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A Caputo power law model predicting the spread of the COVID-19 outbreak in Pakistan

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Abstract This work is devoted to establish a modified population model of susceptible and infected (SI) compartments to predict the spread of the infectious disease COVID-19 in Pakistan. We have formulated the model and derived its boundedness and feasibility. By using next generation matrices method we have derived some results for the global and local stability of different kinds of equilibrium points. Also, by using fixed point approach some results of existence of at least one solution are studied. Furthermore, the numerical simulations for various values of isolation parameters corresponding to different fractional order are investigated by using modified Euler’s method.

1. Introduction

In recent time a threatful outbreak which has been originated from China is spreading throughout the globe very rapidly. About 0.6125 million people have been died due to this disease. More than 14 million people have got the infection in all over the world [1]. Also a large number of infected people have been recovered from this disease. Different researchers and policy makers are struggling to control the disease from further spreading. One big factor of spreading this disease is immigration of infected people from place to another which effect more people and hence cause the spreading of this disease. Therefore, on international level, many countries of the world have banned air traffic for some time and also they have announced lock down in cities so that some precautionary measure should be taken to reduce maximum loss of human lives. Also, each country tries to reduce unnecessary traveling of people which hopefully will reduce the cases of infection in their country [2]. It is remarkable that such like outbreak in the past have caused millions of deaths around the globe. Since scientists and researchers are trying to investigate cure or vaccine for the aforesaid outbreak so that in the future such like pandemic may be controlled. From a medical engineering point of view...
infectious disease can be well understand by using the concept of mathematical model. This concept was started during in 1927. After that various mathematical models have been established for different disease in history. For some famous study, we refer [3–6] and the cited paper therein. Like other countries Pakistan has also affected in different aspects by the aforesaid pandemic. About six thousand people were died and three 0.3 million infected. The number of deaths occur from 22th April to 26th July [7] are shown in Fig. 1.

Keeping in mind the treatment of the present pandemic, researchers have investigated the current disease due to Corona virus –19 from some aspects, for detail see [8–12, 13,16,17]. On the other hand, mathematical models also can help us in understanding and to construct some strategies, how to control the disease from being spreading and to take precautionary necessary measures. In this regards, some models have been very recently studied about the current novel virus disease, see for detail [14]. Since the infection is rapidly transmitting from one person to another, therefore in most of the country the people are advised to keep social distance from each other. Some researchers have taken an SI system that provide the study of change of COVID-19 in a south western European country “Portugal” for 21 days and studies the results of keeping distance from each other, which calculate the approximated increase of pandemic. They investigated the isolation effect on the spreading discussion of this pandemic by considering simple SI model as [14] given by

\[
\begin{align*}
\frac{dS(t)}{dt} &= -k(1-z)S(t)I(t) - a\delta S(t)I(t) - 2k\beta S(t)I(t), \\
\frac{dI(t)}{dt} &= k(1-z)S(t)I(t) + a\delta S(t)I(t) - \frac{I(t)}{C_0} - I(t) - \frac{I(t)}{C_0} - I(t), \\
S(0) &= S_0, \quad I(0) = I_0,
\end{align*}
\]  

(1)

where \(S(t)\) is susceptible people, \(I(t)\) is infected people, \(k\) is rate constant, \(z\) is rate of isolation and \(\beta\) is rate of protection. Here the model (1) was deficient by not containing death rate. Therefore, we insert the death rate \(\delta\) and update the model (1) under fractional order derivative. Keeping in mind that the modified model given in (2) is investigated under fractional order derivative in Caputo sense with \(0 < \theta \leq 1\) as given by

\[
\begin{align*}
(D^\theta S(t)) &= a - k(1-z)S(t)I(t) - a\delta S(t)I(t) - \delta S(t), \\
(D^\theta I(t)) &= k(1-z)S(t)I(t) + a\delta S(t)I(t) - \frac{I(t)}{C_0} - I(t) - \delta I(t), \\
S(0) &= S_0, \quad I(0) = I_0,
\end{align*}
\]  

(2)

where \(S_0\) and \(I_0\) are the initial value for susceptible and infected population, \(a, \delta, k, x, \beta, y\) > 0. The Chart for (2) has been displayed in Fig. 2.

