Special Dio 3 – tuples
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Abstract. We search for three distinct polynomials with integer coefficient such that the product of any two members of the set minus their sum and increased by a non-zero integer (or polynomial with integer coefficient) is a perfect square.

Introduction:
The problem of constructing the set with property that the product of any two its distinct elements is one less than a square has a very long history and such sets were studied by Diophantus. A set of m positive integers \( \{a_1, a_2, \ldots, a_m\} \) is called a Diophantine m-tuple

\[
a_i \cdot a_j + 1 \text{ is a perfect square,}
\]

(1)
a perfect square for all \( 1 \leq i < j \leq m \). Many generalizations of this problem (1) were considered since antiquity, for example by adding a fixed integer \( n \) instead of 1, looking \( k \)th powers instead of squares or considering the powers over domains other than \( \mathbb{Z} \) or \( \mathbb{Q} \). Many mathematicians consider the problem of the existence of Diophantine quadruples with the property \( D(n) \) for any arbitrary integer \( n \) and also for any linear polynomials in \( n \). In this context one may refer [1-16]. The above results motivated us the following definition:

A set of three distinct polynomials with integer coefficient \( \left(a_1, a_2, a_3\right) \) is said to be a special dio 3- tuple with property \( D(n) \) if

\[
a_i \cdot a_j - \left(a_i + a_j\right) + n \text{ is a perfect square for all } 1 \leq i < j \leq 3.
\]

In the above definition \( n \) may be a non – zero integer or polynomial with integer coefficients. In this communication we consider a few special dio 3 tuples of polygonal numbers from \( t_{6,n} \) to \( t_{10,n} \) and centered polygonal numbers from \( Ct_{6,n} \) to \( Ct_{10,n} \) with their corresponding properties.

Notations:

\[
t_{m,n} = n \left(1 + \frac{(n-1)(m-2)}{2}\right) = \text{Polygonal number of rank } n \text{ with sides } m
\]

\[
c_{m,n} = \frac{nm(n+1)}{2} + 1 = \text{Centered Polygonal number of rank } n \text{ with sides } m
\]

Construction of Dio 3 - tuples for Hexagonal numbers

Let \( a = t_{6,n}, b = t_{6,n-2} \) be Hexagonal number of rank \( n \) and \( n-2 \) respectively such that

\[
a b - (a + b) + 4n^2 - 10n + 11 \text{ is a perfect square say } y^2
\]

Let \( c \) be any non zero integer such that
On solving equations (2) and (3), we get
\[
(\alpha - 1) = \frac{c - (a + c) + 4n^2 - 10n + 11}{\alpha} = \beta^2
\]
(4)

Assuming \(\alpha = x + (a - 1)T\) and \(\beta = x + (b - 1)T\), in (4) it reduces to
\[
x^2 = (b - 1)(a - 1)T^2 + (4n^2 - 10n + 10)
\]
(5)

The initial solution of equation (5) is given by
\[
T_0 = 1 \text{ and } x_0 = n^2 - 5n + 1
\]
(6)

Therefore,
\[
\alpha = 4n^2 - 6n
\]
(7)

On substituting the values of \(\alpha\) and \(a\) in equation (2), we get
\[
c = 8n^2 - 20n + 11
\]
\[
= t_{6,2n-2} - 2n + 1
\]

Therefore, triple \(\{t_{6,n}^*, t_{6,n-2}, t_{6,2n-2} - 2n + 1\}\) is Dio 3- tuple with property \(D(4n^2 - 10n + 11)\).

For simplicity, we present in the table below a few Dio 3- tuples for polygonal numbers from \(t_{6,n}\) to \(t_{10,n}\) with suitable properties.

| \(t_{6,n}\) | \(t_{6,n-2}\) | \(t_{6,n-1}\) | \(D(n)\) |
|---|---|---|---|
| \(t_{6,n}\) | \(t_{6,n-2}\) | \(t_{6,n-1}\) | \(D(10)\) |
| \(t_{6,n-2}\) | \(t_{6,n-3}\) | \(t_{6,n-1}\) | \(D(10n^2 - 15n + 7)\) |
| \(2t_{7,n}\) | \(2t_{7,n-2}\) | \(t_{7,n-1}\) | \(D(10n^2 - 26n + 35)\) |
| \(2t_{7,n-2}\) | \(2t_{7,n-3}\) | \(t_{7,n-2}\) | \(D(30)\) |
| \(2t_{7,n-1}\) | \(2t_{7,n-2}\) | \(t_{7,n-1}\) | \(D(25n^2 - 40n + 17)\) |
| \(t_{8,n}\) | \(t_{8,n-2}\) | \(t_{8,n-1}\) | \(D(18n^2 - 48n + 32)\) |
| \(t_{8,n-2}\) | \(t_{8,n-3}\) | \(t_{8,n-1}\) | \(D(17)\) |
| \(t_{8,n-1}\) | \(t_{8,n-2}\) | \(t_{8,n-1}\) | \(D(18n^2 - 30n + 14)\) |
| \(2t_{9,n}\) | \(2t_{9,n-2}\) | \(t_{9,n-1}\) | \(D(28n^2 - 76n + 74)\) |
| \(2t_{9,n-2}\) | \(2t_{9,n-3}\) | \(t_{9,n-2}\) | \(D(54)\) |
| \(2t_{9,n-1}\) | \(2t_{9,n-2}\) | \(t_{9,n-1}\) | \(D(35n^2 - 60n + 28)\) |
| \(t_{10,n}\) | \(t_{10,n-2}\) | \(t_{10,n-1}\) | \(D(24n^2 - 66n + 47)\) |
| \(t_{10,n-2}\) | \(t_{10,n-3}\) | \(t_{10,n-2}\) | \(D(26)\) |
| \(t_{10,n-1}\) | \(t_{10,n-2}\) | \(t_{10,n-1}\) | \(D(20n^2 - 35n + 16)\) |
Construction of Dio 3 - tuples for Centered Hexagonal numbers

