Multivariate Count Time Series Modelling

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Abstract

We review autoregressive models for the analysis of multivariate count time series. In doing so, we discuss the choice of a suitable distribution for a vectors of count random variables. This review focus on three main approaches taken for multivariate count time series analysis: (a) integer autoregressive processes, (b) parameter-driven models and (c) observation-driven models. The aim of this work is to highlight some recent methodological developments and propose some potentially useful research topics.

Keywords: auto-correlation, covariates, copula, estimation, multivariate count distribution, prediction
1 Introduction

This work reviews three main approaches that have been put forward for analysis and inference of multivariate count time series. By now there is an extensive literature for modeling univariate count time series, see the recent volume by Davis et al. (2016) and the review article by Davis et al. (2021), for example. Theoretical and methodological development for multivariate count time series is still on-going research area; see Paul et al. (2008) for a medical application, Pedeli and Karlis (2013b) for a financial study and more recently Ravishanker et al. (2015), for a marketing application, and Livsey et al. (2018) for an environmental study. The interested reader is referred to the review paper by Karlis (2016), for additional literature. The aim of this work is to highlight some recent methodological developments and propose some potentially useful research topics.

Following conventional theory, the standard venue for developing multivariate count time series models requires specification of a joint conditional distribution. Then, likelihood inference, for a given autoregressive model, provides estimation, testing and all type of standard output. However, choosing a joint count distribution is a challenging problem. There are numerous proposals available in the literature generalizing univariate Poisson probability mass function (p.m.f); some of these are reviewed in Sec. 2. The main obstacle is that the p.m.f of a multivariate “Poisson” discrete random vector is often of complicated functional form and therefore maximum likelihood inference is theoretically and numerically burdensome. The choice of joint distribution for modeling multivariate count data is an interesting research topic and some comments will be made throughout this work.

The first modeling approach is based on the theory of integer autoregressive (INAR) models and was initiated by Franke and Rao (1995) and Latour (1997). It was applied more recently by Pedeli and Karlis (2013a,b), Scotto et al. (2014) and Darolles et al. (2019). INAR models fall within the class of observation-driven models but because they are defined by means of thinning operator, (see Def. 3.1 and 3.2), they deserve special attention. Estimation for INAR models is based on least squares methodology and/or likelihood based methods. But, even for univariate INAR models, likelihood theory is quite cumbersome, especially when dealing with higher order autoregressive models. This methodology is reviewed in Sec. 3.

The second model class reviewed is that of parameter driven models whose dynamics–according to the broad categorization introduced by Cox (1981)– are driven by an unobserved process. Such models are also called state space models and have found numerous applications; see Zeger (1988), Harvey and Fernandes (1989), Fahrmeir and Tutz (2001), West and Harrison (1997), Durbin and

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Koopman (2000) and Frühwirth-Schnatter and Wagner (2006), among others, for contributions on univariate time series modeling. Multivariate state space models were studied by Jørgensen et al. (1996) and Jung et al. (2011); see also Ravishanker et al. (2014, 2015), among others, for more recent contributions. We review these models and we illustrate that, even though their specification is simple, they still require extensive computational efforts to be applied (using either frequentist or Bayesian methods). Developments in this area are reviewed in Sec. 4.

Section 5 goes over observation-driven process. This is the third class of models included in this work. Their main characteristic is that dynamics evolve according to past values of the process plus some noise. For example, ordinary autoregressive models belong to this class. Univariate observation-driven models for count time series have been studied by Zeger and Qaqish (1988), Fahrmeir and Tutz (2001), Rydberg and Shephard (2000), Kedem and Fokianos (2002), Fokianos et al. (2009), Fokianos and Tjøstheim (2011), Davis and Liu (2016), Ahmad and Franq (2016), Douc et al. (2017), among others. There is a growing literature within the framework of multivariate observation-driven count time series models; see Heinen and Rengifo (2007), Liu (2012), Andreassen (2013), Ahmad (2016), Lee et al. (2018), Cui and Zhu (2018), Gouriéroux and Lu (2019), Fokianos et al. (2020), Bracher and Held (2020), Opschoor et al. (2020), Piancastelli et al. (2020), Clark and Dixon (2021) for instance. Most of these studies are concerned with linear and/or log-linear count time series models but other alternatives can be developed. Finally, we mention the work of Darolles and Gourieroux (2015) who combine parameter-driven and observation-driven models to predict the number of hedge fund defaults as a function of hedge fund past defaults.

From a personal perspective, I think that this research area is still underdeveloped and there is ample space for exciting new developments. Some more recent works include that of Veraart (2019), who studies continuous-time models for multivariate count time series whose marginal distribution is infinitely divisible. This construction allows for separate modeling of serial correlation and the cross-sectional dependence. Additional work by Hall et al. (2019) considers high-dimensional count time series and studies the issues of inference for autoregressive parameters and the corresponding network structure by developing a sparsity-regularized maximum likelihood estimator. Finally, works by Zhang and Wu (2017) and Dahlhaus (1997) are potentially applicable to develop models for high-dimensional and non-stationary data. It is envisaged that this review will motivate further research on modeling and inference for multivariate count time series. For instance, simple questions like prediction, diagnostics, testing or development of other more suitable models with/without covariates will require further studies and theoretical developments. The list of references is by no means complete but further information is given by the therein and interested readers should
consult them for further details.

2 A Review of Multivariate Count Distributions

We outline some parametric multivariate count distributions for independent data that have been found useful for regression analysis. Simple properties of those models are discussed and their connection to time series data is illustrated in Sec. 5. The goal is to show that some basic multivariate distributions are directly applicable for fitting regression models and develop inference. There are several alternative venues, for instance we can rely on copulas and mixture models; see Johnson et al. (1997, Ch. 37), Joe (1997, Sec 7.2) for numerous multivariate count distributions and Cameron and Trivedi (2013, Ch.8) for an in-depth review of multivariate count regression models. As a general remark, joint p.m.f. of a discrete random vector often has complex functional form which is far from being useful to develop likelihood based inference. Further recent work and good summaries of up-to date parametric models can be found in Zhang et al. (2017), Inouye et al. (2017), Koochemeshkian et al. (2020), among others.

A multinomial distribution, is traditionally employed for analysis of multivariate count data by employing a multinomial logistic regression models. The multinomial distribution and the case of Dirichlet-multinomial distribution are not included in this work (see Zhang et al. (2017) for more). Those distributions are defined by a conditioning argument which might not extend to time series context. It is an elementary exercise to show that if $Y_i$ is independent Poisson distributed with mean $\lambda_i$ for $i = 1, 2, \ldots, d$, then the conditional distribution of $(Y_1, \ldots, Y_d)$ given $\sum_i Y_i = y$ is multinomial with parameters $y_i$ and $\lambda_i/ \sum_j \lambda_j$, for $i = 1, 2, \ldots, d$. So, the multinomial distribution applies to multivariate count modeling but subject to the restriction that it is supported on $\{y = (y_1, \ldots, y_d)^T : \sum_i y_i = y\}$. This support constraint should be considered cautiously in the context of dependent data. Additionally, this approach provides a conditional likelihood function for estimating regression parameters. The full likelihood function requires knowledge of the p.m.f. of $\sum_i Y_i$. Similar remark holds for Dirichlet-multinomial regression.

In what follows, we denote by $Y = (Y_1, \ldots, Y_d)^T$ a $d$-dimensional vector of counts whose components are not necessarily independent.

2.1 Multivariate Poisson Distributions

This class of distributions (Kocherlakota and Kocherlakota (1992) and Johnson et al. (1997)), generalizes the univariate Poisson models. Put $Y_i = W_i + W$, $i = 1, 2, \ldots, d$ where $W_i \sim \text{Poisson}(\lambda_i)$
and \( W \sim \text{Poisson}(\lambda_0) \) and all \( W \)'s are independent. Then, the joint p.m.f of \( Y \) is given

\[
P[Y = y] = \exp\left(-\sum_{i=0}^{d} \lambda_i \right) \sum_{k=0}^{\min_{i} y_i} \left( \prod_{i=1}^{d} \lambda_i \right)^k \frac{y_i!}{k!} \frac{\lambda_0}{\prod_{i=1}^{d} \lambda_i}^k. \tag{1}
\]

The marginals, \( Y_i \), are Poisson with mean \( \lambda_i + \lambda_0 \), for \( i = 1, 2, \ldots, d \) and it holds that \( \text{Cov}(Y_i, Y_j) = \lambda_0 \) which is always positive. In addition, the parameter \( \lambda_0 \) determines all possible pairwise correlations so the resulting model is of limited use. These facts and the complicated form of (1) make this model suitable for relatively low dimensional analysis where the marginals \( Y_i \) are positively correlated. An E-M type algorithm has been proposed by [Karlis (2003)] for inference but this approach is still hard to implement when the dimension \( d \) is large. Given a vector of regressors, say \( X \), an appropriate regression model \( \lambda_i \) as a function of \( X \) (in terms of linear or log-linear link function) and \( \lambda_0 \) is taken as constant. Then using (1) likelihood inference is straightforward, at least for low dimensions, for more see [Cameron and Trivedi (2013, Ch. 8.4.1)].

