The structure of multilinear part of variety $\tilde{V}_3$ of Leibnitz algebras

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We find values of multiplicities and colength variety of Leibniz algebras of almost polynomial growth, which is generated by the algebra constructed with the help of the Heisenberg algebra and its infinite-dimensional irreducible representations.

1. Introduction

The characteristic of base field $\Phi$ throughout the paper will be zero. This article discusses the variety $\tilde{V}_3$ of Leibnitz algebras and its numerical characteristics. This variety is an analogue of the well-known varieties $V_3$ of Lie algebras and is generated by the Leibnitz algebra constructed with help of Heisenberg algebra and its infinite-dimensional irreducible representations. Earlier in the paper [5] it is proved that the variety $\tilde{V}_3$ has almost polynomial growth. So the variety $\tilde{V}_3$ of Leibnitz algebras is extreme in relation to the property "to have polynomial growth." The rest of its numerical characteristics were undefined. The starting point of the study was the result proved in the article [6] about almost finite multilicities of variety $3N$ of Leibnitz algebras, for which $\tilde{V}_3$ is a subvariety. In this paper we study the multiplicities and the colength of variety $\tilde{V}_3$.

The first section of the paper is preliminary and contains the basic definitions and notation needed in the sequel. In the second section we present the structure of the generating algebra of variety $\tilde{V}_3$ of Leibnitz algebras and describe the non-zero elements of a relatively free algebra of this variety, which generate the irreducible modules of the symmetric group. In the third section we obtain the exact values of the multiplicities. This result made it possible to study the question of determining the growth of other numerical characteristics, the so-called colength, for which in the third section we have described only the asymptotics of growth. The fourth section contains the output of the exact formula colength of variety $\tilde{V}_3$. 
2. Basic definitions and notation.

A Leibnitz algebra is a vector space over a field $\Phi$ with multiplication, which satisfies the Leibnitz identity:

$$(xy)z \equiv (xz)y + x(yz).$$

Probably first this class of algebras was introduced in the paper [2] as a generalization of concept of Lie algebra.

The determining identity of Leibnitz algebras can be represented as follows:

$$x(yz) \equiv (xy)z - (xz)y.$$ This form of identity allows us any element from Leibnitz algebra to write in the form of linear combination of elements, in which the brackets are arranged from left to right. Therefore agree omit brackets if they are left-normed arrangement, i.e.:

$$(((x_1x_2)x_3)\ldots x_n) = x_1x_2x_3\ldots x_n.$$ 

For convenience we denote the operator of right multiplication for example an element $z$ by a capital letter $Z$, assuming that $xz = xZ$. In particular, in our notation we obtain $xy\ldots y_m = xY^m$. 

The collection of all algebras over a field $\Phi$ that satisfy a fixed set of identities, called a variety $\mathbf{V}$ of algebras over a field $\Phi$. Note, that the system of identities can be given implicitly. In this case the variety $\mathbf{V}$ is usually defined generating algebra given constructively.

In the paper [4] it is proved that in the case of zero characteristic of the base field, all information about the variety found in the multilinear elements of its relatively free algebra. Let $F(X, \mathbf{V})$ be a relatively free algebra of variety $\mathbf{V}$ from countable set of free generators $X = \{x_1, x_2, \ldots\}$. We will denote by $P_n(\mathbf{V})$ the space of all multilinear elements from generators $x_1, x_2, \ldots, x_n$ of algebra $F(X, \mathbf{V})$. Note that for convenience of presentation we will denote the relatively free algebra generators also other symbols.

Let $q$ be an element of the symmetric group $S_n$. Assume that as a result of the actions of the left permutation $q$ on the element $x_{i_1}x_{i_2}\ldots x_{i_m}$ of space $P_n$ we receive the element $x_{iq(i_1)}x_{iq(i_2)}\ldots x_{iq(i_m)}$. This sets the action of the group $S_n$ on the space $P_n$, in consequence the space $P_n$ becomes a $\Phi S_n$-module. This fact allows for the study varieties of Leibnitz algebras over a field of zero characteristic to use the well-developed theory representations of the symmetric group.

As characteristic of the field $\Phi$ is assumed to be zero, then the module $P_n$ is completely reducible. It is known that up to isomorphism irreducible $\Phi S_n$-modules can be described in terms of representations and Young diagrams.

