SENSITIVITY ANALYSIS OF THE UTILITY MAXIMIZATION
PROBLEM WITH RESPECT TO MODEL PERTURBATIONS

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ABSTRACT. We study the sensitivity of the expected utility maximization problem in a continuous semi-martingale market with respect to small changes in the market price of risk. Assuming that the preferences of a rational economic agent are modeled with a general utility function, we obtain a second-order expansion of the value function, a first-order approximation of the terminal wealth, and construct trading strategies that match the indirect utility function up to the second order. If a risk-tolerance wealth process exists, using it as a numéraire and under an appropriate change of measure, we reduce the approximation problem to a Kunita-Watanabe decomposition.

1. INTRODUCTION

It is well-known, see for example [DS06, HS10], that for a continuous stock price process, the no-arbitrage condition implies that the return of the stock price $S$ has the following representation:

$$ S = M + \lambda \cdot \langle M \rangle, $$

where $M$ is a continuous local martingale, and $\lambda$ is a predictable process, i.e., that the quadratic variation of a stock price has to be absolutely continuous with respect to the quadratic variation of $M$. We analyze the effect of perturbations of the market price of risk $\lambda$, on the utility maximization problem.

In the setting of an incomplete model, where the preferences of a rational economic agent are modeled with a general utility function $U$ with bounded (away from zero and infinity) relative risk-aversion and the stock prices process is continuous, we obtain a
quadratic expansion of the value function, a first-order correction to the optimal terminal wealth, and a construction of the approximate trading strategies that match the value functions up to the second order. For the power-utility case, a first-order asymptotic expansion with respect to perturbations of the market price of risk is obtained in [CR16], whereas a second-order analysis is performed in [LMZ14]. Mathematically, the results in the present paper rely on different techniques. We can summarize our contribution as three-fold:

(1) We first need to increase dimensionality and look at the simultaneous perturbations of the market price of risk and the initial wealth. As the proofs show, the increase of dimensionality is a necessary way of getting the expansions of the value functions up to the second order.\footnote{In the constant relative risk aversion case considered in [LMZ14], as the optimal terminal wealth depends on the initial wealth via a multiplicative constant, the increase of dimensionality is not needed for obtaining quadratic expansions.}

(2) Then, we formulate auxiliary quadratic stochastic control problems and relate the second-order approximations of both primal and dual value functions to these problems.

(3) Finally, if the risk-tolerance wealth process exists, we use it as a numéraire, and change the measure accordingly, to identify solutions to the general quadratic optimization problems above in terms of a Kunita-Watanabe decomposition (of a certain martingale) generated by the perturbation process.

To the best of our knowledge, the closest paper from the mathematical viewpoint is [KS06b], where the authors obtain a second-order expansion of the value function with respect to simultaneous perturbations of the initial wealth and the number of units of random endowment held in the portfolio. We would like to stress that unlike the present setting, in [KS06b], the value function is jointly concave (in both the initial wealth and the number of units of random endowment held in the portfolio), a fact that plays a significant role in the proofs there.

We combine here the increase of dimensionality described in item (1) with a similar change of measure and numéraire to [KS06b] relating them to general quadratic optimization problems. However, one of the main technical difficulties lies in the fact that our value function as a function of two variables is not concave or convex in the perturbation variable $\delta$ (in general). Despite this obstacle, our approach, which relies only partially on convex conjugacy, still produces a quadratic expansion via auxiliary quadratic problems and simultaneous expansions of $u$ in $(x, \delta)$ and $v$ in $(y, \delta)$. In addition to obtaining a quadratic expansion, we also get a relationship between the existence of such an approximation and the existence of the risk-tolerance wealth process, which was
introduced in [KS06b]. We show that the existence of the risk-tolerance wealth process allows for a more explicit form of the correction terms in our approximation coming from a Kunita-Watanabe decomposition under appropriate measure and numéraire that are specified in terms of the risk-tolerance wealth process. Another connection to [KS06b] is given in Lemma 6.1, where the perturbation to the market-price of risk plays the role of a multiplicative (and non-linear) random endowment.

To separate the financial aspects of the problem from the mathematical ones, we state and prove abstract versions of main theorems. After that we reduce the proofs of (some of) the main theorems to verification of the conditions in the abstract theorems.

As an application, we consider models, which admit closed form solutions in incomplete markets, see [KO96, Liu07, GR15] (we also refer to [LMZ14] for more examples and a literature review). These models are sensitive to perturbations of the input parameters: ones they are perturbed even slightly, a close form solution typically ceases to exist. Our results show that even though we do not know how to obtain an exact solution for such perturbed problems, an approximation, which is accurate up to the second order, can still be constructed.

We prove our results under the assumption of no unbounded profit with bounded risk, the weakest no-arbitrage type condition, which allows for the utility maximization problem from terminal wealth to be non-degenerate, see [KK07, Proposition 4.19]. For the perturbation process, we formulate an assumption and give a counterexample, which shows the necessity of the assumption. In addition, we provide a set of sufficient conditions for the integrability assumption on the perturbation process to hold.

For the general utility function, we suppose that its relative risk-aversion is bounded away from zero and infinity. This condition is (essentially) necessary for twice differentiability with respect to the initial wealth to hold, see [KS06a] for counterexamples. On an even more technical side, as we consider perturbations of the initial wealth, we obtain as a by-product here the second-order derivatives of the primal and dual value functions with respect to the spatial variables ($x$ and $y$, correspondingly). Note that, in [KS06a] this result was obtained for discontinuous stock prices, but under NFLVR.

The remainder of the paper is organized as follows: in section 2, we formulate the model and state the expansion theorems, section 4 contains the approximation of optimal trading strategies theorem, section 5 includes abstract versions of Theorems 3.7, 3.8, 3.10 and 3.12 with proofs, section 6 contains proofs of non-abstract theorems and Theorem 4.1, where a construction of corrections to the optimal trading strategies (accurate up to the second order of the value function) is specified. Section 7 includes a counterexample, which shows that without Assumption 3.2 on the perturbation process, the quadratic expansions of the value functions might not exist. In section 8 we relate the asymptotic
expansions from previous sections to the existence of the risk-tolerance wealth process and a Kunita-Watanabe decomposition. We conclude the paper with an illustration of an application of our results to analysis of the perturbations of models that admit closed-form solutions.

2. Model

2.1. Parametrized family of stock prices processes. Let us consider a complete stochastic basis \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P}) \), where \( T \in (0, \infty) \) is the time horizon, \( \mathcal{F} \) satisfies the usual conditions, and \( \mathcal{F}_0 \) is the completion of the trivial \( \sigma \)-algebra. We assume that there are two traded securities, a bank account with zero interest rate and a stock. Let \( M \) be a one-dimensional continuous local martingale and \( \lambda \) is a progressively measurable process, such that

\[
\int_{0}^{T} \lambda_t^2 \, d\langle M \rangle_t < \infty, \quad \mathbb{P} - a.s.
\]

The stock price return process\(^2\) for the unperturbed, or equivalently, 0-model is given by

\[
S^0 \triangleq \lambda \cdot \langle M \rangle + M.
\]

Here we consider a parametric family of semimartingales \( S^\delta \), \( \delta \in \mathbb{R} \), with the same martingale part \( M \) and where the market price of risk \( \lambda \)'s are perturbed

\[
S^\delta \triangleq \lambda^\delta \cdot d\langle M \rangle + M,
\]

where for some progressively measurable process \( \nu \), such that

\[
\int_{0}^{T} \nu_t^2 \, d\langle M \rangle_t < \infty, \quad \mathbb{P} - a.s.
\]

we have

\[
\lambda^\delta \triangleq \lambda + \delta \nu, \quad \delta \in \mathbb{R}.
\]

2.2. Primal problem. Let \( U \) be a utility function that satisfies Assumption 2.1 below.

**Assumption 2.1.** The utility function \( U \) is strictly increasing, strictly concave, two times differentiable on \((0, \infty)\) and there exist positive constants \( c_1 \) and \( c_2 \), such that

\[
c_1 \leq A(x) \triangleq \frac{U''(x)x}{U'(x)} \leq c_2,
\]

i.e. the relative risk aversion of \( U \) is uniformly bounded away from zero and infinity.

\(^2\)We denote the return of the stock by \( S \), since \( R \) is used for different purposes.
The family of primal feasible sets is defined as

\[ X(x, \delta) \triangleq \{ X \geq 0 : X_t = x + H_t S^\delta_t, \quad t \in [0, T] \}, \quad (x, \delta) \in (0, \infty) \times \mathbb{R}, \]

where \( H \) is a predictable and \( S^\delta \)-integrable process representing the amount invested in the stock. The corresponding family of the value functions is given by

\[ u(x, \delta) \triangleq \sup_{X \in X(x, \delta)} \mathbb{E}[U(X_T)], \quad (x, \delta) \in (0, \infty) \times \mathbb{R}. \]

We use the convention

\[ \mathbb{E}[U(X_T)] \triangleq -\infty, \quad if \quad \mathbb{E}[U^-(X_T)] = \infty, \]

where \( U^- \) is the negative part of \( U \).

### 2.3. Dual problem.

The investigation of the primal problem (2.5) is conducted via the dual problem. First, let us define the dual domain as follows:

\[ Y(y, \delta) \triangleq \{ Y : Y \ is \ a \ nonnegative \ supermartingale, \ such \ that \ Y_0 = y \ and \ XY = (X_t Y_t)_{t \geq 0} \ is \ a \ supermartingale \ for \ every \ X \in X(1, \delta) \}, \quad (y, \delta) \in (0, \infty) \times \mathbb{R}. \]

We set the convex conjugate to utility function \( U \) as

\[ V(y) \triangleq \sup_{x > 0} (U(x) - xy), \quad y > 0. \]

Note that for \( y = U'(x) \), we have

\[ V''(y) = -\frac{1}{U''(x)}, \]

and

\[ B(y) \triangleq -\frac{V''(y)y}{V'(y)} = \frac{1}{A(x)}. \]

Therefore, Assumption 2.1 implies that

\[ \frac{1}{c_2} \leq B(y) \leq \frac{1}{c_1}, \quad y > 0. \]

The parametrized family of dual value functions is given by

\[ v(y, \delta) \triangleq \inf_{Y \in Y(y, \delta)} \mathbb{E}[V(Y_T)], \quad (y, \delta) \in (0, \infty) \times \mathbb{R}. \]

We use the convention

\[ \mathbb{E}[V(Y_T)] \triangleq \infty, \quad if \quad \mathbb{E}[V^+(Y_T)] = \infty, \]

where \( V^+ \) is the positive part of \( V \).
3. Technical assumptions

We recall the assumption that $M$ is continuous. The absence of arbitrage opportunities in the 0-model in the sense of no unbounded profit with bounded risk follows from condition (2.1), which implies that $\mathcal{Y}(1, 0) \neq \emptyset$. Note that (2.1) and (2.2) imply no unbounded profit with bounded risk for every $\delta \in \mathbb{R}$, thus

$$\mathcal{Y}(1, \delta) \neq \emptyset, \quad \delta \in \mathbb{R}. $$

In order for the problem (2.5) to be non-degenerate, we also need to assume that

$$(3.1) \quad u(x, 0) < \infty \quad \text{for some } x > 0. $$

Remark 3.1. Conditions (2.1) and (3.1) are necessary for the expected utility maximization problem to be non-degenerate. Note that we only impose them for $\delta = 0$.

As in [KS06a, KS06b], an important role will be played by the probability measures $\mathbb{R}(x, \delta)$, given by

$$d\mathbb{R}(x, \delta) \triangleq \frac{\hat{X}_T(x, \delta)\hat{Y}_T(y, \delta)}{xy}, $$

for $x > 0$ and $y = u_x(x, \delta)$. As Example 7.1 below demonstrates, we need to impose an integrability condition. First, let us define

$$(3.2) \quad \zeta(c, \delta) \triangleq \exp\left( c(|\nu \cdot S^\delta_T| + \langle \nu \cdot S^\delta \rangle_T) \right), \quad (c, \delta) \in \mathbb{R}^2. $$

Assumption 3.2. Let $x > 0$ be fixed. There exists $c > 0$, such that

$$\mathbb{E}^{\mathbb{R}(x, 0)}[\zeta(c, 0)] < \infty. $$

Remark 3.3. The stronger condition

$$(3.3) \quad \sup_{(x', \delta) \in B_{c}(x, 0)} \mathbb{E}^{\mathbb{R}(x', \delta)}[\zeta(c, \delta)] < \infty, $$

for some $\varepsilon > 0$ and $c > 0$, where $B_{c}(x, 0)$ denotes the ball in $\mathbb{R}^2$ of radius $\varepsilon$ centered at $(x, 0)$, implies local semiconcavity of the value function $u(x, \delta)$. Consequently, in the quadratic expansions of $u$ and $v$ given by (5.22) and (5.23), the matrices $H_u(x, 0)$ and $H_v(y, 0)$ defined in (5.20) and (5.21), respectively, are Hessian matrices, i.e. are derivatives of gradients. This will follow from Lemma 5.14. However, the very restrictive condition (3.3) is an assumption that depends on optimal solutions for $\delta \neq 0$, and thus usually impossible to check.

Let us also set

$$(3.4) \quad L^\delta \triangleq \mathcal{E}\left(-\langle \delta \nu \rangle \cdot S^0\right)_T, \quad \delta \in \mathbb{R}. $$

Here and below $\mathcal{E}$ denotes the Doléans-Dade exponential. One can see that $L^\delta$ is a terminal value of an element of $\mathcal{X}(1, 0)$ for every $\delta \in \mathbb{R}$. 
**Sufficient conditions for Assumption 3.2**

**Remark 3.4.** A sufficient condition for Assumption 3.2 to hold is the existence a wealth process under the numéraire $\tilde{X}(x, 0)$, $\tilde{X}$, and a constant $c > 0$, such that

$$\exp \left( c (|\nu \cdot S^0| + \nu^2 \cdot \langle M \rangle) \right)_T \leq \tilde{X}_T, \quad \text{a.s.}$$

**Remark 3.5.** Let us assume that in (2.3), $c_1 > 1$, i.e. that relative-risk aversion of $U$ is strictly greater than 1, (for example, this holds if $U(x) = \frac{x^p}{p}$ with $p < 0$, note that for such a $U$, the conjugate function $V(y) = \frac{y^q}{q}$ for $q \in (-1, 0)$). In this case, a sufficient condition for Assumption 3.2 to hold is the existence of some positive exponential moments under $P$ of

$$|\nu \cdot S^0_T| \quad \text{and} \quad \nu^2 \cdot \langle M \rangle_T.$$ 

This can be shown as follows. Let us set

$$q_i \triangleq -\left( 1 - \frac{1}{c_i} \right), \quad i = 1, 2.$$ 

As $c_2 \geq c_1 > 1$, we deduce that $q_i \in (-1, 0)$, $i = 1, 2$. Using Lemma 5.12, one can find a constant $C > 0$, such that

$$(3.5) \quad -V'(y)y \leq C \left( y^{-q_1} + y^{-q_2} \right), \quad y > 0.$$ 

In order to prove (3.5), let us observe that from Lemma 5.12 we get

$$(3.6) \quad U'(z) \leq z^{-c_2} U'(1), \quad -V'(z) \leq z^{-\frac{1}{c_1}} (-V'(1)), \quad \text{for every } z \in (0, 1].$$

As $(U')^{-1} = -V'$, the first inequality implies that there exists $z_0$, such that

$$-V'(z) \leq (U'(1))^{-\frac{1}{c_2}} z^{-\frac{1}{c_2}}, \quad \text{for every } z \geq z_0.$$ 

Combining this inequality with (3.6) and since $\sup_{z \in [\min(z_0, 1), \max(z_0, 1)]} | -V'(z) | < \infty$, we obtain (3.5). Thus, if some positive exponential moments of $|\nu \cdot S^0_T|$ and $\nu^2 \cdot \langle M \rangle_T$ exist under $P$, using Hölder’s inequality one can find a positive constant $a$, such that

$$(3.7) \quad \mathbb{E} [\zeta(a, 0)] < \infty,$$

where $\zeta(a, 0)$ is defined in (3.2). Let us set

$$c \triangleq a(1 + q_2)$$
and note that \( \frac{c}{1+q_1} = a \frac{1+q_2}{1+q_1} \leq a \). With \( y = u_x(x,0) \), using Hölder’s inequality again (note that \( \frac{1}{1+q_i} \) are the Hölder conjugate of \( \frac{1}{-q_i} \), \( i = 1, 2 \)) and (3.5), we get

\[
xyE^{\mathbb{R}(x,0)}[\zeta(c,0)] \leq CE \left[ \left( \hat{Y}_T(y,0)^{-q_1} + \hat{Y}_T(y,0)^{-q_2} \right) \zeta(c,0) \right] \\
\leq CE \hat{Y}_T(y,0)^{-q_1} E \left[ \zeta \left( \frac{c}{1+q_1}, 0 \right)^{1+q_1} \right] + CE \hat{Y}_T(y,0)^{-q_2} E \left[ \zeta \left( \frac{c}{1+q_2}, 0 \right)^{1+q_2} \right] \\
\leq Cy^{-q_1} E \left[ \zeta(a,0)^{1+q_1} \right] + Cy^{-q_2} E \left[ \zeta(a,0)^{1+q_2} \right] < \infty,
\]

where the last inequality follows from the supermartingale property of \( \hat{Y}(y,0) \) and (3.7).

