Stealthy Cyber-Attack Design Using Dynamic Programming

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Abstract—This paper addresses the issue of data injection attacks on control systems. We consider attacks which aim at maximizing system disruption while staying undetected in the finite horizon. The maximum possible disruption caused by such attacks is formulated as a non-convex optimization problem whose dual problem is a convex semi-definite program. We show that the duality gap is zero using S-lemma. To determine the optimal attack vector, we formulate a soft-constrained optimization problem using the Lagrangian dual function. The framework of dynamic programming for indefinite cost functions is used to solve the soft-constrained optimization problem and determine the attack vector. Using the Karush–Kuhn–Tucker conditions, we also provide necessary and sufficient conditions under which the obtained attack vector is optimal to the primal problem. Finally, we illustrate the results through numerical examples.

I. INTRODUCTION

Research in the security of industrial control systems (ICSs) has received considerable attention due to increased cyber-attacks [1]. The research focus can be largely classified into (a) attack modelling, (b) attack detection, (c) attack impact assessment and (d) attack treatment and prevention [2]. This paper focuses on the attack impact assessment problem for false data injection (FDI) attacks.

Existing literature has discussed different aspects of impact assessment in the finite horizon. The necessary and sufficient condition for the attack impact to be bounded was formulated in [3] as a function of the system (block) Toeplitz matrix (TM). Since TM involves higher powers of the system matrices, its computation is prone to numerical errors. To this end, we provide necessary and sufficient conditions for the attack impact to be bounded without using TM.

The paper [4] determines the highest possible attack impact (infinity norm of the attacked system states) when the attack is constrained to be stealthy. Our paper differs from [4] in the facts that (a) we consider the output-to-output gain (OOG) as an impact metric which is advantageous [5], and (b) [4] solves for the impact as the maximum of $N_h n$ convex optimization problems (here $N_h$ is the horizon length and $n$ is the order of the system), whereas we propose a single convex optimization problem for impact assessment.

An FDI attack from an optimal control framework was proposed in [6]. In particular, it proposes a switching policy for the adversary to increase the efficacy of the attack when the number of actuators that can be attacked simultaneously is limited. However, the attacks are not stealthy, rather a bound is imposed on the attack energy.

The authors in [7] and [8] consider bounded actuator attacks which is similar to a classical $H_\infty$ metric based approach to attack design. However, we have shown previously in [5] that the OOG based design can outperform $H_\infty$ based design in the infinite horizon. In [9], the authors do not consider stealthy attacks and in [10], the authors consider a perfectly undetectable attack whereas we consider FDI attacks in actuators and sensors but not both. Additionally, all of these works assume that the weighting matrices of the cost function are positive (semi-)definite. In this paper, we show that this is a restrictive assumption.

In [11], the authors considers an adversary that maximizes the disruption whilst remaining stealthy. A data-driven adaptive DP algorithm was proposed for stealthy attack design, whereas we alternatively adopt the framework of [12]. To this end, we present the following contributions.

1) Firstly, the worst-case impact caused by an FDI attack on the sensor or actuator channels in the finite horizon is posed as a non-convex optimization problem. It is solved through its convex dual problem, addressing the limitation of [4]. It is shown using S-lemma that the duality gap is zero.

2) Secondly, a soft-constrained optimization problem is formulated using the Lagrange dual function, to determine the optimal attack vector. We observe that the weight matrix of the cost function is indefinite. So, we adopt the recently proposed DP approach [12].

3) Thirdly, we provide the necessary and sufficient conditions under which the attack vector is optimal (and consequently is stealthy) to the primal problem, thus partially addressing the limitation of [6]. We outline the merits of using the framework of DP by providing insights into how the attack impact can be made bounded by means of state-feedback policies instead of using open-loop attacks computed from TM.

4) Finally, this paper serves as a practical application of [12] to the security of ICS.

Outline: The problem is formulated in Section \textsuperscript{II} Section \textsuperscript{III} proposes an optimization framework for determining the attack impact. Section \textsuperscript{IV} adopts the DP framework to determine the optimal attack policy. It also discusses the merits of the DP framework. Section \textsuperscript{V} provides a numerical illustration of the proposed optimizations frameworks. Finally, we provide concluding remarks in Section \textsuperscript{VI}.

