Higher Order Quantum Superintegrability: a new "Painlevé conjecture"

Higher Order Quantum Superintegrability

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Abstract We review recent results on superintegrable quantum systems in a two-dimensional Euclidean space with the following properties. They are integrable because they allow the separation of variables in Cartesian coordinates and hence allow a specific integral of motion that is a second order polynomial in the momenta. Moreover, they are superintegrable because they allow an additional integral of order $N > 2$. Two types of such superintegrable potentials exist. The first type consists of "standard potentials" that satisfy linear differential equations. The second type consists of "exotic potentials" that satisfy nonlinear equations. For $N = 3, 4$ and $5$ these equations have the Painlevé property. We conjecture that this is true for all $N \geq 3$. The two integrals $X$ and $Y$ commute with the Hamiltonian, but not with each other. Together they generate a polynomial algebra (for any $N$) of integrals of motion. We show how this algebra can be used to calculate the energy spectrum and the wave functions.

Keywords Superintegrable systems · Painlevé transcendents · Polynomial algebras and Exact solvability

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1 Introduction

Let us first consider a classical system in an n-dimensional Riemannian space with Hamiltonian

\[ H = \sum_{i,k=1}^{n} g_{ik} p_i p_k + V(x), \quad x \in \mathbb{R}^n. \]  

The system is called integrable (or Liouville integrable) if it allows \( n-1 \) Poisson commuting integrals of motion (in addition to \( H \))

\[ X_a = f_a(x, p), \quad a = 1, \ldots, n-1, \]

\[ \frac{dX_a}{dt} = \{ H, X_a \} = 0, \{ X_a, X_b \} = 0. \]  

This system is superintegrable if it allows further integrals

\[ Y_b = f_b(x, p), \quad b = 1, \ldots, k \]  

\[ 1 \leq k \leq n-1, \]

\[ \frac{dY_b}{dt} = \{ H, Y_b \} = 0. \]

In addition, the integrals must satisfy the following requirements:

1. The integrals \( H, X_a, Y_b \) are well defined functions on phase space, i.e. polynomials or convergent power series on phase space (or an open submanifold of phase space).

2. The integrals \( H, X_a \) are in involution, i.e. Poisson commute as indicated in (2). The integrals \( Y_b \) Poisson commute with \( H \) but not necessarily with each other, nor with \( X_a \).

3. The entire set of integrals is functionally independent, i.e., the Jacobian matrix satisfies

\[ \text{rank} \frac{\partial(H, X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_k)}{\partial(x_1, \ldots, x_n, p_1, \ldots, p_n)} = n + k \]  

In quantum mechanics we define integrability and superintegrability in the same way, however in this case, \( H, X_a \) and \( Y_b \) are operators. The condition on the integrals of motion must also be modified e.g. as follows:

1. \( H, X_a \) and \( Y_b \) are well defined Hermitian operators in the enveloping algebra of the Heisenberg algebra \( H_n \sim \{ x, p, \hbar \} \) or some generalization thereof.

2. The integrals satisfy the Lie bracket relations

\[ [H, X_a] = [H, Y_b] = 0, [X_i, X_k] = 0 \]  

3. No polynomial in the operators \( H, X_a, Y_b \) formed entirely using Lie anticommutators (i.e. Jordan polynomials) should vanish identically.

The two best known superintegrable systems are the Kepler-Coulomb system with potential \( V(r) = \frac{\alpha}{r} \) and the isotropic harmonic oscillator \( V(r) = \alpha r^2 \). In both cases the integrals \( X_a \) correspond to angular momentum, the additional integrals \( Y_b \) to the Laplace-Runge-Lenz
vector for \( V(r) = 2r \) and to the quadrapole tensor \( T_{ik} = p_i p_k + \alpha x_i x_k \), for \( V(r) = \beta r^2 \).

No further ones were discovered until a 1940 paper by Jauch and Hill [36] on the rational anisotropic harmonic oscillator \( V(x) = \alpha \sum_{n=1}^{\infty} n_i x_i^2 \), \( n_i \in \mathbb{Z} \). A systematic search for superintegrable systems was started in 1965 [25, 71, 43] and a real proliferation of them was observed during the last few years [57]. This research program remains very active [61, 59, 12, 63, 44, 5, 11, 40, 21, 42, 33, 6, 32, 34]. The search also been extended to systems with spin, magnetic fields and monopoles. Many families of superintegrable systems have been constructed using combinations of approaches such as the co-algebra and the recurrence method. Let us just list some of the reasons why superintegrable systems are interesting both in classical and quantum physics.

In classical mechanics, superintegrability restricts trajectories to an \( n - k \) dimensional subspace of phase space.

1. For \( k = n - 1 \) (maximal superintegrability), this implies that all finite trajectories are closed and motion is periodic.
2. Moreover, at least in principle, the trajectories can be calculated without any calculus.
3. Bertrand’s theorem states that the only spherically symmetric potentials \( V(r) \) for which all bounded trajectories are closed are \( \frac{\alpha}{r} \) and \( \alpha r^2 \), hence no other maximally superintegrable systems are spherically symmetric.
4. The algebra of integrals of motion \( \{H, X_a, Y_b\} \) is a non-Abelian and interesting one. Usually it is a finitely generated polynomial algebra, only exceptionally a finite dimensional Lie algebra. In the special case of quadratic superintegrability (all integrals of motion are at most quadratic polynomials in the moment), integrability is related to separation of variables in the Hamilton-Jacobi equation.

In quantum mechanics,

1. Superintegrability leads to an additional degeneracy of energy levels, sometimes called ”accidental degeneracy”. The term was coined by Fock and used by Moshinsky and collaborators, though the point of their studies was to show that this degeneracy is certainly no accident. Quadratic integrability is related to separation of variables to the corresponding Schrödinger equation. Quadratic superintegrability implies multiseparability of the Schrödinger equation.
2. A conjecture, born out by all known examples, is that all maximally superintegrable systems are exactly solvable [66]. If the conjecture is true, then the energy levels can be calculated algebraically. The wave functions are polynomials (in appropriately chosen variables) multiplied by some overall gauge factor.
3. The non-Abelian polynomial algebra of integrals of motion has been obtained for various models [28, 7, 20, 27, 40, 21, 42, 33, 6, 32, 34]. In many cases they correspond to higher rank polynomial algebras. They provide energy spectra and information on wave functions via Casimir operators and representation theory. Moreover, it has been demonstrated how Inonu-Wigner and more generally Bocher contractions of quadratic algebras play a role [40, 21] in connecting all quadratically superintegrable models in conformally flat spaces. Interesting relations exist between superintegrability and supersymmetry in quantum mechanics [45] and even more generally other types of operator algebras appear [52].
4. Relation to special function theory: multivariable orthogonal polynomials, new ”nonclassical” orthogonal polynomials, Askey-Wilson classification [40] and exceptional orthogonal
The theory of superintegrable systems has also been formulated in the context of Lie theory and generalized symmetries. As a comment, let us mention that superintegrability has also been called non-Abelian integrability. From this point of view, infinite dimensional integrable systems (soliton systems) described e.g. by the Korteweg-de-Vries equation, the nonlinear Schrödinger equation, the Kadomtsev-Petviashvili equation, etc. are actually superintegrable. Indeed, the generalized symmetries of these equations form infinite dimensional non-Abelian algebras (the Orlov-Shulman symmetries) with infinite dimensional Abelian subalgebras of commuting flows. There is another connection between superintegrable systems in quantum mechanics and soliton theory namely the important role of the Painlevé property and Painlevé transcendents (of second and higher order) in both.

The paper is organized as follows. In Section 2, we present the case of second order superintegrable systems in two-dimensional Euclidean space. In Section 3, we present a summary of results for integrals of motion of order N in $E^2$. In the Section 4, we review the case of $N=4$ with exotic potentials and separable in Cartesian coordinates and present the connection with the Chazy class of equations. We present a summary of the classification of exotic potentials with fourth order integrals separable in cartesian coordinates in Section 5. In Section 6, we present a discussion on the algebraic derivation of the spectrum using cubic algebras and Section 7 is devoted to a connection with supersymmetric quantum mechanics.

2 Second Order Superintegrability

Let us consider the Hamiltonian (1) in the Euclidian space $E_2$ and search for second order integrals of motion [25, 71, 57]. We have

$$ H = \frac{1}{2} (p_1^2 + p_2^2) + V(x_1, x_2), \quad X = \sum_{j+k=0}^2 \left\{ f_{jk}(x_1, x_2), p_1^j p_2^k \right\}. \quad (6) $$

In the quantum case we have

$$ p_j = -i\hbar \frac{\partial}{\partial x_j}, \quad L_3 = x_1 p_2 - x_2 p_1. \quad (7) $$

The commutativity condition $[H, X] = 0$ implies that the even terms $j + k = 0, 2$ and odd terms $j + k = 1$ in $X$ commute with $H$ separately. Hence we can, with no loss of generality, set $f_{10} = f_{01} = 0$. Further we find that the leading (second order) term in $X$ lies in the enveloping algebra of the Euclidian algebra $e(2)$. Thus we obtain

$$ X = a L_3^3 + b_1 (L_3 p_1 + p_1 L_3) + b_2 (L_3 p_2 + p_2 L_3) + c_1 (P_1^2 - P_2^2) + 2c_2 P_1 P_2 + \phi(x_1, x_2) \quad (8) $$

where $a, b_i, c_i$ are constants.

