PERTURBATION THEORY FOR LYAPUNOV EXPONENTS OF A TORAL MAP: EXTENSION OF A RESULT OF SHUB AND WILKINSON.

by David Ruelle*.

Abstract. Starting from a hyperbolic toral automorphism, we obtain, for a small volume preserving perturbation, an exact and rigorous second order perturbation expansion of the Lyapunov exponents.

Keywords: Lyapunov exponent, toral automorphism, hyperbolicity.

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We consider volume preserving perturbations $F$ of a diffeomorphism $F_0 = (\Phi, J)$ of $T^{m+1} = T^m \times T$, where $\Phi$ is a hyperbolic automorphism of $T^m$, and $J$ is a translation of $T$. Writing $F = F_0 + aF'$, we shall show that the Lyapunov exponents for $(F, \text{volume})$ can be expanded to second order in $a$ (Theorem 1). In particular, the central Lyapunov exponent $\lambda^c$ of $(F, \text{volume})$, to second order in $a$, is generally $\neq 0$ (Corollary 11). For a special family of perturbations one obtains particularly simple formulae, first noted by Shub and Wilkinson [10]. We recover their result in Theorem 12. We deviate from [10] mostly in that we don’t have differentiability of $\lambda^c$, only a second order expansion around $a = 0$.

The ideas used here are largely those in Shub and Wilkinson [10], and can be appreciated in the background provided by Hirsh, Pugh and Shub [6], Burns and Wilkinson [3], Ruelle and Wilkinson [9]. For recent work related to Lyapunov exponents, see also Bonatti, Gómez-Mont and Viana [2], Avila and Bochi [1].

After completing the writing of this paper, the author received a preprint by D. Dolgopyat [4], which develops similar ideas in a more general setting, but without the specific formulas we obtain here.

1. Theorem.

Let $\Phi$ be a hyperbolic automorphism of $T^m$, and $J : y \mapsto y + \alpha \pmod{1}$ a translation of $T$. Define $F_0 = (\Phi, J)$, and let $F = F_0 + aF'$ be a $C^2$ perturbation of $F_0$, volume preserving to first order in $a$. (We take $F' : T^{m+1} \mapsto \mathbb{R}^{m+1}$ and $F_0 \xi + aF'(\xi)$ has to be understood (mod 1) in each component). Let $\lambda_1 < \lambda_2 < \ldots$ be the Lyapunov exponents of $(F_0, \text{volume})$ and $m_1, m_2, \ldots$ their multiplicities (the exponent $0$ occurs with multiplicity $1$). Also let $\lambda_a^{(1)} \leq \lambda_a^{(2)} \leq \ldots$ be the Lyapunov exponents of $(F, \text{volume})$ repeated according to multiplicity. Then we have the second order expansion

$$\sum_{\ell = m_1 + \ldots + m_{r-1} + 1}^{m_1 + \ldots + m_r} \lambda_a^{(\ell)} = m_r \lambda_r + a^2 L_r + o(a^2)$$

If $m_r = 1$, and writing $\lambda_r = \lambda_a^{(\ell)}$, we have

$$\lambda_a^{(\ell)} = \lambda_0^{(\ell)} + a^2 L_0^{(\ell)} + o(a^2)$$

(this applies in particular to $\lambda^c = \lambda_0^{(\ell)}$ for $\lambda_0^{(\ell)} = 0$).

An explicit expression for $L_r$ can be obtained (see Proposition 9). We do not assume ergodicity of $(F, \text{volume})$, and therefore we use integrated Lyapunov exponents (averaged over the volume), see however Remark 15(a).

If one likes one may to $F_0 + aF'$ add terms of higher order in $a$ so that the sum $F$ is exactly volume preserving. These higher order terms will not change our results, and are omitted in what follows.

2. Normal hyperbolicity.

As in [10], we invoke the theory of normal hyperbolicity of [6]. We start from the fact that $F_0$ is normally hyperbolic to the smooth fibration of $T^{m+1}$ by circles $\{x\} \times T$. 2
Taking some \( k \geq 2 \) we apply [6] Theorems (7.1), (7.2). Thus we obtain a \( C^1 \) neighborhood \( U \) of \( F_0 \) in the \( C^k \) diffeomorphisms of \( T^{m+1} \) such that, for \( F \in U \), there is an equivariant fibration \( \pi : T^{m+1} \to T^m \) with

\[
\pi F = \Phi \pi
\]

The fibers \( \pi^{-1}\{x\} \) are \( C^k \) circles forming a continuous fibration of \( T^{m+1} \) (this fibration is in general not smooth). Furthermore there is a \( TF \)-invariant continuous splitting of \( TT^{m+1} \) into three subbundles:

\[
TT^{m+1} = E^s + E^u + E^c
\]

such that \( E^c \) is 1-dimensional tangent to the circles \( \pi^{-1}\{x\} \), \( E^s \) is \( m^s \)-dimensional contracting and \( E^u \) is \( m^u \)-dimensional expanding for \( TF \).

If \( \lambda_r < 0 \) (and \( F \) is in a suitable \( C^1 \)-small neighborhood \( U \) of \( F_0 \)), we can introduce a continuous vector subbundle \( E^r \) of \( TT^{m+1} \) which consists of vectors contracting under \( TF^n \) faster than \((\lambda_r + \epsilon)^n\) where \( \epsilon > 0 \) and \( \lambda_r + \epsilon < \lambda_{r+1} \). In fact \( E^r \) is a hyperbolic (attracting) fixed point for the action induced by \( TF^{-1} \) on the bundle of \( m_1 + \ldots + m_r \) dimensional linear subspaces of \( TT^{m+1} \) (over \( F^{-1} \) acting on \( T^{m+1} \)).

