On a Characterization of the Rellich–Kondrachov Theorem on Groups

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Abstract
Motivated by an eigenvalue-eigenfunction problem posed in $\mathbb{R}^n \times \Omega$, where $\Omega$ is a probability space, we are concerned in this paper with the Sobolev space on groups. Hence it is established an equivalence between locally compact Abelian groups and the space of solutions to the associated variational problem. Then, we study some conditions which characterize in a precisely manner the Rellich-Kondrachov Theorem, the principal ingredient to solve the variational problem.

1 Introduction

The main motivation in this paper to study Sobolev spaces on groups, besides being an elegant and modern mathematical theory, is the Rellich–Kondrachov Theorem. More precisely, we are interested on a characterization of it, which is related to the solution of the following eigenvalue-eigenfunction problem: Find $\lambda(\theta) \in \mathbb{R}$ and $\Psi(\theta)$ a complex-value function satisfying for any $\theta \in \mathbb{R}^n$ fixed,

$$
\begin{cases}
L^\Phi(\theta)[\Psi(z, \omega)] = \lambda \Psi(z, \omega), & \text{in } \mathbb{R}^n \times \Omega, \\
\Psi(z, \omega) = \psi(\Phi^{-1}(z, \omega), \omega), & \psi \text{ is a stationary function},
\end{cases}
$$

(1.1)

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where \((\Omega, \mathbb{P})\) is a probability space, and the linear operator \(L^\Phi(\theta)\) is defined by

\[
L^\Phi(\theta)[.] := - \left( \text{div} z + 2i\pi\theta \right) \left( A(\Phi^{-1}(z, \omega), \omega) \left( \nabla z + 2i\pi\theta \right)[.] \right) \\
+ V \left( \Phi^{-1}(z, \omega), \omega \right)[.].
\] (1.2)

Here, \(\Phi : \mathbb{R}^n \times \Omega \to \mathbb{R}^n\) is a stochastic diffeomorphisms, (called stochastic deformations). The \(n \times n\) matrix-value stationary function \(A = (A_{k\ell})\) and the real stationary potential \(V\) are measurable and bounded functions, i.e. \(A_{k\ell}, V \in L^\infty(\mathbb{R}^n \times \Omega)\). Moreover, \(A\) is symmetric and uniformly positive defined, that is, there exists \(a_0 > 0\), such that, for a.a. \((y, \omega) \in \mathbb{R}^n \times \Omega\), and each \(\xi \in \mathbb{R}^n\)

\[
\sum_{k,\ell=1}^{n} A_{k\ell}(y, \omega) \xi_k \xi_\ell \geq a_0 |\xi|^2.
\] (1.3)

The stationarity property of random functions will be precisely defined in Section 1.1 also the definition of stochastic deformations which were introduced by X. Blanc, C. Le Bris, P.-L. Lions (see [2, 3]).

One observes that, the eigenvalue problem (1.1) is called Bloch or shifted spectral cell equation, see Section 2.4 in [4]. Moreover, each \(\theta \in \mathbb{R}^n\) is called a Bloch frequency, \(\lambda(\theta)\) is called a Bloch energy and the corresponded \(\Psi(\theta)\) is called a Bloch wave.

Now, let us consider the following spaces

\[
\mathcal{L}_\Phi := \{ F(z, \omega) = f(\Phi^{-1}(z, \omega), \omega); f \in L^2_{\text{loc}}(\mathbb{R}^n; L^2(\Omega)) \text{ stationary} \} \quad (1.4)
\]

and

\[
\mathcal{H}_\Phi := \{ F(z, \omega) = f(\Phi^{-1}(z, \omega), \omega); f \in H^1_{\text{loc}}(\mathbb{R}^n; L^2(\Omega)) \text{ stationary} \} \quad (1.5)
\]

which are Hilbert spaces, endowed respectively with the inner products

\[
\langle F, G \rangle_{\mathcal{L}_\Phi} := \int_{\Omega} \int_{\Phi((0,1)^n, \omega)} F(z, \omega) \overline{G(z, \omega)} \, dz \, d\mathbb{P}(\omega),
\]

\[
\langle F, G \rangle_{\mathcal{H}_\Phi} := \int_{\Omega} \int_{\Phi((0,1)^n, \omega)} F(z, \omega) \overline{G(z, \omega)} \, dz \, d\mathbb{P}(\omega) \\
+ \int_{\Omega} \int_{\Phi((0,1)^n, \omega)} \nabla z F(z, \omega) \cdot \nabla z \overline{G(z, \omega)} \, dz \, d\mathbb{P}(\omega).
\]
Remark 1.1. Under the above notations, when $\Phi = \text{Id}$ we denote $L_\Phi$ and $H_\Phi$ by $L$ and $H$ respectively. Moreover, a function $F \in H_\Phi$ if, and only if, $F \circ \Phi \in H$. Analogously, $F \in L_\Phi$ if, and only if, $F \circ \Phi \in L$.

Associated to the eigenvalue-eigenfunction problem (1.1) is a variational formulation. To this end, we consider for $F, G \in H_\Phi$, the following functional

$$
\langle L^\Phi(\theta)[F], G \rangle = \int_\Omega \int_{\Phi([0,1]^n,\omega)} A(\Phi^{-1}(z,\omega), (\nabla_z + 2i\pi \theta)F(z,\omega) \cdot (\nabla_z + 2i\pi \theta)G(z,\omega)) \, dz \, d\mathbb{P}(\omega) + \int_\Omega \int_{\Phi([0,1]^n,\omega)} V(\Phi^{-1}(z,\omega), \omega) F(z,\omega) \overline{G(z,\omega)} \, dz \, d\mathbb{P}(\omega).
$$

\[ \text{(1.6)} \]

Then, the variational problem is to minimize $E(\Psi) := \langle L^\Phi(\theta)[\Psi], \Psi \rangle$, subject to the constraint $\|\Psi\|_{L_\Phi}^2 \equiv \langle \Psi, \Psi \rangle_{L_\Phi} = 1$. The minimizing function $\Psi_0$, when it exists, satisfies (1.1) with $\lambda = \lambda_0$.

A standard routine to solve this variational problem relies on a compact embedding. Albeit, from mathematical-physical reasons, see [4], the parameter $\omega \in \Omega$ in (1.1), equivalently (1.6), can not be fixed, that is to say, $\lambda_0$ can not depend on $\omega$. Therefore, we established an equivalence between $H_\Phi$ and the Sobolev space on groups, see Section 3, and then consider a related Rellich-Kondrachov’s Theorem. Indeed, we establish a compactness argument in Section 4, which enable us to show that the space $H_\Phi$ is compactly embedded in $L_\Phi$, in order to solve the associated variational problem.

Finally, we stress that, the subject of Sobolev spaces on Abelian locally compact groups, to the best of our knowledge, was introduced by P. Görka, E. G. Reyes [7].

1.1 Notation and background

We denote by $\mathbb{G}$ the group $\mathbb{Z}^n$ (or $\mathbb{R}^n$), with $n \in \mathbb{N}$. The set $[0,1)^n$ denotes the unit cube, which is also called the unitary cell and will be used as the reference period for periodic functions. The symbol $\lfloor x \rfloor$ denotes the unique number in $\mathbb{Z}^n$, such that $x - \lfloor x \rfloor \in [0,1)^n$.

Let $U \subset \mathbb{R}^n$ be an open set, $p \geq 1$, and $s \in \mathbb{R}$. We denote by $L^p(U)$ the set of (real or complex) $p$–summable functions with respect to the
Lebesgue measure. Given a Lebesgue measurable set $E \subset \mathbb{R}^n$, $|E|$ denotes its $n$–dimensional Lebesgue measure. Moreover, we will use the standard notations for the Sobolev spaces $W^{s,p}(U)$ and $H^s(U) \equiv W^{s,2}(U)$.

