proof. a) First, we remark that $df(z): \mathbb{K}^d \to \overline{E}$ is continuous for every $z \in \Omega$ since $df(z)$ is linear and $\mathbb{K}^d$ a finite dimensional normed space. Let $(z, v) \in \Omega \times \mathbb{K}^d$, $\varepsilon > 0$ and $\alpha \in \overline{\mathbb{R}}$ where $(\overline{E}, (p_\alpha)_{\alpha \in \mathbb{R}})$ is the completion of $E$. For every $(w, x) \in \Omega \times \mathbb{K}^d$ we estimate

$$p_\alpha(df(w)[x] - df(z)[v]) \leq p_\alpha(df(w)[x-v]) + p_\alpha(df(w)[v] - df(z)[v]) \leq \sqrt{d} \sup_{1 \leq j \leq d} p_\alpha(df(w)[e_j])[x-v] + p_\alpha(df(w)[v] - df(z)[v]).$$

Since $df(\cdot)[v]: \Omega \to \overline{E}$ is continuous, there is $\delta = \delta_{\alpha, z, v} > 0$ such that for all $w \in \Omega$ with $|w - z| < \delta$ we have

$$p_\alpha(df(w)[v] - df(z)[v]) < \varepsilon/2.$$ 

As $\Omega$ is open, there is $\delta_0 > 0$ such that $K_z := \overline{B}_{\delta_0}(z) \subset \Omega$. From the compactness of $K_z$ and the continuity of $df(\cdot)[e_j]: \Omega \to \overline{E}$ for every $1 \leq j \leq d$ we deduce that

$$C_{j,z} := \sup_{w \in K_z} p_\alpha(df(w)[e_j]) < \infty.$$ 

Thus we obtain for every $(w, x) \in \Omega \times \mathbb{K}^d$ with

$$|(w, x) - (z, v)| \leq \min\left(\delta, \delta_0, \frac{\varepsilon}{2(1 + \sqrt{d} \sup_{1 \leq j \leq d} C_{j,z})}\right)$$

that

$$p_\alpha(df(w)[x] - df(z)[v]) \leq \sqrt{d} \sup_{1 \leq j \leq d} C_{j,z}[x-v] + (\varepsilon/2) < (\varepsilon/2) + (\varepsilon/2) = \varepsilon.$$ 

b) Let $z \in \Omega$, $\varepsilon > 0$ and $\alpha \in \overline{\mathbb{R}}$. Then there is $\delta > 0$ such that for all $w \in \mathbb{K}^d$ with $0 < |w - z| < \delta$ we have

$$p_\alpha\left(\frac{f(w) - f(z)}{|w - z|}\right) - p_\alpha\left(\frac{df(z)[w-z]}{|w-z|}\right) \leq p_\alpha\left(\frac{f(w) - f(z) - df(z)[w-z]}{|w-z|}\right) < 1.$$ 

It follows from the continuity of $df(z): \mathbb{K}^d \to \overline{E}$ that there is $C > 0$ such that

$$p_\alpha\left(\frac{f(w) - f(z)}{|w - z|}\right) < 1 + p_\alpha\left(\frac{df(z)[w-z]}{|w-z|}\right) \leq 1 + C\frac{|w-z|}{|w-z|} = 1 + C.$$ 

Thus we have for all $w \in \mathbb{K}^d$ with $|w - z| < \min(\varepsilon/(1 + C), \delta)$ that

$$p_\alpha(f(w) - f(z)) \leq (1 + C)|w-z| < \varepsilon$$

implying $f \in C^0(\Omega, E)$. Since $f(\Omega) \subset E$, we derive that $f: \Omega \to E$ is continuous.
Proof. Let \( z \in \Omega, v \in \mathbb{K}^d, \alpha \in \mathbb{K} \) and \( h \in \mathbb{K}, h \neq 0 \), such that \( z + hv \in \Omega \). We observe that
\[
p_a \left( \frac{f(z + hv) - f(z)}{h} \right) = \frac{|v| p_a \left( f(z + hv) - f(z) - df(z)[hv] \right)}{|hv|}
\]
which yields the Gâteaux-differentiability of \( f \) and \( Df(z)[v] = df(z)[v] \) because \( f \) is differentiable. Due to the linearity of \( df(z) \) we obtain for \( v = (v_1, \ldots, v_d) \in \mathbb{K}^d \)
\[
df(z)[v] = \sum_{j=1}^{d} df(z)[e_j]v_j = \sum_{j=1}^{d} Df(z)[e_j]v_j = \sum_{j=1}^{d} \left( \frac{\partial f_j}{\partial z_k} \right) E f(z)v_j.
\]
Finally, let \( 1 \leq j \leq d \) and \( z_0 \in \pi_z^{-1}(\Omega) \subset \mathbb{K} \). Clearly, \( v_0 \mapsto df(\pi_z, z_0)) [e_j] \cdot v_0 \) is a linear map from \( \mathbb{K} \) to \( \mathbb{E} \) and \( df(\pi_z, z_0)) [e_j] \cdot v_0 : \pi_z^{-1}(\Omega) \to \mathbb{E} \) is continuous for every \( v_0 \in \mathbb{C} \) by the differentiability of \( f \) and the continuity of \( \pi_z \). We set \( z := \pi_z(z_0) \) and get for \( v_0 \in \pi_z^{-1}(\Omega), v_0 \neq z_0 \), that
\[
p_a \left( \frac{f(\pi_z z_0) - f(\pi_z z_0)}{w_0 - z_0} \right) = p_a \left( \frac{(\pi_z + (w_0 - z_0)v_j) - f(z)}{w_0 - z_0} \right) = p_a \left( \frac{(\pi_z + (w_0 - z_0)v_j) - f(z)}{w_0 - z_0} \right).
\]
Letting \( w_0 \to z_0 \), we derive that \( f \) is separately differentiable and
\[
d(\pi_z z_0)(v_0) = df(\pi_z z_0)) [e_j] \cdot v_0, \quad v_0 \in \mathbb{K}.
\]

d) We just have to observe that \((\mathbb{E})' = \mathbb{E}' \) by \([24, 3.4.4 \text{ Corollary, p. 63}]\) and get for every \( e' \in \mathbb{E}' \)
\[
d(e' \circ f) = e' \circ df, \quad D(e' \circ f) = e' \circ Df, \quad d(e' \circ f \circ \pi_z) = e' \circ d(f \circ \pi_z)
\]
for differentiable, Gâteaux-differentiable and separately differentiable \( f \), respectively.

c) Follows directly from Definition 3.1 c) and the fact that \( \mathbb{E} \) is Hausdorff by \([24, 3.3.2 \text{ Theorem, p. 60}]\). \( \square \)

We denote by \( \phi : \mathbb{C}^d \to \mathbb{R}^{2d}, \phi(Re z_1 + i Im z_1, \ldots, Re z_d + i Im z_d) := (Re z_1, Im z_1, \ldots, Re z_d, Im z_d) \), the isometric isomorphism between \( \mathbb{C}^d \) and \( \mathbb{R}^{2d} \) with respect to Euclidean norm on both sides and remark the following.

3.3. Remark. Let \( E \) be an lcHs over \( \mathbb{C}, \Omega \subset \mathbb{C}^d \) open and \( f : \Omega \to E \) holomorphic. Then \( f \circ \phi^{-1} \) is real differentiable on \( \phi(\Omega) \) and
\[
d_{\mathbb{R}}(f \circ \phi^{-1})(x)[v] = df(\phi^{-1}(x))\phi^{-1}(v), \quad x \in \phi(\Omega), v \in \mathbb{R}^{2d}.
\] (1)
In particular, \( f \circ \phi^{-1} \in \mathcal{C}^1(\phi(\Omega), \mathbb{E}) \).

Proof. \( E \) is also an lcHs over \( \mathbb{R} \) and equation (1) follows from
\[
\frac{(f \circ \phi^{-1})(y) - (f \circ \phi^{-1})(x) - d_{\mathbb{C}}f(\phi^{-1}(x))\phi^{-1}(y))}{|x - y|}
= \frac{f(\phi^{-1}(y)) - f(\phi^{-1}(x)) - d_{\mathbb{C}}f(\phi^{-1}(x))\phi^{-1}(y)}{|\phi^{-1}(x) - \phi^{-1}(y)|}
\]
for \( x, y \in \phi(\Omega), x \neq y \), and the holomorphy of \( f \) on \( \Omega \). The \( \mathbb{R} \)-linearity of \( d_{\mathbb{R}}(f \circ \phi^{-1})(x) \) for every \( x \in \phi(\Omega) \) and the continuity of \( d_{\mathbb{R}}(f \circ \phi^{-1})(\cdot) [v] : \phi(\Omega) \to \mathbb{E} \) for every \( v \in \mathbb{R}^{2d} \) are a direct consequence of (1).
Since \( f \) is continuous on \( \Omega \) by Remark 3.2(b), the map \( f \circ \phi^{-1} \) is continuous on \( \phi(\Omega) \). Further, if \( e_j \) is the \( j \)th unit vector in \( \mathbb{R}^{2d} \), we obtain

\[
(\partial_R^j)^E(f \circ \phi^{-1})(x) = d_R(f \circ \phi^{-1})(x)[e_j] = dc(f(\phi^{-1})(x))[\phi^{-1}(e_j)], \quad x \in \phi(\Omega).
\]

It follows that \( f \circ \phi^{-1} \) is continuously partially (real) differentiable on \( \phi(\Omega) \) because \( dc(f)[\phi^{-1}(e_j)] \) is continuous for every \( 1 \leq j \leq 2d \).

Our next goal is to derive Cauchy’s integral formula (for derivatives) for a holomorphic function with values in a locally complete lcHs \( E \). We use the notion of a Pettis-integral to define integration of a vector-valued function.

