Analytic representation of $F_K/F_\pi$ in two loop chiral perturbation theory

B. Ananthanarayan, Johan Bijnens, Samuel Friot, and Shayan Ghosh

1Centre for High Energy Physics, Indian Institute of Science, Bangalore-560012, Karnataka, India
2Department of Astronomy and Theoretical Physics, Lund University, Södregatan 14A, SE 223-62 Lund, Sweden
3Institut de Physique Nucléaire d’Orsay, Université Paris-Sud 11, IN2P3-CNRS, F-91405 Orsay Cedex, France
4Institut de Physique Nucléaire de Lyon, Université Lyon 1, IN2P3-CNRS, F-69622 Villeurbanne Cedex, France

We present an analytic representation of $F_K/F_\pi$ as calculated in three-flavour two-loop chiral perturbation theory, which involves expressing three mass scale sunsets in terms of Kampé de Fériet series. We demonstrate how approximations may be made to obtain relatively compact analytic representations. An illustrative set of fits using lattice data is also presented, which shows good agreement with existing fits.

Introduction - The spectrum of QCD contains as lightest particles the pseudo-scalar octet, and their properties provide a delicate test of its non-perturbative features, including that of chiral symmetry breaking in the sector involving the three lightest quarks. Of these, a special place is accorded to the kaon and pion, namely $F_K$ and $F_\pi$. Their ratio has been investigated on the lattice recently [1]. On the other hand, in chiral perturbation theory (ChPT) at two-loops, expressions have been available for nearly two decades, but involving certain integrals (sunsets) that are evaluated numerically [2].

In this work, we provide an analytic expression for $F_K/F_\pi$, which among other things incorporates double series derived using Mellin-Barnes (MB) representations of the sunsets. This allows us to produce a template for easy fitting to lattice simulations.

Methodology - Three-flavour ChPT expressions for the decay constants of the pseudoscalar mesons at two-loops are given in [3]. These may be decomposed as:

$$\frac{F_P}{F_0} = 1 + F_P^{(4)} + (F_P)^{(6)}_{CT} + (F_P)^{(6)}_{\text{loop}} + \mathcal{O}(p^8),$$

where $P$ is the particle in question. The $\mathcal{O}(p^6)$ contribution can be subdivided as:

$$F_P^{(6)} = d_{\text{sunset}}^P + d_{\log \times \log}^P + d_{\log}^P + d_{\log \times L_i}^P + d_{L_i}^P + d_{L_i \times L_i}^P,$$

where $d_{\text{sunset}}^P$, $d_{\log \times \log}^P$, and $d_{\log}^P$ are the pure sunset terms, the terms linear respectively quadratic in chiral logarithms without $L_i$, $d_{\log}^P$, and $d_{L_i \times L_i}^P$ the terms linear respectively quadratic in chiral logarithms with $L_i$. Their ratio has been investigated on the lattice recently [1]. The term $(F_P)^{(6)}_{CT}$ is composed of the $\mathcal{O}(p^6)$ counterterms, i.e. the LECs $C_i^\alpha$, while $d_{\text{sunset}}^P$ are the pure sunset terms.

One determines the ratio $F_K/F_\pi$ using:

$$\frac{F_K}{F_\pi} = 1 + \left( \left. \frac{F_K}{F_0} \right|_{p^4} - \frac{F_\pi}{F_0} \right|_{p^4} \right)_{\text{NLO}}$$

$$+ \left( \left. \frac{F_K}{F_0} \right|_{p^6} - \frac{F_\pi}{F_0} \right|_{p^6} \right)_{\text{NNLO}}.$$

(3)

The terms $d_{\text{sunset}}^P$ are not available fully analytically. Their determination is the goal of this work. The sunset integral is defined as:

$$H_{\{\alpha,\beta,\gamma\}}^d(m_1, m_2, m_3, p^2) =$$

$$\frac{1}{(2\pi)^2} \int \frac{d^d q}{q^2 - m_1^2} \frac{d^d r}{r^2 - m_2^2} [q + r - p]^2 - m_3^2 \gamma.$$

(4)

Aside from the basic scalar integral defined above, tensor integrals in which the momenta $q_\mu$ and $q_\nu q_\mu$ appear in the numerator, and derivatives with respect to the external momentum of both the scalar and tensor integrals contribute to $d_{\text{sunset}}^P$ [3]. The tensor integrals, as well as all the derivatives, may be reduced into a linear combination of scalar integrals using the methods given in [3]. Thus only a smaller set of master integrals (MI) is needed.
The full list of sunset integrals contributing to $d_{\text{sunset}}^\tau$ can thus all be expressed in terms of a set of four MI ($H_{(1,1,1)}^{d}, H_{(2,1,1)}^{d}, H_{(1,2,1)}^{d}$ and $H_{(1,1,2)}^{d}$) and the one-loop tadpole integral. The problem reduces to solving these analytically in the required mass configurations. For the evaluation of $F_K/F_\pi$, seven distinct three mass SL need evaluation.

