Continuous products of matrices

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Abstract
We answer the question if the continuous product of square matrices $M(t)$ over $t \in [0, 1]$ can be correctly defined. The case where all $M(t)$ are taken from a finite set $\Sigma$ is studied. We find necessary and sufficient conditions on $\Sigma$ that ensure the convergence of products $M(t_0 = 0)M(t_1)\ldots M(t_N = 1)$ as the partition $0 < t_1 < \cdots < 1$ refines. These conditions are properties LCP (left convergent product) and RCP (right convergent product) of the set $\Sigma$. That is, it suffices to require the convergence of all finite products $M_1M_2\ldots M_K$ and $M_K\ldots M_2M_1$ as $K \to \infty$, where $M_i \in \Sigma$. The theory of joint spectral radius is heavily used.

Keywords: Sweeping process, oblique projection, left convergent product, joint spectral radius.

1. Introduction

Directional derivatives of polyhedral sweeping processes [8–11] are closely related to finite sets of projection matrices and their infinite products taken in special order. Here we develop the theory of continuous products of matrices keeping in mind its possible applications to sweeping process and similar hysteresis systems.

Suppose we want to define correctly the “multiplicative integral” of a matrix-valued function $M(t) \in \Sigma \subseteq Mn(R)$ (or $Mn(C)$) over the time interval $[0, 1]$. That is, the question is if there exists a limit of finite products $M(t_1)\ldots M(t_K)$ as the partitions $0 \leq t_1 < \cdots < t_K \leq 1$ of $[0, 1]$ refine in the sense of inclusion.

An obvious case where the answer is positive is that of a finite family $\Sigma$ such that $\rho(\Sigma) < 1$, where $\rho(\Sigma)$ is the joint spectral radius, see [1, 3, 4, 6, 7, 12]. It is not surprising that each function $M : [0, 1] \to \Sigma$ has the zero continuous product over $[0, 1]$ (or over any other infinite linearly ordered set).

There are less obvious cases where the limit exists, again, for each map $M : [0, 1] \to \Sigma$ (no continuity or any other regularity is needed) and may be different from zero. As we prove in this paper, for finite families $\Sigma$, this happens exactly if $\Sigma$ is both of left convergent products (LCP) and right convergent products (RCP) type.

Additionally, we demonstrate that LCP and RCP properties together imply transversality (TR) of the family $\Sigma$ and, moreover, that LCP and TR imply RCP and that RCP and TR imply LCP. It happens also that LCP and RCP imply the convergence of any infinite sequence of matrix products where each subsequent product is obtained by insertion of an arbitrary matrix $A \in \Sigma$ in an arbitrary position of the previous product (CP property).

2. Definitions

Let a finite set $\Sigma = \{A_j : j \in J = \{1, \ldots, k\}\}$ of real $n\times n$-matrices be given. Denote by $\Sigma^m$, $m = 1, 2, \ldots$, the set of all products $A_{j_1}\ldots A_{j_m}$, $A_{j_i} \in \Sigma$ for $i = 1, \ldots, m$. Denote also $\Sigma^0 = \{I\}$, where $I$ is the identity matrix.

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\( n \times n \)-matrix. Let

\[ \mathcal{L}(\Sigma) = \cup_{m=0,1,\ldots} \Sigma^m. \]

The set \( \Sigma \) is said to be product bounded if there exists a \( C > 0 \) such that \( \|A\| < C, A \in \mathcal{L}(\Sigma) \).

**Definition 2.1.** A set \( \Sigma \) is called LCP (left convergent products) if, for any sequence \( S = \{ j_i \in J : i = 1, 2, \ldots \} \), there exists a limit matrix \( L_S \) such that

\[ \lim_{m \to \infty} \|L^m_S - L_S\| = 0, \]

where

\[ L^m_S = A_{j_m} \ldots A_{j_1}, \quad m = 1, \ldots, \tag{2.1} \]

The set \( \Sigma \) is called RCP if, for any sequence \( S = \{ j_i \in J : i = 1, 2, \ldots \} \), there exists a limit matrix \( R_S \) such that

\[ \lim_{m \to \infty} \|R^m_S - R_S\| = 0, \]

where

\[ R^m_S = A_{j_1} \ldots A_{j_m}, \quad m = 1, \ldots, \tag{2.2} \]

These properties are not the same as the following simple example demonstrates. Let \( \Sigma = \{ A_1, A_2 \} \) in \( \mathbb{R}^2 \), where

\[ A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}. \]

Since \( A_1 A_2 = A_2 \) and \( A_2 A_1 = A_1 \), we get \( A_{j_1} \ldots A_{j_m} = A_{j_m} \) for any sequence \( j_i \in \{ 1, 2 \}, i = 1, \ldots, m \), and the set \( \Sigma \) is LCP and not RCP.

