Fully Unintegrated Parton Correlation Functions and Factorization in Lowest Order

Hard Scattering

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(Dated: May 1, 2008)

Motivated by the need to correct the potentially large kinematic errors in approximations used in the standard formulation of perturbative QCD, we reformulate deeply inelastic lepton-proton scattering in terms of gauge invariant, universal parton correlation functions which depend on all components of parton four-momentum. Currently, different hard QCD processes are described by very different perturbative formalisms, each relying on its own set of kinematical approximations. In this paper we show how to set up formalism that avoids approximations on final-state momenta, and thus has a very general domain of applicability. The use of exact kinematics introduces a number of significant conceptual shifts already at leading order, and tightly constrains the formalism. We show how to define parton correlation functions that generalize the concepts of parton density, fragmentation function, and soft factor. After setting up a general subtraction formalism, we obtain a factorization theorem. To avoid complications with Ward identities the full derivation is restricted to Abelian gauge theories; even so the resulting structure is highly suggestive of a similar treatment for non-Abelian gauge theories.

Keywords: QCD, factorization

I. INTRODUCTION

The standard leading twist formalism for calculating deeply inelastic lepton-proton scattering (DIS) cross sections provides a foundation for understanding parton distribution functions (PDFs) and perturbative Quantum Chromodynamics (pQCD) in general. At issue in the present paper are the various approximations to parton kinematics that are fundamental to the standard approach, as we will review in Sec. II. These approximations are appropriate for very inclusive cross sections, and they result in a number of appealing conceptual and practical simplifications. The resulting factorization theorems involve standard (fully integrated) PDFs that have rigorous definitions and depend only on the longitudinal component of the parton momentum.

But when the true final states are studied in more detail, problems arise. Although various resummation methods are used to overcome the problems, it has become increasingly clear that it is the standard kinematic approximations that should be questioned. The problems arise because the approximations change momenta of particles in the final state, typically resulting in a final state that does not obey 4-momentum conservation. For a fully inclusive cross section, this is not a critical issue. Furthermore, if one is only concerned with obtaining a leading-logarithm approximation (improved only by the use of an appropriate running coupling), it is legitimate to be imprecise about the details of the kinematic approximations.

But to be able to make systematic improvements by including higher order corrections to the hard scattering, and to the showering and evolution kernels, etc, a precise formulation of the approximations is needed. Of course, one must expect approximations to be needed if one is to obtain tractable factorization results. But for the formalism to be generally applicable, it is necessary to define the approximations in such a way that final-state momenta are unaltered. Attempts to remedy the situation are now stymied by the use of conventional “integrated parton densities”; there is a mismatch between the definitions of integrated parton densities and the imperatives of factorization with correct final-state kinematics. The mismatch extends equally to fragmentation functions. As explained in [3, 4, 5, 6], when we use methods that treat final-state kinematics exactly, we are led to the replacement of conventional parton densities by more general quantities. The range of methods, from standard ones with approximated final-state kinematics to the improved ones that are the subject of this paper, can be characterized by the kind of parton densities, or generalization, that are used:

- **Conventional integrated parton densities:** These are the usual PDFs; they depend only on a longitudinal momentum fraction variable, $x$, and the hard scale, $Q^2$. All other components of parton momentum are integrated over. Correspondingly, the intrinsic external parton transverse momentum $k_t$ and virtuality are neglected in the hard scattering. These parton densities have consistent operator definitions used in the classic proofs of factorization.

- **Unintegrated parton densities:** These are the $k_t$-dependent PDFs obtained when the integral over parton transverse momentum is left undone, while the minus component (or virtuality) is still integrated over. The concept of an unintegrated
parton density appeared quite early \[7\], necessary for the treatment of the transverse momentum distribution in the Drell-Yan process. These quantities are also called transverse-momentum-dependent (TMD) parton densities. There are also TMD fragmentation functions.

Unintegrated parton distribution functions also appear naturally in the high energy limit of QCD \[8\] where one of the key features is the lack of the transverse momentum ordering of subsequent gluon emissions. Small-x resummation then provides an evolution equation in a rapidity variable for this unintegrated gluon distribution function. In this approach the integrated gluon density is defined simply as an integral over the transverse momentum up to a hard scale. However, the definitions of TMD densities that we refer to in this work may not, in general, agree with other definitions appearing in the literature.

As for a definition, the obvious and natural one is given by the hadron expectation value of the parton number operator in light-front quantization, equivalent to a standard simple expression as an expectation value of a bilocal field operator, in light-front gauge. However, this definition suffers \[2, 9\] from where we will call rapidity divergences, from where the rapidity of internal gluons goes to infinity. These divergences occur even when ultraviolet and infra-red divergences are cut off, as reviewed in \[10\].

So a correct definition requires an explicit cutoff for the rapidity divergences. Collins and Soper \[2, 4, 11\] used a non-light-like axial gauge for this purpose. Certain improvements are probably necessary, as we will see. Other work suffers from an imprecision in the definitions. For example, in \[12, 13\] we do not see the operator definitions at all, and the issues concerning rapidity divergences are completely hidden, probably in some application of a leading logarithm approximation, with cutoffs on both real and virtual lines in Feynman graphs. However, it is important to understand the differences in the definitions of TMDs within different theoretical treatments. For example, in a recent analysis of dijet correlations at the Relativistic Heavy Ion Collider (RHIC) \[14\], it is shown that there is a large variation between the predictions obtained using different TMDs from the literature. Therefore it is necessary to construct a precise and unique operator definition of the unintegrated parton distribution functions. Then, hopefully, the different cases discussed above (for example those which emerge in the context of BFKL \[8\] or angular ordered \[13, 14\] evolutions) will follow naturally as particular limits of the general definition, or as changes of factorization scheme.

For the concept of a parton density to be applicable in the real world of hadron physics, it needs to include all the relevant non-perturbative phenomena concerning the actual state of the parent hadron. In particular, the need for TMD densities arises when we treat final states in enough detail to be sensitive to the transverse momenta of partons relative to their parent hadrons, etc.

### Parton Correlation Functions

An even closer examination of final-state kinematics, as was done for small-x physics by Watt, Martin, and Ryskin \[3, 4\], indicates that “doubly unintegrated” PDFs (differential in all components of parton momentum) are more appropriate in many situations. These can also be called “fully unintegrated parton densities”, but in this paper we will simply call them “parton correlation functions” (PCFs). Collins and Jung \[5\] showed very generally that for differential distributions in final-states the use of parton correlation functions is necessary. Collins and Zu \[6\] set up a complete formalism in a model field theory, suitable for Monte-Carlo event generator implementation, with the possibility of incorporating arbitrarily non-leading-order corrections.

Under the label of parton correlation function we will include both the fully unintegrated generalization of parton densities, and related objects for fragmentation, and for soft factors in factorization theorems.

The need in the general case for making no approximations on final-state kinematics impels us to a formalism that uses PCFs. Unfortunately, a complete treatment and derivation of factorization using PCFs does not yet exist for QCD. Therefore, our aim in this paper is to initiate the construction of such a formalism. The new formalism should handle non-leading-order corrections as generally as the standard formalism. Thus it will be as good, if not better, in cases where more traditional approaches are applicable. But it will also apply to more general situations.

To treat kinematics correctly, we need to go back to the foundations, and will see that the basic structure of the derivation in fact needs to be significantly modified even at the lowest-order, parton-model level for DIS. Normally, one starts with the handbag diagram of Fig. \[1\] (a), where the outgoing struck quark is exactly massless and on-shell. But even without any sophisticated treatment of QCD effects, we know that the quark must turn into a jet. So a minimum logical foundation starts from Fig. \[1\] (b), where the outgoing quark line now has acquired a jet subgraph. In fact, examining the kinematics of such graphs gives the motivation \[3, 4, 10\] for searching for methods that use PCFs.

However, in real QCD, this is not sufficient. There must certainly be non-perturbative interactions at late
times to neutralize the color of the outgoing quark. Furthermore, even in perturbation theory, and even without going to the conventional domain for NLO corrections where extra high transverse momentum jets are produced, it is necessary to consider all regions of the form symbolized by the graphs of Fig. 1(c). There are arbitrarily many collinear gluons exchanged between the jet and spectator subgraphs and the hard scattering, and there are arbitrarily many soft (interacting) gluons connecting the jet and spectator subgraphs. We do not have topological factorization \[17\] of the subgraphs. Only after an application of Ward identities to give a coherent sum over different graphs does one obtain factorization for the cross section. A consequence is that, to obtain a correct factorization formula, one needs to make a careful selection of an appropriate gauge-invariant definition of the parton correlation functions. So the key issues center around what such a definition is and why.

These issues are in fact present in conventional factorization, although most treatments do not emphasize them. The reasons why we now need to examine them in much closer and exact detail are coupled to our current aims of treating final states less inclusively. The Ward identities used to derive factorization apply only after the application of certain approximations. In the most obvious way of setting them up, the approximations are only valid in regions of appropriately strongly ordered kinematics. Moreover, there is a general tendency in constructing derivations to neglect power-suppressed corrections whenever it assists the ease of derivation. The Ward identity arguments are therefore typically applied to amplitudes where cutoffs are imposed on the momenta of internal lines of the graphs. But the Ward identities are only exact when applied to amplitudes in the exact theory, i.e., when no cutoffs whatever are applied to the internal momenta other than, possibly, a conventional gauge invariant cutoff for UV divergences.

There is therefore a conflict between the cutoffs used in discussing why factorization occurs and the lack of cutoffs needed for the validity of the Ward identities. Once we go beyond the leading logarithm approximation we cannot restrict the kinematics to strongly ordered regions. A careful and precise formulation of approximations and Ward identities is needed, otherwise there are likely to be present uncontrolled correction terms that can readily violate factorization. A subtraction formalism, such as we will use, avoids the problems with direct cutoffs on the momenta of lines of Feynman graphs. Examination of the literature — e.g., \[17, 18\] — shows that the Ward identities for proving factorization are typically stated and proved too inexplicitly for our purposes.

We stress that the treatment in this paper is limited to
lowest order in the hard scattering, where the final states are rather simple. What we discuss in this paper may be thought of as a formal generalization of the parton model to cases where the details of over-all kinematics are important. This is a critical step toward a complete formalism because higher order calculations rely on the subtraction formalism to avoid double counting the zeroth order contribution. Knowing exactly what is subtracted requires a complete understanding of the zeroth order contribution.

If it turned out that factorization were universally true, these issues would not be pressing. However, recent work indicates otherwise. For example, it has been shown [19, 20, 21, 22, 23] that factorization fails for production of hadrons of high transverse momentum in hadron-hadron collisions. This is in the case that the hadrons are close to back-to-back azimuthally, when unintegrated parton densities are an appropriate tool. See also the recent paper by Bauer and Tackmann [24] for a closely related argument.

In other situations, model calculations can appear to show that factorization fails, but the culprit is an inappropriate definition of parton densities. An example is the transverse single-spin-asymmetry (SSA) in semi-inclusive deep-inelastic scattering (SIDIS), which Brodsky, Hwang and Schmidt [25] found to be non-factorizing. Collins [26] showed that factorization does actually hold, but only if suitable Wilson line operators are used in the definition of the unintegrated parton densities.

These motivations lead us to the detailed treatment in the present paper. The issue that makes the work of Collins and Zu [6] insufficient is the presence of extra gluonic connections between the subgraphs of Fig. 1(c). Further difficulties are associated with the masslessness of the gluon in QCD, and with the complicated nature of Ward identities in a non-Abelian gauge theory. It is useful to solve the difficulties one-by-one. So the full technical results in this paper are restricted to the case of an Abelian gauge theory with a massive gluon. However, much of our treatment applies more generally and includes QCD. Furthermore we will obtain a factorization stated in terms of PCFs with precise definitions as certain operator matrix elements. The statement of factorization is equally applicable to QCD.

In Sec. II we will deconstruct the standard parton model/handbag diagram in the context of pQCD. Then we will discuss in Sec. III what requirements are needed for a formalism that gives a good treatment of parton kinematics. In Sec. IV we discuss the projection operators which enable us to make approximations suitable for obtaining the standard parton model. In Sec. V we list a set of candidate definitions for the PCFs, and discuss the reasoning for our choices. Sec. VI is devoted to a subtraction procedure and in Sec. VII we discuss the kinematic approximations and the rapidity differences. In Secs. VIII and IX we present calculations to support the consistency of the structure outlined in Sec. III. Finally in Sec. X we present all-orders proofs of the the factorization formula with the PCFs, including the necessary Ward identities, but at this point restricted to an Abelian theory.

II. THE BREAKDOWN OF STANDARD KINEMATICAL APPROXIMATIONS

A. The parton-model approximation

In this section we carefully analyze the derivation of the parton-model approximation for DIS, paying special attention to the analysis of parton kinematics.

Although Fig. 1(a), with its single on-shell final-state struck quark, is the usual starting point, we actually need to start with Fig. 1(b), where the quark fragments into a group of final-state particles. We will show in what sense the sum over all graphs of the form of Fig. 1(b) is approximated by Fig. 1(a), i.e., we will analyze how it is possible to neglect all higher-order corrections to the final-state bubble $\mathcal{F}(k + q)$. For this part of the discussion, we will restrict our attention to graphs without the extra gluonic attachments shown in Fig. 1(c).

In Fig. 1(b), $q$ is the incoming virtual photon momentum and $P$ is the momentum of the target (proton). As usual, $Q^2 = -q^2$ is the photon virtuality and $x_{ Bj} \equiv Q^2/(2P \cdot q)$ is the Bjorken scaling variable. The incoming momenta may be expressed as

$$P = \left( P^+, \frac{M_p^2}{2P^+}, 0_\perp \right), \quad q = \left( -xP^+, \frac{Q^2}{2xP^+}, 0_\perp \right).$$

Here, to provide a simple formula we use the Nachtmann variable $x$ instead of the Bjorken variable. Since we want to start with no approximation on kinematics, we note the exact relation between the two variables:

$$x = \frac{2x_{ Bj}}{1 + \sqrt{1 + 4\frac{M_p^2}{Q^2}x_{ Bj}^2}},$$

they are equal up to a power-suppressed correction, as $Q \to \infty$.

The contribution to the hadronic tensor from Fig. 1(b) is

$$W^{\mu\nu}(q, P) = \sum_j \frac{e_j^2}{4\pi} \int \frac{d^4k}{(2\pi)^4} \text{Tr}[\gamma^\mu \mathcal{F}(k + q) \gamma^\nu \Phi(k, P)].$$

Here, the sum is over quark flavors, and $e_j$ is the electric charge of quark $j$ in units of the size of the electron’s charge. We will leave implicit the dependence on the target and jet factors $\mathcal{F}$ and $\Phi$. We will sometimes write the outgoing quark momentum as $l = k + q$. The kinematic constraints, that the final states in $\mathcal{F}(k + q)$ and $\Phi(k, P)$ have positive energy and positive invariant mass, impose limits on the values of $k$ where the integrand is non-vanishing.
To obtain the standard LO DIS expression, we apply a number of approximations valid at the leading power of $Q$. First, we expand both the upper bubble, $\mathcal{J}(k+q)$, and the lower bubble, $\Phi(k, P)$, in a basis of Dirac matrices:

$$\Phi(k, P) = \Phi_S + \gamma^\mu \Phi_\mu + \sigma^{\mu\nu} \Phi_{\mu
u} + \gamma_5 \Phi_5 + \gamma^\mu \gamma_5 \Phi_{5\mu},$$

$$\mathcal{J}(l) = \mathcal{J}_S + \gamma^\mu \mathcal{J}_\mu + \sigma^{\mu\nu} \mathcal{J}_{\mu\nu} + \gamma_5 \mathcal{J}_5 + \gamma^\mu \gamma_5 \mathcal{J}_{5\mu}.$$  

(4)

Then we observe that in the Breit frame, the trace in Eq. (3) is dominated by terms containing $\gamma^-$ in $\Phi$ and $\gamma^+$ in $\mathcal{J}$, i.e., $\gamma^- \Phi^+$ and $\gamma^+ \mathcal{J}^-$, together with terms only relevant for polarized scattering. These terms dominate because the coefficients $\Phi^+$ and $\mathcal{J}^-$ are the large ones after a boost from the rest frames of the final state of each of the bubbles. Therefore, up to power-suppressed corrections,

$$\text{Tr}[\gamma^\mu \mathcal{J}(k+q)\gamma^\nu \Phi(k, P)] \simeq \text{Tr}[\gamma^\mu \gamma^+ \gamma^\nu \gamma^- \mathcal{J}^-(k+q) \Phi^+(k, P)].$$  

(5)

Next, we focus attention on values of the quark momentum where parton model kinematics are good. First, for each parton line, we define corresponding massless momenta:

$$\hat{k} \equiv (x_{Bj} P^+, 0, 0_t) \quad \hat{l} \equiv \left(0, \frac{Q^2}{2x_{Bj} P^+}, 0_t\right).$$  

(6)

These are the parton momenta that normally appear in LO calculations. The parton-model region is where the transverse and minus components of $k$ are small relative to $Q$, and where $k$ and $l$ both have small virtualities relative to $Q^2$. Then we can treat $\hat{k}$ and $\hat{l}$ as good approximations to $k$ and $l$:

$$k \simeq \hat{k}, \quad l \simeq \hat{l},$$  

(7)

with the errors being small compared with the large components of $k$ and $l$. (For example the transverse momentum might be of order $M$, compared with order $Q$ for the large components.)

The standard parton-model approximation is obtained by neglecting the small momentum components, $k^-$, $k_t$, $l^+$, $l_t$, i.e., by replacing $k$ and $l$ by $\hat{k}$ and $\hat{l}$. However, the replacement is only applicable in the hard scattering, where we can neglect the small momentum components with respect to $Q$. It is incorrect to replace $k$ by $\hat{k}$ in $\Phi(k, P)$ and $l$ by $\hat{l}$ in $\mathcal{J}(l)$ because internal virtualities in $\Phi$ and $\mathcal{J}$ may be small. But it is valid to replace $k^+$ by the fixed value $x_{Bj} P^+$ inside the lower bubble, and to replace $k^- + q^-$ by $q^-$ inside the upper bubble. These give only small fractional shifts in the large components $k^+$ and $k^- + q^-$, and they give small fractional shifts in the lines’ virtualities. Furthermore, we can perform a small Lorentz transformation to set to zero the transverse momentum entering $\mathcal{J}(k+q)$. After this we change variables for the $k$ integral to $k^- - k_t$, $k_t$ and $l^+$ to obtain a factorized approximation to $W^{\mu\nu}(q, P)$:

$$T_{PM} W^{\mu\nu}(q, P) = \sum_J \frac{e_J^2}{4\pi} \left[ \int \frac{dk^- d^2k_t}{(2\pi)^4} \Phi^+(x_{Bj} P^+, k^-, k_t; P) \right] \left[ \int dl^+ \text{Tr}(\gamma^\mu \gamma^+ \gamma^\nu \gamma^- \mathcal{J}^-(l^+, q^-, 0_t)) \right].$$  

(8)

The symbol $T_{PM}$, which we call the “parton model approximator”, represents the operation of replacing the integrand in Eq. (3) by the integrand in Eq. (8).

At this stage, we should emphasize a distinction important for a more detailed treatment of final states. While the approximations, Eqs. (7), are clearly good in the hard scattering calculation if $k^-$ and $k_t$ are small, the shift in integration variables needed to get Eq. (8) introduces errors in the evaluation of $\Phi(k, P)$ and $\mathcal{J}(l)$ that need to be examined more carefully. Within the parton-model region of collinear quark momentum, the integrand in Eq. (8) is a good numerical approximation to the original integrand, if it is a smooth enough function. However, because it involves replacing final-state momenta by somewhat different momenta, the approximation will change certain kinds of cross sections differentially in the final state.

Even when we only treat inclusive cross sections, the changes in the kinematics affect the positions of thresholds. Indeed, the approximated nonperturbative factors $\Phi^+(x_{Bj} P^+, k^-, k_t; P)$ and $\mathcal{J}^-(l^+, q^-, 0_t)$ no longer restrict $k$ to the actual kinematically allowed values of the original unapproximated integral. Therefore, the approximations leading to Eq. (8) can lead to unphysical results [27], particularly if one is interested in the details of the final state.

For purely inclusive DIS, the usual formalism includes higher order corrections that provide extra large-transverse-momentum jets, with Eq. (8) corresponding to the first term in a perturbative expansion of the hard scattering. Higher-order terms in the hard scattering include terms that can compensate for kinematic approximations that are particularly bad at large $k_t$.

**B. Parton density and Wilson lines**

We now use the approximation Eq. (8) to explain a definition of a parton density. First, we recognize that
The overall numerical factor of $1/2$ is the standard convention; it ensures that the PDF has exactly the normalization of a number density, at least in field theories where light-front quantization is non-problematic.

Now, the integral on the right-hand side is UV-divergent in a renormalizable theory like QCD so, as it stands, Eq. (10) is ill-defined and needs to be replaced by something else. The divergence comes from regions of the integral where $k^-$ and $k_i$ are large, i.e., from values of $k$ that are far from parton kinematics. This is the domain where higher-order corrections to the hard scattering are important, so it is appropriate to modify the definition while preserving its treatment of the parton-model region.

One possibility is to place some sort of UV cutoff on the integral in Eq. (10) near the hard scale, $k^-,k_i \lesssim Q$ [28]. While physically plausible, such a definition has problems when one tries to make it gauge invariant — the same problems that we will have to solve in our improved treatment with parton correlation functions.

