Wave Reflection and Transmission at the Interface of Convective and Stably Stratified Regions in a Rotating Star or Planet

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Abstract

We use a simplified model to study wave reflection and transmission at the interface of the convective region and stably stratified region (e.g., radiative zone in a star or stratification layer in a gaseous planet). The inertial wave in the convective region and gravito-inertial wave in the stably stratified region are considered. We begin with the polar area and then extend to any latitude. Six cases are discussed (see Table 1), and in Case 2 both waves co-exist in both regions. Four configurations are further discussed for Case 2. The angles and energy ratios of wave reflection and transmission are calculated. It is found that wave propagation and transmission depend on the ratio of buoyancy frequency to rotational frequency. In a rapidly rotating star or planet, the wave propagates across the interface and most of the energy of the incident wave is transmitted to the other region, but in a slowly rotating star or planet, wave transmission is inhibited.

Unified Astronomy Thesaurus concepts: Stellar interiors (1606); Planetary interior (1248)

1. Motivation

The interior of a star or planet has a layered structure. In a solar-type star the convective zone sits on the radiative zone, while in a massive star the radiative zone sits on the convective zone. The radiative zone can be treated as a stably stratified region with the diffusion limit. In some gaseous planets, such as Saturn, a stably stratified layer sits on the convective region (Fuller 2014) and this layer is believed to filter out the non-axisymmetric components of magnetic field (Cao et al. 2011). In a rotating star or planet, an inertial wave (r mode) can be induced by the Coriolis force in the convective region (Ogilvie & Lin 2004; Wu 2005; Goodman & Lackner 2009; Ogilvie 2014). In a non-rotating star or planet, a gravity wave (g mode) can be induced by density stratification in the stably stratified region (Zahn 1977; Goldreich & Nicholson 1989; Goodman & Dickson 1998). In a rotating star or planet, a gravito-inertial wave can be induced in the stably stratified region due to the combined effect of rotation and density stratification (Dintrans et al. 1999).

An interesting problem is how the wave reflects and transmits at the interface of two regions. In the solar interior, it is believed that the g mode in the radiative region propagating outward cannot penetrate deep into the convective zone, such that it is difficult to find the g mode on the solar surface (García et al. 2007). However, in some other stars or planets, can the wave in one region transmit to the other region? If the stratified region is replaced with the rigid region, then the inertial wave in the convective region totally reflects at the interface (Goodman & Lackner 2009). However, for a “soft” stratified region, how much wave energy can be transmitted and how much can be reflected? In this paper I will try to answer this question with a simplified model, qualitatively and quantitatively.

2. Model

A simplified local geometry is studied in this paper. The convective region and stably stratified region (e.g., radiative zone in the stellar interior or stratification layer in a gaseous planet) are on two sides of an infinite plane, which represents the interface of the two regions. The rotational axis is normal to the plane, which simplifies the calculations. This is only valid near the vicinity of the polar area and we will discuss in Section 5 the more general situation that the rotational axis is not necessarily normal to the interface. A sound wave is too fast to interact with the inertial or internal gravity wave, and therefore we consider an incompressible fluid to filter out the sound wave. In this local geometry, the global spherical curvature is not considered. However, we will see in the next text that some interesting physics can be found in this simplified geometry.

In the convective region, the linear perturbation equations of momentum and mass conservation in a rotating frame read

\[
\begin{align*}
\frac{\partial u_r^i}{\partial t} &= -\frac{1}{\rho} \frac{\partial p'}{\partial x} + 2\Omega u_r^i \\
\frac{\partial u_r^i}{\partial t} &= -\frac{1}{\rho} \frac{\partial p'}{\partial x} - 2\Omega u_r^i \\
\frac{\partial u_r^i}{\partial t} &= -\frac{1}{\rho} \frac{\partial p'}{\partial y} \\
\frac{\partial u_r^i}{\partial t} &= -\frac{1}{\rho} \frac{\partial p'}{\partial z} \\
\frac{\partial u_r^i}{\partial x} + \frac{\partial u_r^i}{\partial y} + \frac{\partial u_r^i}{\partial z} &= 0.
\end{align*}
\]

The variables are written in their conventional manner and prime denotes Eulerian perturbation. Equation (1) describes the inertial wave caused by the Coriolis force. In the stably stratified region, the linear perturbation equations read

\[
\begin{align*}
\frac{\partial u_r^s}{\partial t} &= -\frac{1}{\rho} \frac{\partial p'}{\partial x} + 2\Omega u_r^s \\
\frac{\partial u_r^s}{\partial t} &= -\frac{1}{\rho} \frac{\partial p'}{\partial x} - 2\Omega u_r^s \\
\frac{\partial u_r^s}{\partial t} &= -\frac{1}{\rho} \frac{\partial p'}{\partial y} - \frac{\rho'}{\rho} g' \\
\frac{\partial p'}{\partial t} + \beta u_r^s &= 0 \\
\frac{\partial u_r^s}{\partial x} + \frac{\partial u_r^s}{\partial y} + \frac{\partial u_r^s}{\partial z} &= 0.
\end{align*}
\]
Equation (2) describes the gravito-inertial wave. Compared to Equation (1), a gravity term appears in the $z$ direction and an equation of density stratification is added. The parameter

$$\beta = \frac{d\rho}{dz} < 0$$ (3)

measures stratification and it is assumed to be constant in the local geometry, i.e., the WKB approximation due to the short wavelength of perturbations compared to the length scale of stratification. That is, the density is continuous across the interface but the density gradient is not. Here the Boussinesq approximation is used, i.e., in the equation of motion density is constant and density variation is considered only in the gravity term, and density stratification is only considered with $\beta$. This approximation is valid only for small variation of density. Usually in stars or gaseous planets this approximation cannot hold, and a fully compressible fluid or anelastic approximation should be considered. However, in a very small local region near interface the Boussinesq approximation still works.

