ESTIMATES FOR THE CORONA THEOREM ON $H_1^\infty(\mathbb{D})$

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Abstract. Let $\mathcal{I}$ be a proper ideal of $H^\infty(\mathbb{D})$. We prove the corona theorem for infinitely many generators in the algebra $H_1^\infty(\mathbb{D})$. This extends the finite corona results of Mortini, Sasane, and Wick [8]. We also provide the estimates for corona solutions. Moreover, we prove a generalized Wolff’s Ideal Theorem for this sub-algebra.

1. Introduction

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be an open unit disk in the complex plane $\mathbb{C}$ and $H^\infty(\mathbb{D})$ be the set of all bounded analytic functions with the norm $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)| < \infty$. In 1962, Carleson proved his famous corona theorem which states that the ideal, $\mathcal{I}$, generated by a finite set of functions $\{f_i\}_{i=1}^n \subset H^\infty(\mathbb{D})$ is the entire space $H^\infty(\mathbb{D})$, if for some $\epsilon > 0$, $\sum_{i=1}^n |f_i(z)|^2 \geq \epsilon$ for all $z \in \mathbb{D}$. In 1979, Wolff gave a simplified proof of Carleson’s corona theorem, which can be found in [5], that made use of $H^2$-Carleson’s measures and Littlewood-Paley expressions. Both Carleson and Wolff provided the bounds for corona solutions depending on the number of functions $n$. Later, Rosenblum [14], Tolokonnikov [20], and Uchiyama [26], independently, extended the corona theorem for infinitely many functions, where as the best estimate for the corona solution was due to Uchiyama as follows:

Corona Theorem. Let $\{f_i\}_{i=1}^\infty \subset H^\infty(\mathbb{D})$, with

$$0 < \epsilon^2 \leq \sum_{i=1}^\infty |f_i(z)|^2 \leq 1 \text{ for all } z \in \mathbb{D}.$$  

Then there exist $\{g_i\}_{i=1}^\infty \subset H^\infty(\mathbb{D})$ such that

$$\sum_{i=1}^\infty f_i(z)g_i(z) = 1 \text{ for all } z \in \mathbb{D}$$

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and
\[ \sup_{z \in \mathbb{D}} \left\{ \sum_{i=1}^{\infty} |g_i(z)|^2 \right\} \leq \frac{9}{\epsilon^2} \ln \frac{1}{\epsilon^2}, \quad \text{for } \epsilon^2 < \frac{1}{e}. \]

The main purpose of this paper is to extend the corona theorem for infinitely many functions in \( H_\infty^I(\mathbb{D}) \). Moreover, we provide the estimates for the corona solutions. This will completely settle the conjecture of Ryle [15].

The algebra, \( H_\infty^I(\mathbb{D}) \), of our interest is defined as follows:

Let \( I \) be any proper closed ideal in \( H_\infty(\mathbb{D}) \), and define
\[ H_\infty^I(\mathbb{D}) := \{ c + \phi | c \in \mathbb{C} \text{ and } \phi \in I \}. \]

Then \( H_\infty^I(\mathbb{D}) \) is a sub-algebra of \( H_\infty(\mathbb{D}) \). We regard \( (H_\infty^I(\mathbb{D}))_{\ell^2} \) as a sub-algebra of \( H_\infty^I(\mathbb{D}) \), where \( H_\infty^I(\mathbb{D}) \) is a sequence of bounded analytic functions. Also, for \( F = (f_1, f_2, \ldots), f_j \in H_\infty(\mathbb{D}) \), we use the norm
\[ \| F \|_\infty = \sup_{z \in \mathbb{D}} \left( \sum_{i=1}^{\infty} |f_i(z)|^2 \right)^{1/2}. \]

In [8], Mortini, Sasane, and Wick proved the corona theorem for finitely many generators in \( H_\infty^I(\mathbb{D}) \). In fact, [8] provided the estimates on the solutions \( g_j \) in terms of the parameters \( \epsilon \) and \( n \) (the number of functions \( f_j \)). In this paper, we prove an analogous result of Uchiyama for the sub-algebra \( H_\infty^I(\mathbb{D}) \) by removing the dependency of estimates on \( n \).

