A CONNECTION BETWEEN MULTIRESOLUTION
WAVELET THEORY OF SCALE $N$ AND
REPRESENTATIONS OF THE CUNTZ ALGEBRA $\mathcal{O}_N$

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1. Introduction

In this paper we will give a short survey of a connection between the theory of wavelets in $L^2(\mathbb{R})$ and certain representations of the Cuntz algebra on $L^2(\mathbb{T})$. This connection was first pointed out in [12] and has been developed further in [5] and [6], and these references contain complete proofs. Basic reference for wavelet theory is [9] and for the Cuntz algebra [7]. Let us emphasize at the outset that this is a field with more questions than answers, and even quite fundamental questions are wide open. For example, it is hard to pinpoint abstractly which representations of $\mathcal{O}_N$ are obtained (although they can be “written down” as we will see), and it is unclear how various equivalence relations between wavelets that one may envisage (same father function up to scaling and translation, etc.) are reflected in equivalence relations between representations (unitary equivalence, quasi-equivalence, etc.). The decomposition theory of the representations has not been obtained in general, although it has been worked out in great detail for related representations in [2], [4], [5], [6], and [8].

2. From wavelets to representations

Since wavelet theory of scale $N$ seems non-standard in the literature (but see [11]), we will give it a rundown here (see [6, Section 10] for proofs). Define scaling by $N$ on $L^2(\mathbb{R})$ by the unitary operator

$$ (U\xi)(x) = N^{-\frac{1}{2}}\xi(N^{-1}x) \quad \text{for } \xi \in L^2(\mathbb{R}), x \in \mathbb{R}. \quad (2.1) $$
Let the father function be a unit vector $\varphi \in L^2(\mathbb{R})$, and let $\mathcal{V}_0$ be the closed linear span of the translates $T^k\varphi$, $k \in \mathbb{Z}$, where

$$T^k\xi(x) = \xi(x - k)$$

is translation by $k$. One assumes that $\varphi$ has the properties

\begin{enumerate}[(2.3)]
  \item $\{T^k\varphi\}_{k \in \mathbb{Z}}$ is an orthonormal set in $L^2(\mathbb{R})$,
  \item $U\varphi \in \mathcal{V}_0$,
  \item $\bigwedge_{n \in \mathbb{Z}} U^n\mathcal{V}_0 = \{0\}$,
  \item $\bigvee_{n \in \mathbb{Z}} U^n\mathcal{V}_0 = L^2(\mathbb{R})$.
\end{enumerate}

One example is the Haar father function: $\varphi(x) = \chi_{[0,1]}(x)$. By (2.3) we may define an isometry $F_\varphi : \mathcal{V}_0 \to L^2(\mathbb{T}) : \xi \to m$ as follows: if

$$\xi(\cdot) = \sum_n b_n \varphi(\cdot - n)$$

then

$$m(t) = m(e^{-it}) = \sum_n b_n e^{-int}$$

and we have the connection

$$\hat{\dot{\xi}}(t) = m(t)\hat{\varphi}(t),$$

where $\xi \to \hat{\dot{\xi}}$ is the Fourier transform, normalized so that $\|\xi\|_2 = \|\hat{\dot{\xi}}\|_2$. In particular, if $\xi \in \mathcal{V}_{-1} = U^{-1}\mathcal{V}_0$, then $U\xi \in \mathcal{V}_0$, and then

$$m_\xi = F_\varphi(U\xi) \in L^2(\mathbb{T}),$$

and

$$\sqrt{N}\hat{\xi}(Nt) = m_\xi(t)\hat{\varphi}(t).$$

In particular, using (2.4), we define

$$m_0(t) = m_\varphi(t).$$

Now the condition (2.3) is equivalent to

$$\text{PER}(|\hat{\varphi}|^2)(t) = \sum_k |\hat{\varphi}(t + 2\pi k)|^2 = (2\pi)^{-1},$$

and this implies

$$\sum_{k=0}^{N-1} |m_0(t + 2\pi k/N)|^2 = N.$$
If $\xi, \eta \in U^{-1}V_0$, then $\xi$ and $T^k\eta$ are orthogonal for all $k \in \mathbb{Z}$ if and only if

$$
\sum_{k=0}^{N-1} \tilde{m}_\xi(t + 2\pi k/N)m_\eta(t + 2\pi k/N) = 0
$$

for almost all $t \in \mathbb{R}$, and $\{\xi(\cdot - k)\}_{k \in \mathbb{Z}}$ is an orthonormal set if and only if

