RECOVERING A HIDDEN COMMUNITY BEYOND THE SPECTRAL LIMIT IN $O(|E| \log^* |V|)$ TIME

BRUCE HAJEK,* University of Illinois at Urbana-Champaign
YIHONG WU,** Yale University
JIAMING XU,*** Purdue University

Abstract
Community detection is considered for a stochastic block model graph of $n$ vertices, with $K$ vertices in the planted community, edge probability $p$ for pairs of vertices both in the community, and edge probability $q$ for other pairs of vertices. The main focus of the paper is on recovery of the community based on the graph $G$, with $o(K)$ misclassified vertices on average, in the sublinear regime $n^{1-o(1)} \leq K \leq o(n)$. It is shown that such recovery is attainable by a belief propagation algorithm running for $\log^* n + O(1)$ iterations, if $\lambda = K^2(p - q)^2/((n - K)q)$, the signal-to-noise ratio, exceeds $1/e$, with the total time complexity $O(|E| \log^* n)$. Conversely, if $\lambda \leq 1/e$, no local algorithm can asymptotically outperform trivial random guessing. By analyzing a linear message-passing algorithm that corresponds to applying power iteration to the non-backtracking matrix of the graph, we provide evidence to suggest that spectral methods fail to recovery the community if $\lambda \leq 1$. In addition, the belief propagation algorithm can be combined with a linear-time voting procedure to achieve the information limit of exact recovery (correctly classify all vertices with high probability) for all $K \geq \frac{\log n}{\log \log n} (\rho_{BP} + o(1))$, where $\rho_{BP}$ is a function of $p/q$.

* Postal address: Department of ECE and Coordinated Science Lab, University of Illinois at Urbana-Champaign, Urbana, IL 61801
** Postal address: Department of Statistics and Data Science, Yale University, New Haven, CT 06511
*** Postal address: Krannert School of Management, Purdue University, West Lafayette, IN 47907
1. Introduction

The problem of finding a densely connected subgraph in a large graph arises in many research disciplines such as theoretical computer science, statistics, and theoretical physics. To study this problem, the stochastic block model \cite{16} for a single dense community is considered.

**Definition 1.** *(Planted dense subgraph model.)* Given \( n \geq 1 \), \( C^* \subset [n] \), and \( 0 \leq q \leq p \leq 1 \), the corresponding planted dense subgraph model is a random undirected graph \( G = (V, E) \) with \( V = [n] \), such that two vertices are connected by an edge with probability \( p \) if they are both in \( C^* \), and with probability \( q \) otherwise, with the outcomes being mutually independent for distinct pairs of vertices.

The terminology is motivated by the fact that the subgraph induced by the community \( C^* \) is typically denser than the rest of the graph if \( p > q \) \cite{24, 3, 5, 12, 27}. The problem of interest is to recover \( C^* \) based on the graph \( G \).

We consider a sequence of planted dense subgraphs indexed by \( n \) and assume \( p \) and \( q \) depend on \( n \). For a given \( n \), the set \( C^* \) could be deterministic or random. We also introduce \( K \geq 1 \) depending on \( n \), and assume either that \( |C^*| \equiv K \) or \( |C^*|/K \to 1 \) in probability as \( n \to \infty \). Where it matters we specify which assumption holds. For simplicity, we assume the model parameters \((K, p, q)\) are known to the estimators, and impose the mild assumptions that \( K/n \) is bounded away from one and \( p/q \) is bounded.

We primarily focus on two types of recovery guarantees.

**Definition 2.** *(Exact Recovery.)* Given an estimator \( \hat{C} = \hat{C}(G) \subset [n] \), \( \hat{C} \) exactly recovers \( C^* \) if

\[
\lim_{n \to \infty} \mathbb{P}\{\hat{C} \neq C^*\} = 0,
\]

where the probability is taken with respect to the randomness of \( G \) and with respect to possible randomness in \( C^* \) and the algorithm for generating \( \hat{C} \) from \( G \).

It would be nice to remove this assumption. The paper \cite{7} suggests a method for estimating the parameters but we do not know how to incorporate it into our theorems.
Depending on the application, it may be enough to ask for an estimator $\hat{C}$ which almost completely agrees with $C^*$.

**Definition 3.** (Weak Recovery.) Given an estimator $\hat{C} = \hat{C}(G) \subset [n]$, $\hat{C}$ weakly recovers $C^*$ if, as $n \to \infty$, $\frac{1}{n} |\hat{C} \triangle C^*| \to 0$, where the convergence is in probability, and $\triangle$ denotes the set difference.

Exact and weak recovery are the same as strong and weak consistency, respectively, as defined in [29]. Clearly an estimator that exactly recovers $C^*$ also weakly recovers $C^*$. Also, it is not hard to show that the existence of an estimator satisfying Definition 3 is equivalent to the existence of an estimator such that $\mathbb{E}[|\hat{C} \triangle C^*|] = o(K)$ (see [14, Appendix A] for a proof).

Intuitively, if the community size $K$ decreases, or $p$ and $q$ get closer, recovery of the community becomes harder. A critical role is played by the parameter

$$\lambda = \frac{K^2(p-q)^2}{(n-K)q},$$

which can be interpreted as the effective signal-to-noise ratio for classifying a vertex according to its degree. It turns out that if the community size scales linearly with the network size, optimal recovery can be achieved via thresholding in linear time. For example, if $K \asymp n - K \asymp n$ and $p/q$ is bounded, a naïve degree-thresholding algorithm can attain weak recovery in time linear in the number of edges, provided that $\lambda \to \infty$, which is information theoretically necessary when $p$ is bounded away from one. Moreover, one can show that degree-thresholding followed by a linear-time voting procedure achieves exact recovery whenever it is information theoretically possible in this asymptotic regime (see Appendix A for a proof).

Since it is easy to recover a hidden community of size $K = \Theta(n)$ weakly or exactly up to the information limits, we next turn to the sublinear regime where $K = o(n)$. However, detecting and recovering polynomially small communities of size $K = n^{1-\Theta(1)}$ is known [12] to suffer a fundamental computational barrier (see Section 3 for details). In search for the critical point where statistical and computational limits depart, the main focus of this paper is in the slightly sublinear regime of $K = n^{1-o(1)}$ and $np = n^{o(1)}$ and analysis of the belief propagation (BP) algorithm for community recovery.

The belief propagation algorithm is an iterative algorithm which aggregates the
likelihoods computed in the previous iterations with the observations in the current iteration. Running belief propagation for one iteration and then thresholding the beliefs reduces to degree thresholding. Montanari [27] analyzed the performance of the belief propagation algorithm for community recovery in a different regime with \( p = a/n, q = b/n, \) and \( K = \kappa n \), where \( a, b, \kappa \) are assumed to be fixed as \( n \to \infty \). In the limit where first \( n \to \infty \), and then \( \kappa \to 0 \) and \( a, b \to \infty \), it is shown that using a local algorithm, namely belief propagation running for a constant number of iterations, \( \mathbb{E}[|\hat{C} \Delta C^*|] = o(n) \); conversely, if \( \lambda < 1/e \), for all local algorithms, \( \mathbb{E}[|\hat{C} \Delta C^*|] = \Omega(n) \).

However, since we focus on \( K = o(n) \) and weak recovery demands \( \mathbb{E}[|\hat{C} \Delta C^*|] = o(K) \), the following question remains unresolved: Is \( \lambda > 1/e \) the performance limit of belief propagation algorithms for weak recovery when \( K = o(n) \)?

In this paper, we answer positively this question by analyzing belief propagation running for \( \log^* n + O(1) \) iterations. Here, \( \log^*(n) \) is the iterated logarithm, defined as the number of times the logarithm function must be iteratively applied to \( n \) to get a result less than or equal to one. We show that if \( \lambda > 1/e \), weak recovery can be achieved by a belief propagation algorithm running for \( \log^*(n) + O(1) \) iterations, whereas if \( \lambda < 1/e \), all local algorithms including belief propagation cannot asymptotically outperform trivial random guessing without the observation of the graph.

The proof is based on analyzing the analogous belief propagation algorithm to classify the root node of a random tree graph, which is the limit in distribution of the neighborhood of a given vertex in the original graph \( G \). In contrast to the analysis of belief propagation in [27], where the number of iterations is held fixed regardless of the size of graph \( n \), our analysis on the tree and the associated coupling lemmas entail the number of iterations converging slowly to infinity as the size of the graph increases, in order to guarantee adequate performance of the algorithm in the case that \( K = o(n) \). Also, our analysis is mainly based on studying the recursions of exponential moments of beliefs instead of Gaussian approximations as used in [27].

Furthermore, evidence is given to suggest that the spectral limit for weak recovery

Loosely speaking, an algorithm is \( t \)-local, if the computations determining the status of any given vertex \( u \) depend only on the subgraph induced by vertices whose distance to \( u \) is at most \( t \). See [27] for a formal definition. In this paper, \( t \) is allowed to slowly grow with \( n \) so long as \( (2 + np)^t = n^{o(1)} \).
of the community is given by $\lambda > 1$, or, that is, spectral algorithms are inferior to the belief propagation algorithm by a factor of $e$ in terms of the signal-to-noise ratio. The particular algorithm analyzed is a linear message passing algorithm corresponding to applying the power method to the non-backtracking matrix of the graph [23, 4], whose spectrum has been shown to be more informative than that of the adjacency matrix for the purpose of clustering. It is established that this linear message passing algorithm followed by thresholding provides weak recovery if $\lambda > 1$ and it does not improve upon trivial random guessing asymptotically if $\lambda < 1$. The threshold $\lambda = 1$ is also known as the Kesten-Stigum threshold [20, 28].

Finally, we address exact recovery. As shown in [14, Theorem 3], if there is an algorithm that can provide weak recovery even if the community size is random and only approximately equal to $K$, then it can be combined with a linear-time voting procedure to achieve exact recovery whenever it is information-theoretically possible. For $K = o(n)$, we show that both the belief propagation and the linear message-passing algorithms indeed can be upgraded to achieve exact recovery via local voting. Somewhat surprisingly, belief propagation plus voting achieves the information limit of exact recovery if $K \geq \frac{1}{\log n} (\rho_{BP} + o(1))$, where $\rho_{BP}$ is a function of $p/q$.

2. Main results

As mentioned in the introduction, in search for the critical point where statistical and computational limits depart, we focus on the regime where $K$ is slightly sublinear in $n$ and invoke the following assumption.

**Assumption 1.** As $n \to \infty$, $p \geq q$, $p/q = O(1)$, $n^{1-o(1)} \leq K \leq o(n)$, and $\lambda$ is a positive constant.

2.1. Upper and lower bounds for belief propagation

Let $\sigma \in \{0, 1\}^n$ denote the indicator vector of $C^*$ and $A$ denote the adjacency matrix of the graph $G$. To detect whether a given vertex $i$ is in the community, a natural approach is to compare the log likelihood ratio $\log \frac{p(G|\sigma_i=1)}{p(G|\sigma_i=0)}$ to a certain threshold. However, it is often computationally expensive to evaluate the log likelihood ratio. As we show in this paper, when the average degree scales as $n^{o(1)}$, the
neighborhood of vertex $i$ is locally tree-like with high probability; moreover, on the
tree, the log likelihoods can be computed recursively via belief propagation exactly.

These two observations together suggest the following belief propagation algorithm for
approximately computing the log likelihoods for the community recovery problem (See
Lemma 1 for derivation of belief propagation algorithm on tree). Let $\partial i$ denote the set
of neighbors of $i$ in $G$ and

$$
\nu \triangleq \log \frac{n-K}{K}.
$$

Define the message transmitted from vertex $i$ to its neighbor $j$ at $(t+1)$-th iteration
as

$$
R_{i \to j}^{t+1} = -K(p-q) + \sum_{\ell \in \partial i \setminus \{j\}} \log \left( \frac{e^{R_{\ell \to i}^{t+1} - \nu} \left( \frac{\nu}{q} \right) + 1}{e^{R_{\ell \to i}^{t+1} - \nu + 1}} \right).
$$

(2)

for initial conditions $R_i^{0 \to j} = 0$ for all $i \in [n]$ and $j \in \partial i$. Then we approximate
$\log \frac{P(G|\sigma_i=1)}{P(G|\sigma_i=0)}$ by the belief of vertex $i$ at $(t+1)$-th iteration,
$R_i^{t+1}$, which is determined
by combining incoming messages from its neighbors as follows:

$$
R_{i}^{t+1} = -K(p-q) + \sum_{\ell \in \partial i} \log \left( \frac{e^{R_{\ell \to i}^{t} - \nu} \left( \frac{\nu}{q} \right) + 1}{e^{R_{\ell \to i}^{t} - \nu + 1}} \right).
$$

(3)

**Algorithm 1** Belief propagation for weak recovery

1: Input: $n, K \in \mathbb{N}$, $p > q > 0$, adjacency matrix $A \in \{0,1\}^{n \times n}$, $t_f \in \mathbb{N}$
2: Initialize: Set $R_i^{0 \to j} = 0$ for all $i \in [n]$ and $j \in \partial i$.
3: Run $t_f - 1$ iterations of belief propagation as in (2) to compute $R_{i \to j}^{t_f-1}$ for all $i \in [n]$
and $j \in \partial i$.
4: Compute $R_{i}^{t_f}$ for all $i \in [n]$ as per (3).
5: Return $\hat{C}$, the set of $K$ indices in $[n]$ with largest values of $R_{i}^{t_f}$.

**Theorem 1.** Suppose Assumption 1 holds with $\lambda > 1/e$ and $(np)^{\log^* \nu} = n^{o(1)}$. Let
$t_f = t_0 + \log^* (\nu) + 2$, where $t_0$ is a constant depending only on $\lambda$. Let $\hat{C}$ be produced by
Algorithm 1. If the planted dense subgraph model (Definition 1) is such that $|C^*| = K$,
then for any constant $r > 0$, there exists $\nu(r)$ such that for all $\nu \geq \nu(r)$,

$$
E[|C^* \triangle \hat{C}|] \leq n^{o(1)} + 2Ke^{-\nu r}.
$$

(4)
If instead $|C^*|$ is random with $P\{||C^*| - K| \geq \sqrt{3K \log n}\} \leq n^{-1/2+o(1)}$, then
$$E[|C^* \triangle \hat{C}|] \leq n^{1/2+o(1)} + 2Ke^{-\nu r}.$$  \hspace{1cm} (5)

For either assumption about $|C^*|$, weak recovery is achieved: $E[|C^* \triangle \hat{C}|] = o(K)$. The running time is $O(|E(G)| \log^* n)$, where $|E(G)|$ is the number of edges in the graph $G$.

We remark that the same conclusion also holds for the estimator $\hat{C}_o = \{i : R_i^f \geq \nu\}$, but returning a constant size estimator $\hat{C}$ leads to simpler analysis of the algorithm for exact recovery.

Next we discuss how to use the belief propagation (BP) algorithm to achieve exact recovery. The key idea is to attain exact recovery in two steps. In the first step, we apply BP for weak recovery. In the second step, we use a linear-time local voting procedure to clean-up the residual errors made by BP. In particular, for each vertex $i$, we count $r_i$, the number of neighbors in the community estimated by BP, and pick the set of $K$ vertices with the largest values of $r_i$. To facilitate analysis, we adopt the successive withholding method described in [29, 14] to ensure the first and second step are independent of each other. In particular, we first randomly partition the set of vertices into a finite number of subsets. One at a time, one subset is withheld to produce a reduced set of vertices, to which BP is applied. The estimate obtained from the reduced set of vertices is used to classify the vertices in the withheld subset. The idea is to gain independence: the outcome of BP based on the reduced set of vertices is independent of the data corresponding to edges between the withheld vertices and the reduced set of vertices. The full description of the algorithm is given in Algorithm 2.

**Theorem 2.** Suppose Assumption 1 holds with $\lambda > 1/e$ and $(np)^{\log^* \nu} = n^{o(1)}$. Consider the planted dense subgraph model (Definition 1) with $|C^*| \equiv K$. Select $\delta > 0$ so small that $(1 - \delta)\lambda e > 1$. Let $t_f = \tilde{t}_0 + \log^* (\nu) + 2$, where $\tilde{t}_0$ is a constant depending only on $\lambda(1 - \delta)$. Also, suppose $p$ is bounded away from 1 and the following condition is satisfied:
$$\liminf_{n \to \infty} \frac{Kd(\tau^* || q)}{\log n} > 1,$$  \hspace{1cm} (6)

where
$$\tau^* = \frac{\log \frac{1-q}{1-p} + \frac{1}{K} \log \frac{n}{K}}{\log \frac{p(1-q)}{q(1-p)}}.$$  \hspace{1cm} (7)
Algorithm 2 Belief propagation plus cleanup for exact recovery

1: Input: \( n \in \mathbb{N}, K > 0, p > q > 0, \) adjacency matrix \( A \in \{0, 1\}^{n \times n}, t_f \in \mathbb{N}, \) and \( \delta \in (0, 1) \) with \( 1/\delta, n\delta \in \mathbb{N}. \)

2: (Partition): Partition \([n]\) into \( 1/\delta \) subsets \( S_k \) of size \( n\delta, \) uniformly at random.

3: (Approximate Recovery) For each \( k = 1, \ldots, 1/\delta, \) let \( A_k \) denote the restriction of \( A \) to the rows and columns with index in \([n]\) \( \setminus S_k, \) run Algorithm 1 (belief propagation for weak recovery) with input \((n(1-\delta), \lceil K(1-\delta) \rceil, p, q, A_k, t_f)\) and let \( \hat{C}_k \) denote the output.

4: (Cleanup) For each \( k = 1, \ldots, 1/\delta \) compute \( r_i = \sum_{j \in \hat{C}_k} A_{ij} \) for all \( i \in S_k \) and return \( \tilde{C}, \) the set of \( K \) indices in \([n]\) with the largest values of \( r_i. \)

Let \( \tilde{C} \) be produced by Algorithm 2. Then \( \mathbb{P}\{\tilde{C} = C^*\} \to 1 \) as \( n \to \infty. \) The running time is \( O(|E(G)| \log^* n). \)

Note that the condition (6) is shown in [14] to be the necessary (if “>” is replaced by “≥”) and sufficient condition for the success of clean-up procedure in upgrading weak recovery to exact recovery.

We comment briefly on some implementation issues for Algorithm 2. The assumption \( n\delta \in \mathbb{N} \) is an integer is only for notational convenience. If we drop that assumption, and continue to assume \( \frac{1}{\delta} \in \mathbb{N}, \) and if \( n \geq \left(\frac{1}{\delta} + 1\right)^2, \) we could partition \([n]\) into \( \frac{1}{\delta} + 1 \) subsets, the first \( \frac{1}{\delta} \) of which have cardinality \( \lfloor n\delta \rfloor, \) and the last of which has cardinality less than or equal to \( \lfloor n\delta \rfloor. \) The proof of Theorem 2 then goes through with minor modifications. Also, the constant \( \delta \) does not need to be extremely small to allow \( \lambda \) to be reasonably close to \( 1/e. \) For example, if we take \( \delta = 1/11, \) the condition on \( \lambda \) in Theorem 2 becomes \( \lambda > \frac{11}{e}. \)

Next, we provide a lower bound on the error probability achievable by any local algorithm for estimating the label \( \sigma_u \) of a given vertex \( u. \) Let \( p_e = \pi_0 p_{e,0} + \pi_1 p_{e,1} \) for prior probabilities \( \pi_0 = (n-K)/n \) and \( \pi_1 = K/n, \) where \( p_{e,0} = \mathbb{P}\{\hat{\sigma}_u = 1|\sigma_u = 0\} \) and \( p_{e,1} = \mathbb{P}\{\hat{\sigma}_u = 0|\sigma_u = 1\}. \)

Theorem 3. (Converse for local algorithms.) Suppose Assumption 1 holds with \( 0 < \lambda \leq 1/e. \) Let \( t_f \in \mathbb{N} \) depend on \( n \) such that \((2 + np)^{t_f} = n^{o(1)}. \) Consider the planted dense subgraph model (Definition 1) with \( C^* \) random and uniformly distributed over all subsets of \([n]\) such that \( |C^*| = K. \) Then for any estimator \( \hat{C} \) such that for each vertex
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$u$ in $G$, $\sigma_u$ is estimated based on $G$ in a neighborhood of radius $t_f$ from $u$,

$$E[|\hat{C} \triangle C^*|] \geq \frac{K(n-K)}{n} \exp(-\lambda e/4) - n^{o(1)}. \quad (8)$$

and

$$p_{e,0} + p_{e,1} \geq \frac{1}{2} e^{-1/4} - n^{-1+o(1)}. \quad (9)$$

Furthermore, $\liminf_{n \to \infty} \frac{np_e}{K} \geq 1$, or, equivalently,

$$\liminf_{n \to \infty} \frac{E[|\hat{C} \triangle C^*|]}{K} \geq 1. \quad (10)$$

The assumption $(2 + np)^{t_f} = n^{o(1)}$ is needed to ensure the neighborhood of radius $t_f$ from any given vertex $u$ is a tree with high probability.

