REDUCED DISTANCE BASED AT SINGULAR TIME IN THE RICCI FLOW

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Abstract. In this paper, we define a reduced distance function based at a point at the singular time $T < \infty$ of a Ricci flow. We also show the monotonicity of the corresponding reduced volume based at time $T$, with equality iff the Ricci flow is a gradient shrinking soliton. Our curvature bound assumption is more general than the type I condition.

1. Introduction

In [Ham82], Hamilton introduced the evolution of Riemannian manifolds by the Ricci flow
\[
\frac{\partial g(t)}{\partial t} = -2\text{Ric}_g(t).
\]
Solutions to this nonlinear equation generally develop singularities in finite time. An important tool when studying the formation of singularities in geometric evolution equation are monotone quantities. Perelman [Per02] made significant progress to the understanding of singularities in the Ricci flow by finding such quantities. We will consider the 'reduced volume' and extend its applicability to singular time.

Let $(M,g(t))$ be a complete oriented maximal solution to the Ricci flow for $t \in [0,T)$ where $T < \infty$. We further assume a new curvature assumption, which we call 'type A' (see section 3.2 for the definition) and which includes the well-known type I condition. For any smooth $(p,t_0)$ in space-time $M \times [0,T)$, Perelman defines the 'reduced distance' $l_{p,t_0}$. It is a function on $M \times [0,t_0]$ and satisfies the differential inequality
\[
-\frac{\partial}{\partial \bar{t}} l_{p,t_0}(q,\bar{t}) - \Delta g(\bar{t}) l_{p,t_0}(q,\bar{t}) + |\nabla g(\bar{t}) l_{p,t_0}(q,\bar{t})|^2 g(\bar{t}) - R g(\bar{t}) + \frac{n}{2(t_0 - \bar{t})} \geq 0.
\]
This inequality implies the monotonicity (in $\bar{t}$) of the 'reduced volume'
\[
\tilde{V}_{p,t_0}(\bar{t}) := \int_M (4\pi(t_0 - \bar{t}))^{-\frac{n}{2}} e^{-l_{p,t_0}(q,\bar{t})} d\text{vol}_g(\bar{t})(q)
\]
along the Ricci flow. It follows from Perelman’s work that $\tilde{V}_{p,t_0}(\bar{t})$ is constant in $\bar{t}$ if and only if $(M,g(t))$ is isometric to Euclidean space with the flat (non-evolving) metric. In the proof, the equality case corresponds to a gradient shrinking soliton (see Definition 2.1). However, since $t_0$ is a regular time, gradient shrinking solitons other than flat Euclidean space cannot arise. This motivates the generalization considered in the paper, which we outline now.

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For \( p \in M \) and a sequence \( t_i \nearrow T \), we consider \( l_{p,t_i} \) as above. In our main Theorem 3.7, we show the subsequential convergence

\[
l_{p,t_i} \xrightarrow{c_{loc}} l_{p,T},
\]

where the differential inequality survives to the limit function \( l_{p,T} \), i.e. it satisfies

\[
-\frac{\partial}{\partial t} l_{p,T}(q, \bar{t}) - \Delta l_{p,T}(q, \bar{t}) + |\nabla l_{p,T}(q, \bar{t})|^2 - R_{g(q)}(\bar{t}) + \frac{n}{2(T - \bar{t})} \geq 0.
\]

We conclude in Corollary 4.3 that the corresponding reduced volume based at singular time \((p, T)\) satisfies the monotonicity formula with equality if and only if we have a gradient shrinking soliton (see precise statement in section 4).

We would like to point out, that in very recent work \([\text{Nab07}]\), Naber obtained very similar results as we present in this paper. While our type A assumption is weaker than the type I assumption (and \( \kappa \)–noncollapsedness) used in his work, he additionally applies the results to show that the rescaling limit of a type I singularity is a shrinking soliton. This is the application we had in mind and were currently working out the details of. As soon as we learned about the posting of \([\text{Nab07}]\), we decided to type up this paper, in which we independently present the results on the reduced distance and reduced volume monotonicity based at singular time, that we had presented in talks only so far. This explains the incompleteness of this paper in terms of applications. We would like to add that \([\text{Nab07}]\) also classifies noncompact 4-dimensional shrinking solitons with bounded nonnegative curvature operator. As the above discussion shows, classification of gradient shrinking solitons is crucial in understanding singularities, and there are many related recent results, e.g. \([\text{NW07a}],[\text{NW07b}],[\text{Kot07}]\).

2. Background on Ricci flow

In this section we present background material on Ricci flow and the reduced distance needed in the subsequent sections.

2.1. The equation. Let \((M^n, g(t)), t \in [0, T), 0 < T < \infty \) be a 1-parameter family of complete, oriented n-dimensional Riemannian manifolds with bounded (sectional) curvature solving the equation

\[
\frac{\partial g(t)}{\partial t} = -2\text{Ric}_{g(t)},
\]

where \( \text{Ric}_{g(t)} \) denotes the Ricci curvature tensor of the metric \( g(t) \). We call such a family a Ricci flow on \([0, T)\) throughout this paper.

It follows from work by Hamilton \([\text{Ham82}]\), DeTurck \([\text{DeT83}]\), Shi \([\text{Shi89a}],[\text{Shi89b}]\) as well as Chen and Zhu \([\text{CZ06b}]\) that for a given complete Riemannian manifold with bounded curvature \((M, g_0)\), there exists a unique solution to the quasilinear second order weakly parabolic equation (2.1) with \( g(0) = g_0 \) on a time interval \([0, T)\) and the solution is maximal if and only if

\[
\lim_{t \nearrow T} \max_M |\text{Rm}_{g(t)}|_{g(t)} = \infty,
\]

where \( \text{Rm}_{g(t)} \) denotes the full curvature tensor of \( g(t) \).
2.2. Gradient shrinking solitons. We will now discuss special solutions to the Ricci flow (2.1): Gradient shrinking solitons play an important role in the equality case of monotone quantities, and hence in the study of singularities.

Motivated by Perelman’s perspective in [Per02] and [Per03], we make the following

Definition 2.1. A Riemannian manifold \((M, g)\) is called a gradient shrinking soliton if there exists a potential function \(f : M \to \mathbb{R}\) such that

\[
Ric_g + \nabla^g \nabla^g f - \frac{1}{2} g = 0,
\]

where \(\nabla^g \nabla^g f\) denotes the covariant Hessian of \(f\).