$$
\sum_{k=0}^{N-1} |m_\xi(t + 2\pi k/N)|^2 = N.
$$

With $m_0$ already given, we now choose $m_1, \ldots, m_{N-1}$ in $L^2(\mathbb{T})$ such that

$$
\sum_{k=0}^{N-1} \tilde{m}_i(t + 2\pi k/N)m_j(t + 2\pi k/N) = \delta_{ij}N
$$

for all $t \in \mathbb{R}$, $i, j = 0, \ldots, N-1$. If we define $\psi_1, \ldots, \psi_{N-1}$ by

$$
\sqrt{N}\psi_i(NT) = m_i(t)\hat{\phi}(t)
$$

for $t \in \mathbb{R}$, $i = 1, \ldots, N-1$, it follows that $\{T^k\psi_1\}_{k \in \mathbb{Z}, i \in \{1, \ldots, N-1\}}$ form an orthonormal basis for $V_{-1} \cap V_0^+$, and hence $\{U^nT^k\psi_i\}_{k \in \mathbb{Z}, i \in \{1, \ldots, N-1\}}$ form an orthonormal basis for $L^2(\mathbb{R})$. The functions $\psi_1, \ldots, \psi_{N-1}$ are called mother functions. They are not unique, but depend on the choice of the functions $m_1, \ldots, m_{N-1}$ satisfying (2.16).

If $\rho = \rho_N = e^{2\pi i/N}$, the condition (2.16) translates into the requirement that the $N \times N$ matrix

$$
\frac{1}{\sqrt{N}} \begin{pmatrix}
    m_0(z) & m_0(\rho z) & \ldots & m_0(\rho^{N-1} z) \\
    m_1(z) & m_1(\rho z) & \ldots & m_1(\rho^{N-1} z) \\
    \vdots & \vdots & \ddots & \vdots \\
    m_{N-1}(z) & m_{N-1}(\rho z) & \ldots & m_{N-1}(\rho^{N-1} z)
\end{pmatrix}
$$

is unitary for almost all $z \in \mathbb{T}$.

Now, this is again equivalent to saying that the operators $S_i$ defined on $L^2(\mathbb{T})$ by

$$
(S_i\xi)(z) = m_i(z)\xi(z^N)
$$

for $\xi \in L^2(\mathbb{T})$, $z \in \mathbb{T}$, $i = 0, 1, \ldots, N-1$ satisfy the relations

$$
S_j^*S_i = \delta_{ij}1
$$

$$
\sum_{i=0}^{N-1} S_iS_i^* = 1,
$$

which are exactly the Cuntz relations. This defines the map from the $N$-scale multiresolution wavelet $\{\varphi, \psi_1, \ldots, \psi_{N-1}\}$ into representations of $O_N$. 
3. From representations to wavelets

When, conversely, does a representation of the Cuntz algebra $\mathcal{O}_N$ give rise to a multiresolution wavelet $\{\varphi, \psi_1, \ldots, \psi_{N-1}\}$ such that one can recover the representation again from the wavelet by the construction in Section 2? A minimal requirement is that the representation acts on $L^2(T)$ by formula (2.19), and then unitarity of (2.18) is assured from the Cuntz relations. For any representation of $\mathcal{O}_N$ on a Hilbert space $\mathcal{H}$ we may define an associated endomorphism $\sigma$ of $B(\mathcal{H})$ by $\sigma(\cdot) = \sum_{i=0}^{N-1} S_i \cdot S_i^*$ (see, e.g., [2]). When $\mathcal{H} = L^2(T)$, and $S_i$ is given by (2.19), a simple computation, using unitarity of (2.18), shows that

\[(3.1) \quad \sigma(M_f) = M_{\bar{\sigma}(f)}\]

for all $f \in L^\infty(T)$, where $M_f$ is the operator of multiplication by $f$ on $L^2(T)$, and $\bar{\sigma}(f)(z) = f(z^N)$. Conversely,

**Proposition 3.1.** ([6, Proposition 1.1]) If $S_0, \ldots, S_{N-1}$ is a representation of $\mathcal{O}_N$ on $L^2(T)$ and

\[(3.2) \quad \sum_{i=0}^{N-1} S_i M_f S_i^* = M_{\bar{\sigma}(f)}\]

for all $f \in L^\infty(T)$, then $S_i$ has the form (2.19) with

\[(3.3) \quad m_i = S_i \mathbb{1}.\]

**Proof.** If $f \in L^\infty(T) \subset L^2(T)$ then

\[(3.4) \quad M_{\bar{\sigma}(f)} S_j = \sum_i S_i M_f S_i^* S_j = S_j M_f,\]

and applying this to $\mathbb{1}$ and using (3.3) we have

\[(3.5) \quad f(z^N)m_j(z) = (S_j f)(z).\]

As $L^\infty(T)$ is dense in $L^2(T)$, (2.19) follows.

