Newton’s law in braneworlds with an infinite extra dimension

Masato ITO

Department of Physics, Nagoya University, Nagoya, JAPAN 464-8602

Abstract

We study the behavior of the four–dimensional Newton’s law in warped braneworlds. The setup considered here is a $(3+n)$-brane embedded in $(5+n)$ dimensions, where $n$ extra dimensions are compactified and a dimension is infinite. We show that the wave function of gravity is described in terms of the Bessel functions of $(2+n/2)$-order and that estimate the correction to Newton’s law. In particular, the Newton’s law for $n=1$ can be exactly obtained.

*E-mail address: mito@eken.phys.nagoya-u.ac.jp
1 Introduction

Recent ideas of extra dimensions have developed rapidly and its investigations have become a center of particle physics and cosmology. It is satisfactory to consider that our four-dimensional world is embedded in higher dimensional world. This picture which comes from string/M-theory is so-called braneworld which assumes that Standard model fields are confined to a 3-brane with four-dimensional spacetime. Motivated by the warped braneworld [1], in the framework of simple braneworld with two 3-branes embedded in $AdS_5$, Randall and Sundrum proposed a new suggestion to the hierarchy problem owing to warped metric [2]. On the other hand, the warped geometry including a bulk scalar field has various types [4, 5]. Moreover interestingly the gravity in this warped geometry exhibits an interesting behavior. In the Randall-Sundrum model, graviton is localized on the positive tension 3-brane, and usual four-dimensional Newton’s law can be recovered at distance which is much larger than a radius of Anti-de Sitter space [3]. In particular, the four-dimensional effective Planck scale can be finite even if extra dimension is infinite, namely, non-compact [10]. This implies that zero mode of gravity becomes a bound state and massive mode of one has continuous eigenvalue. In various model the localization of gravity is widely discussed [6, 7, 8, 9].

In this paper, we study the four-dimensional Newton’s law on a $(3 + n)$-brane embedded in $(5 + n)$-dimensional world with negative bulk cosmological constant. It is assumed that $n$ extra dimensions on the brane are compactified in same radius $R \sim M_{Pl}^{-1}$, where $M_{Pl}$ is Planck scale, and that a perpendicular direction to brane is non-compact with $Z_2$ symmetry. This setup corresponds to an extension of the five-dimensional Randall-Sundrum model to $(5 + n)$-dimensions. Taking account of four-dimensional gravitational fluctuation about the background metric, we show that wave function of gravitational field is expressed in terms of the Bessel functions of $(2 + n/2)$-order. In order to investigate the behavior of gravity, we calculate the correction to the four-dimensional Newton’s law. We provide a possibility of searching for the remnant of compactification even if the compactified radius with Planck length is invisible in our world.

In section 2 we describe a setup considered here and show the wave function of gravity. In section 3 we obtain the probability for existence of gravity on the brane and estimate the correction to four-dimensional Newton’s law. In final section we describe a summary.

2 Setup

We consider the model of a single $(3+n)$-brane embedded in the bulk $(5+n)$-dimensions, and the action is given by

$$S = \int d^{4+n}x \ dy \sqrt{-G} \left( \frac{1}{2\kappa^2} R - \Lambda \right) - \int d^{4+n}x \ \sqrt{-g} \ V, \quad (1)$$
where $R$ is the higher dimensional scalar curvature, $\Lambda$ is the cosmological constant in the bulk and $V$ is brane tension, and $1/\kappa^2$ is the higher dimensional fundamental scale which has mass dimension $3 + n$. Let $G$ be the metric in the bulk and $g$ be the metric induced on the brane, $g_{\mu\nu} = G_{\mu\nu}(y = 0)$. Here the ansatz for $(5 + n)$-dimensional metric is taken as follows

$$ds^2 = a^2(y) \left( \eta_{\mu\nu}dx^\mu dx^\nu + \sum_{i=1}^{n} dz_i^2 \right) + dy^2 \equiv G_{MN}dx^M dx^N,$$

(2)

where $\eta_{\mu\nu} = (-,+,+,+,+)$. $a(y)$ is warp factor. It is assumed that $z$-directions are compactified in the same radius $R$ of Planck size and $y$-direction is an infinite extra dimension. This setup is as same as one given in Ref [11] which discussed the four-dimensional electrodynamics on the brane embedded in higher dimensions and the localization of $U(1)$ gauge field was investigated in detail. However we are interested in the shape of gravity in this model. This situation corresponds to a special case (isotropic brane tension) described in Ref [12]. Solving the Einstein equation of this model with negative bulk cosmological constant, we have

$$a(y) = e^{-|y|/L}, \quad \frac{1}{L} = \sqrt{\frac{-2\kappa^2\Lambda}{(n+3)(n+4)}},$$

(3)

where $a(y)$ respects the $Z_2$ symmetry $y \sim -y$. The jump condition with respect to the derivative of $a$ at $y = 0$ leads to the value of positive brane tension

$$V = \frac{2(n+3)}{\kappa^2 L}.$$  

(4)

As is evident from the above equations, the case of $n = 0$ is completely consistent with the solution in the original Randall-Sundrum model.

