Quantizing Yang–Mills Theory: From Parisi-Wu Stochastic Quantization to a Global Path Integral

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Based on a generalization of the stochastic quantization scheme we recently proposed a generalized, globally defined Faddeev-Popov path integral density for the quantization of Yang-Mills theory. In this talk first our approach on the whole space of gauge potentials is shortly reviewed; in the following we discuss the corresponding global path integral on the gauge orbit space relating it to the original Parisi-Wu stochastic quantization scheme.

1. MATHEMATICAL SETTING

It is our aim to discuss a globally valid path integral procedure for the quantization of Yang–Mills theory based on a recently introduced generalization of the Parisi–Wu stochastic quantization scheme; for different globally valid stochastic interpretations of the Faddeev–Popov procedure see.

Let $P(M, G)$ be a principal fiber bundle with compact structure group $G$ over the compact Euclidean space time $M$. Let $\mathcal{A}$ denote the space of all irreducible connections on $P$ and let $\mathcal{G}$ denote the gauge group, which is given by all vertical automorphisms on $P$ reduced by the centre of $G$. Then $\mathcal{G}$ acts freely on $\mathcal{A}$ and defines a principal $\mathcal{G}$-fibration $\mathcal{A} \rightarrow \mathcal{M}$ over the paracompact space $\mathcal{M}$ of all inequivalent gauge potentials with projection $\pi$.

Due to the Gribov ambiguity the principal $\mathcal{G}$-bundle $\mathcal{A} \rightarrow \mathcal{M}$ is not globally trivializable.

From it follows that there exists a locally finite open cover $\mathcal{U} = \{U_\alpha\}$ of $\mathcal{M}$ together with a set of background gauge fields $\{A^{(\alpha)}_0 \in \mathcal{A}\}$ such that

$$\Gamma_\alpha = \{B \in \pi^{-1}(U_\alpha) | D^*_{A^{(\alpha)}_0}(B - A^{(\alpha)}_0) = 0\}$$

2. PARISI–WU STOCHASTIC QUANTIZATION

The Parisi–Wu approach for the stochastic quantization of the Yang–Mills theory is defined in terms of the Langevin equation

$$dA = -\frac{\delta S}{\delta A} ds + dW.$$  

Here $S$ denotes the Yang–Mills action without gauge symmetry breaking terms and without accompanying ghost field terms, $s$ denotes the extra time coordinate with respect to which the stochastic process is evolving, $dW$ is the increment of a Wiener process.

Instead of analyzing Yang-Mills theory in the original field space $\mathcal{A}$ we consider the family of trivial principal $\mathcal{G}$-bundles $\Gamma_\alpha \times \mathcal{G} \rightarrow \Gamma_\alpha$, which are locally isomorphic to the bundle $\mathcal{A} \rightarrow \mathcal{M}$, where the isomorphisms are provided by the maps

$$\chi_\alpha : \Gamma_\alpha \times \mathcal{G} \rightarrow \pi^{-1}(U_\alpha), \quad \chi_\alpha(B, g) := B^g$$

with $B \in \Gamma_\alpha$, $g \in \mathcal{G}$ and $B^g$ denoting the gauge transformation of $B$ by $g$.

We transform the Parisi–Wu Langevin equation (3) into the adapted coordinates $\Psi = \begin{pmatrix} B \\ g \end{pmatrix}$. As this transformation is not globally
possible the region of definition of (3) has to be restricted to \( \pi^{-1}(U(A_0^{(a)})) \). Making use of the Ito stochastic calculus the above Langevin equation now reads

\[
d\Psi = \left[ -G_\alpha^{-1}\frac{\delta S_\alpha}{\delta \Psi} + \frac{1}{\sqrt{\det G_\alpha}} \frac{\delta (G_\alpha^{-1} \sqrt{\det G_\alpha})}{\delta \Psi} \right] ds + E_\alpha dW. \tag{4}
\]

where \( S_\alpha = \chi^*_a S \) denotes the gauge invariant Yang-Mills action expressed in terms of the adapted coordinate \( B \). The explicit form of the vielbein \( E_\alpha \) corresponding to the change of coordinates \( A \rightarrow (B, g) \), the induced metric \( G_\alpha \), its inverse and its determinant can be found in \([1]\); for completeness we just recall that

\[
\det G_\alpha = \nu^2 (\det F_\alpha)^2 \det (\Delta \chi^{(a)}). \tag{5}
\]

Here \( \nu = \sqrt{\det (R_x R_y)} \) implies an invariant volume density on \( G \), where \( R_y \) is the differential of right multiplication transporting any tangent vector in \( T_y G \) back to the identity \( id_G \) on \( G \); \( F_\alpha = D_{\chi^{(a)}}^* D_B \) is the Faddeev–Popov operator.