3.4. Definition (Pettis-integral). Let \( \Omega \subset \mathbb{R}^d \), \( E \) an lcHs, \((\Omega, \mathcal{L}(\Omega), \lambda)\) be the measure space of Lebesgue measurable sets and \( \mathcal{L}^1(\Omega, \lambda) \) the space of \( \mathbb{K} \)-valued Lebesgue-integrable (equivalence classes of) functions on \( \Omega \). A function \( f: \Omega \to E \) is called weakly measurable if the function \( e' \circ f: X \to \mathbb{K}, (e' \circ f)(x) := (e', f(x)) \) is Lebesgue measurable for all \( e' \in E' \). A weakly measurable function is said to be weakly integrable if \( e' \circ f \in \mathcal{L}^1(\Omega, \lambda) \). A function \( f: \Omega \to E \) is called Pettis-integrable on \( \Lambda \in \mathcal{L}(\Omega) \) if it is weakly integrable on \( \Lambda \) and

\[
\exists e_\Lambda(f) \in E \forall e' \in E' : (e', e_\Lambda(f)) = \int_\Lambda (e', f(x))dx.
\]

In this case \( e_\Lambda(f) \) is unique due to \( E \) being Hausdorff and we define the Pettis-integral of \( f \) on \( \Lambda \) by

\[
\int_\Lambda f(x)dx := e_\Lambda(f).
\]

A function \( \gamma: [a, b] \to \mathbb{C} \) is called a \( C^1 \)-curve (in \( \mathbb{C} \)) if \( \gamma \) can be extended to a continuously differentiable function on an open set \( X \subset \mathbb{R} \) with \([a, b] \subset X \). For a family \((\gamma_k)_{1 \leq k \leq d} \) of \( C^1 \)-curves \( \gamma_k: [a, b] \to \mathbb{C} \) a function

\[
\gamma: [a, b]^d \to \mathbb{C}^d, \quad \gamma(t_1, \ldots, t_d) := (\gamma_1(t_1), \ldots, \gamma_d(t_d)),
\]

is called a \( C^1 \)-curve (in \( \mathbb{C}^d \)) and we set

\[
l(\gamma) := \int_{[a, b]^d} \prod_{k=1}^d \gamma_k'(t_k)|dt = \prod_{k=1}^d \int_a^b |\gamma_k'(t_k)|dt_k
\]

which is the product of the length of the curves \( \gamma_k \). We say that \( \gamma \) is a \( C^1 \)-curve in \( \Omega \subset \mathbb{C}^d \) if there are open sets \( X_k \subset \mathbb{R} \) such that \([a, b] \subset X_k \) and \( \gamma_k \) can be extended to a continuously differentiable function \( \overline{\gamma}_k \) on \( X_k \) for every \( 1 \leq k \leq d \) and the so-defined extension \( \overline{\gamma} := (\overline{\gamma}_k)_{k} \) of \( \gamma \) on the open set \( X := \prod_{1 \leq k \leq d} X_k \subset \mathbb{R}^d \) fulfills \( \overline{\gamma}(X) \subset \Omega \).

Let \( E \) be an lcHs over \( \mathbb{C}, \Omega \subset \mathbb{C}^d \) and \( \gamma: [a, b]^d \to \mathbb{C}^d \) be a \( C^1 \)-curve in \( \Omega \). We define the (Pettis)-integral of a function \( f: \Omega \to E \) along \( \gamma \) by

\[
\int_\gamma f(z)dz := \int_{[a, b]^d} f(\gamma(t))\prod_{k=1}^d |\gamma_k'(t_k)|dt,
\]

if the Pettis-integral on the right-hand side exists. If the integral exists, we call \( f \) integrable along \( \gamma \). Since \( \gamma \) is a \( C^1 \)-curve in \( \Omega \), there is some open set \( X \subset \mathbb{R}^d \) such that \([a, b]^d \subset X \) and \( \gamma \) can be extended to a \( C^1 \)-function \( \overline{\gamma} \) on \( X \) with \( \overline{\gamma}(X) \subset \Omega \). If the extension of the factor of the integrand given by

\[
g: X \to E, \quad g(t) := f(\overline{\gamma}(t)),
\]

is a weakly \( C^1 \) function on \( X \), we call \( f \) weakly \( \gamma \)-\( C^1 \).
3.5. **Proposition.** Let $E$ be a locally complete lcHs over $\mathbb{C}$, $\Omega \subset \mathbb{C}^d$ open, $\gamma$ a $C^1$-curve in $\Omega$ and $f: \Omega \to E$.

a) If $f$ is weakly $\gamma$-$C^1$, then $f$ is integrable along $\gamma$.

b) If $f \circ \phi^{-1}: \phi(\Omega) \to E$ is weakly $C^1$, then $f$ is weakly $\gamma$-$C^1$ in $\Omega$.

c) If $f: \Omega \to E$ is holomorphic, then $f$ is weakly $\gamma$-$C^1$.

**Proof.** a) As $f$ is weakly $\gamma$-$C^1$, there is some open set $X \subset \mathbb{R}^d$ such that $[a, b]^d \subset X$ and $\gamma$ can be extended to a $C^1$-function $\tilde{\gamma}$ on $X$ with $\tilde{\gamma}(X) \subset \Omega$ so that $f \circ \tilde{\gamma}$ is weakly $C^1$ on $X$. We observe that

\[ |I_f(e')| := \left| \int_{[a, b]^d} (e', f(\gamma(t))) \prod_{k=1}^d \gamma_k'(t_k) dt \right| \leq l(\gamma) \sup_{x \in f([a, b]^d)} |e'(x)|, \quad e' \in E'. \]

The closure of the absolutely convex hull $\overline{\operatorname{aff}} f([a, b]^d)$ is compact by [10, Proposition 2, p. 354] since $f \circ \tilde{\gamma}$ is weakly $C^1$ on $X$. Hence it follows that $I_f \in (E')'$ and we deduce from the Mackey-Arens theorem that there is $e(f \circ \gamma) \in E$ such that

\[ (e', f(\gamma)) = I_f(e') = \int_{[a, b]^d} (e', f(\gamma(t))) \prod_{k=1}^d \gamma_k'(t_k) dt, \quad e' \in E', \]

implying the integrability of $f$ along $\gamma$.

b) Indeed, writing

\[ e' \circ (f \circ \tilde{\gamma}) = (e' \circ (f \circ \phi^{-1})) \circ (\phi \circ \gamma), \quad e' \in E', \]

for a $C^1$-extension $\tilde{\gamma}$ of $\gamma$ on $X$, we see that $f \circ \tilde{\gamma}$ is weakly $C^1$ on $X$ by the scalar version of the chain rule.

c) We just have to notice that $f$ is weakly holomorphic by Remark 3.2(d) implying that $e' \circ (f \circ \phi^{-1}) \in C^\infty(\phi(\Omega))$ for all $e' \in E'$ which proves the claim by part b). \(\square\)

Next, we prove Fubini’s theorem which facilitates the computation of an integral along a curve. We recall the following lemma whose proof is similar to the one of Proposition 3.5(a).

3.6. **Lemma** ([31], 4.7 Lemma, p. 14]). Let $E$ be a locally complete lcHs, $\Omega \subset \mathbb{R}^d$ open and $f: \Omega \to E$. If $f$ is weakly $C^1$, then $f$ is Pettis-integrable (w.r.t. to the Lebesgue measure) on every compact subset of $K \subset \Omega$.

3.7. **Theorem** (Fubini’s theorem). Let $E$ be a locally complete lcHs, $\Omega \subset \mathbb{R}^d$ open, $[a, b] \times [c, d] \subset \Omega$ and $f: \Omega \to E$ weakly $C^1$. Then $f$ is Pettis-integrable on $[a, b] \times [c, d]$ and

\[ \int_{[a, b] \times [c, d]} f(x_1, x_2) dx_1 dx_2 = \int_{[a, b]} \int_{[c, d]} f(x_1, x_2) dx_1 dx_2 = \int_{[a, b]} \int_{[c, d]} f(x_1, x_2) dx_2 dx_1. \]

**Proof.** The function $f$ is Pettis-integrable on $[a, b] \times [c, d]$ by Lemma 3.6. Since $\Omega$ is open, there are $\overline{a} < a$, $b < \overline{b}$, $\overline{c} < c$ and $d < \overline{d}$ such that $[\overline{a}, \overline{b}] \times [\overline{c}, \overline{d}] \subset \Omega$. Further, we observe that

\[ F_1(\overline{c}, \overline{d}) \to E, \quad F_2(x_2) := \int_{[a, b]} f(x_1, x_2) dx_1, \]

is well-defined by Lemma 3.6 since $f(\cdot, x_2)$ is weakly $C^1$ on $(\overline{a}, \overline{b})$ for every $x_2 \in (\overline{c}, \overline{d})$. We claim that $F$ is weakly $C^1$ on $(\overline{c}, \overline{d})$. Indeed, we have $(e' \circ f)(\cdot, x_2) \in L^1([a, b])$ and $(e' \circ f)(x_1, \cdot) \in C^1((\overline{c}, \overline{d}))$ as well as

\[ (e' \circ F)(x_2) = \int_{[a, b]} (e' \circ f)(x_1, x_2) dx_1 \quad (2) \]
for every \( x_1 \in [a, b] \), \( x_2 \in (\overline{c}, \overline{d}) \) and \( \varepsilon' \in E' \). Furthermore, for every \( x_2 \in (\overline{c}, \overline{d}) \) there is \( \varepsilon > 0 \) with \( B_{\varepsilon}(x_2) = [x_2 - \varepsilon, x_2 + \varepsilon] \subset (\overline{c}, \overline{d}) \) and

\[
C_{\varepsilon'} := \sup \{ |d_{x_2}(\varepsilon' \circ f)(x_1, \overline{x}_2)| \mid (x_1, \overline{x}_2) \in [a, b] \times B_{\varepsilon}(x_2) \} < \infty, \quad \varepsilon' \in E',
\]

because \( \varepsilon' \circ f \in C^0(\Omega) \) for every \( \varepsilon' \in E' \). It follows from the scalar Leibniz rule for differentiation under the integral sign and the continuous dependency of a scalar integral on a parameter (see [13, 5.6, 5.7 Satz, p. 147-148]) that \( \varepsilon' \circ F \in C^0(\mathbb{B}_R(x_2)) \) for every \( \varepsilon' \in E' \). As \( x_2 \in (\overline{c}, \overline{d}) \) is arbitrary, we get that \( F \) is weakly \( C^1 \) on \( (\overline{c}, d) \).

Due to Remark 3.2 a)-c) and Remark 3.3 the map \( F \colon \Omega \rightarrow E \) is continuous, the map \( F \circ \phi^{-1} \colon \phi(\Omega) \rightarrow E \) is \( \gamma \)-\( C^1 \)-curve in \( \Omega \) and \( F \colon \Omega \rightarrow E \) holomorphic. Then for every \( 1 \leq j \leq d \)

\[
(\partial_{x_j}^\gamma)^E((F \circ \phi^{-1}) \circ (\phi \circ \gamma))(t) = (\partial_{x_j}^\gamma)^E F(\gamma(t)) \gamma_j^\gamma(t), \quad t \in [a, b].
\]

Fubini’s theorem for a continuous function \( f \colon \Omega \subset \mathbb{R}^2 \rightarrow E \) can also be found in [11], Chap. 3, §4.1, Remark, p. INT III.43] by Bourbaki under the restriction that \( \bar{\alpha} \in \mathbb{R}(f([a, b] \times [c, d])) \) is compact in \( E \). From the condition that \( f \colon \Omega \rightarrow E \) is weakly \( C^1 \) follows that \( f \) is continuous if \( E \) is sequentially complete or more general if \( E \) has metric ccp by [32, 6.4 Corollary, p. 19]. Thus in this case one can also apply Bourbaki’s version of Fubini’s theorem.

