A SANOV-TYPE THEOREM FOR EMPIRICAL MEASURES ASSOCIATED WITH THE SURFACE AND CONE MEASURES ON $\ell^p$ SPHERES

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We prove a large deviations principle (LDP) for the empirical measure of the coordinates of a random vector distributed according to the surface measure on a suitably scaled $\ell^p$ sphere in $\mathbb{R}^n$, as $n \to \infty$. This LDP is established for $p \in [1, \infty]$, with respect to the $q$-Wasserstein topology, for every $q < p$. We prove the result by first establishing an analogous LDP when the random vector is distributed according to the cone measure on the scaled $\ell^p$ sphere. In addition, we combine our LDP with the Gibbs conditioning principle to obtain an asymptotic probabilistic description of the geometry of an $\ell^p$ sphere under certain $\ell^q$ norm constraints, for $q < p$. These results are also of relevance for the study of the large deviations behavior of random projections of $\ell^p$ balls.

1. Introduction. Given a sequence of independent and identically distributed (i.i.d.) random variables $(X_n)_{n \in \mathbb{N}}$ sampled from a probability measure $\mu$ on $\mathbb{R}$, a classical result in probability theory is Sanov’s theorem, which states that the associated sequence of empirical measures $(L_n)_{n \in \mathbb{N}}$, where

$$L_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}, \quad n \in \mathbb{N},$$

(with $\delta_x$ representing the Dirac mass at $x \in \mathbb{R}$) satisfies a large deviations principle (LDP) with rate function given by the relative entropy with respect to $\mu$ (the precise definitions of an LDP, relative entropy, and other relevant notions are recalled in Section 2.1 and Section 2.2).

The goals of this paper are to establish an analog of Sanov’s theorem for empirical measures arising in a geometric context, and to establish a related conditional limit theorem. Specifically, in Theorem 2.5 (resp., Theorem 2.8), for $p \in [1, \infty]$, we prove an LDP for the sequence $(L_{n,p})_{n \in \mathbb{N}}$, where $L_{n,p}$ is the empirical measure of the coordinates of a random vector that is distributed according to the surface measure (resp., the cone measure) on a suitably scaled $\ell^p$ sphere in $\mathbb{R}^n$, and show that the associated rate function takes the form of a perturbed relative entropy functional. We point out that this LDP holds in the

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$q$-Wasserstein topology for every $q < p$. As summarized in Remark 2.4, we crucially utilize the fact that the LDP holds in the $q$-Wasserstein topology, which is stronger than the usual weak topology, in order to prove some interesting geometric consequences of the LDP for $(L_{a,p})_{n \in \mathbb{N}}$. For example, as elaborated below, we apply the LDP to establish the conditional limit result of Theorem 2.12. Consequently, Corollary 2.13 provides a precise version of the following (roughly stated) asymptotic probabilistic description of an $\ell^p$ sphere under an $\ell^q$ norm constraint: for $q < p$, in high dimensions, a random point on the $\ell^p$ sphere conditioned on having small $\ell^q$ norm is close (in the sense of distribution) to a random point drawn from an appropriately scaled $\ell^q$ sphere.

In contrast to the i.i.d. setting of the classical Sanov’s theorem, the coordinates $X_i^{(n,p)}, i = 1, \ldots, n$, of a random vector distributed according to the surface measure (or cone measure) of the $\ell^p$ unit sphere in $\mathbb{R}^n$ are dependent because $X^{(n,p)}$ is constrained to lie in $S_{n,p}$. Note that LDPs have previously been established for empirical measures of random variables with other dependency structures. For example, see [9, §6.3-6.6] for methods which apply to Markov chains or stationary sequences satisfying certain mixing conditions. Alternatively, in random matrix theory, a large deviations principle for the empirical spectral measure (for which there is strong interaction among the eigenvalues) can be found in [3]. However, these settings do not include the dependency structure of $X^{(n,p)}$ induced by the surface measure on the $\ell^p$ sphere.

A further motivation for our study arises from the fact that the empirical measure LDP of this paper will be applied in [12], in the proof of a variational formula relating certain “quenched” and “annealed” large deviations rate functions for random projections of $\ell^p$ balls.

In addition, we believe that our main question is of intrinsic interest, motivated by geometric considerations. Indeed, we are interested in such LDPs due to the classical mantra that LDPs (as opposed to just large deviation bounds) yield not only the asymptotic exponential rate of decay of the probability of a rare event, but also the most likely way in which such a rare event can occur. This general idea has been realized to great effect in statistical mechanics, where large deviations theory can describe the most probable state of a system of particles under an energy constraint (see, e.g., the surveys [4, 11]). Of particular utility is the so-called “Gibbs conditioning principle”, which transforms an LDP for empirical measures into a statement about the most probable behavior of the underlying sequence of random variables, conditional on a rare event. A central motivation for our work is to employ such a conditional probabilistic perspective in a geometric setting, by investigating how LDPs can inform the analysis of “geometric” rare events in high dimensions. One obstacle to this goal is that it is not a priori clear which rare events should be conditioned upon in order to obtain a meaningful result. In this work, we demonstrate that one natural example of a geometric rare event is a deviation of the $\ell^q$ norm, for $q < p$. At a high level, our second set of results, Theorem 2.12 and Corollary 2.13, analyze a geometric rare event by giving an asymptotic description of the surface measure on a high-dimensional $\ell^p$ sphere, conditional on a sufficiently small $\ell^q$ norm.

To be more concrete, first recall the classical i.i.d. setting (see, e.g., [7, 8, 22]): Sanov’s
Sanov-type theorem for empirical measures of $\ell^p$ spheres

Theorem and the associated Gibbs conditioning principle state that under mild assumptions, conditional on a large deviation of $L_n$, the empirical measure of $n$ i.i.d. random variables, the joint law of $k = o(n)$ of the variables is asymptotically (as $n \to \infty$) close to the $k$-fold product of a certain “exponentially tilted” distribution, under which the large deviation conditioning event is in fact typical. For a concrete example, suppose $(X_n)_{n \in \mathbb{N}}$ are i.i.d. exponential random variables with mean 1; then, conditional on the rare event $\{\int_{\mathbb{R}} x L_n(dx) = \frac{1}{n} \sum_{i=1}^{n} X_i > \beta\}$ for some $\beta > 1$, the joint law of $(X_1, \ldots, X_k)$ converges in total variation as $n \to \infty$ to that of $k$ i.i.d. exponential random variables with mean $\beta$.

In our geometric setting of $\ell^p$ spheres, we identify a suitable class of rare events to condition upon, and apply the Gibbs conditioning principle to obtain Theorem 2.12, which addresses the question of “how” an unlikely geometric event occurs (with fixed $k$ instead of $k = o(n)$), and weak convergence instead of total variation convergence. The primary novel consequence in this setting is that Corollary 2.13 lends a geometric meaning to our exponentially tilted conditional limit law.

The remainder of the paper is organized as follows. In Section 2, we recall some basic definitions and state our main results. In Section 3, we prove Theorem 2.8, our LDP result in the case of the cone measure. In Section 4, we show how the preceding result implies Theorem 2.5, our LDP result in the case of the surface measure. Lastly, in Section 5, we prove the conditional limit results, Theorem 2.12 and Corollary 2.13.

2. Preliminaries and main results.

2.1. The sequence of empirical measures. Let $p \in [1, \infty]$ and $x \in \mathbb{R}^n$. Denote the $\ell^p$ norm of $x$ by $\|x\|_{n,p} = (\sum_{i=1}^{n} |x_i|^p)^{1/p}$ for $p < \infty$, and $\|x\|_{n,\infty} = \max_{i=1,\ldots,n} |x_i|$ for $p = \infty$. Note that $\|\cdot\|_{n,2}$ denotes the usual Euclidean norm on $\mathbb{R}^n$. Let $S_{n,p} = \{x \in \mathbb{R}^n : \|x\|_{n,p} = 1\}$ be the unit $\ell^p$ sphere in $\mathbb{R}^n$. Let $\mathcal{B}(\mathbb{R}^n)$ denote the Borel sigma-algebra of $\mathbb{R}^n$, and let $\text{vol}_n(\cdot)$ denote Lebesgue measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Let $\text{area}_{n,p}(\cdot)$ denote the (unnormalized) surface measure on $S_{n,p}$; in other words, if we let $S_{n,p} = \{S = S' \cap S_{n,p} : S' \in \mathcal{B}(\mathbb{R}^n)\}$, then for $S \in S_{n,p}$,

$$\text{area}_{n,p}(S) \doteq \lim_{\varepsilon \downarrow 0} \frac{\text{vol}_n(\{x + t : x \in S, \|t\|_{n,2} \leq \varepsilon\})}{2\varepsilon}.$$ 

We consider the following natural probability measure on $S_{n,p}$.

**Definition 2.1.** Let $\sigma_{n,p}$ denote the (normalized) surface measure on $S_{n,p}$, which is defined as

$$\sigma_{n,p}(S) \doteq \frac{\text{area}_{n,p}(S)}{\text{area}_{n,p}(S_{n,p})}, \quad S \in S_{n,p}.$$ 

Note that for $p = 2$, the surface measure $\sigma_{n,2}$ is the unique rotation invariant probability measure on $S_{n,2}$, the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$.

Suppose all random variables we introduce are supported on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For a random variable $\xi$ and a measure $\mu$, we write $\xi \sim \mu$ if the law of $\xi$ is $\mu$;
that is, if $\mathbb{P} \circ \xi^{-1} = \mu$. For $n \in \mathbb{N}$ and $p \in [1, \infty]$, let $X^{(n,p)} = (X_1^{(n,p)}, \ldots, X_n^{(n,p)})$ be an $n$-dimensional random vector, and let $L_{n,p}$ be the empirical measure of the coordinates of $\mu^{1/p}X^{(n,p)}$.

\begin{equation}
L_{n,p} = \frac{1}{n} \sum_{i=1}^{n} \delta_{n^{1/p}X_i^{(n,p)}}.
\end{equation}

We aim to prove a large deviations principle for the sequence $(L_{n,p})_{n \in \mathbb{N}}$ under the assumption that $X^{(n,p)} \sim \sigma_{n,p}$. To (heuristically) understand why we scale the vector $X^{(n,p)}$ by a factor of $n^{1/p}$ in the definition of $L_{n,p}$, note that $X^{(n,p)}$ lies on the unit $\ell^p$ sphere in $\mathbb{R}^n$; thus, each coordinate $X_i^{(n,p)}$, $i = 1, \ldots, n$ is approximately of order $n^{-1/p}$, so each scaled coordinate $n^{1/p}X_i^{(n,p)}$ is approximately of order 1, which turns out to be the appropriate order of magnitude to analyze large deviations of empirical measures.

