HYDRODYNAMIC LIMIT OF GRADIENT EXCLUSION PROCESSES WITH CONDUCTANCES

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Abstract. Fix a strictly increasing right continuous with left limits function $W : \mathbb{R} \rightarrow \mathbb{R}$ and a smooth function $\Phi : [l, r] \rightarrow \mathbb{R}$, defined on some interval $[l, r]$ of $\mathbb{R}$, such that $0 < b \leq \Phi' \leq b^{-1}$. We prove that the evolution, on the diffusive scale, of the empirical density of exclusion processes, with conductances given by $W$, is described by the weak solutions of the non-linear differential equation $\partial_t \rho = (d/dx)(d/dW)\Phi(\rho)$. We derive some properties of the operator $(d/dx)(d/dW)$ and prove uniqueness of weak solutions of the previous non-linear differential equation.

1. Introduction

Recently attention has been raised to the hydrodynamic behavior of interacting particle systems with random conductances [10, 5, 2, 3]. In [3], for instance, the authors considered, for a double sided $\alpha$–stable subordinator $W$, $0 < \alpha < 1$, the nearest-neighbor one-dimensional exclusion process on $N^{-1} \mathbb{Z}$ in which a particle jumps from $x/N$ (resp. $(x + 1)/N$) to $(x + 1)/N$ (resp. $x/N$) at rate $\{N[W(x + 1/N) - W(x/N)]\}^{-1}$. Their main result can be restated as follows. On the the diffusive scale, as the parameter $N \uparrow \infty$, the empirical density evolves according to the solution of the differential equation

$$\partial_t \rho = \frac{d}{dx} \frac{d}{dW} \rho. \quad (1.1)$$

The interesting feature is that, in constrast with usual homogeneization phenomena, the entire noise survives in the limit and the differential operator itself depends on the specific realization of the Levy process $W$. The second surprising aspect is the differential equation in which appears the derivative with respect to a strictly increasing function $W$ which may have jumps. In fact, in the Levy case, the set of discontinuities is dense on $\mathbb{R}$.

While the operators $(d/dW)(d/dx)$ have attracted much attention, being closely related to the so-called gap diffusions or quasi-diffusions when $W$ has no jumps [9], the operator $(d/dx)(d/dW)$ have not been examined yet in the case where $W$ exhibit jumps. We refer to [7, 8, 4] for recent results on the operators $(d/dx)(d/dW)$ in the case where $W$ are increasing continuous functions.

As we shall see below, non-linear versions of the partial differential equation (1.1) appear naturally as scaling limits of interacting particle systems in inhomogeneous media. They may model diffusions in which permeable membranes, at the points of the discontinuities of $W$, tend to reflect particles, creating space discontinuities in the solutions.

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We present in this paper a gradient exclusion process whose macroscopic evolution is described by the nonlinear differential equation

\[ \partial_t \rho = \frac{d}{dx} \frac{d}{dW} \Phi(\rho) , \]

where \( \Phi \) is a smooth function strictly increasing in the range of \( \rho \). To prove this result, we examine in details the operator \((d/dx)(d/dW)\) in \( L^2(\mathbb{T}) \), where \( \mathbb{T} \) is the one-dimensional torus. We prove, in Theorem 2.1, that \((d/dx)(d/dW)\), defined on an appropriate domain, is non-positive, self-adjoint and dissipative, and thus, the infinitesimal generator of a reversible Markov process. We also prove that the eigenvalues of \(-((d/dx)(d/dW))\) are countable and have finite multiplicity, the associated eigenvectors forming a complete orthonormal system.

2. Notation and Results

We examine the hydrodynamic behavior of a one-dimensional exclusion process with conductances given by a strictly increasing function. Let \( \mathbb{T}_N \) be the one-dimensional discrete torus with \( N \) points. Distribute particles on \( \mathbb{T}_N \) in such a way that each site of \( \mathbb{T}_N \) is occupied by at most one particle. Denote by \( \eta \) the configurations of state space \( \{0,1\}^{\mathbb{T}_N} \) so that \( \eta(x) = 0 \) if site \( x \) is vacant and \( \eta(x) = 1 \) if site \( x \) is occupied.

Fix \( a > -1/2 \) and a strictly increasing right continuous with left limits (cadlag) function \( W : \mathbb{R} \rightarrow \mathbb{R} \), periodic in the sense that \( W(u+1) - W(u) = W(1) - W(0) \) for all \( u \) in \( \mathbb{R} \). To simplify notation assume that \( W \) vanishes at the origin, \( W(0) = 0 \). For \( 0 \leq x \leq N - 1 \), let

\[ c_{x,x+1}(\eta) = 1 + a(\eta(x-1) + \eta(x+2)) , \]

where all sums are modulo \( N \), and let

\[ \xi_x = \frac{1}{N[W(x+1/N) - W(x/N)]} \]

with the convention that \( \xi_{N-1} = \{N[W(1) - W(1-1/N)]\}^{-1} \).

The stochastic evolution can be described as follows. At rate \( \xi_x c_{x,x+1}(\eta) \) the occupation variables \( \eta(x), \eta(x+1) \) are exchanged. Note that if \( W \) is differentiable at \( x/N \), the rate at which particles are exchanged is of order 1, while if \( W \) is discontinuous, the rate is of order \( 1/N \). Assume, to fix ideas, that \( W \) is discontinuous at some point \( x/N \) and smooth on intervals \((x/N, x/N + \epsilon), (x/N - \epsilon, x/N)\). In this case, the rate at which particles jump over the bond \( \{x-1, x\} \) is of order \( 1/N \), while in a neighborhood of size \( N \) of this bond, particles jump at rate 1. Particles are thus reflected over the bond \( \{x-1, x\} \). However, since time will be scaled diffusively and since on a time interval of length \( N^2 \) a particle spends a time of order \( N \) at site \( x \), particle will be able to jump over the slower bond \( \{x-1, x\} \). This bond may model a membrane which difficults the passage of particles.

The effect of the factor \( c_{x,x+1}(\eta) \) is less dramatic. If the parameter \( a \) is positive, the presence of particles at the neighbor sites of the bond \( \{x, x+1\} \) speeds up the exchange by a factor of order one.

The dynamics informally presented describes a Markov evolution. The generator \( L_N \) of this Markov process acts on functions \( f : \{0,1\}^{\mathbb{T}_N} \rightarrow \mathbb{R} \) as

\[ (L_N f)(\eta) = \sum_{x \in \mathbb{Z}} \xi_x c_{x,x+1}(\eta) \{f(\sigma^{x,x+1}\eta) - f(\eta)\} , \quad (2.1) \]
where \( \sigma^{x,x+1} \eta \) is the configuration obtained from \( \eta \) by exchanging the variables \( \eta(x), \eta(x+1) \):

\[
(\sigma^{x,x+1} \eta)(y) = \begin{cases} 
\eta(x+1) & \text{if } y = x, \\
\eta(x) & \text{if } y = x + 1, \\
\eta(y) & \text{otherwise.}
\end{cases}
\]

(2.2)

A simple computation shows that the Bernoulli product measures \( \{\nu_\alpha^N : 0 \leq \alpha \leq 1\} \) are invariant, in fact reversible, for the dynamics. The measure \( \nu_\alpha^N \) is obtained by placing a particle at each site, independently from the other sites, with probability \( \alpha \). Thus, \( \nu_\alpha^N \) is a product measure over \( \{0,1\}^{TN} \) with marginals given by

\[
\nu_\alpha^N (\eta : \eta(x) = 1) = \alpha
\]

for \( x \) in \( \mathbb{T}_N \). We will often omit the index \( N \) of \( \nu_\alpha^N \).

Denote by \( \{\eta_t : t \geq 0\} \) the Markov process on \( \{0,1\}^{TN} \) associated to the generator \( L_N \). Let \( D(\mathbb{R}_+, \{0,1\}^{TN}) \) be the path space of càdlàg trajectories with values in \( \{0,1\}^{TN} \). For a measure \( \mu_N \) on \( \{0,1\}^{TN} \), denote by \( \mathbb{P}_{\mu_N} \) the probability measure on \( D(\mathbb{R}_+, \{0,1\}^{TN}) \) induced by the initial state \( \mu_N \) and the Markov process \( \{\eta_t : t \geq 0\} \). Expectation with respect to \( \mathbb{P}_{\mu_N} \) is denoted by \( \mathbb{E}_{\mu_N} \).

Denote by \( \mathbb{T} \) the one-dimensional torus \( [0,1) \). A sequence of probability measures \( \{\mu_N : N \geq 1\} \) on \( \{0,1\}^{TN} \) is said to be associated to a profile \( \rho_0 : \mathbb{T} \to [0,1] \) if

\[
\lim_{N \to \infty} \mu_N \left\{ \frac{1}{N} \sum_{x \in \mathbb{T}_N} H(x/N) \eta(x) - \int H(u) \rho_0(u) du \right\} > 0
\]

for every \( \delta > 0 \) and every continuous functions \( H : \mathbb{T} \to \mathbb{R} \).

2.1. The operator \( L_W \). Denote by \( \langle \cdot , \cdot \rangle \) the inner product of \( L^2(\mathbb{T}) \):

\[
\langle f , g \rangle = \int_{\mathbb{T}} f(u) g(u) \, du .
\]

Let \( D_W \) be the set of functions \( f \) in \( L^2(\mathbb{T}) \) such that

\[
f(x) = a + bW(x) + \int_{[0,x]} W(dy) \int_0^y f(z) \, dz
\]

for some function \( f \) in \( L^2(\mathbb{T}) \) such that

\[
\int_0^1 f(z) \, dz = 0 , \quad \int_{[0,1]} W(dy) \left\{ b + \int_0^y f(z) \, dz \right\} = 0 .
\]

Define the operator \( L_W : D_W \to L^2(\mathbb{T}) \) such that \( L_W f = f \). Denote by \( I \) the identity operator in \( L^2(\mathbb{T}) \).

Theorem 2.1. The operator \( L_W : D_W \to L^2(\mathbb{T}) \) enjoys the following properties.

(a) \( D_W \) is dense in \( L^2(\mathbb{T}) \);

(b) The operator \( I - L_W : D_W \to L^2(\mathbb{T}) \) is bijective.

(c) \( L_W : D_W \to L^2(\mathbb{T}) \) is self-adjoint and non-positive:

\[
\langle -L_W f , f \rangle \geq 0
\]

(d) \( L_W \) is dissipative;

(e) The eigenvalues of the operator \( -L_W \) form a countable set \( \{\lambda_n : n \geq 0\} \). All eigenvalues have finite multiplicity, \( 0 = \lambda_0 \leq \lambda_1 \leq \cdots \), and \( \lim_{n \to \infty} \lambda_n = \infty \).
(f) The eigenvectors \{f_n\} form a complete orthonormal system.

In view of (a), (b), (d), by the Hille-Yosida theorem, \(L_W\) is the generator of a strongly continuous contraction semi-group \( \{P_t : t \geq 0\} \), \( P_t : L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T}) \). Denote by \( \{G_\lambda : \lambda > 0\} \), \( G_\lambda : L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T}) \), the semi-group of resolvents associated to the operator \(L_W: G_\lambda = (\lambda - L_W)^{-1}\). In terms of the semi-group \( \{P_t\} \), \( G_\lambda = \int_0^\infty e^{-\lambda t} P_t \, dt \).

2.2. The hydrodynamic equation. For a positive integer \( m \geq 1 \), denote by \( C^m(\mathbb{T}) \) the space of continuous functions \( H : \mathbb{T} \rightarrow \mathbb{R} \) with \( m \) continuous derivatives.

