LINEAR ACCELERATION EMISSION. I. MOTION IN A LARGE-AMPLITUDE ELECTROSTATIC WAVE

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ABSTRACT

We consider the motion of a charge in a large-amplitude electrostatic wave with a triangular waveform relevant to an oscillating model of a pulsar magnetosphere. The (one-dimensional) orbit of a particle in such a wave is found exactly in terms of Weierstrass functions. The result is used to discuss linear acceleration emission (at both low and high frequencies) in an oscillating model for pulsars. An explicit expression for the emissivity is derived in an accompanying paper (Paper II), and used here to derive an expression for the absorption coefficient at low frequencies. We show that absorption can be negative, corresponding to maser emission. For the large amplitude required to trigger pair creation in an oscillating model, the rate of the maser growth is too small to be effective. Effective growth requires smaller amplitudes, such that the maximum Lorentz factor gained by acceleration in the wave is \( \lesssim 10 \).

Key words: plasmas – pulsars: general – radiation mechanisms: non-thermal

1. INTRODUCTION

An oscillatory model for a pulsar magnetosphere introduces an emission mechanism that depends explicitly on the oscillating electric field: linear acceleration emission (LAE). Our purpose in this paper, and in an accompanying paper (Melrose & Luo 2009), referred to hereinafter as Paper II, is to develop the theory for LAE in a large-amplitude electrostatic wave (LAEW), and discuss its viability as a pulsar emission mechanism.

In polar-cap models for pulsars, the inductive electric field associated with the rotation of the magnetized neutron star cannot be completely screened everywhere, and the residual field in “gaps” leads to acceleration of particles along the field lines to high enough energies to trigger a pair cascade. The polar-cap region is populated by the resulting secondary particles with a spread in Lorentz factors that is small compared to the maximum Lorentz factor, \( \gamma_{\text{max}} \), achieved in the LAEW, and we neglect this spread for the “background” particles that are described in terms of their oscillating velocity, Lorentz factor, and number density. The results derived here are not sensitive to the frequency of the LAEW, which we estimate to be \( \sim 10^8–10^9 \) s\(^{-1} \), based on \( \omega_p/\gamma_{\text{max}}^{1/2} \) (Levinson et al. 2005), with \( \gamma_{\text{max}} = 10^6–10^7 \) and the plasma frequency determined by the Goldreich–Julian number density times a multiplicity \( M \): \( \omega_p \approx \sqrt{2\Omega\Omega_M} \sim 5 \times 10^{11} \) s\(^{-1} \) for \( M = 100 \), a cyclotron frequency, \( \Omega_p \), for \( B = 10^9 \) T, and a rotation frequency, \( \Omega = 2\pi/P \) with \( P = 0.1 \) s.

In this paper, we discuss the motion of a charge in an LAEW, and the associated emission of radiation. A first integral of the equation of motion is obtained for a particle in an LAEW with an arbitrary waveform. A second integral requires a specific choice of waveform. The solution for a sinusoidal waveform was found by Rowe (1992a, 1992b) in terms of elliptic functions. Here we consider a sawtooth or triangular waveform, which is an excellent approximation to the actual waveform for an LAEW (Luo & Melrose 2008) in the case where the particles are highly relativistic (Luo & Melrose 2008). The solution for the orbit is found in terms of Weierstrass elliptic integrals. The exact solution is used as the basis for a treatment of LAE in an LAEW, which is developed in terms of Airy integrals in Paper...
II, and used here to discuss the possibility of maser LAE in an LAEW as a possible pulsar radio emission mechanism.

The equation of motion for a relativistic particle is solved in two superficially different ways. In the Appendix, a covariant form for the wave equation is used to derive a solution in terms of (Lorentz) invariants; the solution may be written in any inertial frame by expressing the invariants in terms of quantities in that frame. In Section 2, the solution is obtained in the primed frame, defined to be the frame in which the oscillations in the LAEW are purely temporal (there is only a notational difference between the important results in the Appendix and Section 2).

The exact orbit for a triangular waveform is found in Section 3. The application to LAE is discussed in Section 4, concentrating on the possibility of maser LAE at low frequencies. The results are discussed in Section 5, and our conclusions are summarized in Section 6.

2. MOTION IN AN LAEW: PRIMED FRAME

In this section, we find the orbit of a charge in an LAEW in the primed frame. First, we describe the transformation between the inertial frames, and write down the forms of the invariants introduced in the Appendix in the primed frame. The motion is assumed to be restricted to one dimension (the 3-axis), along the direction of the superstrong magnetic field in a pulsar magnetosphere.

2.1. Description of the LAEW

The LAEW is described in terms of its amplitude, $E(\chi) = E_0 T(\chi)$, as a function of phase, $\chi$, with the maximum electric field, $E_0$, written in terms of a frequency, $\omega_E$, and with the waveform described by the function $T(\chi)$, with maxima and minima $\pm 1$:

$$E(\chi) = \epsilon \frac{m \omega_E}{q} T(\chi), \quad \omega_E = \frac{|q| E_0}{mc},$$

where the sign, $\epsilon = q/|q|$, is opposite for electrons and positrons. For an arbitrary waveform, the only specific assumption made is that $T(\chi) = T(\chi + 2\pi)$ is periodic. An LAEW, even with a nonsinusoidal wave profile, is described in terms of its frequency, $\Omega$, and its wavevector, $K$, along the 3-axis. The LAEW is assumed superluminal, implying that its phase velocity, $\Omega/K = \beta_V c$, satisfies $\beta_V > 1$.