The function $\phi(x_1, x_2)$ must satisfy the determining equations

$$ \phi_{x_1} = -2 (ax_1^2 + 2b_1 x_2 + c_1) V_{x_2} + 2 (ax_1 x_2 + b_1 x_1 - b_2 x_2 - c_2) V_{x_1} $$

$$ \phi_{x_2} = -2 (ax_1 x_2 + b_1 x_1 - b_2 x_2 - c_2) V_{x_1} + 2 (-ax_1^2 + 2b_2 x_1 + c_1) V_{x_2}. \quad (9) $$
The compatibility condition $\phi_{x_1 x_2} = \phi_{x_2 x_1}$ implies

$$(-a x_1 x_2 - b_1 x_1 + b_2 x_2 + c_2)(V_{x_1 x_1} - V_{x_2 x_2})$$
$$- (a(x_1^2 + x_2^2) + 2b_1 x_1 + 2b_2 x_2 + 2c_1)V_{x_1 x_1}$$
$$- (a x_2 + b_1)V_{x_2} + 3(a x_1 - b_2)V_{x_2} = 0. \quad (10)$$

Eq. (10) is exactly the same equation that we would have obtained if we had required that the potential should allow the separation of variables in the Schrödinger equation in one of the coordinate system in which the Helmholtz equation allows separation ( $V(x_1, x_2) = 0$ in (6) ). Another important observation is that (9) and (10) do not involve the Planck constant. Indeed, if we consider the classical functions $H$ and $X$ in (6) and require that they Poisson commute, we arrive at exactly the same conclusions and to equations (9) and (10). Thus for quadratic integrability (and superintegrability) the potentials and integrals of motion coincide in classical and quantum mechanics (up to a possible symmetrization of the integrals). The Hamiltonian (1) is form invariant under Euclidian transformations, so we can classify the integrals $X$ into equivalence classes under rotations, translations and linear combinations with $H$. There are two invariants in the space of parameters $a, b_i, c_i$, namely

$$I_1 = a, \quad I_2 = (2ac_1 - b_i^2 + b_2)^2 + 4(ac_2 - b_1b_2)^2 \quad (11)$$

Solving (6) for different values of $I_1$ and $I_2$ we obtain :

$$I_1 = I_2 = 0 \quad V_C = f_1(x_1) + f_2(x_2)$$
$$I_1 = 1, \ I_2 = 0 \quad V_R = f(r) + \frac{1}{r}g(\phi) \quad x_1 = r \cos \phi, \ x_2 = r \sin \phi$$
$$I_1 = 0, \ I_2 = 1 \quad V_P = \frac{f(\xi) + g(\eta)}{\xi^2 + \eta^2} \quad x_1 = \frac{\xi^2 - \eta^2}{2}, \ x_2 = \xi \eta$$
$$I_1 = 1, \ I_2 = l^2 \neq 0 \quad V_E = \frac{f(\sigma) + g(\eta)}{\cos^2 \sigma - \cosh^2 \rho} \quad x_1 = l \cosh \rho \cos \sigma, \ x_2 = l \sinh \rho \sin \sigma \quad 0 < l < \infty \quad (12)$$

We see that $V_C, V_R, V_P$ and $V_E$ correspond to separation of variables in Cartesian, polar, parabolic and elliptic coordinates, respectively and that second order integrability (in $E_2$) is equivalent to the separation of variables in the Hamilton-Jacobi and the Schrodinger equation. For second order superintegrability, two integrals of the form (9) exist and the Hamiltonian separates in at least two coordinate systems. Four three-parameter families of superintegrable systems exist namely

$$V_l = \alpha(x^2 + y^2) + \frac{\beta}{x^2} + \gamma y, \quad V_{ll} = \alpha(x^2 + 4y^2) + \frac{\beta}{x^2} + \gamma y$$
$$V_{lll} = \frac{\alpha}{r} + \frac{1}{r^2}(\frac{\beta}{\cos^2 \frac{\phi}{2}} + \frac{\gamma}{\sin^2 \frac{\phi}{2}}), \quad V_{lv} = \frac{\alpha}{r} + \frac{1}{r^2}(\beta \cos \frac{\phi}{2} + \gamma \sin \frac{\phi}{2}) \quad (13)$$

The classical trajectories, quantum energy levels and wave functions for all of these systems are known. The potentials $V_l$ and $V_{ll}$ are isospectral deformations of the isotropic and anisotropic harmonic oscillator, respectively, whereas $V_{lll}$ and $V_{lv}$ are isospectral deformations of the Kepler-Coulomb potential. In n-dimensional space $E_n$, a set of n commuting second order integrals corresponds to a separable coordinate system. All of the above results on quadratic superintegrability have been generalized to arbitrary dimensions, to spaces of constant curvature and to other real and complex spaces [38, 39, 56, 57].
3 Summary of results for integrals of motion of order N in \( E_2 \)

In quantum mechanics on two-dimensional Euclidean space \( E_2 \) the most general N-th order integral has the form

\[
X = \frac{1}{2} \sum_{l=0}^{\left[ \frac{N}{2} \right]} \sum_{j=0}^{N-2l} \{ f_{j,2l}, p_1^j p_2^{N-j-2l} \} \tag{14}
\]

where \( f_{j,2l} \) are real functions of \( x,y \) and we set \( f_{j,2l} \) for \( j,l < 0 \) or \( j > N-2l \). The brackets \( \{ \, \} \) denote a symmetrization. In classical mechanics the brackets are inessential.

The determining equations following from the commutativity relation \( [H,X] = 0 \) were obtained in \cite{62} for arbitrary \( N \geq 2 \), both in the classical and quantum cases. The equations are quite complicated but completely explicit.

A priori the Lie or Poisson commutator \( [H,X] \) is a polynomial of order \( N+1 \) in the components of the momenta \( p_i \). The terms of order \( N+1 \) are linear and do not involve the potential \( V(x,y) \). All lower order terms are nonlinear since they involve products of the unknown potential and the unknown coefficients \( f_{j,2l} \).

An analysis of the highest and second to highest order determining equations provides several important results.

1. Even and odd parity terms in \( X \) commute with \( H \) separately, so all terms in \( \text{(14)} \) have the same parity (this is already built into eq. \( \text{(14)} \)).

2. The leading terms in \( X \) are polynomials of order \( N \) in the enveloping algebra of the Euclidean Lie algebra i.e.

\[
X = X_L + l.o.t \tag{15}
\]

where the coefficient \( A_{N-m-n,m,n} \) are real constants. Indeed the leading terms are obtained for \( l = 0 \) in \( \text{(14)} \) and are polynomials

\[
f_{00} = \sum_{n=0}^{N-j} \sum_{m=0}^{j} \left( \frac{N-m-n}{j-m} \right) A_{N-m-n,m,n} x^{N-j-m} y^{j-m} \tag{16}
\]

3) The set of determining equations \( f_{j,2l} \) does involve the potential and is nonlinear. However, the equations are in general incompatible. A compatibility condition for arbitrary \( N \) is the linear PDE

\[
\sum_{j=0}^{N-1} \partial_x^{N-1-j} \partial_y^j (-1)^j [(j+1)f_{j+1,0} \partial_x V + (N-j)f_{j,0} \partial_y V] = 0 \tag{17}
\]

This is a linear PDE for \( V \) alone, since the coefficients \( f_{j,0} \) are already known in terms of the constants \( A_{N-m-n,m,n} \). Other compatibility condition exist, but they are nonlinear PDEs for the potential \( V(x,y) \) and are less useful than \( \text{(17)} \).

For \( N = 2 \) the condition \( \text{(17)} \) reduces to the condition \( \text{(10)} \) and provides the connection between second order integrability and the separation of variables.

For \( N \geq 3 \) eq. \( \text{(17)} \) is also the starting point for all further studies. Right from the beginning we distinguish two types of integrable potentials:
Higher Order Quantum Superintegrability: a new "Painlevé conjecture"

(i) Standard potentials. For these the linear compatibility condition LCC (17) is satisfied nontrivially. For $N = 2$ all integrable potentials are standard.

(ii) Exotic potentials. These exist for $N \geq 3$ and for them the LCC is satisfied trivially i.e. all coefficients $A_{N-n-m-m,n}$ that figure in the LCC vanish identically. Surprisingly that does not imply that the integral $X$ vanishes; it does however greatly simplify.

Solving the remaining nonlinear PDEs is still a formidable task for any $N \geq 3$, specially in quantum mechanics. Instead of attempting this task we turn to a simpler problem, namely construct superintegrable systems in $E^2$ with two independent integrals of motion $X$ and $Y$, where $X$ is of first or second order and $Y$ is of the order $N$. The integrals $X$ implies that $V(x, y)$ has one of the form given in (12). The potential in (12) depends on two arbitrary functions of one variable. Hence the LCC (17) is no longer a PDE but reduces to one or several ODEs. The most interesting cases occurs when the potential has the form $V_c$ and $V_R$ of (12) i.e. allows separation in Cartesian coordinates or polar coordinates. Let us now turn to the example of exotic potentials allowing the separation of variables in cartesian coordinates and admitting an additonal independent integral of order $N = 4$.