In order that the representation shall give rise to wavelets, it is not sufficient that it have the form (2.19), however, as we will discuss further in the next section. Let us for the moment assume that the representation comes from a wavelet satisfying the slight regularity condition that $\hat{\psi}(t)$ is continuous near $t = 0$. Then condition (2.5a) implies $\hat{\psi}(0) \neq 0$ (see [9, Remark 3 after Proposition 5.3.2]). It follows from (2.11) and (2.10) that

\[(3.6) \quad \sqrt{N}\hat{\varphi}(Nt) = m_0(t)\hat{\varphi}(t),\]

and hence $m_0(t)$ is continuous near $t = 0$ and $m_0(0) = \sqrt{N}$. Thus it follows from (2.13) that

\[(3.7) \quad m_0(2\pi k/N) = 0\]
for \( k = 1, \ldots, N - 1 \), and combining this with (3.6) and using a recursive argument we deduce that

\begin{equation}
\hat{\phi}(2\pi k) = 0
\end{equation}

for all \( k \in \mathbb{Z} \setminus \{0\} \). It now follows from (2.12) that

\begin{equation}
|\hat{\phi}(0)| = (2\pi)^{-\frac{1}{2}},
\end{equation}

and by changing \( \hat{\phi} \) by an irrelevant phase factor we may assume \( \hat{\phi}(0) = (2\pi)^{-\frac{1}{2}} \). But an iteration of (3.6) gives

\begin{equation}
\hat{\phi}(t) = \prod_{k=1}^{n} (N^{-\frac{1}{2}} m_0(tN^{-k})) \hat{\phi}(tN^{-n}),
\end{equation}

and as \( \lim_{n \to \infty} \hat{\phi}(tN^{-n}) = \hat{\phi}(0) = (2\pi)^{-\frac{1}{2}} \) we deduce

\begin{equation}
\hat{\phi}(t) = (2\pi)^{-\frac{1}{2}} \prod_{k=1}^{\infty} (N^{-\frac{1}{2}} m_0(tN^{-k})).
\end{equation}

Under even stronger regularity properties on \( \varphi \), for example that \( m_0 \) is Lipshitz continuous near 0, the expansion (3.11) converges absolutely and uniformly on compact sets.

If we view functions \( \xi \) in \( L^2(\mathbb{T}) \) as \( 2\pi \)-periodic functions on \( \mathbb{R} \), it follows from (2.19) that

\begin{equation}
(S_{n}^{0} \xi)(t) = \prod_{k=0}^{n-1} m_0(N^k t) \xi(N^n t),
\end{equation}

and if \( E : L^2(\mathbb{T}) \to L^2(\mathbb{R}) \) is the embedding determined by

\begin{equation}
(E \xi)(t) = \begin{cases} 
\xi(t) & \text{if } -\pi \leq t \leq \pi \\
0 & \text{otherwise}
\end{cases}
\end{equation}

then

\begin{equation}
(U^n ES_{0}^{n} \xi)(t) = \chi_{[-\pi, \pi]}(tN^{-n}) \prod_{k=1}^{n} (N^{-\frac{1}{2}} m_0(N^{-k} t)) \xi(t).
\end{equation}

Thus it follows from (3.11) that

\begin{equation}
\lim_{n \to \infty} U^n ES_{0}^{n} \xi = (2\pi)^{\frac{1}{2}} \hat{\phi} \xi,
\end{equation}

where the convergence is uniform on compact subsets of \( \mathbb{R} \) if \( \xi \in L^\infty(\mathbb{T}) \subset L^2(\mathbb{T}) \). In a similar way, using (2.17) and iteration, one deduces

\begin{equation}
\lim_{n \to \infty} U^n ES_{0}^{n-1} S_i \xi = (2\pi)^{\frac{1}{2}} \hat{\psi} \xi.
\end{equation}

Thus, the formulae (3.15) and (3.16) allow us to recover the wavelet system \( \{ \varphi, \psi_1, \ldots, \psi_{N-1} \} \) from the representation.
4. Which representations may occur?

