CONTINUOUS SINGULARITIES IN HAMILTONIAN RELATIVE EQUILIBRIA WITH ABELIAN MOMENTUM ISOTROPY

Dedicated to Professor James Montaldi

MIGUEL RODRÍGUEZ-OLMOS

Abstract. We survey several aspects of the qualitative dynamics around Hamiltonian relative equilibria. We pay special attention to the role of continuous singularities and its effect in their stability, persistence and bifurcations. Our approach is semi-global using extensively the Hamiltonian tube of Marle, Guillemin and Sternberg.

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1. Introduction. This article presents some contributions to the study of the local dynamics of symmetric Hamiltonian systems near singular relative equilibria. Most parts of this research have been strongly influenced by or carried out in collaboration with J. Montaldi along many conversations and professional collaborations spanning the last 15 years. This article surveys the most important results of [24] in a slightly restricted but far more accessible exposition for the non-expert.

The main object of investigation is the effect of continuous singularities of a Lie group action in the qualitative dynamics of symmetric Hamiltonian relative equilibria. By this we mean the stability, persistence and bifurcations, which are strongly influenced by the existence of singularities in the group action at these solutions. We will follow an approach based on the systematic use of the Marle-Guillemin-Sternberg tube for a Hamiltonian group action on a symplectic manifold, which we will use as a model for a symmetric Hamiltonian system. It turns out that for these systems, the differential equations of the Hamiltonian flow have an
expression especially useful for the qualitative analysis of the relative equilibria and the local dynamics near them.

In Section 2 we collect several standard facts about symmetric Hamiltonian systems and their relative equilibria, together with the precise definitions of stability, persistence and bifurcations that will be used in this work. In Section 3 we develop the local model for the flow of a symmetric Hamiltonian system using the semi-global coordinates provided by the Marle-Guillemin-Sternberg Hamiltonian tube. In Section 4 we provide sufficient conditions for the nonlinear stability of relative equilibria with continuous singularities and in Section 5 we provide two basic results about the persistence and bifurcations of relative equilibria.

Throughout this article, with the exception of the review contained in Section 2 which is completely general, we restrict our study to the case when the relative equilibria under consideration have Abelian momentum isotropy groups. Although all our results carry on for more general cases with some extra hypotheses, this would make the exposition more technically demanding and substantially longer without providing a better insight into the questions addressed. We refer the reader to [24] for more details in the general situation.

We end with two sections including brief reviews of the literature on Hamiltonian relative equilibria as well as on J. Montaldi’s contributions to the field.

2. Hamiltonian relative equilibria. A $G$-Hamiltonian system consists in a 5-tuple $(\mathcal{P}, \omega, G, J, h)$ where $(\mathcal{P}, \omega)$ is a symplectic manifold, $G$ a Lie group acting on $\mathcal{P}$ properly and by symplectomorphisms in a strongly Hamiltonian fashion with equivariant momentum map $J : \mathcal{P} \to g^*$ and $h$ a smooth $G$-invariant function on $\mathcal{P}$. Here $g^*$ is the dual of the Lie algebra of $G$ on which the momentum map takes values in a non-exhaustive way in case the $G$-action on $\mathcal{P}$ is not locally free, the main situation of interest in this article. The momentum map is not uniquely defined and can be any $G$-equivariant map, with respect to the coadjoint representation in the target, satisfying

$$\iota_{\xi_\mathcal{P}} \omega = d\langle J(\cdot), \xi \rangle, \quad \forall \xi \in g$$

where $\xi_\mathcal{P} \in \mathfrak{X}(\mathcal{P})$ is the fundamental vector field associated to the Lie algebra element $\xi$ corresponding to the $G$-action on $\mathcal{P}$. It turns out that two momentum maps for the same action must differ on a locally constant function on $\mathcal{P}$.

Hamilton’s equations can be written in geometric form as

$$\iota_{X_h} \omega = dh,$$

which uniquely define the vector field $X_h \in \mathfrak{X}(\mathcal{P})$ due to the non-degeneracy of $\omega$. In addition, the flow of $X_h$ is equivariant and therefore sends $G$-orbits on $\mathcal{P}$ into $G$-orbits. We will now define our main object of study.

Definition 2.1. Let $(\mathcal{P}, \omega, G, J, h)$ be a $G$-Hamiltonian system. A point $z \in \mathcal{P}$ is called a relative equilibrium if its orbit under the flow of $X_h$ is contained in the group orbit $G \cdot z$.

Notice that from the previous definition, it follows that a relative equilibrium corresponds to a fixed point of the reduced dynamics on the (generally not smooth) quotient space $\mathcal{P}/G$, to which the flow of $X_h$ descends to a well defined continuous dynamical system.

Relative equilibria have received a lot of attention since they correspond to important solutions of physical Hamiltonian systems, like the circular solutions in Kepler’s problem or steady state solutions in fluid dynamics. From a geometric
point of view they are interesting since they are regarded as “organizing centres” for the dynamics of the flow of $X_h$, which cannot be in general be obtained analytically. In the study of the qualitative dynamics of a $G$-Hamiltonian system one is typically interested in finding all the relative equilibria as well as the local dynamics around them, which gives an approximation of the global dynamics on the whole manifold. Once solved the problem of the existence of relative equilibria, the most common questions about the local dynamics around them have to do with stability, persistence and bifurcations of the relative equilibria set.

In the theory of relative equilibria, the concept of isotropy is a key element. The different actions of $G$ on $\mathcal{P}$, $\mathfrak{g}$ and $\mathfrak{g}^*$ are central to the geometric properties of the dynamics. For the convenience of the non-expert reader we collect in the next definition a few standard concepts that will also fix the terminology used in the rest of this article.

**Definition 2.2.** Let a group $G$ act smoothly on a manifold $\mathcal{P}$ and let its left adjoint action on $\mathfrak{g}$ and left coadjoint action on $\mathfrak{g}^*$ be denoted by

$$\xi \mapsto \text{Ad}_g \xi \quad \text{and} \quad \mu \mapsto \text{Ad}_{g^{-1}}^* \mu$$

for all $g \in G$, $\xi \in \mathfrak{g}$, $\mu \in \mathfrak{g}^*$.

