Chaos and the quantum: how nonlinear effects can explain certain quantum paradoxes

Wm. C. McHarris
Departments of Chemistry and Physics/Astronomy, Michigan State University, East Lansing, MI 48824, USA
E-mail: mcharris@chemistry.msu.edu

Abstract. In recent years we have suggested that many of the so-called paradoxes resulting from the Copenhagen interpretation of quantum mechanics could well have more logical parallels based in nonlinear dynamics and chaos theory. Perhaps quantum mechanics might not be strictly linear as has been commonly postulated, and indeed, during the past year experimentalists have discovered signatures of chaos in a definitely quantum system. As an illustration of what can go wrong when quantum effects are forced into a linear interpretation, I examine Bell-type inequalities. In conventional derivations of such inequalities, classical systems are found to impose upper limits on the statistical correlations between, say, the properties of a pair of separated but entangled particles, whereas quantum systems allow greater correlations. Numerous experiments have upheld the quantum predictions (greater statistical correlations than allowed classically), which has led to inferences such as the instantaneous transmission of information between effectively infinitely separated particles — Einstein’s “spooky action-at-a-distance,” incompatible with relativity. I argue that there is nothing wrong with the quantum mechanical side of such derivations (the usual point of attack by those attempting to debunk Bell-type arguments), but implicit in the derivations on the classical side is the assumption of independent, uncorrelated particles. As a result, one is comparing uncorrelated probabilities versus conditional probabilities rather than comparing classical versus quantum mechanics, making moot the experimental inferences. Further, nonlinear classical systems are known to exhibit correlations that can easily be as great as and overlap with quantum correlations — so-called nonextensive thermodynamics with its nonadditive entropy has verified this with numerous examples. Perhaps quantum mechanics does contain fundamental nonlinear elements. Nonlinear dynamics and chaos theory could well provide a bridge between the determinism so dear to Einstein and the statistical interpretation of the Copenhagen school. Einstein and Bohr both could have been right in their debates.

1. Introduction
Ever since the Copenhagen interpretation of quantum mechanics gained status as the orthodox interpretation, many so-called paradoxes have accrued. These have perplexed many of us, but on the other hand, many others have taken an almost perverse sense of pride in them, touting to all who will listen that the quantum world is indescribably different from the classical world — in a quantum world indeterminacy reigns, particles can be in several places or states at the same instant, time perhaps can flow backward with the future affecting the present, and, contrary to and incompatible with relativity, not merely superluminal but instantaneous signals can pass from particle to particle in an entangled system. Such quantum peculiarities have led to predictions such as quantum teleportation and, perhaps more realistically, quantum computation.
— which rest firmly on the principle of superposition of states and the fact that quantum mechanics be a strictly linear science.

During the last several decades, however, the very glamour of some of these predictions has led to a reinvestigation of some of the foundations of quantum mechanics. In particular, the burgeoning field of quantum information has led to questioning of some of the fundamental postulates. During the same period chaos theory has finally become somewhat tractable, thanks to the advent of modern high-speed computers with their associated graphics. It should be noted that the founders of quantum mechanics did not have access to modern chaos theory — or even to anything but minimal nonlinear dynamics — so they were forced to cast quantum mechanics in a linear mold. There have been a few, relatively unsuccessful attempts to apply nonlinear dynamics to quantum mechanics, but these have been primarily along the line of nonlinear perturbations applied to basically linear systems — situations where chaos cannot develop. The resulting frustration is succinctly stated by Mielnik [1]: “I cannot help concluding that we do not know truly whether or not nonlinear QM generates superluminal signals — or perhaps it resists embedding into too narrow a scheme of tensor products. After all, if the scalar potentials were an obligatory tool to describe the vector fields, some surprising predictions could as well arise! ...the nonlinear theory would be in a peculiar situation of an Orwellian ‘thoughtcrime’ confined to a language in which it cannot even be expressed. ...A way out, perhaps, could be a careful revision of all traditional concepts...”

In a series of papers I have raised the question as to whether it is possible to explain some quantum paradoxes by nonlinear, even chaotic, parallels [2]. Thus far, progress has been more or less in a “botanical” or species-collecting mode, so I emphasize that here I am primarily raising the possibility of fundamental nonlinearities underlying quantum mechanics — certainly not stating this as a fact. However, certain quantum paradoxes do have nonlinear parallels, and these parallels are considerably more logical (hence, plausible?) than their quantum mechanical counterparts. Most scientists dealing with quantum mechanics have had relatively little exposure to modern nonlinear dynamics and its extreme manifestation, chaos theory. If they examine it closely, they will find that superficially it is every bit as perplexing and counterintuitive as quantum theory; yet, its peculiarities develop out of the strict logic of physics rather than more ephemeral philosophical arguments. In this it is closer in spirit to relativity than to quantum mechanics.

