SQUARE-ROOT CANCELLATION FOR SUMS OF FACTORIZATION FUNCTIONS OVER SHORT INTERVALS IN FUNCTION FIELDS

WILL SAWIN

Abstract. We present new estimates for sums of the divisor function, and other similar arithmetic functions, in short intervals over function fields. (When the intervals are long, one obtains a good estimate from the Riemann hypothesis.) We obtain an estimate that approaches square root cancellation as long as the characteristic of the finite field is relatively large. This is done by a geometric method, inspired by work of Hast and Matei, where we calculate the singular locus of a variety whose $F_q$-points control this sum. This has applications to highly unbalanced moments of $L$-functions.

1. Introduction

In this paper, we study cancellation of sums of arithmetic functions of functions of polynomials over a finite field in “short intervals” - that is, the sum over the set of monic polynomials with a fixed set of leading terms. For the divisor function, our estimates approach square-root cancellation over finite fields of sufficiently large characteristic:

Theorem 1.1 (Corollary 2.11). For natural numbers $n, m$ with $n \geq m$, a finite field $\mathbb{F}_q$ of characteristic $p$, and $c_1, \ldots, c_m \in \mathbb{F}_q$,

$$\left| \sum_{\substack{f \in \mathbb{F}_q[T] \text{monic} \\text{f}=T^n+c_1T^{n-1}+\ldots+c_mT^{n-m}+...}} d_k(f) - \binom{n+k-1}{k-1} q^{n-m} \right| \leq 3 \binom{n+k-1}{k-1} (k+2)^{n+m} \sqrt{q^{n-m+\lfloor \frac{n}{p} \rfloor - \lfloor \frac{m}{p} \rfloor} + 1}.$$

Note that the error term is $O \left( \left( q^{n-m} \right)^{1/2 + \epsilon} \right)$ as long as $p$ and $q^{\left( 1 - \frac{m}{n} \right) \frac{1}{\log k + 2}}$ are large with respect to $1/\epsilon$.

The variance of the sum of the divisor function in short intervals was calculated by Keating, Rodgers, Roditty-Gershon, and Rudnick [2018] in the $q \to \infty$ limit. Thus, their result controls the average size of the error term for this sum, while Corollary 2.11 controls the worst case. A worst case bound with a savings of $\sqrt{q}$ was proven by Bank, Bary-Soroker, and Rosenzweig [2015] for the occurrence of any polynomial factorization type in short intervals, which would in particular imply a similar estimate for the divisor function. For the M"obius function, a worst case result was proved in small characteristic by Shusterman [2018], giving some power savings, but not square-root cancellation.

Similar estimates holds for divisor-like arithmetic functions where we weight the contribution of a given factorization of a polynomial by some function of the degrees of the
factors. For arbitrary “factorization functions”, such as the Möbius and von Mangoldt functions, our results approach square root cancellation in the large $q$ limit.

Using the divisor sum estimate, we also obtain estimates for certain moments of $L$-functions in the large conductor limit. Specifically, these are the moments of the Dirichlet $L$-functions of the set $S_{m,q}$ of primitive even Dirichlet characters of $\mathbb{F}_q[T]$ with modulus $T^m$ (after apply the field automorphism $T \mapsto T^{-1}$, these give Hecke characters constant on short intervals, explaining the relation). Again, we let $\mathbb{F}_q$ be a finite field of characteristic $p$.

**Theorem 1.2** (Corollary 3.2). Let $m \geq 1$ and $r \geq 0$ be natural numbers and let $\alpha_1, \ldots, \alpha_r$ be complex numbers with nonnegative real part.

\[
\frac{1}{q^m - q^{m-1}} \sum_{\chi \in S_{m,q}} \prod_{i=1}^{r} L(1/2 + \alpha_i, \chi) = \prod_{i=1}^{r} \frac{1}{1 - q^{-m(1/2 + \alpha_i)}} + O\left(m^r (r + 2)^{r(m-1)+1} \sqrt{q^{-m+\left\lceil \frac{r(m-1)}{p} \right\rceil}} + 1\right).
\]

**Theorem 1.3** (Corollary 3.3). Let $m \geq 1$, $r \geq 0$, and $s \geq 1$ be natural numbers, and let $\alpha_1, \ldots, \alpha_r$ be complex numbers with nonnegative real part.

\[
\frac{1}{q^m - q^{m-1}} \sum_{\chi \in S_{m,q}} \epsilon^s \prod_{i=1}^{r} L(1/2+\alpha_i, \chi) = O\left(m^r (r + s + 2)^{(r+s)(m-1)+1} \sqrt{q^{-m+\left\lceil \frac{(r+s)(m-1)}{p} \right\rceil}} + 1\right).
\]

Again, note that the error terms are $O\left((q^m)^{\epsilon - \frac{1}{2}}\right)$ as long as $p$ and $q^{(r+s)\log(r+s+2)}$ are large with respect to $\epsilon$.

Katz [2013] proved estimates for the distribution of this $L$-function, and thus in particular on its moments, with a savings of $\sqrt{q}$. While there has been significant work on the moments of $L$-functions over function fields outside the large $q$ aspect, we are not aware of any work on this particular family, nor of any work dealing with arbitrarily high power moments.

These bounds arise from the stable cohomology approach to analytic number theory over function fields (see e.g. Ellenberg, Venkatesh, and Westerland [2016] and Shende and Tsimerman [2017]). In this strategy, some desired sum or counting problem over a polynomial ring $\mathbb{F}_q[T]$ is expressed as the number of $\mathbb{F}_q$-points of some higher-dimensional variety. Using the Grothendieck-Lefschetz fixed point formula, this is viewed as the sum of traces of Frobenius on the cohomology groups of the higher-dimensional variety. The high-degree cohomology groups, which can give the largest contribution, are calculated, shown to vanish, or otherwise controlled, while the low-degree cohomology groups merely need to have their dimension bounded.