### 2.2 Mixed Poisson Models

Mixed models provide a general class of multivariate count distributions, see [Marshall and Olkin 1988]. Assume that \( Y_1, \ldots, Y_d \) are conditionally independent and Poisson distributed with \( \text{E}[Y_i] = \lambda_i \), for \( i = 1, \ldots, d \), given \( \lambda_1, \ldots, \lambda_d \). Suppose that the vector \( \lambda = (\lambda_1, \ldots, \lambda_d)^T \) is distributed according to some distribution \( G(\lambda) \). The mixed Poisson distribution is defined as

\[
P[Y = y] = \int_{(R^+)^d} \prod_{i=1}^{n} \exp(-\lambda_i) \frac{\lambda_i^{y_i}}{y_i!} \, dG(\lambda). \tag{2}
\]

Several choices for the mixing distribution \( G(\cdot) \) exist. But it is always true (provided that appropriate moments exist) that

\[
\text{E}[Y] = \text{E}[\lambda], \quad \text{Var}[Y] = \text{diag}(\text{E}(\lambda)) + \text{Var}[\lambda] = \text{diag}(\text{E}(Y)) + \text{Var}[\lambda], \tag{3}
\]

because of the first equality and \( \text{diag}(x) \) denotes a diagonal matrix whose elements are given by a vector \( x \). A trivial example of (2) is given when \( G(\cdot) \) is the Dirac distribution placing its mass at \( \lambda \). Then, \( Y \) is just a vector which consists of independent Poisson random variables. Finite mixtures of multivariate Poisson distributions, with application to clustering, have been discussed by [Karlis and Meligkotsidou 2007]. Another interesting case is when \( \lambda \) follows the \( d \)-dimensional log-normal distribution with parameters \( \mu = (\mu_1, \ldots, \mu_d)^T \) and \( \Sigma = (\sigma_{ij})_{i,j=1,\ldots,d} \). Though no closed formula
exists for the p.m.f. of \( Y \), eq. 3 implies that for \( i = 1, 2, \ldots, d \)

\[
E[Y_i] = \exp(\mu_i + 0.5\sigma_{ii}),
\]

\[
\text{Var}[Y_i] = E[Y_i] + E^2[Y_i](\exp(\sigma_{ii}) - 1),
\]

\[
\text{Cov}(Y_i, Y_j) = E[Y_i]E[Y_j](\exp(\sigma_{ij}) - 1).
\]

In general, denote by \( y^{(k)} = y(y-1) \cdots (y-k+1) \), the so called falling factorial. Then eq. 2 shows that multivariate factorial moments are computed by using simple properties of Poisson distribution, i.e.

\[
E\left[\prod_{i=1}^{d} Y_{i,(r_i)}\right] = E\left\{\prod_{i=1}^{d} E\left[Y_{i,(r_i)} \mid \lambda_i\right]\right\} = E\left[\prod_{i=1}^{d} \lambda_i^{r_i}\right],
\]

Furthermore, following Johnson et al. (1992) joint moments are given by

\[
E\left[\prod_{i=1}^{d} Y_i^{r_i}\right] = \sum_{l_1}^{r_1} \cdots \sum_{l_d}^{r_d} \prod_{i=1}^{d} s(r_i, l_i) E\left[\prod_{i=1}^{d} \lambda_i^{l_i}\right],
\]

where \( s(r, l) \) are the Stirling numbers of the second kind.

Next, it is shown that the multivariate negative-multinomial distribution is recovered by means of eq. 2. This is well-known in the univariate case. A multivariate negative-multinomial distribution has p.m.f. which is given by

\[
\text{P}[Y = y] = \frac{(r + \sum_{i=1}^{d} y_i)!}{(\prod_{i=1}^{d} y_i!)(n-1)!} p_0^r \prod_{j=1}^{d} p_j^{y_j}, \quad (4)
\]

where \( y_j = 0, 1, 2, \ldots \) for \( j = 1, \ldots, d \), \( r > 0 \) and \( 0 < p_j < 1 \), \( j = 0, 1, \ldots, d \) satisfying \( p_0 = 1 - \sum_{j=1}^{d} p_j \). The parameters \( p_j, j = 1, 2, \ldots, d \) denote the probabilities of obtaining different failures and thus \( p_0 \) is the probability of success in an experiment terminating to \( r \) failures; see Johnson et al. (1997, Ch. 36) and Joe (1997, Ch. 7.2) for more. Note that \( r \) might assume real values in applications; it is short of ”dispersion” parameter though the concept of dispersion—that is when the variance exceeds the mean—is quite vague in the multivariate case we consider. In this case, it can be shown that all pairwise correlations between the components of \( Y \) are positive.

It is shown next that eq. 4 is obtained as mixed Poisson model by using eq. 2 assuming that, conditionally on a Gamma distributed random variable \( \theta \), say \( \theta \sim \text{Gamma}(\beta, \beta) \), \( Y_j \) is conditionally Poisson distributed with mean \( \lambda_j \theta \), \( j \in \{1, \ldots, d\} \). Then

\[
\text{P}[Y = y] = \int_{0}^{\infty} \text{P}[Y = y \mid \theta] dG(\theta)
\]

\[
= \frac{\Gamma(\beta + \sum_{i=1}^{d} y_i)}{(\prod_{i=1}^{d} y_i!}\Gamma(\beta)} \left( \frac{\beta}{\beta + \sum_{i=1}^{d} \lambda_i} \right)^{\beta} \prod_{j=1}^{d} \left( \frac{\lambda_j}{\beta + \sum_{i=1}^{d} \lambda_i} \right)^{y_j}, \quad (5)
\]
where $\Gamma(\cdot)$ denotes the Gamma function. Obviously (4) holds. Clearly, this model implies that the random variable $\theta$ accommodates common unobserved heterogeneity; see Munkin and Trivedi (1999) for simulated maximum likelihood estimation for this particular class.

Recall (4). Then, given a covariate vector $X$, a multinomial logistic regression model (see Agresti (2002)) is employed to link $X$ with the probabilities $p_j$, $j = 1, 2, \ldots, (d + 1)$. Furthermore, the model can be extended to include a log-linear model for $r$ (which can be positive real in general, see the previous case)–for more details see Zhang et al. (2017).

2.3 Copula approaches

Copula-based construction of multivariate count distributions is an active topic of research; see Nikoloulopoulos (2013a) and Inouye et al. (2017) for nice surveys. Copulas are useful because of Sklar’s theorem (Sklar (1959)) which shows that marginal distributions are combined to give a joint distribution when applying a copula, i.e. a $d$-dimensional distribution function all of whose marginals are standard uniforms; the book by Nelsen (1999) gives a thorough introduction to copulas.

Even though copulas provide an appealing methodology for constructing joint distribution functions they pose challenging issues when used for discrete data analysis. First, the presence of ties in count data (several zeroes, for example) makes them non-identifiable–Genest and Nešlehová (2007), in particular pp. 507-508–illustrate the lack of identifiability. An additional issue is that of the likelihood function’s computationally difficulty for estimating unknown parameters. For a discrete random vector, whose cumulative distribution function (c.d.f) is $F$, its p.m.f. involves $2^d$ finite differences of $F$, i.e.

$$P[Y = y] = \sum_{l_1=0,1} \ldots \sum_{l_d=0,1} (-1)^{l_1+\ldots+l_d} P[Y_1 \leq y_1 - l_1, \ldots, Y_d \leq y_d - l_d].$$

Computational methods using Bayesian data augmentation have been developed by Smith and Khaled (2012), among others; the survey of Smith (2013) provides references on Bayesian methodology for discrete data copula modeling.

In the rest of this section I describe a different approach for employing copulas to model multivariate count data. The methodology is based on the work by Fokianos et al. (2020) who advanced a particular data generating process for multivariate count time series analysis; this topic is discussed in Sec. 5. Initially, the idea is illustrated for the case of i.i.d random vectors. The intent is to introduce a data generating process which keeps all marginal distributions of the vector $Y$ to be Poisson distributed and, at the same time, it allows for arbitrary dependence among them. This is
accomplished by appealing to elementary properties of Poisson process. An explicit account of this
construction is given by the following algorithm—recall that $Y_i, i = 1, 2, \ldots, d$ is the $i$‘th-component
of the count vector $Y$ whose mean is $\lambda_i$.

