Stackelberg Pricing is Hard to Approximate within $2 - \epsilon$

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Abstract

Stackelberg Pricing Games is a two-level combinatorial pricing problem studied in the Economics, Operation Research, and Computer Science communities. In this paper, we consider the decade-old shortest path version of this problem which is the first and most studied problem in this family.

The game is played on a graph (representing a network) consisting of fixed cost edges and pricable or variable cost edges. The fixed cost edges already have some fixed price (representing the competitor’s prices). Our task is to choose prices for the variable cost edges. After that, a client will buy the cheapest path from a node $s$ to a node $t$, using any combination of fixed cost and variable cost edges. The goal is to maximize the revenue on variable cost edges.

In this paper, we show that the problem is hard to approximate within $2 - \epsilon$, improving the previous APX-hardness result by Joret [to appear in Networks]. Our technique combines the existing ideas with a new insight into the price structure and its relation to the hardness of the instances.

1 Introduction

A newly startup company has just acquired some links in a network. The company wants to sell these links to a particular client, who will buy a cheapest path from a node $s$ to a node $t$. However, this company is not alone in the market: there are other companies already in the market owning some links with some fixed prices. The goal of this new company is to price its links to maximize its profit, having the complete knowledge of the network and knowing that the client will buy the cheapest $s$-$t$ path (which may consist of links from many companies). Of course, if they price a link too high, the client will switch to other links and if they price a link too low then they unnecessarily reduce their profit.

This problem is called the Stackelberg Shortest Path Game (STACKSP) and can be defined formally as follows. We are given a directed graph $G = (V, E)$, a source vertex $s$ and a sink vertex $t$. The set $E$ of edges is partitioned into two sets: $E_f$, the set of fixed cost edges, and $E_v$, the set of pricable or variable cost edges. Each edge $e$ in $E_f$ already has some price $p(e)$. Our task is to set a price $p(e)$ to each variable cost edge $e$. Once we set the price, the client will buy a shortest path from $s$ to $t$ (i.e., a path $P$ such that $\sum_{e \in P} p(e)$ is minimized). Our goal is to maximize the profit; i.e., maximize $\sum_{e \in P \cap E_v} p(e)$ where $P$ is the path bought by the client. Throughout, we let $m$ denote the number of variable cost edges. It is usually assumed that if there are many shortest paths, the client will buy the one that maximizes our profit.

Due to its connection to road network tolling and bilevel programming, there is an enormous effort in understanding the problem by means of bilevel programming [24, 12, 14, 19, 21, 20, 13, 4], finding polynomial-time solvable cases [24, 29, 18, 27, 10, 3, 5], solving the problem by heuristics [16, 15], and

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approximating the solution. In this paper, we focus on approximability of this problem. In this realm, STACKSP is the first and the most studied problem in the growing family of one-follower (i.e., one client) Stackelberg network pricing games.

The Stackelberg pricing problems belong to the class of two-player two-level optimization problems which is a subclass of the bilevel linear programming. These problems have a rather strange structure, and this makes the standard approximation techniques such as linear programming seemingly inapplicable. For example, a natural LP formulation for STACKSP (and also another version called STACKMST) has an integrality gap of \( \Omega(\log m) \). Moreover, by using the most (and probably the only) natural upper bound for OPT, one cannot obtain approximation factor better than \( O(\log m) \) so the line of attacks considered in [26] and [6] cannot be pushed any further.

Proving the hardness of this problem seems to have an equally big obstacle. In fact, the progress on the hardness side for the family of Stackelberg pricing problems stops at small constant hardness (APX-hardness in [23, 8] and only \( \text{NP} \)-hardness in [2]). Moreover, a reduction from Unique Coverage problem [11], which proved useful for many pricing problems (including STACKSP with multiple followers) apparently does not apply here. In particular, for STACKSP, only \( \text{NP} \)-hardness, strong \( \text{NP} \)-hardness, and APX-hardness (with a constant as small as 1.001) are shown [24, 26, 23]. In fact, even for approximating the general bilevel program, only the constant ratio can be ruled out [17, 22].

We believe that an improvement to upper or lower bound of the problem might shed some light on approximating a larger subclass of bilevel programs, perhaps generating a new set of techniques for attacking the whole family of Stackelberg problems. (The problem seems to require a new technique due to its bizarre behavior.)

Our result and techniques

In this paper, we give the first result beyond a very small constant hardness:

**Theorem 1.1.** For any \( \epsilon > 0 \), it is \( \text{NP} \)-hard to approximate STACKSP to within a factor of \( 2 - \epsilon \).

The key insight in obtaining this result comes from exploring the structure of the edge prices which was not exploited in the previous inapproximability results [24, 26, 23]: The previous results encode the constraints in the constraint satisfaction problems (Max 3SAT in their cases) using certain gadgets and glue these gadgets together in a uniform way (i.e., using the same edge price throughout). However, we study the influence of non-uniform prices to the hardness of the resulting instances. In particular, we study how the prices of the fixed cost edges affect the hardness of the gadgets and found an optimal price which strikes a balance between being too high (which could hugely reduce the revenue but is easy to avoid) and too low (which is likely to be used but do not affect the revenue much). This observation, armed with a stronger constraint satisfaction problem (i.e., Raz verifier for Max 3SAT(5)) and a right parameter of price, leads to a \( (2 - \epsilon) \)-hardness of approximation. The techniques above are strong enough that the hardness result is obtained with only a slight modification of the gadgets. However, due to the non-uniformity of the prices, a more sophisticated analysis is required. In particular, our analysis relies on a technique called Path Decomposition which breaks the shortest path in the optimal solution into subpaths with manageable structure. We will be able to get deeper into the intuition after we describe the hardness construction in the next section.

Related work

STACKSP is first proposed by Labbé et al. [24] who also derive a bilevel LP formulation of the problem and prove \( \text{NP} \)-hardness. On the algorithmic side, Roch et al. present the first, and still the best, approximation
algorithm which attains $O(\log m)$ approximation factor. Another $O(\log m)$ approximation algorithm is obtained by Briest et al, which has a slightly worse approximation guarantee (larger constant in front of $\log m$ term) but is simpler and applicable to a much richer class of Stackelberg pricing problems. Even though the algorithm of Briest et al. does not rely on the specific problems’ structures, it remains unclear whether one can exploit a special structure of each problem to improve the approximation ratio.

