The Relativistic Quantum Law of motion for a Particle with Spin $1/2$.

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Abstract

In this paper, we introduce a deterministic approach of quantum mechanics for particles with spin $\frac{1}{2}$ moving in one dimension. We present a Lagrangian of a spinning particle ($s = \frac{1}{2}$), and deduce the expression of the conjugate momentum related to the velocity of the particle.

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1- Introduction

Recently, in the frame of a deterministic approach of quantum mechanics, we derived the two relativistic Quantum Stationary Hamilton-Jacobi Equation for a particle with spin $\frac{1}{2}$ [1],

\[
\frac{1}{2m_0} \left( \frac{dS_0}{dx} \right)^2 - \frac{\hbar^2}{4m_0} \{S_0, x\} + \frac{\hbar^2}{2m_0} (E - V + m_0c^2)^{\frac{3}{2}} .
\]

\[
\frac{d^2}{dx^2} \left[ (E - V + m_0c^2)^{-\frac{1}{2}} \right] + \frac{1}{2m_0c^2} \left[ m_0^2 c^4 - (E - V)^2 \right] = 0 ,
\]

(1)

and

\[
\frac{1}{2m_0} \left( \frac{dZ_0}{dx} \right)^2 - \frac{\hbar^2}{4m_0} \{Z_0, x\} + \frac{\hbar^2}{2m_0} (E - V - m_0c^2)^{\frac{3}{2}} .
\]

\[
\frac{d^2}{dx^2} \left[ (E - V - m_0c^2)^{-\frac{1}{2}} \right] + \frac{1}{2m_0c^2} \left[ m_0^2 c^4 - (E - V)^2 \right] = 0 ,
\]

(2)

where

\[
\{f(x), x\} = \begin{bmatrix} \frac{3}{2} \left( \frac{df}{dx} \right)^{-2} \left( \frac{d^2f}{dx^2} \right)^2 & -\left( \frac{df}{dx} \right)^{-1} \left( \frac{d^3f}{dx^3} \right) \end{bmatrix}
\]

represent the schwarzian derivative of $f(x)$ with respect to $x$. Eqs. (1) and (2) represent the two Relativistic Quantum Stationary Hamilton Jacobi Equations for Spinning particle ($s = \frac{1}{2}$) (QSHJES$_{\frac{1}{2}}$). One of these equations correspond to the projection $m_s = +\frac{1}{2}$ of the spin, when the other correspond to the projection $m_s = -\frac{1}{2}$. It follows that the reduced actions $S_0$ and $Z_0$ correspond to the two projections of the spin.

To establish these equations, we started from the Dirac Spinors Equation written in one dimension in Ref. [1] as

\[
-i \hbar c \begin{pmatrix} \sigma_x \frac{d\psi}{dx} \end{pmatrix} = (E - V(x) - \sigma_z m_0c^2) \psi
\]

(3)

where

\[
\begin{align*}
\sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \\
\sigma_y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \\
\sigma_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{align*}
\]

(4)

are the Pauli matrix. $\psi = \begin{pmatrix} \theta \\ \phi \end{pmatrix}$ is a the matrix of the wave functions $\theta$ and $\phi$.

In Ref. [1], we established the solutions of Eqs. (1) and (2). We written the reduced actions as

\[
S_0 = \hbar \arctan \left( a \frac{\theta_1(x)}{\theta_2(x)} + b \right) ,
\]

(5)

\[
Z_0 = \hbar \arctan \left( d \frac{\phi_1(x)}{\phi_2(x)} + e \right) ,
\]

(6)

where $a, b, d$ and $e$ are real constants. ($\theta_1, \phi_1$) and ($\theta_2, \phi_2$) are two real and independent solutions sets of the Dirac Spinors Equation (Eq. (3)) [1].
All these results make it possible to introduce a dynamic formulation of relativistic quantum mechanics as it is done in Refs. [3, 4, 5, 1, 6, 7]. Indeed, we have introduced the relativistic quantum Lagrangian written as [6]

\[ L = -m_0c^2\sqrt{1 - f(x)\dot{x}^2} - V(x), \]  

from which, and using the least action principle, we deduce the expression of the conjugate momentum [6]

\[ \dot{x} \frac{\partial S_0}{\partial x} = E - V(x) - \frac{m_0^2c^4}{E - V(x)}, \]

