Calculation and modular properties of multi-loop superstring amplitudes

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Abstract
Multi-loop superstring amplitude is calculated in the conventional gauge where Grassmann moduli are carried by the 2D gravitino field. Generally, instead of the modular symmetry, the amplitudes hold the symmetry under modular transformations added by relevant transformations of the 2D local supersymmetry. If a number of loops are larger than 3, the integration measures are not modular forms. In this case the expression for the amplitude contains an integral over the bound of the fundamental region of the modular group.

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1. Introduction

In the Ramond–Neveu–Schwarz theory the world sheet is often specified [1] as the Riemann surface with a spin structure [2]. The spin structures are not invariant under transformations of the 2D supersymmetry. It leads to the well-known difficulties [1, 3, 4] in the calculation of the multi-loop interaction amplitudes. They depend [1] on the 2D gravitino field [3, 4]. It means that the world-sheet supersymmetry is lost. Indeed, in the superstring theory the vierbein and the 2D gravitino field are the gauge fields of the group of local symmetries on the string world sheet. Owing to gauge invariance, the ‘true’ amplitudes are independent of the choice of the gauge of the above fields.

In [5] true two-loop amplitudes have been obtained. The calculation of the Ramond–Neveu–Schwarz amplitudes with any number of loops has been done [6, 7] in the supercovariant scheme [8–10]. In this case the zweibein and 2D gravitino field are conformally flat. The string world sheet is specified as the (1|1) complex, non-split supermanifold. The supermanifold carries a ‘superspin’ structure [6, 10, 11] instead of the spin structure [2]. The superstring amplitude is obtained by a summation over the superspin structures. The superspin structures are supersymmetric extensions of the spin structures [2]. In this case the twist about (A, B)-cycles is, generally, accompanied by a supersymmetric transformation including fermion–boson mixing. The fermion–boson mixing arises due to the presence of Grassmann moduli that are assigned to the (1|1) complex, non-split supermanifold in addition to the Riemann ones. The fermion–boson mixing differentiates the superspin structures from the
ordinary spin ones. Indeed, the ordinary spin structures [2] imply that boson fields are single valued on Riemann surfaces. Only fermion fields being twisted about \((A, B)\)-cycles may receive the sign. The \(g\)-loop spinning (fermion) string interaction amplitude (with \(g > 1\)) is given by an integral over \((3g - 3)/2g - 2\) complex moduli and over interaction vertex coordinates on the supermanifold. The integrand (the local amplitude) has been explicitly calculated [6] for every superspin structure. The calculation employs only the gauge symmetry of the fermion string. In doing so, the partition functions are computed from equations [6, 12] that are nothing else than Ward identities. These equations realize the requirement that the superstring amplitudes are independent of both the vierbein and the gravitino field. Therefore, the obtained multi-loop amplitudes are consistent with the gauge invariance of the superstring theory. The world-sheet gauge group is so large that the Ward identities fix the partition function up to a constant factor. This factor is determined by the factorization condition following from the unitarity equations. The module space integral of the local amplitude is, however, ambiguous [4, 7] under non-split replacements of the moduli. The ambiguity is also present in the sum over super-structures for the superstring amplitude. The ambiguity in the superstring amplitude is resolved [7] so that the cosmological constant is equal to zero. The obtained superstring amplitudes are finite. The one-, two- and three-point massless boson amplitudes vanish in accordance with the spacetime supersymmetry.

In this paper, we consider the calculation of the same multi-loop amplitudes in the above-mentioned gauge [1] where the \((1|1)\) complex supermanifold is split, in the sense that fermions are not mixed to bosons under twists around non-contractible cycles. The genus-\(g\) supermanifold is specified as the genus-\(g\) Riemann surface \(\Sigma_g\) with the given spin structure. Each spinning string amplitude is represented by an integral of a local amplitude where the integration is performed over vertex coordinates and over the moduli. The superstring amplitude is obtained by summing over spin structures. Grassmann moduli are carried by the 2D gravitino field \(\phi_m\) which is usually specified such that \(\gamma^m \phi_m = 0\) where \(\gamma^m\) is the Dirac 2D matrix. Non-zero components \(\phi_{\pm}(z, \bar{z})\) of the \(\phi_m\) field are given by

\[
\phi_-(z, \bar{z}) = \sum_{s=1}^{2g-2} \lambda_s \phi_{-s}(z, \bar{z}), \quad \phi_+(z, \bar{z}) = \sum_{s=1}^{2g-2} \bar{\lambda}_s \phi_{s}(z, \bar{z}),
\]

\(\lambda_s\) and \(\bar{\lambda}_s\) being Grassmann moduli, and \(\phi_{\pm}(z, \bar{z})\) fields may depend on the Riemann moduli too. As far as the world-sheet supermanifold is split, one seemingly avoids the ambiguities [4, 7] complicating the calculation of the amplitudes in the superconformal gauge. Also, it seems that the amplitudes might possess the modular symmetry and be represented through theta-functions and modular forms. Indeed, in the two-loop calculation [5] the genus-2 integration measures are modular forms, and the GSO projections of the local amplitudes with less than four legs are equal to zero. The GSO projection of the four-point local amplitude does not depend on \(\phi_{\mp}\). The spinning string amplitude ceases to depend on \(\phi_{\mp}\) due to the integration over vertex coordinates. The papers [5] have initiated the efforts [13] to build genus \(g > 2\) amplitudes assuming certain properties of the amplitudes, the modular symmetry being among them. This strategy meets with difficulties [14, 15], at least for \(g > 3\).

The calculation of the multi-loop interaction amplitudes in this paper is similar to the calculation [6] in the supersymmetric gauge. It exploits the gauge symmetry on the string world sheet and uses no assumptions. In this case the spinning string amplitudes are independent of local variations of the \(\phi_{\pm}\) fields, but the local amplitudes depend on \(\phi_{\mp}\) (the latter fact takes place even in the two-loop case [5]). Integration of the local amplitude over the moduli and over vertex coordinates is performed at fixed \(\phi_{\mp}\). The integral must be invariant under re-definitions of the non-contractable cycles on the string world sheet. The re-definitions of the non-contractable cycles are accomplished by modular transformations, but
these transformations, generally, change $\phi_{\bar{z}}$. Returning back to the original $\phi_{\bar{z}}$ is achieved by an extra transformation of a local 2D supersymmetry. Therefore, the symmetry group of the amplitude consists of modular transformations accompanied by the relevant supersymmetric ones. These supermodular transformations are conveniently discussed in the supersymmetric description [12] of the fermion string on the (1|1) complex supermanifold. The period matrix [6, 7, 11] on the above-mentioned supermanifold collects periods of scalar superfunctions which vanish under the supercovariant Laplacian [7]. The supermodular transformation changes [6, 7, 11] this matrix just as the relevant modular transformation changes the period matrix [16] on the Riemann surface $\Sigma_g$.

A loss of the supersymmetry in [1] occurs because the difference between the supermodular and modular symmetries was ignored and, also, because of an incomplete calculation of the ghost zero mode contribution to the integration measure; see sections 2 and 3.

Since the superscalar functions depend on $\phi_{\bar{z}}$, the period matrix on the (1|1) complex supermanifold is, generally, distinguished from the period matrix on $\Sigma_g$ by terms proportional to the Grassmann moduli. In this case the integration of the local amplitude over the fundamental region of the modular group leads to the loss of 2D supersymmetry. As the result, the spinning string amplitudes depend on $\phi_{\bar{z}}$. To restore the supersymmetry, the integration over the fundamental region of the modular group must be supplemented [7] by the integral around the boundary of the region. If $g \leq 3$, the periods of the superscalar functions can be taken [5] as the moduli set. In this case the boundary integral does not arise. If $g \leq 3$ and the moduli setting [1] is used, the boundary integral is removed (see section 4) by a re-definition of the local amplitude. The integration measures are given by modular forms for both $g = 2$ and $g = 3$. Unlike the two-loop case, the GSO projection of the four-point, three-loop amplitudes ceases to depend on $\phi_{\bar{z}}$ due to the integration over vertex coordinates, just as it arises in each of the spin structures. If $g > 3$, periods of superscalar functions depend on Grassmann moduli for any choice of moduli variables. The boundary integral is present in the expression for the amplitude, and the integration measures are not modular forms. It is akin to what occurs in the superconformal gauge. Hence the strategy [13] is not in accord with the 2D supersymmetry.

In section 2 the integration over moduli is discussed. In section 3 local amplitudes are calculated. In section 4 two- and three-loop amplitudes are considered in more detail.

