STOCHASTIC FORMS OF BRUNN’S PRINCIPLE

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Abstract. A number of geometric inequalities for convex sets arising from Brunn’s concavity principle have recently been shown to yield local stochastic formulations. Comparatively, there has been much less progress towards stochastic forms of related functional inequalities. We work towards a stochastic geometry of concave functions to establish local versions of dimensional forms of Brunn’s principle à la Borell, Brascamp-Lieb, and Rinott. To do so, we define shadow systems of convex epigraphs and hypographs, and revisit Rinott’s approach in the context of multiple integral rearrangement inequalities.

1. Introduction

Brunn’s concavity principle underpins a wealth of inequalities in geometry and analysis. One can formulate it as follows: for any convex body $K \subseteq \mathbb{R}^n$ and any direction $\theta$, the $(n-1)$-volume of slices of $K$ by parallel translates of $\theta^\perp$ is $1/(n-1)$-concave on its support, i.e.,

$$f(t) = |K \cap (\theta^\perp + t\theta)|^{1/(n-1)}$$

is concave. A far-reaching extension of this principle in analysis is exemplified by a family of functional inequalities obtained by Borell [5] and Brascamp-Lieb [5, 7], with an alternate approach put forth by Rinott [36]. These inequalities can be formulated in terms of certain means as follows: for $a, b \geq 0$, $s \geq -1/n$ and $\lambda \in (0, 1)$, set

$$\mathcal{M}^s_\lambda(a, b) = \begin{cases} (\lambda a^s + (1 - \lambda)b^s)^{1/s} & \text{if } ab \neq 0 \\ 0 & \text{if } ab = 0 \end{cases}$$

where the cases $s \in \{-1/n, 0, +\infty\}$ are defined as limits

$$\mathcal{M}^{-1/n}_\lambda(a, b) = \min\{a, b\}, \quad \mathcal{M}^\infty_\lambda(a, b) = \max\{a, b\}, \quad \mathcal{M}^0_\lambda(a, b) = a^\lambda b^{1-\lambda}.$$ 

Then for measurable functions $f, g, h : \mathbb{R}^n \to [0, \infty)$, $0 < \lambda < 1$, and $s \geq -1/n$, if

$$h(\lambda x + (1 - \lambda)y) \geq \mathcal{M}^s_\lambda(f(x), g(y))$$

(1.1)
for all \( x, y \in \mathbb{R}^n \), one has
\[
(1.2) \quad \int_{\mathbb{R}^n} h \geq M^{s/(1+ns)} \left( \int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g \right).
\]

The Prékopa-Leindler inequality \cite{23, 33, 34} corresponds to the logarithmically concave case \( s = 0 \); for earlier work on the real line, see Henstock and Macbeath \cite{21}. These principles are now fundamental in analysis, geometry and probability, among other fields. For their considerable impact, we refer the reader to \cite{17} and the references therein.

The inequalities \((1.2)\) stem from principles rooted in convexity. Indeed, the standard approach to Brunn’s principle connects concavity of \( f \) to the convexity of \( K \) through suitable symmetrizations, see e.g., \cite{2}. In \cite{6}, Brascamp and Lieb used symmetrization to prove \((1.2)\). Subsequently, they provided an alternate inductive approach, based on the Brunn-Minkowski inequality \cite{7}. Rinott provided an alternate proof, starting with epigraphs of convex functions \cite{36}. However, the inequalities ultimately do not require convexity as they hold for measurable functions. So convexity (or concavity) of the functions involved seems of no importance. In this paper, our focus is on what more can be said when the functions involved do possess some concavity.

Our motivation stems from recent work for convex sets in which a “local” stochastic dominance accompanies an isoperimetric principle. A concrete example, which can be derived via Brunn’s principle, is the Blaschke-Santaló \cite{39} inequality. The latter says that the volume of the polar of an origin-symmetric convex body \( K \) is maximized by a Euclidean ball \( B \) under a constraint of equal volume. Proofs via symmetrization depend on variants of \((1.2)\), e.g., Meyer-Pajor \cite{27} and Campi-Gronchi \cite{12}. In \cite{15}, the first-named author, together with Cordero-Erausquin, Fradelizi and Paouris, proved a stochastic version in which the dominance applies “locally” to random polytopes that naturally approximate \( K \) and \( B \) from within. By repeated sampling, this recovers the Blaschke-Santaló inequality by the law of large numbers. This example is indicative of recent research on isoperimetric inequalities: when an isoperimetric principle for convex sets can be proved by symmetrization, it is fruitful to instead carry out the symmetrization on product probability spaces. Multiple integral rearrangement inequalities of Rogers \cite{38}, Brascamp-Lieb-Luttinger \cite{8}, and Christ \cite{14} then enter the picture and yield stronger stochastic formulations. This builds on principles in stochastic geometry e.g., \cite{10, 20, 11}; see \cite{15, 30, 31, 35} for further background.

While there is much work on stochastic isoperimetric inequalities for convex sets, there are far fewer results about random functions. In \cite{32}, we
initiated work on the Prékopa-Leindler inequality for random log-concave functions. Here we will show that the full family of inequalities (1.2) actually have “local” stochastic strengthenings for functions $f$ that are $s$-concave, i.e. $f^s$ is concave; when $s < 0$, this means that $f^s$ is convex. To formulate our main result, for each $s$-concave function $f$, we sample independent random vectors $(X_1, Z_1), \ldots, (X_N, Z_N)$ distributed uniformly under the graph of $f$ according to Lebesgue measure. We associate random functions, $[f]_N$, supported on their convex hull $\text{conv}\{X_1, \ldots, X_N\}$ defined by

$$[f]_N(x) = \begin{cases} \sup\{z^{1/s} : (x, z) \in P_{f,N}\}, & \text{if } s > 0, \\ \sup\{e^z : (x, z) \in P_{f,N}\}, & \text{if } s = 0, \\ \inf\{z^{1/s} : (x, z) \in P_{f,N}\}, & \text{if } s < 0, \end{cases}$$

where

$$P_{f,N} = \begin{cases} \text{conv}\{(X_1, Z_1^s), \ldots, (X_N, Z_N^s)\}, & \text{if } s \neq 0, \\ \text{conv}\{(X_1, \log Z_1), \ldots, (X_N, \log Z_N)\}, & \text{if } s = 0. \end{cases}$$

In other words, when $s = 0$ or $s > 0$, $[f]_N$ is the least log-concave or $s$-concave function, respectively, satisfying $[f]_N(X_i) \geq Z_i$; similarly, when $s < 0$, $[f]_N$ is the greatest $s$-concave function with $[f]_N(X_i) \leq Z_i$. See Figures 1 and 2.

