Twisted Dolbeault cohomology of nilpotent Lie algebras

Liviu Ornea¹, Misha Verbitsky²

Abstract
It is well known that the cohomology of any non-trivial 1-dimensional local system on a nilmanifold vanishes (this result, due to J. Dixmier, was also announced and proved in some particular case by Alaniya). A complex nilmanifold is a quotient of a nilpotent Lie group equipped with a left-invariant complex structure by an action of a discrete, co-compact subgroup. We prove a Dolbeault version of Dixmier’s and Alaniya’s theorem, showing that the Dolbeault cohomology $H^{0,p}(\mathfrak{g}, L)$ of a nilpotent Lie algebra with coefficients in any non-trivial 1-dimensional local system vanishes. Note that the Dolbeault cohomology of the corresponding local system on the manifold is not necessarily zero. This implies that the twisted version of Console-Fino theorem is false (Console-Fino proved that the Dolbeault cohomology of a complex nilmanifold is equal to the Dolbeault cohomology of its Lie algebra, when the complex structure is rational). As an application, we give a new proof of a theorem due to H. Sawai, who obtained an explicit description of LCK nilmanifolds. An LCK structure on a manifold $M$ is a Kähler structure on its cover $\tilde{M}$ such that the deck transform map acts on $\tilde{M}$ by homotheties. We show that any complex nilmanifold admitting an LCK structure is Vaisman, and is obtained as a compact quotient of the product of a Heisenberg group and the real line.

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1 Introduction

In complex dimension $> 2$, an LCK (locally conformally Kähler) manifold is a complex manifold equipped with a Hermitian metric (LCK metric) which is locally conformally equivalent to a Kähler manifold. Then the Hermitian form satisfies $d(\omega) = \theta \wedge \omega$, where $\theta$ is a 1-form. Clearly, $d^2(\omega) = 0$ gives $d\theta \wedge \omega = 0$, which is equivalent to $d\theta = 0$ in complex dimension $> 2$. In complex dimension 2, this condition has to be added artificially.

Summing it up, an LCK metric (or an LCK structure) on a complex manifold of complex dimension $\geq 2$ is a Hermitian metric with Hermitian form $\omega$ which satisfies $d(\omega) = \theta \wedge \omega$, where $\theta$ is a closed 1-form.

Consider the differential $d_\theta(\alpha) := d(\alpha) - \theta \wedge \alpha$. Then the equation $d(\omega) = \theta \wedge \omega$ can be written as $d_\theta(\omega) = 0$. The differential $d_\theta$ is called the Morse-Novikov differential. It can be interpreted as de Rham differential for forms with coefficients in a local system, which is done as follows.

Let $L$ be a trivial real bundle of rank 1, and $\nabla_0$ the standard connection. Denote by $\nabla := \nabla_0 - \theta$ the connection $\nabla(b) = \nabla_0(b) - b \otimes \theta$. Since $\nabla^2 = 0$, this connection is flat, and $(L, \nabla)$ defines a local system, which we denote by the same letter. Then $d_\theta$ is the de Rham differential on differential forms with values in $L$, and the cohomology of $d_\theta$ is identified with the cohomology of this local system.

In this paper we study the locally conformally Kähler structures on nilmanifolds.
A nilmanifold is a quotient of a connected, simply connected nilpotent Lie group by a discrete co-compact lattice. It is not hard to see that any nilmanifold is the total space of an iterated family of circle bundles. Its cohomology is expressed through the cohomology of the Chevalley-Eilenberg differential on the corresponding Lie algebra $\mathfrak{g}$, and the Morse-Novikov cohomology is obtained as cohomology of the twisted version of the Chevalley-Eilenberg differential (Section 4).

In the literature, there are several different (and incompatible) ways to define a complex nilmanifold. Before 1980-ies, most people defined complex nilmanifold as a quotient of a complex nilpotent Lie group $G_{\mathbb{C}}$ by a co-compact lattice. This quotient is a parallelizable complex manifold (that is, its tangent bundle is holomorphically trivial). Indeed, the right action of $G_{\mathbb{C}}$ on itself commutes with the left action, and both are holomorphic. If we take the quotient $M := G_{\mathbb{C}}/\Gamma$ by (say) left action of $\Gamma \subset G_{\mathbb{C}}$, the manifold $M$ remains homogeneous with respect to the right action of $G_{\mathbb{C}}$, and this action trivializes $TM$, making $M$ parallelizable.

Now such nilmanifolds are called **parallelizable complex nilmanifolds** or **Iwasawa type nilmanifolds**, after the Iwasawa group (the non-commutative 3-dimensional complex nilpotent Lie group).

Before the paper [CFD], in the published literature “complex nilmanifold” meant the quotient of a complex Lie group; see for example [Fis] or [Oe]. In [CFD], Cordero, Fernández and de León introduced the modern notion of a complex nilmanifold. Since then, a complex nilmanifold is defined as a compact quotient of a nilpotent Lie group equipped with a left-invariant complex structure (Subsection 3.2). The main advantage of this notion is that it can be rephrased in terms of a complex structure operator on the corresponding Lie algebra. However, such a quotient is not homogeneous, unless $G$ is a complex Lie group.

The same approach can be used to describe other important geometric structures on nilmanifolds in terms of linear-algebraic structures on their Lie algebras (Subsection 3.1). In the same paper [CFD], this approach was used to construct an LCK structure on a nilmanifold obtained from the Heisenberg group. Later, H. Sawai proved in [Saw1] that any LCK structure on a complex nilmanifold is obtained this way. In the present paper we give a proof of Sawai’s theorem based on cohomology vanishing.

