On the stability of renormalizable expansions
in three-dimensional gravity

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Preliminary investigations are made for the stability of the $1/N$ expansion in three-dimensional gravity coupled to various matter fields, which are power-counting renormalizable. For unitary matters, a tachyonic pole appears in the spin-2 part of the leading graviton propagator, which implies the unstable flat space-time, unless the higher-derivative terms are introduced. As another possibility to avoid this spin-2 tachyon, we propose Einstein gravity coupled to non-unitary matters. It turns out that a tachyon appears in the spin-0 or -1 part for any linear gauges in this case, but it can be removed if non-minimally coupled scalars are included. We suggest an interesting model which may be stable and possess an ultraviolet fixed point.

PACS number: 04.60.Kz

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I. INTRODUCTION

One of the important issues in modern particle physics is to clarify quantum properties of gravitation. Although most of the recent interest has been focused on two dimensions, efforts have also been made toward constructing quantum gravity in higher dimensions, as statistical models of space-time and matters. Among them, numerical simulations of dynamical random lattices have given evidences of phase transitions in three and four dimensions [1]. The renormalization-group study of $2 + \epsilon$-dimensional gravity has revealed interesting phenomena, suggestive of higher-dimensional properties [2]. However, apart from the formulation as a Chern-Simons gauge theory [3] and the canonical formalism [4], conventional analytical studies of three- and four-dimensional Einstein gravity coupled to matters have to face the problem of non-renormalizability [5].

Perturbative non-renormalizability does not imply the non-existence of continuum field theory, if regarded as a low-energy effective theory of some fundamental one, such as superstring. It may be even important to extract universal low-energy properties of quantum gravity from such theories. To embody this program in the continuum approach, however, the realization of renormalizability is technically necessary for any actual prediction. To consider this problem, it is useful here to learn from other examples of non-renormalizable field theories. It is well known that there exist a class of weak-coupling non-renormalizable field theories which are rendered renormalizable by using resummation methods such as the $1/N$ expansion. We may list, for example, four-fermi and non-linear sigma models in three dimensions [6,7], in which ultraviolet divergences are softened by use of dressed propagators for auxiliary fields. Moreover, with no restriction to coupling strength, this
“non-perturbative” expansion yields reliable predictions such as of phase transitions and associated exponents even for a finite $N$ \[6–8\]. The application of such a method to gravity system is then appealing as an approach to explore quantum properties of gravitation.

The idea of resummation in quantum gravity is not new. Tomboulis first applied the $1/N$ expansion to four-dimensional gravity coupled to massless spinors, where $N$ is the number of fermion species \[9\]. With the inclusion of higher-derivative counterterms, renormalization properties and behaviors of graviton propagator were examined in the large-$N$ limit. Recently Kugo applied this expansion to three-dimensional Einstein gravity coupled to massive scalar fields \[12\]. In three dimensions Einstein gravity coupled to matters were shown to be formally (power-counting) renormalizable via the $1/N$ expansion without introducing higher-derivative terms, although the graviton propagator turned out to develop a tachyonic pole. After this work, no further investigation has been made along this approach.

To examine quantum nature of space-time, one must handle two kinds of quantum effects. One is from a self-interaction of gravity, and another is from quantum fluctuations of matters turning back into gravity through a matter-gravity coupling (back reaction). Roughly speaking, the present approach ($1/N$ expansion) is such a method that the second effect is taken into account to leading order, while the first one is incorporated with it in higher orders. This method may be regarded as complementary to the usual approach, in which the first effect is taken into account from the first. For example, in modern numerical works of quantum gravity one usually first sums over all simplicially decomposed configurations of space-time to explore the phase structure of pure-gravity system, and next takes a few kinds of matters into simulations to see whether their effects are small or not. In other words, the analyses are performed in the regime of pure gravity and its neighborhood
(gravity dominant regime). To the contrary, the $1/N$ expansion gives by definition a priority to matter fluctuations, and hence formally its prediction is relevant in the regime of many matters (matter dominant regime) although the actual applicability limit is not known a priori. We believe that the studies from both directions are useful for the full understanding of quantum gravity with couplings to general matters.

In this paper, we will report our preliminary results for the stability of renormalizable expansions in three-dimensional gravity coupled to various matter fields, as a first step toward constructing stable (tachyon-free) theories. In sect.II we first review the result of ref. [12] with brief discussions added. In sect.III we consider couplings to various unitary matter fields other than minimal scalars, such as spinors, U(1) gauge fields and non-minimal scalars, and examine whether we can have a tachyon-free graviton propagator for such unitary matters. We will see that in all cases above tachyonic poles show up in the leading-order graviton propagators. In these analyses we will find that the spin-2 part of matter one-loop corrections are quantized and proportional to their physical degrees of freedom, the situation being similar to that of conformal anomaly in two dimensions. In sect.IV we will then propose two independent possibilities to circumvent this spin-2 tachyon. The first one is to add higher-derivative terms for gravity. The second is to couple Einstein gravity to non-unitary matters. In the latter case, although the spin-2 part of the graviton propagator is free from a tachyon, it turns out to appear in the gauge-dependent (spin-0,1) part. In sect.V we will prove that this new tachyonic pole is inevitable for any choice of linear gauges, and indicate a class of tachyon-free models by including non-minimal scalars. Sect.VI is devoted to summarize our results, and to give the future prospects expected from them. In appendix A is listed the notation of the tensors which appear in the graviton propagator. Appendix B includes some technical formulas for the tensor
calculus. The Feynman rules for the calculations of matter one-loop vacuum polarizations are summarized in Table I.

We employ in this paper the conventions of the flat Minkowski metric

\[ \eta^{\mu\nu} = \text{diag}(1, -1, -1) \]  

(1)

and the Ricci tensor

\[ R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} = \partial_\nu \Gamma^\lambda_{\mu\lambda} - \partial_\lambda \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\tau\nu} \Gamma^\tau_{\mu\lambda} - \Gamma^\lambda_{\tau\lambda} \Gamma^\tau_{\mu\nu}. \]  

(2)

II. 1/N EXPANSION

A. Gravity coupled to scalar fields

To present the prototype of the 1/N expansion in three-dimensional gravity, we review here the case of the minimal coupling to N massive scalar fields \( \phi_i \) \((i = 1, \ldots, N)\) \([12]\). The lagrangian of the system is given by

\[ \mathcal{L} = \frac{1}{\kappa^2} \sqrt{g} R + \mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{FP+GF}}, \]

(3)

\[ \mathcal{L}_{\text{matter}} = \mathcal{L}_{\text{scalar}} \equiv \frac{1}{2} \sqrt{g} \left( g^{\mu\nu} \partial_\mu \phi_i \partial_\nu \phi_i - m_i^2 \phi_i^2 \right), \]

(4)

where \( \mathcal{L}_{\text{FP+GF}} \) is the gauge-fixing and the FP ghost terms associated with general coordinate gauge symmetry. The construction of the 1/N expansion in this system is most conveniently achieved in a diagrammatic way with the following non-local action:

\[ \Gamma = N \left[ \int d^3 x \left( \frac{1}{\kappa^2} \sqrt{g} R + \frac{1}{2} \sqrt{g} \left( g^{\mu\nu} \partial_\mu \phi_i \partial_\nu \phi_i - m_i^2 \phi_i^2 \right) + \mathcal{L}_{\text{FP+GF}} \right) + i \frac{1}{2} \log \text{Det}(-\Delta(g_{\mu\nu}, m)) + \int d^3 x \mathcal{L}_{\text{count}}^{(0)} \right], \]

(5)

where \( \Delta(g_{\mu\nu}, m) = \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu + \sqrt{g} m^2 \). \( \mathcal{L}_{\text{count}}^{(0)} \) denotes the counterterms for renormalization. In \([3]\) we have rescaled \( \kappa^2 \) and \( \phi \) as \( \kappa^2 \rightarrow N^{-1} \kappa^2 \) and
\( \phi \rightarrow N^{\frac{1}{2}} \phi \), respectively. The ghosts and the gauge parameters are rescaled similarly. The loop expansion based on the Feynman rules read off from (3) provides the expansion of the full effective action in a power series of \( 1/N \) instead of \( \kappa^2 \). To this rules must be added the provision that when one calculates to higher orders in \( 1/N \), one omits the closed scalar-loop graphs, since they are already taken into account in the leading-order graviton propagator and vertices. Indeed, the leading graviton propagator already includes the contributions from an arbitrary number of scalar one-loop self-energy insertions, and the leading vertices also contain one-loop corrections (fig.1). Thus, in essence, the \( 1/N \) expansion is a gauge-invariant rearrangement of the Feynman diagrams.