With this notation, we can state our main result, formulated in terms of the typical sup-convolution

$$h(v) = (f \ast_{\lambda,s} g)(v) = \sup\{M^s_{\lambda}(f(x), g(y)) : v = \lambda x + (1 - \lambda) y\}.$$
Theorem 1.1. Let \( s \in (\frac{-1}{n}, \infty) \), \( f, g : \mathbb{R}^n \to [0, \infty) \) be integrable functions such that \( f \) and \( g \) are \( s \)-concave, and \( N, M > n + 1 \). Denote by \( f^* \) and \( g^* \) the spherically symmetric decreasing rearrangements of \( f \) and \( g \) (cf (2.1)). Then for any \( \alpha > 0 \)
\[
\mathbb{P} \left( \int_{\mathbb{R}^n} ([f]_N \ast_{\lambda,s} [g]_M)(v)dv > \alpha \right) \geq \mathbb{P} \left( \int_{\mathbb{R}^n} ([f^*]_N \ast_{\lambda,s} [g^*]_M)(v)dv > \alpha \right).
\]
When \( N, M \to \infty \) one gets
\[
(1.4) \quad \int_{\mathbb{R}^n} (f \ast_{\lambda,s} g)(v)dv \geq \int_{\mathbb{R}^n} (f^* \ast_{\lambda,s} g^*)(v)dv.
\]
As mentioned, Brascamp and Lieb’s first approach to (1.2) used rearrangements. Recent renewed interest in rearranged strengthenings for this and other means in (1.2) have been studied by Melbourne [26] for general functions. Our focus here imposes \( s \)-concavity because in this case there is a stronger local stochastic dominance.

The special case of \( s = 0 \) was treated in [32]. This was based on multiple integral rearrangement inequalities (as discussed above) and, additionally, built on ideas of Artstein-Avidan, Klartag and Milman [3] on moving from convex sets to log-concave functions. Here a new key step is inspired by Rinott’s approach to (1.2) via epigraphs of convex functions [36]; the latter has recently been used in a dual setting by Artstein-Avidan, Florentin and Segal [1].

As introduced by Rogers and Shephard [37], given a bounded set \( A \subset \mathbb{R}^n \), \( \alpha : A \to \mathbb{R} \) bounded function, and \( t \in [a, b] \subset \mathbb{R} \), a shadow system along a direction \( \theta \in \mathbb{S}^{n-1} \) is a family of convex sets \( K_t \subset \mathbb{R}^n \) such that
\[
(1.5) \quad K_t = \text{conv}\{x + \alpha(x)t\theta : x \in A\}.
\]
These movements generalize Steiner symmetrization. Rogers and Shephard [37, 12] proved that the volume of \( K_t \) is a convex function of \( t \). They have been successfully used in the treatment of many isoperimetric type inequalities and developed by Campi and Gronchi, e.g., [12, 13]. For recent applications, see [16, 29, 28, 44, 4, 10] and the references therein.

On the account of the fact that the tools mentioned above interface well with each other, we adapt Shephard’s idea [42] to define a shadow system of a convex epigraph or hypograph on \( \mathbb{R}^{n+1} \) as a family of projections of an \((n + 2)\)-dimensional convex epigraph or hypograph. That is, let \( E \subset \mathbb{R}^{n+2} \) be a convex epigraph then
\[
E_t = \{(x + zt\theta, r) \in \mathbb{R}^n \times \mathbb{R}e_{n+2} : (x, z, r) \in E, \theta \in \mathbb{S}^{n-1}\}.
\]
is a shadow system of convex epigraphs. Given \( \varphi : \mathbb{R}^n \to [0, \infty) \) a convex function, we define the shadow system \( \varphi_t : \mathbb{R}^n \to [0, \infty) \) by

\[
\varphi_t(x) = \inf \{ z : (x, z) \in (E\varphi)_t \},
\]

where \( E\varphi \) denotes the epigraph of \( \varphi \). Similarly, we define a shadow system of a convex hypograph and a concave function, see (5.6) and (5.7). This provides a path towards stochastic geometry of \( s \)-concave functions.

2. Preliminaries

We will denote by \( \{e_1, \ldots, e_n\} \) the standard basis in \( \mathbb{R}^n \). Let \( K \) be a convex set in \( \mathbb{R}^n \), \( \theta \) on the unit sphere \( S^{n-1} \) and \( P := P_{\theta^\perp} \) the orthogonal projection onto \( \theta^\perp \). We define \( u_K : PK \to \mathbb{R} \) by

\[
u_K(y) := u(K, y) := \sup \{ \lambda : y + \lambda \theta \in K \} \]

and \( \ell_K : PK \to \mathbb{R} \) by

\[
\ell_K(y) := \ell(K, y) := \inf \{ \lambda : y + \lambda \theta \in K \}. \]

Notice that, for \( K \) convex, \( u_K \) and \( \ell_K \) are concave and convex, respectively.

We recall that the Steiner symmetral of a non-empty compact set \( A \subseteq \mathbb{R}^n \) with respect to \( \theta^\perp \), \( S_{\theta^\perp} A \), is the set with the property that for each line \( l \) orthogonal to \( \theta^\perp \) and meeting \( A \), the set \( l \cap S_{\theta^\perp} A \) is a closed segment with midpoint on \( \theta^\perp \) and length equal to that of the set \( l \cap A \). The mapping \( S_{\theta^\perp} : A \to S_{\theta^\perp} A \) is called the Steiner symmetrization of \( A \) with respect to \( \theta^\perp \). In particular, if \( K \) is a convex body

\[
S_{\theta^\perp} K = \{ x + \lambda \theta : x \in PK, -\frac{u_K(x) - \ell_K(x)}{2} \leq \lambda \leq \frac{u_K(x) - \ell_K(x)}{2} \}.
\]

This shows that \( S_{\theta^\perp} K \) is convex, since the function \( u_K - \ell_K \) is concave. Moreover, \( S_{\theta^\perp} K \) is symmetric with respect to \( \theta^\perp \), it is closed, and by Fubini’s theorem it has the same volume as \( K \).

