Response to “Comment on ‘Origin of the Curie–von Schweidler law and the fractional capacitor from time-varying capacitance [J. Power Sources 532 (2022) 231309]’”

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Abstract

We welcome Allagui et al.’s discussions about our recent paper that has proposed revisions to the existing theory of capacitors. It gives us an opportunity to emphasize on the physical underpinnings of the mathematical relations that are relevant for modeling using fractional derivatives. The concerns raised by Allagui et al. are found to be quite questionable when examined in light of the established standard results of fractional calculus. Consequently, the inferences that they have drawn are not true. Finally, we would like to thank Allagui et al. for their Comment because this subsequent Response has actually led to a further consolidation of our results that are supposed to be significant for materials science as well as for fractional control systems and engineering.

Keywords: Curie–von Schweidler law, universal dielectric response, fractional capacitor, fractional calculus, power-laws, memory

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In our recent publication [1], we gave a physical interpretation to the century-old Curie–von Schweidler law of dielectrics which interestingly may also be seen as an electrical analogue of Nutting’s law from rheology [2, 3]. The derivation of the Curie–von Schweidler law required a revision of the classical charge–voltage relation of capacitors, \( Q(t) = C_0 V(t) \), where \( Q \) is the charge, \( C_0 \) is the constant capacitance of the capacitor, \( V \) is the voltage, and \( t \) is the time. It is worthwhile to note that the classical relation is more than two-centuries old [4] and it was originally proposed for a constant capacitance capacitor. In contrast most capacitors of practical applications exhibit a time-varying capacitance but in lack of a better relation the same-old classical relation has been inadvertently used. In order to overcome this lacking we proposed a new charge–voltage relation which is equally applicable for a constant capacitance capacitor as well as for a capacitor with a time-varying capacitance. The relation is expressed by Eq. (9) in our paper as [1]:

\[
Q(t) = C(t) \ast \dot{V}(t),
\]

where \( \ast \) represents the convolution operation and \( C(t) \) is the time-varying capacitance of the capacitor.

In their Comment on our paper, Allagui et al. [5] have raised certain concerns with the use of the convolution integral of linear time-varying capacitance with the time-derivative of voltage. In this Response, for the sake of clarity, completeness, and correctness, we prove that each of their observations and the inferences that they have drawn from them are quite questionable and therefore not valid.

1. According to Allagui et al. capacitances being added in parallel is a frequency domain assertion, and that they cannot be added in the time domain [5]. Moreover, they have the opinion that circuit synthesis is not done in the time domain.

Response: We would like to emphasize that as long as the linearity of the system holds, circuit synthesis could be done in the time domain as well as in the frequency domain. We transform a problem from one domain to another domain depending upon the domain in which finding a solution to the problem is relatively easier. For example, modeling a digital filter is often easier in the frequency domain as it facilitates a convenient framework for the mathematical analysis of its properties. In contrast, a sampling process is meaningful in the time domain. The two domains should be seen as the two different viewpoints from which to understand the problem. It is also worthwhile to respect the fact that the time domain is the domain of physical reality and it is probably more intuitive than the frequency domain.

It has already been shown that our expression for current matches exactly with the predictions
from Kirchhoff’s current law for capacitors \cite{1}. Since Kirchhoff’s law stems from the fundamental principle of conservation of charge, this further gives confidence in all the mathematical relations that have been proposed and validated in our work. We again prove it here through a simple example in which a linearly time-varying voltage is applied to a capacitor with a linearly time-varying capacitance. If we assume, \( V(t) = at \), \( C(t) = C_0 + \phi t \), such that, \( a > 0 \) and \( \phi > 0 \), then the classical relation predicts:

\[
Q(t) = C(t)V(t) = (C_0 + \phi t) \cdot at = C_0 at + \phi at^2,
\]

from which the respective current, \( I \), is obtained after differentiating Eq. (2) as:

\[
I(t > 0) = C_0 \cdot a + 2\phi t \cdot a.
\]

Since, \( a = \dot{V}(t) \), the two terms on the right hand side of Eq. (3) may be interpreted as two capacitive currents that flow through two capacitors arranged in a parallel combination, the first of which has a constant capacitance, \( C_0 \), and the second capacitor has a time-varying capacitance, \( 2\phi t \). This means that according to Eq. (3), the capacitance of the capacitor is, \( C(t) = C_0 + 2\phi t \), which contradicts our initial assumption of a capacitance, \( C(t) = C_0 + \phi t \). To summarize, the classical relation yields an additional capacitor of capacitance, \( \phi t \), which was not initially present in our chosen example. Since this inconsistency that is observed from Eq. (3) has its origin in Eq. (2), it is evident that the classical relation is not valid for capacitors with a time-varying capacitance. This is because the classical relation dictates a term-by-term multiplication of \( C(t) \) and \( V(t) \) which is applicable for a time-invariant system. On the contrary, a capacitor with a time-varying capacitance constitutes a time-variant system.

