Superconcentration, and randomized Dvoretzky’s theorem for spaces with 1-unconditional bases

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Abstract

Let $n$ be a sufficiently large natural number and let $B$ be an origin-symmetric convex body in $\mathbb{R}^n$ in the $\ell$-position, and such that the space $(\mathbb{R}^n, \| \cdot \|_B)$ admits a 1-unconditional basis. Then for any $\varepsilon \in (0, 1/2]$, and for random $c\varepsilon \log n / \log \frac{1}{\varepsilon}$-dimensional subspace $E$ distributed according to the rotation-invariant (Haar) measure, the section $B \cap E$ is $(1 + \varepsilon)$-Euclidean with probability close to one. This shows that the “worst-case” dependence on $\varepsilon$ in the randomized Dvoretzky theorem in the $\ell$-position is significantly better than in John’s position. It is a previously unexplored feature, which has strong connections with the concept of superconcentration introduced by S. Chatterjee. In fact, our main result follows from the next theorem: Let $B$ be as before and assume additionally that $B$ has a smooth boundary and $E_{\gamma_n} \| \cdot \|_B \leq n^c E_{\gamma_n} \| \text{grad}_B(\cdot) \|_2$ for a small universal constant $c > 0$, where $\text{grad}_B(\cdot)$ is the gradient of $\| \cdot \|_B$ and $\gamma_n$ is the standard Gaussian measure in $\mathbb{R}^n$. Then for any $p \in [1, c \log n]$ the $p$-th power of the norm $\| \cdot \|^p_B$ is $\frac{C}{\log n}$-superconcentrated in the Gauss space.

Keywords: Dvoretzky’s theorem, almost Euclidean sections, superconcentration, $\ell$-position

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1 Introduction

The term superconcentration was introduced by S. Chatterjee to describe a situation when the size of typical fluctuations of a function on a probability space is much smaller than the bound provided by “classical” concentration inequalities [4]. In this note, we are concerned with applications of the superconcentration phenomenon in asymptotic geometric analysis; specifically, in the problem of finding large $(1 + \varepsilon)$-Euclidean sections of convex bodies. On the probabilistic level, we derive a concentration inequality for convex positively homogeneous functions in the Gauss space satisfying some additional assumptions. On the geometric level, we show that John’s position may be a “bad” choice as far as dependence of dimension on $\varepsilon$ is concerned in the randomized Dvoretzky’s

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theorem, and, at least for unit balls of normed spaces with a 1-unconditional basis, the \( \ell \)-position allows a substantially better bound on the dimension.

The theorem of A. Dvoretzky \(^8\) asserts that for arbitrary fixed \( k \in \mathbb{N} \) and \( \varepsilon > 0 \), every symmetric convex body of a large enough dimension contains a \((1 + \varepsilon)\)-Euclidean \( k \)-dimensional section. A proof of the theorem based on the concentration of measure was proposed by V. Milman \(^{18}\). In view of results of Y. Gordon \(^{11}\) and G. Schechtman \(^{24}\), who improved dependence of the dimension \( k \) on \( \varepsilon \), the theorem of Milman reads: If \( B \) is an origin-symmetric convex body in \( \mathbb{R}^n \) with the Minkowski functional \( \| \cdot \|_B \) and

\[
k(B) := \left( \frac{\mathbb{E}\|G\|_B}{\text{Lip}(\| \cdot \|_B)} \right)^2
\]

(where \( G \) is the standard Gaussian vector in \( \mathbb{R}^n \) and \( \text{Lip}(\| \cdot \|_B) \) is the Lipschitz constant of \( \| \cdot \|_B \)), then for any \( \varepsilon \in (0, 1] \) and any natural \( k \leq c\varepsilon^2 k(B) \) the random \( k \)-dimensional subspace \( E \subset \mathbb{R}^n \) uniformly distributed according to the rotation-invariant measure, cuts a \((1 + \varepsilon)\)-Euclidean section \( B \cap E \) with probability close to one. The quantity \( k(B) \) is often called the critical, or Dvoretzky’s, dimension. The last statement asserts that most sections of \( B \) (with respect to the rotation-invariant probability measure) of the given dimension are \((1 + \varepsilon)\)-Euclidean; in our note this version of Dvoretzky’s theorem is called “randomized” (as opposed to “existential”). Let us note that Dvoretzky’s theorem as well as numerous questions around it are covered in several monographs and surveys; see, in particular, \(^{19\ 23\ 25\ 2}\).

The Dvoretzky–Rogers lemma implies that for any convex body \( B \) in John’s position (i.e. such that the ellipsoid of maximal volume contained inside \( B \) is the unit Euclidean ball) one has \( \mathbb{E}\|G\|_B \geq c\sqrt{\log n} \), whence \( k(B) \geq c^2 \log n \) for a universal constant \( c > 0 \). This yields

**Theorem 1 (Randomized Dvoretzky’s theorem in John’s position, \(^{18\ 11\ 24}\)).** Let \( B \) be an origin-symmetric convex body in \( \mathbb{R}^n \) in John’s position, and let \( \varepsilon \in (0, 1] \) and \( k \leq c\varepsilon^2 \log n \). Then, for random \( k \)-dimensional subspace \( E \) uniformly distributed according to the rotation-invariant (Haar) measure, one has

\[
\mathbb{P}\{ B \cap E \text{ is } (1 + \varepsilon)\text{-Euclidean} \} \geq 1 - 2n^{-c\varepsilon^2}.
\]

Here, \( c' > 0 \) is a universal constant.

In fact, in the above theorem “\((1 + \varepsilon)\)-Euclidean” can be replaced with a stronger notion of \((1 + \varepsilon)\)-spherical which we define as a section \( B \cap E' \) such that \( \lambda B^n_2 \cap E' \subset B \cap E' \subset (1 + \varepsilon)\lambda B^n_2 \cap E' \), where \( \lambda \) is some positive real number and \( B^n_2 \) is the standard Euclidean ball in \( \mathbb{R}^n \).

It is not difficult to show that dependence on \( \varepsilon \) in the above theorem cannot be improved. The following statement can be verified by elementary geometric arguments combined with basic probability (for completeness, we give a proof in Section 5):

**Proposition 2 (Optimality of randomized Dvoretzky’s theorem in John’s position).** There are universal constants \( c, C > 0 \) and \( n_0 \in \mathbb{N} \) with the following property: For any \( n \geq n_0 \) there is an origin-symmetric convex body \( B \subset \mathbb{R}^n \) in John’s position (and,
moreover, the standard basis in $\mathbb{R}^n$ is 1-unconditional with respect to the norm $\| \cdot \|_B$ such that for all $\varepsilon \in (0, c]$ and $k \geq C \max(\varepsilon^2 \log n, 1)$, the random $k$-dimensional subspace $E$ uniformly distributed according to the rotation-invariant (Haar) measure, satisfies

$$P\{ B \cap E \text{ is } (1 + \varepsilon)\text{-Euclidean} \} \leq \frac{1}{2}.$$

Although Theorem 1 is sharp, it raises the question whether John’s position is a good choice for generating random almost Euclidean sections, or there is another canonical position which yields a better dependence on $\varepsilon$. As an example, let us note that in a recent paper [21] dealing with Dvoretzky’s theorem for subspaces of $L_p$, G. Paouris and P. Valettas used a position of the unit ball other than John’s.

For an origin-symmetric convex body $B$ in $\mathbb{R}^n$ and a linear operator $U : \mathbb{R}^n \to \mathbb{R}^n$ define

$$\ell(B, U) := \left( \int \|U(x)\|^2_B \, d\gamma_n(x) \right)^{1/2} = \left( \mathbb{E} \|U(G)\|^2_B \right)^{1/2},$$

where $\| \cdot \|_B$ is the Minkowski functional of $B$ and $G$ is the standard Gaussian vector in $\mathbb{R}^n$. We say that $B$ is in the $\ell$-position if $\ell(B, Id_n) = 1$ and

$$1 = \det Id_n = \sup \{|\det U| : U \in \mathbb{R}^{n \times n}, \ell(B, U) \leq 1\}.$$

It is not difficult to check that for any body $B \subset \mathbb{R}^n$ and a linear operator $U_0$ with $\ell(B, U_0) = 1$ and such that

$$|\det U_0| = \sup \{|\det U| : U \in \mathbb{R}^{n \times n}, \ell(B, U) \leq 1\},$$

the image $U_0^{-1}(B)$ is in the $\ell$-position. The importance of the $\ell$-position in asymptotic geometric analysis was revealed by T. Figiel and N. Tomczak-Jaegermann in [9] (see also [30, § 12], [23, Chapter 3], [2, Chapter 6]).

The main result of our note is the following theorem:

**Theorem 3 (Randomized Dvoretzky’s theorem in the $\ell$-position).** Let $B$ be an origin-symmetric convex body in $\mathbb{R}^n$ in the $\ell$-position, and such that the space $(\mathbb{R}^n, \| \cdot \|_B)$ has a 1-unconditional basis. Further, let $\varepsilon \in (0, 1/2]$ and $k \leq c \varepsilon \log n / \log \frac{1}{\varepsilon}$. Then for random $k$-dimensional subspace $E \subset \mathbb{R}^n$ uniformly distributed according to the rotation-invariant (Haar) measure, one has

$$P\{ B \cap E \text{ is } (1 + \varepsilon)\text{-spherical} \} \geq 1 - 2n^{-c\varepsilon},$$

with the notion “$(1 + \varepsilon)$-spherical” defined above. Here, $c > 0$ is a universal constant.

A version of the above statement was known in the particular case $B = [-1, 1]^n$ [20]; see also [22] where the randomized Dvoretzky’s theorem for $\ell_p^n$-balls is studied. The dependence on $\varepsilon$ in Theorem 3 is sharp in a sense that the majority of $C \varepsilon \log n / \log \frac{1}{\varepsilon}$-dimensional sections of the standard cube are not $(1 + \varepsilon)$-spherical [29]. Let us emphasize once more that the above statement does not hold in general if the $\ell$-position is replaced with John’s. We conjecture that the assertion of Theorem 3 is true without the assumption that the space $(\mathbb{R}^n, \| \cdot \|_B)$ admits a 1-unconditional basis.
In [27] G. Schechtman proved, by combining random and deterministic arguments with a result of N. Alon and V. Milman [1], that any origin-symmetric convex body $B$ contains a $c\varepsilon \log n / \log^2 \frac{1}{\varepsilon}$-dimensional $(1 + \varepsilon)$-Euclidean section. However, in contrast with Theorem 3, the result of [27] is existential in a sense that it does not provide a canonical position for a convex body in which most of its $c\varepsilon \log n / \log^2 \frac{1}{\varepsilon}$-dimensional sections are $(1 + \varepsilon)$-Euclidean.

Our proof of Theorem 3 is based on the following dichotomy: Given a convex body $B$ satisfying the assumptions of the theorem, either the expectation of the length of the gradient of the norm $E_{\gamma_n} \| \text{grad}_B (\cdot) \|_2$ is very small compared to $E_{\gamma_n} \| \cdot \|_B$ (in which case a simple analysis shows that the assertion of the theorem is true) or the gradient is relatively "large" in which case we involve the superconcentration. Of course, only the second case is of interest.

Let $n$ be a natural number and let $\gamma_n$ be the standard Gaussian measure in $\mathbb{R}^n$. For any sufficiently smooth real-valued function $f$ in $\mathbb{R}^n$ one has

$$\text{Var}_{\gamma_n} (f) := \int f(x)^2 d\gamma_n(x) - \left( \int f(x) d\gamma_n(x) \right)^2 \leq \int \| \text{grad} f(x) \|_2^2 d\gamma_n(x),$$

where grad$f$ is the gradient of $f$ (the Poincaré inequality). A function $f$ in $\mathbb{R}^n$ is called $\delta$-superconcentrated (for some $\delta < 1$) if

$$\text{Var}_{\gamma_n} (f) \leq \delta \int \| \text{grad} f(x) \|_2^2 d\gamma_n(x).$$

The setting of actual interest involves a sequence of functions (indexed by the dimension $n$) such that $\delta = \delta(n)$ tends to zero with $n \to \infty$. We refer to [4] for definition of superconcentration in a more general context, its relation to other properties (called “chaos” and “multiple valleys”), as well as for results dealing with specific probabilistic models. Theorem 3 of this note follows from the next result.

