Optimal control of the coefficient for fractional and regional fractional $p$-Laplace equations: Approximation and convergence *

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Abstract In this paper we study optimal control problems with either fractional or regional fractional $p$-Laplace equation, of order $s$ and $p \in [2, \infty)$, as constraints over a bounded open set with Lipschitz continuous boundary. The control, which fulfills the pointwise box constraints, is given by the coefficient of the involved operator. To overcome the degeneracy of both fractional $p$-Laplacians, we introduce a regularization for both operators. We show existence and uniqueness of solution to the regularized state equations and existence of solution to the regularized optimal control problems. We also prove several auxiliary results for the regularized problems which are of independent interest. We conclude with the convergence of the regularized solutions.

Key Words Fractional $p$-Laplace operator, non-constant coefficient, quasi-linear nonlocal elliptic boundary value problems, optimal control.

AMS subject classification 35R11, 49J20, 49J45, 93C73.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with boundary $\partial \Omega$ and $p \in [2, \infty)$. In this paper we introduce and investigate the existence and approximation of solution to the following optimal control problem (OCP):

Minimize \( \mathcal{I}(\kappa, u) := \frac{1}{2} \int_{\Omega} |u - \xi|^2 \, dx + \int_{\Omega} |\nabla \kappa| \), \hspace{1cm} (1.1)

subject to the state constraints given by either the regional fractional $p$-Laplace equation

\[
\begin{cases}
\mathcal{L}_{\Omega,p}(\kappa, u) + u = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

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or the fractional $p$-Laplace equation

$$\begin{cases}
(-\Delta)^s_p(\kappa, u) + u = f & \text{in } \Omega \\
u = 0 & \text{on } \mathbb{R}^N \setminus \Omega.
\end{cases} \tag{1.3}$$

The control $\kappa$ fulfills the control constraints

$$\kappa \in A_{ad} := \left\{ \eta \in BV(\Omega) : \xi_1(x) \leq \eta(x) \leq \xi_2(x) \text{ a.e. in } \Omega \right\}. \tag{1.4}$$

Here the regional fractional and fractional operators are given for $x \in \Omega$ by:

$$L^{s}_{\Omega,p}(\kappa, u)(x) = C_{N,p,s}P.V. \int_{\Omega} \kappa(x-y)|u(x) - u(y)|^{p-2} \frac{u(x) - u(y)}{|x-y|^{N+sp}} dy, \tag{1.5}$$

and for $x \in \mathbb{R}^N$ by

$$(-\Delta)^s_p(\kappa, u(x)) := C_{N,p,s}P.V. \int_{\mathbb{R}^N} \kappa(x-y)|u(x) - u(y)|^{p-2} \frac{u(x) - u(y)}{|x-y|^{N+sp}} dy, \tag{1.6}$$

respectively. Moreover, $\kappa : \mathbb{R}^N \to [0, \infty)$ is a measurable and even function, that is,

$$\kappa(x) = \kappa(-x), \quad \forall x \in \mathbb{R}^N. \tag{1.7}$$

In addition, $f$ is a given force and $\xi$ is the given data. The functions $\xi_1$ and $\xi_2$ in (1.4) are the control bounds and fulfill $0 < \alpha \leq \xi_1(x) \leq \xi_2(x)$, a.e. $x \in \Omega$, for some constant $\alpha > 0$. The precise regularity requirements for these quantities and the domain $\Omega$ will be discussed in Section 3. Notice that the control $\kappa$ appears in the coefficient of the quasilinear operators $L^{s}_{\Omega,p}$ and $(-\Delta)^s_p$.

For (1.3), we let $0 < s < 1$. We restrict $s$ to $\frac{1}{2} < s < 1$ in the case (1.2), see Remark 3.2 for more details.

Let $a \in L^\infty(\Omega)$ and set

$$\Delta_{p,a}u := \text{div}(a(x)|\nabla u|^{p-2}\nabla u). \tag{1.8}$$

Most recently, in [6, 13] a similar optimal control problem as OCP with $L^{s}_{\Omega,p}$ replaced by $\Delta_{p,a}$ and the control $a(x)$ has been considered.

Even though OCP with the equation (1.2) is a natural extension of [6, 13], however, the nonlocality of $L^{s}_{\Omega,p}$ in comparison to the local operator $\Delta_{p,a}$ makes OCP challenging. Indeed the papers [11, 27], where the authors considered $\kappa = 1$, realized that the standard techniques available for the local $p$-Laplace equation with the operator $\Delta_{p,a}$ are not directly applicable to the regional fractional $p$-Laplace equation (1.2). For the OCP the additional complication occurs due to the fact that the operator $L^{s}_{\Omega,p}$ may degenerate, see subsection 2.3 for details. We also refer to [6, 13] for a discussion related to this topic in case of $\Delta_{p,a}$. Similar complications can occur when the state constraints in OCP are (1.3).

The problem to search for coefficients in case of linear elliptic problems is classical, we refer (but not limited) to [17, 18, 19, 21] and their references. However, this is the first work which provides a mechanism to search for the coefficients in case of a quasilinear, possibly degenerate and fractional nonlocal problem. From a numerical point of view an added attraction of our theory is the fact that it is Hilbert space $L^2$-based instead of $L^p$-based theory.
Subsequently, to tackle this degeneracy in the operators $\mathcal{L}_{\Omega,p}$ (and similarly to $(-\Delta)^s$), we introduce a regularized optimal control problem (ROCP) and we conclude with the convergence of solution of the regularized problem. Notice that in this paper we discuss the convergence of the optimal controls. Due to the possible degeneracy in the state equation it is unclear how to derive the first order stationarity system for OCP. However, ROCP comes to rescue, indeed the latter is build to precisely avoid such degeneracy issues. In a forthcoming paper, we shall derive the limiting stationarity system corresponding to the first order stationarity for ROCP.

Differential equations of fractional order have gained a lot of attraction in recent years due to the fact that several phenomena in the sciences are more accurately modelled by such equations rather than the traditional equations of integer order. Linear and nonlinear equations have been extensively studied. The applications in industry are numerous and cover almost every area. From the long list of phenomena which are more appropriately modelled by fractional differential equations, we mention: viscoelasticity, anomalous transport and diffusion, hereditary phenomena with long memory, nonlocal electrostatics, the latter being relevant to drug design, and Lévy motions which appear in important models in both applied mathematics and applied probability, as well as in models in biology and ecology. We refer to [15, 20, 22] and their references for more details on this topic.

During the course of studying the OCP, we show the well-posedness (existence, uniqueness, and continuous dependence on data) of our state equation (1.2) and the regularized state equation (3.7). We further show several important results for the regularized state equation in subsection 5.3. Thus we not only address many challenging issues associated with the state equation (1.2) but also initiate several new research directions with many possible extensions.

The rest of the paper is organized as follows: In section 2 we introduce the function spaces needed to investigate our problem. We also provide a precise definition of the regional fractional $p$-Laplacian. The results in this section hold for any $0 < s < 1$. Hereafter, we assume that $\frac{1}{2} < s < 1$. We state our main results for OCP with regional fractional $p$-Laplacian in section 3 which is followed by the introduction of the ROCP in subsection 3.2 and a statement of the convergence results. The well-posedness of the state system is discussed in section 4.1. Section 4.2 discusses the existence of solution to OCP. In section 5.1 we discuss well-posedness of the regularized state equation, which is followed by the existence of solution to ROCP in section 5.2. We show the convergence of ROCP solutions to OCP solutions in section 6. We conclude by studying OCP with fractional equation (1.3) in section 7.

2 Notation and Preliminaries

Here we introduce the function spaces needed to investigate our problem and also prove some intermediate results that will be used throughout the paper. The results stated in this section are valid for any $0 < s < 1$.

2.1 The fractional order Sobolev spaces

In this (sub)section, we recall some well-known results on fractional order Sobolev spaces that are needed throughout the article.
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Let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded open set. For $p \in [1, \infty)$ and $s \in (0, 1)$, we denote by

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \, dx \, dy < \infty \right\},$$

the fractional order Sobolev space endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left( \int_\Omega |u|^p \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \, dx \, dy \right)^{\frac{1}{p}}.$$

We let $W^{s,p}_0(\Omega) := D(\Omega)^{W^{s,p}(\Omega)}$. The following result is taken from [12, Theorem 1.4.2.4, p. 25] (see also [4, 25]).

**Theorem 2.1.** Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a Lipschitz continuous boundary. Then the following assertions hold.

(a) If $0 < s \leq \frac{1}{p}$, then $W^{s,p}(\Omega) = W^{s,p}_0(\Omega)$.

(b) If $\frac{1}{p} < s < 1$, then $W^{s,p}_0(\Omega)$ is a proper closed subspace of $W^{s,p}(\Omega)$.

It follows from Theorem 2.1 that for a bounded open set with a Lipschitz continuous boundary, if $\frac{1}{p} < s < 1$, then

$$\|u\|_{W^{s,p}_0(\Omega)} = \left( \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy \right)^{\frac{1}{p}}$$

defines an equivalent norm on $W^{s,p}_0(\Omega)$. Let $p^*$ be given by

$$p^* = \frac{Np}{N - sp} \quad \text{if } N > sp \quad \text{and} \quad p^* \in [p, \infty) \quad \text{if } N = sp. \quad (2.2)$$

Then by [9, Theorems 6.7 and 6.10], there exists a constant $C > 0$ such that for every $u \in W^{s,p}_0(\Omega)$,

$$\|u\|_{L^q(\Omega)} \leq C \|u\|_{W^{s,p}_0(\Omega)}, \quad \forall \ q \in [1, p^*]. \quad (2.3)$$

Moreover, the continuous embedding $W^{s,p}_0(\Omega) \hookrightarrow L^q(\Omega)$ is compact for every $q \in [1, p^*)$ (see e.g. [9, Corollary 7.2]). If $N < sp$, then one has the continuous embedding $W^{s,p}_0(\Omega) \hookrightarrow C^{0,s-N}(\Omega)$ (see e.g. [9, Theorem 8.2]).

We have the following.

**Proposition 2.2.** Let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded open set and $p \in [1, \infty)$. Then the following assertions hold.

(a) If $0 < t \leq s < 1$, then $W^{s,p}_0(\Omega) \hookrightarrow W^{t,p}_0(\Omega)$.

(b) For every $0 < s < 1$, we have that $W^{1,p}_0(\Omega) \hookrightarrow W^{s,p}_0(\Omega)$.

(c) Let $q > p$. If $0 < t < s < 1$, then $W^{s,q}_0(\Omega) \hookrightarrow W^{t,p}_0(\Omega)$.

**Proof.** The proof of the assertions (a), (b) and (c) is contained in [9, Proposition 2.1], [25, Proposition 2.3] and in [2, Proposition 1.2], respectively.
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If $0 < s < 1$, $p \in (1, \infty)$ and $p' := \frac{p}{p-1}$, then the space $W^{-s,p'}(\Omega)$ is defined as usual to be the dual of the reflexive Banach space $W^{s,p}_0(\Omega)$. For $u \in W^{s,p}(\Omega)$ we shall denote by $U_{(p,s)}$ the function defined on $\Omega \times \Omega$ by

$$U_{(p,s)}(x,y) := \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p} + s}}. \quad (2.4)$$

We will always denote by $\chi_E$ the characteristic function of a set $E \subseteq \Omega \times \Omega$.

**Remark 2.3.** Let $u \in W^{s,p}_0(\Omega)$ and $\{u_n\}_{n \in \mathbb{N}}$ a sequence in $W^{s,p}_0(\Omega)$. Then the following assertions hold.

(a) If $u_n$ converges weakly to $u$ in $W^{s,p}_0(\Omega)$ as $n \to \infty$ (that is, $u_n \rightharpoonup u$ in $W^{s,p}_0(\Omega)$ as $n \to \infty$), then for every $\varphi \in \mathcal{D}(\Omega)$,

$$\lim_{n \to \infty} \int_{\Omega} \int_{\Omega} \frac{(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy = \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy.$$

(b) If $u_n \to u$ in $W^{s,p}_0(\Omega)$, then $U_{n,(p,s)} \to U_{(p,s)}$ in $L^p(\Omega \times \Omega)$ as $n \to \infty$.

(c) If $u_n \to u$ in $W^{s,p}_0(\Omega)$ and $U_{n,(p,s)} \to U_{(p,s)}$ in $L^p(\Omega \times \Omega)$ as $n \to \infty$, then $u_n \rightharpoonup u$ in $W^{s,p}_0(\Omega)$.

For more information on fractional order Sobolev spaces we refer the reader to [4, 9, 12, 25] and the references therein.