4 Fourth Order Superintegrability and Exotic Potentials

The article [51] is part of a general program the aim of which is to derive, classify, and solve the equations of motion of superintegrable systems with integrals of motion that are polynomials of finite order $N$ in the components of linear momentum. The search has been performed in two-dimensional Euclidean space. The study of Hamiltonians with integrals of motion of order $N = 3$ was started in [30] and a classification of Hamiltonians separable in Cartesian coordinates with an integrals of order $N = 3$ was performed [29]. The obtained classical and quantum Hamiltonian systems have been studied in [31, 34, 46, 47, 70]. In [51] the case $N = 4$ was considered and all exotic potentials have been classified. The connection with the Painlevé property and Chazy class of equations was also highlighted. Partial results which consist in classifying all doubly exotic potentials were performed for $N = 5$ [2]. Results are known for systems with integrals of arbitrary order $N$ [64] and anisotropic oscillator complemented by Painlevé transcendentents [45]. In this review we concentrate on superintegrable systems with Hamiltonians of the form

$$H = \frac{1}{2}(p_1^2 + p_2^2) + V(x, y),$$

in two dimensional Euclidean space $E_2$. In classical mechanics, $p_1$ and $p_2$ are the canonical momenta conjugate to the Cartesian coordinates $x$ and $y$. In quantum mechanics, we have

$$p_1 = -i\hbar \partial_x, \quad p_2 = -i\hbar \partial_y, \quad L_3 = xp_2 - yp_1.$$

The determining equations for fourth-order classical and quantum integrals of motion were derived earlier and they are a special case of $N$th order ones given in [62]. In the quantum case, the integral is $Y^{(4)} = Y$:

$$Y = \sum_{j+k+l=4} \frac{A_{jkl}}{2} (L_3^j, p_1^k p_2^l) + \frac{1}{2} \{\{g_1(x, y), p_1^2\} + \{g_2(x, y), p_1 p_2\} + \{g_3(x, y), p_2^2\} + l(x, y),$$

where $A_{jkl}$ are coefficients determined by the separation of variables $x$ and $y$.
where \( A_{ijkl} \) are real constants, the brackets \( \{ \ldots \} \) denote anti-commutators and the Hermitian operators \( p_1, p_2 \) and \( L_3 \) are given in [19]. The functions \( g_1(x, y), g_2(x, y), g_3(x, y), \) and \( l(x, y) \) are real and the operator \( Y \) is self-adjoint. Equation (20) is also valid in classical mechanics where \( p_1, p_2 \) are the canonical momenta conjugate to \( x \) and \( y \), respectively (and the symmetrization becomes irrelevant). The commutation relation \( [H, Y] = 0 \) with \( H \) in [18] provides the determining equations

\[
\begin{align*}
g_{1,x} &= 4f_1V_x + f_2V_y, \\
g_{2,x} + g_{1,y} &= 3f_2V_x + 2f_3V_y, \\
g_{3,x} + g_{2,y} &= 2f_3V_x + 3f_4V_y, \\
g_{3,y} &= f_4V_x + 4f_5V_y.
\end{align*}
\]

These 4 equations are linear PDEs and involve 4 unknown functions \( g_1, g_2, g_3, V \). Furthermore we have the following two further equations

\[
\begin{align*}
\ell_x &= 2g_1V_x + g_2V_y + \frac{\hbar^2}{2} \left( (f_2 + f_4)V_{xxx} - 4(f_1 - f_3)V_{xyy} - (f_2 + f_4)V_{yyy} \\
&\quad + (3f_2 - f_5)V_{x} - (13f_1 + f_4)V_{xy} - 4(f_2 - f_5)V_{yy} \\
&\quad - 2(6A_{400}x^2 + 62A_{400}y^2 + 3A_{301}x - 29A_{310}y + 9A_{220} + 3A_{202})V_x \\
&\quad + 2(56A_{400}xy - 13A_{310}x + 13A_{301}y - 3A_{211})V_y \right), (23a) \\
\ell_y &= g_2V_x + 2g_3V_y + \frac{\hbar^2}{2} \left( (f_2 + f_4)V_{xxx} + 4(f_1 - f_3)V_{xyy} + (f_2 + f_4)V_{yyy} \\
&\quad + 4(f_{1,y} - f_{4,x})V_{x} - (f_{2,y} + 13f_{5,x})V_{xy} - (f_{1,y} - 3f_{4,x})V_{yy} \\
&\quad + 2(56A_{400}xy - 13A_{310}x + 13A_{301}y - 3A_{211})V_x \\
&\quad - 2(62A_{400}x^2 + 62A_{400}y^2 + 29A_{301}x - 3A_{310}y + 9A_{202} + 3A_{220})V_y \right). (23b)
\end{align*}
\]

The quantities \( f_i, i = 1, 2, \ldots, 5 \) are polynomials in \( x \) and \( y \). They are obtained from the highest order terms in the following condition \( [H, Y] = 0 \).

These 2 nonlinear PDEs for \( l, g_1, g_2, g_3, V \) will give nonlinear compatibility condition. Explicitly for these polynomials we have

\[
\begin{align*}
f_1 &= A_{400}y^4 - A_{310}y^3 + A_{220}y^2 - A_{130}y + A_{040} \\
f_2 &= -4A_{400}xy^3 - A_{301}xy^2 + 3A_{310}xy^2 + A_{211}y^2 - 2A_{220}xy - A_{121}y \\
&\quad + A_{130}x + A_{031} \\
f_3 &= 6A_{400}x^2y^2 + 3A_{301}x^2y^2 - 3A_{310}x^2y + A_{202}y^2 - 2A_{211}xy + A_{220}x^2 \\
&\quad - A_{112}y + A_{121}x + A_{022} \\
f_4 &= -4A_{400}xy^3 + A_{310}x^3 - 3A_{301}x^2y + A_{211}x^2 - 2A_{202}xy + A_{112}x \\
&\quad - A_{103}y + A_{013} \\
f_5 &= A_{400}x^4 + A_{301}x^3 + A_{202}x^2 + A_{103}x + A_{004}.
\end{align*}
\]

with 15 constants \( A_{ijkl} \). For a known potential the determining equations (22) and (23) form a set of 6 linear PDEs for the functions \( g_1, g_2, g_3, \) and \( l \). If \( V \) is not known, we have
a system of 6 nonlinear PDEs for $g_i, l$ and $V$. In any case the four equations (22) are a priori incompatible. The compatibility equation is a fourth-order linear PDE for the potential $V(x, y)$ alone, namely

$$\partial_{yyy}(4f_1 V_x + f_2 V_y) - \partial_{xxy}(3f_2 V_x + 2f_3 V_y) + \partial_{xzy}(2f_3 V_x + 3f_4 V_y)$$

$$- \partial_{xx}(f_4 V_x + 4f_5 V_y) = 0.$$  

This is a special case of the $N$th order linear compatibility equation (17). We see that the equation (25) does not contain the Planck constant and is hence the same in quantum and classical mechanics (this is true for any $N$). The difference between classical and quantum mechanics manifests itself in the two equations (23). They greatly simplify in the classical limit $\hbar \to 0$. Further compatibility conditions on the potential $V(x, y)$ can be derived for the systems (22) and (23), they will however be nonlinear. We will not go further into the problem of the fourth order integrability of the Hamiltonian (18). Instead, we turn to the problem of superintegrability formulated in the Introduction.

4.1 Potentials separable in Cartesian coordinates

We shall now assume that the potential in the Hamiltonian (18) has the form

$$V(x, y) = V_1(x) + V_2(y).$$  

This is equivalent to saying that a second order integral exists which can be taken in the form

$$X = \frac{1}{2}(p_1^2 - p_2^2) + V_1(x) - V_2(y).$$  

Equivalently, we have two one dimensional Hamiltonians

$$H_1 = \frac{p_1^2}{2} + V_1(x), \quad H_2 = \frac{p_2^2}{2} + V_2(y).$$

We are looking for a third integral of the form (20) satisfying the determining equations (22) and (23). This means that we wish to find all potentials of the form (26) that satisfy the linear compatibility condition (25). Once (25) is substituted, (25) is no longer a PDE and will split into a set of ODEs which we will solve for $V_1(x)$ and $V_2(y)$.

The task thus is to determine and classify all potentials of the considered form that allow the existence of at least one fourth order integral of motion. As in every classification we must avoid triviality and redundancy. Since $H_1$ and $H_2$ of (28) are integrals, we immediately obtain 3 "trivial" fourth order integrals, namely $H_1^2, H_2^2,$ and $H_1 H_2$. The fourth order integral $Y$ of equation (20) can be simplified by taking linear combination with polynomials in the second order integrals $H_1$ and $H_2$ of (28):

$$Y \to Y' = Y + a_1 H_1^2 + a_2 H_2^2 + a_3 H_1 H_2 + b_1 H_1 + b_2 H_2 + b_0, \quad a_i, b_i \in \mathbb{R}. \quad (29)$$

Using the constants $a_1, a_2$ and $a_3$ we set

$$A_{004} = A_{040} = A_{022} = 0,$$

in the integral $Y$ we are searching for. At a later stage we will use the constants $b_0, b_1$ and $b_2$ to eliminate certain terms in $g_1, g_2, g_3$ and $l$. 

