CHARACTER FORMULA FOR CONJUGACY CLASSES IN A COSET

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Abstract. Let $G$ be a finite group and $N \lhd G$ a normal subgroup with $G/N$ abelian. We show how the conjugacy classes of $G$ in a given coset $qN$ relate to the irreducible characters of $G$ that are not identically 0 on $qN$. We describe several consequences. In particular, we deduce that when $G/N$ is cyclic generated by $q$, the number of irreducible characters of $N$ that extend to $G$ is the number of conjugacy classes of $G$ in $qN$.

Let $G$ be a finite group and $N \lhd G$ a normal subgroup with $Q = G/N$ abelian. The character group $\hat{Q}$ acts on the set of irreducible characters $\text{Irr} G$ by tensoring, and it is well known that (see e.g. [5, Thm 1.3])

$$\#(\text{conjugacy classes of } G \text{ inside } N) = \#(\hat{Q}-\text{orbits on } \text{Irr} G).$$

In this note, we give a simple representation-theoretic interpretation of conjugacy classes in other cosets of $N$, and discuss some corollaries. We write $[g]$ for the conjugacy class of $g \in G$, and $[\rho]$ for the $\hat{Q}$-orbit of $\rho \in \text{Irr} G$.

Theorem 1. Let $N \lhd G$ be finite groups with $Q = G/N$ abelian, and $q \in G$. Consider

$$J_q = \text{set of conjugacy classes of } G \text{ inside } qN,$$

$$R_q = \text{set of } \hat{Q}-\text{orbits } [\rho] \text{ on } \text{Irr} G \text{ with } \rho \text{ not identically } 0 \text{ on } qN.$$

Then $\#J_q = \#R_q$, and the following matrix is unitary:

$$M_q = \left( \sqrt{\frac{\#[g]\#[\rho]}{\#G}} \rho(g) \right)_{[\rho] \in R_q, [g] \in J_q}.$$

Here we pick any representative $\rho$ for each orbit in $R_q$.

Through the article, a character $\chi$ of a group $G$ containing a normal subgroup $N \lhd G$ in its kernel is sometimes seen as a character of $G/N$ and vice versa. (See e.g. [4, Lemma 2.22] and the discussion after that.) When $Q$ is abelian, recall that $\hat{Q}$ is a group under $\otimes$, and $\hat{Q} \cong Q$ non-canonically (see [4] Problem 2.7)).

Proof of Theorem 1. Consider the class functions

$$\pi_q = \frac{1}{\#Q} \sum_{\chi \in \hat{Q}} \chi(q)\chi$$

in the character ring of $G$. We prove the theorem in 6 steps:

(i) Claim. $\pi_q(g) = \begin{cases} 0 & \text{if } g \notin qN, \\ 1 & \text{if } g \in qN \end{cases}$, and hence $(\pi_q \otimes \rho)(g) = \begin{cases} 0 & \text{if } g \notin qN, \\ \rho(g) & \text{if } g \in qN \end{cases}$.

Indeed,

$$\pi_q(g) = \frac{1}{\#Q} \sum_{\chi \in \hat{Q}} \chi(q)\chi(gN) = \begin{cases} 0 & \text{if } g \notin qN, \\ 1 & \text{if } g \in qN \end{cases},$$

by column orthogonality in the character table for $Q$.

(ii) Claim. If $q \notin \cap_{\chi \in \text{Stab}(\rho)} \ker \chi$, then $\pi_q \otimes \rho = 0$. Otherwise, $\langle \pi_q \otimes \rho, \pi_q \otimes \rho \rangle = \frac{1}{\#[\rho]}$.