2.2 Functions of bounded variation

Let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded open set. Let

$$BV(\Omega) := \left\{ g \in L^1(\Omega) : \|g\|_{BV(\Omega)} < \infty \right\},$$

be the space of functions of bounded variation, where

$$\|g\|_{BV(\Omega)} := \|g\|_{L^1(\Omega)} + \sup \left\{ \int_{\Omega} g \text{div}(\Phi) \, dx : \Phi \in C^1_0(\Omega, \mathbb{R}^N), |\Phi(x)| \leq 1, x \in \Omega \right\}.$$ 

For $g \in BV(\Omega)$, we denote by $\nabla g$ the distributional gradient of $g$. We notice that $\nabla g$ belongs to the space of Radon measures $\mathcal{M}(\Omega, \mathbb{R}^N)$.

The following notion of convergence is contained in [1, Definition 3.1].

**Remark 2.4.** Let $g \in BV(\Omega)$ and $\{g_n\}_{n \in \mathbb{N}}$ a sequence in $BV(\Omega)$.

(a) We say that $\{g_n\}_{n \in \mathbb{N}}$ converges weakly* ($\overset{*}{\rightharpoonup}$) to $g \in BV(\Omega)$ as $n \to \infty$, if and only if the following two conditions hold.

(i) $g_n \to g$ (strongly) in $L^1(\Omega)$ as $n \to \infty$, and
(ii) \( \nabla g_n \overset{*}{\rightharpoonup} \nabla g \) (weakly*) in \( \mathcal{M}(\Omega, \mathbb{R}^N) \) as \( n \to \infty \), that is,

\[
\lim_{n \to \infty} \int_{\Omega} \phi \, d\nabla g_n = \int_{\Omega} \phi \, d\nabla g, \quad \forall \phi \in C_0(\Omega).
\]

(b) In addition, if \( g_n \) converges strongly to some \( \tilde{g} \) in \( L^1(\Omega) \) as \( n \to \infty \) and satisfies \( \sup_{n \in \mathbb{N}} \int_{\Omega} |\nabla g_n| < \infty \), then

\[
\tilde{g} \in BV(\Omega), \quad \int_{\Omega} |\nabla \tilde{g}| \leq \liminf_{n \to \infty} \int_{\Omega} |\nabla g_n| \quad \text{and} \quad g_n \overset{*}{\rightharpoonup} \tilde{g} \quad \text{in} \quad BV(\Omega) \quad \text{as} \quad n \to \infty.
\]

For more details on functions of bounded variation we refer to [1, Chapter 3].

2.3 The regional fractional \( p \)-Laplacian

Let \( 0 < s < 1 \) and \( p \in (1, \infty) \). The regional fractional \( p \)-Laplacian \( (-\Delta)^s_{\Omega,p} \) is defined for \( x \in \Omega \) by the formula

\[
(-\Delta)^s_{\Omega,p} u(x) = C_{N,p,s} \text{P.V.} \int_{\Omega} |u(x) - u(y)|^{p-2} \frac{u(x) - u(y)}{|x-y|^{N+ps}} \, dy,
\]

where \( C_{N,p,s} \) is a normalized constant (see, e.g., [3, 5, 9, 24] for the linear case \( p = 2 \), and [26, 27, 28] for the general case \( p \in (1, \infty) \)). We notice that if \( 0 < s < \frac{2}{p} \) and \( u \) is smooth (i.e., at least bounded and Lipschitz continuous on \( \Omega \)), then the above integral is in fact not really singular near \( x \) (see e.g. [2, Section 2.1] for more details). If \( \Omega = \mathbb{R}^N \), then \( (-\Delta)^s_{\mathbb{R}^N,p} = (-\Delta)^s_p \) is usually called the fractional \( p \)-Laplace operator, see section 7 for more details.

It has been shown in [2, Formula (2.4)] that for every \( u \in \mathcal{D}(\Omega) \),

\[
\lim_{s \uparrow 1} \int_{\Omega} u(-\Delta)^s_{\Omega,p} u \, dx = \lim_{s \uparrow 1} \int_{\mathbb{R}^N} u(-\Delta)^s_p u \, dx = \int_{\Omega} |\nabla u|^p \, dx = -\int_{\Omega} u \Delta_p u \, dx. \tag{2.5}
\]

It follows from (2.5) that the regional fractional \( p \)-Laplace operator converges (in some sense) to the \( p \)-Laplace operator, as \( s \uparrow 1 \).

Let \( \kappa \) be as in (1.7). For \( 1 < p < \infty \) and \( 0 < s < 1 \) we define the operator \( \mathcal{L}^s_{\Omega,p} \) as in (1.5). We again call this operator, the regional fractional \( p \)-Laplace operator. We mention that elliptic problems associated with the operator \( \mathcal{L}^s_{\Omega,p} (\kappa, \cdot) \) subject to the Dirichlet boundary condition have been investigated in [7, 8, 11, 14] where the authors have obtained some fundamental existence and regularity results. The case of Neumann and Robin type boundary conditions (with \( \kappa = 1 \)) is contained in [28]. We refer to [11, 27] for further results on parabolic problems.

3 The main results

In this section we state the main results of the article. Throughout the remainder of the article, unless stated otherwise, we assume the following.

Assumption 3.1. We shall always assume the following.

(a) \( \Omega \subset \mathbb{R}^N \ (N \geq 1) \) is a bounded open set with Lipschitz continuous boundary.
(b) \( \frac{1}{2} < s < 1 \).
(c) The functions \( \xi_1, \xi_2 \in L^\infty(\Omega) \) and there exists a constant \( \alpha > 0 \) such that
\[
0 < \alpha \leq \xi_1(x) \leq \xi_2(x) \text{ a.e. in } \Omega. \tag{3.1}
\]
(d) The measurable function \( \kappa \) satisfies the assumption given in \((1.7)\).

Recall that it follows from Assumption 3.1(b) that \((2.1)\) defines an equivalent norm on \( W_0^{s,p}(\Omega) \) for every \( p \in [2, \infty) \).

**Remark 3.2** (Regional case: \( 0 < s < 1 \)). The restriction on \( s \) in Assumption 3.1(b) is used to show the uniqueness of solution to \((1.2)\). This step requires the equivalence between \( \|v\|_{W_0^{s,p}(\Omega)} \) and \( \left( \|v\|_{L^2(\Omega)}^2 + \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^p}{|x-y|^{N+sp}} \, dx \, dy \right)^{\frac{1}{p}} \). The equivalence follows immediately when \( \frac{1}{2} < s < 1 \). When \( 0 < s \leq \frac{1}{2} \) the proof will be along the line of [10, Corollary 1.5.2 p.37] (they only discuss \( s = 1 \)) but we do not explore this here.

### 3.1 The optimal control problem

Let \( \xi, f \in L^2(\Omega) \) be given functions. The OCP we consider first is \((1.1), (1.2)\) and \((1.4)\). The following is our notion of solutions to the state system \((1.2)\).

**Definition 3.3.** A function \( u \in W_0^{s,p}(\Omega) \) is said to be a weak solution of the system \((1.2)\) if for every \( \varphi \in W_0^{s,p}(\Omega) \),
\[
\frac{C_{N,p,s}}{2} \int_\Omega \int_\Omega \kappa(x-y)|u(x) - u(y)|^{p-2} \frac{(u(x) - u(y))(|\varphi(x) - \varphi(y)|)}{|x-y|^{N+sp}} \, dx \, dy \\
+ \int_\Omega u \varphi \, dx = \int_\Omega f \varphi \, dx. \tag{3.2}
\]

The following existence result of optimal pair to the OCP is our first main result.

**Theorem 3.4.** Let \( \xi, f \in L^2(\Omega) \) be given. Then the OCP \((1.1), (1.2)\) and \((1.4)\) admits at least one solution \( (\kappa, u) \in BV(\Omega) \times W_0^{s,p}(\Omega) \).

### 3.2 The regularized optimal control problem

Let \( \xi, f \in L^2(\Omega) \) be given functions and \( p \in [2, \infty) \). Let \( n \in \mathbb{N} \) and \( \mathcal{F}_n : [0, \infty) \to [0, \infty) \) be a function in \( C^1([0, \infty)) \) satisfying
\[
\begin{align*}
\mathcal{F}_n(\tau) &= \tau &\text{if } 0 \leq \tau \leq n^2, \\
\mathcal{F}_n(\tau) &= n^2 + 1 &\text{if } \tau > n^2 + 1, \\
\tau \leq \mathcal{F}_n(\tau) &\leq \tau + \delta &\text{if } n^2 \leq \tau < n^2 + 1 \text{ for some } \delta \in (0, 1). \tag{3.3}
\end{align*}
\]

Let \( \varepsilon > 0 \) be a small parameter. The operator \( \Delta_{p,a} \) defined in \((1.8)\) is degenerate if \( p > 2 \). To overcome the degeneracy, an \((\varepsilon, p)\)-regularization \( \Delta_{\varepsilon,n,p,a} \) of \( \Delta_{p,a} \) has been introduced (see e.g. [6]) as follows:
\[
\Delta_{\varepsilon,n,p,a} u = \text{div} \left( a(x)(\varepsilon + \mathcal{F}_n(|\nabla u|^2))^{\frac{p-2}{2}} \nabla u \right).
\]
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where $F_n$ is the function defined in (3.3). Using the classical definition of degenerate elliptic operators, one cannot immediately say that $(-\Delta)^s_{\Omega, p}$ or $L^s_{\Omega, p}(\kappa, \cdot)$ is degenerate for $p > 2$. We refer to [23] for a discussion on this topic.

But inspired by the convergence given in (2.5), we let

$$L^s_{\Omega, p, \varepsilon, n}(\kappa, u)(x) := C_{N, p, s} P.V. \int_{\Omega} \kappa(x-y) \left[ \varepsilon + F_n \left( \frac{|u(x)-u(y)|^2}{|x-y|^{2s}} \right) \right]^{\frac{p-2}{2}} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy. \quad (3.4)$$

We call $L^s_{\Omega, p, \varepsilon, n}$ an $(\varepsilon, p)$-regularization of $L^s_{\Omega, p}$.

Now we consider our so called regularized optimal control problem (ROCP):

$$\text{Minimize } \left\{ I(\kappa, u) := \frac{1}{2} \int_{\Omega} |u - \xi|^2 \, dx + \int_{\Omega} |
\nabla \kappa| \right\} \quad (3.5)$$

subject to the constraints

$$\kappa \in \mathfrak{A}_{ad} = \left\{ \eta \in BV(\Omega) : \xi_1(x) \leq \eta(x) \leq \xi_2(x) \text{ a.e. in } \Omega \right\}, \quad (3.6)$$

and

$$\begin{cases} L^s_{\Omega, p, \varepsilon, n}(\kappa, u) + u = f & \text{in } \Omega, \\
0 & \text{on } \partial\Omega. \quad (3.7)$$

The following is our notion of weak solution to the system (3.7).

**Definition 3.5.** Let $n \in \mathbb{N}$, $\varepsilon > 0$, $\kappa \in \mathfrak{A}_{ad}$ and $f \in L^2(\Omega)$. A function $u \in W^{s, 2}_0(\Omega)$ is said to be a weak solution to the system (3.7) if the equality

$$F^\kappa_{\varepsilon, n, p}(u, v) = \int_{\Omega} f v \, dx \quad (3.8)$$

holds for every $v \in W^{s, 2}_0(\Omega)$, where we have set

$$F^\kappa_{\varepsilon, n, p}(u, v) := \int_{\Omega} uv \, dx \quad (3.9)$$

and

$$G_n(u, s) := F_n \left( \frac{|u(x) - u(y)|^2}{|x-y|^{2s}} \right). \quad (3.10)$$

The following theorem is our second main result.

**Theorem 3.6.** For every $\varepsilon > 0$ and $n \in \mathbb{N}$, the ROCP (3.5)-(3.7) has at least one solution $(\kappa_{\varepsilon, n}, u_{\varepsilon, n}) \in BV(\Omega) \times W^{s, 2}_0(\Omega)$. 

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We conclude this section by stating the convergence of solutions of the ROCP to the solutions of the OCP.