Note that an estimator is said to achieve weak recovery in $[27]$, if $\lim_{n \to \infty} p_{e,0} + p_{e,1} = 0$. Condition (9) shows that weak recovery in this sense is not possible. If $C^*$ is uniformly distributed over $\{C \subset [n] : |C| = K\}$, among all estimators that disregard the graph, the one that minimizes the mean number of classification errors is $\hat{C} \equiv \emptyset$ (declaring no community), which achieves $\frac{E[|\hat{C} \triangle C^*|]}{K} = 1$, or equivalently, $p_e = K/n$. Condition (10) shows that in the asymptotic regime $\nu \to \infty$ with $\lambda < 1/e$, improving upon random guessing is impossible.

2.2. Upper and lower bounds for linear message passing

Results are given in this section to suggest that the spectral limit for weak recovery is given by $\lambda > 1$. Spectral algorithms estimate the communities based on the principal eigenvectors of the adjacency matrix, see, e.g., $[1, 24, 33]$ and the reference therein. Under the single community model, $E[A] = (p-q)(\sigma \sigma^\top - \text{diag} \{\sigma\}) + q(J - I)$, where $\text{diag} \{\sigma\}$ denotes the diagonal matrix with the diagonal entries given by $\sigma$; $I$ denotes the identity matrix and $J$ denotes the all-one matrix. By the Davis-Kahan sin $\theta$ theorem $[6]$, the principal eigenvector of $A - q(J - I)$ is almost parallel to $\sigma$ provided that the spectral norm $\|A - E[A]\|$ is much smaller than $K(p-q)$; thus one can estimate $C^*$ by thresholding the principal eigenvector entry-wise. Therefore, if we apply the spectral method, a natural matrix to start with is $A - q(J - I)$, or $A - qJ$. Finding the principal eigenvector of $A - qJ$ according to the power method is done by starting with some vector and repeatedly multiplying by $A - qJ$ sufficiently many times. We shall consider
the scaled matrix \( \frac{A-qJ}{\sqrt{m}} \) where \( m = (n-K)q \). Of course the scaling doesn’t change the eigenvectors. This suggests the following linear message passing update equation:

\[
\theta_{t+1}^i = -\frac{q}{\sqrt{m}} \sum_{\ell \in [n]} \theta_{t}^\ell + \frac{1}{\sqrt{m}} \sum_{\ell \in \partial i} \theta_{t}^{\ell}. \tag{11}
\]

The first sum is over all vertices in the graph and doesn’t depend on \( i \). An idea is to appeal to the law of large numbers and replace the first sum by its expectation. Also, in the sparse graph regime \( np = o(\log n) \), there exist vertices of high degrees \( \omega(np) \), and the spectrum of \( A \) is very sensitive to high-degree vertices (see, e.g., [13, Appendix A] for a proof). To deal with this issue, as proposed in [23, 4], we associate the messages in (11) with directed edges and prevent the message transmitted from \( j \) to \( i \) from being immediately reflected back as a term in the next message from \( i \) to \( j \), resulting in the following linear message passing algorithm:

\[
\theta_{t+1}^{i \rightarrow j} = -\frac{q((n-K)A_t + KB_t)}{\sqrt{m}} + \frac{1}{\sqrt{m}} \sum_{\ell \in \partial i \setminus \{j\}} \theta_{t}^{\ell \rightarrow i}. \tag{12}
\]

with initial values \( \theta_{t}^{0 \rightarrow i} = 1 \), where \( A_t \approx \mathbb{E}\left[\theta_{t}^{\ell \rightarrow i} | \sigma_{\ell} = 0 \right] \) and \( B_t \approx \mathbb{E}\left[\theta_{t}^{\ell \rightarrow i} | \sigma_{\ell} = 1 \right] \). Notice that when computing \( \theta_{t+1}^{i \rightarrow j} \), the contribution of \( \theta_{t}^{j \rightarrow i} \) is subtracted out. Since we focus on the regime \( np = n^{o(1)} \), the graph is locally tree-like with high probability. In the Poisson random tree limit of the neighborhood of a vertex, the expectations \( \mathbb{E}\left[\theta_{t}^{\ell \rightarrow i} | \sigma_{\ell} = 0 \right] \) and \( \mathbb{E}\left[\theta_{t}^{\ell \rightarrow i} | \sigma_{\ell} = 1 \right] \) can be calculated exactly, and as a result we take \( A_0 = 1, A_t = 0 \) for \( t \geq 1 \), and \( B_t = \lambda^{t/2} \) for \( t \geq 0 \).

The update equation (12) can be expressed in terms of the non-backtracking matrix associated with graph \( G \). It is the matrix \( B \in \{0,1\}^{2m \times 2m} \) with \( B_{ef} = 1_{\{e_2 = f_1, e_1 \neq f_2\}} \), where \( e = (e_1, e_2) \) and \( f = (f_1, f_2) \) are directed edges. Let \( \Theta_t \in \mathbb{R}^{2m} \) denote the messages on directed edges with \( \Theta_t^{e} = \theta_{e_1 \rightarrow e_2}^{t} \). Then, (12) in matrix form reads

\[
\Theta^{t+1} = -\frac{q((n-K)A_t + KB_t)}{\sqrt{m}} \mathbf{1} + \frac{1}{\sqrt{m}} B^{*} \Theta^{t}.
\]

As shown in [4], the spectral properties of the non-backtracking matrix closely match those of the original adjacency matrix. It is therefore reasonable to take the linear update equation (12) as a form of spectral method for the community recovery problem.

Finally, to estimate \( C^* \), we define the belief at vertex \( u \) as:

\[
\theta_{u}^{t+1} = -\frac{q((n-K)A_t + KB_t)}{\sqrt{m}} + \frac{1}{\sqrt{m}} \sum_{\ell \in \partial u} \theta_{t}^{\ell \rightarrow u}, \tag{13}
\]
and select the vertices \( u \) such that \( \theta_u^t \) exceeds a certain threshold. The full description of the algorithm is given in Algorithm 3.

Algorithm 3 Spectral algorithm for weak recovery

1: Input: \( n, K \in \mathbb{N} \), \( p > q > 0 \), adjacency matrix \( A \in \{0,1\}^{n \times n} \)
2: Set \( \lambda = \frac{K^2(p-q)^2}{(n-K)^2} \) and \( T = \left[ 2\alpha \frac{\log \frac{n-K}{\log \lambda}}{\log \lambda} \right] \), where \( \alpha = 1/4 \) (in fact any \( \alpha < 1 \) works).
3: Initialize: Set \( \theta_0^{i/j} = 1 \) for all \( i \in [n] \) and \( j \in \partial i \).
4: Run \( T - 1 \) iterations of message passing as in (12) to compute \( \theta_{i/j}^{T-1} \) for all \( i \in [n] \) and \( j \in \partial i \).
5: Run one more iteration of message passing to compute \( \theta_i^T \) for all \( i \in [n] \) as per (13).
6: Return \( \hat{C} \), the set of \( K \) indices in \([n]\) with largest values of \( \theta_i^T \).

Theorem 4. Suppose Assumption 1 holds with \( \lambda > 1 \) and \((np)\log(n/K) = n^o(1)\). Consider the planted dense subgraph model (Definition 1) with

\[
\mathbb{P} \left\{ \left| C^* \right| - K \geq \sqrt{3K \log n} \right\} \leq n^{-1/2 + o(1)}.
\]

Let \( \hat{C} \) be the estimator produced by Algorithm 3. Then \( \mathbb{E} \left[ \left| C^* \triangle \hat{C} \right| \right] = o(K) \).

One can upgrade the weak recovery result of linear message passing to exact recovery under condition \( \lambda > 1 \) and condition (6), in a similar manner as described in Algorithm 2 and the proof of Theorem 2.

The next converse shows that if \( \lambda \leq 1 \) then estimating better than the random guessing by linear message passing is not possible.

Theorem 5. (Converse for linear message passing algorithm.) Suppose Assumption 1 holds with \( 0 < \lambda \leq 1 \) and consider the planted dense subgraph model (Definition 1) with \( C^* \) random and uniformly distributed over all subsets of \([n]\) such that \( |C^*| \equiv K \). Assume \( t \in \mathbb{N} \), with \( t \) possibly depending on \( n \) such that \((np)^t = n^o(1)\) and \( t = O(\log \frac{n-K}{K}) \).

Let \( \left\{ \theta_u^t : u \in [n] \right\} \) be computed using the message passing updates (12) and (13) and let \( \hat{C} = \{ u : \theta_u^t \geq \gamma \} \) for some threshold \( \gamma \), which may also depend on \( n \). Equivalently, \( \sigma_u \) is estimated for each \( u \) by \( \hat{\sigma}_u = 1_{\{\theta_u^t \geq \gamma\}} \). Then \( \liminf_{n \to \infty} \frac{n}{K} \geq 1 \).

The proofs of Theorem 4 and Theorem 5 are similar to the counterparts for belief propagation and are given in Appendix E.
3. Related work

The problem of recovering a single community demonstrates a fascinating interplay between statistics and computation and a potential departure between computational and statistical limits.

In the special case of $p = 1$ and $q = 1/2$, the problem of finding one community reduces to the classical planted clique problem [18]. If the clique has size $K \leq 2(1 - \epsilon) \log_2 n$ for any $\epsilon > 0$, then it cannot be uniquely determined; if $K \geq 2(1 + \epsilon) \log_2 n$, an exhaustive search finds the clique with high probability. In contrast, polynomial-time algorithms are only known to find a clique of size $K \geq c\sqrt{n}$ for any constant $c > 0$ [1, 11, 8, 2], and it is shown in [9] that if $K \geq (1 + \epsilon)\sqrt{n/e}$, the clique can be found in $O(n^2 \log n)$ time with high probability and $\sqrt{n/e}$ may be a fundamental limit for solving the planted clique problem in nearly linear time. Recent work [25] shows that the degree-$r$ sum-of-squares (SOS) relaxation cannot find the clique unless $K \gtrsim (\sqrt{n}/\log n)^{1/r}$; an improved lower bound $K \gtrsim n^{1/3}/\log n$ for the degree-4 SOS is proved in [10]. Further improved lower bounds are obtained recently in [17, 32].

Another recent work [12] focuses on the case $p = n^{-\alpha}$, $q = cn^{-\alpha}$ for fixed constants $c < 1$ and $0 < \alpha < 1$, and $K = \Theta(n^\beta)$ for $0 < \beta < 1$. It is shown that no polynomial-time algorithm can attain the information-theoretic threshold of detecting the planted dense subgraph unless the planted clique problem can be solved in polynomial time (see [12, Hypothesis 1] for the precise statement). For exact recovery, MLE succeeds with high probability if $\alpha < \beta < \frac{1}{2} + \frac{1}{4}$; however, no randomized polynomial-time solver exists, conditioned on the same planted clique hardness hypothesis.

In sharp contrast to the computational barriers discussed in the previous two paragraphs, in the regime $p = a \log n/n$ and $q = b \log n/n$ for fixed $a, b$ and $K = \rho n$ for a fixed constant $0 < \rho < 1$, recent work [13] derived a function $\rho^*(a, b)$ such that if $\rho > \rho^*$, exact recovery is achievable in polynomial-time via semidefinite programming relaxations of ML estimation; if $\rho < \rho^*$, any estimator fails to exactly recover the cluster with probability tending to one regardless of the computational costs.

In summary, the previous work revealed that for exact recovery, a significant gap between the information limit and the limit of polynomial-time algorithms emerges as the community size $K$ decreases from $K = \Theta(n)$ to $K = n^\beta$ for $0 < \beta < 1$. In
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search of the exact phase transition point where information and computational limits depart, the present paper further zooms into the regime of \( K = n^{1-o(1)} \). We show in Appendix B that belief propagation plus voting attains the sharp information limit if \( K \geq \frac{n}{\log n}(\rho_{BP}(a/b) + o(1)) \). However, as soon as \( \lim_{n \to \infty} K \log n/n \leq \rho_{BP}(a/b) \), we observe a gap between the information limit and the necessary condition of local algorithms, given by \( \lambda > 1/e \). For weak recovery, as soon as \( K = o(n) \), a gap between the information limit and the necessary condition of local algorithms emerges.

4. Inference problem on a random tree by belief propagation

In the regime we consider, the graph is locally tree like, with mean degree converging to infinity. We begin by deriving the exact belief propagation algorithm for an infinite tree network, and then deduce performance results for using that same algorithm on the original graph.

The related inference problem on a Galton-Watson tree with Poisson numbers of offspring is defined as follows. Fix a vertex \( u \) and let \( T_u \) denote the infinite Galton-Watson undirected tree rooted at vertex \( u \). The neighbors of vertex \( u \) are considered to be the children of vertex \( u \), and \( u \) is the parent of those children. The other neighbors of each child are the children of the child, and so on. For vertex \( i \) in \( T_u \), let \( T_i^t \) denote the subtree of \( T_u \) of height \( t \) rooted at vertex \( i \), induced by the set of vertices consisting of vertex \( i \) and its descendants for \( t \) generations. Let \( \tau_i \in \{0, 1\} \) denote the label of vertex \( i \) in \( T_u \). Assume \( \tau_u \sim \text{Bern}(K/n) \). For any vertex \( i \in T_u \), let \( L_i \) denote the number of its children \( j \) with \( \tau_j = 1 \), and \( M_i \) denote the number of its children \( j \) with \( \tau_j = 0 \). Suppose that \( L_i \sim \text{Pois}(Kp) \) if \( \tau_i = 1 \), \( L_i \sim \text{Pois}(Kq) \) if \( \tau_i = 0 \), and \( M_i \sim \text{Pois}((n - K)q) \) for either value of \( \tau_i \).

We are interested in estimating the label of root \( u \) given observation of the tree \( T_i^t \). Notice that the labels of vertices in \( T_i^t \) are not observed. The probability of error for an estimator \( \hat{\tau}_u(T_u^t) \) is defined by

\[
pe^t \triangleq \frac{K}{n} P(\hat{\tau}_u = 0|\tau_u = 1) + \frac{n-K}{n} P(\hat{\tau}_u = 1|\tau_u = 0). \tag{14}
\]

The estimator that minimizes \( pe^t \) is the maximum a posteriori probability (MAP) estimator, which can be expressed either in terms of the log belief ratio or log likelihood
ratio:

\[ \hat{\tau}_{\text{MAP}} = 1_{\{\xi_t^u \geq 0\}} = 1_{\{\Lambda_t^u \geq \nu\}}, \]

(15)

where

\[ \xi_t^u \triangleq \log \frac{P\{\tau_u = 1|T_t^u\}}{P\{\tau_u = 0|T_t^u\}}, \quad \Lambda_t^u \triangleq \log \frac{P\{T_t^u|\tau_u = 1\}}{P\{T_t^u|\tau_u = 0\}}, \]

and \( \nu = \log \frac{n-K}{K} \). By Bayes’ formula, \( \xi_t^u = \Lambda_t^u - \nu \), and by definition, \( \Lambda_0^u = 0 \). By a standard result in the theory of binary hypothesis testing (due to [21], stated without proof in [31], proved in special case \( \pi_0 = \pi_1 = 0.5 \) in [19], and same proof easily extends to general case) the probability of error for the MAP decision rule is bounded by

\[ \pi_1 \pi_0 \rho_B^2 \leq p_e^t \leq \sqrt{\pi_1 \pi_0 \rho_B}, \]

(16)

where the Bhattacharyya coefficient (or Hellinger integral) \( \rho_B \) is defined by \( \rho_B = \mathbb{E}[e^{\Lambda_t^u/2}|\tau_u = 0] \), and \( \pi_1 \) and \( \pi_0 \) are the prior probabilities on the hypotheses.

We comment briefly on the parameters of the model. The distribution of the tree \( T_u \) is determined by the three parameters

\[ \lambda = \frac{K^2(p-q)^2}{(n-K)^q}, \]

\( \nu \), and the ratio, \( p/q \). Indeed, vertex \( u \) has label \( \tau_u = 1 \) with probability \( \frac{K}{1+e^{\nu}} \), and the mean numbers of children of a vertex \( i \) are given by:

\[ \mathbb{E}[L_i|\tau_i = 1] = Kp = \frac{\lambda(p/q)e^{\nu}}{(p/q - 1)^2} \]

(17)

\[ \mathbb{E}[L_i|\tau_i = 0] = Kq = \frac{\lambda e^{\nu}}{(p/q - 1)^2} \]

(18)

\[ \mathbb{E}[M_i] = (n-K)q = \frac{\lambda e^{2\nu}}{(p/q - 1)^2}. \]

(19)

The parameter \( \lambda \) can be interpreted as a signal to noise ratio in case \( K \ll n \) and \( p/q = O(1) \), because \( \text{var}M_i \gg \text{var}L_i \) and

\[ \lambda = \frac{\mathbb{E}[M_i + L_i|\tau_i = 1] - \mathbb{E}[M_i + L_i|\tau_i = 0]}{\text{var}M_i}. \]

In this section, the parameters are allowed to vary with \( n \) as long as \( \lambda > 0 \) and \( p/q > 1 \), although the focus is on the asymptotic regime: \( \lambda \) fixed, \( p/q = O(1) \), and \( \nu \to \infty \). This entails that the mean numbers of children given in (17)-(19) converge to infinity. Montanari [27] considers the case of \( \nu \) fixed with \( p/q \to 1 \), which also leads to the mean vertex degrees converging to infinity.
It is well-known that the likelihoods can be computed via a belief propagation algorithm. Let $\partial i$ denote the set of children of vertex $i$ in $T_u$ and $\pi(i)$ denote the parent of $i$. For every vertex $i \in T_u$ other than $u$, define

$$\Lambda_{i \rightarrow \pi(i)}^t \triangleq \log \frac{\mathbb{P}\{T_{it}^t|\tau_i = 1\}}{\mathbb{P}\{T_{it}^t|\tau_i = 0\}}.$$ 

The following lemma gives a recursive formula to compute $\Lambda_u^t$; no approximations are needed.