1. Let $U^{l} = (U_{1,l}, \ldots, U_{d,l})$ for $l = 1, 2, \ldots, K$, be a sample from a $d$-dimensional copula
$C(u_1, \ldots, u_d)$. Then $U_{i,l}, l = 1, 2, \ldots, K$ follow marginally the uniform distribution on $(0, 1)$, for $i = 1, 2, \ldots, d$.

2. Consider the transformation $X_{i,l} = -\log U_{i,l}/\lambda_i, \ i = 1, 2, \ldots, d$. Then, the marginal distribu-
tion of $X_{i,l}, l = 1, 2, \ldots, K$ is exponential with parameter $\lambda_i, i = 1, 2, \ldots, d$.

3. If $X_{i,1} > 1$ set $Y_i = 0$, otherwise $Y_i = \max \left\{ K : \sum_{l=1}^{K} X_{i,l} \leq 1 \right\}, \ i = 1, 2, \ldots, d$. Then $Y = (Y_1, \ldots, Y_d)^T$ is marginally realization of a Poisson process with parameter $\lambda$.

4. Repeat steps 1-3 $n$ times to generate a sample $Y^j, j = 1, 2, \ldots, n$.

The algorithm generates i.i.d. random vectors whose dependence among their components is
introduced by a copula structure on the waiting times of the Poisson process. In other words, the
copula is imposed on the uniform random variables generating the exponential waiting times. The
end result gives a sample of multivariate discrete random variables with Poisson marginals. The
methodology can be extended to other discrete marginal distributions provided that they can be
generated by continuous inter arrival times. For instance, suppose that $Y_i$ is marginally mixed
Poisson with mean $\theta \lambda_i$ where $\theta$ satisfies $E[\theta] = 1$. Many families of count distributions, including
the negative binomial, can be generated by this construction; see Sec. 2.2 Then steps 1-4 of the
above algorithm are still useful for generating multivariate random vectors whose marginals are not
necessarily Poisson. Indeed, generating at the first step an additional random variable $\theta$, define
again at step 2 the waiting times by $X_{i,l} = -\log U_{i,l}/\theta \lambda_i, \ i = 1, 2, \ldots, d$. Then, the distribution
of $X_i$ is mixed exponential and therefore steps 3-4 deliver a realization of a count vector whose
marginal distribution is mixed Poisson.

The joint p.m.f. of $Y$, based on the above construction, is shown in Fig. 1. Plots (a), (b) and
(c) show the case of independence, positive and negative correlation, respectively, when using a
Gaussian copula. The algorithm delivers desired marginals and, in addition, shows that all type of
different correlations can be achieved. Fig. 1(d) shows that joint p.m.f. of a negative multinomial
vector, see (5) with the same $\lambda$ parameters values used for the previous algorithm and $\beta = 10$. The
positive correlation between the vector components is obvious.
The algorithm, when implemented, requires clear distinction between the resulting copula to the vector of counts and the copula imposed on waiting times. The transformation from waiting times to counts is stochastic, and while the copula as such is invariant to one-to-one deterministic transformations, we do not have such a transformation in this case. Hence, instantaneous correlation among the components of the count vector is not equal to the correlation induced by the copula imposed to the vector of waiting times. *Interpretation of the instantaneous correlation found in data is related to the correlation of the vector of waiting times and should be done with care.* Recalling Fig. 1 and calculating the sample correlation coefficient, it is found that it is equal to 0.34 (-0.34) when the data are generated with copula parameter $\rho > 0$ ($\rho < 0$), respectively.

![Figure 1](image-url)

Figure 1: Joint p.m.f of a bivariate count distribution using the copula construction as outlined by steps 1–4. Results are based on a Gaussian copula with correlation coefficient $\rho$. (a) $\rho = 0$ (independence) (b) $\rho = 0.8$ (positive correlation) (c) $\rho = -0.8$ (negative correlation). Plots are based on 10000 independent observations where the marginals are Poisson with $\lambda_1 = 3$ and $\lambda_2 = 10$. (d) Joint p.m.f of negative multinomial distribution. Results are based on 10000 independent observations with $\beta = 10$ and $\lambda_1 = 3$ and $\lambda_2 = 10$.

This approach is different from the methodology advanced by Demui and Lambert (2005) who employ the continued extension. Those authors add noise of the form $U - 1$, where $U$ is standard uniform, to counts so that those are transformed into continuous random variables. In doing so, the
copula identifiability problem of is bypassed. An analogous approach, based on the distributional transform which adds a random jump to the c.d.f. of the discrete variable, has been studied by Rüschendorf (2013). An interesting decomposition of the joint p.m.f. of a discrete random vector has been discussed by Panagiotelis et al. (2012) using the idea of pair-copula construction (Czado (2010)) by utilizing the concept D-vine copulas (Bedford and Cooke (2001, 2002)).

Further work on copulas, in the context of generalized linear models, is given by Song (2000) and Song et al. (2009). The latter reference employs Gaussian copulas, for multivariate regression analysis of continuous, discrete, and mixed correlated outcomes under the generalized linear models (GLM) framework (Nelder and Wedderburn (1972) and McCullagh and Nelder (1989)). More recently, Yang et al. (2020) consider discrete regression models and copula estimation arguing that inclusion of continuous covariates implies consistent estimation of the unknown copula. In addition, Jia et al. (2021) employ a latent Gaussian process and a distributional transformation to construct stationary univariate count time series models with flexible correlation features such that their marginal distribution can be prespecified.

2.4 Additional models

There are additional approaches for defining a multivariate count distribution; Joe (1997, Ch.9) and Inouye et al. (2017) review construction of multidimensional Poisson p.m.f by appealing to full conditional distributions and Markov random fields; see Besag (1974) for the so called auto-Poisson model. Additional models include the Sarmanov and bivariate Hurdle distributions among others; see Cameron and Trivedi (2013, Ch. 8). These models are mentioned for completeness of presentation but their properties have not been fully explored in the literature, to the best of my knowledge.

3 Integer AR models

Integer Autoregressive (INAR) models deserve special consideration due to the thinning operation. The calculus of thinning operators provides useful insight into the probabilistic properties of those processes by employing the simple device of summing up a random number of integer-valued random variables (see Steutel and van Harn (1979)). The case that has attracted more attention is when the summands consist of an independent and identically distributed (iid) sequence of Bernoulli random variables.
3.1 The thinning operator

Define the generalized Steutel and van Harn operator (see Latour (1997, Def. 1.1)) as follows:

**Definition 3.1** Suppose that $X$ is a non-negative integer random variable. The generalized thinning operator, denoted by $\circ$, is defined as

$$\alpha \circ X = \begin{cases} \sum_{k=1}^{X} I_k, & X > 0; \\ 0, & X = 0. \end{cases}$$

where $\{I_k, k \in \mathbb{N}\}$ is a sequence of iid integer random variables—indepen- dent of $X$—with mean $\alpha$ and variance $\beta$.

The sequence $\{I_k, k \in \mathbb{N}\}$ is called counting series. If $\{I_k\}$ is an iid sequence of Bernoulli random variables, then $\alpha \circ X$ counts the number of successes in a random number of Bernoulli trials where the probability of success $\alpha$ remains constant throughout the experiment so that given $X$, $\alpha \circ X$ is a binomial random variable with parameters $X$ and $\alpha$. In this case, we call the thinning operator as binomial thinning operator. General thinning operators are discussed by Davis et al. (2021, Sec. 2) and Joe (2016), among others.

Numerous properties can be proved for the thinning operator, for instance it can be shown that $E[\alpha \circ X | X] = \alpha X$, $E[\alpha \circ X] = \alpha E[X]$, $\text{Var}[\alpha \circ X | X] = \beta X$ and $\text{Var}[\alpha \circ X] = \alpha^2 \text{Var}[X] + \beta \text{E}[X]$, provided that appropriate moments of $X$ exist.

**Definition 3.1** can be extended to a non-negative integer $d$-dimensional random vectors. For $i, j = 1, 2, \ldots, d$, define $\{I_{ij,k}, k \in \mathbb{N}\}$ an array of counting series such that $E[I_{ij}] = \alpha_{ij}$ and $\text{Var}[I_{ij}] = \beta_{ij}$. Let $A = (\alpha_{ij})$, $B = (\beta_{ij})$ be the corresponding $d \times d$ matrices. Then, **Definition 3.1** can be extended as follows:

**Definition 3.2** Suppose that $X = (X_1, X_2, \ldots, X_d)^T$ is $d$-dimensional integer-valued random vector with all components being non-negative and denote by $A \circ = (\alpha_{ij} \circ)$ a $d \times d$ matrix of thinning operators whose each element is given by Def. 3.1 with corresponding array of counting series $\{I_{ij,k}, k \in \mathbb{N}\}$. Then the multivariate thinning operator is defined as

$$A \circ X = \begin{pmatrix} \sum_{j=1}^{d} \alpha_{1j} \circ X_j \\ \vdots \\ \sum_{j=1}^{d} \alpha_{dj} \circ X_j \end{pmatrix}.$$ 

Similar to the univariate case, it can be shown that $E[A \circ X] = AE[X]$ and $\text{Var}[A \circ X] = \text{diag}(BE[X]) + ACov[X]A^T$. 

