Mar-Co: a new dependence structure to model match outcomes in football

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Abstract

The approaches commonly used to model the number of goals in a football match are characterised by strong assumptions about the dependence between the number of goals scored by the two competing teams and about their marginal distribution. In this work, we argue that the assumptions traditionally made are not always based on solid arguments and sometimes they can be hardly justified. In light of this, we propose a modification of the Dixon and Coles (1997) model by relaxing the assumption of Poisson-distributed marginal variables and by introducing an innovative dependence structure. Specifically, we define the joint distribution of the number of goals scored during a match by means of thoroughly chosen marginal (Mar-) and conditional distributions (-Co). The resulting Mar-Co model is able to balance flexibility and conceptual simplicity. A real data application involving five European leagues suggests that the introduction of the novel dependence structure allows to capture and interpret fundamental league-specific dynamics. In terms of betting performance, the newly introduced Mar-Co model does not perform worse than the Dixon and Coles one in a traditional framework (i.e. 1-X-2 bet) and it outperforms the competing model when a more comprehensive dependence structure is needed (i.e. Under/Over 2.5 bet).

Keywords: Betting; Dixon and Coles model; Football prediction; Marginal distribution; Poisson distribution; Under/Over

1 Introduction

Modelling match outcomes in association football (referred to simply as "football" hereafter) undoubtedly represents an element of primary interest in the field of sports analysis. In order to do this, two different but interconnected strategies can be considered: the results-based (or direct) approach and the goals-based (or indirect) approach. Given a specific match between two competing teams, the former focuses on modelling the categorical ordinal variable taking the three possible result values (home win-draw-away win) typically through a regression model in which the probabilities of the three final outcomes are estimated on the basis of some external variables (see, e.g., Koning 2000, Goddard and Asimakopoulos 2004, Schauberger and Groll 2018, Carpita et al. 2019, Groll et al. 2019). Instead, the goals-based approach considers a broader framework in which the purpose is to model the number of goals scored by the two teams during that specific match.
match. Since estimating the probability of each possible combination of home goal and away goal allows the estimate of the probability of home win-draw-away win to be consequently obtained, the two approaches are nested. The goals-based approach, being more general, has also some intriguing practical consequences, such as allowing types of bet different from traditional 1-X-2 (e.g. Under/Over). In addition, the difference in terms of performance between the two strategies is investigated by Goddard (2005) and Koopman and Lit (2019); whereas the former highlights no relevant differences between them, the latter finds evidence that the goals-based approach provides more precise forecasts. In light of all these reasons, in the rest of the paper we will focus on the goals-based strategy.

The first articles in this field are Moroney (1956) and Reep et al. (1971), in which the Poisson distribution and the negative binomial distribution are proposed to model the aggregated number of goals scored per game. To the best of our knowledge, Maher (1982) represents the first work aimed at modelling the number of goals scored by individual teams: specifically, the number of goals scored by the home team and the away team defines two independent Poisson random variables whose parameters depend on the attack and defence skills of the two teams. In the same paper, the author also carries out a bivariate Poisson model which keeps the two marginal distributions unchanged, driven by the fact that the initial proposal tends to underestimate the proportion of draws. In the nineteen-nineties two different models were proposed in the same year: Lee (1997) keeps the general structure of Maher (1982) assuming independence between the two Poisson random variables, whereas the article of Dixon and Coles (1997) moves from the pioneering work of Maher by introducing some crucial innovations. First of all, the authors introduce a specific dependence structure by specifying a parameter $\rho$ which allows the joint probability to be different from the product of the marginal probabilities. Secondly, they include a weighting function $\phi$ which down-weights old matches in the likelihood in order to obtain estimates of the parameters that are mainly based on recent performances of the teams. In order to improve the aspect just mentioned, Rue and Salvesen (2000) propose a Bayesian dynamic generalized linear model leaving the attack and defence parameters free to change randomly over time, while the aforementioned bivariate Poisson model is reconsidered and extended by Karlis and Ntzoufras (2003). A completely different approach is the one developed by Baio and Blangiardo (2010): in this case, their Bayesian hierarchical model considers two conditionally independent Poisson random variables for the numbers of goal scored, but the dependence is introduced through a careful choice of the hyper-parameters. A recent development of this approach is given by Egidi et al. (2018), where the betting odds are included in the model specification together with other modifications. Owen (2011) implements a dynamic generalized linear model whose evolution component is specified as a random walk for the attack and defence parameters, whereas McHale and Scarf (2011) use copulas to allow dependence between the two Poisson random variables under the assumption that the dependence parameter can be expressed as a linear function of the rank difference of the two teams. As argued by the two authors, the copula model represents a more flexible solution than the bivariate Poisson since it also permits negative correlation. Another interesting proposal able to capture the main features mentioned so far is the one provided by Koopman and Lit (2015), where a non-Gaussian state space model assumes a bivariate Poisson distribution whose attack and defence parameters are allowed to vary stochastically over time.