We apply normal mode analysis to the linear perturbation equations. All the perturbations are expressed as a Fourier plane wave with its complex amplitude dependent on $z$ coordinate, e.g.,

$$u'_z = \Re \{ \hat{u}_z(z) \exp \{ i(k_x x + k_y y - \omega t) \} \},$$ (4)

where $\hat{u}_z$ denotes the complex amplitude and $\Re$ denotes the real part of a complex variable. Then the two sets of amplitude equations are derived as follows,

$$\begin{align}
-i\omega \hat{u}_x &= -\frac{1}{\rho} ik_x \hat{p} + 2\Omega \hat{u}_y \\
-i\omega \hat{u}_y &= -\frac{1}{\rho} ik_y \hat{p} - 2\Omega \hat{u}_x \\
-i\omega \hat{u}_z &= -\frac{1}{\rho} D \hat{p} \\
ik_x \hat{u}_x + ik_y \hat{u}_y + D \hat{u}_z &= 0 
\end{align}$$ (5)

in the convective region, and

$$\begin{align}
-i\omega \hat{u}_x &= -\frac{1}{\rho} ik_x \hat{p} + 2\Omega \hat{u}_y \\
-i\omega \hat{u}_y &= -\frac{1}{\rho} ik_y \hat{p} - 2\Omega \hat{u}_x \\
-i\omega \hat{u}_z &= -\frac{1}{\rho} D \hat{p} - \hat{\beta} \hat{u}_z \\
-\omega \hat{\beta} \hat{u}_z &= 0 \\
ik_x \hat{u}_x + ik_y \hat{u}_y + D \hat{u}_z &= 0 
\end{align}$$ (6)

in the stably stratified region. In Equations (5) and (6) $D$ denotes $d/dz$ and $\Re$ is omitted because of the linear property of the equations.

Either Equation (5) or (6) can be further reduced to a single equation with $\hat{u}_z$ being variable, i.e.,

$$D^2 \hat{u}_z + \frac{\omega^2 k^2}{4\Omega^2 - \omega^2} \hat{u}_z = 0$$ (7)

in the convective region, where $k^2 = k_x^2 + k_y^2$, and

$$D^2 \hat{u}_z + \frac{(\omega^2 - N^2)k^2}{4\Omega^2 - \omega^2} \hat{u}_z = 0$$ (8)

in the stably stratified region, where

$$N^2 = -\frac{\beta g}{\rho}$$ (9)

is the square of buoyancy frequency. In the derivation of Equation (8) the first-order term $D\hat{u}_z$ is neglected under the Boussinesq approximation in which density $\rho$ and density stratification $\beta$ are both constants. When $N = 0$ Equation (8) reduces to Equation (7). Equations (7) and (8) are what we will discuss in the next text. With the solution of $\hat{u}_z$ the two horizontal components are derived as follows,

$$\begin{align}
\hat{u}_x(t^+) &= -\frac{\omega k_x + 2i\Omega k_y}{i\omega k^2} D\hat{u}_z, \\
\hat{u}_y(t^+) &= -\frac{2i\Omega k_x - \omega k_y}{i\omega k^2} D\hat{u}_z. 
\end{align}$$ (10)

The two boundary conditions are imposed. One is that vertical velocity is continuous across the interface, i.e.,

$$\hat{u}_z(0^+) = \hat{u}_z(0^-).$$ (11)

The other is that the Lagrangian perturbation of pressure, which is identical to Eulerian perturbation of pressure, is continuous across the interface. According to the $x$ and $y$ components of the momentum conservation equation and the mass conservation equation, this condition can be translated to

$$D\hat{u}_z(0^+) = D\hat{u}_z(0^-).$$ (12)

### 3. Six Cases

According to Equation (7), to support wave motion in the convective region it is required that $4\Omega^2 - \omega^2 > 0$. If $4\Omega^2 - \omega^2 < 0$ then wave amplitude exponentially decays away from the interface. Similarly, to support wave motion in the stably stratified region it is required that $(\omega^2 - N^2)(4\Omega^2 - \omega^2) > 0$. Consequently, if waves exist in both regions then it is required that

$$N^2 < \omega^2 < 4\Omega^2.$$ (13)

Condition (13) implies that $N^2 < 4\Omega^2$, i.e., rotation wins out stratification. If stratification is so strong to win out rotation, i.e., $4\Omega^2 < N^2$, then the wave could exist only in one region but cannot exist in both regions.

All the six cases are summarized in Table 1. In Cases 1 and 4 the wave does not exist. In Case 2 waves exist in both regions. In Cases 3, 5, or 6 only one wave exists in one region and in the other region wave amplitude exponentially decays away from the interface, i.e., the so-called evanescent region. For example, Cases 5 and 6 take place in the solar interior.

