A RELATION BETWEEN POROSITY CONVERGENCE AND PRETANGENT SPACES

Maya Altınok and Mehmet Küçükaslan

Abstract. The convergence of porosity is one of the relatively new concepts in Mathematical analysis. It is completely structurally different from the other convergence concepts. Here we give a relation between porosity convergence and pretangent spaces.

1. Introduction

The notion of convergence, as one of the fundamental concepts in Mathematical analysis, has many generalizations such as statistical convergence [14][23], ideal convergence [21], convergence in measure [26], A-convergence for a matrix A [15][19][20], etc. Unlike all types of convergences given in the literature with different forms, porosity convergence as relatively new is defined in [2]. The basis of this study lies in the redefinition of the porosity notion from a point in $[0, \infty)$ to infinity in natural numbers [3].

Porosity notion appeared in the papers of Denjoy [7][8] and Khintchine [18] and, Dolzenko [9]. It has many applications such as in theory of free boundary [16], generalized subharmonic functions [11], complex dynamics [22], quasisymmetric maps [25], infinitesimal geometry [5] and some other areas of mathematics.

Let us remember the definitions of right upper porosity for subsets of real numbers at zero. Let $E \subset \mathbb{R}^+$, then the right upper porosity of $E$ at 0 is defined as

$$p^+ (E) = \limsup_{h \to 0^+} \frac{\lambda(E, h)}{h}$$

where $\lambda(E, h)$ is the length of the largest open subinterval of $(0, h)$ that contains no point of $E$ [24].

The notion of right lower porosity of $E$ at 0 is defined similarly.

In [3] the definition of porosity which was given for the subsets of real numbers, have been redefined for the subsets of natural numbers by using a special function which is called scaling function.

2010 Mathematics Subject Classification: 40A05.
Key words and phrases: porosity, convergence, pretangent spaces.
Communicated by Stevan Pilipović.

41
Let \( \mu : \mathbb{N} \to \mathbb{R}^+ \) be a strictly decreasing function such that \( \lim_{n \to \infty} \mu(n) = 0 \) and let \( A \) be a subset of \( \mathbb{N} \). Now, let us recall from [3] that upper and lower porosity of \( A \) at infinity as follows

\[
\bar{p}_{\mu}(A) := \limsup_{n \to \infty} \frac{\lambda_{\mu}(A, n)}{\mu(n)}, \quad \underline{p}_{\mu}(A) := \liminf_{n \to \infty} \frac{\lambda_{\mu}(A, n)}{\mu(n)}
\]

where

\[
\lambda_{\mu}(A, n) := \sup \{ |\mu(n^{(1)}) - \mu(n^{(2)})| : n \leq n^{(1)} < n^{(2)}, (n^{(1)}, n^{(2)}) \cap A = \emptyset \}.
\]

From the definitions of upper and lower porosity of a subset of \( \mathbb{N} \) at infinity, we have the following trivial result [3].

**Remark 1.1.** If \( A \) is a finite subset of \( \mathbb{N} \), that is \( |A| < \infty \), then, for every \( n \in \mathbb{N} \), \( \lambda_{\mu}(A, n) \) is the length of the largest open subinterval of \( (0, \mu(n)) \) that contains no point of \( \mu(A) \) and has a form \( (\mu(n^{(2)}), \mu(n^{(1)})) \) with \( \mu(n^{(1)}) < \mu(n^{(2)}) \). For the case of finite \( A \) we evidently have \( \lambda_{\mu}(A, n) = \mu(n) \) for all sufficiently large \( n \). Consequently the equality \( \bar{p}_{\mu}(A) = \underline{p}_{\mu}(A) = 1 \) holds with every scaling function \( \mu \) for all \( A \subseteq \mathbb{N} \) with \( |A| < \infty \).

Throughout this paper, we will use only the right upper porosity and the following terminology. A set \( A \subseteq \mathbb{N} \) is called

(i) porous at infinity if \( \bar{p}_{\mu}(A) > 0 \);
(ii) strongly porous at infinity if \( \bar{p}_{\mu}(A) = 1 \);
(iii) nonporous at infinity if \( \bar{p}_{\mu}(A) = 0 \).

