Quantum Mechanics and Algorithmic Randomness

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1. Notions of randomness

As a mathematical concept, randomness (or, more precisely, the notion of a random string of bits) has a convoluted history, with various attempts at formalization going through multiple series of refinements prompted by multiple setbacks (see [1], Chap. 1, for a guide to these developments). Current thinking on randomness encompasses two broadly-useful notions: pseudo-randomness and algorithmic randomness. Random bit sequences of the first kind, pseudo-random sequences, are derived from a small seed (initial data) through a deterministic algorithm, and pass as many practical statistical tests of randomness as desired. Such sequences locally “behave as if they are truly random,” and are frequently useful in statistical (e.g. Monte Carlo) simulations. A stronger variant of pseudo-randomness, the notion of a cryptographically-secure pseudo-random sequence, is motivated by applications involving secure communication channels. To be useful in cryptography, a pseudo-random sequence must have a generating algorithm which is secure from eavesdroppers, in the sense that an adversary must be unable to deduce the seed (in polynomial time) by observing a small part of the sequence the seed generates ([2]). However, the strongest precise notion of randomness so far developed is that of algorithmic incompressibility. The central concept that underlies this notion is the Kolmogorov (or algorithmic) complexity of a binary string, defined as the length (in bits) of the minimal program that would output the given string when run on a fixed universal Turing machine. A (long) bit string of length \( N \) is algorithmically random, or “patternless,” if its Kolmogorov complexity is greater than \( N \); if its complexity is significantly less than \( N \), then the string contains patterns which may be exploited to build a description (algorithm) for it shorter than its length, i.e., the string is algorithmically compressible.

An algorithmically random string is not only statistically random (and, in fact, cryptographically secure), but it also captures the intuitive idea of a truly random sequence which underlies the foundations of probability theory (allusions to strict randomness, which are implicit whenever the probability theorist talks about selecting an object “at random” out of a collection of objects, can only be formalized through the notion of an infinite sequence of truly random bits). It is not difficult to show (see [1], Chapter 3) that an algorithmically random bit string is guaranteed to pass all computable statistical tests of randomness. In particular, as the length \( N \) of the string approaches infinity, any computable pattern of \( k \) (not necessarily contiguous) bits has an asymptotic frequency of occurrence \( 2^{-k} \); in other words, there are no correlations between any set of \( k \) bits whose relative positions are computably determined. Moreover, algorithmic randomness is a ubiquitous property of long strings: among all possible infinite binary strings those which are algorithmically random form a subset of full measure; in other words, an infinite string chosen “without bias” is almost certain to be algorithmically random.

Despite these formal strengths (or, more precisely, because of them), the notion of algorithmic randomness has one major weakness: no algorithmically incompressible binary string can ever be constructed via a finitely-prescribed procedure (since, otherwise, such a procedure would present an obvious algorithm to compress the string thus obtained). Furthermore, given a binary string which is, in fact, algorithmically random, it is impossible to prove this fact starting from any set of axioms whose information content is smaller than the length of the string itself ([1], Section 2.7). Hence, for long enough strings, algorithmic randomness is unprovable in any familiar mathematical system based on set theory and logic (a fact closely related to Gödel’s incompleteness theorems; see [3] for a discussion of this connection). This “fundamentally non-constructible” nature of algorithmic randomness raises the following question: Is there any physical process which naturally gives rise to an incompressible string of binary outcomes where both the initial conditions and the dynamics are finitely prescribed and precisely controlled?

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Indeed, it turns out that quantum mechanics naturally provides examples of such processes. It is a natural question to ask whether quantum mechanical probabilities are “genuine”; by the discussion above, showing that quantum probabilities are genuine is equivalent to demonstrating that quantum measurement outcomes are truly (algorithmically) random. Such a demonstration is the basic claim of this paper.

2. Randomness in classical and quantum physics

A long string of pseudo-random bits produced by computer passes all practical statistical tests of randomness (provided the algorithm used is sound, see [4]), but it is not truly (algorithmically) random: the information content of the string (its Kolmogorov complexity) is bounded by the size of the generating algorithm plus a few extra bits which specify the random number seed, typically a much smaller quantity overall than the string’s (arbitrarily large) length. Similarly, provided it is continued to long enough lengths, a random string of bits (such as a sequence of coin tosses) produced by any classical physical system—of which a digital computer, or a Turing machine, is a special example—is not truly random: its complexity is bounded by the size of its “algorithm,” i.e. the deterministic physical laws which govern its evolution, plus the size of a description of the initial conditions on which those laws act. If the sequence of bits is continued much longer than the size of this combined description, the length of the resulting bit string is much larger than its algorithmic complexity, which implies that the string is non-random (compressible) even though, again, all statistical randomness tests may be satisfied (as they would be if, e.g., the underlying dynamical system is chaotic; see [5]). The algorithmic information content of a classically-produced bit string is contained entirely in the description of initial conditions (with a small additional contribution from the dynamical laws of evolution).

