Two new Markov order estimators

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Abstract. We present two new methods for estimating the order (memory depth) of a finite alphabet Markov chain from observation of a sample path. One method is based on entropy estimation via recurrence times of patterns, and the other relies on a comparison of empirical conditional probabilities. The key to both methods is a qualitative change that occurs when a parameter (a candidate for the order) passes the true order. We also present extensions to order estimation for Markov random fields.

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1 Introduction

Fix a finite set $A$ and let $x^n = x^n_m$ denote the sequence $x_m, x_{m+1}, \ldots, x_n$, where $x_i \in A$. A stationary, ergodic, $A$-valued process $X = \{X_n\}$ is Markov of order $M = 0$ if it is i.i.d., and Markov of order $M > 0$ if $M$ is the least positive integer such that $P(a_{k+1} | a_k^k) = P(a_{k+1} | a_{k-M}^k)$, for all $a_k^k$ such that $k \geq M$. A consistent Markov order estimator is a sequence of functions $M_n^* : A^n \mapsto \{0, 1, \ldots\}, n \geq 1$, such that for any $M$ and any Markov process $X$ of order $M$,

$$\lim_n M_n^*(x^n_1) = M, \text{ a.s.}$$

(Here and throughout, “a.s.” always refers to the distribution $P = P_X$ of $X$.) In this paper we introduce two new Markov order estimators. Both use test functions that depend on the sample size and a candidate $k$ for the order. The key to our methods is that as $k$ increases, our test functions exhibit a qualitative change of behavior when $k$ reaches the true order.

Our estimators use the empirical frequencies of overlapping blocks,

$$N_n(a_k^1) = N_n(a_k^1 | x^n_1) \overset{\text{def}}{=} |\{i \in [0, n-k] : x_{i+1}^{i+k} = a_k^1\}|. \quad (1)$$

The corresponding empirical probabilities and conditional probabilities are

$$\hat{P}_n(a_k^{k+1}) \overset{\text{def}}{=} \frac{1}{n-k} N_n(a_k^{k+1}) \quad \text{and} \quad \hat{P}_n(a_{k+1}^1 | a_k^1) \overset{\text{def}}{=} \frac{N_n(a_{k+1}^1 | a_k^1)}{N_n(a_k^1)}/N_{n-1}(a_k^1).$$

We also define the $k$-step conditional empirical entropy,

$$\hat{h}_k(n) \overset{\text{def}}{=} -\sum_{a_k^{k+1}} \hat{P}_n(a_k^{k+1}) \log \hat{P}_n(a_{k+1}^1 | a_k^1).$$

Our first method, which we call the entropy estimator method, compares $\hat{h}_k(n)$ with the entropy estimator $[\ell(n)]^{-1} \log n$, where $\ell(n)$ denotes the length of the longest initial block in $x^n_1$ that repeats in $x^n_1$ (see [14] and Section 2 below).

**Theorem 1** $M_n^*(x^n_1) \overset{\text{def}}{=} \min\{k : \hat{h}_k(n) \leq [\ell(n)]^{-1} \log n + 2(\log n)^{-1/4}\}$ is a consistent Markov order estimator.

Our second method, which we call the maximal fluctuation method, is based on the test function

$$\phi_m(x^n_1) \overset{\text{def}}{=} \max_{m < k < f(n)} \max_{a_k^1 \in A} \left[ \hat{P}_n(a_k | a_k^{k-1}) - \hat{P}_n(a_k | a_k^{k-1}) \right] N_{n-1}(a_k^{k-1}), \quad (2)$$

where $f(n) = \log \log n$. Define $M_n^\#(x^n_1) \overset{\text{def}}{=} \min\{m < n - f(n) : \phi_m(x^n_1) < n^{3/4}\}$; if the set we are minimizing over is empty, then we take $M_n^\#(x^n_1) = n$. 

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Theorem 2 $M_n^\#(x^n)$ is a consistent Markov order estimator.

A more general form of Theorem 1 that allows any entropy estimator with a known rate of convergence is given in Section 2. An extension of Theorem 2 to Markov random fields is given in Section 3.1. Connections to other model selection methods are given in Section 4.

Careful proofs of Theorems 1 and 2 are given in Sections 2 and 3, respectively. For the reader’s convenience, we first present sketches of the proofs.