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3.2 The multivariate INAR model

The multivariate thinning operator serves as basic tool to develop multidimensional integer AR models (abbreviated by MINAR) of order \( p \). A \( d \)-dimensional time series \( \{Y_t, t \in \mathbb{Z}\} \) is called multivariate INAR\((p)\) process if it satisfies

\[
Y_t = \sum_{i=1}^{p} A_i \circ Y_{t-i} + \epsilon_t,
\]

where \( \{\epsilon_t, t \in \mathbb{Z}\} \) is a sequence of iid integer-valued random vectors with mean \( E[\epsilon_t] = \mu_\epsilon \) and \( \text{Var}[\epsilon_t] = \Sigma_\epsilon \) which is independent of all thinning operators \( A_i, i = 1, 2, \ldots, p \) and \( A_p \neq 0 \). Denote by \( I_d \) the \( d \)-dimensional identity matrix. Provided that the roots of polynomial \( \text{det}(I_d - A_1 z - \cdots - A_p z^p) \) are all located outside the unit circle, then Latour (1997, Prop. 3.1) shows that there exists an almost surely unique integer-valued strictly stationary process that satisfies (6) and such that \( \epsilon_t \) is independent of \( Y_s, s < t \). For the case of \( d = 1 \), this condition is equivalent to \( 0 < \sum_{i=1}^{p} A_i < 1 \), see Du and Li (1991). The univariate INAR\((p)\) processes have been introduced by Al-Osh and Alzaid (1987), Alzaid and Al-Osh (1990). In the same vein, multivariate INAR moving average (MINARMA) models can be defined but they will not be discussed any further.

We assume the stability condition for MINAR\((p)\) processes holds true. Recall (6) and consider the special case of \( p = 1 \). By taking expectations in both sides of (6), \( E[Y_t] = (I_d - A_1)^{-1} \mu_\epsilon \). Multiplying both sides of (6) by \( Y_{t+h} \) and taking expectations, it follows that \( \text{Cov}(Y_t, Y_{t+h}) = A_1^h \text{Var}(Y_t) \), where \( \text{Var}(Y_t) = A_1 \text{Var}(Y_t) A_1^T + \text{diag}(BE[Y_t]) + \Sigma_\epsilon \) for \( h = 0, \pm 1, \pm 2 \cdots \). As a final remark, Latour (1997, Prop. 4.1) shows that a MINAR\((p)\), which satisfies the stability condition discussed earlier, has identical second order properties with an ordinary vector AR\((p)\) (VAR) model (see Tsay (2014)). Consider again the case \( p = 1 \) for (6). Then \( \{Y_t\} \) is represented by a VAR\((1)\) process of the form

\[
Y_t = \omega + A_1 Y_{t-1} + \zeta_t
\]

where \( \{\zeta_t\} \) is a white noise process with covariance matrix \( \Sigma_\zeta = \text{diag}(BE[Y_t]) + \Sigma_\epsilon \) and \( \omega = E[Y_t] \). This fact has important consequences on estimation. For instance least squares estimators (LSE) directly applies to this class of models but subject to restriction that all unknown coefficients are positive.

3.3 Estimation

Besides LSE, likelihood estimation has been also developed for estimating the unknown parameters of the model (5). Both methods are discussed next by assuming that \( Y_1, \ldots, Y_n \) is a sample from
a MINAR(1) model—this is done mostly for convenience.

LSE are computed and studied by using (7). Let

$$\begin{align*}
Y \in (n-1) \times d & = X^* \beta + Z \\
\text{where the } i\text{'th row of } Y, X^* \text{ and } Z \text{ is given by } Y^T_{i+1}, (1, Y_i^T) \text{ and } \zeta^T_i, i = 1, 2, \ldots, (n-1), \text{respectively.}
\end{align*}$$

The regression matrix parameter is denoted by $\beta$, i.e. $\beta = (\omega, A_1)^T$. Then the LSE of $\beta$ is denoted by $\tilde{\beta}$ and is equal to

$$\tilde{\beta} = \left(X^*X^*\right)^{-1}X^*Y = \left(\sum_{t=2}^{n} \begin{bmatrix} 1 & Y_t^T & \zeta_t^T \end{bmatrix}\right)^{-1} \left(\sum_{t=2}^{n} \begin{bmatrix} Y_t^T \\ Y_tY_t^T \end{bmatrix}\right).$$

Based on this define the residual matrix $\tilde{Z} = Y - X^*\tilde{\beta}$ to obtain an estimator of $\Sigma_\zeta$ by

$$\tilde{\Sigma}_\zeta = \frac{1}{n - d - 2} \tilde{Z}\tilde{Z}^T,$$

where the numerator $n - d - 2 = (n - 1) - (d + 1)$ is equal to the effective degrees of freedom minus the number of parameters estimated for each component series. More precisely, it can be shown that $E[\tilde{\beta}] = \beta$ and $E[\tilde{\Sigma}_\zeta] = \Sigma_\zeta$. Moreover, as $n \to \infty$, and assuming suitable regularity conditions are fulfilled

$$\sqrt{n} \text{vec}(\tilde{\beta} - \beta) \overset{D}{\to} N(0, \Sigma_\zeta \otimes H^{-1}),$$

where vec(.) denotes the vec operator, $\otimes$ is the Kronecker product and the $(d + 1) \times (d + 1)$ matrix $H$ is the limit (in probability) of $X^*X^*/n$. A rigorous statement is developed along the lines of Latour (1997) and Lütkepohl (2005, Lemma 3.1). This result holds when the true parameters belong to the interior of parameter space employing unconstrained optimization. The asymptotic distribution of LSE, under the constraint that all elements of the matrix $A_1$ are positive, is an open problem.

Besides LSE, conditional likelihood estimation is developed (recall again (6) with $p = 1$) by maximizing the likelihood function

$$L(\beta) = \prod_{t=1}^{n} P_\beta[Y_t = y_t \mid Y_{t-1} = y_{t-1}],$$

when imposing a multivariate distribution on the error term $\epsilon_t$. In general the conditional transition is given by the $d$-dimensional convolution

$$P_\beta[Y_t = y_t \mid Y_{t-1} = y_{t-1}] = \sum_{k=0}^{Y_t} P_\beta[\epsilon_t = k]P[\epsilon_t = k].$$

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Both the above equation implies that the log-likelihood function is given by

\[ l(\beta) \equiv \log L(\beta) = \sum_{t=1}^{n} \log \left( \sum_{k=0}^{\mu_t} P_{\beta}[A_1 \circ Y_{t-1} = y_t - k] P[\epsilon_t = k] \right). \]

This short discussion shows that the task of computing the log-likelihood function is daunting even in the simple case \( p = 1 \). Some simplifications occur when we assume that the matrix \( A_1 \) is diagonal and by applying pairwise likelihood methodology as in Pedeli and Karlis (2013a). Such an approach can deliver some insights for data analysis but it will be difficult to be justified for multivariate dynamic systems. Generally, likelihood methods are not suitable for this class of models because of their complicated structure. The problem complexity increases when both \( p \) and \( d \) grow but sparsity ideas (see Hastie et al. (2015)) will be helpful.

### 3.4 Prediction

Consider again the case of model \( (6) \) for \( p = 1 \). We briefly discuss prediction for this particular case to overcome cumbersome notation. Similar to the case of VAR(\( p \)) models, a MINAR(\( p \)) model is written as a “big” MINAR(1) so these results suffice to develop a general point of view.

The one-step ahead predictor of the MINAR(1) process is easily calculated by

\[ E[Y_{t+1} | Y_t] = E[A_1 \circ Y_t + \epsilon_t | Y_t] = A_1 Y_t + \mu. \]

Therefore, by recursion

\[ E[Y_{t+h} | Y_t] = A_h^T Y_t + (I_d + A_1 + \cdots + A_{h-1}^T) \mu. \]

Define \( V(h) = \text{Var}[Y_{t+h} | Y_t] \). Then

\[ V(1) = \text{Var}(A_1 \circ Y_t + \epsilon_t | Y_t) = \text{diag}(BE[Y_t]) + \Sigma. \]

The law of total variance shows that

\[ V(h) = E[\text{Var}(Y_{t+h} | Y_{t+1}) | Y_t] + \text{Var}[E(Y_{t+h} | Y_{t+1}) | Y_t] \]

\[ = E[V(h-1)] + A_1^T h V(1) (A_1^{h-1})^T. \]

Provided that \( A_1 \) is diagonal, some simplifications of the previous formulas have been proved by Pedeli and Karlis (2013b). The complexity of prediction problem increases as both dimension and model order increase. In addition, care should be taken when unknown parameters are replaced by their estimators.
Closing this section, we mention that properties of multivariate INAR models are well understood for low-dimensional data and under tangible assumptions. It is interesting to consider properties of the multivariate thinning operator in high-dimensions and investigate the problem of estimation and prediction.