Our specific method is based on one applied by Hast and Matei [2018] to the moments of sums of arithmetic functions, including the divisor function, in short intervals. They observed that the relevant variety is an affine cone on a projective complete intersection,
and used general results that estimate the high-degree cohomology of such a complete intersection in terms of the dimension of its singular locus. However, because the dimension was relatively large, the numerical result they obtained was not new - it follows directly from Weil’s Riemann hypothesis. Applying the same idea to sums of arithmetic functions in short intervals directly requires generalizing the results from affine cones to more general affine complete intersections, a straightforward etale cohomology calculation. Once this is done, we can show by a trick involving the logarithmic derivative (Lemma 2.3) that the dimension of the singular locus is much smaller than it was for the variety studied by Hast and Matei, and we instead obtain a new result, one approaching square-root cancellation.

We expect that these results can be generalized to sums in arithmetic progressions and to moments of L-functions over other families of Dirichlet characters. The main difficulty is finding a suitable compactification.

This research was partially conducted during the period I served as a Clay Research Fellow, and partially conducted during the period I was supported by Dr. Max Rössler, the Walter Haefner Foundation and the ETH Zurich Foundation. I would like to thank Emmanuel Kowalski for helpful conversations and Jon Keating for the inspiration to attack this problem.

2. Proofs

We work over a finite field \( \mathbb{F}_q \) of characteristic \( p \). Let \( n \) and \( m \) be natural numbers with \( n \geq m \).

Let \( X_{n,m,(c_1,\ldots,c_m)} \) be the subspace of \( \mathbb{A}^n \) with variables \( (a_1,\ldots,a_n) \) defined by the system of \( m \) equations \( \prod_{i=1}^n (1 - Ta_i) = 1 + c_1 T + c_2 T^2 + \cdots + c_m T^m \) modulo \( T^{m+1} \).

Let \( \overline{X}_{n,m,(c_1,\ldots,c_m)} \) be its projective closure, i.e. the subspace of \( \mathbb{P}^n \) with variables \( (a_1 : \cdots : a_n : z) \) defined by the system of \( m \) equations \( \sum_{|S|=k} \prod_{i\in S} a_i = (-1)^k c_k z^k \) for \( k \) from 1 to \( m \). Let \( D_{n,m,(c_1,\ldots,c_m)} = \overline{X} - X \) be the divisor defined by \( z = 0 \). We will drop these subscripts when the meaning is clear.

**Lemma 2.1.** The dimension of \( \overline{X}_{n,m,(c_1,\ldots,c_m)} \) is \( n - m \). In particular, \( \overline{X}_{n,m,(c_1,\ldots,c_m)} \) is a complete intersection.

**Proof.** It suffices to check that the affine cone on \( \overline{X}_{n,m,(c_1,\ldots,c_m)} \) has dimension \( n - m + 1 \), or that for each value of \( z \), the possible values of \( a_1,\ldots,a_n \) are \( n - m \)-dimensional. The polynomial \( \prod_{i=1}^n (1 - Ta_i) \) lies in an \( n \)-dimensional affine space parameterizing polynomials of degree \( \leq n \) with constant term 1. If \( m \leq n \), then the conditions on the first \( m \) nonconstant terms are linearly independent, so the space of polynomials in \( n - m \)-dimensional. Then the space of factorizations of the polynomial into \( n \) linear factors \( (1 - Ta_i) \) is a finite cover with the same dimension.

\[ \square \]

**Lemma 2.2.** \( \overline{X} \) is a smooth scheme and \( D \) is a smooth divisor away from the locus where \( |\{a_1,\ldots,a_n\}| \leq m - 1 \).
Proof. At a point \((a_1 : \cdots : a_n : z)\) that is a solution of the defining equations, \(X\) is smooth and \(D\) is a smooth divisor unless some linear combination of the equations has vanishing derivatives with respect to the variables \(a_1, \ldots, a_n\). Because the equations have different degrees, we must work with the explicit coordinates \(a_1, \ldots, a_n, z\), not up to scaling, to define this linear combination.

So if \((a_1, \ldots, a_n, z)\) is singular then for some \((\lambda_1, \ldots, \lambda_m)\) we have

\[
\frac{\partial}{\partial a_j} \sum_{k=1}^{m} \lambda_k \sum_{S \subseteq \{1, \ldots, n\}, |S| = k} \prod_{i \in S} a_i = 0
\]

for all \(j\) from 1 to \(n\). Now

\[
\sum_{k=1}^{m} \lambda_k \sum_{S \subseteq \{1, \ldots, n\}, |S| = k} \prod_{i \in S} a_i
\]

is the coefficient of \(T^m\) in

\[
\left( \sum_{k=1}^{m} \lambda_k T^{m-k} \right) \left( \prod_{i=1}^{n} (1 - Ta_i) \right).
\]

So its derivative with respect to \(a_j\) is the coefficient of \(T^m\) in

\[
\left( \sum_{k=1}^{m} \lambda_k T^{m-k} \right) \left( \prod_{i=1}^{n} (1 - Ta_i) \right) \frac{-T}{1 - Ta_j}.
\]

Let \(d_i\) be such that

\[
\left( \sum_{k=1}^{m} \lambda_k T^{m-k} \right) \left( \prod_{i=1}^{n} (1 - Ta_i) \right) (-T) = \sum_{i=1}^{\infty} d_i T^i.
\]

Then this derivative is the coefficient of \(T^m\) in

\[
\frac{\sum_{i=1}^{\infty} d_i T^i}{1 - Ta_j},
\]

which is equal to

\[
\sum_{i=1}^{m} d_i a_j^{m-i}.
\]

If this vanishes for all \(j\) then \(a_j\) is the root of a polynomial equation of degree \(\leq m - 1\). This polynomial equation is nonzero as \((\sum_{k=1}^{m} \lambda_k T^{m-k})\) is not divisible by \(T^m\) so

\[
\left( \sum_{k=1}^{m} \lambda_k T^{m-k} \right) \left( \prod_{i=1}^{n} (1 - Ta_i) \right) (-T)
\]

is not divisible by \(T^{m+1}\). Thus, for all \(j\), \(a_j\) lies in a fixed set of cardinality at most \(m - 1\).

\[\square\]