Consider an oriented rank 1 local system $(L, \nabla_0 + \theta)$ on a nilmanifold $M = G/\Gamma$, where $(L, \nabla_0)$ is a trivial bundle with connection, and $\theta$ is a closed, non-exact 1-form. In [Ala] (see also [Mi]), it was shown that $H^*(M, L) = 0$. The proof goes as follows: first, one identifies $H^*(M, L)$ with the cohomology of the Lie algebra $\mathfrak{g} = \text{Lie}(G)$ with coefficients in a
non-trivial rank one representation. This is done by using the homogeneous $G$-action. Then one takes the Chevalley-Eilenberg complex which computes the Lie algebra cohomology, filters it by the central series, and computes the first page of the corresponding spectral sequence. This first page is identified with the twisted cohomology of a commutative Lie algebra, which always vanishes.

However, this approach will not work for Dolbeault cohomology. Console and Fino ([CF]) proved that the Dolbeault cohomology can, indeed, be computed using the Hodge decomposition on the Chevalley-Eilenberg complex of the corresponding Lie algebra, when the complex structure is rational (in dimension up to 6 the rationality of the complex structure is not needed, see [FRR], but for higher dimensions the question is still open). However, this result is highly non-trivial, because complex nilmanifolds are not homogeneous. Moreover, the extension of Console-Fino theorem to the Dolbeault cohomology with coefficients in a local system (“twisted Dolbeault cohomology”) is false (Subsection 4.2).

The main result of the present paper is the following theorem, which computes the Lie algebra version of the Dolbeault cohomology with coefficients in a local system.

**Theorem 1.1:** Let $\mathfrak{g}$ be a nilpotent real Lie algebra, $I : \mathfrak{g} \rightarrow \mathfrak{g}$ a complex structure operator, $\mathfrak{g}_\mathbb{C} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, and $\mathfrak{g}_\mathbb{C} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$ the corresponding eigenvalue decomposition, called the Hodge decomposition in the sequel. Assume that $\mathfrak{g}^{1,0} \subset \mathfrak{g}_\mathbb{C}$ is a Lie subalgebra (this is equivalent to the left-invariant complex structure on the Lie group $G = \text{Lie}(\mathfrak{g})$ associated with $I$ being integrable, see Definition 3.1). Consider the Hodge decomposition $\Lambda^*(\mathfrak{g}_\mathbb{C}^*) = \bigoplus \Lambda^{p,q}(\mathfrak{g}_\mathbb{C}^*)$ on the Grassman algebra of $\mathfrak{g}_\mathbb{C}$, with

$$\Lambda^{p,q}(\mathfrak{g}^*) = \Lambda^p((\mathfrak{g}^{1,0})^*) \otimes_{\mathbb{C}} \Lambda^q((\mathfrak{g}^{0,1})^*)$$

and define the twisted Dolbeault differentials $\partial_\theta$, $\overline{\partial}_\theta$ as Hodge components of the twisted Chevalley-Eilenberg differential $d_\theta(x) := d(x) - \theta \wedge x$, i.e. $\partial_\theta = \partial - \theta^{1,0}$ and $\overline{\partial}_\theta = \overline{\partial} - \theta^{0,1}$ (Subsection 3.1). Then the cohomology of the complex $(\Lambda^{0,\cdot}(\mathfrak{g}_\mathbb{C}^*), \partial_\theta)$ vanishes.

**Proof:** See Theorem 4.2. \[\blacksquare\]

We apply this result to get a new proof of the classification of LCK structures on complex nilmanifolds.
2 Preliminaries

2.1 Locally conformally Kähler manifolds

Definition 2.1: Let $(M, I)$ be a complex manifold, $\dim \mathbb{C} M \geq 2$. It is called locally conformally Kähler (LCK) if it admits a Hermitian metric $g$ whose fundamental 2-form $\omega(\cdot, \cdot) := g(\cdot, I \cdot)$ satisfies
\[ d\omega = \theta \wedge \omega, \quad d\theta = 0, \tag{2.1} \]
for a certain closed 1-form $\theta$ called the Lee form.

Remark 2.2: Definition (2.1) is equivalent to the existence of a covering $\tilde{M}$ endowed with a Kähler metric $\Omega$ which is acted on by the deck group $\text{Aut}_M(\tilde{M})$ by homotheties. Let
\[ \chi: \text{Aut}_M(\tilde{M}) \to \mathbb{R}^>, \quad \chi(\tau) = \frac{\tau^* \Omega}{\Omega}, \tag{2.2} \]
be the character which associates to a homothety its scale factor. On this cover, the pull-back of the Lee form is exact.

Remark 2.3: For an LCK manifold, coverings with the above property are not unique. The covering for which the character $\chi$ is injective is called the minimal cover.

Remark 2.4: The operator $d_\theta := d - \theta \wedge$ obviously satisfies $d_\theta^2 = 0$ and hence $(\Lambda^* M, d_\theta)$ produces a cohomology called Morse-Novikov or twisted. It can be interpreted as the cohomology of the local system $L$ associated to the line bundle endowed with (flat) connection form $\theta$. See Section 4 for details.

Theorem 2.5: ([Va1]) Let $(M, \omega, \theta)$ be a compact LCK manifold, not globally conformally Kähler (i.e. with non-exact Lee form). Then $M$ does not admit a Kähler metric.