Fix \( l < r \) and a smooth function \( \Phi : [l, r] \rightarrow \mathbb{R} \) whose derivative is bounded below by a strictly positive constant and bounded above by a finite constant:

\[
0 < B^{-1} \leq \Phi'(u) \leq B
\]

for \( u \) in \([l, r]\). Consider a bounded density profile \( \gamma : \mathbb{T} \rightarrow [l, r] \). A bounded function \( \rho : \mathbb{R}_+ \times \mathbb{T} \rightarrow [l, r] \) is said to be a weak solution of the parabolic differential equation

\[
\left\{ \begin{array}{ll}
\partial_t \rho &= L_W \Phi(\rho) \\
\rho(0, \cdot) &= \gamma(\cdot)
\end{array} \right. \tag{2.4}
\]

if for all functions \( H \) in \( C^1(\mathbb{T}) \), all \( t > 0 \) and all \( \lambda > 0 \),

\[
\langle \rho_t, G_\lambda H \rangle - \langle \gamma, G_\lambda H \rangle = \int_0^t \langle \Phi(\rho_s), L_W G_\lambda H \rangle \, ds .
\]

We prove in Section 4 uniqueness of weak solutions. Existence follows from the tightness of the sequence of probability measures \( Q_{\mu, N}^W \) introduced in Section 3.

**Theorem 2.2.** Fix a continuous initial profile \( \rho_0 : \mathbb{T} \rightarrow [0, 1] \) and consider a sequence of probability measures \( \mu_N \) on \( (0, 1)^N \) associated to \( \rho_0 \). Then, for any \( t \geq 0 \),

\[
\lim_{N \rightarrow \infty} \mathbb{P}_{\mu_N} \left\{ \left| \frac{1}{N} \sum_{x \in \mathbb{T}_N} H(x/N) \eta(x) - \int H(u) \rho(t, u) \, du \right| > \delta \right\} = 0
\]

for every \( \delta > 0 \) and every continuous functions \( H \). Here, \( \rho \) is the unique weak solution of the non-linear equation \( (2.4) \) with \( l = 0, r = 1, \gamma = \rho_0 \) and \( \Phi(\alpha) = \alpha + \alpha \alpha^2 \).

Denote by \( \pi_t^N \) the empirical measure at time \( t \). This is the measure on \( \mathbb{T} \) obtained by rescaling space by \( N \) and by assigning mass \( N^{-1} \) to each particle:

\[
\pi_t^N = \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta_t(x) \delta_{x/N} ,
\]

where \( \delta_u \) is the Dirac measure concentrated on \( u \).

Theorem 2.2 states that the empirical measure \( \pi_t^N \) converges, as \( N \uparrow \infty \), to an absolutely continuous measure \( \pi(t, du) = \rho(t, u)du \), whose density \( \rho \) is the solution of \( (2.4) \). In Sections 4 we prove that \( \rho \) has finite energy: for all \( t > 0 \),

\[
\int_0^t ds \int_\mathbb{T} \left\{ \frac{d}{dW} \Phi(\rho(s, u)) \right\}^2 dW < \infty .
\]

The derivative \( d/dW \Phi(\rho(s, u)) \) must be understood in the generalized sense. Details are given in Section 4.
3. The operator $\mathcal{L}_W$

We examine in this section properties of the operator $\mathcal{L}_W$ introduced in the previous section. Recall that we denote by $\langle \cdot, \cdot \rangle$ the inner product of the Hilbert space $L^2(T)$ and by $\| \cdot \|$ its norm.

Let $\mathbb{D}(f)$ be the set of discontinuity points of a function $f : T \to \mathbb{R}$. Denote by $C_W(T)$ the set of càdlàg functions $f : T \to \mathbb{R}$ such that $\mathbb{D}(f) \subset \mathbb{D}(W)$. $C_W(T)$ is provided with the usual sup norm $\| \cdot \|_{\infty}$.

All functions in $C_W(T)$ are bounded. In fact, it is easy to prove that for each fixed $f$ in $C_W(T)$ and $\epsilon > 0$, there exists $n \geq 1$ and $0 \leq z_1 < z_2 < \cdots < z_n < 1$ such that
\[|f(x) - f(y)| \leq \epsilon \quad \text{for all} \quad z_k \leq x, y < z_{k+1}, \quad 1 \leq k \leq n, \quad (3.1)\]
where $z_{n+1} = z_1$.

Define the generalized derivative $\frac{df}{dW}$ as follows
\[\frac{df}{dW}(x) = \lim_{\epsilon \to 0} \frac{f(x + \epsilon) - f(x)}{W(x + \epsilon) - W(x)}, \quad (3.2)\]
if the above limit exists and is finite. Denote by $D_W$ the set of functions $f$ in $C_W(T)$ such that $\frac{df}{dW}(x)$ is well defined and derivable, and $\frac{d}{dx} \left( \frac{df}{dW} \right)$ belongs to $C_W(T)$. Define the operator $\mathcal{L}_W : D_W \to C_W(T)$ by
\[\mathcal{L}_W f = \frac{d}{dx} \left( \frac{df}{dW} \right). \quad (3.3)\]

By [1, Lemma 0.9 in Appendix], given a right continuous function $f$ and a continuous function $h$,
\[\frac{df}{dW}(x) = h(x) \quad \text{for all} \quad x \in T \quad \text{if and only if} \quad f(b) - f(a) = \int_{[a,b]} h(y)dW(y) \quad (3.4)\]
for all $a < b$. Note that the function $h$ has integral equal to zero, $\int_T h \, dW = 0$, because $f(1) = f(0)$.

It follows from this observation and the definition of the operator $\mathcal{L}_W$ that $D_W$ is the set of functions $f$ in $C_W(T)$ such that
\[f(x) = a + bW(x) + \int_{[0,x]} dW(y) \int_0^y g(z) \, dz \quad (3.5)\]
for some function $g$ in $C_W(\mathbb{R})$ and two reals $a, b$ such that
\[bW(1) + \int_T dW(y) \int_0^y g(z) \, dz = 0, \quad \int_T g(z) \, dz = 0. \quad (3.6)\]
The first requirement corresponds to the boundary condition $f(1) = f(0)$ and the second one to the boundary condition $(df/dW)(1) = (df/dW)(0)$. Equivalently, \[\mathbb{L}_W \] follow from the conditions
\[\int_T \frac{df}{dW} \, dW = 0, \quad \int_T \frac{d}{dx} \frac{df}{dW} \, dx = 0. \quad (3.7)\]
One can check that the function $g$, as well as the constants $a, b$, are unique.

**Lemma 3.1.** The following statements hold.
(a) The set $\mathcal{D}_W$ is dense in $L^2(\mathbb{T})$.

(b) The operator $\mathcal{L}_W : \mathcal{D}_W \to L^2(\mathbb{T})$ is symmetric and nonpositive. More precisely,

$$\langle \mathcal{L}_W f, g \rangle = -\int_\mathbb{T} \frac{df}{dW} \frac{dg}{dW} dW$$

for all $f, g$ in $\mathcal{D}_W$.

(c) $\mathcal{L}_W$ satisfies a Poincaré inequality: There exists a finite constant $C_0$ such that

$$\|f\|^2 \leq C_0(-\mathcal{L}_W f, f) + \left( \int_\mathbb{T} f(x) dx \right)^2$$

for all functions $f$ in $\mathcal{D}_W$.

(d) The Green’s function $G$ of $\mathcal{L}_W$ is given by

$$G(x, y) = \begin{cases} -\frac{[W(y) - W(0)] [W(1) - W(x)]}{W(1) - W(0)} & 0 \leq y \leq x \leq 1, \\ -\frac{[W(1) - W(y)] [W(x) - W(0)]}{W(1) - W(0)} & 0 \leq x \leq y \leq 1. \end{cases}$$

Proof. Since the continuous functions are dense in $L^2(\mathbb{T})$, to prove (a) it is enough to show that for each continuous function $f : \mathbb{T} \to \mathbb{R}$ and $\epsilon > 0$, there exists $g$ in $\mathcal{D}_W$ such that $\|f - g\| \leq \epsilon$.

Fix therefore a continuous function $f : \mathbb{T} \to \mathbb{R}$ and $\epsilon > 0$. There exists $\delta > 0$ such that $|f(y) - f(x)| \leq \epsilon$ if $|x - y| \leq \delta$. Choose an integer $n \geq \delta^{-1}$ and consider the function $g : \mathbb{T} \to \mathbb{R}$ defined by

$$g(x) = \sum_{j=0}^{n-1} \frac{f([j+1]/n) - f(j/N)}{W([j+1]/n) - W(j/N)} \mathbf{1}\{[j/n, (j+1)/n]\}(x),$$

where $\mathbf{1}\{A\}$ stands for the indicator of the set $A$. Let $G : \mathbb{T} \to \mathbb{R}$ be given by $G(x) = f(0) + \int_{[0,x]} g(y)W(dy)$. By definition of $g$, $G(j/n) = f(j/n)$ for $0 \leq j < n$. Thus, by our choice of $n$ and by definition of $G$, for $j/n \leq x \leq (j+1)/n$,

$$|G(x) - f(x)| \leq |G(x) - G(j/n)| + |f(x) - f(j/n)| \leq 2\epsilon,$$

so that $\|G - f\|_\infty \leq 2\epsilon$ if $\|\cdot\|_\infty$ stands for the sup norm. Note that

$$\int_{[0,1]} g \, dW = 0. \quad (3.7)$$

It remains to show that the function $G$ may be approximated in $L^2(\mathbb{T})$ by functions in the domain $\mathcal{D}_W$. Note that we were free to choose the set $\{0, 1/n, \ldots, (n-1)/n\}$ as long as the distance between two consecutive points is bounded by $\delta$. We may therefore assume, without loss of generality, that $W$ is continuous at these points. Denote by $\{H_k : k \geq 1\}$ a sequence of smooth functions $H_k : \mathbb{T} \to \mathbb{R}$ absolutely bounded by $\|g\|_\infty$ and such that $\lim_k H_k(x) = g(x)$ for $x \notin \mathbb{Z}$. By the dominated convergence theorem,

$$\lim_{k \to \infty} \int_\mathbb{T} |H_k(y) - g(y)| \, dW(y) = 0. \quad (3.8)$$
Let \( \{F_k : k \geq 1\} \) be the sequence of functions \( F_k : \mathbb{T} \to \mathbb{R} \) defined by

\[
F_k(x) = f(0) + \int_{(0,x)} \left( b_k + \int_0^y H_k'(z) \, dz \right) W(dy)
\]

\[
= f(0) + b_k W(x) + \int_{(0,x]} W(dy) \int_0^y H_k'(z) \, dz,
\]

where \( b_k = H_k(0) - W(1)^{-1} \int_{(0,1]} H_k(y) \, dW(y) \). By (3.7), (3.8), \( F_k \) converges in the uniform topology to \( G \). On the other hand, in view of (3.4) and our choice of \( b_k \), \( F_k \) belongs to \( \mathcal{D}_W \) for each \( k \geq 1 \) because \( H_k' \), being continuous, belongs to \( C_W(\mathbb{T}) \). This concludes the proof of (a).

To prove (b), fix two functions \( f, g \) in \( \mathcal{D}_W \) and let \( F = df/dW \). \( F \) is differentiable with derivative in \( C_W(\mathbb{T}) \). Fix \( \epsilon > 0 \) and denote by \( \{z_1, \ldots, z_n\} \) the finite set given by (3.1) for the function \( g \). Adding extra points if necessary, we may assume that \( \max_{1 \leq k \leq n} \sup_{x \leq z_k, y \leq z_{k+1}} |F(y) - F(x)| \leq \epsilon \) because \( F \) is continuous. Decomposing the integral over \( \mathbb{T} \) on the intervals \([z_k, z_{k+1}]\), we get that

\[
\langle \mathcal{L}_W f, g \rangle = \int_{\mathbb{T}} \frac{dF}{dx}(x) g(x) \, dx = \sum_{k=1}^n \int_{[z_{k-1}, z_k]} \frac{dg}{dW}(x) \, dW(x) + \epsilon \left\| \frac{dF}{dW} \right\|_{\infty}.
\]

Changing the order of summations in the last term, in view of (3.3), we obtain that the previous sum is equal to

\[
- \sum_{k=1}^n \{g(z_k) - g(z_{k-1})\} F(z_k) = - \sum_{k=1}^n F(z_k) \int_{(z_{k-1}, z_k]} \frac{dg}{dW}(x) \, dW(x).
\]

Recall that \( dg/dW \) is continuous and that \( |F(x) - F(z)\) ≤ \( \epsilon \) for \( z_{k-1} \leq x \leq z_k \). The previous sum is thus equal to

\[
- \int_{\mathbb{T}} F(x) \frac{dg}{dW}(x) \, dW(x) + \epsilon \left\| \frac{dg}{dW} \right\|_{\infty} \left[ W(1) - W(0) \right].
\]

This proves the first identity from which it follows that \( \mathcal{L}_W \) is symmetric and nonpositive.

