Symmetry reduction and exact solutions of the non-linear Black–Scholes equation

Sergii Kovalenko\textsuperscript{a}, Oleksii Patsiuk\textsuperscript{b}

\textsuperscript{a}Poltava National Technical Yuri Kondratyuk University, 24 Pershotravnyi Ave., 36011 Poltava, Ukraine, e-mail: kovalenko@imath.kiev.ua
\textsuperscript{b}Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereshchenkivs’ka Str., 01601 Kyiv-4, Ukraine, e-mail: patsyuck@yahoo.com

Abstract

In this paper, we investigate the non-linear Black–Scholes equation:

\[ u_t + ax^2 u_{xx} + bx^3 u_{xx} + c(xu_x - u) = 0, \quad a, b > 0, \quad c \geq 0. \]

and show that one can be reduced to the equation

\[ u_t + (u_{xx} + u_x)^2 = 0 \]

by an appropriate point transformation of variables. For the last equation, we study the group-theoretic properties, namely, we find the maximal algebra of invariance of its in Lie sense, carry out the symmetry reduction and seek for a number of exact group-invariant solutions of this equation. Using the obtained results, we get a number of exact solutions of the Black–Scholes equation.

Keywords: Black–Scholes equation, symmetry reduction, exact solutions.

1 Introduction

In modern mathematical finance, the Black–Scholes equation (BSE) is one of the key equations used in option pricing theory. Note that standard derivative pricing theory is based on the assumption of perfectly liquid markets. In this case, the well studied linear BSE \cite{1,2} is used. But in recent years much attention is paid to illiquid markets. As noted in \cite{3} (see also \cite{4} and \cite{5}), the most comprehensive equation providing the price of a European option is the following non-linear BSE:

\[ u_t + \frac{1}{2} \sigma^2 (S, u_S, u_{SS}) S^2 u_{SS} + r (S u_S - u) = 0, \quad r \geq 0, \quad (1) \]
where \( u \) is the price of the European option under study, \( S \) is the price of the underlying stock, \( r \) is the risk-free interest rate, and \( \tilde{\sigma} \) is the volatility function.

For modeling illiquid markets, one can use \([5]\):

1) transaction-cost models with the volatility function of the form \([1]\):

\[
\tilde{\sigma}^2 = \sigma^2(1 + 2\rho Su_S);
\]

2) reduced-form SDE models with the volatility function

\[
\tilde{\sigma}^2 = \frac{\sigma^2}{(1 - \rho Su_S)^2};
\]

3) equilibrium (reaction-function) models with the volatility function

\[
\tilde{\sigma}^2 = \frac{\sigma^2(1 - \rho u_S)^2}{(1 - \rho u_S - \rho Su_S)^2}.
\]

In all these formulas \( \sigma \) is the constant (historical) volatility, and \( \rho \) is a parameter modeling the liquidity of the market under study \([2]\).

Since \( \rho \) is often considered to be small, we can replace \( \tilde{\sigma}^2 \) with its first order Taylor approximation around \( \rho = 0 \) in the last two formulas. Thus, for small values of \( \rho \) we can restrict our considerations by the transaction-cost models and investigate only the BSE of the form:

\[
\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}S^2u_{SS}(1 + 2\rho Su_S) + r(Su_S - u) = 0, \quad \sigma, \rho > 0, \quad r \geq 0.
\] (2)

Equation (2) with \( r = 0 \) was investigated in Master's Thesis \([8]\) by Bobrov. Using the methods of the Lie group theory, the author determine the symmetry group and find the maximal algebra of invariance of the equation, carry out the symmetry reduction and present examples of exact invariant solutions.

Using the notation \( a = \frac{1}{2}\sigma^2, \quad b = \rho \sigma^2, \quad c = r, \) and \( x = S \), we rewrite (2) in a more convenient form:

\[
\frac{\partial u}{\partial t} + ax^2u_{xx} + bx^3u_{xxx} + c xu_x - cu = 0, \quad a, b > 0, \quad c \geq 0.
\] (3)

In what follows, we consider only the values of independent variables \( t, x \) from the domain \( \mathbb{R}_+ \times \mathbb{R}_+ \) (this is due to the economic sense of these variables).

