y-DEFORMED BPS Dp- BRANES ON A SURFACE IN A CALABI-YAU
THREEFOLD

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ABSTRACT

Using y-deformed algebraic geometric techniques the y-deformed Mukay vector of RR-charges of the y-deformed BPS Dp-branes localized on a surface in a Calabi-Yau threefold. The formulae that are obtained here are generalizations of the formulae of the fourth section of the preprint hep-th/0007243.

1 Introduction: y-deformed BPS Dp-branes on a Calabi-Yau threefold

A BPS D-brane on a Calabi-Yau threefold X can be represented using a coherent $O_X$-module G. The RR charge of G is given by the Mukai vector:

$$ v_X(G) = ch(G) \sqrt{Todd(T_X) \in H_{2*}(X; Q)} := \oplus_{i=0}^{3} H_{2i}(X; Q) $$

where $ch(G) = \sum_{i=0}^{3} ch_i(G)$ is the Chern character with $ch_i(G) \in H_{6-2i}(X; Q)$, which can be computed by the homology-cohomology duality [1]: always one can to have a resolution of G by locally free sheaves $(V_*)$, in such way that one can to set that $ch(G) := \sum_{i=0}^{3} (-1)^i ch(V_i)$, and these result does not depend on the choice of the resolution. Finally $Todd(T_X) = [X] + \frac{c_1[X]}{2} + \frac{c_2[X] + c_1^2[X]}{12} + \frac{c_2[X]c_1[X]}{24}$.

Now when X is a Calabi-Yau threefold one has $c_1[X] = 0$ and then one obtains:

$$ Todd(T_X) = [X] + \frac{c_2[X]}{24} $$

From these the effect of the square root of the Todd Class on the RR charges is to say the geometric version of the Witten effect is given by:

$$ \sqrt{Todd(T_X)} = [X] + \frac{c_2[X]}{24} $$

For the investigation of the topological aspects of D-branes is of the great importance to obtain several basic invariants of BPS D-Branes. One of these invariants is the RR charge of the D-brane. Other invariant is the intersection
form on D-branes on \( X \) \[1\]. This invariant for intersections of two Dp-branes is obtained by multiplication of the Mukay vectors of RR charges corresponding to the intersecting Dp-branes and is given by: \[1\]

\[
I_X(G_1, G_2) = [v_X(G_1)^\vee \cdot v_X(G_2)]_X = \\
[(ch(G_1)\sqrt{Todd(T_X)})^\vee .ch(G_2)\sqrt{Todd(T_X)}]_X = \\
[ch(G_1)^\vee .ch(G_2)Todd(T_X)]_X
\]

where \([...)_X\] evaluates the degree of \( H_0(X;Q)\otimes Q \) component, and \( v^\vee \) flips the sign of \( H_0(X)\oplus H_4(X) \) part of the Mukay vector \( v \). In particular, if \( G \) itself is locally free, then \( ch(G)^\vee = ch(G^\vee) \), where \( G^\vee = Hom_X(G,0_X) \) is the dual sheaf. Finally it is easy to check that: \( I_X(G_1, G_2) = -I_X(G_2, G_1) \). On other hand the invariant of intersection between D-branes is an application of the Hirzebruch-Riemann-Roch and for then you can write\[1\]

\[
I_X(G_1, G_2) = \sum_{i=0}^{3}(-1)^i dim Ext^i_X(G_1, G_2)
\]

For this reason the skew-symmetric property \( I_X(G_1, G_2) = -I_X(G_2, G_1) \) of the intersection form \( I_X \) for the intersection of two Dp-branes may be attributed to the Serre duality: \( Ext^i_X(G_1, G_2)\cong Ext^{3-i}_X(G_1, G_2)^\vee \) \[1\]. Another interesting commentary is that from the integrality theorems for differential and complex manifolds the formula H.R.R. is an integer and this assures that \( I_X \) takes values in \( Z \). \[1\],[2].

Now the result that this work presents is about the \( y \)-deformed Dp-branes on a Calabi Yau threefold. A \( y \)-deformed BPS Dp-brane on a Calabi-yau \( X \) can be represented by a \( y \)-deformed \( O_X - modulo G \). The \( y \)-deformed RR charge of \( G \) is given by the \( y \)-deformed Mukai vector:

\[
v_{X,y}(G) = ch_y(G)\sqrt{X_y(T_X)}\in(H_{2\ast}(X;Q)\otimes Q[y]) := \\
\oplus_{i=0}^{3}(H_{2i}(X;Q)\otimes Q[y])
\]
where $\chi_y$ is the $y$-chi-genus which is a generalization of the Todd class [2,3] and $ch_y(G)$ is the $y$-deformed Chern Character. the total Chern Class for $T_X$ has the following summarization:

$$c(T_X) = \sum_{j=0}^{3} c_j(T_X)$$

also, the total Chern Class for the such bundle has the following factorization:

$$c(T_X) = \prod_{i=1}^{3} (1 + x_i)$$

The CHI-$y$- genus for $T_X$ has the following formal factorisation:

$$\chi_y(T_X) = \prod_{i=1}^{3} \frac{(1+y\exp(-(y+1)x_i))x_i}{1-\exp(-(y+1)x_i)}$$