We now calculate the leading graviton propagator. It is convenient to define the fluctuation around the flat metric as follows

\[
\tilde{g}^{\mu\nu} \equiv \sqrt{g} g^{\mu\nu} = \eta^{\mu\nu} + \kappa \tilde{h}^{\mu\nu},
\]

and

\[
g^{\mu\nu} = \eta^{\mu\nu} + \kappa h^{\mu\nu}.
\]

\( \tilde{h}^{\mu\nu} \) and \( h^{\mu\nu} \) are related with each other by

\[
\tilde{h}^{\mu\nu} = \frac{1}{2} \eta^{\mu\nu} h - h^{\mu\nu} + O(\kappa),
\]

\[
h^{\mu\nu} = \eta^{\mu\nu} \tilde{h} - \tilde{h}^{\mu\nu} + O(\kappa),
\]

where \( h = h^{\mu\mu} \) and \( \tilde{h} = \tilde{h}^{\mu\mu} \). Their indices are raised and lowered by the flat metric \( \eta^{\mu\nu} \). In the following we will take \( \tilde{h}^{\mu\nu} \) as our basic quantum field around the flat space-time. In terms of \( \tilde{h}^{\mu\nu} \), \( \mathcal{L}_{\text{scalar}} \) is expanded as

\[
\mathcal{L}_{\text{scalar}} = \frac{1}{2} \left( \eta^{\mu\nu} \partial_{\mu} \phi_i \partial_{\nu} \phi_i - m^2 \phi_i^2 \right) + \frac{\kappa}{2} \left( \tilde{h}^{\mu\nu} \partial_{\mu} \phi_i \partial_{\nu} \phi_i - m^2 \tilde{h} \phi_i^2 \right) + \kappa^2 m^2 \frac{2}{4} \left( \tilde{h}^{\mu\nu} \tilde{h}_{\mu\nu} - \tilde{h}^2 \right) \phi_i^2 + O(\kappa^3).
\]
All the scalar one-loop diagrams (fig.2) have degrees of divergence three. By using the Pauli-Villars-Gupta regularization, the counterterms are found to be [12]

\[
\mathcal{L}^{(0)}_{\text{count}} = -\frac{1}{24\pi} \left( (2 - \sqrt{2}) \Lambda^3 - \frac{3}{\sqrt{2}} \Lambda m^2 + 2m^3 \right) \sqrt{g} - \frac{1}{24\pi} \left( (\sqrt{2} - 1) \Lambda - m \right) \sqrt{gR}
\]

(11)

with a large regulator mass \( \Lambda \). The coefficients in (11) are so determined that the renormalized graviton \( \tilde{h}^{\mu\nu} \) two-point function \( \Gamma^{(2)}_{\mu\rho,\nu\sigma}(p) \) should satisfy the following renormalization conditions:

\[
\Gamma^{(2)}_{\mu\rho,\nu\sigma}(p) \bigg|_{p=0} = 0,
\]

(12)

\[
\frac{\partial \Gamma^{(2)}_{\mu\rho,\nu\sigma}}{\partial p^2}(p) \bigg|_{p=0} = \frac{1}{2} \left( P^{(2)}_{\mu\rho,\nu\sigma} - Q_{\mu\rho,\nu\sigma} \right),
\]

(13)

where \( P^{(2)}_{\mu\rho,\nu\sigma} \) denotes the physical spin-2 projection operator

\[
P^{(2)}_{\mu\rho,\nu\sigma} = \frac{1}{2} (\theta_{\mu\nu} \theta_{\rho\sigma} + \theta_{\mu\sigma} \theta_{\nu\rho} - \theta_{\mu\rho} \theta_{\nu\sigma}),
\]

(14)

\[
\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{p_{\mu} p_{\nu}}{p^2},
\]

(15)

and the explicit form of the other remaining gauge-dependent parts \( Q_{\mu\rho,\nu\sigma} \) is presented in appendix A. The right-hand side of (13) is just a tree-level contribution from \( \sqrt{gR} \), and the local gauge symmetry is not yet fixed. The general covariance ensures that the choice of counterterms (11) also makes other diagrams in fig.2 finite, and in particular the tadpoles vanish. Since there are no infrared divergences and the scalar mass does not affect the ultraviolet behavior of \( \Gamma^{(2)}_{\mu\rho,\nu\sigma}(p) \), we will henceforth set \( m = 0 \) for simplicity. In this case the renormalized vacuum polarization tensor \( \Pi^{\text{scalar}}_{\mu\rho,\nu\sigma} \) is calculated to be [12]

\[
\Pi^{\text{scalar}}_{\mu\rho,\nu\sigma} = \frac{\kappa^2 (-p^2)^{\frac{3}{2}}}{512} (P^{(2)}_{\mu\rho,\nu\sigma} + 2Q_{\mu\rho,\nu\sigma}).
\]

(16)
The graviton propagator thus reads

$$< \tilde{h}_{\mu\rho}(p)\tilde{h}_{\nu\sigma}(-p) > = i \left[ \frac{p^2}{2} \left( P^{(2)}_{\mu\rho,\nu\sigma} - Q_{\mu\rho,\nu\sigma} \right) + \frac{1}{2} \Pi^{\text{scalar}}_{\mu\rho,\nu\sigma} + (\text{gauge fixing}) \right]^{-1} \ . \quad (17)$$

The tensor structure of $\Pi^{\text{scalar}}_{\mu\rho,\nu\sigma}$ is expressed in terms of $P^{(2)}_{\mu\rho,\nu\sigma}$ and $Q_{\mu\rho,\nu\sigma}$ only since the matter integration is performed in a gauge-invariant way. If we take the harmonic gauge $\partial_{\mu} \tilde{h}^{\mu\nu} = 0$, we have

$$< \tilde{h}_{\mu\rho}(p)\tilde{h}_{\nu\sigma}(-p) > = i \left[ \frac{2 P^{(2)}_{\mu\rho,\nu\sigma}}{p^2 + \frac{\kappa^2}{312}(-p^2)^2} + \frac{2 P^{(0-s)}_{\mu\rho,\nu\sigma}}{-p^2 + \frac{\kappa^2}{256}(-p^2)^2} \right] \ . \quad (18)$$

The first term in (18) is gauge-independent, while the second term depends on the gauge choice. The definition of $P^{(0-s)}_{\mu\rho,\nu\sigma}$ is given in appendix A.

In view of (18), the graviton propagator behaves like $p^{-3}_E$ ($p_E \equiv \sqrt{-p^2}$) for a large momentum. Also, all $n$-point vertices ($n \geq 3$) scale like $p^{-3}_E$ in the ultraviolet regime, as expected from Weinberg’s theorem [13]. One may then easily see that this expansion is power-counting renormalizable. Furthermore, the standard argument in gauge theories may formally apply to prove the all-order renormalizability [12]. Namely, one may first derive the Ward-Takahashi (Slavnov-Taylor) identity as a consequence of the BRS invariance. It leads to a renormalization equation, which governs the possible structure of the divergent part of the proper vertices. Then, by induction, it is shown that the solution of the equation is a BRS-invariant local functional of dimension three or less. In this way the $1/N$ expansion is claimed to be renormalizable [12].

