PARALLEL EDGES IN RIBBON GRAPHS AND INTERPOLATING BEHAVIOR OF PARTIAL-DUALITY POLYNOMIALS

QIYAO CHEN AND YICHAO CHEN

ABSTRACT. Recently, Gross, Mansour and Tucker introduced the partial-twuality polynomials. In this paper, we find that when there are enough parallel edges, any multiple graph is a negative answer to the problem 8.7 in their paper [European J. Combin. 95 (2021), 103329]: Is the restricted-orientable partial-Petrial polynomial of an arbitrary ribbon graph even-interpolating? In addition, we also find a counterexample to the conjecture 8.1 of Gross, Mansour and Tucker: If the partial-dual genus polynomial is neither an odd nor an even polynomial, then it is interpolating.

1. Introduction

We assume that the readers are familiar with the basic knowledge of topological graph theory. The reader is referred to [GMT21a] for the explanation of all terms not defined here.

Let $G^*|_A$ ($G^\times|_A$) be the partial dual (partial Petrial) of $G$ with respect to $A \subseteq E(G)$. Denote by $v(G)$, $e(G)$, $f(G)$ and $c(G)$ the number of vertices, edges, faces and connected components of $G$, respectively. For $\bullet \in \{\times,*,\times*,*,\times\}$, Gross, Mansour, and Tucker [GMT21a] introduced the partial-$\bullet$ polynomial for the ribbon graph $G$, i.e.,

$$\partial E_G^\bullet(z) = \sum_{A \subseteq E(G)} z^{e_u[G^\bullet|_A]},$$

where $e_u[G^\bullet|_A]$ represents the Euler-genus of $G^\bullet|_A$.

They also introduced the restricted orientable partial-$\times$ polynomial of $G$ by enumerating Euler-genus only over edge-subsets $A \subseteq E(G)$ such that $G^\times|_A$ is orientable. We recall that partial duality was introduced by Chmutov in [Chm09]. In [EM12], Ellis-Monaghan and Moffatt extended the partial-duality to include
the Wilson dual, the Petrie dual, and the two kinds of triality operators. In [AE19], Abrams and Ellis-Monaghan called the five operators twualities. We may refer the reader to [EM13, GMT20, GMT21a, GMT21b] for more background about partial-duality and partial Petrial.

A subdivision of $G$ is obtained by replacing an edge $e = uv$ of $G$ by a path $uvw$, and a proper edge is an edge with two different ends. The contraction on edge $e$ is denoted by $G/e$, and we denote by $G - e$ the ribbon graph obtained from $G$ by deleting the ribbon $e$. In [GMT20], Gross, Mantour, and Tucker proved a subdivided edge recursion for partial-∗.

**Theorem 1.1.** [GMT20] Given a ribbon graph $G$ and a ribbon $e$. Let $K$ be a subdivision of $G$, then

\[
\partial E^*_K(z) = \begin{cases} 
2 \partial E^*_G(z), & \text{if $e$ is a cut ribbon}, \\
\partial E^*_G(z) + 2z^2 \partial E^*_{G-e}(z), & \text{if $e$ is non-separating}.
\end{cases}
\]

Gross, Mantour, and Tucker also give subdivided edges and parallel edges recursions for partial-× in [GMT21a].

**Theorem 1.2.** [GMT21a] Let $G$ be a ribbon graph with an $e$-type pp ribbon $e$, and let $G + e'$ be the ribbon graph obtained by adding a ribbon $e'$ parallel to a ribbon $e$ in $G$. Thus

\[
\partial E^*_{G+e'}(z) = (1 + 2z) \partial E^*_{G'/e}(z) + (z^2) \partial E^*_{G-e}(z)
\]

**Theorem 1.3.** [GMT21a] Let $G$ be a ribbon graph with a ribbon $e$, and let $H$ be obtained from $G$ by subdividing $e$ into edges $e_1$ and $e_2$. Then

\[
\partial E^*_{H}(z) = 2 \partial E^*_{G}(z).
\]

The support of a polynomial $f(z) = \sum_{i=0}^{n} a_i z^i$ is the set $\{i|a_i \neq 0\}$. If $\text{supp}(f(z))$ is an integer interval $[m, n]$ of all integers from $m$ to $n$, inclusive, we call $f(z)$ an interpolating polynomial. The size of integer interval $[m, n]$ is the number of elements of $[m, n]$. If $\text{supp}(f(z))$ is the set of all even natural numbers in an integer interval, then $f(z)$ is even-interpolating. If the terms of the non-zero coefficient of the polynomial are even (odd) degree, we call it an even (odd) polynomial.

Conjecture 8.1 and Problem 8.7 in their paper [GMT21a] state that

**Conjecture 1.4.** If the partial-∗ polynomial $\partial E^*_G(z)$ is neither an odd nor an even polynomial, then it is interpolating.
**Problem 1.5.** Is the restricted-orientable partial-$\times$ polynomial of an arbitrary ribbon graph $G$ even-interpolating?

In this paper, we apply the operations of adding parallel edges and subdividing edges to get some counter-examples of Conjecture 8.1 and Problem 1.2 [GMT21a]. We first disprove Conjecture 1.1 by finding an infinite family of ribbon graphs as counterexamples. Then we answer the Problem 1.2 by proving the restricted orientable partial polynomial of any multiple ribbon graph with enough parallel edges (we also have a tight lower bound for this) are not even-interpolating.

**Remark 1.6.** Throughout the paper, we will use the rotation projection [GT87] of $G$ instead of the ribbon graph $G$ itself (See Figure 2.1 - Figure 4.2).

2. THE PARALLEL EDGE RECURSION FOR PARTIAL-$\times$ POLYNOMIALS

In this section, we derive a recursion for the partial-$\times$ polynomials. Given two disjoint ribbon graphs $G_1$ and $G_2$, we let $G_1 \lor G_2$ denote the join of $G_1$ and $G_2$. The complement of $A$ in $E(G)$ is $A^c = E(G) - A$.

