Pumped Dirac magnons paired state

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We study pumping of magnons to the Dirac points of magnon’s Brillouin zone of a ferromagnet on a honeycomb lattice. In particular, we consider second-order Suhl process, when due to interaction between magnons, a pair of magnons is created due to absorption of two electromagnetic wave quanta. We introduce a bosonic analog of the Cooper ladder for the magnon pair, which is shown to enhance the pairing of magnons at the Dirac points. As a result of pairing of the Dirac magnons, the system becomes unstable towards formation of a magnetic state with zero magnetization - the Dirac magnon paired state. In this case the resonant frequency of the pump equals to that of energy of the Dirac points. Our estimates suggest that the Dirac magnon paired state can be found in the CrBr$_3$ ferromagnet in the vicinity of the Curie temperature.

PACS numbers:

Magnons are fluctuations about the spontaneous magnetic order. Typically two types of magnons are distinguished based on the magnetic structure, ferromagnetic or antiferromagnetic. The two have different low-energy, low-momentum dispersion, regardless of the lattice structure of the magnetic structure. Ferromagnetic magnons are quadratic in momentum, while antiferromagnetic are linear (for example, see [1][2]). Recently, because of the progress made in topological properties of fermions, a topology tool has been applied to understand intrinsic transport properties of magnons. With that details of the lattice structure became important. Certain lattices, for example, pyrochlore [3][4], kagome [5][6], and honeycomb [7][8], allow for natural magnons’s momentum-pseudospin locking. In ferromagnets such locking results in Dirac crossing points (degeneracies) at some particular high-energy and finite-momentum points in magnon’s Brillouin zone. It is convenient to call magnons at such crossing points as the Dirac magnons [9]. As a result of the locking, certain types of the Dzyaloshinskii-Moriya interaction, allowed by the lattice symmetry, result in various transverse responses of magnons to the temperature gradient, such as magnon thermal Hall [10][11] and Nernst effects [12], and to fictitious gauge fields [13], such as the magnon Hall effect.

In this Letter we find another unique Dirac magnons property revealed under the second-order Suhl magnon pumping process [1][2][20][21]. A single Dirac magnon can’t be created in a process of absorption of one pumping field quanta. This is because such magnon is located at non-zero momentum in the Brillouin zone, and there is no way to conserve the momentum in the process of absorption, as the experimentally relevant pumping field has a zero wave vector. However, a pair of magnons with opposite momenta can be created when two pump field quanta are absorbed. Such processes are known as the second-order Suhl processes (see Fig. 1B). We show that this process is not present in linearized spin-wave theory of magnons, but appears when the interaction between the magnons is included to the consideration. The frequency of the pump can scan the entire Brillouin zone of the magnons, and the absorption of two magnons can happen at any frequency. However, as we show in this Letter, the pump’s frequency equal to the energy of the Dirac points is the resonant due to the magnon rescattering processes of the Cooper ladder type (see Fig. 2B). This is because at such frequency, the system can accommodate the largest amount of magnon pairs with opposite momenta and frequencies (see Fig. 2B). For frequencies away from the Dirac points, the pairing of two magnons is parametrically weakened by the rescattering processes. The resonance corresponds to an instability of the system towards formation of a zero magnetization state. Below, we refer to such resonance as the Dirac magnons paired state. We hope Dirac magnons paired state can be experimentally observed in ferromagnets with spins on pyrochlore lattice [3], or layered kagome [5] and honeycomb lattices [7][9]. In particular, based on our estimates, we predict that it can be observed in the honeycomb CrBr$_3$ ferromagnet [17] in the vicinity of the Curie temperature.

To demonstrate the effect, let us study a model of insulating ferromagnet in which spins of length $S$ are located on the sites of honeycomb lattice (see Fig. 1A). Near neighbor spins interact with each other via ferromagnetic Heisenberg exchange interaction. Ferromagnetic order is assumed to be in $z$-direction, this can be achieved by applying a small magnetic field in $z$-direction. There is a pumping field which is perpendicular to the order, and which oscillates with a frequency $\Omega$ and has a zero wave vector. Hamiltonian of the system reads,

$$H = -J \sum_{\langle ij \rangle} S_i S_j + \Gamma \sum_i \left[S_i^x \cos(\Omega t) + S_i^y \sin(\Omega t)\right],$$

where $J > 0$ is the exchange coupling energy and $\Gamma$ is the pump’s intensity. In order to study the spin-waves, we use the Holstein-Primakoff presentation of spin operators in terms of bosons, namely for A atoms $S_i^+ = S_i^x + i S_i^y$ and $S_i^z$ read as $S_i^z = \sqrt{2S - a_i^\dagger a_i} a_i$, $S_i^- = \sqrt{2S}$, and $S_i^y = \frac{1}{\sqrt{2S}} (a_i^\dagger + a_i)$.
$a_i^\dagger \sqrt{2S - a_i^\dagger a_i}$, and $S^j_i = S - a_i^\dagger a_i$, with $[a_i, a_j^\dagger] = \delta_{i,j}$ boson commutation relation. The same is performed for the $B$ atoms with the help of $b_i$ and $b_i^\dagger$ boson operators.

In the space of elements of the honeycomb’s unit cell, in which case the boson operators are defined by $\Psi_k = (a_k^\dagger, b_k^\dagger)$, the Hamiltonian of non-interacting spin-waves reads as

$$H_0 = SJ \int \Psi_k^\dagger \left[ \frac{3}{-\gamma_k} \right] \Psi_k \equiv \int \Psi_k^\dagger[H_0] \Psi_k,$$

where $\gamma_k = \sum_{i=1,2,3} e^{ik \gamma_n}$, with corresponding wave functions $\varphi_{\pm} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i k_x} \mp e^{i k_y} \end{pmatrix}$. At special $K = (0, -\frac{\pi}{2})$ and $K' = (0, \frac{\pi}{2})$ points the spectrum is linear and is described by the Dirac Hamiltonian. The energy of magnons at these points is $\epsilon_{\pm,k} = 3SJ$. Terms quartic in boson operators describe the interaction between the magnons. Normal ordered interaction reads

$$H_{\text{int}} = -J \int \langle k \rangle \delta(\langle k \rangle) \gamma_{k_1-k_2} a_{k_1}^\dagger b_{k_2}^\dagger a_{k_3} b_{k_4} + \frac{J}{4} \int \langle k \rangle \left[ \gamma_{k_1-k_2} a_{k_1}^\dagger b_{k_2}^\dagger a_{k_3} + \gamma_{k_3-k_4} a_{k_1}^\dagger b_{k_2}^\dagger a_{k_3} b_{k_4} \right] + \frac{J}{4} \int \langle k \rangle \left[ \gamma_{k_1-k_2} a_{k_1}^\dagger b_{k_2}^\dagger b_{k_3} + \gamma_{k_3-k_4} a_{k_1}^\dagger b_{k_2}^\dagger b_{k_3} b_{k_4} \right],$$

where $\delta(\langle k \rangle) \equiv \delta_{k_1-k_2-k_3-k_4}$ short notation was used, and $\langle k \rangle$ stands for integration over all momenta. The interaction is instantaneous in time. In momentum space describing pump field with a frequency $\Omega$ is

$$H_{\text{pump}} = \frac{\Gamma \sqrt{S}}{\sqrt{2}} \left[ (a_0 + b_0) e^{-i \Omega t} + (a_0^\dagger + b_0^\dagger) e^{i \Omega t} \right],$$

where $a_0 \equiv a_{k_0}$ and the same for $b_0$. In order to understand the effect of the pumping field Eq. (4) on the magnons described by Eq. (2) and (3), we study the system in the Keldysh time space. This space complicates the analysis but gives a clear understanding of all relevant processes. We promote boson fields $a_k, b_k$ to $\Psi_{\alpha,k,\epsilon}, \Psi_{\beta,k,\epsilon}$ fields, in which frequency $\epsilon$ was explicitly used, and $a_k^\dagger, b_k^\dagger$ to $\bar{\Psi}_{\alpha,k,\epsilon}, \bar{\Psi}_{\beta,k,\epsilon}$. Furthermore, the fields are promoted to classical (cl) and quantum components (q) components in accord with the Keldysh technique (see SM for details and, for example, [22]). Let us show how to conveniently capture the process of absorption of the pumping field Eq. (4) by the magnons.

We write the advanced part of the Lagrangian describing non-interacting magnons defined by Eq. (2) with $\epsilon = \Omega$ frequency and $k = 0$ momentum,

$$\mathcal{L}^A_{\alpha,0,\Omega} = \sum_{m,n} \bar{\Psi}_{\alpha,m,0,\Omega}^\dagger \psi_{\beta,m,0,\Omega}^A - \Gamma \sqrt{S} \sum_n \psi_{\alpha,n,0,\Omega}^A,$$

where $\Gamma = \frac{\Omega}{\sqrt{2}} (\tau_2 - 1/2)$. Then, the Hamiltonian of non-interacting magnons, where $\psi_{\alpha,n,0,\Omega}^A = \delta_{\alpha,n} \Omega$, is given by

$$\mathcal{H}^A_{\alpha,0,\Omega} = \sum_{m,n} \bar{\Psi}_{\alpha,m,0,\Omega}^\dagger \psi_{\beta,m,0,\Omega}^A - \Gamma \sqrt{S} \sum_n \psi_{\alpha,n,0,\Omega}^A.$$
\[ \frac{1}{2} (\tilde{\psi}^{cl}/a,_{\alpha k + \mathbf{r}}, \tilde{\psi}^{cl}/a,_{\beta k + \mathbf{r}}, \tilde{\psi}^{cl}/a,_{\beta, -\mathbf{k} - \mathbf{r}}, -\tilde{\psi}^{cl}/a,_{\alpha, -\mathbf{k} - \mathbf{r}}), \]

the spectrum of a pair of magnons is given by a solution of the following secular equation,

\[
\text{det} \begin{bmatrix}
\zeta + \epsilon & SJ\gamma_k & -\Delta^2\gamma_0 & \Delta^2\gamma_k \\
SJ\gamma_k^* & \zeta + \epsilon & \Delta^2\gamma_k^* & -\Delta^2\gamma_0 \\
-\Delta^2\gamma_0 & \Delta^2\gamma_k & \zeta - \epsilon & SJ\gamma_k \\
\Delta^2\gamma_k & -\Delta^2\gamma_0 & SJ\gamma_k^* & \zeta - \epsilon
\end{bmatrix} = 0, \tag{9}
\]

where \( \zeta = \Omega - 3SJ \) is introduced for brevity, \( \gamma_0 = 3 \), and where the pairing strength \( \Delta^2 = \frac{J}{4} \left( \frac{\Gamma_S}{3SJ} \right)^2 \) for \( \zeta = 0 \), and \( \Delta^2 = \frac{J}{4} \left( \frac{\Gamma_S}{3SJ} \right)^2 \) for \( \zeta = \pm 3SJ \) and otherwise according to the shift Eq. \( \text{(6)} \), was defined. The equation is the boson analog of the Bogoliubov-de Gennes Hamiltonian in fermion systems. The difference is in the structure of signs of the frequencies \( \epsilon \) on the main diagonal in Eq. \( \text{(9)} \). The spectrum reads

\[
\epsilon_{\pm: k}^2 = (\zeta \pm SJ|\gamma_k|)^2 - \Delta^4 (\gamma_0 + |\gamma_k|)^2. \tag{10}
\]

Therefore, the system of pumped interacting magnons will become unstable when \( \epsilon_{\pm: k}^2 < 0 \) is satisfied. Let us analyze different parts of the spectrum for such an instability.

Let us first study a special case when pump’s frequency is \( \Omega = 3SJ \) for which \( \zeta = 0 \). Then, at the \( \Gamma = (0, 0) \) point \( |\gamma_k| \approx 3 - \frac{k^2}{4} \), then \( \epsilon_{\pm: k} = (SJ)^2 \left( 3 - \frac{k^2}{4} \right)^2 - 36\Delta^4 \). For the instability to occur at the \( \Gamma \) point, the intensity of the pump should become larger than the exchange coupling energy. However, experimentally reasonable assumption is \( SJ > \Delta^2 \) which means it is impossible to make the system unstable at the \( \Gamma \) point. On the other hand, at the \( K \) and \( K' \) we approximate \( |\gamma_k| \approx \frac{\sqrt{4} \Delta}{2}k \), and get for the spectrum \( \epsilon_{\pm: k} \approx (SJ)^2 \frac{\sqrt{4} \Delta}{2}k \) - 9\( \Delta^4 \). From here we observe that the solution is always unstable for momenta smaller than the threshold value of \( k_{\text{th}} = \frac{2\sqrt{4} \Delta}{3SJ} \), i.e. for \( k < k_{\text{th}} \). For schematics see Fig. \( 1 \).