Let \( f \in H_\infty^I(\mathbb{D}) \), say \( f(z) = c + \phi(z) \), for \( \phi \in I \) and \( c \in \mathbb{C} \). For simplicity, we use the notation: \( f(z) = f_c + \phi_f(z) \), where \( f_c \in \mathbb{C} \) and \( \phi_f \in I \). Similarly, let \( F = (f_1, f_2, \ldots), f_j \in H_\infty^I(\mathbb{D}) \). Then for \( z \in \mathbb{D} \), we write \( F(z) = f_c + \phi_F(z) \).

We are now ready to state our Main Theorem, which extends to the corona theorem for infinitely many functions in \( H_\infty^I(\mathbb{D}) \).

**Theorem 1.1.** Let \( F(z) = (f_1(z), f_2(z), \ldots), f_j \in H_\infty^I(\mathbb{D}) \) and
\[ 0 < \epsilon^2 \leq F(z)F(z)^* \leq 1 \quad \text{for all } z \in \mathbb{D}. \]

Then there exists \( U = (u_1(z), u_2(z), \ldots), u_j \in H_\infty^I(\mathbb{D}) \) such that
\begin{itemize}
  \item[(a)] \( F(z)U(z)^T = 1 \) for all \( z \in \mathbb{D} \) and
  \item[(b)] \( \| U \|_\infty \leq \left( 1 + \frac{1}{\| F_c \|} \right) \frac{9}{\epsilon^2} \ln \left( \frac{1}{\epsilon^2} \right) \).
\end{itemize}
In order to generalize the corona theorem, it is natural to ask if the corona theorem still holds true if we replace the lower bound, $\epsilon$, in the corona condition by any $H^\infty(D)$ functions. Namely, let $h, f_1, f_2, \ldots, f_n \in H^\infty(D)$ such that
\[ |h(z)| \leq \sum_{i=1}^{n} |f_i(z)| \leq 1 \text{ for all } z \in \mathbb{D}. \]  
(1)

Then the question is does (1) always implies $h \in \mathcal{I}(f_1, f_2, \ldots, f_n)$, ideal generated by $f_1, f_2, \ldots, f_n$? Of course, (1) is a necessary condition, but the counter example provided by Rao [12] suggests that it is far from being sufficient.

Rao’s Counter Example: If $B_1$ and $B_2$ are Blaschke products without common zeros for which $\inf_{z \in \mathbb{D}} (|B_1(z)| + |B_2(z)|) = 0$, then $|B_1 B_2| \leq (|B_1|^2 + |B_2|^2)$, but $B_1 B_2 \notin \mathcal{I}(B_1^2, B_2^2)$.

However, T. Wolff’s beautiful proof (see [5], Theorem 2.3 in page 319) showed that the condition (1) is sufficient for $h \in \mathcal{I}(f_1, f_2, \ldots, f_n)$. Wolff’s Theorem can be rephrased as follows:

Wolff’s Theorem. Let $F(z) = (f_1(z), f_2(z), \ldots, f_n(z))$, $f_j \in H^\infty(D)$, $h \in H^\infty(D)$. If
\[ |h(z)| \leq \sqrt{F(z)F(z)^*} \text{ for all } z \in \mathbb{D}, \]
then
\[ h^3 \in \mathcal{I}(\{f_j\}_{j=1}^{n}). \]
But, it was shown by Treil [21] that this is not sufficient for $p = 2$.

