QUANTITATIVE ESTIMATE OF DIAMETER FOR WEIGHTED MANIFOLDS UNDER INTEGRAL CURVATURE BOUNDS AND $\varepsilon$-RANGE

TAKU ITO

Abstract. In this article, we extend the compactness theorems proved by Sprouse [12] and Hwang–Lee [3] to a weighted manifold under the assumption that the weighted Ricci curvature is bounded below in terms of its weight function. With the help of the $\varepsilon$-range, we treat the case that the effective dimension is at most 1 in addition to the case that the effective dimension is at least the dimension of the manifold. To show these theorems, we extend the segment inequality of Cheeger–Colding [1] to a weighted manifold.

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1. INTRODUCTION

One of the most fundamental theorems in Riemannian geometry is the Myers theorem [9], which states that if an $n$-dimensional complete Riemannian manifold $(M, g)$ satisfies $\text{Ric}_g \geq (n - 1)K$ with $K > 0$ and $n \geq 2$, then $M$ is compact and its diameter $\text{diam}(M)$ is at most $\pi/\sqrt{K}$. Moreover, its fundamental group $\pi_1(M)$ is a finite group. Here, we denote by $\text{Ric}_g$ the Ricci curvature of $(M, g)$. For $\kappa \in C^\infty(M)$, $\text{Ric}_g \geq \kappa$ means that $\text{Ric}_g(v, v) \geq \kappa(p)g(v, v)$ holds for $v \in T_pM$. This theorem has been generalized by several ways. As one of these attempts, we introduce the theorems by Sprouse [12] below.

In this article, we always assume that $(M, g)$ is a complete Riemannian manifold and its dimension is at least 2. We denote by $d$ and $\text{vol}_g$ the Riemannian distance function and the volume measure on $(M, g)$, respectively. We write $a_+ = \max\{a, 0\}$ for $a \in \mathbb{R}$.
\( \mathbb{N}_{\geq 2} = \{ n \in \mathbb{N} \mid n \geq 2 \} \). For \( p \in M \) and \( R > 0 \), we denote
\[
U_p M = \{ v \in T_p M \mid g(v, v) = 1 \},
\]
\[
\text{Ric}_{g^-}(p) = \inf\{ \text{Ric}_g(v, v) \mid v \in U_p M \},
\]
\[
B(p, R) = \{ q \in M \mid d(p, q) < R \}.
\]

**Theorem 1.1** ([12, Theorem 1.1]). Let \( n \in \mathbb{N}_{\geq 2} \) and \( \eta > 0 \). There exists a positive constant \( \delta(n, \eta) \) such that if an \( n \)-dimensional compact Riemannian manifold \((M, g)\) satisfies \( \text{Ric}_g \geq 0 \) and
\[
\frac{1}{\text{vol}_g(M)} \int_M ((n - 1) - \text{Ric}_{g^-})_+ d\text{vol}_g < \delta(n, \eta)
\]
then \( \text{diam}(M) < \pi + \eta \).

**Theorem 1.2** ([12, Theorem 1.2]). Let \( n \in \mathbb{N}_{\geq 2}, K \leq 0 \) and \( R, \eta > 0 \). There exists a positive constant \( \delta(n, K, R, \eta) \) such that if an \( n \)-dimensional complete Riemannian manifold \((M, g)\) satisfies \( \text{Ric}_g \geq (n - 1)K \) and
\[
\sup_{p \in M} \frac{1}{\text{vol}_g(B(p, R))} \int_{B(p, R)} ((n - 1) - \text{Ric}_{g^-})_+ d\text{vol}_g < \delta(n, K, R, \eta)
\]
then \( M \) is compact and \( \text{diam}(M) < \pi + \eta \).

Hwang–Lee [3] extended Theorems 1.1 and 1.2 to a weighted manifold having a bounded \( \infty \)-Ricci curvature. A **weighted manifold** is a triple \((M, g, \mu_f)\), where \((M, g)\) is an \( n \)-dimensional Riemannian manifold, endowed with a measure \( \mu_f = e^{-f}d\text{vol}_g \) having smooth positive density with respect to the volume measure \( \text{vol}_g \). We call \( f \in C^\infty(M) \) the **weight function** of \( \mu_f \). For given \( N \in (-\infty, \infty] \), the \( N \)-Ricci curvature is defined for \( v \in TM \) by
\[
\text{Ric}_N(v, v) = \begin{cases} \text{Ric}_g(v, v) + \text{Hess}_g f(v, v) - \frac{1}{N - n} g(\nabla f, v)^2 & \text{if } N \in \mathbb{R} \setminus \{n\}, \\
\text{Ric}_g(v, v) + \text{Hess}_g f(v, v) & \text{if } N = \infty, 
\end{cases}
\]
and for \( N = n \),
\[
\text{Ric}_n(v, v) = \begin{cases} \text{Ric}_g(v, v) & \text{if } f \text{ is a constant function,} \\
-\infty & \text{otherwise.}
\end{cases}
\]
We call the parameter \( N \) the **effective dimension**. We refer to a **weighted Ricci curvature** as a generic term for the \( N \)-Ricci curvature with \( N \in (-\infty, \infty] \). As similar as \( \text{Ric}_- \), we define \( \text{Ric}_{n_-} : M \to \mathbb{R} \) by
\[
\text{Ric}_{n_-}(p) = \inf\{ \text{Ric}_N(v, v) \mid v \in U_p M \}.
\]

**Theorem 1.3** ([3, Theorem 1.4]). Let \( n \in \mathbb{N}_{\geq 2}, k \geq 0 \) and \( \eta > 0 \). There exists a positive constant \( \delta(n + 4k, \eta) \) such that if an \( n \)-dimensional compact weighted manifold \((M, g, \mu_f)\) satisfies \( \text{Ric}_\infty \geq 0 \), \( |f| \leq k \) and
\[
\frac{1}{\mu_f(M)} \int_M ((n - 1) - \text{Ric}_{\infty_-})_+ d\mu_f < \delta(n + 4k, \eta)
\]
then
\[
\text{diam}(M) < \left( \frac{\pi + \eta}{2} \right) \sqrt{1 + \frac{8k}{(n - 1)\pi} + \frac{\eta}{2}}.
\]
Theorem 1.4 \((\text{Theorem 1.5})\). Let \(n \in \mathbb{N}_{\geq 2}, k \geq 0, K \leq 0\) and \(R, \eta > 0\). There exists a positive constant \(\delta(n + 4k, K, R, \eta)\) such that if an \(n\)-dimensional complete weighted manifold \((M, g, \mu_f)\) satisfies \(|f| \leq k, \text{Ric}_\infty \geq (n - 1)K\) and

\[
\sup_{p \in M} \frac{1}{\mu_f(B(p, R))} \int_{B(p, R)} ((n - 1) - \text{Ric}_\infty) \_ d\mu_f < \delta(n + 4k, K, R, \eta)
\]

then \(M\) is compact and

\[
\text{diam}(M) < \left( \pi + \frac{\eta}{2} \right) \sqrt{1 + \frac{8k}{(n - 1)\pi}} + \frac{\eta}{2}.
\]

The aim of this article is to extend Theorems 1.1–1.4 for a weighted manifold under the assumption that the weighted curvature is bounded below in terms of its weight function. The key of the proof is the notion of the following \(\varepsilon\)-range introduced by Lu–Minguzzi–Ohta \([7]\).

Definition 1.5. For \(n \in \mathbb{N}_{\geq 2}\) and \(N \in (-\infty, 1] \cup [n, \infty]\), we say that \(\varepsilon \in \mathbb{R}\) is in the \(\varepsilon(n, N)\)-range if \(\varepsilon \in \mathbb{R}\) satisfies the following conditions:

\[
\varepsilon = 0 \text{ for } N = 1, \quad |\varepsilon| < \sqrt{\frac{N - 1}{N - n}} \text{ for } N \neq 1, n, \quad \varepsilon \in \mathbb{R} \text{ for } N = n.
\]

For \(n, N\) as above and \(\varepsilon\) in the \(\varepsilon(n, N)\)-range, we define the constant \(c = c(n, N, \varepsilon)\) by

\[
c = \frac{1}{n - 1} \left( 1 - \varepsilon^2 \frac{N - n}{N - 1} \right) > 0
\]

for \(N \neq 1\). If \(\varepsilon = 0\), then one can take \(N \to 1\) and set \(c(1, 0) = 1/(n - 1)\).

Notice that \(|\varepsilon| = 1\) happens only if \(N \in [n, \infty)\). For \(N \in (-\infty, 1] \cup [n, \infty]\) and \(K \in \mathbb{R}\), Lu–Minguzzi–Ohta \([8]\) established a curvature bound \(\text{Ric}_N \geq Ke^{4(\varepsilon - 1)f/(n - 1)}\) on a weighted manifold \((M, g, \mu_f)\) by using the \(\varepsilon(n, N)\)-range. This curvature bound is a generalization of a different kind of curvature bounds \(\text{Ric}_1 \geq Ke^{4f/(1 - n)}\) introduced by Wylie–Yeroshkin \([13]\) and for \(N \in (-\infty, 1]\), \(\text{Ric}_N \geq Ke^{4f/(N - n)}\) introduced by Kuwae–Li \([5]\). They presented several comparison theorems with each curvature bounds. Furthermore, Kuwae–Sakurai \([6]\) also provided several comparison theorems with the \(\varepsilon(n, N)\)-range.

To state our theorems, we prepare the following condition.

Definition 1.6. Let \(n \in \mathbb{N}_{\geq 2}, N \in (-\infty, 1] \cup [n, \infty], \varepsilon\) in the \(\varepsilon(n, N)\)-range, \(K \leq 0\) and \(b \geq a > 0\).