**Theorem 4.** There are universal constants $c, C > 0$ and $n_0 \in \mathbb{N}$ with the following property. Let $n \geq n_0$ and let $B$ be an origin-symmetric convex body in $\mathbb{R}^n$ in the $\ell$-position, with a smooth boundary, and such that the space $(\mathbb{R}^n, \| \cdot \|_B)$ admits a $1$-unconditional basis. Further, assume that $E \| G \|_B \leq n^c E \| \text{grad}_B (G) \|_2$, where $G$ denotes the standard Gaussian vector in $\mathbb{R}^n$ and grad$_B (\cdot)$ is the gradient of the norm $\| \cdot \|_B$. Then for any $p \in [1, c \log n]$ the function $\| \cdot \|_B^p$ is $\frac{C}{\log n}$-superconcentrated in the Gauss space.

The main tool in the proof of Theorem 4 is Talagrand’s $L_1 - L_2$ bound (see Theorem 6), which we combine with some special properties of the $\ell$-position (“balancing conditions”). The proof is not difficult and admits various generalizations in a sense that the $\ell$-position can be replaced with other transformations of the convex body that provide appropriate “balancing” in regard to the Gaussian measure (we’ll return to this issue at the end of the paper).
2 Notation and preliminaries

Let $n$ be a natural number. The canonical basis in $\mathbb{R}^n$ will be denoted by $e_1, e_2, \ldots, e_n$ and the standard inner product — by $\langle \cdot, \cdot \rangle$. Given a set of vectors $\{y_1, y_2, \ldots, y_k\}$ in $\mathbb{R}^n$, we denote their linear span by $\text{span}\{y_1, y_2, \ldots, y_k\}$. For a subspace $E \subset \mathbb{R}^n$, $E^\perp$ is its orthogonal complement in $\mathbb{R}^n$ and $\text{Proj}_E$ is the orthogonal projection operator onto $E$. Given a boolean variable $b$, denote by $\chi_b$ the indicator function of $b$, so that $\chi_b = 1$ if and only if $b$ is true. Similarly, for an event $\mathcal{E}$ denote by $\chi_{\mathcal{E}}$ the indicator function of the event.

A convex body in $\mathbb{R}^n$ is any compact convex set with non-empty interior. Everywhere in this note, we say that the boundary $\partial B$ of a convex body $B$ is smooth if every point of $\partial B$ admits a unique tangent hyperplane. Given an origin-symmetric convex body $B$, denote by $\| \cdot \|$ its Minkowski functional. By some abuse of notation, for any subspace $E \subset \mathbb{R}^n$ we denote by $\| \cdot \|_{B \cap E}$ the Minkowski functional of $B \cap E$ considered as a convex body inside $E$. Further, given a $k$-dimensional subspace $E \subset \mathbb{R}^n$, we say that the section $B \cap E$ is $L$–Euclidean (for some $L \geq 1$) if the Banach–Mazur distance from $B \cap E$ to a $k$-dimensional Euclidean ball is bounded from above by $L$.

A basis $y_1, y_2, \ldots, y_n$ of an $n$-dimensional normed space $W$ with a norm $\| \cdot \|$ is 1-unconditional if $\| \sum_{i=1}^n a_i y_i \| = \| \sum_{i=1}^n \sigma_i a_i y_i \|$ for any scalars $a_1, a_2, \ldots, a_n$ and any signs $\sigma_1, \sigma_2, \ldots, \sigma_n \in \{-1, 1\}$. The canonical basis of $\mathbb{R}^n$ is 1-unconditional with respect to a norm $\| \cdot \|$ if and only if the unit ball of $\| \cdot \|$ is symmetric with respect to coordinate hyperplanes.

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The set of all $k$-dimensional subspaces of $\mathbb{R}^n$ admits a unique normalized rotation-invariant Borel measure (the Haar measure). Whenever we speak about a random subspace in this note, we assume it is distributed according to that measure. The standard Gaussian measure in $\mathbb{R}^n$ is denoted by $\gamma_n$, the standard Gaussian vector — by $G$, and the standard real-valued Gaussian variable — by $g$ (or $g_i$'s when there are several of them).

Universal constants will be denoted by $C, c$, etc. and their value may be different on different occasions.

2.1 Gaussian concentration inequalities

The next theorem (with a worse constant in the exponent) is due to G. Pisier.

**Theorem 5** (see, in particular, [23, Chapter 4], [15, p. 12] or [14, Chapter 2]). Let $G$ be a standard Gaussian vector in $\mathbb{R}^N$ and $f : \mathbb{R}^N \to \mathbb{R}$ be a 1-Lipschitz function. Then

$$
\mathbb{P}\left\{ f(G) - \mathbb{E} f(G) \geq t \right\} \leq \exp(-t^2/2), \quad t > 0.
$$

The next statement plays a crucial role in our analysis (a version of this inequality for the uniform measure on discrete cube was proved by M. Talagrand in [28]):

**Theorem 6** (Talagrand’s $L_1-L_2$ bound; see, in particular, [13, Chapter 5]). Let $f$ be an absolutely continuous function in $\mathbb{R}^N$ and let $\partial_i f$ ($i \leq N$) denote $i$-th component of
the gradient of \( f \). Then

\[
\text{Var}(f(G)) \leq C \sum_{i=1}^{N} \frac{\mathbb{E}|\partial_i f(G)|^2}{1 + \log (\sqrt{\mathbb{E}|\partial_i f(G)|^2/\mathbb{E}|\partial_i f(G)|})},
\]

where \( C > 0 \) is a universal constant.

**2.2 Canonical positions of convex bodies**

Given an origin-symmetric convex body \( B \), its position is any convex body \( T(B) \) for some invertible linear transformation \( T \). The two canonical positions we consider in this note are John’s and the \( \ell \)-position, which were defined in the introduction. Recall that a position \( T(B) \) is John’s if the ellipsoid of maximal volume contained in \( T(B) \) is the unit Euclidean ball. The concept was used by F. John [12] to estimate the Banach–Mazur distance of arbitrary convex body to the Euclidean ball (see also [3]).

A viewpoint to canonical positions involving arbitrary norms on the space of linear operators \( \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \) was developed by D.R. Lewis [16] (see also [30, Chapter 3], [23, Chapter 3], as well as an alternative approach of A. Giannopoulos and V. Milman [10] based on isotropic measures on the Euclidean sphere). In particular, the \( \ell \)-position can be defined as a linear transformation \( T(B) \) satisfying

\[
\ell(T(B), \text{Id}_n) = \ell(B, T^{-1}) = 1, \quad \ell^*(T(B), \text{Id}_n) = n,
\]

where \( \ell(\cdot) \) is the norm on the space of linear operators in \( \mathbb{R}^n \) defined in the introduction, and \( \ell^*(\cdot) \) is the norm in trace duality with \( \ell(\cdot) \) [16, 30, 23, 2]. It is easy to see that the \( \ell \)-position is rotation-invariant in a sense that, together with \( T(B) \), any linear image of the form \( UT(B) \) \((U \in O_n)\) is in the \( \ell \)-position. At the same time, the \( \ell \)-position is unique up to an orthogonal transformation [30, Proposition 14.3]. In fact, the following stability result is true:

**Lemma 7** (Stability of the \( \ell \)-position). Let \( n > 1 \) and let \( B \) be an origin-symmetric convex body in \( \mathbb{R}^n \) in the \( \ell \)-position. Then for any \( \delta > 0 \) there is \( \kappa = \kappa(\delta) > 0 \) depending only on \( \delta \) with the following property: whenever \( T \) is an invertible linear operator in \( \mathbb{R}^n \) with \( \ell(T(B), \text{Id}_n) = 1 \) and \( |\det T| \leq 1 + \kappa \), there exists an orthogonal transformation \( U \) of \( \mathbb{R}^n \) such that \( (1 - \delta)B \subset UT(B) \subset (1 + \delta)B \).

**Proof.** The proof to a large extent follows the argument in [30, Proposition 14.3]. Fix a small \( \delta > 0 \) and define \( \kappa := \frac{\delta^2}{4 + 4\delta} \). Now, let \( T \) be an operator satisfying the assumptions of the lemma. Choose an orthogonal operator \( U \) so that \( T = U^{-1}P \), with \( P \) being positive definite (the polar decomposition of \( T \)). Then \( \det P \leq 1 + \kappa \) and \( \ell(P(B), \text{Id}_n) = 1 \). It remains to show that \( 1 - \delta \leq \lambda_{\text{min}}(P) \leq \lambda_{\text{max}}(P) \leq 1 + \delta \). We shall prove this by contradiction. Assume that either \( \lambda_{\text{min}}(P) < 1 - \delta \) or \( \lambda_{\text{max}}(P) > 1 + \delta \). Define an operator \( W \) via its inverse: \( W^{-1} := \frac{1}{2}(P^{-1} + \text{Id}_n) \). Clearly, if \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are eigenvalues of \( P \) then

\[
\det W^{-1} = \prod_{i=1}^{n} \frac{\lambda_i^{-1} + 1}{2}.
\]
Obviously, $\frac{\lambda_i^{-1} + 1}{2} \geq \lambda_i^{-1/2}$ for all $i \leq n$ and, additionally, as at least one of the eigenvalues $\lambda_i$ satisfies $|\lambda_i - 1| > \delta$, we have

$$\frac{\lambda_i^{-1} + 1}{2} = \frac{\lambda_i^{-1/2} + \lambda_i^{1/2}}{2} > \lambda_i^{-1/2} \frac{(1 + \delta)^{-1/2} + (1 + \delta)^{1/2}}{2} = \lambda_i^{-1/2} \sqrt{1 + \frac{\delta^2}{4 + 4\delta}} = \lambda_i^{-1/2} \sqrt{1 + \kappa}.$$ 

Thus,

$$\det W^{-1} > \sqrt{1 + \kappa} \prod_{i=1}^{n} \lambda_i^{-1/2} \geq 1,$$

whence $\det W < 1$. Next, observe that for any vector $x \in \mathbb{R}^n$ we have

$$\|x\|_{W(B)}^2 = \|W^{-1}x\|_B^2 \leq \left( \frac{\|P^{-1}x\|_B + \|x\|_B}{2} \right)^2 \leq \frac{1}{2} \|x\|_{P(B)}^2 + \frac{1}{2} \|x\|_B^2.$$

Thus, $\ell(W(B), \text{Id}_n)^2 \leq \frac{1}{2} \ell(P(B), \text{Id}_n)^2 + \frac{1}{2} \ell(B, \text{Id}_n)^2 = 1$ while $|\det W| < 1$. This contradicts the assumption that $B$ is in the $\ell$-position.

As a simple corollary, we obtain

**Corollary 8.** Let $n > 1$ and let $B$ be an origin-symmetric convex body in the $\ell$-position. Then for any $\delta > 0$ there is an origin-symmetric convex body $B_\delta$ with a smooth boundary, in the $\ell$-position, and such that $(1 - \delta)B \subset B_\delta \subset (1 + \delta)B$. Moreover, if the norm $\| \cdot \|_B$ admits a 1-unconditional basis then $B_\delta$ can be defined so that $\| \cdot \|_{B_\delta}$ is 1-unconditional as well.