Another interesting problem in the family of one-follower Stackelberg network games is Stackelberg Minimum Spanning Trees Game (STACKMST) in which the client aims to buy the minimum spanning tree instead of the shortest path. Cardinal et al. [8] introduce this problem and prove that it is APX-hard but has an $O(\log m)$ approximation algorithm. Very recently, they consider the special cases of planar and bounded-treewidth graphs [9] and prove that even in such graph classes, STACKMST remains NP-hard. There are also many other variations in the family of Stackelberg games, depending on what the client wants to buy. This includes vertex cover [4, 5], shortest path tree [2], and knapsack [5].

Among the known approximation algorithms, the most universal one is an $O((1+\epsilon) \log m)$-approximation algorithm invented by Briest et al. [6]. This elegant algorithm works on a large class of problems, including STACKSP and STACKMST and is coupled with a simple analysis. In the same paper, the case of $k$ clients is also considered. An $O((1+\epsilon)(\log m + \log k))$-approximation algorithm is given, and the problem is shown to be hard to approximate within $O(\log^\epsilon m + \log^\epsilon k)$ for some large $k$. Therefore, the gap is almost closed in the case of many clients while left wide open when $k$ is small (e.g. $k$ is constant, and particularly when $k = 1$).

In Economics and Operation Research literature, STACKSP is also known as a tarification problem. Many special cases are considered and polynomial-time algorithms are given for this problem [24, 29, 18, 27, 10, 3, 6]. It is also sometimes called a bilevel pricing problem due to its connection to the bilevel linear program. (See a formulation in, e.g., [24].) STACKSP is also heavily studied from this perspective [24, 12, 14, 19, 21, 20, 13, 4]. Approximating a solution of bilevel program to within any constant factor is shown to be NP-hard [22, 17]. Unfortunately, these reductions do not extend to the family of Stackelberg games due to specific structures of the constraints used in the reduction of [22, 17]. For more details, we refer the readers to [28, 13, 17] and references therein.

Remark Recently Briest and Khanna [7] discover a similar result to ours using a different approach. They show that STACKSP is hard to approximate within a factor of $2 - o(1)$.

Organization Our construction is a reduction from Raz verifier for Max 3SAT(5). We first give an overview of Raz verifier in Section 2. We then describe our reduction in Section 3 before we are able to give more intuition behind the construction and its analysis. This will be done in Section 4. We then show a formal analysis in Section 5.

2 Raz Verifier

Our reduction uses the Raz verifier for Max 3SAT(5) with $\ell$ repetitions. We explain this framework in this section. The given instance of Max 3SAT(5) is a 3CNF formula with $n$ variables and $5n/3$ clauses where each clause contains exactly 3 different literals, and each variable appears in exactly 5 different clauses.

Let $\epsilon$ be a constant and let $\varphi$ be an instance of Max 3SAT(5). Then $\varphi$ is called a Yes-Instance if there is an assignment that satisfies all the clauses, and it is called a No-Instance if any assignment satisfies at most $(1 - \epsilon)$-fraction of the clauses. The following is a form of the PCP theorem.
Theorem 2.1. There is a constant $\epsilon : 0 < \epsilon < 1$, such that it is NP-hard to distinguish between YES-INSTANCE and NO-INSTANCE of the Max 3SAT(5) problem.

Raz verifier for Max 3SAT(5) with $\ell$ repetitions is a two-provers one-round interactive proof system. The verifier sends one query to each prover simultaneously. The first prover is asked for an assignment to the variables in the given clauses while the second prover is asked for an assignment of the variables that satisfies all the given clauses. The verifier will accept the answers if and only if both provers return consistent assignments. The detailed description of the provers-verifier actions is as follows.

- The verifier first chooses $\ell$ clauses, say $C_1, \ldots, C_\ell$, independently and uniformly at random (with replacement). Next, choose one variable in each of these clauses uniformly at random. Let $x_1, \ldots, x_\ell$ denote the resulting (not necessarily distinct) variables.
- The verifier generates a query $q$ consisting of the indices of $C_1, C_2, \ldots, C_\ell$ and a query $q'$ consisting of the indices of $x_1, x_2, \ldots, x_\ell$. The verifier then sends $q$ and $q'$ to Prover 1 and Prover 2, respectively.
- Prover 1 returns an assignment to all variables associated with clauses $C_1, C_2, \ldots, C_\ell$.
- Prover 2 returns an assignment to variables $x_1, x_2, \ldots, x_\ell$.
- The verifier reads the assignment received from both provers and accepts if and only if the assignments are consistent and satisfy $C_1, C_2, \ldots, C_\ell$.

Intuitively, for the YES-INSTANCE, both provers can ensure that the verifier always accepts by returning the satisfying assignments to the prover. On the other hand, any provers’ strategy fails with high probability in the case of NO-INSTANCE. This is an application of the Parallel Repetition Theorem and Theorem 2.1 and can be stated formally as follows.

Theorem 2.2 ( [25] [1]). There exists a universal constant $\alpha > 0$ (independent of $\ell$) such that

- If $\varphi$ is a YES-INSTANCE, then there is a strategy of the provers that makes the verifier accepts with probability 1.
- If $\varphi$ is a NO-INSTANCE, for any provers’ strategy, the verifier will accept with probability at most $2^{-\alpha \ell}$.

In our reduction, we view Raz verifier as the following constraint satisfaction problem. We have two sets of queries, $Q_1$ and $Q_2$, corresponding to all possible queries sent to Prover 1 and Prover 2, respectively. That is, $Q_1$ consists of all possible choices of $\ell$ clauses sent to Prover 1 (hence, $|Q_1| = (5n/3)^\ell$) and $Q_2$ consists of all possible choices of $\ell$ variables sent to Prover 2 (hence $|Q_2| = n^\ell$). For each $q \in Q_1 \cup Q_2$, let $A(q)$ denote the set of all possible answers to $q$. Notice that $|A(q)| = 7^\ell$ if $q \in Q_1$ (since there are 7 ways to satisfy each of the $\ell$ clauses given to Prover 1) and $|A(q)| = 2^\ell$ if $q \in Q_2$ (since there are 2 possible assignment to each of the $\ell$ variables given to Prover 2). Denote by $A_1$ and $A_2$ the set of all possible answers by Prover 1 and Prover 2, respectively.

