On the thin wall limit of thick planar domain walls

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Abstract

Considering a planar gravitating thick domain wall of the $\lambda\phi^4$ theory, we demonstrate how the Darmois junction conditions written on the boundaries of the thick wall with the embedding spacetimes reproduce the Israel junction condition across the wall when one takes its thin wall limit.

KEY WORDS: Thick planar domain walls; Darmois junction condition; Thin planar domain walls; Israel junction condition.
1 Introduction

Domain walls are solutions to the coupled Einstein-scalar field equations with a potential having a spontaneously broken discrete symmetry and a discrete set of degenerate minima. In the simplest case of two minima, a domain wall having a non-vanishing energy density appears in the separation layer, with the scalar field interpolating between these two values. Domain walls in the cosmological context have a long history [1]. It was realized very early that the formation of domain walls with a typical energy scale of $\geq 1\text{MeV}$ must be ruled out [2], because a network of such objects would dominate the energy of the universe, violating the observed isotropy and homogeneity. Domain walls were reconsidered in a possible late time phase transition scenario at the scale of $\leq 100\text{MeV}$. Such walls were supposed to be thick because of the low temperature of the phase transition [3]. The suggestion that Planck size topological defects could be regarded as triggers of inflation, revived the discussion of thin and thick domain walls [4]. The realization of our universe as a $(3 + 1)$-dimensional domain wall immersed in a higher dimensional spacetime has led to the recent numerous studies [5].

The first attempts to investigate the gravitational properties of domain walls were based on the so called thin wall approximation. In this approach one forgets about the underlying field theory and simply treats the domain wall as a zero thickness $(2+1)$-dimensional timelike hypersurface embedded in a four-dimensional spacetime. The Israel thin wall formalism [6] is then used to continue the solutions of the Einstein equations on both sides of the wall in the embedding spacetime across the thin wall. However, such spacetimes have delta function-like distributional curvature and energy-momentum tensor supported on the hypersurface. Using this procedure, the first vacuum solutions for a spacetime containing a infinitely thin planar domain wall was found by Vilenkin [7], and Ipser and Sikivie [8]. The very interesting feature of such domain walls is that they are not static, but have a de Sitter-like expansion in the wall’s plane. External observers experience a repulsion from the wall, and there is an event horizon at finite proper distance from the wall’s core. These results were initially obtained within the framework of the Israel thin wall formalism which has been shown to be an approximative description of a real thick wall by using an expansion scheme in powers of the wall thickness [9, 10]. Typically, a self gravitating domain wall has two length scale, its thickness $w$ and the distance to event horizon which can be compared to $w$. Since these lengths are expressed in terms of the coupling constants of the theory, thin walls turn out to be an artificial construction in terms of these underlying parameters as mentioned in [11].

The first exact dynamical solution to thick planar domain walls was obtained by Goetz [12] and later a static solution was recovered with the price of sacrificing reflection symmetry [13]. Within the context of a fully nonlinear treatment of a scalar field coupled to gravity, Bonjour, Charmousis and Gregory (BCG) found an approximate but analytic description of the spacetime of a thick planar domain wall of the $\lambda\phi^4$ model by examining the field equations perturbed in a parameter characterizing the gravitation interaction of the scalar field [14]. Recently, the thin wall limit of Goetz’s solution was studied in [15] and it has been shown that this solution has a well-defined limit. But to date the thin wall limit of BCG’s solution has never been investigated. In this paper, we study the thin wall limit of the thick planar domain wall described by BCG spacetime. To do so, we use the formalism developed by Mansouri and Khakhshournia (MK) [16] to treat dust thick shells.

The organization of this paper is as follows. In section 2 we give a brief introduction to BCG
thick wall solution and summarize all the useful equations we will need in the present work. In Section 3 we describe the thick wall formalism of reference [15] and apply it to the thick planar domain wall solved by BCG. Section 4 considers the thin wall limit of the thick domain wall solution followed by the conclusion.