**Proof.** Fix the convex body $B$ and a small positive $\delta$. Define a positive number $\tilde{\kappa} := \min(\kappa(\delta/4), \delta/4)$, where the function $\kappa$ is taken from Lemma 7. First, one can construct a smooth approximation $B'$ of $B$ satisfying the relations

$$(1 + \tilde{\kappa}/4)^{-1/n} B' \subset B \subset (1 + \tilde{\kappa}/4)^{1/n} B',$$

and such that $\| \cdot \|_{B'}$ is 1-unconditional whenever $\| \cdot \|_B$ is (see, for example, [13]). The inclusion relations imply that $(1 + \tilde{\kappa}/4)^{-1/n} \leq \ell(B', \text{Id}_n) \leq (1 + \tilde{\kappa}/4)^{1/n}$, whence, applying an appropriate dilation, we get a smooth convex body $B''$ satisfying

$$(1 + \tilde{\kappa})^{-1/n} B \subset B'' \subset (1 + \tilde{\kappa})^{1/n} B \quad (1)$$

and such that $\ell(B'', \text{Id}_n) = 1$. Now, let $T$ be an invertible linear transformation so that $T(B'')$ is in the $\ell$-position. Obviously, $T(B'') \subset (1 + \tilde{\kappa})^{1/n} T(B)$, and, as $B$ is in the $\ell$-position, we have $\ell(T(B), \text{Id}_n) \geq |\det T|^{-1/n}$. Thus, $1 = \ell(T(B''), \text{Id}_n) \geq (1 + \tilde{\kappa})^{1/n}$.
\[ \kappa^{-1/n} |\det T|^{-1/n}, \text{ i.e. } |\det T^{-1}| \leq 1 + \kappa. \] By Lemma 11 (applied to \( T(B'') \) and operator \( T^{-1} \)), there is an orthogonal transformation \( U \) such that

\[ (1 - \delta/4)T(B'') \subset UT^{-1}T(B'') = U(B'') \subset (1 + \delta/4)T(B''), \]

whence

\[ (1 - \delta/4)U^{-1}T(B'') \subset B'' \subset (1 + \delta/4)U^{-1}T(B''). \]

Together with (1), this implies that \( U^{-1}T(B'') \) is the smooth convex body in the \( \ell \)-position satisfying the required conditions. \( \square \)

**Remark 1.** Corollary 8 will allow us to reduce the proof of Theorem 8 to the case when the underlying convex body is smooth.

The next statement is intuitively obvious; we give its proof for completeness.

**Lemma 9.** Let \( B \) be an origin-symmetric convex body in \( \mathbb{R}^n \) in the \( \ell \)-position, and assume that the normed space \((\mathbb{R}^n, \| \cdot \|_B)\) admits a 1-unconditional basis. Then the basis is orthogonal with respect to the canonical inner product in \( \mathbb{R}^n \).

**Proof.** Let \( x_1, x_2, \ldots, x_n \) be a 1-unconditional basis in \((\mathbb{R}^n, \| \cdot \|_B)\), and suppose that it is not orthogonal. Without loss of generality, we can assume that \( H := \text{span}\{x_1, \ldots, x_{n-1}\} \) and \( x_n \) are not orthogonal. Let \( T \) be the linear transformation of \( \mathbb{R}^n \) given by its action on the basis vectors: \( Tx_i = x_i \) for all \( i \leq n - 1 \), and \( Tx_n = x_n - \text{Proj}_{H}x_n \), where \( \text{Proj}_{H} \) is the orthogonal projection onto \( H \). It is easy to see that the transformation \( T \) is volume-preserving. Further, define a convex body \( B' \) via its Minkowski functional:

\[ \left\| \sum_{i=1}^{n} a_iTx_i \right\|_{B'} := \left\| \sum_{i=1}^{n} a_ix_i \right\|_{B}, \text{ for all } a_i \in \mathbb{R}, \ i \leq n. \]

Thus, \( B' \) is a (non-orthogonal) linear transformation of \( B \) and \( \text{Vol}(B) = \text{Vol}(B') \). We will show that \( \ell(B', \text{Id}_n) \leq \ell(B, \text{Id}_n) \) which, in view of the uniqueness of the \( \ell \)-position mentioned above (see [30], Proposition 14.3) or the last corollary leads to contradiction.

Let \( G' \) be the standard \((n - 1)\)-dimensional Gaussian vector in \( H \) and let \( g_n \) be the standard Gaussian variable independent from \( G' \). We consider three random variables \( \xi, \eta_1, \eta_2 \) on the probability space given by

\[ \xi := \left\| G' + \frac{Tx_n}{\|Tx_n\|_2}g_n \right\|_{B'}; \]
\[ \eta_1 := \left\| G' + \frac{Tx_n}{\|Tx_n\|_2}g_n \right\|_{B}; \]
\[ \eta_2 := \left\| G' - \frac{Tx_n}{\|Tx_n\|_2}g_n \right\|_{B}. \]

Obviously, \( \ell(B, \text{Id}_n) = (\mathbb{E} \eta_1^2)^{1/2} = (\mathbb{E} \eta_2^2)^{1/2} \), and \( \ell(B', \text{Id}_n) = (\mathbb{E} \xi^2)^{1/2} \). At the same
time, using 1-unconditionality of the basis \{x_1, \ldots, x_n\} with respect to \| \cdot \|_B, we obtain

\[
\eta_1 + \eta_2 = \left\| G' + \frac{Tx_n}{\|Tx_n\|_2^2} g_n \right\|_B + \left\| G' - \frac{Tx_n}{\|Tx_n\|_2^2} g_n \right\|_B
\]

\[
= \left\| G' + \frac{x_n - \text{Proj}_H x_n}{\|Tx_n\|_2^2} g_n \right\|_B + \left\| G' - \frac{x_n - \text{Proj}_H x_n}{\|Tx_n\|_2^2} g_n \right\|_B
\]

\[
= \left\| G' + \frac{x_n - \text{Proj}_H x_n}{\|Tx_n\|_2^2} g_n \right\|_B + \left\| G' + \frac{x_n + \text{Proj}_H x_n}{\|Tx_n\|_2^2} g_n \right\|_B
\]

\[
\geq 2 \left\| G' + \frac{x_n}{\|Tx_n\|_2^2} g_n \right\|_B
\]

\[
= 2 \left\| G' + \frac{Tx_n}{\|Tx_n\|_2^2} g_n \right\|_{B'}
\]

\[
= 2 \xi.
\]

Thus, by the triangle inequality we get

\[
2 \left( \mathbb{E} \xi^2 \right)^{1/2} \leq \left( \mathbb{E} \eta_1^2 \right)^{1/2} + \left( \mathbb{E} \eta_2^2 \right)^{1/2},
\]

whence

\[
\ell(B', \text{Id}_n) \leq \ell(B, \text{Id}_n).
\]

This implies that \( B' \) must also be in the \( \ell \)-position contradicting the fact that the position is unique up to an orthogonal transformation. \( \square \)

### 2.3 The gradient

Let \( B \) be an origin-symmetric convex body in \( \mathbb{R}^n \) with a smooth boundary. For any point \( x \in \mathbb{R}^n \setminus \{0\} \), the gradient \( \text{grad}_B(x) \) of the function \( \parallel \cdot \parallel_B \) at point \( x \) is well defined. It is not difficult to check that

\[
\|x\|_B = \langle \text{grad}_B(x), x \rangle = \sup_{y \in \mathbb{R}^n \setminus \{0\}} \langle \text{grad}_B(y), x \rangle,
\]

and that \( \text{grad}_B(\lambda x) = \text{grad}_B(x) = -\text{grad}_B(-x) \) for all \( x \in \mathbb{R}^n \setminus \{0\} \) and \( \lambda > 0 \). Further, the gradient of \( \| \cdot \|_B \) is continuous everywhere on its domain.

The next statement follows from the fact that any 1-unconditional norm in \( \mathbb{R}^n \) is a monotone function in the positive cone, as well as from hyperplane symmetries. We omit the proof.

**Lemma 10.** Let \( B \) be a convex body in \( \mathbb{R}^n \) with a smooth boundary such that the standard basis \( e_1, e_2, \ldots, e_n \) is 1-unconditional with respect to \( \| \cdot \|_B \). Then for every point \( (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \setminus \{0\} \) and every collection of signs \( (\sigma_j)_{j=1}^n \in \{-1, 1\}^n \) we have

\[
0 \leq \langle \text{grad}_B \left( \sum_{j \leq n} x_j e_j \right), x_i e_i \rangle = \langle \text{grad}_B \left( \sum_{j \leq n} \sigma_j x_j e_j \right), \sigma_i x_i e_i \rangle, \quad i \leq n.
\]

As an almost immediate consequence, we obtain
Lemma 11. Let $B$ be a convex body in $\mathbb{R}^n$ with a smooth boundary such that the standard basis $e_1, e_2, \ldots, e_n$ is 1-unconditional with respect to $\| \cdot \|_B$. Then for any $p \geq 1$ we have

$$
\mathbb{E}\|\nabla_B(G)\|_1^p = \mathbb{E}\left( \sum_{i=1}^{n} |\langle \nabla_B(G), e_i \rangle| \right)^p \leq \left( \frac{\pi}{2} \right)^{p/2} \mathbb{E}\|G\|_B^p.
$$

Proof. Let $G'$ be an independent copy of $G$. Then, applying Lemma 10 and formula (2), as well as standard estimates for the moments of Gaussians, we get

$$
\mathbb{E}\left( \sum_{i=1}^{n} |\langle \nabla_B(G), e_i \rangle| \right)^p \leq \left( \frac{\pi}{2} \right)^{p/2} \mathbb{E}\left( \sum_{i=1}^{n} |\langle \nabla_B(G), e_i \rangle \langle G', e_i \rangle| \right)^p \leq \left( \frac{\pi}{2} \right)^{p/2} \mathbb{E}\left( \sum_{i=1}^{n} |\langle \nabla_B(Z), e_i \rangle \langle G', e_i \rangle| \right)^p = \left( \frac{\pi}{2} \right)^{p/2} \mathbb{E}\|Z\|_B^p,
$$

where $Z = \sum_{i=1}^{n} \text{sgn}(\langle G, e_i \rangle \langle G', e_i \rangle) \langle G', e_i \rangle e_i$. It remains to note that $Z$ is the standard Gaussian vector in $\mathbb{R}^n$. \qed

Let us state one more simple geometric property of the gradient:

Lemma 12. Let $B$ be a smooth convex body in $\mathbb{R}^n$ such that the standard basis $e_1, e_2, \ldots, e_n$ is 1-unconditional with respect to $\| \cdot \|_B$. Then for any $i \leq n$ and any fixed numbers $x_j$ ($j \neq i$), the function $|\langle \nabla_B(x_1, x_2, \ldots, x_n), e_i \rangle|$ of one variable $x_i \in \mathbb{R}$ is non-increasing on $(-\infty, 0)$ and non-decreasing on $(0, \infty)$.

The next elementary observation follows directly from property (2).

Lemma 13. Let $B$ be an origin-symmetric convex body in $\mathbb{R}^n$ with a smooth boundary. There is a universal constant $C > 0$ such that for all $x \in \mathbb{R}^n \setminus \{0\}$ we have

$$
\|\nabla_B(x)\|_2 \leq C \mathbb{E}\|G\|_B.
$$

3 Basic properties of the $\ell$-position

Let us note that all auxiliary results proved in this section work for arbitrary origin-symmetric convex bodies with smooth boundaries in the $\ell$-position.

