CERTAIN CIRCLE ACTIONS ON KÄHLER MANIFOLDS

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Abstract. Let the circle act holomorphically on a compact Kähler manifold \( M \) of complex dimension \( n \) with moment map \( \phi: M \to \mathbb{R} \). Assume the critical set of \( \phi \) consists of 3 connected components, the extrema being isolated points. We show that \( M \) is equivariantly biholomorphic to \( \mathbb{C}P^n \), where \( n \geq 2 \), or to \( G_2(\mathbb{R}^{n+2}) \), the Grassmannian of oriented 2-planes in \( \mathbb{R}^{n+2} \), where \( n \geq 3 \), with a standard circle action; we also show that \( M \) is equivariantly symplectomorphic to \( \mathbb{C}P^n \), where \( n \geq 2 \), or to \( G_2(\mathbb{R}^{n+2}) \), where \( n \geq 3 \), with a standard circle action.

1. Introduction

Let \((M, \omega)\) be a connected compact symplectic manifold. Assume that \( M \) admits a nontrivial Hamiltonian circle action with moment map \( \phi: M \to \mathbb{R} \). We call \( M \) a Hamiltonian \( S^1 \)-manifold. The moment map \( \phi: M \to \mathbb{R} \) is a Morse-Bott function whose critical set is exactly the fixed point set \( M^{S^1} \) of the circle action.

Let \((M, \omega)\) be a compact Hamiltonian \( S^1 \)-manifold whose even Betti numbers are minimal, i.e., \( b_{2i}(M) = 1 \) for all \( 0 \leq 2i \leq \dim(M) \); or, whose even Betti numbers are the same as those of a complex projective space. Some recent work studied certain cases of such Hamiltonian \( S^1 \)-manifolds. The papers \([19, 15]\) studied the case when \( \dim(M) = 6 \), \([8]\) studied the case when \( \dim(M) = 8 \) and \( M^{S^1} \) consists of isolated points, and \([13, 12]\) studied the case when \( M \) is of any dimension and \( M^{S^1} \) consists of two connected components. These papers showed that such Hamiltonian \( S^1 \)-manifolds have certain geometrical and topological invariants the same as those of the complex projective spaces, the Grassmannian of oriented two planes in some \( \mathbb{R}^N \), or some other relatively well known Kähler manifolds. In particular, for two important and interesting cases when \( \dim(M) = 6 \) and when the fixed points are isolated, \([15]\) determined the equivariant symplectomorphism type of the manifolds. While for the case when \( M \) is 2n-dimensional and \( M^{S^1} \) consists of two connected components, \([12]\) showed that there are finitely many such manifolds up to equivariant diffeomorphism; only in a few special cases, we...
know the equivariant symplectomorphism type of the manifold, in particular in the case when one fixed component is isolated, by a theorem of Delzant [5], $M$ is equivariantly symplectomorphic to $\mathbb{C}P^n$ with a standard circle action.

Let $(M, \omega)$ be a compact Hamiltonian $S^1$-manifold. In [13, Section 4], it was shown that if $b_{2i}(M) = 1$ for all $0 \leq 2i \leq \dim(M)$, then the fixed components of the circle action satisfy

$$\sum_{F \subset M^{S^1}} (\dim(F) + 2) = \dim(M) + 2,$$

where the sum is over all the fixed set components. While the converse is false as shown by Example 1.3 below (when $n$ is even), it is anticipated that when (1.1) holds, the cohomology groups of $M$ are relatively “simple”.

In this paper, we consider a Kähler manifold satisfying the following assumption:

**Assumption A.** Let $(M, \omega, J)$ be a compact Kähler manifold of complex dimension $n$ which admits a holomorphic Hamiltonian circle action. Assume the critical set of the moment map consists of 3 connected components, the extrema being isolated points.

Under this assumption, by Morse theory (see Lemma 2.5), we can see that the 3 critical components of the moment map satisfy the equality (1.1). Hence this is a special case of a Hamiltonian $S^1$-manifold with fixed point set satisfying (1.1). We have two families of examples satisfying Assumption A.

**Example 1.2.** Consider $\mathbb{C}P^n$, where $n \geq 2$. As a coadjoint orbit of $SU(n + 1)$, it admits a Kähler structure. Consider the effective $S^1 \subset SU(n + 1)$ action on $\mathbb{C}P^n$ given by

$$\lambda \cdot [z_0, z_1, \ldots, z_n] = [\lambda^{-l}z_0, \lambda^{l'}z_1, z_2, \ldots, z_n],$$

where $l, l' \in \mathbb{N}$ and $\gcd(l, l') = 1$. The fixed point set $(\mathbb{C}P^n)^{S^1}$ consists of 3 connected components:

$$X = [z_0, 0, \cdots, 0], \quad Y = [0, 0, z_2, \cdots, z_n], \quad \text{and} \quad Z = [0, z_1, 0, \cdots, 0],$$

where $X$ and $Z$ are isolated points. The action is holomorphic and is Hamiltonian. The weights of the circle action on the normal bundles of $X, Y$ and $Z$ are respectively