2 The thick planar domain wall solution

Domain walls as the regions of varying scalar field separating two vacua with different values of field are usually described by the matter Lagrangian:

\[ L = \nabla_\mu \phi \nabla^\mu \phi - V(\phi), \]  

(1)

where \( \phi \) is a real scalar field and \( V(\phi) \) is a symmetry breaking potential which we take to be \( V(\phi) = \lambda (\phi^2 - \eta^2)^2 \), where \( \lambda \) is a coupling constant and \( \eta \) the symmetry breaking scale. Looking for a static solution of the equation of motion derived from this lagrangian in flat space-time, we get

\[ X = \tanh(\frac{z}{w}), \]  

(2)

where \( X = \frac{\phi}{\eta}, w = \frac{1}{\sqrt{\lambda \eta}} \), and \( z \) is the coordinate normal to the wall. This particular solution represents an infinite planar domain wall centered at \( z = 0 \). From the stress-energy tensor of the wall, one can easily observe that the wall energy density peaked around \( z = 0 \) falls down effectively at \( z = w \). So \( w \) a length scale in the system is called the effective thickness of the wall within the theory.

We now look at the planar gravitating domain wall solutions. The line element of a plane symmetric spacetime may be written in the general form

\[ ds^2 = A^2(z)dt^2 - B^2(z,t)(dx^2 + dy^2) - dz^2, \]  

(3)

which displays reflection symmetry around the wall’s core located at \( z=0 \), where \( z \) is the proper length along the geodesics orthogonal to the wall. In order to obtain a thick domain wall solution one should solve the coupled system of the Einstein and scalar matter field equations as follows

\[ R_{\mu\nu} = 8\pi G \eta^2 \left( 2X_{,\mu}X_{,\nu} - \frac{1}{w^2} g_{\mu\nu}(X^2 - 1)^2 \right), \]  

(4)

\[ \Box X + \frac{2}{w^2} X(X^2 - 1) = 0, \]  

(5)

where \( R_{\mu\nu} \) is the spacetime Ricci tensor. For a static field \( X(z) \), Einstein equations (4) constraint \( B(z,t) \) as \( B(z,t) = A(z) \exp(kt) \).  

BCG investigated the spacetime of a thick gravitating planar domain wall for a \( \lambda \phi^4 \) potential [11]. In the context of their work a dimensionless parameter \( \epsilon \) arisen from equation (4) is singled out to characterize the coupling of gravity to the the scalar field namely

\[ \epsilon = 8\pi G \eta^2. \]  

(6)

Supposing that gravity is weakly coupled to the scalar field, \( A_0(z) \) and \( X(z) \) may be expanded in the powers of \( \epsilon \):

\[ A(z) = A_0(z) + \epsilon A_1(z) + O(\epsilon^2), \]  

(7)

\[ X(z) = X_0(z) + \epsilon X_1(z) + O(\epsilon^2). \]  

(8)
In the $\epsilon \to 0$ limit, the results should be the same as non-gravitating planar wall's which are $A(z) = 1$ and $X(z) = \tanh(\frac{z}{w})$. Using these expansions, BCG solved the coupled Einstein and scalar matter field equations to first order in $\epsilon$ and obtained the following results:

$$A_i(z) = 1 - \frac{\epsilon}{3} \left[ 2 \ln \cosh \left( \frac{z}{w} \right) + \frac{1}{2} \tanh^2 \left( \frac{z}{w} \right) \right] + O(\epsilon^2), \quad (9)$$

$$k_i = \frac{2}{3} \frac{\epsilon}{w} + O(\epsilon^2), \quad (10)$$

$$X_i(z) = \tanh \left( \frac{z}{w} \right) - \frac{\epsilon}{2} \text{sech}^2 \left( \frac{z}{w} \right) \left[ \frac{1}{3} \tanh \left( \frac{z}{w} \right) \right] + O(\epsilon^2). \quad (11)$$

Thus, (9) is a perturbative solution to the spacetime of the thick wall (8) obtained by BCG. In the following sections we will use this solution.