The model is considered for fractional order because in last few years the area of fractional differential and integral calculus have been given much attentions by the scholars. It is because of the significant uses of fractional calculus which produces accurate and realistic results in many physical and biological sciences of real world process. As fractional differential and integral operators are globally used operators which provides greater degree of freedom. Therefore, the aforesaid area have been given much attentions and a lot of research papers, monograph, books, etc address the said area in different aspects [18–21,40–42].

These aspects been investigated for arbitrary order of differential equations are of qualitative theory and numerical analysis. In this regards plenty of research articles, books are available. Modern calculus generalize derivatives and antiderivatives to any arbitrary or complex order and hence the ability to explain many real world process more practically compared to classical calculus. In fact various phenomena of real word have been modeled by using differential equations (DEs) and their systems. Like population and logistic models, Predator–prey (Lotka-Volterra) model, SIR, SIERS models, etc. These all models are of integer order, so researches have represented all these models by fractional order which have been given more realistic results as compared to integer order [22–27,33–35,43,44].

Fractional DEs have been studied by many authors for qualitative analysis [28–31] and different authors used different theories like topological degree theory, “Banach contraction theorem and Leray-Schauder fixed point theorem” [32]. From very beginning scientists always shown keen interest for achievement of exact and approximate evaluation to fractional

![COVID Mortality Trends in Pakistan (n=5,842)](image)

Fig. 1 Daily number of deaths occurs in Pakistan from 22th April to 26th July.
DEs. This is a quiet tired task to evaluate analytical solutions to arbitrary order differential equations. So, authors are trying for optimal and numerical solutions corresponding to which they used numerous approaches, available in the literature. Two type of solutions have been investigated in applied analysis one is called analytical results for which we will use some analytical techniques, while other is computed through numerical method called numerical solutions. In this regards, a lot of contribution has been made by many researchers for solving FDEs using different techniques like perturbation methods, integral transform techniques, spectral techniques, collocation methods, etc [23,36,37]. Here we will study (2) for mathematical analysis using theory of existence and uniqueness, along with approximate solution using the considered techniques of generalized Euler’s iterative method. We will check the effect of isolation on the population dynamics corresponding to different fractional order. We will simulate the results for different values of the isolation rates involve in the model (2) under different fractional order and predicted for future planning.

2. Background materials

Here, we recall some definition from [18,19,22].

Definition 1. Let we assign \( v = t \) then for a function say \( S(v) \) we define fractional integral corresponding to \( v \) as

\[
\mathcal{I}_\theta^\nu S(v) = \frac{1}{\Gamma(\theta)} \int_0^\nu (v - \tau)^{\nu-\theta} S(\tau) d\tau, \quad \theta > 0,
\]
such that antiderivative converges to some value.

Definition 2. Consider a mapping, say \( f(v) \), we define the Caputo arbitrary order derivative corresponding to \( v \) as

\[
\mathcal{D}^\theta_0 f(v) = \frac{1}{\Gamma(n - \theta)} \int_0^\nu (v - \tau)^{n-\theta-1} \frac{d^n}{d\tau^n} [f(\tau)] d\tau, \quad \theta > 0,
\]
with all mappings are point wised define on \( \mathbb{R}^+ \), \( n = [\theta] + 1 \). If \( \theta \in (0,1] \), then one has

\[
\mathcal{D}^\theta_0 f(v) = \frac{1}{\Gamma(1-\theta)} \int_0^\nu (v - \tau)^{-\theta} \frac{d}{d\tau} [f(\tau)] d\tau.
\]

Lemma 1. [18] The solution of

\[
\mathcal{D}^\theta_0 f(v) = x(v), \quad 0 < \theta \leq 1,
\]
is given by

\[
I(v) = c_0 + \frac{1}{\Gamma(1-\theta)} \int_0^\nu (v - \tau)^{\theta-1} x(\tau) d\tau,
\]
where \( c_0 \) is the constant of integration.

To prove boundedness and feasibility of the model we give the following result.