The solution that is in fact used for normal QCD factorization, and that corresponds exactly to what is done with the operator product expansion for DIS, is to apply UV renormalization to the bilocal operator. With the insertion of appropriate Wilson line operators, which give gauge-invariance, we get the usual definition [2]

$$f_j(x_B, \mu) = \int \frac{dw^-}{4\pi} e^{-ikx_B} P^w \gamma^+ \psi(0, w^-) V^r_w(u_j) V^i_0(u_j) \psi(0)|p\rangle_R.$$ (11)

Here, $\psi(w)$ is the field operator for quark $j$, and $|p\rangle_R$ is the proton state vector. The subscript, $R$, indicates that the operator is renormalized using ordinary UV-renormalization techniques. This definition reproduces the basic structure of the integral in Eq. (10), but renormalization removes the UV divergence with a renormalization scale, $\mu$. However, a derivation of factorization must allow for graphs with extra gluon exchanges, as in Fig. (1c). It is known that therefore in the operator defining the parton density, there must be inserted a path-ordered exponential of the gluon field along the light-like direction joining the quark and antiquark fields, as in Eq. (11). This also makes the definition gauge-invariant.

Deriving an appropriate generalization for a parton correlation function, where the separation of the two quark fields is no longer light-like, will be an important part of the present paper.

We will find that we need a Wilson line that goes out to infinity from the origin along one line, not necessarily light-like, and returns along a nearby line to a point $w$. So as a general notation we define $V_w(n)$ to be a Wilson line from $w$ to infinity in the direction, $n$: $$V_w(n) = P \exp \left( -ig \int_0^\infty d\lambda n \cdot A(w + \lambda n) \right).$$ (12)

Here, the symbol $P$ is a path-ordering operator.

In Eq. (11), we use a light-like direction, i.e., we replace $n$ by the vector $u_j = (0, 1, 0_i)$; the separation of the fields, $w$, is in the same direction. Thus in the combination $V_w(u_j) V_0(u_j)$ the segments between $w$ and $\infty$ cancel, so that

$$V_w(u_j) V_0(u_j) = P \exp \left( -ig \int_0^w d\lambda u_j \cdot A(\lambda u_j) \right).$$ (13)

Thus the Wilson line is simply along the straight line joining the quark and antiquark fields.

If we set $\mu \sim Q$, the hard scattering can, as is well known, be usefully calculated as a power series in $\alpha_s(Q)$, which is small because of the asymptotic freedom of QCD. (However, the simple use of the perturbation expansion breaks down at small and large $x$.)

C. Quark fragmentation factor

The last factor in Eq. (8) is an integral over a cut propagator. In the absence of UV problems a simple unitary argument shows that the integral over all values of $l^+$ is equal to the value obtained by integrating just the lowest-order term in the cut propagator. Equivalently, we may replace the final-state jet bubble as follows,

$$\int dl^+ Tr(\gamma^+ \gamma^-) \mathcal{J}^-(\tilde{l})$$

$$\leftrightarrow \int dl^+ Tr(\gamma^+ \frac{l^-}{2k^+} \gamma^-) 2\pi \delta_+(l^2)$$

$$= \frac{2\pi}{l^+ k^+} Tr(\gamma^+ \frac{l^-}{2k^+} \gamma^-).$$ (14)

Here the approximated momentum in $\mathcal{J}$ is $\tilde{l} = (l^+, q^- - \Omega_i)$, and we have normalized the trace so that it corresponds to the trace in the calculation of a partonic cross section. Graphically, (14) corresponds to replacing Fig. (1b), where the full final-state jet is included, by Fig. (1a), with the lowest order quark propagator. It is important to recognize that two assumptions are necessary for this identification to be justified — that it is valid to use the approximate momentum variable, $\tilde{l}$, in the upper bubble, and that it is valid to allow the integrals in Eq. (8) to be unconstrained by kinematical requirements. These assumptions go beyond the use of on-shell parton kinematics in the hard scattering.
There is a delta-function that forces $\hat{l}^2 = 0$ and hence $l^+ = 0$. We use $\hat{l} = (0, q^{-}, 0)$ to denote the resulting light-like momentum, and then we have

$$T_{PM}W^{\mu\nu}(P, q) = \sum_j \frac{e_j^2}{8k^+l^-} f_j(x_{BJ}, \mu) \text{Tr} \left( \gamma^\mu \gamma^\nu \hat{k} \right).$$

(15)

Projected out the $F_2(x_{BJ}, Q^2)$ component produces the familiar expression

$$F_2(x_{BJ}, Q^2) \approx \sum_j e_j^2 x_{BJ} f_j(x_{BJ}).$$

(16)

The diagrammatic representation of Eq. (15) is the familiar handbag diagram in Fig. 1(a). In fact, Fig. 1(a) is the typical starting point for most pedagogical introductions to a pQCD treatment of DIS (e.g., [29]), although we see now that the justification for using Fig. 1(a) involves a number of non-trivial steps.

For our further work it is important to emphasize the distinction between the approximation that one restricts attention to the generalized handbag formula, Eq. (3), and the set of approximations that lead from it to the standard parton-model formula Eq. (15). In writing down Eq. (3), the only approximation is to restrict to certain topologies of graph, whereas to reproduce Eq. (15), we made several very non-trivial kinematic approximations [49]. It is these later approximations that we will find we need to avoid.

![FIG. 2: The amplitude for $\gamma^* p$ scattering into two jets with fixed masses.](image)

![FIG. 3: An event in which the collinear, on-shell matrix element — the photon-quark vertex in this example — is accompanied by initial and final state showers. (In the showers, solid lines denote both quarks and gluons.)](image)

**D. The Limits of Standard Kinematical Approximations**

The approximations at issue change the momenta of final state particles. This can give problems whenever cross sections are investigated that are differential in final-state jets, for example by producing final states that violate energy-momentum conservation [27]. In particular, at large $x$, the true parton kinematics are strongly restricted, whereas Eq. (8) has these restrictions removed.

To see this more explicitly, consider a particular final state consisting of two outgoing jets, Fig. 2 of momenta $l_1$ and $l_2$. Let the invariant masses of the struck quark jet and the spectator jet be $M_J$ and $M_X$. The Mandelstam variable, $s$, is

$$s = (1 - x) M_p^2 + \frac{Q^2}{x} (1 - x).$$

(17)

In the center-of-mass frame, the 3-momenta of $p - k$ and $l$ are equal and opposite, so that the internal parton transverse momentum obeys $k_t^2 = l_t^2 = k_{t,X}^2$. Thus

$$s = (t^0 + k_X^0)^2 = \left( \sqrt{M_J^2 + t^2} + \sqrt{M_X^2 + k_t^2 + k_{t,X}^2} \right)^2.$$  

(18)

Since $M_J^2, M_X^2, t^2, k_{t,X}^2 > 0$, we have $4k_t^2 < s$. Thus, from Eq. (17) we get a strict upper limit on the kinematically allowed values of $k_t^2$,

$$k_t^2 < \frac{(1 - x)}{4} M_p^2 + \frac{Q^2}{4x} (1 - x).$$

(19)

When $x$ is close to one, this limit is much less than $Q^2$. This is in gross contradiction to Eq. (8) where the integral over $k$ is unrestricted. Even if we apply renormalization at a scale $\mu \sim Q$, this implies an effective cutoff of order $Q$, far above the actual kinematic limit. However, it is not sufficient to set $\mu \sim Q \sqrt{1 - x}$, since corrections to the photon vertex still have an external virtuality $Q^2$, for which the scale $\mu \sim Q$ is appropriate. There is a mismatch of scales. Therefore, we have an example where Eq. (8) is inappropriate even for a totally inclusive process.

There are further problems with using the framework of standard collinear factorization for observables that are differential in final state kinematics. This is illustrated in Fig. 3 where the photon-quark vertex is now
accompanied by initial and final state showers. This situation is appropriate not only for the discussion of jet cross sections, but also for the theory of Monte-Carlo event generators (MCEGs). Then the mass of the outgoing jet is given by

\[ m_J^2 = (k + q)^2 = 2(k^+ - xP^+) \left( k^- + \frac{Q^2}{2xP^+} \right) - k_t^2, \]

so that

\[ k^+ = xP^+ + \frac{m_J^2 + k_t^2}{2(q^- + k^-)}, \]

which is strictly greater than \( xP^+ \). (Note that \( k^- \) is always negative.) This shows that away from collinear kinematics, there is a substantial inconsistency between the value of the longitudinal momentum used to evaluate the parton density, namely \( k^+ = xP^+ \approx x \Phi P^+ \) and the correct value of \( k^+ \). Since we must allow the transverse momentum and the parton virtualities to range up to large values, this represents a substantial shift in \( k^+ \).

The value of \( k^+ \) depends on both target-related and jet-related variables, so particularly difficult problems arise in constructing a systematic treatment of higher order corrections in a factorization framework with conventional PDFs, as explained by Collins and Zu [6]. Different numerical values for the same quantity are used at different places in the formalism.

The important conclusion of this section is that the steps that allow one to replace Fig. 1(b) with Fig. 1(a) introduce possibly large errors in certain types of calculation. Since the kinematical approximations that allow us to replace Eq. (3) by Eq. (8) are what normally allow us to replace the final state jet bubble in Fig. 1(b) with the on-shell massless parton in Fig. 1(a), then a unified treatment must improve the approximations to avoid changing momenta in the final state.

III. WHAT IS NEEDED

In the last section, we argued that some of the approximations that lead to the parton-model formula need to be avoided, because they produce large kinematical errors that affect the measured final state. Furthermore, from the analysis of Libby and Sterman [31], we know that for QCD the correct starting point is the sum of regions represented by Fig. (1c), and not just graphs (a) or even (b). In order to get a factorization formula and be able to perform perturbative calculations, we need to rewrite Fig. (1c) in a useful approximation as the product of a hard part that can be calculated directly with ordinary Feynman graphs for on-shell external partons, and a collection of universal parton correlation functions to describe the non-perturbative physics. Therefore, what is needed is a set of approximations and Ward identities that reduce Fig. (1c) to a factorized form, but without the problems with kinematics that we have just discussed.

Each region of the form shown in Fig. (1c) has a set of lines collinear to the target, subgraph \( \Phi \), a set collinear to the outgoing quark, subgraph \( J \), a set of soft lines, \( B \), and two hard subgraphs \( H_L \) and \( H_R \) (on the left and right of the final-state cut). (We use the symbol \( B \) for the soft bubble in a general graph. The symbol \( S \) is reserved for the soft PCF whose definition will arise when we discuss factorization.) For each region, we will define an approximator, in the same spirit as Eq. (8). Our aim is to find definitions of approximators that simplify as much as possible the systematic application of Ward identities without uncontrolled remainder terms.

A. Requirements on approximators

On the basis of the observations in the previous sections, we propose that the approximators should obey the following:

1. The kinematics of the initial and final states must be kept exact. Otherwise, large errors occur in certain types of calculation.

2. The bubbles representing the sums over physical final states must be kept explicit. For example, we must take Fig. (1b) rather than Fig. (1a) as the starting point of the derivation of the parton model. It can be argued that the integral over final-state bubbles such as \( J \) is unity, as in Eq. (13). But this involves a cancellation between final states of different invariant masses, and this violates the first requirement.

3. To avoid making kinematical approximations in the initial and final states, the non-perturbative factors need to be functions of all components of parton four-momentum. Hence all of the non-perturbative factors are fully unintegrated factors, rather than standard PDFs, i.e., they are PCFs. In addition to the fully unintegrated PDF, we need to define a fully unintegrated soft factor and a fully unintegrated jet factor.

4. The hard scattering matrix element should appear as an on-shell parton matrix element in the final factorization formula.

- Setting on-shell the external partons of a hard-scattering subgraph involves no shift of the momenta of observable lines. Thus it is a safe choice, since it merely involves changing the numerical value of the integrand.

- The use of on-shell and massless matrix elements allows the use of already existing Feynman graph calculations. The only changes from the usual case concern the subtraction terms to remove the double counting of collinear and soft contributions.
This is the primary place where explicit higher-order calculations of Feynman graphs are actually used. These calculations are much easier when on-shell and massless.

It is much easier to maintain gauge-invariance in on-shell amplitudes than in off-shell amplitudes. For the PCFs, we make gauge invariant quantities with the aid of Wilson line factors in their definitions as matrix elements of operators. But this is much harder to do in the hard-scattering coefficients unless their external parton lines are on-shell.

5. We must be able to apply Ward identities exactly to the approximated graphs, in order to convert the gluon exchanges in Fig. 1(c) to a factorized form.

- Any approximation on momenta inside the hard scattering matrix element should be consistent with the use of Ward identities. In the process of factorizing soft and collinear gluons, it will be important to identify contributions to the PCFs. The resulting constraints on the organization of the approximations will lead to corresponding constraints on the definitions of the PCFs.

- It is easy to get a situation where Ward identities are applicable only with remainder terms that are of nonleading power in what we will term the core region of an approximator. These are typically of leading power when the integrations are extended, as is always necessary, to a full range of kinematics. As far as possible, therefore, the approximators should be arranged so that the remainder terms are exactly zero. Otherwise, explicit treatment of the remainder terms is needed to get factorization beyond a leading logarithm approximation.

6. Each approximator should give a good approximation in a particular region of momentum space, but, for the purpose of proving factorization, should be well-defined for all momenta for which it is used.

- In the context of a systematic subtraction scheme — Sec. VI — applied to all orders, an approximator $T_R$ for a region $R$ is used to provide a good approximation to a graph in region $R$, with errors suppressed by a power of $\Lambda/Q$ in the core of the region. ($\Lambda$ is a characteristic hadronic mass scale.)

- For a larger region $R_1$, we will apply its approximator $T_{R_1}$ to the graph with contributions from smaller regions subtracted, to compensate for double counting.

- In order for this procedure to work, we need to take as the contribution of the region $R$ the integral of its approximation up to where the error becomes of order 100%.

- Thus it must be possible to extend the formula for $T_R \Gamma$ beyond the core of the region $R$. Therefore, its definition cannot assume the momenta are in the core of the region.

In the next section, we will address the first four points by returning to Fig. 1(b) and demonstrating how the hard scattering part of the graph may be approximated without violating initial and final state kinematics. To address the last three items, we will discuss the definitions of the PCFs. There we give a set of candidate definitions for the PCFs. Having completed this, we will be in a position to approximate Fig. 1(c) to produce a factorization formula in terms of the lowest order on-shell-parton hard scattering amplitude and the PCFs of Sec. VI.

B. Collinear Gluons in the Standard, Integrated Treatment

To clarify the strategy for dealing with soft and collinear gluons, it is useful to recall relevant steps for obtaining the appropriate gauge-invariant definition of the fully integrated PDF, Eq. (11), with LO factorization in the standard formalism. Relevant graphs beyond the standard parton model graph Fig. 1(a) are those of the form Fig. 1(a), where we extend the handbag diagram to allow for an arbitrary number of gluon exchanges connecting the lower bubble with the outgoing quark. We restrict the extra gluons to be collinear to the target. To leading power, the gluons are longitudinally polarized. We thus have a special case of Fig. 1(c). By the use of a simple Ward identity, it is seen that the gluon attachments eikonalize, and may be converted into a Wilson line factor, as illustrated in Fig. 1(b). This leaves the convolution product of the on-shell LO parton scattering amplitude with the gauge-invariant PDF given in Eq. (11). In the light-cone gauge, $A^+ = 0$, the Wilson line operator is unity and we exactly reproduce the most naive graph for lowest order hard scattering, Fig. 1(a), and graphs with extra collinear gluon exchanges are power suppressed.

As we will see, a number of complications arise in extending these ideas to deal with more general cases. Our aim is to make a precise, general-purpose formalization suitable for the generalization.

C. Soft and Collinear Gluons in the Generalized Formalism

Since we do not make any kinematical approximations on initial- or final-state momentum variables, our generalization of the handbag diagram will contain all final state bubbles. In addition to the initial-state-collinear gluons, we also need to show that final-state-collinear and soft gluons also factorize into appropriately defined
gauge-invariant PCFs. We will find that after appropriate approximations, we can apply Ward identities that disentangle the coupled subgraphs in Fig. 1(c) to give the factorized form shown diagrammatically in Fig. 2(c) below, up to power suppressed corrections. After some further manipulations to compensate for double counting, we find that the cross section, \( \sigma \) (or a related object like a structure function), is a convolution product of a hard factor, a PDF, a final state jet factor, and a soft factor:

\[
\sigma = C \otimes F \otimes J \otimes S + \mathcal{O}\left(\frac{\Lambda}{Q}\right) |\sigma| .
\]

Equation (22) establishes our notation for the PCFs — \( F \) for the target PCF, \( J \) for the jet PCF, and \( S \) for the soft PCF. The notation should be carefully distinguished from the notation (\( \Phi, J, \) and \( B \)) that we have used so far in discussing the subgraph bubbles in a particular graph. The last term in Eq. (22) indicates that errors should be suppressed by a power of \( \Lambda/Q \) where \( \Lambda \) is a typical hadronic mass scale and the power is \( a > 0 \).

IV. THE BASIC APPROXIMATION

In this section we reexamine and reformulate the parton model approximation, as appropriate for Fig. 1(b), in a form suitable for our later work. We arrange to use PCFs rather than regular parton densities. As regards parton kinematics, a suitable definition was given by Collins and Zu [6]. We now extend this to convenient projections onto appropriate two-dimensional spaces for on-shell massless Dirac spinors. Although the calculations are quite elementary for tree graphs, a precise formal definition with a convenient graphical notation will greatly assist later work with higher order graphs.

Since we normally work with scalar structure functions, \( F_1 \) etc, we define a projection tensor \( P^{\mu\nu} \) for any chosen structure function. Thus, the projection onto \( F_1 \) is done by

\[
P^{\mu\nu} = \frac{1}{2} \left[ -g^{\mu\nu} + \frac{Q^2 P^\mu P^\nu}{(P \cdot q)^2 + M^2 Q^2} \right],
\]

so that \( F_1 = P^{\mu\nu} W^{\mu\nu}(q, P) \).

We now apply such a projection to Eq. (3):

\[
\Gamma = \frac{P^{\mu\nu}}{4\pi} \int \frac{d^4k}{(2\pi)^4} \epsilon_j^2 \text{Tr} \left[ J(l) \gamma^\mu \Phi(k, P) \gamma^\nu \right].
\]

From here on, a sum over quark flavors, \( j \), is implicit.

The first step is to replace exact parton momentum variables with approximated parton momentum variables (indicated with a hat) inside the hard matrix element:

\[
k \rightarrow \hat{k}, \quad l \rightarrow \hat{l}.
\]

These are the same as in our treatment of the conventional parton model; but now we no longer use the other kinematic approximations in the \( J \) and \( \Phi \) factors. The approximated momenta, defined in Eq. (6), form a particular case of the prescription in Ref. [6]. They are uniquely determined by the following requirements. First, the approximated momenta, \( k \) and \( \hat{l} \), describe a collinear struck parton and an on-shell final state parton,

\[
|\hat{k}| = \hat{k}^- = 0, \quad \hat{l}^2 = 0.
\]

Then we require four-momentum conservation for both the exact and approximated variables:

\[
k + q = l, \quad \hat{k} + q = \hat{l}.
\]

Next we formalize the projection onto the leading power terms in the trace over Dirac matrices by defining projection matrices \( \mathcal{P}_T \) and \( \mathcal{P}_J \):

\[
\mathcal{P}_T = \frac{1}{2} \gamma^- \gamma^+, \quad \mathcal{P}_J = \frac{1}{2} \gamma^+ \gamma^-,
\]

where

\[
\gamma^- = \frac{1}{\sqrt{2}}(\gamma^0 - \gamma^3), \quad \gamma^+ = \frac{1}{\sqrt{2}}(\gamma^0 + \gamma^3).
\]

As elsewhere, we use the subscript \( T \) to denote target-related quantities, and \( J \) to denote jet-related quantities. Some properties of the projection matrices are listed in App. A. It can be readily checked that \( \mathcal{P}_T \) projects a general 4-dimensional spinor onto the 2-dimensional subspace of spinors \( u \) that obey the massless Dirac equation for momentum \( \hat{k} \), i.e., \( \hat{k} u = 0 \). Similarly, \( \mathcal{P}_J \) projects onto spinors for momentum \( \hat{l} \).

We next use the decompositions (4) of \( J(l) \) and \( \Phi(k, P) \) in a basis of Dirac matrices. The terms that appear at the leading power in the trace can be projected out by sandwiching \( \Phi \) and \( J \) between projection matrices:

\[
\mathcal{P}_T \Phi(k, P) \mathcal{P}_T = \mathcal{P}_T \Phi(k, P) \mathcal{P}_J,
\]

\[
\mathcal{P}_J J(l) \mathcal{P}_J = \mathcal{P}_J J(l) \mathcal{P}_T.
\]

FIG. 4: Target collinear gluons explicit in the definition of the standard integrated PDF.