If \(\nabla^g f\) determines a complete vector field (e.g. if it is bounded, or in particular if \(M\) is compact), then as in Theorem 4.1 in [CLN06] we get a corresponding Ricci flow \(g(t)\) on \((-\infty, T)\) with \(g(T - 1) = g\), called gradient shrinking soliton in canonical form, in the following way: The 1-parameter family of diffeomorphisms \(\phi_t\) of \(M\) generated by integrating the vector field \(\frac{1}{T - t} \nabla^g f\) with \(\phi_{T - 1} = id\) gives us

\[
g(t) = (T - t) \phi_t^* g.
\]

With the potential function \(f(t) = \phi_t^* f\) the solution \(g(t)\) satisfies

\[
Ric_{g(t)} + \nabla^{g(t)} \nabla^{g(t)} f(t) - \frac{1}{2(T - t)} g(t) = 0
\]

on \((-\infty, T)\). Also, \(f(t)\) satisfies

\[
\frac{\partial f}{\partial t} = |\nabla^{g(t)} f(t)|^2_{g(t)}.
\]

Remark 2.2. Note that if we have a Ricci flow \(g(t)\) on \([0, T)\), which together with a 1-parameter family of functions \(f(t)\) satisfies (2.4), this clearly defines a gradient shrinking soliton according to Definition 2.1 by considering the equation at \(t = T - 1\). However, it might not be a gradient shrinking soliton in canonical form. If however \(\nabla^{g(T - 1)} f(T - 1)\) is complete, we can conclude by uniqueness of solutions to the Ricci flow that the corresponding Ricci flow in canonical form equals \(g(t)\) on \([T - 1, T)\).

Solitons are ‘self-similar’ solutions, since they evolve only by scaling and diffeomorphism. Their existence is to be expected due to the diffeomorphism invariance of the Ricci flow equation (2.1). Solitons can be regarded as generalized fixed points, i.e. fixed points of the volume normalized Ricci flow on the space of metrics modulo the diffeomorphism group.

By changing the minus sign in (2.2) to “+” (or dropping the metric term), one can analogously define gradient expanding (or steady) solitons. Also, if the vector field generating the diffeomorphisms \(\phi_t\) is not a gradient vector field, one gets a more general notion of solitons. We will skip a more detailed discussion, since those examples are not directly relevant in this paper.

Example 2.3 (Einstein solutions). Let \((M^n, g_0)\) be an Einstein manifold, i.e. \(g_0\) satisfies \(Ric_{g_0} = \frac{g_0}{n} g_0\), where \(R_{g_0}\) denotes the (constant) scalar curvature of \(g_0\). Then

\[
g(t) = \left(1 - \frac{2}{n} R_{g_0} t\right) g_0
\]
is a solution the the Ricci flow, where \( g(t) \) is Einstein for each \( t \). It only changes by scaling and can be seen to be a fixed point of the volume normalized Ricci flow. If \( R_{g(t)} > 0 \) the solution exists on the time interval \((-\infty, \frac{n}{2R_{g_0}})\). With \( T := \frac{n}{2R_{g_0}} \) we then get from (2.6) that

\[
R_{g(t)} = \frac{n}{2(T - t)} \quad \text{and} \quad \text{Ric}_{g(t)} = \frac{1}{2(T - t)} g(t).
\]

It follows from Remark 2.2 that any Einstein solution with positive scalar curvature can be regarded as a gradient shrinking soliton with \( f \equiv 0 \), which is in canonical form.

**Example 2.4.** Consider the non-evolving Ricci flow \((\mathbb{R}^n, g(t) = g_{\mathbb{R}^n})\). If we let

\[
f(x, t) = \frac{|x|^2}{4(T - t)},
\]

which makes flat Euclidean space into a gradient shrinking soliton, called **Gaussian soliton**. It is in canonical form and will arise in section 2.4.

**Example 2.5.** We can construct gradient shrinking solitons as products of Einstein solutions \( N^{n-k} \) of positive scalar curvature with flat Euclidean space \( \mathbb{R}^k \). It is an interesting question to ask when gradient shrinking solitons are of this form [PW07].

We will now discuss important equations satisfied by gradient shrinking solitons: Let \((M, g(t), f(t))\) be a gradient shrinking soliton in canonical form on \((-\infty, T)\). Tracing equation (2.4) gives

\[
R_{g(t)} + \Delta_g f(t) - \frac{n}{2(T - t)} = 0.
\]

We conclude from equations (2.5) and (2.8) that

\[
-\frac{\partial f}{\partial t} - \Delta_g f(t) + |\nabla g(t)|^2_{g(t)} - R_{g(t)} + \frac{n}{2(T - t)} = 0.
\]

In Perelman’s point of view [Per02], if we let

\[
u(x, t) := (4\pi(T - t))^{-\frac{n}{2}} e^{-f(x,t)},
\]

where \( x \in M \) and \( t \in (-\infty, T) \), then a straight forward computation shows that (2.9) is equivalent to

\[
\Box^* u(x, t) = 0.
\]

Here \( \Box^* = -\frac{\partial}{\partial t} - \Delta_g + R_{g(t)} \) denotes the formal adjoint of the heat operator \( \Box = \frac{\partial}{\partial t} - \Delta_g \) under the Ricci flow. This observation plays a key role in the proof of the equality case of the reduced volume monotonicity in Corollaries 2.13 and 4.3.

We will also need the following fact: If \((M, g, f)\) is a gradient shrinking soliton, then it follows from the contracted second Bianchi identity and (2.8) that there exists a constant \( C \in \mathbb{R} \) such that

\[
R_g + |\nabla g|^2_g - f = C.
\]
2.3. **Perelman’s reduced distance.** In this section we will briefly discuss Perelman’s reduced distance for the Ricci flow. Contrary to [Per02], we will use forward time notation rather than backward time, only since it will come more natural when in sections 3 and 4 we consider a sequence of different base-times. The monotonicity of Perelman’s reduced volume will then be described in section 2.4. While the results are due to Perelman, there are by now several references detailing his work, e.g. [STW04], [KL06], [Ye06a][Ye06b], [CZ06a], [MT06], [CCG+07].

**Definition 2.6.** Let \((M^n, g(t))\) be a Ricci flow on \([0, T)\). For any curve \(\gamma : [\bar{t}, t_0] \to M\), where \(0 < \bar{t} < t_0 < T\), we define the \(\mathcal{L}-\text{length}\) of \(\gamma\) by

\[
\mathcal{L}(\gamma) := \int_{\bar{t}}^{t_0} \sqrt{\bar{t} - t}(|\dot{\gamma}(t)|^2_{g(t)} + R_{g(t)}(\gamma(t))) dt.
\]

At a curve \(\gamma\) the first variation of \(\mathcal{L}\) with fixed endpoints is given by

\[
\delta_{\gamma} \mathcal{L}(\gamma) = \left(2\sqrt{\bar{t} - t} \dot{\gamma}(t), Y(t)\right)_{\bar{t}}^{t_0}
\]

\[
-\int_{\bar{t}}^{t_0} 2\sqrt{\bar{t} - t} (\nabla^{g(t)} g_{\delta\gamma(t)})(\dot{\gamma}(t)) - \frac{1}{2(t_0 - \bar{t})} \dot{\gamma}(t) - 2Ric_{g(t)}(\dot{\gamma}(t), \cdot)^\# - \frac{1}{2} \nabla^{g(t)} R_{g(t)}, Y(t) dt.
\]

\(Y(t)\) is the variational vector field along \(\gamma(t)\) with \(Y(\bar{t}) = Y(t_0) = 0\) and \(^\#\) denotes the metric dual. The Euler-Lagrange equation is called \(\mathcal{L}-\text{geodesic equation}\) and given by

\[
\nabla^{g(t)} \dot{\gamma}(t) - \frac{1}{2(t_0 - \bar{t})} \dot{\gamma}(t) - 2Ric_{g(t)}(\dot{\gamma}(t), \cdot)^\# - \frac{1}{2} \nabla^{g(t)} R_{g(t)} = 0.
\]