2.2. Lorentz Transformation

The Lorentz transformation to the primed frame corresponds to a boost with velocity (in units of $c$) $\beta^* = -1/\beta_V$ and Lorentz factor $\gamma^* = \beta_V (\beta_V^2 - 1)^{-1/2}$. The relation between time and position in the two frames is

$$ct' = \gamma^* (ct - \beta^* z), \quad z' = \gamma^* (z - \beta^* ct).$$

In the primed frame, the frequency, $\Omega'$, of the oscillation and the 4-velocity, $u'^\nu = \gamma'[1, 0, 0, \beta']$ of the test charge are given by

$$\Omega' = \frac{\Omega}{\gamma'}, \quad \gamma' = \gamma \sqrt{1 - \beta^2}, \quad \beta' = \frac{\beta}{\beta_V},$$

where the wavevector in the primed frame is zero by construction, $K' = 0$. The frequency and angle of emission of LAE are related in the two frames by

$$\omega = \gamma^* \omega' (1 + \beta^* \cos \theta'), \quad \cos \theta = \frac{\cos \theta' + \beta^*}{1 + \beta^* \cos \theta'}.$$

Relevant invariants introduced in the Appendix have the following values in the primed frame, implied by $K^\mu = (\Omega', 0, 0, 0)$, $K'_\mu = (0, 0, 0, \Omega')$:

$$\Omega' = (K^2)^{1/2}, \quad \chi = \Omega' t', \quad K\tilde{u}(\chi) = \Omega' \gamma', \quad K_d\tilde{u}(\chi) = -\Omega' u',$$

where we assume $K^2 = \Omega^2 - |K|^2 > 0$ and write $u' = \gamma' \beta'$. The electric field, $E(\chi)$, is an invariant, and has the same form in all frames.

2.3. Equation of Motion

The equation of motion may be written in two equivalent forms:

$$\frac{d\gamma'}{d\chi} = \frac{\omega_E}{\Omega} T(\chi) u', \quad \frac{du'}{d\chi} = \frac{\omega_E}{\Omega} T(\chi) \gamma'.$$

Integrating the second part of Equation (6) gives

$$u'(\chi) = u'_0 + (\omega_E/\Omega') F(\chi),$$

where $u'_0 = \gamma_0 \beta_0' c$ contains the initial conditions, and with $F(\chi)$ given by Equation (A8), with $d_0 = 0$ assumed in Equation (A10). Equation (7) implies a Lorentz factor

$$\gamma'(\chi) = [1 + (u'_0 + (\omega_E/\Omega') F(\chi))^2]^{1/2},$$

and a velocity (in units of $c$)

$$\beta'(\chi) = \frac{u'_0 + (\omega_E/\Omega') F(\chi)}{[1 + (u'_0 + (\omega_E/\Omega') F(\chi))^2]^{1/2}},$$

2.4. Background and Test Charges

In Equation (7), the particle has an arbitrary initial velocity, $\beta_0 c$, at $\chi = 0$. It is convenient to separate the particles into two classes: background particles and test charges. The background particles are an idealized class of electrons and positrons that move in opposite directions in phase with the LAEW, with $\beta_0'$ assumed to be identically zero. In the idealized case where there is no intrinsic velocity spread in the background particles, such particles are instantaneously at rest twice per period, and reach their maximum Lorentz factor twice per period. Let the initial phase, $\chi = 0$, be chosen such that the background particles have $\beta_0' = 0$ in Equation (7), so that they are instantaneously at rest at $\chi = n\pi$ for any integer $n$. The background particles have their maximum Lorentz factors, $\gamma_{\text{max}}'$, at the phases $\chi = \pi / 2 + n\pi$. A test charge is then any particle that has $\beta_0' \neq 0$ at $\chi = 0$. In the analytic model (Luo & Melrose 2008), all particles are in effect test charges. The concept of a background particle is useful when considering situations in which the velocity spreads of the electrons or of the positrons are unimportant, and are approximated by zero.

The maximum 4-velocity for the background particles, $u_{\text{max}}' = \gamma_{\text{max}}' \beta_{\text{max}}'$, is determined by Equation (7), which becomes

$$u'(\chi) = u_{\text{max}}' F(\chi) / F(\pi / 2), \quad u'_{\text{max}} = \frac{\omega_E}{\Omega} F(\pi / 2).$$

The Lorentz factor corresponding to Equation (10) is

$$\gamma'(\chi) = [1 + (\omega_E/\Omega)^2 F(\chi)]^{1/2}.$$
The function $F(\chi)$, defined by Equation (A8), is

$$F(\chi) = \frac{\epsilon}{\pi} \left\{ (\chi - \pi/2)^2 - \pi^2/4, \quad 0 < \chi < \pi,\right.$$  
$$- (\chi - 3\pi/4)^2 + \pi^2/4, \quad \pi < \chi < 2\pi. \tag{14}$$

A background particle has extrema in its velocity at $\chi = \pi/2 + n\pi$, so that Equation (10) implies

$$|u'_\text{max}| = \pi \omega_E/4\Omega', \tag{15}$$

with $|u'_\text{max}| \approx \gamma'_{\text{max}} \gg 1$ in the case of most interest here.

