ON THE DE BRUIJN–NEWMAN CONSTANT: A NEW APPROACH

XIAO-JUN YANG

Abstract. The conjecture of Newman, proposed in 1976 by Newman, states that all zeros of \( \Xi_{\alpha}(\lambda) \) are real for \( \alpha \in \mathbb{R} \). Its equivalent statement is that \( M_{\alpha}(\tau) \) has purely imaginary zeros for \( \alpha \in \mathbb{R} \). It is well known that \( M_{\alpha}(\tau) \) is an even entire function of order one. This article addresses the product representation for \( M_{\alpha}(\tau) \) by the works of Hadamard and Csordas, Norfolk and Varga. We establish a new class of \( M_{\alpha}(\tau) \) by its series and product. Based on the obtained result, we prove that it has only purely imaginary zeros for \( \alpha \in \mathbb{R} \). This implies that the conjecture of Newman is true.

1. Introduction

In 1950 de Bruijn [1] introduced a family of the functions \( \Xi_{\alpha}(\lambda) : \mathbb{C} \to \mathbb{C} \) for \( \alpha \in \mathbb{R} \) by the Fourier cosine integral

\[
\Xi_{\alpha}(\lambda) = \int_0^\infty \exp\left(-\alpha x^2\right) G(x) \cos(\lambda x) \, dx,
\]

where

\[
G(x) = \sum_{m=1}^\infty \left[2\pi^2 m^4 \exp(9x) - 3\pi m^2 \exp(5x)\right] \exp\left(-\pi m^2 \exp(4x)\right).
\]

Here, we denote the sets of the real and complex numbers by \( \mathbb{R} \) and \( \mathbb{C} \), respectively.

It has been reported by Rodgers and Tao that the even entire function (1) has the functional equation [2]

\[
\Xi_{\alpha}(\lambda) = \overline{\Xi_{\alpha}(\lambda)}
\]

and satisfies the backward heat equation [3]

\[
\partial_\alpha \Xi_{\alpha}(\lambda) + \partial_{\lambda\lambda} \Xi_{\alpha}(\lambda) = 0.
\]

In 1976 Newman [4] considered that there exists an absolute constant

\[
\infty < \alpha \leq \frac{1}{2},
\]

now known as the de Bruijn–Newman constant \( \alpha \), with the case that \( \Xi_{\alpha}(\lambda) \) has purely real zeros.

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Newman [4] conjectured $\aleph \geq 0$ with the case that $\Xi_\aleph(\lambda)$ has purely real zeros. The recent progress in the lower bound on $\aleph$ is presented in Table 1 [2]. The recent result on the upper bound on $\aleph$ is showed in Table 2. Using the result of Pólya [5], we find that $\aleph = 0$ implies that $\Xi_0(\lambda)$ has purely real zeros. The conjecture of Newman [4] states that all zeros of $\Xi_\aleph(\lambda)$ are real for $\aleph \in \mathbb{R}$.

In this article we consider the following case of (1):

$$\aleph \in (-\infty, \infty)$$

which can be equivalently transferred into a family of the function $M_\aleph(\tau) : \mathbb{C} \to \mathbb{C}$ for $\aleph \in \mathbb{R}$ by the integral [6]

$$M_\aleph(\tau) = \int_0^\infty \exp(-\aleph x^2) G(x) \cosh(\tau x) \, dx$$

with the property that $M_\aleph(\tau)$ has purely imaginary zeros.

Obviously, substituting

$$\tau = i\lambda$$

into (7), this implies that

$$\Xi_\aleph(\lambda) = M_\aleph(i\lambda).$$

It was proved by Csordas, Norfolk and Varga [6] that the function $M_\aleph(\tau)$ is an even entire function of order $\beta = 1$.

### Table 1. The results in the lower bound on $\aleph$ [2]

| Lower bound on $\aleph$ | References                        |
|-------------------------|-----------------------------------|
| $-\infty$               | Newman [4]                        |
| -50                     | Csordas–Norfolk–Varga [6]         |
| -5                      | te Riele [7]                      |
| -0.385                  | Norfolk–Ruttan–Varga [8]          |
| -0.0991                 | Csordas–Ruttan–Varga [9]          |
| -0.379 \times 10^{-6}   | Csordas–Smith–Varga [10]          |
| -5.895 \times 10^{-9}   | Csordas–Odlyzko–Smith–Varga [11] |
| -2.63 \times 10^{-9}    | Odlyzko [12]                      |
| -1.15 \times 10^{-11}   | Saouter–Gourdon–Demichel [13]    |