MB theory leads to representations of these MI where each integral consists of at least one double complex plane integral. These double MI integrals are evaluated using the method proposed in [3] and fully systematized in [6] to obtain results in the form of sums of single and double infinite series [7-9].

**The analytic representation** - Using Eq.(3), we obtain the following representation of $F_K/F_\pi$:

\[
\frac{F_K}{F_\pi} = 1 + 4(4\pi)^2 L_5^r (\xi_K - \xi_\pi) + \frac{5}{8} \xi_\pi \lambda_\pi - \frac{1}{4} \xi_K \lambda_K \\
+ \left( \frac{1}{8} \xi_\pi - \frac{1}{2} \xi_K \right) \lambda_\pi + \xi_2 F F \left( \frac{m_2^2}{m_K^2} \right) + \hat{K}_1 r^2 \\
+ \hat{K}_2 \lambda_\pi \lambda_K + \hat{K}_4 \lambda_\pi \lambda_\eta + \hat{K}_5 \lambda_K \lambda_\eta \\
+ \hat{K}_6 \lambda_\eta^2 + \hat{C}_1 \lambda_\pi + \hat{C}_2 \lambda_K + \hat{C}_3 \lambda_\eta + \hat{C}_4,
\]

where $\xi_\pi = m_\pi^2/(16\pi^2 F_\pi^2)$, $\xi_K = m_K^2/(16\pi^2 F_\pi^2)$, $\lambda_\pi = \log(m_\pi^2/m_\pi^2)$, and:

\[
\hat{K}_1 = \frac{113}{72} \xi_\pi \xi_K - \frac{131}{192} \xi_\pi^2, \\
\hat{K}_2 = -\frac{41}{96} \xi_\pi \xi_K - \frac{3}{32} \xi_\pi^2, \\
\hat{K}_3 = \frac{13}{24} \xi_\pi \xi_K + \frac{59}{96} \xi_\pi^2, \\
\hat{K}_4 = \frac{17}{36} \xi_\pi^2 + \frac{7}{144} \xi_K \xi_\pi, \\
\hat{K}_5 = -\frac{163}{144} \xi_\pi \xi_K - \frac{67}{288} \xi_\pi^2 + \frac{3}{32} \xi_\pi^2, \\
\hat{K}_6 = \frac{241}{288} \xi_\pi \xi_K - \frac{13}{192} \xi_\pi^2.
\]

\[
\hat{C}_1 = -\left( \frac{7}{9} + \frac{11}{2} (4\pi)^2 L_5^r \right) \xi_\pi \xi_K \\
- \left( \frac{113}{72} + (4\pi)^2 (4L_5^r + 10L_5^r + \frac{13}{2} L_5^r - \frac{21}{2} L_5^r) \right) \xi_\pi^2, \\
\hat{C}_2 = \left( \frac{209}{144} + 3(4\pi)^2 L_5^r \right) \xi_\pi \xi_K \\
+ \left( \frac{53}{96} + (4\pi)^2 (4L_5^r + 10L_5^r + 5L_5^r - 5L_5^r) \right) \xi_\pi^2, \\
\hat{C}_3 = \left( \frac{13}{18} + (4\pi)^2 \left( \frac{8}{3} L_5^r - \frac{2}{3} L_5^r - 16L_7^r - 8L_5^r \right) \right) \xi_K^2 \\
- \left( \frac{4}{9} + (4\pi)^2 \left( \frac{4}{3} L_5^r + \frac{25}{6} L_5^r - 32L_5^r - 16L_5^r \right) \right) \xi_\pi \xi_K
\]

\[
F_F = \frac{m_\pi^2}{m_K^2} \left( \frac{m_\pi^2}{m_K^2} \right) \left( \frac{F_\pi^2}{F_\pi^2} \right) + \frac{4}{3} \xi_\pi \xi_K \\
+ \left( \frac{19}{288} + (4\pi)^2 \left( \frac{1}{6} L_5^r + \frac{11}{6} L_5^r - 16L_7^r - 8L_5^r \right) \xi_\pi \right)
\]