2.2. Background on large deviations. We recall below the definition of a large deviations principle.

**Definition 2.2.** Let $\mathcal{X}$ be a topological space equipped with its Borel $\sigma$-algebra. The sequence of probability measures $(\mu_n)_{n \in \mathbb{N}} \subset \mathcal{P}(\mathcal{X})$ is said to satisfy a large deviations principle (LDP) with a rate function $I : \mathcal{X} \to [0, \infty]$ if $I(\cdot)$ is lower semicontinuous, and for all Borel measurable sets $\Gamma \subset \mathcal{X}$,

\[- \inf_{x \in \Gamma^c} I(x) \leq \liminf_{n \to \infty} \frac{1}{n} \log \mu_n(\Gamma^c) \leq \limsup_{n \to \infty} \frac{1}{n} \log \mu_n(\bar{\Gamma}) \leq - \inf_{x \in \Gamma} I(x),\]

where $\Gamma^c$ and $\bar{\Gamma}$ denote the interior and closure of $\Gamma$, respectively. Furthermore, $I$ is said to be a good rate function if it has compact level sets. Similarly, we say the sequence of $\mathcal{X}$-valued random variables $(\xi_n)_{n \in \mathbb{N}}$ satisfies an LDP if the sequence of laws $(\mu_n)_{n \in \mathbb{N}}$ given by $\mu_n = \mathbb{P} \circ \xi_n^{-1}$ satisfies an LDP.

For a broad review of large deviations, we refer to [9]. In particular, suppose $X^{(n)} \sim \mu^\otimes n$ (the $n$-fold product measure) for some $\mu \in \mathcal{P}(\mathbb{R})$. Equivalently, suppose the coordinates $X_i^{(n)}, i = 1, \ldots, n$ of $X^{(n)}$ are independent and identically distributed (i.i.d.) with common distribution $\mu$. Then, Sanov’s theorem states that the sequence of associated empirical measures $L_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i^{(n)}}$ satisfies an LDP with good rate function $H(\cdot \| \mu)$, where $H$ is the relative entropy,

\begin{equation}
H(\nu \| \mu) = \begin{cases} 
\int_{\mathbb{X}} \log \left( \frac{d\nu}{d\mu} \right) d\nu & \text{if } \nu \ll \mu, \\
+\infty & \text{else},
\end{cases}
\end{equation}

where $\nu \ll \mu$ denotes that $\nu$ is absolutely continuous with respect to $\mu$.

Before we state the LDP and associated rate function in our setting, we first set some notation and definitions. For $q \in [1, \infty]$, let $m_q : \mathcal{P}(\mathbb{R}) \to \mathbb{R}_+$ be the map that takes a measure to its $q$-th absolute moment; that is, for $\nu \in \mathcal{P}(\mathbb{R})$,

\begin{equation}
m_q(\nu) = \int_{\mathbb{R}} |x|^q \nu(dx).
\end{equation}
For \( q = \infty \), let
\[
m_\infty(\nu) = \inf\{a > 0 : \nu([-a,a]^c) = 0\}.
\]

Our LDP will hold with respect to the so-called Wasserstein topologies on \( \mathcal{P}(\mathbb{R}) \) introduced below. We refer to [25, §6] for an extensive review of the Wasserstein topology.

**Definition 2.3.** Let \( q \in [1, \infty] \), and let
\[
\mathcal{P}_q(\mathbb{R}) = \{\mu \in \mathcal{P}(\mathbb{R}) : m_q(\mu) < \infty\}.
\]

A sequence of probability measures \((\mu_n)_{n \in \mathbb{N}} \subset \mathcal{P}_q(\mathbb{R})\) is said to converge to \( \mu \in \mathcal{P}_q(\mathbb{R}) \) with respect to the Wasserstein-\( q \) topology (or with respect to \( W_q \)) if \( \mu_n \Rightarrow \mu \) (converges weakly) and \( m_q(\mu_n) \to m_q(\mu) < \infty \) as \( n \to \infty \).

**Remark 2.4.** We consider the \( q \)-Wasserstein topology on probability measures (instead of, e.g., the weak topology or the \( \tau \) topology) because we must consider a topology that is weak enough to allow an LDP to hold, but at the same time strong enough to allow certain moments to be continuous functionals of measures. In particular, we require a topology strong enough such that the moment map \( m_q \) is continuous for \( q < p \), which is used in the proofs of both the Gibbs conditioning result of Theorem 2.12 and the variational formula of [12] in the context of random projections of \( \ell^p \) balls.

Let \( \mu_p \) denote the **generalized Gaussian** distribution with location 0, shape \( p \), and scale \( p^{1/p} \). That is, for \( p \in [1, \infty) \),
\[
\mu_p(dy) = \frac{1}{2p^{1/p}\Gamma(1 + \frac{1}{p})} e^{-|y|^{p}/p} dy, \quad y \in \mathbb{R}.
\]

The case \( p = 2 \) corresponds to the standard Gaussian distribution. For \( p = \infty \), define
\[
\mu_\infty(dy) = \frac{1}{2} 1_{[-1,1]}(y) dy, \quad y \in \mathbb{R}.
\]

As for the rate function itself, for \( p \in [1, \infty) \), let \( \mathbb{H}_p : \mathcal{P}(\mathbb{R}) \to [0, \infty] \) be a version of the relative entropy with respect to \( \mu_p \), perturbed by some \( p \)-th moment penalty:
\[
\mathbb{H}_p(\nu) = \begin{cases} 
H(\nu \| \mu_p) + \frac{1}{p} (1 - m_p(\nu)) & \text{if } m_p(\nu) \leq 1, \\
+\infty & \text{else},
\end{cases}
\]

where we take the convention \( \frac{1}{\infty} = 0 \).

**2.3. Main results.** Our first main result is as follows:

**Theorem 2.5 (LDP under the surface measure).** Let \( p \in [1, \infty] \) and assume \( X^{(n,p)} \sim \sigma_{n,p} \). For \( q < p \), the sequence of empirical measures \((L_{n,p})_{n \in \mathbb{N}} \) of (2.1) satisfies an LDP in \( \mathcal{P}_q(\mathbb{R}) \) equipped with the \( \mathcal{W}_q \) topology, with the convex good rate function \( \mathbb{H}_p \) of (2.5).
The proof of Theorem 2.5 is deferred to Section 4. Our approach is to first prove an LDP under a related measure on $\mathbb{S}_{n,p}$ called the cone measure, defined as follows.

**Definition 2.6.** Let $\gamma_{n,p}$ denote the cone measure on $\mathbb{S}_{n,p}$,

$$\gamma_{n,p}(S) = \frac{\text{vol}_n(\{cx : x \in S, c \in [0,1]\})}{\text{vol}_n(\mathbb{B}_{n,p})}, \quad S \in \mathbb{S}_{n,p}.$$ 

**Remark 2.7.** The cone measure corresponds to the surface measure on $\mathbb{S}_{n,p}$ if and only if $p = 1, 2, \text{ or } \infty$. See [19, §3] and [17, §3] for more extensive discussions.

Then, we establish the following LDP for the sequence of empirical measures $(L_n,p)_{n \in \mathbb{N}}$ when $X^{(n,p)}$ is distributed according to the cone measure $\gamma_{n,p}$.

**Theorem 2.8 (LDP under the cone measure).** Let $p \in [1, \infty]$ and assume $X^{(n,p)} \sim \gamma_{n,p}$. For $q < p$, the sequence of empirical measures $(L_{n,p})_{n \in \mathbb{N}}$ of (2.1) satisfies an LDP in $\mathcal{P}_q(\mathbb{R})$ equipped with the $W_q$ topology, with the convex good rate function $H_p$ of (2.5).

The proof of Theorem 2.8 is given in Section 3.

**Remark 2.9.** The Sanov-type LDP of Theorem 2.8 complements existing Glivenko-Cantelli-type LLN (law of large numbers) and Donsker-type CLT (central limit theorem) for the empirical measure $L_{n,p}$ under the cone measure [21, Theorem 1].

The analysis of the LDP for $(L_{n,p})_{n \in \mathbb{N}}$ when $X^{(n,p)}$ is distributed according to the cone measure is facilitated by a certain probabilistic representation for the cone measure. For example, it is well known that if $Z$ is an $n$-dimensional standard Gaussian random variable, then $Z/\|Z\|_{n,2}$ is uniformly distributed on the Euclidean sphere $\mathbb{S}_{n,2}$. More generally, we have the following:

**Lemma 2.10 ([19, §3] and [20, Lemma 1]).** Fix $n \in \mathbb{N}$ and $p \in [1, \infty]$. Suppose $X^{(n,p)} \sim \gamma_{n,p}$, and let $Y^{(n,p)} \sim \mu_{n,p}$. Then,

$$(2.6) \quad X^{(n,p)} \overset{(d)}{=} \frac{Y^{(n,p)}}{\|Y^{(n,p)}\|_{n,p}}.$$ 

This representation allows us to exploit the underlying independence structure of the cone measure and leverage results from classical large deviations theory in the proof of Theorem 2.8.

Our next set of results provide insight into the most probable asymptotic behavior of $X^{(n,p)}$, conditioned on a certain class of rare events. Before stating our results, recall the following Poincaré-Maxwell-Borel type result for the asymptotic independence induced by $\gamma_{n,p}$ and $\sigma_{n,p}$.
LEMMA 2.11. Fix \( p \in [1, \infty) \) and \( k \in \mathbb{N} \). For \( n \in \mathbb{N} \) such that \( k \leq n \), suppose that either \( X^{(n,p)} \sim \sigma_n \) or \( X^{(n,p)} \sim \gamma_n \). Then, as \( n \to \infty \),

\[
\text{Law}\left[n^{1/p} \left(X_{1}^{(n,p)}, \ldots, X_{k}^{(n,p)}\right)\right] \Rightarrow \nu_{p}^{\otimes k}.
\]

The preceding lemma is classical in the case \( p = 2 \). For general \( p \in [1, \infty] \), Lemma 2.11 is due to Theorem 4.1 and Theorem 4.4 of [19] (in the case of the cone measure) and [16] (in the case of the surface measure). In addition, Theorem 3 and Theorem 4 of [18] offer a simplification of the proof for both the cone and surface measure. In fact, the results we cite are stated in the form of a finite \( n \) bound on the total variation distance, but the asymptotic weak convergence result stated in Lemma 2.11 is all we will need in the sequel.

