Multiplicity, regularity and blow-spherical equivalence of real analytic sets

José Edson Sampaio

Received: 2 August 2021 / Accepted: 25 October 2021 / Published online: 9 January 2022

© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2021

Abstract
This article is devoted to studying multiplicity and regularity of analytic sets. We present an equivalence for analytic sets, named blow-spherical equivalence, which generalizes differential equivalence and subanalytic bi-Lipschitz equivalence and, with this approach, we obtain several applications on analytic sets. On multiplicity, we present a generalization for Gau–Lipman’s Theorem about differential invariance of the multiplicity in the complex and real cases, and we show that the multiplicity mod 2 is invariant under blow-spherical homeomorphisms in the case of real analytic curves and surfaces and also for a class of real analytic foliations and is invariant by (image) arc-analytic blow-spherical homeomorphisms in the case of real analytic hypersurfaces, generalizing some results proved by G. Valette. On regularity, we show that blow-spherical regularity of real analytic sets implies $C^1$ smoothness only in the case of real analytic curves. We present also a complete classification of the germs of real analytic curves.

Keywords  Blow-spherical equivalence · Real analytic sets · Multiplicity

Mathematics Subject Classification  14B05 · 14P25 · 32S50

Contents

1 Introduction .............................................. 3 8 6
2 Preliminaries .............................................. 3 8 8
  2.1 Multiplicity and relative multiplicities .............................. 3 8 8
  2.2 Euler cycles and allowed paths .................................. 3 9 1
  2.3 Allowed cycles .......................................... 3 9 1
3 Blow-spherical equivalence ...................................... 3 9 3
  3.1 Blow-spherical equivalence as a new equivalence .................. 3 9 3
4 Some generalizations of Gau–Lipman’s Theorem  ..................... 3 9 5

The author was partially supported by CNPq-Brazil Grant 303811/2018-8.

José Edson Sampaio
edson@mat.ufc.br

1 Departamento de Matemática, Universidade Federal do Ceará, Rua Campus do Pici, s/n, Bloco 914, Pici, Fortaleza, CE 60440-900, Brazil
1 Introduction

Recently, Birbrair et al. in [4] (see also [3] and [23]) defined a new equivalence, named blow-spherical equivalence, to study some properties of subanalytic sets such as, for example, to generalize thick-thin decomposition of normal complex surface singularity germs introduced in [6]. With the aim to study multiplicity and regularity as well as to present some classifications of real and complex analytic sets, we have the following weaker variation of the blow-spherical equivalence presented in [4], named also blow-spherical equivalence.

Definition 1.1 Let \( X \subset \mathbb{R}^n \) and \( Y \subset \mathbb{R}^p \) be two subsets containing respectively the origin of \( \mathbb{R}^n \) and \( \mathbb{R}^p \). A homeomorphism \( \varphi: (X, 0) \to (Y, 0) \) is called a blow-spherical homeomorphism, if the homeomorphism \( \beta^{-1}_p \circ \varphi \circ \beta_n: \beta^{-1}_n(X\setminus\{0\}) \to \beta^{-1}_p(Y\setminus\{0\}) \)

extends as a continuous mapping \( \varphi': \beta^{-1}_n(X\setminus\{0\}) \to \beta^{-1}_p(Y\setminus\{0\}) \), where \( \beta_k: S^{k-1} \times [0, +\infty) \to \mathbb{R}^k \) is the mapping given by \( \beta_k(x, r) = rx \). In this case, we say that the germs \((X, 0)\) and \((Y, 0)\) are blow-spherical equivalent or blow-spherical homeomorphic.

Roughly speaking, two subset germs of Euclidean spaces are called blow-spherical equivalent, if their spherical modifications are homeomorphic and, in particular, this homeomorphism induces a homeomorphism between their tangent links. In particular, this equivalence lives strictly between topological equivalence and subanalytic bi-Lipschitz equivalence and between topological equivalence and differential equivalence.

In [26], the author presented some results on multiplicity, regularity and blow-spherical geometry of complex analytic sets. In this article, we present similar results in the case of real analytic sets, for example, we obtain some results on real versions of Zariski’s multiplicity conjecture, we have that blow-spherical regular real analytic curves are \( C^1 \) smooth and we obtain a complete classification of real analytic curves.

When the subject is about multiplicity, the most famous open problem is Zariski’s multiplicity conjecture. O. Zariski in 1971 (see [33]) asked the following.

Question A. Let \( f, g: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be two reduced complex analytic functions. If there is a homeomorphism \( \varphi: (\mathbb{C}^n, V(f), 0) \to (\mathbb{C}^n, V(g), 0) \), then is \( m(V(f), 0) = m(V(g), 0) \)?

This is still an open problem. However, several authors approached it, as for example Ephraim in [9] and Trotman in [29] showed that Question A has a positive answer when the homeomorphism \( \varphi \) is a \( C^1 \) diffeomorphism. Another important result about this problem was proved.
by Gau and Lipman in the article [12], they proved that if \( X, Y \subseteq \mathbb{C}^n \) are complex analytic sets and there is a homeomorphism \( \varphi : (\mathbb{C}^n, X, 0) \to (\mathbb{C}^n, Y, 0) \) such that \( \varphi \) and \( \varphi^{-1} \) are differentiable at the origin, then the multiplicities of \( X \) and \( Y \) at the origin are equal (see [7] for a definition of multiplicity). In the case of real analytic sets, Question A has a negative answer, as we can see in the following example.

**Example 1.2** Let \( X = \{(x, y) \in \mathbb{R}^2; y = 0\}, Y = \{(x, y) \in \mathbb{R}^2; y^3 = x^2\}. \varphi : \mathbb{R}^2 \to \mathbb{R}^2 \) given by \( \varphi(x, y) = (x, x^{\frac{3}{2}} - y) \). Then, \( \varphi \) is a homeomorphism such that \( \varphi(X) = Y \), but \( m(X) \equiv 1 \mod 2 \) and \( m(Y) \equiv 0 \mod 2 \).

However, some authors approached some versions of Question A in the real case. For example, Risler in [21] proved that multiplicity mod 2 of a real analytic curve is invariant under bi-Lipschitz homeomorphisms. Fukui et al. in [11] made the following conjecture:

**Conjecture F-K-P.** Let \( h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0) \) be the germ of a subanalytic, arc-analytic, bi-Lipschitz homeomorphism, and let \( X, Y \subseteq \mathbb{R}^n \) be two irreducible analytic germs. Suppose that \( Y = h(X) \), then \( m(X) = m(Y) \).

and they proved that the multiplicity of a real analytic curve is invariant under arc-analytic bi-Lipschitz homeomorphisms. Valette in [31] proved that the multiplicity mod 2 of a real analytic hypersurface is invariant under arc-analytic bi-Lipschitz homeomorphisms and the multiplicity mod 2 of a real analytic surface is invariant by subanalytic bi-Lipschitz homeomorphisms. Recently, the author in [28] proved a real version of Gau–Lipman’s Theorem and in [27] proposed the following conjecture:

**Conjecture \( A_{\mathbb{R}} \) (Lip).** Let \( X, Y \subseteq \mathbb{R}^n \) be two real analytic sets. If there is a bi-Lipschitz homeomorphism \( \varphi : (\mathbb{R}^n, X, 0) \to (\mathbb{R}^n, Y, 0) \), then \( m(X) \equiv m(Y) \mod 2 \).

and he proved that Conjecture \( A_{\mathbb{R}} \) (Lip) has a positive answer when \( n = 3 \).

In this article, we consider the following version of the Conjecture \( A_{\mathbb{R}} \) (Lip).

**Conjecture \( A_{\mathbb{R}} \) (BS).** Let \( X, Y \subseteq \mathbb{R}^n \) be two real analytic sets. If there is a blow-spherical homeomorphism \( \varphi : (\mathbb{R}^n, X, 0) \to (\mathbb{R}^n, Y, 0) \), then \( m(X) \equiv m(Y) \mod 2 \).

In this article, we improve the famous Gau–Lipman’s Theorem [12] (on the differential invariance of the multiplicity of complex analytic sets) with a very much simplified proof (see Theorem 4.4). We provide also a real analogue of this result. Moreover, we prove that Conjecture \( A_{\mathbb{R}} \) (BS) has a positive answer when \( n \leq 3 \) (see Proposition 5.2, Theorems 5.5 and 5.7 and Corollary 5.18) or when \( \varphi \) is also image arc-analytic (see Theorem 5.16). These last results about multiplicity give generalizations of Valette’s results recalled above (see Remarks 5.6 and 5.19). We obtain also a result of invariance of the multiplicity of real analytic foliations in \( \mathbb{R}^2 \) (see Corollary 5.3).

Another subject approached in [26] was the study on regularity of complex analytic sets, for example, it was proved there that any complex analytic set which is blow-spherical regular (see Definition 6.1) must be smooth. This is also a subject of interest for many mathematicians, for instance, Mumford in [19] showed that a topologically regular complex surface in \( \mathbb{C}^3 \), with isolated singularity, is smooth. In high dimension, A’Campo in [1] and Lê D. T. in [17] showed that if \( X \) is a complex analytic hypersurface in \( \mathbb{C}^n \) which is a topological submanifold, then \( X \) is smooth. Recently, the author in [24] (see also [5]) proved a version of Mumford’s Theorem, he showed that if a complex analytic set is Lipschitz regular (see Definition 6.2) then it is smooth. Thus, since we are interested in the real case, for each positive integer \( d \), we have the following question:
**Question BSR(d)** Let $X \subset \mathbb{R}^n$ be a real analytic set with dimension $d$. Suppose that $X$ is blow-spherical regular at $0 \in X$. Is it true that $X$ is $C^1$ smooth at $0$?

In this article, we prove that the above question has a positive answer if and only if $d = 1$ (see Corollary 6.10 and Example 6.12).

On classification, let us remark that the bi-Lipschitz invariance of the multiplicity is an advance about a problem that has been extensively studied in the recent years: the classification of real analytic surfaces under bi-Lipschitz homeomorphisms. Since any subanalytic bi-Lipschitz homeomorphism is a blow-spherical homeomorphism, we believe that the study of blow-spherical equivalence can help in the bi-Lipschitz classification problem. There were presented in [4] some results on classification of real analytic surfaces under blow-spherical homeomorphisms. However, the classification of real analytic curves under blow-spherical homeomorphisms is still not known and this is why we present such a classification here.

### 2 Preliminaries

Here, all real analytic sets are supposed to be pure dimensional.

**Definition 2.1** Let $A \subset \mathbb{R}^n$ be a subset such that $x_0 \in \overline{A}$. We say that $v \in \mathbb{R}^n$ is a tangent vector of $A$ at $x_0 \in \mathbb{R}^n$ if there is a sequence of points $\{x_i\} \subset A \setminus \{x_0\}$ tending to $x_0 \in \mathbb{R}^n$ and there is a sequence of positive numbers $\{t_i\} \subset \mathbb{R}^+$ such that \[
\lim_{i \to \infty} \frac{1}{t_i} (x_i - x_0) = v.
\]

Let $C(A, x_0)$ denote the set of all tangent vectors of $A$ at $x_0 \in \mathbb{R}^n$. We call $C(A, x_0)$ the tangent cone of $A$ at $x_0$.

**Remark 2.2** It follows from the Curve Selection Lemma for subanalytic sets that, if $A \subset \mathbb{R}^n$ is a subanalytic set and $x_0 \in A$ is a non-isolated point, then the following holds true
\[
C(A, x_0) = \{v; \exists \text{ analytic } \alpha : [0, \varepsilon) \to \mathbb{R}^n \text{ s.t. } \alpha(0) = x_0, \alpha((0, \varepsilon)) \subset A \text{ and } \alpha(t) - x_0 = tv + o(t)\}.
\]

**Definition 2.3** The mapping $\beta_n : S^{n-1} \times \mathbb{R}^+ \to \mathbb{R}^n$ given by $\beta_n(x, r) = rx$ is called spherical blowing-up (at the origin) of $\mathbb{R}^n$.

Note that $\beta_n : S^{n-1} \times (0, +\infty) \to \mathbb{R}^n \setminus \{0\}$ is a homeomorphism with inverse $\beta_n^{-1} : \mathbb{R}^n \setminus \{0\} \to S^{n-1} \times (0, +\infty)$ given by $\beta_n^{-1}(x) = \left(\frac{x}{\|x\|}, \|x\|\right)$.

**Definition 2.4** The strict transform of the subset $X$ under the spherical blowing-up $\beta_n$ is $X' := \beta_n^{-1}(X \setminus \{0\})$ and the boundary $\partial X'$ of the strict transform is $\partial X' := X' \cap (S^{n-1} \times \{0\})$.

Remark that $\partial X' = C_X \times \{0\}$, where $C_X = C(X, 0) \cap S^{n-1}$.

#### 2.1 Multiplicity and relative multiplicities

Let $X \subset \mathbb{R}^n$ be a $d$-dimensional real analytic set with $0 \in X$ and $X_C = V(\mathcal{I}_R(X, 0))$. 

Δ Springer
where \( \mathcal{I}_\mathbb{R}(X, 0) \) is the ideal in \( \mathbb{C}[z_1, ..., z_n] \) generated by the complexifications of all germs of real analytic functions that vanish on the germ \((X, 0)\). We have that \( X_C \) is a germ of a complex analytic set and \( \dim \mathcal{I}_\mathbb{C} X_C = \dim \mathcal{I}_\mathbb{R} X \) (see Whitney [32], p. 546, Theorem 1 and p. 552, Lemma 8 and Lemma 9). Then, for each \((n - d)\)-dimensional complex linear subspace \( L \subset \mathbb{C}^n \) such that \( L \cap C(X_C, 0) = \{0\} \), there exists an open neighborhood \( U \subset \mathbb{C}^n \) of \( 0 \) and a complex analytic subset \( \sigma \subset U' = \pi_L(U) \) such that \#(\pi_L^{-1}(x) \cap X_C \cap U) \) is constant for every point \( x \in U \setminus \sigma \), where \( \pi_L: \mathbb{C}^n \rightarrow L^\perp \) is the orthogonal projection onto \( L^\perp \) and \( \dim \sigma < d \). The number \#(\pi_L^{-1}(x) \cap X_C \cap U) \) is the multiplicity of \( X_C \) at the origin and it is denoted by \( m(X_C, 0) \).