Smooth solutions to (2.14) are called \(\mathcal{L}-\text{geodesics}\.\) Multiplying by \(t_0 - t\), we can rewrite (2.14) to get rid of the unbounded coefficient of the second term:

\[
\nabla^{g(t)} \sqrt{\bar{t} - t} \dot{\gamma}(t) - 2\sqrt{\bar{t} - t} \nabla^{g(t)} Ric_{g(t)}(\sqrt{\bar{t} - t} \dot{\gamma}(t), \cdot)^\# - \frac{1}{2} (t_0 - \bar{t}) \nabla^{g(t)} R_{g(t)} = 0.
\]

Using the direct method in the calculus of variations after a change of variables in \(\mathcal{L}\), as well as the ODE (2.15) together with the curvature boundedness assumption and Shi’s derivative estimates [Shi89b], one obtains the following

**Proposition 2.7.** For any \((q, \bar{t})\) and \((p, t_0)\) with \(p, q \in M\) and \(0 < \bar{t} < t_0 < T\) there exists a \(\mathcal{L}-\text{minimizing} \ \mathcal{L}-\text{geodesic} \ \gamma : [\bar{t}, t_0] \to M\) with \(\gamma(\bar{t}) = q\) and \(\gamma(t_0) = p\). Moreover, \(\lim_{t_0, t_0} \mathcal{L}(\gamma) = 0\).

**Definition 2.8.** For \((q, \bar{t})\) and \((p, t_0)\) as in the Proposition 2.7 we define

(i) **the \(\mathcal{L}-\text{distance from} \ \ (q, \bar{t}) \ to \ (p, t_0)\**

\[
L_{p,t_0}(q, \bar{t}) := \inf\{\mathcal{L}(\gamma) \mid \gamma : [\bar{t}, t_0] \to M, \gamma(\bar{t}) = q, \gamma(t_0) = p\},
\]

(ii) **the reduced distance based at \(p, t_0)\**

\[
l_{p,t_0}(q, \bar{t}) := \frac{L_{p,t_0}(q, \bar{t})}{2\sqrt{t_0 - \bar{t}}},
\]

(iii) and

\[
v_{p,t_0}(q, \bar{t}) := (4\pi(t_0 - \bar{t}))^{-\frac{n}{2}} e^{-l_{p,t_0}(q, \bar{t})}.
\]
We will fix \((p, t_0) \in M \times (0, T)\) and regard \(L_{p, t_0}, l_{p, t_0}\) and \(v_{p, t_0}\) as functions on space-time \(M \times (0, t_0)\).

By studying the second variation of \(\mathcal{L}\) and obtaining a Laplacian comparison theorem for the reduced distance, Perelman derives the following three differential (in)equalities:

**Theorem 2.9.** Fix \((p, t_0) \in M \times (0, T)\). Then for all \((q, \bar{t}) \in M \times (0, t_0)\) we have

\[
\frac{\partial}{\partial \bar{t}} l_{p, t_0}(q, \bar{t}) - \Delta_{g(\bar{t})} l_{p, t_0}(q, \bar{t}) - |\nabla g(\bar{t})|_{g(\bar{t})}^2 \leq R_{g(\bar{t})} + \frac{n}{2(t_0 - \bar{t})} \geq 0, \tag{2.16}
\]

or equivalently,

\[
\Box^* v_{p, t_0}(q, \bar{t}) \leq 0. \tag{2.17}
\]

\[
-|\nabla g(\bar{t})|_{g(\bar{t})}^2 l_{p, t_0}(q, \bar{t}) + \nabla g(\bar{t}) \cdot \nabla g(\bar{t}) v_{p, t_0}(q, \bar{t}) - R_{g(\bar{t})} l_{p, t_0}(q, \bar{t}) + \frac{l_{p, t_0}(q, \bar{t}) - n}{t_0 - \bar{t}} + 2|\nabla g(\bar{t})|_{g(\bar{t})}^2 \leq 0. \tag{2.18}
\]

\[
-2 \frac{\partial}{\partial \bar{t}} l_{p, t_0}(q, \bar{t}) + |\nabla g(\bar{t})|_{g(\bar{t})}^2 l_{p, t_0}(q, \bar{t}) - R_{g(\bar{t})} l_{p, t_0}(q, \bar{t}) = 0. \tag{2.19}
\]

**Remark 2.10.** It can be shown (see e.g. [Ye06a]) that \(l_{p, t_0}(q, \bar{t})\) is locally Lipschitz on \(M \times (0, t_0)\). Hence the same is true for \(l_{p, t_0}(q, \bar{t})\) and \(v_{p, t_0}(q, \bar{t})\), and the functions are differentiable a.e. in \(q\) and \(\bar{t}\). The (in)equalities in Theorem 2.9 therefore hold in the barrier sense, and in particular in the sense of distributions. For (2.16) this means that for any nonnegative \(\phi \in C_0^\infty(\{0 \leq \bar{t} < t_0\})\)

\[
\int_0^T \int_M \left(-\frac{\partial}{\partial \bar{t}} l_{p, t_0} + |\nabla g(\bar{t})|_{g(\bar{t})}^2 l_{p, t_0}\right) d\mathcal{L}_{g(\bar{t})}(q) dq \geq 0. \tag{2.20}
\]

### 2.4 Reduced volume and monotonicity

**Definition 2.11.** Let \((M^n, g(t))\) be a Ricci flow on \([0, T)\), and \((p, t_0) \in M \times (0, T)\). Then for each \(\bar{t} \in (0, t_0)\) we define the **reduced volume based at \(p, t_0\)**

\[
\bar{V}_{p, t_0}(\bar{t}) := \int_M v_{p, t_0}(q, \bar{t}) d\mathcal{L}_{g(\bar{t})}(q) = \int_M \left(4\pi(t_0 - \bar{t})ight)^{-\frac{n}{2}} e^{-l_{p, t_0}(q, \bar{t})} d\mathcal{L}_{g(\bar{t})}(q). \tag{2.4}
\]

**Example 2.12.** For the Gaussian soliton (example 2.4) one computes

\[
l_{p, t_0}(q, \bar{t}) = \left|\frac{q - p}{4(t_0 - \bar{t})}\right|^2,
\]

and hence

\[
\bar{V}_{p, t_0}(\bar{t}) = \int_{\mathbb{R}^n} \left(4\pi(t_0 - \bar{t})ight)^{-\frac{n}{2}} e^{-\left|\frac{q - p}{4(t_0 - \bar{t})}\right|^2} dq = 1.
\]

The monotonicity of the reduced volume along the Ricci flow is now essentially a consequence of inequality (2.17) in Theorem 2.9:
Corollary 2.13 (Monotonicity of the reduced volume). Under the same assumptions as in Definition 2.11, we have

(i) \( \frac{d}{dt} \tilde{V}_{p,t_0}(\tilde{t}) \geq 0 \),

(ii) \( \lim_{\tilde{t} \to t_0} \tilde{V}_{p,t_0}(\tilde{t}) = 1 \),

(iii) \( \tilde{V}_{p,t_0}(\tilde{t}_1) = \tilde{V}_{p,t_0}(\tilde{t}_2) \) for \( 0 < \tilde{t}_1 < \tilde{t}_2 < t_0 \) if and only if \((M,g(t))\) is isometric to the Gaussian soliton and \( \tilde{V}_{p,t_0}(\tilde{t}) \equiv 1 \).