\[
\hat{C}_4 = (4\pi)^2 (\xi_K - \xi_\pi) \\
\times \left( \frac{8(4\pi)^2}{2} \left( C_{14}^r + C_{15}^r \right) \xi_K + \left( C_{15}^r + 2C_{17}^r \right) \xi_\pi \right) \\
+ \left( \frac{8(4\pi)^2 L_5^r (8L_5^r + 3L_5^r - 16L_6^r - 8L_5^r) - 2L_1^r \xi_\pi \right) \\
- \frac{1}{18} L_5^r - \frac{4}{3} \xi_\pi \xi_K - \frac{3}{16} \xi_K - 8L_5^r \xi_\pi \right) \\
+ \left( \frac{8(4\pi)^2 L_5^r (4L_5^r + 5L_5^r - 8L_6^r - 8L_8^r) - 2L_1^r \xi_\pi \right) \\
- \frac{5}{18} L_5^r - \frac{4}{3} \xi_\pi \xi_K + 16L_5^r + 8L_8^r \xi_\pi \right)
\]
The MI are denoted by $\overline{\mathcal{H}}_{\pi K \eta}$ (blue region). The red dot marks the physical values of the meson masses.

FIG. 1. Region of convergence of Eqs. (15a–17) (blue region). The red dot marks the physical values of the meson masses.
\[ F_{0.3}^{2.0} \left[ - \frac{1}{2} + \alpha, -\frac{1}{2} ; 1 ; 1, \frac{3}{2} : \frac{m_{\eta}^{2}}{m_{\eta}^{2} - 4m_{K}^{2}} \right] \bigg|_{\alpha=0} - \frac{m_{\eta}^{2}}{m_{K}^{2}} \left( \log \left[ \frac{m_{\eta}^{2}}{m_{\eta}^{2} - 4m_{K}^{2}} \right] + \frac{\partial}{\partial \alpha} \right) \cdot \left( \Gamma \left( \frac{3}{2} - \alpha \right) - \alpha \right) \]
\[
\begin{align*}
- & \frac{7269419973251}{1120324867200} + \frac{145\pi}{72\sqrt{2}} + \frac{38693\pi}{25920\sqrt{3}} + \frac{82\gamma}{405} \\
- & \frac{121}{576} \log^2 \left[ \frac{4}{3} \right] + \left( \frac{6035437}{9797760} + \frac{13\pi}{864\sqrt{3}} \right) \log[3] \\
- & \frac{468002719}{161663040} + \frac{13\pi}{576\sqrt{3}} \log \left[ \frac{4}{3} \right] - \frac{29}{324} \psi \left[ \frac{5}{2} \right] \\
+ & \left( \frac{463\log[3]}{384\sqrt{2}} + \frac{\log \left[ \frac{4}{3} \right]}{2\sqrt{2}} - \frac{11\pi}{18\sqrt{2}} - \frac{13\gamma}{18\sqrt{2}} - 3456\sqrt{2} \right) \\
\times & \csc^{-1} \left[ \sqrt{3} \right] + \frac{11}{48} \csc^{-1} \left[ \sqrt{3} \right]^2, \\
\end{align*}
\]

\[a_3 = \frac{803}{810} + \frac{13\pi}{1728\sqrt{3}} + \frac{7}{48} \log \left[ \frac{4}{3} \right] - \frac{1}{2\sqrt{2}} \csc^{-1} \left[ \sqrt{3} \right],\]

\[a_4 = \frac{11}{24}, \quad a_7 = \frac{337}{384}, \quad a_{10} = -\frac{9}{64}, \quad a_{13} = -\frac{27}{128},\]

\[a_5 = \frac{47}{128} \log^2 \left[ \frac{4}{3} \right] - \frac{845}{648} \left( \frac{\mathrm{Li}_2 \left[ \frac{3}{4} \right]}{4} + \log[4] \log \left[ \frac{4}{3} \right] \right) - \frac{1501}{512} - \frac{1576413731881}{12960} + \frac{5\pi^2}{18} + \frac{3572063\pi}{3585039575040} + \frac{353}{48} \csc^{-1} \left[ \sqrt{3} \right]^2 \\
+ \left( \frac{744674317}{313528320} + \frac{176189\pi}{55296\sqrt{3}} \right) \log[3] + \frac{35}{144} \psi \left[ \frac{5}{2} \right] \\
+ \left( \frac{97621}{55296\sqrt{2}} + \frac{59}{48} \psi \left[ \frac{5}{2} \right] - \frac{3167\gamma}{288\sqrt{2}} - \frac{19589\log[3]}{4096\sqrt{2}} \right) \\
- \frac{115}{48\sqrt{2}} \log \left[ \frac{4}{3} \right] \csc^{-1} \left[ \sqrt{3} \right] \\
+ \left( \frac{4312709021}{1299304320} + \frac{176189\pi}{36684\sqrt{3}} \right) \log \left[ \frac{4}{3} \right],\]

