IMPROVED RESOLVENT ESTIMATES FOR CONSTANT-COEFFICIENT ELLIPTIC OPERATORS IN THREE DIMENSIONS

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Abstract. We prove new $L^p$-$L^q$-estimates for solutions to elliptic differential operators with constant coefficients in $\mathbb{R}^3$. We use the estimates for the decay of the Fourier transform of particular surfaces in $\mathbb{R}^3$ with vanishing Gaussian curvature due to Erdős–Salmhofer to derive new Fourier restriction–extension estimates. These allow for constructing distributional solutions in $L^q(\mathbb{R}^3)$ for $L^p$-data via limiting absorption by well-known means.

1. Introduction

The purpose of this note is to show new $L^p$-$L^q$-estimates for solutions to elliptic differential equations in $\mathbb{R}^3$. Let

$$p(\xi) = \sum_{|\alpha| \leq N} a_\alpha \xi^\alpha$$

be a multi-variate polynomial in $\mathbb{R}^3$ with real coefficients and suppose that $a_{\alpha} \neq 0$ for some $\alpha \in \mathbb{N}_0^3$ with $|\alpha| = N$. We consider partial differential operators

$$P(D) = p(-i\nabla_x) = \sum_{|\alpha| \leq N} a_{\alpha}(-i)^{|\alpha|} \partial^{\alpha}$$

such that for $u \in S'(\mathbb{R}^3)$ we have

$$\mathcal{F}(P(D)u)(\xi) = p(\xi)\hat{u}(\xi).$$

By ellipticity we mean that

$$p_N(\xi) = \sum_{|\alpha| = N} a_{\alpha} \xi^\alpha \neq 0$$

for $\xi \neq 0$. We assume $p_N(\xi) > 0$ for the sake of definiteness. In the following we prove existence of solutions $u \in L^q(\mathbb{R}^3)$ such that

$$P(D)u = f$$

for $f \in L^p(\mathbb{R}^3)$ in a certain range of $p$ and $q$, which satisfy the estimate

$$\|u\|_{L^q(\mathbb{R}^3)} \lesssim \|f\|_{L^p(\mathbb{R}^3)}.$$
under a transversality assumption, which was described by Erdős–Salmhofer [6]. The idea of constructing solutions is to consider approximates

\[ \hat{u}_\delta(\xi) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{e^{ix.\xi} \hat{f}(\xi)}{p(\xi) + i\delta} \, d\xi \]

for \( \delta \neq 0 \) and show uniform bounds

\[ \|u_\delta\|_{L^q(\mathbb{R}^3)} \lesssim \|f\|_{L^p(\mathbb{R}^3)} \]

for fixed \( P(D) \).

Then we shall find distributional limits \( u \in L^q(\mathbb{R}^3) \), which satisfy

\[ P(D)u = f \text{ in } \mathcal{S}'(\mathbb{R}^3) \]

and

\[ \|u\|_{L^q(\mathbb{R}^3)} \lesssim \|f\|_{L^p(\mathbb{R}^3)}. \]

This is referred to as limiting absorption principle. We shall still assume that \( \nabla p(\xi) \neq 0 \) for \( \xi \in \Sigma_0 \).

This is a generic assumption for polynomials. In this case Sokhotsky’s formula yields for solutions as described above

\[ u(x) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{e^{ix.\xi} \hat{f}(\xi)}{p(\xi) \pm i0} \, d\xi \]

\[ = \mp \frac{i\pi}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix.\xi} \hat{f}(\xi) \delta_{\Sigma_0}(\xi) \, d\xi + \frac{1}{(2\pi)^3} \nu.p. \int_{\mathbb{R}^3} e^{ix.\xi} \hat{f}(\xi) \, d\xi. \]

This points out a close connection to Fourier restriction. The most basic \( L^p-L^q \)-results rely on the decay of the Fourier transform of the surface measure. This in turn is caused by the curvature of the surface.