Lemma 14. Let $B \subset \mathbb{R}^n$ be a smooth origin-symmetric convex body in the $\ell$-position. Then

$$
\mathbb{E}\left( \|G\|_B \langle \nabla_B(G), u \rangle \langle G, u \rangle \right) = \frac{1}{n} \mathbb{E}\|G\|_B^2 = \frac{1}{n} \quad \text{for any} \quad u \in S^{n-1}.
$$
Proof. We will show that
\[ \mathbb{E}(\|G\|_B \langle \nabla_B(G), e_i \rangle \langle G, e_i \rangle) = \frac{1}{n} \mathbb{E}\|G\|_B^2 = \frac{1}{n}, \quad i \leq n; \]
the statement will then follow by rotation-invariance of the $\ell$-position. Fix for a moment any $i \leq n$, take $\varepsilon \in (0, 1)$ and define a diagonal operator $D = D_\varepsilon$ via its diagonal entries:
\begin{align*}
d_{jj} &= \begin{cases} (1 - \varepsilon)^{n-1}, & \text{if } j = i; \\ (1 - \varepsilon)^{-1}, & \text{otherwise.} \end{cases}
\end{align*}
We clearly have $\det D = 1$, and, in view of (2),
\begin{align*}
\mathbb{E}\|DG\|_B^2 &= \mathbb{E}\left( \sum_{j=1}^{n} \langle \nabla_B(DG), e_j \rangle d_{jj} \langle G, e_j \rangle \right)^2 \\
&= \mathbb{E}\left( \sum_{j=1}^{n} \langle \nabla_B(DG), e_j \rangle \langle G, e_j \rangle + \sum_{j=1}^{n} \langle \nabla_B(DG), e_j \rangle (d_{jj} - 1) \langle G, e_j \rangle \right)^2 \\
&\leq \mathbb{E}\left( \sum_{j=1}^{n} \langle \nabla_B(G), e_j \rangle \langle G, e_j \rangle + \sum_{j=1}^{n} \langle \nabla_B(DG), e_j \rangle (d_{jj} - 1) \langle G, e_j \rangle \right)^2 \\
&= \mathbb{E}\left( \|G\|_B - \varepsilon(n - 1) \langle \nabla_B(DG), e_i \rangle \langle G, e_i \rangle 
\right. \\
&\quad + \varepsilon \sum_{j \neq i} \langle \nabla_B(DG), e_j \rangle \langle G, e_j \rangle + o(\varepsilon) \|G\|_B 
\left. \right)^2 \\
&= \mathbb{E}\left( \|G\|_B - \varepsilon n \langle \nabla_B(DG), e_i \rangle \langle G, e_i \rangle 
\right. \\
&\quad + \varepsilon \sum_{j=1}^{n} \langle \nabla_B(DG), e_j \rangle \langle G, e_j \rangle + o(\varepsilon) \|G\|_B 
\left. \right)^2 \\
&= \mathbb{E}\left( \|G\|_B + \varepsilon \sum_{j=1}^{n} \langle \nabla_B(DG), e_j \rangle \langle G, e_j \rangle \right)^2 \\
&\quad - 2\varepsilon n \mathbb{E}\left( \|G\|_B \langle \nabla_B(DG), e_i \rangle \langle G, e_i \rangle \right) + o(\varepsilon).
\end{align*}
Further, as $\nabla_B(\cdot)$ is continuous at every point of $\mathbb{R}^n \setminus \{0\}$, we get
\begin{align*}
\mathbb{E}\left( \|G\|_B + \varepsilon \sum_{j=1}^{n} \langle \nabla_B(DG), e_j \rangle \langle G, e_j \rangle \right)^2 &- 2\varepsilon n \mathbb{E}\left( \|G\|_B \langle \nabla_B(DG), e_i \rangle \langle G, e_i \rangle \right) \\
&\leq \mathbb{E}\left( \|G\|_B + \varepsilon \sum_{j=1}^{n} \langle \nabla_B(G), e_j \rangle \langle G, e_j \rangle \right)^2 \\
&\quad - 2\varepsilon n \mathbb{E}\left( \|G\|_B \langle \nabla_B(G), e_i \rangle \langle G, e_i \rangle \right) + o(\varepsilon) \\
&= (1 + 2\varepsilon) \mathbb{E}\|G\|_B^2 - 2\varepsilon n \mathbb{E}\left( \|G\|_B \langle \nabla_B(G), e_i \rangle \langle G, e_i \rangle \right) + o(\varepsilon).
\end{align*}
On the other hand, in view of the definition of the $\ell$-position, we have $\mathbb{E}\|G\|_B^2 \leq \mathbb{E}\|DG\|_B^2$ for any $\varepsilon$. Combining this with the above inequalities, we obtain
\[ \mathbb{E}\|G\|_B^2 \leq (1 + 2\varepsilon) \mathbb{E}\|G\|_B^2 - 2\varepsilon n \mathbb{E}\left( \|G\|_B \langle \nabla_B(G), e_i \rangle \langle G, e_i \rangle \right) + o(\varepsilon). \]
Taking the limit when $\varepsilon \to 0$, we get
\[ n\mathbb{E}(\|G\|_B\langle \nabla B(G), e_i \rangle \langle G, e_i \rangle) \leq \mathbb{E}\|G\|_B^2, \quad i \leq n. \]

At the same time, obviously
\[ \sum_{i=1}^n \mathbb{E}(\|G\|_B\langle \nabla B(G), e_i \rangle \langle G, e_i \rangle) = \mathbb{E}\|G\|_B^2. \]

Thus, the above relations must be equalities for all $i$. \hfill $\Box$

**Lemma 15.** For any $\delta > 0$ and $p \in [1, \infty)$ there are numbers $n_0 = n_0(\delta)$ depending only on $\delta$ and $c_{\delta, p} > 0$ depending on $\delta$ and $p$ with the following property. Let $n \geq n_0$, and let $B$ be an origin-symmetric convex body in $\mathbb{R}^n$ such that
\[ \mathbb{P}\{\|gx\|_B^p \geq \mathbb{E}\|G\|_B^p \} \leq n^{-\delta} \quad \text{for any vector } x \in S^{n-1}. \quad (3) \]

Then
\[ \frac{\mathbb{E}\|G\|_B}{\text{Lip}(\| \cdot \|_B)} \geq c_{\delta, p}\sqrt{\log n}. \]

**Proof.** Without loss of generality, $n^\delta$ is large and $\text{Lip}(\| \cdot \|_B) = 1$. Fix a vector $x \in S^{n-1}$ with $\|x\|_B = 1$. Standard deviation estimates for Gaussian variables imply
\[ \mathbb{P}\{\|gx\|_B \geq \sqrt{\log n^\delta} \} \geq n^{-\delta}, \]

whence, in view of (3), we have
\[ \mathbb{E}\|G\|_B^p \geq (\log n^\delta)^{p/2}. \]

It remains to note that $\mathbb{E}\|G\|_B \geq c_p (\mathbb{E}\|G\|_B^p)^{1/p}$ for some $c_p > 0$ depending only on $p$ (see, for example, [15, Corollary 3.2]). \hfill $\Box$

Together Lemmas 14 and 15 imply

**Proposition 16.** There are universal constants $n_0 \in \mathbb{N}$ and $c > 0$ with the following property. Let $n \geq n_0$, and let $B$ be an origin-symmetric convex body in $\mathbb{R}^n$ with a smooth boundary in the $\ell$-position. Then
\[ \frac{\mathbb{E}\|G\|_B}{\text{Lip}(\| \cdot \|_B)} \geq c\sqrt{\log n}. \]

**Proof.** We can assume that $n$ is large. We will construct an orthogonal basis $y_1, y_2, \ldots, y_n$ in $\mathbb{R}^n$ as follows. First, there is a vector $y_1$ with $\|y_1\|_2 = \text{Lip}(\| \cdot \|_B)$ such that $B \subset \{x \in \mathbb{R}^n : |\langle x, y_1 \rangle| \leq 1\}$. We set $H_1 := \langle x, y_1 \rangle$. Now, assuming that $y_1, y_2, \ldots, y_k$ are constructed, choose a vector $y_{k+1} \in H_k := \text{span}\{y_1, y_2, \ldots, y_k\}$ with $\|y_{k+1}\|_2 = \text{Lip}(\| \cdot \|_{B \cap H_k})$ such that $B \cap H_k \subset \{x \in H_k : |\langle x, y_{k+1} \rangle| \leq 1\}$.

Note that $\|y_k\|_B = \|y_k\|_2^2$ for any $k \leq n$ and that $\|y_{k+1}\|_2 \leq \|y_k\|_2$ for all $k \leq n-1$. Now, set $q := \mathbb{E}\|G\|_B$, $m := \lceil\sqrt{n}\rceil$, and consider two cases.
• Suppose that $\|y_m\|_2 \geq \sqrt{q \operatorname{Lip}(\|\cdot\|_B)/(\log n)^{1/4}}$. In view of the triangle inequality and the definition of $y_i$’s, we have

$$\mathbb{E}\|G\|_B \geq \mathbb{E}\left|\sum_{i=1}^m y_i g_i \right|_B \geq \frac{1}{2} \mathbb{E}\max_{i \leq m} \frac{|g_i|\|y_i\|_B}{\|y_i\|_2} = \frac{1}{2} \mathbb{E}\max_{i \leq m} |g_i|\|y_i\|_2$$

(here and further in the proof $g_i$’s are independent standard Gaussians). Then standard estimates for the maximum of independent Gaussian variables [7, p. 302], together with the assumption on the Euclidean norm of $y_i$’s, imply

$$q = \mathbb{E}\|G\|_B \geq \frac{\sqrt{q \operatorname{Lip}(\|\cdot\|_B)}}{2(\log n)^{1/4}} \mathbb{E}\max_{i \leq m} |g_i| > \frac{\sqrt{q \operatorname{Lip}(\|\cdot\|_B)}(\log n)^{1/4}}{4}.$$ 

Hence, we get

$$q \frac{\operatorname{Lip}(\|\cdot\|_B)}{\sqrt{\log n}} > \frac{1}{16} \sqrt{\log n}.$$ 

• Assume that $\|y_m\|_2 < \sqrt{q \operatorname{Lip}(\|\cdot\|_B)/(\log n)^{1/4}}$. Thus, the Lipschitz constant of $\|\cdot\|_{B \cap H_m}$ is less than $\frac{\sqrt{q \operatorname{Lip}(\|\cdot\|_B)}}{(\log n)^{1/4}}$, and Theorem 5 together with the relation $\mathbb{E}\|G\|_B \geq \mathbb{E}\|\operatorname{Proj}_{H_m} G\|_B$, imply

$$\mathbb{P}\{\|\operatorname{Proj}_{H_m} G\|_{B \cap H_m} \geq q + 4\sqrt{q \operatorname{Lip}(\|\cdot\|_B)}(\log n)^{1/4}\} \leq \frac{1}{n^2}. \quad (4)$$ 

On the other hand, in view of the choice of $y_1$ and standard deviation estimates for a Gaussian variable, we have

$$\mathbb{P}\{\|G\|_B \geq \frac{1}{2} \sqrt{\log n} \operatorname{Lip}(\|\cdot\|_B)\} \geq \frac{1}{2} \mathbb{P}\{\|gy_1\|_B \geq \frac{1}{2} \sqrt{\log n} \operatorname{Lip}(\|\cdot\|_B)\} > \frac{1}{\sqrt{n}}.$$

Clearly, in view of (2) we have

$$\|G\|_B = \sum_{i=1}^n \|y_i\|^2_2 \langle \operatorname{grad}_B(G), y_i \rangle \langle G, y_i \rangle$$

and

$$\|\operatorname{Proj}_{H_m} G\|_{B \cap H_m} = \sum_{i=m+1}^n \|y_i\|^2_2 \langle \operatorname{grad}_B(\operatorname{Proj}_{H_m} G), y_i \rangle \langle G, y_i \rangle \geq \sum_{i=m+1}^n \|y_i\|^2_2 \langle \operatorname{grad}_B(G), y_i \rangle \langle G, y_i \rangle.$$ 

Thus, (4) yields

$$\mathbb{P}\{\|G\|_B - \sum_{i=1}^m \|y_i\|^2_2 \langle \operatorname{grad}_B(G), y_i \rangle \langle G, y_i \rangle \geq q + 4\sqrt{q \operatorname{Lip}(\|\cdot\|_B)}(\log n)^{1/4}\} \leq \frac{1}{n^2}.$$
The last inequality, together with the above deviation estimates for \(\|G\|_B\), implies
\[
\mathbb{P}\left\{ \|G\|_B \sum_{i=1}^m \|y_i\|_2^2 \langle \text{grad}_B(G), y_i \rangle \langle G, y_i \rangle \geq \frac{1}{2} \sqrt{\log n} \text{Lip}(\| \cdot \|_B) \right. \\
\left. \left( \frac{1}{2} \sqrt{\log n} \text{Lip}(\| \cdot \|_B) - q - 4q \sqrt{\text{Lip}(\| \cdot \|_B)(\log n)^{1/4}} \right) \right\} \geq \frac{1}{\sqrt{n}} - \frac{1}{n^2} > \frac{1}{2\sqrt{n}},
\]
whence
\[
\mathbb{E}\left( \|G\|_B \sum_{i=1}^m \|y_i\|_2^2 \langle \text{grad}_B(G), y_i \rangle \langle G, y_i \rangle \right)
> \text{Lip}(\| \cdot \|_B) \sqrt{\log n} \left( \frac{1}{2} \sqrt{\log n} \text{Lip}(\| \cdot \|_B) - q - 4q \sqrt{\text{Lip}(\| \cdot \|_B)(\log n)^{1/4}} \right).
\]

On the other hand, by Lemma 14 and in view of the equivalence of moments of \(\|G\|_B\) (see [15, Corollary 3.2]),
\[
\mathbb{E}\left( \|G\|_B \sum_{i=1}^m \|y_i\|_2^2 \langle \text{grad}_B(G), y_i \rangle \langle G, y_i \rangle \right) = \frac{m}{n} \mathbb{E}\|G\|_B^2 \leq \frac{Cq^2}{\sqrt{n}}
\]
for a universal constant \(C > 0\). Thus,
\[
Cq^2 \geq \text{Lip}(\| \cdot \|_B) \sqrt{\log n} \left( \frac{1}{2} \sqrt{\log n} \text{Lip}(\| \cdot \|_B) - q - 4q \sqrt{\text{Lip}(\| \cdot \|_B)(\log n)^{1/4}} \right).
\]
Solving for \(q\), we get
\[
q \geq c' \text{Lip}(\| \cdot \|_B) \sqrt{\log n}
\]
for some constant \(c' > 0\).

