Standard Cosmology on a Self-Tuning Domain Wall

Conall Kennedy
School of Mathematics, Trinity College, University of Dublin, Ireland

Emil M. Prodanov
National Centre for Scientific Research "Demokritos", Athens, Greece

Abstract

We investigate the cosmology of (4+1)-dimensional gravity coupled to a scalar field and a bulk anisotropic fluid within the context of the single-brane Randall-Sundrum scenario. Assuming a separable metric, a static fifth radius and the scalar to depend only on the fifth direction, we find that the warp factor is given as in the papers of Kachru, Schulz and Silverstein [hep-th/0001206, hep-th/0002121] and that the cosmology on a self-tuning brane is standard. In particular, for a radiation-dominated brane the pressure in the fifth direction vanishes.

PACS numbers: 04.50.+h, 11.27.+d, 98.80.Cq.
Keywords: Randall–Sundrum, Domain Walls, Warp Factor, Cosmology

*e-mail: conall@maths.tcd.ie
**e-mail: prodanov@maths.tcd.ie
1 Introduction

Theories with extra dimensions where our four-dimensional world is a hypersurface (three-brane) embedded in a higher-dimensional spacetime and at which gravity is localised have been the subject of intense scrutiny since the work of Randall and Sundrum [1]. The main motivation for such models comes from string theory where they are reminiscent of the Hořava-Witten solution [2] for the field theory limit of the strongly-coupled $E_8 \times E_8$ heterotic string. The Randall–Sundrum (RS) scenario may be modelled [3,4] by coupling gravity to a scalar field and mapping to an equivalent supersymmetric quantum mechanics problem. A static metric is obtained with a warp factor determined by the superpotential. A generalisation to non-static metrics was considered by Binétruy, Deffayet and Langlois (BDL) who modelled brane matter as a perfect fluid delta-function source in the five-dimensional Einstein equations [5]. However, this resulted in non-standard cosmology in that the square of the Hubble parameter on the brane was not proportional to the density of the fluid. Other cosmological aspects of “brane-worlds” have been considered in [6].

In this letter, we investigate RS-type single brane cosmological solutions of five-dimensional gravity coupled to a scalar field which we assume to depend only on the fifth dimension. We further assume that the fifth dimension is static and infinite in extent. We also include a bulk anisotropic fluid with energy-momentum tensor $\hat{T}^A_B(\rho) = \text{diag}(-\rho, p, p, p, P)$ and equations of state $P = \tilde{\omega} \rho$, $p = \omega \rho$. Assuming a separable metric, we find that the warp factor is given as in the papers of Kachru, Schulz and Silverstein (KSS) [7, 8].

We also find that the cosmology on a self-tuning brane is standard but that the pressure in the fifth direction is constrained by the relation $\tilde{\omega} = \frac{3}{2}\omega - \frac{1}{2}$. In particular, we find that the pressure in the fifth direction vanishes for a radiation-dominated brane with $\omega = 1/3$.

2 The Model

We consider a single, thin brane at $r = 0$, as in KSS [7]. The action for the gravity and scalar part of the model is:

\[
S = S_{\text{gravity}} + S_{\text{bulk}} + S_{\text{brane}},
\]

\[
S_{\text{gravity}} = \frac{1}{2\hat{\kappa}^2} \int d^4x dr \sqrt{-\hat{g}} \hat{R},
\]

\[
S_{\text{bulk}} = \int d^4x dr \sqrt{-\hat{g}} \left( -\frac{4}{3} \hat{g}^{AB} \partial_A \Phi \partial_B \Phi - U(\Phi) \right),
\]
\[ S_{\text{brane}} = \int d^4 x \sqrt{-g^{(4)}} (-V(\Phi)) , \]  

(1)

where \( \hat{g}_{AB} \) is the five-dimensional metric, \( g^{(4)}_{ij} \) is the induced metric on the brane and the tension of the brane is parametrised by \( V(\Phi) \).

We assume a separable metric with flat spatial three-sections on the brane:

\[ ds^2 = \hat{g}_{AB} dy^A dy^B = e^{2A(r)} (-dt^2 + g(t) \delta_{ab} dx^a dx^b) + dr^2 . \]  

(2)

This is a natural generalisation of the 4d flat Robertson-Walker metric to a RS context and is a special case of the BDL ansatz (see [5]) with \( n(t, r) = e^{A(r)} \), \( a(t, r) = e^{A(r)} g^{1/2}(t) \), \( b(t, r) = 1 \) in conventional notation.