Many authors, independently, have considered this question, including Cegrell [2], Pau [11], Trent [23], and Treil [22] for $p = 1$. We refer this as a problem of “ideal membership.” It is Treil who has given the best known sufficient condition for ideal membership. We state Treil’s Theorem as follows:

Ideal Theorem (Treil). Let $F(z) = (f_1(z), f_2(z), \ldots, f_n(z))$, $f_j \in H^\infty(D)$, $F(z)F(z)^* \leq 1$ for all $z \in \mathbb{D}$, and $h \in H^\infty(D)$ such that
\[ F(z)F(z)^* \psi(F(z)F(z)^*) \geq |h(z)| \text{ for all } z \in \mathbb{D}, \]
where $\psi : [0, 1] \rightarrow [0, 1]$ is a non-decreasing function such that $\int_{0}^{1} \frac{\psi(t)}{t} \, dt < \infty$. Then there exists $G \in H^\infty_{\ell_1}(\mathbb{D})$ such that
\[ F(z)G(z)^T = h(z), \text{ for all } z \in \mathbb{D}. \]
An example of a function \( \psi \) that works in the case when \( F(z) \) is an \( n \)-tuple, \( n < \infty \), is
\[
\psi(t) = \frac{1}{(\ln t^{-2})(\ln_2 t^{-2}) \ldots (\ln_n t^{-2})(\ln_{n+1} t^{-2})^{1+\epsilon}},
\]
where \( \ln_k(t) = \ln \ln \ldots \ln \) \( k+1 \) times \( t \) and \( \epsilon > 0 \).

Applying Treil’s result, we extend the analogue of “ideal theorem” on \( H_\infty^I(D) \). Recall that \( H_\infty^I(D) \) is a sub-algebra of \( H_\infty(D) \). Also, for \( F = (f_1, f_2, \ldots), f_j = f_{cj} + \phi f_j \in H_\infty^I(D) \), we denote \( F = F_c + \phi F \). In the case that \( F_c = 0 \), several authors have given sufficient conditions for ideal membership, for example, see \([6], [7], \) and \([13]\). For the case \( F_c \neq 0 \), we provide the following theorem:

**Theorem 1.2.** Let \( F(z) = (f_1(z), f_2(z), \ldots) \), \( f_j \in H_\infty^I(D) \) such that \( F_c \neq 0 \), and suppose
\[
|h(z)| \leq F(z)F(z)^* \psi(F(z)F(z)^*) \leq 1 \text{ for all } z \in D,
\]
where \( \psi \) is the function given in Treil’s theorem. Then there exists \( V = (v_1(z), v_2(z), \ldots) \), \( v_j \in H_\infty^I(D) \) such that
\[
(a) \ F(z)V(z)^T = h(z) \text{ for all } z \in D \text{ and }
(b) \ \|V\|_\infty \leq C_0 \left( 1 + \frac{1}{\|F_c\|} \right),
\]
where \( C_0 \) is the estimate for the \( H_\infty(D) \) solution obtained in \([22]\).

**Corollary 1.** Let \( F(z) = (f_1(z), f_2(z), \ldots) \), \( f_j \in H_\infty^I(D) \) such that \( F_c \neq 0 \), and suppose
\[
|h(z)| \leq \sqrt{F(z)F(z)^*} \leq 1 \text{ for all } z \in D.
\]
Then there exists \( V = (v_1(z), v_2(z), \ldots) \), \( v_j \in H_\infty^I(D) \) such that
\[
(a) \ F(z)V(z)^T = h^3(z) \text{ for all } z \in D \text{ and }
(b) \ \|V\|_\infty \leq C_1 \left( 1 + \frac{1}{\|F_c\|} \right),
\]
where \( C_1 \) is the estimate for the \( H_\infty(D) \) solution obtained in \([23]\).

