Integrable relativistic systems given by Hamiltonians with momentum-spin-orbit coupling

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Abstract

In the paper we investigate the evolution of the relativistic particle (massive and massless) with spin defined by Hamiltonian containing the terms with momentum-spin-orbit coupling. We integrate the corresponding Hamiltonian equations in quadratures and express their solutions in terms of elliptic functions.

1 Introduction

Deformation of Lie algebra structure on the fixed vector space $\mathfrak{g}$ is known to provide a construction of a family of Lie–Poisson brackets on $C^\infty(\mathfrak{g}^*)$, where $\mathfrak{g}^*$ is the vector space dual to $\mathfrak{g}$. If this family contains a pencil of brackets one obtains bi–Hamiltonian structure on $\mathfrak{g}^*$ which leads to the construction of an integrable hamiltonian hierarchy, e.g. see [1], [2]. In the paper [4] we investigated such infinite parameter deformation of the Lie–Poisson structure on the ideal $L_2^2(\mathcal{H})$ of the upper triangular Hilbert–Schmidt operators acting on the real Hilbert space $\mathcal{H}$, which has the properties mentioned above. Applying results obtained in [4] to the case when $\dim \mathcal{H} = 5$ we find an integrable hamiltonian system, see equations (20-23), on the vector space $L_+^2$ of upper triangular $(5 \times 5)$–matrices which one can consider as a dual space to the Lie algebra $\mathfrak{e}_a(1, 3)$. This Lie algebra is defined as a one parameter $a \in [-1, 1]$ deformation of Euclidean Lie algebra $(a = 1)$ in four dimensions. When $a = -1$ we obtain Poincare algebra and the Galileo algebra for $a = 0$. 

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In Section 2 we solve equations (20-23) in quadratures expressing their solutions in terms of the first-coordinate $W_0(t)$ of Pauli–Lubansky four-vector. The dependence of $W_0(t)$ on the evolution parameter $t$ is also studied.

Using the twistor description of Minkowski space-time (e.g. see [7], [3]) we investigate in Section 3 the time-evolution of the position vector $\vec{X}(t)$ of the massive relativistic particle with a spin, see formula (99). In Section 3 we also describe the time dependence of the twistor coordinates for the massless particle with non–zero helicity, see formula (77).

The physical sense of the hamiltonian relativistic systems studied in the paper is hidden in the form of Hamiltonian (19) which gives a coupling between the momentum, angular momentum and spin of the relativistic particle.

In the Euclidean case, i.e. when $a = 1$, one has a model with a rather different geometry. Namely, from (10), (11) and (14) we find that solution of (20-23) describes the evolution of a point on the bundle $T\mathbb{S}^3$ tangent to the three–dimensional sphere.

2 Bi–Hamiltonian structure on $\mathcal{L}_+$ and the related integrable systems

Let us start our consideration with defining the pencil of metric tensors

$$ds_a^2 = \eta^a_{\mu\nu} dx^\mu dx^\nu := a(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2,$$

where $a \in [-1, 1]$, on the four-dimensional affine space $\mathbb{E}^{1,3}_a$ with coordinates $x^\mu$, $\mu = 0, 1, 2, 3$. We will denote the symmetry group of $(\mathbb{E}^{1,3}_a, ds^2_a)$ by $E_a(1,3)$ and its Lie algebra by $\mathcal{E}_a(1,3)$, respectively. Using $(5 \times 5)$–matrix representation we find that $g \in E_a(1,3)$ iff

$$g = \begin{pmatrix} \frac{1}{\tau} & 0^T \\ \tau^T & \Lambda \end{pmatrix},$$

where $\tau \in \mathbb{R}^4$, $\mathbb{R}^4 \ni 0$ is zero vector and $\Lambda \in Mat_{4\times4}(\mathbb{C})$ satisfies