To prove the Poincaré inequality, fix a function \( f \) in \( \mathcal{D}_W \) and observe that by (3.3)

\[
\int_{\mathbb{T}} f(x)^2 \, dx - \left( \int_{\mathbb{T}} f(x) \, dx \right)^2 = \int_{\mathbb{T}} \left( \int_{\mathbb{T}} [f(x) - f(y)] \, dy \right)^2 \, dx
\]

\[
= \int_{\mathbb{T}} dx \left( \int_{\mathbb{T}} dy \int_{(y,x]} \frac{df}{dW}(z) \, dW(z) \right)^2.
\]

To conclude the proof, it remains to apply twice the Schwarz inequality and to change the order of integration. Note that this proof gives \( C_0 = W(1) - W(0) \).

An elementary computation permits to check that the Green’s function is given by the expression proposed. □

Denote by \( \langle \cdot, \cdot \rangle_{1,2} \) the inner product on \( \mathcal{D}_W \) defined by

\[
\langle f, g \rangle_{1,2} = \langle f, g \rangle + \langle -\mathcal{L}_W f, g \rangle = \langle f, g \rangle + \int_{\mathbb{T}} \frac{df}{dW} \frac{dg}{dW} \, dW.
\]

Let \( H_1^2(\mathbb{T}) \) be the set of all functions \( f \) in \( L^2(\mathbb{T}) \) for which there exists a sequence \( \{f_n : n \geq 1\} \) in \( \mathcal{D}_W \) such that \( f_n \) converges to \( f \) in \( L^2(\mathbb{T}) \) and \( f_n \) is Cauchy for the
inner product $\langle \cdot , \cdot \rangle_{1.2}$. Such sequence \{f_n\} is called admissible for f. For f, g in $H^1_T$, define

$$\langle f, g \rangle_{1.2} = \lim_{n \to \infty} \langle f_n, g_n \rangle_{1.2},$$

(3.9)

where \{f_n\}, \{g_n\} are admissible sequences for f, g, respectively. By [14 Proposition 5.3.3], this limit exists and does not depend on the admissible sequence chosen. Moreover, $H^1_T$ endowed with the scalar product $\langle \cdot , \cdot \rangle_{1.2}$ just defined is a real Hilbert space.

Denote by $L^2_W(T)$ the Hilbert space generated by the continuous functions endowed with the inner product $\langle \cdot , \cdot \rangle_W$ defined by

$$\langle f, g \rangle_W = \int_T f(x)g(x)\,dW(x).$$

The norm associated to the scalar product $\langle \cdot , \cdot \rangle_W$ is denoted by $\| \cdot \|_W$.

**Lemma 3.2.** A function f in $L^2(T)$ belongs to $H^1_T$ if and only if there exists F in $L^2(W)$ and a finite constant c such that

$$\int_{(0,1]} F(y)\,dW(y) = 0 \quad \text{and} \quad f(x) = c + \int_{(0,x]} F(y)\,dW(y)$$

Lebesgue almost surely. We denote the generalized W-derivative F of f by $df/dW$. For f, g in $H^1_T$,

$$\langle f, g \rangle_{1.2} = \langle f, g \rangle + \int_T \frac{df}{dW} \frac{dg}{dW} \,dW.$$

**Proof.** Fix f in $H^1_T$. By definition, there exists a sequence \{f_n : n \geq 1\} in $\mathcal{D}_W$ which converges to f in $L^2(T)$ and which is Cauchy in $H^1_T$. In particular, $df_n/dW$ is Cauchy in $L^2_W(T)$ and therefore converges to some function G in $L^2_W(T)$. By ([14],

$$\int_T \frac{df_n}{dW} \,dW = 0$$

for all $n \geq 1$ so that $\int_{(0,1]} G \,dW = 0$. Let $g(x) = \int_{(0,x]} G(y)\,dW(y)$. Since $1\{(x, y)\}$ belongs to $L^2_W(T)$, for all x, y in T,

$$g(y) - g(x) = \int_{(x, y]} G \,dW = \lim_{n \to \infty} \int_{(x, y]} \frac{df_n}{dW} \,dW = \lim_{n \to \infty} \{f_n(y) - f_n(x)\}.$$

We claim that $\int_T \{f_n(y) - f_n(x)\} \,dx$ converges to $\int_T \{g(y) - g(x)\} \,dx$ for all y in T. Indeed, on the one hand, for each fixed y, $f_n(y) - f_n(x)$ converges to $g(y) - g(x)$. On the other hand, by Schwarz inequality,

$$[f_n(y) - f_n(x)]^2 \leq |W(1) - W(0)| \int_T \left( \frac{df_n}{dW} \right)^2 \,dW \leq C_0$$

for some finite constant $C_0$. It remains to apply the dominated convergence theorem to conclude.

Since $f_n$ converges to f in $L^2(T)$, $\int_T f_n(x) \,dx$ converges to $\int_T f(x) \,dx$. By Schwarz inequality, g belongs to $L^2(T)$ so that $\int_T g(x) \,dx$ is finite. Therefore, for all y in T,

$$\lim_{n \to \infty} f_n(y) = \lim_{n \to \infty} \left\{f_n(y) - \int_T f_n(x) \,dx\right\} + \int_T f(x) \,dx$$

$$= g(y) - \int_T g(x) \,dx + \int_T f(x) \,dx.$$
Thus $f_n$ converges pointwisely to the above function. As $f_n$ also converges to $f$ in $L^2(T)$, we deduce that $f = c + g$ a.s., and thus in $L^2(T)$, for $c = \int_T f(x) dx - \int_T g(x) dx$, proving the first statement of the lemma.

The reciprocal is simpler. Let $f = c + \int_{(0,x]} F(y) dW(y)$ for some $F$ in $L^2_W(T)$ such that $\int_{(0,1]} F(y) dW(y) = 0$. There exists a sequence $\{g_n : n \geq 1\}$ of smooth functions converging to $F$ in $L^2_W(T)$ and such that $\int_{(0,1]} g_n(y) dW(y) = 0$. Let $f_n(x) = c + \int_{(0,x]} dW(y) \{g_n(0) + \int_0^y g_n'(z) dz\}$. For each $n \geq 1$, $f_n$ belongs to $D_W$ because $g_n'$ is continuous. Schwarz inequality shows that $f_n$ converges to $f$ in $L^2(T)$.

Finally, $\{f_n : n \geq 1\}$ is a Cauchy sequence for the inner product $\langle \cdot, \cdot \rangle_1,2$ because $df_n/dW = g_n$ converges to $f$ in $L^2_W(T)$. Note that we just proved that the sequence $\{f_n : n \geq 1\}$ is admissible for $f$.

Fix $f, g$ in $H^1_2(T)$ and recall that we denote by $df/dW, dg/dW$ the generalized $W$-derivatives of $f, g$, respectively. Denote by $\{f_n : n \geq 1\}, \{g_n : n \geq 1\}$ the admissible sequences constructed in the previous paragraph for $f$ and $g$, respectively. By definition,

$$\langle f, g \rangle_{1,2} = \lim_{n \to \infty} \langle f_n, g_n \rangle_{1,2} = \lim_{n \to \infty} \left\{ \langle f_n, g_n \rangle + \int_T df_n dW dg_n dW \right\}.$$ 

Since $f_n$ (resp. $g_n$) converges to $f$ (resp. $g$) in $L^2(T)$ and since $df_n/dW$ (resp. $dg_n/dW$) converges to $df/dW$ (resp. $dg/dW$) in $L^2_W(T)$, the previous expression is equal to

$$\langle f, g \rangle + \int_T df dW dg dW.$$ 

This concludes the proof of the lemma. $\square$

**Lemma 3.3.** The embedding $H^1_2(T) \subset L^2(T)$ is compact.

**Proof.** Consider a sequence $\{u_n : n \geq 1\}$ bounded in $H^1_2(T)$. We need to prove the existence of a subsequence $\{u_{n_k} : k \geq 1\}$ which converges in $L^2(T)$.

By the previous lemma, $u_n(x) = c_n + \int_{(0,x]} U_n(y) dW(y)$ for some $U_n$ in $L^2_W(T)$ such that $\int_{(0,1]} U_n(y) dW(y) = 0$. Moreover, $\|U_n\|_W \leq \|u_n\|_{1,2}$. The sequence $\{U_n\}$ is therefore bounded in $L^2_W(T)$. Also, by Schwarz inequality, the sequence $\int_{(0,x]} U_n(y) dW(y)$ is bounded in $L^2(T)$. Since $c_n = u_n(x) - \int_{(0,x]} U_n(y) dW(y)$ and since both sequence of functions on the right hand side are bounded in $L^2(T)$, the sequence of real numbers $\{c_n\}$ is also bounded.

Since $\{U_n\}$ is a bounded sequence in $L^2_W(T)$ and since the sequence of real numbers $\{c_n\}$ is bounded, there exists a subsequence $\{n_k\}$ such that $c_{n_k}$ converges and $U_{n_k}$ converges weakly in $L^2_W(T)$ to a limit denoted by $U$. As constants belong to $L^2_W(T)$, $\int_{(0,1]} U(y) dW(y) = \lim_k \int_{(0,1]} U_{n_k}(y) dW(y) = 0$. Moreover, for all $x$ in $T$, as $1\{0, x]\}$ belongs to $L^2_W(T)$,

$$\lim_{k \to \infty} u_{n_k}(x) = \lim_{k \to \infty} \left\{ c_{n_k} + \int_{(0,x]} U_{n_k}(y) dW(y) \right\} = c + \int_{(0,x]} U(y) dW(y),$$ 

if $c$ stands for the limit of the sequence $c_{n_k}$. The sequence $u_{n_k}$ thus converges pointwisely to $u(x) = c + \int_{(0,x]} U(y) dW(y)$. Since, by Schwarz inequality, $u_{n_k}(x)^2$ is bounded by $2c^2 + 2\|W(1) - W(0)\|_W^2 \|U_{n_k}\|_{1,2}^2$, the dominated convergence theorem, $u_{n_k}$ converges to $u$ in $L^2(T)$. Note that the limit $u$ belongs to $H^1_2(T)$. $\square$
Let $D_W$ be the set of functions $f$ in $H^1_2(\mathbb{T})$ for which there exists $u$ in $L^2(\mathbb{T})$ such that
\[
\langle f, g \rangle_{1,2} = \langle f, g \rangle + \int \frac{df}{dW} \frac{dg}{dW} dW = \langle u, g \rangle \quad (3.10)
\]
for all $g$ in $H^1_2(\mathbb{T})$. By Lemma 3.1(b), $D_W \subset D_W$ and, by definition, $D_W \subset H^1_2(\mathbb{T})$. The function $u$ is uniquely determined because, by Lemma 3.1(a), $H^1_2(\mathbb{T}) \supset D_W$ is dense in $L^2(\mathbb{T})$. By definition of $H^1_2(\mathbb{T})$ and by (3.10), it is enough to check (3.10) for functions $g$ in $D_W$.

