Hulls of Cyclic Codes over $\mathbb{Z}_4$

Somphong Jitman, Ekkasit Sangwisut, and Patanee Udomkavanich

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Abstract

The hulls of linear and cyclic codes over finite fields have been of interest and extensively studied due to their wide applications. In this paper, the hulls of cyclic codes of length $n$ over the ring $\mathbb{Z}_4$ have been focused on. Their characterization has been established in terms of the generators viewed as ideals in the quotient ring $\mathbb{Z}_4[x]/(x^n - 1)$. An algorithm for computing the types of the hulls of cyclic codes of arbitrary odd length over $\mathbb{Z}_4$ has been given. The average 2-dimension $E(n)$ of the hulls of cyclic codes of odd length $n$ over $\mathbb{Z}_4$ has been established. A general formula for $E(n)$ has been provided together with its upper and lower bounds. It turns out that $E(n)$ grows the same rate as $n$.

Keywords: hulls, cyclic codes, reciprocal polynomials, average 2-dimension

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1 Introduction

The hull of a linear code, the intersection of the code and its dual, has been first introduced in [1] to classify finite projective planes. Properties of hulls have been extensively studied since the hull dimension is key to determine the complexity of algorithms for investigating permutation equivalence of two linear codes and calculating the automorphism of a fixed linear code given in [11, 12, 13, 18, 20, 21]. Precisely, most of the algorithms do not work if the hull dimension is large.

Recently, the hulls of linear codes have been applied in the construction of good entanglement-assisted quantum error correcting codes in [4]. Therefore, the study of the hulls and the hull dimensions of linear codes over finite fields has become of
interest. The number of distinct linear codes of length \( n \) over a finite field whose hulls share a given dimension has been established in [19] together with the average hull dimension of linear codes of length \( n \) over a finite field. In [23], this study has been extended to the class of cyclic codes over finite fields and the average hull dimension of cyclic codes has been determined. Later, the hull dimensions of cyclic and negacyclic codes and the number of cyclic codes whose hulls share a given dimension have been established in [17]. The average hull dimension of constacyclic codes over finite fields have been given in [8, 9, 10].

In the early history of coding theory, codes were usually taken over finite fields. In the last three decades, interest has been shown in linear codes over rings. In an important work [2, 5], it has been shown that the Kerdock codes, Preparata codes and Delsarte-Goethals codes can be obtained through the Gray images of linear codes over \( \mathbb{Z}_4 \). Some properties and applications of the hulls of linear codes over finite rings have been introduced and studied in [3, Chapter 5] and [6]. Most of the study of the hulls of codes over rings have been done in the two special cases where the hull is trivial (complementary dual code) in [14] and the hull equals the code itself (self-orthogonal code) in [16] and [22]. It is therefore of interest to investigate properties of the hulls of linear and cyclic codes over rings for arbitrary cases.

In this paper, we focus on the hulls of cyclic codes of odd length \( n \) over \( \mathbb{Z}_4 \). The characterization of the hulls of cyclic codes over \( \mathbb{Z}_4 \) is given in terms of their generators viewed as ideals in \( \mathbb{Z}_4[x]/(x^n - 1) \). Based on this characterization, the types of the hulls of cyclic codes of odd length \( n \) over \( \mathbb{Z}_4 \) are determined. Furthermore, the average 2-dimension and its upper and lower bounds are derived.

The paper is organized as follows. In Section 2, basic concepts and key results on cyclic codes over \( \mathbb{Z}_4 \) are recalled. In Section 3, the characterization of the hull of cyclic codes of odd length \( n \) over \( \mathbb{Z}_4 \) is given in terms of their generators. Subsequently, the types of the hulls of such codes are determined. The formula for the average 2-dimension of the hull of cyclic codes of odd length \( n \) over \( \mathbb{Z}_4 \) is derived in Section 4. In Section 5, upper and lower bounds on \( E(n) \) are given together with asymptotic behaviors of \( E(n) \).

## 2 Preliminaries

In this section, definitions and preliminary results required in the study of the hulls of cyclic codes over \( \mathbb{Z}_4 \) are recalled. Precisely, properties of linear codes, hulls of codes, cyclic codes and polynomials over \( \mathbb{Z}_4 \) are discussed.

### 2.1 Linear Codes and Hulls over \( \mathbb{Z}_4 \)

A linear code \( C \) of length \( n \) over \( \mathbb{Z}_4 \) is defined to be a submodule of the \( \mathbb{Z}_4 \)-module \( \mathbb{Z}_4^n \). As a linear code \( C \) of length \( n \) over \( \mathbb{Z}_4 \) can be viewed as a vector space over \( \mathbb{F}_2 \), the concept of 2-dimension of \( C \) was introduced in [24] to be \( \dim_2(C) = \log_2(|C|) \). Elements \( u = (u_0, u_1, \ldots, u_{n-1}) \) and \( v = (v_0, v_1, \ldots, v_{n-1}) \) in \( \mathbb{Z}_4^n \) are said to be orthogonal if and only if \( \sum_{i=0}^{n-1} u_i v_i = 0 \). Subsets \( U \) and \( V \) of \( \mathbb{Z}_4^n \) are said to be orthogonal if
$u$ is orthogonal to $v$ for all $u \in U$ and $v \in V$. The dual of a linear code $C$ of length $n$ over $\mathbb{Z}_4$ is defined to be the linear code

$$C^\perp = \{ u \in \mathbb{Z}_4^n \mid u \text{ is orthogonal to } c \text{ for all } c \in C \}.$$ 

The hull of a linear code $C$ is defined to be

$$\text{Hull}(C) = C \cap C^\perp.$$ 

2.2 Cyclic Codes over $\mathbb{Z}_4$

A linear code of length $n$ over $\mathbb{Z}_4$ is said to be cyclic if $(c_{n-1}, c_0, \ldots, c_{n-2}) \in C$ for all $(c_0, c_1, \ldots, c_{n-1}) \in C$. A vector $u = (u_0, u_1, \ldots, u_{n-1})$ in $\mathbb{Z}_4^n$ can be represented as its corresponding polynomial $u(x) = u_0 + u_1x + \cdots + u_{n-1}x^{n-1}$ in $\mathbb{Z}_4[x]$. It is well known that each cyclic code $C$ of length $n$ over $\mathbb{Z}_4$ can be viewed as an ideal of the quotient ring $R_n = \mathbb{Z}_4[x]/(x^n - 1)$ (see [7, Chapter 12]). Moreover, if $n$ is odd, the corresponding ideal of a cyclic code $C$ has generators of the form

$$\langle f(x)g(x), 2f(x)h(x) \rangle = \langle f(x)g(x), 2f(x) \rangle,$$

where $f(x), g(x), h(x)$ are unique monic pairwise coprime polynomials such that $x^n - 1 = f(x)g(x)h(x)$ (see [7, Theorem 12.3.13]). Furthermore, $|C| = 4^{\deg f(x)\deg g(x)}$.

Let $f(x) = a_0 + a_1x + \cdots + a_{k-1}x^{k-1} + x^k \in \mathbb{Z}_4[x]$ (resp., $\mathbb{F}_2[x]$) be a monic polynomial such that $a_0$ is a unit in $\mathbb{Z}_4$ (resp., $\mathbb{F}_2$). The reciprocal polynomial of $f(x)$ is defined to be

$$f^\ast(x) = a_0^{-1}x^{\deg f(x)}f \left( \frac{1}{x} \right).$$

Clearly, $(f^\ast)^\ast(x) = f(x)$. Therefore, there are two types of monic polynomials in $\mathbb{Z}_4[x]$ (resp., $\mathbb{F}_2[x]$) whose constant terms are units. A polynomial $f(x)$ is called self-reciprocal if $f(x) = f^\ast(x)$. Otherwise, $f(x)$ and $f^\ast(x)$ are called a reciprocal polynomial pair. Note that $f(x)g(x)h(x) = x^n - 1 = (x^n - 1)^\ast = f^\ast(x)g^\ast(x)h^\ast(x)$.

For a cyclic code $C$ of length $n$ over $\mathbb{Z}_4$ generated by $\langle f(x)g(x), 2f(x) \rangle$, the dual $C^\perp$ is generated by

$$\langle h^\ast(x)g^\ast(x), 2h^\ast(x)f^\ast(x) \rangle = \langle h^\ast(x)g^\ast(x), 2h^\ast(x) \rangle$$

(see [7, Theorem 12.3.20]).

For a positive integer $n$, let $C(n, 4)$ denote the set of all cyclic codes of length $n$ over $\mathbb{Z}_4$. The average 2-dimension of the hull of cyclic codes of length $n$ over $\mathbb{Z}_4$ is defined to be

$$E(n) = \sum_{C \in C(n, 4)} \frac{\dim_2(\text{Hull}(C))}{|C(n, 4)|}.$$ 

Properties of the average 2-dimension $E(n)$ of the hull of cyclic codes of length $n$ over $\mathbb{Z}_4$ are studied in Sections 4–5.
2.3 Factorization of $x^n - 1$ over $\mathbb{Z}_4$

In this subsection, the factorization of $x^n - 1$ over $\mathbb{Z}_4$ for odd positive integers $n$ is recalled. Let $\mu : \mathbb{Z}_4[x] \to \mathbb{F}_2[x]$ be a map defined by $\mu(0) = 0 = \mu(2)$, $\mu(1) = 1 = \mu(3)$ and $\mu(x) = x$. It follows that $\mu(x^n - 1) = x^n - 1$.