Both properties LCP and RCP are stronger than the product boundedness [1]. Transposing the matrix equality (2.1) we get the following assertion.

**Proposition 2.2.** A set \( \Sigma = \{ A_j : j \in J \} \) is LCP if and only if the dual set \( \Sigma^* = \{ A^*_j : j \in J \} \) is RCP.

By discrete linear inclusion DLI(\( \Sigma \)) we will understand the set of all infinite sequences \( \{ x_i \}, i = 0, 1, \ldots, \) of vectors in \( \mathbb{R}^n \) such that

\[ x_i = A_{j_i} x_{i-1}, \quad j_i \in J, \quad i = 1, 2, \ldots, \tag{2.3} \]

These sequences will be called paths of \( \Sigma \). The LCP property is equivalent to convergence of any path \( \{ x_i \} \) of \( \Sigma \), see [1].

There are equivalent definitions of LCP and RCP for finite sets \( \Sigma \), see, for instance, [13]. The family \( \Sigma \) is LCP if and only if all its paths have bounded variation. It is RCP if and only if each family of affine maps

\[ \Psi = \{ A + (A-E)h_A : A \in \Sigma, \quad h_A \in \mathbb{R}^n \} \]

generates a bounded semigroup. They are contraction semigroups if \( \Sigma \) is irreducible. For affine semigroups see [14] and the bibliography within.

For any subset \( J' \subseteq J \) of indices, let us define two subspaces of \( \mathbb{R}^n \):

\[ N_{J'} = \bigcap_{j \in J'} N(I - A_j), \quad R_{J'} = \text{span}_{j \in J'} \{ R(I - A_j) \}, \]

where \( N(A) = \{ x \in \mathbb{R}^n : Ax = 0 \} \) is the nullspace of \( A \) and \( R(A) = \{ Ax : x \in \mathbb{R}^n \} \) is the range of \( A \).

Both \( N_{J'} \) and \( R_{J'} \) are invariant subspaces for paths of the subset \( \Sigma' = \{ A_j \in \Sigma : j \in J' \} \). Moreover, any affine subspace of the form \( x + R_{J'} \), \( x \in \mathbb{R}^n \), is also invariant for these paths because \( x_i - x_{i-1} \in R_{J'} \) whenever \( \{ x_i \} \) is a path of \( \Sigma' \).

**Definition 2.3.** The set \( \Sigma \) is 0-transversal if \( N_{J'} \cap R_{J'} = 0 \) for each \( J' \subseteq J \); it is \( \mathbb{R}^n \)-transversal if \( N_{J'} + R_{J'} = \mathbb{R}^n \) for each \( J' \subseteq J \). The set \( \Sigma \) is transversal if \( N_{J'} \oplus R_{J'} = \mathbb{R}^n \) for each \( J' \subseteq J \), that is, if it is both 0-transversal and \( \mathbb{R}^n \)-transversal.
Obviously, the properties of 0-transversality and $\mathbb{R}^n$-transversality are equivalent to each other: any one of them holds for $\Sigma$ if and only if the other one holds for $\Sigma^* = \{A^* : A \in \Sigma\}$. Let us prove a simple auxiliary assertion.

