Distribution function of the endpoint fluctuations of one-dimensional directed polymers in a random potential

Victor Dotsenko

LPTMC, Université Paris VI, F-75252 Paris, France
and
LD Landau Institute for Theoretical Physics, 119334 Moscow, Russia
E-mail: dotsenko@lptmc.jussieu.fr

Received 11 October 2012
Accepted 13 December 2012
Published 5 February 2013

Online at stacks.iop.org/JSTAT/2013/P02012
doi:10.1088/1742-5468/2013/02/P02012

Abstract. The explicit expression for the probability distribution function of the endpoint fluctuations of one-dimensional directed polymers in a random potential is derived in terms of the Bethe ansatz replica technique by mapping the replicated problem to the N-particle quantum boson system with attractive interactions.

Keywords: rigorous results in statistical mechanics, thermodynamic Bethe ansatz, disordered systems (theory)
1. Introduction

One-dimensional directed polymers in a quenched random potential and the equivalent problem of the solutions of the KPZ equation [1] describing the growth in time of an interface in the presence of noise have been the subject of intense investigations during the past two decades (see, e.g., [2]–[14]). The model of directed polymers describes an elastic string directed along the $\tau$-axis within an interval $[0,t]$. Randomness enters the problem through a disorder potential $V[\phi(\tau), \tau]$, which competes against the elastic energy. The system is defined by the Hamiltonian

$$H[\phi(\tau), V] = \int_{0}^{t} d\tau \left\{ \frac{1}{2} [\partial_{\tau} \phi(\tau)]^2 + V[\phi(\tau), \tau] \right\};$$ (1)

where the disorder potential $V[\phi, \tau]$ is Gaussian distributed with a zero mean $\overline{V(\phi, \tau)} = 0$ and the $\delta$-correlations

$$\overline{V(\phi, \tau)V(\phi', \tau') } = u \delta(\tau - \tau') \delta(\phi - \phi').$$ (2)

Here the parameter $u$ describes the strength of the disorder.

In what follows we consider the problem in which the polymer is fixed at the origin, $\phi(0) = 0$, and it is free at $\tau = t$. In other words, for a given realization of the random potential $V$ the partition function of the considered system is

$$Z = \int_{-\infty}^{+\infty} dx \; Z(x) = \exp\{-\beta F\}$$ (3)

where

$$Z(x) = \int_{\phi(0)=0}^{\phi(t)=x} D\phi(\tau) e^{-\beta H[\phi]}$$ (4)

doi:10.1088/1742-5468/2013/02/P02012
is the partition function of the system with the fixed boundary condition, $\phi(t) = x$ and $F$ is the total free energy. Besides the usual extensive part $f_0 t$ (where $f_0$ is the linear free energy density), the total free energy $F$ of such a system is known to contain the disorder dependent fluctuating contribution, which in the limit of large $t$ scales as $t^{1/3}$ (see, e.g., [5]–[8]). In other words, in the limit of large $t$, the total (random) free energy of the system can be represented as $F = f_0 t + c t^{1/3} f$, where $c$ is a non-universal parameter, which depends on the temperature and the strength of disorder, and $f$ is the random quantity which in the thermodynamic limit $t \to \infty$ is described by a non-trivial universal distribution function $P(f)$. The trivial self-averaging contribution $f_0 t$ to the free energy can be eliminated from further study by the simple redefinition of the partition function, $Z = \exp\{-\beta f_0 t\} \tilde{Z}$, so that $\tilde{Z} = \exp\{-\lambda f\}$, where $\lambda = \beta c t^{1/3}$. Thus, to simplify notations the contribution $f_0 t$ will be just dropped out from the further calculations.

For the problem with the zero boundary conditions, $\phi(0) = \phi(t) = 0$, the distribution function $P(f)$ was demonstrated to be described by the Gaussian unitary ensemble (GUE) Tracy–Widom distribution [15]–[18]. On the other hand, the free energy distribution function of the directed polymers with the free boundary conditions, equations (1)–(4), was shown to be given by the Gaussian orthogonal ensemble (GOE) Tracy–Widom distribution [19, 20] In the course of these derivations a rather efficient Bethe ansatz replica technique has been developed [17]–[20]. Here in terms of this technique we are going to study the statistical properties of the transverse fluctuations of the directed polymers. The scaling properties of the typical value of the endpoint deviations, $\phi(t)$, at large times is well known: $\langle \phi(t)^2 \rangle \propto t^{4/3}$ (here $\langle \cdots \rangle$ denotes the thermal average and $\{ \cdots \}$ is the average over the disorder potential, equation (2)) [5]–[8]. A much more interesting object is the probability distribution function $P(x)$ for the rescaled quantity $x = \phi(t)/t^{2/3}$, which is expected to become a universal function in the limit $t \to \infty$. Recently, this function has been derived in terms of the so called maximal point of the Airy$_2$ process minus a parabola [21]–[23], which is believed to describe the scaling limit of the endpoint of the directed polymers in a random potential. The long-standing conjecture that the top line of the Airy line ensemble minus a parabola attains its maximum at a unique point was recently proved in [24]. The obtained explicit expression for $P(x)$ turned out to be rather complicated and its analytic properties are not so easy to analyze, although the asymptotic behavior of this function is already known: $P(x \to \infty) \sim \exp\{-|x|^{9/12}\}$ [22].

In this work, the explicit form of the distribution function of the directed polymer’s endpoint fluctuations will be derived in terms of the Bethe ansatz replica technique. The distribution function we are going to consider is defined as follows:

$$W(x) = \lim_{t \to \infty} \text{Prob}[\phi(t)t^{-2/3} > x] = \int_x^\infty dx' P(x').$$

This function gives the probability that the rescaled value of the polymer’s right endpoint $\phi(t)/t^{2/3}$ is bigger than a given value $x$. In this paper it will be shown that (see equations (79)–(84) below)

$$W(x) = \int_{-\infty}^{+\infty} df F_1(-f) \int_0^{+\infty} d\omega \int_0^{+\infty} d\omega' (\hat{B}_{-f})^{-1}(\omega, \omega') \Phi(\omega', \omega; f, x).$$

Here $\hat{B}_{-f}$ is the integral operator with the kernel $B_{-f}(\omega, \omega') = Ai(\omega + \omega' - f)(\omega, \omega' > 0)$, the function $F_1(-f) = \det[\hat{1} - \hat{B}_{-f}]$ is the GOE Tracy–Widom distribution and
Directed polymer’s endpoint fluctuation distribution function

$$(\hat{1} - \hat{B}_f)^{-1}(\omega, \omega')$$ denotes the kernel of the inverse operator $(\hat{1} - \hat{B}_f)^{-1}$ in $\omega$ and $\omega'$ (note that, since $F_1(-f) > 0$ for all real $f$, the operator $(\hat{1} - \hat{B}_f)$ is invertible). The function $\Phi(\omega', \omega; f, x)$ is defined as follows:

$$\Phi(\omega', \omega; f, x) = -\frac{1}{2} \int_0^{+\infty} dy \left[ \left( \frac{\partial}{\partial \omega} - \frac{\partial}{\partial \omega'} \right) \Psi \left( \omega - \frac{1}{2} f + y; x \right) \Psi \left( \omega' - \frac{1}{2} f + y; -x \right) \right]$$

where

$$\Psi(\omega; x) = 2^{1/3} \text{Ai} \left[ 2^{1/3} (\omega + \frac{1}{8} x^2) \right] \exp \left\{ -\frac{1}{2} \omega x \right\}. \tag{8}$$

The above result looks quite similar to the one obtained in [21], although at the moment I am not able to provide the proof that these results are indeed the same\(^1\). In any case, the above expressions, equations (6)–(8), for the probability function $P(x)$ look as complicated as the ones obtained in [21]–[23], and for the moment its analytic properties are not clear.

The paper is organized as follows. In section 2 we define the distribution function $W(x)$ via the two-point free energy distribution function $V_x(f_1, f_2)$, which gives the probability that the free energy of the polymer with the endpoint located above a position $x$ is higher than a given value $f_1$, while the free energy of the polymer with the endpoint located below the position $x$ is higher than a given value $f_2$. In section 3 the function $V_x(f_1, f_2)$ is defined by mapping the considered problem to the one-dimensional $N$-particle system of quantum bosons with attractive $\delta$-interactions. In section 4 the explicit expression for the probability function $V_x(f_1, f_2)$ is obtained in terms of the Bethe ansatz replica technique. Finally, in section 5 the result equations (6)–(8) is derived. Conclusions and future perspectives are discussed in section 6.

