ONE CLASS OF CONTINUOUS FUNCTIONS RELATED TO ENGEL SERIES AND HAVING COMPLICATED LOCAL PROPERTIES

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Abstract. In the paper, we construct and study the class of continuous on \([0, 1]\) functions with continuum set of peculiarities (singular, nowhere monotonic, and non-differentiable functions are among them). The representative of this class is the function \(y = f(x)\) defined by the Engel representation of argument:

\[
x = \sum_{n=1}^{\infty} \frac{1}{(2 + g_1)(2 + g_1 + g_2)\ldots(2 + g_1 + g_2 + \ldots + g_n)} = \Delta^E_{g_1, g_2, \ldots}
\]

where \(g_n = g_n(x) \in \{0, 1, 2, \ldots\}\), and convergent real series

\[
\sum_{n=0}^{\infty} u_n = u_0 + u_1 + \ldots + u_n + r_n = 1, \quad |u_n| < 1, \quad 0 < r_n < 1,
\]

by the following equality

\[
f(\Delta^E_{g_1(x), g_2(x), \ldots}) = r_{g_1(x)} + \sum_{k=2}^{\infty} \left( r_{g_k(x)} \prod_{i=1}^{k-1} u_{g_i(x)} \right).
\]

We study local and global properties of function \(f\): structural, extremal, differential, integral, and fractal properties.

1. Introduction

Most of continuous on the unit interval functions have complicated local properties (infinite and even continuum set of peculiarities) \([4, 17, 19, 46]\). In particular, singular functions (their derivative is equal to zero almost everywhere with respect to Lebesgue measure), nowhere monotonic functions (they do not have any arbitrary small monotonicity interval), and non-differentiable functions (they do not have derivative in any point) are among them. This paper is devoted to such functions. To model and study them we need fine tools and methods. Analytic expressions for these functions contain infinite amount of operations or limiting processes. Series \([7]\), infinite products, continued fractions and other are often used to this end. This is true at least for classic nowhere differentiable Weierstraß \([14]\), Takagi \([43]\), Sierpiński \([55]\), Bush–Wunderlich \([8, 45]\) functions etc., singular Cantor \([26]\), Salem \([9, 13, 40, 42]\), Minkowski \([20]\), Sierpiński \([41]\) functions etc.

Recently systems of functional equations \([1, 15, 31–33]\), iterated function systems, various system of representation of numbers (systems of encoding of numbers) \([2, 6, 27, 29, 36, 37]\), and automata with finite memory (converters of digits from one representation to another) \([16, 23, 25, 27]\) are used to this end.

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There exist some methodological problems in development of general as well as individual theory of such functions. First of all the reason is an absence of effective means of their definition (description) and tools for their study.

Ideas of theory of fractals (fractal geometry and fractal analysis), i.e., self-similar, self-affine, scale-invariant properties can be effectively used to this end.

Now several main directions in the study of local and global fractal properties of functions are developed:

1. fractal characteristics of essential sets for function (for example, sets of peculiarities) [26, 44];
2. properties of level sets of function [38, 44];
3. fractal properties of graphs [22–24, 35];
4. preservation or transformation of fractal dimension by function [3].

In this paper, we construct and study the infinite-parameter family of continuous functions with complicated local properties: singular (monotonic or non-monotonic), nowhere monotonic, non-differentiable or almost everywhere non-differentiable functions are among them. To this end we use $E$-representation of real number, i.e., its encoding by infinite alphabet in the form of Engel series (positive series such that their terms are reciprocal to cumulative products of positive integers) [5, 10, 11, 13, 14, 30, 39].

Similar object related to representation of numbers with finite alphabet and self-similar geometry ($s$-adic representation, $Q$-representation) was studied in papers [31, 32]. But $E$-representation has an infinite alphabet and non-self-similar geometry. So some metric and probabilistic problems are essentially complicated for $E$-representation than for self-similar and $N$-self-similar representations. In papers [31, 32], they found expression (not only estimation) for integral of function in terms of parameters of initial system of functional equations and described conditions for non-differentiability of function.

In our previous paper [1] we studied continuous functions such that they are solutions of infinite system of functional equations with countable set of parameters related to representation of real numbers by the first Ostrogradsky series. Unlike [1] in this paper we study properties of level sets as well as scale-invariant and integral properties of function.

2. Object of study

It follows [30] from the known Engel theorem [10] that for any number $x \in (0, 1]$ there exists a unique sequence $(g_n)$ of non-negative integers such that

$$x = \sum_{n=1}^{\infty} \frac{1}{(2 + g_1)(2 + g_1 + g_2)\ldots(2 + g_1 + g_2 + \ldots + g_n)} \equiv \Delta^E_{g_1 g_2 \ldots g_n \ldots} \quad (1)$$

The series (1) is called Engel series, last symbolic notation of number $x$ is called its $E$-representation, and $g_n = g_n(x)$ is $n$th symbol (digit) of this representation.

Let us remark that $E$-representation of number is its encoding by infinite alphabet $A \equiv \mathbb{Z}_0 = \{0, 1, 2, \ldots\}$. If there exist $p \in \mathbb{N}$ such that $g_{m+np+j} = g_{m+j}, \; 1 \leq j \leq p$, for any $n \in \mathbb{Z}_0$, then they say that $E$-representation has a period

$$g_{m+1}g_{m+2}\ldots g_{m+p}.$$  

It is written by $\Delta^E_{g_1 g_2 \ldots g_n (g_{m+1} g_{m+2} \ldots g_{m+p})^p}$.

The number is called $E$-rational if its $E$-representation has a period (0). The $E$-representation of such numbers has the following form: $\Delta^E_{c_1 c_2 \ldots c_n (0)}$. 
Let \((u_n)_{n=0}^\infty\) be an infinite sequence of real numbers having the following properties (initial conditions):

\[
\sum_{n=0}^\infty u_n = u_0 + u_1 + \ldots + u_n + r_n = S_n + r_n = 1; \quad (2)
\]

\[|u_n| < 1 \quad \text{for any } n \in \mathbb{Z}_0; \quad (3)
\]

\[0 < r_n \equiv \sum_{i=n+1}^\infty u_i = r_{n-1} - u_n = 1 - (u_0 + u_1 + \ldots + u_n) < 1, \quad n \in \mathbb{Z}_0. \quad (4)
\]

It follows from property (2) that sequence \((u_n)\) is infinitesimal, and it follows from property (3) that

\[0 < u_0 + u_1 + \ldots + u_n = S_n < 1 \quad \text{for any} \quad n = 0, 1, 2, \ldots
\]

and series (1) has an infinite number of nonzero terms.