**Lemma 2.4.** If $\Sigma$ is LCP then $\Sigma$ is $\mathbb{R}^n$-transversal.

**Proof.** Suppose the contrary, that is, $N_J + R_J \neq \mathbb{R}^n$ for some $J' \subseteq J$. Then there exists an $x_0 \in \mathbb{R}^n$ such that $(x_0 + R_{J'}) \cap N_{J'} = \emptyset$. Consider a path $\{x_0, x_1, \ldots\}$ of matrices $A_j$, $j \in J'$ is used infinitely many times. Then the limit point $x^* = \lim_{i \to \infty} x_i$ belongs to $N_{J'}$. Since $x_i \in x_0 + R_{J'}$, we also have $x^* \in x_0 + R_{J'}$, which is a contradiction. □

Hence, the dual assertion is also true:

**Lemma 2.5.** If $\Sigma$ is RCP then $\Sigma$ is 0-transversal.

Thus, we get the following result.

**Proposition 2.6.** If $\Sigma$ is both LCP and RCP then it is transversal.

### 3. Relations between LCP and RCP

We will need the following assertion from [5].

**Theorem 3.1.** Suppose $\Sigma = \{A_j : j \in J\}$ is LCP and $N_J = \{0\}$. Then there exists a norm $\| \cdot \|_\Sigma$ in $\mathbb{R}^n$ and a constant $0 \leq q < 1$ such that

$$
\|A_j\|_\Sigma \leq 1, \quad j \in J, \quad \text{and}
$$

$$
\|A_{j_1} \ldots A_{j_m}\|_\Sigma \leq q
$$

for any finite product containing each $A_j$ from $\Sigma$.

**Theorem 3.2.** If $\Sigma = \{A_j : j \in J\}$ is LCP and 0-transversal then it is also RCP.

**Proof.** Let us use induction on the cardinality $k$ of $\Sigma$. For $k = 1$, there is no difference between LCP and RCP by definition.

Suppose the assertion is true for all matrix sets of cardinality $1, \ldots, k - 1$ and prove it for $\Sigma = \{A_j : j \in J = \{1, \ldots, k\}\}$. Let us first assume $N_J = \{0\}$ Consider a right-infinite product of matrices $A_{j_1} A_{j_2} \ldots$. Suppose that any $j \in J$ occurs in the sequence $(j_1, j_2, \ldots)$ infinitely many times. Then we can represent each finite product $A_{j_1} \ldots A_{j_m}$ as $B_{1} \ldots B_{s(m)}$, where each $B_p$, $p = 1, \ldots, s(m)$, is a finite product of matrices $A_{j_p}$ containing all $A_{j_p}$, $j_p \in J$, and $s(m) \to \infty$ as $m \to \infty$. Then, according to Theorem 3.1, the sequence of products $A_{j_1} \ldots A_{j_m}$ converges to the zero matrix as $m \to \infty$.

Let us consider the remaining cases, that is, suppose that for some $j \in J$, there exists an $m > 0$ such that $j \not\in \{j_m, j_{m+1}, \ldots\}$. By the induction hypothesis, the product $A_{j_m} A_{j_{m+1}} \ldots$ converges to some matrix $A$ and, hence, the product $A_{j_1} \ldots A_{j_m} A_{j_{m+1}} \ldots$ converges to the matrix $A_{j_1} \ldots A_{j_{m-1}} A$, and the required assertion is proved.

It remains to consider the case $N_J \neq \{0\}$. Because of the 0-transversality assumption and Lemma 2.4 we have $\mathbb{R}^n = N_J \oplus R_J$. Since the subspaces $R_J$ and $N_J$ are invariant to all $A_j \in \Sigma$, we can reduce all matrices $A_j$ to the form

$$
A_j = \begin{pmatrix}
I & 0 \\
0 & \tilde{A}_j
\end{pmatrix}
$$

by the same nonsingular linear change of variables corresponding to the decomposition $\mathbb{R}^n = N_J \oplus R_J$. Obviously, both the LCP and RCP properties of $\Sigma$ are equivalent to those of $\tilde{\Sigma} = \{\tilde{A}_j : j \in J\}$, and for $\tilde{\Sigma}$ we have $\tilde{N}_J = \{0\}$. □

Transposition of matrices $A_j$ gives us the dual assertion:
Theorem 3.3. If \( \Sigma = \{ A_j : j \in J \} \) is RCP and \( \mathbb{R}^n \)-transversal then it is also LCP.

Consider the three properties LCP, RCP and transversality (TR) of sets of matrices. Proposition 2.6 and Theorems 3.2, and 3.3 yield the following result.