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II. PROBLEM BACKGROUND

In this section, we describe the control system structure and the goal of the adversary. Consider the general description [13] of a finite-horizon discrete-time (DT) linear time-invariant (LTI) system with a process \((P)\), output feedback controller \((C)\) and an anomaly detector \((D)\) as shown in Fig. 1. The closed-loop system is represented by

\[
\begin{align*}
\mathcal{P} : & \quad x_p[k+1] = Ax_p[k] + Bu[k] \\
& \quad y[k] = Cx_p[k] \\
& \quad y_p[k] = Cy_p[k] + Dju[k] \\
\mathcal{C} : & \quad z[k+1] = Az[k] + Bu_0[k] + K_r\tilde{y}[k] \\
& \quad u[k] = C_\gamma z[k] + D_\gamma\tilde{y}[k] \\
\mathcal{D} : & \quad s[k+1] = A_s s[k] + B_s u[k] + K_m\tilde{y}[k] \\
& \quad y_r[k] = C_s s[k] + D_s u[k] + E_s\tilde{y}[k]
\end{align*}
\]

\(k = 0, \ldots, N_h\), where the state of the process, controller and the observer are represented by \(x_p[k] \in \mathbb{R}^{n_p}\), \(z[k] \in \mathbb{R}^{n_z}\) and \(s[k] \in \mathbb{R}^{n_s}\) respectively. The control signal generated by the controller and applied to the actuator is denoted by \(\tilde{u}[k] \in \mathbb{R}^{n_u}\), and \(u[k] \in \mathbb{R}^{n_u}\) respectively. The measurement output produced by the process is \(y[k] \in \mathbb{R}^{n_m}\), \(\tilde{y}[k] \in \mathbb{R}^{n_m}\) is the measurement signal received by the controller and the detector, \(y_p[k] \in \mathbb{R}^{n_y}\) is the virtual performance output, \(y_r[k] \in \mathbb{R}^{n_y}\) is the residue generated by the detector. The closed-loop system described above is said to have a good performance over the horizon \(N_h\), when the energy of the performance output \(\|y_p\|_{\ell_2[0,N_h]}^2\) is small and an anomaly is said to be detected when the detector output energy \(\|y_r\|_{\ell_2[0,N_h]}^2\) is greater than a predefined threshold, say \(\epsilon_r\). Without loss of generality, we assume \(\epsilon_r \equiv 1\).

A. Data injection attack scenario

In the closed-loop system described in the previous section, we consider an adversary injecting false data into the sensors or actuators of the process. We next discuss the resources the adversary has access to.

1) Disruption and disclosure resources: The adversary can inject data into the sensor or actuators channels but not both (we do not consider a covert attack). This is represented by

\[
\begin{bmatrix}
\tilde{u}[k] \\
\tilde{y}[k]
\end{bmatrix} = \begin{bmatrix}
u[k] \\
y[k]
\end{bmatrix} + \begin{bmatrix} B_a \\ D_a \end{bmatrix} a[k],
\]

where \(\begin{bmatrix} B_a \\ D_a \end{bmatrix} = \begin{bmatrix} \Gamma_u & 0_{n_u \times n_m} \\ 0_{n_m \times n_a} & \Gamma_y \end{bmatrix}\), and \(a[k] \in \mathbb{R}^{n_a}\) is the data injected by the adversary. In general the adversary cannot access all controller/sensor channels but only a limited number of them which is captured by the rank of the matrix \([B_a^T \quad D_a^T]^T\). The adversary does not have access to any disclosure (eavesdropping) resources.

2) System knowledge: The adversary knows the closed-loop system. This knowledge is used by the adversary to construct the optimal attack policy. Defining \(x[k] \equiv [x_p[k]^T \quad z[k]^T \quad s[k]^T]^T\), the closed-loop system under attack with the performance output and detection output as system outputs becomes

\[
x[k+1] = A_c x[k] + B_c a[k], \\
y_p[k] = C_p x[k] + D_c a[k], \\
y_r[k] = C_r x[k] + D_r a[k],
\]

and the closed-loop system matrices are given by

\[
\begin{bmatrix}
A_c & B_c & 0 \\
B_c & C_c & 0 \\
(B_c D_c + K_c) & B_c C_c & A_r
\end{bmatrix},
\]

\[
\begin{bmatrix}
B B_c A_c & B D_c A_r \\
B D_c A_r & (B_c D_c + K_c) D_d A_r
\end{bmatrix},
\]

\[
\begin{bmatrix}
C_p & C_r & 0 \\
C_r & D_r & C_e
\end{bmatrix},
\]

\[
\begin{bmatrix}
D_p & D_r & B_e C_e \\
(D_r D_e + E_e C_e) & D_r C_e & C_e \\
(D_r D_e + E_e) D_r
\end{bmatrix},
\]