2. Quantum paradoxes having nonlinear parallels
The following so-called paradoxes found in quantum mechanics have parallel explanations in nonlinear dynamics, usually in a realm on the verge of chaos or actually in a chaotic region. Some of them are clearer and better understood than others, so here I list them in descending order of understanding or plausibility. Because of space limitations, I cover them rather briefly and suggest that the reader examine the extensive references found in Ref. [2] for greater details. The important fact to glean here is that classical systems exhibit these properties, properties that are commonly associated only with quantum systems.

• Exponential decay laws. Exponential decay laws for radioactive decay or for any other first-order quantum kinetics process were one of the earliest quantum mysteries encountered. It is impossible to determine ahead of time exactly when an individual radioactive nucleus will decay; yet, given a statistically significant number of such nuclei, one can be confident that they will decay according to a well-behaved exponential decay law, exhibiting a well-defined half-life. The statistical nature of their decay is often likened to actuarial statistics: The mortality of an individual can be difficult to predict; yet, again given a statistically significant number of people, insurance companies can be confident of well-behaved statistical economics. However, the analogy is completely false. Mortality rates depend on a myriad of complex factors — different people, different life styles, different diseases, etc. On the other hand, one of the most
basic tenets of quantum mechanics is the indistinguishability of individual species — a radioactive $^{22}\text{Na}$ nucleus is a radioactive $^{22}\text{Na}$ nucleus...! Why, then, should there be any differences in the time required for different nuclei to decay?

Chaos theory provides a relatively straightforward answer involving the “butterfly effect,” the exponential sensitivity of chaotic systems to minute differences in initial conditions. It is quite well known that differences in initial conditions do exist — the Uncertainty Principle involving energy and time demonstrates one aspect of this, with $\Gamma$, the width in energy of a state varying inversely with its lifetime. One can “derive” an exponential decay law in various ways with chaos theory, but perhaps the simplest is to iterate a map such as the quadratic or logistic map. (This is carried out in considerable detail in Refs. [2] and [3].) With such a map in a well-defined chaotic region, one can iterate and follow points selected from a miniscule initial interval (corresponding to the initial nuclear state) until they “escape” into a second interval (the final state); the process of iteration corresponds to physical processes such as the oscillations of a dipole for $\gamma$ decay or to the collisions of an $\alpha$ particle with a coulomb barrier. Iterating 10,000 randomly chosen points from an initial interval having a width of $10^{-11}$, then plotting the survivors against time (iteration number) produces a well-defined exponential curve. And the slight variation in initial conditions is consistent with the Uncertainty Principle.

• **Bell’s inequality and conditional probabilities.** Bell’s theorem and the various inequalities built around it have been very active fields of research, both theoretically and experimentally, during recent decades, with extreme advocates both for and against the validity and interpretations of Bell’s theorem. In particular, the series of Swedish conferences on “Quantum Theory: Reconsideration of Foundations” has attracted anti-Bell critics [4], whereas the Vienna school, centered around experimental tests of Bell-type inequalities, has strongly advocated pushing nonlocal interpretations to the limit [5,6]. As an example of “anti-Bell” reasoning, Hess and Philipp [7] make an excellent case for the irrelevance of nonrelativistic Bell-type inequalities. In much of the latter part of this paper, however, I take a different approach, examining the possible misuse (or misinterpretation) of conditional probabilities.

• **Quantization itself and preferred eigenstates.** The whole concept of emergent systems is based on self-organization. Even many simple nonlinear systems are “self-quantized,” i.e., have preferred oscillation modes, and their dynamics are governed by eigenvalue equations. Their phase space is characterized by separate (sometimes complicated or “riddled”) basins of attraction for these quantized modes [8]. In addition, most Hamiltonian nonlinear, chaotic systems are characterized by islands of stable (periodic or quasi-periodic) motion in a sea of otherwise chaotic motion. Thus, mesoscopic and macroscopic strictly classical systems can exhibit behavior strikingly similar to that of quantum systems.