Integrating out the extra dimensions, squared Planck scale is identified with the coefficient of the four-dimensional scalar curvature and we obtain

$$M_{Pl}^2 \sim \frac{1}{\kappa^2} \int d^n z \int_0^\infty dy \ [a(y)]^{n+2} \sim \frac{R^n}{\kappa^2 L}. $$

(5)

In order to study the behavior of four-dimensional gravity, the gravitational fluctuation around the background metric is given by

$$G_{\mu\nu} = a^2(y)\eta_{\mu\nu} + h_{\mu\nu}(x,y,z).$$

(6)

By imposing transverse-traceless gauge for these fluctuations, i.e. $\partial_\mu h^\mu = h^\mu = 0$, the wave function of gravity is governed by the familiar eigenvalue problem of the
Schrödinger equation. In the context of this situation we treat the fluctuation of four-dimensional part, the additional fluctuations not of the form in transverse-traceless gauge are neglected. As for this point, we will describe it somewhere.

We calculate the effective four-dimensional static gravitational potential between two masses placed on the positive tension brane at a distance $r$ each other. It is assumed that $r$ is much larger than compactified radius $R \sim M_{Pl}^{-1}$ and that $L$ is set to the intermediate scale, $R \ll L$. Although the gravity propagating in the bulk has massive KK-modes along $n$ compactified $z$-directions, these dimensions are invisible at long distance under the above assumption. Namely, the dependence of $z$ in $h_{\mu\nu}$ is neglected here. Performing a separation of variables $x, y$ in wave function, $h_{\mu\nu}(x, y) = h_{\mu\nu}(x) \psi(y)$, and a change of variable $\psi(y) = a^{-n/2}(y) \hat{\psi}(y)$ yields

$$\left[ \frac{d^2}{dy^2} + m^2 e^{2|y|/L} - \frac{(n+4)^2}{4L^2} + \frac{n+4}{L} \delta(y) \right] \hat{\psi}(y) = 0, \quad (7)$$

where $m^2$ is the four-dimensional mass. Moreover, following the change of variable given in Ref. [3], $|u| = L(e^{|y|/L} - 1)$ and $\hat{\psi}(y) = \hat{\psi}(u)e^{-|y|/2L}$, the above equation leads to the non-relativistic quantum mechanics problem as follows

$$\left[ -\frac{d^2}{du^2} + V(u) \right] \hat{\psi}^{(n)}(u) = m^2 \hat{\psi}^{(n)}(u), \quad (8)$$

where a superscript $n$ in wave function represents the number of extra dimensions in $(3+n)$-brane, a volcano potential given here is given by

$$V(u) = \frac{(n+4)^2 - 1}{4(|u| + L)^2} - \frac{n+3}{L} \delta(u). \quad (9)$$

Here a potential form in the Schrödinger equation for $n = 0$ corresponds to a one in the original Randall-Sundrum model. An attractive potential of delta-function type causes the gravity to have a bound state. From Eq.(8), the zero mode wave function with $m^2 = 0$ can be normalizable, we have

$$\hat{\psi}_0(u) = \sqrt{\frac{n+2}{2}} L^{(n+2)/2} (|u| + L)^{-(n+3)/2}. \quad (10)$$

The wave function with continuous mode can be expressed in terms of superposition of the Bessel functions of $(2 + n/2)$ order

$$\hat{\psi}_m(u) = (|u| + L)^{1/2} \left\{ AJ_{2+n/2}(m(|u| + L)) + BY_{2+n/2}(m(|u| + L)) \right\}, \quad (11)$$

where $J_\nu$ and $Y_\nu$ is the Bessel functions of the first kind and ones of the second kind, respectively; $A$ and $B$ are constants to be determined below. The jump condition at
y = 0 leads to the relation between A and B, so that
\[ A = -BY_{1+n/2}(mL)/J_{1+n/2}(mL). \]
Furthermore, the normalization factor in wave function \( \tilde{\psi}_m \) can be determined by the orthonormalization condition of Bessel functions. Thus we get
\[
\tilde{\psi}_m(u) = N_m [m (|u| + L)]^{1/2} \times \left[ -Y_{1+n/2}(mL)J_{2+n/2}(m(|u| + L)) + J_{1+n/2}(mL)Y_{2+n/2}(m(|u| + L)) \right], \tag{12}
\]
where
\[
N_m = \frac{1}{\sqrt{[J_{1+n/2}(mL)]^2 + [Y_{1+n/2}(mL)]^2}}. \tag{13}
\]

Obviously, it is fact that the behaviors of Bessel functions are quite different forms whether \( n \) even or odd. The Bessel functions of odd order are written in terms of sine or cosine functions, explicitly. Actually, it is important to study the form of \( N_m \) since the correction to the Newton’s law is investigated by integral of mode \( m \). For \( n \) even, the normalization factor \( N_m \) is written by the asymptotic form of Bessel functions depending on the magnitude of argument. On the other hand, in the case that \( n \) is odd, the exact form of \( N_m \) can be described in terms of elementary functions \( \dagger \).