Taking into account the variety of approaches presented in the literature, it is absolutely evident that the definition of a proper dependence structure between the goals scored by the two teams represents an essential issue in modelling the final outcome of a
football game. The aim of this work is to enrich the current literature with an innovative method able to balance flexibility and conceptual simplicity. Specifically, moving from the Dixon and Coles approach, we introduce a more comprehensive dependence structure to improve the overall forecasting performance. In order to do that, we define the joint probability mass function describing the number of goals scored by the two teams by means of carefully chosen marginal probability mass functions (Mar-) and conditional probability mass functions (-Co). In so doing, the resulting model (Mar-Co) still manages the dependence between the home and the away goals by using an univariate parameter as in the Dixon and Coles approach.

The article is organised as follows: in Section 2 a broad discussion about the aforementioned dependence structure (focusing in particular on the one proposed by Dixon and Coles) is proposed; we present our model in Section 3 whereas in Section 4 an application compares our model with the Dixon and Coles approach. Finally, Section 5 provides some concluding remarks.

2 The dependence assumption

2.1 On the existence of dependence

Differently from the results-based strategy, the goals-based one requires a careful management of the dependence between the number of goals scored by the two competing teams, a notoriously tough task (Karlis and Ntzoufras 2009). Indeed, different studies draw very different conclusions depending on the championship, period of time and statistical tool considered. For example, Karlis and Ntzoufras (2000) highlight a small positive correlation combining the evidence from 24 championships of different European countries by means of the method proposed by Hasselblad (1994). In McHale and Scarf (2007), the correlations computed on the English Premier League data from August 2003 to March 2006 suggest that, in contrast to shots, the goals scored by the two teams show only slight positive or no correlation, whereas the work of McHale and Scarf (2011) shows statistically significant negative correlation considering matches between national teams.

In this regard, a fundamental aspect needs to be clarified. Let $X$ be the number of goals scored by the home team in a generic match of a given championship and let $Y$ be the number of goals scored by the opposing team (i.e. the away team). Similarly, let us define $X_{ij}$ and $Y_{ij}$ the number of goals scored by teams $i$ and $j$ respectively in the specific match in which $i$ plays at home and $j$ plays away. Variables $X$ and $Y$ are different random objects from $X_{ij}$ and $Y_{ij}$ (although they are interconnected somehow), and so the development of an empirical study assessing the dependence between $X$ and $Y$ is of limited practical use since the purpose is to model the team-specific variables $X_{ij}$ and $Y_{ij}$. For instance, one is justified in expecting a team scoring many goals during a game to concede a low number of goals (i.e. negative correlation between $X$ and $Y$) since in that specific occasion it totally outclasses the opposing team. On the other hand, a different dynamic can occur when two specific teams - characterised by specific attack and defence skills - play against each other: for example, a strong team playing against a weak one may score less goals than those expected since it mainly focuses on conserving energy for future more challenging matches. A discussion on the topic is also provided by Dixon and Robinson (1998) and Rue and Salvesen (2000).

However, the crucial empirical study of the dependence between $X_{ij}$ and $Y_{ij}$ is practically unfeasible since each couple of teams plays against just twice during a season (and
only once team $i$ plays at home and team $j$ away). Considering more than one season represents an intuitive solution, but it seems to be unsuitable since the strength of a team can vary widely from season to season (due to newly signed players, the sacking of a manager, etc...). In view of this, in the rest of Section 2.1 we will focus on the dependence between the aggregated variables $X$ and $Y$ in order to have an indication about the general behaviour, but all the results provided should be approached with particular caution in light of the considerations just expressed.

In order to investigate the dependence, the data we consider—obtained from [http://www.football-data.co.uk/](http://www.football-data.co.uk/)—refers to the 9130 matches of the English Premier League, the French Ligue 1, the German Bundesliga, the Italian Serie A and the Spanish La Liga played between the 2014-2015 season and the 2018-2019 season. A first, fundamental study concerns the existence of the most familiar kind of dependence—i.e. correlation—between $X$ and $Y$. To do so, we consider a bootstrap test aimed at evaluating the Pearson correlation coefficient with the samples drawn under the hypothesis of independence. The observed value of the Pearson correlation coefficient (computed considering all the 9130 matches) is -0.085 and the p-value is $< 0.001$. The same strong evidence against the hypothesis of independence can be found considering each championship separately, with the only small exception of the Italian Serie A, whose p-value is equal to 0.026. This result confirms the evidence provided by [McHale and Scarf (2011)](http://www.football-data.co.uk/) and consequently the importance of a proper modelling of the dependence between the number of goals scored during a match.

2.2 The Dixon and Coles dependence structure

Among the various goals-based approaches proposed in this field, the model introduced by [Dixon and Coles (1997)](http://www.football-data.co.uk/) represents one of the most famous, innovative and performing. Since the model proposed in this paper moves from it, an overview of the Dixon and Coles approach is provided as follows.