Examples of various sequences \((u_n)\) are following:

1. \(\frac{1}{u_0} = 2, \frac{1}{u_{n+1}} = \frac{1}{u_n} \left(\frac{1}{u_n} - 1\right) + 1, \quad n \in \mathbb{Z}_0; \quad (5)
\]
2. \(u_n = \frac{1}{n+1}, \quad n \in \mathbb{Z}_0; \quad (6)
\]
3. \(u_n = \begin{cases} \frac{1}{2n+1} & \text{if } n = 2k, \\ 0 & \text{if } n = 2k + 1, \quad k \in \mathbb{Z}_0; \end{cases} \quad (7)
\]
4. \(u_0 = \frac{1}{3}, u_1 = -\frac{1}{3}, \quad u_n = \frac{1}{2^n}, \quad n = 2, 3, \ldots; \quad (8)
\]
5. \(u_{2(k-1)} = \frac{a}{2^k}, u_{2k-1} = \frac{1-a}{2^k}, \quad k \in \mathbb{N}, \quad a \in (1, 2). \quad (9)
\]

Remark 1. If sequence \((u_n)\) satisfies initial conditions (5–8), then putting finite or infinite number of zeroes between its terms we obtain new sequence satisfying initial conditions (5–8) too.

The main object of this study is the function

\[y = f(x) = g_1(x) + \sum_{k=2}^\infty \left(r_{g_k(x)} \prod_{i=1}^{k-1} u_{g_i(x)} \right) \equiv \Delta_{g_1 g_2 \ldots g_n}, \quad (10)
\]

where \(g_n = g_n(x)\) is \(n\)th symbol of \(E\)-representation of the number \(x \in (0, 1]\).

It is evident that equality (10) does not define function \(f\) out of left-open interval \((0, 1]\). Moreover,

\[f(1) = f(\Delta_{(0)}) = r_0 + r_0 u_0 + r_0 u_0^2 + \ldots = \frac{r_0}{1 - u_0} = 1,
\]

\[f(\Delta_{c_1 \ldots c_m(x)}) = r_{c_1} + \sum_{k=2}^m \left(r_{c_k} \prod_{i=1}^{k-1} u_{c_i} \right) + \frac{r_{c_m}}{1 - u_{c_m}} \prod_{i=1}^m u_{c_i},
\]

in particular,

\[f(\Delta_{c_1 \ldots c_m(0)}) = r_{c_1} + \sum_{k=2}^m \left(r_{c_k} \prod_{i=1}^{k-1} u_{c_i} \right) + \prod_{i=1}^m u_{c_i}. \quad (11)
\]

Equality (11) gives an expression for value of the function in \(E\)-rational point.

Since any number \(x \in (0, 1]\) has a unique \(E\)-representation, to prove that function \(f\) is well defined, it is enough to show that series (10) is convergent for any sequence of non-negative integers \((g_n)\).

There are negative numbers and zeroes among the terms \(u_n\). So, in general, series (10) is not positive. Thus consider the series with general term

\[v_k = r_{g_k} \prod_{j=1}^{k-1} |u_{g_j}|.
\]
Thus passing to the limit in this inequality we obtain

\[ g - 1 \]

If \( m = 1 \), then we have \( v_k \leq u_k \).

Using the direct comparison test we obtain that convergence of series with general term \( w_k \) implies convergence of series with general term \( v_k \). Moreover,

\[
\sum_{k=1}^{\infty} v_k \leq \sum_{k=1}^{\infty} w_k = \frac{r^*}{1 - u^*}.
\]

Therefore, series (5) is absolutely convergent. So function \( f \) is well defined.

We study local and global properties of function \( f \): structural, extremal, differential, integral and fractal properties, in particular, “symmetries” of graph.

3. RANGE OF THE FUNCTION

**Lemma 1.** Value of the function \( f \) belongs to closed interval \([0, 1] \).

**Proof.** Let

\[ y_m = r_{g_1}(x) + \sum_{k=2}^{m} \left( r_{g_k}(x) \prod_{i=1}^{k-1} u_{g_i}(x) \right) \]

be a partial sum of series (5).

We prove by induction that for any sequence \((g_n)\), \( g_n \in \mathbb{Z}_0 \), and any positive integer \( m \) the following inequality holds:

\[ 0 < y_m < 1. \]  

(7)

For \( m = 1 \), we have \( y_1 = r_{g_1} \in (0, 1) \) by definition of \((u_n)\). Consider \( y_2 = r_{g_1} + r_{g_2} u_{g_1} \).

If \( g_1 = 0 \), then \( 0 < r_0 < y_2 = r_0 + r_{g_2} u_0 < r_0 + w_0 = 1 \), because \( u_0 \in (0, 1) \).

Let \( g_1 > 0 \). If \( u_{g_1} > 0 \), then

\[ 0 < r_{g_1} < y_2 < r_{g_1} + u_{g_1} = r_{g_1-1} < 1. \]

If \( u_{g_1} < 0 \), then

\[ 0 < r_{g_1-1} = r_{g_1} + u_{g_1} \leq y_2 \leq r_{g_1} < 1. \]

Thus \( 0 < y_2 < 1 \) for any sequence \((g_n)\).

Suppose that \( 0 < y_k < 1 \) for any \((g_n)\) and consider \( y_{k+1} \).

Since

\[
y_{k+1} = r_{g_1} + u_{g_1} \left( r_{g_2} + \sum_{l=3}^{k+1} \left( r_l \prod_{i=1}^{l-1} u_{g_i} \right) \right) = r_{g_1} + u_{g_1} y_k^*,
\]

where \( y_k^* \) is a partial sum of series (5) for sequence \((g_2, g_3, \ldots, g_l, \ldots)\), by the inductive assumption, we have \( 0 < y_k^* < 1 \). Hence for \( u_{g_1} > 0 \) we obtain

\[ 0 < r_{g_1} < y_{k+1} < r_{g_1} + u_{g_1} = r_{g_1-1} \leq 1, \]

and for \( u_{g_1} \leq 0 \) we obtain

\[ 0 < r_{g_1-1} < r_{g_1} + u_{g_1} \leq y_{k+1} \leq r_{g_1} < 1. \]

Thus \( 0 < y_{k+1} < 1 \) for any sequence \((g_n)\).

Then by principle of mathematical induction we have the double inequality (7).