A partition of number $n$ is a set of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, where $\lambda_1 \geq \lambda_2 \geq \ldots, \lambda_k > 0$ and $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$. The partition $\lambda$ of number $n$ we will denote by $\lambda \vdash n$. Each partition $\lambda$ of the number $n$ of
one-to-one corresponds to a Young diagram consisting of $n$ cells in $k$ rows and containing in the $i$-th row $\lambda_i$ cells.

Denoted by $\chi_\lambda$ the character of the irreducible representations of the symmetric group, which corresponds to the partition $\lambda$ of the number $n$. Then, as the module $P_n(V)$ is completely reducible, for its character is true a decomposition:

$$\chi_n(V) = \sum_{\lambda \vdash n} m_\lambda(V) \chi_\lambda.$$ 

The number of terms

$$l_n(V) = \sum_{\lambda \vdash n} m_\lambda$$

in this sum is called a co-length of the variety. Important numerical characteristics of variety are also multiplicities $m_\lambda(V)$.

The dimension of the space $P_n(V)$ we denote by $c_n(V)$. Let $d_\lambda$ be a dimension of corresponding to $\lambda$ irreducible module. Then for introduced numerical characteristics is carried out the following relation:

$$c_n(V) = \sum_{\lambda \vdash n} m_\lambda(V) d_\lambda.$$ 

An important concept for varieties is their growth. The growth of variety $V$ is the growth of the sequence of number $c_n(V)$. The sequence of number $c_n(V)$ is called also a sequence of codimensions of verbal ideal. The growth of variety is called polynomial, if there are non-negative integers $C, m$ such that for any $n$ is true the inequality $c_n(V) < Cn^m$. We say that the variety has almost polynomial growth if the growth of this variety is not polynomial, and the growth of any of its proper subvariety is polynomial. As previously noted the variety $\tilde{V}_3$ has almost polynomial growth.

Since we consider the case of zero characteristic of the base field, then every identity is equivalent to the system of multilinear identities, which can be obtained using the standard method of linearization [4]. Here is an example of this process for the identity

$$x_0(xy)(xy) \equiv 0.$$ 

After linearization of the variable $x$ we obtain:

$$x_0(x_1y)(x_2y) + x_0(x_2y)(x_1y) \equiv 0.$$ 

Complete linearization is so:

$$x_0(x_1y_1)(x_2y_2) + x_0(x_1y_2)(x_2y_1) + x_0(x_2y_1)(x_1y_2) + x_0(x_2y_2)(x_1y_1) \equiv 0.$$ 

The space of the multilinear elements of degree $n$ of any variety of Leibnitz algebras over a field of zero characteristic by Mashke’s theorem can be decomposed as a direct sum of irreducible submodules, corresponding to all possible Young diagrams of $n$ cells; moreover two modules are isomorphic if and only if they are correspond to the same diagram. It is well-known (see, for example,
that each from these submodules is generated by linearization of an element $f$, which is constructed according to the Young diagram corresponding to the partition $\lambda$ of number $n$.

To describe the structure of elements, linearization of which generates irreducible submodules of the specified sum, we need to introduce some notation. Recall that the standard polynomial of degree $n$ has the form:

$$St_n(x_1, x_2, \ldots, x_n) = \sum_{q \in S_n} (-1)^q x_{q(1)} x_{q(2)} \cdots x_{q(n)},$$

where the summation is carried out by elements of the symmetric group, and $(-1)^q$ is equal to $+1$ or $-1$ depending on the parity of permutation $q$. Agree variables in standard polynomial denote with special symbols above (below, wave and etc.). For example the standard polynomial of degree $n$ in the variables $x_1, x_2, \ldots, x_n$ we will write as follows: $St_n = \overline{x_1 x_2 \ldots x_n}$. It is clear that the standard polynomial is skew symmetric. Variables in different skew symmetric sets will be denoted by different symbols, for example:

$$\sum_{q \in S_n, p \in S_m} (-1)^q (-1)^p x_{q(1)} x_{q(2)} \cdots x_{q(n)} y_{p(1)} y_{p(2)} \cdots y_{p(m)} = \overline{x_1 x_2 \ldots x_n y_1 y_2 \ldots y_n}.$$ 

Note that when the element has the same variables in different skew symmetric sets, then its sign depends on the parity of the permutation implicitly, therefore the variables in this element will be called alternating. For example, element $\overline{x_1 x_2 \ldots x_n x_1 \ldots x_m}$ has two alternating sets of variables.