**Theorem 2.1.** [Mof12] Let $G$ be a ribbon graph and $A \subseteq E(G)$, then

$$\text{eu}(G^*|A) = 2c(G) + e(G) - f(A) - f(A^c).$$

**Theorem 2.2.** Let $G$ be a ribbon graph with a proper ribbon $e$, let $G + e_1$ be the ribbon graph obtained by adding parallel ribbon $e_1$ to the ribbon $e$ as shown in Figure 2.1. Then

$$\partial E_{G + e_1}(z) = \partial E_G(z) + 2z^2 \partial E_{G/e}(z).$$

![Figure 2.1](image-url)
Proof. Let \( A \subseteq E(G + e_1) \), \( A_1 \subseteq E(G) \), \( A_2 \subseteq E(G/e) \). It is easy to see that \( e(G + e_1) = e(G/e) + 2 = e(G) + 1 \). There are three cases.

**Case 1**: Suppose that \( e_1 \in A \) and \( e \in A \), and let \( A_1 = A - e_1 \). Since the ribbon graph \( A_1 \) is obtained from \( A \) by deleting the multiple edge \( e_1 \), which decreases the number of faces by 1, we have

\[
(2.3) \quad f(A) = f(A_1) + 1.
\]

We use the fact that the complement of \( A \) in \( E(G + e_1) \) is equal to the complement of \( A_1 \) in \( E(G) \), then

\[
(2.4) \quad f(A^c) = f(A_1^c).
\]

Thus, by Theorem 2.1

\[
eu((G + e_1)^{+|A}) = 2e(G + e_1) + e(G + e_1) - f(A) - f(A^c)
\]

\[
= 2e(G) + e(G) + 1 - f(A_1) - 1 - f(A_1^c) \quad \text{by (2.3) and (2.4)}
\]

\[
(2.5) \quad = eu(G^{+|A_1}).
\]

**Case 2**: Assume that \( e_1, e \in A^c \), and let \( A = A_1 \). Obviously,

\[
(2.6) \quad f(A) = f(A_1).
\]

Because \( A_1^c \) is obtained from \( A^c \) by deleting the multiple edge \( e_1 \), which decreases the number of faces by 1. Therefore,

\[
(2.7) \quad f(A^c) = f(A_1^c) + 1.
\]

Hence

\[
eu((G + e_1)^{+|A}) = 2e(G + e_1) + e(G + e_1) - f(A) - f(A^c) \quad \text{by (2.1)}
\]

\[
= 2e(G) + e(G) + 1 - f(A_1) - f(A_1^c) - 1 \quad \text{by (2.6) and (2.7)}
\]

\[
(2.8) \quad = eu(G^{+|A_1}).
\]

Let \( A = \{A|e_1, e \in A\} \cup \{A|e_1, e \in A^c\} \), \( A_1 = \{A_1|e \in A_1\} \), \( A_2 = \{A_1|e \in A_1^c\} \), then

\[
\sum_{A \in A} z^{eu((G + e_1)^{+|A})} = \sum_{A_1 \in A_1} z^{eu(G^{+|A_1})} + \sum_{A_1 \in A_2} z^{eu(G^{+|A_1})} \quad \text{by (2.5) and (2.8)}
\]

\[
(2.9) \quad = \partial e_G^+(z).
\]

**Case 3**: Let \( e_1 \in A^c, e \in A \), and we let \( A_2 = A/e \). Recall that \( A_2 \) is obtained from \( A \) by contracting the proper edge \( e \) and \( A_2^c \) is obtained from \( A^c \) by contracting the proper edge \( e_1 \). Since the contraction does not change the number
of faces, we have
\[ f(A) = f(A_2), \quad f(A^c) = f(A_2^c). \] (2.10)
Therefore,
\[ 
eu((G + e_1)^{*|A}) = 2c(G + e_1) + e(G + e_1) - f(A) - f(A^c) \quad \text{by (2.1)} \\
= 2c(G/e) + e(G/e) + 2 - f(A_2) - f(A_2^c) \quad \text{by (2.10)} \\
\] (2.11)
\[ = \text{eu}((G/e)^{*|A_2}) + 2. \]

Similarly, formula (2.11) also holds for the case: \( e \in A^c \), and \( e_1 \in A \).
Let \( \mathcal{A} = \{ A | e_1 \in A^c, e \in A \} \cup \{ A | e \in A^c, e_1 \in A \} \), and then
\[ \sum_{A \in \mathcal{A}} z^{\text{eu}}((G + e_1)^{*|A}) = 2z^2 \partial E^*_{G/e}(z) \quad \text{by (2.11)} \] (2.12)
Combining cases 1-3, we obtain
\[ \partial E^*_{G + e_1}(z) = \sum_{A \in \mathcal{A}} z^{\text{eu}}[(G + e_1)^{*|A}] + \sum_{A \in \mathcal{A}} z^{\text{eu}}[(G + e_1)^{*|A}] \\
= 2z^2 \partial E^*_{G/e}(z) + \partial E^*_G(z). \]

\[ \square \]

Figure 2.2

**Corollary 2.3.** Let \( G \) be a ribbon graph with a proper ribbon \( e \), and let \( G + \sum_{i=1}^n e_i \) be the ribbon graph obtained by adding \( n \) parallel ribbons \( e_1, e_2, \ldots, e_n \) to the ribbon \( e \) as shown in Figure 2.2. Then
\[ \partial E^*_{G + \sum_{i=1}^n e_i}(z) = \partial E^*_G(z) + (2^{n+1} - 2)z^2 \partial E^*_{G/e}(z). \] (2.13)

**Proof.** By Theorem 2.2
\[ \partial E^*_{G + \sum_{i=1}^n e_i}(z) = \partial E^*_{G + \sum_{i=1}^{n-1} e_i}(z) + 2z^2 \partial E^*_{(G + \sum_{i=1}^{n-1} e_i)/e_{n-1}}(z). \] (2.14)
Recall that the ribbons $e, e_1, e_2, \cdots, e_{n-2}$ in $(G + \sum_{i=1}^{n-1} e_i)/e_{n-1}$ are all loops, and the ribbon graph $G/e$ is obtained from $(G + \sum_{i=1}^{n-1} e_i)/e_{n-1}$ by deleting $n - 1$ loops. Thus, from Proposition 3.2 in [GMT20],

$$\partial E^*_{(G + \sum_{i=1}^{n-1} e_i)/e_{n-1}}(z) = 2^{n-1} \partial E^*_G(z).$$

(2.14), (2.15) and (2.2) clearly imply (2.13).