Now, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For each random variable $f$ in $L^1(\Omega; \mathbb{P})$, $(L^1(\Omega)$ for short), we define its expectation value by

$$\mathbb{E}[f] := \int_{\Omega} f(\omega) \, d\mathbb{P}(\omega).$$

A mapping $\tau : \mathbb{G} \times \Omega \to \Omega$ is said a $n$–dimensional dynamical system, when

(i) (Group Property) $\tau(0, \cdot) = id_\Omega$ and $\tau(x + y, \omega) = \tau(x, \tau(y, \omega))$ for each $x, y \in \mathbb{G}$ and $\omega \in \Omega$.

(ii) (Invariance) The mappings $\tau(x, \cdot) : \Omega \to \Omega$ are $\mathbb{P}$-measure preserving, that is, for all $x \in \mathbb{G}$ and every $E \in \mathcal{F}$, it follows that

$$\tau(x, E) \in \mathcal{F}, \quad \mathbb{P}(\tau(x, E)) = \mathbb{P}(E).$$

We shall use $\tau(k)\omega$ to denote $\tau(k, \omega)$, and it is usual to say that $\tau(k)$ is a discrete (continuous) dynamical system if $k \in \mathbb{Z}^n$ ($k \in \mathbb{R}^n$), although we only stress this when it is not obvious from the context.

A measurable function $f$ on $\Omega$ is called $\tau$-invariant, when for each $k \in \mathbb{G}$

$$f(\tau(k)\omega) = f(\omega) \quad \text{for almost all } \omega \in \Omega.$$

Then, a measurable set $E \in \mathcal{F}$ is $\tau$-invariant, if its characteristic function $\chi_E$ is $\tau$-invariant. It is a straightforward to show that, a $\tau$-invariant set $E$ can be equivalently defined by

$$\tau(k)E = E \quad \text{for each } k \in \mathbb{G}.$$

We say that the dynamical system $\tau$ is ergodic, when all $\tau$-invariant sets $E$ have measure $\mathbb{P}(E)$ of either zero or one. Equivalently, a dynamical system is ergodic if each $\tau$-invariant function is constant almost everywhere, that is

$$\left( f(\tau(k)\omega) = f(\omega) \quad \text{for each } k \in \mathbb{G} \text{ and a.e. } \omega \in \Omega \right) \Rightarrow f(\cdot) = \text{const. a.e.}.$$
Now, let \((\Gamma, \mathcal{G}, \mathbb{Q})\) be a given probability space. We say that a measurable function \(g : \mathbb{R}^n \times \Gamma \to \mathbb{R}\) is stationary, if for any finite set consisting of points \(x_1, \ldots, x_j \in \mathbb{R}^n\), and any \(k \in \mathcal{G}\), the distribution of the random vector
\[
\left( g(x_1 + k, \cdot), \cdots, g(x_j + k, \cdot) \right)
\]
is independent of \(k\). Moreover, subjecting the stationary function \(g\) to some natural conditions it can be showed that, there exists other probability space \((\Omega, \mathcal{F}, \mathbb{P})\), a \(n\)–dimensional dynamical system \(\tau : \mathcal{G} \times \Omega \to \Omega\) and a measurable function \(f : \mathbb{R}^n \times \Omega \to \mathbb{R}\) satisfying

(i) For all \(x \in \mathbb{R}^n, k \in \mathcal{G}\) and \(\mathbb{P}\)–almost every \(\omega \in \Omega\)
\[
f(x + k, \omega) = f(x, \tau(k)\omega).
\]

(ii) For any \(x \in \mathbb{R}^n\) the random variables \(g(x, \cdot), f(x, \cdot)\) have the same law.

**Remark 1.2.** The set of stationary functions forms an algebra, and also is stable by limit process. Therefore, the product of two stationaries functions is a stationary one, and the derivative of a stationary function is stationary. Moreover, the stationarity concept is the most general extension of the notions of periodicity and almost periodicity for a function to have some "self-averaging" behaviour.

To follow, we present the definition of the stochastic deformation as presented in \([\Pi]\).

**Definition 1.3.** A mapping \(\Phi : \mathbb{R}^n \times \Omega \to \mathbb{R}^n, (y, \omega) \mapsto z = \Phi(y, \omega)\), is called a stochastic deformation (for short \(\Phi_\omega\)), when satisfies:

i) For \(\mathbb{P}\)–almost every \(\omega \in \Omega\), \(\Phi(\cdot, \omega)\) is a bi–Lipschitz diffeomorphism.

ii) There exists \(\nu > 0\), such that
\[
\text{ess inf}_{\omega \in \Omega, y \in \mathbb{R}^n} \left( \text{det}(\nabla \Phi(y, \omega)) \right) \geq \nu.
\]

iii) There exists a \(M > 0\), such that
\[
\text{ess sup}_{\omega \in \Omega, y \in \mathbb{R}^n} \left( |\nabla \Phi(y, \omega)| \right) \leq M < \infty.
\]
iv) The gradient of $\Phi$, i.e. $\nabla \Phi(y, \omega)$, is stationary in the sense (1.7).

Connected with the notion of stationarity, we consider now the concept of mean value. A function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is said to possess a mean value if the sequence $\{f(\cdot/\varepsilon)\}_{\varepsilon>0}$ converges in the duality with $L^\infty$ and compactly supported functions to a constant $M(f)$. This convergence is equivalent to

$$
\lim_{t \to \infty} \frac{1}{t^n |A|} \int_{A_t} f(x) \, dx = M(f), \quad (1.8)
$$

where $A_t := \{x \in \mathbb{R}^n : t^{-1}x \in A\}$, for $t > 0$ and any $A \subset \mathbb{R}^n$, with $|A| \neq 0$.

**Remark 1.4.** Unless otherwise stated, we assume that the dynamical system $\tau : G \times \Omega \to \Omega$ is ergodic and we will also use the notation

$$
\int_{\mathbb{R}^n} f(x) \, dx \quad \text{for } M(f).
$$

Then, we state the result due to Birkhoff, see [10].

**Theorem 1.5** (Birkhoff Ergodic Theorem). Let $f \in L^1_{\text{loc}}(\mathbb{R}^n; L^1(\Omega))$ be a stationary random variable. Then, for almost every $\tilde{\omega} \in \Omega$ the function $f(\cdot, \tilde{\omega})$ possesses a mean value in the sense of (1.8). Moreover, the mean value $M(f(\cdot, \tilde{\omega}))$ as a function of $\tilde{\omega} \in \Omega$ satisfies for almost every $\tilde{\omega} \in \Omega$:

i) Discrete case (i.e. $\tau : \mathbb{Z}^n \times \Omega \to \Omega$);

$$
\int_{\mathbb{R}^n} f(x, \tilde{\omega}) \, dx = \mathbb{E} \left[ \int_{[0,1]^n} f(y, \cdot) \, dy \right].
$$

ii) Continuous case (i.e. $\tau : \mathbb{R}^n \times \Omega \to \Omega$);

$$
\int_{\mathbb{R}^n} f(x, \tilde{\omega}) \, dx = \mathbb{E} [f(0, \cdot)].
$$

The Birkhoff Ergodic Theorem holds if a stationary function is composed with a stochastic deformation:

**Lemma 1.6.** Let $\Phi$ be a stochastic deformation and $f \in L^\infty_{\text{loc}}(\mathbb{R}^n; L^1(\Omega))$ be a stationary random variable in the sense (1.7). Then for almost $\tilde{\omega} \in \Omega$ the function $f(\Phi^{-1}(\cdot, \tilde{\omega}) , \tilde{\omega})$ possesses a mean value in the sense of (1.8) and satisfies:
i) Discrete case;
\[
\int_{\mathbb{R}^n} f(\Phi^{-1}(z, \tilde{\omega}), \tilde{\omega}) \, dz = \frac{\mathbb{E} \left[ \int_{[0,1]^n} f(\Phi^{-1}(z, \cdot), \cdot) \, dz \right]}{\det \left( \mathbb{E} \left[ \int_{[0,1]^n} \nabla_y \Phi(y, \cdot) \, dy \right] \right)} \text{ for a.a. } \tilde{\omega} \in \Omega.
\]

ii) Continuous case;
\[
\int_{\mathbb{R}^n} f(\Phi^{-1}(z, \tilde{\omega}), \tilde{\omega}) \, dz = \frac{\mathbb{E} \left[ f(0, \cdot) \det \left( \nabla \Phi(0, \cdot) \right) \right]}{\det \left( \mathbb{E} \left[ \nabla \Phi(0, \cdot) \right] \right)} \text{ for a.a. } \tilde{\omega} \in \Omega.
\]

Proof. See Blanc, Le Bris, Lions [2], also Andrade, Neves, Silva [1].

