Ranks and Pseudo-Ranks
- Paradoxical Results of Rank Tests -

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Abstract

Rank-based inference methods are applied in various disciplines, typically when procedures relying on standard normal theory are not justifiable, for example when data are not symmetrically distributed, contain outliers, or responses are even measured on ordinal scales. Various specific rank-based methods have been developed for two and more samples, and also for general factorial designs (e.g., Kruskal-Wallis test, Jonckheere-Terpstra test). It is the aim of the present paper (1) to demonstrate that traditional rank-procedures for several samples or general factorial designs may lead to paradoxical results in case of unbalanced samples, (2) to explain why this is the case, and (3) to provide a way to overcome these disadvantages of traditional rank-based inference. Theoretical investigations show that the paradoxical results can be explained by carefully considering the non-centralities of the test statistics which may be non-zero for the traditional tests in unbalanced designs. These non-centralities may even become arbitrarily large for increasing sample sizes in the unbalanced case. A simple solution is the use of so-called pseudo-ranks instead of ranks. As a special case, we illustrate the effects in sub-group analyses which are often used when dealing with rare diseases.
1 Introduction

If the assumptions of classical parametric inference methods are not met, the usual recommendation is to apply nonparametric rank-based tests. Here, the Wilcoxon-Mann-Whitney and Kruskal-Wallis (1952) tests are among the most commonly applied rank procedures, often utilized as replacements for the unpaired two-sample $t$-test and the one-way ANOVA, respectively. Other popular rank methods include the Hettmansperger-Norton (1987) and Jonckheere-Terpstra (1952, 1954) tests for ordered alternatives, and the procedures by Akritas et al. (1997) for two- or higher-way designs. In statistical practice, these procedures are usually appreciated as robust and powerful inference tools when standard assumptions are not fulfilled. For example, Whitley and Ball (2002) conclude that “Nonparametric methods require no or very limited assumptions to be made about the format of the data, and they may, therefore, be preferable when the assumptions required for parametric methods are not valid.” In line with this statement, Bewick et al. (2004) also state that “the Kruskal-Wallis, Jonckheere-Terpstra (...) tests can be used to test for differences between more than two groups or treatments when the assumptions for analysis of variance are not held.”

These descriptions are slightly over-optimistic since nonparametric methods also rely on certain assumptions. In particular, the Wilcoxon-Mann-Whitney and Kruskal-Wallis tests postulate homoscedasticity across groups under the null hypothesis, and they have originally only been developed for continuous outcomes. In case of doubt, it is nevertheless expected that rank procedures are more robust and lead to more reliable results than their parametric counterparts. While this is true for deviations from normality, and while by now it is widely accepted that ordinal data should rather be analyzed using adequate rank-based methods than using normal theory procedures, we illustrate in various instances that nonparametric rank tests for more than two samples possess one noteworthy weakness. Namely, they are generally non-robust against changes from balanced to unbalanced designs. In particular, keeping the data generating processes fixed, we provide paradigms under which commonly used rank tests surprisingly yield completely opposite test decisions when rearranging group sample sizes. These examples are in general not artificially generated to obtain paradoxical results, but even include homoscedastic normal models. This effect is completely undesirable, leading to the somewhat heretical question

- Are nonparametric rank procedures useful at all to handle questions for more than two groups?

In order to comprehensively answer this question, we carefully analyze the underlying nonparametric effects of the respective rank procedures. From this, we develop detailed guidelines for an adequate application of rank-based procedures. Moreover, we even state a simple solution for all these problems: Substituting ranks by closely related quantities, the so-called pseudo-ranks that have already been considered by Kulle (1999), Gao and Alvo (2005a, b), Gao, Alvo,
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Chen, and Li (2008), and in more detail by Thangavelu and Brunner (2007), and by Brunner, Konietzchke, Pauly and Puri (2017). It should be noted that the motivation in these references was different, and that their authors had not been aware of the striking paradoxical properties that may arise when using classical rank tests. These surprising paradigms only appear in case of unbalanced designs since all rank procedures discussed below coincide with their respective pseudo-rank analogs in case of equal sample sizes. Pseudo-ranks are easy to compute, share the same advantageous properties of ranks and lead to reliable and robust inference procedures for a variety of factorial designs. Moreover, we can even obtain confidence intervals for (contrasts of) easy to interpret reasonable nonparametric effects. Thus, resolving the commonly raised disadvantage that “nonparametric methods are geared toward hypothesis testing rather than estimation of effects” (Whitley and Ball, 2002).

The paper is organized as follows. Notations are introduced in Section 2. Then in Section 3 some paradoxical results are presented in the one-way layout for the Kruskal-Wallis test and for the Hettmansperger-Norton trend test by means of certain tricky (non-transitive) dice. In the two-way layout, a paradoxical result for the Akritas-Arnold-Brunner test in a simple $2 \times 2$-design is presented in Section 4 using a homoscedastic normal shift model. The theoretical background of the paradoxical results is discussed in Section 5 and a solution of the problem by using pseudo-ranks is investigated in detail. Moreover, the computation of confidence intervals is discussed and applied to the data in Section 4. Section 6 provides a cautionary note for the problem of sub-group analysis where typically unequal sample sizes appear.

The paper closes with some guidelines for adequate application of rank procedures in the discussion and conclusions section. There, it is also briefly discussed that the use of pairwise and stratified rankings would make matters potentially worse.

## 2 Statistical Model and Notations

For $d > 2$ samples of $N = \sum_{i=1}^{d} n_i$ independent observations $X_{ik} \sim F_i = \frac{1}{2}[F_i^- + F_i^+]$, $i = 1, \ldots, d$, $k = 1, \ldots, n_i$, the nonparametric relative effects which are underlying the rank tests are commonly defined as

$$p_i = \int H dF_i, \quad \text{where} \quad H = \frac{1}{N} \sum_{r=1}^{d} n_r F_r$$