2. Integration of local amplitudes

As noted in the introduction, the period matrix on the (1|1) complex supermanifold determines the periods of the scalar superfunctions. The scalar superfunctions vanish under the super-Laplacian $(D_{\sigma L}^{(\phi)} D_{\sigma R}^{(\phi)} - D_{\bar{\sigma R}}^{(\phi)} D_{\bar{\sigma L}}^{(\phi)})/2$, where $T$ denotes transposing. Operators $D_{\sigma L}^{(\phi)}$ and $D_{\sigma R}^{(\phi)}$ depend on the gravitino field (1). We assume that the $\phi_{\bar{z}}$ fields do not overlap. Then [7]

$$D_{\sigma L}^{(\phi)} = D + \frac{1}{2} \phi_+(z, \bar{z}) \left[ \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial \bar{\sigma}} - \frac{\partial}{\partial z} \frac{\partial}{\partial \sigma} \right] - \frac{1}{2} \phi_-(z, \bar{z}) \left[ \frac{\partial}{\partial z} \frac{\partial}{\partial \sigma} - \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial \bar{\sigma}} \right]$$

$$D_{\bar{\sigma} R}^{(\phi)} = D + \frac{1}{2} \phi_+(z, \bar{z}) \left[ \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{\sigma}} - \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial \sigma} \right] - \frac{1}{2} \phi_-(z, \bar{z}) \left[ \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial \bar{\sigma}} - \frac{\partial}{\partial z} \frac{\partial}{\partial \sigma} \right],$$

where $\bar{\sigma}$ is the superpartner of $z$. The scalar superfunctions $J_r^{(\phi)}(z, \bar{z}, \bar{\sigma})$ are associated with the right movers, and $J_r^{(\phi)}(z, \bar{z}, \bar{\sigma})$ are associated with the left movers; $\sigma$ and $\sigma'$ mark spin structures. The desired superfunctions are represented as

$$J_r^{(\phi)}(z, \bar{z}, \bar{\sigma}) = J_r^{(\phi)}(z, \bar{z}) + \theta \eta_r^{(\phi)}(z, \bar{z}), \quad J_r^{(\phi)}(z, \bar{z}, \bar{\sigma}) = J_r^{(\phi)}(z, \bar{z}) + \bar{\theta} \eta_r^{(\phi)}(z, \bar{z}).$$

(3)
They can be found from the equations

\[ D_J^{(0)} J^r_{\sigma}(z, \bar{z}, \theta) = 0, \quad D_J^{(0)} J^s_{\sigma}(z, \bar{z}, \bar{\theta}) = 0. \]  

(4)

If \( \phi_r(z, \bar{z}) = \phi_r(z, \bar{z}) = 0 \), then \( J^r_{\sigma}(z, \bar{z}) \) is reduced to the scalar function \( J_r(z, \bar{z}) \), and \( J^s_{\sigma}(z, \bar{z}) \) is reduced to \( J_s(z, \bar{z}) \). Under \( 2\pi \)-twists about \( B \)-cycles on the Riemann surface \( \Sigma_\eta \) (which specifies the genus-\( g \) supermanifold in question), the \( J^r_{\sigma}(z, \bar{z}) \) functions receive periods forming the \( \Omega^{(R)} \) matrix. Correspondingly, the periods of \( J^s_{\sigma}(z, \bar{z}) \) form the \( \Omega^{(L)} \) matrix. In this case

\[ \Omega^{(R)} = \Omega + \tilde{\Omega}^{(R)}, \quad \Omega^{(L)} = \overline{\Omega} + \tilde{\Omega}^{(L)}, \]

(5)

where \( \tilde{\Omega}^{(R)} \) and \( \tilde{\Omega}^{(L)} \) vanish when all the Grassmann moduli are equal to zero, and \( \Omega \) is the period matrix on \( \Sigma_\eta \). Equations (4) can be transformed to the integral equations. In doing so, the desired equations for the \( (J^r_{\sigma}(z, \bar{z}), \eta^{(R)}_{\sigma}(z, \bar{z})) \) pair and for \( \Omega^{(R)} \) elements of \( \Omega^{(R)} \)-matrix are found to be

\[ J^r_{\sigma}(z, \bar{z}) = J_r(z) - \frac{1}{2\pi} \int \partial_z \ln[E(z, z')] \phi_r(z', \bar{z}') \eta^{(R)}_{\sigma}(z', \bar{z}') d^2z' + \text{const}, \]

\[ \eta^{(R)}_{\sigma}(z, \bar{z}) = \frac{1}{2\pi} \int S_\sigma(z, z') \phi_r(z', \bar{z}') \partial_z J^r_{\sigma}(z', \bar{z}') d^2z', \]

\[ \Omega^{(R)} = \Omega + i \int \partial_z J_r(z) \phi_r(z, \bar{z}) \eta^{(R)}_{\sigma}(z, \bar{z}) d^2z, \]

(6)

where \( J_r(z) \) is the scalar function on \( \Sigma_\eta \) having the same properties as \( E(z, z') \) is the prime form and \( S_\sigma(z, z') \) is the Szego kernel [1, 17]. For the even spin structure \( \sigma = (\sigma_1, \sigma_2) \) it is given by [1]

\[ S_\sigma(z, z') = \frac{\theta_{[\sigma]}(z - z')}{E(z, z') \theta_{[\sigma]}(0)}, \quad z - z' = \int_z^{z'} v(x) \, dx, \quad v = \{v_s(x)\}, \quad v_s(x) = \partial_z J_s(x), \quad \theta_{[\sigma]}(z) = \theta(z + \Omega \sigma_1 + \sigma_2) \exp[i\pi \sigma_1 \Omega \sigma_1 + 2\pi i \sigma_1 (z + \sigma_2)], \quad \sigma_1 = \{\sigma_s\}, \quad 1 \leq s \leq g \]

(7)

where \( z \) is related to \( z \in \Sigma_\eta \) by the Jacobi mapping. Further on, \( \theta_{[\sigma]}(z) \) is the theta-function with characteristics \( \sigma = (\sigma_1, \sigma_2) \) corresponding to the spin structure \( \sigma = (\sigma_1, \sigma_2) \). The \( \theta_{[\sigma]}(z) \) function is related to the Riemann theta-function \( \theta(z) \) as seen from the second line of (7). The first two equations in (6) are equivalent to the first equation in (4). To verify it, every integral equation reduced to the differential equation by the \( \partial_z \) operator. Indeed, \( S_\sigma(z, z') \to 1/(z - z') \), \( \partial_z \ln[E(z, z')] \to -1/(z - z') \) at \( z \to z' \) and \( \partial_z 1/z = i \pi \delta^2(z) \), where \( \delta^2(z) \equiv \delta(\text{Re} z) \delta(\text{Im} z) \). Therefore, the integration equations (4) are reduced to the differential ones. Due to (2), these differential equations are identical to the first equation in (4). The third equation in (6) determines the periods of \( J^r_{\sigma}(z, \bar{z}) \). It is obtained from the first equation by means of the transformation which is assigned to the \( 2\pi \)-twist about the \( B_\eta \)-cycle on \( \Sigma_\eta \). Since the kernels in (6) are proportional to the Grassmann moduli, equations (6) are solved by the iteration procedure. The functions and the period matrix for the left movers are calculated in the similar manner.

Thus \( \Omega^{(R)} \) and \( \Omega^{(L)} \) depend on Grassmann moduli. Under these conditions, the moduli space integral over the fundamental region of the modular group [16] is not invariant under the supermodular transformations that leads to loss of the 2D supersymmetry. To restore the supermodular symmetry, the discussed integral is supplemented by an integral over the boundary of the integration region. To derive this boundary integral, it is useful to define a function which is an extension of the step function \( \rho(x) \) (being \( \rho(x) = 1 \) for \( x > 0 \), and \( \rho(x) = 0 \) for \( x < 0 \)) to the case when \( x = x_0 + x_\eta \) contains the ‘soul’ part \( x_\eta \) that is the part proportional to the Grassmann parameters. Then \( \rho(x) \) is understood in the sense that it is the
Taylor series in $x_b$. In the calculation of the Taylor series one employs the known relation $d \rho(x_b)/dx_b = \delta(x_b)$, where $\delta(x)$ is the Dirac delta-function, and the ‘body’ $x_b$ of $x$ contains no Grassmann parameters. Under this convention the fermion string interaction amplitude $A_{\sigma,\sigma'}$ can be represented as the integral of the $A_{\sigma,\sigma'}$ local amplitude as follows:

$$A_{\sigma,\sigma'} = \int A_{\sigma,\sigma'} \mathcal{O}(\Omega^{(R_\sigma)}, \Omega^{(L_{\sigma'})}) \left( \prod_i \widetilde{G}(\zeta_i, \tilde{\zeta}_i; q, \tilde{q}) d^2 \zeta_i d^2 \tilde{\zeta}_i \right) d^2 q d^2 \lambda,$$

where $A_{\sigma,\sigma'}$ depends on the Riemann moduli $(q, \tilde{q})$, the Grassmann moduli $(\lambda, \tilde{\lambda})$ and the $(\zeta_i, \tilde{\zeta}_i)$ coordinates of the $i$th interaction vertex. In this case $q = \{q_\beta\}$ and $\lambda = \{\lambda_j\}$. Generally, $\Omega^{(R_\sigma)}(q, \lambda)$ and $\Omega^{(L_{\sigma'})}(q, \tilde{q}, \tilde{\lambda})$. The $\widetilde{G}(\zeta_j, \tilde{\zeta}_j; q, \tilde{q})$ factor is a step function product restricting the integration region (that is the fundamental region of the Klein group) on the complex $\zeta_i$-planes. The integration region over the moduli space is ‘restricted’ by the $\mathcal{O}(\Omega^{(R_\sigma)}, \Omega^{(L_{\sigma'})})$ step function product. In this case

$$\mathcal{O}(\Omega^{(R_\sigma)}, \Omega^{(L_{\sigma'})}) = \prod_j \rho(\mathcal{G}_j), \quad \mathcal{G}_j \equiv \mathcal{G}_j(\Omega^{(R_\sigma)}, \Omega^{(L_{\sigma'})}).$$

The set of the $\mathcal{G}_j(\Omega, \bar{\Omega}) = 0$ conditions gives the boundary [16] of the fundamental region of the modular group. The step functions $\rho(\mathcal{G}_j)$ in (9) are treated as the Taylor series in $\mathcal{G}_j(\Omega^{(R_\sigma)})$ and in $\mathcal{G}_j(\Omega^{(L_{\sigma'})})$ matrix elements. Therefore,

$$\mathcal{O}(\Omega^{(R_\sigma)}, \Omega^{(L_{\sigma'})}) = B_{R_\sigma} B_{L_{\sigma'}} \mathcal{O}(\Omega, \bar{\Omega}),$$

where the differential operators $B_{R_\sigma}$ and $B_{L_{\sigma'}}$ are defined as follows:

$$B_{R_\sigma} = 1 + \sum_{p \leq q} \mathcal{G}_j^{(R_\sigma)} \frac{\partial}{\partial \Omega_{pq}} + \frac{1}{2} \sum_{p \leq q, r \leq s} \mathcal{G}_j^{(R_\sigma)} \mathcal{G}_j^{(R_\sigma)} \frac{\partial}{\partial \Omega_{pq}} \frac{\partial}{\partial \Omega_{rs}} + \cdots,$$

$$B_{L_{\sigma'}} = 1 + \sum_{p \leq q} \mathcal{G}_j^{(L_{\sigma'})} \frac{\partial}{\partial \Omega_{pq}} + \frac{1}{2} \sum_{p \leq q, r \leq s} \mathcal{G}_j^{(L_{\sigma'})} \mathcal{G}_j^{(L_{\sigma'})} \frac{\partial}{\partial \Omega_{pq}} \frac{\partial}{\partial \Omega_{rs}} + \cdots.$$

The derivatives in (11) are calculated assuming that the period matrix elements are unrelated to each other up to the transposing operation. Under the operators (11), the step function in (10) receives $\delta$-function-type terms that leads to the appearances of the integral over the boundary of the fundamental region of the modular group.

Under the change of integration variables in (8), the arguments of the step functions are correspondingly replaced. As the result the amplitude (8) is independent of the choice of the integration variables. Strictly speaking, the last statement implies that the integral (8) is properly regularized at the points where the Riemann surface is degenerate, but we do not discuss this matter in this paper. The amplitude (8) can be derived [7] by a change of integration variables in the expression for the same amplitude in the superconformal gauge [8, 9]. Hence (8) is independent of $\phi_{+\mp}$ (we have directly verified it for $g \leq 3$).

3. Local amplitude $A_{\sigma,\sigma'}$

To derive $A_{\sigma,\sigma'}$ in (8), we start [6, 7, 12] with the integral [18] over all the fields, including the zweibein and the world-sheet gravitino field. The integral is divided by the volume of the local group $\mathcal{G}$ of the world-sheet symmetries of the fermion string. So far as the zweibein and the world-sheet gravitino field are arbitrary, we can map [12, 6] the Riemann surface onto the complex plane $\omega$ choosing the same transition group $\mathcal{G}_j$ for all surfaces of the given genus-$n$. There is no integration over any moduli. The zweibein and gravitino fields can be reduced to the full set of the reference fields. It is performed by globally defined transformations of the
$\mathcal{G}$ group that do not change the $\hat{\mathcal{G}}_t$ transition group. The reduction is impossible within the full set of the reference fields. The zweibein and the gravitino field are represented in terms of the reference fields and the gauge functions. Since the gauge functions correspond to the $\mathcal{G}$ group transformations, the $\hat{\mathcal{G}}_t$ transition group is unchanged and, therefore, it is the same for all the genus-$g$ surfaces. In this case the reference fields (for $g > 1$) depend on $(3g - 3|2g - 2)$ complex moduli (defined up to the supermodular transformations). Locally, the reference fields are arbitrary. The integration over the zweibein and the gravitino field is transformed to the integration over the gauge functions and the moduli. In doing so the Jacobian of the transformation is represented by the integral over the ghost fields and over (3g - 3|2g - 2) global complex variables dual to the $(3g - 3|2g - 2)$ complex moduli. Calculating alterations of the integral under infinitesimal local variations of the reference fields, one can derive the Ward identities [6, 7, 12] from the condition that the amplitude (8) is unchanged under the above-mentioned variations of the reference fields. The obtained Ward identities are transformed to the desirable gauge of the reference fields. The Ward identities can be used for the calculation of the local amplitude. Indeed, the direct calculation of the amplitude from the integral over the fields is hampered as determinants of the differential operators appear in the calculation.

Therefore, the integral requires a regularization ensuring the independence of the amplitude (8) from infinitesimal local variations of the reference fields. In the considered gauge [1] the discussed uncertainty, however, appears in the local amplitude as the factor which is independent of the Grassmann moduli and the vertex coordinates. Excepting this factor, the local amplitude can be obtained from the discussed integral over the fields and over the (3g - 3|2g - 2) global complex variables ($A^{\mu}_m|A^f$) dual to the $(3g - 3|2g - 2)$ complex moduli. The integral is as follows [6, 7, 12]:

$$A_{\sigma,\sigma'}^{(v)} = A_{\sigma,\sigma'}^{(v)} \left[ \prod_{j} (V_{j})_{\phi} \right] = \int (DF) \, d^2 \Lambda^b \, d^2 \Lambda^f (V) \exp \left[ S_m + S_{\phi b}^\phi + S^K_{\Lambda b} + S^K_{\Lambda f} \right],$$

where $A_{\sigma,\sigma'}^{(v)}$ is a ‘vacuum local amplitude’ and $\langle \prod_{j} V_j \rangle_{\phi}$ is the vacuum expectation of the vertex product $(V)$. The vacuum expectation is calculated in the gravitino field (1). Furthermore, $(DF)$ is the product of differentials of the fields, the fields being the ghost complex fields and ten scalar $\chi^N$ fields with their superpartners ($\psi^N, \bar{\psi}^N$). The ghost complex fields are (2, -1)-tensor fields $(b, c)$ and (3/2, -1/2)-tensor fields $(\beta, \gamma)$. Further on,

$$\mathcal{S}_m = \frac{2}{\pi} \sum_{N} \int d^2 z \left[ -\bar{\pi} \chi^N \bar{\partial} \chi_N + \psi^N \bar{\psi} \gamma_N + \bar{\psi}^N \partial \chi_N + \phi_- \psi^N \bar{\partial} \chi_N + \phi_+ \bar{\psi}^N \partial \chi_N \right],$$

$$S_{\phi b}^\phi = \frac{1}{\pi} \int d^2 z \left[ -\bar{b} \bar{\partial} c + \bar{\beta} \bar{\partial} y - \frac{1}{2} \phi_- (b y + \beta \partial c) + \beta (\partial \phi_- c) \right],$$

$$S^K_{\Lambda b} = \frac{1}{\pi} \int d^2 z \left[ -\sum_m \left( \partial \bar{\partial} \xi_m + \frac{1}{2} \phi_- \beta \partial \xi_m - \beta \partial \phi_- \xi_m - \beta \frac{\partial \phi_-}{\partial q_m} \Lambda^b_m + \sum_s \beta \phi_s \Lambda^f_s \right) \right],$$

where $\partial f \equiv \partial_z f$ and $\bar{\partial} f \equiv \bar{\partial}_z f$ for any function $f \equiv f(z, \bar{z})$. Also, $\phi_- \equiv \phi_-(z, \bar{z})$ and $\phi_+ \equiv \phi_+(z, \bar{z})$; see (1). $S_{\phi b}^\phi$ and $S^K_{\Lambda b}$ in (12) are obtained by the complex conjugation of (14) and (15) together with the $\phi_-(z, \bar{z}) \rightarrow \phi_+(z, \bar{z})$ replacement.