Theorem 3.4. For a finite set of matrices, any two of the properties LCP, RCP, and TR imply the third one.

Let us say that the set \( \Sigma \) is CP if it is LCP, RCP, and transversal. As an example of a CP set, consider any finite set \( \Sigma \) of orthogonal projections \( P_j \) onto subspaces \( L_j \subseteq \mathbb{R}^n \) of arbitrary dimensions. The transversality of \( \Sigma \) is immediate. The LCP property of \( \Sigma \) is an easy consequence of the following result [13].

Proposition 3.5. If, for any path \( \{ x_i \} \) of a finite set \( \Sigma \), the relation
\[
\lim_{i \to \infty} \| x_{i+1} - x_i \| = 0 \tag{3.3}
\]
holds, then \( \Sigma \) is LCP.

Indeed, for any \( j \in J \), we have
\[
\| P_j x \| = \sqrt{\| x \|^2 - \| P_j x - x \|^2}
\]
and, hence, (3.3) must hold for any path \( \{ x_i \} \) of \( \Sigma \).

4. Continuous products of matrices

In all what follows \( \Sigma \) is a finite CP set. Let \( R \) be an arbitrary linearly ordered set and let \( A \) be a map from \( R \) to \( \Sigma \) (a matrix-valued function on \( R \) ranging in \( \Sigma \)). For any finite subset \( S = \{ s_1 < s_2 < \cdots < s_k \} \subseteq R \), the product
\[
M_S = A(s_1) \cdots A(s_k)
\]
is defined. Since \( \Sigma \) is CP, this definition is extended in a natural way to any countable subset \( S \subseteq R \) of the form \( S = \{ s_1 < s_2 < \cdots \} \) or \( S = \{ s_1 > s_2 > \cdots \} \). We will define the product
\[
M_S = \prod_{r \in S} A(r) \tag{4.1}
\]
for each subset \( S \) of \( R \). It suffices to do this for \( R \) itself because each subset of a linearly ordered set is again a linearly ordered set.

Let us use an iterative procedure on \( k = \# J \) for this definition (by \# we denote the cardinality of the set). Suppose that, for all \( J' \) such that \( \# J' < K \), the products (4.1) are defined. Define \( I_R(A) \) as the maximal number of disjoint intervals \( S_i \subseteq R \) such that \( A(S_i) = \Sigma \) (let us call them complete intervals). By an interval we understand a set \( S \subseteq R \) without holes, that is, conditions \( a, b \in S, g \in R, a < g < b \) should imply \( g \in S \). Note that we can assume without loss of generality that \( \cup_i S_i = R \), that is, intervals \( S_i \) are adjacent. For any incomplete interval \( S \), the product \( M(S) \) is already defined.

Now, we need an auxiliary result.

Lemma 4.1. If \( I_R(A) < \infty \), there exists a partition of \( R \) into no more than \( 3I_R(A) \) adjacent disjoint incomplete intervals.

Proof. First, let \( I_R(A) = 1 \). Let us say that the interval \( S \) is a left interval of \( R \) if
\[
S = \cup_{r \in S} [x \in R : x \leq r].
\]
Let us show that there exists an incomplete left interval of \( R \). Suppose the contrary. Then \( R \) has no minimal element. Let us choose a subset \( P = \{ r_1, \ldots, r_k \} \subseteq R \) such that \( A(P) = \Sigma \). The interval \( [r_1, r_2] \) is complete, and the interval \( Q = \{ x \in R : x < r_1 \} \) is also complete, which is a contradiction.
Now, let $S_1$ be the union of all incomplete left intervals. (It is, obviously, incomplete itself). Denote $R_1 = R\setminus S_1$ and repeat the same argument to construct the interval $R_2$ as the maximal incomplete left interval of $R_1$. If $R_2\setminus (S_1 \cup S_2)$ is nonempty, we repeat the same construction, and so on. Suppose that $R_3$ is nonempty. Then choose some $r' \in S_2$ and $r'' \in R_3$. Obviously, the intervals
\[ I_1 = \{ x \in R : x \leq r' \} \quad \text{and} \quad I_2 = S_3 \cup \{ x \in R : x \leq r'' \} \]
are complete and disjoint. The contradiction finishes the proof for $I_R(A) = 1$. Finally, in the general case $m = I_R(A) < \infty$, the set $R$ can be partitioned into $m$ disjoint adjacent complete intervals $Q_i$ such that $I_{Q_i}(A) = 1$, $i = 1, \ldots, m$, and then each one of these intervals is partitioned into no more than 3 disjoint incomplete intervals.