Using the above designation we give an example of elements corresponding, for example, to the partition $\lambda = (m + k + l, m + k, m)$ of number $n$, where $3m + 2k + l = n$, $m, k, l \geq 1$. First, let us note that this diagram will contain three corner cells. Recall, that the cell of the diagram is corner, if to the right and below it there are no cells. Build a diagram corresponding to this partition:

```
+----------------+    +----------------+
|                | +---|                |
|   m+k+l        |   |   m+k            |
|                |   |                +---
|                |   |                |
```

Now we construct the elements corresponding to this diagram:

$$f_1 = \overline{x_1 x_2 x_3} St_3^{-m} \overline{St_2} X_1^l,$$

$$f_2 = \overline{x_1 x_2} St_3^{-m} St_2^{-k-1} X_1^l,$$

$$f_3 = x_1 St_3^{-m} St_2^{k} X_1^{l-1}.$$

The described structure of elements relatively free algebra of the variety will be used by us in the future in the proof of the results.
3. The structure of generating algebra of variety $\tilde{V}_3$.

Consider the structure of Leibnitz algebra, which generated the variety $\tilde{V}_3$. Let $T = \Phi[t]$ be a ring of polynomial in the variable $t$. Consider three-dimensional Heisenberg algebra $H$ with the basis $\{a, b, c\}$ and multiplication $ba = -ab = c$, the product of the remaining basis elements is zero. Well known and easy to verify that the algebra $H$ is nilpotent of the class two Lie algebra. Transform the polynomial ring $T$ in the right module of algebra $H$, in which the basis elements of algebra $H$ act on the right on the polynomial $f$ from $T$ follows:

$$fa = f', fb = tf, fc = f,$$

where $f'$ is a partial derivative of a polynomial $f$ in the variable $t$. Consider the direct sum of vector spaces $H$ and $T$ with multiplication by the rule:

$$(x + f)(y + g) = xy + fy,$$

where $x, y$ are from $H$; $f, g$ are from $T$. Denote it by the symbol $\tilde{H}$. Direct verification shows that $\tilde{H}$ is an algebra of Leibnitz.

Thus constructed algebra $\tilde{H}$ generates the variety $\tilde{V}_3$.

Determine the general form of non-zero elements of a relatively free algebra $F(X, \tilde{V}_3)$. To do it, we will replace the variables of these polynomials on the basis elements of algebra $\tilde{H}$. As a result of this replacement, we will get the elements of the algebra $\tilde{H}$, of which equality or difference from zero we can check into force of the structure of this algebra. The replacement we will choose so that it allows to perform a reverse replacement. This will mean that the non-zero elements of the algebra $\tilde{H}$ correspond to the nonzero elements of the algebra $F(X, \tilde{V}_3)$.

Since the Heisenberg algebra $H$ is nilpotent of class two, then the product of any its three elements is equal to zero. Consequently, all elements of degree three and above, resulting from this replacement, containing only the elements of the algebra $H$ are zero. From the structure of algebra $\tilde{H}$ follows that the product on the left of the basis elements from $H$ by a polynomial from $T$ is equal to zero. Therefore, the element of the algebra $F(X, \tilde{V}_3)$ not containing in the first alternating set the generator, which is replaced on the polynomial from $T$, is zero. Since $T$ is regarded as a Lie algebra with zero multiplication, then all elements of algebra $\tilde{H}$, which have more than one polynomial from $T$, are zero. Moreover if the element from $F(X, \tilde{V}_3)$ has at least one alternating set of four variables not in the first place, then as a result of the replacement we will have element, which contains twice one of the basic elements of algebra $H$ in the alternating set. It is clear that such an element is zero, in account of its structure.
4. The multiplicities of variety $\tilde{V}_3$.

Consider the variety $\tilde{V}_3$ of Leibnitz algebras and its numerical characteristics. Recall, that by $m_\lambda(\tilde{V}_3)$ we denoted the multiplicities of irreducible $\Phi_{S_n}$-submodules of module $P_n(\tilde{V}_3)$, which correspond to the partition $\lambda$ of the number $n$.