$$(l, \cdots, l, l + l'), (-l, l'), (-l', \cdots, -l', -(l + l')).$$

**Example 1.3.** Consider $\tilde{G}_2(\mathbb{R}^{n+2})$, the Grassmannian of oriented 2-planes in $\mathbb{R}^{n+2}$, where $n \geq 3$. We have $\dim_{\mathbb{R}}(\tilde{G}_2(\mathbb{R}^{n+2})) = 2n$. As a coadjoint orbit of $SO(n + 2)$, it has a Kähler structure. Consider the $S^1 \subset SO(n + 2)$ action on $\tilde{G}_2(\mathbb{R}^{n+2})$ induced by the action on $\mathbb{R}^{n+2} = \mathbb{C} \times \mathbb{R}^n$ given by

$$\lambda \cdot (z, x_1, \ldots, x_n) = (\lambda \cdot z, x_1, \ldots, x_n).$$
The fixed point set \((\widetilde{G}_2(\mathbb{R}^{n+2}))^{S^1}\) consists of 3 connected components:
\[ X = \mathbb{P}(z, 0, \cdots, 0) = pt, \quad Y = \widetilde{G}_2(0 \times \mathbb{R}^n), \quad \text{and} \quad Z = \mathbb{P}(z, 0, \cdots, 0) = pt, \]
where \(X\) and \(Z\) correspond to the two orientations on the real 2-plane \((z, 0, \cdots, 0)\). The \(S^1\) action is holomorphic and Hamiltonian. The weights of the circle action on the normal bundles of \(X\), \(Y\) and \(Z\) are respectively
\[ (1, \cdots, 1), \quad (-1, 1), \quad (-1, \cdots, -1). \]

Our results in Theorems 1.4 and 1.5 equivariantly identify the manifold satisfying Assumption A with one of these examples in the complex and symplectic categories.

**Theorem 1.4.** Under Assumption A, \(M\) is \(S^1\)-equivariantly biholomorphic to \(\mathbb{C}P^n\), where \(n \geq 2\), or to \(\widetilde{G}_2(\mathbb{R}^{n+2})\), where \(n \geq 3\), with a standard circle action.

**Theorem 1.5.** Under Assumption A, \(M\) is \(S^1\)-equivariantly symplectomorphic to \(\mathbb{C}P^n\), where \(n \geq 2\), or to \(\widetilde{G}_2(\mathbb{R}^{n+2})\), where \(n \geq 3\), with a standard circle action.

Under the following assumption, where \(M\) is a symplectic manifold, our next result is that the circle action and the first Chern class of the manifold are exactly as those of an example in Example 1.2 or 1.3.

**Assumption B.** Let \((M, \omega)\) be a compact 2\(n\)-dimensional symplectic manifold which admits an effective Hamiltonian circle action. Assume the critical set of the moment map \(\phi\) consists of 3 connected components \(X, Y\) and \(Z\), where \(\phi(X) < \phi(Y) < \phi(Z)\), and \(X\) and \(Z\) are isolated points.

**Theorem 1.6.** Suppose Assumption B holds. Then \(n\) and the weights of the \(S^1\) action respectively on the normal bundles of \(X, Y\) and \(Z\) are
\[ (1) \quad n \geq 3; \quad (1, \cdots, 1), \quad (-1, 1), \quad (-1, \cdots, -1), \quad \text{or} \]
\[ (2) \quad n \geq 2; \quad (l, \cdots, l, l+l'), \quad (-l, l'), \quad (-l', \cdots, -l', -(l+l')), \quad \text{where} \quad l, l' \in \mathbb{N}, \quad \text{and} \quad \gcd(l, l') = 1. \]
Moreover, if \([\omega]\) is a primitive integral class, then \(c_1(M) = n[\omega]\) in case (1), and \(c_1(M) = (n + 1)[\omega]\) in case (2).

We use equivariant cohomology and Morse theory to prove Theorem 1.6. Then we use the Kähler condition and a result by Kobayashi and Ochiai to prove Proposition 4.2, and we use Theorem 1.6 and Proposition 4.2 to prove Theorem 1.4. Finally, we use Theorem 1.4 to prove Theorem 1.5.

The work of this paper provides a new idea to study Hamiltonian \(S^1\)-manifolds which are Kähler and which have fixed point set satisfying (1.1). We hope to be able to use the method to treat more general cases of the fixed point set for compact Kähler Hamiltonian \(S^1\)-manifolds.

For the case when \((M, \omega)\) is a compact 2\(n\)-dimensional Hamiltonian \(S^1\)-manifold such that the critical set of the moment map consists of two connected components, \(X\) and \(Y\), [13] showed that the condition \(b_2(M) = 1\) for
all $0 \leq 2i \leq \dim(M)$ is equivalent to the condition $\dim(X) + \dim(Y) + 2 = \dim(M)$. Under this assumption, [13, Theorem 1] showed that, $c_1(M) = (n+1)[\omega]$, or $c_1(M) = n[\omega]$ with $n \geq 3$ odd, if $[\omega]$ is primitive integral. If we assume in addition that $M$ is Kähler and the $S^1$ action is holomorphic, then our method (see Proposition 4.2 and our proofs of Theorems 1.4 and 1.5) implies that $M$ is $S^1$-equivariantly biholomorphic to $\mathbb{CP}^n$, or to $G_2(\mathbb{R}^{n+2})$ with $n \geq 3$ odd, with a standard circle action; and $M$ is $S^1$-equivariantly symplectomorphic to $\mathbb{CP}^n$, or to $\tilde{G}_2(\mathbb{R}^{n+2})$ with $n \geq 3$ odd, with a standard circle action.

