Comments on Solutions for Nonsingular Currents in Open String Field Theories

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Introduction

• Witten’s bosonic open string field theory (d=26):

\[ S[\Psi] = -\frac{1}{g^2} \left( \frac{1}{2} \langle \Psi, Q_B \Psi \rangle + \frac{1}{3} \langle \Psi, \Psi \ast \Psi \rangle \right). \]

• There were various attempts to prove Sen’s conjecture since around 1999 using the above.

• Numerically, it has been checked with “level truncation approximation.” [c.f. … Gaiotto-Ratelli “Experimental string field theory”(2002)]

• Analytically, some solutions have been constructed.

• Here, we generalize “Schnabl’s analytical solutions” (2005, 2007) which include “tachyon vacuum solution” in Sen’s conjecture and “marginal solutions.”
• In Berkovits’ WZW-type superstring field theory (d=10) the action in the NS sector is given by

\[ S_{NS}[\Phi] = -\frac{1}{g^2} \int_0^1 dt \langle (\eta_0 \Phi)(e^{-t\Phi} Q_B e^{t\Phi}) \rangle. \]

• There were some attempts to solve the equation of motion.
• Numerically, tachyon condensation was examined using level truncation. [Berkovits(-Sen-Zwiebach)(2000),…]

• Analytically, some solutions have been constructed.
• Recently [April (2007)], Erler / Okawa constructed some solutions, which are generalization of Schnabl / Kiermaier-Okawa-Rastelli-Zwiebach’s marginal solution (2007) in bosonic SFT. We consider generalization of their solutions and examine gauge transformations.
Main claim

Suppose that $\hat{\Psi}$ is BRST invariant and nilpotent:

$$Q_B \hat{\Psi} = 0, \quad \hat{\Psi} \ast \hat{\Psi} = 0.$$  Then,

$$\Psi^{(\alpha,\beta)} = P_{\alpha} \ast \frac{1}{1 + \hat{\Psi} \ast A^{(\alpha+\beta)}} \ast \hat{\Psi} \ast P_{\beta}$$

gives a solution to the EOM:

$$Q_B \Psi^{(\alpha,\beta)} + \Psi^{(\alpha,\beta)} \ast \Psi^{(\alpha,\beta)} = 0,$$

where

$$Q_B P_{\alpha} = 0, \quad P_{\alpha} \ast P_{\beta} = P_{\alpha+\beta}, \quad P_{\alpha=0} = I, \quad Q_B A^{(\gamma)} = I - P_{\gamma}.$$  

In the case $|r = \alpha + 1\rangle = P_{\alpha}$: wedge state, we have $A^{(\gamma)} = \frac{\pi}{2} \int_0^{\gamma} d\alpha B_1^\alpha P_{\alpha}$.

$\hat{\Psi} = U_1^{\dagger} U_1 \lambda eJ(0)|0\rangle$,
$\alpha = \beta = 1/2$  : Schnabl / Kiermaier-Okawa-Rastelli-Zwiebach’s marginal solution for nonsingular current is reproduced.

$\hat{\Psi} = \lambda Q_B U_1^{\dagger} U_1 B_1^\gamma c_1 |0\rangle$,
$\alpha = \beta = 1/2, \quad \lambda = \infty$  : Schnabl’s tachyon vacuum solution is reproduced.
Suppose that $\hat{\phi}$ satisfies following conditions:

\[
\eta_0 Q_B \hat{\phi} = 0, \quad \hat{\phi} \ast \hat{\phi} = 0, \quad \hat{\phi} \ast \eta_0 \hat{\phi} = 0, \quad \hat{\phi} \ast Q_B \hat{\phi} = 0.
\]

Then,

\[
\Phi^{(\alpha, \beta)}_{(1)} = \log(1 + P_\alpha \ast f_{(1)} \ast P_\beta), \quad f_{(1)} = \frac{1}{1 - \eta_0 \hat{\phi} \ast Q_B \hat{A}^{(\alpha+\beta)} \ast \hat{\phi}},
\]

\[
\Phi^{(\alpha, \beta)}_{(2)} = \log(1 - P_\alpha \ast f_{(2)} \ast P_\beta), \quad f_{(2)} = \hat{\phi} \ast \frac{1}{1 - \eta_0 \hat{A}^{(\alpha+\beta)} \ast Q_B \hat{\phi}},
\]

\[
\Phi^{(\alpha, \beta)}_{(3)} = - \log(1 - P_\alpha \ast f_{(3)} \ast P_\beta), \quad f_{(3)} = \frac{1}{1 - Q_B \hat{\phi} \ast \eta_0 \hat{A}^{(\alpha+\beta)} \ast \hat{\phi}},
\]

