Effects of spin on the dynamics of the 2D Dirac oscillator in the magnetic cosmic string background

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In this work the dynamics of a 2D Dirac oscillator in the spacetime of a magnetic cosmic string is considered. It is shown that earlier approaches to this problem have neglected a $\delta$ function contribution to the full Hamiltonian, which comes from the Zeeman interaction. The inclusion of spin effects leads to results which confirm a modified dynamics. Based on the self-adjoint extension method, we determined the most relevant physical quantities, such as energy spectrum, wave functions and the self-adjoint extension parameter by applying boundary conditions allowed by the system.

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I. INTRODUCTION

The Dirac oscillator is a natural model for studying properties of physical systems. This model is based on the dynamics of a harmonic oscillator for spin-1/2 particles by introducing a nominal prescription into free Dirac equation [1]. Because it is a exactly solvable model, several investigations have been developed in the context of this theoretical framework in the last years. The interest in this issue appears in different contexts, such as quantum optics [2–4], supersymmetry [5–7], nuclear reactions [8], Clifford algebra [9, 10] and noncommutative space [11, 12]. Recently, the Dirac oscillator has been verified experimentally by J. A. Franco-Villafañe et al., based on a tight-binding system [13]. A detailed description for the Dirac oscillator is given in Ref. [14] and for other contributions see Refs. [15–22].

Among the various contexts in which the Dirac oscillator can be addressed, we refer to the cosmic string, a linear defect that change the topology of the the medium when viewed globally. This framework has inspired a great deal of investigation in recent years. Such works encompass several distinct aspects to investigate the effects produced by topological defects of this nature [23–29].

In this work, we show rigorously how the dynamics of a 2D Dirac oscillator is modified when addressed in the presence of this topological defect, giving a special attention to the effects of spin. Our approach is based on the self-adjoint extension method which is appropriate to address any system endowed with a singular Hamiltonian (due to localized fields sources or quantum confinement). We determine the most relevant physical quantities from the present model, such as energy spectrum, wave functions and self-adjoint extension parameter by applying boundary conditions allowed by the system.

II. THE 2D DIRAC OSCILLATOR IN THE MAGNETIC COSMIC STRING BACKGROUND

In this section, we study the motion of the particle in the cosmic string background. The cosmic string spacetime with an internal magnetic field is an object described by the following line element in cylindrical coordinates $(t, r, \varphi, z)$:

$$ds^2 = -dt^2 + dr^2 + \alpha^2 r^2 d\varphi^2 + dz^2,$$

with $-\infty < (t, z) < \infty, r \geq 0$ and $0 \leq \varphi \leq 2\pi$. The parameter $\alpha$ is related to the linear mass density $\tilde{m}$ of the string by $\alpha = 1 - 4\tilde{m}$ runs in the interval $(0, 1]$ and corresponds to a deficit angle $\gamma = 2\pi(1 - \alpha)$. Geometrically, the metric (1) corresponds to a Minkowiski space-time with a conical singularity [30]. We start with the Dirac equation in the curved spacetime (with $\hbar = c = G = 1$):

$$[i\gamma^\mu (\partial_\mu + \Gamma_\mu) - e\gamma^\mu A_\mu - M] \psi = 0,$$

where $e$ is the charge, $M$ is mass of the particle, $\psi$ is a four-component spinorial wave function and $\Gamma_\mu$ is the spin connection. The calculation of $\Gamma_\mu$ can be found in some works in the literature. Here, we chose the same tetrad as in Ref. [30] (see also Ref. [28]). The recipe for determining $\Gamma_\mu$ is purely algebraic and the result is found to be

$$\Gamma_\mu = (0, 0, \Gamma_\varphi, 0),$$

with the non-vanishing element given as

$$\Gamma_\varphi = i\frac{(1 - \alpha)}{2} \Sigma^z,$$

where $\Sigma^z$ is the $z$-component of the spin vector $\Sigma$ [29, 30].