Let $X_{ij}^k$ be the number of goals scored by the home team $i$ against the away team $j$ in match $k$, $Y_{ij}^k$ be the number of goals scored by the away team $j$ on the same occasion and let $m$ be the total number of teams considered. In order not to overcomplicate the notation, hereafter $X_k$ ($Y_k$ respectively) will be used instead of $X_{ij}^k$ ($Y_{ij}^k$ respectively) with the home team $i$ and the away team $j$ implicitly assigned to every match $k$. First of all, the two authors define $X_k$ and $Y_k$ as follows:

\[
X_k \sim \text{Pois}(\lambda_k), \\
Y_k \sim \text{Pois}(\mu_k), \\
\log(\lambda_k) = \gamma + \alpha_{i(k)} + \beta_{j(k)}, \\
\log(\mu_k) = \alpha_{j(k)} + \beta_{i(k)},
\]

with $\text{Pois}(\lambda)$ denoting a Poisson distribution with mean $\lambda$, $i(k)$ and $j(k)$ indices which identify the home and away teams playing match $k$, and where $\alpha$, $\beta$, $\gamma$ are the attack, defence and home effect parameter, respectively. By considering the matches chronologically, the two authors divide the seasons into a series of half-weekly time points and
construct the following function for each time point $t$:

$$L_t(\alpha_i, \beta_i, \rho, \gamma; i = 1, \ldots, m) = \prod_{k \in A_t} \left[ \tau_{\lambda_k, \mu_k}(x_k, y_k) e^{-\lambda_k} \lambda_k^{x_k} \mu_k^{y_k} e^{-\mu_k} \right] \delta^{\xi(t-t_k)},$$

with $t_k$ the time that game $k$ is played, $\tau_{\lambda_k, \mu_k}(x_k, y_k)$ the function depending on parameter $\rho$ which manages the dependence between $X_k$ and $Y_k$, and $\xi \geq 0$ the parameter that regulates the down-weighting of old matches. Consistently with the notation of the original paper, the constraint $\sum_{i=1}^{m} \alpha_i = m$ is included for identifiability.

Hence, the two authors obtain the estimates of the parameters by numerically maximising the function in Equation (2) at each time point $t$ after choosing $\xi$. The choice of $\xi$ is particularly tough because Equation (2) defines a sequence of non-independent functions that makes difficult to obtain the value of $\xi$ that maximises the overall predictive capability of the model. In order to overcome this problem, the two authors focus on the prediction of match outcomes rather than match scores and define the value of $\xi$ as the value maximising

$$S(\xi) = \sum_k \left( \delta_k^H \log p_k^H + \delta_k^D \log p_k^D + \delta_k^A \log p_k^A \right)$$

with

$$p_k^r = \sum_{h, a \in B_r} \Pr(X_k = h, Y_k = a),$$

which is implicitly a function of $\xi$ since the score probabilities $\Pr(X_k = h, Y_k = a)$ are estimated from the maximisation of the function in (2) at $t_k$ with weighting parameter set at $\xi$ and with $r = \{H, D, A\}, B_H = \{(h, a) : h > a\}, B_D = \{(h, a) : h = a\}, B_A = \{(h, a) : h < a\}$, and $\delta_k^H$ ($\delta_k^D$, $\delta_k^A$ respectively) the delta function equal to 1 if the final result of game $k$ is a home win (draw, away win respectively).

The last, fundamental aspect introduced by Dixon and Coles concerns the description of the dependence structure, that is defined by means of the function

$$\tau_{\lambda_k, \mu_k}(x, y) = \begin{cases} 1 - \lambda_k \mu_k \rho & \text{if } x = 0, y = 0 \\ 1 + \lambda_k \rho & \text{if } x = 0, y = 1 \\ 1 + \mu_k \rho & \text{if } x = 1, y = 0 \\ 1 - \rho & \text{if } x = 1, y = 1 \\ 1 & \text{otherwise} \end{cases}$$

subject to

$$\max \left( \frac{1}{\lambda_k}, \frac{1}{\mu_k} \right) \leq \rho \leq \min \left( \frac{1}{\lambda_k \mu_k}, 1 \right).$$

By using function (3), the marginal distributions of $X_k$ and $Y_k$ are Poisson with means $\lambda_k$ and $\mu_k$ respectively and the independence between $X_k$ and $Y_k$ is obtained when $\rho = 0$.

Figure provides an intuitive representation of the dependence structure, highlighting the difference between the joint probability mass functions of the goals scored assuming independence and dependence. The only outcomes affected by a change in probability, when $\rho$ varies, are those in the black boxes of the picture. This means that the probability that one team scores at least two goals does not depend on the number of goals scored by
Figure 1: Each panel shows, under the Dixon and Coles model, the difference between the probability of the exact outcomes in match $k$, given a certain value of $\rho$ (equal to -0.1 on the left, 0 in the middle and 0.1 on the right), and the probability of the exact outcomes in match $k$ in case of independence (i.e. $\rho = 0$). The marginal means $\lambda_k$ and $\mu_k$ are set equal to 1 and 1.5, respectively.