\[
\Phi^{(\alpha, \beta)}_{(4)} = - \log(1 - P_\alpha \ast f_{(4)} \ast P_\beta), \quad f_{(4)} = \hat{\phi} \ast \frac{1}{1 - Q_B \hat{A}^{(\alpha+\beta)} \ast \eta_0 \hat{\phi}},
\]

give solutions to the EOM:

\[
\eta_0 (e^{-\Phi^{(\alpha, \beta)}_{(i)}} Q_B e^{\Phi^{(\alpha, \beta)}_{(i)}}) = 0, \quad (i = 1, 2, 3, 4)
\]

where

\[
\eta_0 P_\alpha = 0, \quad Q_B P_\alpha = 0, \quad P_\alpha \ast P_\beta = P_{\alpha+\beta}, \quad P_{\alpha=0} = I,
\]

\[
\eta_0 Q_B \hat{A}^{(\gamma)} = I - P_\gamma.
\]

In the case $P_\alpha :$ wedge state, we find $\hat{A}^{(\gamma)} = \int_0^\gamma d\alpha \log \left(\frac{\alpha}{\gamma}\right) \left(\frac{\pi}{2} j^{1-\gamma}_1 + \frac{\pi^2}{4} g^{1-\gamma}_1 b_1^1\right) P_\alpha.$

\[
\hat{\phi} = \zeta_a U_1^\dagger U_1 \xi e^{-\phi} \psi^a(0) |0\rangle, \quad \zeta_a \zeta_b \Omega^{ab} = 0, \quad \alpha = \beta = 1/2
\]

$\hat{\phi}$: Erler / Okawa’s marginal solutions for nonsingular supercurrents are reproduced.
Witten’s bosonic open string field theory

Action: \[ S[\Psi] = -\frac{1}{g^2} \left( \frac{1}{2} \langle \Psi, Q_B \Psi \rangle + \frac{1}{3} \langle \Psi, \Psi \ast \Psi \rangle \right) \]

String field: \[ |\Psi\rangle = \phi(x)c_1|0\rangle + A_\mu(x)\alpha_\mu^i c_1|0\rangle + iB(x)c_0|0\rangle + \cdots \]

BRST operator: \[ Q_B = \oint \frac{dz}{2\pi i} \left( cT^m + b c \partial c + \frac{3}{2} \partial^2 c \right) \]

Witten star product:

Equation of motion: \[ Q_B \Psi + \Psi \ast \Psi = 0 \]

Gauge transformation: \[ \delta_\Lambda \Psi = Q_B \Lambda + \Psi \ast \Lambda - \Lambda \ast \Psi \]
Preliminary

• “sliver frame”: \( \tilde{z} = \arctan z \) (\( z : \text{UHP} \))

For a primary field \( \phi \) of dim=h,

\[
\tilde{\phi}(\tilde{z}) = \left( \frac{dz}{d\tilde{z}} \right)^h \phi(z) = (\cos \tilde{z})^{-2h} \phi(\tan \tilde{z})
\]

In particular, we often use

\[
\mathcal{L}_0 \equiv \tilde{L}_0 = L_0 + \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{4k^2 - 1} L_{2k}, \quad K_1 \equiv \tilde{L}_{-1} = L_1 + L_{-1},
\]

\[
\mathcal{B}_0 \equiv \tilde{b}_0 = b_0 + \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{4k^2 - 1} b_{2k}, \quad B_1 \equiv \tilde{b}_{-1} = b_1 + b_{-1},
\]

and

\[
\hat{\mathcal{L}} = \mathcal{L}_0 + \mathcal{L}_0^\dagger, \quad K_1^{L/R} = \frac{1}{2} K_1 \pm \frac{1}{\pi} \hat{\mathcal{L}}, \quad \hat{\mathcal{B}} = \mathcal{B}_0 + \mathcal{B}_0^\dagger, \quad B_1^{L/R} = \frac{1}{2} B_1 \pm \frac{1}{\pi} \hat{\mathcal{B}}.
\]
Using $U_r = \left(\frac{2}{r}\right)^{L_0} = \left(\frac{2}{r}\right)^{L_0} e^{-\frac{r^2}{3r^2}} L_2 + \frac{r^2}{30r^4} L_4 + \cdots$ we have a $\star$ product formula:

$$U_r^\dagger U_r \phi_1(\tilde{x}_1) \cdots \phi_n(\tilde{x}_n) \ket{0} \star U_s^\dagger U_s \tilde{\psi}_1(\tilde{y}_1) \cdots \tilde{\psi}_m(\tilde{y}_m) \ket{0} $$

$$= U_{r+s-1}^\dagger U_{r+s-1} \phi_1(\tilde{x}_1 + \frac{\pi}{4}(s-1)) \cdots \phi_n(\tilde{x}_n + \frac{\pi}{4}(s-1)) \tilde{\psi}_1(\tilde{y}_1 - \frac{\pi}{4}(r-1)) \cdots \tilde{\psi}_m(\tilde{y}_m - \frac{\pi}{4}(r-1)) \ket{0}$$

