Harmonic morphisms, conformal foliations and shear-free ray congruences

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Abstract

Equivalences between conformal foliations on Euclidean 3-space, Hermitian structures on Euclidean 4-space, shear-free ray congruences on Minkowski 4-space, and holomorphic foliations on complex 4-space are explained geometrically and twistorially; these are used to show that 1) any real-analytic complex-valued harmonic morphism without critical points defined on an open subset of Minkowski space is conformally equivalent to the direction vector field of a shear-free ray congruence, 2) the boundary values at infinity of a complex-valued harmonic morphism on hyperbolic 4-space define a real-analytic conformal foliation by curves of an open subset of Euclidean 3-space and all such foliations arise this way. This gives an explicit method of finding such foliations; some examples are given.

1 Introduction

This paper consists of two parts: Firstly, we describe natural correspondences between the following four quantities:

(Q1) holomorphic foliations by $\alpha$-planes of an open subset of complex 4-space $\mathbb{C}^4$,
(Q2) positive Hermitian structures $J$ on an open subset of Euclidean 4-space $\mathbb{R}^4$,
(Q3) shear-free ray congruences on an open subset of Minkowski 4-space $\mathbb{M}^4$,
(Q4) conformal foliations by curves of an open subset of Euclidean 3-space $\mathbb{R}^3$.

The correspondences are described both geometrically and twistorially.

Then these correspondences are used to find a relationship between (complex-valued) Minkowski harmonic morphisms, i.e. ones defined on open subsets of Minkowski space, and shear-free ray congruences; in fact, the direction vector field of a shear-free ray congruence defines such a harmonic morphism; conversely, to any Minkowski harmonic morphism $\phi$ without critical points, we can associate a shear-free ray congruence such that every fibre of $\phi$ is the union of parallel rays of the congruence (Theorem 4.5). It follows that, up to conformal transformations of the codomain, any “non-Kähler” Minkowski harmonic morphism is the direction vector field of a shear-free ray congruence (Corollary 4.8). Similar results are given for complex harmonic morphisms (Theorem 4.4 and Corollary 4.5).

Secondly we show (Theorem 5.6) that the level sets of the boundary values at infinity of a complex-valued submersive harmonic morphism from hyperbolic 4-space define a conformal foliation of an open subset of $\mathbb{R}^3$ by curves and that every such (real-analytic) foliation arises in this way. Combining with $[30]$ and $[2]$, this gives us a practical method of finding all conformal foliations by curves of open subsets of $\mathbb{R}^3$; this is illustrated in the final chapter where some explicit examples are calculated.

We now describe these ideas in more detail:

A harmonic morphism is a smooth map $\phi$ between Riemannian manifolds which preserves Laplace’s equation in the sense that if $f$ is a local harmonic function on the codomain manifold then $f \circ \phi$ is a local harmonic function on the domain, see $[2]$ and $[30]$ for some relevant background.

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A conformal foliation by curves (§3.3) of an open subset of Euclidean 3-space $\mathbb{R}^3$ is a foliation by curves such that the Lie transport along the fibres is conformal on the normal bundle. Conformal foliations include Riemannian foliations. A foliation is conformal if and only if its leaves are locally the level sets of a complex-valued submersion $f$ which satisfies the horizontal weak conformality condition:

$$\left( \frac{\partial f}{\partial x_1} \right)^2 + \left( \frac{\partial f}{\partial x_2} \right)^2 + \left( \frac{\partial f}{\partial x_3} \right)^2 = 0. \quad (1)$$

A positive Hermitian structure $J$ (§3.2) on an open subset of Euclidean 4-space $\mathbb{R}^4$ is an almost Hermitian structure which is associated to a positive basis and is integrable.

A shear-free ray congruence (§3.4) on an open subset of Minkowski 4-space $\mathbb{M}^4$ is a foliation by null lines such that Lie transport is conformal on the screen spaces.

A holomorphic foliation by $\alpha$-planes (§3.1) of an open subset of complex 4-space $\mathbb{C}^4$ is a foliation, holomorphic with respect to the standard complex structure on $\mathbb{C}^4$, by planes which are null and are associated to a positively oriented basis.

We show that these quantities are equivalent by establishing correspondences between them. For example:

Given a Hermitian structure $J$ on an open subset of $\mathbb{R}^4 = \{(x_0, x_1, x_2, x_3)\}$, putting $U = J(\partial/\partial x_0)$ on a slice $x_0 = \text{constant}$ defines the tangent vector field of a real-analytic conformal foliation by curves of an open subset of $\mathbb{R}^3$ and any such foliation arises this way giving a bijective correspondence between germs (Corollary 3.14).

Analogously, given a $(C^\infty)$ shear-free ray congruence on an open subset of $\mathbb{M}^4 = \{(t, x_1, x_2, x_3)\}$, projecting its tangent vector onto a slice $t = \text{constant}$ defines the tangent vector field of a $(C^\infty)$ conformal foliation by curves of an open subset of $\mathbb{R}^3$; any such foliation arises this way giving a bijective correspondence between germs (Theorem 3.7). In fact, as $t$ varies, we get a 1-parameter family of “associated” conformal foliations.

Both shear-free ray congruences and Hermitian structures complexify to holomorphic foliations by $\alpha$-planes (see Proposition 3.11) so that all four quantities are in bijective correspondence (Theorem 3.13) and have a Kerr-type representation as a complex hypersurface of the twistor space $\mathbb{C}P^3$ (§3.7).

The quantities (Q1 - Q4) arise as distributions of the quantities at a point in §3 below which satisfy certain conditions; we first show that these quantities are equivalent at a point (Proposition 2.3) and can all be represented by a unit vector in $\mathbb{R}^3$, the direction vector of the quantity, or twistorially (§2.7), this last description extending to compactified spaces. Then in §3 we consider how various conditions on distributions of the quantities in §3 correspond leading to the equivalences of (Q1 - Q4). We spend some time explaining this material as, although some aspects may be known to Mathematical Physicists, we hope that it is worthwhile to give a geometrical (index and spinor-free) description adapted to our applications. In §5 we give our main result (Theorem 5.6) and its application to finding conformal foliations of open subsets of $\mathbb{R}^3$ by curves. The idea is that, given a complex hypersurface $S$ of $\mathbb{C}P^3$, $S$ is the twistor surface of some holomorphic structure $J$ and we can find a hyperbolic harmonic morphism $\phi$ holomorphic with respect to $J$ by solving a first order holomorphic partial differential equation (67) on $S$. Then the level sets of the boundary values on the $\mathbb{R}^3$ at infinity define a conformal foliation. By introducing a parameter $a \in \mathbb{C}^4$ we can actually find a 5-parameter family of associated conformal foliations.

Some of our work can be generalized to more general manifolds but with a loss of explicitness; this will be done in a future article.

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2 Null planes and associated structures

In this section we consider four quantities defined at a point \( p \) of \( \mathbb{C}^4 \) or its conformal compactification and show that they are all in bijective correspondence, namely, with definitions to follow,

\[
\begin{align*}
(PQ1) \quad & \text{\( \alpha \)-planes } \Pi_p \text{ at } p, \\
(PQ2) \quad & \text{positive real almost Hermitian structures } J_p \text{ at } p, \\
(PQ3) \quad & \text{null directions } V_p \text{ at } p \text{ in the Minkowski slice through } p, \\
(PQ4) \quad & \text{unit vectors } U_p \text{ at } p \text{ in the } \mathbb{R}^3\text{-slice through } p.
\end{align*}
\]

(2)

2.1 Null planes

We consider \( \mathbb{C}^4 = \{ (x_0, x_1, x_3, x_4) : x_i \in \mathbb{C} \} \) with its standard complex structure, and conformal structure given by the holomorphic metric \( g = \sum_{i=0}^{3} dx_i^2 \). (For general definitions of holomorphic metrics and related concepts, see [17].) A tangent vector \( V \) at any point is a null plane.

For any \( p \in \mathbb{C}^4 \), let \( \mathbb{R}_p^4 \) be the real 4-dimensional affine subspace (real slice) through \( p = (p_0, p_1, p_2, p_3) \) given by \( \{ (x_0, x_1, x_2, x_3) \in \mathbb{C}^4 : \exists x_i = 3p_i, \ i = 0, 1, 2, 3 \} \) and parametrized by \( \mathbb{R}_p^4 \ni (x_0, x_1, x_2, x_3) \mapsto (p_0 + x_0, p_1 + x_1, p_2 + x_2, p_3 + x_3) \in \mathbb{C}^4 \); if \( p = 0 \equiv (0, 0, 0, 0) \) we write \( \mathbb{R}_p^4 = \mathbb{R}^4 \). We give each \( \mathbb{R}_p^4 \) the standard orientation. The holomorphic metric and conformal structure restrict to the standard metric and conformal structure on each \( \mathbb{R}_p^4 \). Note that \( T_p \mathbb{C}^4 \) can be identified with \( T_p \mathbb{R}_p^4 \otimes \mathbb{C} \); we shall frequently make this identification.

Given any orthonormal basis \( \{ e_0, e_1, e_2, e_3 \} \) of \( T_p \mathbb{R}_p^4 \), the plane \( \Pi_p = \text{span}\{e_0 + i e_1, e_2 + i e_3\} \) is null; we call \( \Pi_p \) an \( \alpha\)-plane (resp. \( \beta\)-plane) according as the basis \( \{ e_0, e_1, e_2, e_3 \} \) is positively (resp. negatively) oriented. This construction induces a bijection

\[
\text{SO}(4)/U(2) \leftrightarrow \{ \alpha\text{-planes at } p \}. \tag{3}
\]

To proceed, it is convenient to introduce new coordinates \( (z_1, \bar{z}_1, z_2, \bar{z}_2) \) on \( \mathbb{C}^4 \) by setting

\[
z_1 = x_0 + i x_1, \quad \bar{z}_1 = x_0 - i x_1, \quad z_2 = x_2 + i x_3, \quad \bar{z}_2 = x_2 - i x_3. \tag{4}
\]

Then \( \mathbb{R}^4 \) is given by \( \bar{z}_1 = z_1, \bar{z}_2 = z_2 \) and the holomorphic metric on \( \mathbb{C}^4 \) by \( g = dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2 \).

2.2 Real almost Hermitian structures and null-planes

By a real almost Hermitian structure \( J_p \) at \( p \in \mathbb{C}^4 \) we mean an endomorphism \( J_p : T_p \mathbb{C}^4 \to T_p \mathbb{C}^4 \) which maps real vectors, i.e. ones in \( T_p \mathbb{R}_p^4 \), to real vectors isometrically and has \( J_p^2 = -I \). Equivalently, a real almost Hermitian structure is an almost Hermitian structure on the real slice \( \mathbb{R}_p^4 \) at \( p \) extended to \( T_p \mathbb{C}^4 = T_p \mathbb{R}_p^4 \otimes \mathbb{C} \) by complex linearity.

Given any orthonormal basis \( \{ e_0, e_1, e_2, e_3 \} \) of \( T_p \mathbb{R}_p^4 \), setting \( J_p(e_0) = e_1, J_p(e_2) = e_3 \) defines a real almost Hermitian structure \( J_p \) at \( p \); we call \( J_p \) positive (resp. negative) according as \( \{ e_0, e_1, e_2, e_3 \} \) is a positively (resp. negatively) oriented basis; this construction gives a bijection

\[
\text{SO}(4)/U(2) \leftrightarrow \{ \text{positive real almost Hermitian structures } J_p \text{ at } p \}. \tag{5}
\]

Combining with \( (3) \) gives a bijection

\[
\{ \alpha\text{-planes } \Pi_p \text{ at } p \} \leftrightarrow \{ \text{positive real almost Hermitian structures } J_p \text{ at } p \}
\]

given by

\[
\Pi_p \mapsto J_p = \begin{cases} -i & \text{on } \Pi_p, \\
+ i & \text{on } \Pi_p \end{cases}
\]

with inverse

\[
J_p \mapsto \Pi_p = (0,1)\text{-tangent space of } J_p = \{ X + i J_p X : X \in T_p \mathbb{R}_p^4 \}.
\]
2.3 Vectors in $\mathbb{R}^3$

For any $p = (p_0, p_1, p_2, p_3) \in \mathbb{C}^4$, define the $\mathbb{R}^3$-slice through $p$ by

$$\mathbb{R}^3_p = \{(x_0, x_1, x_2, x_3) : x_0 = p_0, \Im x_i = \Im p_i \ (i = 1, 2, 3)\}$$

parametrized by $\mathbb{R}^3 \ni (x_1, x_2, x_3) \mapsto (p_0, p_1 + x_1, p_2 + x_2, p_3 + x_3) \in \mathbb{C}^4$. Given any unit vector $U_p \in T_p \mathbb{R}^3_p$, there is a positive orthonormal basis $\{e_0, e_1, e_2, e_3\}$ of $T_p \mathbb{R}^3_p$ with $e_0 = \partial/\partial x_0$, $e_1 = U_p$, so that $\{e_2, e_3\}$ spans $U_p^\perp \cap T_p \mathbb{R}^3_p$. This then defines an $\alpha$-plane $\Pi_p = \text{span}(e_0 + ie_1, e_2 + ie_3)$ and so a positive real almost Hermitian structure $J_p$, and gives rise to a bijection:

$$\{\text{positive real almost Hermitian structures } J_p \text{ at } p\} \leftrightarrow \{\text{unit vectors } U_p \text{ in } T_p \mathbb{R}^3_p\} \quad (6)$$

given by: $J_p \mapsto U_p = J_p(\partial/\partial x_0)$, $U_p \mapsto J_p = \text{the unique positive almost Hermitian structure at } p$ with $J_p(\partial/\partial x_0) = U_p$.

2.4 Null vectors in $\mathbb{M}^4$ and the fundamental bijections at a point

Minkowski space $\mathbb{M}^4$ is defined to be the set $\{(t, x_1, x_2, x_3) \in \mathbb{R}^4\}$ with the semi-Riemannian metric $g^\mathbb{M} = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2$. We include $\mathbb{M}^4$ in $\mathbb{C}^4$ by the map $(t, x_1, x_2, x_3) \mapsto (it, x_1, x_2, x_3)$. (Note that the minus sign is unimportant; it is included simply to avoid minus signs later on.) More generally, for any $p \in \mathbb{C}^4$, let $\mathbb{M}^4_p$ be the Minkowski slice $\mathbb{M}^4_p = \{(x_0, x_1, x_2, x_3) : \Re x_0 = \Re p_0, \Im x_i = \Im p_i \ (i = 1, 2, 3)\}$ parametrized by $\mathbb{M}^4 \ni (t, x_1, x_2, x_3) \mapsto (p_0 - it, p_1 + x_1, p_2 + x_2, p_3 + x_3) \in \mathbb{C}^4$. A vector $v = (v_0, v_1, v_2, v_3) \in T_p \mathbb{M}^4_p$ is called null if $g_M(v, v) = -v_0^2 + v_1^2 + v_2^2 + v_3^2 = 0$, equivalently its image $(-iv_0, v_1, v_2, v_3)$ in $T_p \mathbb{C}^4$ is null in the sense of (2.1). A 1-dimensional subspace $V_p$ of $T_p \mathbb{M}^4_p$ is called a null direction at $p$ if it is spanned by a null vector $v_p$. An (affine) null line or (light) ray of $\mathbb{M}^4_p$ is a line in $\mathbb{M}^4_p$ whose tangent space at any point is null, i.e. spanned by a null vector. Any null plane $\Pi_p \subset T_p \mathbb{M}^4_p$ at $p$ intersects $T_p \mathbb{M}^4_p$ in a null direction $V_p$ at $p$. Indeed, if we write $\Pi_p = \text{span}\{e_0 + ie_1, e_2 + ie_3\}$ where $\{e_0, e_1, e_2, e_3\}$ is a positive orthonormal basis of $T_p \mathbb{R}^4_p$ with $e_0 = \partial/\partial x_0$, then $V_p$ is spanned by $v_p = \partial/\partial t + e_1$. This gives a bijection between null directions at $p$ and $\alpha$-planes at $p$ which, combined with our previous bijections gives correspondences which will be fundamental in this paper:

**Proposition 2.1** For any $p \in \mathbb{C}^4$ we have bijective correspondences between the quantities (QP1) – (QP4) in (6) given by

\[
\begin{align*}
J_p &= J_p(\Pi_p) = \begin{cases} -i & \text{on } \Pi_p \\ +i & \text{on } \overline{\Pi_p} \end{cases} \\
\Pi_p &= \Pi_p(J_p) = (0, 1)\text{-tangent space of } J_p = \{X + iJ_p X : X \in T_p \mathbb{R}^4_p\} \\
U_p &= U_p(J_p) = J_p(\partial/\partial x_0) \\
J_p &= J_p(\Pi_p) = \text{the unique positive almost Hermitian structure at } p \\
& \text{with } J_p(\partial/\partial x_0) = U_p \\
V_p &= V_p(U_p) = \text{span(} \partial/\partial t + U_p \text{)} \\
U_p &= U_p(V_p) = \text{normalized projection of } V_p \text{ onto } T_p \mathbb{R}^3_p \\
& \text{i.e. the unique vector } U_p \text{ such that } V_p = \text{span(} \partial/\partial t + U_p \text{)} \\
\end{align*}
\]

Schematically we have a commutative diagram:

\[
\begin{align*}
\text{(QP2)} &= \{\text{positive real almost Hermitian structures at } p:\} \\
& \quad J_p : T_p \mathbb{R}^3_p \rightarrow T_p \mathbb{R}^3_p \\
\text{(QP3)} &= \{\text{null directions at } p:\} \\
& \quad V_p \subset T_p \mathbb{M}^4_p \\
\text{(QP1)} &= \{\alpha\text{-planes at } p:\} \\
& \quad \Pi_p \subset T_p \mathbb{C}^4 \\
\text{(QP4)} &= \{\text{unit vectors } U_p \text{ in } T_p \mathbb{R}^3_p\} \\
\end{align*}
\]

4
Remark 2.2 There is a fifth quantity in bijective correspondence with the quantities \( \overline{8} \) namely given by (9) – (13)
given, in the notations above, by

For any 2.5 Representations in coordinates

For any \( p \in C^4 \) and \( [w_0, w_1] \in CP^1 \), set

\[
\Pi_p = \text{span} \left\{ w_0 \frac{\partial}{\partial z_1} - w_1 \frac{\partial}{\partial z_2}, w_0 \frac{\partial}{\partial z_2} + w_1 \frac{\partial}{\partial z_1} \right\}.
\]

Writing \( \mu = w_1/w_0 \in C \cup \{\infty\} \), if \( \mu \neq \infty \), \( \Pi_p \) has a basis

\[
\left\{ \frac{\partial}{\partial z_1} - \mu \frac{\partial}{\partial z_2}, \quad \frac{\partial}{\partial z_2} + \mu \frac{\partial}{\partial z_1} \right\};
\]

if \( \mu = \infty \), a basis for \( \Pi_p \) is given by \( \{\partial/\partial z_2, \partial/\partial z_1\} \).