Equation (15) contains a function $\zeta_m \equiv \zeta_m(z, \bar{z})$ that is one valued under rounds about $A$-cycles and has a discontinuity under twists about $B$-cycles on the Riemann surface. Let us assign the $z \rightarrow g(z)$ replacement\(^2\) to the $2\pi$-twist about $B_\tau$-cycle. Then

$$\zeta_m (g(z), g_\tau(z)) = \zeta_m (z, \bar{z}) \left[ \frac{\partial g(z)}{\partial \bar{z}} + \frac{\partial g_\tau(z)}{\partial q_m} \right].$$

\(^2\) For the sake of simplicity we assume the Schottky description of the Klein group, but the usage of the Schottky moduli is not implied.
The last term on the right-hand side of (16) is the discontinuity. Due to the discontinuity in $\zeta_m(z, \bar{z})$, the integration over zero modes of $b$-fields in (12) is convergent. As explained above, the result of the integration in (12) does not depend on a further specification of $\zeta_m(z, \bar{z})$.

The $V$ vertex in (12) is built using the supercovariant operators (2), but it can be verified that the $\phi_\gamma$-dependent terms in (2) do not contribute to the amplitude. It appears due to motion equations following from (13). So the conventional vertex [19] can be employed.

With the exception of the $(\lambda_1, \bar{\lambda}_1)$ independent factor in $A_{\mu_1\nu_1}$, the amplitude (12) is expressed in terms of the correlation functions. They are calculated from the integral (12) at vanishing $(\lambda_1, \bar{\lambda}_1)$. In doing so the linear sources of the fields and of the global variables are added to the exponent (it is the known trick in the calculation of correlation functions). From (14) and (15), it follows that $c \equiv c(z, \bar{z})$ and $\zeta_m(z, \bar{z})$ are combined into the $c(z, \bar{z})$ field, where $\tilde{c}(z, \bar{z}) = c(z, \bar{z}) + \sum_s \zeta_m(z, \bar{z}) \Lambda^s_m$. The $(\tilde{a}b)$ correlator (at $\lambda_s = 0$) is [6, 7, 11]

$$\langle \tilde{c}(z, \bar{z}) b(z', \bar{z'}) \rangle = G_b(z, z'),$$

where $G_b(z, z') \to 1/(z - z')$ at $z \to z'$, and

$$G_b(g_1(z), z') = \frac{\partial g_s(z)}{\partial z} G_b(z, z') + \sum_{m=1}^{(3g-1)} \frac{\partial g_s(z)}{\partial q_m} \chi_m(z').$$

The last term on the right-hand side of the first equation in (18) appears due to the discontinuity (16) of $\zeta_m$. Furthermore, $G_b(z, z')$ is not changed under rounds about $A_1$-cycles. As the function of $z'$, the correlator (17) is the conform two-rank tensor. These properties are sufficient to determine both $G_b(z, z')$ and 2-rank-tensor zero modes $\chi_m(z)$. Henceforth the amplitude (12) does not depend on details of $\zeta_m(z, \bar{z})$. It should be noted that $G_b(z, z')$ is given by a Poincaré series that is not expressed through a local combination of theta-like functions [7, 11].

In [1] the correlator (17) is mistakenly replaced by $\langle c(z, \bar{z}) b(z', \bar{z'}) \rangle \equiv G_b(z, z'; p)$ depending on $3(g - 1)$ arbitrary points $p = \{p_m\}$. The $G_b(z, z'; p)$ correlator is one valued on the Riemann surface, has poles at $z = p_m$ and vanishes at $z' = p_m$. These properties determine $G_b(z, z'; p)$ up to a numerical factor. In our normalization of the fields the correlators are related as follows:

$$G_b(z, z'; p) = G_b(z, z') - \sum_m G_b(z, p_m) \tilde{\chi}_m(z'), \quad \tilde{\chi}_m(p_m) = \delta_m.$$  

where $\tilde{\chi}_m(z)$ are the 2-rank-tensor zero modes which are normalized as shown in (20). Due to equation (18), the right-hand side of (20) is one valued on the Riemann surface. In addition, it has poles at $z = p_m$ vanishes at $z' = p_a$ and goes to $1/(z - z')$ at $z \to z'$. Thus the right-hand side of (20) coincides with $G_b(z, z'; p)$.

To clarify discrepancy between the amplitude (12) and the corresponding amplitude in [1] we transform (12) to an integral where each $(b, \tilde{b})$ field vanishes in $3(g - 1)$ points on the Riemann surface. For this aim the $(\zeta_m, \tilde{\zeta}_m)$ functions are properly specified, and the terms that are proportional to $b \Lambda^s_m$ and $\tilde{b} \Lambda^s_m$ are removed from the exponent in (12) by a relevant shift of the $(\gamma, \bar{\gamma})$ fields. In more detail, $\gamma \to \gamma + \sum_s \tilde{\zeta}_m \Lambda^s_m$, where $\tilde{\zeta}_m \equiv \tilde{\zeta}_m(z, \bar{z}; p)$ and $\zeta_m \equiv \zeta_m(z, \bar{z}; p)$ depend on the $p = \{p_m\}$ set of $3(g - 1)$ points $p_m$ on the Riemann surface. Furthermore,

$$\tilde{\zeta}_m(z, \bar{z}; p) = -\frac{1}{\pi} \int G_s(z, z'; \phi) [\partial_{z'} \phi(z', \bar{z})] \zeta_m(z, \bar{z}; p) - \frac{1}{2} \phi(z', \bar{z}) \partial_{z'} \zeta_m(z, \bar{z}; p)$$