\[a_6 = \frac{17003}{8640} - \frac{176189\pi}{110592\sqrt{3}} - \frac{155}{192} \log \left[ \frac{4}{3} \right] \\
+ \frac{115}{48\sqrt{2}} \csc^{-1} \left[ \sqrt{3} \right],\]

\[a_8 = \frac{265}{864} \left( \frac{\mathrm{Li}_2 \left[ \frac{3}{4} \right]}{4} + \log[4] \log \left[ \frac{4}{3} \right] \right) + \frac{199393\gamma}{138240} \\
+ \frac{25001310633017}{948109639600} + \frac{4753\pi}{13824\sqrt{2}} + \frac{20910563\gamma}{26542080\sqrt{3}} \\
- \frac{29\pi^2}{288} - \frac{101313035}{143327232} + \frac{804611\gamma}{442368\sqrt{3}} \log[3] \\
- \frac{129118553}{117573120} + \frac{804611\pi}{294912\sqrt{3}} \log \left[ \frac{4}{3} \right] - \frac{119}{288} \psi \left[ \frac{5}{2} \right],\]

\[\frac{5}{16} \csc^{-1} \left[ \sqrt{3} \right]^2 + \frac{5\pi}{16} \csc^{-1} \left[ \sqrt{3} \right] \left( \frac{823}{3072\sqrt{2}} \log \left[ \frac{4}{3} \right] \right) \\
+ \frac{5\pi}{16} - \frac{19319\gamma}{9216\sqrt{2}} - \frac{5341499}{3538944\sqrt{2}} + \frac{104075\log[3]}{196608\sqrt{2}}.\]

\[a_9 = -\frac{8327}{138240} + \frac{804611\pi}{884736\sqrt{3}} - \frac{1}{96} \log \left[ \frac{4}{3} \right],\]

\[-\frac{832}{3072\sqrt{2}} \csc^{-1} \left[ \sqrt{3} \right],\]

\[a_{11} = -\frac{5}{192} \left( \frac{\mathrm{Li}_2 \left[ \frac{3}{4} \right]}{4} + \log[4] \log \left[ \frac{4}{3} \right] \right) - \frac{25\pi^2}{192},\]

\[a_{12} = -\frac{1310311\gamma}{6635520} - \frac{10567863311827}{10113169489920} + \frac{4453\sqrt{3}}{65536} \\
+ \left( \frac{12616533707}{45864714240} + \frac{1674775\pi}{7077888\sqrt{3}} \right) \log[3] \\
+ \frac{17720699}{46448640} + \frac{1674775\pi}{4718592\sqrt{3}} \log \left[ \frac{4}{3} \right] \\
- \frac{13905571\pi}{84934656\sqrt{3}} + \frac{2135\pi}{73728\sqrt{2}} + \frac{97}{64} \psi \left[ \frac{5}{2} \right] \\
+ \frac{1}{\sqrt{2}} \left( \frac{605645}{18874368} - \frac{391\gamma}{49152} - \frac{121093\log[3]}{4194304} \right) \\
- \frac{59}{4096} \log \left[ \frac{4}{3} \right] \csc^{-1} \left[ \sqrt{3} \right],\]

\[a_{12} = \frac{5538437}{11612160} - \frac{1674775\pi}{14155776\sqrt{3}} + \frac{1}{64} \log \left[ \frac{4}{3} \right] \\
+ \frac{59}{4096\sqrt{2}} \csc^{-1} \left[ \sqrt{3} \right].\]

The range of validity of Eqs. (13-19) is shown in Fig. 2 in which the exact value of \(F_F\) is plotted against \(x = \sqrt{\rho}\), as are the approximate \(F_F\) retained up to various orders of \(\rho\). The expansion up to \(O(\rho^4)\) approximates the exact value of \(F_F\) to 1% for \(m_\pi/m_K < 3\) and to 6% for \(m_\pi/m_K < 0.5\). One may obtain a representation with greater accuracy by truncating the series with a larger number of terms.