### 3.2. Exact Solution for the Trajectory

The integral over $\chi'$ in Equation (12) with Equation (9) may be carried out exactly for the triangular waveform, in terms of elliptic integrals, specifically, the Weierstrass $\wp(x)$ and $\zeta(x)$ functions (Abramowitz & Stegun 1965). The integral is written as a sum over integrals of the form

$$I = \int_{\chi_1}^{\chi_2} d\chi' \frac{(\chi' - a)^2 - p}{\sqrt{[(\chi' - a)^2 - p]^2 + b}}, \tag{16}$$

where $\chi_1, \chi_2, a, p,$ and $b$ are constants. By making the substitution $y = (\chi' - a)^2 - 2p/3$, the integral (Equation (16)) reduces to the form

$$I = \pm \int_{\gamma_1}^{\gamma_2} dy' \frac{y' - p/3}{\sqrt{4y'^3 - g_2y' - g_3}}, \tag{17}$$

with $g_2 = 4(p^2 - 3b)/3$, $g_3 = -8p(p^2 + 9b)/27$, which is a standard form for the Weierstrass functions (Abramowitz & Stegun 1965)

$$\zeta(x; g_2, g_3) = -\int dx \wp(x; g_2, g_3), \tag{18}$$

denoted $\zeta(x)$ here, with $\wp(x; g_2, g_3)$, denoted $\wp(x)$ here, defined implicitly by $y = \wp(x)$ and

$$x = \int_{\infty}^{y} dy' \frac{1}{\sqrt{4y'^3 - g_2y' - g_3}}. \tag{19}$$

Two sets of choices, $a_i, p_i$ with $i = 1, 2$ are needed to find the full solution, and these define

$$x_i(\chi) = \wp^{-1}((\chi - a_i)^2 - 2p_i/3). \tag{20}$$

In terms of the functions

$$w_i(\chi) = p_i x_i(\chi) + 3\zeta(x_i(\chi)), \tag{21}$$

with $g_{2i}, g_{3i}$ evaluated for $p = p_i$, the solution is

$$Z'(\chi) = \frac{\epsilon}{3} \left\{ -w_1(0) + w_1(\chi), \quad 0 < \chi < \pi/2, \right.$$  
$$w_1(\pi/2) - w_1(\chi), \quad \pi/2 < \chi < \pi, \right.$$  
$$w_2(3\pi/2) - w_2(\chi), \quad \pi < \chi < 3\pi/2, \right.$$  
$$-w_1(3\pi/2) + w_1(\chi), \quad 3\pi/2 < \chi < 2\pi, \tag{22}$$

with $a_1 = \pi/2$, $p_1 = \pi^2/4 - \pi\gamma'_{\text{max}}/u'_{\text{max}}$, $a_2 = 3\pi/2$, $p_2 = \pi^2/4 + \pi\gamma'_{\text{max}}/u'_{\text{max}}$, $b = \pi^2/3$. 

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**Figure 1.** Triangular waveform is illustrated for an LAEW; in the exact theory (Luo & Melrose 2008), the sharply curved parts correspond to a range of phase $\sim 1/\gamma'_{\text{max}}$ where the background particles are briefly nonrelativistic.
3.3. Examples

Examples of the exact solution are shown in Figure 2. A feature of the solutions is that the trajectory is well approximated by straight lines, with one of $z' = ct'$ constant, corresponding to $Z'(\chi) \approx \pm \chi$ in the first half phase, with $Z'$ oscillating between approximately $\pm \pi$. The linear approximation is accurate except for a small range of phase, $\sim 1/\gamma_{\text{max}}'$, where the particles become nonrelativistic and reverse their direction of propagation. As is evident from Figure 2, the trajectory of a background particle ($\beta_0' = 0$) involves an oscillation about a fixed point in the primed frame. A test particle with $|u_0'| < |u_{\text{max}}'|$, reverses its direction of motion periodically in the LAEW, but for $|u_0'| > |u_{\text{max}}'|$, the direction of motion of the test particle does not change, as its 4-velocity oscillates in the LAEW. For $|u_0'| \gg |u_{\text{max}}'|$, the effect of the LAEW on the motion of the test charge may be treated as a perturbation to the motion of a background particle.

For $\gamma_{\text{max}}' > 1$, the orbit of any particle (except for $|\gamma_{\text{max}}' - \gamma_0'| \lesssim 1$) is well approximated by a sequence of light lines, such that $z'$ varies as $\pm ct$ in sequential sections, as shown on the left in Figure 3. For the large value, $\gamma_{\text{max}}' \sim 10^6$, expected in an oscillating pulsar model, the fraction of the phase, $\Delta \chi' \sim 1/\gamma_{\text{max}}'$, where the particles are nonrelativistic is tiny, and the approximation in terms of a sequence of light lines is very accurate. The distance between the turnaround points for a background particle is very nearly the light distance, $\pi c/\Omega'$, corresponding to the half period in which the particle travels in the same direction. The solution in terms of Weierstrass functions shows how the orbit evolves from one section to another; to illustrate this, an example where the motion is only mildly relativistic is shown on the right in Figure 3.

4. PARTICLE CURRENT ASSOCIATED WITH LAE

The motivation for the investigation reported is the treatment of LAE due to motion in an LAEW. In the Appendix, we use the covariant formalism to evaluate the 4-current associated with LAE, and in this section we use the results derived in the Appendix to write down the 3-current associated with LAE in the primed frame.

4.1. LAE in the Primed Frame

The emission of LAE can be calculated in terms of the current associated with the motion of the charge in the LAEW. For periodic motion, the current may be expanded in a Fourier series (Rowe 1995), giving a sum over a harmonic number $s$. The expansion is written down using a covariant formalism in the Appendix. In the primed frame, the current along the direction defined by the LAEW follows from Equation (22):

$$J_s(\omega', k') = -B_s(\omega', k') 2\pi \delta(\omega' - s\Omega'),$$

with $B_s(\omega', k')$ following from Equation (A20). It follows that in the primed frame, emission at the $s$th harmonic is at the frequency, $\omega' = s\Omega'$. That is, the emission is at harmonics of the frequency of the LAEW in the primed frame.

The explicit form for the current obtained by inserting Equation (A20) into Equation (23) involves an integral over a phase factor, with phase $\phi(\chi)$ say. After replacing the sum over $s$ by an integral and performing the integral over the $\delta$-function in Equation (23), the phase becomes

$$\phi(\chi) = \frac{\omega'}{\Omega} \left[ \chi - Z'(\chi) \cos \theta' \right]$$

in the primed frame.