As a consequence of the above proposition, we get

**Lemma 17.** There are universal constants \(n_0 \in \mathbb{N}\) and \(c > 0\) with the following property. Let \(n \geq n_0\), let \(B\) be an origin-symmetric convex body in \(\mathbb{R}^n\) with a smooth boundary in the \(\ell\)-position. Assume that
\[
\mathbb{E}\|G\|_B \leq n^c \mathbb{E}\|\text{grad}_B(G)\|_2.
\]
Then for all \(p \in [1, c \log n]\) we have
\[
\mathbb{E}\|G\|_B^p \leq n^{1/32} \mathbb{E}_n \|\text{grad}(\| \cdot \|_B^p)\|_2^2 = n^{1/32} p^2 \mathbb{E}(\|G\|_B^{2p-2}\|\text{grad}_B(G)\|_2^2),
\]
where \(\text{grad}(\| \cdot \|_B^p)\) is the gradient of the \(p\)-th power of the norm \(\| \cdot \|_B\).
Proof. We can assume that \( n \) is large. Proposition 16 and Theorem 5 imply that \( \|G\|_B \geq \frac{1}{2}\mathbb{E}\|G\|_B \) with probability at least \( 1 - n^{-c'} \) for a universal constant \( c' \in (0, 1/64] \). Now, set \( c := c'/2 \) and assume that \( \mathbb{E}\|G\|_B \leq n^c \mathbb{E}\|\text{grad}_B(G)\|_2 \). In view of Lemma 13 we have \( \|\text{grad}_B(x)\|_2 \leq C\mathbb{E}\|G\|_B \) for all non-zero vectors \( x \) and a universal constant \( C > 0 \). Hence, denoting by \( \mathcal{E} \) the event that \( \|\text{grad}_B(G)\|_2 \geq \frac{1}{2}n^{-c}\mathbb{E}\|G\|_B \), we obtain

\[
\mathbb{E}\|G\|_B \leq n^c \mathbb{E}(\|\text{grad}_B(G)\|_2 \chi_{\mathcal{E}}) + n^c \mathbb{E}(\|\text{grad}_B(G)\|_2 \chi_{\mathcal{E}}^c) \\
\leq \frac{1}{2}\mathbb{E}\|G\|_B + Cn^c \mathbb{P}(\mathcal{E})\mathbb{E}\|G\|_B,
\]

implying \( \mathbb{P}(\mathcal{E}) \geq \frac{1}{2C}n^{-c} \). Thus, with probability at least \( \frac{1}{2C}n^{-c} - n^{-c'} > n^{-c'} \) we have

\[
\|G\|_B \geq \frac{1}{2}\mathbb{E}\|G\|_B \quad \text{and} \quad \|\text{grad}_B(G)\|_2 \geq \frac{1}{2}n^{-c}\mathbb{E}\|G\|_B,
\]

whence for all \( p \geq 1 \) we get

\[
\mathbb{E}(\|G\|_B^{2p-2}\|\text{grad}_B(G)\|_2^2) \geq 2^{-2p}n^{-2c'}(\mathbb{E}\|G\|_B)^{2p}.
\]

On the other hand, the concentration of \( \|G\|_B \) (again, provided by Proposition 16 and Theorem 5) implies that

\[
\mathbb{E}\|G\|_B^{2p} < 2^p(\mathbb{E}\|G\|_B)^{2p}
\]

for all \( p \leq c'' \log n \) for a sufficiently small universal constant \( c'' > 0 \). Thus, for all such \( p \) we have

\[
\mathbb{E}\|G\|_B^{2p} \leq 2^{3p}n^{2c'}(\mathbb{E}\|G\|_B^{2p-2}\|\text{grad}_B(G)\|_2^2),
\]

and the result follows. \( \square \)

4 The superconcentration of \( \| \cdot \|_B^p \)

Let \( B \) be an origin-symmetric convex body in \( \mathbb{R}^n \) with a smooth boundary and let \( p \geq 1 \). For any point \( x \in \mathbb{R}^n \setminus \{0\} \) the \( i \)-th partial derivative of \( \| \cdot \|_B^p \) at \( x \) is equal to \( p\|x\|_B^{-p-1}\langle \text{grad}_B(x), e_i \rangle \). Hence, applying Theorem 6 we get

\[
\text{Var}(\|G\|_B^p) \leq \sum_{i=1}^n C_p^2 \mathbb{E}(\|G\|_B^{2p-2}\langle \text{grad}_B(G), e_i \rangle^2) \left( 1 + \log \left( \frac{\mathbb{E}(\|G\|_B^{2p-2}\langle \text{grad}_B(G), e_i \rangle^2)/\mathbb{E}\|G\|_B^{p-1}\langle \text{grad}_B(G), e_i \rangle}{\mathbb{E}\|G\|_B^{p-1}\langle \text{grad}_B(G), e_i \rangle} \right) \right),
\]

where \( C > 0 \) is a universal constant. For each \( i \leq n \), write

\[
\langle \text{grad}_B(x), e_i \rangle = F_i(B, x) + S_i(B, x), \quad x \in \mathbb{R}^n \setminus \{0\},
\]

where

\[
F_i(B, x) := \langle \text{grad}_B(x), e_i \rangle \chi_{\{\|\text{grad}_B(x), e_i \| \leq n^{-1/8}\mathbb{E}\|G\|_B\}},
\]

\[
S_i(B, x) := \langle \text{grad}_B(x), e_i \rangle \chi_{\{\|\text{grad}_B(x), e_i \| > n^{-1/8}\mathbb{E}\|G\|_B\}}.
\]
Here, “F” stands for “flat” and “S” — for “spiky”. Then the upper bound for the variance can be written as

$$\text{Var}(\|G\|_B^p) \leq C' p^2 \mathbb{E}\left(\|G\|_B^{2p-2} \sum_{i=1}^{n} F_i^2(B, G)\right)$$

$$+ \sum_{i=1}^{n} \frac{C' p^2 \mathbb{E}(\|G\|_B^{2p-2} S_i^2(B, G))}{1 + \log \left(\mathbb{E}(\|G\|_B^{2p-2} \langle \text{grad}_B(G), e_i \rangle^2) / \mathbb{E}\|G\|_B^{2p-1} \langle \text{grad}_B(G), e_i \rangle\right)}.$$ 

(5)

We will treat the flat and the spiky parts separately.

**Lemma 18** (The flat part). There are universal constants $n_0 \in \mathbb{N}$ and $c > 0$ with the following property. Let $n \geq n_0$, let $B$ be a smooth origin-symmetric convex body in $\mathbb{R}^n$ in the $\ell$-position, and assume that the standard basis in $\mathbb{R}^n$ is $1$-unconditional with respect to $\| \cdot \|_B$. Then for all $p$ in the interval $1 \leq p \leq c \log n$ we have

$$p^2 \mathbb{E}\left(\|G\|_B^{2p-2} \sum_{i=1}^{n} F_i^2(B, G)\right) \leq n^{-1/16} \mathbb{E}\|G\|_B^p.$$ 

*Proof.* We will assume that $n$ is large. Note that

$$\sum_{i=1}^{n} F_i^2(B, G) \leq n^{-1/8} \mathbb{E}\|G\|_B \sum_{i=1}^{n} |F_i(B, G)|,$$

whence for all $p \geq 1$ we get

$$\mathbb{E}\left(\|G\|_B^{2p-2} \sum_{i=1}^{n} F_i^2(B, G)\right)$$

$$\leq n^{-1/8} \mathbb{E}\|G\|_B \mathbb{E}\left(\|G\|_B^{2p-2} \sum_{i=1}^{n} |F_i(B, G)|\right)$$

$$\leq n^{-1/8} \mathbb{E}\|G\|_B (\mathbb{E}\|G\|_B^{2p-1})^{(2p-2)/(2p-1)} \left(\mathbb{E}\left(\sum_{i=1}^{n} |F_i(B, G)|\right)^{2p-1}\right)^{1/(2p-1)},$$

where the last relation follows from Hölder’s inequality. Next, in view of Lemma 11

$$\mathbb{E}\left(\sum_{i=1}^{n} |F_i(B, G)|\right)^{2p-1} \leq \mathbb{E}\left(\sum_{i=1}^{n} |\langle \text{grad}_B(G), e_i \rangle|\right)^{2p-1} \leq (\pi/2)^{p-1/2} \mathbb{E}\|G\|_B^{2p-1},$$

whence

$$\mathbb{E}\left(\|G\|_B^{2p-2} \sum_{i=1}^{n} F_i^2(B, G)\right) \leq \sqrt{\pi/2} n^{-1/8} \mathbb{E}\|G\|_B \mathbb{E}\|G\|_B^{2p-1}.$$ 

It remains to note that we can choose the constant $c > 0$ small enough and $n_0$ large enough to guarantee that

$$\sqrt{\pi/2} p^2 n^{-1/8} \leq n^{-1/16}.$$ 

$\square$
Lemma 19 (The spiky part). There are universal constants $n_0 \in \mathbb{N}$ and $c' > 0$ with the following property. Let $n \geq n_0$, let $B$ be as in the last lemma, and let $i \leq n$, $p \geq 1$ and $\tau > 0$ be such that

$$\mathbb{E}(\|G\|_B^{2p-2}F_i^2(B, G)) \leq n^{-\tau}\mathbb{E}(\|G\|_B^{2p-2}\langle \text{grad}_B(G), e_i \rangle^2).$$

Then

$$\left(\mathbb{E}\|G\|_B^{p-1}\langle \text{grad}_B(G), e_i \rangle\right)^2 \leq 2(n^{-c'} + n^{-\tau})\mathbb{E}(\|G\|_B^{2p-2}\langle \text{grad}_B(G), e_i \rangle^2).$$

Proof. We will assume that $n$ is large. In view of Lemma 14 as well as Lemma 10 and the definition of $S_i(B, G)$, we have

$$\mathbb{E}\|G\|_B \mathbb{E}(\|G\|_B\langle G, e_i \rangle | \chi_{\{S_i(B, G) \neq 0\}}) n^{-1/8}) \leq \mathbb{E}(\|G\|_B\langle \text{grad}_B(G), e_i \rangle \langle G, e_i \rangle) = \frac{1}{n} \mathbb{E}\|G\|_B^2.$$ 