For the reader to be able to verify the implementation of these expressions, we give the numerical values of \(F_K/F_\pi\) coming from both exact and approximate expressions and obtained with physical values \(m_\pi = 0.1350\text{GeV}, m_K = 0.4955\text{GeV}, F_\pi = 0.0922\text{GeV},\) as well as the LEC values of the BE14 fit of [11]. We get, using Eq. (5),

\[F_K/F_\pi = 1.19897,\]

(20)
and using the approximation of Eqs. (18)-(19),

\[ F_K / F_\pi = 1.20071. \]  \quad (21)

**Illustrative Lattice Fits**—In this section, we present an exploratory numerical study based on our analytical representation by fitting Eq. (3) with the data of the lattice study [1] to determine best-fit values of the NLO LEC \( L_5^r \) and the NNLO LEC combinations \( C_{14}^r + C_{15}^r \) and \( C_{15}^r + 2C_{17}^r \). We perform the fit (using [12]) on the mass sets for which \( m_\pi < 0.4 \) GeV. We do the fit on the ‘exact’ \( F_K \), i.e. truncating the KdF series after 1000\(^2\) terms, and cross-check by fitting the exact purely numerical version of Eq. (3) with CHIRON [13]. The fit on the approximate version presented in Eq. (15) gives compatible results.

The uncertainties on the values of the LEC given in this section derive from the errors of the \( F_K / F_\pi \) data of the lattice study, but do not take into account other uncertainties. As detailed in [3], systematic effects due to lattice artifacts can arise from correlator time choices, lattice spacings, renormalization and finite volume corrections, among other things. When these effects are taken into account, such as by means of the results presented in [14, 15] to account for the extrapolation to infinite volume, the values of the LEC presented in this section are likely to change. However, determining the exact nature and magnitude of the change involves a detailed study that is outside the scope of this paper. Therefore, the numerical results in this section are given for an illustrative purpose only, to encourage the lattice community to undertake just such a detailed study using the NNLO analytic results presented above.

We fix the renormalization scale \( \mu \) at \( m_\pi = 0.77 \) GeV, and use the values of the BE14 fit [11] for the other \( L_5^r \). In addition we fix \( F_\pi \) in the determination of \( \xi_\pi \) and \( \xi_K \) to 92.2 MeV and obtain:

\[
\begin{align*}
L_5^r &= (3.92 \pm 0.55) \times 10^{-4} \\
C_{14}^r + C_{15}^r &= (2.59 \pm 0.63) \times 10^{-6} \\
C_{15}^r + 2C_{17}^r &= (6.10 \pm 1.41) \times 10^{-6}. \quad (22)
\end{align*}
\]

The correlation parameters are given in Table I and the quality of the fit is shown in Fig. 3 (Left). The correlation is shown graphically in Fig. 3 (Middle, Right) by plotting a number of random points in a distribution given by the correlation matrix of the fit projected on the two different planes.

With these LEC values and the physical meson masses as inputs, we get for the value of \( F_K / F_\pi \):

\[ F_K / F_\pi = 1.194, \]  \quad (23)

which agrees well with the literature value of [11].

The values of Eq. (22) differ from those of the BE14 exact fit (\( L_5 = 10.1 \times 10^{-4}, C_{14} + C_{15} = -4.00 \times 10^{-6}, C_{15} + 2C_{17} = -5.00 \times 10^{-6} \)) significantly, but are more compatible with those of [10] (\( L_5 = 0.76 \times 10^{-3}, C_{14} + C_{15} = 3.15 \times 10^{-6}, C_{15} + 2C_{17} = 10.96 \times 10^{-6} \) in dimensionless units) and [17] (\( L_5 = 0.75 \times 10^{-3}, C_{14} + C_{15} = 1.70 \times 10^{-6}, C_{15} + 2C_{17} = 6.04 \times 10^{-6} \)).

A similar fit, but now with \( F_\pi \) also varied in \( \xi_\pi, \xi_K \) requires the use of lattices common to [1] and [18] to obtain the values of \( F_\pi \) for each lattice. This fit gives:

\[
\begin{align*}
L_5^r &= (0.49 \pm 1.08) \times 10^{-4} \\
C_{14}^r + C_{15}^r &= (5.59 \pm 1.08) \times 10^{-6} \\
C_{15}^r + 2C_{17}^r &= (39.7 \pm 2.10) \times 10^{-6}. \quad (24)
\end{align*}
\]

The change in the values above arises primarily due to the variation of \( F_\pi \). Keeping \( F_\pi \) fixed at 92.2 MeV but with the set of inputs used to calculate Eq. (24) results in changes of \( \approx 20\%, 35\% \) and \( 10\% \) in the Eq. (22) values of the \( L_5^r, C_{14}^r + C_{15}^r \) and \( C_{15}^r + 2C_{17}^r \), respectively. As the difference in the inputs for Eq. (22) and Eq. (21) is primarily the data from the coarsest lattices, it seems that the lattice data has a significant impact on fitting the LECs.