A compact Kähler manifold $M$ with the same Betti numbers as $\mathbb{CP}^n$ but is different from $\mathbb{CP}^n$ is called a fake projective space. For example, $G_2(\mathbb{R}^{n+2})$ with $n \geq 3$ odd are such spaces. There are more fake projective spaces other than these, see [17, 10, 18] and etc. for the classification and study in complex dimensions 2, 3, 4. But with stronger topological conditions on a compact Kähler manifold $M$, one can exclude the fake ones, or show that $M$ is biholomorphic to $\mathbb{CP}^n$: if $M$ has the homotopy type and Pontryagin classes of $\mathbb{CP}^n$ [9]; if $M$ has the homotopy type of $\mathbb{CP}^n$ and $n \leq 6$ [14]; or if $M$ has positive first Chern class, and its integral cohomology ring is isomorphic to that of $\mathbb{CP}^n$ with $n \leq 5$ [6].

Both in the symplectic and Kähler cases, the study of compact Hamiltonian $S^1$-manifolds with minimal even Betti numbers, or with the weaker condition — the fixed point set satisfying (1.1), provides a different perspective to the extensive and interesting study of rational cohomology complex projective spaces.

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2. Some preliminary results

In this section, we state and prove some preliminary results for symplectic Hamiltonian $S^1$-manifolds — for the general case and for the case when the Hamiltonian function has 3 critical components, with isolated extrema.

Let $(M, \omega)$ be a Hamiltonian $S^1$-manifold with moment map $\phi: M \to \mathbb{R}$, and let $F$ be a fixed component of the $S^1$ action. Let us set up the following notations.

- $2\lambda_F$: the Morse index of $F$ as a critical set of the Morse-Bott function $\phi$;
- $2\chi_F$: the Morse coindex of $F$ for $\phi$, or the Morse index of $F$ for $-\phi$;
- $\Gamma_F$: the sum of the weights of the $S^1$ action on the normal bundle $N_F$ to $F$;
- $\Lambda_F$: the product of the weights of the $S^1$ action on the normal bundle $N_F$ to $F$;
• $\Lambda^-_F$: the product of the weights of the $S^1$ action on the negative normal bundle $N^-_F$ to $F$;

• $\Lambda^+_F$: the product of the weights of the $S^1$ action on the positive normal bundle $N^+_F$ to $F$.

For a smooth $S^1$-manifold $M$, let $H^*_S(M; R) = H^*(S^\infty \times S^1; M; R)$ be the $S^1$-equivariant cohomology of the manifold $M$ in $R$ coefficient, where $R$ is a coefficient ring. Let $t \in H^2_{S^1}(pt; \mathbb{Z}) = H^2(\mathbb{C}P^\infty; \mathbb{Z})$ be a generator.

The projection $\pi: S^\infty \times S^1 M \to \mathbb{C}P^\infty$ induces a natural push forward map $\pi_*: H^*_{S^1}(M; Q) \to H^*(\mathbb{C}P^\infty; Q)$, which is given by “integration over the fiber”, denoted $\int_M$. We will use the following theorem due to Atiyah-Bott, and Berline-Vergne [1, 3] in Section 3.

**Theorem 2.1.** Let the circle act on a compact manifold $M$. Fix a class $\alpha \in H^*_{S^1}(M; Q)$. Then as elements of $Q(t)$,

$$\int_M \alpha = \sum_{F \subset M} \int_F \frac{\alpha|_F}{e^{S^1}(N_F)},$$

where the sum is over all fixed components, and $e^{S^1}(N_F)$ is the equivariant Euler class of the normal bundle to $F$.

2.1. Some preliminary results on Hamiltonian $S^1$-manifolds.

First, we have the following “equivariant extension” of the cohomology class represented by the symplectic form.

**Lemma 2.2.** [13, Lemma 2.7] Let the circle act on a connected compact symplectic manifold $(M, \omega)$ with moment map $\phi: M \to \mathbb{R}$. Let $F$ be a fixed component. Then there exists $\tilde{u} \in H^2_{S^1}(M; \mathbb{R})$ so that

$$\tilde{u}|_{F'} = [\omega|_{F'}] + t (\phi(F) - \phi(F'))$$

for any fixed component $F'$. If $[\omega]$ is an integral class, then $\tilde{u}$ is an integral class.

Now, for a Hamiltonian $S^1$-manifold $M$ with $H^2(M; \mathbb{R}) = \mathbb{R}$, we use the above lemma to express the 1st Chern class $c_1(M)$ of $M$ in terms of data related to any two fixed components of the action.

**Lemma 2.3.** Let the circle act on a connected compact symplectic manifold $(M, \omega)$ with moment map $\phi: M \to \mathbb{R}$. Assume $H^2(M; \mathbb{R}) = \mathbb{R}$. Then

$$c_1(M) = \frac{\Gamma_F - \Gamma_{F'}}{\phi(F') - \phi(F)} [\omega],$$

where $F$ and $F'$ are any two fixed components.

**Proof.** Since $H^2(M; \mathbb{R}) = \mathbb{R}$, $[\omega] \in H^2(M; \mathbb{R})$ is the generator. So $t$ and the $\tilde{u}$ in Lemma 2.2 are the generators of $H^2_{S^1}(M; \mathbb{R})$. We can write the equivariant first Chern Class $c_1^{S^1}(M)$ of $M$ as

$$c_1^{S^1}(M) = at + b\tilde{u},$$

where $a, b \in \mathbb{R}$. 
Then
\[ c_1^S(M)|_f = \Gamma_F t = at, \] where \( f \in F \) is a point, and
\[ c_1^S(M)|_{f'} = \Gamma_{F'} t = at + bt \left( \phi(F) - \phi(F') \right), \] where \( f' \in F' \) is a point.