**Lemma 3.4.** The domain $D_W$ consist of all functions $f$ in $L^2(\mathbb{T})$ such that
\[
f(x) = a + bW(x) + \int_{[0,x]} W(dy) \int_0^y f(z) dz
\]
for some function $\xi$ in $L^2(\mathbb{T})$ such that
\[
\int_0^1 f(z) dz = 0, \quad \int_{[0,1]} W(dy) \{ b + \int_0^y f(z) dz \} = 0.
\]
Moreover, in this case,
\[
- \int \frac{df}{dW} \frac{dg}{dW} dW = \langle f, g \rangle
\]
for all $g$ in $H^1_2(\mathbb{T})$.

**Proof.** We first show that any function $f$ in $L^2(\mathbb{T})$ with the properties listed in the statement of the lemma belongs to $D_W$. Fix such a function and consider a sequence $\{ f_n : n \geq 1 \}$ of smooth functions $f_n : \mathbb{T} \to \mathbb{R}$ which converges to $\xi$ in $L^2(\mathbb{T})$ and such that $\int_0^1 f_n(z) dz = 0$. Let
\[
f_n(x) = a + \int_{[0,x]} W(dy) \{ b_n + \int_0^y f_n(z) dz \},
\]
where $b_n$ is chosen so that $\int_{[0,1]} W(dy \{ b_n + \int_0^y f_n(z) dz \} = 0$. Note that $f_n$ belongs to $D_W$ for each $n \geq 1$.

As $n \to \infty$, $b_n$ converges to $b$, $f_n$ converges to $f$ in $L^2(\mathbb{T})$ and $\{ f_n \}$ is Cauchy for the $\| \cdot \|_{1,2}$ norm. Thus, $f$ belongs to $H^1_2(\mathbb{T})$ and $\{ f_n \}$ is an admissible sequence for $f$.

Fix $g$ in $D_W$. We claim that
\[
\langle f, g \rangle_{1,2} = \langle f, g \rangle - \langle \xi, g \rangle.
\]
Indeed, as $g$ belongs to $D_W$, by equation (3.9), $\langle f, g \rangle_{1,2} = \lim_n \langle f_n, g \rangle_{1,2}$ because the sequence $\{ g_n \ : \ n \geq 1 \}$ constant equal to $g$ is admissible for $g$. By definition of the inner product $\langle \cdot, \cdot \rangle_{1,2}$ and since $\mathcal{L}_W f_n = f_n$, $\langle f_n, g \rangle_{1,2} = \langle f_n, g \rangle + \langle - \mathcal{L}_W f_n, g \rangle = \langle f_n, g \rangle + \langle - f_n, g \rangle$. Since $f_n, f_n$ converge in $L^2(\mathbb{T})$ to $f, \xi$, respectively, the claim is proved. In particular (3.10) holds with $u = f - \xi$. This proves that $f$ belongs to $D_W$ and the identity claimed.

Conversely, assume that $f$ belongs to $D_W$ and satisfy (3.10) for some $u$ in $L^2(\mathbb{T})$. Thus, there exists $v$ (equal to $f - u$) in $L^2(\mathbb{T})$ such that
\[
- \int \frac{df}{dW} \frac{dg}{dW} dW = \langle v, g \rangle \quad (3.11)
\]
for all $g$ in $D_W$. Taking $g = 1$ in this equation we obtain that $\int_0^1 v(x) dx = 0$. 

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Since \( f \) belongs to \( H^1_1(\mathbb{T}) \), by Lemma 3.2 \( f(x) = c + \int_{(0,x]} F(y) dW(y) \) for some function \( F \) in \( L^2_W(\mathbb{T}) \) such that \( \int_{(0,1]} F(y) dW(y) = 0 \). To prove the lemma we need to show that

\[
F(y) = b + \int_0^y f(z) dz
\]

for some finite constant \( b \) and some function \( f \) in \( L^2(\mathbb{T}) \) such that \( \int_0^1 f(z) dz = 0 \).

Fix \( g \) in \( \mathcal{D}_W \) so that

\[
g(x) = a + \int_{(0,x]} G(y) dW(y)
\]

for some continuous function \( G : \mathbb{T} \to \mathbb{R} \) such that \( \int_0^1 G(y) dW(y) = 0 \). Since the integral of \( v \) (resp. \( G \)) with respect to the Lebesgue measure (resp. the measure \( dW \)) vanishes, changing the order of integration, we obtain that

\[
\int_0^1 v(x) g(x) dx = - \int_{(0,1]} G(y) \int_0^y v(x) dx dW(y).
\]

Therefore, in view of (3.11),

\[
\int_{(0,1]} G(y) \int_0^y v(x) dx dW(y) = \int_{(0,1]} G(y) F(y) dW(y)
\]

for all functions \( g \) in \( \mathcal{D}_W \). The proof of Lemma 3.1 (a) shows that the set \( \{dg/dW : g \in \mathcal{D}_W \} \) is dense in \( L^2_{W,0} = \{H \in L^2_W(\mathbb{T}) : \int H dW = 0 \} \). In particular, \( F(y) = c + \int_0^y v(x) dx \) for some finite constant \( c \). This concludes the proof of the lemma.

Recall that we denote by \( \mathbb{I} \) the identity in \( L^2(\mathbb{T}) \). By Lemma 3.1 the symmetric operator \( (\mathbb{I} - \mathcal{L}_W) : \mathcal{D}_W \to \mathbb{L}^2(\mathbb{T}) \), is strongly monotone:

\[
\langle (\mathbb{I} - \mathcal{L}_W)f, f \rangle \geq \langle f, f \rangle
\]

for all \( f \) in \( \mathcal{D}_W \). Denote by \( \mathcal{A}_1 : \mathcal{D}_W \to \mathbb{L}^2(\mathbb{T}) \) its Friedrichs extension, defined as \( \mathcal{A}_1 f = u \), where \( u \) is the function in \( L^2(\mathbb{T}) \) given by (3.10). By [11, Theorem 5.5.a], \( \mathcal{A}_1 \) is self-adjoint, bijective and

\[
\langle \mathcal{A}_1 f, f \rangle \geq \langle f, f \rangle
\]

(3.12)

for all \( f \) in \( \mathcal{D}_W \). Note that the Friedrichs extension of the strongly monotone operator \( (\lambda \mathbb{I} - \mathcal{L}_W) \), \( \lambda > 0 \), is \( \mathcal{A}_\lambda = (\lambda - 1)\mathbb{I} + \mathcal{A}_1 : \mathcal{D}_W \to \mathbb{L}^2(\mathbb{T}) \).

Define \( \mathcal{L}_W : \mathcal{D}_W \to L^2(\mathbb{T}) \) by \( \mathcal{L}_W = \mathbb{I} - \mathcal{A}_1 \). In view of (3.10), \( \mathcal{L}_W f = u \) if and only if

\[
- \int \frac{df}{dW} \frac{dg}{dW} dW = \langle u, g \rangle
\]

for all \( g \) in \( H^1_1(\mathbb{T}) \). In particular by Lemma 3.1 (b) \( \mathcal{L}_W f = \mathcal{L}_W f \) for all \( f \) in \( \mathcal{D}_W \). Moreover, if a function \( f \) in \( \mathcal{D}_W \) is represented as in Lemma 3.3 \( \mathcal{L}_W f = f \). This identity together with the identification of the space \( \mathcal{D}_W \) provides the alternative definition of the operator \( \mathcal{L}_W \) presented just before the statement of Theorem 2.1.

Proof of Theorem 2.1 It follows from Lemma 3.1 (a) that the domain \( \mathcal{D}_W \) is dense in \( L^2(\mathbb{T}) \) because \( \mathcal{D}_W \subset \mathcal{D}_W \). This proves (a).

By definition, \( \mathbb{I} - \mathcal{L}_W = \mathcal{A}_1 : \mathcal{D}_W \to \mathbb{L}^2(\mathbb{T}) \), which have been shown to be bijective. This proves (b).
The self-adjointness of \( L_W : D_W \to L^2(\mathbb{T}) \) follows from the one of \( A_1 \) and the definition of \( L_W \) as \( I - A_1 \). Moreover, from \((3.12)\) we obtain that \( \langle -L_W f, f \rangle \geq 0 \) for all \( f \) in \( D_W \).

To prove \((d)\), fix a function \( g \) in \( D_W \), \( \lambda > 0 \) and let \( f = (\lambda I - L_W)g \). Taking the scalar product with respect to \( g \) on both sides of this equation, we obtain that
\[
\lambda (g, g) + \langle -L_W g, g \rangle = \langle g, f \rangle \leq \langle g, f \rangle^{1/2} (f, f)^{1/2}.
\]
Since \( g \) belongs to \( D_W \), by \((c)\), the second term on the left hand side is positive.

Thus, \( \|\lambda g\| \leq \|f\| = \|\lambda (I - L_W)g\| \).

We have already seen that the operator \((I - L_W) : D_W \to L^2(\mathbb{T})\) is symmetric and strongly monotone. By Lemma \(3.3\), the embedding \( H^1(\mathbb{T}) \subset L^2(\mathbb{T}) \) is compact. Therefore, by \[(11, \text{Theorem 5.5.c})\], the Friedrichs extension of \((I - L_W)\), denoted by \( A_1 : D_W \to L^2(\mathbb{T}) \), satisfies claims \((e)\) and \((f)\) with \( 1 \leq \lambda_1 \leq \lambda_2 \leq \cdots \), \( \lambda_n \uparrow \infty \). In particular, the operator \(-L_W = A_1 - I\) has the same property with \( 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \), \( \lambda_n \uparrow \infty \). Since \( 0 \) is an eigenvalue of \(-L_W\) associated at least to the constants, \((e)\) and \((f)\) are in force. \(\square\)

It follows also from \[(11, \text{Theorem 5.5.c})\] that \( f_n \) belongs to \( H^2(\mathbb{T}) \) for all \( n \).

3.1. **Random walk with conductances.** Recall that \( \mathbb{T}_N \) stands for the discrete one-dimensional torus with \( N \) points and recall the definition of the sequence \( \{\xi_x : 0 \leq x \leq N - 1\} \).

Consider the random walk \( \{X^N_t : t \geq 0\} \) on \( N^{-1}\mathbb{T}_N \) which jumps over the bond \( \{x/N, (x + 1)/N\} \) at rate \( N^2 \xi_x = N/(W(x + 1/N) - W(x/N)) \). The generator \( L_N \) of this Markov process writes
\[
(L_N f)(x/N) = N^2 \xi_x (f(x + 1/N) - f(x/N)) + N^2 \xi_{x-1} (f(x - 1/N) - f(x/N)) .
\]

The counting measure \( m_N \) on \( N^{-1}\mathbb{T}_N \) is reversible for this process. Denote by \( \{P_t^N : t \geq 0\} \) (resp. \( \{G^N_\lambda : \lambda > 0\} \)) the semigroup (resp. the resolvent) associated to the generator \( L_N \):
\[
G^N_\lambda H = \int_0^\infty dt e^{-\lambda t} P_t^N H
\]
for \( H : N^{-1}\mathbb{T}_N \to \mathbb{R} \).

Fix a function \( H : N^{-1}\mathbb{T}_N \to \mathbb{R} \). For \( \lambda > 0 \), let \( H^N_\lambda = G^N_\lambda H \) be the solution of the resolvent equation
\[
\lambda H^N_\lambda - L_N H^N_\lambda = H .
\]

Taking the scalar product on both sides of this equation with respect to \( H^N_\lambda \), we obtain that for all \( N \geq 1 \)
\[
\frac{1}{N} \sum_{x \in \mathbb{T}_N} H^N_\lambda(x/N)^2 \leq \frac{1}{\lambda^2} \frac{1}{N} \sum_{x \in \mathbb{T}_N} H(x/N)^2 ,
\]
\[
\frac{1}{N} \sum_{x \in \mathbb{T}_N} \xi_x (\nabla_N H^N_\lambda)(x/N)^2 \leq \frac{1}{\lambda^2} \frac{1}{N} \sum_{x \in \mathbb{T}_N} H(x/N)^2 ,
\]
where \( \nabla_N \) stands for the discrete derivative: \( (\nabla_N H)(x/N) = N[H(x + 1/N) - H(x/N)] \).