(1)

denotes the weighted mean distribution of the distributions $F_1, \ldots, F_d$ in the design. Here we use the so-called normalized version of the the distribution $F_i$ to cover the cases of continuous, as well as non-continuous distributions in a unified approach. Thus, the case of ties does not require a separate consideration. This idea was first mentioned by Kruskal (1952) and later considered in more detail by Ruymgaart (1980). Akritas, Arnold, and Brunner (1997) extended this approach to factorial designs, while Akritas and Brunner (1997) and Brunner, Munzel, and Puri (1999) applied this technique to repeated measures and longitudinal data.
Easily interpreted, \( p_i = P(Z < X_{i1}) + \frac{1}{2} P(Z = X_{i1}) \) is the probability that a randomly selected observation \( Z \) from the weighted mean distribution \( H \) is smaller than a randomly selected observation \( X_{i1} \) from the distribution \( F_i \), plus \( \frac{1}{2} \) times the probability that both observations are equal. Thus, the quantity \( p_i \) measures an effect of the distribution \( F_i \) with respect to the weighted mean distribution \( H \). In the case of two independent random variables \( X_1 \sim F_1 \) and \( X_2 \sim F_2 \), Birnbaum and Klose (1957) had called the function \( L(t) = F_2[F_1^{-1}(t)] \) the “relative distribution function” of \( X_1 \) and \( X_2 \), assuming continuous distributions. Thus, its expectation

\[
\int_0^1 tdL(t) = \int_{-\infty}^{\infty} F_1(s)dF_2(s) = P(X_1 < X_2)
\]

is called a “relative effect” with an obvious adaption of the notation. In the same way, the quantity \( p_i = P(Z < X_{i1}) + \frac{1}{2} P(Z = X_{i1}) \) is called a “relative effect” of \( X_{i1} \sim F_i \) with respect to the weighted mean \( Z \sim H \). This effect \( p_i \) is a linear combination of the pairwise effects \( w_{ri} = \int F_r dF_i \). In vector notation, Equation (1) is written as

\[
p = \int HdF = W'n = \left( \begin{array}{c} w_{11} \\ \vdots \\ w_{dd} \end{array} \right) \left( \begin{array}{c} \frac{n_1}{N} \\ \vdots \\ \frac{n_d}{N} \end{array} \right) = \left( \begin{array}{c} p_1 \\ \vdots \\ p_d \end{array} \right). \tag{2}
\]

Here, \( F = (F_1, \ldots, F_d)' \) denotes the vector of distribution functions, and

\[
W = \int F' dF = \left( \begin{array}{ccc} w_{11} & \cdots & w_{ld} \\ \vdots & \ddots & \vdots \\ w_{dl} & \cdots & w_{dd} \end{array} \right) \tag{3}
\]

is the matrix of pairwise effects \( w_{ri} \). Note that \( w_{ii} = \frac{1}{2} \) and \( w_{ir} = 1 - w_{ri} \) which follows from integration by parts. The relative effects \( p_i \) are arranged in the vector \( p = (p_1, \ldots, p_d)' \) and can be estimated consistently by the simple plug-in estimator

\[
\hat{p}_i = \int \tilde{H}d\tilde{F}_i = \frac{1}{N}(\tilde{R}_i - \frac{1}{2}). \tag{4}
\]

Here, \( \tilde{F}_i \) denotes the (normalized) empirical distribution of \( X_{i1}, \ldots, X_{in} \), \( i = 1, \ldots, d \), and \( \tilde{H} = \frac{1}{N} \sum_{r=1}^{d} n_r \tilde{F}_r \) their weighted mean. Finally, \( \tilde{R}_i = \frac{1}{n} \sum_{k=1}^{n} R_{ik} \) denotes the mean of the ranks

\[
R_{ik} = \frac{1}{2} + NH(X_{ik}) = \frac{1}{2} + \sum_{r=1}^{d} \sum_{\ell=1}^{n} c(X_{ik} - X_{r\ell}), \tag{5}
\]

where the function \( c(u) = 0, 1/2, 1 \) for \( u <, = \) or \( > 0 \), respectively, denotes the count function. The estimators \( \hat{p}_1, \ldots, \hat{p}_d \) are arranged in the vector

\[
\hat{p} = \int \tilde{H}d\tilde{F} = \frac{1}{N}(\tilde{R} - \frac{1}{2}1_d), \tag{6}
\]
where $\hat{F} = (\hat{F}_1, \ldots, \hat{F}_d)'$ is the vector of the empirical distributions, $\hat{R} = (\hat{R}_1, \ldots, \hat{R}_d)'$ the vector of the rank means $\hat{R}_i$, and $1_d = (1, \ldots, 1)_{d \times 1}$ denotes the vector of 1s.

In the following sections we demonstrate that for $d \geq 3$ groups, rank tests may lead to paradoxical results in case of unequal sample sizes. In particular, for factorial designs involving two or more factors, the nonparametric main effects and interactions (defined by the weighted relative effects $p_{ij} = \int HD_{ij}$) may be severely biased.

### 3 Paradoxical Results in the One-Way Layout

To demonstrate some paradoxical results of rank tests for $d \geq 3$ samples in the one-way layout, we consider the vector $p = W'n$ in (2) of the nonparametric effects $p_i$, which are all equal to their mean $\bar{p} = \frac{1}{d} \sum_{i=1}^{d} p_i$ if $\sum_{i=1}^{d} (p_i - \bar{p})^2 = 0$ or in matrix notation $p'T_d p = 0$. Here, $T_d = I_d - \frac{1}{d} J_d$ denotes the centering matrix, $I_d$ the $d$-dimensional unit matrix, and $J_d = 1_d 1_d'$ the $d \times d$-dimensional matrix of 1s. Let $\hat{p} = \int HD\hat{F}$ denote the plug-in estimator of $p$ defined in (6). In order to detect whether the the $p_i$ are different, we study the asymptotic distribution of $\sqrt{N}T_d\hat{p}$. This is obtained from the asymptotic equivalence theorem (see, e.g., Akritas et al., 1997; Brunner and Puri, 2001, 2002 or Brunner et al., 2017),

$$\sqrt{N}T_d\hat{p} \overset{\text{a.s.}}{=} \sqrt{N}T_d \left[ \bar{Y} + \bar{Z} - 2\bar{p} \right] + \sqrt{NT_d p},$$

where the symbol $\overset{\text{a.s.}}{=}$ denotes asymptotic equivalence. Here, $\bar{Y} = \int HD\hat{F}$ and $\bar{Z} = \int HD\hat{F}$ are vectors of means of independent random vectors with expectation $E(\bar{Y}) = E(\bar{Z}) = \bar{p}$. It follows from the central limit theorem that $\sqrt{NT_d} \left[ \bar{Y} + \bar{Z} - 2\bar{p} \right]$ has, asymptotically, a multivariate normal distribution with mean 0 and covariance matrix $T_d \Sigma_N T_d$, where $\Sigma_N = \text{Cov} \left( \sqrt{N}(\bar{Y} + \bar{Z}) \right)$ has a quite involved structure (for details see Brunner et al., 2017). Obviously, the multivariate distribution is shifted by $\sqrt{NT_d} p$ from the origin $0$. Therefore, we call $T_d p$ the “multivariate non-centrality”, and a “univariate non-centrality” may be quantified by the quadratic form $c_p = p'T_d p$. In particular, we have $c_p = 0$ iff $T_d p = 0$. The actual (multivariate) shift of the distribution, depending on the total sample size $N$, is $\sqrt{NT_d} p$, and the corresponding univariate non-centrality (depending on $N$) is then given by $N \cdot c_p$. From these considerations, it should become clear that $N \cdot c_p \to \infty$ as $N \to \infty$ if $T_d p \neq 0$. This defines the consistency region of a test based on $\sqrt{NT_d} p$.