$$+ \gamma \partial_{z} \phi_{-}(\zeta, \bar{z}) \bar{d}_z z'.$
\[ \zeta_m(z, \tilde{z}; p) = \sum_a G_b(z, p_u) \hat{N}_{am} + \frac{1}{2\pi} \int G_b(z, z') \phi(z', \tilde{z}') \xi_m(z, \tilde{z}; p) d^2z', \]
\[ \hat{N}_{am} = \sum_n \hat{N}_{am}^{-1} N_{mn}, \quad \hat{N}_{ma} = \chi_a(p_a), \]
\[ N_{nm} + \frac{1}{2\pi} \int \chi_a(z) \phi(z', \tilde{z}) \xi_m(z, \tilde{z}; p) d^2z' = 1, \quad (21) \]
where \( \hat{N}_{am}^{-1} \) is the element of the matrix \( \hat{N}^{-1} \). The 2-rank-tensor zero modes \( \chi_a(z) \) are the same as in (19). The correlator \( G_a(z, z'; \{ \phi \}) \) function satisfies to the equation as follows:
\[ \partial_z G_a(z, z'; \{ \phi \}) = \pi \delta^2(z - z') - \sum_m \phi_{-m}(z, \tilde{z}) \tilde{\chi}_m(z'), \quad \int \tilde{\chi}_m(z) \phi_{-m}(z, \tilde{z}) d^2z = \delta_{mn}, \quad (22) \]
where \( \delta_{mn} \) is the Kronecker symbol and \( \tilde{\chi}_m(z') \) is 3/2-rank-tensor zero modes. In this case \( G_a(z, z'; \{ \phi \}) = -\langle y(z, \tilde{z}) \beta(z', \tilde{z}') \rangle \), where \( \langle y \beta \rangle \) is the correlator at vanishing Grassmann moduli. Due to the \( \sim A^1 \) terms in (15), it is a functional of \( \phi_{-m} \). The \( (\tilde{\gamma}, \tilde{\bar{\beta}}) \)-dependent part of the exponent is represented in the kindred manner. The \( (\gamma, \beta) \) and \( (\gamma, \bar{\beta}) \) correlators at arbitrary \( \phi_{\pm m} \) are expressed through the correlators in the case when \( \phi_{-m} \) and \( \phi_{+m} \) are localized at \( z = r \) and, respectively, at \( z = r' \), as follows:
\[ \phi_{-m}(z, \tilde{z}) = \delta^2(z - r), \quad \phi_{+m}(z, \tilde{z}) = \delta^2(z - r'). \quad (23) \]
In this case we denote the \( (\gamma, \beta) \) correlator as \( \langle y(z, \tilde{z}) \beta(z', \tilde{z}') \rangle = -G_a(z, z'; r), \) where \( r = \{ r_j \} \). This correlator was calculated in [1]. It has the pole at \( z = r_j \), the residue being the 3/2-rank-tensor zero mode \( \chi_j(z') \) satisfying the condition \( \chi_j(r_j) = -\delta_{ij} \). Then
\[ G_a(z, z'; \{ \phi \}) = G_a(z, z'; r) + \sum_s \frac{1}{\pi} \int G_a(z, z_1; r) \phi_{-s}(z_1, \tilde{z}_1) d^2z_1 \tilde{\chi}_s(z'), \quad (24) \]
\[ \chi_j(z) + \frac{1}{\pi} \sum_s \int \tilde{\chi}_s(z') \phi_{-s}(z', \tilde{z}_1) d^2z' \tilde{\chi}_s(z) = 0. \quad (25) \]
Indeed, one can verify that the right-hand side of (25) satisfies to (22). The kindred expression exists for the \( (\tilde{\gamma}, \tilde{\beta}) \) correlators. Once the integration over the global variables being performed, the vacuum amplitude \( \mathcal{A}^{(v)}_{\gamma, \beta} \) is found to be as follows:
\[ \mathcal{A}^{(v)}_{\gamma, \beta} = \int (DF) W_{\text{R}}(p, \tilde{p}) W_{\text{L}}(p', \tilde{p}') \exp \left[ S_m + S_{gh}^m + S_{gh}^L \right], \quad (26) \]
where
\[ W_{\text{R}}(p, \tilde{p}) = \frac{\det \mathcal{N}}{\det \chi_m(p_a)} \left[ \prod_{a=1}^{3(g-1)} b(p_a, \tilde{p}_a) \right] \left[ \prod_{j=1}^{2(g-1)} \delta \left( \frac{1}{\pi} \int \beta(z, \tilde{z}) \phi_{-j}(z, \tilde{z}) d^2z \right) \right] \quad (27) \]
and \( W_{\text{L}}(p', \tilde{p}') \) is the kindred expression associated with the left movers. Elements \( \mathcal{N}_{am} \) of the \( \mathcal{N} \) matrix are defined in (21). Zero modes \( \chi_m(p_a) \) are defined by (19). The product over \( a \) in \( W_{\text{R}}(p, \tilde{p}) \) provides the vanishing of the \( b(z, \tilde{z}) \) field at \( z = p_a \). Equation (27) differs from the corresponding expression in [1] by the \( \det \mathcal{N} \) factor (apart from the fact that in [1] the set of 2-rank tensor modes is not specified). This factor generates the terms in \( W_{\text{R}}(p, \tilde{p}) \) which are added to the vacuum expectations of the supercurrent products (arising from the expanding of \( \exp[S_{gh}^m] \) in \( \lambda_j \)). As the result, the \( G_a(z, z'; p) \) correlator is replaced by the \( G_a(z, z') \) one. If \( \phi_{-m} \) depends on the Riemann moduli, additional terms in the amplitude also appear due to the \( \partial_{\chi_a} \phi_{-m} \) terms in (15). The kindred thing arises in the integral over the \( (\tilde{\gamma}, \tilde{\beta}) \) fields.
As was noted, the vacuum local amplitude (12) contains an uncertain factor. The factor is independent of the \((\lambda_1, \lambda_2)\) moduli. Nevertheless, it depends on the \(\phi_{\Sigma}Q\) fields (1) because \(S^R_{\Lambda}Q\) and \(S^R_{\Lambda}\) contain the derivatives of the gravitino field with respect to the Grassmann moduli. The factor can be calculated from Ward identities \([7]\) as was mentioned above. For the commonly used setting \((23)\) for \(\phi_{\Sigma}Q\) the discussed factor was already calculated in \([1]\). If this setting is employed, the local vacuum amplitude is represented as follows:

\[
A_{\sigma,\sigma'}^{(v)} = \left[ \frac{1}{\det \Im \Omega} \right]^{5} Z_\sigma (q, r) Z_{\sigma}^{(q, r')} Z_{\sigma}^{\text{mat}} (q, \tilde{q}, \lambda, \tilde{\lambda}, r, \tilde{r}) Z_{\sigma}^{(gh)} (q, \lambda, r) Z_{\sigma}^{(gh)} (q, \lambda, r'),
\]

(28)

where \(r = \{ r_i \}, r' = \{ r'_i \}\). The \((\lambda_1, \tilde{\lambda})\)-independent factor \(Z_\sigma (q, r)\) can be taken from \([1]\). The other three factors differ from the unity only because of proportional \((\lambda, \tilde{\lambda})\) terms. Among them, \(Z_{\sigma}^{\text{mat}} (q, \tilde{q}, \lambda, \tilde{\lambda}, r, \tilde{r})\) is due to the expanding in \((\lambda, \tilde{\lambda})\) of \(\exp S^R_{\Lambda}Q\). The last factors are due to the \((\lambda, \tilde{\lambda})\) expanding of the rest exponential in (12). The calculation of \(Z_{\sigma}^{(gh)} (q, \lambda, r)\) is ambiguous because, as was noted above, the correlator \(\langle \gamma(z, \bar{z}) \beta(\bar{z}', \bar{\gamma}) \rangle = -G_\sigma (z, \bar{z}'; r)\) has the pole at \(z = r_j\). To resolve the ambiguity\(^3\) one treats fields \((23)\) as the limit of a spread fields \(\phi_{\Sigma}Q\) using equation (24) to calculate the \((\gamma, \beta)\) correlator.

4. Two- and three-loop amplitudes

In the discussed case the periods \(\Omega_{\text{au}} = \Omega_{\text{au}}\) of the scalar functions can be taken as moduli. By using equation (10), the integration by parts is performed in (8), and \(A_{\sigma,\sigma'}\) is represented as

\[
A_{\sigma,\sigma'} = \int \Omega(\Omega, \overline{\Omega}) B_{\text{LR}}^{T} B_{\text{LR}}^{T} \left[ A_{\sigma,\sigma'} \prod_i \tilde{O}(z_i, \bar{z}_i) \right] d^2 \omega_i d^2 \lambda_i \prod_i d^2 z_i,
\]

(29)

where \(\tilde{O}(z_i, \bar{z}_i) \equiv \tilde{O}(z_i, \bar{z}_i; \Omega, \overline{\Omega})\), and \((B_{\text{LR}}^{T}, B_{\text{LR}}^{T})\) are obtained by transposing \((B_{\text{LR}}, B_{\text{LR}})\). If the Schottky description is employed, then

\[
\tilde{O}(z_i, \bar{z}_i) = \rho (1 - |g_{s}'|^2) \rho (1 - |g_{s}'|^2),
\]

(30)

where \(g_{s}'(z) = \frac{\partial g_{s}(z)}{\partial z}\); the \(z \to g_{s}(z)\) transformation is assigned to \(2\pi\)-twist about \(B_{\text{LR}}\) cycle. The \(z \to g_{s}(z)\) transformation is inverse to the \(z \to g_{s}(z)\) one, so that \(g_{s}(g_{s}(z)) = g_{s}(z)\). As above, \(\rho(z)\) is the step function. Instead of the boundary integral in the moduli space equation (29) contains the integral along the boundary of the fundamental region of the Klein group. This integral arises due to the action of the \((B_{\text{LR}}^{T}, B_{\text{LR}}^{T})\) operators on the \(\tilde{O}(z_i, \bar{z}_i)\) step function products. This boundary integral can be reduced to the integral over the fundamental region of the Klein group. The above-mentioned reduction is performed employing the set of certain functions \(U_{pq}(z) = U_{pq}(z)\). The \(U_{pq}(z)\) function is unchanged under twists about \(A_{\tau}\)-cycles, but it has the discontinuity under \(2\pi\)-twist \(z \to g_{s}(z)\) about \(B_{\text{LR}}\)-cycle, as follows:

\[
U_{pq}(g_{s}(z)) = \frac{\partial g_{s}(z)}{\partial z} U_{pq}(z) + \frac{\partial g_{s}(z)}{\partial \omega_{pq}},
\]

(31)