**Theorem 3.7.** Let $\frac{1}{2} < t \leq s < 1$ and $p \in [2, \infty)$ with $t = s$ if $p = 2$. Let $n \in \mathbb{N}$ and $\varepsilon > 0$. Let $\{(\kappa_{\varepsilon,n}^*, u_{\varepsilon,n}^*)\}_{\varepsilon > 0, n \in \mathbb{N}} \subset BV(\Omega) \times W_{0}^{t,2}(\Omega)$ be an arbitrary sequence of solutions to the ROCP \[ (3.5) - (3.7) \]. Then $\{(\kappa_{\varepsilon,n}^*, u_{\varepsilon,n}^*)\}_{\varepsilon > 0, n \in \mathbb{N}}$ is bounded in $BV(\Omega) \times W_{0}^{t,2}(\Omega)$ and any cluster point $(\kappa_*, u_*)$ with respect to the (weak*, weak) topology of $BV(\Omega) \times W_{0}^{t,2}(\Omega)$ is a solution to the OCP \[ (1.1), (1.2) \] and \[ (1.3) \]. In addition, if $\kappa_{\varepsilon,n}^* \xrightarrow{\ast} \kappa_*$ in $BV(\Omega)$ and $u_{\varepsilon,n}^* \rightharpoonup u_*$ in $W_{0}^{t,2}(\Omega)$, as $\varepsilon \to 0$ and $n \to \infty$ (that is, as $(\varepsilon, n) \to (0, \infty)$), then the following assertions hold.

\[ \lim_{(\varepsilon, n) \to (0, \infty)} (\kappa_{\varepsilon,n}^*, u_{\varepsilon,n}^*) = (\kappa_*, u_*) \text{ stongly in } L^1(\Omega) \times W_{0}^{t,2}(\Omega). \] \[ (3.11) \]

\[ \lim_{(\varepsilon, n) \to (0, \infty)} \int_{\Omega} |\nabla \kappa_{\varepsilon,n}^*| = \int_{\Omega} |\nabla \kappa_*|. \] \[ (3.12) \]

\[ \lim_{(\varepsilon, n) \to (0, \infty)} \chi(\Omega \times \Omega) \langle (\kappa_{\varepsilon,n}^*, u_{\varepsilon,n}^*) - (\kappa_*, u_*) \rangle = 0 \text{ stongly in } L^p(\Omega \times \Omega). \] \[ (3.13) \]

\[ \lim_{(\varepsilon, n) \to (0, \infty)} \int_{\Omega} \int_{\Omega} \kappa_{\varepsilon,n}^*(x - y) [\varepsilon + G_n(u_{\varepsilon,n}^*, s)]^{p-2} \frac{|u_{\varepsilon,n}^*(x) - u_{\varepsilon,n}^*(y)|^p}{|x - y|^{1 + sp}} dy \]

\[ = \int_{\Omega} \int_{\Omega} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{1 + sp}} dx dy. \] \[ (3.14) \]

\[ \lim_{(\varepsilon, n) \to (0, \infty)} \Pi(\kappa_{\varepsilon,n}^*, u_{\varepsilon,n}^*) = \Pi(\kappa_*, u_*). \] \[ (3.15) \]

where we recall that $G_n$ is given by \[ (3.10) \].

### 4 Proof of Theorem 3.4

To prove the first main result we need some preparations and some intermediate important results.

#### 4.1 The state equation is well-posed

Throughout the remainder of the paper for $u, \varphi \in W_{0}^{t,p}(\Omega)$, we shall let

\[ E_{P,s}(u, \varphi) := \int_{\Omega} u \varphi dx \]

\[ + \frac{C_{N, p, s}}{2} \int_{\Omega} \int_{\Omega} \kappa(x - y)|u(x) - u(y)|^{p-2} \frac{(u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N + sp}} dx dy. \] \[ (4.1) \]

We have the following result of existence of weak solutions to the system \[ (1.2) \].

**Proposition 4.1 (The well-posedness of the state equation).** For every $f \in L^2(\Omega)$, the system \[ (1.2) \] has a unique weak solution $u \in W_{0}^{t,p}(\Omega)$. In addition there exists a constant $C > 0$ such that

\[ \|u\|_{W_{0}^{t,p}(\Omega)}^{p-1} \leq C \|f\|_{L^2(\Omega)}. \] \[ (4.2) \]

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Proof. The proposition follows by showing first that \( E_{p,s}^\kappa(u, \cdot) \in W^{-s,p}(\Omega) \) for every fixed \( u \in W_0^{s,p}(\Omega) \), and then that \( E_{p,s}^\kappa \) is strictly monotone, hemi-continuous and coercive. Finally (4.2) follows by taking \( \varphi = u \) as a test function in (3.2). For more details we refer to [2, Proposition 2.3]. The proof is finished.

Remark 4.2 (The state equation and Minty relation). As a consequence of the proof of Proposition 4.1, we have that \( u \in W_0^{s,p}(\Omega) \) satisfies (3.2) if and only if the Minty relation holds. That is, for every \( \varphi \in W_0^{s,p}(\Omega) \),

\[
E_{p,s}^\kappa(\varphi, \varphi - u) \geq \int_{\Omega} f(\varphi - u) \, dx.
\]

(4.3)

For more details we refer to [2, Remark 2.5].

4.2 The optimal control problem (OCP)

Towards this end we introduce the set of admissible control-state pair for the OCP (1.1)-(1.2), namely,

\[
\Xi := \left\{ (\kappa, u) : \kappa \in \mathfrak{A}_{ad}, u \in W_0^{s,p}(\Omega), (\kappa, u) \text{ are related by (3.2)} \right\}.
\]

(4.4)

Using Proposition 4.1, we get that the set \( \Xi \) is nonempty. With the notation (4.4), we have that the OCP (1.1)-(1.2) can be rewritten as the following minimization problem:

\[
\min_{(\kappa, u) \in \Xi} I(\kappa, u).
\]

(4.5)

Next, we endow the Banach space \( BV(\Omega) \times W_0^{s,p}(\Omega) \) with the norm defined by

\[
\| (\kappa, u) \|_{BV(\Omega) \times W_0^{s,p}(\Omega)} := \| \kappa \|_{BV(\Omega)} + \| u \|_{W_0^{s,p}(\Omega)}.
\]

We have the following result.

Lemma 4.3. Let \( \{ (\kappa_n, u_n) \}_{n \in \mathbb{N}} \subset \Xi \) be such that \( \kappa_n \rightharpoonup \kappa \) in \( BV(\Omega) \) and \( u_n \rightharpoonup u \) in \( W_0^{s,p}(\Omega) \), as \( n \to \infty \). Then for every \( \varphi \in \mathcal{D}(\Omega) \) we have that

\[
\lim_{n \to \infty} \int_{\Omega} \int_{\Omega} \kappa_n(x - y) \frac{(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy = \int_{\Omega} \int_{\Omega} \kappa(x - y) \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy.
\]

(4.6)

Proof. First, since \( \kappa_n \to \kappa \) in \( L^1(\Omega) \) as \( n \to \infty \) and \( \{ \kappa_n \}_{n \in \mathbb{N}} \) is bounded in \( L^\infty(\Omega) \), we have that \( \kappa_n \rightharpoonup \kappa \) in \( L^q(\Omega) \) as \( n \to \infty \), for every \( 1 \leq q < \infty \).

(4.7)

In addition we have that \( \kappa \in \mathfrak{A}_{ad} \). Since \( u_n \rightharpoonup u \) in \( W_0^{s,p}(\Omega) \) as \( n \to \infty \), it follows from Remark 2.3 that for every \( \varphi \in \mathcal{D}(\Omega) \),

\[
\lim_{n \to \infty} \int_{\Omega} \int_{\Omega} \frac{(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy = \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy.
\]
Let $\varphi \in \mathcal{D}(\Omega)$ and define the functions $F_{n,p}^\varphi, F_p^\varphi : \Omega \times \Omega \to \mathbb{R}$ by

$$F_{n,p}^\varphi(x, y) := \frac{(u_n(x) - u_n(y))(\varphi(x) - \varphi(y))}{|x - y|^{\frac{N}{p} + s + 1}},$$

and

$$F_p^\varphi(x, y) := \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{\frac{N}{p} + s + 1}}.$$

Then

$$\int_\Omega \int_\Omega |F_{n,p}^\varphi(x, y)|^p \, dx \, dy \leq ||\varphi||_{C^{0,1}(\Omega)}^p \|u_n\|_{W_0^{s,p}(\Omega)}^p < \infty.$$  \hspace{1cm} (4.8)

Similarly, we get that $F_{n,p}^\varphi, F_p^\varphi \in L^p(\Omega \times \Omega)$. Since $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $W_0^{s,p}(\Omega)$, it follows from [4,3] that $\{F_{n,p}^\varphi\}_{n \in \mathbb{N}}$ is bounded in $L^p(\Omega \times \Omega)$. Thus, after a (sub)sequence if necessary, $F_{n,p}^\varphi$ converges weakly to some function $F$ in $L^p(\Omega \times \Omega)$, as $n \to \infty$. Since $u_n$ converges a.e. to $u$ in $\Omega$ as $n \to \infty$, it follows that $F_{n,p}^\varphi$ converges a.e. to $F_p^\varphi$ in $\Omega \times \Omega$, as $n \to \infty$. By the uniqueness of the limit we have that $F_p^\varphi = F$. We have shown that $F_{n,p}^\varphi \rightharpoonup F_p^\varphi$ in $L^p(\Omega \times \Omega)$ as $n \to \infty$. Let $K_{n,p'}, K_{p'} : \Omega \times \Omega \to \mathbb{R}$ be the functions given by

$$K_{n,p'}(x, y) := \frac{\kappa_n(x - y)}{|x - y|^{\frac{N}{p'} + s - 1}} \quad \text{and} \quad K_{p'}(x, y) := \frac{\kappa(x - y)}{|x - y|^{\frac{N}{p'} + s - 1}}.$$

Let $x \in \Omega$ be fixed. Let $B(x, R)$ be a large ball with center $x$ and radius $R$ such that $\Omega \subset B(x, R)$. Since $\kappa_n \in L^\infty(\Omega)$, then using polar coordinates, we have that there exists a constant $C > 0$ (depending on $\Omega$, $N$, $s$ and $p$) such that

$$\int_\Omega \int_\Omega |K_{n,p'}(x, y)|^{p'} \, dx \, dy \leq \|\kappa_n\|_{L^\infty(\Omega)}^{p'} \int_\Omega \int_\Omega \frac{1}{|x - y|^{N + p'(s - 1)}} \, dx \, dy \leq C \|\kappa_n\|_{L^\infty(\Omega)}^{p'} \int_0^R r^{p'(1-s)-1} \, dr \leq C \|\kappa\|_{L^\infty(\Omega)}^{p'} < \infty.$$

Thus $K_{n,p'} \in L^{p'}(\Omega \times \Omega)$. Similarly, we get that $K_{p'} \in L^{p'}(\Omega \times \Omega)$. Using (4.7) and the Lebesgue Dominated Convergence Theorem, we get that $K_{n,p'} \rightharpoonup K_{p'}$ in $L^{p'}(\Omega \times \Omega)$ as $n \to \infty$. Therefore,

$$\lim_{n \to \infty} \int_\Omega \int_\Omega K_{n,p'}(x, y) F_{n,p}^\varphi(x, y) \, dx \, dy = \int_\Omega \int_\Omega K_{p'}(x, y) F_p^\varphi(x, y) \, dx \, dy = \int_\Omega \int_\Omega \kappa(x - y) \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N + 2s}} \, dx \, dy,$$

for every $\varphi \in \mathcal{D}(\Omega)$. We have shown (4.6) and the proof is finished.

Using Lemma [4,3] we can prove the following theorem which will play an important role in the proof of our first main result.

**Theorem 4.4.** Let $\{(\kappa_n, u_n)\}_{n \in \mathbb{N}} \subset \Xi$ be a bounded sequence. Then there exists $(\kappa, u) \in \Xi$ such that, after a (sub)sequence if necessary, $\kappa_n \rightharpoonup \kappa$ in $BV(\Omega)$, $u_n \rightharpoonup u$ in $W_0^{s,p}(\Omega)$ and $u_n \to u$ in $L^2(\Omega)$, as $n \to \infty$. 