With a view to avoiding cumbersome calculations made by Bobrov in the case \( c = 0 \), we reduce (3) to a simpler form using point transformations of variables. Having made the group analysis of the obtained equation and built a number of its exact invariant solutions, we transform them into solutions of equation (3) using the inverse transformations of variables. We also solve a boundary value problem (BVP) for the equation (3) using the obtained solutions.

1 This model was suggested by Çetin, Jarrow, and Protter \([6]\). Note that in \([4]\) several other transaction-cost models with some different volatility functions are considered.

2 For \( \rho = 0 \) the market is perfectly liquid (and we have the linear BSE), whereas for \( \rho \) large a trade has a substantial impact on the transaction price. For the stock of major U.S. corporations \( \rho \) is a small parameter (of the order of \( 10^{-4} \)) \([7]\ p. 186\).
The structure of this article is as follows. In Section 2, using the simplifying point transformations of variables, we reduce non-linear BSE (3) to a partial differential equation (PDE), which is a special case of an equation from the famous handbook [9]. In Section 3, we present the optimal system of the one-dimensional subalgebras of maximal algebra of invariance (MAI) of the obtained equation, carry out the symmetry reduction, and get a number of exact group-invariant solutions of one. Returning to BSE (3), we obtain a number of its exact solutions in Section 4.

2 Simplifying point transformations of variables

Using the point transformations of variables

\[ \tilde{t} = \begin{cases} t, & c = 0, \\ ct, & c > 0; \end{cases} \]

\[ \tilde{u} = \begin{cases} u, & c = 0, \\ \frac{\log x}{c} - ct, & c > 0; \end{cases} \]

we can reduce equation (3) to the equation

\[ u_t + (u_x + u_{xx})^2 = 0 \]

(hereafter we omit the overlines for convenience).

We get an equation of the form \( u_t = F(u_x, u_{xx}) \). It is known (see [9, Subs. 12.1.1, No. 2]) that the last equation admits traveling-wave solution

\[ u(t, x) = u(\xi), \quad \xi = kx + \lambda t, \]

where the function \( u(\xi) \) is determined by the autonomous ordinary differential equation (ODE)

\[ F(ku_\xi, k^2u_{\xi\xi}) - \lambda u_\xi = 0, \]

and a more complicated solution of the form

\[ u(t, x) = c_1 + c_2t + \varphi(\xi), \quad \xi = kx + \lambda t, \]

where the function \( u(\xi) \) is determined by the autonomous ODE

\[ F(k\varphi_\xi, k^2\varphi_{\xi\xi}) - \lambda \varphi_\xi - c_2 = 0. \]

In the following Section, we find a number of other solutions of equation (5).

3 Symmetry reduction and exact solutions of equation (5)

Using program LIE [10], we obtain that the basis of MAI of equation (5) can be chosen as follows:

\[ X_1 = -\partial_x, \quad X_2 = -e^{-x}\partial_u, \quad X_3 = \partial_t, \quad X_4 = \partial_u, \quad X_5 = t\partial_t - u\partial_u. \]
Table 1: The symmetry reduction of equation \((5)\)

