Abstract

We investigate using Clifford algebra methods the theory of algebraic dotted and undotted spinor fields over a Lorentzian spacetime and their realizations as matrix spinor fields, which are the usual dotted and undotted two component spinor fields. We found that some ad hoc rules postulated for the covariant derivatives of Pauli sigma matrices and also for the Dirac gamma matrices in General Relativity cover important physical meaning, which is not apparent in the usual matrix presentation of the theory of two components dotted and undotted spinor fields. We also discuss some issues related to the previous one and which appear in a proposed "unified" theory of gravitation and electromagnetism which use two components dotted and undotted spinor fields and also paravector fields, which are particular sections of the even subbundle of the Clifford bundle of spacetime.

1 Introduction

In this paper, using the general theory of Clifford and spin-Clifford bundles, as described in [15, 27] we scrutinize the concept of covariant derivatives of algebraic dotted and undotted spinor fields, which have as matrix representatives...
the standard two components spinor fields (dotted and undotted) already introduced long ago, see, e.g., \[1, 19, 20, 21\]. What is new here is that we identify in the theory of algebraic spinor fields an important and nontrivial physical interpretation for some postulated rules that are used in the standard formulation of the matrix spinor fields, e.g., why the covariant derivative of the Pauli matrices must be null. We show that such a rule implies some constraints on the geometry of the spacetime manifold, with admit a very interesting geometrical interpretation. Indeed, a possible realization of that rules in the Clifford bundle formalism is one where the vector fields defining a global tetrad \(\{e_a\}\) must be such that \(D_{e_0} e_a = 0\), i.e., \(e_0\) must be a geodesic reference frame and along each one of its integral lines, say \(\sigma\), the \(e_d (d = 1, 2, 3)\) must be Fermi transported, i.e., they are not rotating relative to the local gyroscope axes. For the best of our knowledge these important facts are here disclosed for the first time. We also examine the genesis of some ad hoc rules that are postulated for the covariant derivatives of some paravector fields\(^2\) \[31, 32, 33\] in some proposed ‘unified’ theories and for the Dirac gamma matrices in General Relativity \[2\].

2 Spacetime, Pauli and Quaternion Algebras

In this section we recall some facts concerning three special real Clifford algebras, namely, the spacetime algebra \(\mathbb{R}_{1,3}\), the Pauli algebra \(\mathbb{R}_{3,0}\) and the quaternion algebra \(\mathbb{R}_{0,2} = \mathbb{H}\) and the relation between them.\(^3\)

2.1 Spacetime Algebra

To start, we recall that the spacetime algebra \(\mathbb{R}_{1,3}\) is the real Clifford algebra associated with Minkowski vector space \(\mathbb{R}^{1,3}\), which is a four dimensional real vector space, equipped with a Lorentzian bilinear form

\[
\eta : \mathbb{R}^{1,3} \times \mathbb{R}^{1,3} \to \mathbb{R}.
\]

Let \(\{m_0, m_1, m_2, m_3\}\) be an arbitrary orthonormal basis of \(\mathbb{R}^{1,3}\), i.e.,

\[
\eta(m_{\mu}, m_{\nu}) = \eta_{\mu\nu} = \begin{cases} 
1 & \text{if } \mu = \nu = 0 \\
-1 & \text{if } \mu = \nu = 1, 2, 3 \\
0 & \text{if } \mu \neq \nu
\end{cases}
\]

As usual we resume Eq.\( (2)\) writing \(\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)\). We denote by \(\{m^0, m^1, m^2, m^3\}\) the reciprocal basis of \(\{m_0, m_1, m_2, m_3\}\), i.e., \(\eta(m^\mu, m_\nu) = \delta^\mu_\nu\). We have in obvious notation \(\eta(m^\mu, m^\nu) = \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)\).

The spacetime algebra \(\mathbb{R}_{1,3}\) is generate by the following algebraic fundamental relation

\[
m^\mu m^\nu + m^\nu m^\mu = 2\eta^{\mu\nu}.
\]

\(^2\)In \[21, 22, 24\] the author states that the basic variables of his ‘unified’ theory are quaternion fields over a Lorentzian spacetime. Well, they are not are will be proved below.

\(^3\)This material is treated in details e.g. in the books \[5, 13, 22, 23\]. See also \[2, 3, 5, 6, 7, 16, 17, 18\].
We observe that in the above formula and in all the text the Clifford product is denoted by juxtaposition of symbols. The spacetime algebra \( \mathbb{R}_{1,3} \) as a vector space over the real field is isomorphic to the exterior algebra \( \bigwedge \mathbb{R}_{1,3} = \sum_{j=0}^{4} \bigwedge^j \mathbb{R}_{1,3} \). We code that information writing \( \bigwedge \mathbb{R}_{1,3} \cong \mathbb{R}_{1,3} \). Also, we make the following identifications:

\[ \bigwedge^0 \mathbb{R}_{1,3} \equiv \mathbb{R} \text{ and } \bigwedge^1 \mathbb{R}_{1,3} \equiv \mathbb{R}_{1,3} \].

Moreover, we identify the exterior product of vectors by

\[ m^\mu \wedge m^\nu = \frac{1}{2} (m^\mu m^\nu - m^\nu m^\mu), \quad (4) \]

and also, we identify the scalar product of of vectors by

\[ \eta(m^\mu, m^\nu) = \frac{1}{2} (m^\mu m^\nu + m^\nu m^\mu). \quad (5) \]

Then we can write

\[ m^\mu m^\nu = \eta(m^\mu, m^\nu) + m^\mu \wedge m^\nu. \quad (6) \]

Now, an arbitrary element \( C \in \mathbb{R}_{1,3} \) can be written as sum of nonhomogeneous multivectors, i.e.,

\[ C = s + \sum_{\mu} c_\mu m^\mu + \frac{1}{2} \sum_{\mu\nu} c_{\mu\nu} m^\mu m^\nu + \frac{1}{3!} \sum_{\mu\nu\rho} c_{\mu\nu\rho} m^\mu m^\nu m^\rho + pm^5 \quad (7) \]

where \( s, c_\mu, c_{\mu\nu}, c_{\mu\nu\rho}, p \in \mathbb{R} \) and \( c_{\mu\nu}, c_{\mu\nu\rho} \) are completely antisymmetric in all indices. Also \( m^5 = m^0 m^1 m^2 m^3 \) is the generator of the pseudo scalars. As matrix algebra we have that \( \mathbb{R}_{1,3} \simeq \mathbb{H}(2) \), the algebra of the \( 2 \times 2 \) quaternionic matrices.

### 2.2 Pauli Algebra

Now, we recall that the Pauli algebra \( \mathbb{R}_{3,0} \) is the real Clifford algebra associated with the Euclidean vector space \( \mathbb{R}^{3,0} \), equipped as usual, with a positive definite bilinear form. As a matrix algebra we have that \( \mathbb{R}_{3,0} \simeq \mathbb{C}(2) \), the algebra of \( 2 \times 2 \) complex matrices. Moreover, we recall that \( \mathbb{R}_{3,0} \) is isomorphic to the even subalgebra of the spacetime algebra, i.e., writing \( \mathbb{R}_{1,3} = \mathbb{R}_{1,3}^{0} \oplus \mathbb{R}_{1,3}^{1} \) we have,

\[ \mathbb{R}_{3,0} \simeq \mathbb{R}_{1,3}^{0}. \quad (8) \]

The isomorphism is easily exhibited by putting \( \sigma^i = m^i m^0 \), \( i = 1, 2, 3 \). Indeed, with \( \delta^{ij} = \text{diag}(1, 1, 1) \), we have

\[ \sigma^i \sigma^j + \sigma^j \sigma^i = 2 \delta^{ij}, \quad (9) \]

which is the fundamental relation defining the algebra \( \mathbb{R}_{3,0} \). Elements of the Pauli algebra will be called Pauli numbers\(^4\). As vector space over the real field,\(^4\)Sometimes they are also called ‘complex quaternions’. This last terminology will be obvious in a while.

