UPPER AND WEAK-LOWER SEMICONTINUITY OF PULLBACK ATTRACTORS TO IMPULSIVE EVOLUTION PROCESSES

MATHEUS C. BORTOLAN*
Departamento de Matemática, Universidade Federal de Santa Catarina
Campus Florianópolis - Brasil
Florianópolis, 88040-900, SC, Brazil

JOSÉ MANUEL UZAL
Departamento de Estatística, Análise Matemática e Optimización
& Instituto de Matemáticas
Universidade de Santiago de Compostela
Santiago de Compostela, Spain

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Abstract. In this paper, following the work done in [11], we deal with the upper and weak-lower semicontinuity of pullback attractors for impulsive evolution processes. We first deal with the upper semicontinuity, presenting the abstract theory and applying it to uniform perturbations of a nonautonomous integrate-and-fire neuron model. We also present the abstract theory of weak-lower semicontinuity, and finish with an improvement of [11, Subsection 4.2], proving an invariance property for impulsive pullback ω-limits with weaker assumptions.

1. Introduction. The study of perturbations of attractors for dynamical systems in the continuous framework is extensive, and passes through four distinct levels of increasing difficulty: the upper semicontinuity, the lower semicontinuity, the topological stability and the geometrical stability. The upper semicontinuity, the simplest one, is present in many real mathematical models and deals with non-explosion of solutions when the system is subjected to small perturbations (see, for instance, [4, 13, 14, 22]). The lower semicontinuity, although similar in definition to the upper semicontinuity, is harder to obtain and deals with non-implosion of solutions (see, for instance, [15, 17, 23, 24]). It requires a deep knowledge of the internal structure of the attractor for the limiting problem, and a control in the behavior of such structure under small perturbations. The topological stability deals with the permanence of invariants inside the attractor, and is closely related to the concept of dynamically gradient dynamical systems (see, for instance, [16, 2, 3]). Finally, the geometrical stability is the permanence of invariants and their

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* Corresponding author: Matheus C. Bortolan.
connections inside the attractor, and it is related to the concept of Morse-Smale systems (see, for instance, [31, 1, 27, 29, 30, 10]).

In the impulsive framework the theory of perturbation of attractors is fairly recent (see, for instance, [8, 7]), and deals mainly with the upper semicontinuity of attractors, together with a few applications, as well as the abstract theory for weak-lower semicontinuity (see, for instance, [8, 7]).

As an application for it is that it requires a good concept of critical elements for the given dynamical system, that is stable under perturbations (for instance, hyperbolic points and normally hyperbolic periodic orbits in the continuous autonomous case, or hyperbolic global solutions in the continuous nonautonomous case), and also such that their local unstable and stable manifolds are also stable (in the $C^1$-topology) under perturbations (see, for instance, [25, 26, 10]).

This issue is challenging, and even in the continuous case it presents difficulties. There are recent results concerning impulsive periodic orbits (see, for instance, [6, 19, 20, 21]), but the problem of robustness under perturbations of these orbits still remains open.

Therefore, in this paper, we follow the same path. Continuing the results of [11], summarized in Section 2, we present results concerning convergence properties for a family of impulsive evolution processes (see Section 3) and the abstract theory of upper semicontinuity for pullback attractors of impulsive evolution processes (see Section 4), which culminates in Theorem 4.4. As an application of this theory we study, in Section 5, uniform perturbations of a nonautonomous integrate-and-fire model for neuron membranes. This model was presented in [28], and was later studied in [12, 9, 11]. This simple one-dimensional model is enough to display the difficulties of verifying conditions of Theorem 4.4. In Section 6, we introduce the abstract framework to study a form of lower semicontinuity of pullback attractors for impulsive evolution processes, called weak-lower semicontinuity (see Definition 6.1), which results in Theorem 6.10. Finally, in Section 7 we improve the work done in [11, Subsection 4.2] and prove an invariance property for impulsive $\omega$-limits without using the so-called tube conditions (see [11, Definition 4.3]).

2. Impulsive evolution processes and their pullback attractors

In this section, we present a summary of the basic concepts and results of [11], which deal with the existence of pullback attractors for impulsive evolution processes. To continue, we refer to [18] for the basic theory of (continuous) evolution processes and their associated pullback attractors.

We begin with a (continuous) evolution process $\mathcal{U}$ in a metric space $(X, d)$, and for each $s \in \mathbb{R}$, $r \geq 0$ and family $\hat{D} = \{D(t)\}_{t \in \mathbb{R}}$ of subsets of $X$, we set

$$F(\hat{D}, r, s) = \{x \in X : U(r + s, x) \in D(r + s)\},$$

with the simpler notations $F(D, r, s) := F(\hat{D}, r, s)$ if $D(t) = D$ for all $t \in \mathbb{R}$ and $F(x, r, s) := F(\{x\}, r, s)$ for $x \in X$.

A family $\hat{D}$ is called collectively closed if for $t_n \to t$, $x_n \in D(t_n)$ with $x_n \to x$, then $x \in D(t)$. Given a collectively closed family $\hat{D}$ in $X$ and a family of functions $I = \{I_t : D(t) \to X\}_{t \in \mathbb{R}}$, we say that $I$ is collectively continuous if given $t_n \to t$, $x_n \in D(t_n)$ for each $n \in \mathbb{N}$ with $x_n \to x$, then $I_{t_n}(x_n) \to I_t(x)$ (note that since $\hat{D}$ is collectively closed, $x \in D(t)$ and $I_t(x)$ is well-defined).

An impulsive evolution process $\mathcal{U} = (\mathcal{U}, X, \hat{M}, I)$ consists of an evolution process $\mathcal{U}$ in $X$, a collectively closed family $\hat{M}$ (called impulsive family) with the
property that for every \( z \in M(s) \) there exists \( \epsilon = \epsilon(z,s) > 0 \) such that
\[
\cup_{r \in (0,\epsilon)} \left( F(z,r,s-r) \cap M(s-r) \right) = \emptyset \quad \text{and} \quad \cup_{r \in (0,\epsilon)} \left( \{ U(r+s,s)z \} \cap M(r+s) \right) = \emptyset,
\]
and a collectively continuous family of functions \( I = \{ I_t : M(t) \to X \}_{t \in \mathbb{R}} \) called impulse function (such function is used to define the impulsive semitrajectories, see [11]). We use the simple notation \( \tilde{\epsilon} \) for the definition of the impulse function and we assume that all impulsive trajectories exist for all time \( t \) and a collectively continuous family of functions \( \tilde{\epsilon} \). From [11, Proposition 2.14] we have
\[
\tilde{\epsilon}(x,s) = \infty \quad \text{for all} \quad (x,s) \in X \times \mathbb{R}.
\]

With the impulsive evolution process and impulsive semitrajectories defined, we can move on to the definition of pullback attractors, but first we must recall a few auxiliary concepts: a universe is a collection \( \mathcal{O} \) of families \( \tilde{D} = \{ D(t) \}_{t \in \mathbb{R}} \) of subsets of \( X \) (that is, \( D(t) \subset X \) for each \( t \in \mathbb{R} \)) which is closed by inclusion, that is, if \( D_1 \subset D_2 \in \mathcal{O} \) then \( D_1 \in \mathcal{O} \). From now on \( \mathcal{O} \) is a universe in \( X \) and \( \tilde{U} \) satisfies (1).

A family \( \tilde{D} \) in \( X \) is called \( \tilde{U} \)-invariant, or positively \( \tilde{U} \)-invariant, or negatively \( \tilde{U} \)-invariant if
\[
\tilde{U}(t,s)D(s) = D(t) \quad \text{or} \quad \tilde{U}(t,s)D(s) \subset D(t) \quad \text{or} \quad \tilde{U}(t,s)D(s) \supset D(t),
\]
for all \( t \geq s \), respectively. Given families \( \tilde{A} \) and \( \tilde{B} \), we say that \( \tilde{A} \) pullback \( \tilde{U} \)-attracts \( \tilde{B} \) if for each \( t \in \mathbb{R} \) we have
\[
d_H(\tilde{U}(t,s)B(s),A(t)) \to 0 \quad \text{as} \quad s \to -\infty,
\]
where \( d_H \) is the Hausdorff semidistance associated with the metric \( d \) in \( X \), that is \( d_H(E,F) = \sup_{e \in E} \inf_{f \in F} d(e,f) \).

Now we can present the notion of a pullback \( \mathcal{O} \)-attractor for the impulsive evolution process \( \tilde{U} \): a family \( \tilde{A} \in \mathcal{O} \) is a pullback \( \mathcal{O} \)-attractor for \( \tilde{U} \) if \( \tilde{A} \) is compact, that is, \( A(t) \) is a compact subset of \( X \) for each \( t \in \mathbb{R} \), \( \tilde{A} \) pullback \( \tilde{U} \)-attracts each \( D \in \mathcal{O} \), and the family \( \tilde{A} \setminus M = \{ A(t) \setminus M(t) \}_{t \in \mathbb{R}} \) is \( \tilde{U} \)-invariant.

To state the result concerning the existence of pullback \( \mathcal{O} \)-attractors, we need the following: an impulsive evolution process \( \tilde{U} \) is called pullback \( \mathcal{O} \)-asymptotically compact if given \( \tilde{D} \in \mathcal{O} \), \( t \in \mathbb{R} \), sequences \( s_n \to -\infty \), \( \epsilon_n \to 0 \), \( x_n \in D(s_n) \) for each \( n \in \mathbb{N} \), then the sequence \( \{ \tilde{U}(t+s_n,s_n)x_n \} \) has a convergent subsequence in \( X \). Also we say that \( \tilde{U} \) is pullback \( \mathcal{O} \)-dissipative if there exists a nonempty collectively closed family \( B_0 \in \mathcal{O} \) such that given \( \tilde{D} \in \mathcal{O} \), \( t \in \mathbb{R} \), \( s_n \to -\infty \) and \( \epsilon_n \to 0 \) there exists \( n_0 = n_0(\tilde{D},t) \in \mathbb{N} \) such that
\[
\tilde{U}(t+s_n,s_n)D(s_n) \subset B_0(t+\epsilon_n) \quad \text{for all} \quad n \geq n_0.
\]

\(^1\)By \( \tilde{D}_1 \subset \tilde{D}_2 \) we mean \( D_1(t) \subset D_2(t) \) for all \( t \in \mathbb{R} \).
A family \( \hat{B}_0 \) with these properties is a **pullback \( \mathcal{D} \)-absorbing family** for \( \hat{U} \).

With these considerations, the main results of \([11]\) concerning the existence of pullback \( \mathcal{D} \)-attractors can be summarized as follows:

**Theorem 2.1.** If \( \hat{U} \) is a pullback \( \mathcal{D} \)-asymptotically compact and pullback \( \mathcal{D} \)-dissipative impulsive evolution process, with a pullback absorbing family \( \hat{B}_0 \), satisfying

\[
I_\tau(M(\tau)) \cap M(\tau) = \emptyset \text{ for all } \tau \in \mathbb{R};
\]

there exists \( \xi > 0 \) such that \( \phi(x,s) \geq 2\xi \) for all \( s \in \mathbb{R} \) and \( x \in I_s(M(s)) \); \hspace{1cm} (I)

\[
\text{given } s \in \mathbb{R}, \ t > s, \ x \in M(t), \ \text{a convergent sequence } \{x_j\}
\]

with \( U(t,s)x_j \to x \), there exist a subsequence \( \{x_{j_k}\} \) of \( \{x_j\} \), and a sequence \( \alpha_k \to 0 \) with \( \alpha_k \geq s \) such that

\[
U(t+\alpha_k, s)x_{j_k} \in M(t+\alpha_k) \text{ for all } k.
\]

Then \( \hat{U} \) satisfies\(^2\) (1) and has a pullback \( \mathcal{D} \)-attractor \( \hat{A} \subset \hat{B}_0 \).

We will need an additional result, which is not presented in \([11]\), that relates the notions of pullback attractors and global solutions. We say that \( \xi : \mathbb{R} \to X \) is a **global solution** of \( \hat{U} \) if \( \hat{U}(t,s)\xi(s) = \xi(t) \) for all \( t \geq s \). Thus we have the following:

**Proposition 2.2.** Let \( \hat{U} = (\mathcal{U}, X, \hat{M}, I) \) be an impulsive evolution process satisfying (1) and (I), \( \mathcal{D} \) be a universe of families in \( X \), \( \hat{A} \) a pullback \( \mathcal{D} \)-attractor for \( \hat{U} \), and for each \( t \in \mathbb{R} \) define

\[
\Xi(t) = \{ \xi(t) : \xi \text{ is a global solution of } \hat{U} \text{ with } \hat{\xi} = \{\xi(t)\}_{t \in \mathbb{R}} \in \mathcal{D} \}.
\]

Then \( A(t) \setminus M(t) = \Xi(t) \) for each \( t \in \mathbb{R} \).

**Proof.** Fix \( t \in \mathbb{R} \) and \( \xi \) a global solution of \( \hat{U} \) with \( \hat{\xi} = \{\xi(t)\}_{t \in \mathbb{R}} \in \mathcal{D} \). Condition (I) implies that \( \xi(t) \notin M(t) \) for all \( t \in \mathbb{R} \) (see \([11\text{, Proposition 2.12}]\)). Also, since \( \hat{\xi} \in \mathcal{D} \), \( \xi \) is a global solution of \( \hat{U} \) and \( \hat{A} \) is a pullback \( \mathcal{D} \)-attractor, we have

\[
d_H(\xi(t), A(t)) = \lim_{s \to -\infty} d_H(\hat{U}(t,s)\xi(s), A(t)) = 0,
\]

which implies that \( \xi(t) \in A(t) \) (since \( A(t) \) is closed). Hence \( \xi(t) \in A(t) \setminus M(t) \).