2. The endpoint probability distribution function

In terms of the partition function $Z(x)$, equation (4), the probability distribution function of the polymer’s endpoint $W(x)$, equation (5), can be defined as follows:

$$W(x) = \lim_{t \to \infty} \left( \frac{\int_0^{+\infty} dx' \frac{Z(x')}{\int_0^{+\infty} dx' Z(x')}}{\int_{-\infty}^{+\infty} dx' Z(x')} \right)$$

$$= \lim_{t \to \infty} \left( \frac{Z^+(x)}{Z^+(x) + Z^-(x)} \right) \tag{9}$$

where

$$Z^-(x) \equiv \int_{-\infty}^{x} dx' Z(x') = \exp \{-\lambda f_-(x)\} \tag{10}$$

$$Z^+(x) \equiv \int_{x}^{+\infty} dx' Z(x') = \exp \{-\lambda f_+(x)\} \tag{11}$$

\(^1\) After acceptance of this paper I have learned that the equivalence of the result of this work, equations (6)–(8), and that of references [21]–[23] has been proved in a recent paper [31].
with the parameter $\lambda \propto t^{1/3}$ and $f(z)$ the free energies of the polymers with the endpoint $\phi(t)$ located correspondingly above and below a given position $x$. According to these definitions we find

$$W(x) = \lim_{\lambda \to \infty} \frac{\exp\{-\lambda f(+)\}}{\exp\{-\lambda f(-)\} + \exp\{-\lambda f(+)\}} = \begin{cases} 0, & \text{for } f(-) < f(+) \\ 1, & \text{for } f(-) > f(+). \end{cases} \quad (12)$$

Let us introduce the joint probability density function $P_x[f(+) ; f(-)]$. By definition, the quantity $P_x[f(+) ; f(-)] \, df(+) \, df(-)$ gives the probability that the free energy of the polymer with the endpoint located below $x$ is equal to $f(-)$ (within the interval $df(-)$), while the free energy of the polymer with the endpoint located above $x$ is equal to $f(+)$(within the interval $df(+)$. Thus, according to equation (12),

$$W(x) = \int_{-\infty}^{+\infty} df(-) \int_{f(+)}^{+\infty} df(+) P_x[f(+) ; f(-)]. \quad (13)$$

Let us introduce one more joint probability distribution function:

$$V_x(f_1, f_2) = \text{Prob}[f(+) > f_1; f(-) > f_2] = \int_{f_1}^{+\infty} df(+) \int_{f_2}^{+\infty} df(-) P_x[f(+) ; f(-)]. \quad (14)$$

This two-point free energy distribution function gives the probability that the free energy of the polymer with the endpoint located above the position $x$ is higher than a given value $f_1$, while the free energy of the polymer with the endpoint located below the position $x$ is higher than a given value $f_2$. According to this definition,

$$P_x[f_1; f_2] = \frac{\partial}{\partial f_1} \frac{\partial}{\partial f_2} V_x(f_1, f_2). \quad (15)$$

Substituting this relation into equation (13) we find

$$W(x) = \int_{-\infty}^{+\infty} df_1 \int_{f_1}^{+\infty} df_2 \frac{\partial}{\partial f_1} \frac{\partial}{\partial f_2} V_x(f_1, f_2). \quad (16)$$

Integrating by parts over $f_2$ and taking into account that $V_x(f_1, f_2)|_{f_2=+\infty} = 0$, we get

$$W(x) = -\int_{-\infty}^{+\infty} df_1 \left( \frac{\partial}{\partial f_1} V_x(f_1, f_2) \right) \bigg|_{f_2=f_1+0}. \quad (17)$$

Thus, to get the distribution function $W(x)$ for the polymer’s endpoint fluctuations, we have to derive the two-point free energy distribution function $V_x(f_1, f_2)$ first. Note that this function is different from the two-point free energy distribution function derived in [25], which describes joint statistics of the free energies of the directed polymers coming to two different endpoints.

3. Mapping to quantum bosons

According to definition (14), the probability distribution function $V_x(f_1, f_2)$ can be defined as follows:

$$V_x(f_1, f_2) = \lim_{\lambda \to \infty} \sum_{L=0}^{\infty} \sum_{R=0}^{\infty} \frac{(-1)^L (-1)^R}{L! R!} \exp(\lambda L f_1 + \lambda R f_2) |Z^{(+)}(x)|^L |Z^{(-)}(x)|^R. \quad (18)$$
Indeed, substituting here definitions (10) and (11), we find

$$V_x(f_1, f_2) = \lim_{\lambda \to \infty} \int_{-\infty}^{+\infty} \frac{d(1)^L}{L!} \int_{-\infty}^{+\infty} \frac{d(1)^R}{R!} \exp[\lambda L (f_1 - f_1^+)] \exp[\lambda R (f_2 - f_2^-)]$$

$$= \lim_{\lambda \to \infty} \int_{-\infty}^{+\infty} \frac{d(1)^L}{L!} \int_{-\infty}^{+\infty} \frac{d(1)^R}{R!} \exp[-\lambda (f_1 - f_1^+) - \lambda (f_2 - f_2^-)]$$

$$= \int_{-\infty}^{+\infty} \frac{d(1)^L}{L!} \int_{-\infty}^{+\infty} \frac{d(1)^R}{R!} \exp[-\lambda (f_1 - f_1^+) - \lambda (f_2 - f_2^-)]$$

where \( \theta(f) \) is the Heaviside step function. We see that the above representation coincides with definition (14).

Further calculations of the two-point distribution function \( V_x(f_1, f_2) \) to a large extent repeat the procedure described in detail in the previous paper [20] for the one-point free energy distribution function. Using definitions (10) and (11), the distribution function, equation (18), can be represented as follows:

$$V_x(f_1, f_2) = \lim_{\lambda \to \infty} \sum_{L, R = 0}^{\infty} \frac{(-1)^L + R}{L! R!} \exp(\lambda L f_1 + \lambda R f_2)$$

$$\times \int_{-\infty}^{+\infty} dx_1 \cdots dx_L \int_{-\infty}^{+\infty} dy_1 \cdots dy_R \Psi(x_1, \ldots, x_L; y_1, \ldots, y_1; t)$$

where

$$\Psi(x_1, \ldots, x_N; t) = \prod_{a=1}^{N} \left[ \int_{\phi_a(0) = 0}^{\phi_a(t) = x_a} D\phi_a(\tau) \right] \exp(-\beta H_N[\phi_1, \phi_2, \ldots, \phi_N])$$

with the replica Hamiltonian

$$H_N[\phi_1, \phi_2, \ldots, \phi_N] = \frac{1}{2} \int_{0}^{t} d\tau \left( \sum_{a=1}^{N} [\partial_\tau \phi_a(\tau)]^2 - \beta u \sum_{a \neq b}^{N} \delta[\phi_a(\tau) - \phi_b(\tau)] \right).$$

The propagator \( \Psi(x; t) \), equation (21), describes \( N \) trajectories \( \phi_a(\tau) \) all starting at zero \( (\phi_a(0) = 0) \), and coming to \( N \) different points \( \{x_1, \ldots, x_N\} \) at \( \tau = t \). One can easily show that \( \Psi(x; t) \) can be obtained as the solution of the imaginary-time Schrödinger equation

$$-\beta \partial_\tau \Psi(x; t) = \hat{H} \Psi(x; t)$$

with the initial condition

$$\Psi(x; 0) = \Pi_{a=1}^{N} \delta(x_a).$$

Here the Hamiltonian is

$$\hat{H} = -\frac{1}{2} \sum_{a=1}^{N} \partial_{x_a}^2 - \frac{1}{2} \kappa \sum_{a \neq b}^{N} \delta(x_a - x_b)$$

\( \text{doi}:10.1088/1742-5468/2013/02/P02012 \)
Directed polymer’s endpoint fluctuation distribution function

and the interaction parameter $\kappa = \beta^3 u$. This Hamiltonian describes $N$ Bose particles interacting via the attractive two-body potential $-\kappa \delta(x)$.