2. Preliminaries

In this section, we discuss the method of our proofs and also provide some required lemmas. To prove Theorem 1.1 and Theorem 1.2 in \( H_\infty^I(D) \), we first find the corresponding solutions in the bigger algebra \( H_\infty(D) \). Then we add some correction terms on the \( H_\infty(D) \) solutions to get the required solutions in our smaller algebra \( H_\infty^I(D) \).
For example, provided the corona condition, using Uchiyama version of corona theorem, we can easily find a solution $G$ in $(H^\infty(\mathbb{D}))_{l^2}$ such
that $F(z)G(z)^T = 1$ for all $z \in \mathbb{D}$. But, our goal is finding a solution $U \in (H^\infty(\mathbb{D}))_{l^2}$ such that $F(z)U(z)^T = 1$ for all $z \in \mathbb{D}$. For this, if we can find an operator $Q$ so that $M_Q(H^\infty(\mathbb{D}))_{l^2} \subseteq (H^\infty(\mathbb{D}))_{l^2}$ and for all $z \in \mathbb{D}$, $\text{ran} Q(z) = \text{ker} F(z)$, then we can construct the required solution $U$ as

$$U^T := G^T + QX^T,$$

with a right choice of $X \in (H^\infty(\mathbb{D}))_{l^2}$. This solves our problem as follows:

$$F(z)U(z)^T = F(z)G(z)^T = 1, \text{ for all } z \in \mathbb{D},$$

and the proper choice of $X$ will make $U \in (H^\infty(\mathbb{D}))_{l^2}$.

The next lemma is a linear algebra result which gives us the desired $Q$ operator and so enables us to write down the most general pointwise solution of $F(z)U(z)^T = 1$. This lemma can be found in Ryle - Trent [16], but we provide a proof for convenience.

**Lemma 2.1.** Let $\{a_j\}_{j=1}^{\infty} \in l^2$ and $A = (a_1, a_2, \ldots) \in B(l^2, \mathbb{C})$. Then there exists a matrix $Q_A$ of order $\infty \times \infty$ such that the entries of $Q_A$ are either $+a_j$ or $0$ and $Q_A$ satisfies:

$$\text{ran} Q_A = \text{ker} A$$

and

$$(AA^*)I_{l^2} - A^*A = Q_A Q_A^* \text{ with } \|Q_A\|_{B(l^2)} \leq \|A\|_{l^2}.$$

Also, if $\{d_j\}_{j=1}^{\infty} \in l^2$ and $D = (d_1, d_2, \ldots)$, then

$$(AD^T)I_{l^2} - D^TA = Q_A Q_D^T.$$  

Following few examples should be helpful to understand the Lemma 2.1 in a simple way.

Let $f_1, f_2, \ldots, f_n \in H^\infty(\mathbb{D})$ and fix $z \in \mathbb{D}$. Take $F = [f_1, f_2, \ldots, f_n]$.

For $n = 2$, $F = [f_1, f_2]$, $Q_F = \begin{bmatrix} f_2 \\ -f_1 \end{bmatrix}$.

Thus, $(FF^*)I_2 - F^*F = \begin{bmatrix} |f_2|^2 & -\bar{f}_1 f_2 \\ \bar{f}_2 f_1 & |f_1|^2 \end{bmatrix} = Q_F Q_F^*$.

Also, for any $D = [d_1, d_2]$, 

$$(FD^T)I_2 - D^TF = \begin{bmatrix} f_2 d_2 & -d_1 f_2 \\ -d_2 f_1 & f_1 d_1 \end{bmatrix} = Q_F Q_D^T.$$
Similarly, for \( n = 3 \), we take \( F = [f_1 \ f_2 \ f_3] \).

So,
\[
Q_F = \begin{bmatrix}
  f_2 & f_3 & 0 \\
- f_1 & 0 & f_3 \\
  0 & - f_1 & - f_2 \\
\end{bmatrix}.
\]

And, for \( n = 4 \), \( F = [f_1 \ f_2 \ f_3 \ f_4] \) and
\[
Q_F = \begin{bmatrix}
  f_2 & f_3 & f_4 & 0 & 0 & 0 \\
- f_1 & 0 & 0 & f_3 & f_4 & 0 \\
  0 & - f_1 & 0 & - f_2 & 0 & f_4 \\
  0 & 0 & - f_1 & 0 & - f_2 & f_3 \\
\end{bmatrix}.
\]

Form the above pattern, it is easy to see that the operators \( Q_F \)'s can be constructed inductively. Also, it is clear from (3), applied to \( A = F(z) \) and \( Q_D = Q_{F(z)} \), that \( \text{ran} \ Q_F(z) = \ker F(z) \).