Let us now finish the definition of product (4.1). If $I_R(A) < \infty$, we define this product by partitioning $R$ into a finite number of adjacent disjoint incomplete intervals; this is possible because of Lemma 4.1. If, on the contrary, $I_R(A) = \infty$, then we define $M$ as the projection $P_J$ on $N_J$ along $R_J$. Because of transversality of $\Sigma$, the projection $P_J(x)$ is well defined as a unique element $y \in N_J$ satisfying $x - y \in R_J$.

Let us study basic properties of the map $M(S) : 2^R \to \Sigma$. First, for finite sets $S$, we have $M(S) = A_{r_1} \ldots A_{r_m}$, where
\[ S = \{ r_1, \ldots, r_m \} \quad \text{and} \quad r_1 < r_2 < \cdots < r_m \]
in the sense of the order on $R$.

**Lemma 4.2.** Let $S_1, S_2 \subseteq R$ and $S_1 < S_2$, that is, $r_1 < r_2$ for each pair $r_1 \in S_1$, $r_2 \in S_2$. Then $M(S_1 \cup S_2) = M(S_1)M(S_2)$.

**Proof.** Let us consider four possible cases:
(i) $I_{S_1}(A) < \infty$ and $I_{S_2}(A) < \infty$,
(ii) $I_{S_1}(A) = \infty$ and $I_{S_2}(A) < \infty$,
(iii) $I_{S_1}(A) < \infty$ and $I_{S_2}(A) = \infty$,
(iv) $I_{S_1}(A) = \infty$ and $I_{S_2}(A) = \infty$.

For instance, if $I_{S_1}(A) < \infty$ and $I_{S_2}(A) = \infty$, we have $I_S(A) = \infty$ and $M(S) = P_J$. Then the required statement follows from the easy fact $P_J A_J = P_J$ for any $A_J \in \Sigma$. The remaining cases are also obvious.

More generally, suppose $F$ is a monotone map from $R$ to another linearly ordered set $Q$, that is, $r_2 \geq r_1$ implies $F(r_2) \geq F(r_1)$. Define $M'(q) = M(F^{-1}(Q))$ for each $q \in Q$.

**Theorem 4.3.** For each subset $S' \subseteq Q$, we have
\[ M'(S') = M(F^{-1}(S')). \]

**Proof.** Let us use induction on $k = \# J$. It suffices to consider the cases $I_{F^{-1}(S')} (A) = \infty$ and $I_{F^{-1}(S')} (A) < \infty$. In each case, the required assertion follows directly from the definitions.

The following assertion, again, follows from elementary induction considerations.

**Theorem 4.4.** Any $M(S)$ can be represented as a limit of matrices $M_i \in \mathcal{L} (\Sigma)$ as $i \to \infty$.

**Theorem 4.5.** The matrix-valued function $M(S^i(r))$ on $R$ has bounded variation for $i = 1, \ldots, 4$, where
\[ S_1(r) = \{ p \in R : p < r \}, \quad S_2(r) = \{ p \in R : p \leq r \}, \quad S_3(r) = \{ p \in R : p > r \}, \quad \text{and} \quad S_4(r) = \{ p \in R : p \geq r \}. \]

**Proof.** As is known [13], there exists an upper bound $V$ on the variation
\[ V(\{ A_i \}) = \sum_{i=1}^{m-1} \| M_i - M_{i+1} \|, \quad \text{where} \quad M_i = A_{1} \ldots A_i, \]
of any finite product $A_1 \ldots A_m$, $A_i \in \Sigma$, $i = 1, \ldots, m$. Let us consider a finite partition of $R$ into intervals $S_i$, $i = 1, \ldots, m$, and prove that the same bound is valid for the product $M(S_1) \ldots M(S_m)$. Indeed, by
Theorem 5.2. There exists a uniform upper bound \( F \) the family of all finite subsets \( S \) of \( \mathbb{R} \) such that
\[
\sum_{i=1}^{m-1} \| M(S_{i+1}) - M(S_i) \| \leq V.
\]

\[ \square \]