The CHI-$y$- genus for $T_X$ has the following formal summarisation in terms of the $y$-deformed Todd polynomials which are formed from the corresponding Chern classes and from the polynomials on $y$:

$$\chi_y(T_X) = \sum_{j=0}^{\infty} T_j(c_1(T_X), ..., c_j(T_X), y)$$

The $y$-Todd polynomials are given by:

$$T_0(c_0(T_X), y) = T_0(1, y) = 1$$

$$T_1(c_1(T_X), y) = \frac{(1-y)c_1(T_X)}{2}$$

$$T_2(c_1(T_X), c_2(T_X), y) = \frac{(y+1)^2c_1(T_X)^2+(y^2-10y+1)c_2(T_X)}{12}$$

$$T_3(c_1(T_X), c_2(T_X), c_3(T_X), y) = \frac{-(y+1)^2(y-1)c_1(T_X)c_2(T_X)+12y(y-1)c_3(T_X)}{24}$$

Then one has:

$$\chi_y(T_X) = 1 + \frac{(1-y)c_1(T_X)}{2} + \frac{(y+1)^2c_1(T_X)^2+(y^2-10y+1)c_2(T_X)}{12} + \frac{-(y+1)^2(y-1)c_1(T_X)c_2(T_X)+12y(y-1)c_3(T_X)}{24}$$
When $X$ is a Calabi-Yau threefold then the chi-y-genus is given by

$$\chi_y(T_X) = 1 + \frac{(y^2-10y+1)c_2(T_X)}{12} + \frac{12y(y-1)c_3(T_X)}{24}$$

From this one can to write the following formula for the y-deformed geometric version of the Witten effect:

$$\sqrt{\chi_y(T_X)} = [X] + \frac{(y^2-10y+1)c_2[X]}{24} + \frac{y(y-1)c_3[X]}{4}$$

when $y=0$ one obtains the usual Witten effect:

$$\sqrt{\chi_0(T_X)} = [X] + \frac{0^2-0+1)c_2[X]}{24} + \frac{0(0-1)c_3[X]}{4} = [X] + \frac{c_2[X]}{24}$$

For the other hand the y-deformed Chern Character $ch_y(G)$ is given by:

$$ch_y(G) = \sum_{i=0}^{3} ch_{i,y}(G)$$

which can be computed using y-deformed homology-cohomology duality: always one can to have a y-deformed resolution of $G$ by y-deformed locally free sheaves $(V_i)$, in such way that one can to set that $ch_y(G) := \sum_{i=0}^{3} (-1)^i ch_y(V_i)$, and these result does not depend on the choice of the y-deformed resolution. The total Chern Class for $G$ has the following summarization:

$$c(G) = \sum_{j=0}^{q} c_j(G)$$

also, the total Chern Class for $G$ has the following factorization:

$$c(G) = \prod_{i=1}^{q}(1 + z_i)$$

The total Chern character of $G$ is defined by:

$$ch(G) = \sum_{j=1}^{q} e^{z_i}$$

The total y-deformed Chern character for $G$ has the following summarization:
\[ \text{ch}_y(G) = \sum_{j=1}^{r} e^{(1+y)z_j} \]

The total \( y \)-deformed Chern character for \( G \) has the following expansion in terms of the Chern class of \( G \) and polynomials for \( y \):

\[ \text{ch}_y(G) = rk(G) + (y + 1)c_1(G) + (y + 1)^2\left(\frac{c_2(G)^2 - c_2(G)}{2}\right) + (y + 1)^3\left(\frac{c_3(G)^3 - 3c_1(G)c_2(G) + 3c_3(G)}{6}\right) \]

