INFINITE REDUCED WORDS, LATTICE PROPERTY AND BRAID GRAPH OF AFFINE WEYL GROUPS

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ABSTRACT. In this paper, we establish a bijection between the infinite reduced words of an affine Weyl group and certain biclosed sets of its positive system and determine all finitely generated biclosed sets in the positive system of an affine Weyl group. Using these results, we show first that the biclosed sets in the standard positive system of rank 3 affine Weyl groups when ordered by inclusion form a complete algebraic ortholattice and secondly that the (generalized) braid graphs of those Coxeter groups are connected, which can be thought of as an infinite version of Tit’s solution to the word problem.

INTRODUCTION

Given a Coxeter system \((W, S)\) one can attach a root system \(\Phi = \Phi^+ \sqcup \Phi^-\) to it in the canonical way. A particular total ordering on \(\Phi^+\) called reflection order plays an important role in understanding the Kazhdan-Lusztig polynomials. And such orderings and their initial sections encapsulate lots of combinatorics of the Coxeter system and the associated Hecke algebra. A long standing conjecture of Dyer is that the initial sections of reflection orders under inclusion form a complete ortholattice. The conjecture is true for all finite Coxeter groups (reduced to the weak order on \(W\)) and it can be easily verified for the infinite dihedral case \((\tilde{A}_1)\). In \cite{5}, Dyer shows that in the case of affine Weyl group, the notion of initial sections of reflection orders and biclosed sets in the positive systems coincide. In order to study this poset, one considers a subposet of it consisting of the inversion sets of \(W\) and the infinite reduced expressions. In this paper we show that such poset is a complete meet semilattice. And we further investigate the infinite reduced words of an affine Weyl group and characterize the biclosed sets which are the inversion sets of infinite reduced words. An immediate consequence of this characterization is that for rank 3 affine Weyl groups all biclosed sets are either the inversion set of some (finite or infinite) word or the complement of an inversion set. Furthermore the poset of the biclosed sets in this case splits into a lower half (consisting of the inversion sets of finite or infinite words) and a top half (consisting of the complements of the inversion sets of finite or infinite word). Now the key ingredient to show that the poset is a lattice is to prove that in the lower half, a family of biclosed sets is either bounded or its orthogonal family is bounded. We prove this property by some case studies. In this way we show that for rank 3 affine Weyl groups Dyer’s conjecture holds. We also classify the finitely generated biclosed sets in the positive system of an affine root system and this allows us to conclude that the lattices under study are algebraic. The paper is organized as follow.

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Date: March 9, 2018.
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In section 2, we consider the set $W$ consisting of the infinite reduced words together with the elements in the group $W$. Then the weak order on $W$ is naturally extended to $W$. We study such order and show that $W$ is a complete meet semilattice and it is a complete lattice if and only if $W$ is the weak direct product of countably many finite or locally finite Coxeter groups. We also show that $W$ has favorable Join Orthogonality Property and admits maximal elements provided the rank of $W$ is at most infinite countable.

In section 3, we specialize in the case of affine Weyl group and establish a ($W-$equivarent) bijection between the infinite reduced word and the certain biclosed sets in the positive system. And by using this bijection we give a characterization of affine Weyl group: A finite-rank irreducible infinite Coxeter system is affine if and only if $W$ admits finitely many maximal elements.

A consequence of the classification obtained in section 3 is that for rank 3 affine positive system, the poset of the biclosed sets splits into two halves where the lower half consists of the inversion sets of the (finite or infinite) words. Section 4 is devoted to the cause of proving that the lower half of the poset has the following favorable property: a family of its elements is either bounded or its orthogonal family is bounded. Then that the poset of biclosed sets form a lattice follows easily from this property. Also we proved that in the positive system of an affine root system if the closure of the union of a family of finite biclosed sets is finite then it is biclosed.

In section 5, we study those biclosed sets in the positive system of an affine root system that are finitely generated. In rank 3 cases they are the compact elements of the lattice of biclosed sets. Our results show that those lattices are algebraic. We also provide some evidences that those finitely generated biclosed sets can be generated by roots whose twisted length is $-1$ based on rank 3 calculation.

In section 6, we show that the (generalized) braid graphs (defined using reflection orders) of rank three affine Weyl groups are all connected, which generalizes the fact that the reduced expressions of a (finite) word are connected by braid moves. In this infinite setting, a biclosed set of the standard positive system can be thought of as an element of “infinite length”. The reflection order is a substitute for reduced expression and a braid move is characterized by the reversal of a dihedral string. The proof makes use of the explicit description of the biclosed sets as either inversion sets or their complements in affine rank 3 cases.

The paper is based on part of author’s dissertation. I wish to thank my advisor Dr. Matthew Dyer for his guidance, many helpful discussions and for reading the manuscript. I also wish to thank Jiefang Xu, Yiqing Zou and Zhisheng Wang for their encouragement.

1. Preliminaries

Let $L$ be a partially ordered set. $L$ is a meet semilattice if any two elements $x, y \in L$ admit a greatest lower bound (meet), denoted $x \land y$. $L$ is called a complete meet semilattice if any subset $A \subset L$ admits a greatest lower bound (meet), denoted $\land A$. $L$ is a join semilattice if any two elements $x, y \in L$ admit a least upper bound (join), denoted $x \lor y$. $L$ is called a complete join semilattice if any subset $A \subset L$ admits a least upper bound (join), denoted $\lor A$. If $L$ is both a meet semilattice and a join semilattice, $L$ is called a lattice. If $L$ is both a complete meet semilattice and a complete join semilattice, $L$ is called a complete lattice. Suppose a lattice
L has a minimum element, denoted 0, and a maximum element, denoted 1. And assume there exists an order-reversing involution on L, denoted \( x \mapsto x^\circ \) such that \( x \wedge x^\circ = 0 \) and \( x \lor x^\circ = 1 \) for all \( x \in L \). We call L an ortholattice.

An element \( x \) in a complete lattice \( L \) is called compact if \( x \leq \bigvee Y \) for some \( Y \subset L \) implies that \( x \leq \bigvee Y_0 \) for some finite \( Y_0 \subset Y \). A complete lattice is called algebraic if every element of it is a join of a set of compact elements.

For a totally ordered set \( T \), a set \( I \subset T \) is called an initial section if for all \( x \in I \), \( y \in T \setminus I \) we have \( x \leq y \). A set \( F \subset T \) is called a final section if for all \( x \in F \), \( y \in T \setminus F \) we have \( x \geq y \).

Let \((W,S)\) be a Coxeter system with root system \( \Phi \), see [14] for example. The set of positive roots and negative roots are denoted \( \Phi^+ \) and \( \Phi^- \) respectively. The set of simple roots is denoted \( \Pi \). An effective way to visualize the infinite root system in lower rank is to consider the projective representation of \( W \). The idea is to project all the roots in the standard reflection representation to an affine hyperplane which is transverse to \( \Phi^+ \). For projective representation see [8],[13] and [16]. Given \( w \in W \) define the inversion set of \( w \) to be \( \Phi_w := \{ \alpha \in \Phi^+ | w^{-1}(\alpha) \in \Phi^- \} \). Let \( T := \{ wsw^{-1} | w \in W, s \in S \} \). \( T \) is called the set of reflections. There exists a canonical bijection between \( T \) and \( \Phi^+ \) and denote \( s_\alpha \) the reflection corresponding to a positive root \( \alpha \). Let \( W' \) be a subgroup of \( W \) generated by a subset of \( T \). Then \( W' \) is called a reflection subgroup. And \((W', \{ t \in T \cap W' | l(t') > l(t), \forall t' \in T \cap W' \setminus \{t\} \})\) is a Coxeter system. For reflection subgroups see [6].

The weak (right) order \( \leq \) on \( W \) is defined as \( x \leq y \) if and only if \( \Phi_x \subset \Phi_y \) for all \( x, y \in W \). \((W, \leq)\) is a complete meet semilattice and is a complete lattice if and only if \( W \) is finite. See [1] Chapter 3.

A set \( \Gamma \subset \Phi^+ \) is called closed (in \( \Phi^+ \)) if for all \( \alpha, \beta \in \Gamma, k_1 \alpha + k_2 \beta \in \Phi^+ \) with \( k_1, k_2 \in \mathbb{R}_{\geq 0} \) implies that \( k_1 \alpha + k_2 \beta \in \Gamma \). A subset \( \Gamma \) is called coclosed (in \( \Phi^+ \)) if \( \Phi^+ \setminus \Gamma \) is closed (in \( \Phi^+ \)). \( \Gamma \subset \Phi^+ \) is called biclosed (in \( \Phi^+ \)) if \( \Gamma \) and \( \Phi^+ \setminus \Gamma \) are both closed (in \( \Phi^+ \)). Replacing \( \Phi^+ \) with \( \Phi \) in the above definition, we define the closed sets, coclosed sets and biclosed sets in \( \Phi \). We usually denote \( \overline{X} \) the closure of a set \( X \).

It is known that the finite biclosed sets in \( \Phi^+ \) are precisely the inversion sets of the elements in the Coxeter group (See [7] Lemma 4.1(d)).

Denote by \( \mathcal{B}(\Phi^+) \) the set of all biclosed sets in \( \Phi^+ \), regarded as a partially ordered set under set inclusion. Two elements \( B_1, B_2 \) of this poset are said to be in the same block if their symmetric difference is finite.

For \( \Gamma \) biclosed in \( \Phi^+ \) and \( x \in W \), define
\[
x \cdot \Gamma := (\Phi_x \setminus x(-\Gamma)) \cup (x(\Gamma) \setminus (-\Phi_x)).
\]

Then \( x \cdot \Gamma \) is again a biclosed set and this is a group action of \( W \) on the set of biclosed sets in \( \Phi^+ \). See [7] Lemma 4.1(a)(c).

Let \( A \in \mathcal{B}(\Phi^+) \). Following [3], denote \( \Omega_{W,A} \) the directed graph with vertex set \( W \) and edge set \{ \( (s_\alpha w, w) | w \in W, \alpha \in w \cdot A \) \} and call it the twisted Bruhat graph. Take \( u, v \in W \). Define \( u \prec_A v \) if and only if there exists \( s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_k} \in T, (\alpha_i \in \Phi^+, 1 \leq i \leq k) \) such that \( v = s_{\alpha_k} s_{\alpha_{k-1}} \cdots s_{\alpha_1} u \) and \( \alpha_i \notin s_{\alpha_{i-1}} \cdots s_{\alpha_2} s_{\alpha_1} u \cdot A \) for \( 1 \leq i \leq k \). Then this is a partial order on \( W \) and is called twisted Bruhat order. Define \( l_A : W \to \mathbb{Z}_{\geq 0} : w \mapsto l(w) - 2|\Phi_w \cap A| \). Call \( l_A \) the \((A)\)-twisted length.

A total order \( \prec \) on \( \Phi^+ \) is called a reflection order if for all \( \alpha, \beta \in \Phi^+, \alpha \prec \beta \) and \( a, b \in \mathbb{R}_{\geq 0} \) such that \( a \alpha + b \beta \in \Phi^+ \) we have \( \alpha \prec a \alpha + b \beta \prec \beta \). Suppose \((W, S)\) is a
dihedral Coxeter system with \( S = \{s_\alpha, s_\beta\} \). Then
\[
\alpha \prec s_\alpha(\beta) \prec s_\beta s_\alpha(\beta) \prec \cdots \prec s_\beta(\alpha) \prec \beta
\]
is a reflection order.

A locally finite Coxeter system \((W, S)\) is the one with \( W \) infinite such that each parabolic subgroup \( W_J \) is finite if \( J \) is finite. Irreducible locally finite Coxeter systems are precisely the ones of type \( A_\infty, A_{\infty, \infty}, B_\infty, D_\infty \). See Exercise 4.14 of [15].

Now let \( \Phi \) be an irreducible crystallographic root system of a Weyl group \( W \) and be contained in the Euclidean space \( V \) as in [2]. One calls a subset \( \Gamma \subset \Phi \mathbb{R} \)-closed if for all \( \alpha, \beta \in \Gamma \) such that \( \alpha + \beta \in \Phi \) then \( \alpha + \beta \in \Gamma \).

Let \( \Delta \) be a simple system of \( \Phi \) with the corresponding positive system \( \Phi^+ \). Let \( \gamma \) be the highest root. Suppose \( \Delta' \) and \( \Delta'' \) are two subsets of \( \Delta \) which are orthogonal (i.e. \( (\alpha, \beta) = 0, \forall \alpha \in \Delta', \beta \in \Delta'' \)). Define
\[
\Phi_{\Delta', \Delta''}^+ = (\Phi^+ \setminus (\mathbb{R} \Delta' \cap \Phi)) \cup (\mathbb{R} \Delta'' \cap \Phi).
\]

By requiring \((\alpha, \beta) = 0, \forall \alpha \in \Delta', \beta \in \Delta''\)
Define a real vector space \( V' = V \oplus \mathbb{R} \delta \) and extend the inner product on \( V \) to \( V' \) by requiring \((\delta, V') = 0\). For \( \alpha \in \Phi^+ \), we define \( \tilde{\alpha} = \{\alpha + n\delta | n \in \mathbb{Z}_{\geq 0}\} \subset V' \). For \( \alpha \in \Phi^+ \), we define \( \tilde{\alpha} = \{\alpha + (n+1)\delta | n \in \mathbb{Z}_{\geq 0}\} \subset V' \). For a set \( \Gamma \subset \Phi \), define \( \tilde{\Gamma} = \bigcup_{\alpha \in \Gamma} \tilde{\alpha} \subset V' \).

Denote \( \hat{W} \) the (irreducible) affine Weyl group corresponding to \( W \). Then \( \hat{W} \) has \( \hat{\Phi} \cup -\hat{\Phi} \) as root system with \( \hat{\Phi}(= \Phi^+) \) as the set of positive roots. \( \hat{\Delta} = \Delta \cup \{\delta - \gamma\} \) is the set of simple roots. For this construction, see [15] or [9].

In [5] all biclosed sets in \( \Phi \) are classified:

**Theorem 1.1.** Let \( \Gamma \subset \Phi \). \( \Gamma \) is biclosed in \( \Phi \) if and only if there exists a simple system \( \Delta \) with the corresponding positive system \( \Phi^+ \) and orthogonal subsets \( \Delta', \Delta'' \subset \Delta \) such that \( \Gamma = \Phi_{\Delta', \Delta''}^+ \).

Define a real vector space \( V' = V \oplus \mathbb{R} \delta \) and extend the inner product on \( V \) to \( V' \) by requiring \((\delta, V') = 0\). For \( \alpha \in \Phi^+ \), we define \( \tilde{\alpha} = \{\alpha + n\delta | n \in \mathbb{Z}_{\geq 0}\} \subset V' \). For \( \alpha \in \Phi^+ \), we define \( \tilde{\alpha} = \{\alpha + (n+1)\delta | n \in \mathbb{Z}_{\geq 0}\} \subset V' \). For a set \( \Gamma \subset \Phi \), define \( \tilde{\Gamma} = \bigcup_{\alpha \in \Gamma} \tilde{\alpha} \subset V' \).

Denote \( \tilde{W} \) the (irreducible) affine Weyl group corresponding to \( W \). Then \( \tilde{W} \) has \( \hat{\Phi} \cup -\hat{\Phi} \) as root system with \( \hat{\Phi}(= \Phi^+) \) as the set of positive roots. \( \hat{\Delta} = \Delta \cup \{\delta - \gamma\} \) is the set of simple roots. For this construction, see [15] or [9].

In [5] all biclosed sets in \( \Phi \) are classified:

**Theorem 1.2.** (1) Let \( \Gamma \subset \Phi \). \( \Gamma \) is biclosed in \( \Phi \) if and only if \( \Gamma = w \cdot \Lambda \) where \( \Lambda \) is biclosed in \( \Phi \) and \( w \in \tilde{W} \).