4 Parameter-Driven Models

A parameter driven model, according to classification introduced by [Cox (1981)], is a time series driven by an unobserved process (as opposed to past process values; see Sec 5). For multivariate count series, state-space models were studied in [Jørgensen et al. (1999) and Jung et al. (2011); see Ravishanker et al. (2014), and Ravishanker et al. (2015), among others, for more recent contributions. The approaches that have been taken for estimation are based either on likelihood or full Bayesian methods. We review some of these works.

The model proposed by [Jørgensen et al. (1999)], is closely related to the theory of mixed Poisson distributions, see Sec. 2.2. It assumes that the conditional distribution of the $i$'th component of the multivariate count series at time $t$, $Y_{i,t}$, given an unobserved univariate time-varying process $\theta_t$, is Poisson distributed with mean $a_{i,t}\theta_t$ such that $a_{i,t} = \exp(c_i^T \alpha_i)$, where $c_i$ are $k$-dimensional time-varying covariate vectors and $\alpha_i$ are $k$-dimensional regression parameters, $i = 1, \ldots, d$. Assume that $\theta_0 = 1$ and the conditional distribution of $\theta_t$ given $\theta_{t-1}$ is Gamma with mean $b_t\theta_{t-1}$ and a squared coefficient of variation of form $\sigma^2/\theta_{t-1}$. The parameters $b_t$ depend on the so called long–term covariates $z_t$ through $b_t = \exp(\Delta z_t^T \beta)$, where $\Delta z_t = z_t - z_{t-1}$ and $z_0 = 0$, $\sigma^2$ denotes a dispersion parameter and $\beta$ is regression coefficient. It can be shown that

$$E[\theta_t] = b_1 \ldots b_t$$

which implies that $\log(E[\theta_t]) = z_t^T \beta$. In addition, for $h \geq 0$,

$$\text{Var}(\theta_t) = \phi_t E[\theta_t] \sigma^2, \quad \text{Cov}(\theta_t, \theta_{t+h}) = \phi_t E[\theta_{t+h}] \sigma^2,$$

where $\phi_t = b_t + b_t b_{t-1} + b_t b_{t-1} \ldots b_1$. Set $a_t = (a_{1,t}, \ldots, a_{d,t})^T$ and $A_t = \text{Diag}(a_{1,t}, \ldots, a_{d,t})$. Then (compare with (3) when $a_{i,t} = 1$ for all $i$ and $t$)

$$E[Y_t] = a_t E[\theta_t], \quad \text{Var}(Y_t) = A_t E[\theta_t] + a_t a_t^T \phi_t \sigma^2 E[\theta_t].$$

1For this part of the text, I replace the notation $\lambda_t$ by $\theta_t$ because it denotes a univariate time-varying mean process.
This last result shows that the variance matrix of $Y_t$ consists of two components: (a) a Poisson variance and (b) a type of "overdispersion" component. The authors discuss Kalman prediction and filtering, for the log-linear model $E[Y_{i,t}] = \exp(c_i^T \alpha_i + z_i^T \beta)$, relying on previous calculations.

The above model is closely related to the model of Jung et al. (2011) which in turn generalizes that of Wedel et al. (2003) who developed a comprehensive class of factor models for multivariate truncated count data. The former authors assume that $Y_{i,t}$, conditionally on $\lambda_{i,t}$, are independent Poisson distributed random variables with mean $\lambda_{i,t}$, $i = 1, 2, \ldots, d$ and for all $t$. By considering the $d$-dimensional time-varying vector process $\lambda_t = (\lambda_{1,t}, \ldots, \lambda_{d,t})^T$, it is assumed that

$$\log \lambda_t = \omega + \Gamma f_t,$$

where $\omega$ is $d$-dimensional vector of parameters, $\Gamma$ is a $d \times s$ matrix of factor loadings, $f_t$ is an $s$-dimensional vector of latent random factors and the log($\cdot$) function is taken componentwise. Further, the components of $f_t$ are decomposed to similar subsets which are assumed to follow independently Gaussian AR(1) model. Jung et al. (2011) develop estimation under Poisson and Negative binomial distribution by employing efficient importance sampling and apply this methodology to numbers of trades, in 5-min intervals, for five New York Stock Exchange stocks from two industrial sectors.

Similarly, Wang and Wang (2018) assume that $E[Y_{i,t}] = E[\lambda_{i,t}] \varepsilon_{i,t}$, where $\varepsilon_{i,t}$ is the $i$'th component of a $d$-dimensional hidden process $\varepsilon_t$ such that $E[\varepsilon_{i,t}] = 1$ (see also Zeger (1988), Davis and Wu (2009) and Christou and Fokianos (2014)). This assumption implies that

$$\text{Cov}[Y_{i,t}, Y_{j,u}] = E[\lambda_{i,t}] E[\lambda_{j,u}] \text{Cov}[\varepsilon_{i,t}, \varepsilon_{j,u}],$$

for any $t, u$ and $i \neq j$ or $t \neq u$ and all $i, j$. Effectively the autocovariance function of the hidden process is identical to the autocovariance function of the standardized process $Y_{i,t}/E[\lambda_{i,t}]$. Wang and Wang (2018) assume further that $E[a_{i,t}] = \exp(c_i^T \alpha_i)$ using the previous notation. To reduce the dimensionality of hidden process $\varepsilon_t$, it is assumed to satisfy $\varepsilon_t = \Gamma f_t$ as in Jung et al. (2011) but with $s = \text{dim}(f_t)$ unknown. Correlation is taken into account by this construction since the dynamics of $f_t$ drive the time-evolution of $\varepsilon_t$. Inference proceeds in two steps: (a) pseudo-maximum-likelihood estimation for regression coefficients and (b) identification of common factor(s) utilizing eigenanalysis on a positive definite matrix.

In a related article, Zhang, Chen, and Li (2017) discuss a model which is based on the the multivariate lognormal mixture Poisson distribution (see Sec. 2.2) and allows for serial correlations by assuming that the Poisson mean vector is a latent process driven by a nonlinear autoregressive model. The authors employ Monte Carlo Expectation Maximization algorithm together with
particle filtering and smoothing methods to develop inference. Similarly, Al-Wahsh and Hussein (2020) motivated by an application concerning asthma related visits to emergency rooms, consider a hidden autoregressive process which drives the dynamics of a positively correlated bivariate time series of counts whose conditional distribution is assumed to be the multivariate Poisson distribution. The authors use a Bayesian data cloning approach to compute maximum likelihood estimators and their standard errors.

From a fully Bayesian point of view, Aktekin et al. (2018) (see also Gamerman et al. (2013)) assume that \( Y_{it} \) are independent, conditionally on univariate parameters \( \alpha_i \) and a process \( \varepsilon_t \), Poisson distributed with mean \( \alpha_i \theta_t \). The parameter \( \alpha_i \) are individual specific rates and \( \varepsilon_t \) is a common process that drives the dynamics of the observed process and satisfies \( \varepsilon_t = (\varepsilon_{t-1}/\gamma)dt \), where \( \gamma \in (0, 1) \) and \( dt \) are independent Beta random variables with suitable parameters. The authors study, in addition, a negative binomial model and they implement inference by particle learning algorithm. Another fully Bayesian approach is that of Berry and West (2020) who introduce models, within the framework of dynamic GLM (see West and Harrison (1997)), that allow use of time-varying covariates for binary and Poisson conditionally distributed time series. The recent review article by West (2020) gives further insight for Bayesian modeling of multivariate count time series. Other works, along these lines include Serhiyenko (2015), Ravishanker et al. (2014), Ravishanker et al. (2015). The previous articles and the recent work of Davis et al. (2021) give further references and list other approaches.