Passing to the limit in this inequality we obtain

\[ 0 \leq y = \lim_{m \to \infty} y_m \leq 1. \]
Lemma 2. The function $f$ satisfies the functional equation
\[
f(x) = r_{g_1(x)} + u_{g_1(x)} f(\omega(x)),
\]
where $\omega(x) = \omega(\Delta^E_{g_1(x)g_2(x)\ldots g_n(x)} \ldots) = \Delta^E_{g_1(x)g_2(x)\ldots g_n(x)} \ldots = x'$ is a shift operator on symbols of $E$-representation of number.

Proof. Proposition follows directly from fact that series \([34, 47, 48]\), because shift operator $\omega(x)$ is not a piecewise-linear function of $x$ (for example, this is true for expansion of numbers in the form of the Lüroth series or the $L$-representation \([34, 47, 48]\), because
\[
\omega(x) = x' = \Delta^E_{g_1(x)g_2(x)\ldots g_n(x)} \ldots = \frac{1}{2 + g_2} + \frac{1}{(2 + g_2)(2 + g_2 + g_1)} + \ldots.
\]
Thus relation between $x$ and $x'$ is more complicated.

Let us remark that shift operator $\omega(x)$ on symbols of $E$-representation for operator (function)
\[
\theta(x) = \Delta^E_{g_1(x)g_2(x)\ldots g_n(x)} \ldots = [2 + g_1(x)]x - 1.
\]
The latter is piecewise-linear.

Theorem 1. The range of the function $f$ defined by equality \([5]\) belongs to interval \([0, 1]\), and value $f(x)$ belongs to interval \([a(x), b(x)]\), where
\[
a(x) = \min\{r_{g_1(x)}, r_{g_1(x)-1}\}, \quad b(x) = \max\{r_{g_1(x)}, r_{g_1(x)-1}\},
\]
\[
r_{-1} \equiv u_0 + r_0 = 1.
\]

Proof. Since equality \([5]\) holds and by Lemma \([4]\)
\[
0 \leq f(\omega(x)) \leq 1,
\]
for $u_{g_1(x)} > 0$ we have
\[
0 < a(x) = r_{g_1(x)} < f(x) \leq r_{g_1(x)} + u_{g_1(x)} = r_{g_1(x)-1} = b(x) \leq 1,
\]
and for $u_{g_1(x)} \leq 0$ we have $g_1(x) \geq 1$ and
\[
0 < a(x) = r_{g_1(x)-1} = r_{g_1(x)} + u_{g_1(x)} \leq f(x) \leq r_{g_1(x)} = b(x) < 1.
\]
Hence $a(x) < f(x) \leq b(x)$. Since for $j \geq 0$
\[
0 < r_j < 1 \quad \text{and} \quad 0 < a(x) < f(x),
\]
we have $f(x) > 0$.

Corollary 2. If $u_{g_1(x)} > 0$, then $r_{g_1(x)} < f(x) \leq r_{g_1(x)-1}$; and if $u_{g_1(x)} \leq 0$, then $r_{g_1(x)-1} \leq f(x) \leq r_{g_1(x)}$. 

\[\square\]
4. Continuity of the function

To extend the definition of the function \( f \) at the point \( x = 0 \), put \( f(0) = 0 \).

**Theorem 2.** The function \( f \) is continuous at any point of interval \((0, 1)\), and it is right-continuous at the point \( x = 0 \), left-continuous at the point \( x = 1 \).

**Proof.** Let \( x_0 \) be any point of \((0, 1)\). To prove the continuity of the function at the point \( x_0 \) it is enough to show that

\[
\lim_{x \to x_0} |f(x) - f(x_0)| = 0. \tag{9}
\]

If \( x \neq x_0 \), then there exists \( m \in \mathbb{N} \) such that

\[
g_m(x) \neq g_m(x_0) \quad \text{but} \quad g_i(x) = g_i(x_0) \quad \text{for} \quad i < m.
\]

Then

\[
|f(x) - f(x_0)| = \left( \prod_{i=1}^{m-1} |u_{g_i(x_0)}| \right) r_{g_m(x)} + \sum_{k=m+1}^{\infty} \left( r_{g_k(x)} \prod_{j=m}^{k-1} u_{g_j(x)} \right) - r_{g_m(x_0)} - \sum_{k=m+1}^{\infty} \left( r_{g_k(x_0)} \prod_{j=m}^{k-1} u_{g_j(x_0)} \right) |.
\]

Whence it follows that

\[
|f(x) - f(x_0)| \leq \prod_{i=1}^{m-1} |u_{g_i(x_0)}| \leq (u^*)^{m-1} \to 0 \quad (m \to \infty),
\]

where \( u^* = \max\{ |u_0|, |u_1|, \ldots, |u_n|, \ldots \} < 1 \).

Since condition \( x \to x_0 \) is equivalent to condition \( m \to \infty \), equality (9) holds.

Hence \( f \) is continuous at the point \( x_0 \).

To prove that \( f \) is left-continuous at the point \( x = 1 \) we use analogous arguments.

Condition \( x \neq 1 \) means that there exists \( m \) such that \( g_m(x_0) \neq 0 \). At the same time \( g_i(x) = 0 \) for \( i < m \).

Condition \( x \to 1 - 0 \) is equivalent to condition \( m \to \infty \). Since

\[
|f(x) - f(1)| < \prod_{i=1}^{m-1} |u_{g_i(1)}| \leq u_0^{m-1} \to 0 \quad (m \to \infty),
\]

we see that function \( f \) is left-continuous at the point \( x = 1 \).

Consider point \( x = 0 \). Condition \( x \to 0 + 0 \) is equivalent to condition \( g_1(x) \to \infty \). Hence,

\[
|f(x) - f(0)| = f(x) \leq b(x) = \max\{ r_{g_1(x)}, r_{g_1(x)-1} \} \to 0,
\]

as \( g_1(x) \to \infty \). Thus \( f \) is right-continuous at the point \( x = 0 \). \( \square \)

**Corollary 3.** The range of the function \( f \) is a closed interval \([0, 1]\).

5. Functional relations

**Lemma 3.** The function \( f \) defined by equality (5) is a unique solution of the system of functional equations

\[
f(x) = r_i + u_i f(\omega(x)), \quad i = 0, 1, 2, \ldots, \tag{10}
\]

in the class of bounded functions defined at every point of \((0, 1)\).
Proof. From the fact that \( f \) satisfies system (10) for any \( x \in (0, 1) \) it follows that
\[
 f(x) = f(\Delta^E_{g_1(x)g_2(x)\ldots g_n(x)}) = r_{g_1} + u_{g_1} f(\Delta^E_{g_3(x)g_4(x)\ldots g_n(x)}) = \\
= r_{g_1} + r_{g_2} u_{g_1} + u_{g_1} u_{g_2} f(\Delta^E_{g_3(x)g_4(x)\ldots g_n(x)}) = \ldots = \\
= r_{g_1} + \sum_{k=2}^{m} r_{g_k} \prod_{i=1}^{k-1} u_{g_i} + \left( \prod_{i=1}^{m} u_{g_i} \right) f(\Delta^E_{g_{m+1}(x)g_{m+2}(x)\ldots g_{m+n}(x)}) .
\]