We shall also make the ansatz that both the potentials \( U(\Phi) \) and \( V(\Phi) \) are of Liouville type (see, for instance, [9]):

\[ U(\Phi) = U_0 e^{\alpha \Phi} , \]
\[ V(\Phi) = V_0 e^{\beta \Phi} , \]  

(3)

where \( U_0 \) and \( V_0 \) are constants.

The stress-tensor for the scalar is

\[ \hat{T}^A_B(\Phi) = \hat{T}^A_B + \tilde{T}^A_B , \]  

(4)

where

\[ \hat{T}^A_B = \frac{8}{3} \partial^A \Phi \partial_B \Phi - \delta^A_B \left( \frac{4}{3} \delta^C \Phi \partial_C \Phi + U(\Phi) \right) , \]  

(5)

and

\[ \tilde{T}^A_B = - \sqrt{-g^{(5)}} V(\Phi) \delta(r) \tilde{g}^{(4)}_{ij} \delta^i_A \delta^j_B , \]  

(6)

where there is no sum over the indices \( i \) and \( j \). We shall assume that \( \Phi \) depends only on \( r \).

The bulk fluid has the stress-tensor [10]:

\[ \hat{T}^A_B(\rho) = \text{diag} (-\rho, p, p, p, P) \]  

(7)

in the comoving coordinates \( y^A \). \( \rho \) is the density and \( p \) and \( P \) are the pressures in the three spatial directions on the brane and in fifth dimension, respectively. The anisotropy can be considered as a result of the mixing of two interacting perfect fluids [11].
3 The Solutions

We now proceed to solve Einstein’s equations \( \hat{G}_{\hat{A}} = \hat{\kappa}_5^2 (\hat{T}_{\hat{B}}(\Phi) + \hat{T}_{\hat{B}}(\rho)) \) given the above ansätze.

If we take a linear combination of the 00- and 11-components of Einstein’s equations then the following equation results:

\[
\frac{\dot{g}^2}{g^2} - \frac{\ddot{g}}{g} - \hat{\kappa}_5^2 e^{2A}(\rho + p) = 0 .
\] (8)

Therefore, we see that \( \rho \) and \( p \) must be of the form

\[
\rho(t, r) = e^{-2A} (\tilde{\rho}(t) + F(t, r)) ,
\] (9)

\[
p(t, r) = e^{-2A} (\tilde{\rho}(t) - F(t, r)) ,
\] (10)

for arbitrary \( F(t, r) \). However, it is normal to assume the equation of state \( p = \omega \rho \), where \( \omega \) is constant in the range \(-1 \leq \omega \leq 1\). In the generic case \( \omega \neq -1 \) this implies that \( F \) should be zero. We shall assume this also to be so in the special case \( \omega = -1 \). Furthermore, we shall also assume \( P = \hat{\omega} \rho \). Equation (8) then reduces to

\[
\frac{\dot{g}^2}{g^2} - \frac{\ddot{g}}{g} - \hat{\kappa}_5^2 (1 + \omega) \tilde{\rho} = 0 .
\] (11)

Given \( F = 0 \), the 00-component of Einstein’s equations separates into

\[
\frac{3}{4} \frac{\dot{g}^2}{g^2} - \hat{\kappa}_5^2 \tilde{\rho} = C ,
\] (12)

\[
(6A'^2 + 3A'') + \frac{4}{3} \hat{\kappa}_5^2 \Phi'^2 + \hat{\kappa}_5^2 U + \hat{\kappa}_5^2 V \delta(r) = C e^{-2A} ,
\] (13)

where \( C \) is the separation constant.

The \( rr \)-equation also splits in two:

\[
\frac{3}{2} \frac{\ddot{g}}{g} + \hat{\kappa}_5^2 \tilde{\omega} \tilde{\rho} = D ,
\] (14)

\[
6A'^2 - \frac{4}{3} \hat{\kappa}_5^2 \Phi'^2 + \hat{\kappa}_5^2 U = D e^{-2A} ,
\] (15)

where \( D \) is another separation constant.

Equation (15) allows us to recast (13) in the form

\[
3A'' + \frac{8}{3} \hat{\kappa}_5^2 \Phi'^2 + (D - C) e^{-2A} + \hat{\kappa}_5^2 V \delta(r) = 0 .
\] (16)
We shall see below that in fact $D = 2C$.

