THE ASYMPTOTICS OF THE CURVATURE OF THE FREE DISCONTINUITY SET NEAR THE CRACKTIP FOR THE MINIMIZERS OF THE MUMFORD-SHAH FUNCTIONAL IN THE PLAIN

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Abstract. We consider in 2D the following special case of the Mumford-Shah functional

\[ J(u, \Gamma) = \int_{B_1 \setminus \Gamma} |\nabla u|^2 \, dx + \lambda^2 \sqrt{\frac{\pi}{2}} H^1(\Gamma). \]

It is known that if the minimizer has a cracktip in the ball \( B_1 \) (assume at the origin), then \( u \approx \lambda \Im \sqrt{z} \) at this point. We calculate higher order terms in the asymptotic expansion, where the homogeneity orders of those terms appear to be solutions to a certain trigonometric relation.

1. Introduction

1.1. The Problem. Let us consider the following minimization problem; Minimize the functional

\[ J(u, \Gamma) = \int_{B_1 \setminus \Gamma} |\nabla u|^2 \, dx + \lambda^2 \sqrt{\frac{\pi}{2}} H^1(\Gamma) \]

over all curves \( \Gamma \subset B_1 \) starting at the point \(-1\) of the complex plane and functions \( u \in H^1(B_1 \setminus \Gamma) \) with boundary values \( g \) on \( \partial B_1 \). We will assume that \( g \) is a little perturbation of the function \( \lambda \Im \sqrt{z} = \lambda r^{1/2} \sin \phi/2 \) on the \( \partial B_1 \), where the discontinuity is taken at the point \(-1\).

Let us agree that throughout the paper, depending on the context, the functions \( \Im \sqrt{z}, r^a \sin(a\phi) \) and \( r^a \cos(a\phi) \) will be defined in \( \mathbb{R}^2 \setminus \{ (x, 0) | -\infty \leq x \leq 0 \} \), in \( B_1 \setminus \Gamma \) or in \( B_1 \setminus \Gamma_r \), where the discontinuity is at \( \{ (x, 0) | -\infty \leq x \leq 0 \} \), at \( \Gamma \) or at \( \Gamma_r \) respectively.

It is proven in [BD] that the pair

\( (\lambda r^{1/2} \sin \phi/2; \{ (x, 0) | -\infty \leq x \leq 0 \}) \)

is the global minimizer of the functional (1) in the plane \( \mathbb{R}^2 \). From this result it follows that for the boundary value function \( g = \lambda \sin \phi/2 \), the interval \( \Gamma_0 = \{ (x, 0) | 1 \leq x \leq 0 \} \) and the function \( u_0 = \lambda r^{1/2} \sin \phi/2 \) give the absolute minimizer.

Key words and phrases. Mumford-Shah functional, cracktip.

2000 Mathematics Subject Classification. Primary 49Q20, secondary 35R35.
We will assume the stability of the problem under small perturbations, i.e. that for small perturbations of $g$ the minimizing set $\Gamma$ is a $C^1$ curve connecting the point $(-1, 0) \in \partial B_1 \cap \mathbb{R}$ with an unknown point inside the ball $B_1 := B_1(0)$ and that this curve is not “far from” the interval $\Gamma_0$.

The authors refer to [D] and [AFP] for further references and as sources of inexhaustible knowledge in the field.

Probably some of the assumptions made in the paper could be avoided but still we believe that the results even in this simplified formulation are new and important.

1.2. The Main Theorem. Assume the minimizer of the functional (1) in $B_1 \subset \mathbb{R}^2$ is given by the pair $(u, \Gamma)$, where $\Gamma = \{(-t, f(t))|0 \leq t < 1\}$, $f \in C^1([0, 1))$ $f(0) = f'(0) = 0$. Further assume there exist a limit in $C^{1, \text{loc}}((0, 1))$

$$g(t) = \lim_{\rho \to 0} \frac{f(\rho t)}{\max_{0 < \tau < \rho} |f(\tau)|} \neq 0.$$ 