We are now ready to prove Lemma 2.1.

Proof of Lemma 2.1. For \( k \in \mathbb{N} \), define
\[
A_k = \begin{bmatrix}
  0 & 0 & 0 & \ldots \\
  \vdots & \vdots & \vdots & \ddots \\
  c_{k+1} & c_{k+2} & c_{k+3} & \ldots \\
- c_k & 0 & 0 & \ldots \\
  0 & - c_k & 0 & \ldots \\
  0 & 0 & - c_k & \ldots \\
  \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}
\]

Multiplying \( A_k \) by \( A_k^* \), we get
\[
A_k A_k^* = \begin{bmatrix}
  0 & 0 & \ldots \\
  0 & 0 & \ldots \\
  \vdots & \vdots & \ddots \\
 0 & \ldots & \ldots \\
 0 & \sum_{j=k+1}^{\infty} |c_j|^2 & - \bar{c}_k c_{k+2} & - \bar{c}_k c_{k+3} & \ldots \\
0 & - c_k \bar{c}_{k+2} & |c_k|^2 & 0 & \ldots \\
0 & - c_k \bar{c}_{k+3} & 0 & |c_k|^2 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}
\]
Hence,
\[ \sum_{k=1}^{\infty} A_k A_k^* = \begin{bmatrix} \sum_{k \neq 1}^{\infty} |c_k|^2 & -\bar{c}_1 c_2 & -\bar{c}_1 c_3 & \ldots \\ -\bar{c}_2 c_1 & \sum_{k \neq 2}^{\infty} |c_k|^2 & -\bar{c}_2 c_3 & \ldots \\ -\bar{c}_3 c_1 & -\bar{c}_3 c_2 & \sum_{k \neq 3}^{\infty} |c_k|^2 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = CC^* I_{l^2} - C^* C. \]

Thus the required operator \( Q_A \) can be defined as
\[ Q_A = [A_1, A_2, \ldots] \in B(\oplus_{l^2}) \]

We note that (3) follows in a similar manner.

\[ \square \]

We also need the following key lemma.

**Lemma 2.2.** Assume that \( \{f_j\}_{j=1}^{\infty} \subset H_\pi^\infty(\mathbb{D}) \) and
\[ 0 < \epsilon^2 \leq \sum_{j=1}^{\infty} |f_j(z)|^2 \leq 1 \quad \text{for all } z \in \mathbb{D}. \]

Then
\[ (a) \quad \epsilon^2 \leq F_c F_c^* = \sum_{j=1}^{\infty} |f_{c_j}|^2 \leq 1 \]
and
\[ (b) \quad \|\phi_F\|_\infty = \sup_{z \in \mathbb{D}} \left( \sum_{j=1}^{\infty} |\phi f_j(z)|^2 \right) \leq 2. \]

**Proof.** Since for all \( z \in \mathbb{D} \),
\[ \epsilon^2 \leq \sum_{j=1}^{\infty} |f_{c_j} + \phi f_j(z)|^2 \leq 1, \]
we have that for each \( N \in \mathbb{N} \),
\[ \sum_{j=1}^{N} |f_{c_j} + \phi f_j(z)|^2 \leq 1. \]
But, \( \{\phi f_j\}_{j=1}^{N} \subset I \) and \( I \) is a proper ideal, so by the corona theorem
\[ \inf_{z \in \mathbb{D}} \sum_{j=1}^{N} |\phi f_j(z)|^2 = 0. \]
This means that for each $N$

$$\sum_{j=1}^{N} |f_{c_j}|^2 \leq 1, \text{ and hence } \sum_{j=1}^{\infty} |f_{c_j}|^2 \leq 1.$$ 