On inserting the exact solution (Equation (22)) for the orbit into Equation (24), it is obvious that some simplifying procedure is needed before one can evaluate the integral over the phase factor. The analogy between LAE and synchrotron emission, discussed in detail in Paper II, suggests that an Airy-integral approximation is appropriate. The Airy integral applies when the phase is approximated by an expansion about a specific phase in which the quadratic term vanishes and only the linear and cubic terms are retained. On making a Taylor expansion of $Z'(\chi)$ about an arbitrary phase, $\chi_0$, say, the quadratic term vanishes when the second derivative of $Z'(\chi)$ is zero at $\chi = \chi_0$. This corresponds to an extremum of $\beta'(\chi)$ and hence of $\gamma'(\chi)$. The only relevant extrema are maxima of $\gamma'(\chi)$, which occur at $\chi_0 = \pi/2 + n\pi$. Assuming the radiating particle to be highly relativistic near this phase, one may make the approximation $\beta'(\chi) \approx \pm \left[ 1 - 1/2 \gamma^2(\chi) \right]$. This approximation leads to a highly relativistic counterpart of Equation (22). For $u_0' = 0$ the integral can be performed exactly, giving

$$Z'(\chi) = \chi - \frac{1}{8\gamma_{\text{max}}'^2} \left( \frac{\chi - \pi/2}{1 - 4(\chi - \pi/2)^2/\pi^2} \right)$$

$$+ \frac{\pi}{4} \ln \left( \frac{1 + 2(\chi - \pi/2)/\pi}{1 - 2(\chi - \pi/2)/\pi} \right),$$

for $0 < \chi < \pi$. Comparison with Equation (22) shows that Equation (25) is an excellent approximation except near $\chi = 0, \pi$ where the particle is briefly nonrelativistic. The Airy-integral approximation is obtained by expanding Equation (25) around $\chi_0 = \pi/2$. The same result is obtained by expanding $\gamma'(\chi)$ around $\pi/2$. The latter procedure is readily generalized to $u_0' \neq 0$, for which the maxima in $\gamma'(\chi)$ also occur at $\chi_0 \approx \pi/2 + n\pi$ and are $\gamma_{\text{max}}' \approx \gamma_{\text{max}}' \pm u_0'$. The Airy-integral form for the phase (Equation (24)) then follows from

$$Z'(\chi) = \chi - \frac{1}{2\gamma_{\text{max}}'} \left[ \left( \chi - \frac{\pi}{2} \right)^2 + \frac{8}{3\pi^2} \gamma_{\text{max}}'^2 \left( \chi - \frac{\pi}{2} \right)^3 \right].$$

The evaluation of the integral is discussed in detail in Paper II, and here we quote the result and discuss some of its implications.
4.2. Emission and Absorption of LAE

The formal treatment of LAE involves both conceptual and mathematical complications, discussed in Paper II. To proceed with the discussion here, we quote the results derived in Paper II for the emissivity (power per unit frequency and per unit solid angle) in the primed frame. In terms of the characteristic frequency \( \omega'_{\pm} = 2K' \gamma'^2 \), one has

\[
\eta'_{\pm}(\omega', \theta') = \frac{3q^2 \omega'^2 \gamma'^2 (1 + \gamma'^2 \theta'^2)}{16\pi^2 \varepsilon_0 c} K^{2}_{1/3}(\xi'_\pm),
\]

where \( K_{1/3}(\xi) \) is a modified Bessel function, with \( \xi'_\pm = \xi'_\pm(1 + \gamma'^2 \theta'^2)^{1/2}, \xi'_{\pm} = (\pi/\sqrt{2})(\omega'/\omega_{\pm}')(\gamma'/\gamma_{\pm}')^{1/2} \), and the \( \pm \) signs correspond to emission in the forward (defined for \( u'_0 > 0 \)) and backward direction. The characteristic properties of LAE implied by Equation (27) are discussed in Paper II.

Let \( g(u'_0)du'_0 \) be the number density of one species of particle, electrons say, with initial (at phase \( \chi = 0 \)) 4-velocity in the range \( u'_0 \) to \( u'_0 + du'_0 \). The volume emissivity is \( \int g(u'_0)u'_0 du'_0 \). The absorption coefficient can be related to the emissivity (Equation (27)) by an argument using detailed balance (Twiss 1958; Wild et al. 1963). In the one-dimensional case of relevance here this gives

\[
\Gamma'_{\pm}(\omega', \theta') = \frac{2(2\pi)^3 c}{m_0 \omega^2} \int du'_0 \eta'_{\pm}(\omega', \theta') \cos \theta' \frac{dg(u'_0)}{du'_0},
\]

with \( \cos \theta' \approx \pm 1 \) such that the signs of \( \cos \theta' \) and \( u'_0 \) are the same. Maser emission corresponds to negative absorption, requiring \( \Gamma'_{\pm}(\omega', \theta') < 0 \). We comment below on the possibility of maser emission at low frequencies.

As discussed in Paper II, a mathematical inconsistency arises when one attempts to consider the low-frequency limit of Equation (28) with Equation (27). One way of avoiding this difficulty is to consider the average of Equation (28) over solid angle, denoted by an overline:

\[
\bar{\Gamma}'_{\pm}(\omega') = \frac{2(2\pi)^3 c}{m_0 \omega^2} \int du'_0 \eta'_{\pm}(\omega') \frac{dg(u'_0)}{du'_0},
\]

where the choice of the sign \( \cos \theta' = \pm 1 \) is implicit. The average of the emissivity is evaluated in Paper II, and at low frequencies for a triangular waveform it reduces to

\[
\bar{\eta}'_{\pm}(\omega') \approx \frac{2q^2 \Omega'}{80\pi^2 \varepsilon_0 c} \left( \frac{\pi \omega'}{\omega_{\pm}'} \right)^{4/3} \left( \frac{\gamma'}{\gamma_{\pm}'} \right)^{2/3} \xi'_{\pm} Ai^2(0),
\]

with \( Ai(0) = 1/3^{2/3} \Gamma(2/3) = 0.355 \). We use Equation (29) rather than Equation (28) in the discussion below.