3.8. **Remark.** Let \( E \) be a locally complete lcHs over \( C, \Omega \subset C^d \) open and \( \gamma \) a \( C^1 \)-curve in \( \Omega \). If \( f \colon \Omega \rightarrow E \) is weakly \( \gamma \)-\( C^1 \), then

\[
\int_{[a, b]} f(z) dz = \int_{[a, b]} \cdots \int_{[a, b]} f(\gamma(t)) \prod_{k=1}^{d} \gamma_k^\gamma(t_k) d\gamma_1 \cdots d\gamma_d.
\]

by Fubini’s theorem.

3.9. **Proposition** (chain rule). Let \( E \) be an lcHs over \( C, \Omega \subset C^d \) open, \( \gamma \colon [a, b]^d \rightarrow C^d \) a \( C^1 \)-curve in \( \Omega \) and \( F \colon \Omega \rightarrow E \) holomorphic. Then for every \( 1 \leq j \leq d \)

\[
(\partial_{x_j}^\gamma)^E((F \circ \phi^{-1}) \circ (\phi \circ \gamma))(t) = (\partial_{x_j}^\gamma)^E F(\gamma(t)) \gamma_j^\gamma(t), \quad t \in [a, b]^d.
\]

**Proof.** Due to Remark 3.2 a)-c) and Remark 3.3 the map \( F \circ \phi^{-1} \colon \phi(\Omega) \rightarrow E \) is continuous, the map \( D_R(F \circ \phi^{-1})(x) \colon \mathbb{R}^{2d} \rightarrow \mathbb{E} \) is \( \gamma \)-linear and the map \( D_R(F \circ \phi^{-1}) \colon \mathbb{R}^{2d} \rightarrow \mathbb{E} \) is continuous. The set \( \phi(\Omega) \) is open, thus for every \( x \in \phi(\Omega) \) there is \( R > 0 \) such that \( B_R(x) \subset \phi(\Omega) \). Hence \( D_R(F \circ \phi^{-1}) \) is uniformly continuous on the compact set \( B_R(x) \times K \) for any compact set \( K \subset \mathbb{R}^{2d} \). Let \( (\mathbb{E}, (p_\alpha)_{\alpha \in \mathbb{N}}) \)
denote the completion of $E$. It follows that for every $\alpha \in \mathbb{K}$ and $\varepsilon > 0$ there is $\delta > 0$ such that for all $y \in \overline{mR(x)}$ and $v \in K$ with $|y - x| = |(y, v) - (x, v)| < \delta$ we have
\[
\sup_{v \in K} \left| D_{\mathbb{R}}(F \circ \phi^{-1})(y)[v] - D_{\mathbb{R}}(F \circ \phi^{-1})(x)[v] \right| < \varepsilon
\]
implies that
\[
D_{\mathbb{R}}(F \circ \phi^{-1})(\phi(\Omega)) \to L_{c}(\mathbb{R}^{2d}, \overline{E})
\]
is continuous. Since $\gamma$ is a $C^{1}$-curve in $\Omega$, there are an open set $X \subset \mathbb{R}^{d}$ with $[a, b]^{d} \subset X$ and a continuously partially differentiable extension $\overline{\gamma}$ of $\gamma$ on $X$ such that $\phi \circ \overline{\gamma} : X \to \mathbb{R}^{2d}$ is continuous, $D_{\overline{\mathbb{R}}}(\phi \circ \overline{\gamma}) : X \to \mathbb{L}(\mathbb{R}^{d}, \mathbb{R}^{2d})$ and by direct computation
\[
D_{\mathbb{R}}(\phi \circ \overline{\gamma})(x)[v] = \sum_{k=1}^{d} ((\text{Re} \overline{\gamma})'(x_{k})e_{2k-1} + (\text{Im} \overline{\gamma})'(x_{k})e_{2k})v_{k}, \quad x \in X, \; v \in \mathbb{R}^{d}.
\]
For $x, y \in X$ we set
\[
u_{k} := (\text{Re} \overline{\gamma})'(y_{k}) - (\text{Re} \overline{\gamma})'(x_{k}) \quad \text{and} \quad w_{k} := (\text{Im} \overline{\gamma})'(y_{k}) - (\text{Im} \overline{\gamma})'(x_{k})
\]
and observe for $v \in \mathbb{R}^{d}$ that
\[
\left| D_{\mathbb{R}}(\phi \circ \overline{\gamma})(y)[v] - D_{\mathbb{R}}(\phi \circ \overline{\gamma})(x)[v] \right| = \left| \sum_{k=1}^{d} (u_{k}e_{2k-1} + w_{k}e_{2k})v_{k} \right| \leq \sum_{k=1}^{d} |u_{k}^{2} + w_{k}^{2}|^{1/2}|v_{k}|
\]
where the second inequality follows from the Cauchy-Schwarz inequality. This implies that $D_{\mathbb{R}}(\phi \circ \overline{\gamma}) : X \to L_{c}(\mathbb{R}^{d}, \overline{\mathbb{R}^{d}})$ is continuous because $\gamma$ is continuously partially differentiable. Summarising, this means that $F \circ \phi^{-1}$ and $\phi \circ \overline{\gamma}$ are of class $C^{1}$ in the notion of $[\mathbb{K}, 1.0.0 \text{Definition}, \text{p. 59.}]$. From (the proof of) $[\mathbb{K}, 1.3.4 \text{Corollary}, \text{p. 80}]$ follows that
\[
D_{\mathbb{R}}(F \circ \phi^{-1})(\phi \circ \overline{\gamma}())[D_{\mathbb{R}}(\phi \circ \overline{\gamma}())] : X \to L_{c}(\mathbb{R}^{d}, \overline{E})
\]
is continuous and thus
\[
D_{\mathbb{R}}((F \circ \phi^{-1}) \circ (\phi \circ \overline{\gamma}))[x][v] = D_{\mathbb{R}}(F \circ \phi^{-1})(\phi \circ \overline{\gamma}(\mathbb{R}))[D_{\mathbb{R}}(\phi \circ \overline{\gamma})(x)][v], \quad x \in X, \; v \in \mathbb{R}^{d},
\]
by the chain rule $[\mathbb{K}, 1.3.0 \text{Theorem}, \text{p. 77.}]$. In combination with Remark $[\mathbb{K}, 1.1.3 \text{c}])$ we obtain for every $x \in X$ and $v \in \mathbb{R}^{d}$ that
\[
\sum_{k=1}^{d} (\partial_{x_{j}}^{\phi})^{E}(\phi \circ \overline{\gamma})(x_{j})v_{k}
\]
and thus with $v = e_{j}, 1 \leq j \leq d$
\[
(\partial_{x_{j}}^{\phi})^{E}(\phi \circ \overline{\gamma})(x) = (\partial_{x_{j}}^{\phi})^{E}(\phi \circ \overline{\gamma})(x_{j})
\]
connoting $[\mathbb{K}, 1.3.0 \text{c}])$ for $x \in [a, b]^{d}$. \hfill \Box

3.10. Theorem (fundamental theorem of calculus). Let $E$ be a locally complete lcHs over $\mathbb{C}, \Omega \subset \mathbb{C}$ open, $\gamma : [a, b] \to \mathbb{C}$ a $C^{1}$-curve in $\Omega$, $f : \Omega \to E$ weakly $\gamma$-$C^{1}$ and let there be a holomorphic function $F : \Omega \to E$ such that $F' = f$. Then
\[
\int_{\gamma} f(z)dz = F(\gamma(b)) - F(\gamma(a)).
\]
Proof. The left-hand side of (3.3) is defined by Proposition 3.5 (a). Due to the chain rule Proposition 3.9 and Remark 3.2 (e) we have
\[ \int_{\gamma} f(z) dz = \int_{[a,b]} f(\gamma(t)) \gamma'(t) dt = \int_{[a,b]} \underbrace{f(\gamma(t))}_{=F(\gamma(t))} \gamma'(t) dt = \int_{[a,b]} ((F \circ \phi^{-1}) \circ (\phi \circ \gamma))'(t) dt. \]
Looking at the last integral, we observe that
\[ \langle e', \int_{[a,b]} ((F \circ \phi^{-1}) \circ (\phi \circ \gamma))'(t) dt \rangle \]
\[ = \int_{[a,b]} \langle e' \circ (F \circ \phi^{-1}) \circ (\phi \circ \gamma) \rangle'(t) dt \]
\[ = \langle e' \circ (F \circ \phi^{-1}) \circ (\phi \circ \gamma) \rangle_{[a,b]} \leq \langle e' \circ (F \circ \phi^{-1}) \circ (\phi \circ \gamma) \rangle_{[a,b]} \]
holds by the scalar fundamental theorem of calculus (applied to the real and the imaginary part of the integrand) where the second integral is a Riemann-integral. Finally, we deduce from the Hahn-Banach theorem that
\[ \int_{\gamma} f(z) dz = F(\gamma(b)) - F(\gamma(a)). \]

3.11. Lemma (Leibniz' rule). Let \( E \) be a locally complete lcHs over \( \mathbb{C}, V, U \subset \mathbb{C}^d \) open and \( \gamma: [a, b]^d \to \mathbb{C}^d \) a \( C^1 \)-curve in \( V \).

a) Let \( T \) be a set and \( f, f_n: V \times T \to E \) such that \( f(., t), f_n(., t): V \to E \) are weakly \( \gamma \cdot C^1 \) for every \( t \in T, n \in \mathbb{N} \) and \( f_n \to f \) uniformly on \( \gamma([a, b]^d) \times T \). Then
\[ \lim_{n \to \infty} \int_{\gamma} f_n(z, t) dz = \int_{\gamma} f(z, t) dz \]
holds uniformly on \( T \).

b) Let \( f: V \times U \to E \) be such that \( f(., \lambda): V \to E \) is weakly \( \gamma \cdot C^1 \) for every \( \lambda \in U, f(z, .): U \to E \) is holomorphic for every \( z \in V \) with \( (\partial_{\lambda_j})^E f: V \times U \to E \) being continuous and \( (\partial_{\lambda_j})^E f(., \lambda): V \to E \) weakly \( \gamma \cdot C^1 \) for every \( \lambda \in U \) and some \( 1 \leq j \leq d \). Then
\[ G: U \to E, G(\lambda) := \int_{\gamma} f(z, \lambda) dz, \]
is well-defined, complex differentiable with respect to \( \lambda_j \) and
\[ (\partial_{\lambda_j})^E G(\lambda) = \int_{\gamma} (\partial_{\lambda_j})^E f(z, \lambda) dz \in E, \quad \lambda \in U. \]

Proof. First, we remark that the integrals appearing in a) and b) are well-defined elements of \( E \) by Proposition 3.5 (a) and the weakly \( \gamma \cdot C^1 \) condition.

a) Let \( \alpha \in \mathfrak{A} \). Then we have
\[ \sup_{t \in T} p_{\alpha} \left( \int_{\gamma} f_n(z, t) dz - \int_{\gamma} f(z, t) dz \right) \leq \ell(\gamma) \sup_{(z, t) \in T} p_{\alpha} (f_n(z, t) - f(z, t)) \to 0, \quad n \to \infty, \]
since $f_n \to f$ uniformly on $\gamma([a,b]^d) \times T$.