3) Attack goals and constraints: Given the resources the adversary has access to, the adversary aims at disrupting the system’s behavior while staying stealthy. The system disruption is evaluated by the increase in energy of performance output whereas, the adversary is stealthy if the energy of the detection output is below the threshold \(\epsilon_r\). This leads to the optimal attack policy discussed next.

B. Problem formulation

From the previous discussions, it can be understood that the goal of the adversary is to maximize the performance cost while staying undetected. The attack policy of the adversary can be formulated as

\[
\gamma^* \equiv \sup_{a \in \ell_2[0,N_h]} \|y_p\|_{\ell_2[0,N_h]}^2
\]

s.t. \(\|y_r\|_{\ell_2[0,N_h]}^2 \leq 1, \quad x[0] = 0\),

where \(\gamma^*\) is the disruption caused by the attack signal on the system and \(N_h\) is the horizon length. In the above optimization problem, the constraint \(x[0] = 0\) is introduced since the system is at equilibrium before the attack.

Assumption 2.1: The closed-loop system \((\mathfrak{3})\) is at equilibrium \(x[0] = 0\) before the attack commences.

The optimization problem \((\mathfrak{4})\) is non-convex since it has a convex objective function (which has to be maximized) with convex constraints. Thus, in the remainder of this paper, we propose methods to solve \((\mathfrak{4})\). In Section III we determine the optimal value of the optimization problem \((\mathfrak{4})\). Whereas in Section IV we determine the optimal attack policy.
III. OPTIMIZATION FRAMEWORK FOR ATTACK IMPACT ASSESSMENT

In this section, we determine the attack impact \( \gamma^* \), which is the value of the optimization problem (4). Let us define \( \mathbf{a} \triangleq [a[0]^T, \ldots, a[N_h]^T]^T \), \( \mathbf{y}_r \triangleq [y_r[,0]^T, \ldots, y_r[N_h]^T]^T \), and \( \mathbf{y}_p \triangleq [y_p[,0]^T, \ldots, y_p[N_h]^T]^T \). Let us additionally define the matrices \( \mathcal{T}_r \in \mathbb{R}^{n_x(N_h + 1) \times n_x(N_h + 1)} \) and \( \mathcal{T}_p \in \mathbb{R}^{n_p(N_h + 1) \times n_p(N_h + 1)} \) similar to [3, (12)] such that \( \mathbf{y}_r = \mathcal{T}_r \mathbf{a} \) and \( \mathbf{y}_p = \mathcal{T}_p \mathbf{a} \). Under these definitions, (4) can be equivalently written as

\[
\gamma^* = \sup_{\mathbf{a}} \mathbf{a}^T \mathcal{T}_p^T \mathcal{T}_r \mathbf{a} \quad \text{s.t.} \quad \mathbf{a}^T \mathcal{T}_r^T \mathcal{T}_r \mathbf{a} \leq 1, \ x[0] = 0. \tag{5}
\]

In the following theorem, we propose a convex dual SDP to solve the optimization problem (5).

**Theorem 3.1:** When Assumption 2.7 holds, the optimal value of (5) can be obtained by solving the convex SDP

\[
\gamma^* = \min_{\mathbf{a}} \gamma \quad \text{s.t.} \quad \mathcal{T}_p^T \mathcal{T}_p - \gamma \mathcal{T}_r^T \mathcal{T}_r \preceq 0. \tag{6}
\]

**Proof:** See Appendix.

Thus, the impact assessment problem (4) was solved by a convex dual SDP (6). Using S-lemma, we also proved that the duality gap is indeed zero. As stated before, this is a convex dual SDP (6). Using S-lemma, we also proved that the duality gap is indeed zero. As stated before, this is