the opposing team in the same match. In other terms, the Dixon and Coles model is based on the strong assumption that the number of goals scored by the home team is independent of the number of goals scored by the away team, conditionally on observing an outcome not included in the set \{(0, 0), (1, 0), (0, 1), (1, 1)\}. Looking at the 9130 matches played during the period from August 2014 to May 2019 in the top five European championships, this assumption seems no longer reasonable, as suggested by testing it as null hypothesis in the following bootstrap procedure. Let $\tilde{X}_k$ and $\tilde{Y}_k$ be the home and away number of goals, but such that scoring either zero or one goal is mapped to the same event of their support. Hence, under the null hypothesis, the variables $\tilde{X}_k$ and $\tilde{Y}_k$ are independent. To perform the test, we generated 1000 data sets with $\tilde{X}_k$ and $\tilde{Y}_k$ independently re-sampled and we computed the sample correlation between the two variables in each data set. The negative correlation computed on the observed data set lies in the left tail of the bootstrap correlation distribution, encouraging the rejection of the null hypothesis with a p-value < 0.001. As a further evidence against the Dixon and Coles dependence structure, we replicated on the above mentioned recent data the study reported in Section 3 of Dixon and Coles (1997), noticing that nowadays, unlike what observed by Dixon and Coles at the time, the frequency of several match outcomes is significantly different from the product of the marginal frequencies of goals scored by home and away teams (see Table I for details).

These empirical results, together with the evidence that a kind of dependence between the home and away goals exists, suggest that a new, more complex dependence structure should be considered. Furthermore, the Dixon and Coles model aims to properly estimate the probabilities of home win-draw-away win, which are directly influenced by the dependence structure proposed, since each of these three probabilities changes according to $\rho$. 

| Home goals | Away goals | Variation in probability |
|------------|------------|--------------------------|
| 0          | 0          | Negative                 |
| 0          | 1          | Absent                   |
| 0          | 2          | Positive                 |
| 0          | 3          |                          |
| 0          | 4          |                          |
| 1          | 0          |                          |
| 1          | 1          |                          |
| 1          | 2          |                          |
| 1          | 3          |                          |
| 1          | 4          |                          |
| 2          | 0          |                          |
| 2          | 1          |                          |
| 2          | 2          |                          |
| 2          | 3          |                          |
| 2          | 4          |                          |
| 3          | 0          |                          |
| 3          | 1          |                          |
| 3          | 2          |                          |
| 3          | 3          |                          |
| 3          | 4          |                          |
| 4          | 0          |                          |
| 4          | 1          |                          |
| 4          | 2          |                          |
| 4          | 3          |                          |
| 4          | 4          |                          |
Vice versa, such dependence structure cannot adjust the probability of events defined by other types of bet, as we will see in details in Section 4. Motivated by these considerations, in the next section, starting from the Dixon and Coles model, we propose a new model characterised by a more general dependence structure, such that the induced joint probability mass function is different from the product of the marginal probability mass functions for each possible match outcome.

3 The Mar-Co model

Moving from the considerations discussed in the previous section, we propose a new model which differs from the Dixon and Coles one by the choice of the dependence structure between $X_k$ and $Y_k$.

Given a certain marginal distribution of $X_k$, an intuitive approach to allow the specification of a sufficiently general dependence structure consists in modelling the conditional distribution $Y_k \mid X_k$. By indicating $\text{pr}(Y_k \leq x)$ as the cumulative distribution function of the Poisson random variable with mean $\mu_k$ defined as in (1), we specify

$$Y_k \mid X_k = x \sim \text{Pois}(\psi_Y(\mu_k, x)),$$

with

$$\psi_Y(\mu_k, x) = \exp\{\theta_1 + \theta_2 \log \mu_k + \theta_3 \text{logit}(\text{pr}(Y_k \leq x))\},$$

where $\text{logit}(p) = \log\{p/(1 - p)\}$ and $\theta = (\theta_1, \theta_2, \theta_3)^\top \in \mathbb{R}^3$. From an interpretative point of view, the mean of $Y_k \mid X_k$ depends on the facility with which the away team, given $\mu_k$, scores more goals than those scored in that occasion by the home team.

As in the Dixon and Coles model, an univariate parameter ($\theta_3$ in this case) regulates the dependence between $X_k$ and $Y_k$, where $\theta_3 \neq 0$ implies dependence existence. In addition, it is possible to obtain the above mentioned independence case corresponding to the Dixon and Coles model with $\rho = 0$ by assuming $\theta = (0, 1, 0)^\top$ and by modelling $X_k$ as the Poisson random variable defined in (1).

Hence, assuming $X_k \sim \text{Pois}(\lambda_k)$, we can exploit the definition of joint probability mass function as product of conditional and marginal probability mass functions to define

$$\text{pr}_A(X_k = x, Y_k = y) = \frac{e^{-\psi_Y(\mu_k, x)} \psi_Y(\mu_k, x)^y e^{-\lambda_k} \lambda_k^x}{y! x!}. \quad (4)$$

Table 1: Ratios between the observed frequencies and the products of the observed marginal frequencies for 25 possible match outcomes. Bootstrap standard errors are reported in parentheses, while grey cells indicate ratios that are significantly different from 100 at level 0.05.