**Lemma 1.** For $t \geq 0$,

$$\Lambda_{u \rightarrow \pi(i)}^{t+1} = -K(p-q) + \sum_{\ell \in \partial u} \log \left( \frac{e^{\Lambda_{\ell \rightarrow u}^t - \nu(p/q) + 1}}{e^{\Lambda_{\ell \rightarrow u}^t - \nu + 1}} \right),$$

$$\Lambda_{i \rightarrow \pi(i)}^{t+1} = -K(p-q) + \sum_{\ell \in \partial i} \log \left( \frac{e^{\Lambda_{\ell \rightarrow i}^t - \nu(p/q) + 1}}{e^{\Lambda_{\ell \rightarrow i}^t - \nu + 1}} \right), \quad \forall i \neq u$$

$$\Lambda_{i \rightarrow \pi(i)}^0 = 0, \quad \forall i \neq u.$$

**Proof.** The last equation follows by definition. We prove the first equation; the second one follows similarly. A key point is to use the independent splitting property of the Poisson distribution to give an equivalent description of the numbers of children with each label for any vertex in the tree. Instead of separately generating the number of children of with each label, we can first generate the total number of children and then independently and randomly label each child. Specifically, for every vertex $i$ in $T_u$, let $N_i$ denote the total number of its children. Let $d_1 = Kp + (n-K)q$ and $d_2 = Kq + (n-K)q = nq$. If $\tau_i = 1$ then $N_i \sim \text{Pois}(d_1)$, and for each child $j \in \partial i$, independently of everything else, $\tau_j = 1$ with probability $Kp/d_1$ and $\tau_j = 0$ with probability $(n-K)q/d_1$. If $\tau_i = 0$ then $N_i \sim \text{Pois}(d_2)$, and for each child $j \in \partial i$, independently of everything else, $\tau_j = 1$ with probability $K/n$ and $\tau_j = 0$ with probability $(n-K)/n$. With this view, the observation of the total number of children $N_u$ of vertex $u$ gives some information on the label of $u$, and then the conditionally independent messages from those children give additional information. To be precise,
we have that
\[
\Lambda_{u}^{t+1} = \log \frac{\mathbb{P}\{T_{u}^{t+1} | \tau_{u} = 1\}}{\mathbb{P}\{T_{u}^{t+1} | \tau_{u} = 0\}} = \log \frac{\mathbb{P}\{N_{u} | \tau_{u} = 1\}}{\mathbb{P}\{N_{u} | \tau_{u} = 0\}} + \sum_{i \in \partial u} \log \frac{\mathbb{P}\{T_{i}^{t} | \tau_{u} = 1\}}{\mathbb{P}\{T_{i}^{t} | \tau_{u} = 0\}}
\]
\[
\overset{(a)}{=} -K(p - q) + N_{u} \log \frac{d_{1}}{d_{0}} + \sum_{i \in \partial u} \log \left[ \frac{\sum_{x \in \{0, 1\}} \mathbb{P}\{\tau_{i} = x | \tau_{u} = 1\} \mathbb{P}\{T_{i}^{t} | \tau_{i} = x\}}{\sum_{x \in \{0, 1\}} \mathbb{P}\{\tau_{i} = x | \tau_{u} = 0\} \mathbb{P}\{T_{i}^{t} | \tau_{i} = x\}} \right]
\]
\[
\overset{(b)}{=} -K(p - q) + \sum_{i \in \partial u} \log \frac{Kp \mathbb{P}\{T_{i}^{t} | \tau_{i} = 1\} + (n - K)q \mathbb{P}\{T_{i}^{t} | \tau_{i} = 0\}}{Kq \mathbb{P}\{T_{i}^{t} | \tau_{i} = 1\} + (n - K)q \mathbb{P}\{T_{i}^{t} | \tau_{i} = 0\}}
\]
\[
\overset{(c)}{=} -K(p - q) + \sum_{i \in \partial u} \log \frac{e^{\Lambda_{i | \tau_{u}}^{t} - \nu} (p/q) + 1}{e^{\Lambda_{i | \tau_{u}}^{t} - \nu} + 1},
\]

where (a) holds because \(N_{u}\) and \(T_{i}^{t}\) for \(i \in \partial u\) are independent conditional on \(\tau_{u}\); (b) follows because \(N_{u} \sim \text{Pois}(d_{1})\) if \(\tau_{u} = 1\) and \(N_{u} \sim \text{Pois}(d_{0})\) if \(\tau_{u} = 0\), and \(T_{i}^{t}\) is independent of \(\tau_{u}\) conditional on \(\tau_{i}\); (c) follows from the fact \(\tau_{i} \sim \text{Bern}(Kp/d_{1})\) given \(\tau_{u} = 1\), and \(\tau_{i} \sim \text{Bern}(Kq/d_{0})\) given \(\tau_{u} = 0\); (d) follows from the definition of \(\Lambda_{i | \tau_{u}}^{t}\).

Notice that \(\Lambda_{i}^{t}\) is a function of \(T_{u}^{t}\) alone; and it is statistically correlated with the vertex labels. Also, since the construction of a subtree \(T_{i}^{t}\) and its vertex labels is the same as the construction of \(T_{u}^{t}\) and its vertex labels, the conditional distribution of \(T_{i}^{t}\) given \(\tau_{i}\) is the same as the conditional distribution of \(T_{u}^{t}\) given \(\tau_{u}\). Therefore, for any \(i \in \partial u\), the conditional distribution of \(\Lambda_{i | \tau_{u}}^{t}\) given \(\tau_{i}\) is the same as the conditional distribution of \(\Lambda_{i | \tau_{u}}^{t}\) given \(\tau_{u}\). For \(i = 0\) or 1, let \(Z_{i}^{t}\) denote a random variable that has the same distribution as \(\Lambda_{i}^{t}\) given \(\tau_{u} = i\). The above update rules can be viewed as an infinite-dimensional recursion that determines the probability distribution of \(Z_{0}^{t+1}\) in terms of that of \(Z_{0}^{t}\).

The remainder of this section is devoted to the analysis of belief propagation on the Poisson tree model, and is organized into two main parts. In the first part, Section 4.1 gives expressions for exponential moments of the log likelihood messages, which are applied in Section 4.2 to yield an upper bound, in Lemma 8 on the error probability for the problem of classifying the root vertex of the tree. That bound, together with a standard coupling result between Poisson tree and local neighborhood of \(G\) (stated in Appendix C), is enough to establish weak recovery for the belief propagation algorithm run on graph \(G\), given in Theorem 1. The second part of this section focuses on lower bounds on the probability of correct classification in Section 4.3. Those bounds, together with the coupling lemmas, are used to establish the converse results.
for local algorithms.

4.1. Exponential moments of log likelihood messages for Poisson tree

The following lemma gives formulas for some exponential moments of $Z^t_0$ and $Z^t_1$, based on Lemma 1. Although the formulas are not recursions, they are close enough to permit useful analysis.

**Lemma 2.** For $t \geq 0$ and any integer $h \geq 2$,

$$E \left[ e^{hZ^t_0} \right] = E \left[ e^{(h-1)Z^{t+1}_0} \right]$$

$$= \exp \left\{ K(p-q) \sum_{j=2}^{h} \binom{h}{j} \left( \frac{\lambda}{K(p-q)} \right)^{j-1} E \left[ \left( \frac{e^{Z^t_1}}{1 + e^{Z^t_1-\nu}} \right)^{j-1} \right] \right\}.$$  \hspace{1cm} (20)

**Proof.** We first illustrate the proof for $h = 2$. By the definition of $\Lambda^t_u$ and change of measure, we have $E \left[ g(\Lambda^t_u)\tau_u = 0 \right] = E[|\Lambda^t_u| e^{-\Lambda^t_u} |\tau_u = 1]$, where $g$ is any measurable function such that the expectations above are well-defined. It follows that

$$E \left[ g(Z^t_0) \right] = E[g(Z^t_1)e^{-Z^t_1}].$$ \hspace{1cm} (21)

Plugging $g(z) = e^z$ and $g(z) = e^{2z}$, we have that $E \left[ e^{Z^t_0} \right] = 1$ and $E \left[ e^{2Z^t_0} \right] = E \left[ e^{Z^t_1} \right]$. Moreover,

$$e^\nu E \left[ g(Z^t_0) \right] + E \left[ g(Z^t_1) \right] = E \left[ g(Z^t_1)(e^{-Z^t_1+\nu} + 1) \right].$$ \hspace{1cm} (22)

Plugging $g(z) = (1 + e^{-z+\nu})^{-1}$ and $g(z) = (1 + e^{-z+\nu})^{-2}$ into the last displayed equation, we have

$$e^\nu E \left[ \frac{1}{1 + e^{-Z^t_1+\nu}} \right] + E \left[ \frac{1}{1 + e^{-Z^t_1+\nu}} \right] = 1,$$ \hspace{1cm} (23)

$$e^\nu E \left[ \frac{1}{(1 + e^{-Z^t_1+\nu})^2} \right] + E \left[ \frac{1}{(1 + e^{-Z^t_1+\nu})^2} \right] = E \left[ \frac{1}{1 + e^{-Z^t_1+\nu}} \right].$$ \hspace{1cm} (24)

In view of Lemma 1, by defining $f(x) = \frac{e^{x(p/q)+1}}{x+1}$, we get that

$$e^{2\Lambda^t_{u+1}} = e^{-2K(p-q)} \prod_{\ell \in \partial u} f^2 \left( e^{\Lambda^t_{\ell-u}} - \nu \right).$$

Since the distribution of $\Lambda^t_{u+1}$ conditional on $\tau_u = 0$ and $\tau_u = 1$ is the same as the distribution of $Z^t_0$ and $Z^t_1$, respectively, it follows that

$$E \left[ e^{2Z^t_0} \right] = e^{-2K(p-q)} \mathbb{E} \left[ \mathbb{E} \left[ f^2 \left( e^{Z^t_1-\nu} \right) \right]^{L_u} \right] \mathbb{E} \left[ \mathbb{E} \left[ f^2 \left( e^{Z^t_1-\nu} \right) \right]^{M_u} \right].$$
Using the fact that \( E[c^X] = e^{\lambda(c-1)} \) for \( X \sim \text{Pois}(\lambda) \) and \( c > 0 \), we have

\[
E[e^{2Z_{t+1}^i}] = e^{-2K(p-q)+Kq \left( E\left[f^2\left(e^{Z_{t+1}^i-\nu}\right)\right] - 1\right) + (n-K)q \left( E\left[f^2\left(e^{Z_{t+1}^i-\nu}\right)\right] - 1\right)}.
\]

Notice that

\[
f^2(x) = \left(1 + \frac{p/q - 1}{1 + x^{-1}}\right)^2 = 1 + \frac{2(p/q - 1)}{1 + x^{-1}} + \frac{(p/q - 1)^2}{(1 + x^{-1})^2}.
\]

It follows that

\[
Kq \left( E\left[f^2\left(e^{Z_{t+1}^i-\nu}\right)\right] - 1\right) + (n-K)q \left( E\left[f^2\left(e^{Z_{t+1}^i-\nu}\right)\right] - 1\right)
\]

\[
= 2Kq(p/q - 1) \left( E\left[\frac{1}{1 + e^{-Z_{t+1}^i+\nu}}\right] + e^{\nu}E\left[\frac{1}{1 + e^{-Z_{t+1}^i}}\right]\right)
\]

\[
+ Kq(p/q - 1)^2 \left( E\left[\frac{1}{(1 + e^{-Z_{t+1}^i+\nu})^2}\right] + e^{\nu}E\left[\frac{1}{(1 + e^{-Z_{t+1}^i})^2}\right]\right)
\]

\[
\overset{(a)}{=} 2K(p-q) + Kq(p/q - 1)^2 E\left[\frac{1}{1 + e^{-Z_{t+1}^i+\nu}}\right]
\]

\[
= 2K(p-q) + \lambda E\left[\frac{e^{Z_{t+1}^i}}{1 + e^{Z_{t+1}^i-\nu}}\right],
\]

where (a) follows by applying (23) and (24). Combining the above proves (20) with \( h = 2 \). For general \( h \geq 2 \), we expand \( f^h(x) = \left(1 + \frac{p/q - 1}{1 + x^{-1}}\right)^h \) using binomial coefficients as already illustrated for \( h = 2 \).

Using the notation

\[
a_t = E\left[e^{Z_{t+1}^i}\right],
\]

\[
b_t = E\left[\frac{e^{Z_{t+1}^i}}{1 + e^{Z_{t+1}^i-\nu}}\right],
\]

(20) with \( h = 2 \) becomes

\[
a_{t+1} = \exp(\lambda b_t).
\]

The following lemma provides upper bounds on some exponential moments in terms of \( b_t \).

**Lemma 3.** Let \( C \triangleq \lambda(2 + \frac{p}{q}) \) and \( C' \triangleq \lambda(3 + 2\frac{p}{q} + (\frac{p}{q})^2) \). Then \( E[e^{2Z_{t+1}^i}] \leq \exp(Cb_t) \) and \( E[e^{3Z_{t+1}^i}] \leq \exp(C'b_t) \). More generally, for any integer \( h \geq 2 \),

\[
E\left[e^{hZ_{t+1}^i}\right] = E\left[e^{(h-1)Z_{t+1}^i}\right] \leq e^{\lambda b_t \sum_{j=2}^{h} (\frac{p}{q})^j (\frac{p}{q} - 1)^{j-2}}.
\]
Proof. Note that $\frac{e^z}{1+e^{-\nu}} \leq e^\nu$ for all $z$. Therefore, for any $j \geq 2$, $\left(\frac{e^z}{1+e^{-\nu}}\right)^{j-1} \leq e^{(j-2)\nu} \left(\frac{e^z}{1+e^{-\nu}}\right)$. Applying this inequality to (20) yields (28). □

4.2. Upper bound on classification error via exponential moments

Note that $b_t \approx a_t$ if $\nu \gg 0$, in which case (27) is approximately a recursion for $\{b_t\}$. The following two lemmas use this intuition to show that if $\lambda > 1/e$ and $\nu$ is large enough, the $b_t$'s eventually grow large. In turn, that fact will be used to show that the Bhattacharyya coefficient mentioned in (16), which can be expressed as $\rho_B = \mathbb{E}[e^{Z_t/2}] = \mathbb{E}[e^{-Z_1/2}]$, becomes small, culminating in Lemma 8, giving an upper bound on the classification error for the root vertex.

Lemma 4. Let $C \triangleq \lambda(2 + \frac{p}{q})$. Then

$$b_{t+1} \geq \exp(\lambda b_t) \left(1 - e^{-\nu/2}\right) \quad \text{if} \quad b_t \leq \frac{\nu}{2(C - \lambda)}.$$ 

Proof. Note that $C - \lambda > 0$. If $b_t \leq \frac{\nu}{2(C - \lambda)}$, we have

$$b_{t+1} \overset{(a)}{=} a_{t+1} - \mathbb{E} \left[ e^{-\nu + 2Z_t} \right] \overset{(b)}{=} e^{\lambda b_t} - e^{-\nu + C b_t}$$

$$= e^{\lambda b_t} \left(1 - e^{-\nu + (C - \lambda)b_t}\right) \overset{(c)}{=} e^{\lambda b_t} \left(1 - e^{-\nu/2}\right).$$

where (a) follows by the definitions (25) and (26) and the fact $\frac{1}{1+x} \geq 1 - x$ for $x \geq 0$; (b) follows from Lemma 3; (c) follows from the condition $b_t \leq \frac{\nu}{2(C - \lambda)}$. □

Lemma 5. The variables $a_t$ and $b_t$ are nondecreasing in $t$ and $\mathbb{E}[e^{Z_t/2}]$ is non-increasing in $t$ over all $t \geq 0$. More generally, $\mathbb{E} \left[ \Upsilon \left(e^{Z_t}\right) \right]$ is nondecreasing (non-increasing) in $t$ for any convex (concave, respectively) function $\Upsilon$ with domain $(0, \infty)$.

Proof. Note that, in view of (21), $\mathbb{E} \left[ \Upsilon \left(e^{Z_t}\right) \right]$ becomes $a_t$ for the convex function $\Upsilon(x) = x^2$, $b_t$ for the convex function $\Upsilon(x) = x^2/(1 + xe^{-\nu})$, and $\mathbb{E}[e^{Z_t/2}]$ for the concave function $\Upsilon(x) = \sqrt{x}$. It thus suffices to prove the last statement of the lemma.

It is well known that for a nonsingular binary hypothesis testing problem with a growing amount of information indexed by some parameter $s$ (i.e. an increasing family of $\sigma$-algebras as usual in martingale theory), the likelihood ratio $\frac{dP}{dQ}$ is a martingale under measure $Q$. Therefore, the likelihood ratios $\{e^{\Lambda_t} : t \geq 0\}$ (where $\Lambda_t$ denotes the log likelihood ratio) at the root vertex $u$ for the infinite tree, conditioned on $\tau_u = 0$,
form a martingale. Thus, the random variables \( \{e^{2t^k} : t \geq 0\} \) can be constructed on a single probability space to be a martingale. The lemma therefore follows from Jensen’s inequality.

Recall that \( \log^+(\nu) \) denotes the number of times the logarithm function must be iteratively applied to \( \nu \) to get a result less than or equal to one.

**Lemma 6.** Suppose \( \lambda > 1/e \). There are constants \( \bar{t}_0 \) and \( \nu_0 > 0 \) depending only on \( \lambda \) such that

\[
 b_{\bar{t}_0 + \log^+(\nu) + 2} \geq \exp(\lambda\nu/(2(C - \lambda)) \left(1 - e^{-\nu/2}\right), 
\]

where \( C = \lambda \left(\frac{\lambda}{e} + 2\right) \), whenever \( \nu \geq \nu_0 \) and \( \nu \geq 2(C - \lambda) \).

**Proof.** Given \( \lambda \) with \( \lambda > 1/e \), select the following constants, depending only on \( \lambda \):

- \( D \) and \( \nu_0 \) so large that \( \lambda e^D \left(1 - e^{-\nu_0/2}\right) > 1 \) and \( \lambda e \left(1 - e^{-\nu_0/2}\right) \geq \sqrt{\lambda e} \).
- \( \nu_0 > 0 \) so large that \( \nu_0 e^D \left(1 - e^{-\nu_0/2}\right) - \lambda D \geq w_0 \).
- A positive integer \( \bar{t}_0 \) so large that \( \lambda((\lambda e)^{\bar{t}_0/2} - 1 - D) \geq w_0 \).

Throughout the remainder of the proof we assume without further comment that \( \nu \geq \nu_0 \) and \( \nu \geq 2(C - \lambda) \). The latter condition and the fact \( \bar{b}_0 = \frac{\nu}{2(C - \lambda)} \) ensure that \( \bar{b}_0 < \frac{\nu}{2(C - \lambda)} \). Let \( t^* = \max \left\{ t \geq 0 : b_t < \frac{\nu}{2(C - \lambda)} \right\} \) and let \( \bar{t}_1 = \log^+(\nu) \). The first step of the proof is to show \( t^* \leq \bar{t}_0 + \bar{t}_1 \). For that purpose we will show that the \( b_t \)'s increase at least geometrically to reach a certain large constant (specifically, so (30) below holds), and then they increase as fast as a sequence produced by iterated exponentiation.

Since \( b_0 \geq 0 \) it follows from (29) and the choice of \( \nu_0 \) that \( b_1 \geq \left(1 - e^{-\nu_0/2}\right) \geq (\lambda e)^{-1/2} \). Note that \( e^u \geq e^u \) for all \( u > 0 \), because \( \frac{e^u - 1}{u} \) is minimized at \( u = 1 \). Thus \( e^{\lambda b_t} \geq \lambda b_t \), which combined with the choice of \( \nu_0 \) and (29) shows that if \( b_t \leq \frac{\nu}{2(C - \lambda)} \) then \( b_{t+1} \geq \sqrt{\lambda b_t} \). It follows that \( b_t \geq ((\lambda e)^{t/2} - 1) \) for \( 1 \leq t \leq t^* + 1 \).

If \( b_{t_0-1} \geq \frac{\nu}{2(C - \lambda)} \) then \( t^* \leq \bar{t}_0 - 1 \) and the claim \( t^* \leq \bar{t}_0 + \bar{t}_1 \) is proved (that is, the geometric growth phase alone was enough), so to cover the other possibility, suppose \( b_{t_0-1} < \frac{\nu}{2(C - \lambda)} \). Then \( \bar{t}_0 \leq t^* + 1 \) and therefore \( b_{t_0} \geq (\lambda e)^{\bar{t}_0/2} - 1 \). Let \( t_0 = \min \{ t : b_t \geq (\lambda e)^{\bar{t}_0/2} - 1 \} \). It follows that \( t_0 \leq \bar{t}_0 \), and, by the choice of \( \bar{t}_0 \) and the definition of \( t_0 \),

\[
 \lambda(b_{t_0} - D) \geq w_0. 
\]
Define the sequence \( (w_t : t \geq 0) \) beginning with \( w_0 \) already chosen, and satisfying the recursion \( w_{t+1} = e^{w_t} \). It follows by induction that
\[
\lambda(b_{t_0+t} - D) \geq w_t \text{ for } t \geq 0, \ t_0 + t \leq t^* + 1.
\] (31)

Indeed, the base case is (30), and if (31) holds for some \( t \) with \( t_0 + t \leq t^* \), then \( b_{t_0+t} \geq \frac{\nu}{\lambda} + D \), so that
\[
\lambda(b_{t_0+t+1} - D) \geq \lambda \left( e^{\lambda b_{t_0+t}} \left( 1 - e^{-\nu/2} \right) - D \right) \\
\geq w_{t+1} \lambda e^{\lambda D} \left( 1 - e^{-\nu/2} \right) - \lambda D \geq w_{t+1},
\]
where the last inequality follows from the choice of \( w_0 \) and the fact \( w_{t+1} \geq w_0 \). The proof of (31) by induction is complete.

Let \( \bar{t}_1 = \log^*(\nu) \). Since \( w_1 \geq 1 \) it follows that \( w_{\bar{t}_1+1} \geq \nu \) (verify by applying the log function \( \bar{t}_1 \) times to each side). Therefore, \( w_{\bar{t}_1+1} \geq \frac{\lambda \nu}{2(C-\lambda)} - \lambda D \), where we use the fact \( C - \lambda \geq 2\lambda \). If \( t_0 + \bar{t}_1 < t^* \) it would follow from (31) with \( t = t_0 + \bar{t}_1 + 1 \) that
\[
b_{t_0+\bar{t}_1+1} \geq \frac{w_{\bar{t}_1+1}}{\lambda} + D \geq \frac{\nu}{2(C-\lambda)},
\]
which would imply \( t^* \leq t_0 + \bar{t}_1 \), which would be a contradiction. Therefore, \( t^* \leq t_0 + \bar{t}_1 \leq t_0 + \bar{t}_1 \), as was to be shown.