b) Let $\lambda \in U$. Then there is $R > 0$ such that $\overline{B}_R(\lambda) \subset U$ as $U$ is open. Let $(h_n)$ be a null sequence in $\mathbb{C} \setminus \{0\}$ with $|h_n| < R/2$ for all $n \in \mathbb{N}$ which implies that the line segment $\Gamma_n$ from $\lambda_j$ to $\lambda_j + h_n$ is a $C^1$-curve in $\pi_{\lambda,j}^{-1}(\overline{B}_R(\lambda))$ that we parametrise by $[0,1]$. Applying Theorem 3.10 to the holomorphic function (in one variable) $f(z,\cdot) \circ \pi_{\lambda,j} : \pi_{\lambda,j}^{-1}(\overline{B}_R(\lambda)) \to E$ for $z \in V$, we get
\[
f(z,\lambda + h_ne_j) - f(z,\lambda) = f(z,\pi_{\lambda,j}(\lambda_j + h_n)) - f(z,\pi_{\lambda,j}(\lambda_j)) = \int_{\Gamma_n} (\partial_{\zeta_j})^Ef(z,\pi_{\lambda,j}(\zeta_j))d\zeta_j
\]
and therefore
\[
|f_n(z,\lambda)| = \left| \frac{f(z,\lambda + h_ne_j) - f(z,\lambda)}{h_n} - (\partial_{\lambda_j})^Ef(z,\lambda) \right| \leq \frac{1}{|h_n|} \left| \int_{\Gamma_n} (\partial_{\zeta_j})^Ef(z,\pi_{\lambda,j}(\zeta_j)) - (\partial_{\lambda_j})^Ef(z,\lambda)d\zeta_j \right| \leq \frac{1}{|h_n|} \sup_{\zeta,\pi_{\lambda,j}(\zeta)} \left| (\partial_{\zeta_j})^Ef(z,\pi_{\lambda,j}(\zeta)) - (\partial_{\lambda_j})^Ef(z,\lambda) \right|.
\]
Hence we obtain
\[
\sup_{z \in \gamma([a,b]^d)} |f_n(z,\lambda)| \leq \sup_{z \in \gamma([a,b]^d)} \sup_{\zeta,\pi_{\lambda,j}(\zeta)} \left| (\partial_{\zeta_j})^Ef(z,\pi_{\lambda,j}(\zeta)) - (\partial_{\lambda_j})^Ef(z,\lambda) \right| \to 0, \quad n \to \infty,
\]
since $(\partial_{\lambda_j})^Ef$ is uniformly continuous on the compact set $\gamma([a,b]^d) \times \overline{B}_R(\lambda)$ meaning $f_n \to 0$ uniformly on $\gamma([a,b]^d) \times \{\lambda\}$. From part a) we conclude \( \int_{\gamma} f_n(z,\lambda)dz \to 0 \) and thus
\[
\int_{\gamma} (\partial_{\lambda_j})^Ef(z,\lambda)dz = \lim_{n \to \infty} \int_{\gamma} \frac{f(z,\lambda + h_ne_j) - f(z,\lambda)}{h_n}dz = \lim_{n \to \infty} \frac{G(z,\lambda + h_ne_j) - G(z,\lambda)}{h_n} = (\partial^E_{\lambda_j})^EG(\lambda).
\]

Now, we want to define complex partial derivatives of higher order for an $E$-valued function $f$. Let $E$ be an lcHs over $\mathbb{C}$ and $\Omega \subset \mathbb{C}^d$ open. A function $f: \Omega \to E$ is called complex partially differentiable on $\Omega$ and we write $f \in D^1_c(\Omega; E)$ if $\partial^E_{\beta}f(z) := (\partial^E_{\beta_1})^E f(z) \in E$ for every $z \in \Omega$ and $1 \leq j \leq d$ (see Definition 3.1 c)). For $k \in \mathbb{N}$, $k \geq 2$, a function $f$ is said to be $k$-times complex partially differentiable and we write $f \in D^k_c(\Omega; E)$ if $f \in D^k_c(\Omega; E)$ and all its first complex partial derivatives are in $D^{k-1}_c(\Omega; E)$. A function $f$ is called infinitely complex partially differentiable and we write $f \in D^\infty_c(\Omega; E)$ if $f \in D^k_c(\Omega; E)$ for every $k \in \mathbb{N}$.

Let $f \in D^k_c(\Omega; E)$. For $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}_0^d$ with $|\beta| := \sum_{j=1}^d \beta_j \leq k$ we set
\[
\partial^E_{\beta}f := (\partial^E_{\beta_1})^E f := f, \quad \text{if } \beta_1 = 0, \quad \text{and}
\]
\[
(\partial^E_{\beta_1})^E f := (\partial^E_{\beta_1})^E f := (\partial^E_{\beta_1})^E \cdots (\partial^E_{\beta_1})^E f,
\]
if $\beta_j \neq 0$, as well as
\[
(\partial^E_{\beta_1})^E f := (\partial^E_{\beta_1})^E f := (\partial^E_{\beta_1})^E \cdots (\partial^E_{\beta_1})^E f.
\]
A holomorphic function \( f : \Omega \rightarrow E \) can be considered as a function from \( \Omega \) to \( \tilde{E} \) which gives us \( f \in D^0_c(\Omega, \tilde{E}) \) (see Remark 2.2 c)). Our goal is to show that we actually have \( f \in D^\infty_c(\Omega, E) \) if \( E \) is locally complete via proving Cauchy’s integral formula for holomorphic functions. For this purpose we recall the definition of a polydisc, its distinguished boundary and define integration along the distinguished boundary. For \( w = (w_1, \ldots, w_d) \in \mathbb{C}^d \) and \( R = (R_1, \ldots, R_d) \in (0, \infty)^d \) we define the polydisc \( D_R(w) := \prod_{k=1}^d B_{R_k}(w_k) \) and its distinguished boundary \( \partial D_R(w) := \prod_{k=1}^d \partial B_{R_k}(w_k) \). For \( R, \rho \in (0, \infty)^d \) we write \( \rho < R \) if \( \rho_k < R_k \) for all \( 1 \leq k \leq d \). For a function \( f : \Omega \rightarrow E \) on a set \( \Omega \subset \mathbb{C}^d \) with \( D_\rho(w) \subset \Omega \) for some \( w \in \mathbb{C}^d \) and \( \rho \in (0, \infty)^d \) we set
\[
\int_{\partial \partial D_\rho(w)} f(z)dz := \int_{\gamma} f(z)dz
\]
if the integral on the right-hand side exists where \( \gamma \) is the \( C^1 \)-curve in \( \Omega \) given by the restriction \( \gamma := \tilde{\gamma} |_{[0, 2\pi]} \) of the map \( \tilde{\gamma} : \mathbb{R}^d \rightarrow \mathbb{C}^d \) defined by \( \tilde{\gamma}_k : \mathbb{R} \rightarrow \mathbb{C}, \tilde{\gamma}_k(t) := w_k + \rho_k e^{it}, \) for \( 1 \leq k \leq d \). Further, we need the usual notation
\[
\beta! := \prod_{j=1}^d (\beta_j!), \quad (z - \zeta)^\beta := \prod_{j=1}^d (z_j - \zeta_j)^{\beta_j}
\]
for \( \beta \in \mathbb{N}_0^d \) and \( z, \zeta \in \mathbb{C}^d \).

3.12. **Theorem** (Cauchy’s integral formula). Let \( E \) be a locally complete lcHs over \( \mathbb{C} \), \( \Omega \subset \mathbb{C}^d \) open, \( w \in \Omega \), \( R \in (0, \infty)^d \) with \( D_R(w) \subset \Omega \) and \( f : \Omega \rightarrow E \) be holomorphic. Then
\[
(\partial^2 \Omega) f(\zeta) = \frac{\beta!}{(2\pi i)^d} \int_{\partial D_\rho(w)} \frac{f(z)}{(z - \zeta)^{\beta+1}} dz, \quad \zeta \in D_\rho(w), \beta \in \mathbb{N}_0^d,
\]
for all \( \rho \in (0, \infty)^d \) with \( \rho < R \).

**Proof.** Let \( \tilde{\gamma} \) and \( \gamma \) be defined as above for \( \partial D_\rho(w) \). First, we consider the case \( \beta = 0 \). We set
\[ g_\zeta : \Omega \setminus \{ \zeta \} \rightarrow E, \quad g_\zeta(z) := \frac{f(z)}{(z - \zeta)^{1 + \gamma}}, \]
for \( \zeta \in D_\rho(w) \) and observe that \( g_\zeta \circ \tilde{\gamma} \) is weakly \( C^1 \) on \( \mathbb{R}^d \) since \( f \) is holomorphic on \( \Omega \) and \( \tilde{\gamma} \in C^1(\mathbb{R}^d, \mathbb{C}^d) \). Thus \( g_\zeta \) is weakly \( \gamma \)-\( C^1 \) and integrable along \( \gamma \) by Proposition 3.5 a). Since \( f \) is weakly holomorphic and \( g_\zeta \) integrable along \( \gamma \), we get by the scalar version of Cauchy’s integral formula that
\[
(e' \circ f)(\zeta) = \frac{1}{(2\pi i)^d} \int_{\partial D_\rho(w)} \frac{(e' \circ f)(z)}{(z - \zeta)^{1 + \gamma}} dz
\]
implies
\[
f(\zeta) = \frac{1}{(2\pi i)^d} \int_{\partial D_\rho(w)} \frac{f(z)}{(z - \zeta)^{1 + \gamma}} dz
\]
by the Hahn-Banach theorem which proves (9) for \( \beta = 0 \).