It is easy to see that when \( y = 0 \), one obtains the usual expansion for the usual Chern character. For the investigation of the topological aspects of the \( y \)-deformed D-branes is of the great importance to obtain several basic \( y \)-deformed invariants of \( y \)-deformed BPS D-Branes. One of these \( y \)-deformed invariants is the \( y \)-deformed RR charge of the \( y \)-deformed D-brane. Other \( y \)-deformed invariant is the \( y \)-deformed intersection form on \( y \)-deformed D-branes on \( X \). This \( y \)-deformed invariant for intersections of two \( y \)-deformed \( Dp \)-branes is obtained by multiplication of the \( y \)-deformed Mukay vectors of the \( y \)-deformed RR charges corresponding to the intersecting \( y \)-deformed \( Dp \)-branes and is given by:

\[ I_{X,y}(G_1, G_2) = [v_{X,y}(G_1)^v. v_{X,y}(G_2)]_X = \]

\[ [(\text{ch}(G_1)^\sqrt{\text{ch}_y(T_X)})^v. \text{ch}(G_2)^\sqrt{\text{ch}_y(T_X)}]_X = [\text{ch}(G_1)^v. \text{ch}(G_2)^\text{ch}_y(T_X)]_X \]

where \([...]\) \( X,y \) evaluates the degree of \((H_0(X;Q)\otimes Q[y])\otimes(Q\otimes Q[y])\) component, and \( v^v \) flips the sign of \((H_0(X)\otimes Q[y])\oplus(H_4(X)\otimes Q[y]))\) \( y \)-deformed part of the \( y \)-deformed Mukay vector \( v \). In particular, if \( G \) itself is locally free, then \( \text{ch}_y(G)^v = \text{ch}_y(G^v) \), where \( G^v = \text{Hom}_X(G, 0_X) \) is the \( y \)-deformed dual sheaf. Finally, it is easy to check that: \( I_{X,y}(G_1, G_2) = -I_{X,y}(G_2, G_1) \).

On the other hand the \( y \)-deformed invariant of intersection between \( y \)-deformed D-branes is an application of the \( y \)-deformed Hirzebruch-Riemann-Roch and for then you can write:
\[ I_{X,y}(G_1, G_2) = \sum_{i=0}^{3} (-1)^i \dim Ext^i_{X,y}(G_1, G_2) \]

For this reason the skew-symmetric property \( I_{X,y}(G_1, G_2) = -I_{X,y}(G_2, G_1) \) of the intersection form \( I_{X,y} \) for the intersection of two \( y \)-deformed Dp-branes may be attributed to the \( y \)-deformed Serre duality: \( Ext^i_{X,y}(G_1, G_2) \cong Ext^{3-i}_{X,y}(G_1, G_2)^\vee \).

Another interesting commentary is that from the \( y \)-deformed integrality theorems for differential and complex manifolds the \( y \)-deformed formula H.R.R. is an polynomial on \( y \) and this assures that \( I_{X,y} \) takes values in \( \mathbb{Q}[y] \).

Now let \( J_{X,y} \in (H_4(X; R) \otimes R[y]) \) be a \( y \)-deformed Kahler form on \( X \), which is here identified with an \( y \)-deformed \( R \)-extended ample divisor. The \( y \)-deformed classical expression of the \( y \)-deformed central charge of the \( y \)-deformed D-brane \( G \) is then given by [1]:

\[
Z_{J_{X,y}}(G) = -[e^{-J_{X,y}} \cdot v_{X,y}(G)]_X = -\sum_{k=0}^{3} \left( -\frac{1}{k!} \right) [J_{X,y}^k \cdot v_{X,y,k}(G)]_X
\]

where \( v_{X,y,k} \) is the \( H_{2k}(X) \otimes \mathbb{Q}[y] \) component of \( v_{X,y} \in (H_{2*}(X; \mathbb{Q}) \otimes \mathbb{Q}[y]) \).

In such way we obtain the three \( y \)-deformed invariants: \( y \)-deformed RR charge, \( y \)-deformed central charge and \( y \)-deformed intersections pairings of two \( y \)-deformed BPS Dp-branes. With this aid of some algebraic geometry-topology techniques we can to begin the study of topological aspects of \( y \)-deformed BPS Dp-branes bounded on a projective algebraic surface in a Calabi-Yau threefold \( X \).

## 2 \( y \)-deformed BPS Dp-branes localized on a surface in a Calabi-Yau threefold

Let \( f \) be an embedding of a projective algebraic surface \( S \) in a Calabi-Yau threefold \( X \). In the limit of infinite elliptic fiber, the \( y \)-deformed BPS Dp-branes for which the \( y \)-deformed central charge remains finite are those \( y \)-deformed BPS Dp-branes which are confined to the algebraic surface \( S \). The physical and topological properties of the \( y \)-deformed BPS D-p-branes localized on the
algebraic surface $S$ then depend on not on the details of the global model $X$, but only on the intrinsic $y$-deformed geometry of $S$ and its $y$-deformed normal bundle $N_{S,y} = N_{S|X,y}$ which is isomorphic to the $y$-deformed canonical line bundle $K_{S,y}$. In particular, this means that we can compute the $y$-deformed central charges of $y$-deformed BPS D-p-branes using $y$-deformed local mirror symmetry principle on $S$.