For coprime positive integers $i$ and $j$, let $\operatorname{ord}_j(i)$ denote the multiplicative order of $i$ modulo $j$. Let $N_2 := \{ \ell \geq 1 : \ell \text{ divides } 2^i + 1 \text{ for some positive integer } i \}$. From [17], the factorization of $x^n - 1$ in $\mathbb{F}_2[x]$ is of the form

$$x^n - 1 = \prod_{j \in N_2} \left( \prod_{i = 1}^{\gamma(j)} h_{ij}(x) \right) \prod_{j \in N_2} \left( \prod_{i = 1}^{\beta(j)} k_{ij}(x)k_j^*(x) \right),$$

where

$$\gamma(j) = \frac{\phi(j)}{\operatorname{ord}_j(2)}, \quad \beta(j) = \frac{\phi(j)}{2 \operatorname{ord}_j(2)},$$

$k_{ij}(x)$ and $k_j^*(x)$ form a monic irreducible reciprocal polynomial pair of degree $\operatorname{ord}_j(2)$ and $h_{ij}(x)$ is a monic irreducible self-reciprocal polynomial of degree $\operatorname{ord}_j(2)$.

By Hensel’s lift (see [7, Theorem 12.3.7]), the factorization of $x^n - 1$ in $\mathbb{Z}_4[x]$ is

$$x^n - 1 = \prod_{j \in N_2} \left( \prod_{i = 1}^{\gamma(j)} g_{ij}(x) \right) \prod_{j \in N_2} \left( \prod_{i = 1}^{\beta(j)} f_{ij}(x)f_j^*(x) \right)$$

$$= \prod_{j = 1}^{s} g_j(x) \prod_{j = 1}^{t} f_j(x)f_j^*(x),$$

where $f_{ij}(x), f_j^*(x)$ form a monic basic irreducible reciprocal polynomial pair and $g_{ij}(x)$ is a monic basic irreducible self-reciprocal polynomial,

$$s = \sum_{j \in N_2} \frac{\phi(j)}{\operatorname{ord}_j(2)}$$

is the number of a monic basic irreducible self-reciprocal polynomial in the factorization of $x^n - 1$, and

$$t = \sum_{j \in N_2} \frac{\phi(j)}{2 \operatorname{ord}_j(2)}$$

is the number of a monic basic irreducible reciprocal polynomial pair in the factorization of $x^n - 1$. Moreover, $\mu(g_{ij}(x)) = h_{ij}(x), \mu(f_{ij}(x)) = k_{ij}(x)$ and $\mu(f_j^*(x)) = k_j^*(x)$.

Let $B_n = \deg \prod_{j \in N_2} \left( \prod_{i = 1}^{\gamma(j)} g_{ij}(x) \right)$. Then

$$B_n = \deg \prod_{j \in N_2} \left( \prod_{i = 1}^{\gamma(j)} g_{ij}(x) \right) = \sum_{j \in N_2} \frac{\phi(j)}{\operatorname{ord}_j(2)} \cdot \operatorname{ord}_j(2) = \sum_{j \in N_2} \phi(j).$$

The number $B_n$ plays an important role in the study of the average 2-dimension of the hull of cyclic codes over $\mathbb{Z}_4$ in Sections 4-5.
3 Hulls of Cyclic Codes over $\mathbb{Z}_4$

In this section, properties of the hulls of cyclic codes of arbitrary odd lengths over $\mathbb{Z}_4$ are focused on. From now on, assume that $n$ is an odd positive integer. The characterization of the hulls of cyclic codes of length $n$ over $\mathbb{Z}_4$ is given in terms of their generators in Subsection 3.1. Subsequently, the types of the hulls of cyclic codes of length $n$ over $\mathbb{Z}_4$ are determined Subsection 3.2.

3.1 Characterization of the Hulls of Cyclic Codes over $\mathbb{Z}_4$

Here, we focus on algebraic structures of the hulls of cyclic codes of odd length $n$ over $\mathbb{Z}_4$. The following lemma is useful in the study of their generators.

**Lemma 3.1 ([7], Theorem 12.3.18).** Let $u = (u_0, u_1, \ldots, u_{n-1})$ and $v = (v_0, v_1, \ldots, v_{n-1})$ be vectors in $\mathbb{Z}_4^n$ with corresponding polynomial $u(x)$ and $v(x)$, respectively. Then $u$ is orthogonal to $v$ and all its shifts if and only if $u(x)v^*(x) = 0$ in $\mathbb{Z}_4[x]/(x^n - 1)$.

The generators of the hull of a cyclic code is determined as follows.

**Theorem 3.2.** Let $C$ be a cyclic code of odd length $n$ over $\mathbb{Z}_4$ generated by

$$\langle f(x)g(x), 2f(x) \rangle,$$

where $x^n - 1 = f(x)g(x)h(x)$ and $f(x)$, $g(x)$ and $h(x)$ are pairwise coprime. Then $\text{Hull}(C)$ is generated by

$$\langle \text{lcm}(f(x)g(x), h^*(x)g^*(x)), 2 \text{lcm}(f(x), h^*(x)) \rangle.$$

Furthermore, $\text{Hull}(C)$ is of type $4^{\text{deg}H(x)/\text{deg}G(x)}$, where

$$H(x) = \gcd(h(x), f^*(x)) \quad \text{and} \quad G(x) = \frac{x^n - 1}{\gcd(h(x), f^*(x)) \cdot \text{lcm}(f(x), h^*(x))}.$$

**Proof.** From Eq. (i), note that $C^\perp$ is generated by

$$\langle h^*(x)g^*(x), 2h^*(x) \rangle.$$

Let $C'$ be a cyclic code of length $n$ over $\mathbb{Z}_4$ whose generators are of the form

$$\langle F(x)G(x), 2F(x) \rangle,$$

where

$$F(x) = \text{lcm}(f(x), h^*(x)),$$

$$G(x) = \frac{\text{lcm}(f(x)g(x), h^*(x)g^*(x))}{\text{lcm}(f(x), h^*(x))} = \frac{x^n - 1}{\gcd(h(x), f^*(x)) \cdot \text{lcm}(f(x), h^*(x))}$$

and

$$H(x) = \frac{x^n - 1}{\text{lcm}(f(x)g(x), h^*(x)g^*(x))} = \gcd(h(x), f^*(x)).$$
It is not difficult to see that \( x^n - 1 = F(x)G(x)H(x) \) and the polynomials \( F(x), G(x) \) and \( H(x) \) are pairwise coprime. Since \( \langle F(x)G(x), 2F(x) \rangle \subseteq \langle f(x)g(x), 2f(x) \rangle \) and \( \langle F(x)G(x), 2F(x) \rangle \subseteq \langle h'(x)g'(x), 2h'(x) \rangle \), we have \( C' \subseteq \mathrm{Hull}(C) \).

Next, we show that \( \mathrm{Hull}(C) \subseteq C' \). Since \( \mathrm{Hull}(C) \) is a cyclic code of length \( n \) over \( \mathbb{Z}_4 \), assume that \( \mathrm{Hull}(C) \) has generators of the form \( \langle A(x)B(x), 2A(x) \rangle \) where \( x^n - 1 = A(x)B(x)C(x) \) and the polynomials \( A(x), B(x) \) and \( C(x) \) are pairwise coprime. Since \( \mathrm{Hull}(C) \subseteq C' \) is orthogonal to \( C \), by Lemma \[3.1\] we have
\[
A(x)B(x) \cdot 2f'(x) = 0 \quad \text{and} \quad 2A(x) \cdot f'(x)g'(x) = 0
\]
which imply that \( h'(x)g'(x)|A(x)B(x) \) and \( h'(x)|A(x) \).

Similarly, \( \mathrm{Hull}(C) \subseteq C \) is orthogonal to \( C' \) which implies that
\[
A(x)B(x) \cdot 2h(x) = 0 \quad \text{and} \quad 2A(x) \cdot h(x)g(x) = 0
\]
by Lemma \[3.1\]. It follows that \( f(x)g(x)|A(x)B(x) \) and \( f(x)|A(x) \).

Consequently, \( \text{lcm}(f(x)g(x), h'(x)g'(x)|A(x)B(x) \) and \( \text{lcm}(h'(x), f(x)|A(x) \) which imply that \( F(x)G(x)|A(x)B(x) \) and \( F(x)|A(x) \). Hence, \( \mathrm{Hull}(C) \subseteq C' \). Therefore, \( \mathrm{Hull}(C) = C' \) as desired. \( \Box \)

An illustrative example of Theorem \[3.2\] is given as follows.