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Proof. First, since \( \{u_n\}_{n \in \mathbb{N}} \) is bounded in \( W_0^{s,p}(\Omega) \) and the continuous embedding \( W_0^{s,p}(\Omega) \hookrightarrow L^2(\Omega) \) is compact, then after a (sub)sequence if necessary, there exists a \( u \in W_0^{s,p}(\Omega) \) such that

\[
u_n \rightharpoonup u \text{ in } W_0^{s,p}(\Omega) \quad \text{and} \quad \nu_n \to u \text{ in } L^2(\Omega), \quad \text{as } n \to \infty.
\]

Next, since \( \{\kappa_n\}_{n \in \mathbb{N}} \) is bounded in \( BV(\Omega) \), it follows from [11 Corollary 3.39] that after a (sub)sequence if necessary, there exists a \( \kappa \in L^1(\Omega) \) such that \( \kappa_n \rightharpoonup \kappa \) in \( L^1(\Omega) \). Since \( \kappa_n \rightharpoonup \kappa \) in \( L^1(\Omega) \) as \( n \to \infty \) and \( \sup_{n \in \mathbb{N}} \int_{\Omega} |\nabla \kappa_n| < \infty \) (this follows from the fact that \( \{\kappa_n\}_{n \in \mathbb{N}} \) is bounded in \( BV(\Omega) \)), then by Remark 2.4(b), this implies that \( \kappa \in BV(\Omega) \) and \( \kappa_n \rightharpoonup \kappa \) in \( BV(\Omega) \) as \( n \to \infty \). We have shown that, as \( n \to \infty \),

\[
k_n \rightharpoonup \kappa \text{ in } BV(\Omega), \quad \nu_n \rightharpoonup u \text{ in } W_0^{s,p}(\Omega) \quad \text{and} \quad \nu_n \to u \text{ in } L^2(\Omega).
\]

It remains to show that \((\kappa, u) \in \Xi \). Since \( \alpha \leq \xi_1(x) \leq \kappa_n(x) \leq \xi_2(x) \) for a.e. \( x \in \Omega \) and every \( n \in \mathbb{N} \), we have that \( \kappa \in \mathcal{A}_{ad} \). It also follows from Lemma 4.3 that

\[
\lim_{n \to \infty} \int_{\Omega} \int_{\Omega} \kappa_n(x-y) \frac{(u_n(x)-u_n(y))((\varphi(x)-\varphi(y)))}{|x-y|^{N+2s}} \, dx \, dy \leq \int_{\Omega} \int_{\Omega} \kappa(x-y) \frac{(u(x)-u(y))((\varphi(x)-\varphi(y)))}{|x-y|^{N+2s}} \, dx \, dy,
\]

for every \( \varphi \in \mathcal{D}(\Omega) \). Let \( \Phi : \Omega \times \Omega \to \mathbb{R} \) be given by \( \Phi(x,y) := \frac{|\varphi(x)-\varphi(y)|^{p-2}}{|x-y|^s} \). Note that for a.e. \( x, y \in \Omega \), we have that \( |\Phi(x,y)| \leq \|\varphi\|_{C^{0,s}_x}^2 \). Since \( \Phi \in L^\infty(\Omega \times \Omega) \), if we multiply the functions under the integrals in both sides of (4.10) by \( \Phi \), then we have the same convergence. This implies that for every \( \varphi \in \mathcal{D}(\Omega) \),

\[
\lim_{n \to \infty} \int_{\Omega} \int_{\Omega} \kappa_n(x-y) \frac{|\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))(u_n(x)-u_n(y))}{|x-y|^{N+2s}} \, dx \, dy
\]

\[
= \int_{\Omega} \int_{\Omega} \kappa(x-y) \frac{|\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))(u(x)-u(y))}{|x-y|^{N+2s}} \, dx \, dy
\]

\[
= \int_{\Omega} \int_{\Omega} \kappa(x-y) |\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))(u(x)-u(y)) \, dx \, dy.
\]

We show that \((\kappa, u) \) is related by (4.3). Since \((\kappa_n, u_n) \) satisfies (4.3) we have that

\[
\mathcal{E}^{\kappa_n}_{p,s}(\varphi, \varphi - u_n) \geq \int_{\Omega} f(\varphi - u_n) \, dx,
\]

for every \( \varphi \in \mathcal{D}(\Omega) \), where we recall that \( \mathcal{E}^{\kappa_n}_{p,s}(\varphi, \varphi - u_n) = \mathcal{E}^{\kappa_n}_{p,s}(\varphi, \varphi) - \mathcal{E}^{\kappa_n}_{p,s}(\varphi, u_n) \). It follows from (4.9) that

\[
\lim_{n \to \infty} \int_{\Omega} f(\varphi - u_n) \, dx = \int_{\Omega} f \varphi \, dx - \lim_{n \to \infty} \int_{\Omega} f u_n \, dx = \int_{\Omega} f(\varphi - u) \, dx.
\]

Using (4.11) and (4.13) we can pass to the limit in (4.12) as \( n \to \infty \) and obtain that \((\kappa, u) \) is related by (4.3) for every \( \varphi \in \mathcal{D}(\Omega) \). Finally, since \( \mathcal{D}(\Omega) \) is dense in \( W_0^{s,p}(\Omega) \), we have that (4.3) also holds for every \( \varphi \in W_0^{s,p}(\Omega) \). Hence, \( u \in W_0^{s,p}(\Omega) \) is a weak solution of (1.2). This, together with \( \kappa \in \mathcal{A}_{ad} \) imply \((\kappa, u) \) \in \Xi \). 

\[\square\]
Now we are able to give the proof of our first main result.

**Proof of Theorem 3.4.** Since the set $\Xi$ is nonempty and the cost functional is bounded from below on $\Xi$, it follows that there exists a minimizing sequence $(\kappa_n, u_n) \in \Xi$ to the problem \eqref{eq:4.5}, that is,

$$
\inf_{(\kappa, u) \in \Xi} I(\kappa, u) = \lim_{n \to \infty} \left[ \frac{1}{2} \int_\Omega |u_n - \xi|^2 \, dx + \int_\Omega |\nabla \kappa_n| \right] < \infty.
$$

This implies that $\{(\kappa_n, u_n)\}_{n \in \mathbb{N}}$ is bounded in $BV(\Omega) \times W_0^{s,p}(\Omega)$. It follows from Theorem 4.3 that after a (sub)sequence if necessary, there exists $(\kappa_\star, u_\star) \in \Xi$ such that $\kappa_n \rightharpoonup \kappa_\star$ in $BV(\Omega)$, $u_n \rightharpoonup u_\star$ in $W_0^{s,p}(\Omega)$ and $u_n \to u_\star$ in $L^2(\Omega)$, as $n \to \infty$. Therefore using also Remark 2.4, we get that

$$
\lim_{n \to \infty} \frac{1}{2} \int_\Omega |u_n - \xi|^2 \, dx = \frac{1}{2} \int_\Omega |u_* - \xi|^2 \, dx \quad \text{and} \quad \int_\Omega |\nabla \kappa_*| \leq \liminf_{n \to \infty} \int_\Omega |\nabla \kappa_n|.
$$

We have shown that $I(\kappa_\star, u_\star) \leq \inf_{(\kappa, u) \in \Xi} I(\kappa, u)$. Thus $(\kappa_\star, u_\star)$ is a solution to \eqref{eq:4.5} and hence, a solution to our initial OCP \eqref{eq:1.1}–\eqref{eq:1.2}. The proof is finished. \hfill \Box

### 5 Proof of Theorem 3.6

Here also in order to be able to prove our theorem we need some preparation. For $\phi \in W_0^{s,2}(\Omega)$, $p \in [2, \infty)$, $n \in \mathbb{N}$ and $\varepsilon > 0$ a small parameter, we shall use the following notation:

$$
\|\phi\|_{\varepsilon, n, \kappa, s, p} := \left( \int_\Omega \int_\Omega \kappa(x - y) \left[ \varepsilon + G_n(\phi, s) \right] \frac{2^{p-2} |\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} \, dxdy \right)^{\frac{1}{p}} \quad (5.1)
$$

where we recall that $G_n(\phi, s) := \mathcal{F}_n \left( \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{2s}} \right)$. We notice that $\| \cdot \|_{\varepsilon, n, \kappa, s, p}$ is a quasi-norm but is not a norm unless $p = 2$.

Let $\omega : \Omega \times \Omega \to \mathbb{R}$ be the function and $\mu$ the measure on $\Omega \times \Omega$ given by

$$
\omega(x, y) := \frac{1}{|x - y|^{N+2s-2}} \quad \text{and} \quad d\mu(x, y) := \omega(x, y) \, dxdy. \quad (5.2)
$$

Let $x \in \Omega$ fixed and $R > 0$ such that $\Omega \subset B(x, R)$. Using polar coordinates we get that there exists a constant $C > 0$ (depending only on $N$ and $s$) such that

$$
\mu(\Omega \times \Omega) := \int_\Omega \int_\Omega \omega(x, y) \, dxdy = \int_\Omega \int_\Omega \frac{1}{|x - y|^{N+2s-2}} \, dxdy \\
\leq \int_\Omega dx \int_{B(x, R)} \frac{1}{|x - y|^{N+2s-2}} \, dy \leq C|\Omega| \int_0^R \frac{1}{r^{2s-1}} \, dr = \frac{C|\Omega|}{2(1-s)} R^{2(1-s)} < \infty.
$$

For $u \in W_0^{s,p}(\Omega)$ or $u \in W_0^{s,2}(\Omega)$ fixed and $n \in \mathbb{N}$, we consider the level set

$$(\Omega \times \Omega)_n(u) := \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(x) - u(y)|}{|x - y|^s} > \sqrt{n^2 + 1} \right\}.$$

We have the following result.
Lemma 5.1. The following assertions hold.

(a) There exists a constant $C > 0$ such that for $u \in W^{s,2}_0(\Omega)$,
\[
\left| (\Omega \times \Omega)_n(u) \right| \leq \frac{Cn^{-1}}{n^p} \|u\|^p_{\varepsilon,n,k,s,p}.
\] (5.3)

(b) There exists a constant $C > 0$ such that for $u \in W^{s,2}_0(\Omega)$,
\[
\mu\left( (\Omega \times \Omega)_n(u) \right) \leq \frac{Cn^{-1}}{n^p} \|u\|^p_{\varepsilon,n,k,s,p}.
\] (5.4)

Proof. Let $u \in W^{s,2}_0(\Omega)$ and $p \in [2, \infty)$.

(a) Using the H"older inequality and (3.3) we get that there exists a constant $C > 0$ such that
\[
\begin{align*}
\left| (\Omega \times \Omega)_n(u) \right| &= \int \int_{(\Omega \times \Omega)_n(u)} 1 \ dxdy \\
&\leq Cn \left| (\Omega \times \Omega)_n(u) \right|^\frac{1}{n} \left( \int \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \ dxdy \right)^\frac{n}{2} \\
&\leq Cn \left( \frac{1}{\varepsilon + n^2 + 1} \right)^\frac{n-2}{2} \left| (\Omega \times \Omega)_n(u) \right|^\frac{1}{n} \alpha^{-\frac{1}{2}} \\
&\quad \times \left( \int \int_{\Omega} \left[ \varepsilon + G_n(u, s) \right] \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \ dxdy \right)^\frac{n}{2} \\
&\leq Cn \frac{1}{\varepsilon + n^2 + 1} \left| (\Omega \times \Omega)_n(u) \right|^\frac{1}{n} \|u\|^p_{\varepsilon,n,k,s,p}.
\end{align*}
\] (5.5)

We have shown (5.3).

(b) Let $\omega$ be the weighted function and $\mu$ the measure given in (5.2). Proceeding as in (5.5) we get that there exists a constant $C > 0$ such that
\[
\begin{align*}
\mu\left( (\Omega \times \Omega)_n(u) \right) &= \frac{1}{\varepsilon + n^2 + 1} \int \int_{(\Omega \times \Omega)_n(u)} \frac{\sqrt{n^2 + 1}}{|x - y|^{N+2s-2}} \ dxdy \\
&\leq Cn \mu\left( (\Omega \times \Omega)_n(u) \right)^\frac{1}{n} \left( \int \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \ dxdy \right)^\frac{n}{2} \\
&\leq Cn \left( \frac{1}{\varepsilon + n^2 + 1} \right)^\frac{n-2}{2} \mu\left( (\Omega \times \Omega)_n(u) \right)^\frac{1}{n} \alpha^{-\frac{1}{2}} \\
&\quad \times \left( \int \int_{\Omega} \left[ \varepsilon + G_n(u, s) \right] \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \ dxdy \right)^\frac{n}{2} \\
&\leq Cn \frac{1}{\varepsilon + n^2 + 1} \mu\left( (\Omega \times \Omega)_n(u) \right)^\frac{1}{n} \|u\|^p_{\varepsilon,n,k,s,p}.
\end{align*}
\] (5.5)

We have shown (5.4) and the proof is finished. \qed
5.1 Well-posedness of the regularized problem

Next, we show the existence of solution to the regularized state equation (3.7).

**Proposition 5.2.** For every \( \varepsilon > 0, n \in \mathbb{N} \), \( \kappa \in \mathfrak{A}_{ad} \) and \( f \in L^2(\Omega) \), the system (3.7) has a unique weak solution \( u_{\varepsilon,n} \in W_0^{s,2}(\Omega) \). In addition, there exists a constant \( C > 0 \) (depending only on \( \Omega \) and \( s \)) such that

\[
\varepsilon \frac{p}{p-2} \| u_{\varepsilon,n} \|_{W_0^{s,2}(\Omega)} \leq C \| f \|_{L^2(\Omega)}.
\]

**Proof.** Here also, the proposition follows by showing that \( \mathbb{F}^\kappa_{\varepsilon,n,p}(u, \cdot) \in W^{-s,2}(\Omega) \) for every fixed \( u \in W_0^{s,2}(\Omega) \), and that \( \mathbb{F}^\kappa_{\varepsilon,n,p} \) is hemicontinuous, strictly monotone and coercive. The estimates (5.6) follows by taking \( v = u_{\varepsilon,n} \) as a test function in (3.8). For more details we refer to [2, Proposition 2.7]. The proof is finished. \( \square \)

**Remark 5.3 (The regularized state equation and Minty relation).** As in Remark 4.2 we have that \( u_{\varepsilon,n} \in W_0^{s,2}(\Omega) \) satisfies (3.8) if and only if the following **Minty relation** holds, that is, for every \( \varphi \in W_0^{s,2}(\Omega) \),

\[
\mathbb{F}^\kappa_{\varepsilon,n,p}(\varphi, \varphi - u_{\varepsilon,n}) \geq \int_{\Omega} f(\varphi - u_{\varepsilon,n}) \, dx.
\]

For more details see [2, Remark 2.9].