A generic eigenstate of such system is characterized by $N$ momenta $\{q_a\}$ ($a = 1, \ldots, N$) which are split into $M$ ($1 \leq M \leq N$) ‘clusters’ described by continuous real momenta $q_a$ ($\alpha = 1, \ldots, M$) and having $n_\alpha$ discrete imaginary ‘components’ (for details see [17], [26]–[30]):

$$q_a \equiv q^a_r = q_a - \frac{i\kappa}{2}(n_\alpha + 1 - 2r); \quad (r = 1, \ldots, n_\alpha)$$

(26)

with the global constraint

$$\sum_{\alpha=1}^{M} n_\alpha = N.$$  

(27)

A generic solution $\Psi(x, t)$ of the Schrödinger equation (23) with the initial conditions (24) can be represented in the form of the linear combination of the eigenfunctions $\Psi^{(M)}_q(x)$:

$$\Psi(x_1, \ldots, x_N; t) = \sum_{M=1}^{N} \frac{1}{M!} \left[ \int \mathcal{D}^{(M)}(q, n) \right] |C^{(M)}_q(q, n)|^2 \Psi^{(M)}_q(x) \Psi^{(M)}_q(x^0)$$

$$\times \exp \{-E^{(M)}_q(q, n)t\}$$

(28)

where we have introduced the notation

$$\int \mathcal{D}^{(M)}(q, n) \equiv \prod_{\alpha=1}^{M} \left[ \int_{-\infty}^{+\infty} \frac{dq_a}{2\pi} \sum_{n_a=1}^{\infty} \delta \left( \sum_{\alpha=1}^{M} n_\alpha, N \right) \right]$$

(29)

and $\delta(k, m)$ is the Kronecker symbol; note that the presence of this Kronecker symbol in the above equation allows us to extend the summations over $n_\alpha$s to infinity. Here (non-normalized) eigenfunctions are [17, 30]

$$\Psi^{(M)}_q(x) = \sum_{\mathcal{P}} \prod_{a<b} \left[ 1 + i\kappa \frac{\text{sgn}(x_a - x_b)}{q_p_a - q_p_b} \right] \exp \left[ \sum_{a=1}^{N} q_p_a x_a \right]$$

(30)

where the summation goes over $N!$ permutations $\mathcal{P}$ of $N$ momenta $q_a$, equation (26), over $N$ particles $x_a$; the normalization factor

$$|C^{(M)}_q(q, n)|^2 = \frac{\kappa^N}{N! \prod_{\alpha=1}^{M} (\kappa n_\alpha)} \prod_{\alpha<\beta} |q_\alpha - q_\beta - (i\kappa/2)(n_\alpha - n_\beta)|^2$$

(31)

and the eigenvalues

$$E^{(M)}_q(q, n) = \frac{1}{2\beta} \sum_{a=1}^{N} q_a^2 = \frac{1}{2\beta} \sum_{a=1}^{M} n_\alpha q_a^2 - \frac{\kappa^2}{24\beta} \sum_{a=1}^{M} (n_\alpha^3 - n_\alpha)$$

$$= \sum_{\alpha=1}^{M} \left[ \frac{1}{2\beta} n_\alpha q_a^2 - \frac{\kappa^2}{24\beta} n_\alpha^3 \right] + \frac{\kappa^2}{24\beta} N.$$  

(32)

The last term in the above expression provides just the trivial contribution to the self-averaging part of the free energy (discussed in section 1) and therefore it will be dropped out of the further calculations.

doi:10.1088/1742-5468/2013/02/P02012
This is done in the standard way by introducing a supplementary parameter $\gamma$ the group $\{P \cup \lambda \}$ left particles Here the summation over all permutations of the $N$-particle bosonic problem, which is parametrized by the set of both continuous, \{q_1, \ldots, q_M\}, and discrete, \{n_1, \ldots, n_M\}; $M = 1, \ldots, N; N = 1, \ldots, \infty$, degrees of freedom.

4. Two-point free energy distribution function

Substituting equations (28)–(33) into (20), we get

$$V_\varepsilon(f_1, f_2) = 1 + \lim_{\lambda \to \infty} \sum_{L+R \geq 1} \left( -1 \right)^{L+R} e^{\lambda L f_1 + \lambda R f_2}$$

$$\times \sum_{M=1}^{L+R} \frac{1}{M!} \prod_{\alpha=1}^{M} \left[ \sum_{n_\alpha=1}^{\infty} \int_{-\infty}^{+\infty} \frac{dq_\alpha}{2\pi n_\alpha} e^{-\frac{(t/2)\beta}{(\beta/2)} n_\alpha q_\alpha^2 + (\beta/24) n_\alpha^2} \right]$$

$$\times \delta \left( \sum_{\alpha=1}^{M} n_\alpha, L + R \right) |\tilde{C}_M(q, n)|^2 I_{L,R}(q, n)$$

where

$$|\tilde{C}_M(q, n)|^2 = \prod_{\alpha<\beta} \left| \frac{q_\alpha - q_\beta - (i\kappa/2)(n_\alpha - n_\beta)}{q_\alpha - q_\beta - (i\kappa/2)(n_\alpha + n_\beta)} \right|^2$$

and

$$I_{L,R}(q, n) = \sum_{P(L) R} \sum_{P(L)} \sum_{P(R)} \prod_{a=1}^{L} \prod_{c<d}^{R} \left[ \frac{q_{P_a}^{(L)} - q_{P_c}^{(R)} - i\kappa}{q_{P_a}^{(L)} - q_{P_c}^{(R)}} \right]$$

$$\times \prod_{a < c} \left[ \frac{q_{P_a}^{(L)} - q_{P_c}^{(L)} + i\kappa}{q_{P_a}^{(L)} - q_{P_c}^{(L)}} \right] \prod_{c < d}^{R} \left[ \frac{q_{P_c}^{(R)} - q_{P_d}^{(R)} - i\kappa}{q_{P_c}^{(R)} - q_{P_d}^{(R)}} \right]$$

$$\times \int_{-\infty}^{x_1} \cdots dx_L \exp \left[ \sum_{a=1}^{L} (q_{P_a}^{(L)} - i\epsilon) x_a \right]$$

$$\times \int_{x_L} y_{R} \cdots y_1 \exp \left[ \sum_{c=1}^{R} (q_{P_c}^{(R)} + i\epsilon) y_c \right].$$

Here the summation over all permutations $P$ of $(L + R)$ momenta $\{q_1, \ldots, q_{L+R}\}$ over $L$ ‘left’ particles $\{x_1, \ldots, x_L\}$ and $R$ ‘right’ particles $\{y_R, \ldots, y_1\}$ are divided into three parts: the permutations $P^{(L)}$ of $L$ momenta (taken at random out of the total list $\{q_1, \ldots, q_{L+R}\}$) over $L$ ‘left’ particles, the permutations $P^{(R)}$ of the remaining $R$ momenta over $R$ ‘right’ particles, and finally the permutations $P^{(L,R)}$ (or the exchange) of the momenta between the group ‘$L$’ and the group ‘$R$’. Note also that the integrations both over the $x_a$ and over the $y_c$ in equation (36) require proper regularization at $-\infty$ and $+\infty$ correspondingly. This is done in the standard way by introducing a supplementary parameter $\epsilon$, which will
be set to zero in final results. The result of the integrations can be represented as follows:

\[ I_{L,R}(\mathbf{q}, \mathbf{n}) = i^{-(L+R)} \exp \left\{ i \varepsilon \sum_{a=1}^{M} n_{a} \mathbf{q}_{a} \right\} \sum_{\mathcal{P}(L,R)} \prod_{a=1}^{L} \prod_{c=1}^{R} \frac{q_{P_{a}^{(L)}} - q_{P_{a}^{(L-1)}} - i \kappa}{q_{P_{a}^{(L-1)}} - q_{P_{a}^{(L-1)}}} \]

\[ \times \sum_{\mathcal{P}(L)} \prod_{a=1}^{L} (q_{P_{1}^{(L)}} q_{P_{2}^{(L)}} + q_{P_{2}^{(L)}} q_{P_{1}^{(L)}}) \cdots (q_{P_{1}^{(L)}} q_{P_{2}^{(L)}} + \cdots + q_{P_{1}^{(L)}} q_{P_{L}^{(L)}}) \prod_{a < b} \frac{1}{q_{P_{a}^{(L)}} q_{P_{b}^{(L)}}} \]

\[ \times \sum_{\mathcal{P}(R)} \prod_{a=1}^{R} (q_{P_{1}^{(R)}} q_{P_{2}^{(R)}} + q_{P_{2}^{(R)}} q_{P_{1}^{(R)}}) \cdots (q_{P_{1}^{(R)}} q_{P_{2}^{(R)}} + \cdots + q_{P_{1}^{(R)}} q_{P_{R}^{(R)}}) \prod_{c < d} \frac{1}{q_{P_{c}^{(R)}} q_{P_{d}^{(R)}}} \]

where

\[ q_{a}^{(\pm)} \equiv q_{a} \pm \varepsilon \]

and where we have used the fact that for any permutation of the momenta, equation (26), one has

\[ \sum_{a=1}^{L+R} q_{a} = \sum_{a=1}^{M} n_{a} q_{a}. \]

Using the Bethe ansatz combinatorial identity [19],

\[ \sum_{\mathcal{P}} \prod_{a=1}^{M} q_{P_{a}^{1}} q_{P_{a}^{2}} \cdots q_{P_{a}^{N}} \prod_{a < b} q_{P_{a}^{1}} q_{P_{b}^{1}} = \frac{1}{\prod_{a=1}^{N} q_{a}^{1}} \prod_{a < b} q_{a} q_{b} \]

(40)

