SAT solving techniques: a bibliography

Louis Abraham∗
École polytechnique

April 24, 2018

Abstract
We present a selective bibliography about efficient SAT solving, focused on optimizations for the CDCL-based algorithms.

1 Introduction

SAT is one of the most famous NP-complete problems. However, today, state-of-the-art solvers are able to solve industrial benchmarks with more than one million variables and 10 million clauses despite using exponential worst case algorithms

The most performant solvers are based on two mechanisms: the boolean constraint propagation (BCP) from the Davis–Putnam–Logemann–Loveland (DPLL) algorithm, and its extension by conflict driven clause learning (CDCL). The whole is commonly refered as CDCL.

The CDCL solvers are high feats of engineering that manage to tackle very hard problems thanks to advanced heuristics. We gathered the main articles that describe such optimizations.

2 Optimizations

We can distinguish three types of optimizations.

2.1 Algorithmic optimizations

Like any complex program, a CDCL solver relies on data structures. Better data structures allow huge performance gains.

For example, detecting unit clauses (all but one literals are assigned to false, the last one is free) and unsatisfied clauses is a problem known as boolean constraint propagation (BCP). One can think of a simple counter-based

∗louis.abraham@yahoo.fr
1https://www.princeton.edu/~chaff/zchaff.html
algorithm, but now the best solvers use lazy data structures like *Head and Tail* or two watched literals (2WL).

Another technique that seems performant, although not widely used, is Early Conflict Detection Based BCP (ECDB).

The algorithm also uses a clause database, and might need priority queues to implement the various heuristics described below.

### 2.2 Search optimizations

Like in any search program, the **order of the literal assignments** is important. There are two types of strategies: static and dynamic. Static strategies choose an order on the variables at the beginning of the code while dynamic strategies make the order evolve during the search.

The interest of dynamic strategies is reinforced by the ability to **restart** the algorithm, that is delete all assignments of variables. A new execution of the algorithm may execute faster because of both the updated order of assignment on the variables, and the learned clauses.

### 2.3 Clause learning and deletion

When the current assignment does not satisfy a clause, one can deduce a **conflicting clause** implied by the known clauses, but that allows to find the conflict more easily if the situation occurs again.

While there are various ways to deduce such a conflicting clause, the first unique implication point (1UIP) strategy is the most widely used.

However, learning too many clauses can also deteriorate the performance because of memory overflow or an overuse of BCP. Therefore, strategies arose to control the size of the clause database and **forget clauses**.

### 3 General references

Knuth [14] gives a really broad vision of SAT solving and discusses a broad range of topics like random SAT instances, symmetry breaking or parallelism.

However, this bibliography is focused on the state-of-the-art solvers that mostly use the CDCL paradigm.

A really good introduction with a lot of references is found in the blog post [24]. It also explains why the lazy BCP data structures cannot use pure literal elimination [25].

Biere et al. [5] formally explains the principle of CDCL and lists the most discussed optimizations areas: backtracking scheme, lazy data structures for BCP, restart strategies, variable selection heuristics and clause deletion strategies.
Zhand and Malik [27], although older, gives an idea of what the state of the art was in 2002.

Some parts of a forthcoming textbook, *Automated Reasoning—The Art of Generic Problem Solving* by Weidenbach, are also available online [23].

Ryan [21] gives detailed insights on the various algorithms and heuristics, particularly for the BCP and the decision strategy.

Fleury et al. [8] presents a totally formalized algorithm using 2WL, thus it should be preferred to other references because it proved the invariants of 2WL that are sometimes insufficiently presented in other references.

### 4 Search-related optimizations

#### 4.1 Decision heuristics

In the absence of conflicts, the DPLL algorithms needs to make boolean decisions on variables. There is no good strategy to choose between the two possibilities (Knuth [14] suggests to default to `false` for human-generated instances). However, the choice of the variable to be decided is really important.

Today, most solvers use the VSIDS (Variable State Independent, Decaying Sum) heuristic that updates priority scores for the literals.

Yi [26] suggests VSIDS does not perform better than random variable selection. However, this paper was neither published nor quoted, according to Google Scholar.

Liang et al. [17] uses the community structures to explain why VSIDS works and improves it further.

Dershowitz et al. [6] compares three families of decision heuristics: VSIDS, Berkmin and their novel Clause-Based Heuristic (CBH).