Lemma 2. The solution of the considered model is bounded in the feasible region given by

\[
T = \{(S,I) \in \mathbb{R}_+^3 : \, 0 \leq N(t) \leq \frac{a}{\delta} \}.
\]

Proof. By adding both equation of (2), we have

\[
\frac{dS}{dt} = a - k(1-x)S(t)I(t) - \delta(S(t) - S(t) - \delta S(t))
\]

\[
+ (k(1-x)S(t)I(t) + \delta S(t) - \delta I(t) - \delta I(t)),
\]

\[
n = a - \delta (S(t) + I(t) - I(t)),
\]

\[
\leq a - \delta S(t),
\]

\[
\leq a - \delta (N(t)),
\]

\[
\frac{dN}{dt} + \delta N \leq a.
\]

Solving (3), we have

\[
N(t) \leq \frac{a}{\delta} + C \text{exp}(-\delta t),
\]
when \( t \to \infty \), \( N(t) < \frac{a}{\delta} \), hence the required result is received.

Definition 3. [38] The generalized Taylor’s formula for \( f(t) \) is given as under

\[
f(t) = \sum_{n=0}^a \frac{t^n}{\Gamma(n+1)} D^n_0 f(0) + \frac{D^{(a+1)}(f(\xi))}{\Gamma((a+1)\theta + 1)},
\]
with \( 0 \leq \xi \leq t \), \( \forall \, t \in (0, a] \), \( 0 < \theta \leq 1 \). From this formula we can calculate the generalized Euler’s method.

Fig. 2 Interaction between susceptible and infected people.
3. Global and local stability analysis

For stability, we have to find the equilibrium points for (2) as

\[ \text{det} \left( FV^2 \right) = 0, \]

\[ \text{det} \left( FV^3 \right) = 0. \]

We have two equilibrium points which are described as: \( E_0 = \left( \frac{c}{1}, 0 \right) \) is the disease free equilibrium point of (2) and the unique pandemic equilibrium point \( E^* = (S^*, I^*) \), where

\[ S^* = \frac{1 + \gamma \delta}{\gamma (1 - x + z k \beta)}, \]

and

\[ I^* = \frac{\nu (k (1 - x) + z k \beta) - \delta (1 + \gamma \delta)}{(1 + \gamma \delta) (1 + \gamma \delta)}. \]

**Theorem 1.** The reproduction number for (2) is computed as

\[ R_0 = \frac{\nu (k (1 - x) + z k \beta)}{(1 + \gamma \delta) \delta}. \]

**Proof.** Let we take 2nd equation of (2) for finding reproduction number as \( X = I \),

\[ \text{det} \left( FV^2 \right) = \text{det} \left( FV^3 \right) = k (1 - x) SI + z k \beta SI - \left( \frac{1}{\gamma} I + \delta I \right), \]

\[ \text{det} \left( FV^4 \right) = F - V, \]

where \( F = k (1 - x) SI + z k \beta SI, V = \frac{1}{\gamma} I + \delta I, F \) is the non-linear term and \( V \) is the linear term. Next, we have to find the next generation matrix as \( FV^{-1} \), where

\[ FV^{-1} = \frac{[\gamma (k (1 - x) S + z k \beta S)]}{1 + \gamma \delta}. \]

Now, \( R_0 \) is equal to leading eigenvalue of the next generation matrix \( FV^{-1} \), at disease free equilibrium point \( E_0 = \left( \frac{c}{1}, 0 \right) \), which can be written as

\[ \rho(FV^{-1})_{E_0} = \frac{\nu (k (1 - x) + z k \beta)}{(1 + \gamma \delta) \delta}. \]

So the reproduction number is given by

\[ R_0 = \frac{\nu (k (1 - x) + z k \beta)}{(1 + \gamma \delta) \delta}. \]

Hence the required result is proved. \( \square \)

**Theorem 2.** Statement "The pandemic free equilibrium point of (2) is locally asymptotically stable if \( R_0 < 1 \) and unstable if \( R_0 > 1 \)."

**Proof.** The “Jacobian matrix” for (2) can be computed as

\[ J = \begin{bmatrix} \frac{\partial \Phi_1}{\partial t} (t, S(t), I(t)) & \frac{\partial \Phi_1}{\partial S} (t, S(t), I(t)) \\ \frac{\partial \Phi_2}{\partial t} (t, S(t), I(t)) & \frac{\partial \Phi_2}{\partial S} (t, S(t), I(t)) \end{bmatrix}, \]

or

\[ J = \begin{bmatrix} -k (1 - x) I - z k \beta I - \delta & -k (1 - x) S - z k \beta S \\ k (1 - x) I + z k \beta I & k (1 - x) S + z k \beta S - \frac{1}{\gamma} - \delta \end{bmatrix}. \]

