Multiscale schemes for stochastic dynamical systems driven by $\alpha$-stable processes

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Abstract

This work is about strong and weak convergence of projective integration schemes for multiscale stochastic dynamical systems driven by $\alpha$-stable processes. Firstly, we analyze a class of “projective integration” methods, which are used to estimate the effect that the fast components have on slow ones. Secondly, we obtain the $p$th moment error bounds between the results of the projective integration method and the slow components of the original system with $p \in (1, \min(\alpha_1, \alpha_2))$. Finally, a numerical experiment is constructed to illustrate this scheme.

Keywords: $\alpha$-stable process, averaging principle, projective integration, error analysis.
1. Introduction

The multiscale models arise widely in various fields [1, 2, 3, 4]. For example, the production of mRNA and proteins occur in a bursty, unpredictable, and intermittent manner, which create variation or noise in individual cells or cell-to-cell interactions. Since the mRNA synthesis process is faster than the protein dynamics, this leads to a multiscale system. Finding a coarse-grained model that can effectively describe the dynamics of the multiscale model has always been a very active research field. Khasminskii et al. [5] developed a stochastic averaging principle driven by Wiener noise that enables one to average out the fast-varying variables. The main idea is as follows: under appropriate conditions, with the slow-varying component fixed, if the fast-varying component has a stationary distribution, it can be shown that the process represented by the slow-changing component converges weakly to a limit averaging system. Motivated by the previous works, averaging principle for various stochastic dynamical systems or stochastic partial differential equations driven by Wiener noise have also drawn much attention, see, e.g., [6, 7, 8, 9, 10]. Some authors also studied the averaging principle of two-scale dynamical systems driven by non-Gaussian noises with finite second moments [12, 13, 14]. This excludes the $\alpha$-stable noise, since its second moment is divergent [15].

Recently, two-scale dynamical systems driven by $\alpha$-stable processes have drawn much attention. Bao et al. [16] studied the averaging principle for stochastic partial differential equation with two-time-scale Markov switching. They showed that under suitable conditions, a limit process that was a solution of either an SPDE or an SPDE with switching was obtained. In [17] and [18], they studied data assimilation and parameter estimation and showed that the averaged, low dimensional filter approximated the original filter, by examining the corresponding Zakai stochastic partial differential equations. Sun et al. [19, 20] studied the averaging principle for stochastic real Ginzburg-Landau equation and stochastic differential equation. They used the classical Khasminskii approach to show the convergence between the slow component and
averaged equation. Moreover, they also studied the strong and weak convergence rates for slow-fast stochastic differential equations and proved that the strong and weak convergent order are $1 - 1/\alpha$ and 1 respectively.

However, it is often impractical to obtain the reduce equations in closed form, since the invariant measure is often unknown. Standard computational schemes may fail due to the separation between the $O(\varepsilon)$ time scale and the $O(1)$. This inspires us to develop new algorithms to estimate the effect that the fast components have on slow ones. Several related techniques have been proposed for two-time stochastic dynamics driven by Wiener noises or non-Gaussian noises with finite second moments. The heterogeneous multi-scale method is a general methodology for efficient numerical computation of problems with multiple scales and/or multi-levels of physics. For example, E. Vanden-Eijnden [21] used HMM to compute the evolution of the slow variables without having to derive explicitly the effective equations beforehand. W. E et al. [22] analyzed a class of numerical schemes for the two-time dynamical systems driven by Wiener noises. A similar idea, also called “projective integration” method was proposed in [23]. D. Givon et al. used the method to analyze multiscale stochastic dynamics driven by noises with finite second moments and obtained explicit bounds for the discrepancy between the results of the multiscale integration method and the slow components of the original system, which excludes the $\alpha$-stable noise.

A natural and important question is the following: for the two-time dynamical systems driven by $\alpha$-stable noises, how to estimate the effect that the fast components have on the slow ones as the invariant measure is unknown from the perspective of computation?

The main technique used in this present manuscript is the framework of “projective integration” method, which consist of a hybridization between a standard solver for the slow components, and short runs for the fast dynamics. The main difficulty is how to deal with the nonlinear term and $\alpha$-stable process.

This paper is organized as follows. In Section 2, we recall the basic concepts about symmetric $\alpha$-stable process and ergodic theory. In Section 3, we formulate the problem and analysis of the projective integration scheme. In Section
4, we give some specific examples to illustrate this method. Some discussion is contain in Section 5.

To end this section, we introduction some notations, $C$ denote positive constants, whose values may change from one place to another. $C_p$ is used to emphasize that the constant only depends on the parameter $p$. We will use $\langle \cdot, \cdot \rangle$ to denote the scalar product in $\mathbb{R}^n$ and $\|\cdot\|$ to denote the norm. $B_b(\mathbb{R}^d)$ denotes the space of all Borel measurable functions. For any $k \in \mathbb{N}_+$ and $\delta \in (0, 1)$, we define

\[ C^k(\mathbb{R}^n) := \{ u : \mathbb{R}^n \to \mathbb{R} : u \text{ and all its partial derivative up to order } k \text{ are continuous} \}, \]

\[ C^k_b(\mathbb{R}^n) := \{ u \in C^k(\mathbb{R}^n) : \text{for } 0 \leq i \leq k, \text{the } i \text{ order partial derivative are bounded} \}, \]

\[ C^{k+\delta}_b(\mathbb{R}^n) := \{ u \in C^k_b(\mathbb{R}^n) : \text{all the } k \text{-th order partial derivative of } u \text{ are } \delta \text{- Hölder continuous} \}. \]

For $k_1, k_2 \in \mathbb{N}_+, 0 \leq \delta_1, \delta_2 < 1$ and a real-valued function on $\mathbb{R}^n \times \mathbb{R}^m$, the notation $C^{k_1+\delta_1, k_2+\delta_2}_b$ denotes (i) for all $0 \leq |\beta| \leq k_1, 0 \leq |\gamma| \leq k_2$ and $|\beta| + |\gamma| \geq 1$ the partial derivative $\partial_\beta x \partial_\gamma y u$ is bounded continuous; (ii) $\partial_\beta x \partial_\gamma y u$ is $\delta_1$-Hölder continuous with respect to $x$ with index $\delta_1$ uniformly in $y$ and $\delta_2$-Hölder continuous with respect to $y$ with index $\delta_2$ uniformly in $x$.