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we have that $\mathbb{R}_{3,0}$ is isomorphic to $\bigwedge \mathbb{R}^{3,0} \hookrightarrow \mathbb{R}_{3,0} \subset \mathbb{R}_{1,3}$. So, any Pauli number can be written as

$$P = s + p^i \sigma^i + \frac{1}{2} p^i_{ij} \sigma^i \sigma^j + pt,$$

where $s, p_i, p_{ij}, p \in \mathbb{R}$ and $p_{ij} = -p_{ji}$ and also

$$1 = \sigma^i \sigma^j \sigma^3 = m^3. \quad (11)$$

Note that $i^2 = -1$ and that $i$ commutes with any Pauli number. We can trivially verify

$$\sigma^i \sigma^j = \varepsilon^i_{jk} \sigma^k + \delta^{ij},$$

$$[\sigma^i, \sigma^j] = \sigma^i \sigma^j - \sigma^j \sigma^i = 2 \sigma^i \wedge \sigma^j = 2 \varepsilon^i_{jk} \sigma^k. \quad (12)$$

In that way, writing $\mathbb{R}_{3,0} = \mathbb{R}_{3,0}^{(0)} + \mathbb{R}_{3,0}^{(1)}$, any Pauli number can be written as

$$P = Q_1 + i Q_2, \quad Q_1 \in \mathbb{R}_{3,0}^{(0)}, \quad i Q_2 \in \mathbb{R}_{3,0}^{(1)}, \quad (13)$$

with

$$Q_1 = a_0 + a_k (i \sigma^k), \quad a_0 = s, \quad a_k = \frac{1}{2} \varepsilon^i_{jk} p_{ij},$$

$$Q_2 = i (b_0 + b_k (i \sigma^k)), \quad b_0 = p, \quad b_k = -p_k. \quad (14)$$

2.3 Quaternion Algebra

Eqs. (14) show that the quaternion algebra $\mathbb{R}_{0,2} = \mathbb{H}$ can be identified as the even subalgebra of $\mathbb{R}_{3,0}$, i.e.,

$$\mathbb{R}_{0,2} = \mathbb{H} \simeq \mathbb{R}_{3,0}^{(0)}. \quad (15)$$

The statement is obvious once we identify the basis $\{1, \hat{i}, \hat{j}, \hat{k}\}$ of $\mathbb{H}$ with

$$\{1, i \sigma^1, i \sigma^2, i \sigma^3\}, \quad (16)$$

which are the generators of $\mathbb{R}_{3,0}^{(0)}$. We observe moreover that the even subalgebra of the quaternions can be identified (in an obvious way) with the complex field, i.e., $\mathbb{R}_{0,2}^{(0)} \simeq \mathbb{C}$.

Returning to Eq. (10) we see that any $P \in \mathbb{R}_{3,0}$ can also be written as

$$P = P_1 + i L_2, \quad (17)$$

where

$$P_1 = (s + p_k \sigma^k) \in \bigwedge^0 \mathbb{R}^{3,0} \oplus \bigwedge^1 \mathbb{R}^{3,0} \equiv \mathbb{R} \oplus \bigwedge^1 \mathbb{R}^{3,0},$$

$$i L_2 = i (p + i l_k \sigma^k) \in \bigwedge^2 \mathbb{R}^{3,0} \oplus \bigwedge^3 \mathbb{R}^{3,0}. \quad (18)$$
with \( l_k = -e_k^i p_{ij} \in \mathbb{R} \). The important fact that we want to emphasize here is that the subspaces \((\mathbb{R} \oplus \bigwedge^1 \mathbb{R}^3, 0)\) and \((\bigwedge^2 \mathbb{R}^3, 0) \oplus \bigwedge^3 \mathbb{R}^3, 0)\) do not close separately any algebra. In general, if \( A, C \in (\mathbb{R} \oplus \bigwedge^1 \mathbb{R}^3, 0) \) then
\[
AC \in \mathbb{R} \oplus \bigwedge^1 \mathbb{R}^3, 0 \oplus \bigwedge^2 \mathbb{R}^3.
\] (19)

To continue, we introduce \( \sigma_i = m_i m_0 = -\sigma^i, \quad i = 1, 2, 3. \) (20)

Then, \( 1 = -\sigma_1 \sigma_2 \sigma_3 \) and the basis \( \{1, i, j, k\} \) of \( \mathbb{H} \) can be identified with \( \{1, -i\sigma_1, -i\sigma_2, -i\sigma_3\} \).

Now, we already said that \( \mathbb{R}_{3,0} \simeq \mathbb{C}(2) \). This permit us to represent the Pauli numbers by \( 2 \times 2 \) complex matrices, in the usual way \( (i = \sqrt{-1}) \). We write \( \mathbb{R}_{3,0} \ni P \mapsto P \in \mathbb{C}(2), \) with
\[
\sigma_1 \mapsto \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\sigma_2 \mapsto \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\
\sigma_3 \mapsto \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\] (21)

2.4 Minimal left and right ideals in the Pauli Algebra and Spinors

It is not our intention to present here the details of the general theory of algebraic spinors. Nevertheless, we shall need to recall some results that we necessary for what follows\(^5\). The elements \( e_\pm = \frac{1}{2}(1 + \sigma_3) = \frac{1}{2}(1 + m_3 m_0) \in \mathbb{R}_{1,3}^{(0)} \simeq \mathbb{R}_{3,0} \), \( e_\pm^2 = e_\pm \) are minimal idempotents of \( \mathbb{R}_{3,0} \). They generate the minimal left and right ideals
\[
I_\pm = \mathbb{R}_{1,3}^{(0)} e_\pm, \quad R_\pm = e_\pm \mathbb{R}_{1,3}^{(0)}.
\] (22)

From now on we write \( e = e_+ \). It can be easily shown (see below) that, e.g., \( I = I_+ \) has the structure of a 2-dimensional vector space over the complex field \( \mathbb{C} \), i.e., \( I \simeq \mathbb{C}^2 \). The elements of the vector space \( I \) are called algebraic contravariant undotted spinors and the elements of \( \mathbb{C}^2 \) are the usual contravariant undotted spinors used in physics textbooks. They carry the \( D(\frac{1}{2}, 0) \) representation of \( SL(2, \mathbb{C}) \) \(^1\). If \( \varphi \in I \) we denote by \( \varphi \in \mathbb{C}^2 \) the usual matrix representative\(^6\) of \( \varphi \) is
\[
\varphi = \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix}, \quad \varphi^1, \varphi^2 \in \mathbb{C}.
\] (23)

\(^5\)For details, see, e.g., \( \mathbb{S}, \mathbb{K}, \mathbb{I}, \mathbb{C} \).

\(^6\)The matrix representation of the elements of the ideals \( I, \bar{I} \), are of course, \( 2 \times 2 \) complex matrices (see, \( \mathbb{S} \), for details). It happens that both columns of that matrices have the same information and the representation by column matrices is enough here for our purposes.
We denote by \(\dot{\mathbf{I}} = eR^{(0)}_{1,3}\) the space of the algebraic covariant dotted spinors. We have the isomorphism, \(\dot{\mathbf{I}} \cong (\mathbb{C}^2)^\dagger \cong \mathbb{C}2\), where \(\dagger\) denotes Hermitian conjugation. The elements of \((\mathbb{C}^2)^\dagger\) are the usual contravariant spinor fields used in physics textbooks. They carry the \(D^{(0,\frac{1}{2})}\) representation of \(Sl(2,\mathbb{C})\) \([14]\). If \(\xi \in \dot{\mathbf{I}}\) its matrix representation in \((\mathbb{C}^2)^\dagger\) is a row matrix usually denoted by

\[\dot{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \quad \xi_1, \xi_2 \in \mathbb{C}.\]  

(24)

The following representation of \(\dot{\xi} \in \dot{\mathbf{I}}\) in \((\mathbb{C}^2)^\dagger\) is extremely convenient. We say that to a covariant undotted spinor \(\xi\) there corresponds a covariant dotted spinor \(\dot{\xi}\) given by

\[\dot{\mathbf{I}} \ni \xi \mapsto \dot{\xi} = \bar{\xi}\varepsilon \in (\mathbb{C}^2)^\dagger, \quad \bar{\xi}_1, \bar{\xi}_2 \in \mathbb{C},\]  

(25)

with

\[\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.\]  

(26)

We can easily find a basis for \(\mathbf{I}\) and \(\dot{\mathbf{I}}\). Indeed, since \(\mathbf{I} = R^{(0)}_{1,3}e\) we have that any \(\varphi \in \mathbf{I}\) can be written as

\[\varphi = \varphi^1 \vartheta_1 + \varphi^2 \vartheta_2\]  

where

\[\vartheta_1 = e, \quad \vartheta_2 = \sigma_1 e, \quad \varphi^1 = a + ib, \quad \varphi^2 = c + id, \quad a, b, c, d \in \mathbb{R}.\]  

(27)

Analogously we find that any \(\dot{\xi} \in \dot{\mathbf{I}}\) can be written as

\[\dot{\xi} = \xi^1 s^1 + \xi^2 s^2\]  

\[s^1 = e, \quad s^2 = es_1.\]  

(28)

Defining the mapping

\[\iota : \mathbf{I} \otimes \dot{\mathbf{I}} \rightarrow R^{(0)}_{1,3} \simeq \mathbb{R}_{3,0},\]

\[\iota(\varphi \otimes \dot{\xi}) = \varphi \dot{\xi},\]  

(29)

we have

\[1 \equiv \sigma_0 = \iota(s_1 \otimes s^1 + s_2 \otimes s^2),\]

\[\sigma_1 = -\iota(s_1 \otimes s^2 + s_2 \otimes s^1),\]

\[\sigma_2 = \iota[i(s_1 \otimes s^1 - s_2 \otimes s^1)],\]

\[\sigma_3 = -\iota(s_1 \otimes s^1 - s_2 \otimes s^2).\]  

(30)
From this it follows the identification
\[ \mathbb{R}_{3,0} \simeq \mathbb{R}_{1,3}^{(0)} \simeq \mathbb{C}(2) = \mathbb{I} \otimes_{\mathbb{C}} \mathbb{I}, \] (31)
and then, each Pauli number can be written as an appropriate sum of Clifford products of algebraic contravariant undotted spinors and algebraic covariant dotted spinors. And, of course, a representative of a Pauli number in \( \mathbb{C}^2 \) can be written as an appropriate Kronecker product of a complex column vector by a complex row vector.