Let us recall the definition of porosity convergence:

**Definition 1.1.** [2] Let \( \bar{x} = (x_n)_{n \in \mathbb{N}} \) be a real valued sequence. We say that, \( \bar{x} \) is \( \bar{p}_{\mu} \)-convergent to \( l \) if for each \( \varepsilon > 0 \),

\[
\bar{p}_{\mu}(A_{\varepsilon}) > 0 \quad \text{and} \quad \bar{p}_{\mu}(A_{\varepsilon}) = 0
\]

where \( A_{\varepsilon} := \{ n : |x_n - l| \geq \varepsilon \} \) and \( A_{\varepsilon}^c \) is the complement of the set \( A_{\varepsilon} \). It is denoted by \( \bar{x} \xrightarrow{\bar{p}_{\mu}} l \).

Let us note that the second condition in Definition 1.1 is necessary for only uniqueness of \( \bar{p}_{\mu} \)-limit.

In [2], it is particularly shown that \( \bar{p}_{\mu} \)-convergence is a regular summability method for real (or complex) valued sequences.

Our aim is to establish the relationship between porosity convergence and tangent space of the set \( \mu(A_{\varepsilon}) \cup \{ 0 \} \subset [0, \infty) \).

The concept of pretangent space was defined by Dovgoshey and Martio in [12][13] for the first time. After this basic studies, tangent spaces are the focus of research [16][10].

Now, let us recall construction of pretangent spaces to \( E \) in the particular case when \( E \subset \mathbb{R}^+ \). Let \( \bar{r} = (r_n)_{n \in \mathbb{N}} \) be a sequence of positive real numbers such that \( \lim_{n \to \infty} r_n = 0 \). The sequence \( \bar{r} \) will be called a normalizing sequence. We define the set

\[
\bar{E} := \{ \bar{x} = (x_n) : x_n \in E, \forall n \in \mathbb{N} \text{ and } \lim_{n \to \infty} x_n = 0 \}.
\]
Definition 1.2. [3] Two sequences $\tilde{x} = (x_n)_{n \in \mathbb{N}} \in \tilde{E}$ and $\tilde{y} = (y_n)_{n \in \mathbb{N}} \in \tilde{E}$ are mutually stable w.r.t. $\tilde{r}$ if the following limit

\[
|\tilde{x} - \tilde{y}|_{\tilde{r}} := \lim_{n \to \infty} \frac{|x_n - y_n|}{r_n}
\]

exists and is finite.

A family $\tilde{F} \subseteq \tilde{E}$ is called self-stable (w.r.t. $\tilde{r}$) if each pair of sequences $\tilde{x}, \tilde{y} \in \tilde{F}$ are mutually stable. A family $\tilde{F} \subseteq \tilde{X}$ is called maximal self-stable if $\tilde{F}$ is self-stable and for an arbitrary $\tilde{z} \in \tilde{E}$ either $\tilde{z} \in \tilde{F}$ or there is a sequence $\tilde{x} \in \tilde{F}$ such that $\tilde{x}$ and $\tilde{z}$ are not mutually stable.

Proposition 1.1. [12][13] Let $E \subseteq \mathbb{R}^+$ be a pointed set with the marked point $0 \in E$. Then, for every normalizing sequence $\tilde{r} = (r_n)_{n \in \mathbb{N}}$, there exists a maximal self-stable family $\tilde{E}_{0,\tilde{r}}$ such that $0 := (0, \ldots, 0, 0, \ldots) \in \tilde{E}_{0,\tilde{r}}$.

Consider a function $|.|_{\tilde{r}} : \tilde{E}_{0,\tilde{r}} \times \tilde{E}_{0,\tilde{r}} \to [0, \infty)$ such that $|\tilde{x}, \tilde{y}|_{\tilde{r}} = |\tilde{x} - \tilde{y}|_{\tilde{r}}$ is defined by (1.1). Obviously, it is nonnegative, symmetric and satisfies the triangle inequality $|\tilde{x} - \tilde{y}|_{\tilde{r}} \leq |\tilde{x} - \tilde{z}|_{\tilde{r}} + |\tilde{z} - \tilde{y}|_{\tilde{r}}$ for all $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{E}_{0,\tilde{r}}$. Therefore, $(\tilde{E}_{0,\tilde{r}}, |.|_{\tilde{r}})$ is a pseudometric space.