**Theorem 1.** Let the decomposition of character $\chi_n(\tilde{V}_3)$ of the module $P_n(\tilde{V}_3)$ into the integer combination of irreducible characters $\chi_\lambda$ corresponding to the partition $\lambda$ of number $n$ has the form

$$\chi_n(\tilde{V}_3) = \sum_{\lambda \vdash n} m_\lambda(\tilde{V}_3) \chi_\lambda.$$ 

Then the multiplicity calculated by the formula:

$$m_\lambda(\tilde{V}_3) = \begin{cases} 
1, & \lambda = (n), \lambda = (p, p), \lambda = (p, p, p), \ n, p \geq 1, \\
\lambda = (p + q + r + 1, p + q + 1, p + 1, 1), & p, q, r \geq 0; \\
2, & \lambda = (p + q, p), \lambda = (p + q, p, p), \\
\lambda = (p + q, p + q, p), & p, q \geq 1; \\
3, & \lambda = (p + q + r, p + q, p), p, q, r \geq 1. \\
0, & 
\end{cases}$$

**Proof.** According to the arguments given above, the space of multilinear elements of degree $n$ any variety of Leibnitz algebras can be decomposed as a direct sum of irreducible submodules, corresponding to all possible Young diagrams of $n$ cells; moreover two modules are isomorphic if and only if they are correspond to the same diagram. In the paper [1] it is proved that the number of isomorphic terms in the specified sum for the space $P_n(3N)$ is equal to the number of corner cells in the corresponded Young diagram. Since the variety $\tilde{V}_3$ is subvariety of variety $3N$, then for its multilinear part the number of isomorphic terms does not exceed the number of corner cells.

Consider the diagrams corresponded to non-zero elements, which generates linearization irreducible $\Phi_{S_n}$-submodules of the space $P_n(\tilde{V}_3)$. In the paper [1] it is proved that these are diagrams, in which the first column has not more than four cells and all other columns have not more than three cells. Here are the elements corresponding to such diagrams given in the article [6].

First, consider the diagram, the first column of which has four cells. These diagrams correspond to partitions

$\lambda = (m + 1, 1, 1, 1)$, where $m + 4 = n$,
$\lambda = (m + 1, m + 1, 1, 1)$, where $2m + 4 = n$,
$\lambda = (m + 1, m + 1, m + 1, 1)$, where $3m + 4 = n$,
$\lambda = (m + k + 1, k + 1, 1, 1)$, where $k \neq 1$ and $m + 2k + 4 = n$,
$\lambda = (m + k + 1, k + 1, k + 1, 1)$, where $k \neq 1$ and $m + 3k + 4 = n$,
$\lambda = (m + k + 1, m + k + 1, k + 1, 1)$, where $k \neq 1$ and $2m + 3k + 4 = n$,.
\[ \lambda = (m+k+p+1, k+p+1, p+1), \text{ where } k, p \neq 1 \text{ and } m+2k+3p+4 = n. \]

