On the images of certain $G_2$-valued automorphic Galois representations

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Abstract

In this paper we study the images of certain families $\{\rho_{\pi,\lambda}\}_\lambda$ of $G_2$-valued Galois representations associated to algebraic regular, self-dual, cuspidal automorphic representations $\pi$ of $\text{GL}_7(\mathbb{A}_\mathbb{Q})$. In particular, we prove that under certain conditions in the weights of $\pi$ the images of the residual representations $\overline{\rho}_{\pi,\lambda}$ are as large as possible for infinitely many primes $\lambda$ and provide some examples where such conditions are satisfied.

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1. Introduction

Let $G_\mathbb{Q}$ be the absolute Galois group of $\mathbb{Q}$ and $\pi$ be an algebraic regular, self-dual, cuspidal automorphic representation of $\text{GL}_n(\mathbb{A}_\mathbb{Q})$. Thanks to the work of Chenevier, Clozel, Harris, Kottwitz, Shin, Taylor and several others, we know that there exists a number field $E_\pi$ and a family $\{\rho_{\pi,\lambda}\}_\lambda$ of semi-simple Galois representations $\rho_{\pi,\lambda} : G_\mathbb{Q} \rightarrow \text{GL}_n(E_{\pi,\lambda})$ where $\lambda$ ranges over all finite places of $E_\pi$ and $E_{\pi,\lambda}$ is an algebraic closure of the completion $E_{\pi,\lambda}$ of $E_\pi$ at $\lambda$. In particular, by the self-duality, the image of $\rho_{\pi,\lambda}$ is contained in $\text{GO}_n(\mathbb{Q}_\ell)$ or $\text{GSp}_n(\mathbb{Q}_\ell)$. See Section 2.1 of [BGGT14] for details and references.

A folklore conjecture, ensures that the images of the residual representations $\overline{\rho}_{\pi,\lambda}$ should be as large as possible for almost all places $\lambda$ (i.e. all but finitely many), unless there is an automorphic reason for it does not happen. In the 2-dimensional case, the conjecture was proven by Momose [Mo81] and Ribet [Ri85] when $\pi$ comes from a classical modular form of weight $k \geq 2$. In this case, modular forms with complex multiplication (the automorphic reason in this case) had to be excluded in order to obtain large image. When $\pi$ comes from a Siegel modular form of genus 2 and weights $(k_1, k_2)$, $k_1 \geq k_2 \geq 2$, the conjecture has been proved recently by Weiss [Wei]. In this case, CAP, endoscopics, automorphic inductions and symmetric cubes need to be excluded to obtain large image.

In a recent work [Ch19], Chenevier has studied certain algebraic regular, self-dual, cuspidal automorphic representations $\pi$ of $\text{GL}_7(\mathbb{A}_\mathbb{Q})$ with weights of the form $0, \pm u, \pm v, \pm (u + v)$, $0 < u < v$, such that the 7-dimensional families of Galois representations $\{\rho_{\pi,\lambda}\}_\lambda$ associated to them, are $G_2$-valued (Theorem 2.1). In this paper, we prove a weak version of the conjecture for these automorphic representations. More precisely, we prove that if the weights of $\pi$ are such that $v \neq 2u$ (condition imposed to exclude sixth symmetric powers), there exists a positive Dirichlet

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density set of primes \( L \) such that for all \( \lambda \) above \( \ell \in L \) the image of \( \rho_{\pi,\lambda} \) is isomorphic to \( G_2(\mathbb{F}_\ell) \) for some positive integer \( s \) (Theorem 5.1). In fact, if we assume that \( \pi \) is such that \( \pi_p \) is Steinberg for some prime \( p \), the set of primes \( L \) has Dirichlet density 1 (Theorem 4.2).

The proof of our result follows the line of [Dj02] and [DZ20], in the sense that our main tools are: some recent results about residual irreducibility of compatible systems of Galois representations [BGGT14], the classification of the maximal subgroups of \( G_2(\mathbb{F}_\ell) \) [Kle88], and Fontaine-Laffaille theory [FLS2].

Finally, we show that 10 examples of cuspidal automorphic representations of \( GL_7(A_\mathbb{Q}) \) of level one, given in [Ch19], satisfy our result (Proposition 4.1). As far as we know, examples of cuspidal automorphic representation of \( GL_7(A_\mathbb{Q}) \) with certain prescribed local ramification, such that the \( G_2 \)-valued Galois representation associated to them, are large as possible for a fixed prime (resp. a set of primes of Dirichlet density at least 1/18) have been studied in [KLS10] (resp. [MS]).