5.2 The regularized optimal control problem (ROCP)

We begin by introducing the set of admissible controls for the ROCP (3.5)-(3.6). That is,

\[
\Xi_{\varepsilon,n} = \left\{ (\kappa, u) : \kappa \in \mathfrak{A}_{ad}, u \in W_0^{s,2}(\Omega), (\kappa, u) \text{ are related by } (3.8) \right\}.
\]

It follows from Remark 5.3 that the set \( \Xi_{\varepsilon,n} \) is nonempty for every \( \varepsilon > 0 \) and \( n \in \mathbb{N} \). Therefore the ROCP (3.5)-(3.6) can be rewritten as the minimization problem:

\[
\min_{(\kappa, u) \in \Xi_{\varepsilon,n}} I(\kappa, u).
\]

Now we are ready to give the proof of our second main result.

**Proof of Theorem 3.6.** Since the set \( \Xi_{\varepsilon,n} \) given in (5.8) is nonempty, then we can take a minimizing sequence \( \{(\kappa_k, u_k)\}_{k \in \mathbb{N}} \subset \Xi_{\varepsilon,n} \). As \( \{\kappa_k\}_{k \in \mathbb{N}} \) is bounded in \( L^\infty(\Omega) \) and \( 0 < \alpha \leq \kappa_k(x) \leq \xi_2(x) \) for a.e. \( x \in \Omega \) and every \( k \in \mathbb{N} \), we have that there exists a constant \( C > 0 \) (independent of \( k \)) such that

\[
\| \kappa_k \|_{BV(\Omega)} \leq \| \xi_2 \|_{L^1(\Omega)} + C.
\]

Using the lower boundedness of \( I \), the above estimate and (5.6), we have that there exists a constant \( C > 0 \) (independent of \( k \)) such that

\[
\| \kappa_k \|_{BV(\Omega)} + \| u_k \|_{W_0^{s,2}(\Omega)} \leq C \left( \| \xi_2 \|_{L^1(\Omega)} + 1 + \varepsilon^{\frac{2p}{2-p}} \| f \|_{L^2(\Omega)} \right).
\]

Thus \( \{(\kappa_k, u_k)\}_{k \in \mathbb{N}} \) is bounded in \( BV(\Omega) \times W_0^{s,2}(\Omega) \). Therefore, proceeding as the proof of Theorem 3.4 we deduce the existence of a (sub)sequence still denoted by \( \{(\kappa_k, u_k)\}_{k \in \mathbb{N}} \), and a pair \( (\kappa, u) \in \Xi_{\varepsilon,n} \) such that \( \kappa_k \rightharpoonup \kappa \) in \( BV(\Omega) \), \( u_k \rightharpoonup u \) in \( W_0^{s,2}(\Omega) \) and \( u_k \to u \) in \( L^2(\Omega) \), as \( k \to \infty \). Thus \( I(\kappa, u) \leq \liminf_{k \to \infty} I(\kappa_k, u_k) \). The proof of the theorem is finished. \( \square \)
5.3 Further a priori estimates for the regularized state

Next we give further a priori estimates of weak solutions to the system (3.7). These results will be useful to show the convergence of solutions to the ROCP in section 6.

**Proposition 5.4.** Let $\kappa \in A_{ad}$, $n \in \mathbb{N}$ and $\varepsilon > 0$ be given. Then there exists a constant $C > 0$ such that for every $f \in L^2(\Omega)$ and $u \in W^{s,2}_0(\Omega)$,

$$\left| \int_{\Omega} fu \, dx \right| \leq C \|f\|_{L^2(\Omega)} \left[ \alpha^{-\frac{1}{p}} \|u\|_{\varepsilon,n,\kappa,s,p} + \alpha^{-\frac{1}{2}} \|u\|_{\varepsilon,n,\kappa,s,p}^p \right].$$  \hspace{1em}(5.10)

**Proof.** We associate with $u \in W^{s,2}_0(\Omega)$ the level set $(\Omega \times \Omega)^n(u)$ given by

$$(\Omega \times \Omega)^n(u) := \left\{ (x, y) \in \Omega \times \Omega : \frac{|u(x) - u(y)|}{|x - y|^s} > n \right\}. $$  \hspace{1em}(5.11)

Let $\frac{1}{2} < t \leq s < 1$ and $p \in [2, \infty)$ with $t = s$ if $p = 2$. Then $W^{s,2}_0(\Omega) \hookrightarrow W^{t,2}_0(\Omega)$ (by Proposition 2.2) and hence, $u \in W^{t,2}_0(\Omega)$. Using (2.3) and (2.1), we get that there exists a constant $C > 0$ such that

$$\left| \int_{\Omega} fu \, dx \right| \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)} \|u\|_{W^{t,2}_0(\Omega)}$$

$$\leq C \|f\|_{L^2(\Omega)} \left[ \left( \int_{(\Omega \times \Omega) \setminus (\Omega \times \Omega)^n(u)} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2t}} \, dx \, dy \right)^{\frac{1}{2}} \right. $$

$$\left. + \left( \int_{(\Omega \times \Omega)^n(u)} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2t}} \, dx \, dy \right)^{\frac{1}{2}} \right].$$  \hspace{1em}(5.12)

Using the Hölder inequality we get that there is a constant $C > 0$ such that

$$\left( \int_{(\Omega \times \Omega) \setminus (\Omega \times \Omega)^n(u)} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2t}} \, dx \, dy \right)^{\frac{1}{2}} \leq \left( \int_{(\Omega \times \Omega) \setminus (\Omega \times \Omega)^n(u)} \frac{1}{|x - y|^{N-\frac{2N}{p}+2(t-s)}} \, dx \, dy \right)^{\frac{p}{p-2}} \times \left( \int_{(\Omega \times \Omega) \setminus (\Omega \times \Omega)^n(u)} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy \right)^{\frac{1}{p}} \leq C \left( \int_{(\Omega \times \Omega) \setminus (\Omega \times \Omega)^n(u)} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy \right)^{\frac{1}{p}}.$$  \hspace{1em}(5.13)
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It follows from (5.13) that there exists a constant $C > 0$ such that

$$
\left( \int \int_{(\Omega\times\Omega)\setminus(\Omega\times\Omega)^n(u)} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dxdy \right)^{\frac{1}{2}}
\leq C \left( \int \int_{(\Omega\times\Omega)\setminus(\Omega\times\Omega)^n(u)} \left( \varepsilon + \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \right)^{\frac{p-2}{2}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dxdy \right)^{\frac{1}{2}}.
$$

(5.14)

Since $F_n \left( \frac{|u(x) - u(y)|^2}{|x - y|^{2s}} \right) = \frac{|u(x) - u(y)|^2}{|x - y|^{2s}}$ a.e. in $(\Omega \times \Omega) \setminus (\Omega \times \Omega)^n(u)$, and $0 < \alpha \leq \kappa(x - y)$ for a.e. $x, y \in \Omega$, it follows from (5.14) that

$$
\left( \int \int_{(\Omega\times\Omega)\setminus(\Omega\times\Omega)^n(u)} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2t}} \, dxdy \right)^{\frac{1}{2}}
\leq C\alpha^{-\frac{1}{p}} \left( \int \int_{(\Omega\times\Omega)\setminus(\Omega\times\Omega)^n(u)} \kappa(x - y) \left[ \varepsilon + G_n(u, s) \right]^{\frac{p-2}{2}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dxdy \right)^{\frac{1}{2}}
\leq C\alpha^{-\frac{1}{p}}\|u\|_{\varepsilon, n, \kappa, s, p}.
$$

(5.15)

Since $n^2 \leq F_n \left( \frac{|u(x) - u(y)|^2}{|x - y|^{2s}} \right) \leq n^2 + 1$ a.e. in $(\Omega \times \Omega)^n(u)$, and $\frac{1}{2} < t \leq s < 1$, we have that there exists a constant $C > 0$ such that

$$
\left( \int \int_{(\Omega\times\Omega)^n(u)} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2t}} \, dxdy \right)^{\frac{1}{2}}
\leq C \left( \int \int_{(\Omega\times\Omega)^n(u)} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2t}} \, dxdy \right)^{\frac{1}{2}}
\leq C\alpha^{-\frac{1}{p}} \left( \int \int_{(\Omega\times\Omega)^n(u)} \kappa(x - y) \left[ \varepsilon + G_n(u, s) \right]^{\frac{p-2}{2}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dxdy \right)^{\frac{1}{2}}
\leq C\alpha^{-\frac{1}{p}}\|u\|_{\varepsilon, n, \kappa, s, p}.
$$

(5.16)

Now (5.10) follows from (5.12), (5.15) and (5.16). The proof is finished.

Proposition 5.5. Let $\varepsilon > 0$ and $n \in \mathbb{N}$. Then for every $\kappa \in \mathcal{A}_{ad}$ and $f \in L^2(\Omega)$, the sequence of weak solution $\{u_{\varepsilon,n}\}_{\varepsilon > 0, n \in \mathbb{N}}$ to the system (3.7) is bounded with respect to the $\| \cdot \|_{\varepsilon, n, \kappa, s, p}$-quasi norm, that is,

$$
\sup_{\varepsilon > 0, n \in \mathbb{N}} \|u_{\varepsilon,n}\|_{\varepsilon, n, \kappa, s, p} < \infty.
$$

(5.17)
Proof. Using (3.8) and (5.10) we get that there is a constant $C > 0$ such that
\[
\|u_{\varepsilon,n}\|_{\varepsilon,n,\kappa,s,p} \leq \mathbb{F}_{\varepsilon,n,p}^\kappa(u_{\varepsilon,n}, u_{\varepsilon,n}) = \int_\Omega fu_{\varepsilon,n} \, dx
\]
\[
\leq C\|f\|_{L^2(\Omega)} \left[\alpha^{-\frac{1}{p}}\|u\|_{\varepsilon,n,\kappa,s,p} + \alpha^{-\frac{1}{2}}\|u\|_{\varepsilon,n,\kappa,s,p}^2\right].
\]  
(5.18)

Let $C_f := C\left(\alpha^{-\frac{1}{p}} + \alpha^{-\frac{1}{2}}\right)\|f\|_{L^2(\Omega)}$. It follows from (5.18) that
\[
\|u_{\varepsilon,n}\|_{\varepsilon,n,\kappa,s,p} \leq \max \left\{C_f^\frac{2}{p}, C_f^{\frac{1}{p-1}}\right\}, \quad \forall \varepsilon > 0, \forall n \in \mathbb{N}, \forall \kappa \in \mathcal{A}_{ad}.
\]  
(5.19)

Now (5.17) follows from (5.19) and the proof is finished. \[\Box\]

We also mention that using (5.10) and (5.19) we get that
\[
\|u_{\varepsilon,n}\|_{L^2(\Omega)} \leq \max \left\{C^2\|f\|_{L^2(\Omega)}, C^\frac{p}{p-1}\|f\|_{L^2(\Omega)}^{\frac{1}{p-1}}\right\}, \quad \forall \varepsilon > 0, \forall n \in \mathbb{N},
\]  
(5.20)

where $C$ is the constant appearing in (5.10).

We conclude this section with the following remark.

**Remark 5.6.** Let $\kappa \in \mathcal{A}_{ad}$, $n \in \mathbb{N}$, $\varepsilon > 0$ and $u_{\varepsilon,n} \in W_0^{s,2}(\Omega)$ be the solution of (3.7). Let $(\Omega \times \Omega)^n(u_{\varepsilon,n})$ be given by (5.11). Let $p \in [2, \infty)$ and $\frac{1}{2} < t \leq s < 1$ with $t = s$ if $p = 2$. We notice that it follows from (5.15) and (5.16) that there exists a constant $C > 0$ (depending on $\Omega, N, s, t$ and $p$) such that
\[
\|u_{\varepsilon,n}\|_{W_0^{t,2}(\Omega)} \leq C\left(\alpha^{-\frac{1}{p}}\|u_{\varepsilon,n}\|_{\varepsilon,n,\kappa,s,p} + \alpha^{-\frac{1}{2}}\|u_{\varepsilon,n}\|_{\varepsilon,n,\kappa,s,p}^2\right).
\]  
(5.21)

We do not know if (5.21) holds with $W_0^{t,2}(\Omega)$ replaced by $W_0^{s,2}(\Omega)$ in the case $p > 2$.