On the other hand, conditioned on any realization of $\langle G, e_j \rangle (j \neq i)$, $\|G\|_B$ is a monotone function of $|\langle G, e_i \rangle|$, and, in view of Lemma 12 and the definition of $S_i$, $\chi_{\{S_i(B, G) \neq 0\}}$ is a monotone function of $|\langle G, e_i \rangle|$. Hence, setting $g_j := \langle G, e_j \rangle (j \leq n)$, we obtain

$$\mathbb{E}(\|G\|_B\langle G, e_i \rangle | \chi_{\{S_i(B, G) \neq 0\}}) = \mathbb{E}_{\{g_j, j \neq i\}} \mathbb{E}_{\{g_i\}} (\|G\|_B\langle G, e_i \rangle | \chi_{\{S_i(B, G) \neq 0\}})$$

$$\geq \mathbb{E}(\|G\|_B \chi_{\{S_i(B, G) \neq 0\}}) \mathbb{E}\langle G, e_i \rangle$$

$$= \sqrt{\frac{2}{\pi}} \mathbb{E}(\|G\|_B \chi_{\{S_i(B, G) \neq 0\}}).$$

Further,

$$\mathbb{E}(\|G\|_B \chi_{\{S_i(B, G) \neq 0\}}) \geq \mathbb{E}(\|G\|_B \chi_{\{S_i(B, G) \neq 0\} \text{ and } 2\|G\|_B \geq \mathbb{E}\|G\|_B})$$

$$\geq \frac{1}{2} \mathbb{E}\|G\|_B \mathbb{P}\{S_i(B, G) \neq 0 \text{ and } 2\|G\|_B \geq \mathbb{E}\|G\|_B\}$$

$$\geq \frac{1}{2} \mathbb{E}\|G\|_B \left(\mathbb{P}\{S_i(B, G) \neq 0\} - \mathbb{P}\{2\|G\|_B < \mathbb{E}\|G\|_B\}\right)$$

$$\geq \frac{1}{2} \mathbb{E}\|G\|_B \left(\mathbb{P}\{S_i(B, G) \neq 0\} - n^{-c'}\right),$$

where the last inequality follows from Proposition 16 and Theorem 5. Combining all the above inequalities, we obtain

$$\frac{1}{n} = \mathbb{E}\|G\|_B^2 \geq \frac{1}{\sqrt{2\pi}} n^{-1/8} (\mathbb{E}\|G\|_B)^2 \left(\mathbb{P}\{S_i(B, G) \neq 0\} - n^{-c'}\right).$$

Taking into consideration the equivalence of moments of $\|G\|_B$, we get that

$$\mathbb{P}\{S_i(B, G) \neq 0\} \leq n^{-c''}$$

for a universal constant $c'' > 0$. Thus,

$$\left(\mathbb{E}\|G\|_B^{p-1}S_i(B, G)\right)^2 \leq n^{-c''}\mathbb{E}(\|G\|_B^{2p-2}S_i^2(B, G)).$$
Finally, we have, by the assumption of the lemma and the above relation,
\[
\mathbb{E}\left(\|G\|_{B}^{2p-2}\langle \text{grad}_{B}(G), e_i \rangle^2 \right) \geq n^{-c''} \left( \mathbb{E}\|G\|_{B}^{p-1}S_i(B, G) \right)^2;
\]
\[
\mathbb{E}\left(\|G\|_{B}^{2p-2}\langle \text{grad}_{B}(G), e_i \rangle^2 \right) \geq n^{-\tau} \left( \mathbb{E}\|G\|_{B}^{p-1}F_i(B, G) \right)^2,
\]
whence
\[
\left( \mathbb{E}\|G\|_{B}^{p-1}\langle \text{grad}_{B}(G), e_i \rangle \right)^2 \leq 2(n^{-c''} + n^{-\tau})\mathbb{E}\left(\|G\|_{B}^{2p-2}\langle \text{grad}_{B}(G), e_i \rangle^2 \right).
\]
\hspace{1cm} \square

**Proof of Theorem 4.** We suppose that \( n \) is large. Moreover, in view of Lemma 9 and rotation-invariance of the Gaussian distribution, we can assume without loss of generality that the standard basis in \( \mathbb{R}^n \) is 1-unconditional with respect to \( \| \cdot \|_B \). Let \( c > 0 \) be minimum of the constants from Lemmas 17 and 18 and let \( p \in [1, c \log n] \). We assume that
\[
\mathbb{E}\|G\|_{B} \leq n^c \mathbb{E}\|\text{grad}_{B}(G)\|_2.
\]
Let us start by applying Lemma 19 with \( \tau := 1/64 \). Note that for those \( i \leq n \) with
\[
\mathbb{E}\left(\|G\|_{B}^{2p-2}F_i^2(B, G) \right) \leq n^{-1/64}\mathbb{E}\left(\|G\|_{B}^{2p-2}\langle \text{grad}_{B}(G), e_i \rangle^2 \right)
\]
we have
\[
\log \left( \sqrt{\mathbb{E}(\|G\|_{B}^{2p-2}\langle \text{grad}_{B}(G), e_i \rangle^2)/\mathbb{E}\|G\|_{B}^{p-1}\langle \text{grad}_{B}(G), e_i \rangle} \right) \geq \tilde{c}\log n
\]
for a universal constant \( \tilde{c} > 0 \). Hence,
\[
\sum_{i=1}^{n} \frac{C'p^2 \mathbb{E}(\|G\|_{B}^{2p-2}S_i^2(B, G))}{1 + \log \left( \sqrt{\mathbb{E}(\|G\|_{B}^{2p-2}\langle \text{grad}_{B}(G), e_i \rangle^2)/\mathbb{E}\|G\|_{B}^{p-1}\langle \text{grad}_{B}(G), e_i \rangle} \right)} \leq C'p^2n^{1/64} \sum_{i=1}^{n} \mathbb{E}(\|G\|_{B}^{2p-2}F_i^2(B, G)) + \frac{C'n^p}{\log n} \sum_{i=1}^{n} \mathbb{E}(\|G\|_{B}^{2p-2}S_i^2(B, G)).
\]
Together with relation (5) this gives
\[
\text{Var}(\|G\|_{B}^p) \leq \tilde{C}p^2n^{1/64} \sum_{i=1}^{n} \mathbb{E}(\|G\|_{B}^{2p-2}F_i^2(B, G)) + \frac{\tilde{C}p^2}{\log n} \sum_{i=1}^{n} \mathbb{E}(\|G\|_{B}^{2p-2}S_i^2(B, G)).
\]
Next, applying Lemma 18 we obtain
\[
\text{Var}(\|G\|_{B}^p) \leq \tilde{C}n^{-3/64} \mathbb{E}\|G\|_{B}^{2p} + \frac{\tilde{C}p^2}{\log n} \sum_{i=1}^{n} \mathbb{E}(\|G\|_{B}^{2p-2}S_i^2(B, G)).
\]
Finally, in view of Lemma 17 and (6), this gives
\[
\text{Var}(\|G\|_{B}^p) \leq \left( \tilde{C}n^{-1/64} + \frac{\tilde{C}p^2}{\log n} \right)\mathbb{E}(\|G\|_{B}^{p})^2\|\text{grad}_{B}(G)\|_{B}^2.
\]
It remains to note that \( p\|G\|_{B}^{p-1}\|\text{grad}_{B}(G)\|_{2} \) equals the Euclidean norm of the gradient of \( \| \cdot \|_{B}^p \) at \( G \), and apply the definition of superconcentration. \hspace{1cm} \square
5 The randomized Dvoretzky theorem

We will show how the variance bound from the previous section is translated into a small deviations inequality for $\|G\|_B$. At a high level, the procedure is rather standard; for example, let us refer to [11, Chapter 3] for a very general scheme that allows to deduce exponential concentration from the Poincaré inequality. On the other hand, as we have better bounds on the variance of $\|G\|^p_B$ than those provided by the Poincaré inequality, our deviation estimates are stronger. Let us remark that the use of superconcentration in our analysis was inspired by a recent paper of G. Paouris, P. Valett as and J. Zinn [22] dealing with almost Euclidean sections of $\ell^p$-balls, which complemented earlier results of A. Naor [20].

Let us start with a simple lemma that follows immediately from Proposition 19 and Theorem 5.

Lemma 20. There are universal constants $n_0 \in \mathbb{N}$ and $C \geq c$ with the following property. Let $n \geq n_0$, let $B$ be an origin-symmetric convex body in $\mathbb{R}^n$ in the $\ell$-position, with a smooth boundary, and let $q \leq \log n$. Then $\mathbb{E}\|G\|^{2q}_B \leq C^q \left(\mathbb{E}\|G\|^2_B\right)^q$.

Lemma 21. There are constants $n_0 \in \mathbb{N}$, $c' > 0$ and $C' > 0$ with the following property. Let $B$ be an origin-symmetric convex body in $\mathbb{R}^n$ in the $\ell$-position, with a smooth boundary, and such that $(\mathbb{R}^n, \| \cdot \|_B)$ admits a $1$-unconditional basis. Then for all $p \in [1, c' \log n]$ we have

$$\text{Var}(\|G\|^p_B) \leq \frac{C' p^2}{(\log n)^2} \mathbb{E}\|G\|^{2p}_B.$$ 

Proof. We can suppose that $n$ is large. Let $c > 0$ be the constant from Theorem 4 (we can safely assume that $c \leq 1/2$), and set $c' := c/(8 \log C)$, where $C$ is taken from Lemma 20. Further, let $p \in [1, c' \log n]$ and consider two cases:

- Suppose that $\mathbb{E}\|G\|_B > n^c \mathbb{E}\|\text{grad}_B(G)\|_2$. Denote by $\mathcal{E}$ the event $\|\text{grad}_B(G)\|_2 \geq n^{-c/2} \mathbb{E}\|G\|_B$. Clearly, $\mathbb{P}(\mathcal{E}) \leq n^{-c/2}$ by Markov’s inequality. Now, the Poincaré inequality for $\| \cdot \|_B$ implies

$$\text{Var}(\|G\|^p_B) \leq p^2 \mathbb{E}\left(\|G\|^{2p-2}_B \|\text{grad}_B(G)\|^2_2 \right) \leq p^2 \mathbb{E}\left(\|G\|^{2p-2}_B \|\text{grad}_B(G)\|^2_2 \chi_{\mathcal{E}} \right) + p^2 n^{-c} \left(\mathbb{E}\|G\|_B\right)^2 \mathbb{E}\|G\|^{2p-2}_B \leq \tilde{C} p^2 \left(\mathbb{E}\|G\|_B\right)^2 \mathbb{E}\left(\|G\|^{2p-2}_B \chi_{\mathcal{E}} \right) + p^2 n^{-c} \mathbb{E}\|G\|^{2p}_B,$$

where at the last step we applied Lemma 13. Further, by the Cauchy–Schwarz inequality and Lemma 20 we get

$$\mathbb{E}\left(\|G\|^{2p-2}_B \chi_{\mathcal{E}} \right) \leq \sqrt{\mathbb{P}(\mathcal{E})} \left(\mathbb{E}\|G\|^{4p-4}_B\right)^{1/2} \leq n^{-c/4} C^{p-1} \mathbb{E}\|G\|^{2p-2}_B.$$ 

Finally,

$$\text{Var}(\|G\|^p_B) \leq 2\tilde{C} p^2 n^{-c/4} C^{p-1} \mathbb{E}\|G\|^{2p}_B \leq 2\tilde{C} p^2 n^{-c/8} \mathbb{E}\|G\|^{2p}_B \leq \tilde{C} n^{-c/16} \mathbb{E}\|G\|^{2p}_B,$$

where the second inequality follows from the choice of $c'$. 

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• If $\mathbb{E}\|G\|_B \leq n^c \mathbb{E}\|\operatorname{grad}_B(G)\|_2$ then, by Theorem 11 we have

$$\operatorname{Var}(\|G\|^p_B) \leq \frac{C'p^2}{\log n} \mathbb{E}(\|G\|^{2p-2}_B \|\operatorname{grad}_B(G)\|^2_2)$$

for a universal constant $C' > 0$. Note that, in view of Proposition 10,

$$\|\operatorname{grad}_B(G)\|_2 \leq \operatorname{Lip}(\| \cdot \|_B) \leq C'' \|G\|_B \sqrt{\log n}$$

for a universal constant $C'' > 0$, whence

$$\operatorname{Var}(\|G\|^p_B) \leq \frac{\tilde{C} p^2}{(\log n)^2} \mathbb{E}\|G\|^{2p}_B.$$ 

Thus in any case we obtain the required bound. 