2. \( G_2 \)-valued automorphic Galois representations

In this section we review some definitions and results about certain 7-dimensional Galois representations associated to cuspidal automorphic representations of \( GL_7(A_\mathbb{Q}) \). We refer the reader to [Ci01, Chapter 3] and [Ci19, Section 6] for more details and references.

Let \( \pi = \pi_\infty \otimes \pi_f \) be a cuspidal automorphic representation of \( GL_7(A_\mathbb{Q}) \). We recall that \( \pi_f \) decomposes into a restricted tensor product \( \otimes \pi_p \) of irreducible smooth representations \( \pi_p \) of \( GL_7(Q_p) \) which are well defined up to isomorphism and unramified for almost all primes \( p \). We will denote by \( S_\pi \) the (finite) set of primes such that \( \pi_p \) is not unramified and we will say that \( \pi \) is of \textit{level one} if \( S_\pi = \emptyset \). According to Satake Parametrization, the set of isomorphism classes of unramified representations of \( GL_7(Q_p) \) is in canonical bijection with the set of conjugacy classes of semi-simple elements in \( GL_7(C) \). So, when \( \pi_p \) is unramified, we denote by \( c(\pi_p) \subset GL_n(C) \) the semi-simple conjugacy class associated to \( \pi_p \) that is called the Satake parameter of \( \pi_p \). On the other hand, we recall that \( \pi_\infty \) is a \( \mathfrak{gl}_7(K) \)-module, where \( K \) is a fixed maximal compact subgroup of \( GL_7(R) \) and \( \mathfrak{gl}_7(R) = M_7(R) \) is the Lie algebra of \( GL_7(R) \). Then, the center \( z \) of the enveloping algebra of \( \mathfrak{gl}_7(R) \) acts by scalars in \( \pi_\infty \). The resulting \( C \)-algebra homomorphism \( j : C \to C \) is called the \textit{infinitesimal character} of \( \pi_\infty \). By the Harish-Chandra isomorphism this character can be viewed as a semi-simple conjugacy class \( c(\pi_\infty) \in M_7(C) \). We will say that \( \pi_\infty \) is \textit{algebraic} if the eigenvalues \( k_1 \leq \cdots \leq k_7 \) of \( c(\pi_\infty) \) are integers and that it is \textit{regular} if such integers are distinct. We will say that \( \pi = \pi_\infty \otimes \pi_f \) is \textit{algebraic regular} if \( \pi_\infty \) is algebraic and regular, and the integers \( k_1, \cdots, k_7 \) will be called the \textit{weights} of \( \pi \).

Let \( \pi \) be an algebraic regular cuspidal automorphic representation of \( GL_7(A_\mathbb{Q}) \). Thanks to algebraic regularity, Clozel [Clo90, Theorem 3.7] proved that there is a number field \( E_\pi \subset \mathbb{C} \), called a \textit{coefficient field for} \( \pi \), such that for each prime \( p \) the representation \( \pi_p \) of \( GL_7(Q_p) \) is defined over \( E_\pi \). In particular, if \( \pi_p \) is unramified, the characteristic polynomial \( \det(X - c(\pi_p)) \) belong to \( E_\pi[X] \). Moreover, if we assume that \( \pi \) is self-dual (i.e., \( \pi^\vee \simeq \pi \)) it can be proved that, for any prime \( \ell \) and any place \( \lambda \) of \( E_\pi \) above \( \ell \), there exists a continuous semi-simple representation

\[
\theta_{\pi,\lambda} : G_\mathbb{Q} \longrightarrow GL(E_{\pi,\lambda})
\]

(unique up to isomorphism) unramified outside \( S_\pi \cup \{ \ell \} \) and such that, for every \( p \notin S_\pi \cup \{ \ell \} \), the characteristic polynomial of a Frobenius element \( Frob_p \) satisfies

\[
\det(X - \theta_{\pi,\lambda}(Frob_p)) = \det(X - c(\pi_p)).
\]

Additionally, if \( \ell \notin S_\pi \), the representation \( \theta_{\pi,\lambda} \) is crystalline at \( \ell \) and its Hodge-Tate numbers are equal to the weights of \( \pi \) [Shi11, Theorem 1.2]. We remark that, the existence of \( E_\pi \) and
The residual representation $\rho_{\pi, \lambda}$ is known in greater generality (e.g. any dimension, without the assumption of self-duality), but we focus on this case because it is the only one that we really use in this paper.