Now, if \( x \in A(t) \setminus M(t) \), since \( \hat{A} \setminus \hat{M} \) is \( \hat{U} \)-invariant we have \( x \in \hat{U}(t,t-1)(A(t-1) \setminus M(t-1)) \). This implies that there exists \( x_{-1} \in A(t-1) \setminus M(t-1) \) such that \( x = \hat{U}(t,t-1)x_{-1} \). Inductively we can construct a sequence \( \{x_{-n}\}_n \) with \( x_{-n} \in A(t-n) \setminus M(t-n) \) and \( \hat{U}(t-n+1,t-n)x_{-n} = x_{-n+1} \). We define \( \xi : \mathbb{R} \to X \) as

\[
\xi(s) = \begin{cases} 
\hat{U}(s,t-n)x_{-n}, & s \in [t-n, t-n+1], \ n \in \mathbb{N} \\
\hat{U}(s,t)x, & s \in [t, \infty).
\end{cases}
\]

The function \( \xi \) is a global solution of \( \hat{U} \) by construction, and \( \xi(s) \in A(s) \setminus M(s) \subset A(s) \) for each \( s \in \mathbb{R} \). Since \( \mathcal{D} \) is a universe and \( \hat{A} \in \mathcal{D} \), we obtain \( \hat{\xi} = \{\xi(s)\}_{s \in \mathbb{R}} \in \mathcal{D} \). Hence \( x = \xi(t) \in \Xi(t) \).

\(^2\)Note that condition (H) implies (1).
3. Family of impulsive evolution processes. From now on we consider a family \(\{\mathcal{U}_\eta\}_{\eta \in [0,1]}\) of impulsive evolution process \(\mathcal{U}_\eta = (\mathcal{U}_\eta, X, M^\eta, I^\eta)\) for each \(\eta \in [0,1]\). We begin with some definitions: we say that the family \(\{\mathcal{U}_\eta\}_{\eta \in [0,1]}\) of (continuous) evolution processes is \textbf{continuous} at \(\eta = 0\) if \(U_\eta(t,s)x \to U_0(t,s)x\) as \(\eta \to 0\), uniformly for \((t, s, x)\) in compact subsets of \(\mathcal{P} \times X\).

\textbf{Definition 3.1.} \(\text{(a) A collection of families } \{\bar{D}^\eta\}_{\eta \in [0,1]} \text{ is called collectively closed at } \eta = 0 \text{ if for } \eta_k \to 0, t_k \to t, x_k \in D^{\eta_k}(t_k) \text{ with } x_k \to k, \text{ then } x \in D^0(t).\)

(b) Given a collectively closed at \(\eta = 0\) collection of families \(\{\bar{D}^\eta\}_{\eta \in [0,1]}\), and a collection of family of functions \(\{I^\eta\}_{\eta \in [0,1]} = \{\{I^\eta_\eta : D^\eta(t) \to X\}_{t \in \mathbb{R}}\}_{\eta \in [0,1]}\), we say that \(\{I^\eta\}_{\eta \in [0,1]}\) is \textbf{collectively continuous at } \(\eta = 0\) if given \(\eta_k \to 0, t_k \to t \text{ in } \mathbb{R}, x_k \in D^{\eta_k}(t_k) \text{ with } x_k \to x\), then \(I^{\eta_k}_{\eta_k}(x_k) \to I^\eta(x).\)

\textbf{Remark 3.2.} Note that the collective closedness at \(\eta = 0\) of a collection \(\{\bar{D}^\eta\}_{\eta \in [0,1]}\) implies the collective closedness of the family \(\bar{D}^0\) (since we could take \(\eta_k = 0\) for all \(k\)). However, it does not imply the collective closedness of \(\bar{D}^\eta\) for \(\eta > 0\).

3.1. \textbf{Collective continuity of the impact time maps.}

\textbf{Lemma 3.3.} Let \(\{\bar{U}_\eta\}_{\eta \in [0,1]}\) be a family of impulsive evolution processes such that \(\bar{U}_\eta\) satisfies (1) and (1) for each \(\eta \in [0,1]\). Assume also that the associated family of (continuous) evolution processes \(\{U_\eta\}_{\eta \in [0,1]}\) is continuous at \(\eta = 0\), and that the collective sets \(\{M^\eta\}_{\eta \in [0,1]}\) is collectively closed at \(\eta = 0\).

Fix \(s \in \mathbb{R}, x_0 \notin M^0(s), x_k \to x_0\) and \(\eta_k \to 0\). Then

\(\text{(a) if } s_k \to s \text{ then } x_k \notin M^{\eta_k}(s_k) \text{ for } k \text{ sufficiently large;}
\)

(b) if \(s_k \to s\) we have

\[\liminf_{k \to \infty} \phi_{\eta_k}(x_k, s_k) \geq \phi_0(x_0, s),\]

where \(\phi_\eta\) is the impact time map of \(\bar{U}_\eta\), for each \(\eta \in [0,1]\);

(c) if \(\alpha_k, \beta_k \to 0\) with \(\beta_k \leq \alpha_k\) then

\[\bar{U}_{\eta_k}(s + \alpha_k, s + \beta_k)x_k \to x_0\]

\textbf{Proof.} (a) If \(x_k \in M^{\eta_k}(s_k)\), for \(k\) in a subsequence of \(\mathbb{N}\), by the collective closedness at \(\eta = 0\) of \(\{M^\eta\}_{\eta \in [0,1]}\) we obtain \(x_0 \in M^0(s)\), which is a contradiction.

(b) Suppose that, up to subsequences, \(\phi_{\eta_k}(x_k, s_k) \to \alpha\). From the definition of \(\phi_\eta\) we have

\[U_{\eta_k}(s_k + \phi_{\eta_k}(x_k, s_k), s_k) x_k \in M^{\eta_k}(s_k + \phi_{\eta_k}(x_k, s_k))\]

Making \(k \to \infty\), our hypotheses imply that \(U_0(s + \alpha, s)x_0 \in M^0(s + \alpha)\). Thus \(\phi_0(x_0, s) \leq \alpha\), and the result follows.

(c) Using item (b), for \(k\) sufficiently large we obtain

\[0 \leq \alpha_k - \beta_k < \frac{\phi_0(x_0, s)}{2} < \phi_{\eta_k}(x_k, s + \beta_k),\]

and hence

\[\bar{U}_{\eta_k}(s + \alpha_k, s + \beta_k)x_k = \bar{U}_{\eta_k}(s + \beta_k + (\alpha_k - \beta_k), s + \beta_k)x_k = U_{\eta_k}(s + \beta_k + (\alpha_k - \beta_k), s + \beta_k)x_k \to U_0(s, s)x_0 = x_0.\]
Before continuing, a collective version of condition (T) is required, as follows:

Given $s \in \mathbb{R}$, $t > s$, $x \in M^0(t)$, a convergent sequence $\{x_j\}$ and $\eta_j \to 0$ with $U_{\eta_j}(t,s)x_j \to x$, there exist subsequences $\{x_{j_k}\}$ of $\{x_j\}$ and $\{\eta_{j_k}\}$ of $\{\eta_j\}$, and a sequence $\alpha_k \to 0$ with $t + \alpha_k \geq s$

such that $U_{\eta_{j_k}}(t + \alpha_k, s)x_{j_k} \in M^n(t + \alpha_k)$ for all $k$.

Lemma 3.4. Let $\{\hat{U}_\eta\}_{\eta \in [0,1]}$ be a family of impulsive evolution processes satisfying (CT) such that $\hat{U}_\eta$ satisfies (1) and (1) for each $\eta \in [0,1]$. Assume also that the associated family of (continuous) evolution processes $\{U_\eta\}_{\eta \in [0,1]}$ is continuous at $\eta = 0$.

Then if $x_0 \in X$, $s \in \mathbb{R}$, $\eta_k \to 0$ and $x_k \to x_0$ we have

$$\lim_{k \to \infty} \sup_s \phi_{\eta_k}(x_k, s) \leq \phi_0(x_0, s).$$

Proof. If $\phi_0(x_0, s) = \infty$, there is nothing to prove. If $t = \phi_0(x_0, s) < \infty$, then $U_0(s+t, s)x_0 \in M^0(t+s)$ and $U_{\eta_j}(t+s, s)x_j \to U_0(t+s, s)x_0$. If, up to subsequences, we have

$$\phi_{\eta_j}(x_k, s) \to \beta = \lim_{k \to \infty} \sup_s \phi_{\eta_k}(x_k, s),$$

and, using (CT), again up to subsequences, there exists $\alpha_k \to 0$ with $s + t + \alpha_k \geq s$ and

$$U_{\eta_k}(s + t + \alpha_k, s)x_k \in M^n(s + t + \alpha_k)$$

for all $k$.

This implies that $\phi_{\eta_k}(x_k, s) \leq \alpha_k = \phi_0(x_0, s) + \alpha_k$, and making $k \to \infty$ we obtain $\beta \leq \phi_0(x_0, s)$, which concludes the proof.

Theorem 3.5. Let $\{\hat{U}_\eta\}_{\eta \in [0,1]}$ be a family of impulsive evolution processes satisfying (CT) such that $\hat{U}_\eta$ satisfies (1) and (1) for each $\eta \in [0,1]$. Assume also that the associated family of (continuous) evolution processes $\{U_\eta\}_{\eta \in [0,1]}$ is continuous at $\eta = 0$, and that the collection of impulse sets $\{M^n\}_{\eta \in [0,1]}$ is collectively closed at $\eta = 0$.

If $s \in \mathbb{R}$, $x_0 \in M^0(s)$, $x_k \to x_0$ and $\eta_k \to 0$ then

$$\lim_{k \to \infty} \phi_{\eta_k}(x_k, s) = \phi_0(x_0, s).$$

3.2. Convergence of impulsive evolution processes.

Proposition 3.6. Let $\{\hat{U}_\eta\}_{\eta \in [0,1]}$ be a family of impulsive evolution processes satisfying (CT) such that $\hat{U}_\eta$ satisfies (1) and (1) for each $\eta \in [0,1]$. Assume also that the associated family of (continuous) evolution processes $\{U_\eta\}_{\eta \in [0,1]}$ is continuous at $\eta = 0$, that the collection of impulse sets $\{M^n\}_{\eta \in [0,1]}$ is collectively closed at $\eta = 0$ and finally that the collection of impulse functions $\{I^n\}$ is collectively continuous at $\eta = 0$.

If $s \in \mathbb{R}$, $t \geq s$, $x_0 \notin M^0(s)$ and $x_k \to x_0$ then there exists $\varepsilon_k \to 0$ such that $\hat{U}_{\eta_k}(t + \varepsilon_k, s)x_k \to \hat{U}_0(t, s)x_0$.

Proof. If $\phi_0(x_0, s) = \infty$, from Lemma 3.3 item (b), $\phi_{\eta_k}(x_k, s) > t - s$ for $k$ sufficiently large. Thus

$$\hat{U}_{\eta_k}(t, s)x_k = U_{\eta_k}(t, s)x_k \to U_0(t, s)x_0 = \hat{U}_0(t, s)x_0,$$
and the result is proven by taking $\varepsilon_k = 0$ for all $k$ sufficiently large. Now if $\phi_0(x_0, s) < \infty$, using Lemma 3.4, we can assume that $\phi_{\eta_k}(x_k, s) < \infty$ for all $k \in \mathbb{N}$. We treat three distinct cases separately.

**Case 1.** $0 \leq t - s < \phi_0(x_0, s)$.

From Lemma 3.3 (b), for $k$ sufficiently large, we have $t - s < \phi_{\eta_k}(x_k, s)$. Taking $\varepsilon_k = 0$ we have

$$\hat{U}_{\eta_k}(t + \varepsilon_k, s)x_k = \hat{U}_{\eta_k}(t, s)x_k = U_{\eta_k}(t, s)x_k - U_0(t, s)x_0 = \hat{U}_0(t, s)x_0.$$

**Case 2.** $t - s = \phi_0(x_0, s)$.

Setting $t_k = s + \phi_{\eta_k}(x_k, s)$, from Theorem 3.5, we have $t_k \to t$. Then

$$\hat{U}_{\eta_k}(t_k, x)x_k = I_{\eta_k}(U_{\eta_k}(t_k, s)x_k) \to I_z^0(U_0(t, s)x_0) = \hat{U}_0(t, s)x_0,$$

and taking $\varepsilon_k = \phi_{\eta_k}(x_k, s) - \phi_0(x_0, s) = t_k - t$ we obtain $\varepsilon_k \to 0$ and $\hat{U}_{\eta_k}(t + \varepsilon_k, s)x_k = \hat{U}_{\eta_k}(t_k, s)x_k \to \hat{U}_0(t, s)x_0$.

**Case 3.** $t - s > \phi_0(x_0, s)$.

Denote $s_0 = s$ and let $s_1, \ldots, s_m$ be the jump times of $\hat{U}_0$ at $(x_0, s)$ in the interval $[s, t]$. Then $t = s_m + t'$, with $t' \geq 0$. Define $\delta = \min\{s_{i+1} - s_i : i = 0, \ldots, m-1\}$ and

$$2a = \begin{cases} \min\{t', \delta\}, & t' > 0, \\ \delta, & t' = 0. \end{cases}$$

This implies that for $z_0^+ = x_0$, we have $s_{i+1} = s_i + \phi_0(z_i^+, s_i)$, where $z_{i+1} = U_0(s_{i+1}, s_i)z_i^+$ and $z_i^+ = I_{\eta_i}(z_{i+1})$, for $i = 0, \ldots, m-1$. Using Case 2, there exists $\gamma_k \to 0$ such that

$$\hat{U}_{\eta_k}(s_1 + \gamma_k, s_0)x_k \to \hat{U}_0(s_1, s)x_0,$$

and we can assume, without loss of generality, that $|\gamma_k| \leq a$ for all $k \in \mathbb{N}$. Using (l) and Lemma 3.3 (b), we can assume that

$$\frac{\phi_0(\hat{U}_0(s_1, s_0)x, s_1)}{2} < \phi_{\eta_k}(\hat{U}_{\eta_k}(s_1 + \varepsilon_k, s_0)x_k, s_1 + \varepsilon_k),$$

for all $k \in \mathbb{N}$, which implies that

$$a \leq \frac{s_2 - s_1}{2} = \frac{\phi_0(\hat{U}_0(s_1, s_0)x, s_1)}{2} < \phi_{\eta_k}(\hat{U}_{\eta_k}(s_1 + \gamma_k, s_0)x_k, s_1 + \gamma_k).$$

Thus, we obtain

$$\hat{U}_{\eta_k}(s_1 + a, s_0)x_k = \hat{U}_{\eta_k}(s_1 + a, s_1 + \gamma_k + \hat{U}_{\eta_k}(s_1 + \gamma_k, s_0)x_k