If \( K \neq 0 \), the estimate

\[ |\hat{\mu}_S(\xi)| = | \int_S e^{ix.\xi} \, dx | \lesssim \langle \xi \rangle^{-1} \]

is classical (cf. [14, 16]). Corresponding \( L^p-L^q \)-estimates for solutions were proved in [15].

In this note we consider vanishing total curvature under additional transversality assumptions. For constructing solutions as laid out above, we also have to consider level sets \( \Sigma_a = \{ p(\xi) = a \} \) for \( |a| \leq \delta_0 \).

We recall the assumptions in Erdős–Salmhofer:

Let \( I \) be a compact interval and let \( D = \exp^{-1}(I) \). Suppose that \( \Sigma_a \) is a two-dimensional submanifold for each \( a \in I \). Let \( f \in C_0^\infty(D) \) and define

\[ \hat{\mu}_a(\xi) = \int_{\Sigma_a} e^{ix.\xi} \, d\sigma_a(\xi) \]

the Fourier transform of the surface carried measure \( f \, d\sigma_a \).

Let \( C_0 = \text{diam}(D) \), \( C_1 = \|p\|_{C^1(D)} \). The following assumptions have to be met:

**Assumption 1:**

\[ C_2 = \min_{\xi \in D} |\nabla p(\xi)| > 0, \]

which means that \( (\Sigma_a)_{a \in I} \) is a regular foliation of \( D \).

Let \( K : D \to \mathbb{R} \) be the Gaussian curvature of the foliation, i.e., for \( \xi \in \Sigma_a \subseteq D \), \( K(\xi) \) denotes the curvature of \( \Sigma_a \) at \( \xi \).

The crucial assumption is that the vanishing set of the Gaussian curvature is a submanifold, which intersects \( (\Sigma_a)_{a \in I} \) transversally:

**Assumption 2:** Let \( C = \{ \xi \in D : K(\xi) = 0 \} \). Then

\[ C_3 = \min_{\xi \in D} (|\nabla p(\xi) \times \nabla K(\xi)| : \xi \in C) > 0. \]
With $\nabla K$ non-vanishing on $C$, it is a two-dimensional submanifold by the regular value theorem. Since $p$ and $K$ are smooth, we find that

$$\Gamma_a = C \cap \Sigma_a$$

is a finite union of disjoint regular curves on $\Sigma_a$ for each $a \in I$.

Let

$$\xi \mapsto w(\xi) = \frac{\nabla p(\xi) \times \nabla K(\xi)}{|\nabla p(\xi) \times \nabla K(\xi)|}$$

be the unit vectorfield tangent to $\Gamma_a$. Denote the normal map $\nu : D \to S^2$ by

$$\nu(\xi) = \frac{\nabla p(\xi)}{|\nabla p(\xi)|}.$$  

Recall that the Gaussian curvature is given by the Jacobian of the normal map restricted to each surface, $\nu : \Sigma_a \to S^2$: $K(\xi) = \det \nu(\xi)$. We further require the following regularity assumption on the Gauss map.

**Assumption 3**: The number of preimages of $\nu : \Sigma_a \to S^2$ is finite, i.e.,

$$C_4 = \sup_{a \in I} \sup_{\omega \in S^2} \text{card}\{p \in \Sigma_a : \nu(p) = \omega\} < \infty.$$  

On the curves $\Gamma_a$, exactly one of the principal curvatures vanish. We define a (local) unit vectorfield $Z \in T\Sigma_a$ along $\Gamma_a$ in the tangent plane of $\Sigma_a$. $Z$ can be extended to a neighbourhood of $\Gamma_a$ as the direction of the principal curvature that is small and vanishes on $\Gamma_a$. We assume that $Z$ is transversal to $\Gamma_a$. Weaker, non-uniform decay estimates were proved in $[6]$ also in the presence of tangential points. To ensure uniform decay, we assume the following:

**Assumption 4**: The set of tangential points

$$\mathcal{T}_a = \{\xi \in \Gamma_a : Z(\xi) \times w(\xi) = 0\},$$

is empty.