Then

(i) If $z$ is a point in $\mathcal{P}$ its isotropy group (or stabilizer) is the subgroup of $G$ defined as

$$G_z = \{ g \in G : g \cdot z = z \}.$$ Analogous definitions hold for the isotropy groups of elements in $\mathfrak{g}$ and $\mathfrak{g}^*$, denoted by $G_\xi$ and $G_{\mu}$ respectively. Their corresponding Lie algebras will be denoted by $\mathfrak{g}_z$, $\mathfrak{g}_\xi$ and $\mathfrak{g}_\mu$.

(ii) We say that the $G$-action at $z$ is singular if $G_z$ is not trivial. We say that $z$ has continuous isotropy if $G_z$ is not trivial.

The properness assumption for the $G$-action on $\mathcal{P}$ that we are using throughout this article is an important technicality that has several ramifications relevant to this work. Most importantly, it guarantees that its isotropy groups are all compact, even if $G$ is not. In particular, compact groups act properly. These facts are standard (see for instance [3]). The main reason for which we are interested in compact isotropy groups is because they admit invariant metrics, a tool used extensively in many of the proofs related to the construction of the Marle-Guillemin-Sternberg form that will be introduced in Section 3.

In the realm of $G$-Hamiltonian systems, there is an important relationship between the several isotropy groups involved that we collect in the following lemma for the convenience of the reader.

**Lemma 2.1.** Let $(\mathcal{P}, \omega, G, J, h)$ be a $G$-Hamiltonian system and $z \in \mathcal{P}$ with momentum $\mu = J(z)$. Then $G_z \subset G_{\mu}$.

**Proof.** Let $g \in G_z$. By equivariance of $J$ we have

$$\text{Ad}_{g^{-1}}^* \mu = \text{Ad}_{g^{-1}}^* (J(z)) = J(g \cdot z) = J(z) = \mu,$$

and therefore $g \in G_{\mu}$. \qed

**Example 2.1.** Let $G$ be $\text{SO}(3)$ as a matrix group acting linearly on $\mathcal{P} = \mathbb{R}^3$ by rotations in the usual way. This action is singular at every point. There are (up to conjugations) two different isotropy groups for this action. If $z = \mathbf{0}$, Then $G_z = \text{SO}(3)$, since by linearity the origin is fixed by any matrix. If $z = \mathbf{x} \neq \mathbf{0}$
then \( G_z = \text{SO}(2)_x \), the subgroup of rotations around \( x \). The Lie algebras for these two groups are isomorphic to \( \mathbb{R}^3 \) and \( \mathbb{R} \) respectively. Therefore, any point at \( P \) has continuous isotropy for this action.

The following characterizations of relative equilibria are standard and can be found for instance in [14].

**Proposition 2.1.** Let \((P, \omega, G, J, h)\) be a \( G \)-Hamiltonian system and \( z \in P \) with momentum \( J(z) = \mu \). The following are equivalent:

(i) \( z \) is a relative equilibrium.

(iii) There is an element \( \xi \in \mathfrak{g}_\mu \) such that \( X_h(z) = e^{t\xi} \cdot z \), where \( \tilde{z}(t) \) is the integral curve of \( X_h \) with \( \tilde{z}(0) = z \).

(v) There is an element \( \xi \in \mathfrak{g}_\mu \) such that \( z \) is a critical point of the augmented Hamiltonian

\[
h_\xi := h - J_\xi,
\]

with \( J_\xi(z) := \langle J(z), \xi \rangle \).

Additionally, if \( z \) is a relative equilibrium with velocity \( \xi \) then the relative equilibrium \( g \cdot z \) has velocity \( \text{Ad}_g \xi \).

In Proposition 2.1 the element \( \xi \) is called a velocity for the relative equilibrium \( z \). This terminology is borrowed from classical mechanics, since if the Lie group is \( \text{SO}(3) \) this element is actually the angular velocity, which is constant for a steady state solution. Notice that in the case of non-free group actions, the main object of interest in this article, the velocity \( \xi \) is not unique. To see this, let \( z \) be a relative equilibrium with velocity \( \xi \). Then the element \( \xi + \eta \), where \( \eta \) is any element in \( \mathfrak{g}_z \), is also a velocity for \( z \). This follows easily from (iii) since

\[
(\xi + \eta)_P(z) = \xi_P(z) + \eta_P(z) = \xi_P(z).
\]

This non-uniqueness of the velocity of a relative equilibrium in the presence of non-free group actions is one of the most important features of relative equilibria with continuous singularities.

Although there are several (and not equivalent) definitions of nonlinear stability for relative equilibria we will consider the widely adopted notion of \( G_\mu \)-stability, which was shown to be the natural one in this setting in [34].

**Definition 2.3.** Let \( z \in P \) be a relative equilibrium with momentum \( J(z) = \mu \). We say \( z \) is nonlinearly stable (or \( G_\mu \)-stable) if for every \( G_\mu \)-invariant neighbourhood \( U \) of \( z \) there is an open neighbourhood \( O \) of \( z \) contained in \( U \) such that the integral curve of \( X_h \) through any point in \( O \) is contained in \( U \) for all time.

Many times relative equilibria come in families, typically parametrized by the momentum. Following the approach of regarding relative equilibria as organizing centres of the dynamics it is important to detect such families, as well as points where more than one of these families meet. We will divide this task into studying persistence and bifurcations of relative equilibria. We start by stating our working definition of persistence.

**Definition 2.4.** A relative equilibrium \( z \) is said to persist, if for every \( G \)-invariant neighbourhood \( U \) containing \( z \), the set \( U \setminus G \cdot z \) contains a relative equilibrium.
The problem of bifurcations is different in nature, since we are interested in knowing when a parametrized branch of relative equilibria bifurcates from another existing branch, at a given point of the latter one. Although the theory of Hamiltonian bifurcations of relative equilibria is not as well developed as for general dynamical systems we will, for the purposes of this article, state the problem in the following way.

**Definition 2.5.** Let $W$ be a neighbourhood of $0$ in some vector space and $z : W \to \mathcal{P}$ an injective parametrization with $z(w)$ being a relative equilibrium for each $w \in W$. We will assume that

$$G_{z(w)} = g G_{z(0)} g^{-1}$$

for every $w$ and some $g \in G$. We say that the family $z(w)$ bifurcates at $z(0)$ if there exist another (possibly not parametrized) family $A$ of relative equilibria satisfying $z(0) \in A$ and

$$G_{z'} \subsetneq g G_{z(0)} g^{-1}$$

for every $z' \in A \setminus \{z(0)\}$ and any $g \in G$.