2.2 Vaisman manifolds

Definition 2.6: An LCK manifold $(M, \omega, \theta)$ is called Vaisman if $\nabla \theta = 0$, where $\nabla$ is the Levi-Civita connection of $g$.

The following characterization, very much used in applications, is available:
Theorem 2.7: ([KO]) Let \((M, \omega, \theta)\) be an LCK manifold equipped with a holomorphic and conformal \(C\)-action without fixed points, which lifts to non-isometric homotheties on the Kähler covering \(\tilde{M}\). Then \((M, \omega, \theta)\) is conformally equivalent to a Vaisman manifold.

The main example of Vaisman manifold is the diagonal Hopf manifold ([OV1]). The Vaisman compact complex surfaces are classified in [Bel], see also [OVV].

Remark 2.8: There exist compact LCK manifolds which do not admit Vaisman metrics. Such are the LCK Inoue surfaces, [Bel], the Oeljeklaus-Toma manifolds, [OT], [Ot], and the non-diagonal Hopf manifolds, [OV1], [OVV].

Remark 2.9: On a Vaisman manifold, the Lee field \(\theta^\xi\) and the anti-Lee field \(I\theta^\xi\) are real holomorphic \(\langle \text{Lie}_{\theta^\xi} I = \text{Lie}_{I\theta^\xi} I = 0 \rangle\) and Killing \(\langle \text{Lie}_{\theta^\xi} g = \text{Lie}_{I\theta^\xi} g = 0 \rangle\), see [DO].

Recall that a Killing vector field \(X\) satisfies the equation:

\[ g(\nabla_A X, B) = -g(A, \nabla_B X). \]

Setting \(A = X\), we get \(g(\nabla_X X, B) = -g(X, \nabla_B X)\). The last term is equal to \(-1/2 \text{Lie}_B (g(X, X))\), hence \(g(\nabla_X X, B) = 0\) for all \(B\). Then \(\nabla_X X = 0\), and hence the trajectories of a Killing field of constant length are geodesics. Therefore, the canonical foliation in a Vaisman manifold is totally geodesic.

Remark 2.10: (i) Note that while the LCK condition is conformally invariant (changing the metric \(g \mapsto e^f g\) changes the Lee form into \(\theta \mapsto \theta + df\)), the Vaisman condition is not conformally invariant. Indeed, on a Vaisman manifold, the Lee form is coclosed (being parallel) and hence a Vaisman metric is a Gauduchon metric; but one knows that on a compact complex manifold, each conformal class contains a unique Gauduchon metric (up to constant multipliers), [Ga].

(ii) Since \(\theta\) is parallel, it has constant norm and thus we can always scale the LCK metric such that \(|\theta| = 1\). In this assumption, the following formula holds, [Va2], [DO]:

\[ d\theta^e = \theta \wedge \theta^e - \omega, \quad \text{where} \quad \theta^e(X) = -\theta(I X). \quad (2.3) \]
Moreover, one can see, [Ve], that the (1,1)-form $\omega_0 := -d^c \theta$ is semi-positive definite, having all eigenvalues\(^1\) positive, except one which is 0.

### 2.3 LCK manifolds with potential

We now introduce a class of LCK manifolds strictly containing the Vaisman manifolds.

**Definition 2.11:** We say that an LCK manifold has **LCK potential** if it admits a Kähler covering on which the Kähler metric has a global and positive potential function which is acted on by holomorphic homotheties by the deck group. In this case, $M$ is called **LCK manifold with potential**.

**Remark 2.12:** Note that in several previous papers of ours we asked the potential function to be proper. It is not the case for the above definition. See [OV2] for a comprehensive discussion about proper and improper LCK potentials.

**Remark 2.13:** One can prove that $(M, I, g, \theta)$ is LCK with potential if and only if equation (2.3) is satisfied.

**Definition 2.14:** A function $\varphi \in C^\infty(M)$ is called **$d \theta d^c \theta$-plurisubharmonic** if $\omega = d \theta d^c \theta(\varphi)$.

**Remark 2.15:** Note that $d \theta d^c \theta$-plurisubharmonic are not plurisubharmonic (they exist on compact manifolds). Moreover, if $\varphi$ is $d \theta d^c \theta$-plurisubharmonic, then $\varphi + \text{const}$ is not necessarily $d \theta d^c \theta$-plurisubharmonic.

Equation (2.3) (and hence the definition of LCK manifolds with potential) can be translated on the LCK manifold itself:

**Theorem 2.16:** ([OV2, Claim 2.8]) $(M, I, \theta, \omega)$ is LCK with potential if and only if $\omega = d \theta d^c \theta(\varphi)$ for a strictly positive $d \theta d^c \theta$-plurisubharmonic function $\varphi$ on $M$.

**Theorem 2.17:** ([OV4]) Let $(M, \theta, \omega)$ be an LCK manifold which is not Kähler, and suppose $\omega = d \theta d^c \theta(\varphi_0)$ for some smooth function $\varphi_0 \in C^\infty(M)$. Then $\varphi_0 > 0$ at some point of $M$.

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\(^1\)The eigenvalues of a Hermitian form $\eta$ are the eigenvalues of the symmetric operator $L_\eta$ defined by the equation $\eta(x, Iy) = g(L_\eta x, y)$. 