Thus, Assumption 3.2 holds.

**Remark 3.6 (On the relationship with existing literature).** Assumption 3.2 is related to the condition on random endowment, Assumption 4 in [KS06b], via the following argument. Assume that, for some \( x > 0 \) and \( c > 0 \), there exists a wealth process \( X \in \mathcal{X}(x,0) \), such that

\[
(3.8) \quad \zeta(c,0) \leq \frac{X_T}{X(x,0)},
\]

where \( X(x,0) \) is the optimal solution to (2.5). Then Assumption 3.2 is satisfied. The wealth process \( \frac{X}{X(x,0)} \) under the numeraire \( X(x,0) \) in condition (3.8) is local martingale under \( \mathbb{R}(x,0) \), i.e., \( X \) can be an arbitrary element of \( \mathcal{X}(x,0) \). In [KS06b] it is assumed that \( \frac{X}{X(x,0)} \) is a square-integrable martingale under \( \mathbb{R}(x,0) \).

**Expansion Theorems**

In Theorem 3.7 we prove finiteness of the value functions and first-order derivatives with respect to \( \delta \).

**Theorem 3.7.** Let \( x > 0 \) be fixed, assume that (2.1) and (3.1) as well as Assumptions 2.1 and 3.2 hold, and denote \( y = u_x(x,0) \), which is well-defined by the abstract theorems in [KS99]. Then there exists \( \delta_0 > 0 \) such that for every \( \delta \in (-\delta_0, \delta_0) \), we have

\[
(3.9) \quad u(x,\delta) \in \mathbb{R}, \ x > 0, \ and \ v(y,\delta) \in \mathbb{R}, \ y > 0.
\]

In addition, \( u \) and \( v \) are jointly differentiable (and, consequently, continuous) at \( (x,0) \) and \( (y,0) \), respectively. We also have

\[
(3.10) \quad \nabla u(x,0) = \begin{pmatrix} y \\ u_\delta(x,0) \end{pmatrix} \quad and \quad \nabla v(y,0) = \begin{pmatrix} -x \\ v_\delta(y,0) \end{pmatrix},
\]

where

\[
(3.11) \quad u_\delta(x,0) = v_\delta(y,0) = xyE^{\mathbb{R}(x,0)}[F].
\]
In order to characterize the second-order derivatives of the value functions, we will need the following notations. Let $S^{X(x,0)}$ be the price process of the traded securities under the numéraire $\hat{X}(x,0)$, i.e.

$$S^{X(x,0)} = \left( \frac{x}{\hat{X}(x,0)}, \frac{xS^0}{\hat{X}(x,0)} \right).$$

For every $x > 0$, let $H^2_0(\mathbb{R}(x,0))$ denote the space of square integrable martingales under $\mathbb{R}(x,0)$, such that

$$M^2(x,0) \triangleq \{ M \in H^2_0(\mathbb{R}(x,0)) : M = H \cdot S^{X(x,0)} \},$$

$$N^2(y,0) \triangleq \{ N \in H^2_0(\mathbb{R}(x,0)) : MN is \mathbb{R}(x,0) martingale for every M \in M^2(x,0) \},$$

where $y = u_x(x,0)$.

**Auxiliary minimization problems**

As in [KS06a], for $x > 0$ let us consider

$$a(x,x) \triangleq \inf_{M \in M^2(x,0)} \mathbb{E}^{\mathbb{R}(x,0)} \left[ A(\hat{X}_T(x,0))(1 + M_T)^2 \right],$$

$$b(y,y) \triangleq \inf_{N \in N^2(y,0)} \mathbb{E}^{\mathbb{R}(x,0)} \left[ B(\hat{Y}_T(y,0))(1 + N_T)^2 \right],$$

where $A$ is the relative risk aversion and $B$ is the relative risk tolerance of $U$, respectively. It is proven in [KS06a] that (3.12) and (3.13) admit unique solutions $M^0(x,0)$ and $N^0(y,0)$, correspondingly, and

$$u_{xx}(x,0) = -\frac{y}{x} a(x,x),$$

$$v_{yy}(y,0) = \frac{x}{y} b(y,y),$$

$$a(x,x)b(y,y) = 1,$$

$$A(\hat{X}_T(x,0))(1 + M^0_T(x,0)) = a(x,x)(1 + N^0_T(y,0)).$$

In order to characterize the derivatives of the value functions with respect to $\delta$, with

$$F \triangleq \nu \cdot S^0_T and \ G \triangleq \nu^2 \cdot \langle M \rangle_T,$$

we consider the following minimization problems:

$$a(d,d) \triangleq \inf_{M \in M^2(x,0)} \mathbb{E}^{\mathbb{R}(x,0)} \left[ A(\hat{X}_T(x,0))(M_T + xF)^2 - 2xFM_T - x^2(F^2 + G) \right],$$

$$b(d,d) \triangleq \inf_{N \in N^2(y,0)} \mathbb{E}^{\mathbb{R}(x,0)} \left[ B(\hat{Y}_T(y,0))(N_T - yF)^2 + 2yFM_N - y^2(F^2 - G) \right].$$

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3Under the assumption of NFLVR. Below we will show that the formulas (3.12), (3.13), and (3.14) can also be obtained in the present setting.
Denoting by \( M^1(x,0) \) and \( N^1(y,0) \) the unique solutions to (3.16) and (3.17) respectively, we also set

\[
(3.18) \quad a(x, d) \triangleq E^{\mathbb{R}(x,0)} \left[ A(\bar{X}_T(x,0))(1 + M^0_T(x,0))(xF + M^1_T(x,0)) - xF(1 + M^0_T(x,0)) \right],
\]

\[
(3.19) \quad b(y, d) \triangleq E^{\mathbb{R}(x,0)} \left[ B(\bar{Y}_T(y,0))(1 + N^0_T(y,0))(N^1_T(y,0) - yF) + yF(1 + N^0_T(y,0)) \right].
\]

Theorems 3.8, 3.10, and 3.12 contain the second-order expansions of the value functions, derivatives of the optimizers, and properties of such derivatives.

**Theorem 3.8.** Let \( x > 0 \) be fixed. Assume all conditions of Theorem 3.7 hold, with \( y = u_x(x,0) \). Define

\[
(3.20) \quad H_u(x,0) \triangleq -\frac{y}{x} \begin{pmatrix}
a(x, x) & a(x, d) \\
a(x, d) & a(d, d)
\end{pmatrix},
\]

where \( a(x, x), a(d, d), \) and \( a(x, d) \) are specified in (3.12), (3.16), and (3.18), and, respectively,

\[
(3.21) \quad H_v(y,0) \triangleq \frac{x}{y} \begin{pmatrix}
b(y, y) & b(y, d) \\
b(y, d) & b(d, d)
\end{pmatrix},
\]

where \( b(y, y), b(d, d), b(y, d) \) are specified in (3.13), (3.17), and (3.19). Then, the value functions \( u \) and \( v \) admit the second-order expansions around \((x, 0)\) and \((y, 0)\), respectively,

\[
(3.22) \quad u(x + \Delta x, \delta) = u(x, 0) + (\Delta x, \delta) \nabla u(x, 0) + \frac{1}{2} (\Delta x, \delta) H_u(x, 0) \begin{pmatrix} \Delta x \\ \delta \end{pmatrix} + o(\Delta x^2 + \delta^2),
\]

and

\[
(3.23) \quad v(y + \Delta y, \delta) = v(y, 0) + (\Delta y, \delta) \nabla v(y, 0) + \frac{1}{2} (\Delta y, \delta) H_v(y, 0) \begin{pmatrix} \Delta y \\ \delta \end{pmatrix} + o(\Delta y^2 + \delta^2).
\]

**Remark 3.9.** Although we only have second order expansions, we may abuse the language and call \( H_u(x,0) \) and \( H_v(y,0) \) the Hessians of \( u \) and \( v \), without having twice differentiability. This causes no confusion, see the discussion e.g., in [LS02]. The meaning of partial derivatives \( u_{xx}(x,0), u_{x\delta}(x,0) \) and so on then becomes apparent by identifying entries in the Hessian matrices.

**Theorem 3.10.** Let \( x > 0 \) be fixed, the assumptions of Theorem 3.7 hold, and \( y = u_x(x,0) \). Then, we have

\[
(3.24) \quad \begin{pmatrix}
a(x, x) & 0 \\
a(x, d) & -\frac{x}{y}
\end{pmatrix} \begin{pmatrix}
b(y, y) & 0 \\
b(y, d) & -\frac{y}{x}
\end{pmatrix} = I_2,
\]
where $I_2$ denotes two-by-two identity matrix. Moreover,

\begin{equation}
\frac{y}{x}a(d,d) + \frac{x}{y}b(d,d) = a(x,d)b(y,d)
\end{equation}

\begin{equation}
U''(\hat{X}_T(x,0))\hat{X}_T^0(x,0) \left( \begin{array}{cc}
M_T^0(x,0) + 1 \\
M_T^1(x,0) + xF
\end{array} \right) = - \left( \begin{array}{cc}
a(x,x) & 0 \\
a(x,d) & -\frac{x}{y}
\end{array} \right) V''(\hat{Y}_T(y,0))\hat{Y}_T^0(y,0) \left( \begin{array}{cc}
N_T^0(y,0) + 1 \\
N_T^1(y,0) - yF
\end{array} \right),
\end{equation}

and the product of any of $\hat{X}(x,0), \hat{X}(x,0)M^0(x,0), \hat{X}(x,0)M^1(x,0)$ and any of $\hat{Y}(y,0), \hat{Y}(y,0)N^0(y,0), \hat{Y}(y,0)N^1(y,0)$ is a martingale under $\mathbb{P}$, where $M_T^0(x,0), M_T^1(x,0), N_T^0(y,0)$, and $N_T^1(y,0)$ are the solutions to (3.12), (3.16), (3.13), and (3.17), correspondingly.

Remark 3.11. Continuing the discussion in Remark 3.9, (3.24) implies that

\[
\begin{pmatrix}
u_{xx}(x,0) & 0 \\ u_{xs}(x,0) & 1
\end{pmatrix}
\begin{pmatrix}
u_{yy}(y,0) & 0 \\ v_{ys}(y,0) & -1
\end{pmatrix} = -I_2,
\]

where

\[
\begin{pmatrix}
u_{xx}(x,0) & 0 \\ u_{xs}(x,0) & 1
\end{pmatrix} = -\frac{y}{x} \begin{pmatrix} a(x,x) & 0 \\ a(x,d) & -\frac{x}{y} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
u_{yy}(y,0) & 0 \\ v_{ys}(y,0) & -1
\end{pmatrix} = \frac{x}{y} \begin{pmatrix} b(y,y) & 0 \\ b(y,d) & -\frac{y}{x} \end{pmatrix}.
\]

Likewise, (3.25) gives

\[-u_{xs}(x,0) + v_{ys}(y,0) = -u_{xs}(x,0)\nu_{ys}(y,0).\]

Theorem 3.12. Let $x > 0$ be fixed, the assumptions of Theorem 3.7 hold, and $y = u_x(x,0)$. Then the terminal values of the wealth processes $M^0(x,0)$ and $M^1(x,0)$, which are the solutions to (3.12) and (3.16), respectively, satisfy

\begin{equation}
\lim_{|\Delta x| + |\delta| \to 0} \frac{1}{|\Delta x| + |\delta|} \left| \hat{X}_T(x + \Delta x, \delta) - \frac{\hat{X}_T(x,0)}{x} (x + \Delta x(1 + M_T^0(x,0)) + \delta M_T^1(x,0)) \right| = 0,
\end{equation}

where the convergence takes place in $\mathbb{P}$-probability and $L^\delta$’s are defined in (5.3). Likewise, let $N_T^0(y,0)$ and $N_T^1(y,0)$, which are solutions to (3.13) and (3.17), correspondingly, satisfy

\begin{equation}
\lim_{|\Delta y| + |\delta| \to 0} \frac{1}{|\Delta y| + |\delta|} \left| \hat{Y}_T(y + \Delta y, \delta) - \frac{\hat{Y}_T(y,0)}{y} (y + \Delta y(1 + N_T^0(y,0)) + \delta N_T^1(y,0)) \right| = 0,
\end{equation}

where the convergence takes place in $\mathbb{P}$-probability.

One can obtain the following corollary.
Corollary 3.13. Let \( x > 0 \) be fixed, the assumptions of Theorem 3.7 hold, and \( y = u_x(0,0) \). Then, if we define
\[
X'_T(x,0) \triangleq \frac{\widehat{X}_T(x,0)}{x}(1 + M_T^0(0,x)), \quad Y'_T(y,0) \triangleq \frac{\widehat{Y}_T(y,0)}{y}(1 + N_T^0(0,y)),
\]
and
\[
X''_T(x,0) \triangleq \frac{\widehat{X}_T(x,0)}{x}(M_T^1(x,0) + xF), \quad Y''_T(y,0) \triangleq \frac{\widehat{Y}_T(y,0)}{y}(N_T^1(y,0) - yF),
\]
we have
\[
\lim_{|\Delta x| + |\delta| \to 0} \frac{1}{|\Delta x| + |\delta|} \left| \widehat{X}_T(x + \Delta x,0) - \widehat{X}_T(x,0) - \Delta x X'_T(x,0) - \delta X''_T(x,0) \right| = 0,
\]
\[
\lim_{|\Delta y| + |\delta| \to 0} \frac{1}{|\Delta y| + |\delta|} \left| \widehat{Y}_T(y + \Delta y,0) - \widehat{Y}_T(y,0) - \Delta y Y'_T(y,0) - \delta Y''_T(y,0) \right| = 0,
\]
where the convergence takes place in \( P \)-probability.

Remark 3.14. Even though Corollary 3.13 gives a more explicit form of the derivatives of the terminal wealth, an approximation given in (3.27) turns out to be more useful in applications.