**Definition 2.5** With the above notation, we define the (real) multiplicity of \( X \) at the origin by \( m(X) := m(X_C, 0) \).

**Remark 2.6** For \( \pi := \pi_L|_{\mathbb{R}^n}: \mathbb{R}^n \rightarrow \pi_L(\mathbb{R}^n) \), we have \( m(X) \equiv \#(\pi^{-1}(x) \cap X \cap U) \mod 2 \), for any \( x \in \pi_L(\mathbb{R}^n) \setminus \sigma \).

**Definition 2.7** Let \( X \subset \mathbb{R}^n \) be a subanalytic set such that \( 0 \in \overline{X} \) is a non-isolated point. We say that \( x \in \partial X' \) is a simple point of \( \partial X' \), if there is an open \( U \subset \mathbb{R}^{n+1} \) with \( x \in U \) such that:

a) the connected components of \((X' \cap U) \setminus \partial X' \), say \( X_1, ..., X_r \), are topological manifolds with \( \dim X_i = \dim X, i = 1, ..., r \);

b) \( (X_i \cup \partial X') \cap U \) are topological manifolds with boundary.

Let \( \text{Smp}(\partial X') \) be the set of simple points of \( \partial X' \).

**Definition 2.8** Let \( X \subset \mathbb{R}^n \) be a subanalytic set such that \( 0 \in X \). We define \( k_X: \text{Smp}(\partial X') \rightarrow \mathbb{N} \), with \( k_X(x) \) the number of connected components of the germ \((\beta_n^{−1}(X\setminus\{0\}), x)\).

**Remark 2.9** It is clear that the function \( k_X \) is locally constant. In fact, \( k_X \) is constant in each connected component \( C_j \) of \( \text{Smp}(\partial X') \). Then, we define \( k_X(C_j) := k_X(x) \) with \( x \in C_j \).

**Remark 2.10** By Theorems 2.1 and 2.2 in [20], we obtain that \( \text{Smp}(\partial X') \) is an open dense subset of the \((d - 1)\)-dimensional part of \( \partial X' \) whenever \( \partial X' \) is a \((d - 1)\)-dimensional subset, where \( d = \dim X \).

**Remark 2.11** The numbers \( k_X(C_j) \) are equal to the numbers \( n_j \) defined by Kurdyka and Raby [16], p. 762.

**Definition 2.12** Let \( X \subset \mathbb{R}^n \) be a real analytic set. We denote by \( C_X \) the closure of the union of all connected components \( C_j \) of \( \text{Smp}(\partial X') \) such that \( k_X(C_j) \) is an odd number. We call \( C_X' \) the odd part of \( C_X \subset \mathbb{S}^{n-1} \).

**Definition 2.13** Let \( X \subset \mathbb{R}^n \) be a \( d \)-dimensional real analytic set with \( 0 \in X \) and let \( \pi: \mathbb{C}^n \rightarrow \mathbb{C}^d \) be a projection such that \( \pi^{-1}(0) \cap C(X_C, 0) = \{0\} \). Let \( \pi': \mathbb{S}^{n-1} \setminus \pi^{-1}(0) \rightarrow \mathbb{S}^{d-1} \) given by \( \pi'(u) = \frac{\pi(u)}{\|\pi(u)\|} \), where we are considering the natural inclusions \( \mathbb{S}^{n-1} \subset \mathbb{R}^n \subset \mathbb{C}^n \) and identifying \( \pi(\mathbb{R}^n) \cap \mathbb{S}^{2d-1} \) with \( \mathbb{S}^{d-1} \). We define \( \varphi_{\pi, C_X'}(x) := \#(\pi'^{-1}(x) \cap C_X') \).

In this case, if \( \varphi_{\pi, C_X'}(x) \) is constant mod 2 for a generic \( x \in \mathbb{S}^{d-1} \), we write \( m_\pi(C_X') := \varphi_{\pi, C_X'}(x) \mod 2 \), for a generic \( x \in \mathbb{S}^{d-1} \).
Proposition 2.14 Let $X \subset \mathbb{R}^n$ be a $d$-dimensional real analytic set and $0 \in X$. Then, $\varphi_{\pi,C'_X}(y)$ is constant for a generic projection $\pi$ and a generic point $y \in S^{d-1}$. Moreover, $m_{\pi}(C'_X) \equiv m(X) \mod 2$.

Proof Firstly, let us assume that $\dim C_X = d - 1$. By Remark 2.10, $Smp(\partial X')$ is an open dense subset of the $(d-1)$-dimensional part of $\partial X' = C_X \times \{0\} \cong C_X$. Let $y \in S^{d-1}$ be a generic point, $u = \#(\pi^{-1}(ty) \cap X) \equiv m(X) \mod 2$, for small enough $t > 0$ and $\pi^{-1}(y) \cap C_X = \pi'^{-1}(y) \cap Smp(\partial X') = \{y_1, ..., y_p\}$. Then, we have the following

$$u = \sum_{j=1}^{p} k_X(y_j).$$

In fact, let $\eta, \varepsilon > 0$ be small enough numbers such that $C_{n,\varepsilon}(y) \cap \pi(Crit(\pi|_X)) = \emptyset$, where $C_{n,\varepsilon}(y) = \{v \in \mathbb{R}^d; \|v - ty\| \leq \eta t, \ t \in (0, \varepsilon)\}$ and $Crit(\pi|_X)$ is the critical locus of the mapping $\pi|_X: X \to \mathbb{R}^d \cong \pi(\mathbb{R}^n)$. Thus, denote the connected components of $(\pi|_X)^{-1}(C_{n,\varepsilon}(y))$ by $Y_1, ..., Y_u$. Hence, $\pi|_{Y_i} : Y_i \to C_{n,\varepsilon}(y)$ is a homeomorphism, for $i = 1, ..., u$. Thus, for each $i = 1, ..., u$, there is a unique $\gamma_i : (0, \varepsilon) \to Y_i$ such that $\pi(\gamma_i(t)) = ty$ for all $t \in (0, \varepsilon)$. We define for each $i = 1, ..., u$, $\tilde{\gamma}_i : [0, \varepsilon) \to \beta^{-1}_n(Y_i)$ given by $\tilde{\gamma}_i(s) = \lim_{t \to s^+} \beta^{-1}_n \circ \gamma_i(t)$, for all $s \in [0, \varepsilon)$.

We remark that $\tilde{\gamma}_i(0) = \lim_{t \to 0^+} \tilde{\gamma}_i(t) \in \{y_1, ..., y_p\}$, for all $i = 1, ..., u$ and, thus, $u \leq \sum_{j=1}^{p} k_X(y_j)$. Shrinking $\eta$, if necessary, we can suppose that each $C_{Y_i}$ contains at most one $y_j$. Thus, fixing $y_j$, if $\gamma : [0, \delta) \to X$ is a subanalytic curve such that $\lim_{t \to 0^+} \beta^{-1}_n \circ \gamma(t) = y_j$, then there exists $\delta_0 > 0$ such that $\pi(\gamma(t)) \in C_{n,\varepsilon}(y)$, for all $t < \delta_0$. So, there is $i \in \{1, ..., u\}$ such that $\gamma(t) \in Y_i$, with $0 < t < \delta_0$. Then, $\tilde{\gamma}_i(0) = y_j$ and we obtain the equality $u = \sum_{i=1}^{p} k_X(y_j)$. Therefore, we obtain

$$u = \sum_{i=1}^{r} k_X(C_i) \cdot \#(\pi'^{-1}(y) \cap C_i),$$

where $C_1, ..., C_r$ are the connected components of $Smp(\partial X')$. Hence,

$$\sum_{i=1}^{r} k_X(C_i) \cdot \#(\pi'^{-1}(y) \cap C_i) \equiv \#(\pi'^{-1}(y) \cap C'_X) \mod 2$$

and since $u \equiv m(X) \mod 2$, we obtain

$$m(X) \equiv \#(\pi'^{-1}(y) \cap C'_X) \mod 2,$$

for a generic $y \in S^{d-1}$.

When $\dim C_X < d - 1$, we have that $C'_X = \emptyset$ and $\dim C(\pi(X), 0) < d$, which implies that there exist $w \in S^{d-1}$ and small enough numbers $\eta, \varepsilon \in (0, 1)$ such that $C_{n,\varepsilon}(w) \cap \pi(X) = \emptyset$. Therefore $\varphi_{\pi,C'_X}(y) = 0$ for any point $y \in S^{d-1}$ and $m(X) \equiv 0 \mod 2$, since $C'_X = \emptyset$ and $\pi^{-1}(v) \cap X = \emptyset$, for all $v \in C_{n,\varepsilon}(w)$. In particular, $m_{\pi}(C'_X)$ is defined and satisfies $0 \equiv m_{\pi}(C'_X) \equiv m(X) \mod 2$. □

Corollary 2.15 Let $X \subset \mathbb{R}^n$ be a $d$-dimensional real analytic set and $0 \in X$. If $m(X) \equiv 1 \mod 2$ then $\dim C(X, 0) = d$. 

\textcopyright Springer
2.2 Euler cycles and allowed paths

**Definition 2.16** An \((n-1)\)-dimensional subanalytic set \(C\) is said to be a **Euler cycle** if it is a closed set and if, for a stratification of \(C\) (and hence for any that refines it), the number of \((n-1)\)-dimensional strata containing a given \((n-2)\)-dimensional stratum in their closure is even.

**Definition 2.17** Let \(k \in \mathbb{N} \cup \{\infty, \omega\}\) and let \(C \subseteq \mathbb{S}^n\) be a subset. We say that a point \(p \in C\) is a **\(C^k\) regular point** of \(C\) if there exists an open \(U \subseteq \mathbb{S}^n\) that contains \(p\) and \(C \cap U\) is a \(C^k\) submanifold of \(\mathbb{S}^n\). We denote by \(\text{Reg}_k(C)\) to be the set of all \(C^k\) regular points of \(C\). We define also \(\text{Sing}_k(C) = C \setminus \text{Reg}_k(C)\).

**Definition 2.18** Let \(C \subseteq \mathbb{S}^n\) be an Euler cycle. A subanalytic \(C^1\) path \(\gamma : [0, 1] \rightarrow \mathbb{S}^n\) is said to be an **allowed path** for \(C\) if for every \(t \in \mathbb{I}_\gamma = \{t \in [0, 1]; \gamma(t) \in C\}\), the point \(\gamma(t)\) is a \(C^1\) regular point of \(C\) at which the mapping \(\gamma\) is transverse to \(C\). In this case, we define \(\lg_C(\gamma) = \#I_\gamma\).

For \(\lambda, \mu \in \mathbb{S}^n \setminus C\), we define
\[
d_C(\lambda; \mu) = \min\{\lg_C(\gamma); \gamma \text{ allowed path joining } \lambda \text{ and } \mu\}.
\]

We define the diameter of an Euler cycle \(C \subseteq \mathbb{S}^2\) as the integer
\[
\delta_C = \sup\{d_C(\lambda; \mu); \lambda, \mu \in \mathbb{S}^2 \setminus C\}.
\]

We have the following result proved by Valette in [31].

**Lemma 2.19** (Proposition 2.4 and Theorem 4.1 in [31]) Let \(X \subseteq \mathbb{R}^3\) be a real analytic surface with \(0 \in X\). Then \(C_X'\) is an Euler cycle and \(\delta_{C_X'} \equiv m(X) \mod 2\).

**Lemma 2.20** ([31], Propositions 2.4, 3.2 and 3.3) Let \(X \subseteq \mathbb{R}^n\) be a real analytic hypersurface with \(0 \in X\). Then, for a generic projection \(\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}\) with \(\pi^{-1}(0) \cap \mathbb{S}^{n-1} = \{-\lambda, \lambda\}\), \(\varphi_{\pi, C_X'}(x) \mod 2\) is constant for a generic \(x \in \mathbb{S}^{n-2}\) and
\[
m(X) \equiv m_{\pi}(C_X') \equiv d_{C_X'}(\lambda; -\lambda) \mod 2.
\]

2.3 Allowed cycles

**Definition 2.21** We say that a subset \(C \subseteq \mathbb{R}^{n+1}\) is a **\(a\)-invariant** if \(a(C) = C\), where \(a : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}\) is given by \(a(x) = -x\).

**Definition 2.22** Let \(C \subseteq \mathbb{S}^2\) be an \(a\)-invariant set. We say that an embedding \(e : \mathbb{S}^1 \rightarrow C\) is an **allowed embedding**, if \(a(\text{Im}(e)) = \text{Im}(e)\) or if there is another embedding \(e' : \mathbb{S}^1 \rightarrow C\) such that \(e' = a \circ e\) and \(\text{Im}(e) \cap \text{Im}(e')\) is a finite set or empty. A subset \(A \subseteq C\) is called an **allowed set** for \(C\) if there exists a subset \(E(A) \subseteq \{e : \mathbb{S}^1 \rightarrow C; e\) is an allowed embedding\} satisfying the following:

1. \(A = \bigcup_{e \in E(A)} e(\mathbb{S}^1)\);
2. if \(e_i, e_j \in E(A)\) with \(e_i \neq e_j\) then \(\text{Im}(e_i) \cap \text{Im}(e_j)\) is a finite set or empty;
3. if \(e \in E(A)\) with \(a(\text{Im}(e)) \neq \text{Im}(e)\) then \(a \circ e \in E(A)\).

In this case, we define the **number of allowed circles of** \(C\) to be \(\text{nac}(C) := \max\{\#E(A); A\) is an allowed set for \(C\}\). If \(A\) is an allowed set such that \(\text{nac}(C) = \#E(A)\), we say that \(A\) is a **maximal allowed set**. (for \(C\)).
When $X \subset \mathbb{R}^3$ is a real analytic surface, we define the **number of allowed circles of** $X$ to be $\text{nac}(X) := \text{nac}(C'_X)$.