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Proof: (Courtesy of Matei Toma.) By absurd, \( \varphi_0 \leq 0 \) everywhere on \( M \). Let \( \tilde{M} \) be the minimal Kähler cover of \( M \) (see Remark 2.3), \( \Gamma \) its deck group, and \( \rho \) a function on \( \tilde{M} \) such that \( d\rho = \theta \). Then the \( d\theta d\bar{\theta} \)-plurisubharmonicity of \( \varphi_0 \) is equivalent to the plurisubharmonicity of \( \varphi := e^{-\rho} \varphi_0 \), see Theorem 2.16.

Since the strict \( d\theta d\bar{\theta} \)-plurisubharmonicity is stable under \( C^2 \)-small deformations of \( \varphi \), the function \( \varphi - \varepsilon \) is also strictly \( d\theta d\bar{\theta} \)-plurisubharmonic. Therefore, we may assume that \( \varphi < 0 \) everywhere.

Define
\[
\psi := -\log(-\varphi).
\]
Since \( x \rightarrow -\log(-x) \) is strictly monotonous and convex, the function \( \psi \) is strictly \( d\theta d\bar{\theta} \)-plurisubharmonic. Moreover, for every element \( \gamma \in \Gamma \), we have
\[
\gamma^* \psi = -\log(-\varphi \circ \gamma) = -\log(\chi(\gamma)) - \log(-\varphi) = \text{const} + \psi.
\]
Therefore, the Kähler form \( d\bar{\psi} \psi \) is \( \Gamma \)-invariant and descends to \( M \), contradiction with Theorem 2.5.

Remark 2.18: All Vaisman manifolds are LCK manifolds with potential. Among the non-Vaisman examples, we mention the non-diagonal Hopf manifolds, \([OV2]\). On the other hand, the Inoue surfaces and their higher dimensional analogues, the Oeljeklaus-Toma manifolds, are compact LCK manifolds which are not LCK manifolds with potential (\([Ot]\)).

We can characterize the Vaisman metrics among the LCK metrics with potential:

Proposition 2.19: ([OV3, Proposition 2.3 & Corollary 2.4])
Let \( (M, \omega, \theta) \) be a compact LCK manifold with potential. Then the LCK metric is Gauduchon if and only if \( \omega_0 = -d^c \theta \) is semi-positive definite, and hence it is Vaisman. Equivalently, a compact LCK manifold with potential and with constant norm of \( \theta \) is Vaisman.

3 LCK structures on nilmanifolds

3.1 Invariant geometric structures on Lie groups
Let \( G \) be a Lie group, \( \Lambda \) a discrete subgroup, and \( I \) a left-invariant complex structure on \( G \). Consider the quotient space \( G/\Lambda \), where \( \Lambda \) acts by left
translations. Since $I$ is left-invariant, the manifold $G/\Lambda$ is equipped with a natural complex structure. This construction is usually applied to solv-manifolds or nilmanifolds, but it makes sense with any Lie group. When we need to refer to this particular kind of complex structures on $G/\Lambda$, we call them \textit{locally $G$-invariant}.

A left-invariant almost complex structure $I$ on $G$ is determined by its restriction to the Lie algebra $\mathfrak{g} = T_e G$, giving the Hodge decomposition $\mathfrak{g} \otimes \mathbb{R} \mathbb{C} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$. This structure is integrable if and only if the commutator of $(1,0)$-vector fields is again a $(1,0)$-vector field, which is equivalent to $[\mathfrak{g}^{1,0}, \mathfrak{g}^{1,0}] \subset \mathfrak{g}^{1,0}$. This allows one to define a complex structure on a Lie algebra.

**Definition 3.1:** A complex structure on a Lie algebra $\mathfrak{g}$ is a subalgebra $\mathfrak{g}^{1,0} \subset \mathfrak{g} \otimes \mathbb{R} \mathbb{C}$ which satisfies $\mathfrak{g}^{1,0} \oplus \mathfrak{g}^{1,0} = \mathfrak{g} \otimes \mathbb{R} \mathbb{C}$.

Indeed, an almost complex structure operator $I$ can be reconstructed from the decomposition $\mathfrak{g}^{1,0} \oplus \overline{\mathfrak{g}^{1,0}} = \mathfrak{g} \otimes \mathbb{R} \mathbb{C}$ by making it act as $\sqrt{-1}$ on $\mathfrak{g}^{1,0}$ and $-\sqrt{-1}$ on $\overline{\mathfrak{g}^{1,0}}$.

In a similar way one could define symplectic structures or LCK structures on a Lie algebra $\mathfrak{g}$. Recall that the Grassmann algebra $\Lambda^\ast(\mathfrak{g}^\ast)$ is equipped with a natural differential, called \textit{Chevalley differential}, which is equal to the de Rham differential on left-invariant differential forms on the Lie group if we identify the space of such forms with $\Lambda^\ast(\mathfrak{g}^\ast)$. In dimensions 1 and 2 it can be written explicitly as follows: if $\lambda \in \mathfrak{g}^\ast$, then $d\lambda(x, y) = -\lambda(x, y)$; if $\beta \in \Lambda^2 \mathfrak{g}^\ast$, then $d\beta(x, y, z) = -\beta([x, y], z) - \beta([y, z], x) - \beta([z, x], y)$. The corresponding complex is called the \textit{Chevalley-Eilenberg complex}.

Further on, we shall always interpret the elements of $\Lambda^p(\mathfrak{g}^\ast)$ as left-invariant differential forms on the corresponding Lie group, and refer to them as to “$p$-forms”, with all the usual terminology (“closed forms”, “exact forms”) as used for the elements of de Rham algebra.