(where the summation goes over all permutations \( \mathcal{P} \) of \( N \) momenta \( \{q_{1}, \ldots, q_{N}\} \)) we get

\[ I_{L,R}(\mathbf{q}, \mathbf{n}) = i^{-(L+R)} \exp \left\{ i \varepsilon \sum_{a=1}^{M} n_{a} \mathbf{q}_{a} \right\} \sum_{\mathcal{P}(L,R)} \prod_{a=1}^{L} \prod_{c=1}^{R} \frac{q_{P_{a}^{(L)}} - q_{P_{a}^{(L-1)}} - i \kappa}{q_{P_{a}^{(L-1)}} - q_{P_{a}^{(L-1)}}} \]

\[ \times \prod_{a=1}^{L} \prod_{a < b} \frac{1}{q_{P_{a}^{(L)}} q_{P_{b}^{(L)}}} \times \frac{(-1)^{R}}{\prod_{c=1}^{R} q_{P_{c}^{(R)}} q_{P_{d}^{(R)}}} \]

(41)

Further simplification comes from the following important property of the Bethe ansatz wave function, equation (30). It has such structure that for ordered particle positions (e.g. \( x_{1} < x_{2} < \cdots < x_{N} \)) in the summation over permutations the momenta \( q_{a} \) belonging to the same cluster also remain ordered. In other words, if we consider the momenta, equation (26), of a cluster \( \alpha \), \( \{q_{1}^{\alpha}, q_{2}^{\alpha}, \ldots, q_{n_{\alpha}}^{\alpha}\} \), belonging correspondingly to the particles \( \{x_{i_{1}} < x_{i_{2}} < \cdots < x_{i_{n_{\alpha}}}\} \), the permutation of any two momenta \( q_{a}^{\alpha} \) and \( q_{b}^{\alpha} \) of this ordered set gives zero contribution. Thus, in order to perform the summation over the permutations \( \mathcal{P}(L,R) \) in equation (41) it is sufficient to split the momenta of each cluster into two parts: \( \{q_{1}^{\alpha}, \ldots, q_{m_{\alpha}}^{\alpha} \parallel q_{m_{\alpha}+1}^{\alpha}, \ldots, q_{n_{\alpha}}^{\alpha}\} \), where \( m_{\alpha} = 0, 1, \ldots, n_{\alpha} \) and where the momenta

doi:10.1088/1742-5468/2013/02/P02012
\[ q_1^\alpha, \ldots, q_{m_\alpha}^\alpha \] belong to the particles of the sector ‘\( L \)’, while the momenta \( q_{m_\alpha + 1}^\alpha, \ldots, q_{n_\alpha}^\alpha \) belong to the particles of the sector ‘\( R \)’.

Let us introduce the numbering of the momenta of the sector ‘\( R \)’ in reverse order:

\[
\begin{align*}
q_{m_\alpha}^\alpha & \rightarrow q_1^\alpha \\
q_{m_\alpha - 1}^\alpha & \rightarrow q_2^\alpha \\
\ldots & \\
q_{m_\alpha + 1}^\alpha & \rightarrow q_{s_\alpha}^\alpha
\end{align*}
\]

(42)

where \( m_\alpha + s_\alpha = n_\alpha \) and (s.f. equation (26))

\[
q_{s_\alpha}^\alpha = q_\alpha + \frac{i\kappa}{2}(n_\alpha + 1 - 2r) = q_\alpha + \frac{i\kappa}{2}(m_\alpha + s_\alpha + 1 - 2r).
\]

(43)

By definition, the integer parameters \( \{m_\alpha\} \) and \( \{s_\alpha\} \) fulfil the global constraints

\[
\begin{align*}
\sum_{\alpha=1}^{M} m_\alpha &= L \\
\sum_{\alpha=1}^{M} s_\alpha &= R.
\end{align*}
\]

(44)

(45)

In this way the summation over permutations \( \mathcal{P}^{(L,R)} \) in equation (33) is changed by the summations over the integer parameters \( \{m_\alpha\} \) and \( \{s_\alpha\} \):

\[
\sum_{\mathcal{P}^{(L,R)}} (\cdots) \rightarrow \prod_{\alpha=1}^{M} \left[ \sum_{m_\alpha + s_\alpha \geq 1} \delta(m_\alpha + s_\alpha, n_\alpha) \delta\left(\sum_{\alpha=1}^{M} m_\alpha, L\right) \delta\left(\sum_{\alpha=1}^{M} s_\alpha, R\right) (\cdots) \right]
\]

(46)

which allows to lift the summations over \( L, R, \) and \( \{n_\alpha\} \) in equation (34). Straightforward but slightly cumbersome calculations result in the following expression (see the appendix):

\[
V_\varepsilon(f_1, f_2) = \lim_{\lambda \to -\infty} \left\{ 1 + \sum_{M=1}^{\infty} \frac{(-1)^M}{M!} \prod_{\alpha=1}^{M} \left[ \sum_{m_\alpha + s_\alpha \geq 1} (-1)^{m_\alpha + s_\alpha - 1} \int_{-\infty}^{+\infty} dq_\alpha \frac{G(q_\alpha, m_\alpha, s_\alpha)}{2\pi\kappa(m_\alpha + s_\alpha)} \right] \right.
\]

\[
\times \exp\left\{ -\frac{t}{2\beta}(m_\alpha + s_\alpha)q_\alpha^2 + \frac{\kappa^2 t}{24\beta}(m_\alpha + s_\alpha)^3 + \lambda m_\alpha f_1
\]

\[
+ \lambda s_\alpha f_2 + i(x(m_\alpha + s_\alpha)q_\alpha - y(m_\alpha + s_\alpha)q_\beta) \right\} |\mathcal{C}_M(q, m + s)|^2 G_M(q, m, s) \right\}
\]

(47)

where

\[
|\mathcal{C}_M(q, m + s)|^2 = \prod_{\alpha < \beta} \left| q_\alpha - q_\beta - (i\kappa/2)(m_\alpha + s_\alpha - m_\beta - s_\beta) \right|^2
\]

(48)

and

\[
G(q_\alpha, m_\alpha, s_\alpha) = \frac{\Gamma(s_\alpha + (2i/\kappa)q_\alpha(-1)) \Gamma(m_\alpha - (2i/\kappa)q_\alpha^{(+1)}) \Gamma(1 + m_\alpha + s_\alpha)}{2^{(m_\alpha + s_\alpha)} \Gamma(m_\alpha + s_\alpha + (2i/\kappa)q_\alpha^{(-)}) \Gamma(m_\alpha + s_\alpha - (2i/\kappa)q_\alpha^{(+)}) \Gamma(1 + m_\alpha) \Gamma(1 + s_\alpha)}
\]

(49)

doi:10.1088/1742-5468/2013/02/P02012

J. Stat. Mech. (2013) P02012
The explicit expression for the factor $G_M(q,m,s)$ is given in the appendix, equation (A.17).

Redefining

$$q_{a} = \frac{\kappa}{2\lambda} p_{a}$$

and

$$x \to \frac{2\lambda^{2}}{\kappa} x$$

with

$$\lambda = \frac{1}{2} \left( \frac{\kappa^{2} t}{\beta} \right)^{1/3} = \frac{1}{2} (\beta^{5} u^{2} t)^{1/3}$$

the normalization factor $|\tilde{C}_M(q,m+s)|^{2}$, equation (48), can be represented as follows:

$$|\tilde{C}_M(q,m+s)|^{2} = \prod_{\alpha<\beta}^{M} \left| \frac{\lambda(m_{\alpha} + s_{\alpha}) - \lambda(m_{\beta} + s_{\beta}) - ip_{\alpha} + ip_{\beta}}{\lambda(m_{\alpha} + s_{\alpha}) + \lambda(m_{\beta} + s_{\beta}) - ip_{\alpha} + ip_{\beta}} \right|^{2}$$

$$= \prod_{\alpha=1}^{M} \left[ 2\lambda(m_{\alpha} + s_{\alpha}) \right] \times \det \left[ \frac{1}{\lambda(m_{\alpha} + s_{\alpha}) - ip_{\alpha} + \lambda(m_{\beta} + s_{\beta}) + ip_{\beta}} \right]_{\alpha, \beta = 1, \ldots, M}$$

where we have used the Cauchy double alternant identity

$$\prod_{\alpha<\beta}^{M}(a_{\alpha} - a_{\beta})(b_{\alpha} - b_{\beta}) = (-1)^{M(M-1)/2} \det \left[ \frac{1}{a_{\alpha} - b_{\beta}} \right]_{\alpha, \beta = 1, \ldots, M}$$

with $a_{\alpha} = p_{\alpha} - i\lambda(m_{\alpha} + s_{\alpha})$ and $b_{\alpha} = p_{\alpha} + i\lambda(m_{\beta} + s_{\beta})$.