We denote the set of constraints by $\Phi$. Each constraint in $\Phi$ corresponds to a pair $(q_1, q_2)$ of queries sent by the verifier. That is, for each random string $r$ of the verifier, there is a constraint $(q_1, q_2) \in Q_1 \times Q_2$ in $\Phi$ where $q_1$ and $q_2$ are queries sent to Prover 1 and Prover 2 respectively. A constraint $(q_1, q_2)$ is satisfied if and only if the assignments to $q_1$ and $q_2$ are consistent. For convenience, we will treat $\Phi$ as the set of all possible random strings, and we denote, for each random string $r$, the corresponding queries by $q_1(r)$ and $q_2(r)$ respectively. Note that each query $q \in Q_1$ is associated with $3^\ell$ constraints in $\Phi$ and each query $q' \in Q_2$ with $5^\ell$ constraints. Moreover, let $M = |\Phi|$. We have $M = (5n)^\ell$. The goal of this problem is to find an assignment $f : Q_1 \rightarrow A_1, Q_2 \rightarrow A_2$ that maximizes the number of satisfied constraints in $\Phi$. 
The following corollary can be directly obtained from Theorem 2.2.

**Corollary 2.3.** If \( \varphi \) is a YES-INSTANCE, then there is an assignment to \( Q_1 \cup Q_2 \) such that all constraints in \( \Phi \) are satisfied. Otherwise, no assignment satisfies more than \( 2^{-\alpha \ell} \)-fraction of the constraints in \( \Phi \).

## 3 The Reduction

Let \( \epsilon > 0 \) be a constant from Theorem 1.1. Recall that we want to prove \((2 - \epsilon)\)-hardness of approximation.

**Overview**  
Starting with an instance \( \varphi \) of \( \text{Max 3SAT}(5) \), we first perform the two-prover protocol with \( \ell = \lceil \log(3/\epsilon)/\alpha \rceil \) rounds, and we enumerate all possible constraints in \( \Phi \). Next we transform \( \Phi \) to an instance of the Stackelberg problem in two steps, as follows. In the first step of the reduction, we order the constraints in \( \Phi \) to get a \((\delta, \gamma)\)-far sequence (see Section 3.1). In the second step, we convert such sequence to an instance of the Stackelberg problem, denoted by \( G \), using the construction explained in Section 3.2.

### 3.1 Obtaining \((\delta, \gamma)\)-far sequence

**Definition 3.1.** \((\delta, \gamma)\)-far constraint sequence) Consider a sequence of all possible constraints \( r_1, \ldots, r_M \) in \( \Phi \). A constraint \( r_i \) is said to be \( \delta\)-far if for every \( j : i < j \leq i + \lceil \delta M \rceil \), \( q_1(r_i) \neq q_1(r_j) \) and \( q_2(r_i) \neq q_2(r_j) \). The sequence \( r_1, \ldots, r_M \) is said to be \((\delta, \gamma)\)-far if at least \((1 - \gamma)\)-fraction of constraints is \( \delta\)-far.

We can obtain \((\delta, \gamma)\)-far sequence with the right parameter for our purpose using probabilistic arguments.

**Theorem 3.2.** For any \( \ell \geq 1, \delta > 1/M \) and \( \gamma \geq (8\delta)^{5\ell} \), there is a polynomial-time algorithm \( A \) that outputs a \((\delta, \gamma)\)-far sequence.

**Proof.** We present a randomized algorithm here. In Appendix, we derandomize it to the desired \( A \) by the method of conditional expectation. Let \( r_1, r_2, \ldots, r_M \) be the constraints. Let \( A' \) be an algorithm that picks random a permutation \( \pi : [M] \to [M] \). We claim that the sequence \( r_{\pi(1)}, \ldots, r_{\pi(M)} \) is \((\delta, \gamma)\)-far with probability at least 1/2.

To prove the above claim, consider each constraint \( r_i \). Let \( J = \{ j \in [M] : q_1(r_j) = q_1(r_i) \text{ or } q_2(r_j) = q_2(r_i) \} \). Notice that \( |J| \leq 3^\ell + 5^\ell < 2 \cdot 5^\ell \) because there are \( 3^\ell \) constraints \( r_j \) in \( \Phi \) with \( q_1(r_j) = q_1(r_i) \) and \( 5^\ell \) constraints \( r_j \) in \( \Phi \) with \( q_2(r_j) = q_2(r_i) \). For each such \( j \in J \), the probability that \( |\pi(i) - \pi(j)| \leq \lceil \delta M \rceil \) is at most \( \delta \). By applying the Union bound for all such \( j \in J \), the probability that \( r_{\pi(i)} \) is not \( \delta\)-far is at most \((4\delta)^{5\ell} \leq \gamma/2\). The expected number of constraints that are not \( \delta\)-far is at most \( \gamma M/2 \), so by Markov’s inequality, the sequence is \((\delta, \gamma)\)-far with probability at least 1/2, and the claim follows.

### 3.2 The Construction

Given a \((\delta, \gamma)\)-far sequence of constraints \( r_1, \ldots, r_M \), we construct an instance of \( \text{STACKSP} \) as follows. For each constraint \( r_i \), construct a gadget \( G_i \) containing source \( s_i \), destination \( t_i \), and a set of intermediate vertices \( \{ u_i^a, v_i^a \}_{a \in A(q_1(r_i))} \). There are \( 2 \cdot 7^\ell \) such intermediate vertices (since \( |A(q_1(r_i))| = 7^\ell \)).

Recall that, for each answer \( a \in A(q_1(r_i)) \), there exists a unique consistent answer \( a' \in A(q_2(r_i)) \). In other words, for each \( a \in A(q_1(r_i)) \) there exists a unique \( a' \in A(q_2(r_i)) \) such that \((a, a')\) satisfies the constraint \( r_i \). From now on, we will use \( \pi_i \) to denote the function that maps each \( a \in A(q_1(r_i)) \) to its consistent answer \( a' \in A(q_2(r_i)) \). Therefore, each pair of \( u_i^a, v_i^a \) corresponds to a pair of possible answer \((a, \pi_i(a))\) that satisfies \( r_i \).

Edges in each gadget \( G_i \) are the following.
Before we move on to the analysis, we explain the intuition behind the hardness construction in the previous section and the analysis in the next section.

**4 Intuition and Overview of the Analysis**

Before we move on to the analysis, we explain the intuition behind the hardness construction in the previous section and the analysis in the next section.

