Tracing a planet’s orbit with a straight hedge and a compass with the help of the hodograph and the Hamilton vector

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Abstract

We describe a geometrical method for tracing a planet’s orbit using its velocity hodograph, that is, the path of the planet’s velocity. The method requires only a straight edge, a compass, and the help of the hodograph. We also obtain analitically the hodograph and some of the features of the motion that can be obtained from it.
I. INTRODUCTION

One way of solving the Kepler problem is to determine the path of the planet in velocity space. This path is called the hodograph.$^1$ Although this way of solving the problem apparently involves a detour, it ends up being one of the simplest ways of finding the orbit.$^2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17$ Moreover, this approach makes it straightforward to obtain an additional constant of the motion, namely the Hamilton vector. A recent contribution to using the hodograph is the interesting paper by Derbes.$^8$ This discussion is beautifully conducted using Euclidean geometry, but some people may be confounded by the use of such arguments.

We discuss a different geometrical approach for tracing a planet’s orbit that also uses the hodograph. Our method can be cast as a series of steps starting from the initial position and velocity of the planet. Then using the Hamilton vector and the hodograph, the method directly leads to the points on the planetary orbit. In addition to constructing the orbits, we discuss other features of the motion that follow naturally from our approach.

The paper is organized as follows. In Sec. II the analytical basis of the hodographic method is expounded. We introduce the Hamilton vector, an interesting conserved quantity that is rarely used today. We then obtain the hodograph and some of the features of the motion that can be obtained from it. The geometrical method for determining the orbit is the subject of Sec. III. We describe a straight edge and compass method for finding both the hodograph and the planet’s orbit starting from its initial position and velocity. The technique can be made simple enough to be used to teach some of the underpinnings of planetary motion — especially if it is used with a computer program for visualizing the constructions.$^{18,19}$ In Sec. IV we show geometrically that the orbit is indeed elliptical and that the direction of the velocities calculated from it correspond to those directly obtained from the hodograph. Finally, in Sec. V we make some concluding remarks and suggest some problems that can be solved with the methods presented in the paper.

II. THE ANALYTIC APPROACH TO THE HODOGRAPH

The geometrical argument$^{5,8}$ used for establishing the circularity of the Kepler problem hodograph can be stated analytically as follows.$^6,17$ The equation of motion of a planet of
mass \( m \) in the gravitational field of the sun is

\[
m \frac{dv}{dt} = - \frac{GMm}{r^2} \hat{e}_r
\]

(1)

where \( M \) is the mass of the sun, \( r \) is the distance between the sun and the planet, and \( \hat{e}_r \) is a unit vector that points in the radial direction from the sun toward the planet. We work in a polar coordinate system where the basis vectors \( \hat{e}_r, \hat{e}_\phi, \) and \( \hat{e}_z \) satisfy \( \hat{e}_r = \hat{e}_\phi \times \hat{e}_z \).

Hence, using the polar identity \( \dot{\hat{e}}_\phi = -\dot{\phi} \hat{e}_r \) and the conservation of angular momentum (so that \( L = mr^2 \dot{\phi} \) is a constant), we can eliminate \( r \) and the unit vector \( \hat{e}_r \), and replace them by \( \phi \) and \( \dot{\hat{e}}_\phi \). In this way we can express Eq. (1) in the form

\[
\frac{d}{dt} (v - u \dot{\hat{e}}_\phi) = \frac{d}{dt} (v - u) = 0,
\]

(2)

where we have defined the constant \( u \equiv GMm/L \) which has dimensions of velocity and the rotating vector of constant magnitude \( u \equiv u \hat{e}_\phi \). Equation (2) shows that the Hamilton vector,

\[
h \equiv v - u \hat{e}_\phi = v + \frac{u}{rL} (r \times L),
\]

(3)

is a constant of the motion in the Kepler problem.

The constancy of \( h \) is an exclusive feature of the \( 1/r^2 \) force and is related to its extraordinary symmetries.\(^\text{20}\) Equation (2) shows that \( h \) is in the plane of the orbit and orthogonal to \( L \). Equation (3) also shows that the hodograph has a dynamical symmetry axis that is defined by \( h \). The axis is dynamical in the sense that the constant Hamilton vector \( h \) is defined using the dynamical variables \( r \), and \( v \). Hence, the orbit also has to have a dynamical symmetry axis which must be orthogonal to \( h \). The orbit’s symmetry axis can be defined by any vector constant orthogonal to \( h \) and in the orbital plane. An obvious choice is \( A = h \times L = v \times L - r(uL/r) \), which is the Laplace or Runge-Lenz vector.\(^\text{20}\) Another consequence of Eq. (3) is that every bounded orbit has to be not only planar but periodic.