### 4. Case 2

We now focus on Case 2, i.e., waves exist in both regions. The general solution in the convective region is

$$u'_z = \Re \{ a_1 \exp \{ i(k_x x + k_y y + qz - \omega t) \} + a_2 \exp \{ i(k_x x + k_y y - qz - \omega t) \} \}$$ (14)
\[ w = W - q k^4, \]

and the general solution in the stably stratified region is

\[ u' = \Re \{ b_1 \exp [i(k_x x + k_y y + sz - \omega t)] \]
\[ + b_2 \exp [i(k_x x + k_y y - sz - \omega t)] \} \]

where

\[ s = \sqrt{\omega^2 - N^2 \over 4\Omega^2 - \omega^2} k. \]

Table 1

| \( N^2 < 4\Omega^2 \) | Inertial Wave | Gravito-inertial Wave |
|----------------------|---------------|----------------------|
| Case 1: \( N^2 < 4\Omega^2 < \omega^2 \) | no | no |
| Case 2: \( N^2 < \omega^2 < 4\Omega^2 \) | yes | yes |
| Case 3: \( \omega^2 < N^2 < 4\Omega^2 \) | yes | no |

\[ 4\Omega^2 < N^2 \]

| Case 4: \( 4\Omega^2 < N^2 < \omega^2 \) | no | no |
| Case 5: \( 4\Omega^2 < \omega^2 < N^2 \) | no | yes |
| Case 6: \( \omega^2 < 4\Omega^2 < N^2 \) | yes | no |

Figure 1. Four configurations. 1 and 4 are categorized together, while 2 and 3 are categorized together.

Phase velocity of the transmitted wave in the stably stratified region is downward. Therefore, \( b_1 = 0 \) and

\[ u'_t = \Re \{ a_2 \exp [i(k_x x + k_y y - qz - \omega t)] \} \]

\[ u'_r = \Re \{ a_1 \exp [i(k_x x + k_y y + qz - \omega t)] \} \]

\[ u'_s = \Re \{ b_2 \exp [i(k_x x + k_y y - sz - \omega t)] \} \]

Then the boundary conditions (11) and (12) yield the relationship

\[ a_1 = \frac{q - s}{q + s} a_2, \quad b_2 = \frac{2q}{q + s} a_2. \]

The three wave solutions can be then obtained as follows, incident wave:

\[ u'_t = \Re \{ q \frac{\omega k_x + 2\Omega k_y}{\omega^2 - k_x^2} a_2 \exp [i(k_x x + k_y y - qz - \omega t)] \} \]

reflected wave:

\[ u'_r = \Re \{ -q \frac{\omega k_x - \Omega k_y}{\omega^2 - k_x^2} a_2 \exp [i(k_x x + k_y y + qz - \omega t)] \} \]

transmitted wave:

\[ u'_s = \Re \{ a_2 \exp [i(k_x x + k_y y - qz - \omega t)] \} \]
reflected wave:
\[
\begin{align*}
\mathbf{u}'_1 &= \Re \left\{ -q \frac{\partial^2}{\partial x^2} a_1 \exp \left[ i (k_x x + k_y y + qz - \omega t) \right] \right\} \\
\mathbf{u}'_2 &= \Re \left\{ q \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} a_1 \exp \left[ i (k_x x + k_y y + qz - \omega t) \right] \right\} \\
\mathbf{u}'_3 &= \Re \{ a_1 \exp \left[ i (k_x x + k_y y + qz - \omega t) \right]\}
\end{align*}
\]

(21)

transmitted wave:
\[
\begin{align*}
\mathbf{u}'_1 &= \Re \left\{ s \frac{\partial^2}{\partial x^2} b_2 \exp \left[ i (k_x x + k_y y - sz - \omega t) \right] \right\} \\
\mathbf{u}'_2 &= \Re \left\{ -s \frac{\partial^2}{\partial x^2} a_1 \exp \left[ i (k_x x + k_y y - sz - \omega t) \right] \right\} \\
\mathbf{u}'_3 &= \Re \{ b_2 \exp \left[ i (k_x x + k_y y - sz - \omega t) \right]\}
\end{align*}
\]

(22)

where \( k = \sqrt{k_x^2 + k_y^2} \).

By the three wave solutions, we can find more information. First we calculate the angles of three waves to the \( z \)-axis. Clearly, incident and reflected waves have the same angle but the transmitted wave has a different one. The reflected and transmitted angles are, respectively,
\[
\arctan \left( \frac{q}{k} \sqrt{1 + \frac{4\Omega^2}{\omega^2}} \right), \quad \arctan \left( \frac{s}{k} \sqrt{1 + \frac{4\Omega^2}{\omega^2}} \right).
\]

(23)

It shows that the transmitted wave has a smaller angle because \( s < q \). The time-averaged energies
\[
\frac{\omega}{2\pi} \int_0^{2\pi} (u_x'^2 + u_y'^2 + u_z'^2) dt = \frac{1}{2} \left\{ |\hat{u}_x|^2 + |\hat{u}_y|^2 + |\hat{u}_z|^2 \right\}
\]

(24)

of three waves are, respectively,
\[
\begin{align*}
a_1^2 &\frac{4\Omega^2}{4\Omega^2 - \omega^2}, \quad a_1^2 \frac{4\Omega^2}{4\Omega^2 - \omega^2}, \\
b_2^2 &\frac{4\Omega^2 - N^2/2 - 2\Omega^2 N^2/\omega^2}{4\Omega^2 - \omega^2}
\end{align*}
\]