The conditions on the stabilizers of the primary and bifurcating branches can be understood as follows: The fact that the stabilizers of all the points in the primary branch are all conjugated is usually interpreted as that all those points are the same type of relative equilibrium. This is the case for instance for the Lagrange top, in which there is a family of rotating relative (sleeping tops) equilibria parametrized by angular momentum in which the symmetry axis of the top and the gravity axis are aligned. For the bifurcating branch, the condition on its points having strictly smaller stabilizers therefore indicates that their relative equilibria are genuinely different from the primary ones, resulting in a true bifurcation. Going back to the Lagrange top, it is a well known fact that all stable sleeping equilibria are bifurcation points for families of less symmetric relative equilibria which both precess around the gravity axis and rotate around the symmetry axis of the top, see [10] for a comprehensive treatment of the Lagrange top.

### 3. The MGS form and the local structure of Hamilton’s equations.

Since we are interested in the local properties of a Hamiltonian flow near a relative equilibrium, we will substitute the phase space $\mathcal{P}$ by the tubular neighbourhood given by the Marle-Guillemin-Sternberg (MGS) model. Originally due to Marle [13] and Guillemin and Sternberg [7], it is now standard and details can be found for instance in [32]. We briefly recall its construction. For the remainder of this article we will assume that $G_\mu$ is Abelian in order to keep the exposition more clear. However, most of the results presented here are valid under more general hypotheses. See [24] for details.

Let $z \in \mathcal{P}$ be an arbitrary point with momentum $\mathbf{J}(z) = \mu$. Let $\mathfrak{g}_z$ and $\mathfrak{g}_\mu$ be the Lie algebras of the stabilizers of $z$ and $\mu$ as before. Notice that $G_z$ is compact by the properness of the $G$-action on $\mathcal{P}$ and also that $G_z \subseteq G_\mu$ by the equivariance of $\mathbf{J}$ (see Lemma 2.1). Therefore we can find a $G_z$-invariant (with respect to the restriction of the coadjoint representation of $G_\mu$ to $G_z$) splitting

$$\mathfrak{g}_\mu = \mathfrak{m} \oplus \mathfrak{g}_z,$$

with associated dual invariant splitting $\mathfrak{g}_\mu^* = \mathfrak{m}^* \oplus \mathfrak{g}_z^*$. For each of these splittings we will denote by $\mathbb{P}_I$ the projection onto the factor $I$. 


Next, choose $N$ to be a complement to $\mathfrak{g}_H \cdot z$ in $\ker T_z \mathcal{J}$, $G_z$-invariant with respect to the induced linear representation of $G_z$ on $T_z \mathcal{P}$. This is always possible due to the compactness of $G_z$. Here we use the notation

$$I \cdot z = \{ \eta \in T_z \mathcal{P} : \eta \in I = \text{Lie}(L) \} \subseteq T_z \mathcal{P}$$

where $L \subseteq G$.

It is a standard fact in symplectic geometry that $(N, \Omega)$ is a symplectic vector space, where $\Omega$ is the restriction to $N$ of $\omega(z)$, and the linear action of $G_z$ on $N$ is Hamiltonian, with associated equivariant momentum map $\mathcal{J}_N : N \to \mathfrak{g}_H^*$ defined by

$$\langle \mathcal{J}_N(v), \eta \rangle = \frac{1}{2} \Omega(\eta \cdot v, v), \quad \forall \eta \in \mathfrak{g}_H^*, v \in N. \quad (2)$$

The space $N$ is usually called the symplectic normal space at $z$ for the $G$-action on $\mathcal{P}$.

The product space $G \times \mathfrak{m}^* \times N$ supports free actions of both $G$ and $G_z$, given by

$$g' \cdot (g, \rho, v) = (g' g, \rho, v) \quad g' \in G$$

$$h \cdot (g, \rho, v) = (gh^{-1}, \rho, h \cdot v) \quad h \in G_z$$

for all $g \in G$, $\rho \in \mathfrak{m}^*$ and $v \in N$. It is clear that these actions are free and commute. Consider the principal bundle associated to the $G_z$-action

$$\pi : G \times \mathfrak{m}^* \times N \rightarrow G \times_{G_z} (\mathfrak{m}^* \times N)$$

$$(g, \rho, v) \mapsto [g, \rho, v],$$

where $[g, \rho, v]$ is the equivalence class of points consisting of the orbit $G_z \cdot (g, \rho, v) \subset G \times \mathfrak{m}^* \times N$. The $G$-action (3) on $G \times \mathfrak{m}^* \times N$ descends to a smooth action on $G \times_{G_z} (\mathfrak{m}^* \times N)$ given by

$$g' \cdot [g, \rho, v] = [g' g, \rho, v] \quad g' \in G. \quad (3)$$

It follows from the MGS construction that there is an open neighbourhood $Y \subset G \times_{G_z} (\mathfrak{m}^* \times N)$ of $[e, 0, 0]$ on which it is possible to define a local symplectic form $\omega_Y$ as well as a local $G$-equivariant symplectomorphism

$$\varphi : Y \subset G \times_{G_z} (\mathfrak{m}^* \times N) \rightarrow \mathcal{P}$$

onto an open $G$-invariant neighbourhood of the orbit $G \cdot z \subset \mathcal{P}$, and $\varphi$ satisfies $\varphi([e, 0, 0]) = z$.

In addition, the induced $G$-action on $(Y, \omega_Y)$ given by (3) is Hamiltonian with momentum map $\mathcal{J}_Y$ satisfying $\mathcal{J}_Y = \mathcal{J} \circ \varphi : Y \rightarrow \mathfrak{g}_H^*$. Therefore $\mathcal{J}_Y$ provides a normal form for $\mathcal{J}$ and its expression is given by (see [32])

$$\mathcal{J}_Y([g, \rho, v]) = \text{Ad}^*_{g^{-1}}(\mu + \rho + \mathcal{J}_N(v)). \quad (4)$$

Because of the symplectic and equivariant nature of $\varphi$, the quintuple $(Y, \omega_Y, G, \mathcal{J}_Y, \varphi^* h)$ is a local model for the $G$-Hamiltonian system $(\mathcal{P}, \omega, G, \mathcal{J}, h)$ in a neighbourhood of the group orbit $G \cdot z$. Let $X_{\varphi^* h}$ denote the corresponding symmetric Hamiltonian vector field. The main idea used in in [36] and [37] in order to apply the MGS form to the analysis of the local dynamics of $G$-Hamiltonian systems near relative equilibria consists in studying the flow of a lift of $X_{\varphi^* h}$ on $G \times \mathfrak{m}^* \times N$.