Are there any physical systems that can generate arbitrarily long, truly random (incompressible) bit strings starting from an initial state with a simple description? Note that the “simplicity” of the initial conditions is crucial: Clearly, in classical physics one can start from initial conditions of high algorithmic complexity and generate data strings of high complexity (e.g. as described in [5]), but then one gets only as much complexity as one is willing to pay for up front, and no more. Even though “almost all” initial conditions in classical physics are algorithmically random, what is required here is to start from a (simple) initial state which is precisely controlled, and generate measurements of unbounded complexity from it via its dynamical evolution. Indeed, in this paper I will present an argument that, if violations of relativistic causality are to be ruled out, a bit string obtained as a result of binary measurements performed on a string of identical copies of the same quantum state (where the measurements yield 0 vs. 1 with equal probability) must be almost surely (i.e. with probability that approaches 1 as the length of the string grows to infinity) incompressible. More generally, my argument shows that when the binary quantum measurements yield 1 with some (non-trivial) probability \( p \), the resulting bit string has (almost surely) the maximal algorithmic complexity consistent with that probability \( p \). Unlike a classical system, the information “contained” in a quantum state cannot be compactly encoded in a description of initial conditions. Note that such complete encoding of a quantum system’s algorithmic complexity in its initial conditions would be possible if quantum mechanics were equivalent to a classical theory with microscopic, local hidden variables; therefore the argument in this paper can be further interpreted as a modest strengthening of Bell’s no-hidden-variables theorem ([6]). In fact, the argument here relies on causality (i.e. locality) and quantum entanglement in a manner similar to Bell’s original argument.

Why is local causality necessary to infer such a basic fact as the true randomness of quantum probabilities? Consider standard quantum theory, consisting of the complex Hilbert space structure, linearity (superposition), unitary evolution, and the standard theory of measurement with the probability interpretation of inner-products. The incompressibility of binary-measurement outcomes cannot be deduced from these principles alone, since a computer-simulated quantum world, where probabilistic outcomes are achieved through the use of random-number generators, will respect these principles (and their observable statistical consequences) just as well: The algorithmic complexity of a string of quantum measurement outcomes is not determined simply by the statistics of those outcomes, which statistics are governed by the density matrix, or wave function, of the system, which is in turn governed by the physical laws encoded in the principles outlined above. For example, the binary expansion of the Champernowne number \( C = .010011011001 \ldots \) is statistically random ([7]), but it is not algorithmically random. If the outcomes from binary measurements on a quantum state were identical to a segment of the binary expansion of \( C \) (with some arbitrary starting point), this would not violate any of the statistical predictions of quantum mechanics. If this were the case repeatedly, or, more precisely, if quantum measurement outcomes were (with some finite probability) algorithmically compressible (but still statistically random) just like the binary expansion of \( C \), then the result claimed in this paper would be violated ([8]).

A more dramatic way to highlight this point is the following: Suppose we simulate quantum mechanics and quantum-measurement outcomes on a digital computer, using a very high-quality random-number generator. The statistical structure of our simulated quantum world would be identical (to arbitrarily high accuracy) with that of the real world, but all of our bit strings from successively-run measurements would be highly compressible. How do we know, then, that there is not
some universal random-number generator which generates the probabilistic outcomes of quantum measurements (with the right statistics) in the real world? It is easy to imagine random-number generators with large enough seeds (with seeds, say, of order 1-megabyte in size) such that it is essentially impossible to discover their existence by experiment (for example, to make sure that we discover—via discovering repetitive occurrences of the same string—that the bit strings from our laboratory quantum measurements are generated by a hidden random-number generator with 1-megabyte seed, it is necessary that we observe as many as $2^{2^{23}} + 1$ measurement sequences).

The result presented in this paper shows that if violations of local causality are to be ruled out there can be no such universal random-number generators in the real quantum world, and that, as a result, it is not possible to simulate quantum mechanics on a digital computer (i.e., a Turing machine); quantum randomness is “uncomputable” in this sense. This fundamental lack of computability of quantum phenomena may have certain far-reaching implications; for example, if quantum-mechanical processes play a significant role in the activities of biological neural systems, then brain activity cannot be simulated faithfully on a digital computer, no matter how elaborate the simulation.