**Assumption 4.1:** The value of (6) is bounded.

**Theorem 4.1:** The optimal attack vector \( \mathbf{a}^*[k] \) which minimizes the cost function of (8), is parameterized as a function of only the vector \( \mathbf{v}[k] \) as

\[
\mathbf{a}^*[k] = -K_k x[k] + G_k \mathbf{v}[k], \quad \forall 0 \leq k \leq N_h \tag{9}
\]

where \( K_k = \left( R + B_{cl}^T X_{k+1} B_{cl} \right)^+ \left( S^T + B_{cl}^T X_{k+1} A_{cl} \right), G_k = I_m - \left( R + B_{cl}^T X_{k+1} B_{cl} \right)^+ \left( R + B_{cl}^T X_{k+1} B_{cl} \right) \) and the matrices \( X_k \), \( \forall k \in \{N_h, \ldots, N_0\} \) are obtained by solving the generalized Riccati equation (GRE)

\[
X_k = Q + A_{cl}^T X_{k+1} A_{cl} - (S + A_{cl}^T X_{k+1} B_{cl})^T \left( R + B_{cl}^T X_{k+1} B_{cl} \right)^+ \left( S^T + B_{cl}^T X_{k+1} A_{cl} \right). \tag{10}
\]

**Proof:** Directly follows from [12, Theorem 2.1].

**Theorem 4.1** describes a recursive method to calculate the optimal attack vector which minimizes the cost function of (8). Next, we discuss the conditions under which the obtained attack vector is the optimal attack vector to (4).

Let us characterize the attack vector of (8) by (9). Let \( x[0] = 0 \) and \( \mathbf{v} \triangleq [v[0]^T, \ldots, v[N_h]^T]^T \). Then (9) becomes a function of only the vector \( \mathbf{v} \) and the system matrices. Let us define \( \mathcal{T}_{pv} \) and \( \mathcal{T}_{rv} \) such that \( \mathbf{y}_r = \mathcal{T}_r \mathbf{v} \) and \( \mathbf{y}_p = \mathcal{T}_p \mathbf{v} \). Let \( \mathcal{A}_k \triangleq A_{cl} - B_{cl} K_k \), then \( \mathcal{T}_{rv}, \mathcal{A}_k \in \{\mathcal{p}, \mathcal{r}\} \) is represented in (11). Then, **Lemma 4.2** states the necessary and sufficient conditions under which the attack vector obtained from (9) is optimal to (4).

**Lemma 4.2:** Let \( \gamma^* \) be the optimal value of (6). Then, any attack vector of the form (9) is optimal to (4) if and only if \( \mathbf{v}^T \mathcal{T}_{rv} \mathcal{T}_{rv}^T \mathbf{v} = 1 \).

**Proof:** See Appendix.

**Lemma 4.2** states that any attack vector of the form (9), which yields the detection output energy as 1, is an optimal attack vector to (4).
\[ T_{\alpha \nu} = \begin{bmatrix} D_\alpha G_0 & 0 & \cdots & 0 \\ (C_\alpha - D_\alpha K_{1})B_{cl}G_0 & D_\alpha G_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (C_\alpha - D_\alpha K_{N_h}) \prod_{k=1}^{N_h-1} A_k B_{cl}G_0 & \cdots & D_\alpha G_{N_h} \end{bmatrix} \] (11)

**B. Conditions for bounded attack impact**

In Section IV-A, we used the DP framework to determine the optimal attack vector. The state feedback matrices for the attack vector \((K_k)\) were obtained from the solution to the GRE \([10]\). The necessary and sufficient conditions for a solution to exist for the GRE is stated Lemma 4.3.

**Lemma 4.3:** There exists a solution to the GRE \((10)\) iff \((12)\) and \((13)\) holds \(\forall k \in \{1, \ldots, N_h\}\).

\[ R + B_{cl}^T X_k B_{cl} \geq 0, \quad (12) \]
\[ \ker(R + B_{cl}^T X_k B_{cl}) \subseteq \ker(S + A_{cl}^T X_k B_{cl}). \quad (13) \]

**Proof:** Refer to the proof of [12, Theorem 2.1].

A consequence of Lemma 4.3 is that, when \((12)\) and \((13)\) does not hold, the value of \((7)\) (and consequently \((4)\) due to strong duality) is unbounded. Thus, the first advantage of the DP framework is that, using the results of Lemma 4.3, we can characterize scenarios under which the impact \((4)\) becomes unbounded. To this end, we use Lemma 4.3 to characterize a scenario where the impact is unbounded in Lemma 4.4.