The following result is the starting point for the local study of relative equilibria for Hamiltonian flows in $G$-Hamiltonian systems. See [24] for a proof.
Lemma 3.1. Let $(Y, \omega_Y, G, J_Y, \varphi^* h)$ be a local model for the $G$-Hamiltonian system $(\mathcal{P}, \omega, G, J, h)$ near $G \cdot z$. The most general lift to $G \times m^* \times N$ of the flow of $X_{\varphi^* h}$ on $Y$ is given by the set of equations

\begin{align}
\dot{g} &= g \cdot (D_m \bar{h}(\rho, v) + \eta) \\
\dot{\rho} &= 0 \\
\dot{v} &= \Omega^2(D_N \bar{h}(\rho, v) - dJ^\eta_N(v)).
\end{align}

where $\eta \in \mathfrak{g}_z$ is arbitrary and the function $J^\eta_N$ is defined by $J^\eta_N(v) := \langle J_N(v), \eta \rangle$ for all $\eta \in \mathfrak{g}_z$ and $v \in N$. The function $\bar{h} \in C^\infty(m^* \times N)$ is defined by

$$\bar{h}(\rho, v) = \bar{h}(\rho, g \cdot v) = \tilde{h}(\rho, v)$$

for any $g \in G_z$. Notice that we can rewrite equation (7) as

$$\dot{v} = \Omega^2(D_N \bar{h}_\eta(\rho, v))$$

where $\bar{h}_\eta(\rho, v) = \bar{h}(\rho, v) - J^\eta_N(v)$.

Therefore the local dynamics of $X_{\bar{h}}$ near $G \cdot z$ can be reconstructed from a $\mathfrak{g}_z$-parametrized system of coupled differential equations on $G \times m^* \times N$ to which we refer as the bundle equations. The precise way to obtain concrete results about the qualitative dynamical properties of a relative equilibrium $z \in \mathcal{P}$ by studying equations (5), (6) and (7) will be the main theme of the remainder of the article. Notice the particular simple expression of equation (6). This is a consequence of the assumption of $G_\mu$ being Abelian that we are using in this article. If this condition is not satisfied, the corresponding equation is given by (2.11) in [24]. The fact that under our assumption (6) is uncoupled from the remaining two equations, although not completely general, greatly simplifies the proofs of the main results in stability, persistence and bifurcations of Hamiltonian relative equilibria, and at the same time is general enough to apply in most important cases of interest in geometric mechanics.

Also, in applications to geometric mechanics, it is difficult to find $G$-Hamiltonian systems of interest for which relative equilibria which are not fixed equilibria have non-Abelian $G_\mu$, and therefore this assumption is not a critical one. On the other hand, for many systems of interest, including all simple mechanical systems (with Hamiltonian function of the form kinetic + potential energy) the momentum value for any fixed equilibria is zero, meaning that if we regard fixed equilibria as relative equilibria with trivial velocity, our assumption would not hold unless $G$ itself is Abelian.

The following is a main technical result in order to transfer properties of the dynamics on $G \times m^* \times N$ to $\mathcal{P}$ (see [24] for a proof).

Lemma 3.2. Consider a point $z \in \mathcal{P}$.

(i) Let $\rho \in m^*$ satisfying $\varphi([e, \rho, 0]) = z$ and $\eta' \in \mathfrak{g}_z$. Then

$$D_N \bar{h}_{\eta'}(\rho, 0) = 0 \quad \text{if and only if} \quad d\bar{h}_{\xi}(z) = 0,$$

where $\xi = D_m \bar{h}(\rho, 0) + \eta'$. Furthermore, if $D_N \bar{h}_{\eta'}(\rho, 0) = 0$, then

$$D^2_{N} \bar{h}_{\eta'}(\rho, 0) = d^2(\varphi^* h)_{\xi}([e, 0, 0])_{N},$$

where $D^2_N \bar{h}_{\eta'}(\rho, 0)$ is the Hessian of the function $\bar{h}_{\eta'}(\rho, \cdot) \in C^\infty(N)$ at $0 \in N$. 
(ii) Let $z \in \mathcal{P}$ be a critical point of $h_\xi$ for some $\xi \in \mathfrak{g}$. Suppose that $\mathfrak{d}^2 h_\xi(z)|_N$ is non-degenerate, degenerate, positive definite or negative definite. Then the corresponding property is satisfied for any other choice of symplectic normal space $N'$.

(iii) The same conclusions hold if in (ii) we replace $N$ by $N^K$ and $N'$ by $(N')^K$ respectively, for any compact subgroup $K \subseteq G_z$.

In Lemma 3.2 $N^K$ denotes the $K$-fixed point space, defined as

$$N^K = \{ v \in N : k \cdot v = v \quad \text{for all } k \in K \}.$$ 

Notice from (i) and the pullback property of Hessians that the information about the degeneracy and definiteness of $\mathfrak{d}^2 h(z)$ restricted to $N$ or any of their fixed point subspaces can be obtained from $D^2_N h_\eta(\rho, 0)$ and its corresponding restrictions.

4. $G_\mu$-stability. We will start with a necessary technical result giving a characterization of relative equilibria in the local model $Y$.

**Proposition 4.1.** A point $[g, \rho, v]$ near $z = [e, 0, 0] \in Y$ is a relative equilibrium for $X_{\phi^\tau}$ if and only if there exists an element $\eta \in \mathfrak{g}_z$ such that

$$D_N h_\eta(\rho, v) = 0. \quad (8)$$

The element of $\mathfrak{g}$ given by $\text{Ad}_g(D_{m^*} h(\rho, v) + \eta)$ is a velocity for the relative equilibrium $[g, \rho, v]$.