Take an arbitrary \( P \in \mathbb{R}_{3,0} \) such that
\[ P = \frac{1}{\sqrt{2}} p^{k_1k_2...k_j} \sigma_{k_1k_2...k_j}, \] (32)
where \( p^{k_1k_2...k_j} \in \mathbb{R} \) and
\[ \sigma_{k_1k_2...k_j} = \sigma_{k_1} \ldots \sigma_{k_j}, \quad \text{and} \quad \sigma_0 \equiv 1 \in \mathbb{R}. \] (33)

With the identification \( \mathbb{R}_{3,0} \simeq \mathbb{R}_{1,3}^{(0)} \simeq \mathbb{I} \otimes_{\mathbb{C}} \mathbb{I}, \) we can also write
\[ P = P_A^{\dagger} s_{sA} s_{s\bar{B}}, \] (34)
where the \( P_A^{\dagger} = X_A^{\dagger} + i Y_A^{\dagger} \), \( X_A, Y_A \in \mathbb{R} \).

Finally, the matrix representative of the Pauli number \( P \in \mathbb{R}_{3,0} \) is \( P \in \mathbb{C}(2) \) given by
\[ P = P_A^{\dagger} s_{sA} s_{s\bar{B}}, \] (35)
with \( P_A^{\dagger} \in \mathbb{C} \) and
\[ s_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \]
\[ s_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \] (36)

It is convenient for our purposes to introduce also covariant undotted spinors and contravariant dotted spinors. Let \( \varphi \in \mathbb{C}^2 \) be given as in Eq. (23). We define the covariant version of undotted spinor \( \varphi \in \mathbb{C}^2 \) as \( \varphi^* \in (\mathbb{C}^2)^* \simeq \mathbb{C}_2 \) such that
\[ \varphi^* = (\varphi_1, \varphi_2) \equiv \varphi_{sA}, \]
\[ \varphi_A = \varphi^B \varepsilon_{BA}, \quad \varphi^B = \varepsilon^B_{sA} \varphi_A, \]
\[ s_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \] (37)
where \( \varepsilon_{AB} = \varepsilon^{AB} = \text{adiag}(1, -1) \). We can write due to the above identifications that there exists \( \varepsilon \in \mathbb{C}(2) \) given by Eq. (26) which can be written also as
\[ \varepsilon = \varepsilon^{AB} s_{sA} \otimes s_{sB} = \varepsilon_{AB} s_{sA} \otimes s_{sB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i \sigma_2 \] (38)

The symbol \( \text{adiag} \) means the antidiagonal matrix.
where \( \otimes \) denote the Kronecker product of matrices. We have, e.g.,
\[
\begin{align*}
\sigma_1 \otimes \sigma_2 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\
\sigma^i \otimes \sigma^i &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\end{align*}
\] (39)

We now introduce the contravariant version of the dotted spinor
\[
\dot{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \mathbb{C}^2
\]
as being \( \dot{\xi}^* \in \mathbb{C}^2 \) such that
\[
\dot{\xi}^* = \begin{pmatrix} \xi_1^* \\ \xi_2^* \end{pmatrix} = \xi^A \sigma_A^*, \\
\dot{\xi} = \varepsilon^B A \xi_A^*, \quad \xi_A^* = \varepsilon_A^B \dot{\xi}^B,
\]
\[
\sigma_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\] (40)

where \( \varepsilon_{AB} = \varepsilon^{AB} = \text{diag}(1, -1) \). Then, due to the above identifications we see that there exists \( \dot{\varepsilon} \in \mathbb{C}(2) \) such that
\[
\dot{\varepsilon} = \varepsilon^A_B \sigma_A^* \otimes \sigma_B^* = \varepsilon_{AB} \sigma^A \otimes \sigma^B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \varepsilon.
\] (41)

Also, recall that even if \( \{\sigma_A\}, \{\sigma_A^*\} \) and \( \{\sigma^A\}, \{\sigma^A^*\} \) are bases of distinct spaces, we can identify their matrix representations, as it is obvious from the above formulas. So, we have \( s_A \equiv s_A^* \) and also \( s^A = s^A \). This is the reason for the representation of a dotted covariant spinor as in Eq.(24). Moreover, the above identifications permit us to write the matrix representation of a Pauli number \( P \in \mathbb{R}_{3,0} \) as, e.g.,
\[
P = P_{AB} \sigma^A \otimes \sigma^B
\] (42)

besides the representation given by Eq.(35).

3 Clifford and Spinor Bundles

3.1 Preliminaries

To characterize in a rigorous mathematical way the basic field variables used in M. Sachs ‘unified’ field theory \cite{32, 33, 34}, we shall need to recall some results of the theory of spinor fields on Lorentzian spacetimes. Here we follow the approach given in \cite{27, 13}.

\footnote{Another important reference on the subject of spinor fields (in the spirit of this work) is \cite{12}, which however only deals with the case of spinor fields on Riemannian manifolds.}
Recall that a Lorentzian manifold is a pair \((M, g)\), where \(g \in \sec T^{2,0}M\) is a Lorentzian metric of signature \((1, 3)\), i.e., for all \(x \in M\), \(T_x M \cong T^*_x M \cong \mathbb{R}^{1,3}\), where \(\mathbb{R}^{1,3}\) is the vector Minkowski space.

Recall that a Lorentzian spacetime is a pentuple \((M, g, D, \tau, \uparrow)\) where \((M, g, \tau_g)\) is an oriented Lorentzian manifold\(^9\) which is also time oriented by an appropriated equivalence relation\(^{10}\) (denoted \(\uparrow\)) for the timelike vectors at the tangent space \(T_x M\), \(\forall x \in M\). \(D\) is a linear connection for \(M\) such that \(Dg = 0\), \(\Theta(D) = 0\), \(\mathcal{R}(D) \neq 0\), where \(\Theta\) and \(\mathcal{R}\) are respectively the torsion and curvature tensors of \(D\).

Now, M. Sachs theory as described in \[32, 33, 34\] uses spinor fields. These objects are sections of so-called spinor bundles, which only exist in spin manifolds. The ones used in Sachs theory are the matrix representation of sections of the bundles of dotted spinor fields, i.e., \(S(M) = P_{Spin^\dagger}(M) \times D(\uparrow, 0) \mathbb{C}^2\) and the matrix representation of the bundle of undotted spinor fields, here denoted \(\bar{S}(M) = P_{Spin^\dagger}(M) \times D(0, \uparrow) \mathbb{C}^2\). In the previous formula \(D(\uparrow, 0)\) and \(D(0, \uparrow)\) are the two fundamental non-equivalent 2-dimensional representations of \(SL(2, \mathbb{C}) \cong Spin^\dagger\), the universal covering group of \(SO^\dagger\), the restriction orthonochronous Lorentz group. \(P_{Spin^\dagger}(M)\) is a principal bundle called the spin structure bundle\(^11\). We recall that it is a classical result (Geroch theorem \[39\]) that a 4-dimensional Lorentzian manifold is a spin manifold if and only if \(P_{SO^\dagger}(M)\) has a global section\(^{12}\), i.e., if there exists a set \(\{e_0, e_1, e_2, e_3\}\) of orthonormal fields defined for all \(x \in M\). In other word, for spinor fields to exist in a 4-dimensional spacetime the orthonormal frame bundle must be trivial.