### Table 2. The results in the upper bound on $\aleph$.

| Upper bound on $\aleph$ | References                     |
|-------------------------|--------------------------------|
| 0.5                     | Ki, Kim and Lee [14]           |
| 0.22                    | Polymath [15]                  |
| $+\infty$               | Rodgers and Tao [2]            |
With the help of (6), (7), and (8), we need to prove the equivalent representation for the conjecture of Newman as follows:

**Theorem 1.** The function $M_\Re(\lambda)$ has only purely imaginary zeros for $\Re \in \mathbb{R}$.

Motivated by the idea, we plan to set up a class of (7), expressed by the product and series, to prove Theorem 1. The outline of the paper is given as follows. In Section 2 we consider the product and series presentations and order of (7). In Section 3 we give the six steps to prove Theorem 1. In Section 4 we finally draw our conclusion.

2. The series and product for (7)

**Theorem 2.** There exists

\[ M_\Re (\tau) = \sum_{\gamma=0}^{\infty} \alpha_{2\gamma} \tau^{2\gamma}, \]

where

\[ \alpha_{2\gamma} = \frac{1}{(2\gamma)!} \int_0^\infty \exp(-\Re x^2) G(x) x^{2\gamma} dx. \]

**Proof.** Using the fact

\[ \cosh(x) = \sum_{\gamma=0}^{\infty} \frac{x^{2\gamma}}{(2\gamma)!}, \]

(7) can be expressed as

\[ M_\Re (\tau) = \int_0^\infty \exp(-\Re x^2) G(x) \cosh(\tau x) dx \]

\[ = \int_0^\infty \exp(-\Re x^2) G(x) \left[ \sum_{\gamma=0}^{\infty} \frac{(\tau x)^{2\gamma}}{(2\gamma)!} \right] dx \]

\[ = \sum_{\gamma=0}^{\infty} \alpha_{2\gamma} \tau^{2\gamma}, \]

where $\alpha_{2\gamma}$ is defined by (11).

We hence complete the proof.

**Theorem 3.** There exists

\[ M_\Re (\tau) = M_\Re (0) \prod_{\Im(\sigma_\ell) > 0} \left( 1 - \frac{\tau^2}{\sigma_\ell^2} \right), \]

where this product runs over all zeros $\sigma_\ell$ of $M_\Re (\tau)$. Moreover, $\sum_{\ell=1}^{\infty} |\sigma_\ell|^{-(1+\varepsilon)}$ is convergent for $\varepsilon > 0$. 
Proof. By using (7) and (10), we know

\[ M_\mathcal{R}(0) = \alpha_0 = \int_0^\infty \exp(-\kappa x^2) G(x) \, dx > 0. \]

By the fact that \( M_\mathcal{R}(\tau) \) is an even entire function of order \( \beta = 1 \) [6] and Hadamard’s factorization theorem (Theorem 13 in [16], p.24-29; also see [17], p.250),

\[ M_\mathcal{R}(\tau) = M_\mathcal{R}(0) e^{\vartheta_0 \tau} \prod_{\sigma_\ell} \left(1 - \frac{\tau}{\sigma_\ell} \right) \exp\left(\frac{\tau}{\sigma_\ell}\right) \]

and \( \sum_{\ell=1}^\infty |\sigma_\ell|^{-(1+\varepsilon)} \) is convergent for \( \varepsilon > 0 \).

Since \( M_\mathcal{R}(\tau) \) is an even function, we have

\[ M_\mathcal{R}(\sigma_\ell) = M_\mathcal{R}(-\sigma_\ell) \]

such that

\[ M_\mathcal{R}(\tau) = M_\mathcal{R}(0) e^{\vartheta_0 \tau} \prod_{\sigma_\ell} \left(1 - \frac{\tau}{\sigma_\ell} \right) \exp\left(\frac{\tau}{\sigma_\ell}\right) \]

\[ = M_\mathcal{R}(0) e^{\vartheta_0 \tau} \prod_{\Im(\sigma_\ell) > 0} \left(1 - \frac{\tau}{\sigma_\ell} \right) \left(1 + \frac{\tau}{\sigma_\ell} \right) \exp\left(\frac{\tau}{\sigma_\ell} - \frac{\tau}{\sigma_\ell} \right) \]

\[ = M_\mathcal{R}(0) e^{\vartheta_0 \tau} \prod_{\Im(\sigma_\ell) > 0} \left(1 - \frac{\tau}{\sigma_\ell} \right) \left(1 + \frac{\tau}{\sigma_\ell} \right). \]