Let \( n \in \mathbb{N}_0 \) and (9) be fulfilled for every \( \beta \in \mathbb{N}_0^d \) with \( |\beta| = n \). Let \( \beta \in \mathbb{N}_0^d \) with \( |\beta| = n + 1 \). Then there is \( j \in \mathbb{N}, 1 \leq j \leq d \), and \( \bar{\beta} \in \mathbb{N}_0^d \) with \( |\bar{\beta}| = n \) such that \( \beta = \bar{\beta} + e_j \),
Let $\zeta \in \mathbb{D}_\rho(w)$. Then there is $0 < r < \rho$ such that $\zeta \in \mathbb{D}_r(w)$. We define the open set $V := \mathbb{D}_r(w) \setminus \mathbb{D}_r(w)$ and the function

$$F_\beta: V \times \mathbb{D}_r(w) \to E, \quad F_\beta(z, \lambda) := \frac{f(z)}{(z - \lambda)^{\beta+1}}.$$  

Further, we compute for $\lambda \in \mathbb{D}_r(w)$

$$\partial_\lambda F_\beta(z, \lambda) = \frac{(\beta_j + 1)f(z)}{(z - \lambda)^{\beta_j+1}} = \frac{\beta_j f(z)}{(z - \lambda)^{\beta_j+1}} \in E, \quad z \in V.$$  

We see that $\gamma$ is a $C^1$-curve in $V$ and $F(\cdot, \lambda) \circ \gamma$ and $\partial_\lambda F_\beta(\cdot, \lambda) \circ \gamma$ are weakly $C^1$ on $\mathbb{R}^d$ for every $\lambda \in \mathbb{D}_r(w)$ since $f$ is holomorphic on $\Omega$. Hence $F_\beta(\cdot, \lambda)$ and $\partial_\lambda F_\beta(\cdot, \lambda)$ are weakly $\gamma$-C$^1$ for every $\lambda \in \mathbb{D}_r(w)$. In addition, $\partial_\lambda F_\beta$ is continuous on $V \times \mathbb{D}_r(w)$ by Remark 3.2 b), $F_\beta(z, \cdot)$ is holomorphic on $\mathbb{D}_r(w)$ for every $z \in V$ and thus we can apply Leibniz’ rule Lemma 3.11 b) yielding

$$\partial_\zeta^j \partial_\zeta^k \circ F_\beta f(\lambda) = \frac{\beta_1}{(2\pi i)^d} \int \partial_\lambda F_\beta(z, \lambda)dz = \frac{\beta_1}{(2\pi i)^d} \int \frac{f(z)}{(z - \lambda)^{\beta_j+1}}dz \in E$$

for every $\lambda \in \mathbb{D}_r(w)$, in particular for $\lambda = \zeta$, where we used the induction hypothesis in the first equation. It remains to be shown that $\partial_\zeta^j (\partial_\zeta^k f) (\lambda) = \partial_\zeta^k f(\lambda)$ for every $\lambda \in \mathbb{D}_r(w)$, i.e. that the order of the partial derivatives does not matter. For $\beta = 0$ this is clear. If $\beta = 1$, then our preceding considerations imply that

$$\partial_\zeta^j \partial_\zeta^k f(\lambda) = \frac{1}{(2\pi i)^d} \int_{\partial \mathbb{D}_r(w)} \frac{f(z)}{(z - \lambda)^{\gamma+1}}dz = \partial_\zeta^k \partial_\zeta^j f(\lambda)$$

for all $1 \leq j, k \leq d$. This yields that $\partial_\zeta^j (\partial_\zeta^k f)(\lambda) = \partial_\zeta^k f(\lambda)$ for every $\lambda \in \mathbb{D}_r(w)$. 

Cauchy’s integral formula for derivatives is usually derived by using the Riemann-integral instead of the Pettis-integral and can be found for holomorphic functions in one variable in [21, Théorème 1, p. 37-38], in several variables in [7, Corollary 3.7, p. 5] and infinitely many variables in [12, Proposition 2.4, p. 55] as well. The Riemann-integrals are elements of $E$ under the condition that $E$ has ccp in [21] or more general if $E$ is sequentially complete in [7] and [12] by [3, Lemma 1.1, p. 79]. In general, they are only elements of the completion $\overline{E}$. From our approach using Pettis-integrals we guarantee that they belong to $E$ even if $E$ is only locally complete.

3.13. Corollary. If $E$ is a locally complete lchs over $\mathbb{C}$, $\Omega \subset \mathbb{C}^d$ open and $f: \Omega \to E$ holomorphic, then $(\partial_\zeta^k f)$ does not depend on the order of the partial derivatives involved and $f \in \mathcal{D}_C^\infty(\Omega, E)$. 

Proof. The independence of the order follows from the proof of Cauchy’s integral formula Theorem 3.12. Combining this formula with the commutativity of the complex partial derivatives, we conclude $f \in \mathcal{D}_C^\infty(\Omega, E)$. 

For an lchs $E$ over $\mathbb{C}$, an open set $\Omega \subset \mathbb{C}^d$ and a function $f: \Omega \to E$, we write $f \in \mathcal{C}_R(\phi(\Omega), E)$ if $f \circ \phi^{-1} \in \mathcal{C}_R(\phi(\Omega), E)$ for $k \in \mathbb{N}_{0,\infty}$. We define the space

$$\mathcal{O}(\Omega, E) := \{ f \in \mathcal{C}_R^k(\Omega, E) \mid \forall \beta \in \mathbb{N}_0^d, z \in \Omega : (\partial_\zeta^k f)(z) \in E \}$$

which we equip with the system of seminorms given by

$$[f]_{K, \alpha} := \sup_{z \in K} \| f(z) \|_{\mathbb{C}^\alpha}, \quad f \in \mathcal{O}(\Omega, E),$$

for $K \subset \Omega$ compact and $\alpha \in \mathbb{A}$. If $E = \mathbb{C}$, we just write $\mathcal{O}(\Omega) := \mathcal{O}(\Omega, \mathbb{C})$. 

HOLOMORPHIC 13
Due to Cauchy’s integral formula and Remark 4.2. c) of Remark 4.2, we already know that every holomorphic function \( f : \Omega \to E \) is an element of \( \mathcal{O}(\Omega, E) \) and we prove in the following that every element of \( \mathcal{O}(\Omega, E) \) is a holomorphic function on \( \Omega \) as well if \( E \) is locally complete. The space \( \mathcal{O}(\Omega) \) coincides with the space of all \( C^\infty \)-valued holomorphic functions on \( \Omega \) in the sense of [23, Definition 1.7.1, p. 47] by [23, Theorem 1.7.6, p. 48-49] and is a Fréchet space by [23, Example 1.10.7 (a), p. 66]. As a start in proving that \( \mathcal{O}(\Omega, E) \) is the space of all holomorphic functions from \( \Omega \) to a locally complete space \( E \), we show that the elements of \( \mathcal{O}(\Omega, E) \) fulfill the Cauchy inequality.

3.14. Corollary (Cauchy inequality). Let \( E \) be a locally complete lcHs over \( \mathbb{C} \) and \( \Omega \subset \mathbb{C}^d \) open.

a) If \( w \in \Omega, \ R \in (0, \infty]^d \) with \( \mathbb{D}_R(w) \subset \Omega \) and \( f \in \mathcal{O}(\Omega, E) \), then
\[
\rho_\alpha(\partial^2_C f(\zeta)) \leq \frac{\beta!}{\rho^\beta} \max_{\zeta \in \partial \mathbb{D}_\rho(w)} \rho_\alpha(f(z)), \quad \zeta \in \partial \mathbb{D}_\rho(w), \ \beta \in \mathbb{N}_0^d, \tag{7}
\]
for every \( \rho \in (0, \infty)^d \) with \( \rho < R \) and \( \alpha \in \mathfrak{A} \).

b) For every compact set \( K \subset \Omega \) there is a compact set \( K' \subset \Omega \) such that for every \( \beta \in \mathbb{N}_0^d \) there is \( C_{K, \beta} > 0 \) such that for every \( \alpha \in \mathfrak{A} \) and every \( f \in \mathcal{O}(\Omega, E) \) holds
\[
\sup_{z \in K} \rho_\alpha(\partial^2_C f(z)) \leq C_{K, \beta} \max_{z \in K} \rho_\alpha(f(z)). \tag{8}
\]

Proof. a) For \( \alpha \in \mathfrak{A} \) we set \( B_\alpha := \{ x \in E \mid p_\alpha(x) < 1 \} \), its polar \( B_\alpha^\circ := \{ e' \in E' \mid \forall x \in B_\alpha : |e'(x)| \leq 1 \} \) and denote by \( \gamma \) the \( C^1 \)-curve on \( [0, 2\pi]^d \) corresponding to \( \partial \mathbb{D}_\rho(w) \). It follows from the scalar version of Cauchy’s integral formula (see [23, Theorem 1.7.6, p. 48-49]) that for all \( \zeta \in \partial \mathbb{D}_\rho(w) \) and \( \beta \in \mathbb{N}_0^d \) we have
\[
\rho_\alpha(\partial^2_C f(\zeta)) = \sup_{e' \in B_\alpha^\circ} (\partial^2_C f(\zeta) \circ e')(\zeta) = \frac{\beta!}{(2\pi)^d} \sup_{e' \in B_\alpha^\circ} \sup_{\rho \in \partial \mathbb{D}_\rho(w)} \frac{|e'(f(z))|}{(z - \zeta)^{\beta + (1, \ldots, 1)}} \leq \frac{\beta!}{(2\pi)^d} \sup_{e' \in B_\alpha^\circ} \sup_{\rho \in \partial \mathbb{D}_\rho(w)} \frac{|e'(f(z))|}{\rho^{\beta + (1, \ldots, 1)}} = \frac{\beta!}{\rho^\beta} \max_{\zeta \in \partial \mathbb{D}_\rho(w)} \rho_\alpha(f(z)) \tag{9}
\]
where we used [33, Proposition 22.14, p. 256] in the first and last equation to get from \( \rho_\alpha \) to \( \sup_{e' \in B_\alpha^\circ} \rho_\alpha(\partial^2_C f(\zeta) \circ e') \) and back. Our statement follows from the continuity of \( f \) on \( \Omega \) by Remark 4.2. b) and the compactness of the distinguished boundary.

b) Is a direct consequence of a) since every compact set \( K \subset \Omega \) can be covered by a finite number \( n \) of open, bounded polydiscs \( \mathbb{D}_{\rho_j}(w_j) \) with \( \overline{\mathbb{D}_{\rho_j}(w_j)} \subset \Omega \) for \( 1 \leq j \leq n \).

For sequentially complete \( E \) Cauchy’s inequality can also be found in [12, Proposition 2.5, p. 57] and as a direct consequence we obtain:

3.15. Remark (Weierstrass). Let \( E \) be a locally complete lcHs over \( \mathbb{C} \) and \( \Omega \subset \mathbb{C}^d \) open. Then the system of seminorms generated by
\[
|f|_{K, m, \alpha} := \sup_{z \in K} \rho_\alpha(\partial^2_C f(z)), \quad f \in \mathcal{O}(\Omega, E), \tag{10}
\]
for $K \subset \Omega$ compact, $m \in \mathbb{N}_0$ and $\alpha \in \mathbb{R}$ induces the same topology on $\mathcal{O}(\Omega, E)$ as the system $(|f|_{K, \alpha})$ by $[5]$

This remark implies $[6]$, Proposition 3.1, p. 85 if $E$ is sequentially complete. We observe the following useful relation between real and complex first partial derivatives.

3.16. Proposition. If $E$ is an lcHs over $\mathbb{C}$, $\Omega \subset \mathbb{C}^d$ open and $f \in \mathcal{O}(\Omega, E)$, then for every $1 \leq j \leq d$ and $x \in \phi(\Omega)$

$\partial^\infty_{R}(f \circ \phi^{-1})(x) = i \partial^\infty_{C}(f(\phi^{-1})(x)) \quad \text{and} \quad \partial_{R}^{2j-1}(f \circ \phi^{-1})(x) = \partial^\infty_{C}f(\phi^{-1}(x))$.