Having pumped the magnons to the Dirac points, let us now study their rescattering processes. In the first order in interaction Eq. \( \text{(3)} \) we get Hartree-Fock type corrections shown in Fig. \( 2 \)a to the magnon’s dispersion \( \text{(23)} \). See SM for the details of their derivation. Interaction Eq. \( \text{(3)} \) treated to second order contributes to the magnon’s lifetime \( \text{(16)} \). Here we study how the pairing interaction strength \( \Delta^2 \) gets renormalized by the interaction. For that we construct a boson analog of the Cooper ladder shown in Fig. \( 2 \)b. Our calculations show (see SM for details) that the operator structure of \( \Delta^2 \) given in Eq. \( \text{(9)} \) gets reproduced at each step of the ladder. Then, summing up the ladder, we replace \( \Delta^2 \) for \( \zeta = 0 \) with

\[
\Delta^2 \rightarrow \frac{\Delta^2}{1 - \frac{J}{4} \left( \frac{\sqrt{4} \Delta}{2} \right)^2 \left( \frac{\Gamma_S}{3SJ} \right)^2} \tag{11}
\]

where \( J = J \left[ 1 - \frac{\pi}{3S} \left( \frac{T}{3SJ} \right)^2 \right] \) includes the Hartree-Fock corrections. The integral defining a step of the ladder is counting the number of pairs which can be created for a given frequency. Clearly the pairing of Dirac magnons is enhanced due to the rescattering processes. The minus sign in the denominator in Eq. \( \text{(11)} \) is due to the repulsive nature of the last two terms in Eq. \( \text{(9)} \). Our estimates suggest that for honeycomb lattice \( \text{CrBr}_3 \) \( S = \frac{3}{2} \) ferromagnet \( \text{(17)} \), the tendency is such that at temperatures in the vicinity of the Curie temperature, i.e. \( T \approx T_c \approx 3SJ \), the expression for the renormalized pairing strength Eq. \( \text{(11)} \) diverges. This signals a transition to a new state, which we call Dirac magnons paired state. It seems natural that the transition occurs in the vicinity of the Curie temperature, as there are plenty of magnons in the system and their rescattering processes are known to become important \( \text{(16)} \text{(17)} \text{(23)} \). If the spin is made more classical by increasing its length \( S \), the denominator in Eq. \( \text{(11)} \) does not become singular for any temperature.

Let us study the effect of Dzyaloshinskii-Moriya interaction of the \( H_{\text{DMI}} = D \sum_{\langle\langle ij\rangle\rangle} \mu_{ij} [S_i \times S_j]_{\perp} \) type on the pairing. Here \( D \) is a constant, \( \langle\langle ij\rangle\rangle \) notation counts second-nearest neighbors, and \( \mu_{ij} = \pm 1 \) is defined by the green dashed arrows in Fig. \( 1 \)b (see SM for more details). In the vicinity of the Dirac points, i.e. \( \zeta = 0 \), the spectrum of magnon pairs is now

\[
\epsilon_{\pm}^2 = (SJ)^2 \frac{3}{4} k^2 + \chi^2 - 9\Delta^4, \tag{12}
\]

where \( \chi = 3\sqrt{3}SD \). We conclude that if \( |\chi| \geq 3\Delta^2 \) there will be no instability in the system. In unpumped ferromagnet such Dzyaloshinskii-Moriya interaction opens up a gap at the Dirac points in the spectrum of the magnons. Then, for the Dirac magnons paired state to occur, pumping strength should overcome this gap.

When \( \zeta < 0 \) only \( \epsilon_{\pm: k}^2 \) can become less than zero and cause instability of the system. For example, close to
the $\Gamma$ point, we expand $|\gamma_k| \approx 3 - \frac{4}{3} \zeta$ and obtain for the threshold $k_{th} = \sqrt{\frac{6}{3S} \frac{\Lambda^2}{3S}}$ of the instability. When $3SJ > \zeta > 0$ only $e_{-k}^2$ can become less than zero. Performing the same approximations as for the $\zeta < 0$ case, we get for the threshold value of the momentum $k_{th} = \frac{6\Delta^2}{S(3SJ-\zeta)}$ for $3SJ - \zeta > \frac{3}{2} \Delta^2$, i.e. away from the $\Gamma$ point, and $k_{th} = \frac{2\sqrt{\Delta}}{\sqrt{3S}}$ for $3SJ - \zeta < \frac{3}{2} \Delta^2$ in the vicinity of the $\Gamma$ point. Rescattering processes shown in Fig. 2B result for the $3SJ > \zeta > 0$ case in

$$\Delta^2 \to \frac{\Delta^2}{1 + \frac{3}{16\pi S} \ln \left(\frac{S\Lambda^2}{4(3SJ-\zeta)}\right) + i \frac{3}{32S}}$$

(13)

where $\Lambda$ is the high-frequency cut-off. Therefore, as $\Omega \to 6SJ$, the pairing of magnons vanishes. We think that this might be natural, as the one pump quanta absorption is the most effective at $\Omega = 6SJ$, and the two pump quanta absorption channel must thus get closed.

In addition to studied rescattering processes, one needs to include magnon decay rate, which originates due to interactions in second-order perturbation theory, to the main diagonal in the secular equation Eq. (9). Then the threshold value is going to be decreased by the decay rate. In particular, [10] showed (see Fig. 2 there) that the decay rate, which is $1/\tau \propto T^2$, for a honeycomb lattice ferromagnet is the smallest for the Dirac magnons and is the largest for the $\varepsilon_{+-k}$ magnons in the vicinity of the $\Gamma$ point. Therefore, the threshold value for the Dirac magnon paired state instability does not get drastically modified by the decay rate. All in all, from Eq. (11) and discussions above we conclude that the Dirac magnon paired state can become the most unstable for temperatures in the vicinity of the Curie temperature.

Let us speculate on the nature of the Dirac magnon paired state. If the system is finite and isolated, the exponential growth of the Dirac magnons pairs in time can’t last forever, and it will be stopped by interactions between the magnons, effects which are beyond studied in the present letter. It is clear that the Dirac magnon paired state is the instability of a ferromagnet towards formation of a zero magnetization state. Note that the resonant frequency of the pump equals $3SJ$, hence, absorption of two quanta describes a flip of one of the spins in the unit cell. To start thinking about such state, one can imagine dynamically generated antiferromagnetic order on the honeycomb lattice. However, such antiferromagnetic order will be fluctuating in time between different configurations with zero magnetization. Unlike in the experiment [24], we restrain ourselves from calling the Dirac magnon paired state as the Bose-Einstein condensate (BEC) of Dirac magnons, because the pumped system is out-of-equilibrium, the Dirac magnons are not at the lowest energy, and more importantly question of phase coherence of Dirac magnon pairs is not understood.

Detailed understanding of the nature of the new state is a question for future research.

In passing, let us discuss another possibility of pumping the magnons. First note that in the honeycomb lattice there are two energy branches at the $\Gamma$ point corresponding to $\epsilon_{+-0} = 6SJ$ and $\epsilon_{--0} = 0$, which are connected by $\Omega = 6SJ$ frequency. Therefore, one can excite a singel magnon by a pump Eq. (1) with a frequency via $\epsilon_{+-0} + \Omega \to \epsilon_{+-0}$ process. Pumping a single magnon will not make the system unstable in a sense of Eqs. (9) and (10).

However, an additional rescattering of the excited magnon with frequency $\epsilon_{+-0}$ in to a pair of Dirac magnons, via $\epsilon_{+-0} + \epsilon_{--0} \to \epsilon_{+-k} + \epsilon_{-k'}$ (schematically) processes, might create the Dirac magnons paired state and may cause an instability in the system. This pumping scheme is the parametric pumping similar to the one in the experiment [24]. We have made a thorough analysis of such pumping, and could not positively conclude that there is an instability of the Dirac magnons. This is because of technical issues arising after the $\epsilon_{+-0}$ and $\epsilon_{--0}$ magnons are integrated out, and resulting interaction between Dirac magnons is obtained. We showed that the interaction is no longer only quartic in magnon operators and there is no small parameter to analyze the higher orders in conventional ways. Detailed analysis of such pumping scheme is a subject for future research.

To conclude, we studied second-order Suhl processes in a honeycomb ferromagnet and showed that under certain conditions the resonant pump’s frequency corresponds to the energy of the Dirac points, causing an instability of the ferromagnet. This is because, as is schematically shown in Fig. 1B, the system can accommodate the largest amount of magnon pairs, and their rescattering processes of the Cooper ladder type shown in Fig. 2B result in a pole structure Eq. (11), which can become singular as a function of temperature. We deduced that the instability is towards formation of the zero magnetization state, and called it as the Dirac magnons paired state. We estimated that the CrBr$_3$ ferromagnet might show this Dirac magnon paired state in the vicinity of the Curie temperature.

Acknowledgements. The author thanks A.M. Finkel’stein and A.Yu. Zyuzin for helpful discussions, and to Pirinem School of Theoretical Physics for hospitality. This work was started in a research group of A.V. Balatsky in Nordita, in an attempt to theoretically find the BEC of Dirac magnons. The author thanks A.V. Balatsky for discussions on the subject. This work is supported by the VILLUM FONDEN via the Centre of Excellence for Dirac Materials (Grant No. 11744), the European Research Council under the European Unions Seventh Framework Program Synergy HERO, and the Knut and Alice Wallenberg Foundation KAW.
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SUPPLEMENTAL MATERIAL FOR "PUMPED DIRAC MAGNONS PAIRED STATE"

FERROMAGNET ON A HONEYCOMB LATTICE

![Diagram of a honeycomb lattice](image)

FIG. 1: Schematics of the honeycomb lattice. Ferromagnetic order is assumed to be in the \( z \)-direction. Vectors connecting the nearest neighbor cites are \( \tau_1 = \frac{1}{2} \left( \frac{1}{\sqrt{3}}, 1 \right) \), \( \tau_2 = \frac{1}{2} \left( \frac{1}{\sqrt{3}}, -1 \right) \), and \( \tau_3 = \frac{1}{\sqrt{3}} (-1, 0) \). Green dashed lines correspond to the sign convention of the \( \nu_{ij} = \pm 1 \), which enter the Dzyaloshinskii-Moriya interaction.

We study spins of the length \( S \) on the honeycomb lattice. The spins interact via the ferromagnetic Heisenberg interaction. We assume the order to be in \( z \)-direction, and wish to understand the spin waves about the order. We follow standard procedure discussed, for example, in books on magnetism [1,2,3]. Holstein-Primakoff bosons for the spin operators \( S^x = S^z \pm i S^y \), and \( S^z \) read

\[
S^+ = \sqrt{2S - a^\dagger a}, \quad S^- = a^\dagger \sqrt{2S - a^\dagger a}, \quad S^z = S - a^\dagger a.
\]

(1)

Exchange interaction is

\[
H_{ex} = -J \sum_{\langle ij \rangle} \left( S_i^x S_j^x + S_i^y S_j^y + S_i^z S_j^z \right) = -J \sum_{\langle ij \rangle} \left( \frac{1}{2} S_i^+ S_j^- + \frac{1}{2} S_i^- S_j^+ + S_i^z S_j^z \right),
\]

(2)

where \( \langle \cdot \rangle \) stands for the nearest-neighbor interaction. We are assuming \( S > 1 \) so that \( \frac{1}{2} \) expansion applies. This allows us to drop out higher orders of interaction between magnons. Hamiltonian of interacting spin-waves reads,

\[
H_{sw} = -JS \sum_{\langle ij \rangle} \left( a_i^\dagger b_j + b_j^\dagger a_i \right) + 3JS \sum_{\langle ij \rangle} \left( a_i^\dagger a_i + b_j^\dagger b_j \right)
+ \frac{J}{4} \sum_{\langle ij \rangle} a_i^\dagger a_i b_j^\dagger b_j + \frac{J}{4} \sum_{\langle ij \rangle} a_i^\dagger a_i b_j + \frac{J}{4} \sum_{\langle ij \rangle} b_j^\dagger b_j b_j - J \sum_{\langle ij \rangle} a_i^\dagger a_i b_j^\dagger b_j.
\]

(3)

(4)

Fourier transform of the Hamiltonian reads as

\[
H_{sw} \approx -JS \int_{k} \left( \gamma_k a_k^\dagger b_k + \gamma_k^* b_k^\dagger a_k \right) + 3JS \int_{k} \left( a_k^\dagger a_k + b_k^\dagger b_k \right) - J \int_{\{k\}} \delta_{k_1-k_2,k_4-k_3} \gamma_{k_4-k_3} a_{k_1}^\dagger b_{k_3}^\dagger b_{k_2} a_{k_4}
+ \frac{J}{4} \int_{\{k\}} \delta_{k_1-k_2,k_4-k_3} \left[ \gamma_{k_3} a_{k_1}^\dagger b_{k_4}^\dagger b_{k_2} a_{k_4} + \gamma_{k_3} a_{k_1}^\dagger b_{k_4}^\dagger b_{k_2} a_{k_4} + \gamma_{k_3} b_{k_1}^\dagger b_{k_4}^\dagger a_{k_3} b_{k_2} a_{k_4} + \gamma_{k_3} b_{k_1}^\dagger b_{k_4}^\dagger a_{k_3} b_{k_2} a_{k_4} \right],
\]