To simply (18), we present

\[ M_\mathcal{R}(-\tau) = M_\mathcal{R}(0) e^{\vartheta_0 \tau} \prod_{\Im(\sigma_\ell) > 0} \left(1 - \frac{\tau}{\sigma_\ell} \right) \left(1 + \frac{\tau}{\sigma_\ell} \right) \]

\[ = M_\mathcal{R}(0) e^{\vartheta_0 \tau} \prod_{\Im(\sigma_\ell) > 0} \left(1 - \frac{\tau^2}{\sigma_\ell^2} \right). \]

By using the functional equation

\[ M_\mathcal{R}(\tau) = M_\mathcal{R}(-\tau), \]

we have

\[ M_\mathcal{R}(\tau) = M_\mathcal{R}(0) e^{\vartheta_0 \tau} \prod_{\Im(\sigma_\ell) > 0} \left(1 - \frac{\tau^2}{\sigma_\ell^2} \right) \]

and

\[ M_\mathcal{R}(-\tau) = M_\mathcal{R}(0) e^{-\vartheta_0 \tau} \prod_{\Im(\sigma_\ell) > 0} \left(1 - \frac{\tau^2}{\sigma_\ell^2} \right). \]
such that
\[ M_\mathcal{R} (0) e^{\theta_0 \tau} \prod_{\Im(\sigma) > 0} \left( 1 - \frac{\tau^2}{\sigma^2} \right) = M_\mathcal{R} (0) e^{-\theta_0 \tau} \prod_{\Im(\sigma) > 0} \left( 1 - \frac{\tau^2}{\sigma^2} \right). \]

From (21) and (23) we obtain \( \theta_0 = 0 \) and
\[ M_\mathcal{R} (\tau) = M_\mathcal{R} (0) \prod_{\Im(\sigma) > 0} \left( 1 - \frac{\tau^2}{\sigma^2} \right). \]

Therefore, we finish the proof. \( \square \)

3. The proof of Theorem 1

We now suggest six steps to prove it.

Step 1 is to suggest a class of the even entire function.

Making use of (10) and (24), a class of \( M_\mathcal{R} (\tau) \) is given as follows:
\[ \sum_{\gamma=0}^{\infty} \alpha_{2\gamma} \tau^{2\gamma} = M_\mathcal{R} (0) \prod_{\Im(\sigma) > 0} \left( 1 - \frac{\tau^2}{\sigma^2} \right), \]
where
\[ \alpha_{2\gamma} = \frac{1}{(2\gamma)!} \int_0^{\infty} \exp (-\Re x^2) G(x) x^{2\gamma} dx. \]

Since \( G(x) > 0 \) for \( x > 0 \) \([18]\), it follows from (26) that
\[ \alpha_{2\gamma} > 0. \]

Step 2 is to set up the first product of \( M_\mathcal{R} (\tau) \).

Obviously, we have
\[ M_\mathcal{R} (\tau) = \sum_{\gamma=0}^{\infty} \alpha_{2\gamma} \tau^{2\gamma} \]
such that
\[ \overline{M_\mathcal{R} (\tau)} = \left[ \sum_{\gamma=0}^{\infty} \alpha_{2\gamma} \tau^{2\gamma} \right] = \sum_{\gamma=0}^{\infty} \alpha_{2\gamma} \tau^{2\gamma} = \sum_{\gamma=0}^{\infty} \alpha_{2\gamma} \tau^{2\gamma}. \]

From (29) we show that
\[ \overline{M_\mathcal{R} (\tau)} = \sum_{\gamma=0}^{\infty} \alpha_{2\gamma} \tau^{2\gamma} = M_\mathcal{R} (\tau). \]
It follows from (25) that we have

\begin{equation}
M_N(\tau) = M_R(0) \prod_{\Im(\sigma_\ell) > 0} \left( 1 - \frac{\tau^2}{\sigma_\ell^2} \right)
\end{equation}

such that (30) can be written as

\begin{equation}
\overline{M_N}'(\tau) = \overline{M_N}'(\tau) = M_R(0) \prod_{\Im(\sigma_\ell) > 0} \left( 1 - \frac{\tau^2}{\sigma_\ell^2} \right).
\end{equation}

Step 3 is to set up the second product of \( M_N(\tau) \).