**B. A spin-2 tachyonic pole in the graviton propagator**

The above proof of renormalizability is formal and does not actually work, since the graviton propagator (18) develops a spin-2 tachyonic pole in $p_E$ at $p_E = 512\kappa^{-2}$ which renders the higher-order calculations ill-defined.
The presence of a tachyonic pole at the Planck scale may be seen as a reflection of the breakdown of the $1/N$ approximation based on (3) at that scale. To see this let us calculate the renormalization-group $\beta$ function. We define the $\beta$ function by setting the following renormalization condition:

$$\left. \frac{\partial \Gamma^{(2)}_{\mu\rho,\nu\sigma}}{\partial p^2} (p) \right|_{p^2 = \mu^2} = \frac{1}{2} \left( P^{(2)}_{\mu\rho,\nu\sigma} + \cdots \right),$$

by which we replace the condition (13). Here $\mu$ is an arbitrary renormalization mass scale, and the ellipsis denotes the terms proportional to $Q_{\mu\rho,\nu\sigma}$. In this case the counterterms are given by

$$L^{(0)}_{\text{count}} = - \frac{1}{24\pi} (2 - \sqrt{2}) \Lambda^3 \sqrt{g} \left( \frac{\mu}{1024} \right) \sqrt{g} R.$$

The $\beta$ function is obtained from the equations

$$\mu \frac{\partial}{\partial \mu} \kappa_0^2 = 0,$$

$$\frac{1}{\kappa_0^2} = \frac{\mu}{\kappa^2} + \left\{ - \frac{1}{24\pi} (\sqrt{2} - 1) \Lambda + \frac{3\mu}{1024} \right\},$$

where $\kappa_0$ is the bare coupling constant. They provide

$$\beta(\kappa^2) \equiv \mu \frac{\partial}{\partial \mu} \kappa^2 = \kappa^2 + \frac{3}{1024} \kappa^4.$$  

This indicates that the theory is asymptotically non-free and has no ultraviolet fixed points in the large $N$ limit. By integrating the leading-order $\beta$ function (24), we obtain the following running coupling constant $\kappa^2(\mu)$:

$$\frac{\kappa^2(\mu)}{\kappa^2(\mu_0) + \frac{1024}{3}} \left/ \frac{\kappa^2(\mu_0)}{\kappa^2(\mu_0) + \frac{1024}{3}} \right. = \frac{\mu}{\mu_0}$$

or

$$\kappa^2(\mu) = \frac{1024}{3} \left( A \mu^{-1} - 1 \right).$$
where $\mu_0$ is an integration constant, and $A = \frac{\kappa^2(\mu_0) + \frac{1024}{\kappa^2(\mu_0)}}{\kappa^2(\mu_0)} \mu_0$. We see from (23) that $\kappa^2(\mu)$ diverges if $\mu$ approaches to some finite value $A$. Hence the expansion in terms of the dressed propagator (18) is not reliable in the ultra-violet regime. Therefore it is plausible to consider that the appearance of the tachyonic pole in (18) is a reflection of the breakdown of this expansion in the present system. The situation is similar to those in other well-known non-asymptotically-free theories without fixed points, such as four-dimensional QED and $\phi^4$ theories [14].

To understand the situation it would be worth comparing it with three-dimensional QCD (QCD$_3$). QCD$_3$ is a super-renormalizable asymptotically free theory, and much reliance cannot be placed on the loop approximation in the low-momentum region. In this case the dressed ghost and gluon propagators also exhibit tachyonic poles in the infrared regime [15], which implies the breakdown of the validity of the dressed propagators. In this sense the $\mu_0$-dependent constant $A$ above may be understood as an analogue of $\Lambda_{\text{QCD}}$ parameter.

III. INCLUSION OF OTHER UNITARY MATTER FIELDS

In sect.II we have seen that the $1/N$ expansion of three-dimensional gravity coupled to $N$ scalar fields is unstable due to the appearance of the tachyonic mode in the graviton propagator. In this section we consider the cases of other unitary matter fields, searching for a stable and renormalizable expansion of three-dimensional gravity. The Feynman rules for the calculations in this section are listed in Table.1.
A. Massless spinors

Let us first consider the coupling with massless spinors. The matter lagrangian is given by

$$L_{\text{spinor}} = \frac{ie}{2} \epsilon_{a}^{\mu} (\bar{\psi} \gamma^{a} \partial_{\mu} \psi - \partial_{\mu} \bar{\psi} \gamma^{a} \psi + \frac{i}{2} \bar{\psi} \omega_{\mu bc} \epsilon^{abc} \psi),$$

(26)

where $\psi$ is an $N$-plet spinor and we omit their indices for a simple notation.

The dreibein $e_{a}^{\mu}$ and the spin-connection $\omega_{\mu}^{ab}$ are defined by

$$g_{\mu\nu} = e_{a}^{\mu} e_{a}^{\nu} = e_{a}^{\mu} e_{b}^{\nu} \eta_{ab} \quad e = \det e_{a}^{\mu}$$

(27)

and

$$\omega_{\mu}^{ab} = \frac{1}{2} e^{av} (\partial_{\mu} e^{b}_{v} - \partial_{v} e^{b}_{\mu}) + \frac{1}{4} e^{av} e^{bc} (\partial_{\lambda} e^{c}_{v} - \partial_{v} e^{c}_{\lambda}) e_{\mu} - (a \leftrightarrow b).$$

(28)

Replacing $L_{\text{matter}}$ in (3) by (26) and integrating over $\psi$ fields, we have the spinor one-loop effective action

$$\Gamma = N \left[ \int d^{3}x \left( \frac{1}{\kappa^{2}} \sqrt{g} R + \frac{e}{2} \epsilon_{a}^{\mu} (\bar{\psi} \gamma^{a} \partial_{\mu} \psi - \partial_{\mu} \bar{\psi} \gamma^{a} \psi + \frac{i}{2} \bar{\psi} \omega_{\mu bc} \epsilon^{abc} \psi) + L_{\text{FP+GF}} \right) 
- i \log \text{Det}(-D(g_{\mu\nu})) + \int d^{3}x L_{\text{count}}^{(0)} \right],$$

(29)

where we have made appropriate rescaling of $\kappa$, $\psi$, ghosts and gauge parameters as in sect.II. $D(g_{\mu\nu})$ denotes the kernel of the spinor bilinear in $L_{\text{spinor}}$.

To obtain the functional determinant we expand the dreibein as

$$e_{a}^{\mu} = \delta_{a}^{\mu} + \kappa f_{a}^{\mu}. $$

(30)

In terms of $f_{a}^{\mu}$, $L_{\text{spinor}}$ is expanded as follows:

$$L_{\text{spinor}} = i \left( \bar{\psi} \gamma^{a} \partial_{a} \psi - \partial_{a} \bar{\psi} \gamma^{a} \psi \right) 
+ \frac{i}{2} \kappa (-f_{a}^{\mu} + f_{b}^{\mu}) (\bar{\psi} \gamma^{a} \partial_{a} \psi - \partial_{a} \bar{\psi} \gamma^{a} \psi) 
+ \text{O}(\kappa^{2}) + \text{(functionals of the antisymmetric part of } f_{a}^{\mu} \text{).}$$

(31)
with \( f = f^a_a \). It suffices to consider only the symmetric part of \( f^a_\mu \), which is related to \( h_{\mu\nu} \) by

\[
f^a_\mu = \frac{1}{2} h^a_{\mu}.
\]

For actual calculations it is more convenient to work in terms of the linear combination

\[
\tilde{f}^a_\mu \equiv -f^a_\mu + f^a_\mu \delta_a^\mu.
\]

We will calculate the spinor one-loop diagram (fig.3) by the dimensional regularization. By this regularization we obtain a finite result without any subtractions because of the odd-dimensionality. Since the spinors are massless, we have no dimensionful constants except for \( \kappa \), and so the renormalization conditions (12) and (13) are automatically satisfied. It is known that in three dimensions massive spinors coupled to gravity induces gravitational Chern-Simons terms if the dimensional regularization is used [16]. One would also obtain those parity-odd terms if one employed the Pauli-Villars regularization for massless spinors, and the graviton becomes massive topologically [17]. In this paper we will make use of the dimensional regularization for massless spinors so that no Chern-Simons terms are induced, and will not consider the effect of those terms.