For the wedge state: $|r = \alpha + 1\rangle = U_{\alpha+1}^\dagger U_{\alpha+1} \ket{0} = P_\alpha$, we have $P_\alpha \star P_\beta = P_{\alpha+\beta}$.
• Associated with the wedge states, we have

\[ A^{(\gamma)} = \frac{\pi}{2} \int_0^\gamma d\alpha \: B_1^{\alpha} P_{\alpha} \] such as \( Q_B A^{(\gamma)} = I - P_\gamma \).

[Ellwood-Schnabl]

With BRST invariant and nilpotent \( \hat{\psi} \):

\[ Q_B \hat{\psi} = 0, \quad \hat{\psi} \ast \hat{\psi} = 0, \]

we have a solution to the equation of motion

\[
\Psi^{(\alpha,\beta)} = P_\alpha \ast \frac{1}{1 + \hat{\psi} \ast A^{(\alpha+\beta)}} \ast \hat{\psi} \ast P_\beta
\]

\[
= \sum_{k=0}^\infty (-1)^k P_\alpha \ast (\hat{\psi} \ast A^{(\alpha+\beta)})^k \ast \hat{\psi} \ast P_\beta.
\]
\[ Q_B \Psi^{(\alpha,\beta)} = P_\alpha \ast Q_B \left( \frac{1}{1 + \hat{\psi} \ast A^{(\alpha+\beta)}} \right) \ast \hat{\psi} \ast P_\beta \]

\[ = -P_\alpha \ast \frac{1}{1 + \hat{\psi} \ast A^{(\alpha+\beta)}} \ast (Q_B (I + \hat{\psi} \ast A^{(\alpha+\beta)})) \ast \frac{1}{1 + \hat{\psi} \ast A^{(\alpha+\beta)}} \ast \hat{\psi} \ast P_\beta \]

\[ = P_\alpha \ast \frac{1}{1 + \hat{\psi} \ast A^{(\alpha+\beta)}} \ast \hat{\psi} \ast (Q_B A^{(\alpha+\beta)}) \ast \frac{1}{1 + \hat{\psi} \ast A^{(\alpha+\beta)}} \ast \hat{\psi} \ast P_\beta \]

\[ = P_\alpha \ast \frac{1}{1 + \hat{\psi} \ast A^{(\alpha+\beta)}} \ast \hat{\psi} \ast (I - P_{\alpha+\beta}) \ast \frac{1}{1 + \hat{\psi} \ast A^{(\alpha+\beta)}} \ast \hat{\psi} \ast P_\beta \]

\[ = P_\alpha \ast \frac{1}{1 + \hat{\psi} \ast A^{(\alpha+\beta)}} \ast \hat{\psi} \ast \hat{\psi} \ast \frac{1}{1 + A^{(\alpha+\beta)}} \ast \hat{\psi} \ast P_\beta \]

\[ = -P_\alpha \ast \frac{1}{1 + \hat{\psi} \ast A^{(\alpha+\beta)}} \ast \hat{\psi} \ast P_\beta \ast P_\alpha \ast \frac{1}{1 + \hat{\psi} \ast A^{(\alpha+\beta)}} \ast \hat{\psi} \ast P_\beta \]

\[ = -\Psi^{(\alpha,\beta)} \ast \Psi^{(\alpha,\beta)} . \]

**Note 1.** \( \lambda \hat{\psi} \) is also BRST invariant and nilpotent. \( \rightarrow \) \( \Psi^{(\alpha,\beta)} \) can naturally include 1-parameter.
Note 2.