From the definition (3) we can express the velocity as a rotating vector with fixed magnitude, \( u \hat{e}_\phi \), superimposed on the constant Hamilton vector

\[
v = h + u \hat{e}_\phi.
\]

(4)

As can be seen from Eq. (1) (by evaluating \( h \) for \( \phi = 0 \)), \( h \) is parallel to the velocity at the perihelion \( v_p \). Equation (1) also shows that the hodograph is an arc of a circle of radius \( u \) with its center located at the tip of \( h \), that is, it is not centered at the origin in velocity.
space. The hodograph is always concave toward the origin when the interaction is attractive as is the case of planetary motion (see Fig. 1).

The velocity vector does not necessarily traverse the entire hodographic circle during the motion; it may just move on a circular arc. To see this, we write Eq. (4) in cartesian coordinates using \( h \) to define the \( x \)-axis: \( v_x = u \sin \phi \) and \( v_y = u \cos \phi + h \), where \( \phi \) is the angle between \( u \) and \( h \). These expressions for \( v_x \) and \( v_y \) imply that the equation of the hodograph can be written as \( v_x^2 + (v_y - h)^2 = u^2 \), which represents a circle of radius \( u \) centered at the point \((0, h)\) in velocity space. If we make a simple substitution for the components, the speed \( v = \sqrt{v_x^2 + v_y^2} \) can be written as

\[
v = u \sqrt{1 + \epsilon^2 - 2\epsilon \cos \phi},
\]

(5)

where \( \epsilon \equiv h/u \). Equation (5) shows that the polar angle is limited to the interval \( |\phi| \leq \phi_{\text{max}} \) when \( \epsilon \geq 1 \), where \( \phi_{\text{max}} \equiv \arccos(1/\epsilon) \). That is, the hodograph coincides with the entire circle only when \( \epsilon < 1 \). When \( |\phi_{\text{max}}| \) is approached, the velocity becomes tangent to the hodograph and the speed reaches a limiting value \( v_\infty = u \sqrt{\epsilon^2 - 1} \). We conclude from this expression that when \( \epsilon > 1 \), the planet is unbounded and moves asymptotically (as \( t \to \infty \)) toward a point in velocity space that we correspondingly call \( v_\infty \).

Another interesting relation exists between the hodograph radius \( u \) and the speeds at the perihelion \( v_p \) and at the aphelion \( v_a \):

\[
u = \frac{1}{2}(v_a + v_p).
\]

(6)

To derive Eq. (6) we need to equate the total energy evaluated at these two special positions on the orbit, solve the resulting equation for \( u \), and then substitute the angular momentum evaluated at the perihelion. Moreover, \( h \) can be expressed in terms of the sum of these two velocities: \( h = (v_p + v_a)/2 \), and its magnitude can be expressed in terms of the difference of the corresponding speeds: \( h = (v_p - v_a)/2 \). These results are easily seen geometrically from Fig. 1.

The orbit of the planet can be obtained by projecting \( h \) onto \( \hat{e}_\phi \) to obtain

\[
r = \frac{L/m}{u + h \cos \phi},
\]

(7)

which is the polar equation of a conic with semilatus rectum \( L/mu \) and eccentricity \( \epsilon = h/u \). The angle \( \phi \), which has the same meaning in both Eqs. (7) and (5), is usually called the
true anomaly in celestial mechanics. Therefore, the possible orbits are ellipses when $h > u$ ($\epsilon < 1$), parabolas when $h = u$ ($\epsilon = 1$), and hyperbolas when $h < u$ ($\epsilon > 1$). If the orbit is elliptical, the hodograph traverses the entire circle. In any other case the hodograph traverses just an arc of the circle — although in the parabolic case, it only misses a single point on it.