Substituting (20) into the compatibility condition (25), we obtain a linear condition, relating the functions $V_1(x)$ and $V_2(y)$

$$
(-60A_{310} + 240yA_{400})V_1'(x) + (-20A_{211} + 60yA_{301} - 60xA_{310} + 240xyA_{400})V_1''(x)
+ (-5A_{112} + 10yA_{202} - 10xA_{211} + 30xyA_{301} - 15x^2A_{310} + 60x^2yA_{400})V_1^{(3)}(x)
+ (-A_{013} + yA_{103} - xA_{112} + 2xyA_{202} - x^2A_{211} + 3x^2yA_{301} - x^3A_{310} + 4x^3yA_{400})V_1^{(4)}(x)
+ (-60A_{301} - 2140xyA_{400})V_2'(y) + (20A_{211} - 60yA_{301} - 60xA_{310} - 240xyA_{400})V_2''(y)
+ (-5A_{121} + 10yA_{211} - 10xA_{220} - 15y^2A_{301} + 30xyA_{310} - 60xy^2A_{400})V_2^{(3)}(y)
+ (A_{031} + yA_{121} + xA_{130} + y^2A_{211} - 2xyA_{220} - y^3A_{301} + 3xyA_{310} - 4xy^3A_{400})V_2^{(4)}(y) = 0.
$$

(31)

It should be stressed that this is no longer a PDE, since the unknown functions $V_1(x)$ and $V_2(y)$ both depend on one variable only. We differentiate (31) twice with respect to $x$ and thus eliminate $V_2(y)$ from the equation. The resulting equation for $V_1(x)$ splits into two linear ODEs (since the coefficients contain terms proportional to $y^0$, and $y^1$), namely

$$
210A_{310}V_1^{(3)}(x) + 42(A_{211} + 3A_{310}x)V_1^{(4)}(x) + 7(A_{112} + 2A_{211}x)
+ 3A_{103}x^2V_1^{(5)}(x) + (A_{013} + A_{112} + A_{211}x^2)A_{310}V_1^{(6)}(x) = 0,
$$

(32a)

$$
840A_{400}V_1^{(3)}(x) + (126A_{301} + 504A_{400}x)V_1^{(4)}(x) + 14(A_{202} + 3A_{301}x)
+ 6A_{400}x^2V_1^{(5)}(x) + (A_{103} + 2A_{202}x + 3A_{301}x^2 + 4A_{400}x^3)V_1^{(6)}(x) = 0.
$$

(32b)

Similarly, differentiating (31) with respect to $y$ we obtain two linear ODEs for $V_2(y)$,

$$
210A_{301}V_2^{(3)}(y) - 42(A_{211} - 3A_{310}y)V_2^{(4)}(y) + 7(A_{121} - 2A_{211}y)
+ 3A_{103}y^2V_2^{(5)}(y) - (A_{031} - A_{121}y + A_{211}y^2 - A_{301}y^3)V_2^{(6)}(y) = 0,
$$

(33a)

$$
840A_{400}V_2^{(3)}(y) - (126A_{310} - 504A_{400}y)V_2^{(4)}(y) + 14(A_{220} - 3A_{310}y)
+ 6A_{400}y^2V_2^{(5)}(y) - (A_{130} - 2A_{220}y + 3A_{310}y^2 - 4A_{400}y^3)V_2^{(6)}(y) = 0.
$$

(33b)

The compatibility condition $\ell_{xy} = \ell_{yx}$, for (25a) and (25b) implies

$$
-gzV_1'(x) + gzV_2'(y) + (2g_y - 2g_x)\mathcal{V}_1'(x) + (g_{2y} - 2g_{2x})V_2'(y) +
\frac{h^2}{4} \left((f_2 + f_4)(V_1^{(4)} - V_2^{(4)}) + (f_2 - 4f'_1(y))V_1^{(3)} + (4f'_5(x) - 5f_2 - f_4)V_2^{(3)}
+ (3f_{2yy} + 4f_{4xx} + 6A_{211} - 2A_{301}y + 2A_{310}x - 12A_{400}xy)V_1''
- (4f_{2yy} + 3f_{4xx} + 6A_{211} - 2A_{310}y + 2A_{310}x - 12A_{400}xy)V_2''
+ (84A_{310} - 360A_{400}y)V_1' + (84A_{310} + 360A_{400}y)V_2' \right) = 0.
$$

(34)
This equation, contrary to (32) and (33), is nonlinear since it still involves the unknown functions \( g_1, g_2, \) and \( g_3, \) (in addition to \( V_1(x) \) and \( V_2(y) \)).

### 4.2 ODEs with the Painlevé property

In order to study exotic potentials \( V(x, y) = V_1(x) + V_2(y) \), allowing fourth order integrals of motion in quantum mechanics we must first recall some known results on Painlevé type equations [41, 31, 14, 15]. Passing the test [1] is a necessary condition for having the Painlevé property. We shall need it only for equations of the form

\[
P''_1(z) = 6P''_1(z) + z, \quad P''_2(z) = 2P_2(z)^3 + zP_2(z) + \alpha, \quad P_3(z)'' = \frac{P''_3(z)}{P_3(z)} - \frac{P'_3(z)}{P_3(z)} + \frac{\alpha P''_2(z) + \beta}{z} + \gamma P_3(z) + \frac{\delta}{P_3(z)}, \quad P_4(z)'' = \frac{P''_4(z)}{2P_4(z)} + \frac{3}{2} P_4'(z) + 4zP_4'(z) + 2(z^2 - \alpha)P_4(z) + \frac{\beta}{P_4(z)},
\]

\[
P''_5(z) = \left( \frac{1}{2P_5(z)} + \frac{1}{P_5(z) - 1} \right) P_5'(z)^2 - \frac{1}{z} P_5'(z) + \frac{(P_5(z) - 1)^2}{z^2} \left( \frac{\alpha P_5''(z) + \beta}{P_5(z)} \right) + \frac{\gamma P_5(z)}{z} + \frac{\delta P_5(z)(P_5(z) + 1)}{P_5(z) - 1},
\]

\[
P''_6(z) = \frac{1}{2} \left[ \frac{1}{P_6(z)} + \frac{1}{P_6(z) - 1} + \frac{1}{P_6(z) - z} \right] P_6'(z)^2 - \frac{1}{z} \left( \frac{1}{z} + \frac{1}{z - 1} + \frac{1}{P_6(z) - z} \right) P_6'(z) + \frac{P_6(z)(P_6(z) - 1)(P_6(z) - z)}{z^2(z - 1)^2} \left( \gamma_1 + \frac{\gamma_2 z}{P_6(z)} + \frac{\gamma_3(z - 1)}{(P_6(z) - 1)^2} + \frac{\gamma_4 z(z - 1)}{(P_6(z) - z)^2} \right)\]

An ODE has the Painlevé property if its general solution has no movable branch points, (i.e. branch points whose location depends on one or more constants of integration). For a review and further developments see [41, 31, 14, 15]. Passing the test [1] is a necessary condition for having the Painlevé property. We shall need it only for equations of the form

\[
W^{(n)} = F(y, W, W', W'', ..., W^{(n-1)}),
\]

where \( F \) is polynomial in \( W, W', W'', ..., W^{(n-1)} \) and rational in \( y \).

The general solution must have the form of a Laurent series with a finite number of negative power terms

\[
W = \sum_{k=0}^{\infty} d_k (y - y_0)^{k+p}, \quad d_0 \neq 0,
\]

satisfying the requirements
1. The constant $p$ is a negative integer.
2. The coefficients $d_k$ satisfy a recursion relation of the form
   \[ P(k)d_k = \phi_k(y_0, d_0, d_1, \ldots, d_{k-1}), \]
   where $P(k)$ is a polynomial that has $n - 1$ distinct nonnegative integer zeros. The values of $k_j$ for which we have $P(k_j) = 0$ are called resonances and the values of $d_k$ for $k = k_j$ are free parameters. Together with the position $y_0$ of the singularity we thus have $n$ free parameters in the general solution \((37)\) of the $n$-th order ODE \((36)\).
3. A compatibility condition, also called the resonance condition:
   \[ \phi_k(y_0, d_0, d_1, \ldots, d_{k-1}) = 0, \]
   must be satisfied identically in $y_0$ and in the values of $d_{k_j}$ for all $k_j; j = 1, 2, \ldots, n - 1$.

This test is a generalization of the Frobenius method used to study fixed singularities of linear ODEs. Passing the Painlevé test is a necessary condition only. To make it sufficient one would have to prove that the series \((37)\) has a nonzero radius of convergence and that the $n$ free parameters can be used to satisfy arbitrary initial conditions. A more practical procedure that we shall adopt is the following. Once a nonlinear ODE passes the Painlevé test one can try to integrate it explicitly.