2. Preliminaries

In this section, we recall some basic definitions for Lévy motions.

2.1. Symmetric $\alpha$-stable process

A Lévy process $L_t$ taking values in $\mathbb{R}^n$ is characterized by a drift vector $b \in \mathbb{R}^n$, an $n \times n$ non-negative-definite, symmetric covariance matrix $Q$ and a Borel measure $\nu$ defined on $\mathbb{R}^n \setminus \{0\}$. We call $(b, Q, \nu)$ the generating triplet of the Lévy motions $L_t$. Moreover, we have the Lévy-Itô decomposition for $L_t$ as follows

\[ L_t = bt + B_Q(t) + \int_{||y|| < 1} y \tilde{N}(t, dy) + \int_{||y|| \geq 1} y N(t, dy), \]  

(2.1)
where $N(dt, dy)$ is the Poisson random measure, $\tilde{N}(dt, dy) = N(dt, dy) - \nu(dx)dt$ is the compensated Poisson random measure, $\nu(A) = EN(1, A)$ is the jump measure, and $B_Q(t)$ is an independent standard $n$-dimensional Brownian motion. The characteristic function of $L_t$ is given by

$$E[\exp(i\langle u, L_t \rangle)] = \exp(t\rho(u)), \quad u \in \mathbb{R}^n, \quad (2.2)$$

where the function $\rho : \mathbb{R}^n \to \mathbb{C}$ is the characteristic exponent

$$\rho(u) = i\langle u, b \rangle - \frac{1}{2}\langle u, Qu \rangle + \int_{\mathbb{R}^n \setminus \{0\}} \{e^{i\langle u, z \rangle} - 1 - i\langle u, z \rangle I_{\{|z| < 1\}}\} \nu(dz). \quad (2.3)$$

The Borel measure $\nu$ is called the jump measure.

**Definition 1.** For $\alpha \in (0, 2)$, an $n$-dimensional symmetric $\alpha$-stable process $L_t^\alpha$ is a Lévy process with characteristic exponent $\rho$

$$\rho(u) = -|u|^\alpha, \quad \text{for } u \in \mathbb{R}^n \quad (2.4)$$

For a $n$-dimensional symmetric $\alpha$-stable Lévy process, the diffusion matrix $Q = 0$, the drift vector $b = 0$, and the Lévy measure $\nu$ is given by

$$\nu(du) = \frac{c(n, \alpha)}{|u|^{n+\alpha}} du, \quad (2.5)$$

where $c(n, \alpha) := \frac{\alpha\Gamma\left(\frac{n+\alpha}{2}\right)}{2^{\frac{n+\alpha}{2}}\pi^{\frac{n}{2}}\Gamma(1-\frac{\alpha}{2})}$. Let $(P_t)_{t \geq 0}$ be a semigroup of bounded linear operators on Banach space $B_b(\mathbb{R}^d)$. Let $\mu$ be a probability measure on Borel space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Use the following standard notation:

$$\langle \mu, \varphi \rangle = \int_{\mathbb{R}^d} \varphi(x) \mu(dx) \quad (2.6)$$

$\mu$ is said to be an invariant probability measure of $P_t$ if

$$\langle \mu, P_t \varphi \rangle = \langle \mu, \varphi \rangle, \quad \forall t > 0, \quad \forall \varphi \in B_b(\mathbb{R}^d). \quad (2.7)$$
One says that $P_t$ is ergodic if $P_t$ admits a unique invariant probability measure $\mu$, which amounts to say that

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t P_s f(x) ds = \langle \mu, f \rangle, \ f \in \mathbb{B}_b(\mathbb{R}^d).$$

(2.8)

**Definition 2.** Let $V : \mathbb{R}^d \to [1, \infty)$ be a measurable function and $\mu$ an invariant probability measure of $P_t$. We say $P_t$ to be $V$-uniformly exponential ergodic if there exist $c_0, \gamma > 0$ such that for all $t \geq 0$ and $x \in \mathbb{R}^d$,

$$\sup_{\|\varphi\|_V \leq 1} |P_t \varphi(x) - \langle \mu, \varphi \rangle| \leq c_0 V(x) e^{-\gamma t},$$

(2.9)

where $\|\varphi\|_V = \sup_{x \in \mathbb{R}^d} |\varphi(x)| < +\infty$. If $V \equiv 1$, then $P_t$ is said to be uniformly exponential ergodic, which is equivalent to

$$\|P_t(x, \cdot) - \mu\|_{\text{Var}} \leq c_0 e^{-\gamma t}, \forall x \in \mathbb{R}^d.$$  

(2.10)

where $P_t(x, \cdot)$ is the kernel of bounded linear operator $P_t$.

3. Strong convergence analysis of the projective integration scheme

3.1. Stochastic averaging principle

Consider the following singularly perturbed systems of stochastic differential equations of the form

$$\begin{cases} dX_t^\varepsilon = f_1(X_t^\varepsilon, Y_t^\varepsilon)dt + \sigma_1 dL_t^\alpha_1, & X_0^\varepsilon = x_0 \in \mathbb{R}^n, \\ dY_t^\varepsilon = \frac{1}{\varepsilon} f_2(X_t^\varepsilon, Y_t^\varepsilon) dt + \frac{\sigma_2}{\varepsilon^{\alpha_2}} dL_t^\alpha_2, & Y_0^\varepsilon = y_0 \in \mathbb{R}^m, \end{cases}$$

(3.1)

where $L_t^{\alpha_1}, L_t^{\alpha_2}$ ($1 < \alpha_1, \alpha_2 < 2$) are independent symmetric $\alpha$-stable Lévy processes with triplets $(0, 0, \nu_1)$ and $(0, 0, \nu_2)$, respectively. The function $f_1 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $f_2 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ are Borel functions. The positive constants $\sigma_1$ and $\sigma_2$ represent the noises intensities. The parameter $\varepsilon$ describing the ratio of the time scale between the slow component and fast component.
We make the following assumptions for the slow-fast stochastic dynamical system (3.1).