(5)

where \( \gamma_k = \sum_{i=1,2,3} e^{i k \tau_i} = 2 e^{i k/2} \cos \left( \frac{k}{\sqrt{3}} \right) + e^{-i k/2} \) is the dispersion (see Fig. 1 for definitions of \( \tau_i \) vectors), \( \{k\} \equiv k_1,k_2,k_3,k_4, \) and \( \delta_{k_1,k_2} = 2\pi \delta(k_1 - k_2) \) is the delta-function. Note that the two first lines of the interaction are written in the convenient for conjugation way. The last line is already Hermitian conjugate to itself. The interaction is instantaneous in time. This implies certain frequency dependence, for example,

\[
- J \int_{\{k\}} \delta_{k_1-k_2,k_4-k_3} \int_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4} a_{\epsilon_1;k_1}^\dagger b_{\epsilon_2;k_2}^\dagger b_{\epsilon_3;k_4} a_{\epsilon_4;k_4} \delta_{\epsilon_1-\epsilon_2,\epsilon_3-\epsilon_4}
\]

(6)

\[
= - J \int_{\{k\}} \delta_{k_1-k_2,k_4-k_3} \int_{\epsilon_1,\epsilon_3,\omega} a_{\epsilon_1;k_1}^\dagger b_{\epsilon_3;k_3}^\dagger a_{\epsilon_1-\omega;k_2} b_{\epsilon_3+\omega;k_4}.
\]

(7)
In the space of unitary cell, in which case the boson operators are defined by \( \Psi^\dagger_k = (a_k^\dagger, b_k^\dagger) \) the Hamiltonian of linear spin-waves reads as

\[
\hat{H} = JS \begin{bmatrix}
3 & -\gamma_k \\
-\gamma_k^* & 3
\end{bmatrix},
\]

(8)

diagonalization immediately gives energy spectrum,

\[
\epsilon_{\pm k} = JS (3 \pm |\gamma_k|)
\]

(9)

with corresponding wave functions

\[
\varphi_+ = \frac{1}{\sqrt{2}} \begin{bmatrix}
-\frac{\gamma_k}{|\gamma_k|} \\
1
\end{bmatrix}, \quad \varphi_- = \frac{1}{\sqrt{2}} \begin{bmatrix}
\frac{\gamma_k}{|\gamma_k|} \\
1
\end{bmatrix},
\]

(10)

Green function is

\[
G_{\alpha\beta}^{R/A}(\epsilon, k) = \frac{\varphi_{+, k} \varphi_{+, k}^\dagger}{\epsilon - \epsilon_{+, k} \pm i0} + \frac{\varphi_{-, k} \varphi_{-, k}^\dagger}{\epsilon - \epsilon_{-, k} \pm i0},
\]

(11)

where \( \alpha \) and \( \beta \) are pseudospins. Green function can be presented in a more convenient way

\[
G_{\alpha\beta}^{R/A}(\epsilon, k) = \frac{1}{2} \left( \frac{1}{\epsilon - \epsilon_{+, k} \pm i0} + \frac{1}{\epsilon - \epsilon_{-, k} \pm i0} \right) - \frac{1}{2} \left( \frac{1}{\epsilon - \epsilon_{+, k} \pm i0} - \frac{1}{\epsilon - \epsilon_{-, k} \pm i0} \right) \begin{bmatrix}
0 & \frac{\gamma_k}{|\gamma_k|} \\
\frac{\gamma_k^*}{|\gamma_k|} & 0
\end{bmatrix}.
\]

(12)

The pumping is

\[
H_{\text{pump}} = \Gamma \sum_i \left[ S_i^x \cos(\Omega t) + S_i^y \sin(\Omega t) \right] = \Gamma \sum_i \left[ S_i^+ e^{-i\Omega t} + S_i^- e^{i\Omega t} \right]
\]

(13)

\[
\approx \sqrt{2} S \Gamma \sum_i \left[ a_i e^{-i\Omega t} + a_i^\dagger e^{i\Omega t} \right] + \sqrt{2} S \gamma \sum_i \left[ b_i e^{-i\Omega t} + b_i^\dagger e^{i\Omega t} \right].
\]

(14)

For the sake of discussion, we also consider Dzyaloshinskii-Moriya interaction

\[
H_{\text{DMI}} = D \sum_{\langle ij \rangle} \nu_{ij} [S_i \times S_j]_z,
\]

(15)

where \( \langle ij \rangle \) stands for the next-nearest neighbor interaction, and \( \nu_{ij} = \pm 1 \) depending on the direction of interaction with the signs defined by green dashed arrows in Fig. 1. In Holstein-Primakoff boson representation of spins, the DMI becomes

\[
H_{\text{DMI}} = SD \int \xi_k \left( a_k^\dagger a_k - b_k^\dagger b_k \right),
\]

(16)

where \( \xi_k = 2 \left[ \sin(k_y) - 2\sin \left( \frac{k_y}{2} \right) \right] \cos \left( \frac{\sqrt{3}k_z}{2} \right) \).

**Keldysh Formalism**

We stress that in the hindsight, the Keldysh technique is certainly not the only choice for the problem at hand. It seems that Matsubara frequency space should work equally well. However, as the system under study is pumped and formally out-of-equilibrium, we decided to be on a safe side and follow non-equilibrium field theory technique - the Keldysh technique. Here we briefly outline steps of the Keldysh technique, which we utilized in analysis of the system. For a detailed review of the Keldysh formalism see book [4], which is going to be followed below. When considering the action of non-interacting magnons, the integral over the Keldysh contour is split as usual in to forward \( \Psi^+, \Psi^+ \) and backward \( \Psi^-, \Psi^- \) parts. For example, a part containing non-interacting Hamiltonian transforms as

\[
\int dt \bar{\Psi}(t) \hat{H} \Psi(t) = \int_{-\infty}^{+\infty} dt \bar{\Psi}^+(t) \hat{H} \Psi^+(t) - \int_{-\infty}^{+\infty} dt \bar{\Psi}^-(t) \hat{H} \Psi^-(t) = \int_{-\infty}^{+\infty} dt \left[ \bar{\Psi}^+(t) \hat{H} \Psi^+(t) + \bar{\Psi}^-(t) \hat{H} \Psi^-(t) \right].
\]

(17)
where
\[
\Psi^{cl/q} = \frac{1}{\sqrt{2}} \left( \Psi^+ \pm \Psi^- \right),
\]  
(18)
and the same for \(\bar{\Psi}\) fields. The action of non-interacting magnons is
\[
iS = i \int_{-\infty}^{+\infty} dt \Psi(t) \left[ \begin{array}{c} 0 \\ \left[ G^{-1} \right]^R \\ \left[ G^{-1} \right]^A \end{array} \right] \Psi(t)
\]  
(19)
where
\[
\Psi = \left[ \begin{array}{c} \Psi^{cl} \\ \Psi^q \end{array} \right], \quad \bar{\Psi} = \left[ \begin{array}{c} \bar{\Psi}^{cl} \\ \bar{\Psi}^q \end{array} \right],
\]  
(20)
and \(\left[ G^{-1}(\epsilon) \right]^{R/A} = \epsilon \pm i0 - \hat{H}\) is the inverse Green function in the Fourier space. Note that the \(\left[ G^{-1} \right]^K\) is the quantum-quantum component of the action, and the classical-classical component of the action is absent. The Green function is
\[
\langle \Psi(t) \bar{\Psi}(t') \rangle_S = i \left[ \begin{array}{cc} G^K(t - t') & G^R(t - t') \\ G^A(t - t') & 0 \end{array} \right],
\]  
(21)
where in particular
\[
\langle \Psi^{cl}(t) \bar{\Psi}^{cl}(t') \rangle_S = \sum_\epsilon iG^K(\epsilon) e^{-i\epsilon(t-t')},
\]  
(22)
\[
\langle \Psi^{cl}(t) \bar{\Psi}^q(t') \rangle_S = \sum_\epsilon iG^R(\epsilon) e^{-i\epsilon(t-t')},
\]  
(23)
\[
\langle \Psi^q(t) \bar{\Psi}^{cl}(t') \rangle_S = \sum_\epsilon iG^A(\epsilon) e^{-i\epsilon(t-t')},
\]  
(24)
In frequency space
\[
\langle \Psi^{cl}(\epsilon_1) \bar{\Psi}^{cl}(\epsilon_2) \rangle_S = iG^K(\epsilon_1) \delta_{\epsilon_1,\epsilon_2},
\]  
(25)
\[
\langle \Psi^{cl}(\epsilon_1) \bar{\Psi}^q(\epsilon_2) \rangle_S = iG^R(\epsilon_1) \delta_{\epsilon_1,\epsilon_2},
\]  
(26)
\[
\langle \Psi^q(\epsilon_1) \bar{\Psi}^{cl}(\epsilon_2) \rangle_S = iG^A(\epsilon_1) \delta_{\epsilon_1,\epsilon_2},
\]  
(27)
where \(\delta_{\epsilon_1,\epsilon_2} = 2\pi\delta(\epsilon_1 - \epsilon_2)\) is the delta-function. The Green function must satisfy unity identity (here everywhere multiplication assumes convolution in time),
\[
\left[ \begin{array}{c} 0 \\ \left[ G^{-1} \right]^R \\ \left[ G^{-1} \right]^A \end{array} \right] \left[ \begin{array}{ccc} G^K & G^R \\ G^A & 0 \end{array} \right] = 1,
\]  
(28)
which gives us a condition on \(G^K\) function
\[
\left[ G^{-1} \right]^R G^K + \left[ G^{-1} \right]^K G^A = 0,
\]  
(29)
which means
\[
\left[ G^{-1} \right]^K = - \left[ G^{-1} \right]^R G^K \left[ G^{-1} \right]^A.
\]  
(30)
With the parametrization
\[
G^K = G^R \mathcal{F} - \mathcal{F} G^A,
\]  
(31)
where \(\mathcal{F}\) is the distribution function, we get
\[
\left[ G^{-1} \right]^K = \left[ G^{-1} \right]^R \mathcal{F} - \mathcal{F} \left[ G^{-1} \right]^A.
\]  
(32)
This is the kinetic equation determining distribution function.

The pumping field is described by
\[
\frac{\sqrt{2 \xi}}{2} \int_c dt \left( \bar{\Psi} e^{-i \Omega t} + \Psi e^{i \Omega t} \right) = \Gamma \sqrt{S} \int_{-\infty}^{+\infty} dt \left( \bar{\Psi} \Omega e^{-i \Omega t} + \Psi \Omega e^{i \Omega t} \right).
\] (33)

This might update the Hamiltonian and the Green functions. To check this, we can use the following identity,
\[
\int d[\bar{\Psi}, \Psi] e^{-\sum_{ij} \bar{\Psi}_i \overline{A}_{ij} \Psi_j + \sum_i (\bar{\Psi}_i \overline{J}_i + \Psi_i J_i)} = \frac{1}{\det A} e^{\sum_i \bar{J}_i (\hat{A}^{-1})_{ij} J_j}
\] (34)

and since there is no q-q element in the \(\hat{A}^{-1}\) matrix, the pumping field will not enter the final result of integration. However, the corresponding classical fields and consequently Green functions are going to be affected by the pumping fields. We are going to go over that in the next subsection.

Now let us include interactions between magnons. Schematically, general four-boson interaction rewritten in terms of Keldysh fields is
\[
\int_c dt \bar{\Psi}_1 \bar{\Psi}_2 \bar{\Psi}_3 \Psi_4 = \int_{-\infty}^{+\infty} dt \bar{\Psi}_1^+ \bar{\Psi}_2^+ \bar{\Psi}_3^+ \Psi_4^+ - \int_{-\infty}^{+\infty} dt \bar{\Psi}_1^- \bar{\Psi}_2^- \bar{\Psi}_3^- \Psi_4^- = \frac{1}{2} \int_{-\infty}^{+\infty} dt \left( \bar{\Psi}_1^c \Psi_2^c + \bar{\Psi}_2^c \Psi_1^c \right) \left( \bar{\Psi}_3^c \Psi_4^c + \bar{\Psi}_4^c \Psi_3^c \right) + \frac{1}{2} \int_{-\infty}^{+\infty} dt \left( \bar{\Psi}_1^c \Psi_2^c + \bar{\Psi}_2^c \Psi_1^c \right) \left( \bar{\Psi}_3^c \Psi_4^c + \bar{\Psi}_4^c \Psi_3^c \right)
\] (35)

where 1, 2, 3, 4 indeces stand for a general frequency-momentum-spin variable. Under relabelling, the two terms after second equality sign double each other, but for the sake of generality kept as they are.