In addition, the equation of motion for the scalar field

$$\frac{8}{3} \hat{\nabla}^2 \Phi - \frac{\partial U(\Phi)}{\partial \Phi} - \sqrt{-\hat{g}_5^{(4)}} \frac{\partial V(\Phi)}{\partial \Phi} \delta(r) = 0 ,$$

results in the equation

$$\frac{8}{3} \Phi'' + \frac{32}{3} A' \Phi' - \alpha U - \beta V \delta(r) = 0 , \quad (18)$$

Note that the scalar field equation of motion implies that $\hat{\nabla} A \hat{T}_{AB} = 0$ (and, conversely, off the brane only). This, in turn, implies that the fluid equations of motion $\hat{\nabla} A \hat{T}_{AB}(\rho) = 0$ are automatically satisfied.

**The Warp Factor**

Equations (15), (16) (with $D = 2C$) and (18) have been extensively studied in [7,8,12–14]. The self-tuning domain wall (solution (I) of [7]) is given by

$$U = C = 0 , \quad \beta \neq \pm \frac{1}{a} , \quad (19)$$

$$\Phi(r) = a \epsilon \log (d - cr) , \quad (20)$$

$$A(r) = \frac{1}{4} \log (d - cr) - e , \quad (21)$$

where $a = 3/(4 \sqrt{2} \hat{\kappa}_5)$ and $\epsilon$ is a sign that takes opposite values either side of the brane at $r = 0$. The parameters $c, d$ and $e$ are constants of integration that can also differ either side of the brane. For (21) to make sense, we require $d > 0$. The continuity of $\Phi$ and $A$ across the brane requires

$$d_+ d_- = 1 , \quad (22)$$

$$e_+ = \frac{1}{4} \log d_+ , \quad e_- = \frac{1}{4} \log d_- , \quad (23)$$

where we have chosen the convention $A(0) = 0$ and denoted constants defined on the right (left) side of the the brane with a $+$ ($-$) subscript. The jump conditions implied by (16) and (18) result in the relations

$$c_+ = - \frac{2}{3} \hat{\kappa}_5^2 d_+ (a \beta \epsilon_+ - 1) V_0 e^{a \beta \epsilon_+ \log d_+} ,$$

$$c_- = - \frac{2}{3} \hat{\kappa}_5^2 d_- (a \beta \epsilon_+ + 1) V_0 e^{a \beta \epsilon_+ \log d_+} . \quad (24)$$
The solution is self-tuning because given $d_+, \epsilon_+ = \pm 1$ and $\beta \neq \pm 1/a$, there is a Poincaré-invariant four-dimensional domain wall for any value of the brane tension $V_0$; $V_0$ does not need to be fine-tuned to find a solution.

Other warp factors are possible both when $C = 0$ and when $C \neq 0$. Solution (II) of [7] with $U = 0$ and solution (III) of the same reference with $U \neq 0$ are examples of the former case. The solution presented in [8] with $U = 0$ provides an example the latter.

### The Cosmology

Adding equations (12) and (14) gives:

$$\dot{g}^2 + 2g\ddot{g} + \frac{4}{3} \kappa^2 g^2 (\tilde{\omega} - 1) \tilde{\rho} = \frac{4}{3} (D + C) g^2 .$$

(25)

On the other hand, using (12) in (11) we obtain:

$$\dot{g}^2 + 2g\ddot{g} + 2 \kappa^2 g^2 (\omega - 1) \tilde{\rho} = 4C g^2 .$$

(26)

Since $\tilde{\rho}$ is generically a function of $t$ (rather than a constant) the above two equations imply the relations

$$D = 2C$$

(27)

$$\omega = \frac{1}{3} (1 + 2\tilde{\omega}) .$$

(28)

Relation (28) previously appeared in [15]. In particular, it implies that an isotropic (perfect) fluid ($P = p$) is stiff, that is, $\omega = \tilde{\omega} = 1$. The attribute “stiff” refers to the fact that the velocity of sound in the fluid is equal to the velocity of light. It should be noted that the case of a bulk cosmological constant ($\omega = \tilde{\omega} = -1$) is not covered here; however, it corresponds to the choice $U(\Phi) = constant$ instead.

Using (12), equation (27) may be expressed alternatively as

$$\tilde{\omega} \dot{g}^2 + 2g \ddot{g} = \frac{4}{3} (\tilde{\omega} + 2) C g^2 ,$$

(29)

with the solutions

$$g \sim \begin{cases} 
\sinh^{2q} \left( \sqrt{\frac{C}{3q^2}} t \right) & C > 0 , \\
\sqrt{\frac{|C|}{3q^2}} t^{2q} & C = 0 , \\
\sin^{2q} \left( \sqrt{\frac{|C|}{3q^2}} t \right) & C < 0 ,
\end{cases}$$

(30)

*These solutions, in the particular case of an isotropic fluid, appeared in a different setting in [16].
where \( q = 1/(2 + \tilde{\omega}) = 2/(3(1 + \omega)) = q_{\text{standard}} \).