Using the values of \( E_0 \), we get

\[ J = \begin{bmatrix} -\delta & -k (1 - x) \frac{\gamma}{\gamma} - z k \beta \frac{\gamma}{\gamma} \\ 0 & k (1 - x) \frac{\gamma}{\gamma} + z k \beta \frac{\gamma}{\gamma} - \frac{1}{\gamma} - \delta \end{bmatrix}. \]

Now the characteristics equation can be find as

\[ \text{det}(J - \lambda I) = \begin{vmatrix} -\delta - \lambda & -k (1 - x) \frac{\gamma}{\gamma} - z k \beta \frac{\gamma}{\gamma} \\ 0 & k (1 - x) \frac{\gamma}{\gamma} + z k \beta \frac{\gamma}{\gamma} - \frac{1}{\gamma} - \delta - \lambda \end{vmatrix} = 0. \]

Thus the eigen values are given by

\[ \lambda_1 = -\delta, \]

\[ \lambda_2 = k (1 - x) \frac{\gamma}{\gamma} + z k \beta \frac{\gamma}{\gamma} - \frac{1}{\gamma} - \delta. \]

Further, \( \lambda_2 \) can be written as

\[ \lambda_2 = \frac{\nu (k (1 - x) + z k \beta)}{(1 + \gamma \delta) \delta} - 1. \]

This implies that

\[ \lambda_2 = R_0 - 1 \]

and \( \lambda_2 \) will be negative if \( R_0 < 1 \). So all eigen values are negative, therefore (2) is locally asymptotically stable at \( E_0 \), otherwise the considered system will be unstable. \( \square \)

**Theorem 3.** Statement "The pandemic equilibrium point \( E^* = (S^*, I^*) \) is locally asymptotically stable if \( R_0 > 1 \) and globally asymptotically stable if the minors of Routh-Hurwitz matrix are positive.

**Proof.** Putting the values of \( E^* = (S^*, I^*) \) in (6), we get

\[ J = \begin{bmatrix} -(k (1 - x) + z k \beta I^* - \delta & -(k (1 - x) S^* + z k \beta) S^* \\ (k (1 - x) + z k \beta I^* & (k (1 - x) + z k \beta) S^* - \frac{1}{\gamma} - \delta \end{bmatrix}. \]

After simplification we get

\[ J = \begin{bmatrix} -\frac{\nu (k (1 - x) + z k \beta)}{(1 + \gamma \delta)} - \delta & -\frac{1 + \gamma \delta}{1 + \gamma \delta} \\ 0 & \frac{\nu (k (1 - x) + z k \beta)}{(1 + \gamma \delta)} \end{bmatrix}. \]

Assigning

\[ Y = \frac{\nu (k (1 - x) + z k \beta) - \delta (1 + \gamma \delta)}{1 + \gamma \delta}. \]

Then the characteristics equation becomes

\[ \text{det}(J - \lambda I) = \begin{vmatrix} -Y - \delta - \lambda & -\frac{1 + \gamma \delta}{1 + \gamma \delta} \\ Y & 0 - \lambda \end{vmatrix} = 0, \]

or
\[ j^2 + (Y + \delta) j + \frac{Y(1 + \gamma \delta)}{\gamma} = 0, \]

or

\[ a_0 j^2 + a_1 j + a_2 = 0. \]

Forming Hurwitz matrix, one has

\[
\begin{bmatrix}
  a_0 & 0 \\
  a_2 & a_1 
\end{bmatrix}.
\]

On using Routh-Hurwitz criteria, the minors must be greater than zero. As \( a_0 = 1 > 0 \), and \( a_0 a_1 > 0 \), by putting values of \( a_0 a_1 = (Y + \delta) \) and simplifying we get as

\[ a_0 a_1 = \frac{\alpha (1 - \alpha) + 2k \beta}{1 + \gamma} = R_0 \delta, \]

which yields that \( R_0 > 1 \), hence the considered system is locally and globally asymptotically stable. \( \square \)

4. Qualitative analysis

Now let us discuss some some properties for solution of the fractional order model given in (2). The existence of a physical problem is guaranteed by fixed point theory. Hence we apply “Schauder and Banach fixed point theorems” for proving of the required results. We considers the fractional order model (2) as

\[
\begin{align*}
\frac{d^\alpha}{dt^\alpha} \tilde{S}(t) &= \Phi_1(t, \tilde{S}(t), \tilde{T}(t)), \\
\frac{d^\beta}{dt^\beta} \tilde{T}(t) &= \Phi_2(t, \tilde{S}(t), \tilde{T}(t)), \\
\tilde{S}(0) &= \xi_0, \quad \tilde{T}(0) = \eta_0, \quad 0 < \alpha, \beta \leq 1.
\end{align*}
\]