By using Eqs. (5) and (7) in the bounded case, we can easily check that $v_p/v_a = r_a/r_p$, where $r_a$ and $r_p$ are, respectively, the distances to the planet at the aphelion and perihelion. This relation also follows from angular momentum conservation. We may also write the energy of the planet as

$$E = \frac{m}{2}(h^2 - u^2).$$

From Eq. (8) we immediately see that the orbit is elliptical if $E < 0$, parabolic if $E = 0$, and hyperbolic if $E > 0$. We may also see from Eq. (8) that in the hyperbolic case, we can write $h^2 = u^2 + v_\infty^2$. That is, the limiting velocity $v_\infty$, the Hamilton vector $h$, and the limiting vector $u$ as $t \to \infty (u_\infty)$, always form a right triangle with $h$ as the hypotenuse. This result comes in handy for deriving geometrically the Rutherford relation for the scattering of celestial bodies off the sun\textsuperscript{15,16} as is the case for comets moving in hyperbolic orbits.

### III. TRACING THE ORBIT FROM THE INITIAL CONDITIONS

Geometrical methods are powerful and intuitive\textsuperscript{22}, although some students may find them unfamiliar and hence confusing. Nevertheless, these methods can be used to find $h$, the hodograph, and then to trace the orbit starting with an initial position $r_0$ and velocity $v_0$. If this method is properly presented, it can be very concrete because students can draw, point by point, any orbit by themselves.

The method can be described as follows: Given $r_0$ and $v_0$, we can obtain the magnitude of the planet’s angular momentum $L = mr_0v_0 \sin \delta$, where $\delta$ is the angle between $r_0$ and $v_0$ (see Fig. 1), or as the area spanned by these same two vectors (area $VOO'C$ in Fig. 1). Once $L$ is known, the hodograph radius $u = GMm/L$ can be calculated.

We next select a point $F$ on the plane as the origin of the coordinates, that is, as the position of the center of force. From this origin we draw a line segment $\overline{FR}$ (parallel to $r_0$) representing the initial position. The line $\overline{FR}$ can be extended to the position we choose for the velocity space origin $O$. From $O$ we also draw the segment $\overline{OV}$ corresponding to
the initial velocity. Then we draw, perpendicular to $\overrightarrow{FR}$, a line segment $\overline{OO'}$ of length $u$. Using the parallelogram rule, we add the segments $\overline{OV}$ and $\overline{OO'}$ to obtain the line $\overline{OC}$ corresponding to Hamilton’s vector. It is now a matter of tracing a circle of radius $u$ centered at point $C$, the tip of Hamilton’s vector. This circle is the hodograph. Notice that the points marked $V_p$ and $V_a$ correspond, respectively, to the velocities at the perihelion and at the aphelion. The velocity vectors at the aphelion and perihelion are necessarily orthogonal to the symmetry axis ($\overrightarrow{FP}$) of the orbit. This construction is illustrated in Fig. 1. The symmetry axis has the direction of the Runge-Lenz vector (shown in Fig. 2).

Figures 2, 3, and 4 include the same information as Fig. 1, but have certain features that have been added or removed to focus the reader on a particular point. We have packed much information in Figs. 1 and 2. Angular momentum conservation is explicitly included because we have assumed that the orbit lies in a plane. The flatness of all the orbits can be shown to imply the central nature of the force.\(^{23}\)

Given the amount of information in Fig. 1, it is not surprising that we can determine from Fig. 1 the bounded or unbounded nature of the orbit stemming from $r_0$ and $v_0$: if the point $O$, the origin in velocity space, is within the hodographic circle, the orbit is necessarily elliptical and hence bounded, otherwise, the orbit is hyperbolic or parabolic (and hence unbounded). The parabolic case only occurs if $O$ sits exactly on the hodographic circle, a property that follows directly from Eq. (8). For a circular orbit the center of the hodograph $C$ coincides with the velocity space origin $O$, that is, $h = 0$, which means that the speed equals the constant hodograph radius $u$.

To completely determine the orbit (elliptical, in this case) with the information shown in Fig. 1, we proceed as follows (see Fig. 2). Trace the line $\overrightarrow{FP}$ that is perpendicular to the line $\overline{OC}$ and passes through $F$. This line is the symmetry axis of the orbit, as follows from the orthogonality property mentioned earlier. By using the holograph and the symmetry axis of the orbit, we can begin to locate points on the planet’s orbit. In Fig. 2 the points marked $O$, $V$, $F$, $R$, and $C$, have the same interpretation as in Fig. 1; for example, the segment $\overline{OC}$ represents the Hamilton vector $\mathbf{h}$.