Now, the so-called tangent \((TM)\) and cotangent \((T^*M)\) bundles, the tensor bundle \((\oplus_{r,s} \otimes^r T M)\) and the bundle of differential forms for the spacetime are the bundles denoted by

\[
TM = P_{SO^\dagger}(M) \times_{\rho, 1} \mathbb{R}^{1,3}, \quad T^*M = P_{SO^\dagger}(M) \times_{\rho^*, 1} \mathbb{R}^{1,3}, \quad (43)
\]

\[
\oplus_{r,s} \otimes^r T M = P_{SO^\dagger}(M) \times_{\otimes^r \rho, 1} \mathbb{R}^{1,3}, \quad \bigwedge T^*M = P_{SO^\dagger}(M) \times_{\Lambda^k \rho^*, 1} \bigwedge \mathbb{R}^{1,3}.
\]

In Eqs. \[43\]

\[
\rho_{1,3} : SO^\dagger \rightarrow SO^\dagger(\mathbb{R}^{1,3}) \quad (44)
\]

is the standard vector representation of \(SO^\dagger\) usually denoted by \(^{13}\) \(D^{(\uparrow, 0)} = D(\uparrow, 0) \otimes D(0, \uparrow)\) and \(\rho^*_{1,3}\) is the dual (vector) representation \(\rho^*_{1,3} \left( l \right) = \rho_{1,3}^*(l^{-1})^t\). Also \(\otimes^r \rho_{1,3}\) and \(\Lambda^k \rho^*_{1,3}\) are the induced tensor product and induced exterior power

\(^9\)Oriented by the volume element \(\tau_g \in \sec \bigwedge^4 T^*M\).

\(^{10}\)See \[38\] for details.

\(^{11}\)It is a covering space of \(P_{SO^\dagger}(M)\). See, e.g., \[15\] for details. A section of \(P_{Spin^\dagger}(M)\) is called a spin frame, which can be identified as pair \((\Sigma, u)\) where for any \(x \in M\), \(\Sigma(x)\) is an orthonormal frame and \(u(x)\) belongs to the \(Spin^\dagger\).

\(^{12}\)In what follows \(P_{SO^\dagger}(M)\) denotes the principal bundle of oriented Lorentz tetrad. We presuppose that the reader is acquainted with the structure of \(P_{SO^\dagger}(M)\), whose sections are the time oriented and oriented orthonormal frames.

\(^{13}\)See, e.g., \[13\] if you need details.
product representations of $SO_{1,3}^e$. We now briefly recall the definition and some properties of the Clifford bundle of multivector fields \[27\]. We have,

\[
\mathcal{C}l(TM) = P_{SO_{1,3}^e}(M) \times_{\ell_{r_{1,3}}} \mathbb{R}_{1,3} = P_{Spin_{1,3}^e}(M) \times_{\text{Ad}} \mathbb{R}_{1,3}.
\] (45)

Now, recall that $Spin_{1,3}^e \subset \mathbb{R}_{1,3}^{(0)}$. Consider the 2-1 homomorphism $h : Spin_{1,3}^e \to SO_{1,3}^e$, $h(\pm u) = l$. Then $c\ell_{r_{1,3}}$ is the following representation of $SO_{1,3}^e$,

\[
c\ell_{r_{1,3}} : SO_{1,3}^e \to \text{Aut}(\mathbb{R}_{1,3}), \\
c\ell_{r_{1,3}}(L) = \text{Ad}_u : \mathbb{R}_{1,3} \to \mathbb{R}_{1,3}, \\
\text{Ad}_u(m) = umu^{-1}
\] (46)
i.e., it is the standard orthogonal transformation of $\mathbb{R}_{1,3}$ induced by an orthogonal transformation of $\mathbb{R}^{1,3}$. Note that $\text{Ad}_u$ act on vectors as the $D(\frac{1}{2}, \frac{1}{2})$ representation of $SO_{1,3}^e$ and on multivectors as the induced exterior power representation of that group. Indeed, observe, e.g., that for $v \in \mathbb{R}^{1,3} \subset \mathbb{R}_{1,3}$ we have in standard notation

\[
Lv = v^\nu L_\mu^\nu m_\mu = v^\nu um_\mu u^{-1} = uvu^{-1}.
\]

The proof of the second line of Eq. (45) is as follows. Consider the representation

\[
\text{Ad} : Spin_{1,3}^e \to \text{Aut}(\mathbb{R}_{1,3}), \\
\text{Ad}_u : \mathbb{R}_{1,3} \to \mathbb{R}_{1,3}, \quad \text{Ad}_w(m) = umu^{-1}.
\] (47)

Since $\text{Ad}_{-1} = 1 (= \text{identity})$ the representation $\text{Ad}$ descends to a representation of $SO_{1,3}^e$. This representation is just $c\ell(\rho_{1,3})$, from where the desired result follows.

Sections of $\mathcal{C}l(TM)$ can be called Clifford fields (of multivectors). The sections of the even subbundle $\mathcal{C}l^{(0)}(TM) = P_{Spin_{1,3}^e}(M) \times_{\text{Ad}} \mathbb{R}_{1,3}^{(0)}$ may be called Pauli fields (of multivectors). Define the real spinor bundles

\[
S(M) = P_{Spin_{1,3}^e}(M) \times_l \mathbb{I}, \quad \bar{S}(M) = P_{Spin_{1,3}^e}(M) \times_r \mathbb{I}
\] (48)

where $l$ stands for a left modular representation of $Spin_{1,3}^e$ in $\mathbb{R}_{1,3}$ that mimics the $D(\frac{1}{2}, 0)$ representation of $Sl(2, \mathbb{C})$ and $r$ stands for a right modular representation of $Spin_{1,3}^e$ in $\mathbb{R}_{1,3}$ that mimics the $D(0, \frac{1}{2})$ representation of $Sl(2, \mathbb{C})$.

Also recall that if $\bar{S}(M)$ is the bundle whose sections are the spinor fields $\bar{\varphi} = (\bar{\varphi}_1, \bar{\varphi}_2) = \bar{\varphi}_\varepsilon = (\varphi^1, \varphi^2)$, then it is isomorphic to the space of contravariant dotted spinors. We have,

\[
S(M) \simeq P_{Spin_{1,3}^e}(M) \times_{D(0, \frac{1}{2})} \mathbb{C}^2, \quad \bar{S}(M) \simeq P_{Spin_{1,3}^e}(M) \times_{\bar{D}(\frac{1}{2}, 0)} \mathbb{C}^2 \simeq \bar{S}(M),
\] (49)
and from our playing with the Pauli algebra and dotted and undotted spinors in section 2 we have that:

\[ S(M) \simeq S(M), \quad \hat{S}(M) \simeq \hat{S}(M) \simeq \tilde{S}(M). \] (50)

Then, we have the obvious isomorphism

\[ \mathcal{C}^{(0)}(TM) = P_{\text{Spin}_{1,3}}(M) \times_{\text{Ad}} \mathbb{R}^{(0)}_{1,3} \]

\[ = P_{\text{Spin}_{1,3}}(M) \times_{t_{\mathbb{R}}} \mathbb{1} \otimes \mathbb{1} \]

\[ = S(M) \otimes \mathbb{C} \hat{S}(M). \] (51)

Let us now introduce the following (complex) bundle,

\[ \mathcal{C}^{(0)}(0) = P_{\text{Spin}_{1,3}}(M) \times_{D} \mathbb{C}(2) \] (52)

It is clear that

\[ \mathcal{C}^{(0)}(0) = S(M) \otimes \mathbb{C} \hat{S}(M) \simeq \mathcal{C}^{(0)}(0). \] (53)

Finally, we consider the bundle

\[ \mathcal{C}^{(0)}(TM) \otimes \bigwedge^2 TM \simeq \mathcal{C}^{(0)}(M) \otimes \bigwedge^2 TM. \] (54)

Sections of \( \mathcal{C}^{(0)}(TM) \otimes \bigwedge^2 TM \) may be called \textit{Pauli valued differential forms} and sections of \( \mathcal{C}^{(0)}(M) \otimes \bigwedge^2 TM \) may be called \textit{matrix Pauli valued differential forms}\(^{14}\).

Denote by \( \mathcal{C}^{(0)}_{(0,2)}(TM) \) the seven dimensional subbundle \( \bigl( \mathbb{R} \oplus \bigwedge^2 TM \bigr) \subset \bigwedge^2 TM \hookrightarrow \mathcal{C}^{(0)}(TM) \subset \mathcal{C}^{(0)}(TM) \). Now, let \( \langle x^\mu \rangle \) be the coordinate functions of a chart of the maximal atlas of \( M \). The fundamental field variable of Sachs theory can be described as

\[ Q = q_\mu(x)dx^\mu = q_\mu(x)dx^\mu \in \text{sec} \mathcal{C}^{(0)}_{(0,2)}(TM) \otimes \bigwedge^2 TM \subset \text{sec} \mathcal{C}^{(0)}(TM) \otimes \bigwedge^2 TM \]

i.e., a Pauli valued 1-form obeying certain conditions to be presented below. If we work (as Sachs did) with \( \mathcal{C}^{(0)}(M) \otimes \bigwedge^2 TM \), a representative of \( Q \) is

\[ Q = q_\mu(x)dx^\mu = h^a_\mu(x)dx^\mu \sigma_a, \] (55)

where \( \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( \sigma_j \) (\( j = 1, 2, 3 \)) are the Pauli matrices. We observe that the notation anticipates the fact that in Sachs theory the variables \( h^a_\mu(x) \) define the set \( \{ \theta^a \} \equiv \{ \theta^0, \theta^1, \theta^2, \theta^3 \} \) with

\[ \theta^a = h^a_\mu(dx^\mu) \in \text{sec} \bigwedge^2 TM, \] (56)

\(^{14}\)A detailed theory of Clifford valued differential forms is given in [29].