After rescaling, equations (50)–(52), for the exponential factor in equation (47) we find

$$-\frac{t}{2\beta} (m_{\alpha} + s_{\alpha}) q_{a}^{2} + \frac{\kappa^{2} t}{24\beta} (m_{\alpha} + s_{\alpha})^{3} + \lambda m_{a} f_{1} + \lambda s_{a} f_{2} + i\lambda x(m_{\alpha} + s_{\alpha}) q_{a}$$

$$\to -\lambda(m_{\alpha} + s_{\alpha}) p_{a}^{2} + \frac{1}{2} \lambda^{2} (m_{\alpha} + s_{\alpha})^{3} + \lambda m_{a} f_{1} + \lambda s_{a} f_{2} + i\lambda x(m_{\alpha} + s_{\alpha}) p_{a}.$$  

The cubic exponential term can be linearized using the Airy function relation

$$\exp \left[ \frac{1}{3} \lambda^{3} (m_{\alpha} + s_{\alpha})^{3} \right] = \int_{-\infty}^{+\infty} dy_{a} \text{Ai}(y_{a}) \exp [\lambda(m_{\alpha} + s_{\alpha}) y_{a}].$$

Substituting equations (53), (55) and (56) into equation (47), and redefining $y_{a} \to y_{a} + p_{a}^{2} - ix p_{a}$, we get

$$V_{\varepsilon}(f_{1}, f_{2}) = \lim_{\lambda \to \infty} \left\{ 1 + \sum_{M=1}^{\infty} \frac{(-1)^{M}}{M!} \right\}$$

$$\times \prod_{\alpha=1}^{M} \left[ \int_{-\infty}^{+\infty} dy_{a} \frac{dp_{a}}{2\pi} \text{Ai}(y_{a} + p_{a}^{2} - ix p_{a}) \sum_{m_{a}+s_{a} \geq 1} (-1)^{m_{a}+s_{a}-1} \right]$$
Directed polymer’s endpoint fluctuation distribution function

Figure 1. The contours of integration in the complex plane used for summing the series: (a) the original contour $C$; (b) the deformed contour $C'$.

$$\times \exp\{\lambda m_\alpha(y_\alpha + f_1) + \lambda s_\alpha(y_\alpha + f_2)\}G\left(\frac{p_\alpha}{\lambda}, m_\alpha, s_\alpha\right)$$

$$\times \det \tilde{K}[\{\lambda m_\alpha, \lambda s_\alpha, p_\alpha\}; \{\lambda m_\beta, \lambda s_\beta, p_\beta\}]_{\alpha,\beta=1,\ldots,M}G_M\left(\frac{p_\lambda}{\lambda}, m, s\right)$$  \hspace{1cm} (57)

where

$$\tilde{K}[\{\lambda m, \lambda s, p\}; \{\lambda m', \lambda s', p'\}] = \frac{1}{\lambda m + \lambda s - ip + \lambda m' + \lambda s' + ip'}.$$  \hspace{1cm} (58)

The crucial point of the further calculations is the procedure of taking the thermodynamic limit $\lambda \to \infty$. In this limit the summations over $\{m_\alpha\}$ and $\{s_\alpha\}$ are performed according to the following algorithm. Let us consider the example of the sum of a general type:

$$R(y, p) = \lim_{\lambda \to \infty} \prod_{\alpha=1}^M \left[ \sum_{n_\alpha=1}^\infty (-1)^{n_\alpha-1} \exp\{\lambda n_\alpha y_\alpha\} \right] \Phi\left(\frac{p_\alpha}{\lambda}, p, \lambda n, n\right)$$  \hspace{1cm} (59)

where $\Phi$ is a function which depends on the factors $\lambda n_\alpha, p_\alpha/\lambda$ as well as on the parameters $n_\alpha$ and $p_\alpha$ (which do not contain $\lambda$). The summations in the above example can be represented in terms of the integrals in the complex plane:

$$R(y, p) = \lim_{\lambda \to \infty} \prod_{\alpha=1}^M \left[ \sum_{n_\alpha=1}^\infty (-1)^{n_\alpha-1} \int_C \frac{dz_\alpha}{\sin(\pi z_\alpha)} \exp\{\lambda z_\alpha y_\alpha\} \right] \Phi\left(\frac{p_\alpha}{\lambda}, p, \lambda z, z\right)$$  \hspace{1cm} (60)

where the integration goes over the contour $C$ shown in figure 1(a). Shifting the contour to the position $C'$ shown in figure 1(b) (assuming that there is no contribution from infinity), and redefining $z \to z/\lambda$, in the limit $\lambda \to \infty$ we get

$$R(y, p) = \prod_{\alpha=1}^M \left[ \int_{C'} \frac{dz_\alpha}{z_\alpha} \exp\{z_\alpha y_\alpha\} \right] \lim_{\lambda \to \infty} \Phi\left(\frac{p_\alpha}{\lambda}, p, z, \frac{z}{\lambda}\right)$$  \hspace{1cm} (61)

where the parameters $y_\alpha, p_\alpha$ and $z_\alpha$ remain finite in the limit $\lambda \to \infty$. 

doi:10.1088/1742-5468/2013/02/P02012
Directed polymer’s endpoint fluctuation distribution function

To perform the summations over \( m_\alpha \) and \( s_\alpha \) in equation (57) it is convenient to represent it in the following way:

\[
V_x(f_1, f_2) = 1 + \sum_{M=1}^{\infty} \frac{(-1)^M}{M!} \prod_{\alpha=1}^{M} \left[ \int_{-\infty}^{+\infty} \frac{dy_\alpha dp_\alpha}{2\pi} \text{Ai}(y_\alpha + p_\alpha^2 - ixp_\alpha) \right] S_M(p, y; f_1, f_2)
\]  

(62)

where

\[
S_M(p, y; f_1, f_2) = \lim_{\lambda \to \infty} \prod_{\alpha=1}^{M} \left[ \sum_{m_\alpha+s_\alpha=1}^{\infty} (-1)^{m_\alpha+s_\alpha-1} \exp\{\lambda m_\alpha(y_\alpha + f_1) + \lambda s_\alpha(y_\alpha + f_2)\} \right]
\]

\[
\times \prod_{\alpha=1}^{M} \left[ G\left(\frac{y_\alpha}{\lambda}, m_\alpha, s_\alpha\right) \right] \det \hat{K}\left[ (\lambda m_\alpha, \lambda s_\alpha, p_\alpha); (\lambda m_\beta, \lambda s_\beta, p_\beta) \right]
\]

\[
\times G_M\left(\frac{p}{\lambda}, m, s\right).
\]

(63)

The summations over \( m_\alpha \) and \( s_\alpha \) in the above expression can be represented as follows:

\[
\sum_{m_\alpha+s_\alpha=1}^{\infty} (-1)^{m_\alpha+s_\alpha-1} = \sum_{m_\alpha=1}^{\infty} (-1)^{m_\alpha-1} \delta(s_\alpha, 0) + \sum_{s_\alpha=1}^{\infty} (-1)^{s_\alpha-1} \delta(m_\alpha, 0)
\]

\[
- \sum_{m_\alpha=1}^{\infty} (-1)^{m_\alpha-1} \sum_{s_\alpha=1}^{\infty} (-1)^{s_\alpha-1}.
\]

(64)

Thus in the integral representation, equations (59)–(61), for the function \( S_M(p, y; f_1, f_2) \), equation (63), we get

\[
S_M(p, y; f_1, f_2) = \prod_{\alpha=1}^{M} \left[ \int_{c^i} \frac{dz_1 \, dz_2}{(2\pi i)^2} (\frac{2\pi i}{z_1} \delta(z_2) + \frac{2\pi i}{z_2} \delta(z_1)) - \frac{1}{z_1z_2} \right]
\]

\[
\times \exp\{z_1(y_\alpha + f_1) + z_2(y_\alpha + f_2)\}
\]

\[
= \lim_{\lambda \to \infty} \left\{ \prod_{\alpha=1}^{M} \left[ G\left(\frac{p_\alpha}{\lambda}, \frac{z_1_\alpha}{\lambda}, \frac{z_2_\alpha}{\lambda}\right) \right] G_M\left(\frac{p}{\lambda}, \frac{z_1}{\lambda}, \frac{z_2}{\lambda}\right) \right\}
\]

\[
\times \det \hat{K}\left[ (z_1_\alpha, z_2_\alpha, p_\alpha); (z_1_\beta, z_2_\beta, p_\beta) \right].
\]

(65)

Taking into account the Gamma function properties, \( \Gamma(z)|_{z=0} = 1/z \) and \( \Gamma(1+z)|_{z=0} = 1 \), for the factors \( G \), equation (49), and \( G \), equation (A.17), we obtain

\[
\lim_{\lambda \to \infty} G\left(\frac{p_\alpha}{\lambda}, \frac{z_1_\alpha}{\lambda}, \frac{z_2_\alpha}{\lambda}\right) = \frac{(z_1_\alpha + z_2_\alpha + ip_\alpha^(-))(z_1_\alpha + z_2_\alpha - ip_\alpha^+) (z_2_\alpha + ip_\alpha^-)(z_1_\alpha - ip_\alpha^+)}{(z_2_\alpha + ip_\alpha^-)(z_1_\alpha - ip_\alpha^+)}
\]