Since \( t^* \) is the last iteration index \( t \) such that \( b_t < \frac{\nu}{2(C-\lambda)} \), either \( b_{t^*+1} = \frac{\nu}{2(C-\lambda)} \), and we say the threshold \( \frac{\nu}{2(C-\lambda)} \) is exactly reached at iteration \( t^*+1 \), or \( b_{t^*+1} > \frac{\nu}{2(C-\lambda)} \), in which case we say there was overshoot at iteration \( t^* + 1 \). First, consider the case the threshold is exactly reached at iteration \( t^* + 1 \). Then, \( b_{t^*+1} = \frac{\nu}{2(C-\lambda)} \), and (29) can be applied with \( t = t^* + 1 \), yielding
\[
b_{t^*+2} \geq \exp(\lambda b_{t^*+1})(1 - e^{-\nu/2}) = \exp(\lambda \nu/(2(C-\lambda)))(1 - e^{-\nu/2}).
\]
Since \( t^* + 2 \leq t_0 + \bar{t}_1 + 2 = t_0 + \log^*(\nu) + 2 \), it follows from Lemma 5 that \( b_{t_0+\log^*(\nu)+2} \geq b_{t^*+2} \), which completes the proof of the lemma in case the threshold is exactly reached at iteration \( t^* + 1 \).

To complete the proof, we explain how the information available for estimation can be reduced through a \textit{thinning} method, leading to a reduction in the value of \( b_{t^*+1} \), so that we can assume without loss of generality that the threshold is always exactly reached at iteration \( t^* + 1 \). Let \( \phi \) be a parameter with \( 0 \leq \phi \leq 1 \). As before, we
we will be considering a total of \( t^* + 2 \) iterations, so consider a random tree with labels, \((T_{u}^{t^∗+2}, \tau_{u}^{t^∗+2})\), with root vertex \( u \) and maximum depth \( t^* + 2 \). For the original model, each vertex of depth \( t^* + 1 \) or less with label 0 or 1 has Poisson numbers of children with labels 0 and 1 respectively, with means specified in the construction. For the thinning method, for each \( \ell \in \partial u \) and each child \( i \) of \( \partial \ell \), (i.e. for each grandchild of \( u \)) we generate a random variable \( U_{\ell,i} \) that is uniformly distributed on the interval \([0, 1]\). Then we retain \( i \) if \( U_{\ell,i} \leq \phi \), and we delete \( i \), and all its decedents, if \( U_{\ell,i} > \phi \). That is, the grandchildren of the root vertex \( u \) are each deleted with probability \( 1 - \phi \). It is equivalent to reducing \( p \) and \( q \) to \( \phi p \) and \( \phi q \), respectively, for that one generation.

Consider the calculation of the likelihood ratio at the root vertex for the thinned tree. The log likelihood ratio messages begin at the leaf vertices at depth \( t^* + 2 \).

For any vertex \( \ell \neq u \), let \( \Lambda_{\ell \rightarrow \pi(\ell), \phi} \) denote the log likelihood message passed from vertex \( \ell \) to its parent, \( \pi(\ell) \). Also, let \( \Lambda_{u, \phi} \) denote the log likelihood computed at the root vertex. For brevity we leave off the superscript \( t \) on the log likelihood ratios, though \( t \) on the message \( \Lambda_{\ell \rightarrow \pi(\ell), \phi} \) would be \( t^* + 2 \) minus the depth of \( \ell \). The messages of the form \( \Lambda_{\ell \rightarrow \pi(\ell), \phi} \) don’t actually depend on \( \phi \) unless \( \ell \in \partial u \). For a vertex \( \ell \in \partial u \), the message \( \Lambda_{\ell \rightarrow u, \phi} \) has nearly the same representation as in Lemma 1, namely:

\[
\Lambda_{\ell \rightarrow u, \phi} = -\phi K(p - q) + \sum_{i \in \partial U_{\ell,i} \leq \phi} \log \left( \frac{e^{\Lambda_{i \rightarrow \ell, \phi}} - p}{e^{\Lambda_{i \rightarrow \ell, \phi}} - p + 1} \right) \tag{32}
\]

The representation of \( \Lambda_{u, \phi} \) is the same as the representation of \( \Lambda_{u}^{t^*+1} \) in Lemma 1, except with \( \Lambda_{u}^{t^*+1} \) replaced both places on the right hand side by \( \Lambda_{\ell \rightarrow u, \phi} \).

Let \( Z_{0, \phi}^{t} \) and \( Z_{1, \phi}^{t} \) denote random variables for analyzing the message passing algorithm for this depth \( t^* + 2 \) tree. Their laws are the following. For \( 0 \leq t \leq t^* + 1 \), \( L(Z_{0, \phi}^{t}) \) is the law of \( \Lambda_{\ell \rightarrow \pi(\ell), \phi} \) given \( \tau_{\ell} = 0 \), for a vertex \( \ell \) of depth \( t^* + 2 - t \). And \( L(Z_{0, \phi}^{t^*+2}) \) is the law of \( \Lambda_{u, \phi} \) given \( \tau_{u} = 0 \). Note that \( Z_{0, \phi}^{0} = 0 \). The laws \( L(Z_{1, \phi}^{t}) \) are determined similarly, conditioning on the labels of the vertices to be one. For \( t \) fixed, \( L(Z_{0, \phi}^{t}) \) and \( L(Z_{1, \phi}^{t}) \) each determine the other because they represent distributions of the log likelihood for a binary hypothesis testing problem.

The message passing equations for the log likelihood ratios translate into recursions for the laws \( L(Z_{0, \phi}^{t}) \) and \( L(Z_{1, \phi}^{t}) \). We have not focused directly on the full recursions of the laws, but rather looked at equations for exponential moments. The basic recursions we’ve been considering for \( L(Z_{0, \phi}^{t}) \) are exactly as before for \( 0 \leq t \leq t^* - 1 \) and for
For the first order phase transition, let us consider the function $f(t) = a_t^\phi$. By the independence of the Poisson process, we have

$$f(t+1) = f(t) + \mathbb{E}[e^{Z_{t+1}^*}] - \mathbb{E}[e^{Z_{t+1}^*}]$$

where $Z_{t+1}^*$ is the size of the thinned subtree at depth $t+1$. This equation implies that $f(t)$ is a martingale and therefore converges in distribution to a limit $\mathbb{E}[e^{Z_{\infty}^*}]$. As $t \to \infty$, the thinning process becomes negligible and we can approximate $f(t)$ by the expected value of the size of the thinned subtree at depth $t$, which is a continuous and nondecreasing function of $\phi$. Thus, we can construct $e^{Z_{\infty}^*}$ on a single probability space for $0 \leq \phi \leq 1$.

On the other hand, if $\phi = 0$, then $\Lambda_{t \to u, \phi} \equiv 0$ for all $t \in \partial u$, so that $Z_{1,t}^* = Z_{1,t}^* \equiv 0$ so that $b_{t^*+1,0} = \frac{\nu - K}{2(C-\lambda)} \leq 1 < \frac{\nu}{2(C-\lambda)}$. On the other hand, by the definition of $t^*$ we know that $b_{t^*+1,0} \geq \frac{\nu}{2(C-\lambda)}$. We shall show that there exists a value of $\phi \in [0,1]$ so that $b_{t^*+1,\phi} = \frac{\nu}{2(C-\lambda)}$. To do so we next prove that $b_{t^*+1,\phi}$ is a continuous, and, in fact, nondecreasing, function of $\phi$, using a variation of the proof of Lemma 5. Let $\ell$ denote a fixed neighbor of the root node $u$. Note that $e^{\Lambda_{t \to u, \phi}}$ is the likelihood ratio for detection of $\tau_{\ell}^*$ based on the thinned subtree of depth $t^* + 1$ with root $\ell$. As $\phi$ increases from 0 to 1 the amount of thinning decreases, so larger values of $\phi$ correspond to larger amounts of information. Therefore, conditioned on $\tau_u = 0$, $(e^{\Lambda_{t \to u, \phi}} : 0 \leq \phi \leq 1)$ is a martingale. Moreover, the independent splitting property of Poisson random variables imply that, given $\tau_u = 0$, the random process $\phi \mapsto |\{i \in \partial \ell : U_{i,i} \leq \phi\}|$ is a Poisson process with intensity $nq$, and therefore the sum in (32), as a function of $\phi$ over the interval $[0,1]$, is a compound Poisson process. Compound Poisson processes, just like Poisson processes, are almost surely continuous at any fixed value of $\phi$, and therefore the random process $\phi \mapsto \Lambda_{t \to u, \phi}$ is continuous in distribution. Therefore, the random variables $e^{\Lambda_{t \to u, \phi}}$ can be constructed on a single probability space for $0 \leq \phi \leq 1$ to form
a martingale which is continuous in distribution. Since \( b_{t^*,1,\phi} \) is the expectation of a bounded, continuous, convex function of \( e^{Z_{t^*,1,\phi}^+} \), it follows that \( b_{t^*,1,\phi} \) is continuous and nondecreasing in \( \phi \). Therefore, we can conclude that there exists a value of \( \phi \) so that \( b_{t^*,1,\phi} = \frac{p}{2(3\phi-1)} \), as claimed.

Since there is no overshoot, we obtain as before (by using (33) for \( t = t^* + 1 \) to modify Lemma 4 to handle \((b_{t+1}, b_t)\) replaced by \((b_{t^*+2,\phi}, b_{t^*+1,\phi})\)):

\[
b_{t^*+2,\phi} \geq \exp(\lambda b_{t^*+1,\phi}) \left(1 - e^{-\nu/2}\right) = \exp(\lambda \nu/(2(C - \lambda)) \left(1 - e^{-\nu/2}\right).\]

The same martingale argument used in the previous paragraph can be used to show that \( b_{t^*+2,\phi} \) is nondecreasing in \( \phi \), and in particular, \( b_{t^*+2} = b_{t^*+2,1} \geq b_{t^*+2,\phi} \) for \( 0 \leq \phi \leq 1 \). Hence, by Lemma 5 and the fact \( t^* + 2 \leq t_0 + \log^* (\nu) + 2 \), we have \( b_{t_0 + \log^* (\nu) + 2} \geq b_{t^*+2} \geq b_{t^*+2,\phi} \), completing the proof of the lemma.

**Lemma 7.** Let \( B = (p/q)^{3/2} \). Then

\[
\exp \left(\frac{-\lambda}{8} b_t\right) \leq \mathbb{E} \left[ e^{Z_0^{t+1}/2} \right] \leq \exp \left(\frac{-\lambda}{8 B} b_t\right).
\]

**Proof.** We prove the upper bound first. In view of Lemma 1, by defining \( f(x) = \frac{x(p/q)+1}{x+1} \), we get that

\[
e^{L_{x+1}/2} = e^{-K(p-q)/2} \prod_{\ell \in \partial u} f^{1/2} \left(e^{L_{\ell-u} - \nu}\right).
\]

Thus,

\[
\mathbb{E} \left[ e^{Z_0^{t+1}/2} \right] = e^{-K(p-q)/2} \mathbb{E} \left[ \left(\mathbb{E} \left[ f^{1/2} \left(e^{Z_0^t - \nu}\right)\right] \right)^{L_{u}} \right] \mathbb{E} \left[ \left(\mathbb{E} \left[ f^{1/2} \left(e^{Z_0^0 - \nu}\right)\right] \right)^{M_{u}} \right].
\]

Using the fact that \( \mathbb{E} [c^X] = e^{\lambda(c-1)} \) for \( X \sim \text{Pois} (\lambda) \) and \( c > 0 \), we have

\[
\mathbb{E} \left[ e^{Z_0^{t+1}/2} \right] = \exp \left[ -K(p-q)/2 + K q \left(\mathbb{E} \left[ f^{1/2} \left(e^{Z_0^0 - \nu}\right)\right] - 1\right)\right] + (n - K)q \left(\mathbb{E} \left[ f^{1/2} \left(e^{Z_0^0 - \nu}\right)\right] - 1\right) \quad (34)
\]

By the intermediate value form of Taylor’s theorem, for any \( x \geq 0 \) there exists \( y \) with \( 1 \leq y \leq x \) such that \( \sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8(1+y)^{3/2}} \). Therefore,

\[
\sqrt{1+x} \leq 1 + \frac{x}{2} - \frac{x^2}{8(1+A)^{3/2}}, \quad \forall 0 \leq x \leq A. \quad (35)
\]
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Letting $A \triangleq \frac{p}{q} - 1$ and noting that $B = (1 + A)^{3/2}$, we have

$$
\left( \frac{e^{z - \nu} (p/q) + 1}{1 + e^{z - \nu}} \right)^{1/2} = \left( \frac{1 + \frac{p/q - 1}{1 + e^{z - \nu}}}{1 + \frac{e^{z - \nu}}{1 + e^{z - \nu}}} \right)^{1/2}
\leq 1 + \frac{1}{2} \frac{(p/q - 1)}{(1 + e^{z - \nu})} - \frac{1}{8B} \left( \frac{p/q - 1}{1 + e^{z - \nu}} \right)^{2}.
$$

It follows that

$$
Kq \left( E \left[ f^{1/2} \left( e^{Z_1^u} \right) \right] - 1 \right) + (n - K)q \left( E \left[ f^{1/2} \left( e^{Z_0^u} \right) \right] - 1 \right) \\
\leq \frac{1}{2} Kq(p/q - 1) \left( E \left[ \frac{1}{1 + e^{Z_1^u + \nu}} \right] + e^{\nu}E \left[ \frac{1}{1 + e^{-Z_0^u + \nu}} \right] \right) \\
- \frac{1}{8B} Kq(p/q - 1)^2 \left( E \left[ \frac{1}{(1 + e^{Z_1^u + \nu})^2} \right] + e^{\nu}E \left[ \frac{1}{(1 + e^{-Z_0^u + \nu})^2} \right] \right)
= K(p - q)/2 - \frac{1}{8B} Kq(p/q - 1)^2 E \left[ \frac{1}{1 + e^{Z_1^u + \nu}} \right]
= K(p - q)/2 - \frac{\lambda}{8B} E \left[ \frac{e^{Z_1^u}}{1 + e^{Z_1^u - \nu}} \right],
$$

where the first equality follows from (23) and (24); the last equality holds due to $Kq(p/q - 1)^2 e^\nu = \lambda$. Combining the last displayed equation with (34) yields the desired upper bound.

The proof for the lower bound is similar. Instead of (35), we use the inequality that $\sqrt{1 + x} \geq 1 + \frac{x}{2} - \frac{x^2}{8}$ for all $x \geq 0$, and the lower bound readily follows by the same argument as above. $\square$

**Lemma 8.** (Upper bound on classification error for the random tree model.) Consider the random tree model with parameters $\lambda$, $\nu$, and $p/q$. Let $\lambda$ be fixed with $\lambda > 1/e$. There are constants $\bar{t}_0$ and $\nu_0$ depending only on $\lambda$ such that if $\nu \geq \nu_0$ and $\nu \geq 2(C - \lambda)$, then after $\bar{t}_0 + \log^*(\nu) + 2$ iterations of the belief propagation algorithm, the average error probability for the MAP estimator $\widehat{\tau}_u$ of $\tau_u$ satisfies

$$
\mathbb{P}_c^l \leq \left( \frac{K(n - K)}{n^2} \right)^{1/2} \exp \left( \frac{\lambda}{8B} \exp(\nu \lambda/(2(C - \lambda)) \left( 1 - e^{-\nu/2} \right) \right),
$$

(36)

where $B = \left( \frac{p}{q} \right)^{3/2}$ and $C = \lambda \left( \frac{p}{q} + 2 \right)$. In particular, if $p/q = O(1)$, and $r$ is any positive constant, then if $\nu$ is sufficiently large,

$$
\mathbb{P}_c^l \leq \frac{Ke^{-\nu r}}{n} = \frac{K}{n} \left( \frac{K}{n - K} \right)^r.
$$

(37)
Proof. We use the Bhattacharyya upper bound in (16) with \( \pi_1 = \frac{K}{n} \) and \( \pi_0 = \frac{n-K}{n} \), and the fact \( \rho = \mathbb{E} \left[ e^{Z_t/2} \right] \). Plugging in the lower bound on \( b_{t_0+\log_*(\nu)+2} \) from Lemma 6 into the upper bound on \( \mathbb{E} \left[ e^{Z_t/2} \right] \) from Lemma 7 yields (36). If \( p/q = O(1) \) and \( r > 0 \), then for \( \nu \) large enough,

\[
\frac{\lambda}{8B} \exp\left(\nu\lambda/(2(C-\lambda))\right) \left(1 - e^{-\nu/2}\right) \geq \nu(r + 1/2),
\]

which, together with (36), implies (37). \( \square \)

4.3. Lower bounds on classification error for Poisson tree

The bounds in this section will be combined with the coupling lemmas of Appendix C to yield converse results for recovering a community by local algorithms.

Lemma 9. (Lower bounds for Poisson tree model.) Fix \( \lambda \) with \( 0 < \lambda \leq 1/e \). For any estimator \( \hat{\tau}_u \) of \( \tau_u \) based on observation of the tree up to any depth \( t \), the average error probability satisfies

\[
p_{e,t} \geq \frac{K(n-K)}{n^2} \exp(-\lambda e/4),
\]

(38)

and the sum of Type-I and Type-II error probabilities satisfies

\[
p_{e,0}^t + p_{e,1}^t \geq \frac{1}{2} \exp(-\lambda e/4).
\]

(39)

Furthermore, if \( p/q = O(1) \) and \( \nu \to \infty \), then

\[
\liminf_{n \to \infty} \frac{n}{K} p_e^t \geq 1.
\]

(40)

Proof. Lemma 7 shows that the Bhattacharyya coefficient, given by \( \rho_B = \mathbb{E}[e^{Z_t/2}] \), satisfies \( \rho_B \geq \exp\left(-\frac{\nu}{8}b_t\right) \). Note that \( b_{t+1} \leq a_{t+1} = e^{\lambda b_t} \) for \( t \geq 0 \) and \( b_0 = \frac{1}{1+e} \). It follows from induction and the assumption \( \lambda e \leq 1 \) that \( b_t \leq e \) for all \( t \geq 0 \). Therefore, \( \rho_B \geq \exp(-\lambda e/8) \). Applying the Bhattacharyya lower bound on \( p_e^t \) in (16) (which holds for any estimator) with \( (\pi_0, \pi_1) = \left(\frac{n-K}{n}, \frac{K}{n}\right) \) yields (38) and with \( (\pi_0, \pi_1) = (1/2, 1/2) \) yields (39), respectively.

It remains to prove (40), so suppose \( p/q = O(1) \) and \( \nu \to \infty \). It suffices to prove (40) for the MAP estimator, \( \hat{\tau}_u = 1_{\{\lambda_t^* \geq \nu\}} \), because the MAP estimator minimizes the average error probability. Lemma 16 implies that, as \( n \to \infty \), the Type-I and Type-II error probabilities satisfy

\[
p_{e,1}^t - Q\left(\frac{\lambda b_{t-1}/2 - \nu}{\sqrt{\lambda b_{t-1}}}\right) \to 0 \quad \text{and} \quad p_{e,0}^t - Q\left(\frac{\lambda b_{t-1}/2 + \nu}{\sqrt{\lambda b_{t-1}}}\right) \to 0,
\]
where $Q$ is the complementary CDF of the standard normal distribution. Recall that $b_t \leq e$ for all $t \geq 0$. Also, $b_t$ is bounded away from zero, because $b_t \geq b_0 = \frac{1}{1+e}$. Since $\nu \to \infty$, we have that $p_{e,1}' \to 1$. By definition, $\frac{n}{K} p_{e}^t \geq p_{e,1}'$ and consequently $\lim inf_{n \to \infty} \frac{n}{K} p_{e}^t \geq 1$. □

5. Proofs of main results of belief propagation

Proof of Theorem 1. The proof basically consists of combining Lemma 8 and the coupling lemma 10. Lemma 8 holds by the assumptions $K^2 (p - q)^2 \equiv \lambda$ for a constant $\lambda$ with $\lambda > 1/e$, $\nu \to \infty$, and $p/q = O(1)$. Lemma 8 also determines the given expression for $t_f$. In turn, the assumptions $(np)^{log^* \nu} = n^{o(1)}$ and $e^{log^* \nu} \leq \nu = n^{o(1)}$ ensure that $(2 + np)^{t_f} = n^{o(1)}$, so that Lemma 10 holds.