| N | Algebra\(^a\) | Ansatz | Reduced equation |
|---|----------------|--------|------------------|
| 1 | \((X_1)\)     | \(u = \varphi(t)\) | \(\varphi' = 0\) |
| 2 | \((X_3)\)     | \(u = \varphi(x)\) | \(\varphi'' + \varphi' = 0\) |
| 3 | \((X_1 + \varepsilon X_3)\) | \(u = \varphi(x + \varepsilon t)\) | \((\varphi'' + \varphi')^2 + \varepsilon \varphi' = 0\) |
| 4 | \((X_1 + \varepsilon X_4)\) | \(u = \varphi(t - \varepsilon x)\) | \(\varphi' = -1\) |
| 5 | \((X_2 + \varepsilon X_3)\) | \(u = \varphi(x) - \varepsilon t e^{-x}\) | \((\varphi'' + \varphi')^2 - \varepsilon e^{-x} = 0\) |
| 6 | \((X_3 + \varepsilon X_4)\) | \(u = \varphi(x) + \varepsilon t\) | \((\varphi'' + \varphi')^2 + \varepsilon = 0\) |
| 7\(^b\) | \((X_1 + k(X_3 + \varepsilon X_4))\) | \(u = \varphi(y) + \varepsilon t\) | \((\varphi'' + \varphi')^2 + \frac{1}{k} \varphi' + \varepsilon = 0\) |
| 8\(^c\) | \((X_2 + k(X_3 + \varepsilon X_4))\) | \(u = \varphi(x) + (\varepsilon - \frac{1}{k} e^{-x}) t\) | \((\varphi'' + \varphi')^2 - \frac{1}{k} e^{-x} + \varepsilon = 0\) |
| 9 | \((X_5)\)     | \(u = t^{-1} \varphi(x)\) | \((\varphi'' + \varphi')^2 - \varepsilon = 0\) |
| 10\(^d\) | \((X_5 + kX_1)\) | \(u = t^{-1} \varphi(y)\) | \((\varphi'' + \varphi')^2 + k \varphi' - \varphi = 0\) |
| 11\(^e\) | \((X_5 - X_1 + \varepsilon X_2)\) | \(u = e^{-x} (\varphi(y) - \varepsilon x)\) | \((\varphi'' - \varphi' + \varepsilon)^2 - e^y \varphi' = 0\) |

\(^a\)In this column, \(\varepsilon = \pm 1\).
\(^b\)In this case, \(k \neq 0, y = x + \frac{k}{t} t\).
\(^c\)In this case, \(0 < |k| < 1\).
\(^d\)In this case, \(k \neq 0, -1, y = x + k \log t\).
\(^e\)In this case, \(y = x - \log t\).

Non-zero commutators of these operators are:

\([X_1, X_2] = X_2, \quad [X_2, X_5] = -X_2, \quad [X_3, X_5] = X_3, \quad [X_4, X_5] = -X_4\).

Hence, our MAI \(A\) can be written as a semidirect sum of a one-dimensional algebra and a four-dimensional ideal:

\(A = \{X_5\} \oplus \{X_1, X_2, X_3, X_4\}\).

The ideal is of the type \(A_2 \oplus 2A_1\). Using this facts and executing the well-known classification algorithm \([11]\) p. 1450], we obtain the following assertion.

**Proposition 1.** The optimal system of the one-dimensional subalgebras of MAI of equation \((5)\) consists of the following ones: \((X_1), (X_2), (X_3), (X_4), (X_5), (X_1 + \varepsilon X_3), (X_1 + \varepsilon X_4), (X_2 + \varepsilon X_3), (X_2 + \varepsilon X_4), (X_3 + \varepsilon X_4), (X_1 + y(\varepsilon_1 X_3 + \varepsilon_2 X_4)), (X_2 + \sin \varphi (\varepsilon_1 X_3 + \varepsilon_2 X_4)), (X_3 + \varepsilon_1 X_1), (X_5 + z X_1), (X_5 - X_1 + \varepsilon X_2)\), where \(\varepsilon = \pm 1, \varepsilon_1 = \pm 1, \varepsilon_2 = \pm 1, y > 0, z \neq 0, -1, and 0 < \varphi < \frac{\pi}{2}\).

