SHORT COMMUNICATION

On the boundedness and integration of non-oscillatory solutions of certain linear differential equations of second order

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ABSTRACT

In this paper, certain system of linear homogeneous differential equations of second-order is considered. By using integral inequalities, some new criteria for bounded and $L^2[0, \infty)$-solutions, upper bounds for values of improper integrals of the solutions and their derivatives are established to the considered system. The obtained results in this paper are considered as extension to the results obtained by Kroopnick (2014) [1]. An example is given to illustrate the obtained results.

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Introduction

Very recently, Kroopnick [1] discussed some qualitative properties of the following scalar linear homogeneous differential equation of second order

$$x'' + a(t)x' + k^2x = 0, \quad (k \in \mathbb{R}).$$

He established sufficient conditions under which all solutions of Eq. (1) are bounded, and the solution and its derivative are both elements in $L^2[0, \infty)$. Furthermore, the author proved that when the solutions are non-oscillatory, they approach 0 as $t \to \infty$ and calculated upper bounds for values of improper integrals of the solutions and their derivatives, that is, for $\int_0^\infty x^2(s)ds$ and $\int_0^\infty [x'(s)]^2ds$. Finally, Kroopnick [1] introduced a short discussion about the $L^2[0, \infty)$-solutions to second order scalar linear homogeneous differential equation $x'' + q(t)x = 0$.

The results obtained by Kroopnick are summarized in Theorems A and B.

Theorem A (Kroopnick [1, Theorem 1]). Given Eq. (1). Suppose $a(.)$ is a positive element in $C[0, \infty)$ such that $A_0 > a(t) > a_0 > 0$ for some positive constants $A_0$ and $a_0$.
then all solutions to Eq. (1) are bounded. Moreover, if any solution $x(.)$ is non-oscillatory, then both $x(t) \to 0$ and $x'(t) \to 0$ as $t \to \infty$. Finally, the solution and its derivative are both elements of $L^2[0, \infty)$.

The second result proved by Kroopnick [1] is the following theorem.

**Theorem B (Kroopnick [1, Theorem II])**. Under the conditions of Theorem I, the following inequalities hold:

$$
\int_0^\infty [x'(s)]^2 \, ds \leq \frac{[x(0)]^2 + k^2|x(0)|^2}{2a_0}
$$

and

$$
\int_0^t [x'(s)]^2 \, ds \leq x(0)x'(0) + \frac{1}{2k^2} a(0)|x(0)|^2 + \frac{[x'(0)]^2 + k^2|x(0)|^2}{2a_0k^2}.
$$

It should be noted that Kroopnick [1] proved both of Theorems A and B by the integral inequalities.

In this paper, in lieu of Eq. (1), we consider the more general vector linear homogeneous differential equation of the second order of the form

$$
X'' + a(t)X' + b(t)X = 0,
$$

where $X \in \mathbb{R}^n$, $t \in \mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$; $a(.)$, $b(.) : \mathbb{R}^+ \to (0, \infty)$ are continuous functions and $a(.)$ and $b(.)$ have also lower and upper positive bounds.

It should be noted that Eq. (2) represents the vector version for the system of real second order non-homogeneous differential equations of the form

$$
x'' + a(t)x_i' + b(t)x_i = 0, \quad (i = 1, 2, \ldots, n).
$$

Then, it is apparent that Eq. (1) is a special case of Eq. (2).

It is worth mentioning that, in the last century, stability, instability, boundedness, oscillation, etc., theory of differential equations has developed quickly and played an important role in qualitative theory and applications of differential equations. The qualitative behaviors of solutions of differential equations of second order, stability, instability, boundedness, oscillation, etc., play an important role in many real world phenomena related to the sciences and engineering technique fields. See, in particular, the books of Ahmad and Rama Mohana Rao [2], Bellman and Cooke [3], Chicone [4], Hsu [5], Kolmanovskii and Myshkis [6], Sanchez [7], Smith [8], Tennenbaum and Pollard [9] and Wu et al. [10]. In the case $n = 1$, $a(t) = 0$ and $b(t) \neq 0$, Eq. (2) is known as Hill equation in the literature. Hill equation is significant in investigation of stability and instability of geodesic on Riemannian manifolds where Jacobi fields can be expressed in form of Hill equation system [11]. The mentioned properties have been used by some physicists to study dynamics in Hamiltonian systems [12]. Eq. (1) is also encountered as a mathematical model in electromechanical system of physics and engineering [2]. By this, we would like to mean that it is worth to work on the qualitative properties of solutions of Eq. (2).