A subtle point is that the performance bound of Lemma 8 is for the MAP rule (15) for detecting the label of the root vertex. The same rule could be implemented at each vertex of the graph $G$ which has a locally tree like neighborhood of radius $t_0 + log^*(\nu) + 2$ by using the estimator $\hat{C}_o = \{i : R_i^{t_f} \geq \nu\}$. We first bound the performance for $\hat{C}_o$ and then do the same for $\hat{C}$ produced by Algorithm 1. (We could have taken $\hat{C}_o$ to be the output of Algorithm 1, but returning a constant size estimator leads to simpler analysis of the algorithm for exact recovery.)

The average probability of misclassification of any given vertex $u$ in $G$ by $\hat{C}_o$ (for prior distribution $(\frac{n}{n}, \frac{n-K}{n})$) is less than or equal to the sum of two terms. The first term is $n^{1+o(1)}$ in case $|C^*| \equiv K$ or $n^{-1/2+o(1)}$ in the other case (due to failure of tree coupling of radius $t_f$ neighborhood–see Lemma 10). The second term is $\frac{n}{K} e^{-\nu r}$ (bound on average error probability for the detection problem associated with a single vertex $u$ in the tree model–see Lemma 8.) Multiplying by $n$ bounds the expected total number of misclassification errors, $E \left(|C^* \triangle \hat{C}_o|\right)$; dividing by $K$ gives the bounds stated in the lemma with $\hat{C}$ replaced by $\hat{C}_o$ and the factor 2 dropped in the bounds.

The set $\hat{C}_o$ is defined by a threshold condition whereas $\hat{C}$ similarly corresponds to using a data dependent threshold and tie breaking rule to arrive at $|\hat{C}| \equiv K$. Therefore, with probability one, either $\hat{C}_o \subset \hat{C}$ or $\hat{C} \subset \hat{C}_o$. Together with the fact $|\hat{C}| \equiv K$ we have

$|C^* \triangle \hat{C}| \leq |C^* \triangle \hat{C}_o| + |\hat{C}_o \triangle \hat{C}| = |C^* \triangle \hat{C}_o| + ||\hat{C}_o| - K|$. 


and furthermore,
\[\|\hat{C}_o| - K| \leq \|\hat{C}_o| - |C^*|\| + \|C^*| - K| \leq |C^* \triangle \hat{C}_o| + \|C^*| - K|\].

So
\[|C^* \triangle \hat{C}| \leq 2|C^* \triangle \hat{C}_o| + \|C^*| - K|\].

If \(|C^*| \equiv K\) then \(|C^* \triangle \hat{C}| \leq 2|C^* \triangle \hat{C}_o|\) and (4) follows from what was proved for \(\hat{C}_o\).

In the other case, \(\mathbb{E}[\|C^*| - K]\| \leq n^{3+o(1)}\), and (5) follows from what was proved for \(\hat{C}_o\).

As for the computational complexity guarantee, notice that in each BP iteration, each vertex \(i\) needs to transmit the outgoing message \(R_{t+1}^i \rightarrow j\) to its neighbor \(j\) according to (2). To do so, vertex \(i\) can first compute \(R_{t+1}^i\) and then subtract neighbor \(j\)'s contribution from it to get the desired message \(R_{t+1}^i \rightarrow j\). In this way, each vertex \(i\) needs \(O(|\partial_i|)\) basic operations and the total time complexity of one BP iteration is \(O(|E(G)|\log^* n)\), where \(|E(G)|\) is the total number of edges. Since \(\nu \leq n\), at most \(O(\log^* n)\) iterations are needed and hence the algorithm terminates in \(O(|E(G)|\log^* n)\) time. \(\square\)

**Proof of Theorem 2.** The theorem follows from the fact that the belief propagation algorithm achieves weak recovery, even if the cardinality \(|C^*|\) is random and is only known to satisfy \(\mathbb{P}\{|C^*| - K| \geq \sqrt{3K \log n}\| \leq n^{-1/2+o(1)}\) and the results in [14]. We include the proof for completeness. Let \(C_k^* = C^* \cap ([n] \setminus S_k)\) for \(1 \leq k \leq 1/\delta\). As explained in Remark 2, \(C_k^*\) is obtained by sampling the vertices in \([n]\) without replacement, and thus the distribution of \(C_k^*\) is hypergeometric with \(\mathbb{E}[|C_k^*|] = K(1 - \delta)\). A result of Hoeffding [15] implies that the Chernoff bounds for the Binom \((n(1 - \delta), \frac{K}{n})\) distribution also hold for \(|C_k^*|\), so (50) and (51) with \(np = K(1 - \delta)\) and \(\epsilon = \sqrt{3\log n/[K(1 - \delta)]}\) imply
\[\mathbb{P}\{|C_k^*| - K(1 - \delta)\| \geq \sqrt{3K(1 - \delta) \log n}\| \leq 2n^{-1} \leq n^{-1/2+o(1)}\].

Hence, it follows from Theorem 1 and the condition \(\lambda > 1/e\) that
\[\mathbb{P}\{|\hat{C}_k \triangle C_k^*| \leq \delta K\| 1 \leq k \leq 1/\delta\| \to 1,\]
as \(n \to \infty\), where \(\hat{C}_k\) is the output of the BP algorithm in Step 3 of Algorithm 2.

Applying [14, Theorem 3] together with assumption (6), we get that \(\mathbb{P}\{\hat{C} = C^*\} \to 1\) as \(n \to \infty\). \(\square\)
Proof of Theorem 3. The average error probability, $p_e$, for classifying the label of a vertex in the graph $G$ is greater than or equal to the lower bound (38) on average error probability for the tree model, minus the upper bound, $n^{-1+o(1)}$, on the coupling error provided by Lemma 10. Multiplying the lower bound on average error probability per vertex by $n$ yields (8). Similarly, $p_{e,0}$ and $p_{e,1}$, for the community recovery problem can be approximated by the respective conditional error probabilities for the random tree model by the last part of the coupling lemma, Lemma 10, so (9) follows from (39).

By Lemma 9, assuming $p/q = O(1)$ and $\nu \to \infty$, $\lim \inf_{n \to \infty} \frac{n}{K} \hat{p}_e \geq 1$, where $\hat{p}_e$ is the average error probability for any estimator for the corresponding random tree network. By the coupling lemma, Lemma 10, $|\hat{p}_e - p_e| \leq n^{-1+o(1)}$. By assumption that $\frac{n}{K} = n^{o(1)}$, $|\frac{n}{K} \hat{p}_e - \frac{n}{K} p_e| \leq n^{-1+o(1)}$. The conclusion $\lim \inf_{n \to \infty} \frac{n}{K} \hat{p}_e \geq 1$ follows from the triangle inequality. \hfill \Box

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Appendix A. Degree-thresholding when $K \asymp n$

A simple algorithm for recovering $C^*$ is degree-thresholding. Specifically, let $d_i$ denote the degree of vertex $i$. Then $d_i$ is distributed as the sum of two independent random variables, with distributions $\text{Binom}(K-1, p)$ and $\text{Binom}(n-K, q)$, respectively, if $i \in C^*$, while $d_i \sim \text{Binom}(n-1, q)$ if $i \notin C^*$. The mean degree difference between these two distributions is $(K-1)(p-q)$, and the degree variance is $O(nq)$. By assuming $p/q$ is bounded, it follows from the Bernstein’s inequality that $|d_i - \mathbb{E}[d_i]| \geq (K-1)(p-q)/2$ with probability at most $e^{-\Omega((K-1)^2(p-q)^2/(nq))}$. Let $\hat{C}$ be the set of vertices with degrees larger than $nq + (K-1)(p-q)/2$ and thus $\mathbb{E}||\hat{C} \triangle C^*|| = ne^{-\Omega((K-1)^2(p-q)^2/(nq))}$. Hence, if $(K-1)^2(p-q)^2/(nq) = \omega(\log \frac{p}{K})$, then $\mathbb{E}||\hat{C} \triangle C^*|| = o(K)$, i.e., weak recovery is achieved. In the regime $K \asymp n - K \asymp n$ and $p$ is bounded away from 1,
the necessary and sufficient condition for the existence of estimators providing weak recovery, is $K^2(p-q)^2/(nq) \to \infty$ as shown in [14]. Thus, degree-thresholding provides weak recovery in this regime whenever it is information theoretically possible. Under the additional condition (6), an algorithm attaining exact recovery can be built using degree-thresholding for weak recovery followed by a linear time voting procedure, as in Algorithm 2 (see [14, Theorem 3] and its proof). In the regime $\frac{K}{n} \log \frac{n}{K} = o(\log n)$, or equivalently $K = \omega(n \log \log n / \log n)$, the information-theoretic necessary condition for exact recovery given by (43) and (45) imply that $K^2(p-q)^2/(nq) = \omega(\log n)$, and hence in this regime the degree-thresholding attains exact recovery whenever it is information theoretically possible.

**Appendix B. Comparison with information theoretic limits**

As noted in the introduction, in the regime $K = \Theta(n)$, degree-thresholding achieves weak recovery and, if a voting procedure is also used, exact recovery whenever it is information theoretically possible. This section compares the recovery thresholds by belief propagation to the information-theoretic thresholds established in [14], in the regime of

$$K = o(n), \quad np = n^{o(1)}, \quad p/q = O(1), \quad \text{(41)}$$

which is the main focus of this paper.

The information-theoretic threshold for weak recovery is established in [14, Corollary 1], which, in the regime (41), reduces to the following: If

$$\liminf_{n \to \infty} \frac{K d(p||q)}{2 \log \frac{n}{K}} > 1, \quad \text{(42)}$$

then weak recovery is possible. On the other hand, if weak recovery is possible, then

$$\liminf_{n \to \infty} \frac{K d(p||q)}{2 \log \frac{n}{K}} \geq 1. \quad \text{(43)}$$

To compare with belief propagation, we rephrase the above sharp threshold in terms of the signal-to-noise ratio $\lambda$ defined in (1). Note that $d(p||q) = (p \log \frac{p}{q} + q - p)(1 + o(1))$ provided that $p/q = O(1)$ and $p \to 0$. Therefore the information-theoretic weak recovery threshold is given by

$$\lambda > (C(p/q) + \epsilon) \frac{K}{n} \log \frac{n}{K}, \quad \text{(44)}$$
for any $\epsilon > 0$, where $C(\alpha) \triangleq \frac{2(\alpha-1)^2}{1-\alpha+\alpha \log \alpha}$. In other words, in principle weak recovery only demands a vanishing signal-to-noise ratio $\lambda = \Theta(\frac{K}{n} \log \frac{n}{K})$, while, in contrast, belief propagation requires $\lambda > 1/e$ to achieve weak recovery. No polynomial-time algorithm is known to succeed for $\lambda \leq 1/e$, suggesting that computational complexity constraints might incur a severe penalty on the statistical optimality in the sublinear regime of $K = o(n)$.

Next we turn to exact recovery. The information-theoretic optimal threshold has been established in [14, Corollary 3]. In the regime of interest (41), exact recovery is possible via the maximum likelihood estimator (MLE) provided that (42) and (6) hold. Conversely, if exact recovery is possible, then (43) and

\[ \liminf_{n \to \infty} \frac{Kd(\tau^*\|q)}{\log n} \geq 1 \]  \hspace{1cm} (45)

must hold. Notice that the information-theoretic sufficient condition for exact recovery has two parts: one is the information-theoretic sufficient condition (42) for weak recovery; the other is the sufficient condition (6) for the success of the linear time voting procedure. Similarly, recall that the sufficient condition for exact recovery by belief propagation also has two parts: one is the sufficient condition $\lambda > 1/e$ for weak recovery, and the other is again (6).

Clearly, the information-theoretic sufficient conditions for exact recovery and $\lambda > 1/e$, which is needed for weak recovery by local algorithms, are both at least as strong as the information theoretic necessary conditions (43) for weak recovery. It is thus of interest to compare them by assuming that (43) holds. If $p/q$ is bounded, $p$ is bounded away from 1, and (43) holds, then $d(\tau^*\|q) \approx d(p\|q) \approx \frac{(p-q)^2}{q}$ as shown in [14]. So under those conditions on $p, q$ and (43), and if $K/n$ is bounded away from 1,

\[ \frac{Kd(\tau^*\|q)}{\log n} \approx \frac{K(p-q)^2}{q \log n} \approx \left( \frac{n}{K \log n} \right) \lambda. \]  \hspace{1cm} (46)

Hence, the information-theoretic sufficient condition for exact recovery (6) demands a signal-to-noise ratio

\[ \lambda = \Theta \left( \frac{K \log n}{n} \right). \]  \hspace{1cm} (47)

Therefore, on one hand, if $K = \omega(n/\log n)$, then condition (6) is stronger than $\lambda > 1/e$, and thus condition (6) alone is sufficient for local algorithms to attain exact
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Figure 1: Phase diagram with $K = \rho n / \log n$, $p = cq$, and $q = b \log^2 n / n$ for $c = 2$ and $n \to \infty$. In region I, exact recovery is provided by the BP algorithm plus voting procedure. In region II, weak recovery is provided by the BP algorithm, but exact recovery is not information theoretically possible. In region III, exact recovery is information theoretically possible, but no polynomial-time algorithm is known for even weak recovery. In region IV, with $b > 0$ and $\rho > 0$, weak recovery, but not exact recovery, is information theoretically possible and no polynomial time algorithm is known for weak recovery.

On the other hand, if $K = o(n / \log n)$, then $\lambda > 1/e$ is stronger than condition (45), and thus for local algorithms to achieve exact recovery, it requires $\lambda > 1/e$, which far exceeds the information-theoretic optimal level (47). The critical value of $K$ for this crossover is $K = \Theta \left( \frac{n}{\log n} \right)$. To illustrate it further, consider the regime $K = \frac{\rho n}{\log n}$, $p = \frac{a \log^2 n}{n}$, $q = \frac{b \log^2 n}{n}$, where $\rho \in (0, 1)$ and $a > b > 0$ are fixed constants. In this regime, $Kd(p || q) \propto \frac{K(p-q)^2}{q} \propto \log n$ and thus (42) is satisfied and weak recovery is always information theoretically possible. Furthermore, $\lambda = \frac{\rho^2 (a-b)^2}{b}$, and weak recovery is possible via BP (Algorithm 1) if $\frac{\rho^2 (a-b)^2}{b} > 1/e$ and is impossible if $\frac{\rho^2 (a-b)^2}{b} \leq 1/e$ for any local algorithm. Also, $\tau^* = [1 + o(1)] \tau_0 \log^2 n / n$, $d(\tau^* || q) = [1 + o(1)] I(b, \tau_0) \log^2 n / n$, where $\tau_0 = (a - b) / \log(a/b)$ and $I(x, y) \triangleq x - y \log(ex/y)$ for $x, y > 0$. Hence,
information-theoretically, exact recovery is possible if \( \rho I(b, \tau_0) > 1 \) and impossible if \( \rho I(b, \tau_0) < 1 \). Fig. 1 shows the curve \( \{(b, \rho) : \rho^2 \frac{(a-b)^2}{b} = 1/e\} \) corresponding to the weak recovery condition by belief propagation, and the curve \( \{(b, \rho) : \rho I(b, \tau_0) = 1\} \) corresponding to the information-theoretic exact recovery threshold. Therefore, BP plus voting (Algorithm 2) achieves optimal exact recovery whenever the former curve lies below the latter, or equivalently, \( K \geq n \log n \left( \rho_{BP}(a/b) + o(1) \right) \), where \( \rho_{BP}(c) \triangleq \frac{1}{e(e-1)} \left( 1 - \frac{c}{\log e} \right) \) is the solution to \( \rho I(\frac{1}{e(e-1)} \rho, \tau_0) = 1 \) for \( c > 1 \).

Appendix C. Coupling lemma

Consider a sequence of planted dense subgraph models \( G = (E, V) \) as described in the introduction. For each \( i \in V \), \( \sigma_i \) denotes the indicator of \( i \in C^* \). For \( u \in V \), let \( G_u^t \) denote the subgraph of \( G \) induced by the vertices whose distance from \( u \) is at most \( t \). Recall from Section 4 that \( T_u^t \) is defined similarly for the random tree graph, and \( \tau_i \) denotes the label of a vertex \( i \) in the tree graph. The following lemma shows there is a coupling such that \( (G, \sigma) = (T_u^t, \tau_u^t) \) with probability converging to 1, where \( t_f \) is growing slowly with \( n \).

**Lemma 10.** (Coupling lemma.) Let \( d = np \). Suppose \( p, q, K \) and \( t_f \) depend on \( n \) such that \( t_f \) is positive integer valued, and \( (2 + d)^t = n^{o(1)} \). Consider an instance of the planted dense subgraph model. Suppose that \( C^* \) is random and all \( \binom{n}{|C^*|} \) choices of \( C^* \) are equally likely give its cardinality, \( |C^*| \). (If this is not true, this lemma still applies to the random graph obtained by randomly, uniformly permuting the vertices of \( G \).) If the planted dense subgraph model (Definition 1) is such that \( |C^*| \equiv K \), then for any fixed \( u \in [n] \), there exists a coupling between \( (G, \sigma) \) and \( (T_u^t, \tau_u^t) \) such that

\[
P \left\{ \left( G_u^t, \sigma_{G_u^t} \right) = \left( T_u^t, \tau_{T_u^t} \right) \right\} \geq 1 - n^{-1+o(1)}. \tag{48} \]

If the planted dense subgraph model is such that \( |C^*| \sim \text{Binom}(n, K/n) \), then for any fixed \( u \in [n] \), there exists a coupling between \( (G, \sigma) \) and \( (T_u, \tau_u) \) such that

\[
P \left\{ \left( G_u^t, \sigma_{G_u^t} \right) = \left( T_u^t, \tau_{T_u^t} \right) \right\} \geq 1 - n^{-1/2+o(1)}. \tag{49} \]
If the planted dense subgraph model is such that $K \geq 3\log n$ and $|C^*|$ is random such that $\mathbb{P}\{||C^*| - K| \geq \sqrt{3K \log n}\} \leq n^{-1/2+o(1)}$, then there exists a coupling between $(G,\sigma)$ and $(T_u,\tau_{T_u})$ such that (49) holds.

Furthermore, the bounds stated remain true if the label, $\sigma_u$, of the vertex $u$ in the planted community graph, and the label $\tau_u$ of the root vertex in the tree graph, are both conditioned to be 0 or are both conditioned to be one.

Remark 1. The condition $(2 + d)t_f = n^{o(1)}$ in Lemma 10 is satisfied, for example, if $t_f = O(\log^* n)$ and $d \leq n^{o(1/\log^* n)}$, or if $t_f = O(\log \log n)$ and $d = O((\log n)^s)$ for some constant $s > 0$. In particular, the condition is satisfied if $t_f = O(\log^* n)$ and $d = O((\log n)^s)$ for some constant $s > 0$.

Remark 2. The part of Lemma 10 involving $||C^*| - K| \geq \sqrt{3K \log n}$ is included to handle the case that $|C^*|$ has a certain hypergeometric distribution. In particular, if we begin with the planted dense subgraph model (Definition 1) with $n$ vertices and a planted dense community with $|C^*| = K$, for a cleanup procedure we will use for exact recovery (See Algorithm 2), we need to withhold a small fraction $\delta$ of vertices and run the belief propagation algorithm on the subgraph induced by the set of $n(1 - \delta)$ retained vertices. Let $C^{**}$ denote the intersection of $C^*$ with the set of $n(1 - \delta)$ retained vertices. Then $|C^{**}|$ is obtained by sampling the vertices of the original graph without replacement. Thus, the distribution of $|C^{**}|$ is hypergeometric, and $\mathbb{E}[|C^{**}|] = K(1 - \delta)$. Therefore, by a result of Hoeffding [15], the distribution of $|C^{**}|$ is convex order dominated by the distribution that would result by sampling with replacement, namely, by Binom$(n(1 - \delta), \frac{K}{n})$. That is, for any convex function $\Psi$, $\mathbb{E}[\Psi(|C^{**}|)] \leq \mathbb{E}[\Psi(\text{Binom}(n(1 - \delta), \frac{K}{n}))]$. Therefore, Chernoff bounds for Binom$(n(1 - \delta), \frac{K}{n})$ also hold for $|C^{**}|$. We use the following Chernoff bounds for binomial distributions [26, Theorem 4.4, 4.5]: For $X \sim \text{Binom}(n, p)$:

$$\mathbb{P}\{X \geq (1 + \epsilon)np\} \leq e^{-\epsilon^2 np/3}, \quad \forall 0 \leq \epsilon \leq 1$$

(50)

$$\mathbb{P}\{X \leq (1 - \epsilon)np\} \leq e^{-\epsilon^2 np/2}, \quad \forall 0 \leq \epsilon \leq 1.$$  

(51)

Thus, if $K(1 - \delta) \geq 3 \log n$, then (50) and (51) with $\epsilon = \sqrt{3 \log n / [K(1 - \delta)]}$ imply

$$\mathbb{P}\{||C^{**}| - K(1 - \delta)| \geq \sqrt{3K(1 - \delta) \log n}\} \leq n^{-1}.$$  

Thus, Lemma 10 can be applied with $K$ replaced by $K(1 - \delta)$. 