3. Reduced distance based at singular time

3.1. Motivation. In the proof of Corollary 2.13 (iii), one actually obtains that if the reduced volume is constant, then \((M,g(t_0 − 1))\) is a gradient shrinking soliton. Considering the corresponding canonical form and using the fact that at time \( t_0 \) the curvature is bounded (in fact zero), one concludes that the soliton must be the Gaussian soliton. If for a maximal Ricci flow on \([0,T)\) we are able to base the reduced distance and volume at singular time \((p,T)\), depending on the base point \( p \in M \), we expect to get gradient shrinking solitons other than the Gaussian soliton whenever this generalized reduced volume is constant. We will prove this in Corollary 4.3 in section 4. Before that, in this section we will define a reduced distance based at a singular time.

Remark 3.1. Results of the type described above are known for other monotone quantities in geometric evolution equations, e.g. for Perelman’s \( \lambda \) and \( \mu \) functionals for the Ricci flow [Per02] or Huisken’s monotonicity formula for the mean curvature flow [Hui90]. Note also that the equality case of Harnack type inequalities similarly identifies gradient expanding and steady solitons, see e.g. [Ham93], [CZ00], [Ni05].

Let \((M,g(t))\) be a maximal Ricci flow on \([0,T)\). Let \( t_i \nearrow T \) and \( p \in M \). Then for all \((q,\tilde{t}) \in M \times (0,T)\) the reduced distance \( l_{p,t_i}(q,\tilde{t}) \) is defined for large enough \( i \) and the differential inequality (2.16) holds for each such \( i \). This raises two questions:

(i) Does there exist a good limit \( l_{p,T} := \lim_{t_i \nearrow T} l_{p,t_i} \)?

(ii) Does the differential inequality (2.16) hold for \( l_{p,T} \)?

Example 3.2. If \((M,g(t))\) is an Einstein solution on \([0,T)\) with \( R(g(0)) > 0 \). For \( p \in M \) and \( t_i \nearrow T \) it follows from an explicit computation (see e.g. [CCG+07]) that

\[
    l_{p,t_i}(q,\tilde{t}) \to \frac{n}{2} =: l_{p,T}(q,\tilde{t})
\]

uniformly on \( M \times [a,b] \subset M \times [0,T) \). Then (2.16) holds for the constant function \( l_{p,T} \) because of equation (2.7).

Example 3.3. Let \((M,g(t),f(t))\) be a gradient shrinking soliton on \((-\infty,T)\) in canonical form and let \( p \in M \) and \( t_i \nearrow T \). Then from the discussion in section 7.3 in [CCG+07] we can conclude that for a subsequence

\[
    l_{p,t_i}(q,\tilde{t}) \to f(q,\tilde{t}) + C =: l_{p,T}(q,\tilde{t}),
\]

where the convergence is uniform on compact subsets of \( M \times (0,T) \) and \( C \) is the constant coming from equation (2.12) satisfied by \((M,g(T − 1),f(T − 1))\). In particular, the limit \( l_{p,T} \) is independent of \( p \) and \( t_i \).

In general, to answer both questions above positively, we need to mildly strengthen the bounded curvature assumption.
3.2. Type A Ricci flows.

**Definition 3.4.** A Ricci flow \((M, g(t))\) on \([0, T)\) is said to be of type A if there exist \(C > 0\) and \(r \in [1, \frac{3}{2})\) such that for all \(t \in [0, T)\)

\[ |Rm_{g(t)}|_{g(t)} \leq \frac{C}{(T - t)^{r}}. \]

**Remark 3.5.** Note that for \(r = 1\) this is known as the type I condition. Our type A assumption is weaker. From the maximum principle for \(|Rm_{g(t)}|_{g(t)}\) it follows that for a maximal Ricci flow on \([0, T)\)

\[ \max_M |Rm_{g(t)}|_{g(t)} \geq \frac{1}{8(T - t)}, \]

which implies that curvature blow-up with \(r < 1\) is impossible. On the other hand, the type I condition is assumed to be generic. To our knowledge it is not known whether there are maximal Ricci flows which are not of type A. The only known example which is not of type I (i.e. type II) is the degenerate neckpinch [GZ07], but its curvature blow-up rate is not known.

**Example 3.6.** Let \((M, g, f)\) be a gradient shrinking soliton where \(\nabla g f\) is a complete vector field. Then it follows from equation (2.3) that the corresponding Ricci flow in canonical form is of type A, in fact of type I.

3.3. Main Theorem. In this section, we state and prove the main

**Theorem 3.7.** Let \((M, g(t))\) be a Ricci flow on \([0, T)\) of type A. Also let \(p \in M\) and \(t_i \nearrow T\). Then there exists a locally Lipschitz function

\[ l_{p,T} : M \times (0, T) \to \mathbb{R}, \]

which is a subsequential limit

\[ l_{p, t_i} \xrightarrow{C^0_{loc}} l_{p,T} \]

and which satisfies the differential inequality analogous to (2.16), i.e. for all \((q, \bar{t}) \in M \times (0, T)\)

\[ (3.1) \quad -\frac{\partial}{\partial \bar{t}}l_{p,T}(q, \bar{t}) - \Delta l_{p,T}(q, \bar{t}) + |\nabla l_{p,T}(q, \bar{t})|^2 - R_{g(\bar{t})}(q) + \frac{n}{2(T - \bar{t})} \geq 0 \]

holds in the sense of distributions. Equivalently,

\[ (3.2) \quad \Box^* v_{p,T} \leq 0, \]

where \(v_{p,T}(q, \bar{t}) := (4\pi(T - \bar{t}))^{-\frac{n}{2}} e^{-l_{p,T}(q, \bar{t})}. \)

**Remark.** This theorem has very recently been independently obtained in [Nab07] under the type I assumption (and \(\kappa\)–noncollapsedness) using very similar techniques.

Theorem 3.7 allows us to make the following

**Definition 3.8.** Under the assumptions of Theorem 3.7 we define

\[ l_{p,T} : M \times (0, T) \to \mathbb{R} \]

to be a reduced distance for \((M, g(t))\) based at singular time \((p, T)\).
We now give the proof of Theorem 3.7.

Pro. To simplify notation let \( l_i := l_{p,t_i}(q, \bar{t}) \) and \( L_i := L_{p,t_i}(q, \bar{t}) \). The proof will be in 3 steps.