Are there other criteria than those in Proposition 3.1 ensuring that a representation of $O_N$ has the form (2.19)? A necessary condition can be formulated in terms of the Wold decomposition of the isometries $S_i$. In general, if $S$ is an isometry, define a decreasing sequence of projections by

\[(4.1)\quad E_k = S^k S^{* k}\]

and let $P_U$ be the limit projection

\[(4.2)\quad P_U = \text{s-lim}_{k \to \infty} E_k.\]

Then $SP_U = P_U S$, $P_U S$ is a unitary operator on $P_U \mathcal{H}$, and $(1 - P_U) S$ is a shift on $(1 - P_U) \mathcal{H}$, i.e.,

\[(4.3)\quad \bigcap_{n} S^n (1 - P_U) \mathcal{H} = \{0\}.

The Wold decomposition is

\[(4.4)\quad S = SP_U \oplus S(1 - P_U).\]

**Theorem 4.1.** Let $S$ be an operator on $L^2(\mathbb{T})$ of the form

\[(4.5)\quad (S \xi)(z) = m(z) \xi(z^N)\]

and assume that $S$ is an isometry, i.e.,

\[(4.6)\quad \sum_{k=0}^{N-1} |m(\rho^k z)|^2 = N\]

for almost all $z \in \mathbb{T}$, where $\rho = e^{2\pi i / N}$. It follows that the projection $P_U$ corresponding to the unitary part of the Wold decomposition of $S$ is one- or zero-dimensional. Furthermore, it is one-dimensional if and only if both conditions (4.7) and (4.8) are satisfied.

\[(4.7)\quad |m(z)| = 1 \quad \text{for almost all } z \in \mathbb{T}.\]

\[(4.8)\quad \text{There exists a measurable function } \xi : \mathbb{T} \to \mathbb{T} \text{ and a } \lambda \in \mathbb{T} \text{ such that } m(z) \xi(z^N) = \lambda \xi(z) \quad \text{for almost all } z \in \mathbb{T}.\]

In this case the range of the projection $P_U$ is $\mathbb{C} \xi$.

**Proof.** See [6, Theorem 3.1]. This paper also contains more general versions of Theorem 4.1.

Now, combining (4.7) with (2.17) and using the ergodicity of $z \mapsto z^N$ one deduces:
Corollary 4.2. The operators $S_0, \ldots, S_{N-1}$ in the representation of $\mathcal{O}_N$ defined by a wavelet $\{\varphi, \psi_1, \ldots, \psi_{N-1}\}$ have zero unitary part in the Wold decomposition, i.e., they are all shifts.

Proof. See Lemma 9.3 in [6].

Corollary 4.2 gives a rather severe restriction on the representations that can be defined by wavelets. In the same way as a single shift is always isomorphic to a multiple of the shift given by multiplication by $z$ on the Hardy space $H_+(L^2(\mathbb{T}))$, one may use the shift property of $S_i$ (actually it suffices that $S_0$ is a shift for the following construction) to realize the representation $\{S_0, \ldots, S_{N-1}\}$ of $\mathcal{O}_N$ on $\mathcal{K} = L^2(\mathbb{T})$ on the Hilbert space.

\begin{equation}
(4.9) \quad \mathcal{H}_+ \left( \bigoplus_{j=1}^{N-1} \mathcal{K} \right) = \bigoplus_{n=1}^{\infty} \left( \bigoplus_{j=1}^{N-1} \mathcal{K} \right),
\end{equation}

where we view the elements $(\xi_n) \in \mathcal{H}_+(\mathcal{K})$ as the $\mathbb{C}^{N-1} \otimes \mathcal{K}$-valued functions\n
\begin{equation}
(4.10) \quad \xi(z) = \sum_{n=1}^{\infty} \xi_n z^n
\end{equation}
on $\mathbb{T}$, such that $S_0$ is represented by the operator $M_z = \text{multiplication by } z$. To this end we define a unitary operator $V : \mathcal{H}_+ \left( \bigoplus_{j=1}^{N-1} \mathcal{K} \right) \rightarrow \mathcal{K}$ by

\begin{equation}
(4.11) \quad V \left( \sum_{k=1}^{\infty} \left( \bigoplus_{j=1}^{N-1} \psi_k^{(j)} \right) z^k \right) = \sum_{k=1}^{\infty} \sum_{j=1}^{N-1} S_k^{k-1} S_j \psi_k^{(j)}
\end{equation}