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Let $S$ be a set of representatives for $\hat{Q}/\ker\chi$. Every $\chi \in \hat{Q}$ can be written uniquely as $\chi_1\chi_2$ with $\chi_1 \in \ker\chi$ and $\chi_2 \in S$. Then

$$\pi_q \otimes \rho = \frac{1}{\#\ker\chi} \sum_{\chi \in \hat{Q}} \chi(q)(\chi \otimes \rho) = \sum_{\chi_2 \in S} (\chi_2 \otimes \rho) \sum_{\chi_1 \in \ker\chi} \chi_1(q).$$

If $q \notin \cap_{\chi \in \ker\chi} \ker\chi$, then the inner sum is 0 by column orthogonality for $q$ and the identity element in the character table of $\ker\chi$. Otherwise, it is $\#\ker\chi$, so

$$\pi_q \otimes \rho = \frac{1}{\#\ker\chi} \sum_{\chi_2 \in S} \chi_2(q) \chi_2(\rho). \quad (i)$$

In that case, the characters $\chi_2 \otimes \rho$ are all distinct, hence orthonormal, and

$$\langle \pi_q \otimes \rho, \pi_q \otimes \rho \rangle = \frac{1}{\#\ker\chi^2} \sum_{\chi_2 \in S} \chi_2(q) \chi_2(q) \sum_{\chi_1 \in \ker\chi} \chi_1(q) = 1 = \frac{1}{\#\ker\chi}.$$

(iii) Claim. $[\rho] \in R_q \iff q \in \cap_{\chi \in \ker\chi} \ker\chi$.

Suppose $[\rho] \in R_q$, so $\rho \neq 0$ on $qN$. If $\chi \in \ker\chi$, then $\chi \otimes \rho = \rho$, and in particular $\chi(q) = 1$ as $\chi$ is constant on $qN$. Therefore, $q \in \cap_{\chi \in \ker\chi} \ker\chi$. Conversely, if $q$ lies in this intersection, then $\langle \pi_q \otimes \rho, \pi_q \otimes \rho \rangle \neq 0$ by (ii). As $\pi_q \otimes \rho$ is zero outside $qN$ by (i), we must have $\rho \neq 0$ on $qN$. In other words $[\rho] \in R_q$.

(iv) Claim. Choose a set of representatives $U$ of orbits of $\hat{Q}$ on $\Irr G$. Then

$$\{ \sqrt{\#\ker\chi}(\pi_q \otimes \rho) \mid \rho \in U, q \in \cap_{\chi \in \ker\chi} \ker\chi \}$$

is an orthonormal basis of class functions for $G$.

From (i) it is clear that $\pi_q \otimes \rho$ and $\pi_q' \otimes \rho'$ are orthogonal whenever $\rho \neq \rho'$, as $[\rho]$ and $[\rho']$ are disjoint. From (i) it follows that they are orthogonal when $q \neq q'$ as well, and (ii) shows orthonormality.

Next, for abelian groups $B < A$, we have

$$\bigcap_{g \in B} \{ \psi \in \hat{A} \mid \psi(g) = 1 \} = \{ \psi \in \hat{A} \mid B \subset \ker\psi \} = \hat{A}/B.$$

Applying this to $B = \ker\chi$, $A = \hat{Q}$ and $\hat{A} = \hat{Q} = Q$ we find that

$$\bigcap_{\chi \in \ker\chi} \ker\chi = \bigcap_{\chi \in \ker\chi} \{ q \in Q \mid \chi(q) = 1 \} = \hat{Q}/\ker\chi.$$

In particular, for for each $\rho \in U$, the left-hand side is a group of order $\#Q/\#\ker\chi = \#\rho$. Thus our set of class functions has cardinality

$$\sum_{\rho \in U} \#\rho = \#\Irr G,$

and is therefore a basis.

(v) Claim. $\#J_q = \#R_q$.

The class functions $\sqrt{\#\rho}(\pi_q \otimes \rho)$ with $q \in \cap_{\chi \in \ker\chi} \ker\chi$ are 0 outside $qN$ and are the only such functions from the basis in (iv). So they are a basis of class functions that are zero outside $qN$, and hence their number is the number of conjugacy classes in $qN$.

(vi) Claim. $M_q$ is unitary.
Choose a set of representatives $U_q$ of $\tilde{Q}$-orbits in $R_q$. By (i), for $\rho, \rho' \in U_q$, we have

$$\sum_{g \in J_q} \sqrt{\frac{\#q}{\#G}} \rho(g) \sqrt{\frac{\#q}{\#G}} \rho'(g) = \frac{1}{\#G} \sum_{g \in J_q} \#q \sqrt{\#\rho} (\pi_q \otimes \rho)(g) \sqrt{\#\rho'} (\pi_q \otimes \rho')(g).$$

By (i), $\pi_q \otimes \rho$ is 0 outside $qN$, so this is just

$$\langle \sqrt{\#\rho} \pi_q \otimes \rho, \sqrt{\#\rho'} \pi_q \otimes \rho' \rangle,$$

which is 1 if $\rho = \rho'$ and 0 otherwise, by (iv). So the rows of $M_q$ are orthonormal. As $M_q$ is a square matrix by (v), it is unitary.