(1) We say that an \(n\)-dimensional weighted manifold \((M, g, \mu_f)\) satisfies a \((N, K, \varepsilon, a, b)\)-condition if one has

\[
\text{Ric}_N \geq Ke^{\frac{4(\varepsilon - 1)f}{n - 1}}, \quad a \leq e^{\frac{2(\varepsilon - 1)f}{n - 1}} \leq b \text{ on } M.
\]
(2) We define the constants \( \tilde{D} = \tilde{D}(n, N, \varepsilon, a, b) \) by

\[
\tilde{D} = \begin{cases} 
\sqrt{\frac{N-1}{n-1}}, & \text{if } \varepsilon = 1, \\
\sqrt{1 + \frac{2}{\varepsilon - 1 - \varepsilon} \log \frac{b}{a}}, & \text{if } \varepsilon \neq 1 \text{ and } N \in [n, \infty], \\
\sqrt{\left(\frac{b}{a}\right)^{\lambda_0} + \frac{2}{\varepsilon - 1 - \varepsilon} \pi \lambda_0 \left(\frac{b}{a}\right)^{\lambda_0} - 1} & \text{otherwise.}
\end{cases}
\]

Moreover, for \( N \in (-\infty, 1] \), we define \( \lambda_0 = \lambda_0(n, N, \varepsilon) \) by

\[
\lambda_0 = \frac{1}{1 - \varepsilon} \left(1 - \sqrt{\frac{N-1}{N-n}}\right).
\]

For a weighted manifold satisfying the \((N, K, \varepsilon, a, b)\)-condition, we estimate its diameter quantitatively.

**Theorem 1.7.** Let \( n \in \mathbb{N}_{\geq 2}, N \in (-\infty, 1] \cup [n, \infty], \varepsilon \) in the \(\varepsilon(n, N)\)-range, \( b \geq a > 0 \) and \( H, \eta > 0 \). There exists a positive constant \( \delta_1(n, N, \varepsilon, a, b, H, \eta) \) such that if an \(n\)-dimensional compact weighted manifold \((M, g, \mu_f)\) satisfies the \((N, 0, \varepsilon, a, b)\)-condition and

\[
\frac{1}{\mu_f(M)} \int_M \left( (n-1)H - \text{Ric}_{N-}^+ \right) d\mu_f \leq \delta_1(n, N, \varepsilon, a, b, H, \eta)
\]

then

\[
\text{diam}(M) \leq \frac{\pi + \eta}{\sqrt{H}} \tilde{D}(n, N, \varepsilon, a, b).
\]

**Theorem 1.8.** Let \( n \in \mathbb{N}_{\geq 2}, N \in (-\infty, 1] \cup [n, \infty] \) and \( \varepsilon \) in the \(\varepsilon(n, N)\)-range. Take real numbers \( K, a, b, H, R \) and \( \eta \) such that \( K \leq 0 \) and \( a, b, H, R, \eta > 0 \) with \( a \leq b \). There exists a positive constant \( \delta(n, N, K, \varepsilon, a, b, H, R, \eta) \) such that if an \(n\)-dimensional complete weighted manifold \((M, g, \mu_f)\) satisfies the \((N, K, \varepsilon, a, b)\)-condition and

\[
\sup_{p \in M} \frac{1}{\mu_f(B(p, R))} \int_{B(p, R)} \left( (n-1)H - \text{Ric}_{N-}^+ \right) d\mu_f \leq \delta(n, N, K, \varepsilon, a, b, H, R, \eta)
\]

then \( M \) is compact and

\[
\text{diam}(M) \leq \frac{\pi + \eta}{\sqrt{H}} \tilde{D}(n, N, \varepsilon, a, b).
\]

**Remark 1.9.** Assume that an \(n\)-dimensional complete weighted manifold \((M, g, \text{vol}_g)\) satisfies \( \text{Ric}_g \geq (n-1)H \) for \( H > 0 \). Then \((M, g, \text{vol}_g)\) satisfies the \((n, 0, 1, 1, 1)\)-condition and

\[
\sup_{p \in M} \frac{1}{\text{vol}_g(B(p, R))} \int_{B(p, R)} \left( (n-1)H - \text{Ric}_{n-}^+ \right) d\text{vol}_g = 0
\]

holds for each \( R > 0 \). Furthermore, we find \( \tilde{D}(n, n, 1, 1, 1) = 1 \). By Theorem 1.8, we see that \( \text{diam}(M) \leq (\pi + \eta)/\sqrt{H} \) for any \( \eta > 0 \). Taking \( \eta \to 0 \), we obtain \( \text{diam}(M) \leq \pi/\sqrt{H} \). This means that Theorem 1.8 recovers the Myers theorem.
This article is organized as follows: In Section 2, we recall a Bishop-type inequality, the volume comparison theorem on a weighted manifold given in [8]. Then we extend the segment inequality of Cheeger–Colding [1] to a weighted manifold under the assumption that the weighted Ricci curvature is bounded below in terms of its weight function. In Sections 3 and 4, we prove Theorems 1.7 and 1.8 respectively. In Sections 5, we analyze the fundamental group of a weighted manifold satisfying the \((N, K, \varepsilon, a, b)\)-condition.

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2. Preliminaries and segment inequality with \(\varepsilon\)-range

We recall some results by Lu–Minguzzi–Ohta [8]. Although they discussed a weighted Finsler manifold, throughout this article, we treat a weighted manifold.

2.1. Preliminaries. Lu–Minguzzi–Ohta [8] introduced the \(\varepsilon\)-range and proved a volume comparison theorem for a weighted manifold \((M, g, \mu_f)\).

For \(p \in M\), let \(\gamma : [0, l) \to M\) be a unit speed geodesic such that \(\gamma(0) = p\). Set \(f_\gamma(t) = f(\gamma(t))\). We denote by

\[\mu_f(t) = e^{-f_\gamma(t)} A_\gamma(t) dtd\theta_{n-1}\]

the weighted measure in the geodesic polar coordinates along geodesics \(\gamma\), where \(d\theta_{n-1}\) is the volume measure of the unit sphere in \(T_p M\).

First we introduce a Bishop-type inequality.

**Proposition 2.1** ([8, Theorem 3.5]). Let \((M, g, \mu_f)\) be an \(n\)-dimensional complete weighted manifold, \(N \in (-\infty, 1] \cup [n, \infty]\), \(\varepsilon\) in the \(\varepsilon(n, N)\)-range and \(c = c(n, N, \varepsilon)\). For a unit speed geodesic \(\gamma : [0, l) \to M\), set

\[h(t) = e^{-cf_\gamma(t)} A_\gamma(t)^c, \quad h_1(\tau) = h(\varphi_\gamma^{-1}(\tau))\]

for \(t \in [0, l)\) and \(\tau \in [0, \varphi_\gamma(l)]\), where

\[\varphi_\gamma(t) = \int_0^t e^{\frac{2(\varepsilon - 1)}{n-1} f_\gamma(s)} ds\]

Then, for all \(\tau \in (0, \varphi_\gamma(l))\),

\[h_1''(\tau) \leq -ch_1(\tau) \text{Ric}_N((\gamma \circ \varphi_\gamma^{-1})'(\tau))\]

We define the comparison function \(\text{sn}_\kappa\) by

\[\text{sn}_\kappa(t) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}t) & \kappa > 0, \\ t & \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}t) & \kappa < 0, \end{cases}\]

where \(t \in [0, \pi/\sqrt{\kappa}]\) for \(\kappa > 0\) and \(t \in \mathbb{R}\) for \(\kappa \leq 0\). Notice that \(\text{sn}_\kappa\) satisfies

\[\begin{align*}
\text{sn}_\kappa''(t) + \kappa \text{sn}_\kappa(t) &= 0, \\
\text{sn}_\kappa(0) &= 0, \\
\text{sn}_\kappa'(0) &= 1.
\end{align*}\]
Let us recall a volume comparison theorem.

**Proposition 2.2** ([8, Theorem 3.11]). Let \((M, g, \mu_f)\) be an \(n\)-dimensional complete weighted manifold satisfying the \((N, K, \varepsilon, a, b)\)-condition. Then

\[
\frac{\mu_f(B(p, R))}{\mu_f(B(p, r))} \leq \frac{b}{a} \cdot \int_0^{R/a} \frac{\text{sn} c K(\tau)^{1/c} d\tau}{\int_0^{r/b} \text{sn} c K(\tau)^{1/c} d\tau}
\]

holds for all \(p \in M\) and \(0 < r \leq R\).

It should be mentioned that Theorem 2.2 was originally proved for all \(K \in \mathbb{R}\) in [8, Theorem 3.11].

**Remark 2.3.** If \(K = 0\), we have \(\text{sn} 0(\tau) = \tau\), which gives

\[
\int_0^r \text{sn}(\tau)^{1/2} d\tau = \frac{c}{c + 1} r^{1 + 1}
\]

for \(r > 0\). In the setting of Theorem 2.2 for \(K = 0\), we find that

\[
\frac{\mu_f(B(p, R))}{\mu_f(B(p, r))} \leq \frac{b}{a} \cdot \left(\frac{b R}{a r}\right)^{1 + 1}.
\]

### 2.2. Segment inequality with \(\varepsilon\)-range.

The segment inequality proved by Cheeger–Colding [1, Theorem 2.11] plays an important role in the proof of Theorems 1.1 and 1.2. We extend this theorem as follows.