Proof. $f \in C^2_{\mathbb{R}}(\Omega, E)$ and for $x = (x_1, \ldots , x_{2d}) \in \phi(\Omega)$ we get

$\partial^\infty_{R}(f \circ \phi^{-1})(x) = \lim_\substack{h \to 0 \\text{h} \neq 0 \text{ or } \infty}{\frac{f(\ldots, x_j - i x_j + ih, . . . ) - f(\ldots, x_j - x_j, . . . )}{ih}}$

as well as

$\partial_{R}^{2j-1}(f \circ \phi^{-1})(x) = \lim_\substack{h \to 0 \\text{h} \neq 0 \text{ or } \infty}{\frac{f(\ldots, x_j + ix_j + ih, . . . ) - f(\ldots, x_j + x_j, . . . )}{ih}}$

for $j \in \mathbb{N}$.

3.17. Proposition. Let $E$ be a locally complete lcHs over $\mathbb{C}$ and $\Omega \subset \mathbb{C}^d$ open. Then the map

$S: \mathcal{O}(\Omega) \subset E \to \mathcal{O}(\Omega, E), \quad u \mapsto [z \mapsto u(\delta_z)]$.

is a (topological) isomorphism where $\delta_z$ is the point evaluation functional at $z$.

Proof. Let $u \in \mathcal{O}(\Omega) \subset E$. Due to $[30]$, 4.12 Proposition, p. 22] and the barrelledness of the Fréchet space $\mathcal{O}(\Omega)$ we have $(\partial^\infty_{C}E)S(u)(z) = u(\delta_z \circ \partial^\infty_{C}) \in E$ for all $\beta \in \mathbb{N}_0^d$ and $z \in \phi(\Omega)$ where one has to replace $\partial^\infty_{C}$ by $\partial^\infty_{C}$ and the space $\mathcal{C}^{k}(\Omega)$ by $\mathcal{O}(\Omega)$ in the proof of $[30]$, 4.12 Proposition, p. 22] Furthermore, $S(u) \in C^\infty_{\mathbb{R}}(\Omega, E)$ by $[30]$, 4.12 Proposition, p. 22] with $k = 1$, Remark $[12]$, and Proposition $[10]$ which implies that $S(u) \in \mathcal{O}(\Omega, E)$.

Let $f \in \mathcal{O}(\Omega, E)$ and $K \subset \Omega$ be compact. It is easily checked that $e' \circ f \in \mathcal{O}(\Omega)$ for every $e' \in E'$. Since $f \circ \phi^{-1}$ is weakly $C^1$ on the open set $\phi(\Omega) \subset \mathbb{R}^{2d}$, it follows from $[10]$, Proposition 2, p. 354] that $K_1 := \text{aff}(f(K))$ is a absolutely convex and compact. The inclusion $N_K(f) := f(K) \subset K_1$ implies that $[30]$, 3.16 Condition d), p. 12] is fulfilled yielding that $S$ is a (topological) isomorphism by $[30]$, 3.17 Theorem, p. 12] in combination with $[30]$, 3.13 Lemma b), p. 10].

Once we have the equivalent conditions for holomorphy from our main Theorem $5.3$, namely the equivalence ‘$a \iff d$’, the preceding proposition is just a consequence of $[8]$, Theorem 9, p. 232].

3.18. Theorem. Let $E$ be a locally complete lcHs over $\mathbb{C}$, $z \in \mathbb{C}^d$ and $R \in (0, \infty[^d]$. Then the tensor product $\mathcal{O}(\mathbb{D}_R(z)) \otimes E$ is sequentially dense in $\mathcal{O}(\mathbb{D}_R(z), E)$ and

$\forall f \in \mathcal{O}(\mathbb{D}_R(z), E)$ where the series converges in $\mathcal{O}(\mathbb{D}_R(z), E)$.
Proof. The monomials \((-z)^\beta, \beta \in \mathbb{N}_0^d\), form an equicontinuous Schauder basis with associated coefficient functionals \(1/(\delta_z \circ \partial^j_z)\) of the barrelled space \(O(D_R(z))\) by [25, Theorem 1.7.6, p. 48-49]. Thus our statement follows from Proposition 3.17 and [31, 3.6 Corollary, p. 7]. □

In the one variable case the theorem above is given in [31, 3.6 Corollary c), p. 7] combined with [31, 4.5 Theorem, p. 13] as well.

3.19. Corollary. Let \(E\) be a locally complete lcHs over \(\mathbb{C}\) and \(\Omega \subset \mathbb{C}^d\) open. Then the following statements are equivalent for a function \(f: \Omega \to E\).

a) \(f\) is holomorphic on \(\Omega\).

b) \(f \in \mathcal{O}(\Omega,E)\).

Proof. We only need to prove the implication ‘(b) \Rightarrow (a)’. We claim that for every \(z \in \Omega\) holds

\[
df(z)[v] = \sum_{j=1}^d \partial^j_z f(z)v_j, \quad v \in \mathbb{C}^d.
\]

(9)

Observe that the right-hand side is already linear in \(v\). Let \(\alpha \in \mathbb{A}\) and \(z \in \Omega\). Then there is \(R \in (0,\infty)^d\) such that \(D_R(z) \subset \Omega\). We fix \(\rho \in (0,\infty)^d\) with \(\rho < R\) and derive from Theorem 3.18 a) that

\[
g(w,z) := f(w) - f(z) - \sum_{j=1}^d \partial^j_z f(z)(w_j - z_j)
\]

\[
= \sum_{|\beta| > 1} \frac{\partial^\beta f(z)}{\beta!}(w - z)^\beta = \sum_{j=1}^d (w_j - z_j) \sum_{|\beta| = 1} \frac{\partial^{\beta + e_j} f(z)}{(\beta + e_j)!}(w - z)^\beta
\]

for every \(w \in D_\rho(z)\).

Let \(0 < \varepsilon \leq 1\) and set \(r := \min_{1 \leq j \leq d} p_j\). We observe that \(B_{\varepsilon r/2}(z)\) is a subset of \(D_\rho(z)\). Applying Cauchy’s inequality (7), we obtain for \(w \in B_{\varepsilon r/2}(z)\)

\[
p_\alpha\left(\frac{\partial^{\beta + e_j} f(z)}{(\beta + e_j)!}(w - z)^\beta\right) \leq \prod_{k=1}^d \frac{|w_k - z_k|^{\beta_k}}{r^{\beta_k + \sigma_j}} \max_{z \in B_{\varepsilon r}(z)} p_\alpha(f(z)) \leq \frac{\prod_{k=1}^d (|w_k - z_k|^{\beta_k})}{2^d \varepsilon r/2} \max_{z \in B_{\varepsilon r}(z)} p_\alpha(f(z)).
\]

Hence we conclude for every \(w \in B_{\varepsilon r/2}(z)\)

\[
p_\alpha\left(\frac{g(w,z)}{|w - z|}\right) \leq \frac{\varepsilon d^d}{r} \max_{z \in B_{\varepsilon r}(z)} p_\alpha(f(z)) \sum_{|\beta| = 1} \frac{1}{2^{\beta}} \leq \frac{2^d \cdot d^d}{r} \max_{z \in B_{\varepsilon r}(z)} p_\alpha(f(z))
\]

where the last estimate follows from [25, Corollary 1.2.14 (a), p. 12-13]. Letting \(\varepsilon \to 0\), proves (1).

Fix \(v \in \mathbb{C}^d\) and let \(z, w \in \Omega\). The estimate

\[
p_\alpha(\partial f(w)[v]) - \partial f(z)[v] \leq \sum_{j=1}^d p_\alpha(\partial^{j}_z f(w) - \partial^{j}_z f(z))[v_j]
\]

implies that \(df(\cdot)[v]\) is continuous on \(\Omega\) since \(f \in \mathcal{C}_B^1(\Omega)\) and by Proposition 3.18. Therefore \(f\) is holomorphic on \(\Omega\). □

We briefly recall the following definitions which enable us to phrase our main theorem concerning holomorphic functions in several variables. Let \(E\) be an lcHs
over $\mathbb{C}$. For an open set $\Omega \subset \mathbb{R}^{2d}$ and $1 \leq j \leq d$ we define the Cauchy-Riemann operator by

$$\overline{\partial}_j f(x) := (\overline{\partial}_j)^E f(x) := \frac{1}{2}(\partial_{x_j}^E + i\partial_{y_j}^E) f(x), \quad f \in C^1(\Omega, E), \ x \in \Omega.$$ 

A function $f: \Omega \to E$ from a topological space $\Omega$ to $E$ is called locally bounded on a subset $\Lambda \subset \Omega$ if for every $z \in \Lambda$ there is a neighbourhood $U \subset \Omega$ of $z$ such that $f$ is bounded on $U$. A subspace $G \subset E'$ is said to be separating if for every $x, y \in E$ there is $e' \in G$ such that $e'(x) \neq e'(y)$. A subspace $G \subset E'$ is said to determine boundedness if every $\sigma(E, G)$-bounded subset of $E$ is already bounded where $\sigma(E, G)$ denotes the weak topology w.r.t. the dual pair $(E, G)$. If $G$ determines boundedness, then $G$ is separating. For instance, $G := E'$ determines boundedness by Mackey’s theorem and further examples may be found in [1, Remark 1.4, p. 781-782] and [9, Remark 11, p. 233].

