THE GROTHENDIECK–SERRE CONJECTURE
OVER SEMILOCAL DEDEKIND RINGS

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Abstract. For a reductive group scheme $G$ over a semilocal Dedekind ring $R$ with total ring of fractions $K$, we prove that no nontrivial $G$-torsor trivializes over $K$. This generalizes a result of Nisnevich–Tits, who settled the case when $R$ is local. Their result, in turn, is a special case of a conjecture of Grothendieck–Serre that predicts the same over any regular local ring. With a patching technique and weak approximation in the style of Harder, we reduce to the case when $R$ is a complete discrete valuation ring. Afterwards, we consider Levi subgroups to reduce to the case when $G$ is semisimple and anisotropic, in which case we take advantage of Bruhat–Tits theory to conclude. Finally, we show that the Grothendieck–Serre conjecture implies that any reductive group over the total ring of fractions of a regular semilocal ring $S$ has at most one reductive $S$-model.

1. Introduction

The Grothendieck–Serre conjecture was proposed by J.-P. Serre ([Ser58, p. 31, Rem.]) and A. Grothendieck ([Gro58, pp. 26–27, Rem. 3]) in 1958, who predicted that for an algebraic group $G$ over an algebraically closed field, a $G$-torsor over a nonsingular variety is Zariski-locally trivial if it is generically trivial. Subsequently, Grothendieck extended the conjecture to semisimple group schemes over regular schemes ([Gro68, Rem. 1.11.a]). By spreading out, the conjecture reduces to its local case whose precise statement is the following.

Conjecture 1 (Grothendieck–Serre). For a reductive group $G$ over a regular local ring $R$ with fraction field $K$, a $G$-torsor that becomes trivial over $K$ is trivial. In other words, the following map between nonabelian étale cohomology pointed sets has a trivial kernel:

$$H^1_{\text{ét}}(R,G) \to H^1_{\text{ét}}(K,G).$$

(1)

In this paper, we prove a variant of Conjecture 1 when $R$ is a semilocal Dedekind ring. Our main result is the following theorem.

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Theorem 1. For a reductive group scheme $G$ over a semilocal Dedekind ring $R$ with a total ring of fractions $K$, the following pullback map of nonabelian étale cohomology pointed sets

$$H^1_{\text{ét}}(R, G) \hookrightarrow H^1_{\text{ét}}(K, G)$$

is injective.

Beyond the trivial case of 0-dimensional $R$, Conjecture 1 has several known cases.

(i) The case when $R$ is a discrete valuation ring with a perfect residue field was addressed in Nisnevich’s PhD thesis [Nis83, 2, Thm. 7.1] and his Comptes Rendus paper [Nis84, Thm. 4.2]. There, he reduced to the complete case that had been considered in unpublished work of Tits that rested on Bruhat–Tits theory (see [BTIII, Lem. 3.9], but still with auxiliary conditions). Various other special cases are based on the discrete valuation ring case, for instance, the case when $R$ is of arbitrary dimension and complete regular local, see [CTS79, 6.6.1]; further, the case when $R$ is Henselian and $G$ splits was proved explicitly in [BB70, Prop. 2].

(ii) The case when $\dim R = 2$ with infinite residue field and $G$ is quasi-split was considered by Nisnevich in [Nis89, Thm. 6.3].

(iii) The case when $G$ is torus was settled by Colliot-Thélène and Sansuc in [CTS87, Thm. 4.1]. This result is often useful for various reductions of more general cases.

(iv) The case when $R$ is a semilocal Dedekind domain and $G$ is a form of $\text{PGL}_n$, $\text{PSp}_n$ and $\text{PO}_n$ ($\text{PO}_n$ is not connected) as a variant of the Grothendieck–Serre conjecture was proved by Beke and Van Geel in [BVG14, Thm. 3.7].

(v) The case when $R$ contains a field $k$ is summarized as follows. When $k$ is algebraically closed and $G$ is defined over $k$, the conjecture was settled by Colliot-Thélène–Ojanguren in [CTO92]. For regular semilocal domains $R$ containing a field $k$, when $k$ is infinite, the conjecture was recently proved by Fedorov and Panin in [FP15]; the case when $k$ is finite was originally proved by Panin in [Pan15], whose exposition was organized in his latter work [Pan17]. For a certain simplification of ibid., see [FR18].

(vi) The case when $R$ is a semilocal Dedekind domain and $G$ is a form of $\text{GL}_n$, $\text{O}_n$ and $\text{Sp}_n$ was recently settled by Bayer-Fluckiger and First in [BF17, Thm. 5.3], where they also proved a variant for non-reductive groups with generic fibers of the form $\text{GL}_n$, $\text{O}_n$ and $\text{Sp}_n$. They also investigated the corresponding problem for $\text{PGL}_n$, $\text{PO}_n$ and $\text{PSp}_n$ in [BFH19], where counterexamples for non-reductive $G$ emerge, even when $R$ is a complete discrete valuation ring.

(vii) The case when $R$ is a semilocal Dedekind domain and $G$ is a semisimple simply connected $R$-group scheme such that every semisimple normal $R$-subgroup scheme of $G$ is isotropic was recently proved by Panin and Stavrova in [PS17, Thm. 3.4]. By induction on the number of maximal ideals of $R$ and a decomposition of groups, they reduce to the case when $R$ is a Henselian discrete valuation ring and use Nisnevich’s result [Nis84, Thm. 4.2] to conclude.

(viii) A stronger conjecture predicts that the map (1) is injective, but it is equivalent to Conjecture 1 due to a twisting technique (Corollary 1), which is standard.
The main result Theorem 1 covers (i) and the corresponding cases of (iv)—(vii) and finishes the remaining cases of the semilocal Dedekind variant of Conjecture 1. To be precise, this article is aimed at working out Nisnevich’s proof [Nis83], extending it to cover 1-dimensional regular semilocal rings, and in particular, generalizing the result of Panin–Stavrova [PS17, Thm. 3.4] by eliminating the semisimple simply connected isotropic condition on the reductive group scheme $G$.

We now summarize Nisnevich’s work ([Nis83], [Nis84]) in the discrete valuation ring case. For a reductive group scheme $G$ over a discrete valuation ring $R$ with fraction field $K$, we let $\hat{R}$ be the completion of $R$ and $\hat{K} := \text{Frac}(\hat{R})$. Built upon Harder’s weak approximation [Har68], Nisnevich proved that $G(\hat{K})$ decomposes as a product of $G(\hat{R})$ and $G(K)$. This reduces one to the case when $R$ is complete. Following an argument of Tits in this case, one uses Bruhat–Tits theory and Galois cohomology to conclude.

Unfortunately, the details of Tits’ argument, which are given in [Nis83], have some unclear points. These details relied on [BT II], which appeared in print two years after [Nis83]. Further, Nisnevich’s proof is written under an auxiliary hypothesis that the residue field of $R$ is perfect ([Nis83, Chap. 2, Thm. 7.1]). Although this hypothesis does not appear in [Nis84], the part of the proof corresponding to the key step [Nis83, Chap. 2, 4.2] is implicit as an unpublished result of Tits. Moreover, in [Nis83, Chap. 2, Thm. 4.2], there is a decomposition of groups, which fails in general and forces us to consider alternative reductions (see Remark 4). Besides, some recent articles such as [PS17, Thm. 3.4] use Nisnevich’s result [Nis84, Thm. 4.2] directly without treating these gaps.

In this article, we first review weak approximation in Section 2. For a semilocal Dedekind domain $R$ with fraction field $K$, we consider a reductive $K$-group $G$ instead of merely semisimple as in Harder’s original setting in [Har68]. The goal of Section 2 is to construct an open normal subgroup of the closure of $G(K)$ in $\prod_v G(K_v)$ (Proposition 3), where $v$ ranges over the nongeneric points of Spec $R$ and $K_v$ is the completion of $K$ at $v$. In Section 3, with the aid of Harder’s construction, we exhibit the decomposition

$$\prod_v G(K_v) = G(K) \prod_v G(R_v)$$

(Theorem 3)

by mildly simplifying the argument in [Nis83, Chap. 2, §6]. This permits us to reduce Theorem 1 to the case of complete discrete valuation rings by a patching technique (Proposition 10).