1. First, we will derive a basic uniform bound on \( l_i \) on compact subsets \( K = K_1 \times [a, b] \subset M \times (0, T) \). By definition of \( l_i \) it suffices to show such a bound for \( L_i(q, \bar{t}) \) on \( K \). Let \( \eta : [0, 1] \to M \) be a \( g(0) \)-geodesic with \( \eta(0) = q \) and \( \eta(1) = p \). Fix \( k \in (b, T) \) and consider

\[
\gamma(t) := \begin{cases} \eta\left(\frac{t-i}{k-t}\right) & t \in [i, k] \\ p & t \in (k, t_i]. \end{cases}
\]

Since \( |\eta'(s)|^2_{g(0)} = c \) for a constant \( c \), the uniform equivalence of the metrics along the Ricci flow on \([0, k]\) yields a constant \( D \) such that \( |\eta'(s)|^2_{g(t)} \leq D \). The type A assumption implies that there exist constants \( C \) and \( r \) such that \( |R| \leq \frac{C}{(T-t)^r} \). Then we get the following estimate:

\[
(3.3) \quad |L_i(q, \bar{t})| \leq \int_{i}^{\bar{t}} \sqrt{t_i-t} (|\gamma_t(t)|^2_{g(t)} + R_{g(t)}(\gamma(t))) dt \\
\leq \int_{i}^{k} \sqrt{t_i-t} \left| \eta'\left(\frac{t-i}{k-t}\right)\right|^2 dt + C \int_{i}^{T} \sqrt{t_i-t} (T-t)^r dt \\
\leq D \sqrt{T} \frac{k-b}{k-b} + \frac{2C}{3} \frac{T^{4-r}}{2} =: E,
\]

i.e. we have uniform bounds in \( i \) on any given compact subset \( K \).

2. Next we derive uniform derivative bounds for \( L_i \) on compact subsets \( K \) of space-time as above. We will first prove the following

**Lemma 3.9.** Under the assumptions of Theorem 3.7 let \( \gamma_i(t) \) be an \( \mathcal{L} \)-minimizing \( \mathcal{L} \)-geodesics from \( (q, \bar{t}) \) to \( (p, t_i) \), where \( (q, \bar{t}) \in K \). Then there exists a constant \( G \) independent of \( i \) such that for all \( t \in [i, k] \)

\[
|\sqrt{t_i-t} \gamma_i(t)|^2_{g(t)} \leq G.
\]

**Proof.** Denote by \( V_i(t) = \sqrt{t_i-t} \gamma_i(t) \). Using the \( \mathcal{L} \)-geodesic equation (2.15) we compute

\[
\frac{d}{dt}|V_i(t)|^2_{g(t)} = -2\text{Ric}(V_i(t), V_i(t)) + 2\langle \nabla_{\gamma_i(t)} V_i(t), V_i(t) \rangle_t \\
= -2\text{Ric}(V_i(t), V_i(t)) + \frac{2}{\sqrt{t_i-t}} \langle \nabla_{V_i(t)} V_i(t), V_i(t) \rangle_t \\
= -2\text{Ric}(V_i(t), V_i(t)) + \frac{2}{\sqrt{t_i-t}} (2\sqrt{t_i-t} \text{Ric}_{g(t)}(V_i(t), \cdot)) + \frac{1}{2} (t_i-t) \langle \text{R}^{g(t)} R_{g(t)}, V_i(t) \rangle_t \\
= 2\text{Ric}(V_i(t), V_i(t)) + \sqrt{t_i-t} \langle \text{R}^{g(t)} R_{g(t)}, V_i(t) \rangle_t \\
\leq 2\frac{C_1}{(T-t)^r} |V_i(t)|^2_{g(t)} + \frac{C_2}{(T-t)^{r-\frac{1}{2}}} |V_i(t)|^2_{g(t)},
\]

where in the last inequality, since \( t \geq \bar{t} > 0 \), we used Shi’s derivative estimates [Shi89b] combined with the type A assumption to bound \( \text{R}^{g(t)} R_{g(t)} \). Note that
$C_{1}, C_{2}$ are constants depending on the type A constant $C$, $n$ and $t$, but are independent of $i$.

Before integrating this ordinary differential inequality to conclude the proof of the lemma, we need to get uniform bounds on each $V_{i}(t)$ for some $t$ in a compact set of time: By definition of $\mathcal{L}$ and using again the type A assumption we estimate

$$
\int_{t}^{t_{1}} \frac{1}{\sqrt{t - t}} |V_{i}(t)|_{g(t)}^{2} dt = \mathcal{L}(\gamma_{i}) - \int_{t}^{t_{1}} \sqrt{t - t} R_{g(t)}(\gamma_{i}(t)) dt \\
\leq \mathcal{L}(\gamma_{i}) + \frac{2C}{3 - 2r} T^{\frac{3}{2}} - r.
$$

Now the integral mean value theorem yields the existence of $\hat{t}_{i} \in [\bar{t}, k]$, such that

$$
\frac{1}{k - t} |V_{i}(\hat{t}_{i})|_{g(\hat{t}_{i})}^{2} = \frac{1}{k - t} \int_{t}^{\hat{t}_{i}} \frac{1}{\sqrt{t - t}} |V_{i}(t)|_{g(t)}^{2} dt \\
\leq \frac{1}{k - t} \int_{t}^{t_{1}} \frac{1}{\sqrt{t - t}} |V_{i}(t)|_{g(t)}^{2} dt \\
\leq \frac{1}{k - t} \mathcal{L}(\gamma_{i}) + \frac{2C}{3 - 2r} T^{\frac{3}{2}} - r,
$$

or equivalently

$$
|V_{i}(\hat{t}_{i})|_{g(\hat{t}_{i})}^{2} = \frac{\sqrt{t - t_{1}}}{k - t} (\mathcal{L}(\gamma_{i}) + \frac{2C}{3 - 2r} T^{\frac{3}{2}} - r) \\
\leq \frac{\sqrt{T}}{k - t} (E + \frac{2C}{3 - 2r} T^{\frac{3}{2}} - r) =: F,
$$

since by choice of $\gamma_{i}$ the bound (3.3) holds for $\mathcal{L}(\gamma_{i})$.

W.l.o.g. we can assume that $|V_{i}(t)|_{g(t)} \geq 1$ and estimate (3.4) for $t \in [a, k]$ to get

$$
\frac{d}{dt} |V_{i}(t)|_{g(t)}^{2} \leq \frac{2}{(T - k)^{r}} \left( \frac{C_{1}}{(T - k)^{r}} + \frac{C_{2}}{(T - k)^{r} - \frac{r}{2}} \right) |V_{i}(t)|_{g(t)}^{2} = C_{3}|V_{i}(t)|_{g(t)}^{2}.
$$

Integrating this, we conclude that for all $t \in [\bar{t}, k]$

$$
|V_{i}(t)|_{g(t)}^{2} \leq F e^{C_{3}(t - \bar{t})} \leq F e^{C_{3} T} =: G.
$$

This proves the Lemma. \hfill \square

To get the gradient bounds for $L_{i}$ recall that from (2.13)

$$
\nabla g(t) L_{i}(q, \bar{t}) = -2 \sqrt{t_{i} - t} \hat{\gamma}_{i}(t),
$$

so with Lemma 3.9 we obtain for $(q, \bar{t}) \in K$

$$
|\nabla g(t) L_{i}(q, \bar{t})|_{g(t)} = 2 \sqrt{t_{i} - t} \hat{\gamma}_{i}(t)|_{g(t)} \leq \sqrt{2G}.
$$