Thus, (b) holds, since for $z \in \mathbb{D}$

$$\left( \sum_{j=1}^{\infty} |\phi_{f_j}(z)|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{j=1}^{\infty} |f_{c_j} + \phi_{f_j}(z)|^2 \right)^{\frac{1}{2}} + \left( \sum_{j=1}^{\infty} |f_{c_j}|^2 \right)^{\frac{1}{2}} \leq 2.$$ 

Now by the Rosenblum- Tolokonnikov-Uchiyama version of the corona theorem, since $\{\phi_{f_j}\}_{j=1}^{\infty} \subset I$ and $I$ is a proper closed ideal and $\sup_{z \in \mathbb{D}} \sum_{j=1}^{\infty} |\phi_{f_j}(z)|^2 \leq 2 < \infty$, we have

$$\inf_{z \in \mathbb{D}} \sum_{j=1}^{\infty} |\phi_{f_j}(z)|^2 = 0.$$ 

Thus there exist $\{z_k\}_{k=1}^{\infty} \subset \mathbb{D}$ so that $\lim_{k \to \infty} \sum_{j=1}^{\infty} |\phi_{f_j}(z_k)|^2 = 0$.

Therefore, from

$$\epsilon \leq \left( \sum_{j=1}^{\infty} |f_{c_j} + \phi_{f_j}(z_k)|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{j=1}^{\infty} |f_{c_j}|^2 \right)^{\frac{1}{2}} + \left( \sum_{j=1}^{\infty} |\phi_{f_j}(z_k)|^2 \right)^{\frac{1}{2}},$$

we deduce that

$$\epsilon^2 \leq \sum_{j=1}^{\infty} |f_{c_j}|^2.$$ 

So (a) follows. \ \Box

Now we are ready to prove our theorems.

3. The Proofs

Proof of Theorem 1.1. Let $F \in (H^\infty_{\mathbb{D}})_{l2}$, and suppose

$$0 < \epsilon^2 \leq F(z)F(z)^* \leq 1 \text{ for all } z \in \mathbb{D}.$$ 

Then we know that there is a corona solution for $F$, say $G$, which lies in $(H^\infty_{\mathbb{D}})_{l2}$ such that

$$F(z)G(z)^T = 1, \text{ for all } z \in \mathbb{D} \text{ and}$$

$$\|G\|_{\infty} \leq \frac{9}{\epsilon^2} \ln \left( \frac{1}{\epsilon^2} \right).$$
Our aim is finding \( U \in (H^\infty_1(\mathbb{D}))_{l^2} \) such that \( F(z)U(z)^T = 1 \) for all \( z \in \mathbb{D} \). For this, we construct a new solution by adding a correction term to \( G(z)^T \).

Write \( F(z) = F_c + \phi_F(z) \), where \( F_c = \{f_{c_1}, f_{c_2}, \ldots\} \in l^2 \) and \( \phi_F = \{\phi_{f_1}, \phi_{f_2}, \ldots\} \in \mathbb{D} \).

Using (3), we have that

\[
I_{l^2} = (F(z)G(z)^T)I = G(z)^TF(z) + Q_{F(z)Q_{G(z)}}^T.
\]

This implies that

\[
I_{l^2} = G(z)^TF_c + Q_{F(z)Q_{G(z)}}^T + G(z)^T\phi_F(z).
\]  

(4)

Applying \( F_c^* \) to (3), we get

\[
F_c^* = G(z)^TF_cF_c^* + Q_{F(z)Q_{G(z)}}^T F_c^* + G(z)^T\phi_F(z)F_c^*.
\]

Also, from Lemma 2.2, we know that \( \|F_c\|^2 > 0 \), so

\[
\frac{F_c^*}{\|F_c\|^2} = G(z)^T + Q_{F(z)Q_{G(z)}}^T \frac{F_c^*}{\|F_c\|^2} + G(z)^T\phi_F(z) \frac{F_c^*}{\|F_c\|^2}.
\]