Lemma 2.3. The locus in \(\overline{X}\) where \(|\{a_1, \ldots, a_n\}| \leq m - 1\) has dimension at most \(\left\lfloor \frac{p}{q} \right\rfloor - \left\lfloor \frac{m}{p} \right\rfloor\).
Proof. It is sufficient to show that the intersection of this locus with $D$ has dimension at most $\left\lfloor \frac{n}{p} \right\rfloor - \left\lfloor \frac{m}{p} \right\rfloor - 1$, because $D$ is a projective hyperplane. Fix a point in this intersection and observe that

$$\frac{\partial}{\partial T} \prod_{i=1}^{n} (1 - Ta_i) = -\sum_{i=1}^{n} \frac{-a_i}{1 - Ta_i}.$$

Because $(a_1 : \cdots : a_n : z)$ lies in $D$, we have $z = 0$ and so $\prod_{i=1}^{n} (1 - Ta_i) = 1$ modulo $T^{m+1}$, and so $\frac{\partial}{\partial T} \prod_{i=1}^{n} (1 - Ta_i)$ is divisible by $T^m$. Hence the numerator of the left side is divisible by $T^m$ while the denominator is prime to $T$. Hence the $T$-adic valuation of the left side is at least $m$. On the other hand, examining the right side, its denominator has degree at most $\left| \{a_1, \ldots, a_n\} \right| \leq m - 1$ and it vanishes at $\infty$, hence its numerator has degree $\leq m - 2$. Thus the $T$-adic valuation of the right side is at most $m - 2$ unless the right side vanishes. Because the first possibility is a contradiction, both sides vanish, and

$$\frac{\partial}{\partial T} \prod_{i=1}^{n} (1 - Ta_i) = 0.$$

Hence $\prod_{i=1}^{n} (1 - Ta_i)$ is a polynomial in $T^p$. Furthermore, because $(a_1 : \cdots : a_n : 0)$ lies in $\overline{X}$, the coefficients of $T$ through $T^m$ must vanish. Thus the coefficient of $T^d$ in $\prod_{i=1}^{n} (1 - Ta_i)$ can only be nonzero in this polynomial if $d = 0$ or $m < d \leq n$ and $p|d$. The coefficient of $T^0$ is necessarily 1 and the number of remaining coefficients is $\left\lfloor \frac{n}{p} \right\rfloor - \left\lfloor \frac{m}{p} \right\rfloor$. Hence the space of such polynomials has dimension $\left\lfloor \frac{n}{p} \right\rfloor - \left\lfloor \frac{m}{p} \right\rfloor$, and the space of possible factorizations has the same dimension. Because the coordinates are only well-defined up to scaling, the dimension of the intersection of the singular locus with $D$ is $\left\lfloor \frac{n}{p} \right\rfloor - \left\lfloor \frac{m}{p} \right\rfloor - 1$, as desired.

The following lemma is a variant of [Hooley, 1991, appendix by Nicholas M. Katz, assertion (2) in proof of Theorem 1] and we prove it by adapting Katz’s method, though other proofs, for instance by applying Katz’s result, are surely available.

Lemma 2.4. Let $\overline{X}$ be a complete intersection in projective space, let $D$ be a hyperplane in $\overline{X}$, and let $X = \overline{X} - D$. Let $Z$ be the complement of the largest open subset of $\overline{X}$ where $\overline{X}$ is smooth and $D$ is a smooth divisor. Then $H^i_c(X, \mathbb{Q}_\ell) = 0$ for $\dim Z + \dim X + 1 < i < 2\dim X$. If $\dim Z < \dim X - 1$, then $H^2_c(X, \mathbb{Q}_\ell) = 0$.

Proof. Let $d = \dim X$.

Let us first check the claims in the case that $Z$ is empty. By convention, in this case $\dim Z = -1$, so we must show that $H^i_c(X, \mathbb{Q}_\ell) = 0$ for $d < i < 2d$ and $H^{2d}_c(X, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-d)$. To do this, we apply the excision long exact sequence to the open set $X$ of $\overline{X}$. In degrees $> d$, $H^i_c(\overline{X}, \mathbb{Q}_\ell)$ is generated by powers of the hyperplane class, and these powers are nonzero exactly in the even degrees from 0 to $2d$. Similarly, in degrees $> d - 1$, $H^i_c(D, \mathbb{Q}_\ell)$ is generated by powers of the hyperplane class, and these powers are nonzero exactly in the even degrees from 0 to $2d - 2$. So the natural map $H^i_c(\overline{X}, \mathbb{Q}_\ell) \to H^i_c(D, \mathbb{Q}_\ell)$ is surjective in degrees $> d - 1$ because the image contains all powers of the hyperplane class, it is injective in degrees $i$ with $d < i < 2d$ because the hyperplane class is sent to itself, and it is zero in degree $2d$. Applying excision, we see that $H^i_c(X, \mathbb{Q}_\ell) = 0$ for
\(d < i < 2d\) and \(H^{2d}(X, \mathbb{Q}_\ell) = H^d_c(\overline{X}, \mathbb{Q}_\ell)\) is generated by the \(d\)th power of the hyperplane class and thus is \(\mathbb{Q}_\ell(-d)\).

Now we prove the claims for an arbitrary complete intersection. A generic such complete intersection is smooth and has \(D\) a smooth divisor. So we may find a one-parameter family \(\overline{X}_t\) of complete intersections containing a family of hyperplanes \(D_t\) such that \(\overline{X}_0 = \overline{X}\) and \(D_0 = D\), such that \(X_t\) is smooth and \(D_t\) is a smooth divisor for generic \(t\).

Let \(j : \overline{X} - D \to \overline{X}\) be the open immersion. We have \(H^i_c(X, \mathbb{Q}_\ell) = H^i_c(\overline{X}_0, j_!\mathbb{Q}_\ell)\) by definition.

Then we have a long exact sequence
\[
H^*(\overline{X}_0, j_!\mathbb{Q}_\ell) \to H^*(\overline{X}_0, R\Psi j_!\mathbb{Q}_\ell) \to H^*(\overline{X}_0, R\Phi j_!\mathbb{Q}_\ell)
\]
with nearby and vanishing cycles taken with respect to \(t\). Because \(R\Phi j_!\mathbb{Q}_\ell[d]\) is perverse \cite{grothendieck1983exposes}, is supported in degree \(\leq \dim X\), its cohomology is supported in degree \(\leq \dim Z\), so \(H^i_c(\overline{X}_0, R\Psi j_!\mathbb{Q}_\ell)\) is supported degree \(\leq \dim Z + d\). Thus the first map is an isomorphism in degrees \(> \dim Z + d + 1\).