Definition 1.3. [12][13] Let $\tilde{E}_{0,\tilde{r}}$ be a maximal self-stable family. A pretangent space to $E \subseteq \mathbb{R}^+$ (at the point $0 \in E$ w.r.t. $\tilde{r}$) is the metric identification of a pseudometric space $(\tilde{E}_{0,\tilde{r}}, |.|_{\tilde{r}})$.

Because the notion of pretangent space is important for the paper, we shall describe the metric identification construction (see, for example, [17]). Define a binary relation $\sim$ on $\tilde{E}_{0,\tilde{r}}$ by $\tilde{x} \sim \tilde{y}$ if and only if $|\tilde{x} - \tilde{y}|_{\tilde{r}} = 0$. It is clear that $\sim$ is an equivalence relation. Let us denote by $\Omega_{0,\tilde{r}}^E$, the set of equivalence classes in $\tilde{E}_{0,\tilde{r}}$ under $\sim$. For an arbitrary $\alpha, \beta \in \Omega_{0,\tilde{r}}^E$, we set

$$
\rho(\alpha, \beta) := |\tilde{x} - \tilde{y}|_{\tilde{r}}, \quad \tilde{x} \in \alpha, \; \tilde{y} \in \beta.
$$

The function $\rho$ is a well-defined metric on $\Omega_{0,\tilde{r}}^E$. By definition, $(\Omega_{0,\tilde{r}}^E, \rho)$ is the metric identification of $(\tilde{E}_{0,\tilde{r}}, |.|_{\tilde{r}})$.

Lemma 1.1. The equality

\[
\Omega_{0,\tilde{r}}^{\mathbb{R}^+} = \mathbb{R}^+
\]

holds for any normalizing sequence $\tilde{r}$.

Proof. Let us note that $0 \in \Omega_{0,\tilde{r}}^{\mathbb{R}^+}$ to prove (1.2). If $\tilde{x} := (hr_n)_{n \in \mathbb{N}}$ for any $h \in (0, \infty)$, then we obviously have $\lim_{n \to \infty} hr_n/r_n = h$. By [3] Corollary 2.5 we obtain $\tilde{x} \in \Omega_{0,\tilde{r}}^{\mathbb{R}^+}$ is a maximal self-stable family corresponding to $\Omega_{0,\tilde{r}}^{\mathbb{R}^+}$. The statements $h \in \Omega_{0,\tilde{r}}^{\mathbb{R}^+}$ is fulfilled by [3] Proposition 2.6]. Consequently, (1.2) holds. □

Definition 1.4. Let $A$ and $B$ be any subsets of $\mathbb{R}^+$. We shall write $A \lessgtr B$, if for every sequence $(a_n)_{n \in \mathbb{N}} \in A \lessgtr \{0\}$, there is a sequence $(b_n)_{n \in \mathbb{N}} \in B \lessgtr \{0\}$, such that $\lim_{n \to \infty} a_n/b_n = 1$ holds [3].
2. Main results

Let \( \hat{x} = (x_n) \) be a real valued sequence and \( \mu : \mathbb{N} \to \mathbb{R}^+ \) be a scaling function. Consider the sets \( A^\mu := \mu(A) \cap \{0\} \subset [0, \infty) \) and \( A^\mu_{\varepsilon} := \mu(A) \cup \{0\} \subset [0, \infty) \) where \( A := \{k : |x_k - l| \geq \varepsilon\} \) for any \( \varepsilon > 0 \).

**Theorem 2.1.** The following statements are equivalent.

(i) The sequence \( \hat{x} = (x_n) \) is not \( \tilde{p}_\mu \)-convergent to \( l \), i.e.,

\[
\hat{x} \not\to l(\tilde{p}_\mu), \quad n \to \infty.
\]

(ii) The equality

\[
\tilde{\Omega}^\mu_{0, \tilde{\varepsilon}} = \mathbb{R}^+
\]

holds for every normalizing sequence \( \tilde{\varepsilon} \).