Then there exists a constant $C \neq 0$ such that

$$f(t) = Ct^{1/2+\alpha_k} + o(t^{1/2+\alpha_k})$$

and

$$(2) \quad u(x, y) = \lambda \mathfrak{A} \sqrt{z} + \Sigma_k(x, y) + C\lambda b_k r^{\alpha_k} \cos(\alpha_k \phi) + o(r^{\alpha_k}),$$

where $k < \alpha_k < k + \frac{1}{2}$ is one of the positive solutions of

$$\tan(\pi \alpha) = \sqrt{\frac{\pi^2}{2} \alpha^2 - \frac{1}{4}},$$

$b_k$ are some other absolute constants ($k = 1, 2, \ldots$), $\Sigma_1 \equiv 0$ and for $k \geq 2$ there are some constants $c_j$ depending on $u$ such that

$$\Sigma_k(x, y) = \sum_{j=1}^{k-1} c_j r^{2j+1} \sin \left(\frac{2j + 1}{2} \phi\right).$$

If

$$\lim_{\rho \to 0} \frac{f(\rho t)}{\max_{0 < \tau < \rho} |f(\tau)|} = 0$$

then $\lim_{t \to 0} t^{-M} f(t) = 0$ for any $M > 0$.

1.3. Acknowledgment. The authors are grateful to MSRI at UC Berkeley for the invitation to attend the Free Boundary Problems Program in 2011 and particularly to Henrik Shahgholian, who was the speaker of the program. The substantial part of the research was carried out during the stay in Berkeley. The second author thanks Stephan Luckhaus for useful discussions.
2. Asymptotic of the minimizing function near the cracktip

In this section we want to calculate the asymptotic expansion for the minimizing function \( u \) near the cracktip.

Let us assume that pair \((u, \Gamma)\) is as above and minimizes the functional \((\mathbf{1})\). Without loss of generality we can assume that the origin is the endpoint of \( \Gamma \) and is tangential to the x-axis at that point, for this reason we allow \( \Gamma \) to "start" from a point different from \((-1, 0)\). We will parametrize \( \Gamma = \{(-t, f(t)), t \in [0, 1]\} \) and at the cracktip we have \( f(0) = f'(0) = 0 \).

\[
\lambda^2 H_\Gamma(-t, f(t)) = \lambda^2 \frac{f''(t)}{(1 + (f'(t))^2)^{3/2}} = \sqrt{\frac{\pi}{2}} \left[ |\nabla u(z)|^2 \right]^\pm,
\]

where \( |\nabla u(z)|^2 = |\nabla u(x, y + 0)|^2 - |\nabla u(x, y - 0)|^2 \) is the difference of the values of \( |\nabla u|^2 \) on the two sides of the discontinuity set and \( H_\Gamma \) is the curvature of \( \Gamma \). Further we know that the first asymptotic term of the function \( u \) near the origin is \( \lambda r^{1/2} \sin \phi/2 \). A natural question arise what are the next terms in the asymptotic?

Let us consider the function

\[
v_\rho(x, y) = S_\rho^{-1} u(\rho x, \rho y) - \rho r^{1/2} \lambda^{1/2} \sin(\phi/2),
\]

where \( S_\rho = \sup_{B_\rho} |u(x, y) - \lambda r^{1/2} \sin(\phi/2)| \) and \( S_\rho \rho^{-1/2} \to 0 \) as \( \rho \to 0 \).

Note that

\[
r^{1/2} \sin(\phi/2) = \frac{1}{\sqrt{2}} \text{sgn}_\Gamma(x, y) \sqrt{x^2 + y^2 - x},
\]

where \( \text{sgn}_\Gamma(x, y) = \text{sgn}(y) \) if \( x \geq 0 \), \( \text{sgn}_\Gamma(x, y) = 1 \) if \( x < 0, y > f(-x) \) and \( \text{sgn}_\Gamma(x, y) = -1 \) if \( x < 0, y < f(-x) \).
Thus
\[ \nabla r^{1/2} \sin(\phi/2) = \frac{\sqrt{x^2 + y^2} + x}{2\sqrt{2} \text{sgn}(x, y)|y|} \left( -1 + \frac{x}{\sqrt{x^2 + y^2}} \cdot \frac{y}{\sqrt{x^2 + y^2}} \right). \]

The rescaled crack \( \Gamma_\rho \) is given by \( \{ (-t, f_\rho(t)) \} \) where \( f_\rho(t) = \rho^{-1} f(\rho t) \).

Let us calculate \( \partial_\nu v_\rho \) and \( \partial_\tau v_\rho \) on the crack, where the vectors \( \tau = (1 + f_\rho'^2)^{-1/2} (1; f_\rho') \) and \( \nu = (1 + f_\rho'^2)^{-1/2} (f_\rho'; 1) \) are the tangential and normal directions on \( \Gamma_\rho \).