**Example 3.3.** In \( \mathbb{Z}_4[x] \), \( x^7 - 1 = (x - 1)(x^3 + 2x^2 + x - 1)(x^3 - x^2 + 2x - 1) \) is the factorization of \( x^7 - 1 \) into a product of monic basic irreducible polynomials. Let \( C \) be the cyclic code of length 7 over \( \mathbb{Z}_4 \) generated by
\[
\langle f(x)g(x), 2f(x) \rangle = \langle (x^3 + 2x^2 + x - 1)(x^3 - x^2 + 2x - 1), 2(x^3 + 2x^2 + x - 1) \rangle
\]
where \( f(x) = x^3 + 2x^2 + x - 1, g(x) = x^3 - x^2 + 2x - 1 \) and \( h(x) = x - 1 \). Moreover, \( f^*(x) = g(x) \) and \( h^*(x) = h(x) \). From Eq \[1\], \( C' \) is of the form
\[
\langle h^*(x)g^*(x), 2h^*(x) \rangle = \langle (x - 1)(x^3 + 2x^2 + x - 1), 2(x - 1) \rangle.
\]

By Theorem \[3.2\], \( \mathrm{Hull}(C) \) is of the form
\[
\langle \text{lcm}(f(x)g(x), h^*(x)g^*(x)), 2 \text{lcm}(f(x), h^*(x)) \rangle = \langle 2(x - 1)(x^3 + 2x^2 + x - 1) \rangle.
\]

### 3.2 Characterization of Cyclic Codes of the same Hull

For a given cyclic code \( D \) of odd length \( n \) over \( \mathbb{Z}_4 \), the cyclic codes of odd length \( n \) over \( \mathbb{Z}_4 \) whose hulls equal \( D \) are determined in this subsection.

**Theorem 3.4.** Let \( D \) be a cyclic code of odd length \( n \) over \( \mathbb{Z}_4 \) generated by
\[
\left\langle \prod_{j_{in}, j_{in} \in \mathbb{Z}_2} \gamma_{j_{in}}(x)^{g_{ij}} \prod_{j_{in}, j_{in} \in \mathbb{Z}_2} \beta_{j_{in}} \prod_{i=1}^{\beta_{j_{in}}} f_{ij}(x)^{b_{ij}} f_{ij}^*(x)^{c_{ij}},
2 \prod_{j_{in}, j_{in} \in \mathbb{Z}_2} \gamma_{j_{in}}(x)^{D_{ij}} \prod_{j_{in}, j_{in} \in \mathbb{Z}_2} \beta_{j_{in}} \prod_{i=1}^{\beta_{j_{in}}} f_{ij}(x)^{E_{ij}} f_{ij}^*(x)^{F_{ij}} \right\rangle,
\]
where \( g_{ij}(x) \) and \( f_{ij}(x) \) are given in Eq (2) and \( A_{ij}, B_{ij}, C_{ij}, D_{ij}, E_{ij}, F_{ij} \in \{0, 1\} \). Then the cyclic codes of length \( n \) over \( \mathbb{Z}_4 \) whose hulls equal \( D \) are generated by

\[
\left\langle \prod_{j_{n, j} \in N_2} g_{ij}(x)^{u_{ij}+b_{ij}} \prod_{j_{n, j} \in N_2} f_{ij}(x)^{v_{ij}+z_{ij}} \prod_{j_{n, j} \in N_2} f_{ij}(x)^{w_{ij}+d_{ij}}, \rightangle
\]

where \( (u_{ij}, b_{ij}) \in \{(0, 1) \} \) if \( D_{ij} = 0 \), \( \{(0, 0), (1, 0) \} \) if \( D_{ij} = 1 \), and

\[
(v_{ij}, z_{ij}, w_{ij}, d_{ij}) \in \begin{cases} 
(0, 0, 0, 0), (1, 0, 1, 0) & \text{if } (A_{ij}, B_{ij}, C_{ij}, E_{ij}, F_{ij}) = (1, 1, 1, 1), \\
(0, 1, 1, 0), (0, 0, 0, 1) & \text{if } (A_{ij}, B_{ij}, C_{ij}, E_{ij}, F_{ij}) = (1, 1, 1, 0), \\
(0, 1, 0, 0), (1, 0, 0, 1) & \text{if } (A_{ij}, B_{ij}, C_{ij}, E_{ij}, F_{ij}) = (1, 1, 0, 1), \\
(0, 0, 0) & \text{if } (A_{ij}, B_{ij}, C_{ij}, E_{ij}, F_{ij}) = (1, 0, 1, 0), \\
(0, 0, 1) & \text{if } (A_{ij}, B_{ij}, C_{ij}, E_{ij}, F_{ij}) = (0, 1, 1, 1), \\
(0, 1, 0, 1) & \text{if } (A_{ij}, B_{ij}, C_{ij}, E_{ij}, F_{ij}) = (1, 1, 1, 0)
\end{cases}
\]

for all \( i \) and \( j \). Otherwise, there are no cyclic codes of length \( n \) over \( \mathbb{Z}_4 \) whose hulls equal \( D \).

**Proof.** Let \( C \) be a cyclic code of odd length \( n \) over \( \mathbb{Z}_4 \) generated by \( \langle f(x)g(x), 2 f(x) \rangle \) where \( f(x) \) and \( g(x) \) are in Eqs (9) and (10). By Theorem 3.2 Hull(\( C \)) is generated by

\[
\langle \text{lcm}\left(f(x)g(x), h^*(x)g^*(x)\right), 2 \text{lcm}(f(x), h^*(x))\rangle = 
\left\langle \prod_{j_{n, j} \in N_2} g_{ij}(x)^{\max[u_{ij}+b_{ij}, 1-u_{ij}]} \prod_{j_{n, j} \in N_2} f_{ij}(x)^{\max[v_{ij}+z_{ij}, 1-w_{ij}]} f_{ij}(x)^{\max[w_{ij}+d_{ij}, 1-v_{ij}]} , \rightangle
\]

\[
2 \prod_{j_{n, j} \in N_2} g_{ij}(x)^{\max[u_{ij}+1-u_{ij}, 1-w_{ij}]} \prod_{j_{n, j} \in N_2} f_{ij}(x)^{\max[v_{ij}+1-w_{ij}]+w_{ij}+d_{ij}1-v_{ij}]} f_{ij}(x)^{\max[w_{ij}+1-v_{ij}]+z_{ij}]} , \rightangle
\]

where \( (u_{ij}, b_{ij}), (v_{ij}, z_{ij}), (w_{ij}, d_{ij}) \in \{(0, 0), (1, 0), (0, 1)\} \).

Comparing the coefficients in Eqs (7) and (8), we have \( A_{ij} = \max\{(u_{ij}+b_{ij}, 1-w_{ij})\} = 1 \) and \( D_{ij} = \max\{u_{ij}, 1-u_{ij}-b_{ij}\} \). Thus \( (u_{ij}, b_{ij}) = (0, 1) \) if \( D_{ij} = 0 \) and \( (u_{ij}, b_{ij}) \in \{(0, 0), (1, 0)\} \) if \( D_{ij} = 1 \).

From Eqs (7) and (8), it can be deduced that \( B_{ij} = \max\{v_{ij}+z_{ij}, 1-w_{ij}\} \), \( C_{ij} = \max\{w_{ij}+d_{ij}, 1-v_{ij}\} \), \( E_{ij} = \max\{v_{ij}, 1-w_{ij}+d_{ij}\} \) and \( F_{ij} = \max\{w_{ij}, 1-v_{ij}-z_{ij}\} \). All 9 values of \( (v_{ij}, z_{ij}, w_{ij}, d_{ij}) \) are illustrated in the following table together with their corresponding values \( (B_{ij}, C_{ij}, E_{ij}, F_{ij}) \).
Since there are only 9 possible values of \((v_{ij}, z_{ij}, w_{ij}, d_{ij})\), there are no cyclic codes of length \(n\) whose hulls equal \(D\) for the other values of \((A_{ij}, B_{ij}, C_{ij}, D_{ij}, E_{ij}, F_{ij})\).

Therefore, the proof is completed. □

### 3.3 Types and 2-Dimensions of Hulls of Cyclic Codes

In this subsection, the types of the hulls of cyclic codes of arbitrary odd length \(n\) over \(\mathbb{Z}_4\) are investigated. Moreover, an algorithm for finding the types of the hulls of cyclic codes of odd length \(n\) over \(\mathbb{Z}_4\) is given. Finally, the 2-dimensions of the hulls of cyclic codes of arbitrary odd length \(n\) over \(\mathbb{Z}_4\) are determined.

The types of the hulls of cyclic codes of odd length over \(\mathbb{Z}_4\) is derived in Theorem 3.6. The following lemma is required in its proof.

**Lemma 3.5.** Let \(\beta\) be a positive integer. For \(1 \leq i \leq \beta\), let \((v_{ij}, z_{ij}), (w_{ij}, d_{ij})\) and \((u_i, b_i)\) be elements in \(((0, 0), (1, 0), (0, 1))\). Let \(a_i = \min\{1 - v_i - z_i, w_i\} + \min\{1 - w_i - d_i, v_i\}\). Then \(a_i \in \{0, 1\}\). Moreover, the following statements hold.

1. \(2 - \min\{1 - v_i - z_i, w_i\} - \max\{v_i, 1 - w_i - d_i\} - \min\{1 - w_i - d_i, v_i\} - \max\{w_i, 1 - v_i - z_i\} = z_i + d_i\).

2. If \(a_i = 0\), then \(z_i + d_i \in \{0, 1, 2\}\).

3. If \(a_i = 1\), then \(z_i + d_i = 0\).

4. Let \(\alpha = \sum_{i=1}^{\beta} a_i\), then \(\sum_{i=1}^{\beta} (z_i + d_i) = c\) for some \(0 \leq c \leq 2(\beta - \alpha)\).