To improve and extend Nisnevich’s proof, after reducing to the complete case, we take a different approach. The passage from $G$ to its minimal parabolic subgroup $P$ and the isomorphism $H^1(R, P) \simeq H^1(R, L)$ for a Levi subgroup $L$ of $P$ facilitate reduction to the case when $G$ is semisimple and anisotropic (Proposition 12). This reduction and subsequent steps rely on properties of anisotropic groups described in Proposition 6, where we use the formalism of Bruhat–Tits theory from [BT I], [BT II].

**Proposition 1 (Proposition 6).** *For a reductive group $G$ over a Henselian discrete valuation ring $R,*
(1) for the strict Henselization $\tilde{R}$ of $R$ with fraction field $\tilde{K}$, the group $G(\tilde{R})$ is a maximal parahoric subgroup of $G(\tilde{K})$;
(2) if $G$ is $K$-anisotropic, then $G(K) = G(R)$.

In the final step, we establish the case when $G$ is semisimple anisotropic and $R$ is a complete discrete valuation ring (Theorem 5) by following the argument of Nisnevich–Tits.

Finally, for a reductive group $G$ over the function field $K$ of a scheme $X$, we consider the number of reductive $X$-models of $G$. In fact, if a variant of Conjecture 1 holds for regular semilocal $X$, then we prove that such models should be unique:

**Proposition 2** (Proposition 14). For a regular semilocal ring $S$ with a total ring of fractions $K$, if

$$H^1_{\text{ét}}(S, G') \to H^1_{\text{ét}}(K, G')$$

is injective,

then any reductive $K$-group $G$ has at most one reductive $S$-model.

In particular, when $S$ contains a field, the condition (*) is satisfied thanks to the corresponding settled case of the Grothendieck–Serre conjecture [FP15], [Pan17], hence follows the uniqueness of reductive models over regular semilocal rings containing a field. On the other hand, the combination of Proposition 2 and our main result Theorem 1 implies the uniqueness of reductive models for semilocal Dedekind rings (Corollary 3), which generalizes [Nis84, Thm. 5.1] where $S$ is a Henselian local ring and $G$ is semisimple.

1.1. Notation and conventions

If not particularly indicated, in the sequel, we let $R$ be a semilocal Dedekind ring and let $K$ be its total ring of fractions. The completion of $R$ at its radical ideal is a direct product of fields and complete discrete valuation rings $R_v$ with fraction fields $K_v$. In the case when $R$ is local, we also denote the strict Henselization of $R$ by $\tilde{R}$ and its fraction field by $\tilde{K}$.

We also assume that $G$ is a reductive group scheme over $R$ (or over $K$ in Section 2), that is, $G$ is a smooth, affine $R$-group scheme with connected reductive algebraic groups as geometric fibers. For an $R$-algebra $R'$, the reductive group $G$ is called $R'$-anisotropic, if $G_{R'}$ contains no copy of $\mathbb{G}_{m, R'}$.

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2. Construction of open normal subgroups

Let $H$ be a reductive group over a field $F$ equipped with a finite set $V$ of nonequivalent nontrivial valuations of rank one. This section is devoted to working out Harder’s construction from [Har68] of an open normal subgroup $N$ contained
in the closure of $H(F)$ in $\prod_{v \in V} H(F_v)$. Here, $F_v$ is the completion of $F$ at $v$; the groups $H(F_v)$ are endowed with their $v$-adic topologies, and $\prod_{v \in V} H(F_v)$ has the product topology. In the sequel, we regard $H(F_v)$ as a subgroup of $\prod_{v \in V} H(F_v)$ by identifying it with the subgroup $H(F_v) \times \prod_{v' \neq v} \{1\}$. We will need the group $N$ in Section 3 when exhibiting the decomposition in Theorem 3, which leads to the reduction of Theorem 1 to the case of complete discrete valuation rings (Proposition 10). To begin, we recall Grothendieck’s Theorem ([SGA 3\text{II}, XIV, 1.1]) that any smooth group of finite type over a field contains a maximal torus.

Let $L_w$ be a minimal splitting field of a maximal $F_v$-torus $T$ of $H_{F_v}$, where $w$ is a valuation extending $v$. It turns out that the image $U$ of the norm map

$$N_{L_w/F_v} : T(L_w) \to T(F_v)$$

is an open subgroup of $T(F_v)$ and contained in $\overline{H(F)} \cap H(F_v)$ (see Lemma 2), where the closure $\overline{H(F)}$ is formed in $\prod_{v \in V} H(F_v)$. We use $U$ to construct an open normal subgroup of $H(F_v)$ contained in $H(F_v) \setminus H(F_v)$ (see Lemma 2), where the closure $H(F_v)$ is formed in $\prod_{v \in V} H(F_v)$. This gives rise to the desired open normal subgroup $N$ constructed in Proposition 3.

**Lemma 2.** For a maximal torus $T$ of $H_{F_v}$, the image $U$ of the norm map

$$N_{L_w/F_v} : T(L_w) \to T(F_v)$$

is an open subgroup of $T(F_v)$ contained in $\overline{H(F)} \cap H(F_v)$.

**Proof.** We recall from [SGA 3\text{II}, XIV, 6.1] that the functor

$$\text{Tor}(H) : \text{Sch}^\text{op}_{/F} \to \text{Sets}, \quad F' \mapsto \{\text{maximal tori of } H_{F'}\}$$

is representable by an $F$-scheme that is a rational variety (that is, by [SP, 0BXP], it has an open dense subscheme isomorphic to an open subscheme of $\mathbb{A}^n_F$). In particular, by the same argument as in [PV, Prop. 7.3], $\text{Tor}(H)$ satisfies weak approximation: $\text{Tor}(H)(F)$ is dense in $\prod_{v \in V} \text{Tor}(H)(F_v)$ under the diagonal embedding.

By [SGA 3\text{INew}, XXII, 5.8.3], the morphism

$$H_{F_v} \to \text{Tor}(H_{F_v}), \quad g \mapsto gTg^{-1}$$

induces an isomorphism $H_{F_v}/\text{Norm}_{H_{F_v}}(T) \simeq \text{Tor}(H_{F_v})$. By [SGA 3\text{II}, XI, 2.4bis], $\text{Norm}_{H_{F_v}}(T)$ is smooth. Therefore, by [Ces15d, Prop. 4.3], the map

$$\phi : H(F_v) \to \text{Tor}(H)(F_v), \quad g \mapsto gTg^{-1}$$

is open. Thus, for any neighbourhood $V$ of the neutral element of $H(F_v)$, by weak approximation for $\text{Tor}(H)$, the image $\phi(V)$ contains a maximal torus $T'$ of $H$. Let $L/F$ be a minimal splitting field of $T'$. Then, we have the decomposition:

$$L \otimes_F F_v \simeq \prod_{i=1}^r L_i.$$
The isomorphism $T'_{F_v} \simeq T$ implies that $L_i \simeq L_w$ for each $i$. The norm map $N_{L_w/F_v}: T'(L_w) \to T'(F_v)$ is the map on the $F_v$-points induced by the Weil restriction $N: \text{Res}_{L_w/F_v} T'_{L_w} \to T'_{F_v}$, whose kernel is a torus. So [Ces15d, Prop. 4.3] implies the openness of the image $U' := N_{L_w/F_v}(T'(L_w)) \subset T'(F_v)$.

Now we prove that $U' \subset H(F) \cap H(F_v)$. For $a \in T'(L_w)$ and $b := N_{L_w/F_v}(a)$, by weak approximation of the split torus $T'_L$, it suffices to choose an $x \in T'(L)$ approximating $a$ at $w$ and approximating $1$ at the other places. We conclude that $b$ is approximated by $N_{L/F}(x) \in H(F)$, and hence lies in $H(F) \cap H(F_v) \subset \prod_{v \in V} H(F_v)$.