Let \( A \subseteq \mathbb{R}^n \) be a Borel set with finite Lebesgue measure. The symmetric rearrangement, \( A^* \), of \( A \) is the open ball with center at the origin whose volume is equal to the measure of \( A \). Since we choose \( A^* \) to be open, \( 1_A^* \) is lower semicontinuous. The symmetric decreasing rearrangement of \( 1_A \) is defined by \( 1_A^* = 1_{A^*} \). We say a Borel measurable function \( f : \mathbb{R}^n \to [0, \infty) \) vanishes at infinity if for every \( t > 0 \), the set \( \{ x \in \mathbb{R}^n : f(x) > t \} \) has finite Lebesgue measure. In such a case, the symmetric decreasing rearrangement \( f^* \) is defined by

\[
f^*(x) = \int_0^\infty 1_{\{ f>t \}^*}(x)dt = \int_0^\infty 1_{\{ f>t \}}(x)dt. \]

(2.1)
Observe that \( f^* \) is radially symmetric, radially decreasing, and equimeasurable with \( f \), i.e., \( \{ f > t \} \) and \( \{ f^* > t \} \) have the same measure for each \( t > 0 \). Let \( \{ e_1, \ldots, e_n \} \) be an orthonormal basis of \( \mathbb{R}^n \) such that \( e_1 = \theta \). Then, for \( f \) vanishing at infinity, the Steiner symmetral \( f(\cdot | \theta) \) of \( f \) with respect to \( \theta^\perp \) is defined as follows: set \( f_{(x_2, \ldots, x_n), \theta}(t) = f(t, x_2, \ldots, x_n) \) and define \( f^*(t, x_2, \ldots, x_n | \theta) := (f_{(x_2, \ldots, x_n), \theta})^*(t) \). In other words, we obtain \( f^*(\cdot | \theta) \) by rearranging \( f \) along every line parallel to \( \theta \). We refer to the books [24, 43] or the introductory notes [9] for further background material on rearrangement of functions.

3. Rinott’s approach to s-concave functions via epigraphs and hypographs

Rinott [36] provides a geometric proof of Borell’s Theorem [5] by deriving integral inequalities for functions using certain higher-dimensional measures. We start this section by recalling his approach.

A function \( f : \mathbb{R}^n \to [0, \infty) \) is called log-concave if \( \log f \) is concave on its support. In accordance with the usage of s-concavity in [5], we say \( f \) is s-concave if \( f^s \) is concave on its support; this is not the same other common uses of the term, e.g., [3, 22, 32], however, it fits with the approach taken here. In particular, any s-concave function, for \( s > 0 \), is also log-concave. Let \( A \subset \mathbb{R}^n \), we define the epigraph of \( f \) at \( A \) in \( \mathbb{R}^n \) by

\[
E_A(f) = \{(x, z) \in A \times \mathbb{R} : f(x) \leq z\}.
\]

Analogously we define the hypograph of \( f \) in \( A \) by

\[
H_A(f) = \{(x, z) \in A \times \mathbb{R} : 0 \leq z \leq f(x)\}.
\]

Let \( f : \mathbb{R}^n \to [0, \infty) \) be an s-concave function for \( s \in (-1/n, \infty) \), and \( \nu \) a measure on \( \mathbb{R}^n \times \mathbb{R} \) such that

\[
d\nu(x_1, \ldots, x_{n+1}) = h(x_{n+1}) dx_1 \cdots dx_{n+1},
\]

for some continuous function \( h : \mathbb{R} \to [0, \infty) \). With this set up, we can express the integral of \( f \) in terms of the \( \nu \)-measure of the epigraph or hypograph of a transformation of it. Moreover,

\[
\int_A f(x) dx = \begin{cases} 
\nu(H_A(f^s)), & \text{for } h(x_{n+1}) = \left( \frac{1-sn}{s} \right)^{\frac{1-sn}{sn}-1}, \text{if } s > 0. \\
\nu(E_A(- \log f)), & \text{for } h(x_{n+1}) = e^{-x_{n+1}}, \text{if } s = 0. \\
\nu(E_A(f^s)), & \text{for } h(x_{n+1}) = \left( \frac{1-sn}{s} \right)^{\frac{1-sn}{sn}-1}, \text{if } s < 0.
\end{cases}
\]

Notice the s-concavity of \( f \), implies the convexity of \( H_A(f^s), E_A(- \log f) \), and \( E_A(f^s) \) respectively.
It is easy to check that for \( \varphi, \psi : \mathbb{R}^n \to \mathbb{R} \) convex functions, defining
\[
(\varphi \Box \psi)(v) = \inf_{v=x+y} \{ \varphi(x) + \psi(y) \}
\]
then one has
\[
(\varphi \Box \psi)(v) = \inf \{ z \in \mathbb{R} : (v, z) \in E(\varphi) + E(\psi) \}.
\]
In particular it follows that
\[
E(\varphi \Box \psi) = E(\varphi) + E(\psi).
\]
Equivalently, defining
\[
(\varphi \Box_{\lambda, s} \psi)(v) = \inf_{v=\lambda x + (1-\lambda)y} \{ (\lambda \varphi^s(x) + (1-\lambda)\psi^s(y))^s \}
\]
we have
\[
E(\varphi \Box_{\lambda, 1} \psi) = \lambda E(\varphi) + (1-\lambda)E(\psi).
\]
Given \( f, g : \mathbb{R}^n \to [0, \infty) \) \( s \)-concave functions with \( s > 0 \), then \( -f^s, -g^s \) are convex and
\[
H((f \star_{\lambda, s} g)^s) = \lambda H(f^s) + (1-\lambda)H(g^s)
\]
Since
\[
-(f \star_{\lambda, s} g)^s(v) = -\left( \sup_{v=\lambda x + (1-\lambda)y} \{ (\lambda f^s(x) + (1-\lambda)g^s(y))^{1/s} \} \right)^s
\]
\[
= \inf_{v=\lambda x + (1-\lambda)y} \{ \lambda(-f^s)(x) + (1-\lambda)(-g^s)(y) \}
\]
\[
= (-f^s \Box_{\lambda, 1} - g^s)(v).
\]
For \( f, g : \mathbb{R}^n \to [0, \infty) \) log-concave functions, \( -\log f \) and \( -\log g \) are convex. Then we have
\[
E(-\log (f \Box_{\lambda, 0} g)) = \lambda E(-\log f) + (1-\lambda)E(-\log g).
\]
Lastly, if \( f, g : \mathbb{R}^n \to [0, \infty) \) are \( s \)-concave functions with \( s < 0 \), then \( f^s \) and \( g^s \) are convex functions. Thus, it follows
\[
E((f \Box_{\lambda, s} g)^s) = \lambda E(f^s) + (1-\lambda)E(g^s)
\]
since

\[(f \square_{\lambda,s} g)^s(v) = \left( \inf_{v=\lambda x+(1-\lambda) y} \{ (\lambda f^s(x) + (1-\lambda) g^s(y) \}^{1/s} \right)^s \]

\[= \inf_{v=\lambda x+(1-\lambda) y} \{ \lambda f^s(x) + (1-\lambda) g^s(y) \} \]

\[= (f^s \square_{\lambda,1} g^s)(v). \]

4. Convex hull and $\mathcal{M}$-addition operations

Let $C \subseteq \mathbb{R}^N$ be a compact convex set; for $x_1, \ldots, x_N \in \mathbb{R}^n$, we view the $n \times N$ matrix $[x_1, \ldots, x_N]$ as an operator from $\mathbb{R}^N$ to $\mathbb{R}^n$. Then

\[(4.1) \quad [x_1, \ldots, x_N]C = \left\{ \sum_i c_i x_i : c = (c_i) \in C \right\} \]

produces a convex set in $\mathbb{R}^n$. This viewpoint was used by the first-named author and Paouris in [30] in randomized isoperimetric inequalities for convex sets; for the special case $C = \text{conv}\{e_1, \ldots, e_N\}$, one has

\[ [x_1, \ldots, x_N]C = \text{conv}\{x_1, \ldots, x_N\}. \]