**Example 2.** Consider $G = F_5 = C_5 \rtimes C_4$ of order 20, and $N = C_5 \triangleleft G$. Pick $h \in N$ and $q \in G$ of order 5 and 4, respectively. The character table of $G$ is given on the right.

For the trivial coset $N$ we have

$$J_N = \{[l], [l]\}, \quad R_N = \{[\rho_4], [\rho_5]\}.$$

Thus, $\#J_N = \#R_N$, and indeed

$$M_N = \begin{pmatrix} \sqrt{\frac{1}{5}} & 1 & \sqrt{\frac{4}{5}} & 1 \\ \sqrt{\frac{1}{5}} & 4 & \sqrt{\frac{4}{5}} &(-1) \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

is unitary. All other cosets have

$$J_{qN} = \{[q^j]\}, \quad R_{qN} = \{[\rho_1]\} \quad \text{and} \quad M_{qN} = (1).$$

**Remark 3.** Generally, Clifford-Fischer theory links cosets of $N$, even when $G/N$ is not abelian, to certain projective representations of $G$; see [3] and the references in [1]. It is quite possible that Theorem [1] is a special case, although usually in Clifford-Fischer theory to deduce results for ordinary (rather than projective) characters, one requires either that $G = N \times Q$ or that every character of $N$ extends to its inertia group. Theorem [1] does not need those assumptions (which fail e.g. when $N = C_2$, $G = Q_8$), and it is probably easier to prove from first principles in any case.

**Remark 4.** When $N = G$, the unitary matrix $M_G = \left( \sqrt{\frac{\#q}{\#G}} \rho(g) \right)$ is the usual modified character table.

**Lemma 5.** In the setting of Theorem [1]

1. $\#\text{Stab}_G(\rho) = \langle \text{Res}_N \rho, \text{Res}_N \rho \rangle$.
2. $[\rho] \in R_q \iff q \in \cap_{\chi \in \text{Stab}(\rho)} \ker \chi$.
3. $R_q \subset R_{qk}$ and $\#J_q \leq \#J_{qk}$ for all $q \in G$ and $k \geq 1$.
4. If $G/N$ is cyclic generated by $q \in G$, then

$$[\rho] \in R_q \iff \text{Stab}_G(\rho) = \{1\} \iff \text{Res}_N \rho \text{ is irreducible}.$$

**Proof.** (1) Since $Q$ is abelian, $\sum_{\psi \in \tilde{Q}} \psi$ is the character of the regular representation of $Q$, in other words $\text{Ind}_Q^G 1$ as a character of $G$. Recall that $\rho \otimes \text{Ind}_Q^G 1 = \text{Ind}_Q^G (\langle \text{Res}_N \rho \rangle \otimes 1)$, see e.g. [4] Problem 5.3. So, by Frobenius reciprocity,

$$\#\text{Stab}_G(\rho) = \langle \rho, \rho \otimes \sum_{\psi \in Q} \psi \rangle = \langle \rho, \rho \otimes \text{Ind}_Q^G 1 \rangle = \langle \rho, \text{Ind}_Q^G (\langle \text{Res}_N \rho \rangle \otimes 1) \rangle = \langle \text{Res}_N \rho, \text{Res}_N \rho \rangle.$$


(2) This is claim (iii) in the proof of Theorem 1. (3) Immediate from (2) and the equality \( \#R_q = \#J_q \). (4) First equivalence follows from (2), noting that \( q \notin \chi \) for any \( 1 \neq \chi \in \bar{Q} \). Second equivalence follows from (1).