3 The thick wall formalism

In this section we first make a short review of the thick wall formalism developed by MK in [15]. Then we apply it to the thick planar domain wall described by the metric (8). In MK formalism a thick wall is modelled with two boundaries $\Sigma_1$ and $\Sigma_2$ dividing a spacetime $\mathcal{M}$ into three regions. Two regions $\mathcal{M}_+$ and $\mathcal{M}_-$ on either side of the wall and region $\mathcal{M}_0$ within the wall itself. Treating the two surface boundaries $\Sigma_1$ and $\Sigma_2$ separating the manifold $\mathcal{M}_0$ from two distinct manifolds $\mathcal{M}_+$ and $\mathcal{M}_-$, respectively, as nonsingular timelike hypersurfaces, we do expect the intrinsic metric $h_{\mu\nu}$ and extrinsic curvature tensor $K_{\mu\nu}$ of $\Sigma_j$ ($j=1,2$) to be continuous across the corresponding hypersurfaces. These requirements named the Darmois conditions are formulated as

$$[h_{\mu\nu}]_{\Sigma_j} = 0 \quad j = 1, 2, \quad (12)$$

$$[K_{\mu\nu}]_{\Sigma_j} = 0 \quad j = 1, 2, \quad (13)$$

where the square bracket denotes the jump of any quantity that is discontinuous across $\Sigma_j$.

To apply the Darmois conditions on two surface boundaries of a given thick wall one needs to know the metrics in three distinct spacetimes $\mathcal{M}_+, \mathcal{M}_-$ and $\mathcal{M}_0$ being jointed at $\Sigma_j$. While the metrics in $\mathcal{M}_+$ and $\mathcal{M}_-$ are usually given in advance, knowing the metric in the wall spacetime $\mathcal{M}_0$ requires a nontrivial work.

Let us now impose these junction conditions for a self gravitating thick planar domain wall described in the previous section. Recalling $w$ is the effective thickness of the wall, we first follow Ref. [10] to introduce a parameter $\Delta \gg 1$ to assure that the scalar field takes its vacuum values on the wall boundaries $\Sigma_1$ and $\Sigma_2$ being located at the proper distances $z = \pm \Delta w/2$ far from the wall's core surface at $z = 0$. We can then think of $\Delta w$ as the proper thickness of the planar domain wall. In the coordinate frame of the metric (8) in which the wall is stationary, the nonvanishing components of the intrinsic metric $h_{\mu\nu}$ and extrinsic curvature $K_{\mu\nu}$ of $\Sigma_j$ take the following simple forms:

$$h_{\mu\nu} = g_{\mu\nu}, \quad \mu, \nu \neq z, \quad (14)$$

$$K_{\mu\nu} = -\frac{1}{2} g_{\mu\nu,z}. \quad (15)$$
In order to find the spacetime metric on both sides of $\Sigma_j$, we first note that within the vacuum region $\mathcal{M}_+ (\mathcal{M}_-)$ in which $\phi = \eta (-\eta)$, the spacetime metric can be easily determined by solving the Einstein equations (6) yielding

$$A_o(z) = -k_o|z| + C.$$  \hfill (16)

Using (14) and (15) we write down junction conditions (12) and (13) as

$$k_i = k_0,$$  \hfill (17)

$$A_i(z)|_{z=\Delta w/2} = A_o(z)|_{z=\Delta w/2},$$  \hfill (18)

$$\frac{\partial A_i(z)}{\partial z}|_{z=\Delta w/2} = \frac{\partial A_o(z)}{\partial z}|_{z=\Delta w/2}.$$  \hfill (19)

We now use the solutions (9) and (16) due to BCG for the wall metric in the region $\mathcal{M}_0$, where the scalar field varies according to (11), and for the metric in the vacuum regions, respectively. Then the junction conditions (18) and (19) lead to the following constraints on the vacuum metric constants $C$ and $k_o$

$$C = 1 + k_o\frac{\Delta w}{2} - \frac{\epsilon}{3} \left( 2 \ln(\cosh(\Delta/2)) + \frac{1}{2} \tanh^2(\Delta/2) \right),$$  \hfill (20)

$$k_o = \frac{\epsilon}{w} \left( \tanh(\frac{\Delta}{2}) - \frac{1}{3} \tanh^3(\frac{\Delta}{2}) \right).$$  \hfill (21)

Note that within the context of BCG work, it is supposed that the boundaries of the wall where the scalar field takes its vacuum values are at infinity. But here we have modelled the thick planar wall in such a way that the wall boundaries $\Sigma_j$ are situated at the finite proper distances $\pm \Delta w/2$ from the core of the wall. Hence, we choose $\Delta$ to be sufficiently large in order to simulate BCG solution within our wall model.