Recall that the Tutte polynomial ([Tut54]) and Bollobas-Riordan polynomial ([Bol02]) have deletion-contraction recursions, as well as the role of series-parallel graphs in the foundations of matroids([Bry71]).

**Remark 2.4.** As pointed out by the anonymous referee, subdividing an edge is adding a parallel edge in the dual ribbon graph. Thus Theorem 2.2 can be seen as the dual form of Theorem 1.1.

Let $C_n$ and $K_n$ denote a $n$-cycle and a complete graph with $n$ vertices, respectively. A graph is *series parallel* ([AVJ99]), if its 2-connected components can be generated by repeatedly adding a parallel edge and subdividing an edge, starting with $K_2$. As an example, we apply Theorem 1.1 and Theorem 2.2 to calculate the partial $*_*$ polynomial for a series parallel graph.

**Example 2.1.** Figure 2.3 shows the series parallel graphs $H$, $G$ and $G_1$. It’s clear that $G$ is isomorphic to $H$ with ribbon $e_1$ subdivided once, and $G_1 = H - e_1$. Moreover, $H$ is obtained from $G_1$ by adding a parallel edge $e_1$ to the ribbon $e_2$, hence, we have

$$\partial E^*_G(z) = \partial E^*_H(z) + 2z^2 \partial E^*_G(z) \quad \text{by (1.1)}$$

$$= (2z^2 + 1) \partial E^*_G(z) + 2z^2 \partial E^*_G/e_2(z) \quad \text{by (2.2)}$$

Figure 2.3
Since $G_1$ is isomorphic to $G_1/e_2$ with ribbon $e_3$ subdivided once, let $G_2 = (G_1/e_2) - e_3$ as in Figure 2.4, it follows that

$$
\partial E^*_G(z) = \partial E^*_{G_1}(z) + 2z^2 \partial E^*_{G_2}(z)$$

by (1.1). Combining (2.16) and (2.17), we have

$$
\partial E^*_G(z) = (4z^2 + 1) \partial E^*_{G_1}(z) - 4z^4 \partial E^*_{G_2}(z).
$$

Similarly, we have

$$
\partial E^*_{G_2}(z) = (4z^2 + 1) \partial E^*_{K_3}(z) - 4z^4 \partial E^*_{K_2}(z)
$$

where $K_3$ is obtained by adding a parallel edge and subdividing an edge to $K_2$ once. It’s obvious that $\partial E^*_{K_3}(z) = 6z^2 + 2$, and $\partial E^*_{K_2}(z) = 2$. Therefore, combining (2.18), (2.19), and (2.20), we have

$$
\partial E^*_G(z) = (32z^6 + 40z^4 + 12z^2 + 1) \partial E^*_{K_3}(z) - (48z^8 + 332z^6 + 4z^4) \partial E^*_{K_2}(z) - 96z^8 + 240z^6 + 144z^4 + 30z^2 + 2.
$$

3. Some counterexamples to the conjecture 1.1.

In this section, we find some infinite families of ribbon graphs as counterexamples to Conjecture 1.1.

Let $G$ and $G + \sum_{i=1}^m \bar{e}_i$ be the ribbon graphs of Figure 3.1. Clearly, $e(A) + e(A^c) = e(G)$ and $e(G^i|A) = e(G^i|A^c)$. Note that $e(G) = 7$, and for each edge set with the number of edges greater than 3 in $G$, we can find the complement with the number of edges less than 4.
Figure 3.1. $G + \sum_{i=1}^{n} \bar{e}_i$ (left), $G$ (middle) and $G/e_1$ (right)

(1) $G$ has partial-$*$ polynomial $8z^2 + 48z^4 + 32z^5 + 40z^6$. By the previous analysis, we need to show the partial-$*$ polynomial of dualizing edge set with the number of edges less than 4 is $4z^2 + 24z^4 + 16z^5 + 20z^6$. The $z^2$ term corresponds to dualizing none, or twisted edge $e$, or all edges in the same $C_3$ (2 choices) and all are isomorphic to $G$. The $z^4$ terms comes from dualizing one edge in the same $C_3$ (6 choices), or a pair in the same $C_3$ (6 choices), or one edge in the same $C_3$ and $e$ (6 choices), or a pair in the same $C_3$ and $e$ (6 choices). Dualizing one edge of $C_3$ and two edges of the other $C_3$ produces $z^5$ and $z^6$ (18 choices). Where dualizing $e_1$, $e_3$, and one edge of $C_3$ ($e_4$, $e_5$, $e_6$) (3 choices), or $e_5$, $e_6$, and one edge of $C_3$ ($e_1$, $e_2$, $e_3$) (3 choices) produces $z^5$, the remaining 12 choices produces $z^6$. Dualizing one edge in each $C_3$ and $e$ produces $z^5$ and $z^6$ (9 choices). Where dualizing $e_2$, $e_4$, and $e$ produces $z^5$, the remaining 8 choices produces $z^6$. Dualizing one edge in each $C_3$ (9 choices) also produces $z^5$.

(2) $G/e_1$ has partial-$*$ polynomial $8z^6 + 16z^5 + 32z^4 + 8z^2$. For each edge set with the number of edges greater than 3 in $G/e_1$, we can find the complement with the number of edges less than 3. Moreover, for the edge set whose number of edges is 3, the edge set with twisted edge $e$ is the complement of the edge set without $e$. Therefore, we need to show the partial-$*$ polynomial of dualizing edge set with the number of edges not greater than 2, and edge set without $e$ where the number of edges is 3 are $4z^6 + 8z^5 + 16z^4 + 4z^2$. The $z^2$ terms comes from dualizing none, or $e$, or $C_3$, or $C_2$ and all are isomorphic to $G/e_1$. The $z^4$ terms comes from dualizing one edge in $C_3$ and $C_2$ (5 choices), or a pair in $C_3$ (3 choices), or $e$ and one edge in $C_3$ and $C_2$ (5 choices), or all edges in $C_2$ and one edge in $C_3$ (3 choices). Dualizing one edge of $C_3$ and one edge of $C_2$ produces $z^5$ and $z^6$ (6 choices). Where dualizing $e_3$, $e_5$, $e_6$, and $e_2$, $e_5$, $e_6$...
produces $z^5$, the remaining 4 choices produces $z^6$. Dualizing one edge in $C_3$ and one edge in $C_2$ (6 choices) also produces $z^5$.