By exploiting the symmetry under $z$-translations, the $(2+1)$-dimensional Dirac equation is obtained from the decoupling of (3+1)-dimensional Dirac equation (2) for the specialized case where $\partial_z = 0$ and $A_z = 0$, into two uncoupled two-component equations [27, 31, 32]. The
relevant equation is the Dirac equation in (2 + 1) dimensions
\[ [\beta \gamma \cdot \pi + \beta M] \psi = E \psi, \] (5)

where
\[ \pi = -i(\nabla_\alpha + \Gamma) - eA, \] (6)
is the generalized momentum, with
\[ \nabla_\alpha = \frac{\partial}{\partial r} \hat{r} + \frac{1}{\alpha r} \frac{\partial}{\partial \varphi} \hat{\varphi}, \] (7)
the Nabla operator in cylindrical coordinates in the conical space and \( \psi \) is a two-component spinor. In (2 + 1) dimensions the \( \gamma \) matrices are given in terms of the Pauli matrices in cylindrical coordinates:
\[ \beta \gamma^r = \sigma^r, \quad \beta \gamma^\varphi = s \sigma^\varphi, \quad \beta = \gamma^0 = \sigma^z, \] (8)
where \( s \) is twice the spin value, with \( s = +1 \) for spin “up” and \( s = -1 \) for spin “down”.

The magnetic flux tube in the background space described by the metric (1) is related to the magnetic field as
\[ eB = e\nabla_\alpha \times A = -\frac{\phi}{\alpha r} \frac{\delta(r)}{r} \hat{z}, \] (9)
where \( \phi = \Phi/\Phi_0 \) is the flux parameter with \( \Phi_0 = 2\pi/e \), and the vector potential in the Coulomb gauge is
\[ eA = -\frac{\phi}{\alpha r} \hat{\varphi}. \] (10)
Note that in the limit as \( \alpha \to 1 \), we obtain the vector potential in Euclidean space.

The 2D Dirac oscillator is introduced by the nonminimal substitution \[1\],
\[ \frac{1}{i} \nabla_\alpha - \frac{1}{i} \nabla_\alpha - iM\omega\beta r, \] (11)
where \( r \) is the position vector and \( \omega \) the frequency of the oscillator (for a comprehensive discussion of the Dirac oscillator see Ref. [14]). In this case, Eq. (5) reads
\[ [\alpha \cdot (\pi - iM\omega\beta r) + \beta M] \psi = E \psi. \] (12)
The second order equation implied by Eq. (12) is obtained by applying the matrix operator \( [\beta M + E + \alpha \cdot (\pi - iM\omega\beta r)] \):
\[ (E^2 - M^2)\psi = [\alpha \cdot (\pi - iM\omega\beta r)] [\alpha \cdot (\pi - iM\omega\beta r)] \psi. \] (13)

By using anticommuting relation \( \{\alpha, \beta\} = 0 \) and the property \( (\sigma \cdot A)(\sigma \cdot B) = A \cdot B + i\sigma \cdot (A \times B) \), Eq. (13) becomes
\[ (E^2 - M^2)\psi = \left\{ \left[ \frac{1}{i} (\nabla_\alpha + \Gamma) - eA \right]^2 - 2M\omega [1 + s\sigma \cdot (L - r \times (i\Gamma + eA))] \sigma^z + M^2 \omega^2 r^2 - es(\sigma \cdot B) \right\} \psi. \] (14)
Inserting Eqs. (9), (10) and the expression for \( \Gamma_\varphi \) in (4), one obtains
\[ (E^2 - M^2)\psi = H \psi, \] (15)
where
\[ H = \left[ -i\nabla_\alpha + \left( \frac{\phi}{\alpha} + \frac{1 - \alpha}{2\alpha} \sigma^z \right) \frac{1}{r} \hat{\varphi} \right]^2 - 2M\omega \left[ \sigma^z + s \left( \frac{1}{\alpha r} \frac{\partial}{\partial \varphi} \hat{\varphi} + \frac{\phi}{\alpha} + \frac{1 - \alpha}{2\alpha} \right) \right] + M^2 \omega^2 r^2 + \frac{\phi s}{\alpha} \frac{\delta(r)}{r} \sigma^z. \] (16)

In Eq. (16), the quantity
\[ \frac{\phi}{\alpha} + \frac{1 - \alpha}{2\alpha} \sigma^z, \] (17)
contributes to the term which depends explicitly of the spin of the particle. The first term is the contribution due to the magnetic flux while the second is due to the spin connection. Note that, by making \( \alpha = 1 \) (flat spacetime) and \( \phi = 0 \) (absence of magnetic field) in Eq. (15), we obtain, for the planar case, the 2D Dirac oscillator as proposed by Moshinsky and Szczepaniak [1] and discussed in Appendix A.