As before, we use the notation \( \Phi(l) \) and \( \Phi(k, P) \), etc., to denote target-related quantities, and \( J \) to denote jet-related quantities.

\[
\mathcal{P}_T \Phi(l) \mathcal{P}_T = \mathcal{P}_T \Phi(l) \mathcal{P}_J,
\]

\[
\mathcal{P}_J J(l) \mathcal{P}_J = \mathcal{P}_J J(l) \mathcal{P}_T.
\]
Using them, we replace Eq. (24) with its parton-model approximation

$$ T_{PM} \Gamma = \frac{P_{\mu \nu}}{4\pi} \int \frac{dk^+ dk^- d^2 k_i}{(2\pi)^4} \times e_j^2 \text{Tr} \left[ \mathcal{J}(l) \mathcal{P}_T \gamma^\mu \mathcal{P}_T \Phi(k, P) \mathcal{P}_J \gamma^\nu \mathcal{P}_J \Phi(k, P) \gamma^\nu \mathcal{P}_J \right], \tag{32} $$

which is changed from its previous definition. The errors incurred by making this substitution are power suppressed. We now restrict to the case of unpolarized scattering, for simplicity, in which case only $\Phi^+$ and $\mathcal{J}^-$ appear, so that Eq. (32) can be rewritten as

$$ T_{PM} \Gamma[W] = \frac{P_{\mu \nu}}{4\pi} \int \frac{dk^+ dk^- d^2 k_i}{(2\pi)^4} \times e_j^2 \text{Tr} \left[ \gamma^+ \mathcal{J}^-(l) \gamma^\mu \gamma^- \Phi^+(k, P) \gamma^\nu \right]. \tag{33} $$

Dividing and multiplying by $\hat{k}^+ = x_B P^+$ and $\hat{l}^- = q^-$, we obtain

$$ T_{PM} \Gamma = \frac{P_{\mu \nu}}{4\pi} \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{4k^+ l^-} \times e_j^2 \text{Tr} \left[ \hat{\gamma}^\mu \hat{k} \gamma^\nu \right] \Phi^+(k, P) \mathcal{J}^-(l). \tag{34} $$

This gives a good approximation to $\Gamma$ so long as the integral is dominated by the region where $|k_i| \ll k^+$. The lowest order hard matrix element squared is immediately identifiable, and we define it as

$$ |H_0(q, \hat{k}, \hat{l})|^2 \equiv \frac{1}{2} e_j^2 P_{\mu \nu} \text{Tr} \left[ \hat{\gamma}^\mu \hat{k} \gamma^\nu \right]. \tag{35} $$

The factor 1/2 ensures that this is normalized just like the Born graph for scattering on a spin-averaged massless quark. This definition then entails a factor of $1/l^- k^+$ outside the hard scattering amplitude. Thus, the hard scattering is evaluated with the on-shell parton amplitude, while the momentum used to evaluate the PDF and the jet factor remain exact. We write the approximation as

$$ T_{PM} \Gamma = \frac{1}{2\pi} \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{4k^+ l^-} \times |H_0(q, \hat{k}, \hat{l})|^2 \Phi^+(k, P) \mathcal{J}^-(l). \tag{36} $$

In the hard matrix element in Eq. (36), the hatted approximate variables should be regarded as functions of the exact variables. All the nonperturbative objects (the PCFs) are evaluated with unapproximated momentum variables. Thus the only kinematical approximation is in the evaluation of the hard matrix element.

Finally, notice that at this order there is no ultraviolet divergence corresponding to $|k^2| \to \infty$ because of the kinematic constraints and positive energy condition on the final state. This is to be contrasted with the situation in Eq. (8).

The symbol, $T_{PM}$, is the “approximator” which acts on Eq. (24) to produce the approximation in Eq. (36). Graphically, we depict the operation of $T_{PM}$ by a circle around the electromagnetic vertex, as in Fig. 5. It is defined as follows:

- Everything outside the circle is left unapproximated. This is the essential change from the standard parton model.
- The label $T$ next to the circle symbolizes where approximations appropriate to the initial-state quark are made:
  - Parton momentum $\hat{k}$ is replaced inside the circle by $\hat{k}$, thus projecting it onto the plus direction.
  - The projection matrix $\mathcal{P}_T$ is applied in Dirac spinor space.
  - In the complex conjugate amplitude, to the right of the final-state cut, the projection matrix is $\mathcal{P}_T = \mathcal{P}_j$.
- Similarly, the label $J$ symbolizes where approximations appropriate to outgoing quark are made:
  - Parton momentum $\hat{l}$ is replaced inside the circle by $\hat{l}$, thus projecting it onto the minus direction.
  - The projection matrix $\mathcal{P}_J$ is applied in Dirac spinor space.
  - In the complex conjugate amplitude, to the right of the final-state cut, the projection matrix is $\mathcal{P}_J = \mathcal{P}_T$.

The approximations that lead to the factorized form in Eq. (36) are shown diagrammatically in Fig. 5. In contrast to the standard formalism for DIS, the momenta used to evaluate $\Phi^+(k, P)$ and $\mathcal{J}^-(l)$ in Eq. (36) are exact. Furthermore, the integrals over $k$ are constrained by the positive energy condition and relativistic kinematics, in contrast to Eq. (8).
V. DEFINING PARTON CORRELATION FUNCTIONS

A major complication in developing a generalized treatment of DIS is in the difficulty of giving appropriate and consistent gauge invariant definitions for the PCFs. In the absence of gauge fields, the definition of the PCFs is clear [6]: they are just the obvious matrix elements of quark fields, Fourier transformed into momentum space.

In a gauge theory, there are two sources of complication that are intimately tied to each other: (a) There are important leading contributions from regions like Fig. 1(c), where gluons connect subgraphs that correspond to very different ranges of rapidity. (b) The operators in the definition of the PCFs must be made gauge-invariant. As we will see in the later sections, the gluon connections can be converted to a factorized form by applying certain approximations, after which Ward identities are used to show that the sum over the gluon exchanges corresponds to contributions associated with Wilson lines inserted in the partonic operators in the definitions of the PCFs and PDFs.

The Wilson lines give gauge-invariant definitions, and there is, a priori, a choice in the path used in the integral over the gluon field in the Wilson lines. However, only certain directions are suitable, i.e., consistent with factorization. Evidently, a complete discussion can only be made within the context of a treatment of factorization of soft and collinear gluons as will be described below in Sec. VIII. But given that a definition has been found or proposed, its properties can be discussed separately from the motivation.

The characteristic difficulty is that the most obvious definitions have divergences. This applies not just to the PCFs but to PDFs as well. There are three basic sources of divergence:

1. UV divergences due to integration to infinite transverse momentum: These appear in graphs for ordinary integrated PDFs in all renormalizable theories. They also appear in virtual graphs contributing to unintegrated PDFs and PCFs. In all cases they can be removed by renormalization counterterms, beyond those needed to renormalize the interactions of the theory.

2. Divergences due to the masslessness of the gluon: These appear in Feynman graphs, but presumably are cut off by confinement effects in real QCD when the PCFs and PDFs are treated non-perturbatively in hadronic targets.

At this stage of our work, we will use a model Abelian gauge theory with a nonzero gluon mass, so that we separate the mass divergences from other issues. Given that color is confined in QCD, we can expect a real physical cutoff of these divergences, which however is difficult to discuss within pure perturbation theory.

3. Divergences due to the use of light-like Wilson lines: We will call these rapidity divergences. In simple Feynman-graph calculations, rapidity divergences are frequently confused with the divergences due to the masslessness of the gluon, since both arise from regions where the plus or minus momentum of a gluon goes to zero.

We will discuss the issues starting with fully integrated PDFs. Then we will examine the intermediate case of unintegrated PDFs, which are differential in $k^+/P^+$ and $k_1$.

Finally we will examine the case of the fully unintegrated functions, the PCFs, of all the three types we will need: target related (like an ordinary PDF), jet related, and soft factor.

A. Integrated PDFs

For the fully integrated PDFs, the definition [2] given in Eq. (11) is entirely satisfactory. The primary parton fields have a light-like separation in the minus direction so that the Wilson line can be taken along the line joining the parton fields, as is needed to be consistent with factorization. UV divergences are canceled by renormalization, and there is a cancellation of all the rapidity divergences and of the final-state divergences associated with masslessness of the gluon. In a color singlet hadron with confinement there should be no other divergences.

B. Unintegrated PDFs

As for a simple unintegrated PDF, a common approach is to define it as a matrix element of parton fields without a Wilson line:

$$P(x, k, t, \mu) = \int \frac{dw - dw_t}{16\pi^2} e^{-ixp^+w^- + ik_1w_t} \times \langle p|\bar{\psi}(0, w^-, w_t)\gamma^+\psi(0)|p\rangle_{lcg}, \quad (37)$$

but with the fields defined to be in the light-cone gauge $A^+ = 0$. This definition is natural, because if the rules of
light-cone quantization are applied naively, the definition becomes exactly invariant. However, as Collins and Soper [2] demonstrated, this definition has a rapidity divergence in the interacting theory.

The rapidity divergence, from gluons whose rapidity goes to $-\infty$, occurs even when all ultraviolet and mass divergences are cutoff. As illustrated in Fig. 7 the important range of gluon rapidity for the actual cross section is between the rapidity of the target and the rapidity of the jet. Naturally gluons to the right of the rapidity of the virtual photon can be considered as associated with the PDF, and gluons to the left as associated with the jet’s fragmentation function, while gluons in the center belong to a soft factor, in the Collins-Soper-Sterman (CSS) factorization method [3]. But the use of light-cone gauge gives contributions to the PDF from gluons not merely from the positive rapidities that are naturally part of the PDF, but also from rapidities running all the way to $-\infty$. Evidently, the parton density needs to be redefined so that it has some kind of rapidity cutoff at around the photon’s rapidity. This can be accomplished by Soper’s definition with a non-light-like gauge-fixing vector $n$ of about the same rapidity as the virtual photon. Collins and Soper (CS) [2] derived an equation for the derivative w.r.t. the direction of $n$; they used this equation to show very generally the existence of a rapidity divergence in the light-cone-gauge limit $n^2/(n \cdot p)^2 \to 0$.

The definition in Eq. (37) is readily converted to a gauge-invariant form by using the fact that a Wilson line operator $V_w(n)$ in direction $n$ is unity in the $n \cdot A = 0$ gauge. But a further complication is uncovered when the definition is made exactly invariant:

$$P(x, k_t, \mu) = \int \frac{dw d\mathbf{w}_t}{16\pi^3} e^{-ixp^+w^- + ik_t \cdot \mathbf{w}_t} \times \langle p|\tilde{\psi}(0, w^-, \mathbf{w}_t) V_w^+(n) I_{n,w,0} |\gamma^+ V_0(n)\psi(0)|p\rangle. \tag{38}$$

Here Wilson lines go out from the parton fields to infinity in the direction $n$. With only these, we would get exact agreement with Eq. (37) in $n \cdot A = 0$ gauge. But as was pointed out by Belitsky et al. [32], strict gauge invariance requires that the Wilson line be completed in the transverse direction by a segment connecting the two points at infinity. This is accomplished in Eq. (38) by a factor of the following operator

$$I_{n,w,0} = P \exp \left( -ig \int_C dz^\mu A_\mu(z) \right), \tag{39}$$

where the contour $C$ is in the transverse direction and connects $(0, \infty, w_t)$ to $(0, \infty, 0_t)$.

Belitsky et al. [32] showed how this gauge link at infinity is essential to get correct physics even in $n \cdot A = 0$ gauge; the naive definition Eq. (37) is wrong. Their demonstration of the failure of Eq. (37) was done in the context of one-gluon-exchange calculations of the Sivers function; this is the single-spin-asymmetry (SSA) of the unintegrated parton density. In Feynman gauge the Sivers function is obtained from from an imaginary part associated with the usual part of the Wilson line, and the gauge link at infinity does not contribute. But in light-cone gauge, the total contribution comes from the link at infinity. The naive definition Eq. (37) gives zero for the Sivers function, as observed by Brodsky, Hwang, and Schmidt [23].

For this particular calculation of the SSA, the problem with rapidity divergences when $n$ is light-like did not appear, so the calculation in [32] was done in light-like gauge. But the problem of rapidity divergences does appear when more gluons are exchanged, and it does appear with one-gluon exchange in the unpolarized case, as is verified by explicit calculation [10]. Furthermore, couplings of multiple gluons to the link at infinity should affect the unpolarized parton density in axial gauge.

A satisfactory solution to all the difficulties is therefore to use Eq. (38) as the definition of an unintegrated quark density, with now a non-light-like vector $n$ for the Wilson lines’ direction and with a gauge-link at infinity. Then $n$ being non-light-like cuts off the rapidity divergence, and the CS evolution equation gives the effect of changing the cutoff. Apart from the gauge-link at infinity, the definition is exactly that of CS [2]. Presumably the complications made evident by the calculation of the Sivers function also infect the CS definition, which makes no allowance for the gauge link at infinity.

Unfortunately, the use of a non-light-like vector to set the gauge condition or the directions of the Wilson lines produces some practical complications: Feynman graph calculations are algorithmically harder than when a light-like vector $n$ is used, and the evolution equations have...
in inhomogeneous terms that are difficult to discuss explicitly. The inhomogeneous terms are of non-leading power in the hard scale $Q$, so that they are neglected in phenomenological applications. The appearance of the non-leading powers in $Q$, however, suggests that the use of non-light-like Wilson lines might be preferred in studies of higher-twist effects (see, for example, the work of [34]). Collins, Hautmann and Metz [10, 32, 35] suggested modified definitions. They take the definition with light-like lines as basic. But to cancel rapidity divergences, it is divided by an extra factor involving the vacuum expectation value of certain Wilson lines. The non-light-like vector needed to specify the physically necessary rapidity cutoff is in this extra factor. This represents a kind of generalized renormalization of the operators whose matrix element is the parton density. But we will not take this route in the present paper.

C. PCFs

Parton correlation functions (PCFs) are defined like parton densities, but without any of the integrals over $k_t$ or $k^-$. The primary issue in constructing a definition is the choice of directions of Wilson lines. So we start by discussing what leads us to our choice. The issues are closely linked to those of rapidity divergences and the important regions of gluon rapidity relevant for a process.

Compared with the case of PDFs (integrated or unintegrated) some simplification occurs because we remove the integral over parton virtuality (or $k^-$). Any rapidity divergence occurs when the rapidity of a gluon goes to $-\infty$. At fixed transverse momentum, the divergence therefore occurs where the gluon’s minus momentum goes to infinity. For emission of real gluons in a PCF, the divergence is therefore cutoff, because the minus momentum on any one gluon is restricted to $P^- - k^-$ by the externally imposed $k^-$. This occurs even if the gluon is “dressed” so that it decays to multiple particles. This contrasts with the case of an ordinary PDF, integrated or unintegrated, with its integral over all values of $k^-$. Although the use of exact parton kinematics in a PCF cuts off rapidity divergences when the external transverse momentum is fixed, it does nothing about virtual gluons. Moreover, when we work with Feynman graphs in a theory with massless gluons, the limit on rapidity expands as the transverse momentum is reduced to zero. So rapidity divergences reappear as part of the infra-red divergences. We regard rapidity divergences as particularly dangerous because a derivation of factorization associates the PCF or PDF with momenta that are related to the target. A rapidity divergence gives important contributions from momenta that are infinitely far away, thereby at least endangering the value of treating parton kinematics more exactly.

Therefore we define all our PCFs with non-light-like Wilson lines. There are three types: (a) A PCF in a target, generalizing the notion of parton density. (b) A corresponding object, a jet PCF, for fragmentation, generalizing the notion of fragmentation function. (c) A soft PCF factor. These will be used to capture the physics associated with partons whose momenta are respectively in the regions: (a) target-collinear, (b) jet-collinear, (c) soft.

For the target and jet PCFs, the direction of the Wilson line is chosen to be approximately at rest in the center-of-mass, so as to correspond to a separation between left- and right-moving momenta, as is natural for factorization. (For our purposes, we have generalized the notion of center-of-mass to any four-vector with zero rapidity so that it is applicable to space-like as well as time-like vectors.) As for the soft factor, it concerns gluons central in rapidity, and is defined in terms of a vacuum expectation of a suitable Wilson line operator. The Wilson line has segments representing the active partons, and that therefore go in almost light-like directions.

There will be an apparently unfortunate profusion of directions of Wilson lines. The associated complications are manageable once one recognizes that the directions provide rapidity cutoffs in each factor to ensure it only concerns momenta appropriate for the given factor. The factors do acquire extra arguments corresponding to differences between the rapidity cutoffs and the external momenta. But then the CS equation for the dependence on the Wilson line directions gives the dependence on the extra arguments. Thus the non-light-like Wilson lines provide a tool for quantifying the evolution with respect to the available rapidity range, and hence with respect to energy. The correspondence between rapidity and angle for a massless particle is presumably the link that relates Collins-Soper evolution to the well-known angular-ordering rule for coherent gluon emission in parton showering.

We leave open the possibility that the definitions may be replaced with definitions that use light-like Wilson lines with rapidity divergences removed by gauge invariant factors as in the treatment of the Sudakov form factor in Refs. [33, 36], but that is left for later work.

The observations made so far in this section motivate our new definitions for the PCFs. In this section, we will simply state the definitions with qualitative remarks to motivate them. The justification for the definitions will come when we find we can set up a detailed set of approximations which result in factorization with our given definitions for the PCFs.

First, we define light-like vectors corresponding to the directions of the primary hadrons in the process:

$$u_T = (1, 0, 0, t), \quad u_J = (0, 1, 0, t). \quad (40)$$

We also define slightly non-light-like vectors,

$$n_T = (1, -e^{-y_T}, 0, t), \quad n_J = (-e^{-|y_J|}, 1, 0, t), \quad (41)$$

with $y_T$ large and positive, $y_J$ large and negative. Our notation is to use the letter $u$ for light-like vectors, and $n$ for non-light-like vectors; the subscript indicates which
direction of a hadron or an active parton is approximated by the vector. To achieve factorization, we will also need a vector to characterize the boundary between left and right-movers; it should correspond to the gauge-fixing vector in the Collins-Soper formalism. Therefore we define:

\[
n_s = (-e^{y_s}, e^{-y_s}, 0).
\]  

(42)

In accordance with the results of Collins and Metz, we use space-like, not time-like, vectors. They found that when virtual gluon emission is included, space-like Wilson lines give the widest and most universal factorization.

The above vectors will appear as directions of Wilson lines in the definitions of PCFs, and they serve to provide cutoffs on the rapidities of momenta in each PCF. Thus, each PCF is restricted to rapidities appropriate to its function in a derivation of the factorization property. During the derivation we will require that the rapidities \( y_T \) \(, y_s \) and \( y_J \) correspond approximately to the target, a rest vector in the center-of-mass, and the outgoing jet. (The precise values will not be relevant.) After we have a factorization, we will wish to exploit the universality of the PCFs to relate processes at different energies. This will involve, for example, boosting the target PCF to change the target state from one energy to another. The boost will also apply to the vector \( n_s \), thereby giving it an inappropriate rapidity for proving factorization at the new energy. The CSS equation gives the dependence on \( y_s \), so that we can convert the PCF to the one appropriate to the new energy. Thus, although a factorization proof can legitimately assume that \( y_s = 0 \), i.e., that \( n_s \) can be considered at rest in the center-of-mass, we leave \( y_s \) as a parameter because we will need to exploit the \( y_s \)-dependence of the PCFs.

In a subtraction scheme, we start the treatment from the smallest region and successively generate terms for larger regions, with subtractions to avoid double counting. Pattern lines give the widest and most universal factorization. (The precise values will not be relevant.) After we have used space-like, not time-like, vectors. They found that when virtual gluon emission is included, space-like Wilson lines give the widest and most universal factorization.

1. Soft factor

Soft gluons couple to the target jet, with its large plus component of momentum, and to the outgoing jet with its large minus component of momentum. This suggests that the soft factor is the vacuum expectation value of Wilson lines that are nearly light-like in the plus and minus directions. In coordinate space we define the soft factor by

\[
S_I(w, y_T, y_J, \mu) = \langle 0| I^I_{n_T;0} V^0_w(n_J) V^I_I(n_J) I^I_{n_J;0} V^0_w(n_T) V^I_I(n_T) |0\rangle_R.
\]

(43)

This expression has non-light-like Wilson lines going out in approximately the plus and minus directions from a particular point in spacetime which we may choose as the origin of our coordinate system, times a conjugate amplitude with emission from a different point \( w \). Fourier transformation then gives a factor for the production of a final state of a given momentum. This represents emission from outgoing eikonalized colored lines in directions appropriate to the quarks \( k \) and \( k+q \) in the parton model. That the Wilson lines are not quite light-like restricts the states to those appropriate for a finite energy process. They also provide cutoffs on rapidity divergences.

Now choosing the \( n_J \) Wilson line to be outgoing naturally matches the idea that gluon radiation from this line concerns emission from the actual \( k + q \) line at the hard scattering. This suggests that, to match the incoming \( k \) line, the direction for the \( n_T \) Wilson line should be incoming from \( -\infty \) rather than outgoing to \( +\infty \). The work of Collins and Metz shows otherwise: the choice of an outgoing line (with color corresponding to an antiquark, if the \( k \) line is an incoming quark) turns out to work better and to give broader universality properties.

The paths for these Wilson lines are illustrated in Fig. where we also indicate the exactly light-like directions. The gauge links, \( I^I_{n_T;w,0} \) and \( I^I_{n_J;w,0} \), at infinity are needed for strict gauge invariance. It should be noted, of course, that the lines representing these gauge links in Fig. should have components in the transverse direction (out of the page). Thus, the soft factor is the vacuum expectation value of a closed Wilson loop.

Renormalized field operators are used in Eq. (43). In a renormalizable theory, Eq. (43) will contain UV divergences, both from the divergences that appear in the Lagrangian, and from the fact that Eq. (43) involve Wilson lines meeting at a cusp. Both types of divergence can be dealt with using standard renormalization techniques (as indicated by the subscript, \( R \)). The renormalization scale is \( \mu \).
The target PCF for a quark should involve a gauge-invariant expectation value of the quark fields inside the target. Therefore, a reasonable first attempt at a definition (in coordinate space) is

$$\tilde{F}(w, y_p, n_s, \mu) = \langle p|\bar{\psi}(w) V_w^\dagger(n_s) I_{n_s,w:0} \frac{\gamma^+}{2} V_0(n_s) \psi(0)|p\rangle_R.$$  \hspace{1cm} (44)

Here the Wilson lines are in direction $n_s$, and in proving factorization we will assume that the rapidity $y_s = 1/2 \ln |n_s^+ / n_s^-|$ is close to zero in the center-of-mass, i.e., that $n_s$ is approximately along the $-z$ direction in the center-of-mass. That $n_s$ is space-like is obtained from the results of Collins and Metz [35]. As we explained earlier, exhibiting the dependence on $y_s$ allows the use of the CSS evolution of the PCFs. The above definition is compatible with the CS definition, where $n_s$ is the gauge-fixing vector for the axial gauge $n_s \cdot A = 0$. For exact gauge invariance we have also inserted a gauge link at infinity. The path for the complete Wilson line is shown in Fig. 9. The factor of $1/2$ with $\gamma^+$ is to keep the normalization the same as for a PDF.