**Proof.** To prove Statement 1, let \(* = 1 - v_i - z_i, \circ = 1 - w_i - d_i\) and \(\bigcirc = 2 - \min\{*, w_i\} - \max\{v_i, \circ\} - \min\{v_i, \circ\} - \max\{*, w_i\}\). Consider the following table, it can be concluded that

\[
2 - \min\{*, w_i\} - \max\{*, w_i\} - \min\{v_i, \circ\} - \max\{v_i, \circ\} = \bigcirc = z_i + d_i.
\]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
v_i & z_i & * & w_i & d_i & \bigcirc \\
\hline
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 2 & 2 \\
\hline
\end{array}
\]
Clearly, \( a_i \) can be either 0 or 1. Statements 2 and 3 can be easily obtained by considering the possible cases.

To prove Statement 4, assume that \( a = \sum_{i=1}^{\beta} a_i \). Without loss of generality, we assume that \( a_i = 1 \) for all \( i = 1, \ldots, a \). Thus \( z_i + d_i = 0 \) for all \( i = 1, \ldots, a \) and

\[
\sum_{i=1}^{\beta} (z_i + d_i) = \left( \sum_{i=1}^{a} z_i + d_i \right) + \left( \sum_{i=a+1}^{\beta} z_i + d_i \right) = 0 + \left( \sum_{i=a+1}^{\beta} z_i + d_i \right) = c,
\]

where \( 0 \leq c \leq 2 (\beta - a) \).

**Theorem 3.6.** Let \( n \) be an odd positive integer. Then the types of the hull of a cyclic code of length \( n \) over \( \mathbb{Z}_4 \) are of the form \( 4^{k_1} 2^{k_2} \), where

\[
k_1 = \sum_{j_n, j \in N_2} \text{ord}_j(2) \cdot a_j \quad \text{and} \quad k_2 = \sum_{j_n, j \in N_2} \text{ord}_j(2) \cdot b_j + \sum_{j_n, j \in N_2} \text{ord}_j(2) \cdot c_j,
\]

\( 0 \leq a_j \leq \gamma(j), \ 0 \leq b_j \leq \gamma(j) \) and \( 0 \leq c_j \leq 2(\beta(j) - a_j) \).

**Proof.** Let \( C \) be a cyclic code of length \( n \) over \( \mathbb{Z}_4 \) generated by \( \langle f(x)g(x), 2f(x) \rangle \), where \( f(x) \), \( g(x) \) and \( h(x) \) are monic polynomials such that \( x^d - 1 = f(x)g(x)h(x) \). By Theorem 3.2, Hull\((C)\) has type \( 4^{\deg H(x)} 2^{\deg G(x)} \), where \( H(x) \) and \( G(x) \) are defined as in Theorem 3.2. By Eq (2), we have

\[
f(x) = \prod_{j_n, j \in N_2} \prod_{i=1}^{\gamma(j)} g_{ij}(x)^{u_{ij}} \prod_{j_n, j \in N_2} \prod_{i=1}^{\beta(j)} f_{ij}(x)^{v_{ij} + f_{ij}(x)^{w_{ij}}},
\]

(9)

\[
g(x) = \prod_{j_n, j \in N_2} \prod_{i=1}^{\gamma(j)} g_{ij}(x)^{b_{ij}} \prod_{j_n, j \in N_2} \prod_{i=1}^{\beta(j)} f_{ij}(x)^{z_{ij} + f_{ij}(x)^{d_{ij}}},
\]

(10)

\[
h(x) = \prod_{j_n, j \in N_2} \prod_{i=1}^{\gamma(j)} g_{ij}(x)^{1-u_{ij} - b_{ij}} \prod_{j_n, j \in N_2} \prod_{i=1}^{\beta(j)} f_{ij}(x)^{1-v_{ij} - z_{ij} + f_{ij}(x)^{1-w_{ij} - d_{ij}}},
\]

(11)

\[
f^*(x) = \prod_{j_n, j \in N_2} \prod_{i=1}^{\gamma(j)} g_{ij}(x)^{u_{ij}} \prod_{j_n, j \in N_2} \prod_{i=1}^{\beta(j)} f_{ij}(x)^{v_{ij} + f_{ij}(x)^{w_{ij}}},
\]

\[
g^*(x) = \prod_{j_n, j \in N_2} \prod_{i=1}^{\gamma(j)} g_{ij}(x)^{b_{ij}} \prod_{j_n, j \in N_2} \prod_{i=1}^{\beta(j)} f_{ij}(x)^{d_{ij} + f_{ij}(x)^{z_{ij}}},
\]

\[
h^*(x) = \prod_{j_n, j \in N_2} \prod_{i=1}^{\gamma(j)} g_{ij}(x)^{1-u_{ij} - b_{ij}} \prod_{j_n, j \in N_2} \prod_{i=1}^{\beta(j)} f_{ij}(x)^{1-v_{ij} - d_{ij} + f_{ij}(x)^{1-w_{ij} - z_{ij}}},
\]

where \((u_{ij}, b_{ij}), (v_{ij}, z_{ij}), (w_{ij}, d_{ij}) \in \{(0, 0), (1, 0), (0, 1)\} \).
First we determine \( \deg H(x) \). Observe that

\[
H(x) = \gcd(h(x), f^*(x))
\]

\[
= \prod_{j \in \mathbb{N}_2} \prod_{i=1}^{\gamma(j)} g_{ij}(x)^{\min\{1-u_i-b_j, w_i\}}
\]

\[
\times \prod_{j \in \mathbb{N}_2} \prod_{i=1}^{\beta(j)} f_{ij}(x)^{\min\{1-v_{ij}, z_{ij}, w_{ij}\}} f^*_{ij}(x)^{\min\{1-w_i-d_j, v_i\}}
\]

\[
= \prod_{j \in \mathbb{N}_2} \prod_{i=1}^{\beta(j)} f_{ij}(x)^{\min\{1-v_{ij}, z_{ij}, w_{ij}\}} f^*_{ij}(x)^{\min\{1-w_i-d_j, v_i\}}
\]

which implies that

\[
\deg H(x) = \deg \gcd(h(x), f^*(x))
\]

\[
= \deg \prod_{j \in \mathbb{N}_2} \prod_{i=1}^{\beta(j)} f_{ij}(x)^{\min\{1-v_{ij}, z_{ij}, w_{ij}\}} f^*_{ij}(x)^{\min\{1-w_i-d_j, v_i\}}
\]

\[
= \sum_{j \in \mathbb{N}_2} \text{ord}_j(2) \sum_{i=1}^{\beta(j)} \left( \min\{1 - v_{ij} - z_{ij}, w_{ij}\} + \min\{1 - w_i - d_j, v_i\} \right) \tag{12}
\]

\[
= \sum_{j \in \mathbb{N}_2} \text{ord}_j(2) \sum_{i=1}^{\beta(j)} a_{ij}, \text{ where } 0 \leq a_{ij} \leq 1.
\]

\[
= \sum_{j \in \mathbb{N}_2} \text{ord}_j(2) \cdot a_j, \text{ where } 0 \leq a_j \leq \beta(j).
\]

Next we compute \( \deg G(x) \). Since

\[
lcm(f(x), h^*(x)) = \prod_{j \in \mathbb{N}_2} \prod_{i=1}^{\gamma(j)} g_{ij}(x)^{\max\{u_i, 1-u_i-b_j\}}
\]

\[
\times \prod_{j \in \mathbb{N}_2} \prod_{i=1}^{\beta(j)} f_{ij}(x)^{\max\{v_{ij}, 1-w_i-d_j\}} f^*_{ij}(x)^{\max\{w_i, 1-v_i-z_{ij}\}}
\]

and

\[
\gcd(h(x), f^*(x)) \cdot \lcm(f(x), h^*(x)) = \prod_{j \in \mathbb{N}_2} \prod_{i=1}^{\gamma(j)} g_{ij}(x)^{\max\{u_i, 1-u_i-b_j\}}
\]

\[
\times \prod_{j \in \mathbb{N}_2} \prod_{i=1}^{\beta(j)} f_{ij}(x)^{\min\{1-v_{ij}-z_{ij}, w_{ij}\} + \max\{v_{ij}, 1-w_i-d_j\}} f^*_{ij}(x)^{\min\{1-w_i-d_j, v_i\} + \max\{w_i, 1-v_i-z_{ij}\}},
\]
of a cyclic code of length \( n \) over \( \mathbb{Z} \).

Let \( n \) be an odd integer such that \( n \) is of the form \( 4^k 2^i \).

Corollary 3.7. Let \( n \) be an odd integer such that \( n \in N_2 \). Then the types of the hull of a cyclic code of length \( n \) over \( \mathbb{Z}_4 \) are of the form \( 4^0 2^{k_2} \), where

\[
k_2 = \sum_{j \in N_2} \text{ord}_j(2) \cdot b_j, \quad 0 \leq b_j \leq \gamma(j).
\]

**Proof.** Since \( n \in N_2 \), we have \( j \in N_2 \) for all \( j|n \). Hence, the result follows. \( \square \)

An algorithm for computing the types of the hull of a cyclic code of odd length \( n \) over \( \mathbb{Z}_4 \) is given as follows.

---

**Algorithm:** The types of the hull of a cyclic code of odd length \( n \) over \( \mathbb{Z}_4 \)
1. For each \( j \mid n \), consider the following cases.
   
   (a) If \( j \in N_2 \), then compute \( \text{ord}_j(2) \) and \( \gamma(j) \).
   
   (b) If \( j \notin N_2 \), then compute \( \text{ord}_j(2) \) and \( \beta(j) \).