In general, for \[ \Psi^{(\alpha,\beta)}(\psi) \equiv P_{\alpha} * \frac{1}{1 + \psi * A^{(\alpha+\beta)}} * \psi * P_{\beta} \]
we have

\[ Q_B \Psi^{(\alpha,\beta)}(\psi) + \Psi^{(\alpha,\beta)}(\psi) * \Psi^{(\alpha,\beta)}(\psi) \]
\[ = P_{\alpha} * \frac{1}{1 + \psi * A^{(\alpha+\beta)}} * (Q_B \psi + \psi * \psi) * \frac{1}{1 + A^{(\alpha+\beta)} * \psi} * P_{\beta}. \]

We can regard \[ \psi \mapsto \Psi^{(\alpha,\beta)}(\psi) = P_{\alpha} * \frac{1}{1 + \psi * A^{(\alpha+\beta)}} * \psi * P_{\beta} \]
as a map \textit{from a solution to another solution}:

\[ Q_B \psi + \psi * \psi = 0 \]
\[ \rightarrow Q_B \Psi^{(\alpha,\beta)}(\psi) + \Psi^{(\alpha,\beta)}(\psi) * \Psi^{(\alpha,\beta)}(\psi) = 0 \]

Composition of maps forms a commutative monoid:

\[ \Psi^{(\alpha,\beta)}(\Psi^{(\alpha',\beta')}(\psi)) = \Psi^{(\alpha+\alpha',\beta+\beta')}(\psi), \quad (\alpha, \beta, \alpha', \beta' \geq 0) \]
\[ \Psi^{(0,0)}(\psi) = \psi. \]
Example of BRST invariant and nilpotent $\hat{\psi}$

\[
\hat{\psi} = \lambda_s \hat{\psi}_s + \lambda_m \hat{\psi}_m ,
\]
\[
\hat{\psi}_s = Q_B \hat{\Lambda}_0 , \quad \hat{\Lambda}_0 \equiv U_1^\dagger U_1 B_1^L c_1 |0\rangle ,
\]
\[
\hat{\psi}_m = U_1^\dagger U_1 c J(0) |0\rangle .
\]

where $J(z) = \zeta_\alpha J^\alpha(z)$ is “nonsingular” matter primary of dimension 1:

\[
\zeta_\alpha \zeta_\beta g^{\alpha \beta} = 0 , \quad J^\alpha(y) J^\beta(z) \sim \frac{g^{\alpha \beta}}{(y-z)^2} + \frac{1}{y-z} i f^{\alpha \beta \gamma} J^\gamma(z) + \cdots .
\]

In particular, $\lambda_s = 0 \quad \Longrightarrow \quad$ marginal solution

$\lambda_m = 0 \quad \Longrightarrow \quad$ tachyon solution

Due to the nonsingular condition for the current, we find nilpotency:

\[
c\zeta_\alpha J^\alpha(\epsilon) c\zeta_\beta J^\beta(0) \sim 0
\]
Marginal solution

From a BRST invariant, nilpotent \( \hat{\Psi}_m = U_1^{\dagger} U_1 c J(0) |0\rangle \) which satisfies

\[(\mathcal{B}_0 - \mathcal{B}_0^{\dagger}) \hat{\Psi}_m = 0, \]

we can generate a solution

\[
\Psi^{(\alpha, \beta)} = \sum_{k=0}^{\infty} (-1)^k \lambda_m^{k+1} P_{\alpha} \ast (\hat{\Psi}_m \ast A^{(\alpha + \beta)})^k \ast \hat{\Psi}_m \ast P_{\beta} = \sum_{n=1}^{\infty} \lambda_m^n \psi_{m,n},
\]

\[
\psi_{m,1} = U_{\alpha + \beta + 1}^{\dagger} U_{\alpha + \beta + 1} c \tilde{J}(\frac{\pi}{4} (\beta - \alpha)) |0\rangle,
\]

\[
\psi_{m,k+1} = \left( -\frac{\pi}{2} \right)^k \int_0^{\alpha + \beta} dr_1 \cdots \int_0^{\alpha + \beta} dr_k U_{\alpha + \beta + 1 + \sum_{l=1}^{k} r_l}^{\dagger} U_{\alpha + \beta + 1 + \sum_{l=1}^{k} r_l} \prod_{m=0}^{k} \tilde{J}(\frac{\pi}{4} (\beta - \alpha - \sum_{l=1}^{m} r_l + \sum_{l=m+1}^{k} r_l)) \right. 
\times \left[ - \frac{1}{\pi} \tilde{J}(\frac{\pi}{4} (\beta - \alpha + \sum_{l=1}^{k} r_l)) \tilde{J}(\frac{\pi}{4} (\beta - \alpha - \sum_{l=1}^{k} r_l)) + \frac{1}{2} \left( \tilde{J}(\frac{\pi}{4} (\beta - \alpha + \sum_{l=1}^{k} r_l)) + \tilde{J}(\frac{\pi}{4} (\beta - \alpha - \sum_{l=1}^{k} r_l)) \right) \right] |0\rangle.
\]

\[
\Psi^{(\alpha, \beta)} \sim \sum_{m} \lambda_m^n \int dr_k
\]
Tachyon solution