The next theorem presents important properties of stationary functions.

Theorem 1.7. For \( p > 1 \), let \( u, v \in L^1_{\text{loc}}(\mathbb{R}^n; L^p(\Omega)) \) be stationary functions. Then, for any \( i \in \{1, \ldots, n\} \) fixed, the following sentences are equivalent:

(A) \[
\int_{[0,1]^n} \int_{\Omega} u(y, \omega) \frac{\partial \zeta}{\partial y_i}(y, \omega) \, d\mathbb{P}(\omega) \, dy = - \int_{[0,1]^n} \int_{\Omega} v(y, \omega) \zeta(y, \omega) \, d\mathbb{P} \, dy,
\]

for each stationary function \( \zeta \in C^1(\mathbb{R}^n; L^q(\Omega)) \), with \( 1/p + 1/q = 1 \). \( (1.9) \)

(B) \[
\int_{\mathbb{R}^n} u(y, \omega) \frac{\partial \varphi}{\partial y_i}(y) \, dy = - \int_{\mathbb{R}^n} v(y, \omega) \varphi(y) \, dy,
\]

for any \( \varphi \in C^1_c(\mathbb{R}^n) \), and almost sure \( \omega \in \Omega \). \( (1.10) \)

Proof. See Blanc, Le Bris, Lions [2].

\[ \square \]

2 Sobolev spaces on groups

To begin, we sum up some definitions and properties of topological groups, which will be used along this section. Most of the material could be found in E. Hewitt, A. Ross [8] and G. B. Folland [6] (with more details).

A nonempty set \( G \) endowed with an application, \( * : G \times G \rightarrow G \), is called a group, when for each \( x, y, z \in G \):

1. \( (x * y) * z = x * (y * z) \);
2. There exists $e \in G$, such that $x * e = e * x = e$;

3. For all $y \in G$, there exists $y^{-1} \in G$, such that $y * y^{-1} = y^{-1} * y = e$.

Moreover, if $x * y = y * x$, then $G$ is called an Abelian group. From now on, we write for simplicity $xz$ instead of $x * z$. A topological group is a group $G$ together with a topology, such that, both the group’s binary operation $(x, y) \mapsto xy$, and the function mapping group elements to their respective inverses $x \mapsto x^{-1}$ are continuous functions with respect to the topology. Unless the contrary is explicit stated, any group mentioned here is a locally compact Abelian (LCA for short) group, and we may assume without loss of generality that, the associated topology is Hausdorff (see G. B. Folland [9, Corollary 2.3]).

A complex value function $\xi : G \to S^1$ is called a character of $G$, when

$$
\xi(xy) = \xi(x)\xi(y), \quad \text{(for each } x, y \in G).
$$

We recall that, the set of characters of $G$ is an Abelian group with the usual product of functions, identity element $e = 1$, and inverse element $\xi^{-1} = \overline{\xi}$. The characters’ group of the topological group $G$, called the dual group of $G$ and denoted by $G^\wedge$, is the set of all continuous characters, that is to say

$$
G^\wedge := \{\xi : G \to S^1 ; \xi \text{ is a continuous homomorphism}\}.
$$

Moreover, we may endow $G^\wedge$ with a topology with respect to which, $G^\wedge$ itself is a LCA group.

We denote by $\mu, \nu$ the unique (up to a positive multiplicative constant) Haar measures in $G$ and $G^\wedge$ respectively. The $L^p$ spaces over $G$ and its dual are defined as usual, with their respective measures. Let us recall two important properties when $G$ is compact:

\begin{enumerate}
  \item If $\mu(G) = 1$, then $G^\wedge$ is an orthonormal set in $L^2(G; \mu)$.
  \item The dual group $G^\wedge$ is discrete, and $\nu$ is the countermeasure.
\end{enumerate}

(2.11)

One remarks that, the study of Sobolev spaces on LCA groups uses essentially the concept of Fourier Transform, then we have the following
Definition 2.1. Given a complex value function \( f \in L^1(G; \mu) \), the function \( \hat{f} : G^\wedge \to \mathbb{C} \), defined by
\[
\hat{f}(\xi) := \int_G f(x) \overline{\xi(x)} \, d\mu(x)
\] (2.12)
is called the Fourier transform of \( f \) on \( G \).

Usually, the Fourier Transform of \( f \) is denoted by \( \mathcal{F}f \) to emphasize that it is an operator, but we prefer to adopt the usual notation \( \hat{f} \). Moreover, we recall that the Fourier transform is an homomorphism from \( L^1(G; \mu) \) to \( C_0(G^\wedge) \) (or \( C(G^\wedge) \) when \( G \) is compact), see Proposition 4.13 in [6]. Also we address the reader to [6], Chapter 4, for the Plancherel Theorem and the Inverse Fourier Transform.

Before we establish the definition of (energy) Sobolev spaces on LCA groups, let us consider the following set
\[
P = \{ p : G^\wedge \times G^\wedge \to [0, \infty) / \\
p \text{ is a continuous invariant pseudo-metric on } G^\wedge \}.
\]
The Birkhoff-Kakutani Theorem (see [8] p.68) ensures that, the set \( P \) is not empty. Any pseudo-metric \( p \in P \) is well defined for each \((x, y) \in G^\wedge \times G^\wedge\), hence we may define
\[
\gamma(x) := p(x, e) \equiv p(x, 1).
\] (2.13)
Moreover, one observes that \( \gamma(1) = 0 \). Then, we have the following

Definition 2.2 (Energy Sobolev Spaces on LCA Groups). Let \( s \) be a non-negative real number and \( \gamma(x) \) be given by (2.13) for some fixed \( p \in P \). The energy Sobolev space \( H^s_\gamma(G) \) is the set of functions \( f \in L^2(G; \mu) \), such that
\[
\int_{G^\wedge} (1 + \gamma(\xi)^2)^s |\hat{f}(\xi)|^2 \, d\nu(\xi) < \infty.
\] (2.14)
Moreover, given a function \( f \in H^s_\gamma(G) \) its norm is defined as
\[
\|f\|_{H^s_\gamma(G)} := \left( \int_{G^\wedge} (1 + \gamma(\xi)^2)^s |\hat{f}(\xi)|^2 \, d\nu(\xi) \right)^{1/2}.
\] (2.15)

Below, taking specific functions \( \gamma \), the usual Sobolev spaces on \( \mathbb{R}^d \) and other examples are considered. In particular, the Plancherel Theorem implies that, \( H^0_\gamma(G) = L^2(G; \mu) \).
Example 2.3. Let $G = (\mathbb{R}^n, +)$ which is known to be a LCA group, and consider its dual group $(\mathbb{R}^n)^\wedge = \{\xi_y \ ; \ y \in \mathbb{R}^n\}$, where for each $x \in \mathbb{R}^n$

$$\xi_y(x) = e^{2\pi i y \cdot x}, \quad (2.16)$$

hence $|\xi_y(x)| = 1$ and $\xi_0(x) = 1$. One remarks that, here we denote (without invocation of vector space structure)

$$a \cdot b = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n, \quad (\text{for all } a, b \in G).$$

For any $x, y \in \mathbb{R}^n$ let us consider

$$p(\xi_x, \xi_y) = 2\pi \| x - y \|,$$

where $\| \cdot \|$ is the Euclidean norm in $\mathbb{R}^n$. Hence $\gamma(\xi_x) = p(\xi_x, 1) = 2\pi \| x \|$.

Since $(\mathbb{R}^n)^\wedge \cong \mathbb{R}^n$, the Sobolev space $H_{\gamma}^s(G)$ coincide with the usual Sobolev space on $\mathbb{R}^n$.