**Hypothesis H.1** The functions $f_1 \in C_b^{1+\gamma,2+\delta}$ and $f_2 \in C_b^{1+\gamma,2+\gamma}$ with some $\gamma \in (\alpha - 1, 1)$ and $\delta \in (0, 1)$.

**Remark 1.** Note that with the help of Hypothesis H.1, there exist positive constants $L$ and $K$ such that

$$|f_1(x_1, y_1) - f_1(x_2, y_2)| \leq L(|x_1 - x_2| + |y_1 - y_2|),$$

$$|f_2(x_1, y_1) - f_2(x_2, y_2)| \leq L(|x_1 - x_2| + |y_1 - y_2|),$$

and

$$|f_i(x, y)| \leq K(1 + |x| + |y|),$$

for all $x, x \in \mathbb{R}^n$, $y, y \in \mathbb{R}^m$, $i = 1, 2$.

**Hypothesis H.2** The function $f_2$ satisfies

$$\sup_{x \in \mathbb{R}^n} |f_2(x, 0)| < \infty.$$  \hspace{1cm} (3.3)

**Hypothesis H.3** There exists a positive constants $\beta$ such that for any $x \in \mathbb{R}^n, y_1, y_2 \in \mathbb{R}^m$,

$$(f_2(x, y_1) - f_2(x, y_2), y_1 - y_2) \leq -\beta|y_1 - y_2|^2,$$  \hspace{1cm} (3.4)

Below, we will state the results concerning the strong convergence for the averaging principle for system (3.1), which comes from [20, Theorem 2.1].

**Theorem 1.** Under Hypotheses H.1-H.3, for any initial value $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$, $T > 0$ and $p \in [1, \alpha)$, we have

$$\mathbb{E} \left( \sup_{t \in [0,T]} |X_t - \bar{X}_t|^p \right) \leq C\varepsilon^{p(1-1/\alpha)},$$  \hspace{1cm} (3.5)
where the effective equation is of the form

\[ \hat{X}_t = \bar{f}_1(X_t)dt + \sigma_1 dL_{\alpha_1}^t \]  

(3.6)

with

\[ \bar{f}_1 = \int_{\mathbb{R}^m} f_1(x, y) \mu_x(dy), \]  

(3.7)

3.2. Numerical scheme

For \( n = 1, 2, \cdots, \lfloor T/\Delta \rfloor \), we assume that the slow component of (3.1) has the numerical solution \( X_n \). The projective integration scheme consists of a macro-solver: an Euler-Maruyama time-stepper,

\[ X_{n+1} = X_n + A(X_n)\Delta t + \sigma_1 \Delta L_{\alpha_1}^n, \]  

(3.8)

where

\[ \Delta L_{\alpha_1}^n = L_{\alpha_1}^{n+1} - L_{\alpha_1}^n. \]  

(3.9)

Remark 2. The function \( A(X_n) \) is the approximation of \( \bar{f}_1(X_n) \). We refer to (3.8) as the macro-solver.

Given the coarse variable at the \( n \)-th time step \( X_n \), we assume that \( Y_m^n, m = 0, 1, \cdots, M \) is the discrete variables associated with the fast dymanics at the \( n \)-th coarse step, which are numerically generated by the Euler-Maruyama scheme with the time step \( \delta t \) (\( 0 < \delta t \ll 1 \)), i.e.,

\[ Y_{m+1}^n = Y_m^n + \frac{1}{\varepsilon} f_2(X_n, Y_m^n)\delta t + \frac{\sigma_2}{\varepsilon \alpha_2} \Delta L_{\alpha_2}^m, \quad Y_0^n = y_0, \]  

(3.10)

where

\[ \Delta L_{\alpha_2}^m = L_{\alpha_2}^{m+1} - L_{\alpha_2}^m. \]  

(3.11)

Remark 3. The sequence \( Y_m^n \) is called the micro-solver. Equations (3.8) and (3.10) define the projective integration scheme.

Let \( \Delta t \) be a fixed time step, and \( \hat{X}_n \) be the numerical approximation to the coarse variable \( \hat{X} \), at time \( t_n = n\Delta t \). Inspired by the effective equation (3.6),
\( \bar{X}_n \) is evolved in time by an Euler-Maruyama step,

\[
\bar{X}_{n+1} = \bar{X}_n + \bar{f}_1(\bar{X}_n) \Delta t + \sigma_1 \Delta L^{\alpha_1}_n,
\]

where \( \Delta L^{\alpha_1}_n \) is \( \alpha \)-stable displacements over a time interval \( \Delta t \).

Indeed, for every \( c > 0 \), \( L^{\alpha_2}_ct \) and \( c^{\alpha_2} \tilde{L}^{\alpha_2}_t \) have the same distribution, then we gain the following lemma.

**Lemma 1.** Let \( Y^\varepsilon_t \) be the solution of the equation

\[
dY^\varepsilon_t = \frac{1}{\varepsilon} f_2(x, Y^\varepsilon_t) dt + \frac{\sigma_2}{\varepsilon^{\alpha_2}} dL^{\alpha_2}_t, (3.13)
\]

then \( Y_t = Y^\varepsilon_{\varepsilon t} \) is a solution of the stochastic differential equation

\[
dY_t = f_2(x, Y_t) dt + \sigma_2 d\tilde{L}^{\alpha_2}_t, (3.14)
\]

where \( \tilde{L}^{\alpha_2}_t = \frac{1}{\varepsilon^{\alpha_2}} L^{\alpha_2}_t \).

Using Lemma 1 we know that the micro-solver (3.10) is a particular realization that uses an Euler-Maruyama time-stepper as well,

\[
Y^{n+1}_{m+1} = Y^n_m + f_2(X^n_n, Y^n_m) \delta t + \sigma_2 \Delta \tilde{L}^{\alpha_2}_{m,n}. (3.15)
\]

Thus \( A(X_n) \) can be estimated by an empirical averaging

\[
A(X_n) = \frac{1}{M} \sum_{m=1}^{M} f_1(X_n, Y^n_m). (3.16)
\]

In the following, we will present a discrete version of Gronwall inequality, which comes from [24].

