Statistical properties of high-lying chaotic eigenstates

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1 Introduction

The field of quantum chaos is developing very fast and there has been substantial progress in our understanding of generic properties of eigenstates in classically nonintegrable and chaotic bound systems. In contrast to the theoretical description of the energy spectra (and of other quantal observables) where we have now a rather complete understanding of the spectral statistical universality classes and also of statistics in the transition region between integrability and ergodicity (i.e. going from Poisson to GOE/GUE), we are still far from a correspondingly complete knowledge of generic and statistical properties of the wavefunctions. For a few recent reviews see contributions in Giannoni et al (1991), Gutzwiller’s book (1990), the papers in Casati et al (1993) and also the review on statistical properties of energy spectra by Robnik (1994).

In order to understand the wavefunctions especially in the semiclassical limit it is intuitively very appealing to use the so-called Principle of Uniform Semiclassical Condensation (PUSC) of the Wigner functions (of the eigenstates) which is implicit in (Berry 1977a): As $\hbar \to 0$ we assume that the Wigner function of a given eigenstate uniformly (ergodically) condenses on the classical invariant object on which the classical motion is ergodic and which supports the underlying quantal state. Such an object can be e.g. an invariant torus, a chaotic region as a proper subset of the energy surface, or the entire energy surface if the system has ergodic dynamics there.

In classically integrable systems the eigenfunctions possess a lot of ordered
structure globally and locally. Applying PUSC the average probability density in the configuration space is seen to be determined by the projection of the corresponding quantized invariant torus onto the configuration space, which implies the global order. Moreover, the local structure is implied by the fact that the wavefunction in the semiclassical limit is locally a superposition of a finite number of plane waves (with the same wavenumber as determined by the classical momentum).

In the opposite extreme of a classically ergodic system PUSC predicts that the average probability density is determined by the microcanonical Wigner function. Its local structure is spanned by the superposition of infinitely many plane waves with random phases and equal wavenumber. The random phases might be justified by the classical ergodicity and this assumption, originally due to Berry (1977b), is a good starting approximation which immediately predicts locally the Gaussian randomness for the probability amplitude distribution. Berry (1977b) has also calculated the autocorrelation function of semiclassical chaotic (ergodic) wavefunctions which we will discuss later on in detail. One major surprise in this research was Heller’s discovery (1984) of scars of unstable classical periodic orbits in classically ergodic systems. The scar phenomenon is of course a consequence of subtle correlations in the quantal phases. This has been analyzed and discussed by Bogomolny (1988) and Berry (1989) in the context of the Gutzwiller periodic orbit theory. The insufficiency of the single-periodic-orbit theory of scars has been discussed by Prosen and Robnik (1993a) in a study of the transition region between integrability and chaos.

In the generic case of a KAM-like system with mixed classical dynamics the application of PUSC is again very useful and has a great predictive power. Here the states can be classified as either regular (they "live" on a quantized invariant torus) or irregular (they "live" on a chaotic invariant region), quite in agreement with Percival’s (1973) speculative prediction, which has been recently carefully re-analyzed by Prosen and Robnik (1994a). In this case PUSC implies asymptotic ($\hbar \rightarrow 0$) statistical independence of level series (subsequences) associated with different regular and irregular components. This picture has been used by Berry and Robnik (1984) to deduce the resulting energy level statistics in such generic Hamilton systems with mixed classical dynamics, especially the level spacing distribution. In the recent
work Prosen and Robnik (1994b) have numerically confirmed the applicability of the Berry-Robnik theory and also explained the Brody-like behaviour (as discovered and described in (Prosen and Robnik 1993b)) before reaching the far semiclassical limit.

2 The definition of the billiard system and the numerical technique

In the present paper we study the chaotic wavefunctions in the 2-dim billiard system whose domain $\mathcal{B}$ (in w-plane) is defined by the complex quadratic conformal map of the unit disk (in z-plane), namely