Let $G_2$ be the automorphism group scheme of the standard split octonion algebra over $\mathbb{Z}$. It is well known that, for any algebraically closed field $k$ of characteristic 0, there is a unique (up to isomorphism) irreducible $k$-linear algebraic representation $\sigma : G_2(k) \to GL_7(k)$. By using the previous result on the existence of Galois representations associated to self-dual cuspidal automorphic representations of $GL_7(\mathbb{A}_Q)$, Chenevier [Ch19 Corollary 6.5, Corollary 6.10] proved the following result:

**Theorem 2.1.** Let $\pi = \pi_{\infty} \otimes \pi_f$ be an algebraic regular, self-dual, cuspidal automorphic representation of $GL_7(\mathbb{A}_Q)$ and assume that, for almost all primes $p$, the Satake parameter $c(\pi_p)$ of $\pi_p$ is the conjugacy class of an element in $\sigma(G_2(\mathbb{C}))$. Then, for any prime $\ell$ and any place $\lambda$ of $E_\pi$ above $\ell$, there exists a continuous semi-simple representation

$$
\rho_{\pi, \lambda} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to G_2(\mathbb{F}_{\pi, \lambda})
$$

(unique up to $G_2(\mathbb{F}_{\pi, \lambda})$-conjugacy) unramified outside $S_{\pi} \cup \{\ell\}$ and such that

$$
det(X - \sigma(\rho_{\pi, \lambda}(\text{Frob}_p))) = det(X - c(\pi_p))
$$

for every $p \notin S_{\pi} \cup \{\ell\}$. Moreover, the weights of $\pi$ are of the form $0, \pm u, \pm v, \pm (u + v)$, with $0 < u < v$.

### 3. Study of the images of the residual representations $\mathfrak{f}_{\pi, \lambda}$

Let $\rho_{\pi, \lambda} : G_2 \to G_2(\mathbb{F}_{\pi, \lambda})$ be a Galois representation as in Theorem 2.1 we will denote by $\mathfrak{f}_{\pi, \lambda} : G_2 \to G_2(\mathbb{F}_s)$ the semi-simplification of its reduction, which is well defined up to conjugacy. This representation is usually called the residual representation of $\rho_{\pi, \lambda}$. The main goal of this paper is to prove the following result.

**Theorem 3.1.** Let $\pi = \pi_{\infty} \otimes \pi_f$ be a cuspidal automorphic representation of $GL_7(\mathbb{A}_Q)$ as in Theorem 2.1 and assume that the weights of $\pi$ are such that $v \neq 2u$. Then, there exists a positive Dirichlet density set of primes $\mathcal{L}$ such that for all $\lambda$ above $\ell \in \mathcal{L}$ the image of $\mathfrak{f}_{\pi, \lambda}$ is isomorphic to $G_2(\mathbb{F}_s)$ for some positive integer $s$.

The proof of this theorem follows the structure of [Di02] and [DZ20]. Then, as in loc. cit., the proof of Theorem 3.1 is done by considering the possible images of $\mathfrak{f}_{\pi, \lambda}$ given by the maximal subgroups of $G_2(\mathbb{F}_s)$. Such subgroups were classified by Kleidman in [Kle88].

**Proposition 3.2.** Let $\mathbb{F}_q$ be a finite field of characteristic $\ell > 3$ and $q = \ell^r$. Then, the maximal proper subgroups of $G_2(\mathbb{F}_q)$ are as follows:

i) maximal parabolic subgroups;

ii) $SL_2(\mathbb{F}_q), SU_3(\mathbb{F}_q)$;

iii) $(SL_2(\mathbb{F}_q) \circ SL_2(\mathbb{F}_q)).2$;

iv) $PGL_2(\mathbb{F}_q), \ell \geq 7, q \geq 11$;

v) $2^4 PSL_3(\mathbb{F}_2), PSL_2(\mathbb{F}_{13}), PSL_2(\mathbb{F}_8), G_2(\mathbb{F}_2)$;

vi) the sporadic Janko group $J_1, \ell = 11$;
Moreover, if \( \rho \) depends on the restriction of \( \text{Inert}(\pi_{\rho, \lambda}) \) as defined as in Section 2 of \[Bar20\], which only depends on the restriction of \( \pi_{\rho, \lambda} \) to the inertia subgroup \( I\ell \). Fontaine-Laffaille theory gives us a way to compute the inertial weights of \( \pi_{\rho, \lambda} \) from the Hodge-Tate numbers of \( \rho_{\pi, \lambda} \) for almost all \( \ell \). More precisely, we have the following result \[Bar20\] Theorem 1.0.1].

**Proposition 3.3.** Let \( \pi \) be a cuspidal automorphic representation of \( \text{GL}_7(A_q) \) as in Theorem 2.1 Then, \( \rho_{\pi, \lambda} \) is crystalline at \( \ell \not\in S_{\pi} \) with Hodge-Tate numbers

\[
\text{HT}(\rho_{\pi, \lambda}) = \{-(u + v), -v, -u, 0, u, v, u + v\}.
\]

Moreover, if \( \ell \not\in S_{\pi} \) and \( \ell \geq 2u + 2v + 1 \), we have that \( \text{HT}(\rho_{\pi, \lambda}) \in \text{Inert}(\pi_{\rho, \lambda}) \).