Thus by Corollary 2.3, we have

$$
\partial E^\ast_{G+\sum_{i=1}^{\mathbb{m}} \ell_i}(z) = \partial E^\ast_{G}(z) + (2^{n+1} - 2)z^2 \partial E^\ast_{G/\ell_1}(z)
$$

$$
= (2^{n+4} - 16)z^8 + (2^{n+5} - 32)z^7 + (2^{n+6} - 24)z^6
$$

$$
+ 32z^5 + (2^{n+4} + 32)z^4 + 8z^2.
$$

It is quite obvious that the polynomial above is neither an odd nor an even polynomial, it’s also not interpolating.

Remark 3.1. The anonymous referee also pointed out that we can take lots of joins of $G$ with itself to get counterexamples, since

$$(z^6 + z^5 + z^4 + z^2)^n = z^{2n} + nz^{2n+2} + nz^{2n+3} + \ldots.$$ 

4. A solution to the Problem 1.2.

In this section, we discuss the effect of adding multiple edges on the restricted orientable partial-$\times$ polynomial. The restricted orientable partial-$\times$ ribbon graph of $G$ is the orientable ribbon graph $G^{\times|A}$. For short, we denote the restricted orientable partial-$\times$ polynomial of ribbon graph $G$ by $\partial E^\times_{G}|O(z)$.

Example 4.1. Let $D_n$ be the dipole ribbon graph in the sphere. Clearly, if one twists any proper subset of the edges, the resulting ribbon graph is non-orientable, there will be circuits of length two containing only one twisted edge. Thus the only orientable partial-$\times$ duals are $D_n$ and $D_n^\times$. The latter has 1 or 2 faces depending on whether $n$ is odd or even and hence has orientable genus $\frac{n-1}{2}$ or $\frac{n-2}{2}$. Furthermore, we have $\partial E_{D_n}^{\times}|O(z) = 1 + z^{n-1}$, when $n$ is odd; $\partial E_{D_n}^{\times}|O(z) = 1 + z^{n-2}$, when $n$ is even. Thus, when $n \geq 5$, the restricted-orientable partial-$\times$ polynomial of $D_n$ is not even-interpolating.

Lemma 4.1. Let $G$ be a ribbon graph, then $\partial E^\times_{G}|O(z) = \partial E^\times_{G^{\times|A}}|O(z)$.

Proof. Let $A_1 \subseteq E(G)$, and $A \subseteq E(G)$. For each orientable ribbon graph $G^{\times|A_1}$, there exists $B \subseteq E(G^{\times|A})$ such that the orientable ribbon graph $(G^{\times|A})^{\times|B}$ and $G^{\times|A_1}$ are the same.

Lemma 4.1 shows that it suffices to consider the case that $G$ is an orientable ribbon graph.
Lemma 4.2. Let \( G \) be a ribbon graph, then the number of the restricted orientable partial-\( \times \) ribbon graphs of \( G \) is \( 2^{\delta(G)-1} \).

Proof. By [GMT21a], we know that the proportion of partial-\( \times \) duals of \( G \) that are orientable is \( \frac{1}{2^{\delta(G)}} \). There are \( 2^{\delta(G)} \) spanning subgraphs in \( G \), thus, the number of the restricted orientable partial-\( \times \) ribbon graphs of \( G \) is

\[
\frac{2^{\delta(G)}}{2^{\delta(G)-1}} = 2^{\delta(G)-1}.
\]

\( \square \)

Let \( e \) be a ribbon of \( G \), and let \( A \) be a subset of \( E(G) \), we define \( eu^0(G \times |A|) \) and \( eu^1(G \times |A|) \) as \( e \in A \) and \( e \notin A \) of the Euler-genus of orientable ribbon graph \( G \times |A| \), respectively. Similarly, we let \( f^0(G \times |A|) \) and \( f^1(G \times |A|) \), respectively, denote the number of faces of orientable ribbon graph \( G \times |A| \) with \( e \in A \) and \( e \notin A \). And let \( f_{\max}(G) = \max \{f^0(G \times |A|) | e \in A\} \), \( f_{\min}(G) = \min \{f^1(G \times |A|) | e \notin A\} \).

Theorem 4.3. Let \( G \) be a ribbon graph with a proper ribbon \( e \), and for even \( n \), let \( G + \sum_{i=1}^{n} e_i \) be the ribbon graph obtained by adding \( n \) parallel ribbons \( e_1, e_2, \cdots, e_n \) to the ribbon \( e \) (see Figure [2,2]). Then the polynomial \( \partial_\mathcal{G} \times_{G+\sum_{i=1}^{n} e_i} \mathcal{O}(z) \) is not even-interpolating for sufficiently large \( n \).

Proof. Suppose that \( \partial_\mathcal{G} \times_{G+\sum_{i=1}^{n} e_i} \mathcal{O}(z) \) is even-interpolating. We need to analysis the number of faces for orientable ribbon graphs \( G \times |A| \) and \( (G + \sum_{i=1}^{n} e_i) \times |A'| \).

If \( e \notin A \), and \( G \times |A| \) is an orientable ribbon graph, then \( f(G \times |A|) = f^1(G \times |A|) \).

We let \( A' = A \), thus \( (G + \sum_{i=1}^{n} e_i) \times |A'| \) is also an orientable ribbon graph, and

\[
\begin{align*}
 f((G + \sum_{i=1}^{n} e_i) \times |A'|) &= f^1((G + \sum_{i=1}^{n} e_i) \times |A'|) \\
 &= f^1(G \times |A|) + n
\end{align*}
\]

(4.1)
because \( G \times |A| \) is obtained from \((G + \sum_{i=1}^{n} e_i) \times |A'| \) by deleting \( n \) multiple ribbons \( e_1, e_2, \cdots, e_n \), and deleting a multiple ribbon will reduce one face.