Our next result yields a “conditional” version of the preceding lemma. Let \( h(\cdot) \) denote the differential entropy of a probability measure \( \nu \in \mathcal{P}(\mathbb{R}) \),

\[
h(\nu) = \begin{cases} 
- \int_{\mathbb{R}} \frac{d\nu}{dx} \log \left( \frac{d\nu}{dx} \right) dx & \text{if } \nu \ll \text{Lebesgue measure on } \mathbb{R}, \\
-\infty & \text{else.}
\end{cases}
\]

Differential entropy arises naturally in the analysis of \( \mathbb{H}_p \), since for measures \( \nu \in \mathcal{P}(\mathbb{R}) \) that are absolutely continuous with respect to Lebesgue measure and satisfy \( m_{p}(\nu) \leq 1 \),

\[
\mathbb{H}_p(\nu) = \int_{\mathbb{R}} \log\left(\frac{d\nu}{d\mu_p}\right)d\nu + \frac{1}{p}(1 - m_p(\nu))
= \int_{\mathbb{R}} \log\left(\frac{d\nu}{dx}\right)d\nu + \int_{\mathbb{R}} \log\left(\frac{d\nu}{d\mu_p}\right)d\nu + \frac{1}{p}(1 - m_p(\nu))
= -h(\nu) + \frac{1}{p}m_p(\nu) + \log(2p^{1/p}\Gamma(1 + \frac{1}{p})) + \frac{1}{p}(1 - m_p(\nu))
= -h(\nu) + c_p,
\]

for a finite constant \( c_p \) that depends only on \( p \), but not on \( \nu \). Using the LDP of Theorem 2.8, we can obtain the following conditional limit theorem, which involves a constrained maximum entropy problem.

THEOREM 2.12 (Conditional limit theorem). Fix \( p \in [1, \infty] \) and suppose that either \( X^{(n,p)} \sim \sigma_n \) or \( X^{(n,p)} \sim \gamma_n \). Fix a closed interval \( C = [\alpha, \beta] \subset \mathbb{R} \), and for \( \varepsilon > 0 \), let \( C_\varepsilon = [\alpha - \varepsilon, \beta + \varepsilon] \). Then, for \( q < p \), the optimizing measure

\[
\nu_\ast = \arg \max \{h(\nu) : m_p(\nu) \leq 1, m_q(\nu) \in C\}
\]

is well defined (i.e., exists and is unique), and

\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \mathbb{P}(L_{n,p} \in \cdot : m_q(L_{n,p}) \in C_\varepsilon) = \delta_{\nu_\ast}.
\]

Moreover, for \( k \in \mathbb{N} \),

\[
\mathbb{P}\left(n^{1/p}(X_{1}^{(n,p)}, \ldots, X_{k}^{(n,p)}) \in \cdot : \frac{1}{n}m_{p}X^{(n,p)} \|q_{n,q} \in \mathcal{C}_{\varepsilon}\right) \Rightarrow \nu_\ast^{\otimes k},
\]

as \( n \to \infty \) followed by \( \varepsilon \to 0 \).
The proof of Theorem 2.12 is given in Section 5.

Note that the conditioning event of (2.11) is merely a restatement of the conditioning event of (2.10). That is, for any \( q, p \in [1, \infty] \), we have the identity

\[
\frac{1}{n} \| n^{1/p} X^{(n,p)} \|_{q,n} = \frac{1}{n} \sum_{i=1}^{n} | n^{1/p} X^{(n,p)} |^q = m_q(L_{n,p}).
\]

We now describe an interesting application of Theorem 2.12 that admits a precise geometric interpretation. Specifically, we show that in high dimensions, when a random sample from the surface measure of a (suitably scaled) \( \ell^p \) sphere is conditioned on having a sufficiently small \( \ell^q \) norm (for \( q < p \)), then it behaves like a sample from the surface measure of a corresponding \( \ell^q \) sphere. That is, the particular conditioning operation specified in (2.14) induces a \textit{probabilistic} change that admits a \textit{geometric} interpretation. A precise statement is given below. First, define for \( q, p \in [1, \infty) \),

\[
\beta_{p,q} = \frac{1}{q} \left( \frac{\Gamma\left(\frac{q}{q-p}+1\right)}{\Gamma\left(\frac{q}{q-p}\right)} \right)^{1/q}.
\]

Roughly speaking, the constant \( \beta_{p,q} \) is chosen such that for \( \beta \leq \beta_{p,q} \) and the interval \( C = [0, \beta] \), the variational problem of (2.9) has a solution with a natural geometric interpretation. Also, note that this constant \( \beta_{p,q} \) is “small”, in the sense that if \( q < p \), then \( \beta_{p,q} < 1 \); we refer to Remark 5.4 for why the following result of Corollary 2.13 is stated only for “small \( \beta \”).

Note that the variational problem of (2.9) does have an explicit solution even for intervals \( C = [0, \beta] \) where \( \beta > \beta_{p,q} \). For further discussion of this issue, we refer to Remark 5.5 for why the “large \( \beta \)” setting is less interesting, and to Remark 5.6 for some comments on an “intermediate \( \beta \)” setting. We now state the aforementioned application of Theorem 2.12, which applies in the “small \( \beta \)” regime.

**Corollary 2.13.** Fix \( p \in [1, \infty) \) and suppose that either \( X^{(n,p)} \sim \sigma_{n,p} \) or \( X^{(n,p)} \sim \gamma_{n,p} \). Fix \( q < p \) and \( \beta \leq \beta_{p,q} \), and let \( \lambda_{n,q} = \lambda_{n,q}^{(\beta,k)} \) denote the law of the \( k \) coordinates \( \beta^{1/q} n^{1/q} (X^{(n,p)}_1, \ldots, X^{(n,p)}_k) \), assuming either \( X^{(n,p)} \sim \sigma_{n,q} \) or \( X^{(n,p)} \sim \gamma_{n,q} \). Furthermore, for \( \varepsilon > 0 \), let \( \hat{\lambda}_{n,p|q,\varepsilon} = \hat{\lambda}_{n,p|q,\varepsilon}^{(\beta,k)} \) denote the conditional law,

\[
\hat{\lambda}_{n,p|q,\varepsilon} = \mathbb{P}\left( n^{1/p} (X^{(n,p)}_1, \ldots, X^{(n,p)}_k) \in \cdot \mid \frac{1}{n} \| n^{1/p} X^{(n,p)} \|_{q,n} \leq \beta + \varepsilon \right).
\]

Lastly, let \( \rho \) be a metric which metrizes the topology of weak convergence of probability measures (e.g., Lévy-Prohorov, bounded Lipschitz). Then, we have

\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \rho\left( \lambda_{n,q}, \hat{\lambda}_{n,p|q,\varepsilon} \right) = 0.
\]

We prove Corollary 2.13 in Section 5.
To clarify the $\beta^{1/q} n^{1/q}$ scaling in Corollary 2.13, note that $\hat{\lambda}_{n,p,q,\varepsilon}$ is the law of the $k$ coordinates $n^{1/p}(X_{1}^{(n,p)},\ldots,X_{k}^{(n,p)})$ conditioned on the event \{\$n^{1/p}X_{1}^{(n,p)} \in A_{n,\varepsilon}\}, where
\[
A_{n,\varepsilon} \doteq \{x \in \mathbb{R}^{n}: \frac{1}{n}\|x\|_{n,q}^{q} \leq \beta + \varepsilon\}.
\]
Conditional on the rare event that $n^{1/p}X_{1}^{(n,p)}$ lies in $A_{n,\varepsilon}$, we intuitively expect that for large $n$, the conditional law $\hat{\lambda}_{n,p,q,\varepsilon}$ will be close to a measure that concentrates on $A_{n,\varepsilon}$. This intuition motivates the introduction of the distribution $\sigma_{n,q}$ (or $\gamma_{n,q}$) and the scaling $\beta^{1/q}$, which induce a distribution that concentrates on $A_{n,\varepsilon}$. That is, it is immediate from the definition of $\sigma_{n,q}$ and $\gamma_{n,q}$ that for all $n \in \mathbb{N}$ and \(\mathbb{P}\text{-a.s.}, \frac{1}{n}\|\beta^{1/q} n^{1/q}X_{1}^{(n,p)}\|_{n,q} = \beta\). It follows that $\mathbb{P}(\beta^{1/q} n^{1/q}X_{1}^{(n,p)} \in A_{n,\varepsilon}) = 1$ for all $\varepsilon > 0$.

3. LDP under the cone measure. Throughout this section, fix $p \in [1, \infty]$ and for $n \in \mathbb{N}$, let $L_{n,p}$ be defined as in (2.1) with $X_{1}^{(n,p)} \sim \gamma_{n,p}$. Also, let $Y_{1}^{(n,p)} \sim \mu_{p}^{\otimes n}$, where $\mu_{p}$ is as defined in (2.4), and let $L_{n,p}^{Y}$ denote the empirical measure of $Y_{1}^{(n,p)}$,

\[
L_{n,p}^{Y} \doteq \frac{1}{n} \sum_{i=1}^{n} \delta_{Y_{i}^{(n,p)}}.
\]

In view of the representation in Lemma 2.10, our proof of Theorem 2.8 consists of the following steps:

1. write $L_{n,p}$ as a mapping of $L_{n,p}^{Y}$ and its $p$-th absolute moment $m_{p}(L_{n,p}^{Y})$, and show that this mapping is continuous with respect to the weak topology (Lemma 3.1);
2. prove a (joint) LDP for $(L_{n,p}^{Y}, m_{p}(L_{n,p}^{Y}))$ in $\mathbb{P}(\mathbb{R}) \times \mathbb{R}_{+}$ (Lemma 3.2 and Lemma 3.3);
3. apply the contraction principle to obtain the desired LDP for $(L_{n,p})_{n \in \mathbb{N}}$ in the weak topology (Proposition 3.4);
4. show convexity of $\mathbb{H}_{p}$ (Lemma 3.6);
5. extend the LDP to the Wasserstein topology, and conclude the proof.