**Lemma 4.4:** If \(\exists s \neq 0\) such that \(D_s = 0\) and \(D_p s \neq 0\), then the solution to the GRE \((10)\) does not exist thus making the impact unbounded.

**Proof:** See Appendix.

Thus, if the system operator could alter the direct feedthrough matrices such that the lemma conditions do not hold, the risk of having an unbounded impact can be, although not eliminated, reduced. Another advantage of using the DP framework is detailed as follows. Let us consider \(\gamma^*\) as a variable. If the defender could find a bounded \(\gamma^*\) such that the conditions of Lemma 4.3 hold, the attack impact will be bounded. If the attack impact is not bounded, the defender could alter the system matrices such that the conditions hold. This would result in a bounded/lowered impact. This relationship on altering the system matrices to lower the worst-case impact was not evident from Lemma 3.2 but now is clearer from the DP framework. A numerical illustration of the proposed approach for stealthy attack design is provided in the next section.

**V. NUMERICAL EXAMPLE**

Consider a power generating system [14, Section 4] as represented by \((14)\) and \((15)\).

\[
\begin{bmatrix}
\eta_1 \\
\eta_2 \\
\eta_3
\end{bmatrix} =
\begin{bmatrix}
-\frac{1}{T_{p}} & K_{lm} & -2K_{lm} \\
0 & \frac{3}{T_{y}} & T_{e} \\
\frac{1}{T_{y}} & 0 & \frac{1}{T_{y}}
\end{bmatrix}
\begin{bmatrix}
\eta_1 \\
\eta_2 \\
\eta_3
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} \tilde{u} \quad (14)
\]

\[
y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \eta_1, 
\quad y_p = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \eta_2
\]

Here \(\eta \triangleq [df; dp + 2dx; dx]\), \(df\) is the frequency deviation in Hz, \(dp\) is the change in the generator output per unit (p.u.), and \(dx\) is the change in the valve position p.u. The constants \(T_{lm} = 6, T_{h} = 4, \) and \(T_{y} = 0.2\) represent the time constants of load and machine, hydro turbine, and governor, respectively, and \(R = 0.05\text{(Hz/p.u.)}\) is the speed regulation due to the governor action. The constant \(K_{lm} = 1\) represents the steady state gain of the load and machine. The DT system matrices are obtained by discretizing the process using zero-order hold with a sampling time \(T_s = 0.1\) seconds. The DT process is stabilized with an output feedback controller of the form \((1)\) with \(D_s = 19\). The detector is of the form \((2)\) where \(A_r = (A_d - K_c C_d)\), \(B_r = B_d, C_r = C_d\) and \(K_c = [0.17 -2.83 -7.43]^T\). The adversary attacks only the actuator, i.e.: \(B_a = 1\) and \(D_a = 0\). The other unspecified matrices are zero. The system is assumed to satisfy \(\eta(0) = 0\). We consider a horizon length of \(N_h = 50\).

By solving the optimization problem \((6)\), we obtain \(\gamma^* = 4733.3\). We formulate the Lagrange dual function similar to \((8)\). The matrices \(K_k, G_k\) and \(X_k\) were obtained by solving the GRE described in Theorem 4.4. Using these matrices, we obtained the matrices \(T_{p}, T_{y}\). According to Lemma 4.4, we found the eigenvector \(v\), corresponding to the eigenvalue \(\gamma^*\) of the matrix pencil \((T_{p}, T_{y}, T_{p}, T_{y})\). This vector is scaled such that \(v^T T_{p}T_{y}v = 1\). The obtained optimal attack vector is

\[
v^*[k] = \begin{cases}
33.9024, & k = 0 \\
0, & \text{otherwise}
\end{cases}
\]

The resulting attack vector obtained from \((9)\) is shown in Fig. 2. Applying this attack signal to \((3)\), the performance of the system is shown in Fig. 3. It can be seen that the detection output energy reaches the value 1 which represents that the constraint of the primal problem \((4)\) is satisfied. Similarly, the performance output energy reaches the value of 4733.3 which shows that the duality gap is zero. Finally, in this example, the matrices \(D_s\) and \(D_r\) are zero and the limitation described in Lemma 4.4 does not occur.