$$\Lambda \eta^a \Lambda^T = \eta^a. \quad (3)$$

Any $\chi \in \mathcal{E}_a(1,3)$ is of the form

$$\chi = \begin{pmatrix} 0 & 0^T \\ y & \vec{X} \end{pmatrix},$$

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where \( y \in \mathbb{R}^4 \) and \( X \in \text{Mat}_{4 \times 4}(\mathbb{R}) \) satisfies
\[
X\eta^a + \eta^a X^\top = 0. \tag{5}
\]
The vector space \( \mathcal{E}_a(1, 3)^* \) dual to the Lie algebra \( \mathcal{E}_a(1, 3) \) we will identify with the vector space \( \mathcal{L}_+ \) of strictly upper triangular \((5 \times 5)\)-matrices. The pairing between \( \rho \in \mathcal{L}_+ \) and \( \chi \in \mathcal{E}_a(1, 3) \) is given by
\[
\langle \chi, \rho \rangle := \text{Tr}(\chi \rho). \tag{6}
\]
Taking
\[
\rho = \begin{pmatrix}
0 & P^0 & P^1 & P^2 & P^3 \\
0 & 0 & L_1 & L_2 & L_3 \\
0 & 0 & 0 & J_3 & -J_2 \\
0 & 0 & 0 & 0 & J_1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} \tag{7}
\]
and defining
\[
M_{0k} = -M_{k0} := L_k, \quad M_{kl} := \epsilon_{kln}J_n, \tag{8}
\]
where \( k, l, n = 1, 2, 3 \), we find that Lie–Poisson bracket \( \{\cdot, \cdot\}_a \) of \( f, g \in C^\infty(\mathcal{L}_+) \) is expressed as follows
\[
\{f, g\}_a = \frac{1}{2} \eta^a_{\varphi \phi} M_{\mu \nu} \left( \frac{\partial f}{\partial M_{0\mu}} \frac{\partial g}{\partial M_{0\nu}} \frac{\partial f}{\partial M_{\varphi \mu}} \frac{\partial g}{\partial M_{\varphi \nu}} \right) + \tag{9}
\]
\[
+ \eta^a_{\varphi \mu} P^\mu \left( \frac{\partial f}{\partial P^\nu} \frac{\partial g}{\partial M_{\varphi \nu}} - \frac{\partial f}{\partial M_{\varphi \nu}} \frac{\partial g}{\partial P^\nu} \right) = \]
\[
= aP^0 \left( \frac{\partial f}{\partial P} \cdot \frac{\partial g}{\partial L} - \frac{\partial f}{\partial L} \cdot \frac{\partial g}{\partial P} \right) + \]
\[
+ \tilde{J} \cdot \left( a \left( \frac{\partial f}{\partial L} \times \frac{\partial g}{\partial L} \right) + \frac{\partial f}{\partial \tilde{J}} \times \frac{\partial g}{\partial \tilde{J}} \right) + \]
\[
+ \frac{\partial g}{\partial P^0} \tilde{P} \cdot \frac{\partial f}{\partial L} - \frac{\partial f}{\partial P^0} \tilde{P} \cdot \frac{\partial g}{\partial L} + \]
\[
+ \tilde{P} \cdot \left( \frac{\partial f}{\partial \tilde{P}} \times \frac{\partial g}{\partial \tilde{J}} + \frac{\partial f}{\partial \tilde{J}} \times \frac{\partial g}{\partial \tilde{P}} \right) + \]
\[
+ \tilde{L} \cdot \left( \frac{\partial f}{\partial \tilde{L}} \times \frac{\partial g}{\partial \tilde{J}} + \frac{\partial f}{\partial \tilde{J}} \times \frac{\partial g}{\partial \tilde{L}} \right).}.
Note that one has the following two invariants (Casimir functions) of the coadjoint representation
\[ c_1 = \eta^a_{\mu\nu} P^\mu P^\nu, \]  
\[ c_2 = \eta^a_{\mu\nu} W^\mu W^\nu, \]
where
\[ W^0 = -\vec{J} \cdot \vec{P}, \]  
\[ \vec{W} = aP^0 \vec{J} + \vec{L} \times \vec{P}, \]
while
\[ \eta^a_{\mu\nu} P^\mu W^\nu = 0. \]

The coadjoint representation of \( E_a(1, 3) \) on the dual space of \( \mathcal{L}_+ \) has the form
\[ Ad^a_g(P, M) = \left( (\eta^a)^{-1} \Lambda \eta^a P, \right) \]
\[ \Lambda \left( \pi_+ (M) - \eta^a \pi_+ (M^\top) \right) (\eta^a)^{-1} \Lambda^{-1} + \tau P^\top \Lambda^{-1} - \Lambda \eta^a P \tau^\top (\eta^a)^{-1}, \]
\[ Ad^a_g(W) = (\eta^a)^{-1} \Lambda \eta^a W, \]
where we represent \( \rho \in \mathcal{L}_+ \cong \mathcal{E}_a(1, 3)^* \) by the four-momentum \( P = (P_\mu) \) and the angular momentum \( M = (M_{\mu\nu}) \) defined in (7) and (8).

We note also that for \( a, b \in [-1, 1] \) the Poisson brackets \( \{\cdot, \cdot\}_a \) and \( \{\cdot, \cdot\}_b \) define bi–Hamiltonian structure on \( \mathcal{L}_+ \), i.e. their linear combination \( \{\cdot, \cdot\}_a + \epsilon \{\cdot, \cdot\}_b, \epsilon \in \mathbb{R}, \) is also a Poisson bracket on \( \mathcal{L}_+ \), see [4].

Thus we obtain that Casimirs of \( \{\cdot, \cdot\}_b \):
\[ h_1 = b(P^0)^2 + \vec{P} \cdot \vec{P}, \]
\[ h_2 = b \left( \vec{P} \cdot \vec{J} \right)^2 + \left( bP^0 \vec{J} + \vec{L} \times \vec{P} \right)^2 \]
are the integrals of motion being in involution with respect to the Poisson bracket \( \{\cdot, \cdot\}_a \).

Hamiltonian equations associated with the Hamiltonian
\[ h = \frac{1}{2} (ch_1 + dh_2) = \]
\[ = \frac{c}{2} \left( b(P^0)^2 + \vec{P} \cdot \vec{P} \right) + \frac{d}{2} \left( b \left( \vec{P} \cdot \vec{J} \right)^2 + \left( bP^0 \vec{J} + \vec{L} \times \vec{P} \right)^2 \right) \]


\[
\frac{1}{2}(b-a) \left( c(P^0)^2 + d(b-a)(P^0)^2 \mathbf{J}^2 - \frac{d}{a} \mathbf{W}^2 + 2dP^0 \mathbf{W} \cdot \mathbf{J} \right),
\]

where \(c,d \in \mathbb{R}\) are as follows

\[
\frac{dP^0}{dt} = \{P^0, h\}_a = 0, \quad \frac{d\mathbf{J}}{dt} = \{\mathbf{J}, h\}_a = 0,
\]

\[
\frac{d\mathbf{P}}{dt} = \{\mathbf{P}, h\}_a = (b-a)dP^0 \left( \mathbf{P} \times \left( \mathbf{P} \times \mathbf{L} \right) + bP^0 \mathbf{J} \times \mathbf{P} \right),
\]

\[
\frac{d\mathbf{L}}{dt} = \{\mathbf{L}, h\}_a = (b-a) \left( cP^0 \mathbf{P} + bdP^0 \mathbf{J}^2 \mathbf{P} + dP^0 \mathbf{L} \times \left( \mathbf{P} \times \mathbf{L} \right) + dP^0 \mathbf{J} \times \mathbf{P} \right) + bd(P^0)^2 \mathbf{J} \times \mathbf{L}.
\]