First of all, note that algebras \((X_2), (X_4), and (X_2 + \varepsilon X_4) | \varepsilon = \pm 1) do not satisfy the necessary conditions for existence of the non-degenerate invariant solutions. Further, we perform the detailed analysis of invariant solutions, which is based on all other algebras from Proposition\([11]\) The results of our investigation are presented in Tables\([1] and [2]\) Table\([1]\) consists of anzatse generated by the subalgebras and corresponding reduced equations, exact solutions of which (or the first order ODEs, if we could not find their solutions) are given in Table\([2]\)
Table 2: The exact group-invariant solutions of equation (5)

| N  | Algebra | Exact solution or first order ODE |
|----|---------|----------------------------------|
| 1  | a       | $u = c_1$                        |
| 2  | b       | $u = c_1 + c_2 e^{-x}$           |
| 3  | c       | $u = c_1 - \varepsilon(x + \varepsilon t) + 4\delta c_2 e^{-\frac{1}{2}(x+\varepsilon t)} + \varepsilon c_2 e^{-(x+\varepsilon t)}$ |
| 4  | d       | $u = c_1 - t - \varepsilon x$    |
| 5  | e       | $u = c_1 + 4\delta e^{-\frac{t}{2}} - (t + c_2)e^{-x}$ |
| 6  | f       | $u = c_1 + c_2 e^{-x} + \delta x - t$ |
| 7  | g       | $u = c_1 + \varepsilon t + \frac{1}{\delta} \left( \delta \sqrt{1 - 4\varepsilon k^2} - 1 \right) (x + \frac{1}{2}t)$ |
| 8  | h       | $u = c_1 + c_2 e^{-x} - (1 + \frac{1}{2}k e^{-x}) t + \frac{\delta}{\varepsilon} \left( k - \frac{1}{2}e^{-x} \right) \arctan \sqrt{\frac{1}{-k} (e^{-x} - k)} - 3k (e^{-x} - k)$ |
| 9  | i       | $u = c_1 + c_2 e^{-x} + (1 - \frac{1}{2}k e^{-x}) t + \frac{\delta}{\varepsilon} \left( k + \frac{1}{2}e^{-x} \right) \arctan \sqrt{\frac{1}{-k} (e^{-x} - k)} - 3k (e^{-x} - k)$ |
| 10 | j       | $w'(z) = 1 - \varepsilon k^2 \frac{2w(z)}{w(z)}$ |
| 11 | k       | $w'(z) = \left( \frac{1}{w(z)} - \sqrt{\frac{k}{3z}} \right) \frac{2w(z)}{w(z)}$ |
| 12 | l       | $w'(z) = \sqrt{\frac{\varepsilon - kw'(\varphi)}{w'(\varphi)}} - 1$ |
| 13 | m       | $w'(z) = \left( 1 - e^{-\frac{x}{2}} \right) \frac{2w(z)}{w(z)}$ |

\(a\) In this column, the numbers of algebras from Table 1 are indicated.

\(b\) In this column, \(\varepsilon, \delta \in \{1, -1\}\); \(c_1, c_2\) are arbitrary real constants.

\(c\) In this case, \(k \neq 0\), if \(\varepsilon = -1\), and \(0 < |k| < \frac{1}{2}\), if \(\varepsilon = 1\).

\(d\) In this case, \(0 < k < 1\).

\(e\) In this case, \(0 < k < 1\), \(x \geq -\log k\).

\(f\) In this case, \(0 < k < 1\), \(x \leq -\log k\).

\(g\) In this case, \(k \neq 0\).

\(h\) In this case, \(k \neq 0, -1\).
Remark 1. Reduced equations 6 and 8 (with $k < 0$) from Table 1 can have real solutions, only if $\varepsilon = -1$.