**VI. CONCLUSION**

In this paper, we considered FDI attacks which aim at maximizing impact while staying undetected in the finite horizon. We formulated the impact assessment problem and it corresponded to a non-convex optimization problem. This problem was shown to be equivalent, using S-lemma, to a...
convex dual SDP. Secondly, we formulated a soft-constrained optimization problem using the Lagrangian dual function, to determine the optimal attack vector. The framework of DP was used to determine the optimal attack vector. We also provided the necessary and sufficient conditions under which the obtained attack vector is optimal to the primal problem. Finally, the results were illustrated through numerical examples. Future works include extending the framework to adaptive dynamic programming.

APPENDIX

PROOF OF THEOREM 3.1

Before presenting the proof, an introduction to S-lemma is provided.

Lemma 1.1: [15, Theorem 2] Let \( q_a(x) \) and \( q_b(x) \) be quadratic functions. Suppose \( \exists \bar{x} \) such that \( q_a(\bar{x}) \geq 0 \). Then the quadratic inequality \( q_b(x) \geq 0 \) is a consequence of \( q_a(x) \geq 0 \) if and only if \( q_b(x) \geq \gamma q_a(x) \).

Proof: [Proof of Theorem 3.1] When Assumption 2.1 hold, we can rewrite (5) as

\[
\begin{align*}
- \inf_t & - a^T \mathcal{P}_p^T \mathcal{P}_r a \\
\text{s.t.} & a^T \mathcal{P}_r^T \mathcal{P}_r a \leq 1
\end{align*}
\]

Using the hypo-graph formulation, (16) can be recast as

\[
- \max_t t \\
\text{s.t.} - a^T \mathcal{P}_p^T \mathcal{P}_r a \geq t \quad \text{whenever } a^T \mathcal{P}_r^T \mathcal{P}_r a \leq 1,
\]

which can again be rewritten as

\[
- \max_t t \\
\text{s.t.} 1 - a^T \mathcal{P}_r^T \mathcal{P}_r a \geq 0 \quad \iff - a^T \mathcal{P}_p^T \mathcal{P}_r a - t \geq 0.
\]

Let us define the quadratic functions \( q_a(a) \equiv 1 - a^T \mathcal{P}_r^T \mathcal{P}_r a \) and \( q_b(a) \equiv - a^T \mathcal{P}_p^T \mathcal{P}_r a - t \). Firstly, if the optimization problem (5) is feasible, then \( \exists \bar{a} \) such that \( q_a(\bar{a}) \geq 0 \) is feasible. Then from Lemma 1.1 it holds that \( q_a(\bar{a}) \geq 0 \quad \iff q_b(\bar{a}) \geq 0 \) if and only if \( q_b(\bar{a}) \geq \gamma q_a(\bar{a}) \). Using this iff relation, (17) can be reformulated as

\[
- \max_t t \\
\text{s.t. } q_b(a) \geq \gamma q_a(a).
\]

Substituting the definition of the quadratic function yields

\[
- \max_t t \\
\text{s.t. } - a^T \mathcal{P}_p^T \mathcal{P}_r a - \gamma + \gamma a^T \mathcal{P}_r^T \mathcal{P}_r a \geq t.
\]

Since (18) resembles an epigraph formulation, it can be rewritten as

\[
- \max_\gamma \left\{ \min_a \{ - a^T \mathcal{P}_p^T \mathcal{P}_r a - \gamma + \gamma a^T \mathcal{P}_r^T \mathcal{P}_r a \} \right\}.
\]

Since \( \kappa \) is a minimization problem, it holds that

\[
\kappa = \begin{cases} 
- \gamma, & \text{iff } \mathcal{P}_p^T \mathcal{P}_r + \gamma \mathcal{P}_r^T \mathcal{P}_r \geq 0 \\
- \infty, & \text{otherwise}
\end{cases}
\]

Therefore (19) can be rewritten as

\[
\min_\gamma \\
\text{s.t. } \mathcal{P}_p^T \mathcal{P}_r + \gamma \mathcal{P}_r^T \mathcal{P}_r \geq 0,
\]

which concludes the proof.