Making use of the underlying rotational symmetry we can express the two component spinor as
\[ \psi(r, \varphi) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} f_m(r) e^{im\varphi} \\ g_m(r) e^{i(m+s)\varphi} \end{pmatrix}, \] (18)
with \( m \in \mathbb{Z} \). By replacing Eq. (18) into Eq. (15), we obtain the radial equation for \( f_m(r) \)
\[ Hf_m(r) = k^2 f_m(r), \] (19)
where
\[ k^2 = E^2 - M^2 + 2M\omega(sj + 1), \] (20)
\[ j = \frac{1}{\alpha} \left( m + \phi + \frac{1 - \alpha}{2} \right), \] (21)
and
\[ H = H_0 + \frac{\phi s}{\alpha} \frac{\delta(r)}{r}. \] (22)
The Hamiltonian in Eq. (22) governs the dynamics of a Dirac oscillator in a magnetic cosmic string background, i.e., a Dirac oscillator problem in the presence of the Aharonov-Bohm effect in a conical spacetime. The presence of a two-dimension \( \delta \) interaction in the radial Hamiltonian \( H \), which is singular at the origin, makes the problem more complicated to solve. The most adequate manner to address this kind of point interaction potential is by making use of the self-adjoint extension approach [33, 34]. This is the method adopted in this work and discussed in the next section.
III. SELF-ADJOINT EXTENSION ANALYSIS

In this section, we review some concepts on the self-adjoint extension approach. An operator \( \mathcal{O} \), with domain \( \mathcal{D}(\mathcal{O}) \), is said to be self-adjoint if and only if \( \mathcal{O} = \mathcal{O}^\dagger \) and \( \mathcal{D}(\mathcal{O}) = \mathcal{D}(\mathcal{O}^\dagger) \), \( \mathcal{O}^\dagger \) being the adjoint of operator \( \mathcal{O} \). For smooth functions, \( \xi \in C_0^\infty(\mathbb{R}^2) \) with \( \xi(0) = 0 \), we should have \( H \xi = H_0 \xi \), and it is possible to interpret the Hamiltonian (22) as a self-adjoint extension of \( H_0|_{C_0^\infty(\mathbb{R}^2(0))} \) [35–37]. The self-adjoint extension approach consists, essentially, in extending the domain of \( \mathcal{D}(\mathcal{O}) \) in order to match \( \mathcal{D}(\mathcal{O}^\dagger) \). From the theory of symmetric operators, it is a well-known fact that the symmetric radial operator \( H_0 \) is essentially self-adjoint for \( |j| \geq 1 \), while for \( |j| < 1 \) it admits an one-parameter family of self-adjoint extensions [38], \( H_{0,\lambda_m} \), where \( \lambda_m \) is the self-adjoint extension parameter. To characterize this family, we will use the approach in [33, 34], which is based in a boundary conditions at the origin. All the self-adjoint extensions \( H_{0,\lambda_m} \) of \( H_0 \) are parametrized by the boundary condition at the origin

\[
\Omega_0 = \lambda_m \Omega_1, \quad (24)
\]

with

\[
\Omega_0 = \lim_{r \to 0^+} r^{|j|} f_m(r), \quad (25)
\]

\[
\Omega_1 = \lim_{r \to 0^+} \frac{1}{r^{|j|}} \left[ f_m(r) - \Omega_0 \frac{1}{r^{|j|}} \right], \quad (26)
\]

where \( \lambda_m \in \mathbb{R} \) is the self-adjoint extension parameter. For \( \lambda_m = 0 \), we have the free Hamiltonian (without the \( \delta \) function) with regular wave functions at the origin, and for \( \lambda_m \neq 0 \) the boundary condition in Eq. (24) permit an \( r^{-|j|} \) singularity in the wave functions at the origin.