Proof. We write $V = V(G)$ and $V^t = V(G) \setminus V(G_u^t)$. Let $V_0^t$ and $V_1^t$ denote the set of vertices $i$ in $V^t$ with $\sigma_i = 0$ and $\sigma_i = 1$, respectively. For a vertex $i \in \partial G_u^t$, let $\tilde{L}_i$ denote the number of $i$'s neighbors in $V_1^t$, and $\tilde{M}_i$ denote the number of $i$'s neighbors in $V_0^t$.

The event $C^t$ is useful to ensure that $V^t$ is large enough so that the binomial random variables $\tilde{M}_i$ and $\tilde{L}_i$ can be well approximated by Poisson random variables with the appropriate means. The following lemma shows that $C^t$ happens with high probability conditional on $C^{t-1}$.

**Lemma 11.** For $t \geq 1$,

$$\mathbb{P}\{C^t|C^{t-1}\} \geq 1 - n^{-4/3}.$$  

Moreover, $P(C^t) \geq 1 - tn^{-4/3}$, and conditional on the event $C^{t-1}$, $|G_u^{t-1}| \leq 4(2 + 2d)^t \log n$.

**Proof.** Conditional on $C^{t-1}$, $|\partial G_{u}^{t-1}| \leq 4(2 + 2d)^{t-1} \log n$. For any $i \in \partial G_{u}^{t-1}$, $\tilde{L}_i + \tilde{M}_i$ is stochastically dominated by $\text{Binom}(n, d/n)$, and $\{\tilde{L}_i, \tilde{M}_i\}_{i \in \partial G_{u}^{t-1}}$ are independent. It follows that $|\partial G_{u}^t|$ is stochastically dominated by (using $d + 1 \geq d$):

$$X \sim \text{Binom}(4(2 + 2d)^{t-1}n \log n, (d + 1)/n).$$  

Notice that $\mathbb{E}[X] = 2(2 + 2d)^t \log n \geq 4 \log n$. Hence, in view of the Chernoff bound (50) with $\epsilon = 1$,

$$\mathbb{P}\{C^t|C^{t-1}\} \geq \mathbb{P}\{X \leq 4(2 + 2d)^t \log n\}$$

$$= 1 - \mathbb{P}\{X > 2\mathbb{E}[X]\} \geq 1 - e^{-\mathbb{E}[X]/3} \geq 1 - n^{-4/3}.$$  

Since $C^0$ is always true, $P(C^t) \geq (1 - n^{-4/3})^t \geq 1 - tn^{-4/3}$. Finally, conditional on $C^{t-1}$,

$$|G_u^{t-1}| = \sum_{s=0}^{t-1} \partial G_u^s \leq \sum_{s=0}^{t-1} 4(2 + 2d)^s \log n$$

$$= 4(2 + 2d)^t \frac{1}{1 + 2d} \log n \leq 4(2 + 2d)^t \log n.$$
Note that it is possible to have \( i, i' \in \partial G^t_u \) which share a neighbor in \( V^t \), or which themselves are connected by an edge, so \( G^t_u \) may not be a tree. The next lemma shows that with high probability such events don’t occur. For any \( t \geq 1 \), let \( A^t \) denote the event that no vertex in \( V^{t-1} \) has more than one neighbor in \( G^{t-1}_u \); \( B^t \) denote the event that there are no edges within \( \partial G^t_u \). Note that if \( A^t \) and \( B^t \) hold for all \( s = 1, \ldots, t \), then \( G^t_u \) is a tree.

**Lemma 12.** For any \( t \) with \( 1 \leq t \leq t_f \),
\[
\Pr \{ A^t | C^{t-1} \} \geq 1 - n^{-1+o(1)} \\
\Pr \{ B^t | C^t \} \geq 1 - n^{-1+o(1)}.
\]

**Proof.** For the first claim, fix any \( i, i' \in \partial G^{t-1}_u \). For any \( j \in V^{t-1} \), \( \Pr \{ A_{ij} = A_{i'j} = 1 \} \leq d^2/n^2 \). Since \( |V^{t-1}| \leq n \) and conditional on \( C^{t-1} \), \( |\partial G^{t-1}_u| \leq 4(2d)^{t-1} \log n = n^{o(1)} \).

It follows from the union bound that, given \( C^{t-1} \),
\[
\Pr \{ \exists i, i' \in \partial G^{t-1}_u, j \in V^{t-1} : A_{ij} = A_{i'j} = 1 \} \leq n16(2d)^{2t-2} \log^2 n \times \frac{d^2}{n^2} = n^{-1+o(1)}.
\]
Therefore, \( \Pr \{ A^t | C^{t-1} \} \geq 1 - n^{-1+o(1)} \). For the second claim, fix any \( i, i' \in \partial G^t_u \). Then \( \Pr \{ A_{i,i'} = 1 \} \leq d/n \). It follows from the union bound that, given \( C^t \),
\[
\Pr \{ \exists i, i' \in \partial G^t_u : A_{i,i'} = 1 \} \leq 16(2d)^{2t} \log^2 n \times \frac{d}{n} \leq n^{-1+o(1)}.
\]
Therefore, \( \Pr \{ B^t | C^t \} \geq 1 - n^{-1+o(1)} \). \( \square \)

In view of Lemmas 11 and 12, in the remainder of the proof of Lemma 10 we can and do assume without loss of generality that \( A_t, B_t, C_t \) hold for all \( t \geq 0 \). We consider three cases about the cardinality of the community, \( |C^*| \):

- \( |C^*| \equiv K \).
- \( K \geq 3 \log n \) and \( \Pr \{ |C^*| - K \leq \sqrt{3K \log n} \} \geq 1 - n^{-1/2+o(1)} \). This includes the case that \( |C^*| \sim \text{Binom}(n, K/n) \) and \( K \geq 3 \log n \), as noted in Remark 2.
- \( K \leq 3 \log n \) and \( \Pr \{ |C^*| \leq 6 \log n \} \geq 1 - n^{-1/2+o(1)} \). This includes the case that \( |C^*| \sim \text{Binom}(n, K/n) \) and \( K \leq 3 \log n \), because, in this case, \( |C^*| \) is
stochastically dominated by a Binom\(n, 3 \log n/n\) random variable, so Chernoff bound \((50)\) with \(\epsilon = 1\) implies: \(\mathbb{P}\{ |C^*| \leq 6 \log n \} \geq 1 - n^{-1}\) if \(K \leq 3 \log n\).

In the second and third cases we assume these bounds (i.e., either \(|C^*| - K| \leq \sqrt{3K \log n}\) if \(K \geq 3 \log n\) or \(|C^*| \leq 6 \log n\) if \(K \leq 3 \log n\) hold, without loss of generality.

We need a version of the well-known bound on the total variation distance between the binomial distribution and a Poisson distribution with approximately the same mean:

\[
d_{\text{TV}}(\text{Binom}(m, p), \text{Pois}(\mu)) \leq mp^2 + \psi(\mu - mp), \tag{52}
\]

where \(\psi(u) = e^{|u|/(1+|u|)} - 1\). The term \(mp^2\) on the right side of \((52)\) is Le Cam’s bound on the variational distance between the Binom\((m, p)\) and the Poisson distribution with the same mean, \(mp\); the term \(\psi(\mu - mp)\) bounds the variational distance between the two Poisson distributions with means \(\mu\) and \(mp\), respectively (see [30, Lemma 4.6] for a proof). Note that \(\psi(u) = O(|u|)\) as \(u \to 0\).

We recursively construct the coupling. For the base case, we can arrange that \(\tau_u^t = 1\) and assume that \((G_u^t, \sigma_{G_u^t}) = (T_u^t, \tau_u^t)\) holds with probability at least \(1 - n^{-1+o(1)}\) if \(|C^*| \equiv K\) and with probability at least \(1 - n^{-1/2+o(1)}\) in the other cases. Each of the vertices \(i\) in \(\partial G_u^{t-1}\) has a random number of neighbors \(\bar{L}_i\) in \(V_1^{t-1}\) and a random number of neighbors \(\bar{M}_i\) in \(V_0^{t-1}\). These variables are conditionally independent given \((G_u^{t-1}, \sigma_{G_u^{t-1}}, |V_1^{t-1}|, |V_0^{t-1}|)\). Thus we bound the total variational distance of these random variables from the corresponding Poisson distributions by using a union bound, summing over all \(i \in \partial G_u^{t-1}\). Since \(C^t-1\) holds, \(|\partial G_u^{t-1}| \leq 4(2+2d)t^{-1} \log n = n^{o(1)}\), so it suffices to show that the variational distance for the numbers of children with each label for any given vertex in \(\partial G_u^{t-1}\) is at most \(n^{-1/2+o(1)}\) (because \(n^{o(1)}n^{-1/2+o(1)} =...\)

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n^{-1/2+o(1)}. Specifically, we need to obtain such a bound on the variational distances for three types of random variables:

- \( L_i \) for vertices \( i \in \partial G_u^{t-1} \) with \( \sigma_i = 1 \)
- \( \tilde{L}_i \) for vertices \( i \in \partial G_u^{t-1} \) with \( \sigma_i = 0 \)
- \( \tilde{M}_i \) for vertices in \( i \in \partial G_u^{t-1} \) (for either \( \sigma_i \)).

The corresponding variational distances, conditioned on \(|V_1^{t-1}| \) and \(|V_0^{t-1}| \), and the bounds on the distances implied by (52), are as follows:

\[
\begin{align*}
    d_{TV}(\text{Binom}(|V_1^{t-1}|, p), \text{Pois}(Kp)) & \leq |V_1^{t-1}|p^2 + \psi((K - |V_1^{t-1}|)p) \\
    d_{TV}(\text{Binom}(|V_1^{t-1}|, q), \text{Pois}(Kq)) & \leq |V_1^{t-1}|q^2 + \psi((K - |V_1^{t-1}|)q) \\
    d_{TV}(\text{Binom}(|V_0^{t-1}|, q), \text{Pois}((n - K)q)) & \leq |V_0^{t-1}|q^2 + \psi((n - K - |V_0^{t-1}|)q)
\end{align*}
\]

The assumption on \( d \) implies \( p \leq o(n^{-1+o(1)}) \) and \( np^2 = dp \leq n^{-1+o(1)} \), and thus also \( |V_1^{t-1}|q^2 \leq |V_1^{t-1}|p^2 \leq n^{-1+o(1)} \) and \( |V_0^{t-1}|q^2 \leq n^{-1+o(1)} \). Also, for use below, \( Kq^2 \leq Kp^2 \leq n^{-1+o(1)} \).

We now complete the proof for the three possible cases concerning \(|C^*|\). Consider the first case, that \(|C^*| \equiv K\). Since we are working under the assumption \( C^{t-1} \) holds, in the case \(|C^*| \equiv K\),

\[
|(|K - |V_1^{t-1}|)|p| \leq p|G_u^{t-1}| \leq p4(2 + 2d)^t \log n \leq n^{-1+o(1)}
\]

and similarly

\[
|(|n - K| - |V_0^{t-1}|)|q| \leq q|G_u^{t-1}| \leq q4(2 + 2d)^t \log n \leq n^{-1+o(1)}
\]

The conclusion (48) follows, proving the lemma in case \(|C^*| \equiv K\).

Next consider the second case: \(||C^*| - K| \leq \sqrt{3K \log n} \) and \( K \geq 3 \log n \). Using \( C^{t-1} \) as before, we obtain

\[
|(|K - |V_1^{t-1}|)|p| \leq \sqrt{3Kp^2} \log n + p4(2 + 2d)^t \log n \leq n^{-1/2+o(1)}
\]

and

\[
|(|n - K| - |V_0^{t-1}|)|q| \leq \sqrt{3Kq^2} \log n + q4(2 + 2d)^t \log n \leq n^{-1/2+o(1)}
\]

which establishes (49) in the second case.
Finally, consider the third case: $|C^*| \leq 6 \log n$ and $K \leq 3 \log n$. Then

$$|(K - |V_1^{t-1}|)| \leq 6p \log n + p(2 + 2d)^t \log n \leq n^{-1/2+o(1)}$$

and

$$|(n - K - |V_0^{t-1}|)| \leq 6q \log n + q(2 + 2d)^t \log n \leq n^{-1/2+o(1)},$$

which establishes (49) in the third case.

Thus, we can construct a coupling so that $(T_u^t, \tau_u^t) = (G_u^t, \sigma_{G_u^t})$ holds with probability at least $1 - n^{-1+o(1)}$ in case $|C^*| \equiv K$, and with probability $1 - n^{-1/2+o(1)}$ in the other cases, at each of the $t_f$ steps, and, furthermore, the $o(1)$ term in the exponents of $n$ are uniform in $t$ over $1 \leq t \leq t_f$. Since $2^{t_f} = n^{o(1)}$, it follows that $t_f = o(\log n)$. So the total probability of failure of the coupling is upper bounded by $t_fn^{-1+o(1)} = n^{-1+o(1)}$ in case $|C^*| \equiv K$ and by $n^{-1/2+o(1)}$ in the other cases.

Finally, we justify the last sentence of the lemma. At the base level of a recursive construction above, the proof uses the fact that the labels can be coupled with high probability because $P\{\sigma_u = 1\} \approx \frac{K}{n} = P\{\tau_u = 1\}$. If instead we let $u$ be a vertex selected uniformly at random from $C^*$, so that $\sigma_u \equiv 1$, and we consider the random tree conditioned on $\tau_u = 1$, the labels of $u$ in the two graphs are equal with probability one (i.e. exactly coupled), and then the recursive construction of the coupled neighborhoods can proceed from there. Similarly, if $u$ is a vertex selected uniformly at random from $[n]\backslash C^*$, then the lemma goes through for coupling with the labeled tree graph conditioned on $\tau_u = 0$. \hfill $\square$

**Appendix D. Analysis of BP on a tree continued—moments and CLT**

This section establishes messages in the BP algorithm are asymptotically Gaussian, a property which is used in the proof of the converse result, Theorem 3. First bounds on the first and second moments are found and then a version of the Berry-Essen CLT is applied.

**D.1. First and second moments of log likelihood messages for Poisson tree**

The following lemma provides estimates for the first and second moments of the log likelihood messages for the Poisson tree model.
Lemma 13. With \( C = \lambda(p/q + 2) \), for all \( t \geq 0 \),
\[
\begin{align*}
\mathbb{E}[Z_0^{t+1}] &= -\frac{\lambda b_t}{2} + O\left(\frac{\lambda^2 e^{Cbt-1}}{K(p-q)}\right) \\
\mathbb{E}[Z_1^{t+1}] &= -\frac{\lambda b_t}{2} + O\left(\frac{\lambda^2 e^{Cbt-1}}{K(p-q)}\right) \\
\text{var}(Z_0^{t+1}) &= \lambda b_t + O\left(\frac{\lambda^2 e^{Cbt-1}}{K(p-q)}\right) \\
\text{var}(Z_1^{t+1}) &= \lambda b_t + O\left(\frac{\lambda^2 e^{Cbt-1}}{K(p-q)}\right)
\end{align*}
\] (53) (54) (55) (56)

Lemma 14. Let \( \psi_2(x) \) and \( \psi_3(x) \) be defined for \( x \geq 0 \) by the relations: \( \log(1 + x) = x + \psi_2(x) \) and \( \log(1 + x) = x - \frac{x^2}{2} + \psi_3(x) \). Then \( 0 \geq \psi_2(x) \geq -\frac{x^2}{2} \), and \( 0 \leq \psi_3(x) \leq \frac{x^3}{3} \). In particular, \( |\psi_2(x)| \leq x^2 \) and \( |\psi_3(x)| \leq x^3 \). Moreover, \( |\log^2(1 + x) - x^2| \leq x^3 \).

Proof of Lemma 14. By the intermediate value form of Taylor's theorem, for any \( x \geq 0 \), \( \log(1 + x) = x + \frac{x^2}{2} \left(-\frac{1}{(1+y)^2}\right) \) for some \( y \in [0, x] \). The fact \(-1 \leq -\frac{1}{(1+y)^2} \leq 0\) then establishes the claim for \( \psi_2 \). Similarly, the claim for \( \psi_3 \) follows from the fact that for some \( z \in [0, x] \) \( \log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} \left(\frac{2}{1+y}\right) \). Finally, the first and second derivatives of \( \log^2(1 + x) \) at \( x = 0 \) are 0 and 2, and
\[
\left|\frac{1}{3!} \left(\frac{d}{dx}\right)^3 \log^2(1 + x)\right| = \left|\frac{4\log(1 + x) - 6}{3!(1 + x)^3}\right| \leq 1 \quad \text{for} \quad x \geq 0,
\]
so the final claim of the lemma also follows from Taylor's theorem. \(\square\)

Proof of Lemma 13. Plugging \( g(z) = \frac{1}{(1+e^{-z})^2} \) into (22) we have
\[
e^{\nu} \mathbb{E}\left[\frac{1}{(1+e^{-Z_0^{t+1}})^3}\right] + \mathbb{E}\left[\frac{1}{(1+e^{-Z_1^{t+1}})^3}\right] = \mathbb{E}\left[\frac{1}{(1+e^{-Z_0^{t+1}})^3}\right].
\] (57)

Applying Lemma 14, we have
\[
\log\left(\frac{e^{z\nu}(p/q) + 1}{e^{z\nu} + 1}\right) = \log\left(1 + \frac{p/q - 1}{1 + e^{-z\nu}}\right) = \frac{p/q - 1}{1 + e^{-z\nu}} + \frac{(p/q - 1)^2}{2(1 + e^{-z\nu})^2} + \psi_3\left(\frac{p/q - 1}{1 + e^{-z\nu}}\right).
\] (58) (59)

Hence,
\[
\Lambda_{u}^{t+1} = -K(p-q) + \sum_{t \in \partial u} \left[\frac{p/q - 1}{1 + e^{-\Lambda^t_{r\to u}+\nu}} - \frac{(p/q - 1)^2}{2(1 + e^{-\Lambda^t_{r\to u}+\nu})^2} + \psi_3\left(\frac{p/q - 1}{1 + e^{-\Lambda^t_{r\to u}+\nu}}\right)\right].
\]
It follows, by considering the case the label of vertex $u$ is conditioned to be zero, that:

$$
E[Z_{q+1}^t] = -K(p - q) + E[L_u]E\left[\frac{p/q - 1}{1 + e^{-Z_{i+1}^t}}\right] + E[M_u]E\left[\frac{p/q - 1}{1 + e^{-Z_{0+1}^t}}\right] - E[L_u]E\left[\frac{(p/q - 1)^2}{2(1 + e^{-Z_{i+1}^t})^2}\right]
$$

$$
- E[M_u]E\left[\frac{(p/q - 1)^2}{2(1 + e^{-Z_{0+1}^t})^2}\right] + E[L_u]E\left[\psi_3\left(\frac{p/q - 1}{1 + e^{-Z_{i+1}^t}}\right)\right] + E[M_u]E\left[\psi_3\left(\frac{p/q - 1}{1 + e^{-Z_{0+1}^t}}\right)\right].
$$

Notice that $E[L_u] = Kq$ and $E[M_u] = (n - K)q$. Thus

$$
E[L_u]E\left[\frac{p/q - 1}{1 + e^{-Z_{i+1}^t}}\right] + E[M_u]E\left[\frac{p/q - 1}{1 + e^{-Z_{0+1}^t}}\right] = Kq(p/q - 1)\left(E\left[\frac{1}{1 + e^{-Z_{i+1}^t}}\right] + e^\nu E\left[\frac{1}{1 + e^{-Z_{0+1}^t}}\right]\right)
$$

$$
= K(p - q),
$$

where the last equality holds due to (23). Moreover,

$$
E[L_u]E\left[\frac{(p/q - 1)^2}{(1 + e^{-Z_{i+1}^t})^2}\right] + E[M_u]E\left[\frac{(p/q - 1)^2}{(1 + e^{-Z_{0+1}^t})^2}\right]
$$

$$
= Kq(p/q - 1)^2\left(E\left[\frac{1}{(1 + e^{-Z_{i+1}^t})^2}\right] + e^{2\nu}E\left[\frac{1}{(1 + e^{-Z_{0+1}^t})^2}\right]\right)
$$

$$
\leq K(q(p/q - 1)^2E\left[\frac{1}{(1 + e^{-Z_{i+1}^t})^2}\right],
$$

$$
\leq Kq(p/q - 1)^2E\left[\frac{e^{Z_{i+1}^t}}{1 + e^{-Z_{i+1}^t}}\right] = \lambda b_i
$$