Moreover, for vectors $x_1, \ldots, x_N, x_{N+1}, \ldots, x_{N+M}$, we have

\[(4.2) \quad [x_1, \ldots, x_N]C_N + [x_{N+1}, \ldots, x_{N+M}]C_M
= [x_1, \ldots, x_N]C_N + [x_{N+1}, \ldots, x_{N+M}]C_M
= [x_1, \ldots, x_{N+M}] (C_N + \hat{C}_M), \]

where $C_k = \text{conv}\{e_1, \ldots, e_k\}$ for $k = N, M$ and $\hat{C}_M = \text{conv}\{e_{N+1}, \ldots, e_{N+M}\}$.

The convex operations on points (4.1) can also be generalized to convex operations on sets by using the notion of $\mathcal{M}$-addition. This notion was introduced by Gardner, Hug, and Weil [19, 18] as a unifying framework for operations in Lutwak, Yang, and Zhang’s $L_p$ and Orlicz Brunn-Minkowski theory (see e.g., [25]). For $\mathcal{M} \subseteq \mathbb{R}^N$ and subsets $K_1, \ldots, K_N$ in $\mathbb{R}^n$, their $\mathcal{M}$-combination is defined by

\[ \oplus_\mathcal{M}(K_1, \ldots, K_N) = \left\{ \sum_{i=1}^N m_i x_i : x_i \in K_i, (m_1, \ldots, m_N) \in \mathcal{M} \right\}. \]

Thus, with this notation, for $C = \mathcal{M}$,

\[ \oplus_C(\{x_1\}, \ldots, \{x_N\}) = [x_1, \ldots, x_N]C. \]

Additionally, when $K_1, \ldots, K_N$ are convex and $\mathcal{M}$ is convex and contained in the positive orthant or origin-symmetric, then $\oplus_\mathcal{M}(K_1, \ldots, K_N)$ is convex [18] Theorem 6.1].
To connect with the epigraphs and hypographs defined in §3, we use \( M \)-combinations of rays in \( \mathbb{R}^{n+1} \). Let \( C \subset \mathbb{R}^N \) be a convex set contained in the positive orthant, \( \rho_1, \ldots, \rho_N \in \mathbb{R} \), and \( x_1, \ldots, x_N \in \mathbb{R}^n \). We define the rays
\[
R_{\rho_i}(x_i) = \{(x_i, r) \in \mathbb{R}^n \times \mathbb{R} : \rho_i \leq r\}
\]
and the line segments
\[
\tilde{R}_{\rho_i}(x_i) = \{(x_i, r) \in \mathbb{R}^n \times \mathbb{R} : \rho_i \geq r \geq 0\}.
\]
Accordingly,
\[
\oplus_C(R_{\rho_1}(x_1), \ldots, R_{\rho_N}(x_N)) \quad \text{and} \quad \oplus_C(\tilde{R}_{\rho_1}(x_1), \ldots, \tilde{R}_{\rho_N}(x_N))
\]
are a convex epigraph and hypograph, respectively. Also when choosing \( C = \text{conv}\{e_1, \ldots, e_N\} \) this is the operation of taking the convex hull of the rays or line segments, respectively.

5. Shadow systems of convex epigraphs and hypographs

As introduced by Rogers and Shephard, given a bounded set \( A \subset \mathbb{R}^n \), \( \alpha : A \rightarrow \mathbb{R} \) bounded functions, and \( t \in [a, b] \subset \mathbb{R} \), a shadow system along a direction \( \theta \in \mathbb{S}^{n-1} \) is a family of convex sets \( K_t \subset \mathbb{R}^n \) such that
\[
K_t = \text{conv}\{x + \alpha(x)t\theta : x \in A\}.
\]
Later on, Shephard [42] showed a shadow system can also be seen as a family of projections of an \((n + 1)\)-dimensional convex set onto \( \mathbb{R}^n \). Let \( K \subset \mathbb{R}^{n+1} \) be a closed convex set, \( \theta \in \mathbb{S}^{n-1} \), and \( t \in [a, b] \). Define the projection, \( \overline{P}_t \), onto \( \mathbb{R}^n \) parallel to \( e_{n+1} - t\theta \) by
\[
\overline{P}_t(x, z) = x + tz\theta, \quad \text{for } (x, z) \in \mathbb{R}^n \times \mathbb{R}e_{n+1}.
\]
Then, the family \( \{K_t\} = \{\overline{P}_tK\} \subset \mathbb{R}^n \) is a shadow system of convex sets. For background on shadow systems, see e.g., [12], [11]. We follow the idea from (5.2) together with equalities (3.2) to define shadow system for epigraphs and, therefore, for \( s \)-concave functions.

Let \( E \subset \mathbb{R}^{n+2} \) be a convex epigraph, \( \theta \in \mathbb{S}^{n-1} \), \( t \in [a, b] \), and \( P_t \) the projection onto \( \mathbb{R}^n \times \mathbb{R}e_{n+2} \) parallel to \( e_{n+1} - t\theta \) defined by
\[
P_t(x, z, r) = (x + tz\theta, r) \in \mathbb{R}^n \times \mathbb{R}e_{n+2}
\]
for \( (x, z, r) \in \mathbb{R}^n \times \mathbb{R}e_{n+1} \times \mathbb{R}e_{n+2} \). Then, the family \( \{E_t\} = \{P_tE\} \) is a shadow system of convex epigraphs where
\[
E_t = \{(x + zt\theta, r) \in \mathbb{R}^n \times \mathbb{R}e_{n+2} : (x, z, r) \in E\}.
\]
Thus, for $\varphi : \mathbb{R}^n \to [0, \infty)$ a convex function, we define the shadow system of convex functions $\varphi_t : \mathbb{R}^n \to [0, \infty)$ by

$$\varphi_t(x) = \inf \{ z : (x, z) \in (E\varphi)_t \},$$

where $(E\varphi)_t = P_t E\Phi$ for some convex function $\Phi : \mathbb{R}^{n+1} \to [0, \infty)$ with $P_0 E\Phi = E\varphi$. Analogously, given a convex hypograph $H \subset \mathbb{R}^{n+2}$ the family \( \{ H_t \} = \{ P_t H \} \) is a shadow system of convex hypographs where

$$H_t = \{(x + zt\theta, r) \in \mathbb{R}^n \times \mathbb{R}_{n+2} : (x, z, r) \in H \}.$$ 

Consequently for a concave function $\phi : \mathbb{R}^n \to [0, \infty)$, we define the shadow system of convex functions $\phi_t : \mathbb{R}^n \to \mathbb{R}$ by

$$\phi_t(x) = \sup \{ z : (x, z) \in (H\phi)_t \}$$

where $(H\phi)_t = P_t H\overline{\Phi}$ for a concave $\overline{\Phi} : \mathbb{R}^{n+1} \to [0, \infty)$ with $P_0 H\overline{\Phi} = H\phi$.