\(^{15}\)Note that a bold index (sub or superscript), say \( a \), take the values 0, 1, 2, 3.
which is the dual basis of \( \{ e_\mu \} \equiv \{ e_0, e_1, e_2, e_3 \}, e_\mu \in \sec TM \). We denote by \( \{ e_\mu \} \equiv \{ e_0, e_1, e_2, e_3 \} \), a coordinate basis associated with the local chart \( \langle x^\mu \rangle \) covering \( U \subset M \). We have \( e_\mu = e^a e_\mu \in \sec TM \), and the set \( \{ e_\mu \} \) is the dual basis of \( \{ dx^\mu \} \equiv \{ dx^0, dx^1, dx^2, dx^3 \} \). We will also use the reciprocal basis to a given basis \( \{ e_a \} \), i.e., the set \( \{ e^a \} \equiv \{ e^0, e^1, e^2, e^3 \}, e^a \in \sec TM \), with \( g(e_a, e^b) = \delta^b_a \) and the reciprocal basis to \( \{ \theta^a \} \), i.e., the set \( \{ \theta_a \} = \{ \theta_0, \theta_1, \theta_2, \theta_3 \} \), with \( \theta_a (e^b) = \delta^b_a \). Recall that since \( \eta_{ab} = g(e_a, e_b) \), we have

\[
g_{\mu\nu} = g(e_\mu, e_\nu) = h^{a_\mu}_\mu h^{b_\nu}_\nu \eta_{ab}. \tag{57}\]

To continue, we define

\[
\sigma_0 = -\sigma_0 \quad \text{and} \quad \sigma_j = \sigma_j, j = 1, 2, 3 \tag{58}\]

and

\[
\tilde{Q} = \tilde{q}_\mu (x) dx^\mu = h^{a_\mu}_\mu (x) dx^\mu \sigma_a. \tag{59}\]

We note that

\[
\sigma_a \tilde{\sigma}_b + \sigma_b \tilde{\sigma}_a = -2 \eta_{ab}. \tag{60}\]

Readers of Sachs’ books \[31, 33\] will recall that he said that \( Q \) is a representative of a quaternion.\(^{16}\) From our previous discussion we see that this statement is not correct.\(^{17}\) Sachs identification is a dangerous one, because the quaternions close a division algebra, also-called a noncommutative field or skew-field and objects like \( Q = q_\mu \otimes dx^\mu \in \sec \mathcal{C}^{(0)}(T M) \otimes \bigwedge T^* M \subset \sec \mathcal{C}^{(0)}(T M) \otimes \bigwedge T^* M \), called paravector fields, did not close a division algebra.

Next we introduce a tensor product of sections \( A, B \in \sec \mathcal{C}^{(0)}(M) \otimes \bigwedge T^* M \). Before we do that we recall that from now on

\[
\{ 1, \sigma_k, \sigma_{k_1 k_2}, i = \sigma_{123} \}, \tag{61}\]

refers to a basis of \( \mathcal{C}^{(0)}(M) \), i.e., they are fields.\(^{18}\)

Recalling Eq.\[34\] we introduce the (obvious) notation

\[
A = \frac{1}{j!} a^{k_1 k_2 \ldots k_j} \sigma_{k_1 k_2 \ldots k_j} dx^\mu, \quad B = \frac{1}{j!} b^{k_1 k_2 \ldots k_j} \sigma_{k_1 k_2 \ldots k_j} dx^\mu, \tag{62}\]

where the \( a^{k_1 k_2 \ldots k_j}, b^{k_1 k_2 \ldots k_j} \) are, in general, real scalar functions. Then, we define

\[
A \otimes B = \frac{1}{j!} a^{k_1 k_2 \ldots k_j} b^{p_1 p_2 \ldots p_i} \sigma_{k_1 k_2 \ldots k_j} \sigma_{p_1 p_2 \ldots p_i} dx^\mu \otimes dx^\nu. \tag{63}\]

\(^{16}\)Note that Sachs represented \( Q \) by dS, which is a very dangerous notation, which we avoid. Sachs notation has lead him in the past \[30\] to identify dS with the element of arc of a curve in a Lorentzian manifold, thus producing unfortunately a lot of misunderstandings, as showed in \[34\]. On this issue see also the erroneous Sachs reply to \[24\] in \[34\]. See also \[25\].

\(^{17}\)Nevertheless most of the calculations done by Sachs in \[24\] are correct because he worked always with the matrix representation of \( Q \). However, his claim of having produce an unified field theory of gravitation and electromagnetism is wrong as we shall prove in a following paper.\[29\].

\(^{18}\)We hope that in using (for symbol economy) the same notation as in section 2 where the \( \{ 1, \sigma_k, \sigma_{k_1 k_2}, \sigma_{123} \} \) is a basis of \( \mathbb{R}^{(0)}_{1,3} \simeq \mathbb{R}_{3,0} \) will produce no confusion.
Let us now compute the tensor product of \( \mathbf{Q} \otimes \tilde{\mathbf{Q}} \) where \( \mathbf{Q} \in \text{sec}\mathcal{C}_{(0,2)}^2(M) \otimes \bigwedge T^*M \). We have,

\[
\mathbf{Q} \otimes \tilde{\mathbf{Q}} = q_{\mu}(x)dx^\mu \otimes \tilde{q}_\nu(x)dx^\nu = q_{\mu}(x)\tilde{q}_\nu(x)dx^\mu \otimes dx^\nu \\
= q_{\mu}(x)\tilde{q}_\nu(x)\frac{1}{2}(dx^\mu \otimes dx^\nu + dx^\nu \otimes dx^\mu) \\
+ \frac{1}{2}q_{\mu}(x)\tilde{q}_\nu(x)(dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu) \\
= \frac{1}{2}(q_{\mu}(x)\tilde{q}_\nu(x) + q_{\nu}(x)\tilde{q}_\mu(x))dx^\mu \otimes dx^\nu \\
+ \frac{1}{2}q_{\mu}(x)\tilde{q}_\nu(x)dx^\mu \wedge dx^\nu
\]  

(64)

In writing Eq.(64) we have used \( dx^\mu \wedge dx^\nu \equiv dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu \). Also, using

\[
g_{\mu\nu} = \eta_{ab}h^a_\mu(x)h^b_\nu(x), \quad g = g_{\mu\nu}dx^\mu \otimes dx^\nu = \eta_{ab}\theta^a \otimes \theta^b
\]

\[
F'_{\mu\nu} = F'_{\mu\nu}i\sigma_k = -\frac{1}{2}(\epsilon^{ijk}h^i_\mu(x)h^j_\nu(x))i\sigma_k; \quad i,j,k = 1,2,3,
\]

\[
F' = \frac{1}{2}F'_{\mu\nu}dx^\mu \wedge dx^\nu = \frac{1}{2}(F'_{\mu\nu}i\sigma_i\sigma_j)dx^\mu \wedge dx^\nu = \frac{1}{2}F'_{\mu\nu}i\sigma_kdx^\mu \wedge dx^\nu \\
= -\epsilon^{ijk}h^i_\mu(x)h^j_\nu(x)dx^\mu \wedge dx^\nu i\sigma_k \in \text{sec} \bigwedge^2 T^*M \otimes \mathcal{C}_{(2)}^0(M)
\]  

(65)

we can write Eq.(64) as

\[
\mathbf{Q} \otimes \tilde{\mathbf{Q}} = \mathbf{Q} \otimes \tilde{\mathbf{Q}} + \mathbf{Q} \wedge \tilde{\mathbf{Q}} \\
= -g + \mathbf{F}.
\]  

(66)

We can also write

\[
\mathbf{Q} \otimes \tilde{\mathbf{Q}} = -\eta_{ab}\sigma_0\theta^a \otimes \theta^b + \epsilon^{ijk}i\sigma_k\theta^i \wedge \theta^j.
\]  

(67)