= U_{\eta_k}(s_1 + a, s_1 + \gamma_k)\hat{U}_{\eta_k}(s_1 + \gamma_k, s_0)x_k

\to U_0(s_1 + a, s_1)\hat{U}_0(s_1, s)x_0 = \hat{U}_0(s_1 + a, s_1)x_0.$$

Also, from [11, Lemma 2.13], we have

$$\phi_0(\hat{U}_0(s_1 + a, s)x_0, s_1 + a) = \phi_0(U_0(s_1 + a, s_1)\hat{U}_0(s_1, s)x_0, s_1 + a)

= \phi_0(\hat{U}_0(s_1, s)x_0, s_1) - a.$$
If \( m = 1 \), then \( 0 < t' < \phi_0(\tilde{U}_0(s_1,s_0)x_0,s_1) \) and (3) implies that \( 0 < t' - a < \phi_0(\tilde{U}(s_1 + a,s_0)x_0,s_1 + a) \). Using Case 1 we obtain
\[
\tilde{U}_{\alpha}(t' + s_1,s_0)x_k = \tilde{U}_{\alpha}(t' + s_1 + a)\tilde{U}_{\alpha}(s_1 + a,s_0)x_k \\
\rightarrow \tilde{U}_0(t' + s_1 + a)\tilde{U}_0(s_1 + a,s_0)x_0 = \tilde{U}_0(t,s)x_0.
\]
If \( m > 1 \), then \( s_2 = s_1 + \phi_0(\tilde{U}_0(s_1,s_0)x_0,s_1) = s_1 + a + \phi_0(\tilde{U}_0(s_1 + a,s_0)x_0,s_1 + a) \). Then, using Case 2 together with (2) and (1), we obtain \( \varepsilon_k \rightarrow 0 \) such that
\[
\tilde{U}_0(s_2 + \varepsilon_k,s_0)x_k = \tilde{U}_0(s_2 + \varepsilon_k + a)\tilde{U}_0(s_1 + a,s_0)x_k \rightarrow \tilde{U}_0(s_2,s_0)x.
\]
If \( m = 2 \) and \( t' = 0 \), then we are done. If \( m = 2 \) and \( t' > 0 \), we proceed as before and use Case 1. Inductively, if \( m > 2 \) we proceed to obtain \( \varepsilon_k \rightarrow 0 \) such that \( \tilde{U}(s_m + \varepsilon_k,s)x_n \rightarrow \tilde{U}(s_m,s)x_0 \). Again, if \( t' = 0 \) we are done. If \( t' > 0 \) we use Case 1, and the proof is complete. \( \square \)

**Corollary 3.7.** With assumptions of Proposition 3.6, \( \varepsilon_k \) can be taken nonnegative.

**Proof.** From Proposition 3.6, there exists \( \varepsilon_k \rightarrow 0 \) such that
\[
y_k := \tilde{U}_{\alpha}(t + \varepsilon_k,s)x_k \rightarrow \tilde{U}_0(t,s)x_0 =: y_0.
\]
Defining \( \alpha_k = \varepsilon_k + |\varepsilon_k| \) and \( \beta_k = \varepsilon_k \), from Lemma 3.3 (c), since \( y_0 \notin M^0(s) \) (using (1)), we obtain
\[
\tilde{U}_{\alpha_k}(t + \alpha_k,s)x_k = \tilde{U}_{\alpha_k}(t + \alpha_k,t + \beta_k)y_k \rightarrow y_0 = \tilde{U}_0(t,s)x_0,
\]
and the result follows noting that \( \alpha_k \geq 0 \) for all \( k \in \mathbb{N} \). \( \square \)

4. **Upper semicontinuity of pullback attractors.** In this section we deal with the abstract theory of upper semicontinuity for pullback attractors of impulsive evolution processes. We begin with the definition of upper semicontinuity for the impulsive case.

**Definition 4.1.** Let \( \{\tilde{U}_{\eta}\}_{\eta \in [0,1]} \) be a family of impulsive evolution processes such that \( \tilde{U}_{\eta} \) satisfies (1) and has a pullback \( \mathfrak{D} \)-attractor \( \hat{A}_{\eta} \) for each \( \eta \in [0,1] \). We say that the family of pullback \( \mathfrak{D} \)-attractors \( \{\hat{A}_{\eta}\}_{\eta \in [0,1]} \) is **upper semicontinuous at** \( \eta = 0 \) if
\[
\lim_{\eta \rightarrow 0} \text{d}_H(A_{\eta}(t) \setminus M^0(t),A_0(t)) = 0 \quad \text{for each } t \in \mathbb{R}.
\]

**Lemma 4.2.** Let \( \{\tilde{U}_{\eta}\}_{\eta \in [0,1]} \) be a family of impulsive evolution processes such that \( \tilde{U}_{\eta} \) satisfies (1) and has a pullback \( \mathfrak{D} \)-attractor \( \hat{A}_{\eta} \) for each \( \eta \in [0,1] \). The family of pullback \( \mathfrak{D} \)-attractors \( \{\hat{A}_{\eta}\}_{\eta \in [0,1]} \) is upper semicontinuous at \( \eta = 0 \) if and only if given \( t \in \mathbb{R} \), sequences \( \eta_k \rightarrow 0 \) and \( \{x_k\} \) with \( x_k \in A_{\eta_k}(t) \setminus M^0(t) \), then there exists a subsequence of \( \{x_k\} \) that converges to a point in \( A_0(t) \).

**Proof.** If \( \{\hat{A}_{\eta}\}_{\eta \in [0,1]} \) is upper semicontinuous at \( \eta = 0 \), since \( A_0(t) \) is compact for each \( t \in \mathbb{R} \), the conclusion follows easily. Now for the converse, if \( \{\hat{A}_{\eta}\}_{\eta \in [0,1]} \) is not upper semicontinuous at \( \eta = 0 \), then there exist \( t \in \mathbb{R}, \epsilon > 0 \) and sequences \( \eta_k \rightarrow 0 \), \( \{x_k\} \) with \( x_k \in A_{\eta_k}(t) \setminus M^0(t) \) such that
\[
\text{d}_H(x_k,A_0(t)) \geq \epsilon \quad \text{for all } k \in \mathbb{N}.
\]
This contradicts the fact that \( \{x_k\} \) has a subsequence that converges to a point in \( A_0(t) \), and concludes the result. \( \square \)
Lemma 4.3. Let \( \{U_{\eta}\}_{\eta \in [0,1]} \) be a family of impulsive evolution processes satisfying (CT) such that \( U_{\eta} \) satisfies (1), (I) and has a pullback \( \mathcal{D} \)-attractor \( A_\eta \) for each \( \eta \in [0,1] \) with \( \cup_{\eta \in [0,1]} A_\eta(t) \) precompact for each \( t \in \mathbb{R} \). Assume also that the associated family of (continuous) evolution processes \( \{U^\eta\}_{\eta \in [0,1]} \) is continuous at \( \eta = 0 \), that the collection of impulsive sets \( \{M^\eta\}_{\eta \in [0,1]} \) is collectively closed at \( \eta = 0 \), finally that the collection of impulse functions \( \{I^\eta\} \) is collectively continuous at \( \eta = 0 \) and that there exists \( \delta > 0 \) such that

\[
\phi_\eta(z,s) \geq 2\delta \quad \text{for all } z \in I^\eta(M^\eta(s)), \ s \in \mathbb{R} \text{ and } \eta \in [0,1]. \quad \text{(CH)}
\]

If \( s \in \mathbb{R}, \eta_k \to 0, x_k \in A_{\eta_k}(s) \setminus M^0(s) \) for each \( k \in \mathbb{N} \) with \( x_k \to x_0 \in M^0(s) \), then \( \phi_{\eta_k}(x_k, t) \to 0 \).

**Proof.** From Proposition 2.2, for each \( k \in \mathbb{N} \) there exists a global solution \( \xi_k \) of \( U_{\eta_k} \) with \( \xi_k(s) = x_k \) and \( \xi_k \in \mathcal{D} \). Define \( s_k \) as the last jump time of \( U_{\eta_k} \) at \( (\xi_k(s-1), s-1) \) in the interval \([s-1, s+\delta/2]\), when it exists. If there are no jump times of \( U_{\eta_k} \) at \( (\xi_k(s-1), s-1) \) in the interval \([s-1, s+\delta/2]\), we set \( s_k = s-1 \). We can distinguish two cases:

**Case 1.** there exists \( 0 < \varepsilon < \min\left\{ \frac{\delta}{2}, \frac{1}{2} \right\} \) such that, up to a subsequence, \( |s_k-s| \geq 2\varepsilon \).

Using (CH), since \( 2\varepsilon < \delta < 2\delta \), there are no jump times of \( U_{\eta_k} \) at \( (\xi_k(s-1), s-1) \) in the interval \([s-\varepsilon, s+\varepsilon]\). If \( \gamma = \varepsilon/2 \) we have

\[
x_k = \tilde{U}_{\eta_k}(s, s-1)\xi_k(s-1) = \tilde{U}_{\eta_k}(s, s-\gamma)\tilde{U}_{\eta_k}(s-\gamma, s-1)\xi_k(s-1)
= U_{\eta_k}(s, s-\gamma)\tilde{U}_{\eta_k}(s-\gamma, s-1)\xi_k(s-1).
\]

Setting \( y_k = \tilde{U}_{\eta_k}(s-\gamma, s-1)\xi_k(s-1), \) we have \( \{y_k\} \subset \cup_{\eta \in [0,1]} A_\eta(s-\gamma), \) which is precompact. Therefore, up to a subsequence, we can assume that \( y_k \to y_0 \). Summing up, we obtain \( y_k \to y_0 \) and \( U_{\eta_k}(s, s-\gamma)y_k = x_k \to x_0 \in M^0(s) \). Using (CT), there exists \( \alpha_k \to 0 \) with \( \gamma + \alpha_k \geq 0 \) and, up to a subsequence, \( U_{\eta_k}(s+\alpha_k, s-\gamma)y_k \in M^0(s+\alpha_k) \). For \( k \) sufficiently large, this is a contradiction with the fact that there are no jump times of \( U_{\eta_k} \) at \( (\xi_k(s-1), s-1) \) in the interval \([s-\varepsilon, s+\varepsilon]\). Thus, Case 1 does not happen.

**Case 2.** \( s_k \to s \).

In this case we can distinguish two subcases: up to subsequences, either \( s_k \leq s \) for all \( k \) or \( s_k > s \) for all \( k \). We prove that the first is not possible. If \( 0 < \varepsilon < \min\left\{ \frac{\delta}{2}, \frac{\gamma}{2} \right\} \), then \( s_k \) is the only jump time of \( U_{\eta_k} \) at \( (\xi_k(s-1), s-1) \) in the interval \([s-\varepsilon, s+\varepsilon]\). As before, setting \( y_k = \tilde{U}_{\eta_k}(s-\delta, s-1)\xi_k(s-1), \) we can assume that up to a subsequence \( y_k \to y_0 \). Then

\[
x_k = \tilde{U}_{\eta_k}(s, s-1)\xi_k(s-1) = \tilde{U}_{\eta_k}(s, s_k)\tilde{U}_{\eta_k}(s_k, s-\gamma)\tilde{U}_{\eta_k}(s-\gamma, s-1)\xi_k(s-1)
= U_{\eta_k}(s, s_k)\tilde{U}_{\eta_k}(s_k, s-\gamma)y_k = U_{\eta_k}(s, s_k)P^\eta_{s_k}(U_{\eta_k}(s_k, s-\gamma)y_k).
\]

Hence

\[
x_0 \leftarrow x_k = U_{\eta_k}(s, s_k)P^\eta_{s_k}(U_{\eta_k}(s_k, s-\delta)y_k) \to I^0_s(U_0(s, s-\gamma)y_0),
\]

which implies that \( M^0(s) \ni x_0 = I^0_s(U_0(s, s-\gamma)y_0) \in I^0_s(M^0(s)) \) and contradicts (I).

Hence, the only scenario left is \( s_k \to s \) and \( s_k > s \) for all \( k \). As \( s_k \) is the only jump time of \( U \) at \( (\xi_k(s-1), s-1) \) in the interval \([s-\varepsilon, s+\varepsilon]\) and \( x_k = \tilde{U}_{\eta_k}(s, s-1)\xi_k(s-1), \) we have \( U_{\eta_k}(s_k, s)x_k \in M^0(s_k) \) and then \( \phi_{\eta_k}(x_k, s) = s_k - s \to 0. \)
We now prove the upper semicontinuity for a collection of pullback attractors of a family of impulsive evolution processes.

**Theorem 4.4.** Let \( \{\mathcal{U}_n\}_{n \in [0, 1]} \) be a family of impulsive evolution processes satisfying (CT) such that \( \mathcal{U}_n \) satisfies (1), (1) and has a pullback \( \mathcal{D} \)-attractor \( \hat{A}_n \) for each \( n \in [0, 1] \). We assume also that \( \hat{A}_0 \) collectively closed, the associated family of (continuous) evolution processes \( \{\mathcal{U}_n\}_{n \in [0, 1]} \) is continuous at \( n = 0 \), the collection of impulsive sets \( \{\mathcal{M}^n\}_{n \in [0, 1]} \) is collectively closed at \( n = 0 \), the collection of impulse functions \( \{\mathcal{I}^n\} \) is collectively continuous at \( n = 0 \) and satisfies (CH) and that for each \( t \in \mathbb{R} \), there exists \( \gamma_t > 0 \) such that

\[
D(t) = \bigcup_{n \in [0, 1]} \bigcup_{s \in [t - \gamma_t, t + \gamma_t]} \mathcal{A}_n(s) \setminus \mathcal{M}^n(s) \text{ is compact. (4)}
\]

If \( \mathcal{D} \in \mathcal{D} \), the family \( \{\hat{A}_n\}_{n \in [0, 1]} \) is upper semicontinuous at \( n = 0 \), in the sense of Definition 4.1.

**Proof.** Fix \( t \in \mathbb{R} \), and consider sequences \( n_k \to 0 \) and \( \{x_k\} \) with \( x_k \in \mathcal{A}_{n_k}(t) \setminus \mathcal{M}^{n_k}(t) \) for each \( k \in \mathbb{N} \). We are going to prove that there exists a convergent subsequence of \( \{x_k\} \) to a point in \( \mathcal{A}_0(t) \), which together with Lemma 4.2 concludes the result.

Since \( \{x_k\} \subset \bigcup_{n \in [0, 1]} \mathcal{A}_n(t) \setminus \mathcal{M}^n(t) \) for \( k \) sufficiently large, there exist \( x_0 \in X \) and a subsequence of \( \{x_k\} \), which we denote the same, such that \( x_k \to x_0 \). It only remains to prove that \( x_0 \in \mathcal{A}_0(t) \).