3.20. Theorem. Let $E$ be a locally complete lcHs over $\mathbb{C}$ and $\Omega \subset \mathbb{C}^d$ be open. Then the following statements are equivalent for a function $f: \Omega \to E$.

a) $f$ is holomorphic (Gâteaux-, separately holomorphic) on $\Omega$.

b) $f \in C^1(\Omega, E)$ and $\partial^E f(z) \exists$ in $\overline{E}$ for every $z \in \Omega$ and $1 \leq j \leq d$.

c) $\partial_j^E f(z) \exists$ in $\overline{E}$ for every $z \in \Omega$ and $1 \leq j \leq d$.

d) $f \in C^\infty(\Omega, E)$ and $\overline{\partial}_j f(\phi^{-1}) = 0$ for all $1 \leq j \leq d$.

e) There is a subspace $G \subset E'$ which determines boundedness such that $e' \circ f$ is holomorphic (Gâteaux-, separately holomorphic) on $\Omega$ for every $e' \in G$.

f) $f$ is locally bounded outside some compact set $K \subset \Omega$ and there is a separating subspace $G \subset E'$ such that $e' \circ f$ is holomorphic (Gâteaux-, separately holomorphic) on $\Omega$ for every $e' \in G$.

g) For every $z \in \Omega$ there are $R \in (0, \infty]^d$ and $(a_\beta)_{\beta \in \mathbb{N}^d} \subset E$ such that

$$f = \sum_{\beta \in \mathbb{N}^d} a_\beta (-z)^\beta \ \text{on} \ D_R(z).$$

If one of the equivalent conditions above is fulfilled, then

$$df(z)[v] = Df(z)[v] = \sum_{j=1}^d \partial_j^E f(z) v_j, \quad z \in \Omega, \ v \in \mathbb{C}^d. \quad (10)$$

Proof. We write $x(i)$ if we consider $x$ for holomorphic functions, $x(ii)$ if we consider $x$ for Gâteaux-holomorphic functions and $x(iii)$ if we consider $x$ for separately holomorphic functions in the cases $x \in \{a, e, f\}$. First, we remark that the endorsement (10) follows from Cauchy’s integral formula and Remark 3.2(c)+e). ‘$a$’(i) $\Rightarrow$ ‘$e$’(i)’; The implication ‘$\Rightarrow$’ is clear with $G := E'$. Let us turn to ‘$\Leftarrow$’. We claim that $O(\Omega, E)$ coincides with the space of functions $f: \Omega \to E$ such that $e' \circ f \in O(\Omega)$ for each $e' \in G$ which then yields the desired equivalence by Corollary 3.19: $O(\Omega)$ is a closed subspace of $C^\infty(\phi(\Omega))$ via the map $f \mapsto f \circ \phi^{-1}$ (see e.g. [18, p. 691]). Thus we can apply the weak-strong principle [9, Corollary 10 (a), p. 233] in combination with [9, Definition 3, p. 229-230] and Proposition 3.14 proving our claim.

‘$a$’(i) $\Rightarrow$ ‘$g$’): Follows from Corollary 3.19 and Theorem 3.18 since for every $f \in O(\Omega, E)$ there is $z \in \Omega$ and $R \in (0, \infty]^d$ such that $f|_{D_R(z)} \in O(D_R(z), E).

‘$g$’ $\Rightarrow$ ‘$e$’(i)’; Let $z \in \Omega$, $R \in (0, \infty]^d$ and $(a_\beta)_{\beta \in \mathbb{N}^d} \subset E$ be such that

$$f(w) = \sum_{\beta \in \mathbb{N}^d} a_\beta (w - z)^\beta, \quad w \in D_R(z).$$
Then we have for every \( e' \in G := E' \) that
\[
(e' \circ f)(w) = \sum_{\beta \in \mathbb{N}_0^n} e'(a_\beta)(w - z)^\beta, \quad w \in \mathbb{D}_R(z),
\]
implying the holomorphy of \( e' \circ f \) by [23, Theorem 1.7.6, p. 48-49].

\(\forall (i) \Rightarrow e(i)(ii) \Rightarrow e(i)(iii) \Rightarrow e(i)'\): The first implication is obvious, the second and the third follow from the scalar version of Hartogs’ theorem (see [23, Theorem 2.2.8, p. 28]).

\(\forall (i) \Rightarrow a(i)(ii) \Rightarrow e(i)'\) and \(\forall (i) \Rightarrow a(i)(iii) \Rightarrow e(i)'\): These implications are obvious.

\(\forall (i) \Rightarrow e(i)'\): This implication holds with \( G := E' \) due to \( E' = (\overline{E})' \) and the scalar version of Hartogs’ theorem.

\(\forall (i) \Rightarrow d\): Let \( f : \Omega \to E \) be such that \( e' \circ f \) is holomorphic on \( \Omega \) for every \( e' \in G \). Then \( e' \circ f \circ \phi^{-1} \) is \( C^\infty \) on \( \phi(\Omega) \) for each \( e' \in G \) and we even obtain \( f \circ \phi^{-1} \in C^\infty(\phi(\Omega), E) \) by the weak-strong principle [4].

Furthermore, the holomorphy of \( e' \circ f \) implies
\[
(e', \overline{f})^E((f \circ \phi^{-1})(x)) = (\overline{f})^E((f \circ \phi^{-1}))(x) = 0, \quad x \in \phi(\Omega), \quad e' \in G,
\]
every \( 1 \leq j \leq d \) by [23, Definition 2.1.1, p. 23]. We conclude that \( d \) is valid since \( G \) is separating.

\(\forall (i) \Rightarrow e(i)'\): Let \( f \in C^\infty(\Omega, E) \) and \( \overline{f}(f \circ \phi^{-1}) = 0 \) for all \( 1 \leq j \leq d \). Then \( f \circ \phi^{-1} \) is weakly \( C^\infty \) on \( \phi(\Omega) \) and
\[
(\overline{f})^E((f \circ \phi^{-1}))(x) = (e', (\overline{f})^E((f \circ \phi^{-1}))(x) = 0, \quad x \in \phi(\Omega), \quad e' \in E',
\]
every \( 1 \leq j \leq d \). We derive \( e(i)' \) with \( G := E' \) from the scalar version of Hartogs’ theorem and [23, Definition 2.1.1, p. 23].

\(\forall (i) \Rightarrow f(i)'\): A consequence of Remark 5.2 \( (b) + (d) \) with \( K := \emptyset \) and \( G := E' \).

\(\forall (i) \Rightarrow f(i) \leftrightarrow f(i)\): Follows from the corresponding equivalences in case \( e \).

\(\forall (i)' \Rightarrow e(i)'\): Fix \( 1 \leq j \leq d \) and \( z \in \Omega \) and consider the map \( f \circ \pi_{j,z} : \overline{\pi_{j,z}^{-1}(\Omega)} \to E \).
Let \( w \in \pi_{j,z}^{-1}(\Omega) \) and \( K \). Then \( \pi_{j,z}(w) \in \Omega \setminus K \) and thus there is a neighbourhood \( U \subset \Omega \) of \( \pi_{j,z}(w) \) such that \( f \) is bounded on \( U \) implying that \( f \circ \pi_{j,z} \) is bounded on the neighbourhood \( \overline{\pi_{j,z}^{-1}(U)} \) of \( w \). Thus we can apply [19, 5.2 Theorem, p. 35] to \( f \circ \pi_{j,z} \) and obtain that \( (f \circ \pi_{j,z})'(w) \) exists in \( E \) for all \( w \in \pi_{j,z}^{-1}(\Omega) \) implying for \( w = z_j \)
\[
\partial_{\overline{E}} C_f(z) = (f \circ \pi_{j,z})'(z_j) \in E.
\]
\(\forall (i) \Rightarrow f(i)'\): Let \( G := E' \) in [23, Satz 10.11, p. 241] for quasi-complete \( E \), in [21, Théorème 1, p. 37-38] (cf. [24, 16.7.2 Theorem, p. 362-363]) for \( E \) with ccp and more general in [19, 2.1 Theory and Definition, p. 17-18] and [19, 5.2 Theorem, p. 35] for locally complete \( E \). In several variables our theorem improves [7, Theorem 3.2, p. 83-84] where \( E \) has to be sequentially complete and even in one variable it is more general than the mentioned ones due to Theorem 3.2 \( (c) \). The equivalence \( \forall (i) \Rightarrow a(i)(iii)' \) is Hartogs’ theorem and can be found for Banach-valued holomorphic functions on an open set \( \Omega \subset \mathbb{C}^d \) in [34, 36.1 Theorem, p. 265] and for holomorphic functions with values in a sequentially complete space in [7, Corollary 3.6, p. 85]. The equivalence \( \forall (i) \Rightarrow e(i)' \) is also contained in [7, Corollary 10 (a), p. 233].

The following two corollaries improve [7, Corollary 3.7, p. 85] from sequentially complete \( E \) to locally complete \( E \).
3.21. Corollary. Let \( E \) be a locally complete lcHs over \( \mathbb{C} \) and \( \Omega \subset \mathbb{C}^d \) open. If \( f: \Omega \to E \) is holomorphic, then \( \partial f \) is holomorphic for all \( \beta \in \mathbb{N}_0^d \).

Proof. Let \( \beta \in \mathbb{N}_0^d \) and \( 1 \leq j \leq d \). Then we deduce from Corollary 3.13 and Cauchy’s integral formula that
\[
\partial^\beta f(z) = \frac{1}{(2\pi i)^n} \int_{|z-z_0| = r} \frac{f(z_0)}{(z-z_0)^{\beta+1}} \, dz,
\]
where \( z_0 \in \Omega \) and \( r > 0 \) is chosen such that \( z \neq z_0 \). Then, for \( \beta \in \mathbb{N}_0^d \),
\[
\partial f = \left( \sum_{j=1}^d \partial_j f \right).
\]
It follows from Theorem 3.20 \( (a) \Leftrightarrow (c) \) that \( \partial f \) is holomorphic.

3.22. Corollary. Let \( E \) be a locally complete lcHs over \( \mathbb{C} \) and \( \Omega \subset \mathbb{C}^d \) open. Then the following statements are equivalent for a function \( f:E \to \mathbb{C} \).

a) \( f \) is holomorphic on \( \Omega \).

b) \( f \in C^\infty(\Omega, E) \).

Proof. The implication \( (b) \Rightarrow (a) \) follows from Theorem 3.20 \( (c) \Rightarrow (a) \). The other implication is a consequence of Corollary 3.13 and Corollary 3.21 which gives that the holomorphic function \( \partial f, \beta \in \mathbb{N}_0^d \), is continuous by Remark 3.20 \( b \).

The following generalisation of Proposition 3.16 describes the relation between higher order real and complex partial derivatives. For convenience we recall the definition of higher real partial derivatives. Let \( k \in \mathbb{N}_0, \Omega \subset \mathbb{R}^d \) open, \( E \) an lcHs and \( f \in C^k(\Omega, E) \). For \( \beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}_0^d \) we define
\[
\partial^\beta f := (\partial^\beta f)(\phi^{-1})(x) := \sum_{j=1}^d \beta_j \partial_j \partial^{\beta_1} \cdots \partial^{\beta_d} f(\phi^{-1})(x), \quad x \in \phi(\Omega), \beta \in \mathbb{N}_0^d.
\]

3.23. Proposition. If \( E \) is a locally complete lcHs over \( \mathbb{C}, \Omega \subset \mathbb{C}^d \) open and \( f:E \to \mathbb{C} \) holomorphic, then \( f \in C^\infty(\Omega, E) \) and
\[
\partial f(\phi^{-1})(x) = \sum_{j=1}^d \beta_j \partial_j \partial f(\phi^{-1})(x) = C \cdot \partial f(\phi^{-1})(x).
\]