Although this is an excellent definition, we will find a different quantity arises when we first obtain a factorization. We will find gluonic effects that appear both in the soft PCF factor and this definition of the target PCF. To remove the double counting, we will use a related definition which our first proposal, Eq. (44), divided by a factor related to the soft PCF:

$$\tilde{F}_{\text{mod}}(w, y_p, y_T, n_s, \mu) = \langle p|\bar{\psi}(w) V_w^\dagger(n_s) I_{n_s,w:0} \frac{\gamma^+}{2} V_0(n_s) \psi(0)|p\rangle_R,$$

$$= \langle 0|f_{n_T:0} V_w(n_T) V_w^\dagger(n_s) I_{n_s,w:0} V_0(n_s) V_0^\dagger(n_T)|0\rangle_R.$$  \hspace{1cm} (45)

The denominator is the soft factor, but with the $n_T$ Wilson line changed to have direction $n_s$. The rapidity argument, $y_p$, is the exact rapidity of the target proton. (This should be distinguished from $y_T$ which parameterizes the direction of the target associated Wilson line and points approximately in the target direction.) Notice that this definition is given in coordinate space. When we Fourier transform to momentum space, the division will be in the sense of a convolution product.

We understand the meaning of the definitions in Eq. (43) and Eq. (45) as follows:

- The PCF for the soft factor treats central gluons accurately, the $n_T$ and $n_J$ Wilson lines providing accurate approximations to the quark lines and associated collinear subgraphs.

- As gluons approach the target rapidity, the accuracy of the $n_T$ Wilson line as an approximation for the incoming quark line is degraded in both the target and soft PCF.

- The numerator in Eq. (45) provides a good approximation for gluons in the target range of rapidity, and the denominator accurately cancels the bad approximation for this same region in the soft factor.

- In the numerator of Eq. (45), central gluons, around the $n_s$ direction, are accurately given by an eikonal approximation of the form also appearing in the denominator, so we get a cancellation. Gluons of intermediate rapidity also cancel.

- The non-lightlike Wilson lines in direction $n_s$ provide strong cut offs in Eq. (45) on gluon rapidities beyond the central region.

- Similar ideas apply to negative rapidities and the corresponding jet factor (next section).

The lack of question marks on the equal sign in Eq. (45) implies that this definition is the main definition. It has acquired a second rapidity argument, which is quite undesirable, but this does correspond to a definition by Idilbi, Ji, Ma and Yuan [37, 38] for unintegrated PDFs in the context of SIDIS. However, various further reorganizations, together with an application of the CS evolution equation can handle the dependence on the extra arguments.

However, to match the derivation of factorization, an excellent choice for $y_T$ is to be close to the target rapidity. In fact the simplest choice is to set $y_T = y_p$. In that case, the important dependence for which we need the CS equation is on $y_s$.

3. Jet PCF

We define a jet PCF, to account for final-state-collinear behavior, just like the target PCF, in versions without the denominator:
\[ \tilde{J}(w, \text{jet direction}, y_J, y_s, \mu) = \langle 0 | \bar{\psi}(w) V^1_w(-n_s) I_{-n_s:w,0} \gamma^0 V_0(-n_s) \psi(0) | 0 \rangle_R, \]  

and with the denominator:

\[ \tilde{J}_{\text{mod}}(w, \text{jet direction}, y_J, y_s, \mu) = \frac{\langle 0 | \bar{\psi}(w) V^1_w(-n_s) I_{-n_s:w,0} \gamma^0 V_0(-n_s) \psi(0) | 0 \rangle_R}{\langle 0 | I_{-n_s:w,0} V_w(-n_s) V^1_w(n_1) I_{n_1:w,0} V_0(n_J) V_0^*(n_s) | 0 \rangle_R}. \]

**D. Connection to factorization**

The momentum-space PCFs are determined by the Fourier transforms of the above definitions. For example, the momentum space PCF in the target is

\[ F_{\text{mod}}(k, y_p, y_T, y_s, \mu) = \int \frac{dw^+ dw^- d^2w_t}{32\pi^4} e^{-ik \cdot w} \tilde{F}_{\text{mod}}(w, y_p, y_T, y_s, \mu). \]

When we come to the issue of factorization it will be important to recall that the momentum space PCFs discussed in this section are well-defined for all values of momentum, even those that lie far outside the range that is meant to be accurately described by the PCF. For example, the momentum-space soft factor \( S(l, y_T, y_J, \mu) \), obtained by Fourier transforming Eq. (43), exists even for values of \( l \) that are far from soft. Of course, when the PCFs appear in the factorization formula for physical processes, they should be large only for appropriate values of momentum. This is partly accomplished by the use of non-light-like Wilson lines in the definitions above, which, as we shall see in the next two sections, cut-off the contribution from light-like gluons. The motivation for writing down Eqs. (13), (15), and (47) will become clearer in the next few sections where we will show explicitly what is needed for a factorization formula. This new factorization formula will be defined with exact over-all kinematics that takes into account soft and collinear gluon emissions. We will show how the definitions for parton correlation functions listed above follow naturally from the factorization of soft and collinear gluons in DIS.

**VI. SUBTRACTIONS**

The overall approach we use is a subtractive approach — e.g., (39) — generalized from the Bogolubov approach to renormalization. Up to power-suppressed terms, each graph \( \Gamma \) is written as a sum over a contribution for each of its leading regions:

\[ \Gamma = \sum_{R \text{ of } \Gamma} C_R \Gamma + \text{power-suppressed}. \]  

A single graph typically has many different regions, each corresponding to a different graphical decomposition of the form of Fig. (1c). As explained below the definitions of the terms \( C_R \Gamma \) employ approximations and then subtractions to eliminate double counting between regions. We denote the chosen approximation corresponding to a particular region by the action of an “approximator”,

![Diagram of Wilson line and momentum-space PCF](image-url)
$T_R$, as in Eq. (52) for the simple case of parton model kinematics.

The different regions can be ranked according to their sizes, e.g., a soft region corresponds to a smaller range of momentum than a collinear region, and is therefore a smaller region in a set-theoretic sense. We define a region as minimal if there are no smaller regions. The contribution from a minimal region $R_0$ is simply the action of the corresponding approximator on the unapproximated graph,

$$C_{R_0} \Gamma = T_{R_0} \Gamma.$$  

(50)

For the contributions from larger regions, we avoid double counting by performing subtractions for the contributions from smaller regions. So we define

$$C_R \Gamma = T_R \left( \Gamma - \sum_{R' \subset R} C_{R'} \Gamma \right).$$  

(51)

For a minimal region, Eq. (51) reduces to Eq. (50), so that it gives a valid recursive definition of $C_{R} \Gamma$ with the terms being constructed sequentially starting from the minimal region(s).

As exhibited in Fig. 1(c), a complication is that the graphical representation of the regions does not directly correspond to factorization, because of multiple gluon connections between the different factors. This contrasts with the case of Fig. 1(b), where we have topological factorization. Therefore an important constraint on choosing the definition of $T_R$ out of the range of possibilities is that (if possible) the graphical factorization results in an actual factorization after a sum over graphs and regions:

$$\sum_{R, \Gamma} C_R \Gamma = \text{factorized form}.$$  

(52)

VII. KINEMATIC APPROXIMATIONS AND RAPIDITY DIFFERENCES

The kinematic approximations that enable factorization to be derived utilize certain properties of Minkowski-space momenta. We now review them with a view to systematizing our later work.

Corresponding approximations in a Euclidean space are much more trivial. Thus if $p$ and $q$ are two spatial momenta with $|p| \ll |q|$, then we can approximate $(p + q)^2$ by $q^2$, up to a power-law correction. This is simply because angles are bounded in a Euclidean space.

In Minkowski space, we have to deal with unbounded rapidity variables instead of angles. (Rapidity is useful to us because our process has a preferred axis.) Suppose we have two 4-momenta $k_1$ and $k_2$ with rapidities defined by $y_i = \frac{1}{2} \ln|k_i^+|/|k_i^-|$, and such that $|k_i^+ k_i^-|$ is comparable to or bigger than $k_{i,t}^2$, as would be the case for an on-shell momentum. These could be, for example, the four-momenta of two internal gluon lines in a graph, $\Gamma$. We write the orders of magnitude of the $(+, -, T)$ components of each momentum as

$$(k_i^+, k_i^-, k_i,t) = O(M e^{y_i}, M e^{-y_i}, M),$$  

(53)

where $M$ is an appropriate mass scale.

The interesting case will be where the rapidities are quite different, let us say $e^{y_1 - y_2} \gg 1$. Then $k_1 \cdot k_2$ is dominated by one term:

$$k_1 \cdot k_2 = k_1^+ k_2^- \left[ 1 + O(e^{-(y_1 - y_2)}) \right].$$  

(54)

We will be able to apply this in Fig. 1(c), for the numerators of the attachments of the gluons from $B$ to the collinear subgraphs $J$ and $\Phi$, and for the attachments of the gluons from the collinear subgraphs to the hard subgraphs.

There is an interesting region, where Eq. (54) fails because one momentum variable has particularly small longitudinal momentum components, i.e., $|k_i^+ k_i^-| \ll k_{i,t}^2$. This is called the Glauber region, and it is a natural case to examine since it corresponds to a virtual particle exchanged in small-angle elastic scattering, as in a final-state interaction. An important part of factorization proofs is to arrange for a contour deformation to get out of the Glauber region, when possible. See Collins and Metz [35] for a recent treatment of issues relevant to our discussion in later sections; for the DIS reactions treated in this paper, a contour deformation out of the Glauber region is possible.

It is worth observing that the word “region” has two slightly different connotations in our discussions. One refers to a locality in the space of 4-momenta, as in the explanation of the Glauber region in the previous paragraph. The other connotation is as a locality in the multi-dimensional space of loop or line momenta for a whole graph, as in Eq. (51). For a graph with a single gluon exchange, we often use the direct correspondence between the regions of the graph and the regions for the gluon’s momentum.

A propagator denominator

$$(k_1 + k_2)^2 - m^2,$$  

(55)

needs a bit more care than a simple product $k_1 \cdot k_2$, since the appropriate approximation depends also on the relative virtualities of $k_1$ and $k_2$. We again assume that we have deformed out of any Glauber region, and that $e^{y_1 - y_2} \gg 1$. Then:

1. If the virtualities of $k_1$ and $k_2$ are comparable to each other, and both are comparable to or bigger than $m^2$, then the denominator is dominated by $k_1^+$ and $k_2^-:

$$(k_1 + k_2)^2 - m^2 \simeq 2k_1^+ k_2^-.$$  

(56)

An elementary application is to the virtual photon in Fig. 1(a) where $Q^2 \simeq 2k^+ t^- \Gamma$ when the initial quark is collinear to the target.
2. But if we also have to treat the case that one momentum, \( k_2 \) say, has virtuality much less than that of the other, then although we can neglect \( k_2^2 \) with respect to \( k_1^2 k_2^2 \), we cannot necessarily neglect \( k_1^2 \) or \( m^2 \). Thus we can only write

\[
(k_1 + k_2)^2 - m^2 \approx k_1^2 - m^2 + 2k_1^2 k_2^2. \tag{57}
\]

This amounts to replacing \( k_2 \) by its minus component. This approximation will be the primary tool in deriving factorization.

**VIII. REAL GLUON EMISSION**

In this section we examine the simplest case of the gluonic corrections to the parton model that were summarized in Fig. 1(c). This is given by the emission of one real dressed gluon — Fig. 11. We will consider virtual gluon radiation in a later section.

**A. Regions for gluon exchange**

The gluon, of momentum \( l_2 \), evolves into a final state represented by the bubble, \( J_2 \lambda \rho_2(l_2) \). It attaches to a jet-associated bubble denoted by \( J \lambda \rho \), and a target-associate bubble \( \Phi^{\rho}(k_1, l_2, P) \). Here \( \lambda \) and \( \rho \) are Lorentz indices, and the subscript, \( g \) indicates that this is the bubble associated with an outgoing gluon.

We will restrict attention to the case that there is no production of extra jets of high transverse momentum. Therefore, by standard power-counting arguments, we only need to consider the contribution from three regions of the gluon’s momentum: soft, target-collinear, and jet-collinear. These are fundamentally distinguished by the rapidity of the gluon’s momentum: soft, target-collinear, and jet-collinear.

A complication is that there are two ways to characterize the regions. One is by the Libby-Sterman [31] analysis in terms of the pinch-singularity surfaces (PSSs) of the corresponding massless graphs. The other is in terms of the very different rapidity ranges of, in this case, the exchanged gluon. The analysis in terms of rapidity is closely related to the use of angular ordering in leading logarithm approximations.

For the Libby-Sterman analysis, the massless singularities, illustrated in Fig. 12, are: A soft gluon singularity at zero gluon momentum \( l_2^\mu = (0, 0, 0, l_2) \), a target-collinear singularity at \( l_2^\mu = (z_T^2, 0, 0, 0) \), and a jet-collinear singularity at \( l_2^\mu = (0, z_j q, 0, 0) \). Here, \( z_T \) and \( z_J \) parameterize the position along the collinear-singularity lines in a frame-independent fashion. The momenta that are actually relevant are in neighborhoods that surround the PSSs in Fig. 12. Therefore, we need to introduce variables that allow us to specify the proximity of the momenta to a PSS. For each PSS, we parameterize its neighboring momenta by what we will term radial and angular coordinates. Thus we write:

- For the soft region

\[
l_2 = \lambda (\hat{l}_2^+, \hat{l}_2^-, \hat{l}_2^3). \tag{58}
\]

The dimensionless “angular” variables \( \hat{l}_2^\mu \) obey some moderately arbitrary normalization condition, e.g., \( \sum \hat{l}_2^{\mu 2} \approx 1 \). These variables parameterize a surface of constant \( \lambda \) surrounding the soft point \( l = 0 \). Then \( l_2^2 \) is of order \( \lambda^2 \), with a coefficient bounded away from zero and infinity. This is the property that enables us to estimate the errors on our approximations systematically.

A formal implementation of the quasi-angular integration can be made as follows:

\[
\int d^4 l_2 \ldots = \int d\lambda \lambda^3 \int d^3 \hat{l}_2 \ldots \equiv \int d\lambda \lambda^3 \int d^4 l_2 \frac{1}{\lambda^3} \delta (\lambda - \sqrt{\sum \hat{l}_2^{\mu 2}}) \ldots, \tag{59}
\]

where we use the normalization condition proposed above. This formula is written in a form equally...
suited for virtual gluon exchange. Any constraints on the invariant mass of the gluon from the nature of the final state are taken to be in the integrand, the part indicated by “...”.

The formula obviously does not give a Lorentz-invariant decomposition of the integration, but is adapted to the needs of the process. It is arranged so that power-counting in \( \lambda \) is straightforward: The angular integral phase space \( \int d^3l_2 \) is independent of \( \lambda \). Moreover, it is unambiguous that a small value for \( \lambda \) corresponds to a small neighborhood of the origin in Fig. 12. This would not be true if we had tried to characterize the soft region by specifying it in terms of a Lorentz invariant quantity, i.e., \( 2l_2^2 - l_{2,t}^2 \).

- For the target-collinear region

\[
l_2 = \left( z_T P^+ + \frac{\lambda^2 i_{2\ell}}{z_T P^+}, \lambda \right).
\]

Our choice for the normalization condition of the dimensionless angular variables is \( |\ell_{2L}| + l_{2,t}^2 = 1 \). The asymmetric scaling is suitable for a momentum highly boosted from a rest frame, and again ensures that \( l_2^2 \) is proportional to \( \lambda^2 \).

A formal definition of the quasi-angular integration is:

\[
\int d^4l_2 \ldots = \int d\lambda \lambda^3 \int \frac{dz_T}{z_T} \int d^2l_2 \ldots
\]

\[
= \int d\lambda \lambda^3 \int \frac{dz_T}{z_T} \int d^2l_2 \frac{z_T}{\lambda^3} \delta(z_T - l_{2}^2/P^+)_\lambda 
\times \delta \left( \lambda - \sqrt{|l_2^2 - l_{2,t}^2|} \right) \ldots.
\]

In the Libby-Sterman terminology, \( z_T \) is an “intrinsic” variable for the PSS, parameterizing the position on the surface. Then \( \lambda \) can be treated as measuring the distance from the surface, while \( l_2 \) parameterizes a (2-dimensional) surface around the PSS, for a given value of the intrinsic variable.

- For the jet-collinear region

\[
l_2 = \left( \frac{\lambda^2 i_{2\ell}}{z_T q^\perp}, z_T q^\perp, \lambda \right),
\]

exactly similarly to the the situation for the target-collinear region.

For any of these cases, when the four-momentum has a virtuality of order a typical hadronic scale \( \Lambda^2 \), then the radial variable \( \lambda \) is itself of order \( \Lambda \), and this can be regarded as the canonical size of \( l_2 \) for the region. This gives a basic intuition about the meaning of collinear and soft momenta. (Here we temporarily assume that the \( z_{T,J} \) variables are of order unity.) But we integrate over all accessible momenta (or up to some limit of order \( Q \)), so it is important to treat \( \lambda \) as ranging from 0 to order \( Q \).

The approximations we use will have typical fractional errors suppressed by a power of the various small mass scales, e.g., \( \lambda/Q \), \( \Lambda/Q \), \( m/Q \). As we move to larger \( \lambda \) than the “canonical” value, the errors become larger. But at the same time, as we will see, the approximations for larger regions become useful, and the overall effect in the subtraction formalism will be that the total error in the sum of all the approximated contributions will be of higher twist. That is, the fractional error will be a power of a fixed hadronic mass scale divided by a large scale like \( Q \). (Logarithmic corrections will slightly weaken an initially determined power-law suppression.)

We can now compare the Libby-Sterman analysis and the rapidity analysis. With the above definitions, there is a large rapidity difference between the two collinear regions and this will be sufficient for us to obtain appropriate approximations. The Libby-Sterman analysis further requires \( z_J \) and \( z_T \) to be of order unity, so that the large component of a collinear momentum is of order the hard scale \( Q \). While this is important for discussing hard scattering, it complicates the treatment of the important subcase where one of these \( z \) variables goes to zero. In that case when \( \lambda \) for a collinear region is sufficiently small (of order a mass times \( z \)), the gluon is then simultaneously collinear, by the rapidity criterion, and soft by the criterion of small size. But the gluon is not collinear by the Libby-Sterman criterion of energy being of order \( Q \). This case is only significant for a massless gluon. To avoid a proliferation of special cases, we unify this case as much as possible with the collinear case. For our initial all-orders treatment in the present paper, we will cut off this region by the use of a gluon mass, as announced in the introduction. But we will not need to do this just yet.

Related issues have arisen in the literature in the form of a distinction between a soft and a supersoft region \([10]\).

In the language of this section, they are distinguished by the numerical values of \( \lambda \). When the components of \( l_2 \) are of order \( \Lambda \) we are in the conventional soft region; when they are of order \( \Lambda/Q^2 \) we are in the supersoft region. (This will be discussed further in Sect. 8.) Again we will unify as much as possible the soft and supersoft regions.

### B. Unapproximated graph

Before any approximations, the formula for Fig. 11 is

\[
\Gamma^{(R)} = \frac{e^2 P_{\mu \nu}}{4\pi} \int d^4l_2 \int d^4k \frac{1}{(2\pi)^4} \frac{1}{(2\pi)^4}
\]

\[
\times \text{Tr} \left[ e^\mu \bar{\mathcal{J}}^\nu (k + q, l_2) e^\rho \bar{\Phi}^\rho (k, l_2, P) \right] J_{\nu,\alpha\beta}(l_2).
\]

Here the superscript \( (R) \) denotes graphs for emission of a real dressed gluon, and, as before, \( P_{\mu \nu} \) represents a...
projection for a particular chosen structure function. We use bars on $\Phi(k_1, l_2)$ and $\bar{J}(k_1, l_2)$ to distinguish bubbles with an extra external gluon from those in Fig. 11(b). The indices $\kappa$ and $\rho$ on $\bar{J}$ and $\Phi$ are now for the coupling to the gluon, not for a decomposition over Dirac matrices. The final-state bubble for the gluon in Fig. 11 is denoted by $J_{g, \kappa, \rho}$. In the approach with standard kinematical approximations this gluon would be put on-shell, and therefore we would have $J_{g, \kappa, \rho} \propto \sum_{p} \epsilon_{c} \epsilon_{\rho} \delta(t_{2}^{2})$, where $\epsilon$ is a gluon polarization vector. Since we keep exact kinematics we do not perform these approximations. Finally, we have left implicit the quark-flavor label $j$ on these quantities.

Our strategy is as follows: We start with the smallest region, the soft region, and construct an approximator $T_{S}$ that is (a) accurate in the soft region, and (b) suitable for the use of a Ward identity argument to bring the total soft contribution, $\sum \Gamma^{(R)}$, into a factorized form. The subtraction method requires us to extend the integration in the soft term to larger regions. Then we follow similar steps to construct approximators $T_{J}$ and $T_{T}$ for the collinear regions. Again, these have to be compatible with Ward identity arguments. Application of the methods of Sec. VII will provide subtractions that compensate double counting between the terms for different regions, so as to ensure that the sum of these terms, $C_{S} \Gamma^{(R)} + C_{T} \Gamma^{(R)} + C_{J} \Gamma^{(R)}$ gives an accurate approximation for the union of the regions, including all intermediate cases. This therefore deals with all regions involving low transverse momentum for $l_2$, i.e., for $l_{2, \perp} \ll Q$, with relative errors being approximately of order $l_{2, \perp}/Q$.