Below, we will demonstrate that for the same vector of distributions $F$, the non-centrality $c_p = p'T_d p$ may be 0 in case of equal sample sizes, while $c_p$ may be unequal to 0 in case of unequal sample sizes. Under $H^F_0 : T_d F = 0$, tests based on $\hat{p}$ (such as the Kruskal-Wallis test) reject the hypothesis $H^F_0$ with approximately the pre-assigned type-I error probability $\alpha$. If, however, the strong hypothesis $H^F_0$ is not true then the non-centrality $c_p = p'T_d p$ may be 0 or unequal to 0 for the same set of distributions $F_1, \ldots, F_d$, since $c_p$ depends on the relative samples sizes $n_1/N, \ldots, n_d/N$. This means that for the same set of distributions $F_1, \ldots, F_d$
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and unequal sample sizes the p-value of the test may be arbitrary small if \( N \) is large enough. However, the p-value for the same test may be quite large for the same total sample size \( N \) in case of equal sample sizes. Some well-known tests which have this paradoxical property are, for example, the Kruskal-Wallis test (1952), the Hettmansperger-Norton trend test (1987), and the Akritas-Arnold-Brunner test (1997).

As an example, consider the case of \( d = 3 \) distributions where straightforward calculations show that

1. in case of equal sample sizes,

\[
p_1 = p_2 = p_3 \iff w_{21} = w_{32} = 1 - w_{31} = w,
\]

2. in general, however,

\[
p_1 = p_2 = p_3 \iff w_{21} = w_{32} = w_{31} = \frac{1}{2}.
\]

This means that \( c_p = p'T_d p = 0 \) in case of equal sample sizes, but \( c_p \neq 0 \) in case of unequal sample sizes if \( w_{21} = w_{32} = 1 - w_{31} = w \neq \frac{1}{2} \).

We note that (9) follows under the strict hypothesis \( H_0^{F} : F_1 = F_2 = F_3 \). However, this null hypothesis is not a necessary condition for (9) to hold. For example, if \( F_i, i = 1, 2, 3 \) are symmetric distributions with the same center of symmetry then \( w_{ri} = \int F_i dF_i = \frac{1}{2} \) for \( i = 1, 2, 3 \). Thus, in this case, \( c_p = 0 \) is also true for all sample sizes.

An example of discrete distributions generating the nonparametric effects \( w_{ri} \) in (8) is given by the probability mass functions

- \( f_1(x) = \frac{1}{6} \) if \( x \in \{9, 16, 17, 20, 21, 22\} \) and \( f_1(x) = 0 \) otherwise,
- \( f_2(x) = \frac{1}{6} \) if \( x \in \{13, 14, 15, 18, 19, 26\} \) and \( f_2(x) = 0 \) otherwise,
- \( f_3(x) = \frac{1}{6} \) if \( x \in \{10, 11, 12, 23, 24, 25\} \) and \( f_3(x) = 0 \) otherwise,

which are derived from some tricky dice (see, e.g., Peterson, 2002). For the distribution functions \( F_i(x) \) defined by \( f_i(x), i = 1, 2, 3 \) above, it is easily seen that

\[
w_{21} = P(X_2 < X_1) = \int F_2 dF_1 = \frac{7}{12},
\]

\[
w_{13} = P(X_1 < X_3) = \int F_1 dF_3 = \frac{7}{12},
\]

\[
w_{32} = P(X_3 < X_2) = \int F_3 dF_2 = \frac{7}{12}.
\]

Thus, \( w_{21} = w_{13} = 1 - w_{31} = w_{32} = w \) and the vector of the weighted relative effects is given by

\[
p = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = Wn = \frac{1}{N} \begin{pmatrix} \frac{1}{2}n_1 + n_3 + (n_2 - n_3)w \\ n_1 + \frac{1}{2}n_2 + (n_3 - n_1)w \\ n_2 + \frac{1}{2}n_3 + (n_1 - n_2)w \end{pmatrix}.
\]
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Table 1: Ratios of relative sample sizes \( n/N \), weighted relative effects \( p_i \), and the non-centralities for the example of the tricky dice where \( w = 7/12 \) and the distributions \( F_1, F_2, \) and \( F_3 \) are fixed.

| Setting | \( n_1/N \) | \( n_2/N \) | \( n_3/N \) | \( p_1 \) | \( p_2 \) | \( p_3 \) | \( \bar{p} \) | \( c_p \) |
|---------|-------------|-------------|-------------|---------|---------|---------|-------|-------|
| (A)     | \( 1/3 \)   | \( 1/3 \)   | \( 1/3 \)   | 0.5     | 0.5     | 0.5     | 0.5   | 0     |
| (B)     | \( 2/3 \)   | \( 1/12 \)  | \( 1/4 \)   | 0.4861  | 0.4653  | 0.5486  | 0.5   | 0.00376 |
| (C)     | \( 1/4 \)   | \( 2/3 \)   | \( 1/12 \)  | 0.5486  | 0.4861  | 0.4653  | 0.5   | 0.00376 |

The weighted relative effects \( p_i \) and the resulting non-centralities \( c_p \) are listed in Table 1 for equal and some different unequal sample sizes.

Since for unequal sample sizes one obtains \( c_p \neq 0 \), it is only a question of choosing the total sample size \( N \) large enough to reject the hypothesis \( H^F_0 : F_1 = F_2 = F_3 \) by the Kruskal-Wallis test with a probability arbitrary close to 1 while in case of equal sample sizes for \( N \to \infty \) the probability of rejecting the hypothesis remains constant equal to \( \alpha^* \) (close to \( \alpha \)) since in this case, \( c_p = 0 \). It may be noted that in general \( \alpha^* \neq \alpha \) since the variance estimator of the Kruskal-Wallis statistic is computed under the strong hypothesis \( H^F_0 : F_1 = F_2 = F_3 \), which is obviously not true here. Thus, the scaling is not correct, and the Kruskal-Wallis test has a slightly different type-I error \( \alpha^* \).