\( \Pi_p \) is easily seen to be an \( \alpha \)-plane at \( p \) and \( \overline{8} \) gives an explicit bijection

\[
CP^1 \longleftrightarrow \{\alpha \text{-planes at } p\}.
\]

Applying the matrix \( \begin{pmatrix} w_0 & -w_1 \\ -w_1 & w_0 \end{pmatrix} \), a multiple of which is unitary, converts the basis of \( \Pi_p \) to an equivalent basis \( \{e_0 + i e_1, e_2 + i e_3\} \). Writing \( u = iw_1/w_0 = i\mu \), and letting \( \sigma : S^2 \rightarrow C \cup \{\infty\} \) denote stereographic projection from \((-1,0,0)\), the positive orthonormal basis \( \{e_0, e_1, e_2, e_3\} \) is given by

\[
\begin{align*}
e_0 &= \partial/\partial x_0, \\
U_p = e_1 &= \frac{1}{1 + |u|^2} \left( 1 - |u|^2, 2\Re u, 2\Im u, \sigma^{-1}(u) \right), \\
e_2 + i e_3 &= \frac{1}{1 + |u|^2} \left( -2u, 1 - u^2, i(1 + u^2) \right).
\end{align*}
\]

Note that \( \{e_2, e_3\} \) gives an orthonormal basis for the screen space \( U_p^\perp = V_p^\perp \cap R_3 \) of the corresponding null direction

\[
V_p = \text{span} \left( \frac{\partial}{\partial t} + U_p \right)
\]

and the corresponding almost Hermitian structure \( J_p \) is determined by

\[
J_p(e_0) = e_1, \quad J_p(e_2) = e_3.
\]

In summary, any of the corresponding quantities \( \Pi_p \in (QP1), J_p \in (QP2), V_p \in (QP3), U_p \in (QP4) \)
can be represented by a point \([w_0, w_1] \in CP^1\) or a number \( \mu = w_1/w_0 \in C \cup \{\infty\} \), the quantities being given by \( \overline{8} \) – (13).
2.6 Compactifications

We form the conformal compactification \( \tilde{\mathbb{C}}^4 \) of \( \mathbb{C}^4 \) as follows (cf. [27] but note that our conventions differ slightly from theirs — they agree with those of [30]): Set \( \tilde{\mathbb{C}}^4 = G_2(\mathbb{C}^4) \), the Grassmannian of all complex 2-planes through the origin with its standard complex structure. Then \( \tilde{\mathbb{C}}^4 \) is embedded holomorphically in \( \mathbb{C}^4 \) by the mapping

\[
\iota : (z_1, \bar{z}_1, z_2, \bar{z}_2) \mapsto \text{column space of } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ z_1 & \bar{z}_2 \\ z_2 & \bar{z}_1 \end{bmatrix}.
\]  

We shall call this the standard coordinate chart for \( \tilde{\mathbb{C}}^4 \). Points of \( \tilde{\mathbb{C}}^4 \setminus \mathbb{C}^4 \) will be called points at infinity.

This embedding gives the image of \( \mathbb{C}^4 \) in \( \tilde{\mathbb{C}}^4 \) a holomorphic conformal structure; this can be extended to the whole of \( \mathbb{C}^4 \) by the action of \( SU(2, 2) \) on \( \mathbb{C}^4 \); this group restricting to the conformal group on \( \mathbb{C}^4 \) which sends (affine) null lines to null lines.

Real slices \( \mathbb{R}^4_p \) compactify to submanifolds \( \tilde{\mathbb{R}}^4_p \) conformally equivalent to 4-spheres; for example the map \( \mathbb{R}^4 = \mathbb{R}^4_0 \hookrightarrow \tilde{\mathbb{C}}^4 \) is given by sending \((z_1, \bar{z}_1, z_2, \bar{z}_2) \in \mathbb{R}^4 \mapsto \text{column space of } \begin{bmatrix} I \end{bmatrix} \) where \( Q = \begin{bmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{bmatrix} \) is the usual representation of the quaternion \( z_1 + z_2 i \); then \( \mathbb{R}^4 \) is compactified by adding the single point at infinity given by the column space of \( \begin{bmatrix} 0 \\ I \end{bmatrix} \).

Each \( \mathbb{R}^3 \)-slice \( \mathbb{R}^3_p \) compactifies in \( \tilde{\mathbb{C}}^4 \) to a submanifold \( \tilde{\mathbb{R}}^3_p \) conformally equivalent to the 3-sphere \( S^3 \).

Each \( \mathbb{M}^4_p \) compactifies in \( \tilde{\mathbb{C}}^4 \) to a subset \( \tilde{\mathbb{M}}^4_p \) topologically equivalent to \( S^3 \times S^1 \). For more details, see, for example, [13, 22, 27].

All the quantities (14) make sense on the compactified spaces and we still have the bijections (8), though to represent them by a vector in \( S^2 \) requires coordinates to be chosen.

2.7 Representations by twistors

We recall the twistor correspondence which parametrizes the set of affine null planes in \( \tilde{\mathbb{C}}^4 \) by \( \mathbb{C}P^3 \) (cf. [27]):

Let \( F_{1, 2} \) be the complex manifold \( \{(w, p) \in \mathbb{C}P^3 \times \tilde{\mathbb{C}}^4 : \text{the line } w \text{ lies in the plane } p \} \), the so-called correspondence space. The restrictions of the natural projections define the double holomorphic fibration:

\[
\begin{aligned}
\mu &\left\langle \begin{array}{c}
(\nu \circ \mu^{-1}) \circ \mu^{-1}(w) \\
(\nu \circ \mu^{-1})^{-1}(w)
\end{array} \right. \quad & p \in \tilde{\mathbb{C}}^4 \\
\end{aligned}
\]

Then for any \( w \in \mathbb{C}P^3 \), \( \tilde{w} = \nu \circ \mu^{-1}(w) \) is an \( \alpha \)-plane in \( \tilde{\mathbb{C}}^4 \) which we call the \( \alpha \)-plane determined by or represented by \( w \).

Conversely, for any point \( p \in \tilde{\mathbb{C}}^4 \) we write \( \hat{p} = \mu \circ \nu^{-1}(p) \); then \( \hat{p} \) is a \( \mathbb{C}P^1 \) in \( \mathbb{C}P^3 \) which represents the \( \mathbb{C}P^1 \)-worth of \( \alpha \)-planes through \( p \).

Explicitly, in the standard coordinates (14) for \( \mathbb{C}^4 \mapsto \tilde{\mathbb{C}}^4 \), \( (w, p) \in F_{1, 2} \) if and only the following incidence relations are satisfied:

\[ w_0 z_1 - w_1 \bar{z}_2 = w_2 \quad w_0 \bar{z}_2 + w_1 z_1 = w_3 \]  

(16)

(for these express the condition that \( w \) is a linear combination of the columns of the matrix (14) and so lies in the plane represented by the point \( p \in G_2(\mathbb{C}^4) \)). Now note that, for any \([w_0, w_1, w_2, w_3] \) with \([w_0, w_1] \neq [0, 0] \), (16) defines an affine \( \alpha \)-plane in \( \mathbb{C}^4 \); indeed, its tangent space at any point is the set of vectors annihilated by \( \{w_0 d z_1 - w_1 d \bar{z}_2, w_0 d \bar{z}_2 + w_1 d z_1\} \) and so is given by (14). Points of \( \mathbb{C}P^3 \) on the projective line \( \mathbb{C}P^3_0 = \{[w_0, w_1, w_2, w_3] : [w_0, w_1] = [0, 0]\} \) correspond to affine \( \alpha \)-planes at infinity, i.e. in \( \tilde{\mathbb{C}}^4 \setminus \mathbb{C}^4 \). Thus \( \mathbb{C}P^3 \) parametrizes all \( \alpha \)-planes in \( \tilde{\mathbb{C}}^4 \) and (14) expresses the condition that a point
\((z_1, \bar{z}_1, z_2, \bar{z}_2) \in \mathbb{C}^4\) lies on the \(\alpha\)-plane determined by \([w_0, w_1, w_2, w_3] \in \mathbb{C}P^3\), i.e. \((16)\) is the equation of that \(\alpha\)-plane.

Note that any \(\alpha\)-plane intersects a slice \(\mathcal{R}_p^4\) in a unique point giving a map \(\pi_p : \mathbb{C}P^3 \to \mathcal{R}_p^4\); for example, if \([w_0, w_1, w_2, w_3] \in \mathbb{C}P^3 \setminus \mathcal{C}P_0^1\), the \(\alpha\)-plane \((16)\) intersects \(\mathcal{R}_p^4 = \mathcal{R}_0^4\) when
\[
\begin{align*}
\frac{w_0 z_1 - w_1 \bar{z}_2}{w_1 z_1 + w_0 \bar{z}_2} &= \frac{w_2}{\bar{w}_3}.
\end{align*}
\]
which has the unique solution
\[
\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{1}{|w_0|^2 + |w_1|^2} \begin{pmatrix} w_0 w_2 + w_1 \bar{w}_3 \\ \bar{w}_0 w_3 - \bar{w}_1 w_2 \end{pmatrix}.
\]

The resulting map
\[
\pi = \pi_0 : \mathbb{C}P^3 \to \mathcal{R}_p^4 = S^4
\]
is just the standard twistor map — in quaternionic notation \([w_0, w_1, w_2, w_3] \to (w_0 + w_1 j)^{-1}(w_2 + w_3 j) \in \mathbb{H} \cup \infty \cong S^4\). The affine \(\alpha\)-plane \((16)\) intersects the Minkowski slice \(\mathcal{M}_p^4\) if and only if it intersects \(\mathcal{R}_p^3\), the intersection is then an affine null line. For \(p = 0\) this holds if and only if the point \((13)\) lies in \(\mathcal{R}_p^3\), i.e. \(\Re z_1 = 0\), i.e. \([w_0, w_1, w_2, w_3] \) lies on the real quadric
\[
N^5 = \pi^{-1}(\mathcal{R}_p^3) = \{|[w_0, w_1, w_2, w_3] : w_0 \bar{w}_2 + \bar{w}_0 w_2 + w_1 \bar{w}_3 + \bar{w}_1 w_3 = 0\} \subset \mathbb{C}P^3.
\]
Points in \(N^5\) thus represent affine null lines of \(\mathcal{M}_p^4\). For general \(p\) we replace \(N^5\) by \(N^5 = \pi^{-1}(\mathcal{R}_p^3)\).

Interchanging \(z_2\) and \(\bar{z}_2\) in \((16)\) gives the standard parametrization of \(\beta\)-planes by \(\mathcal{C}P^3\).

The incidence relations \((16)\) define a fundamental map which gives the point of \(\mathbb{C}P^3\) representing the \(\alpha\)-plane through \((z_1, \bar{z}_1, z_2, \bar{z}_2)\) with direction vector \(\sigma^{-1}(iw_1/w_0)\):
\[
\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^4 \times \mathbb{C}P^1 \quad \xrightarrow{\quad} \quad \mathbb{C}P^3 \setminus \mathcal{C}P_0^1 \subset \mathbb{C}P^3,
\]
\[
\big((z_1, \bar{z}_1, z_2, \bar{z}_2), [w_0, w_1]\big) \quad \xrightarrow{\quad} \quad [w_0, w_1, w_0 z_1 - w_1 \bar{z}_2, w_0 z_2 + w_1 \bar{z}_1].
\]

The restriction of this to \(\mathcal{R}_p^4 \times \mathbb{C}P^1\) gives a trivialisation of the twistor bundle \((19)\) over \(\mathcal{R}_p^4\).

In summary, (i) \(w \mapsto \bar{w}\) defines a bijection from \(\mathbb{C}P^3\) to the set of all affine \(\alpha\)-planes in \(\mathbb{C}^4\); if \(w \in \mathbb{C}P^3 \setminus \mathcal{C}P_0^1\), \(\bar{w}\) is given on \(\mathbb{C}^4\) by \((14)\), (ii) a point \((w, p) \in F_{1,2}\) represents the \(\alpha\)-plane \(\Pi_p\) at \(p\) tangent to \(\bar{w}\) and, so, any of the four quantities \((3)\); at a point \(p \in \mathbb{C}^4\), \(\Pi_p\) is given by \((4)\).

3 Distributions of null planes and associated structures

In this section we study how our four quantities \((4)\) vary when the point is varied and obtain bijections between their germs (Theorem 3.13). As in the last section, for an open set \(A^C\) of \(\mathbb{C}^4\), \(g\) will denote the standard holomorphic metric or any conformal multiple of it, and \(\nabla\) its Levi-Civita connection; these quantities can be restricted to slices \(\mathcal{R}_p^4\), \(\mathcal{R}_p^3\), \(\mathcal{M}_p^4\) to give metrics and their Levi-Civita connections. For any bundle \(E \to A\), we denote by \(C^\infty_A(E)\) the space of \(C^\infty\) sections of \(E\) defined on \(A\).

3.1 Holomorphic foliations by null planes on \(\mathbb{C}^4\)

By a holomorphic distribution of \(\alpha\)-planes on an open set \(A^C\) of \(\mathbb{C}^4\) we mean a map \(\Pi\) which assigns to each point \(p\) of \(A^C\) an \(\alpha\)-plane at \(p\), \(\Pi_p \subset T_p \mathbb{C}^4\), in a holomorphic fashion, i.e. \(\Pi_p = \text{span}(w_1(p), w_2(p))\) where the \(w_i : A^C \to T \mathbb{C}^4\) are holomorphic. We can identify \(\Pi\) with its image, a holomorphic subbundle of \(T \mathcal{C}^4\).

We call a holomorphic distribution \(\Pi\) of \(\alpha\)-planes on \(A^C\) autoparallel if \(\nabla w_1, w_2 \in C^\infty_A(\Pi)\) for all \(w_1, w_2 \in \mathcal{C}A^C(\Pi)\). An autoparallel holomorphic distribution of \(\alpha\)-planes has integral submanifolds which are affine \(\alpha\)-planes; thus, an autoparallel holomorphic distribution of \(\alpha\)-planes on \(A^C\) is equivalent to a holomorphic foliation by affine \(\alpha\)-planes of \(A^C\). Similar definitions can be given replacing ‘\(\alpha\)-plane’ by ‘\(\beta\)-plane’.
3.2 Hermitian structures on $\mathbb{R}^4$

By a smooth almost Hermitian structure on an open subset $A^4$ of $\mathbb{R}^4$, we mean a map $J$ which assigns to each point $p$ of $A^4$ an almost Hermitian structure at $p$ in a smooth fashion, i.e. $p \mapsto J_p(U_p)$ is smooth for all $U \in C^\infty_4(T\mathbb{R}^4)$; equivalently $J$ defines a smooth section on $A^4$ of the bundle $E = \text{End}(T\mathbb{R}^4) \to \mathbb{R}^4$. We call $J$ integrable if there are local complex coordinates on $A^4$ with associated almost complex structure $J$, this is equivalent to the vanishing of the Nijenhuis tensor. A short calculation (see, e.g. [10, p. 42] or [23, p. 169]) shows that this is equivalent to

$$\nabla^E_{JX}J = J\nabla^E_XJ \ \forall X \in C^\infty_4(T\mathbb{R}^4)$$

i.e.

$$(\nabla^E_{JX}J)(Y) = J((\nabla^E_XJ)(Y)) \ \forall X,Y \in C^\infty_4(T\mathbb{R}^4)$$

where $\nabla^E$ is the induced connection of the bundle $E = \text{End}(T\mathbb{R}^4)$ given by $(\nabla^E_YJ)(Y) = \nabla_X(JY) - J(\nabla_XY)$. Such a $J$ is always real-analytic.