On the other hand, if \( H : \mathbb{T} \to \mathbb{R} \) is a continuous function and we denote also by \( H \) its restriction to \( N^{-1}\mathbb{T}_N \), by \[(3, \text{Lemma 4.6})\],
\[
\lim_{\lambda \to \infty} \limsup_{N \to \infty} \frac{1}{N} \sum_{x \in \mathbb{T}_N} |\lambda H^N_\lambda(x/N) - H(x/N)| .
\]
Note that in [3], the function $W$ is of pure jump type, while here it is any strictly increasing càdlàg function. One can check, however, that the proof applies to our general case.

It follows from [3, Lemma 4.5 (iii)] that for every continuous function $H : T \rightarrow \mathbb{R}$,

$$\limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{x \in \mathbb{T}_N} \left| (G_N^x H)(x/N) - (G_x H)(x/N) \right|. \quad (3.15)$$

4. SCALING LIMIT

Let $\mathcal{M}$ be the space of positive measures on $\mathbb{T}$ with total mass bounded by one endowed with the weak topology. Recall that $\pi_t^N \in \mathcal{M}$ stands for the empirical measure at time $t$. This is the measure on $\mathbb{T}$ obtained by rescaling space by $N$ and by assigning mass $N^{-1}$ to each particle:

$$\pi_t^N = \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta_t(x) \delta_{x/N}, \quad (4.1)$$

where $\delta_u$ is the Dirac measure concentrated on $u$. For a continuous function $H : T \rightarrow \mathbb{R}$, $(\pi_t^N, H)$ stands for the integral of $H$ with respect to $\pi_t^N$:

$$\langle \pi_t^N, H \rangle = \frac{1}{N} \sum_{x \in \mathbb{T}_N} H(x/N) \eta_t(x).$$

This notation is not to be confounded with the inner product in $L^2(\mathbb{T})$ introduced earlier. Also, when $\pi_t$ has a density $\rho$, $\pi(t, du) = \rho(t, u) du$, we sometimes write $\langle \rho_t, H \rangle$ for $\langle \pi_t, H \rangle$.

Fix $T > 0$. Let $D([0, T], \mathcal{M})$ be the space of $\mathcal{M}$-valued càdlàg trajectories $\pi : [0, T] \rightarrow \mathcal{M}$ endowed with the uniform topology. For each probability measure $\mu_N$ on $\{0, 1\}^{\mathbb{T}_N}$, denote by $Q_{\mu_N}^{W,N}$ the measure on the path space $D([0, T], \mathcal{M})$ induced by the measure $\mu_N$ and the process $\pi_t^N$ introduced in (4.1).

Fix a continuous profile $\rho_0 : \mathbb{T} \rightarrow [0, 1]$ and consider a sequence $\{\mu_N : N \geq 1\}$ of measures on $\{0, 1\}^{\mathbb{T}_N}$ associated to $\rho_0$ in the sense (2.3). Let $Q_W$ be the probability measure on $D([0, T], \mathcal{M})$ concentrated on the deterministic path $\pi(t, du) = \rho(t, u) du$, where $\rho$ is the unique weak solution of (2.4) with $\gamma = \rho_0$, $l = 0$, $r = 1$ and $\Phi(\alpha) = \alpha + a\alpha^2$.  

**Proposition 4.1.** As $N \uparrow \infty$, the sequence of probability measures $Q_{\mu_N}^{W,N}$ converges in the uniform topology to $Q_W$.

The proof of this result is divided in two parts. In Subsection 4.1 we show that the sequence $\{Q_{\mu_N}^{W,N} : N \geq 1\}$ is tight and in Subsection 4.2 we characterize the limit points of this sequence.

**Proof of Theorem 2.2.** Since $Q_{\mu_N}^{W,N}$ converges in the uniform topology to $Q_W$, a measure which is concentrated on a deterministic path, for each $0 \leq t \leq T$ and each continuous function $H : T \rightarrow \mathbb{R}$, $(\pi_t^N, H)$ converges in probability to $\int_0^t du \rho(t, u) H(u)$, where $\rho$ is the unique weak solution of (2.4) with $l = 0$, $r = 1$, $\gamma = \rho_0$ and $\Phi(\alpha) = \alpha + a\alpha^2$. \qed
4.1. Tightness. Tightness of the sequence \( \{ Q_{W,N}^N : N \geq 1 \} \) is proved as in \cite{5, 3} by considering first the auxiliary \( M \)-valued Markov process \( \{ \Pi_{\lambda,N}^t : t \geq 0 \} \), \( \lambda > 0 \), defined by

\[
\Pi_{\lambda,N}^t(H) = \langle \pi_{N}^t, G_{\lambda}^N H \rangle = \frac{1}{N} \sum_{x \in \mathbb{Z}} (G_{\lambda}^N H)(x/N) \eta_t(x) ,
\]

\( H \) in \( C(\mathbb{T}) \), where \( \{ G_{\lambda}^N : \lambda > 0 \} \) is the resolvent associated to the random walk \( \{ X_{t}^N : t \geq 0 \} \) introduced in Section 3.

We first prove tightness of the process \( \{ \Pi_{\lambda,N}^t : 0 \leq t \leq T \} \) for every \( \lambda > 0 \) and then show that \( \{ \Pi_{\lambda,N}^t : 0 \leq t \leq T \} \) and \( \{ \pi_{N}^t : 0 \leq t \leq T \} \) are not far apart if \( \lambda \) is large.

In contrast with \cite{5, 3}, \( D([0, T], M) \) is here endowed with the uniform topology. It is well known \cite{6} that to prove tightness of \( \{ \Pi_{\lambda,N}^t : 0 \leq t \leq T \} \) it is enough to show tightness of the real-valued processes \( \{ \Pi_{\lambda,N}^t(H) : 0 \leq t \leq T \} \) for a set of smooth functions \( H : \mathbb{T} \to \mathbb{R} \) dense in \( C(\mathbb{T}) \) for the uniform topology.

Fix a smooth function \( H : \mathbb{T} \to \mathbb{R} \). Denote by the same symbol the restriction of \( H \) to \( \mathbb{T} \). Let

\[
\lambda H_{\lambda}^N - L_N H_{\lambda}^N = H .
\]

(4.2)

Keep in mind that \( \Pi_{\lambda,N}^t(H) = \langle \pi_{N}^t, H_{\lambda}^N \rangle \) and denote by \( M_{t}^{N,\lambda} \) the martingale defined by

\[
M_{t}^{N,\lambda} = \Pi_{\lambda,N}^t(H) - \Pi_{0}^{\lambda,N}(H) - \int_{0}^{t} ds N L_N \langle \pi_{s}^t, H_{\lambda}^N \rangle .
\]

(4.3)

Clearly, tightness of \( \Pi_{\lambda,N}^t(H) \) follows from tightness of the martingale \( M_{t}^{N,\lambda} \) and tightness of the additive functional \( \int_{0}^{t} ds N L_N \langle \pi_{s}^t, H_{\lambda}^N \rangle \).

An elementary computation shows that the quadratic variation \( \langle M_{t}^{N,\lambda} \rangle_t \) of the martingale \( M_{t}^{N,\lambda} \) is given by

\[
\frac{1}{N^2} \sum_{x \in \mathbb{Z}} \xi_x ([\nabla_N H_{\lambda}^N](x/N))^2 \int_{0}^{t} c_{x,x+1}(\eta_s) [\eta_s(x + 1) - \eta_s(x)]^2 ds .
\]

In particular, by (3.13),

\[
\langle M_{t}^{N,\lambda} \rangle_t \leq \frac{C_0 t}{N^2} \sum_{x \in \mathbb{T}_N} \xi_x ([\nabla_N H_{\lambda}^N](x/N))^2 \leq \frac{C(H) t}{\lambda N}
\]

for some finite constant \( C(H) \) which depends only on \( H \). Thus, by Doob inequality, for every \( \lambda > 0 \), \( \delta > 0 \)

\[
\lim_{N \to \infty} \mathbb{P}_{\mu_N} \left[ \sup_{0 \leq t \leq T} |M_{t}^{N,\lambda}| > \delta \right] = 0 .
\]

(4.4)

In particular, the sequence of martingales \( \{ M_{t}^{N,\lambda} : N \geq 1 \} \) is tight for the uniform topology.
It remains to examine the additive functional of the decomposition \(1.13\). A long and elementary computations shows that \(N^2L_N\langle \pi^N, H_\lambda^N \rangle\) is equal to
\[
\frac{1}{N} \sum_{x \in \mathbb{T}_N} (L_N H_\lambda^N)(x/N) \eta(x) \\
+ \frac{1}{N} \sum_{x \in \mathbb{T}_N} \{ (L_N H_\lambda^N)(x + 1/N) + (L_N H_\lambda^N)(x/N) \} (\tau_x h_1)(\eta) \\
- \frac{1}{N} \sum_{x \in \mathbb{T}_N} (L_N H_\lambda^N)(x/N) (\tau_x h_2)(\eta),
\]
where \(\{\tau_x : x \in \mathbb{Z}\}\) is the group of translations so that \((\tau_x \eta)(y) = \eta(x + y)\) for \(x, y\) in \(\mathbb{Z}\) and the sum is understood modulo \(N\). Also, \(h_1, h_2\) are the cylinder functions
\[
h_1(\eta) = \eta(0) \eta(1), \quad h_2(\eta) = \eta(-1) \eta(1).
\]
Since \(H_\lambda^N\) is the solution of the resolvent equation \(1.12\), we may replace \(L_N H_\lambda^N\) by \(U_\lambda^N = \lambda \pi^N \cdot H\) in the previous formula. In particular, for all \(0 \leq s < t \leq T\),
\[
\left| \int_s^t \, dr \, N^2 L_N \langle \pi^N_r, H_\lambda^N \rangle \right| \leq \frac{(1 + 3|a|)(t - s)}{N} \sum_{x \in \mathbb{T}_N} \left| U_\lambda^N(x/N) \right|.
\]
It follows from the first estimate in \(5.13\) and from Schwarz inequality that the right hand side is bounded above by \(C(H, a)(t - s)\) uniformly in \(N\), where \(C(H, a)\) is a finite constant depending only on \(a\) and \(H\). This proves that the additive part of the decomposition \(4.3\) is tight for the uniform topology and therefore that the sequence of processes \(\Pi_\lambda^\ast : N \geq 1\) is tight.

**Lemma 4.2.** The sequence of measures \(\{Q_{W,N}^\ast : N \geq 1\}\) is tight for the uniform topology.

**Proof.** It is enough to show that for every smooth function \(U : \mathbb{T} \to \mathbb{R}\) and every \(\epsilon > 0\), there exists \(\lambda > 0\) such that
\[
\lim_{N \to \infty} \mathbb{P}_{\mu_N} \left[ \sup_{0 \leq t \leq T} |\Pi_\lambda^N(LH) - \langle \Pi^N_t, H \rangle| > \epsilon \right] = 0
\]
because in this case the tightness of \(\pi^N_\ast\) follows from the tightness of \(\Pi_\lambda^\ast\). Since there is at most one particle per site the expression inside the absolute value is less than or equal to
\[
\frac{1}{N} \sum_{x \in \mathbb{T}_N} \left| \lambda H_\lambda^N(x/N) - H(x/N) \right|.
\]
By \(5.14\) this expression vanishes as \(N \uparrow \infty, \lambda \uparrow \infty\). \(\square\)

**4.2. Uniqueness of limit points.** We prove in this subsection that all limit points \(Q^\ast\) of the sequence \(Q_{\mu_N}^W\) are concentrated on absolutely continuous trajectories \(\pi(t, du) = \rho(t, u) du\), whose density \(\rho(t, u)\) is a weak solution of the hydrodynamic equation \(2.3\) with \(l = 0, r = 1, \gamma = \rho_0\) and \(\Phi(\theta) = \theta + \theta^2\).