**Theorem 2.4.** Let \((M, g, \mu_f)\) be an \(n\)-dimensional complete weighted manifold, \(N \in (-\infty, 1] \cup [n, \infty]\), \(\varepsilon \in \text{the } \varepsilon(n, N)\)-range and \(K \in \mathbb{R}\). Assume that

\[
\text{Ric}_N \geq K c^{\frac{4-n}{2-n}}
\]

holds. For \(i = 1, 2\), let \(A_i\) be bounded open subsets of \(M\) and \(W\) be an open subset of \(M\) such that, for each two points \(y_i \in A_i\), any unit minimal geodesics \(\gamma_{y_1, y_2}\) from \(y_1\) to \(y_2\) is contained in \(W\). Then for any non-negative integrable function \(F\) on \(W\), we have

\[
\int_{A_1 \times A_2} \left(\int_0^{d(y_1, y_2)} F(\gamma_{y_1, y_2}) \, ds\right) d\mu_{f \times f} \leq C(n, c, K) (\mu_f(A_2) \text{diam}(A_1) + \mu_f(A_1) \text{diam}(A_2)) \int_W F \, d\mu_f,
\]

where \(\mu_{f \times f}\) is the product measure on \(M \times M\) induced by \(\mu_f\) and

\[
C(n, c, K) = \sup_{y_1 \in A_1, y_2 \in A_2} \left(\sup_{0 < s_0 \leq s \leq d(y_1, y_2)} \frac{\text{sn} c K(\varphi_{y_1}(s))^{1/c}}{\text{sn} c K(\varphi_{y_2}(u))^{1/c}}\right).
\]

**Proof.** Set

\[B = \{(y_1, y_2) \in A_1 \times A_2 \mid \text{there exists a unique minimal geodesic from } y_1 \text{ to } y_2\}.
\]

Then \(\mu_{f \times f}(B) = \mu_{f \times f}(A_1 \times A_2)\) holds since the measure of a cut locus of each \(y_i \in A_i\) with respect to \(\mu_f\) is zero for \(i = 1, 2\).
Define maps $E_1, E_2, E : A_1 \times A_2 \to \mathbb{R}$ by

$$E_1(y_1, y_2) = \int_{d(y_1, y_2)/2}^{d(y_1, y_2)} F(\gamma_{y_1, y_2}(u)) \, du,$$

$$E_2(y_1, y_2) = \int_{0}^{d(y_1, y_2)/2} F(\gamma_{y_1, y_2}(u)) \, du,$$

$$E(y_1, y_2) = E_1(y_1, y_2) + E_2(y_1, y_2) = \int_{0}^{d(y_1, y_2)} F(\gamma_{y_1, y_2}(u)) \, du. $$

Fix $y_i \in A_i$ and $v_i \in U_{y_i}M$ for $i = 1, 2$. Set

$$I(y_i, v_i) = \{ s > 0 \mid \exp_{y_i}(sv_i) \in A_{i+1}, d(y_i, \exp_{y_i}(sv_i)) = s \},$$

where we put $A_3 = A_1$. We denote by $|I(y_i, v_i)|$ the 1-dimensional Lebesgue measure of $I(y_i, v_i)$. Since $\{ \exp_{y_i}(sv_i) \mid s \in I(y_i, v_i) \}$ is contained only in $A_{i+1}$, we have

$$|I(y_i, v_i)| \leq \text{diam}(A_{i+1})$$

for $i = 1, 2$.

Fix $s \in I(y_1, v_1)$. For $\gamma(u) = \exp_{y_1}(uv_1)$ on $u \in [0, s]$, by Proposition 2.1, we find that $A_i^T(u)/(\text{sn}_{cK}(\varphi(\gamma(u))))^{1/c}$ is non-increasing, where $\varphi(\gamma(u)) < \pi/\sqrt{cK}$ holds if $K > 0$ (see [2], Proof of Theorem 3.6]). Then we find that

$$\frac{A_i^T(s)}{A_i^T(u)} \leq \frac{\text{sn}_{cK}(\varphi(\gamma(s)))^{1/c}}{\text{sn}_{cK}(\varphi(\gamma(u)))^{1/c}} \leq C(n, c, K)$$

for $0 < s/2 \leq u < s \leq T(v_1)$, where $T(v_1)$ is the supremum of $s$ such that $s \in I(y_1, v_1)$. This yields

$$E_1(y_1, \gamma(s))A_i^T(s) = A_i^T(s) \int_{s/2}^{s} F(\gamma(u)) \, du$$

$$\leq C(n, c, K) \int_{s/2}^{s} F(\gamma(u))A_i^T(u) \, du$$

$$\leq C(n, c, K) \int_{0}^{T(v_1)} F(\gamma(u))A_i^T(u) \, du.$$

Integrating this inequality on $I(y_1, v_1)$ implies that

$$\int_{I(y_1, v_1)} E_1(y_1, \gamma(s))A_i^T(s) \, ds \leq C(n, c, K) \int_{I(y_1, v_1)} ds \int_{0}^{T(v_1)} F(\gamma(u))A_i^T(u) \, du$$

$$\leq C(n, c, K) \text{diam}(A_2) \int_{0}^{T(v_1)} F(\gamma(u))A_i^T(u) \, du.$$
By integrating this inequality over the unit sphere in $T_{y_1}M$, we see that
\[
\int_{A_2} E_1(y_1, y_2) d\mu_f = \int_{U_{y_1}M} d\theta_{n-1} \int_{I(y_1, y_2)} E_1(y_1, \gamma(s)) A_f^1(s) ds \\
\leq C(n, c, K) \text{diam}(A_2) \int_{U_{y_1}M} d\theta_{n-1} \int_0^{T(v_1)} F(\gamma(u)) A_f^1(u) du \\
\leq C(n, c, K) \text{diam}(A_2) \int_W F d\mu_f,
\]
where, in the third inequality, we use the fact that any unit minimal geodesics from $y_1 \in A_1$ to $y_2 \in A_2$ is contained in $W$. Therefore we have
\begin{equation}
\int_{A_1 \times A_2} E_1(y_1, y_2) d\mu_{f \times f} \leq C(n, c, K) \text{diam}(A_2) \mu_f(A_1) \int_W F d\mu_f.
\end{equation}
On the other hand, substituting $t = d(y_1, y_2) - u$ for $E_2(y_1, y_2)$ yields
\[
E_2(y_1, y_2) = \int_0^{d(y_1, y_2)/2} F(\gamma_{y_1, y_2}(u)) du \\
= \int_0^{d(y_1, y_2)/2} F(\sigma_{y_2, y_1}(t)) dt \\
= E_1(y_2, y_1),
\]
where $\sigma_{y_2, y_1}(t) = \gamma_{y_1, y_2}(d(y_1, y_2) - t)$ on $[d(y_2, y_1)/2, d(y_2, y_1)]$. Interchanging the roles of $A_1$ and $A_2$ in (2.1), it turns out that, by (2.2),
\[
\int_{A_1 \times A_2} E_2(y_1, y_2) d\mu_{f \times f} = \int_{A_1 \times A_2} E_1(y_2, y_1) d\mu_{f \times f} \\
\leq C(n, c, K) \text{diam}(A_1) \mu_f(A_2) \int_W F d\mu_f.
\]
Thus we obtain
\[
\int_{A_1 \times A_2} E(y_1, y_2) d\mu_{f \times f} \\
= \int_{A_1 \times A_2} E_1(y_1, y_2) d\mu_{f \times f} + \int_{A_1 \times A_2} E_2(y_1, y_2) d\mu_{f \times f} \\
\leq C(n, c, K) \left[ \text{diam}(A_2) \mu_f(A_1) + \text{diam}(A_1) \mu_f(A_2) \right] \int_W F d\mu_f.
\]
This completes the proof of the theorem. 

Remark 2.5. Set $S = \sup_{y_1, y_2 \in A_2} d(y_1, y_2)$. If we take $N = n$, $f \equiv 0$, $\varepsilon = 0$ and $K = (n - 1)H$ with $H > 0$, then we have
\[
C(n, c, K) = \sup_{0 < \frac{1}{2} \leq u \leq S} \left( \frac{\text{sn}_H(s)}{\text{sn}_H(u)} \right)^{n-1}
\]
and the estimate in Theorem 2.3 coincides with that of [1] Theorem 2.11. Whereas, for $N = \infty$, since we impose the different condition compared with [4] Proposition 2.3, the estimate in Theorem 2.3 differs from that of [4] Proposition 2.3.
Finally, we provide the following lemma towards the proof of Theorems 1.7 and 1.8.

**Lemma 2.6.** Let \((M, g, \mu_f)\) be an \(n\)-dimensional complete weighted manifold satisfying the \((N, K, \epsilon, a, b)\)-condition. Take \(A_1, A_2\) and \(C(n, c, K)\) as in Theorem 2.4. Then

\[
C(n, c, K) \leq \frac{\text{sn} c K(S/a)^{1/c}}{\text{sn} c K(S/2b)^{1/c}}
\]

holds, where \(S = \sup_{y_1 \in A_1, y_2 \in A_2} d(y_1, y_2)\).

**Proof.** The boundedness of the weight function \(f\) yields \(s/b \leq \varphi_\gamma(s) \leq s/a\) for any unit speed minimal geodesic \(\gamma\) from \(y_1 \in A_1\) to \(y_2 \in A_2\). Then for \(K \leq 0\), the monotonicity of \(sn c K(\tau)\) in \(\tau > 0\) gives

\[
C(n, c, K) \leq \sup_{0 < s \leq S} \frac{\text{sn} c K(s/a)^{1/c}}{\text{sn} c K(s/2b)^{1/c}}.
\]

For \(K = 0\), \(\text{sn} 0(s) = s\) implies

\[
C(n, c, K) \leq \sup_{0 < s \leq S} \frac{\text{sn} 0(s/a)^{1/c}}{\text{sn} 0(s/2b)^{1/c}} = \left(\frac{2b}{a}\right)^{\frac{1}{c}}.
\]

When \(K < 0\), we will see the following claim:

**Claim 2.7.** Let \(F: \mathbb{R} \to \mathbb{R}\) be

\[
F(s) = \frac{\sinh(As)}{\sinh(Bs)}
\]

for \(A > B > 0\). Then \(F(s)\) is strictly increasing on \((0, \infty)\).