- From a BRST invariant, nilpotent \( \hat{\psi}_s = Q_B U_1^{\dagger} U_1 B_1^L c_1 |0\rangle \) which satisfies \( (B_0 - B_0^\dagger) \hat{\psi}_s = 0 \), we can generate a solution:

\[
\Psi^{(\alpha, \beta)} = \sum_{k=0}^{\infty} (-1)^k \lambda_s^{k+1} P_\alpha * \hat{\psi}_s * (A^{(\alpha+\beta)} * \hat{\psi}_s)^k * P_\beta = \sum_{n=1}^{\infty} \lambda_s^n \psi_{s,n}.
\]

Each term is computed as

\[
\psi_{s,n} = P_\alpha * (Q_B \hat{\Lambda}_0) * P_\beta * (P_\alpha * \hat{\Lambda}_0 * P_\beta - I)^{n-1} = -\sum_{l=0}^{n-1} \frac{(-1)^{n-1-l}(n-1)!}{l!(n-1-l)!} \partial_t \psi_t^{(\alpha, \beta)} |_{t=0},
\]

\[
\psi_{t,n}^{(\alpha, \beta)} = \frac{2}{\pi} U_{n(\alpha+\beta)+t+\alpha+\beta+1} U_{n(\alpha+\beta)+t+\alpha+\beta+1} \left[ -\frac{1}{\pi} \tilde{c}(\frac{\pi}{4}(\beta - \alpha + t + n(\alpha + \beta))) \tilde{c}(\frac{\pi}{4}(\beta - \alpha - t - n(\alpha + \beta))) \\
+ \frac{1}{2} \left\{ \tilde{c}(\frac{\pi}{4}(\beta - \alpha + t + n(\alpha + \beta))) + \tilde{c}(\frac{\pi}{4}(\beta - \alpha - t - n(\alpha + \beta))) \right\} \right] |0\rangle.
\]

Then, we can re-sum the above as

\[
\Psi^{(\alpha, \beta)} = -\sum_{l=0}^{\infty} \lambda_s^{l+1} \partial_t \psi_{t,l}^{(\alpha, \beta)} |_{t=0}.
\]

Here, expansion parameter is redefined as

\[
\lambda_S \equiv \frac{\lambda_s}{\lambda_s + 1}.
\]
The solution can be rewritten as 

\[ \Psi(\alpha, \beta) = e^{\pi \beta - \alpha} K_1(\alpha + \beta) D^2 \Psi^{(1/2,1/2)}, \]

where \( K_1 = L_1 + L_{-1}, \ D = \mathcal{L}_0 - \mathcal{L}_0^\dagger \) are BPZ odd and derivations w.r.t. \*,

and \( \Psi^{(1/2,1/2)} \) is the Schnabl’s solution for tachyon condensation at 

\[ \lambda_S = 1 \iff \lambda_S = \infty. \]

By regularizing it as 

\[ \Psi(\alpha, \beta)|_{\lambda_S=1} = \lim_{N \to \infty} \left( \frac{1}{\alpha + \beta} \psi^{(\alpha,\beta)}_{t=0,N} - \sum_{n=0}^{N} \partial_t \psi_{t,n}^{(\alpha,\beta)} \right), \]

the new BRST operator around the solution \( Q'_B \) satisfies 

\[ Q'_B A^{(\alpha+\beta)} \equiv Q_B A^{(\alpha+\beta)} + \Psi(\alpha,\beta)|_{\lambda_S=1} \ast A^{(\alpha+\beta)} + A^{(\alpha+\beta)} \ast \Psi(\alpha,\beta)|_{\lambda_S=1} = I, \]

which implies vanishing cohomology and 

\[ S[\Psi(\alpha,\beta)|_{\lambda_S=1}]/V_{26} = \frac{1}{2\pi^2 g^2} = T_{25}. \]

This result is \((\alpha, \beta)\)-independent.
Note

We can evaluate the action as \( S[\Psi^{(\alpha,\beta)}]/V_{26} = 0 \) (\(|\lambda_s| < 1\)).

