Fractal Droplets in Two Dimensional Spin Glasses

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The two-dimensional Edwards-Anderson model with Gaussian bond distribution is investigated at $T = 0$ with a numerical method. Droplet excitations are directly observed. It turns out that the averaged volume of droplets is proportional to $v^D$ with $D = 1.80(2)$ where $v$ is the spanning length of droplets, revealing their fractal nature. The exponent characterizing the $l$ dependence of the droplet excitation energy is estimated as $\theta_D = -0.42(4)$, clearly different from the stiffness exponent for domain wall excitations.

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“Droplet” is a well-known and useful concept in the study of critical phenomena [1]. The droplet argument for spin glasses [2,3] is based on this concept and is one of important working hypotheses in the field. Its validity is being actively discussed, in particular, for the low-temperature phase in three dimensions [4]. Droplets are defined as collective excitations from some pure state below the transition temperature. In the case of a typical spin-glass model in two dimension, however, the system is believed to be critical right at the zero-temperature. Therefore, it is not clear, a priori, if we can treat droplets defined at zero-temperature in two dimensions in the same fashion as we treat the droplets below the critical point in three dimensions. In two dimensions, most of the critical indices can be derived from a single exponent $\theta_D$ that characterizes the size-dependence of the droplet excitation energy, according to the droplet argument or other similar arguments such as the domain wall renormalization group theory [5]. For example, $-\theta_D$ was expected to be equal to $y_D = 1/\nu$ while $-\theta_S$ was identified with $y_T [1]$, where $\theta_D$ is the stiffness exponent which characterizes the system size dependence of the domain wall excitation energy. A deviation from this standard story for two-dimensional spin glass models was first reported based on numerical calculation of magnetization at zero-temperature with finite field [7], in which we found that the thermal exponent $y_T$ does not agree with $-\theta_S$. Several subsequent numerical works [8,9] confirmed this result. Thus the equivalence among three exponents, $y_T$, $-\theta_S$ and $-\theta_D$, should now be a subject to a close re-examination. As for the equivalence between $y_T$ and $-\theta_D$, so far no evidence of its violation has been found. However, this may be simply because identifying droplets is technically so difficult that nobody has actually succeeded in direct observation of droplets large enough to discuss asymptotic behavior. In fact, if one adopts the original definition [2] of droplets and apply a simple combinatorial optimization algorithm, it would be impossible to obtain droplets larger than a few lattice spacings.

In this letter, we focus our attention on the EA model on a square lattice

$$\mathcal{H} = -\sum_{(ij)} J_{ij} S_i S_j$$

where $J_{ij}$'s are quenched Gaussian random variables with the mean value of zero and the standard deviation of $J$. We take $J$ as the unit of energy throughout the present letter. It should be noted that the one-parameter scaling which was derived from various “pictures” such as the droplet picture and the domain wall renormalization group argument can be also derived within the framework of standard finite size scaling theory without any assumptions or pictures. We start with the following standard form for the partition function,

$$\log \hat{Z}(T, H, L) \sim f(T L^{y_T}, H L^{y_H})$$

(1)

where $\hat{Z}, T, H, L$ are the singular part of the partition function, temperature, magnetic field, and system size, respectively. We hold [1] to be a defining equation of the exponent $y_T$. One of the two scaling exponents can be fixed by using the fact that the magnetization at zero temperature and zero magnetic field is proportional to $L^{d/2}$ because of the absence of non-trivial degeneracy. This leads to $y_T = y_t + d/2$. In previous works, we measured the magnetization $M(T = 0, H, L)$ [1] and the spin-glass susceptibility $\chi_{SG}(T = 0, L)$ [10] which are both derivatives of $\log \hat{Z}$ with respect to $T$ and $H$ multiplied by some known factors. By matching the numerical results with the asymptotic form derived from [1], we obtained $y_T \simeq 0.48$ in both calculations.

On the other hand, it is often assumed, or for some models proved, that the excitation energy $E_W$ of domain walls induced by twisting the system has the same scaling behavior as the singular part of the free energy. This means $\theta_S = -y_t$ where $\theta_S$ is defined by

$$E_W(L) \propto L^{\theta_S} \quad (T = 0).$$

In the early stage of the study on the present issue, the domain wall energy was measured for various sizes [11] with the hope of obtaining an estimate of $y_t$. The stiffness
exponent was estimated as \( -\theta_S \simeq 0.29 \). This result was reconfirmed later by other groups with larger scale computations. Therefore, it is now rather clear that \( -\theta_S \) and \( y_i \) are two different exponents.