In present model, in despite of neglecting the dependence of compactified directions in four-dimensional fluctuation, it is found that the form of wave function depends on the number of compactified extra dimensions in the brane.

3 Correction to 4-d Newton’s law

The correction to the four-dimensional Newton’s law between two masses \( M_1 \) and \( M_2 \) is generated by the exchange of the massive modes, it is given by
\[
U_n(r) = \frac{\kappa^2}{R^n} \int_0^\infty dm \ M_1 M_2 \frac{e^{-mr}}{r} \left| \tilde{\psi}_m^{(n)}(0) \right|^2, \tag{14}
\]
where the probability for existence of gravity with continuous mode on the brane at \( y = 0 \) is
\[
\left| \tilde{\psi}_m^{(n)}(0) \right|^2 = \frac{4}{\pi^2 mL} \frac{1}{[J_{1+n/2}(mL)]^2 + [Y_{1+n/2}(mL)]^2}. \tag{15}
\]
\( \dagger \) The exact form of \( N_m \) is determined by using familiar relations between half-integer order Bessel functions of first kind and second kind as follows,
\[
[J_{3/2}(x)]^2 + [Y_{3/2}(x)]^2 = \frac{2}{\pi} (x^{-1} + x^{-3}),
\]
\[
[J_{5/2}(x)]^2 + [Y_{5/2}(x)]^2 = \frac{2}{\pi} (9x^{-3} + (3 - x^2)^2x^{-5}),
\]
\[
[J_{7/2}(x)]^2 + [Y_{7/2}(x)]^2 = \frac{2}{\pi} ((15 - x^2)^2x^{-5} + (15 - 6x^2)^2x^{-7}).
\]
Here we used Eqs. (12), (13) and Lommel’s formula $J_{\nu+1}(x)Y_\nu - J_\nu Y_{\nu+1}(x) = 2/(\pi x)$. The coefficient of the integral in Eq. (14) is the effective five-dimensional gravitational constant because the gravity is essentially five-dimensions with an infinite extra dimension at long distance $r \gg R$. The integral of Eq.(14) depends on the magnitude of argument in the Bessel functions of Eq.(15).

For $n$ even (this implies that Bessel functions of integer order), at $mL \ll 1$, the Bessel function of second kind is obviously dominant in denominator of Eq.(15). On the other hand, at $mL \gg 1$, the asymptotic behaviors of Bessel functions are written in terms of sine or cosine functions, $\sqrt{z} J_\nu(z) \sim \sqrt{2/\pi \cos z}$ and $\sqrt{z} Y_\nu(z) \sim \sqrt{2/\pi \sin z}$ for $z \gg 1$. Namely, Eq.(15) becomes constant. Thus we obtain that $|\tilde{\psi}_m(0)|^2 \sim (mL)^{1+n}$ for $m \ll L^{-1}$ and $|\tilde{\psi}_m(0)|^2 \sim 2/\pi$ for $m \gg L^{-1}$. Consequently, it is necessary to divide this integral into two regions, $mL \ll 1$ and $mL \gg 1$, and Eq.(14) is expressed as

$$U_n(r) = G_N L \frac{M_1 M_2}{r} \left[ \int_0^{L^{-1}} dm (mL)^{1+n} e^{-mr} + \int_{L^{-1}}^\infty dm \frac{2}{\pi} e^{-mr} \right],$$

(16)

Here we used $G_N \sim M_{Pl}^2 \sim \kappa^2/(R^n L)$, where $G_N$ is the four-dimensional Newton constant. The first term in Eq.(16) is the contribution of the light mode ($m \ll L^{-1}$) and the second term is the contribution of the heavy mode ($L^{-1} \ll m \ll R^{-1} \sim M_{Pl}$).

$$U_n(r) = G_N L \frac{M_1 M_2}{r} \left[ L^{1+n} \left( -\frac{\partial}{\partial r} \right)^{1+n} \frac{1 - e^{-r/L}}{r} + \frac{2}{\pi} \frac{e^{-r/L}}{r} \right],$$

(17)