Sketch of proof of Theorem 1: The $2(\log n)^{-1/4}$ term incorporates the rate of convergence of $\ell(n)^{-1}\log n$ and that of $\hat{h}_M(n)$ to their common almost sure limit, the entropy $H(X)$ of $X$. Thus $\hat{h}_M(n) \leq [\ell(n)]^{-1}\log n + 2(\log n)^{-1/4}$, eventually a.s., whence $M_n^* \leq M$ eventually a.s. On the other hand, if $k < M$ then $\hat{h}_k(n)$ converges a.s. to the $k$-step conditional theoretical entropy $H_k(X)$, which exceeds $H(X)$, the almost sure limit of $[\ell(n)]^{-1}\log n + 2(\log n)^{-1/4}$. Therefore $M_n^* \geq M$ eventually a.s.

Sketch of proof of Theorem 2: If $m < M$ then there exists $a_{k+1}^{M+1}$ such that $\mathbb{P}(a_{M+1}|a_{k+1}^{M-1}) > \mathbb{P}(a_{M+1}|a_{k+1}^{M-1})$, and hence $\phi_m(x^n)$ grows a.s. like $cn$, for some $c > 0$. Thus $M_n^\# \geq M$ eventually a.s. On the other hand, classical large deviations theory shows that for any $\epsilon > 0$, we have $\phi_M(x^n) = o(n^{1/2+\epsilon})$, a.s., so $M_n^\# \leq M$ eventually a.s.

2 The entropy estimator method.

We first review some elementary facts about entropy, see [8] or [17] for details. The conditional entropy of the next symbol given $k$ previous symbols is defined by

$$H_k = H(X_{k+1}|X_k^k) \overset{\text{def}}{=} - \sum_{a_i^{k+1}} \mathbb{P}(a_i^{k+1}) \log \mathbb{P}(a_{k+1}|a_i^k).$$

The sequence $\{H_k\}$ is nonincreasing with limit equal to the entropy $H = H(X)$ of the process. Furthermore, the process is Markov of order $M$ if and only if

$$k < M \implies H_k > H(X) \quad \text{and} \quad k \geq M \implies H_k = H(X),$$

that is, if and only if $H_k$ reaches its limit $H$ exactly when $k = M$, see [17, Thm I.6.11].

The conditional $k$-th order empirical entropy $\hat{h}_k(n)$ is defined by replacing theoretical probabilities by the corresponding empirical probabilities. The ergodic theorem implies that for $k$ fixed, $\hat{P}_n(a_i^{k+1}) \to \mathbb{P}(a_i^{k+1})$ and $\hat{P}_n(a_{k+1}|a_i^k) \to \mathbb{P}(a_{k+1}|a_i^k)$, each with probability 1, and hence that $\hat{h}_k(n) \to H_k$, a.s. Furthermore, in the Markov case we have the following iterated logarithm result.
Lemma 1. If $X$ is Markov of finite order $M$ then for each $k$ there is a constant $c_k$ such that

$$|H_k - \hat{h}_k(n)| \leq c_k \sqrt{\frac{\log \log n}{n}}$$

eventually a.s. as $n \to \infty$.

Remark. A slightly weaker inequality (which would suffice for our application here), with an extra factor of $\log n$ on the right-hand side can be obtained by applying [3, Theorem 16.3.2] instead of (4) below.

Proof of Lemma 1. Let $\Psi(x) = x \log x - x + 1$, so that $\Psi(1) = \Psi'(1) = 0$. For $x > 1/2$ we have $\Psi''(x) = 1/x < 2$, whence $|\Psi(x)| < (x - 1)^2$ for all $x \geq 1/2$.

Consider two distributions $P$ and $Q$ on the same alphabet $A$, and suppose that

$$\gamma = \max_{a \in A} \left| \frac{P(a)}{Q(a)} - 1 \right| \leq 1/2. \tag{3}$$

Then the divergence $D(P|Q) = \sum_a P(a) \log \frac{P(a)}{Q(a)}$ satisfies

$$D(P|Q) = \sum_a \left[ Q(a) \Psi \left( \frac{P(a)}{Q(a)} \right) + P(a) - Q(a) \right] = \sum_a \left[ Q(a) \Psi \left( \frac{P(a)}{Q(a)} \right) \right] \leq \gamma^2.$$  

Moreover,

$$\sum_a \left| (P(a) - Q(a)) \log Q(a) \right| \leq \sum_a \left| \gamma Q(a) \log Q(a) \right| = \gamma H(Q) \leq \gamma \log |A|.$$  

Adding the last two inequalities (using positivity of the divergence) gives

$$|H(Q) - H(P)| \leq \gamma^2 + \gamma \log |A|.$$  

under the assumption (3).