Example 2.4. Let us recall that, the set $[0, 1)^n$ endowed with the binary operation

$$(x, y) \in [0, 1)^n \times [0, 1)^n \mapsto x + y - \lfloor x + y \rfloor \in [0, 1)^d$$

is an Abelian group, and the function $\Lambda : \mathbb{R}^n \to [0, 1)^n$, $\Lambda(x) := x - \lfloor x \rfloor$ is an homomorphism of groups. Moreover, under the induced topology by $\Lambda$, that is to say

$$\{ U \subset [0, 1)^n \ ; \ \Lambda^{-1}(U) \text{ is an open set of } \mathbb{R}^n \},$$

$[0, 1)^n$ is a compact Abelian group, which is called $n$–dimensional Torus and denoted by $\mathbb{T}^n$. Its dual group is characterized by the integers $\mathbb{Z}^n$, that is

$$(\mathbb{T}^n)^\wedge = \{\xi_m \ ; \ m \in \mathbb{Z}^n\}, \text{ where } \xi_m(x) \text{ is given by } (2.16) \text{ for all } x \in \mathbb{R}^n.$$ 

For each $m, k \in \mathbb{Z}^n$, we consider

$$p(\xi_m, \xi_k) = 2\pi \sum_{j=1}^n |m_j - k_j|, \quad \text{and thus } \gamma(\xi_m) = 2\pi \sum_{j=1}^n |m_j|.$$ 

Then, the Sobolev space $H_{\gamma}^s(\mathbb{T}^n)$ coincide with the usual Sobolev space on $\mathbb{T}^n$.

Now, following the above discussion let us consider the infinite Torus $\mathbb{T}^I$, where $I$ is an index set. Since an arbitrary product of compact spaces is compact in the product topology (Tychonoff Theorem), $\mathbb{T}^I$ is a compact Abelian
group. Here, the binary operation on $\mathbb{T}^I \times \mathbb{T}^I$ is defined coordinate by coordinate, that is, for each $\ell \in I$

\[ g_\ell + h_\ell := g_\ell + h_\ell - \lfloor g_\ell + h_\ell \rfloor. \]

Moreover, the set $\mathbb{Z}_c^I := \{ m \in \mathbb{Z}^I; \text{supp } m \text{ is compact} \}$ characterizes the elements of the dual group $(\mathbb{T}^I)^\wedge$. Indeed, applying Theorem 23.21 in [8], similarly we have

\[ (\mathbb{T}^I)^\wedge = \{ \xi_m ; m \in \mathbb{Z}_c^I \}, \]

where $\xi_m(k)$ is given by (2.16) for each $m, k \in \mathbb{Z}_c^I$, the pseudo-metric

\[ p(\xi_m, \xi_k) = 2\pi \sum_{\ell \in I} |m_\ell - k_\ell|, \text{ and } \gamma(\xi_m) = 2\pi \sum_{\ell \in I} |m_\ell|. \]

Consequently, we have establish the Sobolev spaces $H^s_\gamma(\mathbb{T}^I)$.

### 3 Groups and Dynamical systems

In this section, we are interested to come together the discussion about dynamical systems considered in Section 1.1 with the theory developed in the last section for LCA groups. To this end, we consider stationary functions in the continuous sense (continuous dynamical systems). Moreover, we recall that all the groups in this paper are assumed to be Hausdorff.

To begin, let $G$ be a locally compact group with Haar measure $\mu$, we know that $\mu(G) < \infty$ if, and only if, $G$ is compact. Therefore, we consider from now on that $G$ is a compact Abelian group, hence $\mu$ is a finite measure and, up to a normalization, $(G, \mu)$ is a probability space. We are going to consider the dynamical systems, $\tau : \mathbb{R}^n \times G \to G$, defined by

\[ \tau(x)\omega := \varphi(x) \omega, \quad (3.17) \]

where $\varphi : \mathbb{R}^n \to G$ is a given (continuous) homomorphism. Indeed, first $\tau(0)\omega = \omega$ and $\tau(x + y, \omega) = \varphi(x)\varphi(y)\omega = \tau(x, \tau(y)\omega)$. Moreover, since $\mu$ is a translation invariant Haar measure, the mapping $\tau(x, \cdot) : G \to G$ is $\mu$–measure preserving. Recall from Remark 1.4 we have assumed that, the dynamical systems we are interested here are ergodic. Then, it is important to characterize the conditions for the mapping $\varphi$, under which the dynamical system defined by (3.17) is ergodic. To this end, first let us consider the following
Lemma 3.1. Let $H$ be a topological group, $F \subset H$ closed, $F \neq H$ and $x \notin F$. Then, there exists a neighborhood $V$ of the identity $e$, such that

$$FV \cap xV = \emptyset.$$ 

Proof. First, we observe that:

i) Since $F \subset H$ is closed and $F \neq H$, there exists a neighborhood $U$ of the identity $e$, such that $F \cap xU = \emptyset$.

ii) There exists a symmetric neighborhood $V$ of the identity $e$, such that $VV \subset U$.

Now, suppose that $FV \cap xV \neq \emptyset$. Therefore, there exist $v_1, v_2 \in V$ and $k_0 \in F$ such that, $k_0v_1 = xv_2$. Consequently, $k_0 = xv_2v_1^{-1}$ and from (ii) it follows that, $k_0 \in xU$. Then, we have a contradiction from (i).

Claim 1: The dynamical system defined by (3.17) is ergodic if, and only if, $\varphi(\mathbb{R}^n)$ is dense in $G$.

Proof of Claim 1: 1. Let us show first the necessity. Therefore, we suppose that $\varphi(\mathbb{R}^n)$ is not dense in $G$, that is $K := \varphi(\mathbb{R}^n) \neq G$. Then, applying Lemma 3.1 there exists a neighborhood $V$ of $e$, such that $KV \cap xV = \emptyset$, for some $x \notin K$. Recall that the Haar measure on open sets are positive, moreover

$$KV = \bigcup_{k \in K} kV,$$

which is an open set, thus we have

$$0 < \mu(KV) + \mu(xV) \leq 1.$$ 

Consequently, it follows that $0 < \mu(\varphi(\mathbb{R}^n)V) < 1$. For convenience, let us denote $E = \varphi(\mathbb{R}^n)V$, hence $\tau(x)E = E$ for each $x \in \mathbb{R}^n$. Then, the dynamical system $\tau$ is not ergodic, since $E \subset G$ is a $\tau$-invariant set with $0 < \mu(E) < 1$.

2. It remains to show the sufficiency. Let $E \subset G$ be a $\mu$–measurable $\tau$-invariant set, hence $\omega E = E$ for each $\omega \in \varphi(\mathbb{R}^n)$. Assume by contradiction that, $0 < \mu(E) < 1$, thus $\mu(G \setminus E) > 0$. Denote by $\mathcal{B}_G$ the Borel $\sigma$–algebra on $G$, and define, $\lambda := \mu|_E$, that is $\lambda(A) = \mu(E \cap A)$ for all $A \in \mathcal{B}_G$. Recall that $G$ is not necessarily metric, therefore, it is not clear if each Borel set is $\mu$–measurable. Then, it follows that:
(i) For any $A \in \mathcal{B}_G$ fixed, the mapping $\omega \in G \mapsto \lambda(\omega A)$ is continuous. Indeed, for $\omega \in G$ and $A \in \mathcal{B}_G$, we have

$$\lambda(\omega A) = \int_G 1_E(\varpi)1_{\omega A}(\varpi) d\mu(\varpi)$$

$$= \int_G 1_E(\varpi)1_A(\omega^{-1}\varpi) d\mu(\varpi) = \int_G 1_E(\omega \varpi)1_A(\varpi) d\mu(\varpi).$$

Therefore, for $\omega, \omega_0 \in G$

$$|\lambda(\omega A) - \lambda(\omega_0 A)| = \left| \int_G (1_E(\omega \varpi) - 1_E(\omega_0 \varpi)) 1_A(\varpi) d\mu(\varpi) \right|$$

$$\leq (\mu(A))^{1/2} \left( \int_G |1_E(\omega \varpi) - 1_E(\omega_0 \varpi)|^2 d\mu(\varpi) \right)^{1/2} \to 0.$$ 

(ii) $\lambda$ is invariant, i.e. for all $\omega \in G$, and $A \in \mathcal{B}_G$, $\lambda(\omega A) = \lambda(A)$. Indeed, for each $\omega \in \varphi(\mathbb{R}^d)$, and $A \in \mathcal{B}_G$, we have

$$(\omega A) \cap E = (\omega A) \cap (\omega E) = \omega(A \cap E).$$

Thus since $\mu$ is invariant, $\mu|_E(\omega A) = \mu|_E(A)$. Consequently, due to item (i) and $\varphi(\mathbb{R}^d) = G$, it follows that $\lambda$ is invariant.