Under the above assumptions, Erdős–Salmhofer [6, Theorem 2.1] proved the following dispersive estimate for the Fourier transform of the surface measure $\mu_a$:

$$|\hat{\mu}_a(\xi)| \leq C|\xi|^{-\frac{d}{2}}$$

with $C = C(C_0, \ldots, C_4, \|f\|_{C^2(D)})$. This morally corresponds to a decay from $\frac{d}{2}$ principal curvatures bounded from below in modulus and thus improves the previous result for one non-vanishing principal curvature (cf. [15, Theorem 1.3]). In this article we record its consequence for solutions to elliptic differential operators. Allowing for tangential points covers generic surfaces in $\mathbb{R}^3$ as pointed out in [6]. However, the decay proved in [6] is not uniform in this case anymore. It might be possible to show the same results for a broader class via the estimates due to Ikromov–Müller [9].

In the first step, we derive a Fourier restriction–extension theorem for surfaces $\Sigma_a$ by following along the lines of the preceding work [15]. We prove strong bounds

$$\|\int_{\mathbb{R}^3} e^{i\xi \cdot \xi} \delta_{\Sigma_a}(\xi) \beta(\xi) \hat{f}(\xi) d\xi\|_{L^q(\mathbb{R}^3)} \lesssim \|f\|_{L^p(\mathbb{R}^3)}$$

within a pentagonal region. Here $\beta \in C_c^\infty$ localizes to a suitable neighbourhood of $\{K = 0\}$ in $\{\Sigma_a\}_{a \in [-\delta_0, \delta_0]}$. Away from $\{K = 0\}$, [15, Theorem 1.3] provides better estimates for $d = 3$, $k = 2$. On part of the boundary of the pentagonal region, we show weak bounds

$$\|\int_{\mathbb{R}^3} e^{i\xi \cdot \xi} \delta_{\Sigma_a}(\xi) \beta(\xi) \hat{f}(\xi) d\xi\|_{L^{q,\infty}(\mathbb{R}^3)} \lesssim \|f\|_{L^p(\mathbb{R}^3)}$$

and

$$\|\int_{\mathbb{R}^3} e^{i\xi \cdot \xi} \delta_{\Sigma_a}(\xi) \beta(\xi) \hat{f}(\xi) d\xi\|_{L^3(\mathbb{R}^3)} \lesssim \|f\|_{L^{p^{-1}}(\mathbb{R}^3)}.$$
and lastly, restricted weak bounds

\[ \| \int_{\mathbb{R}^3} e^{i x \cdot \xi} \delta_{\Sigma_0}(\xi) \beta(\xi) \hat{f}(\xi) d\xi \|_{L^{p',\infty}((\mathbb{R}^3)^*)} \lesssim \|f\|_{L^p((\mathbb{R}^3)^*)} \]

at its inner endpoints. We refer to Figure 2 for a diagram. For \( X, Y \in [0, 1]^2 \) we write \([X, Y] = \{Z: \exists \lambda \in [0, 1]: Z = \lambda X + (1 - \lambda)Y\}\) and correspondingly \((X, Y), (X, Y), \) etc.

**Proposition 1.1.** Let \( p: \mathbb{R}^3 \to \mathbb{R} \) be an elliptic polynomial with \( \delta_0 > 0 \) such that for \( \Sigma_0 = \{p(\xi) = a\}, -\delta_0 \leq a \leq \delta_0 \) Assumptions 1-4 are satisfied in a neighbourhood of \( K = 0 \) in \( \Sigma_0 \). Then, we find \( \| \| \) to hold for \( (p, \frac{1}{p}) \in [0, 1]^2 \) provided that

\[ \frac{1}{p} > \frac{7}{10}, \quad \frac{1}{q} < \frac{3}{10}, \quad \frac{1}{p} - \frac{1}{q} \geq \frac{4}{7}. \]

Let

\[ B = \left( \frac{7}{10}, \frac{9}{70} \right), \quad C = \left( \frac{7}{10}, 0 \right), \quad B' = \left( \frac{61}{70}, \frac{3}{10} \right), \quad C' = \left( 1, \frac{3}{10} \right); \]

Furthermore, we find \( \| \) to hold for \((1/p, 1/q) \in (B', C']\), \( \| \) for \((1/p, 1/q) \in (B, C]\), and \( \| \) for \((1/p, 1/q) \in \{B, B'\} \).