**Remark 5.1.** Let $K \subset \mathbb{R}^n \times \mathbb{R}_{n+1}$ be a convex body, then $E1_K$ is a convex epigraph on the support of $1_K : \mathbb{R}^n \times \mathbb{R}_{n+1} \to \mathbb{R}_{n+2}$. Notice

$$P_t E1_K = \{(x + zt\theta, r) : (x, z) \in K, r \geq 1 \}$$

$$= \{ (\bar{x}, r) : \bar{x} \in P_t K, r \geq 1 \}$$

$$= E1_{\overline{P}_t K}.$$

In addition, Rogers and Shephard [37] introduced a linear parameter system along a given direction $\theta \in S^{n-1}$ as a family $t \mapsto K_t$ of convex sets of the form

$$K_t = \text{conv}\{x_j + \lambda_j t\theta : j \in J\}, \quad t \in I$$

where $I \subset \mathbb{R}$ is an interval, $J$ is an index set, and $\{x_j\}_{j \in J} \subset \mathbb{R}^n$ and $\{\lambda_j\}_{j \in J}$ are bounded sets. Inspired by this definition, we define a linear parameter system of convex epigraphs.

Let $N \geq n + 1$, $\{(x_i, z_i)\}_{i=1}^N \subset \mathbb{R}^n \times \mathbb{R}$, $\{\lambda_i\}_{i=1}^N \subset \mathbb{R}$, $\theta \in S^{n-1}$, and $t \in [a, b] \subset \mathbb{R}$. We denote $E_t$ to the epigraph of the greatest convex function beneath $\{(x_i + \lambda_i t\theta, z_i)\}_{i=1}^N$, i.e., for $C$ as above

$$E_t = \oplus_C (R_{z_1}(x_1 + \lambda_1 t\theta), \ldots, R_{z_N}(x_N + \lambda_N t\theta))$$

$$= \oplus_C (\{ R_i(x_i + \lambda_i t\theta)\}_{i=1}^N).$$

Then, the family $\{E_t\}$ is called a linear parameter system of convex epigraphs. Similarly, we denote $H_t$ to the hypograph of the greatest concave function above $\{(x_i + \lambda_i t\theta, z_i)\}_{i=1}^N$, that is

$$H_t = \oplus_C (\tilde{R}_{z_1}(x_1 + \lambda_1 t\theta), \ldots, \tilde{R}_{z_N}(x_N + \lambda_N t\theta))$$

$$= \oplus_C (\{ \tilde{R}_i(x_i + \lambda_i t\theta)\}_{i=1}^N).$$
Thus, the family \( \{ H_t \} \) is a linear parameter system of convex hypographs.

**Lemma 5.2.** The \( \nu \)-measure of the sets \( E_t \) (resp. \( H_t \)) of a linear parameter system of convex epigraphs (resp. hypographs) is a convex function of \( t \). Moreover, for \( t \in \mathbb{R}^N \) and \( P_t : \mathbb{R}^n \times \mathbb{R}_{n+1}^N \to \mathbb{R}^n \times \mathbb{R}_{n+1}^N \) defined by \( P_t(x, r, y) = (x + \langle y, t \rangle, \theta, r) \). Then \( \mathbb{R}^N \ni t \mapsto \nu(P_t E) \) is a convex function.

**Proof.** Let \( \nu \) be as in (3.2), then \( \nu \) is a finite measure and \( \nu(E_t) \) is a continuous function of \( t \). Thus, it suffices to show
\[
(5.10) \quad \nu(E_{t_1 + t_2}) \leq \frac{1}{2}(\nu(E_{t_1}) + \nu(E_{t_2})),
\]
for all \( t, r \in \mathbb{R} \). Indeed, defining \( H_z = \{(x, z) : x \in \mathbb{R}^n \} \) for \( z \in \mathbb{R} \), one has
\[
\nu(E_t) = \int_{\mathbb{R}} |E_t \cap H_z| e^{-z} \, dz.
\]
Rogers and Shephard showed the convexity of the function \( t \mapsto |K_t| \), for \( K_t \) a shadow system of convex bodies defined as in (5.1). Thus, since the restrictions of the epigraph to the parallel hyperplanes are shadow systems of convex bodies, the convexity of \( \nu(E_t) \) follows from the convexity of the function
\[
t \mapsto |E_t \cap H_z| = |(E \cap H_z)_t|.
\]
For the \( N \)-parameter variation, fix \( t, \tilde{t} \in \mathbb{R}^N \). Thus it suffices to show \([0, 1] \ni \beta \mapsto \nu(P_{t+\beta(\tilde{t}-t)} E) \) is convex. However, note the later is a one-parameter shadow system and we can apply the first part of the statement. Similarly one obtains the one-parameter and \( N \)-parameter variation for hypographs.

\[\square\]

### 6. Random epigraphs and hypographs

We now define our stochastic model. Let \( C \) be a convex set contained in the positive orthant, \( f : \mathbb{R}^n \to [0, \infty) \) an \( s \)-concave function with \( s \in (-1/n, \infty) \), and denote the region under the graph of \( f \) by
\[
(6.1) \quad G_f := \{(x, z) \in \mathbb{R}^n \times [0, \infty) : x \in \text{supp } f, z \leq f(x)\}.
\]
We sample independent random vectors \((X_1, Z_1), \ldots, (X_N, Z_N)\) in \( \mathbb{R}^n \times [0, \infty) \) according to the uniform Lebesgue measure on \( G_f \), and denote by \([f]_N\) the function defined in the introduction, repeated here for the convenience of the reader: for
\[
(6.2a) \quad P_{f,N} = \begin{cases} 
\text{conv}\{(X_1, Z_1^s), \ldots, (X_N, Z_N^s)\}, & \text{if } s \neq 0 \\
\text{conv}\{(X_1, \log Z_1), \ldots, (X_N, \log Z_N)\}, & \text{if } s = 0
\end{cases}
\]
we set

\[ [f]_N(x) = \begin{cases} 
\inf \{ z^{1/s} : (x, z) \in P_{f,N} \}, & \text{if } s < 0. \\
\sup \{ e^z : (x, z) \in P_{f,N} \}, & \text{if } s = 0. \\
\sup \{ z^{1/s} : (x, z) \in P_{f,N} \}, & \text{if } s > 0.
\end{cases} \tag{6.2b} \]