The image $U$ of the norm map $N_{L_w/F_v}: T(L_w) \to T(F_v)$ satisfies $U' = g_0 U g_0^{-1}$ for a $g_0 \in V \subset H(F_v)$ such that $T' = g_0 T g_0^{-1}$, so each $u \in U$ is $g_0^{-1} u_0 g_0$, where $u_0 \in U'$ is approximated by elements in $H(F) \cap H(F_v)$. When shrinking $V$, we find that the associated $g_0$ is approximating the neutral element of $H(F_v)$ such that $u$ is approximated by $u_0 = g_0 u g_0^{-1}$. Therefore, there are elements in $H(F) \cap H(F_v)$ approximating $u$ and $U \subset H(F) \cap H(F_v)$. □

**Proposition 3.** There is a normal open subgroup $N$ of $\prod_{v} H(F_v)$ contained in the closure of $H(F)$.

**Proof.** It suffices to construct a normal open subgroup of $H(F_v)$ contained in $H(F) \cap H(F_v)$ for each $v$. For a fixed maximal torus $T$ of $H_{F_v}$, by Lemma 2, there is an open subgroup $U := (N_{L_w/F_v} T)(L_w) \subset T(F_v)$ contained in $H(F) \cap H(F_v)$. Consider the following morphism

$$f: H_{F_v} \times T \to H_{F_v}, \quad (g,t) \mapsto gt g^{-1}.$$ 

By the construction in the proof of [SGA 3II, XIII, 3.1], there is a principal open dense subscheme $W \subset T$ such that every $t_0 \in W(F_v)$ is a regular element, that is, by [SGA 3II, XIII, 3.0, 2.1, 2.2], the geometric point $\bar{t}_0 \in T_{\overline{F}_v}(F_v)$ over $t_0$ satisfies the following property:

$$f_{\bar{t}_0}: H_{\overline{F}_v} \times T_{\overline{F}_v} \to H_{\overline{F}_v}, \quad (g,t) \mapsto gt g^{-1}, \quad \text{is smooth at } (id, \bar{t}_0).$$

Note that this morphism is not a morphism of group schemes in general. We claim that $U \cap W(F_v) \neq \emptyset$. For the splitting field $L_w$ of $T$, we consider the preimage $\text{Nr}^{-1}(W)$ for the norm map $\text{Nr}: \text{Res}_{L_w/F_v}(T_{L_w}) \to T$. Because $T_{L_w}$ is isomorphic to a dense open of an affine space $\mathbb{A}^n_{L_w}$, the Weil restriction $\text{Res}_{L_w/F_v}(T_{L_w})$ is also isomorphic to a dense open of $\mathbb{A}^m_{F_v}$ for $m = [L_w : F_v]$. Since $F_v$ is infinite, the Zariski density of $(\text{Res}_{L_w/F_v}(T_{L_w}))(F_v)$ implies that $(\text{Res}_{L_w/F_v}(T_{L_w}))(F_v) \cap \text{Nr}^{-1}(W)(F_v) \neq \emptyset$ and

$$U \cap W(F_v) = \text{Nr}((\text{Res}_{L_w/F_v}(T_{L_w}))(F_v)) \cap W(F_v) \neq \emptyset.$$

Therefore, we may choose a $t_0 \in U \cap W(F_v)$ such that $f$ is smooth at $(id, t_0)$. Hence, there is an open neighborhood $W'$ of $(id, t_0)$ such that $f|_{W'}$ is a smooth morphism. Since a smooth morphism is a composite of étale morphism and a projection (see [SP, 054L]), by [Ces15d, 2.8], the map $f|_{W'}(F_v): W'(F_v) \to H(F_v)$ is open for
3. Decomposition of groups

The goal of this section is to prove the following decomposition of groups, which leads to the reduction of Theorem 1 to the case of complete discrete valuation rings (see Proposition 10):

**Theorem 3.** For a reductive group scheme $G$ over a semilocal Dedekind domain $R$ with fraction field $K$, let $\mathcal{V}$ be the set of valuations corresponding to the maximal ideals of $R$. For each $v \in \mathcal{V}$, let $R_v$ be the completion of $R$ at $v$ and its fraction field be $K_v$. Then, we have

$$\prod_{v \in \mathcal{V}} G(K_v) = G(K) \prod_{v \in \mathcal{V}} G(R_v).$$

The proof proceeds in Propositions 4, 5, 7 and 8, by mildly simplifying and improving Nisnevich’s argument. A minimal parabolic subgroup of $G_{R_v}$ is denoted by $P_v$, and its unipotent radical and Levi subgroup are denoted by $U_v$ and $L_v$ respectively. By [SGA 3IInew, XXVI, 6.11, 6.18], the maximal central split torus $T_v$ of $L_v$ is a maximal split torus of $G_{R_v}$.

**Proposition 4.** We have

$$\prod_{v \in \mathcal{V}} T_v(K_v) \subset G(K) \prod_{v \in \mathcal{V}} G(R_v).$$

**Proof.** It suffices to replace each $T_v$ by a maximal torus of $G_{R_v}$ containing it to prove the inclusion. For a fixed $v$, the morphism

$$f : G(K_v) \to \text{Tor}(G)(K_v), \quad g \mapsto gT_v g^{-1}$$

sends a neighbourhood $U$ of the neutral element of $G(K_v)$ to an open subset $V \subset \text{Tor}(G)(K_v)$ containing $T_v$. Therefore, $V \cap \text{Tor}(G)(R_v)$ is an open subset of $\text{Tor}(G)(K_v)$ containing $T_v$. Since $\text{Tor}(G)$ is an affine $R$-scheme ([SGA 3II, XII, 5.4]), we have a Cartesian square with inclusion arrows

$$\begin{array}{ccc}
\text{Tor}(G)(R) & \to & \text{Tor}(G)(R_v) \\
\downarrow & & \downarrow \\
\text{Tor}(G)(K) & \to & \text{Tor}(G)(K_v)
\end{array}$$

and $\text{Tor}(G)(R) = \text{Tor}(G)(K) \cap \text{Tor}(G)(R_v)$.

As in the proof of Lemma 2, $\text{Tor}(G)(K)$ is dense in $\text{Tor}(G)(K_v)$, so the intersection $V \cap \text{Tor}(G)(R_v) \cap \text{Tor}(G)(K) = V \cap \text{Tor}(G)(R) \neq \emptyset$ contains a maximal $R$-torus $T_0$ of $G$. By Proposition 3, there is an open normal subgroup $N$ of $\prod_v G(K_v)$ contained in $\overline{G(K)}$. Assuming that $U \subset N \cap \prod_v G(R_v)$, we have

$$T_v(K_v) = gT_0(K_v)g^{-1} = gT_0(K)T_0(R_v)g^{-1} \subset gG(K)G(R_v)g^{-1} \subset \overline{G(K)}G(R_v),$$

the $v$-adic topology. By shrinking $W'$ if necessary and by translation, we find an open neighborhood $U_0 \subset U$ of $t_0$ such that $E := f(H(F_v) \times U_0)$ is open. Let $N$ be the subgroup of $H(F_v)$ generated by $E$. Since $N$ contains an open subset $E$, it is an open subgroup. Because $E$ is stable under $H(F_v)$-conjugation, $N$ is a normal subgroup of $H(F_v)$. By Lemma 2, the conjugation $gtg^{-1}$ is in $\tilde{U}$ for another maximal torus $\tilde{T}$ of $H(F_v)$, and hence $gtg^{-1} \in \overline{H(F)} \cap H(F_v)$. Consequently, $N$ is the desired normal open subgroup of $H(F_v)$ contained in $\overline{H(F)} \cap H(F_v)$. □
where the second equality is by [CTS87, Prop. 8.1]. By [Con12, Prop. 2.1] and that $G$ is affine, $\prod_v G(R_v)$ is both closed and open in $\prod_v G(K_v)$, so the product $G(K)\prod_v G(R_v)$ contains $G(K)$. Therefore,

$$\overline{G(K)}\prod_v G(R_v) \subset G(K)\prod_v G(R_v) \quad \text{and} \quad \prod_v T_v(K_v) \subset \overline{G(K)}\prod_v G(R_v). \quad \square$$

**Proposition 5.** We have

$$\prod_{v \in \mathcal{V}} U_v(K_v) \subset \overline{G(K)}.$$

**Proof.** The maximal split torus $T_v$ with character lattice $M$ acts on $G_{R_v}$ by conjugation

$$T_v \times G_{R_v} \to G_{R_v}, \quad (t, g) \mapsto t g t^{-1},$$

yielding a weight decomposition $\text{Lie}(G_{R_v}) = \bigoplus_{\alpha \in M} \text{Lie}(G_{R_v})^\alpha$. The subset $\Phi$ of $\alpha \in M - \{0\}$ such that $\text{Lie}(G_{R_v})^\alpha \neq 0$ is the relative root system of $(G_{R_v}, T_v)$. By [SGA 3II\text{new}, XXVI, 6.1; 7.4], the zero-weight space of $\text{Lie}(G_{R_v})$ is $\text{Lie}(L_0)$. The relative root datum $((G_{R_v}, T_v), M, \Phi)$ has a set $\Phi_+$ of positive roots such that

$$\text{Lie}(P_0) = \text{Lie}(L_0) \oplus \text{Lie}(U_v) = \text{Lie}(L_0) \oplus \left( \bigoplus_{\alpha \in \Phi_+} \text{Lie}(G_{R_v})^\alpha \right).$$