In fact, the solution can be rewritten as pure gauge form by evaluating the infinite summation formally

\[
\Psi^{(\alpha,\beta)} = Q_B(\lambda_s P_\alpha \ast \hat{\Lambda}_0 \ast P_\beta) \ast \frac{1}{1 - \lambda_s P_\alpha \ast \hat{\Lambda}_0 \ast P_\beta}.
\]
Berkovits’ WZW-type super SFT

The action for the NS sector is

$$S_{NS}[\Phi] = -\frac{1}{g^2} \int_0^1 dt \langle (\eta_0 \Phi)(e^{-t\Phi} Q_B e^{t\Phi}) \rangle.$$ 

String field $\Phi$: ghost number 0, picture number 0, Grassmann even, expressed by matter and ghosts $b, c, \phi, \xi, \eta$ ($\beta = e^{-\phi} \partial \xi, \gamma = \eta e^\phi$):

$$Q_B = \oint \frac{dz}{2\pi i} (c(T^m - \frac{1}{2} (\partial \phi)^2 - \partial^2 \phi + \partial \xi \eta) + b c \partial c + \eta e^\phi G^m - \eta \partial \eta e^{2\phi} b)(z),$$

$$\eta_0 = \oint \frac{dz}{2\pi i} \eta(z).$$

Equation of motion: $\eta_0 (e^{-\Phi} Q_B e^\Phi) = 0 \iff Q_B (e^\Phi \eta_0 e^{-\Phi}) = 0$

Gauge transformation: $\delta e^\Phi = \Xi_1 \ast e^\Phi + e^\Phi \ast \Xi_2$, $Q_B \Xi_1 = 0$, $\eta_0 \Xi_2 = 0$.

Using the wedge states $|r = \alpha + 1\rangle = P_\alpha$ as in bosonic SFT, we have

$$Q_B P_\alpha = 0, \quad \eta_0 P_\alpha = 0, \quad P_\alpha \ast P_\beta = P_{\alpha+\beta}, \quad P_{\alpha=0} = I.$$
Corresponding to the wedge states, we have constructed \( \hat{A}(\gamma) \):

\[
\hat{A}(\gamma) = \int_0^\gamma d\alpha \log \left( \frac{\alpha}{\gamma} \right) \left( \frac{\pi}{2} J_{1}^{--L} + \frac{\pi^2}{4} \tilde{G}_{1}^{-L} B_{1}^{L} \right) P_\alpha ,
\]

such as \( \eta_0 \hat{A}(\gamma) = -\frac{\pi}{2} \int_0^\gamma d\alpha B_{1}^{L} P_\alpha , \quad Q_B \hat{A}(\gamma) = -\frac{\pi}{2} \int_0^\gamma d\alpha \tilde{G}_{1}^{-L} P_\alpha , \quad \eta_0 Q_B \hat{A}(\gamma) = I - P_\gamma , \)

\( J^{--}(z) = \xi b(z), \quad \tilde{G}^- = [Q_B, J^{--}(z)] \implies J_{1}^{--L}, \tilde{G}_{1}^{-L} \) are defined in the same way as \( B_{1}^{L} \).

Then, we find that

\[
\Phi_{(1)}^{(\alpha, \beta)}(\phi) = \log(1 + P_\alpha \ast f_{(1)} \ast P_\beta) , \quad f_{(1)} = \frac{1}{1 + (e^\phi \eta_0 e^{-\phi}) Q_B \hat{A}^{(\alpha + \beta)}(e^\phi - 1) ,}
\]

\[
\Phi_{(2)}^{(\alpha, \beta)}(\phi) = \log(1 + P_\alpha \ast f_{(2)} \ast P_\beta) , \quad f_{(2)} = (e^\phi - 1) \frac{1}{1 - \eta_0 \hat{A}(\alpha + \beta)(e^{-\phi} Q_B e^\phi)} ,
\]

\[
\Phi_{(3)}^{(\alpha, \beta)}(\phi) = -\log(1 - P_\alpha \ast f_{(3)} \ast P_\beta) , \quad f_{(3)} = \frac{1}{1 - (e^{-\phi} Q_B e^\phi) \eta_0 \hat{A}(\alpha + \beta)(1 - e^{-\phi})} ,
\]

\[
\Phi_{(4)}^{(\alpha, \beta)}(\phi) = -\log(1 - P_\alpha \ast f_{(4)} \ast P_\beta) , \quad f_{(4)} = (1 - e^{-\phi}) \frac{1}{1 + Q_B \hat{A}^{(\alpha + \beta)}(e^\phi \eta_0 e^{-\phi})} ,
\]

map solutions to other solutions because

\[
e^{\Phi_{(1)}^{(\alpha, \beta)}} \eta_0 e^{-\Phi_{(1)}^{(\alpha, \beta)}} = e^{\Phi_{(4)}^{(\alpha, \beta)}} \eta_0 e^{-\Phi_{(4)}^{(\alpha, \beta)}} = P_\alpha \frac{1}{1 + (e^\phi \eta_0 e^{-\phi}) Q_B \hat{A}(\alpha + \beta)(e^\phi \eta_0 e^{-\phi})} P_\beta ,
\]