C. Soft Region

We now define our approximation for the soft region. The method is that of Grammer and Yennie [41], as applied in factorization proofs (e.g., [4]).

In the Breit frame, the target is boosted to have a large plus component of momentum, of order $Q$, while the outgoing jet is boosted to have a large minus component of momentum, also of order $Q$. Therefore, for the coupling of the jet and target bubbles, $\bar{J}^+ \Phi^-$, to the exchanged gluon, we may characterize the sizes of the vector components by their transformations under boosts. The largest components have $\kappa = -$ and $\rho = +$ respectively. Relative to the largest components, the smaller components have sizes

$$\frac{\bar{J}^+}{\bar{J}} = O\left(\frac{\Lambda^2}{Q^2} + \frac{\lambda}{Q}\right), \quad \frac{\bar{J}_-}{\bar{J}} = O\left(\frac{\lambda}{Q} + \frac{\Lambda}{Q}\right), \quad (64)$$

$$\frac{\Phi^-}{\Phi^+} = O\left(\frac{\Lambda^2}{Q^2} + \frac{\lambda}{Q}\right), \quad \frac{\Phi_1}{\Phi^+} = O\left(\frac{\Lambda}{Q} + \frac{\Lambda}{Q}\right). \quad (65)$$

These power laws result both from the size of the components of the momentum $l_2$ of the exchanged gluons and from the sizes of the components of the collinear momenta, which are boosted from their rest frame. For the moment, we treat the collinear momenta as having transverse momenta of order a normal hadronic mass scale $\Lambda$. At first sight, the Lorentz boosts to get collinear momenta indicates that the non-leading longitudinal components, $\bar{J}^+ \Phi^-$ for the collinear subgraph, would be of order $\Lambda^2/Q^2$ relative to the large components. This would in fact be correct if the minus component of the injected soft momentum $l_2$ were sufficiently small, i.e., if $\lambda$ were less than about $\Lambda^2/Q$. But when it has a larger value, e.g., the “natural” value for a soft momentum $l_2 = \Lambda$, some lines of $\Phi$ acquire this size for their minus momentum. Correspondingly $\Phi^-/\Phi^+$ increases to a size $\lambda/Q$.

To leading power, (in either $\lambda/Q$ or $\Lambda/Q$) we need to keep only the leading polarization components, and we make the following string of approximations to the product of bubbles in Eq. (63):

\begin{align*}
\bar{J}^e(k + q, l_2) J_{g, \kappa, \rho}(l_2) \Phi^\rho(k, l_2, P) & \simeq \bar{J}^-(k + q, l_2) J_g^-(l_2) \Phi^+(k, l_2, P) \\
& \simeq \bar{J}(k + q, l_2) \cdot l_2 \frac{1}{l_2} J_g^-(l_2) \frac{1}{l_2} l_2 \cdot \Phi(k, l_2, P).
\end{align*}

(66)

Here, our aim is to obtain a form in which the gluon momentum $l_2$ is contracted with each jet factor, a situation in which we can apply a Ward identity. The critical step is in the last line, where we use the fact that in the soft region $l_2 \cdot \Phi \simeq l_2 \Phi^+ \rho$ and $\bar{J} \cdot l_2 \simeq \bar{J} - l_2^-$. This step requires that the longitudinal components of $l_2$ be comparable to each other, which in turn requires that the rapidity of $l_2$ be small. It also requires that $l_2$ be outside the Glauber region, as is always true for real gluon emission. (Recall that the Glauber region is where $|l_2^\perp l_2^-| \ll l_2^\perp l_2_2$.)

Within the soft region, the relative error in the approximation is then of order $\lambda/Q$. However, in the factorization formula, we will integrate over the whole accessible range of $l_2$. This will of course take us outside the soft region where the above approximations are accurate. By itself this is no problem, since such a contribution will eventually be accommodated by proper double-counting subtractions in the treatment of other regions. But, particularly when we apply the same soft approximation to virtual gluons, the denominators $l_2^+ \Phi^-$ and $l_2^- \Phi^+$ will create rapidity divergences that are completely unphysical. The simplest solution is to replace these denominators by dot products with $n_3$ and
\( n_T \), the vectors defined in Eq. (11):

\[
\tilde{J}^s(k + q, l_2) J_{g,\kappa \rho}(l_2) \tilde{\Phi}^\rho(k, l_2, P) \simeq \tilde{J}(k + q, l_2) \cdot l_2 \frac{n_1^\kappa J_{g,\kappa \rho}(l_2)n_T^\rho}{(l_2 \cdot n_1 - i\epsilon)(l_2 \cdot n_T + i\epsilon)} l_2 \cdot \tilde{\Phi}(k, l_2, P).
\]  

(67)

FIG. 13: Distribution of gluon rapidity. The solid line is the exact distribution, whereas the dotted line represents the distribution obtained from the soft approximator, \( T_s \).

FIG. 14: Corrections to current vertex, and kinematic projection for hard scattering.

In the soft region, \( l_2 \cdot n_1 \simeq l_2^+ \), and \( l_2 \cdot n_T \simeq l_2^- \), so that the accuracy of the approximation is unimpaired. Beyond the soft region, these replacements provide cutoffs in an integral over the rapidity of \( l_2 \). We also make a corresponding change in the \( J_g \) part of the numerator, so that after we apply a Ward identity to sum over all attachments of the gluon to \( \Phi \) and \( \tilde{J} \), we obtain exactly a term where the gluon attaches to a Wilson line operator. Finally, we introduce the \( i\epsilon \) prescriptions appropriate to the directions of the Wilson lines determined in [35].

Let us choose the rapidities \( y_T \) and \( y_1 \) that define the vectors \( n_T \) and \( n_1 \) to match the target and jet rapidities. Then there is a natural correspondence between the rapidity cutoff provided in the soft approximation and that provided by the \( \Phi \) and \( \tilde{J} \) subgraphs before approximation. This is illustrated in Fig. 13.

One further refinement in the exact definition of the approximation is needed to ensure that it works suitably when there are higher-order hard corrections at the electromagnetic vertex, the left-hand-side of Fig. 14. The momentum \( l_2 \) flows through the vertex, so that the hard factor can vary with \( l_2 \). In the soft region (of the \( l_2 \) integral) this is an unimportant power-suppressed effect, but in the complete integral over all \( l_2 \) it can create a big effect. When multiple soft gluons are exchanged, we will be likely to find that the definition of the soft factor needs to be changed from Eq. (15) in an inconvenient way: instead of Wilson lines joining at two point vertices at 0 and \( w \), we will have nonlocal vertices, with the nonlocality governed by detailed properties of the hard scattering. To avoid this issue unambiguously, we define the hard scattering subgraph to be evaluated at suitably projected momenta that stay fixed as \( l_2 \) varies. This is readily done by requiring from the beginning that the external momenta of the hard subgraph always be projected down to the parton model values, as in Fig. 14. There we use the momenta \( \tilde{k} = (-q^+, 0, 0) \) and \( \tilde{l} = q + \tilde{k} = (0, q^- , 0) \) that we defined earlier. Thus the incoming and outgoing momenta of the approximated vertex are exactly massless, on-shell and independent of \( l_2 \). Naturally, we also need to apply the projections \( P_T \) and \( P_J \) in the Dirac algebra, exactly as in the basic parton model.

Putting these elements together gives the definition of the approximator for the soft region for Eq. (63):

\[
T_s \Gamma^{(R)} = \frac{e^2}{4\pi} \int \frac{d^4l_2}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left[ P_3 P_3 \tilde{J}(k + q, l_2) P_T \gamma^\mu P_T l_2 \cdot \Phi(k, l_2, P) \right] \frac{n_1^\kappa J_{g,\kappa \rho}(l_2)n_T^\rho}{(l_2 \cdot n_1 - i\epsilon)(l_2 \cdot n_T + i\epsilon)}.
\]  

(68)

Now apart from the explicit gluon line, the only off-shell external lines of \( \Phi \) and \( \tilde{J} \) are the quark lines at the photon vertex. So an application of a Ward identity to the contraction of \( l_2 \) with these factors, summed over graphs, takes the gluon line outside of \( \Phi \) and \( \tilde{J} \), to give

\[
\sum_{\Gamma} T_s \Gamma^{(R)} = \frac{e^2}{4\pi} \int \frac{d^4l_2}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left[ P_3 P_3 \tilde{J}(k + q, l_2) P_T \gamma^\mu P_T \Phi(k, l_2, P) \right] \frac{g^2 C_F n_1^\kappa J_{g,\kappa \rho}(l_2)n_T^\rho}{(l_2 \cdot n_1 - i\epsilon)(l_2 \cdot n_T + i\epsilon)}.
\]  

(69)

Notice how the bubbles are replaced by those that were used in the parton model, in Eq. (24), but with the momentum in \( \tilde{J} \) shifted by \( -l_2 \). Also, there is an overall minus sign, and a factor \( C_F \) that arises from the color matrices on the quark lines. As for the Ward identity, its derivation is well-known, for the case of the model Abelian gauge theory that we use at the moment — see e.g., [42, p. 339]. We review the derivation in App. [3] in the context of graphs such
as Fig. 11 and Eq. (69). Complications arise in the non-Abelian case which does not yet have a complete treatment. We postpone the issue of the corresponding derivation in a non-Abelian theory (i.e., QCD) to later work.

Finally we rewrite this result using the notation we used for the case of the pure parton model:

$$\sum_{\Gamma} T_{S} \Gamma^{(R)} = \frac{1}{2\pi} \int \frac{d^{4}l_{2}}{(2\pi)^{4}} \int \frac{d^{4}k}{(2\pi)^{4}(-q^{+}q^{-})} |H_{0}(q, \hat{k})|^{2} \Phi^{+}(k, P) J^{-}(k + q - l_{2}) S^{(R,1)}(l_{2}, y_{T}, y_{J}). \tag{70}$$

which we notate diagrammatically in Fig. 15. The indices “+” on $\Phi^{+}$ and “−” on $J^{-}$ now denote, not Lorentz indices for a gluon, but the same projections concerning the leading part of the Dirac matrix structure that we used in the parton model in Sec. IV. We define the one-loop real-gluon contribution to the soft factor as

$$S^{(R,1)}(l_{2}, y_{T}, y_{J}) = \frac{-g^{2}C_{F}}{(2\pi)^{4}} \frac{J_{g,p}(l_{2})n_{T}^{q}}{(l_{2} \cdot n_{J} - i\epsilon)(l_{2} \cdot n_{T} + i\epsilon)}, \tag{71}$$

and as indicated in Fig. 15, this is obtained from the Feynman rules from our definition of the soft factor in Eq. (69). The lowest-order hard-scattering factor squared $|H_{0}|^{2}$ is given by Eq. (69), so it is exactly the same as we found in the parton model approximation. This is necessary if factorization is to hold; the hard scattering does not depend on how many gluons attach to Wilson lines in the soft factor or on their momenta.

D. Accuracy and limits of soft approximation

After application of a Ward identity, we get a simple soft factor in Fig. 15 with one graph for the $l_{2}$-dependent soft factor. However, depending on the precise size and rapidity of $l_{2}$, different kinds of graph will dominate in Fig. 11, before the Ward identities are used. This leads to complications if one wishes to set up factorization by considering individual graphs with on-shell final state partons; as we will now show, different types of graphs dominate depending on how soft the radiated gluon is. Therefore, the unified treatment of the soft region discussed in the previous subsection, with all graphs implicitly included in the final state bubbles, has the notable advantage of dealing with all types of soft gluon behavior at once.

We illustrate this by examining the graphs in Fig. 10 in a model field theory. For this we use an Abelian gauge theory supplemented by a color-singlet scalar field, which we treat as describing the model’s hadrons, and which we use for the measured initial- and final-state particles. By using an Abelian gauge theory, we are allowed a gluon mass $m_{g}$, which can be zero or nonzero; we use this to conveniently illustrate the issues associated with the mass of the gluon.

First, without gluons we have graph (a), which gives a parton-model description of DIS with a parton density and a fragmentation function to describe an observed hadron in the jet — the target splits into a quark and an antiquark, each with a large plus component of momentum, and the incident virtual photon scatters from the quark.

Then we add one extra emitted real soft gluon. To keep the discussion simple, we restrict our attention to the situation where the rest of the final state is near the parton model region. That is, in the final state, the transverse momenta (in the Breit frame) are limited to at most order $\Lambda$, and the hadron and quark fields have masses of order $\Lambda$. The struck quark has a plus momentum of order $Q$, and minus and transverse components less than or of order $\Lambda$. Later we will weaken these conditions.

There are graphs in which the gluon attaches only to the target or jet parts of the graph; these simply provide corrections to the parton density and to the fragmentation. There remain 4 graphs of interest, (b) to (e) in Fig. 10 (plus their Hermitian conjugates).
FIG. 16: Illustrating the different detailed leading regions for an exchanged gluon in a spectator model. The double lines are for a color-singlet scalar field treated as an analog of a hadron. The dashed arrows in graphs (b) and (c) indicate the lines that are pushed far off-shell by the inclusion of the extra gluon.

We first treat the case that the extra emitted gluon has small rapidity and has transverse momentum of order $\Lambda$. Then it pushes some of the quark lines off-shell. For example, in graph (b), these requirements and the other requirements listed above on the spectator antiquark and the outgoing hadron-quark system imply that $(k+q)^2$ and $(k-l_2)^2$ are of order $\Lambda^2$:

\begin{align}
\frac{1}{(q+k)^2 - m^2} &= \frac{1}{(q+k-l_2)^2 - l_2^2 - m^2 + 2l_2 \cdot (k+q)} \sim \frac{1}{\Lambda Q^2}, \tag{72} \\
\frac{1}{(k-l_2)^2 - m^2} &= \frac{1}{k^2 - m^2 + l_2^2 - 2l_2 \cdot k} \sim \frac{1}{\Lambda Q}. \tag{73}
\end{align}

Here we have used the fact that $(k+q)^- \sim Q$ and $k^+ \sim Q$. Although in the upper part of the graph we have routed $l_2$ towards the final state on the quark line of momentum $q+k-l_2$, it is the line $q+k$ that goes off-shell. The two lines listed above are the only such far off-shell propagators in graph (b), so that the two $1/Q$ factors are compensated by a factor of $Q^2$ from the numerator. Thus there is no suppression of graph (b).

In contrast, there are at least three far off-shell propagators in graphs (c) through (e). Therefore, graph (b) dominates, and the others are power suppressed. In graph (c), the lines that are far off-shell are indicated with dashed arrows.

As long as the mass of the gluon is of order $\Lambda$, and the transverse momenta in the target and in the jet system are also of this magnitude then we can regard graph (b) as the only important graph at this order and we are finished.
But if we break with the requirements listed above by allowing the gluon mass to approach zero, then the transverse momentum of the gluon can go much smaller while keeping leading power contributions in graphs (c) through (e). Alternatively, we could raise the transverse momentum in the target and/or jet systems, for example, to order $\sqrt{\Lambda Q}$, while still preserving the essential collinearity. In these situations the relative importance of graphs (b) through (e) in Fig. 14 changes.

Let us use a zero gluon mass and continue to keep its rapidity central. When its transverse momentum is reduced to about $\Lambda^2/Q$, there is no longer a penalty from off-shell propagators: all the denominators are of order $\Lambda^2$. When the gluon’s transverse momentum is reduced much further, graph (b) becomes unimportant, because the decreasing denominators in (72) plateau at order $\Lambda^2$. For such supersoft gluons to contribute we must have denominators that continue to decrease. This happens only for graph (e), where the gluon attaches directly to final-state colored lines. This is in fact just the ordinary IR divergence that appears when a massless gauge boson is emitted from an outgoing on-shell fermion. In a suitable sum over final states, the ordinary IR divergences from vanishing transverse momentum cancel against the graphs for virtual gluon emission.

All of these cases are covered by our general Ward identity argument. We just let a soft gluon couple in all possible ways to the collinear subgraphs. Some of these are smaller than others, but that does not matter.

These results can be generalized to include also collinear gluons, i.e., of large rapidity. This gives Fig. 17(i), where we plot the regions in gluon transverse momentum and rapidity where the different graphs dominate. The figure is labeled by letters that indicate which of the single gluon radiation graphs in Fig. 15 dominate in different kinematical regimes. Naturally, on a boundary between the regions the graphs associated with both sides of the boundary are important. The soft region corresponds to the part of the graph near the vertical axis, the target-collinear region to the part at positive rapidity, and the jet-collinear region to the part at negative rapidities.

We now describe Fig. 17(i) in more detail. We start by describing the top diamond-shaped area where graph (b) gives an important contribution. For a massless real gluon in the final state, we may write its momentum as

$$l_2 = \left( e^y \frac{|l_{2,t}|}{\sqrt{2}}, e^{-y} \frac{|l_{2,t}|}{\sqrt{2}}, l_{2,t} \right).$$

Then the condition that $l_2^2/Q \lesssim 1$ gives one bound on the rapidity for the area where (b) contributes: $y \lesssim \ln(Q/l_2^2)$. Likewise, the condition $l_2^2/Q$ gives a bound $y \gtrsim -\ln(Q/l_2^2)$. These bounds give the top two diagonal lines for the boundary of the graph-(b)-dominant area in Fig. 17(i).

Furthermore, the propagator $q + k - l_2$ should have virtuality at least of order $\Lambda^2$, otherwise we gain by going to graph (d) with the gluon attaching directly to the final-state quark. This gives a bound $y \gtrsim -\ln(Ql_{2,1}/\Lambda^2)$, the lower left edge of the diamond. The corresponding bound, $y \lesssim \ln(Ql_{2,1}/\Lambda^2)$, on the target side gives the remaining side of the diamond. When $l_2$ strongly violates any of these bounds, there is a power-law suppression relative to the largest contribution.

Next we obtain the triangular area where graph (c) becomes important. For graph (c), the upper end of the gluon line still connects to an internal quark line, so the bound $y \gtrsim -\ln(Ql_{2,1}/\Lambda^2)$ also applies to graph (c)’s area, i.e., the continuation of the lower left boundary of the region for graph (b). But on the target side, we need $y \gtrsim \ln(Ql_{2,1}/\Lambda^2)$, to avoid gaining two far off-shell propagators. In addition, there is a suppression of the contribution whenever $y$ is bigger than the target rapidity $y_T$, since then the previous dominance of the $2l_2^2(P^+ - k^+)$ term in the $P - k + l_2$ propagator is cutoff by the term $2l_2^2(P^- - k^-)$. This gives the right-hand vertical line bounding the area for graph (c).

Exactly similar considerations give the area where graph (d) is important.

The bounds relative to target and jet rapidity, $y_T \lesssim y \lesssim y_T$, actually apply to all the graphs, but are automatically implied by the other bounds except for two cases: the target rapidity bound for graph (c) and the jet rapidity bound for graph (d).

Finally, in graph (e), the gluon connects to both of the final-state quarks. Whenever $l_2$ goes far outside the bounds given by the lowest two diagonal lines in Fig. 17(i), pairs of lines on the target or jet side are made off-shell by much more than $\Lambda^2$, and we get a suppression. In addition, the rapidity is restricted to be between the jet and target rapidities. Therefore, graph (e) dominates in the area beneath the wedge shape at the bottom of Fig. 17(i).

Notice that if the gluon had a mass of order $\Lambda$, there would be a suppression of the region with $k_i$ much less than $\Lambda$. Then graph (b) would dominate in the soft region, and graphs (c) and (d) would only contribute in the target- and jet-collinear regions, while graph (e) would always be power suppressed. This shows some of the essential complications caused by the masslessness of the gluon in Feynman graph calculations, even though the regions of supersoft gluon momenta are presumably cut off by non-perturbative confinement in real QCD.

Our definition of the soft approximation, adapted from the Grammer-Yennie paper, is arranged to avoid any need to discuss the details of which graphs are important in different ranges of soft $l_2^2$. The approximations are made in the numerator coupling each jet factor to the soft gluon $l_2$. These approximations become 100% inaccurate when either the energy of the gluon is of order the jet energy or when the rapidity of the gluon is comparable to the jet rapidity. For the coupling to the target, the magnitude of the fractional error is then the maximum of $l_2^2/Q, \Lambda/Q$, and $e^{-(y_T - y)}$, where $y_T$ is the target rapidity. For the coupling to the jet, the fractional error is similarly the maximum of $l_2^2/Q, \Lambda/Q$, etc.
and $e^{-(y-y_3)}$. (The calculation is to be done in the Breit frame, and we have assumed $x_{Bj}$ is not close to unity, otherwise the limits are decreased for accuracy of the coupling to $\Phi$.)

These error estimates are arranged in a form that is equally suitable for the case that $l_2$ is collinear to the jet or target. For the soft approximation, we need to apply all the error estimates simultaneously and then we find its fractional error is the maximum of $\lambda/Q$, $\Lambda/Q$, $e^{-(y_T-y)}$, and $e^{-(y-y_3)}$, where $\lambda$ is defined by Eq. (58). That is the soft approximation is valid when $\lambda \ll Q$, $e^{y_T} \ll e^y \ll e^{y_3}$, and of course when $\Lambda$ and the transverse momentum scale of the jets is much less than the hard scattering scale $Q$. Thus the gluon has to have low momentum and have central rapidity.

Observe that a complication ensues when $x_{Bj}$ gets close to unity. For small or moderate $x_{Bj}$, we expect the outgoing jet to have large positive rapidity as in Fig. 17(i). But, for $x_{Bj} \sim 1$, the target remnant has low energy and rapidity, so that the range of applicability of the soft region is very restricted on the target side, as shown in Fig. 17(ii): The target region has a wide range of rapidity.