If \( \lambda_r > 0 \), replacement of \( F \) by \( F^{-1} \) similarly yields a continuous subbundle \( \bar{E}^r \) of \( m_r + \ldots \) dimensional subspaces.

3. Proposition.

Assume that \( F \) is of class \( C^k \), \( k \geq 2 \), and that \( F \) is \( C^k \) close to \( F_0 \). The bundles \( E^r \), \( \bar{E}^r \), when restricted to a circle \( \pi^{-1}\{x\} \) are of class \( C^{k-1} \), continuously in \( x \).

If \( G \) denotes the (Grassmannian) manifold of \( m_1 + \ldots + m_r \) dimensional linear subspaces of \( \mathbb{R}^{m+1} \), we may identify the bundle of \( m_1 + \ldots + m_r \) dimensional linear subspaces of \( TT^{m+1} \) with \( T^{m+1} \times G \). We denote by \( E \in G \) the spectral subspace of the matrix defining \( \Phi \) corresponding to the smallest \( m_1 + \ldots + m_r \) eigenvalues (in absolute value, and repeated according to multiplicity).

If \( F_0 \) is the action defined by \( TF_0 \) on \( TT^{m+1} \times G \), the circles \( \{x\} \times T \times \{E\} \) form an \( F_0 \) invariant fibration of \( T^{m+1} \times \{E\} \), to which \( F_0 \) is normally hyperbolic. If \( F \) is \( C^k \) close to \( F_0 \), the corresponding \( C^{k-1} \) action \( F \) is normally hyperbolic to a pertubed fibration where \( \{x\} \times T \times \{E\} \) is replaced by \( E^r|\pi^{-1}\{x\} \). According to [6] Theorem 7.4, Corollary (8.3) and the following Remark 2, we find that the \( C^{k-1} \) circle \( E^r|\pi^{-1}\{x\} \subset T^{m+1} \times G \) depends continuously on \( x \in T^{m+1} \). Similarly for \( \bar{E} \). \[ \square \]

Note that in [10], the \( C^r \) section theorem is used in a similar situation, giving estimates uniform in \( x \). However, continuity in \( x \) (not just uniformity) will be essential for us in what follows.

4. Corollary.

The splitting \( TT^{m+1} = E^s + E^u + E^c \) when restricted to a circle \( \pi^{-1}\{x\} \) is of class \( C^{k-1} \), continuously in \( x \).
It is clear that $E^r|\pi^{-1}\{x\}$ is of class $C^{k-1}$ because it is the tangent bundle to the $C^k$ circle $\pi^{-1}\{x\}$. As to $E^s, E^u$, they are special cases of $E^r, E^u$. 

Notation.

Remember that $F = F_0 + aF'$, and fix $F'$. We shall use the notation $\pi_a, E^r_a, \ldots$ to indicate the $a$-dependence of $\pi, E^r, \ldots$

5. Proposition.

For small $\epsilon > 0$ there is a continuous function $x \mapsto \gamma_x$ from $T^m$ to $C^k(T \times (-\epsilon, \epsilon) \to T^m)$ such that $\gamma_x(y, 0) = 0$ and $\pi^{-1}_a\{x\} = \{(x + \gamma_x(y, a), y) : y \in T\}$.

To see this define $\tilde{F} : T^{m+1} \times (-\epsilon, \epsilon) \to T^{m+1} \times (-\epsilon, \epsilon)$ by $\tilde{F}(\xi, a) = ((F_0 + aF')(\xi), a)$ and observe that $\tilde{F}$ is normally hyperbolic to the 2-dimensional manifolds

$$\cup_{a \in (-\epsilon, \epsilon)}(\pi^{-1}_a\{x\}, a)$$

and these are thus $C^k$ 2-dimensional submanifolds of $T^{m+1} \times (-\epsilon, \epsilon)$. 

We may in the same manner replace $\pi^{-1}_a\{x\}$ by $\cup_{a \in (-\epsilon, \epsilon)}(\pi^{-1}_a\{x\}, a)$ in Proposition 3 and Corollary 4. Writing $E_a$ for $E^r_a, \bar{E}^r_a, E^s_a, E^u_a, E^c_a$, we obtain that $(\cdot, a) \mapsto E_a(\cdot)$, when restricted from $T^{m+1} \times (-\epsilon, \epsilon)$ to $\cup_{a \in (-\epsilon, \epsilon)}(\pi^{-1}_a\{x\}, a)$ is of class $C^{k-1}$. We rephrase this as follows:

6. Proposition.

The map

$$x \mapsto \{(y, a) \mapsto E_a(x + \gamma_x(y, a), y)\}$$

where $E_a$ stands for $E^r_a, \bar{E}^r_a, E^s_a, E^u_a, E^c_a$, is continuous $T^m \to C^{k-1}(T \times (-\epsilon, \epsilon) \to \text{Grassmannian of } \mathbb{R}^{m+1})$ where we have used the identification $TT^{m+1} = T^{m+1} \times \mathbb{R}^{m+1}$.

Notation.

From now on we write $E_a$ for $E^r_a, \bar{E}^r_a, E^s_a, E^u_a, E^c_a$. When $a = 0$, $E_0$ is a trivial subbundle of $TT^{m+1} = T^{m+1} \times \mathbb{R}^{m+1}$, and we shall write $E_0 = T^{m+1} \times \mathcal{E}$, denoting thus by $\mathcal{E}$ a spectral subspace of the matrix on $\mathbb{R}^{m+1}$ defining $(\Phi, 1)$. We denote by $\mathcal{E}^\perp$ the complementary spectral subspace.