**Proof.** Differentiating \(F(s)\) gives

\[
F'(s) = \frac{1}{\sinh^2(Bs)} (A \cosh(As) \sinh(Bs) - B \sinh(As) \cosh(Bs)).
\]

We use the following hyperbolic function identities:

\[
2 \cosh(As) \sinh(Bs) = \sinh (A + B) s + \sinh (A - B) s,
\]

\[
2 \sinh(As) \cosh(Bs) = \sinh (A + B) s - \sinh (A - B) s.
\]

These yield

\[
F'(s) = \frac{1}{2 \sinh^2(Bs)} \left\{ (A - B) \sinh (A + B) s - (A + B) \sinh (A - B) s \right\}.
\]

Setting

\[
F_1(s) = (A - B) \sinh (A + B) s - (A + B) \sinh (A - B) s,
\]

we have \(F_1(0) = 0\) and

\[
F_1'(s) = AB \left\{ \cosh (A + B) s - \cosh (A - B) s \right\} > 0
\]

on \((0, \infty)\). Therefore we get \(F_1(s) > 0\) on \((0, \infty)\). Hence we obtain \(F'(s) > 0\) on \((0, \infty)\). This completes the proof of the claim. \(\diamond\)

When \(K < 0\), applying this claim to (2.3) gives

\[
C(n, c, K) \leq \frac{\text{sn} c K(S/a)^{1/c}}{\text{sn} c K(S/2b)^{1/c}}.
\]
3. PROOF OF THEOREM 1.7

We extend Theorem 1.1 to the case that the weighted manifold has the non-negative weighted curvature.

Proof of Theorem 1.7. Set \( D = \text{diam}(M) \) and take \( p_1, p_2 \in M \) such that \( D = d(p_1, p_2) \). We put \( W = M \) and \( A_i = B(p_i, r) \) for \( i = 1, 2 \), where \( r > 0 \) is later determined. Then Theorem 2.4 gives

\[
\begin{aligned}
\int_{A_1 \times A_2} \left( \inf_{(y_1, y_2) \in A_1 \times A_2} \int_0^d(y_1, y_2) \left( (n - 1)H - \text{Ric}_N \right)_+ (\gamma_{y_1, y_2}) ds \right) d\mu_f \times f \\
\leq \int_{A_1 \times A_2} \left( \int_0^d(y_1, y_2) \left( (n - 1)H - \text{Ric}_N \right)_+ (\gamma_{y_1, y_2}) ds \right) d\mu_f \times f \\
\leq 2rC(n, c, K) [\mu_f(A_1) + \mu_f(A_2)] \int_M \left( (n - 1)H - \text{Ric}_N \right)_+ d\mu_f.
\end{aligned}
\]

We observe from Theorem 2.2 with \( K = 0 \) that \( \frac{\mu_f(M)}{\mu_f(A_i)} \leq \left( \frac{bD}{ar} \right)^{\frac{1}{n+1}} \) for \( i = 1, 2 \) and from Lemma 2.6 that

\[
C(n, c, K) \leq \left( \frac{2b}{a} \right)^{\frac{1}{2}}.
\]

Dividing (3.1) by \( \mu_f(A_1)\mu_f(A_2) \) yields

\[
\begin{aligned}
\int_{A_1 \times A_2} \left( \inf_{(y_1, y_2) \in A_1 \times A_2} \int_0^d(y_1, y_2) \left( (n - 1)H - \text{Ric}_N \right)_+ (\gamma_{y_1, y_2}) ds \right) d\mu_f \times f \\
\leq 2rC(n, c, K) \left( \frac{1}{\mu_f(A_1)} + \frac{1}{\mu_f(A_2)} \right) \int_M \left( (n - 1)H - \text{Ric}_N \right)_+ d\mu_f \\
\leq 2r \left( \frac{2b^2D}{a^2r} \right)^{\frac{1}{n+1}} \frac{1}{\mu_f(M)} \int_M \left( (n - 1)H - \text{Ric}_N \right)_+ d\mu_f.
\end{aligned}
\]

There exists a unit speed minimal geodesic \( \gamma \) from \( y_1 \in A_1 \) to \( y_2 \in A_2 \) that attains the infimum of (3.2). Let \( L = d(y_1, y_2) \) and \( \{ E_1, \ldots, E_n = \hat{\gamma} \} \) be a parallel orthonormal frame along \( \gamma \). For a smooth function \( \alpha \in C^\infty([0, L]) \) such that \( \alpha(0) = \alpha(L) = 0 \), we set \( Y_i(t) = \alpha(t)E_i(t), \ i = 1, \ldots, n - 1 \). We denote by \( L_i(s) \) the length functional of a fixed-endpoint variation of a curves \( c(s, t) : (-\epsilon, \epsilon) \times [0, L] \to M \) such that

\[
c(0, t) = \gamma(t), \quad \frac{\partial}{\partial s} c(s, t) \bigg|_{s=0} = Y_i(t).
\]
Then the second variation formula for $L_i(s)$ (see [10] Chapter III Theorem 2.5) provides

$$
(3.3) \sum_{i=1}^{n-1} \frac{d^2 L_i}{ds^2} = \sum_{i=1}^{n-1} \int_0^L \left\{ g(\nabla_i Y_i, \nabla_i Y_i) - R_g(Y_i(t), \gamma'(t), \gamma'(t), Y_i(t)) \right\} dt
$$

$$
= \int_0^L \left\{ (n-1)\alpha'(t)^2 - \alpha(t)^2 \text{Ric}_g(\gamma'(t), \gamma'(t)) \right\} dt
$$

$$
= \int_0^L \left[ - (n-1)H \alpha(t)^2 + (n-1)\alpha'(t)^2 + \alpha(t)^2 \text{Hess}_g f(\gamma'(t), \gamma'(t)) \right.
$$

$$
- \frac{\alpha(t)^2}{N-n} f'_\gamma(t)^2 + \alpha(t)^2 \left\{ (n-1)H - \text{Ric}_N (\gamma'(t), \gamma'(t)) \right\} \right] dt,
$$

where $R_g$ is the Riemannian curvature tensor of $(M, g)$. Note that if $N = n$, the forth term on the right-hand side of (3.3) always vanishes since we set $f'_\gamma(t)^2/(N-n) = 0$.

We take a parameter $\lambda$ satisfying

$$
(3.4) \begin{cases}
|1 - \varepsilon| \lambda - 1 | \leq \sqrt{\frac{N-1}{N-n}} & \text{if } N \in (-\infty, 1] \cup (n, \infty] \text{ and } \varepsilon \neq 1, \\
\lambda \in \mathbb{R} & \text{if either } N = n \text{ or } \varepsilon = 1.
\end{cases}
$$

We define $\Phi(\lambda)$, $\Psi(\lambda)$ by

$$
\Phi(\lambda) = \begin{cases}
b^\lambda & \text{if } \lambda > 0, \\
1 & \text{if } \lambda = 0, \\
a^\lambda & \text{if } \lambda < 0,
\end{cases}
$$

$$
\Psi(\lambda) = \begin{cases}
a^\lambda & \text{if } \lambda > 0, \\
1 & \text{if } \lambda = 0, \\
b^\lambda & \text{if } \lambda < 0.
\end{cases}
$$

If we choose

$$
\alpha(t) = e^{\frac{(1-\varepsilon)\lambda}{n-1} f'_\gamma(t)} \sin \left( \frac{\pi t}{L} \right),
$$

then we have

$$
\alpha'(t) = \frac{(1-\varepsilon)\lambda}{n-1} f'_\gamma(t) \alpha(t) + \frac{\pi t}{L} e^{\frac{(1-\varepsilon)\lambda}{n-1} f'_\gamma(t)} \cos \left( \frac{\pi t}{L} \right).
$$

We estimate the right-hand side of (3.3). The first term constructs to

$$
(3.5) -(n-1)H \int_0^L \alpha(t)^2 dt \leq -(n-1)H \int_0^L \Psi(\lambda) \sin^2 \left( \frac{\pi t}{L} \right) dt
$$

$$
\leq \frac{(n-1)HL}{2} \Psi(\lambda).
$$
The second term is estimated as

\[(3.6)\]
\[
\int_0^L \alpha'(t)^2 \, dt
\]
\[
= (n-1)\pi^2 \int_0^L e^{2(1-\varepsilon)\lambda \int_t L} \cos^2 \left( \frac{\pi t}{L} \right) \, dt
\]
\[
+ \frac{(1-\varepsilon)^2 \lambda^2}{n-1} \int_0^L \alpha(t)^2 f'_\gamma(t)^2 \, dt
\]
\[
= \frac{(n-1)\pi^2}{L^2} \int_0^L e^{2(1-\varepsilon)\lambda \int_t L} \cos^2 \left( \frac{\pi t}{L} \right) \, dt
\]
\[
- \frac{(n-1)\pi^2}{L^2} \int_0^L e^{2(1-\varepsilon)\lambda \int_t L} \cos \left( \frac{2\pi t}{L} \right) \, dt
\]
\[
+ \frac{(1-\varepsilon)^2 \lambda^2}{n-1} \int_0^L \alpha(t)^2 f'_\gamma(t)^2 \, dt
\]
\[
= \frac{(n-1)\pi^2}{2L^2} \int_0^L e^{2(1-\varepsilon)\lambda \int_t L} \left( 1 - \cos \left( \frac{\pi t}{L} \right) \right) \, dt
\]
\[
+ \frac{(1-\varepsilon)^2 \lambda^2}{n-1} \int_0^L \alpha(t)^2 f'_\gamma(t)^2 \, dt
\]
\[
\leq \frac{(n-1)\pi^2}{2L} \Phi(\lambda) + \frac{(1-\varepsilon)^2 \lambda^2}{n-1} \int_0^L \alpha(t)^2 f'_\gamma(t)^2 \, dt.
\]