The above formulas show very clearly the mathematical nature of \( \mathbf{F} \), it is a 2-form with values on the subspace of multivector Clifford fields, i.e., \( \mathbf{F} : \bigwedge^2 TM \to \mathcal{C}_{(2)}^0(TM) \subset \mathcal{C}_{(2)}^0(TM) \). In [31, 32, 33] the author identified erroneously \( \mathbf{F} \) with an electromagnetic field. We discuss in detail that issue in a sequel paper [29]. Now, we write the formula for \( \mathbf{Q} \otimes \tilde{\mathbf{Q}} \) where \( \mathbf{Q} \in \mathbb{C}(2) \otimes \bigwedge T^*M \) given by Eq.(66) is the matrix representation of \( \mathbf{Q} \in \text{sec}\mathcal{C}_{(0,2)}^2(M) \otimes \bigwedge T^*M \).
We have,

\[ Q \otimes \tilde{Q} = Q \otimes \tilde{Q} + Q \wedge \tilde{Q} = (-g_{\mu \nu}dx^\mu \otimes dx^\nu)\sigma_0 + (\varepsilon^k_{ij}v^\mu_i(x)v^\nu_j(x) \, dx^\mu \wedge dx^\nu)(-i\sigma_k) = -g\sigma_0 + F^k \varepsilon k, \number{68} \]

with

\[ F^k = \frac{1}{2}F^k_{\mu \nu}dx^\mu \wedge dx^\nu = \varepsilon^k_{ij}v^\mu_i(x)v^\nu_j(x)dx^\mu \wedge dx^\nu. \number{69} \]

For future reference we also introduce

\[ F'_\mu \nu = \frac{1}{2}F^k_{\mu \nu}dx^\mu \wedge dx^\nu \number{70} \]

3.2 Covariant Derivatives of Spinor Fields

We now briefly recall the concept of covariant spinor derivatives \[2, 12, 15, 27\]. The idea is the following:

(i) Every connection on the principal bundle of orthonormal frames \( P_{SO^e_1,3}(M) \) determines in a canonical way a unique connection on the principal bundle \( P_{Spin^e_1,3}(M) \).

(ii) Let \( D \) be a covariant derivative operator acting on sections of an associated vector bundle to \( P_{SO^e_1,3}(M) \), say, the tensor bundle \( \tau M \) and let \( D^* \) be the corresponding covariant spinor derivative acting on sections of associated vector bundles to \( P_{Spin^e_1,3}(M) \), say, e.g., the spinor bundles \( S(M) \), \( \tilde{S}(M) \) and \( P(M) \simeq C(0)^e(M) \), which may be called Pauli spinor bundle. The matrix representations of the above bundles are:

\[ S(M) = P_{Spin^e_1,3}(M) \times _{D'(\frac{1}{2})} C^2, \quad \tilde{S}(M) = P_{Spin^e_1,3}(M) \times _{D''(\frac{1}{2})} C^2 \]

\[ P(M) = S(M) \otimes \tilde{S}(M) = P_{Spin^e_1,3}(M) \times _{D'(\frac{1}{2}) \otimes D''(\frac{1}{2})} C^2 \otimes C^2, \number{71} \]

and \( P(M) \) may be called matrix Pauli spinor bundle. Of course, \( P(M) \simeq C(0)^e(M) \).

(iii) We have for \( T \in \text{sec} \bigwedge T M \to C(0)^e(M) \) and \( \xi \in \text{sec} S(M) \), \( \xi \in \text{sec} \tilde{S}(M) \), \( P \in \text{sec} P(M) \) and \( v \in \text{sec} T M \),

\[ D^\nu_v (T \otimes \xi) = D^\nu_v T \otimes \xi + T \otimes D^\nu_v \xi, \]

\[ D^\nu_v (T \otimes \tilde{\xi}) = D^\nu_v T \otimes \tilde{\xi} + T \otimes D^\nu_v \tilde{\xi}. \number{72} \]
where (see [27] for details)

\begin{align*}
D_v T &= \partial_v T + \frac{1}{2} [\omega_v, T], \\
D_v \xi &= \partial_v \xi + \frac{1}{2} \omega_v \xi, \\
D_v \dot{\xi} &= \partial_v \dot{\xi} - \frac{1}{2} \xi \omega_v, \\
D_v P &= \partial_v P + \frac{1}{2} \omega_v P - \frac{1}{2} P \omega_v = \partial_v P + \frac{1}{2} [\omega_v, P].
\end{align*}

(73)

(iv) For \(T \in \sec \bigwedge T M \hookrightarrow \mathcal{C}l^{(0)}(TM)\) and \(\xi \in \sec S(M)\), \(\bar{\xi} \in \sec \bar{S}(M)\), \(P \in \sec P(M)\) and \(v \in \sec T M\), we have

\begin{align*}
D_v (T \otimes \xi) &= D_v T \otimes \xi + T D_v \xi, \\
D_v (T \otimes \bar{\xi}) &= D_v T \otimes \bar{\xi} + T D_v \bar{\xi}
\end{align*}

and (see [27] for details)

\begin{align*}
D_v T &= \partial_v T + \frac{1}{2} [\omega_v, T], \\
D_v \xi &= \partial_v \xi + \frac{1}{2} \Omega_v \xi, \\
D_v \dot{\xi} &= \partial_v \dot{\xi} - \frac{1}{2} \xi \Omega_v, \\
D_v P &= \partial_v P + \frac{1}{2} \Omega_v P - \frac{1}{2} P \Omega_v = \partial_v P + \frac{1}{2} [\Omega_v, P].
\end{align*}

(75)

In the above equations \(\omega_v \in \sec \mathcal{C}l^{(0)}(TM)\) and \(\Omega_v \in \sec P(M)\). Writing as usual, \(v = e^a e_a\), \(D_e e_b = -\omega_{bc} e_c\), \(\omega_{abc} = -\omega_{cab}\), \(\omega_{a}^c = -\omega_{b}^a\), \(\sigma_b = e_b e_0\) and\(^{19}\) \(i = -\sigma_1 \sigma_2 \sigma_3\), we have

\begin{align*}
\omega_{e_a} &= \frac{1}{2} \omega_{bc} e_b e_c = \frac{1}{2} \omega_{a}^c e_b \wedge e_c \\
&= \frac{1}{2} \omega_{a}^c \sigma_b \sigma_c \\
&= \frac{1}{2} (-2 \omega_{b}^0 i_1 + \omega_{b}^j i_1 \sigma_j) \\
&= \frac{1}{2} (-2 \omega_{a}^0 i_1 - i_1 j a k \sigma_k) = \Omega_{a}^b \sigma_b.
\end{align*}

(76)

Note that the \(\Omega_{a}^b\) are ‘formally’ complex numbers. Also, observe that we can write the ‘formal’ Hermitian conjugate \(\omega_{e_a}^\dagger\) of \(\omega_{e_a}\) as

\begin{equation}
\omega_{e_a}^\dagger = -e^0 \omega_{e_a} e^0.
\end{equation}

\(^{19}\)Have in mind that \(i\) is a Clifford field here.
Also, write $\Omega_{e_a}$ for the matrix representation of $\omega_{e_a}$, i.e.,

$$\Omega_{e_a} = \Omega_{e_a}^b \sigma_b,$$

where $\Omega_{e_a}^b$ are complex numbers with the same coefficients as the ‘formally’ complex numbers $\Omega_{e_a}^b$. We can easily verify that

$$\Omega_{e_a} = \varepsilon \Omega_{e_a}^\dagger \varepsilon. \quad (78)$$

We can prove the third line of Eq.(75) as follows. First, take the Hermitian conjugation of the second line of Eq.(75), obtaining

$$D_v \bar{\xi} = \partial_v \bar{\xi} + \frac{1}{2} \bar{\xi} \Omega_v^\dagger \varepsilon.$$

Next multiply the above equation on the left by $\varepsilon$ and recall that $\dot{\xi} = \bar{\xi} \varepsilon$ and Eq.(78). We get

$$D_v \dot{\xi} = \partial_v \dot{\xi} - \frac{1}{2} \dot{\xi} \Omega_v^\dagger \varepsilon.$$

Note that this is compatible with the identification $\mathcal{Ct}(0)(TM) \simeq S(M) \otimes_\mathbb{C} S(M)$ and $\mathcal{Ct}(0)(M) \simeq S(M) \otimes_\mathbb{C} \bar{S}(M)$.

Note moreover that if $q_{\mu} = e_{\mu} e_0 = h_\mu^a e_a e_0 = h_\mu^a \sigma_a \in \mathcal{Ct}(0)(TM) \simeq S(M) \otimes_\mathbb{C} \bar{S}(M)$ we have,

$$D_v q_{\mu} = \partial_v q_{\mu} + \frac{1}{2} \omega_v q_{\mu} + \frac{1}{2} q_{\mu} \omega_v^\dagger. \quad (79)$$

For $q_{\mu} = h_\mu^a \sigma_a \in \text{sec} \mathcal{Ct}(0)(M) \simeq S(M) \otimes_\mathbb{C} \bar{S}(M)$, the matrix representative of the $q_{\mu}$ we have for any vector field $v \in \text{sec} TM$

$$D_v q_{\mu} = \partial_v q_{\mu} + \frac{1}{2} \Omega_v q_{\mu} + \frac{1}{2} q_{\mu} \Omega_v^\dagger \quad (80)$$

which is the equation used by Sachs for the spinor covariant derivative of his ‘quaternion’ fields. Note that M. Sachs in [31, 33] introduced also a kind of total covariant derivative for his would be ‘quaternion’ fields. That ‘derivative’ denoted in this text by $D_v^S$ will be discussed below.