**NP-hardness** First, let us understand what happens when we apply the construction in Section 3.2 to Raz verifier’s $\Phi$ without applying Algorithm $A$ (cf. Section 3.1) to get a $(\delta, \gamma)$-far sequence; in other words, the sequence of constraints is arbitrary.

We use the following example to convey the idea. Consider a Max 2SAT instance with three variables $x_1, x_2, x_3$ and two clauses $C_1 = (x_1 \lor x_2)$ and $C_2 = (x_1 \lor x_3)$. (For the sake of simplicity, we consider an instance of Max 2SAT instead of Max 3SAT.) The constraints of the Raz verifier with $\ell = 1$ repetition are $r_1 = (C_1, x_1)$, $r_2 = (C_1, x_2)$, $r_3 = (C_2, x_3)$, and $r_4 = (C_2, x_1)$. If we construct the graph $G$ from the sequence of constraints $r_1, r_2, r_3, r_4$ according to the construction in Section 3.2, then we will get the graph $G$ as in Figure 1.

Consider any pricing $p$ and let $P$ be the corresponding shortest path from $s$ to $t$. We classify the shortcuts

- **Fixed cost edges**: There is a fixed cost edge of cost 1 from $s_i$ to $t_i$. There are also fixed cost edges of cost 0 from $s_i$ to each of $u_i^a$, and from each of $u_i^a$ to $t_i$.

- **Variable cost edges**: There is a variable cost edge from $u_i^a$ to $v_i^a$ for each $a \in A(q_1(r_i))$.

Now we link all the gadgets together. First, for all $1 \leq i < M$, we create a fixed cost edge of cost 0 from $t_i$ to $s_{i+1}$. We denote the source of instance $s = s_1$ and the sink $t = t_M$ (i.e., we want to buy a shortest path from $s_1$ to $t_M$).

Next, we add another set of fixed cost edges, called shortcuts, whose job is to put constraints between pairs of edges that represent inconsistent assignment. We only have shortcuts between far gadgets. (Gadget $G_i$ is called a far gadget if its corresponding constraint $r_i$ is a $\delta$-far constraint.) Consider any pair of far constraints $r_i, r_j$ for $i < j$ such that $r_i$ shares a query with $r_j$; i.e., either $q_1(r_i) = q_1(r_j)$ or $q_2(r_i) = q_2(r_j)$. If $q_1(r_i) = q_1(r_j)$, we add a shortcut from $u_i^{a_i}$ to $u_j^{a_j}$ for every pair of $a_i \in A(q_1(r_i))$ and $a_j \in A(q_1(r_j))$ such that $a_i \neq a_j$. For the case when $q_2(r_i) = q_2(r_j)$, we add a shortcut from $v_i^{a_i}$ to $v_j^{a_j}$ for every pair of $a_i, a_j$ such that $\pi_i(a_i) \neq \pi_j(a_j)$. We define the cost of this shortcut to be $(j - i)/2$.

This completes the hardness construction. It is easy to see that the instance size is polynomial (for completeness, we add the proof in Appendix).
whose both endpoints are in \( P \) into two types, edges that are \textit{contained} in \( P \) and edges that are \textit{induced} by \( P \), as follows.

\textbf{Definition 4.1.} We say that \( P \text{ contains} \) an edge \( e \) if \( e \) is an edge on \( P \), and we say that \( P \text{ induces} \) \( e \) if \( e \) is not an edge on \( P \) but both end vertices of \( e \) are on \( P \). We say that \( P \text{ involves} \) \( e \) if \( P \) contains or induces \( e \).

Observe that if \( P \) involves no shortcuts then we can construct a satisfying assignment from \( P \). For example, a path \( s_1u_{11}v_{11}t_1s_2u_{11}v_{11}t_2s_3u_{10}v_{10}t_3s_4u_{10}v_{10}t_4 \) involves no shortcuts and could be converted to an assignment \( x_1 = 1, x_2 = 1 \) and \( x_3 = 0 \). Conversely, a satisfying assignment of \( \Phi \) can also be converted to a solution (a price function) with respect to which the corresponding shortest path involves no shortcut edges. Moreover, observe that if \( P \) involves no shortcuts then we can get a revenue of \( M \) by setting price of all variable edges to 1 and we always get a revenue less than \( M \) otherwise. The following observation follows: \( \Phi \) has a satisfying assignment if and only if there is a solution that gives a revenue of \( M \) in the corresponding graph \( G \). This observation, along with the reduction from \textsc{Max 3SAT}, already lead to the NP-hardness of \textsc{StackSP}. This is in fact the essential idea used in the previous hardness results \cite{26, 23}.

\textbf{Beyond NP-hardness} To extend the above idea to a constant-hardness, we further observe an effect of the shortcuts on the revenue. In particular, we observe that if there are many “parts” of the shortest path that either contain or induce too many shortcuts then the revenue can be essentially at most \( M/2 \). To be more precise, let us first make the following two observations.

First, observe that if \( P \) contains shortcuts \( e_1, e_2, \ldots, e_k \), for some \( k \), with costs \( c_1, c_2, \ldots, c_k \) then we can collect a revenue of at most \( M - \sum_{i=1}^{k} c_i \) from \( P \). This is because there is a path of length \( M \) from \( s \) to \( t \) and, for each \( i \), once edge \( e_i \) with fixed cost \( c_i \) is used, the revenue on \( P \) decreases by \( c_i \). For example, the path \( P_1 = s_1u_{11}v_{11}u_{12}v_{10}t_2s_3u_{10}v_{10}t_3u_{11}v_{11}v_{10}t_4 \) contains two shortcuts \( u_{11}^1u_{10}^2 \) and \( u_{10}^3v_{11}^4 \) of cost of 1/2 each. Therefore, any solution in which such path is the corresponding shortest path gives a revenue of at most \( 4 - 1/2 - 1/2 = 3 \).

Secondly, consider when \( P \) induces a shortcut edge \( e' \) from gadget \( G_i \) to gadget \( G_j \) with cost \( c' \) \textit{and}, for some reason, the edges in the gadgets \( G_i \) and \( G_j \) have price at most \( 1 \) each. Then we can collect a revenue of roughly \( M - (j - i) + c' + 2 \). This is because we cannot collect more than \( c' + 2 \) on the subpath of \( P \) from gadget \( G_i \) to gadget \( G_j \). For example, consider a path \( P_2 = s_1u_{11}v_{11}t_1s_2u_{11}v_{11}t_2s_3u_{11}v_{11}t_3u_{11}v_{11}v_{11}t_4 \) which induces a shortcut \( v_{11}^1u_{11}^2 \) of cost 1. For a pricing that \( P_2 \) is the shortest path, we can collect a revenue of at most \( 3 \) for the following reason. First, we can collect at most 1 from edge \( u_{11}^1v_{11}^1 \) because edge \( s_1t_1 \) would be used otherwise. Similarly, we can collect at most 1 from edge \( u_{11}^3v_{11}^3 \) because edge \( \text{shortest path} \) \( v_{11}^1u_{11}^4 \) would be used otherwise.