To begin with, let $G_{p} : \mathbb{P}(\mathbb{R}) \times \mathbb{R}_{+} \to \mathbb{P}(\mathbb{R})$ be defined as

\[
G_{p}(\nu, c) \doteq \nu(\cdot \times c^{1/p}).
\]

Note that by Lemma 2.10, for $S \in \mathbb{B}(\mathbb{R})$,

\[
L_{n,p}(S) = \frac{1}{n} \sum_{i=1}^{n} \delta_{n^{1/p}X_{i}^{(n,p)}}(S) = \frac{1}{n} \sum_{i=1}^{n} \delta_{n^{1/p}Y_{i}^{(n,p)}}(S) = \frac{1}{n} \sum_{i=1}^{n} \delta_{Y_{i}^{(n,p)}}(\frac{\|Y_{i}^{(n,p)}\|_{n,p}}{n^{1/p}}S).
\]

Rewriting this in terms of the $p$-th moment $m_{p}$ defined in (2.3), we have

\[
L_{n,p} \doteq (d) G_{p}(L_{n,p}^{Y}, m_{p}(L_{n,p}^{Y})).
\]

When not otherwise specified, we equip $\mathbb{P}(\mathbb{R})$ with the weak topology, $\mathbb{R}_{+}$ with the Euclidean topology, and $\mathbb{P}(\mathbb{R}) \times \mathbb{R}_{+}$ with the induced product topology.
Lemma 3.1. The map $G_p : \mathcal{P}(\mathbb{R}) \times \mathbb{R}_+ \to \mathcal{P}(\mathbb{R})$ of (3.1) is continuous.

Proof. It is sufficient to prove sequential continuity since the product topology is metrizable, and thus, sequential. Fix $(\nu, c) \in \mathcal{P}(\mathbb{R}) \times \mathbb{R}_+$ and a convergent sequence of probability measures $\nu_n \Rightarrow \nu$ and positive reals $c_n \to c$. For $n \in \mathbb{N}$, let $\xi_n \sim v_n$ and $\xi \sim v$. Then, by Slutsky’s theorem, $c_n^{-1/p} \xi_n \Rightarrow c^{-1/p} \xi$. Since $c_n^{-1/p} \xi_n \sim v_n(\cdot \times c_n^{1/p})$ and, likewise, $c^{-1/p} \xi \sim v(\cdot \times c^{1/p})$, this proves that $G_p(\nu_n, c_n) \Rightarrow G_p(\nu, c)$. 

Lemma 3.2. The sequence $(L_{n,p}Y, m_p(L_{n,p}Y))_{n \in \mathbb{N}}$ satisfies an LDP in $\mathcal{P}(\mathbb{R}) \times \mathbb{R}_+$ with the good rate function $J$ defined as follows: for $\nu \in \mathcal{P}(\mathbb{R})$ and $c \in \mathbb{R}_+$, let

\[
J(\nu, c) = \sup_{f \in C_b(\mathbb{R}), \nu \in \mathbb{R}_+} \left\{ \int f(y) \nu(dy) + tc - \log \int e^{f(y)+|y|^p} \mu_p(dy) \right\},
\]

where $C_b(\mathbb{R})$ denotes the space of all bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

Proof. The $n$-th term in our sequence of interest, $S_n \equiv (L_{n,p}Y, m_p(L_{n,p}Y))$ is the empirical mean of the i.i.d. random variables $\xi_i \equiv (\delta_{\xi_i(\cdot, p)}, |Y_i(\cdot, p)|^p)$, $i = 1, \ldots, n$, which take values in the Polish space $\mathcal{P}(\mathbb{R}) \times \mathbb{R}_+$. To establish the LDP for $(S_n)_{n \in \mathbb{N}}$, we will first apply Cramér’s theorem for general Polish spaces, as can be found in Theorem 6.1.3 of [9], to show that $(S_n)_{n \in \mathbb{N}}$ satisfies a weak LDP with rate function $J$. To verify the conditions of that theorem, let $\mathcal{M}(\mathbb{R})$ be the space of finite regular Borel measures on $\mathbb{R}$, equipped with the weak topology (i.e., the coarsest topology such that the functionals $\nu \mapsto \int f d\nu$ are continuous for all $f \in C_b(\mathbb{R})$), and let $\mathcal{X} = \mathcal{M}(\mathbb{R}) \times \mathbb{R}$ be endowed with the product topology. Note that $\mathcal{X}$ is a locally convex Hausdorff topological space (being the product of two such spaces), $\mathcal{P}(\mathbb{R}) \times \mathbb{R}_+$ is a closed convex subset of $\mathcal{X}$, and the relative topology coincides with the product topology on $\mathcal{P}(\mathbb{R}) \times \mathbb{R}_+$. This shows that Assumption 6.1.2(a) of [9] is satisfied. In view of Remark (b) on p. 253 of [9], to verify the remaining condition, which is stated in Assumption 6.1.2(b) of [9], it suffices to show that there exists a metric $d(\cdot, \cdot)$ on $\mathcal{P}(\mathbb{R}) \times \mathbb{R}_+$ (compatible with its topology) that satisfies the following convexity condition: for all $\alpha \in [0, 1]$ and $x_1, x_2, y_1, y_2 \in \mathcal{X}$,

\[
d(\alpha x_1 + (1 - \alpha) x_2, \alpha y_1 + (1 - \alpha) y_2) \leq \max\{d(x_1, y_1), d(x_2, y_2)\}.
\]

It is known that the convexity condition (3.4) holds for $d_{lp}$, the Lévy-Prohorov metric which metrizes weak convergence on $\mathcal{P}(\mathbb{R})$ [9, p. 261], and it is easy to see that it also holds for $d_{eu}$, the Euclidean metric on $\mathbb{R}_+$. Elementary calculations show that (3.4) also holds for the product metric $d_{oe}$, defined by

\[
d_{oe}(\mu, s, (\nu, t)) = \max\{d_{lp}(\mu, \nu), d_{eu}(s, t)\},
\]

for $\mu, \nu \in \mathcal{P}(\mathbb{R})$ and $s, t \in \mathbb{R}$.

We can now apply Theorem 6.1.3 of [9] to conclude that $(S_n)_{n \in \mathbb{N}}$ satisfies a weak LDP with rate function

\[
\Lambda^*(x) = \sup_{\lambda \in \mathcal{X}} \{\langle \lambda, x \rangle - \Lambda(\lambda)\}, \quad x \in \mathcal{X},
\]
where $\mathcal{X}^*$ is the topological dual of $\mathcal{X}$, $\langle \cdot, \cdot \rangle$ denotes the duality pairing of $\mathcal{X}^*$ and $\mathcal{X}$, and

$$\Lambda (\lambda) = \log \mathbb{E} [e^{\langle \lambda, \xi \rangle}], \quad \lambda \in \mathcal{X}^*.$$ 

To show that $\Lambda^* = J$ as given in (3.3), note that for $\mathcal{X} = \mathcal{M}(\mathbb{R}) \times \mathbb{R}$,

$$\mathcal{X}^* = (\mathcal{M}(\mathbb{R}) \times \mathbb{R})^* = (\mathcal{M}(\mathbb{R}))^* \times \mathbb{R}^* \simeq C_b(\mathbb{R}) \times \mathbb{R},$$

where $\simeq$ means isomorphic. That is, for $\lambda \in \mathcal{X}^*$, there exists a unique $(f, t) \in C_b(\mathbb{R}) \times \mathbb{R}$ such that for all $x = (v, y) \in \mathcal{M}(\mathbb{R}) \times \mathbb{R} = \mathcal{X}$, we have $\langle \lambda, x \rangle = \int f dv + ty$. Therefore,

$$\Lambda (\lambda) = \log \mathbb{E} \left[ \exp \left( \langle \lambda, (\delta_{Y_1^{(n,p)}}, |Y_1^{(n,p)}|^p) \rangle \right) \right] = \log \int_\mathbb{R} \exp(f(y) + ty)^p \mu_p(dy).$$

To strengthen the weak LDP for $(S_n)_{n \in \mathbb{N}}$ to a full LDP and show that $J$ is a good rate function, by Lemma 1.2.18 of [9], it suffices to show that the sequence $(S_n)_{n \in \mathbb{N}}$ is exponentially tight, or equivalently, that the sequences $(L_{n,p})_{n \in \mathbb{N}}$ and $(m_p(L_{n,p}))_{n \in \mathbb{N}}$ are both exponentially tight. However, it follows from Lemma 6.2.6 of [9] that the sequence $(L_{n,p})_{n \in \mathbb{N}}$ is exponentially tight. Moreover, note that 0 is in the interior of the domain of the log moment generating function of $|Y_1^{(n,p)}|^p$. Hence, it follows from Corollary 6.1.6 of [9] and Lemma 2.6 of [15] that the sequence $(m_p(L_{n,p}))_{n \in \mathbb{N}}$ is also exponentially tight. To conclude, the sequence $(L_{n,p}, m_p(L_{n,p}))_{n \in \mathbb{N}}$ is exponentially tight and satisfies a weak LDP, and hence, satisfies a full LDP with good rate function $J$. 

**Lemma 3.3.** The good rate function $J$ of (3.3) can be written as follows: for $v \in \mathcal{P}(\mathbb{R})$ and $c \in \mathbb{R}_+$,

$$J(v, c) = \begin{cases} H(v\|\mu_p) + \frac{1}{p}(c - m_p(v)) & \text{if } m_p(v) \leq c, \\ +\infty & \text{else.} \end{cases} \tag{3.5}$$

**Proof.** For $t \in (-\infty, \frac{1}{p})$, define $\mu_p^{(t)} \in \mathcal{P}(\mathbb{R})$ as follows:

$$\mu_p^{(t)}(dy) = \frac{(1 - pt)^{1/p}}{2p^{1/p} \Gamma(1 + \frac{1}{p})} e^{-(1-pt)|y|^p/p} dy. \tag{3.6}$$

Note that $\mu_p = \mu_p^{(0)}$, and $d\mu_p^{(t)} / d\mu_p = (1 - pt)^{1/p} e^{(y)^p}$.