**Corollary 6.** Let \( N \triangleleft G \) with \( G/N \) cyclic generated by \( q \in G \). The following are equal:

- The number of conjugacy classes of \( G \) inside \( qN \).
- The number of \( \bar{Q} \)−orbits \( \lbrack \rho \rbrack \) on \( \text{Irr} \ G \) with \( \rho \) not identically 0 on \( qN \).
- The number of \( \bar{Q} \)−orbits on \( \text{Irr} \ G \) of length \( (G : N) \).
- \( \frac{1}{(G:N)} \) times the number of \( \rho \in \text{Irr} \ G \) whose restriction to \( N \) is irreducible.
- The number of \( \tau \in \text{Irr} N \) that extend to a character of \( G \).

**Proof.** The first equivalence follows from Theorem 1 \( (\#R_q = \#J_q) \). The second and third are the two equivalences in Lemma 5 \( (2) \). For the last one, observe that \( \tau \in \text{Irr} N \) extends to a character of \( G \) if and only if \( \tau = \text{Res} \rho \) for some \( \rho \in \text{Irr} G \). Suppose \( \tau = \text{Res} \rho \). Then

\[
\text{Res}_N \rho' = \tau \implies 1 = \langle \text{Res}_N \rho', \tau \rangle = \langle \rho', \text{Ind}_N^G \text{Res} \rho \rangle = \langle \rho', \rho \otimes \text{Ind}_N^G 1_N \rangle \implies \rho' \in \lbrack \rho \rbrack.
\]

Conversely, if \( \rho' \in \lbrack \rho \rbrack \), then clearly \( \tau = \text{Res} \rho' \). In other words, the characters that restrict to \( \tau \) are exactly those in \( \lbrack \rho \rbrack \), so there are \( (G : N) \) of them. The last equivalence now follows. \( \square \)

**Corollary 7.** Let \( N \triangleleft G \) with \( G/N \) cyclic. Then \( N \) has a non-trivial irreducible character that extends to \( G \) if and only if \( G \) has no conjugacy classes of size \( \#N \).

**Proof.** Let \( q \in G \) generate \( G/N \). Then \( G \) has a conjugacy class of size \( \#N \) if and only if \( qN \) is such a class, by Lemma 5 \( (3) \). Equivalently, by Corollary 6 only \( 1_N \) extends to a character of \( G \). \( \square \)

**Example 8.** For \( p > 2 \) there are exactly \( p \) irreducible representations of \( N = SL_2(\mathbb{F}_p) \) that extend to \( G = GL_2(\mathbb{F}_p) \). Indeed, fix a generator \( qN \in G/N \cong \mathbb{F}_p^\times \), a primitive root mod \( p \). Note that if the determinant of a matrix in \( GL_2(\mathbb{F}_p) \) generates \( \mathbb{F}_p^\times \), then the element is automatically semisimple, provided \( p > 2 \). Thus there are \( p \) conjugacy classes in the coset \( qN \), characterised by their trace, and hence, by Corollary 6 exactly \( p \) irreducible representations that extend to \( GL_2(\mathbb{F}_p) \).

We end with an ‘inversion formula’, which reconstructs a character \( \Theta \) on \( G \) from its values on a sufficient number of cosets. This was our original motivation in the number-theoretic setting when \( G \) is a local Galois group and \( N \triangleleft G \) its inertia subgroup. In that case, this formula explicitly reconstructs a representation of \( G \) from characteristic polynomials of Frobenius over sufficiently many intermediate fields. We refer the reader to \( [2] \) for the applications of the formula.

**Corollary 9** (Inversion formula). Suppose \( N \triangleleft G \) with \( Q = G/N \) cyclic, generated by \( q \in G \). Let \( U \) be a set of representatives of orbits of \( Q \) on \( \text{Irr} \ G \), and denote \( m_\rho = \langle \text{Res}_N \rho, \text{Res}_N \rho \rangle \) for \( \rho \in U \). Let \( \Theta \) be a character of \( G \).

(i) \( \Theta \) can be written as

\[
\Theta = \sum_{\rho \in U} \Psi_\rho \otimes \rho \quad \text{for some character } \Psi_\rho \text{ of } G \text{ with } N \subset \ker \Psi_\rho.
\]
(ii) The eigenvalues of the matrix associated to $\Psi_\rho(q)$ are well-defined up to multiplication by $m_\rho$th roots of 1.