(8)

On using integral with order \( x \in [0, 1] \) on both sides of (8), to obtain the nonlinear integral equations as under:

\[
\begin{align*}
\tilde{S}(t) &= \tilde{S}_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} \Phi_1(\xi, \tilde{S}(\xi), \tilde{T}(\xi))d\xi, \\
\tilde{T}(t) &= \tilde{T}_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t - \xi)^{\beta-1} \Phi_2(\xi, \tilde{S}(\xi), \tilde{T}(\xi))d\xi.
\end{align*}
\]

(9)

Further let \( 0 \leq t \leq T < \infty \), we take the Banach space by \( \mathcal{E}_1 = C([0, T] \times \mathbb{R}^2, \mathbb{R}_+) \), clearly the product \( \mathcal{E} = \mathcal{E}_1 \times \mathcal{E}_2 \) is also “Banach space” under the norm \( \|S, T\| = \sup_{t \in [0, T]} |S(t)| + \sup_{t \in [0, T]} |T(t)| \). Expressing the system (9) as

\[
\begin{align*}
\tilde{X}(t) &= \tilde{X}_0(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} \Psi(\xi, \tilde{X}(\xi))d\xi, \\
\tilde{T}(t) &= \tilde{T}_0(t) + \frac{1}{\Gamma(\beta)} \int_0^t (t - \xi)^{\beta-1} \Psi(\xi, \tilde{T}(\xi))d\xi,
\end{align*}
\]

(10)

where

\[ \tilde{X}(t) = \begin{bmatrix} S(t) \\ T(t) \end{bmatrix}, \quad \tilde{X}_0(t) = \begin{bmatrix} S_0(t) \\ T_0(t) \end{bmatrix}, \quad \Psi(t, \tilde{X}(t)) = \begin{bmatrix} \Phi_1(t, S(t), T(t)) \\ \Phi_2(t, S(t), T(t)) \end{bmatrix}. \]

(11)

For obtaining the “existence and uniqueness”, we assume some growth conditions on function vector \( \Psi: [0, T] \times \mathbb{R}_+^2 \rightarrow \mathbb{R}_+ \) as:

(E1) \( \exists \ L_\Psi > 0 \) for each \( \tilde{X}(t), \tilde{X}(t) \in \mathbb{R} \times \mathbb{R} \) such that

\[ |\Psi(t, \tilde{X}(t)) - \Psi(t, \tilde{X}(\xi))| \leq \frac{L_\Psi}{2} |\tilde{X}(t) - \tilde{X}(\xi)|, \]

(E2) \( \exists \ C_\Psi > 0 \) and \( M_\Psi > 0 \) such that

\[ |\Psi(t, \tilde{X}(t))| \leq C_\Psi |\tilde{X}| + M_\Psi. \]

Theorem 4. Under the continuity of \( \Psi \) together with (E2), system (8) has one or more than one solution.

Proof. By “Schauder fixed point theorem”, we will derive the required result. Let us take a closed subset \( A \) of \( \mathcal{E} \) as

\[ B = \{ \tilde{X} \in \mathcal{E} : \|\tilde{X}\| \leq R, \quad R > 0 \}. \]

We define an operator \( B : A \rightarrow A \) by using (10) as

\[ B(\tilde{X}) = \tilde{X}_0(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} \Psi(\xi, \tilde{X}(\xi))d\xi, \]

(12)

For any \( \tilde{X} \in A \), we have

\[ |B(\tilde{X})(t)| \leq |\tilde{X}_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} |\Psi(\xi, \tilde{X}(\xi))|d\xi, \]

\[ \leq |\tilde{X}_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} (C_\Psi |\tilde{X}| + M_\Psi)d\xi, \]

\[ \leq |\tilde{X}_0| + \frac{1}{\Gamma(\alpha + 1)} (C_\Psi R + M_\Psi) |\tilde{X}| + M_\Psi R, \]