In this paper, stemmed from the ideas in Kroopnick [1,13], Tunc [14,15] and Tunc and Tunc [16], etc., we obtain here some new criteria related to the bounded and $L^2[0, \infty)$-solutions, upper bounds for values of improper integrals of solutions of Eq. (2) and their derivatives, where the functions $a(.)$ and $b(.)$ do not need to be differentiable at any point and the Gronwall inequality is avoided which are the usual cases. The technique of proofs involves the integral test and an example is included to illustrate the obtained results. This work has a new contribution to the topic in the literature. This case shows the novelty of this work. The results to be established here may be useful for researchers working on the qualitative theory of solutions of differential equations.

**The main results**

In this section, we introduce the main results. We arrive at the following theorem:

**Theorem 1.** Given Eq. (2). Suppose $a(.)$ and $b(.)$ are positive elements in $C[0, \infty)$ such that

$$
A_0 > a(t) > a_0 > 0 \quad \text{and} \quad B_0 > b(t) > b_0 > 0
$$

for some positive constants $A_0$, $a_0$, $B_0$ and $b_0$ and for all $t \in \mathbb{R}^+$.

Then all solutions of Eq. (2) are bounded. Moreover, if any solution $X(.)$ of Eq. (2) is non-oscillatory, then both $\|X(t)\| \to 0$ and $\|X'(t)\| \to 0$ as $t \to \infty$. Finally, the solution and its derivative are both elements of $L^2[0, \infty)$.

**Proof.** First, we prove boundedness of solutions of Eq. (2). When we multiply Eq. (2) by $2X'(t)$, it follows that

$$
2\langle X'(t), X''(t) \rangle + 2\langle a(t)X'(t), X'(t) \rangle + 2\langle b(t)X(t), X'(t) \rangle = 0. \quad (3)
$$

Integrating estimate (3) from 0 to $t$ and then applying integration by parts to the first term on the left hand side of (3), we find

$$
2 \int_0^t \langle X'(s), X''(s) \rangle \, ds + 2 \int_0^t \langle a(s)X'(s), X'(s) \rangle \, ds + 2 \int_0^t \langle b(s)X(s), X'(s) \rangle \, ds = 0,
$$

and

$$
\|X'(t)\|^2 - \|X(0)\|^2 + 2 \int_0^t \langle a(s)\|X'(s)\|^2 \, ds + 2 \int_0^t \langle b(s)X(s), X'(s) \rangle \, ds = 0, \quad (4)
$$

respectively.

In view of the last two terms included in estimate (4), first apply the mean value theorem for integrals and then use the assumptions of Theorem 1, it follows that

$$
2 \int_0^t \langle a(s)\|X'(s)\|^2 \, ds = 2a(t') \int_0^t \|X'(s)\|^2 \, ds \geq 2a_0 \int_0^t \|X'(s)\|^2 \, ds,
$$

$$
2 \int_0^t \langle b(s)X(s), X'(s) \rangle \, ds = 2b(t') \int_0^t \langle X(s), X'(s) \rangle \, ds = b_0 \|X(t)\|^2 - b_0 \|X(0)\|^2,
$$

where $X(0)$ and $X'(0)$ are the initial values.
where $0 < t^* < t$.

On gathering the obtained estimates in (4), we have

$$2a_0\int_0^t \|X'(s)\|^2 ds + \|X'(t)\|^2 \leq \|X(0)\|^2 + b_0\|X(t)\|^2$$

$$- b_0\|X(t)\|^2 \leq \|X'(t)\|^2 - \|X(0)\|^2 + 2\int_0^t a(s)\|X'(s)\|^2 ds$$

$$+ 2\int_0^t \langle b(s)X(s), X'(s)\rangle ds.$$

Hence, in view of (4) and the last estimate, it is obvious that

$$2a_0\int_0^t \|X'(s)\|^2 ds + \|X'(t)\|^2 + b_0\|X(t)\|^2$$

$$\leq \|X(0)\|^2 + b_0\|X(0)^2\|^2. \tag{5}$$

It follows from estimate (5) that all terms on the left hand side of (5) are positive and the right hand side of (5) is bounded as $t \to \infty$. Hence, we can conclude that both $\|X(t)\|$ and $\|X'(t)\|$ must remain bounded when $t \to \infty$. Otherwise, the left hand side of (5) would become infinite, which is impossible. Furthermore, it can be seen from (5) that $\|X(\cdot)\|$ is in $L^2[0, \infty)$ since the integral

$$\int_0^t \|X(s)\|^2 ds$$

must be bounded when $t \to \infty$.