$$\mathcal{B}_\lambda = \{ w | w = z + \lambda z^2, \ |z| \leq 1 \}. \quad (1)$$

as introduced by Robnik (1983,1984) and further studied by Prosen and Robnik (1993b). See also (Hayli et al 1987) and (Bruus and Stone 1994, Stone and Bruus 1993, 1994). Following most people in the field we shall refer to it as Robnik billiard. As the shape parameter $\lambda$ changes from 0 to 1/2 this system goes from the integrable case of the circular billiard continuously through a KAM-like regime to an almost ergodic regime at large $\lambda$. At $\lambda \leq 1/4$ the boundary is convex and therefore the Lazutkin like caustics and invariant tori (of boundary glancing orbits) exist. At $\lambda \geq 1/4$ the billiard was speculated (based on numerical evidence in (Robnik 1983)) to become ergodic, which has been disproved by Hayli et al (1987): Close to $\lambda \geq 1/4$ there are still some stable periodic orbits surrounded by very tiny stability islands. On the other hand, for $\lambda = 1/2$ (the cardioid billiard) the ergodicity and mixing have been rigorously proved by Markarian (1993). Nevertheless, at large values of $\lambda$, say $\lambda = 0.375$ (which we study exclusively in the present paper) the numerical evidence does not exclude the possibility of ergodicity: If there are some tiny regions of stability, then they must be so small that they cannot be detected at large scales.

We want to calculate and analyze the high-lying states far in the semiclassical limit, as high as 100,000th eigenfunction (of even parity which is about 200,000th when counting all states) and above, in the regime where the classical dynamics is almost completely ergodic (within the numerical
resolution of the Poincaré Surface Of Section). As mentioned above, the latter condition is satisfied at $\lambda = 0.375$. However in order to reach the said high-lying eigenstates using the available supercomputer facilities we had to abandon the conformal mapping diagonalization technique developed in (Robnik 1984) and further employed by Prosen and Robnik (1993b). Instead we have implemented the Heller’s method of the plane wave decomposition of the wavefunctions (e.g. see Heller 1991). Heller’s method enables one to go very high in the semiclassical limit (high energies) where we can then calculate a few consecutive levels, whereas the diagonalization method (with the conformal mapping technique) has the advantage of yielding many levels from the ground state upwards. So, if one is interested in significant statistical analysis the latter method is superior, whilst when studying the individual high-lying eigenstates the former method is the better one.

Let us spend just a few words on the technical aspects of this difficult task, since to the best of our knowledge many crucial ingredients have not been discussed in the literature so far. To solve the Schrödinger equation with Dirichlet boundary condition,

$$\Delta \Psi + E\Psi = 0, \quad \Psi = 0 \text{ at the boundary,} \quad (2)$$

we use the superposition of plane waves with the wavevectors of the same magnitude $k$ but with different directions. The wavefunction we used for the even parity is

$$\Psi(u, v) = \sum_{j=1}^{N} a_j \cos(k_{ju}u + \phi_j) \cos(k_{jv}v), \quad (3)$$

where $k_{ju} = k \cos(\theta_j), \quad k_{jv} = k \sin(\theta_j), \quad k^2 = E$ the eigenenergy, $N$ the number of plane waves and $\phi_j$ are random phases drawn from the interval $[0, 2\pi)$, assuming uniform distribution, and $\theta_j = 2j\pi/N$ (i.e. the direction angles of the wavevectors are chosen equidistantly). The ansatz (3) solves the Schrödinger equation (2) in the interior of the billiard region, so that we have only to satisfy the Dirichlet boundary condition. Taking the random phases, as we discovered, is equivalent to spreading the origins of plane waves all over the billiard region, and at the same time this results in reducing the CPU-time by almost a factor of ten. For a given $k$ we put the wavefunction equal to zero at a finite number $M$ of boundary points (primary nodes).
and equal to 1 at an arbitrarily chosen interior point. Of course, \( M \geq N \).
This gives an inhomogeneous set of equations which can be solved by matrix inversion. Usually the matrix is very singular, thus the *Singular Value Decomposition* (SVD) method has been invoked (Press *et al* 1986). After obtaining the coefficients \( a_j \) we calculate the wavefunctions at other boundary points (secondary nodes). The sum of the squares of the wavefunction at all the secondary nodes (Heller called this sum ”tension”) would be ideally zero if \( k^2 \) is an eigenvalue. In practice it is a positive number. Therefore the eigenvalue problem now is to find the minimum of the ”tension”. In our numerical procedure we have looked for the zeros of the first derivative of the tension; namely the derivative is available analytically/explicitely from (3) once the amplitudes \( a_j \) have been found. In fact, since the SVD-method is based on finding the least square solution of the linear equations, we can choose \( M \) larger than \( N \) without running into the overdetermination problem. This has been done indeed, with a typical choice \( M = 5N/3 \). It must be pointed out that the wavefunctions obtained in this way are not (yet) normalized, due to the arbitrary choice of the interior point where the value of the wavefunction has been arbitrarily set equal to unity. We therefore explicitly normalize these wavefunctions before embarking to the analysis of their statistical properties.