Theorem 4.6. The matrix \( M(R) \) can be found as an inductive limit of all finite products \( M(G) \), \( G \subseteq R \), that is, For each \( \varepsilon > 0 \), there exists a finite set \( F \subseteq R \) such that
\[
\| M(F') - M(R) \| < \varepsilon \quad \text{for all finite } F' \supseteq F.
\]

\textbf{Proof.} If \( I_R(A) = \infty \), this is obvious. If \( I_R(A) < \infty \), we use the induction on \#\( J \) again.

Hence, the product \( M(R) \) can also be defined by means of the following formal constructions. Let \( F \) be the family of all finite subsets \( F \subseteq R \). Considering \( F \) as a directed set with respect to the order relation \( F \leq F' \Leftrightarrow F \subseteq F' \), we can define \( M(R) \) as the limit of the net \( \{ M(F) : F \in F \} \).

5. Insertions

The LCP property means that if, at each discrete time instant, a matrix from \( \Sigma \) is added to the current product at the left, then the resulting procedure converges. For RCP left should be replaced by right. It is also easy to see that, for a CP set \( \Sigma \), one can add matrices alternately at the left and at the right, in any sequence, and the resulting product still converges. Now, we are going to prove that, for CP sets, this procedure can be generalized so that any subsequent matrix can be inserted at any place of the current product, the front and the rear positions included.

Theorem 5.1. Let a sequence \( M_i \in \Sigma^j \), \( i = 1, \ldots \), possess the following property. For each \( i \), there exists an \( m_i \), \( 0 \leq m_i \leq i \) such that
\[
M_i = M_{m_i} M_+ \quad \text{and} \quad M_{i+1} = M_+ A_{j_i} M_+,
\]
where \( M_- \in \Sigma^{m_i} \), \( M_+ \in \Sigma^{i-m_i} \), and \( A_{j_i} \in \Sigma \). Then \( M_i \) converges to some \( n \times n \)-matrix as \( i \to \infty \).

\textbf{Proof.} Let us introduce a linear order relation \( \succ \) on the set of indices \( 1, 2, \ldots \) as follows. We will write \( i' \succ i'' \) if the matrix \( A_{j_i} \), takes position to the right of \( A_{j_{i''}} \) in the product \( M_i \), where \( i = \max\{i', i''\} \).

Let us choose a countable number of reals \( x_i \) such that \( x_i > x_j \) if and only if \( i \succ j \). This kind of choice is possible because, if at step \( i \) the set \( \{ x_j : j \succ i \} \) satisfies this requirement, the next point \( x_{i+1} \) can also be chosen in a way that the requirement still holds for the set \( \{ x_j : j \succeq i+1 \} \). The assertion of the theorem follows now from Theorem 4.6, where \( R = \{ x_i : i = 1, 2, \ldots \} \) equipped with the order induced by the natural order on \( \mathbb{R} \), and \( A(x_i) = A_{j_i} \).

Moreover, the following stronger assertion holds.

Theorem 5.2. There exists a uniform upper bound \( V \) on the variation of any sequence \( M_i \) from the hypothesis of Theorem 5.1.

\textbf{Proof.} Let us again use induction on \#\( J \) and suppose that the assertion is proved for all proper subsets of \( \Sigma \) (denote the corresponding upper bound by \( B_h \)). Let \( R \) be the set defined in the proof of Theorem 5.1 and \( A(x_i) = A_{j_i} \). Denote by \( R_j \) the subset \( \{ x_i : i = 1, \ldots, j \} \) of \( R \). Denote also \( k(j) = I_R(A) \), \( j = 1, 2, \ldots \).

The sequence \( k(j) \) is nondecreasing and \( k(1) = 0 \) if \#\( J > 1 \). Now, denote by \( l_m \) the maximal index \( j \) such that \( k(j) = m \) (if it exists).