### 3. Generalized Stochastic Quantization

It is the basic idea of the stochastic quantization scheme to study in addition to a given Langevin equation the associated Fokker–Planck equation for the probability density \( \rho \) interpreting the equilibrium limit of this density as Euclidean path integral measure. It is well known that such a procedure breaks down in the case of gauge theories, as due to the gauge invariance no normalizable equilibrium limit emerges. A generalization of the Parisi–Wu scheme was introduced in \([14]\) and extended recently in \([1]\) by performing special, geometrically distinguished modifications of both the drift and diffusion terms such that -most essentially- all expectation values of gauge invariant observables are left unchanged. This lead to a well defined Fokker–Planck formulation and the equilibrium density was derivable straightforwardly. The Langevin equation (2) expressed in the adapted coordinates \( \Psi \) thus gets recast into

\[
d\Psi = \left[ -\tilde{G}_\alpha^{-1}\delta S_\alpha^{\text{tot}} + \frac{1}{\sqrt{\det G_\alpha}} \frac{\delta (\tilde{G}_\alpha^{-1} \sqrt{\det G_\alpha})}{\delta \Psi} \right] ds + \tilde{E}_\alpha dW. \tag{6}
\]

Here \( \tilde{E}_\alpha \) and \( \tilde{G}_\alpha^{-1} = \tilde{E}_\alpha \tilde{E}_\alpha^* \) denote a specific vielbein and a (inverse) metric, respectively, which are associated to the above mentioned modifications of the drift and diffusion term of (2) or (4), respectively. The geometric interpretation of these modifications was discussed in full length in \([1]\) and \([1]\) and will not be repeated here. \( S_\alpha^{\text{tot}} \) denotes a total Yang-Mills action

\[
S_\alpha = \chi^*_a S + pr_G^* S_G \tag{7}
\]

defined by the original Yang-Mills action \( S \) and by \( \tilde{S}_G \in C^\infty(G) \) which is an arbitrary functional on \( \tilde{G} \) such that \( e^{-S} \) is integrable with respect to the invariant volume density \( \nu \), \( pr_G \) is the projector \( \Gamma \times G \rightarrow G \).

The Fokker–Planck equation associated to (6) can easily be deduced

\[
\frac{\partial \rho[\Psi, s]}{\partial s} = L[\Psi] \rho[\Psi, s], \tag{8}
\]

where the Fokker-Planck operator \( L[\Psi] \) is appearing in just factorized form

\[
L[\Psi] = \frac{\delta}{\delta \Psi} \tilde{G}_\alpha^{-1} \left[ \frac{\delta S_\alpha^{\text{tot}}}{\delta \Psi} - \frac{1}{\sqrt{\det G_\alpha}} \frac{\delta (\sqrt{\det G_\alpha})}{\delta \Psi} + \frac{\delta}{\delta \Psi} \right]. \tag{9}
\]

Due to the positivity of \( \tilde{G}_\alpha \) the fluctuation dissipation theorem applies and the equilibrium distribution (for a proper normalization condition see the next section) is obtained by mere inspection as

\[
\mu_\alpha e^{-S_\alpha^{\text{tot}}}, \quad \mu_\alpha = \sqrt{\det G_\alpha}. \tag{10}
\]

Although our result implies unconventional finite contributions along the gauge group (arising from the \( pr_G^* S_G \) term) it is equivalent to the usual Faddeev–Popov prescription \([8]\) for Yang–Mills theory. This follows from the fact that for expectation values of gauge invariant observables
these contributions along the gauge group are exactly canceled out due to the normalization of the path integral. We stress once more that due to the Gribov ambiguity the usual Faddeev–Popov approach as well as -presently- our modified version are valid only locally in field space.

4. GLOBAL PATH INTEGRAL

In order to compare expectation values on different patches we consider the diffeomorphism \( \phi_{\alpha_1,\alpha_2} \) in the overlap of the two patches \( (\Gamma_{\alpha_1} \cap \pi^{-1}(U_{\alpha_2})) \times G \) and \( (\Gamma_{\alpha_2} \cap \pi^{-1}(U_{\alpha_1})) \times G \) which is given by

\[
\phi_{\alpha_1,\alpha_2}(B,g) := (B^{\alpha_2}(B)^{-1}, g).
\]

Here the field dependent gauge transformation \( \omega_{\alpha_2} : \pi^{-1}(U_{\alpha_2}) \rightarrow G \) is uniquely defined by \( A^{\omega_{\alpha_2}(A)^{-1}} \in \Gamma_{\alpha_2} \). To the density \( \mu_\alpha \) there is associated a corresponding twisted top form \( \Omega \) on \( \Gamma_x \times G \) which for simplicity we denote by the same symbol. Using for convenience a matrix representation of \( G_\alpha \) we straightforwardly verify that

\[
\phi^*_{\alpha_1,\alpha_2} \mu_{\alpha_2} = \mu_{\alpha_1}.
\]  

(12)

This immediately implies that in overlap regions the expectation values of gauge invariant observables \( f \in C^\infty(A) \) are equal when evaluated in different patches. Let \( \gamma_\alpha \) be a partition of unity of \( M \). We propose the definition of the global expectation value of a gauge invariant observable \( f \in C^\infty(A) \) by summing over all \( \gamma_\alpha \) such that

\[
\langle f \rangle = \frac{\sum_{\alpha} \int_{\Gamma_x \times G} \mu_\alpha e^{-S^{\text{tot}}_\alpha}(f \pi^* \gamma_\alpha)}{\sum_{\alpha} \int_{\Gamma_x \times G} \mu_\alpha e^{-S^{\text{tot}}_\alpha} \pi^* \gamma_\alpha}.
\]

(13)

Due to (12) it is trivial to prove that the global expectation value \( \langle f \rangle \) is independent of the specific choice of the locally finite cover \( \{U_\alpha\} \), of the choice of the background gauge fields \( \{A_0^{\alpha}\} \) and of the choice of the partition of unity \( \gamma_\alpha \), respectively.