To locate any point on the orbit, extend the line $\overrightarrow{VO}$ back until it intercepts the hodograph at point $V_s$. Trace a segment that is perpendicular to $\overline{CV_s}$ and passes through $R$. This line intercepts the symmetry axis at the point $F'$. To locate the point on the orbit corresponding to any given point on the hodograph, we notice that we already have one such pair of points,
namely the initial conditions (points $R$ and $V$ in Fig. 2). We choose another point, $V'$, on the hodograph, and extend the straight line $OV'$ until it again intersects the hodograph at point $V''$. Draw two perpendiculrals, one to $CV'$ and the other to $CV''$, passing through $F$ and $F'$, respectively. The intersection, $R'$, of these two perpendiculrals is the required point on the orbit. This construction is similar to the case of lines $CV$ and $CV''$ that meet at the initial condition $R$.

This process is repeated for every point on the hodograph and in this way we can trace the complete orbit starting with the initial conditions and using only a straight edge and a compass. This last feature is a manifestation of the extreme regularity of the orbit. We should note that parabolic and hyperbolic orbits can also be traced using variants of the method described above.

IV. THE SHAPE OF THE ORBIT AND THE ASSOCIATED VELOCITIES

Our method for drawing an orbit is fully contained in Sec. III. Here we will address two loose ends that are not important if you are only interested in tracing the orbit. What is the shape of the orbit and how can we be sure that the points we have found on the orbit are the required points for the velocity vectors as required by the dynamics?

That the loci of the points found by the method of Sec. III is indeed an ellipse can be seen as follows. We first draw a circular arc $FW'$ centered at point $R$ with radius $RF'$; this arc helps trace an auxiliary circle centered at $F$ with a radius equal to the sum of the lengths of the lines $FR$ and $RF'$, that is, the length of $FW$. This radius equals the length of the orbit’s major axis, that is, the line $P_aP_p$, in Fig. 3. The points $W$ and $W'$ are, respectively, the intersections of lines $FR$ (the initial condition) and $FW'$ (the calculated point on the orbit) with the auxiliary circle.

The isosceles triangles $\triangle CV'V''$ and $\triangle R'W'F'$ in Fig. 3 are similar to each other, because the line $FW'$ makes the same angle with line $F'F''$ as the line $CV'$ makes with the line $OC$. Thus the point $R'$ on the orbit is at the same distance from both $W'$ and $F'$. So we must have that $FW' + F'W'' = FR + RF' = P_aP_p$. This description shows that the sum of the distances from points on the orbit, such as $R$ and $R'$, to the points $F$ and $F'$ is a constant; this equality is precisely the defining property of an ellipse. Therefore, the planets travel on elliptical paths and the sun (the origin) is located at the position of one of the foci of the...
ellipse, $F$ in Figs. 1, 2, and 3; $F'$ is the other focus of the ellipse.

Once the shape of the orbit has been established, we can check that the velocities, as defined by points on the hodograph, are parallel to the tangents at the corresponding points on the ellipse. For example, in Fig. 3, point $V$ on the hodograph and the tangent to the ellipse at point $R$ are parallel.

Figure 4 is similar to Fig. 3 with certain lines added and others removed with the purpose of explaining what follows. The argument relies on identifying the three similar triangles, $\triangle WFW'$, $\triangle WRF'$, and $\triangle F'R_sW_s$, which by construction are isosceles, and on the fact that a triangle inscribed in a circle (for example, $\triangle WW'W_s$) whose diameter coincides with one of the sides of the triangle, is necessarily a right triangle. Trace the lines $\overline{WW_s}$ through $F'$, $\overline{W_sW''}$ through $F$, and $\overline{W''W}$ which closes the right triangle. Now trace lines $\overline{MR}$ and $\overline{M_sR_s}$. These are, respectively, perpendicular bisectors of the lines $\overline{WF'}$ and $\overline{F'W_s}$, and at the same time, bisectors of the angles $\angle WRF'$ and $\angle F'R_sW_s$. These properties guarantee that $\overline{MR}$ and $\overline{M_sR_s}$ are parallel to the tangents to the orbit at the points $R$ and $R_s$. To see that $\overline{MR}$ and $\overline{M_sR_s}$ are parallel to $\overline{VO}$ and $\overline{OV_s}$, respectively, it suffices to understand that $\overline{WF'}$ is perpendicular to $\overline{VV_s}$. This argument establishes that the tangent to any point on the orbit is necessarily parallel to the corresponding velocity on the hodograph.