In summary, the observations above imply that a shortcut from gadget \( i \) to gadget \( j \) (either contained or induced) causes the revenue on the subpath from gadget \( G_i \) to gadget \( G_j \) to be bounded by \( (j - i)/2 + 2 \).

\textbf{The role of} \((\delta, \gamma)\)-far sequence \ Before we proceed to show the consequence of these observations, we would like to eliminate the effect of the the constant “+2” in the bound of the revenue above since it will be an obstacle in the analysis. In particular, to get the factor of 2 hardness, we would like to say that we can get a revenue of roughly \( (j - i)/2 \) and somehow conclude that the graph reduced from \textsc{No-Instance} gives a revenue of at most \( M/2 \). (Recall that we can get a revenue of \( M \) in \textsc{Yes-Instance}.) However, the constant +2 is a problem when \( j - i \) is small.
We eliminate the above effect in a straightforward way: instead of including the shortcuts for every constraint, we consider only the shortcuts with large cost \((j - i)/2\). The problem is, when we throw away some constraints, the constraint satisfaction problem becomes easier, and we should be able to satisfy more fraction of the constraints. We do not want this to happen. We want to somehow make sure that by neglecting a particular set of “bad” constraints, the soundness parameter does not grow by much. Roughly speaking, Section 5.1 shows that we can get the desired properties while the soundness parameter remains comparatively small. In particular, we lose an additive factor of \(\gamma\) in the soundness parameter. (Please refer to Section 5.1 for more details.)

**Getting 2-approximation hardness**  Now that we can eliminate the effect of the constant +2, let us see how we can use the above two observations to conclude the 2-approximation hardness. Intuitively, the two observations above imply that if the shortest path \(P\) involves many shortcuts then the revenue we can collect on \(P\) is essentially at most \(M/2\). To prove this intuitive assertion, we argue in the next section that we can always decompose \(P\) into three types of paths – paths that look like \(P_1\), paths that look like \(P_2\) and paths that can be converted to the solution for \(\Phi\) such that the number of satisfied constraints is equal to the number of variable cost edges in such paths altogether. This decomposition needs to be carefully designed to maintain the properties of the three types of paths and will be elaborated in Section 5.3.

Using the above decomposition and the fact that paths of the first two types give a revenue of at most half of their lengths, we conclude that the revenue is at most \(M/2 + c\) where \(c\) is the number of edges in the paths of the third type. Using the fact that \(\Phi\) is \((\delta, \gamma)\)-far, we conclude that \(c\) is at most \((\gamma + \epsilon/3)M\) where \(\epsilon\) is the constant as in Theorem 1.1. By considering large enough \(n\) (and thus, large enough \(|\Phi|\)) and choosing an appropriate value of \(\delta\) and \(\gamma\) so that \(c \leq \epsilon M\), we have that the revenue is at most \((1/2 + \epsilon)M\). This implies the gap of \(2 - \epsilon\), and Theorem 1.1 thus follows. We formalize these ideas in the next section.

### 5 Analysis

Now we prove Theorem 1.1 using the reduction in Section 3. Recall that \(\epsilon\) is a constant as in Theorem 1.1 and we let \(\ell = \lceil \log(3/\epsilon)/\alpha \rceil\) (where \(\alpha\) is as in Theorem 2.2), \(\delta = (\epsilon/10)5^{-\ell}\) and \(\gamma = \epsilon/3\). It follows that the soundness parameter of the Raz verifier is \(2^{-\alpha \ell} \leq \epsilon/3\). (I.e., if \(\varphi\) is a NO-INSTANCE, then at most \(\epsilon/3\) fraction of constraints in \(\Phi\) can be satisfied.)

In this section, we show that when the size of \(\varphi\) (denoted by \(n\)) is large enough, the reduction gives a \((2 - \epsilon)\)-gap between the case when \(\varphi\) is satisfiable and when it is not. In particular, in section 5.1 we show that if \(\varphi\) is satisfiable, then there is a price function that collects a revenue of \(M\). Moreover, in Section 5.2 we show that if \(\varphi\) is not satisfiable and \(n\) is large enough, there is no pricing strategy which collects a revenue of more than \((1/2 + \epsilon)M\). The value of \(n\) will be specified in Section 5.2.

#### 5.1 Yes-Instance

Let \(f : Q_1 \to A_1, Q_2 \to A_2\) be an assignment that satisfies every constraint in \(\Phi\). For gadget \(G_i\) corresponding to the variable \(r_i\), set price 1 to the edge from \(u^a_i\) to \(v^a_i\) for \(a = f(q_1(r_i))\). Other variable cost edges in \(G_i\) are assigned the price of \(\infty\). We now show that we can collect a revenue of \(M\) in this case.

Let \(P\) be the shortest path on this graph with respect to the above pricing. Notice that path \(P\) does not contain any shortcut since a shortcut only goes between two edges that represent inconsistent assignments. (I.e., if there is a shortcut from \(v^{a_i}_j\) to \(u^{a_j}_i\) on \(P\) then either \(a_i\) is not consistent with \(a_j\) or \(\pi_i(a_i)\) is not consistent with \(\pi_j(a_j)\). Specifically, either \(q_1(r_i) = q_1(r_j)\) and \(a_i \neq a_j\), or \(q_2(r_i) = q_2(r_j)\) and \(\pi_i(a_i) \neq \pi_j(a_j)\). However, this is impossible since if \(q_1(r_i) = q_1(r_j)\) then \(a_i = a_j = f(q_1(r_i))\) and, similarly, if
\[ q_2(r_i) = q_2(r_j) \] then \( \pi_i(a_i) = \pi_j(a_j) = f(q_2(r_i)). \)

Since the shortcut is not used, the length of \( P \) is exactly \( M \). Moreover, observe that the path that uses all variable edges of price 1 also has length \( M \). This path is a shortest path and gives a total revenue of \( M \).