A straightforward spinor one-loop calculation provides the following graviton-graviton vacuum polarization tensor:

\[
\Pi^{\text{spinor}}_{\mu\rho,\nu\sigma} = \frac{\kappa^2 (-p^2)^2}{256} P^{(2)}_{\mu\rho,\nu\sigma},
\]

where, by using the formulas in appendix B, the two external graviton fields of (34) have been transformed from \( \tilde{f}^a_\mu \) to \( \tilde{h}^{\mu\nu} \) so that we may compare this

\[1\] The expressions of \( \Pi^{\text{scalar}}_{\mu\rho,\nu\sigma} \) and \( \Pi^{\text{spinor}}_{\mu\rho,\nu\sigma} \) may be found in refs. [18,19].
with (16). Note that in this case are induced no terms proportional to $Q_{\mu \rho, \nu \sigma}$. We see that the sign of $(-p^2)^{\frac{3}{2}} F^{(2)}_{\mu \rho, \nu \sigma}$ is plus, meaning the presence of a spin-2 tachyon.²

**B. U(1) gauge fields**

We next consider the coupling to the $(U(1))^N$ gauge fields, i.e. $N$ independent abelian gauge fields. The matter lagrangians are the following:

\[
\mathcal{L}_{U(1)} = -\frac{1}{4} \sqrt{g} g^{\mu \nu} g^{\lambda \sigma} F_{\mu \lambda} F_{\nu \sigma} + \mathcal{L}_{U(1)GF},
\]

\[
\mathcal{L}_{U(1)GF} = -\frac{1}{2\alpha} \sqrt{g} (\nabla^\mu A_\mu)^2,
\]

\[
\mathcal{L}_{U(1)FP} = \sqrt{g} \tau \nabla^\mu \partial_\mu c,
\]

where $A_\mu = A^{(i)}_\mu$, $c = c^{(i)}$ and $\tau = \tau^{(i)}$ ($i = 1, \ldots, N$). $\alpha$ is a gauge parameter. If the space-time is flat, U(1) FP ghosts are free and completely decoupled. In our case, however, the ghosts do interact with gravitational field and cannot be neglected.

Repeating a similar procedure, we have only to calculate two diagrams shown in fig.4 to examine the pole structure of the dressed graviton propagator. In this case we expand $\mathcal{L}_{U(1)}$ in terms of $h_{\mu \nu}$:

\[
\mathcal{L}_{U(1)} = \frac{1}{2} A^\lambda (\eta_{\mu \lambda} \Box - \partial_\mu \partial_\lambda) A^\mu - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 \\
- \kappa \hat{h}_{\alpha \beta} \left[ \frac{1}{2} \eta^{\alpha \beta} \partial_\mu A_\lambda (\eta^{\lambda \sigma} \partial^\mu - \eta^{\mu \sigma} \partial^\lambda) A_\sigma \\
- \partial^\alpha A_\lambda (\eta^{\lambda \sigma} \partial^\beta - \eta^{\beta \sigma} \partial^\lambda) A_\sigma \\
+ \partial_\mu A_\lambda (\eta^{\lambda \alpha} \partial^\beta - \eta^{\beta \alpha} \partial^\lambda) A_\sigma \\
- \frac{\kappa}{2} \frac{1}{2\alpha} \left( h \partial^\lambda A_\lambda \partial^\sigma A_\sigma + 2 \partial^\lambda h \cdot A_\lambda \partial^\sigma A_\sigma \right)
\]

² We were informed by T. Kugo that he had also obtained this fact [20].
The most convenient gauge-choice is the Feynman gauge $\alpha = 1$. After some straightforward calculations we obtain

$$\Pi_{\mu\rho,\nu\sigma}^{U(1)\text{gauge}} = + \frac{3\kappa^2 (-p^2)^{3/2}}{512} (P_{\mu\rho,\nu\sigma}^{(2)} + 2Q_{\mu\rho,\nu\sigma}).$$  \hfill (39)

Again, this expression is that for the external $\tilde{h}^{\mu\nu}$ fields. As for the ghosts, $\mathcal{L}_{U(1)\text{FP}}$ is conveniently expanded by $\tilde{h}^{\mu\nu}$:

$$\mathcal{L}_{U(1)\text{FP}} = i\bar{c} \gamma^\mu \gamma^\nu c - \kappa \tilde{h}^{\mu\rho} i\partial_\rho \bar{c} \partial_\mu c + O(\kappa^2).$$  \hfill (40)

A similar calculation gives

$$\Pi_{\mu\rho,\nu\sigma}^{U(1)\text{FP}} = - \frac{2\kappa^2 (-p^2)^{3/2}}{512} (P_{\mu\rho,\nu\sigma}^{(2)} + 2Q_{\mu\rho,\nu\sigma}).$$  \hfill (41)

Combining (39) and (41), we have the total vacuum polarization tensor

$$\Pi_{\mu\rho,\nu\sigma}^{U(1)\text{total}} = + \frac{\kappa^2 (-p^2)^{3/2}}{512} (P_{\mu\rho,\nu\sigma}^{(2)} + 2Q_{\mu\rho,\nu\sigma}).$$  \hfill (42)

We find that the spin-2 part is the same as that of the scalar case (16).

\section*{C. Non-minimal scalar fields}

Finally, let us consider $N$ scalar fields with a non-minimal coupling to gravity. The lagrangian is given by

$$\mathcal{L}_{\text{non-minimal}} = \frac{1}{2} \sqrt{g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \lambda \sqrt{g} R \phi^2$$

$$= - \frac{1}{2} \phi \Box \phi + \kappa \left[ \frac{1}{2} \tilde{h}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \lambda \left( \Box \tilde{h} + \partial_\mu \partial_\nu \tilde{h}^{\mu\nu} \right) \phi^2 \right]$$

$$+ O(\kappa^2),$$  \hfill (43)

where $\lambda$ is a free real parameter. In this case the vacuum polarization tensor is modified from (16) as follows:
\[ \Pi_{\mu\nu,\rho\sigma}^\text{non-minimal} = + \frac{\kappa^2(-p^2)^{\frac{3}{2}}}{512} \left( P_{\mu\nu,\rho\sigma}^{(2)} + 2(1 + 16\lambda)^2 Q_{\mu\nu,\rho\sigma} \right). \] 

Comparing (44) with (16), we see that the difference is only in the \( Q_{\mu\rho,\nu\sigma} \) part. Therefore the inclusion of a non-minimal coupling can not cure the spin-2 tachyon disease. However, we will discuss in subsect. V.B its special role in avoiding the tachyon in the gauge-dependent (spin-0 or -1) part of the graviton propagator.

D. Remarks

We have seen that the 1/N expansion of three-dimensional Einstein gravity coupled to realistic matters such as (non-minimal) scalars, spinors and U(1) gauge fields, is unstable; in all cases above the dressed graviton propagators possess tachyonic poles in the spin-2 part. The vacuum polarization tensor for gravitons is proportional to \((-p^2)^{\frac{3}{2}}\) and the coefficient of \( P_{\mu\nu,\rho\sigma}^{(2)} \) is quantized to \( \kappa^2(-p^2)^{\frac{3}{2}}/512 \) times the physical degrees of freedom. For instance, in the case of U(1) gauge field the contributions to \( \Pi_{\mu\nu,\rho\sigma}^{U(1)\text{total}} \) of photon itself is three in this unit, but the FP ghosts cancel two unphysical contributions from longitudinal and scalar modes, and that leaves precisely one: the physical degrees of freedom of photon in three dimensions. On the other hand, as we have seen in the previous section, a tachyonic pole in the dressed propagator necessarily follows from such positive contributions in this unit.

Concluding sects. II and III, in the 1/N expansion of three-dimensional Einstein gravity coupled to N-plet unitary matters (the matters with positive degrees of freedom) a tachyonic pole exists in the spin-2 mode of the leading graviton propagator. Physically, this implies that the flat space-time \( g_{\mu\nu} = \eta_{\mu\nu} \), having been taken as the classical vacuum of space-time \( \Box \), is
quantum-mechanically unstable if the number \( N \) of unitary matters is very large. Theoretically, the presence of a tachyon prevents us to include graviton-loop corrections consistently.

As mentioned in subsect.II.B, the presence of a tachyon in the spin-2 part of the graviton propagator has a close correspondence with the absence of an ultraviolet fixed point for \( \kappa \). Note here that in the \((2 + \epsilon)\)-dimensional gravity the one-loop \( \beta \) function \((16\pi G = \kappa^2)\)

\[
\beta(G) = \epsilon G - \frac{25 - c}{24\pi} G^2, \quad (45)
\]

exhibits an ultraviolet fixed point only when the central charge \( c \) of matters satisfies \( c < 25 \). This fact may suggest that in three dimensions also, the existence of the non-trivial ultraviolet fixed point provides an upper limit for the number of physical degrees of freedom for matters. This conjecture is at least consistent with our results that three-dimensional gravity coupled to infinite number of unitary matters has no ultraviolet fixed point. Thus, if we adhere to Einstein gravity \( \sqrt{g}R \) and want to stabilize the flat space-time in the present \( 1/N \) scheme, it would be necessary to consider in some sense a coupling to matters with negative degrees of freedom; otherwise we should modify the gravity part of the lagrangian.