To prove the equality (3.5), we apply the Donsker-Varadhan variational formula for relative entropy (see, e.g., Lemma 1.4.3(a) of [10]), and use the expression (3.6) for $\mu_p^{(t)}$. Note that for $t \geq \frac{1}{p}$, we have $\log \int_\mathbb{R} e^{f(y) + ty^p} \mu_p(dy) = +\infty$, so it suffices to take the supremum of (3.3) over $t < \frac{1}{p}$. Thus, for $(v, c) \in \mathcal{P}(\mathbb{R}) \times \mathbb{R}_+$,

$$J(v, c) = \sup_{t < \frac{1}{p}} \left\{ tc + \frac{1}{p} \log(1 - pt) + \sup_{f \in C_b(\mathbb{R})} \left[ \int_\mathbb{R} f(y) v(dy) - \log \int_\mathbb{R} e^{f(y)} \mu_p^{(t)}(dy) \right] \right\}$$

$$= \sup_{t < \frac{1}{p}} \left\{ tc + \frac{1}{p} \log(1 - pt) + H(v\|\mu_p^{(t)}) \right\}.$$
We can rewrite each term in the supremum as follows:

\[
tc + \frac{1}{p} \log(1 - pt) + H(v|\mu_p(t)) = tc + \frac{1}{p} \log(1 - pt) + \int_{\mathbb{R}} \log \left( \frac{d\nu}{d\mu_p}(y) \right) v(dy) - \int_{\mathbb{R}} \log \left( \frac{d\mu_p(t)}{d\mu_p}(y) \right) v(dy) = tc + \frac{1}{p} \log(1 - pt) + H(v|\mu_p) - \left( \frac{1}{p} \log(1 - pt) + tm_p(v) \right) = tc + m_p(v) + H(v|\mu_p).
\]

Thus, if \(m_p(v) > c\), then \(J(v, c) = +\infty\); otherwise, if \(m_p(v) \leq c\), then

\[
J(v, c) = H(v|\mu_p) + \sup_{t < 1/p} \{tc - m_p(c)\},
\]

which proves the expression in (3.5).

\[\Box\]

**Proposition 3.4.** Let \(p \in [1, \infty]\) and assume \(X^{(n,p)} \sim \gamma_{n,p}\). The sequence \((L_{n,p})_{n \in \mathbb{N}}\) of (2.1) satisfies an LDP in \(\mathcal{P}(\mathbb{R})\) equipped with the weak topology, with the good rate function \(J_p\) of (2.5).

**Proof.** Due to the representation (3.2), the continuity of the map \(G_p\) established in Lemma 3.1, and the LDP for the sequence \((L_{n,p}, m_p(L_{n,p}))_{n \in \mathbb{N}}\) from Lemma 3.2, the contraction principle yields an LDP for \((L_{n,p})_{n \in \mathbb{N}}\) with the good rate function

\[
J_p(v) = \inf \{J(\lambda, c): \lambda \in \mathcal{P}(\mathbb{R}), c \in \mathbb{R}_+, G_p(\lambda, c) = v\}.
\]

It remains to show that the rate function \(J_p\) is identical to \(J_p\) of (2.5).

Let \(v \in \mathcal{P}(\mathbb{R})\). From the definition of \(G_p\) in (3.1), we see that if \(G_p(\lambda, c) = v\) for some \(\lambda \in \mathcal{P}(\mathbb{R})\) and \(c \in \mathbb{R}_+\), then \(\lambda = v(\cdot \times c^{-1/p})\) and \(m_p(\lambda) = cm_p(v)\). Therefore, using the representation for \(J(v, c)\) in Lemma 3.3, we have

\[
J_p(v) = \inf \left\{ \int_{\mathbb{R}} H(v(\cdot \times c^{-1/p})|\mu_p) + \frac{1}{p} (c - cm_p(v)) \right\} = \inf_{c \geq 0} \left\{ \int_{\mathbb{R}} H\left( \frac{dv(\cdot \times c^{-1/p})}{d\mu_p}(x) \right) v(dx) + \frac{1}{p} (c - cm_p(v)) \right\}
\]

if \(m_p(v) \leq 1\),

else.

If \(m_p(v) > 1\), then \(J_p(v) = \infty\). If \(m_p(v) \leq 1\), then use the fact that

\[
H(v(\cdot \times c^{-1/p})|\mu_p) = \int_{\mathbb{R}} \log \left( \frac{dv(\cdot \times c^{-1/p})}{d\mu_p}(x) \right) v(dx) = \int_{\mathbb{R}} \log \left( \frac{dv(\cdot \times c^{-1/p})(c^{1/p}y)}{d\mu_p(c^{1/p}y)} \right) v(dy) = \int_{\mathbb{R}} \log \left( \frac{dv}{d\mu_p(\cdot \times c^{1/p})}(y) \right) v(dy) = H(v|\mu_p(\cdot \times c^{1/p})).
\]
to find that
\[
\mathbb{J}_p(v) = \inf_{c \geq 0} \left\{ H(v \| \mu_p(\cdot \times c^{1/p})) + \frac{c}{p} (1 - m_p(v)) \right\}
\]
\[
= H(v \| \mu_p) + \inf_{c \geq 0} \left\{ \int_{\mathbb{R}} \log \left( \frac{d\mu_p}{d\mu_p(\cdot \times c^{1/p})}(y) \right) v(dy) + \frac{c}{p} \right\}
\]
\[
= H(v \| \mu_p) + \inf_{c \geq 0} \left\{ -\frac{1}{p} \log c - \frac{1-c}{p} m_p(v) + \frac{c}{p} \right\}
\]
\[
= H(v \| \mu_p) - \frac{1}{p} m_p(v) + \frac{1}{p} \inf_{c \geq 0} \{ c - \log c \}
\]
\[
= \mathbb{H}_p(v).
\]

Note that Proposition 3.4 is non-trivial because it cannot be obtained via a simple application of the contraction principle to the map \( v \mapsto (v, m_p(v)) \). Indeed, there does not appear to be a standard topology on \( \mathcal{P}(\mathbb{R}) \) such that both the sequence \((L_{n,p})_{n \in \mathbb{N}}\) satisfies an LDP with good rate function and the map \( m_p(\cdot) \) is continuous with respect to that topology. For example, consider \( v_n = \frac{n-1}{n} \delta_0 + \frac{1}{n} \delta_{n^{1/p}} \); then, \( v_n \Rightarrow \delta_0 \) in the weak topology and \( m_p(\delta_n) = 1 \) for all \( n \), but \( m_p(\delta_0) = 0 \). The same counterexample holds for the \( \tau \) topology on \( \mathcal{P}(\mathbb{R}) \) (which is defined analogously to the weak topology, except that the test functions in \( C_b(\mathbb{R}) \) are replaced by bounded measurable functions). On the other hand, \( m_p(\cdot) \) is continuous with respect to the \( \mathcal{W}_p \) topology, but in this case, \((L_{n,p}^Y)_{n \in \mathbb{N}}\) does not satisfy an LDP with respect to the \( \mathcal{W}_p \) topology. In particular, \([14]\) and \([26]\) develop strong exponential integrability conditions that show that while the sequence \((L_{n,p}^Y)_{n \in \mathbb{N}}\) satisfies an LDP with good rate function with respect to the \( \mathcal{W}_q \) topology for all \( q < p \), it does not satisfy such an LDP for \( q = p \).

An alternative approach to the contraction principle that is used in the theory of large deviations is Varadhan’s lemma, which allows one to transfer LDPs from one sequence of probability measures to another sequence which is (termwise) absolutely continuous with respect to the former. However, this approach is not applicable in our setting because the laws of \( L_{n,p} \) and \( L_{n,p}^Y \) are mutually singular.

In a different direction, instead of appealing to the LDP for \((L_{n,p}^Y)_{n \in \mathbb{N}}\), we could attempt to analyze the LDP for \((L_{n,p})_{n \in \mathbb{N}}\) directly. In particular, note that \((X^{(1,p)}, X^{(2,p)}, \ldots)\) is an infinite triangular array where each row \( X^{(n,p)} \) consists of exchangeable random variables, so \( L_{n,p} \) is the empirical measure of a finite exchangeable sequence. Some large deviations results for the empirical measure of a row-wise exchangeable triangular array can be found in \([24]\), but on their p. 653, they acknowledge that “Even in the simple case of binary valued finite exchangeable random variables there is no general result for the LD behavior of [the row-wise empirical measure].” That is, exchangeable structure on its own is not sufficient for a general LDP result (and in particular, not sufficient for an LDP for \((L_{n,p})_{n \in \mathbb{N}}\)).

Yet another approach to the LDP of Theorem 2.8 is through the machinery of a “non-continuous” version of the contraction principle developed in \([2, \S 6.2-6.3]\). In particular, the authors of \([2]\) explicitly state the result for \( p = 2 \) in the weak topology. However, Theorem
2.5 and Theorem 2.8 improves this result in multiple ways. First, our results are for general $p \in [1, \infty]$. Second, we have results for both the cone measure and surface measure, whereas the $p = 2$ case is one of only three cases ($p = 1, 2, \infty$) where the two measures coincide. Third, in Lemma 3.3 we provide an explicit proof by computing the rate function $J$ of Lemma 3.2 directly from the variational formula given by Cramér’s theorem in $\mathcal{P}(\mathbb{R}) \times \mathbb{R}_+$, instead of appealing to higher level results. Lastly, as shown below, our result extends from the weak topology to the stronger Wasserstein topology on $\mathcal{P}(\mathbb{R})$.

**Lemma 3.5.** Let $K_p = \{ v \in \mathcal{P}(\mathbb{R}) : m_p(v) \leq 1 \}$. Then, for all $q < p$, the set $K_p \subset \mathcal{P}_q(\mathbb{R})$ is compact with respect to the $\mathcal{W}_q$ topology. In addition, $K_p$ is convex and non-empty.

**Proof.** The properties of convexity and non-emptiness are elementary. As for compactness, we first prove that $K_p$ is weakly compact. For $v \in K_p$, for all $M > 0$, Chebyshev’s inequality yields

$$v([-M, M]^c) \leq \frac{m_p(v)}{M^p} \leq \frac{1}{M^p}.$$ 

Thus, $K_p$ is tight, and by Prokhorov’s theorem, precompact. Note that $K_p$ is weakly closed, since it is a level set of the map $v \mapsto \int x^p v(dx)$, which is lower semicontinuous due to the Portmanteau theorem for weak convergence. Therefore, $K_p$ is weakly compact.