| Home goals | 0  | 1  | 2  | 3  | 4  |
|------------|----|----|----|----|----|
|            |    |    |    |    |    |
| 0          | 99.47 (2.25) | 92.95 (2.02) | 100.68 (3.10) | 114.96 (5.13) | 139.57 (9.53) |
| 1          | 94.46 (1.77) | 102.11 (1.65) | 102.75 (2.35) | 103.21 (3.91) | 107.72 (6.86) |
| 2          | 99.01 (2.10) | 103.40 (1.98) | 102.69 (2.92) | 89.23 (4.55)  | 84.39 (7.75)  |
| 3          | 108.13 (3.18) | 101.23 (2.95) | 91.24 (4.12)  | 99.11 (7.13)  | 59.27 (10.11) |
| 4          | 121.20 (5.29) | 98.78 (4.73)  | 88.63 (6.65)  | 68.19 (9.49)  | 52.58 (14.97) |

To facilitate the reading, the numbers are multiplied by 100.
Figure 2: Each panel shows, under the Mar-Co model, the difference between the probability of the exact outcomes in match $k$, given a certain value of $\theta_3$ (equal to $-0.05$ on the left, $0$ in the middle and $0.05$ on the right), and the probability of the exact outcomes in match $k$ in case of independence (i.e. $\theta_3 = 0$). The marginal means $\lambda_k$ and $\mu_k$ are set equal to 1 and 1.5, respectively, while $\theta_1 = 0$ and $\theta_2 = 1$.

On the other hand, if we follow symmetrical steps specifying

$$Y_k \sim \text{Pois}(\mu_k),$$

$$X_k \mid Y_k \sim \text{Pois}\{\psi_X(\lambda_k, y)\},$$

$$\psi_X(\lambda_k, y) = \exp\{\theta_1 + \theta_2 \log \lambda_k + \theta_3 \logit (\text{pr}(X_k \leq y))\},$$

we can define the symmetrical joint probability mass function

$$\text{pr}_B(X_k = x, Y_k = y) = \frac{e^{-\psi_X(\lambda_k, y)} \psi_X(\lambda_k, y)^x e^{-\mu_k} \mu_k^y}{x! y!}. \tag{5}$$

As we do not have formal reasons to favour one of the two specifications, we define the distribution of the joint outcome $(X_k, Y_k)$ as an equally weighted mixture of the two distributions defined by $\text{pr}_A$ and $\text{pr}_B$. Therefore, under the Mar-Co model, the likelihood related to the $k$-th match can be expressed as

$$\mathcal{L}_k^M(\alpha_i, \beta_i, \alpha_j, \beta_j, \gamma, \theta) \propto \text{pr}_A(X_k = x, Y_k = y; \alpha_i, \beta_i, \alpha_j, \beta_j, \theta, \gamma) + \text{pr}_B(X_k = x, Y_k = y, \alpha_i, \beta_i, \alpha_j, \beta_j, \theta, \gamma).$$

As a merely football-based argument in favour of this model construction, the presence of two symmetrical data generation processes appears to be plausible, since one is justified in expecting sometimes the away team to react to the performance of the home team, sometimes vice versa.

The key difference between the Mar-Co model and the Dixon and Coles one lies in the specification of a joint distribution that permits $X_k$ and $Y_k$ to be dependent also
conditionally on observing an outcome not included in the set \{(0, 0), (1, 0), (0, 1), (1, 1)\}. This aspect is clearly displayed in Figure 2, where the grey scale identifies the difference between the Mar-Co joint probability mass function considering three possible values of \(\theta_3\) and the probability mass function obtained when \(\theta_3 = 0\). A comparison of the left and right panels of Figure 2 with the corresponding panels in Figure 1 immediately highlights the discrepancy between the two models in terms of outcomes affected by the introduction of dependence.

As regards the estimation process, let \(\alpha\) and \(\beta\) denote the vectors including the parameters \(\alpha_i\) and \(\beta_i\) for \(i = 1, \ldots, m\). The estimates of the parameters at time point \(t\) are obtained by maximising the function

\[
L_t^M(\alpha, \beta, \gamma, \theta) = \prod_{k \in A_t} \left[ pr_A(x_k; y_k; \alpha, \beta, \gamma, \theta) + pr_B(x_k; y_k; \alpha, \beta, \gamma, \theta) \right] e^{-\xi(t-t_k)},
\]