IV. THE BOUND STATE ENERGY AND WAVE FUNCTION

In this section, we determine the energy spectrum for the Dirac oscillator in the cosmic string background by solving Eq. (19). For \( r \neq 0 \), the equation for the component \( f_m(r) \) can be transformed by the variable change \( \rho = M \omega r^2 \) resulting in

\[
\rho f_m''(\rho) + f_m'(\rho) - \left( \frac{\omega^2}{4\rho} + \frac{\rho}{4} - \frac{k^2}{4\gamma} \right) f_m(\rho) = 0, \quad (27)
\]

with \( \gamma = M \omega \). Due to the boundary condition in Eq. (24), we seek for regular and irregular solutions for Eq. (27). Studying the asymptotic limits of Eq. (27) leads us to the following regular (+) (irregular (−)) solution

\[
f_m(\rho) = \rho^{|j|/2} e^{-\rho/2} M(\rho), \quad (28)
\]

With this, Eq. (27) is rewritten as

\[
\rho M''(\rho) + (1 \pm |j| - \rho) M'(\rho) - \left( \frac{1 \pm |j|}{2} - \frac{k^2}{4\gamma} \right) M(\rho) = 0. \quad (29)
\]

Equation (27) is of the confluent hypergeometric equation

\[
z M''(z) + (b - z) M'(z) - a M(z) = 0. \quad (30)
\]

In this manner, the general solution for Eq. (27) is

\[
f_m(r) = a_m \rho^{|j|/2} e^{-\rho/2} M(d_+, 1 + |j|, \rho) + b_m \rho^{-|j|/2} e^{-\rho/2} M(d_-, 1 - |j|, \rho), \quad (31)
\]

with

\[
d_\pm = \frac{1 \pm |j|}{2} - \frac{k^2}{4\gamma}. \quad (32)
\]

In Eq. (31), \( M(a, b, z) \) is the confluent hypergeometric function of the first kind [39] and \( a_m \) and \( b_m \) are, respectively, the coefficients of the regular and irregular solutions.

In this point, we apply the boundary condition in Eq. (24). Doing this, one finds the following relation between the coefficients \( a_m \) and \( b_m \)

\[
\lambda_m |j| = b_m a_m \left[ 1 + \frac{\lambda_m k^2}{4(1 - |j|)} \lim_{r \to 0^+} r^{2-2|j|} \right]. \quad (33)
\]

We note that \( \lim_{r \to 0^+} r^{2-2|j|} \) diverges if \( |j| \geq 1 \). This condition implies that \( b_m \) must be zero if \( |j| \geq 1 \) and only the regular solution contributes to \( f_m(r) \). For \( |j| < 1 \), when the operator \( H_0 \) is not self-adjoint, there arises a contribution of the irregular solution to \( f_m(r) \) [29, 40–45]. In this manner, the contribution of the irregular solution for system wave function steams from the fact that the operator \( H_0 \) is not self-adjoint.

For \( f_m(r) \) be a bound state wave function, it must vanish at large values of \( r \), i.e., it must be normalizable. So, from the asymptotic representation of the confluent hypergeometric function, the normalizability condition is translated in

\[
\frac{b_m}{a_m} = - \frac{\Gamma(1 + |j|) \Gamma(d_-)}{\Gamma(1 - |j|) \Gamma(d_+)} \quad (34)
\]

From Eq. (33), for \( |j| < 1 \) we have \( b_m/a_m = \lambda_m |j| \).

Using this result into Eq. (34), one finds

\[
\frac{\Gamma(d_+)}{\Gamma(d_-)} = - \frac{1}{\lambda_m |j|} \frac{\Gamma(1 + |j|)}{\Gamma(1 - |j|)}. \quad (35)
\]

Equation (35) implicitly determines the bound state energy for the Dirac oscillator in the cosmic string background for different values of the self-adjoint extension parameter. Two limiting values for the self-adjoint extension parameter deserves some attention. For \( \lambda_m = 0 \), when the \( \delta \) interaction is absent, only the regular solution contributes for the bound state wave function. In the other side, for \( \lambda_m = \infty \) only the irregular solution contributes for the bound state wave function. For all other values of the self-adjoint extension parameter, both regular and irregular solutions contributes for the bound
state wave function. The energy for the limiting values are obtained from the poles of gamma function, namely,

\[
\begin{align*}
    \phi_{\pm} = -n & \quad \text{for} \quad \lambda_m = 0, \quad \text{(regular solution)}, \\
    \phi_{\pm} = -n & \quad \text{for} \quad \lambda_m = \infty, \quad \text{(irregular solution)},
\end{align*}
\]