As already indicated in [1] these structures can equally be translated into the original field space \( \mathcal{A} \). With the help of the partition of unity the locally defined densities \( \mu_\alpha \) as well as \( e^{-S^{\text{tot}}_\alpha} \) can be pieced together to give a globally well defined twisted top form \( \Omega \) on \( \mathcal{A} \)

\[
\Omega = \sum_{\alpha} \chi^{-1}_\alpha \ast (\mu_\alpha e^{-S^{\text{tot}}_\alpha}) \pi^* \gamma_\alpha
= \sum_{\alpha} \mu e^{-S - \chi^{-1}_\alpha \ast pr_G \gamma} \pi^* \gamma_\alpha.
\]

(14)

The second equation follows as a further consequence of (12) and \( \mu \) denotes the flat measure on \( \mathcal{A} \). We remark the absence of functional determinants in the path integral measure and point out the additional unconventional interactions implied by the \( \chi^{-1}_\alpha \ast pr_G \gamma \) terms. The global expectation value (13) then reads

\[
\langle f \rangle = \frac{\int_{\mathcal{A}} \Omega f}{\int_{\mathcal{A}} \Omega}
\]

(15)

which due to the discussion from above is independent of all the particular local choices.

5. GAUGE ORBIT SPACE FORMULATION

In addition to the global expressions (13) and (15) for the path integral in the whole space of connections the generalized stochastic quantization scheme also offers the possibility of deriving the corresponding formulation on the gauge orbit space \( \mathcal{M} \): We consider the projections of either the original Parisi–Wu Langevin equation (4) or of the modified equation (6) onto the gauge invariant subspaces \( \Gamma_\alpha \) described by the coordinate \( B \); in both cases we obtain

\[
\delta B = \left[-(G^{-1}_\alpha)^{\Gamma_\alpha \Gamma_\alpha} \frac{\delta S}{\delta B} + \frac{1}{\sqrt{\det G_\alpha}} \sqrt{\det G_\alpha} \right] ds + E^{\Gamma_\alpha} dW.
\]

(16)

Notice that in local coordinates \( (G^{-1}_\alpha)^{\Gamma_\alpha \Gamma_\alpha} \) is the pullback of the restriction on \( U_\alpha \) of the inverse of a globally defined metric on the gauge orbit space \( \mathcal{M} \) induced by the natural metric on \( \mathcal{A} \); similarly \( E^{\Gamma_\alpha} \) is defined. Since the locally defined equations (16) are transforming covariantly under the local diffeomorphisms issued by the coordinate transformations, using [13] it is straightforward
to check that their further projections onto $\mathcal{M}$ are yielding a globally defined stochastic process.

In direct analogy to our derivation of the local Fokker–Planck densities (8) we obtain that the Fokker–Planck equation associated to the projected Langevin equations (16) has an equilibrium distribution given by just the gauge invariant part of the densities (8)

$$\det F_\alpha (\det \Delta A_0^{(\alpha)})^{-1/2} e^{-\chi^*_\alpha S}.$$  \hspace{1cm} (17)

By using (12) we can prove explicitly that their projections onto $\mathcal{M}$ on overlapping sets of $\mathcal{U}$ agree giving rise to a globally well defined top form $\tilde{\Omega}$ on $\mathcal{M}$. Furthermore we can show that the above expectation values (13) and (15) of gauge invariant observables $f$ can identically be rewritten as corresponding integrals over the gauge orbit space $\mathcal{M}$ with respect to $\tilde{\Omega}$

$$\langle f \rangle = \frac{\int_{\mathcal{M}} \tilde{\Omega} f}{\int_{\mathcal{M}} \tilde{\Omega}}.$$  \hspace{1cm} (18)

We note that this last expression shows agreement with the formulation proposed by Stora [15] upon identification of $\tilde{\Omega}$ with the Ruelle-Sullivan form. Whereas in [13], however, this form of the expectation values on $\mathcal{M}$ appeared as the starting point for a global formulation of Yang-Mills theory in the whole space of gauge potentials it appears in our case as the final result; we aimed at its direct derivation within the generalized stochastic quantization approach.

We see that the projections onto the local gauge fixing surfaces $\Gamma_\alpha$ of in specific the original Parisi-Wu stochastic process induce a globally defined stochastic process on the gauge orbit space yielding the construction of the globally defined path integral density $\tilde{\Omega}$. We conclude that the globally defined Parisi-Wu Langevin equation (2) on the whole field space $\mathcal{A}$ is intimately related to the globally defined path integral density (18) on the gauge orbit space $\mathcal{M}$.

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