The last term on the right-hand side of (31) is the discontinuity of \(U_{pq}(z)\). Then

\[
\int A_{\sigma,\sigma'} \frac{\partial \tilde{O}(z_i, \bar{z}_i)}{\partial \omega_{pq}} d^2 z_i = \int A_{\sigma,\sigma'} U_{pq}(z_i) \frac{\partial \tilde{O}(z_i, \bar{z}_i)}{\partial z_i} d^2 z_i.
\]

(32)

Indeed, from (30), the left- and right-hand sides of equation (32) are the integrals along the \((A_1 + A_2)\) contour where the \(A_1\) contour is given by the \(|g_{s}'(z)|^2 = 1\) condition, and the \(A_2\) contour is given by the \(|g_{s}'(z)|^2 = 1\) condition. By the \(z \to g(z)\) replacement the integration

\(^3\) In [1] this matter is treated inexactly.
along $\hat{A}_1$ is reduced to the integration along $A_1$. The right-hand side of (32) is calculated using equation (18) and taking into account that $A_{\sigma,\sigma'}$ is $(1, 1)$-tensor in $(z_1, \bar{z}_1)$. The left-hand side of (32) is calculated using the relation
\[
\frac{\partial \rho(1 - |g'_v(z)|^2)}{\partial \Omega_{pq}} \bigg|_{z \to g_v(z)} = \frac{\partial \rho(1 - |g'_v(z)|^2)}{\partial \Omega_{pq}} + \frac{\partial \rho(1 - |g'_v(z)|^2)}{g'_v(z) \partial z}.
\]
(33)
As the result, the same expression appears for both the left- and right-hand sides of (32) that proves the validity of equation (32). The integration by parts being performed, equation (29) is reduced to the integral over the fundamental regions of the modular group and the Klein group (there is not any boundary integral), as follows:
\[
A_{\sigma,\sigma'} = \int \tilde{A}_{\sigma,\sigma'} \prod_{m \in n} d^2 \Omega_{mn} \prod_j d^2 z_j, \quad \tilde{A}_{\sigma,\sigma'} = \int A_{\sigma,\sigma'}^{(\text{mod})} \prod_i d^2 \lambda_i, A_{\sigma,\sigma'}^{(\text{mod})} = \tilde{B}_{R_0}^T \tilde{B}_{L_0}^T A_{\sigma,\sigma'},
\]
(34)
where $A_{\sigma,\sigma'}$ is given by (12). Operators $\tilde{B}_{R_0}^T$ and $\tilde{B}_{L_0}^T$ are obtained by the $\partial/\partial \Omega_{pq} \to D_{pq}$ replacement in $B_{R_0}^T$ and, respectively, by the $\partial/\partial \Omega_{pq} \to D_{pq}$ replacement in $B_{L_0}^T$. In this case
\[
D_{pq} = \frac{\partial}{\partial \Omega_{pq}} - \sum_i \frac{\partial}{\partial z_i} U_{pq}(z_i).
\]
(35)
The $U_{pq}(z)$ function is constructed using $G_{ij}(z, z')$ and zero modes $\chi_i(z)$; see (17)–(19). In $g = 2$ or $g = 3$ cases $n$-index is replaced by a pair $(j l)$ of indices listing the period matrix elements. Then we use the notation $\chi_{(j l)}(z)$ instead of $\chi_i(z)$. As shown below
\[
\chi_{(j l)}(z) = -2\pi iv_j(z),
\]
(36)
where $v_j(z) = \partial_z J_z(z)$ is 1-form. For $U_{pq}(z)$ one can use a sum of $G_{ij}(z, w_j) N_{ij(pq)}^{-1}$ over $3(g - 1)$ arbitrary points $w_j$, where $N^{-1}$ is the inverse to the $N$ matrix $(\det N \neq 0)$, whose matrix elements $N_{ij(pq)}$ are $N_{ij(pq)} = \chi_{(pq)}(w)$, This $U_{pq}(z)$ depends on $3(g - 1)$ arbitrary points $w_j$. For the calculation of GSO projection it is more convenient to use alternative functions which depend on $(2g - 3)$ points $w_j$ (these points can be identified with the 2D gravitino location points). These functions are built using 2-rank-tensor modes $\zeta_j(z), \tilde{\zeta}_j(z)$ and $\tilde{\tau}(u)$ which are
\[
\zeta_j(z) = \tau_j(z) - \sum_{i=1}^{e-1} \tau'_j(w_i) \tilde{\tau}_i(z) - \left[ \tau''_j(w_1) - \sum_{i=1}^{e-1} \tau''_j(w_i) \tilde{\tau}_i''(w_1) \right] \tilde{\tau}(z),
\]
\[
\tilde{\zeta}_j(z) = \tilde{\tau}_i(z) - \tilde{\tau}_i''(w_1) \tilde{\tau}(z)
\]
where 2-rank-tensor modes $\tau_j(z)$ and $\tilde{\tau}(z)$ satisfy conditions that $\tau_j(w_i) = \delta_{ij}$, $\tilde{\tau}_i(w_j)$ for all $w_i$, while $\tilde{\tau}_i(w_l) = \delta_{il}$ for $l = 1$ and (if $g = 3$) for $l = 2$. In this case, for any $f(x)$ function,
\[
\tilde{\tau}(x) = \partial_z \tilde{\tau}(x) \quad \text{and} \quad \tilde{\tau}(x) = \partial_z^2 \tilde{\tau}(x).
\]
For $g = 2$ one can set $\tau_1(z), \tilde{\tau}_1(z)$ and $\tilde{\tau}(z)$ as follows:
\[
\tau_1(z) = \frac{v_1^2(z)}{v_1^2(w_1)}, \quad \tilde{\tau}_1(z) = \frac{v_1(z) d(z, w_1)}{v_1(w_1) d'(w_1, w_1)}, \quad \tilde{\tau}(z) = \frac{1}{2} \left( \frac{d(z, w_1)}{d'(w_1, w_1)} \right)^2,
\]
(38)
where $d'(x, y) = \partial_x d(x, y)$ and $d(x, y) = v_1(x) v_2(y) - v_1(y) v_2(x)$. For $g = 3$ one can set
\[
\tau_1(z) = \frac{d(z, w_2) d(z, w_3)}{d(w_1, w_2) d(w_1, w_3)}, \quad \tilde{\tau}_1(z) = \frac{d(z, w_2) d(z, w_1, w_2)}{d(w_1, w_2) d'(w_1, w_1, w_2)}, \quad \tilde{\tau}(z) = \frac{1}{2} \left( \frac{d(z, w_1, w_2)}{d'(w_1, w_1, w_2)} \right)^2,
\]
(39)
where $d(x_1, x_2, x_3) = \det v_1(x_1)$, $d'(x, y, z) = \partial_i d(x, y, z)$ and $d(x, y) = v_1(x) v_2(y) - v_1(y) v_2(x)$. Functions $\tau_2(z)$, $\tau_3(z)$ and $\bar{\tau}_2(z)$ are obtained by a replacement of indices in $\tau_1(z)$ and $\bar{\tau}_1(z)$. In this case the $U_{nm}(z)$ functions are defined by

$$-2\pi i U_{nm}(z) = \sum_{j=1}^{2e-3} G_b(z, w) \zeta_{j,m,n} + \sum_{s=1}^{g-1} \partial_{w_s} G_b(z, w_s) \tilde{\zeta}_{s,m,n} + \partial^2_{w_s} G_b(z, w_1) \tilde{\tau}_{s,m,n},$$

(40)

where $\zeta_{j,m,n}$, $\tilde{\zeta}_{s,m,n}$ and $\tilde{\tau}_{s,m,n}$ are coefficients of expansion of $\zeta_j(u)$, $\tilde{\zeta}_s(u)$ and $\tilde{\tau}(u)$ in the $\chi_{(n)}(z) / (-2\pi i)$ modes. Derivatives with respect to the Riemann moduli of the correlators and the functions are calculated in line with [6, 7]. As proved below

$$\sum_{m=1}^{2\pi} \chi_{(m)}(w) \frac{\partial J_1(z)}{\partial \Omega_m} = -\frac{\partial^2 R(z, w)}{\partial w} v_i(w) - v_i(z) G_R(z, w) + \mu_s(w),$$

(41)

where $R(z, w)$ is the scalar, holomorphic Green function [6] for the $\partial_\bar{z} / \partial z$ operator. This $R(z, w)$ is not changed under twists about $A_s$-cycles, where $2\pi$-twist $z \to g^1_s(z)$ about $B_s$ cycle, it is changed as follows:

$$R(g^1_s(z), w) = R(z, w) + 2\pi i J_1(z), \quad R(z, g^1_s(w)) = R(z, w) + 2\pi i J_1(z).$$

(42)

To derive equation (36), the $z \to g_s(z)$ replacement in (41) is performed and the relation $J_s(g_s(z)) = J_s + \Omega_m$ is used, along with equation (18) and with the following relations:

$$\frac{\partial J_s}{\partial q_m} \bigg|_{z \to g_s(z)} = \bigg( \frac{\partial J_s}{\partial q_m} \bigg)_{z} - \frac{\partial g_s(z)}{\partial q_m} v_i(z),$$

(45)

$$\bigg( \frac{\partial J_s}{\partial q_m} \bigg)_{z} = \frac{\partial J_s}{\partial q_m} + \frac{\partial \Omega_m}{\partial q_m}.$$

(46)

In equation (45) the derivative is calculated under fixed $z$.