(iii) For every $\rho \in U$ and every $d \geq 0$,

$$\Psi_\rho(q^{dm_\rho}) = \frac{1}{\#N\cdot m_\rho} \sum_{g \in q^{dm_\rho}N} \rho(g) \Theta(g).$$

(iv) $\Psi_\rho \otimes \rho$ is uniquely determined by (iii). Concretely, suppose $\dim \Psi_\rho \leq B$. There is a unique $0 \leq n \leq B$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^\times$ such that $\sum_k \lambda_k^d = \Psi_\rho(q^{dm_\rho})$ for $d = 1, \ldots, B$. Then $\Psi_\rho(q) = m\sqrt[\rho]{\lambda_1} + \ldots + m\sqrt[\rho]{\lambda_n}$ for some choice of the roots. The character $\Psi_\rho \otimes \rho$ is independent of this choice.

Proof. (i) Clear. (ii) As the decomposition into irreducibles is unique, the terms $\Psi_\rho \otimes \rho$ are uniquely determined by $\Theta$. It remains to show that for $\psi, \psi' \in \hat{Q}$,

$$\rho \otimes \psi \cong \rho \otimes \psi' \iff (\psi/\psi')^{m_\rho} = 1.$$ 

By Lemma 5 (1), we have $m_\rho = \# \text{Stab}_g(\rho)$. Because $\hat{Q}$ is cyclic, Stab$_g(\rho)$ is exactly the subgroup of characters of order $m_\rho$, as required.

(iii) Fix $\rho \in U$ and a multiple $j = dm_\rho$. Let $U_{q^j} \subset U$ be the subset of characters that are not identically 0 on $q^jN$. Define the matrix $M_{q^j}$ as in the theorem with these representatives for $R_{q^j}$. For $g \in J_{q^j}$, we have

$$\Theta(g) = \sum_{\rho \in U} \Psi_\rho(g) \rho(g) = \sum_{\rho \in U_{q^j}} \Psi_\rho(q^j) \rho(g) = \sum_{\rho \in U_{q^j}} \Psi_\rho(q^j) \rho(g) = \sqrt{\#G/\#|\rho|} \sum_{\rho \in U_{q^j}} M_{q^j, \rho} \Psi_\rho(q^j) / \sqrt{\#|\rho|}.$$

In other words, we have a matrix equation

$$\left(\sqrt{\#|\rho|/\#G} \Theta(g)\right)_{g \in J_{q^j}} = M_{q^j} \cdot \left(\Psi_\rho(q^j) / \sqrt{\#|\rho|}\right)_{\rho \in U_{q^j}}.$$

As $M_{q^j}$ is unitary, we can rewrite it as

$$\left(\Psi_\rho(q^j) / \sqrt{\#|\rho|}\right)_{\rho \in U_{q^j}} = M_{q^j}^t \cdot \left(\sqrt{\#|\rho|/\#G} \Theta(g)\right)_{g \in J_{q^j}}.$$

Using the fact that $m_\rho = \# \text{Stab}_g(\rho)$ proved in (ii), and the orbit-stabiliser equality $m_\rho [\rho] = (G : N)$, we get

$$\Psi_\rho(q^j) = \sqrt{\#|\rho|} \sum_{g \in J_{q^j}} M_{q^j, \rho} \sqrt{\#|\rho|/\#G} \Theta(g) = \sum_{g \in J_{q^j}} \frac{\#|\rho|/\#G}{\#N\rho} \rho(g) \Theta(g)$$

as claimed.

(iv) Let $\mu_1, \ldots, \mu_n$ be the eigenvalues of the matrix associated to $\Psi_\rho(q)$, so that $\Psi_\rho(q^{dm_\rho}) = \sum_k \mu_k^{dm_\rho}$ for any $d \geq 0$. Generally, the Vandermonde system of equations $\sum_{d=1}^B \nu_k^d = a_q$ for $d = 1, \ldots, B$ has a unique (unordered) solution $\nu_1, \ldots, \nu_B$. Thus, the unique solution to $\sum_k \lambda_k^d = \Psi_\rho(q^{dm_\rho})$ is $\mu_1^{m_\rho}, \ldots, \mu_n^{m_\rho}, 0, \ldots, 0$. The formula for $\Psi_\rho(q)$ follows, and independence of the choice is proved in (ii). □
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