Since \( \partial_\nu u = 0 \) on \( \Gamma \) we have that
\[ \partial_\nu v_\rho(-t, f_\rho(t)) = \nu \cdot \nabla r^{1/2} \sin(\phi/2) = \]
\[ (1 + f_\rho'^2)^{-1/2} \lambda \rho^{1/2} \sqrt{t^2 + f_\rho'^2} \]
\[ \left( f_\rho' \left( -1 - \frac{t}{\sqrt{t^2 + f_\rho'^2}} \right) + \frac{f_\rho}{\sqrt{t^2 + f_\rho'^2}} \right). \]
\[ (1 + f_\rho'^2)^{-1/2} \lambda \sqrt{t} \left( 1 + \left( \frac{f_\rho}{t} \right)^2 \right) \frac{1}{\sqrt{1 + \left( \frac{f_\rho}{t} \right)^2}} \]
\[ \left( f_\rho' \left( -1 - \frac{1}{\sqrt{1 + \left( \frac{f_\rho}{t} \right)^2}} \right) + \frac{f_\rho}{t} \frac{1}{\sqrt{1 + \left( \frac{f_\rho}{t} \right)^2}} \right). \]

If we now denote by \( \sigma_\rho = \sup_{t \in (0,1)} |f_\rho(t)| = \rho^{-1} \sup_{t \in (0,\rho)} |f(t)| \) and \( g_\rho = \frac{f_\rho}{\sigma_\rho} \) we obtain that
\[ \partial_\nu v_\rho(-t, f_\rho(t)) = \]
\[ (1 + o(f'(1)/t)) \frac{\sigma_\rho}{\rho^{-1/2} S_\rho} \frac{\lambda}{4\sqrt{t}} \left( f_\rho' \left( -1 - \frac{1}{\sqrt{1 + \left( \frac{f_\rho}{t} \right)^2}} \right) + \frac{g_\rho}{t} \frac{1}{\sqrt{1 + \left( \frac{f_\rho}{t} \right)^2}} \right). \]

In the Appendix A at the end of the paper we prove that \( \liminf_{\rho \to 0} \frac{\sigma_\rho}{\rho^{-1/2} S_\rho} < +\infty \). Assume for a sequence \( \rho_k \to 0 \)
\[ \frac{\sigma_{\rho_k}}{\rho_k^{-1/2} S_{\rho_k}} \to A \geq 0 \]
and pass to the limit taking if necessary a subsequence of \( \rho_k \) to have \( g_{\rho_k} \to g_0 \) and \( v_{\rho_k} \to v_0 \); we obtain that for \( x < 0 \)
\[ \partial_y v_0(x, \pm 0) = \pm \frac{\lambda A}{4\sqrt{-x}} \left(-2g' + \frac{g_0}{-x}\right) \]

Let us now rewrite the condition in terms of \( f \) and \( v_r \). After rescaling we have

\[ \lambda^2 \rho H(\Gamma_\rho)(-t, f_\rho(t)) = \sqrt{\frac{\pi}{2}} \frac{\partial_r u_\rho(-t, f_\rho(t))}{2} [\tau \cdot \nabla r^{1/2} \sin(\phi/2)](-t, f_\rho(t)). \]

The mean curvature can be expressed in terms of function \( f \)

\[ H(\Gamma_\rho)(-t, f_\rho(t)) = \frac{f_\rho''(t)}{(1 + (f_\rho')^2)^{3/2}} \]

and \( \partial_r u_\rho^\pm \) in terms of \( v_\rho \)

\[ \partial_r u_\rho^\pm(-t, f_\rho(t)) = S_\rho \partial_r v_\rho(-t, f_\rho(t)) \pm \rho^{1/2} \lambda [\tau \cdot \nabla r^{1/2} \sin(\phi/2)](-t, f_\rho(t)). \]