In order to solve these equations it suffices to possess four functionally independent integrals of motion. We choose \(P^0\) and \(\mathbf{J}\) as these integrals. Using the variables \((\mathbf{P}, \mathbf{W})\) we rewrite (22) and (23) in the form

\[
\frac{d\mathbf{P}}{dt} = (b-a)dP^0 \left( -\mathbf{P} \times \mathbf{W} + (b-a)P^0 \mathbf{J} \times \mathbf{P} \right),
\]

\[
\frac{d\mathbf{W}}{dt} = (b-a)d \left( \left( \mathbf{J} \cdot \mathbf{P} \right) \mathbf{P} \times \mathbf{W} + bP^0 \mathbf{J} \times \mathbf{W} + aP^0 \left( \mathbf{J} \cdot \mathbf{P} \right) \mathbf{J} \times \mathbf{P} \right).
\]

Now let us introduce new variables

\[
y := \mathbf{J} \cdot \mathbf{W},
\]

\[
z := \mathbf{J} \cdot \left( \mathbf{P} \times \mathbf{W} \right).
\]

From (24) and (25) we find that these variables and \(W^0\) satisfy the following equations

\[
\frac{dW^0}{dt} = (b-a)dP^0 z,
\]

\[
\frac{dy}{dt} = -(b-a)dW^0 z,
\]

\[
\frac{dz}{dt} = -(b-a)dW^0 \left( c_2P^0 + c_1aP^0 \mathbf{J}^2 - (c_1 + a(P^0)^2) y \right),
\]

In order to solve these equations it suffices to possess four functionally independent integrals of motion. We choose \(P^0\) and \(\mathbf{J}\) as these integrals.
which can be integrated in quadratures:

\[
t + t_0 = \int \frac{dW^0}{(b - a)d\sqrt{\frac{c_1}{4}(W^0)^4 + \frac{c_1(h_2 - c_2 - (b^2 - a^2)(P^0)^2J^2)}{2(b - a)}}(W^0)^2 + \beta}
\]

(31)

\[
y(t) = -\frac{1}{2P^0}(W^0)^2(t) + \frac{h_2 - c_2}{2P^0(b - a)} - \frac{b - a}{2}P^0J^2,
\]

(32)

\[
z(t) = \frac{1}{P^0} \sqrt{-\frac{(c_1 + a(P^0)^2)}{4}(W^0)^4(t) + \frac{c_1(h_2 - c_2 - (b^2 - a^2)(P^0)^2J^2)}{2(b - a)}(W^0)^2(t) + \beta}.
\]

(33)

Without loss of generality we can assume \(\vec{J} = (0, 0, J)\) and obtain

\[
P^3 = -\frac{1}{J}W^0,
\]

\[
W^3 = \frac{1}{J} \left( -\frac{1}{2P^0}(W^0)^2 + \frac{h_2 - c_2}{2P^0(b - a)} - \frac{b - a}{2}P^0J^2 \right),
\]

(34)

\[
(P^1)^2 + (P^2)^2 = c_1 - a(P^0)^2 - \frac{1}{J^2}(W^0)^2,
\]

\[
(W^1)^2 + (W^2)^2 = c_2 - a(W^0)^2 - \frac{1}{J^2} \left( -\frac{1}{2P^0}(W^0)^2 + \frac{h_2 - c_2}{2P^0(b - a)} - \frac{b - a}{2}P^0J^2 \right)^2.
\]

After passing to polar coordinates

\[
P^1 = \sqrt{(P^1)^2 + (P^2)^2}\cos\varphi, \quad P^2 = \sqrt{(P^1)^2 + (P^2)^2}\sin\varphi,
\]

\[
W^1 = \sqrt{(W^1)^2 + (W^2)^2}\cos\psi, \quad W^2 = \sqrt{(W^1)^2 + (W^2)^2}\sin\psi
\]

from (24), (25) we get

\[
\frac{d\varphi}{dt} = (b - a)d_2P^0 \left( bP^0J + \frac{y - aP^0J^2}{J} \right) - \frac{(W^0)^2(y + aP^0J^2)}{(W^0)^2 - c_1J^2 + aJ^2(P^0)^2},
\]

(36)

\[
\frac{d\psi}{dt} = (b - a)d_2P^0 \left( bP^0J + \frac{y^2 - a^2(P^0)^2J^4}{JP^0(c_2J^2 - aJ^2(W^0)^2 - y^2)} \right) - \frac{1}{P^0J(W^0)^2}.
\]

(37)

Using formulas (34), (35), (36) and (37) we find that the solutions \(\vec{W}(t), \vec{P}(t)\) of equations (24), (25) are expressed by first–coordinate of the spin four–vector \(W^0(t)\) which is an elliptic function of \(t\) defined in (31).
Now, we express the solution $\vec{L} = \vec{L}(t)$ by the functions $\vec{W}(t)$ and $\vec{P}(t)$. To this end we take the vector product of both sides of (13) and apply the identity

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}. \quad (38)$$

This gives

$$\vec{L} = \frac{1}{\vec{J} \cdot \vec{P}} \left( \vec{J} \times \vec{W} + (\vec{J} \cdot \vec{L}) \vec{P} \right) = -\frac{1}{W^0} \left( \vec{J} \times \vec{W} + \xi \vec{P} \right), \quad (39)$$

where $\xi := \vec{J} \cdot \vec{L}$ and $-W^0 = \vec{J} \cdot \vec{P}$ is found in (31). From (23) and (39) we obtain

$$\frac{d}{dt} (W^0 \xi) = -d(b - a) P^0 \left( -\frac{1}{4(P^0)^2}(W^0)^4 + c_2^2 \vec{J}^2 - \left( \frac{h_2 - c_2}{2P^0(b - a)} - \frac{b - a}{2} P^0 \vec{J}^2 \right) \right). \quad (40)$$

Solution of (40) is given by

$$\xi(t) = -\frac{d(b - a) P^0}{W^0} \int_{t_0}^{t} \left( -\frac{1}{4(P^0)^2}(W^0)^4(s) + c_2^2 \vec{J}^2(s) - \left( \frac{h_2 - c_2}{2P^0(b - a)} - \frac{b - a}{2} P^0 \vec{J}^2(s) \right) \right) ds. \quad (41)$$

Substituting (41) into (39) we obtain $\vec{L} = \vec{L}(t)$.