Let us turn now to the presentation of the main argument:

3. Entangled spin pairs and spacelike communication

Consider a pair of spin-½ particles in the singlet (zero total spin) quantum state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow_i \rangle \otimes |\downarrow_2 \rangle - |\downarrow_i \rangle \otimes |\uparrow_2 \rangle) ,$$

where, with respect to spin measurements along an arbitrary direction axis, $|\uparrow_i \rangle$ and $|\downarrow_i \rangle$ denote the standard (orthonormal) eigenstates (spin-up and spin-down) for spin $i$ ($i = 1, 2$). Consider a long stream of such pairs produced by some (stationary) source, all pairs created in exactly the same entangled quantum state $|\psi\rangle$ given by Eq. (1), and each particle in the pair flying away from the source in opposite directions. I will assume that two observers, Bob and Alice, are positioned to perform observations at the opposite ends of this pair of particle beams, where their measurements are performed only after the particles have flown apart across a spatial distance so large that any pair of observations at the respective ends of the beam during the observers’ lifetime are spacelike-separated events in (flat) spacetime. It has always been a useful question to ask whether the observers can make use of the correlations inherent in the entangled state $|\psi\rangle$ to transmit information to each other, thereby violating relativistic causality. This paper is no exception.

As is well known, standard laws of quantum mechanics reveal that measurements performed on a single pair of spins in the quantum state Eq. (1) can never be used to transmit information between spacelike-separated observers ([(9)], i.e., local causality does not teach us anything new about quantum mechanics in this case. However, a long stream of identical copies of the same state provide significantly greater opportunities for communication, and it turns out that imposing the no-spacelike-communications requirement here leads to new knowledge on the structure of outcomes from a string of quantum measurements.

To examine this structure, assume that Bob and Alice have agreed beforehand (at some point in the distant past when they were in causal contact) on a common axis (e.g., one which points towards some distant quasar), and to measure each spin arriving at their end in the orthonormal bases $\{|\uparrow_1 \rangle , |\downarrow_1 \rangle \}$ along that axis (where $i = 1$ for Bob and $i = 2$ for Alice). Upon performing a measurement, Bob records his result as a 1-bit if the measured spin is in the up direction (along $|\uparrow_1 \rangle$) and as a 0-bit otherwise, and Alice records her result as a 1-bit if her measured spin is in the down direction (along $|\downarrow_2 \rangle$) and as a 0-bit otherwise. The nature of the singlet state Eq. (1) guarantees that, as long as both observers keep their measurements along the predetermined axis, the bit strings Bob and Alice obtain at each end of the singlet-pair stream are identical. The observers can now attempt to manipulate these two bit strings to build a spacelike-separated communications channel.

The simplest strategy for communication Bob and Alice might think of involves varying the frequency of 1’s and 0’s observed at one end by varying the measurement procedure at the other. For example, Alice and Bob know from standard quantum theory that as long as they both follow the above measurement procedure (which they previously agreed on), their measured bit strings would each contain asymptotically equal numbers of 1-bits and 0-bits, reflecting a probability of $\frac{1}{2}$ for either outcome. Now Bob can attempt to transmit a single bit of information to Alice in the following way: to send a 0-bit, Bob does nothing special, preserving the probabilistic structure of Alice’s string the way she expects it; to send a 1-bit, Bob changes his spin-measurement direction to point at some new direction ($\theta, \phi$) away from the original axis, and he keeps this modified axis during a large (predetermined) number of measurements, reverting back to the original axis only at the end of his bit-transmission period. Can Alice reliably detect the transmission of this single bit of information from Bob by examining the minute changes (if any) in the probability distribution of 0’s and 1’s in her bit string? It is easy to see that the answer is no: although by changing his spin-measurement direction Bob will cause Alice’s bit string to be different from his (in case Bob decides to send a 1-bit), there is no way for Alice to reliably discover this difference (other than a direct comparison with Bob’s string, copied via a conventional communication channel); no
matter which bit Bob decides to send, Alice’s string has exactly the same probability distribution of 1’s and 0’s (namely, probability precisely \( \frac{1}{2} \) for each outcome).

Although the proof of this result is relatively easy for the singlet state Eq. (1), it will be useful to briefly review it for the more general case (where it is considerably less obvious) in which the spins in the pair stream are in a generic entangled quantum state \( |\psi\rangle \) given, instead of Eq. (1), by

\[
|\psi\rangle = \alpha |\uparrow_1\rangle \otimes |\downarrow_2\rangle + \beta |\downarrow_1\rangle \otimes |\uparrow_2\rangle ,
\]

where \( \alpha \) and \( \beta \) are complex numbers satisfying the normalization condition

\[
|\alpha|^2 + |\beta|^2 = 1
\]

with the singlet state corresponding to the special case 

\( \alpha = -\beta = 1/\sqrt{2} \). From Bob’s point of view, \( |\psi\rangle \) corresponds to a pure state \( \rho = |\psi\rangle \langle \psi | \) which, when averaged (“traced”) over all possible spin states of particle 2, reduces to an effective density matrix

\[
\text{Tr}_2 \rho = \langle \uparrow_2 \rho | \uparrow_2 \rangle + \langle \downarrow_2 \rho | \downarrow_2 \rangle = |\alpha|^2 \langle \uparrow_1 \rangle \langle \uparrow_1 | + |\beta|^2 \langle \downarrow_1 \rangle \langle \downarrow_1 |
\]

\[
= \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix}
\]