(2) The biclosed sets in \( \Phi \) that are in the same block as \( \Phi^+_{\Delta_1, \Delta_2} \) are those of the form \( w \cdot \Phi^+_{\Delta_1, \Delta_2} \) for some \( w \in W' \subset \tilde{W} \) where \( W' \) is the reflection subgroup generated by \( \{s_\alpha | \alpha \in \Delta_1 \cup \Delta_2\} \cup \{s_{\Delta - \rho} | \rho \) is the highest root of an irreducible component of the root subsystem \( \mathbb{R}(\Delta_1 \cup \Delta_2) \cap \Phi \) of \( \Phi \) with respect to the standard positive system \( \Phi \).

We also need the following theorem which is proved in [5]:

**Theorem 1.3.** Any maximal totally ordered ordered subset of \( \mathcal{B}(\hat{\Phi}) \) is the set of all initial sections of some reflection order of \( \hat{\Phi} \).

In this paper we denote the disjoint union by \( \cup \).

2. Infinite Reduced Words of a Coxeter System

Let \( (W, S) \) be a Coxeter system with \( W \) infinite.

**Definition 2.1.** An infinitely long reduced expression is a sequence
\[
s_1 s_2 s_3 \cdots
\]
with $s_i \in S$ such that for all $k \geq 1$, $s_1s_2\cdots s_k$ is a reduced expression.

If $s_1s_2s_3\cdots$ is an infinitely long reduced expression then immediately we have

$$\Phi_{s_1} \subset \Phi_{s_1s_2} \subset \Phi_{s_1s_2s_3} \subset \cdots$$

**Definition 2.2.** Let $s_1s_2s_3\cdots$ be an infinitely long reduced expression. Define

$$\Phi_{s_1s_2s_3\cdots} = \bigcup_{i=1}^{\infty} \Phi_{s_1s_2\cdots s_i}$$

and

$$\Phi'_{s_1s_2s_3\cdots} = \Phi^+ \setminus \Phi_{s_1s_2s_3\cdots}$$

It is clear from the definition that $\Phi_{s_1s_2s_3\cdots}$ is infinite and biclosed. Given two infinitely long reduced expressions $s_1s_2s_3\cdots$ and $r_1r_2r_3\cdots$. Write $s_1s_2s_3\cdots \sim r_1r_2r_3\cdots$ if $\Phi_{s_1s_2s_3\cdots} = \Phi_{r_1r_2r_3\cdots}$. It is clear that this is an equivalence relation.

**Definition 2.3.** Let $L$ be the set of all infinitely long reduced expressions. Define $W_l = L/\sim$. An element in $W_l$ is called an infinite reduced word of $(W, S)$. Denote $\overline{W} = W_l \bigcup W$. Given $w = [s_1s_2s_3\cdots] \in W_l$, $s_1s_2s_3\cdots$ is called a reduced expression of $w$.

In case of no confusion, we write $w = s_1s_2s_3\cdots$ if $w = [s_1s_2s_3\cdots]$. We can extend the weak order on $W$ to $\overline{W}$ in an obvious way. Such construction has been studied by various authors, for example see [3], [12] and [18]. In [12], the authors study infinite reduced words and the weak order from the point of view of projective representation and imaginary cone and raise a few questions.

**Definition 2.4.** Given $x, y \in \overline{W}$, define $x \leq y$ if and only if $\Phi_x \subset \Phi_y$. Call this order the weak order on $\overline{W}$. Let $x \in W_l$. An element $w \in W$ is called a prefix of $x$ if $w \leq x$. Say two elements $x, y \in \overline{W}$ are orthogonal, denoted $x \perp y$ if $\Phi_x \cap \Phi_y = \emptyset$.

**Definition 2.5.** Let $s \in S$ and $w$ be an infinite reduced word of $W$. We define $sw$ in the following way:

Case I: if $s \perp w$, we define $sw$ to be the word obtained by concatenating $s$ with a reduced expression $w$.

Case II: if $s \not\perp w$, choose one reduced expression $s_1s_2s_3\cdots$ of $w$ and one can find $u = s_1s_2\cdots s_k$ such that $l(sw) = l(u) - 1$. We define $sw$ to be the word obtained by concatenating one reduced expression of $su$ with the word $s_{k+1}s_{k+2}\cdots$.

Lemma 2.7 implies that this multiplication is well-defined.

**Definition 2.6.** Let $u \in W$ and $w$ be an infinite reduced word. Let $s_1s_2\cdots s_k$ be any (reduced or non-reduced) expression of $u$. Define $uw$ to be

$$s_1(\cdots(s_{k-1}(s_kw))\cdots)$$

Again Lemma 2.7 ensures that this multiplication is well-defined.

We summarize the basic properties of infinite reduced words in the following lemma and their proofs reduced easily to the analogous statements of Coxeter groups and are therefore omitted.

**Lemma 2.7.** (a) Let $w_1 = s_1s_2s_3\cdots$ and $w_2 = r_1r_2r_3\cdots$ be two infinite reduced words of $W$. $w_1 \leq w_2$ if and only if any prefix of $w_1$ is a prefix of $w_2$.

(b) Let $x, y \in \overline{W}$. $x \perp y$ if and only if for any prefix $u$ of $x$ and any prefix $v$ of $y$ we have $l(u^{-1}v) = l(u) + l(v)$. 

(c) The multiplications in Definition 2.5 and 2.6 are all well-defined, i.e. in case I the product $sw$ is independent of the choice of the reduced expression of $w$, and in case II the product $sw$ is independent of the choice of reduced expression of $w$, the choice of $u$ and the choice of the reduced expression of $su$, and in case II the product $uw$ does not depend on the choice of the expression of $u$.

(d) Let $x, y \in W$ and $z \in W$. Then $x \cdot \Phi_z = \Phi_{xz}$ and $x \cdot \Phi'_z = \Phi'_{xz}$. And $(xy)z = x(yz)$.

Now we study the weak order on $\overline{W}$. First we prove an easy but useful lemma.

**Lemma 2.8.** Let $A \subset W$ and $|A| < \infty$. Suppose $A$ is bounded in $\overline{W}$. Then $A$ is bounded in $W$ and $\bigvee A$ exists in $W$.

**Proof.** Suppose $A$ is bounded by $s_1s_2s_3\cdots$. Then every element of $a$ is bounded by some $s_1s_2\cdots s_k \in W$. Since $A$ is finite, we can take the longest of these bounds and it is a bound in $W$ of $A$. The second assertion follows from the fact $W$ is a complete meet semilattice. \hfill $\square$

**Theorem 2.9.** $(\overline{W}, \preceq)$ is a complete meet semilattice.

**Proof.** We first show

Claim: Let $w$ be an infinite reduced word. Then the set of its prefixes is countably infinite.

Let $s_1s_2s_3\cdots$ be a reduced expression of $w$. Then the set of the prefixes of $w$ is $\bigcup_{i=1}^{\infty} \{u \in W | u \leq s_1s_2\cdots s_i \}$ and each set in this union is finite. Therefore the claim is true.

We first show that any non-empty subset of $W_l$ has a meet. Take $\emptyset \neq A \subset W_l$.

As a consequence of the claim, the set $B := \{w \in W | w < a, \forall a \in A \}$ is countable. So we can denote $B = \{b_1, b_2, \cdots \}$. And we obtain an ascending chain under weak order:

$$b_1 \leq b_1 \lor b_2 \leq b_1 \lor b_2 \lor b_3 \leq \cdots$$

Each term in this chain exists in $W$ by Lemma 2.8. Therefore this chain gives rise to a well-defined infinitely long word or an element in $W$. Denote it $v$. We shall show that $v$ is the meet of $A$.

First we note that it is immediate that $v \geq b, \forall b \in B$. Then take some $w \in \overline{W}$ which is a lower bound of $A$. So any prefix of $w$ is in $B$ and therefore is bounded by $v$. This shows that $w \leq v$. So $v$ is the meet of $A$.

Since $W$ is a complete meet semilattice, any subset of $W$ admits a meet in $W$ and it is clear that such meet remains to be the meet in $\overline{W}$.

Now we consider $A \subset \overline{W}$ with $A \cap W_l \neq \emptyset$ and $A \cap W \neq \emptyset$. We denote $a = \wedge (A \cap W_l)$ and $b = \wedge (A \cap W)$. Fix the reduced expressions $s_1s_2s_3\cdots$ (finite if $a \in W$) for $a$. Then we have a chain:

$$b \land s_1 \leq b \land s_1s_2 \leq b \land s_1s_2s_3 \leq \cdots$$

This chain defines an infinite reduced word or an element in the group. Denote it $v$. One readily sees that all lower bounds of $A$ are less than or equal to $v$. \hfill $\square$

The poset $\overline{W}$ was considered by various authors, for example it was proved to be a complete meet semilattice in [17] in the affine case. There is no previously known proof for the general case to the knowledge of the author.

It is natural to ask when $\overline{W}$ is a complete lattice. This question is answered in the following
Theorem 2.10. \( \overline{W} \) is a complete lattice if and only if \( W \cong \overset{\infty}{\underset{i=1}{\Pi}} W_i \) where each \( W_i \) is either finite or locally finite irreducible Coxeter group and \( \Pi \) denotes the weak direct product.

Proof. We first prove the only if part. \( W \) can always be written as the weak direct product of a family of irreducible Coxeter groups, called its components. Since any two elements of \( W \) have an upper bound in \( \overline{W} \), they have an upper bound in \( W \) by Lemma 2.8. So each group of that family must either be locally finite or finite. If the family consists of uncountably many non-trivial Coxeter groups, then we take one generator from each of these Coxeter groups. Clearly their join cannot possibly be an infinite reduced word or an element in \( W \).

To prove the converse, we first note that for a finite Coxeter group \( W \) there exists a longest element \( w_0(W) \). For a locally finite Coxeter group \( W \), one can find a sequence of parabolic subgroups 
\[
W_1 < W_2 < W_3 < \cdots
\]
such that each \( W_i \) is a finite and \( \bigcup_{i=1}^{\infty} W_i = W \). Thus we have a chain
\[
w_0(W_1) \leq w_0(W_2) \leq w_0(W_3) \leq \cdots.
\]
This chain defines an infinite reduced word \( w_0(W) \) which is the unique maximal element in \( (\overline{W}, \leq) \). Now suppose \( W \cong \overset{\infty}{\underset{i=1}{\Pi}} W_i \) and \( w_0(W_i) = s_{i,1}s_{i,2}s_{i,3} \cdots \). Listing these elements
\[
\begin{align*}
s_{1,1}s_{1,2}s_{1,3} & \cdots \\
s_{2,1}s_{2,2}s_{2,3} & \cdots \\
s_{3,1}s_{3,2}s_{3,3} & \cdots \\
& \cdots
\end{align*}
\]
and then enumerating \( s_{i,j} \)'s in a zig-zag way (i.e. \( s_{1,1}s_{1,2}s_{2,1}s_{3,1}s_{2,2}s_{1,3}s_{1,4}s_{2,3} \cdots \)) gives rise to a well-defined infinite reduced word which is maximal under the weak order. The existence of such an element makes the complete semilattice a complete lattice.

□

Remark. It can be shown that for a group as in the above theorem any biclosed set in \( \Phi^+ \) is the inversion set of a (finite or infinite) word. Thus the above theorem proves that the poset of biclosed sets in \( \Phi^+ \) form a lattice for those types of Coxeter groups.

Next we show that \( \overline{W} \) has the following favorable property and we refer to it as Join Orthogonal Property.

Theorem 2.11. Let \( A \subset \overline{W} \) and \( A \) admits a join \( w \in \overline{W} \). If \( \forall a \in A, a \perp v \) for some \( v \in \overline{W} \), then \( w \perp v \).

Proof. Suppose \( w \) is in \( W \) (then \( A \subset W \)). If \( v \in W \), the assertion follows from Lemma 4.2(b) in [7]. If \( v \in W_i \), then every prefix of \( v \) is orthogonal to \( w \) by the above argument and thus \( v \perp w \). Suppose \( w \) is in \( W_i \). Consider \( B = \{ b \in W \mid \exists a \in A, b \leq a \} \). Clearly \( \bigvee B = \bigvee A = w \). And \( B \) is contained in the set of prefixes of \( w \) and hence countable by the claim in the proof of Theorem 2.9. Then write \( B = \{ b_1, b_2, \cdots \} \). We also have \( b_i \perp v \) for all \( i \). Therefore any prefix of \( w \) must be smaller than \( \bigvee_{i=1}^{n} b_i \) for some \( n \) and thereby orthogonal to \( v \). □
The following proposition will be used in proving a characterization of affine Weyl groups in the next section. Now suppose \( W \) is countably generated (i.e. \(|S| \leq \aleph_0\)). As a consequence \( W \) is countable.

**Proposition 2.12.** \((\overline{W}, \leq)\) contains maximal elements. And every element \( w \in \overline{W} \) is less than or equal to some maximal element.

**Proof.** To prove the first assertion it suffices to show that the condition of Zorn’s Lemma is satisfied. Suppose we have an ascending chain in \( \overline{W} \):
\[
\{w_i\}_{i \in I}.
\]
Then the set \( B = \{b \in W | b \leq w_i \text{ for some } i\} \) is countable since \( W \) is countable. Denote \( B = \{b_1, b_2, b_3, \ldots\} \). For \( b_1, b_2, \ldots, b_k \), one can find some \( w_j \) which is their bound. Hence \( b_1, b_2, \ldots, b_k \) admit a join in \( W \) by Lemma 2.8. So we have an ascending chain of elements in \( W \):
\[
b_1 \leq b_1 \lor b_2 \leq b_1 \lor b_2 \lor b_2 \leq \cdots.
\]
This defines an element in \( \overline{W} \) which is easily seen to be the join the original chain \( \{w_i\}_{i \in I} \). The second assertion follows by the same argument applied to \( \{x \in \overline{W} | x \geq w\}\).

3. **infinite reduced words And Biclosed Sets In an Affine Root System**

The main aim of this section is to obtain a description of \( \Phi_w \) for \( w \in \overline{W} \) in the affine case. Related description has been studied by other authors, for example in [3]. Our description is from the point of view of loop extension of finite root system. The proof relies on an interesting property of finite root system which is unobserved previously to the knowledge of the author and this property will be used again later in the paper.

Let \( \Phi \) be an irreducible crystallographic root system of a Weyl group \( W \). Let \( \Pi \) be a simple system of \( \Phi \) and \( L \subset \Pi \). \( \Phi^+ \) is the positive system corresponding to \( \Pi \).

To begin with, we define the function
\[
h : \Phi^+ \to \mathbb{N}
\]
Suppose \( \alpha, \beta, \gamma \in \Phi^+ \). Suppose \( \alpha + \beta \) and \( \alpha + \beta + \gamma \) are both in \( \Phi^+ \). Then either \( \alpha + \gamma \) or \( \beta + \gamma \) is in \( \Phi^+ \).