For the time derivative bound for $L_{i}$ we compute

$$
\frac{\partial}{\partial t} L_{i}(q, \bar{t}) = \frac{d}{dt} L_{i}(q, \bar{t}) - \langle \nabla g(t) L_{i}(q, \bar{t}), \gamma(t) \rangle_{\bar{t}} \\
= -\frac{1}{\sqrt{t_{i} - t}} \left( |\hat{\gamma}_{i}(t)|_{g(t)}^{2} + R_{g(t)}(\gamma(t)) \right) + 2 \sqrt{t_{i} - t} \hat{\gamma}_{i}(t)|_{g(t)}^{2} \\
= \frac{1}{\sqrt{t_{i} - t}} \left( \sqrt{t_{i} - t} \hat{\gamma}_{i}(t)|_{g(t)}^{2} - \sqrt{t_{i} - t} R_{g(t)}(\gamma(t)) \right)
$$
Using the type A assumption and Lemma 3.9 we get the time derivative bound for $(q, \bar{t}) \in K$

\begin{equation}
|\frac{\partial}{\partial t} L_i(q, \bar{t})| \leq \frac{G}{\sqrt{k-b}} + \frac{C}{(T - b)^{r - \frac{1}{2}}}.
\end{equation}

Recall that by Remark 2.10 each $l_i$ is locally Lipschitz on $M \times (0, t_i)$. The above bounds show that $l_i$ are in fact uniformly locally bounded and Lipschitz on $M \times (0, T)$. Hence there exists a locally Lipschitz function

$$l_{p,T} : M \times (0, T) \to \mathbb{R}$$

such that a subsequence, denoted $l_i$ again, converges to a $l_{p,T}$ in $C^0_{\text{loc}}(M \times (0, T))$. This proves the first part of the theorem.

To prove that the differential inequality (3.1) holds in the sense of distributions, we first note that $l_i \in W^{1,2}_{\text{loc}}(M \times (0, T))$ and the bounds derived above imply

$$|l_i|_{W^{1,2}_{\text{loc}}(M \times (0, T))} < C.$$  

W.l.o.g. we can assume that

$$l_i \to l_{p,T}$$

weakly in $W^{1,2}_{\text{loc}}(M \times (0, T))$. This implies for all $(V, \psi) \in W^{1,2}_{\text{loc}}(M \times (0, T), \mathbb{R}^{n+1})$

$$\int_0^T \int_M \nabla^{g(t)} i_l \cdot V + \frac{\partial}{\partial t} l_i \psi \, dvol_{g(t)} dt \to \int_0^T \int_M \nabla^{g(t)} l_{p,T} \cdot V + \frac{\partial}{\partial t} l_{p,T} \psi \, dvol_{g(t)} dt$$

In particular, if for nonnegative $\phi \in C^\infty_c(M \times (0, T))$, we let $\psi = -\phi$ and $V = \nabla^{g(t)} \phi$, we get the distributional convergence

$$\int_0^T \int_M \nabla^{g(t)} i_l \cdot \nabla^{g(t)} \phi - \frac{\partial}{\partial t} l_i \phi \, dvol_{g(t)} dt$$

$$\to \int_0^T \int_M \nabla^{g(t)} l_{p,T} \cdot \nabla^{g(t)} \phi - \frac{\partial}{\partial t} l_{p,T} \phi \, dvol_{g(t)} dt.$$  

Comparing with the distributional formulation (2.20), we see that we now only need to show that for all nonnegative $\phi \in C^\infty_c(M \times (0, T))$

$$\int_0^T \int_M |\nabla^{g(t)} i_l|^2_{g(t)} \phi \, dvol_{g(t)} dt \to \int_0^T \int_M |\nabla^{g(t)} l_{p,T}|^2_{g(t)} \phi \, dvol_{g(t)} dt.$$  

It suffices to show for each $t \in (0, T)$ and nonnegative $\phi \in C^\infty_c(M)$

$$\int_M |\nabla^{g(t)} i_l|^2_{g(t)} \phi \, dvol_{g(t)} \to \int_M |\nabla^{g(t)} l_{p,T}|^2_{g(t)} \phi \, dvol_{g(t)}.$$  

Since this is weak $L^2$ convergence of $\sqrt{\phi} \nabla^{g(t)} i_l$, we have

$$\int_M |\nabla^{g(t)} l_{p,T}|^2_{g(t)} \phi \, dvol_{g(t)} \leq \liminf_{i \to \infty} \int_M |\nabla^{g(t)} i_l|^2_{g(t)} \phi \, dvol_{g(t)}.$$  

We now show the other direction

$$\limsup_{i \to \infty} \int_M |\nabla^{g(t)} i_l|^2_{g(t)} \phi \, dvol_{g(t)} \leq \int_M |\nabla^{g(t)} l_{p,T}|^2_{g(t)} \phi \, dvol_{g(t)}.$$
using an argument similar to Lemma 9.21 in [MT06]. We rewrite

\[
\limsup_{i \to \infty} \int_M \left( |\nabla g(t) l_i|^2_{g(t)} - |\nabla g(t) l_{p,T}|^2_{g(t)} \right) \phi \, dvol_{g(t)}
\]

(3.8)

\[
= \limsup_{i \to \infty} \left( \int_M \langle \nabla g(t) (l_i - l_{p,T}), \phi \nabla g(t) l_{p,T} \rangle_t \, dvol_{g(t)} \right.
\]

\[
+ \int_M \langle \nabla g(t) (l_i - l_{p,T}), \phi \nabla g(t) l_i \rangle_t \, dvol_{g(t)}. \]

We can approximate \( \phi \nabla g(t) l_{p,T} \) by \( V_j \in C^\infty_{cp}(M, \mathbb{R}^n) \) and conclude by weak \( L^2 \)

convergence of \( \nabla g(t) l_i \) that the first integral goes to zero as \( i \to \infty \). For the second integral, we first use the \( C^0_{loc} \) convergence of \( l_i \to l_{p,T} \) to get a sequence \( \epsilon_i \downarrow 0 \), such that on \( \text{supp}(\phi) \) we have

\[
|l_{p,T} - l_i + \epsilon_i| > 0.
\]

Then the second integral above equals

\[
\limsup_{i \to \infty} \int_M \langle \nabla g(t) (l_i - l_{p,T} - \epsilon_i), \phi \nabla g(t) l_i \rangle_t \, dvol_{g(t)}.
\]

We multiply Perelman’s differential inequality (2.18) for \( l_i \) by \( \phi \) and write it in the
distributional sense for a nonnegative \( \psi \in C^\infty_{cp}(M) : \)

\[
- \int_M \langle \nabla g(t) (\psi \phi), \nabla g(t) l_i \rangle_t \, dvol_{g(t)} \leq \int_M \frac{\psi \phi}{2} \left( |\nabla g(t) l_i|^2_{g(t)} - R_g(t) - \frac{l_i - n}{t_i - \tau} \right) \, dvol_{g(t)}.
\]

By approximation in \( W^{1,2} \), we can take \( \psi = l_{p,T} - l_i + \epsilon_i \geq 0 \) to be only locally