(iii) There exists a subsequence \( \varepsilon' \) of normalizing sequence \( \varepsilon \) such that the pre-tangent space \( \Omega^\varepsilon_{0, \varepsilon'} \) includes a dense subset of \((0,1)\).

**Proof.** (i) \( \Rightarrow \) (ii) Assume that \( \hat{x} \not\to l(\tilde{p}_\mu), \ n \to \infty \). Then, the set \( A_\varepsilon \) is a nonporous subset of \( \mathbb{N} \) at infinity for every \( \varepsilon \). So, the set \( A_\varepsilon \) has infinitely many elements, and it can be represented as \( A_\varepsilon = \{n_1, n_2, \ldots, n_k, n_{k+1}, \ldots\} \) where \( (n_k) \) is strictly increasing sequence of natural numbers.

Since \( \tilde{p}_\mu(A_\varepsilon) = 0 \), then from [3] Proposition 3.5] we have that

\[
\lim_{k \to \infty} \frac{\mu(n_{k+1})}{\mu(n_k)} = 1.
\]

Let \( \tilde{l} = \{t_m\}_{m \in \mathbb{N}} \) be a sequence of positive reals such that \( \lim_{m \to \infty} t_m = 0 \). For every \( m \in \mathbb{N} \), define the number \( k(m) \) as follows \( k(m) := \min\{k \in \mathbb{N} : \mu(n_k) \leq t_m\} \). Then, the double inequalities

\[
\mu(n_{k(m)} - 1) \leq t_m < \mu(n_{k(m)})
\]

hold for all sufficiently large \( m \). It follows from (2.3) and (2.4) that

\[
1 \leq \liminf_{m \to \infty} \frac{t_m}{\mu(n_{k(m)})} \leq \limsup_{m \to \infty} \frac{t_m}{\mu(n_{k(m)})} \leq \limsup_{m \to \infty} \frac{\mu(n_{k(m)} - 1)}{\mu(n_{k(m)})} = 1
\]

holds. Hence, we conclude that \( \lim_{m \to \infty} \frac{t_m}{\mu(n_{k(m)})} = 1 \) holds. Since \( \{t_m\} \subset \mathbb{R}^+ \) and \( \mu(n_{k(m)}) \subset \mu(A_\varepsilon) \cup \{0\} \), then we have

\[
\mathbb{R}^+ \preceq \mu(A_\varepsilon) \preceq A^\mu_{\varepsilon} \preceq \mathbb{R}^+.
\]

Let \( \tilde{r} = (r_n)_{n \in \mathbb{N}} \) be any normalizing sequence. By considering (2.5) with [3 Proposition 2.9] we have \( \tilde{\Omega}^\mu_{0, \tilde{\varepsilon}} = \tilde{\Omega}^\mu_{0, \tilde{r}} \). Consequently, from Lemma [1.1 \( 2.2 \) holds.

(ii) \( \Rightarrow \) (iii) is trivial. Let prove (iii) \( \Rightarrow \) (i). Now assume that (iii) holds. Using [3] Theorem 3.6] we obtain that

\[
\tilde{p}(A^\mu_{\varepsilon}) = 0.
\]

Since \( \tilde{p}(A^\mu_{\varepsilon}) = \tilde{p}(\mu(A_\varepsilon)) \), then equality (2.6) implies that \( \tilde{p}(\mu(A_\varepsilon)) = 0 \). By the equality of \( \tilde{p}(\mu(E)) = \tilde{p}_\mu(E) \) for \( E \subseteq \mathbb{N} \), we have \( \tilde{p}(\mu(A_\varepsilon)) = \tilde{p}_\mu(A_\varepsilon) \). Consequently (2.4) holds. \( \square \)
POROSITY CONVERGENCE AND PRETANGENT SPACES 45

Theorem 2.2. The following statements are equivalent.
(i) The sequence $\hat{x} = (x_n)$ is $(\hat{p}_\mu)$-convergent to $l$, i.e., $x_n \to l(\hat{p}_\mu)$.
(ii) There is a normalizing sequence $\hat{r}$ and an interval $(a, b) \subseteq (0, 1)$ with $|a - b| > 0$ such that the equalities $\hat{\Omega}_0^{\hat{r}} \cap (a, b) = \emptyset$ and $\hat{\Omega}_0^{\hat{r}} = \mathbb{R}^+$ holds for every $\hat{r}$ and $\varepsilon > 0$.