**Lemma 3.1.** Let \( \alpha, \beta, \gamma \in \Phi^+ \). Suppose \( \alpha + \beta \) and \( \alpha + \beta + \gamma \) are both in \( \Phi^+ \). Then either \( \alpha + \gamma \) or \( \beta + \gamma \) is in \( \Phi^+ \).
Proof. Assume to the contrary, neither $\alpha + \gamma$ nor $\beta + \gamma$ is a (positive) root. Then by the property of a crystallographic root system we have $(\alpha, \gamma) \geq 0$ and $(\beta, \gamma) \geq 0$. Also we have neither $(\alpha + \beta + \gamma) + (-\beta)$ nor $(\alpha + \beta + \gamma) + (-\alpha)$ is a (positive) root. Again we have $(\alpha + \beta + \gamma, -\beta) \geq 0$ and $(\alpha + \beta + \gamma, -\alpha) \geq 0$. Let $v' = \frac{2\alpha}{(v, v)}$. Then we have

\[
\begin{align*}
(\alpha', \gamma) &\geq 0 \\
(\beta', \gamma) &\geq 0 \\
(\alpha + \beta + \gamma, -\alpha') &\geq 0 \\
(\alpha + \beta + \gamma, -\beta') &\geq 0
\end{align*}
\]

This gives that

\[
\begin{align*}
(\beta, \alpha') &\leq -2 \\
(\alpha, \beta') &\leq -2.
\end{align*}
\]

This yields

\[
(\alpha, \beta)^2 \geq (\alpha, \alpha)(\beta, \beta).
\]

But by Cauchy-Schwarz inequality this forces $\alpha = \beta$, which again contradicts the assumption $\alpha + \beta$ is a root. \qed

**Proposition 3.2.** $d(\alpha) = h(\alpha)$.

**Proof.** (M. Dyer) If $\alpha \not\in \Phi_{L,0}^+$, then the both sides are 0. So we assume $\alpha \in \Phi_{L,0}^+$.

Suppose $\alpha$ can be written as a sum of $n$ elements in $\Phi_{L,0}^+$. When writing such element as the positive linear combination of simple roots, there must be at least one simple root outside $L$ appearing in the sum. Hence we see $d(\alpha) \leq h(\alpha)$.

Now we prove the $h(\alpha) \leq d(\alpha)$. For simple roots we clearly have $h(\alpha) = d(\alpha)$. Also note that any positive root $\alpha$ can be written as $\alpha_1 + \alpha_2 + \cdots + \alpha_k$ where for each $i$ $\alpha_i$ is simple and $\alpha_1 + \alpha_2 + \cdots + \alpha_i$ is a root. So it suffices to show that $d(\alpha + \beta) \geq d(\alpha) + d(\beta)$ for $\alpha, \beta, \alpha + \beta$ all positive roots.

If $d(\alpha) = d(\beta) = 0$, the inequality is trivial. If $d(\alpha) > 0$ and $d(\beta) > 0$, then the inequality follows from the definition of the function $d$. So it suffices to prove the case $d(\alpha) = 0$ and $d(\beta) > 0$. We prove this by the induction on $d(\beta)$. The case that $d(\beta) = 1$ is trivial. Suppose that for $d(\beta) \leq m$, $d(\alpha) + d(\beta) \leq d(\alpha + \beta)$.

Assume $d(\beta) = m + 1$ and $\beta = \beta_1 + \beta_2 + \cdots + \beta_{m+1}$ with each $\beta_i \not\in \Phi_L$. We can even assume that for all $i$ $\beta_1 + \beta_2 + \cdots + \beta_i$ is a root by Chapter VI §1 Proposition 19 in [2]. By Lemma 3.1 either $\alpha + \beta_{m+1}$ or $\alpha + \beta_1 + \beta_2 + \cdots + \beta_m$ is a (positive) root. In the first case

\[
\alpha + \beta = (\alpha + \beta_{m+1}) + (\beta_1) + (\beta_2) + \cdots + (\beta_m)
\]

So we have $d(\alpha + \beta) \geq m + 1$. In the second case,

\[
\alpha + \beta = (\alpha + \beta_1 + \beta_2 + \cdots + \beta_m) + \beta_{m+1}
\]

We have $d(\alpha + \beta_1 + \beta_2 + \cdots + \beta_m) > 0$ and so by the previous discussion $d(\alpha + \beta) \geq d(\alpha + \beta_1 + \beta_2 + \cdots + \beta_m) + 1$. But by induction $d(\alpha + \beta_1 + \beta_2 + \cdots + \beta_m) \geq m$ and we are done. \qed

**Corollary 3.3.** If $\alpha = k_1 \beta + k_2 \gamma$ with $\alpha, \beta, \gamma \in \Phi^+$ and $k_1, k_2 \in \mathbb{R}_{\geq 0}$, then $d(\alpha) = k_1 d(\beta) + k_2 d(\gamma)$.

**Proof.** By Proposition 3.2 it suffices to show $h(\alpha) = k_1 h(\beta) + k_2 h(\gamma)$. But this is clear from the definition of the $h$ function. \qed
From now on we consider the associated affine root system $\tilde{\Phi}$ with positive system $\Phi$. The corresponding (irreducible) affine Weyl group is denoted $\tilde{W}$.

**Definition 3.4.** Denote

$$(\Phi^+_L, 0) = \{ \alpha + k\delta | \alpha \in \Phi^+_L, 0 \leq k \leq n \}.$$  

**Lemma 3.5.**

$$(\Phi^+_L, 0) = \{ \alpha + k\delta | \alpha \in \Phi^+_L, 0 \leq k \leq nd(\alpha) \}.$$  

**Proof.** Let $0 \leq k \leq nd(\alpha)$. Then we can find $0 \leq a_1, a_2, \cdots, a_{d(\alpha)} \leq n$ such that

$$a_1 + a_2 + \cdots + a_{d(\alpha)} = k.$$  

By definition of the function $d$, we have $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_{d(\alpha)}$ where $\alpha_i \in \Phi^+_L$. Therefore

$$\alpha + k\delta = (\alpha_1 + a_1\delta) + (\alpha_2 + a_2\delta) + \cdots + (\alpha_{d(\alpha)} + a_{d(\alpha)}\delta).$$

This combined with Chapter VI §1 Proposition 19 in [2] shows $\alpha + k\delta \in (\Phi^+_L, 0)$.

Conversely assume $\alpha_1, \alpha_2 \in \Phi^+_L$ and $0 \leq k_1 \leq nd(\alpha_1), 0 \leq k_2 \leq nd(\alpha_2)$ and $m_1, m_2 \geq 0$ (where $k_1, k_2 \in \mathbb{Z}$ and $m_1, m_2 \in \mathbb{R}$) and $m_1(\alpha_1 + k_1\delta) + m_2(\alpha_2 + k_2\delta)$ is a root in $\tilde{\Phi}$. Then $m_1\alpha_1 + m_2\alpha_2 \in \Phi$. Since $h(\alpha_i) > 0$ for $i = 1, 2$ and $m_1, m_2$ cannot both be zero so we have $h(m_1\alpha_1 + m_2\alpha_2) > 0$ and therefore $m_1\alpha_1 + m_2\alpha_2 \in \Phi^+_L$.

By Corollary 3.3 $d(m_1\alpha_1 + m_2\alpha_2) = m_1d(\alpha_1) + m_2d(\alpha_2) \geq m_1\frac{k_1}{n} + m_2\frac{k_2}{n}$. So

$$nd(m_1\alpha_1 + m_2\alpha_2) \geq m_1k_1 + m_2k_2.$$  

This implies $m_1\alpha_1 + m_2\alpha_2 + (m_1k_1 + m_2k_2)\delta$ is still in $(\alpha + k\delta | \alpha \in \Phi^+_L, 0 \leq k \leq nd(\alpha))$.  

An immediate consequence of this lemma is

**Lemma 3.6.** $(\Phi^+_L, 0)$ is a finite set.

We also have

**Lemma 3.7.** $(\Phi^+_L, 0)$ is a biclosed set.

**Proof.** We only need to show the set is coclosed.

Case I:

Consider $\alpha + k_1\delta, \beta + k_2\delta$ where $\alpha, \beta \notin \Phi^+_L$ and $k_1 > nd(\alpha), k_2 > nd(\beta)$. Suppose $t_1\alpha + t_2\beta + (t_1k_1 + t_2k_2)\delta, t_1, t_2 \geq 0$ is a root of $\tilde{\Phi}$. So $d(t_1\alpha + t_2\beta) = t_1d(\alpha) + t_2d(\beta) < t_1\frac{k_1}{n} + t_2\frac{k_2}{n}$. Therefore

$$t_1k_1 + t_2k_2 > nd(t_1\alpha + t_2\beta).$$

That implies $t_1\alpha + t_2\beta + (t_1k_1 + t_2k_2)\delta \notin (\Phi^+_L, 0)$.

Case II:

Consider $\alpha + k_1\delta, \beta + k_2\delta$ where $\alpha, \beta \notin \Phi^+_L$. Suppose $t_1\alpha + t_2\beta + (t_1k_1 + t_2k_2)\delta, t_1, t_2 \geq 0$ is a root of $\tilde{\Phi}$. Then $t_1\alpha + t_2\beta \not\in \Phi^+_L$ as $\Phi^+_L$ is a biclosed set in $\Phi$.

Case III:

Consider $\alpha + k_1\delta, \beta + k_2\delta$ where $\alpha \notin \Phi^+_L$ and $\beta \in \Phi^+_L$ and $k_2 > nd(\beta)$. Suppose $t_1\alpha + t_2\beta + (t_1k_1 + t_2k_2)\delta, t_1, t_2 \geq 0$ is a root of $\tilde{\Phi}$. We only need to examine the case
when \( t_1\alpha + t_2\beta \in \Phi_{L,\emptyset}^+ \). In this case \( d(t_1\alpha + t_2\beta) \leq t_2d(\beta) < t_2k_2/n \) (If \( \alpha \) is positive then the first \( \leq \) is \( = \)). Therefore
\[
 t_1k_1 + t_2k_2 > nd(t_1\alpha + t_2\beta).
\]
That implies \( t_1\alpha + t_2\beta + (t_1k_1 + t_2k_2)\delta \notin (\Phi_{L,\emptyset}^+)_n \). □

**Corollary 3.8.** \( \Phi_{L,\emptyset}^+ \) is of the form \( \Phi_w \) where \( w \) is an infinite reduced word (in \( \tilde{W} \)).

**Proof.** This follows from the above Lemma 3.6 and Lemma 3.7 and
\[
\Phi_{L,\emptyset}^+ = \bigcup_{n=0}^{\infty} (\Phi_{L,\emptyset}^+)_n.
\]

**Lemma 3.9.** Let \( \Psi^+ \) be a positive system of \( \Phi \), \( \Delta \) be its simple system, \( N, M \subset \Delta \) and \( M \neq \emptyset \). Then \( \Phi_{N,M}^+ \) is not of the form \( \Phi_w \) where \( w \) is an infinite reduced word.

**Proof.** Let \( \Phi_{N,M}^+ = \Phi_u \) where \( w \) is an infinite reduced word and \( \alpha \in M \). Then \( \alpha + m\delta \) and \( -\alpha + n\delta \) are both in \( \Phi_{N,M}^+ \) for some \( m, n \in \mathbb{Z}_{\geq 1} \). Then they must be both in some \( \Phi_u, u \in \tilde{W} \). Because \( \Phi_u \) is biclosed so we have
\[
2(\alpha + m\delta) + (-\alpha + n\delta) = \alpha + (2m + n)\delta \in \Phi_u
\]
\[
3(\alpha + m\delta) + 2(-\alpha + n\delta) = \alpha + (3m + 2n)\delta \in \Phi_u
\]
\[
\ldots
\]
This means \( \Phi_u \) is infinite which is impossible. □

**Theorem 3.10.** The map
\[
u \mapsto \Phi_u
\]
gives

(1) a bijection between \( \tilde{W}_1 \) and the following subset of \( B(\Phi) \)
\[
\{w \cdot \Phi_{L,\emptyset}^+ | w \in \tilde{W}, L \subseteq \Pi \}.
\]

(2) a bijection between \( \tilde{W} \) and the following subset of \( B(\Phi) \)
\[
\{w \cdot \Phi_{L,\emptyset}^+ | w \in \tilde{W}, L \subseteq \Pi \}.
\]

**Proof.** Suppose \( u \in \tilde{W} \). In view of Theorem 1.1 and 1.2, we need to examine whether a set \( \Phi_{M,N}^+ \) where \( M, N \subset \Pi \) is in the image of the given map. Corollary 3.8 shows that \( \Phi_{L,\emptyset}^+ \) is in the image of the map \( u \mapsto \Phi_u \) and \( w \cdot \Phi_{L,\emptyset}^+ \) is also in the image by Lemma 2.7 (d). Lemma 3.9 shows that \( \Phi_{N,M}^+ \) where \( M \neq \emptyset \) is not in the image. Then neither is \( w \cdot \Phi_{N,M}^+ \) by Lemma 2.7 (d). Finally we see that \( w \cdot \Phi_{L,\emptyset}^+ \) is finite and thus not in the image if \( u \in \tilde{W}_1 \).

**Corollary 3.11.** Let \( \Psi^+ \) be a positive system in \( \Phi \) and \( \Delta \) be the simple system of it. Suppose \( M \subset \Delta \) and \( v \in \tilde{W} \). Then \( v \cdot \Psi_{M,\emptyset}^+ \) is equal to \( \Phi_x \) for some \( x \in \tilde{W} \).
Proof. There exists \( w \in W \) such that \( w(\Psi^+) = \Phi^+ \). So \( w(\Psi_{M,0}^+) = \Phi_{w(M),0}^+ \). Hence
\[
\Psi_{M,0}^+ = w^{-1}(\Phi_{w(M),0}^+).
\]
Now by Corollary 5.13(b) of [2] one can find \( z \in \tilde{W} \) such that \( \Psi_{M,0}^+ = w^{-1}(\Phi_{w(M),0}^+) = z \cdot \Phi_{w(M),0}^+ \). Then the assertion follows from Theorem [3.10] and Lemma [2.7(d)]. \( \square \)

As a consequence of the bijection discussed above, we provide a characterization of affine Weyl group. Let \( B \) be a subset of \( \Phi \) then define \( I_B = \{ \alpha \in \Phi | \hat{\alpha} \cap B = \emptyset \} \). Define \( A_B = \{ \alpha \in \Phi | \hat{\alpha} \cap B \neq \emptyset \} \).

**Lemma 3.12.** Let \( \Gamma \subset \Phi \) be \( Z \)-closed. If \( \Gamma \cap -\Gamma = \emptyset \) then \( \Gamma \) is contained in a positive system of \( \Phi \).

**Proof.** This is [2] Chapter IV, §1, Proposition 22. \( \square \)

**Lemma 3.13.** Let \( \Gamma \) be a closed subset of \( \Phi^+ \). Then \( A_{\Gamma} \) is \( Z \)-closed in \( \Phi \).

**Proof.** Let \( \alpha, \beta \in A_{\Gamma} \) and \( \alpha + \beta \) is a root. This implies \( \alpha + m\delta \) and \( \beta + n\delta \) are in \( \Gamma \) for some \( m, n \in \mathbb{Z}_{\geq 0} \). Then \( m + n \in \mathbb{Z}_{\geq 0} \). If \( \alpha + \beta \in \Phi^- \) then at least one of \( \alpha \) and \( \beta \) is in \( \Phi^- \), in which case \( m + n \in \mathbb{Z}_{>0} \). So \( \alpha + \beta + (m+n)\delta \in \Phi^+ \), and therefore in \( \Gamma \). \( \square \)

**Theorem 3.14.** Let \((W, S)\) be a finite rank, irreducible, infinite Coxeter system. Then \( W \) is affine if and only if \( \tilde{W} \) admits finitely many maximal elements.

**Proof.** The existence of such maximal elements is guaranteed by Proposition [2.12]. For only if, we prove that the maximal element in \( \tilde{W} \) where \( W \) is a affine Weyl group corresponds to \( \Psi^+ \) where \( \Psi^+ \) is a positive system of \( \Phi \) (and therefore there are only finitely many of them). First from Corollary 5.11 every \( \Psi^+ \) is indeed the inversion set of an infinitely long word. Exactly same argument as in the proof of Lemma 3.9 shows any biclosed set properly containing \( \Psi^+ \) is not the inversion set of an element in \( \tilde{W} \) (such biclosed sets must contain some \( \alpha + t\delta \) and \( -\alpha + s\delta \) for some \( \alpha \in \Phi^+ \) and \( t, s \in \mathbb{Z}_{\geq 0} \)). Now it suffices to show that any other element \( u \) in \( \tilde{W} \) is not maximal. By Lemma 3.13 \( A_{\Phi_u} \) is \( Z \)-closed. We claim that \( A_{\Phi_u} \cap -A_{\Phi_u} = \emptyset \). To see the claim suppose \( \alpha \in A_{\Phi_u} \cap -A_{\Phi_u} \). So some \( \alpha + t\delta \) and \( -\alpha + s\delta \) are both in \( \Phi_u \) for \( t, s \in \mathbb{Z}_{>0} \). Then the reasoning of the proof of Lemma 3.9 shows that this is not possible. Hence by Lemma 3.12 \( A_{\Phi_u} \subset \Psi^+ \) where \( \Psi^+ \) is a positive system in \( \Phi \) and \( \Phi_u \subset \Psi^+ \).