By the law of the iterated logarithm for finite-order Markov chains, there is a constant $\tilde{c}_k$ such that

$$\left| \frac{\hat{P}_n(a_k^n)}{P(a_k^n)} - 1 \right| \leq \tilde{c}_k \sqrt{\frac{\log \log n}{n}},$$

eventually a.s. so an application of (4) to $\hat{P}_n$ and $P$ proves the lemma. \hfill $\square$

The Ornstein-Weiss recurrence theorem, [14], states that for any ergodic finite alphabet process $X$, the time until the opening $n$-block occurs again,

$$R_n = \min \{ r \geq n : x_{r+1}^n = x_1^n \},$$
grows like $e^{nH(X)}$, that is, $(1/n) \log R_n(x) \to H(X)$ a.s. (Earlier, Wyner and Ziv, [19], established convergence-in-probability for a related recurrence idea.) In our setting $\ell(n) = \max\{k: R_k \leq n\}$ and the Ornstein-Weiss recurrence theorem gives

$$\lim_{n \to \infty} \frac{1}{\ell(n)} \log R_{\ell(n)}(x) = H(X), \text{ a.s.}$$

Let $\mathcal{M}$ denote the set of ergodic, $A$-valued processes $X$ that are finite-order Markov. To obtain a rate of convergence for $X \in \mathcal{M}$ we use Kontoyiannis’ second-order result, [12, Corollary 1], that for any $\beta > 0$ and $X \in \mathcal{M}$

$$\log[R_n(x)P(x^n)] = o(n^\beta), \text{ a.s.} \quad (5)$$

The statement and proof were for Wyner-Ziv recurrence but can easily be adapted to Ornstein-Weiss recurrence. We use it to prove

**Lemma 2** \(\forall \beta > 1/2 \text{ and } X \in \mathcal{M}, \log R_n(x) = nH + o(n^\beta), \text{ a.s.}\)

**Proof.** Suppose $X$ has order $M$ and $\beta > 1/2$. The Markov property and the law of the iterated logarithm yield

$$\log P(x^n_1) = \log P(x^n_M) + \sum_{a_{i}^{M+1}} N(a_{i}^{M+1}) \log P(a_{M+1}|a_{i}^{M})$$

$$= (n-M) \sum_{a_{i}^{M+1}} P(a_{i}^{M+1}) \log P(a_{M+1}|a_{i}^{M}) + o(n^\beta)$$

$$= -nH + o(n^\beta), \text{ a.s.}$$

which, combined with (5), yields the lemma. \qed

**Lemma 3** For all $X \in \mathcal{M}$,

$$\frac{1}{\ell(n)} \log R_{\ell(n)} \leq \frac{1}{\ell(n)} \log n \to H(X), \text{ a.s.}$$

**Proof.** Since $R_{\ell(n)}(x) \leq n \leq R_{\ell(n)+1}(x)$, the lemma follows from

$$\frac{1}{\ell(n)} \log R_{\ell(n)}(x) \leq \frac{1}{\ell(n)} \log n \leq \frac{\ell(n)}{\ell(n)} \left[ \frac{1}{\ell(n)+1} \log R_{\ell(n)+1}(x) \right],$$

and the fact that both the left-hand and right-hand terms go to $H(X)$, a.s. \qed

We also need a lower bound on the growth of $\ell(n)$.

**Lemma 4** For any $X \in \mathcal{M}$ there is a constant $C > 0$ such that $\ell(n) \geq C \log n$, eventually a.s.
Proof. By the Ornstein-Weiss recurrence theorem,
\[ R_k \leq e^{k(H+1)} \leq e^{k(1+\log |A|)} \text{, eventually a.s.} \]

Thus we can take \( C = (1 + \log |A|)^{-1} \).

The lemmas yield

**Proposition 1** For any \( X \in \mathcal{M} \),
\[ \frac{1}{\ell(n)} \log n \geq H(X) - \frac{1}{\sqrt{\log n}}, \text{ eventually a.s.} \]

Proof. The following chain of inequalities holds eventually a.s.
\[
\frac{1}{\ell(n)} \log n \overset{(a)}{\geq} \frac{1}{\ell(n)} \log R_{\ell(n)} \overset{(b)}{\geq} H(X) - \frac{1}{\ell(n)^{3/8}} \overset{(c)}{\geq} H(X) - \frac{1}{[C\log n]^{3/8}} \overset{(d)}{\geq} H(X) - \frac{1}{\sqrt{\log n}};
\]

inequality (a) by Lemma 3, inequality (b) by Lemma 2 for \( \beta = 5/8 \) and inequality (c) by Lemma 4, while inequality (d) is clear.

We are now ready to prove Theorem 1 which for ease of reference we restate here.

**Theorem 1** \( M_n(x^n) \overset{\text{def}}{=} \min\{k: \hat{h}_k(n) \leq [\ell(n)]^{-1} \log n + 2(\log n)^{-1/4}\} \) is a consistent Markov order estimator.