**Definition 3.2:** A \textbf{symplectic structure} on a Lie algebra $\mathfrak{g}$ is a non-degenerate, closed 2-form $\omega \in \Lambda^2(\mathfrak{g}^\ast)$. A \textbf{Kähler structure} on a Lie algebra $\mathfrak{g}$ is a complex structure $I$ and a Hermitian form $h$ on $\mathfrak{g}$ such that the fundamental 2-form $\omega(\cdot, \cdot) := h(\cdot, I(\cdot))$ is closed.

**Remark 3.3:** In [BG] (see also [Has]) it was shown that any nilpotent Lie algebra admitting a Kähler structure is actually abelian.
3.2 LCK nilmanifolds

**Definition 3.4:** An LCK structure on a Lie algebra is a complex structure $I$ and a Hermitian form $h$ on $\mathfrak{g}$ such that the fundamental 2-form $\omega(\cdot,\cdot) := h(\cdot, I(\cdot))$ satisfies $d(\omega) = \theta \wedge \omega$, where $\theta \in \Lambda^1(\mathfrak{g}^*)$ is a closed 1-form.

**Definition 3.5:** A nilmanifold is a quotient of a simply connected nilpotent Lie group by a co-compact discrete subgroup. Alternatively ([Ma]), one can define nilmanifolds as manifolds which admit a homogeneous action by a nilpotent Lie group.

**Definition 3.6:** Let $(G, I)$ be a nilpotent Lie group equipped with a left-invariant complex structure. For any co-compact discrete subgroup $\Gamma \subset G$, the (left) quotient $G/\Gamma$ is equipped with a natural complex structure. This quotient is called a complex nilmanifold. Similarly, if $G$ is a Lie group with left-invariant LCK structure, the quotient $G/\Gamma$ is called an LCK nilmanifold. This structure is clearly locally homogeneous. When we need to refer to this particular kind of structures, we call them locally $G$-invariant.

**Remark 3.7:** The existence of a co-compact lattice already imposes strong restrictions: the group should be unimodular, [Mo, §1.1, Exercise 14b]. All nilpotent Lie groups are unimodular, but this is not sufficient for the existence of a cocompact lattice. According to [Ma], a nilpotent Lie group admits a cocompact lattice if and only if its Lie algebra admits a basis in which the structural constants are all rational.

In this paper we present results concerning LCK structures on nilmanifolds. A classification of nilpotent Lie algebras which admit LCK structures is given.

**Remark 3.8:** As shown by Mal’cev, [Ma], any nilmanifold is uniquely determined by its fundamental group, which is a discrete nilpotent torsion-free group, and any such group uniquely determines a nilmanifold.

**Remark 3.9:** Note that right translations are not holomorphic with respect to left invariant complex structures. This means that for a lattice $\Lambda$, the complex structure induced on the manifold $G/\Lambda$ is not necessarily invariant, and hence $G/\Lambda$ is not a homogeneous LCK manifold.
Remark 3.10: Note that it may happen that a Kähler manifold be diffeomorphic (as real manifold) with a nilmanifold but the Kähler structure need not be invariant. For example, let $E$ be an elliptic curve which is a 2-fold ramified covering of $\mathbb{C}P^1$, and take the fibered product $E \times \mathbb{C}P^1$, $\text{Tw}(T^4)$, where $\text{Tw}(T^4)$ is the twistor space of a 4-torus. This fibered product is actually a torus, but with an inhomogeneous complex structure (since it has non-trivial canonical bundle, in fact its anticanonical bundle is semi-positive, with many non-trivial sections); compare with [Ca, §5]. This example suggests that the same phenomenon could appear for LCK manifolds (to be diffeomorphic with a nilmanifold without having an invariant LCK structure), but for the moment we don’t have a concrete example.

Example 3.11: Let $H_{2n-1}$ be the Heisenberg group of matrices of the form
\[
\begin{pmatrix}
1 & A & c \\
0 & I_{2n-1} & B^t \\
0 & 0 & 1
\end{pmatrix}, \quad c \in \mathbb{R}, \quad A, B \in \mathbb{R}^{n-1}.
\]

Its Lie algebra $h_{2n-1}$ can be defined in terms of generators and relations as follows: it has a basis $\{X_i, Y_i, Z\}$, $i = 1, \ldots, n-1$, with $[X_i, Y_i] = Z$, and the rest of Lie brackets trivial. Clearly, it is a nilpotent Lie algebra. After taking the quotient of the corresponding Lie group by a co-compact discrete subgroup $\Lambda$, one obtains a nilmanifold called the Heisenberg nilmanifold. It has an invariant Sasakian structure given by requiring the above basis to be orthonormal, defining the contact form as $\sum y_i dx_i + dz$ (where $dx_i, dy_i, dz$ are the dual invariant one-forms). Then $Z$ is its Reeb field.