The emissivity (Equation (27)) is in the primed frame, and it is straightforward to transform it to the pulsar frame, in which the LAEW is an outward propagating wave. The Lorentz transformation (Equation (4)) implies

\[
\eta_{\pm}(\omega, \theta) = \omega \eta'_{\pm}(\omega', \theta'), \quad \omega = \gamma^*(1 \pm \beta^*)\omega', \quad \theta = \gamma^*(1 \mp \beta^*)\theta'.
\]

However, it is usually more convenient to discuss the emission and absorption in the primed frame before transforming to the pulsar frame.

5. DISCUSSION

In this section, we discuss four aspects of the foregoing results in further detail: systematic acceleration, the displacement of individual charges in an LAEW, drift motion and its relevance to LAE in an LAEW, and maser LAE in an LAEW.

5.1. Systematic Acceleration

In solving the equation of motion, we find that an LAEW with an asymmetric waveform, specifically one with \( a_0 \neq 0 \) in Equation (A10), leads to systematic acceleration of all particles. According to Equation (7) with Equation (A10), the 4-velocity increasing by \( 2\pi a_0 \) with each wave period. This corresponds to a systematic acceleration by a static electric field \( \vec{E} = a_0 \Omega mc/q \). No uniform electric field is included in the analytic model for an LAEW (Luo & Melrose 2008), and we assume \( a_0 = 0 \) in the foregoing.

It is interesting to speculate on the possible significance of assuming \( a_0 \neq 0 \). One could interpret \( \vec{E} \) associated with \( a_0 \neq 0 \) as one way of modeling an unscreened component of the pulsar’s inductive field. The systematic acceleration transfers electromagnetic energy into particles. In a conventional polar-cap model, such acceleration occurs in a gap, and screening of the inductive field results from a net charge density in a pair formation front (Harding & Muslimov 1998). One motivation for an oscillating model is that such screening is unstable to temporal perturbations, resulting in large-amplitude electric oscillations (Levinson et al. 2005). The build up of an LAEW in an oscillating model relies on acceleration by an incompletely screened electric field. As in a stationary model, the resulting
pair creation should lead to screening of the electric field, and whereas this occurs locally in a stationary model, it corresponds to a systematic reduction in \( \hat{E} \), and hence of \( a_0 \), in the oscillating model. This effect needs to be included in a detailed theory of the instability leading to the LAEW, but we do not attempt a quantitative treatment of this here.

### 5.2. Displacement of Charges in an LAEW

A particle in an LAEW oscillates about a center that is drifting (except for a background particle in the primed frame). The displacement about this center is by \( \pm \pi c/\Omega \) in the primed frame. A particle is constrained to move along a magnetic field line that is curved, and the neglect of this curvature is valid only if this distance is small compared with the radius of curvature of the field line. For an LAEW with frequency \( \sim 10^9 \text{ s}^{-1} \), \( \pm \pi c/\Omega \) is less than a meter, and this condition is well satisfied.

A test particle has a drift velocity in the primed frame. This velocity may be estimated from the distance, \( \Delta z' \) say, its position advances in each period of the LAEW:

\[
\Delta z' = \frac{c}{\Omega} \int_0^{2\pi} d\chi \beta'(\chi),
\]

with \( \beta'(\chi) \) given by Equation (9). The mean drift velocity is \( \Omega \Delta z'/2\pi \), which depends on \( u'_{\text{D}} \) and \( u_{\text{max}} \), and is zero for \( u'_{\text{D}} = 0 \). The 4-velocity constructed from the 3-velocity \( \Omega \Delta z'/2\pi \) is rather cumbersome. Alternatively, one might consider a drift 4-velocity found by dividing the displacement (32) by the proper time elapsed in a wave period, \( \Delta t \) say. One has

\[
\Delta \tau = \frac{1}{\Omega} \int_0^{2\pi} d\chi \frac{1}{\gamma'(\chi)},
\]

with \( \gamma'(\chi) \) given by Equation (8). The ratio \( \Delta z'/\Delta \tau \) is some measure of the mean 4-velocity, but describes a different quantity. We note that neither the integral of Equation (32) nor the integral of Equation (33) appear in the detailed theory for LAE developed in Paper II.

### 5.3. Comparison with Rowe (1995)

The solution found here for the orbit of a particle in an LAEW with a triangular waveform is similar to the solution found by Rowe (1995) for a sinusoidal waveform. However, there are important differences in the results for LAE. An important difference is the dependence of the emission formula found by Rowe (1995) on a drift velocity, and we comment on this below. First, however, we comment on similarities in the results, to eliminate them as possible reasons for the differences in the treatment of LAE.

As in the present paper, Rowe (1995) found that when the particles are highly relativistic over most of the phase, their orbit is well approximated by a sequence of light lines, with the direction of motion reversing at the phase where the particles are briefly nonrelativistic. Rowe (1995) also found that the emission of LAE is dominated by the phase where the particles have their maximum Lorentz factor. From one perspective this is a surprising result, which is clearly insensitive to the assumed waveform. Based on the generalized Larmor formula, one expects the emission to be maximum when the acceleration is a maximum, that is, at the phase where the electric field is maximum. In contrast, it is found that the emission of LAE maximizes around the phase where the acceleration is zero. Clearly, the important differences in the treatment of LAE between the present paper and that of Rowe (1995) are not due to the assumed waveform, to the detailed form of the orbit or to the phase that determines the properties of the emission.