(66)

and

\[
\lim_{\lambda \to \infty} G\left(\frac{p}{\lambda}, \frac{z_1}{\lambda}, \frac{z_2}{\lambda}\right) = 1.
\]

(67)

doi:10.1088/1742-5468/2013/02/P02012

J. Stat. Mech. (2013) P02012
Thus, in the limit $\lambda \to \infty$ the expression for the probability distribution function, equation (62), takes the form of the Fredholm determinant

$$V_x(f_1, f_2) = 1 + \sum_{M=1}^{\infty} \frac{(-1)^M}{M!} \prod_{\alpha=1}^{M} \left[ \int_{-\infty}^{\infty} \frac{dy_\alpha}{2\pi} \frac{dp_\alpha}{2\pi} \right. \left. \frac{\partial}{\partial y_\alpha} \frac{\partial}{\partial p_\alpha} \right] \exp \left\{ z_1 \delta(z_1) + z_2 \delta(z_2) - \frac{1}{z_1 z_2} \right\}$$

with the kernel

$$\hat{A}[(z_1, z_2, p); (z_1', z_2', p')] = \int_{-\infty}^{\infty} \frac{dy}{2\pi} \frac{dp}{2\pi} \exp \left\{ z_1 \delta(z_1) + z_2 \delta(z_2) - \frac{1}{z_1 z_2} \right\}$$

In the exponential representation of this determinant we get

$$V_x(f_1, f_2) = \exp \left[ -\sum_{M=1}^{\infty} \frac{1}{M} \text{Tr} \hat{A}^M \right]$$

where

$$\text{Tr} \hat{A}^M = \prod_{\alpha=1}^{M} \left[ \int_{-\infty}^{\infty} \frac{dy_\alpha}{2\pi} \frac{dp_\alpha}{2\pi} \right. \left. \frac{\partial}{\partial y_\alpha} \frac{\partial}{\partial p_\alpha} \right] \exp \left\{ z_1 \delta(z_1) + z_2 \delta(z_2) - \frac{1}{z_1 z_2} \right\}$$

$$\times \left( 1 + \frac{z_1}{z_2 + ip_{\alpha}(-)} \right) \left( 1 + \frac{z_2}{z_1 - ip_{\alpha}(+)} \right) \exp \left\{ z_1(y + f_1) + z_2(y + f_2) \right\}$$

$$\times \frac{1}{z_1 + z_2 - ip + z_1' + z_2' + ip'}.$$
Here, by definition, it is assumed that \( z_{iM+1} \equiv z_i \) \((i = 1, 2)\) and \( p_{M+1} \equiv p_1 \). Substituting
\[
\frac{1}{z_1 + z_2 - ip_1 + z_1 + z_2 + ip_1 + i p_{M+1}}
= \int_0^\infty d\omega \exp[-(z_1 + z_2 - ip_1 + z_1 + z_2 + ip_1 + i p_{M+1})\omega]
\] into equation (71), we obtain
\[
\text{Tr} \hat{A}^M = \int_0^\infty d\omega_1 \cdots d\omega_M \prod_{a=1}^M \left[ \int \int_{-\infty}^{+\infty} \frac{dy dp}{2\pi} \text{Ai}(y + p^2 + \omega + \omega_{a-1} - ip) \times \exp[ip(\omega - \omega_{a-1})] S(p, y; f_1, f_2) \right]
\]
where, by definition, \( \omega_0 \equiv \omega_M \), and
\[
S(p, y; f_1, f_2) = \int \int \frac{dz_1 dz_2}{(2\pi)^2} \left( \frac{2\pi i}{z_1} \delta(z_2) + \frac{2\pi i}{z_2} \delta(z_1) - \frac{1}{z_1 z_2} \right) \left( 1 + \frac{z_1}{z_2 + ip(+) \right) \exp[z_1(y + f_1) + z_2(y + f_2)]
\] Simple integrations provide the following result:
\[
S(p, y; f_1, f_2) = \theta(y + f_1) + \theta(y + f_2) - \theta(y + f_1)\theta(y + f_2)
- \theta(y + f_1)\theta(y + f_2) \exp[ip(f_1 - f_2) - 2\epsilon y]
+ \frac{i}{p + i\epsilon} \delta(y + f_2) - \frac{i}{p - i\epsilon} \delta(y + f_1)
- \frac{i}{p + i\epsilon} \delta(y + f_2)\theta(f_1 - f_2)[1 - \exp[i(p + i\epsilon)(f_1 - f_2)]]
+ \frac{i}{p - i\epsilon} \delta(y + f_1)\theta(f_2 - f_1)[1 - \exp[i(p - i\epsilon)(f_1 - f_2)]].
\]
According to equation (17), in what follows we will be dealing with the sector \( f_2 > f_1 \) only. In this case the above expression simplifies to
\[
S(p, y; f_1, f_2)\big|_{f_2 > f_1} = \left( \frac{i}{p + i\epsilon} - \frac{i}{p - i\epsilon} \right) \delta(y + f_2)
+ \frac{i}{p - i\epsilon} [\delta(y + f_2) - \delta(y + f_1) \exp[ip(f_1 - f_2)]]
+ \theta(y + f_2) - \theta(y + f_1) \exp[ip(f_1 - f_2) - 2\epsilon y].
\]
Note that at edge of the sector \( f_2 > f_1 \) for \( f_2 = f_1 + 0 \) (in the limit \( \epsilon \to 0 \))
\[
S(p, y; f_1, f_2)\big|_{f_2 = f_1 + 0} = 2\pi \delta(p) \delta(y + f_2).
\]
Thus, according to equations (73) and (76), the two-point free energy distribution function \( V_p(f_1, f_2) \), equation (14) (in the sector \( f_2 > f_1 \)), is given by the Fredholm
\[\text{doi:10.1088/1742-5468/2013/02/P02012}\]
det, equation (70), with the kernel
\[
A(\omega, \omega') = \text{Ai}(\omega + \omega' - f_2) - \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \times \left[ \text{Ai}(\omega + \omega' + p^2 - ipx - f_2) - \text{Ai}(\omega + \omega' + p^2 - ipx - f_1) \exp\{ip(f_1 - f_2)\} \right] \\
\times \exp\{ip(\omega - \omega')\} + \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} dy \ \text{Ai}(\omega + \omega' + p^2 - ipx + y) \\
- \int_{-f_1}^{+\infty} dy \ \text{Ai}(\omega + \omega' + p^2 - ipx + y) \exp\{ip(f_1 - f_2)\} \exp\{ip(\omega - \omega')\}
\]
(78)
with \(\omega, \omega' > 0\).

5. The endpoint probability distribution function

Substituting the above result, equations (70) and (78), into equation (17) for the endpoint distribution function, one obtains the following expression:
\[
W(x) = \int_{-\infty}^{+\infty} df F_1(-f) \int_{0}^{+\infty} d\omega \int_{0}^{+\infty} d\omega' (\hat{1} - \hat{B}_{-f})^{-1}(\omega, \omega') \Phi(\omega', \omega; f, x)
\]
(79)
where
\[
F_1(-f) = \det[\hat{1} - \hat{B}_{-f}] = \exp\left[-\sum_{M=1}^{\infty} \frac{1}{M} \text{Tr} \hat{B}_{-f}^{M} \right]
\]
(80)
is the GOE Tracy–Widom distribution with the kernel
\[
B_{-f}(\omega, \omega') = \text{Ai}(\omega + \omega' - f), \quad (\omega, \omega' > 0)
\]
(81)
and
\[
\Phi(\omega, \omega'; f, x) = i \int_{-\infty}^{+\infty} \frac{dp}{2\pi} p \frac{1}{p - i\epsilon} \text{Ai}(\omega + \omega' + p^2 - ipx - f) \exp\{ip(\omega - \omega')\} \\
- i \int_{-f}^{+\infty} dy \int_{-\infty}^{+\infty} \frac{dp}{2\pi} p \text{Ai}(\omega + \omega' + p^2 - ipx + y) \exp\{ip(\omega - \omega')\}.
\]
(82)
Using the standard integral representation of the Airy function, one can easily reduce the above function \(\Phi(\omega, \omega'; f, x)\) to the following sufficiently simple form:
\[
\Phi(\omega, \omega'; f, x) = -\frac{1}{2} \int_{0}^{+\infty} dy \left[ \left( \frac{\partial}{\partial \omega} - \frac{\partial}{\partial \omega'} \right) \Psi \left( \omega - \frac{1}{2} f + y; x \right) \Psi \left( \omega' - \frac{1}{2} f + y; -x \right) \\
+ \left( \frac{\partial}{\partial \omega} + \frac{\partial}{\partial \omega'} \right) \Psi \left( \omega - \frac{1}{2} f - y; x \right) \Psi \left( \omega' - \frac{1}{2} f - y; -x \right) \right]
\]
(83)
where
\[
\Psi(\omega; x) = 2^{1/3} \text{Ai}[2^{1/3}(\omega + \frac{1}{8} x^2)] \exp\{-\frac{1}{2}\omega x\}.
\]
(84)
Thus, equations (79), (83) and (84) complete the derivation of the probability distribution function for the directed polymer’s endpoint. Unfortunately, at present stage.
the analytical properties of this function are not quite clear. The study of this function requires the special analysis and it will be done elsewhere.