From item (ii) the Radon measure $\lambda$ is a Haar measure on $G$. By the uniqueness of the Haar measure on $G$, there exists $\alpha > 0$, such that for all $A \in \mathcal{B}_G$, $\alpha \lambda(A) = \mu(A)$. In particular, $\alpha \lambda(G \setminus E) = \mu(G \setminus E)$. But $\lambda(G \setminus E) = 0$ by definition and $\mu(G \setminus E) > 0$, which is a contradiction and hence $\tau$ is ergodic.

**Remark 3.2.** 1. One remarks that, in order to show that $\tau$ given by (3.17) is ergodic, it was not used that $\varphi$ is continuous, nor that $G$ is metric. Compare with the statement in [7] p.225 (after Theorem 7.2).

2. From now on, we assume that $\varphi(\mathbb{R}^n)$ is dense in $G$.

Now, for the dynamical system established before, the main issue is to show how the Sobolev space $H^1_\gamma(G)$ is related with the space $\mathcal{H}_\Phi$ given by (1.5) for $\Phi = \text{Id}$, that is

$$\mathcal{H} = \{ f(y, \omega); f \in H^1_{\text{loc}}(\mathbb{R}^n; L^2(G)) \text{ stationary} \},$$

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which is a Hilbert space endowed with the following inner product
\[
\langle f, g \rangle_H = \int_G f(0, \omega) \overline{g(0, \omega)} d\mu(\omega) + \int_G \nabla_y f(0, \omega) \cdot \nabla_y g(0, \omega) d\mu(\omega).
\]

Let \( \chi \) be a character on \( G \), i.e. \( \chi \in G^\wedge \). Since \( \varphi : \mathbb{R}^n \to G \) is a continuous homomorphism, the function \( (\chi \circ \varphi) : \mathbb{R}^n \to \mathbb{C} \) is a continuous character in \( \mathbb{R}^n \). More precisely, given any fixed \( \chi \in G^\wedge \) we may find \( y \in \mathbb{R}^n \), \( (y \equiv y(\chi)) \), such that, for each \( x \in \mathbb{R}^n \)
\[
(\chi \circ \varphi)(x) =: \xi_{y(\chi)}(x) = e^{2\pi i y(\chi) \cdot x}.
\]

Following Example 2.3 we define the pseudo-metric \( p_\varphi : G^\wedge \times G^\wedge \to [0, \infty) \) by
\[
p_\varphi(\chi_1, \chi_2) := p(\xi_{y_1(\chi_1)}, \xi_{y_2(\chi_2)}) = 2\pi \|y_1(\chi_1) - y_2(\chi_2)\|.
\]

Then, we have
\[
\gamma(\chi) = p_\varphi(\chi, 1) = 2\pi \|y(\chi)\|.
\]

Let us observe that, we have used in the above construction of \( \gamma \) the continuity of the homomorphism \( \varphi : \mathbb{R}^n \to G \), that is to say, it was essential the continuity of \( \varphi \). In fact, the function \( \gamma \) was given by the pseudo-metric \( p_\varphi \), which is not necessarily a metric. Although, we have the following

**Claim 2:** The pseudo-metric \( p_\varphi : G^\wedge \times G^\wedge \to [0, \infty) \) given by (3.18) is a metric if, and only if, \( \varphi(\mathbb{R}^n) \) is dense in \( G \).

Proof of Claim 2: 1. First, let us assume that \( \varphi(\mathbb{R}^n) \neq G \), and then show that \( p_\varphi \) is not a metric. Therefore, we have the necessity proved. From Corollary 24.12 in [8], since \( \varphi(\mathbb{R}^n) \) is a closer proper subgroup of \( G \), hence there exists \( \xi \in G^\wedge \setminus \{1\} \), such that \( \xi(\varphi(\mathbb{R}^n)) = \{1\} \). Hence there exists \( \xi \in G^\wedge \setminus \{1\} \), such that, \( \xi(\varphi(x)) = 1 \), for each \( x \in \mathbb{R}^n \), i.e. \( y(\xi) = 0 \). Therefore, we have \( p_\varphi(\xi, 1) = 0 \), which implies that \( p_\varphi \) is not a metric.

2. Now, let us assume that \( \varphi(\mathbb{R}^n) = G \), and it is enough to show that if \( p_\varphi(\xi, 1) = 0 \), then \( \xi = 1 \). Indeed, if \( 0 = p_\varphi(\xi, 1) = 2\pi \|y(\xi)\| \), then \( y(\xi) = 0 \). Therefore, \( \xi(\varphi(x)) = 1 \) for each \( x \in \mathbb{R}^d \), since \( \xi \) is continuous and \( \varphi(\mathbb{R}^n) = G \), it follows that, for each \( \omega \in G \), \( \xi(\omega) = 1 \), which finishes the proof of the claim.

**Remark 3.3.** Since we have already assumed that \( \varphi(\mathbb{R}^n) \) is dense in \( G \), it follows that \( p_\varphi \) is indeed a metric, which does not imply necessarily that \( G \), itself, is metric.
Under the assumptions considered above, we have the following

**Lemma 3.4.** If \( f \in \mathcal{H} \), then for \( j \in \{1, \ldots, d\} \) and all \( \xi \in G^\wedge \)

\[
\partial_j f(0, \xi) = 2\pi i \, y_j(\xi) \hat{f}(0, \xi). \tag{3.19}
\]

**Proof.** First, for each \( x \in \mathbb{R}^d \) and \( \omega \in G \), define

\[
\xi_\tau(x, \omega) := \xi(\tau(x, \omega)) = \xi(\varphi(x)) \xi(\omega) = \xi(\varphi(x)) \xi(\omega).
\]

Therefore \( \xi_\tau \in C^\infty(\mathbb{R}^d, L^2(G)) \), and we have for \( j \in \{1, \ldots, d\} \)

\[
\partial_j \xi_\tau(0, \omega) = 2\pi i \, y_j(\xi) \xi(\omega). \tag{3.20}
\]

Finally, applying Theorem 1.7 we obtain

\[
\int_G \partial_j f(0, \omega) \xi_\tau(0, \omega) d\mu(\omega) = - \int_G f(0, \omega) \partial_j \xi_\tau(0, \omega) d\mu(\omega)
\]

\[
= 2\pi i \, y_j(\xi) \int_G f(0, \omega) \xi(\omega) d\mu(\omega),
\]

where we have used (3.20). From the above equation and the definition of the Fourier transform on groups we obtain (3.19), and the lemma is proved. \( \square \)

Now we are able to state the equivalence between the spaces \( \mathcal{H} \) and \( H^1_\gamma(G) \), which is to say, we have the following

**Theorem 3.5.** A function \( f \in \mathcal{H} \) if, and only if, \( f(0, \cdot) \in H^1_\gamma(G) \), and

\[
\|f\|_\mathcal{H} = \|f(0, \cdot)\|_{H^1_\gamma(G)}.
\]

**Proof.** 1. Let us first show that, if \( f \in \mathcal{H} \) then \( f \in H^1_\gamma(G) \). To follow we observe that

\[
\int_{G^\wedge} (1 + \gamma(\xi)^2) |\hat{f}(0, \xi)|^2 \, d\nu(\xi) = \int_{G^\wedge} |\hat{f}(0, \xi)|^2 \, d\nu(\xi)
\]

\[
+ \int_{G^\wedge} |2\pi i \, y(\xi) \hat{f}(0, \xi)|^2 \, d\nu(\xi)
\]

\[
= \int_{G^\wedge} |\hat{f}(0, \xi)|^2 \, d\nu(\xi) + \int_{G^\wedge} |\nabla_y \hat{f}(0, \xi)|^2 \, d\nu(\xi),
\]

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where we have used (3.19). Therefore, applying Plancherel theorem
\[
\int_{G^\wedge} (1+\gamma(\xi)^2)|\hat{f}(0,\xi)|^2\,d\nu(\xi) = \int_G |f(0,\omega)|^2\,d\mu(\omega) + \int_G |\nabla_y f(0,\omega)|^2\,d\mu(\omega) < \infty,
\]
and thus \(f(0,\cdot) \in H_1^\gamma(G)\).