3.3 Conformal foliations by curves on $\mathbb{R}^3$

By a $C^\infty$ (resp. $C^\omega$) non-zero vector field on an open subset $A^3$ of $\mathbb{R}^3$ we mean a $C^\infty$ (resp. $C^\omega$) section $U : A^3 \to T\mathbb{R}^3 \setminus \{\text{zero section}\}$. Without loss of generality we may choose a conformal Euclidean metric $g$ on $\mathbb{R}^3$ and scale $U$ to be a section of the unit tangent bundle $T^1\mathbb{R}^3|A^3$ for that metric. To such a distribution corresponds a $C^\infty$ (resp. $C^\omega$) (oriented) foliation $\mathcal{C}$ of $A^3$ by curves given by integrating $U$. Note that $U$ can be recovered from $\mathcal{C}$ as its field of (positive) unit tangents.

Let $U^\perp$ be the distribution of (oriented) subspaces of $T\mathbb{R}^3$ perpendicular to $U$. Then the distribution $U$ is called shear-free and the corresponding foliation $\mathcal{C}$ conformal if $\text{Lie}$ transport along $U$ of vectors in $U^\perp$ is conformal. In concrete terms, letting $J^\perp$ denote rotation through $+\pi/2$ on each oriented plane $U^\perp_p$, (p $\in A^3$), then $U$ is shear-free if and only if $\mathcal{L}_U J^\perp = 0$ on $A^3$ where $\mathcal{L}$ denotes Lie derivative. Now, for any $X \in C^\infty(U^\perp)$,

$$(\mathcal{L}_U J^\perp)(X) = (\{\mathcal{L}_U (J^\perp X)\}^\perp - J^\perp \{\mathcal{L}_U (X)\}^\perp)$$

$$= \nabla_U (J^\perp X)^\perp - \nabla_{J^\perp X}U - J^\perp \{\nabla_U X\}^\perp + J^\perp \nabla_X U$$

where $\{\}^\perp$ denotes orthogonal projection onto $U^\perp$ (noting that $\nabla_X U \in C^\infty(U^\perp)$ since $g(\nabla_X U, U) = \frac{1}{2}Xg(U,U) = 0$). Further, since $U^\perp$ is a Hermitian connected bundle of rank 2, as for all such bundles we have

$$\nabla_U (J^\perp X)^\perp = J^\perp \{\nabla_U X\}^\perp = (\nabla_{J^\perp U} J^\perp)(X) = 0,$$

hence $U$ is shear-free if and only if

$$\nabla_{J^\perp X}U = J^\perp \nabla_X U.$$  \hfill (23)

Remark 3.1 This can be interpreted as follows: A CR structure [13, 15] on an odd-dimensional manifold $M = M^{2k+1}$ is a choice of rank $k$ complex subbundle $V$ of the complexified tangent bundle $T^cM = TM \otimes \mathbb{C}$ with $V \cap \overline{V} = \{0\}$ and

$$[C^\infty(V), C^\infty(V)] \subset C^\infty(V).$$

Given $V$ we define the Levi subbundle $H$ of $TM$ by $H \otimes \mathbb{C} = V \oplus \overline{V}$ and a real endomorphism $J : H \otimes \mathbb{C} \to H \otimes \mathbb{C}$ by multiplication by $+i$ (resp. $-i$) on $V$ (resp. $\overline{V}$). Conversely $(H, J)$ determines $V$ so that a CR structure can be specified by giving the pair $(H, J)$.

Any real hypersurface $M$ of a complex manifold $\tilde{M}$ has a canonical CR structure called the hypersurface CR structure given by $V = T^{1,0} \tilde{M} \cap T^cM$.

We give the unit tangent bundle $T^1\mathbb{R}^3 = \mathbb{R}^3 \times S^2$ a CR structure $(H, J)$ as follows: At each point $(p, U) \in \mathbb{R}^3 \times S^2$ use the canonical isomorphism $\mathbb{R}^3 \cong T_p\mathbb{R}^3$ to regard $U^\perp$ as a subspace $U^\perp_p$ of $T_p\mathbb{R}^3$; then define $H_{(p, U)} = U^\perp_p \oplus T_pS^2$ and define $J$ as rotation through $+\pi/2$ on $U^\perp_p$ together with the standard
complex structure $J^S$ on $T_U S^2$. This is the hypersurface CR structure given by regarding $\mathbb{R}^3 \times S^2$ as a real hypersurface of the manifold $(\mathbb{R}^3 \times S^2, J)$ where the complex structure $J$ is given by

$$\mathbb{R}^4 \times S^2 \to \text{End}(T(\mathbb{R}^4 \times S^2)), \quad (p, U) \mapsto (J(U)_p, J^S),$$

with $J(U)_p$ the unique positive almost Hermitian structure with $J(U)_p(\partial/\partial x_0) = U_p$ as in (3). A calculation using (2) shows that $J$ is integrable.

More generally, for any oriented Riemannian 3-manifold $M^3$, we can give the unit tangent bundle $T^1 M^3$ a CR structure $(H, J)$ as follows [24, 18]. At each point $(p, U) \in T^1 M^3$ (where $p \in M^3$ and $U \in T_p M^3$) the Levi-Civita connection on $M^3$ defines a splitting $T_{(p, U)}(T^1 M^3) = T_p M^3 \oplus T_U (T^1_p M^3)$. Since $T_p M^3$ is isometric to a 2-sphere, it has a canonical Kähler structure $J^S$. We define $H_{(p, U)} = U_p^+ \oplus T_U (T^1_p M^3)$ and $J$ as rotation through $+\pi/2$ on $U_p^+$ together with $J^S$ on $T_U (T^1_p M^3)$.

For $M^3 = S^3$ this can be described more explicitly: The differential of the canonical embedding $S^3 \to \mathbb{R}^4$ defines an embedding $i : T^1 S^3 \to \mathbb{R}^4 \times \mathbb{R}^4$. At a point $(p, U) \in T^1 S^3, \ (p \in S^3, \ U \in T_p S^3)$, we have

$$di(T_{(p, U)}T^1 S^3) = \{(X, U) \in p^+ \times U^+ : (X, U) + (p, U) = 0\} \subset \mathbb{R}^4 \times \mathbb{R}^4 \cong T_{(p, U)}(\mathbb{R}^4 \times \mathbb{R}^4),$$

then we choose $H_{(p, U)} = (U \oplus p)^+ \times (U \oplus p)^-$ and $J$ = rotation through $\pi/2$ on each plane $(U \oplus p)^\perp$.

That this is integrable can be seen by noting that any stereographic projection $S^3 \setminus \{\text{point}\} \to \mathbb{R}^3$ is conformal and induces a CR diffeomorphism between $T^1(S^3 \setminus \{\text{point}\})$ and $T^1 \mathbb{R}^3$.

Next give $A^3$ the CR structure defined by $(H, J) = (U^\perp, J^1)$. Then (23) says that $U : A^3 \to T^1 S^3$ is CR (cf. [13]).

A concrete way of obtaining conformal foliations is given as follows: Let $f : A^3 \to N^2$ be a $C^\infty$ (resp. $C^\omega$) submersion from an open subset of $\mathbb{R}^3$ to a Riemann surface $N^2$, usually $\mathbb{C}$ or $\mathbb{C} \cup \{\infty\}$. Then $f$ is called horizontally conformal if the restriction of its differential $df_p$ to its horizontal space $(\ker df_p)^\perp$ is conformal and surjective for all $p \in A^3$. Explicitly, $f$ is horizontally conformal if and only if, in any local complex coordinate on $N^2$, we have on $A^3$,

$$\left( \frac{\partial f}{\partial x_1} \right)^2 + \left( \frac{\partial f}{\partial x_2} \right)^2 + \left( \frac{\partial f}{\partial x_3} \right)^2 = 0. \quad (26)$$

Writing $f = f_1 + i f_2$ this is equivalent to

$$g(\text{grad} f_1, \text{grad} f_2) = 0 \quad \text{and} \quad g(\text{grad} f_1, \text{grad} f_1) = g(\text{grad} f_2, \text{grad} f_2).$$

Then we have the simple lemma (cf. [Vai]):

**Lemma 3.2** 1) If $f$ is $C^\infty$ (resp. $C^\omega$) and horizontally conformal then the foliation defined by (the fibres of) $f$ is $C^\infty$ (resp. $C^\omega$) and conformal.

2) All $C^\infty$ (resp. $C^\omega$) conformal foliations are given locally in this way.

Note that the shear-free unit vector field $U$ tangent to the conformal foliation defined by $f = f_1 + i f_2$ is given by

$$U = \text{grad} f_1 \times \text{grad} f_2 / |\text{grad} f_1 \times \text{grad} f_2|. \quad (27)$$

**Example 3.3** Let $f(x_1, x_2, x_3) = (x_2 \pm i x_1)/(|x| - x_1)$. It is easily checked that $f$ is a horizontally conformal map from $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$ to $\mathbb{C} \cup \{\infty\}$. Its level curves are radii from the origin and give the leaves of a conformal foliation $C$ of $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$ whose tangent vector field is the shear-free unit vector field

$$U(x_1, x_2, x_3) = \pm \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} (x_1, x_2, x_3). \quad (28)$$

Note that $U : \mathbb{R}^3 \setminus \{0\} \to S^2$ is surjective. Note further that, when $\mathbb{R}^3$ is compactified to $S^3$, this example becomes projection from the poles: $S^3 \setminus \{(\pm 1, 0, 0, 0)\} \to S^2$ given by the formula $(x_0, x_1, x_2, x_3) \mapsto \pm (x_1, x_2, x_3)/\sqrt{x_1^2 + x_2^2 + x_3^2}$. \footnote{This is not the CR structure on the unit tangent bundle discussed, for example, in [4, Chapter 7] or [25].}
Example 3.4 Let \( f(x_1, x_2, x_3) = -ix_1 \pm \sqrt{x_2^2 + x_3^2} \). This is a horizontally conformal map from \( \mathbb{A}^3 = \mathbb{R}^3 \setminus \{(x_1, x_2, x_3) : x_2 = x_3 = 0\} \) to \( \mathbb{C} \). Its level curves are circles in planes parallel to the \((x_2, x_3)\)-plane and centred on points of the \(x_1\)-axis; these give a conformal foliation of \( \mathbb{A}^3 \) whose tangent vector field is the shear-free unit vector field \( U : \mathbb{A}^3 \to S^2 \) given by

\[
U(x_1, x_2, x_3) = \pm \frac{1}{\sqrt{x_2^2 + x_3^2}} (0, -x_3, x_2). \tag{29}
\]

Note that \( U \) has 1-dimensional image — the equator of \( S^2 \).

3.4 Shear-free ray congruences on \( M^4 \)

By a \( C^\infty \) (resp. \( C^\omega \)) distribution of null directions on a subset \( A^M \) of Minkowski space \( M^4 \) we mean a map \( V \) which assigns to each point \( p \) of \( A^M \) a 1-dimensional null subspace of \( T_p M^4 \) in a smooth fashion, i.e. \( V_p = \text{span}(v(p)) \) where \( v : A^M \to TM^4 \) is a smooth nowhere zero null vector field. We can identify \( V \) with its image, a smooth null rank one subbundle of \( TM^4|A^M \). Such a distribution integrates to a \( C^\infty \) (resp. \( C^\omega \)) foliation \( \ell \) by null curves (i.e. curves whose tangent vectors are all null). We call \( V \) autoparallel if

\[
\nabla_v v \in C^\infty_{A^M}(V) \quad \text{for all} \quad v \in C^\infty_{A^M}(V), \tag{30}
\]
equivalently the integral curves of \( V \) are null lines (“light rays”) and then \( \ell \) is called the foliation by null lines or ray congruence \( \ell \) on \( A^M \) corresponding to \( V \). Given \( \ell \), the autoparallel distribution \( V \) of null directions can be recovered as its tangent field.

For a foliation \( \ell \) by null lines, or the corresponding autoparallel distribution \( V \) of null directions, we define the shear-free condition as follows: The distribution \( V^\perp \) orthogonal to \( V \) (with respect to the conformal Minkowski metric \( g^M \)) is three-dimensional and contains \( V \) so that the quotient \( V^\perp / V \) is 2-dimensional and \( g^M \) factors to give a positive definite inner product on it. Let \( J^\perp \) denote rotation through \( \pi/2 \) (with any orientation) on \( V^\perp / V \). We say that \( V \) is shear-free if

\[
\mathcal{L}_v J^\perp = 0 \tag{31}
\]
for any \( v \in C^\infty(V) \). As in the last section this can be interpreted as saying that Lie transport along \( V \) in \( V^\perp / V \) is conformal. For another interpretation, note that \( \mathcal{B} \) is equivalent to

\[
(\nabla_{J^\perp X}v)^{V^\perp / V} = J^\perp (\nabla_X v)^{V^\perp / V} \quad \text{for all} \quad X \in C^\infty(V^\perp) \tag{32}
\]
where \( (\ )^{V^\perp / V} \) denotes the projection \( V^\perp \to V^\perp / V \). More concretely, choose any complement \( \Sigma \) of \( V \) in \( V^\perp \); \( \Sigma \) is called a screen space. Then \( V \) is shear-free if and only if

\[
(\nabla_{J^\perp X}v)^\Sigma = J^\perp (\nabla_X v)^\Sigma \quad \text{for all} \quad X \in C^\infty(\Sigma) \tag{33}
\]
where now \( (\ )^\Sigma \) indicates projection onto \( \Sigma \) along \( V \).

Remarks 3.5

1. The twistorial representation of a foliation by null lines is as follows: Recall that the affine \( \alpha \)-plane \( [\mathbf{4}] \) determined by \( w = [w_0, w_1, w_2, w_3] \in \mathbb{C}P^3 \) intersects the Minkowski space \( M^4 \) if and only if \( w \in N^5 \) (cf. Eqn. \( [\mathbf{2}] \)). Thus a foliation by null lines on an open subset \( A^M \) of \( M^4 \) is given by a map

\[
w : A^M \to N^5. \tag{34}
\]

For a CR interpretation of condition \( [\mathbf{3}] \), see Remarks \( [\mathbf{8}] \).

2. The \textit{shear tensor} (cf. \( [\mathbf{3}] \)) of a foliation \( \ell \) by null lines is defined by \( S(X, Y) = \text{the trace-free part of} \ 
\frac{1}{2} g(\nabla_X Y + \nabla_Y X, v) \) for \( X, Y \in C^\infty(\Sigma) \). It is easy to see that \( \ell \) is shear-free if and only if this vanishes. The \textit{rotation or twist tensor} is defined by \( T(X, Y) = \frac{1}{2} g(\nabla_X Y - \nabla_Y X, v) = \frac{1}{2} g([X, Y], v) = \frac{1}{2} \left( g(\nabla_X Y, X) - g(\nabla_Y X, Y) \right). \) At a point \( p \) it measures how much (infinitesimally) nearby null lines passing through the screen space \( \Sigma_p \) twist around the null line through \( p \).

---

\(^2\)The term \textit{ray congruence} is often used to describe a three-parameter family of rays which may not form a foliation everywhere.
3.5 Shear-free ray congruences and conformal foliations

In this section we describe a geometrical correspondence between $C^\infty$ shear-free ray congruences on $M^4$ and $C^\infty$ conformal foliations by curves on $\mathbb{R}^3$. Firstly, recall the correspondence of Proposition 3.6 at any point $p \in C^4$ between unit vectors $U_p \in T_p R^3_p$ and null directions $V_p$ in $T_p M^4_p$ given by

\[
\begin{align*}
V_p &= V_p(U_p) = \text{span}(\partial/\partial t + U_p), \\
U_p &= U_p(V_p) = \text{normalized projection of } V_p \text{ onto } T_p R^3_p,
\end{align*}
\]

i.e. the unique vector $U_p$ such that $V_p = \text{span}(\partial/\partial t + U_p)$.

Now let $\ell$ be a $C^\infty$ foliation by null lines on an open subset $A^M$ of $M^4$ and let $V$ be its tangent (autoparallel) distribution of null directions as in the last section. Let $p \in A^M$. Then we define the projection of $\ell$ onto the slice $R^3_p$ as (i) the $C^\infty$ unit vector field $U$ on $A^3 = A^M \cap R^3_p$ given by $U_q = U_q(V_q)$, $(q \in A^3)$, or, equivalently, (ii) the $C^\infty$ foliation $\mathcal{C}$ by curves with tangent vector field $U$.

Conversely, given a $C^\infty$ foliation $\mathcal{C}$ by curves or, equivalently, a $C^\infty$ unit vector field $U$ on an open subset $A^3$ of $R^3_p$, setting $V_q = V_q(U_q) = \text{span}(\partial/\partial t + U_q)$ for $q \in A^3$ defines a distribution of null directions on $A^3$. We extend this to a foliation by null lines as follows: For each $q \in A^3$, set $\ell_q = \text{the affine null line of } M^4$ through $q$ tangent to $V_q$; this defines a foliation $\ell$ by null lines of some neighbourhood $A^M$ of $A^3$ in $M^4$; we define $V$ to be its tangent distribution on $A^M$, this is an autoparallel distribution of null directions. We call $\ell$ (or $V$) the extension of $\mathcal{C}$ (or of $U$). We have the following relation between properties of $\mathcal{C}$ and $\ell$ (equivalently $U$ and $V$):

**Proposition 3.6** Let $\mathcal{C}$ be a foliation by curves of an open subset $A^3$ of $R^3_p$ and let $\ell$ be its extension to a foliation by null lines of a neighbourhood $A^M$ of $A^3$ in $M^4$ as defined above. Then $\ell$ is shear-free on $A^M$ if and only if $\mathcal{C}$ is conformal.