Taking $k = 2$ we have then:

7. Corollary.

There are linear maps $G(x, y), R(x, y, a) : \mathcal{E} \to \mathcal{E}^\perp$ such that $G(x, y)$ depends continuously on $(x, y) \in T^m \times T$, $R(x, y, a)$ on $(x, y, a) \in T^m \times T \times (-\epsilon, \epsilon)$,

$$E_a(x + \gamma_x(y, a), y) = \{X + aG(x, y)X + R(x, y, a)X : X \in \mathcal{E}\}$$

and $\|R(x, y, a)\|$ is $o(a)$ uniformly in $x, y$. 

Notice now that, if \( \tilde{x} = \pi_a(x, y) \), then \( x = \tilde{x} + \gamma_{\tilde{x}}(y, a) \), where \( \gamma_{\tilde{x}}(y, a) = O(a) \). Now

\[
E_a(x, y) = E_a(\tilde{x} + \gamma_{\tilde{x}}(y, a), y) = \{X + aG(\tilde{x}, y)X + R(\tilde{x}, y, a)X : X \in \mathcal{E}\}
\]
diffs from

\[
E_a(x + \gamma_x(y, a), y) = \{X + aG(x, y)X + R(x, y, a)X : X \in \mathcal{E}\}
\]
by the replacement \( \tilde{x} \to x \) in the right-hand side, and since \( \text{dist}(\tilde{x}, x) = O(a) \), we find that \( \text{dist}(E_a(x, y), E_a(x + \gamma_x(y, a), y)) = o(a) \). Therefore, changing the definition of \( R \), we can again write:

8. Corollary.

There are linear maps \( G(x, y), R(x, y, a) : \mathcal{E} \to \mathcal{E}^\perp \), depending continuously on their arguments, such that

\[
E_a(x, y) = \{X + aG(x, y)X + R(x, y, a)X : X \in \mathcal{E}\}
\]
and \( ||R(x, y, a)|| \) is \( o(a) \) uniformly in \( x, y \). \( \square \)

We may write \( T_\xi F = T_\xi(F_0 + aF') = D_0 + aD'(\xi) \) where \( D_0 \) does not depend on \( \xi \) and preserves the decomposition \( T_\xi M = \mathcal{E} + \mathcal{E}^\perp \). If we apply \( TF \) to an element \( X + aGX + RX \) of \( E_a \) (as in Corollary 8) we obtain \( X_1 + \) element of \( \mathcal{E}^\perp \in E_a \), with \( X_1 \in \mathcal{E} \):

\[
X_1 = D_0X + aD'X + a^2D'GX + aD'RX \quad \text{projected on } \mathcal{E} \quad (1)
\]
Under \( (TF)^\wedge \), the volume element \( \theta \) in \( E_a(\xi) \) is multiplied by a factor \( M(\xi, a) \), and the projection in \( \mathcal{E} \) of \( (TF)^\wedge \theta \) is equal to the projection in \( \mathcal{E} \) of \( \theta \) multiplied by a factor \( N(\xi, a) \) such that

\[
M(\xi, a) = N(\xi, a) + \ell_a(\xi) - \ell_a(F\xi)
\]
for suitable \( \ell_a \). We may compute \( N \) from (1):

\[
N(\xi, a) = N(0) + aN(1)(\xi) + a^2N(2)(\xi) + o(a^2)
\]

To proceed we take now \( E_a = E_a^r \), and assume \( \lambda_r < 0 \). We have then, writing \( d\xi \) for the volume element in \( T^{m+1} \),

\[
L_a = \sum_{\ell=1}^{m_1 + \ldots + m_r} \chi_{a}^{(\ell)} = \int d\xi \log M(\xi, a) = \int d\xi \log N(\xi, a)
\]

\[
= L(0) + aL(1)(\xi) + a^2L(2)(\xi) + o(a^2) \quad (2)
\]

More precisely, we shall prove
9. Proposition.

If \( \lambda_r < 0 \), we have

\[
\sum_{\ell=1}^{m_1+\ldots+m_r} \lambda_a^{(\ell)} = \sum_{k=1}^r m_k \lambda_k + a^2 L + o(a^2)
\]

where

\[
L = \frac{1}{2} \sum_{n=-\infty}^\infty \int d\xi \text{Tr}_\mathcal{E}(D_0^{-1}D'(\xi))\text{Tr}_\mathcal{E}(D_0^{-1}D'(F_0^n\xi)) \geq 0
\]

and \( \mathcal{E} \) is the spectral subspace of the matrix defining \( \Phi \) corresponding to the smallest \( m_1 + \ldots + m_r \) eigenvalues (in absolute value, and repeated according to multiplicity).

The proof that \( L \geq 0 \) is postponed to Remark 15(b).

The proposition is obtained by comparing formula (2) with the formula (5) below, which we shall obtain by a second order perturbation calculation.

To first order in \( a \) we have

\[
F^n = (F_0 + aF')^n = F_0^n + a \sum_{j=1}^n F_0^{n-j} \circ F' \circ F_0^{j-1}
\]

hence

\[
T_\xi F^n = D_0^n + a \sum_{j=1}^n D_0^{n-j} D'(F^{j-1}\xi)D_0^{j-1}
\]

If we apply \( TF^n \) to \( X + aGX + RX \in E_a \) we obtain \( X_n + \text{element of } \mathcal{E}_\perp \in E_a \), with \( X_n \in \mathcal{E} \). To zero-th order in \( a \), \( X_n = D_0^nX \), so we may write to first order \( X_n = D_0^nX + aY_n(\xi) \).