The Cuntz relation together with $s\text{-lim}_{k \rightarrow \infty} S_0^k S_0^* = 0$ ensures that $V$ is unitary (see Lemma 6.1 in [6]), and if $S_i^+ = V^* S_i V$ for $i = 0, \ldots, N-1$, one verifies that $S_0^+ = M_z = \text{multiplication by } z$

and

\[ S_i^+ \psi = z \left( \bigoplus_{j=1}^{i-1} 0 \oplus V \psi \oplus \left( \bigoplus_{j=i+1}^{N-1} 0 \right) \right) \quad \text{for } i = 1, \ldots, N - 1. \]

Now, the space $\mathcal{H}_+ \left( \bigoplus_{j=1}^{N-1} \mathcal{K} \right) = \mathbb{C}^{N-1} \otimes \mathcal{K} \otimes H_2^2(\mathbb{T})$, where $H_2^2(\mathbb{T})$ consists of the functions $\xi \in L^2(\mathbb{T})$ with a Fourier expansion of the form $\sum_{n=1}^{\infty} a_n z^n$, has an obvious embedding in $\mathbb{C}^{N-1} \otimes \mathcal{K} \otimes L^2(\mathbb{T})$. If the representation comes from a wavelet $\{\varphi, \psi_1, \ldots, \psi_N\}$, so that $\mathcal{K} = L^2(\mathbb{T})$, one may define a unitary map

\[ J : L^2(\mathbb{R}) \rightarrow \mathbb{C}^{N-1} \otimes \mathcal{K} \otimes L^2(\mathbb{T}) \]

by the requirement

\[ J(U^n T^k \psi_m) (e^{-it}, z) = e_m \otimes e^{-ikt} \otimes z^n \]
for $n, k \in \mathbb{Z}$, $m = 1, \ldots, N - 1$, where $\{e_m\}_{m=1}^{N-1}$ is the standard basis in $\mathbb{C}^{N-1}$. One can then establish that the diagram

\[
\begin{array}{c}
V_0 \xrightarrow{\mathcal{F}_\varphi} \mathcal{K} = L^2(\mathbb{T}) \xrightarrow{V} \mathbb{C}^{N-1} \otimes \mathcal{K} \otimes H^2_+(\mathbb{T}) \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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where the $m_i$ satisfy the unitarity condition (2.18). Let $M_{L^\infty(\mathbb{T})}$ be the image of $L^\infty(\mathbb{T})$ acting as multiplication operators on $L^2(\mathbb{T})$. The following conditions are equivalent:

$$
\begin{align}
(5.2) & \quad \pi(D_N)^{\prime\prime} \subset M_{L^\infty(\mathbb{T})}, \\
(5.3) & \quad \pi(D_N)^{\prime\prime} = M_{L^\infty(\mathbb{T})}, \\
(5.4) & \quad m_j(z) = \sqrt{N}\chi_{A_j}(z)u(z),
\end{align}
$$

where $u : \mathbb{T} \to \mathbb{T}$ is a measurable function, and $A_0, A_1, \ldots, A_{N-1}$ are $N$ measurable subsets of $\mathbb{T}$ with the property that if $\rho = \rho_N = e^{\frac{2\pi i}{N}}$, then for almost all $z \in \mathbb{T}$ the $N$ equidistant points $\rho z, \rho^2 z, \ldots, \rho^{N-1} z$ lie with one in each of the sets $A_0, A_1, \ldots, A_{N-1}$ (i.e., $A_0, \ldots, A_{N-1}$ form a partition of $\mathbb{T}$ up to null-sets, and for each $k$ the sets $A_k, \rho A_k, \ldots, \rho^{N-1} A_k$ form a partition of $\mathbb{T}$). Any $m_i$ of this form does indeed define a representation of $O_N$.