Conversely, by [11], for \( W \) finite rank, irreducible, infinite and non-affine, there exists a universal reflection subgroup of arbitrarily large rank. Suppose \( \tilde{W} \) has \( k \) maximal elements. Choose a rank \( k+1 \) universal reflection subgroup generated by \( \{t_1, t_2, \cdots, t_{k+1}\} \) (as canonical generators). These reflections correspond to a set of positive roots \( \{\alpha_1, \alpha_2, \cdots, \alpha_{k+1}\} \). There must be a maximal elements \( w \) such that \( \Phi_w \) contains \( \alpha_i \) and \( \alpha_j, i \neq j \) by pigeonhole. But \( \mathbb{R}_{>0}\{\alpha_i, \alpha_j\} \cap \Phi \) is infinite and therefore they cannot be contained in any finite biclosed set. This is a contradiction. \( \square \)

4. **A Lattice of Biclosed Sets in Rank 3 Affine Root System**

Let \( W \) be a Weyl group with the irreducible crystallographic root system \( \Phi \). \( \Phi^+ \) is a positive system with \( \Pi \) the corresponding simple system. \( \tilde{W} \) is the corresponding (irreducible) affine Weyl group. Then \( \tilde{\Phi} := \tilde{\Phi} \cup -\tilde{\Phi} \) is the root system of \( \tilde{W} \) and
$\tilde{\Phi}^+ = \tilde{\Phi}$. For $\alpha \in \Phi^+$, define $\alpha_0 = \alpha$. For $\alpha \in \Phi^-$, define $\alpha_0 = \alpha + \delta$. The following lemma in fact reveals the so called dominance order in affine root system.

**Lemma 4.1.** Let $x \in \tilde{W}$. If $\alpha \in \Phi$ and $\alpha_0 + k\delta \in \Phi_x$ with $k \geq 0$, then $\alpha_0, \alpha_0 + \delta, \cdots, \alpha_0 + k\delta$ are all contained in $\Phi_x$.

*Proof.* Let $\alpha_0 \in \Phi$ and $\alpha_0 + k\delta \in \Phi_x$. Suppose $x^{-1}(\alpha_0) = \beta + m\delta$. One has $x^{-1}(\alpha_0 + k\delta) = \beta + (k + m)\delta \in \Phi^-$. If $\beta \in \Phi^+$ then $k + m \leq -1$ i.e. $m \leq -k - 1$. Then for $0 \leq p \leq k$, $x^{-1}(\alpha_0 + p\delta) = \beta + (m + p)\delta$. $m + p \leq m + k \leq -1$. So $x^{-1}(\alpha_0 + p\delta) \in \Phi^-$. If $\beta \in \Phi^-$ then $k + m \leq 0$ i.e. $m \leq -k$. Then for $0 \leq p \leq k$, $x^{-1}(\alpha_0 + p\delta) = \beta + (m + p)\delta$. $m + p \leq m + k \leq 0$. So $x^{-1}(\alpha_0 + p\delta) \in \Phi^-$. □

The following theorem answers a question in [7] in the case of (irreducible) affine Weyl group (and easily generalized to affine Coxeter groups). See remark after Theorem 1.5 of [7].

**Theorem 4.2.** Let $X \subset \tilde{W}$. If $\bigcup_{x \in X} \Phi_x$ is finite, then it is biclosed and the join of $X$ exists.

*Proof.* We note that it could not happen that $\alpha + k\delta$ and $-\alpha + m\delta$ are both in $\bigcup_{x \in X} \Phi_x$ since that will cause the set to be infinite. Therefore by Lemma 3.13 all $\alpha \in \Phi$ with $\alpha + p\delta \in \bigcup_{x \in X} \Phi_x$ for some $p$ form a $\mathbb{Z}$-closed set and this set has empty intersection with its negative. So by Lemma 3.12 they belong to some positive system of $\Phi$, called it $\Xi^+$. So we have $\bigcup_{x \in X} \Phi_x \subset \tilde{\Xi}^+$. And in fact by the finiteness of $\bigcup_{x \in X} \Phi_x$, $\bigcup_{x \in X} \Phi_x \subset \Phi_u$ where $u \in \tilde{W}$ where $u$ is a prefix of the infinite reduced word corresponding to $\tilde{\Xi}^+$. So $X$ has a bound in $\tilde{W}$ and the assertion follows from Theorem 1.5(a) of [7]. □

**Lemma 4.3.** Let $\Gamma$ be a closed subset of $\tilde{\Phi}^+$ and $\alpha \in \Phi$. If $\alpha$ is in $I_\Gamma$ and $\alpha + N\delta \in \Gamma$ then $\alpha + n\delta$ is in $\Gamma$ for all $n \geq N$.

*Proof.* There must be some $M > N$ such that $\alpha + M\delta \in \Gamma$ since $\alpha \in I_{\Gamma}$. Then

$$\frac{1}{M - N} (\alpha + M\delta) + \frac{M - N - 1}{M - N} (\alpha + N\delta) = \alpha + (N + 1)\delta.$$

□

**Lemma 4.4.** Let $x, y \in \tilde{W}$. Then it cannot happen that $\Phi'_x \subset \Phi'_y$.

*Proof.* By Lemma 3.3 Corollary 3.11 and Lemma 2.7(d) the biclosed sets $w \cdot \Psi^+_{0, \alpha, M}$ where $w \in \tilde{W}$ are precise those $\Phi'_x, x \in \tilde{W}$. Suppose $M \neq \emptyset$. Then combining Lemma 2.8(f) and Lemma 1.9(f) of [5], one can see that all these sets contain some infinite $\delta$ chain through some $\alpha$ and $-\alpha$ (where $\alpha \in \Phi$). This implies that these sets cannot be possibly contained in a biclosed set coming from $\Phi_x, x \in \tilde{W}$ by the same reasoning as in the proof of Lemma 3.3. Now suppose $M = \emptyset$. Then $w \cdot \Psi^+_{0, \alpha} = \tilde{\Xi}^+$ for some positive system $\Xi^+$ in $\Phi$. Clearly if there is a biclosed set properly containing $\tilde{\Xi}^+$, again it must have infinite $\delta$-chains through some $\alpha, -\alpha \in \Phi$, which implies it is not of the form $\Phi_x, x \in \tilde{W}$. □

**Remark.** One can easily see that $\Phi'_x = \Phi_y$ if and only if $x, y$ are both maximal in $\tilde{W}$. 
From now on, let \( \widetilde{W} \) be of type \( \tilde{A}_2 \) or \( \tilde{B}_2 = \tilde{C}_2 \) or \( \tilde{G}_2 \).

**Lemma 4.5.** Suppose \( x, y \in \widetilde{W} \) and \( x \wedge y = e \). Let \( \Gamma = \Phi_e \cup \Phi_y \) and assume that \( I_\Gamma = A_\Gamma = \Psi^+_\theta\{\alpha\} \) for some positive system \( \Psi^+ \) in \( \Phi \) and a simple root \( \alpha \) of \( \Psi^+ \).

Then \( \Gamma = \Psi^+_{\theta\{\alpha\}} \).

**Proof.** The proof will be a case-by-case analysis. But before that we made a few case-free observations.

(1) \( \tilde{\alpha} \) and \( \tilde{\beta} \) are both contained in \( \Gamma \).

This is because some \( \alpha + s\delta \) and \( -\alpha + t\delta \) must be in \( \Phi_e \cup \Phi_y \). Otherwise roots in \( \tilde{\alpha} \) or \( \tilde{\beta} \) cannot be generated. Therefore \( \alpha_0 \) and \( -\alpha_0 \) are in \( \Phi_e \cup \Phi_y \) by Lemma 4.1. And by Lemma 4.3, we see the assertion.

(2) There is at least one simple root \( \rho_0 \in \Phi^+ \) such that \( \rho \neq \pm \alpha \) and \( \tilde{\rho}_0 \) is contained in \( \Gamma \).

Because \( \alpha_0 \) and \( -\alpha_0 \) cannot both be simple roots in \( \Phi^+ \) and we assumed that \( x \) and \( y \) have trivial meet, there must be \( \rho_0 \in \Phi_x \cup \Phi_y, \rho_0 \neq \alpha_0, -\alpha_0 \) and \( \rho_0 \) is simple in \( \Phi^+ \). This will cause \( \tilde{\rho}_0 \) to be contained in \( \Gamma \) by Lemma 4.3.

(3) If \( \alpha_0 \) and \( -\alpha_0 \) are both not simple in \( \Phi^+ \), there are two simple roots \( \rho_0, \gamma_0 \in \Phi^+ \) such that \( \tilde{\rho} \) and \( \tilde{\gamma} \) are both contained in \( \Gamma \).

Again because \( x \) and \( y \) have trivial meet, we must have at least two simple roots in \( \Phi_x \cup \Phi_y \). And then Lemma 4.3 forces the assertion.

Now we carry out the case study.

**A\(_2\):**

The two simple roots of \( \Phi^+ \) are denoted \( \alpha, \beta \). By making use of the symmetry of \( A_2 \), it is sufficient to treat the case

\[ I_\Gamma = A_\Gamma = \{ \alpha, \beta, \alpha + \beta, -\alpha \}. \]

By the previous discussion \( \tilde{\alpha}, \tilde{\beta} \) have to be contained in \( \Gamma \). Then \( \tilde{\alpha} + \tilde{\beta} \) can clearly be generated and is in \( \Gamma \).

**B\(_2\):**

The short simple root is denoted \( \alpha \) and the long simple root is denoted \( \beta \).

(1) \( I_\Gamma = A_\Gamma = \{ \alpha, \beta, \alpha + \beta, 2\alpha + \beta, -\alpha \}. \)

By the previous discussion \( \tilde{\alpha}, \tilde{\beta} \) have to be contained in \( \Gamma \). Then \( \tilde{\alpha} + \tilde{\beta} \) and \( 2\alpha + \beta \) can clearly be generated and are in \( \Gamma \).

By making use of the symmetry of \( B_2 \), the case that \( I_\Gamma = A_\Gamma = \{ -2\alpha - \beta, -\alpha, -\beta, -\alpha - \beta, \alpha \} \) can be proved in the same way.

(2) \( I_\Gamma = A_\Gamma = \{ -2\alpha - \beta, \beta, \alpha + \beta, 2\alpha + \beta, -\alpha \}. \)

By the previous discussion \( 2\alpha + \beta, -2\alpha - \beta, \tilde{\beta} \) have to be contained in \( \Gamma \). Then note that \( \frac{1}{2}(2\alpha + \beta) + \frac{1}{2}\beta = \alpha + \beta, -\alpha + \delta = \frac{1}{2}(-2\alpha - \beta + \delta) + \frac{1}{2}(\beta + \delta) \). Therefore \( \tilde{\alpha} + \tilde{\beta} \) and \( \tilde{-\alpha} \) are in \( \Gamma \).

By making use of the symmetry of \( B_2 \), the case that \( I_\Gamma = A_\Gamma = \{ -2\alpha - \beta, \beta, -\alpha, -\beta, -\alpha - \beta \} \) can be proved in the same way.

(3) \( I_\Gamma = A_\Gamma = \{ -2\alpha - \beta, \beta, \alpha + \beta, -\alpha - \beta, -\alpha \}. \)

By the previous discussion \( \alpha + \beta, -\alpha - \beta, \beta, -2\alpha - \beta \) have to be contained in \( \Gamma \). Then note that \( -\alpha + \delta = (-\alpha - \beta + \delta) + \beta \). So \( \tilde{-\alpha} \) is in \( \Gamma \).

(4) \( I_\Gamma = A_\Gamma = \{ -2\alpha - \beta, 2\alpha + \beta, -\beta, -\alpha - \beta, \alpha \}. \)
By the previous discussion $2\alpha + \beta, \alpha, -2\alpha - \beta$ have to be contained in $\Gamma$. Then note that $-\alpha - \beta + \delta = (-2\alpha - \beta + \delta) + \alpha$ and $-\beta + \delta = (-\alpha - \beta + \delta) + \alpha$. So $-\alpha - \beta$ and $-\beta$ are in $\Gamma$.

By making use of the symmetry of $B_2$, the case that $I_\Gamma = A_\Gamma = \{\alpha + \beta, 2\alpha + \beta, -\alpha - \beta, \alpha\}$ can be proved in the same way.

(5) $I_\Gamma = A_\Gamma = \{\alpha + \beta, 2\alpha + \beta, -\beta, -\alpha - \beta, \alpha\}$.

By the previous discussion this situation cannot happen as there will at most be one simple root in $\Gamma$. 

$G_2$:

The long simple root is denoted $\beta$ and the short simple root is denoted $\alpha$.

(1) $I_\Gamma = A_\Gamma = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta, -\alpha\}$.

By the previous discussion $\alpha, -\alpha, \beta$ have to be contained in $\Gamma$. Then $\alpha + \beta$, $2\alpha + \beta$, $3\alpha + \beta$ and $3\alpha + 2\beta$ can clearly be generated and are all in $\Gamma$.

(2) $I_\Gamma = A_\Gamma = \{-3\alpha - \beta, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta, -\alpha\}$.

By the previous discussion this situation cannot happen as there will at most be one simple root in $\Gamma$.

(3) $I_\Gamma = A_\Gamma = \{-3\alpha - \beta, \beta, \alpha + \beta, 2\alpha + \beta, -2\alpha - \beta, 3\alpha + 2\beta, -\alpha\}$.

By the previous discussion this situation cannot happen as there will at most be one simple root in $\Gamma$.

(4) $I_\Gamma = A_\Gamma = \{-3\alpha - \beta, \beta, \alpha + \beta, -3\alpha - 2\beta, -2\alpha - \beta, 3\alpha + 2\beta, -\alpha\}$.

By the previous discussion $3\alpha + 2\beta, -3\alpha - 2\beta, \hat{\beta}$ have to be contained in $\Gamma$. Then note that $\alpha + \beta = \frac{1}{4}(3\alpha + 2\beta) + \frac{1}{4}\hat{\beta}$, $\alpha + 2\alpha + \beta = \frac{1}{4}(-3\alpha - 2\beta + \delta) + \frac{3}{4}(\beta + \delta)$, $-3\alpha - \beta + \delta = (-3\alpha - 2\beta + \delta) + \beta$ and $-2\alpha - \beta + \delta = (-3\alpha - 2\beta + \delta) + (\alpha + \beta)$. Hence $\alpha + \beta$, $\alpha$, $\alpha + \beta$, $-3\alpha - 2\beta$ and $-2\alpha - \beta$ are all in $\Gamma$.

(5) $I_\Gamma = A_\Gamma = \{-3\alpha - \beta, \beta, \alpha + \beta, -3\alpha - 2\beta, -2\alpha - \beta, -\alpha - \beta, -\alpha\}$.

By the previous discussion $\alpha + \beta$, $-3\alpha - 2\beta$ and $-\alpha - \beta$ have to be contained in $\Gamma$. Then note that $\alpha + \delta = \frac{1}{3}(-3\alpha - 2\beta + \delta) + \frac{1}{3}(\beta + \delta)$, $-3\alpha - \beta + \delta = (-3\alpha - 2\beta + \delta) + \beta$, $-2\alpha - \beta + \delta = \frac{1}{3}(-3\alpha - 2\beta + \delta) + \frac{1}{3}(\beta + \delta)$ and $\frac{1}{3}(-3\alpha - 2\beta + \delta) + \frac{1}{3}(\beta + \delta) = -\alpha - \beta + \delta$. Hence $\alpha$, $-3\alpha - \beta$, $-2\alpha - \beta$ and $-\alpha - \beta$ are all in $\Gamma$.