\[
e^{-\Phi_{(2)}^{(\alpha, \beta)}} Q_B e^{\Phi_{(2)}^{(\alpha, \beta)}} = e^{-\Phi_{(3)}^{(\alpha, \beta)}} Q_B e^{\Phi_{(3)}^{(\alpha, \beta)}} = P_\alpha (e^{-\phi} Q_B e^\phi) \frac{1}{1 - \eta_0 \hat{A}(\alpha + \beta)(e^{-\phi} Q_B e^\phi)} P_\beta .
\]
If $\hat{\phi}$ satisfies $\eta_0 Q_B \hat{\phi} = 0$, $\hat{\phi} \ast \hat{\phi} = 0$, $\hat{\phi} \ast \eta_0 \hat{\phi} = 0$, $\hat{\phi} \ast Q_B \hat{\phi} = 0$, $\hat{\phi}$ is a solution: $\eta_0 (e^{-\hat{\phi}} Q_B e^{\hat{\phi}}) = 0$.

\[ \Phi^{(\alpha, \beta)}_{(i)} (\hat{\phi}), \ (i = 1, 2, 3, 4) \] are also solutions.

Example of $\hat{\phi}$ using nonsingular matter supercurrent:

\[ J^a(z, \theta) = \psi^a(z) + \theta J^a(z) \]

\[ \hat{\phi} = \zeta_a U_1^\dagger U_1 c \xi e^{-\phi} \psi^a(0) |0\rangle, \quad \zeta_a \zeta_b \Omega^{ab} = 0, \]

where we suppose

\[ \psi^a(y)\psi^b(z) \sim (y - z)^{-1} \Omega^{ab}, \]
\[ J^a(y)\psi^b(z) \sim (y - z)^{-1} i f^{abc} \psi^c(z), \]
\[ J^a(y)J^b(z) \sim (y - z)^{-2} \Omega^{ab} + (y - z)^{-1} i f^{abc} J^c(z). \]

More explicitly, on the flat background, we can take

\[ J^\mu(z, \theta) = \psi^\mu(z) + \theta i \partial X^\mu(z), \quad \zeta_\mu \zeta_\nu \eta^{\mu\nu} = 0. \]
Gauge transformations

Using path-ordering, we found

\[
\Psi^{(\alpha,\beta)} = V^{(\alpha,\beta)-1} \ast \psi \ast V^{(\alpha,\beta)} + V^{(\alpha,\beta)-1} \ast Q_B \ast V^{(\alpha,\beta)},
\]

\[
V^{(\alpha,\beta)} = \mathcal{P} \exp \int_0^1 dt G^{(\alpha,\beta)}(t),
\]

\[
G^{(\alpha,\beta)}(t) \equiv \frac{-\pi}{2} \left( \alpha (B_1^L P_t \alpha) \ast \frac{1}{1 + \psi \ast A(t(\alpha+\beta))} \ast \psi \ast P_{t\beta} + \beta P_t \alpha \ast \frac{1}{1 + \psi \ast A(t(\alpha+\beta))} \ast \psi \ast B_1^R P_{t\beta} \right),
\]

for bosonic SFT.

(In the case \( \alpha = \beta \), this form coincides with Ellwood’s one.)

In this sense,

\[\Psi^{(\alpha,\beta)} \sim \hat{\psi}\]

Without the identity state, including Schnabl's marginal and scalar solutions

Based on the identity state, BRST inv. and nilpotent
• Similarly, in super SFT, we have found