**Proof** We use notations as above, and write $v = \partial/\partial t + U$ so that $V = \text{span}(v)$. Taking the screen space $\Sigma$ at a point $q \in A^3$ to be $V_q \cap R^3 = U_q \cap R^3$ since $\partial/\partial t$ is parallel, the conditions (33) and (23) coincide so that $V$ is shear-free at points of $A^3 = A^M \cap R^3_p$ if and only if $U$ is shear free on $A^3$. But now the Sachs’ equations [22] show that if $V$ is shear-free on a slice cutting each null geodesic, it is shear-free everywhere and the proposition follows.

Recall that by a germ of $V$ at $A^3$ we mean an equivalence class of distributions $V$ on open neighbourhoods of $A^3$, deemed equivalent if they agree on a neighbourhood of $A^3$. Then extension and projection defined above are inverse at the level of germs, precisely:

**Theorem 3.7** Let $p \in M^4$ and let $A^3$ be an open subset of $R^3_p$. Then projection onto the slice $R^3_p$ defines a bijective correspondence between germs at $A^3$ of $C^\infty$ shear-free ray congruences $\ell$ (equivalently shear-free autoparallel null distributions $V$) defined on an open neighbourhood of $A^3$ in $M^4$ and $C^\infty$ conformal foliations $\mathcal{C}$ by curves (equivalently shear-free unit vector fields $U$) on $A^3$. The inverse of projection is extension.

**Proof** Evident from our description of the maps and the last proposition.

**Remarks 3.8** 1. Thus any $C^\infty$ conformal foliation $\mathcal{C}$ by curves on an open subset of $R^3$ extends to a shear-free ray congruence on an open subset $A^M$ of $M^4$. But then, for any point $p = (t, x_1, x_2, x_3) \in A^M$, this projects to a $C^\infty$ conformal foliation by curves on an open set of the $R^3$-slice $R^3_p = R^3_p$ of $M^4$ through $p$, so that, to $\mathcal{C}$ is associated a 1-parameter family $\mathcal{C}_t$ of other conformal foliations of open subsets of $R^3$ by curves.

   Note that the transformation $\mathcal{C} \rightarrow \mathcal{C}_t$ moves the vector field $U$ a distance $t$ in direction $U$ so that the value of the unit tangent vector field $U_t$ of $\mathcal{C}_t$ at a point $q$ is equal to the value of the unit tangent vector field $U$ of $\mathcal{C}$ at the point $q - tU_t(q)$.

   2. The correspondence of the theorem can be interpreted as follows (taking $p = 0$ for simplicity): Give $A^3$ the CR structure $(U^1, J^1)$ as in Remark 3.7. Then $U$ is shear-free if and only if $U : A^3 \rightarrow T^1 R^3$ is CR. Similarly, by (33), $\ell$ is shear-free if and only if the map $w : A^M \rightarrow N^5$ representing it (Remark
is CR when restricted to $A^3$. Now recall that the conformal compactification $\tilde{R}^3$ of $R^3$ may be identified with $S^3$ and that this is contained in the conformal compactification $\tilde{M}^4$ of $M^4$. We have a CR isomorphism: $k: T^1 S^3 \to N^5$ given by sending a unit tangent vector $U$ at a point $p$ of $S^3 = R^3$ to the point of $C P^3$ representing the affine geodesic through $p$ tangent to $\partial/\partial t + U$. This is given on $R^3$ by the map

$$T^1 R^3 \cong R^3 \times S^2 \to N^5$$

which is the restriction of the fundamental map (3). (Since that map is holomorphic with respect to the complex structure $\mathcal{J}$ of (23) the map (33), and so also the map $k$, is CR). Then $\ell$ is represented by a map $w = k \circ U$ so that $w$ is CR if and only if $U$ is, which provides another proof of Proposition 3.6.

3. It is easily seen that $\ell$ is twist-free (Remarks 3.5) if and only if $\mathcal{C}$ has integrable horizontal spaces.

**Example 3.9** Consider as in Example 3.3 the conformal foliation of $R^3 \setminus \{(0, 0, 0)\}$ given by the shear-free vector field

$$U(x_1, x_2, x_3) = \pm \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} (x_1, x_2, x_3).$$

Then it is easily checked that the shear-free ray congruence $\ell$ extending $U$ is defined on $M^4 \setminus \{(t, 0, 0, 0) : t \in R\}$ and has tangent vector field $v = \partial/\partial t + U$ where

$$U(t, x_1, x_2, x_3) = \pm \frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} (x_1, x_2, x_3).$$

Now for each $t$, $\ell$ projects to a conformal foliation on $R^3_t$ with tangent vector field given by (36): note that these conformal foliations are independent of $t$.

**Example 3.10** Consider as in Example 3.3 the conformal foliation with tangent field the shear-free vector field

$$U(x_1, x_2, x_3) = \frac{1}{\sqrt{x_2^2 + x_3^2}} (0, -x_3, x_2).$$

We compute the tangent vector field to the shear-free ray congruence $\ell$ extending $U$. The affine null geodesic of $\ell$ in $M^4$ through $(x_1, x_2, x_3)$ with direction $\partial/\partial t + U$ is given parametrically by

$$T \mapsto \left(T, x_1, x_2 + T \left(-\frac{x_3}{\sqrt{x_2^2 + x_3^2}}\right), x_3 + T \left(\frac{x_2}{\sqrt{x_2^2 + x_3^2}}\right)\right) (37)$$

$$= (T, X_1, X_2, X_3), \text{ say.}$$

Conversely, given $(T, X_1, X_2, X_3) \in M^4$ the null geodesic of $\ell$ hits $R^3 \subset M^4$ at $(x_1, x_2, x_3)$ where

$$x_1 = X_1, \quad x_2 + T \left(\frac{-x_3}{\sqrt{x_2^2 + x_3^2}}\right) = X_2, \quad x_3 + T \left(\frac{x_2}{\sqrt{x_2^2 + x_3^2}}\right) = X_3.$$

Solving this gives

$$(x_1, x_2, x_3) = \frac{X_2^2 + X_3^2}{R^2} \left(X_1, X_2 + \frac{T}{R}, X_3 - \frac{T}{R}\right)$$

where $R = \sqrt{X_2^2 + X_3^2 - T^2}$. Hence the tangent to the null geodesic of $\ell$ through $(t, x_1, x_2, x_3) \in M^4$ is given by $v = \partial/\partial t + U$ with

$$U = U_t(x_1, x_2, x_3) = U(t, x_1, x_2, x_3) = \frac{r}{\sqrt{x_2^2 + x_3^2}} \left(0, -x_3 + \frac{t}{r} x_2, x_2 + \frac{t}{r} x_3\right)$$

where $r = \sqrt{x_2^2 + x_3^2 - t^2}$.

Of course, again the image of $U$ is the equator of $S^2$. For each $t$, (38) gives a conformal, in fact Riemannian, foliation $\mathcal{C}_t$ of $R^3$. To see what the leaf $L$ through a point $p = (x_1, x_2, x_3)$ looks like, note
that, they lie in the planes \( x_3 = \text{constant} \). Fixing \( x_3 \), note that, as in Remark 3.3, the value of \( U_t \) at a point \( q = (x_2, x_3) \) is equal to the value of \( U \) at the point \( \bar{q} = q - tU_t(q) \). Since \( U(\bar{q}) \) is orthogonal to the position vector of \( \bar{q} \), the points \( \{0, \bar{q}, q, q - \bar{q}\} \) form a rectangle so that the normal to \( L \) at \( q \) is tangent to the circle centred at the origin with radius \( |t| \). It follows that the leaves of \( C_t \) are the involutes of that circle; two such leaves are pictured in Fig. 1.

### 3.6 Unification of the four distributions

In 

\\[ (\text{D1}) \quad \text{holomorphic distributions } \Pi \text{ of } \mathbb{C}^4, \]

\\[ (\text{D2}) \quad \text{positive almost Hermitian structures } J \text{ on an open subset } A^4 \text{ of } \mathbb{R}^4_p, \]

\\[ (\text{D3}) \quad \text{distributions of null directions } V \text{ on an open subset } A^M \text{ of } \mathbb{M}^4_p, \]

(39)

(Here \( p \) is a point of \( \mathbb{C}^4 \) which we could take to be the origin, but for later purposes we prefer not to.)

We now observe that, under complexification, these are equivalent and that the conditions discussed in §§3.1, 3.2 and 3.3 then correspond.

Let \( \Pi \in (\text{D1}) \). Then, the pointwise equivalences of Proposition 2.1 give us corresponding distributions \( J = J(\Pi) \) and \( V = V(\Pi) \) defined on \( A^C \) by \( J_p = J_{\Pi}(p) \) etc. as in Equations (7). Note that these distributions are also holomorphic (with respect to the standard holomorphic structure on \( \mathbb{C}^4 \) given by multiplication by \( i \)). They may be \textit{restricted to slices} to give distributions \( J \in (\text{D2}) \) and \( V \in (\text{D3}) \) defined on the open sets \( A^4 = A^C \cap \mathbb{R}^4_p \), \( A^M = A^C \cap \mathbb{M}^4_p \). Conversely, a given \( J \in (\text{D2}) \) or \( V \in (\text{D3}) \) may extended to an open subset of \( \mathbb{C}^4 \) by analytic continuation, i.e. by insisting that the extended quantity be holomorphic with respect to the standard complex structure on \( \mathbb{C}^4 \), and then \( \Pi = \Pi(J) \) or \( \Pi = \Pi(V) \) is in (D1). These operations are clearly inverse at the level of germs.

Then we have

**Proposition 3.11** Let \( A^C \) be a connected open subset of \( \mathbb{C}^4 \), and let \( \Pi \in (\text{D1}) \). Let \( p \in A^C \) and let \( A^4 = A^C \cap \mathbb{R}^4_p \), \( A^M = A^C \cap \mathbb{M}^4_p \). Let \( J \in (\text{D2}) \), \( V \in (\text{D3}) \) be defined by restriction to slices through \( p \) as above.

Then the following conditions are equivalent:

1) \( \Pi \) is autoparallel on \( A^C \),
2) \( J \) is integrable on \( A^4 \),
3) \( V \) is autoparallel and shear-free on \( A^M \).

**Proof** Firstly we show that \( \Pi \) is autoparallel at points of \( A^4 \) if and only if \( J \) is integrable on \( A^4 \). To do this, let \( q \in A^4 \). Then \( \Pi \) is autoparallel at \( q \) if and only if \( (\nabla_v w)_q \in \Pi_q \) for all \( v, w \in C_{\mathbb{C}^4}(\Pi) \). But since \( J \) is the \((0, 1)\)-subspace of \( \Pi \) this is clearly equivalent to \( \nabla_v J = 0 \) for all \( v \in \Pi_q \), i.e.

\[
\nabla_{x+iJx} J = 0 \quad \text{for all} \quad X \in T_q \mathbb{R}^4_p, \tag{40}
\]

Now since \( \Pi \) is holomorphic,

\[
\nabla_{iX} J = J\nabla_X J \quad \text{for all} \quad X \in T_q \mathbb{R}^4_p,
\]

so that (40) is equivalent to

\[
\nabla_{JX} J = J\nabla_X J \quad \text{for all} \quad X \in T_q \mathbb{R}^4_p, \tag{41}
\]

which is just the integrability condition (22) as required.

Next we show that \( \Pi \) is autoparallel at points of \( A^M \) if and only if \( V \) is autoparallel and shear-free on \( A^M \). To do this, take \( q \in A^M \). Then with \( U = J(\partial/\partial x_0) \), since \( \partial/\partial x_0 \) is parallel, (41) is equivalent to

\[
\nabla_{JX} U = J\nabla_X U \quad \text{for all} \quad X \in T_q \mathbb{R}^4_p, \tag{42}
\]

and so for \( v = \partial/\partial t + U \),

\[
\nabla_{x+iJx} v = 0 \quad \text{for all} \quad X \in T_q \mathbb{R}^4_p. \tag{43}
\]

Choosing \( X = \partial/\partial x_0 = i\partial/\partial t \) this reads

\[
\nabla_v v = 0
\]
which is the autoparallel condition (30); choosing instead $X$ in the screen space $V_q^\perp \cap \mathbb{R}_q^3 = U_q^\perp \cap \mathbb{R}_q^3$ gives the shear-free condition (33). Conversely, since the two choices of $X$ give vectors $X + iJX$ spanning $\Pi_q$, \{ (30) and (33) \} $\Rightarrow$ (43).

To finish the proof, note that if any of the above conditions holds at all points of $A^A$ or $A^M$, by analytic continuation it holds throughout $A^C$.

**Remark 3.12** The proof shows that the equivalent conditions 1) to 3) of the Proposition on quantities (D1)–(D3) are all equivalent to the condition (42) on their common direction vector field $U$ defined by (7).

Consider now the following (sets of) quantities:

(Q1) holomorphic foliations $\mathcal{F}$ by $\alpha$-planes (equivalently, autoparallel holomorphic distributions $\Pi$ of $\alpha$-planes) on an open subset $A^C$ of $\mathbb{C}^4$,

(Q2) positive Hermitian structures $J$ on an open subset $A^4$ of $\mathbb{R}_p^4$,

(Q3) real-analytic shear-free ray congruences $\ell$ (equivalently, autoparallel distributions $V$ of null directions) on an open subset $A^M$ of $\mathbb{M}_p^4$,

(Q4) real-analytic conformal foliations $\mathcal{C}$ by curves (equivalently, real analytic shear-free unit vector fields $U$) on an open subset $A^3$ of $\mathbb{R}_p^3$.

In the last section we showed that (germs of) the quantities (Q3) and (Q4) are equivalent in the $C^\infty$ category. We now show that all four quantities are equivalent in the $C^\omega$ category:

**Theorem 3.13** Let $\Pi \in (Q1)$. Then restriction to slices defines surjections $(Q1) \rightarrow (Q2)$, $(Q1) \rightarrow (Q3)$ and $(Q1) \rightarrow (Q4)$.

In fact, for a fixed open set $A^3$ of $\mathbb{R}_p^3$, these maps define bijections between germs at $A^3$ of quantities $(Q1)$, $(Q2)$, $(Q3)$ and $(Q4)$.

**Proof** This follows by combining Theorems 3.7 and 3.11.

Our results may be summarized by the commutative diagram where the arrows represent restrictions to slices:

\[
\begin{align*}
(Q1) &= \{ \text{holomorphic foliations } \mathcal{F} \text{ by } \alpha\text{-planes} \} \\
(Q2) &= \{ \text{positive Hermitian structures } J \text{ on } A^4 \} \\
(Q3) &= \{ \text{real-analytic shear-free ray congruences } \ell \text{ on } A^M \} \\
(Q4) &= \{ \text{real-analytic conformal foliations } \mathcal{C} \text{ by curves} \}
\end{align*}
\]

The inverses to the maps $(Q1) \rightarrow (Q2)$ and $(Q1) \rightarrow (Q3)$ are given by analytic continuation, the inverse to $(Q3) \rightarrow (Q4)$ is described in Theorem 3.7, for the inverse to $(Q2) \rightarrow (Q4)$, see below.

This last map expresses a particular consequence of Theorem 3.13 which will be central to our work, namely that any $C^\omega$ conformal foliation $U$ of an open subset of $\mathbb{R}_p^3$ comes from a positive Hermitian structure $J$ by projection. Precisely:

**Corollary 3.14** Let $p \in \mathbb{R}^4$ and let $A^3$ be an open subset of $\mathbb{R}_p^3$. Then projection $J \mapsto U = J(\partial/\partial x_0)$ onto the slice $\mathbb{R}_p^3$ defines a bijective correspondence between germs at $A^3$ of Hermitian structures $J$ defined on an open neighbourhood of $A^3$ in $\mathbb{R}_p^4$ and shear-free unit vector fields $U$ on $A^3$ (equivalently $C^\omega$ conformal foliations $\mathcal{C}$ by curves).
Remarks 3.15  1. As a consequence of the theorem, given any one of the quantities \( (Q1) \)–\( (Q4) \) we get all four distributions \( \Pi \in (Q1), J \in (Q2), V \in (Q3), U \in (Q4) \) defined on an open subset of \( \mathbb{C}^4 \) and related at each point by \( \bar{P} \) — we call these (or the foliations \( F, \ell \) or \( C \) to which they are tangent) extended quantities.

2. The correspondence \( (Q2) \to (Q4) \) in the Corollary is the “real analogue” of the correspondence \( (Q3) \to (Q4) \) of Theorem 3.7. However note that the latter applies in the \( C^\infty \) case too and has a inverse which is geometrically described. To describe the inverse of \( (Q2) \to (Q4) \) we must follow the route \( (Q4) \to (Q3) \to (Q1) \to (Q2) \): explicitly, given \( U \in (Q4) \), extend \( J = J(U) \) to an open subset of \( M^4 \) by insisting that it be constant along the null lines of the shear-free null geodesic congruence \( \ell \) which extends \( U \) \( \left( \text{[3.33]} \right) \), then extend to an open subset of \( \mathbb{C}^4 \) by analytic continuation (with respect to the standard complex structure on \( \mathbb{C}^4 \)) given by \( \bar{p} \), then restrict to \( R^4_p \).