Because
\[
H^i_c(\overline{X}_0, R\Psi j_!\mathbb{Q}_\ell) = H^i_c(\overline{X}_0, j_!\mathbb{Q}_\ell) = H^i_c(X_\eta, \mathbb{Q}_\ell)
\]
and the cohomology groups are as stated for \(X_\eta\), they are also as stated for \(X\).

\[\square\]

**Proposition 2.5.** We have \(H^i_c(X_{n,m,(c_1,\ldots,c_m)}, \mathbb{Q}_\ell) = 0\) for \(n - m + \left\lfloor \frac{n}{p} \right\rfloor - \left\lfloor \frac{m}{p} \right\rfloor + 1 < i < 2(n - m)\). Furthermore, as long as \(\left\lfloor \frac{n}{p} \right\rfloor - \left\lfloor \frac{m}{p} \right\rfloor + 1 < n - m\), \(H^{2n-2m}_c(\mathbb{X}_{n,m,(c_1,\ldots,c_m)}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-n-m)\) with trivial \(S_n\) action.

**Proof.** By Lemma 2.4, the assumptions of Lemma 2.4 are satisfied, with dimension \(\dim X = n - m\). By Lemmas 2.2 and 2.3, \(\dim Z \leq \left\lfloor \frac{n}{p} \right\rfloor - \left\lfloor \frac{m}{p} \right\rfloor\). Hence by Lemma 2.4 we get vanishing in the stated degrees, and also the calculation of the top cohomology. The \(S_n\) action is trivial because the top cohomology factors through the trace map, which is \(S_n\)-invariant.

\[\square\]

Let \(\pi\) be a representation of \(S_n\). For \(f\) a polynomial of degree \(n\) over \(\mathbb{F}_q\), let \(V_f\) be the free \(\mathbb{Q}_\ell\) vector space generated by tuples \((a_1, \ldots, a_n)\in \mathbb{F}_q^n\) with \(\prod_{i=1}^n (T - a_i) = f\). Then \(V_f\) admits a natural \(S_n\) action, as well as an action of Frobenius. Let \(F_\pi(f) = \text{tr}(\text{Frob}_q, (V_f \otimes \pi)^{S_n})\).

Let \(B(\pi) = \sum_{i=0}^{2\dim X} \dim \left( (H^i_c(X_{n,m,(c_1,\ldots,c_m)}, \mathbb{Q}_\ell) \otimes \pi)^{S_n} \right)\).

**Remark 2.6.** The function \(F_\pi\) is a “factorization function” on monic polynomials of degree \(n\) associated to a representation \(\pi\) of \(S_n\). In the case when \(f\) is squarefree, \(V_f\) is simply \(\mathbb{Q}_\ell[S_n]\), with Frobenius acting by multiplication by some element in its conjugacy class, so \((V_f \otimes \pi)^{S_n}\) is simply \(V_f\) and \(F_\pi\) recovers the character of \(\pi\) evaluated on the conjugacy class of Frobenius. The function \(F_\pi\) is from several perspectives the most natural extension of this function from squarefree polynomials to all polynomials. It matches exactly \cite{gadishthesis, hast2018, hast2018arithmetic}.
Proposition 2.7. Let \( n > m \) be natural numbers. Let \( c_1, \ldots, c_m \) be elements of \( \mathbb{F}_q \) and let \( \pi \) be a representation of \( S_n \). Then

\[
\left| \sum_{f \in \mathbb{F}_q[T] \text{ monic}} F_\pi(f) - q^{-n-m} \dim(\pi^{S_n}) \right| \leq B(\pi) \sqrt{q^{n-m} + \left\lfloor \frac{n}{p} \right\rfloor + 1}.
\]

Proof. Let \( \rho : X_{n,m,(c_1,\ldots,c_m)} \to X_{n,m,(c_1,\ldots,c_m)}/S_n \) be the projection map. Then

\[
(H^\pi_c(X_{n,m,(c_1,\ldots,c_m)}/S_n, \rho_* \mathbb{Q}_\ell) \otimes \pi)^{S_n} = (H^\pi_c(X_{n,m,(c_1,\ldots,c_m)}/S_n, \rho_* \mathbb{Q}_\ell) \otimes \pi)^{S_n} = H^\pi_c(X_{n,m,(c_1,\ldots,c_m)}/S_n, (\rho_* \mathbb{Q}_\ell \otimes \pi)^{S_n}).
\]

By the Grothendieck-Lefschetz fixed point formula

\[
\sum_i (-1)^i \sum_{x \in X_{n,m,(c_1,\ldots,c_m)}/S_n(\mathbb{F}_q)} \text{tr}(\text{Frob}_q, (\rho_* \mathbb{Q}_\ell \otimes \pi)^{S_n}) = \sum_{f \in \mathbb{F}_q[T] \text{ monic}} \text{tr}(\text{Frob}_q, (\rho_* \mathbb{Q}_\ell \otimes \pi)^{S_n}).
\]

Because \( X_{n}/S_n \) is the moduli space of degree \( n \) monic polynomials, \( X_{n,m,(c_1,\ldots,c_m)}/S_n \) is the moduli space of degree \( n \) monic polynomials whose leading terms are \( T^n + c_1 T^{n-1} + \cdots + c_m T^{n-m} \). Hence we can view \( X_{n,m,(c_1,\ldots,c_m)}/S_n(\mathbb{F}_q) \) as the set of \( f \in \mathbb{F}_q[T] \), monic, with \( f = T^n + c_1 T^{n-1} + \cdots + c_m T^{n-m} + \cdots \). Thus to check that

\[
\sum_{x \in X_{n,m,(c_1,\ldots,c_m)}/S_n(\mathbb{F}_q)} \text{tr}(\text{Frob}_q, (\rho_* \mathbb{Q}_\ell \otimes \pi)^{S_n}) = \sum_{f \in \mathbb{F}_q[T] \text{ monic}} \text{tr}(\text{Frob}_q, (\rho_* \mathbb{Q}_\ell \otimes \pi)^{S_n})
\]

it suffices to show that, for the point \( x \) corresponding to the polynomial \( f \),

\[
\text{tr}(\text{Frob}_q, (\rho_* \mathbb{Q}_\ell \otimes \pi)^{S_n}) = F_\pi(f).
\]

The stalk of \( \rho_* \mathbb{Q}_\ell \) at \( f \) is the free vector space generated by the fiber of \( \rho \) over \( f \), which is \( V_f \). The actions of \( S_n \) and \( \text{Frob}_q \) on this vector space match what we defined for \( V_f \). So the trace function on the stalk is given by the same formula, as desired.