## IV. REMOVING TACHYON IN THE SPIN-2 PART

Based on the above results, we will discuss in sects. IV and V what could be the possible modification or extension of models to stabilize the flat space-time, i.e. to remove tachyons in the graviton propagator. In this section we indicate two possibilities to circumvent the presence of a tachyon in the \textit{spin-2 part} of the propagator.
A. Higher-derivative term for gravity

The first possibility is the generalization of gravity by adding higher-derivative terms such as $\beta R^2$ with matter contents unchanged. In the history of quantum gravity, higher-derivative gravity has sometimes been rejected by the reason that it may not keep unitarity. However, in the broad viewpoint that we may look at gravity as a certain low-energy effective field theory, this point need not be seriously taken. The unitarity problem should be addressed in an original complete theory such as superstrings, valid in all energy scales. In the low-energy renormalizable theory, however, we can still legitimately argue infrared properties such as a phase structure of space-time.

Due to this generalization the pole structure of a dressed graviton propagator shall be changed and a possibility arises to have a tachyon-free propagator by choosing a suitable value for a new parameter $\beta$. This generalization may indeed be a most natural direction for treating gravity coupled to unitary matters. The graviton fluctuations in higher orders can be consistently incorporated to search for an ultraviolet fixed point. Although its absence to leading order remains, it is very interesting to see to which direction the graviton-loop effect works in the renormalization of $\kappa$.

Although this approach may in fact lead to a tachyon-free theory, we shall instead lose the theoretically fascinating possibility that quantum gravity in three dimensions could be controllable by a single parameter $\kappa$, namely that Einstein gravity itself may be renormalizable. In this paper we mainly stick to Einstein gravity $\sqrt{g}R$, and will not pursue the higher-derivative theory which we would like to consider in a future.
B. Non-unitary matters

The second possibility to avoid a spin-2 tachyon is to couple Einstein gravity to non-unitary matters. If the conjecture presented in subsect.III.D is correct, an ultraviolet fixed point should exist in the \( N_{\text{matter}} \to -\infty \) limit. This limit is analogous to the \( c \to -\infty \) limit in the two- and \((2+\epsilon)\)-dimensional gravity. In two dimensions the \( c \to -\infty \) limit is indeed known as the semi-classical limit of gravity \(^{22}\) and its knowledge has been useful when the correct exact solution is chosen from two branches \(^{23}\)\(^3\). In our three-dimensional model this formal limit can be realized by coupling gravity to \( N \)-plet ghost matters and taking the \( N \to \infty \) limit. The physical degrees of freedom of matters are negative, and from our preceding analyses follows that the tachyonic pole in the spin-2 part of the graviton propagator should disappear. Although the positivity of the Hilbert space is lost, this model may actually serve as an interesting theoretical model that could also be simulated numerically.

As an explicit example, consider the minimal coupling of gravity to mass-less fermionic scalar fields, described by the lagrangian

\[
L_{\text{matter}} = \sqrt{g} g^{\mu \nu} \partial_{\mu} \bar{c}_i \partial_{\nu} c_i, \tag{46}
\]

where \( c_i \) and \( \bar{c}_i \) are the N-plet anti-commuting scalar and their conjugate fields. Integrating over these non-unitary (ghost) matters one can formulate the \( 1/N \) expansion of the system \( L \) in eq.(3) where \( L_{\text{matter}} \) is replaced by

\(^3\)As one of the other interesting examples of two-dimensional gravity coupled to non-unitary matters, a numerical simulation has recently been performed for \( c = -2 \) and the fractal scaling has been clearly observed \(^{24}\).
If we use the dimensional regularization and normalize the leading graviton two-point function at $p^2 = 0$ by eqs. (12) and (13), the cosmological constant is renormalized to zero and we get $(\kappa_0^2 = \kappa^2 \mu^{-1})$

$$\Gamma^{(2)}_{\mu\rho,\nu\sigma}(p) = \frac{p^2}{2} \left[ 1 + \frac{\kappa^2 \mu^{-1}}{512} \left( -p^2 \right)^{\frac{1}{2}} \right] P^{(2)}_{\mu\rho,\nu\sigma} + \cdots .$$

Due to the square root term $(-p^2)^{1/2}$, $p^2 = 0$ is a branch point and $\Gamma^{(2)}_{\mu\rho,\nu\sigma}(p)$ is a double-valued function on the complex $p^2$ plane made up of two Riemann sheets. Apart from the expected zero at $p^2 = 0$, no other zeros are in the first sheet. No tachyonic poles then exist in the spin-2 part of the propagator. If we instead normalize $\Gamma^{(2)}_{\mu\rho,\nu\sigma}(p)$ by introducing a finite renormalization mass scale $\mu$ like in (19), we shall obtain the renormalization-group $\beta$ function exhibiting the non-trivial ultraviolet fixed point for $\kappa^2(\mu)\frac{4}{3}$, which would suggest the existence of two gravitational phases. This issue seems very interesting, but includes a non-trivial subtlety with respect to the metric $g_{\mu\nu}$ redefinition ambiguity. The detailed analyses will be given elsewhere [25].

V. REMOVING TACHYON IN THE GAUGE-DEPENDENT (SPIN-1 OR -0) PART

As we have argued above, even if we keep Einstein gravity $\sqrt{g}R$, there exists a case where a tachyon can be avoided in the spin-2 part, i.e. the gauge-invariant part of the graviton propagator. Although the gauge-dependent piece of the propagator indicated by the ellipsis above, i.e. the part dependent on the choice of gauge fixing, can not affect any gauge-invariant quantity, the

\footnote{More correctly speaking, a fixed point should exist for the original coupling constant ($=\kappa^2 N$ for the present $\kappa^2$), as is usual for standard analyses using $1/N$ expansion.}
presence of a tachyonic pole in the gauge part also spoils the calculability to higher orders. It is therefore important to study whether one can avoid tachyonic poles in the gauge-dependent part as well. In this section we first investigate explicitly the pole structure of the gauge-dependent part of the propagator in the second possibility (sect.II B). From all the results we will finally indicate the models which are expected to be completely free from tachyons.

A. Tachyon in the gauge-dependent part

On the assumption that we have chosen the second case (sect.II B) so that a tachyon does not exist in the spin-2 part of the graviton propagator, we will examine here whether we may take any gauge-fixing such that the graviton propagator would not possess any spin-0 or -1 tachyonic poles, either. For this purpose, let us explicitly write the FP ghost and the gauge-fixing lagrangians for the general coordinate gauge invariance

\[ \mathcal{L}_{\text{FP}} = i \mathcal{C}_\mu \partial_\nu D^\mu_\rho \mathcal{O}_\rho, \]

\[ \mathcal{L}_{\text{GF}} = - \frac{1}{2\xi} g_\mu_\nu \mathcal{E}(\square) \left( \partial_\lambda \tilde{g}^{\mu_\lambda} - \zeta \partial^\mu \sqrt{g} \right) \cdot \mathcal{E}(\square) \left( \partial_\rho \tilde{g}^{\mu_\rho} - \zeta \partial^\nu \sqrt{g} \right), \]

\[ D^\mu_\rho = \tilde{g}^{\mu_\sigma} \delta^\nu_\rho \partial_\sigma + \tilde{g}^{\nu_\sigma} \delta^\mu_\rho \partial_\sigma - \tilde{g}^{\mu_\nu} \partial_\rho - (\partial_\rho \tilde{g}^{\mu_\nu}), \]

where we have adopted the following linear gauge-fixing condition:

\[ \mathcal{E}(\square) \left( \partial_\nu \tilde{g}^{\mu_\nu} - \zeta \partial^\mu \sqrt{g} \right) + \frac{1}{2} \xi B^\mu = 0 \]  

with some function \( \mathcal{E}(\square) \) of \( \square \) and an auxiliary field \( B^\mu \). If one takes \( \mathcal{E}(\square) = \text{const.}, \zeta = 0 \) and \( \xi \to 0 \), the gauge is reduced to the familiar de Donder gauge. We will take \( \mathcal{E}(\square) \) to be a non-constant function in order to improve the high-momentum behavior of the gauge-dependent part of the propagator. For this purpose, let us explicitly write the FP ghost and the gauge-fixing lagrangians for the general coordinate gauge invariance.