**Shifting the pump field away**

Lagrangian describing non-interacting magnons with the pump’s frequency \(\Omega\) and momentum \(k = 0\) is schematically written as
\[
\mathcal{L}_{0,\Omega} = \sum_{m,n} \bar{\Psi}_m^{\Omega} \lambda_{mn,0,\Omega} \Psi_n^{\Omega} + \sum_{m,n} \bar{\Psi}_m^{cl} \lambda_{mn,0,\Omega}^{cl} \Psi_n^{\Omega} + \sum_{m,n} \bar{\Psi}_m^{\Omega} \lambda_{mn,0,\Omega}^{R} \Psi_n^{\Omega} - \Gamma \sqrt{S} \sum_{m,n} \bar{\Psi}_m^{\Omega} \lambda_{mn,0,\Omega}^{L} \Psi_n^{\Omega},
\] (38)

where \(\lambda_{mn,0,\Omega}^{K/R/A}\) is the Lagrangian density corresponding to Keldysh, retarded or advanced part correspondingly. For example \(\lambda_{mn,k,\Omega} = (\Omega + i0) \delta_{mn} - [\hat{H}_0]_{mn,k}\). The advanced part of the Lagrangian is
\[
\mathcal{L}_{0,\Omega}^{A} = \sum_{m,n} \bar{\Psi}_m^{cl} \lambda_{mn,0,\Omega}^{A} \Psi_n^{\Omega} - \Gamma \sqrt{S} \sum_{m,n} \bar{\Psi}_m^{\Omega} \lambda_{mn,0,\Omega}^{A} \Psi_n^{\Omega},
\] (40)

in which we would like to shift away terms linear in \(\Psi_n^{\Omega}\). We achieve it with
\[
\bar{\Psi}_\alpha^{cl,0,\Omega} \rightarrow \bar{\Psi}_\alpha^{cl,0,\Omega} + x_A,
\] (41)
\[
\bar{\Psi}_\beta^{cl,0,\Omega} \rightarrow \bar{\Psi}_\beta^{cl,0,\Omega} + y_A,
\] (42)

with
\[
x_A = \frac{\lambda_{\alpha,0,\Omega}^{A} - \lambda_{\beta,0,\Omega}^{A}}{\lambda_{\alpha,\beta,0,\Omega}^{L} \lambda_{\beta,0,\Omega}^{L} - \lambda_{\alpha,\beta,0,\Omega}^{L} \lambda_{\beta,0,\Omega}^{L}} \Gamma \sqrt{S},
\] (43)
\[
y_A = \frac{\lambda_{\alpha,0,\Omega}^{A} - \lambda_{\alpha,0,\Omega}^{L}}{\lambda_{\alpha,\beta,0,\Omega}^{L} \lambda_{\beta,0,\Omega}^{L} - \lambda_{\alpha,\beta,0,\Omega}^{L} \lambda_{\beta,0,\Omega}^{L}} \Gamma \sqrt{S}.
\] (44)
For the retarded analog of the Lagrangian, 

\[ \mathcal{L}_{\Omega, \Omega}^R = \sum_{m,n} \bar{\Psi}^q_{m,0, \Omega} \mathcal{L}_{m,0, \Omega}^R \Psi_{n,0, \Omega}^c - \Gamma \sqrt{S} \sum_n \bar{\Psi}^q_{n,0, \Omega}, \]  

in which we would like to shift away terms linear in \( \bar{\Psi}^q_{n,0, \Omega} \). We achieve it with 

\[ \Psi_{\alpha,0, \Omega}^c \rightarrow \Psi_{\alpha,0, \Omega} + x_R, \]  

\[ \Psi_{\beta,0, \Omega}^c \rightarrow \Psi_{\beta,0, \Omega} + y_R, \]

with 

\[ x_R = \frac{\mathcal{L}_{\beta,0, \Omega}^R - \mathcal{L}_{\beta,0, \Omega}^R - \mathcal{L}_{\alpha,0, \Omega}^R - \mathcal{L}_{\alpha,0, \Omega}^R}{\mathcal{L}_{\alpha,0, \Omega}^R - \mathcal{L}_{\beta,0, \Omega}^R} \Gamma \sqrt{S}, \]  

\[ y_R = \frac{\mathcal{L}_{\beta,0, \Omega}^R - \mathcal{L}_{\alpha,0, \Omega}^R - \mathcal{L}_{\beta,0, \Omega}^R}{\mathcal{L}_{\alpha,0, \Omega}^R - \mathcal{L}_{\beta,0, \Omega}^R} \Gamma \sqrt{S}. \]  

**PUMPING TO THE DIRAC POINTS WITH A \( \Omega = 3SJ \) FREQUENCY PUMP**

Two quanta pumping to Dirac points

Here we discuss off-resonance pumping, when the frequency of the pump is half the bandwidth, namely \( \Omega = 3SJ \). There are no mass-shell states with \( k = 0 \) at this frequency. Thus, there is no possibility to pump single magnon to this point, but due to the interactions, there is a possibility to pump a pair of magnons. See Fig. 2 for the schematics of the process of absorption of two pump field quanta. This processes is known in the literature as the second-order Suhl process [3,5]. One can see it by absorbing the pumping field by shifting corresponding classical (only) fields, 

\[ \Psi_{\alpha,0, \Omega}^c \rightarrow \bar{\Psi}_{\alpha,0, \Omega} - \frac{\Gamma \sqrt{S}}{3SJ}, \]  

\[ \Psi_{\alpha,0, \Omega}^c \rightarrow \Psi_{\alpha,0, \Omega} - \frac{\Gamma \sqrt{S}}{3SJ}. \]

The shift means that a physical state with corresponding quantum numbers acquires a classical value. For example, if it was a Bose-Einstein condensate we were talking about, it would mean that the magnon accumulate in the state. However, since the shifted state is off-shell, one would not expect any magnon accumulation in it. Instead, the magnons can rescatter from this virtual state to the on-shell states according to the frequency and momentum conservation. To describe these effects, we notice that the interaction part of the action will be affected by the shift.

\[ iS_{\text{interaction}} = -\frac{i}{4} \int_{\{\omega\}} \int_{\{k\}} \gamma k_4 a_k^\dagger a_k^\dagger b_k^\dagger b_k^\dagger \delta_{k_1 - k_2, k_4 - k_3} \delta_{\omega_1 - \omega_2, \omega_4 - \omega_3} \]  

\[ -\frac{i}{8} \int_{\{\omega\}} \int_{\{k\}} \gamma k_1 \left( \bar{\Psi}_{\alpha k_1; \omega_1}^c \bar{\Psi}_{\alpha k_3; \omega_3}^c \bar{\Psi}_{\beta k_2; \omega_2}^c \bar{\Psi}_{\beta k_4; \omega_4}^c + \bar{\Psi}_{\alpha k_1; \omega_1}^c \bar{\Psi}_{\alpha k_2; \omega_2}^c \bar{\Psi}_{\alpha k_3; \omega_3}^c \bar{\Psi}_{\alpha k_4; \omega_4}^c \bar{\Psi}_{\beta k_1; \omega_1}^c \bar{\Psi}_{\beta k_2; \omega_2}^c \bar{\Psi}_{\beta k_3; \omega_3}^c \bar{\Psi}_{\beta k_4; \omega_4}^c \right) \delta_{k_1 - k_2, k_4 - k_3} \delta_{\omega_1 - \omega_2, \omega_4 - \omega_3} \]  

\[ -\frac{i}{8} \int_{\{\omega\}} \int_{\{k\}} \gamma k_2 \left( \Gamma \sqrt{S} \right)^2 \bar{\Psi}_{\alpha k_2; \omega_2}^c \bar{\Psi}_{\beta k_4; \omega_4}^c \delta_{-k_2, k_4} \delta_{\omega_2, \omega_4} \]  

\[ -\frac{i}{8} \int_{\{\omega\}} \int_{\{k\}} \gamma k_1 \left( \Gamma \sqrt{S} \right)^2 \bar{\Psi}_{\alpha k_1; \omega_1}^c \bar{\Psi}_{\alpha k_3; \omega_3}^c \delta_{k_1 - k_2} \delta_{\omega_1, \omega_3} \]  

\[ -\frac{i}{8} \int_{\{\omega\}} \int_{\{k\}} \gamma k_2 \left( \Gamma \sqrt{S} \right)^2 \bar{\Psi}_{\alpha k_2; \omega_2}^c \bar{\Psi}_{\beta k_4; \omega_4}^c \delta_{k_2 - k_4} \delta_{\omega_2, \omega_4} \]  

\[ -\frac{i}{8} \int_{\{\omega\}} \int_{\{k\}} \gamma k_1 \left( \Gamma \sqrt{S} \right)^2 \bar{\Psi}_{\alpha k_1; \omega_1}^c \bar{\Psi}_{\alpha k_3; \omega_3}^c \delta_{k_1 - k_2} \delta_{\omega_1, \omega_3} \right). \]
Regarding cubic terms, in experimentally relevant limit of \( \left( \frac{\gamma S}{3SJ} \right)^2 < 1 \) they can be ignored. They will contribute to the interaction between magnons, but will have \( \left( \frac{\gamma S}{3SJ} \right)^2 < 1 \) small factor as compared to the original interaction. It is not possible to generate \( \propto \bar{\Psi}^q_{\alpha;k_1\omega_1} \Psi^\dagger_{\alpha;k_2\omega_2} \) or other similar terms as they all sum up to zero. This cancellation occurs between all terms in the interaction (between \( \propto -J \) and \( \propto \frac{1}{4} \) terms in the interaction). We give an example of such cancellation in the end of this subsection.

Below we list four remaining terms in the interaction.

\[
iS_{\text{interaction};2} = -i \frac{J}{4} \int_\omega \int_k \gamma^*_{k_3} a^\dagger_{k_1} a_{k_2} b^\dagger_{k_3} a_{k_4} \delta_{k_1-k_2,k_4-k_3} \delta_{\omega_1-\omega_2,\omega_4-\omega_3}
\]

\[
- i \frac{J}{8} \int_\omega \int_k \gamma^*_{k_3} \left( \bar{\Psi}^\dagger_{\alpha;k_1\omega_1} \Psi^\dagger_{\beta;k_3\omega_3} \Psi^q_{\alpha;k_2\omega_2} \Psi^q_{\alpha;k_4\omega_4} + \bar{\Psi}^q_{\alpha;k_1\omega_1} \Psi^\dagger_{\beta;k_3\omega_3} \Psi^q_{\alpha;k_2\omega_2} \Psi^\dagger_{\alpha;k_4\omega_4} \right) \delta_{k_1-k_2,k_4-k_3} \delta_{\omega_1-\omega_2,\omega_4-\omega_3} \tag{55}
\]

\[
- i \frac{J}{8} \int_\omega \int_k \gamma^*_{k_3} \left( i \frac{\sqrt{S}}{3SJ} \right)^2 \Psi^\dagger_{\alpha;k_2\omega_2} \Psi^q_{\alpha;k_4\omega_4} \delta_{-k_2,k_4} \delta_{\Omega-\omega_2,\omega_4-\Omega} \tag{56}
\]

\[
- i \frac{J}{8} \int_\omega \int_k \gamma^*_{k_3} \left( i \frac{\sqrt{S}}{3SJ} \right)^2 \bar{\Psi}^q_{\alpha;k_1\omega_1} \Psi^\dagger_{\beta;k_3\omega_3} \delta_{k_1-k_3} \delta_{\omega_1-\omega_3,\Omega-\Omega} \tag{57}
\]

\[
- i \frac{J}{8} \int_\omega \int_k \gamma^*_{k_3} \left( i \frac{\sqrt{S}}{3SJ} \right)^2 \Psi^q_{\alpha;k_1\omega_1} \bar{\Psi}^\dagger_{\beta;k_3\omega_3} \delta_{k_1-k_3} \delta_{\omega_1-\Omega,\Omega-\omega_3} \tag{58}
\]

\[
- i \frac{J}{8} \int_\omega \int_k \gamma^*_{k_3} \left( i \frac{\sqrt{S}}{3SJ} \right)^2 \bar{\Psi}^q_{\alpha;k_1\omega_1} \Psi^\dagger_{\beta;k_3\omega_3} \delta_{k_1-k_3} \delta_{\omega_1-\Omega,\Omega-\omega_3} \tag{59}
\]