6. Conclusions

In this paper the explicit expression for the probability distribution function of the endpoint of one-dimensional directed polymers in random potential is derived in terms of the Bethe ansatz replica technique. The result obtained, equations (79)–(84), looks quite similar to the one derived in terms of a completely different method in which the maximal point of the Airy$_2$ process minus a parabola is considered [21]–[23]. Unfortunately, at the present stage the final expression for the probability distribution function obtained both here and in Refs. [21]–[23] is rather sophisticated, so the study of its analytical properties would require special efforts. Hopefully this problem will be solved in the near future.

One more conclusion of the present study is that the approach used, namely the Bethe ansatz replica technique, has once again (following the works [17]–[20]) demonstrated its efficiency. Hopefully it will also be fruitful for the studies of more serious problems in this scope, such as joint statistical properties of the free energy fluctuations at different times.

Acknowledgment

This work was supported in part by the grant IRSES DCPA PhysBio-269139.

Appendix

In terms of the parameters \( \{m_{\alpha}\} \) and \( \{s_{\alpha}\} \), the product factors in equation (41) are expressed as follows:

\[
\prod_{a=1}^{L} q_{p_a}^{(-)} = \prod_{a=1}^{M} \prod_{r=1}^{m_{\alpha}} q_{r}^{\alpha(-)} 
\]

\[
\prod_{a=1}^{R} q_{p_a}^{(+)} = \prod_{a=1}^{M} \prod_{r=1}^{s_{\alpha}} q_{r}^{\alpha(+)} 
\]

\[
\prod_{a<b} \left[ \frac{q_{p_a}^{(-)} + q_{p_b}^{(-)} + i\kappa}{q_{p_a}^{(-)} + q_{p_b}^{(-)}} \right] = \prod_{a=1}^{M} \prod_{1 \leq r < r'}^{m_{\alpha}} \left[ \frac{q_{r}^{\alpha(-)} + q_{r'}^{\alpha(-)} + i\kappa}{q_{r}^{\alpha(-)} + q_{r'}^{\alpha(-)}} \right] 
\]

\[
\times \prod_{1 \leq \alpha < \beta} \prod_{r=1}^{m_{\alpha}} \prod_{r'=1}^{m_{\beta}} \left[ q_{r}^{\alpha(-)} + q_{r'}^{\beta(-)} + i\kappa \right] 
\]

\[
\prod_{c<d} \left[ \frac{q_{p_c}^{(+)} + q_{p_d}^{(+)} - i\kappa}{q_{p_c}^{(+)} + q_{p_d}^{(+)}} \right] = \prod_{\alpha=1}^{M} \prod_{1 \leq r < r'}^{s_{\alpha}} \left[ \frac{q_{r}^{\alpha(+)} + q_{r'}^{\alpha(+)} - i\kappa}{q_{r}^{\alpha(+)} + q_{r'}^{\alpha(+)}} \right] 
\]

\[
\times \prod_{1 \leq \alpha < \beta} \prod_{r=1}^{s_{\alpha}} \prod_{r'=1}^{s_{\beta}} \left[ q_{r}^{\alpha(+)} + q_{r'}^{\beta(+)} - i\kappa \right] 
\]
The product factors in equation (A.6) can be easily expressed in terms of the Gamma functions. We obtain equation (47), where

\[
\prod_{a=1}^{L} \prod_{c=1}^{R} \left[ \frac{q_{p(a)} - q_{p(c)}}{q_{p(a)} - q_{p(c)}} - i\kappa \right] = \prod_{1 \leq \alpha < \beta} \prod_{r=1}^{M} \prod_{r'=1}^{M} \left[ \frac{q_{r}^{\alpha} - q_{r'}^{\alpha} - i\kappa}{q_{r}^{\alpha} + q_{r'}^{\alpha} - i\kappa} \right] \times \prod_{a=1}^{M} \prod_{r=1}^{M} \prod_{r'=1}^{M} \left[ \frac{q_{r}^{\alpha} - q_{r'}^{\alpha} - i\kappa}{q_{r}^{\alpha} + q_{r'}^{\alpha} - i\kappa} \right].
\]  

(A.5)

Substituting equations (A.1)–(A.5) into equation (41), and then substituting the resulting expression into equation (34), we obtain equation (47), where

\[
G(q_{\alpha}, m_{\alpha}, s_{\alpha}) = \frac{(-1)^{s_{\alpha}}(-i\kappa)^{m_{\alpha}+s_{\alpha}}}{\prod_{r=1}^{m_{\alpha}} q_{r}^{\alpha(-)} \prod_{r=1}^{s_{\alpha}} q_{r}^{\alpha(+)} \prod_{r'<r} \prod_{r'=1}^{m_{\alpha}} \left[ \frac{q_{r}^{\alpha(-)} + q_{r'}^{\alpha(-)} + i\kappa}{q_{r}^{\alpha(-)} + q_{r'}^{\alpha(-)} - i\kappa} \right] \prod_{r=1}^{s_{\alpha}} \prod_{r'=1}^{m_{\alpha}} \left[ \frac{q_{r}^{\alpha(+)} + q_{r'}^{\alpha(+)} - i\kappa}{q_{r}^{\alpha(+)} + q_{r'}^{\alpha(+)} + i\kappa} \right] \prod_{r=1}^{m_{\alpha}} \prod_{r'=1}^{s_{\alpha}} \left[ \frac{q_{r}^{\alpha(-)} - q_{r'}^{\alpha(-)} - i\kappa}{q_{r}^{\alpha(-)} + q_{r'}^{\alpha(-)} - i\kappa} \right] \prod_{r=1}^{s_{\alpha}} \prod_{r'=1}^{m_{\alpha}} \left[ \frac{q_{r}^{\alpha(+)} - q_{r'}^{\alpha(+)} - i\kappa}{q_{r}^{\alpha(+)} + q_{r'}^{\alpha(+)} - i\kappa} \right].
\]  

(A.6)

and

\[
G_{M}(q, m, s) = \prod_{1 \leq \alpha < \beta} \prod_{r=1}^{M} \prod_{r'=1}^{M} \left[ \frac{q_{r}^{\alpha(-)} + q_{r'}^{\alpha(-)} + i\kappa}{q_{r}^{\alpha(-)} + q_{r'}^{\alpha(-)} - i\kappa} \right] \prod_{r=1}^{s_{\alpha}} \prod_{r'=1}^{m_{\alpha}} \left[ \frac{q_{r}^{\alpha(+)} + q_{r'}^{\alpha(+)} + i\kappa}{q_{r}^{\alpha(+)} + q_{r'}^{\alpha(+)} - i\kappa} \right] \prod_{r=1}^{m_{\alpha}} \prod_{r'=1}^{s_{\alpha}} \left[ \frac{q_{r}^{\alpha(-)} - q_{r'}^{\alpha(-)} - i\kappa}{q_{r}^{\alpha(-)} + q_{r'}^{\alpha(-)} - i\kappa} \right] \prod_{r=1}^{s_{\alpha}} \prod_{r'=1}^{m_{\alpha}} \left[ \frac{q_{r}^{\alpha(+)} - q_{r'}^{\alpha(+)} - i\kappa}{q_{r}^{\alpha(+)} + q_{r'}^{\alpha(+)} - i\kappa} \right].
\]  

(A.7)

The product factors in equation (A.6) can be easily expressed in terms of the Gamma functions:

\[
\prod_{r=1}^{m_{\alpha}} q_{r}^{\alpha(-)} = \prod_{r=1}^{m_{\alpha}} \left[ q_{r}^{\alpha(-)} - \frac{i\kappa}{2} (m_{\alpha} + s_{\alpha} + 1) + i\kappa r \right] = (i\kappa)^{m_{\alpha}} \frac{\Gamma(\frac{1}{2} - \frac{s_{\alpha} - m_{\alpha}}{2} - \frac{iq_{\alpha(-)}}{\kappa})}{\Gamma(\frac{1}{2} - \frac{s_{\alpha} + m_{\alpha}}{2} + \frac{iq_{\alpha(-)}}{\kappa})}
\]  