Next, we show that $\|X(\cdot)\|$, too, is in $L^2[0, \infty)$ if the solution is non-oscillatory. Multiply Eq. (2) by $X(t)$ and integrate from $0$ to $t$ and then integrate the first term by parts to obtain the following estimate

$$\langle X(t), X'(t) \rangle - \langle X(0), X'(0) \rangle - \int_0^t \langle X'(s), X'(s) \rangle ds$$

$$+ 2\int_0^t \langle a(s)X(s), X'(s) \rangle ds + 2\int_0^t \langle b(s)X(s), X(s) \rangle ds = 0. \tag{6}$$

Applying the mean value theorem for integrals to the fourth and fifth terms on the left hand side of (6), we have

$$\langle X(t), X'(t) \rangle - \langle X(0), X'(0) \rangle - \int_0^t \langle X'(s), X'(s) \rangle ds + 2a(t')$$

$$+ 2\int_0^t \langle X'(s), X(s) \rangle ds + 2b(t') - \int_0^t \langle X(s), X(s) \rangle ds = 0$$

so that

$$\langle X(t), X'(t) \rangle - \int_0^t \|X'(s)\|^2 ds + a(t')\|X(0)\|^2 + 2b(t')$$

$$\int_0^t \|X(s)\|^2 ds = a(t')\|X(0)\|^2 + \langle X(0), X'(0) \rangle, \tag{7}$$

where $0 < t^* < t$.

Now, if we can show that $X^\prime(\cdot)$ eventually does not change sign, then $X(\cdot)$ must eventually be monotone. We will then show that $\|X(\cdot)\|$ is also an element of $L^2[0, \infty)$. These two facts imply along with what has been proven before will show that both $\|X(\cdot)\|$ and $\|X^\prime(\cdot)\|$ must approach $0$ as $t \to \infty$. Otherwise, the $L^2[0, \infty)$-convergence of the solution and its derivative could not occur. We assume that $X(\cdot)$ does change sign infinitely often, then it is oscillatory. Consequently, $X^\prime(\cdot) = 0$ infinitely often. However, this means that if $X(t) > 0$, then $X^\prime(t) < 0$. So, $X(t)$ has an infinite number of consecutive critical points which are all relative maxima which is impossible. Likewise, if $X(t) < 0$, then we have an infinite number of consecutive relative minima which is also impossible. Consequently, $X^\prime(\cdot)$ must be non-oscillatory. This

implies that $\langle X(t), X^\prime(t) \rangle$ does not change sign. In view of estimate (7), it follows that since the right hand side of (7) is both positive and bounded and all terms on the left hand side of (7) are either positive or bounded, we may conclude that $\|X(\cdot)\|$ is in $L^2[0, \infty)$ and therefore both $\|X(t)\| \to 0$ and $\|X^\prime(t)\| \to 0$ as $t \to \infty$.

The proof is complete. \(\square\)

The second main result of this paper is the following theorem:

**Theorem 2.** If the conditions of Theorem 1 hold, then the following estimates are satisfied:

$$\int_0^t \|X'(s)\|^2 ds \leq \frac{1}{2a_0} \|X(0)^2\|^2 + \frac{b_0}{2a_0} \|X(0)\|^2 \tag{8}$$

and

$$\int_0^t \|X'(s)\|^2 ds \leq \frac{1}{2b_0} \|X(0)^2\|^2 + \frac{1}{2b_0(b^*)} \|X(0)^2\|^2$$

$$+ \frac{b_0}{2a_0(b^*)} \|X(0)\|^2. \tag{9}$$

**Proof.** Consider estimate (5), that is,

$$2a_0\int_0^t \|X'(s)\|^2 ds + \|X'(t)\|^2 + b_0\|X(t)\|^2$$

$$\leq \|X(0)\|^2 + b_0\|X(0)^2\|^2.$$

Taking into consideration the assumptions of Theorem 2, we can conclude that

$$\|X(t)\| \to 0 \quad \text{and} \quad \|X^\prime(t)\| \to 0 \quad \text{as} \quad t \to \infty.$$