Fisher and Huse defined a droplet of the scale \( \lambda \) including a given site \( i \) as a cluster of spins (with \( i \) among them) with the smallest excitation energy that contains more than \( \lambda^d \) and less than \( (2\lambda)^d \) spins. The basis of the argument then is the following scaling form that describes the excitation-energy distribution of droplets of the scale \( \lambda \):

\[
P_\lambda(E_\lambda) = \frac{1}{\Upsilon^{\theta_D}} \tilde{P}\left( \frac{E_\lambda}{\Upsilon^{\theta_D}} \right)
\]

where \( \Upsilon \) is some constant, \( \theta_D \) is the droplet exponent, and \( \tilde{P}(x) \) is the scaling function which is continuous and non-vanishing at \( x = 0 \). Since the droplet “size”, \( \lambda \), is defined to be proportional to \( \text{(volume)}^{1/d} \), it is not necessarily a spanning length of droplets because the volume of a droplet may in general have a non-trivial fractal dimension. In fact, in what follows we demonstrate that this is the case, i.e., a typical large droplet in two dimensions occupies only an infinitesimal fraction of the volume of the box that contains it. Therefore, for a given droplet, we have at least two different length scales, \( \lambda \) and the spanning length. We refer hereafter to the fractal dimension of the droplet volume as \( D \). Then, we have the relation

\[
V = \lambda^d \propto l^D
\]

where \( l \) is the spanning length of the droplet.

Since the above-mentioned definition of droplets is inconvenient from computational point of view, we have adopted the following alternative definition. First we consider an \( L \times L \) system with free boundary condition. We define a droplet of scale \( L \) as the cluster of spins that has the smallest excitation energy among those which contain the central spin and contains no spins on the boundary.

If we assume the conventional droplet argument, the central spin is surrounded by droplets (in the original definition) of various scales. Since perimeters of smaller droplets are far from the boundary of the system they must be relatively free from the influence of the constraint imposed on the boundary. Therefore, smaller droplets in a finite system should have the same scaling properties as they would in an infinite system. On the other hand, the fixed boundary imposes strong restrictions upon the shapes that larger droplets can take. As a result, they have larger excitation energy than they would in an infinite non-restricted system. Therefore, a crossover size of droplets must exist and it separates the region of smaller sizes in which one observes a correct asymptotic behavior from the other region of larger sizes that is strongly affected by the boundary.

In the spirit of effective mutual independence among the droplets of different scales discussed in \( \lambda \), it is natural to expect that the crossover size corresponds to the largest droplet that can fit into a box of \( L \times L \), i.e., the droplet whose spanning length is of the order of \( L \). Then, considering the fact that larger droplets tend to have smaller excitation energy, one may also expect that the spanning length of the droplet identified by the procedure described above would be proportional to \( L \). We here define \( \theta_D \) by the following \( L \) dependence of the average excitation energy of the droplets in our definition,

\[
|E_D(L)| \propto L^{\theta_D},
\]

where \([\cdots]\) stands for the average over samples. Comparing this to the original definitions of the droplet size and droplet exponent, we obtain

\[
\theta'_D = \frac{D}{d} \theta_D.
\]

because \( L \) is proportional to the spanning length of the droplet.

In order to observe droplets in the new definition, we first compute the ground state with free boundary condition, and take it as the reference spin configuration. Then we compute the ground state with the constraint that the spins on the boundary are to be fixed as they are in the reference state while the central spin is to be fixed opposite, thereby forcing a cluster of spins including the central spin to flip. For a system with the free boundary condition, polynomial-time optimization algorithms are available whereas for the systems with constraints no such algorithm is known. In fact, the two dimensional spin glass problem with general constraints
has been proven to be NP hard [12]. Therefore, we have employed the replica optimization [16], which is a heuristic optimization algorithm based on the idea of renormalization group. The details of the algorithm are described elsewhere [16].

Since the algorithm is only heuristic, we need to estimate the rate of obtaining false ground states and how much such states may affect the average values of various quantities of interest. For this purpose, we compared our results with exact solutions when they are available. We have actually found no error with the settings of algorithmic parameters that are used in the computation presented in this letter. We detected some errors only when we loosened the termination condition. Even in that case, the rate and the amount of the errors turned out to be so small that their effect is well within the final statistical errors. Therefore, we believe that the errors are negligible even for the largest systems \((L = 49)\), for which no exact result is available for comparison.