Remark 2. In Table 2:
1) solution 1 is trivial and can be included to solution 2;
2) solution 3 is the traveling-wave one, which can be obtained from (4), if we put $k = 1$, $\lambda = \varepsilon$;
3) solution 4 can be obtained from solution 3, if we put $c_2 = 0$;
4) solution 7 is of the form (6), and one can be obtained, if we put $k = 1$, $\lambda = \frac{1}{k}$, $c_2 = \varepsilon$;
5) ODE 11 is obtained, if we put in ODE 7 from Table 1
$$z = -\frac{1}{k} e^{\frac{\tau}{2}} \eta, \quad \omega = e^{\frac{\tau}{2}} \sqrt{-(\varepsilon + \frac{1}{k} \varphi' (y))},$$
and admits the solution in the parametric form (see [12, Subs. 1.3.1, No. 2]):
$$z = z(\tau), \quad w = \tau \cdot z(\tau),$$
where $z(\tau)$ is defined as:

a) $z(\tau) = c_1 \left( 2\tau - 1 + \sqrt{4k^2 + 1} \right)^{1-\frac{1}{\sqrt{4k^2+1}}} \left( 2\tau - 1 - \sqrt{4k^2 + 1} \right)^{1+\frac{1}{\sqrt{4k^2+1}}} \left( \varepsilon + \frac{1}{k} \varphi' (y) \right)^{-\frac{1}{2}},$
if $\varepsilon = -1$, and $k \neq 0$;
b) $z(\tau) = c_1 \left( 2\tau - 1 + \sqrt{1-4k^2} \right)^{1-\frac{1}{\sqrt{1-4k^2}}} \left( 2\tau - 1 - \sqrt{1-4k^2} \right)^{1+\frac{1}{\sqrt{1-4k^2}}} \left( \varepsilon + \frac{1}{k} \varphi' (y) \right)^{-\frac{1}{2}},$
if $\varepsilon = 1$, and $0 < |k| < \frac{1}{2}$;
c) $z(\tau) = \frac{c_1}{2} e^{\frac{\tau}{2} \sqrt{1-k^2}}, \text{ if } \varepsilon = 1, \text{ and } k = \pm \frac{1}{2}$;
d) $z(\tau) = c_1 (\tau^2 - k^2)^{-\frac{1}{2}} e^{-\frac{1}{\sqrt{4k^2-1}}} \frac{1}{\sqrt{2(2k-1)}} \arctan \frac{2\tau - 1}{\sqrt{4k^2-1}}, \text{ if } \varepsilon = 1, \text{ and } |k| > \frac{1}{2}$;
6) ODE 12 is obtained, if we put in ODE 9 from Table 1
$$z = \frac{1}{6} \sqrt{\varphi^3}, \quad \omega = \frac{1}{2} \varphi';$$
this is the Abel equation of the second kind;
7) ODE 13 is obtained, if we put in ODE 10 from Table 1 $w(\varphi) = \varphi'$;
8) ODE 14 is obtained, if we put in ODE 11 from Table 1
$$z = \frac{1}{2} y, \quad \omega = e^{-\frac{1}{2} y} \sqrt{\varphi' (y)}.$$

4 Exact solutions of equation (3)

Using solutions 2–3, 5–10 of equation (5) from Table 2 and point transformations of variables [14], we obtain a number of exact solutions of equation (3) presented in Tables 3 and 4.
Table 3: The exact solutions of equation (3) with $c = 0$