PROOF OF LEMMA 1.2

Before presenting the proof, an introduction to optimality conditions is presented for general quadratically constrained quadratic problem (QCQP).

Lemma 1.2: [16, Proposition 3.3] Consider the inequality constrained QCQP

\[
\min x^T A_0 x \\
\text{s.t. } x^T A_1 x + c_1 \leq 0.
\]

where \( A_0, A_1 \) are symmetric matrices and \( c_1 \in \mathbb{R} \). Suppose \( \exists x_0 \) such that \( x_0^T A_1 x_0 + c_1 < 0 \). Then \( x_0 \) is a global minimizer of (20) if and only if \( \exists x_* \) and \( \lambda_* \) such that the following Karush–Kuhn–Tucker (KKT) conditions hold:

\[
\begin{align*}
0 &= A_0 x_0 \\
0 &= A_1 x_0 + \lambda_* \mathbb{1} \\
\lambda_* &> 0 \\
\lambda_* (x_0^T A_1 x_0 + c_1) &= 0
\end{align*}
\]
1) Primal feasibility: \( x^T A_1 x_a + c_1 \leq 0 \).
2) Dual feasibility: \( \lambda_a \geq 0 \).
3) Complementary slackness: \( \lambda_a (x^T A_1 x + c_1) = 0 \).
4) Stationarity: \( (\lambda_a A_1 + A_0) x_a = 0 \).
5) \( (\lambda_a A_1 + A_0) \geq 0 \)

**Proof:** [Proof of Lemma 4.2] Consider the optimization problem \( 2 \) which can be reformulated as \( 16 \). To use the result of [Lemma 4.2] let \( A_0 = -T^T p \), \( A_1 = T^T \), and \( c_1 = -1 \). We know that \( \exists a_0 \neq 0 \), such that \( a_0^T T^T a_0 \neq -1 < 0 \). It then follows that any primal argument (the attack vector \( a \)) satisfying all the KKT conditions, along with a dual argument (the Lagrange multiplier \( \gamma^* \)), is a globally optimal primal attack vector.

We show in the proof below that, when \( \gamma^* \) is the optimal value of \( 6 \), any attack vector of the form \( 9 \), which fulfills the conditions of [Lemma 4.2] satisfies all the KKT conditions and thus proving the statement of [Lemma 4.2]. To begin with, with an abuse of notation, let the stacked attack vector resulting from \( 9 \) be represented by \( a \).

**Primal feasibility** requires that
\[
a^T T^T a \leq 1.
\]
Given that the attack vector is of the form \( 9 \), \( 21 \) can be rewritten as \( v^T T^T a \leq 1 \). From the lemma statement, we know that \( v^T T^T a = 1 \).

**Dual feasibility** requires that \( \gamma^* \geq 0 \). This is satisfied since it is a constraint to the optimization problem \( 6 \).

**Complementary slackness** holds if \( \gamma^* (a^T T^T a - 1) = 0 \). We showed in KKT condition 1 that \( a^T T^T a - 1 = 0 \). Thus complementary slackness holds.

**Stationarity** requires that \( (\gamma^* T^T a - T^T a) \leq 0 \). Simplifying it further, we obtain
\[
(\gamma^* T^T a - T^T a) v = 0.
\]
To this end, consider the term \( v^T (\gamma^* T^T a - T^T a) v \). This term can be rewritten as the cost function of \( 8 \).

From [Theorem 4.1] we know that the optimal value of the cost function can be characterized as \( -x^T [0] X_0 [0] \). Since \( x^T = 0 \), \( 23 \) holds from which \( 22 \) follows.

\[
v^T T^T a = a^T T^T a = 0.
\]

**KKT condition 5** requires that requires that
\[
\gamma^* T^T a - T^T a = 0.
\]
Since \( 24 \) is a constraint of \( 9 \), KKT condition 5 holds. We have thus proven that that the attack vector of the form \( 9 \), when satisfying the condition of [Lemma 4.2] is a global maximizer to the optimization problem \( 5 \) (or minimizer to the optimization problem \( 10 \)). This concludes the proof.

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