Construct the corresponding elements. For partition \( \lambda = (m+1, 1, 1, 1) \):
\[
h_1^{(1)} = \tilde{x}_1 \tilde{x}_2 \tilde{x}_3 \tilde{x}_4 X_1^m, \quad h_1^{(2)} = x_1 \tilde{S}_1 X_1^{m-1};
\]
for partition \( \lambda = (m+1, m+1, 1, 1) \):
\[
h_2^{(1)} = \tilde{x}_1 \tilde{x}_2 \tilde{x}_3 \tilde{x}_4 \tilde{S}_2^m, \quad h_2^{(2)} = \tilde{x}_1 \tilde{x}_2 \tilde{S}_1 \tilde{S}_2^{m-1};
\]
for partition \( \lambda = (m+1, m+1, m+1, 1) \):
\[
h_3^{(1)} = \tilde{x}_1 \tilde{x}_2 \tilde{x}_3 \tilde{x}_4 \tilde{S}_3^m, \quad h_3^{(2)} = \tilde{x}_1 \tilde{x}_2 \tilde{x}_3 \tilde{S}_1 \tilde{S}_3^{m-1};
\]
for partition \( \lambda = (m+k, k, 1, 1) \):
\[
h_4^{(1)} = \tilde{x}_1 \tilde{x}_2 \tilde{x}_3 \tilde{x}_4 \tilde{S}_2^k \tilde{X}_1^m, \quad h_4^{(2)} = \tilde{x}_1 \tilde{x}_2 \tilde{S}_1 \tilde{S}_2^{k-1} \tilde{X}_1^m \text{ and } h_4^{(3)} = x_1 \tilde{S}_1 \tilde{S}_2^{k-1} X_1^{m-1};
\]
for partition \( \lambda = (m+k, k+1, 1, 1) \):
\[
h_5^{(1)} = \tilde{x}_1 \tilde{x}_2 \tilde{x}_3 \tilde{x}_4 \tilde{S}_2^k \tilde{X}_1^m, \quad h_5^{(2)} = \tilde{x}_1 \tilde{x}_2 \tilde{x}_3 \tilde{S}_1 \tilde{S}_3^{k-1} \tilde{X}_1^m \text{ and } h_5^{(3)} = x_1 \tilde{S}_1 \tilde{S}_3^{k-1} X_1^{m-1};
\]
for partition \( \lambda = (m+k, k+1, k+1, 1) \):
\[
h_6^{(1)} = \tilde{x}_1 \tilde{x}_2 \tilde{x}_3 \tilde{x}_4 \tilde{S}_3^{k-1} \tilde{X}_2^m, \quad h_6^{(2)} = \tilde{x}_1 \tilde{x}_2 \tilde{x}_3 \tilde{S}_1 \tilde{S}_2^{k-1} \tilde{X}_2^m \text{ and } h_6^{(3)} = \tilde{x}_1 \tilde{x}_2 \tilde{S}_1 \tilde{S}_3^{k-1} \tilde{X}_2^{m-1};
\]
for partition \( \lambda = (m+k, p+1, k+p+1, 1) \):
\[
h_7^{(1)} = \tilde{x}_1 \tilde{x}_2 \tilde{x}_3 \tilde{x}_4 \tilde{S}_2^k \tilde{S}_2^{p-k-1} \tilde{S}_1 X_1^m, \quad h_7^{(2)} = \tilde{x}_1 \tilde{x}_2 \tilde{x}_3 \tilde{S}_1 \tilde{S}_2^{p-k-1} \tilde{S}_1 X_1^m, \quad h_7^{(3)} = \tilde{x}_1 \tilde{x}_2 \tilde{S}_1 \tilde{S}_3 \tilde{S}_2 \tilde{X}_1^m \text{ and } h_7^{(4)} = x_1 \tilde{S}_1 \tilde{S}_3 \tilde{S}_2 \tilde{X}_1^m.
\]

Any replacement the generators of each constructed elements, the upper index of which is different from (1), on the elements from algebra \( \tilde{H} \) will nullify these elements according to the arguments of the second paragraph. In the elements with the upper index (1) we will make the following replacement: \( x_1 = a \), \( x_2 = b \), \( x_3 = c \) and \( x_4 = f \). We obtain non-zero elements. Thus we see that to each diagram with four cells in the first column correspond a unique irreducible submodule of the space \( P_n(\tilde{V}_3) \).

A similar conclusion come, having considered the diagrams of height not more than three with two corner cells. Indeed, such diagrams correspond to partitions \( \lambda = (n) \), \( \lambda = (m, m) \), where \( 2m = n \) and \( \lambda = (m, m, m) \), where \( 3m = n \). To their correspond elements \( h_8 = x_1 X_1^{n-1}, h_9 = \tilde{x}_1 \tilde{x}_2 \tilde{S}_2^{m-1}, h_{10} = \tilde{x}_1 \tilde{x}_2 \tilde{x}_3 \tilde{S}_3^{m-1} \). In this case, we can use next replacement: \( x_1 = a + f \), \( x_2 = b \), \( x_3 = c \). So we get a nonzero elements. Thus, the linearization of each considered element generates one irreducible submodule of the space \( P_n(\tilde{V}_3) \).

Now we consider diagrams of height not more than three with two corner cells. Such diagrams corresponds to partitions \( \lambda = (m+k, k) \), where \( k \geq 1 \) and \( m+2k = n \), \( \lambda = (m+k, k, k) \), where \( k \geq 1 \) and \( m+3k = n \) and at last \( \lambda = (m+k, m+k, k, k) \), where \( k \geq 1 \) and \( 2m+3k = n \). Construct elements corresponding to these diagrams. To partition \( \lambda = (m+k, k) \) responsible elements \( h_{11} = \)
\[ \widetilde{x}_1 \widetilde{x}_2 \widetilde{St}_2^{k-1} X_1^m \] and \[ h_{11}^2 = x_1 \widetilde{St}_2^k X_1^{m-1}. \] We show that the elements \( h_{11}^{(1)} \) and \( h_{11}^{(2)} \) are linearly independent. Assume the contrary. Suppose that there is a linear relationship