Now, we have

**Theorem 22.** There are universal constants $n_0 \in \mathbb{N}$ and $c > 0$ with the following property. Let $n \geq n_0$, let $B$ be an origin-symmetric convex body in $\mathbb{R}^n$ in the $\ell$-position, and assume that $(\mathbb{R}^n, \| \cdot \|_B)$ admits a 1-unconditional basis. Then for any $\varepsilon \in (0, 1/2]$ we have

$$\mathbb{P}\{|\|G\|_B - \operatorname{Med}\|G\|_B| \geq \varepsilon \operatorname{Med}\|G\|_B\} \leq 2n^{-c\varepsilon^2}.$$ 

**Proof.** As before, we assume that $n$ is large. Note that, in view of Corollary 8, our convex body $B$ can be approximated with arbitrary precision by a smooth convex body in the $\ell$-position. Thus, without loss of generality we can assume that $B$ itself is smooth. Let $c'$ and $C'$ be the constants from Lemma 21 and set $\tilde{c} := \min(c', \frac{1}{2\sqrt{C'}})$ and $p := \frac{\tilde{c}}{\log n}$. Note that Lemma 21 then implies

$$\operatorname{Var}(\|G\|^p_B) = \mathbb{E}\|G\|^{2p}_B - (\mathbb{E}\|G\|^p_B)^2 \leq \frac{1}{4} \mathbb{E}\|G\|^{2p}_B,$$

whence $\mathbb{E}\|G\|^{2p}_B \leq \frac{4}{3}(\mathbb{E}\|G\|^p_B)^2$. On the other hand,

$$\operatorname{Var}(\|G\|^p_B) \geq \frac{1}{2}(\operatorname{Med}\|G\|^p_B - \mathbb{E}\|G\|^p_B)^2,$$

which, together with the above inequality, gives

$$|\operatorname{Med}\|G\|^p_B - \mathbb{E}\|G\|^p_B| \leq \sqrt{\frac{2}{3}} \mathbb{E}\|G\|^p_B.$$ 

Now, for any $\varepsilon > 0$ we get

$$\mathbb{P}\{|G\|_B \geq (1 + \varepsilon)\operatorname{Med}\|G\|_B\} = \mathbb{P}\{\|G\|^p_B \geq (1 + \varepsilon)^p \operatorname{Med}\|G\|^p_B\} \leq \mathbb{P}\{\|G\|^p_B \geq 0.18(1 + \varepsilon)^p \mathbb{E}\|G\|^p_B\}.$$
Assume that $0.09(1 + \varepsilon)^p \geq 1$. Then, by Chebyshev’s inequality and the above, we get

\[
\mathbb{P}\{\|G\|_B \geq (1 + \varepsilon)\text{Med}\|G\|_B\} \leq \frac{\text{Var}(\|G\|_B^p)}{0.09^2(1 + \varepsilon)^{2p}(\mathbb{E}\|G\|_B^p)^2} \\
< \frac{200\text{Var}(\|G\|_B^p)}{(1 + \varepsilon)^{2p}\|G\|_B^{2p}} \\
< \frac{100}{(1 + \varepsilon)^{2p}}.
\]

To get lower deviation estimates, we apply a theorem of D. Cordero-Erausquin, M. Fradelizi and B. Maurey [5]. According to the theorem, the function

\[t \rightarrow \mathbb{P}\{\|G\|_B \leq e^t\text{Med}\|G\|_B\}\]

is log-concave on the real line. Hence, assuming $\varepsilon_0 > 0$ is the number satisfying $\mathbb{P}\{\|G\|_B \leq (1 + \varepsilon_0)\text{Med}\|G\|_B\} = \frac{3}{4}$, we get

\[
\log \mathbb{P}\{\|G\|_B \leq (1 - \varepsilon)\text{Med}\|G\|_B\} \leq \log \mathbb{P}\{\|G\|_B \leq e^{-\varepsilon}\text{Med}\|G\|_B\} \\
\leq -\frac{\varepsilon}{\varepsilon_0}\left(\log \mathbb{P}\{\|G\|_B \leq e^{\varepsilon_0}\text{Med}\|G\|_B\} - \log \frac{1}{2}\right) \\
\leq -\frac{\varepsilon}{\varepsilon_0}\left(\log \frac{3}{4} - \log \frac{1}{2}\right),
\]

whence

\[
\mathbb{P}\{\|G\|_B \leq (1 - \varepsilon)\text{Med}\|G\|_B\} \leq \exp\left(-\frac{\varepsilon}{\varepsilon_0} \log \frac{3}{2}\right)
\]

for all $\varepsilon > 0$. Note that the upper deviation estimates obtained in the first part of the proof imply that $\varepsilon_0 \leq \tilde{C}(\log n)^{-1}$ for a universal constant $\tilde{C} > 0$. Finally,

\[
\mathbb{P}\{\|G\|_B - \text{Med}\|G\|_B| \geq \varepsilon\text{Med}\|G\|_B\} \leq \frac{100}{(1 + \varepsilon)^{2\varepsilon\log n}} + \exp\left(-\tilde{C}^{-1} \log \frac{3}{2} \varepsilon \log n\right)
\]

for all $\varepsilon \geq \frac{c''}{\log n}$. The result follows.

The main result of this note — Theorem 3 — is obtained from Theorem 22 via a simple covering argument. We prefer to omit this (completely standard by now) part of the proof; we refer, in particular, to [19, 25, 2] for information on this matter.

6 Proof of Proposition 2

In this section, we prove Proposition 2 from the introduction, by providing an example of a convex set in $\mathbb{R}^n$ in John’s position which shows that in general one cannot expect a better than quadratic dependence on $\varepsilon$ in Theorem 1. In this connection it is natural to recall an example by T. Figiel which highlights the limitations of the existential Dvoretzky theorem (we refer to [25] Lecture 3 for more details). However, our example operates in a different regime as we bound the dimension of almost Euclidean sections by $\varepsilon^2\log n$ whereas Figiel’s convex set admits $\varepsilon^2n$–dimensional sections.
Let us start by stating two facts. The first of the two lemmas below can be verified by combining a concentration inequality for the $\ell_\infty$-norm of the Gaussian vector with a standard covering argument (see [26]), whereas the second one is a simple (and rather crude) corollary of Theorem 5 (again, combined with a covering procedure).

**Lemma 23.** There are universal constants $\kappa > 0$ and $r_0 \in \mathbb{N}$ with the following property: Let $r \geq r_0$, $\varepsilon \in (0, 1/2]$ and $1 \leq k \leq (\kappa \log r)/\log \varepsilon$. Let $X_1, X_2, \ldots, X_k$ be i.i.d. standard Gaussian vectors in $\mathbb{R}^r$, and set $M_r := \text{Med}\|X_1\|_\infty$. Then

$$
\mathbb{P}\left\{(1 - \varepsilon)M_r \leq \left\| \sum_{i=1}^k \alpha_i X_i \right\|_\infty \leq (1 + \varepsilon)M_r \mid \text{for all } (\alpha_1, \ldots, \alpha_k) \in S^{k-1}\right\} \geq \frac{7}{8}.
$$

**Lemma 24.** There are universal constants $n_0 > 0$ and $n_0 > 0$ such that for any $n \geq n_0$ and $k \leq n_24$ the following holds: Let $X_1, X_2, \ldots, X_k$ be i.i.d. standard Gaussian vectors in $\mathbb{R}^n$. Then

$$
\mathbb{P}\left\{(1 - n^{-24})\sqrt{n} \leq \left\| \sum_{i=1}^k \alpha_i X_i \right\|_2 \leq (1 + n^{-24})\sqrt{n} \mid \text{for all } (\alpha_1, \ldots, \alpha_k) \in S^{k-1}\right\} \geq \frac{7}{8}.
$$

The following lemma is a trivial planimetric observation; we provide the proof for reader’s convenience.

**Lemma 25.** Let $0 < a < 1 < b$, and let a figure $F$ in the plane be given by

$$
F := B_2^2 \cap \left\{ (x_1, x_2) \in \mathbb{R}^2 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \leq 1 \right\},
$$

where $B_2^2$ is the unit Euclidean ball in the plane. Then the Banach–Mazur distance from $F$ to $B_2^2$ can be estimated as

$$
d(F, B_2^2) \geq \sqrt{\frac{2b^2 - a^2b^2 - 1}{b^2 - a^2}} = \sqrt{1 + \frac{(b^2 - 1)(1 - a^2)}{b^2 - a^2}}.
$$

**Proof.** Clearly, the convex figure is isometric to the intersection of the disk $B_2^2$ and ellipse $E := \left\{ (x_1, x_2) : a^2x_1^2 + b^2x_2^2 \leq 1 \right\}$. Further, the four points of intersection of the boundaries $\partial B_2^2$ and $\partial E$ have coordinates

$$
\left( \pm \sqrt{\frac{b^2 - 1}{b^2 - a^2}}, \pm \sqrt{\frac{1 - a^2}{b^2 - a^2}} \right).
$$

Now, assume that $T$ is a linear transformation of $\mathbb{R}^2$ such that

$$
\frac{1}{d(B_2^2, E)} B_2^2 \subset T(B_2^2 \cap E) \subset B_2^2.
$$

In view of the symmetries in $B_2^2 \cap E$, we can assume that $T$ is diagonal, with $T e_1 =: \kappa e_1$ and $T e_2 =: \beta e_2$ for some $\kappa < 1 < \beta$. Since $T(\partial B_2^2 \cap \partial E) \subset B_2^2$, we have the inequality

$$
\kappa^2 \frac{b^2 - 1}{b^2 - a^2} + \beta^2 \frac{1 - a^2}{b^2 - a^2} \leq 1.
$$
On the other hand, the distance $d(B_2^2 \cap \mathcal{E}, B_2^2)$ is bounded from below by $\max(\frac{1}{\kappa}, \frac{b}{\beta})$, whence, for any $\lambda \in [0, 1]$, we have

$$d(B_2^2 \cap \mathcal{E}, B_2^2)^{-2} \leq \lambda \kappa^2 + \frac{1 - \lambda}{b^2} \beta^2.$$  

Choose $\lambda := s \frac{b^2 - 1}{b^2 - a^2}$, where $s := \frac{b^2 - a^2}{2b^2 - a^2 b - 1}$. It is easy to check that in this case $1 - \lambda = sb^2 \frac{1-a^2}{b^2-a^2}$, so that

$$d(B_2^2 \cap \mathcal{E}, B_2^2)^{-2} \leq s \left( \kappa^2 \frac{b^2 - 1}{b^2 - a^2} + \beta^2 \frac{1 - a^2}{b^2 - a^2} \right) \leq s.$$

Thus,

$$d(B_2^2 \cap \mathcal{E}, B_2^2)^2 \geq \frac{2b^2 - a^2 b^2 - 1}{b^2 - a^2} = 1 + \frac{(b^2 - 1)(1 - a^2)}{b^2 - a^2}.$$

In the following statement, we estimate the extreme singular values of a standard rectangular Gaussian matrix. The lemma is by no means new; however, we prefer to give an elementary proof based only on the standard concentration inequalities and not involving any spectral theory.

**Lemma 26.** There are universal constants $C_{26}, C_{26} > 0$ with the following property: Let $C_{26} \leq k \leq m$, and let $A$ be the $m \times k$ standard Gaussian matrix. Then $\mathbb{P}\{s_{\max}(A) \leq \sqrt{m + C_{26} \sqrt{k}}\} \leq \frac{1}{16}$ and $\mathbb{P}\{s_{\min}(A) \geq \sqrt{m - C_{26} \sqrt{k}}\} \leq \frac{1}{16}$.