(60)

where (a) holds due to (24), and (b) holds due to the fact $\nu = \log\frac{n - K}{n}$. Also,

$$
E[L_u]E\left[\psi_3\left(\frac{p/q - 1}{1 + e^{-Z_{i+1}^t}}\right)\right] + E[M_u]E\left[\psi_3\left(\frac{p/q - 1}{1 + e^{-Z_{0+1}^t}}\right)\right]
$$

$$
\leq E[L_u]E\left[\frac{(p/q - 1)^3}{(1 + e^{-Z_{i+1}^t})^3}\right] + E[M_u]E\left[\frac{(p/q - 1)^3}{(1 + e^{-Z_{0+1}^t})^3}\right]
$$

$$
= Kq(p/q - 1)^3\left(E\left[\frac{1}{(1 + e^{-Z_{i+1}^t})^3}\right] + e^{3\nu}E\left[\frac{1}{(1 + e^{-Z_{0+1}^t})^3}\right]\right)
$$

$$
\leq Kq(p/q - 1)^3e^{-2\nu}E\left[e^{2Z_{i+1}^t}\right] \leq \frac{\lambda^2 e^{C\nu - 1}}{K(p - q)}.
$$

(61)

where (a) holds due to (57); the last inequality holds because, as shown by Lemma 3, $E\left[e^{2Z_{i+1}^t}\right] \leq e^{C\nu - 1}$. Assembling the last four displayed equations yields (53).
Similarly,
\[
\mathbb{E} \left[ Z_{1}^{t+1} \right] = \mathbb{E} \left[ Z_{0}^{t+1} \right] + K(p - q) \mathbb{E} \left[ \log \left( \frac{e^{Z_{1}^{t+\nu}(p/q) + 1}}{e^{Z_{1}^{t+\nu} + 1}} \right) \right]
\]
\[
= \mathbb{E} \left[ Z_{0}^{t+1} \right] + \lambda b_t + K(p - q) \mathbb{E} \left[ \psi_2 \left( \frac{(p/q) - 1}{e^{-Z_{1}^{t+\nu}} + 1} \right) \right].
\]
and, using \(|\psi_2(x)| \leq x^2\) and the definition of \(\nu\),
\[
\left| K(p - q) \mathbb{E} \left[ \psi_2 \left( \frac{(p/q) - 1}{e^{-Z_{1}^{t+\nu}} + 1} \right) \right] \right| \leq \frac{\lambda^2 \mathbb{E} \left[ e^{2Z_{1}^{t}} \right]}{K(p - q)} \leq \frac{\lambda^2 e^{Cb_{b-1}}}{K(p - q)}
\]
It follows that (54) holds.

Next, we calculate the variance. For \(Y = \sum_{i=1}^{L} X_i\), where \(L\) is Poisson distributed and \(\{X_i\}\) are i.i.d. with finite second moments, it is well-known that \(\text{var}(Y) = \mathbb{E}[L] \mathbb{E}[X_1^2]\).

It follows that
\[
\text{var} \left( Z_{0}^{t+1} \right) = \mathbb{E} \left[ L_u \right] \mathbb{E} \left[ \log^2 \left( \frac{e^{Z_{1}^{t+\nu}(p/q) + 1}}{e^{Z_{1}^{t+\nu} + 1}} \right) \right]
\]
\[
+ \mathbb{E} \left[ M_u \right] \mathbb{E} \left[ \log^2 \left( \frac{e^{Z_{0}^{t+\nu}(p/q) + 1}}{e^{Z_{0}^{t+\nu} + 1}} \right) \right].
\]
Using (58) and the fact \(|\log^2(1 + x) - x^2| \leq x^3\) (see Lemma 14) yields
\[
\text{var} \left( Z_{0}^{t+1} \right) = \mathbb{E} \left[ L_u \right] \mathbb{E} \left[ \frac{(p/q - 1)^2}{(1 + e^{-Z_{1}^{t+\nu}})^2} \right] + \mathbb{E} \left[ M_u \right] \mathbb{E} \left[ \frac{(p/q - 1)^2}{(1 + e^{-Z_{0}^{t+\nu}})^2} \right]
\]
\[
+ O \left( \mathbb{E} \left[ L_u \right] \mathbb{E} \left[ \frac{(p/q - 1)^3}{(1 + e^{-Z_{1}^{t+\nu}})^3} \right] + \mathbb{E} \left[ M_u \right] \mathbb{E} \left[ \frac{(p/q - 1)^3}{(1 + e^{-Z_{0}^{t+\nu}})^3} \right] \right).
\]
Applying (60) and (61) yields (55).

Similarly, applying (60) and the fact \(|\log^2(1 + x) \leq x^2\), yields
\[
\text{var} \left( Z_{1}^{t+1} \right) = \text{var} \left( Z_{1}^{t+1} \right) + K(p - q) O \left( \frac{(p/q - 1)^2}{(1 + e^{-Z_{1}^{t+\nu}})^2} \right)
\]
\[
= \text{var} \left( Z_{0}^{t+1} \right) + O \left( \frac{\lambda^2 e^{Cb_{b-1}}}{K(p - q)} \right)
\]
\[
= \lambda b_t + O \left( \frac{\lambda^2}{K(p - q)} \right) e^{Cb_{b-1}},
\]
which together with (55) implies (56).

\(\square\)

D.2. Asymptotic Gaussian marginals of log likelihood messages

The following lemma is well suited for proving that the distributions of \(Z_0^t\) and \(Z_1^t\) are asymptotically Gaussian.
Lemma 15. (Analog of Berry-Esseen inequality for Poisson sums [22, Theorem 3].)
Let \( S_\lambda = X_1 + \cdots + X_{N_\lambda} \), where \( (X_i : i \geq 1) \) are independent, identically distributed random variables with mean \( \mu \), variance \( \sigma^2 \) and \( \mathbb{E} [|X_i|^3] \leq \rho^3 \), and for some \( \lambda > 0 \), \( N_\lambda \) is a Pois(\( \lambda \)) random variable independent of \( (X_i : i \geq 1) \). Then

\[
\sup_x \left| \mathbb{P} \left\{ \frac{S_\lambda - \lambda \mu}{\sqrt{\lambda (\mu^2 + \sigma^2)}} \leq x \right\} - \Phi(x) \right| \leq \frac{C_{BE} \rho^3}{\sqrt{\lambda (\mu^2 + \sigma^2)^3}}
\]

where \( C_{BE} = 0.3041 \).

Lemma 16. Suppose \( \lambda > 0 \) is fixed, and the parameters \( p/q \) and \( \nu \) vary such that \( p/q = O(1) \), \( \nu \) is bounded from below (i.e. \( K/n \) is bounded away from one) and \( K(p - q) \to \infty \). (The latter condition holds if either \( \nu \to \infty \) or \( p/q \to 1 \); see Remark 4.) Suppose \( t \in \mathbb{N} \) is fixed, or more generally, \( t \) varies with \( n \) such that \( e^{C'b_{t'}} = o(b_t) \) as \( n \to \infty \), where \( C' = \lambda \left( 3 + 2\frac{p}{q} + \left(\frac{p}{q}\right)^2 \right) \). Then

\[
\sup_x \left| \mathbb{P} \left\{ \frac{Z_{t+1} - \lambda b_t}{\sqrt{\lambda b_t}} \leq x \right\} - \Phi(x) \right| \to 0 \quad (62)
\]

\[
\sup_x \left| \mathbb{P} \left\{ \frac{Z_{t+1} - \lambda b_t}{\sqrt{\lambda b_t}} \leq x \right\} - \Phi(x) \right| \to 0 \quad (63)
\]

Remark 3. Note that in the case of \( \lambda \leq 1/e, b_t \leq e \) for all \( t \geq 0 \). As a consequence, (62) and (63) hold for all \( t \), and, as can be checked from the proof, the limits hold uniformly in \( t \). Also, in the case \( b_t \) is bounded independently of \( n \), (63) is a consequence of (62) and the fact that \( Z_{t+1}^0 \) is the log likelihood ratio. In the proof below, (63) is proved directly.

Remark 4. The condition \( K(p - q) \to \infty \) in Lemma 16 is essential for the proof; we state some equivalent conditions here. Equations (17)-(19) express \( Kp \), \( Kq \), and \( (n - K)q \) in terms of the parameters \( \lambda, \nu \), and \( p/q \). Similarly,

\[
K(p - q) = \frac{\lambda e^\nu}{p/q - 1}
\]

\[
np = \frac{\lambda(p/q)e^\nu(e^\nu + 1)}{(p/q - 1)^2}
\]

\[
\frac{(n - K)q}{K(p - q)} = \frac{e^\nu}{p/q - 1}.
\]

It follows that if \( \frac{K(p - q)^2}{(n - K)q} = \lambda \) for a fixed \( \lambda > 0 \), \( p/q = O(1) \), and \( \nu \) is bounded below (i.e. \( K/n \) is bounded away from one) then the following seven conditions are equivalent:
(K(p - q) \to \infty), (\nu \to \infty \text{ or } \frac{\nu}{q} \to 1), (Kp \to \infty), (Kq \to \infty), ((n - K)q \to \infty), (np \to \infty), (K(p - q) = o((n - K)q)).

Proof of Lemma 16. Throughout the proof it is good to keep in mind that \( b_0 = \frac{1}{1 + e^{-t}} \), so that \( b_0 \) is bounded from below by a fixed positive constant, and, as shown in Lemma 5, \( b_t \) is nondecreasing in \( t \). For \( t \geq 0 \), \( Z_0^{t+1} \) can be represented as follows:

\[
Z_0^{t+1} = -K(p - q) + \sum_{i=1}^{N_{nq}} X_i,
\]

where \( N_{nq} \) has the Pois\((nq)\) distribution, the random variables \( \{X_i, i \geq 0\} \) are mutually independent and independent of \( N_{nq} \), and the distribution of \( X_i \) is a mixture of distributions: \( \mathcal{L}(X_i) = \frac{(n-K)q}{nq} \mathcal{L}(f(Z_0^i)) + \frac{Kq}{nq} \mathcal{L}(f(Z_1^i)), \) where \( f(z) = \log \left( \frac{e^{t_1} - x^{1/(p/q)} + 1}{e^{t_1} + 1 + x} \right) \).

By (55) of Lemma 13 and the formula for the variance of the sum of a Poisson distributed number of iid random variables,

\[
nq \mathbb{E} \left[ X_i^2 \right] = \text{var}(Z_0^{1+t}) = \lambda b_t + O \left( \frac{\lambda^2 e^{Cb_{t-1}}}{K(p - q)} \right).
\]

The function \( f \), and therefore the \( X_i \)'s, are nonnegative. Using the fact \( \log^3(1+x) \leq x^3 \) for \( x \geq 0 \), and applying (58) we find \( f^3(z) \leq \left( \frac{p/q - 1}{1 + e^{-Z_1^i + \nu}} \right)^3 \). Applying (61) yields

\[
nq \mathbb{E} \left[ |X_i|^3 \right] = \mathbb{E} \left[ |L_u| \mathbb{E} \left[ \frac{(p/q - 1)^3}{(1 + e^{-Z_1^i + \nu})^3} \right] + \mathbb{E} \left[ |M_u| \mathbb{E} \left[ \frac{(p/q - 1)^3}{(1 + e^{-Z_1^i + \nu})^3} \right] \right] \leq \frac{\lambda^2 e^{Cb_{t-1}}}{K(p - q)}.
\]

(64)

Therefore, the ratio relevant for application of the Berry-Esseen lemma satisfies:

\[
\frac{\mathbb{E} \left[ |X_i|^3 \right]}{nq \mathbb{E} \left[ X_i^2 \right]^3} \leq \frac{\lambda^2 e^{Cb_{t-1}}}{K(p - q) \sqrt{\left( \mathbb{E} \left[ X_i^2 \right] \right)^3} \left( \lambda b_t + O \left( \frac{\lambda^2 e^{Cb_{t-1}}}{K(p - q)} \right) \right)^3} \to 0.
\]

The Berry-Esseen lemma, Lemma 15, implies

\[
\sup_x \left\{ \frac{Z_0^{t+1} - \mathbb{E} \left[ Z_0^{t+1} \right]}{\sqrt{\text{var}(Z_0^{t+1})}} \leq x \right\} - \Phi(x) \leq \frac{C_{BE} \mathbb{E} \left[ |X_i|^3 \right]}{\sqrt{nq \mathbb{E} \left[ X_i^2 \right]^3}}.
\]

Applying Lemma 13 completes the proof of (62).
The proof of (63) given next is similar. For $t \geq 0$, $Z_{t+1}^1$ can be represented as follows:

\[
Z_{t+1}^1 = K(p - q) + \frac{1}{\sqrt{(n - K)q}} \sum_{i=1}^{N_{(n-K)q+Kp}} Y_i
\]

where $N_{(n-K)q+Kp}$ has the Pois($(n-K)q + Kp$) distribution, the random variables \( \{Y_i, i \geq 0\} \) are mutually independent and independent of $N_{(n-K)q+Kp}$, and the distribution of $Y_i$ is a mixture of distributions:

\[
L(Y_i) = \frac{(n-K)q}{(n-K)q+Kp} \mathcal{L}(f(Z_0^1)) + \frac{Kp}{(n-K)q+Kp} \mathcal{L}(f(Z_1^1)),
\]

where $f(z) = \log \left(\frac{e^{z-(p/q)+1}}{e^{z-\nu(p/q)+1}}\right)$.

By (56) of Lemma 13 and the formula for the variance of the sum of a Poisson distributed number of iid random variables,

\[
((n-K)q+Kp)\mathbb{E}[Y_i^2] = \text{var}(Z_{i+1}^1) = \lambda b_t + O\left(\frac{\lambda^2}{K(p-q)}\right) e^{Cb_t-1}.
\]

We again use $f^3(z) \leq \left(\frac{p/q-1}{1+e^{-z(q)}}\right)^3$. Applying (61) and Lemma 3 yields

\[
((n-K)q+Kp)\mathbb{E}[|Y_i|^3] = nq \mathbb{E}[|X_i|^3] + K(p-q)\mathbb{E}\left[\frac{(p/q-1)^3}{(1+e^{-Z_i^1+\nu})^3}\right] \leq \frac{\lambda^2 e^{Cb_{t-1}}}{K(p-q)} + \frac{\lambda^3 \mathbb{E}[e^{3Z_i^1}]}{K(p-q)^2} \leq \frac{\lambda^2 e^{Cb_{t-1}}}{K(p-q)} + \frac{\lambda^3 e^{C'b_{t-1}}}{(K(p-q))^2},
\]

where $C' = \lambda(3 + 2p/q + (p/q)^2)$.

Therefore, the ratio relevant for application of the Berry-Esseen lemma satisfies:

\[
\frac{\mathbb{E}[|Y_i|^3]}{\sqrt{((n-K)q+Kp)\mathbb{E}[Y_i^2]}} \leq \frac{\lambda^2 e^{Cb_{t-1}} + \lambda^3 e^{C'b_{t-1}}}{K(p-q)\sqrt{\left(\lambda b_t + O\left(\frac{\lambda^2}{K(p-q)}\right) e^{Cb_{t-1}}\right)^3}} \to 0.
\]

Therefore, the Berry-Esseen lemma, Lemma 15, along with Lemma 13, completes the proof of (63). \(\square\)
Appendix E. Linear message passing on a random tree

E.1. Linear message passing on a random tree—exponential moments

To analyze the message passing algorithms given in (12) and (13), we first study an analogous message passing algorithm on the tree model introduced in Section 4:

$$
\xi^{t+1}_{i \to \pi(i)} = \frac{q((n-K)A_i + KB_i)}{\sqrt{m}} + \frac{1}{\sqrt{m}} \sum_{\ell \in \partial i} \xi^t_{\ell \to i},
$$

(65)

$$
\xi^{t+1}_u = -\frac{q((n-K)A_i + KB_i)}{\sqrt{m}} + \frac{1}{\sqrt{m}} \sum_{i \in \partial u} \xi^t_{i \to u},
$$

(66)

with initial values $\xi^0_{\ell \to \pi(\ell)} = 1$ for all $\ell \neq u$, where $\pi(\ell)$ denotes the parent of $\ell$, and $m = (n-K)q$. Let $Z^t_0$ denote a random variable that has the same distribution as $\xi^t_0$ given $\tau_u = 0$, and let $Z^t_1$ denote a random variable that has the same distribution as $\xi^t_u$ given $\tau_u = 1$. Equivalently, $Z^t_b$ for $b \in \{0,1\}$ has the distribution of $\xi^t_{\ell \to \pi(\ell)}$ for any vertex $\ell \neq u$, given $\tau_\ell = b$. Let $A_t = \mathbb{E}[Z^t_0]$ and $B_t = \mathbb{E}[Z^t_1]$. Then $A_0 = B_0 = 1$.

Given $\tau_u = 0$, the mean of the sum in (65) is subtracted out, so $A_t = \mathbb{E}[Z^t_0] = 0$ for all $t \geq 1$. Compared to the case $\tau_u = 0$, if $\tau_u = 1$, then on average there are $K(p-q)$ additional children of node $u$ with labels equal to 1, so that $B_{t+1} = \lambda B_t$, which gives $B_t = \lambda^{t/2}$ for $t \geq 0$.

We consider sequences of parameter triplets $(\lambda, p/q, K/n)$ indexed by $n$. Let $\psi^t_i(\eta) = \mathbb{E}[e^{\eta Z^t_i}]$ for $i = 0, 1$ and $t \geq 1$. Expressions are given for these functions when $t = 1$ in (72) and (73) below. Following the same method used in Section 4 for the belief propagation algorithm, we find the following recursions for $t \geq 1$:

$$
\psi^{t+1}_0(\eta) = \exp \left\{ m \left( \psi^t_0 \left( \frac{\eta}{\sqrt{m}} \right) - 1 \right) + Kq \left( \psi^t_1 \left( \frac{\eta}{\sqrt{m}} \right) - 1 - \frac{\eta}{\sqrt{m}} \lambda^{t/2} \right) \right\},
$$

(67)

$$
\psi^{t+1}_1(\eta) = \psi^{t+1}_0(\eta) \exp \left\{ \sqrt{\lambda m} \left( \psi^t_1 \left( \frac{\eta}{\sqrt{m}} \right) - 1 \right) \right\}.
$$

(68)

**Lemma 17.** Assume that as $n \to \infty$, $\lambda$ is fixed, $K = o(n)$, and $p/q = O(1)$. (Consequently, $m \to \infty$; see Remark 4.) Let $\gamma$ be a constant such that $\gamma > 1$ and $\gamma \geq \lambda$.

Let $T = 2\alpha \log \frac{n-K}{\log \gamma}$, where $\alpha = 1/4$ (in fact any $\alpha < 1$ works). Let $c = \frac{1}{4} \log \gamma$ (in fact any $c \in (0, \log \sqrt{\gamma})$ works). For sufficiently large $n$, $t \in [T]$, and $\eta$ such that
\( \gamma^{(t-1)/2}(\frac{\lambda^2}{m} + \frac{\gamma}{\sqrt{m}}) \leq c, \)

\[
\psi_t^0(\eta) \leq \exp(\gamma^{t/2}\eta^2), \quad (69)
\]

\[
\psi_t^1(\eta) \leq \exp(\lambda^{t/2}\eta + \gamma^{t/2}\eta^2). \quad (70)
\]

**Proof of Lemma 17.** Recall that \( m = (n - K)q \) and \( K(p - q) = \sqrt{m} \). Since \( K = o(n) \), it follows that \( (nq)q \rightarrow 1 \). Also, because \( \lambda \) is fixed, we have that \( \lambda/m \rightarrow 0 \).