**Proposition 1.** Let \( u_n \) and \( \omega_n \) be nonnegative sequences, and \( c \) a nonnegative constant. If

\[
u_n \leq \sum_{l=0}^{n-1} \omega_l u_l + c, (3.17)
\]
then we have

$$u_n \leq c e^{\sum_{i=0}^{n-1} \omega_i}. \quad (3.18)$$

Before proceeding the strong convergence of projective integration schemes for slow-fast stochastic dynamical systems under $\alpha$-stable noises, we need to provide some estimates for the processes $Y_n$ and $X_n$.

**Lemma 2.** For small enough $\delta t$ and $1 < p < \alpha_2$, we have

$$\sup_{0 \leq n \leq \lfloor T/\Delta t \rfloor} \mathbb{E}|Y^p_n| \leq C (\delta t)^{\frac{p}{\alpha_2}}, \quad Y^0_n = Y^{n-1}_n. \quad (3.19)$$

**Proof.** By (3.15), Hypothesis H.1 and Hypothesis H.2, we have

$$\mathbb{E}|Y^p_{m+1}| \leq C_p \mathbb{E}|Y^p_m| + C_p \mathbb{E}|f_2(X_n, Y^p_m)\delta t|^p + C_p \sigma_2^p \mathbb{E}|\hat{L}^{\alpha_2,n}_m|^p$$

$$= C_p \mathbb{E}|Y^p_m| + C_p \mathbb{E}|f_2(X_n, Y^p_m)\delta t|^p + C_p \sigma_2 \mathbb{E}|\hat{L}^{\alpha_2,n}_m|^p$$

$$\leq C_p \mathbb{E}|Y^p_m| + C_p \mathbb{E}|Y^p_m| (\delta t)^p + C_p \mathbb{E}|f_2(X_n, 0)|^p (\delta t)^p + C_p \sigma_2 \mathbb{E}|\hat{L}^{\alpha_2,n}_m|^p$$

$$\leq C_p, L (1 + (\delta t)^p) \mathbb{E}|Y^p_m| + C_p, L (\delta t)^p + C_p, \sigma_2 (\delta t)^\frac{p}{\alpha_2}. \quad (3.20)$$

By the discrete Gronwall inequality, we have

$$\mathbb{E}|Y^p_m| \leq C (\delta t)^{\frac{p}{\alpha_2}}. \quad (3.21)$$

**Lemma 3.** For the small enough $\Delta t < 1$, we have

$$\sup_{0 \leq n \leq \lfloor T/\Delta t \rfloor} \mathbb{E}|X^p_n| \leq C|\Delta t|^{\frac{p}{\alpha_1}}. \quad (3.22)$$
Proof. By (3.8), Hypothesis H.1, Lemma 2 and \( \delta t \ll 1 \), we have

\[
\begin{align*}
E|X_{n+1}|^p & \leq C_p E|X_n|^p + C_p E(A(X_n))\Delta t|^p + C_{p, \sigma_1} |\Delta t|^\frac{p}{\alpha_1} \\
& \leq C_p E|X_n|^p + C_p M (\Delta t)^p \sum_{m=1}^{M} E|f_1(X_n, Y^n_m)|^p + C_{p, \sigma_1} |\Delta t|^\frac{p}{\alpha_1} \\
& \leq C_p E|X_n|^p + C_p M \sum_{m=1}^{M} E|f_1(X_n, Y^n_m) - f_1(0, 0)|^p \\
& \quad + C_{p, M} (\Delta t)^p \sum_{m=1}^{M} E|f_1(0, 0)|^p + C_{p, \sigma_1} |\Delta t|^\frac{p}{\alpha_1} \\
& \leq C_p E|X_n|^p + C_{p, M} E|X_n|^p (\Delta t)^p + C_{p, M, K} (\Delta t)^p \left( 1 + E|Y^n_m|^p \right) + C_{p, \sigma_1} |\Delta t|^\frac{p}{\alpha_1} \\
& \leq C_{p, M} \left( 1 + (\Delta t)^p \right) E|X_n|^p + C_{p, \sigma_1, M, K} |\Delta t|^\frac{p}{\alpha_1} \\
& \leq C_{p, M} E|X_n|^p + C_{p, \sigma_1, M, K} |\Delta t|^\frac{p}{\alpha_1} 
\end{align*}
\]

(3.23)

By the discrete Gronwall inequality, we have

\[
\sup_{0 \leq n \leq \lfloor T/\Delta t \rfloor} E|X_n|^p \leq C|\Delta t|^\frac{p}{\alpha_1}.
\]

(3.24)

\[ \square \]

Lemma 4. For the small enough \( \delta t \) and \( 1 < p < \alpha_2 \), the deviation between two successive iterations of the microsolver satisfies

\[
\sup_{0 \leq m \leq M} E|Y^n_{m+1} - Y^n_m|^p \leq C(\delta t)^\frac{p}{\alpha_2}.
\]

(3.25)

Proof. By Lemma 2, Lemma 3 and Hypothesis H.1, we have

\[
\begin{align*}
E|Y^n_{m+1} - Y^n_m|^p & \leq C_p E|f_2(X_n, Y^n_m)|^p (\delta t)^p + C_{p, \sigma_2}^p |I_{\delta t}^{\alpha_2, n}|^p \\
& \leq C_{p, K} (1 + E|X_n|^p + E|Y^n_m|^p) (\delta t)^p + C_{p, \sigma_2} (\delta t)^p/\alpha_2 \\
& \leq C (\delta t)^\frac{p}{\alpha_2}.
\end{align*}
\]

(3.26)
Therefore we have

\[
\sup_{0 \leq n \leq \lfloor \frac{T}{\Delta t} \rfloor} E|Y^m_{m+1} - Y^m_m|^p \leq C(\delta t)^{\frac{p}{2}}. \tag{3.27}
\]

Define the stochastic process \( z^k_t \) which satisfies the following stochastic differential equation,

\[
dz^n_t = f_2(X^n_t, z^n_t)dt + \sigma_2 \tilde{L}^\alpha_t, \quad z^n_0 = y_0, \tag{3.28}
\]

where \( \tilde{L}^\alpha_t \) is independent of \( L^\alpha_t \).