So
\[ a = \Gamma_F, \quad b = \left( \Gamma_{F'} - \Gamma_F \right) / \left( \phi(F) - \phi(F') \right). \]

Hence
\[ c_1^S(M) = \Gamma_F t + \frac{\Gamma_{F'} - \Gamma_F}{\phi(F) - \phi(F')} \tilde{u}. \]
Taking the restriction map \( H^2_S(M; \mathbb{R}) \to H^2(M; \mathbb{R}) \), we get
\[ c_1(M) = \frac{\Gamma_{F'} - \Gamma_F}{\phi(F) - \phi(F')}[\omega]. \]

The following lemma gives an expression of the equivariant Euler class of the negative normal bundle of a fixed component \( F \) when its Morse index is the sum \( \sum \phi(F') < \phi(F) (\dim(F') + 2) \).

**Lemma 2.4.** [13, Remark 4.12] Let the circle act on a connected compact symplectic manifold \( (M, \omega) \) with moment map \( \phi: M \to \mathbb{R} \). Let \( F \) be a fixed component and assume \( \sum \phi(F') < \phi(F) (\dim(F') + 2) = 2\lambda_F \). Then
\[ e^S(N^-_F) = \Lambda^-_F \prod_{\phi(F') < \phi(F)} \left( t + \frac{[\omega]_F}{\phi(F') - \phi(F)} \right)^{\frac{1}{2} \dim(F') + 1}, \]

where \( e^S(N^-_F) \) is the equivariant Euler class of the negative normal bundle \( N^-_F \) of \( F \).

**2.2. The case when the Hamiltonian function has 3 critical components, the extrema being isolated.**

First we show that, the index of the non-extremal critical component of the moment map \( \phi \) satisfies the condition of Lemma 2.4 and \( H^2(M; \mathbb{R}) \cong \mathbb{R} \).

**Lemma 2.5.** Let the circle act on a compact symplectic manifold \( (M, \omega) \) with moment map \( \phi: M \to \mathbb{R} \). Let \( F \) be a fixed component and assume \( \sum \phi(F') < \phi(F) (\dim(F') + 2) = 2\lambda_F \). Then
\[ 2\lambda_Y = 2\lambda_F = 2, \quad \text{and} \quad H^2(M; \mathbb{R}) \cong \mathbb{R}. \]

**Proof.** Since \( \phi \) has 3 critical components, we have \( \dim(M) > 2 \). Since \( Z \) is isolated, we have \( 2\lambda_Z = \dim(M) > 2 \). The fact that \( \phi \) is a perfect Morse-Bott function gives
\[ \dim H^i(M) = \sum_{F \subset M^S} \dim H^{i-2\lambda_F}(F). \]

Then since \( X \) is isolated, and \( H^2(M; \mathbb{R}) \neq 0 \), (2.6) gives \( 2\lambda_Y = 2 \). Similarly, using \(-\phi\), we get \( 2\lambda_Y = 2 \). Now the fact \( H^2(M; \mathbb{R}) \cong \mathbb{R} \) follows from these facts and (2.6). \( \square \)
Next, for the case stated, we write \( c_1(M) \) slightly differently from that in Lemma 2.3 so that we can identify an integral class.

**Lemma 2.7.** Let the circle act on a compact symplectic manifold \((M, \omega)\) with moment map \( \phi: M \to \mathbb{R} \) such that its critical set consists of 3 connected components \( X, Y \) and \( Z \), where \( \phi(X) < \phi(Y) < \phi(Z) \), with \( X \) and \( Z \) being isolated points. Then

\[
c_1(M) = \frac{\Gamma_X - \Gamma_Y}{\Lambda_Y} \left[ \Lambda_Y \frac{\omega}{\phi(Y) - \phi(X)} \right],
\]

where \( \Lambda_Y \frac{\omega}{\phi(Y) - \phi(X)} \) is an integral class.

**Proof.** By Lemma 2.2 we can take \( \bar{u} \in H^2_{S^1}(M; \mathbb{R}) \) so that \( \bar{u}|_Y = \left[ \omega|_Y \right] \) and \( \bar{u}|_X = t(\phi(Y) - \phi(X)) \). Using this together with Lemmas 2.5 and 2.4, we get

\[
e^{S^1}(N^-_Y) = \Lambda_Y \left( t + \frac{[\omega|_Y]}{\phi(X) - \phi(Y)} \right) = \Lambda_Y \left( t + \frac{\bar{u}|_Y}{\phi(X) - \phi(Y)} \right).
\]

Since \( M^{S^1} \) consists of 3 connected components and \( Z \) is isolated, \( \dim(M) = 2\lambda_Z \geq 4 \), so \( \deg(e^{S^1}(N^-_Y)) = 2 < 2\lambda_Z \). Let \( M^- = \{ x \in M | \phi(x) < \phi(Z) \} \). Consider the long exact sequence for the pair \((M, M^-)\) in equivariant cohomology with \( \mathbb{Z} \) coefficients, since \( H^2_{S^1}(M, M^-; \mathbb{Z}) = H^{2-2\lambda_Z}_{S^1}(Z; \mathbb{Z}) = 0 \) and \( H^3_{S^1}(M, M^-; \mathbb{Z}) = H^{3-2\lambda_Z}_{S^1}(Z; \mathbb{Z}) = 0 \), the long exact sequence splits into the short exact sequence

\[
0 \to H^2_{S^1}(M; \mathbb{Z}) \to H^2_{S^1}(M^-; \mathbb{Z}) \to 0.
\]