Once again, from (25) we have

\begin{equation}
M_N(\tau) = M_R(0) \prod_{\Im(\sigma_\ell) > 0} \left( 1 - \frac{\tau^2}{\sigma_\ell^2} \right)
\end{equation}

such that

\begin{equation}
\overline{M_N}'(\tau) = \left[ M_R(0) \prod_{\Im(\sigma_\ell) > 0} \left( 1 - \frac{\tau^2}{\sigma_\ell^2} \right) \right].
\end{equation}

Recall (15) such that

\begin{equation}
M_N(0) = \alpha = \int_0^\infty \exp \left( -\mathcal{R} x^2 \right) G(x) \, dx > 0.
\end{equation}

Combining (34) and (35) implies that

\begin{equation}
\overline{M_N}'(\tau) = \left[ M_R(0) \prod_{\Im(\sigma_\ell) > 0} \left( 1 - \frac{\tau^2}{\sigma_\ell^2} \right) \right] = M_N(0) \prod_{\Im(\sigma_\ell) > 0} \left( 1 - \frac{\tau^2}{\sigma_\ell^2} \right)
\end{equation}

Hence, we have

\begin{equation}
\overline{M_N}'(\tau) = \left[ M_N(0) \prod_{\Im(\sigma_\ell) > 0} \left( 1 - \frac{\tau^2}{\sigma_\ell^2} \right) \right] = M_N(0) \prod_{\Im(\sigma_\ell) > 0} \left( 1 - \frac{\tau^2}{\sigma_\ell^2} \right)
\end{equation}

such that (36) can be rewritten as

\begin{equation}
\overline{M_N}'(\tau) = \left[ M_N(0) \prod_{\Im(\sigma_\ell) > 0} \left( 1 - \frac{\tau^2}{\sigma_\ell^2} \right) \right].
\end{equation}
Step 4 is to set up the products of $M(\varsigma)$.

On combination of (32) and (38) implies that

\begin{align}
M(\tau) &= M(0) \prod_{\Im(\sigma) > 0} \left(1 - \frac{\tau^2}{\sigma^2}\right) = M(0) \prod_{\Im(\sigma) > 0} \left(1 - \frac{\tau^2}{\sigma^2}\right).
\end{align}

Applying (30), we have

\begin{align}
M(\tau) &= M(\tau) \quad \text{such that (39) becomes}
\end{align}

\begin{align}
M(\tau) &= M(0) \prod_{\Im(\sigma) > 0} \left(1 - \frac{\tau^2}{\sigma^2}\right) = M(0) \prod_{\Im(\sigma) > 0} \left(1 - \frac{\tau^2}{\sigma^2}\right).
\end{align}

Taking $\varsigma = \tau \in \mathbb{C}$ into (41), we present

\begin{align}
M(\varsigma) &= M(0) \prod_{\Im(\sigma) > 0} \left(1 - \frac{\varsigma^2}{\sigma^2}\right) = M(0) \prod_{\Im(\sigma) > 0} \left(1 - \frac{\varsigma^2}{\sigma^2}\right).
\end{align}

Step 5 is to investigate the convergence of $M(1)$.

Putting $\varsigma = 1$ into (42), we suggest

\begin{align}
M(1) &= M(0) \prod_{\Im(\sigma) > 0} \left(1 - \frac{1}{\sigma^2}\right) = M(0) \prod_{\Im(\sigma) > 0} \left(1 - \frac{1}{\sigma^2}\right).
\end{align}

From (7) we may get

\begin{align}
M(1) &= \int_0^{\infty} \exp \left(-\Re x^2\right) G(x) \cosh(x) \, dx < \infty.
\end{align}

With use of Theorem 3, we have $\varepsilon > 0$ such that $\sum_{\ell=1}^{\infty} |\sigma|^{-(1+\varepsilon)}$ is convergent. Taking $\varepsilon = 1$ implies that $\sum_{\ell=1}^{\infty} |\sigma|^{-2}$ is convergent.

In view of Theorem 5 in Knopp’s monograph (see [19], p.10),

\begin{align}
M(0) \prod_{\Im(\sigma) > 0} \left(1 - \frac{1}{\sigma^2}\right) = M(0) \prod_{\Im(\sigma) > 0} \left(1 - \frac{1}{\sigma^2}\right)
\end{align}

is absolutely convergent.

Making use of (45), Theorem 3 in Knopp’s monograph (see [19], p.10) implies that $\sum_{\Im(\sigma) > 0} \sigma^{-2}$ and $\sum_{\Im(\sigma) > 0} \overline{\sigma}^{-2}$ are convergent.