To analyze these representations further, note that by a decoding on $\mathbb{T}$, i.e., a measure-preserving transformation of $\mathbb{T}$, we may assume that $A_k$ is the segment of $\mathbb{T}$ between $\rho^k$ and $\rho^{k+1}$, and then we put

$$
\chi_k(z) = \chi_{A_k}(z).
$$

Now let $S_k^{(j)}$, $j = 1, 2$, $k = 0, \ldots, N-1$ be two representations of this kind, i.e., there exist measurable functions $u^{(j)} : \mathbb{T} \to \mathbb{T}$ such that

$$
(S_k^{(j)}\xi)(z) = \sqrt{N}\chi_k(z)u^{(j)}(z)\xi(z^N).
$$

Then one can show that $T \in B(L^2(\mathbb{T}))$ intertwines the two representations, i.e.,

$$
T\pi^{(1)}(x) = \pi^{(2)}(x)T \quad \text{for all } x \in O_N,
$$

if and only if

$$
T = M_f \quad \text{where } f \in L^\infty(\mathbb{T}) \quad \text{is a function satisfying}
$$

$$
f(z)u^{(1)}(z) = u^{(2)}(z)f(z^N)
$$

for almost all $z \in \mathbb{T}$ (i.e., the cocycles $u^{(1)}, u^{(2)}$ cobound with the coboundary $f$ for the action $z \mapsto z^N$).

(See [6, Proposition 8.1].) Since the map $z \mapsto z^N$ is ergodic (w.r.t. Haar measure on $\mathbb{T}$), it follows that $f$ is unique up to a scalar multiple of a function $T \to \mathbb{T}$ if a nonzero $f$ exists at all. In particular, if $u^{(1)} = u^{(2)}$ then $f(z) = f(z^N)$ and $f$ is a scalar multiple of 1. Thus (see [6, Corollary 8.3]):

**Corollary 5.2.** If the representations $\pi^{(j)}$, $j = 1, 2$, are defined by (5.6), then $\pi^{(j)}$ are irreducible, and the following conditions are equivalent.

$$
\begin{align}
(5.9) & \quad \pi^{(1)} \text{ and } \pi^{(2)} \text{ are unitarily equivalent.} \\
(5.10) & \quad \text{The cocycles } u^{(1)}, u^{(2)} \text{ cobound, i.e., there exists a measurable function } \Delta : \mathbb{T} \to \mathbb{T} \text{ such that}
\end{align}
$$

$$
\Delta(z)u^{(1)}(z) = u^{(2)}(z)\Delta(z^N)
$$

for almost all $z \in \mathbb{T}$.
Let us end this section by mentioning a completely different way of describing states and representations of \( \mathcal{O}_N \) from [4] and [3]. If \( \hat{\omega} \) is a state of \( \mathcal{O}_N \), \( \pi \) the associated representation on \( \mathcal{H} \) with cyclic vector \( \Omega \) and \( S_i = \pi(s_i) \), let \( \mathcal{K} \) be the closed linear span of vectors of the form \( S_{i_1}^* \cdots S_{i_k}^* \Omega \), for \( k = 0, 1, \ldots, i_j \in \{0, \ldots, N-1\} \). For example, if \( \omega \) is a Cuntz state (coherent state), \( \mathcal{K} \) is one-dimensional, and conversely. Let \( P \) be the projection from \( \mathcal{H} \) onto \( \mathcal{K} \), and define

\[
V_i^* = PS_i^*P = S_i^*P \in \mathcal{B}(\mathcal{H}).
\]

The Cuntz relations (2.20)–(2.21) imply that

\[
\sum_{i=0}^{N-1} V_i V_i^* = \mathbb{1}_\mathcal{K},
\]

and if \( \omega = \hat{\omega}|_{\mathcal{B}(\mathcal{K})} \), then

\[
\hat{\omega}(s_{i_1} \cdots s_{i_n} s_{j_m}^* \cdots s_{j_1}^*) = \omega(V_{i_1} \cdots V_{i_n} V_{j_m}^* \cdots V_{j_1}^*)
\]

so \( \hat{\omega} \) is completely determined by the pair \( \omega, \{V_i\} \). Conversely:

**Theorem 5.3.** (Popescu’s reconstruction theorem [13], [14].) If \( V_0, \ldots, V_{N-1} \in \mathcal{B}(\mathcal{K}) \), where \( \mathcal{K} \) is a Hilbert space, \( \Omega \in \mathcal{K} \) is a unit vector cyclic under the polynomials in \( V_0^*, \ldots, V_{N-1}^* \), and \( \sum_{i=0}^{N-1} V_i V_i^* = \mathbb{1}_\mathcal{K} \), then there exists a state \( \hat{\omega} \) on \( \mathcal{O}_N \) such that

\[
\hat{\omega}(s_{i_1} \cdots s_{i_n} s_{j_m}^* \cdots s_{j_1}^*) = \langle V_{i_n}^* \cdots V_{i_1}^* \Omega | V_{j_m}^* \cdots V_{j_1}^* \Omega \rangle,
\]

that is, the \( V_i \)’s have a dilation to a representation of the \( s_i \)’s.