The accuracy of this method of course depends on the number of plane waves \( (N) \) and on the number of the primary nodes \( (M) \), and we have a considerable freedom in choosing \( N \) and \( M \geq N \). In order to reach a sufficient accuracy the experience shows that we should take at least \( N = 3L/\lambda_{\text{de Broglie}} \), and \( M = 5N/3 \), where \( L \) is the perimeter of the billiard and \( \lambda_{\text{de Broglie}} \) is the de Broglie wavelength = \( 2\pi/k \). With this choice we reach the double precision accuracy (sixteen digits) for all levels of integrable systems like rectangular billiard (where the eigenenergies can be given trivially analytically) and the circular billiard, but also for the billiard \( B_3 \) for \( \lambda \leq 0.2 \). Also, the same choice enabled us to calculate the 100,000th even parity eigenstate and a few nearby eigenstates for our billiard at \( \lambda = 0.375 \) within an accuracy of 1% of the mean level spacing (seven valid digits). These accuracy checks were based on very careful selfconsistent checks of the method and also on comparison of the eigenvalues with those obtained by using Robnik’s diagonalization method.

The advantage of this method is that, at one side, it is very flexible to
calculate the eigenvalues, and on the other hand, it is self-checkable: The accuracy and the reliability can be checked by changing the interior point and by changing $N$ as well as $M$. The drawback of the method is that with unlucky choice of the interior point and unlucky energy step size some eigenstates may be — and typically are! — missed, so that the calculation must be repeated by using different interior points to finally collect all the levels. The Weyl formula (with perimeter and curvature corrections) can be used to detect the missing of levels (c.f. Bohigas 1991). A similar numerical experience has been reported in (Frisk 1990).

### 3 The wavefunctions and the probability amplitude distribution

All the wavefunctions that we calculated and discuss here are the even parity eigenstates of the billiard $B_\lambda$ at $\lambda = 0.375$. In figure 1 we plot the even parity eigenfunction of energy $E = 625084.5$, which is about 100,010th eigenstate of even parity, as estimated by using the Weyl formula (with perimeter and curvature corrections),

$$N_{\text{even}}(E) = \frac{1 + 2\lambda^2}{8} E - \frac{(1 + 2\lambda)E(\sqrt{8\lambda} / (1 + 2\lambda))}{\frac{2\pi}{24}} \sqrt{E} - \frac{1}{24} \tag{4}$$

where $E(x)$ is the complete elliptic integral of second kind (c.f. Prosen and Robnik 1993b).

This is a good example of a chaotic quantum eigenstate, which exhibits the characteristic filamentary structure as noticed already by Heller et al (1987,1991), which is a consequence of the fact that in the ansatz all plane waves (with random phases) have the same magnitude $k$ of the wavevector. Also, as judged by the naked eye, the average probability density is constant only if the local averaging region is sufficiently large in units of de Broglie wavelength: Probably we need the typical size of at least several ten wavelengths. The local and global average value $<\Psi^2>$ is theoretically expected to be equal to $1/\mathcal{A}$, where $\mathcal{A} = \pi(1 + 2\lambda^2)$ is the area of $B_\lambda$. This is a direct consequence of the microcanonically uniform Wigner function for this eigenstate (Berry 1977, Voros 1979, Shnirelman 1979, see also Berry 1983), which in turn is a consequence of PUSC as explained and discussed in the
introduction. Indeed, the theoretical value of $\langle \Psi^2 \rangle$ is 0.24844, whereas the numerical evaluation yields 0.24832 (after averaging over 1,145,294 grid points distributed uniformly inside the interior of the billiard region), which can be considered as an excellent agreement. In Table 1 we compare the theoretical values and the numerical estimates for a number of eigenstates to show that quite generally this agreement is very good. There we give the numerical estimate for all the lowest four moments of $\Psi$-distribution, namely the average $m_1 = \langle \Psi \rangle$, the variance $m_2 = \langle (\Psi - m_1)^2 \rangle$, the skewness $m_3 = \langle (\Psi - m_1)^3 \rangle / m_2^{3/2}$ and the kurtosis $m_4 = \langle (\Psi - m_1)^4 \rangle / m_2^2 - 3$.