### 5.2 No-instance

We assume for contradiction that there is a pricing function which collects a revenue of \( (1/2 + \epsilon)M \). Let \( p \) be such pricing function and let \( P \) be the corresponding shortest path. Our goal is to construct an assignment that satisfies more than \( \epsilon M/3 \) constraints in \( \Phi \). This will contradict the soundness parameter \( \epsilon/3 \) of the Raz verifier.

**Definition 5.1.** A subpath \( Q \subseteq P \) is said to be a source-sink subpath of \( P \) if it starts at some source \( s_i \) and ends at some sink \( t_j \) for \( i \leq j \). For any source-sink subpath \( Q \), denote by \( s(Q) \) and \( t(Q) \) the gadget index to which the source and sink of \( Q \) belong respectively.

Now, let \( Q \) be any source-sink subpath and let \( s_i \) and \( t_j \) be its source and sink, respectively. Let \( \mathcal{S} = \{Q_1, \ldots, Q_k\} \) be a set of source-sink subpaths of path \( Q \). We say that \( \mathcal{S} \) is a source-sink partition of path \( Q \) if \( s(Q_1) = i, t(Q_k) = j \), and for all \( p < k \), we have \( t(Q_p) + 1 = s(Q_{p+1}) \).

The following theorem is the key idea to proving the result.

**Theorem 5.2 (Path Decomposition).** Let \( p : E_v \rightarrow R^+ \cup \{0\} \) be the optimal pricing of the variable edges and \( P \) be the corresponding shortest path in the graph. Then we can find sets \( \mathcal{R} \) and \( \mathcal{R}' \) such that the following properties hold.

- **D1.** \( \mathcal{R} \cup \mathcal{R}' \) is a source-sink partition of \( P \).
- **D2.** The total revenue collected from edges on paths in \( \mathcal{R}' \) is at most \( M/2 + O(1/\delta) \). In other words, \( \sum_{e \in E_v \cap (\bigcup_{P \in \mathcal{R}'} P)} p(e) \leq M/2 + O(1/\delta) \).
- **D3.** The price of any variable cost edge in \( \mathcal{R} \) is at most 1. That is, \( p(e) \leq 1 \) for any \( e \in E_v \cap (\bigcup_{P \in \mathcal{R}} P) \).
- **D4.** There is no shortcut between any two variable cost edges in \( \mathcal{R} \).

We defer the proof of this theorem to the next section. Meanwhile we show how the theorem implies that we can construct an assignment that satisfies more than \( \epsilon/3 \) fraction of the constraints in \( \Phi \), thus a contradiction to the soundness parameter. First, we consider only when \( n \) is sufficiently large so that we can collect at most \( M/2 + O(1/\delta) < M/2 + \epsilon M/3 \) from edges in \( \mathcal{R}' \) (from Property **D2**). Consequently, at least \( 2\epsilon M/3 \) must be collected from edges in \( \mathcal{R} \).

Let \( E' \) be the set of all variable cost edges that lie on some paths in \( \mathcal{R} \). From Property **D2** we have \( |E'| \geq 2\epsilon M/3 \). Let \( F \subseteq E' \) be the set of edges in \( E' \) that lie in far gadgets. Recall that we have at most \( \epsilon M/3 \) gadgets that are not far (after we run an algorithm \( A \) in Theorem **3.2**), so \( |F| \geq \epsilon M/3 \).

We are now ready to describe how we get an assignment that satisfies a large fraction of constraints in \( \Phi \). For each edge \( e \in F \), edge \( e \) can be written as \( w_i^{a_i}v_i^{a_i} \) for some gadget \( i \). We assign the answer \( a_i \) for query \( q_1(r_i) \) and \( a_i \) for query \( q_2(r_i) \). This assignment satisfies the constraint \( r_i \). This process satisfies at least \( \epsilon M/3 \) constraints corresponding to the edges in \( F \) provided that there is no conflict in assignment.

We argue that there is no such conflict since there is no shortcut between the edges in \( F \). I.e., assume that the above process creates a conflict assignment to the same query \( q \). This means that there are two constraints \( r_i, r_j \in \Phi \) for \( i < j \) with \( q = q_1(r_i) = q_1(r_j) \) or \( q = q_2(r_i) = q_2(r_j) \) and such query \( q \) was assigned different answers \( a_i \) and \( a_j \) when processing gadgets \( i \) and \( j \). Since both \( r_i \) and \( r_j \) are far gadgets, by construction, there must be a shortcut between two vertices \( v_i^{a_i} \) and \( v_j^{a_j} \). This contradicts the fact that there is no shortcut in \( \mathcal{R} \)
5.3 Proof of Theorem 5.2

Consider any source-sink subpath $Q$. Since there is a fixed-cost path of length $t(Q) - s(Q) + 1$ from $s_a(Q)$ to $t_a(Q)$, the revenue collected on $Q$ is at most $t(Q) - s(Q) + 1$, which will be denoted by $\text{len}(Q)$. We let $\text{rev}(Q)$ be the revenue collected on subpath $Q$, i.e. $\text{rev}(Q) = \sum_{e \in Q \cap E_v} p(e)$. First, observe the following lemma whose proof is simple and is deferred to Appendix.

**Lemma 5.3.** If $S = \{Q_1, \ldots, Q_k\}$ is a source-sink partition of $Q$, then $\sum_{j=1}^{k} \text{len}(Q_j) = \text{len}(Q)$.

We now explain the decomposition of the shortest path $P$ (from Theorem 5.2) into several source-sink subpaths. Each subpath is contained in one of the sets $R, S$ and $T$. In the end, we let $R'$ in the Theorem 5.2 equal to $T \cup S$. The composition consists of two phases. We next describe each phase and prove the properties in Theorem 5.2 along the way.

In the first phase, our goal is to make sure that $R$ contains only source-sink subpaths that do not contain any shortcut. Initially, we set $R, S$, and $T$ to $R = \{P\}$, and $S = T = \emptyset$. We then remove the portion of paths $P$ which contains the shortcut edges and add them to set $S$. We ensure that paths are always cut into source-sink subpaths. In particular, we do the following.