Equivalently, we could sample according to the uniform \( \nu \) measure on \( E(f^s) \), \( E(-\log f) \), or \( H(f^s) \), respectively, and define the random epigraph or hypograph

\[ (6.3a) \quad [E]_N = \begin{cases} 
\oplus_C(R_{Z^s_1}(X_1), \ldots, R_{Z^s_N}(X_N)), & \text{if } s < 0. \\
\oplus_C(R_{-\log Z^s_1}(X_1), \ldots, R_{-\log Z^s_N}(X_N)), & \text{if } s = 0.
\end{cases} \]

\[ (6.3b) \quad [H]_N = \oplus_C(\tilde{R}_{Z^s_1}(X_1), \ldots, \tilde{R}_{Z^s_N}(X_N)), \quad \text{if } s > 0. \]

It follows then the correspondence

\[ [E]_N = E([f]^s_N), \quad \text{if } s < 0. \]

\[ [E]_N = E(-\log [f]_N), \quad \text{if } s = 0. \]

\[ [H]_N = H([f]^s_N), \quad \text{if } s > 0. \]

In addition, in view of (5.2) one has

\[ (6.4a) \quad \int [f]_N = \begin{cases} 
\nu([E]_N), & \text{for } h(x_{n+1}) = \left( \frac{1-sn}{s} \right)^{\frac{1-sn}{s}} x_{n+1}^{\frac{1-sn}{s}} - 1, \text{if } s < 0. \\
\nu([F]_N), & \text{for } h(x_{n+1}) = e^{-x_{n+1}}, \quad \text{if } s = 0. \\
\nu([H]_N), & \text{for } h(x_{n+1}) = \left( \frac{1-sn}{s} \right)^{\frac{1-sn}{s}} x_{n+1}^{\frac{1-sn}{s}} - 1, \text{if } s > 0.
\end{cases} \]

7. Multiple integral rearrangement inequalities

7.1. Rearrangements and Steiner convexity. In this section, we show that the multiple integral rearrangement inequalities of Rogers [38], and Brascamp, Lieb, and Luttinger [8] interfaces well with the embedding into \( \mathbb{R}^{n+1} \) using epigraphs and hypographs. In particular, Christ’s version [14] of the latter is especially well-suited for stochastic forms of isoperimetric inequalities; as in [31], the following formulation is convenient for our purpose.
Theorem 7.1. Let $f_1, \ldots, f_N$ be non-negative integrable functions on $\mathbb{R}^n$ and $F : (\mathbb{R}^n)^N \to [0, \infty)$. Then

$$
\int_{(\mathbb{R}^n)^N} F(x_1, \ldots, x_N) \prod_{i=1}^{N} f_i(x_i) dx_1 \ldots dx_N
\geq \int_{(\mathbb{R}^n)^N} F(x_1, \ldots, x_N) \prod_{i=1}^{N} f_i^*(x_i) dx_1 \ldots dx_N,
$$

whenever $F$ satisfies the following condition: for each $\theta \in S^{n-1}$ and all $Y := \{y_1, \ldots, y_N\} \subseteq \theta^\perp$, the function $F_Y : \mathbb{R}^N \to [0, \infty)$ defined by

$$F_Y(\theta_1, \ldots, t_N) := F(y_1 + t_1\theta, \ldots, y_N + t_N\theta)$$

is even and quasi-convex.

The condition on $F$, namely Steiner convexity, allows the theorem to be proved via iterated Steiner symmetrization; notice this terminology differs from the one in [14]. Of special interest, this condition interfaces well with shadow systems, e.g., [37, 11]; see [31] for further background and references.

Proposition 7.2. Let $\rho_1, \ldots, \rho_N \in \mathbb{R}$ and $C$ a convex set contained in the positive orthant. Then the function $F : (\mathbb{R}^n)^N \to [0, \infty)$ defined by

$$(7.1) \quad F(x_1, \ldots, x_N) = \nu(\oplus_C(R_{\rho_1}(x_1), \ldots, R_{\rho_N}(x_N)))$$

is Steiner convex.

Proof. Let $\theta \in S^{n-1}$ and $y_1, \ldots, y_N \in \theta^\perp$. Write $x_i = y_i + t_i\theta \in \mathbb{R}^n$ and let

$$C = \oplus_C(R_{\rho_1}(y_1 + e_{n+1}), \ldots, R_{\rho_N}(y_N + e_{n+1})).$$

Then $C$ is a closed convex epigraph in $\mathbb{R}^n \times \mathbb{R} e_{n+1} \times \mathbb{R}^N$ which is symmetric with respect to $\theta^\perp$ in $\mathbb{R}^n \times \mathbb{R} e_{n+1} \times \mathbb{R}^N$ since $C \subseteq \theta^\perp$. Let $P_t : \mathbb{R}^n \times \mathbb{R} e_{n+1} \times \mathbb{R}^N \to \mathbb{R}^n \times \mathbb{R} e_{n+1}$ be defined by

$$P_t(x, r, y) = (x + \langle y, t \rangle \theta, r) \in \mathbb{R}^n \times \mathbb{R} e_{n+1}.$$ 

Then

$$P_t(C) = \oplus_C(R_{\rho_1}(y_1 + t_1\theta), \ldots, R_{\rho_N}(y_N + t_N\theta)).$$

We apply Proposition 5.2 to obtain the convexity claim.

Lastly notice for each $\theta \in S^{n-1}$ and $y_1, \ldots, y_N \in \theta^\perp$, the sets

$$\oplus_C(R_{\rho_1}(y_1 + t_1\theta), \ldots, R_{\rho_N}(y_N + t_N\theta))$$

and

$$\oplus_C(R_{\rho_1}(y_1 - t_1\theta), \ldots, R_{\rho_N}(y_N - t_N\theta))$$
are reflections of one another and so the evenness condition for the Steiner convexity follows.

Notice, the proof of the following proposition follows similarly to the one above, where we consider instead the line segments (4.3b).

**Proposition 7.3.** Let \( \rho_1, \ldots, \rho_N \in \mathbb{R} \) and \( C \) a convex set contained in the positive orthant. Then the function \( F: (\mathbb{R}^n)^N \to [0, \infty) \) defined by

\[
F(x_1, \ldots, x_N) = \nu \left( \oplus_C (\tilde{R}_{\rho_1}(x_1), \ldots, \tilde{R}_{\rho_N}(x_N)) \right)
\]

is Steiner convex.