Let $\overline{K_v}/K_v$ be a Galois field extension splitting $G_{K_v}$. We denote the base change of $P_v, L_v, U_v$ and $T_v$ over $\overline{K_v}$ by $\overline{P}, \overline{L}, \overline{U}$ and $\overline{T}$ respectively. Since $T_v$ is central in $L_v$, by [SGA 3II\text{new}, XXVI, 2.4; XIX, 1.6.2(iii)], there is a split maximal torus $T' \subset \overline{P}$ of $G_{\overline{K_v}}$ containing $\overline{T}$. The centralizer of $T'$ in $G_{\overline{K_v}}$ is itself and is a Levi subgroup of a Borel subgroup $\overline{B} \subset \overline{P}$. Let $M'$ be the character lattice of $T'$. Then, the adjoint action of $T'$ on $G_{\overline{K_v}}$ induces a decomposition $\text{Lie}(G_{\overline{K_v}}) = \bigoplus_{\alpha \in M'} \text{Lie}(G_{\overline{K_v}})^\alpha$, whose coarsening is $\text{Lie}(G_{R_v}) = \bigoplus_{\alpha \in M} \text{Lie}(G_{R_v})^\alpha$. Let $\Phi'$ be the root system with positive set $\Phi'_+$. For the Borel $\overline{B}$, now [SGA 3II\text{new}, XXVI, 7.12] gives us a surjection $u: M' \to M$ such that $\Phi_+ \subset u(\Phi'_+) \subset \Phi_+ \cup \{0\}$. By op. cit. 1.12, we have decompositions

$$\overline{U} = \prod_{\alpha \in \Phi'_+} \overline{U}_\alpha, \quad \text{Lie}(\overline{U}) = \bigoplus_{\alpha \in \Phi'_+} \text{Lie}(G_{\overline{K_v}})^\alpha,$$

where $\Phi'' \subset \Phi'_+$. Further, we have isomorphisms of group schemes $f_\alpha: G_{a,\overline{K_v}} \sim \overline{U}_\alpha$. The zero-weight space for $T_v$-action on $\text{Lie}(G_{R_v})$ is $\text{Lie}(L_v)$, so the restriction to $T$ of weights in $\text{Lie}(\overline{U})$ must be nonzero, that is $u(\Phi'') \subset \Phi_+$. For a cocharacter $\xi: G_m \to T_v$ and its base change $\xi_{\overline{K_v}}: G_{m,\overline{K_v}} \to \overline{T}$, the composite of $\xi_{\overline{K_v}}$ with $\overline{T} \to T'$ is the image $u^*(\xi)$ of $\xi$ under the dual map $u^*: M^* \to M''^*$. The action of $G_m$ on $U_v$ induced by $\xi$ is

$$\text{ad}: G_m(K_v) \times U_v(K_v) \to U_v(K_v), \quad (t, y) \mapsto \xi(t)y\xi(t)^{-1}.$$
For a $y \in U_v(K_v)$, let $\tilde{y} \in U_v(\widetilde{K}_v)$ denote the image of $y$. Let $N$ be the open normal subgroup of $\prod_v G(K_v)$ constructed in Proposition 3. We consider the commutative diagram

$$
\begin{array}{c}
\mathbb{G}_m(K_v) \times (N \cap U_v(K_v)) \xrightarrow{\xi \times \text{id}} T_v(K_v) \times (N \cap U_v(K_v)) \xrightarrow{\text{ad}} N \cap U_v(K_v) \\
\mathbb{G}_m(K_v) \times U_v(K_v) \xrightarrow{\xi \times \text{id}} T_v(K_v) \times U_v(K_v) \xrightarrow{\text{ad}} U_v(K_v) \\
\mathbb{G}_m(\widetilde{K}_v) \times U_v(\widetilde{K}_v) \xrightarrow{\xi \times \text{id}} T_v(\widetilde{K}_v) \times U_v(\widetilde{K}_v) \xrightarrow{\text{ad}} U_v(\widetilde{K}_v).
\end{array}
$$

Let $\varpi$ be a uniformizer of $K_v^\times = \mathbb{G}_m(K_v)$. For an integer $n$, the action of $\varpi^n$ on $y$ via $\xi$ is denoted by $(\varpi^n) \cdot y$. Decompose $\tilde{y} = \prod_{\alpha \in \Phi'} f_\alpha(z_\alpha)$ for $z_\alpha \in \widetilde{K}_v$. The image $(u^*(\xi)(\varpi^n))\tilde{y}(u^*(\xi)(\varpi^n))^{-1} \in U_v(\widetilde{K}_v)$ of $(\varpi^n) \cdot y$ is the following

$$
\prod_{\alpha \in \Phi''} (u^*(\xi)(\varpi^n))f_\alpha(z_\alpha)(u^*(\xi)(\varpi^n))^{-1} = \prod_{\alpha \in \Phi''} f_\alpha(\xi, u(\alpha))z_\alpha
$$

$$
= \prod_{\alpha \in \Phi''} f_\alpha(\xi, u(\alpha))z_\alpha.
$$

Because $u(\Phi'') \subset \Phi_+$, we can choose a cocharacter $\xi$ such that $(\xi, u(\alpha))$ is positive for all roots $\alpha \in \Phi''$. Then, when $n$ grows, the element $(\varpi^n) \cdot y \in U_v(\widetilde{K}_v)$ gets close to identity and so the same holds in $U_v(K_v)$. Thus, since $N \cap U_v(K_v)$ is an open neighbourhood of $\text{id}_{G(K_v)}$, every orbit of the $T_v$-action on $U_v(K_v)$ intersects with $N \cap U_v(K_v)$ nontrivially, so

$$
U_v(K_v) = \bigcup_{t \in T_v(K_v)} t(N \cap U_v(K_v))t^{-1} = N \cap U_v(K_v),
$$

which implies that $U_v(K_v) \subset N$. Combining Proposition 3, we conclude that $\prod_{v \in V} U_v(K_v) \subset \mathcal{G}(K)$. \qed

Before the next step of decomposing the group, we gather some recollections on anisotropic groups.

**Lemma 4.** For a reductive group $G$ over a discrete valuation ring $R$ with fraction field $K$,

$$
G \text{ is } R\text{-anisotropic if and only if } G \text{ is } K\text{-anisotropic.}
$$

**Proof.** For an $R$-algebra $R'$, by [SGA 3II\text{\text{new}}, XXVI, 6.14], the group $G$ is $R'$-anisotropic if and only if $G_{R'}$ has no proper parabolic subgroups and $\text{rad}(G_{R'})$ has no split subtorus. The proof proceeds in the following two steps.

(a) For the existence of parabolic subgroups, by [SGA 3II\text{\text{new}}, XXVI, 3.5], the functor

$$
\text{Par}(G) : \text{Sch}_R \to \text{Sets}, \quad R' \mapsto \{\text{parabolic subgroups of } G_{R'}\}
$$

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is representable by a projective scheme over $R$ and satisfies the valuative criterion of properness:

$$\text{Par}(G)(K) = \text{Par}(G)(R).$$

(b) If $\text{rad}(G_K) = \text{rad}(G)_K$ is $K$-anisotropic, then $\text{rad}(G)$ is $R$-anisotropic. For the other direction, recall that every torus over a Noetherian normal local ring $S$ is isotrivial, see [SGA 3II, X, 5.16]. Let $\pi^\text{et}_1(S)$ be the étale fundamental group of $S$ whose base point is the geometric generic point of $S$. We use the anti-equivalence in [SGA 3II, X, 1.2]:

\[
\begin{align*}
\{ \text{the category of } S\text{-tori} \} & \iff \{ \text{the category of finite } \mathbb{Z}\text{-lattices with continuous } \pi^\text{et}_1(S)\text{-action} \}.
\end{align*}
\]

Both $\text{rad}(G)$ and $\text{rad}(G_K)$ correspond to a $\mathbb{Z}$-lattice $M$ acted by étale fundamental groups $\pi^\text{et}_1(R)$ and $\pi^\text{et}_1(K)$ respectively. A nontrivial split torus in $\text{rad}(G_K)$ corresponds to a quotient lattice $N$ of $M$ with trivial $\pi^\text{et}_1(K)$-action. By the surjectivity of $\pi^\text{et}_1(K) \to \pi^\text{et}_1(R)$ (see [SP, OBSM]), the action of $\pi^\text{et}_1(R)$ on $N$ is also trivial and the latter corresponds to a nontrivial split torus in $\text{rad}(G)$. □

**Proposition 6.** For a reductive group $G$ over a Henselian discrete valuation ring $R$, the following hold:

1. for the strict Henselization $\tilde{R}$ of $R$ with fraction field $\tilde{K}$, the group $G(\tilde{R})$ is a maximal parahoric subgroup of $G(\tilde{K})$;

2. if $G$ is $K$-anisotropic, then $G(K) = G(R)$.