2. Compute
   \[
   k_1 = \sum_{j \mid n, j \notin N_2} \text{ord}_j(2) \cdot a_j \text{, where } 0 \leq a_j \leq \beta(j).
   \]

3. For a fixed \( a_j \) in 2, compute
   \[
   k_2 = \sum_{j \mid n, j \in N_2} \text{ord}_j(2) \cdot b_j + \sum_{j \mid n, j \notin N_2} \text{ord}_j(2) \cdot c_j,
   \]
   where \( 0 \leq b_j \leq \gamma(j) \) and \( 0 \leq c_j \leq 2(\beta(j) - a_j) \).

Illustrative examples of the above algorithm are given as follows.

**Example 3.8.** Let \( n = 7 \). The types of the hulls of cyclic codes of length 7 over \( \mathbb{Z}_4 \) are determined in the following steps.

1. The divisors of 7 are 1 and 7.
   
   (a) 1 \( \in N_2 \). We have \( \text{ord}_1(2) = 1 \) and \( \gamma(1) = 1 \).
   
   (b) 7 \( \notin N_2 \). We have \( \text{ord}_7(2) = 3 \) and \( \beta(7) = 1 \).

2. Thus \( k_1 = 3a_7 \) where \( 0 \leq a_7 \leq 1 \).
   
   For \( a_7 = 0 \), we have \( k_1 = 0 \) and
   \[
   k_2 = b_7 + 3c_7,
   \]
   where \( 0 \leq b_7 \leq 1 \) and \( 0 \leq c_7 \leq 2(1 - 0) = 2 \). Thus \( k_2 \in \{0, 1, 3, 4, 6, 7\} \).
   
   Hence, the types are of the form \( 4^{k_1}2^{k_2} \), where \( (k_1, k_2) \in \{(0, 0), (0, 1), (0, 3), (0, 4), (0, 6), (0, 7)\} \).
   
   For \( a_7 = 1 \), we have \( k_1 = 3 \) and
   \[
   k_2 = b_7 + 3c_7 = b_7,
   \]
   where \( 0 \leq b_7 \leq 1 \) and \( 0 \leq c_7 \leq 2(1 - 1) = 0 \). Hence, \( k_2 \in \{0, 1\} \). Therefore the types are of the form \( 4^{k_1}2^{k_2} \), where \( (k_1, k_2) \in \{(3, 0), (3, 1)\} \).

Altogether, we conclude that the types of the hulls of cyclic codes of length 7 over \( \mathbb{Z}_4 \) are of the form \( 4^{k_1}2^{k_2} \), where \( (k_1, k_2) \in \{(0, 0), (0, 1), (0, 3), (0, 4), (0, 6), (0, 7), (3, 0), (3, 1)\} \).

**Example 3.9.** Let \( n = 21 \). The types of the hulls of cyclic codes of length 21 are given as follows.

1. The divisors of 21 are 1, 3, 7 and 21.
(a) $1, 3 \in N_2$. We have $\text{ord}_1(2) = 1, \text{ord}_3(2) = 2$ and $\gamma(1) = 1 = \gamma(3)$.
(b) $7, 21 \not\in N_2$. We have $\text{ord}_7(2) = 3, \text{ord}_{21}(2) = 6$ and $\beta(7) = 1 = \beta(21)$.

2. It follows that $k_1 = 3a_7 + 6a_{21}$, where $0 \leq a_7, a_{21} \leq 1$.
   For $(a_7, a_{21}) = (0, 0)$, we have $k_1 = 0$ and
   \[ k_2 = b_1 + 2b_3 + 3c_7 + 6c_{21}, \]
   where $0 \leq b_1, b_3 \leq 1$ and $0 \leq c_7, c_{21} \leq 2$. So $k_2 \in \{0, 1, \ldots, 21\}$.
   For $(a_7, a_{21}) = (1, 0)$, we have $k_1 = 3$ and
   \[ k_2 = b_1 + 2b_3 + 3c_7 + 6c_{21}, \]
   where $0 \leq b_1, b_3 \leq 1$, $c_7 = 0$ and $0 \leq c_{21} \leq 2$. Hence, $k_2 \in \{0, 1, 2, 3, 6, 7, 8, 9, 12, 13, 14, 15\}$.
   For $(a_7, a_{21}) = (0, 1)$, we have $k_1 = 6$ and
   \[ k_2 = b_1 + 2b_3 + 3c_7 + 6c_{21}, \]
   where $0 \leq b_1, b_3 \leq 1$, $0 \leq c_7 \leq 2$ and $c_{21} = 0$. Thus $k_2 \in \{0, 1, \ldots, 9\}$.
   For $(a_7, a_{21}) = (1, 1)$, then $k_1 = 9$ and
   \[ k_2 = b_1 + 2b_3 + 3c_7 + 6c_{21}, \]
   where $0 \leq b_1, b_3 \leq 1$, $c_7 = 0$ and $c_{21} = 0$. Hence, $k_2 \in \{0, 1, 2, 3\}$.

For each odd integer $3 \leq n \leq 35$, the types $4^{k_1}2^{k_2}$ of the hulls of cyclic codes of length $n$ over $\mathbb{Z}_4$ are given in Table I based on the above algorithm.

A formula for the 2-dimensions of the hulls of cyclic codes of odd length $n$ over $\mathbb{Z}_4$ is given as follows.

**Theorem 3.10.** Let $n$ be an odd positive integer. Then the 2-dimensions of the hull of cyclic codes of length $n$ over $\mathbb{Z}_4$ are of the form
\[
\sum_{j|m, j \in N_2} \text{ord}_j(2) \cdot \triangle_j + \sum_{j|m, j \not\in N_2} \text{ord}_j(2) \cdot \triangle_j, \tag{14}
\]
where $0 \leq \triangle_j \leq \gamma(j)$ and $0 \leq \triangle_j \leq 2\beta(j)$.

**Proof.** Let $C$ be a cyclic code of odd length $n$ over $\mathbb{Z}_4$ generated by $(f(x)g(x), 2f(x))$, where $x^n - 1 = f(x)g(x)h(x)$. By Theorem 3.2 we have that Hull$(C)$ has type $4^{\deg H(x)}2^{\deg G(x)}$ and the 2-dimension of Hull$(C)$ is
\[
2 \deg H(x) + \deg G(x),
\]
where
\[
H(x) = \gcd(h(x), f'(x))
\]
| $n$ | $k_1$ | $k_2$                  | $n$ | $k_1$ | $k_1$                  |
|-----|-------|------------------------|-----|-------|------------------------|
| 3   | 0     | 0, 1, 2, 3             | 27  | 0     | 0, 1, 2, 3, 6, 7, 8, 9, 18, |
| 5   | 0     | 0, 1, 4, 5             | 19  | 20, 21, 24, 25, 26, 27 |
| 7   | 0     | 0, 1, 3, 4, 6, 7       | 29  | 0     | 0, 1, 28, 29            |
| 9   | 0     | 0, 1, 2, 3, 6, 7, 8, 9 | 31  | 0     | 0, 1, 5, 6, 10, 11, 15, 16, |
| 11  | 0     | 0, 1, 10, 11           | 5   | 0     | 0, 1, 5, 6, 10, 11, 15, 16, |
| 13  | 0     | 0, 1, 12, 13           |     |       | 20, 21                  |
| 15  | 0     | 0, 1, ..., 15          | 10  | 0     | 0, 1, 5, 6, 10, 11       |
| 17  | 0     | 0, 1, 8, 9, 16, 17     | 15  |       | 0, 1                    |
| 19  | 0     | 0, 1, 18, 19           | 33  | 0     | 0, 1, 2, 3, 10, 11, 12, 13, 20, |
| 21  | 0     | 0, 1, 2, ..., 21       | 21, 22, 23, 30, 31, 32, 33 |
| 3   | 0, 1, 2, 3, 6, 7, 8, 9 | 35  | 0     | 0, 1, 3, 4, 5, 6, 7, 8, 10, |
|     | 9     | 0, 1, 2, 3, 6, 7, 8, 12, 13, 14, 15 | 11, 12, 13, 15, 16, 17, 18, |
| 6   | 0, 1, ..., 9          | 19, 20, 22, 23, 24, 25, 27, |
| 9   | 0, 1, 2, 3            | 28, 29, 30, 31, 32, 34, 35 |
| 23  | 0     | 0, 1, 11, 12, 22, 23   | 3   | 0     | 0, 1, 4, 5, 12, 13, 16, 17, |
| 11  |       | 0, 1                   | 24  | 25, 28, 29 |
| 25  | 0     | 0, 1, 4, 5, 20, 21, 24, 25 | 15  | 0     | 0, 1, 4, 5               |

and

$$G(x) = \frac{x^n - 1}{\gcd(h(x), f^*(x)) \cdot \text{lcm}(f(x), h^*(x))}.$$
By Eqs (12) and (13), it can be deduced that

\[ \dim_2 (\text{Hull}(C)) = 2 \deg H(x) + \deg G(x) \]

\[ = 2 \sum_{j_n, j \in \mathbb{F}_2} \text{ord}_j (2) \sum_{i=1}^{\beta(j)} \left( \min\{1 - v_{ij} - z_{ij}, w_{ij}\} + \min\{1 - w_{ij} - d_{ij}, v_{ij}\} \right) \]

\[ + \sum_{j_n, j \in \mathbb{F}_2} \text{ord}_j (2) \sum_{i=1}^{\gamma(j)} \left( 1 - \max\{u_{ij}, 1 - u_{ij} - b_{ij}\} \right) \]