5 Observation-Driven Models

In this section, we discuss observation-driven models, that is processes whose dynamics are driven by past observations plus noise. A convenient model would postulate a multivariate conditional count distribution to the observed process such that likelihood inference is feasible. But the discussion in Sec. 2 illustrated the obstacles of choosing appropriate count distribution. In this section we will be studying the GLM approach. It will be argued that this framework generalizes the traditional ARMA methodology (see Shumway and Stoffer (2011) for example) to the count time series framework. Model fitting is based, in general, on quasi-likelihood inference; Godambe (1991) Heyde (1997); therefore testing, diagnostics and all type of likelihood arguments are directly applicable for this case.
5.1 Linear Models

To initiate the discussion, consider the standard VAR(1) model, but in the context of a multivariate Poisson autoregression as it is discussed next. Denote by $\mathcal{F}_t$ the $\sigma$–field generated by all past values of the process $\{Y_s, s \leq t\}$. Let $\{\lambda_t = (\lambda_{i,t}), i = 1, 2, \ldots, d, t \in \mathbb{Z}\}$ be the corresponding $d$-dimensional intensity process, vis. $\mathbb{E}[Y_t | \mathcal{F}_{t-1}] = \lambda_t$. The univariate linear autoregressive model discussed by Rydberg and Shephard (2000), Heinen (2003), Ferland et al. (2006) and Fokianos et al. (2009), among others, serves as basic building block to construct a multivariate Poisson linear VAR(1) process by defining

$$Y_{i,t} | \mathcal{F}_{t-1} \sim \text{marginally Poisson}(\lambda_{i,t}), \quad \lambda_t = \omega + B_1 Y_{t-1}, \quad (9)$$

where $\omega$ is a $d$-dimensional vector and $B_1$ is a $d \times d$ unknown matrices. The elements of $\omega$ and $B_1$ are assumed to be positive such that $\lambda_{i,t} > 0$, for all $i$ and $t$. It is instructive to consider (9) in more detail. For the simple case $d = 2$, it implies that

$$\lambda_{1t} = \omega_1 + b_{11} Y_{1,t-1} + b_{12} Y_{2,t-1},$$

$$\lambda_{2t} = \omega_2 + b_{21} Y_{1,t-1} + b_{22} Y_{2,t-1},$$

where $\omega_i$ is the $i$’th element of $\omega$ and $b_{ij}$ is the $(i, j)$th element of $B_1$. Then setting $b_{12} = 0$ implies that the past values of $Y_{2,t}$ do not affect the evolution of $Y_{1,t}$. Similarly, $b_{21} = 0$ shows that past values of $Y_{1,t}$ do not affect the evolution of $Y_{2,t}$. These arguments extend naturally to the case $d > 2$.

Generating data using (9) is accomplished by imposing (1), for instance; see Liu (2012) and Pedeli and Karlis (2013b) for some examples. But the discussion in Sec. 2 shows the challenges of fitting model (9) to data assuming a bivariate Poisson (and more generally multivariate Poisson) distribution. To overcome this challenge we appeal to copulas–but other suitable p.m.f are applicable–by introducing a joint distribution constructed by utilizing the data generating process described in Sec. 2.3, but taking into account (9). The algorithm is repeated for completeness of presentation. Let $\lambda_0$ be a starting value and assume that $\omega, B_1$ are given. Then

1. Let $U^l = (U_{1,l}, \ldots, U_{d,l})$ for $l = 1, 2, \ldots, K$, be a sample from a $d$-dimensional copula $C(u_d, \ldots, u_d)$. Then $U_{i,l}, l = 1, 2, \ldots, K$ follow marginally the uniform distribution on $(0,1)$, for $i = 1, 2, \ldots, d$.

2. Consider the transformation $X_{i,l} = -\log U_{i,l}/\lambda_{i,0}, \quad i = 1, 2, \ldots, d$. Then, the marginal distribution of $X_{i,l}, l = 1, 2, \ldots, K$ is exponential with parameter $\lambda_{i,0}, i = 1, 2, \ldots, d$. 

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3. If $X_{i,1} > 1$, set $Y_{i,0} = 0$, otherwise $Y_{i,0} = \max \left\{ K : \sum_{l=1}^{K} X_{i,l} \leq 1 \right\}$, $i = 1, 2, \ldots, d$. Then $Y_0 = (Y_{1,0}, \ldots, Y_{d,0})^T$ is marginally a realization of a Poisson process with parameter $\lambda_0$.

4. Use (9) to obtain $\lambda_1$.

5. Return back to step 1 to obtain $Y_1$, and so on.

Figure 2: Lag 1 autocorrelation matrix of a bivariate count time series as a function of the Gaussian copula parameter $\rho$. Results are based on 5000 data points.

Figure 2 shows plots of the sample autocorrelation matrix $\text{Cov}(Y_t, Y_{t+1})$ where $Y_t$ is a bivariate count time series generated using the above algorithm and model (9) with a Gaussian copula having parameter $\rho$. The plot reinforces the point raised in Sec. 2.3 where it is noted that interpretation of the instantaneous correlation found in data is related to the correlation of the vector of waiting times and should be done with care. This simple example shows that, as $\rho$ varies between -1 and 1, all the lag 1 correlation functions do not exceed 0.45. But the plot also shows that this approach takes into account properly the correlation found in data because by varying the parameter $\rho$ we obtain different type of autocorrelation matrices.
It is worth pointing out that this approach is different when compared to the work of Heinen and Rengifo (2007). These authors replace the original counts by employing the continued extension method of Denuit and Lambert (2005), as it was discussed in Sec. 2.3. The continued extension method of Heinen and Rengifo (2007) has been investigated in a simulation study by Nikoloulopoulos (2013b). Other copula-based models for multivariate count distributions with serial dependence are discussed in Joe (1997, Ch. 8).

From now on, define (9) as vector Integer Autoregressive Conditional Heteroscedastic model of order 1, i.e. V-INARCH(1) model. The reason for choosing this terminology will be explained below. Following identical arguments as those of Fokianos (2012, Sec. 3.1), model (9) is rewritten as

\[ Y_t = \lambda_t + (Y_t - \lambda_t) = \omega + B_1 Y_{t-1} + \zeta_t, \]  

(10)

which shows that the values of \( Y_t \) depend on \( Y_{t-1} \) plus the white noise sequence \( \{\zeta_t\} \). Indeed, if \( \{Y_t\} \) is assumed to be stationary, then it is easily shown that \( E[\zeta_t] = 0 \), \( \text{Var}[\zeta_t] = E[\Sigma_t] \), where \( \Sigma_t = \text{Var}[Y_t | F_{t-1}] \) and \( \text{Cov}(\zeta_i, \zeta_{t+k}) = 0 \) for \( k \in \mathbb{Z} \). The matrix \( \Sigma_t \) is not determined as in the univariate case whereby the conditional variance of \( Y_t \) is \( \lambda_t \) (with some abuse of notation). Model (9) implies that the diagonal elements of \( \Sigma_t \) are \( \lambda_{i,t} \) but the off-diagonal entries depend on the copula employed to generate data but are unknown because of the contemporaneous correlation between \( Y_{i,t} \) and \( Y_{j,t}, i \neq j \).

Because of the assumed stationarity, (9) shows that \( E[Y_t] = \omega + B_1 E[Y_{t-1}] \). Then \( E[Y_t] = (I_d - B_1)^{-1}\omega \), provided that \( \rho(B_1) < 1 \), where \( \rho(.) \) denotes the spectral radius of a matrix. Furthermore, (see Lütkepohl (2005, Ch. 2))

\[ Y_t = \omega + B_1 Y_{t-1} + \zeta_t \]
\[ = \omega + B_1(\omega + B_1 Y_{t-2} + \zeta_{t-1}) + \zeta_t \]
\[ = (I_d + B_1)\omega + B_1^2 Y_{t-2} + B_1 \zeta_{t-1} + \zeta_t \]
\[ = \cdots \cdots \cdots \cdots \]
\[ = (1 + B_1 + B_1^2 + \cdots + B_1^t)\omega + \sum_{i=0}^{t} B_1^i \zeta_{t-i}. \]

(11)

Therefore, as in the case of ordinary VAR(1) model, assuming that \( \rho(B_1) < 1 \), we obtain (in mean square sense) by (11) and for large \( t \), the useful one-sided infinite order moving average representation

\[ Y_t = (I_d - B_1)^{-1}\omega + \sum_{i=0}^{\infty} B_1^i \zeta_{t-i}. \]

(12)
In addition
\[ \Gamma_Y(h) \equiv \text{Cov}(Y_t, Y_{t+h}) = \sum_{i=0}^{\infty} B_i^{i+h-1} E[\Sigma_i](B_i)^T, \quad h \geq 0, \]

Several other results are readily available because of \((10)\), see Lütkepohl (2005) or Tsay (2014).

Generalizations of \((9)\), such as the V-INARCH\((p)\) model

\[ \lambda_t = \omega + \sum_{i=1}^{p} B_i Y_{t-i}, \]

or the vector V-INGARCH\((p, q)\) (where “G” stands for Generalized)

\[ Y_{i,t} | \mathcal{F}_{t-1} \text{ is marginally Poisson}(\lambda_{i,t}), \quad \lambda_t = \omega + \sum_{i=1}^{p} B_i Y_{t-i} + \sum_{j=1}^{q} A_j \lambda_{t-j}, \quad (13) \]

where \((A_j)_{j=1}^{p}, (B_i)_{i=1}^{q}\) are \(d \times d\) unknown matrices and all the elements of \(\omega, (A_j)_{j=1}^{p}, (B_i)_{i=1}^{q}\) are positive such that \(\lambda_{i,t} > 0\), for all \(i\) and \(t\), are studied along the previous arguments. The abbreviation "INGARCH" for model \((13)\) just indicates its structural connection to ordinary GARCH model, Bollerslev (1986), because each component of the vector process \(Y_t\) is distributed as a Poisson random variable. But the mean of a Poisson random variable equals its variance; therefore the structure of \((13)\) bears some resemblance to that of a multivariate GARCH model, see Francq and Zakoïan (2010). Though the term "V-INGARCH" does not reflect accurately the true data generating process it will be used as a generalization of the terminology introduced for univariate models by Ferland et al. (2006).