Thus we obtain

\[
\sum_i (-1)^i \sum_{f \in \mathbb{F}_q[T] \text{ monic}} \text{tr}(\text{Frob}_q, (H^\pi_c(X_{n,m,(c_1,\ldots,c_m)}, \mathbb{Q}_\ell) \otimes \pi)^{S_n}) = \sum_{f \in \mathbb{F}_q[T] \text{ monic}} F_\pi(f).
\]

We have \( H^{2(n-m)}(X_{n,m,(c_1,\ldots,c_m)}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-(n-m)) \) with trivial \( S_n \) action. So the contribution of \( i = 2(n-m) \) is \( q^{n-m} \pi^{S_n} \).

For each other \( i \), by Deligne’s theorem, all eigenvalues of \( \text{Frob}_q \) on \( H^i \) are bounded by \( q^{i/2} \), so the contribution of \( H^i \) is bounded by \( q^{i/2} \dim \left( (H^\pi_c(X_{n,m,(c_1,\ldots,c_m)}, \mathbb{Q}_\ell) \otimes \pi)^{S_n} \right) \). By
Proposition 2.5 we may assume \( i \leq n - m + \lfloor \frac{n}{p} \rfloor - \lfloor \frac{m}{p} \rfloor + 1 \). Summing over all \( i \), we obtain the stated bound.

**Proposition 2.8.** Let \( n_1, \ldots, n_k \) be natural numbers summing to \( n \) and let \( \pi \) be a summand of \( \text{Ind}_{S_{n_1} \times \cdots \times S_{n_k}}^{S_n} \mathbb{Q}_\ell \). Then \( B(\pi) \leq 3(k + 2)^{n + m} \).

**Proof.** We have

\[
\left( H_c^i(X_{n,m,(c_1,\ldots,c_m)}, \mathbb{Q}_\ell) \otimes \text{Ind}_{S_{n_1} \times \cdots \times S_{n_k}}^{S_n} \mathbb{Q}_\ell \right)^{S_n} = \left( H_c^i(X_{n,m,(c_1,\ldots,c_m)}, \mathbb{Q}_\ell) \right)^{S_{n-1}} \times \cdots \times \left( H_c^i(X_{n,m,(c_1,\ldots,c_m)}, \mathbb{Q}_\ell) \right)^{S_{n_k}} = H_c^i(X_{n,m,(c_1,\ldots,c_m)}/(S_{n_1} \times \cdots \times S_{n_k}), \mathbb{Q}_\ell).
\]

Now \( X_{n,m,(c_1,\ldots,c_m)}/(S_{n_1} \times \cdots \times S_{n_k}) \) is the moduli space of \( k \)-tuples of monic polynomials, of degrees \( n_1, \ldots, n_k \) whose product has leading terms \( T^{n_1} + c_1 T^{n_2} + \cdots + c_m T^{n_m} \). So it is a variety defined by \( m \) equations in \( n \) variables of degree at most \( k \). The statement then follows from \cite[Theorem 12]{Katz2001} (specifically, we take \( s = 0, f = 0, N = n, r = m, \) and \( F_1, \ldots, F_r \) to be these \( n \) equations).

**Definition 2.9.** For natural numbers \( n_1, \ldots, n_k \) with \( \sum_{i=1}^k n_i = n \), let \( d_k^{(n_1,\ldots,n_k)}(f) \) be the function that takes a monic polynomial \( f \in \mathbb{F}_q[t] \) of degree \( n \) to the number of tuples \( g_1, \ldots, g_k \) of monic polynomials in \( \mathbb{F}_q[t] \), with \( g_i \) of degree \( n_i \) for all \( i \), such that \( \prod_{i=1}^k g_i = f \).

It is clear that \( \sum_{\sum_{i=1}^k n_i = n} d_k^{(n_1,\ldots,n_k)}(f) \) is the ordinary \( k \)th divisor function \( d_k(f) \) of \( f \).

**Theorem 2.10.** For natural numbers \( n, m \) with \( n \geq m \), a finite field \( \mathbb{F}_q \) of characteristic \( p \), \( c_1, \ldots, c_m \in \mathbb{F}_q \), and natural numbers \( n_1, \ldots, n_k \) with \( \sum_{i=1}^k n_i = n \),

\[
\left| \sum_{f=\text{monic} \atop f=T^n+c_1 T^{n-1}+\cdots+c_m T^{n-m}+\cdots+1} d_k^{(n_1,\ldots,n_k)}(f) - q^{n-m} \right| \leq 3(k + 2)^{n + m} \sqrt{q^{n-m+\lfloor \frac{n}{p} \rfloor - \lfloor \frac{m}{p} \rfloor + 1}}.
\]

**Proof.** This follows from Propositions 2.7 and 2.8 once we check that \( F_{\text{Ind}_{S_{n_1} \times \cdots \times S_{n_k}}^{S_n} \mathbb{Q}_\ell}(f) = d_k^{(n_1,\ldots,n_k)}(f) \). To do this, observe that \( F_{\text{Ind}_{S_{n_1} \times \cdots \times S_{n_k}}^{S_n} \mathbb{Q}_\ell}(f) \) is the trace of \( \text{Frob}_q \) on \( (V_f \otimes \text{Ind}_{S_{n_1} \times \cdots \times S_{n_k}}^{S_n} \mathbb{Q}_\ell)^{S_n} = V_f^{S_{n_1} \times \cdots \times S_{n_k}} \). Now \( V_f^{S_{n_1} \times \cdots \times S_{n_k}} \) is the free vector space on the \( S_{n_1} \times \cdots \times S_{n_k} \)-orbits of factorizations of \( f \) into linear factors, which are simply the factorizations of \( f \) into factors of degree \( n_1, \ldots, n_k \). The Frobenius element acts by permuting these, hence its trace is equal to the number of Frobenius-fixed factorizations, which is the number of factorizations defined over \( \mathbb{F}_q \), as desired.