**Lemma 5.** Under Hypotheses \( H.1-H.3 \), the process \( z^n_t \) satisfies

\[
\sup_{0 \leq t \leq T} E|z^n_t|^p \leq C(1 + |y_0|^p). \tag{3.29}
\]

**Proof.** By (3.28) and Hypotheses \( H.1 \), we have

\[
|z^n_t| \leq |y_0| + L \int_0^t |z^n_s|ds + \int_0^t |f_2(X^n_s, 0)|ds + \sigma_2 \tilde{L}^\alpha_t. \tag{3.30}
\]

This implies that

\[
E|z^n_t|^p \leq C_p|y_0|^p + C_{p,L,T} \int_0^t E|z^n_s|^pds + C_{p,T} E|f_2(X^n_s, 0)|^p + C_{p,\sigma_2}t^{\frac{p}{2}} \tag{3.31}
\]

\[
\leq C_p|y_0|^p + C_{p,L,T} \int_0^t E|z^n_s|^pds + C_{p,T} E|f_2(X^n_s, 0)|^p + C_{p,\sigma_2,T}
\]

By Gronwall inequality, we have

\[
\sup_{0 \leq t \leq T} E|z^n_t|^p \leq C(1 + |y_0|^p). \tag{3.32}
\]

The following theorem illustrate that the dynamic \( z^n_t \) is exponential er-
godicity with invariant measure $\mu^{X_n}$, which comes from [20, Proposition 3.5]

**Theorem 2.** Under Hypotheses H.1-H.3, there exists a positive constant $C$ such that for each fixed $X_n$ and $F \in C^1_b(\mathbb{R}^n)$, we have

$$\left| \mathbb{E}[F(z^n_t)] - \int_{\mathbb{R}^n} F(y)\mu^{X_n}(dy) \right| \leq C \left( 1 + |y_0| + |X_n| \right) e^{-\beta t}. \quad (3.33)$$

The next lemma establishes the mixing properties of the auxiliary process $z^n_t$.

**Lemma 6.** Under Hypotheses H.1-H.3, for the small enough $\delta t$ and $1 < p \leq \min(\alpha_1, \alpha_2)$, we have

$$\left( \mathbb{E} \left| \frac{1}{M} \sum_{m=1}^{M} f_1(X_n, z^n_m) - \bar{f}_1(X_n) \right|^p \right)^{\frac{1}{p}} \leq 2 \left[ \frac{\ln(M\delta t) + \beta}{M\beta\delta t} + \frac{1}{M} \right]. \quad (3.34)$$

**Proof.** By the property of expectation, we have

$$\left( \mathbb{E} \left| \frac{1}{M} \sum_{m=1}^{M} f_1(X_n, z^n_m) - \bar{f}_1(X_n) \right|^p \right)^{\frac{1}{p}} \leq \left( \mathbb{E} \left| \frac{1}{M} \sum_{m=1}^{M} f_1(X_n, z^n_m) - \bar{f}_1(X_n) \right|^2 \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{M} \left( \sum_{m=1}^{M} \sum_{i=1}^{M} \left( \mathbb{E} \left[ f_1(X_n, z^n_m) - \bar{f}_1(X_n) \right] \cdot \left[ f_1(X_n, z^n_i) - \bar{f}_1(X_n) \right] \right) \right)^{\frac{1}{2}}$$

$$\leq \frac{\sqrt{2}}{M} \left( \sum_{m=1}^{M} \sum_{i=m+1}^{M} \left( \mathbb{E} \left[ f_1(X_n, z^n_m) - \bar{f}_1(X_n) \right] \cdot \left[ f_1(X_n, z^n_i) - \bar{f}_1(X_n) \right] \right) \right)^{\frac{1}{2}}$$

$$+ \frac{1}{M} \left( \sum_{m=1}^{M} \left( \mathbb{E} \left[ f_1(X_n, z^n_m) - \bar{f}_1(X_n) \right] \cdot \left[ f_1(X_n, z^n_m) - \bar{f}_1(X_n) \right] \right) \right)^{\frac{1}{2}}$$

$$:= \frac{\sqrt{2}}{M} \sqrt{J_1} + \frac{1}{M} \sqrt{J_2}. \quad (3.35)$$
For $J_1$, by the Markov property and Hypotheses $\text{H.2}$, we have

\[
\sum_{m=1}^{M} \sum_{l=m+1}^{M} \{ E \left[ f_1(X_n, z^n_m) - \bar{f}_1(X_n) \right] \cdot \left[ f_1(X_n, z^n_l) - \bar{f}_1(X_n) \right] \} \leq \sum_{m=1}^{M} \sum_{l=m+1}^{M} E \left[ f_1(X_n, z^n_m) - \bar{f}_1(X_n) \right] \cdot E_{z^n_m} \left[ f_1(X_n, z^n_l - m) - \bar{f}_1(X_n) \right] \\
\leq \sum_{m=1}^{M} \sum_{l=m+1}^{m+N} e^{-\beta(l-m)\delta t} + \sum_{m=1}^{M} \sum_{l=m+N+1}^{M} e^{-\beta(l-m)\delta t} \\
\leq MN + M^2 e^{-\beta N\delta t}.
\]  

(3.36)

Set $N = \frac{\ln(M\delta t)}{\beta \delta t}$, then we have

\[
\sum_{m=1}^{M} \sum_{l=m+1}^{M} \{ E \left[ f_1(X_n, z^n_m) - \bar{f}_1(X_n) \right] \cdot \left[ f_1(X_n, z^n_l) - \bar{f}_1(X_n) \right] \} \leq \frac{Mln(M\delta t)}{\beta \delta t} + \frac{M}{\delta t}.
\]  

(3.37)

Similarly, for $J_2$, we have

\[
J_2 \leq \sum_{m=1}^{M} e^{-\beta m\delta t} \leq M.
\]  

(3.38)

Combined with (3.37) and (3.38), we have

\[
\left( E \left[ \frac{1}{M} \sum_{m=1}^{M} f_1(X_n, z^n_m) - \bar{f}_1(X_n) \right]^p \right)^{\frac{1}{p}} \leq 2 \sqrt{\frac{\ln(M\delta t) + \beta}{M\beta \delta t}} + \sqrt{\frac{1}{M}}
\]  

(3.39)

\[
\blacksquare
\]

In the following, we will establish the deviation between (3.28) and its numerical approximation (3.15).