(6) $I_\Gamma = A_\Gamma = \{-3\alpha - \beta, \beta, -\beta, -3\alpha - 2\beta, -2\alpha - \beta, -\alpha - \beta, -\alpha\}$.

By the previous discussion $\beta$, $-3\alpha - 2\beta$ and $-2\beta$ have to be contained in $\Gamma$. Then note that $\alpha - \beta + \delta = \frac{1}{3}(-3\alpha - 2\beta + \delta) + \frac{1}{3}(\alpha + \delta)$, $-3\alpha - \beta + \delta = (-3\alpha - 2\beta + \delta) + \beta$, $-2\alpha - \beta + \delta = \frac{1}{3}(-3\alpha - 2\beta + \delta) + \frac{1}{3}(\alpha + \delta)$ and $\frac{1}{3}(-3\alpha - 2\beta + \delta) + \frac{1}{3}(\alpha + \delta) = -\alpha - \beta + \delta$. Hence $\alpha$, $-\alpha - \beta$, $-2\alpha - \beta$ and $-\alpha - \beta$ are all in $\Gamma$.

(7) $I_\Gamma = A_\Gamma = \{-3\alpha - \beta, \alpha, -\beta, -3\alpha - 2\beta, -2\alpha - \beta, -\alpha - \beta, -\alpha\}$.

By the previous discussion $\alpha, -3\alpha - 2\beta$ and $-\beta$ have to be contained in $\Gamma$. Then note that $-\alpha - \beta + \delta = \frac{1}{3}(-3\alpha - 2\beta + \delta) + \frac{1}{3}(\beta + \delta)$, $-3\alpha - \beta + \delta = (-3\alpha - 2\beta + \delta) + \beta$, $-2\alpha - \beta + \delta = \frac{1}{3}(-3\alpha - 2\beta + \delta) + \frac{1}{3}(\alpha + \delta)$ and $\frac{1}{3}(-3\alpha - 2\beta + \delta) + \frac{1}{3}(\alpha + \delta) = -\alpha - \beta + \delta$. Hence $-\alpha$, $-\alpha - \beta$, $-\beta$ and $-\alpha - \beta$ are all in $\Gamma$. Finally we note that in this case the two elements are of the form $s_\alpha \cdots$ and $s_\beta \cdots$. Because $\Gamma$ is infinite they could not be $s_\alpha$ and $s_\beta \cdots$ because the two elements have trivial meet we cannot have $s_\alpha s_\beta \cdots$ or $s_\alpha s_\beta \cdots$. Finally if we have $s_\alpha s_\beta \cdots$ then we will have $3\alpha + \beta \in A_\Gamma$, which is a contradiction. So we must have one element being equal to $s_{3\alpha - 3\beta} s_\beta \cdots$. Therefore we have $-3\alpha - \beta + \delta \in \Phi_2 \cup \Phi_3$. And so we have $-3\alpha - \beta \in \Gamma$.

(8) $I_\Gamma = A_\Gamma = \{-3\alpha - \beta, \alpha, -\beta, -3\alpha - 2\beta, -2\alpha - \beta, -\alpha - \beta, 3\alpha + \beta\}$. 
By the previous discussion, \(-3\alpha - \hat{\beta}, 3\alpha + \beta, \hat{\alpha}, -3\alpha - 2\beta\) have to be in \(\Gamma\). Then note that 
\[-2\alpha - \beta + \delta = (-3\alpha - \hat{\beta} + \delta) + \alpha, \quad -\alpha - \beta + \delta = (-2\alpha - \beta + \delta) + \alpha, \quad \text{and} \quad -\beta + \delta = (-\alpha + \beta + \delta) + \alpha. \]
Hence \(-2\alpha - \beta, -\alpha - \beta\) and \(-\beta\) are all in \(\Gamma\).

(9) \(I_{\Gamma} = A_{\Gamma} = \{2\alpha + \beta, \alpha - \beta, -3\alpha - 2\beta, -2\alpha - \beta, -\alpha - \beta, 3\alpha + \beta\}\).

By the previous discussion, \(-2\alpha - \beta, 2\alpha + \beta, \hat{\alpha}, -3\alpha - 2\beta\) have to be in \(\Gamma\). Then note that 
\[3\alpha + \beta = (2\alpha + \beta) + \beta, \quad -\alpha - \beta + \delta = (-2\alpha - \beta + \delta) + \alpha, \quad \text{and} \quad -\beta + \delta = (-\alpha - \beta + \delta) + \alpha. \]
Hence \(3\alpha + \beta, -\alpha - \beta\) and \(-\beta\) are all in \(\Gamma\).

(10) \(I_{\Gamma} = A_{\Gamma} = \{2\alpha + \beta, \alpha - \beta, -3\alpha - 2\beta, 3\alpha + 2\beta, -\alpha - \beta, 3\alpha + \beta\}\).

By the previous discussion, \(-3\alpha - 2\beta, 3\alpha + 2\beta\) and \(\hat{\alpha}\) have to be in \(\Gamma\). Then note that 
\[2\alpha + \beta = \frac{1}{2}(3\alpha + 2\beta) + \frac{1}{2}\alpha, \quad 3\alpha + \beta = (2\alpha + \beta) + \alpha, \quad -\alpha - \beta + \delta = \frac{1}{3}(-3\alpha - 2\beta + \delta) + \frac{1}{2}(\alpha + \delta) \quad \text{and} \quad -\beta + \delta = (-\alpha - \beta + \delta) + \alpha. \]
Therefore \(2\alpha + \beta, 3\alpha + \beta, -\alpha - \beta\) and \(-\beta\) are all in \(\Gamma\).

(11) \(I_{\Gamma} = A_{\Gamma} = \{2\alpha + \beta, \alpha - \beta, -3\alpha - 2\beta, -\alpha - \beta, 3\alpha + \beta\}\).

By the previous discussion \(\hat{\alpha}, -\hat{\beta}, \hat{\beta}\) have to be contained in \(\Gamma\). Then \(\alpha + \hat{\beta}, 2\alpha + \beta, 3\alpha + \beta\) and \(3\alpha + 2\beta\) can clearly be generated and are all in \(\Gamma\). \(\square\)

**Lemma 4.6.** Suppose \(x, y \in \hat{W}\) and \(x \wedge y = e\). Let \(\Gamma = \hat{\Phi}_x \cup \hat{\Phi}_y\) and assume 
\(I_{\Gamma} = A_{\Gamma} = \Phi\). Then \(\Gamma \supset \Psi_{\hat{\Phi}_x,\{\alpha\}}\) for some positive system \(\Psi^+\) and a simple root \(\alpha\) of \(\Psi^+\).

**Proof.** Again we made a few general observations at the beginning and they are then followed by the case-by-case analysis. First since \(x\) and \(y\) have trivial meet, there are two simple roots \(\rho_0, \gamma_0 \in \hat{\Phi}_x \cup \hat{\Phi}_y\). In addition, there is no closed half-plane in \(\mathbb{R}^2\) with \(0\) in its boundary containing all \(\eta \in \Phi\) such that \(\gamma_0 \in \hat{\Phi}_x \cup \hat{\Phi}_y\) (therefore there exist \(\tilde{\rho}_0, \tilde{\gamma}_0\) and \(\tilde{\eta}_0\) all completely in \(\Gamma\) with \(\rho_0, \gamma_0\) simple but there is no closed half-plane in \(\mathbb{R}^2\) with \(0\) in its boundary containing all \(\rho, \gamma\) and \(\eta\) by Lemma 4.3).

Now we carry out a case-by-case analysis.

\(A_2:\)

The two simple roots are denoted \(\alpha\) and \(\beta\). By symmetry of \(\overline{A}_2\) it is enough to prove this in the case that \(\hat{\alpha}\) and \(\hat{\beta}\) are in \(\Gamma\). If any of \(\overline{\hat{\alpha}}\) and \(\overline{\hat{\beta}}\) is in \(\Gamma\) then by the same reasoning as in the proof of Lemma 4.5 we are done. If \(-\alpha - \beta \in \Gamma\), then whole \(\hat{\Phi}^+\) is in \(\Gamma\). So we see the assertion holds in this case.

\(B_2:\)

Suppose \(\beta\) is the long simple root and \(\alpha\) is the short simple root.

Case (I). Suppose \(\hat{\beta}\) and \(\hat{\alpha}\) are both contained in \(\Gamma\). If any of \(\overline{\hat{\beta}}\) and \(\overline{\hat{\alpha}}\) is in \(\Gamma\) we are done by the same reasoning as in the proof of Lemma 4.5. Now suppose \(-\beta - 2\alpha + \delta \in \Gamma\), then \(\hat{\Phi}^+ \subset \Gamma\). Finally suppose \(-\beta - \alpha + \delta \in \Gamma\), then \(-\beta + \delta = -\beta - \alpha + \delta + \alpha \in \Gamma\). So we see the assertion holds in this case.

Case (II). Suppose \(-\beta - 2\alpha\) and \(\hat{\beta}\) are both contained in \(\Gamma\). Then if any of \(\overline{\hat{\beta}}\) and \(\beta + 2\alpha\) is in \(\Gamma\) we are done by the same reasoning as in the proof of Lemma 4.5. If \(\hat{\alpha} \in \Gamma\) then the whole \(\hat{\Phi}^+\) is in \(\Gamma\). So we see the assertion holds in this case.

Case (III). Suppose \(-\beta - 2\alpha\) and \(\alpha\) are both contained in \(\Gamma\). By the symmetry of \(\overline{\hat{B}_2}\) this case can be proved by using the same reasoning as in case (I).
The long simple root is denoted $\beta$ and the short simple root is denoted $\alpha$.

Case (I). Suppose $\alpha$ and $\beta$ are both in $\Gamma$. If any of $-\alpha$ and $-\beta$ is in $\Gamma$ then we are done by the same reasoning as in the proof of Lemma 3.8. If $-3\alpha - 2\beta \in \Gamma$, then the whole $\Phi^+$ is in $\Gamma$. Now suppose $-3\alpha - 2\beta \subset \Gamma$. Then $-\beta + \delta = (3\alpha - 3\beta + 3\alpha) \in \Gamma$. Now suppose that $2\alpha - \beta \subset \Gamma$. Then $-\beta + \delta = (-2\alpha - \beta + \delta) + 2\alpha \in \Gamma$. Now suppose that $\alpha - \beta \subset \Gamma$. Then $-\beta + \delta = (-\alpha - \beta + \delta) + \alpha \in \Gamma$. So we see the assertion holds in this case.

Case (II). Suppose $-3\alpha - 2\beta$ and $\beta$ are both in $\Gamma$. If any of $3\alpha + 2\beta$ and $-\beta$ is in $\Gamma$ then we are done by some the reasoning as in the proof of Lemma 4.5. If $\tilde{\alpha} \subset \Gamma$, then the whole $\Phi^+$ is in $\Gamma$. Now suppose $3\alpha + \beta \subset \Gamma$. Then $-\beta + \delta = (-3\alpha - 2\beta + \delta) + 3\alpha + \beta$. Now suppose that $2\alpha + \beta \subset \Gamma$. Then $3\alpha + \beta = \frac{1}{2}(2\alpha + \beta) + \frac{1}{2}\beta$. So we see the assertion holds in this case.

Case (III). Suppose $-3\alpha - 2\beta$ and $\tilde{\alpha}$ are both in $\Gamma$. If any of $3\alpha + 2\beta$ and $-\tilde{\alpha}$ is in $\Gamma$ then we are done by the same reasoning as in the proof of Lemma 3.9. If $\hat{\beta}, 3\alpha + 2\beta$ and $\alpha + \beta$ are all not completely in $\Gamma$, then $2\alpha + \beta$ and $-3\alpha - \beta + \delta$ must both be in $\Phi(x, y)$. This causes $-\alpha + \delta = (-3\alpha - \beta + \delta) + (2\alpha + \beta)$ in $\Gamma$. So $-\alpha \subset \Gamma$. Therefore we see the assertion holds in this case.

Lemma 4.7. Let $u, v \in \hat{W}$ and $u, v$ don’t have an upper bound in $W$ and $u \wedge v = e$. Suppose that $B$ is a biclosed set in $\Phi^+$ such that $B \cap \Phi = \emptyset$ and $B \cap \Phi = \emptyset$. Then $B \subset \Phi^+$ for some positive system $\Psi^+$ in $\Phi$ which is independent of $B$.

Proof. By Lemma 3.13 $A := \Phi_u \cup \Phi_v$ is $Z-$closed. If $A \cap -A = \emptyset$, by Lemma 3.12 $A \subset \Omega^+$ where $\Omega^+$ is a positive system in $\Phi$. This causes $\Phi_u \cup \Phi_v \subset \Omega^+$. So $u$ and $v$ are both less than or equal to an infinite reduced word by Theorem 3.10 and consequently less than or equal to a word in $\hat{W}$, contradicting the assumption. This shows that there exists $\alpha \in \Phi$ such that $\alpha$ and $-\alpha$ are both in $\hat{A}$. By the closedness of $\Phi_u \cup \Phi_v$ and Lemma 3.3 this implies an infinite $\delta$ chain through $\alpha$ and an infinite $\delta$ chain through $-\alpha$ are both in $\Phi_u \cup \Phi_v$. That is $\alpha, -\alpha \in \Phi_u \cup \Phi_v$.

Now we claim that there must be at least one $\beta$ and $-\beta$ and $\beta$ and $-\beta$ in $A$. If not, $A = \{\alpha, -\alpha\}$. Suppose $\hat{\alpha} \cap \Phi_v \neq \emptyset$. But one of $\hat{\alpha}$ and $-\hat{\alpha}$ has to intersect $\Phi_u$ trivially due to the same reasoning as in Lemma 3.8. So $-\hat{\alpha} \cap \Phi_u = \emptyset$. Then we must have $-\hat{\alpha} \cap \Phi_v \neq \emptyset$ and $\hat{\alpha} \cap \Phi_v = \emptyset$. But $\alpha_0$ and $(-\alpha)_0$ cannot both be simple. This is a contradiction. (Note that the case that $\hat{\alpha} \cap \Phi_u$ and $-\hat{\alpha} \cap \Phi_u$ are both empty is clearly impossible.) So we established the claim.

Then $\alpha, -\alpha, \beta$ determines a closed half-plane $H$ with $0$ in its boundary in $V = \mathbb{R}^2$ and we shall show that $\Phi \cap H \subset \Phi_\alpha \cup \Phi_{-\alpha}$. There exists at least one root $\gamma$ other than $\alpha, -\alpha, \beta$ in this closed half-plane (clear by inspecting the graphs of rank 2 irreducible crystallographic root systems). Without loss of generality we assume $\gamma = ma + n\beta$ where $m, n \in \mathbb{Q}_{>0}$. And for sufficiently large $k$, $\alpha + k\delta \in \Phi_u \cup \Phi_v$. Then there are infinitely many $k$ such that $\alpha + k\delta \in \Phi_u \cup \Phi_v$ and $km \in \mathbb{Z}_{>0}$. This shows $\gamma \in \Phi_\alpha \cup \Phi_{-\alpha}$. Then $\beta$ is a positive rational linear combination of $-\alpha$ and $\gamma$. By the same reasoning as above we have $\beta \in \Phi_\alpha \cup \Phi_{-\alpha}$. Then apply the same reasoning to other roots in this closed half plane we obtain the desired assertion.
Now we distinguish two cases:

Case (I): \( A = \Phi \cap H = (\Phi_u \cup \Phi_v) \). By Lemma 4.5, \( \Phi_u \cup \Phi_v = \Phi_{\emptyset,\{\alpha\}} \) for some positive system \( \Psi^+ \) in \( \Phi \) and some simple root \( \alpha \) of \( \Psi^+ \). Now \( B \) is biclosed and \( B \cap \Phi_u = \emptyset \) and \( B \cap \Phi_v = \emptyset \). Then \( B \cap (\Phi_u \cup \Phi_v) = \emptyset \). So this gives \( B \subseteq \Phi_{\emptyset,\{\alpha\}} = (-\Psi^+)(-\alpha,\emptyset) \).