Let us first investigate the cases that may lead to "exotic potentials", that is potentials which do not satisfy any linear differential equations. That means that either \((32)\) or \((33)\) (or both) must be satisfied trivially. The linear ODEs \((32)\) are satisfied identically if we have
\[ A_{400} = A_{310} = A_{301} = A_{211} = A_{202} = A_{112} = A_{103} = A_{013} = 0. \]
(38)
The linear ODEs \((33)\) are satisfied identically if we have
\[ A_{400} = A_{310} = A_{301} = A_{211} = A_{220} = A_{121} = A_{130} = A_{031} = 0. \]
(39)
If \((38)\) and \((39)\) both hold then the only fourth order integrals are the trivial ones $H^2_1, H^2_2$ and $H_1 H_2$. Their existence does not imply superintegrability, it is simply a consequence of second order integrability. In other words, no fourth order superintegrable systems, satisfying \((38)\) and \((39)\) simultaneously, exist. This means that at most one of the functions $V_1(x)$ or $V_2(y)$ can be "exotic". The other one will be a solution of a linear ODE. For third order integrals both $V_1(x)$ and $V_2(y)$ could be exotic.

Let us consider the case when, \((39)\) is valid and \((38)\) not. The leading-order term for the nontrivial fourth order integral has the form
\[ Y_L = A_{202} \{L^2_3, p_2^2\} + A_{112} \{L_3, p_1 p_2^2\} + A_{103} \{L_3, p_2^3\} + 2 A_{013} p_1 p_2^3. \]
(40)
We proceed in several steps. (i) Let us classify the integrals \((40)\) under translations (they leave the form of the potential \((26)\) invariant). The three classes are:
\[
\begin{align*}
I. & A_{202} \neq 0, A_{112} = A_{103} = 0, \\
II. & A_{202} = 0, A_{112}^2 + A_{103}^2 \neq 0, A_{013} = 0, \\
IIa. & A_{103} \neq 0, \\
IIb. & A_{103} = 0, A_{112} \neq 0, \\
III. & A_{202} = A_{112} = A_{103} = 0, A_{013} \neq 0.
\end{align*}
\]
(41)
(ii) Let us solve the linear ODE for $V(x)$

The functions $f_i$ in (24) reduce to

$$f_1 = f_2 = 0,$$
$$f_3(y) = A_{202} y^2 - A_{112} y,$$
$$f_4(x,y) = -2A_{202} xy + A_{112} x - A_{103} y + A_{013},$$
$$f_5(x) = A_{202} x^2 + A_{103} x. \quad (42)$$

we obtain two equations for $V_1(x)$ namely

$$5 A_{112} V_1^{(3)}(x) + (A_{013} + A_{112} x) V_1^{(4)}(x) = 0, \quad (43a)$$
$$10 A_{202} V_1^{(3)}(x) + (A_{103} + 2 A_{202} x) V_1^{(4)}(x) = 0. \quad (43b)$$

(They replace equations (32)). These two equations imply $V_1^{(3)} = V_1^{(4)} = 0$ unless we have

$$A_{112} A_{103} - 2 A_{202} A_{013} = 0. \quad (44)$$

The result is that $V_1(x)$ can have one of the following forms:

$$V_1(x) = 0, \ ax, \ ax^2, \frac{a}{2} x + bx + cx^2 \text{ (where } bc = 0 \text{)}$$

(iii) Let us solve the nonlinear ODEs for $V_2(y)$. We first introduce an auxiliary function:

$$W(y) = \int V_2(y) \, dy,$$

$$\tilde{W} \equiv W + \alpha y + \beta.$$

Case I. $A_{202} \neq 0, A_{112} = 0; Y_L = A_{202} \{ L_3^2, p_2^2 \}.$

Let $A_{202} = 1$. We obtain

$$\frac{1}{2} h^2 y^4 W^{(4)} + 2 h^2 W^{(3)} - 6g y W'' + 2 W + \frac{8}{3} e_{2g} y^3 W'' - 8 W''^2 + 16 e_{2g} y^2 W'$$
$$+ 16 e_{2g} W - \frac{16}{3} e_{2g} y^4 + k_1 = 0, \quad (45)$$

integrating once we get

$$h^2 y^2 W^{(3)} + 2 h^2 y W'' - 6g y^2 W'' - 49 W^{(2)} + \left(\frac{16}{3} e_{2g} y^4 - 2h^2\right) W' + 2 W^2 + \frac{32}{3} e_{2g} y^3 W$$
$$- \frac{16}{9} e_{2g} y^6 + k_1 y^2 + k_2 = 0. \quad (46)$$

The equation (46) passes the Painlevé test. Substituting the Laurent series (37) into (46), we find $p = -1$. The resonances are $r = 1$, and $r = 6$, and we obtain $d_0 = -h^2$. The constants $d_1$ and $d_6$ are arbitrary, as they should be. We now proceed to integrate (46). Using the results of Chazy, Bureau, Cosgrove and Scoufis [14, 16, 17, 18].

By the following transformation

$$Y = y^2, \ U(Y) = -\frac{y}{2h^2} W(y) + \frac{e_2}{6h^2} y^3 + \frac{1}{16},$$
we transform (46) to
\[ Y^2 U^{(3)} = -2\left(U'(3YU'' - 2U) - \frac{c_2}{\hbar^2} Y(YU'' - U) + k_3 Y k_4\right) - YU''', \] (47)
where \( k_3 = -\frac{2k_1 - 12\sqrt{2}\Delta^2}{64\Delta^2}, \ k_4 = \frac{k_1}{2k_2}. \) The equation (47) is a special case of the Chazy class I equation. It admits the first integral
\[ Y^2 U'' = -4(U'^2(YU'' - U) - \frac{c_2}{2\hbar^2}(YU'' - U)^2 + k_3(YU'' - U) + k_4 U' + k_5), \] (48)
where \( k_5 \) is the integration constant. The equation of the canonical form SD-I.b.

When \( c_2 \) and \( k_3 \) are both nonzero the solution is
\[
U = \frac{1}{4} \left( \frac{1}{P_5^2} \frac{Y P_5'}{P_5 - 1} - P_3\right)^2 - (1 - \sqrt{2}\alpha)^2(P_5 - 1) - 2\beta \frac{P_5 - 1}{P_5} \]
\[ + \gamma Y P_5 + \frac{1}{P_5 - 1} + 2\delta Y^2 P_5, \]
\[ U' = -\frac{1}{4} \left( \frac{P_5'}{P_5 - 1} \right)^2 - \frac{\beta}{2Y} \frac{P_5 - 1}{P_5}, \]
\[ - \frac{1}{2} \delta Y \frac{P_5}{P_5 - 1} - \frac{1}{4} \gamma; \] (49)
where \( P_5 = P_5(Y); Y = y^2, \) satisfies the fifth Painlevé equation
\[ P_5'' = \left( \frac{1}{2P_5^2} + \frac{1}{P_5 - 1} \right) P_5'^2 - \frac{1}{Y} Y P_5' + \frac{(P_5 - 1)^2}{Y^2} \left( \alpha P_5 + \frac{\beta}{P_5} \right) + \gamma \frac{P_5}{P_5 - 1} + \delta \frac{P_5}{P_5 + 1}, \]
with
\[ c_2 = -h^2 \delta, \ k_3 = -\frac{1}{4} \left( \frac{1}{4} \gamma^2 + 2\beta \delta - \delta(1 - \sqrt{2}\alpha)^2\right), \]
\[ k_4 = -\frac{1}{4} \left( \beta \gamma + \frac{1}{2} \gamma(1 - \sqrt{2}\alpha)^2\right), \]
\[ k_5 = -\frac{1}{32}(\gamma^2((1 - \sqrt{2}\alpha)^2 - 2\beta) - \delta((1 - \sqrt{2}\alpha)^2 + 2\beta)^2). \]

The solution for the potential up to a constant is
\[ V(x, y) = \frac{c_2 - \beta}{x^2} + \delta \hbar^2 (x^2 + y^2)^2 + \hbar^2 \left( \frac{\gamma}{P_5 - 1} + \frac{1}{y^2}(P_5 - 1)(\sqrt{2}\alpha + \alpha(2P_5 - 1) + \beta) \right) \]
\[ + y^2 \left( \frac{P_5'^2}{2P_5} + \delta P_5 \right) \left( \frac{2P_5 - 1}{(P_5 - 1)^2} - \frac{P_5'}{P_5 - 1} - 2\sqrt{2}\alpha P_5' \right) + \frac{3\hbar^2}{8y^2}. \] (50)

The list of exotic superintegrable quantum potentials in quantum case that admit one second order Cartesian and one fourth order integral is given below. We also give their fourth order integrals by listing the leading terms \( Y'_L \) and the functions \( g_i(x, y); i = 1, 2, 3; \) and \( l(x, y). \) Each of the exotic potentials has a non-exotic part that comes from \( V_1(x). \) By construction \( V_2(y) \) is exotic, however in 4 cases a non-exotic part proportional to \( y^2 \) splits...
off from \( V_2(y) \) and can be combined with an \( x^2 \) term in \( V_1(x) \). We order the final list below in such a manner that the first two potentials are isotropic harmonic oscillators (possibly with an additional \( \frac{1}{x^2} \) term) with an added exotic part. The next two are 2 : 1 anisotropic harmonic oscillators, plus an exotic part (in \( y \)). Based on previous experience, we expect these harmonic terms to determine the bound state spectrum. The remaining 8 cases have either \( \frac{a}{x^2} \) or \( c_1x \) as their non-exotic terms and we expect the energy spectrum to be continuous.