We calculate the third term as

\[(3.7)\]
\[
\int_0^L \alpha(t)^2 f''_\gamma(t) \, dt
\]
\[
= -\int_0^L 2\alpha(t)\alpha'(t) f'_\gamma(t) \, dt
\]
\[
= -\frac{\pi}{L} \int_0^L f'_\gamma(t) e^{2(1-\varepsilon)\lambda \int_t L} \sin \left( \frac{2\pi t}{L} \right) \, dt
\]
\[
- \frac{2(1-\varepsilon)\lambda}{n-1} \int_0^L \alpha(t)^2 f'_\gamma(t)^2 \, dt.
\]

The last term constructs to

\[(3.8)\]
\[
\int_0^L \alpha(t)^2 \left\{(n-1)H - \text{Ric}_{N-}(\gamma'(t), \gamma'(t))\right\} \, dt
\]
\[
\leq \int_0^L \alpha(t)^2 \left((n-1)H - \text{Ric}_{N-}\right) + \, dt
\]
\[
\leq \Phi(\lambda) \int_0^L \left((n-1)H - \text{Ric}_{N-}\right) + \, dt.
\]

Combining (3.5), (3.6), (3.7), (3.8) gives

\[(3.9)\]
\[
\sum_{i=1}^{n-1} \left| \frac{d^2 L_i}{ds^2} \right|_{s=0} \leq \frac{(n-1)HL}{2} \Phi(\lambda) + \frac{(n-1)\pi^2}{2L} \Phi(\lambda)
\]
\[
+ \bar{D}_1(\lambda) + \Phi(\lambda) \int_0^L \left((n-1)H - \text{Ric}_{N-}\right) + \, dt,
\]
where
\[ \bar{D}_1(\lambda) = -\frac{\pi}{L} \int_0^L f_\gamma'(t)e^{\frac{2(1-\varepsilon)\lambda}{n-1}f_\gamma(t)} \sin \left(\frac{2\pi t}{L}\right) dt \]
\[ + \left(\frac{(1-\varepsilon)^2 \lambda^2}{(n-1)^2} - \frac{2(1-\varepsilon)\lambda}{n-1} - \frac{1}{N-n}\right) \int_0^L \alpha(t)^2 f_\gamma'(t)^2 dt. \]

First we consider the case of \( \varepsilon = 1 \). We see that
\[ \bar{D}_1(\lambda) = -\frac{\pi}{L} \int_0^L f_\gamma'(t)e^{\frac{2\lambda}{n-1}f_\gamma(t)} \sin \left(\frac{2\pi t}{L}\right) dt - \frac{1}{N-n} \int_0^L \alpha(t)^2 f_\gamma'(t)^2 dt \]
\[ = (N-n)\frac{\pi^2}{L^2} \int_0^L e^{\frac{2\lambda}{n-1}f_\gamma(t)} \cos^2 \left(\frac{\pi t}{L}\right) dt \]
\[ - (N-n) \int_0^L e^{\frac{2\lambda}{n-1}f_\gamma(t)} \left( \frac{\pi}{L} \cos \left(\frac{\pi t}{L}\right) + \frac{f_\gamma'(t)}{N-n} \sin \left(\frac{\pi t}{L}\right) \right)^2 dt \]
\[ \leq (N-n)\frac{\pi^2}{2L} \Phi(\lambda). \]

If \( \varepsilon \neq 1 \), then \( \lambda \neq 0 \) and it follows from [3.4] that
\[ \bar{D}_1(\lambda) = -\frac{\pi}{L} \int_0^L f_\gamma'(t)e^{\frac{2(1-\varepsilon)\lambda}{n-1}f_\gamma(t)} \sin \left(\frac{2\pi t}{L}\right) dt \]
\[ + \left(\frac{(1-\varepsilon)^2 \lambda^2}{(n-1)^2} - \frac{2(1-\varepsilon)\lambda}{n-1} - \frac{1}{N-n}\right) \int_0^L \alpha(t)^2 f_\gamma'(t)^2 dt \]
\[ \leq (n-1)\frac{\pi^2}{(1-\varepsilon)L^2 \lambda} \int_0^L e^{\frac{2(1-\varepsilon)\lambda}{n-1}f_\gamma(t)} \cos \left(\frac{2\pi t}{L}\right) dt \]
\[ \leq (n-1)\frac{\pi}{|1-\varepsilon|L\lambda} (\Phi(\lambda) - \Psi(\lambda)). \]

Moreover, if we define
\[ \bar{D}(\lambda) = \begin{cases} \sqrt{\frac{N-1}{n-1} \left(\frac{\Phi(\lambda)}{\Psi(\lambda)}\right)} & \text{if } \varepsilon = 1, \\ \frac{\Phi(\lambda)}{\Psi(\lambda)} + \frac{2}{|1-\varepsilon|\pi \lambda} \left( \frac{\Phi(\lambda)}{\Psi(\lambda)} - 1 \right) & \text{if } \varepsilon \neq 1, \end{cases} \]
then we find that
\[ \sum_{i=1}^{n-1} \frac{d^2 L_i}{ds^2} \bigg|_{s=0} \leq -\frac{(n-1)HL}{2} \left( 1 - \frac{\pi^2}{HL^2} \bar{D}(\lambda)^2 \right) \]
\[ + \Phi(\lambda) \int_0^L ((n-1)H - \text{Ric}_{N-}) dt. \]
Substituting (3.2) for (3.10) gives

\[
\sum_{i=1}^{n-1} \frac{d^2 L_i}{ds^2} \bigg|_{s=0} \leq -\frac{(n-1)HL\Psi(\lambda)}{2} \left( 1 - \frac{\pi^2}{HL^2} \tilde{D}(\lambda)^2 \right) \\
+ 2r \left( \frac{2b^2D}{a^2r} \right)^{\frac{1}{2}+1} \frac{\Phi(\lambda)}{\mu_f(M)} \int_M \left( (n-1)H - \text{Ric}_{N^-} \right) + d\mu_f.
\]

Choose \( T = T(\eta) > 2 \) such that

\[
\frac{1}{1 - \frac{\eta}{T}} \leq \frac{\pi + \eta}{\pi + \eta^2}
\]

and let \( r = D/T \). By the assumption \( y_i \in \overline{B}(p_i, r) \) for \( i = 1, 2 \) together with the triangle inequality, we find that

\[
L = d(y_1, y_2) \geq d(p_1, p_2) - 2r = D \left( 1 - \frac{2}{T} \right)
\]

Therefore we have

\[
\sum_{i=1}^{n-1} \frac{d^2 L_i}{ds^2} \bigg|_{s=0} \leq -\frac{(n-1)HL\Psi(\lambda)}{2} \left( 1 - \frac{\pi^2}{HL^2} \tilde{D}(\lambda)^2 \right) \\
+ 2 \left( \frac{2b^2D}{a^2} \right)^{\frac{1}{2}+1} \frac{L}{1 - \frac{\eta}{T}} \frac{\Phi(\lambda)}{\mu_f(M)} \int_M \left( (n-1)H - \text{Ric}_{N^-} \right) + d\mu_f.
\]

We consider the limit as

\[
\lambda \to \lambda_0 = \begin{cases} 
0 & \text{if } N \in [n, \infty], \\
\frac{1}{1 - \varepsilon} \left( 1 - \sqrt{\frac{N-1}{N-n}} \right) & \text{if } N \in (-\infty, 1].
\end{cases}
\]

Then we have \( \tilde{D}(\lambda) \to \tilde{D}(n, N, \varepsilon, a, b) \) as \( \lambda \to \lambda_0 \). We set

\[
\delta_1(n, N, \varepsilon, a, b, H, \eta) = \frac{H(n-1)(1 - \frac{2}{\pi})}{2^{\frac{1}{2}+3T^2}} \left( 1 - \frac{\pi^2}{(\pi + \frac{\eta}{2})^2} \right) \left( \frac{a}{b} \right)^{\frac{2}{2}+2+\lambda_0}.
\]

If we assume that

\[
\frac{1}{\mu_f(M)} \int_M \left( (n-1)H - \text{Ric}_{N^-} \right) + d\mu_f \leq \delta_1(n, N, \varepsilon, a, b, H, \eta),
\]

then we get

\[
\sum_{i=1}^{n-1} \frac{d^2 L_i}{ds^2} \bigg|_{s=0} \leq -\frac{(n-1)HL\lambda_0}{2} \left( 1 - \frac{\pi^2}{HL^2} \tilde{D}^2 \right) + \frac{(n-1)HL\lambda_0}{2} \left( 1 - \frac{\pi^2}{(\pi + \frac{\eta}{2})^2} \right) \\
= -\frac{(n-1)H\pi^2\alpha}{2L^2(\pi + \frac{\eta}{2})^2} \left( L^2 - \frac{1}{H} \left( \pi + \frac{\eta}{2} \right)^2 \tilde{D}^2 \right).
\]

Since \( \gamma \) is a minimal geodesic, it follows

\[
\sum_{i=1}^{n-1} \frac{d^2 L_i}{ds^2} \bigg|_{s=0} \geq 0.
\]
Therefore we obtain
\[ L \leq \frac{1}{\sqrt{H}} \left( \pi + \frac{\eta}{2} \right) \tilde{D}(n, N, \varepsilon, a, b). \]

We observe from (3.11), (3.12) that
\[ D \leq \frac{L}{1 - \frac{2}{\pi}} \leq \frac{\pi + \eta}{\sqrt{H}} \tilde{D}(n, N, \varepsilon, a, b). \]