From Proposition 2.2, there exists a global solution \( \xi_k \) of \( \hat{U}_{n_k} \) with \( \xi_k(t) = x_k \) and \( \check{\xi}_k \in \mathcal{D} \) for each \( k \in \mathbb{N} \). Now, \( \{\xi_k(t - 1)\} \subset \bigcup_{n \in [0, \gamma_{t-1}]} \mathcal{A}_n(t - 1) \setminus \mathcal{M}^n(t - 1) \) for \( k \) sufficiently large. Hence there exist a infinite subset \( N_1 \) of \( \mathbb{N} \) and a point \( x_{-1} \in X \) with

\[
\lim_{N_1 \ni k \to \infty} \xi_k(t - 1) = x_{-1}.
\]

But \( \{\xi_k(t - 2)\}_{k \in N_1} \subset \bigcup_{n \in [0, \gamma_{t-2}]} \mathcal{A}_n(t - 2) \setminus \mathcal{M}^n(t - 2) \) for \( k \) sufficiently large, and thus there exist a infinite subset \( N_2 \) of \( N_1 \) and a point \( x_{-2} \in X \) with

\[
\lim_{N_2 \ni k \to \infty} \xi_k(t - 2) = x_{-2}.
\]

Inductively, we can construct a subsequence of \( \{\xi_k\} \), which we denote the same for simplicity, and a sequence \( \{x_{-m}\}_{m \in \mathbb{N}} \) in \( X \) such that

\[
\lim_{k \to \infty} \xi_k(t - m) = x_{-m} \text{ for all } m \in \mathbb{N}.
\]

We split the prove into two cases.

**Case 1.** \( x_0 \notin \mathcal{M}^0(t) \), and in this case we consider two subcases.

**Subcase 1.** There exists an strictly increasing sequence \( \{m_j\} \) of positive integers such that \( x_{-m_j} \notin \mathcal{M}^0(t - m_j) \) for all \( j \in \mathbb{N} \).

In this subcase, using Corollary 3.7, for each \( j \in \mathbb{N} \) there exists a sequence of nonnegative real numbers \( \{\varepsilon'_k\}_{k \in \mathbb{N}} \) with \( \varepsilon'_k \to 0 \) as \( k \to \infty \) such that

\[
\lim_{k \to \infty} \hat{U}_{n_k}(t - m_j + \varepsilon'_k, t - m_j + 1) \xi_k(t - m_j + 1) = \hat{U}_0(t - m_j, t - m_j + 1)x_{-m_j+1}.
\]

Since

\[
\hat{U}_{n_k}(t - m_j + \varepsilon'_k, t - m_j + 1) \xi_k(t - m_j + 1) = \hat{U}_{n_k}(t - m_j + \varepsilon'_k, t - m_j) \xi_k(t - m_j),
\]

it follows from item (c) of Lemma 3.3 that

\[
\lim_{k \to \infty} \hat{U}_{n_k}(t - m_j + \varepsilon'_k, t - m_j + 1) \xi_k(t - m_j + 1) = x_{-m_j},
\]
therefore we conclude that $\tilde{U}_0(t - m_j, t - m_{j+1})x_{-m_{j+1}} = x_{-m_j}$.

Setting $m_0 = 0$, we define $\xi_0 : \mathbb{R} \to X$ as

$$
\xi_0(r) = \begin{cases} 
\tilde{U}_0(r, t)x_0, & r \geq t, \\
\tilde{U}_0(r, t - m_j)x_{-m_j}, & r \in [t - m_j, t - m_{j-1}], j \in \mathbb{N}.
\end{cases}
$$

It is clear that $\xi_0$ is a global solution of $\tilde{U}_0$ with $\xi_0(t) = x_0$. If we prove that $\xi_0 \in \mathcal{D}$, then from Proposition 2.2 it follows that $x_0 \in A_0(t) \setminus M^0(t) \subset A_0(t)$, and this subcase will be complete.

To prove that $\xi_0 \in \mathcal{D}$, first consider $r \geq t$. Since $x_0 \not\in M^0(t)$, Corollary 3.7 implies that there exists a sequence $\{\varepsilon_k\}$ of nonnegative real numbers with $\varepsilon_k \to 0$ such that $\tilde{U}_{\eta_k}(r + \varepsilon_k, t)x_k \to \tilde{U}_0(r, t)x_0 = \xi_0(r)$. But $x_k \in A_{\eta_k}(t) \setminus M^0(t)$, hence $\tilde{U}_{\eta_k}(r + \varepsilon_k, t)x_k \in A_{\eta_k}(r + \varepsilon_k) \setminus M^0(r + \varepsilon_k)$ (from the $\tilde{U}_{\eta_k}$-invariance of $A_{\eta_k} \setminus M^0$), and therefore $\xi_0(r) \in \cup_{\eta \in [0, \gamma]} \cup_{r \in [r - \gamma, r + \gamma]} A_{\eta}(s) \setminus M^0(s) = D(r)$.

Now if $r \in [t - m_j, t - m_{j-1})$ for some $j \in \mathbb{N}$, Corollary 3.7 implies that there exists a sequence $\{\varepsilon_k\}$ of nonnegative real numbers with $\varepsilon_k \to 0$ such that

$$
\tilde{U}_{\eta_k}(r + \varepsilon_k, t - m_j)\xi_k(t - m_j) \to \tilde{U}_0(r, t - m_j)x_{-m_j} = \xi_0(r),
$$

and using the same reasoning as above we conclude that $\xi_0(r) \in D(r)$. Since $\tilde{D} \in \mathcal{D}$ and $\mathcal{D}$ is a universe, we obtain $\xi_0 \in \mathcal{D}$.

**Subcase 2.** There exists $m_0 \in \mathbb{N}$ such that $x_{-m} \in M^0(t - m)$ for all $m \geq m_0$.

We have $\xi_k(t - m) \to x_{-m}$ for all $m \geq m_0$. Using Lemma 4.3, up to a subsequence we obtain $s_{k, m} := \phi_{\eta_k}(\xi_k(t - m), t - m) \to 0$ for all $m \geq m_0$. Fix $m > m_0$, $\beta \in (0, \min\{\frac{4}{7}, \frac{1}{4}\})$ and define $w_{k, m} = U_{\eta_k}(t - m + s_{k, m}, t - m)\xi_k(t - m) \in M^0(t - m + s_{k, m})$. Thus

$$
\lim_{k \to \infty} w_{k, m} = \lim_{k \to \infty} U_{\eta_k}(t - m + s_{k, m}, t - m)\xi_k(t - m) = x_{-m} \in M^0(t - m),
$$

and hence $I_{t-m}^{\eta_k}(w_{k, m}) \to I_{t-m}^{\eta_k}(x_{-m}) \not\in M^0(t - m)$, with $t_{k, m} = t - m + s_{k, m}$, using the collectively continuity of the family of impulse functions and condition (I).

From item (b) of Lemma 3.3 and the fact that $s_{k, m} \to 0$ as $k \to \infty$, without loss of generality we can assume that for all $k \in \mathbb{N}$ we have

$$
\phi_{\eta_k}(I_{t_k}^{\eta_k}(w_{k, m}), t - m + s_{k, m}) \geq \phi_0(I_{t-m}^{\eta_k}(x_{-m}), t - m) \geq \delta > \beta > s_{k, m}.
$$

Therefore

$$
\xi_k(t - m + \beta) = \tilde{U}_{\eta_k}(t - m + \beta, t - m + s_{k, m})\xi_k(t - m + s_{k, m}) = \tilde{U}_{\eta_k}(t - m + \beta, t - m + s_{k, m})I_{t_k}^{\eta_k}(w_{k, m}) = \tilde{U}_{\eta_k}(t - m + \beta, t - m + s_{k, m})I_{t_k}^{\eta_k}(w_{k, m}),
$$

which converges to

$$
U_0(t - m + \beta, t - m)I_{t-m}^{\eta_k}(x_{-m}) = \tilde{U}_0(t - m + \beta, t - m)I_{t-m}^{\eta_k}(x_{-m}) \not\in M^0(t - m + \beta).
$$

Now we can repeat the construction in the beginning of the proof, considering the sequence $\{\xi_k(t - m + \beta)\}_{k \in \mathbb{N}}$ instead of $\{\xi_k(t - m)\}$ to construct a sequence $\{x_{-m}\}$ such that $x_{-m} \not\in M^0(t - m + \beta)$ for all $m > m_0$, and use analogously the
construction of Subcase 1 to obtain a global solution ξ_0 of \( \bar{U}_0 \) with \( ξ_0(t) = x_0 \) and \( \xi_0 \in \mathcal{D} \).

**Case 2.** \( x_0 \in M^0(t) \).

Using Lemma 4.3 we have \( φ_{η_k}(x_k, t) \to 0 \), and we can assume without loss of generality that \( 0 < φ_{η_k}(x_k, t) < \frac{1}{2} \) for all \( k \in \mathbb{N} \). Furthermore, there exists \( ε = ε(x_0, t) > 0 \) such that

\[
\bigcup_{r \in (0, ε)} F_0(x_0, r, t - r) \cap M^0(t - r) = \emptyset. \tag{5}
\]

Choose \( m_0 \in \mathbb{N} \) such that \( 1/m_0 < \min \{ ε, \frac{δ}{2} \} \), fix \( m \geq m_0 \) and define

\[ w_{k,m} = ξ_k(t - 1/m) \in A_{η_k}(t - 1/m) \setminus M^0(t - 1/m) \text{ for each } k \in \mathbb{N}. \]

Since \( \cup_{η \in [0, γ_{t-1/m}]} A_{η}(t - 1/m) \) is precompact, we can assume that \( w_{k,m} \to w^0 \) as \( k \to ∞. \)

We claim that there exists \( k_1 \in \mathbb{N} \) such that \( s_{k,m} = φ_{η_k}(w_{k,m}, t - \frac{1}{m}) > \frac{1}{m} \) for \( k \geq k_1 \). In fact, if this is not true, there exists a subsequence, which we denote the same, such that \( s_{k,m} \leq 1/m \) for all \( k \in \mathbb{N} \). Hence, up to a subsequence, we can assume that \( s_{k,m} \to α \in [0, 1/m] \). In this way

\[ z_{k,m} = U_{η_k}(t - \frac{1}{m} + s_{k,m}, t - \frac{1}{m})w_{k,m} \to U_0(t - \frac{1}{m} + α, t - \frac{1}{m})w^0 =: v^0_m. \]

Since \( z_{k,m} \in M^0(t - \frac{1}{m} + s_{k,m}) \), we conclude that \( z^0_m \in M^0(t - \frac{1}{m} + α) \). This implies that

\[ v_{k,m} := I^0_{t-1/m+s_{k,m}}(z_{k,m}) \to I^0_{t-1/m+α}(z^0_m) =: v^0_m. \]

By hypothesis, \( δ > 0 \) and Lemma 3.3 implies that for \( k \) sufficiently large we have

\[ φ_{η_k}(v_{k,m}, t - \frac{1}{m} + s_{k,m}) \geq \frac{1}{2} φ_{η_k}(v^0_m, t - \frac{1}{m} + α) \geq δ > \frac{1}{m}. \]

Thus we obtain

\[ U_{η_k}(t + φ_{η_k}(x_k, t), t - \frac{1}{m} + s_{k,m})w_{k,m} = U_{η_k}(t + φ_{η_k}(x_k, t), t - \frac{1}{m} + s_{k,m})v_{k,m} \]

\[ = U_{η_k}(t + φ_{η_k}(x_k, t), t - \frac{1}{m} + s_{k,m})U_{η_k}(t - \frac{1}{m} + s_{k,m})ζ_k(t - \frac{1}{m})w_{k,m} \]

\[ = U_{η_k}(t + φ_{η_k}(x_k, t), t - \frac{1}{m} + s_{k,m})U_{η_k}(t - \frac{1}{m} + s_{k,m})ζ_k(t - \frac{1}{m})w_{k,m} \]

\[ = U_{η_k}(t + φ_{η_k}(x_k, t), t - \frac{1}{m} + s_{k,m})U_{η_k}(t - \frac{1}{m} + s_{k,m})ζ_k(t - \frac{1}{m})w_{k,m} \]

\[ = U_{η_k}(t + φ_{η_k}(x_k, t), t - \frac{1}{m} + s_{k,m})U_{η_k}(t - \frac{1}{m} + s_{k,m})ζ_k(t - \frac{1}{m})w_{k,m} \]

where in \((*)\) we used the fact that \( \frac{1}{m} - s_{k,m} < \frac{1}{2} < 2δ \) and \( U_{η_k}(t - \frac{1}{m} + s_{k,m}, t - \frac{1}{m})w_{k,m} \in I^0_{t-1/m+s_{k,m}}(M^0(t - \frac{1}{m} + s_{k,m})) \). This implies that

\[ φ_{η_k}(v_{k,m}, t - \frac{1}{m} + s_{k,m}) < φ_{η_k}(x_k, t) + \frac{1}{m} - φ_{η_k}(w_{k,m}, t - \frac{1}{m}) \]

\[ < φ_{η_k}(x_k, t) + \frac{1}{m} < \frac{δ}{2} + \frac{δ}{2} = δ, \]

which contradicts the fact that \( φ_{η_k}(v_{k,m}, t - \frac{1}{m} + s_{k,m}) > δ \), and proves the claim.

From this claim, for \( k \geq k_1 \), we have

\[ U_{η_k}(t, t - \frac{1}{m})w_{k,m} = U_{η_k}(t, t - \frac{1}{m})w_{k,m} = U_{η_k}(t, t - \frac{1}{m})ζ_k(t - \frac{1}{m}) = ζ_k(t) = x_k, \tag{6} \]
and since \( \phi_{\eta_k}(w_{k,m}, t - \frac{1}{m}) > \frac{1}{m} \), we have \( U_{\eta_k}(t, t - \frac{1}{m})w_{k,m} \notin M^{\eta_k}(t) \). Making \( k \to \infty \) in (6) we obtain \( U_{\eta_k}(t, t - 1/m)w_{m}^0 = x_0 \). This holds for each \( m \geq m_0 \).