**Phase 1:** Initially, $R = \{P\}$ and $T = S = \emptyset$. While there exists a path $P' \in R$ that contains a shortcut edge, do the following. Let $v$ be any shortcut edge. Let $s_i$ be the last source vertex that appears before $v$ in $P'$ and let $t_j$ be the first sink vertex that appears after $v'$ in $P'$. We note that $i, j$ denote the gadget indices to which the vertices belong. First remove $P'$ from $R$. Denote by $Q$ the source-sink subpath of $P'$ from $s_i$ to $t_j$. We break $P'$ into three (possibly empty) source-sink subpaths $Q_1, Q$, and $Q_r$; (i) $Q_1$ starts at $s(Q)$ and ends at vertex $t_{i-1}$, (ii) $Q$ starts and ends at $s_i$ and $t_j$, respectively, and (iii) $Q_r$ starts at $t_{j+1}$ and ends at $t(Q)$. We then add $Q$ to $S$ and add $Q_1, Q_r$ back to $R$.

Consider the set $R' = S \cup T$. We show that, after this phase, the output satisfies properties [D1, D2] and [D3]. After the second phase, property [D4] will be satisfied while other properties remain to hold. Observe that property [D1] holds simply because the way we break path $P'$ guarantees that $s(Q_1) = s(P')$, $t(Q_1) + 1 = s(Q)$, $t(Q) + 1 = s(Q_r)$, and $t(Q_r) = t(P')$. The next two lemmas prove properties [D3] and [D2].

**Lemma 5.4** (Property [D3]). After Phase 1, $p(e) \leq 1$ for any variable edge $e \in E_v$ that belongs to some path $Q$ in $R$.

*Proof:* Since path $Q$ does not contain shortcuts, vertices $s_i$ and $t_i$ lie on $Q$ for all $s(Q) \leq i \leq t(Q)$. Recall that edge $e$ can be written in the form $u^a_i v_j^a$ for some $j$ and $a \in A(q_1(r_i))$. If $p(e) > 1$, we can obtain a path shorter than $P$ by using the fixed cost edge $s_j t_j$ of cost 1 instead of $s_j u^a_i v_j^a t_j$. This contradicts the fact that $P$ is a shortest path.

**Lemma 5.5** (Property [D2]). After the first phase, the revenue in $R' = S \cup T$ is at most $M/2 + O(1/\delta)$. In particular, $\sum_{Q \in S} \text{rev}(Q) \leq \frac{1}{2} \left( \sum_{Q \in S} \text{len}(Q) \right) + O(1/\delta)$.

*Proof:* We will need the following claim.

**Claim 5.6.** For each path $Q \in S$, we have $\text{rev}(Q) \leq (\text{len}(Q) + 1)/2$.  

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Proof. Consider path $Q \in S$ from $s_i$ to $t_j$. Recall that there is a path of length $\text{len}(Q)$ in $G$ from $s_i$ to $t_j$, so the total cost of $Q$ is at most $\text{len}(Q)$. It is, therefore, sufficient to prove that the total cost of the shortcuts contained in $Q$ is at least $(\text{len}(Q) - 1)/2$. The way we construct paths in $S$ guarantees that path $Q$ must be of the form

$$s_i \rightarrow u_{i_1}^{a_1} \Rightarrow v_{i_1}^{a_1} \rightarrow u_{i_2}^{a_2} \Rightarrow v_{i_2}^{a_2} \rightarrow \ldots \Rightarrow v_{i_q}^{a_q} \rightarrow t_j$$

where $i_1 = i$, $i_q = j$, and edges of the form $u_{i_x}^{a_x} \Rightarrow v_{i_x}^{a_x}$ are the variable cost edges, from which we can collect a revenue. Other edges of the form $v_{i_x}^{a_x} \rightarrow v_{i_{x+1}}^{a_{x+1}}$, for $1 \leq x < q$, are shortcuts. Hence the total cost of shortcuts can be written as a telescopic sum. $\sum_{x=1}^{q-1} (i_{x+1} - i_x) = (j - i)/2 = (\text{len}(Q) - 1)/2$. □

By the claim, $\sum_{Q \in S} \text{rev}(Q) \leq \sum_{Q \in S} (\text{len}(Q)/2 + 1/2) \leq \frac{1}{2} \left( \sum_{Q \in S} \text{len}(Q) \right) + |S|/2$. It then suffices to bound the size of set $S$ by $O(1/\delta)$. Notice that each path in $S$ contains at least one shortcut. Recall that, by the construction (cf. Section 3.2), each shortcut only goes from $v^t_i$ to $u^t_j$ if $|j - i| \geq \delta M$. Since the intervals in the set $\{(s(Q), t(Q)) : Q \in S\}$ are disjoint (by definition of source-sink partition), we can have at most $O(1/\delta)$ paths in $S$. □

This completes the description and the proof of Phase 1. Now every path in $R$ contains no shortcut. In phase 2, our goal is to eliminate the shortcuts between paths in $R$. (Note that these shortcuts are not contained in $P$.) Roughly speaking, we scan the gadgets from left to right and once we find such shortcut, we move the whole path that induces this shortcut to the set $T$. The detail is as follows.

Phase 2: Initially, we have $R$ and $S$ from Phase 1, and $T = \emptyset$. We proceed in iterations starting from iteration 1. The description of iteration $i$ is as follows:

- We first check if source $s_i$ belongs to some path in $R$. If not, we proceed to iteration $i + 1$.
- If $s_i$ does belong to any path $Q$ in $R$, we do the following. We check if there is a shortcut (that is not contained in $Q$) leaving from some vertex $v^t_i$ on $Q$ to some vertex $u^t_j$ on some path $Q' \in R$. Note that $Q$ and $Q'$ may be the same. Let $P' \subseteq P$ be the source-sink subpath from $s_i$ to $t_j$. We first remove from $R$ and $S$, all paths $Q''$ such that $Q'' \cap P' \neq \emptyset$. Let $Q_1$ be the source-sink subpath of $Q$ with $s(Q_1) = s(Q)$ and $t(Q_1) = t(P') - 1$. Also, we let $Q_r$ be the source-sink subpath of $Q'$ with $s(Q_r) = t(P') + 1$ and $t(Q_r) = t(Q')$. We add $P'$ to $T$, and add $Q_1$ and $Q_r$ back to $R$.

We now check the properties. Property [D1] holds simply because, in each iteration, we remove only subpaths of what we will add (i.e., we may add paths $Q$, $P'$ and $Q'$ to $R$ and $T$ and remove only subpaths of $Q \cup P' \cup Q'$). Since paths in $R$ only get chopped off, Lemma [5.4] still holds, and so does property [D3]. Properties [D4] and [D2] follow from the following Lemmas whose proofs are in Appendix.