**Conclusions**—The ratio \( F_K / F_\pi \) is a quantity at the heart of chiral symmetry breaking, a fundamental

| \( L_5 \) | \( C_{14} + C_{15} \) | \( C_{15} + 2C_{17} \) |
|---|---|---|
| -0.93 | 1.00 | 0.35 | -0.66 |

TABLE I. Correlation values of the fit in (22).

![FIG. 2. Comparison of the exact and approximate \( F_K \).](image-url)
property of the strong interactions that is measured in ab initio calculations on the lattice. Tuning of the quark masses to physical values is now possible. Thus an analytic expansion for this quantity in masses of the quarks or the mesons is the order of the day. Using modern loop calculation techniques, we have achieved this goal. At present, two-loop precision is sufficient to fit the lattice data; this might change when the lattice precision improves in the future. While there exist three-loop results in two-flavour ChPT [19], in three-flavour ChPT two-loops is the state of the art, making our method and results all the more significant.

This work is a product of combining techniques developed independently in various branches of elementary particle physics and field theory, and represents an important advance on the results that appeared nearly two decades ago, when many sunsets were evaluated numerically. We hope this work will pave the way for detailed comparisons of other similar quantities with lattice simulations, and help improve our understanding of both ChPT and lattice studies.

Acknowledgements - We thank Pere Masjuan for helpful correspondence regarding the LECs. JB is supported in part by the Swedish Research Council grants contract numbers 621-2013-4287, 2015-04089 and 2016-05996 and by the European Research Council under the European Union’s Horizon 2020 research and innovation programme (grant agreement No 668679). BA is partly supported by the MSIL Chair of the Division of Physical and Mathematical Sciences, Indian Institute of Science.

[1] S. Dürr et al., Phys. Rev. D 95 (2017) no.5, 054513 doi:10.1103/PhysRevD.95.054513 [arXiv:1601.05998 [hep-lat]].
[2] J. Gasser and H. Leutwyler, Nucl. Phys. B 250, 465 (1985). doi:10.1016/0550-3213(85)90492-4
[3] G. Amoros, J. Bijnens and P. Talavera, Nucl. Phys. B 568 (2000) 319 [hep-ph/9907264].
[4] O. V. Tarasov, Nucl. Phys. B 502 (1997) 455 [hep-ph/9703319].
[5] J. P. Aguilar, D. Greynat and E. De Rafael, Phys. Rev. D 77 (2008) 093010 doi:10.1103/PhysRevD.77.093010 [arXiv:0802.2618 [hep-ph]].
[6] J. Bijnens, Eur. Phys. J. C 75 (2015) no.1, 27 doi:10.1140/epjc/s10052-014-3249-9 [arXiv:1412.0887 [hep-ph]]. http://home.thep.lu.se/~bijnens/chiron/
[14] G. Colangelo, S. Durr and R. Sommer, Nucl. Phys. Proc. Suppl. 119 (2003) 254 doi:10.1016/S0920-
[15] G. Colangelo, S. Durr and C. Haefeli, Nucl. Phys. B 721 (2005) 136 doi:10.1016/j.nuclphysb.2005.05.015 [hep-lat/0503014].

[16] G. Ecker, P. Masjuan and H. Neufeld, Phys. Lett. B 692 (2010) 184 doi:10.1016/j.physletb.2010.07.037 [arXiv:1001.3122 [hep-ph]].

[17] G. Ecker, P. Masjuan and H. Neufeld, Eur. Phys. J. C 74 (2014) no.2, 2748 doi:10.1140/epjc/s10052-014-2748-z [arXiv:1310.8322 [hep-ph]].

[18] S. Dürr et al. [Budapest-Marseille-Wuppertal Collaboration], Phys. Rev. D 90, no. 11, 114504 (2014) doi:10.1103/PhysRevD.90.114504 [arXiv:1310.3620 [hep-lat]].

[19] J. Bijnens and N. H. Truedsson, JHEP 1711 (2017) 181 doi:10.1007/JHEP11(2017)181 [arXiv:1710.01901 [hep-ph]].