Now, the product $H_{2n-1}/\Lambda \times S^1$ is a Vaisman nilmanifold whose universal cover is the product $H_{2n-1} \times \mathbb{R}$, a Lie group with Lie algebra $g := h_{2n-1} \times \mathbb{R}$, $\mathbb{R} = \langle T \rangle$, in which the brackets of the basis $\{X_i, Y_i, Z, T\}$ are the above, and $T$ is central. The linear LCK structure on $g$ is given by asking the basis to be orthonormal, and by defining the complex structure $IX_i = Y_i$, $IZ = -T$. The fundamental form is $\omega = \sum (X_i^* \wedge Y_i^*) - Z^* \wedge T^*$. A direct computation shows that: $d\omega = T^* \wedge \omega$, and hence the Lee form is $T^*$ which can be seen directly to be $g$-parallel.
4 Twisted Dolbeault cohomology on nilpotent Lie algebras

4.1 Twisted Dolbeault cohomology

Let $\theta$ be a closed 1-form on a complex manifold. Then $d - \theta$ defines a connection on the trivial bundle. We denote the corresponding local system by $L$. The twisted Dolbeault differentials $\partial_\theta := (d_\theta)^{1,0} = \partial - \theta^{1,0}$ and $\overline{\partial}_\theta := (d_\theta)^{0,1} = \overline{\partial} - \theta^{0,1}$ are Morse-Novikov counterparts to the usual Dolbeault differentials. The cohomology of these differentials corresponds to the Dolbeault cohomology with coefficients in the holomorphic line bundle $L_C$ obtained as complexification of the local system $L$.

Working with LCK nilmanifolds and nilpotent Lie algebras, it is natural to consider their twisted Dolbeault cohomology. As shown by Console and Fino ([CF]), the usual Dolbeault cohomology of complex nilmanifolds is equal to the cohomology of the Dolbeault version of the corresponding Chevalley-Eilenberg complex on the Lie algebra. This result does not hold in the twisted Dolbeault cohomology, as we shall see in Example 4.4.

However, as we show in Theorem 4.2 below, the corresponding Lie algebra cohomology vanishes.

Definition 4.1: Let $g$ be a Lie algebra and $\theta \in g^*$ a closed 1-form and

$$d_\theta := d - \theta \in \text{End}(\Lambda^*(g^*)),$$

where $d$ is the Chevalley differential and $\theta$ denotes the operation of multiplication by $\theta$. The twisted Dolbeault differentials on $\Lambda^*(g^*)$ are $\partial_\theta := (d_\theta)^{1,0}$ and $\overline{\partial}_\theta := (d_\theta)^{0,1}$.

The cohomology of $d_\theta$ is always zero for non-zero $\theta$ ([Ala, Mi]). This theorem is due to J. Dixmier, [Di]. The main result of the present section is the following theorem, which is the Dolbeault version of Dixmier’s and Alaniya’s theorem.

Theorem 4.2: Let $g$ be a nilpotent Lie algebra with a complex structure (Definition 3.1), and $\theta \in g^*$ a non-zero, closed real 1-form. Then the cohomology of $(\Lambda^0,*(g^*),\overline{\partial}_\theta)$ vanishes.

Proof: Consider the central series of the Lie algebra $g^{0,1}$, with $W_0 = g^{0,1}, W_1 = [W_0, W_0], ..., W_k = [W_0, W_{k-1}].$ Let $A_k \subset (g^{0,1})^*$ be the annihilator of $W_k$. For any 1-form $\lambda \in (g^{0,1})^*$, one has $\overline{\partial}(\lambda)(x, y) = \lambda([x, y]),$
hence $\overline{\partial}(A_k) \subset \Lambda^2(A_{k-1})$. Consider the filtration $\Lambda^*(A_1) \subset \Lambda^*(A_2) \subset ... \quad \text{on} \quad \Lambda \in (\mathfrak{g}^{0,1})^*$. Since $\overline{\partial}(A_k) \subset \Lambda^2(A_{k-1})$, the operator $\overline{\partial}$ shifts the filtration by 1: $\overline{\partial}(\Lambda^*(A_k)) \subset \Lambda^*(A_{k-1})$.

Consider the spectral sequence of the complex $(\Lambda^*(\mathfrak{g}^*), \overline{\partial}_\theta)$ filtered by $V_0 \subset V_1 \subset V_2 \subset ..., \quad \text{where} \quad V_k := \Lambda^*(A_k)$. Since $\overline{\partial}(V_k) \subset V_{k-1}$, the operator $\overline{\partial}_\theta$ acts on $\bigoplus_k V_{k}/V_{k-1}$ as multiplication by $\theta^{0,1}$. The corresponding associated graded complex, which is $E_1^{0,*}$ of this spectral sequence, is identified with $(\bigoplus_k V_{k}/V_{k-1}, \theta^{0,1})$. We identify $\bigoplus_k V_{k}/V_{k-1}$ with the Grassmann algebra $\bigoplus_k V_{k}/V_{k-1} = \Lambda^*(\mathfrak{g}^*)$. After this identification, the multiplication

$$\theta^{0,1} : \bigoplus_k V_{k}/V_{k-1} \rightarrow \bigoplus_k V_{k}/V_{k-1}$$

becomes multiplication by a 1-form, obtained from $\theta^{0,1}$. The cohomology of multiplication by a 1-form always vanishes. Then the $E_1^{0,*}$-page of the spectral sequence vanishes, which implies vanishing of $H^*(\Lambda^*(\mathfrak{g}^{0,*}), \overline{\partial}_\theta)$. ■

The following corollary will be used in the classification of LCK structures on nilpotent Lie algebras.

**Corollary 4.3:** Let $(\mathfrak{g}, I)$ be a $2n$-dimensional nilpotent Lie algebra with complex structure, $\theta \in \Lambda^1(\mathfrak{g}^*)$ a non-zero, closed real 1-form, and $\omega \in \Lambda^{1,1}(\mathfrak{g}^*)$ a $d_\theta$-closed (1,1)-form. Then there exists a 1-form $\tau \in \Lambda^1(\mathfrak{g}^*)$ such that $\omega = d_\theta(\tau)$, where $d_\theta(I\tau) = 0$.