These experimental values are compared with the theoretical values of the Gaussian random model (see introduction) which predicts

$$P(\Psi) = \frac{1}{\sqrt{2\pi} \sigma} \exp\left\{ -\frac{\Psi^2}{2\sigma^2} \right\},$$

where again according to PUSC $\sigma^2 = 1/A = 1/(\pi (1 + 2\lambda^2))$, and ideally it should be equal to $m_2$. In Figure 2(a-b) we plot the numerical histogram for $P(\Psi)$ (merely for illustrative purposes), and — more importantly - the cumulative distribution $I(\Psi) = \int_{-\infty}^{\Psi} P(x) dx$ which is compared with the theoretical model (5). The agreement is seen to be excellent, even in Figure 2(b) where, as we shall see and discuss in section 5, in the probability density plot of the underlying wavefunction a scar-like feature is observed.

In addition, we have estimated the significance levels of the cumulative distribution $I(\Psi)$ according to the Kolmogorov-Smirnov test (Press et al. 1986 p.472) with respect to the Gaussian distribution for all eigenstates listed in Table 1. It is found to be exactly 1 within five digits. This shows again that indeed our results agree excellently with the theoretical prediction.

Similar study of chaotic eigenfunctions has been published in Chirikov et al. (1989), where even the differences between the numerics and Gaussian random model due to the finite dimensionality of the system have been seen.

Our results are comparable to the findings of Aurich and Steiner (1993) who studied the chaotic wavefunctions of the quantum system whose classical counterpart is the geodesic motion on a compact surface of constant negative curvature, although with our numerical wavefunctions we are considerably farther in the semiclassical limit. So far we have not found any examples of scars in these high-lying states around 100,000th. (However, see section 5.) The conclusion is that scars are difficult to find since they "live" on smaller
and smaller support as $\hbar \to 0$, or $E \to \infty$, and consequently asymptotically no longer influence the $P(\Psi)$ distribution, as further explained in section 5.

4 The autocorrelation function of the wavefunctions

The mean statistical properties of chaotic wavefunctions have been discussed, analyzed and described in the previous section. However the question of space-correlations of a wavefunction is far from trivial. The correlations exist on different scales and their strength can vary substantially. For example, in figure 1 we clearly see that there is some kind of clustering on the scale of a few ten de Broglie wavelengths: there are regions of this size with enhanced probability density, and there are also regions of this size with notably depleted probability density (holes). Not every state is like that and in figure 3 we show another chaotic even parity eigenstate with energy $E = 625,118.4$, approximate number $N_{ev} = 100,015$, again for $\lambda = 0.375$. Here the above mentioned clustering is much less pronounced and this property is well captured in the autocorrelation function of the eigenstates as we shall see in a moment.

The definition of the autocorrelation function of the probability amplitude of a given eigenstate is (Berry 1977b, 1983)

$$C(X; q) = \langle \Psi(q + X/2)\Psi^*(q - X/2) / |\Psi(q)|^2 \rangle \quad (6)$$

where the local average denoted by $\langle \cdots \rangle$ is taken over sufficiently large region around $q$ whose size is typically many de Broglie wavelengths but still small compared with the geometrical size. In our case the wavefunction is of course real, i.e. $\Psi^* = \Psi$. It should be noted that the nominator in (6) is actually the Fourier transform of the Wigner function $W(q, p)$,

$$W(q, p) = \frac{1}{(2\pi)^2} \int d^2X \exp(-i p \cdot X)\Psi(q - X/2)\Psi(q + X/2) \quad (7)$$

where we have specialized to our real $\Psi$ case, and also two degrees of freedom and $\hbar = 1$. Now using the PUSC for a chaotic state following Berry (1977b)
we assume that the Wigner function of such a classically ergodic state is microcanonical, i.e.

\[ W(q, p) = \frac{\delta(E - H(q, p))}{\int d^2q d^2p \delta(E - H(q, p))} \]  

(8)

Thus substituting (8) into the inversion of (7) and then into (6) we immediately obtain the special case of Berry’s (1977b) result, namely

\[ C(X; q) = J_0(ks) \]  

(9)

where \( J_0 \) is the Bessel function of zero order, \( k^2 \) is the eigenenergy and \( s \) is the length of \( X \). So the autocorrelation function is isotropic, and we are going to check numerically the validity of this theoretical prediction.