For the Hettmansperger-Norton trend test, the situation gets worse since for different ratios of sample sizes the nonparametric effects \( p_1, p_2, \) and \( p_3 \) may change their order. In setting (B) in Table 1 we have \( p_2 < p_1 < p_3 \), while in setting (C) we have \( p_3 < p_2 < p_1 \). Now consider the non-centrality of the Hettmansperger-Norton trend test which is a linear rank test. Let \( c = (c_1, \ldots, c_d)' \) denote a vector reflecting the conjectured pattern. Then it follows from (7) that

\[
\sqrt{N}c' T_d \bar{p} \cong \sqrt{N}c' T_d \left[ \bar{Y} + \bar{Z} - 2p \right] + \sqrt{N}c' T_d p, \tag{14}
\]

where \( c_{HN} = c' T_d p \) is a univariate non-centrality. If \( T_d F = 0 \) then it follows that \( T_d p = 0 \) and \( c_{HN} = c' T_d p = 0 \). If, however, \( T_d F \neq 0 \) then \( c_{HN} < 0 \) indicates a decreasing trend and \( c_{HN} > 0 \) an increasing trend. In the above discussed example, we obtain for setting (B) and for a conjectured pattern of \( c = (1, 2, 3)' \) for an increasing trend the non-centrality \( c_{HN} = \sum_{i=1}^{3} c_i (p_i - \frac{1}{3}) = 1/16 > 0 \), indeed indicating an increasing trend. For setting (C) however, we obtain \( c_{HN} = -1/12 \), indicating a decreasing trend. In case of setting (A) (equal sample sizes), \( c_{HN} = 0 \) since \( p_1 = p_2 = p_3 = 1/3 \), and thus indicating no trend. Again it is a question of the total sample size \( N \) to obtain the decision of a significantly decreasing trend for the first setting (B) of unequal sample sizes and for the second setting (C) the decision of a significantly increasing trend with a probability arbitrary close to 1 for the same distributions \( F_1, F_2, \) and \( F_3 \). In the first case, \( \sqrt{N} \cdot c_{HN} \to \infty \) for \( N \to \infty \) and in the second case, \( \sqrt{N} \cdot c_{HN} \to -\infty \) for \( N \to \infty \). In case of equal sample sizes, the hypothesis of no trend is only rejected with a type-I error probability \( \alpha^{**} \). Regarding \( \alpha^{**} \neq \alpha \), a similar remark applies as above for the Kruskal-Wallis test.
4 Paradoxical Results in the Two-Way Layout

In the previous section, paradoxical decisions by rank tests in case of unequal sample sizes were demonstrated for the one-way layout using large sample sizes and particular distributions leading to non-transitive decisions. In this section, we will show that in two-way layouts paradoxical results are already possible with rather small sample sizes and even in simple homoscedastic normal shift models. To this end, we consider the simple $2 \times 2$-design with two crossed factors $A$ and $B$, each with two levels $i = 1, 2$ for $A$ and $j = 1, 2$ for $B$. The observations $X_{ijk} \sim F_{ij}$, $k = 1, \ldots, n_{ij}$, are assumed to be independent.

The hypotheses of no nonparametric effects in terms of the distribution functions $F_{ij}(x)$ are expressed as (see Akritas et al., 1997)

(1) no main effect of factor $A$ - $H_{0}^{F}(A)$: $F_{11} + F_{12} - F_{21} - F_{22} = 0$

(2) no main effect of factor $B$ - $H_{0}^{F}(B)$: $F_{11} - F_{12} + F_{21} - F_{22} = 0$

(3) no interaction $AB$ - $H_{0}^{F}(AB)$: $F_{11} - F_{12} - F_{21} + F_{22} = 0,$

where in all three cases, 0 denotes a function which is identical 0.

Let $F = (F_{11}, F_{12}, F_{21}, F_{22})'$ denote the vector of the distribution functions. Then the three hypotheses formulated above can be written in matrix notation as $H_{0}^{F}(\mathbf{c}) : \mathbf{c}'F = 0$, where $\mathbf{c} = c_{A} = (1, 1, -1, -1)'$ generates the hypothesis for the main effect $A$, $\mathbf{c} = c_{B} = (1, -1, 1, -1)'$ for the main effect $B$, and $\mathbf{c} = c_{AB} = (1, -1, -1, 1)'$ for the interaction $AB$.

For testing these hypotheses, Akritas et al. (1997) derived rank procedures based on the statistic

$$T_{N}(\mathbf{c}) = \sqrt{N}\mathbf{c}'\hat{\mathbf{p}} = \frac{1}{\sqrt{N}} \mathbf{c}'\bar{\mathbf{R}}, \quad (15)$$

where $\bar{\mathbf{R}} = (\bar{R}_{11}, \bar{R}_{12}, \bar{R}_{21}, \bar{R}_{22})'$ denotes the vector of the rank means $\bar{R}_{ij}$ within the four samples. They showed that under the hypothesis $H_{0}^{F}(\mathbf{c})$, the statistic $T_{N}(\mathbf{c})$ has, asymptotically, a normal distribution with mean 0 and variance

$$\sigma_{0}^{2} = \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{N}{n_{ij}} \sigma_{ij}^{2}, \quad (16)$$

where the unknown variances $\sigma_{ij}^{2}$ (see Akritas et al., 1997, for their explicit form) can be consistently estimated by

$$\frac{1}{N^{2}}S_{ij}^{2} = \frac{1}{N^{2}(n_{ij} - 1)} \sum_{k=1}^{n_{ij}} (R_{ijk} - \bar{R}_{ij})^{2} \quad (17)$$
same simulated data sets from Table 2 and compare the results for testing the three hypotheses

Table 2: Comparison of the empirical distributions in the four factor level combinations $A_1B_1$, $A_2B_1$, $A_1B_2$, and $A_2B_2$ sampled from homoscedastic normal distributions with $\mu_{11} = 10, \mu_{12} = \mu_{21} = 9, \mu_{22} = 8$, and standard deviation $\tau = 0.4$. Within each factor level combination $A_iB_j$, $i, j = 1, 2$, the unadjusted $p$-values of a t-test comparing the location of balanced and unbalanced samples, more specifically testing $\mu_{ij}(\text{balanced}) = \mu_{ij}(\text{unbalanced})$, are listed in the last column.

| Level Combinations | Means | Standard Deviations | t-Tests |
|--------------------|-------|---------------------|---------|
|                    | Balanced | Unbalanced | Balanced | Unbalanced | p-Values |
| $A_1B_1$           | 10.07   | 9.91    | 0.311    | 0.314    | 0.178    |
| $A_2B_1$           | 9.04    | 8.95    | 0.380    | 0.313    | 0.409    |
| $A_1B_2$           | 9.05    | 8.99    | 0.408    | 0.480    | 0.684    |
| $A_2B_2$           | 8.07    | 8.02    | 0.371    | 0.359    | 0.562    |

and $\hat{\sigma}^2 = N \sum_{i=1}^{2} \sum_{j=1}^{2} S_{ij}^2/n_{ij}$. For small sample sizes, the null distribution of $L_N(c) = T_N(c)/\hat{\sigma}_0$ can be approximated by a $t_f$-distribution with estimated degrees of freedom

$$\hat{f} = \frac{S_0^4}{\sum_{i=1}^{2} \sum_{j=1}^{2} (S_{ij}^2/n_{ij})^2/(n_{ij} - 1)},$$

(18)

where $S_0^2 = \sum_{i=1}^{2} \sum_{j=1}^{2} S_{ij}^2/n_{ij}$. The non-centrality of $T_N(c)$ is given by $c_T = c'p$, and under the restrictive null hypothesis $H_F^0(c) : c'F = 0$ it follows that $c_Tp = 0$.