### 3 The case of relativistic particle with spin

In this section we will present in details the physical interpretation of Hamiltonian system (20-23) integrated in the previous section. We will restrict ourselves to the case when the deformation parameter $a \in [-1, 0]$, i.e. when $E_a(1, 3)$ is Poincare or Galilean group.
In our considerations we will use the twistor description of the space–time $E_{a,1,3}$, when $a = -1$. So, let us begin from some necessary facts concerning the twistor theory approach to the symplectic geometry of the relativistic particle phase spaces, for the details see [3].

Let us recall that the twistor space $T$ is $C^4$ equipped with the Hermitian form $\Phi$ of the signature $(+ + - -)$. The Grassmannian $G(2, T) =: M$ of the two–dimensional subspaces $z \subset T$ of the twistor space $T$ is the complexification $M^{C(1,3)}$ of the conformal compactification $M^{1,3}$ of the Minkowski space $M^{1,3}$, which in our notation corresponds to $E_{a,1,3}$, with $a = -1$. One can enumerate the orbits $M_{k,l}$ of the action of the conformal group $SU(2, 2)$ on $M$ by signatures $\text{sign} \Phi |_z =: (k, l)$ of the restrictions $\Phi |_z$ of twistor forms $\Phi$ to subspace $z \in M$. The orbit $M^{00}$ is identified with $M^{1,3}$.

In the following considerations we will take

$$\Phi = i \left( \begin{array}{cc} 0 & -\sigma_0 \\ \sigma_0 & 0 \end{array} \right),$$

assume

$$\infty := \left\{ \left( \begin{array}{c} \eta \\ 0 \end{array} \right) \in T : \eta \in C^2 \right\} \in M^{00},$$

$$o := \left\{ \left( \begin{array}{c} 0 \\ \xi \end{array} \right) \in T : \xi \in C^2 \right\} \in M^{00},$$

where $0 = \left( \begin{array}{ccc} 0 & 0 \\ 0 & 0 \end{array} \right)$, $\sigma_0 = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \in \text{Mat}_{2 \times 2}(C)$, and let us identify Lie algebra $su(2, 2)$ with its dual $su(2, 2)^*$ by

$$\langle X, \rho \rangle := \text{Tr} (X \rho), \quad \rho, X \in su(2, 2).$$

The decomposition $T = \infty \oplus o$ of the twistor space defines the corresponding decomposition of $su(2, 2) \subset \text{End} T$:

$$su(2, 2) \cong su(2, 2)^* = T_{\infty} \oplus L_{o,\infty} \oplus D_{o,\infty} \oplus A_o,$$

where $X \in su(2, 2)$ belongs to: the translation Lie subalgebra $T_{\infty} \subset su(2, 2)$ iff $\text{Im} X \subset o \subset \text{Ker} X$, the acceleration Lie subalgebra $A_{\infty}$ iff $\text{Im} X \subset o \subset \text{Ker} X$, while the Lorentz subalgebra $L_{o,\infty}$ and dilatation subalgebra $D_{o,\infty}$ are the commutant and centralizer of $su(2, 2)_o \cap su(2, 2)_\infty$, respectively. Using the pairing (44) we obtain the following vector space isomorphism:

$$T^*_{\infty} \cong A_o, \quad L^*_{o,\infty} \cong L_{o,\infty}, \quad D^*_{o,\infty} \cong D_{o,\infty}, \quad \text{and} \quad A^*_o \cong T_{\infty}.$$
Now, let \((\mathbb{P}, \omega)\) be \(SU(2, 2)\)–symplectic manifold and let \(\mathcal{J} : \mathbb{P} \longrightarrow su(2, 2)^* \cong su(2, 2)\) be the corresponding momentum map of the phase space \(\mathbb{P}\) into Lie–Poisson space \(su(2, 2)^*\). Matrix block notation consistent with the decomposition (45) allows us to express \(\mathcal{J}(p)\) at \(p \in \mathbb{P}\) as follows

\[
\mathcal{J}(p) = \begin{pmatrix}
\frac{1}{2}d(p)\sigma_0 + M(p) \\
\frac{1}{2}d(p)\sigma_0 - M^\dagger(p)
\end{pmatrix},
\]

where \(d(p)\) is the dilatation and

\[
P(p) = P^\mu(p)\sigma_\mu, \quad A(p) = A^\mu(p)\sigma_\mu, \quad M(p) = \frac{1}{2}M^0k\sigma_k + \frac{i}{2}M^{kl}\epsilon_{kn}\sigma_n,
\]

where \(L_k(p) = M^0k, J_k(p) = \epsilon_{kn}M^{ln}\), are the four–momentum, four–acceleration, and relativistic angular momentum, respectively. In (48) we use summation convention for \(\mu = 0, 1, 2, 3\) and \(k, l, n = 1, 2, 3\) and \(\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\) are the Pauli matrices. The Poincare Lie-Poisson space \(\mathcal{P}^*(1, 3) \cong T^*_\infty \oplus L^*_0\) is distinguished in \(su(2, 2)^*\) by \(d = 0\) and \(A = 0\). So, the momentum map \(\mathcal{J} : p \longrightarrow \mathcal{J}(p) \in \mathcal{P}^*(1, 3)\) of the phase space \(\mathbb{P}\) into \(\mathcal{P}^*(1, 3)\) is defined by \(\mathcal{J} : p \longmapsto (P(p), M(p)) \in \text{Mat}_{2 \times 2}(\mathbb{C}) \times \text{Mat}_{2 \times 2}(\mathbb{C})\). By \(\mathcal{P}(1, 3)\) we will denote Poincare–Lie algebra which corresponds to \(a = -1\).