Therefore, to first order in \( a \),

\[
D_0^nX + aY_n(\xi) + aG(F^n\xi)D_0^nX = D_0^nX + a \sum_{j=1}^n D_0^{n-j} D'(F^{j-1}\xi)D_0^{j-1}X + aD_0^nG(\xi)X
\]

and, taking the components along \( \mathcal{E}_\perp \),

\[
G(F^n\xi)D_0^nX = \sum_{j=1}^n D_0^{n-j} D'_\perp(F^{j-1}\xi)D_0^{j-1}X + D_0^nG(\xi)X
\]

where \( D'_\perp(.) \) is \( D'(.) \) followed by taking the component along \( \mathcal{E}_\perp \), or

\[
\sum_{j=1}^n D_0^{n-j} D'_\perp(F^{j-1}\xi)D_0^{j-1}X + G(\xi)X = D_0^{-n}G(F^n\xi)D_0^nX
\]
When \( n \to \infty \), the right-hand side tends to zero (exponentially fast, remember that \( X \in \mathcal{E}, \ GX \in \mathcal{E}^\perp \)). Therefore

\[
G(\xi)X = -\sum_{j=1}^{\infty} D_0^{-j} D'_\perp (F^{j-1}_\perp \xi) D_0^{j-1}X
\]

which we shall use in the form

\[
G(\xi)X = -\sum_{n=0}^{\infty} D_0^{-n-1} D'_\perp (F^n_\perp \xi) D_0^nX
\]

where we have written \( F^n_0 \) instead of \( F^n \) since \( G \) is evaluated to order 0 in \( a \). (The right-hand side is an exponentially convergent series).

Returning to (1) we see that, to second order in \( a \),

\[
X_1 = D_0X + aD'(\xi)X + a^2 D'(\xi)G(\xi)X \quad \text{projected on } \mathcal{E}
\]

\[
= D_0(1 + aD^{-1}_0 D'(\xi) + a^2 D^{-1}_0 D'(\xi)G(\xi))X \quad \text{projected on } \mathcal{E}
\]

Let now \((u^{(i)})\) and \((u^{(i)}\perp)\) be conjugate bases of \( \mathcal{E} \). Also let \( \delta^{(i)} \) for \( i = 1, \ldots, m_1 + \ldots + m_r \) be the eigenvalues of \( D_0 \) restricted to \( \mathcal{E} \). Then, to second order in \( a \),

\[
N(\xi, a) \wedge_{1}^{m_1 + \ldots + m_r} u^{(\ell)}
\]

is, up to a factor of absolute value 1,

\[
\prod_{\ell=1}^{m_1 + \ldots + m_r} \delta^{(\ell)}[1 + a \sum_{i=1}^{m_1 + \ldots + m_r} (u^{(i)}\perp, D^{-1}_0 D'(\xi) u^{(i)})
\]

\[
+ a^2 \sum_{i<j} ((u^{(i)}\perp, D^{-1}_0 D'(\xi) u^{(i)})(u^{(j)}\perp, D^{-1}_0 D'(\xi) u^{(j)})
\]

\[
- (u^{(i)}\perp, D^{-1}_0 D'(\xi) u^{(j)})(u^{(j)}\perp, D^{-1}_0 D'(\xi) u^{(i)}) + a^2 \sum_{i} (u^{(i)}\perp, D^{-1}_0 D'(\xi) G(\xi) u^{(i)})] \wedge_{\ell} u^{(\ell)}
\]

so that

\[
N(\xi, a) = \prod_{\ell=1}^{m_1 + \ldots + m_r} \delta^{(\ell)}[1 + \{a \sum_{i} (u^{(i)}\perp, D^{-1}_0 D'(\xi) u^{(i)})
\]

\[
+ a^2 \sum_{i<j} ((u^{(i)}\perp, D^{-1}_0 D'(\xi) u^{(i)})(u^{(j)}\perp, D^{-1}_0 D'(\xi) u^{(j)})
\]

\[
- (u^{(i)}\perp, D^{-1}_0 D'(\xi) u^{(j)})(u^{(j)}\perp, D^{-1}_0 D'(\xi) u^{(i)}) + a^2 \sum_{i} (u^{(i)}\perp, D^{-1}_0 D'(\xi) G(\xi) u^{(i)})]\]

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Since $\log |\delta^{(0)}| = \lambda_0^{(0)}$ we obtain, to second order in $a$,

$$L_a = \int d\xi \log N(\xi, a) = m_1 \lambda_1 + \ldots + m_r \lambda_r + \int d\xi \left\{ \ldots - \frac{a^2}{2} \left( \sum_i (u^{(i)}_\perp, D_0^{-1} D'(\xi) u^{(i)})^2 \right) \right\}$$

where $\ldots$ has the same meaning as above. Write

$$\Psi_i (\sum_\ell \xi_\ell u^{(\ell)}) = (u^{(i)}_\perp, D_0^{-1} F'(\sum_\ell \xi_\ell u^{(\ell)}))$$