Now, since \( \frac{1}{m} < \varepsilon \), (5) implies that \( w_{m}^0 \notin M^0(\varepsilon - 1/m) \) and \( w_{k,m} \to w_{m}^0 \). Using Case 1 we conclude that \( w_{m}^0 \in A_0(\varepsilon - 1/m) \setminus M^0(\varepsilon - 1/m) \).

The sequence \( \{w_{m}^0\} \), for \( m \) sufficiently large, is contained in \( D(t) \), and we can assume that up to a subsequence \( w_{m}^0 \to w^0 \) as \( m \to \infty \). Since \( U_0(t, t - 1/m)w_{m}^0 = x_0 \) for all \( m \geq m_0 \), making \( m \to \infty \) we obtain \( w^0 = x_0 \), that is, \( w_{m}^0 \to x_0 \). Since \( w_{m}^0 \in A_0(\varepsilon - 1/m) \) and \( A_0 \) is collectively closed, \( x_0 \in A_0(t) \). \( \square \)

**Remark 4.5.** Considering the special universe \( D_\varepsilon \) of **union bounded families**, which consists of families \( D = \{D(t)\}_{t \in \mathbb{R}} \) with \( \cup_{t \in \mathbb{R}} D(t) \) bounded in \( X \), condition (4) of Theorem 4.4 can be replaced by

\[
\overline{\cup_{t \in [0,1]} \cup_{t \in \mathbb{R}} A(t) \setminus M(t)} \quad \text{is compact.}
\]

5. **Uniform perturbations of a nonautonomous integrate-and-fire neuron model.** The nonautonomous integrate-and-fire neuron model, considered in [28] and more recently in [12], models an accumulation of potential in a neuron cell, and when this potential reaches a given threshold, it fires the energy to another neuron. In the nonautonomous context, the model is described in [11, Example 3.1]. Here we consider small uniform nonautonomous perturbations of such a model. That is, we consider for each \( \eta \in [0,1] \), the problem

\[
  u'(t) = -\gamma_\eta(t)u(t) + S_\eta(t),
\]

with the condition

\[
  \text{if } u(t) = \theta_\eta(t) \text{ then } u(t) \text{ resets to } u_{r,\eta}(t) < \theta_\eta(t).
\]

**Remark 5.1.** According to [28], this integrate-and-fire model describe leaky, current-clamped membranes in terms of a state variable \( u(t) \) (membrane potential), a dissipation \( \gamma_\eta(t) \), an applied stimulus \( S_\eta(t) \), a firing threshold \( \theta_\eta(t) \) and a reset state \( u_{r,\eta}(t) \).

We assume the following:

- (B1) \( \gamma_\eta : \mathbb{R} \to (0, \infty) \) is continuous for each \( \eta \in [0,1] \), with
  \[
  \lim_{\eta \to 0} \sup_{t \in \mathbb{R}} |\gamma_\eta(t) - \gamma_0(t)| = 0,
  \]

  and

  \[
  \gamma_{0-} = \inf_{t \in \mathbb{R}} \gamma_0(t) > 0, \quad \gamma_{0+} = \sup_{t \in \mathbb{R}} \gamma_0(t) < \infty;
  \]

- (B2) \( S_\eta : \mathbb{R} \to (0, \infty) \) is continuous for each \( \eta \in [0,1] \), with
  \[
  \lim_{\eta \to 0} \sup_{t \in \mathbb{R}} |S_\eta(t) - S_0(t)| = 0;
  \]

  and

  \[
  S_{0+} = \sup_{t \in \mathbb{R}} S_0(t) < \infty;
  \]

- (B3) \( \theta_\eta : \mathbb{R} \to (0, \infty) \) is continuously differentiable for each \( \eta \in [0,1] \), with
  \[
  \lim_{\eta \to 0} \sup_{t \in \mathbb{R}} (|\theta_\eta(t) - \theta_0(t)| + |\theta_\eta(t) - \theta_0(t)|) = 0.
  \]
Also
\[ \theta_0^+ = \sup_{t \in \mathbb{R}} \theta_0(t) < \infty \]
and there exists \( \varepsilon > 0 \) such that for all \( t \in \mathbb{R} \) we have
\[ |\theta_0'(t) + \gamma_0(t)\theta_0(t) - S_0(t)| > \varepsilon; \]
(B4) \( u_{r,\eta} : \mathbb{R} \to [0, \infty) \) is continuous for each \( \eta \in [0, 1] \), with
\[ \lim_{\eta \to 0} \sup_{t \in \mathbb{R}} |u_{r,\eta}(t) - u_{r,0}(t)| = 0, \]
and there exist \( a, \delta > 0 \) such that for all \( s \in \mathbb{R} \) and \( \mu \in [0, \delta] \) we have
\[ u_{r,0}(s) + a < \theta_0(s + \mu). \]

For each \( \eta \in [0, 1] \) the initial value problem
\[
\begin{cases}
  u'(t) = -\gamma_0(t)u(t) + S_\eta(t), & \text{for } t > s, \\
  u(s) = z \in \mathbb{R},
\end{cases}
\]
can be solved explicitly, and gives us the evolution process \( U_\eta = \{U_\eta(t,s) : (t,s) \in \mathcal{P}\} \), where \( U_\eta \) is given by
\[ U_\eta(t,s)z = ze^{-\int_s^t \gamma_\eta(v)dv} + \int_s^t S_\eta(x)e^{-\int_s^x \gamma_\eta(v)dv}dx, \]
for \( t > s \) and \( z \in \mathbb{R} \). We define \( M_\eta(t) = \{\theta_\eta(t)\} \) and \( I^n_\eta(\theta_\eta(t)) = u_{r,\eta}(t) \) for each \( t \in \mathbb{R} \) and \( \eta \in [0, 1] \).

We use some of the results of [11], and to that end, we must be able to prove that conditions (A1)-(A4) of their work (see [11, Subsection 3.1]) are satisfied. To that end, using (B1)-(B4), we can find \( \eta_1 \in (0, 1) \) such that for all \( \eta \in [0, \eta_1] \) we have
\[ \left\{ \begin{array}{l}
  \sup_{t \in \mathbb{R}} |\gamma_\eta(t) - \gamma_0(t)| < \frac{\gamma_0^-}{2}, \\
  \sup_{t \in \mathbb{R}} |S_\eta(t) - S_0(t)| < \frac{S_0^+}{2}, \\
  \sup_{t \in \mathbb{R}} |\theta_\eta(t) - \theta_0(t)| < \min \left\{ \frac{\theta_0^+}{2}, \frac{a}{4} \right\}, \\
  \sup_{t \in \mathbb{R}} |u_{r,\eta}(t) - u_{r,0}(t)| < \frac{a}{2}, \\
  \text{and} \quad \sup_{t \in \mathbb{R}} |\theta_\eta'(t) + \gamma_\eta(t)\theta_\eta(t) - S_\eta(t) - \theta_0'(t) - \gamma_0(t)\theta_0(t) + S_0(t)| < \frac{\varepsilon}{2}, \end{array} \right. \]
(10)

**Proposition 5.2.** Assume that (B1)-(B4) hold and \( \eta_1 \) is as in (10). Then \( \hat{U}_\eta = (U_\eta, \mathbb{R}, M_\eta, I^n) \) satisfies conditions (A1)-(A4) of [11, Subsection 3.1] for each \( \eta \in [0, \eta_1] \).

**Proof.** **Proof of (A1).** This condition is trivial for each \( \eta \in [0, 1] \).

**Proof of (A2).** For \( \eta \in [0, \eta_1] \), defining \( k_\eta : \mathbb{R} \to (0, \infty) \) by
\[ k_\eta(t) = \int_0^\infty S_\eta(t-u)e^{-\int_u^t \gamma_\eta(v)dv}du, \]
then for each \( t \in \mathbb{R} \) we have
\[ k_\eta(t) \leq \int_0^\infty 3S_0^+e^{-\frac{\gamma_0^-}{2}u}du = \frac{3S_0^+}{\gamma_0} < \infty, \]
and
\[ k_\eta(s)e^{-\int_s^t \gamma_\eta(v)dv} \leq \frac{3S_0^+}{\gamma_0}e^{-\frac{\gamma_0^-}{2}(t-s)} \to 0 \quad \text{as } s \to -\infty. \]
Furthermore,
\[ \int_0^\tau S_\eta(t-u)e^{-\int_0^u \gamma_0(v)dv}du \leq \frac{3S_0^+\tau}{2} \to 0 \quad \text{as} \quad \tau \to 0^+. \]

Proof of (A3). For \( \eta \in [0, \eta_1] \) and \( t \in \mathbb{R} \) we have
\[ \theta_\eta(s)e^{-\int_0^t \gamma_0(v)dv} \leq \frac{3\theta_0^+}{2}e^{-\frac{\gamma_0}{2}(t-s)} \to 0 \quad \text{as} \quad s \to -\infty, \]
and for all \( t \in \mathbb{R} \) and \( \eta \in [0, \eta_1] \) we obtain
\[ |\theta'_\eta(t) + \gamma_\eta(t)\theta_\eta(t) - S_\eta(t)| > \frac{\varepsilon}{2} > 0, \]
and, in particular, \( \theta'_\eta(t) + \gamma_\eta(t)\theta_\eta(t) - S_\eta(t) \neq 0. \)

Proof of (A4). For \( \eta \in [0, \eta_1] \) we have
\[ u_{r,\eta}(s) + \frac{a}{2} \leq u_{r,\eta}(s) + a < \theta_\eta(s + \mu) \leq \theta_\eta(s + \mu) + \frac{a}{4}, \]
and hence for all \( s \in \mathbb{R} \) and \( \mu \in [0, \delta] \) we obtain
\[ u_{r,\eta}(s) + \frac{a}{4} \leq \theta_\eta(s + \mu). \]

From [11, Propositions 3.3, 3.4 and 3.5] we obtain directly the following result.

**Corollary 5.3.** Assume that (B1)-(B4) hold and \( \eta_1 \) is as in (10). Then \( \hat{U}_\eta = (\hat{U}_\eta, \mathbb{R}, \hat{M}_\eta, \hat{I}^\eta) \) defines an impulsive evolution process satisfying (I), (H) and (T) for each \( \eta \in [0, \eta_1] \).

In this example, we consider the universe of union bounded families \( \mathcal{D}_b \), consisting of families \( \hat{D} = \{D(t)\}_{t \in \mathbb{R}} \) with \( \cup_{t \in \mathbb{R}} D(t) \) bounded in \( \mathbb{R} \). Thus, we must modify slightly the proof of [11, Proposition 3.6], in order to obtain a pullback \( \mathcal{D}_b \)-attractor for \( \hat{U}_\eta \), since they consider a different universe \( \mathcal{D} \).

**Proposition 5.4.** Assume that (B1)-(B4) hold and \( \eta_1 \) is as in (10). Then \( \hat{U}_\eta \) is pullback \( \mathcal{D}_b \)-asymptotically compact for each \( \eta \in [0, \eta_1] \), and there exists \( \omega > 0 \) such that the interval \([-\omega, \omega]\) is a pullback \( \mathcal{D}_b \)-absorbing family for \( \hat{U}_\eta \) for all \( \eta \in [0, \eta_1] \).

**Proof.** If \( \hat{D} \in \mathcal{D}_b \) there exists \( M > 0 \) such that \( \cup_{t \in \mathbb{R}} D(t) \subset [-M, M] \). For \( z \in D(s) \) and \( t \geq s \) we then have
\[ |U(t, s)z| \leq Me^{-\frac{\gamma_0}{2}(t-s)} + \frac{3S_0^+}{\gamma_0} \left(1 - e^{-\frac{\gamma_0}{2}(t-s)}\right) \to \frac{3S_0^+}{\gamma_0} \quad \text{as} \quad s \to -\infty. \]

Hence, for \( \omega = 2\max\{\theta_0^+, \frac{3S_0^+}{\gamma_0}\} \), the interval \([-\omega, \omega]\) is pullback \( \mathcal{D}_b \)-absorbing family. It follows directly from [11, Proposition 2.23] that \( \hat{U}_\eta \) is pullback \( \mathcal{D}_b \)-asymptotically compact.

With these results, we obtain from Theorem 2.1 the following:

**Corollary 5.5.** Assume that (B1)-(B4) hold, \( \eta_1 \) is as in (10) and \( \omega \) as in Proposition 5.4. Then for each \( \eta \in [0, \eta_1] \) the impulsive evolution process \( \hat{U}_\eta = (\hat{U}_\eta, \mathbb{R}, \hat{M}_\eta, \hat{I}^\eta) \) has a collectively compact pullback \( \mathcal{D}_b \)-attractor \( \hat{A}_\eta \), with \( \hat{A}_\eta(t) \subset [-\omega, \omega] \) for each \( t \in \mathbb{R} \).

Now we verify that the conditions of Theorem 4.4 hold.
Proof. We can choose $\alpha > 0$ such that for $\lambda \in [0, \alpha]$ we have
\[
\int_0^\lambda S_\eta(s + \lambda - u)\gamma(s)\,du \leq \frac{3S_0^+\lambda}{2} \leq \frac{a}{4}
\]
for all $s \in \mathbb{R}$ and $\eta \in [0, \eta_1]$. Hence
\[
U_\eta(s + \lambda, s) u_{r, \eta}(s) = u_{r, \eta}(s) e^{-\int s + \lambda - u\gamma(s)\,du} + \int_s^{s + \lambda} S_\eta(x)e^{-\int s + \lambda - u\gamma(s)\,du}\,dx
\]
\[
\leq u_{r, \eta}(s) + \int_0^\lambda S_\eta(s + \lambda - u)\gamma(s)\,du \leq u_{r, \eta}(s) + \frac{a}{4},
\]
for all $s \in \mathbb{R}$. Therefore, if $0 \leq \lambda \leq \min\{\delta, \alpha\}$ we obtain $U_\eta(s + \lambda, s) u_{r, \eta}(s) < \theta_\eta(s + \lambda)$ (see the proof of Proposition 5.2), and hence $\phi(u_{r, \eta}(s), s) > \min\{\delta, \alpha\} > 0$ for all $s \in \mathbb{R}$. The result is proved by taking $\xi = \frac{\min\{\delta, \alpha\}}{2} > 0$. \hfill \Box

Before continuing, we note that if $\eta_k \to 0$, $t_k \to t$, then (B3) and (B4) imply that
\[
\theta_{\eta_k}(t_k) \to \theta_0(t) \quad \text{and} \quad u_{r, \eta_k}(t_k) \to u_{r, 0}(t),
\]
which means that the family $\{M_\eta\}_{\eta \in [0, 1]}$ is collectively closed at $\eta = 0$, the family $\{P_\eta\}_{\eta \in [0, 1]}$ is collectively continuous at $\eta = 0$. Moreover, given $\epsilon > 0$ we have
\[
\sup_{\eta \in [0, 1]} |\gamma_\eta(t) - \gamma_0(t)| < \epsilon \quad \text{and} \quad \sup_{t \in \mathbb{R}} |S_\eta(t) - s_0(t)| < \epsilon
\]
for $\eta$ sufficiently small, and from (9) we obtain
\[
|U_\eta(t, s)x - U_0(t, s)x| \leq |x|(e^{\epsilon(t-s)} - 1) + \epsilon(t-s) + S_0^+(t-s)(e^{\epsilon(t-s)} - 1),
\]
and for $(t, s, x)$ in compact subsets of $\mathcal{P} \times \mathbb{R}$, we obtain $U_\eta(t, s)x \to U_0(t, s)x$, which shows that the family $\{U_\eta\}_{\eta \in [0, 1]}$ of continuous evolution processes is continuous at $\eta = 0$.