From (12) we see that the density \( \tilde{\rho} \) (which is actually the density of the fluid on the brane since we have defined \( A(0) = 0 \)) is positive:

\[
\tilde{\rho}(t) = \begin{cases} 
\hat{\kappa}_5^{-2} C \sinh^{-2}(\sqrt{\frac{C}{3q^2}} t) & C > 0 , \\
\frac{3 \hat{\kappa}_5^{-2} q^2}{t^2} & C = 0 , \\
\hat{\kappa}_5^{-2} |C| \sin^{-2}(\sqrt{\frac{|C|}{3q^2}} t) & C < 0 .
\end{cases}
\] (31)

When \( C \geq 0 \), equation (29) also allows the de-Sitter solutions \( g = e^{\pm 2\sqrt{C/3} t} \). These solutions have vanishing density \( \tilde{\rho} \) and were discussed in \([3,4,8,17]\). For the case \( C = 0 \), we obtain conventional cosmology \( H = \dot{a}/a \propto \sqrt{\tilde{\rho}} \) on the brane with evolution at the standard rate.

Of particular note is the case of radiation-dominated fluid on the brane (\( \omega = 1/3 \)). From (28) we see that the pressure in the fifth direction vanishes and the stress tensor is then:

\[
T^A_B(\rho) = e^{-2A(r)} \tilde{\rho}(t) \text{diag}(-1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0) ,
\] (32)

with \( q_{\text{standard}} = 1/2 \).

4 Summary

The self-tuning domain wall, with warp factor given by (21), has vanishing separation constant \( C \) and therefore expands according to the power law (30) at the standard rate and exhibits conventional cosmology when coupled to a bulk anisotropic fluid. The pressure of the fluid in the fifth direction, \( P \), is related to the isotropic pressure on the brane, \( p \), via equation (28) and vanishes for a radiation-dominated brane.

Acknowledgements

We are grateful to George Savvidy, Siddhartha Sen and Andy Wilkins for useful discussions and a critical reading of the manuscript. E.M.P. is supported by EU grant HPRN-CT-1999-00161.

References

[1] L. Randall and R. Sundrum: *A Large Mass Hierarchy from a Small Extra Dimension*. Phys. Rev. Lett. 83, 3370–3373 (1999), hep-ph/9905221.
L. Randall and R. Sundrum: *An Alternative to Compactification*. Phys. Rev. Lett. **83**, 4690–4693 (1999), [hep-th/9906064](http://arxiv.org/abs/hep-th/9906064).

[2] P. Hořava and E. Witten: *Heterotic and Type I String Dynamics from Eleven Dimensions*. Nucl. Phys. **B460**, 506–524 (1996), [hep-th/9510209](http://arxiv.org/abs/hep-th/9510209);

P. Hořava and E. Witten: *Eleven-Dimensional Supergravity on a Manifold with Boundary*. Nucl. Phys. **B475**, 94–114 (1996), [hep-th/9603142](http://arxiv.org/abs/hep-th/9603142).

[3] O. DeWolfe, D. Z. Freedman, S. S. Gubser and A. Karch: *Modelling the Fifth Dimension With Scalars and Gravity*. [hep-th/9909134](http://arxiv.org/abs/hep-th/9909134).

[4] C. Csaki, J. Erlich, T. J. Hollowood and Yu. Shirman: *Universal Aspects of Gravity Localised on Thick Branes*. [hep-th/0001033](http://arxiv.org/abs/hep-th/0001033).

[5] P. Binétruy, C. Deffayet and D. Langlois: *Non-conventional Cosmology from a Brane-Universe*. [hep-th/9905012](http://arxiv.org/abs/hep-th/9905012).