Proof. Let us assume that $x_n \to l(\hat{p}_\mu)$ holds. So, $\hat{p}_\mu(A_\varepsilon) > 0$ and $\hat{p}_\mu(A_\varepsilon^c) = 0$ hold for any $\varepsilon > 0$. From [3] Theorem 3.4 we have $\hat{p}(\mu(A_\varepsilon)) > 0$ and $\hat{p}(\mu(A_\varepsilon^c)) = 0$. Also, it is clear that $\hat{p}(A_\varepsilon^c) > 0$ and $\hat{p}(A_\varepsilon^{\mu}) = 0$ hold. If we use [3] Theorems 2.1 and 2.12, then we obtain that $(i) \iff (ii)$.

Corollary 2.1. Let $\hat{x} = (x_n)$ be a real valued sequence. If $x_n \to l(\hat{p}_\mu)$, then there exists a normalizing sequence $\hat{r}$ such that $\mathbb{R}^+ \setminus \hat{\Omega}_0^{\hat{r}} \neq \emptyset$ holds.

3. Some examples

In this section we give two examples as application of last section. We take here $\mu(n) = \frac{1}{n}$ as a scaling function only for simplicity.

Example 3.1. Consider the sequence $\hat{x} = ((-1)^n)$ for $n \in \mathbb{N}$. It is clear that it is not porosity convergent i.e., $(-1)^n \not\to 1(\hat{p}_\mu)$ and $(-1)^n \not\to -1(\hat{p}_\mu)$. So, from Theorem 2.1 we have
\[(3.1) \quad \hat{\Omega}_0^{\hat{r}} = \mathbb{R}^+ \quad \text{and} \quad \hat{\Omega}_0^{\hat{r}} = \mathbb{R}^+ \]
for $A_\varepsilon = \{n : |(-1)^n - 1| \geq \varepsilon\}$ and $A_\varepsilon^c = \{n : |(-1)^n - 1| < \varepsilon\}$, respectively.

Indeed, $A_\varepsilon = \mathbb{N}_0$ and $A_\varepsilon^c = \mathbb{N}_E$. So, (3.1) hold. The third condition of Theorem 2.1 is obvious from second.

Example 3.2. Consider the sequence $\hat{x} = \left(\frac{1}{n}\right)$ for $n \in \mathbb{N}$. It is clear that $x_n \to 0(\hat{p}_\mu)$ because $x_n \to 0$, $n \to \infty$. So, from Theorem 2.2 we can infer that
\[(3.2) \quad \hat{\Omega}_0^{\hat{r}} \cap (a, b) = \emptyset \quad \text{and} \quad \hat{\Omega}_0^{\hat{r}} = \mathbb{R}^+ \]
for $(a, b) \subseteq (0, 1)$ where $A_\varepsilon = \{n : \frac{1}{n} \geq \varepsilon\}$ and $A_\varepsilon^c$ is the complement of $A_\varepsilon$.

Indeed, let $\varepsilon = 1/2$. From the definition of porosity convergence the set $A_{1/2} = \{n : \frac{1}{n} \geq \frac{1}{2}\}$ is porous at infinity. Also the set $A_{1/2}^c = \{n : \frac{1}{n} < \frac{1}{2}\}$ is nonporsous at infinity.

$A_{1/2}^c = \mu(A_{1/2}) \cup \{0\}$ is a finite set and 0 is not an accumulation point of this set. So, $\hat{\Omega}_0^{\hat{r}} = \emptyset$. Therefore, $\hat{\Omega}_0^{\hat{r}} \cap (a, b) = \emptyset$ for any interval $(a, b) \subseteq (0, 1)$.

$\mu(A_{1/2}^c) = \mu(A_{1/2}^c) \cup \{0\} = \mu(\mathbb{N}) \cup \{0\} = \mathbb{R}^+$. Then $\hat{\Omega}_0^{\hat{r}} = \hat{\Omega}_0^{\hat{r}} = \mathbb{R}^+$ is obtained.