Case (II): If \( A \nsubseteq \Phi \cap H \), then by the same reasoning as the one in the paragraph preceding case (I) one sees that \( A = (\Phi_u \cup \Phi_v) = \Phi \). Then Lemma 4.4 says \( \Phi_u \cup \Phi_v \supset \Phi^+_{\emptyset,\{\alpha\}} \) for some positive system \( \Psi^+ \) in \( \Phi \) and some simple root \( \alpha \) of \( \Psi^+ \). So this gives \( B \subseteq \Phi^+_{\emptyset,\{\alpha\}} = (-\Psi^+)(-\alpha,\emptyset) \).

\[ \square \]

**Corollary 4.8.** Let \( u, v \in \overline{W} \) and do not admit a join in \( \overline{W} \). The set \( \{x \in \overline{W} | x \perp u, x \perp v\} \) admits a join in \( \overline{W} \).

**Proof.** First we assume \( u, v \in \overline{W} \). If \( u, v \) have trivial meet then the assertion follows from Lemma 4.7 and Corollary 3.11 and the fact that \( \overline{W} \) is a complete meet semilattice. Now suppose \( u \land v = w \neq e \). Then by the previous discussion the set \( \{x | x \perp w^{-1} u, x \perp w^{-1} v\} \) admits a join \( y \) in \( \overline{W} \). So \( w^{-1} \leq y \). Then we claim that \( wy \) is the join of \( \{x | x \perp u, x \perp v\} \). We show that for \( p \in \{x | x \perp u, x \perp v\} \), \( w^{-1}p \in \{x | x \perp w^{-1} u, x \perp w^{-1} v\} \) and \( l(w^{-1}p) = l(w^{-1}) + l(p) \):

\[
\begin{align*}
\text{l}(p^{-1}w) + l(w^{-1}u) & \geq l(p^{-1}w^{-1}u) = l(p^{-1}u) + l(u) \\
& \geq l(p)^{-1} + l(w^{-1}u) = l(p)^{-1} + l(w) + l(w^{-1}u) \geq l(p^{-1}w) + l(w^{-1}u)
\end{align*}
\]

(and same for \( v \) in place of \( u \)).

So \( w^{-1}p \leq y \). Since \( w^{-1} \leq w^{-1}p \) and \( w^{-1} \leq y \), we can multiply \( w \) on both sides of the inequality \( w^{-1}p \leq y \) and have \( p \leq wy \) (See Proposition 3.12(vi) of [1]). By Theorem 2.11, \( y \perp w^{-1}u \) and \( y \perp w^{-1}v \). Therefore \( wy \perp u \) and \( wy \perp v \):

\[
\begin{align*}
l(y^{-1}w^{-1}) + l(u) & \geq l(y^{-1}w^{-1}u) = l(y^{-1}) + l(w^{-1}u) \\
& = l(y^{-1}) - l(w) + l(u) = l(y^{-1}w^{-1}) + l(u)
\end{align*}
\]

(and same for \( v \) in place of \( u \)).

This shows \( wy \) indeed is the join. Now if at least one of \( u \) and \( v \) is an infinite word, then some prefix \( x \) of \( u \) and some prefix \( y \) of \( v \) do not have bound in \( \overline{W} \). And one notes that \( \{p | p \perp u, p \perp v\} \subseteq \{p | p \perp x, p \perp y\} \) and thus the former is bounded in \( \overline{W} \). Then the existence of join follows from the fact \( \overline{W} \) is a complete meet semilattice. \( \square \)

Now we have made all the preparations for proving \( \mathcal{B}(\Phi^+) \) is a lattice.

**Proposition 4.9.** Let \( x, y \in \overline{W} \).

1. Suppose \( x \lor y \) exists in \( \overline{W} \). Then the join of \( \Phi_x \) and \( \Phi_y \) exists in \( \mathcal{B}(\Phi^+) \) and is given by \( \Phi_{x \lor y} \).

2. Suppose \( x \) and \( y \) have no upper bound in \( \overline{W} \). Let \( M = \overline{\{w | w \in \overline{W}, w \perp x, w \perp y\}} \). Then the join of \( M \) exists in \( \overline{W} \) (denoted \( z \)). Then the join of \( \Phi_x \) and \( \Phi_y \) is given by \( \Phi_z' \) in \( \mathcal{B}(\Phi^+) \).

3. The join of \( \Phi_x' \) and \( \Phi_y' \) exists in \( \mathcal{B}(\Phi^+) \) and is given by \( \Phi_{x \land y}' \).
(4) The join of $\Phi_x$ and $\Phi'_y$ exists in $\mathcal{B}(\tilde{\Phi}^+)$ and is given by $\Phi'_z$ where $z = \bigvee\{w|w \in W, w \perp x, w \leq y\}$.

**Proof.** One first notes that the biclosed sets in this (rank 3) case are either $\Phi_x$ or $\Phi'_z$ where $x \in W$ by Theorem 1.2, Corollary 3.11 and Lemma 2.7(d).

(1) if for some $w \in W$, $\Phi_x, \Phi_y \subset \Phi_w$, by Join Orthogonality Property (Theorem 2.11) $\Phi'_w \supset \Phi_x \vee y$.

(2) if for some $w \in W$, $\Phi_x, \Phi_y \subset \Phi'_w$, then this is equivalent to $w \perp x, w \perp y, w \leq z$. So $\Phi'_w \supset \Phi'_z$.

(3) by Lemma 4.4 no $\Phi_z$ can properly contain $\Phi_x'$ and $\Phi_y'$. If for some $w \in W$, $\Phi'_x, \Phi'_y \subset \Phi'_w$ then $x \geq w$ and $y \geq w$ and thus $x \wedge y \geq w$ and therefore $\Phi'_x \wedge y \subset \Phi'_w$.

(4) $e \in \{w|w \in W, w \perp x, w \leq y\}$ and therefore $\{w|w \in W, w \perp x, w \leq y\}$ is not empty. The existence of $z$ is guaranteed by the fact that $W$ is a complete meet semilattice and the set $\{w|w \in W, w \perp x, w \leq y\}$ is bounded by $y$. By Join Orthogonality Property (Theorem 2.11), $z \perp x$. Hence $\Phi_x \subset \Phi'_x$. And it is clear that $z \leq y$. Hence $\Phi_z \subset \Phi_y$ and thus $\Phi'_z \supset \Phi'_y$. If there is $\Phi_u'$ such that $\Phi'_u \supset \Phi'_x$ and $\Phi'_u \supset \Phi'_y$, then it follows $u \in \{w|w \in W, w \perp x, w \leq y\}$. So $u \leq z$ and $\Phi'_u \supset \Phi'_x$. If $\Phi_w \supset \Phi_x$ and $\Phi_w \supset \Phi'_y$, then by Lemma 4.4 $\Phi_w = \Phi_y'$ and thus $\Phi_y' \supset \Phi_x$. Therefore $x \perp y$. In this case $z = \bigvee\{w|w \in W, w \perp x, w \leq y\} = y$. $\Phi_w = \Phi_y' = \Phi'_z$. $\square$

**Theorem 4.10.** $\mathcal{B}(\tilde{\Phi}^+)$ is a complete ortholattice.

**Proof.** Proposition 4.3 guarantees that $\mathcal{B}(\tilde{\Phi}^+)$ is lattice. By Defining $B^{oc} = \tilde{\Phi}^+ \setminus B$ one easily sees that $\mathcal{B}(\tilde{\Phi}^+)$ is a ortholattice. Now let $A \subset \mathcal{B}(\tilde{\Phi}^+)$. Let $C$ be the set of biclosed sets bounded by all elements in $A$. Since the union of a chain of biclosed sets is a biclosed set, the condition of Zorn’s Lemma is satisfied. $C$ has at least one maximal element. If the maximal element is not unique, take two of them and their join again is bounded by all elements in $A$, which is a contradiction. This shows the maximal element is unique and $A$ admits a meet. Hence $\mathcal{B}(\tilde{\Phi}^+)$ is a complete meet semilattice with a unique maximal element($\tilde{\Phi}^+$). Therefore it is a complete lattice. $\square$

**Remark.** In [16] the author replaced the closure used in this paper with convex closure and proved the lattice property for biconvex sets in the positive system of rank three Coxeter groups by using the projective representation and geometric method.

**Remark.** Call a biclosed set $\Gamma$ in $\tilde{\Phi}$ a quasi-positive system if $\Gamma \cup -\Gamma = \tilde{\Phi}$. One might ask whether $\mathcal{B}(\Gamma)$ (all the biclosed sets in $\Gamma$ under inclusion) is a complete lattice. This is conjectured in [7]. We show in general this is not true as illustrated by the following example.

Consider the root system of $\tilde{A}_2$. $\alpha, \beta$ are two simple roots of a chosen positive system of $\Phi$, the root system of the corresponding Weyl group.

$$\Psi = \{-\alpha + l\delta, -\beta + m\delta, -\alpha - \beta + n\delta, l, m, n \in \mathbb{Z}\}$$ is a quasi-positive system.

The following four sets are all biclosed in $\Psi$.
\[ B_1 = \{-\alpha + a\delta, -\beta + b\delta, -\alpha - \beta + c\delta\, | \, a, c \in \mathbb{Z}, b \in \mathbb{Z}_{\leq 0}\}. \]
\[ B_2 = \{-\alpha + d\delta, -\beta + e\delta, -\alpha - \beta + f\delta\, | \, e, f \in \mathbb{Z}, d \in \mathbb{Z}_{\geq 0}\}. \]
\[ B_3 = \{-\alpha + g\delta\, | \, g \in \mathbb{Z}_{\geq 0}\}. \]
\[ B_4 = \{-\beta + h\delta\, | \, h \in \mathbb{Z}_{\leq 0}\}. \]

\( B_3 \) and \( B_4 \) are both contained in \( B_1 \cap B_2 \). If \( B_1 \) and \( B_2 \) admit a meet, say \( C \), we have \( B_3 \cup B_4 \subseteq C \subseteq B_1 \cap B_2 \). But the first and third term coincide, which is \( \{-\alpha + i\delta, -\beta + j\delta, -\alpha - \beta + k\delta\, | \, i, j, k \in \mathbb{Z}_{\geq 0}, j \in \mathbb{Z}_{\leq 0}, k \in \mathbb{Z}\} \). This forces \( C \) to be this set. However this set is not biclosed in \( \Psi \).

5. Compact Biclosed Set And Algebraic Lattice

For the first part of this section \( \tilde{W} \) is a general (irreducible) affine Weyl group without the restriction on the rank.

A compact biclosed set in \( \tilde{\Phi}^+ \) is meant to be a compact element in \( \mathcal{B}(\tilde{\Phi}^+) \) provided \( \mathcal{B}(\tilde{\Phi}^+) \) is a lattice. A finitely generated biclosed set is the one equal to the closure of finite many roots.

**Lemma 5.1.** If \( \mathcal{B}(\tilde{\Phi}^+) \) is a lattice, a biclosed set in \( \tilde{\Phi}^+ \) is compact if it is finitely generated.

**Proof.** According to the classification of the biclosed sets in an affine positive system, \( \mathcal{B}(\tilde{\Phi}^+) \) is an infinite countable set. Let \( B = \{\alpha_1, \ldots, \alpha_n\} \) be biclosed in \( \tilde{\Phi}^+ \). Suppose \( B \subseteq \bigvee_{i=1}^\infty B_i \). So \( \alpha_j \in \bigvee_{i=1}^\infty B_i, 1 \leq j \leq n \). This forces that \( \alpha_j \in \bigvee_{i=1}^t B_i, 1 \leq j \leq n \) for some \( t_j \). Take \( t = \max\{t_j | 1 \leq j \leq n\} \) and one sees that \( B \subseteq \bigvee_{i=1}^t B_i \). \( \square \)

Let \( \Psi^+ \) be a positive system in \( \Phi \) and \( \Pi \) be its simple system. \( M \subseteq \Pi \).

**Theorem 5.2.** If \( L = \{\alpha \in \Pi | (\alpha, M) = 0\} \), a biclosed set \( \Gamma \) with \( I_{\Gamma} = \Psi^+_{L, M} \) is finitely generated. If \( L \not\subseteq \{\alpha \in \Pi | (\alpha, M) = 0\} \), a biclosed set \( \Gamma \) with \( I_{\Gamma} = \Psi^+_{L, M} \) is not finitely generated.

**Proof.** A biclosed set \( \Gamma \) with \( I_{\Gamma} = \Psi^+_{L, M} \) consists of the following three disjoint subsets:

(1) For each \( \alpha_i \in \mathbb{R}_{>0} L \cap \Phi \), there may be a \( \delta \)-string through \( \alpha_i \) of finite length beginning from \( (\alpha_i)_0 \). These form a finite set, denoted \( \Gamma_L \).

(2) \( \Psi^+_{L, M, \emptyset} \).

(3) For each \( \gamma_i \in \mathbb{R} M \cap \Phi \), there is a \( \delta \)-string through \( \gamma_i \) of infinite length beginning from \( \gamma_i + n(\gamma_i)\delta \).

(This is a consequence of [3] Lemma 5.10.)

Now suppose \( L = \{\alpha \in \Pi | (\alpha, M) = 0\} \). We show that

\[ \Gamma_L = \{\alpha_0 | \alpha \in \Psi^+_{L, M, \emptyset}\} \cup \{\gamma_i + n(\gamma_i)\delta | \gamma_i \in \mathbb{R} M \cap \Phi\} \]

is a finite generating set of \( \Gamma \).

Since part (1) has been included in the set so we only need to worry about (2) and (3).

We first show part (3) can be generated.

Take \( \gamma, -\gamma \in \mathbb{R} M \cap \Phi \). By assumption \( \gamma + n(\gamma)\delta, -\gamma + n(-\gamma)\delta \) are in the generating set. Then

\[ \frac{n(\gamma) + n(-\gamma) + 1}{n(\gamma) + n(-\gamma)} (\gamma + n(\gamma)\delta) + \frac{1}{n(\gamma) + n(-\gamma)} (-\gamma + n(-\gamma)\delta) = \gamma + (n(\gamma) + 1)\delta. \]
Proceeding in this way one sees that all elements in part (3) can be generated.

Now we check that part (2) can be generated.

We look at $\alpha \in \Psi^+\backslash(\mathbb{R}_{\geq 0}(L \cup M))$. We can define $h$ function as in Section 3, i.e. $h(\alpha) := \sum_{\alpha_i \notin L} k_i$ if $\alpha = \sum_{\alpha_i \in L} k_i \alpha_i$. By Proposition 3.2, $\alpha$ can be written as a sum of $h(\alpha)$ roots in $\Psi^+\backslash(\mathbb{R}_{\geq 0}L)$. We first show that for $\alpha$ such that $h(\alpha) = 1$, we have that $\widehat{\alpha}$ can be generated. Suppose $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_p$ with $\alpha_i \in \Pi$ and $\alpha_j \in \Pi \backslash L$ for some $j$ and the rest $\alpha_i$'s are in $L$. In fact this forces $\alpha_j \in \Pi \backslash (L \cup M)$.