(25)

for the incident wave, reflected wave, and transmitted wave. We then calculate the reflection ratio, i.e., the ratio of reflected wave energy to incident wave energy,
\[
\left( \frac{a_1}{a_2} \right)^2 = \left( \frac{q - s}{q + s} \right)^2 = \left( 1 - \sqrt{\frac{1 - N^2/\omega^2}{1 + \sqrt{1 - N^2/\omega^2}}} \right)^2.
\]

(26)

It is very interesting that the energy ratio of the reflected wave in the convective region is independent of rotation rate but is influenced by the buoyancy frequency in the stably stratified region. When \( N \ll \omega < 2\Omega \) the reflection ratio is approximately equal to \( N^2/16\omega^4 \approx 0 \). The transmission ratio can be also calculated to be the ratio of transmitted wave energy to incident wave energy,
\[
\left( \frac{b_2}{a_2} \right)^2 = \frac{4\Omega^2 - N^2/2 - 2\Omega^2 N^2/\omega^2}{4\Omega^2 - \omega^2} = \frac{4}{1 + \sqrt{1 - N^2/\omega^2}} \left( 1 - N^2/8\Omega^2 - N^2/2\omega^2 \right).
\]

(27)

When \( N \ll \omega < 2\Omega \) the transmission ratio is approximately equal to 1. Therefore, we obtain an important result, i.e., when stratification is very weak or rotation is very strong, the transmitted wave from the convective region can be almost transmitted to the stably stratified region and changes to a gravito-inertial wave.

4.2. Configuration 2

In this configuration, \( a_2 = 0 \) and
\[
\begin{align*}
\text{incident wave: } u'_1 &= \Re \{ b_1 \exp \left[ i (k_x x + k_y y + sz - \omega t) \right]\} \\
\text{reflected wave: } u'_1 &= \Re \{ b_2 \exp \left[ i (k_x x + k_y y - sz - \omega t) \right]\} \\
\text{transmitted wave: } u'_1 &= \Re \{ a_1 \exp \left[ i (k_x x + k_y y + qz - \omega t) \right]\}.
\end{align*}
\]

(28)

The relation of coefficients is
\[
b_2 = -\frac{q}{s} b_1, \quad a_1 = \frac{2s}{q} b_1.
\]

(29)

We do not show readers the three wave solutions but directly give the wave angles to the \( z \)-axis and the time-averaged wave energies. The reflected and transmitted angles to the \( z \)-axis are, respectively,
\[
\arctan \left( \frac{q}{k} \sqrt{1 + \frac{4\Omega^2}{\omega^2}} \right), \quad \arctan \left( \frac{s}{k} \sqrt{1 + \frac{4\Omega^2}{\omega^2}} \right).
\]

(30)

In this configuration the transmitted angle is larger. The time-averaged wave energies are, respectively,
\[
\begin{align*}
b_1^2 \frac{4\Omega^2 - N^2/2 - 2\Omega^2 N^2/\omega^2}{4\Omega^2 - \omega^2}, \\
b_2^2 \frac{4\Omega^2 - N^2/2 - 2\Omega^2 N^2/\omega^2}{4\Omega^2 - \omega^2}, \\
a_1^2 \frac{4\Omega^2}{4\Omega^2 - \omega^2}
\end{align*}
\]

(31)

for incident wave, reflected wave, and transmitted wave. The reflection ratio can be readily calculated,
\[
\left( \frac{b_2}{b_1} \right)^2 = \left( \frac{q - s}{q + s} \right)^2 = \left( 1 - \sqrt{\frac{1 - N^2/\omega^2}{1 + \sqrt{1 - N^2/\omega^2}}} \right)^2.
\]

(32)

When \( N \ll \omega < 2\Omega \) the reflection ratio is approximately equal to \( N^2/16\omega^4 \approx 0 \). The transmission ratio is
\[
\left( \frac{a_1}{b_1} \right)^2 = \frac{4\Omega^2}{4\Omega^2 - N^2/2 - 2\Omega^2 N^2/\omega^2} = \frac{4}{1 + \sqrt{1 - N^2/\omega^2}} \left( 1 - N^2/8\Omega^2 - N^2/2\omega^2 \right).
\]

(33)
4.3. Configuration 3

As before, we show readers the results. \( a_i = 0 \) and the three waves are

\[
\begin{aligned}
\text{incident wave: } u'_i &= \Re \{ b_2 \exp[i(k_s x + k_y y - sz - \omega t)] \} \\
\text{reflected wave: } u'_i &= \Re \{ b_1 \exp[i(k_s x + k_y y + sz - \omega t)] \} \\
\text{transmitted wave: } u'_i &= \Re \{ a_2 \exp[i(k_s x + k_y y - qz - \omega t)] \}.
\end{aligned}
\]

The boundary conditions yield

\[
b_1 = -\frac{q - s}{q + s} b_2, \quad a_2 = \frac{2s}{q + s} b_2.
\]

The reflected and transmitted angles, as well as the reflection and transmission ratios, are identical to those of Configuration 2. When \( N \ll \omega < 2\Omega \), the gravito-inertial wave almost transmits into the convective region and changes to the inertial wave.