### 3.3 Geometrical Meaning of $D_v q_{\mu} = \Gamma^\alpha_{\nu\mu} q_\alpha$

We recall that Sachs wrote \footnote{See, e.g., Eq.(3.69) in [31].} without any mathematically justified argument that

$$D_v q_{\mu} = \Gamma^\alpha_{\nu\mu} q_\alpha. \quad (81)$$
where $\Gamma^\alpha_{\nu\mu}$ are the connection coefficients of the coordinate basis $\{e_\mu\}$, i.e.,
\[ D_{e_\nu} e_\mu = \Gamma^\alpha_{\nu\mu} e_\alpha. \tag{82} \]

How can Eq. (81) be true? Well, let us calculate $D_{e_\nu} q_\mu \in \mathcal{C}(\mathcal{T}M)$. We have,
\[ D_{e_\nu} q_\mu = D_{e_\nu} (e_\mu e_0) = (D_{e_\nu} e_\mu) e_0 + e_\mu (D_{e_\nu} e_0) = \Gamma^\alpha_{\nu\mu} q_\alpha + e_\mu (D_{e_\nu} e_0). \tag{83} \]

So, Eq. (81) follows if, and only if
\[ D_{e_\nu} e_0 = 0. \tag{84} \]

To understand the physical meaning of Eq. (84) let us recall the following. In Relativity Theory reference frames are represented by time like vector fields $Z \in \text{sec} \mathcal{T}M$ pointing to the future \cite{28, 35}. If we write the $\alpha Z = g(Z, \cdot) \in \bigwedge^1 T^* M$ for the physically equivalent 1-form field, we have the well known decomposition
\[ D\alpha Z = a Z \otimes \alpha Z + \varpi Z + \sigma Z + \frac{1}{3} E Z p, \tag{85} \]
where
\[ p = g - \alpha Z \otimes \alpha Z \tag{86} \]
is called the projection tensor (and gives the metric of the rest space of an instantaneous observer \cite{35}), $a Z = g(DZ Z, \cdot)$ is the (form) acceleration of $Z$, $\varpi Z$ is the rotation of $Z$, $\sigma Z$ is the shear of $Z$ and $E Z$ is the expansion ratio of $Z$. In a coordinate chart $(U, x^\mu)$, writing $Z = Z^\mu \partial / \partial x^\mu$ and $p = (g_{\mu\nu} - Z_\mu Z_\nu) dx^\mu \otimes dx^\nu$ we have
\[ \varpi Z_{\mu\nu} = Z_{[\alpha;\beta]} p_{\alpha\mu} p_{\beta\nu}, \]
\[ \sigma Z_{\alpha\beta} = [Z_{(\mu;\nu)} - \frac{1}{3} E Z h_{\mu\nu}] p_{\alpha\mu} p_{\beta\nu}, \]
\[ E Z = Z^\mu ;_\mu. \tag{87} \]

Now, in Special Relativity where the space time manifold is the structure $(M = \mathbb{R}^4, g = \eta, D^0, \tau^0, \uparrow)^{21}$ an inertial reference frame (IRF) $\mathbf{I} \in \text{sec} \mathcal{T}M$ is defined by $D^0 \mathbf{I} = 0$. We can show very easily (see, e.g., \cite{13}) that in General Relativity Theory (GRT) where each gravitational field is modelled by a space-time\textsuperscript{22} $(M, g, D, \tau, \uparrow)$ there is in general no shear free frame ($\sigma_Z = 0$) on any
textsuperscript{21}$\eta$ is a constant metric, i.e., there exists a chart $(x^\mu)$ of $M = \mathbb{R}^4$ such that $\eta(\partial / \partial x^\mu, \partial / \partial x^\nu) = \eta_{\mu\nu}$, the numbers $\eta_{\mu\nu}$ forming a diagonal matrix with entries $(1, -1, -1, -1)$. Also, $D^0$ is the Levi-Civita connection of $\eta$.

\textsuperscript{22}More precisely, by a diffeomorphism equivalence class of Lorentzian spacetimes, according to current dogma.
open neighborhood $U$ of any given spacetime point. The reason is clear if we use local coordinates $\langle x^\mu \rangle$ covering $U$. Indeed, $\sigma_\Omega = 0$ implies five independent conditions on the components of the frame $Q$. Then, we arrive at the conclusion that in a general spacetime model\footnote{We take the opportunity to correct an statement in \cite{28}. There it is stated that in General Relativity there are no inertial frames. Of, course, the correct statement is that in a general spacetime model there are in general no inertial frames. But, of course, there are spacetime models where there exist frames $Q \in \text{sec} TU \subset \text{sec} TM$ satisfying $DQ = 0$. See below.} there is no frame $Q \in \text{sec} TU \subset \text{sec} TM$ satisfying $DQ = 0$, and in general there is no IRF in any model of GRT. Saying that, if there exists in a model of General Relativity a frame $Q$ satisfying $DQ = 0$, we agree in calling $Q$ an inertial frame.

The following question arises naturally: which characteristics a reference frame on a GRT spacetime model must have in order to reflect as much as possible the properties of an IRF of SRT? The answer to that question \cite{28} is that there are two kind of frames in GRT such that each frame in one of these classes share some important aspects of the IRFs of SRT. Both concepts are useful and it is important to distinguish between them in order to avoid misunderstandings. These frames are the \textit{pseudo inertial reference frame} (PIRF) and the \textit{and the local Lorentz reference frames} (LLRF$\gamma$s), but we do not need to enter the details here.

On the open set $U \subset M$ covered by a coordinate chart $\langle x^\mu \rangle$ of the maximal atlas of $M$ multiplying Eq.\cite{34} by $h_\alpha^\nu$ such that $e_a = h_\alpha^\nu e_{\nu}$, we get

$$D_{e_a} e_0 = 0; \ a = 0,1,2,3. \tag{88}$$

Then, it follows that

$$D_X e_0 = 0, \ \forall X \in \text{sec} TM \tag{89}$$

which characterizes $e_0$ as an inertial frame. This imposes several restrictions on the spacetime described by the theory. Indeed, if Eq.\cite{34} holds, we must have

$$\text{Ric}(e_0, X) = 0, \ \forall X \in \text{sec} TM, \tag{90}$$

where, $\text{Ric}$ is the Ricci tensor of the manifold modelling spacetime \footnote{See, exercise 3.2.12 of \cite{35}.}. In particular, this condition cannot be realized in Einstein-de Sitter spacetime. This fact is completely hidden in the matrix formalism used in M. Sachs theory, where no restriction on the spacetime manifold (besides the one of being a spin manifold) need to be imposed.

### 3.4 Geometrical Meaning of $D_{e^\mu} \sigma_1 = 0$ in General Relativity

We now discuss what happens in the usual theory of dotted and undotted two component matrix spinor fields in general relativity, as described, e.g., in \cite{1}.
In that formulation it is postulated that the covariant spinor derivative of Pauli matrices must satisfy

\[ D_{e_\mu} \sigma_i = 0, \ i = 1, 2, 3. \]  \hspace{1cm} (91)

Eq. (91) translates in our formalism as

\[ D_{e_\mu} \sigma_i = D_{e_\mu} (e_i e_0) = 0. \]  \hspace{1cm} (92)

Differently from the case of Sachs theory, Eq. (92) can be satisfied if

\[ D_{e_\mu} e_i = e_i (D_{e_\mu} e_0) e_0 \]  \hspace{1cm} (93)

or, writing \( D_{e_\mu} e_a = \omega^b_{\mu a} e_b \), we have

\[ \omega^b_{\mu i} = e_b \mathbf{\cdot} (\omega^a_{\mu 0} e_i e_a e_0), \]  \hspace{1cm} (94)

where \( \mathbf{\cdot} \) is the left contraction operator in the Clifford bundle (see, e.g., [27], for details). This certainly implies some restrictions on possible spacetime models, but that is the price, necessary to be paid, in order to have spinor fields. At least we do not need to necessarily have \( D e_0 = 0 \).