In Fig. 1, we present 16 randomly chosen droplets in the system of size \(L = 49\). We can see that the size of the droplets varies depending on the sample. Some of them contains only one spin. This reflects the fact that \(|\theta_D|\) is small and the excitation energy does not very strongly depend on the droplet size. Frequency of having small droplets decreases as a function of the system size. On the other hand, we also see that larger droplets whose size is comparable even to the system itself occupy a considerable fraction in the whole population of droplets. In these larger droplets, many fingers and overhangs can be observed, which already suggests the fractal nature of the droplets.

In order to check that the spanning length of a typical droplet is really proportional to the system size as we expect, we compute two length scales \(l_0\) and \(l_2\). The length \(l_0\) is the spanning length itself, i.e., the difference in \(x\)-coordinates of the left-most site and that of the right-most site in a droplet, assuming that the \(x\)-axis stretches from left to right. On the other hand, the length \(l_2\) is defined as

\[
l_2 = \sqrt{\langle (x - \langle x \rangle)^2 \rangle}.
\]

Here,

\[
\langle x^n \rangle = \sum_{i \in \text{droplet}} x_i^n / \sum_{i \in \text{droplet}} 1
\]

where \(x_i\) is the \(x\)-coordinate of the site \(i\). In Fig. 2, these two length scales averaged over many samples are shown as a function of the system size \(L\). A correction to scaling can be observed for small system sizes \(L \leq 17\). In particular the slope for \(l_0\) may seem slightly larger than unity at a first glance. However, since the slope cannot asymptotically exceed unity for an obvious reason, we consider that \(l_0\) is proportional to \(L\) in the large \(L\) limit. The linearity for \(l_2\) is much better and indicates proportionality to \(L\).

The averaged length, \(P\), of the boundary of the droplet and the averaged volume, \(V\) are also plotted in Fig. 2. The linearity in the logarithmic scale is rather good for both \(P\) and \(V^{1/2}\) and both of them have slopes different from unity. For \(P\), we obtain \(P \propto L^{D_s}\), with the surface fractal dimension

\[
D_s = 1.10(2).
\]

For \(V\), we estimate the fractal dimension as

\[
D = 1.80(2). \tag{4}
\]

The fractal dimension \(D\) of droplets is certainly smaller than \(d = 2\). Thus, we conclude that the droplets at the critical point \(T = 0\) have fractal nature in the volume as well as in the perimeter.

Finally, we measure the droplet excitation energy \(E_D(L)\). Similar to the size and the shape, the droplet excitation energy has a broad distribution. When rescaled with the average value \(E_D(L)\), the histograms of excitation energies for various system sizes fit on top of each other, showing the validity of the form (6) (See Fig. 3). In addition, we observe that the scaling function \(\tilde{P}(X)\) has a non-vanishing value at \(X = 0\), satisfying a necessary condition for the droplet argument to be valid.

Since the average droplet excitation energy does not strongly depend on the system size, it is subject to relatively large correction to scaling and the estimate of \(\theta_F\) can be affected by the deviation from the asymptotics in small \(L\) region. Therefore, the estimate depends on which we choose among the above-mentioned length scales that differ from each other for small sizes. When
log $E_D$ is plotted against log $L$ the slope is estimated to be $\theta'_D \sim -0.42$, whereas it becomes $-0.38$ and $-0.45$ when $l_0$ and $l_2$, respectively, are chosen instead of $L$. These three plotings are shown in Fig. 4 together with straight lines with slopes of corresponding estimates of $\theta'_D$. From these results we quote the following value as the estimate of the droplet exponent $\theta'_D$:

$$\theta'_D = 0.42(4).$$

Assuming (3) and using the value of $D$ in (4), we obtain the following estimate of $\theta_D$,

$$\theta_D = 0.47(5)$$

in good agreement with previous estimates of $y_t$ such as $y_t = 0.48(1)$ [8].

To summarize, we have generated droplets in two dimensions for the EA model with Gaussian bond distribution. We have found the droplets non-compact in contrast to what is assumed in the conventional droplet argument for higher dimensions. The exponent $-\theta'_D$ has been estimated to be slightly smaller than the thermal exponent in the absolute value, and this difference can be well understood within the framework of the finite-size scaling and the standard droplet argument, by taking into account the fractal nature of droplets. It is rather surprising that the droplet argument seems to be valid for two dimensional critical behavior in spite of the qualitative difference between fractal droplets at the critical point shown in the present letter and the off-critical, presumably compact, droplets in higher dimensions. The disagreement between the droplet exponent and the stiffness exponent also remains puzzling. At least, however, the present result suggests that a domain wall of a $L \times L$ system may not necessarily be considered as an object of scale $L$ because of the existence of different ways for measuring the size of the same object. In other words, at present we do not know which way of measuring we should use for comparing the size of a domain wall and that of a droplet.

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