\[ + \sum_{j_n, j \in \mathbb{F}_2} \text{ord}_j (2) \sum_{i=1}^{\beta(j)} \left( 2 - \min\{1 - v_{ij} - z_{ij}, w_{ij}\} - \max\{v_{ij}, 1 - w_{ij} - d_{ij}\} \right. \]

\[ - \min\{1 - w_{ij} - d_{ij}, v_{ij}\} - \max\{w_{ij}, 1 - v_{ij} - z_{ij}\} \bigg) \]

\[ = \sum_{j_n, j \in \mathbb{F}_2} \text{ord}_j (2) \sum_{i=1}^{\gamma(j)} \left( 1 - \max\{u_{ij}, 1 - u_{ij} - b_{ij}\} \right) \]

\[ + \sum_{j_n, j \in \mathbb{F}_2} \text{ord}_j (2) \sum_{i=1}^{\beta(j)} \left( 2 + \min\{1 - v_{ij} - z_{ij}, w_{ij}\} - \max\{v_{ij}, 1 - w_{ij} - d_{ij}\} \right. \]

\[ + \min\{1 - w_{ij} - d_{ij}, v_{ij}\} - \max\{w_{ij}, 1 - v_{ij} - z_{ij}\} \bigg) \]  \hspace{1cm} (15)

\[ = \sum_{j_n, j \in \mathbb{F}_2} \text{ord}_j (2) \sum_{i=1}^{\gamma(j)} \Delta_{ij} + \sum_{j_n, j \in \mathbb{F}_2} \text{ord}_j (2) \sum_{i=1}^{\beta(j)} \mathbf{\Delta}_{ij}, \]  \hspace{1cm} (16)

where \( \Delta_{ij} = 1 - \max\{u_{ij}, 1 - u_{ij} - b_{ij}\} \) and

\[ \mathbf{\Delta}_{ij} = 2 + \min\{1 - v_{ij} - z_{ij}, w_{ij}\} - \max\{v_{ij}, 1 - w_{ij} - d_{ij}\} \]

\[ + \min\{1 - w_{ij} - d_{ij}, v_{ij}\} - \max\{w_{ij}, 1 - v_{ij} - z_{ij}\}. \]

It is not difficult to see that \( 0 \leq \Delta_{ij} \leq 1 \) and \( 0 \leq \mathbf{\Delta}_{ij} \leq 2 \). From Eq (16), we have

\[ \dim_2 (\text{Hull}(C)) = \sum_{j_n, j \in \mathbb{F}_2} \text{ord}_j (2) \sum_{i=1}^{\gamma(j)} \Delta_{ij} + \sum_{j_n, j \in \mathbb{F}_2} \text{ord}_j (2) \sum_{i=1}^{\beta(j)} \mathbf{\Delta}_{ij} \]

\[ = \sum_{j_n, j \in \mathbb{F}_2} \text{ord}_j (2) \cdot \Delta_j + \sum_{j_n, j \in \mathbb{F}_2} \text{ord}_j (2) \cdot \mathbf{\Delta}_j, \]

where \( \Delta_j = \sum_{i=1}^{\gamma(j)} \Delta_{ij} \) and \( \mathbf{\Delta}_j = \sum_{i=1}^{\beta(j)} \mathbf{\Delta}_{ij} \). It follows that \( 0 \leq \Delta_j \leq \gamma(j) \) and \( 0 \leq \mathbf{\Delta}_j \leq 2\beta(j) \). Hence, the 2-dimension of \( C \) is of the form in Eq (14). \( \square \)

### 3.4 Enumeration of Cyclic Codes of the same 2-Dimension

The 2-dimensions of the hulls of cyclic codes of odd length \( n \) over \( \mathbb{Z}_4 \) are determined in the previous subsection. Here, the number of cyclic codes of odd length \( n \) over \( \mathbb{Z}_4 \) whose hulls share a fixed 2-dimension is investigated.
Let $\ell$ denote a 2-dimension of the hull of cyclic codes of odd length $n$ over $\mathbb{Z}_4$ given in Eq (14). Later, the number of cyclic codes of odd length $n$ over $\mathbb{Z}_4$ whose hulls have 2-dimension $\ell$ will be obtained in terms of the solutions $\Delta_{ij}$’s and $\Delta_{ij}$’s of

$$\sum_{j,n,j \in \mathbb{N}_2} \text{ord}_j(2) \sum_{i=1}^{\gamma(j)} \Delta_{ij} + \sum_{j,n,j \in \mathbb{N}_2} \text{ord}_j(2) \sum_{i=1}^{\beta(j)} \Delta_{ij},$$

where $0 \leq \Delta_{ij} \leq 1$ and $0 \leq \Delta_{ij} \leq 2$. For convenience, let $((\Delta_{ij}))$ be a vector whose entries are $0 \leq \Delta_{ij} \leq 1$ and the indices satisfy $j \mid n$, $j \in \mathbb{N}_2$ and $1 \leq i \leq \gamma(j)$, i.e.,

$$((\Delta_{ij})) := (\Delta_{ij})_{j,n,j \in \mathbb{N}_2,1 \leq i \leq \gamma(j)}.$$

Similarly, let

$$((\Delta_{ij})) := (\Delta_{ij})_{j,n,j \in \mathbb{N}_2,1 \leq i \leq \beta(j)},$$

where $0 \leq \Delta_{ij} \leq 2$. Denote by $((((\Delta_{ij}))),((\Delta_{ij})))$ the concatenation of the vectors $((\Delta_{ij}))$ and $((\Delta_{ij}))$.

**Theorem 3.11.** Let $n$ be an odd positive integer and $\ell$ be in the form of Eq (14). The number of cyclic codes of length $n$ over $\mathbb{Z}_4$ whose hulls have 2-dimension $\ell$ is

$$\sum_{((\Delta_{ij}))),((\Delta_{ij})) \in h(\ell)} \left( \prod_{j,n,j \in \mathbb{N}_2} \gamma(j) \prod_{j,n,j \in \mathbb{N}_2} \beta(j) \prod_{i=1}^{\gamma(j)} (2 - \Delta_{ij}) \prod_{j,n,j \in \mathbb{N}_2} \sum_{i=1}^{\beta(j)} \left( \frac{3}{2} \Delta_{ij}^2 + \frac{7}{2} \Delta_{ij} + 2 \right) \right),$$

where

$$h(\ell) = \left\{ \left( (\Delta_{ij}))),((\Delta_{ij})) \right) \bigg| \sum_{j,n,j \in \mathbb{N}_2} \text{ord}_j(2) \sum_{i=1}^{\gamma(j)} \Delta_{ij} + \sum_{j,n,j \in \mathbb{N}_2} \text{ord}_j(2) \sum_{i=1}^{\beta(j)} \Delta_{ij} = \ell \right\}. $$

**Proof.** For a fixed $((\Delta_{ij}))),((\Delta_{ij})))$, we want to find the polynomials $f(x), g(x)$ and $h(x)$ in Eqs (9), (10) and (11) such that the 2-dimension of the hull of a cyclic code generated by $(f(x)g(x), 2f(x))$ is

$$\sum_{j,n,j \in \mathbb{N}_2} \text{ord}_j(2) \sum_{i=1}^{\gamma(j)} \Delta_{ij} + \sum_{j,n,j \in \mathbb{N}_2} \text{ord}_j(2) \sum_{i=1}^{\beta(j)} \Delta_{ij} = \ell. \quad (17)$$

By Eqs (16), it can be deduced that

$$\dim_2(\text{Hull}(C)) = 2 \deg H(x) + \deg G(x)$$

$$= \sum_{j,n,j \in \mathbb{N}_2} \text{ord}_j(2) \sum_{i=1}^{\gamma(j)} \Delta_{ij} + \sum_{j,n,j \in \mathbb{N}_2} \text{ord}_j(2) \sum_{i=1}^{\beta(j)} \Delta_{ij},$$

where $\Delta_{ij} = 1 - \max\{u_{ij}, 1 - u_{ij} - b_{ij}\}$ and

$$\Delta_{ij} = 2 + \max\{1 - v_{ij} - z_{ij}, w_{ij}\} - \max\{v_{ij}, 1 - w_{ij} - d_{ij}\}$$

$$+ \min\{1 - w_{ij} - d_{ij}, v_{ij}\} - \max\{w_{ij}, 1 - v_{ij} - z_{ij}\}.$$
For given $\triangle_{ij}$ and $\triangle_{ij}$, the values of $(u_{ij}, b_{ij})$ and $(v_{ij}, z_{ij}, w_{ij}, d_{ij})$ are listed respectively in the following tables.

Thus, for a given $((\triangle_{ij}), ((\triangle_{ij})))$, the number of cyclic codes of odd length $n$ over $\mathbb{Z}_4$ whose hulls have 2-dimension in the form of Eq (17) is

$$\prod_{j|n, j \in \mathbb{N}} \gamma(j) \prod_{j|n, j \in \mathbb{N}} \beta(j) \left( \frac{-3}{2} \bigtriangleup_{ij}^2 + \frac{7}{2} \bigtriangleup_{ij} + 2 \right).$$

(18)

Therefore, the number of cyclic codes of odd length $n$ over $\mathbb{Z}_4$ having hulls of 2-dimension $\ell$ is the summations of (18) where all $((\triangle_{ij}), ((\triangle_{ij})))$ runs in the set $h(\ell)$. □

We note that the average 2-dimension $E(n)$ of the hull of cyclic codes of length $n$ over $\mathbb{Z}_4$ can be given in terms of the fraction of the sum of the number of cyclic codes whose hulls have 2-dimension $\ell$ in Theorem 3.11, where $\ell$ runs over all the 2-dimensions in Theorem 3.10 and the number of cyclic codes $|C(n, 4)|$. Using this direction, it might lead to a very tedious calculation. Here, an alternative simpler way to determine $E(n)$ is given in the next section using probability theory.