To prove equation (41), the difference $\Delta_s(z, w)$ of the left- and right-hand sides of (41) is represented as

$$\Delta_s(z, w) = \int_{\gamma} \Delta_s(u, w) \partial_u R(u, z) \frac{du}{2\pi i},$$

(47)

where the integration is performed along the contour surrounding the point $u = z$. The integral (47) is reduced to the integral along the boundary (30) of the integration region. Furthermore, by the $z \to g^1_s(z)$ replacement, the integral along the $|g^1_s(u)|^2 = 1$ contour is reduced to the integral along the contour $|g^1_s(z)|^2 = 1$. Then, by using equations (18) and (46), the integral (47) is reduced to the sum of expressions, each being proportional to the integral of $\partial_u R(u, z)$ along the $|g^1_s(u)|^2 = 1$ contours. So far as $R(u, z)$ is unchanged under twists about $A_s$-cycles, the integral vanishes that proves equation (41).

The calculation of the same derivatives of the scalar field correlator $\langle \chi(z_1, \bar{z}_1) \chi(z_2, \bar{z}_2) \rangle$

$$= -\chi(z_1, \bar{z}_1; z_2, \bar{z}_2)$$

is simplified, if it is chosen as follows:

$$X(z_1, \bar{z}_1; z_2, \bar{z}_2) = \frac{1}{2} \Re R(z_1, z_2) + \pi \sum_{s,i} \text{Im} J_s(z_1) \left[ \frac{1}{\text{Im} \Omega} \right]_{s,i} \text{Im} J_s(z_2).$$

(48)
This correlator differs from the scalar field correlator in [1] by a scalar zero mode contribution. In the calculation of the amplitude both the correlators can be used on equal terms. Furthermore, it can be proved that
\[
\sum_{m \leq n} \chi_{(m,n)}(w) \frac{\partial R(z, u)}{\partial \Omega_{mn}} = - \frac{\partial R(z, w)}{\partial w} \frac{\partial R(w, u)}{\partial w} - \frac{\partial R(u, w)}{\partial w} \frac{\partial R(w, z)}{\partial w} - G_b(z, w) \frac{\partial R(z, u)}{\partial z}
\]
\[+ G_b(u, w) \frac{\partial R(z, u)}{\partial w} + \mu(w) + (z + u) \widetilde{\mu}(w). \tag{49}\]

An explicit form of \(\mu(w)\) and of \(\widetilde{\mu}(w)\) is not used in this paper. The proof of equation (49) is similar to the proof of equation (41). From (41) and (49), it follows that
\[
\sum_{m \leq n} \chi_{(m,n)}(w) \frac{\partial X(z, \tilde{z}, u, \tilde{u})}{\partial \Omega_{mn}} = -4 \tilde{X}(z, \tilde{z}; w) \tilde{X}(u, \tilde{u}; w) - \tilde{X}(z, \tilde{z}; u) G_b(z, w) \]
\[- \tilde{X}(u, \tilde{u}; z) G_b(z, w), \quad \tilde{X}(z, \tilde{z}; u) \equiv \partial_u X(z, \tilde{z}; u, \tilde{u}). \tag{50}\]

If the relation
\[
\sum_{n \leq m} \chi_{(m,n)}(z) \frac{\partial \Upsilon}{\partial \Omega_{nm}} = \mathcal{H}(z) \tag{51}\]
takes place for certain \(\Upsilon(z)\), then the derivatives on its left-hand side are expressed through \(\mathcal{H}(w)\jmath\) as follows:
\[
-2\pi i \frac{\partial \Upsilon}{\partial \Omega_{nm}} = \sum_{j=1}^{2g-3} \mathcal{H}(w_j) \xi_{j,nm} + \sum_{j=1}^{g-1} \partial_{w_j} \mathcal{H}(w_j) \tilde{\zeta}_{j,nm} + \tilde{\delta}_{w_1}^2 \mathcal{H}(w_1) \tilde{\tau}_{nm}. \tag{52}\]

To calculate GSO projections, the \(r_1\) points (and the \(r_1^j\) ones in (23) are submitted to the condition
\[
\sum_{j=1}^{2g-2} r_j - 2\Delta = 0, \tag{53}\]
where \(\Delta\) denotes the vector of the Riemann constants and \(r_j\) is related to \(r_j\) by the Jacobi mapping. Any \((g - 1)\) points in the \(\{r_j\}\) set can be taken at will. It follows from (53) that
\[
d(r_{p_1}, \ldots, r_{p_{g}}) \equiv \det[v_1(r_{p_1}), \ldots, v_{g-1}(r_{p_{g}})] = 0, \quad v_j(r_{p_j}) = \sum_{s=1}^{g-1} \alpha_s v_s(r_{p_j}), \tag{54}\]
where \(\alpha_s\) are the same for every \(r_{p_s} \subset \{r_j\}\). Furthermore,
\[
d(r_{s_1}, \ldots, r_{s_{g-1}}, z) = f(z; g) d_{gg}(r_{s_1}, \ldots, r_{s_{g-1}}), \quad f(z; g) = v_g(z) - \sum_{s=1}^{g-1} \alpha_s v_s(z), \tag{55}\]
where \(d_{gg}(r_{s_1}, \ldots, r_{s_{g-1}})\) is the \((g, g)\)-minor of the \(d(r_{s_1}, \ldots, r_{s_{g-1}}, z)\) determinant. Under (53), the \(Z_q(g, r)\) factor in (28) is simplified essentially. Besides, the \(\langle \gamma(z, \tilde{z}) \beta(z', \tilde{z}') \rangle\) correlator can be represented as follows:
\[
\langle \gamma(z, \tilde{z}) \beta(z', \tilde{z}') \rangle = - \frac{f(z'; g)}{f(z; g)} S_n(z, z'). \tag{56}\]

The sums over spin structures (GSO projections) are calculated in the known manner using Riemann relations and Fay identities (see [17, equation (45)]). The sums (GSO projections) of the amplitudes vanish, if the vertex number \(n < 4\). If \(n = 4\), then only that part of \((V)\) in (12) contributes to the sum, which contains the product of all the fermion fields. Using (41)
and \((50)\), one can show that the discussed contribution to \(A^{(\text{mod})}_{\sigma,\sigma'}\) is factorized in \([r_j]\) and \([r'_j]\) so that \(A^{(\text{mod})}_{\sigma,\sigma'}\) in \((34)\) is replaced by an expression

\[
A^{(\text{mod})}_{\sigma,\sigma'} \rightarrow \left[ \frac{1}{\det[\text{Im } \Omega]} \right]^5 \langle V_0, A^{(5)}_{\sigma} (\Omega, \bar{\Omega}, \lambda, r, z_V, \bar{z}_V), A^{(5)}_{\sigma'} (\Omega, \bar{\Omega}, \lambda', r', z_V, \bar{z}_V) \rangle
\]

with \(z_V = [z_j]\), and \((V_0)\) to be the vacuum average of \(V_0\) where

\[
V_0 = \exp \left[ \sum_j i k_j \cdot x(z_j, \bar{z}_j) \right], \quad \ln (V_0) = - \sum_{i \neq j} k_i \cdot k_j (x(z_i, \bar{z}_i) x(z_j, \bar{z}_j)),
\]