**FIG. 2:** Schematics of two pump quanta absorption. Here the dashed lines correspond to the pump field, while the wavy line to interaction between the magnons.
\[ iS_{\text{interaction};3} = -i \frac{J}{4} \int_{\omega} \int_{\{k\}} \gamma k_{1} a_{k_{1}}^{\dagger} b_{k_{2}} b_{k_{3}} \delta k_{1} - k_{2}, k_{4} - k_{3} \delta \omega_{1} - \omega_{2}, \omega_{4} - \omega_{3} \]  
\[ \rightarrow -i \frac{J}{8} \int_{\omega} \int_{\{k\}} \gamma k_{1} \left( \bar{\psi}_{\alpha k_{1};\omega_{1}}^{\dagger} \psi_{\beta k_{2};\omega_{2}}^{\dagger} \psi_{\beta k_{3};\omega_{3}}^{\dagger} \psi_{\beta k_{4};\omega_{4}} + \bar{\psi}_{\alpha k_{1};\omega_{1}} \psi_{\beta k_{2};\omega_{2}} \psi_{\beta k_{3};\omega_{3}} \psi_{\beta k_{4};\omega_{4}} \right) \delta k_{1} - k_{2}, k_{4} - k_{3} \delta \omega_{1} - \omega_{2}, \omega_{4} - \omega_{3} \]  
\[ = -i \frac{J}{8} \int_{\omega} \int_{\{k\}} \gamma_{0} \left( \frac{\Gamma \sqrt{S}}{3SJ} \right) \left( \psi_{\beta k_{2};\omega_{2}}^{\dagger} \psi_{\beta k_{4};\omega_{4}} \right)^{2} \left( \psi_{\beta k_{2};\omega_{2}} \psi_{\beta k_{4};\omega_{4}} \right)^{2} \delta k_{1} - k_{2}, k_{4} \delta \Omega - \omega_{2}, \omega_{4} - \Omega \right) \]  
\[ \rightarrow -i \frac{J}{8} \int_{\omega} \int_{\{k\}} \gamma_{0} \left( \frac{\Gamma \sqrt{S}}{3SJ} \right)^{2} \left( \psi_{\beta k_{2};\omega_{2}}^{\dagger} \psi_{\beta k_{4};\omega_{4}} \right)^{2} \left( \psi_{\beta k_{2};\omega_{2}} \psi_{\beta k_{4};\omega_{4}} \right)^{2} \delta k_{1} - k_{2}, k_{4} \delta \Omega - \omega_{2}, \omega_{4} - \Omega \right) \]  
\[ \rightarrow -i \frac{J}{8} \int_{\omega} \int_{\{k\}} \gamma_{1} \left( \frac{\Gamma \sqrt{S}}{3SJ} \right)^{2} \left( \psi_{\beta k_{2};\omega_{2}}^{\dagger} \psi_{\beta k_{4};\omega_{4}} \right)^{2} \left( \psi_{\beta k_{2};\omega_{2}} \psi_{\beta k_{4};\omega_{4}} \right)^{2} \delta k_{1} - k_{2}, k_{4} \delta \Omega - \omega_{2}, \omega_{4} - \Omega \right) \]  

and

\[ iS_{\text{interaction};4} = -i \frac{J}{4} \int_{\omega} \int_{\{k\}} \gamma^{a} a_{k_{2}} b_{k_{3}} b_{k_{4}} \delta k_{1} - k_{2}, k_{4} - k_{3} \delta \omega_{1} - \omega_{2}, \omega_{4} - \omega_{3} \]  
\[ \rightarrow -i \frac{J}{8} \int_{\omega} \int_{\{k\}} \gamma^{a} \left( \bar{\psi}_{\beta k_{2};\omega_{2}}^{\dagger} \psi_{\beta k_{3};\omega_{3}}^{\dagger} \psi_{\beta k_{4};\omega_{4}} + \bar{\psi}_{\beta k_{2};\omega_{2}} \psi_{\beta k_{3};\omega_{3}} \psi_{\beta k_{4};\omega_{4}} \right) \delta k_{1} - k_{2}, k_{4} \delta \Omega - \omega_{2}, \omega_{4} - \Omega \]  
\[ \rightarrow -i \frac{J}{8} \int_{\omega} \int_{\{k\}} \gamma_{0} \left( \frac{\Gamma \sqrt{S}}{3SJ} \right) \left( \psi_{\beta k_{2};\omega_{2}}^{\dagger} \psi_{\beta k_{4};\omega_{4}} \right)^{2} \left( \psi_{\beta k_{2};\omega_{2}} \psi_{\beta k_{4};\omega_{4}} \right)^{2} \delta k_{1} - k_{2}, k_{4} \delta \Omega - \omega_{2}, \omega_{4} - \Omega \]  
\[ \rightarrow -i \frac{J}{8} \int_{\omega} \int_{\{k\}} \gamma_{1} \left( \frac{\Gamma \sqrt{S}}{3SJ} \right)^{2} \left( \psi_{\beta k_{2};\omega_{2}}^{\dagger} \psi_{\beta k_{4};\omega_{4}} \right)^{2} \left( \psi_{\beta k_{2};\omega_{2}} \psi_{\beta k_{4};\omega_{4}} \right)^{2} \delta k_{1} - k_{2}, k_{4} \delta \Omega - \omega_{2}, \omega_{4} - \Omega \]
There is also $\propto -J$ interaction term, which also gets shifted accordingly.

\begin{equation}
\begin{aligned}
\int \{k\} \gamma_4 & \left[ \psi^{\dagger}_{\alpha} k_1 \psi^{\dagger}_{\beta} k_2 \delta_{k_1 - k_2} \delta_{\omega_1 - \omega_2} \right] \\
\rightarrow i & \frac{1}{2} \int \{k\} \gamma_4 \left[ \left( \psi^{\dagger}_{\alpha} k_1 \psi^{\dagger}_{\beta} k_3 \delta_{k_1 - k_3} \delta_{\omega_1 - \omega_2} \right) + \psi^{\dagger}_{\alpha} k_1 \psi^{\dagger}_{\beta} k_2 \delta_{k_1 - k_2} \delta_{\omega_2 - \omega_3} \right]
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
\rightarrow & \frac{i}{2} \int \{k\} \gamma_4 \left[ \left( \psi^{\dagger}_{\alpha} k_1 \psi^{\dagger}_{\beta} k_3 \delta_{k_1 - k_3} \delta_{\omega_1 - \omega_2} \right) + \psi^{\dagger}_{\alpha} k_1 \psi^{\dagger}_{\beta} k_2 \delta_{k_1 - k_2} \delta_{\omega_3 - \omega_1} \right]
\end{aligned}
\end{equation}

Collecting now terms quadratic in fields, we get for the pump

\begin{equation}
\begin{aligned}
H_{\text{pump}} &= \int \{k\} \left[ -\frac{i}{4} \gamma_4 \left( \frac{\Gamma \sqrt{S}}{3SJ} \right)^2 \psi^{\dagger}_{\alpha} k_2 \psi^{\dagger}_{\beta} k_4 \delta_{k_2 - k_4} \delta_{\omega_1 - \omega_2} - \frac{i}{4} \gamma_4 \left( \frac{\Gamma \sqrt{S}}{3SJ} \right)^2 \psi^{\dagger}_{\alpha} k_2 \psi^{\dagger}_{\beta} k_4 \delta_{k_4 - k_2} \delta_{\omega_2 - \omega_3} \right]
\end{aligned}
\end{equation}

Let us now demonstrate that indeed terms of the $\propto \bar{\Psi}^{\dagger}_{\alpha} q_1 \bar{\Psi}^{\dagger}_{\alpha} q_2$ type sum up to zero and, hence, can't be generated by the pump process. Recall, that overall there are five interaction terms listed in this subsection. We refer to them in the order they have appeared. From the first interaction term we have

\begin{equation}
\begin{aligned}
\int \{k\} \gamma_4 \left[ \left( \psi^{\dagger}_{\alpha} k_1 \psi^{\dagger}_{\beta} k_3 \delta_{k_1 - k_3} \delta_{\omega_1 - \omega_2} \right) + \psi^{\dagger}_{\alpha} k_1 \psi^{\dagger}_{\beta} k_2 \delta_{k_1 - k_2} \delta_{\omega_2 - \omega_3} \right]
\end{aligned}
\end{equation}

From the second interaction term we have

\begin{equation}
\begin{aligned}
\int \{k\} \gamma_3 \left[ \left( \psi^{\dagger}_{\alpha} k_1 \psi^{\dagger}_{\beta} k_3 \delta_{k_1 - k_3} \delta_{\omega_1 - \omega_2} \right) + \psi^{\dagger}_{\alpha} k_1 \psi^{\dagger}_{\beta} k_2 \delta_{k_1 - k_2} \delta_{\omega_2 - \omega_3} \right]
\end{aligned}
\end{equation}
From the fifth interaction term we have

\[ - \frac{J}{2} \int \langle \omega \rangle \int \langle \mathbf{k} \rangle \gamma_{k_4 - k_3} \left( \bar{\psi}_{\alpha \beta, k_1} \left( \bar{\psi}_{\alpha \beta, k_2} \psi_{\alpha \beta, k_3} \psi_{\alpha \beta, k_4} + \bar{\psi}_{\alpha \beta, k_1} \left( \bar{\psi}_{\alpha \beta, k_2} \psi_{\alpha \beta, k_3} \psi_{\alpha \beta, k_4} + \bar{\psi}_{\alpha \beta, k_1} \left( \bar{\psi}_{\alpha \beta, k_2} \psi_{\alpha \beta, k_3} \psi_{\alpha \beta, k_4} \right) \right) \right) \right) \delta_{k_1 - k_2, k_3 - k_4} \delta_{\omega_1 - \omega_2, \omega_3 - \omega_4}. \]

Three terms sum up to zero. The same can be proven for the other combinations of the same type.

**Hartree-Fock corrections**

![Hartree-Fock corrections](image)

FIG. 3: Hartree-Fock corrections to the magnon dispersion.

In order to understand possible instabilities in the system due to the magnon pair creation, we also need to take into account Hartree-Fock corrections to the magnon dispersion [6,7]. They are expected to give temperature dependent correction, and, thus, might be important when discussing the experimental details. For example, let us pick the first interaction term,

\[ \langle i S_{\text{Interaction}} \rangle = - \frac{J}{8} \int \langle \omega \rangle \int \langle \mathbf{k} \rangle \gamma_{k_4 - k_3} \left( \bar{\psi}_{\alpha \beta, k_1} \left( \bar{\psi}_{\alpha \beta, k_2} \psi_{\alpha \beta, k_3} \psi_{\alpha \beta, k_4} + \bar{\psi}_{\alpha \beta, k_1} \left( \bar{\psi}_{\alpha \beta, k_2} \psi_{\alpha \beta, k_3} \psi_{\alpha \beta, k_4} + \bar{\psi}_{\alpha \beta, k_1} \left( \bar{\psi}_{\alpha \beta, k_2} \psi_{\alpha \beta, k_3} \psi_{\alpha \beta, k_4} \right) \right) \right) \right) \delta_{k_1 - k_2, k_3 - k_4} \delta_{\omega_1 - \omega_2, \omega_3 - \omega_4}. \]

We found that for the task at hand it is more convenient to come back to time domain rather to work in frequency domain. In this way, equal-time commutation relations

\[ [\psi_{\alpha, k_1}(t), \psi_{\alpha, k_2}(t)] = \delta_{n,m} \delta_{k_1, k_2}. \]

are written in the most transparent way. For example, picking the first term in Eq. (72),

\[ \frac{J}{8} \int \langle \omega \rangle \int \langle \mathbf{k} \rangle \gamma_{k_4 - k_3} \left( \bar{\psi}_{\alpha \beta, k_1} \left( \bar{\psi}_{\alpha \beta, k_2} \psi_{\alpha \beta, k_3} \psi_{\alpha \beta, k_4} + \bar{\psi}_{\alpha \beta, k_1} \left( \bar{\psi}_{\alpha \beta, k_2} \psi_{\alpha \beta, k_3} \psi_{\alpha \beta, k_4} + \bar{\psi}_{\alpha \beta, k_1} \left( \bar{\psi}_{\alpha \beta, k_2} \psi_{\alpha \beta, k_3} \psi_{\alpha \beta, k_4} \right) \right) \right) \right) \delta_{k_1 - k_2, k_3 - k_4} \delta_{\omega_1 - \omega_2, \omega_3 - \omega_4}. \]

\[ \frac{J}{8} \int \langle \omega \rangle \int \langle \mathbf{k} \rangle \gamma_{k_4 - k_3} \left( \bar{\psi}_{\alpha \beta, k_1} \left( \bar{\psi}_{\alpha \beta, k_2} \psi_{\alpha \beta, k_3} \psi_{\alpha \beta, k_4} + \bar{\psi}_{\alpha \beta, k_1} \left( \bar{\psi}_{\alpha \beta, k_2} \psi_{\alpha \beta, k_3} \psi_{\alpha \beta, k_4} + \bar{\psi}_{\alpha \beta, k_1} \left( \bar{\psi}_{\alpha \beta, k_2} \psi_{\alpha \beta, k_3} \psi_{\alpha \beta, k_4} \right) \right) \right) \right) \delta_{k_1 - k_2, k_3 - k_4} \delta_{\omega_1 - \omega_2, \omega_3 - \omega_4} \delta_{\omega_1 - \omega_2, \omega_3 - \omega_4} \delta_{\omega_1 - \omega_2, \omega_3 - \omega_4}. \]