Consider the class \( e^{S^1}(N^-_Y) \). \( \bar{u} \in H^2_{S^1}(M; \mathbb{R}) \). By (2.8), its restriction to \( Y \) is \( e^{S^1}(N^-_Y) \); clearly, its restriction to \( X \) is 0. By (2.9), \( e^{S^1}(N^-_Y) \) is the unique class on \( M \) having this property. So \( e^{S^1}(N^-_Y) \) is integral implies \( e^{S^1}(N^-_Y) \) is integral. Taking the restriction map \( H^*_S(M; \mathbb{R}) \to H^*(M; \mathbb{R}) \) for the class \( e^{S^1}(N^-_Y) \), we get that \([\Lambda_Y \frac{\omega}{\phi(X) - \phi(Y)}]\) is an integral class.

By Lemma 2.6 \( H^2(M; \mathbb{R}) = \mathbb{R} \). Now apply Lemma 2.3 using the fixed components \( X \) and \( Y \).

\[\square\]

3. **Proof of Theorem 1.6**

Let \((M, \omega)\) be a symplectic \( S^1 \)-manifold. An *isotropy submanifold* \( M_{\mathbb{Z}_k} \subseteq M \) is a symplectic submanifold which is not fixed by the \( S^1 \) action, but is fixed by the \( \mathbb{Z}_k \) action for some \( k > 1 \). We need the following lemma in the proof of Theorem 1.6.

**Lemma 3.1.** [19, Lemma 2.6] Let the circle act symplectically on a compact symplectic manifold \((M, \omega)\). Let \( p \) and \( q \) be fixed points which lie on the same
component $N$ of $M^{Z_k}$ for some $k > 1$. Then the weights of the action at $p$ and at $q$ are equal modulo $k$.

Proof of Theorem 1.6. By Lemma 2.5, $2\lambda_Y = 2\lambda_Y^+ = 2$. We have $2\lambda_Z = 2n \geq 4$.

Case (1). Assume the action is semifree. Then the weights of the $S^1$ action on the normal bundles of $X$ and $Y$ are as in (1) of Theorem 1.6.

Suppose $Y$ is an isolated point, then $\dim(M) = \dim(Y) + 2\lambda_Y + 2\lambda_Y^+ = 4$, $e^{S^1}(N_X) = t^2$, $e^{S^1}(N_Y) = t \cdot (-t)$ and $e^{S^1}(N_Z) = (-t)^2$. Using Theorem 2.1 to integrate 1 on $M$, we get a contradiction. Hence $\dim(M) \geq 6$.

Using Lemma 2.7, we get
\[c_1(M) = n \left[ \frac{\omega}{\phi(Y) - \phi(X)} \right], \text{ where } \left[ \frac{\omega}{\phi(Y) - \phi(X)} \right] \in H^2(M; \mathbb{Z}).\]

If $[\omega]$ is primitive integral, then $\phi(Y) - \phi(X) \in \mathbb{N}$ by Lemma 2.2 so
\[\left[ \frac{\omega}{\phi(Y) - \phi(X)} \right] = [\omega] \text{ and } c_1(M) = n[\omega].\]

Case (2). Assume the action is effective and is not semifree.

Case (2a). Assume there is an isotropy submanifold $M_Y^X$ between $X$ and $Y$, which is an $M^{Z_l}$, where $l > 1$, and there is an isotropy submanifold $M_Y^Z$ between $Y$ and $Z$, which is an $M^{Z_{l'}}$, where $l' > 1$. We have $\dim(M_Y^X) = \dim(Y) + 2\lambda_Y = \dim(Y) + 2 + \dim(X)$, and similarly $\dim(M_Y^Z) = \dim(Y) + 2 + \dim(Z)$; moreover, the weights of the normal representation at $Y$ are $(-l, l')$, where gcd($l, l'$) = 1 since the action is effective. Then the weights of the normal representation at $X$ are $(l, \ldots, l, s$) for some $s \in \mathbb{N}$. The weights of the normal representation at $Z$ must be $(-l', \ldots, -l', -s)$.

By Lemma 2.5 $H^2(M; \mathbb{R}) \cong \mathbb{R}$. By rescaling, we may assume that $[\omega]$ is primitive integral. Since $2 < 2\lambda_Z$, $[\omega]|_{\{m \in M \mid \phi(m) < \phi(Z)\}}$ is primitive integral. Since the manifold $\{m \in M \mid \phi(m) < \phi(Z)\}$ deformation retracts to $M_Y^X$ (the retract is given by the gradient flow of $-\phi$), $[\omega]|_{M_Y^X}$ is primitive integral. Similarly, since $2 < 2\lambda_Y^+$, $[\omega]|_{M_Y^Z}$ is primitive integral. Using [13 Proposition 6.1] on $M_Y^X$ and on $M_Y^Z$, we get
\[\phi(Y) - \phi(X) = l \quad \text{and} \quad \phi(Z) - \phi(Y) = l'.\]

Then using Lemma 2.9 to compute $c_1(M)$ respectively in terms of $X$, $Y$ and $Y, Z$, we get
\[s = l + l' \quad \text{and} \quad c_1(M) = (n + 1)[\omega].\]