3. The restriction \( F \in (Q1) \to \ell \in (Q3) \) can be described geometrically: the null lines of \( \ell \) are the intersections of the \( \alpha \)-planes of \( F \) with \( M^4_p \). Note however, that only those \( \alpha \)-planes represented by points of \( N^5_p \) (see \( \left( \text{[2.7]} \right) \)) have non-empty intersection with \( M^4_p \) so that the inverse \( (Q3) \to (Q1) \) is not completely described in this way.

4. The inverse of the map \( (Q1) \to (Q2), F \to J \) can be described in a purely geometrical way similar to the inverse of \( (Q3) \to (Q4) \) (see \( \left( \text{[3.3]} \right) \): given \( J \in (Q2) \), define \( \Pi = \Pi(J) \) on \( A^4 \) by \( \bar{F} \). Then the null planes of \( F \) are those tangent to the distribution \( \Pi \).

5. Given a real-analytic conformal foliation \( F \) by curves of an open subset \( A^R \) of \( R^3 \) by curves, there corresponds a 5-parameter family of real-analytic conformal foliations by curves of open subsets of \( R^3 \) got by extending \( F \) to a holomorphic foliation by \( \alpha \)-planes and then projecting this to \( R^3 \)-slices.

3.7  Other formulations of the equations and a Kerr theorem

Given any quantity \( (Q1) \)–\( (Q4) \) and so all four extended quantities \( \Pi, J, V, U \) related by \( \bar{p} \) writing \( U = \sigma^{-1}(\mu) \) where \( \mu = w_1/w_0 \in \mathbb{C} \cup \{\infty\} \), we see that \( \left( \text{[42]} \right) \) is equivalent to the equation

\[
\nabla_Z \mu = 0 \quad \text{for all} \quad Z \in \Pi_p .
\]

(46)

Recall that, in the coordinates \((z_1, \bar{z}_1, z_2, \bar{z}_2) \) (see \( \left( \text{[4]} \right) \)), a basis for \( \Pi = \Pi(J) \) is given by \( \left( \text{[3]} \right) \) or \( \left( \text{[14]} \right) \). Since \( \Pi \) is the \((0, 1)\)-tangent space of \( J = J(\Pi) \) it follows that \((z_1 - \mu \bar{z}_2, z_2 + \mu \bar{z}_1) \) are holomorphic with respect to \( J \) and, so, provide local complex coordinates for \( J \). Furthermore \( \left( \text{[14]} \right) \) reads

\[
\left( \frac{\partial}{\partial \bar{z}_2} - \mu \frac{\partial}{\partial z_2} \right) \mu = 0 , \quad \left( \frac{\partial}{\partial z_1} + \mu \frac{\partial}{\partial \bar{z}_1} \right) \mu = 0 .
\]

(47)

These equations restrict on real slices to the equations \( \left( \text{[3]} \right), \text{Eqns. (6.1)} \) or \( \left( \text{[2]} \right), \text{Eqns. (2.1)} \):

\[
\left( \frac{\partial}{\partial \bar{z}_2} - \mu \frac{\partial}{\partial z_2} \right) \mu = 0 , \quad \left( \frac{\partial}{\partial z_1} + \mu \frac{\partial}{\partial \bar{z}_1} \right) \mu = 0 ,
\]

(48)

and on Minkowski slices, writing \( v' = x_1 + t, v = x_1 - t \), to

\[
\left( \frac{\partial}{\partial v'} - \mu \frac{\partial}{\partial v_2} \right) \mu = 0 , \quad \left( \frac{\partial}{\partial v_2} + \mu \frac{\partial}{\partial v'} \right) \mu = 0 ,
\]

(49)

which are the equations for a shear-free ray congruence as given by \( \left( \text{[4]} \right), \left( \text{[22]} \right), \text{II (7.4.6)} \) or \( \left( \text{[13]} \right) \text{, p. 50} \) modulo conventions.

Now recall \( \left( \text{[2.7]} \right) \) that the set of affine \( \alpha \)-planes in \( \mathbb{C}^4 \) can be identified with \( \mathbb{C}P^3 \) so that a distribution \( \Pi \) of \( \alpha \)-planes may be represented by a map \( w : A^C \to \mathbb{C}P^3 \) with \( w(p) \in \hat{p} \) for all \( p \in A^C \). Then it is not hard to see that \( \left( \text{[42]} \right) \) is equivalent to

\[
\text{holomorphic part of } dw(Z) = 0 \quad \text{for all } p \in A^C \text{ and } Z \in \Pi_p .
\]

(50)

This formulation needs no coordinates and makes sense on the compactified space \( \bar{\mathbb{C}}^4 \) hence Proposition \( \left( \text{[3.14]} \right) \), Theorem \( \left( \text{[3.13]} \right) \) and Corollary \( \left( \text{[3.14]} \right) \) extend to compactified spaces. We thus have a unified Kerr Theorem for our quantities:
Proposition 3.16 Given any of the quantities (Q1)–(Q4), representing its extension by a holomorphic map \( w : A^C \to \mathbb{CP}^3 \) on an open subset of \( C^4 \) with \( w(p) \in \bar{p} \) for all \( p \in A \), there is a unique complex hypersurface \( S \) of \( \mathbb{CP}^3 \) which is the image of \( w \).

Conversely, given a complex hypersurface \( S \) of \( \mathbb{CP}^3 \), any holomorphic map \( w : A^C \to \mathbb{CP}^3 \) with \( w(p) \in \bar{p} \) and image in \( S \) defines related (extended) quantities (Q1)–(Q4).

Further, if \( S \) is given by

\[
\psi(w_0, w_1, w_2, w_3) = 0
\]

where \( \psi \) is homogeneous and holomorphic, then the direction vector field of the quantity away from points at infinity is given by \( U = \sigma^{-1}(iw_1/w_0) \) where \([w_0, w_1] = \mu(z_1, z_2, z_3)\) is a solution to

\[
\psi(w_0, w_1, w_0z_1 - w_1z_2, w_0z_2 + w_1z_1) = 0.
\]

Proof Given \( w \), set \( S = \text{the image of} \ w \). That \( S \) is a complex hypersurface follows from (50). Indeed, in local coordinates, if \([w_0, w_1] = \mu(z_1, z_2, z_3)\) represents the direction field of the quantity, then by (47), \([w_0, w_1]\) is a holomorphic function of \((z_1 - \mu z_2, z_2 + \mu z_1)\). The latter pair equals \((w_2, w_3)\) by the incidence relations \((10)\). Hence \([w_0, w_1]\) are holomorphic functions of \([w_2, w_3]\) and an equation of the form \((51)\) is satisfied explicitly that \( S \) is a complex hypersurface. Conversely, if \( S \) is a complex hypersurface, then it is locally of the form \((51)\) and reversing the above argument shows that \( \mu = w_1/w_0 \) satisfies (47) and so \( U = \sigma^{-1}(\mu) \in (Q4) \) giving related quantities \( \Pi \in (Q1) \), \( J \in (Q2) \) and \( V \in (Q3) \).

The surface \( S \) is called the twistor surface of the quantity.

Remark 3.17 The bijections (Q3) or (Q4) to (Q1) or (Q2) can be described twistorially as follows (taking \( p = 0 \) for simplicity): By \((8, 3)\), \( \ell \in (Q3) \) or \( C \in (Q4) \) defines a CR map with image a 3-dimensional CR submanifold \( N^3 \) of \( N^5 \); this is \((22)\) the intersection of a complex hypersurface \( S \) with \( N^5 \); \( S \) then determines \( J \in (Q1) \) and \( J \in (Q2) \) as above.

Note that if \( \ell \) or \( C \) is only \( C^\infty \) but has non-integrable horizontal spaces everywhere, then it can be extended to a Hermitian structure \( J \) on one side of \( \mathbb{R}^3 \); precisely, let \( R^2_+ \) (resp. \( R^2_- \)) be the set \( \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : x_0 > 0 \ (\text{resp.} < 0)\} \), then there exists a \( C^\infty \) Hermitian structure on an open subset \( A^4 \) of \( R^2_+ \cup R^2_- \) or \( R^2_+ \cup R^2_- \) with \( A^4 \cap R^3 = A^3 \) which is \( C^\infty \) on \( A^4 \setminus A^3 \). To see this note that the CR manifold \( N^3 \subset \mathbb{CP}^3 \) has non-zero Levi form \((22)\), II, p.220 ff. and so, by a theorem of Harvey and Lawson \((42)\), is the boundary of a complex hypersurface \( S \). This means that \( J \) can be defined on one side of \( \mathbb{R}^3 \).

4 Harmonic morphisms, null planes, and SFR congruences

In this section we define Minkowski and complex- harmonic morphisms and show that, except in trivial cases, their equations can be reduced to the equations \((45), (47)\) for shear-free ray (SFR) congruences or holomorphic foliations by \( \alpha \)-planes.

4.1 Harmonic morphisms on Euclidean and Minkowski space

For any (semi-)Riemannian manifolds \((M^m, g), (N^n, h)\), a harmonic morphism \( \phi : M^m \to N^n \) is a map which pulls back germs of harmonic functions on \( N^n \) to germs of harmonic functions on \( M^m \). By \((3, 4)\) for the Riemannian case and \((3, 5)\) for the semi-Riemannian case, these can be characterized as harmonic maps \( \phi \) which are horizontally weakly conformal, i.e. at each point \( p \in M \) either the differential \( d\phi_p = 0 \) or the restriction of \( d\phi_p \) to the horizontal space \( (ker d\phi_p)^\perp \) is conformal and surjective. In particular, a smooth map \( \phi : A^4 \to C \) from an open subset of Euclidean space \( R^4 \) is a harmonic morphism if it satisfies Laplace’s equation

\[
\Delta \phi = \frac{\partial^2 \phi}{\partial x_0^2} + \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} = 0
\]

and the horizontal weak conformality condition:

\[
\left( \frac{\partial \phi}{\partial x_0} \right)^2 + \left( \frac{\partial \phi}{\partial x_1} \right)^2 + \left( \frac{\partial \phi}{\partial x_2} \right)^2 + \left( \frac{\partial \phi}{\partial x_3} \right)^2 = 0
\]

(53)
whereas a smooth map $\phi : A^M \rightarrow \mathbb{C}$ from an open subset of Minkowski space $\mathbb{M}^4$ is a *Minkowski harmonic morphism*, i.e. a harmonic morphism with respect to the Minkowski metric, if and only if it satisfies the wave equation:

$$\Box \phi = -\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} = 0$$

and the horizontal weak conformality condition:

$$\left(\frac{\partial \phi}{\partial t}\right)^2 + \left(\frac{\partial \phi}{\partial x_1}\right)^2 + \left(\frac{\partial \phi}{\partial x_2}\right)^2 + \left(\frac{\partial \phi}{\partial x_3}\right)^2 = 0$$

at all points $(t, x_1, x_2, x_3) \in \mathbb{M}^4$, i.e. a Minkowski harmonic morphism is a “null” solution to the wave equation (cf. [8]). Note that both (53) and (55) express horizontal weak conformality with respect to any metric conformally equivalent to the standard metrics.

For a harmonic morphism between (semi-)Riemannian manifolds there are three types of point $p$:

a) $d\phi_p = 0$; we call such points *critical points*;

b) $d\phi_p \neq 0$ and $\ker d\phi_p$ is non-degenerate;

c) $d\phi_p \neq 0$ and $\ker d\phi_p$ is degenerate. Then $\ker (\ker d\phi_p)^{\perp} \subset \ker d\phi_p$.

If the last case occurs at some point $p$ we call $\phi$ *degenerate* (at $p$); a simple example of a harmonic morphism $\phi : \mathbb{M}^4 \rightarrow \mathbb{C}$ which is degenerate everywhere is given by $\phi(t, x_1, x_2, x_3) = t - x_1$ or, more generally, $f(t - x_1)$ for any smooth function $f : \mathbb{R} \rightarrow \mathbb{C}$. Note that this has 1-dimensional image. For more theory and examples, see, for example [8, 22] for the Riemannian case, and [19, 24] in the semi-Riemannian case. Note, in particular, that, in the Riemannian case, a smooth harmonic morphism is always real-analytic [7], but the above example shows that this is not so in the semi-Riemannian case.

The equations for a harmonic morphism to a 2-dimensional codomain are conformally invariant in the codomain, i.e. if $\phi : M^m \rightarrow N^2$ is a harmonic morphism to a 2-dimensional Riemannian manifold and $\rho : N^2 \rightarrow N'^2$ is a weakly conformal map to another 2-dimensional Riemannian manifold, the composition $\rho \circ \phi$ is a harmonic morphism. Thus the pair of equations (54,55) makes sense for a map $\phi : \mathbb{M}^4 \rightarrow N^2$ to a Riemann surface.

### 4.2 Complex-harmonic morphisms

By a complex-harmonic function $\phi : A^C \rightarrow \mathbb{C}$ on an open subset $A^C$ of $\mathbb{C}^4$ we mean a holomorphic map satisfying the complex Laplace’s equation:

$$\sum_{i=0}^{3} \frac{\partial^2 \phi}{\partial x_i^2} = 0, \quad (x_0, x_1, x_2, x_3) \in A^C.$$  \hspace{1cm} (56)

By a complex-harmonic morphism $\phi : A^C \rightarrow \mathbb{C}$ we mean a holomorphic map satisfying (56) and a complex version of the horizontal weak conformality condition:

$$\sum_{i=0}^{3} \left(\frac{\partial \phi}{\partial x_i}\right)^2 = 0, \quad (x_0, x_1, x_2, x_3) \in A^C.$$  \hspace{1cm} (57)

We may adapt the arguments of [8, 22] to characterize complex-harmonic morphisms $A^C \rightarrow \mathbb{C}$ as those holomorphic maps which pull back holomorphic functions to complex-harmonic functions. Note also that, since these equations are conformally invariant in $\phi$, we can replace $\mathbb{C}$ by any Riemann surface.

**Proposition 4.1** a) Let $\phi : A^C \rightarrow N^2$ be a complex-harmonic morphism from an open subset $A^C \subset \mathbb{C}^4$ to a Riemann surface. Then, for any $p \in A^C$,

1) $\phi | A^C \cap \mathbb{R}^4_+$ is a harmonic morphism (w.r.t. the Euclidean metric);
2) $\phi | A^C \cap \mathbb{M}^4_+$ is a harmonic morphism (w.r.t. the Minkowski metric);

b) All harmonic morphisms from open subsets of $\mathbb{R}^4$ to Riemann surfaces and real-analytic harmonic morphisms from open subsets of $\mathbb{M}^4$ to Riemann surfaces arise in this way.
Proof
a) Immediate from the equations.
b) As noted above, any harmonic morphism from an open subset of $\mathbb{R}^4$ to a Riemann surface is $C^\omega$.
By analytic continuation, this is the restriction of a holomorphic map on an open subset of $\mathbb{C}^4$ and this holomorphic map is complex-harmonic. The $M^4$ case is similar.

An important example of a harmonic morphism is the following:

**Proposition 4.2** Let $\mu : A \to \mathbb{C} \cup \{\infty\}$ where $A$ is an open subset of $\mathbb{C}^4$ (resp. $\mathbb{R}^4$, $M^4$) satisfy equations (47) (resp. (44), (43)). Then $\mu$ is a complex-harmonic morphism (resp. harmonic morphism, Minkowski harmonic morphism).

**Proof** We give the proof for the complex case only. To establish horizontal weak conformality (51) we have, from (47),

$$\frac{\partial \mu}{\partial z_1} \frac{\partial \mu}{\partial z_2} + \frac{\partial \mu}{\partial z_1} \frac{\partial \mu}{\partial \bar{z}_2} = \frac{\partial \mu}{\partial z_2} \frac{\partial \mu}{\partial \bar{z}_1} - \mu \frac{\partial \mu}{\partial z_1} \frac{\partial \mu}{\partial \bar{z}_2} = 0;$$

complex-harmonicity (50) follows from

$$\frac{\partial^2 \mu}{\partial z_1 \partial \bar{z}_2} + \frac{\partial^2 \mu}{\partial z_2 \partial \bar{z}_1} = \frac{\partial}{\partial z_1} \left( \mu \frac{\partial \mu}{\partial z_2} \right) - \frac{\partial}{\partial z_2} \left( \mu \frac{\partial \mu}{\partial \bar{z}_1} \right) = 0.$$

Thus the direction field $U$ (or its representative $\mu = -i \sigma(U)$) of any of our quantities (Q1) – (Q4) in (44) gives a harmonic morphism; further examples are provided by postcomposition of one of these with a holomorphic or antiholomorphic map to a Riemann surface. We shall see that, away from critical points, this gives all non-trivial harmonic morphisms locally. Firstly, recall

**Theorem 4.3** Let $\phi : A^R \to N^2$ be a harmonic morphism without critical points from an open subset $A^R$ of $\mathbb{R}^4$ to a Riemann surface. Then there exists a Hermitian structure on $A^R$ such that $J$ is parallel along each connected component of a fibre of $\phi$. Further, for any $p \in A^R$, there is a neighbourhood $A_1^R$ of $p$ in $A^R$, and a holomorphic map $p : V \to \mathbb{C} \cup \{\infty\}$ from an open subset $V$ of $N^2$ such that $\mu = p \circ \phi$ represents (32, 23) $J$.