In the case \( n = m \), note that the only \( f \) appearing in the sum on the left is \( T^m + c_1 T^{m-1} + \cdots + c_m \), whose \( d_k^{(n_1,\ldots,n_k)} \) is between 0 and \( k^n \), and \( q^{n-m} \) is 1, so the left side is at most \( k^n - 1 \), and the right side is \( 3(k + 2)^{2n} \sqrt{q} \) so the inequality holds. 

\( \square \)
Corollary 2.11. For natural numbers $n, m$ with $n \geq m$, a finite field $\mathbb{F}_q$ of characteristic $p$, and $c_1, \ldots, c_m \in \mathbb{F}_q$,

\[
\left| \sum_{f \in \mathbb{F}_q[T] \text{monic}} d_k(f) - \binom{n+k-1}{k-1} q^{n-m} \right| \leq 3 \binom{n+k-1}{k-1} (k+2)^{n+m} \sqrt{q^{n-m} - \frac{n}{p}} + 1.
\]

Proof. This follows from Theorem 2.10 by summing over all possible $n_1, \ldots, n_k$. \qed

3. Moments of L-functions

Using the divisor function estimates, we immediately obtain results on moments of $L$-functions for the “short interval” family of characters. This is the family of Hecke characters of $\mathbb{F}_q(t)$ ramified only at $\infty$, and split at 0. Viewed this way, it seems like an analogue of the characters $n \mapsto n^t$ of the integers, which are “ramified at $\infty$” in some sense, and so its moments might appear like the moments of the Riemann zeta function. However, the fact that the characters are split at 0 makes this more difficult, as if we fixed $t$ to lie in the arithmetic progression $\frac{2\pi i}{\log 2} \mathbb{Z}$.

This problem, though sometimes studied [Li and Radziwiłł, 2015], may seem artificial, but if we flip 0 and $\infty$, we get an equivalent family of characters, the even Dirichlet characters mod $T^{m+1}$ of monic polynomials at $\mathbb{F}_q[t]$. Here, the natural definition of Dirichlet characters over $\mathbb{F}_q[t]$ forces them to be split at $\infty$. The comparison to Dirichlet characters probably provides the best analogy for the difficulty of the moments of this family by traditional analytic means, though we invite anyone interested to study these moments (and others we can’t handle, such as the fourth absolute moment of $L(1/2, \chi)$ in this family) from a classical perspective and determine their difficulty more precisely.

Definition 3.1. For $m$ a natural number and $\mathbb{F}_q$ a finite field, let $S_{m,q}$ be the set of all primitive even Dirichlet characters $\mathbb{F}_q[x]/x^{m+1} \rightarrow \mathbb{C}^\times$, which has cardinality $q^m - q^{m-1}$. View elements of $S_{m,q}$ as characters of monic polynomials in $\mathbb{F}_q[T]$ by sending a monic $f$ of degree $d$ to $\chi(f(x^{-1})x^{\deg f})$, i.e. as characters depending on the $m + 1$ leading terms. Form the associated $L$-functions

\[
L(s, \chi) = \sum_{\substack{f \in \mathbb{F}_q[T] \text{monic} \\deg f = m-1}} \chi(f)|f|^{-s}
\]

where $|f| = q^{\deg f}$.

Then it is easy to see that $L(s, \chi)$ is a polynomial in $q^{-s}$ of degree at most $m - 1$, and not much harder to verify the functional equation $L(s, \chi) = \epsilon_\chi q^{(m-1)(s-1/2)} L(1-s, \overline{\chi})$ where $\epsilon_\chi = q^{-(m-1)/2} \sum_{\substack{f \in \mathbb{F}_q[T] \text{monic} \\deg f = m-1}} \chi(f)$.

In the proofs of the below results, it will be convenient to use big $O$ notation, where $f = g + O(T^n)$ denotes that $f - g$ is a polynomial of degree at most $n$. 
Corollary 3.2. Let \( \alpha_1, \ldots, \alpha_r \) be complex numbers with nonnegative real part.

\[
\frac{1}{q^m - q^{m-1}} \sum_{\chi \in S_{m,q}} \prod_{i=1}^{r} L(1/2 + \alpha_i, \chi) = \prod_{i=1}^{r} \frac{1}{1 - q^{-n_i(1/2+\alpha_i)}} + O \left( m^r (r + 2)^{r(m-1)+m} \sqrt{q^{-m+\left\lfloor \frac{r(m-1)}{p} \right\rfloor - \left\lfloor \frac{m}{p} \right\rfloor + 1}} \right).
\]

Proof. We have

\[
\prod_{i=1}^{r} L(1/2 + \alpha_i, \chi) = \sum_{n_1, \ldots, n_r \in \{0, \ldots, m-1\}} \left( \prod_{i=1}^{r} q^{-n_i(1/2+\alpha_i)} \right) \sum_{\substack{f_1, \ldots, f_r \in \mathbb{F}_q[t] \text{ monic} \\text{deg}(f_i)=n_i}} \chi(f_1 f_2 \cdots f_r).
\]

Thus we have

\[
\sum_{\chi \in S_{m,q}} \chi(f) = q^m \quad \text{if the leading } m + 1 \text{ terms of } f \text{ are } q^{\text{deg } f} + 0 \text{ and } 0 \text{ otherwise, minus } q^{m-1} \text{ if the leading } m \text{ terms of } f \text{ are degree } m - 1 \text{ and } 0 \text{ otherwise.}
\]

We split these up and obtain

\[
q^m \sum_{n_1, \ldots, n_r \in \{0, \ldots, m-1\}} \left( \prod_{i=1}^{r} q^{-n_i(1/2+\alpha_i)} \right) \quad \sum_{\substack{f_1, \ldots, f_r \in \mathbb{F}_q[t] \text{ monic} \\text{deg}(f_i)=n_i}} \chi(f_1 f_2 \cdots f_r) \quad 1
\]

\[
- q^{m-1} \sum_{n_1, \ldots, n_r \in \{0, \ldots, m-1\}} \left( \prod_{i=1}^{r} q^{-n_i(1/2+\alpha_i)} \right) \quad \sum_{\substack{f_1, \ldots, f_r \in \mathbb{F}_q[t] \text{ monic} \\text{deg}(f_i)=n_i}} \chi(f_1 f_2 \cdots f_r) \quad 1
\]