4.4. Configuration 4

In this configuration, \( b_2 = 0 \) and the three waves are

\[
\begin{aligned}
\text{incident wave: } u'_i &= \Re \{ a_1 \exp[i(k_s x + k_y y + qz - \omega t)] \} \\
\text{reflected wave: } u'_i &= \Re \{ a_2 \exp[i(k_s x + k_y y - qz - \omega t)] \} \\
\text{transmitted wave: } u'_i &= \Re \{ b_1 \exp[i(k_s x + k_y y + sz - \omega t)] \}
\end{aligned}
\]

The boundary conditions yield

\[
a_2 = \frac{q - s}{q + s} a_1, \quad b_1 = \frac{2q}{q + s} a_1.
\]

The three angles to the \( z \)-axis are

\[
\arctan \left( \frac{q}{k \sqrt{1 + \frac{4\Omega^2}{\omega^2}}} \right), \quad \arctan \left( \frac{q}{k \sqrt{1 + \frac{4\Omega^2}{\omega^2}}} \right), \quad \arctan \left( \frac{s}{k \sqrt{1 + \frac{4\Omega^2}{\omega^2}}} \right).
\]

The transmitted wave has a smaller angle. The three energies are

\[
a'^2 = \frac{4\Omega^2}{4\Omega^2 - \omega^2}, \quad a'^2 = \frac{4\Omega^2}{4\Omega^2 - \omega^2}, \quad b'^2 = \frac{4\Omega^2 - N^2/2 - 2\Omega N^2/\omega^2}{4\Omega^2 - \omega^2}.
\]

The angles and ratios are identical to those of Configuration 1. When \( N \ll \omega < 2\Omega \), the inertial wave almost transmits into the stratified region and changes to a gravito-inertial wave.

4.5. Summary

We studied the six cases as listed in Table 1, and in Case 2 \((N^2 < \omega^2 < 4\Omega^2)\) both convective and stratified regions support wave motion. We then focused on the four configurations in Case 2 for solar-type and massive stars. It is found that

1. Configurations 1 and 4 are symmetric (wave propagation from the convective region to stratified region), while Configurations 2 and 3 are symmetric (wave propagation from the stratified region to convective region);
2. the wave angle in the convective region is larger than that in the stratified region;
3. in a rapidly rotating star or planet \((N \ll \Omega)\) the wave reflects very little and almost transmits to the other region.

5. At Any Latitude

As mentioned in Section 2, in our simplified model the rotational axis is normal to the interface, which is valid only in the vicinity of the polar area. We now discuss the situation at any latitude \( \theta \). In the local plane model, we assign that the \( x \)-axis points east, the \( y \)-axis points north, and the \( z \)-axis points radially outward. At latitude \( \theta \), the angular velocity in the local Cartesian coordinates is written as \((0, \Omega \cos \theta, \Omega \sin \theta)\). Then the Coriolis force depends on \( \theta \) and the two sets of perturbation equations are

\[
\begin{aligned}
\frac{\partial u'_i}{\partial t} &= -\frac{1}{\rho} \frac{\partial p'}{\partial x} + 2\Omega(u'_i \sin \theta - u'_i \cos \theta) \\
\frac{\partial u'_i}{\partial t} &= -\frac{1}{\rho} \frac{\partial p'}{\partial y} + 2\Omega u'_i \sin \theta \\
\frac{\partial u'_i}{\partial t} &= -\frac{1}{\rho} \frac{\partial p'}{\partial z} + 2\Omega u'_i \cos \theta
\end{aligned}
\]

in the convective region and

\[
\begin{aligned}
\frac{\partial u'_i}{\partial t} &= -\frac{1}{\rho} \frac{\partial p'}{\partial x} + 2\Omega u'_i \sin \theta - u'_i \cos \theta \\
\frac{\partial u'_i}{\partial t} &= -\frac{1}{\rho} \frac{\partial p'}{\partial y} - 2\Omega u'_i \sin \theta \\
\frac{\partial u'_i}{\partial t} &= -\frac{1}{\rho} \frac{\partial p'}{\partial z} + 2\Omega u'_i \cos \theta - \rho' \sin \theta
\end{aligned}
\]

in the stratified region. Next, following the above procedure we are led to

\[
D^2 \hat{u}_c + \frac{8\Omega^2 k_z \sin \theta \cos \theta}{4\Omega^2 \sin^2 \theta - \omega^2} D\hat{u}_c - \frac{\omega^2 k_z^2 - 4\Omega^2 k_z^2 \cos^2 \theta}{4\Omega^2 \sin^2 \theta - \omega^2} \hat{u}_c = 0
\]

in the convective region and

\[
D^2 \hat{u}_c + \frac{8\Omega^2 k_z \sin \theta \cos \theta}{4\Omega^2 \sin^2 \theta - \omega^2} D\hat{u}_c - \frac{\omega^2 - N^2}{4\Omega^2 \sin^2 \theta - \omega^2} \hat{u}_c = 0
\]

in the stratified region. When \( \theta = 90^\circ \) Equations (42) and (43) reduce to Equations (7) and (8), respectively.
In Equations (42) and (43), $k_a$ appears, which suggests the importance of the direction of wave vector. The angle of wave vector to the east (i.e., $x$ direction) on the local horizontal plane is denoted by $\alpha$, then

$$k_x = k \cos \alpha, \quad k_y = k \sin \alpha.$$  

(44)