(A.8)

\[
\prod_{r=1}^{s_{\alpha}} q_{r}^{\alpha(+)} = \prod_{r=1}^{s_{\alpha}} \left[ q_{r}^{\alpha(+)} + \frac{i\kappa}{2} (m_{\alpha} + s_{\alpha} + 1) - i\kappa r \right] = (-i\kappa)^{s_{\alpha}} \frac{\Gamma(\frac{1}{2} - \frac{m_{\alpha} - s_{\alpha}}{2} + \frac{iq_{\alpha(+)}}{\kappa})}{\Gamma(\frac{1}{2} - \frac{m_{\alpha} + s_{\alpha}}{2} + \frac{iq_{\alpha(+)}}{\kappa})}
\]  

(A.9)

\[
\prod_{r<r'} q_{r}^{\alpha(-)} q_{r'}^{\alpha(-)} + i\kappa = 2^{-m_{\alpha} + 1} \frac{\Gamma(m_{\alpha} - s_{\alpha} - \frac{2iq_{\alpha(-)}}{\kappa})}{\Gamma(m_{\alpha} - s_{\alpha} + \frac{2iq_{\alpha(-)}}{\kappa})} \frac{\Gamma(1 - \frac{m_{\alpha} + s_{\alpha}}{2} + \frac{iq_{\alpha(-)}}{\kappa})}{\Gamma(1 - \frac{m_{\alpha} + s_{\alpha}}{2} + \frac{iq_{\alpha(-)}}{\kappa})}
\]  

(A.10)

\[
\prod_{r<r'} q_{r}^{\alpha(+)} q_{r'}^{\alpha(+)} - i\kappa = 2^{-s_{\alpha} + 1} \frac{\Gamma(s_{\alpha} - m_{\alpha} + \frac{2iq_{\alpha(+)}}{\kappa})}{\Gamma(s_{\alpha} - m_{\alpha} - \frac{2iq_{\alpha(+)}}{\kappa})} \frac{\Gamma(1 - \frac{m_{\alpha} + s_{\alpha}}{2} + \frac{iq_{\alpha(+)}}{\kappa})}{\Gamma(1 - \frac{m_{\alpha} + s_{\alpha}}{2} + \frac{iq_{\alpha(+)}}{\kappa})}
\]  

(A.11)

\[
\prod_{r=r'} q_{r}^{\alpha(+)} q_{r'}^{\alpha(+)} - i\kappa = \frac{\Gamma(1 + m_{\alpha} + s_{\alpha})}{\Gamma(1 + m_{\alpha}) \Gamma(1 + s_{\alpha})}
\]  

(A.12)

Substituting the above expressions into equation (A.6) and using the standard relations for the Gamma functions,

\[
\Gamma(z) \Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}
\]  

(A.13)
\[\Gamma(1+z) = z \Gamma(z) \quad (A.14)\]
\[\Gamma\left(\frac{1}{2} + z\right) = \frac{\sqrt{\pi} \Gamma(1+2z)}{2^{2z} \Gamma(1+z)} \quad (A.15)\]

for the factor \(G\), equation (A.6), we get
\[G(q_{\alpha}, m_{\alpha}, s_{\alpha}) = \frac{\Gamma(s_{\alpha} + \frac{2}{\kappa} q_{\alpha}^{(-)}) \Gamma(m_{\alpha} - \frac{2}{\kappa} q_{\alpha}^{(+)}) \Gamma(1 + m_{\alpha} + s_{\alpha})}{2^{(m_{\alpha} + s_{\alpha})} \Gamma(m_{\alpha} + s_{\alpha} + \frac{2}{\kappa} q_{\alpha}^{(-)}) \Gamma(m_{\alpha} + s_{\alpha} - \frac{2}{\kappa} q_{\alpha}^{(+)}) \Gamma(1 + m_{\alpha}) \Gamma(1 + s_{\alpha})}. \quad (A.16)\]

Similar calculations for the factor \(G_M\), equation (A.7), yield the following expression:
\[G_M(q, m, s) = \prod_{1 \leq \alpha < \beta} \left\{ \frac{\Gamma[1 + \frac{m_{\alpha} + m_{\beta} - s_{\alpha} - s_{\beta}}{2} - \frac{1}{\kappa} (q_{\alpha}^{(-)} + q_{\beta}^{(-)})] \Gamma[1 - \frac{m_{\alpha} + m_{\beta} + s_{\alpha} + s_{\beta}}{2} - \frac{1}{\kappa} (q_{\alpha}^{(-)} + q_{\beta}^{(-)})]} {\Gamma[1 - \frac{m_{\alpha} + m_{\beta} - s_{\alpha} - s_{\beta}}{2} + \frac{1}{\kappa} (q_{\alpha}^{(+)} + q_{\beta}^{(+))})] \Gamma[1 - \frac{m_{\alpha} + m_{\beta} + s_{\alpha} + s_{\beta}}{2} + \frac{1}{\kappa} (q_{\alpha}^{(+)} + q_{\beta}^{(+))})]} \times \right. \frac{\Gamma[1 + \frac{m_{\alpha} + m_{\beta} + s_{\alpha} + s_{\beta}}{2} + \frac{1}{\kappa} (q_{\alpha} - q_{\beta})]} {\Gamma[1 + \frac{-m_{\alpha} + m_{\beta} + s_{\alpha} - s_{\beta}}{2} + \frac{1}{\kappa} (q_{\alpha} - q_{\beta})]} \left. \times \frac{\Gamma[1 + \frac{m_{\alpha} + m_{\beta} + s_{\alpha} + s_{\beta}}{2} - \frac{1}{\kappa} (q_{\alpha} - q_{\beta})]} {\Gamma[1 + \frac{-m_{\alpha} + m_{\beta} - s_{\alpha} + s_{\beta}}{2} - \frac{1}{\kappa} (q_{\alpha} - q_{\beta})]} \right\}. \quad (A.17)\]

References

[1] Kardar M, Parisi G and Zhang Y-C, 1986 Phys. Rev. Lett. 56 889
[2] Halpin-Healy T and Zhang Y-C, 1995 Phys. Rep. 254 215
[3] Burgers J M, 1974 The Nonlinear Diffusion Equation (Dordrecht: Reidel)
[4] Kardar M, 2007 Statistical Physics of Fields (Cambridge: Cambridge University Press)
[5] Huse D A, Henley C L and Fisher D S, 1985 Phys. Rev. Lett. 55 2924
[6] Huse D A and Henley C L, 1985 Phys. Rev. Lett. 54 2708
[7] Kardar M and Zhang Y-C, 1987 Phys. Rev. Lett. 58 2087
[8] Kardar M, 1987 Nucl. Phys. B 290 582
[9] Bouchaud J P and Orland H, 1990 J. Stat. Phys. 61 877
[10] Brunet E and Derrida B, 2000 Phys. Rev. E 61 6789
[11] Johansson K, 2000 Commun. Math. Phys. 209 437
[12] Prahofer M and Spohn H, 2002 J. Stat. Phys. 108 1071
[13] Ferrari P L and Spohn H, 2000 Commun. Math. Phys. 265 1
[14] Corwin I, 2012 Random Matrices: Theory Appl. 1 1130001
[15] Sasamoto T and Spohn H, 2010 Phys. Rev. Lett. 104 230602
[16] Amir G, Corwin I and Quastel J, 2011 Comm. Pure Appl. Math. 64 466
[17] Dotsenko V and Khunov B, 2010 J. Stat. Mech. P03022
[18] Dotsenko V, 2010 Europhys. Lett. 90 20003
[19] Dotsenko V, 2010 J. Stat. Mech. P07010
[20] Calabrese P, Le Doussal P and Rosso A, 2010 Europhys. Lett. 90 20002
[21] Calabrese P and Le Doussal P, 2011 Phys. Rev. Lett. 106 250603 arXiv:1204.2607
[22] Dotsenko V, 2012 J. Stat. Mech. P11014
[23] Flores G M, Quastel J and Remenik D, 2013 Commun. Math. Phys. 317 363
Directed polymer’s endpoint fluctuation distribution function

[22] Schehr G, 2012 J. Stat. Phys. 149 385
[23] Baik J, Liechty K and Schehr G, On the joint distribution of the maximum and its position of the Airy$_2$ process minus a parabola, 2012 J. Math. Phys. 53 083303 arXiv:1205.3665
[24] Corwin I and Hammond A, Brownian Gibbs property for Airy line ensembles, 2011 arXiv:1108.2291
[25] Prolhac S and Spohn H, 2011 J. Stat. Mech. P01031
[26] Lieb E H and Liniger W, 1963 Phys. Rev. 130 1605
[27] McGuire J B, 1964 J. Math. Phys. 5 622
[28] Yang C N, 1968 Phys. Rev. 168 1920
[29] Calabrese P and Caux J-S, 2007 Phys. Rev. Lett. 98 150403
[30] Dotsenko V S, Universal randomness, 2011 Phys.—Usp. 54 259
[31] Bothner T and Liechty K, 2012 arXiv:1212.3816v2

doi:10.1088/1742-5468/2013/02/P02012