Lipschitz, i.e. conclude

\[
\int_M \langle \nabla g(t) ((l_i - l_{p,T} - \epsilon_i) \phi), \nabla g(t) l_i \rangle_t \, dvol_{g(t)}
\]

\[
\leq \int_M \frac{l_{p,T} - l_i + \epsilon_i}{2} \phi \left( |\nabla g(t) l_i|^2_{g(t)} - R_g(t) - \frac{l_i - n}{t_i - \tau} \right) \, dvol_{g(t)}. \]

Since the right-hand integrand is bounded on \( \text{supp}(\phi) \) and \( l_{p,T} - l_i + \epsilon_i \to 0 \) uniformly, we obtain

\[
\limsup_{i \to \infty} \int_M \langle \nabla g(t) ((l_i - l_{p,T} - \epsilon_i) \phi), \nabla g(t) l_i \rangle_t \, dvol_{g(t)} \leq 0.
\]

Now inserting the characteristic function \( \chi_{\text{supp}(\phi)} \), we can rewrite

\[
\int_M \langle \nabla g(t) ((l_i - l_{p,T} - \epsilon_i) \phi), \nabla g(t) l_i \rangle_t \, dvol_{g(t)}
\]

\[
= \int_M \langle \nabla g(t) (l_i - l_{p,T} - \epsilon_i) \phi, \nabla g(t) l_i \rangle_t \, dvol_{g(t)}
\]

\[
+ \int_M \langle (l_i - l_{p,T} - \epsilon_i) \nabla g(t) \phi, \chi_{\text{supp}(\phi)} \nabla g(t) l_i \rangle_t \, dvol_{g(t)}
\]

\[
= \int_M \langle \nabla g(t) (l_i - l_{p,T}), \phi \nabla g(t) l_i \rangle_t \, dvol_{g(t)}
\]

\[
+ \int_M \frac{1}{2} (l_i - l_{p,T} - \epsilon_i) \langle |\nabla g(t) \phi|^2_{g(t)} + |\chi_{\text{supp}(\phi)} \nabla g(t) l_i|^2_{g(t)} \rangle \, dvol_{g(t)}. \]
As before, the last integral tends to zero because of the uniform convergence of 
\( l_{p,T} - l_i + \epsilon_i \to 0 \). This implies that the second term in (3.8) satisfies
\[
\limsup_{i \to -\infty} \int_M \langle \nabla g(t)(l_i - l_{p,T}), \phi \nabla g(t) l_i \rangle d\text{vol}_g(t) \leq 0
\]
and finishes the proof. \( \square \)

4. Reduced volume and monotonicity based at singular time

**Definition 4.1.** Let \((M, g(t))\) be a Ricci flow on \([0, T)\) of type A. Let \( p \in M, \) \( t_i \not\to T, \) and \( l_{p,T} \) and \( v_{p,T} \) as in Theorem 3.7. Then we define a **reduced volume based a singular time** \((p, T)\) by
\[
\tilde{V}_{p,T}(\bar{t}) := \int_M v_{p,T}(q, \bar{t}) d\text{vol}_g(q) = \int_M ((4\pi(T - \bar{t}))^{-\frac{n}{2}} e^{-l_{p,T}(q, \bar{t})}d\text{vol}_g(q).
\]

**Remark 4.2.** The finiteness of \( \tilde{V}_{p,T}(\bar{t}) \) for any Ricci flow \((M, g(t))\) and any fixed \( \bar{t} \in (0, T) \) follows from Fatou’s lemma and the finiteness of Perelman’s reduced volume as follows:
\[
\tilde{V}_{p,T}(\bar{t}) = \int_M \lim_{t_i \not\to T} \left( ((4\pi(T - \bar{t}))^{-\frac{n}{2}} e^{-l_{p,T}(q, \bar{t})}d\text{vol}_g(q) \right)
\]
\[
\leq \liminf_{t_i \not\to T} \tilde{V}_{p,T}(\bar{t}) \leq 1.
\]

Now we state the analogue of Perelman’s monotonicity (Corollary 2.13) for the reduced volume based at singular time.

**Corollary 4.3** (Monotonicity of the reduced volume based at singular time). *Under the assumptions as in Definition 4.1 we have*

(i) \( \frac{d}{dt} \tilde{V}_{p,T}(\bar{t}) \geq 0 \),

(ii) \( \lim_{t_i \not\to T} \tilde{V}_{p,T}(\bar{t}) \leq 1 \),

(iii) a) If \( \tilde{V}_{p,T}(\bar{t}_1) = \tilde{V}_{p,T}(\bar{t}_2) \) for 0 < \( \bar{t}_1 < \bar{t}_2 < T \), then \((M, g(T - 1))\) is a gradient shrinking soliton with potential function \( l_{p,T}(\cdot, T - 1) \).

b) If \((M, g(t), f(t))\) is a gradient shrinking soliton in canonical form and \( p \in M \), then any \( \tilde{V}_{p,T}(\bar{t}) \) is constant in \( \bar{t} \).

**Remark.** A similar statement has very recently also been obtained by [Nab07] under the type I assumption (and \( \kappa - \)noncollapsedness).

**Proof.** (i) If \( M \) is compact and \( l_{p,T} \) is smooth the proof follows directly from (3.2) in Theorem 3.7:
\[
\frac{d}{dt} \tilde{V}_{p,T}(\bar{t}) = \int_M \left( \frac{\partial}{\partial t} v_{p,T} - R \right) d\text{vol}_g(q)
\]
(4.1)
\[
= \int_M \left( \frac{\partial}{\partial t} v_{p,T} + \Delta v_{p,T} - R \right) d\text{vol}_g(q)
\]
\[
= \int_M -\Box^* v_{p,T} d\text{vol}_g(q)
\]
\[
\geq 0.
\]
In the general case we need to justify the differentiation under the integral and the adding in of the Laplacian term. For both arguments, we need to bound
\begin{equation}
  \int_M e^{-l_{p,T}}|l_{p,T}|\text{dvol}_{g(t)} < \infty
\end{equation}
for fixed time \( \bar{t} \in (0, T) \). From Remark 4.2 we know that
\[ \int_M e^{-l_{p,T}}\text{dvol}_{g(t)} < \infty, \quad \text{and hence} \quad \int_M e^{-|l_{p,T}|}\text{dvol}_{g(t)} < \infty. \]
Since the proof of the finiteness of Perelman’s reduced volume \( V_{p,t} \), relies on comparison of the reduced distance \( l_{p,t} \) with the square of the distance function, we also get that
\[ \int_M e^{-\frac{1}{2}l_{p,T}}\text{dvol}_{g(t)} < \infty, \quad \text{and hence} \quad \int_M e^{-\frac{1}{2}|l_{p,T}|}\text{dvol}_{g(t)} < \infty. \]

Now let \( N := \{ q \in M \mid l_{p,T}(q, \bar{t}) \geq 0 \} \). Then
\[ \int_M e^{-l_{p,T}}|l_{p,T}|\text{dvol}_{g(t)} = \int_{M-N} e^{-l_{p,T}}|l_{p,T}|\text{dvol}_{g(t)} + \int_N e^{-l_{p,T}}|l_{p,T}|\text{dvol}_{g(t)}, \]
where the second term is finite since \( \frac{1}{2}|l_{p,T}| \leq e^{\frac{1}{2}|l_{p,T}|} \). The first integral is bounded because the type A curvature bound yields an upper bound on \( |l_{p,T}| \) on \( N \) by dropping the energy part in the \( L \)-functional. This proves (4.2).