To verify Wasserstein compactness, it suffices to show that the set of probability measures in $K_p$ have uniformly integrable $q$-th moments [1, Proposition 7.1.5]. This latter condition follows from the de la Vallée-Poussin theorem, since for $g(x) \equiv |x|^{p/q}$ (which satisfies the superlinear growth condition $\lim_{|x| \to \infty} \frac{g(x)}{x} = \infty$), we have

$$\sup_{v \in K_p} \int_{\mathbb{R}} g(|x|^q) v(dx) = \sup_{v \in K_p} m_p(v) \leq 1 < \infty.$$ 

This uniform integrability implies that $K_p$ is $\mathcal{W}_q$-compact. □

**Lemma 3.6.** Let $p \in [1, \infty]$. Then, $\mathbb{H}_p$ of (2.5) is convex.

**Proof.** Note that the domain of $\mathbb{H}_p$ is convex, since $m_p$ is linear. Moreover, within its domain, the function $\mathbb{H}_p$ is the sum of the relative entropy $H(\cdot \| \mu_p)$, which is convex, and the $p$-th moment $m_p$, which is linear, so $\mathbb{H}_p$ is convex. □

**Proof of Theorem 2.8.** Fix $q < p$. In view of the fact that $(L_{n,p})_{n \in \mathbb{N}}$ satisfies an LDP in $\mathcal{P}(\mathbb{R})$ with respect to the weak topology due to Proposition 3.4, in order to establish the LDP in $\mathcal{P}_q(\mathbb{R})$ with respect to the Wasserstein topology, it suffices to show exponential tightness of $(L_{n,p})_{n \in \mathbb{N}}$ in the $\mathcal{W}_q$ topology (see Corollary 4.2.6 of [9]). Let $K_p$ be the compact set (with respect to topology) defined in Lemma 3.5. Note that $m_p(L_{n,p}) = 1$ a.s., so $\mathbb{P}(L_{n,p} \in K_p^c) = 0$ for all $n$, which yields

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(L_{n,p} \in K_p^c) = -\infty.$$
Thus, \((L_{n,p})_{n \in \mathbb{N}}\) is exponentially tight, and hence, satisfies the desired LDP with respect to the \(W_q\) topology. Finally, the convexity of \(H_p\) is given by Lemma 3.6.

4. LDP under the surface measure. In this section, we show how Theorem 2.5, the LDP for the surface measure, can be obtained from the LDP for the cone measure. We first recall the following fundamental relationship between the cone and surface measure.

**Lemma 4.1 ([18, Lemma 2]).** Let \(p \in [1, \infty)\). Then,

\[
\frac{d\sigma_{n,p}}{d\gamma_{n,p}}(x) = C_{n,p} \left( \sum_{i=1}^{n} |x_i|^{p-2} \right)^{1/2}, \quad x \in S_{n,p},
\]

where \(C_{n,p}\) is the normalizing constant,

\[
C_{n,p} = \left[ \int_{S_{n,p}} \left( \sum_{i=1}^{n} |x_i|^{p-2} \right)^{1/2} \gamma_{n,p}(dx) \right]^{-1}.
\]

For \(p = \infty\), let \(C_{n,\infty} = 1\).

Next, we state a general result about LDPs for two sequences of measures that satisfy a particular absolute continuity relation.

**Lemma 4.2.** Let \(\mathcal{X}\) be a Polish space, and for \(n \in \mathbb{N}\), let \(\psi_n : \mathcal{X} \to \mathbb{R}\). Moreover, suppose that there exists a sequence of finite constants \((M_n)_{n \in \mathbb{N}}\) satisfying \(\lim_{n \to \infty} \frac{M_n}{n} = 0\), such that for all \(n \in \mathbb{N}\),

\[
|\psi_n(\lambda)| \leq M_n, \quad \mu_n\text{-a.e. } \lambda \in \mathcal{X}.
\]

Suppose \((\mu_n)_{n \in \mathbb{N}} \subset \mathcal{P}(\mathcal{X})\) satisfies an LDP with a good rate function \(I(\cdot)\). Define

\[
\nu_n(d\lambda) = \frac{1}{\int_{\mathcal{X}} e^{\psi_n(\lambda)} \mu_n(d\lambda)} e^{\psi_n(\lambda)} \mu_n(d\lambda).
\]

Then, \((\nu_n)_{n \in \mathbb{N}}\) satisfies an LDP with the same good rate function \(I(\cdot)\).

**Proof.** For \(\phi : \mathcal{X} \to \mathbb{R}\) continuous and bounded, define

\[
\Lambda_\phi = \lim_{n \to \infty} \frac{1}{n} \log \int_{\mathcal{X}} e^{n \phi(\lambda)} \mu_n(d\lambda),
\]

where the limit exists due to the LDP for \((\mu_n)_{n \in \mathbb{N}}\) and Varadhan’s Lemma (see, e.g., Theorem 4.3.1 of [9]). Next, note that the bound on \(\psi_n\) implies that for every \(n \in \mathbb{N}\),

\[
-M_n \leq \log \int_{\mathcal{X}} e^{\psi_n(\lambda)} \mu_n(d\lambda) \leq M_n.
\]

Since \(M_n/n \to 0\), this implies

\[
\lim_{n \to \infty} \frac{1}{n} \log \int_{\mathcal{X}} e^{\psi_n(\lambda)} \mu_n(d\lambda) = 0.
\]
Together with the definition (4.1), the bound on $\psi_n$, and the assumption on $M_n$, this implies that
\[
\liminf_{n \to \infty} \frac{1}{n} \log \int_{X} e^{n\phi(\lambda)} v_n(d\lambda) \geq \liminf_{n \to \infty} \frac{1}{n} \log \int_{X} e^{n\phi(\lambda)} - M_n \mu_n(d\lambda) = \Lambda_\phi - \lim_{n \to \infty} \frac{M_n}{n} = \Lambda_\phi,
\]
\[
\limsup_{n \to \infty} \frac{1}{n} \log \int_{X} e^{n\phi(\lambda)} v_n(d\lambda) \leq \limsup_{n \to \infty} \frac{1}{n} \log \int_{X} e^{n\phi(\lambda)} + M_n \mu_n(d\lambda) = \Lambda_\phi + \lim_{n \to \infty} \frac{M_n}{n} = \Lambda_\phi.
\]
Thus, we have shown that
\[
\Lambda_\phi = \bar{\Lambda}_\phi \overset{\Delta}{=} \lim_{n \to \infty} \frac{1}{n} \log \int_{X} e^{n\phi(\lambda)} v_n(d\lambda).
\]
Note that $(\mu_n)_{n \in \mathbb{N}}$ is exponentially tight since it satisfies an LDP with good rate function and $X$ is Polish [15, Lemma 2.6]. We claim that $(v_n)_{n \in \mathbb{N}}$ is also exponentially tight. Let $L < \infty$, and let $K_L \subset X$ be a compact set such that
\[
\limsup_{n \to \infty} \frac{1}{n} \log \mu_n(K_L^c) < -L. \tag{4.2}
\]
Then, given $L < \infty$, note that
\[
\log v_n(K_L^c) = \log \int_{K_L^c} e^{\psi_n(\lambda)} \mu_n(d\lambda) - \log \int_{X} e^{\psi_n(\lambda)} \mu_n(d\lambda) \\
\leq M_n + \log \mu_n(K_L^c) + M_n - \log \mu_n(X) \\
= 2M_n + \log \mu_n(K_L^c).
\]
Taking the limit supremum as $n \to \infty$, using (4.2), and applying the fact that $\frac{M_n}{n} \to 0$, we find that
\[
\limsup_{n \to \infty} \frac{1}{n} \log v_n(K_L^c) \leq \limsup_{n \to \infty} \frac{1}{n} \log \mu_n(K_L^c) < -L.
\]
Since $\Lambda_\phi = \bar{\Lambda}_\phi$, and both sequences $(\mu_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ are exponentially tight, Bryc’s inverse lemma (see, e.g., Theorem 4.4.2 of [9]) implies that the two sequences satisfy the same LDP with the good rate function $\Pi(\lambda) = \sup_{\phi \in C_b(X)} \{\phi(\lambda) - \Lambda_\phi\}$. \qed

We apply the preceding lemma to the absolute continuity relation of Lemma 4.1 to prove the LDP under the surface measure.

**Proof of Theorem 2.5.** The result is trivial for $p = 1, 2, \infty$, since then $\sigma_{n,p} = \gamma_{n,p}$. Therefore, we restrict to $p \in (1, 2)$ or $p \in (2, \infty)$. In this setting, we apply Lemma 4.2 to the following: let $\mathcal{L}_\sigma : \mathbb{R}^n \to \mathcal{P}(\mathbb{R})$ be the map that takes a vector $x \in \mathbb{R}^n$ to the measure $\frac{1}{n} \sum_{i=1}^{n} x_i \delta_{x_i} \in \mathcal{P}(\mathbb{R})$, and set
- $X = \mathcal{P}_{q}(\mathbb{R})$ (equipped with the $q$-Wasserstein topology, for some $q < p$),
- $\mu_n = \gamma_{n,p} \circ \mathcal{L}_\sigma^{-1}$,
- $\psi_n = \psi = \frac{1}{p} \log m_{2p-2},$
• \( v_n = \sigma_{n,p} \circ \mathcal{L}_n^{-1} \).

We now show that our setup satisfies the hypotheses of Lemma 4.2. For \( n \in \mathbb{N} \), let

\[
A_n = \left\{ \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} : x_i \in \mathbb{R}^n \right\} \subset \mathcal{P}(\mathbb{R}).
\]

Note that for all \( n \in \mathbb{N} \), \( \mu_n(A_n) = 1 \). For \( \lambda \in A_n \), using Lemma 4.1, we can write the Radon-Nikodym derivative of \( v_n \) with respect to \( \mu_n \) (defined on a set of \( \mu_n \) measure 1), as

\[
\frac{dv_n}{d\mu_n}(\lambda) = \frac{d\sigma_{n,p}}{d\gamma_{n,p}}(\mathcal{L}_n^{-1} \lambda) = C_{n,p}(nm_{2p-2}(\lambda))^{1/2} = n^{1/2}C_{n,p}e^{\psi(\lambda)}.
\]

We know from Theorem 2.8 that \((\mu_n)_{n \in \mathbb{N}}\) satisfies an LDP with respect to the \( \mathcal{W}_p \) topology (for \( q < p \)), with good rate function \( \mathbb{H}_p \).