This completes the proof of the theorem. □

Remark 3.1. In Theorem [1.7] when \( \varepsilon \neq 1 \), if we take \( a', b' \) such that
\[ \frac{b'}{a'} = \exp \left( \frac{2|1 - \varepsilon|}{n - 1}(\sup f - \inf f) \right), \]
then \( \delta_1(n, N, \varepsilon, a, b, H, \eta) \leq \delta_1(n, N, \varepsilon, a', b', H, \eta) \) holds. This implies that the assumption of Theorem [1.7] also holds when we replace \( a, b \) by \( a', b' \). We find that
\[
\tilde{D}(n, N, \varepsilon, a', b') = \begin{cases} 
\sqrt{1 + \frac{4(\sup f - \inf f)}{(n - 1)\pi}} & \text{if } N \in [n, \infty], \\
\sqrt{\lambda_1 + \frac{2(n - N)(\lambda_1 - 1)}{(n - 1)\pi} \left( 1 + \sqrt{\frac{N - 1}{N - n}} \right)} & \text{if } N \in (-\infty, 1],
\end{cases}
\]
where
\[ \lambda_1 = \exp \left( \frac{2(\sup f - \inf f)}{n - 1} \left( 1 - \sqrt{\frac{N - 1}{N - n}} \right) \right). \]

4. Proof of Theorem [1.8]

We provide the following lemma to prove Theorem [1.8]

Lemma 4.1. Let \( n, N, \varepsilon, K, a, b, H, R \) and \( \eta \) as in Theorem [1.8] Assume \( R, \eta \) satisfy
\[ R > \frac{\pi}{\sqrt{H}} \tilde{D}(n, N, \varepsilon, a, b) \quad \text{and} \quad 0 < \eta < \eta_*(H, R, \tilde{D}) = \frac{4}{7} \left( \frac{R\sqrt{H}}{D} - \pi \right). \]

There exists a positive constant \( \delta_2(n, N, K, \varepsilon, a, b, H, R, \eta) \) such that if an \( n \)-dimensional complete weighted manifold \( (M, g, \mu_f) \) satisfies the \((N, K, \varepsilon, a, b)\)-condition and
\[ \sup_{p \in M} \frac{1}{\mu_f(B(p, R))} \int_{B(p, R)} ((n - 1)H - \text{Ric}_{N-})_+ \, d\mu_f \leq \delta_2(n, N, K, \varepsilon, a, b, H, R, \eta) \]
then \( M \) is compact and
\[ \text{diam}(M) \leq \frac{\pi + \eta}{\sqrt{H}} \tilde{D}(n, N, \varepsilon, a, b). \]

Proof. The proof goes by contradiction, that is, there exist points \( p_1, q \in M \) such that the distance from \( p_1 \) to \( q \) is greater than \((\pi + \eta)\tilde{D}/\sqrt{H}\). Then there exists \( p_2 \in M \) such that \( p_2 \) lies in a unit minimal geodesic from \( p_1 \) to \( q \) and
\[ \frac{\pi + \eta}{\sqrt{H}} \tilde{D} < d(p_1, p_2) < R - \frac{3\eta\tilde{D}}{4\sqrt{H}}. \]
First we set $W = B(p_1, R)$ for $p_1 \in M$ and $r = \eta \tilde{D}/4\sqrt{H}$ for $\eta < \eta_\ast(H, R, \tilde{D})$. We put $A_i = B(p_i, r) \subset W$ for $i = 1, 2$. The triangle inequality that for $y_i \in A_i$ with $i = 1, 2$ yields
\[
d(y_1, y_2) \leq d(y_1, p_1) + d(p_1, p_2) + d(p_2, y_2) < R - r.
\]
On the other hand, the distance from $y_1 \in A_1$ to the boundary of $W$ is greater than $R - r$. This means that all unit minimal geodesics from $y_1 \in A_1$ to $y_2 \in A_2$ lie in $W$. Using Theorem 2.2 we see that
\[
\inf_{(y_1, y_2) \in \overline{A}_1 \times \overline{A}_2} \int_0^{d(y_1, y_2)} ((n-1)H - \text{Ric}_N)_+ (\gamma_{y_1, y_2}) \, ds \leq 2r C(n, c, K) \left( \frac{1}{\mu_f(A_1)} + \frac{1}{\mu_f(A_2)} \right) \int_{B(p, R)} ((n-1)H - \text{Ric}_N)_+ \, d\mu_f.
\]
We set
\[
v_{cK,a}(R) = \int_0^{R/a} \text{sn}_{cK}(\tau)^{1/c} \, d\tau.
\]
Theorem 2.2 implies
\[
\frac{\mu_f(B(p_1, R))}{\mu_f(B(p_1, r))} \leq \frac{b}{a} \cdot \frac{v_{cK,a}(R)}{v_{cK,b}(r)}
\]
and Lemma 2.6 gives
\[
C(n, c, K) \leq \tilde{C} = \frac{\text{sn}_{cK}((R - r)/a)^{1/c}}{\text{sn}_{cK}((R - r)/b)^{1/c}}.
\]
We observe that
(4.3)
\[
\inf_{(y_1, y_2) \in \overline{A}_1 \times \overline{A}_2} \int_0^{d(y_1, y_2)} ((n-1)H - \text{Ric}_N)_+ (\gamma_{y_1, y_2}) \, ds \leq 2r C(n, c, K) \left( \frac{v_{cK,a}(R)}{v_{cK,b}(r)} \cdot \frac{1}{\mu_f(B(p_1, R))} + \frac{v_{cK,a}(2R)}{v_{cK,b}(r)} \cdot \frac{1}{\mu_f(B(p_2, 2R))} \right) \times \int_{B(p, R)} ((n-1)H - \text{Ric}_N)_+ \, d\mu_f \leq 2r \tilde{C} \left( \frac{v_{cK,a}(R) + v_{cK,a}(2R)}{v_{cK,b}(r)} \right) \left( \frac{1}{\mu_f(B(p_1, R))} \int_{B(p_1, R)} ((n-1)H - \text{Ric}_N)_+ \, d\mu_f \right),
\]
where we use $B(p_1, R) \subset B(p_2, 2R)$ in the second inequality.

We find a unit speed minimal geodesic $\gamma$ from $y_1 \in \overline{A}_1$ to $y_2 \in \overline{A}_2$ that attains the infimum of (4.3). We put $L = d(y_1, y_2)$. With the same argument of Theorem 1.7, we utilize (3.10) and take the limit as $\lambda \to \lambda_0$ again, we see that
\[
\sum_{i=1}^{n-1} \frac{d^2 L_i}{ds^2} \bigg|_{s=0} \leq -\frac{(n-1)HL\lambda_0}{2} \left( 1 - \frac{\pi^2}{HL^2 \tilde{D}^2} \right) + \frac{\eta \tilde{D}}{2\sqrt{H}} b^{\lambda_0 + 1} \tilde{C} \left( \frac{v_{cK,a}(R) + v_{cK,a}(2R)}{v_{cK,b}(\frac{a D}{\sqrt{H}})} \right) \times \frac{1}{\mu_f(B(p, R))} \right) \int_{B(p, R)} ((n-1)H - \text{Ric}_N)_+ \, d\mu_f.
\]
We set
\[ \delta_2(n, N, K, \varepsilon, a, b, H, R, \eta) \]
\[ = H(n - 1)(\pi + \eta) \frac{v_{cK,b}(\eta \tilde{D})}{2\sqrt{H}} \frac{v_{cK,a}(R) + v_{cK,a}(2R)}{\eta C} \left( 1 - \frac{\pi^2}{(\pi + \frac{\eta}{2})^2} \right) \left( \frac{a}{b} \lambda_0 + 1 \right). \]

If we assume that
\[ \sup_{p \in M} \frac{1}{\mu_f(B(p, R))} \int_{B(p, R)} ((n - 1)H - \text{Ric}_N)_+ d\mu_f \leq \delta_2(n, N, K, \varepsilon, a, b, H, R, \eta), \]
then we have
\[ \sum_{i=1}^{n-1} \left. \frac{d^2 L_i}{ds^2} \right|_{s=0} \leq -\frac{(n - 1)HLa^{\lambda_0}}{2} \left( 1 - \frac{\pi^2}{HL^2 \tilde{D}^2} \right) \]
\[ + \frac{(n - 1)(\pi + \eta)\sqrt{H}a^{\lambda_0}\tilde{D}}{2} \left( 1 - \frac{\pi^2}{(\pi + \frac{\eta}{2})^2} \right) \]
\[ = -\frac{n - 1}{2} \left( L - \frac{1}{\sqrt{H}} \left( \pi + \frac{\eta}{2} \right) \tilde{D} \right) \left( Ha^{\lambda_0} + \frac{(n - 1)\pi^2 \tilde{D}a^{\lambda_0} \sqrt{H}}{2L(\pi + \frac{\eta}{2})} \right). \]

Since \( \gamma \) is a minimal geodesic, it follows
\[ \sum_{i=1}^{n-1} \left. \frac{d^2 L_i}{ds^2} \right|_{s=0} \geq 0. \]

Therefore we obtain
\[ L \leq \frac{1}{\sqrt{H}} \left( \pi + \frac{\eta}{2} \right) \tilde{D}. \]

By the triangle inequality, we have
\[ (4.5) \quad d(p_1, p_2) \leq L + 2r \leq \frac{\pi + \eta}{\sqrt{H}} \tilde{D}. \]

On the other hand, we assumed that
\[ \frac{\pi + \eta}{\sqrt{H}} \tilde{D} < d(p_1, p_2) < R - \frac{3\eta \tilde{D}}{4\sqrt{H}}, \]
but by \( (4.5) \), no geodesic starting from \( p_1 \in M \) of a length greater than \( (\pi + \eta)\tilde{D}/\sqrt{H} \) can be minimal. This completes the proof of the lemma. \( \square \)

Finally, we show Theorem 1.8.