Step 6 is to obtain $\Re(\sigma) = 0$. 
Adopting (45), we suggest

\[
\sum_{\Im(\sigma)>0} \infty \sigma^{-2} = \sum_{\Im(\sigma)>0} \infty \overline{\sigma}^{-2}.
\]

Since \(\sum_{\Im(\sigma)>0} \sigma^{-2}\) and \(\sum_{\Im(\sigma)>0} \overline{\sigma}^{-2}\) are convergent, it follows from (46) that

\[
\sum_{\Im(\sigma)>0} \infty (\sigma^{-2} - \overline{\sigma}^{-2}) = 0,
\]

which yields that

\[
\sigma^{-2} - \overline{\sigma}^{-2} = 0.
\]

From (48) we have

\[
\sigma_{\ell}^{-2} - \overline{\sigma_{\ell}}^{-2} = 0
\]

and \(\Im(\sigma_{\ell}) > 0\) such that

\[
(\sigma_{\ell} - \overline{\sigma_{\ell}}) (\sigma_{\ell} + \overline{\sigma_{\ell}}) = 4 \Im(\sigma_{\ell}) \Re(\sigma_{\ell}) = 0.
\]

With the aid of (50), we obtain

\[
\Re(\sigma_{\ell}) = 0.
\]

Putting \(\Im(\sigma_{\ell}) = \rho_{\ell} > 0\) into (25) and using (46), we may give

\[
\sum_{\gamma=0}^{\infty} \alpha_{2\gamma} \tau^{2\gamma} = M_{\Re}(0) \prod_{\ell=1}^{\infty} \left(1 + \frac{\tau^2}{\rho_{\ell}^2}\right)
\]

and

\[
\sum_{\Im(\sigma_{\ell})>0} \infty \sigma_{\ell}^{-2} = -\sum_{\ell=1}^{\infty} \rho_{\ell}^{-2} < \infty.
\]

This implies that Theorem 1 is true.

As a direct consequence of (52), we have the following:

**Corollary 1.** There is

\[
\Xi_{\Re}(\lambda) = \Xi_{\Re}(0) \prod_{\ell=1}^{\infty} \left(1 - \frac{\lambda^2}{\rho_{\ell}^2}\right),
\]

where the product takes over all positive zeros \(\rho_{\ell} > 0\) of \(\Xi_{\Re}(\lambda)\).
Proof. Combining (7), (10) and (52), we present

\[ M_\alpha (\tau) = \sum_{\gamma=0}^{\infty} \alpha_{2\gamma} \tau^{2\gamma} = \int_0^{\infty} \exp (-\alpha x^2) \, G(x) \cosh (\tau x) \, dx \]

(55)

\[ = M_\alpha (0) \prod_{\ell=1}^{\infty} \left( 1 + \frac{\tau^2}{\rho_\ell^2} \right). \]

Substituting (9) into (55), we have

(56) \[ \Xi_\alpha (0) = M_\alpha (0) \]

such that

\[ \Xi_\alpha (\lambda) = M_\alpha (i\lambda) = \sum_{\gamma=0}^{\infty} \alpha_{2\gamma} (i\lambda)^{2\gamma} = \int_0^{\infty} \exp (-\lambda x^2) \, G(x) \cosh (i\lambda x) \, dx \]

(57)

\[ = M_\alpha (0) \prod_{\ell=1}^{\infty} \left[ 1 + \frac{(i\lambda)^2}{\rho_\ell^2} \right] = M_\alpha (0) \prod_{\ell=1}^{\infty} \left( 1 - \frac{\lambda^2}{\rho_\ell^2} \right) \]

\[ = \Xi_\alpha (0) \prod_{\ell=1}^{\infty} \left( 1 - \frac{\lambda^2}{\rho_\ell^2} \right). \]

We therefore complete the proof. \(\square\)

Remark. Obviously, (54) implies that all zeros of (1) are purely real and that Corollary 1 is the truth of the conjecture of Newman. Readers also follow the technology reported in [20] to prove it. The special case for \(\alpha = 0\) was proved in [21]. We also allow that Corollary 1 and Theorem 1 are valid for \(\alpha = 0\). Compared with the method of Rodgers and Tao [2], our work agrees with the pair correlation estimates of Montgomery [22] because the product and series representations of \(\Xi_\alpha (\lambda)\) in Corollary 1 are analogous of the Riemann zeta function.

4. Conclusion

In the present task we have established a class of the even entire function of order \(\beta = 1\) with its series and product formulae. Based on the obtained results, we have obtained the truth of the conjecture of Newman by the functional equation. Compared with the method of Rodgers and Tao, our result is consistent with the well-known pair correlation estimates of Montgomery. This may be proposed as a new approach to deal with a family of the even entire functions of order one.

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Email address: dyangxiaojun@163.com; xjyang@cumt.edu.cn

1 School of Mathematics, and State Key Laboratory for Geo-Mechanics and Deep Underground Engineering, China University of Mining and Technology, Xuzhou 221116, China

2 Department of Mathematics, Faculty of Science, King Abdulaziz University P.O. Box 80257, Jeddah 21589, Saudi Arabia