Thus,

\[
G(z)^T + Q_{F(z)Q_{G(z)}}^T \frac{F_c^*}{\|F_c\|^2} = \frac{F_c^*}{\|F_c\|^2} - G(z)^T\phi_F(z) \frac{F_c^*}{\|F_c\|^2}.
\]  

(5)

Define

\[
U(z)^T := G(z)^T + Q_{F(z)Q_{G(z)}}^T \frac{F_c^*}{\|F_c\|^2}.
\]

Using (2), we can clearly see that

\[
F(z)U(z)^T = F(z)G(z)^T + F(z)Q_{F(z)Q_{G(z)}}^T \frac{F_c^*}{\|F_c\|^2} = F(z)G(z)^T = 1, \text{ for all } z \in \mathbb{D}.
\]

Also, the right side of (5) shows that the solution \( U \) is in \( (H^\infty_1(\mathbb{D}))_{l^2} \).

For the norm estimate, we have that \( \|U\|_\infty \leq \left(1 + \frac{1}{\|F_c\|}\right)\|G\|_\infty \).

Hence,

\[
\|U\|_\infty \leq \left(1 + \frac{1}{\|F_c\|}\right)\frac{9}{e^2}\ln\left(\frac{1}{e^2}\right).
\]

This completes the proof of Theorem 1. \( \square \)

**Proof of Theorem 1.2.** Let \( F \in H^\infty_1(\mathbb{D})_{l^2} \), and suppose

\[ |h(z)| \leq F(z)F(z)^*\psi(F(z)F(z)^*) \leq 1 \text{ for all } z \in \mathbb{D} \]

By Treil’s theorem, there exists \( G \in H^\infty_1(\mathbb{D})\) such that
\[ F(z)G(z)^T = h(z) \quad \text{for all } z \in \mathbb{D} \]
and \( \|G\|_\infty \leq C_0 \), where \( C_0 \) is the estimate for the \( H^\infty(\mathbb{D}) \)-solution obtained in \( \cite{22} \).

Writing \( F(z) = F_c + \phi_F(z) \), \( h(z) = h_c + \phi_h(z) \) and using the relation (3) as in the proof of Theorem 1.1, we get

\[
h_c \frac{F_c^*}{\|F_c\|^2} + \left( \phi_h - G(z)^T \phi_F(z) \right) \frac{F_c^*}{\|F_c\|^2} = G(z)^T + Q_{F(z)}^T Q_{G(z)} \frac{F_c^*}{\|F_c\|^2}. \tag{6}
\]

Define

\[ V(z)^T := G(z)^T + Q_{F(z)}^T Q_{G(z)} \frac{F_c^*}{\|F_c\|^2}. \]

It’s clear that

\[ F(z)V(z)^T = h(z), \quad \text{for all } z \in \mathbb{D}. \]

Since \( G \in (H^\infty(\mathbb{D}))_{\mathbb{I}^2} \) and the elements of \( \phi_F \) are in \( \mathbb{I} \), the left side of the equation \( \text{(6)} \) shows that the solution \( V \) is in \( (H^\infty(\mathbb{D}))_{\mathbb{I}^2} \).

As in the corona theorem, for the norm estimate, we have that

\[ \|V\|_\infty \leq \left( 1 + \frac{1}{\|F_c\|} \right) \|G\|_\infty \leq C_0 \left( 1 + \frac{1}{\|F_c\|} \right), \]

where \( C_0 \) is the norm of the \( H^\infty(\mathbb{D}) \) solution, \( G \), obtained in \( \cite{22} \).

\[ \square \]

Proof of Corollary 1. The proof of this corollary follows similarly as the proof of Theorem 1.2 by using Wolff’s Theorem instead of Treil’s Theorem.

\[ \square \]