Otherwise, put \( e \in A_0 \), suppose that the ribbon graph \( G \times |A_0| \) is an orientable ribbon graph, then \( f(G \times |A_0|) = f^0(G \times |A_0|) \). Due to \( e_i \cup e \) is a 2-cycle in \( G + \sum_{i=1}^{n} e_i \), let \( A'_0 = A_0 \cup \sum_{i=1}^{n} e_i \), thus \((G + \sum_{i=1}^{n} e_i) \times |A'_0| \) is also an orientable ribbon graph. Since the ribbon graph \( G \times |A_0| \) is obtained from \((G + \sum_{i=1}^{n} e_i) \times |A'_0| \) by
deleting \( n \) twisted multiple ribbons \( e_1, e_2, \ldots, e_n \), and the deletion of the even number of twisted ribbons does not change the number of faces, it follows that

\[
f((G + \sum_{i=1}^{n} e_i) \times |A'_0|) = f^0((G + \sum_{i=1}^{n} e_i) \times |A'_0|)
\]

\[
= f^0(G \times |A_0|).
\]

(4.2)

According to the relationship between \( f^1(G \times |A|) \) and \( f^0(G \times |A_0|) \), we have the following two cases.

**Case 1**: If \( f^1(G \times |A|) \geq f^0(G \times |A_0|) \). By Euler’s formula,

\[
eu^0(G \times |A_0|) = 2c(G) + e(G) - v(G) - f^0(G \times |A_0|)
\]

\[
\geq 2c(G) + e(G) - v(G) - f^1(G \times |A|)
\]

\[
= eu^1(G \times |A|),
\]

(4.3)

\[
eu^0((G + \sum_{i=1}^{n} e_i) \times |A'_0|) = 2c(G) + e(G) + n - v(G) - f^0(G \times |A_0|) \quad \text{by (4.2)}
\]

(4.4)

\[
= eu^0(G \times |A_0|) + n,
\]

\[
eu^1((G + \sum_{i=1}^{n} e_i) \times |A'|) = 2c(G) + e(G) + n - v(G) - f^1(G \times |A|) - n \quad \text{by (4.1)}
\]

(4.5)

Moreover, by (4.3)-(4.5), we get \( eu^0((G + \sum_{i=1}^{n} e_i) \times |A'_0|) \geq eu^1((G + \sum_{i=1}^{n} e_i) \times |A'|) \).

So by the hypothesis that \( \partial G^{\times}_{G + \sum_{i=1}^{n} e_i} |O(z) \) is even-interpolating, there is a subinterval \([eu^1((G + \sum_{i=1}^{n} e_i) \times |A'|), eu^0((G + \sum_{i=1}^{n} e_i) \times |A'_0|)]\) in \( supp(\partial G^{\times}_{G + \sum_{i=1}^{n} e_i} |O(z)) \).

Since there are only even numbers in \([eu^1((G + \sum_{i=1}^{n} e_i) \times |A'|), eu^0((G + \sum_{i=1}^{n} e_i) \times |A'_0|)]\) (the Euler-genus of an orientable ribbon graph is even), it follows that the size of \([eu^1((G + \sum_{i=1}^{n} e_i) \times |A'|), eu^0((G + \sum_{i=1}^{n} e_i) \times |A'_0|)]\) is

\[
\frac{eu^0(G \times |A_0|) - eu^1(G \times |A|) + n}{2} + 1 \quad \text{by (4.4) and (4.5)}
\]

(4.6)

By Lemma 4.2 there are \( 2^{f^0(G) - 1} \) orientable ribbon graphs in \((G + \sum_{i=1}^{n} e_i) \times |A'|\).

Even if the number of faces of the restricted orientable partial-\( \times \) ribbon graph
of \((G + \sum_{i=1}^{n} e_i)\) are different, there are only \(2^{v(G)} - 1\) different numbers. By (4.6), for sufficiently large \(n\), we have

\[
\frac{\text{eu}^0(G^{\times |A_0|}) - \text{eu}^1(G^{\times |A|}) + n}{2} + 1 \geq 2^{v(G)} - 1 + 1.
\]

Hence, contradicting the assumption that \(\partial E^\times_{G + \sum_{i=1}^{n} e_i} |O(z)\) is even-interpolating. We have proved the case 1.

**Case 2:** If \(f^1(G^{\times |A|}) < f^0(G^{\times |A_0|})\). By Euler’s formula, we have \(\text{eu}^0(G^{\times |A_0|}) < \text{eu}^1(G^{\times |A|})\), however, when \(n\) is large enough, by (4.4)-(4.5), we still obtain \(\text{eu}^0((G + \sum_{i=1}^{n} e_i)^{\times |A_0|}) \geq \text{eu}^1((G + \sum_{i=1}^{n} e_i)^{\times |A'|})\). The remainder of the proof is analogous to that in case 1 and so is omitted.

□

Next, we give a lower bound for \(\partial E^\times_{G + \sum_{i=1}^{n} e_i} |O(z)\) is not even-interpolating for the number \(n\).

**Theorem 4.4.** Let \(G\) be a ribbon graph with a proper ribbon \(e\), and let \(G + \sum_{i=1}^{n} e_i\) be the ribbon graph obtained by adding \(n\) parallel ribbons \(e_1, e_2, \ldots, e_n\) to the ribbon \(e\) (see Figure 2.2).

1. If \(\partial E^\times_{G + \sum_{i=1}^{n} e_i} |O(z)\) has only one term, and

\[
n \geq \begin{cases} 
4, & \text{if } f_{\text{max}}^0(G + e_1) = f^0(G^{\times |A|}) + 1, \\
3, & \text{if } f_{\text{max}}^0(G + e_1) = f^0(G^{\times |A|}) - 1,
\end{cases}
\]

then \(\partial E^\times_{G + \sum_{i=1}^{n} e_i} |O(z)\) is not even-interpolating.