Observe, defining \([f]_N\) as in (6.2b) and considering \( C = \text{conv}\{e_1, \ldots, e_N\} \), it follows by (6.4a)

\[
\int_{\mathbb{R}^n} [f]_N(x)dx = \begin{cases} 
\nu \left( \oplus_C (\tilde{R}_{Z_1}(X_1), \ldots, \tilde{R}_{Z_N}(X_N)) \right), & \text{if } s > 0. \\
\nu \left( \oplus_C (R_{-\log Z_1}(X_1), \ldots, R_{-\log Z_N}(X_N)) \right), & \text{if } s = 0. \\
\nu \left( \oplus_C (R_{Z_1}(X_1), \ldots, R_{Z_N}(X_N)) \right) & \text{if } s < 0.
\end{cases}
\]

8. **Main proof**

**Proof of Theorem 1.1.** Let \( f, g: \mathbb{R}^n \to [0, \infty) \) be integrable \( s \)-concave functions for \( s \in (-1/n, \infty) \). Sample independent random vectors \( W_i = (X_i, Z_i), i = 1, \ldots, N + M \), according to the uniform Lebesgue measure on \( G_f \) for \( i = 1, \ldots, N \) and \( G_g \) for \( i = N + 1, \ldots, N + M \). Then the random functions \([f]_N, [g]_M\) defined as in (6.2b) satisfy

\[
\mathbb{P} \left( \int_{\mathbb{R}^n} [f]_N \star_{\lambda,s} [g]_M(v) dv > \alpha \right)
= \frac{1}{\prod_{i=1}^{M+N} ||k_i||} \int_{N+M} 1_{\{F > \alpha\}}(\overline{w}) \prod_{i=1}^{N+M} 1_{[0,k_i(x_i)]}(z_i)d\overline{w},
\]

where \( \int_{N+M} \) is the integral on \((\mathbb{R}^n \times [0, \infty))^{N+M}\), \( k_i = f_i \) for \( i = 1, \ldots, N \), \( k_i = g_i \) for \( i = N + 1, \ldots, N + M \), and

\[
\overline{w} = (w_1, \ldots, w_{N+M}), \quad d\overline{w} = dw_1 \ldots dw_{N+M}.
\]

Also we write \( C_N = \text{conv}\{e_1, \ldots, e_N\} \) and \( \widehat{C}_M = \text{conv}\{e_{N+1}, \ldots, e_{N+M}\} \), and consider \( \nu \) be defined as in (3.2) for each case.
Case $s > 0$: By (6.3b) and (3.3b) it follows

$$ H([f]_N *_{\lambda,s} [g]_M)^s = \oplus_{C_N} \{ \{ \tilde{R}Z^*_i(X_i) \}_{i=1}^N \} + \lambda \oplus_{C_M} \{ \{ \tilde{R}Z^*_i(X_i) \}_{i=N+1}^M \} $$

therefore

$$ \int_{\mathbb{R}^n} [f]_N *_{\lambda,s} [g]_M(v) dv = \nu \left( \oplus_{C_N + \lambda \bar{C}_M} \{ \{ \tilde{R}Z^*_i(X_i) \}_{i=1}^{N+M} \} \right). $$

By (8.1), Fubini, Proposition 7.2 and Theorem 7.1 we have

$$ \mathbb{P} \left( \int_{\mathbb{R}^n} [f]_N *_{\lambda,s} [g]_M(v) dv > \alpha \right) $$

$$ = \frac{1}{\| k \|^N \| k \|^N} \int_{[0,\infty)^{N+M}} 1_{\{ F > \alpha \}}(w) \prod_{i=1}^{N+M} 1_{[0,k(x_i)]}(z_i) d\bar{w} $$

$$ = \frac{1}{\| k \|^N \| k \|^N} \int_{[0,\infty)^{N+M}} \left( \int_{[0,\infty)^{N+M}} 1_{\{ F > \alpha \}}(w) \prod_{i=1}^{N+M} 1_{[0,k(x_i)]}(z_i) d\bar{x} \right) d\bar{z} $$

$$ \geq \frac{1}{\| k \|^N \| k \|^N} \int_{[0,\infty)^{N+M}} \left( \int_{[0,\infty)^{N+M}} 1_{\{ F > \alpha \}}(w) \prod_{i=1}^{N+M} 1_{[0,k(x_i)]}(z_i) d\bar{x} \right) d\bar{z} $$

$$ = \frac{1}{\| k \|^N \| k \|^N} \int_{[0,\infty)^{N+M}} 1_{\{ F > \alpha \}}(w) \prod_{i=1}^{N+M} 1_{[0,k(x_i)]}(z_i) d\bar{w} $$

$$ = \mathbb{P} \left( \int_{\mathbb{R}^n} [f^*]_N *_{\lambda,s} [g^*]_M(v) dv > \alpha \right). $$

Case $s = 0$: Using (6.3a), (3.3c), we have

$$ E(-\log ([f] N \square_{\lambda,s} [g]_M)) = \oplus_{C_N + \lambda \hat{C}_M} \{ \{ R_{-\log} Z_i(X_i) \}_{i=1}^{N+M} \}, $$

so

$$ \int_{\mathbb{R}^n} [f]_N *_{\lambda,s} [g]_M(v) dv = \nu \left( \oplus_{C_N + \lambda \hat{C}_M} \{ \{ R_{-\log} Z_i(X_i) \}_{i=1}^{N+M} \} \right). $$

It follows as before

$$ \mathbb{P} \left( \int_{\mathbb{R}^n} [f]_N *_{\lambda,s} [g]_M(v) dv > \alpha \right) \geq \mathbb{P} \left( \int_{\mathbb{R}^n} [f^*]_N *_{\lambda,s} [g^*]_M(v) dv > \alpha \right). $$

Case $s < 0$: It follows by (6.3a) and (3.3c) that

$$ E((f \square_{\lambda,s} [g]_M)^s) = \oplus_{C_N + \lambda \hat{C}_M} \{ \{ R Z^*_i(X_i) \}_{i=1}^{N+M} \}. $$
hence

$$\int_{\mathbb{R}^n} [f]_N \ast_{\lambda,s} [g]_M(v) dv = \nu \left( \bigoplus_{C_{N+\lambda}} \hat{C}_M(\mathcal{R}_{Z_i}(X_i)) \right).$$

It follows as before

$$\mathbb{P} \left( \int_{\mathbb{R}^n} [f]_N \ast_{\lambda,s} [g]_M(v) dv > \alpha \right) \geq \mathbb{P} \left( \int_{\mathbb{R}^n} [f^*]_N \ast_{\lambda,s} [g^*]_M(v) dv > \alpha \right).$$

\[ \square \]

REFERENCES

[1] S. Artstein-Avidan, D. I. Florentin, and A. Segal. Functional Brunn-Minkowski inequalities induced by polarity. *Adv. Math.*, 364:107006, 2020.

[2] S. Artstein-Avidan, A. Giannopoulos, and V. D. Milman. *Asymptotic geometric analysis. Part I*, volume 202 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.