**Proof:** Denote by $H^{1,0}_{d_\theta d_\partial}(\mathfrak{g}^*)$ the twisted Bott-Chern cohomology, that is, all $d_\partial$-closed (1,1)-forms up to the image of $d_\theta d_\partial$. The standard exact sequence

$$H^{0,1}_{d_\partial}(\mathfrak{g}^*) \oplus H^{1,0}_{d_\theta}(\mathfrak{g}^*) \xrightarrow{d} H^{1,1}_{d_\theta d_\partial}(\mathfrak{g}^*) \xrightarrow{\mu} H^2(M)$$

([OV0], equation (4.7)) implies that the kernel of $\mu$ vanishes when $H^{0,1}_{d_\partial}(\mathfrak{g}^*) = H^{1,0}_{d_\theta}(\mathfrak{g}^*) = 0$. The last equation follows from **Theorem 4.2**, hence $\mu$ is injective. The LCK form $\omega$ belongs to the kernel of $\mu$, because the $d_\theta$-cohomology of $\Lambda^*(\mathfrak{g}^*)$ vanishes by Diximier and Alanya theorem ([Ala, Mj]). Therefore, $\omega$ is twisted Bott-Chern exact, i.e. $\omega = d_\theta d_\partial f$ for some $f \in \Lambda^0(\mathfrak{g}^*) = \mathbb{R}$. Set $\tau := d_\theta f$. Then $I\tau = d_\theta f$ and $d_\theta(I\tau) = 0$ as stated. ■

### 4.2 Console-Fino theorem with coefficients in a local system

**Example 4.4:** Let $E_1, E_2$ be elliptic curves. Consider a Kodaira surface $M$,
which is a non-Kähler, locally conformally Kähler complex surface, obtained as the total space of a principal holomorphic $E_1$-bundle $\pi : M \rightarrow E_2$. Such bundles are classified by the first Chern class of the fibration $c_1(\pi)$. This class can be identified with the $d_2$-differential of the corresponding Leray spectral sequence, mapping from $H^1(E_1)$ to $H^2(E_2)$. By Blanchard’s theorem ([Blan]), $M$ is non-Kähler if and only if $c_1(\pi)$ is non-zero. The Kodaira surface is an example of a nilmanifold obtained from the Heisenberg group (Example 3.11).

Let $\omega_{E_2}$ be a Kähler form on $E_2$. Then $\pi^*(\omega_{E_2}) \in \text{im} \; d_2$ is exact, giving $\pi^*(\omega_{E_2}) = d\xi$. We chose $\xi$ in such a way that $d(\theta) = 0$, where $\theta := I(\xi)$ (Corollary 4.3). Consider a trivial complex line bundle $(L, \nabla_0)$ with connection defined by $\nabla_0 - \theta$; this line bundle is flat, hence equipped with a holomorphic structure operator $\bar{\nabla} := \nabla_0^{0,1} - \theta^{0,1}$. Using the standard Hermitian metric, we express its Chern connection as $\nabla_0 + \theta^{1,0} - \theta^{0,1}$, hence the curvature of this bundle satisfies $\Theta_L = -\sqrt{-1} \pi^*(\omega_{E_2})$. Rescaling $\theta$ and $\omega_{E_2}$ if necessary, we can assume that the cohomology class of $\omega_{E_2}$ is integer. Then $L$ is a pullback of an ample bundle $B$ on $E_2$. Now, $H^0(M, L) = H^0(E_2, B) \neq 0$, but this space can be interpreted as $H^0(\Lambda^{0,*}(M), \bar{\nabla}_0)$. In this example the twisted Dolbeault cohomology of $M$ is non-zero, but the corresponding twisted Dolbeault cohomology of the Lie algebra vanishes (Theorem 4.2). This gives a counterexample to Console-Fino theorem for twisted cohomology.

5 Classification of LCK nilmanifolds

5.1 Ugarte’s conjecture

The main question concerning nilmanifolds with LCK structure is to decide if the following conjecture is true or not.

Conjecture 5.1: ([U]) Let $M$ be a differentiable nilmanifold admitting an LCK structure (not necessarily locally $G$-invariant, see Definition 3.6). Then $M$ is conformally biholomorphic to a quotient $S^1 \times H_{2n-1}/\Lambda$ with the Vaisman structure described above.

5.2 Classification theorem for LCK nilmanifolds

For LCK nilmanifolds, Ugarte’s conjecture was proven by H. Sawai in the following form:
**Theorem 5.2:** ([Saw1]) Let \((M, I)\) be a complex nilmanifold, \(M = G/\Lambda\). If \((M, I)\) admits an LCK structure, then it is conformally equivalent to Vaisman. Moreover, it is biholomorphic to a quotient of the product \((H_{2n-1} \times \mathbb{R}, I)\).

We give a new proof of this result, different from the one in [Saw1].

**Proof. Step 1:** We replace the LCK metric by a locally \(G\)-invariant LCK metric. To this purpose, we apply the averaging trick that Belgun used for the Inoue surface \(S^+\) ([Bel, Proof of Theorem 7]); later, it was generalized by Fino and Grantcharov ([FinGra]). This approach does not work, in general, for solvmanifolds, but it works well for all nilmanifolds. For LCK structures on nilmanifolds, this construction is due to L. Ugarte:

**Theorem 5.3:** ([U, Proposition 34])
Let \(M = G/\Lambda\) be a complex nilmanifold admitting an LCK structure \((\omega, \theta)\). Then it also admits a structure of LCK nilmanifold. In other words, there exists a left-invariant LCK structure on \(G\) which induces an LCK structure on \(M\).