\[ \frac{J}{4} \int \langle \omega \rangle \int \langle \mathbf{k} \rangle \gamma_{k_4 - k_3} \left( \bar{\psi}_{\alpha \beta, k_1} \left( \bar{\psi}_{\alpha \beta, k_2} \psi_{\alpha \beta, k_3} \psi_{\alpha \beta, k_4} + \bar{\psi}_{\alpha \beta, k_1} \left( \bar{\psi}_{\alpha \beta, k_2} \psi_{\alpha \beta, k_3} \psi_{\alpha \beta, k_4} + \bar{\psi}_{\alpha \beta, k_1} \left( \bar{\psi}_{\alpha \beta, k_2} \psi_{\alpha \beta, k_3} \psi_{\alpha \beta, k_4} \right) \right) \right) \right) \delta_{k_1 - k_2, k_3 - k_4} \delta_{\omega_1 - \omega_2, \omega_3 - \omega_4}. \]

where we used \( \mathcal{F}(\epsilon) = 1 + \frac{2}{\epsilon^2 - 1} \equiv 1 + 2n_B(\epsilon) \) identity, and where \( -1 \) in the \( \left[-1 + i \int G_{\alpha \alpha}^K(\epsilon; \mathbf{q})\right] \) factor is due to the
Now picking the second term in Eq. (72),
\[
\frac{J}{8} \int_{(t)} \psi^{a}_{\alpha; k}(t) \psi^{\dagger}_{\beta; k}(t) \int_{q} iG_{\alpha \beta}(e; q) \gamma_{q} + \frac{J}{8} \int_{t} \gamma_{k} \psi^{a}_{\alpha; k}(t) \psi^{\dagger}_{\beta; k}(t) \int_{q} \left[ -1 + i \int_{q} iG_{\alpha \beta}(e; q) \right] \tag{79}
\]
which essentially doubles the second term in Eq. (72). Collecting all the four terms, we get
\[
\frac{J}{8} \int_{(t)} \psi^{a}_{\alpha; k}(t) \psi^{\dagger}_{\beta; k}(t) \int_{q} \gamma_{q} \left[ n_{B}(e+q) - n_{B}(e-q) \right] + \frac{J}{8} \int_{t} \gamma_{k} \psi^{a}_{\alpha; k}(t) \psi^{\dagger}_{\beta; k}(t) \int_{q} \left[ n_{B}(e+q) + n_{B}(e-q) \right]. \tag{81}
\]
Third term in Eq. (72) reads,
\[
\frac{J}{8} \int_{(t)} \gamma_{k} \psi^{a}_{\alpha; k}(t) \psi^{\dagger}_{\beta; k}(t) \int_{q} \gamma_{q} \left[ n_{B}(e+q) - n_{B}(e-q) \right] + \frac{J}{8} \int_{t} \gamma_{k} \psi^{a}_{\alpha; k}(t) \psi^{\dagger}_{\beta; k}(t) \int_{q} \left[ n_{B}(e+q) + n_{B}(e-q) \right], \tag{82}
\]
Finally, the last term in Eq. (72) reads,
\[
\frac{J}{8} \int_{(t)} \gamma_{k} \psi^{a}_{\alpha; k}(t) \psi^{\dagger}_{\beta; k}(t) \int_{q} \gamma_{q} \left[ n_{B}(e+q) - n_{B}(e-q) \right] + \frac{J}{8} \int_{t} \gamma_{k} \psi^{a}_{\alpha; k}(t) \psi^{\dagger}_{\beta; k}(t) \int_{q} \left[ n_{B}(e+q) + n_{B}(e-q) \right], \tag{84}
\]
which essentially doubles the second term in Eq. (72). Collecting all the four terms, we get
\[
\langle iS_{\text{interaction;1}} \rangle = -i \frac{J}{4} I_{1} \int_{t} \gamma_{k} \psi^{a}_{\alpha; k}(t) \psi^{\dagger}_{\beta; k}(t) - i \frac{J}{4} I_{2} \int_{t} \gamma_{k} \psi^{a}_{\alpha; k}(t) \psi^{\dagger}_{\beta; k}(t) \tag{87}
\]
where
\[
I_{1} = \int_{q} \left[ -1 + i \int_{q} iG_{\alpha \beta}(e; q) \right] = \int_{q} \left[ n_{B}(e+q) + n_{B}(e-q) \right], \tag{89}
\]
\[
I_{2} = -\int_{q} iG_{\alpha \beta}(e; q) \gamma_{q} = -\int_{q} \gamma_{q} \left[ n_{B}(e+q) - n_{B}(e-q) \right]. \tag{90}
\]
Expressions for the three other interaction terms, i.e. \(iS_{\text{interaction;2,3,4}}\), are similar to the obtained one. Fifth interaction is
\[
\langle iS_{\text{interaction;5}} \rangle = i \frac{J}{2} \int_{(t)} \gamma_{k} \psi^{a}_{\alpha; k}(t) \psi^{\dagger}_{\beta; k}(t) + i \frac{J}{2} I_{3} \int_{t} \gamma_{k} \psi^{a}_{\alpha; k}(t) \psi^{\dagger}_{\beta; k}(t) \tag{91}
\]
which essentially doubles the second term in Eq. (72). Collecting all the four terms, we get
\[
\langle iS_{\text{interaction;5}} \rangle = i \frac{J}{2} \int_{(t)} \gamma_{k} \psi^{a}_{\alpha; k}(t) \psi^{\dagger}_{\beta; k}(t) + i \frac{J}{2} I_{3} \int_{t} \gamma_{k} \psi^{a}_{\alpha; k}(t) \psi^{\dagger}_{\beta; k}(t), \tag{92}
\]
where
New integral appeared above is

\[ I_3(k) = \int_q iG_{\alpha\beta}^K(\epsilon; q)\gamma_{k-q} = -\int_q \frac{\gamma_{k-q}q}{|q|} [n_B(\epsilon_{+q}) - n_B(\epsilon_{-q})]. \]  

(96)

Overall, we have for the Hartree-Fock corrections

\[
\sum_{j=1}^5 (iS_{\text{interaction}, j}) = -i\frac{J}{2} \int_k \left[ [I_1 \gamma_k - I_3(k)] \bar{\Psi}^\dagger_{\alpha;k}(t)\Psi^q_{\beta;k}(t) - i\frac{J}{2}(I_2 - I_1\gamma_0) \int_k \bar{\Psi}^\dagger_{\alpha;k}(t)\Psi_{\beta;k}(t) \right] 
- i\frac{J}{2}(I_2 - I_1\gamma_0) \int_k \bar{\Psi}^\dagger_{\alpha;k}(t)\Psi^q_{\beta;k}(t) - i\frac{J}{2} \int_k [I_1 \gamma_k - I_3(k)] \bar{\Psi}^\dagger_{\alpha;k}(t)\Psi_{\beta;k}(t) 
- i\frac{J}{2}(I_2 - I_1\gamma_0) \int_k \bar{\Psi}^\dagger_{\alpha;k}(t)\Psi^q_{\beta;k}(t) - i\frac{J}{2} \int_k [I_1 \gamma_k^* - I_3^*(k)] \bar{\Psi}^\dagger_{\alpha;k}(t)\Psi_{\beta;k}(t). 
\]  

(97)

Integrals are

\[ I_2 - I_1\gamma_0 = -\int_q (\gamma_0 + |q|)n_B(\epsilon_{+q}) - \int_q (\gamma_0 - |q|)n_B(\epsilon_{-q}) \approx -\left(\frac{T}{3SJ}\right)^2 \frac{\pi}{2} \gamma_0, \]  

(101)

and

\[ I_1\gamma_k - I_3(k) = \int_q \left( \gamma_k + \frac{\gamma_{k-q}q}{|q|} \right) n_B(\epsilon_{+q}) + \int_q \left( \gamma_k - \frac{\gamma_{k-q}q}{|q|} \right) n_B(\epsilon_{-q}) \approx \left(\frac{T}{3SJ}\right)^2 \frac{\pi}{2} \gamma_k, \]  

(102)

which are approximated at low temperatures, \( T < 3SJ \), under assumption that only the \( \epsilon_{-q} \) magnon band contributes to the integrals. At temperatures \( T \sim 3SJ \) (in the vicinity of the Curie temperature) both magnon bands will contribute, and, hence, the magnitude of integrals increase.

We then get Hartree-Fock corrected Hamiltonian describing the magnons

\[ \hat{H} = JS \left[ 1 - \frac{\pi}{4S} \left(\frac{T}{3SJ}\right)^2 \right] \left[ \begin{array}{cc} 3 & -\gamma_k^* \\ -\gamma_k^* & 3 \end{array} \right] \equiv \tilde{J}S \left[ \begin{array}{cc} 3 & -\gamma_k^* \\ -\gamma_k^* & 3 \end{array} \right], \]  

(103)

where \( \tilde{J} = J \left[ 1 - \frac{\pi}{4S} \left(\frac{T}{3SJ}\right)^2 \right] \). Exactly this Hamiltonian will be used below when calculating the ladder equation.

**Instability due to pumping**

We neglect the Hartree-Fock corrections by setting \( T = 0 \). Collecting all generated pumping terms, we construct a secular equation for \( \Omega = 3SJ \),

\[
\det \begin{bmatrix}
\Omega + \epsilon - 3SJ & SJ\gamma_k & -\Delta^2\gamma_0 & \Delta^2\gamma_k \\
SJ\gamma_k^* & \Omega - \epsilon - 3SJ & \Delta^2\gamma_k^* & -\Delta^2\gamma_0 \\
-\Delta^2\gamma_0 & \Delta^2\gamma_k & \Omega + \epsilon - 3SJ & SJ\gamma_k \\
\Delta^2\gamma_k^* & -\Delta^2\gamma_0 & SJ\gamma_k^* & \Omega - \epsilon - 3SJ
\end{bmatrix} = 0.
\]  

(104)

The Hamiltonian is similar to that of the BdG model, but only due to the presence of the anomalous terms. The frequency structure is different because of the boson commutation relation the fields obey in our case. We get

\[ \epsilon_{\pm}^2 = (\Omega - 3SJ \pm SJ|\gamma_k|^2)^2 - \Delta^4|\gamma_0 |^2 \]  

(105)

Let us study the effect of Dzyaloshinskii-Moriya interaction Eq. (15) on the magnon pairing in the vicinity of the Dirac points, i.e. for \( \zeta = 0 \). This is motivated by the fact that the DMI is the largest at the Dirac points. The secular equation is now

\[
\det \begin{bmatrix}
\chi + \epsilon & SJ\gamma_k & -3\Delta^2 & 0 \\
SJ\gamma_k^* & -\chi + \epsilon & 0 & -3\Delta^2 \\
-3\Delta^2 & 0 & \chi - \epsilon & SJ\gamma_k \\
0 & -3\Delta^2 & SJ\gamma_k^* & -\chi - \epsilon
\end{bmatrix} = 0,
\]  

(106)
where $\chi = 3\sqrt{3}SD$, and $|\gamma_k| \approx \frac{\sqrt{3}}{2}k$. The spectrum of magnon pairs is now

$$
\epsilon^2_{\pm} = (SJ)^2 \frac{3}{4} k^2 + \chi^2 - 9\Delta^4.
$$

(107)

We conclude that if $|\chi| \geq 3\Delta^2$ there will be no instability in the system. In unpumped ferromagnet such Dzyaloshinskii-Moriya interaction opens up a gap at the Dirac points in the spectrum of the magnons. Then, for the Dirac magnons paired state to occur, pumping should overcome this gap.

**Ladder equation**

![Diagram](image)

FIG. 4: Graphic equation for the pairing interaction strength. Here empty triangle stands for the initial pairing interaction strength $\Delta_{ij}^2$ defined in accordance with Eq. (104). $\Delta_{ab}^2 = -\Delta^2 / \chi$, and $\Delta_{ab}^2 = (\Delta_{ab}^2)^\ast = \Delta^2 / \chi$. Black triangle is intermittently renormalized pairing interaction strength, and the wavy lines stand for the interaction. Lined triangle is the overall renormalized pairing interaction strength.