**Lemma 7.** Let $z^n_t$ be the family of process defined by (3.28). For small enough $\delta t$ and $1 < p \leq \min(\alpha_1, \alpha_2)$, we have

\[
\max_{0 \leq n \leq \lfloor \frac{T}{\delta t} \rfloor} E \left[ |Y^n_m - z^n_m|^p \right] \leq C(\delta t)^{\frac{p}{2}}
\]  

(3.40)
Proof. Set
\[ Y^n_t = Y^n_0 + \int_0^t f_2(X^n_s, Y^n_{[s/dt]} \delta t) ds + \sigma_2 \tilde{L}^{\alpha, n}_t. \] (3.41)

Then \( Y^n_t \) is the Euler-Maruyama approximation of \( Y^n_m \). Define \( v_t = Y^n_t - z^n_t \), then we have
\[
\mathbb{E}|v_t|^p \leq \int_0^t \mathbb{E}|f_2(X^n_s, Y^n_{[s/dt]} \delta t) - f_2(X^n_s, z^n_s)|^p ds + C_p \int_0^t \mathbb{E}|f_2(X^n_s, Y^n_s) - f_2(X^n_s, z^n_s)|^p ds
\leq C_{p,L,T} \int_0^t \mathbb{E}|Y^n_{[s/dt]} \delta t - Y^n_s|^p ds + C_p \int_0^t \mathbb{E}|v_s|^p ds. (3.42)
\]

By Gronwall’s inequality, we have
\[
\mathbb{E}|v_t|^p \leq C_{p,L,T} \int_0^t \mathbb{E}|Y^n_{[s/dt]} \delta t - Y^n_s|^p ds. (3.43)
\]

Using Lemma 2 and Lemma 3, we have
\[
\mathbb{E}|Y^n_t - Y^n_{[t/dt]} \delta t|^p \leq C_{p,T} \int_{[t/dt]}^{[t]} \mathbb{E}|f_2(X^n_s, Y^n_{[s/dt]} \delta t)|^p ds + C_{p,\sigma_2} \mathbb{E}|L^{\alpha,n}_{t-[t/dt]} \delta t|^p
\leq C_{p,T} \int_{[t/dt]}^{[t]} \mathbb{E}|f_2(X^n_s, Y^n_{[s/dt]} \delta t)|^p ds + C_{p,\sigma_2} |\delta t|^{\frac{p}{\alpha_2}}
\leq C \left( 1 + \mathbb{E}|X^n| + \mathbb{E}|Y^n_{[s/dt]} \delta t|^p \right) ds + C_{p,\sigma_2} |\delta t|^{\frac{p}{\alpha_2}}
\leq C \left( 1 + (\delta t)\frac{p}{\alpha_2} + (\Delta t)\frac{p}{\alpha_1} \right) \delta t + C_{p,\sigma_2} |\delta t|^{\frac{p}{\alpha_2}}
\leq C (\delta t)^{\frac{p}{\alpha_2}}. (3.44)
\]

Take (3.44) into (3.43), we have
\[
\mathbb{E}|Y^n_t - z^n_t|^p \leq C_{p,T} \int_{[t/dt]}^{[t]} \mathbb{E}|Y^n_{[s/dt]} \delta t - z^n_s|^p ds + C_{p,\sigma_2} |\delta t|^{\frac{p}{\alpha_2}}
\leq C (\delta t)^{\frac{p}{\alpha_2}}. (3.45)
\]

Lemma 8. Under Hypotheses H.1-H.3, for all \( 0 \leq n \leq [T/\Delta t] \) and \( 1 < p \leq 15 \).
\[
\min(\alpha_1, \alpha_2), \text{ we have }
\]
\[
\mathbb{E} |A(X_n) - \bar{f}_1(X_n)|^p \leq C \left( (\delta t)^{\frac{\alpha_1}{2}} + \left( \frac{\ln(M\delta t) + \beta}{M\beta \delta t} \right)^{\frac{\alpha_2}{2}} + \left( \frac{1}{M} \right)^{\frac{\alpha_2}{2}} \right). \quad (3.46)
\]

**Proof.** By definition of \( A(X_n) \), Lemma 6 and Lemma 7, we have

\[
\mathbb{E} |A(X_n) - \bar{f}_1(X_n)|^p \leq C_p \mathbb{E} \left| \frac{1}{M} \sum_{m=1}^{M} f_1(X_n, Y_{m}^n) - \frac{1}{M} \sum_{m=1}^{M} f_1(X_n, z_{m}^n) - \bar{f}_1(X_n) \right|^p
\]

\[
\leq C_{p,t} \max_{m \leq M} \mathbb{E} |Y_{m}^n - z_{m}^n|^p + C_p \left( 2 \sqrt{\frac{\ln(M\delta t) + \beta}{M\beta \delta t}} + \sqrt{\frac{1}{M}} \right)^p
\]

\[
\leq C(\delta t)^{\frac{\alpha_1}{2}} + C_p \left( 2 \sqrt{\frac{\ln(M\delta t) + \beta}{M\beta \delta t}} + \sqrt{\frac{1}{M}} \right)^p
\]

\[
\leq C(\delta t)^{\frac{\alpha_1}{2}} + C \left( \frac{\ln(M\delta t) + \beta}{M\beta \delta t} \right)^{\frac{\alpha_2}{2}} + \left( \frac{1}{M} \right)^{\frac{\alpha_2}{2}} + C \left( \frac{\ln(M\delta t) + \beta}{M\beta \delta t} \right)^{\frac{\alpha_2}{2}} + \left( \frac{1}{M} \right)^{\frac{\alpha_2}{2}}.
\]