### 6 Proof of Theorem 3.7

Before we give the proof of our last main result, i.e., Theorem 3.7 we need some intermediate results.

**Lemma 6.1.** Let $\frac{1}{2} < t \leq s < 1$ and $p \in [2, \infty)$ with $t = s$ if $p = 2$. Let $\{\kappa_{\varepsilon,n}\}_{\varepsilon>0,n\in\mathbb{N}} \subset \mathcal{A}_{ad}$ be an arbitrary sequence of admissible control associated with the states $\{u_{\varepsilon,n}\}_{\varepsilon>0,n\in\mathbb{N}} \subset W_0^{s,2}(\Omega)$. Then $\{u_{\varepsilon,n}\}_{\varepsilon>0,n\in\mathbb{N}}$ is bounded in $W_0^{t,2}(\Omega)$. In addition, each cluster point $u$ of $\{u_{\varepsilon,n}\}_{\varepsilon>0,n\in\mathbb{N}}$ with respect to the weak topology in $W_0^{t,2}(\Omega)$, belongs to $W_0^{s,p}(\Omega)$.

**Proof.** Recall that the estimate (5.21) in Remark 5.6 holds. Now using (5.19) we get from (5.21) that $\{u_{\varepsilon,n}\}_{\varepsilon>0,n\in\mathbb{N}}$ is bounded in $W_0^{t,2}(\Omega)$. Let $\{u_{\varepsilon_k,n_k}\}_{k \in \mathbb{N}}$ be a (sub)sequence and $u \in W_0^{t,2}(\Omega)$ be such that $u_{\varepsilon_k,n_k} \rightharpoonup u$ in $W_0^{t,2}(\Omega)$ and $u_{\varepsilon_k,n_k} \rightarrow u$ in $L^2(\Omega)$, as $k \rightarrow \infty$. Let $k \in \mathbb{N}$ be fixed and set
\[
\mathbb{B}_k := \bigcup_{j=k}^{\infty} (\Omega \times \Omega)_{n_j}(u_{\varepsilon_j,n_j}),
\]  
(6.1)
where we recall that
\[(\Omega \times \Omega)_{n_j}(u_{\varepsilon_j,n_j}) := \{(x,y) \in \Omega \times \Omega : \frac{|u_{\varepsilon_j,n_j}(x) - u_{\varepsilon_j,n_j}(y)|}{|x-y|^s} > \sqrt{n_j^2 + 1}\}. \tag{6.2}\]

Using \((5.4)\) and \((5.19)\) we get that
\[
\left| E_k \right| = \alpha^{-1} \sum_{j=k}^{\infty} \frac{1}{n_j^p} \|u_{\varepsilon_j,n_j}\|^p_{\varepsilon_j,n_j,\kappa_{\varepsilon_j,s}} \leq \alpha^{-1} \sup_{j \in \mathbb{N}} \|u_{\varepsilon_j,n_j}\|^p_{\varepsilon_j,n_j,\kappa_{\varepsilon_j,s}} \sum_{j=k}^{\infty} \frac{1}{n_j^p}
\leq \alpha^{-1} \max \left\{ C_f^p, C_f^{\frac{p}{p-1}} \right\} \sum_{j=k}^{\infty} \frac{1}{n_j^p} < \infty.
\]

Therefore,
\[
\lim_{k \to \infty} \left| E_k \right| = \limsup_{k \to \infty} \left| E_k \right| = 0. \tag{6.3}\]

Using \((5.19)\) again we get that for all \(j \geq k,\)
\[
\int \int_{(\Omega \times \Omega) \setminus E_k} \frac{|u_{\varepsilon_j,n_j}(x) - u_{\varepsilon_j,n_j}(y)|^p}{|x-y|^{N+sp}} \, dx \, dy
\leq \frac{1}{\alpha} \int \int_{(\Omega \times \Omega) \setminus E_k} \kappa_{\varepsilon_j,n_j}(x-y) \left[ \varepsilon_j + G_{n_j} \left( u_{\varepsilon_j,n_j}, s \right) \right]^{\frac{p-2}{2}} \frac{|u_{\varepsilon_j,n_j}(x) - u_{\varepsilon_j,n_j}(y)|^2}{|x-y|^{N+2s}} \, dx \, dy
\leq \alpha^{-1} \max \left\{ C_f^p, C_f^{\frac{p}{p-1}} \right\}. \tag{6.4}\]

Let \(U_{\varepsilon,n_j,(p,s)}, U_{(p,s)} : \Omega \times \Omega \rightarrow \mathbb{R}\) be given as in \((2.4)\). It follows from \((6.4)\) that \(\left\{ U_{\varepsilon,n_j,(p,s)} \right\}_{j \in \mathbb{N}}\) is bounded in \(L^p((\Omega \times \Omega) \setminus E_k)\). Since \(u_{\varepsilon_j,n_j} \rightharpoonup u\) in \(W_0^{t,2}(\Omega)\) as \(j \to \infty\), then by Remark \((2.3)\)
\[
U_{\varepsilon_j,n_j,(2,t)} \rightharpoonup U_{(2,t)} \text{ in } L^2(\Omega \times \Omega) \text{ as } j \to \infty. \tag{6.5}\]

Let \(\beta := \frac{N}{2} - \frac{N}{p} + t - s\). Since \(U_{\varepsilon_j,n_j,(p,s)}(x,y) = |x-y|^\beta U_{\varepsilon_j,n_j,(2,t)}(x,y)\) and \(U_{(p,s)}(x,y) = |x-y|^\beta U_{(2,t)}(x,y)\), it follows from \((6.5)\) that
\[
\chi_{(\Omega \times \Omega) \setminus E_k} U_{\varepsilon_j,n_j,(p,s)} \rightharpoonup U_{(p,s)} \text{ in } L^p(\Omega \times \Omega) \text{ as } j \to \infty.
\]

Using \((6.3)\) and \((6.4)\) we get that
\[
\int \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy = \lim_{k \to \infty} \int \int_{(\Omega \times \Omega) \setminus E_k} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy
\leq \lim_{k \to \infty} \liminf_{j \to \infty} \int \int_{(\Omega \times \Omega) \setminus E_k} \frac{|u_{\varepsilon_j,n_j}(x) - u_{\varepsilon_j,n_j}(y)|^p}{|x-y|^{N+sp}} \, dx \, dy
\leq \alpha^{-1} \max \left\{ C_f^p, C_f^{\frac{p}{p-1}} \right\} < \infty. \tag{6.6}\]

Since \(u \in W_0^{t,2}(\Omega)\) and by assumption \(\Omega\) has a Lipschitz continuous boundary, then proceeding as in \((16)\) Corollary 1.5.2 p.37, we can conclude from \((6.6)\) that \(u \in W_0^{s,p}(\Omega)\). The proof of the lemma is finished. \(\Box\)
Remark 6.2. We mention that we do not know if \( \{u_{\varepsilon,n}\}_{\varepsilon > 0, n \in \mathbb{N}} \) given in Lemma 6.1 is bounded in \( W^{s,2}_0(\Omega) \) if \( p > 2 \). We just know that it is bounded in \( W^{s,2}_0(\Omega) \) for \( \frac{1}{2} < s < 1 \). In fact, by (5.6), we have that for \( \varepsilon > 0 \) fixed, \( \{u_{\varepsilon,n}\}_{n \in \mathbb{N}} \) is bounded in \( W^{s,2}_0(\Omega) \). But for \( n \in \mathbb{N} \) fixed, we are not able to show that \( \{u_{\varepsilon,n}\}_{\varepsilon > 0} \) is bounded in \( W^{s,2}_0(\Omega) \) if \( p > 2 \).

Lemma 6.3. Let \( \frac{1}{2} < t \leq s < 1 \) and \( p \in [2, \infty) \) with \( t = s \) if \( p = 2 \). Let \( \{\varepsilon_k\}_{k \in \mathbb{N}}, \{n_k\}_{k \in \mathbb{N}} \) and \( \{\kappa_k\}_{k \in \mathbb{N}} \subset \mathbb{A}_{ad} \) be sequences such that

\[
\varepsilon_k \to 0, \quad n_k \to \infty \quad \text{and} \quad \kappa_k \to \kappa \quad \text{in} \quad L^1(\Omega) \quad \text{as} \quad k \to \infty. \tag{6.7}
\]

Let \( u_k = u_{\varepsilon_k,n_k}(\kappa_k) \) and \( u = u(\kappa) \) be the solutions of (3.7) and (1.2), respectively. Let \( (\Omega \times \Omega)_k(u_k) \) be given in (6.2). Let \( U_{k,(p,s)}, U_{(p,s)} \) be given as in (2.4). Then the following assertions hold.

\[
u_k \to u \quad \text{in} \quad W^{t,2}_0(\Omega) \quad \text{as} \quad k \to \infty. \tag{6.8}
\]

\[
\chi(\Omega \times \Omega) \setminus (\Omega \times \Omega)_k(u_k) \to U_{(p,s)} \quad \text{in} \quad L^p(\Omega \times \Omega) \quad \text{as} \quad k \to \infty. \tag{6.9}
\]

\[
\lim_{k \to \infty} \int_\Omega \int_\Omega \kappa_k(x-y) \left[ \varepsilon_k + G_{n_k}(u_k, s) \right] \frac{|u_k(x) - u_k(y)|^2}{|x-y|^{N+2s}} \, dx \, dy
\]

\[
= \int_\Omega \int_\Omega \kappa(x-y) \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} \, dx \, dy, \tag{6.10}
\]

where we recall that \( G_{n_k} \) is given by (3.10).

Proof. We prove the lemma in several steps.

Step 1. It follows from Lemma 6.1 that after a (sub)sequence if necessary, there exists \( u \in W^{t,2}_0(\Omega) \) such that \( u_k \to u \) in \( W^{t,2}_0(\Omega) \) and \( u_k \to u \) in \( L^2(\Omega) \), as \( k \to \infty \). In addition we have that \( u \in W^{s,p}_0(\Omega) \). We show that \( u \) is a weak solution of (1.2). Let \( \varphi \in \mathcal{D}(\Omega) \) be fixed. Recall that \( u_k \in W^{s,2}_0(\Omega) \) and satisfies the Minty relation

\[
\mathcal{F}_{\varepsilon_k,n_k,p}(\varphi, \varphi - u_k) := \mathcal{F}_{k,p}(\varphi, \varphi - u_k) \geq \int_\Omega f(\varphi - u_k) \, dx. \tag{6.11}
\]

Define the functions \( \mathcal{G}_k, \mathcal{G} : \Omega \times \Omega \to \mathbb{R} \) by

\[
\mathcal{G}_k(x,y) := \left[ \varepsilon_k + \mathcal{F}_k \left( \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{2s}} \right) \right]^{\frac{p-2}{2}} \frac{\varphi(x) - \varphi(y)}{|x-y|^s}^2
\]

and

\[
\mathcal{G}(x,y) := \left( \frac{|\varphi(x) - \varphi(y)|}{|x-y|^s} \right)^{p-2} \frac{\varphi(x) - \varphi(y)}{|x-y|^p}.
\]
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Since $0 \leq F_k(\tau) \leq \tau + 1$ for all $\tau \geq 0$ and $k \in \mathbb{N}$, and $p'(p - 1) = p$, we have that

$$
\|G_k\|_{L^{p'}(\Omega \times \Omega)}^{p'} := \int_\Omega \int_\Omega |G_k|^{p'} \, dxdy \\
\leq \int_\Omega \int_\Omega \left( \varepsilon_k + 1 + \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{2s}} \right)^{p'-2} \frac{|\varphi(x) - \varphi(y)|^{p'}}{|x-y|^{N+sp'}} \, dxdy \\
\leq C \left( \int_\Omega \int_\Omega (\varepsilon_k + 1)^{p'-2} \frac{|\varphi(x) - \varphi(y)|^{p'}}{|x-y|^{N+sp'}} \, dxdy + \int_\Omega \int_\Omega \frac{|\varphi(x) - \varphi(y)|^{p'(p-1)}}{|x-y|^{N+sp'(p-1)}} \, dxdy \right) \\
\leq C \left( \|\varphi\|_{W_0^{s,p'}(\Omega)}^p + \|\varphi\|_{W_0^{s,p'}(\Omega)}^{p'} \right) < \infty,
$$

where we have also used the fact that there exists a constant $C > 0$ such that

$$
(\varepsilon_k + 1 + \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{2s}})^{p'-2} \leq C \left( (\varepsilon_k + 1)^{p'-2} + \frac{|\varphi(x) - \varphi(y)|^{p'(p-2)}}{|x-y|^{sp'(p-2)}} \right),
$$

which follows from the well-known inequalities

$$
\begin{align*}
(a + b)^q &\leq 2^{q-1}(a^q + b^q), & \forall a, b \geq 0, q > 1 \\
(a + b)^q &\leq a^q + b^q, & \forall a, b \geq 0, q \in (0, 1].
\end{align*}
$$