The condition that the wave propagates in both regions (Case 2) is that Equations (42) and (43) admit the wave solution, which yields two inequalities,

$$\omega^2 < \omega_0^2$$

(45)

in the convective region and

$$\omega_1^2 < \omega < \omega_2^2$$

(46)

in the stratified region, where

$$\omega_0^2 = 4\Omega^2(\sin^2 \theta + \cos^2 \theta \sin^2 \alpha),$$

(47)

$$\omega_{1,2}^2 = \frac{1}{2}[\omega_0^2 + \Omega^2 \pm \sqrt{(\omega_0^2 + \Omega^2)^2 - 16\Omega^2\Omega^2 \sin^2 \theta}].$$

(48)

Note that the frequency ranges $\omega_0$, $\omega_1$, and $\omega_2$ all depend on both $\theta$ and $\alpha$. More analysis shows that $\omega_1^2 < \omega_2^2 < \omega_0^2$. Therefore, the condition for the wave propagation across the interface is

$$|\omega_0| < |\omega| < |\omega_1|.$$  

(49)

Figure 2 shows the contours of $(|\omega_0| - |\omega_1|)/2\Omega$ versus $\theta$ and $\alpha$ at $N/2\Omega = 0.1$, $1$, and $10$. The darker region of a wider waveband $|\omega_0| - |\omega_1|$ implies more possibility for wave transmission. Fast rotation (small $N/2\Omega = 0.1$) tends to support wave transmission anywhere, whereas slow rotation (large $N/2\Omega = 10$) tends to support wave transmission only in the equatorial area. Moreover, the wave with a longer longitudinal wavelength (larger $\alpha$) seems more possible to transmit.

In Case 2, we can further discuss the energy ratios of reflected and transmitted waves. Take Configuration I for an example. The wave solutions are

\[
\begin{aligned}
\text{incident wave: } u'_i &= \mathcal{M}(a_1 \exp[i(k_x x + k_y y + q_1 z - \omega t)]) \\
\text{reflected wave: } u'_r &= \mathcal{M}(a_2 \exp[i(k_x x + k_y y + q_2 z - \omega t)]) \\
\text{transmitted wave: } u'_t &= \mathcal{M}(b_2 \exp[i(k_x x + k_y y + q_2 z - \omega t)])
\end{aligned}
\]

(50)

where $q_{1,2}$ and $s_{1,2}$ depend on $\theta$ and $\alpha$,

$$q_{1,2} = \frac{k}{4\Omega^2 \sin^2 \theta - \omega^2} \times (-4\Omega^2 \sin \theta \cos \theta \sin \alpha \pm \sqrt{\omega^2(\omega_0^2 - \omega^2)}),$$

(51)

$$s_{1,2} = \frac{k}{4\Omega^2 \sin^2 \theta - \omega^2} \times (-4\Omega^2 \sin \theta \cos \theta \sin \alpha \pm \sqrt{\omega^2(\omega_0^2 - \omega^2 + N^2) - 4\Omega^2 N^2 \sin^2 \theta}).$$

(52)

At $\theta = 90^\circ$ the above two expressions of $q$ and $s$ reduce to Equations (15) and (17). By the boundary conditions (11) and (12) we can derive

$$a_1 = \frac{q_2 - q_1}{q_1 - q_2} a_2, \quad b_2 = \frac{q_1 - q_2}{q_1 - q_2} a_2.$$  

(53)

Similar to Equation (10) we can derive

$$\hat{u}_x = \frac{2\Omega k_x \cos \theta}{i\omega k^2} \hat{u}_c - \frac{\omega k_x + 2i\Omega k_y \sin \theta}{i\omega k^2} D \hat{u}_c,$$

$$\hat{u}_y = -\frac{2\Omega k_y k_y \cos \theta}{i\omega k^2} \hat{u}_c + \frac{2i\Omega k_x \sin \theta - \omega k_y}{i\omega k^2} D \hat{u}_c.$$  

(54)

Substitution of Equation (54) into Equation (24) leads to the reflection ratio

$$\frac{a_1^2}{\tilde{a}_1^2} = \frac{4\Omega^2(k_x \cos \theta + q_1 \sin \theta)^2 + \omega^2(k_x^2 + q_1^2)}{4\Omega^2(k_x \cos \theta + q_1 \sin \theta)^2 + \omega^2(k_x^2 + q_1^2)}$$

$$= \frac{(s_2 - q_2)^2 - 4\Omega^2(k_x \cos \theta + q_1 \sin \theta)^2 + \omega^2(k_x^2 + q_1^2)}{q_1 - q_2} \cdot \frac{4\Omega^2(k_x \cos \theta + q_1 \sin \theta)^2 + \omega^2(k_x^2 + q_1^2)}{q_1 - q_2}.$$  

(55)

At $\theta = 90^\circ$, $q_1 = q$ and $q_2 = -q$ the reflection ratio reduces to (26). We now investigate (55). First, the reflection ratio is independent of the magnitude $k$ of wave vector, because $q_{1,2}$ and $s_{1,2}$ are all proportional to $k$, but depends on the direction $\alpha$ of wave vector. Second, we have already known that to support wave propagation across interface it is required that $\omega_1^2 < \omega_1^2 < \omega_0^2$, and so we take the average of reflection ratio with $\omega_1^2$ uniformly spacing between $\omega_1^2$ and $\omega_0^2$. Consequently, the reflection ratio depends on $\theta$, $\alpha$ and $N/2\Omega$.