Let \( a = ct_{6,n} \), \( b = ct_{6,n-2} \) be Centered Hexagonal number of rank \( n \) and \( n-2 \) respectively such that \( ab - (a+b) + 24n^2 - 30n + 2 \) is a perfect square say \( \gamma^2 \).

Let \( c \) be any non zero integer such that

\[
ac - (a + c) + 24n^2 - 30n + 2 = \alpha^2 \tag{8}
\]

\[
bc - (b + c) + 24n^2 - 30n + 4 = \beta^2 \tag{9}
\]

On solving equations (8) and (9), we get

\[
(b - 1)\alpha^2 - (a - 1)\beta^2 = (a - b) + \left(24n^2 - 30n + 2\right)(b - a) \tag{10}
\]

Assuming \( \alpha = x + (a - 1)T \) and \( \beta = x + (b - 1)T \) in (10), it reduces to

\[
x^2 = (b - 1)(a - 1)T^2 + \left(24n^2 - 30n + 1\right) \tag{11}
\]

The initial solution of equation (11) is given by

\[
T_0 = 1 \text{ and } x_0 = 3n^2 - 3n + 1 \tag{12}
\]

Therefore,

\[
\alpha = 6n^2 + 1 \tag{13}
\]

On substituting the values of \( \alpha \) and \( a \) in equation (8), we get

\[
c = 12n^2 - 12n + 9 = ct_{6,2n-2} + 6n + 8
\]

Therefore, triple \( \left(ct_{6,n}, ct_{6,n-2}, ct_{6,2n-2} + 6n + 8\right) \) is Dio 3- tuple with property \( D(24n^2 - 30n + 2) \).

For simplicity, we present in the table below a few Dio 3- tuples for Centered polygonal numbers from \( ct_{6,n} \) to \( ct_{10,n} \) with suitable properties.

| \( a \)     | \( b \)             | \( C \)                  | \( D(u) \)                        |
|------------|----------------------|--------------------------|-----------------------------------|
| \( ct_{6,n} \) | \( ct_{6,n-2} \)    | \( ct_{24,n} - 24n \)  | \( D(10) \)                       |
|           | \( ct_{6,n-1} \)    | \( ct_{6,2n-2} + 14n - 4 \) | \( D(-12n^3 + 19n^2 - 4n + 2) \) |
| \( 2ct_{7,n} \) | \( 2ct_{7,n-2} \)    | \( 2ct_{7,2n-2} + 14n + 9 \) | \( D(140n^3 - 140n + 2) \) |
|           | \( ct_{7,2n-2} + 14n + 9 \) | \( 2ct_{7,2n-2} + 14n + 9 \) | \( D(22) \)                       |
|           | \( ct_{5,6n-5} - 56n + 4 \) | \( 2ct_{7,2n-2} + 32n - 9 \) | \( D(-70n^3 + 88n^2 - 20n + 4) \) |
| \( ct_{8,n} \) | \( ct_{8,n-2} \)    | \( ct_{8,2n-2} + 8n - 8 \) | \( D(40n^3 - 40n + 2) \)          |
|           | \( ct_{8,n-1} \)    | \( ct_{8,2n-2} + 32n \)  | \( D(17) \)                       |
| \( 2ct_{9,n} \) | \( 2ct_{9,n-2} \)    | \( 2ct_{9,2n-2} + 18n + 11 \) | \( D(234n^2 - 234n + 7) \) |
|           | \( ct_{9,2n-2} + 18n + 11 \) | \( 2ct_{9,2n-2} + 32n - 13 \) | \( D(46) \)                       |
|           | \( ct_{7,2n-2} - 72n + 4 \) | \( 2ct_{9,2n-2} + 24n - 13 \) | \( D(-126n^3 + 148n^2 - 28n + 4) \) |
| \( ct_{10,n} \) | \( ct_{10,n-2} \)   | \( ct_{10,2n-2} + 10n + 2 \) | \( D(60n^3 - 60n + 2) \)          |
|           | \( ct_{10,n-1} \)   | \( ct_{10,2n-2} + 26n - 8 \) | \( D(-20n^3 + 39n^2 - 4n + 2) \) |
Conclusion:
In this paper we have presented a few examples of constructing a special Dio 3 tuples for polygonal numbers and centered polygonal numbers with suitable properties. To conclude one may search for Dio 3 – tuples for higher order polygonal numbers and centered polygonal numbers with their corresponding suitable properties.

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