4. APPROXIMATION OF THE OPTIMAL TRADING STRATEGIES

Below in this section we will suppose that \( x > 0 \) is fixed. Let us denote
\[
M^R \triangleq S^0 - \widehat{\pi}(x,0) \cdot \langle M \rangle,
\]
where \( \widehat{\pi}(x,0) = (\widehat{\pi}_t(x,0))_{t \in [0,T]} \) is the optimal proportion invested in stock corresponding the initial wealth \( x \) and \( \delta = 0 \). Note that for every predictable pair of processes \( G^1 \) and \( G^2 \), such that both integrals \( G^1 \cdot \left( \frac{1}{X(1,0)} \right) \) and \( G^2 \cdot \left( \frac{S^0}{X(1,0)} \right) \) are well-defined, by direct computations, we can find a process \( G \), such that
\[
G^1 \cdot \left( \frac{1}{X(1,0)} \right) + G^2 \cdot \left( \frac{S^0}{X(1,0)} \right) = G \cdot M^R.
\]

Let \( \gamma^0 \) and \( \gamma^1 \) be such that
\[
\gamma^0 \cdot M^R = \frac{M^0(x,0)}{x} \quad \text{and} \quad \gamma^1 \cdot M^R = \frac{M^1(x,0)}{x}.
\]
We need to define the following families of stopping times.
\[
\sigma_\varepsilon \triangleq \inf \left\{ t \in [0,T] : |M^0_t(x,0)| \geq \frac{\varepsilon}{\varepsilon} \text{ or } \langle M^0_t(x,0) \rangle_t \geq \frac{\varepsilon}{\varepsilon} \right\},
\]
\[
\tau_\varepsilon \triangleq \inf \left\{ t \in [0,T] : |M^1_t(x,0)| \geq \frac{\varepsilon}{\varepsilon} \text{ or } \langle M^1_t(x,0) \rangle_t \geq \frac{\varepsilon}{\varepsilon} \right\}, \quad \varepsilon > 0,
\]
we also set
\[
\gamma^{0,\varepsilon} = \gamma^01_{[0,\sigma_\varepsilon]} \quad \text{and} \quad \gamma^{1,\varepsilon} = \gamma^11_{[0,\tau_\varepsilon]}, \quad \varepsilon > 0.
\]
Finally, for every \((\Delta x, \delta, \varepsilon) \in (-x, \infty) \times \mathbb{R} \times (0, \infty)\), let us define

\[
X_{\Delta x, \delta, \varepsilon} \triangleq (x + \Delta x)E \left( \left( \hat{\pi}(x, 0) + \Delta x \gamma^0\varepsilon + \delta (\nu + \gamma^1\varepsilon) \right) \cdot S^\delta \right).
\]

**Theorem 4.1.** Assume that \(x > 0\) is fixed and the assumptions of Theorem 3.7 hold. Then, there exists a function \(\varepsilon = \varepsilon(\Delta x, \delta)\), \((\Delta x, \delta) \in (-x, \infty) \times \mathbb{R}\), such that

\[
\mathbb{E} \left[ U \left( X_{\Delta x, \delta, \varepsilon}(\Delta x, \delta) \right) \right] = u(x + \Delta x, \delta) - o(\Delta x^2 + \delta^2),
\]

where \(X_{\Delta x, \delta, \varepsilon}\) is defined in (4.2).

**Remark 4.2.** Theorem 4.1 shows how to correct the optimal proportion in order to match the primal value function up to the second order jointly in \((\Delta x, \delta)\).

**Remark 4.3.** Proportions have a nicer representation of the corrections to optimal trading strategies in terms of the quadratic optimization problems (3.12) and (3.16) because the optimal wealth process was used as numéraire, i.e., \(\hat{X}(x, 0)/x\) has a multiplicative structure. The result in Theorem 4.1 compliments the results in [KS06a] and (in a different additive random endowment framework) those in [KS06b] in the context of a one-dimensional and continuous stock model.

### 5. Abstract version

**Abstract version for 0-model**

We begin with the formulation of the abstract version for 0-model. As in [Mos15], let \((\Omega, \mathcal{F}, \mathbb{P})\) be a measure space and we define the sets \(\mathcal{C}\) and \(\mathcal{D}\) to be subsets of \(L^0_+\) that satisfy the following assumption. Note that Assumption 5.1 is the abstract version of no unbounded profit with bounded risk condition (2.1).

**Assumption 5.1.** Both \(\mathcal{C}\) and \(\mathcal{D}\) contain a strictly positive element and

\[
\xi \in \mathcal{C} \quad if f \quad \mathbb{E}[\xi \eta] \leq 1 \quad for \ every \ \eta \in \mathcal{D},
\]

as well as

\[
\eta \in \mathcal{D} \quad if f \quad \mathbb{E}[\xi \eta] \leq 1 \quad for \ every \ \xi \in \mathcal{C}.
\]

We also set \(\mathcal{C}(x, 0) \triangleq x\mathcal{C}\) and \(\mathcal{D}(x, 0) \triangleq x\mathcal{D}, \ x > 0\). Now we can state the abstract primal and dual problems as

\[
u(x, 0) \triangleq \sup_{\xi \in \mathcal{C}(x, 0)} \mathbb{E}[U(\xi)], \quad x > 0,
\]

\[
u(y, 0) \triangleq \inf_{\eta \in \mathcal{D}(y, 0)} \mathbb{E}[V(\eta)], \quad y > 0.
\]
Under finiteness of both primal and dual value functions on $\mathbb{R}$, existence and uniqueness of solutions to \eqref{eq:1} and \eqref{eq:2} follow from \cite[Theorem 3.2]{Mos15}. Likewise, with a deterministic utility function that has reasonable asymptotic elasticity, if $u(x, 0) < \infty$ for some $x > 0$, standard conclusions of the utility maximization theory also follow from the abstract theorems in \cite{KS99} (see the discussion in \cite[Remark 2.5]{CCFM15}).

**Abstract version for $\delta$-models**

For some random variables $F$ and $G \geq 0$, let us set

\begin{equation}
L^\delta \triangleq \exp \left( -\left( \delta F + \frac{1}{2} \delta^2 G \right) \right),
\end{equation}

\begin{equation}
C(x, \delta) \triangleq C(x, 0) \frac{1}{L^\delta} \text{ and } D(y, \delta) \triangleq D(y, 0) L^\delta, \quad \delta \in \mathbb{R}.
\end{equation}

Now, we can state the abstract versions of the perturbed optimization problems.

\begin{equation}
u(y, \delta) \triangleq \inf_{\eta \in D(y, \delta)} E \left[ V(\eta) \right] = \inf_{\eta \in D(y, 0)} E \left[ V(\eta L^\delta) \right], \quad (y, \delta) \in (0, \infty) \times \mathbb{R}.
\end{equation}

Under an appropriate integrability assumption specified below, existence and uniqueness of solutions to \eqref{eq:5} and \eqref{eq:6} as well as conjugacy relations between $u(\cdot, \delta)$ and $v(\cdot, \delta)$ for every $\delta$ sufficiently close to 0 will follow from \cite[Theorem 3.2]{Mos15}.

**Condition on perturbations**

Let $\xi(x, \delta)$ and $\eta(y, \delta)$ denote the solutions to \eqref{eq:5} and \eqref{eq:6}, respectively, if such solutions exist. By $\mathbb{R}(x, \delta)$ we denote the probability measure on $(\Omega, \mathcal{F})$, whose Radon-Nikodym derivative with respect to $\mathbb{P}$ is given by

\begin{equation}
\frac{d\mathbb{R}(x, \delta)}{d\mathbb{P}} \triangleq \frac{\xi(x, \delta) \eta(y, \delta)}{x y},
\end{equation}

where $x > 0$, $\delta \in \mathbb{R}$, and $y = u_x(x, \delta)$.

**Assumption 5.2.** Let there exists $c > 0$, such that

$$\mathbb{E}^{\mathbb{R}(x, 0)} \left[ \exp \left( c \left( |F| + G \right) \right) \right] < \infty.$$ 

Note that, $\mathbb{R}(x, 0)$ is well-defined for every $x > 0$. 


Expansion theorems

Auxiliary sets \( \mathcal{A} \) and \( \mathcal{B} \)

As in \cite{KS06}, for every \( x > 0 \) and \( \delta \in \mathbb{R} \), we denote by \( \mathcal{A}^\infty(x, \delta) \) the family of bounded random variables \( \alpha \), such that \( \xi(x, \delta)(1 + c\alpha) \) and \( \xi(x, \delta)(1 - c\alpha) \) belong to \( \mathcal{C}(x, \delta) \) for some constant \( c = c(\alpha) > 0 \), that is

\[
\mathcal{A}^\infty(x, \delta) = \{ \alpha \in \mathcal{L}^\infty : \xi(x, \delta)(1 + c\alpha) \in \mathcal{C}(x, \delta) \text{ for some } c > 0 \}.
\]

Likewise, for \( y > 0 \) and \( \delta \in \mathbb{R} \), we set

\[
\mathcal{B}^\infty(y, \delta) = \{ \beta \in \mathcal{L}^\infty : \eta(y, \delta)(1 + c\beta) \in \mathcal{D}(y, \delta) \text{ for some } c > 0 \}.
\]

It follows from the Assumption 5.1 that for every \( x > 0 \), \( \mathcal{A}^\infty(x, \delta) \) and \( \mathcal{B}^\infty(u_x(x, \delta), \delta) \) are orthogonal linear subspaces of \( \mathcal{L}^2(\mathbb{R}(x, \delta)) \).

Let us denote by \( \mathcal{A}^2(x, \delta) \) and \( \mathcal{B}^2(y, \delta) \) the respective closures of \( \mathcal{A}^\infty(x, \delta) \) and \( \mathcal{B}^\infty(y, \delta) \) in \( \mathcal{L}^2(\mathbb{R}(x, \delta)) \). One can see that \( \mathcal{A}^2(x, \delta) \) and \( \mathcal{B}^2(y, \delta) \) are closed orthogonal linear subspaces of \( \mathcal{L}^2(\mathbb{R}(x, \delta)) \). In order to make these sets related to the concrete versions of the expansion theorems, we need the following assumption.

Assumption 5.3. For every \( \delta \in \mathbb{R} \) and \( x > 0 \), with \( y = u_x(x, \delta) \), the sets \( \mathcal{A}^2(x, \delta) \) and \( \mathcal{B}^2(y, \delta) \) are complimentary linear subspaces in \( \mathcal{L}^2(\mathbb{R}(x, \delta)) \), i.e.

\[
\alpha \in \mathcal{A}^2(x, \delta) \iff \alpha \in \mathcal{L}^2(\mathbb{R}(x, \delta)) \text{ and } \mathbb{E}^{\mathbb{R}(x,0)}[\alpha \beta] = 0, \text{ for every } \beta \in \mathcal{B}^2(y, \delta),
\]

\[
\beta \in \mathcal{B}^2(y, \delta) \iff \beta \in \mathcal{L}^2(\mathbb{R}(x, \delta)) \text{ and } \mathbb{E}^{\mathbb{R}(x,0)}[\alpha \beta] = 0, \text{ for every } \alpha \in \mathcal{A}^2(x, \delta).
\]

The following theorem shows joint continuity, and differentiability, and is a consequence of the second-order expansion.

Theorem 5.4. Let \( x > 0 \) be fixed. Suppose that assumptions 2.1, 5.1, 5.2, and 5.3 hold, \( u(z, 0) < \infty \) for some \( z > 0 \), and \( y = u_x(x, \delta) \), which is well-defined by the abstract theorems in \cite{KS99}. Then there exists \( \delta_0 > 0 \) such that for every \( \delta \in (-\delta_0, \delta_0) \), we have

\[
u(x, \delta) \in \mathbb{R}, \ x > 0, \text{ and } \nu(y, \delta) \in \mathbb{R}, \ y > 0.
\]

In addition, \( u \) and \( v \) are jointly differentiable (and, consequently, continuous) at \( (x, 0) \) and \( (y, 0) \), respectively. We also have

\[
\nabla u(x, 0) = \begin{pmatrix} y \\ u_\delta(x, 0) \end{pmatrix} \quad \text{and} \quad \nabla v(y, 0) = \begin{pmatrix} -x \\ v_\delta(y, 0) \end{pmatrix},
\]

where

\[
\begin{align*}
\nabla u_\delta(x, 0) &= v_\delta(y, 0) = xy\mathbb{E}^{\mathbb{R}(x,0)}[F].
\end{align*}
\]
Remark 5.5. It is possible to prove Theorem 5.4 without Assumption 5.3. We do not present such a proof for brevity of exposition.

Auxiliary minimization problems

As in [KS06a], for \( x > 0 \), let us consider

\[
a(x, x) \triangleq \inf_{\alpha \in A^2(x, 0)} \mathbb{E}^{R(x, 0)} [A(\xi(x, 0))(1 + \alpha)^2],
\]

\[
b(y, y) \triangleq \inf_{\beta \in B^2(y, 0)} \mathbb{E}^{R(x, 0)} [B(\eta(y, 0))(1 + \beta)^2], \quad y = u_x(x, 0),
\]

where \( A \) is the relative risk aversion and \( B \) is the relative risk tolerance of \( U \), respectively. It is proven in [KS06a] that

\[
\begin{align*}
\alpha(x, x) &= -\frac{x}{y} a(x, x), \\
\beta(y, y) &= \frac{y}{x} b(y, y), \\
a(x, x)b(y, y) &= 1,
\end{align*}
\]

\[
A(\eta(x, 0))(1 + \alpha(x, 0)) = a(x, x)(1 + \beta(y, 0)),
\]

where \( \alpha(x, 0) \) and \( \beta(y, 0) \) are the unique solutions to (5.13) and (5.14) respectively. In order to characterize derivatives of the value functions with respect to \( \delta \), we consider the following minimization problems:

\[
a(d, d) \triangleq \inf_{\alpha \in A^2(x, 0)} \mathbb{E}^{R(x, 0)} [A(\xi(x, 0))(\alpha + xF)^2 - 2xF\alpha - x^2(F^2 + G)],
\]

\[
b(d, d) \triangleq \inf_{\beta \in B^2(y, 0)} \mathbb{E}^{R(x, 0)} [B(\eta(y, 0))(\beta - yF)^2 + 2yF\beta - y^2(F^2 - G)],
\]

Denoting by \( \alpha_d(x, 0) \) and \( \beta_d(y, 0) \) the unique solutions to (5.16) and (5.17) respectively, we also set

\[
a(x, d) \triangleq \mathbb{E}^{R(x, 0)} [A(\xi(x, 0))(1 + \alpha(x, 0))(xF + \alpha_d(x, 0)) - xF(1 + \alpha(x, 0))],
\]

\[
b(y, d) \triangleq \mathbb{E}^{R(x, 0)} [B(\eta(y, 0))(1 + \beta(y, 0))(-yF + \beta_d(y, 0)) + yF(1 + \beta(y, 0))].
\]

We are ready to state the following theorem.