For \((2 \times 2)\)–matrix calculus it is useful to introduce the following operation on \(B \in \text{Mat}_{2 \times 2}(\mathbb{C})\):

\[
\tilde{B} := \sigma_2B^\dagger\sigma_2.
\]

For example using this operation we arrive at the identities

\[
B\tilde{B} = \det B\sigma_0 \quad \tilde{A}\tilde{B} = \tilde{B}\tilde{A}
\]

and

\[
\det(A + B) = \det A + \det B + \text{Tr}\tilde{A}\tilde{B}.
\]

We define the Pauli–Lubansky four–vector \(W = W^\mu\sigma_\mu\) in the following way

\[
M\tilde{P} := R - iW,
\]

where \(R^\dagger = R\) and \(W^\dagger = W\).
Now we apply the formalism investigated in the above preliminaries to the description of the Hamilton dynamics given in Section 2 in terms of the Minkowski space geometry. Let us begin from the case of massless particle with non-zero helicity. The phase space of such particle is the manifold the positive defined projective twistors

\[ \mathbb{PT}^+ := \{ [v] \in \mathbb{CP}(3) : v^\dagger \Phi v > 0 \} , \]  

where \([v] := \mathbb{C}v\) is a one-dimensional complex subspace of \(T\) spanned by \(0 \neq v \in T\). The \(SU(2,2)\)-invariant symplectic form \(\omega_\alpha^+\) on \(\mathbb{PT}^+\) is the Kähler form

\[ \omega_\alpha^+ := i\alpha \delta \bar{\delta} \log v^\dagger \Phi v. \]  

In the subsequent we will use spinor coordinates \((\eta, \xi) \in \mathbb{C}^2 \times \mathbb{C}^2\) for the twistor \(v = (\eta, \xi) \in T\) defined by the decomposition \(T = \infty \oplus o\). After passing to the homogeneous coordinates \(\zeta_1 := \frac{\eta_1}{\xi_2}, \zeta_2 := \frac{\eta_2}{\xi_2}, \zeta := \frac{\xi_1}{\xi_2}\), where \(\xi_2 \neq 0\), we obtain the coordinate representation for symplectic form \((54)\):

\[ \omega_\alpha^+ = \frac{-i\alpha}{\Delta^2} \left( d\zeta_1 \ d\zeta_2 \ d\bar{\zeta}_1 \ d\bar{\zeta}_2 \right) \wedge \begin{pmatrix} -\zeta \bar{\zeta} & -\zeta & -\zeta \bar{\zeta}_1 + \zeta_2 - \bar{\zeta}_2 \\ -\bar{\zeta} & -1 & \bar{\zeta}_1 \\ \zeta_1 - \zeta_2 + \bar{\zeta}_2 & \zeta_1 & -\zeta_1 \zeta_1 \end{pmatrix} \begin{pmatrix} d\zeta_1 \\ d\zeta_2 \\ d\bar{\zeta}_1 \\ d\bar{\zeta}_2 \end{pmatrix}, \]  

where

\[ \Delta = v^\dagger \Phi v = i \left( \zeta \bar{\zeta}_1 - \zeta_1 \bar{\zeta} + \zeta_2 - \bar{\zeta}_2 \right). \]  

The momentum map \(J_\alpha^+ : (\mathbb{PT}^+, \omega_\alpha^+) \rightarrow su(2,2)^*\) for the symplectic manifold \((\mathbb{PT}^+, \omega_\alpha^+)\) is the following one

\[ J_\alpha^+ ([v]) = i\alpha \left( \frac{1}{4} - \frac{vv^\dagger \Phi}{\Delta} \right), \]  

and it leads to the formulas for four-momentum and relativistic angular momentum

\[ P([v]) = \frac{i\alpha}{\eta^\dagger \xi - \xi^\dagger \eta} \xi \xi^\dagger, \]  

\[ M([v]) = \frac{i\alpha}{\eta^\dagger \xi - \xi^\dagger \eta} \left( \eta \xi^\dagger - \frac{1}{2} \xi^\dagger \eta \sigma_0 \right). \]
Using (58) and (59) we obtain $P^\mu([v])$, $\bar{L}([v])$, $\bar{J}([v])$ and $W^\mu([v])$ in the $(\zeta_1, \zeta_2, \zeta)$–coordinate representation:

$$P^0 = -\frac{i\alpha}{2\Delta}(\zeta \bar{\zeta} + 1),$$

$$P^1 = -\frac{i\alpha}{2\Delta}(\zeta + \bar{\zeta}),$$

$$P^2 = -\frac{\alpha}{2\Delta}(\bar{\zeta} - \zeta),$$

$$P^3 = -\frac{i\alpha}{2\Delta}(\zeta \bar{\zeta} - 1),$$

$$L_1 = \frac{\alpha}{2\Delta}(\zeta_2 \bar{\zeta} + \zeta_2 \bar{\zeta} + \zeta_1 + \zeta_1),$$

$$L_2 = \frac{i\alpha}{2\Delta}(-\zeta_2 \bar{\zeta} + \zeta_2 \bar{\zeta} + \zeta_1 - \zeta_1),$$

$$L_3 = \frac{\alpha}{2\Delta}(\zeta_1 \bar{\zeta} + \zeta_1 \bar{\zeta} - \zeta_2 - \zeta_2),$$

$$J_1 = -\frac{i\alpha}{2\Delta}(\zeta \bar{\zeta} - \bar{\zeta} \zeta + \zeta_1 - \zeta_1),$$

$$J_2 = -\frac{\alpha}{2\Delta}(\bar{\zeta} \zeta + \bar{\zeta} \zeta - \zeta_1 - \zeta_1),$$

$$J_3 = -\frac{i\alpha}{2\Delta}(\zeta_1 \bar{\zeta} - \bar{\zeta}_1 \zeta - \zeta_2 + \zeta_2),$$