(4)

living in the (spin) Hilbert space of particle 1. As long as Bob and Alice both follow the measurement procedure they agreed on (i.e., as long as they measure repeatedly along the direction axis which defines the spin bases \( \{ |\uparrow_1\rangle, |\downarrow_1\rangle \} \) and \( \{ |\uparrow_2\rangle, |\downarrow_2\rangle \} \)), it is clear from Eqs. (2) and (4) that Bob’s probability of observing a 1-bit (spin up) is \( |\alpha|^2 \), which is exactly equal to Alice’s probability of observing a 1-bit (spin down). In fact, just as in the special case of the singlet [Eq. (1)], so also here the bit strings Alice and Bob obtain under the standard measurement procedure are identical. Now suppose that Bob, in his attempt to transmit a 1-bit to Alice, modifies his spin-measurement axis to point in a new direction along which the spin-up eigenstate is given by

\[
|\varphi_1\rangle = c |\uparrow_1\rangle + d |\downarrow_1\rangle ,
\]

where \( c \) and \( d \) are complex numbers with \( |c|^2 + |d|^2 = 1 \).

The new \{spin-up, spin-down\} eigenbasis (for the Hilbert space of particle 1) along this modified direction is then given by the orthonormal state vectors

\[
|u\rangle \equiv |\varphi_1\rangle = c |\uparrow_1\rangle + d |\downarrow_1\rangle ,

|v\rangle \equiv |\varphi_1\rangle = -\overline{d} |\uparrow_1\rangle + \overline{c} |\downarrow_1\rangle .
\]

The probability for Bob to observe a 1-bit (spin-up) in this new eigenbasis is the expectation value [with respect to the effective mixed state Eq. (4)] of the projection operator \( |u\rangle \langle u| \) on the eigenstate \( |u\rangle = |\varphi_1\rangle \):

\[
\text{Prob}(1_1) = \text{Tr}_1 \left[ \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix} \right] \left( c |\uparrow_1\rangle + d |\downarrow_1\rangle \right) \left( \overline{c} |\uparrow_1\rangle + \overline{d} |\downarrow_1\rangle \right)
\]

\[
= \text{Tr} \left[ \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix} \right] \left( |\alpha|^2 \overline{d}^2 + |\beta|^2 \overline{c}^2 \right)
\]

\[
= |\alpha|^2 |\alpha|^2 + |\beta|^2 |\beta|^2 ,
\]

(6)

where the subscripts “1” denote that the corresponding objects are associated with particle 1. Bob’s new probability for observing a “1-bit” is manifestly different from the original probability \( |\alpha|^2 \), and his modified bit-string will reflect this difference in the new asymptotic distribution of 1’s and 0’s. But even so, Alice still has no way of detecting the change which Bob’s decision to modify his axis has induced on her bit string: Despite the drastic change in the statistics at Bob’s side, Alice will continue to observe a bit string where 1’s still have the original asymptotic frequency of \( |\alpha|^2 \).

Indeed, rewriting the entangled state Eq. (2) in the new basis (for the Hilbert space of particle 1) Eq. (5)

\[
|\psi\rangle = \alpha \left( (\overline{c} |u\rangle - d |v\rangle) \otimes |\downarrow_2\rangle + \beta (\overline{d} |u\rangle + c |v\rangle) \otimes |\uparrow_2\rangle \right)
\]

\[
= |u\rangle \otimes (\beta \overline{d} |\uparrow_2\rangle + c \overline{c} |\downarrow_2\rangle)

+ |v\rangle \otimes (\beta c |\uparrow_2\rangle - \beta d |\downarrow_2\rangle)
\]

(7)

and making use of Eq. (6), it is straightforward to compute Alice’s new probability of observing a “1-bit” in her bit string (recall: for Alice “1-bit” \( \equiv \) spin-down):

\[
\text{Prob}(1_2) = \text{Prob}(1_1) \frac{|\alpha \overline{c}|^2}{|\beta c|^2 + |\alpha \overline{c}|^2}

+ \text{Prob}(0_1) \frac{|\beta d|^2}{|\beta c|^2 + |\alpha \overline{d}|^2}
\]

\[
= |\alpha|^2 + (1 - |\alpha|^2 - |\beta|^2) \frac{|\alpha d|^2}{|\beta c|^2 + |\alpha d|^2}
\]

\[
= |\alpha|^2 ,
\]

(8)

where the final equality follows at once from the normalization condition \( c^2 + d^2 = 1 \) after one notices the identity

\[
1 - |\alpha|^2 - |\beta|^2 = |\beta c|^2 + |\alpha d|^2
\]

which follows from Eq. (3).