2. If \(\partial E^\times_{G + \sum_{i=1}^{n} e_i} |O(z)\) has more than one term, and \(n \geq \min\{f_{\text{max}}^0(G + e_1) - f_{\text{min}}^1(G) + 4, f_{\text{min}}^0(G) - f_{\text{max}}^1(G) + 4\}\), then \(\partial E^\times_{G + \sum_{i=1}^{n} e_i} |O(z)\) is not even-interpolating.

**Proof.** Suppose that \(\partial E^\times_{G + \sum_{i=1}^{n} e_i} |O(z)\) is even-interpolating. We observe that the number of faces of \(\text{eu}^0_{\text{min}}(G + \sum_{i=1}^{n} e_i)\) is \(f_{\text{max}}^0(G + \sum_{i=1}^{n} e_i)\), and the number of faces of \(\text{eu}^1_{\text{max}}(G + \sum_{i=1}^{n} e_i)\) is \(f_{\text{min}}^1(G + \sum_{i=1}^{n} e_i)\). Let \(e \in A_0, e \notin A_1\), assume that both \(G^{\times |A_0|}\) and \(G^{\times |A_1|}\) are orientable ribbon graphs, then \(f(G^{\times |A_0|}) = f^0(G^{\times |A_0|}), f(G^{\times |A_1|}) = f^1(G^{\times |A_1|})\).

For Item (1), we have

\[
(4.7) \quad f^1(G^{\times |A_1|}) = f^0(G^{\times |A_0|}) = f_{\text{min}}^1(G) = f_{\text{max}}^0(G).
\]

Observe that \(f_{\text{max}}^0(G + e_1) = f_{\text{max}}^0(G) + 1\), or \(f_{\text{max}}^0(G + e_1) = f_{\text{max}}^0(G) - 1\), which implies that we should consider the following two cases.
Case 1: \( f^0_{\text{max}}(G + e_i) = f^0_{\text{max}}(G) + 1 \). Note that deleting an even number of twisted multiple edges does not change the number of faces. Then, when \( n \) is even, the maximum value of \( f^0(G + \sum_{i=1}^n e_i) \) is the maximum value of \( f^0(G) \), that is

\[
f^0_{\text{max}}(G + \sum_{i=1}^n e_i) = f^0_{\text{max}}(G). \tag{4.8}
\]

When \( n \) is odd, and deleting \( n - 1 \) twisted multiple edges does not change the number of faces. We have

\[
f^0_{\text{max}}(G + \sum_{i=1}^n e_i) = f^0_{\text{max}}(G + e_1)
= f^0_{\text{max}}(G) + 1. \tag{4.9}
\]

By Euler’s formula, it follows that

\[
eu^0_{\text{min}}(G + \sum_{i=1}^n e_i) = 2c(G) + e(G) + n - v(G) - f^0_{\text{max}}(G) \quad \text{by (4.8)}
= 2c(G) + e(G) + n - v(G) - f^0(G \times |A_0|) \quad \text{by (4.7)}
= eu^0(G \times |A_0|) + n, \tag{4.10}
\]

for even \( n \), and

\[
eu^0_{\text{min}}(G + \sum_{i=1}^n e_i) = 2c(G) + e(G) + n - v(G) - f^0_{\text{max}}(G) - 1 \quad \text{by (4.9)}
= 2c(G) + e(G) + n - v(G) - f^0(G \times |A_0|) - 1 \quad \text{by (4.7)}
= eu^0(G \times |A_0|) + n - 1, \tag{4.11}
\]

for odd \( n \).

Let \( A'_1 = A_1 \). It is easy to see that \( (G + \sum_{i=1}^n e_i) \times |A'_1| \) is an orientable ribbon graph, and \( f((G + \sum_{i=1}^n e_i) \times |A'_1|) = f^1((G + \sum_{i=1}^n e_i) \times |A'_1|) \). Clearly, for any ribbon subset \( A'_1 \) of \( G + \sum_{i=1}^n e_i \), we have \( f^1((G + \sum_{i=1}^n e_i) \times |A'_1|) = f^1(G \times |A_1|) + n \). Furthermore,

\[
f^1_{\text{min}}(G + \sum_{i=1}^n e_i) = f^1_{\text{min}}(G) + n, \tag{4.12}
\]
Similarly, we have
\[\text{eu}_{max}^1(G + \sum_{i=1}^{n} e_i) = 2c(G) + e(G) + n - v(G) - f_{min}^1(G) - n \text{ by (4.12)}\]
\[= 2c(G) + e(G) + n - v(G) - f_0^0(G \times |A_0|) - n \text{ by (4.7)}\]
(4.13)
\[= \text{eu}_0^0(G \times |A_0|).\]

Recall that \(\partial \mathcal{E}_G^\times \frac{\partial \mathcal{E}_{G+\sum_{i=1}^{n} e_i}}{G+\sum_{i=1}^{n} e_i} |\mathcal{O}(z)\) is even-interpolating, so, there is a subinterval \([\text{eu}^1_{max}(G + \sum_{i=1}^{n} e_i), \text{eu}^0_{min}(G + \sum_{i=1}^{n} e_i)]\) in \(\text{supp}(\partial \mathcal{E}_G^\times \frac{\partial \mathcal{E}_{G+\sum_{i=1}^{n} e_i}}{G+\sum_{i=1}^{n} e_i} |\mathcal{O}(z))\). Note that the Euler-genus of the orientable ribbon graph \((G + \sum_{i=1}^{n} e_i) \times |A|\) is either smaller than \(\text{eu}^1_{max}(G + \sum_{i=1}^{n} e_i)\) or larger than \(\text{eu}^0_{min}(G + \sum_{i=1}^{n} e_i)\), so, there are only two numbers \(\text{eu}^1_{max}(G + \sum_{i=1}^{n} e_i), \text{eu}^0_{min}(G + \sum_{i=1}^{n} e_i)\) in \([\text{eu}^1_{max}(G + \sum_{i=1}^{n} e_i), \text{eu}^0_{min}(G + \sum_{i=1}^{n} e_i)]\).