**E. Target-Collinear Region**

Now that we have characterized the smallest region, a treatment of the collinear regions follows naturally with the aid of the subtraction method described in Sect. VII. For the contribution of the target-collinear gluons to a graph, such as the single gluon emission graph of Fig. 11, we now construct an approximator, $T_T$. To avoid double counting, the subtraction formalism requires the contribution of the target-collinear region to be obtained by applying $T_T$ to the graph minus its soft approximation:

$$C_T \Gamma^{(R)} = T_T (\Gamma^{(R)} - T_3 \Gamma^{(R)}).$$

(75)

At the top end of the $l_2$ gluon, it attaches to a subgraph where the rapidity is much more negative, when the gluon is in the target-collinear region. All the same issues about leading polarizations apply as for the case that the gluon is soft, so at this end of the gluon the approximator is very similar to the soft approximator $T_S$.

As we discussed earlier, there are two ways we could define the target-collinear regions. One is that the gluon has low transverse momentum (e.g., of order $\Lambda$) and its plus momentum is of order $Q$ in the Breit frame. In the case of the gluon-exchange graphs in Fig. 16 only graphs (b) and (c) are then important; graphs (d) and (e) are suppressed because they have extra off-shell propagators. There is one far off-shell line: the quark $k + q$ in between the gluon attachment to the upper subgraph and the hard vertex has virtuality of order $Q^2$. We could correctly consider this line as part of the hard subgraph. (Note that although $l_2$ is not routed through this line, we have imposed a low-mass requirement on the hadron-quark part of the final state, immediately to its right. So the large plus momentum of $l_2$ actually flows on the $k + q$ line.)

However, when the gluon is massless, its transverse momentum can be very small, and then it can be useful to define the target-collinear regions by the gluon rapidity being comparable to the target rapidity, $e^y \sim e^{y_T}$. As shown in Fig. 17(i), which graphs contribute depends on exactly how small the transverse momentum is. In all cases, the same method of approximation applies for the coupling of the gluon to the upper subgraph $\mathcal{J}$. The Ward identity argument for summing the contributions will work: it will extract the gluon from the upper subgraph and convert to couple to a Wilson line independently of which of the graphs are involved.

In view of these issues, it is not totally obvious where to apply the projection on the Dirac matrices. So we formulate the approximator in two stages. The first stage just involves writing a Ward-identity-compatible form for the coupling to $\mathcal{J}$, just as in Eq. (67), to obtain...
\[ T_T \Gamma^{(R)} = \frac{\alpha_s^2}{4\pi} \int \frac{d^4l_2}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left[ \gamma^\nu l_2 \cdot \mathcal{J} (k + q, l_2) \gamma^\mu \phi^\rho (k, l_2, P) \right] \frac{n_s^\rho}{l_2 \cdot n_s - i\epsilon} J_{g,\kappa \rho}(l_2). \] (76)

Here we use the vector \( n_s \) of nearly zero rapidity, rather than the vector \( n_J \) of very negative rapidity that we used in the corresponding part of the soft approximation. This provides a cutoff at central rapidities, as appropriate for a collinear region. When \( l_2 \) has a large positive rapidity, both vectors agree in projecting out the plus component: \( l_2 \cdot n_s \simeq l_2 \cdot n_J \simeq l_2^+ \).

Again, as before, we apply a Ward identity, but now only on the jet side, to obtain

\[ \sum \Gamma T_T \Gamma^{(R)} = \frac{1}{2\pi} \int \frac{d^4k}{(2\pi)^4} \frac{1}{-q^+ q^-} \mathcal{J}^-(k + q) |H_0(q, \hat{k})|^2 \left\{ \int \frac{d^4l_2}{(2\pi)^4} \text{Tr} \left[ \gamma^\nu \mathcal{J} (k + q, l_2) \gamma^\mu \frac{g n_s^\rho J_{g,\kappa \rho}(l_2)}{l_2 \cdot n_s - i\epsilon} \phi^\rho (k, l_2, P) \right] \right\}. \] (77)

We identify the square bracket factor in Eq. (77) as a contribution to the numerator of the PCF defined in Eq. (45) with the gluon coupling to the Wilson line. The remaining factor in the integrand is exactly the same as the jet factor in the ordinary handbag diagram, Eq. (24), aside from a different labeling of the loop momenta.

The definition of the collinear approximator is completed by applying the parton-model approximator at the electromagnetic vertex, as illustrated in Fig. 18. We also make a shift of integration variable, replacing \( k \) by \( k - l_2 \), so as to make the correspondence with the simple handbag diagram clear.

\[ \sum \Gamma T_T \Gamma^{(R)} = \frac{1}{2\pi} \int \frac{d^4k}{(2\pi)^4} \frac{1}{-q^+ q^-} \mathcal{J}^-(k + q) |H_0(q, \hat{k})|^2 \left\{ \int \frac{d^4l_2}{(2\pi)^4} \text{Tr} \left[ \gamma^\nu \mathcal{J} (k + q, l_2) \gamma^\mu \frac{g n_s^\rho J_{g,\kappa \rho}(l_2)}{l_2 \cdot n_s - i\epsilon} \phi^\rho (k, l_2, P) \right] \right\}. \] (78)

Here \( \mathcal{J}^- \) and \( |H_0|^2 \) are exactly the same jet factor and hard scattering coefficient as in the simplest handbag diagram Fig. (11a), and the factor in braces corresponds to the numerator in the definition of the PCF Eq. (45) with one gluon connecting to the Wilson line.

Finally we need the double-counting-subtraction term in Eq. (75). The new item needed is the result of first applying the soft approximator \( T_S \) and then the collinear approximator \( T_T \). The collinear approximator simply modifies the upper vertex of \( l_2 \) by replacing the vector \( n_J \) by the vector \( n_s \):

\[ T_T T_S \mathcal{J}^\kappa (k + q, l_2) = T_T \mathcal{J} (k + q, l_2) \cdot l_2 \frac{n_s^\kappa}{l_2 \cdot n_s - i\epsilon} \]

\[ = \mathcal{J} (k + q, l_2) \cdot l_2 \frac{n_1^\kappa}{l_2 \cdot n_1 - i\epsilon} \frac{n_s^\kappa}{l_2 \cdot n_s - i\epsilon} \]

\[ = \mathcal{J} (k + q, l_2) \cdot l_2 \frac{n_s^\kappa}{l_2 \cdot n_s - i\epsilon}. \] (79)

Applying Ward identities now gives us a term just like that for the soft approximation Fig. (11a) except with the change from \( n_J \) to \( n_s \). However, we now must identify the eikonal factor as part of the PDF.

We therefore write the contribution of the target-collinear region as

\[ \sum \Gamma C_T \Gamma^{(R)} = \frac{1}{2\pi} \int \frac{d^4k}{(2\pi)^4} \frac{1}{-q^+ q^-} |H_0(q, \hat{k})|^2 \mathcal{J}^- (k + q) F_{(R, 1)}(k, P), \] (80)
where

\[
F(R,1)(k, P) \equiv \int \frac{d^4 l_2}{(2\pi)^4} \left\{ J_{g,\rho}(l_2) \frac{g n_s^\rho}{l_2 \cdot n_s - i\epsilon} \text{Tr} \left[ \frac{\gamma^+}{4} \tilde{\Phi}^\rho(k, l_2, P) \right] - \Phi^+(k + l_2, P) S^{(R,1)}(l_2, n_T, n_s) \right\}. \tag{81}
\]

We now recognize \( F(R,1)(k, P) \) as a part of the PCF due to one-gluon exchange with the Wilson lines in its definition, Eq. \( (45) \). The one-gluon exchange term is found from expanding the Wilson line operators in powers of the coupling. Then the first term in \( (81) \), as we have already noted, is from the numerator of Eq. \( (45) \). The second term arises from the \( O(g^2) \) term in the denominator. In coordinate space, this multiplies the lowest order term in the numerator. Fourier transformation gives the convolution product in the second term in Eq. \( (81) \), as illustrated in Fig. 19.

These results support the correctness of Eq. \( (45) \) as the definition of the PCF.

### F. Jet-collinear region

The treatment of the remaining region, the jet-collinear region, is very similar — one follows the same steps as in the target collinear case. For completeness we state the result here. The result for the contribution of this region is

\[
\sum_{i} C_i \Gamma^{(R)} = \sum_{i} T_j \left[ \Gamma^{(R)}(k, l_2) - T_s \Gamma^{(R)}(k, l_2) \right] - \left[ \Gamma^{(R)}(k, l_2) + T_j \Gamma^{(R)}(k, l_2) + T_s \Gamma^{(R)}(k, l_2) \right]. \tag{82}
\]

where the one-gluon-exchange contribution to the fragmentation PCF is

\[
J^{(R)}(k + q) = \int \frac{d^4 l_2}{(2\pi)^4} \left\{ \text{Tr} \left[ \frac{\gamma^-}{4} \tilde{J}^\rho(k + q, l_2) \right] \frac{-g J_{g,\rho}(l_2) n_s^\rho}{l_2 \cdot n_s - i\epsilon} - \frac{\gamma^-}{4} \tilde{J}^-(k + q - l_2) S^{(R,1)}(l_2, n_T, -n) \right\}. \tag{83}
\]

In obtaining this by modifying the derivation for the target-collinear region, we replaced \( n_s \) by \(-n_s\), in accordance with the results of \[35\] and the discussion in sect. \[V\].

### G. Hard Gluons

Although the issue of extracting the NLO hard scattering component is beyond the scope of this article, it is worth describing the general method we would follow. The NLO hard contribution from the single gluon emission diagrams will follow naturally in the subtraction formalism:

\[
\Gamma^{R,NLO}_{H}(k, l_2) = \Gamma^{(R)}(k, l_2) - \left[ \Gamma^{(R)}(k, l_2) + \Gamma^{(R)}_T(k, l_2) + \Gamma^{(R)}_J(k, l_2) \right]. \tag{84}
\]

Here, \( \Gamma^{(R)}_{S,T,J}(k,l) \) represents the result of applying the soft, hard, or jet approximator to the unapproximated graph.

### IX. Virtual Corrections

In this section we analyze in detail the diagrams with virtual corrections for the case of one gluon attachment.

First, we note that all situations with gluons entirely confined to the jet or target subgraphs are already covered by the basic parton-model argument in Sec. \[IV\].
These gluons are already included in what we mean by the jet and target subgraphs.

Therefore we need to analyze graphs with a gluon exchanged between the jet and target subgraphs, Fig. 20. The general procedure follows that for real gluon emission, except that the gluon is on the left of the final-state cut in both bubbles. The subscript ‘vc’ means virtual correction. This formula is just like (63) except that the gluon is too small, and the second assertion becomes invalid.

\[
\Gamma^{\text{V}}_{\text{vc}} = \frac{e^2 P_{\mu\nu}}{4\pi} \int \frac{d^4 l_2}{(2\pi)^4} \int \frac{d^4 k}{(2\pi)^4} \text{Tr}[\gamma^\nu \tilde{J}^\mu(k+q,l_2)\gamma^\mu \tilde{\Phi}^\rho(k,l_2,P)] G_{\kappa\rho}(l_2),
\]

where \(\tilde{\Phi}^\rho\) and \(\tilde{J}^\rho\) are the bubbles with one extra gluon attached. These bubbles are the same as for real gluon emission, except that the gluons are on the left of the final-state cut in both bubbles. The subscript ‘vc’ means virtual correction. This formula is just like (63) except that the cut dressed gluon propagator is replaced by an uncut propagator:

\[
J_{g:\kappa\rho}(l_2) \rightarrow G_{\kappa\rho}(l_2).
\]

### A. Soft region

We start with the smallest region, the soft one, which was appropriately defined in Sec. VII C. Following the procedure used for real-gluon emission, we find that the Grammer-Yennie method applied to (85) gives

\[
\Gamma^{\text{V}}_{\text{vc,soft}} = \frac{e^2 P_{\mu\nu}}{4\pi} \int \frac{d^4 l_2}{(2\pi)^4} \int \frac{d^4 k}{(2\pi)^4} \text{Tr}[\gamma^\nu \tilde{J}^\mu(k+q,l_2)\gamma^\mu \tilde{\Phi}^\rho(k,l_2,P) \cdot l_2] \times \frac{n_1^a G_{\kappa\rho}(l_2)n_2^a}{(n_1 \cdot l_2 - i\epsilon)(n_2 \cdot l_2 + i\epsilon)}.
\]

which exactly corresponds to Eq. (83) for the real-gluon case. Unlike the real-gluon case, we now have to worry about the Glauber region, \(|l_2^+ l_2^-| \ll l_2^\perp\), since the integration over a virtual gluon momentum includes arbitrarily small values of \(l_2^\perp\). To obtain the soft approximation we needed the assertions \(l_2 \cdot \tilde{\Phi} \simeq l_2^- \tilde{\Phi}^+\) and \(\tilde{J} \cdot l_2 \simeq \tilde{J}^\perp l_2^\perp\). The first assertion becomes invalid when \(l_2^\perp\) is too small, and the second assertion becomes invalid when \(l_2^-\) is too small. The conditions for the validity of both parts of the soft approximation can be deduced from Sec. VII C.

\[
\frac{\Lambda^2}{(q^-)^2} \ll \frac{|l_2^+|}{l_2^\perp} \ll \frac{(P^+)^2}{\Lambda^2},
\]

\[
\frac{|l_2^+|}{l_2^\perp} \gg \frac{\Lambda}{q^-}, \quad \frac{|l_2^-|}{l_2^\perp} \gg \frac{\Lambda}{P^+}.
\]

These estimates assume that the transverse momentum in the jet and target subgraphs are of order \(\Lambda\). The first pair of conditions simply state that the rapidity of the gluon must be well inside the range between the jet and target rapidities, as is natural for the soft region. The second pair of conditions are that the longitudinal components of \(l_2\) should not be too much smaller than the transverse momentum; from them can be deduced that we need

\[
|l_2^\perp| \gg l_2^\perp \frac{\Lambda^2}{Q^2}.
\]

A breakdown only of the conditions on the rapidity of \(l_2\) simply takes us to one of the collinear regions, and that need not concern us here since we will treat the collinear regions separately. However a breakdown of the other conditions is problematic. We see from (90) that such a breakdown brings us to the Glauber region, i.e., to \(|l_2^\perp l_2^-| \ll l_2^\perp\).

As explained in [35, 44], we can apply a contour deformation to get out of the Glauber region. The contour deformation is to be applied to \(l_2^\perp\) only, since the only significant dependence on \(l_2^\perp\) in the Glauber region is in the jet subgraph. All the relevant singularities are final-state singularities, and are therefore all in the upper half-plane; thus the same deformation works for all graphs. This is shown graphically in the complex \(l_2^\perp\) plane shown in Fig. 21 where the crosses represent the final state poles. An attempt to apply a corresponding argument to the other longitudinal component \(l_2^\perp\) would fail, because there can be both initial- and final-state singularities for \(l_2^\perp\) in the target subgraph. For the application of a Ward identity, it is essential to have a single contour deformation applied to \(l_2\) for every graph that is summed by the Ward identity. The choice of \(i\epsilon\) in the eikonal denominators, particularly \(n_3 \cdot l_2 - i\epsilon\), was determined by compatibility with the contour deformation of \(l_2^\perp\) away from final-state poles in the jet subgraph. Depending on the size of \(l_2^\perp\), the contour deformation may take \(l_2\) to the
conventional soft region or to the target-collinear region. In either case we have a situation for which we have an applicable technique.

Now that we have deformed the contour integration out of the Glauber region, the soft approximation is valid over the whole of the soft region, as defined by the rapidity condition. Therefore we can apply Ward identities to the sum over graphs, just as with real-gluon emission, to obtain

$$\Gamma_{\text{vc,soft}}^{(V)} = \frac{e^2 P_{\mu\nu}}{4\pi} \int \frac{d^4k}{(2\pi)^4}$$

$$\text{Tr} \left[ P_{\gamma^\nu} P_{\gamma} J(k + q, P_T) \Phi(k, P) \right] \times$$

$$\times \int \frac{d^4l_2}{(2\pi)^4} \frac{g^2 C_F}{n_T^2 G_{\kappa\rho} r_T^2}$$

where we have performed summation over colors and the strong coupling constant has been taken out of the vertices $\gamma_\kappa$ and $\Phi_\rho$. The soft gluon has factorized from the rest of the graph. Effectively the soft gluon only sees the total color charge and direction of the target and jet lines, and has no sensitivity to the details.

The expression in the last line of (91)

$$\tilde{S}^{(V,1)}(y_T - y_1)$$

$$= g^2 C_F \int \frac{d^4l_2}{(2\pi)^4} \frac{n_T^2 G_{\kappa\rho} n_T^2}{(n_1 \cdot l_2 - i\epsilon)(n_T \cdot l_2 + i\epsilon)}$$

is the lowest order term of the vacuum expectation value of two Wilson lines, exactly corresponding to the relevant term in our definition of the soft factor in Eq. (43) when the intermediate state between the left three Wilson lines and the right three is the vacuum state.

The use of non-lightlike lines, in directions $n_T$ and $n_1$ cuts off the rapidity divergences that would occur if light-like lines were used (45). We could follow an alternative procedure, suggested in (45) of using light-like Wilson lines, but with extra generalized renormalization factors in the definition of the soft factor to cancel the rapidity divergences. But we will not follow this idea here.

We do remark that that Eq. (92) does have a UV divergence. We define it to be removed by applying renormalization, as usual: this is an ordinary UV divergence associated with the cusp joining the two segments of the Wilson line.

**B. Target-collinear region**

The treatment of the target-collinear region for $l_2$ works exactly as in the real emission case. But we must apply the approximators and the subtraction to the graph with the contour deformed (to avoid the Glauber region), in order for the double-counting subtraction for the soft region to be correct. This requires that in the Grammer-Yennie approximation for the collinear region we apply the $i\epsilon$ prescription to the eikonal denominator that is compatible with the contour deformation:

$$\gamma_\nu \tilde{J}(k + q, l_2) \gamma_\mu \tilde{\Phi}(k, l_2, P) G_{\kappa\rho}$$

$$\simeq \gamma_\nu \tilde{J}(k + q, l_2) \cdot l_2 \gamma_\mu \frac{n_T^2 G_{\kappa\rho}}{l_2 \cdot n_s - i\epsilon} \tilde{\Phi}(k, l_2, P) .$$

This leads to a first form of the approximator:

$$\text{Tr} \Gamma^{(V)} = \frac{e^2 P_{\mu\nu}}{4\pi} \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4l_2}{(2\pi)^4}$$

$$\text{Tr} \left[ \gamma_\nu \tilde{J}(k + q, l_2) \cdot l_2 \gamma_\mu \tilde{\Phi}(k, l_2, P) \right] \frac{n_T^2 G_{\kappa\rho}}{l_2 \cdot n_s - i\epsilon} .$$

Naturally, we must use the same eikonal denominator for both real and virtual gluon emission in order that all the contributions to the target PCF arise from the same Wilson line.

The remaining steps follow exactly as for real emission: (i) Subtraction of a soft term as in Eq. (75) to compensate double counting with the smaller region. (ii) Application of a leading-power approximation to the hard scattering. (iii) Use of Ward identities. The result is:

$$\sum \Gamma C_T \Gamma^{(V)} = \frac{1}{2\pi} \int \frac{d^4k}{(2\pi)^4} \frac{1}{-q^+ q^-}$$

$$|H_0(q, k)|^2 \tilde{J}^\gamma(k + q) F_{(V,1)}(k, P) ,$$

with the 1-gluon virtual gluon contribution to the parton correlation function defined as

$$F_{(V,1)}(k, P) = \int \frac{d^4l_2}{(2\pi)^4} \left[ \tilde{\Phi}(k, l_2, P) \frac{-g C_F}{l_2 \cdot n_s - i\epsilon} \right]$$

$$- \tilde{\Phi}^+(k, P) S^{(V,1)}(y_T - y_1) .$$

This is exactly the sum of the contributions caused by one virtual gluon coupling to the Wilson line(s) in the target PCF.
C. Jet-collinear region

The case in which the gluon is collinear to the outgoing jet can be worked out in complete analogy with the previous cases. The result is

$$\sum_{\Gamma} C_{\Gamma} \Gamma^{(V)} = \frac{1}{2\pi} \int \frac{d^4k}{(2\pi)^4} \frac{1}{-q^+ q^-}$$

$$|H_0(q, \hat{k})|^2 J_{(V,1)}(k + q) \Phi^+(k, P), \quad (97)$$

where

$$J_{(V,1)}(k, P) = \int \frac{d^4l_2}{(2\pi)^4} \tilde{J}^\alpha(k, l_2, P) \frac{-g C_F G_{\rho\kappa} n^\rho}{l_2 \cdot n_s - i\epsilon}$$

$$- J^-(k, P) g^{(V,1)}(y_s - y_1). \quad (98)$$

To summarize, a graphical depiction of the resulting factorization for the virtual contribution is shown in Fig. 22.

D. Hard vertex correction

We have simplified our work by restricting to final states without particles or jets of extra high transverse momentum. Thus we did not need to treat real corrections where extra partons are emitted from the hard scattering. We treated collinear and soft gluons, but not hard gluons.

For virtual corrections, the situation is different: loop momenta are not restricted by the external states, and can be arbitrarily large. However, in the graphs entailed by Fig. 20, the only one in which a hard gluon gives a leading-power contribution is in the vertex graph of Fig. 23. This graph is, of course, to be considered as a particular subgraph for Fig. 20.