It is proved that \((13)\) is a VARMA\((\max(p, q), q)\) process. Recall that \(\zeta_t = Y_t - \lambda_t\). Then assuming first order stationarity of \(Y_t\) and taking expectations on both sides of \((13)\), we obtain that \(E[Y_t] = (I_d - \sum_{i=1}^{p} A_i - \sum_{j=1}^{q} B_j)^{-1} \omega\), provided that \(\rho(\sum_{i=1}^{p} A_i + \sum_{j=1}^{q} B_j) < 1\). Then, by manipulating \((10)\) but for model \((13)\), it is easily shown that

\[ (Y_t - E[Y_t]) = \sum_{i=1}^{\max(p,q)} \left( A_i + B_i \right) (Y_{t-i} - E[Y_t]) + \zeta_t - \sum_{j=1}^{q} A_j \zeta_{t-j}, \quad (14) \]

where we set \(A_i = 0_d\) if \(q < p\) for \(i = q + 1, \ldots, p\) or \(B_i = 0_d\) if \(q > p\) for \(i = p + 1, \ldots, q\). In the case of \(p = q = 1\) the one-sided MA\((\infty)\) is given by \(Y_t = \mu + \sum_{j=0}^{\infty} \Phi_j \zeta_{t-j}\) with \(\Phi_0 = I_d\) and \(\Phi_j = (A_1 + B_1)^{j-1} B_1\), for \(j \geq 1\), a fact that shows for any \(h > 0\)

\[ \Gamma_Y(h) = \sum_{j=0}^{\infty} (A_1 + B_1)^{i+j-1} B_1 E[\Sigma_i] B_1^T (A_1^T + B_1^T)^{i+h-1}, \]

by using properties of the linear multivariate processes.

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5.2 Log-linear Models

The log-linear model we consider is the multivariate analogue of the univariate log-linear model proposed by Fokianos and Tjøstheim (2011). In a more general form, assume that for each \( i = 1, 2, \ldots, d \)

\[
    Y_{i,t} | \mathcal{F}_{t-1} \text{ is marginally Poisson}(\lambda_{i,t}), \quad \nu_t = \omega + \sum_{i=1}^{p} B_i \log(Y_{i-1} + 1_d) + \sum_{j=1}^{q} A_j \nu_{t-j} \tag{15}
\]

where \( \nu_t \equiv \log \lambda_t \) is defined component wise (i.e. \( \nu_{i,t} = \log \lambda_{i,t} \)) and \( 1_d \) denotes the \( d \)-dimensional vector which consists of ones. For this model, there is no need to impose any constraints on the matrix coefficients. Additionally, the log-linear model accommodates covariates much easier than the linear model which requires that any such inclusion has to satisfy \( \lambda_{i,t} > 0 \). In the case of (15) though, if \( Z_t \) is a covariate vector of dimension \( d \), then the second equation of (15) becomes

\[
    \nu_t = \omega + \sum_{i=1}^{p} B_i \log(Y_{i-1} + 1_d) + \sum_{j=1}^{p} A_j \nu_{t-j} + CZ_t \text{ for a } d \times d \text{ matrix } C.
\]

Interpretation of model parameters for the log-linear model (15) is identical to the case of the linear model but in terms of the vector process \( \nu_t \).

It is more challenging to derive formulas for the mean and autocovariances of model (15). However, some approximations are possible by considering the process \( W_t \equiv \log(Y_t + 1_d) \). Indeed, define now \( \zeta_t = W_t - \nu_t \) and use the results of Fokianos and Tjøstheim (2011); Fokianos et al. (2020) to see that \( W_t \) is approximated by a VARMA model of the form

\[
    (W_t - E[W_t]) = \max(p,q) \sum_{i=1}^{p} (A_i + B_i) (W_{t-i} - E[W_t]) + \zeta_t - \sum_{j=1}^{q} A_j \zeta_{t-j},
\]

similar to (14). An approximate formula for the sequence of autocovariance matrices for \( W_t \) (but not for \( Y_t \)) is then derived but with suitable adjustments. This representation should be used cautiously because it is approximate and it can be applied for developing a model for \( W_t \) (using standard time series methodology) but not for \( Y_t \).

**Remark 5.1** Stability conditions for model (13) have been developed by Liu (2012) (under the framework of multivariate Poisson distribution (1)) and Fokianos et al. (2020) under the copula construction as outlined in Sec. 2.3 for the case \( p = q = 1 \). Recently Debaly and Truquet (2019) have improved these conditions again considering the copula-based data generating process, as outlined before. Without introducing any further notation, we note that the condition \( \rho(\sum_i A_i + \sum_j B_j) < 1 \) guarantees stability of the process. For the log-linear model (15) the desired conditions are more complicated; see Fokianos et al. (2020) who consider the case \( p = q = 1 \) and prove that
either $\|A_1\|_2 + \|B_1\|_2 < 1$ or $\|A_1\|_1 + \|B_1\|_1 < 1$, where $\|A\|_d = \max_{\|x\|_d=1} \|Ax\|_d$, guarantee ergodicity of the process. Related stability conditions are discussed in Debaly and Truquet (2019).

The main notions used to derive such conditions are those of Markov chain theory (Meyn and Tweedie (1993)), weak dependence (Doukhan and Louhichi (1999), Dedecker et al. (2007)) and convergence of backward iterations of random maps (Wu and Shao (2004)). Following the discussions of Neumann (2011) and Tjøstheim (2012, 2015), the main difficulty is that the process itself consists of integer valued random variables; however the mean process takes values on the positive real line and therefore it is quite challenging to prove stability of the joint process (see also Andrews (1984)). The study of theoretical properties of univariate models was initiated by the perturbation method suggested in Fokianos et al. (2009) and was further developed in Neumann (2011) (using the notion of $\beta$-mixing), Doukhan et al. (2012) (weak dependence approach), Woodard et al. (2011) and Douc et al. (2013) (Markov chain theory without irreducibility assumptions) and Liu (2012), Wang et al. (2014) (based on the theory of $e$-chains).

**Remark 5.2** Models (13) and (15) are related to Hawkes processes (Hawkes (1971a,b)) because they can be obtained by suitable discretization of the continuous time process. This connection has been explored in detail by Kirchner (2016) for univariate models in the context of INAR($\infty$) process which in turn is related to the linear model (see Ferland et al. (2006)). Hawkes processes have been found useful in modeling and inference in several scientific areas—see Embrechts et al. (2011) and Bacry et al. (2015) among others.

### 5.3 Quasi-Likelihood Inference

Suppose that $\{Y_t, t = 1, 2, \ldots, n\}$ is an available sample from a count time series and for the sake of presentation assume model (13) for $p = q = 1$. Inference is analogously developed to the case of log-linear model and for $p, q > 1$. Denote by $\theta = (d^T, \text{vec}^T(A_1), \text{vec}^T(B_1))$, $\dim(\theta) \equiv \kappa = d(1+2d)$. Following Fokianos et al. (2020), the estimation problem is approached by employing the theory of estimating functions. Consider the following conditional quasi-likelihood function, given a starting value $\lambda_0$, for the parameter vector $\theta$,

$$L(\theta) = \prod_{t=1}^n \prod_{i=1}^d \exp\left(-\lambda_{i,t}((\theta))\frac{\lambda_{y_{i,t}}(\theta)}{y_{i,t}!}\right),$$

which is identical to consider (13) assuming independence among time series. This is a strong assumption yet it simplifies computation of estimators and their respective standard errors. At the
same time, it guarantees consistency and asymptotic normality of the maximizer. Furthermore, the
dependence structure in (13) and (15) is taken into account through because of the dependence of
the likelihood function on the matrices $A_1$ and $B_1$; see Fokianos et al. (2020) for more. The quasi
log-likelihood function is equal to

$$l(\theta) = \sum_{t=1}^{n} \sum_{i=1}^{d} \left( y_{i,t} \log \lambda_{i,t}(\theta) - \lambda_{i,t}(\theta) \right).$$