**Theorem 5.6.** Let \( x > 0 \) be fixed, the conditions of Theorem 5.4 hold, and \( y = u_x(x, 0) \). Define

\[
H_u(x, 0) \triangleq -\frac{y}{x} \begin{pmatrix}
a(x, x) & a(x, d) \\
a(x, d) & a(d, d)
\end{pmatrix},
\]

where \( a(x, x) \), \( a(d, d) \), and \( a(x, d) \) are specified in (5.13), (5.16), and (5.18) respectively; and

\[
H_v(y, 0) \triangleq \frac{x}{y} \begin{pmatrix}
b(y, y) & b(y, d) \\
b(y, d) & b(d, d)
\end{pmatrix},
\]
Then, we have

\[ y = (5.25) \]

Moreover,

\[ \text{partial derivatives of the solution} \]

\[ \xi = (5.18) \]

where the convergence takes place in \( \mathbb{P} \)

\[ \text{solutions to} \]

\[ (5.14) \]

ingly. Using the formula for the gradients \( b \), and in turn,

\[ (5.24) \]

\[ (5.27) \]

\[ \text{equivalently} \]

\[ (5.28) \]

\[ \lim_{|\Delta x|+|\delta| \to 0} \frac{1}{|\Delta x|+|\delta|} \left| \frac{\xi(x+\Delta x, \delta)}{x} - \frac{\xi(x,0)}{x} (x + \Delta x (1 + \alpha(x,0)) + \delta \alpha_d(x,0)) \right| L^\delta = 0, \]

where the convergence takes place in \( \mathbb{P} \)-probability. Likewise, let \( \beta \) and \( \beta_d \), which are solutions to \( (5.14) \) and \( (5.17) \), correspondingly, are the partial derivatives of the solution \( \hat{\eta}(y,0) \) to \( (5.6) \) evaluated at \( (y,0) \), where \( y = u_x(x,0) \), in the sense that

\[ \lim_{|\Delta y|+|\delta| \to 0} \frac{1}{|\Delta y|+|\delta|} \left| \frac{\hat{\eta}(y+\Delta y, \delta)}{y} - \frac{\hat{\eta}(y,0)}{y} (y + \Delta y (1 + \beta(y,0)) + \delta \beta_d(y,0)) \right| L^\delta = 0, \]
where the convergence takes place in $\mathbb{P}$-probability.

From Theorem 5.8 we obtain the following Corollary.

**Corollary 5.9.** Under the conditions of Theorem 5.8, \( (5.28) \) is equivalent to

\[
\lim_{\|\Delta x\|+\|\delta\| \to 0} |\xi(x+\Delta x, \delta) - \xi(x, 0) - \frac{\xi(x, 0)}{x} (\Delta x(\alpha_s(x, 0) + 1) + \delta(\alpha_d(x, 0) +xF))| = 0.
\]

Likewise, \( (5.29) \) holds if and only if

\[
\lim_{\|\Delta y\|+\|\delta\| \to 0} |\eta(y+\Delta y, \delta) - \eta(y, 0) - \frac{\eta(y, 0)}{y} (\Delta y(\beta_s(y, 0) + 1) + \delta(\beta_d(y, 0) -yF))| = 0,
\]

where the convergence takes place in $\mathbb{P}$-probability.

**Proofs**

We begin the proofs with technical lemmas.

**Lemma 5.10.** Let Assumption 2.1 hold and \( d \in (\max (\exp(-1/c_2), \exp(-c_1)), 1] \). Then for every \( x > 0 \), we have

\[
U'(dx) \leq \frac{1}{1+c_2 \log(d)} U'(x),
\]

\[
-V'(dx) \leq \frac{1}{1+c_1 \log(d)} (-V'(x)).
\]

**Proof.** Let us fix an arbitrary \( x > 0 \) and \( d \in (\max (\exp(-1/c_2), \exp(-c_1)), 1] \). Then using Assumption 2.1 and monotonicity of \( U' \), we get

\[
U'(dx) - U'(x) = \int_d^1 (-U''(tx)) x dt
\]

\[
= \int_d^1 (-U''(tx)) x \frac{dt}{t}
\]

\[
\leq c_2 \int_d^1 U'(tx) \frac{dt}{t}
\]

\[
\leq c_2 U'(dx)(- \log(d)).
\]

Therefore, we obtain

\[
U'(dx)(1 + c_2 \log(d)) \leq U'(x),
\]

This implies the first assertion of the lemma. The other one can be shown entirely similarly. \( \square \)

**Corollary 5.11.** Under the conditions of Lemma 5.10, for every \( k \in \mathbb{N} \), we have

\[
U'(d^k x) \leq \frac{1}{(1+c_2 \log(d)^k)} U'(x),
\]

\[
-V'(d^k x) \leq \frac{1}{(1+c_1 \log(d)^k)} (-V'(x)).
\]

Below \( 1_E \) denotes the indicator function of a set \( E \).

**Lemma 5.12.** Let Assumption 2.1 holds. Then for every \( z \in (0, 1] \) and \( x > 0 \), we have

\[
U'(zx) \leq z^{-c_2} U'(x),
\]

\[
-V'(zx) \leq z^{-\frac{1}{c_1}} (-V'(x)).
\]
Let us fix an arbitrary $d \in (\exp(-1/c_2), 1)$. Using monotonicity of $U''$ and Corollary 5.11 for every $z \in (0, 1]$ and $x > 0$, we get

$$U''(zx) = \sum_{k=1}^{\infty} U''(d^k x) 1_{\{z \in (d^k, d^{k-1})\}}$$

(5.30)

$$\leq \sum_{k=1}^{\infty} U''(d^k x) 1_{\{z \in (d^k, d^{k-1})\}}$$

$$\leq U'(x) \sum_{k=1}^{\infty} \frac{1}{(1+c_2 \log(d))} 1_{\{z \in (d^k, d^{k-1})\}}.$$

Let us set

$$a_1(d) \triangleq \frac{1}{1 + c_2 \log(d)} > 1 \quad \text{and} \quad a_2(d) \triangleq \log(1 + c_2 \log(d)) = -\frac{\log(a_1(d))}{\log(d)} > 0.$$

As $a_1(d) > 1$ and for every $k \in \mathbb{N}$

$$d^k < z \leq d^{k-1} \quad \text{is equivalent to} \quad \frac{\log(z)}{\log(d)} < k \leq \frac{\log(z)}{\log(d)} + 1,$$

we deduce that for every $z \in (0, 1]$, we have

$$\frac{1}{(1+c_2 \log(d))} 1_{\{z \in (d^k, d^{k-1})\}} \leq a_1(d) a_1(d) \frac{\log(z)}{\log(d)} 1_{\{z \in (d^k, d^{k-1})\}}$$

(5.31)

$$= a_1(d) \left( a_1(d) \frac{1}{\log(d)} \right) \log(z) 1_{\{z \in (d^k, d^{k-1})\}}$$

$$= a_1(d) z^{-a_2(d)} 1_{\{z \in (d^k, d^{k-1})\}}.$$

Plugging (5.31) in (5.30), we get

$$U'(zx) \leq U'(x) \sum_{k=1}^{\infty} a_1(d) z^{-a_2(d)} 1_{\{z \in (d^k, d^{k-1})\}} = a_1(d) z^{-a_2(d)} U'(x), \quad \text{for every} \quad z \in (0, 1] \text{and} \quad x > 0.$$

As $\lim_{d \uparrow 1} a_1(d) = 1$ and

$$\lim_{d \uparrow 1} a_2(d) = \lim_{y \uparrow 0} \frac{\log(1 + c_2 \log(d))}{\log(d)} = \lim_{y \uparrow 0} \frac{\log(1 + c_2 y)}{y} = \lim_{y \uparrow 0} \frac{c_2}{1 + c_2 y} = c_2,$$

taking the limit in the latter inequality, we obtain that

$$U'(zx) \leq \lim_{d \uparrow 1} a_1(d) z^{-a_2(d)} U'(x) = z^{-c_2} U'(x),$$

for every $z \in (0, 1]$ and $x > 0$. The other assertion can be proven similarly. This completes the proof of the lemma.

Corollary 5.13. Under Assumption 2.7, for every $z > 0$ and $x > 0$, we have

$$U''(zx) \leq \max (z^{-c_2}, 1) U'(x) \leq (z^{-c_2} + 1) U'(x),$$

$$-V''(zx) \leq \max \left( z^{-\frac{1}{b}}, 1 \right) (-V'(x)) \leq (z^{-\frac{1}{b}} + 1) (-V'(x)).$$
Lemma 5.14. Let $x > 0$ be fixed and the conditions of Theorem 5.4 hold, and $y = u_x(x, 0)$. For arbitrary random variables $\alpha^0$ and $\alpha^1$ in $\mathcal{A}^\infty(x, 0)$, let us define

\begin{align}
\psi(s, t) &\triangleq \frac{1}{x} (s + (1 + \alpha^0) + t\alpha^1) \frac{1}{x}, \\
w(s, t) &\triangleq \mathbb{E} [U(\xi_\psi(s, t))], \quad (s, t) \in \mathbb{R}^2,
\end{align}

where $\xi = \tilde{\xi}(x, 0)$ is the solution to \((5.35)\) corresponding to $x > 0$ and $\delta = 0$. Then $w$ admits the following second-order expansion at $(0, 0)$.

\begin{equation}
w(s, t) = w(0, 0) + (s \ t) \nabla w(0, 0) + \frac{1}{2} (s \ t) H_w \begin{pmatrix} s \\ t \end{pmatrix} + o(s^2 + t^2),
\end{equation}

where\[ w_x(0, 0) = u_x(x, 0), \]
\[ w_t(0, 0) = x y \mathbb{E}^R(x, 0) [F], \]
and\[ H_w \triangleq \begin{pmatrix} w_{ss}(0, 0) & w_{st}(0, 0) \\ w_{st}(0, 0) & w_{tt}(0, 0) \end{pmatrix}, \]
where the second-order partial derivatives of $w$ at $(0, 0)$ are given by\[ w_{ss}(0, 0) = -\frac{2}{x} \mathbb{E}^R(x) [A(\xi)(1 + \alpha^0)^2], \]
\[ w_{st}(0, 0) = -\frac{2}{x} \mathbb{E}^R(x) [A(\xi)(1 + \alpha^0)(xF + \alpha^1) - xF(1 + \alpha^0)], \]
\[ w_{tt}(0, 0) = -\frac{2}{x} \mathbb{E}^R(x) [A(\xi)(\alpha^1 + xF)^2 - 2xF\alpha^1 - x^2(F^2 + G)]. \]

Proof. As $\alpha^0$ and $\alpha^1$ are in $\mathcal{A}^\infty$, there exists constant $\varepsilon \in (0, 1)$, such that

\begin{equation}
|\alpha^0| + |\alpha^1| \leq \frac{x}{6\varepsilon} - 1, \quad \mathbb{P} - a.s.
\end{equation}

Let us fix an arbitrary $(s, t) \in B_\varepsilon(0, 0)$ and define\[ \tilde{\psi}(z) \triangleq \psi(sz, tz), \quad z \in (-1, 1). \]

Note that\[ \frac{2}{3} \leq \tilde{\psi}(z) L^x \leq \frac{4}{3}, \quad z \in (-1, 1). \]

As\[ \psi_t(s, t) = \frac{\alpha^1}{xL^t} + \psi(s, t) (F + tG) \quad \text{and} \quad \psi_s(s, t) = \frac{1 + \alpha^0}{xL^s}, \]
we get\[ \tilde{\psi}'(z) = \psi_s(sz, tz) s + \psi_t(sz, tz) t = \frac{1 + \alpha^0}{2L^s} s + \left( \frac{\alpha^1}{xL^t} + \tilde{\psi}(z) (F + tzG) \right) t. \]

Similarly, since\[ \psi_{tt}(s, t) = \frac{2\alpha^0}{xL^t} (F + tG) + \psi(s, t) ((F + tG)^2 + G), \]
\[ \psi_{st}(s, t) = \frac{1 + \alpha^0}{xL^t} (F + tG), \quad \text{and} \quad \psi_{ss}(s, t) = 0, \]
we obtain
\[
\tilde{\psi}''(z) = \psi_u(zs,zt)t^2 + 2\psi_{st}(zs,zt)ts + \psi_{ss}(zs,zt)s^2 \\
= \left(\frac{2\alpha}{xL}(F + ztG) + \tilde{\psi}(z)\left((F + ztG)^2 + G\right)\right)t^2 + 2\frac{1}{xL^2\alpha}(F + ztG)ts.
\]

Setting \(W(z) \triangleq U(\xi\tilde{\psi}(z)), z \in (-1,1)\), by direct computations, we get
\[
W'(z) = U'(\xi\tilde{\psi}(z))\xi\hat{\psi}'(z), \\
W''(z) = U''(\xi\tilde{\psi}(z))\left(\hat{\psi}'(z)\right)^2 + U'(\xi\tilde{\psi}(z))\hat{\psi}''(z).
\]

Let us define
\[
a_2 \triangleq 2^{c_2+2} \quad \text{and} \quad J \triangleq 1 + |F| + G.
\]

From (5.36) using (5.34) and (5.35), we get
\[
|\tilde{\psi}'(z)| \leq 2J \exp(\varepsilon J), \quad \tilde{\psi}'(z) = 2^{c_2+1} \exp(c_2 \varepsilon J), \quad z \in (-1,1).
\]

Therefore, from (5.37) using Corollary 5.13, we obtain
\[
\sup_{z \in (-1,1)} |W'(z)| \leq \sup_{z \in (-1,1)} U'(|\xi|\tilde{\psi}'(z) - c_2 + 1) \left|\tilde{\psi}'(z)\right| \\
\leq a_2 U'(\xi)J \exp((c_2 + 1)\varepsilon J) \\
\leq a_2 U'(\xi)J \exp(a_2 \varepsilon J).
\]

Similarly, from (5.37) applying Assumption 2.1 and Corollary 5.13, we deduce the existence of a constant \(a_3 > 0\), such that
\[
\sup_{z \in (-1,1)} |W''(z)| \leq a_3 U'(\xi)J^2 \exp(a_3 \varepsilon J).
\]

Combining (5.38) and (5.39), we obtain
\[
\sup_{z \in (-1,1)} \left(|W'(z)| + |W''(z)|\right) \leq U'(|\xi|\tilde{\psi}'(z) - a_2 J \exp(a_2 \varepsilon J) + a_3 J^2 \exp(a_3 \varepsilon J))
\]

Consequently, as \(1 \leq J \leq J^2\), by setting \(a_1 \triangleq \max(a_2, a_3)\), for every \(z_1\) and \(z_2\) in \((-1,1)\), we get
\[
\left|\frac{W(z_1) - W(z_2)}{z_1 - z_2}\right| + \left|\frac{W'(z_1) - W'(z_2)}{z_1 - z_2}\right| \leq 4a_1 U'(\xi)J^2 \exp(a_1 \varepsilon J).
\]

By passing to a smaller \(\varepsilon\), if necessary, and by applying Hölder’s inequality, we deduce from Assumption 5.2 that the right-hand side of (5.40) integrable. As the right-hand side of (5.40) depends on \(\varepsilon\) (and not on \((s,t)\), the assertion of the lemma follows from the dominated convergence theorem.

\(\square\)
Corollary 5.15. Let let $x > 0$ be fixed, the conditions of Theorem 5.4 hold, and $y = u_x(x,0)$. Then, we have
\begin{equation}
(5.41)
\end{equation}
$$
\begin{align*}
u(x + \Delta x, \delta) & \geq u(x,0) + \Delta xy + \delta xy \mathbb{E}^{R(x,0)} [\mathcal{F}] + \frac{1}{2}(\Delta x \delta) H_u(x,0) \left( \frac{\Delta x}{\delta} \right) + o(\Delta x^2 + \delta^2),
\end{align*}
$$
where $H_u(x,0)$ is given by (5.20).

Proof. The result follows from Lemma 5.14 via the approximation of the solutions to (5.13) and (5.16), which are the elements of $A^2(x,0)$, by the elements of $A^\infty(x,0)$. □

Similarly to Lemma 5.14 and Corollary 5.15, we can establish the following results.