### 5.4. Absence of Doppler Shift in LAE

In the theory of LAE developed by Rowe (1995), the drift velocity plays an important role, in the sense that the frequency of LAE is at harmonics of the Doppler-shifted frequency of the LAEW. The emission in the theory of Rowe (1995) occurs at \( (\omega_k - \epsilon_k \beta_0) - s(\Omega - K \epsilon_0 \beta_0) = 0 \), or \( ku_D - s(\Omega) = 0 \) in invariant form, where \( u_D^k \) is the drift 4-velocity, \( \gamma_D, \beta_D \) are the associated Lorentz factor and 3-velocity, and \( s \) is the harmonic number. In contrast, we find no explicit dependence on a drift velocity in our treatment of LAE.

A drift velocity does appear naturally in the analogous theory for emission due to motion in a large-amplitude transverse wave (LATW; Gunn & Ostriker 1971; Arons 1972). In an LATW, the oscillatory motion is in the transverse plane and there is a drift motion along the direction of wave propagation. The frequency of emission by such a particle is at harmonics of the Doppler-shifted wave frequency, with the Doppler shift corresponding to the drift velocity. However, the analogy between the transverse and longitudinal cases is misleading when considering the emission of radiation. Semiquantitatively, one may regard the motion in an LATW as consisting of two effectively independent motions: that in the transverse plane causing the radiation, and that along the axis providing the Doppler shift. In an LAEW both the drift and the oscillatory motion are along the axis, and the two motions cannot be separated in this way.

No Doppler shift associated with the drift motion of a test particle appears explicitly in the phase (Equation (24)). As shown in the Appendix, explicit inclusion of the drift motion involves extracting a term, \( a_1 \chi \) from \( Z'(\chi) \), such that \( \tilde{Z}'(\chi) = Z'(\chi) - a_1 \chi \) is periodic. This does introduce a Doppler shift, as assumed by Rowe (1995). In the continuum approximation, when \( s \) is replaced by a continuous variable, the sum over \( s \) is replaced by an integral that is performed over the resonance condition, \( s \) is replaced by the Doppler-shifted frequency divided by the frequency of the LAEW. The phase is given by Equation (24) irrespective of whether the Doppler shift is included or not. If the Doppler shift is ignored, one has \( s = \omega / \Omega^r \) and Equation (24) follows by the argument given; if the Doppler shift is included, the term \( a_1 \chi \) is subtracted from \( Z'(\chi) \), added to \( s \), and restored to \( Z'\chi \) to give \( Z'(\chi) \) after integration of \( s \). When the integral of the phase factor over \( \chi \) is evaluated in terms of an Airy integral, as in Paper II, no separation of the term \( a_1 \chi \) associated with the drift is necessary or appropriate. It follows that no Doppler shift appears explicitly in LAE.

A physical explanation for the absence of an explicit Doppler shift in LAE (in the primed frame) is as follows. The emission of LAE is concentrated around the phase where \( y'(\chi) \) reaches a maximum, with \( dy'(\chi)/d\chi = 0 \). This is the phase where the acceleration reverses sign and is instantaneously zero; all particles have their maximum Lorentz factor at the same phase. The theory of LAE depends only on the maximum value of \( y'(\chi) \) and the way that it varies with \( \chi \) around this maximum. The emission does depend on the initial 4-velocity, \( u'_{\text{D}} \), which also determines the drift motion, but the two dependences on \( u'_{\text{D}} \) are quite different. In the emission of LAE, \( u'_{\text{D}} \) affects the shape of the variation of \( y'(\chi) \) about its maximum. In contrast, the drift velocity is determined by the integral of Equation (32) for the
displacement in a period of the LAEW, and this is dominated by the asymmetry between the forward and backward paths, as shown in Figure 3; in this case, \( u_0 \) affects the turnaround points, where \( y'(\chi) \) passes through its minimum of unity. The dependence of LAE on \( u_0 \) cannot be described in terms of the drift velocity.

5.5. Maser LAE at Low Frequencies

Maser LAE is known to be possible for small-amplitude oscillations, when the frequency of the maser is determined by a characteristic frequency in the radio range (Melrose 1978). Here the characteristic frequency, \( \omega_{\chi} \sim 2\Omega \gamma_{\text{max}}^2 \), is at a much higher frequency, which is estimated in Paper II to be in the hard X-ray range for the largest-amplitude LAEWs. For maser LAE in an LAEW to be relevant as a pulsar emission radio emission mechanism, the maser needs to operate at low frequencies, \( \omega \ll \omega_{\chi} \). It follows from Equation (28) that negative absorption requires two conditions be satisfied. First, the distribution function must satisfy \( dg(u')/du_0 > 0 \), over at least some range of \( u_0 \). This condition is plausibly satisfied for pairs generated in a cascade. Numerical models for pair creation (Zhang & Harding 2000; Hibschman & Arons 2001; Arendt & Eilek 2002) suggest as shown in Figure 3; in this case, the waveform of the LAEW is well approximated by the triangular form, Figure 1. The orbit of a particle is described in terms of its displacement, \( z' \), along the axis defined by the LAEW, as a function of phase \( \chi \). We choose the zero of the phase such that the maxima and minima of the waveform for the electric amplitude at \( \chi = 0 \) are at \( \pm \pi n \), with \( n \) an integer.