**Proof of Theorem 1.8.** We divide the cases into three parts:

1. \( R \leq \pi \tilde{D}(n, N, \varepsilon, a, b)/\sqrt{H} \),
2. \( R > \pi \tilde{D}(n, N, \varepsilon, a, b)/\sqrt{H} \) and \( \eta \geq \eta_*(H, R, \tilde{D}) \),
3. \( R > \pi \tilde{D}(n, N, \varepsilon, a, b)/\sqrt{H} \) and \( \eta < \eta_*(H, R, \tilde{D}) \).

The case (3) is already discussed in Lemma 4.1. Thus it is enough to consider the cases (1) and (2).
Let $R' = R'(\eta) > \pi \hat{D}/\sqrt{H}$ be fixed such that $\eta < \eta_\ast(H, R', \hat{D})$. For each $p \in M$, we consider the discrete subset $\{x_i\} \subset B(p, R')$ for $i = 1, \ldots, T_2$ such that

$$B(p, R') \subset \bigcup_{i=1}^{T_2} B(x_i, R)$$

and $d(x_i, x_j) > R$ for $i \neq j$, where $T_2$ is the maximal number of the $R$-discrete net of $B(p, R')$ (see [11, Definition 3.1] for the definition of the $R$-discrete net). We now claim:

**Claim 4.2.** Let $(M, g, \mu_f)$ be an $n$-dimensional complete weighted manifold satisfying the $(N, K, \varepsilon, a, b)$-condition. Set $T_2 > 0$ as above. Then it follows

$$T_2 \leq \frac{b}{a} \cdot \frac{v_{cK,b}(2R' + R)}{v_{cK,a}(R/2)}.$$  

**Proof.** Take $i_0 \in \{1, \ldots, T_2\}$ such that

$$\mu_f(B(x_{i_0}, R/2)) = \min_{1 \leq i \leq T_2} \mu_f(B(x_i, R/2)).$$

Since $B(x_i, R/2) \cap B(x_j, R/2) = \emptyset$ for any $i \neq j$ and $B(p, R' + R) \subset B(x_{i_0}, 2R' + R)$ hold, we have

$$1 = \frac{\mu_f(B(p, R' + R))}{\mu_f(B(p, R' + R))} \leq \frac{\mu_f(B(p, R' + R))}{\mu_f(B(p, R' + R) \cap \bigcup_{i=1}^{T_2} B(x_i, R/2))} \leq \frac{\mu_f(B(p, R' + R))}{T_2 \cdot \mu_f(B(x_{i_0}, R/2))} \leq \frac{\mu_f(B(x_{i_0}, 2R' + R))}{T_2 \cdot \mu_f(B(x_{i_0}, R/2))}.$$  

By using Theorem 2.2, we obtain

$$T_2 \leq \frac{b}{a} \cdot \frac{v_{cK,b}(2R' + R)}{v_{cK,a}(R/2)}.$$

\[\Box\]
Then we find that, for any \( z \in M \),
\[
\frac{1}{\mu_f(B(z, R))} \int_{B(z, R)} (n - 1)H - \operatorname{Ric}_- \, d\mu_f \\
\leq \frac{T_2}{\mu_f(B(z, R'))} \sup_{x_i \in B(z, R')} \int_{B(x_i, R)} (n - 1)H - \operatorname{Ric}_- \, d\mu_f \\
\leq T_2 \frac{b}{a} \frac{1}{\mu_f(B(z, R' + R))} \frac{v_{cK,b}(R' + R)}{v_{cK,a}(R')} \\
\times \sup_{x_i \in B(z, R')} \int_{B(x_i, R)} (n - 1)H - \operatorname{Ric}_- \, d\mu_f \\
\leq \frac{b^2}{a^2} \frac{v_{cK,b}(2R' + R)}{v_{cK,a}(R'/2)} \frac{v_{cK,b}(R' + R)}{v_{cK,a}(R')} \\
\sup_{x \in M} \frac{1}{\mu_f(B(x, R))} \int_{B(x, R)} (n - 1)H - \operatorname{Ric}_- \, d\mu_f.
\]
where we use the estimate of \( T_2 \) and \( B(x, R) \subset B(z, R' + R) \) for \( 1 \leq i \leq T_2 \) in the third inequality. This provides that
\[
\sup_{x \in M} \frac{1}{\mu_f(B(x, R))} \int_{B(x, R)} (n - 1)H - \operatorname{Ric}_- \, d\mu_f \\
\leq \frac{b^2}{a^2} \frac{v_{cK,b}(2R' + R)}{v_{cK,a}(R'/2)} \frac{v_{cK,b}(R' + R)}{v_{cK,a}(R')} \sup_{x \in M} \frac{1}{\mu_f(B(x, R))} \int_{B(x, R)} (n - 1)H - \operatorname{Ric}_- \, d\mu_f.
\]
We set
\[
\delta_2'(n, N, K, \varepsilon, a, b, H, R, \eta) \\
= \frac{a^2}{b^2} \frac{v_{cK,a}(R'/2)}{v_{cK,b}(2R'(\eta) + R)} \frac{v_{cK,a}(R'(\eta))}{v_{cK,b}(R'(\eta) + R)} \delta_2(n, N, K, \varepsilon, a, b, H, R'(\eta), \eta).
\]
If we assume that
\[
\sup_{x \in M} \frac{1}{\mu_f(B(x, R))} \int_{B(x, R)} (n - 1)H - \operatorname{Ric}_- \, d\mu_f \leq \delta_2'(n, N, K, \varepsilon, a, b, H, R, \eta),
\]
we obtain
\[
\sup_{x \in M} \frac{1}{\mu_f(B(x, R))} \int_{B(x, R)} (n - 1)H - \operatorname{Ric}_- \, d\mu_f \leq \delta_2(n, N, K, \varepsilon, a, b, H, R'(\eta), \eta).
\]
By Lemma 4.1 this completes the proof of the cases (1), (2) and the theorem. \( \square \)

We prepare the following lemma to provide a slightly weak result compared with Lemma 4.1.

**Lemma 4.3.** Let \( \delta_2 = \delta_2(n, N, K, \varepsilon, a, b, H, R, \eta) \) be given as in (4.4). Then \( \delta_2 \) is strictly increasing in \( \eta \in (0, \eta_*) \).

**Proof.** Set
\[
G(\eta) = \left( \frac{1}{2} + \frac{\pi}{\eta} \right) \int_{0}^{\eta/4b\sqrt{n}} \text{sn}_{cK}(t) dt.
\]

(4.6)
Note that the monotonicity of $\text{sn}_{cK}$. We observe that

\[
G'(\eta) = \frac{\tilde{D}}{4b\sqrt{H}} \left( \frac{1}{2} + \frac{\pi}{\eta} \right) \left( \text{sn}_{cK} \left( \frac{\eta\tilde{D}}{4b\sqrt{H}} \right) \right)^{\frac{1}{\varepsilon}} - \frac{\pi}{\eta^2} \int_0^{\eta\tilde{D}/4b\sqrt{H}} \text{sn}_{cK}(t)^{\frac{1}{\varepsilon}} dt
\]

\[
> \frac{\tilde{D}}{8b\sqrt{H}} \left( \text{sn}_{cK} \left( \frac{\eta\tilde{D}}{4b\sqrt{H}} \right) \right)^{\frac{1}{\varepsilon}}
\]

\[
= \frac{\tilde{D}}{8b\sqrt{H}} \left( \text{sn}_{cK} \left( \frac{\eta\tilde{D}}{4b\sqrt{H}} \right) \right)^{\frac{1}{\varepsilon}}
\]

Thus $G(\eta)$ is strictly increasing on $(0, \eta_*)$. Moreover

\[
(4.7) \quad 1 - \frac{\pi^2}{(\pi + \eta^2)^2}
\]

and, by Claim 2.7

\[
(4.8) \quad \frac{\text{sn}_{cK} \left( \frac{1}{\pi} \left( R - \eta\tilde{D} \right) \right)^{\frac{1}{\varepsilon}}}{\text{sn}_{cK} \left( \frac{1}{\pi} \left( R - \eta\tilde{D} \right) \right)^{\frac{1}{\varepsilon}}}
\]

are also strictly increasing with respect to $\eta$. Since $\delta_2$ was given by multiplication of $(4.6)$, $(4.7)$, $(4.8)$ and a positive constant independent of $\eta$, it follows that $\delta_2$ is strictly increasing in $\eta \in (0, \eta_*)$. \hfill $\square$

We take the limit of $(4.4)$ as $\eta \to \eta_*$, where $\eta_*$ is given in $(4.1)$. Then we have a diameter estimate of a weighted manifold.

**Corollary 4.4.** Let $n, N, \varepsilon, K, a, b, H, R$ and $\eta_*$ as in Lemma 4.1. There exists a positive constant $\delta_2(n, N, K, \varepsilon, a, b, H, R, \eta_*)$ such that if an $n$-dimensional complete weighted manifold $(M, g, \mu_f)$ satisfies the $(N, K, \varepsilon, a, b)$-condition and

\[
(4.9) \quad \sup_{p \in M} \frac{1}{\mu_f(B(p, R))} \int_{B(p, R)} \left( (n - 1)H - \text{Ric}_{N-} \right)_+ d\mu_f < \delta_2(n, N, K, \varepsilon, a, b, H, R, \eta_*)
\]

then $M$ is compact and $\text{diam}(M) < R$.