Then, we derived the Relativistic Quantum Newton’s Law

\[ [(E - V)^2 - m_0^2c^4]^2 + \left( \frac{x^2}{c^2} - \frac{\dot{x}^2}{c^2} \right) \frac{\partial \mathcal{S}_1}{\partial x} = \frac{\hbar^2}{2} \left( \frac{3}{E - V} \right)^2 - \frac{\dot{h}^2}{E - V} \]  

In this paper, we introduce an analogous formalism for a half spinning particle. In Sec. 2, we present such a formalism, and in Sec. 3 we conclude and discuss the results.

2- Dynamical approach of the motion of a particle with Spin 1/2

First, let us introduce the function \( f \) defined in Ref. [8] as

\[ f(x) = \left[ 1 - \frac{\hbar^2}{2} \left( \frac{dh}{dx} \right)^{-2} \{h(x), x\} \right]^{-1}, \]

where \( h \) correspond to the reduced action of the particle (\( S_0 \) or \( Z_0 \)). Via the function \( f \), Eqs. (1) and (2) will be written as

\[ \frac{1}{2m_0} \left( \frac{dS_0}{dx} \right)^2 + \frac{1}{2m_0} \left( \frac{dc}{dx} \right)^2 \left( E - V + m_0c^2 \right)^{\frac{1}{2}} \]  

and

\[ \frac{1}{2m_0} \left( \frac{dZ_0}{dx} \right)^2 + \frac{1}{2m_0} \left( \frac{dc}{dx} \right)^2 \left( E - V - m_0c^2 \right)^{\frac{1}{2}} \]  

with

\[ \frac{1}{2m_0} \left( \frac{dS_0}{dx} \right)^2 \left[ \left( E - V + m_0c^2 \right)^{-\frac{1}{2}} \right] + \frac{1}{2m_0c^2} \left[ m_0^2c^4 - (E - V)^2 \right] = 0, \]  

\[ \frac{1}{2m_0} \left( \frac{dZ_0}{dx} \right)^2 \left[ \left( E - V - m_0c^2 \right)^{-\frac{1}{2}} \right] + \frac{1}{2m_0c^2} \left[ m_0^2c^4 - (E - V)^2 \right] = 0, \]
which give

\[ f_1(x) = \frac{c^2 (dS_0/dx)^2}{(E - V)^2 - m_0^2 c^4 - \hbar^2 c^2 (E - V + m_0 c^2)^2 \frac{d^2}{dx^2} \left( E - V + m_0 c^2 \right)^{-\frac{3}{2}}} \],

and

\[ f_2(x) = \frac{c^2 (dZ_0/dx)^2}{(E - V)^2 - m_0^2 c^4 - \hbar^2 c^2 (E - V - m_0 c^2)^2 \frac{d^2}{dx^2} \left( E - V - m_0 c^2 \right)^{-\frac{3}{2}}} \],

(13)

Now, in order to establish a dynamical approach of the motion, we introduce the following expressions of the two Lagrangians \( L_1 \) and \( L_2 \) corresponding to the two RQSHJES

\[ L_1 = -m_0 c^2 \sqrt{1 - f_1(x) \frac{\dot{x}^2}{c^2}} - V(x), \]  

(15)

\[ L_2 = -m_0 c^2 \sqrt{1 - f_2(x) \frac{\dot{x}^2}{c^2}} - V(x). \]  

(16)

\( L_1 \) and \( L_2 \) describe the motion of particle in the two projections of the spin cases \( (m_s = \pm \frac{1}{2}) \).

Using the Least action principle and after integrating, we get

\[ \frac{m_0 c^2}{\sqrt{1 - f_1(x) \frac{\dot{x}^2}{c^2}}} + V(x) = E, \] 

(17)

and

\[ \frac{m_0 c^2}{\sqrt{1 - f_2(x) \frac{\dot{x}^2}{c^2}}} + V(x) = E, \] 

(18)

where \( E \) is an integrating constant representing the total energy of the particle.