Since \( \prod_{i=1}^{m} u_{g_i} \to 0 \) for \( m \to \infty \) and \( f \) is bounded and defined at every point of \((0, 1)\) (in particular, expression \( f(\Delta^E_{g_{m+1}(x)g_{m+2}(x)\ldots g_{m+n}(x)}) \) is well defined), we see that remainder
\[
\left( \prod_{i=1}^{m} u_{g_i} \right) f(\Delta^E_{g_{m+1}(x)g_{m+2}(x)\ldots g_{m+n}(x)}) \to 0
\]
as \( m \to 0 \). Thus solution of system (10) can be expressed by series (5) uniquely defining function \( f \). \( \square \)

Corollary 4. The function \( f \) is a unique continuous solution of system of functional equations (10).

Remark 2. Lemma 3 gives an equivalent definition of the function \( f \) as a continuous solution of system of functional equations (10).

Remark 3. Condition (2) provides that function \( f \) is bounded and its range is an interval \([0, 1]\). If condition (3) is not fulfilled, then system of functional equations (10) does not have solutions in the class of functions defined on \((0, 1]\), because series is divergent for some \( x \in (0, 1]\). If condition (4) is not fulfilled, the function defined by (5) is not continuous. Accordance of sequences \((u_i)\) and \((r_i)\) provide continuity of solution, and if there is no accordance, then solution is an discontinuous function even if it exists.

6. Monotonicity intervals

Let us recall the definition of useful notion of cylinder for \( E \)-representation of number.

A cylinder of rank \( m \) with base \( c_1c_2\ldots c_m \) is the set \( \Delta^E_{c_1c_2\ldots c_m} \) of all numbers \( x \in (0, 1] \) having \( E \)-representation with first \( m \) symbols \( c_1, c_2, \ldots, c_m \) respectively, i.e.,
\[
\Delta^E_{c_1c_2\ldots c_m} = \left\{ x : g_i(x) = c_i, \ i = 1, m \right\} .
\]

It is known that cylinder \( \Delta^E_{c_1c_2\ldots c_m} \) is a left-open interval with endpoints
\[
a_m = \Delta^E_{c_1c_2\ldots c_{m-1}c_{m+1}}(0) = \sum_{k=1}^{m} \frac{1}{(2 + \sigma_1)(2 + \sigma_2)\ldots(2 + \sigma_k)} ,
\]
\[
b_m = \Delta^E_{c_1c_2\ldots c_m}(0) = a_m + \frac{1}{(2 + \sigma_1)(2 + \sigma_2)\ldots(2 + \sigma_m)(1 + \sigma_m)} ,
\]

where \( \sigma_k \equiv c_1 + c_2 + \ldots + c_k \).

As we can see the endpoints of cylinder \( \Delta^E_{c_1c_2\ldots c_m} \) are \( E \)-rational points:
\[
\Delta^E_{c_1c_2\ldots c_{m-1}c_{m+1}}(0) \text{ and } \Delta^E_{c_1c_2\ldots c_m}(0).
\]

Remark 4. Any \( E \)-rational point is a common point of two cylinders of some rank belonging to the same cylinder of previous rank (we consider that \((0, 1]\) is a cylinder of zero rank).

The cylinders have the following properties:
(1) $\Delta^E_{c_1\ldots c_m} = \bigcup_{c=0}^{\infty} \Delta^E_{c_1\ldots c_m}$

(2) $\max \Delta^E_{c_1\ldots c_m(c+1)} = \inf \Delta^E_{c_1\ldots c_m}$

(3) $|\Delta^E_{c_1\ldots c_m}| = \frac{(2 + \sigma_1)(2 + \sigma_2)\ldots(2 + \sigma_m)(1 + \sigma_m)}{1 + \sigma_m}$

(4) $\frac{|\Delta^E_{c_1\ldots c_m}|}{\Delta^E_{c_1\ldots c_m+1}} = \frac{(2 + \sigma_m+1)(1 + \sigma_m)}{(2 + \sigma_m)}$

(5) For any sequence $(c_n)$, $c_n \in \mathbb{Z}_0$, the following equality holds:

$$\bigcap_{m=1}^{\infty} \Delta^E_{c_1\ldots c_m} = \Delta^E_{c_1\ldots c_m} = x \in (0, 1].$$

Lemma 4. If $u_p = 0$, then function $f$ is constant on every cylinder $\Delta^E_{c_1\ldots c_m p}$.

Proof. Since $E$-representation of any number $x \in \Delta^E_{c_1\ldots c_m}$ has a form

$$x = \Delta^E_{c_1\ldots c_m p} \prod_{i=1}^{m} u_{c_i} + 1,$$

we have

$$f(x) = r_{c_1} + \sum_{k=2}^{m} \left( r_{c_k} \prod_{i=1}^{k-1} u_{c_i} \right) + r_p \prod_{i=1}^{m} u_{c_i} + 0,$$

because $\prod_{i=1}^{m+k} u_{c_i} = 0$ for all $k \in \mathbb{N}$. \(\square\)

Corollary 5. If $(c_1, c_2, \ldots, c_m)$ is any tuple of non-negative integers such that

$$u_{c_1} u_{c_2} \ldots u_{c_m} = 0,$$

then function $f$ is constant on cylinder $\Delta^E_{c_1\ldots c_m}$.