Lemma 5.16. Let $x > 0$ be fixed, the conditions of Theorem 5.4 hold, and $y = u_x(x,0)$. For arbitrary random variables $\beta^0$ and $\beta^1$ in $\mathcal{B}^\infty(y,0)$, let us define
$$
\begin{align*}
\phi(s,t) & \triangleq \frac{1}{y} (y + s(1 + \beta^0) + t\beta^1) \mathcal{L}, \\
\bar{w}(s,t) & \triangleq \mathbb{E} [V(\eta \phi(s,t))], \quad (s,t) \in \mathbb{R}^2,
\end{align*}
$$
where $\eta = \hat{\eta}(y,0)$ is the solution to (5.6) corresponding to $y > 0$ and $\delta = 0$. Then at $(0,0)$, $\bar{w}$ admits the following second-order expansion
$$
\bar{w}(s,t) = \bar{w}(0,0) + (s \ t) \nabla \bar{w}(0,0) + \frac{1}{2}(s \ t) \tilde{H}_{\bar{w}} \left( \begin{array}{c} s \\ t \end{array} \right) + o(s^2 + t^2),
$$
where
$$
\begin{align*}
\bar{w}_x(0,0) & = v_y(y,0), \\
\bar{w}_y(0,0) & = xy \mathbb{E}^{R(x,0)} [\mathcal{F}],
\end{align*}
$$
and
$$
\tilde{H}_{\bar{w}} \triangleq \begin{pmatrix}
\tilde{w}_{xx}(0,0) & \tilde{w}_{xt}(0,0) \\
\tilde{w}_{xt}(0,0) & \tilde{w}_{tt}(0,0)
\end{pmatrix},
$$
where the second-order partial derivatives of $\bar{w}$ at $(0,0)$ are given by
$$
\begin{align*}
\tilde{w}_{xx}(0,0) & = \frac{y}{y} \mathbb{E}^{R(x,0)} \left[ B(\eta)(1 + \beta^0)^2 \right], \\
\tilde{w}_{xt}(0,0) & = \frac{y}{y} \mathbb{E}^{R(x,0)} \left[ B(\eta)(1 + \beta^0)(-yF + \beta^1) + yF(1 + \beta^0) \right], \\
\tilde{w}_{tt}(0,0) & = \frac{y}{y} \mathbb{E}^{R(x,0)} \left[ B(\eta)(\beta^1 - yF)^2 + 2yF\beta^1 - y^2(F^2 - G) \right].
\end{align*}
$$

Lemma 5.17. Let let $x > 0$ be fixed, the conditions of Theorem 5.4 hold, and $y = u_x(x,0)$. Then, we have
\begin{equation}
(5.42)
\end{equation}
$$
\begin{align*}
v(y + \Delta y, \delta) & \leq v(y,0) - \Delta xy + \delta xy \mathbb{E}^{R(x,0)} [\mathcal{F}] + \frac{1}{2}(\Delta y \delta) H_v(y,0) \left( \frac{\Delta y}{\delta} \right) + o(\Delta y^2 + \delta^2),
\end{align*}
$$
where $H_v(y,0)$ is given by (5.21).
CLOSING THE DUALITY GAP

We begin from the proof of Theorem 5.7.

Proof of Theorem 5.7. It follows from [KS06a, Lemma 2] that

\begin{align*}
A(\xi)(1 + \alpha) &= a(x, x)(1 + \beta), \\
B(\eta)(1 + \beta) &= b(y, y)(1 + \alpha).
\end{align*}

Using standard techniques from calculus of variations, we can show that the solutions to (5.16) and (5.17) satisfy

\begin{align*}
A(\xi)(\alpha d + xF) - xF &= c + \tilde{\beta}, \\
B(\eta)(\beta d - yF) + yF &= d + \tilde{\alpha},
\end{align*}

where \(\tilde{\beta} \in B^2(y, \delta)\), \(\tilde{\alpha} \in A^2(x, \delta)\), and \(c\) and \(d\) are some constants. We will characterize \(\tilde{\beta}, \tilde{\alpha}\), and \(d\) below. Let us set

\begin{align*}
\tilde{\alpha} &\equiv \tilde{\alpha} - da \in A^2(x, 0).
\end{align*}

It follows from the second equation in (5.44) that

\begin{align*}
\alpha d - yF &= A(\xi) (d - yF + \tilde{\alpha}) \\
&= A(\xi) (d + da - yF + \tilde{\alpha} - da) \\
&= da(x, x)(1 + \beta) + A(\xi) (-yF + \tilde{\alpha}),
\end{align*}

where we have used (5.43) in the last equality. Multiplying by \(-\frac{x}{y}\), we obtain

\begin{align*}
A(\xi) \left( xF - \frac{x}{y} \tilde{\alpha} \right) &= -\frac{x}{y}(\beta d - yF) - \frac{x}{y}da(x, x)(1 + \beta),
\end{align*}

and thus

\begin{align*}
A(\xi) \left( xF - \frac{x}{y} \tilde{\alpha} \right) - xF &= \tilde{d} + \tilde{\beta},
\end{align*}

where

\begin{align*}
\tilde{d} = -\frac{x}{y}da(x, x) \in \mathbb{R} \quad \text{and} \quad \tilde{\beta} = -\frac{x}{y}da(x, x)\beta - \frac{x}{y}d \in B^2(y, 0).
\end{align*}

It follows from characterization of the unique solution to (5.16) given by (5.44) that

\begin{align*}
-\frac{x}{y} \tilde{\alpha} = \alpha d, \quad \text{equivalently} \quad \tilde{\alpha} = -\frac{y}{x} \alpha d.
\end{align*}

From (5.45), we obtain

\begin{align*}
\tilde{\alpha} = \tilde{\alpha} + da = -\frac{y}{x} \alpha d + da.
\end{align*}

Plugging this back into the second equality in (5.44), we get

\begin{align*}
B(\eta)(\beta d - yF) &= d(1 + \alpha) - \frac{y}{x}(\alpha d + xF).
\end{align*}

Multiplying by \(\frac{x}{y} A(\xi)\), we obtain

\begin{align*}
A(\xi)(\alpha d + xF) &= \frac{x}{y}da(x, x)(1 + \beta) - \frac{x}{y}(\beta d - yF).
\end{align*}
Setting \( d' \triangleq \frac{x}{y} a(x, x) \), we claim that
\[
(5.48) \quad d' = a(x, d),
\]
where \( a(x, d) \) is defined in (5.18). Multiplying both sides of (5.47) by \((1 + \alpha)\), taking expectation under \( \mathbb{R}(x, 0) \), and using orthogonality of the elements of \( \mathcal{A}^2(x, 0) \) and \( \mathcal{B}^2(y, 0) \), we get
\[
\mathbb{E}^{\mathbb{R}(x, 0)} \left[ A(\xi)(\alpha_d + x F)(1 + \alpha) \right] = d' \mathbb{E}^{\mathbb{R}(x, 0)} \left[ (1 + \beta)(1 + \alpha) \right] - \frac{x}{y} \mathbb{E}^{\mathbb{R}(x, 0)} \left[ (\beta_d - y F)(1 + \alpha) \right] = d' + \mathbb{E}^{\mathbb{R}(x, 0)} \left[ x F(1 + \alpha) \right].
\]
Therefore,
\[
d' = \mathbb{E}^{\mathbb{R}(x, 0)} \left[ A(\xi)(\alpha_d + x F)(1 + \alpha) - x F(1 + \alpha) \right] = a(x, d),
\]
where in the last equality we have used (5.18). Thus, (5.48) holds. Now, (5.47) with \( \frac{x}{y} a(x, x) = a(x, d) \) and (5.43) prove (5.26). (5.27) can be shown similarly. As \( A(\xi) = \frac{1}{\beta_0} \), (5.26) and (5.27) imply (5.24).

It remains to prove (5.25). Let us set
\[
\bar{\beta} \triangleq \beta + 1, \quad \bar{\alpha} \triangleq \alpha + 1,
\]
\[
\bar{\beta}_d \triangleq \beta_d - y F, \quad \bar{\alpha}_d \triangleq \alpha_d + x F.
\]
Then from (5.16) using (5.26), we get
\[
(5.49) \quad \frac{x}{y} a(\delta, \delta) = \mathbb{E}^{\mathbb{R}(x, 0)} \left[ \frac{x}{y} a(x, d) \bar{\beta} \bar{\alpha} - \bar{\beta}_d \bar{\alpha}_d \right] - \frac{y}{x} \mathbb{E}^{\mathbb{R}(x, 0)} \left[ 2x F \bar{\alpha}_d \right] - xy \mathbb{E}^{\mathbb{R}(x, 0)} \left[ F^2 + G \right].
\]
Likewise, from (5.17) via (5.49), we obtain
\[
(5.50) \quad \frac{x}{y} b(d, d) = \mathbb{E}^{\mathbb{R}(x, 0)} \left[ \frac{x}{y} b(y, d) \bar{\beta}_d - \bar{\beta}_d \bar{\alpha}_d + 2 \bar{\beta}_d x F - xy (F^2 - G) \right].
\]
Let us define
\[
T_1 \triangleq \mathbb{E}^{\mathbb{R}(x, 0)} \left[ \frac{x}{y} a(x, d) \bar{\beta} \bar{\alpha} + \frac{x}{y} b(y, d) \bar{\beta}_d \right],
\]
and
\[
T_2 \triangleq \mathbb{E}^{\mathbb{R}(x, 0)} \left[ -2 \bar{\beta}_d \bar{\alpha}_d - 2 y F \bar{\alpha}_d + 2 x F \beta_d - 2 x y F^2 \right].
\]
Then, adding (5.49) and (5.50), we deduce that
\[
(5.51) \quad \frac{x}{y} a(\delta, \delta) + \frac{x}{y} b(d, d) = T_1 + T_2.
\]
Let us rewrite \( T_2 \) as
\[
T_2 = \mathbb{E}^{\mathbb{R}(x, 0)} \left[ -2 \bar{\beta}_d \bar{\alpha}_d - 2 y F \bar{\alpha}_d + 2 x F \beta_d - 2 x y F^2 \right] + \mathbb{E}^{\mathbb{R}(x, 0)} \left[ -2 \beta_d \bar{\alpha}_d + 2 \beta_d x F + 2 y F \bar{\alpha}_d + 2 x y F^2 - 2 y F \bar{\alpha}_d + 2 x F \beta_d - 2 x y F^2 \right] = \mathbb{E}^{\mathbb{R}(x, 0)} \left[ -2 \beta_d \bar{\alpha}_d \right] = 0.
\]
as all the terms under the expectation cancel except for $-2\beta_d\alpha_d$, which has still 0 expectation by orthogonality of $A^2(x,0)$ and $B^2(y,0)$. Let us consider $T_1$. First, from (5.24), we get

$$x \frac{y}{y} b(y, d) = \frac{a(x, d)}{a(x, x)} = a(x, d)b(y, y).$$

Therefore, we can rewrite $T_1$ as

$$T_1 = \mathbb{E}^{R(x,0)} \left\{ \frac{2}{a} a(x, d) \bar{\beta} \alpha_d + a(x, d) b(y, y) \bar{\alpha} \bar{\beta}_d \right\} = a(x, d) \mathbb{E}^{R(x,0)} \left\{ \frac{2}{a} \bar{\beta} \alpha_d + b(y, y) \bar{\alpha} \bar{\beta}_d \right\}.
$$

On the other hand, from (5.19) we can express $b(y, d)$ in terms of $\bar{\beta}$, $\bar{\beta}_d$, $\bar{\alpha}$, and $\bar{\alpha}_d$ as follows.

$$b(y, d) = \mathbb{E}^{R(x,0)} \left\{ B(\eta) \bar{\beta} \bar{\beta}_d + \frac{y}{a} \bar{\beta} \alpha_d \right\} = \mathbb{E}^{R(x,0)} \left\{ b(y, y) \bar{\alpha} \bar{\beta}_d + \frac{y}{a} \bar{\beta} \alpha_d \right\},$$

where in the last equality we have used (5.43). Comparing (5.55) with (5.54), we get

$$T_1 = a(x, d)b(y, d).$$

Plugging this into (5.51) and using (5.52), we deduce that

$$\frac{y}{x} a(d, d) + \frac{x}{y} b(d, d) = a(x, d)b(y, d),$$

i.e. (5.25) holds. This completes the proof of the lemma. \qed

**Lemma 5.18.** Let $x > 0$ be fixed, the conditions of Theorem 5.4 hold, and $y = u_x(x,0)$. Then, for

$$\Delta y = -\frac{y}{xb(y, y)} \left( \frac{x}{y} b(y, d) \delta + \Delta x \right),$$

we have

$$\left( \frac{\Delta y}{\delta} \right)^T H_v(y, 0) \left( \frac{\Delta y}{\delta} \right) + 2\Delta x \Delta y = \left( \frac{\Delta x}{\delta} \right)^T H_u(x, 0) \left( \frac{\Delta x}{\delta} \right) - \frac{x}{y} b(d, d) \delta^2.$$ 

**Proof.** First, note that $b(y, y) > 0$ in (5.56). By direct computations, proving (5.57) is equivalent to establishing the following equality.

$$-\frac{y}{xb(y, y)} \left( \frac{x}{y} b(y, d) \delta + \Delta x \right)^2 = \left( \frac{\Delta x}{\delta} \right)^T H_u(x, 0) \left( \frac{\Delta x}{\delta} \right) - \frac{x}{y} b(d, d) \delta^2.$$ 

Now, let us consider the right-hand side or (5.58). By direct computations, it can be rewritten as follows.

$$-\frac{y}{2} \Delta x^2 a(x, x) + 2\Delta x \delta \left( -\frac{y}{x} a(x, d) \right) - \delta^2 \left( \frac{y}{x} a(d, d) + \frac{x}{y} b(d, d) \right) = -\frac{y}{xb(y, y)} \Delta x^2 + 2\Delta x \delta \left( -\frac{y}{x} a(x, d) \right) - \delta^2 a(x, d)b(y, d).$$
where the last equality follows from (5.15) and (5.25). We deduce from (5.24) that
\[(5.60)\]
\[a(x, d) = \frac{x b(y, d)}{y b(y, y)}.\]
Plugging (5.60) into (5.59), we can rewrite the right-hand side of (5.59) as
\[-\frac{y}{x b(y, y)} \Delta x^2 - 2 \Delta x \delta x \frac{b(y, d)}{b(y, y)} - \delta^2 \frac{x (b(y, d))^2}{y b(y, y)} = -\frac{y}{x b(y, y)} \left( \Delta x + \frac{\delta x}{y} b(y, d) \right)^2,
\]
which is precisely the left-hand side of (5.58). We have just shown that (5.58) holds. By the argument preceding (5.58), this implies that (5.57) is valid as well. This completes the proof of the lemma.

**Lemma 5.19.** Let \(x > 0\) be fixed, the conditions of Theorem 5.4 hold, and \(y = u_x(x, 0)\).

Then, we have
\[(5.61)\]
\[u(x + \Delta x, \delta) = u(x, 0) + \Delta xy + \delta xy \mathbb{E}^{\mathbb{R}(x, 0)} [F] + \frac{1}{2}(\Delta x \quad \delta) H_u(x, 0) \left( \begin{array}{c} \Delta x \\ \delta \end{array} \right) + o(\Delta x^2 + \delta^2),
\]
where \(H_u(x, 0)\) is given by (5.20). Likewise
\[(5.62)\]
\[v(y + \Delta y, \delta) = v(y, 0) - \Delta yx + \delta xy \mathbb{E}^{\mathbb{R}(x, 0)} [F] + \frac{1}{2}(\Delta y \quad \delta) H_v(y, 0) \left( \begin{array}{c} \Delta y \\ \delta \end{array} \right) + o(\Delta y^2 + \delta^2),
\]
where \(H_v(y, 0)\) is given by (5.21).