(3.47)

**Lemma 9.** Under Hypotheses H.1-H.3, the functions \( \bar{f}_1 \) satisfies Lipschitz condition, where

\[
\bar{f}_1(x) = \int_{\mathbb{R}^m} f_1(x, y) \mu_x(dy).
\]

**Proof.** As \( \mu_x \) is ergodic, for any \( h \in \mathbb{R}^n \), \( x_1, x_2 \in \mathbb{R}^n \) and \( t > 0 \), by Hypothesis H.1, we have

\[
\frac{1}{t} |\langle f_1(x_1, Y_{t, x_1}^\varepsilon) - f_1(x_2, Y_{t, x_2}^\varepsilon), h \rangle| \leq \frac{L}{t} \int_0^t \left[ |x_1 - x_2| + |Y_{s,x_1}^\varepsilon - Y_{s,x_2}^\varepsilon| \right] ds \cdot |h|.
\]

Hence, thank to [26, Theorem 1.1], it is immediate to check that for any \( t \in \mathbb{R} \),
we have
\[ \sup_{x,y} |\nabla_x Y_{t,x}^\varepsilon| \leq C_T, \quad \mathbb{P} \text{- a.s.} \tag{3.50} \]

Combined with (3.49) and (3.50), we have
\[ \frac{1}{t} \left| \left( f_1(x_1, Y_{t,x_1}^\varepsilon) - f_1(x_2, Y_{t,x_2}^\varepsilon), h \right) \right| \leq C |h| |x_1 - x_2|. \tag{3.51} \]

Therefore we can conclude that \( \bar{f}_1(x) \) is Lipschitz.

Next we will give the rate of strong convergence for the multiscale scheme.

**Theorem 3.** Under Hypotheses H.1-H.3, for all \( 0 \leq n \leq [T/\Delta t] \) and \( 1 < p \leq \min(\alpha_1, \alpha_2) \), we have
\[ \sup_{0 \leq n \leq [T/\Delta t]} \mathbb{E} \left| X_n - \bar{X}_n \right|^p \leq C \left( (\Delta t)^{\frac{\alpha_2}{2}} + \left( \frac{\ln(M\delta t) + \beta}{M\beta \delta t} \right)^{\frac{p}{2}} + \left( \frac{1}{M} \right)^{\frac{p}{2}} \right). \tag{3.52} \]

**Proof.** Set \( E_n = \mathbb{E} |X_n - \bar{X}_n|^p \), by Lemma 8 and Lemma 9, we have
\begin{align*}
E_n = & \mathbb{E} \left| \sum_{i=0}^{n-1} \left[ A(X_i) - \bar{f}_1(\bar{X}_i) \right] \Delta t \right|^p \\
\leq & C_p \mathbb{E} \left| \sum_{i=0}^{n-1} \left[ A(X_i) - \bar{f}_1(X_i) \right] \Delta t \right|^p + C_p \mathbb{E} \left| \sum_{i=0}^{n-1} \left[ \bar{f}_1(X_i) - \bar{f}_1(\bar{X}_i) \right] \Delta t \right|^p \\
\leq & C_p \max_{0 < i < n} \mathbb{E} |A(X_i) - \bar{f}_1(\bar{X}_i)|^p + C_{p,L} \sum_{i=0}^{n-1} E_i (\Delta t)^p \\
\leq & C(\delta t)^{\frac{\alpha_2}{2}} + C \left( \frac{\ln(M\delta t) + \beta}{M\beta \delta t} \right)^{\frac{p}{2}} + C \left( \frac{1}{M} \right)^{\frac{p}{2}} + C_{p,L} \sum_{i=0}^{n-1} E_i (\Delta t). \tag{3.53} 
\end{align*}

By a discrete version of Gronwall inequality, we have
\[ E_n \leq C \left( (\delta t)^{\frac{\alpha_2}{2}} + \left( \frac{\ln(M\delta t) + \beta}{M\beta \delta t} \right)^{\frac{p}{2}} + \left( \frac{1}{M} \right)^{\frac{p}{2}} \right). \tag{3.54} \]
Therefore we have
\[
\sup_{0 \leq n \leq \lfloor T/\triangle t \rfloor} \mathbb{E} |X_n - \bar{X}_n|^p \leq C \left( (\delta t)^{\frac{\alpha}{2}} + \left( \frac{\ln(M\delta t) + \beta}{M \beta \delta t} \right)^{\frac{\gamma}{2}} + \left( \frac{1}{M} \right)^{\frac{\gamma}{2}} \right)^{\frac{1}{p}}. 
\]
(3.55)

\[
\square
\]

4. Weak convergence analysis of the projective integration scheme

Next we will present the rate of weak convergence for the two-time scale stochastic dynamical systems driven by \(\alpha\)-stable processes, which comes from [20, Theorem 2.3].

**Theorem 4.** Suppose that the assumptions in Theorem 1 holds. Further assume that \(f_1, f_2 \in C_b^{2+\gamma, 2+\gamma}\) with \(\gamma \in (\alpha - 1, 1)\). Then for any \(\phi \in C_b^{2+\gamma}(\mathbb{R}^m)\) and initial value \((x, y) \in \mathbb{R}^n \times \mathbb{R}^m\), we have

\[
\sup_{t \in [0, T]} |\mathbb{E} \phi(X_t^\varepsilon) - \mathbb{E} \phi(\bar{X}_t)| \leq C \varepsilon. 
\]
(4.1)

where \(C\) is a positive constant depending on \(T\), \(\|\phi\|_{C_b^{2+\gamma}}\), \(|x|\) and \(|y|\), and \(\bar{X}_t\) is the solution of the averaged equation (3.6).

Next we will give the rate of weak convergence for the multiscale scheme.