We analyze some possibilities of satisfying Eq. (91):

(i) Suppose that \( e_0 \) satisfy \( D_{e_\mu} e_0 = 0 \), i.e., \( D e_0 = 0 \). Then, a necessary and sufficient condition for the validity of Eq. (92) is that

\[ D_{e_\mu} e_i = 0. \]  \hspace{1cm} (95)

Multiplying Eq. (95) by \( h^\mu_a \) we get

\[ D_{e_\mu} e_i = 0, \ i = 1, 2, 3; \ a = 0, 1, 2, 3. \]  \hspace{1cm} (96)

In particular,

\[ D e_0 e_i = 0, \ i = 1, 2, 3. \]  \hspace{1cm} (97)

Eq. (97) means that the fields \( e_i \) following each integral line \( \sigma \) of \( e_0 \) are Fermi transported\(^{25}\). Physicists interpret that equation saying that the \( e_i|_{\sigma (I)} \) are physically realizable by gyroscopic axes, which gives the local standard of no rotation.

The above conclusion sounds fine. However it follows from Eq. (89) and Eq. (96) that

\[ D_{e_a} e_b = 0, \ a = 0, 1, 2, 3; \ b = 0, 1, 2, 3. \]  \hspace{1cm} (98)

Recalling that existence of spinor fields implies that \( \{ e_a \} \) is a global tetrad\(^{9}\), Eq. (98) implies that the connection \( D \) must be teleparallel. Then, under the above conditions the curvature tensor of a spacetime admitting spinor fields must be null. This, is in particular, the case of Minkowski spacetime.

\(^{25}\) An original approach to the Fermi transport using Clifford bundle methods has been given in [26]. There an equivalent spinor equation to the famous Darboux equations of differential geometry is derived.
(ii) Suppose now that \( e_0 \) is a geodesic frame, i.e., \( D_{e_0} e_0 = 0 \). Then, \( h_0 D_{e_0} e_0 = 0 \) and Eq. (93) implies only that

\[
D_{e_0} e_i = 0; \quad i=1, 2, 3
\]

(99)

If we take an integral line of \( e_0 \), say \( \gamma \), then the set \( \{ e_a | \gamma \} \) may be called an inertial moving frame along \( \gamma \). The set \( \{ e_a | \gamma \} \) is also Fermi transported (as can be easily verified) since \( \gamma \) is a geodesic worldline. They define the standard of no rotation along \( \gamma \).

In conclusion, a consistent definition of spinor fields in General Relativity using the Clifford and spin-Clifford bundles formalism of this paper needs not only the triviality of the frame bundle, i.e., existence of a global tetrad, say \( \{ e_a \} \). It also needs the validity of Eq.(93). A nice physical interpretation follows moreover if the tetrad satisfies

\[
D_{e_0} e_a = 0; \quad a =0,1,2,3.
\]

(100)

Of course, as it is the case in Sachs theory, the matrix formulation of spinor fields do not impose any constrains in the possible spacetime models, besides the one needed for the existence of a spinor structure. Saying that we have an important comment, presented in the next section.

### 3.5 Covariant Derivative of the Dirac Gamma Matrices

If we use a real spin bundle where we can formulate the Dirac equation, e.g., one where the typical fiber is the ideal of (algebraic) Dirac spinors, i.e., the ideal generated by a idempotent \( \frac{1}{2}(1 + E_0) \), \( E_0 \in \mathbb{R}_{1,3} \), \( E_0 \cdot E_0 = 1 \), then no restriction is imposed on the global tetrad field \( \{ e_a \} \) defining the spinor structure of spacetime (see [27, 15]). In particular, since

\[
D_{e_a} e_b = \omega^c_{ab} e_c,
\]

(101)

we have,

\[
D_{e_a} e_b = \frac{1}{2} [\omega_{e_a}, e_b]
\]

(102)

Then,

\[
\omega^c_{ab} e_c - \frac{1}{2} \omega_{e_a} e_b + \frac{1}{2} e_b \omega_{e_a} = 0.
\]

(103)

The matrix representation of the real spinor bundle, of course, sends \( \{ e_a \} \mapsto \{ \gamma_a \} \), where the \( \gamma_a \)'s are the standard representation of the Dirac matrices. Then, the matrix translation of Eq. (103) is

\[
\omega^a_{bc} \gamma_c - \frac{1}{2} \omega_{e_a} \gamma_b + \frac{1}{2} \gamma_b \omega_{e_a} = 0.
\]

(104)

For the matrix elements \( \gamma^A_{bB} \) we have

\[
\omega^a_{bc} \gamma^A_{cB} - \frac{1}{2} \omega^A_{e_a} \gamma^C_{bB} + \frac{1}{2} \gamma^A_{bB} \omega^C_{e_a} = 0.
\]

(105)
In [2] this last equation is confused with the covariant derivative of $\gamma_{AB}^C$. Indeed in an exercise in problem 4, Chapter Vbis [2] ask one to prove that

$$\nabla_c \gamma_{AB}^C = \omega_{ab} \gamma_{AB}^C - \frac{1}{2} \omega_{ab}^C \gamma_{AB}^C + \frac{1}{2} b_{AC}^b \omega_{bB} = 0.$$  

Of course, the first member of the above equation does not define any covariant derivative operator. Confusions as that one appears over and over again in the literature, and of course, is also present in Sachs theory in a small modified form, as shown in the next subsubsection.

### 3.6 $D^S_{e\nu} q_{\mu} = 0$

Taking into account Eq.(80) and Eq.(81) we can write:

$$\partial_\nu q_{\mu} + \frac{1}{2} \omega_\nu q_{\mu} + \frac{1}{2} q_{\mu} \omega_\nu - \Gamma^\alpha_{\nu\mu} q_{\alpha} = 0. \tag{106}$$

Write,

$$D^S_{e\nu} q_{\mu} = \partial_\nu q_{\mu} + \frac{1}{2} \omega_\nu q_{\mu} + \frac{1}{2} q_{\mu} \omega_\nu - \Gamma^\alpha_{\nu\mu} q_{\alpha}. \tag{107}$$

from where

$$D^S_{e\nu} q_{\mu} = 0. \tag{108}$$

Of course, the matrix representation of the last two equations are:

$$D^S_{e\nu} q_{\mu} = \partial_\nu q_{\mu} + \frac{1}{2} \Omega_\nu q_{\mu} + \frac{1}{2} q_{\mu} \Omega^\dagger_\nu - \Gamma^\alpha_{\nu\mu} q_{\alpha}. \tag{109}$$

Sachs call $D^S_{e\nu} q_{\mu}$ the covariant derivative of a $q_{\mu}$ field. The nomination is an unfortunate one, since the equation $D^S_{e\nu} q_{\mu} = 0$ is a trivial identity and do not introduce any new connection in the game.27

After this long exercise we can derive easily all formulas in chapters 3-6 of [31] without using any matrix representation at all. In particular, for use in the sequel paper [29] we collect some formulas,

$$q^\mu \tilde{q}_{\mu} = -4, \quad q^\mu \tilde{q}_{\mu} = -4 \sigma_0$$

$$q^\mu \omega \tilde{q}_{\mu} = 0, \quad q^\mu \Omega \tilde{q}_{\mu} = 0,$$

$$\omega_{\rho} = -\frac{1}{2} \tilde{q}_{\mu} (\partial_\rho q^\mu + \Gamma^\mu_{\rho\tau} q^\tau), \quad \Omega_{\rho} = -\frac{1}{2} \tilde{q}_{\mu} (\partial_\rho q^\mu + \Gamma^\mu_{\rho\tau} q^\tau). \tag{110}$$

As a last remark, please keep in mind that our ‘normalization’ of $\omega_{\rho}$ (and of $\Omega_{\rho}$) here differs from Sachs one by a factor of 1/2. We prefer our normalization, since it is more natural and avoid factors of 2 when we perform contractions.

26See Eq.(3.69) in [31].

27The equation $D^S_{e\nu} q_{\mu} = 0$ (or its matrix representation) is a reminiscence of an analogous equation for the components of tetrad fields often printed in physics textbooks and confused with the metric compatibility condition of the connection. See, e.g., comments on page 76 of [10].
4 Conclusions

In this paper we recalled the concept of covariant derivatives of algebraic dotted and undotted spinor fields, when these objects are represented as sections of real spinor bundles ([12, 15, 27]) and study how this theory has as matrix representative the standard spinor fields (dotted and undotted) already introduced long ago, see, e.g., [1, 19, 20, 21]. Through our approach is that was possible to identify a profound physical meaning concerning some of the rules used in the standard formulation of the (matrix) formulation of spinor fields, e.g., why the covariant derivative of the Pauli matrices must be null. Those rules implies in constraints for the geometry of the spacetime manifold. A possible realization of that constraints is one where the fields defining a global tetrad must be such that $e_0$ is a geodesic field and the $e_i | \gamma$ are Fermi transported (i.e., are not rotating relative to the local gyroscopes axes) along each integral line $\gamma$ of $e_0$.

For the best of our knowledge this important fact is here disclosed for the first time.

We use our formalism to disclose the mathematical nature of the basic variables of Sachs "unified" theory as discussed recently in [33] and as originally introduced in [31]. More on that theory will be discussed in a sequel paper [29].

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