The first term of $\int d\xi \{ \ldots \}$ is

$$a \sum_i \int d\xi (u^{(i)}_\perp, D_0^{-1} F'(\xi)) u^{(i)}) = a \sum_i \int d\xi \frac{\partial}{\partial \xi_i} \Psi_i$$

which vanishes because $\int d\xi \frac{\partial}{\partial \xi_i} \ldots = 0$. The next term in $\int d\xi \{ \ldots \}$ is

$$a^2 \sum_{i < j} \int d\xi \left\{ \left( \frac{\partial \Psi_i}{\partial \xi_i} \right) \left( \frac{\partial \Psi_j}{\partial \xi_j} \right) - \left( \frac{\partial \Psi_i}{\partial \xi_j} \right) \left( \frac{\partial \Psi_j}{\partial \xi_i} \right) \right\} = a^2 \sum_{i < j} \int d\xi \left( \frac{\partial}{\partial \xi_i} (\Psi_i \frac{\partial \Psi_j}{\partial \xi_j}) - \frac{\partial}{\partial \xi_j} (\Psi_j \frac{\partial \Psi_i}{\partial \xi_i}) \right)$$

which vanishes as above. Thus we are left with

$$L_a - (m_1 \lambda_1 + \ldots + m_r \lambda_r)$$

$$= a^2 \int d\xi \left\{ \sum_i (u^{(i)}_\perp, D_0^{-1} D'(\xi) G(\xi) u^{(i)}) - \frac{1}{2} \left( \sum_i (u^{(i)}_\perp, D_0^{-1} D'(\xi) u^{(i)})^2 \right) \right\}$$

and we may write, using (3),

$$\sum_i (u^{(i)}_\perp, D_0^{-1} D'(\xi) G(\xi) u^{(i)}) = - \sum_{n=0}^{\infty} \sum_i (u^{(i)}_\perp, D_0^{-1} D'(\xi) D_0^{-n-1} D'_\perp (F^n_0 D^n_0 u^{(i)}))$$

$$= - \sum_{n=0}^{\infty} \sum_i \sum_j^* (u^{(i)}_\perp, D_0^{-1} D'(\xi) u^{(j)}) (u^{(j)}_\perp, D_0^{-n-1} D'(F^n_0 \xi) D^n_0 u^{(i)})$$

where we have introduces conjugate bases $(u^{(j)})$, $(u^{(j)})_\perp$ of $E$, indexed by $j = m_1 + \ldots + m_r + 1, \ldots, m + 1$, and $\sum_i$ is over $i \leq m_1 + \ldots + m_r + 1$, $\sum_j$ is over $j > m_1 + \ldots + m_r + 1$. The above expression is also

$$= - \sum_{n=0}^{\infty} \sum_i \sum_j (u^{(i)}_\perp, D_0^{-1} F'(\sum_\ell \xi_\ell u^{(\ell)})) (u^{(j)}_\perp, D_0^{-n-1} D'(F^n_0 \sum_\ell \xi_\ell u^{(\ell)}))$$

and integration by part gives thus

$$\int d\xi \sum_i (u^{(i)}_\perp, D_0^{-1} D'(\xi) G(\xi) u^{(i)})$$
The fact that $F = F_0 + aF'$ is volume preserving is expressed by $\text{Tr}_{\mathbb{R}^{m+1}}(D_0^{-1}D'(\xi)) = 0$ hence
\[
\int d\xi \sum_{i=1}^{\infty} (u^{(i)}, D_0^{-1}D'(\xi)G(\xi)u^{(i)}) = \sum_{n=0}^{\infty} \int d\xi \text{Tr}_E(D_0^{-1}D'(\xi))\text{Tr}_E(D_0^{-1}D'(F_0^n\xi))
\]
and introducing this in (4) yields
\[
L_a - (m_1\lambda_1 + \ldots + m_r\lambda_r)
\]
\[
= \frac{a^2}{2} \sum_{n=-\infty}^{\infty} \int d\xi \text{Tr}_E(D_0^{-1}D'(\xi))\text{Tr}_E(D_0^{-1}D'(F_0^n\xi))
\]
where the last step used the invariance of $d\xi$ under $F_0^n$. □

10. Proof of Theorem 1.

We use Proposition 9, the corresponding result with $F$ replaced by $F^{-1}$, and the fact that $\sum_{\ell=1}^{m} \lambda_\ell = 0$ (because $F$ is volume preserving). This gives an estimate of all the sums of $\lambda_\ell$ that occur in Theorem 1. □

11. Corollary.

In the situation of Theorem 1, the central Lyapunov exponent is
\[
\lambda^c = \frac{a^2}{2} \sum_{n=-\infty}^{\infty} \int d\xi [\text{Tr}^u(D_0^{-1}D'(\xi))\text{Tr}^u(D_0^{-1}D'(F_0^n\xi)) - \text{Tr}^s(D_0^{-1}D'(\xi))\text{Tr}^s(D_0^{-1}D'(F_0^n\xi))]
\]
\[
= \frac{a^2}{2} \sum_{n=-\infty}^{\infty} \int d\xi [\text{Tr}^s(D_0^{-1}D'(\xi)) - \text{Tr}^u(D_0^{-1}D'(\xi))]\text{Tr}^c(D_0^{-1}D'(F_0^n\xi))
\]
where $\text{Tr}^s$, $\text{Tr}^u$, $\text{Tr}^c$ denote the traces over the spectral subspaces $\mathcal{E}^s$, $\mathcal{E}^u$, $\mathcal{E}^c$ of $D_0$ corresponding to eigenvalues $< 1$, $> 1$, or $= 1$ in absolute value ($\mathcal{E}^c$ is one dimensional).
Since $F$ preserves the volume, the sum of all Lyapunov exponents vanishes. Therefore $\lambda^c$ is minus the sum of the negative Lyapunov exponents, given by (5), minus the sum of the positive Lyapunov exponents. Note that replacing $F$ by $F^{-1}$, $\mathcal{E}^s$ by $\mathcal{E}^u$ (and, to the order considered, $D'(\xi)$ by $-D'(\xi)$) replaces the sum of the negative Lyapunov exponents by minus the sum of the positive exponents. This gives the first formula for $\lambda^c$.