Lemma 5.7 (Property [D4]). After Phase 2, there is no shortcut between any two subpaths in $R$.

Lemma 5.8 (Property [D2]). $\sum_{Q \in T} \text{rev}(Q) \leq \frac{1}{2} \left( \sum_{Q \in T} \text{len}(Q) \right) + O(1/\delta)$.

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APPENDIX

A Derandomization of Algorithm \( A' \) in Theorem 3.2

Now we derandomize \( A' \) to get a deterministic algorithm \( A \) by the method of conditional expectation. Let \( Y \) denote the number of constraints that are not \( \delta \)-far with respect to a random permutation \( \pi \). For a fixed permutation \( \pi \)'s, let \( \mathcal{E}(\pi', I) \) be the event that \( \pi \) agrees with \( \pi' \) on set \( I \) (i.e., \( \pi'(i) = \pi(i) \) for all \( i \in I \)). Notice that, we can efficiently compute \( \mathbb{E}[Y \mid \mathcal{E}(\pi', I)] \) for any \( \pi' \) and \( I \) where the expectation is over random permutation \( \pi \). Therefore, for \( i = 1, 2, \ldots \), we deterministically pick the value of \( \pi'(i) \) that maximizes the value of \( \mathbb{E}[Y \mid \mathcal{E}(\pi', \{1, \ldots, i - 1\})] \).

B Construction Size

We first calculate the size of each gadget \( G_i \). There are \( O(7^\ell) \) vertices and \( O(7^\ell) \) edges for each gadget. Next, we count the number of shortcuts. For each pair of constraints \( r_i \) and \( r_j \), there are at most \( O(7^{2\ell}) \) shortcuts between their intermediate vertices. Since there are \((5n)^\ell \) gadgets, the graph size is at most \( O(n^{1+\ell}) \).

Since \( \ell = \left\lceil \frac{\log(3/\epsilon)}{\alpha} \right\rceil \), the construction size is \( O(n^{O(1/\epsilon)}) \) which is polynomial in \( n \) if \( \epsilon \) is a constant.

C Omitted Proofs from Section 5

C.1 Proof of Lemma 5.3

\[
\sum_{j=1}^{k} \text{len}(Q_j) = \sum_{j=1}^{k} (t(Q_j) - s(Q_j) + 1) = t(Q_k) - s(Q_1) + 1 = t(Q) + 1 - s(Q) = \text{len}(Q)
\]

where the second equality is because \( t(Q_j) + 1 = s(Q_{j+1}) \) for all \( j \leq k \) and the third equality is because \( t(Q_k) = t(Q) \) and \( s(Q_1) = s(Q) \).

C.2 Proof of Lemma 5.7

Notice that once a shortcut leaving gadget \( i \) is found, the whole part of gadget \( i \) is removed completely from \( \mathcal{R} \). Therefore, after iteration \( i \), there is no shortcut leaving the vertex in \( P \cap G_i \) to other vertices lying on some path in \( \mathcal{R} \). (In fact, the vertex in \( P \cap G_i \) is not in any path in \( \mathcal{R} \) anymore.)

C.3 Proof of Lemma 5.8

Similarly to Claim 5.6, we can also bound the revenue on paths in \( T \) as summarized in the following claim whose proof can be found in Appendix.

Claim C.1. For each path \( Q \in T \), we have \( \text{rev}(Q) \leq \frac{1}{2} \text{len}(Q) + 2 \)

Proof. Consider path \( Q \in T \) from \( s_i \) to \( t_j \). Path \( Q \) can be written in the form:

\[
s_i \rightarrow u_i^{a_i} \Rightarrow v_i^{a_i} \rightarrow \ldots \rightarrow u_j^{a_j} \Rightarrow v_j^{a_j} \rightarrow t_j.
\]

Note that we do not assume any structure of the path from \( v_i^{a_i} \) to \( u_j^{a_j} \). Also, recall that edges \( u_i^{a_i} v_i^{a_i} \) and \( u_j^{a_j} v_j^{a_j} \) were in \( \mathcal{R} \) after Phase 1 and moved to \( T \) in Phase 2. Moreover, there is a shortcut edge from \( v_i^{a_i} \) to \( u_j^{a_j} \) (which is not in \( Q \)).

Now, let \( Q' \) be the subpath of \( Q \) from \( v_i^{a_i} \) to \( u_j^{a_j} \), and \( e_i, e_j \) be the edges \( u_i^{a_i} v_i^{a_i} \) and \( u_j^{a_j} v_j^{a_j} \), respectively. Then \( Q = s_i e_i Q' e_j t_j \). The revenue collected on \( Q \) comes from edges in \( Q' \) and \( e_i \) and \( e_j \). Since both \( e_i \) and \( e_j \) belonged to some paths in \( \mathcal{R} \) after Phase 1, we have \( p(e_i) + p(e_j) \leq 2 \) (cf. Lemma 5.4). Path \( Q' \)
can collect revenue of at most \((j - i)/2\) due to the fact that there is a shortcut edge \(v_i^a \to u_j^a\) of cost \((j - i)/2\). Overall, the revenue on \(Q\) is at most \((j - i)/2 + 2 < \frac{1}{2} \text{len}(Q) + 2\).

Since every path \(Q \in \mathcal{T}\) induces some shortcut edges (i.e., there is a shortcut edge between some pairs of vertices in \(Q\)), the length of such path is at least \(\delta M\). Therefore, \(|\mathcal{T}| \leq O(1/\delta)\). We apply Claim C.1 for every path in \(\mathcal{T}\) and sum them up. This immediately gives the lemma.
Figure 2: Example of graph $G$ constructed from Max 2SAT $(x_1 \lor x_2) \land (x_1 \lor x_3)$ with $\ell = 1$ repetition. Each gadget $G_i$ is noted with the corresponding constraints $r_i$ and each variable edge $u_i v_i$ is noted with the corresponding answer from Prover 1. Note that the corresponding answer from Prover 2 can be identified easily. For example, an edge $u_i v_i$ corresponds to assigning $x_i = 0$ and $x_j = 1$. The corresponding answer from Prover 1 is $x_1 = 1$ and $x_2 = 0$. Therefore, Prover 2’s corresponding answer for $u_1 v_1$ is $x_1 = 1$. Each edge $e$ is noted with the fixed cost $c(e)$. Unless otherwise specified, the shortcut edges have cost $1/2$.