Biere and Fröhlich [4] is the most up-to-date reference: it compares 8 scoring schemes and designs a generic queue data structure suitable for any scoring scheme. The ACIDS (average conflict-index decision score) they introduce seems to be competitive against EVSIDS (exponential VSIDS), the most commonly used implementation of VSIDS, and VMTF (variable move-to-front) also named Berkmin because it was introduced by the BERKMIN solver [9]. Furthermore, ACIDS does not involve any parameters.

#### 4.2 Restart policies

A solver can lose a lot of time exploring a barren part of the search space. The search is mostly determined by the decision heuristic and the learned clauses. After a reasonable number of conflicts, the decision literals on the trail will not be the maximal literals with respect to the decision heuristic, and some useful clauses have been learnt.

Hence, restarting the solver by removing all assignments leads to a difference in the execution because the decisions will be made in a different order.
Huang [10] compares several static restart policies, and suggests the universally optimal policy for Las Vegas algorithms introduced by Luby [18] is the best. However, as stated in the article, “The theoretical relevance of this property to clause learning remains an interesting question though”.

Kautz et al. [13] studies dynamic restart policies that improve the performances by 40% to 65% over Luby’s.

Biere and Fröhlich [3] compares the performances of several restart policies in the state-of-the-art solver Glucose [22]. They show their static-256 uniform policy performs similarly to Luby’s policy. Furthermore, they present the performances of various dynamic policies that outperform static ones.

5 CDCL optimizations

5.1 Conflict analysis

Zhang et al. [28] explains the basic principle of learning: backtrack in a non-chronological way while learning clauses. They compare several learning schemes and introduce the 1UIP (first unique implication point) scheme. Their experiments showed that the performance is not enhanced by decision schemes generating smaller clauses; and that the 1UIP scheme clearly outperforms the other learning schemes.

5.2 Deleting clauses

The principle of CDCL is to learn useful clauses that will enhance the quality of the search. The learned clauses are consequences of the clauses given in the input, but some are more useful than others.

On the other hand, adding clauses has a cost for BCP, thus the need for clause deletion.

Most strategies tend to keep smaller clauses and delete bigger ones more aggressively because the BCP overload depends on the clause size.

Audemard and Simon [1] compares strategies based on clause activity (similar to VSIDS for variables) with their own strategy based on Literals Blocks Distance (LBD). They detect special clauses that are always kept independently of their size and named “Glue Clauses”. Their algorithm was implemented in the Glucose solver [22].

Jabbour et al. [11] introduces a simple size-based randomized policy and proves it yields better results than the LBD policy of Glucose.

6 Implementation experiments

Many solvers were implemented to test different optimizations.
Moskewicz et al. [20] presents CHAFF, a solver that brought a revolution in the world of SAT-solving by introducing two major optimizations that we discussed: the 2WL BCP algorithm and the VSIDS decision heuristic. Eén and Sörensson [7] explains the design of MiniSat [19], an implementation inspired by CHAFF. They also introduced the VSIDS activity for clauses. Katebi et al. [12] modified MiniSat to rank the “usefulness” of its features. Thus, clause learning is the most useful, followed by VSIDS. 2WL (compared to counter-based BCP) and Luby’s restart policy come after but are still responsible for major performance improvements. Glucose [22] is a modification of MiniSat based on the alternative scoring scheme for the clause learning mechanism presented in [1].

The Berkmin solver [9] is mainly known for the VMTF branching decision heuristic it used. It is mainly based on the idea that recently deduced clauses are the most important to satisfy.

The MIRA solver [15, 16] implemented two novel optimizations: Implication Queue Sorting (IQS) and Early Conflict Detection Based BCP (ECDB). It also combined the VSIDS and VMTF branching heuristics. According to the benchmarks, it outperformed the state-of-the-art version of zChaff that was available at the time. However, the techniques it introduced were not reused, and the article itself was not widely cited. The details of ECDB are developed in Lewis et al. [15], and cover an heuristic to first propagate the implications that are the most likely to cause a conflict.

7 Conclusion

This overview of the evolution of SAT solving allows us to make some observations. SAT solvers are a very empirical field and most techniques are proved useful through experiments without being explained on a theoretical point of view. The power of clause learning as a proof system was only partially explained in 2004 by Beame et al. [2], years after the advent of powerful CDCL solvers. It is difficult to conclude on the efficiency of a technique because the performance can differ between the implementations and the hardware evolution changed the way algorithms are executed. The performance of a technique can also depend on other aspects of a solver, thus explaining differences in the results.