To demonstrate a paradoxical result, we assume that the observations $X_{ijk}$ are coming from the normal distributions $N(\mu_{ij}, \tau^2)$ with equal standard deviations $\tau = 0.4$ and expectations $\mu = (\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22})' = (10, 9, 9, 8)'$. From the viewpoint of linear models, there is a main effect $A$ of $c_A^\mu = \mu_{11} + \mu_{12} - \mu_{21} - \mu_{22} = 2$, a main effect $B$ of $c_B^\mu = \mu_{11} - \mu_{12} + \mu_{21} - \mu_{22} = 2$, and no $AB$-interaction since $c_{AB}^\mu = \mu_{11} - \mu_{12} - \mu_{21} + \mu_{22} = 0$. Since this is a homoscedastic linear model, the classical ANOVA should reject the hypotheses $H_0^0(c_A) : c_A^\mu = 0$ and $H_0^0(c_B)$ : $c_B^\mu = 0$ with a high probability if the total sample size is large enough. In contrast to that, the hypothesis $H_0^H(c_{AB}) : c_{AB}^\mu = 0$ of no interaction is only rejected with the pre-selected type-I error probability $\alpha$. The non-centralities are given by $c_A^\mu = c_A^\hat{\mu} = 2$, $c_B^\mu = c_B^\hat{\mu} = 2$, and $c_{AB}^\mu = c_{AB}^\hat{\mu} = 0$. The following two settings of samples sizes $n_{ij}$ are considered:

1. $n_{11} = 10$, $n_{12} = 20$, $n_{21} = 20$, $n_{22} = 50$, - (unbalanced)
2. $n_{11} = n_{12} = n_{21} = n_{22} = 25$, - (balanced).

First we demonstrate that the empirical characteristics of the two data sets, which are sampled from the same distributions, are nearly identical. Thus, potentially different results could not be explained by substantially different empirical distributions obtained by an “unhappy randomization”. The results of the comparisons are listed in Table 2.

We apply the classical ANOVA $F$-statistic and the rank statistic $L_N(c) = T_N(c)/\hat{\sigma}_0$ to the same simulated data sets from Table 2 and compare the results for testing the three hypotheses
Comparison of the results obtained by an ANOVA and by the rank test \( L_N(e) \) in case of a balanced (left) and an unbalanced (right) 2 × 2-design. Surprising is the fact that in the balanced case, the decisions of both procedures coincide while in the unbalanced case the decisions for testing the interaction \( AB \) are totally different from that obtained by the parametric ANOVA, as well as that obtained for the parametric hypothesis \( \mu_{AB} = \mu' \). We note that the decisions for \( H_0^F(A) \), \( H_0^F(B) \), and \( H_0^F(AB) \) obtained by the ANOVA as well as by the rank tests based on \( L_N(e) \) are identical in all cases in the balanced setting. In the unbalanced setting, all decisions obtained by the parametric ANOVA are comparable to those in the balanced setting. However, the decision on the interaction \( AB \) based on the rank test is totally different from that obtained by the parametric ANOVA, as well as that obtained for the rank test in the balanced setting. The results are summarized in Table 3.

On the surface, the difference of the decisions in the unbalanced case could be explained by the fact that the nonparametric hypothesis \( H_0^F(AB) \) and the parametric hypothesis \( H_0^F(AB) \) are not identical and that this particular configuration of normal distributions falls into the gap between \( H_0^F(AB) \) and \( H_0^F(AB) \). That is, here \( H_0^F(AB) \) is true, but \( H_0^F(AB) \) is not. It is surprising, however, that this explanation does not hold for the balanced case. The difference of the two \( p \)-values 0.9541 and 0.0065 in Table 3 calls for an explanation.

The reason becomes clear when computing the vector \( p \) in (2) for this particular example of the 2 × 2-design. To avoid fourfold indices we re-label the distributions \( F_{11}, F_{12}, F_{21}, \) and \( F_{22} \) as \( F_1, F_2, F_3, \) and \( F_4 \), respectively, and the sample sizes accordingly as \( n_1, n_2, n_3, \) and \( n_4 \). In the example, \( F_1 = N(10, \tau^2) \), \( F_2 = F_3 = N(9, \tau^2) \), and \( F_4 = N(8, \tau^2) \), where \( \tau = 0.4 \). Thus, the probabilities \( w_{ri} = \int F_r dF_i \) of the pairwise comparisons are

\[
\begin{align*}
w &= w_{12} = w_{13} = w_{24} = w_{34} = \Phi\left(-1\frac{1}{\tau\sqrt{2}}\right) = 0.0392, \\
w_{23} &= w_{32} = \frac{1}{2}, \\
v &= w_{14} = \Phi\left(-\frac{\sqrt{2}}{\tau}\right) \approx 0.
\end{align*}
\]

| Effect | ANOVA | Rank Test | ANOVA | Rank Test |
|--------|--------|-----------|--------|-----------|
|        | \( F \) | \( p \)-Value | \( L_N(e) \) | \( p \)-Value |
|        |        |            |        |            |
| \( A \) | 184.43 | < 10\(^{-4}\) | 168.19 | < 10\(^{-4}\) |
| \( B \) | 182.60 | < 10\(^{-4}\) | 169.32 | < 10\(^{-4}\) |
| \( AB \) | 0.12 | 0.7317 | 0.00 | 0.9541 |
Finally, by observing $w_{ri} = 1 - w_{ip}$, we obtain

$$p = W'n = \frac{1}{N} \begin{pmatrix} \frac{1}{2}n_1 + n_2 + n_3 + n_4 - (n_2 + n_3)w - n_4v \\ \frac{1}{2}(n_2 + n_3) + n_4 + (n_1 - n_4)w \\ \frac{1}{2}(n_2 + n_3) + n_4 + (n_1 - n_4)w \\ \frac{1}{2}n_4 + (n_2 + n_3)w + n_1v \end{pmatrix},$$

and the nonparametric $AB$-interaction is described by

$$c_{AB}^p = c_{AB}'p = p_1 - p_2 - p_3 + p_4$$

$$= \frac{n_1 - n_4}{N} \left( \frac{1}{2} - 2w + v \right).$$

In this example, we obtain for equal samples sizes $c_{AB}^p = 0$, while for $n = (10, 20, 20, 50)$ we obtain $c_{AB}^p = -\frac{2}{5} \left( \frac{1}{2} - 2w + v \right) \approx -0.1686$ and $\sqrt{N}c_{AB}^p \approx -1.686$. This explains the small $p$-value for unequal sample sizes in the example.