$$W^0 = \frac{i\alpha^2}{4\Delta}(\zeta \bar{\zeta} + 1),$$

$$W^1 = -\frac{i\alpha^2}{4\Delta}(\zeta + \bar{\zeta}),$$

$$W^2 = -\frac{\alpha^2}{4\Delta}(\bar{\zeta} - \zeta),$$

$$W^3 = -\frac{i\alpha^2}{4\Delta}(\zeta \bar{\zeta} - 1).$$

The momentum map (57) is a Poisson map from the symplectic manifold $(PT^+, \omega^+\alpha)$ into Lie–Poisson space $(P(1,3)^*, \{\cdot,\cdot\}_a)$, i.e. for $f, g \in C^\infty(P(1,3)^*)$ we have

$$\{f, g\}_a \circ J^+ = \{f \circ J^+, g \circ J^+\}_{a,+},$$

where Poisson bracket $\{\cdot,\cdot\}_{a,+}$ is defined by the symplectic form $\omega^+\alpha$. In the
coordinates \((\zeta_1, \zeta_2, \zeta)\) it takes the form

\[
\{F, G\}_{\alpha,+} = -\frac{\Delta}{\alpha} \left( \zeta_1 \left( \frac{\partial F}{\partial \zeta_2} \frac{\partial G}{\partial \zeta_1} - \frac{\partial F}{\partial \zeta_1} \frac{\partial G}{\partial \zeta_2} \right) \right. \\
+ \left. \left( \zeta_2 - \zeta \right) \left( \frac{\partial F}{\partial \zeta_1} \frac{\partial G}{\partial \zeta_2} - \frac{\partial F}{\partial \zeta_2} \frac{\partial G}{\partial \zeta_1} \right) + \frac{\partial F}{\partial \zeta} \frac{\partial G}{\partial \zeta} - \left( \frac{\partial F}{\partial \zeta} \frac{\partial G}{\partial \zeta} \right) \right) + \zeta \left( \frac{\partial F}{\partial \zeta} \frac{\partial G}{\partial \zeta} - \frac{\partial F}{\partial \zeta} \frac{\partial G}{\partial \zeta} \right) + \zeta \left( \frac{\partial F}{\partial \zeta} \frac{\partial G}{\partial \zeta} - \frac{\partial F}{\partial \zeta} \frac{\partial G}{\partial \zeta} \right) \right),
\]

for \(F, G \in C^\infty(\mathbb{P}T^+)\). The four-momentum \(P^\mu([v])\) and Pauli–Lubansky vector \(W^\mu([v])\) defined in (58), (59) and (52) satisfies the relationships

\[
\eta_{\mu\nu} P^\mu([v]) P^\nu([v]) = 0 \quad \text{and} \quad W^\mu([v]) = \frac{\alpha}{2} P^\mu([v]).
\]

The above conditions confirm that we are in the case of the massless particle with the helicity \(\frac{\alpha}{2}\) which phase space is given by the symplectic leaves of the Lie–Poisson bracket \(\{\cdot, \cdot\}_a\) defined by the conditions \(c_1 = c_2 = 0\). So, one can pull back the solution \(\left(\vec{P}(t), \vec{L}(t)\right)\) of Hamilton equations (20-23) on the symplectic manifold \((\mathbb{P}T^+, \omega^+_a)\) by the map

\[
\zeta_1 = \frac{L_2 - J_1 + i(L_1 + J_2)}{2(P^3 - P^0)},
\]
\[
\zeta_2 = \frac{(P^1 + iP^2)(L_2 - J_1 + i(L_1 - J_2)) - (P^3 - P^0)(J_3 - iL_3)}{-2(P^3 - P^0)^2},
\]
\[
\zeta = \frac{P^1 - iP^2}{P^0 - P^3}.
\]

So, if \(P^0(t), \vec{P}(t), \vec{L}(t), \vec{J}(t)\) satisfy equations (20-23) then \(\zeta_1(t), \zeta_2(t), \zeta(t)\),
given by (77), are the solution of Hamilton equations
\[
\frac{d}{dt} \zeta_1(t) = \{ \zeta_1, h \circ \mathcal{J}_\alpha^+ \}_{\alpha,+} = -\frac{\alpha}{4\Delta} (b - a) (\zeta \zeta + 1) \left( c - \frac{d\alpha^2}{4} \right) \zeta - \frac{\alpha^2}{4\Delta^2} (b - a) (-6\zeta_2 \zeta_2 \zeta^2 \zeta + 4\zeta_1 \zeta_2 \zeta \zeta - 4\zeta_1 \zeta_1 \zeta^2 \zeta + 4\zeta_1 \zeta_2 \zeta \zeta + 3\zeta_1^2 \zeta^2 + 2\zeta_1 \zeta_2 \zeta^2 - 4\zeta_2 \zeta_2 \zeta + \zeta_1^2 \zeta^3 + \zeta_2^2 \zeta + \zeta_2^2 \zeta + 2\zeta_1^2 \zeta + 2\zeta_1 \zeta_2 \zeta^2 \zeta + 2\zeta_1 \zeta_1 \zeta^2 \zeta \right),
\]
\[
\frac{d}{dt} \zeta_2(t) = \{ \zeta_2, h \circ \mathcal{J}_\alpha^+ \}_{\alpha,+} = \frac{i}{2} (b - a) P^0 \left( c - \frac{d\alpha^2}{4} + d(b - a) \right) \left( \zeta \zeta - 1 \right) + \frac{\alpha d}{4\Delta} (b - a)^2 (P^0)^2 \left( -2\zeta_1 \zeta_1 \zeta \zeta - 2\zeta_1 \zeta_1 \zeta \zeta - 4\zeta_2 \zeta_2 \zeta \zeta - 2\zeta_2 \zeta_2 \zeta \zeta - 2\zeta_1 \zeta_2 \zeta \zeta - \zeta_2 + \zeta_2^2 + \zeta_2^2 \zeta + 2\zeta_1 \zeta_2 \zeta + \zeta_2^2 \zeta^2 + \zeta_2^2 \zeta^2 + 2\zeta_2 \zeta - 2\zeta_1 - \zeta_1 \zeta + \zeta_1 \zeta - \zeta_2 \right),
\]
\[
\frac{d}{dt} \zeta(t) = \{ \zeta, h \circ \mathcal{J}_\alpha^+ \}_{\alpha,+} = -\frac{\alpha^3}{8\Delta^3} d(b - a)^2 (\zeta \zeta + 1)^3 (\zeta_2 \zeta - \zeta_1),
\]
derived on \((\mathbb{P}\mathbb{T}^+, \omega_\alpha^+)\) by the Hamiltonian \(h \circ \mathcal{J}_\alpha^+\), where \(h\) is given by (18). Equations (78-80) prove to form a system with rational nonlinearities. Due to (76), it follows from the above considerations that after passing to the coordinates \((\vec{P}, \vec{W})\) we linearize the hamiltonian system under consideration.