Hence, to apply Theorem 4.4, it only remains to show that condition (CT) holds.

Proposition 5.7. Assume that (B1)-(B4) hold then the family $\{U_\eta\}_{\eta \in [0, 1]}$ satisfies condition (CT).

Proof. We fix $t > s$, $\eta_j \to 0$ and $z_j \to z$ such that $U_{\eta_j}(t, s)z_j \to \theta_0(t)$ (and hence $U_0(t, s)z = \theta_0(t)$). We have to prove that there exist subsequences $\{\eta_jk\}$ of $\{\eta_j\}$ and $\{z_jk\}$ of $\{z_j\}$, and a sequence $\{\alpha_k\}$ such that $t + \alpha_k \geq s$, $\alpha_k \to 0$ and $U_{\eta_jk}(t + \alpha_k, s)z_{jk} = \theta_{\eta_jk}(t + \alpha_k)$ for each $k$.

By (B3) either $\theta'_0(t) + \gamma_0(t)\theta_0(t) + S_0(t) > 0$ for all $t \in \mathbb{R}$ or $\theta'_0(t) + \gamma_0(t)\theta_0(t) + S_0(t) < 0$ for all $t \in \mathbb{R}$. We assume that the first case holds (the second case is proved analogously). Define $f(r) = U_0(r, s)z - \theta_0(r)$ for $r \in [s, \infty)$. Thus $f$ is differentiable, $f(t) = 0$ and
\[
f'(r) = \frac{d}{dr}(U_0(r, s)z - \theta_0(r)) = -\gamma_0(r)U_0(r, s)z + S_0(r) - \theta'_0(r).
\]
Thus $f'(t) = -\gamma_0(t)u_0(t, s)z + S_0(t) - \theta'_0(t) = -\gamma(t)\theta_0(t) + S_0(t) - \theta'_0(t) < 0$. This implies that there exists $\lambda_0 > 0$ such that $t - s > \lambda_0$ and
\[
U_0(t - \lambda, s)z > \theta_0(t - \lambda) \quad \text{and} \quad U_0(t + \lambda, s)z < \theta_0(t + \lambda),
\]
for all $\lambda \in (0, \lambda_0]$. Then, for $j$ sufficiently large we have $U_{n_j}(t - \lambda_0, s)z_j > \theta_{n_j}(t - \lambda_0)$ and $U_{n_j}(t + \lambda_0, s)z_j < \theta_{n_j}(t + \lambda_0)$. So there exists $\alpha_j \in [-\lambda_0, \lambda_0]$ such that $U_{n_j}(t + \alpha_j, s)z_j = \theta_{n_j}(t + \alpha_j)$.

It only remains to show that $\alpha_j \to 0$ (up to a subsequence). Since $\alpha_j \in [-\epsilon_0, \epsilon_0]$, we have $\alpha_j \to \tilde\alpha$, up to a subsequence, which we call the same. Since $\{M_\eta\}_{\eta \in [0,1]}$ is collectively closed at $\eta = 0$, this implies that $U_0(t + \tilde\alpha, s)z = \theta_0(t + \tilde\alpha)$. However since $U_0(t + \lambda, s)z \neq \theta_0(t + \lambda)$ for all $\lambda \in [-\lambda_0, \lambda_0] \setminus \{0\}$ we must have $\tilde\alpha = 0$. □

Hence all the conditions of Theorem 4.4 are verified and the family $\{\hat{A}_\eta\}_{\eta \in [0,1]}$ of pullback $\mathcal{D}$-attractors is upper semicontinuous at $\eta = 0$, in the sense of Definition 4.1.

The particular case when $\gamma_0, S_0, \theta_0$ and $u_{r,0}$ are positive constants is interesting (see [9]). In this case, the model is autonomous at $\eta = 0$, and the evolution process $\mathcal{U}_0$ defines a semigroup $T_0$ given by $T_0(t) = U_0(t, 0)$ for all $t \geq 0$. In this case, the pullback $\mathcal{D}$-attractor of the impulsive semigroup $T_0 = (T_0, \mathbb{R}, M_0, I^0)$ is actually a single set $A_0$. Moreover, when $S_0 < \gamma_0 \theta_0$, $A_0 = \{\frac{S_0}{\gamma_0}\}$ and when $S_0 > \gamma_0 \theta_0$ we have $A_0 = \{\frac{S_0}{\gamma_0}\} \cup [u_{r,0}, \theta_0]$. If we are in the second case, from the particular definition of $M_\eta$ in our model, we obtain $d_H(A_\eta(t), A_0) \to 0$ as $\eta \to 0$.

6. Weak-lower semicontinuity of pullback attractors. In this section we consider the problem of weak-lower semicontinuity in the abstract framework, which although similar, is considerably harder than the upper semicontinuity. We deal with the following concept of weak-lower semicontinuity:

**Definition 6.1.** Let $\{\hat{U}_\eta\}_{\eta \in [0,1]}$ be a family of impulsive evolution processes such that $\hat{U}_\eta$ satisfies (1) and has a pullback $\mathcal{D}$-attractor $\hat{A}_\eta$ for each $\eta \in [0,1]$. We say that the family of pullback $\mathcal{D}$-attractors $\{\hat{A}_\eta\}_{\eta \in [0,1]}$ is weak-lower semicontinuous at $\eta = 0$ if given $t \in \mathbb{R}$ and $x_0 \in A_0(t)$ there exist sequences $\eta_k, \varepsilon_k \geq 0$ convergent to zero and $x_k \in A_{\eta_k}(t + \varepsilon_k) \setminus M^n(t + \varepsilon_k)$ such that $x_k \to x_0$.

**Remark 6.2.** The usual notion of lower semicontinuity at $\eta = 0$ for a family $\{\hat{A}_\eta\}_{\eta \in [0,1]}$ would be translated to the impulsive case ideally as

$$d_H(A_\eta(t), A_\eta(t) \setminus M^n(t)) = 0 \quad \text{for each } t \in \mathbb{R},$$

and is equivalent to property that given $t \in \mathbb{R}$ and $x_0 \in A_0(t)$ there exist sequences $\eta_k \to 0$ and $x_k \in A_{\eta_k}(t) \setminus M^n(t)$ such that $x_k \to x_0$ (analogous to Lemma 4.2 to the upper semicontinuity).

However, as usual in the impulsive nonautonomous case, convergence with fixed final time $t$ is not always possible (see Corollary 3.7), and hence the need of the corrective sequence $\{\varepsilon_k\}$ in Definition 6.1. Although this problem can be worked out for the upper semicontinuity, we were not able to solve it for the lower semicontinuity.

**Definition 6.3.** Consider $\hat{U}$ an impulsive evolution process satisfying (1) and $\mathcal{D}$ a universe in $X$. For a $\hat{U}$-invariant family $\hat{E} \in \mathcal{D}$ we define:

(a) the **unstable set of $\hat{E}$ at time** $t \in \mathbb{R}$ as the set

$$W^u(\hat{E})(t) = \{\xi(t) \mid \xi : \mathbb{R} \to X \text{ is a global solution of } \hat{U} \text{ with } \lim_{s \to -\infty} d(\xi(s), E(s)) = 0\};$$
Proposition 6.6. Let \( \hat{x} \) be the pullback of \( \hat{y} \) with a pullback \( U \) in \( D \).

\[ W^u(\hat{x}) = \{ W^u(\hat{y})(t) \}_{t \in \mathbb{R}}; \]

Proof. From (11), there exists \( \delta > 0 \) such that \( E(t) \subset D(t) \) for all \( t \leq \tau \),

\[ d(\xi(s), E(s)) < \delta \quad \text{for all} \quad s \leq t \quad \text{and} \quad \lim_{s \to -\infty} d(\xi(s), E(s)) = 0; \]

(d) the \((\delta-) local unstable family\) of \( \hat{E} \) as the family

\[ W^u(\hat{E}) = \{ W^u(\hat{E})(t) \}_{t \in \mathbb{R}}. \]

Definition 6.4. Let \( \hat{E} = \{ E(t) \}_{t \in \mathbb{R}} \) be a family in \( X \) and \( \mathcal{D} \) a universe. We say that \( \hat{E} \) is \textbf{backwards in} \( \mathcal{D} \) if there exists \( \tau \in \mathbb{R}, \epsilon > 0 \) and \( \hat{D} \in \mathcal{D} \) such that

\[ E(t) \subset D(t) \quad \text{for all} \quad t \leq \tau. \]

Lemma 6.5. Let \( \hat{U} \) be an impulsive evolution process satisfying (1) and (I), with a pullback \( \mathcal{D} \)-attractor \( \hat{A} \). If \( \hat{E} \) is a backwards in \( \mathcal{D} \) \( \hat{U} \)-invariant family, then \( E(t) \subset A(t) \) for all \( t \in \mathbb{R} \). In particular, if \( \hat{E} = \xi \) where \( \xi \) is a global solution of \( \hat{U} \), then \( \xi(t) \in A(t) \setminus M(t) \) for all \( t \in \mathbb{R} \).

Proof. Fix \( t \in \mathbb{R} \). Since \( E \) is backwards in \( \mathcal{D} \), there exists \( \tau \in \mathbb{R} \) and \( \hat{D} \in \mathcal{D} \) such that \( E(s) \subset D(s) \) for all \( s \leq \tau \). For \( s \leq \min\{t, \tau\} \) we have \( E(t) \subset \hat{U}(t, s)E(s) \subset \hat{U}(t, s)D(s) \), and since \( \hat{A} \) pullback \( \hat{U} \)-attracts \( D \) we obtain

\[ d_H(E(t), A(t)) \leq d_H(\hat{U}(t, s)D(s), A(t)) \to 0 \quad \text{as} \quad s \to -\infty. \]

Thus \( E(t) \subset A(t) \) for all \( t \in \mathbb{R} \). When \( \hat{E} = \xi \), since (I) holds, we have \( \xi(t) \in A(t) \setminus M(t) \).

We consider from now on a universe \( \mathcal{D} \) satisfying the following condition:

\[ \forall \epsilon > 0 \text{ there exists } \epsilon > 0 \text{ such that } \{ O_\epsilon(D(t)) \}_{t \in \mathbb{R}} \in \mathcal{D}, \quad (11) \]

where \( O_\epsilon(D(t)) = \{ x \in X : d(x, D(t)) < \epsilon \} \) and \( d(x, D(t)) = \inf_{y \in D(t)} d(x, y) \) for each \( t \in \mathbb{R} \).

Proposition 6.6. Let \( \hat{U} \) be an impulsive evolution process satisfying (1) and (I), with a pullback \( \mathcal{D} \)-attractor \( \hat{A} \), where \( \mathcal{D} \) satisfies (11). If \( \hat{E} \) is a backwards in \( \mathcal{D} \) \( \hat{U} \)-invariant family in \( X \), then \( W^u(\hat{x})(t) \subset A(t) \setminus M(t) \) for all \( t \in \mathbb{R} \).

Proof. Let \( \xi \) be a global solution of \( \hat{U} \) with \( \lim_{s \to -\infty} d(\xi(s), E(s)) = 0 \). Since \( \hat{E} \) is backwards in \( \mathcal{D} \), there exists \( \tau \in \mathbb{R} \) and \( \hat{D} \in \mathcal{D} \) such that \( E(t) \subset D(t) \) for all \( t \leq \tau \). From (11), there exists \( \epsilon > 0 \) such that \( \{ O_\epsilon(D(t)) \}_{t \in \mathbb{R}} \in \mathcal{D} \). Also, there exists \( s_0 \leq \tau \) such that

\[ \xi(t) \in O_\epsilon(E(t)) \subset O_\epsilon(D(t)) \quad \text{for all} \quad t \leq s_0, \]

and hence \( \xi \) is a backwards in \( \mathcal{D} \) global solution. Thus Lemma 6.5 implies that \( \xi(t) \in A(t) \setminus M(t) \) for each \( t \in \mathbb{R} \), and the conclusion follows.
Corollary 6.7. Let \( \mathcal{U} \) be an impulsive evolution process satisfying (1) and (I), with a pullback \( \mathcal{D} \)-attractor \( \hat{A} \), with \( \mathcal{D} \) satisfying (11). Let also \( \mathcal{B} \) be the collection of all backwards in \( \mathcal{D} \) global solutions of \( \mathcal{U} \). Then
\[
A(t) \setminus M(t) = \bigcup_{\xi \in \mathcal{B}} W^u(\xi)(t) \quad \text{for each } t \in \mathbb{R}. \tag{12}
\]

Proof. The inclusion \( \cup_{\xi \in \mathcal{B}} W^u(\xi)(t) \subset A(t) \setminus M(t) \) follows from Proposition 6.6. For the other inclusion, we know that if \( x \in A(t) \setminus M(t) \), using Proposition 2.2, there exists a global solution \( \xi \) of \( \mathcal{U} \) with \( \xi(t) = x \) and \( \xi \in \mathcal{D} \). In particular, \( \xi \) is backwards in \( \mathcal{D} \) and \( x \in W^u(\xi)(t) \).

This last result states, roughly speaking, that in order to describe the pullback \( \mathcal{D} \)-attractor (aside from the impulsive family \( \hat{M} \)) of an impulsive evolution process \( \mathcal{U} \), we need only the backwards in \( \mathcal{D} \) global solutions of \( \mathcal{U} \) and their unstable sets.