We denote by $\hat{\theta} \equiv \arg \max_{\theta} l(\theta)$, the QMLE of $\theta$. The score function is given by

$$S_n(\theta) = \sum_{t=1}^{n} \sum_{i=1}^{d} \left( \frac{y_{i,t}}{\lambda_{i,t}(\theta)} - 1 \right) \frac{\partial \lambda_{i,t}(\theta)}{\partial \theta} = \sum_{t=1}^{n} \frac{\partial \lambda_t^T(\theta)}{\partial \theta} D_t^{-1}(\theta)(Y_t - \lambda_t(\theta))$$

where $\frac{\partial \lambda_t}{\partial \theta^T}$ is a $d \times \kappa$ matrix and $D_t$ is the $d \times d$ diagonal matrix with the $i$'th diagonal element
equal to $\lambda_{i,t}(\theta)$, $i = 1, 2, \ldots, p$. Furthermore

$$\frac{\partial \lambda_t}{\partial \theta^T} = \frac{\partial \lambda_t}{\partial \text{vec}^T(A_1)} = (\lambda_{t-1} \otimes I_d)^T + A_1 \frac{\partial \lambda_{t-1}}{\partial \text{vec}^T(A_1)};$$

$$\frac{\partial \lambda_t}{\partial \text{vec}^T(B_1)} = (Y_{t-1} \otimes I_d)^T + A_1 \frac{\partial \lambda_{t-1}}{\partial \text{vec}^T(B_1)}.$$  

The Hessian matrix is given by

$$H_n(\theta) = \sum_{t=1}^{n} \sum_{i=1}^{d} \left( \frac{y_{i,t}}{\lambda_{i,t}(\theta)} \frac{\partial \lambda_{i,t}(\theta)}{\partial \theta} \frac{\partial \lambda_{i,t}(\theta)}{\partial \theta^T} - \frac{1}{\lambda_{i,t}(\theta)} \frac{\partial^2 \lambda_{i,t}(\theta)}{\partial \theta \partial \theta^T} \right).$$

Therefore, the conditional information matrix is equal to

$$G_n(\theta) = \sum_{t=1}^{n} \frac{\partial \lambda_t^T(\theta)}{\partial \theta} D_t^{-1}(\theta) \Sigma_t(\theta) D_t^{-1}(\theta) \frac{\partial \lambda_t(\theta)}{\partial \theta^T},$$

where the matrix $\Sigma_t(\cdot)$ denotes the true covariance matrix of the vector $Y_t$. In case that the process
\{Y_i\} consists of uncorrelated components then $\Sigma_t(\theta) = D_t(\theta)$. In the case that of $A_1$ being a
diagonal matrix then $\hat{\theta}$ is computed by equation by equation using existing software.

Under suitable conditions, Fokianos et al. (2020) show that

$$\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, H^{-1}GH^{-1})$$

where the matrices $H$ and $G$ are defined by the limits (in probability) of (17) and (18), respectively.
The same result is true for the log-linear model (15); details are omitted. To estimate the copula
parameter, it is desirable to compare the conditional distribution of $Y_t \mid \lambda_t$ to that of $Y_{t^*} \mid \lambda_t$, where
$Y_{t^*}$ is a count time series generated by a suitable choice of a copula. There are several ways of
comparing such distributions and this topic is still under investigation. In Fokianos et al. (2020) an initial approach, based on the newly developed concept of local Gaussian correlation (see Berentsen et al. (2014)) was shown to be satisfactory. But the problem of estimating the copula parameter remains unexplored; see Debaly and Truquet (2021) for recent progress in the framework of mixed time series models.

**Remark 5.3** Equation (16) motivates a more general framework that can be applied to the analysis of multivariate count time series modes. A natural generalization, is to consider the following estimating functions

\[
S_v(\theta) = \sum_{t=1}^{n} \frac{\partial \lambda_t^T(\theta)}{\partial \theta} V_t^{-1}(\rho, \lambda_t(\theta))(Y_t - \lambda_t(\theta)),
\]

(19)

where the notation is completely analogous to (16) and \( V_t(\rho, \lambda_t(\theta)) \) is a \( d \times d \) "working" conditional covariance matrix which depend upon the process \( \{\lambda_t\} \) and possibly some other parameters \( \rho \). Several choices for the working conditional covariance matrix are available in the literature; we list some possibilities. If \( V = I_d \), then (19) corresponds to a least squares minimization problem for estimating \( \theta \). If \( V = D_t \) then we obtain (16). More generally, the choice

\[
V(\rho, \lambda) = \begin{pmatrix}
\lambda_{1,t} & \rho_{12}\sqrt{\lambda_{1,t}\lambda_{2,t}} & \cdots & \rho_{1d}\sqrt{\lambda_{1,t}\lambda_{d,t}} \\
\rho_{12}\sqrt{\lambda_{1,t}\lambda_{2,t}} & \lambda_{2,t} & \cdots & \rho_{2d}\sqrt{\lambda_{2,t}\lambda_{d,t}} \\
\cdots & \cdots & \cdots & \cdots \\
\rho_{1p}\sqrt{\lambda_{1,t}\lambda_{p,t}} & \rho_{2p}\sqrt{\lambda_{2,t}\lambda_{p,t}} & \cdots & \lambda_{d,t}
\end{pmatrix}
\]

yields to a constant conditional correlation type of model for multivariate count time series, see Teräsvirta et al. (2010), among others. This topic deserved further research; a possible method might rely on the Francq and Zakoïan (2016) who consider estimation of multivariate volatility models equation by equation.

5.4 High-Dimensional Models

The advent of technology to economics, biological and social sciences, has given rise to interesting and exciting application of high-dimensional time series models. Some examples include multiple transactions of several stocks, gene regulatory network reconstruction from time course gene expression data, brain connectivity analysis and others. Such applications have revived methodology which is useful for the purpose of modeling and inference. Though the concept of sparsity attracted a lot of attention over the last two decades and proved its usefulness for modeling and inference (see Hastie et al. (2015) among others) research on time series methods is still in progress. Because such
a review is out of the scope of this article, we refer the reader to recent work by Basu and Matteson (2021) who provide an overview about several methods, in the context of large autoregressions and stochastic regression, and Hallin et al. (2020) who provide a concise overview of factor models.

In the rest, we outline a recent methodological contribution related to inference for high-dimensional count time series observation driven models; see Hall et al. (2019). Those authors assume the pure autoregressive model

\[ Y_{i,t} | Y_{t-1} \text{ Poisson}(\lambda_{i,t}), \quad \nu_t = \omega + B_1 Y_{t-1} \tag{20} \]

where \( \nu_t \) defined as in eq. (15). The constant term \( \omega \) is assumed to be known and the \( d \times d \) matrix \( B_1 \) belongs to a compact subset, say \( B \) of the set of all \( d \times d \) matrices with real elements such that \( \|B_1\|_0 \equiv \sum_{l=1}^d \sum_{m=1}^d 1(|B_1^{(l,m)}|) \leq s\). The notation \( 1(.) \) denotes the indicator function. Comparing \( \text{(20)} \) to \( \text{(15)} \) we note that the restriction of \( B_1 \) belonging to a compact set assures the stability of the joint process \( Y_t \) because the components of \( Y_{t-1} \) are unbounded, in general. So when some regression coefficients are positive the conditional expectation of the response given the past of the process tends to grow in an exponential rate. To estimate \( B_1 \) when \( d \) is much larger than \( n \), the authors propose the \( l_1 \) regularized QMLE defined by

\[ \hat{B}_1 = \arg \max_{B_1 \in B} \frac{1}{n} \sum_{t=1}^n \sum_{i=1}^d \left( y_{i,t} \nu_{i,t}(B_1) - \exp(\nu_{i,t}(B_1)) \right) + \hat{\lambda} \sum_{l=1}^d \sum_{m=1}^d |B_1^{(l,m)}|, \]

where \( \hat{\lambda} \) is a regularization parameter and \( \nu_{i,t}(.) \) is defined by \( \text{(20)} \). The authors study mean square error bounds for the proposed estimators and show that they are closely connected with the bounds obtained in the \( n \) Gaussian case. Further work along these lines was developed (for AR(\( p \)) type models) by Pandit et al. (2020) where the interested reader can obtain more references.

In another related work, Armillotta and Fokianos (2021) studied network autoregressive models for high-dimensional count time series with a fixed neighborhood structure. Assessing the effect of a network to multivariate time series processes has attracted considerable attention over the last years. In particular, Zhu et al. (2017) proposed a Network Autoregressive model (NAR) and studied least squares inference under two asymptotic regimes (a) with increasing time sample size \( n \) and fixed network dimension \( d \) and (b) with both \( n, d \) increasing. These ideas are extended to high-dimensional count time series by Armillotta and Fokianos (2021) who propose linear and log-linear Poisson network autoregressions (PNAR) for count processes and by establishing the two related types of asymptotic inference for the QMLE as discussed before.
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