Theorem 3. The function $f$ takes the maximal and minimal values at endpoints of cylinder $\Delta^E_{c_1\ldots c_m}$. Moreover, if

$$D_m = \prod_{i=1}^{m} u_{c_i} \neq 0, \quad y_m = r_{c_1} + \sum_{k=2}^{m} \left( r_{c_k} \prod_{i=1}^{k-1} u_{c_i} \right),$$

then for $D_m > 0$ we have

$$\max f(x) = f(\Delta^E_{c_1\ldots c_m}(0)) = y_m + D_m,$$
$$\min f(x) = f(\Delta^E_{c_1\ldots c_m-1}[c_m+1](0)) = y_m;$$

and for $D_m < 0$ we have

$$\max f(x) = f(\Delta^E_{c_1\ldots c_m-1}[c_m+1](0)) = y_m,$$
$$\min f(x) = f(\Delta^E_{c_1\ldots c_m}(0)) = y_m + D_m;$$

Proof. Since

$$f(x) = y_m + D_m f(\omega^m(x)),$$

where

$$0 \leq f(\omega^m(x)) = r_{g_{m+1}}(x) + \sum_{k=m+2}^{\infty} \left( r_{g_k}(x) \prod_{i=m+1}^{k-1} u_{g_i}(x) \right) \leq 1,$$

we see that for $D_m > 0$ function $f$ takes the maximal value if $f(\omega^m(x)) = 1$, i.e., if $x = \Delta^E_{c_1\ldots c_m}(0)$, and minimal value if $f(\omega^m(x)) = 0$, i.e., if $x = \Delta^E_{c_1\ldots c_m-1}[c_m+1](0)$.

For $D_m < 0$ we have the opposite situation. \(\square\)
Corollary 6. Change in function $f$ on cylinder $\Delta^E_{\xi_1, \xi_2, \ldots, \xi_m}$

$$\mu_f(\Delta^E_{\xi_1, \xi_2, \ldots, \xi_m}) \equiv f(\Delta^E_{\xi_1, \xi_2, \ldots, \xi_m}(0)) - f(\Delta^E_{\xi_1, \xi_2, \ldots, \xi_m-1}(0))$$

can be calculated by formula

$$\mu_f(\Delta^E_{\xi_1, \xi_2, \ldots, \xi_m}) = \prod_{i=1}^{m} u_{\xi_i}. \quad (11)$$

Corollary 7. If there are no zeroes among terms of sequence $(u_n)$, then function $f$ does not have constancy intervals.

Proof. Indeed, suppose that under conditions of this proposition there exists interval $(a, b)$, where $f$ is constant. Then it is easy to find cylinder $\Delta^E_{\xi_1, \xi_2, \ldots, \xi_m}$ completely belonging to $(a, b)$. Hence, by Corollary 6 change in function $f$ on this cylinder as well as on interval $(a, b)$ is nonzero. This contradiction completes the proof. \qed

7. LEBESGUE STRUCTURE OF THE FUNCTION

The known Lebesgue theorem \cite{12} states that any function of bounded variation (in particular, probability distribution function) can be represented in the form of linear combination

$$F(x) = \alpha_1 F_d(x) + \alpha_2 F_{a.c.}(x) + \alpha_3 F_s(x), \quad (12)$$

where $\alpha_i \geq 0$, $\alpha_1 + \alpha_2 + \alpha_3 = 1$, $F_d$, $F_{a.c.}$, $F_s$ is discrete, absolutely continuous and singular function respectively, i.e.,

1. $F_d$ is a function increasing only by jumps (jump function);
2. $F_{a.c.}(x) = \int_{-\infty}^{x} F'(t)dt$;
3. $F_s$ is a continuous function but $F'_s(x) = 0$ Lebesgue-almost everywhere.

Equality (12) is called Lebesgue structure of function $F$.

Moreover, if one of the numbers $\alpha_i$ is equal to 1, then function is called pure. If $\alpha_1 = 1$, then it is called pure discrete, if $\alpha_2 = 1$, then it is called pure absolutely continuous, if $\alpha_3 = 1$, then it is called pure singular (or singularly continuous).

Theorem 4. If all terms of sequence $(u_n)$ are non-negative, then $f$ is

1. a probability distribution function on $[0, 1]$, moreover, this is a distribution function of random variable $\xi = \Delta^E_{\eta_1, \eta_2, \ldots}$ such that its E-symbols $\eta_k$ are independent identically distributed random variables having the distribution $P(\eta_k = n) = u_n$;
2. a strictly increasing function if $u_0 > 0$ for any $n \in \mathbb{Z}_0$;
3. a pure absolutely continuous or pure singular function.

Proof. First of all let us prove that under conditions of the theorem function $f$ is non-decreasing.

Let $x_1 < x_2$. Then there exists $m \in \mathbb{N}$ such that

$$g_m(x_1) > g_m(x_2) \quad \text{but} \quad g_i(x_1) = g_i(x_2) \quad \text{for} \quad i < m. \quad (13)$$

Consider the difference

$$f(x_2) - f(x_1) = \left(\prod_{i=1}^{m-1} u_{g_i(x_1)}\right) \left(r_{g_m(x_2)} - r_{g_m(x_1)}\right) +$$

$$+ \sum_{k=m+1}^{\infty} \left(r_{g_k(x_2)} \prod_{i=m}^{k-1} u_{g_i(x_2)}\right) - \sum_{k=m+1}^{\infty} \left(r_{g_k(x_1)} \prod_{i=m}^{k-1} u_{g_i(x_1)}\right).$$
If \( D = \prod_{i=1}^{m-1} u_{g_i(x_1)} = 0 \), then \( f(x_2) - f(x_1) = 0 \). Let \( D \neq 0 \). Then from condition \( u_i \geq 0 \) it follows that

\[ r \equiv r_{g_m(x_2)} - r_{g_m(x_1)} = u_{g_m(x_2)} + \ldots + u_{g_m(x_1)} \geq 0. \]

If \( r = 0 \), i.e., \( u_{g_m(x_2)+1} = \ldots = u_{g_m(x_1)} = 0 \), then \( \prod_{i=m}^{m+j} u_{g_i(x_1)} = 0 \) for all \( j \in \mathbb{N} \).

So,

\[ f(x_2) - f(x_1) = D \cdot \sum_{k=m+1}^{\infty} r_{g_k(x_2)} \prod_{i=m}^{k-1} u_{g_i(x_2)} \geq 0. \]

If \( r > 0 \), then \( r \geq u_{g_m(x_2)+1} \). Hence,

\[ f(x_2) - f(x_1) \geq D \left( r_{g_m(x_2)} - r_{g_m(x_1)} - \sum_{k=m+1}^{\infty} r_{g_k(x_1)} \prod_{i=m}^{k-1} u_{g_i(x_1)} \right) \geq D(r_{g_m(x_2)} - r_{g_m(x_1)-1}). \]

The last inequality follows from Corollary 2 of Theorem 1.

By \( u_n \geq 0 \) for all \( n \in \mathbb{N} \) and \( g_m(x_2) < g_m(x_1) \), we have

\[ r_{g_m(x_2)} - r_{g_m(x_1)-1} \geq 0. \]

Therefore, \( f(x_2) - f(x_1) \geq 0 \) always. Thus \( f \) is non-decreasing continuous function taking value 0 at \( x = 0 \) and value 1 at \( x = 1 \), i.e., it is a probability distribution function on \([0, 1]\).