**Proof.** Let $[g, \rho, v]$ be in the domain $Y$ of the diffeomorphism $\phi$. First, since relative equilibria come in $G$-orbits and regarding the form of the $G$-action on $Y$ given by (3) it follows that $[g, \rho, v]$ is a relative equilibrium if and only if $[e, \rho, v]$ also is, so we can restrict the study to points of this form. Let $[e, \rho, v](t)$ be the integral curve of $X_{\phi^\tau}$ with initial condition $[e, \rho, v](0) = [e, \rho, v]$. Therefore, we can find representative integral curves of equations (5), (6) and (7) of the form $g(t), \rho, v(t)$ with initial conditions $g(0) = e$ and $v(0) = v$, such that $[e, \rho, v](t) = [g(t), \rho, v(t)]$. According to Proposition 2.1, $[e, \rho, v]$ is a relative equilibrium for the local $G$-Hamiltonian system on $G \times G_\xi (m^* \times N)$ corresponding to the Hamiltonian vector field $X_{\phi^\tau}$, with velocity $\xi \in \mathfrak{g}$ if and only if

$$\frac{d}{dt}|_0 [e, \rho, v](t) = e^{i\xi} \cdot [e, \rho, v].$$

This is equivalent, in view of (3) to

$$\dot{\xi}(0) = \xi + \eta'$$
$$\dot{\rho}(0) = 0$$
$$\dot{v}(0) = -\eta' \cdot v.$$

for some $\eta' \in \mathfrak{g}_z$. Since in the equations (5),(6),(7), $\eta$ is arbitrary, we can absorb $\eta'$ into $\eta$ so that the relative equilibrium conditions are

$$\xi = D_{m^*} h(\rho, v) + \eta$$
$$0 = D_N h_\eta(\rho, v)$$

Therefore we have obtained that $[e, \rho, v]$ is a relative equilibrium on $G \times G_\xi (m^* \times N)$ near $[e, 0, 0]$ with velocity $\xi$ if and only if there exists $\eta \in \mathfrak{g}_z$ such that $\xi = D_{m^*} h(\rho, v) + \eta$ and $D_N h_\eta(\rho, v) = 0$. And by the final remark in Proposition 2.1, this is equivalent to $[g, \rho, v]$ being a relative equilibrium with velocity $\text{Ad}_g(D_{m^*} h(\rho, v) + \eta)$.
Our main stability result is contained in the following theorem, which gives a sufficient condition for the nonlinear stability of a relative equilibrium in the sense of Definition 2.3.

**Theorem 4.1.** Let \((P, \omega, G, J, h)\) be a \(G\)-Hamiltonian system and \(z \in P\) a relative equilibrium with momentum \(J(z) = \mu\). Suppose that \(z\) admits a velocity \(\xi \in g_\mu\) such that \(\mathbf{d}^2 h_\xi(z)|_N\) is definite, where \(N\) is any \(G_z\)-invariant complement to \(g_\mu\). Then \(z\) is nonlinearly stable.

**Proof.** By making \(\rho = 0\) in Lemma 3.2 we have that \(\xi = \xi^\perp + \eta'\) according to the splitting (1), with \(\xi^\perp = D_{m^*} h(0,0)\). And we also have \(D_N h_{\eta'}(0,0) = 0\) and \(D_N^2 h_{\eta'}(0,0) = 0\). Because of the non-degeneracy of \(\Omega\), it follows from equations (6) and (7) that \((0,0) \in m^* \times N\) is Lyapunov stable. Also, since \(D_{m^*} h(\rho, v) + \eta' \in g_\mu\) for all \((\rho, v) \in m^* \times N\), it follows from (5) that \(g(t) \in g_0 G_\mu\) for any initial condition \(g(0) = g_0\) and all \(t\).

Notice that by the symplectic \(G\)-equivariant nature of \(\varphi\), the nonlinear stability of \(z\) is equivalent to the nonlinear stability of \([e, 0, 0] \in Y\) for the flow of \(X_{\varphi^* h}\). In order to prove that this is indeed the case under the above conditions, let \(O_G \subset G\) be an open neighbourhood of \(e\). Any \(G_\mu\)-invariant neighbourhood of \([e, 0, 0] \in Y\) is given by \(U = \{G_\mu \cdot [g, \rho, v]: g \in U_G, \rho \in U_{m^*}, v \in U_N\} \subset G \times G_z (m^* \times N)\) where \(U_G \subset G, U_{m^*} \subset m^*\) and \(U_N \subset N\) are neighbourhoods of \(e, 0\) and \(0\) respectively. It follows from the above discussion, and from the Lyapunov stability of \((0,0)\) in \(m^* \times N\) that we can open neighbourhoods \(O_{m^*} \subset U_{m^*}\) and \(O_N \subset U_N\) of the origins such that \(\rho(t) \in U_{m^*}\) and \(v(t) \in U_N\) for all \(t\) if \(\rho(0) \in O_{m^*}\) and \(v(0) \in O_N\).

Therefore, calling \(O = \{[g, \rho, v]: g \in U_G, \rho \in O_{m^*}, v \in O_N\} \subset G \times G_z (m^* \times N)\), we have that the integral curves of \(X_{\varphi^* h}\) through points in \(O\) always lie inside \(U\), proving the nonlinear stability of \([e, 0, 0] \in Y\). It follows from Lemma 3.3(ii) that this is independent of the choice of symplectic normal space \(N\).

Notice that in the statement of Theorem 4.1 we require the existence of some velocity \(\xi \in g_\mu\) for which the restricted Hessian under consideration is definite. Depending on the properties of \(G_\mu\) and \(G_z\) it could very well be the case that this is achieved only for some choices of velocities and not for others. This is especially relevant when testing for the nonlinear stability of a parametrized family of relative equilibria, since depending on the particular choice of the chosen velocity we may not detect the stability of some elements of the branch that could be stable. Therefore in these problems it is important to test for all possible velocities in order to obtain optimal results. This process has been illustrated in [31]. Also notice that in general we cannot prove or disprove the stability of relative equilibria when the hypotheses of Theorem 4.1 are not satisfied. In these cases one needs to carry out a case by case study, typically using the linearization of the Hamiltonian flow or a normal form analysis. These issues will not be addressed in this article.

### 5. Local persistence and bifurcations.

In this section we study the problem of persistence and bifurcations of Hamiltonian relative equilibria in the sense of Definitions 2.4 and 2.5. Although similar results to the ones contained here can be obtained under more general conditions (and also more restrictive hypotheses), the fact that we are only concerned here with Abelian \(G_\mu\) will allow us to simplify both the exposition and the proofs. Again, for a more in depth investigation of these and other related dynamical properties on the persistence and bifurcations of...
relative equilibria we refer the reader to [24]. We now state our main result on the persistence in the presence of a non-degeneracy condition.