Let \(Q^\ast\) be a limit point of the sequence \(Q_{\mu_N}^W\) and assume, without loss of generality, that \(Q_{\mu_N}^W\) converges to \(Q^\ast\).

Since there is at most one particle per site, it is clear that \(Q^\ast\) is concentrated on trajectories \(\pi_\ast(du)\) which are absolutely continuous with respect to the Lebesgue measure, \(\pi_\ast(du) = \rho(t, u) du\), and whose density \(\rho\) is non-negative and bounded by 1.
Fix a function $H : T \to \mathbb{R}$ continuously differentiable and $\lambda > 0$. Recall the definition of the martingale $M^N_t\epsilon^\lambda$ introduced in the previous section. By (4.1), for every $\delta > 0$,

$$\lim_{N \to \infty} \mathbb{P}_{\mu_N} \left[ \sup_{0 \leq t \leq T} |M^N_t\epsilon^\lambda| > \delta \right] = 0 .$$

The martingale $M^N_t\epsilon^\lambda$ can be written in terms of the empirical measure as

$$\langle \pi^N_t, G^N_\lambda H \rangle - \langle \pi^N_0, G^N_\lambda H \rangle - \int_0^t ds N^2 L_N \langle \pi^N_s, G^N_\lambda H \rangle .$$

Therefore, for fixed $0 < t \leq T$ and $\delta > 0$,

$$\lim_{N \to \infty} \mathbb{Q}^W_{\mu_N} \left[ \left| \int_0^t ds \frac{1}{N} \sum_{x \in \pi_N} U_\lambda(x/N) \left\{ \tau_x h_j(\eta_s) - \left[ \eta^N_s(x) \right]^2 \right\} \right| > \delta \right] = 0 .$$

Since there is at most one particle per site, by (3.15), we may replace $G^N_\lambda H$ by $G_\lambda H$ in the expression $\langle \pi^N_t, G^N_\lambda H \rangle, \langle \pi^N_0, G^N_\lambda H \rangle$ above.

On the other hand, the expression $N^2 L_N \langle \pi^N_s, G^N_\lambda H \rangle$ has been computed in the previous subsection. Recall that $L_N G^N_\lambda H = \lambda G^N_\lambda H - H$. As before, we may replace $G^N_\lambda H$ by $G_\lambda H$. Let $U_\lambda = \lambda G_\lambda H - H$. Since $E_\nu[h_j] = \alpha^2, j = 1, 2$, in view of (3.13) and by Corollary 4.4, for every $t > 0, \lambda > 0, \delta > 0, j = 1, 2$,

$$\lim_{\delta \to 0} \mathbb{P}_{\mu_N} \left[ \left| \int_0^t ds \frac{1}{N} \sum_{x \in \pi_N} U_\lambda(x/N) \left\{ \tau_x h_j(\eta_s) - \left[ \eta^N_s(x) \right]^2 \right\} \right| > \delta \right] = 0 .$$

Since $\eta^N_s(x) = \varepsilon^{-1}\pi^N_s([x/N, x/N + \varepsilon])$, we obtain from the previous considerations that

$$\lim_{\delta \to 0} \mathbb{Q}^W_{\mu_N} \left[ \left| \langle \pi^N_t, G_\lambda H \rangle - \langle \pi^N_0, G_\lambda H \rangle - \int_0^t ds \langle \Phi(\varepsilon^{-1}\pi^N_s([\cdot, + \varepsilon])), U_\lambda \rangle \right| > \delta \right] = 0 .$$

Since $H$ is a smooth function, $G_\lambda H$ and $U_\lambda$ can be approximated in $L^1(T)$ by continuous functions. Since we assumed that $\mathbb{Q}^W_{\mu_N}$ converges in the uniform topology to $\mathbb{Q}^*$, we have that

$$\lim_{\delta \to 0} \mathbb{Q}^* \left[ \left| \langle \pi_t, G_\lambda H \rangle - \langle \pi_0, G_\lambda H \rangle - \int_0^t ds \langle \Phi(\varepsilon^{-1}\pi_s([\cdot, + \varepsilon])), U_\lambda \rangle \right| > \delta \right] = 0 .$$

As $\mathbb{Q}^*$ is concentrated on absolutely continuous paths $\pi_t(du) = \rho(t, u)du$ with positive density bounded by $1, \varepsilon^{-1}\pi_s([\cdot, + \varepsilon])$ converges in $L^1(T)$ to $\rho(s, u)$ as $\varepsilon \downarrow 0$. Thus,

$$\mathbb{Q}^* \left[ \left| \langle \pi_t, G_\lambda H \rangle - \langle \pi_0, G_\lambda H \rangle - \int_0^t ds \langle \Phi(\rho_s), L_W G_\lambda H \rangle \right| > \delta \right] = 0$$

because $U_\lambda = L_W G_\lambda H$. Letting $\delta \downarrow 0$, we see that $\mathbb{Q}^*$ a.s.

$$\langle \pi_t, G_\lambda H \rangle - \langle \pi_0, G_\lambda H \rangle = \int_0^t ds \langle \Phi(\rho_s), L_W G_\lambda H \rangle .$$

This identity can be extended to a countable set of times $t$. Taking this set to be dense, by continuity of the trajectories $\pi_t$, we obtain that it holds for all $0 \leq t \leq T$.

In the same way, it holds for any countable family of continuous functions. Taking
a countable set of continuous functions, dense for the uniform topology, we extend this identity to all continuous function $H$ because $G_{\lambda}H_n$ converges to $G_{\lambda}H$ in $L^1(T)$ if $H_n$ converges to $H$ in the uniform topology. Similarly, we can show that it holds for all $\lambda > 0$, since, for any continuous function $H$, $G_{\lambda}H_n$ converges to $G_{\lambda}H$ in $L^1(T)$, as $\lambda_n \to \lambda$.

Proof of Proposition 4.1. In the previous subsection we showed that the sequence of probability measures $Q^W,N_{\mu,N}$ is tight for the uniform topology. We just proved that all limit points of this sequence are concentrated on weak solutions of the parabolic equation (2.4). The statement of the proposition follows from the uniqueness of weak solutions proved in Section 6. □

4.3. Replacement lemma. Denote by $H_N(\mu_N|\nu_{\alpha})$ the entropy of a probability measure $\mu_N$ with respect to a stationary state $\nu_{\alpha}$. We refer to [6, Section A1.8] for a precise definition. By the explicit formula given in [6, Theorem A1.8.3], we see that there exists a finite constant $K_0$, depending only on $\alpha$, such that

$$H_N(\mu_N|\nu_{\alpha}) \leq K_0N$$

for all measures $\mu_N$.

Denote by $\langle \cdot, \cdot \rangle_{\nu_{\alpha}}$ the scalar product of $L^2(\nu_{\alpha})$ and denote by $I^\xi_N$ the convex and lower semicontinuous [6, Corollary A1.10.3] functional defined by

$$I^\xi_N(f) = \langle -L_N \sqrt{f}, \sqrt{f} \rangle_{\nu_{\alpha}},$$

for all probability densities $f$ with respect to $\nu_{\alpha}$ (i.e., $f \geq 0$ and $\int f d\nu_{\alpha} = 1$). An elementary computation shows that

$$I^\xi_N(f) = \sum_{x \in T_N} I^\xi_{x,x+1}(f),$$

where

$$I^\xi_{x,x+1}(f) = (1/2) \xi_x \int c_{x,x+1}(\eta) \left\{ \sqrt{f(\sigma^{x,x+1}\eta)} - \sqrt{f(\eta)} \right\}^2 d\nu_{\alpha}.$$

By [6, Theorem A1.9.2], if $\{S^N_t : t \geq 0\}$ stands for the semi-group associated to the generator $N^2L_N$,

$$H_N(\mu_N S^N_t|\nu_{\alpha}) + N^2 \int_0^t I^\xi_N(f^N_s) ds \leq H_N(\mu_N|\nu_{\alpha}),$$

provided $f^N_s$ stands for the Radon-Nikodym derivative of $\mu_N S^N_s$ with respect to $\nu_{\alpha}$.

For a local function $g : [0,1]^Z \to \mathbb{R}$, let $\tilde{g} : [0,1] \to \mathbb{R}$ be the expected value of $g$ under the stationary states:

$$\tilde{g}(\alpha) = E_{\nu_{\alpha}}[g(\eta)].$$

For $\ell \geq 1$, let $\eta^\ell(x)$ be the density of particles on the interval $\{x, \ldots, x + \ell - 1\}$:

$$\eta^\ell(x) = \frac{1}{\ell} \sum_{y=x}^{x+\ell-1} \eta(y).$$
Lemma 4.3. Fix a function $F : N^{-1}T_N \to \mathbb{R}$. There exists a finite constant $C_0$, depending only on $a$, $g$ and $W$, such that

$$\frac{1}{N} \sum_{x \in T_N} F(x/N) \int \{ \tau_x g(\eta) - \tilde{g}(\eta\xi N(x)) \} f(\eta) \nu_\alpha(d\eta)$$

$$\leq \frac{C_0}{\varepsilon N^2} \sum_{x \in T_N} |F(x/N)| + \frac{C_0 \varepsilon}{\delta N} \sum_{x \in T_N} F(x/N)^2 + \delta N I_N^\xi(f)$$

for all $\delta > 0$ and all probability density $f$ with respect to $\nu_\alpha$.

Proof. Any local function can be written as a linear combination of functions of type $\prod_{\alpha \in A} \eta(x)$, for finite sets $A$’s. It is therefore enough to prove the lemma for such functions. We prove the result for $g(\eta) = \eta(0)\eta(1)$. The general case can be handled in a similar way.

We estimate first

$$\frac{1}{N} \sum_{x \in T_N} F(x/N) \int \eta(x) \left\{ \eta(x + 1) - \frac{1}{\varepsilon N} \sum_{y=x}^{x+\varepsilon N-1} \eta(y) \right\} f(\eta) \nu_\alpha(d\eta)$$

in terms of the functional $I_N^\xi(f)$. The integral can be rewritten as

$$\frac{1}{\varepsilon N} \sum_{y=x+2}^{x+\varepsilon N-1} \sum_{z=x+1}^{y-1} \int \eta(x) \{ \eta(z) - \eta(z + 1) \} f(\eta) \nu_\alpha(d\eta) + O(\frac{1}{\varepsilon N})$$

where the remainder comes from the contribution $y = x$. Writing last integral as twice the same expression and performing the change of variables $\eta' = \sigma^{x,z+1} \eta$ in one of them, the previous integral becomes

$$(1/2) \int \eta(x) \{ \eta(z) - \eta(z + 1) \} \{ f(\eta) - f(\sigma^{x,z+1} \eta) \} \nu_\alpha(d\eta).$$

Since $a - b = (\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})$, by Schwarz inequality the previous expression is less than or equal to

$$\frac{A}{16(1 - 2a^-) \xi_{\varepsilon}} \int \eta(x) \{ \eta(z) - \eta(z + 1) \}^2 \left\{ \sqrt{f(\eta)} + \sqrt{f(\sigma^{x,z+1} \eta)} \right\} \nu_\alpha(d\eta)$$

$$+ \frac{\xi_{\varepsilon}}{A} \int c_{z,z+1}(\eta) \left\{ \sqrt{f(\eta)} - \sqrt{f(\sigma^{x,z+1} \eta)} \right\} \nu_\alpha(d\eta)$$

for every $A > 0$. In this formula we used the fact that $c_{z,z+1}$ is bounded below by $1 - 2a^-$. Since $f$ is a density with respect to $\nu_\alpha$, the first expression is bounded by $A/4(1 - 2a^-) \xi_{\varepsilon}$, while the second one is equal to $2A^{-1} I_{z,z+1}(f)$. Adding together all previous estimates, we obtain that (4.6) is less than or equal to

$$\frac{1}{\varepsilon N^2} \sum_{x \in T_N} |F(x/N)| + \frac{A}{4(1 - 2a^-) N} \sum_{x \in T_N} F(x/N)^2 \sum_{z=x+1}^{x+\varepsilon N} \xi_{\varepsilon}^{-1} + \frac{2\varepsilon}{A} \sum_{x \in T_N} I_{z,z+1}(f).$$

By definition of the sequence $\{\xi_{\varepsilon}\}$, $\sum_{z+1 \leq \tau \leq \varepsilon N} \xi_{\varepsilon}^{-1} \leq N[W(1) - W(0)]$. Thus, choosing $A = 2\varepsilon N^{-1} \delta^{-1}$, for some $\delta > 0$, we obtain that the previous sum is bounded above by

$$\frac{1}{\varepsilon N^2} \sum_{x \in T_N} |F(x/N)| + \frac{C_0 \varepsilon}{\delta N} \sum_{x \in T_N} F(x/N)^2 + \delta N I_N^\xi(f).$$
Up to this point we have replaced $\eta(x)\eta(x + 1)$ by $\eta(x)\eta^{\varepsilon N}(x)$. The same arguments permit to replace this latter expression by $\left[\eta^{\varepsilon N}(x)\right]^2$, which concludes the proof of the lemma.