In the second step we foliate a neighbourhood \( U \) of \( \Sigma_0 \) with level sets of \( p \) to show bounds

\[ \|A_\delta f\|_{L^p} \lesssim \|f\|_{L^p((\mathbb{R}^3)^*)} \]

independent of \( \delta \). Here, \( p, q \) are as in Proposition 1.1, and \( |p(\xi)| \leq \delta_0 \) for \( \xi \in \text{supp}(\beta_1) \) with \( \Sigma_0 \subseteq \text{supp}(\beta_1) \). Away from the singular set, estimates for

\[ B_\delta f(x) = \int_{\mathbb{R}^3} \frac{e^{i x \cdot \xi} \beta_1(\xi)}{p(\xi) + i \delta} \hat{f}(\xi) d\xi \]

with \( \beta_1 + \beta_2 \equiv 1 \) follow from Young’s inequality and properties of the Bessel potential. The estimate of \( \|B_\delta\|_{L^p \to L^q} \) depends on the order of the elliptic operator. The method of proof is well-known and detailed in [15]; see also [12] and references therein. We shall be brief. It turns out that one can follow along the lines of [15] very closely, substituting \( k = \frac{1}{2} \) non-vanishing principal curvatures. We prove the following:

**Theorem 1.2.** Let \( p: \mathbb{R}^3 \to \mathbb{R} \) be an elliptic polynomial of degree \( N \geq 2 \). Let \( 1 < p_1, p_2, q < \infty \) and \( f \in L^{p_1}(\mathbb{R}^3) \cap L^{p_2}(\mathbb{R}^3) \). Suppose that there is \( \delta_0 > 0 \) such that Assumptions 1-4 are satisfied for \( (\Sigma_0)_{a \in [-\delta_0, \delta_0]} \). Then, there is \( u \in L^q(\mathbb{R}^3) \) satisfying

\[ P(D)u = f \]

in the distributional sense and the estimate

\[ \|u\|_{L^q(\mathbb{R}^3)} \lesssim \|f\|_{L^{p_1} \cap L^{p_2}(\mathbb{R}^3)} \]

holds true provided that

\[ \frac{1}{p_1} > \frac{7}{10}, \quad \frac{1}{q} < \frac{3}{10}, \quad \frac{1}{p_1} - \frac{1}{q} \geq \frac{4}{7} \]

and for \( N \leq 3 \)

\[ 0 \leq \frac{1}{p_2} - \frac{1}{q} \leq \frac{N}{3}, \quad \left( \frac{1}{p_2}, \frac{1}{q} \right) \notin \begin{cases} \{0, \frac{2}{3}\}, \left( \frac{4}{3}, 1 \right) \right) & \text{for } N = 2, \\ \{0, 1\} \right) & \text{for } N = 3. \end{cases} \]
The purpose of this section is to prove Proposition [11]. We shall follow the argument of [15, Section 4]. In the first step, we localize to a small neighbourhood of the vanishing set \{K = 0\}, which by assumptions is a two-dimensional manifold in \(D\). In the complementary set, by compactness, we can apply [15, Theorem 1.3], which gives uniform \(L^p-L^q\)-estimates in a broader range. Thus, it is enough to suppose that Assumptions 1-4 are valid in a neighbourhood of \{K = 0\}. The proof follows [15, Section 4] closely. In the first step, by finite decomposition and rotations, we change to parametric representation of \(\Sigma = \{(\xi', \psi(\xi')) : \xi' \in B(0, c)\}\). We show bounds \(T : L^p(\mathbb{R}^3) \to L^q(\mathbb{R}^3)\) for
\[
Tf(x) = \int_{\mathbb{R}^3} \delta(\xi_3 - \psi(\xi')) e^{ix \cdot \xi} \chi(\xi') \hat{f}(\xi) d\xi.
\]
The following decay estimate, which is (5), is central.
\[
\left\| \int e^{i(x \cdot \xi' + x_3 \psi(\xi'))} \beta(\xi') d\xi' \right\| \lesssim (1 + |x_3|)^{-\frac{2}{3}}.
\]
Applying the \(TT^*\) argument (cf. [17, 7, 11]), we find the following Strichartz estimate:
\[
\left\| \int e^{i(x \cdot \xi' + x_3 \psi(\xi'))} \beta(\xi') \hat{f}(\xi') d\xi' \right\|_{L^q_t L^p_x(\mathbb{R}^3)} \lesssim \|f\|_{L^\infty_t L^2_x(B(0, c))}.
\]
We recall the following lemma to decompose the delta distribution:

Lemma 2.1 ([11, Lemma 2.1]). There is a smooth function \(\phi\) satisfying \(\text{supp}(\phi) \subseteq \{t : |t| \sim 1\}\) such that for all \(f \in \mathcal{S}(\mathbb{R}^3)\),
\[
\langle \delta(\xi_3 - \psi(\xi')), f \rangle = \sum_{j \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} 2^j \int_{\mathbb{R}^3} \phi(2^j(\xi_3 - \psi(\xi'))) \chi(\xi') f(\xi) d\xi.
\]
By this, we can write
\[
Tf(x) = \sum_{j \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} 2^j \int_{\mathbb{R}^3} \phi(2^j(\xi_3 - \psi(\xi'))) e^{ix \cdot \xi} \chi(\xi') \hat{f}(\xi) d\xi := \sum_{j \in \mathbb{Z}} 2^j T_{2^{-j}} f.
\]
As pointed out in [11], the contribution of \(j \leq 0\) is easier to estimate.

The contribution of \(j \geq 0\), i.e., close to the singularity, is estimated by Strichartz and kernel estimates:

Lemma 2.2 (cf. [15, Lemma 4.3]). Let \(q \geq \frac{14}{3}\). Then, we find the following estimate to hold:
\[
\|T_{2^{-j}} f\|_{L^q_x(\mathbb{R}^3)} \lesssim 2^{-j} \|f\|_{L^2_x(\mathbb{R}^3)}.
\]
This estimate does not admit summation. For this purpose, we interpolate with the kernel estimate:

Lemma 2.3 (cf. [15, Lemma 4.4]). Let
\[
K_\delta(x) = \int_{\mathbb{R}^3} e^{ix \cdot \xi} \beta(\xi') \phi\left(\frac{\xi_3 - \psi(\xi')}{\delta}\right) d\xi.
\]
Then \(K_\delta\) is supported in \(\{(x', x_3) : |x_3| \sim \delta^{-1}\}\), and we find the following estimates to hold:
\[
|K_\delta(x)| \lesssim_N \delta^N (1 + \delta|x|)^{-N}, \text{ if } |x'| \geq c|x_3|,
\]
\[
|K_\delta(x)| \lesssim \delta^2 \text{, if } |x'| \leq c|x_3|.
\]

The last ingredient to show (restricted) weak endpoint estimates is Bourgain’s summation argument (cf. [12] and [13, Lemma 2.3] for an elementary proof):
Figure 1. Pentagonal region, within which strong $L^p$-$L^q$-Fourier restriction extension estimates hold.