2. Now, let \(f(x,\omega)\) be a stationary function, such that \(f(0,\cdot) \in H_1^\gamma(G)\), then we show that \(f \in \mathcal{H}\). Given a stationary function \(\zeta \in C^1(\mathbb{R}^d; L^2(G))\), applying the Palncherel theorem and polarization identity
\[
\int_{G^\wedge} \partial_j \zeta(0,\omega) \overline{f(0,\omega)}\,d\mu(\omega) = \int_{G^\wedge} \partial_j \overline{\zeta(0,\xi)} \hat{f}(0,\xi)\,d\nu(\xi)
\]
for \(j \in \{1,\ldots,d\}\). Due to (3.19), we may write
\[
\int_{G^\wedge} \partial_j \zeta(0,\omega) \overline{f(0,\omega)}\,d\mu(\omega) = \int_{G^\wedge} 2\pi i \, y_j(\xi) \overline{\zeta(0,\xi)} \hat{f}(0,\xi)\,d\nu(\xi)
\]
\[
= -\int_{G^\wedge} \zeta(0,\xi) 2\pi i \, y_j(\xi) \hat{f}(0,\xi)\,d\nu(\xi).
\]
(3.21)

For \(j \in \{1,\ldots,d\}\) we define, \(g_j(\omega) := (2\pi i \, y_j(\xi) \hat{f}(0,\xi))^\vee\), then \(g_j \in L^2(G)\). Indeed, we have
\[
\int_G |g_j(\omega)|^2\,d\mu(\omega) = \int_{G^\wedge} |g_j(\xi)|^2\,d\nu(\xi) \leq \int_{G^\wedge} (1+\gamma(\xi)^2)|\hat{f}(0,\xi)|^2\,d\nu(\xi) < \infty.
\]
Therefore, we obtain from (3.21)
\[
\int_G \partial_j \zeta(0,\omega) \overline{f(0,\omega)}\,d\mu(\omega) = -\int_G \zeta(0,\omega) \overline{g_j(\omega)}\,d\mu(\omega)
\]
for any stationary function \(\zeta \in C^1(\mathbb{R}^d; L^2(G))\), and \(j \in \{1,\ldots,d\}\). Then \(f \in \mathcal{H}\) due to Theorem 1.7.

**Corollary 3.6.** Let \(f \in L^2_{\text{loc}}(\mathbb{R}^d; L^2(G))\) be a stationary function and \(\Phi\) a stochastic deformation. Then, \(f \circ \Phi^{-1} \in \mathcal{H}_\Phi\) if, and only if, \(f(0,\cdot) \in H_1^\gamma(G)\), and there exist constants \(C_1, C_2 > 0\), such that
\[
C_1 \|f \circ \Phi^{-1}\|_{\mathcal{H}_\Phi} \leq \|f(0,\cdot)\|_{H_1^\gamma(G)} \leq C_2 \|f \circ \Phi^{-1}\|_{\mathcal{H}_\Phi}.
\]

**Proof.** Follows from Theorem 3.5 and Remark 1.7. \(\square\)
4 Rellich–Kondrachov Theorem

The aim of this section is to characterize when the Sobolev space $H^1_\gamma(G)$ is compactly embedded in $L^2(G)$, written $H^1_\gamma(G) \subset \subset L^2(G)$, where $G$ is considered a compact Abelian group and $\gamma : G^\wedge \to [0, \infty)$ is given by (2.13). We observe that, $H^1_\gamma(G) \subset \subset L^2(G)$ is exactly the Rellich–Kondrachov Theorem on compact Abelian groups, which was established under some conditions on $\gamma$ in [7]. Nevertheless, as a byproduct of the characterization established here, we provide the proof of this theorem in a more precise context.

To start the investigation, let $(G, \mu)$ be a probability space and consider the operator $T : L^2(G^\wedge) \to L^2(G^\wedge)$, defined by

$$[T(f)](\xi) := \frac{f(\xi)}{\sqrt{1 + \gamma(\xi)^2}}.$$  \hspace{1cm} (4.22)

We remark that, $T$ as defined above is a bounded linear, $(\|T\| \leq 1)$, self-adjoint operator, which is injective and satisfies for each $f \in L^2(G^\wedge)$

$$\int_{G^\wedge} (1 + \gamma(\xi)^2) |[T(f)](\xi)|^2 d\nu(\xi) = \int_{G^\wedge} |f(\xi)|^2 d\nu(\xi).$$  \hspace{1cm} (4.23)

Moreover, a function $f \in H^1_\gamma(G)$ if, and only if, $\hat{f} \in T(L^2(G^\wedge))$, that is to say

$$f \in H^1_\gamma(G) \iff \hat{f} \in T(L^2(G^\wedge)).$$  \hspace{1cm} (4.24)

Indeed, if $f \in H^1_\gamma(G)$ then, we have $f \in L^2(G)$ and

$$\int_{G^\wedge} (1 + \gamma(\xi)^2) |\hat{f}(\xi)|^2 d\nu(\xi) = \int_{G^\wedge} |\sqrt{1 + \gamma(\xi)^2} \hat{f}(\xi)|^2 d\nu(\xi) < \infty.$$

Therefore, defining $g(\xi) := \sqrt{1 + \gamma(\xi)^2} \hat{f}(\xi)$, hence $g \in L^2(G^\wedge)$ and we have $\hat{f} \in T(L^2(G^\wedge))$.

Now, if $\hat{f} \in T(L^2(G^\wedge))$ let us show that, $f \in H^1_\gamma(G)$. First, there exists $g \in L^2(G^\wedge)$ such that, $\hat{f} = T(g)$. Thus from equation (4.23), we obtain

$$\int_{G^\wedge} (1 + \gamma(\xi)^2) |\hat{f}(\xi)|^2 d\nu(\xi) = \int_{G^\wedge} |g(\xi)|^2 d\nu(\xi) < \infty,$$

that is, by definition $f \in H^1_\gamma(G)$.

Then we have the following Equivalence Theorem:
Theorem 4.1. The Sobolev space $H^1_\gamma(G)$ is compactly embedded in $L^2(G)$ if, and only if, the operator $T$ defined by (4.22) is compact.

Proof. 1. First, let us assume that $H^1_\gamma(G) \subset \subset L^2(G)$, and take a bounded sequence $\{f_m\}$, $f_m \in L^2(G^\wedge)$ for each $m \in \mathbb{N}$. Thus $T(f_m) \in L^2(G^\wedge)$, and defining $g_m := T(f_m)^\wedge$, we obtain by Plancherel Theorem that $g_m \in L^2(G)$ for each $m \in \mathbb{N}$. Moreover, from equation (4.23), we have for any $m \in \mathbb{N}$

\[
\int_{G^\wedge} |f_m(\xi)|^2 \, d\nu(\xi) = \int_{G^\wedge} (1 + \gamma(\xi)^2) \|T(f_m)(\xi)\|^2 \, d\nu(\xi)
\]

\[
= \int_{G^\wedge} (1 + \gamma(\xi)^2) |\widehat{g_m}(\xi)|^2 \, d\nu(\xi).
\]

Therefore, the sequence $\{g_m\}$ is uniformly bounded in $H^1_\gamma(G)$, with respect to $m \in \mathbb{N}$. By hypothesis there exists a subsequence of $\{g_m\}$, say $\{g_{m_j}\}$, and a function $g \in L^2(G)$ such that, $g_{m_j}$ converges strongly to $g$ in $L^2(G)$ as $j \to \infty$. Consequently, we have

\[T(f_{m_j}) = \widehat{g_{m_j}} \to \widehat{g} \quad \text{in} \ L^2(G^\wedge) \quad \text{as} \quad j \to \infty,
\]

that is, the operator $T$ is compact.