In an elementary physical configuration you have a $y$-deformed BPS D$p$-brane sticking to $S$. Such $y$-deformed D-brane sticking to $S$ can be described mathematically by a $y$-deformed $O_S - moduleE$. For this configuration an important $y$-deformed topological invariant is the $y$-deformed Euler number of $E$ (the Euler $y$-polynomial for $E$) which is defined by $\chi_y(E) = \sum_{i=0}^{2}(-1)^i h^i(S, E, y)$, where $h^i(S, E, y) = dim(H^i(S, E))_y$. For to obtain the $y$-deformed Euler number of $E$ or the Euler polynomial of $E$ the first thing that one needs is the $y$-deformed Todd class of $S$ or $\chi_y$ class of $S$:

$$\chi_y(T_S) = [S] + \frac{(1-y)c_1(S)}{2} + \frac{(y+1)^2 c_1(S)^2 + (y^2 - 10y + 1)c_2(S)}{12}$$

this expansion can be written as:

$$\chi_y(T_S) = [S] + \frac{(1-y)c_1(S)}{2} + \chi_y(O_S)[pt]$$

where:

$$\chi_y(O_S) = \frac{(y+1)^2 c_1(S)^2 + (y^2 - 10y + 1)c_2(S)}{12}$$

The second thing for to do is to apply the $y$-deformed H.R.R formula, and then one get:

$$\chi_y(E) = [ch_y(E)\chi_y(T_S)]_S = [ch_y(E)[[S] + \frac{(1-y)c_1(S)}{2} + \chi_y(O_S)]_S =$$

$$[(rk(E) + (y + 1)c_1(E) + (y + 1)^2 (\frac{c_1(E)^2 - c_2(E)}{2}))[[S] + \frac{(1-y)c_1(S)}{2} +$$
\[
\chi_y(O_S)|_S = \]

\[
rk(E)\chi_y(O_S) + [(y + 1)^2(\frac{c_1(E)^2 - c_2(E)}{2})] + \frac{(y+1)(1-y)c_1(S)\cdot c_1(E)}{2}
\]

From the other side, there is \(y\)-deformed canonical push-forward homomorphism \(f_*\) from \(H_2(S; Q)\otimes Q[y]\) to \(H_2(X; Q)\otimes Q[y]\), which maps a \(y\)-deformed cycle on \(S\) that on \(X\). Also, one can define the \(y\)-deformed coherent sheaf \(f_!E\) on \(X\) by extending \(E\) by zero to \(X/S\). Now using the \(y\)-deformation of the celebrated Grothendieck-Riemman-Roch formula for the embedding of \(S\) in \(X\), one can to relate the \(y\)-deformed chern characters of \(E\) and \(f_!E\) as follows:

\[
ch_y(f_!E) = f_*(ch_y(E)\frac{1}{chi_y(N_S)})
\]

Multiplying the both sides of the \(y\)-deformed GRR formula by \(\sqrt{\chi_y(T_X)}\), one has:

\[
ch_y(f_!E)\sqrt{\chi_y(T_X)} = f_*(ch_y(E)\sqrt{\frac{\chi_y(T_S)}{chi_y(N_S)}})
\]

where we have used the \(y\)-deformed projection formula:

\[
f_*(a\cdot f^*b) = f_*a\cdot b
\]

with \(a\in(H_2(S; Q)\otimes Q[y])\), \(b\in(H_2(X; Q)\otimes Q[y])\)

and \(f^*chi_y(T_X) = chi_y(T_S)\cdot chi_y(N_S)\), which follows from the \(y\)-deformed short exact sequence of bundles on \(S\):

\[
0 \longrightarrow T_S \longrightarrow f^*T_X \longrightarrow N_S \longrightarrow 0
\]

combined with the multiplicative property of the chi-\(y\)-genus.