We distinguish between background particles and test particles. Background particles are assumed to come to rest at the maxima and minima in the waveform, and reach a common maximum Lorentz factor, \( y'_{\text{max}} \) at the intermediate phases, \( \chi = \pi/2 + n\pi \). A test particle is any particle with a nonzero 4-velocity \( (u_0') = y'_{\text{max}} y_0' \); \( y_0' = (1 - \beta_0^2)^{-1/2} \) at the initial phase, \( \chi = 0 \). The Lorentz factor of a test charge oscillates between maxima \( y'_{\text{max}} \approx |y'_{\text{max}} \pm u_0'| \) at \( \chi = \pi/2 + n\pi \); a test charge comes to rest for \( y_0' < y'_{\text{max}} \) twice per period, at phases that depend on \( u_0' \). We derive an explicit expression for the orbit of a particle in an LAEW with a triangular waveform in terms of elliptic integrals (Weierstrass functions). In the highly relativistic case, the exact solution for the orbit is well approximated by a set of light lines, with the velocity changing rapidly between \( \pm z' \) at phases \( \chi = \pi/2 + n\pi \). This leads to a triangular form for the orbit of a background particle, which oscillates about a fixed point. A test charge oscillates about a moving center, with the drift speed of the center of oscillation depending on \( \beta_0' \), but there is no simple analytic expression for the drift speed.

Our motivation for this investigation was to explore the properties of LAE in an LAEW, particularly the possibility of maser LAE in an LAEW as a radio emission mechanism, and high-energy LAE as an intermediate step in a pair cascade. The treatment of LAE involves some conceptual and mathematical difficulties and is discussed separately in Paper II (Melrose & Luo 2009). Here we write down the emissivity derived in Paper II, and use it to consider the absorption coefficient for LAE, with negative absorption implying maser emission. A mathematical difficulty is avoided by averaging the absorption coefficient over the small cone of emission. Maser emission is found to be possible for LAE in LAEW, as in earlier theories: Melrose (1978) found maser LAE in a small-amplitude electrostatic oscillation with \( y_0' \gg y'_{\text{max}} \) in the present notation, and Rowe (1995) found maser LAE in a theory for LAE in a sinusoidal LAEW. We note that the latter theory (Rowe 1995) depends explicitly on a drift velocity, and in this sense is incompatible with the theory developed here and in Paper II. We discuss this difference and explain why there should be no explicit dependence on a drift velocity in LAE.

Our estimate of the growth rate for maser LAE at low frequencies implies that it is too small for effective growth in an LAEW with an amplitude needed to trigger pair creation in an oscillating model for pulsars. For LAE in an LAEW to be viable as a radio emission mechanism, one requires a smaller amplitude LAEW, with \( y'_{\text{max}} \ll 10 \). It seems plausible that LAEWs with a wide range of amplitudes are present in different regions of the magnetosphere, and one needs to assume this to be the case for maser LAE to be effective. We conclude that maser LAE in an LAEW is a possible radio emission mechanism, but that the mere presence of an LAE does not ensure effective maser LAE. We discuss the possible relevance of LAE to high-energy processes in pulsars in Paper II.

6. CONCLUSIONS

In this paper, we solve the equation of motion for a charged particle in an LAEW, both in an arbitrary frame (Appendix) and in the (primed) frame in which the oscillations are purely temporal (Section 2). The case of interest for pulsars is when the electrons and positrons are accelerated to highly relativistic energies in a small fraction of a period of the wave, and in this case the waveform of the LAEW is well approximated by the triangular form, Figure 1. The orbit of a particle is described in terms of its displacement, \( z' \), along the axis defined by the LAEW, as a function of phase \( \chi \). We choose the zero of the phase such that the maxima and minima of the waveform for the electric amplitude at \( \chi = 0 \) are at \( \pm \pi n \), with \( n \) an integer.

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APPENDIX

MOTION IN AN LAEW: COVARIANT TREATMENT

In this appendix, we solve the equation of motion to find the orbit of a charge in an LAEW using a covariant formalism.

A.1. Covariant Formalism

In the covariant formalism used here, Greek indices run over 0–3, with signature +, −, −, −, and units with c = 1 are adopted. An electromagnetic field is described by the Maxwell tensor, $F^\mu\nu$, and for an LAEW with a phase $\chi$, which is an invariant, this is of the form $F^\mu\nu = E(\chi) f^\mu\nu$, where $E(\chi)$ is the invariant electric field defined by $2E^2(\chi) = -F^\mu\nu F^\mu\nu$. For an LAEW along the 3-axis, the tensor $f^\mu\nu$ has components $f^\mu0 = -1$, $f^\mu3 = 1$, $f^{0\nu} = 0$ otherwise.

The orbital differential equations are written in the covariant form

$4$-vector, $K^\mu = K\mu u^\mu(\tau)$, which are orthogonal, $K^\nu K_\nu = 0$. The invariants $K_2 = K^\mu K^\mu = K^\nu K_\nu$ satisfied $K^2 > 0$ for a superluminal LAEW.

The one-dimensional motion is in the two-dimensional 0–3 subspace. The equation of motion then reduces to two invariant components, by choosing two orthogonal 4-vectors that span this subspace. We choose the 4-vectors $K^\mu$ and $K_D^\mu = f^\mu\nu K_\nu$, which are orthogonal, $K_D^\mu K_D^\mu = 0$. The invariants $K_2^2 = K_D^\mu K_D^\mu = K^\nu K_\nu$ satisfied $K^2 > 0$ for a superluminal LAEW.