Proof. Suppose \( X \in \mathcal{M} \) has order \( M \) and entropy \( H = H(X) \). We first show that underestimation does not occur, eventually a.s. For \( m < M \), the simple facts

(a) \( \hat{h}_m(n) \overset{\text{a.s.}}{\to} H_m \text{ as } n \to \infty \) and \( H_m > H \),
(b) \( [\ell(n)]^{-1} \log n \overset{\text{a.s.}}{\to} H \) and \( 2(\log n)^{-1/4} \to 0 \),

immediately imply that \( \hat{h}_m(n) > [\ell(n)]^{-1} \log n + 2(\log n)^{-1/4} \), eventually a.s.

The following chain of inequalities holds eventually a.s.
\[
\begin{align*}
\hat{h}_M(n) &\overset{(a)}{\leq} H + c\sqrt{\frac{\log \log n}{n}} \overset{(b)}{\leq} \frac{1}{\ell(n)} \log n + \frac{1}{\sqrt{\log n}} + c\sqrt{\frac{\log \log n}{n}} \\
&\overset{(c)}{\leq} \frac{1}{\ell(n)} \log n + \frac{2}{\sqrt{\log n}},
\end{align*}
\]
inequality (a) by Lemma 1 and the fact that $H_M = H$, and inequality (b) by Proposition 1, while inequality (c) is obvious. We conclude that $M_n^*(x_1^n) \leq M$, eventually a.s.

As the above proof suggests, the entropy estimator $[\ell(n)]^{-1} \log n$ can be replaced by any consistent entropy estimator $\hat{H}(x_1^n)$ that has an $o(1)$ underestimation bound, i.e., a function $u(n) \to 0$ such that for all $X \in \mathcal{M}$

$$\hat{H}(x_1^n) \geq H(X) - u(n), \text{ eventually a.s.,}$$  

provided we replace $2/\sqrt{\log n}$ by $|u(n)| + (1/n) \log n$.

**Theorem 1 (General form)**

Let $\hat{H}(x_1^n)$ be a consistent entropy estimator with $o(1)$ underestimation bound $u(n)$. Then $M_n^*(x_1^n) \overset{\text{def}}{=} \min \{k: \hat{h}_k(n) < \hat{H}(x_1^n) + |u(n)| + (1/n) \log n\}$ is a consistent Markov order estimator.

We used the recurrence-based entropy estimator as it is one of the simplest to describe and compute, it easily updates as $n$ increases, and its second order properties are easy to determine. Its underestimation bound $1/\sqrt{\log n}$ goes to 0 very slowly, however, which suggests that its associated order estimator $M_n^*(x_1^n)$ converges slowly to $M$. Furthermore, though the recurrence idea does generalize to higher dimensions, see [15], a useful rate theory for it has not been established. In Section 4.1, we present another entropy estimator that has a more rapidly convergent underestimation bound and is extendable to higher dimensions.

### 3 The maximal fluctuation method.

We now prove the second theorem stated in the introduction, namely,

**Theorem 2** $M_n^#(x_1^n) \overset{\text{def}}{=} \min \{m < n - f(n): \phi_m(x_1^n) < n^{3/4}\}$ is a consistent Markov order estimator. (Recall that we defined $M_n^#(x_1^n) = n$ if this set is empty).

**Proof.** Let

$$\delta_m(a^k_1|x_1^n) \overset{\text{def}}{=} N_n(a^k_1) - N_{n-1}(a^{k-1}_1)\tilde{P}_n(a_k|a^{k-1}_{k-m})$$

and note that

$$\phi_m(x_1^n) = \max_{m<k<f(n)} \max_{a^k_1} \delta_m(a^k_1|x_1^n),$$  

where $f(n) = \log \log n$. We first show that eventually a.s. underestimation does not occur. Suppose $X \in \mathcal{M}$ has order $M$ and $m < M$. Choose $a^{M+1}_1$ such that

$$P(a_{M+1}|a^M_1) > P(a_{M+1}|a^M_{M-m+1}).$$  


By the ergodic theorem there exists \( \epsilon > 0 \) such that, eventually a.s.,
\[
N_{n-1}(a_1^M) > \epsilon n \quad \text{and} \quad \tilde{P}_n(a_{M+1}|a_1^M) - \tilde{P}_n(a_{M+1}|a_{M-m+1}^M) \geq \epsilon.
\]
This implies that \( \phi_m(x^n_i) \geq \epsilon^2 n \) and hence that \( M_n^\#(x^n_i) \geq M \), eventually a.s.