where \(pr_A\) and \(pr_B\) refer to the functions in (4) and (5) respectively, while the set \(A_t\) and the parameter \(\xi\) are defined consistently with the notation of the Dixon and Coles model presented in Section 2.2. Given the shape of the mixture distribution, the presence of local modes cannot be excluded and, at the same time, the maximisation algorithms could encounter computational troubles in evaluating \(L_t^M\) at points that are far from the mode. In view of these issues, we propose the following maximisation procedure. Firstly, we obtain a first estimate of the parameters through a plug-in estimation process (Gong and Samaniego 1981). Specifically, after computing \(\hat{\alpha}(0), \hat{\beta}(0), \hat{\gamma}(0)\) by maximising \(L_t^M(\alpha, \beta, \gamma, \theta = (0, 1, 0)^T)\) with \(\theta\) fixed—which is equivalent to maximise the Dixon and Coles function \(L_t\) with \(\rho\) set to 0—we find \(\hat{\theta}(0)\) as the argument that maximises \(L_t^M(\hat{\alpha}(0), \hat{\beta}(0), \hat{\gamma}(0), \theta)\) over \(\mathbb{R}^3\). Secondly, we obtain the final estimates \(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\theta}\) by optimising the function \(L_t^M\) in the neighbourhood of \((\hat{\alpha}(0), \hat{\beta}(0), \hat{\gamma}(0), \hat{\theta}(0))\), which is used as the starting point of the maximisation step. Such algorithm, although unconventional, presents advantages both from a computational and an interpretative point of view. Indeed, it avoids computational troubles related to the evaluation of \(L_t^M\) where such function assumes low values, and, at the same time, it allows an easy interpretation of the estimates of the team fixed effects, since they are close to those obtained under the independence assumption. Furthermore, empirical results suggest that \(L_t^M(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\theta})\) is greater than the values obtained by evaluating the function at other local modes after randomly initialising the algorithm.

On the face of a more comprehensive dependence structure, the Mar-Co model does not present the desired properties in terms of marginal distributions that characterise the Dixon and Coles model. Indeed, in case of dependence under the Mar-Co model, both the conditional and the marginal distributions of \(X_k\) and \(Y_k\) are not Poisson, since each of these distributions can be described as the average of a Poisson distribution obtained from one of the two elements of the mixture and a non-Poisson distribution obtained from the other one. In particular, the lack of Poisson marginal distributions appears in contrast with one of the most widely accepted assumptions in the literature. As mentioned in the Introduction, several authors assume Poisson marginal distributions on the basis of the empirical behaviour of the marginal home and away number of goals in a championship (see, e.g., Dixon and Coles 1997). Nevertheless, a parametric bootstrap test conducted on the same data presented in Section 2.1 questions the validity of this assumption, suggesting to reject the null hypothesis of Poisson-distributed variables \(X\) and \(Y\). In details, we compared the ratio between the sample variance and the sample mean—i.e. an estimate of the dispersion parameter—with those computed on 1000 data
sets drawn from the Poisson distribution with mean equal to the observed sample mean. In so doing, we observed a p-value < 0.001 both for the home and the away number of goals. By replicating the test on each championship separately, we obtained similar strong indications against the null hypothesis for each marginal distribution considered. In addition, the same evidence, although slighter, was obtained by performing an alternative parametric bootstrap test which considers as test statistic the Kullback-Leibler divergence between the empirical distribution of each sample and the Poisson distribution with mean equal to the corresponding observed sample mean.

However, following the considerations discussed in Section 2.1, a further crucial element should be taken into account. The studies of the marginal behaviour of the number of goals, such as that reported above, generally investigate the properties of the distributions of $X$ and $Y$, i.e. the number of goals observed in a generic match of a championship, while the goals-based models usually make assumptions on the distributions of $X_{ij}$ and $Y_{ij}$, i.e. the number of goals scored in a match between two specific teams $i$ and $j$. The article of Karlis and Ntzoufras (2000) represents a progress in this discussion, since the authors analyse the dispersion parameter of the distribution of the number of goals in matches played by a given team $i$, i.e. $X_i$ and $Y_i$, providing an evidence that we can expect to be more similar to that we could obtain by analysing the distribution of $X_{ij}$ and $Y_{ij}$. The study, conducted on 456 teams in 24 different European leagues, suggests the presence of overdispersion, providing a further element against the hypothesis of Poisson-distributed variables $X_{ij}$ and $Y_{ij}$. Therefore, although we cannot conclude that $X_{ij}$ and $Y_{ij}$ are undoubtedly non-Poisson random variables, it is hard to argue that the assumption of marginal Poisson distributions is usually a convenient choice in terms of modelling, rather than a consequence of well established behaviours. In view of this perspective, it could be worth having non-Poisson marginal distributions, if it allows to include a more flexible dependence structure able to improve the overall predictive capability of the model.

4 An application to European leagues

In this section, a detailed comparison between the model presented in Section 3 and the Dixon and Coles one is provided. The two models are evaluated on a real data application which considers the five most important European leagues (the English Premier League, the French Ligue 1, the German Bundesliga, the Italian Serie A, the Spanish La Liga) and different betting types. Although sports betting is not the primary tool to assess the performance of a model, in this specific case it is particularly useful since it allows to compare the two models in different frameworks and to highlight strengths and weaknesses of each approach.

First of all, we focus on the most famous betting type, i.e. Home Win-Draw-Away Win (known simply as 1-X-2). In order to compare the two models, the first step is the choice of the parameter $\xi$, that is made for each combination of league and model separately by means of the procedure proposed by Dixon and Coles and described in Section 2.2. In fact, consistently with the work of Diquigiovanni and Scarpa (2019), hereafter the time point $t$ will refer to the specific day of the year on which a given match takes place. This is due to the fact that nowadays, unlike when Dixon and Coles carried out the study, the teams play almost every day and so a more precise subdivision of the season is required. Specifically, the matches of three consecutive seasons (2012-2013, 2013-2014, 2014-2015) are used to choose $\xi$ and the probabilities of home win-draw-away win are not estimated for matches played between May and September in order to include promoted teams and
to avoid misleading results due to lack of effort of some teams in the final part of the season. In so doing, the estimates of the probabilities are obtained starting from the first game played in October 2013 on the basis of all the information available at that time and \( S(\xi) \) is computed considering that game together with all the subsequent matches.