These results also highlight how the study of higher order Painlevé equations plays a role in the classification of superintegrable systems with higher order integrals of motion. Classes of such equations of third, fourth and fifth order have been studied by Chazy, Bureau, Cosgrove and Scoufis [13, 19, 16, 17, 18].

5 Summary of the classification of exotic potentials with fourth order integrals separable in cartesian coordinates

In this section we give a list of some of these exotic potentials and their fourth order integrals. There are 12 cases that are divided into three types. We present one case among each of them.

I. Isotropic harmonic oscillator: \( Q^1_1 \):

\[
V(x, y) = -\delta h^2(x^2 + y^2) + \frac{a}{x^2} + h^2\left(\frac{\gamma}{x^2} - \frac{1}{y^2}(P_5 - 1)(\sqrt{2\alpha} + \alpha(2P_5 - 1) + \frac{\beta}{P_5})
\right. \\
\left. + \frac{y^2}{2P_5} + \delta P_5\right)(2P_5 - 1)\frac{P_5}{P_5 - 1} - 2\sqrt{2\alpha}P_5') + \frac{3h^2}{8y^2}.
\]

\[Y_L = \{L_3, p_2\} \]

\[g_1(x, y) = 2y(yW' + W + \frac{1}{3}h^2\delta y^3), \]

\[g_2(x, y) = -2x(3yW' + W + \frac{4}{3}h^2\delta y^3) \]

\[l(x, y) = h^2(x^2 + \frac{1}{4}yW^{(4)} + W^{(3)}) - x^2(3yW' + W)W'' - h^2y\left(\frac{4}{3}x^2y^2 + \frac{3}{2}\right)W'' + (4\frac{a}{x^2} - h^2\delta x^2)W''
\]

\[+ 4y\left(\frac{a}{x^2} - h^2\delta x^2\right)W + \frac{4\delta y}{3x^2}h^2\delta y^4 - 2h^2\delta x^2\left(\frac{2}{3}h^2\delta y^4 - h^2\right) - 2h^4\delta y^4. \]

\[W(y) = \frac{-h^2}{2y}\left(\frac{1}{P_5} - \frac{Y P_5'}{P_5 - 1} - P_5\right)^2 - (1 - \sqrt{2\alpha})^2(P_5 - 1) - 2\beta P_5 - 1 + \gamma Y P_5 + 1 + \frac{2\delta y^2 P_5}{(P_5 - 1)^2} \]

\[+ \frac{h^2}{8y} = \frac{\delta h^2}{3}y^3, \text{ where } P_5 = P_5(Y); Y = y^2. \]

II. Anisotropic harmonic oscillator:
\[ Q^1_3 : \]

\[ V(x, y) = c_2(x^2 + 4y^2) + \frac{a}{x^2} - 4\sqrt{2c_2^3}h^2yP_4 + \sqrt{2c_2h}(\epsilon P'_4 + P_4^2) \]

\[ Y_L = \{L_3, p_1, p_2^2\}, \]

\[ g_1(x, y) = -2yW' - W + \frac{4}{3}c_2y^3, \]

\[ g_2(x, y) = 3xW' - 4c_2xy^2, \]

\[ g_3(x, y) = 2c_2x^2y - 2a\frac{y}{x^2}, \]

\[ l(x, y) = -\frac{1}{8}y^2x^4W^{(4)} + \frac{3}{2}x^2yP'' - (2c_2x^2y^2 - 3\frac{h^2}{4})W'' - 2(2a\frac{y}{x^2} + 2c_2x^2y)W' - 2(\frac{a}{x^2} + c_2x^2)W' + \frac{8}{3}c_2y^3(c_2x^2 + \frac{a}{x^2}) - 2c_2h^2y. \]

\[ W(y) = \sqrt[8]{2c_2h^3}(\frac{1}{8}P_4^2 - \frac{1}{8}P_4^2 - \frac{1}{2}3P_4^2 - \frac{1}{2}(Y^2 - \alpha + \epsilon)P_4 + \frac{1}{3}(\alpha - \epsilon)Y + \frac{\beta}{4P_4} + \frac{4c_2}{3}y^3, \]

where \( P_4 = P_4(Y); Y = -\sqrt[8]{2c_2h^3}y. \)

III. Potentials with no confining (harmonic oscillator) term: 8 cases occur involving \( P_1, P_2, P_3 \) or elliptic functions.

For confining potentials the potentials involve \( P_4 \) and \( P_5 \). \((P_6 \) appears in the case of separation in polar coordinates.\)

\[ Q^1_3 : \]

\[ V(x, y) = \frac{a}{x^2} + \frac{\hbar^2}{2}(\sqrt{2}P_4^2 + \frac{3}{4}\alpha(P_4)^2 + \frac{\delta}{4P_4^2} + \frac{\beta P_4}{2y} + \frac{\gamma}{2yP_3} - \frac{P_4^2}{2yP_3} + \frac{P_3^2}{4P_3^2}). \]

\[ Y_L = \{L_3, p_2^2\}, \]

\[ g_1(x, y) = 2y^3W' + 2yW, \]

\[ g_2(x, y) = -6xyW' - 2xW, \]

\[ g_3(x, y) = 4x^2W' + 2a\frac{y^2}{x^2}, \]

\[ l(x, y) = h^2x^2(\frac{1}{y}W^{(4)} + W^{(3)}) - x^2(3yW' + W)W'' - \frac{3}{2}h^2yW'' + (4\frac{a}{x^2}y^2 - 3h^2)W' + 4\frac{a}{x^2}yW. \]

\[ W(y) = -\frac{\hbar^2}{2y}(\frac{1}{4}(P_4^2 - 1) - \frac{1}{2}\alpha y^2 P_3^2 - \frac{1}{8}(\beta + 2\sqrt{\alpha})yP_3 + \frac{\gamma}{8P_3} - \frac{\delta}{16P_3^2}y^2) + \frac{\hbar^2}{8y}. \]

The potentials \( Q^1_2, Q^0_3 \) and \( Q^3_3 \) are in the list of quantum potentials obtained by Gravel \(29\) respectively \( Q^{18}, Q_{19}, Q_{21} \). Among the integrals of motion we have \( \{L_3, p_2^2\} \) and \( \{L_3, p_3^2\} \). These can not be obtained by commuting a third and a second order integral.

For a complete list of exotic potentials of the form \( V(x, y) = V_1(x) + V_2(y) \) with fourth order integrals we refer to the original article \(51\).

The results can be summed up as follows:

(i) For \( N = 4 \) one of the two \( V_a( a=1,2) \) must be standard, i.e. satisfy a linear ODE.

We choose \( V_2(y) \) to be exotic.
(ii) The exotic part satisfies a nonlinear ODE that not only passes the Painlevé test but actually has the Painlevé property. Moreover $V_2(y)$ can always be expressed in terms of either elliptic functions or one of the original Painlevé-Gambier transcendents $P_1,...,P_5$. The sixth transcendent does not occur.

(iii) The exotic potentials may have a nonexotic part that makes them confining. For $N = 4$ this occurs in one of 3 versions

\[
V(x, y) = a(x^2 + y^2 + \frac{b}{x^2}) + \frac{c}{y^2} + V_E(y)
\]

\[
V(x, y) = a(x^2 + 4y^2) + V_E(y)
\]

\[
V(x, y) = a(x^2 + y^2) + V_E(y)
\]

where $V_E$ is expressed in terms of $P_4$ or $P_5$. The nonexotic parts in other cases are nonconfining like

\[
V = \frac{a}{x} + V_E(y), V = ax + V_E(y)
\]

with $V_E(y)$ expressed in terms of $P_1$, $P_2$, $P_3$ or an elliptic function. We expect the confining potentials to correspond to a bound spectrum in quantum mechanics.

6 Example of Schrödinger equation with Painlevé potential

Let us consider the example of an exotic potential expressed in terms of $P_4$

The Hamiltonian and two integrals of motion in this case are

\[
H = \frac{\hbar}{2} [p^2 + p^2 + \omega^2(x^2 + y^2)] + V(x)
\]

\[
A = p^2 - p^2 + \omega^2(x^2 - y^2) + V(x)
\]

\[
B = \frac{1}{2} \{L_3, P^2_1\} + \frac{1}{2} \{\frac{\omega^2}{2} x^2 y - 3xy - 3y V_E', P_1\} - \frac{1}{\omega^2} \left( \frac{\hbar^2}{4} V'' + (-\omega^2 x^2 - 3V_E)(\omega x + V_E'), P_1 \right) \tag{51}
\]

with

\[
V_E = \epsilon \frac{\hbar \omega}{2} P_{IV} \left( \sqrt{\frac{\omega}{\hbar}} x \right) + \frac{\omega \hbar}{2} P_{IV} \left( \sqrt{\frac{\omega}{\hbar}} x \right)
\]

\[
+ \omega \sqrt{\hbar \omega \epsilon} P_{IV} \left( \sqrt{\frac{\omega}{\hbar}} x \right) + \frac{\hbar \omega}{3} (-\alpha + \epsilon), \quad \epsilon = \pm 1
\]

\[
P_{IV} = P_{IV} \left( \sqrt{\frac{\omega}{\hbar}} x, \alpha, \beta \right) \tag{52}
\]