**Theorem 5.1.** Let \( z \in \mathcal{P} \) be a relative equilibrium with momentum \( \mathbf{J}(z) = \mu \), and let \( \xi \in \mathfrak{g}_\mu \) be a velocity for \( z \). Write \( \xi = \xi^L + \eta \in \mathfrak{m} \oplus \mathfrak{g}_Z \) according to the splitting (1). Suppose there exists a subgroup \( K \subseteq G_z \) such that

\[
\mathbf{d}^2 h_\xi(z)_{|N^K}
\]

is non-degenerate. Then, there is a smooth map

\[
\tilde{z} : \mathfrak{m}^* \times \mathfrak{g}_Z \longrightarrow \mathcal{P}^K
\]

\( (\rho, \eta') \longmapsto \tilde{z}(\rho, \eta') \)

defined in a neighbourhood of \((0,0)\) such that \( \tilde{z}(0,0) = z \), and for each \( \eta' \) the map \( \rho \mapsto \tilde{z}(\rho, \eta') \) is an immersion, such that for each \((\rho, \eta')\) in the domain, the point \( \tilde{z}(\rho, \eta') \) is a relative equilibrium with velocity of the form \( \xi + \eta' \in \mathfrak{g}_\mu \) and stabilizer \( G_{\tilde{z}(\rho, \eta')} \) satisfying

\[
K \subseteq G_{\tilde{z}(\rho, \eta')} \subseteq G_z
\]

for some \( g \) in \( G \).

**Proof.** Since \( z \) is a relative equilibrium with velocity \( \xi \in \mathfrak{g}_\mu \), we can write \( \xi = \xi^L + \eta \) according to the splitting (1). Moreover, we have that if we write \( f(\eta', \rho, v) = h_{\eta + \eta'}(\rho, v) \) the hypotheses imply that \( D_N f(0, 0, 0) = 0 \) and \( D^2_N f(0, 0, 0) \) is non-degenerate. By calling \( f^K \) the restriction of \( f \) on its third argument to \( N^K \), and by the pullback property of Hessians, we have that the above non-degeneracy condition is equivalent to \( D^2_{N^K} f^K(0, 0, 0) \) is non-degenerate (notice that simply by restricting the directional derivatives of \( f \) we also have \( D_{N^K} 0, 0, 0) = 0 \). Therefore, by the Implicit Function Theorem we have that there is a unique map \( v : \mathfrak{m} \times \mathfrak{g}_Z \times N^K \rightarrow N^K \) satisfying \( D_{N^K} f^K(\eta', \rho, v(\rho, \eta')) = 0 \). By the Principle of Symmetric Criticality (see [33]) this is equivalent to \( D_N f(\eta', \rho, v(\rho, \eta')) = D_N h_{\eta + \eta'}(\rho, v(\rho, \eta')) = 0 \). Therefore (8) is satisfied for the isotropy Lie algebra element \( \eta + \eta' \) and the pair \((\rho, v(\rho, \eta'))\).

Using the local model given by \( Y \), we therefore put \( \tilde{z}(\rho, \eta') = \varphi([e, v(\rho, \eta')]) \), which for each \( \eta' \) is an immersion. By the properties of \( \varphi \), \( \tilde{z}(\rho, \eta') \) is a continuous family of relative equilibria with velocities \( D_{N^K} h(\rho, v(\rho, \eta')) + \eta + \eta' = \xi + \eta' \) and satisfying \( \tilde{z}(0, 0) = z \).

According to the equivariant nature of \( \varphi \) it follows that

\[
G_{\tilde{z}(\rho, \eta')} = G_{[e, v(\rho, \eta')]} = (G_z)_\rho \cap (G_z)_v = (G_z)_v \subseteq G_z
\]

and also, since \( \tilde{z}(\rho, \eta') \in \mathcal{P}^K \) we have that \( K \subseteq G_{\tilde{z}(\rho, \eta')} \).

Our last result is concerned with the bifurcation of parametrized branches of relative equilibria as in Definition 2.5

**Theorem 5.2.** Let \( z \in \mathcal{P} \) be a relative equilibrium with momentum \( \mathbf{J}(z) = \mu \) and velocity \( \xi \in \mathfrak{g}_\mu \) written \( \xi = \xi^L + \eta \in \mathfrak{m} \oplus \mathfrak{g}_Z \) according to (1). Let \( W \subset \mathfrak{m}^* \times \mathfrak{g}_Z \) be a vector subspace and \( \tilde{z} : W \rightarrow \mathcal{P} \), a parametrized branch of relative equilibria satisfying \( \tilde{z}(0,0) = z \), \( \mathbf{J}(\tilde{z}(\rho, \eta')) = \mu + \rho \), \( G_{\tilde{z}(\rho, \eta')} = G_\mu \) and \( G_{\tilde{z}(\rho, \eta')} = G_z \). Let \( \xi(\rho, \eta') \) be a family of velocities for points of this branch chosen such that \( \mathbb{P}_{G_z}(\xi(\eta, \eta')) = \eta + \eta' \) (this is always possible). Now suppose that there exists a subgroup \( L \subseteq G_z \) satisfying

(i) There is one eigenvalue \( \sigma(\rho, \eta') \) with \((\rho, \eta') \in W \), of \( \mathbf{d}^2 h_{\xi(\rho, \eta')}(\tilde{z}(\rho, \eta')) \) that crosses 0 at \( 0 \in W \), and
(ii) The eigenvector associated to $\sigma(0,0)$ spans $N := \ker d^2h_\xi(z)_{|_{\lambda L}}$.

Then, for every $v \in N$ close enough to the origin, there is a relative equilibrium $\tilde{z}_v$ near $z$ with velocity $\xi_v \in g_\mu$ close to $\xi$. The stabilizer of $\tilde{z}_v$ is

$$L \subseteq G_{\tilde{z}_v} = (G_z)_v.$$  

Proof. The given branch $\tilde{z}(\rho, \eta')$ can be written as

$$\tilde{z}(\rho, \eta) = \varphi([g(\rho, \eta'), \bar{\rho}(\rho, \eta'), v(\rho, \eta')]),$$

with $[g(0,0), \bar{\rho}(0,0), v(0,0)] = [e,0,0]$, for $(\rho, \eta') \in W \subseteq \mathfrak{m}^* \times g_z$. Since relative equilibria come in group orbits, we can choose in our analysis $g(\rho, \eta') = e$ without loss of generality.