\[ \mathcal{L}_{\text{FP}} = i \mathcal{C}_\mu \partial_\nu D^\mu_\rho \mathcal{O}_\rho, \]

\[ \mathcal{L}_{\text{GF}} = - \frac{1}{2\xi} g_\mu_\nu \mathcal{E}(\square) \left( \partial_\lambda \tilde{g}^{\mu_\lambda} - \zeta \partial^\mu \sqrt{g} \right) \cdot \mathcal{E}(\square) \left( \partial_\rho \tilde{g}^{\mu_\rho} - \zeta \partial^\nu \sqrt{g} \right), \]

\[ D^\mu_\rho = \tilde{g}^{\mu_\sigma} \delta^\nu_\rho \partial_\sigma + \tilde{g}^{\nu_\sigma} \delta^\mu_\rho \partial_\sigma - \tilde{g}^{\mu_\nu} \partial_\rho - (\partial_\rho \tilde{g}^{\mu_\nu}), \]

where we have adopted the following linear gauge-fixing condition:

\[ \mathcal{E}(\square) \left( \partial_\nu \tilde{g}^{\mu_\nu} - \zeta \partial^\mu \sqrt{g} \right) + \frac{1}{2} \xi B^\mu = 0 \]  

with some function \( \mathcal{E}(\square) \) of \( \square \) and an auxiliary field \( B^\mu \). If one takes \( \mathcal{E}(\square) = \text{const.}, \zeta = 0 \) and \( \xi \to 0 \), the gauge is reduced to the familiar de Donder gauge. We will take \( \mathcal{E}(\square) \) to be a non-constant function in order to improve the high-momentum behavior of the gauge-dependent part of the propagator. For this
purpose it suffices to choose $\hat{e}(\Box)$ to be a linear function of $\Box$. Note that the operator $\hat{e}(\Box)$ in the gauge-fixing does not affect the kinetic part of the FP ghosts since the jacobian of the field redefinition $\Box C_\mu \to \tilde{C}_\mu$ is trivial \[10\]. The effective action is invariant under the BRS transformation

$$
\delta_B \tilde{g}^{\mu\nu} = \kappa D^{\mu\nu}_\rho C^\rho,
$$

$$
\delta_B C^\mu = -\kappa C^\nu \partial_\nu C^\mu,
$$

$$
\delta_B \tilde{C}_\mu = i B^\mu,
$$

$$
\delta_B B_\mu = 0.
$$

(52)

Indeed, up to irrelevant non-propagating terms, $L_{\text{FP}} + L_{\text{GF}}$ is written in the following BRS-exact form \[21\]:

$$
L_{\text{FP}} + L_{\text{GF}} = -i\kappa^{-1} \delta_B \left[ \tilde{C}_\mu \left( \hat{e}(\Box)(\partial_\nu \tilde{g}^{\mu\nu} - \zeta \partial^\mu \sqrt{\tilde{g}}) + \frac{1}{2} \xi B^\mu \right) \right].
$$

(53)

As understood from the results of previous sections, the quadratic parts of the Einstein action and of the one-loop effective action induced by the non-unitary matters specified by (46), reads in momentum space as

$$
\int d^3 x \frac{1}{\kappa^2} \sqrt{g} R \quad \text{(quadratic in } \tilde{h}^{\mu\nu})
$$

$$
= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2} \tilde{h}^{\mu\rho}(p) \tilde{h}^{\nu\sigma}(-p) \frac{p^2}{2} (P^{(2)}_{\mu\rho,\nu\sigma} - Q_{\mu\rho,\nu\sigma})
$$

(54)

and

$$
-i \log \det (-D_{\text{matter}}(g_{\mu\nu})) + \int d^3 x L^{(0)}_{\text{count}} \quad \text{(quadratic in } \tilde{h}^{\mu\nu})
$$

$$
= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2} \tilde{h}^{\mu\rho}(p) \tilde{h}^{\nu\sigma}(-p) \left( -\kappa^2 \frac{(-p^2)^2}{512} \right) (P^{(2)}_{\mu\rho,\nu\sigma} + 2Q_{\mu\rho,\nu\sigma}),
$$

(55)

respectively. The algebraic structure of the tensors in the quadratic part in $\tilde{h}^{\mu\nu}$ can conveniently be described by $4 \times 4$ matrices (See appendix A.). The coefficient of $\frac{1}{2} \tilde{h}^{\mu\rho}(p) \tilde{h}^{\nu\sigma}(-p)$ in the integrand of (54) can be written as
\[
\frac{p^2}{2} (P^{(2)}_{\mu\nu,\sigma} - Q_{\mu\rho,\nu\sigma}) = \frac{p^2}{2} \begin{bmatrix}
1 \\
0 \\
-1 & -\sqrt{2} \\
-\sqrt{2} & -2
\end{bmatrix},
\] (56)

and that of (55) reads

\[
-\frac{\kappa^2 (-p^2)^{\frac{3}{2}}}{512} (P^{(2)}_{\mu\nu,\sigma} + 2Q_{\mu\rho,\nu\sigma}) = -\frac{\kappa^2 (-p^2)^{\frac{3}{2}}}{512} \begin{bmatrix}
1 \\
0 \\
2 & 2\sqrt{2} \\
2\sqrt{2} & 4
\end{bmatrix}.
\] (57)

The addition of (57) to (56) does not change the rank of the matrix (56), as it should be, due to the (general coordinate transformation) gauge-invariance of matter integration. Similarly, the quadratic part of \(\mathcal{L}_{GF}\) is written as

\[
\int d^3x \mathcal{L}_{GF} \quad \text{(quadratic in } \tilde{h}^{\mu\nu})
\]

\[
= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \tilde{h}^{\mu\nu}(p) \tilde{h}^{\nu\sigma}(-p) \cdot \left\{ -\frac{\kappa^2}{\xi} \tilde{e}(-p^2)^2 p^2 \cdot \left[ \frac{1}{2} P^{(1)}_{\mu\rho,\nu\sigma} + 2\zeta^2 P^{(0-s)}_{\mu\rho,\nu\sigma} + \sqrt{2}(\zeta^2 - \zeta) (P^{(0-sw)}_{\mu\rho,\nu\sigma} + P^{(0-ws)}_{\mu\rho,\nu\sigma}) + (1 - \zeta)^2 P^{(0-w)}_{\mu\rho,\nu\sigma} \right] \right\}.
\] (58)

In the matrix notation \{\cdots\} is represented by

\[
-\frac{\kappa^2}{\xi} \tilde{e}(-p^2)^2 p^2 \begin{bmatrix}
0 \\
\frac{1}{2} \\
2\zeta^2 & \sqrt{2}(\zeta^2 - \zeta) \\
\sqrt{2}(\zeta^2 - \zeta) & (1 - \zeta)^2
\end{bmatrix}.
\] (59)

The propagator is given by

\[
< \tilde{h}_{\mu\rho}(p) \tilde{h}_{\nu,\sigma}(-p) >= i [(56) + (57) + (59)]^{-1}.
\] (60)

Hence no tachyonic pole appears in the spin-1 part if \(\tilde{e}(-p^2)\) has no zeroes in \(p^2 < 0\). On the other hand, the spin-0 part of the propagator (60) turns out to be the following form:
In view of (61), the tachyonic pole appears at $p_E = 128 \kappa^{-2}$. In fact, this pole can never be evaded by any choice of gauge parameters $(\xi, \zeta)$ or by any choice of $\hat{e}(-p^2)$. This is because each matrix of the first line of (61) has vanishing determinant, and consequently the determinant of their sum always contains the factor $\left(-p^2/2 - 2\kappa^2(-p^2)^2/512\right)$. Thus we conclude that we can never avoid a tachyonic pole in the spin-0 or -1 part of the graviton propagator if we consider the $1/N$ expansion of three-dimensional gravity coupled to non-unitary matters only.