Let us first find an upper bound on the variation of the finite sequence \( M_1, \ldots, M_l \). By Lemma 4.1, the set \( R_{l+1} \) can be partitioned into 3 incomplete intervals. At each step \( i < l \), the matrix \( A_{j_i} \) is inserted into
one of these subintervals. By the induction assumption, variation of the product matrix for each subinterval
does not exceed $B_0$ and, hence, because of (3.1), the variation of $\{M_1, \ldots, M_i\}$ does not exceed $3B_0$.

Now, let us find an upper bound for the variation of

$$M_m = \{M_{l_{m-1}+1}, \ldots, M_{l_m}\}$$

for an arbitrary $m$. The set $R_{l_m}$ can be divided into $m$ disjoint adjacent intervals $S_i$ such that $I_{S_i}(A) = 1$, $i = 1, \ldots, m$. Whenever a matrix is inserted into one of these intervals, the variation of the whole product
at this step does not exceed $q^{m-1}V'$, where $V'$ is the variation of the current product, at which the insertion
occurs ($q < 1$ is the constant from Theorem 3.1). Thus, the variation of $M_m$ is bounded from the above by $s_m = 3mq^{m-1}$. But the sum $\sum_{m=1}^{\cdots} s_m$ is bounded since $q < 1$, hence, the theorem is proved. \(\square\)

Let us formulate a generalization of Theorem 5.2. The proof is completely analogous to that of Theorem
5.2, so we leave it to the reader.

**Theorem 5.3.** Let $R$ be a linearly ordered set and let $A$ be a map from $R$ to a CP set $\Sigma$. Suppose $Q$
is another linearly ordered set and $B$ is a monotone map from $Q$ to $2^R$, that is, $B(q_2) \supseteq B(q_1)$ whenever $q_2 \geq q_1$. Then the map $M'(q) = M(B(q))$ possesses the bounded variation property in the sense of Theorem 4.5.

**6. Linear control systems**

In this section we assume that $R$ has the minimal element $r_-$ and the maximal element $r_+$. Suppose
$f(r)$ is a bounded function from $R$ to $\mathbb{R}^n$. Let us define the integral

$$\int_R f(r) dM(r), \text{ where } M(r) = M(\{s \in R : s \leq r\}),$$

(6.1)
as follows. For each finite subset $F \subseteq R$ of the form

$$F = \{r_1, \ldots, r_m\}, \quad r_1 = r_- < r_2 < \cdots < r_{m-1} < r_m = r_+,$$

define

$$H(F) = \sum_{i=1}^{m-1} (M(r_{i+1}) - M(r_i))f(r_i).$$

Then define

$$\int_R f(r) dM(r) = \lim_{F \to R} H(F),$$

(6.2)
where the limit for the net of all finite subsets of $R$ containing $r_-$ and $r_+$ is considered in the right-hand
side. The expression (6.2) is well defined because of Theorem 4.5.

The integral (6.2) is an analogue of the standard Lebesgue–Stiltjes integral. Finally, let us define formally

$$\int_R M(r) df(r) = M(r_+)f(r_+) - M(r_-)f(r_-) - \int_R f(r) dM(r).$$

(6.3)
The variable integral $x(r) = \int_R M(s) df(s), r \in R$, can be interpreted as the output of a linear control
system with the input $f(r), r \in R$. 7
7. Recognizing CP property

Let us say a few words on the general structure of CP families and on the hardness of their recognition. First of all, the finiteness property holds for CP families that have \( \rho(\Sigma) = 1 \). Moreover, if \( \rho(\Sigma) = 1 \), then there exists a matrix \( A \in \Sigma \) such that \( \rho(A) = 1 \).

Indeed, if this is not the case, there exists a sequence \( A_i \in \Sigma \) that contains infinite number of entries of at least two different matrices \( A \) and \( B \) and such that \( \lim_{k \to \infty} A_k \ldots A_1 \neq 0 \). Then any non-zero column of the limit matrix is invariant for \( A \) and \( B \) which is impossible by assumption. For each \( A \in \Sigma \) the limit \( \lim_{m \to \infty} A_m \) exists.

Thus we have two options for \( \Sigma \). If \( \rho(\Sigma) < 1 \), it is a CP family. The hardness of recognition of this case is still an open problem as far as we know. It is conjectured that the problem is algorithmically unsolvable for matrices with rational entries, the same way as it happens for a similar problem \( \rho(\Sigma) \leq 1 \), see [2].