(13)

which implies that

\[ \|B(\tilde{X})\| \leq \|\tilde{X}_0\| + \frac{1}{\Gamma(\alpha + 1)} (C_\Psi R + M_\Psi) \|\tilde{X}\| + M_\Psi R. \]

From (13), one implies that \( \tilde{X} \in A \). Thus \( B(A) \subseteq A \). Also it reveals that operator \( B \) is bounded. For completely continuity we proceed as:

Let \( t_1 < t_2 \in [0, T] \), then consider

\[ |B(\tilde{X})(t_2) - B(\tilde{X})(t_1)| \leq \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - \xi)^{\alpha-1} |\Psi(\xi, \tilde{X}(\xi)) - \Psi(\xi, \tilde{X}(\xi))|d\xi, \]

\[ \leq \frac{1}{\Gamma(\alpha)} \left[ \int_0^{t_2} |\Psi(\xi, \tilde{X}(\xi)) - \Psi(\xi, \tilde{X}(\xi))|d\xi \right], \]

\[ \leq \left[ C_\Psi R + M_\Psi \right] |\tilde{X}| + (t_2 - t_1)^\alpha. \]

(14)

Now from (14), as \( t_1 \rightarrow t_2 \), then right side tends to zero. Hence we see that

\[ |B(\tilde{X})(t_2) - B(\tilde{X})(t_1)| \rightarrow 0, \quad \text{as} \quad t_1 \rightarrow t_2. \]

Consequently we claim that

\[ \|B(\tilde{X})(t_2) - B(\tilde{X})(t_1)\| \rightarrow 0, \quad \text{as} \quad t_1 \rightarrow t_2. \]

Hence \( B \) is equi- continues operator. applying “Arzelá-Ascoli theorem”, the function \( B \) is completely continues function and showed that it is bounded uniformly. By “Schauder’s fixed point theorem” the given model (2) has one or more than one solution. \( \square \)

Now here we provide results about “uniqueness” of solution as:

Theorem 5. Along with (E1), the consider system has unique solution if \( \frac{1}{\Gamma(\alpha + 1)} L_\Psi < 1 \).

Proof. As \( B : \mathcal{E} \rightarrow \mathcal{E} \) defined already, we take \( \tilde{X} \) and \( \tilde{X} \in \mathcal{E} \) and consider as
\[ \|B(x) - B(\tilde{x})\| = \sup_{t \in [0,T]} \left| \frac{T}{\Gamma(\theta + 1)} \int_0^t (t-s)^{\theta-1} \Psi(N, \tilde{x}(N))ds \right|, \]

(15), implies

\[ \|B(x) - B(\tilde{x})\| \leq \frac{T^\theta}{\Gamma(\theta + 1)} L_\Psi \|x - \tilde{x}\|. \]

Hence \( B \) is “contraction”. By “Banach contraction theorem” the considered system has one solution. \( \square \)

Further some concluding remark for stability of numerical results, we take a small change \( \phi \in C[0, T] \) as \( \phi : \phi(0) = 0 \) depends only on the solution \( x \) as.

- \( \|\phi(t)\| \leq \varepsilon, \) for \( \varepsilon > 0; \)
- \( \frac{d}{dt}x(t) = \Psi(t, x(t)) + \phi(t). \)

**Lemma 3.** The solution of the above changed problem

\[ \frac{d}{dt}x(t) = \Psi(t, x(t)) + \phi(t), \]

\[ x(0) = x_0, \]

satisfies the given relation

\[ \|x(t) - (x_0(t) + \frac{T}{\Gamma(\theta + 1)} \int_0^t (t-s)^{\theta-1} \Psi(N, x(N))ds)\| \leq \frac{T^\theta}{\Gamma(\theta + 1)} \varepsilon = \Omega_{x,x_0} \varepsilon. \]

**Proof.** The proof is straightforward so we omit. \( \square \)

**Theorem 6.** Under assumption (E2) together with (18), the solution of the integral Eq. (10) is “Ulam-Hyers stable” and consequently, the numerical results of the considered model are “Ulam-Hyers stable” if \( \Delta = \frac{T^\theta}{\Gamma(\theta + 1)} L_\Psi < 1. \)