**B. Tachyon-free theories**

Technically, the above tachyon observed in the gauge-dependent part comes from the fact that the coefficients of $P_{\mu\nu\sigma}^{(2)}$ and $Q_{\mu\nu\sigma}$ in the vacuum polarization tensor have the same signs, as is seen in (57). On the other hand, the vacuum polarization tensor $\Pi_{\mu\nu\sigma}^{\text{non-minimal}}$ from the non-minimal scalar one-

\[^5\] $\zeta = -1$ is not allowed since, if so, the spin-0 part of (57) becomes proportional to $Q_{\mu\nu\sigma}$, and hence the gauge-fixing is incomplete.
loop (44) has the $Q_{\mu \rho, \nu \sigma}$ term that depends on the value of a (non-minimal) coupling constant $\lambda$. Hence we can tune the coupling constant $\lambda$ so that the sum of $Q_{\mu \rho, \nu \sigma}$ terms from non-unitary matters and from unitary non-minimal scalars may possess the minus sign relative to that of $P_{\mu \rho, \nu \sigma}^{(2)}$ terms. As an example, we first integrate both $N_s$ unitary non-minimal scalars and $N$ non-unitary matters defined in (46) (or $N U(1)$ FP ghost fields). Then, according to the degrees-of-freedom rule, the dressed graviton propagator turns out to have no spin-2 tachyons if

$$r \equiv \frac{N}{N_s} > \frac{1}{2},$$  \hspace{1cm} (62)

Moreover, if we choose

$$\lambda \leq \frac{-1 - \sqrt{2r}}{16} \quad \text{or} \quad \lambda \geq \frac{-1 + \sqrt{2r}}{16},$$  \hspace{1cm} (63)

there are no tachyons in the gauge part, either. Eqs. (62) and (63) are not severe restrictions. One may also include fermions with an appropriate modification of the constraint for $\lambda$.

\section*{VI. SUMMARY AND PROSPECT}

We have presented preliminary results for our work that aims to get insight into universal properties of three- and four-dimensional quantum gravity coupled to matter fields. Our general spirit in the studies of quantum gravity is the following modest one. Although the description of full quantum properties such as of the unitarity problem may require a final theory such as superstring theories, it is reasonably expected that low-energy properties such as the phase structure or the stability of space-time vacuum and even some universal behaviors near a scaling region, if any, may be well described by renormalizable field theories. In the continuum approach, renormalizability is
technically important to make actual predictions especially for gauge theories like gravity, where it is difficult to introduce a cut-off scale consistently. Use of the $1/N$ expansion in quantum gravity is one possible direction to realize renormalizability by making much of matter fluctuations (back reaction) rather than space-time fluctuations. It could also allow non-perturbative predictions such as a non-trivial fixed point in renormalization-group, and is thus worth studying.

As a preliminary step we have reported some results for the applicability of the $1/N$ expansion to three-dimensional gravity. Continuing Kugo's work, we have first confirmed the generality as to the presence of tachyon in the spin-2 part of the graviton propagator dressed by unitary matters, and observed that the spin-2 part of matter one-loop corrections are quantized to $\kappa^2(-p^2)^{3/2}/512$ times the physical degrees of freedom. This "degrees-of-freedom rule" for matter one-loop corrections is an analogue of the one for the conformal anomaly in two dimensions. The graviton called here is the fluctuation around the flat space-time and the result implies that the flat space-time is quantum-mechanically unstable for large $N$ and higher-order analyses cannot work. In other words, in three-dimensional gravity coupled to unitary matters, the $N \to \infty$ limit of matter fields is not a stable zero-th order approximation for the flat space-time. The $\beta$ function for $\kappa$ exhibits no ultraviolet fixed point in these cases, either.

To stabilize the flat space-time in the $1/N$ approach, the results then require the modification of the theory. We have suggested two possible cures (the higher-derivative gravity and the coupling to non-unitary matters) by which a tachyon may be circumvented in the spin-2 part of a graviton propagator. For unitary matters, it seems quite natural and appealing to introduce higher-derivative terms and to investigate further the renormalization effects from the "dressed" graviton-loop appearing in the next-to-leading order.
From the purely theoretical viewpoint, the simplest and interesting case may be Einstein gravity coupled to non-unitary matters to which the $1/N$ expansion can be applied without introducing any new parameters for the gravity part. Although being an unrealistic model, it allows a manifestly gauge-invariant expansion keeping the renormalizability of Einstein gravity and does not have a tachyonic pole at least in the spin-2 part of the graviton propagator. Another fascinating point is that the theory is expected to possess an ultraviolet fixed point for $\kappa$ even to leading order \[25\], which means that it could serve as an interesting theoretical laboratory for studying analytically the critical behavior of gravitation. In this theory, however, a tachyonic pole exists in the gauge-dependent (spin-1 or -0) part of the graviton propagator and we have verified that it can never be evaded by any choice in the class of linear gauges. To cure this difficulty we are led to include unitary non-minimal scalars in addition. Under the conditions \(62\) and \(63\), the full theory is thus completely free from the tachyon disaster; we can have a tractable quantum gravity model in which the flat space-time is stable at $N \to \infty$. For example, we can consider the following model lagrangian:

$$
\mathcal{L} = \frac{1}{\kappa^2} \sqrt{g} R + \Lambda \sqrt{g} + \mathcal{L}_{FP+GF} + \sum_{i=1}^{N} \sqrt{g} g^{\mu \nu} \left( \partial_{\mu} \bar{c}_i \partial_{\nu} c_i + \frac{1}{2} \partial_{\mu} \phi_i \partial_{\nu} \phi_i + \lambda R \phi_i^2 \right),
$$

(64)

where $\bar{c}_i, c_i$ and $\phi_i$ are $N$-plet (non-unitary) anti-commuting scalar fields and (unitary) non-minimal scalar fields respectively and the strength of the non-minimal coupling $\lambda$ (not renormalized to leading order) is assumed to satisfy \(63\). This model, although unrealistic, contains only the minimal parameter $\kappa$ for self-gravity dynamics and allows a renormalizable tachyon-free $1/N$ expansion. Further, the model is anticipated to possess an ultraviolet fixed point for $\kappa$ and will serve as an fascinating statistical model possessing two phases of space-time; to the leading order the transition is driven solely by
matter fluctuations \cite{25} and may also be confirmed by numerical simulations of models with many kinds of non-unitary matters. Further, this model allows the second-order calculation of renormalization-group functions, where we may now consistently take the graviton-loop effects into account in three dimensions. The detailed analyses will appear elsewhere. We hope that the results in this paper provide useful bases that can be developed in several directions.

ACKNOWLEDGEMENTS

A part of the present work was carried out during our stay at Uji Research Center, Yukawa Institute for Theoretical Physics. We gratefully acknowledge the kind hospitality we enjoyed there.

We would like to thank M. Ninomiya for useful discussions, comments and reading the manuscript. We are also grateful to T. Kugo, S. Sawada and S. Uehara for discussions. The work of S.M. is supported by Soryuushi-Shougakukai and the Alexander von Humboldt Foundation.

APPENDIX A

In this appendix we describe the definitions of the projection operators in the space of symmetric rank-two tensors in three-dimensional spacetime \cite{26}. The projection operators in the spaces of spin-2, -1 and -0 are given by

\[ P_{\mu\rho,\nu\sigma}^{(2)} = \frac{1}{2} (\theta_{\mu\nu} \theta_{\rho\sigma} + \theta_{\mu\sigma} \theta_{\nu\rho} - \theta_{\mu\rho} \theta_{\nu\sigma}), \]

\[ P_{\mu\rho,\nu\sigma}^{(1)} = \frac{1}{2} (\theta_{\mu\nu} \omega_{\rho\sigma} + \theta_{\rho\sigma} \omega_{\mu\nu} + \theta_{\rho\sigma} \omega_{\mu\nu} + \theta_{\mu\sigma} \omega_{\rho\nu}), \]

\[ P_{\mu\rho,\nu\sigma}^{(0-s)} = \frac{1}{2} \theta_{\mu\rho} \theta_{\nu\sigma}, \]

\[ P_{\mu\rho,\nu\sigma}^{(0-w)} = \omega_{\mu\rho} \omega_{\nu\sigma}, \]
where
\[ \theta_{\mu\nu} = \eta_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2}, \quad \omega_{\mu\nu} = \frac{p_{\mu}p_{\nu}}{p^2}. \] (69)

They satisfy the completeness condition
\[ P^{(2)}_{\mu\rho,\nu\sigma} + P^{(1)}_{\mu\rho,\nu\sigma} + P^{(0-s)}_{\mu\rho,\nu\sigma} + P^{(0-w)}_{\mu\rho,\nu\sigma} = 1. \] (70)

The tensors which intertwine the two spin-0 subspaces are
\[ P^{(0-sw)}_{\mu\rho,\nu\sigma} = \frac{1}{\sqrt{2}} \theta_{\mu\rho} \omega_{\nu\sigma}, \] (71)
\[ P^{(0-ws)}_{\mu\rho,\nu\sigma} = \frac{1}{\sqrt{2}} \omega_{\mu\rho} \theta_{\nu\sigma}. \] (72)

