Quantum Defragmentation Algorithm

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In this addendum of our paper [D. Burgarth and V. Giovannetti, Phys. Rev. Lett. 99, 100501 (2007)] we prove that during the transformation that allows one to enforce control by relaxation on a quantum system, the ancillary memory can be kept at a finite size, independently from the fidelity one wants to achieve. The result is obtained by introducing the quantum analog of defragmentation algorithms which are employed for efficiently reorganizing classical information in conventional hard-disks. Our result also implies that the reduced dynamics in any noisy system can be simulated with finitely many resources.

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![Figure 1](image)

Figure 1. Schematics of the control by relaxation scheme. The control on the large system \(V = C \cup \overline{C}\) is exerted through an auxiliary (fully controllable) quantum memory \(M\) which is directly coupled to the subsystem \(C\).

Accomplishing controllability of quantum mechanical systems is one of the main hurdles towards a large scale quantum computer. In recent years, increasing attention has been devoted in developing schemes that allows on to achieve global control on a large many-body quantum system \(V = C \cup \overline{C}\) by only having direct access to a relatively small subsystem \(C\). In this context, the majority of results obtained so far have been derived within the general framework of “algebraic” approach to control theory, e.g. see Ref. [12]. Here the allowed operations are parametrized by specifying which (local) components of the system Hamiltonian can be manipulated via proper choices of classical control pulses.

An independent approach was recently proposed by us in Refs. [7,8], introducing the notion of local controllability of quantum systems via “relaxation”. In this scheme additional control of a large ancillary system \(M\) was assumed and a method of controlling \(V = C \cup \overline{C}\) by acting on \(C\) and \(M\) was suggested. This is essentially achieved by transferring the states of \(V\) into \(M\) through a sequence of iterative operational steps (see Fig. 1) which induces an effective relaxation of \(V\) into the memory degree of freedom. The states are then controlled in \(M\) and, using the inverted sequence of steps, transferred back to \(V\).

On one hand, such new method can be important in inhomogeneous scenarios, where some parts of the system are easier to control than others. It also allows for an easy-to-check criterion if a given system is controllable which can be applied analytically to large systems [7,8] and which was subsequently generalized to the algebraic control scenario [6]. Formally compared to algebraic control it has the advantage that the control protocol is constructive and follows a clear physical intuition.

On the other hand, the main drawback of the controllability by relaxation approach stems from the fact that it cannot reduce the size of the controlled system (in contrast to algebraic control). Indeed to be able to store arbitrary states from \(V\) to the ancillary memory \(M\) the latter must be at least as large as the former (i.e. \(\text{dim}\ M \geq \text{dim}\ V\)). Even more problematic is the fact that up to now no upper bounds were known on the minimal size of \(M\) which is needed to accomplish the control. In this paper we fix this problem by showing that \(M\) can be kept at a finite size, which is maximally twice as large as \(V\).

This is a major improvement to [7,8], where \(M\) was arbitrarily large. The result is derived by introducing the quantum analogs of defragmentation algorithms. In computer science, defragmentation is a process that allows one to reduce the amount of fragmentation in file systems. This is obtained by reorganizing the contents of the disk to store the pieces of each file close together and contiguously while creating larger regions of free space. Here we use a similar idea to (coherently) compress quantum information in the quantum memory \(M\) during its transferring from \(V\). This results in a more efficient storing of messages, which saves valuable memory space for the subsequent data processing transformations.

I. THE ALGORITHM

Whilst referring to Refs. [7,8] for details, the scheme of control by relaxation can be summarized by saying that it consists in a downloading stage in which \(C\) is iteratively coupled to a fixed, finite-dimensional subspace (say a qubit) \(M_1\) of \(M\) that is re-prepared into a fiduciary state \(|0\rangle_{M_1}\) after each iteration. The \(\ell\)-th step of this process is described by a unitary downloading operation \(W_\ell\), which for large \(\ell\) moves arbitrary states \(|\psi\rangle_{C\overline{C}}\) of the system into the memory, i.e.,

\[
W_\ell|\psi\rangle_{C\overline{C}} \otimes |0\rangle_M \approx |0\rangle_{C\overline{C}} \otimes |\Phi(\psi)\rangle_M,
\]

with \(|\Phi(\psi)\rangle_M\) being a linear function of the input state \(|\psi\rangle_{C\overline{C}}\).

They are then controlled in \(M\) and moved back to the system in an uploading stage that reverses the process [1]. It is worth

\[
M \leq (C\overline{C})^2 + C
\]
stressing that the transformations $W_\ell$ are known and are independent from the input state of the system.

The above introduction seems to indicate that indeed $M = C\mathcal{C}$ is large enough to contain images of all possible states. This is not the case as states are only transferred asymptotically and for intermediate $\ell$, the downloading operator $W_\ell$ is generating entanglement between $C\mathcal{C}$ and $M$. Introducing an orthonormal basis $\{|k\rangle_{C\mathcal{C}}\}$ of $C\mathcal{C}$, a generic state $|\psi\rangle = \sum_k \alpha_k |k\rangle_{C\mathcal{C}}$ after $\ell$ steps can be written as

$$W_\ell \sum_k \alpha_k |k\rangle_{C\mathcal{C}} \otimes |0\rangle_M = \sum_{kk',\ell} \alpha_k \omega_{kk'}^{(\ell)} |k\rangle_{C\mathcal{C}} \otimes |s_{kk'}^{(\ell)}\rangle_M,$$

(2)

with $|s_{kk'}^{(\ell)}\rangle_M$ being a set of $(\dim C\mathcal{C})^2$ not-necessarily orthogonal vectors of $M$. Indepedently of the value of $\ell$, the states $\{|s_{kk'}^{(\ell)}\rangle_M\}_{kk'}$ span a space of dimension smaller than or equal to $(\dim C\mathcal{C})^2$: they can thus be fitted into a subsystem $M_0$ of $M$ which is twice as large as $C\mathcal{C}$. Therefore, by including an extra defragmentation step into the protocol of Ref. [7], the memory can be kept at a finite size. In detail, write $M = M_0 \otimes M_1$. The defragmentation consists then in operating on the memory with a unitary transformation which maps the $|s_{kk'}^{(\ell)}\rangle_M$ into states of the form $|s_{kk'}^{(\ell)}\rangle_{M_0} \otimes |0\rangle_{M_1}$, with $|0\rangle_{M_1}$ being the fiduciary state of the downloading stage, while the $|s_{kk'}^{(\ell)}\rangle_{M_0}$ are instead characterized by having the same mutual scalar product as the $|s_{kk'}^{(\ell)}\rangle_M$, i.e.

$$M_0 \langle s_{kk'}^{(\ell+1)}| s_{kk'}^{(\ell)}\rangle_{M_0} = M \langle s_{kk'}^{(\ell+1)}| s_{kk'}^{(\ell)}\rangle_M,$$

for all $k, k', k''$, and $k'''$. The whole procedure can be iterated easily by observing that at the $(\ell + 1)$-th step the state of the system can be still described as in Eq. (2) for a proper choice of the vectors $|s_{kk'}^{(\ell+1)}\rangle_M$.

It is worth noticing that the defragmentation procedure presented here finds also useful application in the context of spin chain communication [13]. Indeed by generalizing the result of the end-gate protocol of Ref. [5, 14] to the multi-excitation sector case, it shows that the memory-assisted transmission scheme of Ref. [13] can be implemented with finite resources.

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