[6] N. Kaloper and A. Linde: *Inflation and Large Internal Dimensions*. Phys. Rev. **D59**, 101303 (1999), [hep-th/9811141](http://arxiv.org/abs/hep-th/9811141);

J. M. Cline, C. Grojean and G. Servant: *Cosmological Expansion in the Presence of an Extra Dimension*. Phys. Rev. Lett. **83**, 4245–4247 (1999), [hep-ph/9906523](http://arxiv.org/abs/hep-ph/9906523);

D. J. H. Chung and K. Freese: *Cosmological Challenges in Theories with Extra Dimensions and Remarks on the Horizon Problem*. Phys. Rev. **D61**, 023511 (2000), [hep-ph/9906542](http://arxiv.org/abs/hep-ph/9906542);

H. B. Kim and H. D. Kim: *Inflation and Gauge Hierarchy in Randall–Sundrum Compactification*. Phys. Rev. **D61**, 064003 (2000), [hep-th/9909053](http://arxiv.org/abs/hep-th/9909053);

P. Kanti, I. I. Kogan, K. A. Olive and M. Pospelov: *Cosmological 3-Brane Solutions*. Phys. Lett. **B468**, 31–39 (1999), [hep-ph/9909481](http://arxiv.org/abs/hep-ph/9909481);

E. E. Flanagan, S.-H. Henry Tye and I. Wasserman: *Cosmological Expansion in the Randall-Sundrum Brane World Scenario*. [hep-ph/9910498](http://arxiv.org/abs/hep-ph/9910498);

U. Ellwanger: *Cosmological Evolution in Compactified Hořava–Witten Theory Induced by Matter on the Branes*. [hep-th/0001126](http://arxiv.org/abs/hep-th/0001126);

R. N. Mohapatra, A. Pérez-Lorenzana and C. A. de S. Pires: *Inflation in Models with Large Extra Dimensions Driven by a Bulk Scalar Field*. [hep-ph/0003089](http://arxiv.org/abs/hep-ph/0003089);

M. Brändle, A. Lukas and B. A. Ovrut: *Heterotic M-Theory Cosmology in Four and Five Dimensions*. [hep-th/0003256](http://arxiv.org/abs/hep-th/0003256);

B. Grinstein, D. R. Nolte and W. Skiba: *Adding Matter to Poincaré-Invariant Branes*. Phys. Rev. **D62**, 086006 (2000), [hep-th/0005001](http://arxiv.org/abs/hep-th/0005001).
[7] S. Kachru, M. Schulz and E. Silverstein: *Self-tuning Flat Domain Walls in 5d Gravity and String Theory*. Phys. Rev. D62 (2000) 045021, hep-th/0001206.

[8] S. Kachru, M. Schulz and E. Silverstein: *Bounds on Curved Domain Walls in 5-D Gravity*. Phys. Rev. D62 085003 (2000), hep-th/0002121.

[9] H. A. Chamblin and H. S. Reall: *Dynamic Dilatonic Domain Walls*. Nucl. Phys. B562, 133–157 (1999), hep-th/9903225.

[10] H. Liu and P. S. Wesson: *Exact Solutions of General Relativity derived from 5-D Black-Hole Solutions of Kaluza-Klein Theory*. J. Math. Phys. 33 (1992) 3888.

[11] S. R. Oliveira: *Model of Two Perfect Fluids for an Anisotropic and Homogeneous Universe*. Phys. Rev. D40 (1989) 3976.

[12] D. Youm: *Bulk Fields in Dilatonic and Self-Tuning Flat Domain Walls*. Nucl. Phys. B589, 315-336 (2000), hep-th/0002147.

[13] P. Kanti, K. A. Olive and M. Pospelov: *Static Solutions for Brane Models with a Bulk Scalar Field*. Phys. Lett. B481 (2000) 386, hep-ph/0002229.

[14] C. Csaki, J. Erlich, C. Grojean and T. J. Hollowood: *General Properties of the Self-tuning Domain Wall Approach to the Cosmological Constant Problem*. Nucl. Phys. B584, 359-386 (2000), hep-th/0004133.

[15] C. Kennedy and E. M. Prodanov: *Standard Cosmology from Sigma-Model*. Phys. Lett. B488 11-16 (2000), hep-th/0003293.

[16] K. Enqvist, E. Keski-Vakkuri and S. Räsänen: *Constraints on Brane and Bulk Ideal Fluid in Randall-Sundrum Cosmologies*. hep-th/0007254.

[17] N. Kaloper: *Bent Domain Walls as Braneworlds*. Phys. Rev. D60 (1999) 123506, hep-th/9905210.

T. Nihei: *Inflation in the Five-dimensional Universe With an Orbifold Extra Dimension*. Phys. Lett. B465, 81–85 (1999), hep-ph/9905487.

M. Gremm: *Thick Domain Walls and Singular Spaces*. Phys. Rev. D62 (2000) 044017, hep-th/0002040.

J. E. Kim and B. Kyae: *Exact Cosmological Solution and Modulus Stabilization in the Randall-Sundrum Model with Bulk Matter*. Phys. Lett. B486 (2000) 165, hep-th/0005139.