\(k_j\) being 10-momentum of the \(j\)th boson. Furthermore, terms due to the differentiation of \(\det \text{Im } \Omega\) in \((28)\) are canceled with that part of \(Z^{(\text{mod})}_{\sigma,\sigma'} (q, \bar{q}, \lambda, \bar{\lambda}, r, \bar{r})\) in \((12)\) which arises due to the last term on the right-hand side of \((48)\). As the result, \(A^{(5)}_{\sigma} (\Omega, \bar{\Omega}, \lambda, r, z_V, \bar{z}_V)\) is represented as

\[
A^{(5)}_{\sigma} (\Omega, \bar{\Omega}, \lambda, r, z_V, \bar{z}_V) = \frac{1}{\langle V_0 \rangle} \hat{B}_x (\Omega, \lambda, r, z_V) \langle [V_0], \hat{A}^{(5)}_{\sigma} (\Omega, \lambda, r, z_V) \rangle,
\]

where \(\hat{A}^{(5)}_{\sigma} (\Omega, \lambda, r, z_V)\) is holomorphic in its arguments. The superstring amplitude \(A_g\) (that is, GSO projection of \((34)\)) at \(g = 2\) and \(g = 3\) is given by

\[
A_g = \int \left[ \frac{1}{\det \text{Im } \Omega} \right]^5 \langle V_0, A_g (\Omega, \bar{\Omega}, r, z_V, \bar{z}_V), A_g (\Omega, \bar{\Omega}, r', z_V, \bar{z}_V) \rangle \prod_{m \leq n} d^2 \Omega_{mn} \prod_j d^2 z_j,
\]

\[
A_g (\Omega, \bar{\Omega}, r, z_V, \bar{z}_V) = \sum_\sigma \int A^{(5)}_{g\sigma} (\Omega, \bar{\Omega}, \lambda, r, z_V, \bar{z}_V) \prod_j d\lambda_j.
\]

The calculation of \((61)\) is simplified drastically, if one sums up over \(\sigma\) before the \(\Omega_{mn}\) derivatives will be taken. Under condition \((53)\), there is a quite simplified \(Z_{\sigma} (q, r)\) factor in \((28)\). The dependence on boson polarizations in \((61)\) is extracted in the form of the factor \(|K|^2\) that is the same for \(g = 2\) and \(g = 3\), and the same as in \([5]\).

For the two-loop amplitude the result \([5]\) is reproduced. In this case \(A_g (\Omega, \bar{\Omega}, r, z_V, \bar{z}_V) = \hat{A}_2 (\Omega, \bar{\Omega}, r, z_V, \bar{z}_V)\) is found to be

\[
\hat{A}_2 (\Omega, \bar{\Omega}, r, z_V, \bar{z}_V) = \frac{K}{16 \pi^2} Z_2 (\Omega, r, z_V) [\hat{D} \ln (V_0) + \langle \partial (x(r)) \rangle \cdot \langle \partial (x(r)) \rangle + \hat{A}_2],
\]

where \(\hat{A}_2\) is independent of boson 10-momenta, \(x(r, \bar{r}) = \langle x^M (r, \bar{r}) \rangle\) and

\[
Z_2 (\Omega, r, z_V) = 2 \frac{\nu_1 (r_1) \nu_1 (r_2)}{f' (r_1; 2) f' (r_2; 2)} \int f (z_V; 2), \quad \langle \partial (x(r)) \rangle \cdot \langle \partial (x(r)) \rangle = \frac{\langle V_0 \partial_0 x (r, \bar{r}) \rangle}{\langle V_0 \rangle}.
\]

In this case \(\nu_1 (r)\) is 1-form \(\nu_1 (r)\) for \(s = 1\) and \(\nu' (r; 2) = \partial_s f (r; 2)\). The \(f (r; 2)\) function is defined in \((55)\), and \(\langle V_0 \partial_0 x (r, \bar{r}) \rangle\) is the vacuum average of \(V_0 \partial_0 x (r, \bar{r})\). The \(\hat{D}\) operator in \((62)\) is given by

\[
\hat{D} = \frac{\pi i}{2} \sum_{\rho \in \hat{Q}} \left[ v_p (r_1) v_q (r_2) + v_p (r_2) v_q (r_1) \right] \left[ \frac{\partial}{\partial \Omega_{pq}} - \sum_j U_{pq} (z_j) \frac{\partial}{\partial z_j} \right],
\]

where \(U_{pq} (z)\) is defined by \((40)\). The \(\hat{A}_2\) term in \((62)\) is actually equal to zero. The proof of this statement will be presented in a future publication where the summation over spin structures is planned to be given in detail. The other terms in \((62)\) are calculated using \((48)\), \((50)\), \((52)\) and \((58)\). In this case \(w_1\) in \((38)\) is chosen among \(r_j\), say, \(w_1 = r_1\), and the same
point \( w_1 \equiv r_1 \) is chosen in (52). Then \( A_2(\Omega, \overline{\Omega}, r, zV, \overline{zV}) \equiv \overline{A}_2(\Omega, zV) \) in (62) is found to be (below \( Z_2 \equiv Z_2(\Omega, r, zV) \) in equation (63))

\[
\overline{A}_2(\Omega, zV) = \frac{K}{16\pi^2} Z_2 \sum_{i,j} k_i \cdot k_j \left[ \frac{v_1(r_2)}{v_1(r_1)} \tilde{X}(z_i, \overline{z}_i; r_1) \tilde{X}(z_j, \overline{z}_j; r_1) - \tilde{X}(z_i, \overline{z}_i; r_1) \tilde{X}(z_j, \overline{z}_j, r_2) \right].
\]

see (50) and (58) for notations. Furthermore, using equations (44) into (48) and (52) (for \( g = 2 \), one can derive some number of identities due to the fact that the left-hand side of (52) is independent of \( w_1 \). In the explicit form the identities will be given in the future publication. Using these identities, one can represent (65) as

\[
\overline{A}_2(\Omega, zV) = \frac{K Z_2}{16\pi^2} \sum_{f,j} k_j \cdot k_i \left[ f'(r_2; 2)/v_1^2(r_1) v_1(z_i) v_1(z_j) \right] + \frac{1}{(V_0)} \sum_j \frac{f'(r_2; 2)}{4f(z_j; 2)} \frac{\partial (V_0)}{\partial z_j},
\]

where notations are given in (58) and (63). A detailed deriving of (66) is planned in the future publication. The last sum on the right-hand side of (66) originates in (60) terms corresponding globally defined derivatives with respect to \( z_j \). Thus it does not contribute to the superstring amplitude. The first sum on the right-hand side of (66) is calculated using (63) and the equation \( f'(r_1; 2)/v_1^2(r_1) = -f'(r_2; 2)/v_1^2(r_2) \) which is obtained by differentiating \( \det[v_1(r_1)] = 0 \) with respect to \( r_1 \). In this calculation one employs that \( \partial r_2/\partial r_1 = -v_1(r_1)/v_1(r_2) \). The last equation is obtained by the differentiation of (53) with respect to \( r_1 \). Once the last sum on the right-hand side being omitted and the 10-momentum conservation being taken into account, \( \overline{A}_2(\Omega, zV) \) becomes to be

\[
\overline{A}_2(\Omega, zV) = \frac{K}{64\pi^2} \sum_{j,k,l} k_j \cdot k_i \cdot v_1(z_j) v_1(z_l) v_2(z_m) v_2(z_n) = \frac{K}{64} Y([z, \overline{z}, k]),
\]

\[
Y([z, \overline{z}, k]) = \frac{1}{6} [\mathcal{Y}(1, 2; 3, 4) + \mathcal{Y}(1, 3; 2, 4) + \mathcal{Y}(1, 4; 3, 2)],
\]

where \( \mathcal{Y}(i, j; l, m) = (k_i - k_j) \cdot (k_l - k_m)d(z_i, \overline{z}_i)d(z_j, \overline{z}_j) \).

\[
d(d(z_j, \overline{z}_i) = v_1(z_j) v_2(z_j) - v_1(z_j) v_2(z_i). \]

The superstring amplitude \( A_2 \) is obtained by the substitution of \( \overline{A}_2(\Omega, zV) \equiv A_2(\Omega, \overline{\Omega}, r, zV, \overline{zV}) \) to (60). It coincides with the amplitude in (5).

In the three-loop amplitude, \( A_4(\Omega, \overline{\Omega}, r, zV, \overline{zV}) = A_3(\Omega, \overline{\Omega}, r, zV, \overline{zV}) \equiv A_3 \) in (61) is calculated in the kindred manner. Potentially, \( A_4 \) might contain terms of fourth-order, second-order and zero-order in 10-momentum components \( k_j^M \) of the interaction bosons. It can be proved that the fourth-order terms disappear in (60) due to the integration over \( z_j \). It seems plausible that zero-order terms are absent in \( A_3 \), but it needs a further study. The calculation of quadratic in \( k_j^M \) terms in \( A_3 \) has analogy with the calculation for \( g = 2 \). Unlike the \( g = 2 \) case, these terms in \( A_3 \) depend on \( r_j \), though the whole amplitude (60) is independent of \( r_j \). Besides, the discussed terms in \( A_3 \) contain the \( G_b \) function (17) that, as was noted above, cannot be expressed in terms of a local combination of theta-like functions.

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