V. CONCLUDING REMARKS

The geometrical approach described in this paper is simple and direct and can serve to explain, even to beginning students, how to trace a planet’s orbit from the initial conditions. If presented with no further explanations, the approach may be regarded as being similar to the idea described by the Mayan astronomer analogy as told by Richard Feynman. Our method is an attempt to exhibit in simple terms the geometric beauty of dynamics — beauty that captured the heart of Newton himself.

Our method can be profitably applied to other related problems. For example, what if we wish to describe the trajectory of a comet? How can the hodographic method be used when the initial conditions lead to a hodographic circle that does not surround the velocity space origin $O$? The extension of the method to parabolic or hyperbolic orbits can be a relatively simple project for interested students. Would it be possible to account for the untraversed branch of the hyperbola? Can this branch have some physical interpretation?
We also can take advantage of the right triangle formed by \( \mathbf{h}, \mathbf{u}, \text{and} \mathbf{v}_\infty \) to derive the cross section for comets bouncing off the sun. To begin with, the angle \( \xi \) between the vectors \( \mathbf{v}_{+\infty} \text{ and } \mathbf{v}_{-\infty} \) is the scattering angle. Then, after drawing the hodograph, the Hamilton vector \( \mathbf{h}, \) and the two velocity vectors \( \mathbf{v}_{\pm\infty}, \) it is a simple matter of geometry to obtain the relation

\[
L = \frac{GM}{v_\infty} \cot(\xi/2),
\]

from which the scattering cross section follows.\(^{24}\) Equation (9) is usually called the Rutherford relation.\(^{16}\)

A slightly more ambitious project would be to obtain astronomical data for a planet and use this information to determine the initial position and the initial velocity of the planet. Then the student could trace the orbit with the method described in this paper and then to check such theoretical orbit against the experimental data. That is, to compare the orbit determined directly from the astronomical data with the orbit traced with our method starting from just the initial conditions. A direct way of performing this comparison is to use widely distributed data as that in the Orbit of Mars Experiment of the Project Physics Course\(^{26}\), which includes photography of the night sky containing planet Mars and detailed instructions for reconstructing its orbit from such images.

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FIG. 1: The circular hodograph and the Hamilton vector $h$ for planetary motion. We exhibit the initial position $r_0$, the initial velocity $v_0$, and the rotating vector with constant magnitude $u$. We also show the polar unit vectors $\hat{e}_r$ and $\hat{e}_\phi$, the vector $\hat{e}_z$ points upward from the plane of the paper. $V_p$ and $V_a$ are the points on the hodograph corresponding to the velocities at the perihelion and at aphelion, respectively. The figure shows that $u = (v_a + v_p)/2$, $h = (v_a + v_p)/2$, and $h = (v_a - v_p)/2$. Note that by looking at the figure you can determine the shape of the planet’s orbit. If $O$ is within the hodographic circle, as in the case illustrated here, $E < 0$ and the orbit is elliptical. If $O$ sits on the hodographic circle, then the energy vanishes $E = 0$ and the orbit is parabolic. And if $O$ is outside the circle, then $E > 0$ and the orbit is hyperbolic.

FIG. 2: The geometrical method for tracing the orbit. $F$ is the position of the center of force, $C$ is the hodograph center, $O$ is the origin in velocity space, and the line $FF'$ is the orbit axis of symmetry, that is, the direction of the Runge-Lenz vector $A$. Note that every point on the hodograph (for example, $v$) corresponds to a point on the orbit ($r$). The symbols have the same meaning as the corresponding ones in Fig. 1. $P$ is not necessarily an apsidal point on the ellipse.

FIG. 3: We redraw Fig. 2 to show that the orbit is an ellipse. The shaded triangles are both isosceles and similar to each other. These properties are useful for showing that the sum of the distances from any point on the orbit to the points $F$ and $F'$ is a constant.
FIG. 4: Illustration of the fact that the velocity, as taken from the hodograph, is always parallel to the tangent at the corresponding point on the orbit. For example, the velocity corresponding to the line $\overline{OV}$ is parallel to the tangent line $\overline{MR}$ at the point $R$, and the to velocity, corresponding to the line $\overline{OV_s}$, is parallel to the line $\overline{M_sR_s}$ which is tangent to the orbit at point $R_s$. The three shaded triangles are similar to each other.