The integrals of motion form a polynomial (cubic) algebra, satisfying

\[
[A, B] = C \quad [A, C] = 16 \omega^2 \hbar^2 B
\]

\[
[B, C] = -2 \hbar^2 A^3 - 6 \hbar^2 HA^2 + 8 \hbar^2 H^3
\]

\[
+ \frac{\omega^2 \hbar^4}{3} \left( 4 \alpha^2 - 20 - 6 \beta - 8 \alpha \right) A - 8 \omega^2 \hbar^4 H
\]

\[
+ \frac{\hbar^5 \omega^3}{27} \left( -8 \alpha^3 - 24 \alpha - 36 \alpha \beta + 24 \epsilon \alpha^2 + 8 \epsilon + 36 \epsilon \beta \right) \tag{53}
\]
\begin{equation}
K = -16\hbar^2 H^4 + \frac{4\hbar^4 \omega^2}{3} (4\alpha^2 - 8\alpha + 4 - \alpha\beta) H^2 \\
- \frac{4\hbar^5 \omega^{3\beta}}{27} (8\alpha^3 - 24\epsilon\alpha^2 + 24\alpha + 36\alpha\beta - 8\epsilon - 36\epsilon\beta) H \\
- \frac{4\hbar^6 \omega^4}{3} (4\alpha - 8\epsilon\alpha - 8 - 6\beta).
\end{equation}

The algebra has a Casimir operator that is a 4th order polynomial in the Hamiltonian $H$ (with constant coefficients). The representation theory of the algebra \[53\] and its realization in terms of a deformed oscillator algebra is used to calculate the energy spectrum and wave functions of the system. A connection with "higher order supersymmetry" also gives the wave functions. One obtains 3 series of states with energies

\begin{align*}
E_1 &= \hbar \omega \left( p + \frac{\epsilon + 3}{3} - \frac{\alpha}{3} \right), \\
E_2 &= \hbar \omega \left( p - \frac{\epsilon + 6}{6} + \frac{\alpha}{6} + \sqrt{-\beta} \right), \quad \beta < 0 \\
E_3 &= \hbar \omega \left( p - \frac{\epsilon + 6}{6} + \frac{\alpha}{6} - \sqrt{-\beta} \right),
\end{align*}

and 3 "zero modes", all in terms of the Painlevé transcendent $\mathcal{P}_{IV}$.

It has been demonstrated that this construction may not provide the appropriate number of degeneracies via algebraic approaches and these case are associated with parameters of the fourth Painlevé transcendents related to exceptional orthogonal polynomials. The connection has been established via generalized Hermite and Okamoto polynomials \[49\]. Construction involving other set of integrals and their higher order polynomial algebras have been presented. It has been shown how more complicated patterns of finite dimensional unitary representations can provide the degeneracies of these cases \[50\].

7 SUSYQM construction and wavefunctions

The wave functions can be calculated using another approach that is also in essence algebraic. Supersymmetric quantum mechanics has been studied using many approaches and intertwining of differential operators can be traced back to Darboux and Moutard \[37\]. Second order supersymmetric quantum mechanics has been introduced in \[4\] and has been exploited to generate ladder operators of third order \[8,10,55,47,45\]. Let us present a construction using first and second order supersymmetry given by the following intertwining relation

\begin{equation}
H_1 q^\dagger = q^\dagger (H_2 + 2\lambda), \quad H_1 M^\dagger = M^\dagger H_2
\end{equation}

These relations correspond to a third order ladder operator

\begin{equation}
H_1 a^\dagger = a^\dagger (H_1 + 2\lambda),
\end{equation}

where $a^\dagger$ and $a$ are third order operators.
Higher Order Quantum Superintegrability: a new "Painlevé conjecture"

\[ a^\dagger = q^\dagger M, \quad a = M^\dagger q \]  

(58)

similarly

\[ H_2 a^\dagger = a^\dagger (H_2 + 2\lambda) , \]  

(59)

where \( a^\dagger \) and \( a \) are third order operators.

\[ a^\dagger = M q^\dagger, \quad a = q M^\dagger . \]  

(60)

The explicit form is the following

\[
H_i = \frac{P_i^2}{2} + V_i(x) , \\
q^\dagger = \sqrt{\frac{\hbar}{2}} \partial + W_3(x) , \\
q = -\sqrt{\frac{\hbar}{2}} \partial + W_3(x) , \\
M^\dagger = (\sqrt{\frac{\hbar}{2}} \partial + W_1(x)) (\sqrt{\frac{\hbar}{2}} \partial + W_2(x)) , \\
M = (-\sqrt{\frac{\hbar}{2}} \partial + W_2(x)) (-\sqrt{\frac{\hbar}{2}} \partial + W_1(x))
\]  

(61)

The potential \( V_1 \) and \( V_2 \) correspond up to an additive constant the one given by (52) with \( \epsilon = 1 \) and \( \epsilon = -1 \). Moreover, the functions \( W_1, W_2 \) and \( W_3 \) that appear in the intertwining operators (or supercharges) are also expressed in terms of the fourth Painlevé transcendent

\[
W_{1,2} = \sqrt{\frac{\omega}{8}} P_{IV} (\sqrt{\frac{\omega}{\hbar}} x) \pm \sqrt{\frac{\hbar}{2}} P_{IV}' (\sqrt{\frac{\omega}{\hbar}} x) - 2\sqrt{-\beta} \frac{\omega}{\omega} , \\
W_3 = \sqrt{\frac{\omega}{2}} P_{IV}' (\sqrt{\frac{\omega}{\hbar}} x) - \omega \frac{x}{2\hbar}
\]  

(62)

The spectrum is obtained for cases when normalizable zero modes of the annihilation operator exist

\[ a\psi_k^{(0)} = 0. \]

The energy of the zero modes are for \( \epsilon = 1 \) associated with the three solution from cubic algebra.
\[\psi_0^a(x) = e^{\int \sqrt{2} \hbar \psi(x') dx'},\]
\[\psi_0^b(x) = (\sqrt{\frac{2}{\hbar}} W_2(x) - \sqrt{\frac{2}{\hbar}} W_3(x)) e^{-\int \sqrt{2} \hbar \psi(x') dx'},\]
\[\psi_0^c(x) = \frac{4 \sqrt{-\beta}}{\omega} + (\sqrt{\frac{2}{\hbar}} W_2(x) - \sqrt{\frac{2}{\hbar}} W_3(x)) (\sqrt{\frac{2}{\hbar}} W_1(x) + \sqrt{\frac{2}{\hbar}} W_2(x)) e^{-\int \sqrt{2} \hbar \psi(x') dx'} .\]

with the corresponding zero modes for \(\epsilon = -1\)

\[\psi_a^0(x) = \left(\frac{\omega}{\hbar} \left(\alpha - 1\right) - \frac{2 \sqrt{-\beta}}{\omega}\right) + \left(\sqrt{\frac{2}{\hbar}} W_1(x) + \sqrt{\frac{2}{\hbar}} W_2(x)\right),\]
\[\psi_b^0(x) = e^{\int \sqrt{2} \hbar \psi(x') dx'},\]
\[\psi_c^0(x) = \left(\sqrt{\frac{2}{\hbar}} W_1(x) + \sqrt{\frac{2}{\hbar}} W_2(x)\right) e^{-\int \sqrt{2} \hbar \psi(x') dx'} .\]

In both cases \(\epsilon = 1\) and \(\epsilon = -1\) the complete spectrum is recovered by acting with the raising operators. In addition the raising ladder operators also admit zero modes. However due to conflicting asymptotic we can have in total three, two or one infinite sequence of levels. When a potential allows only one infinite sequence of energies, this potential may also possess a singlet state or doublet states

\[a^\dagger \psi(x) = a^- \psi(x) = 0, \quad (a^\dagger)^2 \psi(x) = a^- \psi(x) = 0\]

It has been shown for case with singlet and doublet related to exceptional orthogonal polynomial one can find a new ladder operator of higher order than 3 and generate the same spectrum only with infinite sequences of level (greater number than 3) \([50]\).

8 Conclusion

This review is devoted to superintegrable quantum systems with Hamiltonians of the form \((6)\) with a potential satisfying \((26)\). They allow 2 integrals of motion \(\{X,Y\}\) with \(X\) (of order 2) as in \((27)\) and \(Y\) (of order \(N\)) as in \((26)\). So far the cases \(N = 3, 4\) and 5 have been investigated in detail \([52, 30, 29, 51, 2]\). Some conclusions for general \(N\) can already be drawn. The general situation can be summed up as follows.

1. The commutator \([H,Y]\) is a priori a linear differential operator of order \(N + 1\). The coefficients of all powers must vanish simultaneously. From terms of order \(N + 1\) we deduce
that the terms of order $N$ in $Y$ are contained in the enveloping algebra of the Euclidean Lie algebra $e(2)$. Moreover, all terms in $Y$ have the same parity (after an appropriate symmetrisation), [62].

2. Terms of order $N-1$ in the commutator provide nonlinear determining equations for the potential $V(x, y) = V_1(x) + V_2(y)$. However, for any $N > 2$ a linear compatibility condition must be satisfied. It amounts to linear ODEs for $V_1(x)$ and $V_2(y)$. These may be satisfied trivially (all coefficients equal to zero). Then we obtain “exotic potentials”. If the linear compatibility condition is satisfied nontrivially, we obtain “standard potentials”. So far, for $N < 7$ all standard potentials are expressed in terms of elementary functions and all exotic ones pass the Painlevé test [1]. We conjecture that this is true for all $N$.