We have the following analogue for complex-harmonic morphisms:

**Theorem 4.4** Let $\phi : A^C \to N^2$ be a complex-harmonic morphism without critical points from an open subset $A^C$ of $\mathbb{C}^4$ to a Riemann surface. Then there exists a holomorphic foliation $F$ of $A^C$ by $\alpha$-planes or by $\beta$-planes such that each connected component of the fibres of $\phi$ is the union of parallel null planes of $F$. Further, for any $p \in A^C$ there is a neighbourhood $A_1^C$ of $p$ in $A^C$, and a holomorphic map $p : V \to \mathbb{C} \cup \{\infty\}$ from an open subset $V$ of $N^2$ such that $\mu = p \circ \phi$ represents (32, 23) the direction field of $F$.

**Proof** Let $p \in A^C$. By Proposition (14), $\phi$ restricts to a harmonic morphism on $A^4 = A^C \cap \mathbb{R}^4$ which is easily seen to be submersive. By (31), $\phi|A^4$ is holomorphic with respect to some Hermitian structure $J$ which is constant along each connected component of a fibre of $\phi$. Replacing $z_2$ by $\bar{z}_2$ if necessary, we can assume that $J$ is positively oriented. Representing $J$ by $\mu : A^4 \to \mathbb{C} \cup \{\infty\}$, we have

$$\frac{\partial \phi}{\partial z_1} - \mu \frac{\partial \phi}{\partial z_2} = 0, \quad \frac{\partial \phi}{\partial \bar{z}_2} + \mu \frac{\partial \phi}{\partial \bar{z}_1} = 0,$$

(58)

$$\frac{\partial \mu}{\partial z_1} - \mu \frac{\partial \mu}{\partial z_2} = 0, \quad \frac{\partial \mu}{\partial \bar{z}_2} + \mu \frac{\partial \mu}{\partial \bar{z}_1} = 0,$$

(59)

at points of $A^4$, the first equation expressing holomorphicity of $\phi$ with respect to $J$ (cf. (26)) and the second, integrability of $J$ (cf. (18)). Extend $\mu$ to $A^C$ by (38), noting that this well-defines $\mu$ since not all the partial derivatives of $\phi$ can vanish simultaneously; $\mu$ is then a holomorphic function. By analytic continuation, (59) holds at all points of $A^C$ so that $\mu$ defines a holomorphic foliation $F$ by $\alpha$-planes. By (38) $\phi$ is constant along any $\alpha$-plane of $F$ so that each fibre of $\phi$ is the union of $\alpha$-planes of $F$. Further, $J$ and $\mu|A^4$ is constant along the connected components of fibres of $\phi|A^4$; by analytic continuation,
\( \mu : A^C \to \mathbb{C} \cup \{\infty\} \) is constant along the connected components of the fibres of \( \phi : A^C \to \mathbb{C} \cup \{\infty\} \) so that the \( a \)-planes of \( F \) making up a connected component of a fibre of \( \phi \) are all parallel.

Lastly, since \( \mu \) is constant on the leaves of the foliation given by the fibres of \( \phi \), it factors through local leaf spaces as \( \mu = \rho \circ \phi \). Since \( \phi \) and \( \mu \) are both holomorphic (with respect to \( i \) and \( J \)), \( \rho \) must be holomorphic.

We have an analogue for Minkowski harmonic morphisms:

**Theorem 4.5** Let \( \phi : A^M \to N^2 \) be a real-analytic Minkowski harmonic morphism without critical points from an open subset \( A^M \) of \( M^4 \) to a Riemann surface. Then there is a shear-free ray congruence \( \ell \) on \( A^M \) such that each connected component of a fibre of \( \phi \) is the union of parallel null lines of \( \ell \). Further, for any \( p \in A^C \) there is a neighbourhood \( A^M_p \) of \( p \) in \( A^M \), and a holomorphic map \( \rho : V \to \mathbb{C} \cup \{\infty\} \) from an open subset \( V \) of \( N^2 \) such that \( \mu = \rho \circ \phi \) represents the direction field of \( \ell \).

**Proof** Extend \( \phi \) to an open subset \( A^C \) of \( C^4 \) with \( A^C \cap M^4 = A^M \). Let \( F \) be the holomorphic foliation by null planes of Theorem 4.4. For each \( q \in A^M \), set \( \ell_q = \) the intersection of the leaf of \( F \) through \( q \) with \( M^4 \). By Theorem 3.6, \( \ell \) is shear-free and is the desired ray congruence. Lastly, \( \rho \) is given as before.

Note that in the real and complex cases the condition “without critical points” is equivalent to “\( \phi \) is submersive”. This is not so in the Minkowski case where \( \phi \) may be degenerate. We can actually be more precise in this case:

**Corollary 4.6** Let \( \phi : A^M \to N \) be a real-analytic harmonic morphism from an open subset \( A^M \) of \( M^4 \). Suppose that \( \phi \) is degenerate at \( p \) with \( d\phi_p \neq 0 \). Then there exists a unique null direction \( V_p \in T_p M^4 \) such that \( V_p \subset \ker d\phi_p \). Furthermore, \( \ker d\phi_p = V_p^\perp \). If, further, at each point \( q \) in the connected component of the fibre through \( p \), \( \phi \) is degenerate with \( d\phi_q \neq 0 \), then that connected component is the affine null 3-space tangent to \( V_p^\perp \).

**Proof** By [1], \( (\ker d\phi_p)^\perp \subset \ker d\phi_p \). This means that \( \ker d\phi_p \) must be three-dimensional. But then \( (\ker d\phi_p)^\perp \) is 1-dimensional and null, say \( (\ker d\phi_p)^\perp = V_p \). So \( \ker d\phi_p = V_p^\perp \).

To prove uniqueness of \( V_p \), suppose that \( V_p' \subset \ker d\phi_p \) is another null direction. Then \( V_p' \subset V_p^\perp \) which is easily seen to imply that \( V_p' = V_p \).

This means that the distribution \( p \mapsto V_p \) must be tangent to the SFR congruence of the theorem, and so each \( V_p \) is parallel for all \( p \) in a connected component of a fibre. The last assertion follows from the fact that the connected component of the fibre is 3-dimensional and has every tangent space parallel to \( V_p \).

**Remark 4.7** If \( \phi \) is non-degenerate at \( p \), the kernel of \( d\phi_p \) has Minkowski signature and so contains two null lines. If the fibre at \( p \) is totally geodesic, then it is the affine Minkowski plane tangent to that kernel. If all the fibres are totally geodesic, then the two null lines both give SFR congruences \( \ell \) as in the theorem; if, however, at least one fibre is not totally geodesic, the SFR congruence \( \ell \) of the theorem is unique. Similar remarks hold in the \( \mathbf{R}^4 \) and \( \mathbf{M}^4 \) cases.

Call a smooth map from an open subset \( A^4 \) of \( \mathbf{R}^4 \) to a Riemann surface \( \mathbf{Kähler} \) if it is holomorphic with respect to a Kähler structure on \( A^4 \), such maps are always harmonic morphisms [3]. Analogously, call a smooth map \( \phi \) from an open subset \( A^C \) of \( C^4 \) (resp. \( \mathbf{M}^4 \)) to a Riemann surface \( \mathbf{Kähler} \) if the restriction of \( \phi \) (resp. of the analytic continuation of \( \phi \)) to real slices is Kähler. Again such maps are always complex- (resp. Minkowski) harmonic morphisms. Clearly, all such maps are explicitly known. If the \( \mu \) in any of the last three theorems is constant, then \( \phi \) is Kähler (the converse is true unless \( \phi \) has totally geodesic fibres in which case it can be holomorphic w.r.t. a Kähler structure of one orientation and also w.r.t. a non-Kähler Hermitian structure of the other orientation, Example 4.9 (4) is of this type). Then we have a converse to Proposition 4.2.
Corollary 4.8 Let \( \phi : A \to N^2 \) be a non-Kähler harmonic (resp. complex-harmonic, real-analytic Minkowski harmonic) morphism from an open subset of \( \mathbb{R}^4 \) (resp. \( \mathbb{C}^4, \mathbb{M}^4 \)) to a Riemann surface. Then, for any \( p \in A \) with \( d\phi_p \neq 0 \), there is a neighbourhood \( A_1 \) of \( p \) and a non-constant holomorphic map \( \rho : V \to \mathbb{C} \cup \{\infty\} \) from an open subset of \( N^2 \) such that \( \rho = \rho \circ \phi \) satisfies Equations (48) (resp. (47), (49)).

Thus, except in the trivial Kähler case, the harmonic morphism equations can be reduced to the first order equations (48), (47) or (49).

Examples 4.9 1) The simplest examples of Minkowski harmonic morphisms from \( \mathbb{M}^4 \) to \( \mathbb{C} \) are given by a) \( (t, x_1, x_2, x_3) \mapsto x_2 + ix_3 \) which is non-degenerate everywhere and surjective and b) \( (t, x_1, x_2, x_3) \mapsto x_1 - t \) which is degenerate everywhere and has 1-dimensional image \( \mathbb{R} \). Note that in both cases, the fibres are totally geodesic; in a) the SFR congruences of Theorem 4.3 (two by Remark 4.7) have leaves with (null) directions \((1, \pm 1, 0, 0)\), in b) the SFR congruence (just one by Corollary 4.6) has leaves with (null) direction \((1, 1, 0, 0)\).

2) Consider the harmonic morphism \( \phi : \mathbb{M}^4 \to \mathbb{R}^2 \) given by \( (t, x_1, x_2, x_3) \mapsto (x_1 - t + x_2^2 - x_3^2, 2x_2x_3) \). For any real \( c \), the fibre \( \phi = c \) is the union of two surfaces intersecting on the line \( x_1 = t, x_2 = x_3 = 0 \); that line consists of degenerate points; other points on the fibre are non-degenerate.

3) The direction field \( U \) of the SFR extending the simple conformal foliation of Example 4.4 is given by (48). It defines a Minkowski harmonic morphism from the cone \( A^M = \{(t, x_1, x_2, x_3) \in \mathbb{M}^4 : x_2^2 + x_3^2 > t^2 \} \) to \( S^2 \) which is degenerate everywhere and has image the equator of \( S^2 \). Note that \( U = \sigma(\tilde{t}) \) with

\[
\mu(t, x_1, x_2, x_3) = \frac{r}{x_2^2 + x_3^2} \left( x_2 + \frac{t}{r}x_2 \right) + i \left( x_3 - \frac{t}{r}x_2 \right)
\]

where \( r = \sqrt{x_2^2 + x_3^2 - t^2} \) so that \( \mu \) is a harmonic morphism \( \mu : \mathbb{M}^4 \to \mathbb{C} \), also degenerate, with image the unit circle. The fibre of \( U \) or \( \mu \) through any point \( p \) is the affine plane perpendicular to \( U_p \) tangent to \( U_p \), \( \partial/\partial x_1 \) and the vector in the \( (x_2, x_3) \)-plane perpendicular to \( U_p \).

4) Define \( \mu : M^4 \setminus \{(t, x_1, x_2, x_3) : x_2 = x_3 = x_1 + t = 0\} \to \mathbb{C} \cup \{\infty\} \) by

\[
\mu = \frac{x_2 + ix_3}{t(x_1 + t)}.
\]

This defines an SFR and so a Minkowski harmonic morphism. It is non-degenerate except at points on the 3-plane \( \{(t, x_1, x_2, x_3) : x_1 + t = 0\} \).

5) Take the twistor surface \( S = \{[w_0, w_1, w_2, w_3] \in \mathbb{C}P^3 : w_0w_1 + w_2w_3 = 0\} \). Then \( \mu \) satisfies

\[
\mu + (z_1 - \mu \bar{z}_2)(z_2 + \mu \bar{z}_1) = 0
\]

and any local smooth solution gives the direction field of an SFR and thus a Minkowski harmonic morphism on an open subset of \( \mathbb{M}^4 \). It is clear that this harmonic morphism does not have totally geodesic fibres.

Note that in Examples 3) to 5) the \( \rho \) of Theorem 4.5 is the identity map whereas for Examples 1) and 2) it is the constant map these examples being Kähler.

5 Finding conformal foliations

5.1 The boundary of a hyperbolic harmonic morphism

According to Corollary 4.14, given a positive Hermitian structure \( J \) on an open subset \( A^4 \) of \( \mathbb{R}^4 \), if \( U \) is its direction vector field, i.e. \( U = J(\partial/\partial x_0) \), then, for any \( p \in A^4 \), \( U|A^3 \) is the tangent vector field of a \( C^\omega \) conformal foliation on the open set \( A^3 = A^4 \cap \mathbb{R}_p^3 \) and all \( C^\omega \) conformal foliations by curves of open subsets of \( \mathbb{R}^3 \) arise this way. To get an explicit description of all conformal foliations we need to find the horizontally conformal functions \( f \) on \( \mathbb{R}^3 \) whose level sets give the leaves of the foliation. To this end note the
**Proposition 5.1** (i) Let \( \phi : A^4 \rightarrow C \) be a submersive map from an open subset of \( R^4 \) which is holomorphic with respect to a positive Hermitian structure \( J \) on \( A^4 \). Let \( p \in A^4 \), \( U = J(\partial/\partial x_0) \) be the corresponding shear-free vector field on the open subset \( A^4 = A^4 \cap R^4_0 \) of \( R^4_0 \) and \( C \) the conformal foliation of \( A^4 \) given by its integral curves. If

\[
\frac{\partial \phi}{\partial x_0} = 0 \text{ on } A^3
\]

then \( f = \phi | A^3 \) is a real-analytic horizontally conformal submersion which is constant on the leaves of \( C \).

(ii) All real-analytic horizontally conformal submersions (and so, all real-analytic conformal foliations by curves) on open subsets of \( R^3 \) are in this way.

**Proof** (i) Let \( q \in A^3 \). By holomorphicity, since \( U_q = J_q(\partial/\partial x_0) \), the directional derivative \( U_q(f) = 0 \) so that \( f \) is constant on the leaves of \( C \). If \( \{e_2, e_3 = Je_2\} \) is a basis for \( U_q^1 \cap R^4_0 \), holomorphicity of \( \phi \) implies that \( e_3(f) = ie_2(f) \) so that \( f \) is horizontally conformal. Submersivity of \( f \) easily follows from that of \( \phi \).

(ii) Given a real-analytic horizontally conformal submersion \( f : A^3 \rightarrow C \) on an open subset of \( R^3 \), let \( U \) be the unit vector field tangent to its level curves. By Corollary 3.14, there is an open subset \( A^4 \) of \( R^4 \) with \( A^3 = A^4 \cap R^3 \) and a positive Hermitian structure \( J \) on \( A^4 \) such that \( U = J(\partial/\partial x_0) \) on \( A^3 \). Then \( f \) is CR with respect to the hypersurface structure on \( A^3 \) induced by \( J \) and so may be extended to a function \( \phi \) holomorphic with respect to \( J \) on a subset \( A^4_1 \) of \( A^4 \) with \( A^3 = A^4_1 \cap R^3 \). Then, since at points of \( A^3 \), \( U = J(\partial/\partial x_0) \) and \( U(\phi) = 0 \), we have \( \partial \phi/\partial x_0 = 0 \) on \( A^3 \) establishing (ii).

**Remark 5.2** The extension of \( f \) to \( \phi \) can be described more geometrically in the same way as the extension of \( U \) to \( J \) (see Remarks 3.13), namely: Extend \( f \) to an open subset of \( M^4 \) by insisting that it be constant along the null lines of the shear-free null geodesic congruence \( \ell \) which extends \( U \) (see 3.3), then extend to an open subset of \( C^4 \) by analytic continuation (with respect to the standard complex structure on \( C^4 \) given by multiplication by \( i \)) and restrict to \( R^3 \). The extension of \( f \) can also be described in a twistorial way as follows: As in Remark 3.17, \( U \) defines a 3-dimensional CR submanifold \( N^3 \) of \( N^5 \) which is the intersection of \( N^5 \) and a complex hypersurface \( S \) of \( CP^3 \) — the twistor surface of \( J \). Then \( f \) defines a CR function \( F = f \circ \pi \) on \( N^3 \) which we may extend to a holomorphic function \( \Phi \) on a neighbourhood of \( N^3 \) in \( S \); on a possibly smaller neighbourhood, \( \Phi \) is of the form \( \phi \circ \pi \) for a unique function \( \phi \) on an open subset of \( R^4 \). This is the desired extension.

We now show how to find such functions \( \phi \) using harmonic morphisms. Equip \( \tilde{R}^4 \equiv R^4 \setminus R^3 \) with the hyperbolic metric \( g^H = \left( \sum_{i=0}^{3} dx_i^2 \right) / x_0^2 \) so that each component \( R^4_0 \), \( R^1 \) is isometric to hyperbolic 4-space \( H^4 \). Let \( \tilde{A}^4 \) be an open subset of \( \tilde{R}^4 \), then we can call a smooth map \( \phi : \tilde{A}^4 \rightarrow C \) a **hyberbolic harmonic map** if it is a harmonic map with respect to the hyperbolic metric \( g^H \). This holds if and only if

\[
\sum_{i=0}^{4} \frac{\partial^2 \phi}{\partial x_i^2} - 2 \frac{\partial \phi}{\partial x_0} = 0
\]

at all points of \( \tilde{A}^4 \).