Since points in the branch satisfy (according to (4))

$$J(\tilde{z}(\rho, \eta')) = \mu + \bar{\rho}(\rho, \eta') + J_N(v(\rho, \eta')) = \mu + \rho$$

it follows that

$$\bar{\rho}(\rho, \eta') = \rho \quad \text{and} \quad G_{\mu+\rho} = G_\mu.$$  

And also $J_N(v(\rho, \eta')) = 0$ which implies, due to the quadratic form of $J_N$ (2), that $v(\rho, \eta') \in N^{G_z}$.

Let $\varphi_{\rho,\eta'} : N \to N$ be the $W$-parametrized family of diffeomorphisms of $N$ given by

$$\varphi_{\rho,\eta'}(v) = v + v(\rho, \eta').$$

These maps are obviously $G_z$-equivariant and therefore $K$-equivariant. Notice also that $J_N(\varphi_{\rho,\eta'}(v)) = J_N(v)$ since $v(\rho, \eta') \in N^{G_z}$.

Notice that $d\hat{h}_{\xi(\rho,\eta')}(\tilde{z}(\rho, \eta')) = 0$ if and only if $d\hat{h}_{\eta+\eta'}(\rho, v(\rho, \eta')) = 0$ which, putting $\hat{h}'_{\eta+\eta'}(\rho, v) = \hat{h}_{\eta+\eta'}(\rho, \varphi_{\rho,\eta'}(v))$, is in turn equivalent to

$$D_N\hat{h}'_{\eta+\eta'}(\rho, v) = 0.$$  

Let us call $f(\eta', \rho, v) = \tilde{h}'_{\eta+\eta'}(\rho, v)$. The function $f \in C^\infty(W \times N)$ is by construction $G_z$-invariant (notice that since $G_\mu$ and therefore $G_z$ is Abelian, the $G_z$-action on $W$ is trivial).

We have according to Lemma 3.2 (using $\tilde{h}'$ as the function $\tilde{h}$ in the statement of the lemma) that, for every $(\rho, \eta') \in W$,

$$D_N f(\eta', \rho, 0) = 0,$$

and that for any $L \subseteq G_z$, we also have

$$\ker D_N^2 f(0,0,0)_{|_{N^L}} = \ker D_N^2 \tilde{h}_{\eta}(0,0)_{|_{N^L}} = \ker d^2(\varphi^*\tilde{h})_{|_{\lambda L}}.$$  

We have that $f$ satisfies the hypotheses of Lemma 2.10 in [24] with $K = G_z$. Therefore we can conclude that for each $v \in \ker D_N^2 \tilde{h}_{\eta}(0,0)_{|_{N^L}} \setminus \{0\}$ there is a pair $(\rho(v), \eta'(v)) \in W$ satisfying

$$D_N \hat{h}'_{\eta+\eta'(v)}(\rho(v), v) = 0.$$  

This shows that the point $[e, \rho(v), v]$ satisfies (8). Therefore for each $v \in \ker N \setminus \{0\}$ close enough to the origin we have that $\tilde{z}(v) = \varphi([e, \rho(v), v])$ is a relative equilibrium with velocity $D_m \tilde{h}(\rho(v), v) + \eta + \eta'(v) \in g_\mu$. In addition, we have $G_{\tilde{z}(v)} = (G_z)_v \cap (G_z)_v = (G_z)_v$. Since $v \in N^L$ we also have $L \subseteq G_{\tilde{z}(v)}$.  

$\square$
6. A historical account of Hamiltonian relative equilibria in geometric mechanics. This section presents a quick historical survey of the geometric theory of qualitative dynamics for relative equilibria of Hamiltonian systems with symmetry. Hopefully, it will help the reader that is not familiar with geometric mechanics to see how the work presented in this article fits in the larger picture of the historical development of the subject from a geometric viewpoint. This survey is necessarily incomplete and biased, since it is based on the research interests, particular tastes and scientific trajectory of the author. In the next section we will present, in the same free vein, an account of the influential work of professor Montaldi in these topics.

Relative equilibria in Hamiltonian systems with symmetry is a vast topic, which in one form or another can probably be traced back to the XVIIth and XVIIIth centuries in the pioneering works of Newton, Euler, Lagrange and others. Some celebrated examples include the identification of the equilibrium solutions of rotating homogenous fluid masses [30] and the central configurations of the 3-body problem [4] and [8]. In the context of geometric mechanics, some of the seminal works on Hamiltonian relative equilibria and their relationship with the geometry of the momentum map are due to Smale [39] and Abraham and Marsden [1]. A key aspect to take into account is that the geometric theory of relative equilibria since those works has been closely related to the development of symplectic reduction, introduced in its more standard form in the work of Marsden and Weinstein [15]. It is to be noticed that in this influential reference, Hamiltonian systems with symmetry and in particular relative equilibria play a prominent role in 4 of its 7 sections. This includes geometric criteria for their existence and some results on their stability, that will be a reference point for the work developed in the subsequent decades.

Another aspect in which the theory of symplectic reduction has greatly influenced the study of Hamiltonian relative equilibria is in regarding them as fixed equilibria points for the reduced dynamics. Indeed, one of the main contributions of [15] is to show that a $G$-Hamiltonian system descends to a Hamiltonian system on the symplectic quotient, the reduced space. Since the reduced space is obtained by quotienting the symmetries of the dynamical system and the manifold that plays the role of the phase space, by Definition 2.1 it turns out that relative equilibria are nothing but fixed points of the reduced dynamics. This fact has been elevated to the principle that in a $G$-Hamiltonian system all the relevant dynamical information is that of the reduced Hamiltonian system on the reduced space. From a topological viewpoint, these reduced space provide richer examples of symplectic manifolds, and a lot of work has been done on their study. From the dynamical viewpoint however, the more complicated geometric nature of the reduced spaces imply that the actual computations involved in the study of the qualitative dynamics is more involved. This last observation is responsible for the development of criteria for the existence and stability of relative equilibria that are equivalent to those for the existence or Lyapounov stability of their corresponding reduced equilibria but that are computable directly on the original, unreduced phase space. Therefore the existence criterion (v) in Proposition 2.1, which is equivalent to the one in [39] is nothing but the condition for the reduced Hamiltonian vector field being zero on the reduced phase space at the projected point. Also, the second variation stability criterion contained in Theorem 6 of [15] is equivalent to the Lyapounov stability of the projected point on the reduced phase space.
A significant turn from this idea happened with the important work of Patrick [34], where the actual meaning on the original phase space of the Lyapounov stability on the reduced space was investigated, resulting in the central concept of $G_\mu$-stability, or more generally stability modulo a subgroup (see Definition 2.3). From this point, it was realized that a full theory of relative equilibria could be developed entirely within the original phase space geometry, without needing any reference to the reduced space. It follows that this has been an extremely important approach, as proved by the emerging of singular symplectic reduction by the same time. The reason for this importance is that, as elegant as a geometric formulation of relative equilibria on reduced spaces is, in many dynamical and geometric interesting situations these reduced spaces are not smooth, generally having structure of orbifolds or stratified spaces. This is a serious problem for the at the time prevailing methods using traditional techniques of differential geometry on reduced spaces, since these spaces are no longer manifolds. Because of this, the first results on the stability of relative equilibria for non-free group actions [9] and [31] extend to the singular case sufficient conditions for $G_\mu$-stability. In particular, [31] is probably the first place where the isotropy Lie algebra and the non-uniqueness of a velocity is systematically used in order to obtain “optimal” stability tests.