Focusing on Table 2, the values of \( \xi \) maximising \( S(\xi) \) seem to show on the one hand no relevant dissimilarities between the two models, and on the other different dynamics in the leagues considered. Although it is not an aspect of primary interest, this evidence suggests a fascinating insight into European leagues: as the greater the value of \( \xi \), the less importance is given to the oldest matches, then the current physical and psychological condition of the two competing teams seems to particularly affect the outcome of a match in the German Bundesliga and the Italian Serie A.

Once the value of \( \xi \) is set, we use the Ranked Probability Score (or RPS; Epstein 1969) to compare the performance of the two models. Despite the well known limitations of it (Wheatcroft 2019), RPS still represents one of the most famous and used scoring rules in this field due to its conceptual simplicity and easiness of implementation. Formally, the RPS related to a specific match and betting type is defined as

\[
\frac{1}{r-1} \sum_{i=1}^{r-1} \left( \sum_{j=1}^{i} (\hat{p}_j - e_j) \right)^2
\]

with \( \hat{p}_j \) the estimate of the probability of the \( j \)-th possible betting outcome and \( e_j = 1 \) if the outcome \( j \) is observed and 0 otherwise. In the 1-X-2 framework, we therefore obtain: \( r = 3, \hat{p}_1 (\hat{p}_2, \text{respectively}) \) equal to the estimate of the probability of home win (draw, respectively) and \( e_1 = 1 (e_2 = 1, \text{respectively}) \) if the final result of the match is a home win (draw, respectively) and 0 otherwise. A detailed description of the RPS and its properties can be found in Constantinou and Fenton (2012) and references therein. We compute the RPSs for the matches of the 2015-2016, 2016-2017, 2017-2018, 2018-2019, 2019-2020 (until interruption due to the COVID-19 outbreak) seasons. As for the choice of \( \xi \), the first RPSs are obtained starting from the second season and they are computed only for matches played between October and April.

Figure 3 shows the cumulative difference over time between the RPSs computed for the Dixon and Coles model and the RPSs computed for the Mar-Co model considering the matches of all the leagues (solid blue line) and considering each league separately (dashed coloured lines). Since a smaller RPS indicates better predictive performance, values above the horizontal white line suggest that the Mar-Co model outperforms the Dixon and Coles one, and vice versa. The trend over time seems to be quite promising: after an initial period characterised by some fluctuations, the Mar-Co model provides more accurate predictions than those outputted by the Dixon and Coles one considering both all the leagues together and each league separately, with the sole exception of the French Ligue 1. Although the overall evidence seems to suggest that the modification

| argmax_{\xi} S(\xi) | England | France | Germany | Italy | Spain |
|----------------------|---------|--------|---------|-------|-------|
| Dixon and Coles Model | 0       | 0      | 0.0046  | 0.0053| 0.0021|
| Mar-Co Model          | 0       | 0      | 0.0025  | 0.0045| 0.0021|
Figure 3: Cumulative difference between the RPSs computed for the two competing models. Betting type: 1-X-2.

introduced by the Mar-Co model can be used profitably in the long run, the analysis of Figure 3 should be accompanied by a study that verifies whether the two models are statistically different or not. To do that, for every match considered, the RPS computed for the Mar-Co model and the RPS computed for the Dixon and Coles one are switched with probability $0.5$ in order to obtain a new sample. By replicating this procedure $n_b$ times, it is possible to compute the $n_b$ mean differences of the reshuffled samples: in so doing, if the evidence in favour of the Mar-Co model is frequently stronger than the observed one, we conclude that the observed difference between the two models is mainly due to chance. By applying this procedure with $n_b = 10000$, we obtain that $14\%$ of the time a stronger evidence is obtained: we conclude that, although it cannot be excluded that the two models are identical in terms of performance, the Mar-Co model seems to represent a promising modification of the starting model.