2. Now, let us assume that the operator $T$ is compact and then show that $H^1_\gamma(G) \subset \subset L^2(G)$. To this end, we take a sequence $\{f_m\}_{m \in \mathbb{N}}$ uniformly bounded in $H^1_\gamma(G)$. Then, due to the equivalence (4.24) there exists for each $m \in \mathbb{N}$, $g_m \in L^2(G^\wedge)$, such that $\widehat{f_m} = T(g_m)$. Thus for any $m \in \mathbb{N}$, we have from equation (4.23) that

\[
\int_{G^\wedge} |g_m(\xi)|^2 \, d\nu(\xi) = \int_{G^\wedge} (1 + \gamma(\xi)^2) \|T(g_m)(\xi)\|^2 \, d\nu(\xi)
\]

\[
= \int_{G^\wedge} (1 + \gamma(\xi)^2) |\widehat{f_m}(\xi)|^2 \, d\nu(\xi) < \infty.
\]

Then, the sequence $\{g_m\}$ is uniformly bounded in $L^2(G)$. Since the operator $T$ is compact, there exist $\{m_j\}_{j \in \mathbb{N}}$ and $g \in L^2(G^\wedge)$, such that

\[{\widehat{f_{m_j}}} = T(g_{m_j}) \underset{j \to \infty}{\longrightarrow} g \quad \text{in} \ L^2(G^\wedge).
\]

Consequently, the subsequence $\{f_{m_j}\}$ converges to $g^\wedge$ strongly in $L^2(G)$, and thus $H^1_\gamma(G)$ is compactly embedded in $L^2(G)$.
Remark 4.2. Due to Theorem 4.1 the compactness characterization, that is $H_1^1(G) \subset \subset L^2(G)$, follows once we show the conditions that the operator $T$ is compact. The study of the dual space of $G$, i.e. $G^\wedge$, and $\gamma$ it will be essential for this characterization.

Recall from (2.11) item (ii) that, $G^\wedge$ is discrete since $G$ is compact. Then, $\nu$ is a countermeasure, and $\nu(\{\chi\}) = 1$ for each singleton $\{\chi\}, \chi \in G^\wedge$. Now, for any $\chi \in G^\wedge$ fixed, we define the point mass function at $\chi$ by

$$\delta_\chi(\xi) := 1_{\{\chi\}}(\xi), \quad \text{for each } \xi \in G^\wedge.$$ 

Hence the set $\{\delta_\xi; \xi \in G^\wedge\}$ is an orthonormal basis for $L^2(G^\wedge)$. Indeed, we first show the orthonormality. For each $\chi, \pi \in G^\wedge$, we have

$$\langle \delta_\chi, \delta_\pi \rangle_{L^2(G^\wedge)} = \int_{G^\wedge} \delta_\chi(\xi) \delta_\pi(\xi) \, d\nu(\xi) = \begin{cases} 1, & \text{if } \chi = \pi, \\ 0, & \text{if } \chi \neq \pi. \end{cases} \quad (4.25)$$

Now, let us show the density, that is $\{\delta_\xi; \xi \in G^\wedge\} = L^2(G^\wedge)$, or equivalently $\{\delta_\xi; \xi \in G^\wedge\}^\perp = \{0\}$. For any $w \in \{\delta_\xi; \xi \in G^\wedge\}^\perp$, we obtain

$$0 = \langle \delta_\xi, w \rangle_{L^2(G^\wedge)} = \int_{G^\wedge} \delta_\xi(\chi) w(\chi) \, d\nu(\chi) = \int_{\{\xi\}} w(\chi) \, d\nu(\chi) = w(\xi)$$

for any $\xi \in G^\wedge$, which proves the density.

From the above discussion, it is important to study the operator $T$ on elements of the set $\{\delta_\xi; \xi \in G^\wedge\}$. Then, we have the following

**Theorem 4.3.** If the operator $T$ defined by (4.22) is compact, then $G^\wedge$ is an enumerable set.

**Proof.** 1. First, let $\{\delta_\xi; \xi \in G^\wedge\}$ be the orthonormal basis for $L^2(G^\wedge)$, and $T$ the operator defined by (4.22). Then, the function $\delta_\xi \in L^2(G^\wedge)$ is an eigenfunction of $T$ corresponding to the eigenvalue $(1 + \gamma^2)^{-1/2}$, that is $\delta_\xi \neq 0$, and

$$T(\delta_\xi) = \frac{\delta_\xi}{\sqrt{1 + \gamma(\xi)^2}}. \quad (4.26)$$

2. Now, since $T$ is compact and $\{\delta_\xi; \xi \in G^\wedge\}$ is a basis for $L^2(G^\wedge)$, it must be enumerable from (4.26). On the other hand, the function $\xi \in G^\wedge \mapsto \delta_\xi \in L^2(G^\wedge)$ is injective, hence $G^\wedge$ is enumerable. \qed
Corollary 4.4. If the operator $T$ defined by (4.22) is compact, then $L^2(G)$ is separable.

Proof. First, the Hilbert space $L^2(G^\wedge)$ is separable, since $\{\delta_\xi : \xi \in G^\wedge\}$ is an enumerable orthonormal basis of it. Then, the proof follows applying the Plancherel Theorem. \hfill \Box

Corollary 4.5. Let $G_B$ be the Bohr compactification of $\mathbb{R}^n$. Then $H^1_1(G_B)$ is not compactly embedded in $L^2(G_B)$.

Proof. Indeed, $G^\wedge_B$ is non enumerable. \hfill \Box

Consequently, $G^\wedge$ be enumerable is a necessarily condition for the operator $T$ be compact, which is not sufficient as shown by the Example 4.8 below. Indeed, it might depend on the chosen $\gamma$, see also Example 4.11.

To follow, we first recall the

Definition 4.6. Let $G$ be a group (not necessarily a topological one) and $S$ a subset of it. The smallest subgroup of $G$ containing every element of $S$, denoted $\langle S \rangle$, is called the subgroup generated by $S$. Equivalently, see Dummit, Foote [5] p.63,

$$\langle S \rangle = \{g_1^{\varepsilon_1}g_2^{\varepsilon_2}\ldots g_k^{\varepsilon_k} \mid k \in \mathbb{N} \text{ and for each } j, g_j \in S, \varepsilon_j = \pm 1\}.$$ 

Moreover, if a group $G = \langle S \rangle$, then $S$ is called a generator of $G$, and in this case when $S$ is finite, $G$ is called finitely generated.

Theorem 4.7. If the operator $T$ defined by (4.22) is compact and there exists a generator of $G^\wedge$ such that $\gamma$ is bounded on it, then $G^\wedge$ is finite generated.

Proof. Let $S_0$ be a generator of $G^\wedge$, such that $\gamma$ is bounded on it. Therefore, there exists $d_0 \geq 0$ such that,

for each $\xi \in S_0$, $\gamma(\xi) \leq d_0$.

Now, since $T$ is compact and $\|T\| \leq 1$, there exists $0 < c \leq 1$ such that, the set of eigenvectors

$$\left\{\delta_\xi : \xi \in G^\wedge \text{ and } \frac{1}{\sqrt{1 + \gamma(\xi)^2}} \geq c\right\} \equiv \left\{\delta_\xi : \xi \in G^\wedge \text{ and } \gamma(\xi) \leq \sqrt{\frac{1}{c^2} - 1}\right\}$$
is finite, where we have used the Spectral Theorem for bounded compact operators. Therefore, since
\[ \{ \delta_\xi ; \xi \in S_0 \} \subset \{ \delta_\xi ; \xi \in G^\wedge \text{ and } \gamma(\xi) \leq d_0 \} \]
it follows that \( S_0 \) is a finite set, and thus \( G^\wedge \) is finite generated.