\[ \alpha_1 h_{11}^{(1)} + \alpha_2 h_{11}^{(2)} = 0, \]

where at least one of \( \alpha_j, j = 1, 2 \) is different from zero. For these elements use the following replacement: \( x_1 = a, x_2 = b + f. \) This substitution leads to the conclusion that the element \( h_{11}^{(2)} \) is zero, and the element \( h_{11}^{(1)} \) is different from zero. Hence, \( \alpha_1 = 0. \) Then it is clear that the assumption is wrong and the elements \( h_{11}^{(1)} \) and \( h_{11}^{(2)} \) are linearly independent.

To partition \( \lambda = (m + k, k, k) \) correspond the elements

\[ h_{12}^1 = \widetilde{x}_1 \widetilde{x}_2 \widetilde{x}_3 \widetilde{St}_3^{k-1} X_1^m \]
\[ h_{12}^2 = x_1 \widetilde{St}_3^k X_1^{m-1}, \]

and to partition \( \lambda = (m + k, m + k, k) \) — elements

\[ h_{13}^1 = \widetilde{x}_1 \widetilde{x}_2 \widetilde{x}_3 \widetilde{St}_3^{k-1} \widetilde{St}_2^m \]
\[ h_{13}^2 = \widetilde{x}_1 \widetilde{x}_2 \widetilde{x}_3 \widetilde{St}_3^{k-1} \widetilde{St}_2^{m-1}. \]

Show that the elements \( h_i^{(1)} \) and \( h_i^{(2)}, \) where \( i = 12, 13, \) are linearly independent. Assume the contrary. Suppose that there is a linear relationship

\[ \alpha_1 h_i^{(1)} + \alpha_2 h_i^{(2)} = 0, \]

where at least one of \( \alpha_j, j = 1, 2 \) is different from zero. On these elements, we introduce the following replacement: \( x_1 = a, x_2 = b \) and \( x_3 = c + f. \) This exchange also resets the elements \( h_i^{(2)}, \) and elements \( h_i^{(1)} \) leaves non-zero. \((i = 12, 13).\) Thus \( h_i^{(1)} \) and \( h_i^{(2)} \) are also linearly independent. Therefore the linearization of each element, corresponded to the diagram of height not more than three with two corner cells, generates two isomorphic irreducible submodules of space \( P_n(\hat{V}_3). \)

And finally, we consider the diagrams of the last fourth type. These include diagrams of height not more than three with three corner cells. Such diagrams correspond to the partition \( \lambda = (m + k + p, k + p, p) \), where \( k, p \geq 1 \) and \( m + 2k + 3p = n. \) They correspond to the following elements:

\[ h_{14}^{(1)} = \widetilde{x}_1 \widetilde{x}_2 \widetilde{x}_3 \widetilde{St}_3^{p-1} \widetilde{St}_2^k X_1^m, \]
\[ h_{14}^{(2)} = \widetilde{x}_1 \widetilde{x}_2 \widetilde{x}_3 \widetilde{St}_2^k \widetilde{St}_2^{p-1} X_1^m \]
\[ h_{14}^{(3)} = x_1 \widetilde{St}_3 \widetilde{St}_2^k X_1^{m-1}. \]

Similarly to the previous cases we have to prove their linear independence. This was done in the paper \( \Pi, \) where for the case of three rows was used three-dimensional Heisenberg algebra. Therefore, the proof remains valid in the case of algebra \( \hat{H}. \) So \( h_{14}^{(1)}, h_{14}^{(2)}, \) and \( h_{14}^{(3)} \) are linearly independent and their linearization generates three isomorphic irreducible submodules of the space \( P_n(\hat{V}_3). \) The theorem is proved.
5. The asymptotic of the colength of variety $\tilde{V}_3$.

Recall that the colength of variety $V$ is the sum of multiplicities $m_{\lambda}(V)$ of this variety $V$.

Since we know the multiplicity of variety $\tilde{V}_3$, we can now determine the nature of the changes its colength.

**Theorem 2.** For any $\varepsilon > 0$ for colength variety $\tilde{V}_3$ of Leibnitz algebras we have the following equality:

$$l_n(\tilde{V}_3) = \frac{n^2}{3} + o(n^{1+\varepsilon}).$$

**Proof.** Consider the case where $n$ is large enough, for example, $n > 100$. In the proof we use Theorem 1. We consider only the diagrams with non-zero multiplicities.