(6.12)

Thus $G_k \in L^{p'}(\Omega \times \Omega)$. Similarly, we get that $G \in L^{p'}(\Omega \times \Omega)$. Using the Lebesgue Dominated Convergence Theorem, we get that

$$
G_k \to G \text{ in } L^{p'}(\Omega \times \Omega) \text{ as } k \to \infty. \quad (6.13)
$$

Let $\mathbb{K}^\varphi_{k,p}, \mathbb{K}^\varphi_{p} : \Omega \times \Omega \to \mathbb{R}$ be defined by

$$
\mathbb{K}^\varphi_{k,p}(x,y) := \kappa_k(x-y) \frac{\varphi(x) - \varphi(y)}{|x-y|^\frac{N}{p+s}} \text{ and } \mathbb{K}^\varphi_{p}(x,y) := \kappa(x-y) \frac{\varphi(x) - \varphi(y)}{|x-y|^\frac{N}{p+s}}.
$$

Then

$$
\int_\Omega \int_\Omega |\mathbb{K}^\varphi_{k,p}(x,y)|^p \, dxdy \leq \|\kappa\|_{L^\infty(\Omega)}^p \|\varphi\|_{W_0^{s,p}(\Omega)}^p.
$$

Proceeding similarly, we get that $\mathbb{K}^\varphi_{k,p}, \mathbb{K}^\varphi_{p} \in L^p(\Omega \times \Omega)$. Since $\{\kappa_k\}_{k \in \mathbb{N}}$ is bounded in $L^\infty(\Omega)$, then using (6.13) and the Lebesgue Dominated Convergence Theorem, we get that

$$
\mathbb{K}^\varphi_{k,p} \to \mathbb{K}^\varphi_{p} \text{ in } L^p(\Omega \times \Omega) \text{ as } k \to \infty. \quad (6.14)
$$

Using (6.13) and (6.14) we get that

$$
\lim_{k \to \infty} \int_\Omega \int_\Omega \kappa_k(x-y) G_{c_k,k,p}(\varphi) \frac{(\varphi(x) - \varphi(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dxdy \\
= \lim_{k \to \infty} \int_\Omega \int_\Omega G_{c_k}(x,y) \mathbb{K}^\varphi_{k,p}(x,y) \, dxdy = \int_\Omega \int_\Omega G(x,y) \mathbb{K}^\varphi_{p}(x,y) \, dxdy \\
= \int_\Omega \int_\Omega \kappa(x-y) \left( \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^s} \right)^{p-2} \frac{(\varphi(x) - \varphi(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \, dxdy. \quad (6.15)
$$
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Proceeding similarly and using $u_k \to u$ in $W^{1,2}_0(\Omega)$ as $k \to \infty$, we get that

$$\lim_{k \to \infty} \int \int _\Omega \kappa(x-y)G_{\varepsilon_k,k,p} (\varphi)(\frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s}})(u_k(x) - u_k(y)) dxdy$$

$$= \int \int _\Omega \kappa(x-y) \left(\frac{|\varphi(x) - \varphi(y)|}{|x-y|^s}\right)^{p-2}(\varphi(x) - \varphi(y))(u(x) - u(y)) dxdy. \quad (6.16)$$

Since $u_k \to u$ in $L^2(\Omega)$ as $k \to \infty$, we have that

$$\lim_{k \to \infty} \int \Omega \varphi(\varphi - u_k) dx = \int \Omega \varphi(\varphi - u) dx. \quad (6.17)$$

Combining (6.15), (6.16) and (6.17), we can pass to the limit in (6.11) to get that $u$ satisfies the Minty relation (4.3) for every $\varphi \in D(\Omega)$. Since $D(\Omega)$ is dense in $W^{s,p}_0(\Omega)$, we have that (4.3) also holds for every $\varphi \in W^{s,p}_0(\Omega)$. We have shown that $u$ is a weak solution to the system (4.2). From the uniqueness of solution to (4.2), we can deduce that the whole sequence $\{u_k\}$ converges weakly to $u = u(\kappa)$ in $W^{1,2}_0(\Omega)$, and hence converges strongly to $u = u(\kappa)$ in $L^2(\Omega)$, as $k \to \infty$.

**Step 2.** We show (6.9). Using (5.19) we get that for every $k \in \mathbb{N},$

$$\int \int _\Omega \chi(\Omega \times \Omega) \chi(\Omega \times \Omega)(x,y) \frac{|u_k(x) - u_k(y)|^p}{|x-y|^{N+sp}} dxdy$$

$$\leq \alpha^{-1} \int \int _\Omega \chi(\Omega \times \Omega) \chi(\Omega \times \Omega)(x,y) \kappa_k(x-y) \left[\varepsilon_k + G_k(u_k, s)\right] \frac{p-2}{|x-y|^{N+2s}} dxdy$$

$$\leq \alpha^{-1} \|u_k\|_{\varepsilon_k,k\kappa_k,s,p} \leq C < \infty. \quad (6.18)$$

It follows from (6.18) that $\{\chi(\Omega \times \Omega) \chi(\Omega \times \Omega)(x,y) U_{k,p}(\cdot, p, s)\}_{k \in \mathbb{N}}$ is bounded in $L^p(\Omega \times \Omega)$. Therefore, after (a sub)sequence if necessary, there exists a $G \in L^p(\Omega \times \Omega)$ such that

$$\chi(\Omega \times \Omega) \chi(\Omega \times \Omega)(x,y) U_{k,p}(\cdot, p, s) \to G \text{ in } L^p(\Omega \times \Omega) \text{ as } k \to \infty. \quad (6.19)$$

Let $K_{k,p'}, p' : \Omega \times \Omega \to \mathbb{R}$ be given by

$$K_{k,p'}(\varphi)(x,y) := \frac{\kappa_k(x-y)}{|x-y|^{N+sp-1}} \text{ and } K_{p'}(\varphi)(x,y) := \frac{\kappa(x-y)}{|x-y|^{N+sp-1}}.$$

Since $\kappa \in L^\infty(\Omega)$, we have that

$$\int \int _\Omega |K_{k,p'}(\varphi)(x,y)| dxdy \leq \|\kappa\|_{L^\infty(\Omega)} \int \int _\Omega \frac{1}{|x-y|^{N+sp-p'}} dxdy < \infty.$$

Proceeding similarly we get that $K_{k,p'}, p' \in L^p(\Omega \times \Omega)$. Using the Lebesgue Dominated Convergence Theorem, we get that $K_{k,p'} \to K_{p'}$ in $L^p(\Omega \times \Omega)$ as $k \to \infty$. Let $\varphi \in D(\Omega)$ and define $\Phi : \Omega \times \Omega \to \mathbb{R}$ by $\Phi(x-y) := \frac{\varphi(x) - \varphi(y)}{|x-y|}$. Then, for every $\varphi \in D(\Omega)$ we have that

$$\lim_{k \to \infty} \int \int _\Omega \chi(\Omega \times \Omega) \chi(\Omega \times \Omega)(x,y) U_{k,p}(\varphi)(x,y) K_{k,p'}(x-y) dxdy$$

$$= \int \int _\Omega G(x,y) K_{p'}(\varphi)(x,y) \frac{\varphi(x) - \varphi(y)}{|x-y|} dxdy, \quad (6.20)$$

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where $G$ is the function mentioned in (6.19). We notice that for every $\varphi \in D(\Omega)$,

$$
\int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \kappa(x - y) \, dx \, dy
= \lim_{k \to \infty} \int_{\Omega} \int_{\Omega} \frac{(u_k(x) - u_k(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \kappa_k(x - y) \, dx \, dy
+ \lim_{k \to \infty} \int_{\Omega} \int_{\Omega} \frac{(u_k(x) - u_k(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \kappa_k(x - y) \, dx \, dy.
$$

Using (5.16), (5.4), (5.19) and the Hölder inequality we get that there exists a constant $C > 0$ (depending only on $\Omega, N, p$ and $s$) such that

$$
\left| \int_{\Omega} \int_{\Omega} \kappa_k(x - y) \frac{(u_k(x) - u_k(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy \right|
\leq C \|\varphi\|_{C^1(\Omega)} \|\kappa\|_{L^\infty(\Omega)} \left( \int \int_{\Omega} \frac{1}{|x - y|^{N+2s-2}} \, dx \, dy \right)^{\frac{1}{2}}
\times \left( \int \int_{\Omega} \frac{|u_k(x) - u_k(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{\frac{1}{2}}
\leq C \|\xi_2\|_{L^\infty(\Omega)} \|\varphi\|_{C^1(\Omega)} \frac{1}{n_k} \left( \frac{\pi}{\xi_2} + n_k^2 + 1 \right) \frac{p-2}{4} \sqrt{\mu(\Omega \times \Omega)k(u_k)} \|u_k\|^\frac{p}{n_k \kappa_k}
\leq C \|\xi_2\|_{L^\infty(\Omega)} \|\varphi\|_{C^1(\Omega)} \frac{1}{n_k} \to 0 \quad \text{as} \quad k \to \infty.
$$

Using (6.22) we get from (6.20) and (6.21) that for every $\varphi \in D(\Omega)$,

$$
\int_{\Omega} \int_{\Omega} G(x,y)K_{p'}(x,y) \frac{\varphi(x) - \varphi(y)}{|x - y|} \, dx \, dy
= \int_{\Omega} \int_{\Omega} \kappa(x - y) \frac{u(x) - u(y) \varphi(x) - \varphi(y)}{|x - y|^{\frac{N}{p} + s}} \, dx \, dy.
$$

Thus $G(x,y) = U_{(p,s)}(x,y) = \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p} + s}}$ for a.e. $(x, y) \in \Omega \times \Omega$. We have shown that

$$
\chi(\Omega \times \Omega) \setminus (\Omega \times \Omega) u_k \to U_{(p,s)} \quad \text{in} \quad L^p(\Omega \times \Omega) \quad \text{as} \quad k \to \infty.
$$

Using (6.23) and the fact that $u_k$ is a solution of (6.7) we get that for every $k \in \mathbb{N}$,

$$
\frac{C_{N,p,s}}{2} \int_{\Omega} \int_{\Omega} \kappa(x - y) \left[ \varepsilon_k + G_n_k(u_k) \right] \frac{|u_k(x) - u_k(y)|^2}{|x - y|^{N+2s}} \, dx \, dy
+ \int_{\Omega} |u_k|^2 \, dx = \int_{\Omega} f u_k \, dx
$$

and

$$
\frac{C_{N,p,s}}{2} \int_{\Omega} \int_{\Omega} \kappa(x - y) \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy + \int_{\Omega} |u|^2 \, dx = \int_{\Omega} f u \, dx.
$$
It follows from (6.24), (6.25) and the fact that $u_k \to u$ in $W_0^{1,2}(\Omega)$ (as $k \to \infty$) that
\[
\lim_{k \to \infty} \frac{C_{N,p,s}}{2} \int_{\Omega} \int_{\Omega} \kappa_k(x - y) \left[ \varepsilon_k + \mathcal{G}_{n_k}(u_k, s) \right] \frac{|u_k(x) - u_k(y)|^2}{|x - y|^{N+2s}} \, dx \, dy
= \int_{\Omega} f u \, dx - \int_{\Omega} |u|^2 \, dx = \frac{C_{N,p,s}}{2} \int_{\Omega} \int_{\Omega} \kappa(x - y) \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy.
\] (6.26)
Moreover,
\[
\int_{\Omega} \int_{\Omega} \kappa(x - y) \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy
= \lim_{k \to \infty} \int_{\Omega} \int_{\Omega} \kappa_k(x - y) \left[ \varepsilon_k + \mathcal{G}_{n_k}(u_k, s) \right] \frac{|u_k(x) - u_k(y)|^2}{|x - y|^{N+2s}} \, dx \, dy
\geq \limsup_{k \to \infty} \int_{\Omega} \int_{\Omega} \chi(\Omega\times\Omega) \chi(\Omega\times\Omega)_k \kappa_k(x - y) \frac{|u_k(x) - u_k(y)|^p}{|x - y|^{N+sp}} \, dx \, dy
\geq \liminf_{k \to \infty} \int_{\Omega} \int_{\Omega} \chi(\Omega\times\Omega) \chi(\Omega\times\Omega)_k \kappa_k(x - y) \frac{|u_k(x) - u_k(y)|^p}{|x - y|^{N+sp}} \, dx \, dy.
\] (6.27)