As for the boundedness properties of \( \psi = \frac{1}{2} \log m_{2p-2} \) stipulated in Lemma 4.2, first note that due to Hölder’s inequality, for any \( 0 < r < s < \infty \) and \( \lambda \in \mathcal{P}(\mathbb{R}) \),

\[
m_r(\lambda) \leq [m_s(\lambda)]^{r/s}.
\]

Now, fix \( 1 < p < 2 \). Then \( 0 < 2p - 2 < p \), and so applying the preceding inequality with \( r = 2p - 2 \) and \( s = p \), and invoking the equivalence (2.12) with \( q = p \), we see that for \( \lambda \in A_n \),

\[
\mu_n(m_{2p-2}(\lambda) > 1) \leq \mu_n(m_p(\lambda)^{2-(2/p)} > 1) = \mathbb{P}\left( \|X^{(n,p)}\|_{n,p} > 1 \right) = 0,
\]

since \( X^{(n,p)} \in \mathbb{S}_{n,p} \), \( \mathbb{P} \)-a.s. On the other hand, to lower bound \( m_{2p-2}(\lambda) \), recall that for \( 0 < r < s < \infty \) and \( x \in \mathbb{R}^n \),

\[
\|x\|_{n,r} \leq \|x\|_{n,s}.
\]

Applying this inequality with \( r = 2p - 2 \) and \( s = p \), and recalling \( L_{n,p} \) from (2.1), we have

\[
m_{2p-2}(L_{n,p}) = \frac{1}{n} \sum_{i=1}^{n} \left( 2p-2 \right)/p |X_i^{(n,p)}|_{2p-2}^2/n_{2p-2}
\]

\[
= n^{1-(2/p)} \|X^{(n,p)}\|_{n,2p-2}^{2p-2}
\]

\[
\geq n^{1-(2/p)} \|X^{(n,p)}\|_{n,p}^{2p-2}
\]

\[
= n^{1-(2/p)} \quad \text{\( \mathbb{P} \)-a.s.,}
\]

where the last equality once again uses the fact that \( \mathbb{P} \)-a.s. (under the cone measure), \( X^{(n,p)} \) lies on the unit \( \ell^p \) sphere \( \mathbb{S}_{n,p} \). In a similar fashion, for \( 2 < p < \infty \), we have \( 2p - 2 > p \) and hence, for \( \lambda \in A_n \),

\[
\mu_n(m_{2p-2}(\lambda) < 1) \leq \mu_n(m_p(\lambda)^{2-(2/p)} < 1) = \mathbb{P}\left( \|X^{(n,p)}\|_{n,p} < 1 \right) = 0,
\]

and

\[
m_{2p-2}(L_{n,p}) \leq n^{1-(2/p)} \|X^{(n,p)}\|_{n,p}^{2p-2} = n^{1-(2/p)} \quad \text{\( \mathbb{P} \)-a.s.}
\]
Since $\mu_n(\mathcal{A}_n) = 1$, we have shown that for $\mu_n$-a.e. $\lambda \in \mathcal{P}(\mathbb{R})$,

$$|\psi(\lambda)| = \left| \frac{1}{2} \log m_{2p-2}(\lambda) \right| \leq M_n,$$

where

$$M_n = \left| \frac{1}{2} - \frac{1}{p} \right| \log n.$$

Since $M_n/n \to 0$, we can apply Lemma 4.2, which shows that the sequence of empirical measures $(L_n, p)_{n \in \mathbb{N}}$ under the surface measure $\sigma_{n, p}$ satisfies an LDP with the same good rate function as under the cone measure $\gamma_{n, p}$. □

5. Gibbs conditioning. We first recall a general version of the Gibbs conditioning principle. The following theorem is one of the results of [13]. For the sake of completeness, and also due to the fact that our statement below imposes slightly different conditions than the result in [13] (although the proof is essentially unaffected), we include a short proof.

**Proposition 5.1** (Gibbs conditioning principle, [13, Theorem 7.1]). Let $X$ be a topological space, and let $(\xi_n)_{n \in \mathbb{N}}$ be a sequence of $X$-valued random variables that satisfies an LDP with good rate function $I$. In addition, let $F \subset X$ be a subset such that

1. $I(F) = \inf_{x \in F} I(x) < \infty$;
2. $F$ is closed;
3. $F = \bigcap_{\varepsilon > 0} F_{\varepsilon}$ for a family of sets $(F_{\varepsilon})_{\varepsilon > 0}$ such that $F_{\varepsilon} \subset X$ for all $\varepsilon > 0$ and $n \in \mathbb{N}$;
4. $F \subset (F_{\varepsilon})^\circ$ for all $\varepsilon > 0$.

Let $\mathcal{M}_F$ be the set of $x \in F$ which minimize $I$. That is,

$$\mathcal{M}_F = \{ x \in F : I(x) = I(F) \}.$$

Then, for all open $G \subset X$ such that $\mathcal{M}_F \subset G$, we have

$$\limsup_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\xi_n \not\in G | \xi_n \in F_{\varepsilon}) < 0. \tag{5.1}$$

As a consequence, if $\mathcal{M}_F = \{ \bar{x} \}$ is a singleton, then

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \mathbb{P}(\xi_n \in : | \xi_n \in F_{\varepsilon}) = \delta_{\bar{x}} \tag{5.2}.$$ 

**Proof.** Due to the LDP for $(\xi_n)_{n \in \mathbb{N}}$, for all $\varepsilon > 0$, with $F$, $F_{\varepsilon}$ and $G$ as in the statement of the proposition, and $G^c$ representing the complement of $G$,

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\xi_n \in G^c | \xi_n \in F_{\varepsilon}) \leq \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\xi_n \in G^c \cap F_{\varepsilon}) - \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(\xi_n \in (F_{\varepsilon})^\circ) \leq -I(G^c \cap F_{\varepsilon}) + I((F_{\varepsilon})^\circ) \leq -I(G^c \cap F_{\varepsilon}) + I(F),$$
where the last inequality follows from the assumption that $F \subset (F_\varepsilon)$. Furthermore, the assumptions of the proposition state that $\mathbb{I}$ is a good rate function and imply the equality \( \cap_{\varepsilon > 0}(G^c \cap F_\varepsilon) = G^c \cap F \). Therefore, by Lemma 4.1.6 of [9],

\[
\lim_{\varepsilon \to 0} \mathbb{I}(G^c \cap F_\varepsilon) = \mathbb{I}(G^c \cap F).
\]

Then, (5.1) follows from the last two displays and the assumption that $M_F$ (the set of minimizers of $\mathbb{I}$ in $\mathbb{F}$) lies within $G$, which implies that

\[
-\mathbb{I}(G^c \cap F) + \mathbb{I}(F) < -\mathbb{I}(G^c \cap F) + \mathbb{I}(G^c \cap F) = 0.
\]

Note that (5.1) implies that conditional on $F_\varepsilon$, the law of $\xi_n$ concentrates on the set $M_F$ (as $n \to \infty$ and $\varepsilon \to 0$), in the sense that for every open set $G$ such that $M_F \subset G$, we have

\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \mathbb{P}(\xi_n \in G|\xi_n \in F_\varepsilon) = 1.
\]

As this holds for all such $G$, the limit (5.2) follows as a simple consequence if $M_F$ is a singleton. □

**Remark 5.2.** The conditions on $F$ and $F_\varepsilon$ of Proposition 5.1 are not the only conditions under which such a result holds. For example, in the case where $\xi_n$ is the empirical measure of $n$ i.i.d. random variables, it is possible to replace condition 4. with a different condition on $F_\varepsilon$ (see, e.g., [9, p. 324, A-1]).

Our proof of the conditional limit theorem stated in Theorem 2.12 follows from the Gibbs conditioning principle of Proposition 5.1. In our proof, it is essential that we appeal to the LDPs of Theorem 2.5 and Theorem 2.8, which hold in a Wasserstein topology, which is stronger than the usual weak topology.

**Proof of Theorem 2.12.** First, we show that Proposition 5.1 applies in the setting of Theorem 2.12. Fix $p \in [1, \infty]$ and $q < p$. Due to Theorem 2.8 (for the cone measure) and Theorem 2.5 (for the surface measure), the sequence $(L_{n,p})_{n \in \mathbb{N}}$ satisfies an LDP in $\mathcal{P}_q(\mathbb{R})$ (equipped with the $\mathcal{W}_q$ topology) with good rate function $H_p$. Fix $C = [\alpha, \beta] \subset \mathbb{R}$ and for $\varepsilon > 0$, define the sets

\[
F_\varepsilon = \{ \nu \in \mathcal{P}_q(\mathbb{R}) : m_q(\nu) \in C_\varepsilon \},
\]

where $C_\varepsilon = [\alpha - \varepsilon, \beta + \varepsilon]$. It is immediate from Definition 2.3 that for $q < p$, the moment map $m_q$ of (2.3) is continuous on $\mathcal{P}_q(\mathbb{R})$, with respect to the $\mathcal{W}_q$ topology. Since the set $C_\varepsilon$ is closed, this implies that $F_\varepsilon$ is closed, and hence, the set

\[
F = \bigcap_{\varepsilon > 0} F_\varepsilon = \{ \nu \in \mathcal{P}(\mathbb{R}) : m_q(\nu) \in C \}
\]

is also closed. Moreover, for $\varepsilon > 0$, we also have

\[
F = m_q^{-1}(C) \subset m_q^{-1}(C_\varepsilon^0) \subset [m_q^{-1}(C_\varepsilon)]^0 = (F_\varepsilon)^0,
\]
where the second inclusion makes another use of the continuity of \( m_q \). Next, let us show that \( M_F \) is a singleton (i.e., \( \nu^* \) of (2.9) is well defined). Recall that
\[
M_F = \left\{ \nu \in F : H_\nu = \min_{\nu \in F} H_\nu \right\}.
\]
Note that \( F \) is closed and convex, since it is the preimage of a closed, convex set \( C \) under a continuous linear map \( m_q \). Because \( H_\nu \) is lower semicontinuous and has compact level sets, it achieves its minimum within \( F \). That the minimum is achieved at a unique \( \nu \in F \) follows from the fact that \( H_\nu \) is strictly convex because it is the sum of the strictly convex relative entropy \( H(\|\mu_p\|) \) and a linear function \( m_p \). The representation for \( \nu^* \) given in (2.9) follows from (2.8) and the definition (2.5) of \( H_\nu \). Thus, (2.10) follows from (5.2) of Proposition 5.1.