Now let us show that expression for distribution function \( F_\xi \) of random variable \( \xi \) coincides with expression for function \( f \).

Find expression for distribution function \( F_\xi = \mathbb{P}\{\xi < x\} \). An event \( \{\xi < x\} \) has the form

\[ \{\xi < x\} = \{\eta_1 > g_1(x)\} \cup \{\eta_1 = g_1(x), \eta_2 > g_2(x)\} \cup \ldots \cup \{\eta_1 = g_1(x), \ldots, \eta_{k-1} = g_{k-1}(x), \eta_k > g_k(x)\} \cup \ldots \]

From disjointness of events in the last union it follows that

\[ \mathbb{P}\{\xi < x\} = \sum_{k=1}^{\infty} \mathbb{P}\{\eta_k = g_k(x), i = 1, \ldots, \eta_k > g_k(x)\}. \]

Since symbols \( \eta_k \) of \( E \)-representation of random variable \( \xi \) are independent random variables, we have

\[ \mathbb{P}\{\eta_i = g_i(x), i = 1, \ldots, \eta_k > g_k(x)\} = \sum_{j=g_k(x)+1}^{\infty} \mathbb{P}\{\eta_j = j\} \prod_{i=1}^{k-1} \mathbb{P}\{\eta_i = g_i(x)\} = \prod_{i=1}^{k-1} u_{g_i(x)} \]

and

\[ \mathbb{P}\{\xi < x\} = g_1(x) + \sum_{k=2}^{\infty} \left( \prod_{i=1}^{k-1} u_{g_i(x)} \right) \cdot \left( r_{g_k(x)} \prod_{i=m}^{k-1} u_{g_i(x)} \right). \]

2. If \( u_n > 0 \) for all \( n \in \mathbb{N} \), then \( D > 0 \) and

\[ \sum_{k=m+1}^{\infty} r_{g_k(x_2)} \prod_{i=m}^{k-1} u_{g_i(x_2)} > 0. \]

Hence,

\[ f(x_2) - f(x_1) > D(r_{g_m(x_2)} - r_{g_m(x_1)-1}) \geq 0, \]

and thus \( f \) is strictly increasing.
Thus $f$ is a distribution function of random variable $\xi = \Delta^E_{m_1m_2\ldots m_n}$ with independent identically distributed $E$-symbols $\eta_k$. By Theorem 2, $f$ is a continuous function. So it is enough to prove that it cannot be a mixture of singular and absolutely continuous distributions.

Let $x = \Delta^E_{g_1(x)g_2(x)\ldots g_n(x)}$ and let $t_1, t_2, \ldots, t_n$ be a fixed tuple of non-negative integers. Denote

$$\bar{\Delta}_{t_1t_2\ldots t_n}(x) = \Delta^E_{t_1t_2\ldots t_n g_{n+1}(x)g_{n+2}(x)}$$

and for any subset $E$ of closed interval $[0, 1]$ 

$$\bar{T}_n(E) = \bigcup_{t_1, t_2, \ldots, t_n} \bar{\Delta}_{t_1t_2\ldots t_n}(E), \quad T(E) = \bigcup_{n} \bar{T}_n(E).$$

Consider event $A = \{\xi \in T(E)\}$. Since $\eta_k$ are independent, we see that event $A$ generated by the sequence of random variables $\eta_k$ does not depend on all $\sigma$-algebras $\mathfrak{B}_m$ generated by $\eta_1, \ldots, \eta_m$. So $A$ is a residual event. Thus, by Kolmogorov’s 0 and 1 law, we have $P(A) = 0$ or $P(A) = 1$.

Since $T(E) \supset E$, we see that from inequality $P\{\xi \in E\} > 0$ it follows that $P\{\xi \in T(E)\} \geq P\{\xi \in E\} > 0$.

Thus $P\{\xi \in T(E)\} = 1$.

Let us consider two cases:

1. There exists $E$ such that $\lambda(E) = 0$ and $P\{\xi \in E\} > 0$.
2. For any set $E$ such that $\lambda(E) = 0$, we have $P\{\xi \in E\} = 0$.

In the first case from equality $\lambda(E) = 0$ it follows that $\lambda(T(E)) = 0$. This means that there exists set $T(E)$ such that $\lambda(T(E)) = 0$ and $P\{\xi \in T(E)\} = 1$, i.e., distribution of $\xi$ is pure singular by definition.

In the second case distribution function of random variable $\xi$ has an $N$-property. This is equivalent to its absolute continuity [21].

8. Conditions of nowhere monotonicity

**Theorem 5.** If sequence $(u_n)$ does not contain zeroes but contains negative terms, then function $f$ is nowhere monotonic on closed interval $[0, 1]$, i.e., it does not have any arbitrary small monotonicity interval.

**Proof.** By Corollary [4] from Theorem 3, function $f$ does not have constancy intervals.

To prove that it does not have any monotonicity interval, it is enough to show that $f$ is not monotonic on any cylinder. To this end, for any cylinder

$$\Delta^E_{c_1c_2\ldots c_m} = (x_0, x_3],$$

it is enough to give two points $x_1$ and $x_2$, where $x_1 < x_2$, such that values

$$f(x_0), f(x_1), f(x_2), f(x_3)$$

does not form a monotonic tuple of numbers. Remark that it is enough even three points

$$x_0, x_1, x_2 \, \text{ or } \, x_1, x_2, x_3.$$
Moreover, Theorem 3 describes the values of extrema. u if \( u < 0 \) is such cylinder because change in function \( f \) on \( \Delta^{E}_{c_1c_2...c_m} \) is equal to
\[
D_m = \prod_{i=1}^{m} u_{c_i},
\]
and on \( \Delta^{E}_{c_1c_2...c_m} \) is equal to \( D_m \cdot u_c \). They have different signs.

If \( D_m > 0 \), then change in function \( f \) is positive on cylinder \( \Delta^{E}_{c_1c_2...c_m} \) and negative on cylinder \( \Delta^{E}_{c_1c_2...c_m} \). If \( D_m < 0 \), then we have the opposite situation. \( \square \)

9. Extrema of the Function

Lemma 4 and Theorem 3 give an exhaustive answer on the question on maximal and minimal value of the function on cylinder. It is enough to study function in the endpoints of cylinders, that is in \( E \)-rational points (points having the representation \( \Delta^{E}_{c_1c_2...c_m}(l(0)) \)).