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References

[1] L. Carleson, Interpolation by bounded analytic functions and the corona problem, Annals of Math. 76 (1962), 547-559.
[2] U. Cegrell, A generalization of the corona theorem in the unit disc, Math. Z. 203 (1990), 255-261
[3] \underline{\text{---------}}, Generalizations of the corona theorem in the unit disc, Proc. Royal Irish Acad. 94 (1994), 25-30.
[4] K. R. Davidson, V. I. Paulsen, and M. Ragupathi, and D. Singh, A constrained Nevanlinna-Pick theorem, Indiana Math. J. 58 (2009), no.2, 709–732.
[5] J. B. Garnett, Bounded Analytic Functions, Academic Press, (2007)
[6] P. Gorkin, R. Mortini, and A. Nicolau, The generalized corona theorem, Math. Annalen 301 (1995), 135-154.
[7] R. Mortini, Generating sets for Ideals of finite type in $H^\infty$, Bull. Sci. Math. 136 (2012), 687 - 708.
[8] R. Mortini, A. Sasane, and B. Wick, The corona theorem and stable rank for $\mathbb{C} + BH^\infty(D)$, Houston J. Math. 36 (2010), no. 1, 289-302.
[9] N. K. Nikolski, Treatise on the Shift Operator, Springer-Verlag, New York (1985).
[10] M. Ragupathi, Nevanlinna-Pick interpolation for $\mathbb{C} + BH^\infty(D)$, Integral Equa. Oper. Theory 63 (2009), 103-125.
[11] J. Pau, On a generalized corona problem on the unit disc, Proc. Amer. Math. Soc. 133 (2004) no. 1, 167-174.
[12] K. V. R. Rao, On a generalized corona problem, J. Analyse Math. 18 (1967), 277-278.
[13] M. V. Renteln, Finitely generated ideals in the Banach algebra $H^\infty$, Collectanea Mathematica 26 (1975), 3-14.
[14] M. Rosenblum, A corona theorem for countably many functions, Integral Equa. Oper. Theory 3 (1980), no. 1, 125-137.
[15] J. Ryle, A corona theorem for certain subalgebras of $H^\infty(D)$, Dissertation, The University of Alabama, (2009).
[16] J. Ryle and T. Trent, A corona theorem for certain subalgebras of $H^\infty(D)$, Houston J. Math 37 (2011), no. 4, 1211-1226.
[17] S. Scheinberg, Cluster sets and corona theorems in Banach spaces of analytic functions, Lecture Notes in Mathematics, Springer, New York, 1976
[18] J. Ryle and T. Trent, A corona theorem for certain subalgebras of $H^\infty(D)$ II, Houston J. Math 38 (2012), no. 4, 1277-1295.
[19] S. Scheinberg, Cluster sets and corona theorems in Banach spaces of analytic functions, Lecture Notes in Mathematics, Springer, New York, 1976.
[20] V. A. Tolokonnikov, The corona theorem in algebras of smooth functions, Translations (American Mathematical Society), 149 (1991) no. 2, 61-95.
[21] S. R. Treil, Estimates in the corona theorem and ideals of $H^\infty$: A problem of T. Wolff, J. Anal. Math 87 (2002), 481-495
[22] ______, The problem of ideals of $H^\infty(D)$: Beyond the exponent $3/2$, J. Fun. Anal. 253 (2007), 220-240.
[23] T. Trent, An estimate for ideals in $H^\infty(D)$, Integr. Equat. Oper. Th. 53 (2005), 573-587.
[24] ______, An $H^2$ corona theorem on the bidisk for infinitely many functions, Linear Alg. and App. 379 (2004), 213-227.
[25] ______, A note on multiplication algebras on reproducing kernel Hilbert spaces, Proc. Amer. Math. Soc. 136 (2008), 2835-2838.
[26] A. Uchiyana, Corona theorems for countably many functions and estimates for their solutions, preprint, UCLA, 1980.
[27] T. Wolff, A refinement of the corona theorem, in Linear and Complex Analysis Problem Book, by V. P. Havin, S. V. Hrusc, and N. K. Nikolski (eds.), Springer-Verlag, Berlin (1984).

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