In order to describe the phase space of a massive particle with the spin \(s \neq 0\) let us consider twistor flag space
\[
\mathbb{F} := \{ ([v], z) \in \mathbb{P}\mathbb{T} \times \mathbb{M} : [v] \subset z \}.
\]
Similarly to the case of Grassmannian \(\mathbb{M}\) we will enumerate the orbits \(\mathbb{F}^{k,lm}\) of the natural action of \(SU(2,2)\) on \(\mathbb{F}\) by the signatures \(k = \text{sign } \Phi \mid [v]\), \(lm = \text{sign } \Phi \mid z\) of the restrictions of twistor form to the flag \([v] \subset z\). We restrict our interest to the orbit \(\mathbb{F}^{+,++}\) consisting of the positive flags. For physical interpretation of the flag spaces of different signatures, see [3]. One has the following double fibration
\[
\begin{array}{c}
\mathbb{F}^{+,++} \\
\downarrow \sigma_1 \\
\mathbb{P}\mathbb{T}^+ \\
\downarrow \sigma_2 \\
\mathbb{M}^{++}
\end{array}
\]
of \(\mathbb{F}^{+,++}\) over \(\mathbb{P}\mathbb{T}^+\) and \(\mathbb{M}^{++}\). According to [3] we show now that \(\mathbb{M}^{++}\) is
the phase space of massive spinless particle and \( F^{+,+} \) is the phase space of a massive particle with non-zero spin. For these reasons let us pass to the coordinate description of \( M^{++,} \) and \( F^{+,+} \) consistent with the decomposition \( T = \infty \oplus o \) given by (43). We have

\[
v = \begin{pmatrix} Z \xi \\ \xi \end{pmatrix} \quad \text{and} \quad z = \left\{ \begin{pmatrix} Z \xi \\ \xi \end{pmatrix} : \xi \in \mathbb{C}^2 \right\}
\]  

(82)

for \([v] \subset z\), where \( Z \in \text{Mat}_{2\times 2}(\mathbb{C})\). The flag \([v] \subset z\) belongs to \( F^{+,+} \) iff the "imaginary" part of \( Z = X + iY, \quad X^\dagger = X \) and \( Y^\dagger = Y \), is positive definite, i.e.

\[
\det Y > 0 \quad \text{and} \quad \text{Tr} \ Y > 0.
\]  

(83)

Let us take the product \( \mathbb{P}T^+ \times \mathbb{P}T^+ \) of the one–twistor phase spaces with the symplectic form

\[
\omega^{12} = \pi_1^* \omega_{\alpha_1}^+ \oplus \pi_2^* \omega_{\alpha_2}^+;
\]  

(84)

where \( \pi_i : \mathbb{P}T^+ \times \mathbb{P}T^+ \to \mathbb{P}T^+ \) is the projection on the \( i \)-th component of the product. The symplectic form \( \omega^{12} \) is invariant with respect to the natural action of \( SU(2, 2) \) on \( \mathbb{P}T^+ \times \mathbb{P}T^+ \) and the momentum map \( J^{1,2} : \mathbb{P}T^+ \times \mathbb{P}T^+ \to su(2, 2)^* \) for \( (\mathbb{P}T^+ \times \mathbb{P}T^+, \omega^{12}) \) is given by

\[
J^{1,2} = J_{\alpha_1}^+ \circ \pi_1 + J_{\alpha_2}^+ \circ \pi_2.
\]  

(85)

The function \( s^2 : \mathbb{P}T \times \mathbb{P}T \to \mathbb{R} \) defined by

\[
s^2([v_1], [v_2]) := \left( \frac{\alpha_1 - \alpha_2}{4} \right)^2 + \frac{\alpha_1 \alpha_2}{4} \frac{|v_1^\dagger \Phi v_2|^2}{v_1^\dagger \Phi v_1 v_2^\dagger \Phi v_2}
\]  

(86)

in an invariant of the conformal group \( SU(2, 2) \). The projective twistors are orthogonal \([v_1] \perp [v_2]\) with respect to the twistor form \( \Phi \) iff

\[
s^2([v_1], [v_2]) := \left( \frac{\alpha_1 - \alpha_2}{4} \right)^2.
\]  

(87)

Any flag \([v] \subset z\) one can identify with the pair of twistors \(([v_1], [v_2]) \in \mathbb{P}T^+ \times \mathbb{P}T^+ \) satisfying condition (87). Namely, one puts \( z = \text{span}\{[v_1], [v_2]\} \in M \) and \([v] = [v_1]\). Reducing symplectic form \( \omega^{1,2} \) to the level submanifold of \( \mathbb{P}T^+ \times \mathbb{P}T^+ \) defined by (87) we obtain \( SU(2, 2) \)– invariant symplectic form
(Kähler form) ω_{s,δ} on \mathbb{F}^{+,++} which in the coordinates ([ξ], Z) ∈ \mathbb{CP}(1) × Mat_{2×2}(\mathbb{C}) is given by

\[ \omega_{s,δ} = i\partial\overline{\partial}\log \left( \left( \det (Z - Z^\dagger) \right)^{s+2\delta} \left( \eta^\dagger (Z - Z^\dagger) \eta \right)^{4s} \right), \]

(88)

where \( s := \frac{α_1 - α_2}{4} \) and \( δ := -\frac{α_1 + α_2}{4} \). The symplectic form (88) rewrited in the variables \( P^μ, W^μ, X^μ \) is the Souriau symplectic form, see [7].