**Proof.** By Lemma 4.3 if $(4.9)$ holds, then there exists $\eta > 0$ such that $\eta < \eta_*(H, R, \tilde{D})$ satisfying $(4.2)$. Hence Lemma 4.1 implies

\[
\text{diam}(M) \leq \frac{\pi + \eta}{\sqrt{H}} \tilde{D}(n, N, \varepsilon, a, b) < R.
\]

\hfill $\square$

5. **Fundamental group under integral curvature bound and $\varepsilon$-range**

We shall show a finiteness of the fundamental group of $M$. Compared with Corollary 4.4, we need to assume a slightly strong condition about integral curvature bound.
Corollary 5.1. Let \((M, g, \mu_f)\) be an \(n\)-dimensional complete weighted manifold satisfying the \((N, K, \varepsilon, a, b)\)-condition. If there exists \(H > 0, R > \pi \tilde{D}(n, N, \varepsilon, a, b)/\sqrt{H}\) such that
\[
\sup_{p \in M} \frac{1}{\mu_f(B(p, R))} \int_{B(p, R)} \left((n-1)H - \text{Ric}_{N^-}\right) d\mu_f < \frac{a}{b} \frac{\nu_{K,b}(R)}{\nu_{K,a}(3R)} \delta_2,
\]
where \(\delta_2 = \delta_2(n, N, K, \varepsilon, a, b, H, R, \eta_\ast)\) is given in (4.3), then the universal cover of \(M\) is compact, and hence \(\pi_1(M)\) is a finite group.

Proof. We find that, by \(3R/a \geq R/b\),\(a/b \nu_{K,b}(R)/\nu_{K,a}(3R) < 1\).

Since \((M, g, \mu_f)\) satisfies the assumption of Corollary 4.4, we have \(\text{diam}(M) < R\).

We denote by \(k : \tilde{M} \to M\) the universal Riemannian covering. Set \(\tilde{g} = k^* g, \tilde{f} = f \circ k\).

Fix \(\tilde{x} \in \tilde{M}\). Let \(F\) be the fundamental domain of \(k\) that contains \(\tilde{x} \in \tilde{M}\). We find that
\[
\mu_f(M) = \tilde{\mu}_f(F), \quad \tilde{M} = \bigcup_{\alpha \in \Gamma} \alpha F,
\]
where \(\Gamma\) is the deck transformation group of \(\tilde{M}\). We set
\[
T = \inf \left\{ \#G_0 \left| B(\tilde{x}, R) \subset \bigcup_{\alpha \in G_0} \alpha F \right. \right\}.
\]

Fix \(w \in B(\tilde{x}, R) \cap \alpha F\) and \(z \in \alpha F\) for each \(\alpha \in G_0\). Since \(d(y, z) < 2R\) for all \(y \in \alpha F\), we see that
\[
d(\tilde{x}, z) \leq d(\tilde{x}, w) + d(w, z) < 3R.
\]

Hence this implies
\[
\bigcup_{i=1}^{T} \alpha_i F \subset B(\tilde{x}, 3R)
\]
and we find that
\[
(5.1) \quad T \cdot \mu_f(M) = T \cdot \tilde{\mu}_f(F) \leq \tilde{\mu}_f(B(\tilde{x}, 3R)).
\]

Then it follows from (5.1) that
\[
\frac{1}{\mu_f(B(\tilde{x}, R))} \int_{B(\tilde{x}, R)} \left((n-1)H - \text{Ric}_{N^-}\right) d\mu_f \\
\leq \frac{T}{\mu_f(B(\tilde{x}, R))} \int_{F} \left((n-1)H - \text{Ric}_{N^-}\right) d\mu_f \\
= \frac{T}{\mu_f(B(\tilde{x}, R))} \int_{M} \left((n-1)H - \text{Ric}_{N^-}\right) d\mu_f \\
\leq \frac{b}{a} \frac{\nu_{K,b}(3R)}{\nu_{K,a}(R)} \frac{T}{\mu_f(B(\tilde{x}, 3R))} \int_{M} \left((n-1)H - \text{Ric}_{N^-}\right) d\mu_f \\
\leq \frac{b}{a} \frac{\nu_{K,b}(3R)}{\nu_{K,a}(R)} \frac{1}{\mu_f(M)} \int_{M} \left((n-1)H - \text{Ric}_{N^-}\right) d\mu_f.
\]
Thus we observe that
\[
\sup_{x \in M} \frac{1}{\mu_f(B(x, R))} \int_{B(x, R)} ((n-1)H - \text{Ric}_{N-})_+ \, d\mu_f \\
\leq \frac{b}{a} \frac{v_{cK,b}(3R)}{v_{cK,a}(R) \mu_f(M)} \sup_{x \in M} \frac{1}{\mu_f(B(x, R))} \int_{B(x, R)} ((n-1)H - \text{Ric}_{N-})_+ \, d\mu_f \\
< \delta_2.
\]
Since \((\tilde{M}, \tilde{g}, \mu_f)\) satisfies the assumption of Corollary 5.1, \(\tilde{M}\) is compact and \(\pi_1(M)\) is a finite group.

Finally, we extend Corollary 5.1 to the cases that \(R \leq \pi \tilde{D}/\sqrt{H}\).

**Corollary 5.2.** Let \(n, N, \varepsilon, K, a, b, H\) and \(R\) as in Theorem 1.8. Set \(R' > 0\) satisfies \(R' > \pi \tilde{D}(n, \varepsilon, a, b)/\sqrt{H}\). There exists a positive constant \(\tilde{\delta}(n, K, \varepsilon, a, b, H, R, R')\) such that if an \(n\)-dimensional complete weighted manifold \((M, g, \mu_f)\) satisfies the \((\varepsilon, a, b, N, K)\)-condition and
\[
\sup_{x \in M} \frac{1}{\mu_f(B(x, R'))} \int_{B(x, R')} ((n-1)H - \text{Ric}_{N-})_+ \, d\mu_f < \tilde{\delta}(n, K, \varepsilon, a, b, H, R, R')
\]
then the universal cover of \(M\) is compact, and hence \(\pi_1(M)\) is a finite group.

**Proof.** Let \(R \leq \pi \tilde{D}/\sqrt{H}\). With the same argument in the proof of Theorem 1.8 we find that
\[
\sup_{x \in M} \frac{1}{\mu_f(B(x, R'))} \int_{B(x, R')} ((n-1)H - \text{Ric}_{N-})_+ \, d\mu_f \\
\leq \frac{b^2}{a^2} \frac{v_{cK,b}(2R + R')}{v_{cK,a}(R/2)} \frac{v_{cK,b}(R' + R)}{v_{cK,a}(R')} \sup_{x \in M} \frac{1}{\mu_f(B(x, R))} \int_{B(x, R)} ((n-1)H - \text{Ric}_{N-})_+ \, d\mu_f.
\]
If we assume that
\[
\sup_{x \in M} \frac{1}{\mu_f(B(x, R))} \int_{B(x, R)} ((n-1)H - \text{Ric}_{N-})_+ \, d\mu_f \\
< \frac{a^3}{b^2} \frac{v_{cK,a}(R/2)}{v_{cK,b}(2R + R)} \frac{v_{cK,a}(R')}{v_{cK,b}(R' + R)} \frac{v_{cK,b}(3R')}{v_{cK,a}(3R')} \delta_2(n, K, \varepsilon, a, b, H, R', \eta_*)
\]
holds, we obtain
\[
\sup_{x \in M} \frac{1}{\mu_f(B(x, R'))} \int_{B(x, R')} ((n-1)H - \text{Ric}_{N-})_+ \, d\mu_f \\
< \frac{a}{b} \frac{v_{cK,b}(R')}{v_{cK,a}(3R')} \delta_2(n, K, \varepsilon, a, b, H, R', \eta_*).
\]
Therefore, by Corollary 5.1 this completes the proof of the corollary. □

**References**

[1] J. Cheeger and T. H. Colding, *Lower bounds on Ricci curvature and the almost rigidity of warped products*, Ann. of Math. (2) **144** (1996), no. 1, 189–237.

[2] S. Hwang and S. Lee, *Integral curvature bounds and bounded diameter with Bakry-Émery Ricci tensor*, Differential Geom. Appl. **66** (2019), 42–51.

[3] , *Erratum to: “Integral curvature bounds and bounded diameter with Bakry-Émery Ricci tensor”* [Differ. Geom. Appl. **66** (2019) 42–51], Differential Geom. Appl. **70** (2020), 101627, 3.
[4] M. Jaramillo, **Fundamental groups of spaces with Bakry-Emery Ricci tensor bounded below**, J. Geom. Anal. **25** (2015), no. 3, 1828–1858.

[5] K. Kuwae and X.-D. Li, **New Laplacian comparison theorem and its applications to diffusion processes on Riemannian manifolds** (2020), arXiv 2001.00444.

[6] K. Kuwae and Y. Sakurai, **Rigidity phenomena on lower $N$-weighted Ricci curvature bounds with $\varepsilon$-range for nonsymmetric Laplacian**, Illinois J. Math. **65** (2021), no. 4, 847–868.

[7] Y. Lu, E. Minguzzi, and S.-i. Ohta, **Geometry of weighted Lorentz–Finsler manifolds I: singularity theorems**, Journal of the London Mathematical Society **104** (2021), 362–393.

[8] _____, **Comparison theorems on weighted Finsler manifolds and spacetimes with $\varepsilon$-range** (2020), arXiv 2007.00219.

[9] S. B. Myers, **Riemannian manifolds with positive mean curvature**, Duke Math. J. **8** (1941), 401–404.

[10] T. Sakai, **Riemannian geometry**, Translations of Mathematical Monographs, vol. 149, American Mathematical Society, Providence, RI, 1996. Translated from the 1992 Japanese original by the author.

[11] T. Shioya, **Metric measure geometry**, IRMA Lectures in Mathematics and Theoretical Physics, vol. 25, EMS Publishing House, Zürich, 2016.

[12] C. Sprouse, **Integral curvature bounds and bounded diameter**, Comm. Anal. Geom. **8** (2000), no. 3, 531–543.

[13] W. Wylie and D. Yeroshkin, **On the geometry of Riemannian manifolds with density** (2016), arXiv 1602.08000.

*Department of Mathematical Sciences, Tokyo Metropolitan University, Tokyo 192-0397, Japan. Email address: size14sphere@gmail.com*