**Proof.** (1) Because $G_{\tilde{R}}$ is a Chevalley group scheme and the residue field of $\tilde{R}$ is separably closed, by [BTII, 4.6.22, 4.6.31], the group $G(\tilde{R})$ is the stabilizer of a special point in the Bruhat–Tits building of $G(\tilde{K})$. Since a special point is a minimal facet (see [BTII, 4.6.15]), by the connectedness of each fiber of $G_{\tilde{R}}/\tilde{R}$ and the definition of parahoric subgroups (see [BTII, 5.2.6]), $G(\tilde{R})$ is a maximal parahoric subgroup of $G(\tilde{K})$.

(2) First, we prove that $G(K) \subseteq G(\tilde{R})$. For the Bruhat–Tits building $\mathcal{F}$ of $G(\tilde{K})$, by [BTII, 5.1.27], the Galois group $\Sigma := \text{Gal}(\tilde{K}/K)$ acts on the enlarged Bruhat–Tits building $\mathcal{F}^{\text{ext}} := \mathcal{F} \times X^*_{\tilde{K}}(G)^R_{\tilde{R}}$ of $G(\tilde{K})$ and has a unique fixed point $x$ located at $\mathcal{F} \times \{0\}$. For $x$, by [BTII, 5.2.6], its connected pointwise stabilizer (see [BTII, 4.6.28]) is a $\Sigma$-invariant parahoric subgroup $P$. This parahoric subgroup $P$, by the $K$-anisotropicity of $G$ and [BTII, 5.2.7], is the unique $\Sigma$-invariant parahoric subgroup and thus, due to (a), coincides with $G(\tilde{R})$. By connectedness of each fiber of $G_{\tilde{R}}/\tilde{R}$, the parahoric subgroup $G(\tilde{R})$ is also the stabilizer of $x$. By [BTI, 9.2.1 (DI 1)], the fixed point $x$ is also stabilized by $G(K)$. Therefore, $G(K) \subseteq G(\tilde{R})$.

Lastly, by the affineness of $G$ and $R = \tilde{R} \cap K$, we obtain the following cartesian square

$$
\begin{array}{ccc}
G(R) & \to & G(\tilde{R}) \\
\downarrow & & \downarrow \\
G(K) & \to & G(\tilde{K})
\end{array}
$$

and combine it with $G(K) \subseteq G(\tilde{R})$ to conclude that $G(K) = G(R)$. □
Remark 1. When using Bruhat–Tits theory for general reductive groups, one needs to check the conditions listed in [BT II, 5.1.1]. Fortunately, these conditions are automatically satisfied when $R$ is a Henselian discrete valuation ring, and $G$ is defined and reductive over $R$ (by [SGA 3\textit{IIInew}, XXII, 2.3], $G$ splits over $\tilde{R}$, hence is split (a fortiori quasi-split) over $\tilde{K}$).

**Proposition 7.** We have

$$\prod_{v \in \mathcal{V}} P_v(K_v) \subset G(K) \prod_{v \in \mathcal{V}} G(R_v).$$

**Proof.** The quotient $H_v := L_v/T_v$ is $R_v$-anisotropic and by Lemma 4 is $K_v$-anisotropic. By Proposition 6, we have $H_v(K_v) = H_v(R_v)$, which fits into the commutative diagram with exact rows:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & T_v(R_v) & \longrightarrow & L_v(R_v) & \longrightarrow & H_v(R_v) & \longrightarrow & H^1(R_v, T_v) = 0 \\
\downarrow & & \downarrow & & \downarrow \lambda_v & & \downarrow & & \\
0 & \longrightarrow & T_v(K_v) & \longrightarrow & L_v(K_v) & \longrightarrow & H_v(K_v) & \longrightarrow & H^1(K_v, T_v) = 0
\end{array}
$$

The image of $a \in L_v(K_v)$ in $H_v(K_v)$ has a preimage $b \in L_v(R_v)$. Hence $a \cdot \lambda_v(b)^{-1} \in T_v(K_v)$ and

$$L_v(K_v) = T_v(K_v)L_v(R_v).$$

By Propositions 4 and 5, we obtain the conclusion. \qed

**Proposition 8.**

$$G(K) \prod_{v \in \mathcal{V}} P_v(K_v) = \prod_{v \in \mathcal{V}} G(K_v).$$

**Proof.** For each $P_v$, there is a unique parabolic subgroup $Q_v$ of $G_{R_v}$ such that the canonical morphism

$$\text{rad}^u(P_v)(K_v) \cdot \text{rad}^u(Q_v)(K_v) \to G(K_v)/P_v(K_v)$$

is surjective ([SGA 3\textit{IIInew}, XXVI, 4.3.2, 5.2]). We conclude by combining this surjectivity with Proposition 5 and deducing

$$\prod_{v \in \mathcal{V}} G(K_v) \subset G(K) \prod_{v \in \mathcal{V}} P_v(K_v).$$

Proof of Theorem 3. By Propositions 7 and 8, we have

$$\prod_{v \in \mathcal{V}} G(K_v) \subset G(K) \prod_{v \in \mathcal{V}} G(R_v) = G(K) \prod_{v \in \mathcal{V}} G(R_v).$$

\qed
4. Reductions: twisting, passage to completion, and to anisotropic groups

This section exhibits a sequence of methods to reduce Theorem 1. First, using twisting technique of torsors, one can prove Theorem 1 by showing that the map

\[ H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G) \]

has a trivial kernel. Secondly, we improve and extend Nisnevich’s argument by a patching technique for the semilocal case and reduce to the case when \( R \) is a complete discrete valuation ring (Proposition 10). One motivation for reducing to the complete case is that in Bruhat-Tits theory [BTII, §5], to define parahoric subgroups for more general cases than the quasi-split case, we need additional conditions (see [BTII, 5.1.6]), which are automatically satisfied in our case when \( R \) is complete. Lastly, with crucial results and properties derived from Bruhat-Tits theory (for instance, Proposition 6), we reduce further to the case when \( G \) is semisimple and anisotropic (Proposition 12). This case has enough advantages such as the uniqueness of Galois-invariant parahoric subgroups of \( G(K) \) (see [BTII, 5.2.7]), for us to prove Theorem 1 in §5.

**Proposition 9** ([Gir71, III, 2.6.1 (i)]). For a group scheme \( G \) over a scheme \( S \), we let \( T \) be a right \( G \)-torsor and let \( \tau G := \text{Aut}_G(T) \). Then twisting by \( T \) induces an isomorphism

\[ H^1_{\text{ét}}(S, G) \overset{\sim}{\to} H^1_{\text{ét}}(S, \tau G), \quad X \mapsto \text{Hom}_G(T, X) \]

such that the image of the class of \( T \) is the neutral class. In fact, there is an equivalence between the gerbe of \( G \)-torsors and the gerbe of \( \tau G \)-torsors.

**Corollary 1.** Conjecture 1 is equivalent to showing that for all reductive groups \( G \) over \( R \),

\[ H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G) \]

is injective.

In particular, in Theorem 1, it suffices to prove that the map

\[ H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G) \]

has a trivial kernel.

The following shows that we can reduce Theorem 1 to the complete case.