It takes somewhat more effort to show that, eventually a.s., \( \phi_M(x^n_i) \leq n^{3/4} \). We first note that for fixed \( k \geq M \),
\[
Z_k(n) \overset{\text{def}}{=} N_n(a_k|x^n_1) - N_{n-1}(a_{k-1}|x^{n-1}_1)P(a_k|a_{k-1-M}), \quad n \geq k,
\]
(8)
is a martingale with bounded differences. Indeed, with \( \chi(B) \) denoting the indicator of \( B \), we can write \( Z_k(n) = \sum_{j=k}^n \Delta_k(j) \), where
\[
\Delta_k(j) = \chi(X^j_{j-k+1} = a_k^1) - \chi(X^j_{j-k+1} = a_k^{k-1})P(a_k|a_{k-1-M}),
\]
and direct calculation shows that \( E(\Delta_k(j)|X^j_1) = 0 \) and \( \|\Delta_k(j)\|_\infty \leq 1 \) for \( j > k \).
From the Hoeffding-Azuma large deviations bound for martingales with bounded differences, [11, 11], the probability that \( |Z_n| \geq n^{3/4} \) is at most \( 2 \exp(-n^{1/2}/2) \).

A similar argument also shows that for
\[
Z_k^*(n) \overset{\text{def}}{=} N_n(a_{k-M}^1|x^n_1) - N_{n-1}(a_{k-M}^{k-1}|x^{n-1}_1)P(a_k|a_{k-M}^{k-1}), \quad n \geq k,
\]
the probability that \( |Z_k^*(n)| \geq n^{3/4} \) is at most \( 2 \exp(-n^{1/2}/2) \).

Next we note that
\[
Z_k(n) - \delta_M(a_k^1|x^n_1) = \frac{N_{n-1}(a_{k-1}^{k-1})}{N_{n-1}(a_{k-M}^{k-1})} Z_k^*(n),
\]
which has absolute value at most \( |Z_k^*(n)| \). Thus, the probability that \( \delta_M(a_k^1|x^n_1) \geq 2n^{3/4} \) is less than \( 4 \exp(-n^{1/2}/2) \). Since there are at most \( |A|^{f(n)+1} = n^{o(1)} \) possible sequences \( a_k^1 \), it follows from [1] and an application of Borel-Cantelli that eventually a.s., \( \phi_M(x^n_i) \leq n^{3/4} \). This completes the proof of Theorem 2.

\[ \square \]

**Remark 1** After one of us lectured on these results [18], B. Weiss noted that in recent joint work he did with G. Morvai, they independently developed the estimator \( M_n^\# \) discussed in Theorem 2.

### 3.1 Markov Random Fields

The method of maximum fluctuations extends in modified form to Markov random fields, where order is usually called range. We confine our discussion to the two dimensional (2-d) case; the extension to higher dimensions is straightforward.

We use the following notation.

1. \( S_t \overset{\text{def}}{=} \{(i, j); -t \leq i \leq t, -t \leq j \leq t\} = \) the square of width \( 2t + 1 \), centered at the origin. (Note that \( S_{t+s} \setminus S_t \) is a square “ annulus” of thickness \( s \).)
2. $S_t(\bar{u}) \overset{\text{def}}{=} \text{the square of width } 2t + 1 \text{ with center at } \bar{u} \in \mathbb{Z}^2$.

3. $\Lambda_n \overset{\text{def}}{=} \text{the square of width } n \text{ with lower left corner at } (1,1)$.

4. A configuration $a(\Lambda)$ is a function $a : \Lambda \mapsto A$; if no confusion results its restriction to $\Lambda' \subseteq \Lambda$ will be denoted by $a(\Lambda')$.

A random field is a collection $X = \{X(\bar{n}) : \bar{n} \in \mathbb{Z}^2\}$ of random variables with values in $A$. Unless stated otherwise, random fields are assumed to be stationary and ergodic. We use the conditional probability notation

$$P(a(\Lambda) | b(\Lambda')) \overset{\text{def}}{=} \frac{\text{Prob}(X(\Lambda) = a(\Lambda), X(\Lambda') = b(\Lambda'))}{\text{Prob}(X(\Lambda') = b(\Lambda'))}.$$   

A random field is said to be Markov with range $R = 0$ if it is i.i.d, and Markov with range $R \geq 1$ if $R$ is the least positive integer $r$ such that for all $\ell \geq 0$ and $t \geq 0$

$$P(a(S_\ell) | b(S_{\ell+r+t} \setminus S_\ell)) = P(a(S_\ell) | b(S_{\ell+r} \setminus S_\ell)),$$

for all configurations $a(S_\ell)$ and $b(S_{\ell+r+t} \setminus S_\ell)$. That is, $R$ is the least $r$ such that the random variables $X(S_\ell)$ on the inner square and $X(S_{\ell+r+t} \setminus S_{\ell+r})$ on the outer annulus are conditionally independent, given the values $X(S_{\ell+r} \setminus S_\ell)$ on the inner annulus. The range of a finite-range random field $X$ is denoted by $R = R(X)$.