So we now present the correction to the hard scattering coefficient, associated with the vertex graph. To it is to be added the complex conjugate contribution from the same correction on the current vertex on the right of the final-state cut. Let us use $V$ to denote the vertex graph. Its contributions where the gluon is soft or collinear have already been allowed for, so there remains only the contribution from the hard region. In accordance with our subtraction formalism, this contribution is

$$C_H V = T_H [1 - T_S - T_T (1 - T_S) - T_J (1 - T_S)] V$$

$$= T_H [1 - T_T - T_J (1 - T_T - T_J) T_J] V$$

$$= -ig^2 \int \frac{d^4r}{(2\pi)^4} \mathcal{P}_T \left\{ \frac{\gamma^\alpha \gamma \cdot (\hat{l} - r) \gamma^\mu \gamma \cdot (\hat{k} - r) \gamma_\alpha}{[(\hat{l} - r)^2 + i\epsilon][\hat{k} - r)^2 + i\epsilon]} \right\}$$

$$+ \mathcal{P}_T \left\{ \frac{\gamma^\mu \gamma \cdot (\hat{l} - r) \gamma \cdot n_s}{(-r \cdot n_s + i\epsilon)[\hat{k} - r)^2 + i\epsilon]} \right\} + \mathcal{P}_T \left\{ \frac{\gamma \cdot n_s \gamma \cdot (\hat{l} - r) \gamma^\mu}{[(\hat{l} - r)^2 + i\epsilon](-r \cdot n_s + i\epsilon)} \right\} \right\}$$

$$+ \text{UV counterterm.} \quad (99)$$

Our normal recipe for a hard scattering required us to insert the projection matrix $\mathcal{P}_T$ at each side, and to use the massless on-shell external momenta $k$ and $l$ that were defined earlier. We also used the fact that in Feynman gauge the three subtraction terms involving $T_S$ exactly cancel. These terms $-T_H (1 - T_T - T_J) T_J V$ are like those in Eq. (99) but with the factor in braces replaced by

$$\frac{u_J \cdot u_T}{(-u_J \cdot r + i\epsilon)(-u_T \cdot r - i\epsilon)} + \frac{n_s \cdot u_T}{(-n_s \cdot r + i\epsilon)(-u_T \cdot r - i\epsilon)} + \frac{u_J \cdot n_s}{(-u_J \cdot r + i\epsilon)(-n_s \cdot r + i\epsilon)}, \quad (100)$$

which is exactly zero. Since we take the massless limit in the hard scattering, we have replace the non-lightlike vectors $n_T$ and $n_J$ by their lightlike counterparts $u_T$ and $u_J$.

X. FULL FACTORIZATION

We now have enough techniques to obtain a full factorization formula, valid at the leading power in $Q$ including
all logarithmic corrections. However, we will restrict our
derivation to a model theory with massive Abelian gluons. This avoids certain complications with Ward identities
in a non-Abelian theory and with actual IR divergences associated with the masslessness of the gluon. These
complications we leave to later work. The model exhibits the issues of gluons coupling subgraphs with differ-
ent kinds of momenta and the need for appropriate Wilson lines in the definitions of the PCFs. Even so, the
formulation of factorization, with the definitions of the PCFs already exhibited, is equally applicable to QCD.

As we have done throughout this paper, we also restrict to final states with low transverse momenta in the
Breit frame. Thus the leading regions do not involve extra groups (or jets) of collinear partons emitted from the
hard scattering. Since this is a restriction on the final state, and our approximation methods leave unaltered
all final state momenta, this is a safe restriction appropriate for exhibiting the simplest and some of the most
important cases where PCFs are important.

In Fig. 24 we give a graphical overview of the proof:

- In graph (a) we symbolize the most general leading region. It has a connected hard subgraph associated
  with each of the external vertices for electromagnetic current. It has a connected collinear sub-
  graph for the target and for the jet, each of which has, in addition to the standard quark connection,
  arbitrarily many gluons connecting it to the hard subgraphs. It has a soft subgraph connected by
  gluons to both collinear subgraphs. The soft subgraph has arbitrarily many connected components
  (including zero as a possibility), and each component must couple to both collinear subgraphs.

Let $R$ denote any such region of a general graph $\Gamma$
for DIS.

- We then apply the operation $CR\Gamma$ to obtain the contribution associated with this region. We can
  now understand graph (a) to denote $CR\Gamma$ with a sum over $R$ and $\Gamma$ to give the complete leading-
  power contribution. Subtractions, as defined in Eq. (51), ensure that the contributions from smaller regions
  are suppressed. That is, when the loop momenta are close to the defining surface of a smaller
  region $R'$, there is an actual power-law suppression, and when the loop momenta are far from this
  surface, there is a subtraction to prevent double counting between $CR\Gamma$ and $CR'\Gamma$.

- In accordance with our earlier discussions, we do not apply any restriction on the internal momenta
  of the graph; any suppression comes purely from overall momenta conservation, from the nature of the
  subtractions, and/or from any restrictions explicitly imposed in the definition of the approxima-
  tors.

- Then we apply Ward identities to the connections of the soft subgraph to the collinear subgraphs.

One of the reasons we have been insisting on not applying restrictions to the internal momenta in a
graph, is that this is necessary in order that Ward identities work exactly. The derivations of Ward
identities involve shifts of loop momenta, so that any internal restriction on loop momenta is liable to
violate the Ward identity: A shift of an integration variable near a boundary can move a momentum
across the boundary.

This contrasts with the situation in Soft-Collinear Effective Theory (SCET) [46], and other methods
related to a Wilsonian renormalization group. In SCET, fields are defined with explicit restrictions
to particular localities in momentum space, and it is not at all obvious how this is to be actually im-
plemented, at least not with the exact preservation of Ward identities.

In the present context of deriving a proper factorization formula, with the anticipated power sup-
pressed corrections, a non-exact Ward identity presents a serious problem. In a particular region,
one may be tempted to apply an approximate Ward identity to the sum over graphs. Close to
the particular region under investigation, the terms that violate the exact Ward identity may be
power suppressed. However, within the subtrac-
tion formalism (which allows for a proper derivation of a factorization formula) we encounter equations
like (51), where the approximated graphs are used far outside their corresponding regions, where the
leftover terms in the Ward identities are not small.

- In an Abelian theory, application of a Ward identity is entirely straightforward, and after the sum over
all relevant graphs the gluons from the soft subgraph are attached to vertices for Wilson lines, one
Wilson line for each quark at the hard scattering. We must define the approximator for the region $R$
so that the approximated hard subgraphs are ex-
actly independent of the soft momenta. Then the
Wilson lines meet at a point and we have separated out a soft factor — Fig. 24(b). The soft factor is a
PCF defined by Eq. (43).

There are some subtleties in defining the approxima-
tors so that the hard subgraphs become exactly independent of the soft momenta; we will return to
this issue.

- The Ward identities apply most directly to the approximated graph $TR\Gamma$; then the collinear sub-
graphs are just ordinary Green functions. How-
ever, the double counting subtractions for smaller regions — Eq. (51) — change this situation, and
we must also ensure, as we will do later, that the double counting subtractions are compatible with the
Ward identities.

- Finally, we apply Ward identities to the gluons con-
necting the collinear subgraphs to each of the two
hard subgraphs. This gives Fig. 24(c), giving the jet and target PCF factors. We will have to investigate more carefully the subtractions for smaller regions, and we need to show that these produce the denominators in the definitions (45) and (47).

- The hard subgraphs are now restricted to having the same external lines as in the parton model.

Loop graphs have subtractions for non-hard regions, so that we have a properly defined hard coefficient.

A. Factorization formula

The resulting factorization formula is...
\[
P_{\mu \nu} W^{\mu \nu} = \int \frac{d^4 k_T}{(2 \pi)^4} \frac{d^4 k_J}{(2 \pi)^4} \frac{d^4 k_S}{(2 \pi)^4} (2 \pi)^4 \delta^{(4)}(q + P - k_T - k_J - k_S) \times \nonumber \\
\times |H(Q, \mu)|^2 S(k_S, y_T, y_J, \mu) F_{\text{mod}}(k_T, y_T, y_S, \mu) J_{\text{mod}}(k_J, y_J, y_S, \mu),
\]

where \( S, F_{\text{mod}} \) and \( J_{\text{mod}} \) are the Fourier transforms [Eq. (18)] into momentum space of the soft, target, and jet PCFs defined in Eqs. (43), (45), and (47). The variables \( k_T, k_J, \) and \( k_S \) are the momenta of the final states in the target, jet, and soft PCFs.

This formula can be simplified. The target (and jet) PCFs have two rapidity arguments for Wilson lines, which is in contrast to the situation in the CS formalism. This can be changed by moving the denominators from the target and jet PCFs to the soft PCF, so that the soft PCF is redefined to

\[
\tilde{S}_1(w, y_T, y_J, y_S, \mu) = \frac{\tilde{S}(w, y_T, y_J, \mu)}{S(w, y_T, y_S, \mu) \tilde{S}(w, y_S, y_J, \mu)}.
\]

In this soft factor, the denominators remove the contributions from large positive and negative rapidities. We expect to be able to take the limits that the vectors \( n_T \) and \( n_J \) become light-like:

\[
\tilde{S}_2(w, y_S, \mu) = \tilde{S}_1(w, +\infty, -\infty, y_S, \mu).
\]

This corresponds to the soft factor defined by Collins and Soper [2].

Then the factorization formula becomes

\[
P_{\mu \nu} W^{\mu \nu} = \int \frac{d^4 k_T}{(2 \pi)^4} \frac{d^4 k_J}{(2 \pi)^4} \frac{d^4 k_S}{(2 \pi)^4} (2 \pi)^4 \delta^{(4)}(q + P - k_T - k_J - k_S) \times \nonumber \\
\times |H(Q, \mu)|^2 S_2(k_S, y_S, \mu) F(k_T, y_T, y_S, \mu) J(k_J, y_S, \mu),
\]

where we use the target PCF \( F \) defined by Eq. (44) instead of \( F_{\text{mod}} \) (and similarly for the jet PCF) defined by Eq. (45). Note that there is a difference from the CS case, where the soft factor is independent of \( y_S \). The CS soft factor is defined appropriately for \( k_J \) factorization, so that it is the integral over \( k_S^+ \) and \( k_S^- \) of the soft factor defined here, to give a function of transverse momentum alone. This quantity is invariant under boosts in the \( z \) direction. In contrast, our soft factor depends on longitudinal momenta as well.

### B. Momentum routing in approximators

It is important for factorization that the hard scattering coefficient should depend only on \( Q^2 \). It should not depend on the loop momenta in the soft factor. Therefore in constructing the approximator for a region, we need to arrange that the hard subgraphs are approximated as independent of the soft momenta. In addition, so that the hard scattering coefficients can be treated as on-shell matrix elements (modified by subtractions), we will approximate their external parton momenta by on-shell massless momenta. So that we may be sure that this can be done in general, for a graph with arbitrarily many soft and collinear gluons, we must be certain that the approximators that we use do not introduce any anomalous dependence on soft gluon momenta, and that all approximations are exactly consistent with the application of Ward identity relations.

The difficulty in constructing a general prescription obeying these requirements can be seen from the simple case of Fig. 11. There, the trouble comes from the correct choice for “routing” momenta. By a choice of routing we mean the choice of which momenta are to be treated as independent variables. Different choices correspond to different explicit appearances of momentum variables around different loops in the graph. Of course, the choice of momentum labeling is arbitrary and has no effect on an unapproximated Feynman graph. However, the approximators are defined with respect to a certain set of variables. Hence, the same instructions for approximating a graph will lead to different results for different routings of momentum. In other words, we can say that the approximators are not completely defined until a choice of momentum routing is made.

For example, in Fig. 11 the soft momentum \( l_2 \) is routed through the left-hand vertex. Thus, it appears natural to define the approximated momenta for the hard vertex by replacing \( l_2 \) by zero and by then keeping only the minus and plus components of the quark momentum. That is, in the hard vertex we perform the replacements \( q + k - l_2 \mapsto (0, q^- + k^-, 0_\ell) \) and \( k - l_2 \mapsto (k^+, 0, 0_\ell) \) on the external momenta of the left hand current vertex. At the right-hand vertex there is no soft momentum, so we simply make the replacements \( q + k \mapsto (0, q^- + k^-, 0_\ell) \).
and $k \mapsto (k^+, 0, 0)$.  

Observe that the value of minus momentum on the quark line going to the jet was changed on the left-hand side, but not on the right-hand side, and similarly for the quark from the target subgraph. We can reverse this situation by simply changing the routing of $l_2$ to go through the right-hand current vertex, for example by changing variables to $k_1 = k - l_2$. In that case the momenta at the left-hand vertex are $q + k_1$ and $k_1$, while the momenta at the right-hand vertex are $q + k_1 + l_2$ and $k_1 + l_2$. Thus the definition of the approximation varies depending on the routing of the loop momenta, and the two routings we have shown are equally legitimate.

In the true soft region, the components of $l_2$ are small compared with $Q$ and so the difference between the definitions is a small, power-suppressed effect. But we perform an integral over all kinematically accessible momenta, so the difference amounts to a genuine inconsistency.

Nevertheless, the inconsistency did not affect our treatment of Fig. 11 because its hard subgraphs are single vertices and hence independent of momentum. But if we consider a more general situation, as in Fig. 21 with non-trivial hard subgraphs, the momentum dependence of the hard subgraphs creates an inconsistency between (at least) two possible definitions of approximation for the hard subgraph.

A further problem appears when we observe that the approximated minus momentum on the $q + k - l_2$ and $q + k$ lines is not $q^-$, and that the approximated plus momentum on the $k - l_2$ and $k$ lines is not $-q^+$. This creates the situation that, beyond lowest order in the hard scattering, the hard scattering coefficient does not depend just on $Q$. Again, in a situation of really collinear momenta this is a small effect, but we integrate the PCFs out beyond this region.

These issues are closely related to the kinematic inconsistencies in parton showering algorithms that were found by Bengtsson and Sjöstrand [27], and that led [3, 4, 5, 6] to the proposal to use PCFs rather than regular parton densities.

The essential difficulty in defining the approximations is that we have to know what are the independent variables in the various subgraphs. However, the momenta of different lines are constrained by momentum conservation and are not all independent. The dangers are made even worse by our pervasive use of Ward identities. Graphical proofs of Ward identities require shifts of integration variables. Consistency of Ward identities with the approximations requires that the approximations be invariant under certain reroutings of loop momenta. As we will now see, a consistent prescription for labeling momenta involves treating all of the outgoing parton momenta as independent variables, but treating the photon momentum as a dependent variable.

Our solution is two fold: In the first step we route all the momenta from the soft subgraph out through the hard vertex; i.e., the photon momentum is not treated as an independent variable, and is not fixed to $q$. See Fig. 23 for a graphical depiction of the basic setup. This choice of independent variables is sufficient to treat both hard subgraphs the same way, and gives a routing consistent with all the Ward identities we apply. The second step, after approximating the external lines of the hard subgraph as before, is to define a new set of variables by rescaling the approximated minus and plus momenta so that outgoing jet momenta sum to $(0, q^-, 0, 0)$ and the incoming jet momenta sum to $(-q^+, 0, 0)$. We end up with a treatment of momenta that is similar to the prescription of Collins and Zü [8], but is more complicated because of the need to treat soft subgraphs.

Now let us go through the steps described above explicitly. We let $L_{S2}$ denote the collection of momenta entering the jet-collinear subgraph from the soft subgraph. Then we write the collection of momenta leaving the hard subgraph to the jet-collinear subgraph as

$$\begin{align*}
L_t - M_{ij}L_{S2}. \tag{105}
\end{align*}$$

Here $M_{ij}$ is an incidence matrix. Thus for a particular outgoing jet line, labeled by an index, $j$, we have

$$\begin{align*}
l_j - \sum_k M_{ijk}l_{S2k}. \tag{106}
\end{align*}$$

where the sum is over the soft lines outgoing from the subgraph $\mathcal{J}$. Similarly, we write the collection of momenta entering the hard subgraph from the target-collinear subgraph as

$$\begin{align*}
L_t + M_{tj}L_{S1}. \tag{107}
\end{align*}$$

Thus for a particular incoming target line, we have

$$\begin{align*}
l_t + \sum_k M_{tjk}l_{S1k}. \tag{108}
\end{align*}$$

where the sum is over the soft lines outgoing from the subgraph $\Phi$. Momentum conservation on the final state will impose some constraints between the momenta.
That is, we project down to the minus and plus components of the collinear momenta only. After this projection of momentum components, the next step is to apply a rescaling so that the momenta sum to \( q \) in the hard vertex. More specifically, (109) and (110) mean that for each of the outgoing and incoming lines, we project onto a new set of variables as follows:

\[
\begin{align*}
\tilde{l}_{1j} &= \sum_k M_{1jk} l_{S2k} \mapsto \lambda_j (0, l_{1j}, 0_t), \\
\tilde{l}_{Tj} &= \sum_k M_{Tjk} l_{S1k} \mapsto \lambda_T (l_{Tj}^+, 0, 0_t).
\end{align*}
\]

As before, a “hat” on a momentum variable indicates the approximation that arises from the replacement of exact momenta with approximated momenta in the hard subgraph. For a gluon coupling to the hard subgraph from the target-collinear subgraph, we make the replacement

\[
\begin{align*}
\left( J \right)_{\kappa} &= \left( \tilde{J} \right)_{\kappa}, \\
\kappa &= l_{S2j} \cdot n_s - i \epsilon l_{s2j} \cdot J (l_{S2j}, \ldots).
\end{align*}
\]

Hence we get the following definitions for the scalings:

\[
\begin{align*}
\lambda_j &= \frac{q^-}{\sum_j \tilde{l}_{1j}^-}, \\
\lambda_T &= \frac{-q^+}{\sum_j \tilde{l}_{Tj}^+}.
\end{align*}
\]

We also define the approximator so that the fermion lines are equipped with projection matrices, \( \mathcal{P} \), in the amplitude and \( \mathcal{P} \) in the complex conjugate. At the gluons we apply an appropriate Grammer-Yennie-style approximation, as in Eq. (76).

To summarize, we have generalized (and modified) the replacement scheme of Sect. 11 and Ref. 2 in such a way that we now consistently deal with the presence of soft gluons.

There is one non-obvious step for the Grammer-Yennie approximation that arises from the replacement of exact momenta with approximated momenta in the hard subgraph. For a gluon coupling to the hard subgraph from the target-collinear subgraph, we make the replacement

\[
\mathcal{H}^c (l_{Tj}, \ldots) \mapsto \frac{n_s^c}{l_{Tj} \cdot n_s - i \epsilon} \tilde{l}_{Tj} \cdot \mathcal{H} (\tilde{l}_{Tj}, \ldots).
\]

Here \( l_{Tj} \) denotes the momentum of the line flowing into the hard subgraph on the gluon line. In the hard part we replace this momentum by its approximation defined in Eq. (110). So that the relevant Ward identity is exactly valid, we have performed the same replacement in the factor of momentum contracted with the hard subgraph. In contrast the denominator is left unaltered. This enables the \( n_s \) vector to fulfill its purpose of cutting off the integral over the rapidity of \( l_{Tj} \). This unaltered denominator also ensures that we get exactly the expected Wilson line operator for the target PCF.

Similar definitions apply to the gluonic connections from the other collinear subgraph. To summarize, the sequence of replacements listed in this subsection defines an approximator which (a) leaves the hard scattering subgraph independent of soft momenta, and (b) allows for the exact application of Ward identities.

C. Ward identities for soft into collinear subgraphs

We first examine the Ward identity argument for the connection of soft gluons from subgraph \( B \) to the jet subgraph \( J \) in Fig. 24(a). For each soft gluon the Grammer-Yennie approximation is applied, by a replacement of the form

\[
J^c (l_{S2j}, \ldots) \mapsto \frac{n_s^c}{l_{S2j} \cdot n_s - i \epsilon} l_{S2j} \cdot J (l_{S2j}, \ldots),
\]

where \( l_{S2j} \) is the soft-gluon momentum oriented to enter subgraph \( J \). Now the sum over regions and graphs for the whole process can be written as independent sums over the different subgraphs, subject to consistency on the kinds of lines joining them. So we now sum over subgraphs \( J \) with the other subgraphs fixed, and also apply the same argument for the connection of the soft subgraph to the target-collinear subgraph. In the absence of the double-counting subtractions, a standard Ward identity applied to every gluon connecting the soft subgraph to the collinear subgraphs would result in Fig. 24(b).

However, to each graph is applied a series of subtractions from smaller regions, as in Eq. (51), so we must examine their compatibility with the Ward identities. So let \( R \) denote a particular region of the form of Fig. 24(a), for which we wish to construct its contribution as in Eq. (51). The direct Ward identity argument applies to every gluon connecting the soft subgraph to the collinear subgraphs would result in Fig. 24(b).

The following discussion entails considering the relation between the two different regions \( R \) and \( R' \) for a single graph \( \Gamma \) and also treating a sum over the possibilities for \( \Gamma \), \( R \), and \( R' \), as in

\[
\sum_{\Gamma, R} C_{R\Gamma} = \sum_{\Gamma, R} T_R \left( \Gamma - \sum_{R' < R} C_{R' \Gamma} \right) = \sum_{\Gamma, R} \left( T_R \Gamma - \sum_{R' < R} T_{R'} T_R \Gamma + \ldots \right).
\]
Each region can be specified in terms of its $J$, $\Phi$, $H$ and $B$ subgraphs, so we will use a notation $J(R)$, $B(R)$ etc for the jet and soft subgraphs, etc, associated with the region $R$. The region $R'$ has its corresponding $J(R')$, $B(R')$ subgraphs etc.