**Proof.** Let us prove only the first assertion of the lemma (the argument for $s_{\min}(A)$ is very similar). We assume that $k \leq m$, and that $k$ is sufficiently large. Let $Y_1, Y_2, \ldots, Y_k$ be i.i.d. standard Gaussian vectors in $\mathbb{R}^m$. We set $\rho_1 := 1$ and define random signs $\rho_2, \rho_3, \ldots, \rho_k$ inductively as follows:

$$\rho_i := \text{sign}(Y_i, \sum_{j=1}^{i-1} \rho_j Y_j), \quad i = 2, 3, \ldots, k.$$

We shall estimate the norm of the linear combination $\sum_{i=1}^k \rho_i Y_i$. Clearly, for any $u \leq k$ we have

$$\left\| \sum_{i=1}^u \rho_i Y_i \right\|_2^2 = \sum_{i=1}^u \left\| Y_i \right\|_2^2 + 2 \sum_{i=2}^u \langle \rho_i Y_i, \sum_{j=1}^{i-1} \rho_j Y_j \rangle$$

$$= \sum_{i=1}^u \left\| Y_i \right\|_2^2 + 2 \sum_{i=2}^u \left| \langle Y_i, \sum_{j=1}^{i-1} \rho_j Y_j \rangle \right|$$

$$= \sum_{i=1}^u \left\| Y_i \right\|_2^2 + 2 \sum_{i=2}^u \left\| \sum_{j=1}^{i-1} \rho_j Y_j \right\|_2 |g_i|,$$
where \( g_i := \left\| \sum_{j=1}^{i-1} \rho_j Y_j \right\|_2^{-1} \langle Y_i, \sum_{j=1}^{i-1} \rho_j Y_j \rangle \) \((i = 2, 3, \ldots, k)\) are standard Gaussian variables. A rough estimate gives (provided that \( k \) is sufficiently large):

\[
P\left\{ \left\| \sum_{i=1}^u \rho_i Y_i \right\|_2 \geq \frac{\sqrt{mk}}{4} \right\} \geq \frac{63}{64}.
\]

Next, the principal observation is that, conditioned on any realization of \( g_2, \ldots, g_{i-1} \), the variable \( g_i \) is distributed according to the normal law, whence \( g_i \)'s are jointly independent. It follows that, provided that \( k \) is sufficiently large,

\[
\sum_{i=\lceil k/2 \rceil + 1}^k |g_i| \geq \frac{k}{8}
\]

with probability at least 63/64. Together with the above estimates, this gives

\[
\left\| \sum_{i=1}^k \rho_i Y_i \right\|_2 \geq \sqrt{mk} + \frac{k \sqrt{mk}}{16}
\]

with probability at least 31/32. Combined with Theorem 5 applied to \((\sum_{i=1}^k \|Y_i\|_2^2)^{1/2}\) viewed as 1-Lipschitz function of \( mk \) i.i.d. standard Gaussian variables, this yields

\[
\left\| \sum_{i=1}^k \rho_i Y_i \right\|_2^2 \geq mk + \frac{k \sqrt{mk}}{17} > k\left(\sqrt{m} + \frac{\sqrt{k}}{64}\right)^2
\]

with probability at least 15/16. It remain to note that, for the \( m \times k \) Gaussian matrix \( A \) with columns \( Y_1, Y_2, \ldots, Y_k \), we have \( \sqrt{k} s_{\text{max}}(A) \geq \left\| \sum_{i=1}^k \rho_i Y_i \right\|_2 \) deterministically. \(\square\)

Remark 2. Note that in the last lemma we estimate \( s_{\text{max}} \) from below and \( s_{\text{min}} \) from above. A lower bound on \( s_{\text{max}} \) and upper bound on \( s_{\text{min}} \) of a random matrix with i.i.d. centered entries can be derived as a corollary of the Marchenko–Pastur theorem for the limiting spectral distribution [17]. However, the Marchenko–Pastur theorem requires that the ratio \( k/m \) converges to a fixed number, and its applicability in the case \( k = o(m) \) is unclear.

On the other hand, the above proof of the lemma is based on an elementary argument which gives rather crude bounds but remains valid under very mild assumptions on \( k, m \).

Proof of Proposition 2. To simplify working with constants, we adopt the following convention in this proof: by writing “\( a \ll b \)” we mean that \( a \leq cb \) for some universal constant \( c > 0 \) which can be made arbitrarily small at expense of changing the constants in the final statement.

Let us fix a (large) \( n \), and define \( m := \lceil \text{Med max}_{i \leq n} g_i^2 \rceil \), where \( g_1, \ldots, g_n \) are i.i.d. standard Gaussians. Note that \( m = O(\log n) \) (see, for example, [26] or [27, p. 302]). Further, define two cylinders

\[
B' := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \max_{i \leq n-m} |x_i| \leq 1\}
\]

\[
B'' := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=n-m+1}^n x_i^2 \leq 1\}.
\]
and set $B := B' \cap B''$. It is easy to see that $B_2^2 \subset B$, and that $\pm e_i$ ($i = 1, 2, \ldots, n$) are contact points of $\partial B$ and $\partial B_2^2$, whence, by John’s theorem [12, 3], $B_2^2$ is the maximal volume ellipsoid inside $B$. Let $\varepsilon \in ((\log n)^{-1/2}, c]$, $k := [c^{-1} \varepsilon^2 \log n]$ (where $c > 0$ is a sufficiently small universal constant whose value can be recovered from the proof), and set $E := \text{span}\{X_1, X_2, \ldots, X_k\}$ where $X_1, X_2, \ldots, X_k$ are i.i.d. standard Gaussian vectors in $\mathbb{R}^n$. We will show that with probability at least $1/2$, the random section $B \cap E$ is not $(1 + \varepsilon)$-Euclidean.

Set $r := n - m$. Note that, by the definition of $k$, we have $k \ll \varepsilon \log r / \log \varepsilon$. Then, applying Lemma 23 with $\varepsilon/2$ in place of $\varepsilon$, we obtain

$$\mathbb{P}\left\{ (1 - \varepsilon/2) M_r \leq \left\| \sum_{i=1}^{k} \alpha_i X_i \right\|_{B'} \leq (1 + \varepsilon/2) M_r \text{ for all } (\alpha_1, \ldots, \alpha_k) \in S^{k-1} \right\} \geq \frac{7}{8},$$

where $M_r = \text{Med} \max_{i \leq r} \|X_i\| = \text{Med}\|X_1\|_{B'}$. Next, observe that, in view of the choice of $m$ and asymptotic estimates of the median of the $\| \cdot \|_\infty$-norm of Gaussian vectors (see, in particular, [21, p. 302]), we have $\left| \sqrt{m}/M_r - 1 \right| \ll (\log n)^{-1/2}$. Together with Lemma 24, the above relation and the assumption that $\varepsilon$ is small, gives

$$\mathbb{P}\left\{ (1 - \varepsilon) \sqrt{n/m} \leq \left\| \sum_{i=1}^{k} \alpha_i X_i \right\|_{B'}^{-1} \cdot \left\| \sum_{i=1}^{k} \alpha_i X_i \right\|_2 \leq (1 + \varepsilon) \sqrt{n/m} \right. \right.$$

for all $(\alpha_1, \ldots, \alpha_k) \in S^{k-1}$ \}

$$\geq \frac{3}{4},$$

or, geometrically,

$$\mathbb{P}\left\{ (1 - \varepsilon) \sqrt{n/m} B_2^n \subseteq B' \cap E \subseteq (1 + \varepsilon) \sqrt{n/m} B_2^n \right\} \geq \frac{3}{4}. \quad (7)$$

Further, the intersection $B'' \cap E$ is clearly an ellipsoid. Let $Y_1, Y_2, \ldots, Y_k$ be the orthogonal projections of $X_i$’s onto the linear span of $\{e_{n-m+1}, \ldots, e_n\}$, and let $A$ be the $m \times k$ random Gaussian matrix with columns $Y_1, \ldots, Y_k$. Note that

$$\sup \left\{ \left\| \sum_{i=1}^{k} \alpha_i Y_i \right\|_2^{-1} \cdot \left\| \sum_{i=1}^{k} \alpha_i X_i \right\|_2 : (\alpha_1, \ldots, \alpha_k) \in S^{k-1} \right\}$$

is the length of the largest semi-axis of $B'' \cap E$ (let us denote the corresponding random vector by $W$, i.e. $W \in \partial(B'' \cap E)$ is the largest vector in $\mathbb{R}^n$ with the end-point on the boundary of $B'' \cap E$). Similarly, we let $Z$ be the smallest semi-axis of $B'' \cap E$, i.e. the shortest vector in $\mathbb{R}^n$ with the end-point on the boundary of $B'' \cap E$. In view of Lemma 24, we have

$$\mathbb{P}\left\{ \left\| W \right\|_2 \geq (1 - n^{-c'}) s_{\min}(A)^{-1} \sqrt{n} \text{ and } \left\| Z \right\|_2 \leq (1 + n^{-c'}) s_{\max}(A)^{-1} \sqrt{n} \right\} \geq \frac{7}{8}$$

for some universal constant $c' > 0$. Hence, by Lemma 26 we get

$$\mathbb{P}\left\{ \left\| W \right\|_2 \geq \frac{\sqrt{n}}{\sqrt{m - c'' k}} \text{ and } \left\| Z \right\|_2 \leq \frac{\sqrt{n}}{\sqrt{m + c'' k}} \right\} \geq \frac{3}{4}, \quad (8)$$

where $c'' > 0$ is a universal constant. Denote by $\tilde{E}$ the (random) 2-dimensional span of $W$ and $Z$. In view of (7), with probability at least $3/4$ the random figure $B \cap \tilde{E} =$

\[25\]
\((B' \cap \bar{E}) \cap (B'' \cap \bar{E})\) is at the distance at most \(\frac{1+\epsilon}{1-\epsilon}\) from \((\sqrt{n/m}B_n^2) \cap B'' \cap \bar{E}\). On the other hand, applying Lemma 25, we get
\[
d((\sqrt{n/m}B_n^2) \cap B'' \cap \bar{E}, B_2^2) \geq 1 + \frac{\|W\|_2^2 - n/m)(n/m - \|Z\|_2^2)}{(n/m)(\|W\|_2^2 - \|Z\|_2^2)}.
\]
Note that, conditioned on the event \(\|W\|_2 \geq \sqrt{n/m} \cdot \sqrt{k} \cdot \epsilon\) and \(\|Z\|_2 \leq \sqrt{n/m} + \sqrt{k} \cdot \epsilon\), we have
\[
\frac{(\|W\|_2^2 - n/m)(n/m - \|Z\|_2^2)}{(n/m)(\|W\|_2^2 - \|Z\|_2^2)} \geq \frac{(\|W\|_2 - \sqrt{n/m})(\sqrt{n/m} - \|Z\|_2)}{2\sqrt{n/m}\|W\|_2 - \|Z\|_2) \geq \tilde{c} \sqrt{k} \gg \epsilon,
\]
as long as \(c\) is chosen to be sufficiently small. Hence, in view of (8),
\[
P\{d((\sqrt{n/m}B_n^2) \cap B'' \cap \bar{E}, B_2^2) \geq 1 + 4\epsilon\} \geq 3/4.
\]
Finally,
\[
P\{d(B \cap E, B_2^2) \geq (1 + 4\epsilon)(1 - \epsilon)/(1 + \epsilon)\} \geq 1/2,
\]
and the result follows.

## 7 Remarks and open questions

- A question of importance is whether the assertion of Theorem 3 holds without assuming existence of a 1-unconditional basis in \((\mathbb{R}^n, \| \cdot \|_B)\). This seems quite plausible, although absence of a preferred orthogonal basis in this case suggests that Talagrand’s \(L_1-L_2\) bound will likely be inapplicable. In our paper the phenomenon of superconcentration is presented as a black box: we do not attempt to investigate the matters that lie beneath the \(L_1-L_2\) bound. Proving the assertion of the theorem in full generality should require new tools.

- The assumption of Theorem 3 that the convex body \(B\) is in the \(\ell\)-position is not something absolutely necessary. Rather, what we need is a sort of a “balancing” condition for the norm \(\| \cdot \|_B\). In particular, it is natural to expect that the assertion of the main theorem remains valid if the \(\ell\)-position is replaced with the one given by
\[
\mathbb{E}\|G\|_B^q = 1 \quad \text{and} \quad 1 = |\det \text{Id}_n| = \sup \{ |\det U| : U \in \mathbb{R}^{n \times n}, \mathbb{E}\|G\|_{U^{-1}(B)}^q \leq 1 \}
\]
for some fixed \(q \geq 1\). One may further ask what are other \textit{natural} positions in which the superconcentration phenomenon guarantees better than quadratic dependence on \(\epsilon\) in the randomized Dvoretzky theorem.

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