The paper [3] contains characterizations of pure states \( \hat{\omega} \), and states \( \hat{\omega} \) with \( \pi_{\hat{\omega}}(\text{UHF}_N)'' = \mathcal{B}(\mathcal{H}_{\hat{\omega}}) \), in terms of ergodicity properties of the completely positive map \( \varphi(x) = \sum_k V_k \cdot x \cdot V_k^* \) of \( \mathcal{B}(\mathcal{K}) \). For example, \( \hat{\omega} \) is pure if and only if \( \varphi \) is ergodic in the sense that \( \varphi(X) = X \) implies \( X \in \mathbb{C} \mathbb{1}_\mathcal{K} \), and this is again true if and only if \( \{V_i, V_i^*\} \) acts irreducibly on \( \mathcal{K} \) and the projection \( P : \mathcal{H} \to \mathcal{K} \) is contained in \( \pi_{\hat{\omega}}(\mathcal{O}_N)'' \). Furthermore, \( \pi_{\hat{\omega}}(\text{UHF}_N)'' = \mathcal{B}(\mathcal{H}_{\hat{\omega}}) \) if and only if \( \text{Tail}(\varphi) = \mathbb{C} \mathbb{1}_\mathcal{K} \), i.e., all w*-limit points of sequences of the form \( \varphi^{n_k}(X_k) \), where \( n_k \to \infty \) and \( X_k \in \mathcal{B}(\mathcal{K}) \) are uniformly bounded, are multiples of \( \mathbb{1}_\mathcal{K} \). Let us end by citing a proof of Reinhard Werner of Theorem 5.3, which is substantially more direct than the original proof in [13], [14]: With the assumptions in Theorem 5.3, it suffices by Stinespring’s theorem to show that the map \( \hat{R} : \mathcal{O}_d \to \mathcal{B}(\mathcal{K}) \) defined by \( \hat{R}(\mathbb{1}) = \mathbb{1} \) and

\[
\hat{R}(s_{i_1} \cdots s_{i_n} s_{j_m}^* \cdots s_{j_1}^*) = V_{i_1} \cdots V_{i_n} V_{j_m}^* \cdots V_{j_1}^*
\]

is completely positive. Let \( \mathcal{T}_N \) be the Cuntz-Toeplitz algebra, i.e., \( \mathcal{T}_N \) is the *-algebra generated by \( N \) isometries \( s_0, \ldots, s_N \) with orthogonal ranges. It is well known that \( \mathcal{T}_N \) is an extension of \( \mathcal{O}_N \) by the compact operators; see [10], \( \mathcal{T}_N \) has a realization on the unrestricted Fock space \( \mathcal{H} = \bigoplus_{k=0}^\infty (\mathbb{C}^N)^\otimes k \) by
π₀(σ_i,ξ) = |eᵢ⟩ ⊗ ξ, where \{|eᵢ⟩\}_{i=0}^{N-1} is the standard basis of \(\mathbb{C}^N\). Let \(λ ∈ \mathbb{C}\) with \(|λ| < 1\). Define

\[
W_λ : \mathcal{K} → \hat{\mathcal{H}} ⊗ \mathcal{K}
\]

by

\[
W_λ φ = \sqrt{1 - |λ|^2} \bigoplus_{k=0}^{∞} \lambda^k \sum_{i_1 \cdots i_k} |i_1⟩ ⊗ \cdots ⊗ |i_k⟩ ⊗ V_{i_k}^* \cdots V_{i_1}^* φ.
\]

One checks that \(W_λ\) is an isometry, and

\[
(π₀(σ_i)^* ⊗ \mathbb{1}_\mathcal{K})W_λ = λW_λ V_1^*.
\]

Define

\[
R_λ(s_{i_1} \cdots s_{i_n} s_{j_m}^* \cdots s_{j_1}^*) := W_λ^*(π₀(s_{i_1} \cdots s_{i_n} s_{j_m}^* \cdots s_{j_1}^*) ⊗ \mathbb{1}_\mathcal{K})W_λ
\]

\[
= \lambda^n λ^m V_{i_1} \cdots V_{i_n} V_{j_m}^* \cdots V_{j_1}^*.
\]

It follows from this explicit Stinespring representation that \(R_λ\) is completely positive for each \(λ\) with \(|λ| < 1\), and letting \(λ → 1\) it follows that \(R\) is completely positive as a map from \(T_N\) into \(\mathcal{B}(\mathcal{K})\). It remains to show that \(R\) defines a map of the quotient \(\mathcal{O}_N\) of \(T_N\), i.e., that \(R\) annihilates the ideal generated by the projection \(\mathbb{1} - \sum_{i=0}^{N-1} s_i s_i^*\). But this is easily checked from \(\sum_i V_i V_i^* = \mathbb{1}_\mathcal{K}\).