Realizing that their attempts at communication via manipulating the statistics of each other’s measurements are doomed to fail, Bob and Alice may turn, in desperation, to the only remaining structural signature their bit strings have: algorithmic complexity ([10, 1]). The algorithmic (or Kolmogorov) complexity of a bit string \( S \) is the length (in bits) of the shortest program that would output \( S \) when run on a fixed Universal Turing Machine (UTM). When a string \( S_n \) of length \( n \) is algorithmically random (patternless), its Kolmogorov complexity is comparable to \( n \): \( K(S_n) \sim n \); if its complexity is significantly less than \( n \), then \( S_n \) contains patterns which may be exploited to build an algorithmic description for it shorter
than its actual length, i.e., such a string is compressible. In general, the quantity \( K(S_n) \) is meaningful only in the limit of very long strings \( (n \to \infty) \), since only in this limit independence from a specific choice of UTM is assured ([11]). While an incompressible string necessarily has the same asymptotic fraction of 0-bits as 1-bits (i.e. \( \frac{1}{2} \)), more generally, a bit string \( S \) in which the asymptotic frequency of 1’s is \( p \) has an algorithmic complexity of at most

\[
K(S_n) \sim nH(p) ,
H(p) \equiv -p \log_2 p - (1-p) \log_2 (1-p) ,
\quad (9)
\]

where \( S_n \) denotes the first \( n \) bits of \( S \), and \( H(p) \) is the Shannon entropy of the probability \( p \). A bit string which satisfies Eq. (9) has the maximal algorithmic complexity (randomness) subject to the statistical constraint imposed by the asymptotic 1-bit-frequency \( p \). I will refer to this property (or its negation) as \( p \)-incompressibility (or \( p \)-compressibility) in what follows (the usual notions obtain when \( p = \frac{1}{2} \)).

To avoid any confusion, it is perhaps appropriate here to re-emphasize that the notions of algorithmic compressibility used in this paper refer strictly to finite (albeit very long) binary strings. Nevertheless, the notions of incompressibility for infinite binary strings are essentially the same notions (save for unavoidable formal differences). For practical reasons, I will always use the finite-substring approach to incompressibility of infinite binary strings as hinted in the previous paragraph [Eq. (9)]. Accordingly, an infinite string \( S \) is (by definition) incompressible if and only if

\[
\lim_{n \to \infty} \frac{K(S_n)}{n} = 1
\]

(where \( S_n \) denotes the finite substring consisting of the first \( n \) bits of \( S \)), and it is \( p \)-incompressible if and only if

\[
\lim_{n \to \infty} \frac{K(S_n)}{n} = H(p) .
\]

For a discussion of other (equivalent) definitions of incompressibility for infinite binary strings the reader should consult Ref. [1].

Let \( p_N \) denote the probability that an \( N \)-bit-long segment of Bob’s (or Alice’s) string of quantum measurements is \( p \)-compressible [where \( p = |\alpha|^2 \) in the context of the discussion between Eqs. (2) and (8)]. I will now sketch a proof that if the statement of the main result of this paper [see the second paragraph in Sect. 2 above] is false, in other words, if the probability \( p_N \) is bounded away from zero as \( N \to \infty \), then a reliable (spacelike) communications channel can be constructed through which Bob—using each \( N \)-bit-long block as a carrier of one data bit—can send information to Alice. Here is how the construction of this communications channel might proceed:

First, Alice and Bob agree at the outset that Alice should interpret any compressible \( N \)-bit block in her string as a 0-bit, and any incompressible block as a 1-bit. Next, they agree on an approximate value for the Universal Halting Probability \( \Omega \) [computed with respect to a common choice of UTM; see Eq. (10) in Sect. 4 below], so that Alice can determine, with a probability of error \( p_\Omega \) less than 1, whether or not a given segment of her bit string is compressible (this is necessary because both the Kolmogorov complexity \( K(\cdot) \) and the map which takes \( n \) to the \( n \)’th bit of \( \Omega \) are nonrecursive functions, therefore Alice cannot make her determinations with absolute certainty; see Sect. 4 below for more details). Now, to send a 0-bit to Alice, Bob does nothing (i.e., keeps his spin-measurement axis unchanged, pointing along the original predetermined direction). To send a 1-bit, Bob “scrambles” his measurement sequence in the following way: first he generates a random “template,” a random bit string \( T \) of length greater than \( N \) which is (almost surely) incompressible [Bob can obtain such a string, among other ways, by using the evolution from random initial conditions of a classical system (such as a roulette wheel) with greater than \( N \) degrees of freedom]. Then, Bob prepares a sequence of \( N \) random measurement directions using \( T \) as a random-number generator, and performs his next \( N \) measurements along the successive orthonormal bases associated with the successive directions from this random sequence. This procedure of scrambling with the random template \( T \) guarantees that Bob’s modified \( N \)-bit long string of quantum measurements is almost surely \( p \)-incompressible [with \( p = \frac{1}{3}(|\alpha|^2 + 2|\beta|^2) \), in the notation of Eqs. (2) — (8)], and that Alice’s corresponding string (which is now different from Bob’s) is also (almost surely) \( p \)-incompressible [with \( p = |\alpha|^2 \), as in Eq. (8)]. The crucial requirement for Bob’s choice of \( N \) measurements is that it should be guaranteed to be free of any regularities that might be present in the \( N \)-bit block (representing the 1-bit) he wishes to send Alice. His scrambling procedure is designed to destroy any nascent correlations that might exist in the original bit stream of measurements, without introducing any extra correlations of quantum-mechanical origin (hence his use of the classical roulette-wheel). There are other “scrambling” procedures Bob might use to ensure this; the procedure just described is only one natural choice.