| N | Sol. $^a$   | Exact solution                                                                 |
|---|-------------|--------------------------------------------------------------------------------|
| 1 | 2           | $u = c_1 + \frac{a}{2b}x (c_2 + \frac{a}{2}t - \log x)$                       |
| 2 | 5           | $u = c_1 - t + 4\delta \sqrt{\frac{a}{b}} + \frac{a}{2b}x (c_2 + \frac{a}{2}t - \log x)$ |
| 3 | 6           | $u = c_1 + \frac{a+2x}{2b}x (c_2 + \frac{a-2x}{2b}t - \log x)$                |
| 4 | 3           | $u = \varepsilon \varepsilon_1 e^{-\varepsilon t} + 4\delta c_1 e^{-\frac{\varepsilon t}{2}} \sqrt{\frac{a}{b}} + \frac{a+2x}{2b}x (c_2 + \frac{a-2x}{2b}t - \log x)$ |
| 5 $^c$ | 7          | $u = \frac{x}{2} \left[ c_1 + \left( \varepsilon + \frac{\varepsilon^2}{4} \right) t - \varepsilon \log x + \frac{1}{\varepsilon} (\delta \sqrt{1 - 4\varepsilon k^2} - 1) \left( \frac{\varepsilon}{2} t + \log x \right) \right]$ |
| 6 $^d$ | 8           | $u = c_1 - \frac{1}{k} t + \frac{c_2}{6} \left[ c_2 + \left( \frac{a^2}{4} - 1 \right) t - \frac{a}{2} \log x \right.$ $+ \delta \left[ (1 - \frac{1}{2kx}) \log \left( \frac{2kx}{b} \left( 1 + \sqrt{1 - \frac{1}{k^2}} \right) + 1 \right) - 3 \sqrt{1 - \frac{1}{k^2}} \right] \right]$ |
| 7 $^e$ | 9           | $u = c_1 + \frac{1}{k} t + \frac{c_2}{6} \left[ c_2 + \left( \frac{a^2}{4} - 1 \right) t - \frac{a}{2} \log x \right.$ $+ \delta \left[ (1 + \frac{1}{2kx}) \log \left( \frac{2kx}{b} \left( 1 + \sqrt{1 - \frac{1}{k^2}} \right) - 1 \right) - 3 \sqrt{1 + \frac{1}{k^2}} \right] \right]$ |
| 8 $^f$ | 10          | $u = c_1 - \frac{1}{k} t + \frac{c_2}{6} \left[ c_2 + \left( \frac{a^2}{4} + 1 \right) t - \frac{a}{2} \log x \right.$ $+ \delta \left[ 2 \left( 1 + \frac{1}{2kx} \right) \arctan \sqrt{\frac{1-k^2}{1-k^2}} - 1 \right] \left( \frac{1}{k} t + \log x \right) \right]$ |

$^a$In this column, the numbers of solutions of equation (4) from Table 2 are indicated.

$^b$In this column, $\varepsilon, \delta \in \{1, -1\}$; $c_1, c_2$ are arbitrary real constants.

$^c$In this case, $k \neq 0$, if $\varepsilon = -1$, and $0 < |k| \leq \frac{1}{2}$, if $\varepsilon = 1$.

$^d$In this case, $0 < k < 1$.

$^e$In this case, $0 < k < 1$, $x \geq \frac{b}{k}$.

$^f$In this case, $0 < k < 1$, $x \leq \frac{b}{k}$.
Table 4: The exact solutions of equation [3] with $c \neq 0$