**Proposition 10.** It suffices to prove Theorem 1 when \( R \) is a complete discrete valuation ring.

**Proof.** If a \( G \)-torsor \( \mathcal{X} \) becomes trivial over \( K \), then it is trivial over \( \coprod_v \text{Spec} \, K_v \) and by assumption, trivial over \( \coprod_v \text{Spec} \, R_v \). Because \( \coprod_v \text{Spec} \, R_v \to \text{Spec} \, R \) is flat and induces isomorphisms on closed points, by patching technique in [MB14, Thm. 1.1], we have an equivalence of categories

\[
\left\{ \text{affine schemes over } \text{Spec} \, R \right\} \leftrightarrow \left\{ \coprod_v \text{Spec} \, R_v, \, \mathcal{X}'', \, \iota \right\} \times \left\{ \text{an isomorphism } \iota : \mathcal{X}'|_{\coprod_v \text{Spec} \, K_v} \overset{\sim}{\to} \mathcal{X}''|_{\coprod_v \text{Spec} \, K_v} \right\}. 
\]
Let $\mathcal{X}'$ be a $G_{\text{Spec } R_v}$-torsor and $\mathcal{X}''$ a $G_{\text{Spec } K}$-torsor. Since $G$ is affine and flat, the resulting scheme $\tilde{\mathcal{X}}$ is a $G$-torsor with the glued structural isomorphism

$$G \times_R \tilde{\mathcal{X}} \cong \tilde{\mathcal{X}} \times_R \tilde{\mathcal{X}}, \quad (g, s) \mapsto (gs, s).$$

Now let $\mathcal{T}$ be an arbitrary $G$-torsor obtained by gluing the trivial torsors $\mathcal{X}_{\text{Spec } R_v}$ and $\mathcal{X}_{\text{Spec } K}$ along $\mathcal{X}_{\text{Spec } R_v}^\text{new}$. The torsor class of $\mathcal{T}$ is determined by the gluing isomorphism $\mathcal{I} : (\mathcal{X}_{\text{Spec } R_v})_{\mathcal{X}_{\text{Spec } K}} \cong (\mathcal{X}_{\text{Spec } K})_{\mathcal{X}_{\text{Spec } K}}$. This isomorphism between trivial torsors is nothing but a multiplication by $a \in \prod_v G(K_v)$. Since automorphisms of $\mathcal{X}_{\text{Spec } K}$ and of $\mathcal{X}_{\text{Spec } R_v}$ are induced by multiplications by elements in $G(K)$ and in $\prod_v G(R_v)$ respectively, the isomorphic classes of the possible $G$-torsors $\mathcal{T}$ are in correspondence with

$$G(K) \setminus \prod_v G(K_v)/\prod_v G(R_v),$$

which is trivial by Theorem 3. Consequently, $\mathcal{X}$ is in the class of trivial torsors. \hfill \square

Now we reduce Theorem 1 to the case when $G$ is anisotropic and semisimple.

**Proposition 11.** Let $R$ be a valuation ring with fraction field $K$ and let $G$ be a reductive $R$-group scheme. If for every reductive $R$-group scheme $H$ without a proper parabolic subgroup, the map $H^1_{\text{ét}}(R, H) \to H^2_{\text{ét}}(K, H)$ is injective, then so is $H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G)$.

**Proof.** If there is a proper minimal parabolic subgroup $P$ of $G$ and $L$ is its Levi subgroup, then we consider the following commutative diagram

$$
\begin{array}{ccc}
H^1_{\text{ét}}(R, L) & \longrightarrow & H^1_{\text{ét}}(R, P) \longrightarrow & H^1_{\text{ét}}(R, G) \\
\downarrow & & \downarrow & \downarrow \\
H^1_{\text{ét}}(K, L) & \longrightarrow & H^1_{\text{ét}}(K, P) \longrightarrow & H^1_{\text{ét}}(K, G)
\end{array}
$$

and show that the kernel of the third column comes from the kernel of the first column as follows.

By [SGA 3III new, XXVI, 2.3], the rows in the left square are isomorphisms. For the second square, we use the argument in [Nis83, Prop. 5.1]. Let $E$ be a $G$-torsor that becomes trivial over $K$. The quotient sheaf $E/P$ is fpqc locally isomorphic to $G/P$ and represented by a scheme projective over $R$ (see [SGA 3III new, XXVI, 3.3; 3.20]). By the valuative criterion of properness, we have

$$(E/P)(K) = (E/P)(R).$$

For a $K$-point $x$ of $E$, its image $\overline{x}$ in $E/P$ is also an $R$-point of $E/P$. Subsequently, the fiber of $E$ over $\overline{x}$

$$F := E \times_{E/P} \text{Spec } R$$

is a $P$-torsor over $R$ such that $F(K) \neq \emptyset$ and its image under the map $H^1_{\text{ét}}(R, P) \to H^1_{\text{ét}}(R, G)$ is the class of $E$. Therefore, the kernel of $H^1_{\text{ét}}(R, G) \to H^1_{\text{ét}}(K, G)$ is in the image of the kernel of $H^1_{\text{ét}}(R, P) \to H^1_{\text{ét}}(K, P)$.

By [SGA 3III new, XXVI, 1.20], parabolic subgroups of $L$ are intersections of $L$ with parabolic subgroups contained in $P$. Therefore, $L$ contains no proper parabolic subgroup. \hfill \square
Proposition 12. In order to prove Theorem 1, it suffices to prove that

\[ H^1_{\text{ét}}(R, G) \rightarrow H^1_{\text{ét}}(K, G) \]

has a trivial kernel when \( R \) is a complete discrete valuation ring and \( G \) is semisimple and \( R \)-anisotropic.

Proof. By Propositions 10 and 11, it suffices to prove Theorem 1 in the case when \( G \) is a reductive group over a complete discrete valuation ring \( R \) and has no nontrivial parabolic subgroup. The quotient \( G/\text{rad}(G) \) has no proper parabolic subgroups and is semisimple, hence by [SGA 3_{III new}, XXVI, 6.14] is \( R \)-anisotropic and by Lemma 4 is \( K \)-anisotropic. We assume that \( l(G/\text{rad}(G)) : H^1_{\text{ét}}(R, G/\text{rad}(G)) \rightarrow H^1_{\text{ét}}(K, G/\text{rad}(G)) \) is injective. Since \( R \) is complete, by Proposition 6, we have

\[ (G/\text{rad}(G))(R) = (G/\text{rad}(G))(K) \]

fitting into the following commutative diagram

\[
\begin{array}{cccc}
(G/\text{rad}(G))(R) & \rightarrow & H^1_{\text{ét}}(R, \text{rad}(G)) & \rightarrow & H^1_{\text{ét}}(R, G) & \rightarrow & H^1_{\text{ét}}(R, G/\text{rad}(G)) \\
\| & & l(\text{rad}(G)) & & l(G) & & l(G/\text{rad}(G)) \\
(G/\text{rad}(G))(K) & \rightarrow & H^1_{\text{ét}}(K, \text{rad}(G)) & \rightarrow & H^1_{\text{ét}}(K, G) & \rightarrow & H^1_{\text{ét}}(K, G/\text{rad}(G))
\end{array}
\]

with exact rows and the map \( l(\text{rad}(G)) \) is injective by [CTS87, Thm. 4.1]. By diagram chasing, the map \( l(G) \) has a trivial kernel. By Corollary 1, we complete the proof. □

5. The semisimple and anisotropic case

By Proposition 12, it suffices to prove Theorem 1 for the case when \( G \) is semisimple anisotropic and \( R \) is a complete discrete valuation ring. When \( R \) has a perfect residue field, this case is contained in [Nis83, 2, Thm. 4.2] for general reductive groups. However, the proof has several unclear points, see Remarks 2-4. These gaps motivate us to first reduce to the semisimple and anisotropic case, where we have the uniqueness of Galois-invariant parahoric subgroups. Further, in our proof, we add supplementary details for normalizers of parahoric subgroups. Theorem 5 also extends a special case implied by [BT_{II}, Lem. 3.9] when the residue field is perfect and \( G \) is semisimple simply connected.