**Proof.** Let \( x \in \mathcal{E} \) be any solution and \( \tilde{x} \in \mathcal{E} \) be at most one solution of (10), then

\[ |x(t) - \tilde{x}(t)| \leq \left| x_0(t) - (x_0(t) + \frac{T}{\Gamma(\theta + 1)} \int_0^t (t-s)^{\theta-1} \Psi(N, \tilde{x}(N))ds)\right|, \]

\[ \leq |x(t) - (x_0(t) + \frac{T}{\Gamma(\theta + 1)} \int_0^t (t-s)^{\theta-1} \Psi(N, \tilde{x}(N))ds)\| \]

\[ + \frac{T}{\Gamma(\theta + 1)} \int_0^t (t-s)^{\theta-1} \Psi(N, \tilde{x}(N))ds \]

\[ \leq \Omega_{x,x_0} \varepsilon + \Delta \|x - \tilde{x}\|. \]

From (10), we have

\[ \|x - \tilde{x}\| \leq \frac{\Omega_{x,x_0}}{1 - \Delta} \varepsilon. \]

Hence from (10), we conclude that the solution of (10) is “Ulam-Hyers stable” and consequently the solution of the considered system is “Ulam - Hyers stable”. \( \square \)

5. Numerical solution

Now here we have to evaluate approximate solution of the Caputo fractional order model (2) and the numerical simulations will be achieved by the considered iterative method. For this, we apply the arbitrary order Caputo derivative to establish a numerical procedure for the simulation of our considered model (2). To develop a numerical scheme, we go ahead with the model (2) as

\[ \begin{aligned}
 &\frac{d^\theta}{dt^\theta} S(t) = \Phi_1(t, S(t), I(t)) = a - k_1 S(t) I(t) - k_2 S(t) I(t) - k_3 S(t) I(t) - \delta S(t), \\
 &\frac{d^\theta}{dt^\theta} I(t) = \Phi_2(t, S(t), I(t)) = k_1 S(t) I(t) + k_2 S(t) I(t) + k_3 S(t) I(t) - I(t) - R(t), \\
 &S(0) = S_0, \quad I(0) = I_0, \quad 0 < \theta \leq 1, \quad t > 0.
\end{aligned} \]

(21)

Let \( [0, \theta] \) be a set of points, on which we must have to evaluate the series solution of the model (21). Actually, we cannot not evaluate the functions \( S(t), I(t), \) which will be the solution the IVP (21). Instead of this, an interval \( [t_p, t_{p+1}] \) of equal difference \( h = \theta/n \) only using the nodes \( t_p = ph \), for \( p = 0, 1, \cdots, n. \) Consider that

\[ \begin{aligned}
 &S(t), I(t), \frac{d}{dt}S(t), \frac{d^2}{dt^2}S(t), \frac{d^3}{dt^3}S(t), \frac{d^4}{dt^4}S(t), \frac{d^5}{dt^5}S(t), \frac{d^6}{dt^6}S(t),
\end{aligned} \]

are continuous on \( [0, T]. \) Applying the generalized Euler’s or Taylor’s method about \( t = t_0 = 0 \) to the considered model expressed in (21) and for each value \( t \) taking \( a \in (0, T), \) the expression for \( t_1, \) we have (see Table 1)

\[ \begin{aligned}
 S(t_1) = S(t_0) + \Phi_1(t_0, S(t_0), I(t_0)) \frac{d^\theta}{dt^\theta} S(t_0) + \frac{d^2}{dt^2}S(t_0) + \frac{d^3}{dt^3}S(t_0) + \frac{d^4}{dt^4}S(t_0) + \frac{d^5}{dt^5}S(t_0) + \frac{d^6}{dt^6}S(t_0), \\
 I(t_1) = I(t_0) + \Phi_2(t_0, S(t_0), I(t_0)) \frac{d^\theta}{dt^\theta} I(t_0) + \frac{d^2}{dt^2}I(t_0) + \frac{d^3}{dt^3}I(t_0) + \frac{d^4}{dt^4}I(t_0) + \frac{d^5}{dt^5}I(t_0) + \frac{d^6}{dt^6}I(t_0).
\end{aligned} \]

(22)