(36)

with \( n \) a nonnegative integer, \( n = 0, 1, 2, \ldots \). By manipulation of Eq. (36), we obtain

\[
E = \pm \left\{ M^2 + 2M\omega \left[ 2n \pm \frac{1}{\alpha} \left| m + \phi + \frac{1 - \alpha}{2} \right| - \frac{s}{\alpha} \left| m + \phi + \frac{1 - \alpha}{2} \right| \right] \right\}^{1/2}.
\]

(37)

In particular, it should be noted that for the case when \(|j| \geq 1\) or when the \( \delta \) interaction is absent, only the regular solution contributes for the bound state wave function \((b_m = 0)\), and the energy is given by Eq. (37) using the plus sign. Note that, for \( \alpha = 1 \) (flat space) and \( \phi = 0 \) (no magnetic flux), Eq. (37) coincides with the energy found for the usual 2D Dirac oscillator (cf. Eq. (A6) in Appendix A). Without loss of generality, let us suppose \( 0 < \phi < 1 \) [37, 46, 47]. In this interval, and recalling that we are interested in the case where \( 0 < \alpha \leq 1 \), another interesting feature is present in the energy eigenvalues. For the regular solution, the eigenvalues are independent of \( m, \phi \), and \( \alpha \) for \( s = 1 \). This situation is shown in Fig. 1(a) for \( n = 1 \) and \( m = 1 \). However, this independence is absent for \( s = -1 \), as shown in Fig. 1(b) for \( n = 1 \) and \( m = 1 \). In the other hand, for the irregular solution, the eigenvalues are independent of \( m, \phi \), and \( \alpha \) for \( s = -1 \) and dependent for \( s = 1 \). Also, for \( s = -1 \), decreasing the value of \( \alpha \), the energy increase as an effect of the quantum localization.

The unnormalized bound state wave functions for our problem are

\[
f_m(r) = \rho^{\pm |j|/2} e^{-\rho/2} M (-n, 1 \pm |j|, \rho).
\]

(38)

The self-adjoint extension is related with the presence of the \( \delta \) interaction. In this manner, the self-adjoint extension parameter must be related with the \( \delta \) interaction coupling constant \( \phi s/\alpha \). In fact, as shown in Refs. [28, 29] (see also Refs. [40, 48]), from the regularization of the \( \delta \) interaction, it is possible to find such a relationship. Using the regularization method, one obtains the following equation for the bound state energy

\[
\Gamma(d_+) = \frac{1}{\Gamma(d_-)} \left( \frac{\phi s + \alpha |j|}{\phi s - \alpha |j|} \right) \frac{1}{\Gamma(1 + |j|)} \frac{1}{\Gamma(1 - |j|)}.
\]

(39)

By comparing Eqs. (35) and (39), this relation is found to be

\[
\frac{1}{\lambda_m} = \frac{1}{r_0^{2|j|}} \left( \frac{\phi s + \alpha |j|}{\phi s - \alpha |j|} \right)
\]

(40)

where \( r_0 \) is a very small radius which comes from the \( \delta \) regularization [28, 29].

V. NONRELATIVISTIC LIMIT

We shall now take the nonrelativistic limit of Eq. (14). Using \( E = M + \mathcal{E} \) with \( M \gg \mathcal{E} \), we obtain

\[
2M\mathcal{E}\psi = \left\{ \frac{1}{\ell^2} \left[ (\mathbf{\nabla}_\phi + \mathbf{A}) - e\mathbf{A} \right]^2 - 2M\omega \left[ 1 + s\sigma \cdot (\mathbf{L} - r \times (i\mathbf{\Gamma} + e\mathbf{A})) \right] \right\} \psi.
\]

(41)

Performing the same steps as for the relativistic case, one obtains the shifted energy levels (cf. Appendix A)

\[
\mathcal{E} + \omega = \left[ 1 + 2n \pm \frac{1}{\alpha} \left| m + \phi + \frac{1 - \alpha}{2} \right| - \frac{s}{\alpha} \left| m + \phi + \frac{1 - \alpha}{2} \right| \right] \omega.
\]

(42)
In this equation, the + (−) sign is for \( \lambda_m = 0 \) (\( \lambda_m = \infty \)) when one has regular (irregular) solution. We note that the energy in Eq. (42) corresponds to the equation (54) of Ref. [49] (cf. also Eq. (A8) in Appendix A) with two additional contributions, the spin-orbit coupling and the spin connection.