We find $\beta \in M$ such that $(\beta, \alpha_j) < 0$, then

$$s_\beta(\alpha) = \alpha - \frac{2(\alpha_j, \beta)}{(\beta, \beta)} \beta \in \Psi^+$$

We note that $\alpha - \frac{2(\alpha_j, \beta)}{(\beta, \beta)} \beta$ is still in $\Psi^+\backslash(\mathbb{R}_{\geq 0}(L \cup M))$ and therefore $(\alpha - \frac{2(\alpha_j, \beta)}{(\beta, \beta)} \beta)_0$ is in the generating set. Suppose $\alpha - \frac{2(\alpha_j, \beta)}{(\beta, \beta)} \beta \in \Phi^+$. Take $k \geq n(-\beta)

\alpha - \frac{2(\alpha_j, \beta)}{(\beta, \beta)} \beta + \left(-\frac{2(\alpha_j, \beta)}{(\beta, \beta)}\right)(-\beta + k \delta)

= \alpha - k \frac{2(\alpha_j, \beta)}{(\beta, \beta)} \delta

Then $(\alpha)_0, \alpha - \frac{2n(-\beta)(\alpha_j, \beta)}{(\beta, \beta)} \delta, \alpha - \frac{2n(-\beta+1)(\alpha_j, \beta)}{(\beta, \beta)} \delta, \alpha - \frac{2n(-\beta+2)(\alpha_j, \beta)}{(\beta, \beta)} \delta, \cdots$ are all generated.

Otherwise suppose $\alpha - \frac{2(\alpha_j, \beta)}{(\beta, \beta)} \beta \in \Phi^-$. Take $k \geq n(-\beta)

\alpha - \frac{2(\alpha_j, \beta)}{(\beta, \beta)} \beta + \left(-\frac{2(\alpha_j, \beta)}{(\beta, \beta)}\right)(-\beta + k \delta)

= \alpha - k \frac{2(\alpha_j, \beta)}{(\beta, \beta)} \delta + \delta

Then $(\alpha)_0, \alpha - \frac{2n(-\beta)(\alpha_j, \beta)}{(\beta, \beta)} \delta + \delta, \alpha - \frac{2n(-\beta+1)(\alpha_j, \beta)}{(\beta, \beta)} \delta + \delta, \alpha - \frac{2n(-\beta+2)(\alpha_j, \beta)}{(\beta, \beta)} \delta + \delta, \cdots$ are all generated.

The “gaps” in those sequences can be filled. Consider $\lambda + m \delta$ and $\lambda + n \delta$ are both in a closed set with $n > m$, then

$$\lambda + (m + 1) \delta = \frac{n - m - 1}{n - m}(\lambda + m \delta) + \frac{1}{n - m}(\lambda + n \delta).$$

So we can assume that $\widehat{\lambda}$ for $h(\alpha) = 1$ can be generated.

For $h(\alpha) = k > 1$ we can assume $\alpha = \gamma_1 + \gamma_2 + \cdots + \gamma_k$ with $\gamma_i \in \Psi^+\backslash\mathbb{R}_{\geq 0}L$ and $h(\gamma_i) = 1$ and $\gamma_1 + \gamma_2 + \cdots + \gamma_{k-1} \in \Psi^+$ by Chapter VI §1 Proposition 19 in [2]. We have shown either that $\gamma_k$ is the first element of the generating set since $\gamma_1 + \gamma_2 + \cdots + \gamma_{k-1} \in \Psi^+\backslash\mathbb{R}_{\geq 0}L$ and $\gamma_k$ is the second element of the generating set depending whether $\gamma_k$ lies in $\mathbb{R} M \cap \Phi$. So we can suppose that $\gamma_k$ can be generated or that there is an infinite $\delta$-chain through $\gamma_k$ beginning from $n(\gamma_k)$ can be generated (depending whether $\gamma_k$ lies in $\mathbb{R} M \cap \Phi$). So we can suppose that there exists $N$ such that $\gamma_k + nd$ is in the generating set for any $n \geq N$. Note

$$\alpha + (N + l) \delta = (\gamma_1 + \gamma_2 + \cdots + \gamma_{k-1} + l \delta) + \gamma_k + N \delta.$$

Hence $\alpha_0, \alpha + (l + N) \delta, \alpha + (l + N + 1) \delta, \cdots$ are all generated. Again by the previous “gap-filling” method we see that $\widehat{\alpha}$ can be generated.

If $L \subseteq \{\alpha \in \Pi | (\alpha, M) = 0\}$, assume $\{\alpha \in \Pi \backslash L | (\alpha, M) = 0\} = \{\alpha_1, \alpha_2, \cdots, \alpha_k\} \neq \emptyset$. Also assume $\Gamma$ is generated by a finite set $\Lambda$. Suppose $t_i = \max\{|p| \alpha_i + p \delta \in \Lambda\}$. Then we show $\alpha_i + (t_i + 1) \delta$ cannot be generated by $\Lambda$ (which is a contradiction):
if there are two roots $\beta_1, \beta_2$ in $\Psi_{A_0, M}$ and $k_1, k_2 \geq 0$ such that $k_1\beta_1 + k_2\beta_2 = \alpha$, the only possibility is $\beta_1 = \beta_2 = \alpha$, and $k_1 + k_2 = 1$.

Now let $\tilde{W}$ be of type $\tilde{A}_2, \tilde{B}_2$ or $\tilde{G}_2$.

**Corollary 5.3.** The complete lattice $B(\tilde{\Phi}^+)$ is algebraic.

**Proof.** By Corollary 2.13(a) of [3] for every biclosed $B$ in $\tilde{\Phi}^+$, $I_B$ is a biclosed set in $\tilde{\Phi}$. Now the rank 2 nature of $\tilde{\Phi}$, Corollary 5.3 and Theorem 5.2 ensures that the only non-finitely generated biclosed sets are $\Phi_w, w \in \tilde{W}$. But they are the union of an ascending chain of finite biclosed sets. □

To finish the section, we show that in rank 3 affine case, those compact biclosed sets can be generated by a smaller set, i.e. the set of roots whose corresponding reflections having twisted length $-1$. Such property is conjectured to be true in general. This results have the interesting geometric interpretation. Consider the projective representation of the root system. In the case of rank 3, the projected reflections which are the extreme points of the convex closure of the projective roots for a compact biclosed $\Gamma$ correspond to the reflection having twisted $\Gamma$–length $-1$.

The following proposition is explained to me by Matthew Dyer and for this proposition $(W, S)$ is an arbitrary Coxeter system with root system denoted $\Phi$. $T$ is the set of reflections of $W$, whose elements are in bijection with $\tilde{\Phi}^+$ via $\alpha \mapsto s_\alpha$.

**Proposition 5.4.** If $A$ is a finite or cofinite biclosed set in $\Phi^+$, then

$$A = \{ \alpha \in A | l_A(s_\alpha) = -1 \}.$$  

**Proof.** The assertion is trivially true for $A = \emptyset$ so we assume $A \neq \emptyset$. First we note by Proposition 1.2 of [4] for all $\alpha \in A$ we have $l_A(s_\alpha) \leq -1$.

Suppose $\alpha \in A$ and $l_A(s_\alpha) < -1$. By [4] (1.2.1) we have

$$l_{s_\alpha A}(s_\alpha) - 1 = \sum_{W'} (l_{W', \Phi_{W'}} \cap s_\alpha A)(s_\alpha) - 1)$$

where the sum is over all maximal dihedral reflection subgroups containing $s_\alpha$. This is equivalent to

$$-l_A(s_\alpha) - 1 = \sum_{W'} (-l_{W', \Phi_{W'}} \cap A)(s_\alpha) - 1).$$

The left hand side is strictly positive and therefore we conclude there is a maximal dihedral reflection subgroup $W'$ containing $s_\alpha$ and $l_{W', \Phi_{W'}} \cap A(s_\alpha) < -1$. Now we fix this subgroup $W'$.

Since $A \cap \Phi_{W'}$ is either finite or cofinite, the Bruhat (sub)graph $\Omega_{\{W', A \cap \Phi_{W'}\}}(\{s_\alpha, 1\})$ is isomorphic to the Bruhat graph of any closed interval of length $-l_A(s_\alpha)$ in a dihedral Coxeter group, i.e. isomorphic to the graph with vertex set $U$ where $U$ is a dihedral Coxeter group of order $-2l_A(s_\alpha)$ and edge set $\{(x, y) \in U \times U | l(x) < l(y), l(y) = l(x) + 1(\text{mod } 2)\}$ by [4] 2.6. Then one sees that there are two reflections $s_\beta_1, s_\beta_2 \in \{s_\alpha, 1\} W', \Phi_{W'}, \cap A$ with twisted length $l_{W', \Phi_{W'}} \cap A(s_\beta_1) = l_{W', \Phi_{W'}} \cap A(s_\beta_2) = -1$ (as coatoms of the interval). And then one can easily see that under a (dihedral) reflection order of $W'$, $\beta_1 \prec \alpha \prec \beta_2$ and therefore we see $\alpha \in \{\beta_1, \beta_2\}$. Finally we have $s_\beta_i \succ_{W', \Phi_{W'}} \cap A s_\alpha, i = 1, 2$. This implies that $s_\beta_i \succ_{A} s_\alpha, i = 1, 2$ and hence $l_A(s_\beta_i) > l_A(s_\alpha), i = 1, 2$. Therefore $\alpha$ can be generated by $\beta_1, \beta_2 \in A$ which have twisted lengths strictly greater than that of $\alpha$. Then an inductive argument finishes the proof. □
Now again $W$ is of type $\tilde{A}_2, \tilde{B}_2$ or $\tilde{G}_2$. We introduce a compact notation describing a set consisting of $\delta$ chains. Denote
\[
\left(\alpha_1\right)_{a_1}^{b_1} \left(\alpha_2\right)_{a_2}^{b_2} \cdots \left(\alpha_p\right)_{a_p}^{b_p} \alpha_{p+1} \alpha_{p+2} \cdots \alpha_{q}
\]
\[
:= \bigcup_{i=1}^{p} \{\alpha_i + k_i \delta | a_i \leq k_i \leq b_i, \text{if } b_i \neq \infty, a_i \leq k_i, \text{if } b_i = \infty\} \cup \alpha_{p+1} \cup \alpha_{p+2} \cup \cdots \cup \alpha_{q}.
\]

**Theorem 5.5.** Let $A$ be a compact biclosed set in $\mathcal{B} (\tilde{\Phi}^+)$. Then
\[
A = \{ \alpha \in A | I_A(s_{\alpha}) = -1 \}.
\]

**Proof.** In view of Theorem 5.2 and Corollary 2.13(a), we conclude that the compact biclosed sets of these rank 3 affine Weyl groups are finite biclosed sets, cofinite biclosed sets and those biclosed sets $B$ with
\[
I_B = \Psi^+_{\delta, \{\alpha\}},
\]
where $\Psi^+$ is a positive system of $\Phi$ and $\alpha$ is a simple root of $\Psi^+$. The finite and cofinite cases are solved by Proposition 5.4. So we need to show those compact biclosed sets of the third type can be generated by roots whose reflections have twisted length $-1$.

Let $\alpha + n\delta$ be a positive root. We describe the set $\Phi_{s_{\alpha + n\delta}}$.

It consists of four types:

Let $(\beta, \alpha) > 0$. By noting that
\[
s_{\alpha + p\delta}(\beta + q\delta) = s_{\alpha}(\beta) + (q - (\beta, \alpha')p)\delta,
\]
we have

(i) $\beta \in \tilde{\Phi}^+, s_{\alpha}(\beta) \in \tilde{\Phi}^-$. Then let $0 \leq k < \frac{2(\alpha, \beta)}{(\alpha, \alpha)} n$. $\beta + k\delta \in \Phi_{s_{\alpha + n\delta}}$.

(ii) $\beta \in \tilde{\Phi}^+, s_{\alpha}(\beta) \in \tilde{\Phi}^-$. Then let $0 \leq k \leq \frac{2(\alpha, \beta)}{(\alpha, \alpha)} n$. $\beta + k\delta \in \Phi_{s_{\alpha + n\delta}}$.

(iii) $\beta \in \tilde{\Phi}^-, s_{\alpha}(\beta) \in \tilde{\Phi}^+$. Then let $0 < k < \frac{2(\alpha, \beta)}{(\alpha, \alpha)} n$. $\beta + k\delta \in \Phi_{s_{\alpha + n\delta}}$.

(iv) $\beta \in \tilde{\Phi}^-, s_{\alpha}(\beta) \in \tilde{\Phi}^-$. Then let $0 < k \leq \frac{2(\alpha, \beta)}{(\alpha, \alpha)} n$. $\beta + k\delta \in \Phi_{s_{\alpha + n\delta}}$.

Now we study each of the three groups. Note we need to show for a compact biclosed set
\[
\left(\alpha_1\right)_{a_1}^{b_1} \left(\alpha_2\right)_{a_2}^{b_2} \cdots \left(\alpha_p\right)_{a_p}^{b_p} \alpha_{p+1} \alpha_{p+2} \cdots \alpha_{q}
\]
$\alpha_i + a_i \delta, 1 \leq i \leq q$ and $(\alpha_j) \alpha, p+1 \leq j \leq q$ can be generated by roots whose twisted length being $-1$ by Theorem 5.2.

Case $\tilde{A}_2$.

Let $\alpha, \beta$ be two simple roots of $\Phi^+$ and by above we then have explicitly
\[
\Phi_{s_{\alpha + n\delta}} = \alpha_0^{2n}(\alpha + \beta)^{n-1}(-\beta)_0^n,
\]
\[
\Phi_{s_{\alpha + \beta + n\delta}} = (\alpha + \beta)_0^{2n} \alpha_0^n(\beta)_0^n.
\]
\[
\Phi_{s_{\beta + n\delta}} = \beta_0^{2n}(\alpha + \beta)^{n-1}(-\alpha)_0^n,
\]
\[
\Phi_{s_{\alpha + -\beta + n\delta}} = (-\alpha)^{2n-1}(-\alpha - \beta)_0^n(-\beta)_0^n,
\]
\[
\Phi_{s_{\alpha - \beta + n\delta}} = (-\beta)^{2n-1}(-\alpha - \beta)_0^n(\alpha)_0^n.
\]

Using the symmetry of $\tilde{A}_2$, it suffices to consider two cases.
\[
B_1 = \alpha_k^\infty \beta^\infty \alpha + \beta^\infty
\]
The roots whose corresponding reflections have $B_1$-twisted length $-1$ include $\beta, -\alpha + \delta, \alpha + k\delta$ and when $k > 0$, $\alpha + \beta$ as well. In each case, $\beta, -\alpha + \delta, \alpha + k\delta, \alpha + \beta$ can be generated by them.

$$B_2 = (-\alpha)_k^\infty \hat{\beta} \alpha + \hat{\beta} \alpha$$

The roots whose corresponding reflections have $B_2$-twisted length $-1$ include $\beta, \alpha, -\alpha + k\delta$. Then $\beta, \alpha, -\alpha + k\delta, \alpha + \beta$ can all be generated.

The same technique can be carried out to verify the assertion for $\tilde{B}_2, \tilde{G}_2$. But in view of their length, we omit the arguments. \hfill \Box

### 6. Braid Operation On The Reflection Orders

First let $(W, S)$ be an arbitrary Coxeter system and $\Phi, \Phi^+$ be its root system and a positive system. If $s$ is a simple reflection denote $\alpha_s$ the corresponding simple root. Dyer made the following conjecture on generalizing the braid relations on $W$ to the reflection orders.

Let $T$ be a finite set contained in $\Phi^+$. We construct a finite undirected graph with vertices being $T$ together with a total order on $T$ obtained by restricting some reflection order to $T$. Note that it could happen that there are different reflection orders $\leq r$ and there exists a reduced expression such that when restricted to $T$ they are identical. In that case $(T, \leq_1), (T, \leq_2)$ are regarded as the same element (vertex).

To define the edges of the graph, we first define the dihedral substring of $(T, \leq)$. Suppose $(T, \leq)$ is given by $\beta_1 < \beta_2 < \cdots < \beta_n$.