\( \tau \cdot \nabla r^{1/2} \sin(\phi/2) \) can be expressed in terms of \( f \) as well

\[ [\tau \cdot \nabla r^{1/2} \sin(\phi/2)](-t, f_\rho(t)) = \\
\left(1 + \frac{f_\rho'^2}{t}\right)^{-\frac{3}{2}} \lambda \sqrt{\frac{t}{2}} \frac{\sqrt{1 + \left(\frac{f_\rho}{t}\right)^2} - 1}{2\sqrt{2} f_\rho} \\
\left(1 + \frac{f_\rho'^2}{t}\right) \left(1 + \frac{1}{\sqrt{1 + \left(\frac{f_\rho}{t}\right)^2}}\right) + \frac{f_\rho f_\rho'}{t} \frac{1}{\sqrt{1 + \left(\frac{f_\rho}{t}\right)^2}} \right) = \\
(1 + o(f^{1/2}/t)) \frac{\lambda}{4\sqrt{t}} \left(1 + \frac{1}{\sqrt{1 + \left(\frac{f_\rho}{t}\right)^2}}\right) + \frac{f_\rho f_\rho'}{t} \frac{1}{\sqrt{1 + \left(\frac{f_\rho}{t}\right)^2}} \right) = \\
(1 + o(f^{1/2}/t)) \frac{\lambda}{2\sqrt{t}} (1 + o(f_\rho/t)) \]
Putting together we rewrite (9) as follows

\begin{equation}
\lambda^2 \rho_\sigma \rho \frac{g^\rho_\sigma(t)}{(1 + (f^\rho_\sigma))^3/2} = \\
\sqrt{\frac{\pi}{2}} [ (S_\rho \partial_t v_\rho(-t, f_\rho(t)) \pm \rho^{1/2} \lambda \left[ \tau \cdot \nabla \rho \sin(\phi/2) \right] (-t, f_\rho(t))^2 ]^\pm = \\
\lambda \sqrt{\frac{\pi}{2}} \left[ S_\rho \left( (\partial_t v_\rho^+(t) - t, f_\rho(t))^2 - (\partial_t v_\rho^-(t) - t, f_\rho(t))^2 \right) + \\
2S_\rho \lambda^{1/2} (1 + o(f(t)/t)) \frac{\lambda}{2\sqrt{t}} (1 + o(f(t)/t)) \left( \partial_t v_\rho^+(t) - t, f_\rho(t) + \partial_t v_\rho^-(t) - t, f_\rho(t) \right) \right]
\end{equation}

Thus

\begin{equation}
\lambda^2 \frac{\sigma_\rho}{\rho^{-1/2} S_\rho} \frac{g^\rho_\sigma(t)}{(1 + (f^\rho_\sigma))^3/2} = \\
\sqrt{\frac{\pi}{2}} \left[ S_\rho \left( (\partial_t v_\rho^+(t) - t, f_\rho(t))^2 - (\partial_t v_\rho^-(t) - t, f_\rho(t))^2 \right) + \\
2\lambda (1 + o(f(t)/t)) \frac{\lambda}{2\sqrt{t}} (1 + o(f(t)/t)) \left( \partial_t v_\rho^+(t) - t, f_\rho(t) + \partial_t v_\rho^-(t) - t, f_\rho(t) \right) \right]
\end{equation}

and passing to a limit

\begin{equation}
\lambda A g^\rho_\sigma(-x) = (-x)^{-1/2} \left[ \partial_x v_\sigma^0(x, 0) + \partial_x v_\sigma^0(x, 0) \right]
\end{equation}

Let us now denote by

\[ W_1(x, y) := \frac{1}{2} (v_0(x, y) - v_0(x, -y)) \]

and by

\[ W_2(x, y) := \frac{1}{2} (v_0(x, y) + v_0(x, -y)). \]

Then we obtain

\[ 2\partial_y W_1(x, 0) = 0, \text{ for } x < 0, \]

which means \( W_1(x, y) \) is a harmonic polynomial of the form \( \Im z^{k+1} \), \( k \in \mathbb{N} \).

For \( W_2 \) we obtain

\begin{equation}
2\partial_y W_2(x, 0) = \frac{\lambda A}{2\sqrt{-x}} \left( -2g^0_\rho(-x) + \frac{g_0(-x)}{-x} \right)
\end{equation}

and

\[ 2(-x)^{-1/2} \sqrt{\frac{\pi}{2}} \partial_x W_2(x, 0) = A \lambda g^\rho_\sigma(-x). \]