First we would like to check the isotropy of the autocorrelation function. To this end we have evaluated (6) by taking the local average on a small strip of 20 \( \times \) 100 wavelengths situated at the center of the billiard as far as possible from the boundaries. The results for the wavefunctions of figure 1 and figure 3 are shown in figures 4 and 5 correspondingly. Because of the inversion symmetry of the autocorrelation function w.r.t. \( X \) and the reflection symmetry of the wavefunctions \( \Psi \) w.r.t. \( \nu \) we can restrict ourselves to the angles within the interval \([0, \pi/2]\), and we have chosen the values between 0 and \( \pi/2 \) in equal steps of \( \pi/12 \), as indicated in the upper right corner of the figures. The autocorrelation function is obviously strongly direction dependent (please notice that the statistical noise is practically zero) and in the case of more uniformly chaotic wavefunction of figure 3 agrees better with the theoretical prediction (9) than for the less chaotic eigestate of figure 1.

We believe that the semiclassical periodic orbit theory (see e.g. Casati et al 1993, Tél and Ott 1993 and references therein) could explain the deviations from the isotropy. Our results agree qualitatively with (Aurich and Steiner 1993) although we are considerably higher in the semiclassical limit (by a factor of 10 or so), and also with somewhat old results in (McDonald and Kaufman 1988, Shapiro and Goelman 1984, Shapiro et al 1988).

It is interesting that after averaging over many directions we get a considerable agreement with (9). This is shown in figure 6(a-d) where we vary the size of the averaging disk and also the number of the directions over which
the average is taken. These plots are for the eigenfunction shown in figure 1. Both effects are clearly visible, namely the increasingly better agreement with (9) as we increase the radius of the averaging disk and/or as we increase the number of directions. The same aspects are shown in figure 7(a-d) for the more uniformly chaotic state of figure 3. By comparing the figures 6(a) and 7(a) we see that in the latter plot the agreement with theory is better.

5 Scar-like features in wavefunctions

As we know since Heller’s (1984) discovery of scars (of unstable classical periodic orbits) in chaotic quantum eigenfunctions of classically ergodic systems we do expect such scars to exist in all chaotic systems, but according to the single-periodic-orbit theory (Bogomolny 1988, Berry 1989) the scar supporting region should shrink as $\sqrt{\hbar}$ as $\hbar \to 0$, whilst the probability density contrast remains fixed since it is predicted to be $\hbar$-independent (Heller 1984). The many-orbits theory (Robnik 1989) would speculatively predict the linear scaling of the scar area with $\hbar$ as a consequence of the interference effects. Some phenomenological material on this topic has been recently published in (Prosen and Robnik 1993a).

In this short section we would like to draw attention to an interesting scar-like feature seen in figure 3: Near the boundary, about ten de Broglie wavelengths away, there is a thin scar-like feature which has no simple explanation, because due to the nonconvexity of the billiard boundary there are no Lazutkin-like caustics and invariant tori and also no such glancing periodic orbits. The only classical object that might be relevant for this feature is possibly the glancing orbit which survives many bounces while going round the boundary until reaching the non-convexity region and flying away, becoming completely chaotic afterwards. We have observed a few similar features in quite a few other eigenfunctions, but we cannot offer any definite theoretical explanation so far. However, the formalism offered and discussed in (Müller et al 1993) might just be right to quantitatively describe the role of such orbits which are recurrent in configuration space but not periodic.
6 Discussion and conclusions

In this paper we have numerically calculated the high-lying chaotic states in the Robnik billiard as high as 200,000th eigenstate and investigated their semiclassical morphology and their statistical properties. To achieve this we have implemented and adapted Heller’s method of plane wave decomposition which has been further developed and its accuracy carefully checked. Similar to other workers (e.g. Aurich and Steiner 1993) we reach the following conclusions. In such high-lying eigenstates the scars are hardly detectable since so far we have not found any of them. The average probability density is globally in excellent agreement with the theoretical semiclassical (and classical!) prediction. The Gaussian random model for the local statistical properties of the wavefunctions is generally excellent, in spite of the characteristic filamentary structure and the relevant clustering of probability density on the scales of a few ten de Broglie wavelengths. This has been found by comparing the theoretical and the numerical distributions and also by the comparison of the lowest four moments and the evaluation of the Kolmogorov-Smirnov test. The autocorrelation function nicely captures the clustering property, is found to be strongly direction dependent in contradistinction with Berry’s (1977b) isotropic prediction, but after averaging over many directions the agreement with Berry’s theory is recovered. Finally we should mention that in some of the eigenstates we discovered scar-like features resembling the whispering gallery modes for which we do not have a proper theoretical explanation.