4 The Average 2-Dimension $E(n)$

Recall that $n$ is an odd positive integer, $C(n, 4)$ is the set of all cyclic codes of length $n$ over $\mathbb{Z}_4$ and the average 2-dimension of the hull of cyclic codes of length $n$ over $\mathbb{Z}_4$ is

$$E(n) = \sum_{C \in C(n, 4)} \frac{\dim_2(\text{Hull}(C))}{|C(n, 4)|}.$$

In this section, an explicit formula of $E(n)$ and its upper bounds are given in terms of $B_n$ and the length $n$ of the codes.

First, we prove the following useful expectations.

**Lemma 4.1.** Let $(v, z), (w, d), (u, b) \in \{(0, 0), (1, 0), (0, 1)\}$. Then

1. $E(1 - \max(u, 1 - u - b)) = \frac{1}{3}$.

2. $E(2 + \min(1 - v - z, w) - \max(v, 1 - w - d) + \min(1 - w - d, v) - \max(w, 1 - v - z)) = \frac{10}{9}$. 

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Proof. To prove 1, consider the values in the following table. Let \( \triangle = 1 - \max\{u, 1 - u - b\} \).

It follows that \( E(\triangle) = 0 \cdot \frac{2}{3} + 1 \cdot \frac{1}{3} = \frac{1}{3} \).

To prove 2, let \( * = 1 - v - z, \circ = 1 - w - d \) and \( \triangledown = 2 + \min\{*, w\} - \max\{*, w\} + \min\{v, \circ\} - \max\{v, \circ\} \).

From the above table, it can be concluded that

\[
E(\triangledown) = E(2 + \min\{*, w\} - \min\{*, w\} + \min\{v, \circ\} - \max\{v, \circ\}) = 10 \cdot \frac{2}{9} + 1 \cdot \frac{4}{9} + 2 \cdot \frac{3}{9} = \frac{10}{9}.
\]

The proof is completed. □

From Eqs (9), (10) and (11), the following lemma can be deduced directly.

Lemma 4.2. There is a bijection between \( C(n, 4) \) and the set

\[
S = \{(u_1, b_1), \ldots, (u_s, b_s), (v_1, z_1), \ldots, (v_t, z_t), (w_1, d_1), \ldots, (w_t, d_t)\} \setminus \{(0, 0), (1, 0), (0, 1)\} \text{ for all } 0 \leq i \leq s \text{ and } 0 \leq j \leq t
\]

where \( s \) and \( t \) are given in Eqs (4) and (5).

Based on Lemma 4.1, the formula for the average 2-dimension of the hull of cyclic codes of odd length \( n \) over \( \mathbb{Z}_4 \) can be determined using the expectation \( E(Y) \), where \( Y \) is the random variable of the 2-dimension \( \dim_2(\text{Hull}(C)) \) and \( C \) is chosen randomly from \( C(n, 4) \) with uniform probability.

Theorem 4.3. Let \( n \) be an odd positive integer. Then the average 2-dimension of the hull of cyclic codes of length \( n \) over \( \mathbb{Z}_4 \) is

\[
E(n) = \frac{5}{9}n - \frac{2}{9}B_n
\]

where \( B_n \) is defined in Eq (6).
Thus from Eqs (15) and (19), we have

\[ \text{Proof.} \]

Let \( C \) be a cyclic code of length \( n \) over \( \mathbb{Z}_4 \) generated by

\[ \langle f(x)g(x), 2f(x) \rangle, \]

where \( x^n - 1 = f(x)g(x)h(x) \) and \( f(x), g(x) \) and \( h(x) \) are monic polynomials. By Theorem 3.2, we have \( \text{Hull}(C) \) has type \( 4^{\deg H(x)}2^{\deg G(x)} \) and the 2-dimension of \( \text{Hull}(C) \) is

\[ 2 \deg H(x) + \deg G(x), \]

where

\[ H(x) = \gcd(h(x), f^*(x)) \]

and

\[ G(x) = \frac{x^n - 1}{\gcd(h(x), f^*(x)) \cdot \text{lcm}(f(x), h^*(x))}. \]

Let \( Y \) be the random variable of the 2-dimension \( \text{dim}_2(C) \), where \( C \) is chosen randomly from \( C(n, 4) \) with uniform probability. Let \( E(Y) \) be the expectation of \( Y \). Thus \( E(n) = E(Y) \). Therefore, choosing a cyclic code \( C \) from \( C(n, 4) \) with uniform probability \( \frac{1}{K(n,4)} = \frac{1}{3^{n+1}} \) and choosing an element in \( S \) defined in Lemma 4.2 with uniform probability \( \frac{1}{3^{2n}} \) are identical. By Theorem 3.2 we obtain

\[ Y = \text{dim}_2(\text{Hull}(C)) = 2 \deg H(x) + \deg G(x). \]  

(19)

From Eqs (15) and (19), we have

\[ E(n) = E(Y) \]

\[ = E(2 \deg H(x) + \deg G(x)) \]

\[ = E \left( \sum_{j,n \in \mathbb{Z}_2} \text{ord}_j(2) \sum_{i=1}^{y(j)} (1 - \max\{u_{ij}, 1 - u_{ij} - b_{ij}\}) \right) \]

\[ + \sum_{j,n \in \mathbb{Z}_2} \text{ord}_j(2) \sum_{i=1}^{\beta(j)} \left( 2 + \min\{1 - v_{ij} - z_{ij}, w_{ij}\} - \max\{v_{ij}, 1 - w_{ij} - d_{ij}\} \right) \]

\[ + \min\{1 - w_{ij} - d_{ij}, v_{ij}\} - \max\{w_{ij}, 1 - v_{ij} - z_{ij}\} \right) \]

\[ = \sum_{j,n \in \mathbb{Z}_2} \text{ord}_j(2) \cdot \gamma(j) \cdot E \left( 1 - \max\{u_{ij}, 1 - u_{ij} - b_{ij}\} \right) \]

\[ + \sum_{j,n \in \mathbb{Z}_2} \text{ord}_j(2) \cdot \beta(j) \cdot \left( 2 + \min\{1 - v_{ij} - z_{ij}, w_{ij}\} - \max\{v_{ij}, 1 - w_{ij} - d_{ij}\} \right) \]

\[ + \min\{1 - w_{ij} - d_{ij}, v_{ij}\} - \max\{w_{ij}, 1 - v_{ij} - z_{ij}\} \right) \]

\[ = \sum_{j,n \in \mathbb{Z}_2} \phi(j) \cdot \frac{1}{3} + \sum_{j,n \in \mathbb{Z}_2} \phi(j) \cdot \frac{10}{9} \]

by Lemma 4.1

\[ = \frac{B_n}{3} + \frac{5(n - B_n)}{9} \]

by Eq (6).

\[ = \frac{5n}{9} - \frac{2B_n}{9}. \]

The proof is completed. □
The next corollary is a direct consequence of Theorem 4.3.

**Corollary 4.4.** Assume the notations as in Theorem 4.3. Then \( E(n) < \frac{5n}{9}\).

The average 2-dimension \( E(n) \) of the hull of cyclic codes of odd length up to 53 over \( \mathbb{Z}_4 \) are given in Table 2. The row is highlighted in gray when \( n \in N_2 \) and \( n \notin N_2 \) otherwise.

| \( n \) | \( B(n) \) | \( E(n) = \frac{5n-2B}{9} \) | \( n \) | \( B(n) \) | \( E(n) = \frac{5n-2B}{9} \) |
|-------|--------|-------------------|-------|--------|-------------------|
| 3     | 3      | 1                 | 29    | 29     | \( \frac{29}{3} \) |
| 5     | 5      | \( \frac{5}{3} \) | 31    | 1      | 17                |
| 7     | 1      | \( \frac{11}{3} \) | 33    | 33     | 11                |
| 9     | 9      | 3                 | 35    | 5      | \( \frac{55}{3} \) |
| 11    | 11     | \( \frac{11}{3} \) | 37    | 37     | \( \frac{37}{3} \) |
| 13    | 13     | \( \frac{13}{3} \) | 39    | 15     | \( \frac{55}{3} \) |
| 15    | 7      | \( \frac{61}{9} \) | 41    | 41     | \( \frac{41}{3} \) |
| 17    | 17     | \( \frac{17}{3} \) | 43    | 43     | \( \frac{43}{3} \) |
| 19    | 19     | \( \frac{19}{3} \) | 45    | 13     | \( \frac{199}{9} \) |
| 21    | 3      | 11                | 47    | 1      | \( \frac{233}{9} \) |
| 23    | 1      | \( \frac{113}{9} \) | 49    | 1      | 27                |
| 25    | 25     | \( \frac{25}{3} \) | 51    | 19     | \( \frac{217}{9} \) |
| 27    | 27     | 9                 | 53    | 53     | \( \frac{53}{3} \) |

**Table 2:** \( E(n) \) of cyclic codes of odd length \( n \) up to 53 over \( \mathbb{Z}_4 \).

## 5 \( N_2 \)-factorization and Bounds on \( E(n) \)

In this section, a simplified formula of \( B_n \) is derived. Lower and upper bounds for \( E(n) \) can be obtained using this formula of \( B_n \).