If \( A = 0 \) then \( W_2 \equiv 0 \), and then we have \( v_0(x, y) = W_1(x, y) = \Im z^{k+1} \) in \( \mathbb{R}^2 \setminus \{ (x, 0) | -\infty < x \leq 0 \} \), \( k \in \mathbb{N} \). We will come back to this case at the end of this section.
Let us now consider the more interesting case $A > 0$. First assume $W_2 = br^\alpha \cos \alpha \phi$, (we know that $\alpha > 1/2$). We obtain

$$g''(r) = \frac{2b}{\lambda A} \sqrt{\frac{\pi}{2}} \alpha r^{\alpha - \frac{3}{2}} \cos \pi \alpha$$

and

$$g(r) = \frac{2b}{\lambda A} \sqrt{\frac{\pi}{2}} \alpha \cos \pi \alpha r^{\alpha + \frac{1}{2}}.$$  

From here we first obtain, since $\sup_{t \in (0,1)} g(t) = 1$ and $g(r) = r^{\alpha + \frac{1}{2}}$

$$2b \sqrt{\frac{\pi}{2}} \frac{\alpha \cos \pi \alpha}{\alpha^2 - \frac{1}{4}} = A \lambda$$

and then from [14]

$$-2\alpha \sin \pi \alpha r^{\alpha - 1} = \frac{\lambda A}{2r^{1/2}} \left( -2(\alpha + \frac{1}{2})r^{\alpha - 1/2} + r^{\alpha - 1/2} \right) = -\alpha \lambda Ar^{\alpha - 1}$$

thus

$$\sin \pi \alpha = \sqrt{\frac{\pi}{2}} \frac{\alpha \cos \pi \alpha}{\alpha^2 - \frac{1}{4}}$$

or

$$\tan \pi \alpha = \sqrt{\frac{\pi}{2}} \frac{\alpha}{\alpha^2 - \frac{1}{4}}.$$ 

Thus we obtain that $\alpha = \alpha_k$, where $k < \alpha_k < k + \frac{1}{2}$ are the positive solutions of [16]. Note that the smallest positive solution $\alpha_1 \approx 1.2739$. 

![Picture 2]
We proved that if we assume that
\[ W_2(x, y) = \pm r^\alpha \cos(\alpha \phi), \]
then \( \alpha = \alpha_k \) for some \( k = 1, 2, \ldots \) and
\[ g(t) = \pm t^{1/2 + \alpha}. \]
In the Appendix B we prove that the case \( W_2 = br^\alpha \cos \alpha \phi \) is the only case we need to consider.

Since the orders of possible solutions for \( W_1 \) and \( W_2 \) are different we will always have that one of them must be identically zero. This means that \( b = \pm 1 \) and from (7) we obtain that if
\[ f(t) = Ct^{\alpha_k + 1/2} + o(t^{\alpha_k + 1/2}) \]
then
\[ u(x, y) = \lambda \Im \sqrt{z} \pm \frac{C \lambda}{\Delta \lambda} r^{\alpha_k} \cos(\alpha_k \phi) + o(r^{\alpha_k}), \]
and we see from (15) that \( |b_k| = \frac{1}{\Delta} > 0 \) depends only on \( k \).

Let us now come back to the case \( A = 0 \): that means \( W_2 \) vanishes and \( W_1 \) is not. If we have that
\[ u(x, y) = \lambda \Im \sqrt{z} + \Sigma_k(x, y) + o(r^{2k-1/2}), \]
where \( \Sigma_k \) is as in (2), then we can repeat the arguments above with the function
\[ v_\rho(x, y) = S_\rho^{-1}(u(\rho x, \rho y) - \rho^{1/2} \lambda r^{1/2} \sin(\phi/2) - \Sigma_k(\rho x, \rho y)), \]
where \( S_\rho = \sup_{B_\rho} |u(x, y) - \lambda r^{1/2} \sin(\phi/2) - \Sigma_k(\rho x, \rho y)| \), instead of (4). This means that we can iterate this procedure until we get the first non-zero \( W_2 \) term. If the iteration goes to infinity without giving any non-vanishing \( W_2 \), then we have that \( t^{-M} f(t) \to 0 \) as \( t \to 0 \) for any \( M > 0 \). Which would correspond the case \( \lim_{\rho \to 0} f(\rho t) \max_{0 < r < \rho} |f(r)| = 0. \)

3. The missing Euler-Lagrange condition

In this section we present some heuristic arguments to justify the existence of a missing Euler-Lagrange condition.