The second case is \( \rho(\Sigma) = 1 \) and then the hardness of recognition is the same as in the first case. Indeed, let \( \Sigma = \{A, B, C\} \), where

\[
A = \begin{pmatrix} E & 0 & 0 \\ M_1 & 0 & 0 \\ M_1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & M_2 & 0 \\ 0 & E & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & M_3 \\ 0 & 0 & 0 \\ 0 & 0 & E \end{pmatrix}
\]

and \( M_i \) are square \( n/3 \times n/3 \)-matrices.

Then \( \Sigma \) is a CP family if and only if the set \( \Psi \) of two matrices \( M_2M_1 \) and \( M_3M_1 \) is asymptotically stable, that is, if \( \rho(\Sigma) < 1 \) (otherwise the variation of paths is not upper bounded). These are arbitrary matrices of size \( n/3 \times n/3 \), hence the hardness of recognition is the same as in the first case, but in a space of lower dimension.

8. Infinite bounded families

If \( \Sigma \) is an infinite bounded subset of \( M_n(\mathbb{R}) \), we conjecture that, again, CP is equivalent to LCP and RCP together. Note that even if we restrict matrices to orthogonal projections in \( \mathbb{R}^2 \) (they are self-conjugated, hence, LCP=RCP), infinite families can be either CP or not.

The following assertion can be extended to higher dimensions.

Theorem 8.1. A family \( \Sigma \) of orthogonal projections on straight lines \( \mathbb{R}h, \ h \in H \subseteq S_1 \) in \( \mathbb{R}^2 \) is CP if and only if, for any sector \( K \) with non-empty interior in \( \mathbb{R}^2 \) there exists a sector \( K' \subseteq K \) with non-empty interior such that \( K' \cap H = \emptyset \).

The crucial point here is that there are no non-trivial limit cycles for paths of \( \Sigma \).

References

[1] M. Berger and Y. Wang. Bounded semigroups of matrices. Linear Algebra Appl., 166:21–27, 1992.
[2] V. Blondel and J. N. Tsitsiklis. The boundedness of all products of a pair of matrices is undecidable. Systems Control Lett., 41:135–140, 2000.
[3] Vincent D. Blondel. The birth of the joint spectral radius: An interview with Gilbert Strang. Linear Algebra and its Applications, 428:2261 – 2264, 2008.
[4] I. Daubechies and J. C. Lagarias. Sets of matrices all infinite products of which converge. Linear Algebra Appl., 161:227–263, 1992.
[5] L. Elsner and S. Friedland. Norm conditions for convergence of infinite products. Linear Algebra Appl., 250:133–142, 1997.
[6] R. Jungers. The joint spectral radius. Theory and applications, volume 385 of Lecture Notes in Control and Information Sciences. Springer-Verlag, London, 2009.
[7] V. Kozyakin. An explicit Lipschitz constant for the joint spectral radius. Linear Algebra and its Applications, 433(1):12–18, 2010.
[8] P. Krejci and A. Vladimirov. Polyhedral sweeping processes with oblique reflection in the space of regulated functions. Set-Valued Anal., 11:91–110, 2003.
[9] M. Kunze and M.D.P. Monteiro Marques. An introduction to Moreau’s sweeping process. In Impacts in Mechanical Systems - Analysis and Modelling, volume 551 of Lecture Notes in Physics, pages 1–60. Springer, Berlin-New York, 2000.

[10] A. Mandelbaum and K. Ramanan. Directional derivatives of oblique reflection maps. Mathematics of Operations Research, 35(3):527–558, 2010.

[11] J. J. Moreau. Evolution problem associated with a moving convex set in a Hilbert space. Journ. of Dif. Eq., 20:347–374, 1977.

[12] G.-C. Rota and W. G. Strang. A note on the joint spectral radius. Indag. Math., 22:379–381, 1960.

[13] A. A. Vladimirov, L. Elsner, and W.-J. Beyn. Stability and paracontractivity of discrete linear inclusions. Linear Algebra Appl., 312:125–134, 2000.

[14] A. S. Voynov and V. Yu. Protasov. Compact noncontraction semigroups of affine operators. Sbornik: Mathematics, 206(7):921, 2015.