If \(n\) is odd, from (4.11) and (4.13), the size of \([\text{eu}^1_{max}(G + \sum_{i=1}^{n} e_i), \text{eu}^0_{min}(G + \sum_{i=1}^{n} e_i)]\) is \(\text{eu}_0^0(G \times |A_0|) + n - 1 - \text{eu}_0^0(G \times |A_0|) + 1\). Otherwise, from (4.10) and (4.13), the size of \([\text{eu}^1_{max}(G + \sum_{i=1}^{n} e_i), \text{eu}^0_{min}(G + \sum_{i=1}^{n} e_i)]\) is \(\text{eu}_0^0(G \times |A_0|) + n - \text{eu}_0^0(G \times |A_0|) + 1\). When the size of \([\text{eu}^1_{max}(G + \sum_{i=1}^{n} e_i), \text{eu}^0_{min}(G + \sum_{i=1}^{n} e_i)]\) is greater than or equal to 3, it is contrary to the hypothesis that \(\partial \mathcal{E}_G^\times \frac{\partial \mathcal{E}_{G+\sum_{i=1}^{n} e_i}}{G+\sum_{i=1}^{n} e_i} |\mathcal{O}(z)\) is even-interpolating. Therefore, \(\partial \mathcal{E}_G^\times \frac{\partial \mathcal{E}_{G+\sum_{i=1}^{n} e_i}}{G+\sum_{i=1}^{n} e_i} |\mathcal{O}(z)\) is not even-interpolating for \(n \geq 4\).

**Case 2**: \(f_{max}^0(G + e_1) = f_{max}^0(G) - 1\). Similarly, when \(n\) is odd,
\[f_{max}^0(G + \sum_{i=1}^{n} e_i) = f_{max}^0(G + e_1)\]
\[= f_{max}^0(G) - 1.\]
(4.14)
\[\text{eu}_{min}^0(G + \sum_{i=1}^{n} e_i) = 2c(G) + e(G) + n - v(G) - f_{max}^0(G) + 1 \text{ by (4.14)}\]
\[= 2c(G) + e(G) + n - v(G) - f_0^0(G \times |A_0|) + 1 \text{ by (4.7)}\]
(4.15)
\[= \text{eu}_0^0(G \times |A_0|) + n + 1,\]

If \(n\) is odd, from (4.15) and (4.13), \(\frac{\text{eu}_0^0(G \times |A_0|) + n + 1 - \text{eu}_0^0(G \times |A_0|)}{2} + 1\) is the size of \([\text{eu}^1_{max}(G + \sum_{i=1}^{n} e_i), \text{eu}^0_{min}(G + \sum_{i=1}^{n} e_i)]\). The remainder of the argument is analogous to that in case 1 and omitted. Hence, \(\partial \mathcal{E}_G^\times \frac{\partial \mathcal{E}_{G+\sum_{i=1}^{n} e_i}}{G+\sum_{i=1}^{n} e_i} |\mathcal{O}(z)\) is not even-interpolating for \(n \geq 3\).

For item (2), we shall adopt the same procedure as in the proof of case 1.
When $n$ is odd, by (4.9), we have $f^0_{\text{max}}(G + \sum_{i=1}^n e_i) = f^0_{\text{max}}(G + e_1)$, and

$$
\epsilon_{\text{min}}^0 (G + \sum_{i=1}^n e_i) = 2c(G + e_1) + e(G + e_1) + n - 1 - v(G + e_1) - f^0_{\text{max}}(G + e_1) \\
= 2c(G + e_1) + e(G + e_1) - v(G + e_1) - f^0_{\text{max}}(G + e_1) + n - 1 \\
(4.16) \\
= \epsilon_{\text{min}}^0 (G + e_1) + n - 1.
$$

When $n$ is even, by (4.8), we have

$$
\epsilon_{\text{min}}^0 (G + \sum_{i=1}^n e_i) = 2c(G) + e(G) + n - 1 - v(G) - f^0_{\text{max}}(G) \\
= 2c(G) + e(G) - v(G) - f^0_{\text{max}}(G) + n \\
(4.17) \\
= \epsilon_{\text{min}}^0 (G) + n.
$$

By (4.13), we have

$$
\epsilon_{\text{max}}^1 (G + \sum_{i=1}^n e_i) = 2c(G) + e(G) + n - v(G) - f^0_{\text{min}}(G) - n \\
= 2c(G) + e(G) - v(G) - f^0_{\text{min}}(G) \\
(4.18) \\
= \epsilon_{\text{max}}^1 (G).
$$

Recall that there exists a subinterval $[\epsilon_{\text{max}}^1 (G + \sum_{i=1}^n e_i), \epsilon_{\text{min}}^0 (G + \sum_{i=1}^n e_i)]$ in $\text{supp} (\partial_{\sum_{i=1}^n e_i} G + \sum_{i=1}^n e_i)$. By (4.16)-(4.18), there exist two subintervals $[\epsilon_{\text{max}}^1 (G), \epsilon_{\text{min}}^0 (G + e_1) + n - 1]$ and $[\epsilon_{\text{max}}^1 (G), \epsilon_{\text{min}}^0 (G) + n]$ in $\text{supp} (\partial_{\sum_{i=1}^n e_i} G + \sum_{i=1}^n e_i)$. When

$$
\frac{\epsilon_{\text{min}}^0 (G) + n - \epsilon_{\text{max}}^1 (G)}{2} + 1 \geq 3,
$$

or

$$
\frac{\epsilon_{\text{min}}^0 (G + e_1) + n - 1 - \epsilon_{\text{max}}^1 (G)}{2} + 1 \geq 3,
$$

there are at least three numbers in $[\epsilon_{\text{max}}^1 (G), \epsilon_{\text{min}}^0 (G + e_1) + n - 1]$ or $[\epsilon_{\text{max}}^1 (G), \epsilon_{\text{min}}^0 (G) + n]$ respectively. However, there are only two numbers $\epsilon_{\text{min}}^0 (G)$, $\epsilon_{\text{min}}^0 (G + e_1) + n - 1$ in $[\epsilon_{\text{max}}^1 (G), \epsilon_{\text{min}}^0 (G + e_1) + n - 1]$ and two numbers $\epsilon_{\text{min}}^0 (G), \epsilon_{\text{min}}^0 (G) + n$ in $[\epsilon_{\text{max}}^1 (G), \epsilon_{\text{min}}^0 (G) + n]$, it is impossible.