The action describing the pump is

$$
i S_{\text{pump}} = -i \int \hat{H}_{\text{pump}} \to -i \int_J \int J_4 \left[ \frac{\Gamma \sqrt{S}}{3S J} \bar{\varphi}^{cl\alpha (p_1)} \varphi^{cl\beta (p_3)} \delta_{p_1, p_3} \delta_{\epsilon_1, \epsilon_3} - \Omega \delta_{\epsilon_1, \epsilon_3} \right] \right) (108)
$$

$$
+ i \int_J \int J_4 \left[ \frac{\Gamma \sqrt{S}}{3S J} \bar{\varphi}^{cl\alpha (p_1)} \varphi^{cl\beta (p_3)} \delta_{p_1, p_3} \delta_{\epsilon_1, \epsilon_3} - \Omega \delta_{\epsilon_1, \epsilon_3} \right] \right) (109)
$$

$$
- i \int_J \int J_4 \left[ \frac{\Gamma \sqrt{S}}{3S J} \bar{\varphi}^{cl\alpha (p_1)} \varphi^{cl\beta (p_3)} \delta_{p_1, p_3} \delta_{\epsilon_1, \epsilon_3} - \Omega \delta_{\epsilon_1, \epsilon_3} \right] \right) (110)
$$

$$
+ i \int_J \int J_4 \left[ \frac{\Gamma \sqrt{S}}{3S J} \bar{\varphi}^{cl\alpha (p_1)} \varphi^{cl\beta (p_3)} \delta_{p_1, p_3} \delta_{\epsilon_1, \epsilon_3} - \Omega \delta_{\epsilon_1, \epsilon_3} \right] \right) (111)
$$

where by right arrow we mean picking a particular term from the overall expression. Below, as an example, we wish to see how structure of Eq. (108) gets renormalized by the interactions. For that we construct a ladder equation shown in Fig. 4. It turns out that only

$$
i S_{\text{interaction}} = -i \int \hat{H}_{\text{interaction}} \to -i \int \hat{J} \int J_4 \left[ \gamma_k \bar{\varphi}^{cl\alpha (k_1, \omega_1)} \varphi^{cl\beta (k_3, \omega_3)} \right] (112)
$$

part of the interaction can reproduce selected by us part of the pump. Contraction of the interaction Eq. (112) with the first term, namely Eq. (108), in the pump’s Hamiltonian, gives the following expression

$$
\left( \bar{\varphi}^{cl\alpha (k_1, \omega_1)} \varphi^{cl\beta (k_3, \omega_3)} \right) (113)
$$

$$
= \left( \bar{\varphi}^{cl\beta (k_4, \omega_4)} \varphi^{cl\alpha (k_2, \omega_2)} \right) (114)
$$

$$
= -[G^{K}_{\beta \alpha} (k_4; \omega_4)G^{R}_{\alpha \beta} (k_2; \omega_2)\delta_{k_4, k_2} \delta_{\omega_4, \omega_2} + G^{R}_{\beta \alpha} (k_4; \omega_4)G^{K}_{\alpha \beta} (k_2; \omega_2)\delta_{k_4, k_2} \delta_{\omega_4, \omega_2}] \varphi^{cl\alpha (k_1, \omega_1)} \varphi^{cl\beta (k_3, \omega_3)} (115)
$$
Contraction of the interaction Eq. \([112]\) with the second term in the pump’s Hamiltonian, namely Eq. \([109]\), results in the following expression

\[
\gamma_p \left( \bar{\Psi}^{cl}_{\alpha_1 k_1 \omega_1} \bar{\Psi}^{q}_{\alpha_3 k_3 \omega_3} \Psi^{cl}_{\alpha_2 k_2 \omega_2} \Psi^{\beta}_{\beta k_4 \omega_4} \bar{\Psi}^{cl}_{\alpha_1 p_1 \epsilon_1} \bar{\Psi}^{q}_{\beta p_3 \epsilon_3} \right) = -\gamma_p \left[ G^K_{\alpha \omega} (k_2; \omega_2) G^R_{\beta \omega} (k_4; \omega_4) \delta_{k_2, p_1} \delta_{\omega_2, \epsilon_1} \delta_{k_4, p_3} \delta_{\omega_4, \epsilon_3} + G^R_{\alpha \omega} (k_2; \omega_2) G^K_{\beta \omega} (k_4; \omega_4) \delta_{k_2, p_3} \delta_{\omega_2, \epsilon_3} \delta_{k_4, p_1} \delta_{\omega_4, \epsilon_1} \right] \bar{\Psi}^{cl}_{\alpha_1 k_1 \omega_1} \bar{\Psi}^{q}_{\alpha_3 k_3 \omega_3}. \tag{116}
\]

Contraction of the interaction Eq. \([112]\) with the third term, namely Eq. \([110]\), in the pump’s Hamiltonian, gives the following bracket

\[
\left\langle \bar{\Psi}^{cl}_{\alpha_1 k_1 \omega_1} \bar{\Psi}^{q}_{\alpha_3 k_3 \omega_3} \Psi^{cl}_{\alpha_2 k_2 \omega_2} \Psi^{\beta}_{\beta k_4 \omega_4} \bar{\Psi}^{cl}_{\alpha_1 p_1 \epsilon_1} \bar{\Psi}^{q}_{\beta p_3 \epsilon_3} \right\rangle = \left\langle \bar{\Psi}^{cl}_{\beta k_2 \omega_2} \bar{\Psi}^{\beta}_{\beta k_4 \omega_4} \bar{\Psi}^{q}_{\alpha_3 k_3 \omega_3} \right\rangle + \left\langle \bar{\Psi}^{cl}_{\alpha_1 k_1 \omega_1} \bar{\Psi}^{cl}_{\alpha_2 k_2 \omega_2} \bar{\Psi}^{q}_{\beta p_3 \epsilon_3} \right\rangle \left\langle \bar{\Psi}^{cl}_{\beta k_4 \omega_4} \bar{\Psi}^{q}_{\alpha_3 k_3 \omega_3} \right\rangle - \left\langle G^R_{\beta \omega} (k_2; \omega_2) G^K_{\alpha \omega} (k_4; \omega_4) \delta_{k_2, p_1} \delta_{\omega_2, \epsilon_1} \delta_{k_4, p_3} \delta_{\omega_4, \epsilon_3} \right\rangle \bar{\Psi}^{cl}_{\alpha_1 k_1 \omega_1} \bar{\Psi}^{q}_{\alpha_3 k_3 \omega_3}. \tag{118}
\]

Contraction of the interaction Eq. \([112]\) with the second term in the pump’s Hamiltonian, namely Eq. \([111]\), results in the following expression

\[
\gamma_p^* \left( \bar{\Psi}^{cl}_{\alpha_1 k_1 \omega_1} \bar{\Psi}^{q}_{\alpha_3 k_3 \omega_3} \Psi^{cl}_{\alpha_2 k_2 \omega_2} \Psi^{\beta}_{\beta k_4 \omega_4} \bar{\Psi}^{cl}_{\alpha_1 p_1 \epsilon_1} \bar{\Psi}^{q}_{\beta p_3 \epsilon_3} \right) = -\gamma_p^* \left[ G^K_{\beta \omega} (k_2; \omega_2) G^R_{\alpha \omega} (k_4; \omega_4) \delta_{k_2, p_1} \delta_{\omega_1, \epsilon_1} \delta_{k_4, p_3} \delta_{\omega_2, \epsilon_3} + G^R_{\beta \omega} (k_2; \omega_2) G^K_{\alpha \omega} (k_4; \omega_4) \delta_{k_2, p_3} \delta_{\omega_2, \epsilon_3} \delta_{k_4, p_1} \delta_{\omega_4, \epsilon_1} \right] \bar{\Psi}^{cl}_{\alpha_1 k_1 \omega_1} \bar{\Psi}^{q}_{\alpha_3 k_3 \omega_3}. \tag{121}
\]

Summing all four contributions, we get

\[
\left\langle (iS_{\text{interaction}}) (iS_{\text{pump}}) \right\rangle \rightarrow \left( \frac{J}{4} \right)^2 \left( \frac{\sqrt{S}}{3SJ} \right)^2 \int_{\omega_1} \int_{\epsilon_1} \delta_{k_1, k_2 - k_3} \delta_{\omega_1 - \omega_2 - \omega_3} \delta_{\epsilon_1 - \epsilon_3} \Omega - \epsilon_3 \right) \tag{123}
\]

\[
\times \left( -3 \left\langle \bar{\Psi}^{cl}_{\alpha_1 k_1 \omega_1} \bar{\Psi}^{q}_{\alpha_3 k_3 \omega_3} \Psi^{cl}_{\alpha_2 k_2 \omega_2} \Psi^{\beta}_{\beta k_4 \omega_4} \bar{\Psi}^{cl}_{\alpha_1 p_1 \epsilon_1} \bar{\Psi}^{q}_{\beta p_3 \epsilon_3} \right\rangle + \gamma_p \left( \bar{\Psi}^{cl}_{\alpha_1 k_1 \omega_1} \bar{\Psi}^{q}_{\alpha_3 k_3 \omega_3} \Psi^{cl}_{\alpha_2 k_2 \omega_2} \Psi^{\beta}_{\beta k_4 \omega_4} \bar{\Psi}^{cl}_{\alpha_1 p_1 \epsilon_1} \bar{\Psi}^{q}_{\beta p_3 \epsilon_3} \right) \right) \tag{124}
\]

\[
- \left( 3 \left\langle \bar{\Psi}^{cl}_{\alpha_1 k_1 \omega_1} \bar{\Psi}^{q}_{\alpha_3 k_3 \omega_3} \Psi^{cl}_{\alpha_2 k_2 \omega_2} \Psi^{\beta}_{\beta k_4 \omega_4} \bar{\Psi}^{cl}_{\alpha_1 p_1 \epsilon_1} \bar{\Psi}^{q}_{\beta p_3 \epsilon_3} \right\rangle \right) \tag{125}
\]

\[
= \left( \frac{J}{4} \right)^2 \left( \frac{\sqrt{S}}{3SJ} \right)^2 \left\{ \int_{k} \int_{\epsilon} 3 \gamma_k \left[ G^K_{\alpha \omega} (k; \Omega - \epsilon) G^R_{\alpha \omega} (k; \Omega + \epsilon) + G^R_{\beta \omega} (k; \Omega - \epsilon) G^K_{\alpha \omega} (k; \Omega + \epsilon) \right] \right. \tag{126}
\]

\[
- \int_{k} \int_{\epsilon} \left[ |\gamma_k|^2 G^K_{\alpha \omega} (k; \Omega + \epsilon) G^R_{\beta \omega} (k; \Omega - \epsilon) + \gamma_k^2 G^R_{\alpha \omega} (k; \Omega + \epsilon) G^K_{\beta \omega} (k; \Omega - \epsilon) \right] \right\} \tag{127}
\]

\[
\times \int_{\omega_1} \int_{\epsilon_1} \bar{\Psi}^{cl}_{\alpha_1 k_1 \omega_1} \bar{\Psi}^{q}_{\alpha_3 k_3 \omega_3} \delta_{k_1, k_2 - k_3} \delta_{\omega_1 - \Omega - \omega_3} \tag{128}
\]

\[
+ \left( \frac{J}{4} \right)^2 \left( \frac{\sqrt{S}}{3SJ} \right)^2 \left\{ \int_{k} \int_{\epsilon} 3 \gamma_k \left[ G^R_{\beta \omega} (k; \Omega - \epsilon) G^K_{\alpha \omega} (k; \Omega + \epsilon) + G^K_{\beta \omega} (k; \Omega - \epsilon) G^R_{\alpha \omega} (k; \Omega + \epsilon) \right] \right. \right\}. \tag{129}
\]

\[
\times \int_{\omega_1} \int_{\epsilon_1} \bar{\Psi}^{cl}_{\alpha_1 k_1 \omega_1} \bar{\Psi}^{q}_{\alpha_3 k_3 \omega_3} \delta_{k_1, k_2 - k_3} \delta_{\omega_1 - \Omega - \omega_3}. \tag{130}
\]

It can be shown that the two terms simply double each other. We will be using

\[
G^R(k; \epsilon) = G^R(k; \epsilon) F_\epsilon - F_\epsilon G^A(k; \epsilon) \tag{132}
\]

identity, and a generalization of \(G^R(k; \epsilon) - G^A(k; \epsilon) = -2\pi i(\epsilon - \epsilon_k)\) identity for the honeycomb lattice.
Case of $\Omega = 3S\tilde{J}$

Let us calculate the step of the ladder for the $\Omega = 3S\tilde{J}$. Recall, that $\tilde{J} = J \left[ 1 - \frac{\pi}{4S} \left( \frac{T}{3S\tilde{J}} \right)^2 \right]$. First integral reads

$$
\int_k \int_{k_1} \left[ 3 \gamma_k \left[ G^{K}_{\beta\alpha}(-k; \Omega - \epsilon)G^{R}_{\alpha\alpha}(k; \Omega + \epsilon) + G^{R}_{\beta\alpha}(-k; \Omega + \epsilon)G^{K}_{\alpha\alpha}(k; \Omega - \epsilon) \right] \right]
= \frac{i}{2} \int_k \left[ 2\gamma_k \left[ \frac{\mathcal{F}_{\epsilon+\gamma_k}}{2\Omega - 6S\tilde{J} - 2S\tilde{J}\gamma_k + i0} - \frac{\mathcal{F}_{\epsilon-\gamma_k}}{2\Omega - 6S\tilde{J} + 2S\tilde{J}\gamma_k + i0} \right] \right] = -\frac{i}{4S\tilde{J}} \int_k \left[ 3 \left[ \mathcal{F}_{\epsilon+\gamma_k} + \mathcal{F}_{\epsilon-\gamma_k} \right] \right].
$$