Now the \(y\)-deformed BPS Dp-brane on a Calabi-Yau threefold \(X\) is represented by \(G\) and \(y\)-deformed BPS Dp-brane sticking to \(S\) can be described by \(E\) then one has \(G = f_!E\) and following formula for the \(y\)-deformed Mukai vector of the \(y\)-deformed RR charges of \(G = f_!E\)
\[ v_{X,y}(f_1 E) = ch_y(f_1 E) \sqrt{\chi_y(T_X)} \in (H_{2*}(X; Q) \otimes Q[y]) := \bigoplus_{i=0}^{3} (H_{2i}(X; Q) \otimes Q[y]) \]

The you have:

\[ v_{X,y}(f_1 E) = f_* (ch_y(E) \sqrt{\chi_y(N_C)}) = f_*(v_{S,y}(E)) \]

In such way the y-deformed RR charge of the y-deformed BPS Dp-brane represented by E on S regarded as a y-deformed BPS Dp-brane on X can written in the following intrinsic description (of the y-deformed RR charge on S):

\[ v_{S,y}(E) = ch_y(E) \sqrt{\frac{\chi_y(T_S)}{\chi_y(N_S)}} = ch_y(E) \sqrt{\frac{\chi_y(T_S)}{\chi_y(K_S)}} \]

The y-deformed gravitational correction factor for S admits the following expansion:

\[ \sqrt{\frac{\chi_y(T_S)}{\chi_y(K_S)}} = [S] + \frac{(1-y)c_1(S)}{2} + \frac{(-10y+1+y^2)c_2(S)+3(y-1)^2c_1(S)}{24} \in (H_{2*}(S; Q) \otimes Q[y]) \]

As a simple exercise one can to compute the y-deformed RR charge of a y-deformed sheaf on S. For this let i: C\rightarrow S be an embedding of a smooth genus g algebraic curve in S with the normal bundle \( N_C = N_{C\rightarrow S} \). Then from a lin bundle \( L_C \) on C, one obtains a y-deformed torsion sheaf \( i_!L_C \) on S and \( ch_y(i_!L_C) \) can be computed from the y-deformed G.R.R. formula:

\[ ch_y(i_!L_C) = i_*(ch_y(L_C) \frac{1}{\chi_y(N_C)}) = i_*((rk(L_C) + (y+1)c_1(L_C)(1 + \frac{(y-1)c_1(N_C)}{2})) = i_*[C] + ((y+1)c_1(L_C) + \frac{(y-1)c_1(N_C)}{2})[pt] = i_*[C] + ((y+1)deg(L_C) + \frac{(y-1)deg(N_C)}{2})[pt] \]
where \( \text{deg}(L) := [c_1(L)]_C \) for a line bundle on \( C \). Then \( y \)-deformed RR charge of the \( y \)-deformed BPS Dp-brane bounded on \( S \) represented by the \( y \)-deformed \( O_S \) module \( i_! L_C \) can be computed as follows:

\[
\nu_{S,y}(i_! L_C) = \text{ch}_y(i_! L_C) \sqrt{\frac{\text{ch}_y(T_C)}{\text{ch}_y(K_C)}} = (i_* [C] + ((y + 1) \text{deg}(L_C) + \frac{(y-1) \text{deg}(N_C)}{2})[pt])[(C) + (1-y)c_1(C)] = (i_* [C] + ((y + 1) \text{deg}(L_C) + (1-y)c_1(C))[pt]) \in \oplus(H_0(S) \otimes \mathbb{Q}[y])
\]

I now turn again to intersection pairings of the \( y \)-deformed BPS Dp-branes one has the question about what is the most appropriate intersection for \( y \)-deformed D-branes on \( S \). Here we will describe only \( y \)-deformed candidate. The \( y \)-deformed candidate uses the intrinsic \( y \)-deformed Mukay vector \( \nu_{S,y} \) and defines a \( y \)-deformed symmetric form:

\[
I_{S,y}(E_1, E_2) = -[\nu_{S,y}(E_1) \langle . \nu_{S,y}(E_2) \rangle]_S = \frac{r_1 r_2 (y^2 - 10y + 1) \chi(S)}{12} + [r_1 \chi_2(E_2) + r_2 \chi_2(E_1) - c_1(E_1) \cdot c_1(E_2)]_S
\]

where \( \chi(E) = r[E] + c_1(E) + \chi_2(E), \chi(S) = [c_2(S)]_S \) is the Euler number, and \( \nu_y = -v_{0,y} + v_{1,y} - v_{2,y} \) with \( v_{i,y} \) being the \( y \)-deformed \( H_2(S) \otimes \mathbb{Q}[y] \) component of the \( y \)-deformed vector \( \nu_y \).

In contrast with \( I_X \) that have values in \( \mathbb{Q}[y] \) and when \( y = 0 \) then takes values in \( \mathbb{Z} \), now \( I_S \) also have values in \( \mathbb{Q}[y] \) but in this case when \( y = 0 \) \( I_S \) is not \( \mathbb{Z} \)-valued in general.

### 3 References

[1] hep-th/0007243

[2] F. Hirzebruch, Topological Methods in Algebraic Geometry, 1978