Case (2b). Assume there is an isotropy submanifold $M_Y^X$ between $X$ and $Y$, which is an $M^{Z_l}$, where $l > 1$, and there is no isotropy submanifold between $Y$ and $Z$. Then the weights of the normal representation at $Y$ are $(-l, 1)$, and the weights of the normal representation at $X$ are $(l, \ldots, l, s$) for some $s \in \mathbb{N}$. Similarly as in Case (2a), assume $[\omega]$ is primitive integral, then
\[\phi(Y) - \phi(X) = l.\]
First assume there is no isotropy submanifold between \( X \) and \( Z \). Then \( s = 1 \) and the weights of the normal representation at \( Z \) are \((-1, \cdots, -1)\). Using Lemma 2.3 to compute \( c_1(M) \) respectively in terms of \( X, Y \) and \( Y, Z \), it gives a contradiction. Hence there is an isotropy submanifold between \( X \) and \( Z \), and there is only one, which corresponds to the weight \( s \) (hence \( s > 1 \)). So the weights of the normal representation at \( Y \) are \((-1, \cdots, -1, -s)\). Using Lemma 3.1 for the fixed sets \( X \) and \( Z \), we have \( l + 1 = as \), where \( a > 0 \). Similarly, using the lemma for the fixed sets \( X \) and \( Y \), we get \( s = 1 + bl \), where \( b > 0 \). These together give \( a = 1 \) and \( b = 1 \), so

\[ s = l + 1. \]

Using Lemma 2.5 to compute \( c_1(M) \), we get

\[ c_1(M) = (n + 1)[[\omega]] \quad \text{and} \quad \phi(Z) - \phi(Y) = 1. \]

For the case when there is an isotropy submanifold between \( Y \) and \( Z \) and there is no isotropy submanifold between \( X \) and \( Y \), we can discuss similarly and we have similar conclusion as above.

Case (2c). Assume there are no isotropy submanifolds between \( X \) and \( Y \) and between \( Y \) and \( Z \). Then the weights of the normal representation at \( Y \) are \((-1, 1)\). Since the action is not semifree, there is an isotropy submanifold between \( X \) and \( Z \). We will show that there is only one isotropy submanifold between \( X \) and \( Z \), which is an \( M^1 \). If \( \dim(M) = 4 \), using Lemma 3.1, this is clear.

Now assume \( \dim(M) > 4 \). Since \( X \) and \( Z \) are isolated points, each isotropy submanifold between \( X \) and \( Z \) is a sphere. If \( W \) is the set of weights at \( X \), then the set of weights at \( Z \) is \(-W\). Then \( \Gamma_X = -\Gamma_Z \) and \( \Lambda_X = \pm \Lambda_Z \). We also have \( \Lambda_Y = -1 \), \( \Lambda_Y^+ = 1 \), and \( \Gamma_Y = 0 \). Assume \([\omega]\) is primitive integral. Then by Lemmas 2.7 and 2.2 (using a similar argument as in Case (1)),

\[ c_1(M) = \Gamma_X [\omega], \quad \text{and} \quad \phi(Y) - \phi(X) = 1. \]

So \( c_1(M)|_Y = \Gamma_X [\omega|_Y] \). By symmetry or using a similar argument as the above, \( \phi(Z) - \phi(Y) = \phi(Y) - \phi(X) = 1 \). By Lemmas 2.5 and 2.4, \( e^{S_1}(N_Y^-) = -t + \frac{[\omega|_Y]}{\phi(Y) - \phi(X)} = -t + [\omega|_Y] \). Using these lemmas for \( -\phi \), we get \( e^{S_1}(N_Y^+) = t + \frac{[\omega|_Y]}{\phi(Z) - \phi(Y)} = t + [\omega|_Y] \). First assume \( \dim(Y) = 2(2k + 1) \), where \( k \geq 0 \).

Let \( c_1^{S_1}(M) \) be the equivariant first Chern class of \( M \). Then \( c_1^{S_1}(M)|_X = \Gamma_X t, c_1^{S_1}(M)|_Y = c_1(M)|_Y + t - \Gamma_X [\omega|_Y], c_1^{S_1}(M)|_Z = \Gamma_Z t \). We have \( e^{S_1}(N_X) = \Lambda_X t^n \), \( e^{S_1}(N_Y) = e^{S_1}(N_Y^-) \cdot e^{S_1}(N_Y^+) \) and \( e^{S_1}(N_Z) = \Lambda_Z t^n \).

Using Theorem 2.1 to compute

\[ \int_M c_1^{S_1}(M) = 0, \]

we get

\[ \frac{\Gamma_X}{\Lambda_X} - \frac{\Gamma_Z}{\Lambda_Z} = 0, \quad \text{where} \quad a = \int_Y [\omega|_Y]^\frac{\dim(Y)}{2} = \int_Y [\omega|_Y]^{2k+1} \in \mathbb{Z}. \]
Since $Y$ is symplectic, $a \neq 0$. Since $\dim(M) = 2(2k + 1) + 4 = 2(2k' + 1)$, $\Lambda_X = -\Lambda_Z$. Solving the above equation, we get

$$a = \frac{2}{\Lambda_X} \in \mathbb{Z}.$$ 

Hence $\Lambda_X = 2$ ($\Lambda_X \neq 1$ since the action is not semifree) which implies that there can only be one isotropy submanifold between $X$ and $Z$, which is fixed by $\mathbb{Z}_2$. Next, assume $\dim(Y) = 2(2k)$, where $k > 0$. We use Theorem 2.1 to compute

$$\int_M \left( c_1^{S_1}(M) \right)^2 = 0.$$ 

A similar argument as the above shows that there is only one isotropy submanifold between $X$ and $Z$, which is fixed by $\mathbb{Z}_2$. Hence, the weights of the action at $X$ are $(1, \cdots, 1, 2)$ and the weights of the action at $Z$ are $(-1, \cdots, -1, -2)$. Then (3.2) gives $c_1(M) = (n + 1)[\omega]$. \hfill \Box

4. Proof of Theorem 1.4

In this section, we first prove Proposition 4.2 below, then we use Proposition 4.2 and Theorem 1.6 to prove Theorem 1.4.