Now, we are ready to prove Theorem 3.1. Our proof will be given by showing that the image of \( \pi_{\rho, \lambda} \) is not contained in any subgroup lying in cases \( i) - vii) \) of Proposition 3.2. In this case, \( \text{PGL}_2(F_{\ell}) \) fits into \( G_2(F_{\ell}) \) via \( \text{Sym}^6 : \text{PGL}_2 \to G_2 \). So, if \( G_\lambda := \text{Im}(\pi_{\rho, \lambda}) \) is contained in \( \text{Sym}^6(\text{PGL}_2(F_{\ell})) \), the elements of \( G_\lambda \) are of the form

\[
\text{Sym}^6 \left( \begin{array}{c c c c c c c}
  x^6 & * & * & * & * & * & * \\
* & x^5y & * & * & * & * & * \\
* & * & x^4y^2 & * & * & * & * \\
* & * & * & x^3y^3 & * & * & * \\
* & * & * & * & x^2y^4 & * & * \\
* & * & * & * & * & x^6y & * \\
* & * & * & * & * & * & y^6
\end{array} \right)
\]

where \( x, y \in F_{\ell} \). Then, we can deduce that

\[
(x^{(6-a)}y^a)(x^{(6-a)-2y^a+2}) = (x^{(6-a)-1}y^{a+1})^2
\]

for \( 0 \leq a \leq 4 \). In particular, from these equalities, the inertial weights \( (k_1, \ldots, k_7) \in \text{Inert}(\pi_{\rho, \lambda}) \) should satisfy the following relation

\[
k_i + k_{i+2} = k_{i+1}
\]

for \( 1 \leq i \leq 5 \) and \( \ell \) sufficiently large. Then, by Proposition 3.3 if \( \ell \not\in S_{\pi} \) and \( \ell \geq 2u + 2v + 1 \), the Hodge-Tate numbers \( \text{HT}(\rho_{\pi, \lambda}) = \{-(u + v), -v, -u, 0, u, v, u + v\} = \{k_1, \ldots, k_7\} \) should satisfy

\[
k_i + k_{i+2} = k_{i+1}
\]
Theorem 4.2. 

Example 4.1. If \((u, v) \in \{(5, 8), (3, 10), (5, 9), (4, 10), (2, 12), (7, 8), (4, 11), (1, 14), (1, 16), (1, 17)\}\) then \(|G_2(u, v)| = 1\).

We remark that if we allow certain local ramification behavior in our automorphic representations, we can obtain a strong version of Theorem 3.1.

Theorem 4.2. Let \(\pi\) be a cuspidal automorphic representation of \(GL_7(\mathbb{A}_\mathbb{Q})\) as in Theorem 3.1, and assume that for some prime \(p\), \(\pi_p\) is square integrable. Then there exists a set of primes \(\mathcal{L}\) of Dirichlet density 1 such that for all \(\lambda\) above \(\ell \in \mathcal{L}\) the image of \(\pi_{\pi,\lambda}\) is isomorphic to \(G_2(\mathbb{F}_\ell)\) for some positive integer \(s\).

Proof. As we are assuming that \(\pi_p\) is square integrable, from Corollary B of [TY07], we have that \(\rho_{\pi,\lambda}\) is irreducible for all \(\lambda\) above \(\ell \neq p\). By Proposition 5.3.2 of [BGGT14], there exists a set of primes \(\mathcal{L}'\) of Dirichlet density 1, such that for all \(\lambda\) above \(\ell \in \mathcal{L}'\), \(\pi_{\pi,\lambda}\) is irreducible.

The rest of the proof is exactly the same as the proof of Theorem 3.1. In particular, the set of primes \(\mathcal{L}\) of Dirichlet density 1 is obtained by removing a finite number of primes from \(\mathcal{L}'\) as in the proof of Theorem 3.1.

Finally, we remark that Magaard and Savin [MS] have used (before the appearance of Chenevier’s work) this kind local behavior to construct a self-dual cuspidal automorphic representation \(\pi\) of \(GL_7(\mathbb{A}_\mathbb{Q})\) (unramified outside 5 and such that \(\pi_5\) is Steinberg), such that the image of the residual representations \(\pi_{\pi,\lambda}\) : \(G_3(\mathbb{F}_\ell) \to GL_7(\mathbb{F}_\ell)\) associated to \(\pi\) are equal to \(G_2(\mathbb{F}_\ell)\) for an explicit set of primes of Dirichlet density at least 1/18.
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