The reduced momentum map \( J_{s,δ} : \mathbb{F}^{+,++} → su(2,2)^* \) is of the form

\[ J_{s,δ}(ξ, Z) = \left( \begin{array}{cc} ZP - iδσ_0 & -ZPZ^\dagger \\ P & -PZ^\dagger - iδσ_0 \end{array} \right), \]

(89)

where

\[ P = -\frac{iα_1}{ξ^\dagger (Z - Z^\dagger)ξ} ξ^\dagger - \frac{iα_2}{\det(Z - Z^\dagger)ξ^\dagger (Z - Z^\dagger)ξ} \left( \widetilde{Z} - \widetilde{Z}^\dagger \right)ξ^\dagger (\widetilde{Z} - \widetilde{Z}^\dagger). \]

(90)

Comparing (47) with (89) we obtain

\[ M = ZP - \frac{1}{2} \text{Tr}(ZP)σ_0. \]

(91)

Substituting (90) into (91) we also express \( M \) in the coordinates ([ξ], Z = X + iY):

\[ M = -\frac{iα_1}{ξ^\dagger (Z - Z^\dagger)ξ} Zξ^\dagger + \frac{iα_1}{2ξ^\dagger (Z - Z^\dagger)ξ} ξ^\dagger Zξσ_0 - \]

\[ -\frac{iα_2}{\det(Z - Z^\dagger)ξ^\dagger (Z - Z^\dagger)ξ} \left( \widetilde{Z} - \widetilde{Z}^\dagger \right)ξ^\dagger (\widetilde{Z} - \widetilde{Z}^\dagger)^* + \]

\[ + \frac{iα_2}{2det(Z - Z^\dagger)ξ^\dagger (Z - Z^\dagger)ξ} \xi^\dagger (Z - Z^\dagger)σ_2 Zσ_2 (Z - Z^\dagger)ξσ_0. \]

(92)

From equalities (90,91), definition (52) and identities (50) we find that

\[ W = -\frac{iδα_1}{ξ^\dagger (Z - Z^\dagger)ξ} \tilde{ξ}^\dagger - \frac{iα_1α_2}{2det(Z - Z^\dagger)} (Z - Z^\dagger) - \]

\[ -\frac{iδα_2}{\det(Z - Z^\dagger)ξ^\dagger (Z - Z^\dagger)ξ} (Z - Z^\dagger)ξ^\dagger (Z - Z^\dagger). \]

15
and
\[
\begin{align*}
\text{Tr } PW &= 0, \\
\det W &= -s^2 \det P, \\
\text{Tr } PY &= 2\delta.
\end{align*}
\] (94)

Using vector notation for $M$, see (48), we can rewrite (91) as follows
\[
\vec{L} = X_0 \vec{P} + P^0 \vec{X} - \vec{Y} \times \vec{P},
\] (95)
\[
\vec{J} = Y_0 \vec{P} + P^0 \vec{Y} + \vec{X} \times \vec{P}.
\]

Inverting (91) we find
\[
Y_0 = -\frac{1}{\det P} \left( W^0 - \delta P^0 \right),
\] (96)
\[
\vec{Y} = -\frac{1}{\det P} \left( \vec{W} + \delta \vec{P} \right),
\] (97)
\[
\vec{X} = \frac{1}{\det P} \vec{J} \times \vec{P} + \frac{P_0}{\det P} \vec{L} - \frac{1}{P^0 \det P} \left( \left( \vec{P} \cdot \vec{L} \right) + \det PX_0 \right) \vec{P}.
\] (98)

The formula given above allows us to obtain the time evolution $Y_0 = Y_0(t)$, $\vec{Y} = \vec{Y}(t)$ and $\vec{X} = \vec{X}(t)$ described by the Hamiltonian (19). For this reason we only need to assume that the evolution parameter $t$ appearing in the Hamilton equations (20-23) is the time related to the space–time coordinate $X_0$ by $X_0 = ct$, where $c$ is the light velocity. From (98) we have
\[
\vec{X} = \frac{1}{(mc)^2} \vec{J} \times \left( \vec{P}(t) - \frac{P^0}{W^0(t)} \vec{W}(t) \right) + \]
\[
+ \left( ct + \frac{1}{(a-b)dP^0 dt} \ln W^0(t) + (mc)^2 \xi(t) \right) \frac{\vec{P}(t)}{P^0},
\] (99)

where $m$ is the relativistic particle mass defined by $-(mc)^2 = c_1$, $cP^0$ and $\vec{J}$ are its energy and angular momentum, being integral of motions in the case under consideration. Note that $\xi(t)$ is expressed in terms of $W^0(t)$ in (41) while elliptic function $W^0(t)$ is defined in (31). Time evolution of the momentum $\vec{P}(t)$ and spin $\vec{W}(t)$ is also given in terms of elliptic function $W^0(t)$, see formula (34-37).

The physical sense of the integrated Hamiltonian dynamics one can recognize from the form of the total energy function (19)
4 Concluding remarks

The hamiltonian integrable system investigated in the paper describes time evolution of the relativistic particle (massive and massless) with non–zero spin. The Hamiltonian governing this evolution is invariant with respect to the translation of Minkowski space-time and the rotation of the space, which means that the particle does not interact with the external field. However, the dynamics has rather complicated nonlinear character that is caused by the coupling between the momentum, angular momentum and spin of the particle.

The Galilean case we do not discuss separately. It can be obtained from relativistic one in the $a$–vanishing limit.

The Euclidean case needs an approach different from the twistor one. Here we can consider four-vectors $W^\mu$ and $P^\mu$ as a dynamical variables which satisfy the conditions (10), (11) and (14). As a result one obtains a hamiltonian system on the bundle $T\mathbb{S}^3$ of vector spaces tangent to three–dimensional sphere $\mathbb{S}^3$. In the paper we do not discuss this case in details.

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