The tensor of spin-0 and -1 parts in the graviton two-point function is given by
\[ Q_{\mu\rho,\nu\sigma} = P^{(0-s)}_{\mu\rho,\nu\sigma} + \sqrt{2}(P^{(0-sw)}_{\mu\rho,\nu\sigma} + P^{(0-ws)}_{\mu\rho,\nu\sigma}) + 2P^{(0-w)}_{\mu\rho,\nu\sigma}. \] (73)

It is convenient to represent the tensor algebra by 4 \times 4 matrices. Let \( r \) be a linear map such that
\[
\begin{bmatrix}
A & B \\
C & D \\
E & F
\end{bmatrix}
\begin{bmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{bmatrix} =
\begin{bmatrix}
7 & 8 \\
9 & 10 \\
11 & 12
\end{bmatrix}.
\] (74)

Then if \( r(T_{\mu\rho,\nu\sigma}) = M \) and \( r(T'_{\mu\rho,\nu\sigma}) = M' \), \( r(T_{\mu\rho,\alpha\beta} T'^{\alpha\beta}_{\lambda\nu}) \) is equal to \( MM' \).

In other words, \( r \) is a representation of this algebra.

**APPENDIX B**
In this appendix we summarize some formulas for the transformations from the bilinear forms in $\tilde{f}_a^\mu$ (33) and $h_{\mu\nu}$ (9) to that in $\tilde{h}_{\mu\nu}$. Let

$$U_{\alpha\beta\gamma\delta}^\mu\nu\sigma(x,y) = (x\eta_{\alpha\beta}\eta_{\gamma\delta} + y\delta_\alpha^\mu\delta_\lambda^\nu\delta_\delta^\beta)(x\eta_{\gamma\delta}\eta_{\nu\sigma} + y\delta_\gamma^\nu\delta_\delta^\sigma).$$

(75)

$(x,y) = (\frac{1}{2}, \frac{1}{2})$ and $(1,-1)$ correspond to the transformations $\tilde{f}_a^\mu \to \tilde{h}_{\mu\nu}$ and $h_{\mu\nu} \to \tilde{h}_{\mu\nu}$, respectively. Then

$$\eta_{\mu\nu}\eta_{\rho\sigma}U_{\alpha\beta\gamma\delta}^\mu\nu\sigma(x,y) = (nx^2 + 2xy)\eta_{\alpha\beta}\eta_{\gamma\delta} + y^2\eta_{\alpha\gamma}\eta_{\beta\delta},$$

(76)

$$\eta_{\mu\rho}\eta_{\nu\sigma}U_{\alpha\beta\gamma\delta}^\mu\nu\sigma(x,y) = (nx + y)^2\eta_{\alpha\beta}\eta_{\gamma\delta},$$

(77)

$$\eta_{\mu\rho}p_\rho p_\sigma U_{\alpha\beta\gamma\delta}^\mu\nu\sigma(x,y) = x^2p^2\eta_{\alpha\beta}\eta_{\gamma\delta} + xy(\eta_{\alpha\beta}p_{\gamma}p_{\delta} + \eta_{\gamma\delta}p_{\alpha}p_{\beta}) + y^2\eta_{\alpha\gamma}p_{\beta}p_{\delta},$$

(78)

$$\eta_{\mu\rho}p_\rho p_\sigma U_{\alpha\beta\gamma\delta}^\mu\nu\sigma(x,y) = (nx + y)xp^2\eta_{\alpha\beta}\eta_{\gamma\delta} + (nx + y)y\eta_{\alpha\beta}p_{\gamma}p_{\delta},$$

(79)

$$p_\mu p_\nu p_\rho p_\sigma U_{\alpha\beta\gamma\delta}^\mu\nu\sigma(x,y) = x^2(p^2)^2\eta_{\alpha\beta}\eta_{\gamma\delta} + xy p^2(\eta_{\alpha\beta}p_{\gamma}p_{\delta} + \eta_{\gamma\delta}p_{\alpha}p_{\beta}) + y^2 p_{\alpha}p_{\beta}p_{\gamma}p_{\delta},$$

(80)

where $n$ is the dimensions of spacetime ($n = 3$).

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FIGURE CAPTIONS

Figure 1: (a) The dressed graviton propagator. (b) The dressed $N$-point vertex.

Figure 2: The scalar one-loop diagrams. Bold and wavy lines stand for scalar and graviton, respectively.

Figure 3: The spinor one-loop vacuum polarization.

Figure 4: The $U(1)$ gauge boson (curly lines) and $U(1)$ FP ghost (broken lines) one-loop vacuum polarizations.

Table 1: The Feyman rules. The momenta of the vertices are taken in-going.
Figure 1: (a) The dressed graviton propagator. (b) The dressed $N$-point vertex.

\[
(a) \quad \begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{figure1a.png}}
\end{array}
= \quad \text{\includegraphics[width=0.2\textwidth]{figure1b.png}} + \text{\includegraphics[width=0.2\textwidth]{figure1c.png}} + \text{\includegraphics[width=0.2\textwidth]{figure1d.png}} \\
+ \text{\includegraphics[width=0.2\textwidth]{figure1e.png}} + \cdots
\]

(b) \quad \begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{figure2a.png}}
\end{array}
= \text{\includegraphics[width=0.2\textwidth]{figure2b.png}} + \sum_{N \text{ legs}} \text{\includegraphics[width=0.2\textwidth]{figure2c.png}}

Figure 2: The scalar one-loop diagrams. Bold and wavy lines stand for scalar and graviton, respectively.

\[
\begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{figure3a.png}} \\
\text{\includegraphics[width=0.1\textwidth]{figure3b.png}} \\
\text{\includegraphics[width=0.1\textwidth]{figure3c.png}}
\end{array}
\]
Figure 3: The spinor one-loop vacuum polarization.

Figure 4: The U(1) gauge boson (curly lines) and U(1) FP ghost (broken lines) one-loop vacuum polarizations.
Table 1: The Feynman rules. The momenta of the vertices are taken in-going.

\[
\begin{align*}
\begin{array}{c}
\text{vertex} \\
\hline
\end{array}
\end{align*}
\begin{align*}
\bar{f}_a & \rightarrow \\
p & \rightarrow \\
k & \rightarrow \\
\end{align*}
\begin{align*}
\frac{i}{k^2} \\
\frac{i}{k} \\
-i\left(\eta_{\mu\nu} - (1 - \alpha) \frac{k_\mu k_\nu}{k^2}\right) \\
\frac{1}{k^2} \\
\end{align*}
\begin{align*}
q & \rightarrow \\
\bar{f}_a^\mu & \rightarrow \\
k & \rightarrow \\
\end{align*}
\begin{align*}
i\gamma^\alpha (k_\mu - q_\mu) \\
\frac{ie}{2} [k \cdot q (\eta^{\alpha\beta} \eta^{\lambda\sigma} - \eta^{\lambda\alpha} \eta^{\beta\sigma} - \eta^{\sigma\alpha} \eta^{\beta\lambda})] \\
- \eta^{\alpha\beta} k_\alpha q_\beta + \eta^{\lambda\sigma} (k_\sigma q_\beta + k_\beta q_\sigma) \\
+ \eta^{\beta\sigma} k_\beta q_\alpha + \eta^{\beta\lambda} k_\beta q_\sigma + \eta^{\sigma\alpha} k_\beta q_\lambda \\
- \frac{i\alpha}{2}\eta^{\alpha\beta} (k_\lambda q_\sigma + k_\lambda k_\sigma + q_\lambda q_\sigma) \\
- 2 (\eta^{\beta\lambda} q_\sigma q_\alpha + \eta^{\beta\sigma} k_\alpha k_\lambda) \\
\end{align*}
\begin{align*}
h_{\alpha\beta} & \rightarrow \\
\bar{h}^{\mu\nu} & \rightarrow \\
\end{align*}
\begin{align*}
-k k_\mu q_\rho \\
-i\kappa [k_\mu q_\rho + 2\lambda (p_\mu \eta_{\rho\sigma} + p_\mu p_\rho)] \\
\end{align*}
\begin{align*}
\text{(non-minimal)}
\end{align*}