Theorem 5. For a semisimple and anisotropic group scheme \( G \) over a complete discrete valuation ring \( R \) with fraction field \( K \), the following map has a trivial kernel:

\[ H^1_{\text{ét}}(R, G) \rightarrow H^1_{\text{ét}}(K, G). \]

Proof. We fix some notations:

- let \( K^{\text{sep}} \) be a separable closure of \( K \) that contains \( \tilde{K} \);
- \( \Gamma := \text{Gal}(\tilde{K}/R) \simeq \text{Gal}(\tilde{K}/K) ; \Gamma_{\tilde{R}} := \text{Gal}(K^{\text{sep}}/\tilde{K}) ; \Gamma_{K} := \text{Gal}(K^{\text{sep}}/K). \)
Since $R$ is Henselian, we can view $G$ as a sheaf over the site of profinite $\Gamma$-sets, whose one-point set with trivial $\Gamma$-action is $\text{Spec}(R/\mathfrak{m}_R)$. Therefore, a variant of the Cartan–Leray spectral sequence [Sch13, 3.7(iii)] gives an isomorphism to the Galois cohomology $H^1_{\acute{e}t}(R, G) \simeq H^1(\Gamma, G(\bar{R}))$. Similarly (or by [SGA 4_{II}, VIII, 2.1]), we have $H^1_{\acute{e}t}(K, G) \simeq H^1(\Gamma_K, G(K_{\text{sep}}))$. It suffices to prove that both $\alpha$ and $\beta$ in the decomposition

$$H^1(\Gamma, G(\bar{R})) \xrightarrow{\alpha} H^1(\Gamma, G(\tilde{K})) \xrightarrow{\beta} H^1(\Gamma_K, G(K_{\text{sep}}))$$

have a trivial kernel. We show this in the following two steps.

(i) The injectivity of

$$\alpha: H^1(\Gamma, G(\bar{R})) \to H^1(\Gamma, G(\tilde{K})).$$

For a cocycle $z \in H^1(\Gamma, G(\bar{R}))$ that becomes trivial in $H^1(\Gamma, G(\tilde{K}))$, there is $h \in G(\tilde{K})$ such that

$$z(s) = h^{-1}s(h) \quad \text{in} \quad G(\tilde{R}) \quad \text{for each} \quad s \in \Gamma.$$ 

To prove that $h \in G(\tilde{R})$, we consider the subgroups $G(\tilde{R})$ and $hG(\tilde{R})h^{-1}$ of $G(\tilde{K})$. We have seen that $G(\tilde{R})$ is a parahoric subgroup of $G(\tilde{K})$ in Proposition 6. The conjugation by $g \in G(\tilde{K})$ of a parahoric subgroup $P_F$ associated to the facet $F$ satisfies the definition of the parahoric subgroup associated to the facet $g \cdot F$ (see [BT, 5.2.6]). In particular, $hG(\tilde{R})h^{-1}$ is also a parahoric subgroup. Now we show that $hG(\tilde{R})h^{-1}$ is invariant under $\Gamma$: for every $s \in \Gamma$,

$$s(hG(\tilde{R})h^{-1}) = s(h)G(\tilde{R})s(h^{-1}) = hzs(s)G(\tilde{R})z(s)^{-1}h^{-1} = hG(\tilde{R})h^{-1},$$

since $G(\tilde{R})$ is $\Gamma$-invariant and $z(s) \in G(\tilde{R})$. Because $G$ is anisotropic over $K$, the uniqueness of a $\Gamma$-invariant parahoric subgroup ([BT, 5.2.7]) implies that

$$hG(\tilde{R})h^{-1} = G(\tilde{R}).$$

By [BT, 4.6.22, 4.6.31], because the residue field of $\tilde{R}$ is separably closed and $G$ is a Chevalley group scheme over $\tilde{R}$, the group $G(\tilde{R})$ is the group of $\tilde{R}$-points of a connected pointwise stabilizer $\mathfrak{G}_x^0$ of a special point $x$ (which is a minimal facet of the extended Bruhat–Tits building $\tilde{\mathcal{F}}^{\text{ext}} := \tilde{\mathcal{F}} \times X^+_K(G)_{\tilde{R}}$) and is the stabilizer $\mathfrak{G}_x^1$ of $x$ in $\tilde{\mathcal{F}}$ (see [BT, 4.6.28]). On the other hand, since $G$ is semisimple, by [SGA 3_{II\text{new}}, XXII, 6.2.1], the $\tilde{K}$-character group of $G$ is

$$X^+_K(G) := \text{Hom}_{\tilde{K} \text{-gr.}}(G, \mathbb{G}_m, \tilde{K}) \simeq \text{Hom}_{\tilde{K} \text{-gr.}}(G/G^{\text{der}}, \mathbb{G}_m, \tilde{K}) = 1.$$ 

Subsequently, $\tilde{\mathcal{F}}^{\text{ext}} = \tilde{\mathcal{F}}$ and $x$ is a point in the Bruhat–Tits building, in which the stabilizer of $x$ in $G(\tilde{K})$ is the normalizer of $G(\tilde{R})$ in $G(\tilde{K})$ (see [BT, 4.6.17]), so $G(\tilde{R}) = \text{Norm}_{G(\tilde{K})}(G(\tilde{R}))$. Thus, we have $h \in G(\tilde{R})$ and $z$ is trivial in $H^1(\Gamma, G(\tilde{R}))$. 
(ii) The injectivity of 
\[ \beta: H^1(\Gamma, G(\tilde{K})) \to H^1(\Gamma_K, G(K^{\text{sep}})) . \]

Recall the inflation-restriction exact sequence in [Ser02, 5.8 a]):
\[ 0 \to H^1(G_1/G_2, A^{G_2}) \to H^1(G_1, A) \to H^1(G_2, A)^{G_1/G_2} , \]
where \( G_1 \) is a group with a closed normal subgroup \( G_2 \) and \( A \) is a \( G_1 \)-group. It suffices to take
\[ G_1 := \Gamma_K, \quad G_2 := \Gamma_{\tilde{K}} , \quad \text{and} \quad A := G(K^{\text{sep}}). \]

The following remarks explain why our approach to proving Theorem 1 in the case of a complete discrete valuation ring would not have worked without reducing first to the semisimple anisotropic case. They also highlight several problems in the argument of [Nis83, 2, Thm. 4.2].

Remark 2. When \( G \) is a semisimple, simply-connected group over a Henselian discrete valuation field \( K \) (for instance, a local field), parahoric subgroups of \( G(K) \) are their own normalizers ([BT II, 5.2.9]). However, this is not true in the semisimple adjoint case. If \( G := \text{PGL}_2(Q_p) \), then the parahoric subgroup
\[ P = \left\{ \text{classes of } \begin{pmatrix} Z_p^\times & pZ_p \\ Z_p & Z_p^\times \end{pmatrix} \right\} \]
is normalized by \( A = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \)

but \( A \notin P \): the diagonals of \( P \) are nonzero. This also provides an example of the fact that maximal bounded subgroups are not necessarily stabilizers of vertices; the converse is true (see [Yu09, p. 14]).

Remark 3. Maximal parahoric subgroups are not their own normalizers in the reductive case. For instance, we consider \( \text{GL}_n(Q_p) \) with a maximal parahoric subgroup \( \text{GL}_n(Z_p) \). The normalizer of \( \text{GL}_n(Z_p) \) contains \( Q_p^\times \cdot 1 \), which is not in \( \text{GL}_n(Z_p) \). To explain this, we note that \( \text{GL}_n(Z_p) \) stabilizes \( x \times \{0\} \subset J^\text{ext} = J \times X_p^* (\text{GL}_n)_{\mathbb{R}}^\vee \), but the stabilizer of the special point \( y := x \times X_p^* (\text{GL}_n)_{\mathbb{R}}^\vee \)
contains \( Q_p^\times \cdot 1 \), which acts on \( y \) by translation only in the second component. This motivates us to reduce to the semisimple case before proving Theorem 5: the reduced Bruhat–Tits building coincides with the extended one (see [BT II, 4.2.16]), so that \( G(\tilde{K}) \) is its own normalizer.

Remark 4. In the proof of [Nis83, Thm. 4.2], the author denoted by \( T \) the central torus of \( G \) and by \( G^{\text{der}} \) the derived group of \( G \), saying that \( G \) is the “almost direct product of \( T \) and \( G^{\text{der}} \)”, without further information about \( T \). There is certainly an isogeny \( G^{\text{der}} \times \text{rad}(G) \to G \) ([SGA 3\text{II}new, XXII, 6.2.4]). Nevertheless, the equation
\[ G(\tilde{K}) = G^{\text{der}}(\tilde{K}) \cdot \text{rad}(G)(\tilde{K}) \]
used in [Nis83, Thm. 4.2] fails in general. For instance, we take $G = \text{GL}_n$ and $\widetilde{K} = Q_p^{ur}$ to find

$$\text{GL}_n(Q_p^{ur}) \neq \text{SL}_n(Q_p^{ur}) \cdot \{\text{diag}(a, \ldots , a)\}, \text{ where } a \text{ ranges over } Q_p^{ur}.$$  

In fact, for $a \in Q_p^{ur} \setminus (Q_p^{ur})^n$, e.g., $a = p$, the matrix $\text{diag}(a,1,\ldots,1)$ has determinant $a$ and is not a product of matrices on the right.