**Theorem 5.** Let \(X_n\) be the Euler approximation for \(X_t\) and \(\bar{X}_n\) be the Euler approximation for \(\bar{X}_t\), then for any \(\phi \in C_b^{2+\gamma}(\mathbb{R}^m)\), we have

\[
\sup_{0 \leq n \leq \lfloor T/\Delta t \rfloor} |\mathbb{E} \phi(X_n) - \mathbb{E} \phi(\bar{X}_n)| \leq C \left( (\delta t)^{\frac{\alpha}{2}} + \left( \frac{\ln(M\delta t) + \beta}{M \beta \delta t} \right)^{\frac{\gamma}{2}} + \left( \frac{1}{M} \right)^{\frac{\gamma}{2}} \right)^{\frac{1}{p}} \Delta t.
\]
(4.2)

**Proof.** For any \(n \leq \lfloor T/\Delta t \rfloor\), we construct the following auxiliary function \(u(k, x_0)\), i.e.,

\[
u(k, x_0) = \begin{cases} 
\phi(x_0), & k = n, \\
\mathbb{E} \left[ u(k+1, x_0 + \bar{f}_1(x_0) \Delta t + \sigma_1 \Delta L^{\alpha_1}) \right], & k < n,
\end{cases}
\]
(4.3)
then we have
\[ u(0, x_0) = \mathbb{E}\phi(\bar{X}_n). \]  
(4.4)

By the smoothness of \( \phi \), it is easy to show that \( \sup_{k,x} \frac{\partial u(k,x)}{\partial x} \) is uniformly bounded. Therefore we have
\[
|\mathbb{E}\phi(X_n) - \mathbb{E}\phi(\bar{X}_n)| \\
= |\mathbb{E}u(n, X_n) - u(0, x_0)| \\
= \left| \mathbb{E} \left( \sum_{l=0}^{n-1} (u(l+1, X_{t+1}) - u(l, X_l)) \right) \right| \\
= \sum_{l=0}^{n-1} \mathbb{E} \left( u(l+1, X_{t+1}) - u(l+1, X_l) - \left( u(l+1, \bar{X}_{t+1}^n) - u(l+1, X_l) \right) \right) \\
\leq \sum_{l=0}^{n-1} \mathbb{E} \left\{ \sup_{l,x} \left| \frac{\partial u}{\partial x} \right| \left( X_{t+1}^l - X_l - (\bar{X}_{t+1}^l - X_l) \right) \right\} \\
\leq C \sum_{l=0}^{n-1} \Delta t |X_l - \bar{f}_1(X_l)|. 
\]  
(4.5)

By Lemma 8, we have
\[
\sup_{0 \leq n \leq \lfloor T/\Delta t \rfloor} |\mathbb{E}\phi(X_n) - \mathbb{E}\phi(\bar{X}_n)| \leq C \left( (\delta t)^{\frac{\beta}{2}} + \left( \frac{\ln(M\delta t) + \beta}{M\delta t} \right)^{\frac{\gamma}{2}} + \left( \frac{1}{M} \right)^{\frac{\gamma}{2}} \right) \Delta t. 
\]  
(4.6)

5. Numerical Experiment

**Example 1.** Consider the following slow-fast stochastic dynamical systems
\[
\begin{align*}
\left\{ 
\begin{array}{ll}
\frac{dX_t}{\varepsilon} & = -X_t + \sin(X_t)e^{-(Y_t)^2} dt + dL_{t}^{n_1}, \\
\frac{dY_t}{\varepsilon} & = -\frac{Y_t}{\varepsilon} dt + \frac{1}{\varepsilon^n} dL_{t}^{n_2}.
\end{array}
\right.
\end{align*}
\]  
(5.1)

where \( f_1(x, y) = -x + \sin xe^{-y^2} \), \( f_2(x, y) = -y \) and \( \sigma_1 = \sigma_2 = 1 \). It is easy to justify that \( f_1, f_2 \) satisfy Hypotheses H.1-H.3. Using a result in [24], we find
the invariant measure $\mu(dx) = \rho(x)dx$ with density

$$\rho(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} e^{-\frac{1}{4\varepsilon^2} |\xi|^2} d\xi.$$  \hspace{1cm} (5.2)

Then the effective equation for $X_\varepsilon^t$ is

$$d\bar{X}(t) = -\bar{X}(t) + \bar{a} \sin(\bar{X}(t)) dt,$$  \hspace{1cm} (5.3)

where

$$\bar{a} = \frac{1}{\sqrt{4\pi}} \int_{\mathbb{R}} e^{-\frac{1}{4\varepsilon^2} - \frac{1}{\alpha} |\xi|^\alpha} d\xi.$$  

The numerical study is performed for this method. In Fig. 1, we plot the original slow-fast systems \([5.1]\) for $\alpha_1 = \alpha_2 = 1.5$. In Fig. 2 and Fig. 4, we compare the original slow sample paths $X_\varepsilon^t$ with the projective integration scheme \((3.8)\) with $\varepsilon = 0.1$ and $\varepsilon = 0.01$, respectively. Here we take the average of 100 sample paths with $\varepsilon = 0.1$, $M = 100$, $N = 1000$, the time step $\Delta t = 0.001$, $\delta t = \Delta t/M$ and initial value $x_0 = 10, y_0 = 10$. To verify the strong convergence for the multiscale scheme, we compute the $L^p$ error between $X_n$ and $\bar{X}_n$ with $p = 1.2$ in Fig. 3 and 5 where $L^p$ error $= \sum_{k=1}^l |X_n^k - \bar{X}_n^k|^p / l$. As seen in Fig. 3 and 5, it is clear that if the larger $\varepsilon$ is, the larger the error between
Figure 2: Compare the original slow sample paths $X^\varepsilon$ with the projective integration scheme for $\varepsilon = 0.1$.

Figure 3: The $L^p$ error between $X_n$ and $\bar{X}_n$ for $\varepsilon = 0.1$. 
Figure 4: Compare the original slow sample paths $X^\varepsilon_t$ with the projective integration scheme \( (3.8) \) for $\varepsilon = 0.01$.

Figure 5: The $L^p$ error between $X_n$ and $\bar{X}_n$ for $\varepsilon = 0.01$. 
the results of the projective integration method and the slow components of the original system is.

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