The LAEW is described by the waveform (1), with a waveform $X^\mu(\tau)$, the phase $\chi(\tau)$, $K_D^\mu u^\mu(\tau)$, and $K_D^\mu u^\mu(\tau)$ are related by

$\chi(\tau) = K X^\mu(\tau), \quad d\chi(\tau)/d\tau = K u(\tau).$ (A5)

Using Equation (A5), the independent variable in Equation (A1) is changed from $\tau$ to $\chi$, and all quantities are regarded as functions of $\chi$. Projecting Equation (A1) onto the 4-vectors $K^\mu, K_D^\mu$ gives

$\frac{dKu(\tau)}{d\chi} = e\omega_E T(\chi) \frac{K_D u(\tau)}{K u(\tau)}, \quad dK_D u(\tau)/d\chi = e\omega_E T(\chi),$ (A6)

where we use Equation (1), and where the dependence of $\chi$ on $\tau$ is implicit.

Integrating the second part of Equation (A6) gives

$K_D u(\tau) = K_D u(0) + e\omega_E F(\chi),$ (A7)

where $K_D u(0)$ is a constant of integration, and with

$F(\chi) = e\int_0^\chi d\chi' T(\chi').$ (A8)

The solution of the first part of Equation (A6) is

$K u(\tau) = [K^2 + (K_D u(\tau))^2]^{1/2},$ (A9)

with only the positive square root allowed for $K^2 > 0$.

The function $F(\chi)$ may be separated into a periodic part and a systematically increasing part,

$F(\chi) = \tilde{F}(\chi) + a_0 \chi, \quad a_0 = \frac{e}{2\pi} \int_0^{2\pi} d\chi T(\chi).$ (A10)

with $\tilde{F}(\chi + 2\pi) = \tilde{F}(\chi)$. The parameter $a_0$ is zero for any waveform that is symmetric, in the sense $T(\chi + \pi) = -T(\chi)$, and we restrict our discussion to such waveforms. The significance of $a_0 \neq 0$ is discussed in Section 5.

The orbit is found by integrating $dX^\mu(\tau)/d\tau = u^\mu(\tau)$, which becomes

$\frac{dX^\mu(\tau)}{d\chi} = K^\mu - K_D u(\tau) K_D^\mu K_D^\nu u^\nu(\tau),$ (A11)

Integrating the first form in Equation (A11) gives

$X^\mu(\tau) = x^\mu_0 + \chi K^\mu - Z(\chi) K_D^\mu,$ (A12)

where an invariant, dimensionless form of the displacement is given by

$Z(\chi) = \int_0^\chi d\chi' K_D u(\tau) + e\omega_E F(\chi'),$ (A13)

where Equations (A7) and (A9) are used. The proper time of the particle depends on phase through

$\tau(\chi) = \int_0^\chi d\chi' \frac{1}{[K^2 + (K_D u(\tau))^2]^{1/2}}.$ (A14)

The source term for emission of radiation by any particle is identified as the Fourier transform of the current density associated with the particle (Melrose & McPhedran 1991). In a covariant formulation, this becomes the 4-current density

$J^\mu(k) = q \int d\tau u^\mu(\tau) e^{ikX(\tau)}.$ (A15)

For motion in an LAEW, the only nonzero components of $J^\mu(k)$ are in the 0–3 plane. The formalism introduced in Equation (A4)
allows one to write
\[ J^\mu(k) = \frac{K J(k) K^\mu - K_D J(k) K_D^\mu}{K^2}. \] (A16)

Using Equation (A12), the phase factor in Equation (A15) becomes
\[ kX(\tau) = kx_0 + \frac{kK}{K^2} + Z(\chi) \frac{kK_D}{K^2}. \] (A17)

Using Equation (A4), Equation (A15) gives
\[ \left[ \frac{K J(k)}{K_D J(k)} \right] = q \int d\chi \left[ \frac{1}{\beta(\chi)} \right] e^{i kX(\chi)}. \] (A18)

with \( \beta(\chi) = K_D u(\chi)/K u(\chi) \), and with \( kX(\chi) \) now regarded as a function of \( \chi \).

The integral in Equation (A18) may be evaluated after expanding in a Fourier series (Rowe 1995). For a background particle, \( Z(\chi) \) is a periodic function and the expansion in Fourier series may be written
\[ \left[ \frac{1}{\beta(\chi)} \right] e^{iZ(\chi)kK_D/k^2} = \sum_{s=-\infty}^{\infty} \left[ \frac{U_s(k)}{B_s(k)} \right] e^{-is\chi}, \] (A19)

with the Fourier coefficients given by
\[ \left[ \frac{U_s(k)}{B_s(k)} \right] = \frac{1}{2\pi} \int_0^{2\pi} d\chi \left[ \frac{1}{\beta(\chi)} \right] e^{i(\chi + Z(\chi)kK_D/k^2)}. \] (A20)

Then Equation (A18) becomes
\[ \left[ \frac{K J(k)}{K_D J(k)} \right] = \sum_{s=-\infty}^{\infty} \left[ \frac{K J_s(k)}{K_D J_s(k)} \right], \] (A21)

with the Fourier coefficient determined by
\[ \left[ \frac{K J_s(k)}{K_D J_s(k)} \right] = q \left[ \frac{U_s(k)}{B_s(k)} \right] 2\pi \delta(s - kK/k^2). \] (A22)

For a test particle, one may write \( Z(\chi) = a_1\chi + \tilde{Z}(\chi) \), with a linear term, \( a_1\chi \) associated with the drift motion, and with \( \tilde{Z}(\chi) \) a periodic function. The Fourier series may be defined in terms of coefficients \( U_s(k) \), \( B_s(k) \) defined by Equation (A20) with \( Z(\chi) \rightarrow \tilde{Z}(\chi) \). The result (Equation (A22)) is modified by replacing the sum over \( s \) by an integral, performed over the \( \delta \) function, the result (Equation (24)) is found to apply to both background and test particles. The separation of \( Z(\chi) \) into a drift term and an oscillatory term is irrelevant when treating LAE.

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