Then after dividing both sides of last estimate by $2a_0$, it can be followed that

$$\int_0^t \|X'(s)\|^2 ds \leq \frac{1}{2a_0} \|X(0)^2\|^2 + \frac{b_0}{2a_0} \|X(0)\|^2 \quad \text{as} \quad t \to \infty.$$

Thus, estimate (8) easily follows.

To arrive at estimate (9), we rearrange estimate (7) as

$$\langle X(t), X'(t) \rangle + \frac{1}{2} a(t') \|X(t)\|^2 + b(t') \int_0^t \|X(s)\|^2 ds$$

$$\leq \frac{1}{2} a(t') \|X(0)\|^2 + \int_0^t \|X'(s)\|^2 ds + \langle X(0), X'(0) \rangle. \tag{10}$$

Let $t \to \infty$. By removing the positive terms $\langle X(t), X'(t) \rangle$ and $\frac{1}{2} a(t') \|X(t)\|^2$ in estimate (10) and using estimate (8), we can write from (10) that

$$b(t') \int_0^t \|X(s)\|^2 ds \leq \frac{1}{2} a(t') \|X(0)\|^2 + \int_0^t \|X'(s)\|^2 ds$$

$$+ \langle X(0), X'(0) \rangle \leq \frac{1}{2} a(t') \|X(0)\|^2 + \langle X(0), X'(0) \rangle$$

$$+ \frac{1}{2} a(t') \|X(t)\|^2 + \frac{1}{2a_0} \|X(0)\|^2$$

$$+ \frac{b_0}{2a_0} \|X(0)\|^2.$$
Finally, when we divide last estimate by $b(t')$, we obtain estimate (9). The proof of Theorem 2 is now complete. □

**Example.** Let $n = 2$. Consider non-homogeneous linear differential system given by

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} + \left(1 + \frac{1}{r+1}\right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \left(2 + \exp(-t)\right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad t \geq 0.$$ (11)

Here,

$$a(t) = 1 + \frac{1}{r+1}, \quad b(t) = 2 + \exp(-t).$$

Then, we have

$$a_0 = \frac{1}{2} < a(t) = 1 + \frac{1}{r+1} < 3 = A_0,$$

$$b_0 = 1 < b(t) = 2 + \exp(-t) < 4 = B_0,$$

$$q(.) \in L^1(0, \infty).$$ Hence, all the conditions of Theorems 1 and 2 hold to system (11).

**Remark.** Kroopnick [1] proved Theorem A and Theorem B by the integral test to scalar linear homogenous differential equation of second order, $x'' + a(t)x' + b(t)x = 0$. In defiance of the results of Kroopnick [1], which are not new, the proofs presented in [1] are new and simplify some previous related works in the literature since the Gronwall inequality is avoided and $a(.)$ does not need to be differentiable at any point, which are the usual cases. It should be noted that the equation discussed in [1] is a special case of our equation

$$X'' + a(t)X' + b(t)X = 0.$$ When we take $n = 1$, then Eq. (2) and the assumptions of Theorems 1 and 2 reduce to those of Kroopnick [1, Theorem 1, Theorem 2]. Since the Gronwall inequality is avoided and $a(.)$ and $b(.)$ do not need to be differentiable, the proofs of this paper are new and the results of this paper simplify previous works in the literature (see [1]). Furthermore, our results extend the results of Kroopnick [1, Theorem 1, Theorem 2] and that in the literature.

**Conclusion**

A linear homogeneous differential system is considered. Some sufficient conditions are established which guarantee to the bounded and $L^2[0, \infty)$-solutions, give upper bounds for values of improper integrals of the solutions and their derivatives for the considered system. To prove the main results, we benefited from well-known integral inequalities. The results obtained essentially complement and extend some known results in the literature. An example is introduced to illustrate the main results of this paper.

**Conflict of interest**

The authors have declared no conflict of interest.

**Compliance with Ethics Requirements**

This article does not contain any studies with human or animal subjects.

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