**Theorem 6.** 1. If \( u_{i-1}u_i < 0 \) for some \( i \), then any point
\[
\Delta^{E}_{c_1c_2...c_m}(l(0)), \quad \text{where} \quad D_m = \prod_{i=1}^{m} u_{c_i} \neq 0,
\]
is an extreme point of the function \( f \). Moreover, it is a maximum point if \( D_mu_i > 0 \), and minimum point if \( D_mu_i < 0 \).

2. If \( u_{i-1}u_i \geq 0 \), any point
\[
\Delta^{E}_{c_1c_2...c_m}(l(0))
\]
is not an extreme point of the function \( f \).

**Proof.** 1. Let \( D_m > 0 \). Then from Corollary of Theorem 3 follows that change in function \( f \) on cylinder \( \Delta^{E}_{c_1c_2...c_m} \) is positive.

If \( u_i > 0 \), then from the same corollary change in function is positive on cylinder \( \Delta^{E}_{c_1c_2...c_m}(l) \) and negative on cylinder \( \Delta^{E}_{c_1c_2...c_m}[l-1] \) lying to the right. So common endpoint of these cylinders, that is point \( x_i \equiv \Delta^{E}_{c_1c_2...c_m}(l(0)) \), is a maximum point.

If \( u_i < 0 \), then change in function \( f \) is negative on cylinder \( \Delta^{E}_{c_1c_2...c_m}(l) \) and positive on cylinder \( \Delta^{E}_{c_1c_2...c_m}[l-1] \). Thus point \( x_i \) is a minimum point.

If \( D_m < 0 \), then we obtain the result analogously.

2. If \( u_{i-1}u_i = 0 \), then using Lemma 4 we have that function is constant at least on one of cylinders
\[
\Delta^{E}_{c_1c_2...c_m}, \quad \Delta^{E}_{c_1c_2...c_m}[l-1].
\]
Thus, their common endpoint \( \Delta^{E}_{c_1c_2...c_m}(l(0)) \) is not extreme point.

If \( u_{i-1}u_i > 0 \), then change in function \( f \) has the same sign on both cylinders (13).

Thus, their common endpoint is not extreme point. \( \square \)

**Corollary 8.** If sequence \( (u_n) \) has negative terms, then function \( f \) has a countable set of extreme points, and they are \( E \)-rational numbers.

**Corollary 9.** If sequence \( (u_n) \) does not contain zeroes, but contains negative terms, then the set of extreme points of function \( f \) is everywhere dense.

In fact, in this case extreme points exist in every cylinder.

**Remark 5.** From Remark 4 and Theorem 6 follows that extreme points form empty set if \( u_n \geq 0 \) for all \( n \in \mathbb{N} \) or countable subset of the set of \( E \)-rational points if at least one \( u_i < 0 \) exists and coincides with it if sequence \( (u_n) \) is alternating. Moreover, Theorem 3 describes the values of extrema.
10. LEVEL SETS OF THE FUNCTION

Let us recall that level set \( y_0 \) of function \( f \) is a set
\[
  f^{-1}(y_0) = \{ x : f(x) = y_0 \}.
\]

If \( u_n > 0 \) for all \( n \in \mathbb{N} \), then \( f \) is a continuous strictly increasing function as proved above. Thus, any its level consists of one point.

If \( u_n \geq 0 \) for all \( n \in \mathbb{N} \), but there are exist \( u_p = 0 \), then level
\[
y = r_{c_1} + \sum_{k=2}^{m} r_{c_k} \prod_{i=1}^{k-1} u_{c_i} + r_f \prod_{i=1}^{m} u_{c_i}
\]
contains cylindrical closed interval \( \Delta_{E}^{E} \). In this case any level is either point or closed interval (due to continuity).

Situation is more complex if there are negative terms in the sequence \( u_n \). Results of the previous section suggest that function \( f \) does not have a continuum level set, so there are not exist levels of the function having fractal properties. Moreover, the properties of the level set essentially depend on the sequence \( (u_n) \).

**Theorem 7.** If there exist negative terms in the sequence \( (u_n) \) and \( E \)-representation of number \( x = \Delta_{g_1(x)g_2(x)...g_n(x)...} \) has the following property:
\[
  u_{g_i(x)}u_{g_{i+1}(x)} < 0
\]
for infinite set of values \( i \in \mathbb{N} \), then level \( f^{-1}(y_0) \), where \( y_0 = f(x) \), is a countable set.

**Proof.** Using Corollary 6 from Theorem 3 we have that changes in function \( f \) have different signs on cylinders
\[
  \Delta_{g_1(x)g_2(x)...g_i(x)} \quad \text{and} \quad \Delta_{g_1(x)g_2(x)...g_i(x)g_{i+1}(x)},
\]
and by Theorem 8 values of function \( f \) on a cylinder form a closed interval such that its endpoints are values of function of cylinder’s endpoints. Thus, taking into account a continuity of function we have that line \( y = y_0 \) intersects the graph of function \( f \) at least at two points belonging to cylinder \( \Delta_{g_1(x)g_2(x)...g_i(x)g_{i+1}(x)} \) and not belonging to cylinder \( \Delta_{g_1(x)g_2(x)...g_i(x)} \).

The pattern repeats for the next \( i \) such that condition (14) holds. This will repeats infinitely many times. So, level \( f^{-1}(y_0) \) is an infinite set. It cannot be continuum as mentioned above, since the set of local maximums and minimums is countable. Thus, \( f^{-1}(y_0) \) is a countable set. \( \square \)

**Remark 6.** If condition (14) holds, the \( E \)-representation of number \( x \) does not have a simple period (i.e., period consisting of one symbol), and therefore, it is not an \( E \)-rational number.

11. “Symmetries” of the graph. Scale invariance

**Theorem 8.** Graph \( \Gamma_f \) of the function \( f \) is a scale-invariant set, namely:
\[
  \Gamma_f = \Delta_{E(0)} \cup \bigcup_{i=0}^{\infty} \Gamma_i, \quad \text{where} \quad \Gamma_i = \varphi_i(\Gamma_f),
\]
and
\[
  \varphi_i: \begin{cases} 
    x' = \Delta_{g_1(x)g_2(x)...g_n(x)...} = \delta_i(x), \\
    y' = r_i + u_i f(x).
  \end{cases}
\]
Proof. 1. First of all we prove that
\[ \Gamma_f \subset \Delta^E_{(0)} \cup \bigcup_{i=0}^{\infty} \Gamma_i \equiv F. \]
Let \( \Delta^E_{(0)} \neq M(x,y) \in \Gamma_f \) that is \( y = f(x) \). We show that \( M \in F \) that is there exist \( i \) such that \( M \in \Gamma_i \).