Since $R'$ is smaller than $R$, it imposes tighter constraints on the momentum categories of the lines. Hence, as regards the soft lines, the set of lines categorized (or labeled) as soft for $R'$ is at least as big as the set of lines labeled as soft for $R$. That is, the soft subgraph of $R'$ is at least as big as the soft subgraph of $R$, i.e., $B(R') \supseteq B(R)$, as shown in Fig. 26. In forming the term $T_R C_{R'} \Gamma$ in Eq. (119), first the approximator $T_{R'} \Gamma$ is applied and then $T_R$. (There can be further subtractions inside $C_{R'} \Gamma$, but that need not concern us here.)

In the representation in Fig. 26 the approximator $T_R \Gamma$ applies some operation at the boundary of $B(R)$, while $T_{R'}$ applies some operation at the boundary of $B(R')$. (There are also operations applied at the boundaries between the collinear and hard subgraphs, but that will be our topic later.) Since $B(R') \supseteq B(R)$, the operation $T_{R'} \Gamma$ has no effect on the soft subgraph $B(R)$ for the original region $R$. But some lines of jet subgraph $J(R)$ may be soft according to $R'$, so the approximator $T_{R'} \Gamma$ has an effect inside $J(R)$, when we form the term $T_R T_{R'} \Gamma$.

An example of the situation we wish to discuss is shown in Fig. 27. For each graph $\Gamma_1$ and $\Gamma_2$ in the left-hand column, suppose we have constructed for each graph the term $C_{R'} \Gamma_j$ for the region shown in the right-hand column where the two gluons are both soft. Next we wish to construct the the term $C_{R'} \Gamma_j$ for the bigger region shown in the middle column, where only one gluon is soft and the other is jet-collinear. After application of the soft approximation, a Ward identity, discussed below, will let us combine the two ways of attaching the red gluon to the upper quark line. There are also subtractions to prevent double-counting. The nature of the soft approximation lets us combine the subtraction terms related to the right-hand column in the same way as in the middle column, thus ensuring that the Ward identities are compatible with subtractions.

Suitably viewed, this pair of graphs actually leads us to the general case. First, let us examine the standard graphical derivation of the Ward identity. The basic step is the following identity applied to the connection of a soft gluon to one quark line, after the replacement $\{118\}$ is applied:

$$\frac{-i g n_j^3}{l \cdot n_1 - i \epsilon \left( \frac{k}{k} - m \right)} = \frac{i}{k \cdot l - m} \left( \frac{i}{k - m} - \frac{i}{k - l - m} \right),$$

which can be written graphically as

$$\begin{align*}
\begin{array}{c}
k - l \\
l \end{array} & \quad \begin{array}{c}
k \\
\end{array} \\
& \quad \begin{array}{c}
k - l \\
\end{array}
\end{align*}
$$

(120b)

On the left-hand side, the double line and the arrow at the end of the gluon denote, respectively, the first factor and the $l_{S2j}$ factor in $\{118\}$. The double-line notation is to exhibit that its factor corresponds to Feynman rules for a Wilson line. The big dots at the vertices on the right hand side of (120b) are used to emphasize the special vertex where the Wilson-line component attaches to the rest of the graph.

We now examine what happens when we take two different graphs for $J(R)$ that are related by having the gluon $l$ attached on opposite sides of some other vertex, with another gluon $l_1$. Embedded inside the subgraph $J(R)$, we have the following situation:

$$\begin{align*}
\begin{array}{c}
k - l - l_1 \\
l_1 \end{array} & \quad \begin{array}{c}
k \\
\end{array} \\
& \quad \begin{array}{c}
k - l \\
\end{array}
\end{align*}
$$

(121a)

$$\begin{align*}
\begin{array}{c}
k - l - l_1 \\
l \end{array} & \quad \begin{array}{c}
k \\
\end{array} \\
& \quad \begin{array}{c}
k - l \\
\end{array}
\end{align*}
$$

(121b)
An example of this situation is provided by Fig. 27, with the gluon \( l \) being the red gluon in the middle column.

After we sum over the two graphs, on the left-hand-side, there is a cancellation between the two terms where the special vertex is at the second gluon, i.e., of the first term on the right of Eq. (121a) with the second term on the right of Eq. (121b):

\[
- = 0
\]

(122)

Then follows a cascade of such cancellations when we sum over all ways of attaching gluon \( l \) to the unsubtracted jet-collinear subgraph, i.e., when we sum over the relevant possibilities for \( J(R) \), e.g., over the the middle column in Fig. 27:

There remain only terms at the end of fermion lines, and those at on-shell ends give zero. Summing over all graphs for \( J(R) \) gives

\[
\text{\ldots} = \text{\ldots}
\]

(123)

where the dots indicate arbitrarily many gluon connections, both between \( J(R) \) and the hard subgraph, and between \( J(R) \) and the soft subgraph \( B(R) \). Applying the same argument to all the gluons from \( B(R) \) to \( J(R) \), and then summing over all cases for \( B(R) \), gives accumulated Wilson line components at the left which exactly correspond to the Feynman rules for the \( n_J \) Wilson line.
shown in Eq. (121) we have subregions \( R \) to work in the presence of subtractions, we make a 1-to-1 correspondence between the regions \( R \) and \( R' \) appropriate to the two graphs in \( \mathcal{J}(R) \), considered as embedded in graphs for \( \mathcal{J}(R) \).

For example, in Eq. (121a) we can consider two regions: The larger region we call \( R_a \) for the case that the other gluon \( l_1 \) is inside \( \mathcal{J}(R_a) \), i.e., is collinear to the outgoing jet — we refer to \( R_a \) as the outer region. There can be a subregion \( R'_a \) for which this gluon is labeled soft, but such that the fermions are still labeled collinear. The subtraction for this region involves applying a Grammer-Yennie approximation at the end of gluon \( l_1 \). But when we exchange the two gluons, so that we are instead considering Eq. (121b), the same assignment of momentum types still corresponds to a leading region. Thus, we have a outer region \( R_b \) and a subregion \( R'_b \) for Eq. (121a) analogous to \( R_a \) and \( R'_a \) for Eq. (121a). In both of the cases shown in Eq. (121) we have subregions \( R'_j \) (embedded in some graph for \( \mathcal{J}(R_j) \)), where the fermions are collinear and \( l_1 \) is soft, and exactly the same approximation is applied at the end of the gluon \( l_1 \) in the second line. Note that the outer regions, \( R_a \) and \( R_b \), are not literally the same because they apply to a different overall graphs for \( \mathcal{J}(R_j) \); the same applies to the subregions, of course.

Of course, we have to sum over all possibilities for leading-power subregions. There is only a very limited set of cases, and the same argument applies to all the other possibilities. Hence the cancellation (122) also applies in the presence of subtractions. This means that, the Ward identities apply both to the unsubtracted graphs and the subtracted graphs, given our definitions of the approximators.

### D. Ward identities for collinear into hard subgraphs

The situation is somewhat more complicates with the Ward identities for collinear gluons connecting to a hard subgraph.

First let us examine a lowest-order connection of a target-collinear gluon to the hard subgraph, Fig. 28. The vector in the approximation is now \( n_s \) instead of \( n_J \). Of the two terms in the elementary Ward identity (120), only the first survives. The second term is zero because in the hard subgraph the quark \( k_J \) is replaced by a massless one with an on-shell momentum, and the associated Dirac-matrix projector \( P_T \) is equivalent to using an on-shell Dirac wave function. Although the hard subgraph is like an on-shell matrix element, one graph is missing, Fig. 29(a), which has the target-collinear gluon connecting to the target-collinear quark. Thus the sum over graphs — really just one graph here — gives the appropriate Wilson line factor for the target PCF. This procedure readily generalizes to all the unsubtracted graphs.

The only subregion and hence the only subtraction for Fig. 28 is where the gluon is soft. As can be seen from Eq. (79), this has no effect on the approximator as applied at the hard subgraph (the hard vertex), so the subtraction leaves the Ward identity unaltered.

A complication concerning subtractions does arise because there is no longer always 1-to-1 correspondence between the subregions \( R' \) used in the subtractions for different terms in the sum (over \( R \) and \( \Gamma \)) to which we wish to apply a Ward identity argument, of the kind we
applied to soft gluons in Eq. (121a). To illustrate the situation explicitly, we analyze the case of two gluons, labeled $l_1$ and $l_2$ attaching to the jet quark line — the two graphs in Fig. 30. For each of these graphs, we will consider two regions. One is the region where both gluons are target collinear. We will call this region $R_{CC}$ (the outer region). The second, smaller region is where $l_1$ remains target collinear, but $l_2$ becomes soft. We will call this region $R_{CS}$, and we will need to take this region into account in setting up subtractions. To connect with the previous discussion, we note that $R_{CC}$ corresponds to the outer region, $R$, and $R_{CS}$ corresponds to the smaller region, $R'$. (Of course, there are other subregions we could consider. We consider this collection of regions for the purpose of illustration.)

Looking first at the outer region, $R_{CC}$, we can immediately sum over the two possible graphs (a) and (b), giving the appropriate two-gluon coupling to the Wilson line for the target PCF. This is shown graphically in Fig. 31(a). Note that in the hard subgraph, the approximator replaces the collinear momenta by their large components. Thus $k_3-l_1-l_2$ is replaced by $(-l_1^+, -l_2^+, k_j^-, 0_t)$ (with some possible scaling as well). Next we consider the smaller region, $R_{CS}$, which will be relevant for subtractions. We will refer to the contribution from a particular graph to $R_{CS}$ as the subregion for that graph. For the subtractions obtained from subregions for graph (a) the one possibility we consider here, $R_{CS}$, is that gluon $l_1$ is collinear and gluon $l_2$ is soft, and this generates a subtraction for graph (a). But for graph (b) this region is not leading, and there is no subtraction. Thus the subregions do not correspond between graphs. (A similar situation occurs with the graphs and momentum assignments reversed.) However, as far as the exact application of the Ward identity is concerned, the contribution from graph (b) is not needed in the subtraction terms. This is because in the subtraction term for graph (a), depicted in graph (c), the approximator $T_{R_{CS}}$ for the subregion sets the intermediate quark line, that propagates between the two gluon attachments, on-shell, i.e., $k_3-l_2$ is replaced by $(0, k_j^-, 0_t)$. (Note the extra “hook” that appears on the intermediate line connecting the two gluon attachments in graph (c).) In order for the Ward identity to work, there is, therefore, no need to also consider graph (b) in the subtractions. In other words, after the application of $T_{R_{CS}}$, graph (c) is already in the desired form. The application of the Ward identity in the smaller region gives the graphical equation in Fig. 31(b).

For the subregion subtraction, we also need to apply Ward identities to the gluon $l_2$ that is labeled soft in the subregion. Without the approximator $T_{R_{CS}}$, we would need a cancellation with something related to the crossed graph (b). But since the approximator discussed in the previous section makes the inner subgraph exactly independent of $l_2$, this does not matter. It should be recalled that we specifically chose the definitions of our soft approximators so that these steps would work, i.e., so that Figs. 31(a) and 31(b) would both be exact.

A fully detailed and explicit treatment is left to a future work.

XI. CONCLUSIONS, OUTLOOK, AND FUTURE WORK

In this paper, we have set up a factorization framework for deep inelastic scattering in QCD in which the exact kinematics of the partons is conserved. We have shown that this seemingly simple requirement leads to a non-trivial and significant conceptual shift even at the lowest, parton model level. The requirement of exact kinematics means that one has to abandon standard integrated parton densities and fragmentation functions, and instead use the parton correlation functions.

The exact treatment of kinematics imposes particularly stringent requirements on the methods by which factorization is formulated and derived in the presence of gluon exchanges between the different subgraphs (collinear and hard) with different kinds of parton momentum. We have shown in detail how this works for one gluon exchange. Then we extended the results quite generally, to obtain a factorization property with defi-
nite gauge invariant operator definitions for the PCFs as well as for the soft factor. These definitions possess additional parameters (related to rapidity) which are introduced via non-light-like Wilson lines, comparable to those in the Collins-Soper-Sterman formalism for TMD distributions.

So far, the factorization formula we have derived involves a rather trivial hard scattering coefficient. However, having precisely formulated the factorization for the zeroth order hard scattering, the structure of higher order hard scattering coefficients can be readily determined using the subtraction formalism. A calculation of this type is in progress.

We expect that a reasonably straightforward generalization of the methods of Ref. \[6\] will be suitable. Thus the hard scattering will be obtained from on-shell partonic cross sections with subtractions that compensate double counting with regions associated with lower-order hard scattering. The subtractions remove singular contributions where some of the partons in the hard scattering become collinear or soft. Because of our use of exact parton kinematics, the subtractions will be applied point-by-point in parton momentum, and will result in hard scattering coefficients that are ordinary integrable functions of external parton momenta. Thus they will correctly represent the corrections to parton probabilities differential in parton kinematics. In contrast, standard methods of collinear factorization result in hard-scattering coefficients that contain non-trivial generalized functions, e.g., the well-known plus distributions. Such distributions are very singular and cannot represent the detailed differential distribution of parton kinematics. They only give correct results for cross sections that involve a broad average over parton kinematics.

There is still much work to do. First, the evolution equations for the PCFs need to be derived, presumably a natural generalization of the the CSS equations. Then we need to obtain methods for higher-order corrections to the hard scattering matrix elements. We need to extend the derivations to a non-Abelian gauge theory. In addition, we need to determine if and how this formalism can be recast in terms of PCFs defined with light-like Wilson lines, but with factors that cancel light-cone divergences, as in \[10\].

Acknowledgments

We would like to thank Andreas Metz for discussions. We thank Markus Diehl for proof reading. Feynman diagrams were made using JaxoDraw \[48\]. A.M.S. was supported by the Polish Committee for Scientific Research grant No. KBN 1 P03B 028 28. All the authors were supported by the U.S. D.O.E. under grant number DE-FG02-90ER-40577.

APPENDIX A: PROJECTION MATRICES

The projection matrices are defined as follows:

\[ P_T = \frac{1}{2} \gamma^- \gamma^+, \quad P_J = \frac{1}{2} \gamma^+ \gamma^- , \] \hspace{1cm} (A1)

where

\[ \gamma^- = \frac{1}{\sqrt{2}} (\gamma_0 - \gamma_3), \quad \gamma^+ = \frac{1}{\sqrt{2}} (\gamma_0 + \gamma_3). \] \hspace{1cm} (A2)

The projectors satisfy the following relations

\[ P_T + P_J = 1, \] \hspace{1cm} (A3)
\[ P_J \gamma^- = \gamma^+ P_J = 0, \] \hspace{1cm} (A4)
\[ P_T \gamma^+ = \gamma^- P_T = 0, \] \hspace{1cm} (A5)
\[ P_J = \gamma^0 P_J^\dagger \gamma^0 = P_T, \] \hspace{1cm} (A6)
\[ P_T \gamma^- P_J = \gamma^-, \] \hspace{1cm} (A7)
\[ P_J \gamma^+ P_T = \gamma^+, \] \hspace{1cm} (A8)
\[ P_T^2 = P_T, \] \hspace{1cm} (A9)
\[ P_J^2 = P_J. \] \hspace{1cm} (A10)

APPENDIX B: THE ELEMENTARY WARD IDENTITY

We have made frequent use of Ward identities to disentangle soft and collinear gluons from the soft and collinear bubbles such as those in Fig. 11(c). In this appendix we review the derivation of the elementary Ward identity for an Abelian gauge theory. We treat the case where a soft gluon attaches to the outgoing jet bubble – the upper bubble in Fig. 11. Analogous results hold for the other situations we have considered that require Ward identities (such as when the gluon attaches at the target bubble) and follow from similar arguments.

Consider the contribution to the outgoing jet bubble represented by the following sum of graphs:

\[ (B1) \]
Here the outgoing quark attaches to two additional gluons as it enters the final state on the left side of the final state cut. (The other ends of these extra gluons may attach anywhere else in the jet bubble). In this example, we assume that the final state quark is on shell. The soft gluon insertion is represented by the attachment with a double line and arrow, indicating that we have made the approximations discussed in the main text that lead to Eq. (68). In particular, the soft momentum, \( l_2 \) has been contracted with the soft gluon vertex, and there is a division by \( l_2 \cdot n_J \).

The sum of graphs in (B1) contributes the following factor to the full graph:

\[
\frac{ig^3}{l_2 \cdot n_J} \bar{u}(l_1) l_2 \left( \frac{i}{l_1 - l_2 - m} \right) \gamma_{\rho_1} \left( \frac{i}{l_1 - l_2 - k_1 - m} \right) \gamma_{\rho_2} \left( \frac{i}{l_1 - l_2 - k_1 - k_2 - m} \right) + \frac{ig^3}{l_2 \cdot n_J} \bar{u}(l_1) \gamma_{\rho_1} \left( \frac{i}{l_1 - l_2 - k_1 - m} \right) l_2 \left( \frac{i}{l_1 - l_2 - k_1 - m} \right) \gamma_{\rho_2} \left( \frac{i}{l_1 - l_2 - k_1 - k_2 - m} \right) + \frac{ig^3}{l_2 \cdot n_J} \bar{u}(l_1) \gamma_{\rho_1} \left( \frac{i}{l_1 - k_1 - m} \right) \gamma_{\rho_2} \left( \frac{i}{l_1 - k_1 - k_2 - m} \right) l_2 \left( \frac{i}{l_1 - l_2 - k_1 - k_2 - m} \right). \tag{B2}
\]

Let us call the first term A, the second term B and the third term C. Now we notice that we can eliminate quark propagators and utilize the Dirac equation by substituting the following trivial identities for \( f_2 \):

\[
\begin{align*}
I_2 &= -(I_1 - I_2 - m) + (I_1 - m) & \text{in term A} \\
I_2 &= -(I_1 - I_2 - k_1 - m) + (I_1 - k_1 - m) & \text{in term B} \\
I_2 &= -(I_1 - I_2 - k_1 - k_2 - m) + (I_1 - k_1 - k_2 - m) & \text{in term C}.
\end{align*}
\]

These are each of the form of a difference of the denominators for the two quark lines next to the vertex for the \( l_2 \) gluon, so that

\[
\begin{align*}
A &= \frac{g^3}{l_2 \cdot n_J} \bar{u}(l_1) \gamma_{\rho_1} \left( \frac{i}{l_1 - l_2 - k_1 - m} \right) \gamma_{\rho_2} \left( \frac{i}{l_1 - l_2 - k_1 - k_2 - m} \right), \tag{B3} \\
B &= -\frac{g^3}{l_2 \cdot n_J} \bar{u}(l_1) \gamma_{\rho_1} \left( \frac{i}{l_1 - I_2 - k_1 - m} \right) \gamma_{\rho_2} \left( \frac{i}{l_1 - I_2 - k_1 - k_2 - m} \right) + \frac{g^3}{l_2 \cdot n_J} \bar{u}(l_1) \gamma_{\rho_1} \left( \frac{i}{l_1 - k_1 - m} \right) \gamma_{\rho_2} \left( \frac{i}{l_1 - I_2 - k_1 - k_2 - m} \right), \tag{B4} \\
C &= -\frac{g^3}{l_2 \cdot n_J} \bar{u}(l_1) \gamma_{\rho_1} \left( \frac{i}{l_1 - k_1 - m} \right) \gamma_{\rho_2} \left( \frac{i}{l_1 - I_2 - k_1 - k_2 - m} \right) + \frac{g^3}{l_2 \cdot n_J} \bar{u}(l_1) \gamma_{\rho_1} \left( \frac{i}{l_1 - k_1 - m} \right) \gamma_{\rho_2} \left( \frac{i}{l_1 - k_1 - k_2 - m} \right). \tag{B5}
\end{align*}
\]

In the sum, all the intermediate terms cancel, to leave only the last term,

\[
A + B + C = \frac{g^3}{l_2 \cdot n_J} \bar{u}(l_1) \gamma_{\rho_1} \left( \frac{i}{l_1 - k_1 - m} \right) \gamma_{\rho_2} \left( \frac{i}{l_1 - k_1 - k_2 - m} \right). \tag{B6}
\]

The soft gluon \( l_2 \) has been factored out of the rest of the graph leaving only an over-all factor corresponding to the eikonal line propagator and eikonal vertex. (See Ref. [47] for a review of Feynman rules involving eikonal lines.)

Exactly the same pattern applies no matter how many gluons attach to the quark line. This results in the general identity for \( N_J \) extra gluons shown graphically in Fig. 32.

As desired, the sum of graphs with a soft gluon insertion is replaced by the graph with no soft gluon, and an over-all factor giving the expected eikonal line.

If we repeat the above argument for multiple soft gluon insertions, we obtain the graph shown in Fig. 33. Each eikonal line gives a propagator factor, \( 1/(h_{j} \cdot n_J) \) for \( j \) running from 1 to \( N_J \) (From here on, the variables \( \{h_1, h_2, \ldots, h_{N_J}\} \) will denote the collection of soft gluon momenta). Figure 33 can be re-written in a way that corresponds more directly with the Feynman rules of Wilson lines if we note that the product of eikonal propagators can be written as,

\[
\prod_{j=1}^{N_J} \frac{1}{h_{j} \cdot n_J} = \sum_{\{1,2,\ldots,N_J\}} \frac{1}{[n_J \cdot h_1] [n_J \cdot (h_1 + h_2)] \times \ldots \times [n_J \cdot (h_1 + h_2 + \ldots + h_{N_J})]}.
\tag{B7}
\]

The summation sign means that we sum over all permutations of the momenta \( \{h_1, h_2, \ldots, h_{N_J}\} \) in the denominator on the right side. That is, Fig. 33 is equivalent to summing all possible ways of attaching the soft gluons to a single eikonal line. To illustrate, we show this graphically in Fig. 34 for the simple case of two soft gluons.
FIG. 32: Applying the Ward identity to factorize a single soft gluon insertion.

FIG. 33: Result of applying the Ward identity with multiple soft gluon insertions.

FIG. 34: Using Eq. (B7) to rewrite Fig. 33 for the case of two soft gluon insertions.

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