The transmission protocol described above establishes a noisy (asymmetric) binary communication channel from Bob to Alice, connecting them across a spacelike spacetime separation: Bob can now transmit a 1-bit to Alice (almost-)reliably, and he can transmit a 0-bit unreliably. For communication to be possible through this channel, the channel capacity must remain nonzero in the limit \( N \to \infty \); Shannon’s noisy-channel coding theorem ([12]) would then ensure that coding schemes can be found which will allow transmission of messages with arbitrarily small probability of error. Indeed, it is not difficult to show that the capacity of the channel just
constructed remains nonzero in the limit \( N \to \infty \) if it is assumed, contrary to the assertion of this paper, that the probability \( p_N \) remains bounded away from zero in this limit.

4. Detailed analysis of the entangled-spin communication channel

I will denote the vanishing of a quantity \( q \) in the limit \( N \to \infty \) by the expression \( q = O(\epsilon) \). For example, the argument of this paper will show that to preserve relativistic causality it is necessary to have \( p_N = O(\epsilon) \), where \( p_N \) denotes the probability that an \( N \)-bit-long segment of Bob’s (or Alice’s) string of quantum measurements is \( p \)-compressible [where \( p = |\alpha|^2 \) in the context of the discussion between Eqs. (2) and (8) in Sect. 3 above]. In the present discussion I will not try to quantify the rate at which \( p_N \) must decay to zero; a more detailed and rigorous analysis, to be presented separately in [13], is needed to provide sharper estimates for the asymptotic decay rate of \( p_N \). However, for all probabilities of order \( O(\epsilon) \) to be discussed here, including for \( p_N \), the true decay rates can be shown to be exponential (see [13] for details).

The “Universal Halting Probability” alluded to in Sect. 3 above is defined by

\[
\Omega = \sum_{\pi: \pi \text{ halts}} 2^{-l(\pi)} ,
\]

where the sum is over all prefix-free (i.e. no \( \pi \) is the prefix of another \( \pi' \)) programs \( \pi \) which halt when run on a fixed UTM, and \( l(\pi) \) denotes the length of \( \pi \) in bits (convergence of the sum in Eq. (10) to a real number less than 1 is ensured by Kraft’s inequality; see Refs. [1] and [14]). Thus defined, \( \Omega \) is the probability that a randomly chosen program \( \pi \) will halt when run on the given UTM (for further details see Refs. [10, 1]). Complete knowledge of \( \Omega \) would allow one to solve the halting problem \((\lceil 15 \rceil)\), consequently \( \Omega \) cannot be recursively evaluated (this is related to the fact that the Kolmogorov complexity \( K \) is a nonrecursive function); moreover, \( \Omega \) is an incompressible real number (i.e. its binary expansion is an incompressible bit string; for a lucid discussion of these and other magical properties of \( \Omega \) consult [16]).

Returning now to the construction described in the previous section (Sect. 3), if Alice wanted to decide with certainty whether or not a given \( N \)-bit block in her string is \( p \)-compressible, she would need to know \( \Omega \) (or at least the first \( N \) bits in the binary expansion of \( \Omega \)) with absolute certainty \((\lceil 17 \rceil)\). However, since Alice and Bob’s construction merely attempts to create a noisy communication channel, all Alice really needs to know is an approximation to \( \Omega \) such that she can make her decisions with a fixed probability of error \( p_\Omega < 1 \) (see [13] for details). The error probabilities for Alice are then

\[
\text{Prob}(\text{Alice falsely decides a } p\text{-compressible string}
\text{ to be } p\text{-incompressible}) = p_\Omega ,
\]

while

\[
\text{Prob}(\text{Alice falsely decides a } p\text{-incompressible string}
\text{ to be } p\text{-compressible}) = 0 .
\]