**Proof:** As \(G\) admits a co-compact lattice, it is unimodular, hence it admits a bi-invariant measure. Let \(d\mu\) be a bi-invariant volume element on \(M\), and suppose \(\text{vol}(M) = 1\). Consider a left-invariant 1-form \(\tilde{\theta}\) on \(g = \text{Lie}(G)\) such that the corresponding 1-form \(\theta_0\) on \(G/\Lambda\) is cohomologous to \(\theta\). Such a 1-form exists because \(H^1(G/\Lambda, \mathbb{R}) = H^1(\Lambda^*(g^*), d)\) ([No]).

Replacing \(\omega\) by a conformally equivalent LCK form \(\omega_0\), we can assume that \(d\omega_0 = \omega_0 \wedge \theta_0\). Let \(D \subset G\) be the fundamental domain of the action of \(\Lambda\). Given left-invariant vector fields \(X, Y\) on \(G\), the 2-form

\[\tilde{\omega}(X, Y) := \int_D \omega_0(X, Y) d\mu\]

defines an Hermitian structure on the Lie algebra \(g\) of \(G\). Moreover, as

\[
d(\tilde{\omega})(X, Y, Z) = - \int_D \omega_0([X, Y], Z) d\mu - \int_D \omega_0(X, [Y, Z]) d\mu + \int_D \omega_0(Y, [X, Z]) d\mu
\]

\[
= - \int_D d(\omega_0)(X, Y, Z) d\mu = - \int_D \theta_0 \wedge \omega_0(X, Y, Z) d\mu = - \tilde{\omega} \wedge \tilde{\theta}(X, Y, Z),
\]

\[\text{version 1.5, March 4, 2020}\]
it follows that \( \tilde{\omega} \) is an LCK form on \( \mathfrak{g} \).

**Proof of Theorem 5.2, Step 2:** Now we prove that any LCK nilmanifold is Vaisman. Using Corollary 4.3, we obtain that \( \omega = d\theta(\tau) \), where \( d\theta(I\tau) = 0 \).

Applying Dixmier and Alaniya’s theorem again, we obtain that \( d\theta(I\tau) = 0 \) implies \( I\tau = d\theta(v) \), where \( v \in \Lambda^0(\mathfrak{g}^*) \) is a constant. Therefore, \( \omega = d\theta d^\mathfrak{g}_0(\text{const}) \), which is the equation for the LCK manifold with potential.\(^1\)

However, as shown in Proposition 2.19, an LCK manifold with potential and constant \( |\theta| \) is Vaisman.

**Step 3:** We show that the Lee field \( \theta^2 \in \mathfrak{g} \) and the anti-Lee field \( I(\theta^2) \in \mathfrak{g} \) generate an ideal in \( \mathfrak{g} \). This observation also follows from [FGV, Theorem 3.12].

Let \( \Sigma = \langle \theta^2, I(\theta^2) \rangle \) be the canonical foliation on \( M \), and \( \mathfrak{s} \subset \mathfrak{g} \) the corresponding subspace of the Lie algebra of \( G \). Since \( \theta^2 \) and \( I(\theta^2) \) are Killing and have constant length, their trajectories are geodesics (Remark 2.9). Let \( S \subset \text{Iso}(M) \) be the group of isometries of \( M \) generated by exponentials of \( \mathfrak{s} \in \mathfrak{s} \).

By construction, \( S \) is a subgroup of \( G \) which acts on \( M = G/\Lambda \). Therefore, it is invariant under conjugation with \( \Lambda \), and the Lie algebra \( \mathfrak{s} \subset \mathfrak{g} \) of \( S \) is invariant under the adjoint action of \( \Lambda \) on \( \mathfrak{g} \). Since \( G \) is the Mal’cev completion of \( \Lambda \), its image in \( \text{End}(\mathfrak{g}) \) coincides with the Zariski closure of the image of \( G \) under the adjoint action (see Property 1.5 for the Mal’cev completion functor in [GH]). Therefore, \( \mathfrak{s} \) is \( G \)-invariant.

**Step 4:** Now we prove that for any Vaisman nilmanifold \( M = G/\Lambda \), the group \( G \) is the product of \( \mathbb{R} \) with the Heisenberg group. Consider the ideal \( \mathfrak{s} \subset \mathfrak{g} \) associated with the canonical foliation \( \Sigma \) (Step 3). The leaf space of \( \Sigma \) is Kähler because \( M = G/\Lambda \) is Vaisman. However, this leaf space is a nilmanifold associated with the Lie algebra \( \mathfrak{g}/\mathfrak{s} \). A Kähler nilpotent Lie algebra is abelian ([BG]), hence \( \mathfrak{g} \) is a central extension of an abelian algebra. Since \( M \) is Vaisman, it is locally isometric to a product of \( \mathbb{R} \) and a Sasakian manifold, hence the corresponding Lie algebra is a product of \( \mathbb{R} \) and a 1-dimensional central extension of \( \mathbb{R}^{2n-2} \). This finishes the proof.

**Remark 5.4:** The general case of Conjecture 5.1 was proven only for nilmanifolds of Vaisman type, by G. Bazzoni, [Baz].

\(^1\)Note that the constant \( \text{const} \) should be positive by Theorem 2.17.
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