Now, we extract the correct results from the proposed convolution relation, Eq. (1), that is also in accordance with the established laws of current-electricity. Evaluating the convolution of \( C(t) \) and \( \dot{V}(t) \) in time-domain, we have:

\[
Q(t) = C(t) * \dot{V}(t) = a \int_0^t [C_0 + \phi \tau] d\tau = C_0 at + \frac{\phi}{2} at^2.
\]

On comparing Eq. (2) with Eq. (4), we observe that both the classical relation and the convolution relation yield a similar quadratic dependence of charge on time but they are not exactly identical.
As expected, this difference also reflects from the respective expression for the current which is,

\[ I(t > 0) = C_0 \cdot a + \phi t \cdot a. \] (5)

It is seen that Eqs. (4) and (5) obtained from the convolution relation are different from their respective counterparts, Eqs. (2) and (3), that are obtained from the classical relation. The result obtained from the convolution relation has two distinct merits over the result obtained from the classical relation. First, since, \( a = \dot{V}(t) \), the two terms on the right hand side of Eq. (5) may be interpreted as two capacitive currents that flow through two capacitors arranged in a parallel combination, the first of which has a constant-capacitance, \( C_0 \), and the second capacitor has a time-varying capacitance, \( \phi t \). It is emphasized that we did not encounter the additional capacitor here which had mysteriously emerged from Eq. (3). Second, the application of Kirchhoff’s current law for capacitors yields exactly the same equation as expressed by Eq. (5). This agreement with Kirchhoff’s current law provides a physical validity to the convolution relation, Eq. (1).

2. Next, Allagui et al. claim that our proposed convolution relation, i.e., Eq. (1), generates a constant current and hence a zero charge if the input is a step voltage.\[^5\]

Response: We refer the readers to the definition of a fractional derivative which is expressed by Eq. (2) in Ref. [1] and the sentence that precedes the respective equation. We reiterate that particular excerpt from the paper here.

“The Caputo definition for the fractional derivative of a causal, continuous function, \( f(t) \), is the convolution of an integer-order derivative with a power-law memory kernel, \( \phi_\alpha(t) \), as Ref. [33]:

\[ \frac{d^\alpha}{dt^\alpha} f(t) \stackrel{\text{def.}}{=} \dot{f}(t) \ast \phi_\alpha(t), \quad \phi_\alpha(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad 0 < \alpha < 1, \] (6)

where the number of over-dots represent the order of differentiation with respect to time, \( t \).”

Rewriting this expression in its respective integral form with its generic limits of integration, we have:

\[ \frac{d^\alpha}{dt^\alpha} f(t) \stackrel{\text{def.}}{=} \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{t} \frac{\dot{f}(\tau)}{(t-\tau)^\alpha} d\tau, \quad 0 < \alpha < 1. \] (7)

Further, we draw attention to the standard results mentioned in Chapter 1, in particular on p. 16 of the textbook [6] which emphasizes on causal functions and the care required in solving
convolution-integrals that involve them. If \( f(t) \) is a causal function, it necessitates \( f(t) = 0 \) at all times, \( t < 0 \). Further, if there is a finite jump discontinuity of the integrand at \( t = 0 \), the following holds:

\[
\int_{-\infty}^{t} (\cdots) \, d\tau = \int_{0^{-}}^{t} (\cdots) \, d\tau.
\]  

(8)

In the case of, \( 0 < \alpha < 1 \), following Eq. (8), the right hand side of Eq. (7) is decomposed in accordance with the following identity:

\[
\frac{1}{\Gamma(1-\alpha)} \int_{0^{-}}^{t} \frac{\dot{f}(\tau)}{(t-\tau)^{\alpha}} \, d\tau = \frac{f(0^+)}{\Gamma(1-\alpha)} t^{-\alpha} + \frac{1}{\Gamma(1-\alpha)} \int_{0}^{t} \frac{\dot{f}(\tau)}{(t-\tau)^{\alpha}} \, d\tau.
\]  

(9)

Since Allagui et al.’s concerns are centered on the term, \( \dot{V}(t) \), independent of the nature of the capacitance, \( C(t) \), in Eq. (1), for simplicity we assume the capacitor has a constant capacitance, \( C_0 \). Mathematically, this means the memory kernel, \( \phi_{\alpha}(t) \), is scaled by \( C_0 \), and \( \alpha \to 0 \).

Identifying, \( f \) as \( V \), we express the convolution relation, Eq. (1), in light of Eqs. (6)–(9), as:

\[
Q(t) = C_0 V(0^+) + C_0 \int_{0}^{t} \dot{V}(\tau) \, d\tau.
\]  

(10)

On imposing Allagui et al.’s condition, \( \dot{V}(t) = 0 \), i.e., \( V(t) = V_0 \geq 0 \), for \( t \geq 0 \), where \( V_0 \) is the constant voltage applied to the capacitor, the integral-term in Eq. (10) vanishes, leaving behind,

\[
Q(t) = C_0 V(0^+) = C_0 V_0 \neq 0, \text{ if, } C_0 \neq 0, \text{ and } V_0 \neq 0.
\]

If the voltage at time, \( t = 0 \), is, \( V_0 = 0 \), then the charge at all times is zero as long as, \( \dot{V}(t) = 0 \), is maintained. This is expected from a discharged capacitor. In contrast, if the voltage at time, \( t = 0 \), is, \( V_0 > 0 \), then the charge at all times is \( C_0 V_0 \) as long as, \( \dot{V}(t) = 0 \), is maintained. This is expected from a charged capacitor. These results match exactly with those obtained from the classical charge–voltage relation for cases in which a constant voltage is fed to a constant-capacitance capacitor.