**Lemma 2.4.** Let $\varepsilon_1, \varepsilon_2 > 0$, $1 \leq p_1, p_2 \leq \infty$, $1 \leq q_1, q_2 < \infty$. For every $j \in \mathbb{Z}$ let $T_j$ be a linear operator, which satisfies

$$
\|T_j(f)\|_{q_1} \leq M_1 2^{\varepsilon_1 j} \|f\|_{p_1},
$$

$$
\|T_j(f)\|_{q_2} \leq M_2 2^{\varepsilon_2 j} \|f\|_{p_2}.
$$

Then, for $\theta, q$ and $p_1$ defined by

$$
\theta = \frac{\varepsilon_1}{\varepsilon_2}, \quad \frac{1}{q_1} = \frac{\theta}{q_2} + \frac{1}{q_2} \quad \text{and} \quad \frac{1}{p_1} = \frac{\theta}{p_2} + \frac{1}{p_2},
$$

the following hold:

$$
\left\| \sum_j T_j(f) \right\|_{q, \infty} \leq C M_1^\theta M_2^{1-\theta} \|f\|_{p, 1},
$$

(13)

$$
\left\| \sum_j T_j(f) \right\|_q \leq C M_1^\theta M_2^{1-\theta} \|f\|_{p, 1} \quad \text{if} \quad q_1 = q_2 = q,
$$

(14)

$$
\left\| \sum_j T_j(f) \right\|_{q, \infty} \leq C M_1^\theta M_2^{1-\theta} \|f\|_{p} \quad \text{if} \quad p_1 = p_2.
$$

(15)

We interpolate the bounds

$$
2^j \|T_2^{-j} f\|_{L^q(R^3)} \lesssim 2^j \|f\|_{L^2(R^3), \quad \frac{14}{3} \leq q \leq \infty},
$$

and

$$
2^j \|T_2^{-j} f\|_{L^\infty(R^3)} \lesssim 2^{-\frac{1}{j}} \|f\|_{L^1(R^3)}
$$

as above together with duality to find restricted weak endpoint bounds

$$
\|T f\|_{L^{q, \infty}(R^3)} \lesssim \|f\|_{L^{p, 1}(R^3)}
$$

for $(1/p, 1/q) \in \{B, B'\}$, weak bounds

$$
\|T f\|_{L^q} \lesssim \|f\|_{L^p}, \quad \|T f\|_{L^\infty} \lesssim \|f\|_{L^p, 1}
$$

for $(1/p, 1/q) \in (B', C')$, respectively, $(1/p, 1/q) \in (B, C]$, and strong bounds in the interior of the pentagon $\text{conv}(A, B, C, C', B')$ with $A = (1, 0)$,

$$
B = \left(\frac{7}{10}, \frac{9}{70}\right), \quad C = \left(\frac{7}{10}, 0\right), \quad B' = \left(\frac{61}{70}, \frac{3}{10}\right), \quad C' = \left(1, \frac{3}{10}\right).
$$
Real interpolation of the weak bounds at $B$ and $B'$ gives strong bounds on $(B, B')$. This finishes the proof of Proposition 1.1. \hfill \Box

3. $L^p$-$L^q$-estimates for solutions to elliptic differential operators

In this section we prove Theorem 1.2 relying on Proposition 1.1. The argument parallels [15, Section 5.2] very closely, to avoid repetition we shall be brief. Let $A_3$ and $B_3$ be as in (10) and (11). We start with the more difficult estimate of $A_3$. We show boundedness of $A_3: L^p(\mathbb{R}^3) \to L^q(\mathbb{R}^3)$ independently of $\delta$ with $p, q$ as in Proposition 1.1. For this it is enough to show restricted weak type bounds

$$
\|A_3\|_{L^{q_0, \infty}} \lesssim \|f\|_{L^{p_0, 1}}
$$

for $(1/p_0, 1/q_0) = (61/70, 3/10)$ and the bounds

$$
\|A_3f\|_{L^q} \lesssim \|f\|_{L^p}\n$$

for $(1/p, 1/q) \in ((61/70, 3/10), (1, 3/10)]$ as strong bounds for $A_3$ with $p, q$ as in Proposition 1.1 are recovered by interpolation and duality. As $\nabla p(\xi) \neq 0$ for $\xi \in \text{supp}(\beta_1)$ by construction, we can change to generalized polar coordinates. Let $\xi = (p, q)$, where $p$ and $q$ are complementary coordinates. Write