**Example 2.23** (a) Let $C = \{(x, y, z) \in S^2; z = 0\}$. Then $C$ is an allowed set (for $C$) and $\text{nac}(C) = 1$.

(b) Let $\ell_1, \ldots, \ell_k : \mathbb{R}^3 \to \mathbb{R}$ be linear functions such that $\bigcap_{i=1}^k \text{Ker}(\ell_i)$ is a line and $p = \ell_1 \cdots \ell_k$ is a reduced homogeneous polynomial. Then, for $X = p^{-1}(0)$, we have that $C'_X = X \cap S^2$ is an allowed set for $C'_X$ and $\text{nac}(X) = \text{nac}(C'_X) = k$.

**Remark 2.24** Let $C \subset S^2$ be an $\alpha$-invariant closed subanalytic set. Then $\text{nac}(C) < +\infty$.

**Lemma 2.25** Let $S_1, \ldots, S_r \subset S^2$ be subanalytic subsets such that each one of them is homeomorphic to $S^1$. If $S_i \cap S_j$ is a finite set whenever $i \neq j$, then $C = \bigcup_{i=1}^r S_i$ is an Euler cycle.

**Proof** It is clear that $C$ is a 1-dimensional closed subanalytic subset. Consider a stratification $S$ of $C$ such that each $S_i$ has at least two 0-dimensional strata. Since $S_i$ is homeomorphic to $S^1$, then it is easy to verify that the number of 1-dimensional strata containing a given 0-dimensional stratum in their closure is even. $\square$

**Lemma 2.26** ([31], Proposition 3.5) **Let $C \subset S^n$ be an $\alpha$-invariant Euler cycle. Then $d_C(-\lambda, \lambda)$ mod 2 is independent of $\lambda \in S^n \setminus C$.**

Thus, for $C \subset S^n$ being an $\alpha$-invariant Euler cycle and $\lambda \in S^n \setminus C$, we denote $d_C(-\lambda, \lambda)$ mod 2 by $m(C)$.

**Lemma 2.27** Let $S_1, S_2 \subset S^2$ be subanalytic subsets. Suppose that $S_1$ and $S_2$ are subanalytically homeomorphic to $S^1$ and $C = S_1 \cup S_2$ is an $\alpha$-invariant set. Then we have the following:

(a) If $S_1 = S_2$, then $m(C) \equiv 1$ mod 2; 

(b) If $S_1 \cap S_2$ is a finite set, then $m(C) \equiv 0$ mod 2.

**Proof** Let $\lambda \in S^2 \setminus C$ be a generic point. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection such that $\pi^{-1}(0) \cap S^n = \{-\lambda, \lambda\}$. Thus, we consider the stereographic projection $p_\lambda : S^2 \setminus \{\lambda\} \to \mathbb{R}^2$ and, then by Lemma 2.26, for a generic point $t \in S^1$, $q^{-1}(t)$ is a ray starting at the origin and $\#(q^{-1}(t) \cap p_\lambda(C)) \equiv m(C)$ mod 2, where $q = \pi' \circ p_\lambda^{-1} : \mathbb{R}^2 \setminus \{0\} \to S^1$ and $\pi' : S^2 \setminus \{-\lambda, \lambda\} \to S^1$ is given by $\pi'(u) = \frac{\pi(u)}{||\pi(u)||}$. For each $i = 1, 2$ let $B_i$ be the bounded connected component of $\mathbb{R}^2 \setminus p_\lambda(S_i)$.

Let us choose $\lambda$ such that 0 belongs to the bounded component of $B_1$. By the proof of the Jordan Curve Theorem (see [30]), $\#(q^{-1}(t) \cap p_\lambda(S_1)) \equiv 1$ mod 2. Therefore, we have proven the item (a), since $C = S_1$.

For item (b), we have two cases:

1. $a(S_1) = S_1$. In this case, $a(S_2) = S_2$ and by item (a), $m(S_1) \equiv 1$ mod 2 and $m(S_2) \equiv 1$ mod 2. Therefore, $m(C) \equiv 0$ mod 2, since in this case $m(C) \equiv m(S_1) + m(S_2)$ mod 2;
2. $a(S_1) \neq S_1$. In this case, $a(S_1) = S_2$ and we have also that 0 belongs to the bounded component of $B_2$. By the proof of the Jordan Curve Theorem once again, $\#(q^{-1}(t) \cap p_\lambda(S_1)) \equiv 1$ mod 2 and $\#(q^{-1}(t) \cap p_\lambda(S_2)) \equiv 1$ mod 2. But $\#(q^{-1}(t) \cap p_\lambda(C)) = \#(q^{-1}(t) \cap p_\lambda(S_1)) + \#(q^{-1}(t) \cap p_\lambda(S_2))$ and this finishes the proof. $\square$
3 Blow-spherical equivalence

Definition 3.1 Let \( X \subset \mathbb{R}^n \) and \( Y \subset \mathbb{R}^p \) be two subsets containing respectively the origin of \( \mathbb{R}^n \) and \( \mathbb{R}^p \).

- A continuous mapping \( \varphi: (X, 0) \to (Y, 0) \), with \( 0 \notin \varphi(X\setminus\{0\}) \), is a blow-spherical morphism (shortened as blow-morphism), if the mapping
  \[
  \beta_p^{-1} \circ \varphi \circ \beta_n: X'\setminus\partial X' \to Y'\setminus\partial Y'
  \]
  extends as a continuous mapping \( \varphi': X' \to Y' \).

- A blow-spherical homeomorphism (shortened as blow-isomorphism) is a blow-morphism \( \varphi: (X, 0) \to (Y, 0) \) such that the extension \( \varphi' \) is a homeomorphism. In this case we say that the germs \((X, 0)\) and \((Y, 0)\) are blow-spherical equivalent or blow-spherical homeomorphic (or blow-isomorphic).

When \( \varphi: (X, 0) \to (Y, 0) \) is a blow-spherical homeomorphism, we denote by \( v_\varphi: C_X \to C_Y \) the homeomorphism such that \( \varphi'(x, 0) = (v_\varphi(x), 0) \) for all \((x, 0) \in \partial X'\).

Remark 3.2 We have the following.

1. \( id: X \to X \) is a blow-spherical homeomorphism for any \( X \subset \mathbb{R}^n \) with \( 0 \in X \);
2. Let \( X \subset \mathbb{R}^n \), \( Y \subset \mathbb{R}^p \) and \( Z \subset \mathbb{R}^k \) be subsets containing respectively the origin of \( \mathbb{R}^n \), \( \mathbb{R}^p \) and \( \mathbb{R}^k \). If \( f: (X, 0) \to (Y, 0) \) and \( g: (Y, 0) \to (Z, 0) \) are blow-spherical morphisms then \( g \circ f: (X, 0) \to (Z, 0) \) is a blow-spherical morphism.

Thus, we have a category, denoted here by BS\(_0\), such that its objects are all subsets of Euclidean spaces that contain the origin and its morphisms are all blow-spherical morphisms.

We have also the following result.

Theorem 3.3 Let \( \varphi: (X, 0) \to (Y, 0) \) be a blow-spherical homeomorphism. Then, \( \varphi'(\text{Smp}(\partial X')) = \text{Smp}(\partial Y') \) and \( k_X(v) = k_Y(\varphi'(v)) \) for all \( v \in \text{Smp}(\partial X') \). In particular, \( \varphi'(C_X') = C_Y' \).

Proof Let \( v \in \text{Smp}(\partial X') \) be a point and let \( U \subset X' \) be a small neighborhood of \( v \). Since \( \varphi': X' \to Y' \) is a homeomorphism, we have that \( V = \varphi'(U) \) is a small neighborhood of \( \varphi'(v) \in \partial Y' \). Moreover, \( \varphi'(U \setminus \partial X') = V \setminus \partial Y' \), since \( \varphi'|_{\partial X'}: \partial X' \to \partial Y' \) is a homeomorphism, as well. Using once more that \( \varphi' \) is a homeomorphism, we obtain that the number of connected components of \( U \setminus \partial X' \) is equal to the number of connected components of \( V \setminus \partial Y' \), showing that \( k_X(v) = k_Y(\varphi'(v)) \) for all \( v \in \text{Smp}(\partial X') \). In particular, we obtain that \( C_Y' = \varphi'(C_X') \).

In the next sections, we give several applications of Theorem 3.3.

3.1 Blow-spherical equivalence as a new equivalence

In this Subsection, we show that Blow-Spherical equivalence is different from some other equivalences studied in Singularity Theory.

Let Lip\(_{0,outer}\) (resp. Lip\(_{0,inner}\)) be the subcategory of Lip, whose objects are all subsets of Euclidean spaces that contain the origin endowed with the induced metric (resp. intrinsic metric) and whose morphisms are all Lipschitz mappings with respect to the induced metric (resp. intrinsic metric) such that their inverse image of the origin of the target is only the origin of the source.
Let $\text{Top}_0$ be the subcategory of $\text{Top}$ whose objects are all subsets of Euclidean spaces that contain the origin and whose morphisms are all continuous mappings such that their inverse image of the origin of the target is only the origin of the source.

**Example 3.4** It is clear that $\text{BS}_0$ is a subcategory of $\text{Top}_0$. $X = \{(x, y) \in \mathbb{R}^2; y = 0\}$ and $Y = \{(x, y) \in \mathbb{R}^2; y^2 = x^3\}$ are homeomorphic and by Theorem 6.10, $X$ and $Y$ are not blow-spherical homeomorphic, since $X$ is not a $C^1$ submanifold of $\mathbb{R}^2$. Thus, $\text{BS}_0 \neq \text{Top}_0$.

The above example also shows that blow-spherical equivalence is also different from blow-analytic equivalence (see the definition in [15]), since $(x, y) \in \mathbb{R}^2; y = 0$ and $(x, y) \in \mathbb{R}^2; y^2 = x^3$ are blow-analytic equivalent (see [13]).

**Definition 3.5** For each rational number $\beta \in [1, +\infty)$, we define $X_\beta = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 = z^{2\beta}\}$ and $X_\beta^\pm = \{(x, y, z) \in X_\beta; \pm z \geq 0\}$.

We remark that for each $\beta \in [1, +\infty)$, $X_\beta^+$ is (outer) bi-Lipschitz homeomorphic to $X_\beta^-$. This implies that if $X_{\beta_1}$ and $X_{\beta_2}$ are inner bi-Lipschitz homeomorphic then $X_{\beta_1}^+$ and $X_{\beta_2}^+$ are inner bi-Lipschitz homeomorphic as well. However, it is known that $X_{\beta_1}^+$ and $X_{\beta_2}^+$ are not inner bi-Lipschitz homeomorphic whenever $\beta_1 \neq \beta_2$ (see [2]).

**Example 3.6** Let $\beta_1, \beta_2 \in (1, +\infty)$ be two different rational numbers. Then, the mapping $\varphi: X_{\beta_1} \rightarrow X_{\beta_2}$ given by $\varphi(x, y, z) = (x, y, \text{sign}(z)|z|^{1/2\beta_2})$ is a blow-spherical homeomorphism, but $X_{\beta_1}$ and $X_{\beta_2}$ are not inner bi-Lipschitz homeomorphic. In particular, $\text{BS}_0 \neq \text{Lip}_0,_{\text{inner}}$ and $\text{BS}_0 \neq \text{Lip}_0,_{\text{outer}}$.

In fact, we can find an example with an embedded blow-spherical homeomorphism.

**Example 3.7** By taking $\beta_1 = \frac{5}{2}$ and $\beta_2 = \frac{7}{2}$ in Definition 3.5, the mapping $\varphi: (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0)$ given by $\varphi(x, y, z) = (xg(x, y, z), yg(x, y, z), z)$ is a blow-spherical homeomorphism such that $\varphi(X_{\beta_1}) = X_{\beta_2}$, where

$$g(x, y, z) = \begin{cases} 1, & \text{if } z^3 \leq x^2 + y^2 \\ \frac{x^2 + y^2 + z^4}{z^3(z+1)}, & \text{if } z^5 < x^2 + y^2 < z^3 \\ z, & \text{if } x^2 + y^2 \leq z^5. \end{cases}$$

In [14], the authors presented the following definition.

**Definition 3.8** We say that a homeomorphism $h: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ satisfies condition semiline-(SSP), if $h(\ell)$ has a unique direction for all semilines $\ell$.

Here, we say that such an $h$ is a **semiline homeomorphism**. Thus, for each semiline $\ell$, there is a unique vector $u \in S^{n-1}$ such that for any sequence $\{x_j\} \subset h(\ell) \setminus \{0\}$ with $\lim_{j \to +\infty} x_j = 0$, we have $\lim_{j \to +\infty} \frac{x_j}{\|x_j\|} = u$.

**Example 3.9** The definition of blow-spherical homeomorphism is intrinsic, but a semiline homeomorphism has to be defined in some open neighborhood of 0. By Theorem 6.1 in [26], $X = \{(x, y) \in \mathbb{C}^2; y^2 = x^3\}$ and $Y = \{(x, y) \in \mathbb{C}^2; y^2 = x^5\}$ are blow-spherical homeomorphic. Since $X$ and $Y$ have different Puiseux pairs, there is no semiline homeomorphism $h: (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$ such that $h(X) = Y$. 

 Springer
4 Some generalizations of Gau–Lipman’s Theorem

Definition 4.1 Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be closed subsets and let $k \in \mathbb{N} \cup \{\infty, \omega\}$. We say that a mapping $f : X \to Y$ is $C^k$ (resp. differentiable) at $x \in X$, if there exist an open $U \subset \mathbb{R}^n$ and a mapping $F : U \to \mathbb{R}^m$ such that $x \in U$, $F|_{X \cap U} = f|_{X \cap U}$ and $F$ is $C^k$ (resp. differentiable) at $x$.