To obtain the second formula, express $\text{Tr}^u \text{Tr}^u - \text{Tr}^s \text{Tr}^s$ in terms of $\text{Tr}^u \pm \text{Tr}^s$, and remember that (because $F$ preserves the volume) $\text{Tr}^s + \text{Tr}^u + \text{Tr}^c = 0$ when applied to $D_0^{-1} D'(\xi)$. □

The above formula (5) takes a particularly simple form in a special case described in the next theorem.

12. Theorem.

Let $\Phi$ be a hyperbolic automorphism of $T^m$, with stable and unstable dimensions $m^s$ and $m^u = m - m^s$, and with entropy $\lambda^s$. Let $J : y \to y + \alpha \pmod{1}$ be a translation of $T$, and $\phi : T^m \to T$ a morphism $\neq 0$. Finally let $\psi : T \to R^m$ be a nullhomotopic $C^2$ function.

Define $h, g_a : T^m \times T \to T^m \times T$ by

$$h\left(\frac{x}{y}\right) = (Jy + \phi \Phi x - \phi x), \quad g_a\left(\frac{x}{y}\right) = \left(x + a\psi(y) \pmod{1}\right)$$

and let $f_a = g_a \circ h$.

Denote by $\lambda^s_a$ (resp. $\lambda^u_a$) the sum of the smallest $m^s$ (resp. the largest $m^u$) Lyapunov exponents for $(f_a, \text{volume})$. Also let $\lambda^c_a = -\lambda^s_a - \lambda^u_a$ be the “central exponent”. Then $\lambda^s_a, \lambda^u_a, \lambda^c_a$ have expansions of order 2 in $a$:

$$\lambda^s_a = -\lambda^s_0 + \frac{a^2}{2} \int_T dy \left( (\nabla \phi) \psi'^s(y) \right)^2 + o(a^2)$$

$$\lambda^u_a = \lambda^u_0 - \frac{a^2}{2} \int_T dy \left( (\nabla \phi) \psi'^u(y) \right)^2 + o(a^2)$$

$$\lambda^c_a = \frac{a^2}{2} \int_T dy \left[ \left( (\nabla \phi) \psi'^u(y) \right)^2 - \left( (\nabla \phi) \psi'^s(y) \right)^2 \right] + o(a^2)$$

Here $\psi'^s(y)$ and $\psi'^u(y)$ are the components of the derivative $\psi'(y) \in R^m$ in the stable and unstable subspaces $\mathcal{E}^s$ and $\mathcal{E}^u$ for $\Phi$. Also, we have used $\nabla \phi : R^m \to R$ to denote the derivative of the map $\phi : T^m \to T$ with the obvious identifications.

This theorem is a simple (but nontrivial) extension of the result proved by Shub and Wilkinson [10]. In the situation that they consider $\Phi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $J =$identity, $\phi = (1,0)$, $\psi' = \psi'^u$. [Remark that, in the notation of [10], $u_0 = ((1,1).v_0)/(m-1) = ((1,0).v_0)$ so that the formula given in Proposition II of [10] agrees with our result above].
Notation.

We shall henceforth omit the (mod 1). We shall keep $\nabla$ to denote the derivative in $T^m$. With obvious abuses of notation, the reader may find it convenient to think of $\Phi$ or $\nabla \Phi$ as an $m \times m$ matrix (with integer entries and determinant $\pm 1$), and $\phi$ or $\nabla \phi$ as a row $m$-vector (with integer entries not all zero).

13. Reformulation of the problem.

Note that $f_a^{-1} = h^{-1} \circ g_a^{-1}$ where $h^{-1}, g_a^{-1}$ are obtained from $h, g_a$ by the replacements $\Phi, J, \phi, \psi \to \Phi^{-1}, J^{-1}, \phi, -\psi$. These replacements also interchange the stable and unstable subspaces for $\Phi$ and replace $\lambda^s, \lambda^u$ by $-\lambda^u, -\lambda^s$. Therefore the formula for $\lambda^u$ in the theorem follows from the formula for $\lambda^s$. And the formula for $\lambda^c = -\lambda^s - \lambda^u$ also follows. To complete the proof of the theorem we turn now to the formula for $\lambda$ for $\Phi$ as an $m$-vector (with integer entries not all zero).

Define

$$\hat{\phi}(x) = (x \ y + \phi x)$$

then

$$F_0(x) = \hat{\phi}^{-1} h \hat{\phi}(x) = (\Phi x \ Jy)$$

$$\hat{g}_a(x) = \hat{\phi}^{-1} g_a \hat{\phi}(x) = (x + a\psi(y + \phi x) \ y - a(\nabla \phi)\psi(y + \phi x))$$

so that

$$F(x) = \hat{\phi}^{-1} f_a \hat{\phi}(x) = \hat{g}_a F_0(x) = (\Phi x + a\psi(Jy + \phi x) \ Jy - a(\nabla \phi)\psi(Jy + \phi x))$$

Finally, $F = F_0 + aF'$ with

$$F_0(x) = (\Phi x \ Jy) \quad , \quad F'(x) = (\psi(Jy + \phi x) \ -(\nabla \phi)\psi(Jy + \phi x))$$

Since $F$ is conjugate (linearly) to $f_a$, we may compute $\lambda^s$ from $F$ instead of $f_a$.