| N | Sol. | Exact solution |
|---|------|---------------|
| 1 | 2 | $u = c_1 e^{ct} + \frac{a}{2b} x \left( e^{ct} \right.$ $+ \frac{a e^{ct}}{2} t - \log x)$ |
| 2 | 5 | $u = (c_1 - ct) e^{ct} + 4 \delta e^{ct} \sqrt{\frac{c_2}{a}} + \frac{a}{2b} x \left( e^{ct} \right.$ $+ \frac{a e^{ct}}{2} t - \log x)$ |
| 3 | 6 | $u = c_1 e^{ct} + \frac{a + 2 c t}{2b} x \left( e^{ct} \right.$ $+ \frac{a + 2(1-t)e}{2} t - \log x)$ |
| 4 | 3 | $u = \varepsilon c_1^2 e^{(1-t)ct} + 4 \delta c_1 e^{(1-t)}} \sqrt{\frac{c_2}{a}} + \frac{a + 2 c t}{2b} x \left( e^{ct} \right.$ $+ \frac{a + 2(1-t)e}{2} t - \log x)$ |
| 5 | 7 | $u = \frac{c t}{a} \left( c_1 + \left[ \varepsilon c + \frac{a}{2} \left( 1 + \frac{a}{2} \right) \right] t - \frac{a}{2b} \log x + \right.$ $+ \left. \frac{1}{2} \left[ \delta \sqrt{1 - 4 \varepsilon k^2} - 1 \right] \left( \frac{1}{k} - 1 \right) ct + \log x \right)$ |
| 6 | 8 | $u = \left( c_1 - \frac{c t}{k} \right) e^{ct} + \frac{c t}{k} \left( c_2 + \left[ \frac{a}{2} \left( 1 + \frac{a}{2} \right) \right] t - \frac{a}{2b} \log x + \right.$ $+ \left. \delta \left[ \left( 1 - \frac{b}{k c e^{ct}} \right) \log \left( \frac{2 k e^{ct}}{b} e^{-ct} \left( 1 + \sqrt{1 + \frac{b}{k c e^{ct}}} \right) + 1 \right) - 3 \sqrt{1 + \frac{b}{k c e^{ct}}} \right) \right)$ |
| 7 | 9 | $u = \left( c_1 + \frac{c t}{k} \right) e^{ct} + \frac{c t}{k} \left( c_2 + \left[ \frac{a}{2} \left( 1 + \frac{a}{2} \right) \right] t - \frac{a}{2b} \log x + \right.$ $+ \left. \delta \left[ \left( 1 + \frac{b}{k c e^{ct}} \right) \log \left( \frac{2 k e^{ct}}{b} e^{-ct} \left( 1 + \sqrt{1 - \frac{b}{k c e^{ct}}} \right) - 1 \right) - 3 \sqrt{1 - \frac{b}{k c e^{ct}}} \right) \right)$ |
| 8 | 10 | $u = \left( c_1 - \frac{c t}{k} \right) e^{ct} + \frac{c t}{k} \left( c_2 + \left[ \frac{a}{2} \left( 1 + \frac{a}{2} \right) \right] t - \frac{a}{2b} \log x + \right.$ $+ \left. \delta \left[ 2 \left( 1 + \frac{b}{k c e^{ct}} \right) \arctan \sqrt{\frac{b}{k c e^{ct}} - 1} - 3 \sqrt{\frac{b}{k c e^{ct}} - 1} \right) \right)$ |

*a In this column, the numbers of solutions of equation [4] from Table 2 are indicated.
*b In this column, $\varepsilon, \delta \in \{1, -1\}; c_1, c_2$ are arbitrary real constants.
*c In this case, $k \neq 0$, if $\varepsilon = -1$, and $0 < |k| \leq \frac{1}{2}$, if $\varepsilon = 1$.
*d In this case, $0 < k < 1$.
*e In this case, $0 < k < 1, x \geq \frac{a}{2b} e^{ct}$.
*f In this case, $0 < k < 1, x \leq \frac{b}{k c e^{ct}}$. 

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Remark 3. Solutions 7 and 8 from Table 3 and also 8 and 9 from Table 4 are not defined for all values \( x \in \mathbb{R}^+ \). Thus, they cannot be considered as the solutions of any BVP determined for equation (3) as \( x \in \mathbb{R}^+ \).

Compare solutions obtained by us with the solutions found in [8]. Changing constants, we can rewrite the Bobrov solutions in the following form:

\[
\begin{align*}
    u(t, x) &= c_1 + c_3 x \left\{ c_2 + (a - bc_4) t - \log x \right\}; \\
    u(t, x) &= c_1 + c_3 t + x \left\{ c_2 + c_4 t - \frac{a}{2b} \log x - \frac{3\delta \sqrt{-c_3 b K}}{b} \sqrt{1 + \frac{1}{K x}} - \\
    &- \frac{\delta c_3 K}{\sqrt{-c_3 b K}} \left( 1 - \frac{1}{2K x} \right) \log \left[ 2K x \left( 1 + \sqrt{1 + \frac{1}{K x}} \right) + 1 \right] \right\}, \\
    K &= \frac{4c_4 b - a^2}{4c_3 b}, \quad \delta = \pm 1.
\end{align*}
\]

It is easy to see that solutions 1 and 3 (and also 5 with \( a = 1 \)) from Table 3 are of the form (8), and solution 6 is of the form (9).

5 Conclusions

In this paper, we found a number of exact group-invariant solutions of the nonlinear Black–Scholes equation.

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