Definition 4.2 Let $X, Y \subset \mathbb{R}^n$ be two sets with $0 \in X \cap Y$ and let $\varphi : X \to Y$ be a blow-spherical homeomorphism. We say that $\varphi$ is blow-spherical differentiable (at 0) if there is a linear isomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\nu_{\varphi}(x) = \frac{\phi(x)}{||\phi(x)||}$ for all $x \in C_X$ (see definition of $\nu_{\varphi}$ in Definition 3.1).

4.1 Complex case

When $X$ is a complex analytic set, there is a complex analytic set $\Sigma$ with $\dim \Sigma < \dim X$ such that for each irreducible component $X_j$ of the tangent cone $C(X, 0)$, $X_j \setminus \Sigma$ intersects only one connected component $C_i$ of $\text{Smp}(\partial X')$ (see [7], pp. 132–133). Then, we define $k_X(X_j) := k_C(C_i)$.

Remark 4.3 ([7, p. 133, Proposition]) Let $X$ be a complex analytic set of $\mathbb{C}^n$ with $0 \in X$ and let $X_1, \ldots, X_r$ be the irreducible components of $C(X, 0)$. Then

$$m(X, 0) = \sum_{j=1}^{r} k_X(X_j) \cdot m(X_j, 0).$$

Theorem 4.4 Let $X, Y \subset \mathbb{C}^n$ be two complex analytic sets with $0 \in X \cap Y$. If there exists a mapping $\varphi : X \to Y$ which is blow-spherical differentiable at 0, then $m(X, 0) = m(Y, 0)$.

Proof Since $\varphi$ is blow-spherical differentiable at 0, then there exists an $\mathbb{R}$-linear isomorphism $\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $\nu_\varphi(x) = \frac{\phi(x)}{||\phi(x)||}$ for all $x \in C_X$. Since $\nu_\varphi$ is a homeomorphism between $C_X$ and $C_Y$, then $\phi$ is a homeomorphism between $C(X, 0)$ and $C(Y, 0)$, which implies that $\phi$ maps bijectively the irreducible components of $C(X, 0)$ to the irreducible components of $C(Y, 0)$. Indeed, for $A := C(X, 0)$ and $B := C(Y, 0)$, we have $\phi(\text{Sing}(A)) = \text{Sing}(B)$ (see [18, page 13]) and then $\phi|_{A \setminus \text{Sing}(A)} : A \setminus \text{Sing}(A) \to B \setminus \text{Sing}(B)$ is, in particular, a homeomorphism. Moreover, we know that if $Z$ is a complex analytic set of pure dimension, then each connected component of $Z \setminus \text{Sing}(Z)$ is open and dense in exactly one irreducible component of $Z$. Let $D$ be an irreducible component of $A$ and let $D_\ast$ be the connected component of $A \setminus \text{Sing}(A)$ such that $D_\ast \subset D$ and $D_\ast = D$. Thus, $\phi(D_\ast)$ is a connected component of $B \setminus \text{Sing}(B)$. Therefore, $\phi(D_\ast)$ is an irreducible component of $B$. Since $\phi$ is a homeomorphism, we obtain that $\phi(D)$ is an irreducible component of $B$.

Let $X_1, \ldots, X_r$ and $Y_1, \ldots, Y_r$ be the irreducible components of $C(X, 0)$ and $C(Y, 0)$, respectively, such that $Y_j = \phi(X_j)$, $j = 1, \ldots, r$. Since $\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is an $\mathbb{R}$-linear isomorphism (with the usual identification $\mathbb{C}^n = \mathbb{R}^{2n}$), it is easy to see that its complexification $\phi_C : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is a $\mathbb{C}$-linear isomorphism and $\phi_C(X_jC) = Y_jC$ for all $j = 1, \ldots, r$.

Let $c_n : \mathbb{C}^n \to \mathbb{C}^n$ be the conjugation map given by $c_n(z_1, \ldots, z_n) = (\bar{z}_1, \ldots, \bar{z}_n)$. It is easy to see that if $Z \subset \mathbb{C}^n$ is a complex analytic set, then $c_n(Z)$ is a complex analytic set as well and $m(c_n(Z), 0) = m(Z, 0)$. In particular, $m(Z \times c_n(Z), 0) = m(Z, 0)^2$. By Proposition 2.9 in [10], for each $j \in \{1, \ldots, r\}$, $X_jC$ is complex analytic isomorphic to $X_j \times c_n(X_j)$ and $Y_jC$ is analytic isomorphic to $Y_j \times c_n(Y_j)$. Since the multiplicity is an analytic invariant,
for each \( j \in \{1, ..., r\} \), we have \( m(X_j \times c_n(X_j), 0) = m(Y_j \times c_n(Y_j), 0) \). Therefore, \( m(X_j, 0) = m(Y_j, 0) \), for all \( j = 1, ..., r \). By Theorem 3.3, we obtain \( k_X(X_j) = k_Y(Y_j) \), for all \( j = 1, ..., r \). The proof follows from Remark 4.3.

As a direct consequence, we obtain Gau–Lipman’s Theorem [12].

**Corollary 4.5** (Gau–Lipman’s Theorem [12]) Let \( X, Y \subset \mathbb{C}^n \) be two complex analytic sets. If there exists a homeomorphism \( \varphi: (\mathbb{C}^n, X, 0) \to (\mathbb{C}^n, Y, 0) \) such that \( \varphi \) and \( \varphi^{-1} \) have a derivative at the origin (as mappings from \((\mathbb{R}^{2n}, 0)\) to \((\mathbb{R}^{2n}, 0)\)), then \( m(X, 0) = m(Y, 0) \).

In the next example, we show that Theorem 4.4 is really a generalization of Gau–Lipman’s Theorem.

**Example 4.6** Let \( X = \{(x, y) \in \mathbb{C}^2; y^2 = x^3\} \) and \( Y = \{(x, y) \in \mathbb{C}^2; y^2 = x^5\} \). By Theorem 6.1 and its proof in [26] there exists a blow-spherical homeomorphism \( \varphi: X \to Y \) such that \( \nu_\varphi = id_{\mathbb{C}}_X \). Thus, \( \varphi \) is a blow-spherical differentiable mapping. However, there is no homeomorphism \( \psi: (\mathbb{C}^2, X, 0) \to (\mathbb{C}^2, Y, 0) \).

**4.2 Real case**

**Proposition 4.7** Let \( X \) and \( Y \) be two real analytic sets with \( 0 \in X \cap Y \) and let \( \varphi: (X, 0) \to (Y, 0) \) be a real analytic diffeomorphism. Then \( m(X) = m(Y) \).

**Proof** We have that the complexification of \( \varphi \), denoted by \( \varphi_\mathbb{C} \), is a complex diffeomorphism between \( X_\mathbb{C} \) and \( Y_\mathbb{C} \). Thus, by the Proposition in [7], Section 11, p. 120), \( m(X_\mathbb{C}, 0) = m(Y_\mathbb{C}, 0) \). Therefore, \( m(X) = m(Y) \).

In fact, we can obtain a stronger version of the above result.

**Proposition 4.8** Let \( X \) and \( Y \) be two real analytic sets with \( 0 \in X \cap Y \) and let \( \varphi: (X, 0) \to (Y, 0) \) be a \( C^\infty \) diffeomorphism. Then \( m(X) = m(Y) \).

**Proof** By Proposition 1.1 in [8], \( (X, 0) \) and \( (Y, 0) \) are real analytic diffeomorphic and by Proposition 4.7, \( m(X) = m(Y) \).

**Theorem 4.9** Let \( X, Y \subset \mathbb{R}^n \) be two analytic sets with \( 0 \in X \cap Y \). If there exists a mapping \( \varphi: X \to Y \) which is blow-spherical differentiable at 0, then \( m(X) \equiv m(Y) \) mod 2.

**Proof** Since \( \varphi \) is blow-spherical differentiable 0, then there exists an \( \mathbb{R} \)-linear isomorphism \( \phi: \mathbb{R}^n \to \mathbb{R}^n \) such that \( \nu_\phi(x) = \frac{\phi(x)}{\|\phi(x)\|} \) for all \( x \in C_X \). Then \( A = \phi(X) \) is a real analytic set and by Proposition 4.7, \( m(X) = m(A) \).

Thus, it is enough to show that \( m(Y) \equiv m(A) \) mod 2. In order to do this, we consider the mapping \( \psi: (Y, 0) \to (A, 0) \) given by \( \psi = \phi \circ \varphi^{-1} \). Thus, \( \psi \) is a blow-spherical homeomorphism such that \( \nu_\psi = id \), i.e., the mapping \( \psi': Y' \to A' \) is given by

\[
\psi'(x, t) = \begin{cases} 
\left( \frac{\psi(\phi(x))}{\|\psi(\phi(x))\|}, \|\psi(\phi(x))\| \right), & t \neq 0 \\
(x, 0), & t = 0.
\end{cases}
\]

Then, we obtain that \( C_Y' = \psi'(C_Y') = C_A' \). By Proposition 2.14, we obtain \( m(Y) \equiv m(A) \) mod 2, which finishes the proof.

As a consequence, we obtain the main result of [28].
Corollary 4.10 (Theorem 3.1 in [28]) Let $X, Y \subset \mathbb{R}^n$ be two real analytic sets with $0 \in X \cap Y$. Assume that there exists a mapping $\varphi : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ such that $\varphi : (X, 0) \rightarrow (Y, 0)$ is a homeomorphism. If $\varphi$ has a derivative at the origin and $D\varphi_0 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an isomorphism, then $m(X) \equiv m(Y) \mod 2$.

In particular, we obtain the real version of Gau-Lipman’s Theorem in [12].

Corollary 4.11 (Corollary 3.2 in [28]) Let $X, Y \subset \mathbb{R}^n$ be two analytic sets with $0 \in X \cap Y$. If there exists a homeomorphism $\varphi : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, Y, 0)$ such that $\varphi$ and $\varphi^{-1}$ are differentiable at $0$, then $m(X) \equiv m(Y) \mod 2$.

5 Invariance of the multiplicity under blow-isomorphisms

5.1 Invariance of the multiplicity for real curves

We start this Subsection stating that the multiplicity mod 2 is not a topological invariant even in the case of real analytic curves, even in a topologically trivial family of real analytic curves, as we can see in the next example.

Example 5.1 For each $t \in \mathbb{R}$, we consider $X_t = \{(x, y) \in \mathbb{R}^2 ; t^2y^2 = x^3\}$ and $\varphi_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by $\varphi_t(x, y) = ((x^3 - t^2y^2)^{\frac{1}{3}}, y)$. Then, $\varphi_t$ is a homeomorphism such that $\varphi_t^{-1}(x, y) = ((x^3 + t^2y^2)^{\frac{1}{3}}, y)$ and $\varphi_t(X_t) = X_0$ for all $t \in \mathbb{R}$. However, $m(X_0) \equiv 1 \mod 2$ and $m(X_t) \equiv 0 \mod 2$ for all $t \in \mathbb{R} \setminus \{0\}$.

In the reality, if $\varphi_t$ is given as in the above example and $f_t : \mathbb{R}^2 \rightarrow \mathbb{R}$ is given by $f_t(x, y) = x^3 - t^2y^2$, we have $f_t = f_0 \circ \varphi_t$ for all $t \in \mathbb{R}$.

Proposition 5.2 The multiplicity mod 2 is an invariant for blow-spherical homeomorphic real analytic curves.

Proof Let $X, Y \subset \mathbb{R}^n$ be two real analytic curves. Suppose that $X$ and $Y$ are blow-spherical homeomorphic. By Theorem 3.3, $C'_X$ and $C'_Y$ are homeomorphic.

We have that

$$\# C'_X = \# (\pi'^{-1}(1) \cap C'_X) + \# (\pi'^{-1}(-1) \cap C'_X)$$

and by Proposition 2.14, we obtain that

$$m(C'_X) \equiv \# (\pi'^{-1}(1) \cap C'_X) \equiv \# (\pi'^{-1}(-1) \cap C'_X) \equiv m(X) \mod 2.$$ 

Then $m(X) \equiv \frac{1}{2}\# C'_X \mod 2$. Similarly, we obtain that $m(Y) \equiv \frac{1}{2}\# C'_Y \mod 2$. Since $C'_X$ and $C'_Y$ are homeomorphic, we have that $m(X) \equiv m(Y) \mod 2$. \hfill \Box

5.2 Invariance of the multiplicity for analytic foliations in the plane

As a consequence of Proposition 5.2, we obtain a result of invariance of the multiplicity for analytic foliations in $\mathbb{R}^2$ (in order to know more about the notation and definitions of the next result, see [21]).

Let $Z$ and $Z'$ be real analytic germs of plane vector fields at $0 \in \mathbb{R}^2$ inducing germs of foliations $\mathcal{F}$ and $\mathcal{F}'$. Assume that in the resolution process of $\mathcal{F}$ (resp. $\mathcal{F}'$) there is no real dicritical component and that $\mathcal{F}$ (resp. $\mathcal{F}'$) is a real generalized curve, i.e., the real
desingularization of $F$ (resp. $F'$) has no singularity with zero eigenvalue. We denote by $v$ (resp. $v'$) the order of $Z$ (resp. $Z'$) at 0.

**Corollary 5.3** Let $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a homeomorphism which is a topological equivalence between $F$ and $F'$. If $h$ is a blow-spherical homeomorphism then $v \equiv v' \mod 2$.

**Proof** Let $Z_C$ (resp. $Z'_C$) be the complexification of $Z$ (resp. $Z'$) and let $S_C$ (resp. $S'_C$) be the separatrix of $Z_C$ (resp. $Z'_C$). As it was remarked in the proof of Proposition 4.4 in [21], the real separatrix is the union of the real components of the complex separatrix. Then,

$$m(S) \equiv m(S_C, 0) \mod 2$$

and

$$m(S') \equiv m(S'_C, 0) \mod 2,$$

where $S$ (resp. $S'$) is the separatrix of $Z$ (resp. $Z'$). Moreover, by remark after Lemma 3.4 in [21], we have

$$m(S_C, 0) \equiv v + 1 \mod 2$$

and

$$m(S'_C, 0) \equiv v' + 1 \mod 2.$$ 

By Proposition 5.2, $m(S) \equiv m(S') \mod 2$, since the hypotheses imply that $S$ and $S'$ are blow-spherical homeomorphic. Therefore, $v \equiv v' \mod 2$. $\square$

Thus, we obtain also a real version of Theorem 1.1 in [22].