$$
A_3f(x) = \int e^{ix \cdot \xi} \beta_1(\xi) \hat{f}(\xi) d\xi = \int dp dq e^{ix \cdot \xi} \beta(\xi(p, q))h(p, q)\hat{f}(\xi(p, q)),
$$

where $h$ denotes the Jacobian. We can suppose that $|\partial^\alpha h| \lesssim 1$ choosing $\text{supp}(\beta)$ small enough. The expression is estimated as in [15, Subsection 5.2] by suitable decompositions in Fourier space and crucially depending on the Fourier restriction estimates for Proposition 1.1 see [12] for $p(\xi) = |\xi|^\alpha$. We write

$$
\frac{1}{p(\xi) + i\delta} = \frac{p(\xi)}{p^2(\xi) + \delta^2} - i\frac{\delta}{p^2(\xi) + \delta^2} = R(\xi) - i\mathcal{J}(\xi).
$$

As in [15], $\mathcal{J}(D)$ is estimated by Minkowski’s inequality and Fourier restriction–extension estimates, in the present context from Proposition 1.1. The only difference in the estimate of $R(D)$ is that [15, Lemma 5.1] is applied for $k = \frac{3}{2}$ according to the dispersive estimate [3]. For details we refer to [15, Section 4]. This finishes the proof of the estimate for $A_3$.

For the estimate of $B_3$, we carry out a further decomposition in Fourier space: By ellipticity, there is $R \geq 1$ such that

$$
|p(\xi)| \gtrsim |\xi|^N
$$

provided that $|\xi| \gtrsim R$. Let $\beta_2(\xi) = \beta_{21}(\xi) + \beta_{22}(\xi)$ with $\beta_{21}, \beta_{22} \in C^\infty$ and $\beta_{22}(\xi) = 0$ for $|\xi| \leq R$, $\beta_{22}(\xi) = 1$ for $|\xi| \geq 2R$. We can estimate

$$
\|B_3(\beta_{21}(D)f)\|_{L^q} \lesssim \|f\|_{L^p}
$$

for any $1 \leq p \leq q \leq \infty$ by Young’s inequality uniform in $\delta$. This gives no additional assumptions on $p$ and $q$. We estimate the contribution of $\beta_{22}$ by properties of the Bessel kernel (cf. [5, Theorem 30])

$$
\|B_3(\beta_{22}(D)f)\|_{L^q(\mathbb{R}^3)} \lesssim \|\beta_{22}(D)f\|_{L^p(\mathbb{R}^3)}
$$

for $1 \leq p, q \leq \infty$ and $0 \leq \frac{1}{p} - \frac{1}{q} \leq \frac{N}{2}$ with the endpoints excluded for $N \leq 3$. For $N \geq 4$ this estimate holds true for $1 \leq p \leq q \leq \infty$. This corresponds to the second assumption on $p$ and $q$ in Theorem 1.2.

Lastly, we give the standard argument for constructing solutions: For $\delta > 0$, consider the approximate solutions $u_\delta \in L^q(\mathbb{R}^3)$

$$
\hat{u}_\delta(\xi) = \frac{\hat{f}(\xi)}{p(\xi) + i\delta}.
$$

By the above, we have uniform bounds

$$
\|u_\delta\|_{L^q(\mathbb{R}^3)} \lesssim \|f\|_{L^{p_1}(\mathbb{R}^3) \cap L^{p_2}(\mathbb{R}^3)}.
$$
By the Banach–Alaoglu–Bourbaki theorem, we find a weak limit $u_\delta \to u$, which satisfies the same bound. We observe that

$$P(D)u_\delta = f - i\frac{\delta}{P(D) + i\delta}f.$$  

Since

$$\|\frac{\delta}{P(D) + i\delta}f\|_{L^q} \lesssim \|f\|_{L^p_1 \cap L^p_2},$$

we find that $P(D)u_\delta \to f$ in $L^q(\mathbb{R}^3)$. Since $P(D)u_\delta \to P(D)u$ in $S'(\mathbb{R}^3)$, this shows that

$$P(D)u = f$$

in $S'(\mathbb{R}^3)$. The proof is complete. \qed

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