Similarly, \( \pi : \tilde{A}^4 \rightarrow C \) will be called a **hyberbolic harmonic morphism** if it is a harmonic morphism with respect to the metric \( g^H \), such maps are characterized as satisfying (61) and (63).

Recall (38), see also (3)

**Theorem 5.3** (i) Any submersive hyperbolic harmonic morphism \( \phi : \tilde{A}^4 \rightarrow C \) is holomorphic with respect to some Hermitian structure \( J \) on \( \tilde{A}^4 \) and has superminiml fibres with respect to \( J \), i.e. \( \ker d \phi \subset \ker \nabla^H J \) on \( \tilde{A}^4 \) where \( \nabla^H \) is the Levi-Civita connection of the hyperbolic metric on \( \tilde{R}^4 \).

(ii) Conversely, let \( J \) be a Hermitian structure on an open subset \( \tilde{A}^4 \) of \( \tilde{R}^4 \), and \( \phi : \tilde{A}^4 \rightarrow C \) a non-constant map which is holomorphic with respect to \( J \), then \( \phi \) is hyperbolic harmonic if and only if, at points where \( d \phi \neq 0 \), its fibres are superminimal with respect to \( J \).
To formulate this analytically, let $\Theta$ be the homogeneous holomorphic contact form
\[
\Theta = w_1 dw_2 - w_2 dw_1 - w_0 dw_3 + w_3 dw_0
\] on $\mathbb{C}P^3$. Then $\ker \Theta$ gives the horizontal spaces of the restriction of the twistor map $\pi : \mathbb{C}P^3 \setminus N^5 \to (\mathbb{R}^4, g^H)$. Set $\Phi = \phi \circ \pi$. Then $\phi$ is a hyperbolic harmonic morphism if and only if
\[
\ker d\Phi \subset \ker \Theta.
\]
Equivalently, let $w : \tilde{A}^4 \to \mathbb{C}P^3$ be the section of $\pi$ corresponding to $J$, i.e. with image the twistor surface $S$ of $J$ (see §3.2); then we may pull back $\Theta$ to a 1-form $\theta = w^*(\Theta)$ on $\tilde{A}^4$. Condition (35) now reads $\ker d\phi \subset \ker \theta$, this condition ensuring that the fibres of $\phi$ are superminimal.

Note that if $A^4$ is an open subset of $\mathbb{R}^4$ or of $\mathbb{R}^2_+ \cup \mathbb{R}^3$ (or $\mathbb{R}^2_- \cup \mathbb{R}^3$) with $A^4 \cup \mathbb{R}^3$ non-empty, any $(C^2, \text{say})$ map which is a hyperbolic harmonic map on $\tilde{A}^4 = A^4 \setminus \mathbb{R}^3$ satisfies (60) and so (63) at the boundary $A^3 = A^4 \cup \mathbb{R}^3$. A key property for us is the following converse:

**Proposition 5.4** Let $A^4$ be a connected open subset of $\mathbb{R}^4$, $\mathbb{R}^2_+ \cup \mathbb{R}^3$ or $\mathbb{R}^2_- \cup \mathbb{R}^3$ with $A^3 = A^4 \cap \mathbb{R}^3$ non-empty, and let $\phi : A^4 \to C$ be a non-constant $C^1$ map which is holomorphic with respect to a Hermitian structure $J$ on $\tilde{A}^4 = A^4 \setminus \mathbb{R}^3$ and submersive at almost all points of $A^3$. Then $\phi$ satisfies (60) on $A^3$ if and only if $\phi|A^4$ is a hyperbolic harmonic morphism.

**Proof** It suffices to work at points where $\phi$ is submersive. At such points note that (60) holds if and only if $\ker \phi = \text{span}(\partial/\partial x_0, J\partial/\partial x_0)$. Now let $S$ be the twistor surface of $J$ and let $\Phi : S \to C$ be defined by $\Phi = \phi \circ \pi$.

We have the

**Lemma 5.5** The pull-back $\theta = w^*\Theta$ to $A^4$ satisfies $\text{span}(\partial/\partial x_0, J\partial/\partial x_0) \subset \ker \theta$ at all points of $A^3$.

**Proof** Since the (complexified) normal to $\mathbb{R}^3$ is given by the annihilator of $\text{span}(dz_1 - d\bar{z}_1, dz_2, d\bar{z}_2)$, it suffices to show that on $\mathbb{R}^3 = \{z_1 + \bar{z}_1 = 0\}$, $\theta$ is a linear combination of those three forms. To do this, taking differentials in (58) gives
\[
\begin{align*}
&dw_2 = z_1 dw_0 + w_0 dz_1 - \bar{z}_2 dw_1 - w_1 d\bar{z}_2, \\
&dw_3 = z_1 dw_0 + w_0 dz_1 - \bar{z}_2 dw_1 - w_1 d\bar{z}_2.
\end{align*}
\]
Substituting these into (62) and rearranging gives
\[
\theta = (w_3 - w_0 z_2 + w_1 z_1) dw_0 - (w_2 + w_0 \bar{z}_1 + w_1 \bar{z}_2) dw_1 + w_0 w_1 (dz_1 - d\bar{z}_1) - w_0^2 d\bar{z}_1 - w_1^2 d\bar{z}_2.
\]
But, by (60), the coefficients of $dw_0$ and $dw_1$ vanish when $z_1 + \bar{z}_1 = 0$ and the lemma follows.

By the lemma the condition (60) is equivalent to the superminimality of the fibres of $\phi$ at points of $A^3$, viz.: $\ker d\phi \subset \ker \theta$, or, equivalently $\ker d\Phi \subset \ker \Theta$ on the real hypersurface $N^3 = w(A^3)$ of $S$. But this is a holomorphic condition, so by analytic continuation, if $\phi$ has superminimal fibres at points of $A^3$ then it has superminimal fibres on the whole of $A^4$ and we are done.

Combining Propositions 5.1 and 5.4 we obtain the basis of our method:

**Theorem 5.6** Let $f : A^3 \to \mathbb{C}$ be a real-analytic horizontally conformal submersion on an open subset of $\mathbb{R}^3$. Then there is an open subset $A^4$ of $\mathbb{R}^4$ with $A^4 \cap \mathbb{R}^3 = A^3$ and a real-analytic submersion $\phi : A^4 \to \mathbb{C}$ with $\phi|A^3 = f$ such that $\phi|A^4 \setminus \mathbb{R}^3$ is a hyperbolic harmonic morphism. In fact $\phi \mapsto f = \phi|A^3$ defines a bijective correspondence between germs at $A^3$ of real-analytic submersions $\phi : A^4 \to \mathbb{C}$ on open neighbourhoods of $A^3$ in $\mathbb{R}^4$ which are hyperbolic harmonic on $A^4 \setminus \mathbb{R}^3$ and real-analytic horizontally conformal submersions $f : A^3 \to \mathbb{C}$. 

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Remarks 5.7 1. The hyperbolic harmonic morphism \( \phi \) has totally geodesic fibres if and only if the level sets of \( f \) are circles, see [3].

2. Since the map \( \phi \) is found geometrically (Remark 5.2), there is no problem with Kähler points as there was in [6], [7].

3. If \( f \) is only \( C^\infty \), then, as in Remark 3.13, if the distribution \( \text{span}\{\text{grad}f_1, \text{grad}f_2\} \) is nowhere integrable, we can extend \( f \) to one side of \( \mathbb{R}^4 \); precisely, there is an open subset \( A^4 \) of \( \mathbb{R}^4 \setminus \mathbb{R}^3 \) or \( \mathbb{R}^4 \setminus \mathbb{R}^3 \) with \( A^4 \cap \mathbb{R}^3 = \mathbb{A}^3 \) and a \( C^\infty \) map \( \phi : A^4 \to \mathbb{C} \) with \( \phi|A^3 = f \) such that \( \phi|A^4 \setminus \mathbb{R}^3 \) is a hyperbolic harmonic morphism.

Corollary 5.8 Let \( c \) be a embedded real-analytic curve in \( \mathbb{R}^3 \). Then there is an embedded real-analytic surface \( s \) in an open subset \( A^4 \) of \( \mathbb{R}^4 \) which is minimal on \( A^4 \setminus \mathbb{R}^3 \) with respect to the hyperbolic metric, hits \( \mathbb{R}^3 \) orthogonally and has \( s \cap \mathbb{R}^3 = c \). In fact there is a conformal real-analytic foliation of an open subset \( A^4 \) of \( \mathbb{R}^4 \) by such surfaces with \( s \) a leaf.

Proof Firstly, \( c \) can be embedded in a conformal foliation by curves of an open subset of \( \mathbb{R}^3 \) as follows: construct the normal planes to \( c \) and integrate the vector field given by the normals to these. This gives a foliation on an open neighbourhood of \( c \) in \( \mathbb{R}^3 \) which has totally geodesic integrable horizontal spaces and so (see, for example, [2]) is Riemannian. (To get a conformal foliation which is not Riemannian, replace the planes by spheres, possibly of varying radii.) Representing this foliation as the level curves of a real-analytic horizontally conformal submersion \( f : \mathbb{A}^3 \to \mathbb{C} \) on an open subset of \( \mathbb{R}^3 \), construct a hyperbolic harmonic morphism \( \phi \) as in the theorem: its fibres give the desired real-analytic foliation.

Remark 5.9 It is easily seen from the equations that any \( C^1 \) surface in an open subset \( A^4 \) of \( \mathbb{R}^4 \), \( \mathbb{R}^4 \setminus \mathbb{R}^3 \) or \( \mathbb{R}^4 \setminus \mathbb{R}^3 \) which is minimal with respect to the hyperbolic metric on \( A^4 = A^4 \setminus \mathbb{R}^3 \) hits \( \mathbb{R}^3 \) orthogonally.

5.2 Finding horizontally conformal functions from twistor surfaces

We now show how to use Theorem 5.3 to find explicitly the horizontally conformal submersion and conformal foliation by curves on an open subset of \( \mathbb{R}^3 \) corresponding to a given complex hypersurface \( S \) of \( \mathbb{C}P^3 \). In fact, by introducing a parameter \( a \in \mathbb{C}^4 \), we can obtain a 5-parameter family of horizontally conformal submersions whose level sets give the 5-parameter family of conformal foliations of open subsets of \( \mathbb{R}^3 \) of in Remarks 3.13. For this we need to translate the hyperbolic metric to different slices: First recall ([2]) the map \( \pi_a : \mathbb{C}P^3 \to \mathbb{R}^4_a \) defined by \( w \mapsto \) the intersection of the \( \alpha \)-plane \([\overline{\alpha}]\) determined by \( w \) with \( \mathbb{R}^4_a \). Then, denoting the first component of \( a \) by \( a_0 \), we have

**Lemma 5.10** Let \( a \in \mathbb{C}^4 \). Equip \( \hat{\mathbb{R}}^4_a = \mathbb{R}^4_a \setminus \mathbb{R}^3_a \) with the hyperbolic metric \( g^H_a = \left( \sum_{i=0}^3 dx_i^2 \right) / (x_0 - \Re a_0)^2 \) and let \( N^5_a = \pi_a^{-1}(\mathbb{R}^4_a) \). Then the kernel of the holomorphic contact form

\[
\Theta_a = \Theta_{a_0} = -2a_0(w_1dw_0 - w_0dw_1) + w_1dw_2 - w_2dw_1 - w_0dw_3 + w_3dw_0
\]

restricted to \( \mathbb{C}P^3 \setminus N^5_a \) gives the horizontal distribution of \( \pi_a : \mathbb{C}P^3 \setminus N^5_a \to (\hat{\mathbb{R}}^4_a, g^H_a) \).

**Proof** Writing \( a \) in coordinates \([\overline{\alpha}]\) as \( (a_1, \overline{a}_1, a_2, \overline{a}_2) \), it can easily be checked that translation \( T_a : x \mapsto x + a \) in \( \mathbb{C}^4 \) corresponds to the map \( T_a : \mathbb{C}P^3 \to \mathbb{C}P^3 \) given by \( [w_0, w_1, w_2, w_3] \mapsto [w_0, w_1, w_2 + a_1z_0 - \overline{a}_2z_1, z_0 + a_2z_0 + \overline{a}_1z_1] \), that is, \( \pi_a \circ T_a = T_a \circ \pi_a \). Then \( \Theta_a = (T^{-1}_a) \circ \Theta = \Theta_{T^{-1}_a} \circ \Theta \); on calculating this, \([\overline{\alpha}]\) follows.

Now let \( S \subset \mathbb{C}P^3 \) be a given complex hypersurface and \( a \in \mathbb{C}^4 \). Let \( U \) be an open set in \( S \) such that \( \pi_a \) maps \( U \) diffeomorphically onto an open set \( A^4 \) of \( \mathbb{R}^4_a \). Then \( U \) defines a Hermitian structure \( J \) on \( A^4 \) represented by the section \( \psi : A^4 \to U \) of \( \pi_a \). If \( S \) is given by

\[
\psi(w_0, w_1, w_2, w_3) = 0
\]

(65)
(where ψ is homogeneous holomorphic) w is given by solving \(\Box\). Given a holomorphic map \(ζ : U \to C \cup \{∞\}\), set \(φ_a = ζ \circ w : A^4 \to C \cup \{∞\}\). Then \(φ_a\) is holomorphic with respect to \(J\) and so, by Theorem \[5.3\], is a harmonic morphism with respect to the hyperbolic metric on \(R^4_a\) if and only if the level surfaces of \(ζ\) are horizontal, i.e. tangent to \(ker Θ_a\).

Set \(Σ^S_\sigma = \{w ∈ S : ker Θ_a = TS\}\). Then on \(S \setminus Σ^S_\sigma\), \(ker Θ_a \cap TS\) is a one-dimensional holomorphic distribution so that its integral (complex) curves foliate \(S \setminus Σ^S_\sigma\). Note that \(π_a(Σ^S_\sigma) \cap A^4\) is the set of Kähler points of \(J = \{p ∈ A^4 : ∇^H p = 0 \forall v ∈ T_p A^4\}\) where \(∇^H\) is the Levi-Civita connection of the hyperbolic metric on \(R^4_a\).

To find \(φ_a\) we proceed as follows: Firstly let \(c : V \to C^2, [w_0, w_1, w_2, w_3] \mapsto (ζ, η)\) be complex coordinates for \(S\) on an open subset \(V\) of \(U\). Then we can solve \([16]\) locally to find the composition \(c \circ w, p \mapsto (ζ(p), η(p))\).

Next let \(ζ = ˜ζ(ζ, η)\) be a holomorphic function with \(∂ ˜ζ/∂ζ ≠ 0\) and set \(η = ˜η\). Then \((ζ, η) \mapsto (˜ζ, ˜η)\) is locally a complex analytic diffeomorphism. Then, \(φ_a(p) = ˜ζ(ζ(p), η(p))\) restricts to a hyperbolic harmonic morphism on \(R^4_a\) if and only if the level sets of \(ζ\) are superminimal, the condition for this is

\[Θ_a \left( \frac{∂}{∂ ˜η} \right) = 0. \tag{66} \]

By the chain rule,

\[\frac{∂}{∂ ˜η} = \frac{∂}{∂ η} + \frac{∂ζ}{∂ η} \frac{∂}{∂ζ} \frac{∂ζ}{∂ ˜ζ} \frac{∂}{∂ζ} \] and \[\frac{∂ ˜ζ}{∂ η} = -\frac{∂ζ}{∂ η} \frac{∂ζ}{∂ ˜ζ} \]

so that \([16]\) reads

\[Θ_a \left( \frac{∂}{∂ζ} \right) \frac{∂ζ}{∂ ˜η} - Θ_a \left( \frac{∂}{∂ ˜η} \right) \frac{∂ζ}{∂ζ} = 0. \tag{67} \]

This equation can be solved to get a holomorphic function \(ζ = ˜ζ_a(ζ, η)\) such that \(φ_a : p \mapsto ˜ζ_a(ζ(p), η(p))\) restricts to a hyperbolic harmonic morphism on \((R^4_a, g^H_a)\). Note that the solution is not unique, but, as the level sets of \(ζ_a\) are uniquely determined on \(V\), \(φ_a\) is unique up to postcomposition with a holomorphic function. Restriction of \(φ_a\) to \(R^3_a\) then gives a horizontally conformal submersion \(f\) on an open subset of \(R^3_a\) and hence the conformal foliation by curves corresponding to \(S\) and \(a\). Note that the method actually finds a holomorphic function \(φ_a\) on an open subset of \(C^4\),

In summary, given a complex hypersurface \(S\) of \(C^p\) there is defined locally a real Hermitian structure \(J\) and all the related quantities \([14]\), in particular, \(U = J(∂/∂x_0)\). Then given \(a ∈ C^4\), the integral curves of \(U\) defined a \(C^\infty\) conformal foliation on an open subset of the slice \(R^3_a\) and all such foliations are given this way. Then we have shown above how to find a holomorphic function \(φ_a : A^C \to C\) on an open subset of \(C^4\) such that the level curves of \(φ_a | R^3_a\) give the leaves of the conformal foliation.