Since the two models differ with regard to their dependence structure, parameter $\theta$ represents the key element in determining the satisfactory results obtained by the Mar-Co model. As a consequence, the estimate $\hat{\theta}$ is a quantity of deep interest that allows to discover fundamental league-specific dynamics, although any analysis based on point estimates should be approached with particular caution. For example, the Italian Serie A and the Spanish La Liga are characterised by a positive value of $\hat{\theta}_3$ at the end of the period considered ($\hat{\theta}_3 = 0.0309$, $\hat{\theta}_3 = 0.0307$ respectively). This evidence seems to confirm the hypothesis made in Section 2.1, namely that a team tends to underperform when it plays against a team able to score just few goals (e.g., $\text{pr}(X \leq y) < 0.5$). Correspondingly, a team tends to overperform when it plays against a team able to score many goals (e.g., $\text{pr}(X \leq y) > 0.5$), thus creating a more balanced match. Conversely, the opposite dynamics characterise the English Premier League ($\hat{\theta}_3 = -0.0077$), the French Ligue 1 ($\hat{\theta}_3 = -0.0268$) and the German Bundesliga ($\hat{\theta}_3 = -0.0742$). In view of this, the ability to properly manage these different behaviours according to the specific league considered seems to represent the key feature of the Mar-Co model.
Another betting type able to highlight the differences in terms of dependence modelling between the two approaches is the so-called Under/Over bet. In this case, the aim is to predict whether the overall number of goals scored during a game will be less (Under) or greater than (Over) a certain threshold. For the sake of clarity, Figure 4 shows the combinations of home and away goals characterising the two possible betting outcomes when the threshold is set equal to 1.5 (in the middle) and 2.5 (on the right). This type of bet is particularly interesting since, as noted in Section 2.2, the Dixon and Coles parameter $\rho$ allows only the probabilities of the results (0,0), (1,0), (0,1), (1,1) to be reshuffled. Conversely, the newly introduced parameter $\theta_3$ allows also the probabilities of the other results to vary. As a result, assuming the other parameters as known, one is justified in expecting both dependence structures to have an impact on the prediction of 1-X-2 (see the left of Figure 4) and Under/Over 1.5, while only the dependence structure of the Mar-Co model is supposed to affect the prediction of Under/Over 2.5 since in that specific case the four cells contained in the black boxes in Figure 4 belong to just one of the two possible events, i.e. Under 2.5.

Figure 5 and Figure 6 show the cumulative difference between the RPSs for Under/Over 1.5 and Under/Over 2.5, respectively. Focusing on Figure 5, the Dixon and Coles model appears to outperform the Mar-Co one, with the sole exception of the Italian Serie A. However, the difference seems to be mainly due to chance: indeed, more than 40% of the time the reshuffled samples obtained by replicating the aforementioned procedure provide a stronger evidence in favour of the Dixon and Coles model than the observed one. In view of this, the observed difference is not sufficient to draw conclusions about the Under/Over 1.5 framework.

As regards the Under/Over 2.5 betting type, Figure 6 highlights a totally different behaviour as the Mar-Co model largely outperforms the Dixon and Coles one, again with the sole exception of the French Ligue 1. In this case, only $\sim 2\%$ of the time the reshuffled
Figure 5: Cumulative difference between the RPSs computed for the two competing models. Betting type: Under/Over 1.5.

Figure 6: Cumulative difference between the RPSs computed for the two competing models. Betting type: Under/Over 2.5.
samples provide a stronger evidence in favour of the Mar-Co model than the observed one, and so the Mar-Co model seems to represent an interesting improvement of the Dixon and Coles model in the Under/Over 2.5 framework. This evidence is not entirely surprising given the specific betting type taken into account as any value of $\rho$ does not modify the probability of the event Under 2.5 (and consequently also of the event Over 2.5) obtained with $\rho = 0$. As a consequence, the fact that, given the estimates of the fixed effects and of $\gamma$, the estimates of the probabilities of Under 2.5 and Over 2.5 obtained by the Dixon and Coles model are equal to the estimates obtained assuming independence regardless the value of $\hat{\rho}$ represents an undeniable limit in the Dixon and Coles procedure that may explain the improved performance obtained by the newly introduced Mar-Co model.

5 Final remarks

This work aimed to enrich the literature debate about goals-based models, by providing in-depth analyses on the distribution of the number of goals and by proposing an innovative model.

According to our contribution, nowadays the structure of the dependence between the number of goals scored by two competing teams in a match cannot be simplified to an adjustment of the probabilities of the outcomes 0-0, 1-0, 0-1, 1-1. This finding motivates the alternative model proposed in this paper, which, moving from the Dixon and Coles approach, presents a more flexible and comprehensive representation of the dependence structure. The particular specification of the term regulating the dependence, which is related to the conditional strength of the two competing teams, also plays a key role in providing interesting insights and interpretations of league dynamics. The encouraging results in terms of predictive capability highlighted by the comparison between the Mar-Co model and the Dixon and Coles one confirm the validity of our proposal. This is particularly true in case of the Under/Over 2.5 bet, where the probability of the two betting outcomes, under the Dixon and Coles model, is not influenced by the dependence parameter.

However, some aspects must be pointed out. First of all, alternative types of bet (e.g. Asian Handicap bet) are worth exploring in order to obtain further evidences about the relation between dependence structure and prediction effectiveness. Secondly, the inclusion of external covariates, such as information on the players conditions or on the teams motivations, should be considered to make the model even more attractive for an effective use in the world of betting. Finally, the lack of a recursive estimation process represents an undeniable limit in terms of computational effort required, since new crucial information is provided after each match.

However, although an extended analysis on a longer period and on more leagues would be desirable, the promising results achieved shed new light on the importance of properly choosing a suitable dependence structure to model the number of goals in a football match.

Declarations

Availability of data and material

Data are publicly available at http://www.football-data.co.uk/
Code availability
The R code used in this study is available from the corresponding author upon reasonable request.

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