However, an impulsive evolution process \( \mathcal{U} \) may have an infinite (and possibly, uncountable) number of backwards in \( \mathcal{D} \) global solutions, and the description (12) is not so helpful. But obviously, if two backwards in \( \mathcal{D} \) global solutions tend to one another as time goes to \(-\infty\), their unstable sets are equal, and we only need one of these two in (12). In what follows, we formalize this idea.

Definition 6.8. Let \( \mathcal{U} \) be an impulsive evolution process satisfying (1) and \( \xi_1, \xi_2 \) global solutions of \( \mathcal{U} \). We say that \( \xi_1 \) and \( \xi_2 \) are backwards-separated (or separated in the past) if
\[
\lim_{s \to -\infty} \sup d(\xi_1(s), \xi_2(s)) > 0.
\]

When \( \xi_1 \) and \( \xi_2 \) are not backwards-separated then \( W^u(\xi_1)(t) = W^u(\xi_2)(t) \) for all \( t \in \mathbb{R} \).

With this, we have the following

Corollary 6.9. Let \( \mathcal{U} \) be an impulsive evolution process satisfying (1) and (I), with a pullback \( \mathcal{D} \)-attractor \( \hat{A} \), with \( \mathcal{D} \) satisfying (11). Let also \( \mathcal{B}_s \) be the collection of all backwards in \( \mathcal{D} \) global solutions of \( \mathcal{U} \), which are two-by-two backwards-separated. Then
\[
A(t) \setminus M(t) = \bigcup_{\xi \in \mathcal{B}_s} W^u(\xi)(t) \quad \text{for each } t \in \mathbb{R}.
\]

Theorem 6.10. Let \( \{\mathcal{U}_\eta\}_{\eta \in [0,1]} \) be a family of impulsive evolution processes satisfying (1), (I) and (CT). Assume that \( \mathcal{U}_\eta \) has a pullback \( \mathcal{D} \)-attractor \( \hat{A}_\eta \) for each \( \eta \in [0,1] \), where \( \mathcal{D} \) satisfies (11). Assume also that the associated family of (continuous) evolution processes \( \{\mathcal{U}_\eta\}_{\eta} \) is continuous at \( \eta = 0 \), that the collection of impulsive sets \( \{M^\eta\}_\eta \) is collectively closed at \( \eta = 0 \), and that the collection of impulse functions \( \{I^\eta\}_\eta \) is collectively continuous at \( \eta = 0 \). Additionally, suppose that

(i) there exists a sequence \( \{\xi_{j,0}\}_{j \in \mathbb{N}} \) of backwards in \( \mathcal{D} \) global solutions of \( \mathcal{U}_0 \) such that
\[
A_0(t) = \bigcup_{j \in \mathbb{N}} W^u(\xi_{j,0})(t);
\]

(ii) for each \( j \in \mathbb{N} \), there exists a family of backwards in \( \mathcal{D} \) global solutions \( \{\xi_{j,\eta}\}_{\eta} \) and a sequence \( \{t_j\}_{j \in \mathbb{N}} \subset \mathbb{R} \) such that
\[
\lim_{\eta \to 0} \sup \{d(\xi_{j,\eta}(t), \xi_{j,0}(t)) : t \leq t_j\} = 0.
\]
(iii) for each $j \in \mathbb{N}$, there exist $\delta_j > 0$ and $t_j \in \mathbb{R}$ such that
\[
\lim_{\eta \to 0} \sup_{t \leq t_j} \{d_H(W^u_j(\hat{\xi}_j,0)(t), W^u_j(\hat{\xi}_j,\eta)(t)) : t \leq t_j \} = 0.
\]

Then the family $\{\hat{A}_\eta\}_{\eta \in [0,1]}$ is weak-lower semicontinuous at $\eta = 0$, in the sense of Definition 6.1.

Proof. Fix $\varepsilon > 0$. Since $x \in A_0(t)$, from (i), there exists $j \in \mathbb{N}$ and $x_\varepsilon \in W^u(\hat{\xi}_j,0)(t)$ such that $d(x_\varepsilon, x) < \varepsilon/2$. Thus, there exists a global solution $\xi$ of $\hat{U}_0$ such that $\xi(t) = x_\varepsilon$ and $d(\xi(s), \xi_j,0(s)) \to 0$ as $s \to -\infty$.

For $\delta_j > 0$ as in Theorem 6.10, there exists $\tau > 0$ such that $z := \xi(t - \tau) \in W^s_j(\hat{\xi}_j,0)(t - \tau)$. Now Theorem 6.10 implies that there exist sequences $\eta_k \to 0$ and $z_k \to z$ with $z_k \in W^s_j(\hat{\xi}_j,\eta_k)(t - \tau) \subset A_{\eta_k}(t - \tau)$. Note that $z = \xi(t - \tau) \notin M^0(t - \tau)$.

Using Corollary 3.7, there exists $\varepsilon_k \to 0$ such that
\[
x_k := \hat{U}_{\eta_k}(t + \varepsilon_k, t - \tau)z_k \to \hat{U}_0(t, t - \tau)z = \hat{U}_0(t, t - \tau)\xi(t - \tau) = \xi(t) = x_\varepsilon.
\]
Also $z_k = \xi_k(t - \tau)$ for some backwards in $\hat{D}$ global solution $\xi_k$ of $\hat{U}_{\eta_k}$ for each $k \in \mathbb{N}$. By Lemma 6.5, $x_k := \xi_k(t + \varepsilon_k) \in A_{\eta_k}(t + \varepsilon_k) \setminus M^{\eta_k}(t + \varepsilon_k)$, and for $k$ sufficiently large, we obtain
\[
d(x_k, x) \leq d(x_k, x_\varepsilon) + d(x_\varepsilon, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

\[\square\]

Remark 6.11. The study of lower semicontinuity is challenging. Even for continuous evolution processes, the majority of papers deal with autonomous and nonautonomous perturbations of autonomous problems (see, for instance, [15, 17, 23, 24]). It becomes even harder when we consider nonautonomous perturbations of nonautonomous problems.

The abstract framework is somewhat easier to study, and follow the lines of Theorem 6.10. The hard part is, in applications, to verify conditions (i)-(iii) of Theorem 6.10. In the continuous framework we need to know the inner structure of the pullback attractor of the limiting problem to verify (i). For (ii) the concept of hyperbolicity is crucial, and it implies the permanence of critical elements and the continuity of their local unstable manifold in a few scenarios, which is condition (iii). One of these scenarios appears when we make small nonautonomous perturbations of autonomous problems (see [5, 25]).

There is, so far, no proof of the lower semicontinuity with its usual definition (see Remark 6.2) for the nonautonomous context and no analogous concept of hyperbolicity to impulsive systems (even in the autonomous setting). One logical next step in this theory is to study the behavior of critical elements and their local unstable and stable manifolds, in order to obtain the conditions of Theorem 6.10.

7. Invariance property for impulsive pullback $\omega$-limits without tube conditions. In this section we improve the work done in [11, Subsection 4.2], where the authors achieve the invariance property for $\omega$-limits using the tube conditions (see [11, Definition 4.3]) together with conditions (H) and (I). Here we obtain this invariance with condition (T), (H) and (I), and we recall that condition (T) is weaker than the tube conditions (see [11, Theorem 4.16]).

To begin we recall that given an impulsive evolution process $\hat{U} = (\mathcal{U}, X, \hat{M}, \hat{I})$, a universe $\mathcal{D}$, a family $\hat{D} \in \mathcal{D}$ and $t \in \mathbb{R}$, the impulsive pullback $\omega$-limit set of $\hat{D}$ at time $t$ is the set $\hat{\omega}(\hat{D}, t)$ given by: $x \in \hat{\omega}(\hat{D}, t)$ iff there exists sequences $s_n \to -\infty,$
Lemma 7.5. Let $\hat{U}$ be a pullback $\mathcal{D}$-asymptotically compact evolution process satisfying conditions (H), (I) and (T). For $\hat{D} \in \mathcal{D}$ and $t \in \mathbb{R}$ we have

$$\hat{\omega}(\hat{D}, t) \setminus M(t) \subset \hat{U}(t, t - \xi)(\hat{\omega}(\hat{D}, t - \xi) \setminus M(t - \xi)).$$

Remark 7.2. The hypotheses of [11, Theorem 2.31] are the same as Proposition 7.1, with the addition of the pullback $\mathcal{D}$-dissipativity of $\hat{U}$. This theorem states that

$$A(t) \setminus M(t) \subset \hat{U}(t, t - \xi)(A(t - \xi) \setminus M(t - \xi)),$$

where $\hat{A} \in \mathcal{D}$ is a collectively closed pullback $\mathcal{D}$-semi attractor for $\hat{U}$. This family is given by $\hat{A} = \hat{\omega}(\hat{B})$, where $B_0$ is a dissipative set for $U$.

The proof only uses the hypotheses of Proposition 7.1 and the fact that $\hat{A}$ is the impulsive pullback $\omega$-limit of a family in $\mathcal{D}$.

Clearly, Proposition 7.1 is still valid if we replace $\xi$ by $u$, where $0 < u \leq \xi$ (since condition (I) also holds with $u$ instead of $\xi$). Therefore, we have the next result.

Corollary 7.3. Let $\hat{U}$ be a pullback $\mathcal{D}$-asymptotically compact evolution process satisfying conditions (H), (I) and (T). For $\hat{D} \in \mathcal{D}$, $t \in \mathbb{R}$ and $0 < u \leq \xi$ we have

$$\hat{\omega}(\hat{D}, t) \setminus M(t) \subset \hat{U}(t, t - \xi)(\hat{\omega}(\hat{D}, t - \xi) \setminus M(t - \xi)).$$

We then obtain the negative invariance property for impulsive pullback $\omega$-limit sets.

Proposition 7.4 (Negative invariance). Let $\hat{U}$ be a pullback $\mathcal{D}$-asymptotically compact evolution process satisfying conditions (H), (I) and (T). If $\hat{D} \in \mathcal{D}$, the set $\hat{\omega}(\hat{D}) \setminus \hat{M} = \hat{\omega}(\hat{D}) \setminus M$ is negatively $\hat{U}$-invariant, that is,

$$\hat{\omega}(\hat{D}, t) \setminus M(t) \subset \hat{U}(t, s)(\hat{\omega}(\hat{D}, s) \setminus M(s)) \quad \text{for all} \quad t \geq s.$$

Proof. The case $t = s$ is trivial, and it suffices to assume $t > s$. Choosing $n \in \mathbb{N}$ such that $0 < u = \frac{ct}{n} \leq \xi$, from Corollary 7.3 we obtain

$$\hat{\omega}(\hat{D}, t - ku) \setminus M(t - ku) \subset \hat{U}(t - ku, t - (k + 1)u)(\hat{\omega}(\hat{D}, t - (k + 1)u) \setminus M(t - (k + 1)u)),$$

for $k = 0, \ldots, n - 1$. Noting that $s = t - nu$ and that

$$\hat{U}(t, s) = \hat{U}(t, t - u) \cdots \hat{U}(t - (n - 1)u, s),$$

we conclude that $\hat{\omega}(\hat{D}, t) \setminus M(t) \subset \hat{U}(t, s)(\hat{\omega}(\hat{D}, s) \setminus M(s))$. \hfill $\square$

Now we deal with the positive invariance. In [11, Proposition 2.30], the invariance for $\hat{A} \setminus M$ is obtained using the pullback attractor property of the pullback $\mathcal{D}$-semi attractor $\hat{A}$. Hence, the same proof does not work for us, and we must prove the positive invariance of $\hat{\omega}(\hat{B}) \setminus \hat{M}$ directly. We begin with a lemma.

Lemma 7.5. Let $\hat{U}$ be a pullback $\mathcal{D}$-asymptotically compact evolution process satisfying conditions (H), (I) and (T). For $\hat{D} \in \mathcal{D}$, $s \in \mathbb{R}$ and $0 < t - s < \xi$, we have

$$\hat{U}(t, s)(\hat{\omega}(\hat{D}, s) \setminus M(s)) \subset \hat{\omega}(\hat{D}, t) \setminus M(t).$$
Proof. Let \( x \in \hat{\omega}(\hat{D}, s) \setminus M(s) \), and \( s_n \to -\infty, \varepsilon_n \to 0 \), and \( x_n \in D(s_n) \) for each \( n \in \mathbb{N} \) such that \( \hat{U}(s + \varepsilon_n, s_n)x_n \to x \). We can assume, without loss of generality, that \( s + \varepsilon_n < t \) for all \( n \in \mathbb{N} \).

We need to prove that \( \hat{U}(t, s)x \in \hat{\omega}(\hat{D}, t) \setminus M(t) \). It is enough to check that \( \hat{U}(t, s)x \in \hat{\omega}(\hat{D}, t) \), since condition (I) implies that \( \hat{U}(t, s)x \notin M(t) \) (see [11, Proposition 2.12]). For each \( n \in \mathbb{N} \) we consider the map \([s_n, s + 5\xi/4] \ni u \mapsto \hat{U}(u, s_n)x_n \in X \), and let \( \tau_n \) be the last jump time of \( \hat{U} \) at \( (x_n, s_n) \) (we set \( \tau_n = s_n \) if there are no such jump times).

We split the proof into three cases.

Case 1. There exist \( 0 < \varepsilon < \xi/2 \) and a subsequence of \( \tau_n \) (still denoted by \( n \)) such that \( \tau_n < s - \varepsilon \).