5 Explanation of the Paradoxical Results

5.1 Unweighted Effects and Pseudo-Ranks

The simple reason for the paradoxical results is the fact that even when all distribution functions underlying the observations are specified, the consistency regions $C'p \neq 0$ of the rank tests based on $C'\hat{p}$ are not fixed. Indeed, the consistency regions are defined by the weighted nonparametric relative effects $p_i$ which are not fixed model quantities by which hypotheses could be formulated or for which confidence intervals could be reasonably computed since the $p_i$ themselves generally depend on the sample sizes $n_i$.

Thus, it appears reasonable to define different nonparametric effects which are fixed model quantities not depending on sample sizes. To this end let $G = \frac{1}{d} \sum_{r=1}^{d} F_r$ denote the unweighted mean distribution, and let $\psi_i = \int GdF_i$. Easily interpreted, this nonparametric effect $\psi_i$ measures an effect of the distribution $F_i$ with respect to the unweighted mean distribution $G$ and is therefore a “fixed relative effect”. As $\psi_i = \frac{1}{d} \sum_{r=1}^{d} w_{ri}$ is the mean of the pairwise nonparametric effects $w_{1i}, \ldots, w_{di}$, it can be written in vector notation as the vector of row means of $W'$, that is,

$$\psi = \int GdF = W' \cdot \frac{1}{d}1_d = \begin{pmatrix} w_{11}, & \cdots & w_{dl} \\ \vdots & \ddots & \vdots \\ w_{1d}, & \cdots & w_{dd} \end{pmatrix} \cdot \frac{1}{d}1_d = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_d \end{pmatrix}. \quad (20)$$

The fixed relative effects $\psi_i$ can be estimated consistently by the simple plug-in estimator

$$\hat{\psi}_i = \int \hat{G}d\hat{F}_i = \frac{1}{N}(\hat{R}^0_k - \frac{1}{2}). \quad (21)$$
where $\hat{G} = \frac{1}{d} \sum_{r=1}^{d} \hat{F}_r$ denotes the unweighted mean of the empirical distributions $\hat{F}_1, \ldots, \hat{F}_d$, and $\overline{R}_i = \frac{1}{n_i} \sum_{k=1}^{n_i} R_{ik}^\phi$ the mean of the so-called pseudo-ranks

$$\text{ps-rank}(X_{ik}) = R_{ik}^\phi = \frac{1}{2} + N\hat{G}(X_{ik}) = \frac{1}{2} + \frac{N}{d} \sum_{r=1}^{d} \frac{1}{n_r} \sum_{\ell=1}^{n_r} c(X_{ik} - X_{i\ell}). \quad (22)$$

Finally, the estimators $\hat{\psi}_i$ are arranged in the vector

$$\hat{\psi} = \begin{pmatrix} \hat{\psi}_1 \\ \vdots \\ \hat{\psi}_d \end{pmatrix} = \int \hat{G} d\hat{F} = \frac{1}{N} \left( \overline{R}_. - \frac{1}{2} \mathbf{1}_d \right), \quad (23)$$

where $\overline{R}_. = (\overline{R}_1^\phi, \ldots, \overline{R}_d^\phi)'$ is the vector of the pseudo-rank means $\overline{R}_i^\phi$.

It may be noted that the pseudo-ranks $R_{ik}^\phi$ have similar properties as the ranks $R_{ik}$. The properties given below follow from the definitions of the ranks and pseudo-ranks and by some straightforward algebra.

**Lemma 1** Let $X_{ik}$ denote $N = \sum_{i=1}^{d} n_i$ observations arranged in $i = 1, \ldots, d$ groups each involving $n_i$ observations. Then, for $i, r = 1, \ldots, d$ and $k = 1, \ldots, n_i, \ell = 1, \ldots, n_r$,

1. If $X_{ik} < X_{i\ell}$ then $R_{ik} < R_{i\ell}$ and $R_{ik}^\phi < R_{i\ell}^\phi$.
2. If $X_{ik} = X_{i\ell}$ then $R_{ik} = R_{i\ell}$ and $R_{ik}^\phi = R_{i\ell}^\phi$.
3. $\frac{1}{N} \sum_{i=1}^{d} \sum_{k=1}^{n_i} R_{ik} = \frac{1}{d} \sum_{i=1}^{d} \frac{1}{n_i} \sum_{k=1}^{n_i} R_{ik}^\phi = \frac{N+1}{2}$.
4. If $m(u)$ is a strictly monotone transformation of $u$ then,
   (a) $R_{ik} = \text{rank}(X_{ik}) = \text{rank}(m(X_{ik}))$
   (b) $R_{ik}^\phi = \text{ps-rank}(X_{ik}) = \text{ps-rank}(m(X_{ik}))$.
5. $1 \leq R_{ik} \leq N$.
6. $\frac{d+1}{2d} \leq \frac{1}{2} + \frac{N}{2dn_i} \leq R_{ik}^\phi \leq N + \frac{1}{2} - \frac{N}{2dn_i} \leq N + \frac{d-1}{2d}$
7. If $n_i = n$, $i = 1, \ldots, d$, then $R_{ik} = R_{ik}^\phi$.
8. Let $X_{ik}, i = 1, \ldots, d; k = 1, \ldots, n_i$ be independent and identically distributed random variables, then
   $$E(R_{ik}) = E(R_{ik}^\phi) = \frac{N+1}{2}.$$
5.2 Consistency Regions of Pseudo-Rank Procedures

As a solution to the paradoxical results discussed in Sections 3 and 4, we demonstrate that replacing the ranks $R_{ik}$ with the pseudo-ranks $R_{ik}^\psi$ leads to procedures that do not have these undesirable properties. The main reason is that pseudo-rank procedures are based on the (un-weighted) relative effects $\psi_i$ which are fixed model quantities by which hypotheses can be formulated and for which confidence intervals can be derived. In case of equal sample sizes $n_i \equiv n$, $i = 1, \ldots, d$, we do not obtain paradoxical results since in this case ranks and pseudo-ranks coincide, $R_{ik} = R_{ik}^\psi$ (see Lemma 1).

Pseudo-rank based inference procedures are obtained in much the same way as the common rank procedures, by using relations (21) and (22), which generally means substituting ranks by the corresponding pseudo-ranks. Only in the computation of the confidence intervals (see Section 5.3), there is a minor change in the variance estimator. For details we refer to Brunner, Bathke, and Konietschke (2018, Result 4.16 in Section 4.6.1).

Below, we examine the behavior of pseudo-rank procedures in the situations where the use of rank tests led to paradoxical results.