Here and below $\epsilon_{\pm k} = \tilde{J} S (3 \pm |\gamma_k|)$, unperturbed energy of the magnons. Second integrals reads

$$
\int_k \int_{k_1} \left[ |\gamma_k|^2 G^{K}_{\beta\alpha}(k; \Omega + \epsilon)G^{R}_{\alpha\alpha}(-k; \Omega - \epsilon) + \gamma_k^2 G^{K}_{\beta\alpha}(k; \Omega + \epsilon)G^{R}_{\alpha\alpha}(-k; \Omega - \epsilon) \right]
= -\frac{i}{2} \int_k \left[ |\gamma_k|^2 \left[ \frac{\mathcal{F}_{\epsilon+\gamma_k}}{2\Omega - 6S\tilde{J} - 2S\tilde{J}\gamma_k + i0} + \frac{\mathcal{F}_{\epsilon-\gamma_k}}{2\Omega - 6S\tilde{J} + 2S\tilde{J}\gamma_k + i0} \right] \right] = \frac{i}{4S\tilde{J}} \int_k \left[ |\gamma_k| \left[ \mathcal{F}_{\epsilon+\gamma_k} - \mathcal{F}_{\epsilon-\gamma_k} \right] \right].
$$

Summing the two, we get

$$
\int_k \left[ 3 \gamma_k \left[ G^{K}_{\beta\alpha}(-k; \Omega - \epsilon)G^{R}_{\alpha\alpha}(k; \Omega + \epsilon) + G^{R}_{\beta\alpha}(-k; \Omega + \epsilon)G^{K}_{\alpha\alpha}(k; \Omega - \epsilon) \right] \right] - \int_k \left[ |\gamma_k|^2 G^{K}_{\beta\alpha}(k; \Omega + \epsilon)G^{R}_{\alpha\alpha}(-k; \Omega - \epsilon) + \gamma_k^2 G^{K}_{\beta\alpha}(k; \Omega + \epsilon)G^{R}_{\alpha\alpha}(-k; \Omega - \epsilon) \right]
= -\frac{i}{4S\tilde{J}} \int_k \left[ (3 + |\gamma_k|) \mathcal{F}_{\epsilon+\gamma_k} + (3 - |\gamma_k|) \mathcal{F}_{\epsilon-\gamma_k} \right]
\approx -\frac{i}{4S\tilde{J}} \left[ 24\sqrt{3} + \pi \left( \frac{T}{3S\tilde{J}} \right)^2 \right],
$$

where

$$
\int_k \left[ (3 + |\gamma_k|) \mathcal{F}_{\epsilon+\gamma_k} + (3 - |\gamma_k|) \mathcal{F}_{\epsilon-\gamma_k} \right] \approx \int_k \left[ 6 + \frac{2}{e^{T/(3S\tilde{J})}} \right] = 24\sqrt{3} + \frac{1}{8\pi} \left( \frac{T}{S\tilde{J}} \right)^2 \int_0^\infty \frac{zd\epsilon}{\epsilon^{3/2} - 1}
$$

and

$$
(3 + |\gamma_k|) \mathcal{F}_{\epsilon+\gamma_k} + (3 - |\gamma_k|) \mathcal{F}_{\epsilon-\gamma_k} = 6 + \frac{2}{e^{T/(3S\tilde{J})}} \left( \frac{T}{3S\tilde{J}} \right)^2 \int_k \int_{(k)} \tilde{\Psi}_{\alpha ; \omega_1}^\dagger \tilde{\Psi}_{\alpha ; \omega_3} \delta_{k_1, -k_3} \delta_{\omega_1 - \omega_3 - \Omega - \omega_3}
$$

which is a natural approximation, as only the low-energy magnons with $\epsilon_{-\gamma_k}$ dispersion can contribute to the integral. The $\epsilon_{+\gamma_k}$ are exponentially suppressed at small temperatures. Then we have for the step of the ladder,

$$
\langle (iS_{\text{interaction}})(iS_{\text{pump}}) \rangle \approx -\frac{i}{4S\tilde{J}} \left( \frac{J}{4} \right)^2 \left( \Gamma S \frac{24\sqrt{3} + 3\pi \left( \frac{T}{3S\tilde{J}} \right)^2}{3S\tilde{J}} \right) \int_k \int_{(k)} \tilde{\Psi}_{\alpha ; \omega_1}^\dagger \tilde{\Psi}_{\alpha ; \omega_3} \delta_{k_1, -k_3} \delta_{\omega_1 - \omega_3 - \Omega - \omega_3}
$$

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Summing the original pumping term, the first step of the ladder, and iterating the steps further, we get,

$$iS_{\text{pump}} + ((iS_{\text{interaction}})(iS_{\text{pump}}))$$

$$= -i3 \frac{J}{4} \left( \frac{\Gamma \sqrt{S}}{3SJ} \right)^2 \left[ 1 + \frac{\sqrt{3} J}{S} \frac{\pi J}{7} \left( \frac{T}{3SJ} \right)^2 \right] \int_{\{k\}} \int_{\{\omega\}} \bar{\Psi}^{cl}_{\alpha; k_1; \omega_1} \Psi^{q}_{\alpha; k_3; \omega_3} \delta k_1 - k_3 \delta \omega_1 - \Omega - \omega_3$$

$$\rightarrow -i3 \frac{J}{4} \left( \frac{\Gamma \sqrt{S}}{3SJ} \right)^2 \left( \frac{1}{1 - \frac{\sqrt{3} J}{S} - \frac{\pi J}{7} \left( \frac{T}{3SJ} \right)^2} \right) \int_{\{k\}} \int_{\{\omega\}} \bar{\Psi}^{cl}_{\alpha; k_1; \omega_1} \Psi^{q}_{\alpha; k_3; \omega_3} \delta k_1 - k_3 \delta \omega_1 - \Omega - \omega_3,$$

clearly there is an enhancement of pairing.

**Case of \( \Omega \neq 3SJ \)**

Here we demonstrate that for \( \Omega \neq 3SJ \) each step of the ladder acquires an imaginary part. Besides, we are going to show that the pumping gets suppressed by the rescattering processes described by the ladder as the frequency approaches \( 6SJ \). To see the general tendency of the renormalization of the pairing strength away from the Dirac points, we disregard Hartree-Fock corrections to the magnon dispersion. We have for the step of the ladder,

$$M(\Omega) \equiv \int_{k} \int_{\omega} 3\gamma_{-k} \left[ G^{K}_{\alpha\alpha}(k; \Omega - \epsilon)G^{R}_{\alpha\alpha}(k; \Omega + \epsilon) + G^{R}_{\beta\alpha}(k; \Omega - \epsilon)G^{K}_{\alpha\alpha}(k; \Omega + \epsilon) \right]$$

$$- \int_{k} \int_{\omega} [\gamma_{\alpha}^{2} G^{K}_{\alpha\alpha}(k; \Omega + \epsilon)G^{R}_{\alpha\beta}(k; \Omega - \epsilon) + \gamma_{\beta}^{2} G^{K}_{\alpha\beta}(k; \Omega + \epsilon)G^{R}_{\alpha\alpha}(k; \Omega - \epsilon)]$$

$$= \frac{i}{2} \int_{k} |\gamma_{\alpha}| \left[ \frac{3 + |\gamma_{\alpha}|}{2\Omega - 6SJ - 2SJ|\gamma_{\alpha}| + i0} - \frac{3 - |\gamma_{\alpha}|}{2\Omega - 6SJ + 2SJ|\gamma_{\alpha}| + i0} \right]$$

$$= \frac{i}{4} PV \int_{k} |\gamma_{\alpha}| \left[ \frac{3 + |\gamma_{\alpha}|}{\zeta - S|\gamma_{\alpha}|} - \frac{3 - |\gamma_{\alpha}|}{\zeta + S|\gamma_{\alpha}|} \right]$$

$$+ \frac{i}{4} \left( -\frac{i\pi}{2} \right) \int_{k} |\gamma_{\alpha}| \delta(\zeta - S|\gamma_{\alpha}|)(3 + |\gamma_{\alpha}|)F_{\epsilon + k} - \frac{i}{4} \left( -\frac{i\pi}{2} \right) \int_{k} |\gamma_{\alpha}| \delta(\zeta + S|\gamma_{\alpha}|)(3 - |\gamma_{\alpha}|)F_{\epsilon - k},$$

where \( PV \) is the principal value of the integral, and where \( \zeta = \Omega - 3SJ \). The imaginary part for \( \zeta > 0 \) is evaluated as

$$- \frac{i\pi}{2} \int_{k} |\gamma_{\alpha}| \delta(\zeta - S|\gamma_{\alpha}|)(3 + |\gamma_{\alpha}|)F_{\epsilon + k} + \frac{i\pi}{2} \int_{k} |\gamma_{\alpha}| \delta(\zeta + S|\gamma_{\alpha}|)(3 - |\gamma_{\alpha}|)F_{\epsilon - k},$$

where we kept the integral as it is. The imaginary part is non-zero and works towards weakening of the pairing between magnons.

Let us estimate the step of the ladder when the pump frequency is \( \Omega = 6SJ - \alpha \) and \( \alpha \) is small. Then \( \zeta = 3SJ - \alpha \), we approximate \( |\gamma_{\alpha}| \approx 3 - \frac{\alpha}{S} \), and we write for the step of the ladder

$$\text{Im}[M(\Omega)] \approx \frac{i}{4} PV \int_{k} |\gamma_{\alpha}| \frac{3 + |\gamma_{\alpha}|}{\zeta - S|\gamma_{\alpha}|} \approx \frac{9}{2} PV \int_{k} \frac{1}{\zeta - S + \frac{\alpha}{S} + \alpha} = i \frac{9}{2\pi SJ} PV \int_{0}^{\Delta_{\alpha}} \frac{dz}{z - \alpha} \approx i \frac{9}{2\pi SJ} \ln \left( \frac{SJ\Delta_{\alpha}^2}{4\alpha} \right),$$

$$\text{Re}[M(\Omega)] \approx \frac{\pi}{8} \int_{k} |\gamma_{\alpha}| \delta(\zeta - S|\gamma_{\alpha}|)(3 + |\gamma_{\alpha}|)F_{\epsilon + k} \approx \frac{9}{4SJ}.$$  

We then get for the renormalization of the pairing

$$\Delta^2 \rightarrow \frac{\Delta^2}{1 + \frac{3}{16\pi SJ} \ln \left( \frac{SJ\Delta^2}{4\alpha} \right) + i \frac{3}{32SJ}}.$$

Importantly, for frequencies away from the Dirac points, the structure of the renormalization due to the rescattering processes drastically changes. Namely, the sign of each ladder changes as compared to the Dirac magnons case, and, as a result, there is no way for the divergency to occur. Moreover, away from \( \Omega = 6SJ \), the pairing is only weakly suppressed by the rescattering processes. However, when pump’s frequency \( \Omega \) approaches \( 6SJ \), \( \alpha \rightarrow 0 \), the pairing vanishes.
Example: shifting the rescattered field away for $\Omega = 6SJ$

When pump’s frequency is $\Omega = 6SJ$ there is a resonant absorption of magnons. This can be see from

$$L^{R/A}_{\alpha\beta,0,\Omega}L^{R/A}_{\beta\alpha,0,\Omega} - L^{R/A}_{\beta\beta,0,\Omega}L^{R/A}_{\alpha\alpha,0,\Omega} = 0 \mp i0$$

for non-interacting magnons. Upon inserting life-time of magnons at $\omega = 6SJ$ and $k = 0$, the quantity becomes finite, imaginary and can be large. Let us call it

$$L^{R/A}_{\alpha\beta,0,\Omega}L^{R/A}_{\beta\alpha,0,\Omega} - L^{R/A}_{\beta\beta,0,\Omega}L^{R/A}_{\alpha\alpha,0,\Omega} = \mp \frac{i}{2\tau_6} (6SJ \pm \frac{i}{2\tau_6}).$$

Also

$$L^{R/A}_{\alpha\beta,0,\Omega}L^{R/A}_{\beta\alpha,0,\Omega} - L^{R/A}_{\beta\beta,0,\Omega}L^{R/A}_{\alpha\alpha,0,\Omega} = \mp \frac{i}{2\tau_6},$$

and, hence, we get

$$L^{R/A}_{\alpha\beta,0,\Omega}L^{R/A}_{\beta\alpha,0,\Omega} - L^{R/A}_{\beta\beta,0,\Omega}L^{R/A}_{\alpha\alpha,0,\Omega} = \frac{1}{6SJ} \mp \frac{i}{2\tau_6}.$$ 

Therefore, the shift of the $\omega = 6SJ$, $k = 0$ fields reads as

$$\Psi^{cl}_{n;0;6SJ} \rightarrow \Psi^{cl}_{n;0;6SJ} + \frac{\Gamma\sqrt{S}}{6SJ - \frac{i}{2\tau_6}},$$

$$\Psi^{cl}_{n;0;6SJ} \rightarrow \Psi^{cl}_{n;0;6SJ} + \frac{\Gamma\sqrt{S}}{6SJ + \frac{i}{2\tau_6}}.$$ 

For physically relevant scenario, $6SJ > \frac{1}{2\tau_6}$, thus, we can neglect the inverse life-time, and recover the claim made in the Main Text.

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