Figure 3 shows the contours of the reflection ratio versus $\theta$ and $\alpha$ at $N/2\Omega = 0.1$, $1$, and $10$. It provides two results. The first is about fast rotation at $N/2\Omega = 0.1$. The small reflection
ratio in panel (a) indicates that fast rotation indeed favors wave transmission as already found at \( \theta = 90^\circ \) with \( N/2\Omega < 1 \). The second is about slow rotation at \( N/2\Omega = 10 \). Panel (c) of Figure 2 implies that at slow rotation the equatorial area may possibly support wave propagation because this area has a wide waveband for wave propagation. However, panel (c) of Figure 3 denies this possibility. The dark area in the low-latitude area indicates that most of the wave is reflected but not transmitted. Instead, it indicates that wave transmission may occur in the high-latitude area. Nevertheless, in the high-latitude area only a narrow band of wave is allowed to transmit so that the total transmission cannot be strong. In a word, slow rotation inhibits wave transmission.

Similarly, the reflection ratio of Configuration 2 is

\[
\left( \frac{s_1 - q_1}{q_1 - s_2} \right)^2 \cdot \frac{4\Omega^2(k_x \cos \theta + s_x \sin \theta)^2 + \omega^2(k_x^2 + s_x^2)}{4\Omega^2(k_x \cos \theta + s_1 \sin \theta)^2 + \omega^2(k_x^2 + s_1^2)}
\]  

(56)

that of Configuration 3 is

\[
\left( \frac{q_2 - s_2}{s_1 - q_2} \right)^2 \cdot \frac{4\Omega^2(k_x \cos \theta + s_x \sin \theta)^2 + \omega^2(k_x^2 + s_x^2)}{4\Omega^2(k_x \cos \theta + s_1 \sin \theta)^2 + \omega^2(k_x^2 + s_1^2)}
\]  

(57)

and that of Configuration 4 is

\[
\left( \frac{q_1 - s_1}{s_1 - q_2} \right)^2 \cdot \frac{4\Omega^2(k_x \cos \theta + q_x \sin \theta)^2 + \omega^2(k_x^2 + q_x^2)}{4\Omega^2(k_x \cos \theta + q_1 \sin \theta)^2 + \omega^2(k_x^2 + q_1^2)}
\]  

(58)

At some \( \theta \) and \( \alpha \), especially with strong stratification \( N/2\Omega = 10 \), the reflection ratio is calculated to be greater than 1. This might arise from the Boussinesq approximation in which constant \( \rho \) in the perturbation equations brings an error when stratification is strong. We then set the threshold at 1.
The contours of the three reflection ratios are shown in Figures 4–6. These figures provide two results. The first is that the symmetry at $\theta = 90^\circ$ is lost, namely the reflection ratios of Configurations 1 and 4 or of Configurations 2 and 3 are no longer identical. The second is that, again, fast rotation at $N/2\Omega = 0.1$ favors wave transmission, whereas slow rotation at $N/2\Omega = 10$ inhibits wave transmission. Slow rotation of Configuration 2 (Figure 4) corresponds to the detection of the g mode on the solar surface. Therefore, in the slowly rotating Sun ($N/2\Omega$ at the order of 10–100) the gravito-inertial wave in the radiative region seems unlikely to transmit to the convective region.

6. Discussions

Our results may provide guidance for the detection of modes, e.g., modes in the inner region of a slowly rotating star or planet are hardly detected on the surface. Our results may be also useful for stellar oscillation observations. For a rapidly rotating solar-type star, the gravito-inertial wave can propagate outward to the convective region such that we can know more about the structure of the inner radiative region. Similarly, for a rapidly rotating massive star, the inertial wave can propagate outward to the radiative region such that we can know more about the differential rotation of the inner convective region. Moreover, since the wave can propagate across the interface to the other region, angular momentum and energy can be carried out farther away than expected before. For example, in Goldreich & Nicholson (1989), in a non-rotating early-type star, it was believed that the gravity wave excited at the interface propagates toward the stellar surface and damps just below the photosphere. According to our model, much of the wave can propagate in an opposite way toward the convective region provided that star rotates sufficiently fast.

Certainly, our simplified model should be numerically validated. One can perform numerical calculations in a global spherical geometry by imposing interface with an overshooting profile (Zhang & Schubert 2002; Augustson & Mathis 2019). Anelastic (Rogers et al. 2013) or fully compressible fluid can be used for numerical study, the latter will bring numerical difficulty due to fast sound speed. In addition, the magnetic field plays an important role in wave propagation, which radically changes the wave dispersion relation (Wei 2016, 2018; Lin & Ogilvie 2018), so that the magnetic effect on wave reflection and transmission also needs to be studied.