As we have seen throughout this article, the notion of persistence and bifurcations is specially relevant in $G$-Hamiltonian systems with symmetry in the case of singular actions. Even more in the case of solutions with continuous isotropy. This has been extensively investigated since the late 90’s, and some references are [9] and [2], although there are many others along these lines.

Before concluding this short survey, it is important to acknowledge two more important aspects of the theory: The first one is a very important particular family of $G$-Hamiltonian systems, the so called simple mechanical systems. These are Hamiltonian systems for which the phase space is the cotangent bundle $T^*Q$ of a smooth manifold $Q$ usually called the configuration space. Points in the base correspond to positions and elements of the fibers are momenta. The Hamiltonian is constructed purely in terms of $Q$, from a function $V$ (potential energy) and a Riemannian metric $K$ (kinetic energy). Both $V$ and $K$ are invariant under the action of a Lie group acting on $Q$, and by natural lifts on $T^*Q$, therefore producing a complete $G$-Hamiltonian system from the data $(Q, K, V, G)$. It is natural then to try to keep all the study of symmetric simple mechanical systems and their relative equilibria purely in terms of these foundational objects, and this strategy has produced the reduced energy-momentum method [38], which is an adaptation of the stability criterion of [15] to this family of systems, as well as its implications to the persistence and bifurcations of relative equilibria, as for instance [10] and [11].

The second one is contained in the two foundational articles [36] and [37], and is the main motivation for the results and approach used in the present article. The authors develop a general framework for the study of relative equilibria in $G$-Hamiltonian systems based completely on the semi-global normal form for the geometry and dynamics provided by the MGS form. This gives the theory a self-contained status, proving that all the local dynamics around a relative equilibrium is determined by the explicit set of coupled differential equations (5), (6) and (7). The first two are expressed in terms of the geometry of the Lie group of symmetries while the third one is related to the geometry of the symplectic normal space. This
provides a very convenient framework for a systematic use of tools from representation theory and symplectic linear algebra. Although it is true that all these ideas have been used in one form or another in the literature before and after, most of the results obtained for relative equilibria of $G$-Hamiltonian systems have used various approaches from different areas and there is not a unified common ground that systematizes the theory. Although the approach introduced in these two references has not been completely adopted, this author regards it as a fundamentally important step towards a self-contained theory of Hamiltonian relative equilibria.

7. The work of J. Montaldi on Hamiltonian relative equilibria. This section highlights some aspects of the trajectory of professor Montaldi in the field of Hamiltonian relative equilibria and the study of $G$-Hamiltonian systems in general. His work in this area is unique since he is one of the few researchers in geometric mechanics with a strong background in singularity and bifurcation theory. This is an ingredient propagating to most of his scientific production, together with topological methods, representation theory and differential geometry, in a not so usual mixture among colleagues from this field.

The first works of Montaldi on $G$-Hamiltonian systems are a series of articles with long time collaborators M. Roberts and I. Stewart during the late 80’s [21], [22] and [23]. They are devoted to the study of a closely related subject: the equivariant version of the well known Moser-Weinstein theorem [29] and [40]. This result gives an estimate of the number of periodic orbits of a nonlinear Hamiltonian system near an equilibrium point generalizing Liapounov’s centre theorem. Montaldi et. al study this problem in the symmetric setting, obtaining results on the existence and spectral stability of relative normal modes near relative equilibria.

Another of his main contributions to the theory of $G$-Hamiltonian systems and in particular to the theory of bifurcation of relative equilibria is given by the influential paper [16]. In this reference he proves an analogue of a result by Arnold that guarantees the persistence of relative equilibria when the momentum is varied, removing Arnold’s strong regularity conditions and replacing them by the topological condition of maximality of the relative equilibrium. This result has been further improved in [27] using the notion of $G$-openness of momentum maps (see also [17]).

Some of Montaldi’s favourite $G$-Hamiltonian systems are given by the dynamics of molecules and specially of point vortices in different geometries. He has a large amount of work on the subject with M. Roberts and other co-workers in which they approach these classical systems with all the geometric techniques from the field of $G$-Hamiltonian systems. The two foundational articles of this line of work are [19] and [12]. Other works with collaborators and students include [26], [35], [28], [18] and [25].

Another aspect of Montaldi’s work is the use not only of Hamiltonian symmetries but also of those transformations that act anti-symplectically, incorporating this effect into the group theoretic aspects of the Hamiltonian dynamics. This idea of merging these two kind of actions, in what is known as semi-symplectic group actions, was introduced first in [20].

Finally, Montaldi has also been interested in the phenomenon of symmetry breaking in $G$-Hamiltonian systems. This happens when the Hamiltonian function $H$ depends on a parameter $\epsilon$ in such a way that for values of $\epsilon$ other than 0 only a subgroup of $G$ is the symmetry group for the corresponding dynamical system. Typical questions in equivariant symmetry breaking theory include: which relative
equilibria for $H_0$ still exist for $H_\epsilon$? And, what is the change in the stability and bifurcations properties of the surviving relative equilibria? In [6] and [5] he has investigated these and other problems with some of his PhD students employing a semi-global approach very similar to the one used in this article.

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E-mail address: miguel.rodriguez.olmos@gmail.com