**Corollary 5.4** Let $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a homeomorphism which is a topological equivalence between $F$ and $F'$. If $h$ has a derivative at the origin and $Dh_0 : \mathbb{R}^2 \to \mathbb{R}^2$ is an isomorphism, then $v \equiv v' \mod 2$.

### 5.3 Invariance of the multiplicity under blow-isomorphism for real surfaces: the embedded case

In this Subsection, we show that the multiplicity (mod 2) of surfaces in $\mathbb{R}^3$ is invariant by embedded blow-spherical homeomorphisms.

**Theorem 5.5** Let $X, Y \subset \mathbb{R}^3$ be two real analytic surfaces with $0 \in X \cap Y$. If there exists a blow-spherical homeomorphism $\phi : (\mathbb{R}^3, X, 0) \to (\mathbb{R}^3, Y, 0)$, then $m(X) \equiv m(Y) \mod 2$.

**Proof** By Theorem 3.3, we have a homeomorphism $\psi : (S^2, C_\lambda') \to (S^2, C'_\mu)$.

Given $\lambda, \mu \in S^2 \setminus C'_X$, let $\gamma : [0, 1] \to S^n$ be an allowed path for $C'_X$ connecting $\lambda$ and $\mu$ such that $lg_{C'_Y}(\gamma) = d_{C'_Y}(\lambda; \mu)$. Let $\beta = \psi \circ \gamma$. Thus, the set $I_\beta := \{t \in [0, 1]; \beta(t) \in C'_Y\}$ is finite and for every $t \in I_\beta$, the point $\beta(t)$ is a $C^0$ regular point of $C'_Y$ at which the mapping $\beta$ is topologically transverse to $C'_Y$ (i.e., for each point $t \in I_\beta$, $p = \beta(t)$ is a $C^0$ regular point of $C$ and there exist open subsets $U \subset S^2$ and $V \subset \mathbb{R}^2$ such that $(p, 0) \in U \times V$ and there exists a homeomorphism $\varphi : U \to V$ satisfying $\varphi(p) = 0, \varphi(U \cap C'_Y) = V \cap \{x_1, x_2 \in \mathbb{R}^2; x_2 = 0\}$ and $\varphi \circ \beta(s) = (0, s)$ in $(t - \delta, t + \delta)$ for some $\delta > 0$).

Since $\text{Reg}_1(C'_Y)$ is dense in $C'_Y$, we can find a subanalytic path $\alpha : [0, 1] \to S^2$ connecting $\bar{\lambda} = \psi(\lambda)$ and $\bar{\mu} = \psi(\mu)$, which is an allowed path for $C'_Y$ and $lg_{C'_Y}(\alpha) = \# I_\beta$ (see Figs. 1 and 2 below).
Thus, we obtain that $d_{C_X}(\lambda; \mu) \geq d_{C_Y}(\psi(\lambda); \psi(\mu))$. Similarly, we obtain also $d_{C_X}(\lambda; \mu) \leq d_{C_Y}(\psi(\lambda); \psi(\mu))$. Therefore, $\delta_{C_X} = \delta_{C_Y}$ and by Lemma 2.19, $m(X) \equiv m(Y) \mod 2$.

**Remark 5.6** In Theorem 5.1 in [31], G. Valette showed that if $X, Y \subset \mathbb{R}^3$ are two real analytic surfaces and $\varphi: (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ is a subanalytic outer bi-Lipschitz homeomorphism such that $\varphi(X) = Y$ then $m(X) \equiv m(Y) \mod 2$. Theorem 5.5 is more general than Theorem 5.1 in [31], since any subanalytic outer bi-Lipschitz homeomorphism is a blow-spherical homeomorphism (see Proposition 3.8 in [26]) and by Examples 3.6 and 3.7, we know that $X_5$ and $X_7$ are not outer bi-Lipschitz homeomorphic, but they have the same multiplicity and there exists a blow-spherical homeomorphism $\varphi: (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ such that $\varphi(X_5) = X_7$. 

\[ \square \]
5.4 Invariance of the multiplicity under blow-isomorphism for real surfaces: the non-embedded case

In this Subsection, we show that the multiplicity (mod 2) of surfaces in \( \mathbb{R}^3 \) is invariant under blow-spherical homeomorphisms which are not necessarily embedded, but with an additional hypothesis.

**Theorem 5.7** Let \( X, \tilde{X} \subset \mathbb{R}^3 \) be two real analytic surfaces. Assume that there exists a blow-spherical homeomorphism \( \varphi : (X, 0) \to (\tilde{X}, 0) \) such that \( \nu \varphi (-v) = -v \) whenever \( v \in C'_X \). Then \( \text{nac}(X) = \text{nac}(\tilde{X}) \) and \( m(X) \equiv m(\tilde{X}) \) mod 2.

**Proof** It follows from Theorem 3.3 that \( C'_X \neq \emptyset \) if and only if \( C'_X \neq \emptyset \). By hypothesis \( \varphi = \nu \varphi : C'_X \to C'_X \) is a homeomorphism such that \( \nu \varphi (-v) = -v \) for all \( v \in C'_X \). Thus, there is no embedding \( e : S^1 \to C'_X \) if and only if there is no embedding \( \tilde{e} : S^1 \to C'_X \).

Therefore, we may assume that there are embeddings \( e : S^1 \to C'_X \) and \( \tilde{e} : S^1 \to C'_X \).

**Claim 1** There is an allowed set \( S \) of \( C'_X \).

**Proof of Claim 1** We are assuming that there exists a set \( S_1 \subset C'_X \) which is homeomorphic to \( S^1 \). Let \( S_2 = a(S_1) \). If \( S_2 = S_1 \) or \( S_1 \cap S_2 = \emptyset \), then \( S = S_1 \cup S_2 \) is an allowed set of \( C'_X \). Thus, let us suppose that \( S_1 \cap S_2 \neq \emptyset \) and \( S_2 \neq S_1 \) and fix \( x_0 \in S_1 \cap S_2 \). In particular, \( x_0 \in S_1 \cap S_2 \). For each \( i = 1, 2 \) we consider a parametrization \( e_i : [0, 1] \to S_i \) such that \( e_i(0) = e_i(1) = x_0 \). Then, there exist open intervals \( I_j = (a_j, b_j) \) (resp. \( J_j = (c_j, d_j) \)) such that \( e_i^{-1}(S_1 \setminus (S_1 \cap S_2)) = \bigsqcup_{j=1}^{k} I_j \) (resp. \( e_2^{-1}(S_2 \setminus (S_1 \cap S_2)) = \bigsqcup_{j=1}^{k} J_j \)), where \( b_j \leq a_{j+1} \) (resp. \( d_j \leq c_{j+1} \)) for all \( j = 1, \ldots, k-1 \). By changing the orientation of \( e_2 \), if necessary, we can suppose that \( e_1(a_j) = e_2(c_j) \) and \( e_1(b_j) = e_2(d_j) \) for all \( j = 1, \ldots, k \). For each \( j \), we consider a positive oriented homeomorphism \( \phi_j : [a_j, b_j] \to [c_j, d_j] \). We define \( \tilde{e}_1, \tilde{e}_2 : [0, 1] \to C'_X \) in the following way

\[
\tilde{e}_1(t) = \begin{cases} e_1(t), & \text{if } t \notin I_j, \forall j; \\ e_1(t), & \text{if } t \in [a_j, b_j] \text{ and } a(e_1(I_j)) \subset e_1([0, 1]); \\ e_2 \circ \phi_j(t), & \text{if } t \in [a_j, b_j] \text{ and } a(e_1(I_j)) \nsubseteq e_1([0, 1]) \end{cases}
\]

and

\[
\tilde{e}_2(t) = \begin{cases} e_2(t), & \text{if } t \notin J_j, \forall j; \\ e_2(t), & \text{if } t \in [c_j, d_j] \text{ and } a(e_2(J_j)) \subset e_2([0, 1]); \\ e_1 \circ \phi_j^{-1}(t), & \text{if } t \in [c_j, d_j] \text{ and } a(e_2(J_j)) \nsubseteq e_2([0, 1]). \end{cases}
\]

Now, we note that \( \tilde{S}_1 = \tilde{e}_1([0, 1]) \) and \( \tilde{S}_2 = \tilde{e}_2([0, 1]) \) are allowed sets of \( C'_X \).

**Claim 2** There is a decomposition \( C := C'_X = L \cup A \) satisfying the following:

1. \( A \) and \( L \) are \( a \)-invariant closed subanalytic sets and \( L \cap A \) is a finite set;
2. \( A \) is an allowed set with \( \text{nac}(A) = \text{nac}(C) \);
3. \( S^2 \setminus L \) is path connected;

**Proof of Claim 2** Since \( \text{nac}(C) < +\infty \), there is a subset \( A \subset C \) such that it is a maximal allowed set, let us write \( E(A) \) as \( E(A) = \{e_1, \ldots, e_{2k}, e_{2k+1}, \ldots, e_{2k+m}\} \) such that \( \text{nac}(C) = 2k + m, a(1m(e_{k+j})) = 1m(e_j) \) for \( j \in \{1, \ldots, k\} \) and \( a(1m(e_r)) = 1m(e_r) \) for \( r \in \{2k+1, \ldots, 2k+m\} \). Since \( C \) and \( A \) are \( a \)-invariant subanalytic sets and \( A \) is maximal, there is no embedding \( e : S^1 \to C \setminus A \). Then, we define \( L := C \setminus A \). Now, it is easy to verify that \( L \) and \( A \) satisfy the claim.
Claim 3 $A = C'_{\tilde{X}}$.

Proof of Claim 3 Suppose that $L \not\subset A$. Since $C'_{\tilde{X}}$ has no isolated point, there exists a subset $I \subset L \setminus A$ that is homeomorphic to $(0, 1)$. Let $\lambda \in S^2 \setminus C'_{\tilde{X}}$ and let $\gamma : [0, 1] \to S^2$ be an allowed path for $C'_{\tilde{X}}$ such that $\gamma(0) = \lambda$ and $\gamma(1) = -\lambda$. Since $S^2 \setminus L$ is path connected, we can assume that $\Gamma \subset S^2 \setminus (L \setminus I)$ and $\gamma$ meets transversally $I$ at exactly one point, where $\Gamma = \gamma([0, 1])$. Let $\beta : [0, 1] \to S^m$ be an allowed path for $C'_{\tilde{X}}$ such that $\beta(0) = \lambda$, $\beta(1) = -\lambda$ and $\beta([0, 1]) \subset S^3 \setminus L$. By Lemma 2.25, $A$ is an Euler cycle and, then, $\gamma$ and $\beta$ are also allowed paths for $A$ and by Lemma 2.20,

\[ \tilde{d}_{A}(-\lambda, \lambda) \equiv l_{gA}(\gamma) \equiv l_{gA}(\beta) \mod 2 \]

and

\[ \tilde{d}_{C'_{\tilde{X}}}(\lambda, \lambda) \equiv l_{gC'_{\tilde{X}}}(\gamma) \equiv l_{gC'_{\tilde{X}}}(\beta) \mod 2. \]

However, by the choice of $\gamma$ and $\beta$, we have $l_{gC'_{\tilde{X}}}(\gamma) \equiv l_{gA}(\gamma) + 1 \mod 2$ and $l_{gC'_{\tilde{X}}}(\beta) \equiv l_{gA}(\beta) \mod 2$, which is a contradiction. Therefore $L \subset A$ and this finish the proof of Claim 3.

\[ \square \]

Similarly, $\tilde{A} = C'_{\tilde{X}}$ is a maximal allowed set of $C'_{\tilde{X}}$.

Claim 4 $nac(X) = nac(\tilde{X})$.

Proof of Claim 4 We write $C'_{\tilde{X}} = \bigcup_{r=1}^{2k+m} A_r$ (resp. $\tilde{X} = \bigcup_{r=1}^{2k+m} \tilde{A}_r$) such that each $A_r$ (resp. $\tilde{A}_r$) is homeomorphic to $S^1$, $A_r \cap A_{r'}$ (resp. $\tilde{A}_r \cap \tilde{A}_{r'}$) is a finite set whenever $r \neq r'$ (resp. $\tilde{r} \neq \tilde{r}'$), $a(A_r) = A_{k+r}$ whenever $r \in \{1, \ldots, k\}$ (resp. $a(\tilde{A}_r) = \tilde{A}_{k+\tilde{r}}$ whenever $\tilde{r} \in \{1, \ldots, \tilde{k}\}$), $a(\tilde{A}_{r}) = A_r$ whenever $r \in \{2k+1, \ldots, 2k+m\}$ (resp. $a(\tilde{A}_{\tilde{r}}) = \tilde{A}_{\tilde{r}}$ whenever $\tilde{r} \in \{2\tilde{k}+1, \ldots, 2\tilde{k}+m\}$) and $nac(X) = 2k + m$ (resp. $nac(\tilde{X}) = 2\tilde{k} + \tilde{m}$).

Since $a \circ \psi = \psi \circ a$, $\bigcup_{r=1}^{2k+m} \psi(A_r)$ is an allowed set of $C'_{\tilde{X}}$, which implies that $nac(X) \leq nac(\tilde{X})$. By using $\psi^{-1}$ instead of $\psi$, we obtain $nac(\tilde{X}) \leq nac(X)$.