In the case of a capacitor characterized with a time-varying capacitance, i.e., \( 0 < \alpha < 1 \), if the voltage, \( V(t) = V_0 > 0 \), and \( \dot{V}(t) = 0 \), for \( t \geq 0 \), the corresponding last term from Eq. (9) vanishes. Consequently, the resulting charge is obtained from the corresponding non-integral
term of Eq. (9), which in this case turns out to be, $Q(t) \propto t^{-\alpha}$. Clearly, neither the charge is zero nor the current is a constant in such a case. This is in accordance with the experimental observations but contrary to the incorrect inferences drawn by Allagui et al. Therefore, the proposed convolution relation, Eq. (1), completes the bigger picture and yet retains the significance of the classical relation for constant-capacitance capacitors.

3. Allagui et al. claim that the question that we raised about the dimensional consistency of their equations in Ref. [7] is not correct.

Response: Although a detail discussion on this can be found in Ref. [8], we briefly mention it here for the completeness of this Response. To summarize, the authors of Ref. [7] seem to have given conflicting information about the same symbol used in their paper. We reproduce the relevant excerpts from their paper here. First, they mention in the paragraph after Eq. (4):

"By applying the principle of causality, the operation by which the charge $q(t)$ is created in the time-domain is therefore a convolution operation of capacitance $c(t)$ and voltage $v(t)$ such that $q(t) = c(t) \ast v(t)$ which is contrary to the usual assumption of a multiplication operation [19]."

It is evident that the authors have declared $c(t)$ as the capacitance, and $v(t)$ as the voltage, but the unit of charge, $q(t)$, according to their relation, $q(t) = c(t) \ast v(t)$, turns out to be of the quantity, $\text{capacitance} \cdot \text{voltage} \cdot \text{time}$, which is, Coulomb $\cdot$ s. Since the correct unit for charge is Coulomb, their proposed equation is dimensionally inconsistent which we had indicated in Ref. [1]. Later, the authors have mentioned after Eq. (5):

"For ideal capacitors, the capacitance is considered to be a geometric constant independent of the applied frequency which implies that in the time-domain $c(t) = C\delta(t)$, which leads to $q(t) = C\delta(t) \ast v(t) = Cv(t)$.”

We know that the unit of the Dirac-delta function is inverse of its argument which implies that in the time domain it is $1/s$. Since Allagui et al. had already declared, $c$, as the capacitance, it is therefore inferred that, $C = c(t) / \delta(t)$, has the unit of Farad $\cdot$ s, which is not the unit of a capacitance. On the contrary, they had introduced the term, $C$, as a capacitance in the first paragraph of the Introduction section of their paper as:

"The constant of proportionality between charge and voltage $C = eA/d$ is the capacitance in units of Farads.”

We observe that the authors of Ref. [7] have used the same symbol, $C$, to represent different physical quantities. This makes their treatment and the respective results ambiguous for readers.
It is to be noted that the origin of this dimensional inconsistency can be traced back to the paper that is cited as Ref. [19] in [7], where again the dimension of the Dirac-delta function has been overlooked.

We take this opportunity to clarify that the range of $\alpha$ in Eq. (13) of our paper:

$$I_{C_f}(t) = C_f \left[ \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \ast \ddot{V}(t) \right] = C_f \frac{d^{\alpha}}{dt^{\alpha}} V(t),$$

is, $0 < \alpha < 1$, where $C_f$ is the pseudocapacitance of the capacitor. Here we prove it through simple mathematical steps. Applying the differentiation property of convolution of two functions on the two terms that are inside the square bracket of Eq. (11) above, we have,

$$\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \ast \ddot{V}(t) = \frac{1}{\Gamma(1+1-\alpha)} \left[ \frac{d}{dt} t^{1-\alpha} \ast \dot{V}(t) \right].$$

Differentiating the power-law term inside the square bracket followed by the use of the functional equation for Gamma functions, $\Gamma(1+z) = z\Gamma(z)$, we have:

$$\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \ast \ddot{V}(t) = \frac{(1-\alpha)t^{-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} \ast \dot{V}(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \ast \dot{V}(t),$$

which when compared with Eq. (2) from Ref. [1], yields,

$$\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \ast \ddot{V}(t) = \frac{d^{\alpha}}{dt^{\alpha}} V(t), \ 0 < \alpha < 1.$$

Consequently, Eq. (11), i.e., Eq. (13) from Ref. [1], effectively implies:

$$I_{C_f}(t) = C_f \frac{d^{\alpha}}{dt^{\alpha}} V(t), \ 0 < \alpha < 1. \quad (12)$$

Lastly, we convey our thanks to Allagui et al. because even though their Comment lacked mathematical consistency, this Response to their Comment has made us even more confident about our results.

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