As an application of the Grothendieck–Serre conjecture for the case of discrete valuation rings, we prove the following proposition.

**Proposition 13.** For a reductive group scheme $G$ over a field $k$, we let $k((X))$ be the field of Laurent power series in the variable $X$. Then the following map is injective:

$$H^1(k, G) \to H^1(k((X)), G).$$

**Proof.** The projection $p: \text{Spec} \, k[X] \to \text{Spec} \, k$ has a section $s: \text{Spec} \, k \to \text{Spec} \, k[X]$ given by $X = 0$. We have the composition $H^1(k, G) \xrightarrow{p^*} H^1(k[X], G) \xrightarrow{s^*} H^1(k, G)$ such that $s^* \circ p^* = \text{id}$. So $p^*: H^1(k, G) \to H^1(k[X], G)$ is injective (in fact, by [SGA 3^{\text{IInew}}, XXIV, 8.1], we have $H^1(k, G) \simeq H^1(k[X], G)$). Since $k[X]$ is a discrete valuation ring, by Theorem 1, the map $H^1(k[X], G) \to H^1(k((X)), G)$ is injective. 

\[ \square \]

### 6. Uniqueness of reductive models

As an application of Theorem 1, we consider a scheme $X$ with function field $K$, over which a reductive group $G$ is defined. We call a model of $G$ a flat, affine, and finite type $X$-group scheme $\mathcal{G}$ such that $\mathcal{G}_K \simeq G$. The question is, how many reductive models does $G$ have? When $X$ is local strictly Henselian, the uniqueness of models is a special case of [PY06, Thm. 8.5], where they classified quasi-reductive models. In fact, in the following, we show that if a variant of Conjecture 1 holds, then a reductive group scheme over a semilocal regular base is determined by its generic fiber. In particular, Corollary 3 implied by Proposition 14 generalizes the special case in [Nis84, Thm. 5.1] where $X$ is the spectrum of a discrete valuation ring and $G$ is semisimple.

**Proposition 14.** For a regular semilocal ring $S$ with a total ring of fractions $K$, if for each reductive group scheme $G'$ over $S$, the map $H^1(S, G') \to H^1(K, G')$ is injective, then any reductive $K$-group $G$ has at most one reductive $S$-model.

**Proof.** For a reductive group scheme $\mathcal{G}$ over $S$, by [SGA 3^{\text{IInew}}, XXIV, 1.17], the following functor defines a one-to-one correspondence (up to isomorphisms) of sets:

$$\left\{ \begin{array}{c}
\text{S-group schemes that are} \\
\text{fpqc locally isomorphic to } \mathcal{G}
\end{array} \right\} \to H^1_{\text{ét}}(S, \text{Aut}_{S\text{-gr.}}(\mathcal{G})), \quad \mathcal{G}' \mapsto \text{Isom}_{S\text{-gr.}}(\mathcal{G}, \mathcal{G}').$$

Let $\mathcal{G}$ and $\mathcal{G}'$ be two reductive $S$-models of $G$. By [SGA 3^{\text{IInew}}, XXII, 2.8], the root datum of $\mathcal{G}$ at each fiber of $\mathcal{G} \to \text{Spec} \, S$ is locally constant, so it is the root datum of
$G = \mathcal{G}_K$. Therefore, $\mathcal{G}$ and $\mathcal{G}'$ have the same root datum. By [SGA 3, II, new, XXIII, 5.1], étale locally we have $\mathcal{G} \simeq \mathcal{G}'$ and $\mathcal{G}$ corresponds to $0 \in H^1_{\text{ét}}(S, \text{Aut}_{S, \text{gr}}(\mathcal{G}))$. We let $\alpha$ be the class of $\mathcal{G}'$ in $H^1_{\text{ét}}(S, \text{Aut}_{S, \text{gr}}(\mathcal{G}))$. By [SGA 3, II, new, XXIV, 1.3], there is an exact sequence of étale $S$-sheaves:

$$1 \to \mathcal{G}/\text{Centr}(\mathcal{G}) \to \text{Aut}(\mathcal{G}) \to \text{Out}(\mathcal{G}) \to 1,$$

where $\text{Out}(\mathcal{G})$ is the group scheme of outer automorphisms of $\mathcal{G}$ and is represented by a locally constant group scheme, whose stalk at every geometric point of $S$ is a finitely generated group. We consider the following commutative diagram of pointed sets with exact rows

$$
\begin{array}{ccc}
H^0(S, \text{Out}(\mathcal{G})) & \to & H^1_{\text{ét}}(S, \mathcal{G}/\text{Centr}(\mathcal{G})) \\
\downarrow f_0 & & \downarrow f_1 \\
H^0(K, \text{Out}(\mathcal{G})) & \to & H^1_{\text{ét}}(K, \mathcal{G}/\text{Centr}(\mathcal{G})) \\
\end{array}
\quad
\begin{array}{ccc}
& & H^1_{\text{ét}}(S, \text{Aut}(\mathcal{G})) \\
& & \downarrow f_2 \\
& & H^1_{\text{ét}}(K, \text{Aut}(\mathcal{G})) \\
& & \downarrow f_3 \\
H^0(K, \text{Out}(\mathcal{G})) & \to & H^1_{\text{ét}}(K, \mathcal{G}/\text{Centr}(\mathcal{G})) \\
\end{array}
\quad
\begin{array}{c}
H^1_{\text{ét}}(S, \text{Out}(\mathcal{G})) \\
\downarrow f_3 \\
H^1_{\text{ét}}(K, \text{Out}(\mathcal{G}))
\end{array}
$$

where $f_2(\alpha) = 0$. By assumption, the map $f_1$ is injective. In order to show that $\alpha = 0$, we prove that the kernel of $f_3$ is trivial. Let $\mathcal{T}$ be a $\text{Out}(\mathcal{G})$-torsor over $S$ that becomes trivial over $K$. Then $\mathcal{T}(K) \neq \emptyset$ contains a section $t_0$. For a surjective étale covering $X' \to \text{Spec} S$ such that $\mathcal{T}|X'$ is a constant sheaf, the base change $X'_K$ satisfies $\mathcal{T}(X'_K) = \mathcal{T}(X')$ and $\mathcal{T}(X'_K \times_K X'_K) = \mathcal{T}(X' \times_{\text{Spec} S} X')$, which fit into the following commutative diagram with exact rows:

$$
\begin{array}{ccc}
\mathcal{T}(K) & \to & \mathcal{T}(X'_K) \\
\uparrow & & \downarrow \\
\mathcal{T}(S) & \to & \mathcal{T}(X') \\
\end{array}
\quad
\begin{array}{ccc}
& & \mathcal{T}(X'_K \times_K X'_K) \\
& & \downarrow \\
& & \mathcal{T}(X' \times_{\text{Spec} S} X')
\end{array}
$$

such that the two images of $t_0$ under the double arrows in $\mathcal{T}(X')$ coincide in $\mathcal{T}(X' \times_{\text{Spec} S} X')$. It follows that $\mathcal{T}(S) = \mathcal{T}(K) \neq \emptyset$ and $f_3$ has a trivial kernel. This argument also shows that $f_0$ is an isomorphism. By diagram chasing, we conclude that $f_2$ has a trivial kernel and $\mathcal{G} \simeq \mathcal{G}'$. $\square$

In fact, the argument above also proves the following corollary.

**Corollary 2.** For a reductive group scheme $G$ over a regular semilocal ring $S$ with a total ring of fractions $K$, if every form $G'$ of $G/\text{Centr}(G)$ satisfies the injectivity of $H^1_{\text{ét}}(S, G') \to H^1_{\text{ét}}(K, G')$, then

$$H^0(S, \text{Aut}(G)) \to H^1_{\text{ét}}(K, \text{Aut}(G))$$

is injective.

By Theorem 1 and the main theorems in [FP15, Pan17], we obtain the following corollaries.

**Corollary 3.** For a semilocal Dedekind ring $S$ with a total ring of fractions $K$, any reductive $K$-group has at most one reductive $S$-model.

**Corollary 4.** For a regular semilocal ring $S$ containing a field with a total ring of fractions $K$, any reductive $K$-group has at most one reductive $S$-model.
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