VI. CONCLUSIONS

In this contribution, we have addressed the Dirac oscillator interacting with a topological defect and in the presence of the Aharonov-Bohm potential. This system has been studied in Ref. [17]. However, the authors not take into account the effects of spin. In other words, the term proportional to the \( \delta \) interaction was discarded, by considering only the regular solution of the problem. The presence of this term has direct implications in the energy spectrum and wave functions of the oscillator. Although being singular at the origin, this term reveals that both regular and irregular solutions contributes for the bound state energies are given explicitly in Eq. (37). We also verified that, for the flat space (\( \alpha = 1 \)) and no magnetic flux (\( \phi = 0 \)), the results of usual 2D Dirac oscillator are recovered.

Appendix A: 2D Dirac oscillator

In this appendix, we briefly discuss the usual 2D Dirac oscillator. We mention that although fully equivalent, the present is slightly different from the previous construction in the literature. Let us consider Eq. (5) with \( \mathbf{p} = \mathbf{p} - iM\omega \mathbf{\hat{r}} \). By using the representation for the \( \gamma \) matrices in terms of the Pauli matrices in Euclidean space

\[
\beta \gamma^1 = \sigma^1, \quad \beta \gamma^2 = s\sigma^2, \quad \beta = \sigma^3, \quad (A1)
\]

we are left with

\[
[\sigma^1 \pi_1 + \sigma^2 \pi_2 + \sigma^3 M - E] \psi = 0, \quad (A2)
\]

with \( \pi_i = p_i - iM\omega r_i \). By squaring Eq. (A2), one obtains

\[
[p^2 + M^2 \omega^2 r^2 - 2M\omega(\sigma^3 + sL_3)] \psi = (E^2 - M^2)\psi. \quad (A3)
\]

Equation (A3), restoring the factors \( h \) and \( c \), in terms of components, provides

\[
2Mc^2 \left[ H^{2D}_{\text{ho}} - h\omega - s\omega L_3 \right] \psi_1 = (E^2 - M^2c^4)\psi_1, \quad (A4a)
\]

\[
2Mc^2 \left[ H^{2D}_{\text{ho}} + h\omega - s\omega L_3 \right] \psi_2 = (E^2 - M^2c^4)\psi_2, \quad (A4b)
\]

where

\[
H^{2D}_{\text{ho}} = \frac{p^2}{2M} + \frac{1}{2}M\omega^2 r^2. \quad (A5)
\]

Equation (A4) for \( s = 1 \) agreed with the expressions found in Eq. (A2) of Ref. [50] and Eqs. (9) and (22) of Ref. [51]. Using the ansatz in Eq. (18) the energy eigenvalues are determined:

\[
E = \pm \sqrt{M^2 + 2M\omega (2n + |m| - sm)}, \quad (A6)
\]

showing that the energy eigenvalues are spin dependent. It should be noted that for \( s = 1 \) (\( s = -1 \)) and \( m > 0 \) (\( m < 0 \)) the energy eigenvalues are independent of the quantum number \( m \).

From Eq. (A3), in the nonrelativistic limit \( E = M + \mathcal{E} \) with \( M \gg \mathcal{E} \), we have

\[
[H^{2D}_{\text{ho}} - \omega(\sigma^3 + sL_3)] \psi = \mathcal{E}\psi. \quad (A7)
\]

The first term on the left side of Eq. (A7) is the Hamiltonian of the nonrelativistic circular harmonic oscillator [52], explaining why this system is called Dirac oscillator. The second term is a constant which shifts all energy levels. The last term is the spin-orbit coupling, which (restoring the factor \( h \)) is of strength \( \omega/h \). Summarizing, the nonrelativistic limit of the 2D Dirac oscillator is the circular harmonic oscillator with a strong spin-orbit coupling term with all levels shifted by the factor \( \omega \). Indeed, the shifted energy levels are

\[
\mathcal{E} + \omega = (1 + 2n + |m| - sm)\omega. \quad (A8)
\]

As for the relativistic case, for \( s = 1 \) (\( s = -1 \)) and \( m > 0 \) (\( m < 0 \)) the energy eigenvalues are independent of the quantum number \( m \).

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