Hopefully, the strong interest of the community allowed a standardization of the benchmarks and the emergence of modular solvers like MiniSat that can be easily modified to conduct meaningful experiments.

8 Acknowledgments

We would like to thank Stéphane Graham-Lengrand for valuable discussions and comments.
References

[1] Gilles Audemard and Laurent Simon. “Predicting Learnt Clauses Quality in Modern SAT Solvers.” In: *IJCAI*. Vol. 9. 2009, pp. 399–404. URL: http://www.ijcai.org/Proceedings/09/Papers/074.pdf (cit. on pp. 4, 5).

[2] Paul Beame, Henry Kautz, and Ashish Sabharwal. “Towards understanding and harnessing the potential of clause learning.” In: *Journal of Artificial Intelligence Research* 22 (2004), pp. 319–351. URL: https://www.jair.org/media/1410/live-1410-2304-jair.pdf (cit. on p. 5).

[3] Armin Biere and Andreas Fröhlich. “Evaluating CDCL restart schemes”. In: *Proceedings POS-15. Sixth Pragmatics of SAT workshop*. 2015. URL: http://fmv.jku.at/papers/BiereFroehlich-POS15.pdf (cit. on p. 4).

[4] Armin Biere and Andreas Fröhlich. “Evaluating CDCL variable scoring schemes”. In: *International Conference on Theory and Applications of Satisfiability Testing*. Springer. 2015, pp. 405–422. URL: http://fmv.jku.at/papers/BiereFroehlich-SAT15.pdf (cit. on p. 3).

[5] Armin Biere et al. “Conflict-driven clause learning SAT solvers”. In: *Handbook of Satisfiability, Frontiers in Artificial Intelligence and Applications* (2009), pp. 131–153. URL: https://www.cis.upenn.edu/~ahur/CIS673/sat-cdcl.pdf (cit. on p. 2).

[6] Nachum Dershowitz, Ziyad Hanna, and Alexander Nadel. “A clause-based heuristic for SAT solvers”. In: *International Conference on Theory and Applications of Satisfiability Testing*. Springer. 2005, pp. 40–60. URL: https://www.cs.tau.ac.il/~nachum/papers/LNCS/ClauseBasedHeuristic.pdf (cit. on p. 3).

[7] Niklas Eén and Niklas Sörensson. “An extensible SAT-solver”. In: *International conference on theory and applications of satisfiability testing*. Springer. 2003, pp. 502–518. URL: http://minisat.se/downloads/MiniSat.pdf (cit. on p. 5).

[8] Mathias Fleury, Jasmin Christian Blanchette, and Peter Lammich. “A Verified SAT Solver with Watched Literals Using Imperative HOL”. In: (2018). URL: http://matryoshka.gforge.inria.fr/pubs/sat_2wl_paper.pdf (cit. on p. 3).

[9] E Goldberg and Y Novikov. “BerkMin: A Fast and Robust Sat-Solver”. In: *Proceedings of the conference on Design, automation and test in Europe*. IEEE Computer Society. 2002, p. 142. URL: https://www.inf.ed.ac.uk/teaching/courses/propm/papers/goldberg_novikov_date02.pdf (cit. on pp. 3, 5).

[10] Jinbo Huang et al. “The Effect of Restarts on the Efficiency of Clause Learning.” In: *IJCAI*. Vol. 7. 2007, pp. 2318–2323. URL: http://users.eecs.anu.edu.au/~jinbo/07-ijcai-restarts.pdf (cit. on p. 4).

[11] S. Jabbour et al. “Revisiting the Learned Clauses Database Reduction Strategies”. In: *ArXiv e-prints* (Feb. 2014). arXiv: 1402.1956 [cs.AI]. URL: https://arxiv.org/pdf/1402.1956.pdf (cit. on p. 4).

[12] Hadi Katebi, Kareem A Sakallah, and João P Marques-Silva. “Empirical study of the anatomy of modern sat solvers”. In: *International Conference on Theory and Applications of Satisfiability Testing*. Springer. 2011, pp. 343–356. URL: http://www.cs.toronto.edu/~fbacchus/csc2512/Lectures/2013Readings/Skallah_Empirical_Study_SAT_Solvers.pdf (cit. on p. 5).
[13] Henry Kautz et al. “Dynamic restart policies”. In: *Aaai/iaai* 97 (2002), pp. 674–681. URL: ftp://ftp.research.microsoft.com/pub/ehb/drestart.pdf (cit. on p. 4).