1. For testing the hypothesis $H_0^F : F_1 = F_2 = F_3 \iff T_3 F = 0$ in case of the tricky dice in the example in Section 3, one obtains for the non-centrality of the Kruskal-Wallis statistic, when the ranks $R_{ik}$ are replaced with the pseudo-ranks $R_{ik}^\psi$, the value $c^\psi = \psi' T_3 \psi = 0$. This is easily seen from (20), since in this case it follows from (10), (11), and (12) that $\psi = W' \frac{1}{2} 1_3 = \frac{1}{2} 1_3$ and $c^\psi = \psi' T_3 \psi = \frac{1}{2} 1_3^' T_3 \frac{1}{2} 1_3 = 0$ by noting that $T_3$ is a contrast matrix and thus, $T_3 1_3 = 0$.

2. The non-centrality for the Hettmansperger-Norton trend test when substituting ranks with pseudo-ranks becomes $c^\psi_{HN} = c' T_3 \psi = c' T_3 W' \frac{1}{4} 1_3 = 0$ for all trend alternatives $c = (c_1, c_2, c_3)'$.

3. In the two-way layout, we reconsider the example of the four shifted normal distributions we obtain the unweighted relative effects $\psi_{ij}$

$$\psi = W' \cdot \frac{1}{4} 1_4 = \begin{pmatrix} 7/2 - 2w - v \\ 2 \\ 2 \\ 1/2 + 2w + v \end{pmatrix}$$

and the non-centrality of the statistic for the interaction

$$c^\psi_{AB} = \psi_1 - \psi_2 - \psi_3 + \psi_4 = 0. \quad (24)$$

In the analysis of the unbalanced data in Table 3 using pseudo-ranks, one obtains for the interaction statistic $L_N^2(c_{AB}) = 0.08$ and the resulting $p$-value is 0.7832 which agrees with the result for the balanced sample from the same distributions in Table 3.
In all cases, paradoxical results obtained by changing the ratios of the sample sizes cannot occur since the non-centralities $c_{\psi}$, $c_{\psi}^{\phi}$, and $c_{\psi}^{\phi AB}$ are equal to 0 for all constellations of the relative sample sizes.

In case of $d = 2$ samples, it is easily seen that $p_2 - p_1 = \psi_2 - \psi_1 = p = \int F_1 dF_2$ which does not depend on sample sizes. Thus, paradoxical results for rank-based tests – such as presented in the previous sections – can only occur for $d \geq 3$ samples.

5.3 Confidence Intervals

Here we briefly explain the details on how to compute confidence intervals for fixed nonparametric effects which have an intuitive and easy to understand interpretation. This shall be demonstrated by means of the example involving the four shifted normal distributions considered in Section 4.

The quantity $\hat{\psi}_i$ estimates the probability that a randomly drawn observation from the mean distribution $G = \frac{1}{4} \sum_{i=1}^{4} F_i$ is smaller than a randomly drawn observation from distribution $F_i$ (plus $\frac{1}{2}$ times the probability that they are equal). The estimator is obtained from (21), and the limits of the confidence interval are obtained from formula (25) in Brunner et al. (2017) or Result 4.16 in Chapter 4 of Brunner, Bathke, and Konietschke (2018). The results are listed in Table 4.

Table 4: Estimates and two-sided 95%-confidence intervals $[\psi_{i,L}, \psi_{i,U}]$ for the (unweighted) nonparametric relative effects $\psi_i$ of the two data sets considered in Table 3. The results for the balanced case (upper part) and the unbalanced case (lower part) are quite similar.

| Sample | Equal Sample Sizes $n_1 = n_2 = n_3 = n_4 = 25$ | Unequal Sample Sizes $n_1 = 10, n_2 = n_3 = 20, n_4 = 50$ |
|---------|---------------------------------|---------------------------------|
|         | Lower Limit $\psi_{i,L}$ | Estimator $\psi_i$ | Upper Limit $\psi_{i,U}$ |
| 1       | 0.84          | 0.86          | 0.88          |
| 2       | 0.45          | 0.50          | 0.55          |
| 3       | 0.45          | 0.50          | 0.55          |
| 4       | 0.12          | 0.14          | 0.16          |
|         | Lower Limit $\psi_{i,L}$ | Estimator $\psi_i$ | Upper Limit $\psi_{i,U}$ |
| 1       | 0.85          | 0.86          | 0.88          |
| 2       | 0.45          | 0.51          | 0.57          |
| 3       | 0.43          | 0.48          | 0.53          |
| 4       | 0.13          | 0.15          | 0.16          |

Obviously, the results for equal and unequal sample sizes are nearly identical. Let samples 1 and 2 refer to the factor level combinations $A_1B_1$ and $A_2B_1$ within level $B_1$ and samples 3 and
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4 to the factor level combination $A_1B_2$ and $A_2B_2$ within level $B_2$, respectively. The differences $\hat{\psi}_1 - \hat{\psi}_2$ and $\hat{\psi}_3 - \hat{\psi}_4$ between factor levels $A_1$ and $A_2$ of the estimates in factor level $B_1$ and in factor level $B_2$ are identical in the balanced case ($\hat{\psi}_1 - \hat{\psi}_2 = 0.36$, $\hat{\psi}_3 - \hat{\psi}_4 = 0.36$) and nearly identical ($\hat{\psi}_1 - \hat{\psi}_2 = 0.35$, $\hat{\psi}_3 - \hat{\psi}_4 = 0.33$) in the unbalanced case. These results do not indicate any interaction. This finding agrees with the analysis of the balanced data in Table 3 when using ranks.

For the weighted relative effects $p_i$ in (1), the estimators $\hat{p}_i$ based on ranks, as well as 95%-intervals for $p_i$ are listed in Table 5. The 95%-intervals for $p_i$, however, cannot strictly be interpreted as confidence intervals since the quantities $p_i$ depend on the relative sample sizes, unless the design is balanced. In case of equal sample sizes we have $p_i = \psi_i$. Therefore only the results for the unbalanced case are listed in Table 5. The 95%-intervals for $p_i$ are obtained from formulas (1.14) - (1.17) in Brunner and Puri (2001) and formulas (2.2.33) - (2.2.35) in Section 2.2.7 in Brunner and Munzel (2013).

Comparing the differences of the estimates within the two factor levels $B_1$ and $B_2$ using the usual ranks, we obtain $\hat{p}_1 - \hat{p}_2 = 0.25$ and $\hat{p}_3 - \hat{p}_4 = 0.41$ which are quite different. Such a difference indicates an interaction effect, which agrees with the results of the analysis of the unbalanced data in Table 3. Moreover, the corresponding 95%-intervals for the weighted effects $p_i$ are totally different from the 95%-confidence intervals for the unweighted effects $\psi_i$ in Table 4. They even do not overlap with the confidence intervals for the $\psi_i$. Thus, only the nonparametric effects $\psi_i$ offer the possibility of computing estimates and confidence intervals for fixed model quantities with an intuitive interpretation.