First, consider the diagram with one corner cell. Their number is not more than three so they do not participate in the asymptotics. The number of the diagram with two corner cell does not exceed $n$, and therefore is a part of the $o(n^{1+\varepsilon})$.

Estimate the number of diagrams height of three. Their number is $\frac{n^2}{3} + o(n^{1+\varepsilon})$. It is a known fact. However, describe it in detail: $\binom{n-2}{2}$ is a number of partition $n$ into three summands. Take into account that the number of partitions that match the two or three terms, limited to a linear function. Then the number of partitions into three different terms is equal to $\binom{n-2}{2} + o(n^{1+\varepsilon})$ for any $\varepsilon > 0$ or $\frac{n^2}{3} + o(n^{1+\varepsilon})$. The number of different ordered partitions $3!$ times less, that is the number of diagrams of height three with three corner cells is $\frac{n^2}{12} + o(n^{1+\varepsilon})$. Given the multiplicity received contribution to the colength $3 \cdot \frac{n^2}{12} + o(n^{1+\varepsilon})$.

Consider the case where the diagrams have the column of height four. If they contain two corner of the cell, then due to the fixity of the fourth row, their number does not exceed three. For these same reasons, the number of such diagrams with three corner cells is limited by a linear function. And finally, of proved earlier, the number of diagrams with four corner cells is equal to $\frac{n^2}{12} + o(n^{1+\varepsilon})$. In this case, the multiplicity are equal to unity and the contribution to the colength asymptotically will be $\frac{n^2}{12}$. Summarizing the results, we obtain the assertion of the theorem.
6. The colength of variety \( \tilde{V}_3 \).

Let’s go find the exact formula of colength variety \( \tilde{V}_3 \) of Leibnitz algebras. Note that theorem 2 implies that she colength can not be expressed by a polynomial.

**Theorem 3.** For the colength of the variety \( \tilde{V}_3 \) of Leibnitz algebras holds following equality:

\[
  l_n(\tilde{V}_3) = \frac{n^2 + n + \delta}{3},
\]

where \( \delta = \begin{cases} 
  1, & \text{if } n = 3k + 1, \\
  0, & \text{if } n \neq 3k + 1.
\end{cases} \)

**Proof.** Consider the sum of multiplicities \( m_\lambda(\tilde{V}_3) \), where \( \lambda = (n), \lambda = (m + k, k) \) \((k \geq 1)\) or \( \lambda = (p, p) \), which correspond to the diagrams of not more than two parts. The number of such partitions denote by \( a(n) \), and the sum of relevant multiplicities by \( l_n^{(2)} \). Then, if \( n = 2m \), are possible the partitions of the form: \((n), (n-1,1),..., (m,m)\). Thus we see that the number of such diagram is \( m + 1 \). If \( n = 2m + 1 \), then are possible follows partitions: \((n), (n-1,1),..., (m+1, m)\). The number of such partitions is \( m + 1 \). Suchwise,

\[
a(n) = \begin{cases} 
  m + 1, & \text{if } n = 2m, \\
  m + 1, & \text{if } n = 2m + 1,
\end{cases}
\]

or \( a(n) = \left\lceil \frac{n}{2} \right\rceil + 1 \).

Find the sum of the corresponding multiplicities. At the same time, we note that the diagrams corresponding partitions \((n)\) and \((m,m)\), have one corner cell, that is, for them \( m_\lambda(\tilde{V}_3) = 1 \). The diagrams corresponding to the remaining partitions have two corner cells, that is, for them \( m_\lambda(\tilde{V}_3) = 2 \). Direct calculation in both cases, we find that \( l_n^{(2)} = n \).