\begin{align*}
&\quad e^{\Phi_{(3)}^{(\alpha,\beta)}} = W_1 \ast e^\phi \ast W_2, \quad Q_B W_1 = 0, \quad \eta_0 W_2 = 0, \\
&\quad W_1 \equiv P' \exp \int_0^1 dt G_1^{(\alpha,\beta)}(t), \quad W_2 \equiv P \exp \int_0^1 dt G_2^{(\alpha,\beta)}(t), \\
&\quad G_1^{(\alpha,\beta)}(t) \equiv \frac{\pi}{2} \left[ -\alpha K_1^L I + (\alpha + \beta) Q_B B_1^R \left( P_{t\alpha} - \frac{1}{1 - Q_B ((e^\phi - 1) \eta_0 \hat{A}^{(t,\alpha+\beta)}(1 - e^\phi))} P_{t\beta} \right) \right], \\
&\quad G_2^{(\alpha,\beta)}(t) \equiv \frac{\pi}{2} \left[ \alpha K_1^L I + (\alpha + \beta) B_1^R \left( P_{t\alpha}(e^{-\phi} Q_B e^\phi) - \frac{1}{1 - \eta_0 \hat{A}^{(t,\alpha+\beta)}(1 - e^{-\phi})} P_{t\beta} \right) \right], \\
&\quad e^{\Phi_{(1)}^{(\alpha,\beta)}} = W_3 \ast e^\phi \ast W_4, \quad Q_B W_3 = 0, \quad \eta_0 W_4 = 0, \\
&\quad W_3 \equiv P' \exp \int_0^1 dt G_4^{(\alpha,\beta)}(t), \quad W_4 \equiv P \exp \int_0^1 dt G_3^{(\alpha,\beta)}(t), \\
&\quad G_3^{(\alpha,\beta)}(t) \equiv \frac{\pi}{2} \left[ \alpha K_1^L I - (\alpha + \beta) \eta_0 \hat{G}_1^{-R} \left( P_{t\alpha} - \frac{1}{1 - \eta_0 ((1 - e^\phi) Q_B \hat{A}^{(t,\alpha+\beta)}(1 - e^\phi))} P_{t\beta} \right) \right], \\
&\quad G_4^{(\alpha,\beta)}(t) \equiv \frac{\pi}{2} \left[ -\alpha K_1^L I + (\alpha + \beta) \hat{G}_1^{-R} \left( P_{t\alpha}(e^{-\phi} \eta_0 e^{-\phi} Q_B \hat{A}^{(t,\alpha+\beta)}(e^\phi - 1)) - \frac{1}{1 + (e^\phi \eta_0 e^{-\phi}) Q_B \hat{A}^{(t,\alpha+\beta)}(e^\phi - 1))} P_{t\beta} \right) \right].
\end{align*}

\begin{align*}
&\quad e^{\Phi_{(2)}^{(\alpha,\beta)}} = U_{23} \ast e^{\Phi_{(3)}^{(\alpha,\beta)}}, \quad e^{\Phi_{(1)}^{(\alpha,\beta)}} = e^{\Phi_{(4)}^{(\alpha,\beta)}} \ast V_{41}, \\
&\quad U_{23} \equiv 1 - Q_B \left( P_{t\alpha}(e^\phi - 1)) - \frac{1}{1 - \eta_0 \hat{A}^{(\alpha+\beta)}(e^\phi - 1))} \eta_0 \hat{A}^{(\alpha+\beta)}(1 - e^\phi) P_{t\beta} \right), \\
&\quad V_{41} \equiv 1 + \eta_0 \left( P_{t\alpha}(1 - e^{-\phi}) - \frac{1}{1 + Q_B \hat{A}^{(\alpha+\beta)}(e^\phi - 1))} Q_B \hat{A}^{(\alpha+\beta)}(e^\phi - 1)) P_{t\beta} \right). 
\end{align*}
In this sense, \( \Phi^{(\alpha, \beta)}_{(i)} \sim \hat{\phi} \)

Based on the identity state,

\[
\eta_0 Q_B \hat{\phi} = 0, \quad \hat{\phi} \ast \hat{\phi} = 0, \\
\hat{\phi} \ast \eta_0 \hat{\phi} = 0, \quad \hat{\phi} \ast Q_B \hat{\phi} = 0.
\]

Note:
The above gauge equivalence relations seem to be *formal* and might not be well-defined.
The gauge parameter string fields might become “singular,” as well as Schnabl or Takahashi-Tanimoto’s tachyon solution.
Future problems

• How about general (super)currents? Namely, $\zeta_a \zeta_b g^{ab} \neq 0$, $\zeta_a \zeta_b \Omega^{ab} \neq 0$.

C.f. [KORZ], [Fuchs-Kroyter-Potting], [Fuchs-Kroyter], [Kiermaier-Okawa]

In [Takahashi-Tanimoto, Kishimoto-Takahashi] some solutions based on the identity state for general (super)current were already constructed.

At least formally, $\Psi^{(\alpha,\beta)}(\Psi^{TT})$ and $\Phi^{(\alpha,\beta)}(\Phi^{KT})$ with $\alpha, \beta > 0$

give solutions which are not based on the identity state!

Zeze’s talk!

So far, various computations seem to be rather formal.

• Definition of the “regularity” of string fields?

It is very important in order to investigate “regular solutions,”
gauge transformations among them and cohomology around them.
理研シンポジウム

弦の場の理論 07

10月6日（土），7日（日）

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http://www.riken.jp/lab-www/theory/sft/