**Corollary 4.4.** Fix a cylinder function $g$ and a sequence of functions $\{F_N : N \geq 1\}$, $F_N : N^{-1} \mathbb{T}_N \to \mathbb{R}$ such that

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{x \in \mathbb{T}_N} F_N(x/N)^2 < \infty.$$  

Then, for any $t > 0$ and any sequence of probability measures $\{\mu_N : N \geq 1\}$ on $\{0, 1\}^\mathbb{T}_N$,

$$\limsup_{\varepsilon \to 0} \limsup_{N \to \infty} \mu_N \left[ \left| \int_0^t \frac{1}{N} \sum_{x \in \mathbb{T}_N} F_N(x/N) \{\tau_x g(\eta_s) - \tilde{g}(\eta_s^{\varepsilon N}(x))\} \right| \right] = 0.$$

**Proof.** Fix $0 < \alpha < 1$. By the entropy and Jensen inequalities, the expectation appearing in the statement of the lemma is bounded above by

$$\frac{H_N(\mu_N; \nu_\alpha)}{\gamma N} + \frac{1}{\gamma N} \log \mathbb{E}_{\nu_\alpha} \left[ \exp \left\{ \gamma \left| \int_0^t \sum_{x \in \mathbb{T}_N} F_N(x/N) \{\tau_x g(\eta_s) - \tilde{g}(\eta_s^{\varepsilon N}(x))\} \right| \right\} \right]$$

for all $\gamma > 0$. In view of (4.5), to prove the corollary it is enough to show that the second term vanishes as $N \uparrow \infty$ and then $\varepsilon \downarrow 0$ for every $\gamma > 0$. We may remove the absolute value inside the exponential because $e^{\varepsilon x} \leq e^{\varepsilon x} + e^{-\varepsilon x}$ and because $\limsup_{N \to \infty} N^{-1} \log \{a_N + b_N\} \leq \max\{\limsup_{N \to \infty} N^{-1} \log a_N, \limsup_{N \to \infty} N^{-1} \log b_N\}$. Thus, to prove the corollary, we need to show that

$$\limsup_{\varepsilon \to 0} \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{E}_{\nu_\alpha} \left[ \exp \left\{ \gamma \left| \int_0^t \sum_{x \in \mathbb{T}_N} F_N(x/N) \{\tau_x g(\eta_s) - \tilde{g}(\eta_s^{\varepsilon N}(x))\} \right| \right\} \right] = 0$$

for every $\gamma > 0$.

By Feynman-Kac formula, for each fixed $N$ the previous expression is bounded above by

$$\limsup_{\varepsilon \to 0} \sup_f \left\{ \int \frac{\gamma}{N} \sum_{x \in \mathbb{T}_N} F_N(x/N) \{\tau_x g(\eta) - \tilde{g}(\eta^{\varepsilon N}(x))\} f(\eta) \, d\nu_\alpha - N \tilde{I}_N^\varepsilon(f) \right\},$$

where the supremum is carried over all density functions $f$ with respect to $\nu_\alpha$. Letting $\delta = 1$ in Lemma 4.3 we obtain that the previous expression is less than or equal to

$$\frac{C_0 \gamma \varepsilon}{\varepsilon N^2} \sum_{x \in \mathbb{T}_N} |F_N(x/N)| + \frac{C_0 \gamma \varepsilon}{N} \sum_{x \in \mathbb{T}_N} F_N(x/N)^2$$

for some finite constant $C_0$ which depends on $g$ and $W$. By assumption on the sequence $\{F_N\}$, for every $\gamma > 0$, this expression vanishes as $N \uparrow \infty$ and then $\varepsilon \downarrow 0$. This concludes the proof of the lemma. □

### 5. Energy estimate

We prove in this section that any limit point $Q^*_W$ of the sequence $Q_{\mu_N}^{W,N}$ is concentrated on trajectories $\rho(t, u)du$ with finite energy. Though not needed in the proof of the law of large numbers of the empirical measure $\pi^N$, this estimate plays an important role in the proof of the large deviations principle.
Let $Q^*_W$ be a limit point of the sequence $Q^{W,N}_{\mu}$ and assume without loss of generality that the sequence $Q^{W,N}_{\mu}$ converges to $Q^*_W$. Denote by $\partial_u$ the partial derivative of a function with respect to the space variable. Let $L^2_W([0,T] \times \mathbb{T})$ be the Hilbert space of measurable functions $H : [0,T] \times \mathbb{T} \to \mathbb{R}$ such that
\[
\int_0^T ds \int_T dW(u) H(s,u)^2 < \infty ,
\]
endowed with the scalar product $\langle H, G \rangle_W$ defined by
\[
\langle H, G \rangle_W = \int_0^T ds \int_T dW(u) H(s,u) G(s,u) .
\]

**Proposition 5.1.** The measure $Q^*_W$ is concentrated on paths $\rho(t,u)du$ with the property that there exists a function in $L^2_W([0,T] \times \mathbb{T})$, denoted by $d\Phi/dW$, such that
\[
\int_0^T ds \int_T du \langle \partial_u H(s,u) \Phi(\rho(s,u)) = - \int_0^T ds \int_T dW(u) \langle d\Phi/dW)(s,u) H(s,u)
\]
for all functions $H$ in $C^{0,1}([0,T] \times \mathbb{T})$.

The previous result follows from the next lemma. Recall the definition of the constant $K_0$ given in (4.3).

**Lemma 5.2.** There exists a finite constant $K_1$, depending only on $\alpha$, such that
\[
E_{Q^*_W} \left[ \sup_H \left\{ \int_0^T ds \int_T du \langle \partial_u H(s,u) \Phi(\rho(s,u)) \right. \right.
\]
\[
\left. \left. \left. \quad - K_1 \int_0^T ds \int_T H(s,u)^2 dW(u) \right\} \right\} \leq K_0 ,
\]
where the supremum is carried over all functions $H$ in $C^{0,1}([0,T] \times \mathbb{T})$.

**Proof of Proposition 5.1.** Denote by $\ell : C^{0,1}([0,T] \times \mathbb{T}) \to \mathbb{R}$ the linear functional defined by
\[
\ell(H) = \int_0^T ds \int_T du \langle \partial_u H(s,u) \Phi(\rho(s,u)) .
\]
Since $C^{0,1}([0,T] \times \mathbb{T})$ is dense in $L^2_W([0,T] \times \mathbb{T})$, by Lemma 5.2 $\ell$ is $Q^*_W$-almost surely finite in $L^2_W([0,T] \times \mathbb{T})$. In particular, by Riesz representation theorem, there exists a function $G$ in $L^2_W([0,T] \times \mathbb{T})$ such that
\[
\ell(H) = - \int_0^T ds \int_T dW(u) H(s,u) G(s,u) .
\]
This concludes the proof of the proposition. \hfill \square

The proof of Lemma 5.2 relies on the following result. For a smooth function $H : \mathbb{T} \to \mathbb{R}$, let $\delta > 0$, $\varepsilon > 0$ and a positive integer $N$, define $W_N(\varepsilon, \delta, H, \eta)$ by
\[
W_N(\varepsilon, \delta, H, \eta) = \sum_{x \in \mathbb{T}} H(x/N) \quad \frac{1}{\varepsilon N} \left\{ \Phi(\eta^{\delta N}(x)) - \Phi(\eta^{\delta N}(x + \varepsilon N)) \right\}
\]
\[
- \frac{K_1}{\varepsilon N} \sum_{x \in \mathbb{T}} H(x/N)^2 \left\{ W([x + \varepsilon N + 1]/N) - W(x/N) \right\} .
\]
Lemma 5.3. Consider a sequence \( \{H_\ell, \ell \geq 1\} \) dense in \( C^{0,1}([0,T] \times \mathbb{T}) \). For every \( k \geq 1 \), and every \( \varepsilon > 0 \),

\[
\limsup_{\delta \to 0} \limsup_{N \to \infty} \mathbb{E}_{\mu_N} \left[ \max_{1 \leq i \leq k} \left\{ \int_0^T W_N(\varepsilon, \delta, H_i(s, \cdot), \eta_s) \, ds \right\} \right] \leq K_0.
\]

**Proof.** It follows from the replacement lemma that in order to prove the lemma we just need to show that

\[
\limsup_{N \to \infty} \mathbb{E}_{\mu_N} \left[ \max_{1 \leq i \leq k} \left\{ \int_0^T W_N(\varepsilon, H_i(s, \cdot), \eta_s) \, ds \right\} \right] \leq K_0,
\]

where

\[
W_N(\varepsilon, H, \eta) = \frac{1}{\varepsilon N} \sum_{x \in \mathbb{Z}^N} H(x/N) \left\{ \tau_x g(\eta) - \tau_{x+\varepsilon N} g(\eta) \right\}
\]

\[= -\frac{K_1}{\varepsilon N} \sum_{x \in \mathbb{Z}^N} H(x/N)^2 \{ W([x+\varepsilon N+1]/N) - W(x/N) \}, \]

and \( g(\eta) = \eta(0) + 2a\eta(0)\eta(1) \).

By the entropy and the Jensen inequality, for each fixed \( N \), the previous expectation is bounded above by

\[
\frac{H(\mu_N^N|\nu_\alpha)}{N} + \frac{1}{N} \log \mathbb{E}_{\nu_\alpha} \left[ \exp \left\{ \max_{1 \leq i \leq k} \left\{ N \int_0^T ds W_N(\varepsilon, H_i(s, \cdot), \eta_s) \right\} \right\} \right].
\]

By Lemma 5.3, the first term is bounded by \( K_0 \). Since \( \exp(\max_{1 \leq j \leq k} a_j) \) is bounded above by \( \sum_{1 \leq j \leq k} \exp(a_j) \) and since \( \limsup_N N^{-1} \log \{ a_N + b_N \} \) is less than or equal to the maximum of \( \limsup_N N^{-1} \log a_N \) and \( \limsup_N N^{-1} \log b_N \), the limit, as \( N \to \infty \), of the second term of the previous expression is less than or equal to

\[
\max_{1 \leq i \leq k} \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{E}_{\nu_\alpha} \left[ \exp \left\{ N \int_0^T ds W_N(\varepsilon, H_i(s, \cdot), \eta_s) \right\} \right].
\]

We now prove that for each fixed \( i \) the above limit is nonpositive.