[3] S. Artstein-Avidan, B. Klartag, and V. Milman. The Santaló point of a function, and a functional form of the Santaló inequality. *Mathematika*, 51(1-2):33–48 (2005), 2004.

[4] C. Bianchini and A. Colesanti. A sharp Rogers and Shephard inequality for the $p$-difference body of planar convex bodies. *Proceedings of the American Mathematical Society*, pages 2575–2582, 2008.

[5] C. Borell. Convex set functions in $d$-space. *Periodica Mathematica Hungarica*, 6(2):111–136, 1975.

[6] H. J. Brascamp and E. H. Lieb. Best constants in Young’s inequality, its converse, and its generalization to more than three functions. *Advances in Math.*, 20(2):151–173, 1976.

[7] H. J. Brascamp and E. H. Lieb. On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. *J. Functional Analysis*, 22(4):366–389, 1976.

[8] H. J. Brascamp, E. H. Lieb, and J.M. Luttinger. A general rearrangement inequality for multiple integrals. *Journal of functional analysis*, 17(2):227–237, 1974.

[9] A. Burchard. A short course on rearrangement inequalities. Availble at [http://www.math.utoronto.ca/almut/rearrange/pdf](http://www.math.utoronto.ca/almut/rearrange/pdf).

[10] H. Busemann. Volume in terms of concurrent cross-sections. *Pacific J. Math.*, 3:1–12, 1953.

[11] S. Campi, A. Colesanti, and P. Gronchi. A note on Sylvester’s problem for random polytopes in a convex body. *Rend. Istit. Mat. Univ. Trieste*, 31(1-2):79–94, 1999.

[12] S. Campi and P. Gronchi. On volume product inequalities for convex sets. *Proc. Amer. Math. Soc.*, 134(8):2393–2402, 2006.

[13] S. Campi and P. Gronchi. Volume inequalities for sets associated with convex bodies. In *Integral geometry and convexity*, pages 1–15. World Scientific, 2006.

[14] M. Christ. Estimates for the $k$-plane transform. *Indiana Univ. Math. J.*, 33(6):891–910, 1984.

[15] D. Cordero-Erausquin, M. Fradelizi, G. Paouris, and P. Pivovarov. Volume of the polar of random sets and shadow systems. *Math. Ann.*, 362(3-4):1305–1325, 2015.
STOCHASTIC FORMS OF BRUNN’S PRINCIPLE

[16] M. Fradelizi, M. Meyer, and A. Zvavitch. An application of shadow systems to
mahler’s conjecture. Discrete & Computational Geometry, 48(3):721–734, 2012.

[17] R. J. Gardner. The Brunn-Minkowski inequality. Bull. Amer. Math. Soc. (N.S.),
39(3):355–405, 2002.

[18] R. J. Gardner, D. Hug, and W. Weil. Operations between sets in geometry. J. Eur.
Math. Soc. (JEMS), 15(6):2297–2352, 2013.

[19] R. J. Gardner, D. Hug, and W. Weil. The Orlicz-Brunn-Minkowski theory: a general
framework, additions, and inequalities. J. of Differential Geom, 97(3):427–476, 2014.

[20] H. Groemer. On the mean value of the volume of a random polytope in a convex set.
Arch. Math. (Basel), 25:86–90, 1974.

[21] R Henstock and A.M Macbeath. On the measure of sum-sets.(i) the theorems of
Brunn, Minkowski, and Lusternik. Proceedings of the London Mathematical Society,
3(1):182–194, 1953.

[22] B. Klartag. Marginals of geometric inequalities. In Geometric aspects of functional
analysis, volume 1910 of Lecture Notes in Math., pages 133–166. Springer, Berlin,
2007.

[23] L. Leindler. On a certain converse of Hölder’s inequality. II. Acta Sci. Math. (Szeged),
33(3-4):217–223, 1972.

[24] E. H. Lieb and M. Loss. Analysis, volume 14 of Graduate Studies in Mathematics.
American Mathematical Society, Providence, RI, second edition, 2001.

[25] E. Lutwak, Yang D., and Zhang G. Orlicz centroid bodies. Journal of Differential
Geometry, 84(2):365–387, 2010.

[26] J. Melbourne. Rearrangement and Prékopa-Leindler type inequalities. Available at
https://arxiv.org/abs/1806.08837.

[27] M. Meyer and A. Pajor. On the Blaschke-Santaló inequality. Arch. Math. (Basel),
55(1):82–93, 1990.

[28] M. Meyer and S. Reisner. Shadow systems and volumes of polar convex bodies.
Mathematika, 53(1):129–148 (2007), 2006.

[29] M. Meyer and S. Reisner. On the volume product of polygons. Abhandlungen aus
dem Mathematischen Seminar der Universität Hamburg, 81(1):93–100, 2011.

[30] G. Paouris and P. Pivovarov. A probabilistic take on isoperimetric-type inequalities.
Adv. Math., 230(3):1402–1422, 2012.

[31] G. Paouris and P. Pivovarov. Randomized isoperimetric inequalities. In Convexity
and concentration, volume 161 of IMA Vol. Math. Appl., pages 391–425. Springer,
New York, 2017.

[32] P. Pivovarov and J. Rebollo Bueno. A stochastic Prékopa-Leindler inequality for
log-concave functions. Communications in Contemporary Mathematics, 2020. Accepted
for publication.

[33] A. Prékopa. Logarithmic concave measures with application to stochastic program-
mming. Acta Sci. Math. (Szeged), 32:301–316, 1971.

[34] A. Prékopa. On logarithmic concave measures and functions. Acta Sci. Math. (Szeged),
34:335–343, 1973.

[35] J. Rebollo Bueno. Stochastic reverse isoperimetric inequalities in the plane. Preprint.
Available at bit.ly/StockPlan.

[36] Y. Rinott. On convexity of measures. Ann. Probability, 4(6):1020–1026, 1976.

[37] C. A. Rogers and G. C. Shephard. Some extremal problems for convex bodies. Math-
ematika, 5:93–102, 1958.
[38] C. A. Rogers. A single integral inequality. *Journal of the London Mathematical Society*, 1(1):102–108, 1957.

[39] L. A Santaló. Un invariante afin para los cuerpos convexos del espacio de \( n \) dimensiones. *Portugaliae Mathematica*, 8(4):155–161, 1949.

[40] C. Saroglou. Characterizations of extremals for some functionals on convex bodies. *Canadian Journal of Mathematics*, 62(6):1404–1418, 2010.

[41] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 151 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, expanded edition, 2014.

[42] G. C. Shephard. Shadow systems of convex sets. *Israel Journal of Mathematics*, 2(4):229–236, 1964.

[43] B. Simon. *Convexity*, volume 187 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2011. An analytic viewpoint.

[44] M. Weberndorfer. Shadow systems of asymmetric \( l_p \) zonotopes. *Advances in Mathematics*, 240:613–635, 2013.

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