For the first term, if \( \sum_{i=1}^{r} n_i \leq m \), this equation is only satisfied if \( f_i = T^{n_i} \) for all \( i \). Hence these terms contribute

\[
q^m \sum_{\sum_{i=1}^{r} n_i \leq m} \left( \prod_{i=1}^{r} q^{-n_i(1/2+\alpha_i)} \right) = q^m \prod_{i=1}^{r} \frac{1}{1 - q^{-n_i(1/2+\alpha_i)}} + O \left( q^m m^{r-1} q^{-(m+1)/2} \right).
\]

Each of the remaining terms is exactly the sum of Theorem 2.10 with \( c_1, \ldots, c_m = 0 \). Hence these terms contribute

\[
q^m \sum_{\sum_{i=1}^{r} n_i > m} \left( \prod_{i=1}^{r} q^{-n_i(1/2+\alpha_i)} \right) \left( q^{\sum_{i=1}^{r} n_i - m} + O \left( (r + 2)^{n+m} \sqrt{q^{\sum_{i=1}^{r} n_i - m + \left\lfloor \frac{r(m-1)}{p} \right\rfloor - \left\lfloor \frac{m}{p} \right\rfloor + 1}} \right) \right).
\]
Similarly, for \( m - 1 \), the terms with \( \sum_{i=1}^r n_i \leq m - 1 \) contribute \( q^{m-1} \prod_{i=1}^r \frac{1}{1-q^{-n_i(1/2+\alpha_i)}} + O(q^{m-1}m^{-1}q^{-(m/2)}) \) and the remaining terms contribute

\[
q^{-1} \sum_{n_1, \ldots, n_r \in \{0, \ldots, m-1\} \atop \sum_{i=1}^r n_i > m-1} \left( \prod_{i=1}^r q^{-n_i(1/2+\alpha_i)} \right) \left( q^{\sum_{i=1}^r n_i+1-m} + O\left( (r+2)^{n+m-1} \sqrt{q^{\sum_{i=1}^r n_i+1-m+\left[ \sum_{i=1}^r n_i \right]/p} - \left[ \frac{m}{p} \right] + 1 \right) \right)
\]

The \( q^{\sum_{i=1}^r n_i-m} \) terms cancel, except when \( \sum_{i=1}^r n_i \) is exactly \( m \), in which case they contribute \( O(m^{-1}q^{m/2}) \). This leaves only the error terms, which are of size

\[
q^m \sum_{n_1, \ldots, n_r \in \{0, \ldots, m-1\}} O\left( \left( \prod_{i=1}^r q^{-n_i(1/2+\alpha_i)} \right) (r+2)^{\sum_{i=1}^r n_i+m} \sqrt{q^{\sum_{i=1}^r n_i-m+\left[ \sum_{i=1}^r n_i \right]/p} - \left[ \frac{m}{p} \right] + 1} \right)
\]

\[
= \sum_{n_1, \ldots, n_r \in \{0, \ldots, m-1\}} O\left( (r+2)^{\sum_{i=1}^r n_i+m} \sqrt{q^{m+\left[ \frac{r(m-1)}{p} \right] - \left[ \frac{m}{p} \right] + 1}} \right)
\]

\[
= O\left( m^r (r+2)^{r(m-1)+m} \sqrt{q^{m+\left[ \frac{r(m-1)}{p} \right] - \left[ \frac{m}{p} \right] + 1}} \right).
\]

Since the other error terms are also bounded by this one, this is only error term we need.

\[
\square
\]

We can prove also a similar estimate for a moment twisted by a positive power of \( \epsilon_\chi \).

**Corollary 3.3.** \( m \geq 1, r \geq 0, \) and \( s \geq 1 \) be natural numbers, and let \( \alpha_1, \ldots, \alpha_r \) be complex numbers with nonnegative real part.

\[
\frac{1}{q^m - q^{m-1}} \sum_{\chi \in S_{m,q}} \epsilon_\chi^s \prod_{i=1}^r L(1/2+\alpha_i, \chi) = O\left( m^r (r + s + 2)^{(r+s)(m-1)+m} \sqrt{q^{-m+\left[ \frac{(r+s)(m-1)}{p} \right] - \left[ \frac{m}{p} \right] + 1}} \right).
\]

**Proof.** Letting \( n_{r+1}, n_{r+s} = m - 1 \), we have

\[
\epsilon_\chi^s \prod_{i=1}^r L(1/2+\alpha_i, \chi) = \sum_{n_1, \ldots, n_r \in \{0, \ldots, m-1\}} q^{-s(m-1)/2} \left( \prod_{i=1}^r q^{-n_i(1/2+\alpha_i)} \right) \sum_{\text{monic } f_1, \ldots, f_{r+s} \in F_q[t] \atop \deg(f_i) = n_i} \chi(f_1f_2 \cdots f_{r+s}).
\]

Summing over \( \chi \), we obtain

\[
\sum_{\chi \in S_{m,q}} \epsilon_\chi^s \prod_{i=1}^r L(1/2+\alpha_i, \chi) = \sum_{n_1, \ldots, n_r \in \{0, \ldots, m-1\}} q^{-s(m-1)/2} \left( \prod_{i=1}^r q^{-n_i(1/2+\alpha_i)} \right)
\]
\[
\left( q^m \sum_{f_1, \ldots, f_{r+s} \in \mathbb{F}_q[t]} 1 - q^{m-1} \sum_{f_1, \ldots, f_{r+s} \in \mathbb{F}_q[t]} 1 \right).
\]

