Longitudinal Walls Oscillations of an Annular Channel Filled with Pulsating Viscous Fluid

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Abstract. The mathematical model for longitudinal oscillations study of elastically fixed walls of an annular channel filled with a viscous fluid is reviewed. We assume that the channel walls are formed by two coaxial cylinders. These cylinders were elastically fixed at the edges and could make only longitudinal oscillations. The fluid motion in the channel takes place due to assigned pulsating pressure difference. The creeping liquid motion is studied and the hydrodynamic parameters distribution laws of the annular fluid layer as well as the motion laws of the channel walls are estimated. The steady-state harmonic oscillations regime is considered. Amplitude frequency characteristics of the annular channel walls are constructed.

1. Introduction
The interaction problems of structural elements with fluid are of theoretical and practical interest [1]. For example, in [2] the stationary interaction problem of a viscous incompressible fluid layer with rigid channels walls is considered. The bending vibrations of an infinite beam resting on a viscous liquid layer are studied in [3]. Reference [4] considers bending vibrations of a cantilever beam surrounded by a viscous incompressible liquid. Bending vibrations of a cantilever beam in a viscous fluid flow applied to piezoelectric elements are studied in [5]. The transverse hydroelastic oscillations problems of the channel walls formed by parallel rectangular or circular plates in plane and axisymmetric formulations are investigated in [6, 7]. Reference [8] considers longitudinal oscillations of the elastic fixed wall of the tapering channel. The study of longitudinal vibrations of the plate in the viscous fluid flow in a channel formed by two parallel rigid walls was carried out in [9].

One of the first studies of pulsating axisymmetric motion of a viscous fluid along an elastic tube was made in [10]. The study of pulsating viscous fluid motion in an elastic annular channel of finite size is performed in [11, 12]. References [13, 14] consider the dynamics and stability problems of coaxial cylinder shells interacting with an internal ideal fluid flow with account of the stationary viscous friction forces. The similar study for a cylindrical shell was performed in [15]. Natural transverse vibrations of the pipeline segment with a moving fluid taking into account the inertia forces of the pipe and fluid, as well as the moments of Coriolis forces and centrifugal forces due to the motion of the medium are considered in [16]. Small transverse vibrations of a straight-line elastic
pipeline with a transported ideal fluid are studied in [17]. The pipeline oscillations are described in the linear setting as the oscillations of a beam. Hydroelastic bending oscillations of a cylinder surrounded by a viscous liquid layer under vibration and shock effects are studied in [18, 19]. Transverse walls oscillations of an annular channel surrounded by an elastic medium are studied in [21, 22]. In this paper, we consider longitudinal oscillations of elastically fixed walls of an annular channel filled with a viscous fluid.

2. Statement of the Problem
Let us consider a narrow annular channel, channel’s length is $\ell$ (Fig. 1). The channel is formed by two coaxial cylinders. These cylinders are elastically fixed at their edges and can make only longitudinal oscillations. The inner wall radius of the channel is $R_2$, and the outer wall radius of the channel is $R_1$. The annular channel is filled with a viscous pulsating incompressible fluid. The thickness of the fluid layer in the channel is $\delta = R_1 - R_2$. The fluid movement is due to an assigned harmonic pulsating pressure difference $\Delta p = p^- - p^+$ at the channel edges. Here $p^- = p_0 + p^*(\omega t)$ is the harmonic pulsating pressure at the left channel edge, $p^+$ is the static pressure at the right channel edge, $\omega$ is the pulsating pressure frequency, $t$ is the time.

We suppose the oscillations amplitudes of the channel walls are significantly less than the fluid layer thickness $\delta$, as well as $\delta << R_1$. Let us introduce a cylindrical coordinate system $r\theta y$ and its corresponding Cartesian coordinate system $xyz$. The coordinate systems centers are connected with the channel geometric center. We consider the axisymmetric problem and assume the fluid to be highly viscous. In this case, transient processes quickly damp, and the regime of steady-state harmonic oscillations is estimated [23].

3. The Theory and Solution
The law of pressure pulsation at the left edge of the channel is given as:

$$p^*(\omega t) = p_m f_p(\omega t), \quad f_p(\omega t) = \sin \omega t,$$

where $p_m$ is the pressure pulsations amplitude.