Now, a binary communication channel with asymmetric bit-error probabilities given by

\[
p_0 \equiv \text{Prob}(\text{a 0-bit is flipped in transmission}) = \text{Prob}(1_{\text{out}} | 0_{\text{in}}) ,
\]

\[
p_1 \equiv \text{Prob}(\text{a 1-bit is flipped in transmission}) = \text{Prob}(0_{\text{out}} | 1_{\text{in}})
\]

\[
\text{can be shown to have a channel capacity (see [13] for a detailed derivation)}
\]

\[
The \text{ capacity } C(p_0, p_1) = \log_2 \left[ 1 + 2^{r(p_0, p_1)} \right] - p_0 r(p_0, p_1) - H(p_0) ,
\]

where

\[
r(p_0, p_1) = \frac{H(p_0) - H(p_1)}{p_0 + p_1 - 1} .
\]

For the binary channel which Bob and Alice would obtain with their communication protocol, it is easy \((\lceil 13 \rceil)\) to show that

\[
p_0 = \text{Prob}(1_{\text{out}} | 0_{\text{in}}) = p_N p_\Omega + (1 - p_N) = 1 - (1 - p_\Omega) p_N ,
\]

while

\[
p_1 = \text{Prob}(0_{\text{out}} | 1_{\text{in}}) = 0 [1 - O(\epsilon)] + (1 - p_\Omega) O(\epsilon) = O(\epsilon) .
\]

Inspection of Eqs. (14)—(15) reveals that, in general, the binary asymmetric channel capacity \( C(p_0, p_1) \) vanishes along the diagonal \( \{ p_0 + p_1 = 1 \} \), and is second order in the distance \( 1 - p_0 - p_1 \) away from this diagonal in its vicinity \((\lceil 13 \rceil)\). Consequently, substitution of Eqs. (16)—(17) in Eqs. (14)—(15) makes it straightforward to verify the main claim of this paper, namely: as long as \( p_N > O(\epsilon) \), the (spacelike) channel capacity from Bob to Alice remains bounded away from zero in the limit \( N \to \infty \) \( [\text{in other words}, C_N > O(\epsilon) \text{ as long as } p_N > O(\epsilon)] \).

5. Conclusions

The argument presented above proves that, if relativistic causality is to be preserved, a bit string generated by binary measurements performed on a string of identical copies of the quantum state Eq. (2) must be almost surely \( (i.e. \text{ with probability that approaches 1 as the length of the string grows to infinity}) \) maximally algorithmically random. This result follows directly from the most basic
laws of standard quantum mechanics and quantum measurement theory, and those laws do not grant any privileged status to the specific entangled form of the state Eq. (2). Given an arbitrary quantum state \( |\Psi\rangle \), any binary measurement determines a choice between two projections: a projection either onto a special state, \( |1\rangle \), say, or to the orthogonal complement \( |0\rangle ^\perp \) of this state in the Hilbert space of the system ([18]). But the true binary decision is between the state \( |1\rangle \) and the projection of \( |\Psi\rangle \) on \( |1\rangle ^\perp \); denote this projection by \( |0\rangle \), and we are back in the two-(complex)-dimensional Hilbert-space geometry (e.g. the geometry of the subspace spanned by \( |1\rangle = |\uparrow_1\rangle \otimes |\downarrow_2\rangle \) and \( |0\rangle = |\downarrow_1\rangle \otimes |\uparrow_2\rangle \)) of Eq. (2) and the discussion which follows it. Combined with the general unitary invariance of quantum mechanics, this argument shows that my result on the algorithmic randomness of binary-measurement outcomes applies just as well to the arbitrary state \( |\Psi\rangle \) as to the specific entangled state Eq. (2).

Furthermore, since one cannot build a physical system which can make copies of an arbitrary ensemble of quantum states (a quantum “copier” can make duplicates of no more than as many different states as would fit within an orthogonal set, as explained, e.g., in Ref. [19]), “a string of identical copies” of a given quantum state is a meaningful construction only in the context of a physical process which creates such copies in unlimited succession, such as the process I discussed in Sect. 3 above immediately following Eq. (1). Nevertheless, the argument from unitary equivalence described in the previous paragraph can be used once again to further enlarge the domain of application of the present result, namely: any string of binary quantum measurements which can be mapped unitarily onto another must give rise to a bit string of outcomes with the same statistical and algorithmic structure as the string it is unitarily mapped onto. Consequently, given any sequence of measurements unitarily equivalent to successive binary measurements on a fixed state \( |\Psi\rangle \) the bit string of successive outcomes is subject to the incompressibility result of this paper.