Hence, the choice of \( c \) ensures that for \( n \) sufficiently large,

\[
\left( \frac{nq}{m} + \frac{\sqrt{m}}{\sqrt{m}} \right) e^c \leq \sqrt{\gamma}. \quad (71)
\]

By (65), \( \xi_{i-\pi(i)}^1 = \frac{-nq+q|\eta_i|}{\sqrt{m}} \). Hence, for \( t = 1 \) and \( \eta \in (-\infty, \sqrt{mc}] \)

\[
\psi_t^0(\eta) = \exp(nq(e^{\eta/\sqrt{m}} - 1 - \eta/\sqrt{m})) \leq \exp \left( \frac{nq}{2m} e^c \eta^2 \right) \leq \exp(\sqrt{\gamma} \eta^2), \quad (72)
\]

where we used the fact that \( e^x \leq 1 + x + e^{c\eta^2} \) for all \( x \in (-\infty, c] \). Similarly,

\[
\psi_t^1(\eta) = \psi_t^0(\eta) \exp(K(p-q)(e^{\eta/\sqrt{m}} - 1)) \leq \exp \left( \frac{nq}{2m} e^c \eta^2 \right) \exp \left( \sqrt{\lambda m} \left( \frac{\eta}{\sqrt{m}} + \frac{e^c \eta^2}{2} \right) \right) \leq \exp \left( \sqrt{\lambda \eta} + \left( \frac{nq}{m} + \sqrt{\frac{\lambda}{m}} \right) e^c \eta^2 \right) \leq \exp(\sqrt{\lambda \eta} + \sqrt{m} \eta^2). \quad (73)
\]

Thus, (69) and (70) hold for \( t = 1 \) and \( \eta \) as described in the lemma.

Observe that

\[
\gamma^{T/2} = \frac{1}{\sqrt{m}} = o(1), \quad (74)
\]

because \( \gamma^{T/2} = (\frac{n-K}{n})^{1/2} \lambda^{-\alpha/2} \lambda^{-\alpha/2} = o(1) \). In addition, the choice of \( c \) guarantees that, for \( n \) sufficiently large,

\[
e^c + \left( \frac{Kq}{m} + \sqrt{\frac{\lambda}{m}} \right) \left( 1 + \frac{e^c}{2} \left( 3e + \gamma^{T/2} \right) \right) \leq \sqrt{\gamma}, \quad (75)
\]

because \( \frac{Kq}{m} = o(1) \), \( m \rightarrow \infty \), \( \frac{Kq}{m} \gamma^{T/2} = (\frac{K}{n-K})^{1-\alpha} = o(1) \), and (74) holds. Assume for the sake of proof by induction that, for some \( t \) with \( 1 \leq t < T \), (69) and (70) hold for all \( \eta \in \Gamma_t \equiv \{ \eta : \gamma^{(t-1)/2}(\frac{\lambda^2}{m} + \frac{\eta}{\sqrt{m}}) \leq c \} \). Now fix \( \eta \in \Gamma_{t+1} \). Since \( \Gamma_t \) is an interval
containing zero for each $t$ and $\Gamma_{t+1} \subset \Gamma_t$, it is clear that $\frac{\eta}{\sqrt{m}} \in \Gamma_t$ for $m \geq 1$. By (67), we have

$$\log \psi^{t+1}_0(\eta) = m \left( \psi^t \left( \frac{\eta}{\sqrt{m}} \right) - 1 \right) + Kq \left( \psi^t \left( \frac{\eta}{\sqrt{m}} \right) - 1 - \frac{\eta}{\sqrt{m}} \lambda^{t/2} \right)$$

$$\leq m \left( e^{\frac{\gamma t}{2} \frac{\eta^2}{m}} - 1 \right) + Kq \left( e^{\frac{\gamma t}{2} \frac{\eta^2}{m} + \frac{\lambda}{\sqrt{m}} \frac{\eta}{\sqrt{m}} \lambda^{t/2}} - 1 - \frac{\eta}{\sqrt{m}} \lambda^{t/2} \right)$$

$$\leq e^{\gamma t} \frac{\eta^2}{m} + Kq \left( \frac{\gamma t}{2} \frac{\eta^2}{m} + \frac{\lambda}{\sqrt{m}} \frac{\eta}{\sqrt{m}} \lambda^{t/2} \right)^2$$

$$\leq \gamma t \frac{\eta^2}{m} + \frac{Kq}{m} \left( 3c\gamma t^{1/2} + \gamma t \right) \eta^2$$

\(\leq \gamma \frac{(t+1)}{2} \frac{\eta^2}{m}\),

where the first inequality holds due to the induction hypothesis; the second inequality holds due to $e^x \leq 1 + e^x x$ for all $x \in [0, c]$ and $e^x \leq 1 + x + \frac{c^2}{2} x^2$ for all $x \in (-\infty, c]$; the third inequality holds due to the fact that $\eta \in \Gamma_{t+1}$ and $\lambda \leq \gamma$. Similarly,

$$\sqrt{\lambda m} \left( \psi^t_1 \left( \frac{\eta}{\sqrt{m}} \right) - 1 \right)$$

$$\leq \sqrt{\lambda m} \left( \gamma t \frac{\eta^2}{m} + \frac{e^c}{2} \left( \gamma t \frac{\eta^2}{m} + \lambda^{t/2} \frac{\eta}{\sqrt{m}} \right)^2 \right)$$

$$= \sqrt{\frac{\lambda}{m}} \left( \gamma t \frac{\eta^2}{m} + \frac{e^c}{2} \left( 3c\gamma t^{1/2} + \gamma t \right) \right) \eta^2 + \lambda \frac{(t+1)}{2} \eta$$

and hence by (68),

$$\log \psi^{t+1}_1(\eta) = \log \psi^{t+1}_0(\eta) + \sqrt{\lambda m} \left( \psi^t_1 \left( \frac{\eta}{\sqrt{m}} \right) - 1 \right)$$

$$\leq \gamma t \frac{\eta^2}{m} + \frac{Kq}{m} \left( \sqrt{\lambda m} \right) + \left( \frac{Kq}{m} \right) \left( \sqrt{\lambda m} \right) e^c \frac{\eta^2}{m} + \lambda \frac{(t+1)}{2} \eta$$

\(\leq \lambda \frac{(t+1)}{2} \frac{\eta^2}{m} + \gamma \frac{(t+1)}{2} \frac{\eta^2}{m}\).

\[\square\]

**Corollary 1.** Assume that as $n \to \infty$, $\lambda$ is fixed with $\lambda > 1$, $K = o(n)$, and $p/q = O(1)$. Let $T = 2\alpha \frac{\log m}{\log \lambda}$, where $\alpha = 1/4$. If $\tau = \frac{1}{2} \lambda^{T/2}$, then $\mathbb{P} \left\{ Z^T_0 \geq \tau \right\} = o \left( \frac{K}{n-K} \right)$ and $\mathbb{P} \left\{ Z^T_1 \leq \tau \right\} = o \left( \frac{K}{n-K} \right)$.

**Proof.** Since $\lambda > 1$ we can let $\gamma = \lambda$ in Lemma 17 so that $T$ here is the same as $T$ in Lemma 17. Equation (74) implies that the interval of $\eta$ values satisfying the condition
of Lemma 17 for \( t = T \) converges to all of \( \mathbb{R} \). By Lemma 17 and the Chernoff bound for threshold at \( \tau = \frac{1}{2} \lambda T^{1/2} \), for any \( \eta > 0 \), if \( n \) is sufficiently large

\[
P \{ Z_0^T \geq \tau \} \leq \psi_0^T(\eta) \exp(-\eta \tau) \leq \exp(\lambda T^{1/2}(\eta^2 - \eta/2)) \frac{\eta^{1/4}}{\eta} \exp(-\lambda T^{1/2}/16). \tag{76}
\]

Similarly, for any \( \eta < 0 \) and \( n \) sufficiently large,

\[
P \{ Z_1^T \leq \tau \} \leq \psi_1^T(\eta) \exp(-\eta \tau) \leq \exp(\lambda T^{1/2}(\eta^2 + \eta/2)) \frac{\eta^{1/4}}{\eta} \exp(-\lambda T^{1/2}/16). \tag{77}
\]

By the choice of \( T \), we have \( \lambda T^{1/2} = \frac{n - K K}{2} \alpha \) and hence \( \exp(-\lambda T^{1/2}/16) = o(\frac{K}{n - K}). \)

**E.2. Gaussian limits of messages**

In this section we apply the bounds derived in Section E.1 and a version of the Berry-Esseen central limit theorem for compound Poisson sums to show the messages are asymptotically Gaussian. As in Section E.1, the result allows the number of iterations to grow slowly with \( n \).

Let \( \alpha_t = \text{var}(Z_0^t) \) and \( \beta_t = \text{var}(Z_1^t) \). Using the usual fact \( \text{var}(\sum_{i=1}^X Y_i) = \mathbb{E}[X] \text{var}(Y) + \text{var}(X)\mathbb{E}[Y]^2 \) for iid \( Y \)'s, we find

\[
\alpha_{t+1} = \alpha_t + A_t^2 + \frac{Kq m}{m} \beta_t + \frac{Kq m}{m} B_t^2
\]

\[
\beta_{t+1} = \alpha_t + A_t^2 + \frac{Kp m}{m} \beta_t + \frac{Kp m}{m} B_t^2
\]

with the initial conditions \( \alpha_0 = \beta_0 = 0 \). Comparing the recursions (without using induction) shows that \( \alpha_t \leq \beta_t \leq \frac{q}{q} \alpha_t \) for \( t \geq 0 \). Note that \( \alpha_1 = \frac{K}{n - K} \geq 1 \), and \( \alpha_t \) is nondecreasing in \( t \). Thus \( 1 \leq \alpha_t \leq \beta_t \) for all \( t \). Therefore, if \( \lambda < 1 \), the signal to noise ratio \( \frac{(B_t - A_t)^2}{\alpha_t} \leq \lambda^t \to 0 \) as \( t \to \infty \). Also, under the assumption \( K = o(n) \) and \( p/q = O(1) \), the coefficients in the recursions (78) and (79) satisfy \( Kq m \to 0 \) and \( Kp m \to 0 \) as \( n \to \infty \). Thus, \( \alpha_t \to 1 \) and \( \beta_t \to 1 \) for \( t \) fixed as \( n \to \infty \).

The following lemma proves that the distributions of \( Z_0^t \) and \( Z_1^t \) are asymptotically Gaussian.

**Lemma 18.** Suppose that as \( n \to \infty \), \( \lambda \) is fixed with \( \lambda > 0 \), \( K = o(n) \), \( p/q = O(1) \), and \( t \) varies with \( n \) such that \( t \in \mathbb{N} \) and the following holds: If \( \lambda > 1 \) then \( \lambda^{t/2} \leq \left( \frac{n - K}{K} \right)^{\alpha} \), where \( \alpha = 1/4 \) (any \( \alpha \in (0, 1/3) \) works), and if \( \lambda \leq 1 \): \( t = O(\log \left( \frac{n - K}{K} \right)) \). Then as
\[ n \to \infty, \]
\[
\sup_x \left| \mathbb{P} \left\{ \frac{Z_0^t}{\sqrt{\alpha_t}} \leq x \right\} - \Phi(x) \right| \to 0 \tag{80}
\]
\[
\sup_x \left| \mathbb{P} \left\{ \frac{Z_1^t - \lambda t^{2/3}}{\sqrt{\beta_t}} \leq x \right\} - \Phi(x) \right| \to 0. \tag{81}
\]

**Proof.** Select a constant \( \gamma > 1 \) as follows. If \( \lambda > 1 \), let \( \gamma = \lambda \). If \( \lambda \leq 1 \), select \( \gamma > 1 \) so that \( \gamma^{t/2} \leq \left( \frac{n-K}{K} \right)^2 \) for all \( n \) sufficiently large, which is possible by the assumptions. Then no matter what the value of \( \lambda \) is, \( \gamma^{t/2} \leq \left( \frac{n-K}{K} \right)^2 \). Let \( T \) be defined as in Lemma 17. Since \( \gamma^{t/2} \leq \left( \frac{n-K}{K} \right)^2 \) it follows that \( t \leq T \).

For \( t \geq 0 \), \( Z_0^{t+1} \) can be represented as follows:
\[
Z_0^{t+1} = -Kq\lambda^{t/2} + (n-K)q1_{t=0} + \frac{1}{\sqrt{\alpha_{t+1}}} \sum_{i=1}^{N_{nq}} X_i
\]
where \( N_{nq} \) has the Pois\((nq)\) distribution, the random variables \( X_i, i \geq 0 \) are mutually independent and independent of \( N_{nq} \), and the distribution of \( X_i \) is a mixture of distributions: \( \mathcal{L}(X_i) = \left( \frac{n-K}{n} \right) \mathcal{L}(Z_0^i) + \frac{K}{n} \mathcal{L}(Z_1^i) \).

Note that \( \mathbb{E} \left[ |X_1|^3 \right] \leq \max \{ \mathbb{E} \left[ |Z_0^i|^3 \right], \mathbb{E} \left[ |Z_1^i|^3 \right] \} \leq \rho^3 \). By Lemma 15,
\[
\sup_x \left| \mathbb{P} \left\{ \sqrt{\alpha_{t+1}} Z_0^{t+1} + Kq\lambda^{t/2} + (n-K)q1_{t=0} - nq\mathbb{E} [X_1] \leq x \right\} - \Phi(x) \right| \leq \frac{C \rho^3}{\sqrt{nq\mathbb{E} [X_1]^3}}.
\]

Using the fact \( \mathbb{E} [X_1^2] \geq 1 \), \( \mathbb{E} [X_1] = \frac{K}{n} \lambda^{t/2} + \frac{n-K}{n} 1_{t=0} \), and \( \frac{n}{n-K} \mathbb{E} [X_1^2] = \alpha_{t+1} \), we obtain
\[
\sup_x \left| \mathbb{P} \left\{ \frac{Z_0^{t+1}}{\sqrt{\alpha_{t+1}}} \leq x \right\} - \Phi(x) \right| \leq \frac{C \rho^3}{\sqrt{nq}}.
\]
Equation (74) implies that the interval of \( \eta \) values satisfying the condition of Lemma 17 for \( t \leq T \) converges to all of \( \mathbb{R} \).

In view of Lemma 17 and the fact \( \gamma = \max \{\lambda, 1\} \), we have that for \( n \) sufficiently large, \( \psi_1^t(\pm \gamma^{t/2}) \leq 1 \) and \( \psi_1^t(\pm \gamma^{-t/2}) \leq e^2 \). Applying \( e^x + e^{-x} \geq |x|^3/6 \) with \( x = Z_0^t/\gamma^{t/2} \) or \( x = Z_1^t/\gamma^{t/2} \) yields:
\[ \mathbb{E} \left[ |Z_0^t|^3 \right] \leq 6 \gamma^{3t/2} \left( \psi_0^t(\gamma^{t/2}) + \psi_0^t(-\gamma^{t/2}) \right) \leq 12 \gamma^{3t/2}, \]
\[ \mathbb{E} \left[ |Z_1^t|^3 \right] \leq 6 \gamma^{3t/2} \left( \psi_1^t(\gamma^{t/2}) + \psi_1^t(-\gamma^{t/2}) \right) \leq 12 e^2 \gamma^{3t/2}. \]
Since \( \lambda \leq \left( \frac{K}{n-K} \right)^2 nq \left( \frac{K}{q} \right)^2 \) it follows that \( \sqrt{nq} = \Omega(n/K) \). Hence, \( \frac{\sqrt{3} \sqrt{nq}}{\sqrt{nq} + (n-K)q} = O\left( \frac{K}{n-K} \right)^{3\alpha} \frac{K}{n} \) = \( O\left( \frac{K}{n} \right)^{1-3\alpha} \) → 0 and (80) follows.

The proof of (81) given next is similar. For \( t \geq 0 \), \( Z_{t+1}^t \) can be represented as follows:

\[
Z_{t+1}^t = -Kq\lambda^{t/2} + (n-K)q \mathbf{1}_{\{t=0\}} + \frac{1}{\sqrt{m}} \sum_{i=1}^{N(n-K)q+Kp} Y_i
\]

where \( N(n-K)q+Kp \) has the \( \text{Pois}((n-K)q+Kp) \) distribution, the random variables \( Y_i, i \geq 0 \) are mutually independent and independent of \( N(n-K)q+Kp \), and the distribution of \( Y_i \) is a mixture of distributions:

\[
\mathcal{L}(Y_i) = \frac{m}{m+Kp} \mathcal{L}(Z_{0}) + \frac{Kp}{m+Kp} \mathcal{L}(Z_{1}).
\]

Note that \( \mathbb{E}[|Y_i|^3] \leq \max\{\mathbb{E}[|Z_{0}|^3], \mathbb{E}[|Z_{1}|^3]\} = \rho^3 \). Lemma 15 therefore implies

\[
\sup_x \left| \mathbb{P}\left\{ \frac{\sqrt{m}Z_{t+1}^t + Kq\lambda^{t/2} + m \mathbf{1}_{\{t=0\}}}{\sqrt{(m+Kp)\mathbb{E}[Y_i^2]}} - (m+Kp)\mathbb{E}[Y_i] \right| \leq x \} - \Phi(x) \right| \leq \frac{C \rho^3}{\sqrt{(m+Kp)\mathbb{E}[Y_i^2]}}
\]

Using the facts \( \mathbb{E}[Y_i^2] \geq 1, p > q \), \( \mathbb{E}[Y_i] = \frac{Kp}{m+Kp} \lambda^{t/2} + \frac{m}{m+Kp} \mathbf{1}_{\{t=0\}} \), and \( \frac{(m+Kp)\mathbb{E}[Y_i^2]}{m} \), we obtain

\[
\sup_x \left| \mathbb{P}\left\{ \frac{Z_{t+1}^t - \lambda^{(t+1)/2}}{\sqrt{\beta_{t+1}}} \leq x \} - \Phi(x) \right| \leq \frac{C \rho^3}{\sqrt{nq}}
\]

and the desired (81) follows. \( \square \)

### E.3. Proofs for linear message passing

**Proof of Theorem 4.** The proof consists of combining Corollary 1 and the coupling lemma. Let \( T = \frac{1}{2} \log \frac{2K}{\lambda} \). By the assumption that \( np^{\log(n/K)} = n^{o(1)} \) and \( \nu = n^{o(1)} \), it follows that Therefore, \((2 + np)^T = n^{o(1)}\); the coupling lemma can be applied. The performance bound of Corollary 1 is for a hard threshold rule for detecting the label of the root node. The same rule could be implemented at each vertex of the graph \( G \) which has a locally tree like neighborhood of radius \( T \) by using the estimator \( \widehat{C}_o = \{i : \theta_i^T \geq \lambda^{T/2}/2\} \). We first bound the performance for \( \widehat{C}_o \) and then do the same for \( \widehat{C} \) produced by Algorithm 3.
The average probability of misclassification of any given vertex \( u \) in \( G \) by \( \hat{C}_o \) (for prior distribution \((K_n, n-K_n)\)) is less than or equal to the sum of two terms. The first term is less than or equal to \( n^{-1/2+o(1)} \) (due to coupling error) by Lemma 10. The second term is \( o(\frac{K_n}{n-K_n}) \) (due to error of classification of the root vertex of the Poisson tree graph of depth \( T \)) by Corollary 1. Multiplying the average error probability by \( n \) bounds the expected total number of misclassification errors, \( \mathbb{E} |C^* \triangle \hat{C}_o| \). By the assumption that \( K = n^{1+o(1)} \), so \( n^{-1/2+o(1)} \frac{K_n}{n-K_n} = n^{-1/2+o(1)} = o(1) \), and of course \( o(\frac{K_n}{n-K_n}) \frac{K_n}{n-K_n} = o(1) \). It follows that \( \mathbb{E} |C^* \triangle \hat{C}_o| \rightarrow 0 \). The set \( \hat{C}_o \) is defined by a threshold condition whereas \( \hat{C} \) similarly corresponds to using a data dependent threshold and tie breaking rule to arrive at \( |\hat{C}| \equiv K \). By the same method used in the proof of Theorem 1, the conclusion for \( \hat{C} \) follows from what was proved for \( \hat{C}_o \). \[ \square \]

The proof of the converse result for linear message passing are quite similar to the proofs of converse results for belief propagation, and thus are omitted. The main differences are that the means here are 0 and \( \lambda^t/2 \) instead of \( \pm b_t/2 \), and the variances here are unequal: \( \alpha_t \) and \( \beta_t \). However, since \( \alpha_t \leq \beta_t \leq \frac{2\mu_t}{\eta} \) and we assume \( p/q = O(1) \), the same arguments go through. Finally, the messages in the linear message passing algorithm do not correspond to log likelihood messages, and the number of iterations needs to satisfy the extra constraint: \( t = O(\log \frac{n-K}{K}) \).