Our 2-d maximum fluctuation method tests whether configurations on a square are conditionally independent of those outside a square that is expanded by $r$ in each axis direction, given the configuration in the annulus between the two squares. Not only do we need to test over a (slowly) growing interval of possible orders $r$, but now we also need to examine a (slowly) growing interval of sizes $\ell$ for the inner square, as order can depend on square size, though it eventually becomes constant as square size increases. Counting overlapping blocks as in (1) will not be used because the higher dimensional analogue of (8) need not be a martingale. We focus instead on counting nonoverlapping blocks, to which classical large deviations is applicable, but now we must also consider translates.

Given $n > 8$, let $\ell, r, t$ be integers in the closed interval $[0, \log \log n]$ and put

$$k \overset{\text{def}}{=} \ell + r + t, \text{ and } T \overset{\text{def}}{=} \left\lceil \frac{n}{2k + 1} \right\rceil - 1.$$  

We assume the integer $n$ is large enough to guarantee that $T > 0$ for all $k \leq 3 \log \log n$. Let

$$\Pi_k = \{S_k(\bar{u}_1), S_k(\bar{u}_2), \ldots, S_k(\bar{u}_{T^2})\}$$

be the partition of the square $\Lambda(2k+1)^2$ into squares of width $2k+1$. For each $\bar{v} \in \Lambda_{2k+1}$, let $\Pi_k(\bar{v}) = \{S_k(\bar{v} + \bar{u}_j), 1 \leq j \leq T^2\}$ be the translated partition of the square
\[ \bar{v} + \Lambda_{(2k+1)T} \subseteq \Lambda_n. \]

Given a configuration \( x(\Lambda_n) \) and a configuration \( a(\Lambda) \) on a centrally symmetric subset \( \Lambda \subset S_k \), and given a vector \( \bar{v} \in \Lambda_{2k+1} \), put

\[ N_{\bar{v}}(a(\Lambda)) = N_{\bar{v}}(a(\Lambda)|x(\Lambda_n)) \overset{\text{def}}{=} \# \{ j : x(\bar{v} + \bar{u}_j + \bar{w}) = a(\bar{w}), \forall \bar{w} \in \Lambda \}, \]

that is, the number of times the configuration \( a(\Lambda) \) appears in \( x(\cdot) \), centered at a member of the translated partition \( \Pi_k(\bar{v}) \). Our 2-d test function is

\[ \delta_{\ell,r,t}(a(S_k)|x(\Lambda_n)) \overset{\text{def}}{=} N_{\bar{v}}(a(S_k)) - N_{\bar{v}}(a(S_k \setminus S_\ell)) \frac{N_{\bar{v}}(a(S_{\ell+r}))}{N_{\bar{v}}(a(S_{\ell+r} \setminus S_\ell))}. \]  

(9)

This is maximized over configurations \( a(S_k) \) and translates \( \bar{v} \) to produce

\[ \delta_{\ell,r,t}(x(\Lambda_n)) = \max_{\bar{v} \in \Lambda_{2k+1}} \max_{a(S_k)} \delta_{\ell,r,t}(a(S_k)|x(\Lambda_n)). \]

For \( \ell = \lfloor \log \log n \rfloor \) define

\[ \phi_r(x(\Lambda_n)) \overset{\text{def}}{=} \max_{0 < t < \log \log n} \delta_{\ell,r,t}(x(\Lambda_n)). \]

Our 2-d order estimator is

\[ R^*_n(x(\Lambda_n)) \overset{\text{def}}{=} \min \{ r < n - 3 \log \log n : \phi_r \leq n^{3/2} \}, \]

where, if there is no such \( r < n - 3 \log \log n \), we set \( R^*_n(x(\Lambda_n)) = n \).

**Theorem 3** Let \( X \) be a stationary, ergodic, finite range random field on \( \mathbb{Z}^d \). Then \( R^*_n(x(\Lambda_n)) = R(X) \), eventually a.s.
Proof. If \( r < R = R(X) \), an argument similar to the 1-dimensional case shows that \( \phi_r(x(\Lambda_n)) \geq Cn^2 \), eventually a.s., for a some \( C > 0 \). Thus, underestimation eventually a.s. does not occur.