[14] Donald E Knuth. *Fascicle 6: Satisfiability, volume 19 of The Art of Computer Programming*. 2015. URL: http://www.cs.utsa.edu/~wagner/knuth/fasc6a2015.09.23.pdf (cit. on pp. 2, 3).

[15] Matthew DT Lewis, Tobias Schubert, and Bernd Becker. “Early Conflict Detection Based BCP for SAT Solving.” In: *SAT*. Citeseer. 2004. URL: http://www.satisfiability.org/SAT04/programme/22.pdf (cit. on p. 5).

[16] Matthew DT Lewis, Tobias Schubert, and Bernd W Becker. “Speedup techniques utilized in modern SAT solvers”. In: *International Conference on Theory and Applications of Satisfiability Testing*. Springer. 2005, pp. 437–443. URL: https://www.researchgate.net/profile/Tobias_Schubert/publication/220944540_Speedup_Techniques_Utillized_in_Modern_SAT_Solvers/links/02e7e536cc5be07003000000.pdf (cit. on p. 5).

[17] J. H. Liang et al. “Understanding VSIDS Branching Heuristics in Conflict-Driven Clause-Learning SAT Solvers”. In: *ArXiv e-prints* (June 2015), arXiv: 1506.08905 [cs.LO]. URL: https://arxiv.org/pdf/1506.08905.pdf (cit. on p. 3).

[18] Michael Luby, Alistair Sinclair, and David Zuckerman. “Optimal speedup of Las Vegas algorithms”. In: *Information Processing Letters* 47.4 (1993), pp. 173–180. URL: https://pdfs.semanticscholar.org/2d0e/4df4de47ec38a2da161a8506d62164e62864.pdf (cit. on p. 4).

[19] MiniSat Page. http://minisat.se/ (cit. on p. 5).

[20] Matthew W Moskewicz et al. “Chaff: Engineering an efficient SAT solver”. In: *Proceedings of the 38th annual Design Automation Conference*. ACM. 2001, pp. 530–535. URL: https://www.princeton.edu/~chaff/publication/DAC2001v56.pdf (cit. on p. 5).

[21] Lawrence Ryan. “Efficient algorithms for clause-learning SAT solvers”. PhD thesis. Citeseer, 2004. URL: http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.68.2512&rep=rep1&type=pdf (cit. on p. 3).

[22] The Glucose SAT Solver. http://www.labri.fr/~perso/lsimon/glucose/ (cit. on pp. 4, 5).

[23] Christoph Weidenbach. “Automated Reasoning—The Art of Generic Problem Solving”. https://www.mpi-inf.mpg.de/fileadmin/inf/rg1/script5ws1617.pdf and https://www.mpi-inf.mpg.de/fileadmin/inf/rg1/script6ws1617.pdf. Forthcoming (cit. on p. 3).

[24] Archy Wilhes. *The Boolean Satisfiability Problem [SAT] and SAT solvers in 5 mins (or more).* http://0a.io/boolean-satisfiability-problem-or-sat-in-5-minutes/. Blog. 2015 (cit. on p. 2).

[25] Archy Wilhes. *Why is pure literal elimination absent in DPLL-based algorithms like Chaff?* Computer Science Stack Exchange. URL:https://cs.stackexchange.com/q/44924 (version: 2015-08-02). 2015 (cit. on p. 2).

[26] Jaeheon Yi. “The Effect of VSIDS on SAT Solver Performance”. In: (2007). URL: https://classes.soe.ucsc.edu/cmps217/Fall07/Project/jaeheon/final_paper/final_paper/input-dist-subm.pdf (cit. on p. 3).

[27] Lintao Zhang and Sharad Malik. “The quest for efficient boolean satisfiability solvers”. In: *International Conference on Computer Aided Verification*. 2005.
Lintao Zhang et al. “Efficient conflict driven learning in a boolean satisfiability solver”. In: Proceedings of the 2001 IEEE/ACM international conference on Computer-aided design. IEEE Press. 2001, pp. 279–285. URL: https://www.princeton.edu/~chaff/publication/iccad2001_final.pdf (cit. on p. 4).