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NOTES AND REFERENCES

1. M. Li and P. Vitanyi, An Introduction to Kolmogorov Complexity and its Applications (Springer-Verlag, New York 1993).
2. A. Shamir, On the generation of cryptographically strong pseudo-random sequences in Proc. ICALP (Springer, 1981). Also see M. Bellare and S. Goldwasser, Lecture Notes on Cryptography, available on the world-wide-web at http://www-cse.ucsd.edu/users/mihir/papers/gb.html.
3. G. J. Chaitin, Int. Jour. Th. Phys. 22, 941 (1982).
4. D. E. Knuth, The Art of Computer Programming. Vol. 2: Seminumerical Algorithms (Addison Wesley Longman, Reading, Massachusetts 1998).
5. A trivial exception to this would be a classical chaotic system with initial conditions given (to arbitrarily high precision) by real numbers which are themselves incompressible (algorithmically random). Because of the exponential loss of accuracy in its chaotic evolution, to describe an \( N \)-bit string of “coin tosses” in this case one would need to know the initial conditions to an accuracy of approximately a part in \( 2^N \) (an accuracy of \( N \) binary digits), which implies an algorithmic complexity of order \( N \) for the resulting bit string. Obviously, such a string can be algorithmically random (incompressible). Note, however, that the initial conditions do not have a finite (or short) description in this case.
6. J. S. Bell, Physics 1, 195 (1964). Also reprinted in Speakable and Unspeakable in Quantum Mechanics (Cambridge University Press, Cambridge 1987).
7. The Champernowne number \( C \) is defined to be the real number in \([0,1]\) whose binary expansion is given by the Champernowne sequence 0100011011000001010011100101110111... a provably statistically random bit string which is not algorithmically random, where the lexicographically ordered list of all strings of length \( k \) are followed by the list of all those of length \( k + 1 \) as \( k \) ranges from 1 to \( \infty \).
8. Of course, one can always add the algorithmic incompressibility of successive measurement outcomes as a basic principle (an “axiom”) to standard quantum mechanics without contradicting observation in any way; this would presumably allow one to deduce causality as a theorem. From this viewpoint, the result claimed here suggests that algorithmic incompressibility and relativistic causality are interchangeable as principles of quantum mechanics. Nevertheless, local causality is arguably more natural than algorithmic incompressibility as a candidate axiom for quantum (or, for that matter, any other physical) theory.
9. Measurements performed on one member of an entangled pair cannot alter the expectation values of operators acting on the other; see, e.g., D. Bohm, Quantum Theory (Prentice Hall, Englewood Cliffs 1951).
10. G. J. Chaitin, IBM Journal of Research and Development, 21, 350 (1977); Advances in Applied Mathematics 8, 119 (1987).

11. I will be somewhat cavalier in my quantitative treatment of Kolmogorov complexity. For example, a proper treatment of the complexity measure $K$ would define it in terms of “prefix-free” (or, equivalently, self-delimiting) UTM programs, and accordingly replace Eq. (9) with the more accurate $K(S_n) \sim nH(p) + 2 \log_2 n$ for a maximally-random string $S$. Also, the symbol “$\sim$” has a rather precise meaning in the algorithmic-information-theory literature which I will gloss over. These technical details are not essential to the flow of my argument, and they can be filled in from the references [10] and [1], especially the book by Li and Vitanyi; I will give a more detailed and rigorous account of my analysis elsewhere ([13]).

12. C. E. Shannon, Bell Sys. Tech. Journal 27, 379; 623 (1948). See also C. E. Shannon and W. W. Weaver, The Mathematical Theory of Communication (University of Illinois Press, Urbana 1949).

13. U. Yurtsever, manuscript in preparation.

14. T. M. Cover and J. A. Thomas, Elements of Information Theory (Wiley-Interscience, New York 1991).

15. Given the exact value of $\Omega$ and a program $\pi_0$ whose halting is to be decided, begin running the given UTM with all possible programs $\{\pi\}$ arranged in a “dove-tailed” input configuration, and simply wait until either $\pi_0$ halts, or the sum Eq. (10) over all programs which already halted accumulates to a value greater than $\Omega - 2^{-\ell(\pi_0)}$ (which, when it happens, will guarantee that $\pi_0$ does not halt).

16. M. Gardner, Scientific American 241, 20 (1979).

17. To make her decision about a bit string $S$, Alice would simply run all possible “short” programs in the same manner as described above (in [15]), wait until she is sure every program which will ever halt has already done so [by monitoring the accumulating sum Eq. (10) until it comes close enough to her value of $\Omega$], and finally see if the string $S$ is contained among the outputs of the halted programs (if it is, then $S$ is compressible; otherwise, $S$ is incompressible).

18. Although this argument assumes $|\Psi\rangle$ is a pure state, it is not difficult to generalize its conclusion to more general mixed states via reduction to the pure-state case; see Ref. [13].

19. C. M. Caves and C. A. Fuchs, Quantum Information: How Much Information in a State Vector? quant-ph/9601025 (1996).