Our current and future work deals with the systematic search for the scars and the analysis of their geometry and scaling properties with $\hbar$. On theoretical side the present paper stimulates further work on the scar theory for which we expect improvement when many-orbits theory will be set up following the suggestions in (Robnik 1989, Prosen and Robnik 1993a). Moreover, we believe that the application of the Gutzwiller’s (one-) periodic orbit theory could explain in detail the anisotropies of the autocorrelation function. Our work also shows that there is still much interesting structure in the range of a few ten de Broglie wavelengths in chaotic wavefunctions which calls for a more refined statistical description.
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Table

Table 1: The average, variance, skewness and kurtosis of a few eigenstates nearby 100,000th eigenstate of even parity in comparison with theoretical values. \( N_{ev}(E) \) is given by the Weyl formula [4]. The significance levels of the Kolmogorov-Smirnov test for all eigenstates listed in this table are exactly one within 5 digits.

| \( E \)     | \( N_{ev}(E) \) | Average | Variance | Skewness | Kurtosis |
|------------|-----------------|---------|----------|----------|----------|
| 625040.6   | 100003          | 0.00002 | 0.24828  | -0.00211 | 0.11607  |
| 625058.4   | 100006          | 0.00001 | 0.24836  | 0.00441  | 0.06576  |
| 625084.5   | 100010          | 0.00001 | 0.24832  | -0.00160 | 0.06809  |
| 625099.5   | 100012          | -0.00001| 0.24834  | -0.00179 | 0.03638  |
| 625118.4   | 100015          | 0.00001 | 0.24838  | 0.00460  | -0.01415 |
| 625161.9   | 100022          | 0.00003 | 0.24837  | 0.00347  | 0.02424  |
| 625172.8   | 100024          | 0.00007 | 0.24839  | -0.00248 | 0.04836  |
| 625182.1   | 100025          | 0.00000 | 0.24854  | -0.00135 | 0.03624  |
| **Gaussian** |                | 0.0     | 0.24844  | 0.0       | 0.0      |

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Figure 1: The probability density plot for the even parity eigenstate with $E = 625,084.5$, for $\lambda = 0.375$, and with the estimated sequential number using the Weyl formula (4) equal to 100,010. The contours are plotted at ten equally spaced steps between zero and the maximum value. In this geometry the unit length is 126 de Broglie wavelengths.

Figure 2: The probability distribution function and cumulative distribution function of eigenstates $E = 625,084.5$ (approximately 100,010th even parity state) (a), and $E = 625,118.4$ (approximately 100,015th even parity state) (b), in comparison with the Gaussian random model (5). In the top diagrams we show the histograms compared with theoretical curve (5), and in the lower diagrams we show the cumulative amplitude distribution function $I(\Psi)$. Three small boxed regions are displayed in the corresponding magnified windows. Here the difference between the theoretical and the numerical curves is hardly visible since the agreement is so good.

Figure 3: The probability density plot for the even parity eigenstate with $E = 625,118.4$, for $\lambda = 0.375$, and the estimated sequential number using the Weyl formula (4) equal to 100,015. The contours are plotted at ten equally spaced steps between zero and the maximum value. In this geometry the unit length is 126 de Broglie wavelengths.

Figure 4: The autocorrelation function $C(X, q)$ of the eigenstate in figure 1. $C(X, q)$ is plotted against $ks$ for seven different angles of $X$ w.r.t the abscissa; here $s = |X|$. The angle and the averaging strip are indicated in the upper right corner of the figure. From (a) to (g) the angle goes from 0 to $\pi/2$ with the increment of $\pi/12$. The dashed curve is the theoretical prediction (4), namely $J_0(ks)$, whilst the full curve denotes the numerical result. The local average has been taken on a strip of $20 \times 100$ de Broglie wavelengths. For a fixed $ks$ about 150,000 grid points inside the strip have been used to calculate $C$. The reference point $q$ is fixed at $(0.4, 0.0)$ which is probably sufficiently far away from the billiard boundary.
**Figure 5:** The same as figure 4 but for the eigenstate of figure 3.

**Figure 6:** The autocorrelation function after averaging over many directions $X$ of the eigenstate in figure 1. $C(X,q)$ is plotted against $ks$ for three different averaging disks and different number of directions, where $s = |X|$. The averaging disk in (a), (b), (c) and (d) has the diameter of 200, 100, 50 and 100 de Broglie wavelengths, respectively. In (a), (b) and (c) the number of directions is 200, whilst in (d) it is 400.

**Figure 7:** The same as in figure 6 but for the eigenstate in figure 3.