The viscous fluid flow in the narrow annular channel formed by two coaxial cylinders can be considered as creeping one [24]. The creeping flow dynamic equations for axisymmetric problem are:

$$V_r \frac{\partial V_r}{\partial r} + V_y \frac{\partial V_y}{\partial y} = \frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left( \frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r} \frac{\partial V_r}{\partial r} + \frac{\partial^2 V_y}{\partial y^2} - \frac{V_r}{r^2} \right),$$

$$V_r \frac{\partial V_y}{\partial r} + V_y \frac{\partial V_r}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 V_y}{\partial r^2} + \frac{1}{r} \frac{\partial V_y}{\partial r} + \frac{\partial^2 V_y}{\partial y^2} \right), \quad \frac{\partial V_y}{\partial r} + \frac{V_r}{r} + \frac{\partial V_y}{\partial y} = 0,$$
where \( V_y, V_r \) are fluid velocity in the \( y \) and \( r \) directions, respectively, \( \rho \) is the fluid density, \( \nu \) is the coefficient of kinematic viscosity, \( p \) is the pressure.

Boundary conditions of equations (2) are the no-slip conditions and the conditions for the pressure at the channel edges:

\[
V_r = 0, \quad V_y = \frac{dy}{dt}, \quad \text{at} \quad r = R_2 + \delta, \quad V_r = 0, V_y = \frac{dy}{dt}, \quad \text{at} \quad r = R_2, \quad (3)
\]

\[
p = p_0 + p^* (\omega \tau) \quad \text{at} \quad y = -\ell, \quad p = p_0 \quad \text{at} \quad y = \ell.
\]

The equations of channel walls longitudinal oscillations are written as:

\[
m_1 \ddot{y}_1 + n_1 y_1 = N_1, \quad m_2 \ddot{y}_2 + n_2 y_2 = N_2. \quad (4)
\]

Here \( m_1 \) is the mass of outer channel wall, \( m_2 \) is the mass of inner channel wall, \( n_1 \) is the elastic fixing stiffness of outer wall, \( n_2 \) is the elastic fixing stiffness of inner wall, \( N_1 \) is the force acting on the outer channel wall, \( N_2 \) is the force acting on the inner channel wall, \( y_1 = y_{1m} f_1 (\omega \tau) \), \( y_2 = y_{2m} f_2 (\omega \tau) \) are the longitudinal oscillations laws of the outer and inner walls, respectively. \( y_{1m}, y_{2m} \) are the walls oscillations amplitudes. We suppose \( y_{1m} \) and \( y_{2m} \) are the same order, i.e. \( y_{1m} / y_{2m} = O(1) \), as well as, \( y_{1m} << \delta, y_{2m} << \delta \).

Forces acting on the outer and inner channel walls are determined by the expression:

\[
N_1 = 2\pi R_2 (1 + \delta / R_2) \int_{-\delta/2}^{\delta/2} q_{ry} |_{r=R_2+\delta} dy, \quad q_{ry} |_{r=R_2} = \rho \nu \left( \frac{\partial V_y}{\partial r} + \frac{\partial V_r}{\partial y} \right) |_{r=R_2+\delta}, \quad (5)
\]

\[
N_2 = 2\pi R_2 \int_{-\delta/2}^{\delta/2} q_{ry} |_{r=R_2} dy, \quad q_{ry} |_{r=R_2} = \rho \nu \left( \frac{\partial V_y}{\partial r} + \frac{\partial V_r}{\partial y} \right) |_{r=R_2},
\]

Here \( q_{ry} \) is the shear stress of a viscous fluid.

Let us introduce dimensionless variables and small parameters:

\[
\xi = (r - R_2) / \delta, \quad \zeta = 2y / \ell, \quad \tau = \omega \tau, \quad \lambda = y_{2m} / \delta << 1, \quad \psi = \delta / R_2 << 1, \quad V_r = y_{2m} \omega \psi U_\zeta, \quad (6)
\]

\[
V_y = y_{2m} \omega \sigma U_\zeta, \quad p = p_0 + \rho \nu \lambda \omega \psi^{-1} P, \quad \sigma = \ell / (2R_2), \quad p^* = \rho \nu \lambda \omega \psi^{-1} P^*,
\]

where \( \psi, \lambda \) are the small parameters.