**Proof.** For small \(\Delta x\) and \(\delta\) and with \(\Delta y\) given by (5.56), we get from conjugacy of \(u\) and \(v\) and Lemma 5.17 that
\[(5.63)\]
\[u(x + \Delta x, \delta) \leq v(y + \Delta y, \delta) + (x + \Delta x)(y + \Delta y)
\leq v(y, 0) - \Delta yx + \delta xy \mathbb{E}^{\mathbb{R}(x, 0)} [F] + \frac{1}{2}(\Delta y \quad \delta) H_v(y, 0) \left( \begin{array}{c} \Delta y \\ \delta \end{array} \right)
+ xy + y \Delta x + x \Delta y + \Delta x \Delta y + o(\Delta y^2 + \delta^2),
\]
where \(H_v(y, 0)\) is given in (5.21). As \(y = u_x(x, 0)\) and \(x = -v_y(y, 0)\), collecting terms in the right-hand side of (5.63), we obtain
\[(5.64)\]
\[u(x + \Delta x, \delta) \leq u(x, 0) + \Delta xy + \delta xy \mathbb{E}^{\mathbb{R}(x, 0)} [F] + \frac{1}{2}(\Delta x \quad \delta) H_u(x, 0) \left( \begin{array}{c} \Delta x \\ \delta \end{array} \right) + \Delta x \Delta y + o(\Delta x^2 + \delta^2).
\]
Likewise, using Corollary 5.15, we get
\[(5.65)\]
\[u(x + \Delta x, \delta) \geq u(x, 0) + \Delta xy + \delta xy \mathbb{E}^{\mathbb{R}(x, 0)} [F] + \frac{1}{2}(\Delta x \quad \delta) H_u(x, 0) \left( \begin{array}{c} \Delta x \\ \delta \end{array} \right) + o(\Delta x^2 + \delta^2).
\]
By Lemma 5.18, the quadratic terms in (5.64) and (5.65) are equal. Therefore, (5.64) and (5.65) imply that \( u \) admits a second-order expansion at \((x, 0)\) given by (5.61). Similarly we can prove (5.62).

\[ \square \]

**Proof of Theorem 5.4.** The assertions of Theorem 5.4 follow from Lemma 5.19.

\[ \square \]

**Proof of Theorem 5.6.** Expansions (5.22) and (5.23) follow from Lemma 5.19 and Theorem 5.4.

\[ \square \]

**Derivatives of the optimizers**

We begin with a technical lemma.

**Lemma 5.20.** Let \( x > 0 \) be fixed, the conditions of Theorem 5.4 hold, \( y = u_x(x, 0) \), and let \((\delta^n)_{n \geq 1}\) be a sequence, which converges to 0. Then, we have

\[
\lim_{n \to \infty} \mathbb{E} \left[ V(\tilde{\eta}(y, 0) L^{\delta^n}) \right] = v(y, 0).
\]

**Proof.** The proof goes along the lines of the proof of Lemma 5.14, it is therefore skipped.

\[ \square \]

**Lemma 5.21.** Let \( x > 0 \) be fixed, the conditions of Theorem 5.4 hold, \( y = u_x(x, 0) \), and \((y^n, \delta^n)_{n \in \mathbb{N}}\) be a sequence, which converges to \((y, 0)\). Then \( \eta^n \triangleq \tilde{\eta}(y^n, \delta^n) \), \( n \geq 1 \), converges to \( \eta \triangleq \tilde{\eta}(y, 0) \) in probability and \( V(\eta^n) \), \( n \geq 1 \), converges to \( V(\eta) \) in \( L^1(\mathbb{P}) \).

**Proof.** In view of Theorem 5.4, without loss of generality, we may assume that \( v(y^n, \delta^n) \) is finite for every \( n \in \mathbb{N} \). Let us assume by contradiction that \((\eta^n)_{n \in \mathbb{N}}\) does not converge in probability to \( \eta \). Then there exists \( \varepsilon > 0 \), such that

\[
\limsup_{n \to \infty} \mathbb{P} \left[ |\eta^n - \eta| > \varepsilon \right] > \varepsilon.
\]

Let us define \( \tilde{\eta}^n \triangleq \frac{y^n}{L^{\delta^n}} \), \( n \geq 1 \), and \( \tilde{y} \triangleq \sup y^n \). As \((\tilde{\eta}^n)_{n \in \mathbb{N}} \subset \mathcal{D}(\tilde{y}, 0) \) and \((L^{\delta^n})_{n \in \mathbb{N}}\) converges to 1 in probability (therefore, in particular \((L^{\delta^n})_{n \in \mathbb{N}}\) is bounded in \( L^0 \)), by possibly passing to a smaller \( \varepsilon \), we may assume that

\[
\limsup_{n \to \infty} \mathbb{P} \left[ |\eta^n - \eta| > \varepsilon, \ |\tilde{\eta}^n - \eta| L^{\delta^n} \leq \frac{1}{2} \right] > \varepsilon.
\]

Let us define

\[
h^n \triangleq \frac{1}{2} (\tilde{\eta}^n + \eta) L^{\delta^n} = \frac{1}{2} \left( \eta^n + \eta L^{\delta^n} \right) \in \mathcal{D} \left( \frac{y_0 + \tilde{y}}{2}, \delta^n \right), \quad n \geq 1.
\]

From convexity of \( V \), we have

\[
V(h^n) \leq \frac{1}{2} \left( V(\eta^n) + V(\eta L^{\delta^n}) \right),
\]

\[ \square \]
and from the strict convexity of $V$, we deduce the existence of a positive constant $\varepsilon_0$, such that
\[
\limsup_{n \to \infty} \mathbb{P} \left[ V(h^n) \leq \frac{1}{2} \left( V(\eta^n) + V(\eta L^{\delta^n}) \right) - \varepsilon_0 \right] > \varepsilon_0.
\]
Therefore, using Lemma 5.20, we obtain
\[
\limsup_{n \to \infty} \mathbb{E} \left[ V(h^n) \right] \leq \frac{1}{2} \limsup_{n \to \infty} \mathbb{E} \left[ V(\eta^n) \right] + \frac{1}{2} \limsup_{n \to \infty} \mathbb{E} \left[ V(\eta L^{\delta^n}) \right] - \varepsilon_0^2
\]
(5.66)
\[
= \frac{1}{2} \limsup_{n \to \infty} v(y^n, \delta^n) + \frac{1}{2} v(y, 0) - \varepsilon_0^2
\]
\[
= v(y, 0) - \varepsilon_0^2,
\]
where in the last equality we have also used continuity of $v$ at $(y, 0)$, which follows from Theorem 5.4. On the other hand, as $h^n \in D \left( \frac{u_n + y}{2}, \delta^n \right)$, $n \geq 1$, we get
\[
\limsup_{n \to \infty} v \left( \frac{u_n + y}{2}, \delta^n \right) \leq \limsup_{n \to \infty} \mathbb{E} \left[ V(h^n) \right].
\]
(5.67)
Combining (5.66) and (5.67) and using continuity of $v$ at $(y, 0)$ again, we get
\[v(y, 0) = \limsup_{n \to \infty} v \left( \frac{u_n + y}{2}, \delta^n \right) \leq \limsup_{n \to \infty} \mathbb{E} \left[ V(h^n) \right] \leq v(y, 0) - \varepsilon_0^2,
\]
which is a contradiction as $\varepsilon_0 \neq 0$. Thus, $(\eta^n)_{n \in \mathbb{N}}$ converges to $\eta$ in probability. In turn, this and continuity of $v$ at $(y, 0)$ imply the other assertion of the lemma. \qed

Proof of Theorem 5.8. We will only prove (5.29), as (5.28) can be shown similarly. In view of Theorem 5.4 without loss of generality we will assume that for every $n \in \mathbb{N}$, $u(\cdot, \delta^n)$ and $v(\cdot, \delta^n)$ are finite-valued functions. The rest of the proof goes along the lines of the proof of Theorem 2 in [KS06a]. Let $(y^n, \delta^n)_{n \in \mathbb{N}}$ be a sequence, which converges to $(y, 0)$, where $y = u(x, 0) > 0$. Let $\hat{\eta}^n = \hat{\eta}(y^n, \delta^n)$, $n \in \mathbb{N}$, denote the corresponding dual optimizers and set
\[
\phi_1 \triangleq \frac{1}{2} \min \left\{ \hat{\eta}(y, 0), \inf_{n \in \mathbb{N}} \hat{\eta}_n \right\} > 0, \quad \mathbb{P} - a.s.
\]
\[
\phi_2 \triangleq 2 \max \left\{ \hat{\eta}(y, 0), \sup_{n \in \mathbb{N}} \hat{\eta}_n \right\} < \infty, \quad \mathbb{P} - a.s.
\]
\[
\theta \triangleq \inf_{\phi_1 \leq \phi \leq \phi_2} V_n(t).
\]
Note that the construction of $\phi_1$ and $\phi_2$ implies that $\theta > 0$, $\mathbb{P} - a.s.$ Let us also fix $\beta^0$ and $\beta^1$ in $B^\infty(y, 0)$ and define
\[
\eta^n \triangleq \frac{\hat{\eta}(y, 0)}{y} \left( y + \Delta y^n (\beta^0 + 1) + \delta^n (\beta^1) \right) \in D(y^n, \delta^n), \quad n \in \mathbb{N},
\]
where $\Delta y^n \triangleq y^n - y$. As $\beta^0$ and $\beta^1$ are bounded, without loss of generality we will assume that
\[
\frac{1}{2} \hat{\eta}(y, 0) \leq \eta^n \leq 2 \hat{\eta}(y, 0), \quad n \in \mathbb{N},
\]
SENSITIVITY ANALYSIS OF THE EXPECTED UTILITY MAXIMIZATION PROBLEM

which implies that

\[ \phi_1 \leq \eta^n \leq \phi_2. \]

Using the definition of \( \theta \), we get

\[ (5.68) \quad V(\eta^n) - V(\tilde{\eta}^n) \geq V'(\tilde{\eta}^n)(\eta^n - \tilde{\eta}^n) + \theta (\eta^n - \tilde{\eta}^n)^2. \]

By [Mos15, Theorem 3.2], \(-V'(\tilde{\eta}^n) = \hat{\xi}(x^n, \delta^n)\) is the optimal solution to (5.5) at \( x^n = -v_y(y^n, \delta^n) \), such that

\[ E[\hat{\xi}(x^n, \delta^n)\tilde{\eta}^n] = x^ny^n. \]

Moreover, the bipolar construction of the sets \( \mathcal{C}(x^n, \delta^n) \) and \( \mathcal{D}(y^n, \delta^n) \) implies that

\[ E[\hat{\xi}(x^n, \delta^n)\eta^n] \leq x^ny^n. \]

Therefore, we obtain

\[ E[V'(\tilde{\eta}^n)(\eta^n - \tilde{\eta}^n)] \geq 0. \]

Combining this with (5.68), we get

\[ (5.69) \quad E[\theta(\eta^n - \tilde{\eta}^n)^2] \leq E[V(\eta^n)] - v(y^n, \delta^n). \]

From Lemma 5.16, we deduce

\[ E[V(\eta^n)] = E[V(\eta^n)] = v(y, 0) - x\Delta y^n + v_\delta(y, 0)\delta^n + \frac{1}{2}(\Delta y^n - \delta^n)H_\bar{w}\left(\frac{\Delta y^n}{\delta^n}\right) + o((\Delta y^n)^2 + (\delta^n)^2). \]

Combining this with (5.69) and using the expansion for \( v \) from Theorem 5.6, we obtain

\[ (5.70) \quad \limsup_{n \to \infty} \frac{1}{(\Delta y^n)^2 + (\delta^n)^2} (E[\eta^n] - v(y^n, \delta^n)) \leq \frac{1}{2}\|H_\bar{w} - H_v(y, 0)\|, \]

where for a vector \( a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \) and a two-by-two matrix \( A \), we define their norms as

\[ \|a\| \triangleq \sqrt{a_1^2 + a_2^2} \quad \text{and} \quad \|A\| \triangleq \sup_{a \in \mathbb{R}^2} \|Aa\| / \|a\|. \]

In view of Lemma 5.16 (by the choice of \( \beta^0 \) and \( \beta^1 \)), we can make the right-hand side of (5.70) arbitrarily small. Combining this with (5.69), we deduce that

\[ \limsup_{n \to \infty} \frac{1}{(\Delta y^n)^2 + (\delta^n)^2} E[\theta(\eta^n - \tilde{\eta}^n)^2] \]

can also be made arbitrarily small. As \( \theta > 0 \), \( \mathbb{P} - a.s. \), the assertion of the theorem follows. \( \square \)

6. PROOFS OF THEOREMS 3.7, 3.8, 3.10, 3.12 AND 4.1

In order to link abstract theorems to their concrete counterparts, we will have to establish some structural properties of the perturbed primal and dual admissible sets first.
Characterization of primal and dual admissible sets

The following lemma gives a useful characterization of the primal and dual admissible sets after perturbations.

**Lemma 6.1.** Under Assumption (2.1), for every \( \delta \in \mathbb{R} \), we have

\[
Y(1, \delta) = Y(1, 0) \mathcal{E} \left( -\delta \nu \cdot S^0 \right), \\
X(1, \delta) = X(1, 0) \mathcal{E} \left( -\delta \nu \cdot S^0 \right).
\]

**Proof.** Let us fix \( \delta \in \mathbb{R} \). Then, for an arbitrary predictable and \( S^\delta \)-integrable process \( \pi \), let \( X^\delta \triangleq \mathcal{E} \left( \pi \cdot S^\delta \right) \). Then \( X^\delta \in \mathcal{X}(1, \delta) \). Let us consider \( X^0 \triangleq X^\delta \mathcal{E} \left( -\delta \nu \cdot S^0 \right) \). One can see that \( X^0 \in \mathcal{X}(1, 0) \). The remainder of the proof is straightforward, it is therefore skipped. \( \square \)

**Proof of Theorem 3.7** Condition (2.1) implies that the respective closures of the convex solid hulls of \( \{X_T : X \in \mathcal{X}(1, 0)\} \) and \( \{Y_T : Y \in \mathcal{Y}(1, 0)\} \) satisfy (abstract) Assumption 5.1. In view of Lemma 6.1 we have

\[
\left\{ \frac{X_T}{L^\delta} : X \in \mathcal{X}(1, 0) \right\} = \left\{ X_T : X \in \mathcal{X}(1, \delta) \right\},
\]

likewise

\[
\left\{ Y_T L^\delta : Y \in \mathcal{Y}(1, 0) \right\} = \left\{ Y_T : Y \in \mathcal{Y}(1, \delta) \right\}, \quad \delta \in \mathbb{R}.
\]

Therefore, the respective closures of convex solid hulls of

\[
\{X_T : X \in \mathcal{X}(1, \delta)\} \quad \text{and} \quad \{Y_T : Y \in \mathcal{Y}(1, \delta)\}
\]
satisfy abstract condition (5.4). The relationship between (abstract) Assumption 5.2 and Assumption 5.2 is apparent. It remains to show that the sets \( \mathcal{M}^2(x) \) and \( \mathcal{N}^2(x) \) satisfy (abstract) Assumption 5.3 However this follows from continuity of \( S^0 \) and [KS06a, Lemma 6]. Therefore, the assertions of Theorem 3.7 follow from (abstract) Theorem 5.4. \( \square \)

**Proof of Theorem 3.8** As in the proof of Theorem 3.8 the assertions of Theorem 3.8 follow from (abstract) Theorem 5.6. \( \square \)

**Proof of Theorem 3.10** Similarly to the proof of Theorem 3.10 the assertions of Theorem 3.10 follow from (abstract) Theorem 5.7. \( \square \)

**Proof of Theorem 3.12** As above, the affirmations of this theorem follow from (abstract) Theorem 5.8. \( \square \)
For the proof of Theorem 4.1, we will need the following technical lemma. First, for 
\( (\delta, \Delta x, \varepsilon) \in \mathbb{R} \times (-\infty, \infty) \times (0, \infty) \), let us set

\[
(6.1) \quad f(\delta, \Delta x, \varepsilon) \triangleq u(x, 0) + (\Delta x \delta) \nabla u(x, 0) + \frac{1}{2} (\Delta x \delta) H_u(x, 0) \left( \frac{\Delta x}{\delta} \right) - \mathbb{E} \left[ U \left( X^x_{\Delta x, \delta, \varepsilon} \right) \right],
\]

where \( \nabla u(x, 0), H_u(x, 0), \) and \( X^x_{\Delta x, \delta, \varepsilon} \)'s are defined in (3.10), (3.20), and (4.2), respectively.