In addition, during the past year Chadhury et al. [9] published the first experimental evidence for chaotic behavior in a clearly quantum system, further blurring the distinction between classical and quantum realms. Using an ensemble of supercooled $^{133}\text{Cs}$ atoms in the $F = 3$ hyperfine ground state, they performed kicked-top experiments. A magnetic field induced spin about the $y$-axis, while laser pulses produced kicked twists about the $x$-axis. Because spin magnitude is conserved, the phase space for this experiment is a spherical surface, in which islands of regularity are separated by a sea of chaos. Each initial state was followed for 40 periods of the kicked-top Hamiltonian. They found that initial states lying in an island of regularity were relatively stable to perturbations, although tunneling did occur between islands. Conversely, initial states lying in the sea of chaos were extremely sensitive to perturbations. Also, there was very little mixing between regular, periodic orbits and chaotic orbits. The implication is that here the quantum regime respects “classical” boundaries that were previously thought to apply only to mesoscopic or macroscopic systems.

• **Spontaneous symmetry breaking — nonconservation of parity.** Spontaneous symmetry breaking is a common feature of nonlinear systems. It has been thoroughly studied
through the iteration of odd-order maps, including the cubic and sine maps. Shinbrot and Muzzio [10] give some vivid examples of this, showing the separation of mixed powders in a nonlinear tumbler into bands and/or their components.

- **Barrier penetration.** Although not emphasized in most elementary physics books, barrier penetration is a fairly common phenomenon in classical optics. For example, when light is beamed into a glass plate, striking its far surface at an angle greater than the critical angle, it is totally reflected. However, if a second plate is brought close to the first, some “extraordinary” light penetrates through the back surface into the second plate — barrier penetration. And it obeys the same type of exponential relation to barrier height and width as is common in quantum systems, such as $\alpha$ decay.

- **Duality and wave versus particle behavior.** This is currently more speculative than the previous parallels, but there are intriguing similarities between chaotic scattering and wave-like behavior. Bleher, Grebogi, and Ott [11] give an excellent overview of this in the initial article for an entire issue of *Physica D*, and an even clearer exposition is given by José and Saletan [12] in their book, *Classical Dynamics: A Contemporary Approach*. Chaotic scattering produces patterns that look remarkably like diffraction patterns. This comes about partly because of the close association of windows of (periodic) stability within chaotic regions of phase space. Thus, it seems possible to obtain “constructive interference” within regions of stability as opposed to “destructive interference” within the chaotic regions. The double-slit experiment is often touted as the very epitome of quantum puzzlement; yet, the double-slit experiment is commonly presented in a much-oversimplified geometrical optics picture. Remember that even a single-slit produces some diffraction if examined closely enough, so, if the double-slit experiment were re-examined on a small-enough (microscopic or atomic?) scale, it is quite conceivable that one might obtain chaotic scattering. However, at present this remains a speculative task for future work.

- **Decoherence and reduction of wave-functions.** Most nonlinear Hamiltonian systems not only contain islands of stability scattered about in a sea of chaos, but also these islands often have a knife-edged stability — the very slightest perturbation can and will drive them into dissipative behavior. An elaborate theory of this “collapse” has been developed, involving the orderly disintegration of KAM tori [13], with the “most irrational” becoming the last to decohere. It is worth noting that in 1917 Einstein touched upon the rather similar problem of how to quantize what are now known as nonintegrable (chaotic) systems [14].

In short, nonlinear classical systems are now known to behave in seemingly paradoxical manners in many ways analogous to how quantum systems behave. It is thus short-sighted to emphasize the “peculiar” behavior of quantum systems while treating all classical mechanics as the well-behaved linear systems depicted in textbooks. Of course, arguments by analogy can be dangerous; however, the fact that certain so-called quantum paradoxes do have analogs in nonlinear classical systems should raise the question as to whether quantum and classical systems are so diametrically opposite and incompatible as is so commonly touted.

3. Bell’s inequality and conditional probabilities.

As mentioned above, in recent years there has been an enormous amount of interest in Bell’s theorem and Bell-type inequalities. Several dozen elegant and intricate experiments have been performed to test Bell-type inequalities, and in essentially all cases (except for some stumbling in the earliest attempts), these experiments have evinced the greater statistical correlations allowed by quantum mechanics rather than the lesser correlations of “classical” mechanics. Interpretations have varied, but the most common inferences have centered around eliminating the *local reality* defined by Einstein, Podolsky, and Rosen in their 1935 EPR paper [15]. Any extension of quantum theory involving *hidden variables* [16] must be non-local; otherwise, nature must not have a well-defined, predictable existence independent of and existing before
measurement. In other words, Schrödinger’s cat arguments must make sense on the microscopic scale even though they were designed to be nonsense on the macroscopic scale! — and Einstein’s “spooky action-at-a-distance” must be an actuality. Zeilinger’s new book, Dance of the Photons [6], gives an excellent summary of this extreme point of view.