As for the second result (2.11), this follows from Proposition 2.2 of [23], which, under the assumption of exchangeability, establishes the equivalence of statements like (2.10) (regarding convergence in probability of the empirical measure to a deterministic measure) and statements like (2.11) (regarding joint convergence in distribution of any fixed \( k \) of the underlying random variables). To be precise, let \( X \) be a Polish space, and for \( n \in \mathbb{N} \), let \( (\xi_1, \ldots, \xi_m) \) be a sequence of \( n \) exchangeable \( X \)-valued random variables. Then, Proposition 2.2 of [23] states that the empirical measure \( \frac{1}{n} \sum_{i=1}^n \delta_{\xi_i} \) converges in law to some \( \nu \in \mathcal{P}(X) \) if and only if the law of \( (\xi_1, \ldots, \xi_m) \) converges weakly to the \( k \)-fold product of \( \nu \). Given that \( (X_1^{(n,p)}, \ldots, X_m^{(n,p)}) \) is an exchangeable sequence, it is still exchangeable conditional on the event \( \frac{1}{n} \sum_{i=1}^n \delta_{\xi_i} \in C_k \), because \( \|n^{1/p}X^{(n,p)}\|_{n,q} \) is a symmetric function of the random variables \( X_i^{(n,p)}, i = 1, \ldots, n \). Since (2.10) establishes convergence in law of the conditioned empirical measure \( L_{n,p} \) to \( \nu^* \), the joint convergence of (2.11) follows. The preceding discussion essentially also outlines the approach followed in Corollary 7.3.5 of [9], which states a Gibbs conditioning result for the empirical measure of i.i.d. random variables, but in fact the i.i.d. assumption is only invoked to establish exchangeability. \( \square \)

As a prerequisite for the proof of Corollary 2.13, recall the following basic information-theoretic fact.

**Lemma 5.3.** Fix \( r_i : \mathbb{R} \to \mathbb{R} \), \( \alpha_i \in \mathbb{R} \) for \( i = 1, \ldots, m \), and \( s_j : \mathbb{R} \to \mathbb{R} \), \( \beta_j \in \mathbb{R} \), for \( j = 1, \ldots, n \). Consider the following maximization problem:

Maximize
\[
\nu \in \mathcal{P}(\mathbb{R}) \quad h(\nu)
\]
subject to
\[
\int_{\mathbb{R}} r_i(x) \nu(dx) = \alpha_i \quad \text{for } i = 1, \ldots, m,
\]
\[
\int_{\mathbb{R}} s_j(x) \nu(dx) \leq \beta_j \quad \text{for } j = 1, \ldots, n.
\]

Then, a probability measure \( \nu^* \in \mathcal{P}(\mathbb{R}) \) attains the maximum in (5.3) if and only if it is of the following form:

\[
\nu^*_e(dx) = \exp \left( -1 - \kappa_0^* - \sum_{i=1}^m \lambda_i^* r_i(x) - \sum_{j=1}^n \mu_j^* s_j(x) \right) dx,
\]
with non-negative constants \( \kappa^*_i \), \((\lambda^*_i)_{i=1}^m\), and \((\mu^*_j)_{j=1}^n\) chosen such that \( \nu_* \) lies in \( \mathcal{P}(\mathbb{R}) \) and satisfies the constraints in (5.3). Moreover, if \( \nu_* \) attains the maximum in (5.3), then

\[
\mu^*_j \left( \int_{\mathbb{R}} s_j(x) \nu_* (dx) - \beta_j \right) = 0
\]

for all \( j = 1, \ldots, n \).

The preceding lemma is standard. See [6, §12.1] for a slight simplification of (5.3) with only equality constraints. Alternatively, see [5, Ex. 5.3] for a version of (5.3) with discrete entropy and both equality and inequality constraints. The claim (5.4) is the so-called “complementary slackness” condition (see, e.g., [5, §5.5.2]).

As a final preliminary, define the following family of probability measures for \( q \in [1, \infty), \beta > 0 \):

\[
\mu_{q,\beta}(dx) = \frac{1}{2(\beta q)^{1/q}(1 + \frac{1}{q})} e^{-(|x|^q/\beta q)} dx, \quad x \in \mathbb{R}.
\]

Note that \( \mu_{p,1} \) corresponds to \( \mu_p \) of (2.4).

**Proof of Corollary 2.13.** Due to the unconditional limit theorem Lemma 2.11, under either the surface measure or the cone measure, we have \( \lambda_{n,q} \Rightarrow \mu^\otimes\beta_k \) where \( \beta \leq \beta_{p,q} \) of (2.13), and \( \mu_{q,\beta} \) is as in (5.5). Therefore, it suffices to show that under either the surface measure or the cone measure, the conditional distribution \( \hat{\lambda}_{n,p/q,q} \) also converges weakly to the same limit \( \mu^\otimes\beta_k \). In view of (2.11) of Theorem 2.12, it suffices to show that when \( C = [0, \beta] \), the unique maximizer \( \nu_* \) of (2.9) satisfies \( \nu_* = \mu_{q,\beta} \). Note that due to the basic maximum entropy calculations of Lemma 5.3, we have for \( \beta > 0 \),

\[
\mu_{q,\beta} = \arg \max \{ h(\nu) : m_q(\nu) \leq \beta \}.
\]

To show that \( \mu_{q,\beta} = \nu_* \), it suffices to show that \( m_p(\mu_{q,\beta}) \leq 1 \). After some elementary calculations, we find that since \( \beta \leq \beta_{p,q} \), and using the expression for \( \beta_{p,q} \) in (2.13),

\[
m_p(\mu_{q,\beta}) = \beta^{p/q} q^{p/q} \frac{\Gamma(\frac{p+1}{q})}{\Gamma(\frac{1}{q})} \leq \frac{1}{q^{p/q}} \left( \frac{\Gamma(\frac{1}{q})}{\Gamma(\frac{p+1}{q})} \right) q^{p/q} \frac{\Gamma(\frac{p+1}{q})}{\Gamma(\frac{1}{q})} = 1.
\]

\[\square\]

**Remark 5.4.** To understand the reasoning behind the choice of \( \beta_{p,q} \) of (2.13), note that the primary criterion for an interval \( C \) in Theorem 2.12 to yield an insightful result is for \( C \) to induce an optimizing measure \( \nu_* \) of (2.9) with an explicit and familiar form. For example, the explicit solution to the simpler variational problem (5.6) holds for all \( \beta > 0 \). However, the original variational problem of (2.9) involves not only the \( q \)-th moment constraint \( m_q(\nu) \in C \), but also an additional constraint on the \( p \)-th moment, \( m_p(\nu) \leq 1 \). To
simplify the variational problem (2.9), it suffices to consider values of $\beta$ small enough such that $m_p(\mu_{q,\beta}) \leq 1$, so that the maximizer of (5.6) is also the maximizer of (2.9) when $C = [0, \beta]$. Elementary calculations show that

$$
(5.7) \quad m_p(\mu_{q,\beta_p}) = \beta_{p,q}^{p/q} q^{p/q} \frac{\Gamma(p+1)}{\Gamma(1/q)} = 1.
$$

That is, $\beta_{p,q}$ is the threshold value such that for $\beta \leq \beta_{p,q}$, we have $m_p(\mu_{q,\beta}) \leq 1$. Note however, that this discussion of $\beta_{p,q}$ (and thus, Corollary 2.13) is valid only for $p \in [1, \infty)$, since for $p = \infty$, we have $m_\infty(\mu_{q,\beta}) = \infty$ for all $q < \infty$ and $\beta > 0$. That is, for $q < \infty$, there is no value of $\beta_{\infty,q}$ greater than 0 such that an analog of (5.7) can hold for $p = \infty$.

**Remark 5.5.** The preceding remark addresses the case of “small enough” $\beta$. We now consider “large” $\beta$: suppose $\beta \geq \overline{\beta}_{p,q} \doteq m_q(\mu_p)$. Note that an easy consequence of Lemma 5.3 is that

$$
(5.8) \quad \mu_p = \arg \max \{ h(\nu) : m_p(\nu) \leq 1 \}.
$$

For $\beta \geq \overline{\beta}_{p,q}$ and $C = [0, \beta]$, the $q$-th moment constraint of (2.9) is automatically satisfied by $\nu = \mu_p$, so the maximizer of (5.8) is also the maximizer of (2.9). In this case, the “conditional” limit of Theorem 2.12 is equivalent to the “unconditional” limit of Lemma 2.11. In other words, for sufficiently large $\beta$, the $\ell^q$ norm conditioning event of (2.14) is extraneous.

**Remark 5.6.** As for $C = [0, \beta]$ and $\beta_{p,q} < \beta < \overline{\beta}_{p,q}$, this regime is less amenable to an immediate geometric interpretation. Whereas the small $\beta$ regime of Corollary 2.13 and Remark 5.4 allows an $\ell^q$ sphere interpretation via the measure $\mu_{q,\beta}$, and the large $\beta$ regime of Remark 5.5 allows an $\ell^p$ sphere interpretation via the measure $\mu_p$, we have different behavior in the intermediate $\beta$ regime. Recall that the maximum entropy considerations of Lemma 5.3 imply that the unique maximizer of Theorem 2.12 is of the form

$$
(5.9) \quad \nu_*(dx) = \exp \left( -1 - \kappa_0 - \kappa_p |x|^p - \kappa_q |x|^q \right),
$$

with constants $\kappa_0, \kappa_p, \kappa_q$ such that $\nu_*$ is a probability measure that satisfies $m_p(\nu_*) \leq 1$ and $m_q(\nu_*) \leq \beta$. To gain insight on these constants, consider the following cases:

- if $\kappa_p = 0$ and $\kappa_q > 0$, then the complementary slackness condition of (5.4) implies $m_q(\nu_*) = \beta$, so the form (5.9) implies $\nu_* = \mu_{q,\beta}$. But this is not a feasible solution, since $m_p(\mu_{q,\beta}) > 1$ for $\beta > \beta_{p,q}$;
- similarly, if $\kappa_p > 0$ and $\kappa_q = 0$, then we have $\nu_* = \mu_p$, but this is not a feasible solution since $m_q(\mu_p) = \beta_{p,q} > \beta$;
- lastly, if $\kappa_p = 0$ and $\kappa_q = 0$, then $\nu_*$ is not a probability measure for any choice of $\kappa_0$.

Therefore, it must be the case that $\kappa_p > 0$ and $\kappa_q > 0$, which implies that $\nu_*$ of (5.9) is not of the form $\mu_{r,b}$ for any $r \in [1, \infty), b > 0$. Instead, $\nu_*$ is of an exponential family that is genuinely different from the generalized normal distributions.
In this paper, we have only discussed applications of Theorem 2.12 to intervals $C$ of the form $C = [0, \beta]$, and primarily for small $\beta$, with some discussion of larger $\beta$ in Remark 5.5 and Remark 5.6. We leave open the question of finding other examples of intervals $C \subset \mathbb{R}$ which lead to an explicit and meaningful maximizing measure $\nu_\beta$.

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