Table 5: Estimates and 95%-intervals $[p_{i,L}, p_{i,U}]$ for the weighted effects $p_i$ of the data set in the unbalanced case in Table 2. The results for the balanced case are identical to those in Table 4 since $p_i = \psi_i$ for equal sample sizes.

| Sample | Lower Limit $p_{i,L}$ | Estimate $\hat{p}_i$ | Upper Limit $p_{i,U}$ |
|--------|-----------------------|----------------------|-----------------------|
| 1      | 0.93                  | 0.94                 | 0.95                  |
| 2      | 0.62                  | 0.69                 | 0.74                  |
| 3      | 0.63                  | 0.68                 | 0.72                  |
| 4      | 0.25                  | 0.27                 | 0.28                  |

6 Cautionary Note for Sub-Group Analysis

The fact that the (weighted) relative effects in (2), which are estimated by the ranks, depend on the relative sample sizes $n_i/N$ is important in sub-group analyses, for example in clinical trials. Here, typically, the total group of patients is quite large while the sub-group may be small. The design of such a trial can be considered as a $2 \times 2$-design where $F_{11}$ and $F_{12}$ are the distributions
of the outcome for the patients who received treatment 1 (e.g., standard treatment with distribution $F_{11}$) or treatment 2 (experimental treatment with distribution $F_{12}$), without the particular sub-group of interest. The corresponding larger sample sizes, $n_{11}$ and $n_{12}$ are approximately the same when using an appropriate randomization procedure. The smaller sample sizes $n_{21}$ and $n_{22}$ in the sub-group may be equal or quite different depending on whether or not the randomization was also stratified for the sub-group.

The main question regarding the sub-group in this design is whether the treatment effect is the same as in the population of patients without the sub-group. Here we consider only the case where the sub-group is known in advance and is not identified on the basis of the data. In order to demonstrate the advantage of using the unweighted relative effects in (20) estimated by the pseudo-ranks, we consider the homoscedastic two-way design from Section 4. To this end, let $c_{AB} = (1, -1, -1, 1)'$ denote the contrast vector for the interaction in the $2 \times 2$-design as given in Section 4 and let $p$ denote the vector of the weighted relative effects estimated by $\hat{p}$ using the ranks. Moreover, let $\psi$ denote the vector of the unweighted relative effects estimated by $\hat{\psi}$ using pseudo-ranks, and finally let $\mu$ denote the vector of the expectations. Then the corresponding non-centralities are given by $c_{AB}^p = c_{AB}'p$, $c_{AB}^\psi = c_{AB}'\psi$, and $c_{AB}^\mu = c_{AB}'\mu$. The total sample size $N$ is increased such that the sample sizes $n_{21} = n_{22} = 50$ in the sub-group remain constant while the samples sizes $n_{11} = n_{12}$ obtained from the population without the sub-group increases from 50 to 2000. For the interaction term of the procedure based on ranks, we also consider the quantity $\sqrt{N}c_{AB}^p = \sqrt{N}c_{AB}'p$ describing the shift of the asymptotic multivariate normal distribution. It is noteworthy that with increasing total sample size $N$ not only $\sqrt{N}c_{AB}^p$ is increasing but also the basic non-centrality $c_{AB}^p$ increases while the corresponding non-centralities $c_{AB}^\psi$ and $c_{AB}^\mu$ of the two other procedures remain constant equal to 0. Note that for equal sample sizes, all non-centralities are 0. The results are listed in Table 6.

Analyses of this type are very common in biostatistical practice. In particular, the sub-group’s proportion of the total sample size is often very small, as, for example, in case of rare disease studies with large control groups. The above example further illustrates the need for changing the usage of classical rank procedures to more favorable pseudo-rank procedures.

### 7 Discussion and Conclusions

We have demonstrated that inference based on ranks may lead to paradox results in the case of at least three samples and an unbalanced design. These results may occur in rather natural situations and are not restricted to artificial configurations of the population distributions.

The reason for this behavior of rank-based tests (including some classical and frequently used tests) can be explained by the non-centralities of the test statistics which are functions of weighted nonparametric relative effects. Moreover, it is worth to note that neither pairwise nor stratified rankings provide a solution to the problem. In fact, it would only cause new problems. For a discussion of these types of rankings see Brunner et al., (2018). As a remedy, the use of unweighted nonparametric relative effects, and thus, the use of tests based on pseudo-ranks instead of ranks, is recommended.
Table 6: Non-centralities $c_{AB}^p$, $c_{AB}^\phi$, and $c_{AB}^\mu$ of the interaction statistics in the $2 \times 2$-design with increasing sample sizes in only one stratum. This reflects the situation of a sub-group analysis where the sample size in the sub-group is small compared to the total population. For the procedure based on ranks, the quantity $\sqrt{N}c_{AB}^p$ is also listed to demonstrate the shift of its asymptotic multivariate normal distribution generated only by the unequal sample sizes in the two strata. The non-centralities $c_{AB}^\phi$ and $c_{AB}^\mu$ of the two statistics based on the pseudo-ranks or on the expectations are both equal to 0, independently of the sample sizes.

| Sample Sizes | Non-Centralities |
|--------------|------------------|
| $n_{11} = n_{12} \quad n_{21} = n_{22}$ | $c_{AB}^\mu$ | $c_{AB}^\phi$ | $c_{AB}^\mu$ | $\sqrt{N}c_{AB}^p$ |
| 50 50 | 0 | 0 | 0 | 0 |
| 100 50 | 0 | 0 | 0.071 | 1.22 |
| 200 50 | 0 | 0 | 0.127 | 2.84 |
| 300 50 | 0 | 0 | 0.151 | 4.00 |
| 400 50 | 0 | 0 | 0.165 | 4.94 |
| 500 50 | 0 | 0 | 0.173 | 5.74 |
| 1000 50 | 0 | 0 | 0.191 | 8.77 |
| 2000 50 | 0 | 0 | 0.201 | 12.89 |

A completely different approach to circumventing the above mentioned problem could be given by so-called aligned rank procedures (see Hodges and Lehmann, 1962 or Puri and Sen, 1985). However, these procedures are restricted to semi-parametric models and are not applicable to ordinal data since the hypotheses are formulated in terms of artificially introduced parameters of metric data. In addition, the typical invariance under strictly monotone transformations of the data is lost by the two different transformations.

On the contrary, the proposed pseudo-rank procedures share the typical invariance of rank-based procedures and do not lead to the paradoxical results described in Sections 3 and 4. They can additionally be used to construct confidence intervals for meaningful nonparametric effect sizes as described in Section 5.3. In case of factorial designs with independent observations pseudo-rank procedures are already implemented in the R-package `rankFD` by choosing the option for `unweighted` effects.
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