Therefore, $nac(X) = nac(\tilde{X})$.

\[ \square \]

Let $\lambda \in S^2 \setminus C'_{\tilde{X}}$ and let $\gamma : [0, 1] \to S^2$ be an allowed path for $C'_{\tilde{X}}$ such that $\gamma(0) = \lambda$ and $\gamma(1) = -\lambda$. Since $S^2 \setminus L$ is path connected, we can assume that $\gamma([0, 1]) \subset S^2 \setminus L$. By Lemma 2.25, $A$ is an Euler cycle and, then, by Lemma 2.26, $\tilde{d}_{A}(-\lambda, \lambda) \equiv \tilde{d}_{C'_{\tilde{X}}}(\lambda, \lambda) \equiv m(X) \mod 2$ and $\gamma$ is an allowed path for $A$. Then, by Lemma 2.27, we have

\[ \tilde{d}_{A}(-\lambda, \lambda) \equiv \sum_{i=1}^{2k+m} \tilde{d}_{A_i}(-\lambda, \lambda) \equiv m \mod 2. \]

Thus, $nac(X) \equiv m(X) \mod 2$, since $nac(X) = 2k + m$. Similarly, we obtain $nac(\tilde{X}) \equiv m(\tilde{X}) \mod 2$. Since $nac(X) = nac(\tilde{X})$, we have that $m(X) \equiv m(\tilde{X}) \mod 2$.

A consequence of the above result is presented in the next section (see Corollary 5.18).

The next example shows that Theorem 5.7 is not a direct consequence of the results in [31].

Example 5.8 Let \( X = \{x, y, z\} \in \mathbb{R}^3; (x^2 + y^2 - z^2)(x^2 + y^2 - \frac{z^2}{4}) = 0 \) and \( Y = \{x, y, z\} \in \mathbb{R}^3; (x^2 + y^2 - z^2)(x^2 + y^2 - \frac{z^2}{4}) = 0 \). The mapping \( \psi : X \to Y \) given by

\[
\psi(x, y, z) = \begin{cases} 
(x, y, z), & \text{if } x^2 + y^2 - z^2 = 0 \\
(x, z, y), & \text{if } x^2 + z^2 - \frac{y^2}{4} = 0 
\end{cases}
\]
is a blow-spherical homeomorphism such that $v_\phi(-v) = -v_\phi(v)$ for all $v \in X$. However, there is no bi-Lipschitz homeomorphism $\varphi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ such that $\varphi(X) = Y$.

### 5.5 Invariance of the multiplicity under arc-analytic blow-isomorphism

#### Definition 5.9
Let $C \subset \mathbb{S}^n$ be an Euler cycle. A path $\gamma : [0, 1] \to \mathbb{S}^n$ is said to be an almost allowed path for $C$ if the set $I_\gamma = \{ t \in [0, 1] : \gamma(t) \in C \}$ is finite and for every $t \in I_\gamma$, the point $\gamma(t)$ is a $C^0$ regular point of $C$ at which the mapping $\gamma$ is topologically transverse to $C$ (i.e., for each point $t \in I_\gamma$, $p = \gamma(t)$ is a $C^0$ regular point of $C$ and there exist open subsets $U \subset \mathbb{S}^n$ and $V \subset \mathbb{R}^n$ such that $(p, 0) \in U \times V$ and there exists a homeomorphism $\varphi : U \to V$ satisfying $\varphi(p) = 0$, $\varphi(U \cap C) = V \cap \{(x_1, ..., x_n) \in \mathbb{R}^n ; x_n = 0 \}$ and $\varphi \circ \gamma(s) = (0, ..., 0, s)$ in $(t - \delta, t + \delta)$ for some $\delta > 0$).

#### Remark 5.10
If $\gamma : [0, 1] \to \mathbb{S}^n$ is an allowed path (for $C$) then it is an almost allowed path (for $C$).

If $\gamma$ is an almost allowed path (for $C$), we define also

$$
\lg_C(\gamma) = \#I_\gamma.
$$

For $\lambda, \mu \in \mathbb{S}^n \setminus C$, we define

$$
\hat{d}_C(\lambda; \mu) = \min\{\lg_C(\gamma) : \gamma \text{ is an almost allowed path joining } \lambda \text{ and } \mu\}.
$$

#### Remark 5.11
Let $C \subset \mathbb{S}^n$ be an Euler cycle and let $\lambda, \mu \in \mathbb{S}^n \setminus C$. It is clear that $\hat{d}_C(\lambda, \mu) \leq d_C(\lambda, \mu)$. If $\gamma : [0, 1] \to \mathbb{S}^n$ is an almost allowed path for $C$ connecting $\lambda$ and $\mu$ such that $\hat{d}_C(\lambda, \mu) = \lg_C(\gamma)$, since $\text{Reg}_1(C)$ is dense in $C$, we can find a path $\tilde{\gamma} : [0, 1] \to \mathbb{S}^n$ connecting $\lambda$ and $\mu$, which is an allowed path for $C$ and $\lg_C(\tilde{\gamma}) = \lg_C(\gamma)$. Therefore, $\hat{d}_C = d_C$.

#### Definition 5.12
Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. We say that $f$ is arc-analytic if for any analytic curve $\alpha : (-1, 1) \to \mathbb{R}^n$ we have that there exists $0 < \varepsilon \leq 1$ such that $f \circ \alpha$ is analytic in $(-\varepsilon, \varepsilon)$. We say that $f$ is image arc-analytic if for any analytic curve $\alpha : (-1, 1) \to \mathbb{R}^n$ we have that there exist $0 < \varepsilon \leq 1$ and an analytic curve $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^m$ such that $\gamma(0) = f \circ \alpha(0)$ and $(\text{Im}(\gamma), 0) = (\text{Im}(f \circ \alpha), 0)$.

#### Remark 5.13
It is clear that if $f$ is arc-analytic then $f$ is image arc-analytic. However, the converse is not true in general. For example, it is easy to see that $f : \mathbb{R} \to \mathbb{R}$ given by $f(t) = t^4$ is image arc-analytic, but it is not arc-analytic.

Moreover, there are image arc-analytic and bi-Lipschitz mappings that are not arc-analytic, as we can see in the next example.

#### Example 5.14
Let $\phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be the mapping given by $\phi(x, y) = (x, y + x^4)$. It is clear that $\phi$ is bi-Lipschitz around the origin and $t \mapsto \phi(t, 0) = (t, t^4)$ is not analytic. Then $\phi$ is not arc-analytic. However, for any analytic curve $\alpha : (-1, 1) \to \mathbb{R}^2$, we have that $\gamma(t) := \phi \circ \alpha(t^3)$ is analytic in $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.

#### Remark 5.15
Let $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be an arc-analytic and bi-Lipschitz homeomorphism. Then, $\varphi$ is a blow-spherical homeomorphism such that $v_{\varphi}(-v) = -v_{\varphi}(v)$ for any $v \in \mathbb{S}^{n-1}$.

The next result is a generalization of Theorem 3.6 in [31].
Theorem 5.16 Let $X$ and $Y$ be two real analytic hypersurfaces in $\mathbb{R}^n$ with $0 \in X \cap Y$. If there exists a blow-spherical homeomorphism $\varphi : (\mathbb{R}^n, X, 0) \to (\mathbb{R}^n, Y, 0)$ such that $\varphi$ is image arc-analytic, then $m(X) \equiv m(Y) \mod 2$.

Proof For fixed $v \in \mathbb{R}^n \setminus \{0\}$, by the hypotheses, there exists an analytic curve $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ such that $(Im(\gamma), 0) = (Im(\beta), 0)$, where $\beta(t) := \varphi(tv)$. Thus, there exists $\delta > 0$ such that $\beta((-\delta, \delta)) \subset \Gamma := Im(\gamma)$. Moreover, since $\varphi$ is a homeomorphism, we can assume that $\gamma$ is injective, since $Im(\beta), 0 \neq (0, 0)$ and also $Im(\beta), 0$ is not homeomorphic to $(0, 1, 0)$. In particular, $\Gamma$ is blow-spherical regular at 0, then by Proposition 6.9,

$$w := \lim_{t \to 0^+} \frac{\gamma(t)}{\|\gamma(t)\|} = -\lim_{t \to 0^-} \frac{\gamma(t)}{\|\gamma(t)\|}.$$ 

Let $\Gamma_1$ and $\Gamma_2$ be the connected components of $\Gamma \setminus \{0\}$ such that $\beta((-\delta, 0)) \subset \Gamma_1$ and $\beta((0, \delta)) \subset \Gamma_2$. We define $a = 1$ and $b = 0$ if $\Gamma_1 = \gamma((-\varepsilon, 0))$ and, $a = 0$ and $b = 1$ if $\Gamma_1 = \gamma((0, \varepsilon))$. Then, for each sequence $\{s_k\} \subset (0, \delta)$ such that $\lim s_k = 0$ there exists a sequence $\{\tilde{s}_k\} \subset \Gamma_2$ that $\lim \tilde{s}_k = 0$ and $\gamma(\tilde{s}_k) = \beta(s_k)$ for $k \gg 1$. Then,

$$\lim_{k \to +\infty} \frac{\beta(s_k)}{\|\beta(s_k)\|} = \lim_{k \to +\infty} \frac{\gamma(\tilde{s}_k)}{\|\gamma(\tilde{s}_k)\|} = (-1)^a w.$$ 

Therefore $\lim_{t \to 0^+} \frac{\beta(t)}{\|\beta(t)\|} = (-1)^a w$ and by the same reason, $\lim_{t \to 0^-} \frac{\beta(t)}{\|\beta(t)\|} = (-1)^b w$. In particular, $\lim_{t \to 0^+} \frac{\beta(t)}{\|\beta(t)\|} = (-1)^a w$ and $\lim_{t \to 0^-} \frac{\beta(t)}{\|\beta(t)\|} = (-1)^b w$ for any $\lambda \in (0, +\infty)$. This implies that the homeomorphism $\nu_\varphi : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ satisfies $\nu_\varphi(-x) = -\nu_\varphi(x)$ for any $x \in \mathbb{S}^{n-1}$. Thus, for a generic $v \in \mathbb{S}^{n-1}$, by Lemma 2.20, we have

$$m(X) \equiv d_{C^1_X}(v; -v) = d_{C^1_Y}(\nu_\varphi(v); -\nu_\varphi(v)) \equiv m(Y) \mod 2.$$ 

□

In fact, we have the following.

Corollary 5.17 Let $X$ and $Y$ be two real analytic hypersurfaces in $\mathbb{R}^n$ with $0 \in X \cap Y$. If $\varphi : (\mathbb{R}^n, X, 0) \to (\mathbb{R}^n, Y, 0)$ is a blow-spherical homeomorphism such that $\nu_\varphi(-v) = -\nu_\varphi(v)$ for any $v \in \mathbb{S}^{n-1}$, then $m(X) \equiv m(Y) \mod 2$.

A consequence of Theorem 5.7 and the proof of Theorem 5.16 is the following:

Corollary 5.18 Let $X$ and $Y$ be two real analytic surfaces in $\mathbb{R}^3$ with $0 \in X \cap Y$. If there exists an image arc-analytic bi-Lipschitz homeomorphism $\varphi : (X, 0) \to (Y, 0)$, then $m(X) \equiv m(Y) \mod 2$.

Remark 5.19 Theorem 5.16 is really a generalization of Theorem 3.6 in [31], as we can see in the next example.

Example 5.20 Let $\phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be the bi-Lipschitz homeomorphism of Example 5.14. If $X = \{(x, y) \in \mathbb{R}^2; \ y = 0\}$ and $Y = \{(x, y) \in \mathbb{R}^2; \ y^3 = x^4\}$ then $\phi(X) = Y$. However, by Proposition 3.2 in [11], there is no arc-analytic bi-Lipschitz homeomorphism $\varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $\varphi(X) = Y$. 

\[\text{Springer}\]
6 Blow-spherical geometry of curves

6.1 Regularity of real analytic curves

Definition 6.1 A subset \( X \subseteq \mathbb{R}^n \) is called **blow-spherical regular** at \( 0 \in X \) if there is an open neighborhood \( U \subseteq \mathbb{R}^n \) of \( 0 \) such that \( X \cap U \) is blow-spherical homeomorphic to an Euclidean ball.

Definition 6.2 A subset \( X \subseteq \mathbb{R}^n \) is called **Lipschitz regular** (resp. \( C^k \) regular) at \( x_0 \in X \) if there is an open neighborhood \( U \subseteq \mathbb{R}^n \) of \( x_0 \) such that \( X \cap U \) is bi-Lipschitz homeomorphic to an Euclidean ball (resp. \( X \cap U \) is a \( C^k \) submanifold of \( \mathbb{R}^n \)), where \( k \in \mathbb{N} \cup \{ \infty, \omega \} \).

Thus, we obtain the following result.

**Proposition 6.3** Let \( X \subseteq \mathbb{R}^n \) be a real analytic set. Then, the below statements are equivalent.

1. \( m(X) = 1 \);
2. \( X \) is \( C^\omega \) regular at \( 0 \);
3. \( X \) is \( C^\infty \) regular at \( 0 \).

**Proof** It is clear that \((2) \Rightarrow (3)\) and by Propositions 4.7 and 4.8, we have that \((2) \Rightarrow (1)\) and \((3) \Rightarrow (1)\). Since \( \text{Sing}(X)_C = \text{Sing}(X)_C \) (see [32, Lemma 9]) and \( m(X_C, 0) = 1 \) if and only if \( X_C \) is the germ of a complex analytic submanifold, we obtain that if \( m(X) := m(X_C, 0) = 1 \) then \( \text{Sing}(X_C, 0) = \emptyset \) and, thus, \( \text{Sing}(X, 0) = \emptyset \), which implies that \( X \) is \( C^\omega \) regular at \( 0 \). Therefore, \((1) \Rightarrow (2)\), which finishes the proof.

However, in contrast with the complex case, Proposition 6.3 does not hold true when we consider \( C^1 \) instead of \( C^\infty \).

**Example 6.4** Let \( X = \{ (x, y) \in \mathbb{R}^2; y^3 = x^4 \} \). Then, \( X \) is \( C^1 \) diffeomorphic to \( \{ (x, y) \in \mathbb{R}^2; y = 0 \} \). Moreover, \( X \) is \( C^1 \) regular at \( 0 \), but it is not \( C^\infty \) regular at \( 0 \).