Without loss of generality, we can assume that \( \varepsilon_n > -\varepsilon/2 \) for all \( n \in \mathbb{N} \). The pullback \( \mathcal{D} \)-asymptotical compactness of \( \hat{U} \) implies that, up to a subsequence (which we denote the same), \( y_n := \hat{U}(s - \varepsilon/2, s_n)x_n \to y \in X \). Then, since \( \tau_n \) is the last jump time in \([s_n, s + 5\xi/4]\) and \( \tau_n < s - \varepsilon/2 \) we obtain

\[
\hat{U}(s + \varepsilon_n, s_n)x_n = \hat{U}(s + \varepsilon_n, s - \varepsilon/2)\hat{U}(s - \varepsilon/2, s_n)x_n = \hat{U}(s + \varepsilon_n, s - \varepsilon/2)y_n \\
= \hat{U}(s, s - \varepsilon/2)y = x.
\]

Furthermore

\[
\hat{U}(s, s_n)x_n = \hat{U}(u, s)\hat{U}(s, s_n)x_n = \hat{U}(u, s)\hat{U}(s, s_n)x_n \to \hat{U}(u, s)x,
\]

and hence

\[
\hat{U}(u, s_n)x_n = \hat{U}(u, s)\hat{U}(s, s_n)x_n = \hat{U}(u, s)\hat{U}(s, s_n)x_n \to \hat{U}(u, s)x,
\]

for each \( u \in (s, s + \xi) \). In particular, for \( u = t \) we obtain \( \hat{U}(t, s_n)x_n \to \hat{U}(t, s)x \). The proof of this case will be complete if we can show that \( \hat{U}(t, s)x = \hat{U}(t, s)x \). If that is not the case, then there exists \( u \in (s, t) \subset (s, s + \xi) \) such that \( \hat{U}(u, s)x \in M(u) \).

Using condition (T), there exist a sequence \( \alpha_k \to 0 \), with \( u + \alpha_k \geq s \) for all \( k \), and a subsequence \( \{ \hat{U}(u, s_n_k)x_{n_k} \} \) of \( \{ \hat{U}(u, s_n)x_n \} \) such that \( \hat{U}(u + \alpha_k, s)\hat{U}(s, s_{n_k})x_{n_k} \in M(u + \alpha_k) \) for all \( k \). But this implies that \( u + \alpha_k \) is a jump point of \( \hat{U} \) at \( (x_{n_k}, s_{n_k}) \), and contradicts the definition of \( \tau_{n_k} \), since \( \tau_{n_k} < s - \varepsilon < s \leq u + \alpha_k \). Thus we conclude that \( \hat{U}(t, s_n)x_n \to \hat{U}(t, s)x \), and hence \( \hat{U}(t, s)x \in \hat{\omega}(\hat{D}, t) \).

Case 2. There exist \( \varepsilon > 0 \) and a subsequence of \( \tau_n \) (still denoted by \( n \)) such that \( \tau_n > s + \varepsilon \).

Without loss of generality, we can assume that \( \varepsilon_n \in (-\xi/4, \varepsilon) \) for all \( n \in \mathbb{N} \). The pullback \( \mathcal{D} \)-asymptotical compactness of \( \hat{U} \) implies that, up to a subsequence (which we denote the same), \( y_n := \hat{U}(s - \xi/4, s_n)x_n \to y \in X \). Since \( \tau_n \) is the unique jump time in \([s - \xi/2, s + 5\xi/4]\) and \( \tau_n > s + \varepsilon > s + \varepsilon_n \) we have

\[
\hat{U}(s + \varepsilon_n, s_n)x_n = \hat{U}(s + \varepsilon_n, s - \xi/4)\hat{U}(s - \xi/4, s_n)x_n = \hat{U}(s + \varepsilon_n, s - \xi/4)y_n \\
= \hat{U}(u, s - \xi/4)y,
\]

which implies that \( x = \hat{U}(s, s - \xi/4)y \). Furthermore,

\[
\hat{U}(s, s_n)x_n = \hat{U}(s, s - \xi/4)\hat{U}(s - \xi/4, s_n)x_n = \hat{U}(s, s - \xi/4)y_n \\
= \hat{U}(u, s - \xi/4)y = x.
\]
We have to distinguish three subcases:

**Subcase 1.** There exist $\delta > 0$ and a subsequence of $\tau_n$ (still denoted by $n$) such that $\tau_n > t + \delta$.

In this subcase, for each $u \in (s,t]$ we obtain

$$\hat{U}(u, t, s) \rightarrow \hat{U}(u, s) = U(u, s) \hat{U}(s, s_n) x_n = U(u, s) \hat{U}(s, s_n) x_n \rightarrow U(u, s)x,$$

and in particular, for $u = t$ we obtain $\hat{U}(t, s_n) x_n \rightarrow U(t, s)x$. As before, it remains only to prove that $\hat{U}(t, s)x = U(t, s)x$. If that is not the case, there exists $u \in (s, t] \subset (s, s + \xi)$ such that $U(u, s)x \notin M(u)$, then condition (T) implies that there exist a sequence $\alpha_k \rightarrow 0$, with $u + \alpha_k \geq s$ for all $k$, and a subsequence $\{\hat{U}(s, s_n) x_{n_k}\}$ of $\{\hat{U}(s, s) x_n\}$ such that $\hat{U}(u + \alpha_k, s) \hat{U}(s, s_n) x_{n_k} \in M(u + \alpha_k)$ for all $k$. But for $k$ sufficiently large, $s \leq u + \alpha_k < t + \delta < \tau_{n_k}$, which contradicts the fact that $\tau_{n_k}$ is the unique jump time of $\hat{U}$ at $(x_{n_k}, s_{n_k})$ in the interval $[s - \xi / 2, s + 5\xi / 4]$. Then we conclude that $\hat{U}(t, s_n)x_n \rightarrow \hat{U}(t, s)x$, and so $\hat{U}(t, s)x \in \hat{w}(\hat{D}, t)$.

**Subcase 2.** There exist $\delta > 0$ and a subsequence of $\tau_n$ (still denoted by $n$) such that $\tau_n < t - \delta$.

Since in this subcase we have $s + \varepsilon < \tau_n < t - \delta$, taking subsequences if necessary, we can assume that $\tau_n \rightarrow \bar{\tau} \in [s + \varepsilon, t - \delta]$. Recall that since $\hat{M}$ is collectively closed and $U(\tau_n, s) x_n \in M(\tau_n)$ for all $n$, we obtain $U(\bar{\tau}, s)x \in M(\bar{\tau})$ and the collective continuity of $\hat{I}$ imply that for each $u \in (t - \delta, t]$ we have

$$\hat{U}(u, s_n) x_n = \hat{U}(u, \tau_n) \hat{U}(\tau_n, s) \hat{U}(s, s_n) x_n = U(u, \tau_n) I_{\tau_n}(U(\tau_n, s) \hat{U}(s, s_n) x_n)$$

$$\rightarrow U(u, \bar{\tau}) I_{\bar{\tau}}(U(\bar{\tau}, s)x),$$

and in particular $\hat{U}(t, s_n)x_n \rightarrow \hat{U}(t, \bar{\tau}) I_{\bar{\tau}}(U(\bar{\tau}, s)x)$. The proof of this subcase is complete if we can show that this last point is equal to $\hat{U}(t, s)x$.

First, we show that $I_{\bar{\tau}}(U(\bar{\tau}, s)x) = \hat{U}(t, \bar{\tau}) I_{\bar{\tau}}(U(\bar{\tau}, s)x)$. Note that for each $u \in (s, \bar{\tau})$, $U(u, s) \hat{U}(s, s_n) x_n \rightarrow U(u, s)x$. If $U(u, s)x \in M(u)$ for some $u \in (s, \bar{\tau})$, from condition (T) there exists a sequence $\alpha_k \rightarrow 0$, with $u + \alpha_k \geq s$ for all $k$, and a subsequence $\{\{U(u, s) \hat{U}(s, s_n) x_n\}\}$ of $\{(U(u, s) \hat{U}(s, s) x_n)\}$ such that $U(u + \alpha_k, s) \hat{U}(s, s_n) x_{n_k} \in M(u + \alpha_k)$ for all $k$. For $k$ sufficiently large we obtain $s + \varepsilon < u + \alpha_k < \tau_{n_k} < t - \bar{\tau} < s + \xi$, which contradicts the fact that $\tau_{n_k}$ is the unique jump time of $\hat{U}$ at $(x_{n_k}, s_{n_k})$ in the interval $[s - \xi / 2, s + 5\xi / 4]$. Hence $I_{\bar{\tau}}(U(\bar{\tau}, s)x) = \hat{U}(t, \bar{\tau}) I_{\bar{\tau}}(U(\bar{\tau}, s)x)$.

Now since $|t - \bar{\tau}| < 2\xi$, condition (H) implies that

$$\hat{U}(t, s)x = \hat{U}(t, \bar{\tau}) I_{\bar{\tau}}(U(\bar{\tau}, s)x) = U(t, \bar{\tau}) I_{\bar{\tau}}(U(\bar{\tau}, s)x),$$

and we conclude that $\hat{U}(t, s_n) x_n \rightarrow \hat{U}(t, s)x$, so $\hat{U}(t, s)x \in \hat{w}(\hat{D}, t)$.

**Subcase 3.** $\tau_n \rightarrow t$.

In this subcase, as before, we obtain $U(t, s)x \in M(t)$. This and the collectively continuity of $\hat{I}$ gives us

$$\hat{U}(t_n, s_n) x_n = \hat{U}(\tau_n, s) \hat{U}(s, s_n) x_n = I_{\tau_n} \big( U(\tau_n, s) \hat{U}(s, s_n) x_n \big)$$

$$\rightarrow I_t(U(t, s)x).$$

Again, the proof will be complete if we can show that $\hat{U}(t, s)x = I_t(U(t, s)x)$.

If $U(u, s)x \in M(u)$ for some $u \in (s, t)$, proceeding as in Subcase 2 above, using condition (T) we obtain a contradiction. Then $U(u, s)x \notin M(u)$ for $u \in (s, t)$ and...
we conclude that \( \tilde{U}(t, s)x = I_t(U(t, s)x) \). Hence \( \tilde{U}(t + (\tau_n - t), s_n)x_n \to \tilde{U}(t, s)x \), so \( \tilde{U}(t, s)x \in \hat{\omega}(D, t) \).

**Case 3.** \( \tau_n \to s \).

As in Case 2, up to a subsequence, \( y_n := \tilde{U}(s - \xi/4, s_n)x_n \) converges to some point \( y \in X \). We can assume, without loss of generality, that \( s - \xi/4 < \tau_n < s + \varepsilon \). We split the proof into two subcases.

**Subcase 1.** There exists a subsequence of \( \tau_n \) (still denoted by \( n \)) such that \( s + \varepsilon_n < \tau_n \).

In this subcase we have
\[
\tilde{U}(s + \varepsilon_n, s_n)x_n = \tilde{U}(s + \varepsilon_n, s - \xi/4)\tilde{U}(s - \xi/4, s_n)x_n = U(s + \varepsilon_n, s - \xi/4)y_n
\to U(s, s - \xi/4)y = x.
\]

Also, since \( \tau_n \) is the unique jump time of \( \tilde{U} \) at \( (x_n, s_n) \) in the interval \([s - \xi/2, s + 5\xi/4] \)
we obtain
\[
U(\tau_n, s - \xi/4)y_n = U(\tau_n, s - \xi/4)\tilde{U}(s - \xi/4, s_n)x_n \in M(\tau_n),
\]
and since \( \tau_n \to s \) and \( M \) is collectively closed, \( x = U(s, s - \xi/4)y \in M(s) \), which gives us a contradiction (recall that \( x \in \omega(B, s) \setminus M(s) \)). Hence, this case does not happen.

**Subcase 2.** There exists a subsequence of \( \tau_n \) (still denoted by \( n \)) such that \( s + \varepsilon_n \geq \tau_n \).

In this subcase, we first note that
\[
\tilde{U}(s + \varepsilon_n, s_n)x_n = \tilde{U}(s + \varepsilon_n, \tau_n)\tilde{U}(\tau_n, s - \xi/4)y_n
= U(s + \varepsilon_n, \tau_n)I_{\tau_n}(U(\tau_n, s - \xi/4)y_n) \to I_s(U(s, s - \xi/4)y),
\]
which implies that \( x = I_s(U(s, s - \xi/4)y) \in I_s(M(s)) \). Furthermore, condition \((H)\) implies that \( \tilde{U}(t, s)x = U(t, s)x \), since \( 0 < t < s < \xi \). Finally,
\[
\tilde{U}(t, s_n)x_n = \tilde{U}(t, s + \varepsilon_n)\tilde{U}(s + \varepsilon_n, s_n)x_n = U(t, s + \varepsilon_n)\tilde{U}(s + \varepsilon_n, s_n)x_n
\to U(t, s)x = \tilde{U}(t, s)x,
\]
and thus \( \tilde{U}(t, s)x \in \tilde{\omega}(D, t) \), which completes the proof.

With this lemma, we are able to prove the positive invariance for impulsive pullback \( \omega \)-limits.

**Proposition 7.6** (Positive invariance). Let \( \tilde{U} \) be a pullback \( \mathcal{D} \)-asymptotically compact evolution process satisfying conditions \((H), (I)\) and \((T)\). For \( D \in \mathcal{D} \), the family \( \tilde{\omega}(D) \setminus M \) is positively \( \tilde{U} \)-invariant, that is
\[
\tilde{U}(t, s)(\tilde{\omega}(D, s) \setminus M(s)) \subset \tilde{\omega}(D, t) \setminus M(t) \quad \text{for all } t \geq s.
\]

**Proof.** For \( t = s \) the result is trivial, so we assume \( t > s \). Choose \( n \in \mathbb{N} \) such that \( 0 < u = \frac{t-s}{n} < \xi \). From Lemma 7.5 we obtain
\[
\tilde{U}(s + ku, s + (k-1)u)(\tilde{\omega}(D, s + (k-1)u) \setminus M(s + (k-1)u)) \subset \tilde{\omega}(D, s + ku) \setminus M(s + ku),
\]
for \( k = 1, \ldots, n \). Furthermore, since \( t = s + nu \) and
\[
\tilde{U}(t, s) = \tilde{U}(t, s + (n-1)u) \tilde{U}(s + u, s),
\]
we conclude that \( \tilde{U}(t, s)(\tilde{\omega}(D, s) \setminus M(s)) \subset \tilde{\omega}(D, t) \setminus M(t) \).

\( \square \)
When put all together, these results give us the following theorem:

**Theorem 7.7** (Invariance). Let $\hat{U}$ be a pullback $\mathcal{D}$-asymptotically compact evolution process satisfying conditions (H), (I) and (T). If $\hat{D} \in \mathcal{D}$, the set $\hat{\omega}(\hat{D}) \setminus \hat{M}$ is $\hat{U}$-invariant, that is

$$\hat{U}(t,s)(\hat{\omega}(\hat{D}, s) \setminus M(s)) = \hat{\omega}(\hat{D}, t) \setminus M(t)$$

for all $t \geq s$.

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E-mail address: m.bortolan@ufsc.br
E-mail address: josemanuel.uzal.couselo@usc.es