Now consider the diagram of three rows. Their number we will denote by \( b(n) \), and the sum of relevant multiplicities — by \( l_n^{(3)} \). Each of the considered diagrams contains at least one column of length three. Remove these diagrams the first column. In the result of detachment will remain diagrams with two or three rows. The number of first is \( a(n-3) \), and the number of seconds — \( b(n-3) \). Thus, we get the following recurrent relationship for \( b(n) \):

\[
b(n) = a(n - 3) + b(n - 3).
\]

Omitting rather cumbersome calculations, we write the formula for \( b(n) \):

\[
b(n) = \begin{cases} 
  3m^2, & \text{if } n = 6m, \\
  3m^2 + m, & \text{if } n = 6m + 1, \\
  3m^2 + 2m, & \text{if } n = 6m + 2, \\
  3m^2 + 3m + 1, & \text{if } n = 6m + 3, \\
  3m^2 + 4m + 1, & \text{if } n = 6m + 4, \\
  3m^2 + 5m + 2, & \text{if } n = 6m + 5.
\end{cases}
\]
If \( n = 6m \) or \( n = 6m + 3 \), then among \( b(n) \) diagrams there are diagrams with one corner cell, corresponded to the partitions \((m, m, m)\) or \((m+1, m+1, m+1)\) relatively, for which \( m_\lambda(\nabla_3) = 1 \). Also the number \( b(n) \) of diagrams includes diagrams with two corner cells, for which \( m_\lambda(\nabla_3) = 2 \). To find this number, consider the partition to which they correspond. Such partitions are divided into two types: \((m, m, m)\) diagrams with two corner cells, for which \( m \) corresponded multiplicities:

\[
\begin{align*}
\psi_n^{(3)} & = \begin{cases} 9m^2 - 3m, & \text{if } n = 6m, \\
9m^2, & \text{if } n = 6m + 1, \\
9m^2 + 3m, & \text{if } n = 6m + 2, \\
9m^2 + 6m + 1, & \text{if } n = 6m + 3, \\
9m^2 + 9m + 3, & \text{if } n = 6m + 4, \\
9m^2 + 12m + 4, & \text{if } n = 6m + 5. 
\end{cases}
\end{align*}
\]

And finally, consider the diagram of height four, number of which we denote by \( c(n) \). According to theorem 1 for them \( m_\lambda(\nabla_3) = 1 \). Explain that their number is equal to the sum \( a(n - 4) \) and \( b(n - 4) \). Remove these diagrams the first column. Since \( n \) assumes a sufficiently large, while the number of new diagrams is the same as the previous (enough to \( n > 4 \)). And new diagrams will be divided into two types: the diagrams with exactly three rows, the number of which is

\[
b(n - 4) = \begin{cases} 3m^2 - 4m + 1, & \text{if } n = 6m, \\
3m^2 - 3m + 1, & \text{if } n = 6m + 1, \\
3m^2 - 2m, & \text{if } n = 6m + 2, \\
3m^2 - m, & \text{if } n = 6m + 3, \\
3m^2, & \text{if } n = 6m + 4, \\
3m^2 + m, & \text{if } n = 6m + 5, 
\end{cases}
\]

and the diagrams with not more then three rows, which number is

\[
a(n - 4) = \begin{cases} m - 1, & \text{if } n = 2m, \\
m - 1, & \text{if } n = 2m + 1. 
\end{cases}
\]

Then

\[
(n) = \begin{cases} 3m^2 - m, & \text{if } n = 6m, \\
3m^2, & \text{if } n = 6m + 1, \\
3m^2 + m, & \text{if } n = 6m + 2, \\
3m^2 + 2m, & \text{if } n = 6m + 3, \\
3m^2 + 3m + 1, & \text{if } n = 6m + 4, \\
3m^2 + 4m + 1, & \text{if } n = 6m + 5, 
\end{cases}
\]
Because of such diagrams $m_\lambda(\tilde{V}_3) = 1$, the sum of corresponding multiplicities is equal to the number of such diagrams.

Since there are not other diagrams with non-zero multiplicities, then summing up the results we obtain the assertion of the theorem.

$$l_n(\tilde{V}_3) = \begin{cases} 
12m^2 + 2m, & \text{if } n = 6m, \\
12m^2 + 6m + 1, & \text{if } n = 6m + 1, \\
12m^2 + 10m + 2, & \text{if } n = 6m + 2, \\
12m^2 + 14m + 4, & \text{if } n = 6m + 3, \\
12m^2 + 18m + 7, & \text{if } n = 6m + 4, \\
12m^2 + 22m + 10, & \text{if } n = 6m + 5,
\end{cases}$$

or

$$l_n(\tilde{V}_3) = \frac{n^2 + n + \delta}{3},$$

where $\delta = \begin{cases} 
1, & \text{if } n = 3k + 1, \\
0, & \text{if } n \neq 3k + 1.
\end{cases}$ The theorem is proved.

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