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Erratum: “Wave Reflection and Transmission at the Interface of Convective and Stably Stratified Regions in a Rotating Star or Planet” (ApJ, 2020, 890, 20)

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An error is found in calculating the wave reflection and transmission ratios at the interface. I used kinetic energy to calculate the ratios such that in some regions the reflection ratio appears to be greater than 1, which is unphysical. To find the correct physical quantity we derive the total energy equation. We perform the dot product $u^i$ with the momentum perturbation equation (note that the $u^i$ Coriolis force vanishes), multiply $(g^2/N^2\rho^2)\rho'$ with the density perturbation equation (only in the stratified region), and add them together. Thus, the total energy equation is derived to be

$$\frac{\partial}{\partial t} \left( \frac{u^2}{2} + \frac{g^2}{N^2\rho^2} \right) = -\frac{1}{\rho} \nabla \cdot (p'u'),$$

where in the left-hand side brackets the first term is kinetic energy per unit mass and the second term is the buoyancy energy per unit mass, and in the right-hand side brackets the term is energy flux. It should be noted that this equation is for the stratified region. In the convective region the buoyancy energy is simply zero. Through the total energy equation, we can find that the correct quantity to measure the reflection and transmission ratios should be the energy flux $\langle p'u' \rangle$. Moreover, the boundary condition shows that the vertical energy flux $\langle p'u'_v \rangle$ of an incident wave is equal to the sum of those of reflected and transmitted waves. Therefore, with the vertical energy flux to measure ratios, the sum of two ratios is equal to 1. Next, we calculate the vertical energy flux $\langle p'u'_v \rangle$. We express $p' = \Re \{p'\}$ and $u'_v = \Re \{u'_v\}$ where the tilde denotes the complex variables of perturbations. The complex pressure perturbation $\tilde{p}'$ can be calculated from perturbation Equations (40) or (41) to be

\[
i\omega^2 \tilde{p}' = 2\Omega \cos \theta (2i\Omega k \sin \theta - \omega k) \tilde{u}' + (4\Omega^2 \sin^2 \theta - \omega^2) \frac{\partial \tilde{u}'_v}{\partial z},
\]

\[
= [-2\Omega \omega k \cos \theta \cos \alpha + i(4\Omega^2 k \cos \theta \sin \alpha + 4\Omega^2 \gamma \sin^2 \theta - \gamma \omega^2)] \tilde{u}'_v,
\]

where $\gamma$ is the vertical wavenumber, i.e., $q_{1.2}$ or $s_{1.2}$. The time-averaged vertical energy flux $\langle p'u'_v \rangle$ is then

\[
\langle p'u'_v \rangle = \langle \Re \{p'\} \Re \{u'_v\} \rangle = \frac{1}{2} \rho |\tilde{u}'_v|^2 \frac{4\Omega^2}{\omega k} \left[ \cos \theta \sin \theta \sin \alpha + \frac{\gamma}{k} \left( \sin^2 \theta - \frac{\omega^2}{4\Omega^2} \right) \right].
\]

Interestingly, by the expressions of $\gamma$ (Equations (51) and (52)), the reflection ratio $\eta = \langle p'u'_v \rangle / \langle p'u'_v \rangle$ in the four configurations is identical,

\[
\eta = \left( \frac{\sqrt{\omega^2 - \omega^2 - \omega^2}}{\sqrt{\omega^2 - \omega^2 - \omega^2 + N^2}} - \frac{4\Omega^2 N^2 \sin^2 \theta}{4\Omega^2 N^2 \sin^2 \theta} \right)^2.
\]

The transmission ratio is $\langle p'u'_v \rangle / \langle p'u'_v \rangle = 1 - \eta$. It is reasonable that the partition of energy flux at the interface depends on the wave properties, i.e., frequency and wavevector, and the medium properties, i.e., rotation rate and stratification strength, so that the four configurations have the same ratios. The above expression shows that the two ratios depend on four parameters: $N/2\Omega\omega/2\Omega$, latitude $\theta$, and the direction of the wavevector $\alpha$ ($\omega_\alpha$ depends on $\alpha$, see Equation (47)). We take three values of $N/2\Omega = (0.1, 1, 10)$ and three values of $\sin^2 \alpha = (0.1, 0.5, 0.9)$ to plot the contours of the reflection ratio $\eta$ versus wave frequency $\omega/2\Omega$ and latitude $\theta$ in Figure 1. In each panel, a wave exists between the top and bottom curves, i.e., $\omega_1 < \omega < \omega_\theta$, and outside the two curves the wave cannot propagate across the interface. In each row, the panels from left to right show that stronger stratification or slower rotation leads to stronger reflection and hence weaker transmission. At $N/2\Omega = 10$ the wave hardly transmits, at $N/2\Omega = 1$ it transmits at lower latitudes with a wider wave band, and at $N/2\Omega = 0.1$ the wave transmits at any latitudes with a wide wave band. This result is consistent with the published article, i.e., rotation favors transmission whereas stratification inhibits transmission. In each column, the panels from top to bottom show that larger $\alpha$, i.e., wavevectors that are aligned with the north–south direction rather than the east–west direction, lead to a wider wave band of transmission.

In summary, in the published article I used an inappropriate physical quantity, i.e., kinetic energy $\langle u'^2/2 \rangle$, to measure the reflection and transmission ratios, and in this Erratum I use the vertical energy flux $\langle p'u'_v \rangle$ to measure the ratios to guarantee the sum of the two ratios to be 1. However, the major result still holds, i.e., rotation favors the wave transmission whereas stratification inhibits the wave transmission.
Figure 1. Contours of the reflection ratio $\eta$ vs. wave frequency $\omega/2\Omega$ and latitude $\theta$ at different stratification strengths $N/2\Omega$ and directions of the wavevector $\sin^2 \alpha$. The grayscale corresponds to the value of the reflection ratio as indicated by the bars on the right.