Fix \( 1 \leq i \leq k \). By Feynman–Kac formula and the variational formula for the largest eigenvalue of a symmetric operator, for each fixed \( N \), the previous expression is bounded above by

\[
\int_0^T ds \sup_f \left\{ \int W_N(\varepsilon, H_i(s, \cdot), \eta) f(\eta(\nu_\alpha(\rho_N^s)) - N I_N^s(f) \right\}.
\]

In this formula the supremum is taken over all probability densities \( f \) with respect to \( \nu_\alpha \).

It remains to rewrite \( \eta(x)\eta(x+1) - \eta(x+\varepsilon N)\eta(x+\varepsilon N+1) \) as \( \eta(x)\{ \eta(x+1) - \eta(x+\varepsilon N+1) \} + \eta(x)\{ \eta(x+\varepsilon N+1) - \eta(x+\varepsilon N) \} \) and to repeat the arguments presented in the proof of Lemma 5.3 to conclude. \( \square \)

**Proof of Lemma 5.2.** Assume without loss of generality that \( \mathcal{Q}^{W,N}_{\mu_N} \) converges to \( \mathcal{Q}^W_W \). Consider a sequence \( \{H_\ell, \ell \geq 1\} \) dense in \( C^{0,1}([0,T] \times \mathbb{T}) \). By Lemma 5.3 for every \( k \geq 1 \)

\[
\limsup_{\delta \to 0} \mathbb{E}_{\mathcal{Q}^W_W} \left[ \max_{1 \leq i \leq k} \left\{ \frac{1}{\varepsilon} \int_0^T ds \int_T \Phi(\rho_N^s(u)) - \Phi(\rho_N^s(u+\varepsilon)) \right\} \right.
\]

\[= -\frac{K_1}{\varepsilon} \int_0^T ds \int_T \Phi(\rho_N^s(u+\varepsilon)) - \Phi(\rho_N^s(u)) \right\} \leq K_0,
\]

for every \( k \geq 1 \).
where $\rho^\delta_t(u) = (\rho * \iota^\delta)(u)$ and $\iota^\delta$ is the approximation of the identity $\iota^\delta(\cdot) = (2\delta)^{-1}\mathbf{1}\{[-\delta, \delta]\}(\cdot)$.

Letting $\delta \downarrow 0$, changing variables and then letting $\varepsilon \downarrow 0$, we obtain that
\[
E_{GW} \left[ \max_{1 \leq k \leq k} \left\{ \int_0^T ds \int_T (\partial_u H_l)(s, u) \Phi(\rho(s, u)) du \right\} \right] - K_1 \int_0^T ds \int_T H_l(s, u)^2 dW(u) \right\} \leq K_0.
\]

To conclude the proof it remains to apply the monotone convergence theorem and recall that $\{H_\ell, \ell \geq 1\}$ is a dense sequence in $C^{0,1}([0, T] \times \mathbb{T})$ for the norm $\|H\|_\infty + \|\nabla H\|_\infty$.

6. Uniqueness of weak solutions of (2.4)

Recall that we denote by $\langle \cdot, \cdot \rangle$ the inner product of the Hilbert space $L^2(\mathbb{T})$ and that $\{G_\lambda : \lambda > 0\}$ stands for the resolvents associated to $L_W$.

Let $\rho$ be a weak solution of the hydrodynamic equation (2.4). Since $\rho$, $\Phi(\rho)$ are bounded, since the smooth functions are dense in $L^2(\mathbb{T})$ and since $L_W G_\lambda = -\mathbb{I} + \lambda G_\lambda$ are bounded operators, for any function $H$ in $L^2(\mathbb{T})$,
\[
\langle \rho_t, G_\lambda H \rangle - \langle \gamma, G_\lambda H \rangle = \int_0^t \langle \Phi(\rho_s), L_W G_\lambda H \rangle \, ds
\]
for all $t > 0$ and all $\lambda > 0$.

Let $\rho : \mathbb{R}_+ \times \mathbb{T} \to [0, T]$ be a weak solution of (2.4). We claim that
\[
\langle \rho_t, G_\lambda \rho_t \rangle - \langle \rho_0, G_\lambda \rho_0 \rangle = 2 \int_0^t \langle \Phi(\rho_s), L_W G_\lambda \rho_s \rangle \, ds \quad (6.1)
\]
for all $t > 0$ and $\lambda > 0$.

To prove this claim, fix $\lambda > 0$, $t > 0$ and consider a partition $0 = t_0 < t_1 < \cdots < t_n = t$ of the interval $[0, t]$ so that
\[
\langle \rho_{t_k}, G_\lambda \rho_{t_k} \rangle - \langle \rho_0, G_\lambda \rho_0 \rangle = \sum_{k=0}^{n-1} \langle \rho_{t_{k+1}}, G_\lambda \rho_{t_{k+1}} \rangle - \langle \rho_{t_{k+1}}, G_\lambda \rho_{t_k} \rangle + \sum_{k=0}^{n-1} \langle \rho_{t_{k+1}}, G_\lambda \rho_{t_k} \rangle - \langle \rho_{t_k}, G_\lambda \rho_{t_k} \rangle.
\]

We handle the first term, the second one being similar. Since $G_\lambda$ is self-adjoint in $L^2(\mathbb{T})$, since $\rho_{t_{k+1}}$ belongs to $L^2(\mathbb{T})$ and since $\rho$ is a weak solution of (2.4),
\[
\langle \rho_{t_{k+1}}, G_\lambda \rho_{t_{k+1}} \rangle - \langle \rho_{t_{k+1}}, G_\lambda \rho_{t_{k}} \rangle = \int_{t_k}^{t_{k+1}} \langle \Phi(\rho_s), L_W G_\lambda \rho_s \rangle \, ds.
\]
Add and subtract on the right hand side $\langle \Phi(\rho_s), L_W G_\lambda \rho_{t_k} \rangle$. The time integral of this term is exactly the expression announced in (6.1) and the remainder is given by
\[
\int_{t_k}^{t_{k+1}} \{\langle \Phi(\rho_s), L_W G_\lambda \rho_{t_{k+1}} \rangle - \langle \Phi(\rho_s), L_W G_\lambda \rho_s \rangle \} \, ds.
\]

Since $L_W G_\lambda = -\mathbb{I} + \lambda G_\lambda$, where $\mathbb{I}$ is the identity, and since $G_\lambda$ is self-adjoint, we may rewrite the previous difference as
\[
-\{\langle \Phi(\rho_s), \rho_{t_{k+1}} \rangle - \langle \Phi(\rho_s), \rho_s \rangle \} + \lambda \{\langle G_\lambda \Phi(\rho_s), \rho_{t_{k+1}} \rangle - \langle G_\lambda \Phi(\rho_s), \rho_s \rangle \}.
\]
The time integral between \( t_k \) and \( t_{k+1} \) of the second term is equal to
\[
\lambda \int_{t_k}^{t_{k+1}} ds \int_s^{t_{k+1}} \langle \mathcal{L}_W G_\lambda \Phi(\rho_s) , \Phi(\rho_r) \rangle \, dr
\]
because \( \rho \) is a weak solution of (2.4) and \( \Phi(\rho_s) \) belongs to \( L^2(T) \). It follows from the boundness of the operator \( \mathcal{L}_W G_\lambda \) and from the boundness of \( \Phi(\rho) \) that this expression is of order \( (t_{k+1} - t_k)^2 \).

To conclude the proof of claim (6.1) it remains to show that
\[
\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left\{ \langle \Phi(\rho_s), \rho_{t_{k+1}} \rangle - \langle \Phi(\rho_s), \rho_s \rangle \right\} ds
\]
vanesishes as the mesh of the partition tends to 0. Fix \( \varepsilon > 0 \) and choose \( \beta \) large enough for
\[
\int_0^t ds \int_T \left\{ \beta G_\beta \Phi(\rho(s,u)) - \Phi(\rho(s,u)) \right\}^2 du \leq \varepsilon .
\]
This is possible because \( \Phi(\rho) \) is bounded, \( \{ \beta G_\beta : \beta > 0 \} \) are uniformly bounded operators, and \( \beta G_\beta \Phi(\rho(s,\cdot)) \) converges to \( \Phi(\rho(s,\cdot)) \) in \( L^2(T) \), as \( \beta \uparrow \infty \), for all \( 0 \leq s \leq t \).

Paying a price of order \( \sqrt{\varepsilon} \), because \( t \leq \rho \leq r \), we may replace \( \Phi(\rho_s) \) in the penultimate formula by \( \beta G_\beta \Phi(\rho_s) \). After this replacement, since \( \rho \) is weak solution, we may rewrite the sum as
\[
\beta \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} ds \int_s^{t_{k+1}} \langle \mathcal{L}_W G_\beta \Phi(\rho_s), \Phi(\rho_r) \rangle \, dr .
\]
We have already seen that this expression vanishes as the mesh of the partition tends to 0. This proves (6.1).

Recall the definition of the constant \( B \) given at the beginning of Subsection 2.2

**Lemma 6.1.** Fix two density profiles \( \gamma^1, \gamma^2 : T \rightarrow [l,r] \) and denote by \( \rho^1, \rho^2 \) weak solutions of (2.4) with initial value \( \gamma^1, \gamma^2 \), respectively. Then,
\[
\langle \rho^1 - \rho^2 , G_\lambda [\rho_1^2 - \rho_1^2] \rangle \leq \langle \gamma^1 - \gamma^2 , G_\lambda [\gamma_1^2 - \gamma_2^2] \rangle e^{BM/2}
\]
for all \( \lambda > 0, t > 0 \). In particular, there exists at most one weak solution of (2.4).

**Proof.** Fix two density profiles \( \gamma^1, \gamma^2 : T \rightarrow [l,r] \). Let \( \rho^1, \rho^2 \) be two weak solutions with initial value \( \gamma^1, \gamma^2 \), respectively. By (6.1), for any \( \lambda > 0 \),
\[
\langle \rho^1 - \rho_2^2 , G_\lambda [\rho_1^2 - \rho_1^2] \rangle - \langle \gamma^1 - \gamma^2 , G_\lambda [\gamma_1^2 - \gamma_2^2] \rangle =
-2 \int_0^t \langle \Phi(\rho^1_s) - \Phi(\rho^2_s) , \rho^1_s - \rho^2_s \rangle \, ds + 2 \lambda \int_0^t \langle \Phi(\rho^1_s) - \Phi(\rho^2_s) , G_\lambda [\rho_s^1 - \rho_s^2] \rangle \, ds .
\]
By Schwarz inequality, the second term on the right hand side is bounded above by
\[
\frac{1}{\lambda} \int_0^t \langle \Phi(\rho^1_s) - \Phi(\rho^2_s) , G_\lambda [\Phi(\rho^1_s) - \Phi(\rho^2_s)] \rangle \, ds + A \lambda^2 \int_0^t \langle \rho^1_s - \rho^2_s , G_\lambda [\rho_s^1 - \rho_s^2] \rangle \, ds
\]
for every $A > 0$. Since the operator $G_\lambda$ is bounded by $\lambda^{-1}$, and since $\Phi'$ is bounded by $B$, the first term of the previous expression is less than or equal to

$$\frac{B}{A\lambda} \int_0^t \langle \rho_s^1 - \rho_s^2, \Phi(\rho_s^1) - \Phi(\rho_s^2) \rangle \, ds.$$  

Choosing $A = B/2\lambda$, this expression cancels with the first term on the right hand side of the first formula. In particular, the left hand side of this formula is bounded by

$$\frac{B\lambda}{2} \int_0^t \langle \rho_s^1 - \rho_s^2, G_\lambda[\rho_s^1 - \rho_s^2] \rangle \, ds.$$  

It remains to recall Gronwall’s inequality to conclude. □

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