Let us consider point \( M_1(\Delta^E_{g_2(\omega^2(x))} \ldots g_n(\omega^2(x))) \). It is evident that \( M_1 \in \Gamma_f \). Then
\[ \varphi_{g_1(\omega^2(\omega^2(x)))} (M_1) = M'_1(\Delta^E_{g_2(\omega^2(x)))} \ldots g_n(\omega^2(x))) \]
and \( M'_1 = M \). Then \( M \in \Gamma_{1+g_1(\omega^2(x)))} \), and thus \( M \in F \) and \( \Gamma_f \subset F \).

2. Show that \( F \subset \Gamma_f \). Let \( M'(x',y') \in F \). If \( M'(x',y') = \Delta^E_{(0)} \), then there is nothing to prove; otherwise there exist \( j \) such that \( M' \in \Gamma_j \). Consider point \( M(x,y) \) such that \( M' = \varphi_j(M) \). Then \( M \in \Gamma_f \) that is \( y = f(x) \), and \( M' \) has coordinates
\[ (\Delta^E_{1+g_1(\omega^2(x)))} \ldots g_n(\omega^2(x))) \]
that is \( y' = f(x') \).

Thus \( M' \in \Gamma_f \) and \( F \subset \Gamma_f \).

Equality \( \Gamma_f = F \) follows from inclusions \( \Gamma_f \subset F \) and \( F \subset \Gamma_f \).

\( \square \)

Remark 7. Transformation \( \delta_i \) of left-open interval \( (0,1] \) defined by the formula
\[ x' = \delta_i(x) = \Delta^E_{1+g_1(\omega^2(x)))} \ldots g_n(\omega^2(x))) \]
has non-trivial properties, since
\[ x' = \frac{1}{2 + i} + \frac{1}{2+i} x^* \]
where \( x^* = \Delta^E_{1+g_1(\omega^2(x)))} \ldots g_n(\omega^2(x))) \)
and requires special study.

12. INTEGRAL PROPERTIES OF THE FUNCTION

Lemma 5. If \( i \) is a fixed number belonging to \( \mathbb{Z}_0 \), then mapping
\[ x' = \delta_i(x) = \Delta^E_{1+g_1(\omega^2(x)))} \ldots g_n(\omega^2(x))) \]
is contractive with coefficient \( \frac{1}{2+i} \).

Proof. This proposition follows from the facts that image of cylinder \( \Delta^E_{i+1,2\ldots c_m} \) of rank \( m \) under mapping \( \delta_i \) is cylinder \( \Delta^E_{i+1,2\ldots c_m} \) of rank \( m+1 \) and the following relations hold:
\[ \frac{|\Delta^E_{i+1,2\ldots c_m}|}{|\Delta^E_{c_1,2\ldots c_m}|} = \frac{(2 + \sigma_1)(2 + \sigma_2) \ldots (2 + \sigma_m)(1 + \sigma_m)}{(2 + i)(2 + \sigma_1 + i)(2 + \sigma_2 + i) \ldots (2 + \sigma_m + i)(1 + \sigma_m + i)} = \frac{1}{2 + i} \cdot \frac{1 + \sigma_m + i}{\prod_{k=1}^{m} \frac{2 + \sigma_k}{2 + \sigma_k + i} < \frac{1}{2 + i} < \frac{1}{2},} \]
where \( \sigma_k = c_1 + c_2 + \ldots + c_k \).

\( \square \)

Corollary 10. Let \( E \) be a set of zero Lebesgue measure. Then measure of its image under mapping \( \delta_i \) is equal to zero too. That is
\[ \lambda(E) = 0 \Rightarrow \lambda(\delta_i(E)) = 0 \]
for any \( i \in \mathbb{Z}_0 \).
Lemma 6. For Lebesgue integral the following equality holds:

\[ I \equiv \int_0^1 f(x) \, dx = \lim_{m \to \infty} \sum_{n=0}^m I_n = \sum_{n=0}^\infty I_n, \]

where

\[ I_n = \frac{\Delta^E_{n+1}(0)}{\Delta^E_{n}(0)} \int_{\Delta^E_{n+1}(0)} f(x) \, dx. \]

Proof. This proposition follows from integrability of the function, additive property of the Lebesgue integral, and fact that \((0, 1)\) is a union of countable set of disjoint intervals \((\Delta^E_{n+1}(0), \Delta^E_{n}(0)), n = 0, 1, 2, \ldots\). \(\square\)

Remark 8. “Symmetries” of graph of function studied in the previous section should help to express \(I_n\) in terms of \(\int_0^1 f(x) \, dx\).

But non-self-similar geometry of \(E\)-representation essentially complicates this problem.

Theorem 9. For Lebesgue integral

\[ \int_0^1 f(x) \, dx \]

the following estimate holds:

\[ \int_0^1 f(x) \, dx \leq \left( 1 - \sum_{n=0}^\infty \frac{u_n}{2 + n} \right)^{-1} \sum_{n=0}^\infty \frac{r_n}{2 + n}. \]  \hspace{1cm} (15)

Proof. Taking into account Theorem 8 and Lemma 6 we have

\[ I_n = \int_{\Delta^E_{n+1}(0)} f(x) \, dx = \int_0^1 (r_n + u_n f(x)) \, d\delta_n(x). \]

Then from the Lemma 6 it follows that

\[ d\delta_n(x) \leq \frac{1}{2 + n} \, dx \]

and

\[ I_n \leq \frac{1}{2 + n} \int_0^1 (r_n + u_n f(x)) \, dx = \frac{r_n}{2 + n} + \frac{u_n}{2 + n} \int_0^1 f(x) \, dx. \]

Thus

\[ \int_0^1 f(x) \, dx \leq \sum_{n=0}^\infty \left( \frac{r_n}{2 + n} + \frac{u_n}{2 + n} \int_0^1 f(x) \, dx \right) = \sum_{n=0}^\infty \frac{r_n}{2 + n} + \left( \sum_{n=0}^\infty \frac{u_n}{2 + n} \right) \int_0^1 f(x) \, dx. \]
and
\[
\left(1 - \sum_{n=0}^{\infty} \frac{u_n}{2 + n}\right) \int_{0}^{1} f(x) \, dx \leq \sum_{n=0}^{\infty} \frac{r_n}{2 + n}.
\]
This inequality is equivalent to (15). \qed

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