$\beta_{p+1} < \beta_{p+2} < \cdots < \beta_q$ is called a dihedral substring of $(T, \leq)$ if there exists a maximal dihedral reflection subgroup $W'$ such that $T \cap \Phi_{W'} = \{\beta_{p+1}, \beta_{p+2}, \cdots, \beta_q\}$.

Now $(T, \leq_1)$ and $(T, \leq_2)$ has an edge connecting them if and only if $(T, \leq_2)$ is obtained by reversing a dihedral substring of $(T, \leq_1)$. We call the graph thus defined the braid graph.

**Conjecture 6.1.** (Dyer) The braid graph is connected for arbitrary $T$ and $(W, S)$.

For $W$ finite the conjecture is reduced to the fact that any two reduced expressions of an element can be connected by braid moves.

We call $(T, \leq_1, \leq_2, \Phi_w)$ be a finite quadruple if

(i) $\{\beta_1, \beta_2, \cdots, \beta_m\} = \{\gamma_1, \gamma_2, \cdots, \gamma_m\} = T \subset \Phi^+$ and $w \in W$,

(ii) $\leq_1, \leq_2$ are two reflection orders, and $\beta_1 <_1 \beta_2 <_1 \cdots <_1 \beta_m, \gamma_1 <_2 \gamma_2 <_2 \cdots <_2 \gamma_m$.

(iii) There exists a reduced expression $s_1s_2 \cdots s_k$ of $w$ such that $\beta_i = s_1s_2 \cdots s_{j_i-1}(\alpha_{s_{j_i}}), j_1 < j_2 < \cdots < j_m$ and there exists a reduced expression $r_1r_2 \cdots r_k$ of $w$ such that $\gamma_i = r_1r_2 \cdots r_{l_i-1}(\alpha_{r_{l_i}}), l_1 < l_2 < \cdots < l_m$.

We call $(T, \leq_1, \leq_2, \Phi'_w)$ be a cofinite quadruple if (i) and (ii) as above hold and

(iii)' There exists a reduced expression $s_1s_2 \cdots s_k$ of $w$ such that $\beta_{m+1-i} = s_1s_2 \cdots s_{j_i-1}(\alpha_{s_{j_i}}), j_1 < j_2 < \cdots < j_m$ and there exists a reduced expression $r_1r_2 \cdots r_k$ of $w$ such that $\gamma_{m+1-i} = r_1r_2 \cdots r_{l_i-1}(\alpha_{r_{l_i}}), l_1 < l_2 < \cdots < l_m$. 

Lemma 6.2. Suppose $W$ is an affine Weyl group. If $(T, \leq_1, \leq_2, \Phi_w)$ is a finite quadruple or $(T, \leq_1, \leq_2, \Phi'_w)$ is a cofinite quadruple, then $(T, \leq_1)$ and $(T, \leq_2)$ are in the same connected component of the braid graph.

Proof. Suppose $(T, \leq_1, \leq_2, \Phi_w)$ is a finite quadruple. Since $W$ is affine, the reflection orders are in bijection with the maximal totally ordered subsets of the set of initial sections by Theorem 1.3. Therefore we can find a reflection order $\leq_1'$ whose initial sections include $\Phi_{s_1}, \Phi_{s_1s_2}, \ldots, \Phi_{s_1s_2\ldots s_k} = \Phi_w$. Then $(T, \leq_1) = (T, \leq_1')$. Similarly we can find a reflection order $\leq_2'$ whose initial sections include $\Phi_{r_1}, \Phi_{r_1r_2}, \ldots, \Phi_{r_1r_2\ldots r_k} = \Phi_w$. Then $(T, \leq_2) = (T, \leq_2')$.

Then we can apply ordinary braid moves to convert $s_1s_2\ldots s_k$ to $r_1r_2\ldots r_k$. Each ordinary braid move is equivalent to reversing the sub-string of the positive roots of a (possibly non-standard) maximal dihedral parabolic subgroup. Therefore we conclude that $(T, \leq_1')$ and $(T, \leq_2')$ are in the same connected component.

The case that $(T, \leq_1, \leq_2, \Phi'_w)$ is a cofinite quadruple is proved in the essentially same way and is omitted. $\square$

Now let $\tilde{W}$ be a rank 3 affine Weyl group with $\Phi, \tilde{\Phi}$ defined as in the previous sections.

Lemma 6.3. Let $u, v \in \overline{W}$ and $\Phi'_v \supset \Phi_u$. Then $|\Phi'_v \setminus \Phi_u| \neq 1$.

Proof. The lemma is an easy consequence of Theorem 3.10 and the rank 2 nature of $\Phi$. $\square$

Theorem 6.4. For $\tilde{W}$ and any finite $T \subset \tilde{\Phi}$, the braid graph is connected.

Proof. First we show that given a reflection order $\leq$ of $\tilde{\Phi}$, there exists some positive system $\Psi^+$ of $\Phi$ such that $\tilde{\Psi}^+$ is an initial section of $\leq$ and $\tilde{\Psi}^-$ is then a final section of $\leq$ and $\tilde{\Phi} = \tilde{\Psi}^+ \cup \tilde{\Psi}^-$. To see this, one notes that under the inclusion the initial sections of $\leq$ form a chain $\{ B_j \}_{j \in J}$ of biclosed sets which is also a maximal chain of $\mathcal{P}(\tilde{\Phi})$ and is chain complete. Consider the sub-chain consisting of those $B_j$ contained in some $\tilde{\Psi}^+$ for some positive system $\Psi^+$ in $\Phi$. (For different $B_j$, such $\Psi^+$ could be different though eventually we will see that a same $\tilde{\Psi}^+$ can be chosen.) This sub-chain is indeed an initial section of the original chain $\{ B_j \}_{j \in J}$ because of Lemma 4.3. Also this sub-chain can be viewed as an ascending chain $C$ of $\tilde{W}$. Hence the union of the sets in this sub-chain (denoted $B$) is the inversion set of the join of the elements in $C$. So $B$ is still in this sub-chain and $B$ is the unique maximal element of this sub-chain. Now consider the sub-chain consisting of those $B_j$ whose complement is contained in some $\tilde{\Psi}^+$ for some positive system $\Psi^+$ in $\Phi$. Again by Lemma 4.3 they are the final section of the original chain $\{ B_j \}_{j \in J}$.

By ordering the complements of them under inclusion, which effectively reverses the total order on this final section and applying the same argument as above, we can see that the intersection of $B_j$’s in this final section, denoted $B'$, is in this final section and is the unique minimal element of this final section. So we have $B \subset B'$. Since any $B_j$ is in one of the two sub-chains constructed above, so there is no $B_j$ in-between $B$ and $B'$. So by maximality either $B = B'$ or $B = B' \setminus \{ \alpha \}$.

But $B = \Phi_u$ and $B' = \Phi'_v$ for some $u, v \in \tilde{W}$, the latter is impossible by Lemma 6.3. Hence we have the first situation and $B = B'$ is both the inversion set and the
complement of the inversion set. This force it to be \( \hat{\Psi}^+ \) for some positive system \( \Psi^+ \) in \( \Phi \).

Now let \( (T, \leq_1) \) and \( (T, \leq_2) \) be two vertices of the braid graph. Thus suppose \( \hat{\Psi}^+_1 \) is an initial section of \( \leq_1, \hat{\Psi}^+_2 \) is a final section of \( \leq_1, \hat{\Psi}^+_2 \) is an initial section of \( \leq_2 \) and \( \hat{\Psi}^+_2 \) is a final section of \( \leq_2 \) where \( \Psi^+_1, \Psi^+_2 \) are positive systems of \( \Phi \).

Case I. \( \Psi^+_1 = \Psi^+_2 \).

Denote \( \Psi^+_1 = \Psi^+_2 \). We show that we can find some \( w \in \hat{W} \) such that \( (T \cap \hat{\Psi}^+_1, \leq_1, \leq_2, \Phi_w) \) is a finite quadruple.

Let \( \rho \in \hat{\Phi} \) and \( \leq \) be a reflection order. Define \( I_{\leq, \beta} := \{ \gamma \in \hat{\Phi} | \gamma \leq \rho \} \).

Suppose \( T \cap \hat{\Psi}^+ = \{ \beta_1, \beta_2, \ldots, \beta_\ell \} \) with \( \beta_1 < \beta_2 < \cdots < \beta_\ell \). If \( I_{\leq, \beta_i} \) is finite then it is finite biclosed and is \( \Phi_x \) for some \( x \in \hat{W} \). Such \( x \) has a reduced expression \( s_1s_2 \cdots s_l = x \) such that

\[
\beta_i = s_1s_2 \cdots s_{j_i-1}(\alpha_{s_i}), j_1 < j_2 < \cdots < j_\ell.
\]

Otherwise \( I_{\leq, \beta_i} \) is infinite but is contained in \( \hat{\Psi}^+ \). Now let \( m \) be maximal such that \( I_{\leq, \beta_m} \) is finite and equals \( \Phi_u, v \in \hat{W} \). So \( I_{\leq, \beta_{m+1}} = \Phi_u \) where \( u \) is an infinite reduced word. So \( v \) is a prefix of \( u \) such that \( \beta_{m+1} \not\in \Phi_u \). So \( v \) has a finite prefix \( v' \) such that \( v \) is a prefix of \( v' \), \( \beta_{m+1} \in \Phi_v' \) and all \( \beta_{m+2}, \ldots, \beta_\ell \not\in \Phi_v \). Proceed this way and we can also find \( x \in \hat{W} \) such that there exists a reduced expression \( s_1s_2 \cdots s_l = x \) such that

\[
\beta_i = s_1s_2 \cdots s_{j_i-1}(\alpha_{s_i}), j_1 < j_2 < \cdots < j_\ell.
\]

Same argument as above works for \( T \cap \hat{\Psi}^+ \) with \( \leq_2 \). So suppose \( T \cap \hat{\Psi}^+ = \{ \gamma_1, \gamma_2, \ldots, \gamma_\ell \} \) with \( \gamma_1 < \gamma_2 < \cdots < \gamma_\ell \). We can find \( y \in \hat{W} \) such that there exists a reduced expression \( r_1r_2 \cdots r_\ell = y \) and

\[
\gamma_i = r_1r_2 \cdots r_{k_i-1}(\alpha_{r_i}), k_1 < k_2 < \cdots < k_\ell.
\]

Since \( \Phi_x, \Phi_y \subset \hat{\Psi}^+ \), they admit a join in \( \hat{W} \) say \( w \). Then one sees that \( (T \cap \hat{\Psi}^+, \leq_1, \leq_2, w) \) is a finite quadruple and by Lemma 6.2 we see that \( (T, \leq_1) \) and \( (T, \leq_3) \) are in the same connected component of the braid graph where \( \leq_3 \mid_{T \cap \hat{\Psi}^+} = \leq_2 \mid_{T \cap \hat{\Psi}^+} \) and \( \leq_3 \mid_{T \cap \hat{\Psi}^+} = \leq_1 \mid_{T \cap \hat{\Psi}^+} \).

Essentially same argument as above shows that one can find \( z \in \hat{W} \) such that \( (T \cap \hat{\Psi}^+, \leq_3, \leq_2, \Phi_z) \) is a cofinite quadruple. Then by Lemma 6.2 we see that \( (T, \leq_3) \) and \( (T, \leq_2) \) are in the same connected component of the braid graph and we are done with this case.

Case II. \( \Psi^+_1 \neq \Psi^+_2 \).

In this case we show that by successively reversing the dihedral substrings we can reduce this case to case I.

Suppose \( \Psi^+_1 = \{ \alpha_1, \alpha_2, \ldots, \alpha_n \} \) where \( \alpha_1 < \alpha_2 < \cdots < \alpha_n \) and \( < \) is one of the two reflection orders on \( \Psi^+_1 \). So \( \alpha_1, \alpha_n \) are simple. Similarly assume \( \Psi^+_2 = \{ -\alpha_1, -\alpha_2, \ldots, -\alpha_k, \alpha_{k+1}, \ldots, \alpha_n \} \), \( 0 \leq k \leq n \).

Suppose \( T \cap \hat{\Psi}^+_1 = \{ \beta_1, \beta_2, \ldots, \beta_\ell \} \) with \( \beta_1 < \beta_2 < \cdots < \beta_\ell \). By the reasoning in case one we can find some \( w \in \hat{W} \) such that \( s_1s_2 \cdots s_l = w \) is a reduced expression of \( w \) and

\[
\beta_i = s_1s_2 \cdots s_{j_i-1}(\alpha_{s_i}), j_1 < j_2 < \cdots < j_\ell.
\]
\[ \{\beta_1, \beta_2, \ldots, \beta_t\} \subseteq T \]

\[ \Phi_u = \Phi_w \cap \Phi_y \]

Hence \( w, y \) admit a join \( z \in W \). Pick a reflection order \( \leq_3 \) which has \( \Phi^+_1, \Phi^+_2 \) and \( \Phi^+_y \) as initial sections. Then consider the reflection order \( \leq_4 \) such that \( \leq_4 \Phi^+_1 = \leq_3 \Phi^+_1 \), \( \leq_4 \Phi^+_2 = \leq_3 \Phi^+_2 \), \( \leq_4 \Phi^+_y = \leq_3 \Phi^+_y \) and thus \( \hat{\Phi}_1 \) is an initial section of \( \leq_4 \). Then \( (T \cap \hat{\Phi}_1, \leq_1, \leq_4, \Phi^+_2) \) is a finite quadruple. So by Lemma 6.2 and the fact \( \leq_4 \Phi^+_1 = \leq_3 \Phi^+_1 \), \( (T, \leq_1) \) and \( (T, \leq_4) \) are in the same connected component in the braid graph. Comparing \( T|_{\leq_1} \) and \( T|_{\leq_4} \), one sees that \( \hat{\alpha}_1 \cap (T \cap \hat{\Phi}_1) \) is moved to the end of \( (T \cap \hat{\Phi}_1)_{\leq_4} \) (while the total order on \( (T \cap \hat{\Phi}_1) \) remains unchanged and \( (\alpha_1)_{0} < (\alpha_1)_{0} + \delta < (\alpha_1)_{0} + 2\delta < \cdots \)). Now we can apply the same technique to \( T \cap \hat{\Phi}_1 \) and obtain a reflection \( \leq_5 \) with the following properties: (1) \( (T, \leq_4) \) and \( (T, \leq_5) \) are in the same connected component of the braid graph; (2) \( \hat{\Phi}_1 \) is an initial section of \( \leq_5 \); (3) \( \leq_4 \mid_{T \cap \hat{\Phi}_1} \) is \( \leq_5 \mid_{T \cap \hat{\Phi}_1} \) and (4) comparing \( \leq_4 \mid_{T \cap \hat{\Phi}_1} \) and \( \leq_5 \mid_{T \cap \hat{\Phi}_1} \) one has that \( \neg \hat{\alpha}_1 \cap (T \cap \hat{\Phi}_1) \) is moved to the beginning of \( (T \cap \hat{\Phi}_1)_{\leq_5} \) and \( \cdots < (\neg \alpha_1)_{0} + \delta < \cdots < (\neg \alpha_1)_{0} \).

Then by reversing the dihedral substring of \( (T, \leq_5) \) corresponding to the reflection subgroup generated by \( s_{(\alpha_1)_{0}}, s_{(\neg \alpha_1)_{0}} \) we obtain a reflection order \( \leq_6 \). Then \( \leq_6 \) has an initial section \( \{\neg \alpha_1, \alpha_2, \ldots, \alpha_n\} \). We can proceed this procedure and eventually convert \( \leq_1 \) to a reflection order having an initial section being \( \hat{\Phi}_2 \).

**Remark 6.5.** The case I part of the proof of the previous theorem works in the case when \( W \) is locally finite and this proves the conjecture in that specific case.

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