The above example tells us also that Propositions 4.7 and 4.8 do not hold true when we consider \( C^1 \) instead of \( C^\omega \) or \( C^\infty \), since \( X = \{ (x, y) \in \mathbb{R}^2; y^3 = x^4 \} \) is \( C^1 \) diffeomorphic to \( Y = \{ (x, y) \in \mathbb{R}^2; y = 0 \} \), but \( m(X) = 3 \) and \( m(Y) = 1 \).

**Definition 6.5** Let \( X \subseteq \mathbb{R}^n \) and \( Y \subseteq \mathbb{R}^m \) be closed subsets such that \( 0 \in X \times Y \). We say that \( (X, 0) \) and \( (Y, 0) \) are **diffeomorphic equivalent** if there exists a homeomorphism \( \varphi: (X, 0) \to (Y, 0) \) such that \( \varphi \) and \( \varphi^{-1} \) are differentiable at \( 0 \). In this case, we say that \( \varphi \) is a **diffeomorphic equivalence** (between \( (X, 0) \) and \( (Y, 0) \)).

**Lemma 6.6** (Proposition 1 in [25]) Let \( X, Y \subseteq \mathbb{R}^m \) be subsets. If \( (X, 0) \) and \( (Y, 0) \) are diffeomorphic equivalent at the origin, then \( (X, 0) \) and \( (Y, 0) \) are blow-spherical homeomorphic.

**Lemma 6.7** (Proposition 3.3 in [26]) If \( X \) and \( Y \) are blow-spherical homeomorphic, then \( C(X, 0) \) and \( C(Y, 0) \) are also blow-spherical homeomorphic at \( 0 \).

**Lemma 6.8** (Milnor [18], Lemma 3.3) Let \( V \subseteq \mathbb{R}^n \) be a real analytic curve and \( x_0 \in V \) a non-isolated point. Then, there are an open neighborhood \( U \subseteq \mathbb{R}^n \) of \( x_0 \) and \( \Gamma_1, \ldots, \Gamma_r \subseteq \mathbb{R}^n \) such that \( \Gamma_i \cap \Gamma_j = \{ x_0 \} \) whenever \( i \neq j \) and

\[
V \cap U = \bigcup_{i=1}^{r} \Gamma_i.
\]

Moreover, for each \( i \in \{ 1, \ldots, r \} \), there is an analytic homeomorphism \( \gamma_i : (-\varepsilon, \varepsilon) \to \Gamma_i \).
Each $\Gamma_i$ in the above proposition is called an analytic branch of $X$ at $x_0$ and $\Gamma_1, \ldots, \Gamma_r$ is called a decomposition in analytic branches for $(X, x_0)$.

**Proposition 6.9** Let $\gamma: (\varepsilon, \varepsilon) \to \mathbb{R}^n$ be an analytic curve and $\Gamma = \gamma((-\varepsilon, \varepsilon))$. Suppose that $\gamma: (\varepsilon, \varepsilon) \to \Gamma$ is a homeomorphism and $\gamma(0) = 0$. Then the following statements are equivalent:

1. $\Gamma$ is blow-spherical regular at 0;
2. $C(\Gamma, 0)$ is homeomorphic to $\mathbb{R}$;
3. $C(\Gamma, 0)$ is a real line;
4. $\text{ord}_0 \gamma$ is an odd number and, in particular, $\lim_{t \to 0^+} \frac{\gamma(t)}{\|\gamma(t)\|} = -\lim_{t \to 0^-} \frac{\gamma(t)}{\|\gamma(t)\|}$;
5. $\Gamma$ is $C^1$ regular at 0;
6. $\Gamma$ is Lipschitz regular at 0.

**Proof** Since $\gamma'$ also is an analytic curve, by shrinking $\varepsilon$, if necessary, we can assume that $\gamma'(t) \neq 0$ for all $t \in (-\varepsilon, \varepsilon) \setminus \{0\}$ and changing $\gamma$ by $\gamma \circ r$, where $r: (-1, 1) \to (-\varepsilon, \varepsilon)$ is given by $r(t) = \varepsilon t$, we can suppose that $\varepsilon = 1$. Moreover, for $k = \text{ord}_0 \gamma$, there exist $w \in \mathbb{R}^n$ such that $\gamma(t) = t^k w + o(t^k)$ and an analytic curve $\alpha: (-1, 1) \to \mathbb{R}^n$ satisfying $\alpha(0) \neq 0$ and $\gamma(t) = t^k \alpha(t)$, for all $t \in (-1, 1)$. Hence, we have the following

$$u = \lim_{t \to 0^+} \frac{\gamma(t)}{\|\gamma(t)\|} = \frac{w}{\|w\|}.$$ 

and

$$v = \lim_{t \to 0^-} \frac{\gamma(t)}{\|\gamma(t)\|} = (-1)^k \frac{w}{\|w\|}.$$ 

(1) $\Rightarrow$ (2). If $\Gamma$ is blow-spherical regular at 0 then $(\Gamma, 0)$ and $(\mathbb{R}, 0)$ are blow-spherical homeomorphic. Thus, by Lemma 6.7, $C(X, 0)$ is homeomorphic to $\mathbb{R}$.

(2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (4). Suppose that $C(\Gamma, 0)$ is homeomorphic to $\mathbb{R}$. Suppose that $k$ is an even natural number. Then by definitions of $u$ and $v$, we have $u = v$ and in this case $C(\Gamma, 0) = \{\lambda w; \lambda \geq 0\}$. Therefore, it is clear that $C(\Gamma, 0)$ is not homeomorphic to $\mathbb{R}$, which is a contradiction.

(4) $\Rightarrow$ (5). Suppose that $k$ is an odd number. Then, the curve $\beta: (-1, 1) \to X$ given by $\beta(s) = \gamma(s^k)$ is well defined. Moreover, $\beta: (-1, 1) \to X \cap U$ is a homeomorphism, since the function $h: (-1, 1) \to (-1, 1)$ given by $h(s) = s^\frac{1}{k}$ and $\gamma$ are homeomorphisms. Thus, we obtain that $\beta(s) = s\alpha(s^\frac{1}{k})$ and, in this form, we obtain that $\beta$ is a $C^1$ function with $\beta'(0) \neq 0$. Therefore, $\Gamma$ is $C^1$ regular at 0.

(5) $\Rightarrow$ (1). If $\Gamma$ is $C^1$ regular at 0, then $(\Gamma, 0)$ and $(\mathbb{R}, 0)$ are $C^1$ diffeomorphic and by Lemma 6.6, $(\Gamma, 0)$ and $(\mathbb{R}, 0)$ are blow-spherical homeomorphic.

(5) $\Rightarrow$ (6). If $\Gamma$ is $C^1$ regular at 0, then $(\Gamma, 0)$ and $(\mathbb{R}, 0)$ are $C^1$ diffeomorphic. Therefore, $(\Gamma, 0)$ and $(\mathbb{R}, 0)$ are bi-Lipschitz homeomorphic.

(6) $\Rightarrow$ (2). If $\Gamma$ is Lipschitz regular at 0, then $(\Gamma, 0)$ and $(\mathbb{R}, 0)$ are bi-Lipschitz homeomorphic. By Theorem 3.2 in [24], $C(X, 0)$ is homeomorphic to $\mathbb{R}$. \qed

**Corollary 6.10** Let $X \subset \mathbb{R}^n$ be a real analytic curve. Then, $X$ is blow-spherical regular at 0 if and only if $X$ is $C^1$ regular at 0.

**Proof** Suppose that $X$ is blow-spherical regular at 0. By Lemma 6.8, there are an open neighborhood $U \subset \mathbb{R}^n$ of 0 and $\Gamma_1, \ldots, \Gamma_r \subset \mathbb{R}^n$ such that $\Gamma_i \cap \Gamma_j = \{0\}$, if $i \neq j$ and

$$X \cap U = \bigcup_{i=1}^{r} \Gamma_i.$$
Moreover, for each \( i \in \{1, ..., r\} \), there is an analytic homeomorphism \( \gamma_i : (-\varepsilon, \varepsilon) \to \Gamma_i \).
Since \( X \) is blow-spherical regular at 0, then \( r = 1 \), \( X \cap U = \Gamma_1 \) and \( \gamma_1 : (-\varepsilon, \varepsilon) \to X \cap U \) is an analytic homeomorphism. Since \( X \) is blow-spherical regular at 0, then by Proposition 6.9, \( X \) is \( C^1 \) regular at 0.

Reciprocally, if \( X \) is \( C^1 \) regular at 0, by Lemma 6.6, we obtain that \( X \) is blow-spherical regular at 0.

This result is sharp. Firstly, the hypothesis of \( X \) to be blow-spherical regular at 0 cannot be removed.

**Example 6.11** Let \( X = \{(x, y) \in \mathbb{R}^2; \ y^2 = x^3\} \). Then \( X \) is homeomorphic to \( \mathbb{R} \), but is not \( C^1 \). In fact, \( X \) is a topological submanifold of \( \mathbb{R}^2 \).

Secondly, the hypothesis of \( X \) to be a curve (i.e. \( \dim X = 1 \)) also cannot be removed.

**Example 6.12** Let \( V = \{(x, y, z) \in \mathbb{R}^3; z^3 = x^5 y + xy^5\} \). Then \( C(V, 0) = \{z = 0\} \) is a plane and \( V \) is a topological submanifold of \( \mathbb{R}^3 \). Moreover, \( V \) is the graph of a differentiable function at the origin and, thus, \( V \) is blow-spherical regular at 0. However, \( V \) is not \( C^1 \) regular at 0.

Finally, the hypothesis of \( X \) to be an analytic set cannot be removed as well.

**Example 6.13** The set \( V = \{(x, y) \in \mathbb{R}^2; \ y = |x|\} \) is semi-algebraic and the mapping \( \varphi : \mathbb{R} \to V \) given by \( \varphi(x) = (x, |x|) \) is a blow-spherical homeomorphism and, in particular, \( V \) is blow-spherical regular at 0, but clearly \( V \) is not \( C^1 \) regular at 0.

### 6.2 Classification of real analytic curves modulo blow-isomorphisms

Let \( p_1 : \mathbb{Z}_{>0} \times \mathcal{F}(\mathbb{Z}_3; \mathbb{Z}_{>0}) \to \mathbb{Z}_{>0} \) and \( p_2 : \mathbb{Z}_{>0} \times \mathcal{F}(\mathbb{Z}_3; \mathbb{Z}_{>0}) \to \mathcal{F}(\mathbb{Z}_3; \mathbb{Z}_{>0}) \) be the canonical projections, where \( \mathcal{F}(\mathbb{Z}_3; \mathbb{Z}_{>0}) \) denotes the set of all non-null functions from \( \mathbb{Z}_3 \cong \{-1, 0, 1\} \) to \( \mathbb{Z}_{>0} \). Let \( \mathcal{A} \) be the subset of \( \mathbb{Z}_{>0} \times \mathcal{F}(\mathbb{Z}_3; \mathbb{Z}_{>0}) \) formed by finite and non-empty subsets \( A \) satisfying the following:

- (i) \( p_1(A) = \{1, ..., N\} \) for some \( N \in \mathbb{Z}_{>0}; \)
- (ii) \( p_2(p_1^{-1}(\ell) \cap A) = \{r_\ell\} \) and \( r_\ell(-1) \leq r_\ell(1) \) for all \( \ell \in \{1, ..., N\}; \)
- (iii) \( r_\ell(0) \leq r_{\ell+1}(0) \) for all \( \ell \in \{1, ..., N - 1\} \). Moreover, if \( r_\ell(0) = r_{\ell+1}(0) \) then \( \sum_{i=-1}^{1} r_\ell(i) \leq \sum_{i=-1}^{1} r_{\ell+1}(i) \).

Let \( A \in \mathcal{A} \) be a set as above. For \( j \in \{-1, 0, 1\} \) and \( r_\ell(j) > 0 \), we define the following curves:

\[
X_{A,j} = \bigcup_{\ell=1}^{N} \{(x, y) \in \mathbb{R}^2; \prod_{r=1}^{r_\ell(j)} ((y - \ell x)^2 - j r(y + \ell x)^3) = 0\}
\]

whenever \( j \in \{-1, 1\} \) and

\[
X_{A,j} = \bigcup_{\ell=1}^{N} \{(x, y) \in \mathbb{R}^2; \prod_{r=1}^{r_\ell(0)} (y - \ell x - r(y + \ell x)^2) = 0\}
\]

whenever \( j = 0 \). Moreover, if \( r_\ell(j) = 0 \) we define \( X_{A,j} = \{0\} \). We define the realization of \( A \) to be the curve \( X_A := X_{A,-1} \cup X_{A,0} \cup X_{A,1} \).
Remark 6.14 Let \( \gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n \) be an analytic curve and \( \Gamma = \gamma((-\varepsilon, \varepsilon)) \). Suppose that \( \gamma : (-\varepsilon, \varepsilon) \to \Gamma \) is a homeomorphism and \( \gamma(0) = 0 \). Then, by Proposition 6.9 and the definition of blow-spherical equivalence, \( \Gamma \) and \( \{(x, y) \in \mathbb{R}^2; y^2 = x^3\} \) are blow-spherical homeomorphic if and only if \( C(\Gamma, 0) \) is a half-line.

Definition 6.15 Let \( X \subset \mathbb{R}^n \) and \( \tilde{X} \subset \mathbb{R}^m \) be two analytic sets. We say that \((X, 0)\) is branch by branch blow-spherical homeomorphic to \((\tilde{X}, 0)\) if there are decompositions in analytic branches \( \Gamma_1, ..., \Gamma_r \) for \((X, 0)\) and \( \tilde{\Gamma}_1, ..., \tilde{\Gamma}_r \) for \((\tilde{X}, 0)\) and there is a blow-spherical homeomorphism \( \varphi : (X, 0) \to (\tilde{X}, 0) \) such that \( \varphi((\Gamma_i, 0)) = (\tilde{\Gamma}_i, 0) \) for all \( i \in \{1, ..., r\} \).