14. Proof of Theorem 12.

Write $R^{m+1} = E^s + E^u + R$. We shall apply Proposition 9 with $E = E^s, E^\perp = E^u + R$. Using $\xi = (x, y)$ and $X \in E^s, Y \in E^u, Z \in R$ we may write

$$D_0(X + Y \ Z) = ((\nabla \Phi)(X + Y) \ )$$

$$D'(\xi)(X + Y \ Z) = (\psi'(Jy + \phi x)((\nabla \phi \Phi)(X + Y) + Z) \ -((\nabla \phi)\psi'(Jy + \phi x)((\nabla \phi \Phi)(X + Y) + Z)$$

where $\psi'$ denotes the derivative of $\psi$. Therefore

$$\text{Tr}_E(D'(\xi)D_0^{-1}) = (\nabla \phi)\psi'_{ls}(Jy + \phi x)$$
and (5) contains the integrals
\[\int d\xi \text{Tr}_E(D_0^{-1}D'(\xi))\text{Tr}_E(D_0^{-1}D'(F_0^n\xi))\]
\[= \int d\xi [(\nabla \phi)\psi^s(Jy + \phi \Phi x)][(\nabla \phi)\psi^s(J^{n+1}y + \phi \Phi^{n+1}x)]\]
Performing a change of variables \(\bar{x} = \Phi x, \bar{y} = Jy + \phi \Phi x\) we find that this is
\[= \int d\bar{x} d\bar{y} [(\nabla \phi)\psi^s(\bar{y})][(\nabla \phi)\psi^s(J^n\bar{y} + \phi \Phi^n\bar{x} - \phi \bar{x})]\]
We claim that this last integral vanishes unless \(n = 0\). This is because, if \(n \neq 0\),
\[\int d\bar{x} \psi'(J^n\bar{y} + \phi \Phi^n\bar{x} - \phi \bar{x}) = 0\]
Indeed, \(\phi \Phi^n\bar{x} - \phi \bar{x}\) is a linear combination with integer coefficients of the components \(\bar{x}_1, \ldots, \bar{x}_m\) of \(\bar{x}\), and the coefficients do not all vanish because \(\phi \Phi^n = \phi\) is impossible (\(\Phi\) is hyperbolic and \(\phi \neq 0\)). Integrating the derivative \(\psi'\) with respect to a variable \(\bar{x}_j\) really occurring in \(\phi \Phi^l\bar{x} - \phi \bar{x}\) gives zero as announced.

Returning to (5) we have thus
\[\lambda^s_0 + \lambda^u_0 = \frac{a^2}{2} \int d\xi (\text{Tr}_E(D_0^{-1}D'(\xi)))^2\]
\[= \frac{a^2}{2} \int d\bar{y}((\nabla \phi)\psi^s(\bar{y}))^2\]
which is the formula given for \(\lambda^s_0\) in Theorem 12. And according to Section 13 this completes our proof. □

15. Final remarks.

(a) Shub and Wilkinson [10] showed that close to a diffeomorphism (hyperbolic automorphism \(\Phi\) of \(T^2\) × (identity on \(T\)) there is a \(C^1\) open set of ergodic volume preserving \(C^2\) diffeomorphisms of \(T^3\) with central Lyapunov exponent \(\lambda^c > 0\). They remark that their result extends to the situation where \(\Phi\) is a hyperbolic automorphism of \(T^m\) with one-dimensional expanding eigenspace. More generally, if \(\Phi\) is any hyperbolic automorphism of \(T^m\), Theorem 12 gives close to \((\Phi, \text{rotation of } T)\) in \(C^2(T^{m+1})\) a diffeomorphism \(F\) with \(\lambda^c > 0\). Since \(\lambda^c\) is given by an integral over the volume of a local “central” stretching exponent, we have \(\lambda^c > 0\) in a \(C^1\) neighborhood of \(F\). But by a result of Dolgopyat and Wilkinson [5] (Corollary 0.5), stable ergodicity is here \(C^1\) open and dense in the \(C^2\) volume preserving diffeomorphisms: we have center bunching and stable dynamical coherence because we consider perturbations of \((\Phi, \text{rotation of } T)\) for which the center foliation is \(C^1\), see [6], [7]. In conclusion, close to (hyperbolic automorphism \(\Phi\) of \(T^m\)) × (rotation on \(T\)) there is a \(C^1\) open set \(V\) of ergodic volume preserving \(C^2\) diffeomorphisms of \(T^{m+1}\) with...
central Lyapunov exponent \( \lambda^c > 0 \) (or also with \( \lambda^c < 0 \)). In particular, if \( F \in V \), the conditional measures of the volume on the circles \( \pi^{-1}\{x\} \) are atomic, as discussed in [9].

(b) The coefficient \( L \) in Proposition 9 is \( \geq 0 \). Consider indeed the unitary operator \( U \) defined by \( U \psi = \psi \circ F \) on \( L^2(\mathbb{T}^{m+1}, \text{volume}) \), and let \( E(.) \) be the corresponding spectral measure, so that

\[
U = \int_{\mathbb{T}} e^{2\pi i \theta} E(d\theta)
\]

If \( \psi(\xi) = \text{Tr} \varepsilon(D_0^{-1}D'(\xi)) \) we have a measure \( \nu \geq 0 \) on \( \mathbb{T} \) defined by

\[
\nu(d\theta) = (\psi, E(d\theta)\psi)
\]

and the Fourier coefficients

\[
c_n = \int e^{2\pi i n \theta} \nu(d\theta) = \int d\xi \text{Tr} \varepsilon(D_0^{-1}D'(\xi))(D_0^{-1}D'(F_0^n\xi))
\]

of this measure tend to zero exponentially. Therefore \( \nu(d\theta) = \rho(\theta)d\theta \) has a smooth density \( \rho \) and

\[
L = \frac{1}{2} \sum_{n=-\infty}^{\infty} c_n = \frac{1}{2} \rho(0) \geq 0
\]