3. For a different approach to superintegrable systems in $E_2$ where such systems separating in Cartesian coordinates are obtained from operator algebras in one dimension we refer to [52].

4. For recent results on superintegrable systems in $E_2$ separable in polar coordinates we refer to the original articles [22, 23, 24].

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References

1. M.J. Ablowitz, A. Ramani, and H. Segur. Non-linear evolution equations and ordinary differential-equations of Painlevé type. Lett. al Nuovo Cimento, 23 333, 1978.
2. I. Abouamal, P. Winternitz. Fifth-order superintegrable quantum system separating in Cartesian coordinates. Doubly exotic potentials, J. Math. Phys. 59 022104 2018.
3. A. Andrianov, F. Cannata, M. Ioffe and D. Nishnianidze, Systems with higher-order shape invariance: spectral and algebraic properties. Phys. Lett. A, 266,341-349 (2000).
4. A. Andrianov, M. Ioffe and V.P. Spiridonov, Higher-derivative supersymmetry and the Witten index. Phys. Lett. A174, 273 (1993).
5. A. Ballesteros, A. Enciso, F.J. Herranz, D. Latini, O. Ragnisco, D. Righi. The classical Darboux III oscillator: factorization, Spectrum Generating Algebra and solution to the equations of motion, J. Phys.: Conf. Ser. 670: 012031 (2016)
6. H. De Bie, V.X. Genest, J.-M. Lemay and L. Vinet, A superintegrable model with reflections on $S^{n−1}$ and the higher rank Bannai-Ito algebra. J. Phys. A: Math. Theor. 50 (19) 195202 (2017)
7. D. Bonatsos, C. Daskaloyannis and K. Kokkotas, Quantum algebraic description of quantum superintegrable systems in 2 dimensions. Phys. Rev. A 48(5), R23407-R3410 (1993).
8. F.J. Bureau. Differential equations with fixed critical points. Annali di Mat. pura ed applicata, LXIV:229-364, 1964.
9. F.J. Bureau. Differential equations with fixed critical points. Annali di Mat. pura ed applicata, LXVI:1-116, 1964.
10. J.M. Carballo, D.J. Fernandez C, J. Negro, and L.M. Nieto, Polynomial Heisenberg algebras, J. Phys. A 37, 10349 (2004).
11. J.F. Carinena, F.J. Herranz and M.F. Ranada. Superintegrable systems on 3-dimensional curved spaces: Eisenhart formalism and separability. J. Math. Phys. 58 022701 (2017).
12. E. Celeghini, S. Kuru, J. Negro and M.A. del Olmo. A unified approach to quantum and classical TTW systems based on factorization Ann. Phys. 332 27-37(2013)
13. J. Chazy. Sur les équations différentielles du troisiem ordre et d’ordre supérieur dont l’intégrale générale a ses points critiques fixes, Acta Math. 34:317-385 (1911).
14. R. Conte. The Painlevé Approach to nonlinear Ordinary Differential Equations. The Painlevé property, one century later, 77–180. Springer, New York, 1999.
15. R. Conte and M. Musette. The Painlevé Handbook. Springer, Berlin, 2008.
16. C.M. Cosgrove, Higher-order Painlevé equation in the polynomial class II: Bureau Symbol P1, Studies in applied mathematics, 116 321-413 (2006).
17. C.M. Cosgrove, Higher-order Painlevé equation in the polynomial class I: Bureau Symbol P2, Studies in applied mathematics, 104:1-65 (2000)
18. C.M. Cosgrove, Chazy classes IX–XI of third-order differential equations, Stud. Appl. Math., 104:3:171-228 (2000)
19. C.M. Cosgrove and G. Scoutis, Painlevé classification of a class of differential equations of the second order and second degree. Stud. Appl. Math., 88(1):25–87, 1993.
20. C. Daskaloyannis, quadratic Poisson algebras of two-dimensional classical superintegrable systems and quadratic algebras of quantum superintegrable systems, J. Math. Phys. 42:1100-1119 (2001)
21. A.M. Escobar Ruiz, E.G. Kalnins, W. Miller Jr. and E. Subag, Bocher and Abstract Contractions of 2nd Order Quadratic Algebras, SIGMA 13:013, 38 pages (2017)
22. A.M. Escobar-Ruiz, J.C. Lopez Vieyra, P. Winternitz. Fourth order superintegrable systems separating in Polar Coordinates. I. Exotic Potentials, J. Phys. A 50(49): 495206 (2017).
23. A.M. Escobar-Ruiz, J.C. Lopez Vieyra, P. Winternitz and I. Yurdusen. Fourth order superintegrable systems separating in Polar Coordinates. II. Standard Potentials, arXiv 1804.05751
24. A.M. Escobar-Ruiz, P. Winternitz, I. Yurdusen, General Nth order superintegrable systems separating in polar coordinates, arXiv:1806.06849
25. I. Fris, V. Mandrosov, J. Smorodinsky, M. Uhlíř, and P. Winternitz. On higher symmetries in quantum mechanics. Phys. Lett. 16:354–356 (1965).
26. B. Gambier. Sur les équations différentielles du second ordre et du premier degré dont l’intégrale générale est à points critiques fixes. Acta Mathematica, 33(1):1–55, 1910.
27. V. Genest, L. Vinet and A. Zhedanov, Superintegrability in two dimensions and the Racah-Wilson algebra. Lett. Math. Phys. 104(8):931–952 (2011).
28. Y. Granovskii, I. Lutzenko and A.Z. Zhedanov, Mutual integrability, quadratic algebras and dynamic symmetry. Ann. of Phys., 217(1), 1–20, 1992
29. S. Gravel. Hamiltonians separable in Cartesian coordinates and third-order integrals of motion. J. Math. Phys. 45:1003-19 (2004).
30. S. Gravel and P. Winternitz. Superintegrability with third order integrals in quantum and classical mechanics. J. Math. Phys. 43(12):5902–5912, 2002.
31. A. N. W. Hone. Painlevé Tests, Singularity Structure and Integrability. Integrability 245–277. Springer, Berlin Heidelberg, 2009.
32. M.F. Hoque, Superintegrable systems, polynomial algebra structures and exact derivations of spectra. PhD Thesis, School of Mathematics and Physics, The University of Queensland, Australia, January, 175 pages. arXiv:1802.08410 (2018)
33. M.F. Hoque, I. Marquette and Y.-Z. Zhang, Quadratic algebra structure in the 5D Kepler system with non-central potentials and Yang-Coulomb monopole interaction. Ann. of Phys. 380:121-134 (2017)
34. P. Iliev, Symmetry algebra for the generic superintegrable system on the sphere. J. High Energy Phys. 2, 44 22 pages (2018)
35. Ince E L 1956 Ordinary differential equations Dover, New York, 574p.
36. J. M. Jauch and E. L. Hill, The problem of degeneracy in quantum mechanics. Phys. Rev. 57, 641-645 (1940).
37. G. Junker, Supersymmetric Methods in Quantum and Statistical Physics, Springer, New York, (1995).
38. E.G. Kalnins, Separation of Variables for Riemannian Spaces of Constant Curvature, Addison-Wesley, Reading, Massachusetts (1986) p.196
39. E.G Kalnins, J.M Kress and W. Miller, Separation of Variables and Superintegrability. The symmetry of solvable systems. JOP (2018)
40. G.E. Kalnins, W. Miller Jr, S. Post. Contractions of 2D 2nd Order Quantum Superintegrable Systems and the Askey Scheme for Hypergeometric Orthogonal Polynomials. SIGMA 9:057 28 pages (2013)
41. M.D. Kruskal, and P. A. Clarkson. The Painlevé-Kowalevski and Poly-Painlevé Tests for Integrability. Studies in Applied Mathematics, 86(2):87–165, 1992.
42. Y. Liao, I. Marquette and Y.-Z. Zhang, Quantum superintegrable system with a novel chain structure of quadratic algebras. J. Phys. A: Math. Theor. 51:255201 (13pp) (2018)
43. A. Makarov, J. Smorodinsky, Kh. Valiev, and P. Winternitz. A systematic search for non-relativistic systems with dynamical symmetries. Nuovo Cimento A, 52:1061–1084 (1967)
44. A. Marchesini, L. Šnobl and P. Winternitz. Three-dimensional superintegrable systems in a static electromagnetic field. J. Phys. A 48(39):395206, 2015. (24 pages).
45. I. Marquette, An infinite family of superintegrable systems from higher order ladder operators and supersymmetry, J. Phys. Conf. Ser. 264:012047, 2011
46. I. Marquette, “Superintegrability with third order integrals of motion, cubic algebras, and supersymmetric quantum mechanics. I. Rational function potentials.” J. Math. Phys. 50, 012101 (2009).
47. I. Marquette, “Superintegrability with third order integrals of motion, cubic algebras, and supersymmetric quantum mechanics. II. Painlevé transcendent potentials,” J. Math. Phys. 50, 095202 (2009).
Higher Order Quantum Superintegrability: a new "Painlevé conjecture"