Most of the experimental tests have involved the CHSH inequality [17], which is a more specific, experimentally friendly version of Bell’s original inequality. It involves generating a pair of entangled particles (e.g., a pair of electrons with their spins in the singlet state), then sending one each to Alice and Bob, the stereotypical quantum information experimentalists separated at an incommunicado (effectively infinite) distance. The entangled pair has an overall spin, $J = 0$, but the spin of each individual particle is undefined — the particles have not yet “determined” their individual spins, in the sense of the Copenhagen interpretation. Alice can measure her particle’s spin with respect to, say, a vertical (measurement $Q$) or a horizontal (measurement $R$) axis, getting a result of up or down (+1 or −1) with respect to her axis of measurement. Similarly, Bob can make measurement $S$ or $T$ on his particle. After accruing a statistically meaningful number of measurements, Alice and Bob get together to compare notes. I show a straightforward derivation of the CHSH inequality in Refs. [2,3], with the result:

$$E(QS) + E(RS) + E(RT) - E(QT) \leq 2$$

This is the inequality they use for the comparison. Each of the measurement can be +1 or −1, and the efficiencies ($E$) lower the correlations. This is the so-called classical CHSH inequality. For the quantum case, we start with the particles in the Bell singlet state:

$$|\Psi> = \frac{|01> - |10>}{\sqrt{2}}$$

Working out the expectation values, using the Pauli spin matrices, we get a higher degree of correlation:

$$<QS> + <RS> + <RT> - <QT> = 2\sqrt{2}$$

Most of the experiments have involved polarized photons, but the basic math is the same. And essentially all of the experiments have vindicated quantum mechanics.

As mentioned earlier, I see nothing wrong with the quantum mechanical predictions, although that has most often been the point of attack for those who wish to discredit this sort of Bell-type arguments. Instead, I would point out that implicit in the so-called classical side is the idea of independent — not entangled! — particles. This can be seen most clearly by going back to the original Bell inequality [18], which, after all, was derived from a classical viewpoint.

Stated simply, if the elements of a system can have three properties, $A$, $B$, and $C$, we can derive an inequality concerning having or not having these properties. Here we say that $N(A)$, $N(B)$, and $N(C)$ are the numbers of elements in the set that do have the properties, while $N(\overline{A})$, $N(\overline{B})$, and $N(\overline{C})$ are the numbers of elements that do not have the properties. Bell’s inequality thus is:

$$[N(A) + N(\overline{B})] + [N(B) + N(\overline{C})] \geq [N(A) + N(\overline{C})]$$

This can be seen most easily by using a diagram [19], as in Figure 1. Here the elements having each property are represented by the (overlapping) circles. For example, region 1 has property $A$ but not $B$ or $C$; region 2 has both $A$ and $C$ but not $B$, and region 3 has all three properties. It is rather clear from the diagram that the areas of the regions obey the inequality:

$$[1 + 2] + [7 + 4] \geq [1 + 4]$$

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Figure 1. Diagram to aid in demonstrating Bell’s inequality.

Figure 2. Diagram altered to demonstrate pairs of twins having different heights, hair colors, and eye colors.

To make the picture even more concrete, we can consider pairs of twins having different attributes. Figure 2 shows the same diagram for twins sorted according to height, hair color, and eye color [20]. (Note that short = not tall, brunet = not blond, and blue = not brown.) There are eight possible combinations:

- Tall, blue-eyed, blond
- Tall, blue-eyed, brunet
- Tall, brown-eyed, blond
- Tall, brown-eyed, brunet
- Short, blue-eyed, blond
- Short, blue-eyed, brunet
- Short, brown-eyed, blond
- Short, brown-eyed, brunet

It is obvious that the areas in Figure 2 add up in the same manner as those in Figure 1, resulting in a specific example for applying Bell’s inequality:

\[ [N(\text{tall}) + N(\text{brunet})] + [N(\text{blond}) + N(\text{blue})] \geq [N(\text{tall}) + N(\text{blue})] \]

Implicit in these diagrams is the fact that the properties are independent: Height, hair color, and eye color are independent variables. Note that the sizes of the sets and the degree of overlap do not affect the validity of the inequality, since it is the presence of the extra overlap areas on the left that force the inequality in the first place.