**Lemma 6.2.** Assume that \( x > 0 \) is fixed and the assumptions of Theorem 3.7 hold. Then, for \( f \) defined in (6.1), there exists a monotone function \( g \), such that

\[
(6.2) \quad g(\varepsilon) \geq \lim_{|\Delta x| + |\delta| \to 0} f(\delta, \Delta x, \varepsilon), \quad \varepsilon > 0,
\]

and

\[
(6.3) \quad \lim_{\varepsilon \to 0} g(\varepsilon) = 0.
\]

**Proof.** The proof goes along the lines of the proof of Lemma 5.14. We only outline the main steps for brevity of exposition. For a fixed \( \varepsilon > 0 \), let us define

\[
(6.4) \quad \psi(\Delta x, \delta) \triangleq \frac{e^{\Delta x}}{\Delta x} \exp \left( \left( \Delta x \gamma_{0, \varepsilon} + \delta \gamma_{1, \varepsilon} \right) \cdot M^R_T - \frac{1}{2} (\Delta x \gamma_{0, \varepsilon} + \delta \gamma_{1, \varepsilon})^2 \cdot (M_T) \frac{1}{L} \right),
\]

\[
(6.5) \quad w(\Delta x, \delta) \triangleq \mathbb{E} \left[ U(X^x_T(x, 0) \psi(\Delta x, \delta)) \right], \quad (\Delta x, \delta) \in \mathbb{R}^2,
\]

where \( M^R \) is defined in (4.1). Let us first fix \( \varepsilon' > 0 \), then fix \( (\Delta x, \delta) \in B_{\varepsilon'}(0, 0) \), and set

\[
\tilde{\psi}(z) \triangleq \psi(z \Delta x, z \delta), \quad z \in (-1, 1).
\]

By direct computations, we get

\[
(6.6) \quad \tilde{\psi}'(z) = \psi_{\Delta x}(z \Delta x, z \delta) \Delta x + \psi_{\delta}(z \Delta x, z \delta) \delta,
\]

where

\[
\psi_{\Delta x}(\Delta x, \delta) = \psi(\Delta x, \delta) \left( \frac{1}{\Delta x} + (\Delta x \gamma_{0, \varepsilon} \cdot M^R_T - \left( (\Delta x \gamma_{0, \varepsilon} + \delta \gamma_{1, \varepsilon}) \gamma_{0, \varepsilon} \right) \cdot (M_T) \right),
\]

\[
\psi_{\delta}(\Delta x, \delta) = \psi(\Delta x, \delta) \left( \gamma_{1, \varepsilon} \cdot M^R_T - \left( (\Delta x \gamma_{0, \varepsilon} + \delta \gamma_{1, \varepsilon}) \gamma_{1, \varepsilon} \right) \cdot (M_T) + F + \delta G \right),
\]

where \( F \) and \( G \) are defined in (3.15). Similarly, we obtain

\[
\tilde{\psi}''(z) = \psi_{\Delta x \Delta x}(z \Delta x, z \delta) \Delta x^2 + 2 \psi_{\Delta x \delta}(z \Delta x, z \delta) \Delta x \delta + \psi_{\delta \delta}(z \Delta x, z \delta) \delta^2,
\]
where

\[
\psi_{\Delta x \Delta x}(\Delta x, \delta) = \psi(\Delta x, \delta) \left( \frac{1}{x + \Delta x} + (\Delta x \gamma^{0,\varepsilon} \cdot M_T^R - ((\Delta x \gamma^{0,\varepsilon} + \delta \gamma^{1,\varepsilon}) \gamma^{0,\varepsilon}) \cdot \langle M_T \rangle \right)^2 \\
+ \psi(\Delta x, \delta) \left( \gamma^{0,\varepsilon} \cdot M_T^R + (\gamma^{0,\varepsilon})^2 \cdot \langle M_T \rangle - \frac{1}{(x + \Delta x)^2} \right),
\]

\[
\psi_{\Delta x \delta}(\Delta x, \delta) = \psi(\Delta x, \delta) \left( \frac{1}{x + \Delta x} + (\Delta x \gamma^{0,\varepsilon} \cdot M_T^R - ((\Delta x \gamma^{0,\varepsilon} + \delta \gamma^{1,\varepsilon}) \gamma^{0,\varepsilon}) \cdot \langle M_T \rangle \right) \times \\
\times (\gamma^{1,\varepsilon} \cdot M_T^R - ((\Delta x \gamma^{0,\varepsilon} + \delta \gamma^{1,\varepsilon}) \gamma^{1,\varepsilon}) \cdot \langle M_T \rangle + F + \delta G) \\
+ \psi(\Delta x, \delta) ((\gamma^{1,\varepsilon} \gamma^{0,\varepsilon}) \cdot \langle M_T \rangle), \\
\psi_{\delta \delta}(\Delta x, \delta) = \psi(\Delta x, \delta) (\gamma^{1,\varepsilon} \cdot M_T^R - ((\Delta x \gamma^{0,\varepsilon} + \delta \gamma^{1,\varepsilon}) \gamma^{1,\varepsilon}) \cdot \langle M_T \rangle + F + \delta G)^2 \\
+ \psi(\Delta x, \delta) ((\gamma^{1,\varepsilon})^2 \cdot \langle M_T \rangle + G).
\]

Setting \( W(z) \triangleq U(\hat{X}_T(x, 0)\bar{\psi}(z)), z \in (-1, 1), \) by direct computations, we get

\[
W'(z) = U'(\hat{X}_T(x, 0)\bar{\psi}(z))\hat{X}_T(x, 0)\bar{\psi}'(z), \\
W''(z) = U''(\hat{X}_T(x, 0)\bar{\psi}(z)) \left( \hat{X}_T(x, 0)\bar{\psi}'(z) \right)^2 + U''(\hat{X}_T(x, 0)\bar{\psi}(z))\hat{X}_T(x, 0)\bar{\psi}''(z).
\]

As in Lemma 5.14 from boundedness of \( \gamma^{0,\varepsilon} \cdot M_T^R, \gamma^{1,\varepsilon} \cdot M_T^R, (\gamma^{0,\varepsilon})^2 \cdot \langle M_T \rangle, \) and \( (\gamma^{1,\varepsilon})^2 \cdot \langle M_T \rangle, \) via Corollary 5.13 and Assumption 5.2, one can show that

\[
\frac{|W(z_1) - W(z_2)|}{z_1 - z_2} + \frac{|W'(z_1) - W'(z_2)|}{z_1 - z_2} \leq \eta,
\]

for some random variable \( \eta, \) which depend on \( \varepsilon' \) and which is integrable for a sufficiently small \( \varepsilon' \). By direct computations, the derivatives of \( W \) plugged inside the expectation lead to the “exact” gradient \( \nabla u(x, 0) \) and the “approximate” Hessian \( H_u(x, 0) \). This results in (6.2). Now, approximation by \( \varepsilon \to 0 \) leads to \( H_u(x, 0) \to H_u(x, 0) \), and, therefore we obtain (6.3). Finally, one can choose \( g \) to be monotone. \( \square \)

**Proof of Theorem 4.1.** First, for \( f \) defined in (6.1), via Lemma 6.2, we deduce the existence of a monotone function \( g \), such that (6.2) and (6.3) hold. Let us define

\[
\phi(\varepsilon) \triangleq \{ (\delta, \Delta x) : f(t\delta, t\Delta x, \varepsilon) \leq 2g(\varepsilon), \text{ for every } t \in [0, 1] \}, \quad \varepsilon > 0,
\]

\[
r(\varepsilon) \triangleq \frac{1}{2} \sup \{ r \leq \varepsilon : B_r(0, 0) \subseteq \phi(\varepsilon) \}, \quad \varepsilon > 0.
\]

Note that \( r(\varepsilon) > 0 \) for every \( \varepsilon > 0 \). With

\[
\varepsilon(\delta, \Delta x) \triangleq \inf \left\{ \varepsilon : r(\varepsilon) \geq \sqrt{\Delta x^2 + \delta^2} \right\}, \quad (\delta, \Delta x) \in \mathbb{R} \times (-x, \infty),
\]

we have

\[
\lim_{|\Delta x| + |\delta| \to 0} \frac{u(x + \Delta x, \delta) - \mathbb{E} \left[ U(\hat{X}_T^{\Delta x, \delta, \varepsilon(\delta, \Delta x)}) \right]}{\Delta x^2 + \delta^2} = 0.
\]

\( \square \)
7. Counterexample

In the following example we show that even when 0-model is nice, but Assumption 3.2 fails, we might have

\[ u(z, \delta) = v(z, \delta) = \infty \quad \text{for every } \delta \neq 0 \text{ and } z > 0. \]

**Example 7.1.** Consider the 0-model, where

\[ T = 1, \quad M = B, \quad \lambda \equiv 1, \quad \text{and} \quad U(x) = \frac{x^p}{p}, \quad p \in (0, 1). \]

Let assume that \( B \) is a Brownian motion defined on a filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where the filtration \((\mathcal{F}_t)_{t \in [0, T]}\) is generated by \( B \). We recall that for the utility function \( U(x) = \frac{x^p}{p} \), the convex conjugate is \( V(y) = \frac{y^q}{q} \), where \( q = \frac{p}{1-p} \). Let \( Z^0 \) denote the martingale deflator for \( S^0 \). The direct computations yield

\[ \mathbb{E} \left[ (Z^0_t)^{-q} \right] = \exp \left( \frac{1}{2} q(q + 1) \right) \in \mathbb{R}. \]

Therefore by [KS03], the standard conclusions of the utility maximization theory hold. The primal and dual optimizers are

\[ \hat{X}_1(x, 0) = x \exp \left((q + 1)B_1 + \frac{1}{2}(1 - q^2)\right) \quad \text{and} \quad \hat{Y}_1(y, 0) = y \exp \left(-B_1 - \frac{1}{2}\right). \]

Now, let us consider a process \( \nu \) such that

\[ (7.1) \quad \nu \cdot B_1 = B_1^3, \quad \mathbb{P}-a.s. \]

Let us denote \( I_t \equiv t, \ t \in [0, 1] \). As

\[ \frac{d\mathbb{E}(x, 0)}{d\mathbb{P}} = \exp \left(-\frac{q(q + 1)}{2} \right) \exp (qB_1 + q\frac{1}{2}) = \exp \left(qB_1 - \frac{q^2}{2}\right), \quad x > 0, \]

with notations \([3.15]\), for every \( c > 0 \), we get

\[ (7.2) \quad \mathbb{E}^{(x, 0)} [\exp (|F| + G)] = \mathbb{E} \left[ \exp \left( qB_1 - \frac{q^2}{2} \right) \exp (c |\nu \cdot B_1 + \nu \cdot I_1| + c\nu^2 \cdot I_1) \right] \]

\[ = \mathbb{E} \left[ \exp \left( qB_1 - \frac{q^2}{2} + c |B_1^3 + \nu \cdot I_1| + c\nu^2 \cdot I_1 \right) \right] \]

\[ \geq \mathbb{E} \left[ \exp \left( qB_1 - \frac{q^2}{2} + c |B_1^3| - c|\nu| \cdot I_1 + c\nu^2 \cdot I_1 \right) \right] \]

\[ \geq \exp \left(-\frac{q^2}{2} - \frac{q}{4}\right) \mathbb{E} \left[ \exp \left( qB_1 + c |B_1^3| + c (|\nu| - \frac{1}{2})^2 \cdot I_1 \right) \right] \]

\[ \geq \exp \left(-\frac{q^2}{2} - \frac{q}{4}\right) \mathbb{E} \left[ \exp (qB_1 + c |B_1^3|) \right] \]

\[ = \exp \left(-\frac{q^2}{2} - \frac{q}{4}\right) \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp (qy + c|y^3| - y^2/2) dy = \infty, \]

i.e. Assumption 3.2 does not hold.

For every \( \delta \in \mathbb{R} \), we can express the local martingale deflator \( Z^\delta \) as follows

\[ Z^\delta_t = \exp \left(-(\lambda + \delta \nu) \cdot B_t - \frac{1}{2}(\lambda + \delta \nu)^2 \cdot I_t \right), \quad t \in [0, 1]. \]
For \( p \in (0, 1) \), as \( q > p > 0 \), we have
\[
E\left[ (Z_\delta^p)^{-q} \right] = E\left[ \exp \left( q(\lambda + \delta \nu) \cdot B_1 + \frac{q}{2}(\lambda + \delta \nu)^2 \cdot I_1 \right) \right] \geq E\left[ \exp \left( q(\lambda + \delta \nu) \cdot B_1 \right) \right].
\]

Therefore, using (7.1), we get
\[
E\left[ (Z_\delta^p)^{-q} \right] \geq E\left[ \exp \left( q(\lambda + \delta \nu) \cdot B_1 \right) \right] = E\left[ \exp \left( qB_1 + \frac{q}{2}(\delta \nu + 1)^2 \cdot I_1 \right) \right].
\]

for every \( \delta \neq 0 \). Consequently, \( v(1, \delta) = \infty \) for every \( \delta \neq 0 \). Moreover, one can find a constant \( D > 0 \), such that
\[
u(1, \delta) \geq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp \left( -\frac{y^2}{2} + q(y + \delta y^3) \right) dy = \infty,
\]
for every \( \delta \neq 0 \).

8. Relationship to the risk-tolerance wealth process

Following [KS06b], we recall that for an initial wealth \( x > 0 \) and \( \delta \in \mathbb{R} \), the risk-tolerance wealth process is a maximal wealth process \( R(x, \delta) \), such that
\[
R_T(x, \delta) = -U'(\tilde{X}_T(x, \delta)) \frac{U''(\tilde{X}_T(x, \delta))}{U''(\tilde{X}_T(x, \delta))},
\]
i.e. it is a replication process for the random payoff given by the right-hand side of (8.1). In general the risk-tolerance wealth process \( R(x, \delta) \) may not exist. It is shown in [KS06b] that the existence of the risk-tolerance wealth process is closely related to some important properties of the marginal utility-based prices and to the validity of the second-order expansions of the value functions under the presence of random endowment. Below we establish a relationship between the existence of \( R(x, 0) \) and the second-order expansions of the value functions in the present context.

The following theorem is a version of [KS06b, Theorem 4]. Despite the fact that the assertions of [KS06b, Theorem 4] are obtained under the existence of an equivalent martingale measure assumption in [KS06b], the proof goes through also under condition (2.1), no changes are needed. Therefore, the proof of the following theorem is not presented.

**Theorem 8.1.** Let \( x > 0 \) be fixed, assume that (2.1), (3.1), and Assumption 2.1 hold, and denote \( y = u_x(x, 0) \). Then the following assertions are equivalent:
The risk-tolerance wealth process $R(x, 0)$ exists.