Combining (4.16)-(4.20), we finish the proof. \qed
Theorem 4.5. [GMT21a] Let $G = G_1 \vee G_2$. Then
\[
\partial E^\times_G(z) = \partial E^\times_{G_1}(z) \partial E^\times_{G_2}(z).
\]

Remark 4.6. It is easy to see that Theorem 1.3 still holds for $\partial E^\times_G|_O(z)$.

Now, let’s take two examples to illustrate the above results.

Example 4.2. Let $C_m$ be the planar ribbon $m$-cycle. Evidently, $\partial E^\times_{C_2}|_O(z) = 2$. Using the fact that $C_m$ can be obtained from $C_2$ by subdividing a ribbon $m-2$ times, we have
\[
\partial E^\times_{C_m}|_O(z) = 2^{m-2} \partial E^\times_{C_2}|_O(z) \quad \text{by Theorem 1.3}
\]
\[
= 2^{m-1}.
\]

The ribbon graph $C_m + \sum_{i=1}^n e_i$ is obtained from $C_m$ by attaching $n$ parallel edges $e_1, e_2, \cdots, e_n$ to the ribbon $e$, as shown in Figure 4.1. It is clear that $f^0(C_m|A) = 2$. Since the number of faces of orientable ribbon graph $(C_m + e_1)^{\times|B}$ with $e \in B$ is 1, it follows that $f^0_{\max}(C_m + e_1) = f^0(C_m|A) - 1 = 1$. By Theorem 4.4 (1), when $n \geq 3$, the restricted-orientable partial-$\times$ polynomial of $C_m + \sum_{i=1}^n e_i$ is not even-interpolating.

Let $H = D_n \vee C_2$, then by Theorem 4.5 and Example 4.1, the restricted-orientable partial-$\times$ polynomial of $H$ is not even-interpolating for $n \geq 5$.

Example 4.3. Let $G$ and $G + \sum_{i=1}^n \bar{e}_i$ be the ribbon graphs in Figure 4.2. When $A \in \{\{e_1, e_3, e\}, \{e_2, e_3, e\}, \{e_1, e_3, e_4\}, \{e_2, e_3, e_4\}\}$, the number of faces of orientable ribbon graph $G^{\times|A}$ is 1. When $A \in \{\{e_1, e_2\}, \{e_4, e\}, \{e_1, e_2, e_4, e\}, \emptyset\}$, the number of faces of orientable ribbon graph $G^{\times|A}$ is 3. We infer that $\partial E^\times_G|_O(z) =$
4 + 4z^2, f_{max}^0(G) = 3, and f_{min}^1(G) = 1. When B ∈ \{\{e_1, e_3, e, \bar{e}_1\}, \{e_2, e_3, e, \bar{e}_1\}, \\
\{e_1, e_2, e_4, e, \bar{e}_1\}, \{e_4, e, \bar{e}_1\}\}, we have \(f((G + \bar{e}_1) × |B|) = 2\), thus,
\[f_{max}^0(G + \bar{e}_1) = 2 < f_{max}^0(G).\]
Therefore, by Theorem 4.4 (2), when \(n ≥ 2 − 1 + 4 = 5\), the restricted-orientable partial-× polynomial of \(G + \sum_{i=1}^{n} \bar{e}_i\) is not even-interpolating.

4.1. Acknowledgments. We are grateful to the anonymous referees for their valuable comments.

References

[AE19] L. Abrams, and J. Ellis-Monaghan, New dualities from old: generating geometric, Petrie, and Wilson dualities and trialities of ribbon graphs, arXiv:1901.03739v2 [math.CO], 9 Aug (2019).

[AVJ99] A. Brandstadt, V. B. Le, and J. P. Spinrad, Graph Classes: a Survey, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, (1999).

[Bol02] B. Bollobás, O. Riordan, A polynomial of graphs on surfaces, Math. Ann. 323 (2002) 81-96.

[Bry71] T. Brylawski, A combinatorial model for series-parallel networks, Trans. Amer. Math. Soc., 154 (1971), 1-22.

[Chm09] S. Chmutov, Generalized duality for graphs on surfaces and the signed Bollobás-Riordan polynomial, J. Combin. Theory Ser. B 99 (2009), 617-638.

[EM12] J. Ellis-Monaghan and I. Moffatt, Twisted duality for embedded graphs, Trans. Amer. Math. Soc. 364 (2012), 1529–1569.

[EM13] J. Ellis-Monaghan and I. Moffatt, Graphs on Surfaces: Dualities, Polynomials, and Knots. Springer, (2013).

[GT87] J. L. Gross and T. W. Tucker, Topological Graph Theory, John Wiley & Sons, Inc. New York, (1987).

[GMT20] J. L. Gross, T. Mansour and T. W. Tucker, Partial duality for ribbon graphs, I:Distributions, European J. Combin. 86 (2020), 103084.
[GMT21a] J. L. Gross, T. Mansour and T. W. Tucker, Partial duality for ribbon graphs, II: Partial-duality polynomials and monodromy computations, *European J. Combin.* 95 (2021), 103329.

[GMT21b] J. L. Gross, T. Mansour and T. W. Tucker, Partial duality for ribbon graphs, III: a Gray code algorithm for enumeration, *J Algebr. Comb.* (2021). https://doi.org/10.1007/s10801-021-01040-y.

[Mof12] I. Moffatt, A characterization of partially dual graphs, *Journal of Graph Theory* 67(3) (2010), 198-217.

[Tut54] W. T. Tutte, A contribution to the theory of chromatic polynomials, *Can. J. Math.,* 6 (1954), 80-91

**College of Mathematics, Hunan University, 410082 Changsha, China**

*Email address: chen1812020@163.com*

**College of Mathematics, Hunan University, 410082 Changsha, China**

*Email address: chengraph@163.com*