**Example 4.8** (Infinite enumerable Torus). Let us recall the Sobolev space \( H_1^1(T^N) \), where \( T^N \) is the infinite enumerable Torus. We claim that: \( H_1^1(T^N) \) is not compactly embedded in \( L^2(T^N) \), for \( \gamma \) defined in Example 4.4. Indeed, given \( k \in \mathbb{N} \) we define \( 1_k \in \mathbb{Z}^N \), such that it is zero for any coordinate \( \ell \neq k \), and one in \( k \)-coordinate. Therefore, the set
\[ S_0 := \{ \xi_k ; k \in \mathbb{N} \} \]
is an infinite generator of the dual group \((T^N)^\wedge\). Since for each \( k \in \mathbb{N} \), \( \gamma(\xi_k) = 1 \), i.e. bounded in \( S_0 \), applying Theorem 4.7 it follows that \( H_1^1(T^N) \) is not compactly embedded in \( L^2(T^N) \).

**Remark 4.9.** The above discussion in the Example 4.8 follows as well to the Sobolev space \( H_1^1(T^I) \), where \( I \) is an index set (enumerable or not). Clearly, the Sobolev space \( H_1^1(T^I) \) is not compactly embedded in \( L^2(T^I) \), when \( I \) is a non enumerable index set. Indeed, the set \((T^I)^\wedge\) is non enumerable.

Now, we characterize the condition on \( \gamma : G^\wedge \to [0, \infty) \), in order to \( T \) be compact. More precisely, let us consider the following property:

\[ \text{C.} \quad \text{For each } d > 0, \text{ the set } \{ \xi \in G^\wedge ; \gamma(\xi) \leq d \} \text{ is finite.} \quad \text{(4.27)} \]

**Theorem 4.10.** If \( \gamma : G^\wedge \to [0, \infty) \) satisfies \( \text{C} \), then the operator \( T \) defined by (4.22) is compact.

**Proof.** By hypothesis, \( \{ \xi \in G^\wedge ; \gamma(\xi) \leq d \} \) is finite, then we have
\[ G^\wedge = \bigcup_{k \in \mathbb{N}} \{ \xi \in G^\wedge ; \gamma(\xi) \leq k \} . \]
Consequently, the set \( G^\wedge \) is enumerable and we may write \( G^\wedge = \{ \xi_i \}_{i \in \mathbb{N}} \).

Again, due to condition \( \text{C} \) for each \( c \in (0, 1) \) the set
\[ \left\{ \xi \in G^\wedge ; \frac{1}{\sqrt{1 + \gamma(\xi)^2}} \geq c \right\} \quad \text{(4.28)} \]
is finite. Since the function $\xi \in G^\wedge \mapsto \delta_\xi \in L^2(G^\wedge)$ is injective, the set $\{\delta_\xi ; \ i \in \mathbb{N}\}$ is an enumerable orthonormal basis of eigenvectors for $T$, which corresponding eigenvalues satisfy

$$\lim_{i \to \infty} \frac{1}{1 + \gamma(\xi_i)^2} = 0,$$

where we have used (4.28). Consequently, $T$ is a compact operator.

**Example 4.11** (Bis: Infinite enumerable Torus). There exists a function $\gamma_0$ such that, $H^1_{\gamma_0}(\mathbb{T}^N)$ is compactly embedded in $L^2(\mathbb{T}^N)$. Indeed, we are going to show that, $\gamma_0$ satisfies $\mathcal{C}$. Let $\alpha \equiv (\alpha_\ell)_{\ell \in \mathbb{N}}$ be a sequence in $\mathbb{R}^N$, such that for each $\ell \in \mathbb{N}$, $\alpha_\ell \geq 0$ and

$$\lim_{\ell \to \infty} \alpha_\ell = +\infty. \quad (4.29)$$

Then, we define the following pseudo-metric in the dual group $(\mathbb{T}^N)^\wedge$ as follows

$$p_0(\xi_m, \xi_n) := 2\pi \sum_{\ell=1}^{\infty} \alpha_\ell \left| m_\ell - n_\ell \right|, \quad (m, n \in \mathbb{Z}_c^N),$$

and consider $\gamma_0(\xi_m) = p_0(\xi_m, 1)$. Thus for each $d > 0$, the set

$$\{m \in \mathbb{Z}_c^N ; \ \gamma_0(\xi_m) \leq d\} \ \text{is finite.}$$

Indeed, from (4.29) there exists $\ell_0 \in \mathbb{N}$, such that $\alpha_\ell > d$, for each $\ell \geq \ell_0$. Therefore, if $m \in \mathbb{Z}_c^N$ and the support of $m$ is not contained in $\{1, \ldots, \ell_0-1\}$, that is to say, there exists $\tilde{\ell} \geq \ell_0$, such that, $m_{\tilde{\ell}} \neq 0$. Then,

$$2\pi \sum_{\ell=1}^{\infty} \alpha_\ell \left| m_\ell \right| \geq \alpha_{\tilde{\ell}} > d.$$ 

Consequently, we have

$$\{m \in \mathbb{Z}_c^N ; \ \gamma_0(\xi_m) \leq d\} \subset \{m \in \mathbb{Z}_c^N ; \ \text{supp} \ m \subset \{1, \ldots, \ell_0 - 1\}\},$$

which is a finite set. Finally, applying Theorem 4.10 we obtain that, the Sobolev space $H^1_{\gamma_0}(\mathbb{T}^N)$ is compactly embedded in $L^2(\mathbb{T}^N)$.
4.1 Application. On a class of quasiperiodic functions

In this section we consider the important class of quasiperiodic functions (see [9]), which includes the class of $[0, 1)^n$—periodic functions.

Let $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}^n$ be $m$—linear independent vectors with respect to $\mathbb{Z}$, and consider the following matrix

$$\Lambda := \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{pmatrix}_{m \times n}$$

such that, for each $d > 0$ the set

$$\{ k \in \mathbb{Z}^m : |\Lambda^T k| \leq d \} \text{ is finite.}$$

(4.30)

Therefore, we are considering the class of quasiperiodic functions satisfying condition (4.30). This set is not empty, for instance let us define the matrix $B := \Lambda \Lambda^T$, such that $\det B > 0$, which is called here positive quasi-periodic functions. It is not difficult to see that, positive quasiperiodic functions satisfies (4.30). Indeed, it is sufficiently to observe that, for each $k \in \mathbb{Z}^m$, we have

$$|k| = |B^{-1}Bk| \leq \|B^{-1}\|\|\Lambda\|\|\Lambda^T k|.$$

Moreover, since $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{R}^n$ are $m$—linear independent vectors with respect to $\mathbb{Z}$, (this property does not imply $\det B > 0$), the dynamical system

$$\tau : \mathbb{R}^n \times \mathbb{T}^m \to \mathbb{T}^m,$$

given by

$$\tau(x)\omega := \omega + \Lambda x - [\omega + \Lambda x]$$

(4.31)

is ergodic.

Now we remark that, the application $\varphi : \mathbb{R}^n \to \mathbb{T}^m$, $\varphi(x) := \Lambda x - [\Lambda x]$, is a continuous homeomorphism of groups. Then, we have

$$\tau(x)\omega = \varphi(x)\omega \equiv \omega + \Lambda x - [\omega + \Lambda x].$$

Consequently, under the conditions of the previous sections, we obtain for each $k \in \mathbb{Z}^m$, $\gamma(\xi_k) = 2\pi|\Lambda^T k|$, and applying Theorem 4.10 (recall (4.30)), it follows that

$$H_{\gamma}^1(\mathbb{T}^m) \subset L^2(\mathbb{T}^m).$$

Therefore, given a stochastic deformation $\Phi$, we have $\mathcal{H}_\Phi \subset \mathcal{L}_\Phi$ for the class of quasiperiodic functions satisfying (4.30), and it follows a solution of the Bloch’s spectral cell equation.
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