Let the difference between two successive point is \( h \) will be chosen small enough, then we may ignore the higher-order term from involving \( h^{10} \) and get the results from (22) as

\[ \begin{aligned}
 S(t_1) = S(t_0) + \Phi_1(t_0, S(t_0), I(t_0)) \frac{d^\theta}{dt^\theta} S(t_0) + \frac{d^2}{dt^2}S(t_0) + \frac{d^3}{dt^3}S(t_0) + \frac{d^4}{dt^4}S(t_0) + \frac{d^5}{dt^5}S(t_0) + \frac{d^6}{dt^6}S(t_0), \\
 I(t_1) = I(t_0) + \Phi_2(t_0, S(t_0), I(t_0)) \frac{d^\theta}{dt^\theta} I(t_0) + \frac{d^2}{dt^2}I(t_0) + \frac{d^3}{dt^3}I(t_0) + \frac{d^4}{dt^4}I(t_0) + \frac{d^5}{dt^5}I(t_0) + \frac{d^6}{dt^6}I(t_0).
\end{aligned} \]

(23)

On repeating the same fashion, a sequence of points that approximates the solution \( (S(t), I(t)) \) is formed. A general formula about \( t_{p+1} = t_p + h \)

| Parameters | Description | Numerical value |
|------------|-------------|-----------------|
| \( S_0 \)  | Initial value of susceptible class | 220 millions [39] |
| \( I_0 \)  | Initial value of infected class | 0.142 million [39] |
| \( a \)    | Recruitment rate | 0.00009 |
| \( \alpha \) | Rate of isolating the people | 50%, 70%, 90% |
| \( \beta \) | Protective measures rate | 0.00078 |
| \( \delta \) | Natural death rate | 0.019 [39] |
| \( \gamma \) | Removal rate of infection | 100 |
| \( k \)    | Total infection rate | 0.0009 |
where $p = 0, 1, 2, \cdots, n - 1$.

6. Numerical results and discussion

Here we take the numerical values for the proposed model in the given table. The concerned data we have taken for Pakistan. Further the total population of the country is $N = 220.0142$ millions.

6.1. Case-I, when $\alpha = 0.50$

Corresponding to this data we provide the dynamics of arbitrary order model 2 using the given numerical scheme in (24). From Fig. 3, we see that in coming few months the susceptibility class will decrease with very rapid rate. The concerned decay will be faster at smaller fractional order as compared to larger arbitrary order. While on the given data
in coming three months the infection may raised up to 0.9 million in Pakistan. The respective raise is high at smaller fractional order and as the order increases the concerned rate of infection becomes slow (see Fig. 4).

6.2. Case-II, when $a = 0.70$

Here we increase the isolation rate from 0.50 to 0.70 and present the solution through graphs Figs. 5 and 6. We see that as the susceptibility is decreasing, then the concerned infection is increasing but to increase isolation among the people. Hence, we predict that in coming three months under the 70% isolation the maximum infection may reach nearly 0.25 million. This amount is very less than the previous case which demonstrate the effect of isolation or implementing strictness following the precautionary measure.

6.3. Case-III, when $a > 0.70$

Here we remark that for higher degree to percent of social distance $a > 70\%$, the considered system shows a rapid decrease in the population of infections class and take more time to reach
the maximum (peak), stopping the COVID-19 epidemic. Hence our approximate solutions would result in much less total mortality and hospitalization requirements on peak in comparison to the current situation in the country. In this duration, this comes with extending the time of disease to take a year. The concerned dynamical behavior has been demonstrated in Figs. 7 and 8 respectively. Next in Fig. 9, we provide the comparison between simulated data for infected class for the last 120 days in Pakistan and real data. From Fig. 9, we observe exponential growth of infection in Pakistan in the last 120 days.

7. Conclusion

Here in this study we have considered an SI model for the prediction of COVID-19 in Pakistan and its evolution under Caputo fractional order derivative. We have proved the global and local stability for the concerned equilibrium points by using the tools of next generation matrix and Routh-Hurwitz criteria. Further the feasibility region for the models and boundedness has been proved by using nonlinear analysis. Some fixed point results for the existence of at least one solution and its Hyers-Ulam stability results have been developed for the model under consideration. Developing modified Euler method, we have established numerical scheme for the graphing of the considered system. By applying actual statistics available for Pakistan, we have simulated the results and its dynamics under the variation of the isolation parameter corresponding to different fractional order. On increasing the isolation parameters means that following the strict precautionary measure will produce good significant effect on the reduction or slowing the transmission of the current infection exponentially. It has been observed that for $x > 70\%$, the consider model showed substantial decrease of the current infection.

Declaration of Competing Interest

There exist no competing interest regarding this manuscript.

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