To complete the proof it is enough to show that \( \phi_R < n^{3/2} \), eventually a.s. Towards this end, we fix \( \ell > 0 \) and \( t > 0 \), put \( r = R \) and \( k = \ell + R + t \), fix \( a(S_k) \) and \( \bar{v} \in \Lambda_{2k+1} \), and put \( N = N_\bar{v} \). Our 2-d test function \([\mathbf{9}]\) can then be expressed as the sum

\[
N(a(S_k)) - N(a(S_k \setminus S_\ell)) \frac{N(a(S_{\ell+R}))}{N(a(S_{\ell+R} \setminus S_\ell))} = \Delta_1 + \Delta_2,
\]

where

\[
\Delta_1 = N(a(S_k)) - N(a(S_k \setminus S_\ell))P(a(S_\ell) \mid a(S_{\ell+R} \setminus S_\ell)),
\]

and

\[
\Delta_2 = \frac{N(a(S_k \setminus S_\ell))}{N(a(S_{\ell+R} \setminus S_\ell))} \left[ N(a(S_{\ell+R} \setminus S_\ell))P(a(S_\ell) \mid a(S_{\ell+R} \setminus S_\ell)) - N(a(S_{\ell+R})) \right]
\]

\[
\leq \left| N(a(S_{\ell+R} \setminus S_\ell))P(a(S_\ell) \mid a(S_{\ell+R} \setminus S_\ell)) - N(a(S_{\ell+R})) \right|.
\]

(10)

Denote \( \tilde{p} = P(a(S_\ell) \mid a(S_{\ell+R} \setminus S_\ell)) \) and \( \bar{w}_j = \bar{u}_j + \bar{v} \). Then we can write \( \Delta_1 = \sum_{j=1}^{T^2} \Delta_{1,j} \), where with \( \chi(\cdot) \) denoting the indicator function,

\[
\Delta_{1,j} \overset{\text{def}}{=} \chi \left( X(S_k(\bar{w}_j)) = a(S_k) \right) - \chi \left( X(S_k(\bar{w}_j)) = a(S_k \setminus S_\ell) \right) \tilde{p}.
\]

Therefore, conditioned on the values \( a(S_k \setminus S_\ell) \) in the square annulus, \( \Delta_1 \) is a sum of \( N(a(S_k \setminus S_\ell)) \leq T^2 \) binary i.i.d. mean 0 random variables. The classical Hoeffding large deviations bound, \([\mathbf{11}]\), implies that the probability that \( |\Delta_1| > \frac{1}{2}n^{3/2} \) is at most \( 2 \exp(-n/4) \). The inequality \([\mathbf{10}]\) implies that the same result holds for \( |\Delta_2| \). Since there are only subexponentially many \( a(S_k \setminus S_\ell) \) and \( \bar{v} \in \Lambda_{2k+1} \) to consider, the Borel-Cantelli lemma implies that \( \phi_R \leq n^{3/2} \), eventually a.s. This completes the proof of Theorem \([\mathbf{3}]\). \hfill \Box

Remark 2 To simplify the discussion we focused on squares rather than diamonds which are more natural in Ising models. Our concepts and results can easily be converted to the latter setting.

Remark 3 Csiszár and Talata, \([\mathbf{7}]\), have recently shown the existence of a consistent range estimator for a restricted class of Markov random fields, namely, those for which, conditioned on any boundary, probabilities in a square are positive, a condition that allows them to focus only on squares of size 1, rather than squares of growing size as we did. They assume no bound on the range and use a variant of the BIC in which maximum likelihood is replaced by maximum pseudolikelihood.
4 Extensions and related work.

4.1 Other entropy estimators.

There are many known consistent entropy estimators, for most of which $o(1)$ underestimation bounds for the Markov case have not been established. In addition to the recurrence estimator such an underestimation bound can be shown to hold for $\hat{h}_{f(n)}(n)$, where for example $f(n) = \frac{\log \log n}{\log |A|}$.

**Proposition 2** There exists a positive constant $C$ such that for any $X \in \mathcal{M}$,

(a) $\hat{h}_{f(n)}(n) \rightarrow H(X)$, a.s.

(b) $\hat{h}_{f(n)}(n) \geq H(X) - C\frac{\log^2 n}{n}$, eventually a.s.

**Proof.** By the Ornstein-Weiss entropy estimation theorem, [13], the per-symbol empirical block entropy $\frac{1}{f(n)}H(P_{f(n)}(\cdot)) \rightarrow H$, a.s. as $f(n) \rightarrow \infty$, provided only that $f(n) \leq \frac{\log n}{H + \epsilon}$, for some $\epsilon > 0$. It is easy to see that this implies $\hat{h}_{f(n)}(n) \rightarrow H$, a.s., for the case $f(n) = \frac{\log \log n}{\log |A|}$. This proves (a).