Recall that \( N_2 = \{ \ell \geq 1 : \ell \text{ divides } 2^i + 1 \text{ for some positive integer } i \} \).

**Lemma 5.1.** Let \( \ell \) be a positive integer. If \( \ell \in N_2 \), then \( \text{ord}_\ell(2) \) is even.
Proof. Assume that $\ell \in N_2$. Then there exists the smallest positive integer $k$ such that $\ell(2^k + 1)$, which implies $\ell(2^{2k} - 1)$. So $\text{ord}_\ell(2)|2k$. Since $\text{ord}_\ell(2) \nmid k$, $\text{ord}_\ell(2)$ is even. □

Let $P_\alpha := \{\ell \in N_2 : 2^\alpha | \text{ord}_\ell(2)\}$, where the notation $2^\alpha | k$ means that $\alpha$ is the non-negative integer such that $2^\alpha|k$ but $2^{\alpha+1} \nmid k$. Clearly, $P_0 = \{1\}$.

**Theorem 5.2** (23 Theorem 4]). Let $\ell > 1$ be an odd positive integer. Let $\ell = p_1^{e_1} \ldots p_k^{e_k}$ be the prime factorization of $\ell$. Then $\ell \in N_2$ if and only if there exists $\alpha \geq 1$ such that $p_i \in P_\alpha$ for all $i$. In this case, we have $\ell \in P_\alpha$.

**Lemma 5.3.** Let $\alpha \geq 1$ an integer and let $\ell$ be a positive integer. If $\ell \in P_\alpha$, then $\ell \geq 2^\alpha + 1$.

**Proof.** Note that $\ell \geq 3$. Since $\ell \in P_\alpha$, it follows that $2^\alpha | \text{ord}_\ell(q)$. By Fermat’s Little Theorem, we have $\text{ord}_\ell(q)\phi(\ell)$. Then $2^\alpha \phi(\ell)$. Hence, $2^\alpha \leq \phi(\ell) \leq \ell - 1$. □

Let $\ell = p_1^{e_1} \ldots p_k^{e_k}$ be the prime factorization of $\ell$, where $p_1, \ldots, p_k$ are distinct odd primes and $e_i \geq 1$ for all $1 \leq i \leq k$. Partition the index set $\{1, \ldots, k\}$ into $K', K_1, K_2, \ldots$ as follows:

1. $K' = \{i \mid p_i \notin N_2\}$,
2. $K_\alpha = \{i \mid p_i \in N_2 \text{ and } p_i \in P_\alpha\}$.

Let $d' = \prod_{i \in K'} p_i^{e_i}$ and $d_\alpha = \prod_{i \in K_\alpha} p_i^{e_i}$. For convenience, the empty product will be regarded as 1. Therefore, we have $\ell = d'd_1d_2 \ldots$ which is called the $N_2$-factorization of $\ell$.

**Lemma 5.4** ([23 Lemma 9]). Let $\ell$ be an odd integer and let $\ell = d'd_1d_2 \ldots$ be the $N_2$-factorization of $\ell$. If $\ell \notin N_2$, then at least one of the following conditions hold.

1. $d' > 1$.
2. $d_{\alpha_1} > 1$ and $d_{\alpha_2} > 1$ for two distinct $\alpha_1 \geq 1$ and $\alpha_2 \geq 1$.

**Proposition 5.5.** Let $n$ be an odd integer and let $n = d'd_1d_2 \ldots$ be an $N_2$-factorization of $n$. If $n \notin N_2$, then $B_n = d_1 + \sum_{\alpha \geq 2} (d_\alpha - 1)$.

**Proof.** Since $\sum_{i \in d} \phi(i) = d$, we have

$$B_n = \sum_{j \in d, j \notin N_2} \phi(j) + \sum_{\alpha \geq 1} \sum_{k|d_\alpha, k \neq 1} \phi(k) = 1 + \sum_{\alpha \geq 1} (d_\alpha - 1) = d_1 + \sum_{\alpha \geq 2} (d_\alpha - 1)$$

by Eq (6). □

Applying Lemma 5.3, Lemma 5.4 and Proposition 5.5 some upper and lower bounds of $E(n)$ can be concluded in the following theorem.

**Theorem 5.6.** Let $n$ be an odd integer. The following statements hold.
1. $n \in N_2$ if and only if $E(n) = \frac{4}{9}$.

2. If $n \notin N_2$, then $\frac{11n}{27} \leq E(n) \leq \frac{5n}{9}$.

Proof. To prove 1, let $n \in N_2$. Then $B_n = \sum_{j \in j \in N_2} \phi(j) = \sum_{j \in N_2} \phi(j) = n$.

Conversely, we assume that $E(n) = \frac{4}{9}$. By Theorem 5.3, we have $\frac{n}{3} = \frac{5n}{9} - \frac{2n}{9}$.

Thus $n = B_n$, which implies $n \in N_2$.

To prove 2, let $n \notin N_2$. Let $n = d'd_1d_2 \ldots$ be an $N_2$-factorization of $n$ and

$$n = d'd_1d_2 \ldots d'a_1 \ldots a_j,$$

where $d_{ai} > 1$ for all $1 \leq i \leq j$ and $a_1 < a_2 < \ldots < a_j$. Note that if $d_{ai}$ and $d'$ are greater than 1 then they are greater than or equal to 3. By Proposition 5.5, we obtain

$$B_n = \frac{d_1 + \sum_{i=1}^{j} (d_i - 1)}{d'd_1d_2 \ldots} = \frac{1 - j + \sum_{i=1}^{j} a_i}{d'd_{a_1} \ldots a_j}.$$

By Lemma 5.3, we have the following 4 cases.

Case 1 $d' > 1$, $j = 0$. Then $\frac{B_n}{n} = \frac{1}{d'} \leq \frac{1}{3}$.

Case 2 $d' > 1$, $j = 1$. So $\frac{B_n}{n} = \frac{d_1}{d'd_{a_1}} = \frac{1}{d'} \leq \frac{1}{3}$.

Case 3 $j = 2$. Without loss of generality, we may assume that $d_{a_2} \leq d_{a_1}$. Hence, we have

$$B_n = \frac{-1 + d_1 + d_{a_2}}{d'd_{a_1}d_{a_2}} \leq \frac{2d_{a_1}}{d'd_{a_1}d_{a_2}} = \frac{2}{d'd_{a_2}} \leq \frac{2}{3}$$

Case 4 $j \geq 3$. Let $d_{a_r} = \max_{1 \leq i \leq j} d_{a_i}$.

$$B_n = \frac{1 - j + \sum_{i=1}^{j} d_{a_i}}{d'd_{a_1} \ldots d_{a_j}} \leq \frac{\sum_{i=1}^{j} d_{a_i}}{d'd_{a_1} \ldots d_{a_j}} \leq \frac{j}{d'd_{a_1} \ldots d_{a_j}} \leq \frac{j}{\prod_{1 \leq i \leq j, i \neq r} d_{a_i}}.$$}

Let $s$ be an index such that $j - 1 \leq s \leq j$ and $s \neq r$. Then $j < 2^{j-1} \leq 2^s \leq 2^{a_r}$. Since $d_{a_r} \in P_{a_r}$, we have $d_{a_r} \geq 2^{a_r} + 1$ by Lemma 5.3. Hence, $j < 2^{a_r} < d_{a_r}$. Therefore,

$$B_n \leq \frac{j}{\prod_{1 \leq i \leq j, i \neq r} d_{a_i}} \leq \frac{d_{a_r}}{\prod_{1 \leq i \leq j, i \neq r} d_{a_i}} = \frac{1}{\prod_{1 \leq i \leq j, i \neq r} d_{a_i}} \leq \frac{1}{3}.$$}

Altogether, we obtain $B_n \leq \frac{2n}{9}$, and hence

$$E(n) = \frac{5n}{9} - \frac{2B_n}{9} \geq \frac{5n}{9} - \frac{4n}{27} = \frac{11n}{27}.$$}

From Theorem 5.6, it can be concluded that $E(n)$ grows at the same rate with $n$ as $n$ is odd and tends to infinity.
6 Conclusion and Remarks

The hulls of cyclic codes of odd lengths over $\mathbb{Z}_4$ has been studied. The characterization of the hulls has been given in terms of their generators. The types of the hulls of cyclic codes of arbitrary odd length have been determined as well. Subsequently, the 2-dimension of the hulls of cyclic codes of odd length over $\mathbb{Z}_4$ has been determined together with the average 2-dimension of the hull of cyclic codes of odd length $n$ over $\mathbb{Z}_4$. Upper and lower bounds for the average 2-dimension of the hull have been given. Asymptotically, it has been shown that the average of 2-dimension of the hull of cyclic codes of odd length over $\mathbb{Z}_4$ grows the same rate as the length of the codes.

It would be interesting to study the properties the hulls of cyclic codes of even lengths over $\mathbb{Z}_4$. An extension of this paper to the case of the hulls of cyclic or constacyclic codes over finite chain rings is an interesting research problem as well.

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