6. Further Results and Problems

Consider \(m_0, m_1 ∈ L^∞(\mathbb{T})\) given, and assume that the matrix

\[
C(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} m_0(z) & m_1(z) \\ m_0(−z) & m_1(−z) \end{pmatrix}
\]

is unitary for almost all \(z ∈ \mathbb{T}\). Then consider the spectral problem of finding \(L^2(\mathbb{T})\)-solutions \(φ\) to

\[
\frac{1}{\sqrt{2}} (m_0(z) φ(z^2) + m_1(z) φ(−z^2)) = λφ(z).
\]

Recall that, with unitarity of (6.1) assumed, the operators

\[
(S,ξ)(z) = m_i(z)ξ(z^2)
\]

define a representation of \(\mathcal{O}_2\) acting on the Hilbert space \(L^2(\mathbb{T})\), and we are concerned, in [5], with the possible decompositions of this class of representations. For our analysis in [6], we introduce a new index,

\[
\text{ind}(π) := \sum_λ \text{ (the dimension of the space of solutions to (6.2))}.
\]

It is motivated in part by Arveson’s index [1]. We denote by \(π\) the representation given by \(π(σ_i) = S_{i_1}\), where \(S_i\) are defined by (6.3). As further motivation, note
that, for any two solutions \( \varphi, \psi \) to (6.2), the associated function \( \{ \varphi, \psi \} \) defined by
\[
z \mapsto \bar{\varphi}(z)\psi(z) + \bar{\varphi}(-z)\psi(-z)
\]
is necessarily constant on \( \mathbb{T} \) (a.e.). We show that the index must take on values as follows:
\[
\text{ind}(\pi) = p \in \{0, 1, 2\}
\]
and then the representation \( \pi \) contains \( \rho_1 \oplus \cdots \oplus \rho_p \) where each representation \( \rho_i \) is isomorphic to one given by the Haar wavelet. (It is understood that the sum is empty if \( p = 0 \).)

In a future paper, we plan to study and refine our new index, with a view to picking up copies of isomorphism classes of wavelets other than the Haar one. Certainly the various families of wavelets due to Daubechies are good candidates. Our analysis so far is based on matrix versions of (6.2) of the form
\[
C(z)\Psi(z^N) = \lambda \Psi(z), \quad z \in \mathbb{T}, \; \lambda \text{ some constant matrix},
\]
where \( C \) is related to (6.1), and \( \Psi \) is a matrix-valued function. Thus as an added problem for future research, we will study further the unitary part in the Wold decomposition of \( L^2(\mathbb{T}) \) corresponding to the given isometric operator
\[
\xi \mapsto \frac{1}{\sqrt{2}}(m_0(z)\xi(z^2) + m_1(z)\xi(-z^2)).
\]

Our preliminary examination indicates that the wavelets, which correspond to the pairs \( m_0, m_1 \) (high pass/low pass filters) for which the isometry in (6.5) has a nonzero unitary part of its Wold decomposition, are precisely the wavelets in \( L^2(\mathbb{R}) \) which are equivalent to the familiar Haar wavelet. But we plan to continue and extend this research, as the idea appears to be also applicable (with modifications and work) to other wavelets. A second line of research, connected with (6.5), is to study the solutions \( \xi \neq 0 \) which (for given \( m_0, m_1 \) as described above, see (6.1)) correspond to the unitary part of the Wold decomposition. It turns out that these solutions \( \xi \) themselves generate quadrature mirror filters and therefore correspond to orthogonal wavelets in \( L^2(\mathbb{R}) \). We hope later to clarify this new form of duality for wavelets in \( L^2(\mathbb{R}) \). As the idea seems basic, it should also be useful (with further modifications) for understanding wavelets in \( L^2(\mathbb{R}^d) \) when \( d > 1 \).

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