(c) Suppose now that \( F \) is not necessarily a volume preserving perturbation of \( F_0 \). We may still hope that \( F \) has an SRB measure \( \rho_a \). If \( F_0 \) were hyperbolic, we would have an expansion

\[
\rho_a = \rho_0 + a\delta + o(a)
\]

(see [8]) with \( \rho_0 = \text{Lebesgue measure} \) and \( \delta \) a distribution. For smooth \( \Psi, \delta(\Psi) \) is given (because \( \rho_0 \) is Lebesgue measure) by the simple formula (see [8])

\[
\delta(\Psi) = -\sum_{0}^{\infty} \rho_0((\Psi \circ F_0^n) \text{div}(F' \circ F_0^{-1}))
\]

Similarly (replacing \( F \) by \( F^{-1} \), hence \( F_0, D_0^{-1}D'(\xi) \) by \( F_0^{-1}, -D'(F_0^{-1}\xi)D_0^{-1} \) we see that the anti-SRB state has an expansion

\[
\bar{\rho}_a = \rho_0 + a\bar{\delta} + o(a)
\]

with

\[
\bar{\delta}(\Psi) = \sum_{n=1}^{\infty} \int d\xi \Psi(F_0^{-n}\xi)\text{Tr}_{\mathbb{R}^{m+1}}(D'(F_0^{-1}\xi)D_0^{-1})
\]

\[
= \sum_{n=0}^{\infty} \int d\xi \Psi(F_0^{-n}\xi)\text{Tr}_{\mathbb{R}^{m+1}}(D_0^{-1}D'(\xi))
\]

We can now estimate the Lyapunov exponents for \((F, \rho_a)\) to second order in \( a \) even though we are not sure of the existence of the SRB measure \( \rho_a \). We simply assume that we can use the formula for \( \delta(\Psi) \). Going through the proof of Proposition 9 we have to replace
\[ \int d\xi \log N(\xi, a) \text{ by } \rho_a(\log N(., a)) \text{ and (to second order in } a) \text{ this adds to the right-hand side of (4) a term} \]
\[ -a^2 \sum_{n=1}^{\infty} \int d\xi \operatorname{Tr}_E(D_0^{-1}D'(\xi)) \operatorname{Tr}_{R^{m+1}}(D_0^{-1}D'(\xi)) \]

Taking into account the integrations by part we obtain now instead of (5) the formula
\[ L_a - (m_1 \lambda_1 + \ldots + m_r \lambda_r) = \frac{a^2}{2} \sum_{n=-\infty}^{\infty} \int d\xi \operatorname{Tr}_E(D_0^{-1}D'(\xi)) \operatorname{Tr}(D_0^{-1}D'(F_0^n\xi)) \]
\[ -a^2 \sum_{n=-\infty}^{\infty} \int d\xi \operatorname{Tr}_E(D_0^{-1}D'(\xi)) \operatorname{Tr}_{R^{m+1}}(D_0^{-1}D'(F_0^n\xi)) \] (6)

Let \( a^2L^s, a^2L^u, a^2L^c \) be the \( a^2 \) contributions to the sum of the noncentral negative, noncentral positive, and the central Lyapunov exponents for the SRB measure. We obtain \( a^2L^s \) from (6) when \( n_r = n^s \). A similar calculation gives \( a^2L^u \) (it is convenient here to work via the anti-SRB measure, then replace \( F \) by \( F^{-1} \)). Estimating the average expansion coefficient gives \( a^2(L^s + L^u + L^c) = \rho_a(\log \det(D_0 + aD'(.)), \text{ hence } L^s + L^u + L^c \), hence \( L^c \). The results are
\[ L^s = \frac{1}{2} \sum_{n=-\infty}^{\infty} \int d\xi \operatorname{Tr}^s(D_0^{-1}D'(\xi)) \operatorname{Tr}^s(D_0^{-1}D'(F_0^n\xi)) \]
\[ - \sum_{n=-\infty}^{\infty} \int d\xi \operatorname{Tr}^s(D_0^{-1}D'(\xi)) \operatorname{Tr}_{R^{m+1}}(D_0^{-1}D'(F_0^n\xi)) \]
\[ L^u = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \int d\xi \operatorname{Tr}^u(D_0^{-1}D'(\xi)) \operatorname{Tr}^u(D_0^{-1}D'(F_0^n\xi)) \]
\[ L^c = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \int d\xi \operatorname{Tr}^c(D_0^{-1}D'(\xi)) \operatorname{Tr}^c(D_0^{-1}D'(F_0^n\xi)) \]
\[ - \sum_{n=-\infty}^{\infty} \int d\xi \operatorname{Tr}^c(D_0^{-1}D'(\xi)) \operatorname{Tr}_{R^{m+1}}(D_0^{-1}D'(F_0^n\xi)) \]
\[ L^s + L^u + L^c = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \int d\xi \operatorname{Tr}_{R^{m+1}}(D_0^{-1}D'(\xi)) \operatorname{Tr}_{R^{m+1}}(D_0^{-1}D'(F_0^n\xi)) \]

which can be rewritten variously.

The existence of second order expansions for the Lyapunov exponents gives added interest to the question whether the SRB measure \( \rho_a \) really exists for (small) finite \( a \).

(d) The author has not looked seriously into possible extensions of the results presented here. Generalizations are thus left for the reader to formulate, and to prove.
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