As it is proven in [BD], for the boundary function \( g = \lambda \sin \phi/2 \), the interval \( \Gamma = \{(x, 0) \mid -1 < x \leq 0\} \) and the function \( u = \lambda r^{1/2} \sin \phi/2 \) give the absolute minimizer of (1). Let us take a point \( w \in B_\delta \) and minimize the functional (1) among all curves starting at the discontinuity point at the boundary data \( g \), i.e. \( (-1, 0) \) and ending at the point \( w \). Heuristically it is natural to assume that for small \( \delta \) the minimizers \( (u_w, \Gamma_w) \) will exist and \( \Gamma_w \) will be a smooth curve. This would mean in particular that the first term in the asymptotic of \( u_w \) will be, up to a rotation, of the form \( \lambda_w \Im \sqrt{z - w} \). The authors think that \( \lambda_w \) is monotone in the horizontal direction and this would mean that there
exists an interface $\Xi \subset B_\delta$ on which $\lambda_w = \lambda$, where $\lambda$ is the "right" constant from the functional [1].

If we now look at the pairs $(u_w, \Gamma_w)$, where $w \in \Xi$, then all of them satisfy the three known Euler-Lagrange conditions

(i) $\Delta u_w = 0$ in $B_1 \setminus \Gamma_w$
(ii) $\lambda^2 H_{\Gamma_w} = \frac{\pi}{2}[|\nabla u_w|^2]^\pm$ on $\Gamma_w$
(iii) $\lambda_w = \lambda$

On the other hand we see that only one of them is the absolute minimizer. Since $\Xi$ (if the heuristics is true) is an interface near the origin the missing condition should be of first order.

From the **Main Theorem** one can see that the curvature at the cracktip can be either zero or $\infty$ (in case the coefficient of the $\alpha_1$-term is not zero). One possible candidate for the missing Euler-Lagrange condition could be the following:

The curvature of the crack vanishes at the cracktip.

or in terms of asymptotic, that the coefficient $C$ of the $r^{\alpha_1} \cos(\alpha_1 \phi)$-term in the expansion (2) vanishes.

**APPENDIX A**

Here we will prove that the expression

$$\frac{\sigma_\rho}{\rho^{-\frac{1}{2}} S_\rho}$$

is bounded at least on a subsequence near the origin.

To make the problem easier to handle let us "open" the crack. We consider $\mathbb{R}^2$ as complex plane and apply $\sqrt{z}$ conformal transformation given by

$$(x, y) \mapsto \left(\frac{\text{sgn}_\Gamma(x, y)}{\sqrt{2}} \sqrt{\sqrt{x^2 + y^2} + x}; \frac{\text{sgn}_\Gamma(x, y)}{\sqrt{2}} \sqrt{\sqrt{x^2 + y^2} - x}\right).$$
The new function \( w(z) := u(z^2) \) will be harmonic in \( \Omega \), where \( \Omega \) is the new domain. The free discontinuity set \( \Gamma \) will be mapped to the symmetric set \( \Sigma = \partial \Omega \setminus \partial B_1 \). Another important point is that the Neumann derivatives of \( w \) on \( \Sigma \) will be zero as it was the case with \( \Gamma \).

Since the set \( \Gamma \) was parametrized by \( \{(-t, f(t)) : 0 \leq t < 1\} \) the set \( \Sigma \) will be given by
\[
(F(\tau); \pm \tau) = \left( \frac{1}{\sqrt{2}} \sqrt{\sqrt{t^2 + f^2} - t; \pm \frac{1}{\sqrt{2}} \sqrt{\sqrt{t^2 + f^2} + t} \right).
\]

If we now calculate the \( \tilde{\sigma}_\rho = \sup_{\tau \in (0, \rho)} \rho^{-1} |F(\tau)| \) we will see that
\[
\tilde{\sigma}_\rho \approx \sup_{t \in (0, \rho^2)} \frac{\sqrt{1 + (\frac{f}{t})^2 - 1}}{\sqrt{1 + (\frac{f}{t})^2 + 1}} = \sup_{t \in (0, \rho^2)} \frac{|f|}{\sqrt{1 + (\frac{f^2}{t^2})}} \approx \frac{1}{2} \sigma_\rho^2.
\]

On the other hand
\[
\tilde{S}_\rho := \sup_{B_\rho} |w - \lambda y| = S_\rho^2
\]
then we need to prove that
\[
\liminf_{\rho \to 0} \frac{\tilde{\sigma}_\rho}{\rho^{-1} \tilde{S}_\rho} < +\infty.
\]