References
1. F. Abdullayev, O. Dovgoshey, Küçüksalan, Metric spaces with unique pretangent spaces. Conditions of the uniqueness, Ann. Acad. Sci. Fenn., Math. 36(2) (2011), 353–392.
2. M. Altunok, M. Küçüksalan, On porosity-convergence of real valued sequences, An. Ştiinţ. Univ. Al. I. Cuza Iaşi, Ser. Nouă, Mat. 65(2) (2019), 181–193.
3. M. Altınok, O. Dovgoshey, M. Küçükaslan, Local one-sided porosity and pretangent spaces, Analysis 36 (2016), 147–171.
4. _____, Unions and ideals of locally strongly porous sets, Turk. J. Math. 41 (2017), 1510–1534.
5. V. Bilet, O. Dovgoshey, Boundedness of pretangent spaces to general metric spaces, Ann. Acad. Sci. Fenn. Math. 39 (2009), 73–82.
6. _____, Finite spaces pretangent to metric spaces at infinity, J. Math. Sci., New York 242(3) (2019), 360–380.
7. A. Denjoy, Sur une propriété des séries trigonométriques, Verlag v.d. G. V. der Wie-en Natuur. Afd.29 (1920), 220–232.
8. A. Denjoy, Leçons sur le calcul des coefficients d’une série trigonométrique, Part II, Métrie et topologie d’ensembles parfaits et de fonctions, Gauthier-Villars, Paris, 1941.
9. E.P. Dolženko, Boundary properties of arbitrary functions, Faz. Akad. Nauk SSSR, Ser. Mat. 31 (1967), 3–14. (in Russian)
10. O. Dovgoshey, F. Abdullayev, M. Küçükaslan, Compactness and boundedness of tangent spaces to metric spaces, Beitr. Algebra Geom. 51(2) (2010), 547–576.
11. O. Dovgoshey, J. Rähentaus, Mean value type inequalities for quasinearly subharmonic functions, Glasg. Math. J. 55(2) (2013), 349–368.
12. O. Dovgoshey, O. Martio, Tangent spaces to metric spaces and to their subspaces, Ukr. Mat. Visn. 5(4) (2008), 470–487.
13. _____, Tangent spaces to general metric spaces, Rev. Roum. Math. Pures Appl. 56(2) (2011), 137–155.
14. H. Fast, Sur la convergence statistique, Colloq. Math. 2(3-4) (1951), 241–244.
15. A.P. Freedman, J. J. Sember, Densities and summability, Pac. J. Math. 95 (1981), 293–305.
16. L. Karp, T. Kilpeläinen, A. Petrosyan, H. Shahgholian, On the porosity of free boundaries in degenerate variational inequalities, J. Differ. Equations 164 (2000), 110–117.
17. J. L. Kelly, General Topology, Van Nostrand, Prienceton, 1965.
18. A. Khintchine, An investigation of the structure of measurable functions, Mat. Sb. 31 (1924), 265–285. (in Russian)
19. E. Kolk, Statistically convergent sequences in normed spaces, Reports of conference, Methods of algebra and analysis, Tartu, 1988, 63–66. (in Russian)
20. _____, The statistical convergence in Banach spaces, Tartu Ulik. Toim. 928 (1991), 41-52.
21. P. Kostyrko, T. Salat, W. Wilczysnski, I-convergence, Real Anal. Exchange 26(2) (2000–2001), 669–686.
22. F. Przytycki, S. Rohde, Porosity of Collet-Eckmann Julia sets, Fundam. Math. 155 (1998), 189–199.
23. H. Steinhaus, Sur la convergence ordinate et la convergence asymptotique, Colloq. Math. 2 (1951), 73–84.
24. B. S. Thomson, Real Functions, Lect. Notes Math. 1170, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985.
25. J. Väisälä, Porous sets and quasisymmetric maps, Trans. Am. Math. Soc. 299 (1987), 525–534.
26. A. Zygmund, Trigonometric Series, Cambridge Univ. Press., Cambridge, 1979.

Department of Natural and Mathematical Sciences
Tarsus University, Mersin, Turkey
mayalatinok@tarsus.edu.tr

Department of Mathematics
Mersin University, Mersin, Turkey
mkucukaslan@mersin.edu.tr