Therefore we obtain the following classification.

Theorem 6.16 For each real analytic curve \( X \subset \mathbb{R}^n \) such that \( 0 \in X \) there exists a unique set \( A \in A \) such that \((X_A, 0)\) is branch by branch blow-spherical homeomorphic to \((X, 0)\).

Proof Given a real analytic curve \( X \subset \mathbb{R}^2 \), by Lemma 6.8, there are an open neighborhood \( U \subset \mathbb{R}^n \) of 0 and \( \Gamma_1, ..., \Gamma_r \subset \mathbb{R}^n \) such that \( \Gamma_i \cap \Gamma_j = \{0\} \) whenever \( i \neq j \) and \( X \cap U = \bigcup_{i=1}^{r} \Gamma_i \), where for each \( i \in \{1, ..., r\} \), there is an analytic homeomorphism \( \gamma_i : (-\varepsilon, \varepsilon) \to \Gamma_i \). Let \( L_1, ..., L_N \) be all the lines such that each \( L_j \) contains the tangent cone of some \( \Gamma_i \). By reordering the indices, if necessary, we may assume that for each \( \ell \in \{1, ..., N-1\} \) we have
\[
\#\{i; C(\Gamma_i, 0) = L_\ell\} \leq \#\{i; C(\Gamma_i, 0) = L_{\ell+1}\}
\]
and, moreover, if happens the equality in (2) then
\[
\#\{i; C(\Gamma_i, 0) \subset L_\ell\} \leq \#\{i; C(\Gamma_i, 0) \subset L_{\ell+1}\}.
\]
Thus, for each \( \ell \in \{1, ..., N\} \) we write \( L_\ell = L^-_\ell \cup L^+_\ell \), where \( L^-_\ell \) and \( L^+_\ell \) are half-lines such that \( L^-_\ell \cap L^+_\ell = \{0\} \) and
\[
\#\{i; C(\Gamma_i, 0) = L^-_\ell\} \leq \#\{i; C(\Gamma_i, 0) = L^+_\ell\}.
\]
and, thus, we define the function \( r_\ell : \{-1, 0, 1\} \to \mathbb{Z}_{\geq 0} \) given by \( r_\ell(-1) = \#\{i; C(\Gamma_i, 0) = L^-_\ell\}, r_\ell(0) = \#\{i; C(\Gamma_i, 0) = L_\ell\} \) and \( r_\ell(1) = \#\{i; C(\Gamma_i, 0) = L^+_\ell\} \).

Then \( A = \bigcup_{\ell=1}^{N}\{\ell, r_\ell\} \in A \) and it is not hard to verify that \((X_A, 0)\) is branch by branch blow-spherical homeomorphic to \((X, 0)\). The uniqueness of \( A \) follows directly from the definition of \( A \).

The uniqueness of \( A \) in Theorem 6.16 does not hold true for blow-spherical homeomorphisms, as we can see in the next example.

Example 6.17 Let \( A = \{(1, r_1)\} \) and \( \tilde{A} = \{(1, \tilde{r}_1)\} \), where \( r_1, \tilde{r}_1 : \{-1, 0, 1\} \to \mathbb{Z}_{\geq 0} \) satisfy \( (r_1(-1), r_1(0), r_1(1)) = (0, 2, 0) \) and \( (\tilde{r}_1(-1), \tilde{r}_1(0), \tilde{r}_1(1)) = (1, 0, 1) \). We have that \( X_A = \{(x, y) \in \mathbb{R}^2; y = x^2 \text{ or } y = 2x^2\} \) and \( X_{\tilde{A}} = \{(x, y) \in \mathbb{R}^2; y^2 = x^3 \text{ or } y^2 = -x^3\} \) are blow-spherical homeomorphic but, by Corollary 6.10, they are not branch by branch blow-spherical homeomorphic.

Thus, we are going to present a complete characterization of blow-spherical homeomorphisms.
Theorem 6.19 Let $X, \tilde{X} \subset \mathbb{R}^2$ be two real analytic curves. Let $\text{Smp}(\partial X') = \{a_1, a_2, \ldots, a_r\}$ and $\text{Smp}(\partial \tilde{X}') = \{\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_s\}$. Then the following statements are equivalents:

1. $(X,0)$ and $(\tilde{X},0)$ are blow-spherical homeomorphic;
2. There is a bijection $\sigma: \{1, \ldots, r\} \to \{1, \ldots, s\}$ such that for each $j \in \{1, \ldots, r\}$, $k_X(a_j) = k_{\tilde{X}}(\tilde{a}_{\sigma(j)})$;
3. There is an isomorphism between the real blow-spherical trees at 0 of $X$ and $\tilde{X}$.

Proof The items (2) and (3) are clearly equivalent. Moreover, by Theorem 3.3 and Lemma 6.7, we have that (1) implies (2).

Let us prove that (2) $\Rightarrow$ (1). We assume that (2) holds true. By Lemma 6.8, there are an open neighborhood $U \subset \mathbb{R}^n$ (resp. $\tilde{U}$) of 0 and $\Gamma_1, \ldots, \Gamma_m \subset \mathbb{R}^n$ (resp. $\tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_k \subset \mathbb{R}^n$) such that $\Gamma_i \cap \Gamma_j = \{0\}$ (resp. $\tilde{\Gamma}_j \cap \tilde{\Gamma}_j = \{0\}$) whenever $i \neq j$ and

$$X \cap U = \bigcup_{i=1}^m \Gamma_i \quad \text{and} \quad \tilde{X} \cap \tilde{U} = \bigcup_{i=1}^k \tilde{\Gamma}_i,$$

where for each $i \in \{1, \ldots, m\}$ (resp. $i \in \{1, \ldots, k\}$), there is an analytic homeomorphism $\gamma_i: (-\varepsilon, \varepsilon) \to \Gamma_i$ (resp. $\tilde{\gamma}_i: (-\varepsilon, \varepsilon) \to \tilde{\Gamma}_i$). In particular, we have that $2m = \sum_{j=1}^r k_X(a_j)$ and $2k = \sum_{j=1}^s k_{\tilde{X}}(\tilde{a}_j)$. Then $m = k$.

For each $i$, let $\alpha_{2i-1}, \alpha_{2i}: [0, \delta) \to \Gamma_i$ (resp. $\alpha_{2i-1}, \alpha_{2i}: [0, \delta) \to \tilde{\Gamma}_i$) be the reparametrizations by the distance to the origin, resp., of $(\gamma_i \circ \iota)|_{[0,\varepsilon)}$ and $\gamma_i|_{[0,\varepsilon)}$ (resp. $(\tilde{\gamma}_i \circ \iota)|_{[0,\varepsilon)}$ and $\tilde{\gamma}_i|_{[0,\varepsilon)}$), where $\iota: \mathbb{R} \to \mathbb{R}$ is given by $\iota(t) = -t$.

For each $j \in \{1, \ldots, r\}$, let $\beta_{j,1}, \ldots, \beta_{j,k_X(a_j)} = \{a_j; (\lim_{t \to 0^+} \frac{a_j(t)}{t}, 0) = a_j\}$ (resp. $\tilde{\beta}_{j,1}, \ldots, \tilde{\beta}_{j,k_{\tilde{X}}(\tilde{a}_{\sigma(j)})} = \{\tilde{a}_j; (\lim_{t \to 0^+} \frac{\tilde{a}_j(t)}{t}, 0) = \tilde{a}_{\sigma(j)}\}$). Thus, we define $\varphi: (X,0) \to (\tilde{X},0)$ by $\varphi(x) = \tilde{\beta}_{j,i} \circ \beta_{j,i}^{-1}(x)$ whenever $x \in \text{Im}(\beta_{j,i})$. We have that $\varphi$ is a homeomorphism.
and $\varphi': X' \to \tilde{X}'$ is the homeomorphism given by

$$
\varphi'(x,t) = \begin{cases} 
\left( \frac{\hat{\beta}_{j,i}(t)}{t}, t \right), & t \neq 0 \text{ and } tx \in Im(\beta_{j,i}) \\
(\hat{a}_{\sigma(j)}, 0), & t = 0 \text{ and } x = a_j 
\end{cases}
$$

which finishes the proof. □

Acknowledgements The author wishes to thank Eurípedes C. da Silva for his interest in this research. The author also wishes to thank the anonymous referee for an accurate reading of the paper and essential suggestions and comments, which allowed to improve the exposition and added clarifications at several points.

References

1. A’Campo, N.: Le nombre de Lefschetz d’une monodromie. (French) Nederl. Akad. Wetensch. Proc. Ser. A 76 Indag. Math. 35, 113–118 (1973)
2. Birbrair, L.: Lipschitz geometry of curves and surfaces definable in o-minimal structures. Ill. J. Math. 52(4), 1325–1353 (2008)
3. Birbrair, L., Fernandes, A., Grandjean, V.: Collapsing topology of isolated singularities. arXiv:1208.4328 [math.MG] (2012)
4. Birbrair, L., Fernandes, A., Grandjean, V.: Thin–thick decomposition for real definable isolated singularities. Indiana Univ. Math. J. 66, 547–557 (2017)
5. Birbrair, L., Fernandes, A., Lê, D.T., Sampaio, J.E.: Lipschitz regular complex algebraic sets are smooth. Proc. Am. Math. Soc. 144, 983–987 (2016)
6. Birbrair, L., Neumann, W.D., Pichon, A.: The thick-thin decomposition and the bilipschitz classification of normal surface singularities. Acta Math. 212(2), 199–256 (2014)
7. Chirka, E.M.: Complex Analytic Sets. Kluwer Academic Publishers (1989)
8. Ephraim, R.: $C^\infty$ and analytic equivalence of singularities. Complex analysis, 1972, Vol. I: Geometry of singularities. Rice Univ. Stud. 59(1), 11–32 (1973)
9. Ephraim, R.: $C^1$ preservation of multiplicity. Duke Math. 43, 797–803 (1976)
10. Ephraim, R.: The cartesian product structure and $C^\infty$ equivalences of singularities. Trans. Am. Math. Soc. 224(2), 299–311 (1976)
11. Fukui, T., Kurdyka, K., Paunescu, L.: An inverse mapping theorem for arc-analytic homeomorphisms. In: Geometric Singularity Theory, vol. 65, pp. 49–56. Banach Center Publ., Polish Acad. Sci. Inst. Math., Warsaw (2004)
12. Gau, Y.-N., Lipman, J.: Differential invariance of multiplicity on analytic varieties. Inventiones mathematicae 73(2), 165–188 (1983)
13. Kobayashi, M., Kuo, T.-C.: On blow-analytic equivalence of embedded curve singularities. Real analytic and algebraic singularities. Pitman Res. Notes Math. Ser. 381, 30–37 (1998)
14. Koike, S., Paunescu, L.: On the geometry of sets satisfying the sequence selection property. J. Math. Soc. Jpn. 67(2), 721–751 (2015)
15. Kuo, T.-C.: On classification of real singularities. Invent. Math. 82(2), 257–262 (1985)
16. Kurdyka, K., Raby, G.: Densité des ensembles sous-analytiques. Ann. Inst. Fourier (Grenoble) 39(3), 753–771 (1989)
17. Lê, D.T.: Calcul du nombre de cycles évanouissants d’une hypersurface complexe. (French) Ann. Inst. Fourier (Grenoble) 23(4), 261–270 (1973)
18. Milnor, J.: Singular Points of Complex Hypersurfaces. Princeton University Press, Princeton (1968)
19. Mumford, D.: The topology of normal singularities of an algebraic surface and a criterion for simplicity. Inst. Hautes Études Sci. Publ. Math. 9, 5–22 (1961)
20. Pawłucki, W.: Quasi-regular boundary and Stokes’ formula for a sub-analytic leaf. In: Ławrynowicz, J. (ed.) Seminar on Deformations. Lecture Notes in Mathematics, vol. 1165, pp. 235–252. Springer, Berlin (1985)
21. Risler, J.-J.: Invariant curves and topological invariants for real plane analytic vector fields. J. Differ. Equ. 172, 212–226 (2001)
22. Rosas, R.: The differentiable-invariance of the algebraic multiplicity of a holomorphic vector field. J. Differ. Geom. 83(2), 337–396 (2009)
23. Sampaio, J.E.: Regularidade Lipschitz, invariância da multiplicidade e a geometria dos cones tangentes de conjuntos analíticos. Ph.D. thesis, Universidade Federal Do Ceará (2015). http://www.repositorio.ufc.br/bitstream/riufc/12545/1/2015_tese_jesampaio.pdf

24. Sampaio, J.E.: Bi-Lipschitz homeomorphic subanalytic sets have bi-Lipschitz homeomorphic tangent cones. Selecta Math. (N.S.) 22(2), 553–559 (2016)

25. Sampaio, J.E.: A proof of the differentiable invariance of the multiplicity using spherical blowing-up. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas 113(4), 3913–3920 (2019)

26. Sampaio, J.E.: Multiplicity, regularity and blow-spherical equivalence of complex analytic sets. Asian J. Math. 24(5), 803–820 (2020)

27. Sampaio, J.E.: Multiplicity, regularity and Lipschitz Geometry of real analytic hypersurfaces. To appear in the Israel Journal of Mathematics (2022)

28. Sampaio, J.E.: Differential invariance of the multiplicity of real and complex analytic sets. To appear in Publicacions Matemàtiques (2022)

29. Trotman, D.: Multiplicity is a $C^1$ invariant. University Paris 11 (Orsay), Preprint (1977)

30. Tverberg, H.: A proof of the Jordan curve theorem. Bull. Lond. Math. Soc. 12, 34–38 (1980)

31. Valette, G.: Multiplicity mod 2 as a metric invariant. Discrete Comput. Geom. 43, 663–679 (2010)

32. Whitney, H.: Elementary structure of real algebraic varieties. Ann. Math. 66(3), 545–556 (1957)

33. Zariski, O.: Some open questions in the theory of singularities. Bull. Am. Math. Soc. 77(4), 481–491 (1971)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.