Let us first recall the following theorem which was proved in [11].

**Theorem 4.1.** Let $M$ be a compact Kähler manifold of complex dimension $n$. If there exists a positive element $\alpha \in H^{1,1}_{\mathbb{Z}}(M)$ such that $c_1(M) = (n + 1)\alpha$, then $M$ is biholomorphic to $\mathbb{C}P^n$; if there exists a positive element $\alpha \in H^{1,1}_{\mathbb{Z}}(M)$ such that $c_1(M) = n\alpha$, then $M$ is biholomorphic to a quadratic hypersurface in $\mathbb{C}P^{n+1}$.

**Proposition 4.2.** Let $(M, \omega, J)$ be a compact Kähler manifold of complex dimension $n$, which admits a holomorphic Hamiltonian circle action. Assume that $[\omega]$ is an integral class. If $c_1(M) = (n + 1)[\omega]$, then $M$ is $S^1$-equivariantly biholomorphic to $\mathbb{C}P^n = \mathbb{P}(H^0(M; L))$, and if $c_1(M) = n[\omega]$, then $M$ is $S^1$-equivariantly biholomorphic to a quadratic hypersurface in $\mathbb{C}P^{n+1} = \mathbb{P}(H^0(M; L))$, where $L$ is a holomorphic line bundle over $M$ with first Chern class $[\omega]$ and $H^0(M; L)$ is its space of holomorphic sections.

The proof of Proposition 4.2 is to incorporate the circle action into the proof of Theorem 4.1. The idea of the proof of Theorem 4.1 is as follows. Since $\alpha \in H^{1,1}_{\mathbb{Z}}(M)$, by a result of Kodaira and Spencer, there is a holomorphic line bundle $L$ over $M$ such that $c_1(L) = \alpha$; and since $c_1(L)$ is positive, $L$ is ample by a theorem of Kodaira. Let $H^0(M; L)$ be the complex vector space of holomorphic sections of $L$ over $M$. The authors of [11] showed the following fact.

**Lemma 4.3.** When $c_1(M) = (n + 1)c_1(L)$, $\dim H^0(M; L) = n + 1$; and when $c_1(M) = nc_1(L)$, $\dim H^0(M; L) = n + 2$. 


Moreover, they showed that $H^0(M; L)$ has no base points, i.e., the elements of $H^0(M; L)$ have no common zeros. Hence, if $\mathbb{P}(H^0(M; L))$ is the complex projective space defined as the set of hyperplanes through the origin in the vector space $H^0(M; L)$, then we can define a holomorphic mapping

$$j : M \to \mathbb{P}(H^0(M; L))$$

by

$$j(x) = \{s \in H^0(M; L) \mid s(x) = 0\} \quad \text{for each } x \in M.$$

The authors showed that the map $j$ from $M$ to its image is biholomorphic. In the case when $c_1(M) = (n+1)[\omega]$ or $c_1(M) = n[\omega]$, where $[\omega] \in H^{1,1}(M; \mathbb{Z})$ is positive. So we have the ample line bundle $L$ as above with $c_1(L) = [\omega]$ and the results in Theorem 4.1.

We choose a Hermitian structure $h$ on $L$ and a Hermitian connection $\nabla$ such that its curvature form is $\omega$. Then $(L, h, \nabla)$ is a (pre)quantum line bundle over $M$.

Let $X_M$ be the vector field generated by the circle action on $M$, and let $X_M$ be the horizontal lift of $X_M$ to $TL$. Then

$$X_L = X_M + \phi \frac{\partial}{\partial \theta}$$

gives an action of the Lie algebra of $S^1$ on $L$, where $\frac{\partial}{\partial \theta}$ is the vector field on $L$ generated by the fiberwise multiplication by $e^{i\theta}$.

Let $X, Y, Z$ be the fixed components of the circle action such that $\phi(X) < \phi(Y) < \phi(Z)$. Since $[\omega]$ is an integral class, Lemma 2.2 implies that

$$\phi(Y) - \phi(X) \in \mathbb{N} \quad \text{and} \quad \phi(Z) - \phi(Y) \in \mathbb{N}.$$  

Since the moment map is defined up to translation by a constant, (4.5) implies that we may assume the moment map values of the fixed point set components are in the integral lattice of $\mathbb{R}$. By [7, Example 6.10], the above action of the Lie algebra of $S^1$ on $L$ integrates to an $S^1$ action on $L$ which is compatible with the $S^1$ action on $M$. Then $S^1$ acts on $H^0(M; L)$ which is given by

$$\lambda \cdot s(x) = \lambda \cdot s(\lambda^{-1} \cdot x), \quad \text{where } \lambda \in S^1 \text{ and } s \in H^0(M; L);$$

or infinitesimally given by

$$X \cdot s = -\nabla_{X_M} s + \sqrt{-1} \phi \cdot s, \quad \text{where } X \in \text{Lie}(S^1) \text{ and } s \in H^0(M; L).$$

The $S^1$ action on $H^0(M; L)$ induces an $S^1$ action on $\mathbb{P}(H^0(M; L))$.