Inserting (6) to equations (2), (3) we obtain the dimensionless fluid dynamics problem in a zero-term approximation for small parameters \( \psi \) and \( \lambda \):

\[
\frac{\partial P}{\partial \xi} = 0, \quad \frac{1}{\sigma^2} \frac{\partial P}{\partial \zeta} = \frac{\partial^2 U_\zeta}{\partial \xi^2} + \frac{\partial U_\zeta}{\partial \xi} + \frac{\partial U_\zeta}{\partial \zeta} = 0, \quad (7)
\]

\[
U_\zeta = 0, \quad U_\xi = \frac{1}{\sigma} \frac{y_{im}}{y_{2m}} \frac{df_1}{dt} \quad \text{at} \quad \xi = 1, \quad U_\zeta = 0, U_\xi = \frac{1}{\sigma} \frac{df_2}{d\tau} \quad \text{at} \quad \zeta = 0, \quad (8)
\]

\[
P = P^* \quad \text{at} \quad \zeta = -1, \quad P = 0 \quad \text{at} \quad \zeta = 1.
\]

Solving the equations (7) with boundary conditions (8) we obtained:
\[ U_\xi = \frac{\xi}{\sigma} \left( \frac{y_{1m}}{y_{2m}} \frac{df_1}{d\tau} - \frac{df_2}{d\tau} \right) + \frac{1}{\sigma} \frac{df_2}{d\tau} - P + \frac{\xi^2 - \xi}{4\sigma^2}, \quad U_\xi \equiv 0, \quad P = \frac{1}{2} \left[ P^* - \xi P^* \right]. \quad (9) \]

In a zero-term approximation for small parameter \( \psi \), expressions (5) bearing in mind (6) take the form of

\[
N_1 = \pi R_z \frac{y_{2m}}{\psi} \frac{\partial U_\xi}{\partial \xi} \left. \frac{d\xi}{\partial \psi} \right|_{\psi=1} \quad d\xi = \pi R_z \left( \frac{2\psi \pi}{\psi} \left( y_{1m} \frac{df_1}{d\tau} - y_{2m} \frac{df_2}{d\tau} \right) - \delta P^* \right), \quad (10)
\]

\[
N_2 = \pi R_z \frac{y_{2m}}{\psi} \frac{\partial U_\xi}{\partial \xi} \left. \frac{d\xi}{\partial \psi} \right|_{\psi=0} \quad d\xi = \pi R_z \left( \frac{2\psi \pi}{\psi} \left( y_{1m} \frac{df_1}{d\tau} - y_{2m} \frac{df_2}{d\tau} \right) + \delta P^* \right). \quad (11)
\]

Inserting (10) to equations (4) we obtain:

\[ m_1 y_1 - K y_1 + K y_2 + n_1 y_1 = -p^* S, \quad m_2 y_2 - K y_1 + K y_2 + n_2 y_2 = p^* S, \quad (12) \]

where \( K = 2\pi R_y \psi / \delta \), \( S = \pi R_y \delta \).

For the steady-state harmonic oscillations case, the ordinary differential equations system (11) transforms into an algebraic one. Solving this system we obtain the following expressions:

\[ y_i = p_m A_i(\omega) \sin(\omega t + \phi_i(\omega)), \quad (12) \]

where \( i = 1, 2 \) and the symbols are introduced:

\[ A_i(\omega) = S \sqrt{(a_1^2 + b_1^2)/(a^2 + b^2)}, \quad A_2(\omega) = S \sqrt{(a_2^2 + b_2^2)/(a^2 + b^2)}, \]

\[ \phi_i(\omega) = \text{arctg}((b_i a - a_i b)/(a_i a + b_i b)), \quad \phi_2(\omega) = \text{arctg}((b_2 a - a_2 b)/(a_2 a + b_2 b)), \]

\[ a = (n_1 - m_1 \omega^2)(m_2 - m_2 \omega^2), \quad b = K \omega(n_1 - n_2 + \omega^2 (m_2 - m_1)), \quad a_i = (n_i - m_i \omega^2), \quad b_i = 2K \omega, \]

\[ a_2 = (m_1 \omega^2 - n_1) \]

Here \( A_i(\omega) \) is amplitude-frequency characteristic of i-th channel wall, \( \phi_i(\omega) \) phase response characteristic of i-th channel wall.

4. Conclusion

The presented mathematical model can be used for investigating the longitudinal oscillations of elastically fixed walls of a narrow annular channel filled with a viscous pulsating fluid flow. The obtained expressions for channel walls amplitude-frequency characteristics and phase response characteristics provides a possibility to determine the oscillations resonance frequencies relying on the given frequency rate pulsating pressure at the channel edges. Thus, the obtained results can be used to analyze the causes of vibrations in the elements of heat exchange systems, hydraulic drive, fuel supply systems and lubrication, etc.

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