The proof of Lemma 3.9 implies that in fact there exist constants \( C_1 \) and \( C_2 \) (depending on \( \bar{t} \)) such that
\[ |\nabla g(t) l_{p,T}|_{g(t)}^2 \leq C_1 |l_{p,T}|_{g(t)} + C_2. \]
(Note that \( L_{p,T} \) gets bounded on compact sets by the constant \( E \) in that proof.) This implies that
\begin{equation}
  \int_M e^{-l_{p,T}}|\nabla g(t) l_{p,T}|_{g(t)}^2\text{dvol}_{g(t)} < \infty.
\end{equation}

We lastly show that also
\begin{equation}
  \int_M e^{-l_{p,T}}|\frac{\partial}{\partial t} l_{p,T}|\text{dvol}_{g(t)} < \infty.
\end{equation}
The differential equality analogous to (2.19)
\[ -2 \frac{\partial}{\partial t} l_{p,T}(q, \bar{t}) + |\nabla g(t) l_{p,T}(q, \bar{t})|^2_{g(t)} - R_{g(t)} + \frac{l_{p,T}(q, \bar{t})}{T-\bar{t}} = 0 \]
is satisfied by \( l_{p,T} \) for the same reasons that (3.1) in Theorem 3.7 holds. We can use it to bound \( |\frac{\partial}{\partial t} l_{p,T}| \). Then (4.4) follows from (4.2) and (4.3) and the bounded curvature assumption.

Now we can justify what corresponds to the adding in of the Laplacian term in (4.1) in the distributional setting. As in (2.20), Theorem 3.7 implies that for fixed \( \bar{t} \in (0, T) \) and for any nonnegative \( \phi \in C^\infty(M) \)
\[ \int_M \left( \left( -\frac{\partial}{\partial t} l_{p,T} + |\nabla g(t) l_{p,T}|^2_{g(t)} - R_{g(t)} + \frac{n}{2(T-\bar{t})} \right) \phi + \nabla g(t) l_{p,T} \cdot \nabla g(t) \phi \right)\text{dvol}_{g(t)} \geq 0. \]
Note that
\[ \nabla g(t) (e^{-l_{p,T}}) = -e^{-l_{p,T}} \nabla g(t) l_{p,T}, \]
so using the bounds for the integrals proved above and approximating \( \phi = e^{-l_{p,T}} \) by \( \phi_j \in C_\text{cpt}^\infty(M) \), we conclude that

\[
\int_M \left( - \frac{\partial}{\partial t} l_{p,T} - R_{g(t)} + \frac{n}{2(T-t)} \right) e^{-l_{p,T}} \text{dvol}_{g(t)} \geq 0.
\]

Up to the factor of \((4\pi(T-t))^{-\frac{n}{2}}\) this is the integral we obtain when differentiating the integrand in the reduced volume based at \((p, T)\). To conclude (i), we only have to justify the differentiation under the integral sign. We will do so using the standard dominated convergence argument. We bound, at \( t \in (0, T) \), the reduced volume integrand difference quotient \( D(q, \bar{t}, h) \), rewritten in integral form, by an integrable function:

\[
D(q, \bar{t}, h) = \frac{1}{h} \int_0^h \left( - \frac{\partial}{\partial \bar{t}} l_{p,T}(q, \bar{t} + s) - R_{g(\bar{t}+s)}(q) + \frac{n}{2(T-t-s)} \right) (4\pi(T-\bar{t} - s))^{-\frac{n}{2}} e^{-l_{p,T}(q, \bar{t} + s)} \frac{\text{dvol}_{g(\bar{t}+s)}(q)}{\text{dvol}_{g(\bar{t})}(q)} d\bar{t}
\]

(compare e.g. [CCG+07] for the argument in Perelman’s case) Now from Remark 4.2 and the bound (4.4) as well as the exponential growth bound of the volume form under Ricci flow, which makes the quotient bounded near \( \bar{t} \) where we have bounded curvature), we conclude that \( D(q, \bar{t}, h) \) is dominated by an integrable function for \( h \) near 0.

(ii) This follows from (i) and Remark 4.2.

(iii) a) From the initial computation in (i), which we have justified for noncompact \( M \) and in the distributional setting, we get

\[
0 = \tilde{V}_{p,T}(\bar{t}_2) - \tilde{V}_{p,T}(\bar{t}_1) = \int_{\bar{t}_1}^{\bar{t}_2} d\tilde{V}_{p,T}(\bar{t}) d\bar{t} = \int_{\bar{t}_1}^{\bar{t}_2} \int_M -\Box^* v_{p,T} \text{dvol}_{g(\bar{t})} d\bar{t}.
\]

From (3.2) we conclude that in the sense of distributions

\[
\Box^* v_{p,T} \equiv 0,
\]

and hence by parabolic regularity \( l_{p,T} \) is smooth. It follows from the proof of Theorem 3.7 that (2.19) holds for \( l_{p,T} \):

\[
-2 \frac{\partial}{\partial \bar{t}} l_{p,T}(q, \bar{t}) + |\nabla_{g(\bar{t})} l_{p,T}(q, \bar{t})|^2_{g(\bar{t})} - R_{g(\bar{t})} + \frac{l_{p,T}(q, \bar{t})}{T - \bar{t}} = 0.
\]

Combining (4.5) and (4.6) we get

\[
w_{p,T} := \left((T - \bar{t})\left(2\Delta_{g(\bar{t})} l_{p,T} - |\nabla_{g(\bar{t})} l_{p,T}|^2_{g(\bar{t})} + R\right) + l_{p,T} - n\right) v_{p,T} \equiv 0.
\]

Hence

\[
0 \equiv \Box^* w_{p,T} = -2(T - \bar{t}) \left| \text{Ric}_{g(\bar{t})} + \nabla_{g(\bar{t})} \nabla_{g(\bar{t})} l_{p,T} - \frac{1}{2(T - \bar{t})} \right|_{g(\bar{t})}^2 v_{p,T},
\]

where the last equality follows from Perelman’s Proposition 9.1 [Per02]. Because of \( v_{p,T} \geq 0 \) we conclude that \((M, g(T - 1))\) is a gradient shrinking soliton with potential function \( l_{p,T}(\cdot, T - 1)\).
b) Let \( p \in M \). From Example 3.6 we get that the type A assumption is satisfied. Then Example 3.3 shows that there exists a constant \( C \) such that
\[
l_{p,T}(q, \tilde{t}) = f(q, \tilde{t}) + C,
\]
and in particular \( l_{p,T} \) is smooth. Hence the inequality (3.1) becomes the equality (2.9), i.e. \( \Box^* v_{p,T} = 0 \). From the arguments in (i), we conclude
\[
\frac{d}{dt} \tilde{V}_{p,T}(\tilde{t}) = 0
\]
for all \( \tilde{t} \in (0, T) \), which finishes the proof of Corollary 4.3.

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