To establish part (b), suppose $X \in \mathcal{M}$ has order $M$. The BIC consistency theorem, see [6], implies that

$$\frac{|A|^{f(n)}(|A| - 1)}{2} \log n + n\hat{h}_{f(n)}(n) > \frac{|A|^M(|A| - 1)}{2} \log n + n\hat{h}_M(n),$$

eventually a.s. Using the relation $|A|^{|f(n)|} = \log n$ and the bound $n\hat{h}_M(n) \geq nH - c \log \log n$, which holds eventually a.s. by Lemma [11], we obtain

$$n\hat{h}_{f(n)}(n) \geq nH - c \log \log n + \frac{|A|^M(|A| - 1)}{2} \log n - \frac{(|A| - 1)}{2} \log^2 n,$$

from which (b) follows. \qed

**Remark 4** The empirical entropy estimator $\hat{h}_{f(n)}(n)$ converges to entropy faster than the recurrence-based estimator, which is not surprising as the latter uses so little about the sample path. We suspect there may be a more direct proof of Proposition 2(b) than the one we gave.

**Remark 5** An important example for which an $o(1)$ underestimation bound is not known is the Lempel-Ziv entropy estimator, [20]. An $O((1/n) \log n)$ underestimation bound for the class $\mathcal{M}_0$ of i.i.d. processes has been established, see [8], a result we suspect can be extended to the class $\mathcal{M}$. 

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4.2 The “flat spot” problem.

For the Markov order estimation problem, it is tempting to take as order estimator the first \( k \) for which \( \hat{h}_k(n) - \hat{h}_{k+1}(n) < n^{-1/4} \). This eventually a.s. gets stuck at the first \( k \) for which \( H_k = H_{k+1} \). Such flat spots can occur for \( k < M - 1 \). This shows, incidentally, why we needed to take the maximum over a growing interval of possible orders in the definition (2) of our maximal fluctuation test function.

Remark 6 The “no flat spot” case is “generic” for it is easy to see that in the usual parametrization of the set of \( X \in \mathcal{M} \) of order \( M \) as a subset of \( |A|^M(|A| - 1) \)-dimensional Euclidean space, the set of \( X \) of order \( M \) whose conditional entropy has flat spots before \( M \) has Lebesgue measure 0. This is a good example where genericity is not an interesting concept.

4.3 The BIC, MDL, and related methods.

Two important and related methods, the Bayesian Information Criterion (BIC) and the Minimum Description Length (MDL) Principle are the basis for many model selection methods, see [2, 4, 6] for discussion and references to these and other methods. Both the BIC and the MDL focus on selecting the correct class from a nested sequence of parametric model classes, \( \mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \ldots \), based on a sample path drawn from some \( P \in \bigcup \mathcal{M}_k \).

The BIC, introduced by Schwarz [16], is based on Bayesian principles and leads to the model estimator

\[
M_{\text{BIC}}^*(x^n_1) \overset{\text{def}}{=} \arg\min_k \left( -\log P_{\text{ML}(k)}(x^n_1) + \frac{\phi(k)}{2} \log n \right),
\]

where \( P_{\text{ML}(k)}(x^n_1) \) is the \( k \)-th order maximum likelihood, i.e., the largest probability given to \( x^n_1 \) by distributions in \( \mathcal{M}_k \), and \( \phi(k) \) is the number of free parameters needed to describe members of \( \mathcal{M}_k \). For the Markov order estimation problem, \( \mathcal{M}_k = \{ X \in \mathcal{M} : M(X) \leq k \} \), \(-\log P_{\text{ML}(k)}(x^n_1) = (n - k)\hat{h}_k(n) \), and \( \phi(k) = |A|^k(|A| - 1) \). Schwarz [16] proved consistency if the model classes are i.i.d. exponential families and a bound on the number of models is assumed, a result later extended to the Markov case by Finesso [10]. The first consistency proofs for the Markov case without an order bound assumption are given in [6]. The proofs are surprisingly complicated, though they have been simplified somewhat in [4], which focuses on MDL consistency.

The MDL principle, introduced by Rissanen (see [24]), is based on universal coding ideas. For each \( k \leq n \), the sequence \( x^n_1 \) is encoded using a binary code that is “optimal” for the class \( \mathcal{M}_k \) and the model that has the shortest code length is chosen, that is,

\[
M_{\text{MDL}}^*(x^n_1) \overset{\text{def}}{=} \arg\min_k \mathcal{L}_k(x^n_1) \tag{11}
\]

where \( \mathcal{L}_k(x^n_1) \) is the length of the code word assigned to \( x^n_1 \). Different concepts of “optimal” lead to different estimators. For a discussion of consistency for such estimators without a prior order bound, see [6] and [4].
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