Observe that in the sum

\[
\sum_{f_1, \ldots, f_{r+s} \in \mathbb{F}_q[t] \atop \text{monic } \deg(f_i) = n_i} \frac{1}{\prod_{i=1}^{r+s} f_i = T^{\sum_{i=1}^{r} n_i} + O(T^{\sum_{i=1}^{r} n_i} - m)}
\]

there is always at exactly one value of \( f_{r+s} \) that satisfies the equation for any \( f_1, \ldots, f_{r+s-1} \) and thus the sum is equal to \( q^{\sum_{i=1}^{r} n_i + (s-1)(m-1)} \). Hence the difference has the form

\[
\sum_{n_1, \ldots, n_r \in \{0, \ldots, m-1\}} q^{-s(m-1)/2} \left( \prod_{i=1}^{r} q^{-n_i(1/2 + \alpha_i)} \right)
\]

Each term is \( q^m \) times the left side of Theorem 2.10 with \( k = r + s \) and \( c_1, \ldots, c_m = 0 \). Applying Theorem 2.10 we see that in each term with \( (\sum_{i=1}^{r} n_i) + s(m - 1) \geq m \), we have

\[
q^m \sum_{f_1, \ldots, f_{r+s} \in \mathbb{F}_q[t] \atop \text{monic } \deg(f_i) = n_i} \frac{1 - q^{\sum_{i=1}^{r} n_i + s(m-1)}}{\prod_{i=1}^{r+s} f_i = T^{\sum_{i=1}^{r} n_i} + O(T^{\sum_{i=1}^{r} n_i} - m - 1)}
\]

\[
\leq 3(r + s + 2) \sum_{i=1}^{r} n_i + s(m-1) + m \sqrt{q^{\sum_{i=1}^{r} n_i + s(m-1) + m + \frac{\sum_{i=1}^{r} n_i + s(m-1)}{p} - \left\lfloor \frac{m}{p} \right\rfloor + 1}}.
\]

Multiplying by \( q^{-s(m-1)/2} \left( \prod_{i=1}^{r} q^{-n_i(1/2 + \alpha_i)} \right) \), this term is at most

\[
3(r + s + 2) \sum_{i=1}^{r} n_i + s(m-1) + m \sqrt{q^{m + \frac{\sum_{i=1}^{r} n_i + s(m-1)}{p} - \left\lfloor \frac{m}{p} \right\rfloor + 1}}.
\]

Summing over \( n_1, \ldots, n_r \), we get

\[
3m^r (r + s + 2)^{r + s(m-1) + m} \sqrt{q^{m + \frac{(r+s)(m-1)}{p} - \left\lfloor \frac{m}{p} \right\rfloor + 1}}.
\]
The only remaining term occurs when \( s = 1 \) and \( n_1, \ldots, n_r = 0 \). This term is simply \( q^{(-m-1)/2}(q^m - q^{m-1}) \) and is bounded by big \( O \) term. So we obtain.

\[
\frac{1}{q^m - q^{m-1}} \sum_{\chi \in \mathcal{S}_{m,q}} \epsilon_{\chi} \prod_{i=1}^{r} L(1/2 + \alpha_i, \chi) = O \left( m^r (r + s + 2)^{(r+s)(m-1)+m} \sqrt{q}^{-m+\left[\frac{(r+s)(m-1)}{p}\right]-\left[\frac{m}{p}\right]+1} \right). 
\]

\[\square\]

References

Efrat Bank, Lior Bary-Soroker, and Lior Rosenzweig. Prime polynomials in short intervals and in arithmetic progressions. *Duke Mathematical Journal*, 164(2), 2015. URL https://projecteuclid.org/euclid.dmj/1422627049.

Jordan S. Ellenberg, Akshay Venkatesh, and Craig Westerland. Homological stability for Hurwitz spaces and the Cohen-Lenstra conjecture over function fields. *Annals of Mathematics*, 183:729–786, 2016. URL https://doi.org/10.4007/annals.2016.183.3.1.

Nir Gadish. A trace formula for the distribution of rational \( G \)-orbits in ramified covers, adapted to representation stability. *New York Journal of Mathematics*, to appear. URL https://arxiv.org/abs/1703.01710.

Alexander Grothendieck. *Séminaire de Géométrie Algébrique du Bois Marie - 1967-69 - Groupes de monodromie en géométrie algébrique - (SGA 7) - vol. 1*, volume 288 of *Lecture Notes in Mathematics*. Springer-Verlag, 1983. URL http://library.msri.org/books/sga/sga/pdf/sga7-1.pdf.

Daniel Rayor Hast and Vlad Matei. Higher moments of arithmetic functions in short intervals: a geometric perspective. *International Mathematics Research Notices*, 2018, 2018. URL https://doi.org/10.1093/imrn/rnx310.

C. Hooley. On the number of points on a complete intersection over a finite field. *Journal of Number Theory*, 38:338–358, 1991. URL https://doi.org/10.1016/0022-314X(91)90023-5.

Nicholas M. Katz. Sums of Betti numbers in arbitrary characteristic. *Finite Fields and Their Applications*, 7:29–44, 2001. URL https://web.math.princeton.edu/~nmk/BettiSum14.pdf.

Nicholas M. Katz. Witt vectors and a question of Keating and Rudnick. *International Mathematics Research Notices*, 2013(15):361373638, 2013. doi: 10.1093/imrn/rns144. URL http://imrn.oxfordjournals.org/content/early/2012/06/20/imrn.rns144.abstract.

Jon Keating, Brad Rodgers, Edva Roditty-Gershon, and Zeev Rudnick. Sums of divisor functions in \( \mathbb{F}_q[t] \) and matrix integrals. *Mathematische Zeitschrift*, 288:167–198, 2018. URL https://doi.org/10.1007/s00209-017-1884-1.

Xiannan Li and Maksym Radziwiłł. The Riemann zeta function on vertical arithmetic progressions. *International Mathematics Research Notices*, 2015(2):325–354, 2015. URL http://dx.doi.org/10.1093/imrn/rnt197.
Vivek Shende and Jacob Tsimerman. Equidistribution on the space of rank two vector bundles over the projective line. *Duke Mathematical Journal*, 166(18):3461–3504, 2017. URL [https://arxiv.org/abs/1307.8237](https://arxiv.org/abs/1307.8237).

Mark Shusterman. The geometric Möbius function and quadratic character sums. [https://arxiv.org/abs/1808.04001](https://arxiv.org/abs/1808.04001), 2018.