The value function $u$ admits the expansion (5.22) at $(x, 0)$ and $u_{xx}(x, 0) = -\frac{2}{x} a(x, x)$ satisfies

$$
\frac{(u_x(x, 0))^2}{u_{xx}(x, 0)} = \mathbb{E} \left[ \frac{\left( U'(\hat{X}_T(x, 0)) \right)^2}{U''(\hat{X}_T(x, 0))} \right],
$$

$$
u_{xx}(x, 0) = \mathbb{E} \left[ U''(\hat{X}_T(x, 0)) \left( \frac{R_T(x, 0)}{R_0(x, 0)} \right)^2 \right].
$$

The value function $v$ admits the expansion (5.23) at $(y, 0)$ and $v_{yy}(y, 0) = \frac{x}{y} b(y, y)$ satisfies

$$
y^2 v_{yy}(y, 0) = \mathbb{E} \left[ \left( \hat{Y}_T(y, 0) \right)^2 V''(\hat{Y}_T(y, 0)) \right] = xy \mathbb{E} R(x, 0) \left[ B(\hat{Y}_T(y, 0)) \right].
$$

In addition, if these assertions are valid, then the initial value of $R(x)$ is given by

$$
R_0(x, 0) = -\frac{u_x(x, 0)}{u_{xx}(x, 0)} = \frac{x}{a(x, x)},
$$

the product $R(x, 0)Y(y, 0) = (R_t(x, 0)Y_t(y, 0))_{t \in [0, T]}$ is a uniformly integrable martingale and

$$
\lim_{\Delta x \to 0} \frac{\hat{X}_T(x + \Delta x, 0) - \hat{X}_T(x, 0)}{\Delta x} = \frac{R_T(x, 0)}{R_0(x, 0)},
$$

$$
\lim_{\Delta y \to 0} \frac{\hat{Y}_T(y + \Delta y, 0) - \hat{Y}_T(y, 0)}{\Delta y} = \frac{\hat{Y}_T(y, 0)}{y},
$$

where the limits in (8.4) and (8.5) take place in $\mathbb{P}$-probability.

As in [KS06b], for $x > 0$ and with $y = u_x(x, 0)$, let us define

$$
\frac{d\tilde{M}(x, 0)}{d\mathbb{P}} \triangleq \frac{R_T(x, 0)\hat{Y}_T(y, 0)}{R_0(x, 0)y},
$$

and choose $\frac{R(x, 0)}{R_0(x, 0)}$ as a numéraire, i.e., let us set

$$
S^{R(x, 0)} \triangleq \left( \frac{R_0(x, 0)}{R(x, 0)}, \frac{R_0(x, 0)S}{R(x, 0)} \right).
$$

We define the spaces of martingales

$$
\tilde{\mathcal{M}}^2(x, 0) \triangleq \left\{ M \in \mathcal{H}_0^2(\tilde{\mathbb{M}}(x, 0)) : M = H \cdot S^{R(x, 0)} \right\},
$$

and $\tilde{\mathcal{N}}^2(y, 0)$ its the orthogonal complement in $\mathcal{H}_0^2(\tilde{\mathbb{M}}(x, 0))$. We start with the following simple lemma (stated without a proof) relating the change of numéraire to the structure of martingales:
Lemma 8.2. Let \( x > 0 \) be fixed, assume that the conditions of Theorem 8.1 hold, and denote \( y = u_x(x, 0) \). Then, we have

\[
M \in M^2(x, 0) \text{ if and only if } M \frac{\hat{X}_T(x, 0)}{R_T(x, 0)} \in \tilde{M}^2(x, 0),
\]

and

\[
N \in N^2(y, 0) \text{ if and only if } N \in \tilde{N}^2(y, 0).
\]

The following theorem describes the structural properties the approximations in Theorems 3.8, 3.10, and 3.12 under the assumption that the risk-tolerance process exists. In words, the second order approximation of the value function optimal strategies amounts to a Kunita-Watanabe decomposition under the changes of measure and numéraire described above.

Theorem 8.3. Let \( x > 0 \) be fixed, assume that the conditions of Theorem 8.1 hold, and denote \( y = u_x(x, 0) \). Let us also assume that the risk-tolerance process \( R(x, 0) \) exists. Consider the Kunita-Watanabe decomposition of the square integrable martingale

\[
P_t \triangleq \mathbb{E}^{(x,0)} \left[ \left( A(\hat{X}_T(x, 0)) - 1 \right) xF | \mathcal{F}_t \right], \quad t \in [0, T]
\]

given by

\[
P = P_0 - \tilde{M}^1 - \tilde{N}^1,
\]

where \( \tilde{M}^1 \in \tilde{M}^2(x, 0) \), \( \tilde{N}^1 \in \tilde{N}^2(y, 0) \), \( P_0 \in \mathbb{R} \).

Then, the optimal solutions \( M^1(x, 0) \) and \( N^1(y, 0) \) of the quadratic optimization problems (3.16) and (3.17) can be obtained from the Kunita-Watanabe decomposition (8.10) by reverting to the original numéraire, according to Lemma 8.2 through the identities

\[
\tilde{M}^1_t = \frac{\hat{X}_t(x, 0)}{R_t(x, 0)} M^1_t(x, 0), \quad \tilde{N}^1_t = \frac{x}{y} N^1_t(y, 0), \quad t \in [0, T].
\]

In addition, the Hessian terms in the quadratic expansion of \( u \) and \( v \) can be identified as

\[
a(d, d) = \frac{R_0(x, 0)}{x} \inf_{M \in M^2(x, 0)} \mathbb{E}^{(x,0)} \left[ \left( \tilde{M}_T + xF \left( A \left( \hat{X}_T(x, 0) \right) - 1 \right) \right)^2 \right] + C_a,
\]

where

\[
C_a \triangleq x^2 \mathbb{E}^{(x,0)} \left[ F^2 A(\hat{X}_T(x, 0)) - 1 \right],
\]

and

\[
b(d, d) = \frac{R_0(x, 0)}{x} \inf_{N \in N^2(y, 0)} \mathbb{E}^{(y,0)} \left[ \left( \tilde{N}_T + yF \left( A \left( \hat{X}_T(x, 0) \right) - 1 \right) \right)^2 \right] + C_b.
\]

where

\[
b(d, d) = \frac{R_0(x, 0)}{x} \left( \frac{d}{y} \right)^2 \mathbb{E}^{(y,0)} \left[ \left( \tilde{N}_T \right)^2 \right] + \frac{R_0(x, 0)}{x} \left( \frac{d}{y} \right)^2 P_0^2 + C_b,
where
\begin{equation}
(8.15) \quad C_b \triangleq y^2 \mathbb{E}^{\mathbb{R}(x,0)} \left[ G + F^2 \left( 1 - A \left( \hat{X}_T(x,0) \right) \right) \right].
\end{equation}

The cross terms in the Hessians of \( u \) and \( v \) are identified as
\[ a(x,d) = P_0 \]
and \( b(y,d) \) is given by
\[ b(y,d) = \frac{y}{x a(x,x)} P_0. \]

With these identifications, all the conclusions of Theorem 3.8 and Corollary 3.13 hold true.

**Proof.** Let us prove (8.11) first. Completing the square in (3.16), we get
\begin{equation}
(8.16) \quad a(d,d) = \inf_{M \in M^2(x,0)} \mathbb{E}^{\mathbb{R}(x,0)} \left[ A \left( \hat{X}_T(x,0) \right) \left( M_T + x F \left( 1 - \frac{1}{A(\hat{X}_T(x,0))} \right) \right) \right]^2 + C_a,
\end{equation}

where \( C_a \) is defined in (8.13). As
\[ \frac{d \mathbb{R}(x,0)}{d \mathbb{R}(x,0)} = \frac{A \left( \hat{X}_T(x,0) \right) R_0(x,0)}{R_T(x,0)x}, \]

using Lemma 8.2, we can reformulate (8.16) as
\begin{equation}
(8.17) \quad a(d,d) = \frac{R_0(x,0)}{x} \inf_{M \in M^2(x,0)} \mathbb{E}^{\mathbb{R}(x,0)} \left[ \left( M_T \frac{\hat{X}_T(x,0)}{R_T(x,0)} + x F \left( A \left( \hat{X}_T(x,0) \right) - 1 \right) \right) \right]^2 + C_a,
\end{equation}

Likewise, completing the square in (3.17), we obtain
\begin{equation}
(8.18) \quad b(d,d) = \inf_{N \in N^2(y,0)} \mathbb{E}^{\mathbb{R}(y,0)} \left[ B \left( \hat{Y}_T(y,0) \right) \left( N_T + y F \left( \frac{B \left( \hat{Y}_T(y,0) \right)}{A \left( \hat{Y}_T(y,0) \right)} \right) \right) \right]^2 + C_b,
\end{equation}

where \( C_b \) is defined in (8.15). Now, decomposition (8.11) (where the constant \( P_0 \) is still to be determined) results from (8.17), (8.18), and optimality of \( M^1(x,0) \) and \( N^1(y,0) \) for (3.16) and (3.17), respectively. As \( A \left( \hat{X}_T(x,0) \right) = \frac{\hat{X}_T(x,0)}{R_T(x,0)} \), taking the expectation in (3.20) under \( \hat{R}(x,0) \), we deduce that \( P_0 = a(x,d) \). Therefore, using (3.24), we deduce that \( b(y,d) = \frac{y}{x a(x,x)} P_0 \). \( \square \)

**Remark 8.4.** Applying Itô formula, one can find expressions for the corrections of the optimal proportions in terms of the Kunita-Watanabe decomposition under risk-tolerance wealth process as numérique, in the spirit of Theorem 4.1. However, in the general case...
when \( \frac{R(x,0)}{R_0(x,0)} = X'(x,0) \neq \hat{X}(x,0)/x \) such a correction to proportions also contains the terms \( \hat{X}(x,0)/R(x,0) \) and \( \tilde{M}^1(x,0) \).

Remark 8.5. Theorem 8.3 gives an interpretation of \( a(x,d) \) as an utility-based price. Let us start by observing that

\[
a(x,d) = \mathbb{E} \left[ \left( A(\hat{X}_T(x,0)) - 1 \right) xF \right] = \mathbb{E} \left[ \frac{(\hat{X}_T(x,0) - R_T(x,0))}{R_0(x,0)} xF \hat{Y}_T(y,0) \right].
\]

If there exists a wealth process \( X' \) such that

\[
X_T' \geq \left| \left( \hat{X}_T(x,0) - R_T(x,0) \right) F \right|,
\]

and \( X'\hat{Y} \) is a uniformly integrable martingale\(^4\), according to [HK04, HKS05], \( a(x,d) \) represents the marginal utility-based price of the “random endowment” \( \frac{(\hat{X}_T(x,0) - R_T(x,0))}{R_0(x,0)} xF \).

**AN EXTENDED REMARK**

Below we will consider an application of our results. As was pointed out in the introduction, there is a number of models, or rather classes of models, which admit a closed form solution, see for example [Liu07] and references therein. Once we perturb the input parameters, the solution typically halts to exist in the closed form. However, Theorems 3.7, 3.8, 3.12, and 4.1 give approximations to the value function, the optimizer, and the optimal trading strategy. We will assume that \( U(x) = \frac{x^p}{p} \), \( p \in (-\infty, 0) \cup (0, 1) \) and there are two traded securities, a money market account with zero interest rate and one traded stock that satisfies conditions of [Liu07]. In this case the optimal strategy can be obtained explicitly, see [Liu07, Proposition 2].

**Explicit form of the correction terms.** As we are in power-utility settings, it is enough to consider \( x = 1 \). We assume that 0-model admits a solution \( \hat{X}(1,0) \), where \( \hat{X}(1,0) = 1 + (\hat{X}(1,0)\hat{\pi}(1,0)) \cdot S^0 \), for some predictable and \( S^0 \)-integrable process \( \hat{\pi}(1,0) \). Let us set recall that \( M^R \) (specified in (4.1)) is given by

\[
M^R = M + (\lambda - \hat{\pi}(1,0)) \cdot \langle M \rangle = S^0 - \hat{\pi}(1,0) \cdot \langle S^0 \rangle.
\]

Let us consider perturbations of \( \lambda \) by a process \( \nu \), such that Assumption 3.2 holds. In these settings, the solutions to (3.12) and (3.16), respectively, are

\[
M^0(1,0) \equiv 0 \quad \text{and} \quad M^1(1,0) = \gamma^1 \cdot M^R.
\]

\(^4\)In particular, such a process \( X' \) satisfying both conditions exists if \( |F| \leq C \) a.s. for some constant \( C > 0 \). In this case \( X' = C(R(x,0) + \hat{X}(x,0)) \) satisfies (8.19) and \( X'\hat{Y}(y,0) \) is a \( \mathbb{P} \)-martingale by Theorem 8.1.
Following the argument in section 4, we specify
\[
\tau_\varepsilon = \inf \{ t \in [0, T] : |M^1_t(1,0)| \geq \frac{1}{\varepsilon} \text{ or } \langle M^1(1,0) \rangle_t \geq \frac{1}{\varepsilon} \}, \quad \varepsilon > 0,
\]
and
\[
\gamma^{1,\varepsilon} = \gamma^1 1_{[0, \tau_\varepsilon]}, \quad \varepsilon > 0.
\]
In (4.2), we have
\[
X^{\Delta x, \delta, \varepsilon} = (1 + \Delta x) E \left( \left( \tilde{\pi}(1, 0) + \delta (\nu + \gamma^{1,\varepsilon}) \right) \cdot S^\delta \right).
\]
Following the argument of Theorem 4.1, we can find \( \varepsilon(\Delta x, \delta) \), such that
\[
E_U \left( X^{\Delta x, \delta, \varepsilon(\Delta x, \delta)} T \right) = u(1 + \Delta x, \delta) - o(\Delta x^2 + \delta^2).
\]
Using Theorem 3.10, we deduce that the Kunita-Watanabe decomposition of \( (\mathbb{E}^{R(x,0)} \left[ \frac{p}{1-p} F|F_t \right] )_{t \in [0,T]} \), where \( F \) is specified in (3.15), is:
\[
(8.20) \quad \frac{p}{1-p} F = -a(x,d) \frac{1}{1-p} + \gamma^1 \cdot M^R_T + \frac{1}{(1-p)y} N^1(y, 0),
\]
where \( y = u_x(1,0) \) and \( N^1(y, 0) \) is the solution to (3.17).

Moreover, in this case the corresponding coefficients \( a(x,x), a(x,d), \) and \( a(d,d) \) from (3.12), (3.18), and (3.16), respectively, are given by
\[
a(x,x) = 1 - p, \quad a(x,d) = -p \mathbb{E}^{R(1.0)} [F], \quad a(d,d) = \frac{1}{1-p} (a(x,d))^2 + \frac{1}{y^2(1-p)} \mathbb{E}^{R(1.0)} [N^1_T(1,0)^2] - \mathbb{E}^{R(1.0)} \left[ \frac{p}{1-p} F^2 + G \right].
\]

**Relation to the risk-tolerance process.** In this case, risk-tolerance wealth process exists for every \( x > 0 \), and is given by \( R(x,0) = \frac{\tilde{X}(x,0)}{1-p} = \frac{\tilde{\pi}(1,0)}{1-p} \tilde{X}(1,0) \), whereas \( \tilde{R}(x,0) = \mathbb{R}(x,0), x > 0 \), where \( \tilde{R}(x,0) \) is defined in (8.6). Theorem 8.1 implies that
\[
u_{xx}(x,0) = -\frac{\tilde{\pi}(x,0)}{1-p}, \quad \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left( \tilde{X}_T(x + \Delta x, 0) - \tilde{X}_T(x, 0) \right) = \frac{\tilde{\pi}(x,0)}{x}, \quad \lim_{\Delta y \to 0} \frac{1}{\Delta y} \left( \tilde{Y}_T(y + \Delta y, 0) - \tilde{Y}_T(y, 0) \right) = \frac{\tilde{\pi}(y,0)}{y},
\]
where the convergence take place in \( \mathbb{P} \)-probability. In turn, (8.20) also asserts that \( M^1_T(x,0) \) and \( N^1_T(y,0) \) form (up to multiplicative constants) an orthogonal decomposition of \( F + \frac{a(x,d)}{p} \) under \( \mathbb{R}(1,0) \), in accordance with Theorem 8.3. In particular, \( \mathbb{E}^{R(1.0)} [F] = \frac{a(x,d)}{1-p} \).
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