At distance $r \gg L$, the first term in Eq.(17) is dominant. From Eqs.(14) and (15), the usual four-dimensional Newton’s law can be recovered via the contribution of zero mode $\tilde{\psi}_0(0) \sim 1/\sqrt{L}$. This means that the four-dimensional effective Planck scale is finite, as indicated in Eq.(5). Consequently, the four-dimensional Newton’s law $V_n(r)$ by adding the contribution of zero mode is

$$V_n(r) \sim G_N M_1 M_2 \left( 1 + C \left( \frac{L}{r} \right)^{2+n} \right),$$

(18)

where $C$ is a numerical constant. The second term in the bracket is the contribution of short distance correction to Newton’s law, which is consistent with the original Randall-Sundrum model. Moreover, at distance $r \ll L$, the second term in Eq.(17) is dominant and we have

$$V_n(r) \sim G_N L \frac{M_1 M_2}{r^2},$$

(19)

where corresponds to the five-dimensional Newton’s law. In this situation, we cannot see the four-dimensional world.
For $n$ odd, as shown in footnote †, we can obtain the exact form of normalization factor $N_m$ in wave function. Consequently, the probability for existing gravity on the brane is given by

$$
\left| \tilde{\psi}_m(0) \right|^2 = \frac{2}{\pi} \frac{(mL)^2}{(mL)^2 + 1},
$$

$$
\left| \tilde{\psi}_m(3) \right|^2 = \frac{2}{\pi} \frac{(mL)^4}{(mL)^4 + 3(mL)^2 + 9},
$$

$$
\left| \tilde{\psi}_m(5) \right|^2 = \frac{2}{\pi} \frac{(mL)^6}{(mL)^6 + 6(mL)^4 + 45(mL)^2 + 225}.
$$

for $n = 1, 3, 5$. Note that the asymptotic form with respect to $mL$ in Eq.(20) is completely consistent with the case of $n$ even. Namely, the forms of the correction to the Newton’s law are Eqs.(18) and (19) with regardless of $n$ odd or even. Although the integral of Eq.(14) is performed by using the asymptotic form of Bessel functions, it is straightforward to calculate the integral Eq.(14) in the case of $n = 1$. Consequently, the Newton’s law on compactified 4-brane embedded in six-dimensions with an infinite extra dimension is exactly given by

$$
V_1(r) = G_N \frac{M_1 M_2}{r} + G_N L M_1 M_2 \int_0^\infty dm \frac{e^{-mr}}{r} \frac{2}{\pi} \frac{(mL)^2}{(mL)^2 + 1}
$$

$$
= G_N \frac{M_1 M_2}{r} \left( 1 + \frac{2L}{\pi r} \left\{ 1 - \frac{r}{L} \left[ \sin \left( \frac{r}{L} \right) \text{ci} \left( \frac{r}{L} \right) - \cos \left( \frac{r}{L} \right) \text{si} \left( \frac{r}{L} \right) \right] \right\} \right)
$$

(21)

where $\text{ci}(x) = - \int_x^\infty \cos t/t \, dt$ and $\text{si}(x) = - \int_x^\infty \sin t/t \, dt$ are cosine and sine integrals, respectively. Obviously, the behavior of Eq.(21) at long distance $r \gg L$ is consistent with the case of $n = 1$ in Eq.(18). On the other hand, at distance $r \ll L$, we can obtain the form up to sub-leading order

$$
V_1(r) \sim G_N L \frac{M_1 M_2}{r^2} \left( 1 - \frac{r^2}{L^2} \left( \gamma - 1 + \log \frac{r}{L} \right) \right),
$$

(22)

where $\gamma$ is the Euler constant. As indicated in Eq.(18), in this model at long distance the power of distance $r$ is a clue to searching for the invisible compactified radius with Planck length.

Since the possibility of theories with extra dimensions was indicated, the experiment of searching for the presence of extra dimensions are increasingly performed. From recent gravitational experiments, it is found that the gravitational force $1/r^2$ law is maintained up to 0.218 mm [13]. However it is unknown whether $1/r^2$ law is violated or not at about micrometer range. In the near future it is expected that the sophisticated equipment of gravitational experiment will confirm the presence of extra dimensions.
4 Summary

In the framework of a \((3 + n)\)-brane embedded in \((5 + n)\) dimensions, where \(n\) extra dimensions are compactified in Planck length and a dimension is infinite, we have shown that the wave function of gravity is described in terms of superposition of the Bessel functions of \((2 + n/2)\)-order. We estimated the small correction to the four-dimensional Newton’s law on the brane, and the correction term is \(1/r^{n+2}\). In particular, for \(n = 1\) we can obtain the exact form of Newton’s law. We presented a model indicated that a tiny remnant of higher dimensional world is observable in our world.

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