Remark that, for purely relativistic cases (\( \hbar \rightarrow 0 \)), \( f_1 \rightarrow 1 \) and \( f_2 \rightarrow 1 \), Eqs. (17) and (18) reduce to the well known conservation equation

\[ \frac{m_0 c^2}{\sqrt{1 - \frac{\dot{x}^2}{c^2}}} + V(x) = E, \] 

(19)

Now, if we replace the expressions of \( f_1 \) and \( f_2 \) given by Eqs. (13) and (14) into Eqs. (17) and (18), we find

\[ \dot{x} \frac{dS_0}{dx} = \left( E - V(x) - \frac{m_0^2 c^4}{E - V(x)} \right) \sqrt{1 - \frac{\hbar^2 c^2}{(E - V)^2 - m_0^2 c^4} \sqrt{E - V + m_0 c^2} \frac{d^2}{dx^2} \left( \frac{1}{\sqrt{E - V + m_0 c^2}} \right)} \],

(20)

and

\[ \dot{x} \frac{dZ_0}{dx} = \left( E - V(x) - \frac{m_0^2 c^4}{E - V(x)} \right) \sqrt{1 - \frac{\hbar^2 c^2}{(E - V)^2 - m_0^2 c^4} \sqrt{E - V - m_0 c^2} \frac{d^2}{dx^2} \left( \frac{1}{\sqrt{E - V - m_0 c^2}} \right)} \].

(21)
$dS_0/dx$ and $dZ_0/dx$ represent the two conjugate momenta of the particles with spin $\frac{1}{2}$, one particle with spin projection $m_s = +\frac{1}{2}$, the other with $m_s = +\frac{1}{2}$. Remark that for the constant potentials, the two momenta reduce to the relativistic quantum momentum (Eq. (8)), so the degenerating expression disappear for the momenta as well as for the RQSHJE\textsubscript{$\frac{1}{2}$} and the reduced actions \textsuperscript{11}.

Note that, taking the purely relativistic limit ($\hbar \to 0$) Eqs. (20) and (21) reduce to the well known conservation equation (Eq. (19)).

Remark also that, for the purely quantum cases $T \ll m_0c^2$ ($T = E - V - m_0c^2$ is the kinetic energy of the particle), Eqs. (20) and (21) reduce to Eq. (8) given the expression of the purely quantum conjugate momentum.

Both relativistic quantum Newton’s Law can be derived after replacing both conjugate momenta, given by Eqs. (20) and (21), into Eqs. (1) and (2). So, we get to the first integral of the quantum Newton’s Law for Spinning particles ($s = \frac{1}{2}$) (FIQNLS\textsubscript{$\frac{1}{2}$}), equation which is a third order derivative of coordinate $x$ with respect to time $t$, containing the constant $E$. Deriving this equation with respect to $x$, we deduce the QNLS\textsubscript{$\frac{1}{2}$} which is a fourth order derivative of $x$ with respect to $t$. Because the overflowing of the FIQNLS\textsubscript{$\frac{1}{2}$}, we do not present it in the present paper. However, if one want to plot the trajectories of the spinning particle, he can use the expression of the conjugate momenta (Eqs. (20) and (21)) grounding himself on the solutions of the Dirac equation (Eq. (3)). This can be done with the same manner as it is done in Refs. \textsuperscript{11,10} for the particles without spinning behaviour.

3- Conclusion

We present, in this paper, a Lagrangian formulation of a deterministic dynamics of the particle with spin $\frac{1}{2}$. We demonstrate that, as for the no spinning particle case, it is possible to investigate the dynamical law of motion for the relativistic quantum phenomena. This is another important step to build a deterministic approach of quantum mechanics.

The relativistic quantum conjugate momenta reduce to the classical one, when $\hbar \to 0$. We demonstrate also that, each projection of the spin $\frac{1}{2}$ is described by its own Lagrangian, conjugate momentum, RQSHJE\textsubscript{$\frac{1}{2}$} and reduced action. This means that, for such a particle there is two classes of trajectories corresponding to the two projections of spin $\frac{1}{2}$.

For the purely quantum limit ($T \ll m_0c^2$), the conjugate momenta reduce to two distinguished momenta, which make the description of the spinning particles with our deterministic approach always possible. So the spin can be introduced in our approach even for non relativistic and purely quantum cases. This point will be more investigated in next works.

Finally, we stress that, for more understanding and completing of this approach, one must generalize to more than one dimension problems.
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