Since $\{\kappa_k\}_{k \in \mathbb{N}}$ is bounded in $L^{\infty}(\Omega)$ and $\kappa_k(x) \geq \alpha$ for a.e. $x \in \Omega$ and every $k \in \mathbb{N}$, then using (4.7) we get that
\[
\chi(\Omega\times\Omega)\chi(\Omega\times\Omega)_k U_{k,(p,s)} \kappa_k^{1/p} \to U_{(p,s)} \kappa_k^{1/p} \text{ in } L^p(\Omega \times \Omega) \text{ as } k \to \infty,
\] (6.28)
where $\tilde{\kappa}_k(x,y) = \kappa_k(x-y)$ and $\tilde{\kappa} = \kappa(x-y)$. Using (6.27) we get that
\[
\int_{\Omega} \int_{\Omega} \kappa(x - y) \frac{|u_k(x) - u_k(y)|^p}{|x - y|^{N+sp}} \, dx \, dy
\geq \limsup_{k \to \infty} \int_{\Omega} \int_{\Omega} \kappa_k(x - y) \chi(\Omega\times\Omega) \chi(\Omega\times\Omega)_k \frac{|u_k(x) - u_k(y)|^p}{|x - y|^{N+sp}} \, dx \, dy
\geq \liminf_{k \to \infty} \int_{\Omega} \int_{\Omega} \kappa_k(x - y) \chi(\Omega\times\Omega) \chi(\Omega\times\Omega)_k \frac{|u_k(x) - u_k(y)|^p}{|x - y|^{N+sp}} \, dx \, dy
= \liminf_{k \to \infty} \left\| \chi(\Omega\times\Omega) \chi(\Omega\times\Omega)_k U_{k,(p,s)} \kappa_k^{1/p} \right\|_{L^p(\Omega \times \Omega)}^p
= \left\| U_{(p,s)} \kappa_k^{1/p} \right\|_{L^p(\Omega \times \Omega)}^p \left\| \chi(\Omega\times\Omega) \chi(\Omega\times\Omega)_k U_{k,(p,s)} \kappa_k^{1/p} \right\|_{L^p(\Omega \times \Omega)}^p.
\] (6.29)

We have shown that
\[
\lim_{k \to \infty} \left\| \chi(\Omega\times\Omega) \chi(\Omega\times\Omega)_k U_{k,(p,s)} \kappa_k^{1/p} \right\|_{L^p(\Omega \times \Omega)}^p = \left\| U_{(p,s)} \kappa_k^{1/p} \right\|_{L^p(\Omega \times \Omega)}^p.
\] (6.29)

The weak convergence (6.28) and the norm convergence (6.29) imply the strong convergence. From this we get that $\chi(\Omega\times\Omega) \chi(\Omega\times\Omega)_k U_{k,(p,s)} \to U_{(p,s)}$ in $L^p(\Omega \times \Omega)$ as $k \to \infty$ and we have shown (6.9).

**Step 3.** Now we show (6.10). Notice that it follows from (6.9) and (6.27) that
\[
\lim_{k \to \infty} \int_{(\Omega\times\Omega)_k(u_k)} \kappa_k(x - y) \left[ \varepsilon_k + \mathcal{G}_{n_k}(u_k, s) \right] \frac{|u_k(x) - u_k(y)|^2}{|x - y|^{N+2s}} \, dx \, dy = 0.
\] (6.30)
Next we claim that
\[
\lim_{k \to \infty} \int \int_{(\Omega \times \Omega) \setminus (\Omega \times \Omega)(u_k)} \kappa_k(x - y) \left[ \varepsilon_k + G_n(u_k, s) \right] \frac{|u_k(x) - u_k(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy = \int \int_{\Omega \times \Omega} \kappa(x - y) \frac{|u_k(x) - u_k(y)|^p}{|x - y|^{N + sp}} \, dx \, dy. \tag{6.31}
\]
Indeed, it follows from (3.3) that
\[
\lim_{k \to \infty} \int \int_{(\Omega \times \Omega) \setminus (\Omega \times \Omega)(u_k)} \frac{|u_k(x) - u_k(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy = \int \int_{\Omega \times \Omega} \frac{|u_k(x) - u_k(y)|^p}{|x - y|^{N + sp}} \, dx \, dy.
\]
Using (6.9), (6.32) and the Lebesgue Dominated Convergence Theorem we get (6.31) and the claim is proved. Now (6.10) follows from (6.30) and (6.31).

Step 4. It remains to show that \(u_k \to u\) in \(W^{1,2}_0(\Omega)\) as \(k \to \infty\). Recall that by Step 1, \(u_k \to u\) in \(W^{1,2}_0(\Omega)\) as \(k \to \infty\). Applying (6.30) we can deduce that
\[
\lim_{k \to \infty} \int \int_{(\Omega \times \Omega) \setminus (\Omega \times \Omega)(u_k)} \frac{|u_k(x) - u_k(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy = 0. \tag{6.33}
\]
Now combining (6.33) and (6.9) we get that
\[
U_{k,(2,t)} = \chi(\Omega \times \Omega)(u_k)U_{k,(2,t)} + \chi(\Omega \times \Omega)(\Omega \times \Omega)(u_k)U_{k,(2,t)} \to U_{(2,t)} \quad \text{in} \quad L^2(\Omega \times \Omega),
\]
as \(k \to \infty\). By Remark 2.3 this implies (6.8). The proof of the lemma is finished.

Now we are ready to give the proof of our last main result.

**Proof of Theorem 3.7.** It follows from Lemma 6.1 and the fact that the set \(A_{ad}\) is bounded in \(BV(\Omega)\) that \(\{\kappa_{\varepsilon,n}^*, u_{\varepsilon,n}^*\}_{\varepsilon > 0, n \in \mathbb{N}}\) is bounded in \(BV(\Omega) \times W^{1,2}_0(\Omega)\). Hence, after a (sub)sequence if necessary, \(\kappa_{\varepsilon,n}^* \rightharpoonup \kappa_* \) in \(BV(\Omega)\), \(u_{\varepsilon,n}^* \rightharpoonup u_* \) in \(W^{1,2}_0(\Omega)\) and \(u_{\varepsilon,n}^* \to u_* \) in \(L^2(\Omega)\), as \((\varepsilon, n) \to (0, \infty)\). It follows from Remark 2.4 that
\[
\lim_{(\varepsilon,n) \to (0, \infty)} \kappa_{\varepsilon,n}^* = \kappa_* \quad \text{in} \quad L^1(\Omega) \quad \text{and} \quad \int \Omega |\nabla u_*| \leq \liminf_{(\varepsilon,n) \to (0, \infty)} \int \Omega |\nabla u_{\varepsilon,n}^*|.
\tag{6.34}
\]
This implies that \(\kappa_* \in A_{ad}\). It also follows from Lemmas 6.1 and 6.3 that \(u_* \) is a weak solution to (1.2) with \(\kappa = \kappa_*\). Thus \((\kappa_*, u_*) \in \Xi\). Now combining (6.8) and (6.34) we get (6.11). The convergences in (3.13) and (3.14) follow from (6.9) and (6.10), respectively. Next we claim that
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$(\kappa_*, u_*)$ is a solution of (4.5). Given $(\kappa, u) \in \Xi$, $n \in \mathbb{N}$ and $\varepsilon > 0$, we define $\kappa_{\varepsilon, n} = \kappa$ and $u_{\varepsilon, n}$ as the solution of (3.7). Hence, $(\kappa_{\varepsilon, n}, u_{\varepsilon, n}) \in \Xi_{\kappa, n}$. It follows from (6.8) and (6.10) that

$$
\mathbb{I}(\kappa, u) = \lim_{(\varepsilon, n) \to (0, \infty)} \mathbb{I}(\kappa_{\varepsilon, n}, u_{\varepsilon, n}) = \lim_{(\varepsilon, n) \to (0, \infty)} \mathbb{I}(\kappa_{\varepsilon, n}, u_{\varepsilon, n}).
$$

(6.35)

Now using (6.8), (3.11), (6.34), (6.35) and the fact that $u_{\varepsilon, n}$ is the solution of (3.7), we get that

$$
\mathbb{I}(\kappa_*, u_*) \leq \liminf_{(\varepsilon, n) \to (0, \infty)} \mathbb{I}(\kappa_{\varepsilon, n}, u_{\varepsilon, n}) \leq \limsup_{(\varepsilon, n) \to (0, \infty)} \mathbb{I}(\kappa_{\varepsilon, n}, u_{\varepsilon, n})
$$

$$
\leq \limsup_{(\varepsilon, n) \to (0, \infty)} \mathbb{I}(\kappa_{\varepsilon, n}, u_{\varepsilon, n}) = \mathbb{I}(\kappa, u).
$$

Since $(\kappa, u)$ is arbitrary in $\Xi$, it follows that $(\kappa_*, u_*)$ is a solution of (4.5). Moreover, taking $(\kappa, u) = (\kappa_*, u_*)$ in the above inequality we get (3.15). Finally (3.12) follows directly from (3.15) and the fact that $\kappa_{\varepsilon, n} \overset{\ast}{\rightharpoonup} u_*$ in $BV(\Omega)$ and $u_{\varepsilon, n} \to u_*$ in $L^2(\Omega)$, as $(\varepsilon, n) \to (0, \infty)$. The proof of the theorem is finished. \qed

\section{Optimal control of fractional $p$-Laplacian for $0 < s < 1$}

We conclude the article by mentioning that all our results are also valid if one replaces $\mathcal{L}_{\Omega, p}^s(\kappa, \cdot)$ with $(-\Delta)^s_{p, \cdot}(\kappa, \cdot)$ given in (1.6) with $0 < s < 1$. In that case one replaces the state system (1.2) by (1.3) and $W_{0, \kappa, p}(\Omega)$ by the space

$$
W_{0, \kappa, p}(\Omega) := \left\{ u \in W^{\kappa, p}(\mathbb{R}^N) : u = 0 \text{ on } \mathbb{R}^N \setminus \Omega \right\}.
$$

Under Assumption 3.1(a), it has been shown in [10, Theorem 6] that $\mathcal{D}(\Omega)$ is dense in $W_{0, \kappa, p}(\Omega)$. Moreover, for every $0 < s < 1$,

$$
\|u\|_{W_{0, \kappa, p}(\Omega)} := \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N + sp}} \, dxdy \right)^{\frac{1}{p}}
$$

defines an equivalent norm on $W_{0, \kappa, p}(\Omega)$. In addition we have that $W_{0, \kappa, p}(\Omega) = W_{0, \kappa, p}(\Omega)$ with equivalent norms if $\frac{1}{p} < s < 1$ (see e.g. [2, Section 1.1]). With the above setting, a weak solution of (1.3) is defined to be a function $u \in W_{0, \kappa, p}(\Omega)$ such that for every $\varphi \in W_{0, \kappa, p}(\Omega)$ the equality

$$
\frac{C_{N, p, s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (x - y)|u(x) - u(y)|^{p - 2} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N + sp}} \, dxdy
$$

$$
+ \int_{\Omega} u\varphi \, dx =: \mathcal{E}_{p, s}(u, \varphi) = \int_{\Omega} f\varphi \, dx,
$$

holds. The associated regularized problem is given by

$$
\begin{cases}
(\Delta)^s_{p, \varepsilon, n}(\kappa, u) + u = f & \text{in } \Omega \\
u = 0 & \text{on } \mathbb{R}^N \setminus \Omega,
\end{cases}
$$

with

$$
(\Delta)^s_{p, \varepsilon, n}(\kappa, u) := C_{N, p, s, \text{P.V.}} \int_{\mathbb{R}^N} (x - y) \left[ (\varepsilon + \mathcal{G}_{n}(u, s)) \frac{u(x) - u(y)}{|x - y|^{N + 2s}} \right] \, dy,
$$

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and a weak solution is defined to be a $u \in W^{s,2}_0(\Omega)$ such that for every $\varphi \in W^{s,2}_0(\Omega)$,

$$
\frac{C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \kappa(x - y) \left[ \varepsilon + G_n(u, s) \right]^{\frac{p-2}{2}} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}}
$$

$$
dx dy + \int_\Omega u \varphi dx =: \tilde{E}_{\varepsilon,n,p}(u, \varphi) = \int_\Omega f \varphi dx.
$$

All our results hold with very minor changes in the proofs, if one replaces the expressions of $E_{\kappa}^{p,s}$ and $F_{\varepsilon,n,p}$ given in (4.1) and (3.9), respectively, by $\tilde{E}_{\kappa}$ and $\tilde{F}_{\varepsilon,n,p}(u, \varphi)$ for $u, \varphi \in W^{s,2}_0(\Omega)$, respectively. In this case $\xi_1, \xi_2 \in L^\infty(\mathbb{R}^N)$ and

$$
\kappa \in \tilde{A}_{ad} := \left\{ \eta \in BV(\Omega) : 0 < \alpha \leq \xi_1(x) \leq \eta(x) \leq \xi_2(x) \text{ a.e. in } \mathbb{R}^N \right\}.
$$

In addition in this situation, all the results holds for every $0 < s < 1$.

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