**Lemma 4.7.** Under the assumptions of Proposition 4.2, the holomorphic map $j$ in (4.4) can be made to be $S^1$-equivariant.
Proof. The above analysis shows that \( S^1 \) acts on \( H^0(M; L) \) and on \( \mathbb{P}(H^0(M; L)) \).
For any \( g \in S^1 \), and \( s \in H^0(M; L) \), by \( \text{(4.6)} \), we have
\[
(g \cdot s)(x) = g \cdot s(g^{-1} \cdot x).
\]
So
\[
j(g \cdot x) = \{ s \in H^0(M; L) \mid s(g \cdot x) = 0 \} \text{ for each } x \in M,
\]
and
\[
g \cdot j(x) = \{ g \cdot s \in H^0(M; L) \mid g \cdot s(x) = 0 \} = \{ g \cdot s \in H^0(M; L) \mid g \cdot s(g^{-1} g \cdot x) = 0 \}
\]
\[
= \{ g \cdot s \in H^0(M; L) \mid (g \cdot s)(g \cdot x) = 0 \} \text{ for each } x \in M.
\]
Hence
\[
j(g \cdot x) = g \cdot j(x).
\]

Proposition \( \text{(4.2)} \) follows from Theorem \( \text{(4.1)} \) and Lemma \( \text{(4.7)} \).

Remark 4.8. The embedding \( j \) in \( \text{(4.3)} \) may not be a Kähler embedding. In the case when \( M \) is a compact homogeneous Kähler manifold which admits a homogeneous very ample quantum line bundle \( L \), for example, when \( M \) is \( \mathbb{C}P^n \) or \( G_2(\mathbb{R}^{n+2}) \), the embedding is Kähler \( \text{(2.4)} \).

Proof of Theorem \( \text{(4.4)} \). By Lemma \( \text{(2.5)} \), we can rescale the Kähler form \( \omega \) so that \( [\omega] \) is primitive integral.

By quotienting out a finite subgroup of \( S^1 \) that acts trivially, we may assume that the \( S^1 \) action is effective. Then by Theorem \( \text{(4.6)} \), \( c_1(M) = (n + 1)[\omega] \) with \( n \geq 2 \), or \( c_1(M) = n[\omega] \) with \( n \geq 3 \).

By Proposition \( \text{(4.2)} \), \( M \) is equivariantly biholomorphic to \( \mathbb{P}(H^0(M; L)) = \mathbb{C}P^n \) or to a quadratic hypersurface in \( \mathbb{P}(H^0(M; L)) = \mathbb{C}P^{n+1} \) through the equivariant holomorphic map \( j \) in \( \text{(4.4)} \). In particular, the circle action on \( j(M) \) has 3 fixed components biholomorphic to those on \( M \).

Since the \( S^1 \) action on the complex vector space \( H^0(M; L) \) is Hamiltonian, the induced \( S^1 \) actions on \( \mathbb{P}(H^0(M; L)) \) and on the invariant complex manifold \( j(M) \) are Hamiltonian. In the case when \( j(M) \) is a quadratic hypersurface in \( \mathbb{C}P^{n+1} \), \( j(M) \) can be identified with \( G_2(\mathbb{R}^{n+2}) \). \( \square \)

5. PROOF OF THEOREM \( \text{(4.5)} \)

With the help of Theorem \( \text{(4.4)} \), we can prove Theorem \( \text{(4.5)} \) as follows.

Proof of Theorem \( \text{(4.5)} \). By Theorem \( \text{(4.4)} \), there is an \( S^1 \)-equivariant biholomorphism
\[
f: (M, \omega, J) \rightarrow (M', \omega', J'),
\]
where \( (M', \omega', J') \) represents the Kähler manifold \( \mathbb{C}P^n \) with \( n \geq 2 \) or \( G_2(\mathbb{R}^{n+2}) \) with \( n \geq 3 \), with the standard structures. By Lemma \( \text{(2.5)} \), \( H^2(M; \mathbb{R}) = \mathbb{R} \). So by rescaling \( \omega \), we may assume that \( \omega \) and \( f^*\omega' \) represent the same cohomology class. Consider the family of forms \( \omega_t = (1 - t)\omega + tf^*\omega' \), where \( t \in [0, 1] \). Each \( \omega_t \) is clearly closed. Each \( \omega_t \) is also nondegenerate: for any point \( x \in M \), suppose \( X \in T_xM \) is such that \( \omega_t(X, Y) = 0 \) for all
$Y \in T_x M$. In particular, if we take $Y = JX$, then we get $\omega_t(X, JX) = 0$. Using the facts $\omega(X, JX) \geq 0$, $f_*(JX) = J'f_* X$, and $\omega'(f_*X, J'f_*X) \geq 0$, we get $X = 0$. So $\omega_t$ is a family of symplectic forms. It represents the same cohomology class and it is $S^1$-invariant. By Moser’s theorem [16] (Moser’s argument without the presence of a group action can be adapted to the case when a compact group acts by choosing invariant objects), there is an $S^1$-equivariant isotopy $\Phi_t$ such that $\Phi_t^* \omega_t = \omega$; in particular, $\Phi_1^* f^* \omega' = \omega$. □

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