### 4 From the thick to thin domain walls

We now turn our attention to the thin wall limit of our thick wall model. First let us define the process of passing from a thick gravitating domain wall to a thin one by letting $\epsilon$ and $w$ go to zero while keeping their ratio $\frac{\epsilon}{w}$ fixed. This has the effect that the distance of the event horizon to the domain wall remains finite. We then rewrite the Darmois junction condition (13) demanding the continuity of the extrinsic curvature tensor $K_{\mu \nu}$ across the thick wall boundary, say $\Sigma_1$, located at the proper distance $z = \Delta w/2$ as

$$K^o_{\mu \nu}|_{z=\Delta w/2} = K^i_{\mu \nu}|_{z=\Delta w/2}.$$  \hfill (22)

From the formula (15) one can evaluate the right hand side of the $(tt)$ component of the equation (22) for the metric (3) using the BCG wall metric solution (9). This yields

$$K^o_{tt}|_{z=\Delta w/2} = \frac{\epsilon}{w} \left( \tanh(\frac{\Delta}{2}) - \frac{1}{3} \tanh^3(\frac{\Delta}{2}) \right) \left( 1 - \frac{\epsilon}{3} \left[ 2 \ln[\cosh(\Delta/2)] + \frac{1}{2} \tanh^2(\frac{\Delta}{2}) \right] \right).$$  \hfill (23)
Imposing the above thin wall limit prescription, the equation (23) reduces to

\[ K_{tt}^\circ|_{z=0} = \frac{\epsilon}{w} \left( \tanh\left(\frac{\Delta}{2}\right) - \frac{1}{3} \tanh^3\left(\frac{\Delta}{2}\right) \right). \]  

(24)

To identify the right hand side of the equation (24) we recall the definition of the surface energy density \( \sigma \) of an infinitely thin wall. Within our thick wall model it takes the form

\[ \sigma = \lim_{(w\to0,\epsilon\to0)} \int_{-\Delta w/2}^{\Delta w/2} \rho dz, \]  

(25)

where \( \rho = \rho(z) \) is the energy density of the scalar field which is computed for the BCG scalar field solution (11). Finally, we get the following expression for \( \sigma \)

\[ \sigma = \frac{\epsilon}{2\pi Gw} \left( \tanh\left(\frac{\Delta}{2}\right) - \frac{1}{3} \tanh^3\left(\frac{\Delta}{2}\right) \right), \]  

(26)

where we used the definition \( \epsilon \) given by (6). Comparing the results (24) and (26) one immediately obtains

\[ K_{tt}^\circ|_{z=0} = 2\pi G\sigma. \]  

(27)

Now, consider a planer thin domain wall placed at \( z = 0 \). The Israel thin wall approximation treats the wall as a singular hypersurface with the surface energy \( \sigma \) separating the two plane symmetric vacuum spacetimes \( M_+ \) and \( M_- \) from each other. Then the Israel junction condition across the wall is written as

\[ K_{\mu\nu}^\circ|_{z=0} = 2\pi G\sigma h_{\mu\nu}|_{z=0}. \]  

(28)

Not surprisingly, we now see that the equation (27) is just the same as the \((tt)\) component of the Israel’s equation (28), since from the equations (14) and (16) it follows that \( h_{tt}|_{z=0} = A_0^2 = C \), where in the thin wall limit, the equation (20) reduces to \( C = 1 \). Using the condition (17) being held at \( z = 0 \), one can easily show the same results for the \((xx)\) and \((yy)\) components of the equation (22).

5 Conclusion

We have studied the thin wall limit of the thick planar domain wall solution obtained by Bonjour, Charmousis and Gregory in Ref. [11]. Treating the thick planar wall as a defect having two boundaries at the same proper distance from the wall’s core, as formulated by Mansouri-Khakshournia for the case of a dust shell in Ref. [15], we have shown that the Darmois junction conditions for the extrinsic curvature tensor at the wall boundaries with the two embedding spacetimes generate the well-known Israel jump condition at the separating boundary of the corresponding thin wall with the same embedding spacetimes. We have realized that in the process of passing from a thick planar domain wall to the thin one all the information about the internal structure of the wall are squeezed in the parameter \( \sigma \) characterizing the wall surface energy density as introduced in (26).

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