Suppose
\[
\frac{-\tilde{S}_\rho}{\rho^{-1} \tilde{S}_\rho} \to_{\rho \to 0} 0
\]
This means that
\[
V_\rho(x, y) = \frac{w(\rho x, \rho y) - \rho \lambda y}{\rho \tilde{\sigma}_\rho} \to_{\rho \to 0} 0.
\]
On the other hand we can find a sequence of points \( z_k \in \Sigma \) such that \( z_k \to 0 \) and the angle between the normal vector \( \nu|\Sigma(z_k) = (\nu_{|\Sigma}^x(z_k), \nu_{|\Sigma}^y(z_k)) \) and the \( x \)-axis is large enough, i.e.
\[
|\nu_{|\Sigma}^y(z_k)| \geq \frac{1}{2} \tilde{\sigma}_{|z_k|}.
\]
Thus
\[
\partial_x V_{|z_k|} \left( \frac{z_k}{|z_k|} \right) > \frac{1}{4} \lambda > 0,
\]
which is in contradiction with the fact of \( V_{|z_k|} \to 0 \).
Appendix B

Lemma 1. [Müntz Theorem] The space spanned by \{1, r^{\alpha_1}, r^{\alpha_2}, \ldots, r^{\alpha_j}, \ldots\} is dense in \(C([0, 1])\) the uniform norm if and only if
\[\sum_{j=1}^{\infty} \frac{1}{\alpha_j} = \infty.\]

For a proof see [C].

Lemma 2. Let \(\Sigma = \{(t, 0); \ t \in [-1, 0]\}\) and \(w \in W^{1,2}(B_1 \setminus \Sigma)\) be a function even in \(y\) that solves
\[
\begin{align*}
\Delta w &= 0 \quad \text{in } B_1 \setminus \Sigma \\
\frac{\partial w}{\partial y} &= \frac{AA}{2\sqrt{-x}} \left( -2y_0 + \frac{g_0}{-x} \right) \quad \text{on } \Sigma \\
\frac{\partial w}{\partial x} &= \frac{AA}{\sqrt{2\pi}} \frac{g_0}{x} \frac{g''}{g_0} \quad \text{on } \Sigma.
\end{align*}
\]
Moreover we assume that \(g \in C^1([-1,0])\) and that \(g(0) = g'(0) = 0\). Then there exist \(a_j \in \mathbb{R}\) such that in polar coordinates \((r, \phi)\)
\[w = a_0 + \sum_{j=1}^{\infty} a_j r^{\alpha_j} \cos(\alpha_j \phi)\]
where \(\alpha_j\) are the positive solutions to
\[
\tan(\pi\alpha) = \sqrt{\frac{\pi}{2}} \frac{\alpha}{\alpha^2 - \frac{1}{4}}.
\]

Proof: From the intermediate value theorem it follows that \([19]\) has a solution \(\alpha_j\) in each interval \((j, j + 1/2)\) (see picture [2]). Therefore it follows that
\[\sum_{j=1}^{\infty} \frac{1}{\alpha_j} = \infty.\]

Also,
\[\sum_{j=2}^{\infty} \frac{1}{\alpha_j - 1} = \infty.\]

Using Lemma [1] we can write
\[g'_0 = \tilde{a}_0 + \sum_{j=1}^{\infty} \alpha_j a_j r^{\alpha_j - 1}\]
for some sequence of real numbers \(a_j\). By assumption \(g'_0(0) = 0\) from which it follows that \(\tilde{a}_0 = 0\). If we define
\[u = \sum_{j=1}^{\infty} a_j r^{\alpha_j} \cos(\alpha_j \phi)\]
simple verification to see that \(u\) is a solution to \([17]\).
Next we consider \( v = w - u \). Then \( v \) solves
\[
\begin{align*}
\Delta v &= 0 \quad \text{in } B_1 \setminus \Sigma \\
\frac{\partial v}{\partial y} &= 0 \quad \text{on } \Sigma \\
\frac{\partial v}{\partial x} &= 0.
\end{align*}
\]

From well known unique continuation properties of harmonic functions (see \([AE]\) for instance) it follows that \( v \equiv \text{constant} \). The Lemma follows.

\[\Box\]

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