But what happens if the properties are not truly independent? Again, a concrete example perhaps can clarify matters. It is well known that many blue-eyed white cats are deaf. This
Figure 3. Diagram altered to show the conditional probabilities involved with blue-eyed white cats. There is a subtle link between blue-eyes and deafness, thought to involve a mutation on a specific gene. It is clear that the areas of overlap between blue eyes and deaf are not well defined, so one cannot apply the same straightforward reasoning here to obtain a variant of Bell’s inequality.

We are not innately responsive to conditional probabilities. A rather frivolous example of this can be found in the so-called Monty Hall paradox [21], named for the host of a long-gone television game show. It goes like this: There are three closed doors, behind two of which are goats and behind the third is a new sports car, but who knows which is which? The contestant picks one of the doors but does not open it. The host then opens one of the two remaining doors to reveal a goat. The question then is — should the contestant stick to his or her original choice, or should she or he switch to the remaining unopened door? Almost everyone will say it does not make any difference – the chances are one in three that he/she will get the car. This is not true. If the choice of which door to open were independent or random, it would make no difference. However, the host may not choose which door to open or do so at random. The conditional probability involved is that he must open the door that has the goat behind it. Here are the final odds: If the contestant first chose the door with the car behind it, it would make no difference whether she/he switched or not — the odds of this happening are one in three. However, had the contestant chosen first a door with a goat behind it, that would have forced the host to open the remaining door with a goat behind it, and it pays the contestant to switch. The odds of this are two in three, so switching doubles the chance of winning. Think about it for a moment. This rather trivial example contains the essence of conditional probabilities.

On a more serious note, classical nonlinear systems are known to exhibit correlations, ranging from the directions of particles in tornadoes to the distribution of energies in cosmic rays — and at times these can be great enough to overlap with quantum correlations. A more quantitative understanding of these can be had with the use of “nonextensive” thermodynamics, including nonextensive entropy introduced by Tsallis in 1988 [22]. He defined a generalized entropy:

$$S_q = \frac{1 - \sum_{i=1}^{W} p_i^q}{q - 1}$$

Here the phase space has been divided into $W$ cells of equal measure, with $p_i$ being the probability of occupying cell $i$. When the exponent (the “entropic index”), $q = 1$, this reduces to the standard Boltzmann entropy:

$$S_1 = -\sum_{i=1}^{W} p_i \ln p_i$$
The greater the difference of $q$ from 1, the greater the deviation from standard distributions, with apparent long-range correlations resulting in long tails in the distributions. When $q < 1$, the entropy of a combined system is greater (superextensive) than the sum of the individual entropies, and when $q > 1$, it is less. Thus, $q > 1$ indicates that a system has long-range correlations that interfere constructively, characteristic, e.g., of many emergent systems.

The concept of nonextensive entropy has found rather widespread applications in classical systems, especially in biological systems, and there have been several international conferences related to its use — see, e.g., Ref. [23] for examples. And it has been found that non-ergodic distributions in phase space can easily masquerade as long-range interactions. The quantitative expression for adding the entropies of two nonlinear systems includes feedback or interference between the systems as a cross-term:

$$\frac{S_q(A + B)}{k} = \frac{S_q(A)}{k} + \frac{S_q(B)}{k} + (1 - q)\left(\frac{S_q(A)}{k}\right)\left(\frac{S_q(B)}{k}\right)$$

In summary, one could almost as well use the term “entangled” for some interacting nonlinear classical systems as for quantum systems.

4. Conclusions.

Because of space limitations the arguments in this paper are necessarily somewhat sketchy. Also, I would emphasize once again that its intent is more to raise questions about the possibility of nonlinear correlations in quantum systems rather than to try to prove things quantitatively. Nevertheless, arguments about the fundamentals of quantum mechanics are far from over. The originators and early developers of quantum theory did not have access to modern nonlinear dynamics and chaos theory, so they were forced to deal with quantum ideosyncracies within a linear, if perturbative framework. The result has been beautiful insofar as current practical quantitative applications are concerned. However, the specter of nonlinear contributions could wreak havoc with some of the basic ideas involved with quantum computing, and even more so with more far-out ideas such as quantum teleportation.

Finally, perhaps both Einstein and Bohr were fundamentally correct, and their disagreement was only apparent. Chaos and nonlinear dynamics as applied to quantum mechanics has nothing to do with hidden variables, but it does provide the fundamental determinism so dear to Einstein’s heart. On the other hand, it can be interpreted only statistically, in line with Bohr’s Copenhagen school of thought. Perhaps it can provide a bridge over the chasm between the two sides.

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