6 Examples

6.1 Linear Examples

Let \(S\) be the \(CP^2\) given by the homogeneous linear function

\[ψ(w_0, w_1, w_2, w_3) = b_0 w_0 + b_1 w_1 + b_2 w_2 + b_3 w_3\]

where \((b_0, b_1, b_2, b_3) ≠ (0, 0, 0, 0)\). We consider a representative example where \(ψ\) is given by

\[b_0, b_1, b_2, b_3 = [0, s, 0, 1]\]

with \(s\) a real parameter. Note that \(s = 0\) if and only if \([b_0, b_1, b_2, b_3] ∈ N^5\), this will be a special case. Equation \([12]\) reads

\[sμ (z_2 + μ ˜z_1) = 0\]

so that the direction field \(U\) of the quantities corresponding to \(S\) is given by \(U = σ^{-1}(iμ)\) where

\[μ = -z_2/(z_1 + s)\].
Parametrize $S \setminus \{z_0 = 0\}$ by

$$\quad (\zeta, \eta) \mapsto [1, w_1, w_2, w_3] = [1, \zeta, \eta, -s\zeta].$$

Equation (62) reads

$$\theta_a = 2a_0 dw_1 + w_1 dw_2 - w_2 dw_1 - dw_3$$

so that Equation (67) reads

$$(-2a_0 + \eta - s) \frac{\partial \tilde{z}}{\partial \eta} + \zeta \frac{\partial \tilde{z}}{\partial \zeta} = 0$$

which has a solution

$$\tilde{z} = (-2a_0 + \eta - s)/\zeta.$$  

The incidence relations (16) read

$$\left\{ \begin{array}{l}
z_1 - \zeta \bar{z}_2 = \eta \\
z_2 + \zeta \bar{z}_1 = -s \zeta
\end{array} \right.$$  

with solution

$$\zeta = -\frac{z_2}{z_1 + s}, \quad \eta = \frac{z_1 \bar{z}_1 + z_2 \bar{z}_2 + z_1 s}{z_1 + s}$$

so that

$$\phi_a = -\frac{z_1 \bar{z}_1 + z_2 \bar{z}_2 - 2a_0(\bar{z}_1 + s) + (z_1 - \bar{z}_1)s - s^2}{z_2}.$$  

Putting $a_0 = -ic$ gives the horizontally conformal function on the slice $R^3_c = R^3_{0i}$:

$$\phi_c = -\frac{(x_1 + c)^2 + x_2^2 + x_3^2 - s^2 + 2i(x_1 + c)s}{x_2 + ix_3}.$$  

Writing $\rho = |s|$, if $s \geq 0$, this is the composition of

$$R^3 \xrightarrow{T^1} R^3 \xrightarrow{\sigma_3^{-1}} S^3(\rho) \xrightarrow{\overline{\mathbf{F}}} S^2 \xrightarrow{\sigma} C \cup \{\infty\}$$

where $T^1$ is the translation $(x_1, x_2, x_3) \mapsto (x_1 + c, x_2, x_3)$, $\sigma_3^{-1}$ is the inverse of stereographic projection from $(-\rho, 0, 0, 0)$ given by $(X_1, X_2, X_3) \mapsto ((\rho^2 - |X|^2, 2\rho X_1, 2\rho X_2, 2\rho X_3)/(\rho^2 + |X|^2)$ and $\overline{\mathbf{F}}$ is the ("conjugate") Hopf map given by $C^2 \supset S^3(\rho) \ni (z_1, z_2) \mapsto -\overline{z_1}/z_2$. If $s < 0$ we replace $H$ by the Hopf map $(z_1, z_2) \mapsto -z_1/z_2$. In either case the fibres of $\phi$ give the conformal foliation of $R^3$ by the circles of Villarceau, see [28] for a description, [22, II p. 62] or the cover of Twistor Newsletter for a picture, and compare with a different treatment in [3]. For $s = 0$ the foliation degenerates to a bunch of circles tangent to the $x_1$ axis at $(-c, 0, 0)$.

We remark that an arbitrary linear function $\psi$ gives a foliation by circles but not all foliations by circles are given this way.

### 6.2 Quadratic examples

We now give three examples where the twistor surface is quadratic so that we can solve all equations explicitly.

**Example 6.1** Let the twistor surface be the quadratic surface $S = \{[w_0, w_1, w_2, w_3] \in CP^3 : w_0w_3 - w_1w_2 = 0\}$. Then the direction field $U$ of the corresponding quantities is given by $U = \sigma^{-1}(i\mu)$ where

$$z_2 + \mu \bar{z}_1 - \mu(z_1 - \bar{z}_2) = 0$$

which has solutions

$$\mu = \frac{z_1 - \bar{z}_1 + \sqrt{(z_1 - \bar{z}_1)^2 - 4z_2\bar{z}_2}}{2\bar{z}_2}.$$  

(69)
so that
\[ i\mu = \frac{-x_1 \pm |x|}{x_2 - ix_3} = \frac{x_2 + ix_3}{x_1 \pm |x|} \]
where \( |x| = \sqrt{x_1^2 + x_2^2 + x_3^2} = \sigma^{-1}(i\mu) \). Thus on any slice \( \mathbb{R}^3 \), \( U \) is given by
\[ U(t, x_1, x_2, x_3) = \pm (x_1, x_2, x_3)/|x| \]
which gives the tangent vector field of foliation by radial lines of Example 3.3.

Carrying out the calculations of \( \S.3.4 \) gives \( \phi_a = \mu \) for all \( a \) — we omit the details which are similar to the next example. Then, for each \( a \), \( \sigma^{-1} \circ \phi_a \) restricted to \( \mathbb{R}^3 \setminus \{x_0\text{-axis}\}, g_a^H \rightarrow S^2 \) given by orthogonal projection to \( \mathbb{R}^3 \) followed by radial projection.

**Example 6.2** This time let the twistor surface be the quadratic surface \( S = \{ [w_0, w_1, w_2, w_3] \in \mathbb{C}P^3 : w_0w_3 + w_1w_2 = 0 \} \). Then the direction field \( U \) of the corresponding quantities is given by \( U = \sigma^{-1}(i\mu) \) where
\[ z_2 + \mu \bar{z}_1 + \mu(z_1 - \mu \bar{z}_2) = 0 \]
which has solutions
\[ \mu = \frac{z_1 + \bar{z}_1 \pm \sqrt{(z_1 + \bar{z}_1)^2 + 4z_2 \bar{z}_2}}{2\bar{z}_2} = \frac{x_0 \pm s}{x_2 - ix_3} \]
where \( s = \sqrt{x_0^2 + x_2^2 + x_3^2} \). Note that
\[ \mu|\mathbb{R}^3_0 = \pm \frac{\sqrt{x_0^2 + x_3^2}}{x_2 - ix_3} = \pm \frac{x_2 + ix_3}{\sqrt{x_2^2 + x_3^2}} \]
so that on \( \mathbb{R}^3_0 \),
\[ U(x_1, x_2, x_3) = \pm \frac{1}{\sqrt{x_2^2 + x_3^2}}(0, -x_3, x_2) \]
which gives the tangent vector field of the conformal foliation \( C \) by circles around the \( x_1 \)-axis of Example 3.4. Further note that \( \mu|\mathbb{M}^4 \) is given by
\[ \mu(t, x_1, x_2, x_3) = \frac{-it \pm r}{x_2 - ix_3} = \frac{r}{x_2^2 + x_3^2} \left\{ \left( \pm x_2 + \frac{t}{r}x_3 \right) + i \left( \mp x_3 - \frac{t}{r}x_2 \right) \right\} \]
where \( r = \sqrt{x_2^2 + x_3^2} - t^2 \) and so, on the open set \( x_2^2 + x_3^2 > t^2 \), we have
\[ U(t, x_1, x_2, x_3) = \frac{r}{\sqrt{x_2^2 + x_3^2}} \left( 0, \pm x_3 + \frac{t}{r}x_2, \pm x_2 + \frac{t}{r}x_3 \right) ; \]
then \( v = U + \partial/\partial t \) gives the tangent field of the shear-free congruence \( \ell \) extending \( C \) discussed in Example 3.10.

Parametrizing \( S \) away from \( w_0 = 0 \) by \( (\zeta, \eta) \mapsto \{1, w_1, w_2, w_3\} = \{1, \eta, -\zeta, \zeta\eta\} \), the incidence relations \( (16) \) read
\[ \begin{align*}
  z_1 - \eta \bar{z}_2 &= -\zeta \\
  z_2 + \eta \bar{z}_1 &= \zeta \eta
\end{align*} \]
which have solutions
\[ \zeta = -ix_1 \pm s \quad \text{and} \quad \eta = \frac{x_0 \pm s}{x_2 - ix_3} . \]
On \( \mathbb{M}^4 \),
\[ \zeta = -ix_1 \pm r \quad \text{and} \quad \eta = \frac{-it \pm r}{x_2 - ix_3} . \]
These solutions are real analytic except on the cone $x_1^2 + x_2^2 = t^2$. We obtain smooth solutions if we avoid this cone. Note that $S \cap N^5$ has equation: $(\zeta + \bar{\zeta})(|\eta|^2 - 1) = 0$ and so consists of the union of two submanifolds: $S_1 : \Re \zeta = 0$ and $S_2 : |\eta| = 1$. Smooth branches of $(\zeta, \eta)$ will correspond to CR maps $A \to S \cap N^5$ with image lying in $S_1 \setminus S_2$ or $S_2 \setminus S_1$: $S_1 \cap S_2$ corresponding to the branching set of $(\zeta, \eta)$ which is the subset of $A$ on which the square root $r$ vanishes. Specifically, note from (70) that if $t > |z_2|$, $\Re \zeta = 0$ corresponding to points of $S_1$ and if $t < |z_2|$, $|\eta| = 1$ corresponding to points of $S_2$.

We have

$$\theta = 2a_0 dw_1 + w_1 dw_2 - w_2 dw_1 - dw_3$$
$$2a_0 d\eta - \eta d\zeta + \zeta d\eta - \zeta d\eta + \eta d\zeta$$

so that (71) reads

$$\frac{\partial \bar{\zeta}}{\partial \eta} + a_0 \frac{\partial \bar{\zeta}}{\partial \zeta} = 0$$

which has a solution

$$\bar{\zeta}_a = \zeta - a_0 \ln \eta.$$  

Setting $\phi_a = \bar{\zeta}_a (\zeta(p), \eta(p))$ gives the complex-valued map on a dense subset of $C^4$:

$$\phi_a(x_0, x_1, x_2, x_3) = -i x_1 \pm s - a_0 \ln \frac{x_0 \pm s}{x_2 - i x_3},$$  

(72)

For any $a \in C^4$ this restricts to a complex-valued harmonic morphism on a dense subset of $(\bar{R}_s^4, g^H)$. In particular, when $a = 0$, this simplifies to the hyperbolic harmonic morphism on $R^4 \setminus \{x_1\text{-axis}\}$ given by

$$\phi(x_0, x_1, x_2, x_3) = -i x_1 \pm \sqrt{r^2 - x_2^2 - x_3^2},$$  

(73)

which further restricts on $R^3$ to

$$\phi(x_1, x_2, x_3) = -i x_1 \pm \sqrt{x_2^2 + x_3^2},$$  

(74)

the level surfaces of this giving the conformal foliation by circles round the $x_1$-axis of Example 3.4. The harmonic morphism (74) has fibres given by the Euclidean spheres having these circles as diameters, these spheres are totally geodesic in $(\bar{R}_s^4, g^H)$.

For definiteness, now take plus signs in the above. Then, putting $a_0 = -it$ in (72) and restricting to the open set $\{(x_1, x_2, x_3) : t^2 + x_2^2 > t^2\} \subset R^3_t = R^3$ gives the horizontally conformal map

$$\phi_t = \phi_a = -i x_1 + r + it \ln \frac{r - it}{x_2 - i x_3}$$
$$= -i x_1 + r - t \arg \frac{r - it}{x_2 - i x_3},$$

the level curves of this lie in the planes $x_1 = \text{constant}$ and are the involutes of the unit circle pictured in Fig. 1.

Example 6.3 This time, as in Example 1.9 let the twistor surface be the quadratic surface $S = \{[w_0, w_1, w_3, w_4] \in CP^3 : w_0 w_1 + w_2 w_3 = 0\}$. Then $\mu$ satisfies

$$\mu + (z_1 - \mu \bar{z}_2)(z_2 + \mu \bar{z}_1) = 0$$

so that

$$\mu = \frac{(1 + z_1 \bar{z}_1 - z_2 \bar{z}_2) \pm s}{2 \bar{z}_1 \bar{z}_2}$$

where $s = \sqrt{(1 - z_1 \bar{z}_1 - z_2 \bar{z}_2)^2 + 4z_1 \bar{z}_1 z_2 \bar{z}_2}$. Parametrize $S$ away from $w_0 = 0$ by

$$(\zeta, \eta) \mapsto [1, w_1, w_2, w_3] = [1, \zeta \eta, -\eta, \zeta].$$
The incidence relations \((6)\) read
\[
\begin{align*}
  z_1 - \zeta \bar{z}_2 &= \eta \\
  z_2 + \zeta \bar{z}_1 &= \zeta .
\end{align*}
\]
Solving,
\[
\zeta = \frac{1 + z_1 \bar{z}_1 + z_2 \bar{z}_2 \pm s}{2 \bar{z}_2} .
\] (75)

This has branch points when the square root \(s\) is zero; on \(\mathbb{R}_0^3\) this occurs on \(C : z_1 = 0, |z_1|^2 + |z_2|^2 = 1\), a circle in \(\mathbb{R}_0^3\). On any open set \(A \subset \mathbb{R}_0^3 \setminus C\), on fixing the \(\pm \) sign in \((75)\), we obtain a smooth solution.

Next note that
\[
\theta_a = 2a_0 dw_1 + w_1 dw_2 - w_2 dw_1 - dw_3
\]
\[
= 2a_0 (\eta d\zeta + \zeta d\eta) - \zeta d\eta + \eta (\eta d\zeta + \zeta d\eta) - d\zeta
\]
\[
= (\eta^2 + 2a_0 \eta - 1) d\zeta + 2a_0 \zeta d\eta .
\]

So \((67)\) reads
\[
(\eta^2 + 2a_0 \eta - 1) \frac{\partial \tilde{\zeta}}{\partial \eta} - 2a_0 \tilde{\zeta} \frac{\partial \tilde{\zeta}}{\partial \zeta} = 0
\]
which has a solution
\[
\tilde{\zeta} = \zeta \left\{ \begin{array}{l}
\frac{\eta + a_0 + \sqrt{a_0^2 + 1}}{\eta + a_0 - \sqrt{a_0^2 + 1}} \\
- a_0 / \sqrt{a_0^2 + 1}
\end{array} \right. .
\]

When \(a_0 = 0\) this gives \(\tilde{\zeta} = \zeta\) so that, for any smooth branch \(\zeta : A \to \mathbb{C}\) of \((73)\) the level curves of \(\phi = \zeta |A\) give a conformal foliation. Explicitly,
\[
\phi(x_1, x_2, x_3) = \frac{1 + x_1^2 + x_2^2 + x_3^2 \pm s}{2(x_2 + ix_3)} .
\]
with \(s = \sqrt{1 - (x_1^2 + x_2^2 + x_3^2)^2 + 4x_1^2(x_2^2 + x_3^2)}\), giving a rotationally symmetric foliation.

Sometimes we can solve \((67)\) explicitly even when the twistor surface has higher degree than two; here is a cubic example:

**Example 6.4** Let the twistor surface \(S\) be
\[
S = \{(w_0, w_1, w_2, w_3) \in \mathbb{C}P^3 : w_1 w_2^2 + iw_0^2 w_3 = 0\} .
\]

We can parametrize this away from \(w_0 = 0\) by
\[
(\zeta, \eta) \mapsto [1, z_1, z_2, z_3] = [\zeta, i \zeta, \eta, i \zeta \eta^2] .
\]

The incidence relations \((6)\) read
\[
\begin{align*}
  z_1 - i \zeta \bar{z}_2 &= \eta \\
  z_2 + i \zeta \bar{z}_1 &= \zeta \eta^2 .
\end{align*}
\] (76)

Then
\[
\theta_a = 2a_0 dw_1 + w_1 dw_2 - w_2 dw_1 - dw_3
\]
\[
= (2ia_0 - \eta^2 + \eta) d\zeta + (-2 \zeta \eta - \zeta) d\eta
\]
so \((67)\) reads
\[
(2ia_0 - \eta^2 + \eta) \frac{\partial \tilde{\zeta}}{\partial \eta} + (2 \zeta \eta + \zeta) \frac{\partial \tilde{\zeta}}{\partial \zeta} = 0 .
\]

This can be solved by the product method to yield
\[
\tilde{\zeta} = -\zeta \left( 2ia_0 - \eta^2 + \eta \right) \left( \frac{\eta - \alpha_1}{\eta - \alpha_2} \right)^{2/\sqrt{1 + 8ia_0}}
\]
where \(\alpha_1 = 1 + \sqrt{1 + 8ia_0}/2\) and \(\alpha_2 = 1 - \sqrt{1 + 8ia_0}/2\). When \(a_0 = 0\), \(\tilde{\zeta} = \zeta \eta - 1/\eta\) where \((\zeta, \eta)\) is a solution to \((76)\). For any \(a \in \mathbb{C}^1\), \(\phi_a = \zeta |\mathbb{R}_a^3\) gives a conformal foliation by curves.
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Figure 1: Two leaves of the Riemannian foliation of Example 3.10 with $t = 1$. 