Proof. Due to Theorem 3.20 \( (a) \Leftrightarrow (d) \) we have \( f \in C^\infty(\Omega, E) \) implying that the left-hand side of (11) is defined whereas the right-hand side is defined by Cauchy’s integral formula. Now, for \( \beta = 0 \) equation (11) is trivial. Let \( n \in \mathbb{N}_0 \) and assume that (11) holds for all \( \beta \in \mathbb{N}_0^d \) with \( |\beta| = n \). Let \( \beta \in \mathbb{N}_0^d \) with \( |\beta| = n + 1 \). Then there is \( j \in \mathbb{N}, 1 \leq j \leq 2d \), and \( \beta \in \mathbb{N}_0^d \) with \( |\beta| = n \) such that \( \beta = \beta + e_j \). We set
\[
g := \partial(\beta_1, \beta_2, \ldots, \beta_{2d-1}, \beta_{2d}) f \quad \text{and} \quad C := \partial(\beta_1, \beta_2, \ldots, \beta_{2d}) f
\]
and obtain for \( x \in \phi(\Omega) \) by Schwarz’ theorem
\[
\partial f(\phi^{-1})(x) = \partial(\beta_1, \beta_2, \ldots, \beta_{2d}) f(\phi^{-1})(x) = C \cdot \partial f(\phi^{-1})(x).
\]

We deduce from Corollary 3.21 that \( g \) is holomorphic and from Corollary 3.19 that \( g \in \mathcal{O}(\Omega, E) \). Thus we can apply Proposition 3.16 to \( g \) and compute
\[
\partial f(\phi^{-1})(x) = iD f(\phi^{-1})(x),
\]
if \( j \) is even, and
\[
\partial f(\phi^{-1})(x) = D f(\phi^{-1})(x),
\]
if \( j \) is odd, which implies by Corollary 3.13

\[
\partial_z^j (f \circ \phi^{-1})(x) = C \cdot \partial_z^j (g \circ \phi^{-1})(x) = i^{\sum_{k=1}^{j} \beta_k} \partial_z^j (f(\phi^{-1}(x))).
\]

The identity theorem for vector-valued holomorphic functions in several variables takes the following form where the vector-valued one variable case can be found in [9, Corollary 10 (c), p. 233] and the scalar-valued several variables case for example in [25, Proposition 1.7.10, p. 50]. Its version for Banach-valued holomorphic functions on an open subset of a Banach space is given in [24, 5.7 Proposition, p. 37].

3.24. Theorem (Identity theorem). Let \( E \) be a locally complete lcHs over \( \mathbb{C} \), \( F \subset E \) a locally closed subspace, \( \Omega \subset \mathbb{C}^d \) open and connected and \( f : \Omega \to E \) holomorphic. If

(i) the set \( \Omega_F := \{ z \in \Omega \mid f(z) \in F \} \) has an accumulation point in \( \Omega \), or if

(ii) there exists \( z_0 \in \Omega \) such that \( \partial_z^\beta f(z_0) \in F \) for all \( \beta \in \mathbb{N}^d \),

then \( f(z) \in F \) for all \( z \in \Omega \).

Proof. This follows from Proposition 3.17 and [9, Corollary 8, p. 232] with the Fréchet-Schwartz space \( Y := \mathcal{O}(\Omega) \) and the separating subspace \( X := \text{span}\{\partial_z \mid z \in \Omega_F \} \subset Y \) in (i) resp. \( X := \text{span}\{\partial_z \circ \phi \mid \beta \in \mathbb{N}^d \} \subset Y \) in (ii).

For the definition of local closedness see [35, Definition 5.1.14, p. 154-155]. In particular, every locally complete subspace of \( E \) is locally closed by [35, Proposition 5.1.20 (i), p. 155).

3.25. Theorem (Liouville). Let \( E \) be a locally complete lcHs over \( \mathbb{C} \), \( f : \mathbb{C}^d \to E \) holomorphic and \( k \in \mathbb{N}_0 \). Then the following assertions are equivalent.

a) \( f \) is a polynomial of degree \( \leq k \).

b) \( \forall \alpha \in \mathbb{A} \exists C, R > 0 \forall z \in \mathbb{C}^d, |z| \geq R : p_\alpha(f(z)) \leq C|z|^k\)

Proof. The implication ‘a) \( \Rightarrow \) b)’ is obvious and the converse implication holds due to the power series expansion Theorem 3.18 of \( f \) around zero and the Cauchy inequality.

Let \( \Omega \subset \mathbb{C}^d \) open and connected. A set \( A \subset \Omega \) is called thin if for every \( z \in A \) there are \( R > 0 \) with \( \partial_R(z) \subset \Omega \) and \( f \in \mathcal{O}(\partial_R(z)) \), \( f \neq 0 \), such that \( f(z) = 0 \) on \( A \cap \partial_R(z) \) (see e.g. [22, Chap. 1, Sec. C, 1, Definition, p. 19]). A thin set \( A \subset \Omega \) is nowhere dense by [22, p. 19] and thus the complement \( \Omega \setminus A \) contains a dense open subset.

3.26. Theorem (Riemann’s removable singularities theorem). Let \( E \) be an lcHs over \( \mathbb{C} \), \( G \subset E \) a subspace, \( \Omega \subset \mathbb{C}^d \) open and connected, \( A \subset \Omega \) thin and closed and \( f : \Omega \setminus A \to E \) holomorphic. If for every \( z \in \Omega \) there is a polydisc \( \mathbb{D}_R(z) \subset \Omega \) such that \( f \) is bounded on \( \mathbb{D}_R(z) \setminus A \) and

(i) \( G \) is separating and \( E \mathcal{B}_l \)-complete, or if

(ii) \( G \) is dense in \( E \mathcal{B}_l \) and \( E \) locally complete,

then \( f \) extends holomorphically to \( \Omega \).

Proof. First, we remark that \( e^i \circ f \) is holomorphic and bounded on \( \mathbb{D}_R(z) \setminus A \) for some \( R \) and each \( z \in \Omega \). Due to the scalar version of Riemann’s removable singularities theorem (see [22, Chap. 1, Sec. C, 3, Theorem, p. 19]) \( e^i \circ f \) extends to a holomorphic function \( e^i \circ f \) on \( \Omega \) for each \( \epsilon \in G \). Let \((\Omega_n)_{n \in \mathbb{N}}\) be any exhaustion of \( \Omega \) with relatively compact, open and connected sets such that \( \Omega_n \subset \Omega_{n+1} \) for every \( n \in \mathbb{N} \). Since \( M := \Omega \setminus A \) is dense in \( \Omega \), we have \( \partial \Omega_n \subset \Omega_{n+1} = A \setminus \Omega_{n+1} \). Hence our statement is true by [9, Corollary 18, p. 238] with \( \mathcal{F}(\Omega) := \mathcal{O}(\Omega) \).
3.27. Corollary. Let \( E \) be a locally complete lcHs over \( \mathbb{C} \), \( z \in \mathbb{C}^d \) and \( R \in (0, \infty)^d \). Then the following statements hold.

a) The tensor product \( \mathcal{P}(\mathbb{D}_R(z), \mathbb{C}) \otimes E \) is dense in \( \mathcal{A}(\mathbb{D}_R(z), E) \).

b) If \( E \) is complete, then

\[
\mathcal{A}(\mathbb{D}_R(z), E) \equiv \mathcal{A}(\mathbb{D}_R(z)) \otimes E \equiv \mathcal{A}(\mathbb{D}_R(z)) \hat{\otimes}_\varepsilon E
\]

where \( \equiv \) stands for topologically isomorphic and \( \mathcal{A}(\mathbb{D}_R(z)) \otimes E \) is the completion of the injective tensor product \( \mathcal{A}(\mathbb{D}_R(z)) \hat{\otimes}_\varepsilon E \).

c) \( \mathcal{A}(\mathbb{D}_R(z)) \) has the approximation property.

Proof. The map

\[
S: \mathcal{A}(\mathbb{D}_R(z)) \otimes E \to \mathcal{A}(\mathbb{D}_R(z), E), \quad u \mapsto [w \mapsto u(\delta w)]
\]

is a (topological) isomorphism into, i.e., to its range, by \( \ref{3.1} \) Bemerkung, p. 141 with \( Y := \mathcal{A}(\mathbb{D}_R(z)) \) and Theorem \( \ref{3.20} \) ‘(a) \( \Leftrightarrow \) (c)’). Let \( \alpha \in \mathfrak{A} \), \( \varepsilon > 0 \) and \( f \in \mathcal{A}(\mathbb{D}_R(z), E) \). Since \( \mathbb{D}_R(z) \) is compact, \( f \) is uniformly continuous on \( \mathbb{D}_R(z) \) and thus there is \( \delta > 0 \) such that \( p_\alpha(f(w) - f(x)) < \varepsilon \) for all \( x, w \in \mathbb{D}_R(z) \) with \( |w - x| < \delta \). Choosing \( r > 0 \) such that

\[
1 - \frac{\delta}{\sqrt{d} \max_{1 \leq j \leq d}(R_j + |z_j|)} < r < 1,
\]

we get \( 1/r > 1 \) and thus \( \mathbb{D}_R(z) \subset \mathbb{D}_{R/r}(z) \). Furthermore, for every \( w \in \mathbb{D}_R(z) \) we have

\[
|w| \leq \sqrt{d} \max_{1 \leq j \leq d}|w_j| \leq \sqrt{d} \max_{1 \leq j \leq d}(|w_j - z_j| + |z_j|) \leq \sqrt{d} \max_{1 \leq j \leq d}(R_j + |z_j|)
\]

connoting

\[
|w - rw| = (1 - r)|w| < \frac{\delta}{\sqrt{d} \max_{1 \leq j \leq d}(R_j + |z_j|)}|w| \leq \delta.
\]

Therefore

\[
\sup_{w \in \mathbb{D}_R(z)} p_\alpha(f(w) - f(rw)) \leq \varepsilon
\]
and we set $g : \mathbb{D}_{R_1}(z) \to E$, $g(w) := f(rw)$, which is function in $O(\mathbb{D}_{R_1}(z), E)$. By Theorem 3.18 there is $N \in \mathbb{N}$ such that for all $n \geq N$

$$\|g - \sum_{|\beta| \leq n} (-z)^\beta \otimes \left(\frac{\partial^\beta_C f(z)}{\beta!}\right)\|_\alpha = \sup_{w \in \overline{\mathbb{D}}(z)} p_\alpha(g(w)) - \sum_{|\beta| \leq n} \frac{(\partial^\beta_C f(z))(w - z)^\beta}{\beta!} < \varepsilon.$$


We observe that the restriction of $T_N$ to $\mathbb{D}(z)$ is an element of $\mathcal{P}(\overline{\mathbb{D}}(z), \mathbb{C}) \otimes E = \mathcal{P}(\overline{\mathbb{D}}(z), E)$ and thus of $\mathcal{A}(\overline{\mathbb{D}}(z)) \otimes E$ as well. We conclude from (12) that

$$\|f - T_N\|_\alpha \leq \|f - g\|_\alpha + \|g - T_N\|_\alpha < 2\varepsilon$$

which proves our first statement. The second follows from the first by [31, 3.5 Remark, p. 7] because $\mathcal{A}(\overline{\mathbb{D}}(z))$ is a Banach space and thus complete. The last statement results from the second by [24, 18.1.8 Theorem, p. 400]. □

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