In this short report, a brief introduction to Arkani-Hamed, Cohen, Georgi model (ACG-model, (de)constructing dimensions model), whose main characters are that extra-dimensional space-time are generated dynamically from a four-dimensional gauge theory and that extra dimensions are lattices, will be given first. Then after a concise review of NCG on cyclic groups, actions for gauge fields along extra dimensions will be constructed by virtue of NCG and classical (vacuum) solutions will be solved, with low energy phenomenology being classified accordingly. As a conclusion, the behavior of spontaneous symmetry broken within ACG-model can be determined by noncommutative Yang-Mills theory.

PACS: 11.10.Kk, 11.15.Ex, 02.40.Gh

Keywords: (de)constructing dimensions, spontaneous symmetry broken, noncommutative geometry, Yang-Mills action, cyclic group

1 Arkani-Hamed, Cohen, Georgi (ACG) model: (de)constructing dimensions

1.1 Motivations

The intuition of the existence of some silent extra dimensions has been renewed time to time; see literatures listed in [1] for example. However as a quantum field theory in its usual formalism, this idea suffers the problem of dimensional couplings; namely, the perturbative series is \textit{out of control} once the energy scale approaches upwards the cutoff. One outlet, being reasonable from both technical and physical senses, is to \textit{complete} the higher-dimensional theory with a more fundamental theory at its ultraviolet end, with or without gravity being included. To be another candidate of these \textit{UV-completions}, ACG-model is characterized as being fully \textit{under control}; in out understanding, this feature is the initial, if not unique, motivation of this speculation from ACG.

1.2 Pictures

Suppose that the \textit{Platonic} physics without including gravity is a four-dimensional quantum gauge theory. The gauge group is of a direct product form $\prod_{i=0}^{N-1} G_i \otimes \prod_{i=0}^{N-1} G'_i$ where $N$ is a generic (large) integer. A collection of Weyl fermions
transform according to $\chi^i \sim R_{def}(G^i) \otimes \overline{R}_{def}(G^i)$, $\xi^i \sim R_{def}(G^i) \otimes \overline{R}_{def}(G^{i+1})$, $i = 0, 1, 2, \ldots, N - 1$ in which $R_{def}(G)$ is the definition representation of $G$ and $\overline{R}$ is the complex conjugate representation of a representation $R$. Note that $i + 1$ is computed modulo $N$, which is taken to be a convention henceforth in corresponding context. Let $G^i = U(m), G^i_s = SU(n)$ for all $i = 0, 1, 2, \ldots, N - 1$ with generic $m, n$, where the choice of $G^i$ here is enlarged compared with the original ACG’s consideration. A cyclic symmetry is assumed to set gauge couplings of $G^i_s$ to be $g_s$ and those of $G^i$ to be $g_d$ and $g$ for the center subgroups and the quotients respectively. $g_d$ is supposed to be weak for all energy scales that we are interested. By the mechanism of dimensional transmutation, the gauge couplings $g$, $g_s$ can be described as well by two energy scales $\Lambda$, $\Lambda_s$ respectively; here it is required that $\Lambda_s \gg \Lambda$.

When energy scale is lower downwards $\Lambda_s$, gauge interactions of $G^i_s$ become strong and fermion condensates happen in a way that $\langle \chi^i \xi^i \rangle \sim 4\pi f_s^3 U_{i,i+1}$ where $f_s$ is a decay constant of mass dimension and $U_{i,i+1}$ is $\sigma$-model scalar transformed as $U_{i,i+1} \sim R_{def}(G^i) \otimes \overline{R}_{def}(G^{i+1})$. Below, let $m = 1$ for simplicity hence the subscripts “d” are omitted. The low energy effective theory is read as

$$S = \int d^4x \sum_{i=0}^{N-1} \left( -\frac{1}{4g^2} F^i \cdot F^i + f_s^2 (D^\mu U_{i,i+1})^\dagger D_\mu U_{i,i+1} + \text{(irrelevant terms)} \right)$$

where the covariant derivative is defined to be $D_\mu U_{i,i+1} = \partial_\mu U_{i,i+1} - iA^i_\mu U_{i,i+1} + iU_{i,i+1} A^i_{\mu+1}$. The center observation of ACG is that action Eq.(1) is on the other hand a five-dimensional gauge theory where the fifth direction is a periodic lattice with $N$ sites. The following geometric and physical quantities can be computed, as low energy phenomenology provided by ACG-model. Lattice spacing $a = 1/\sqrt{2gf_s}$, circumference $R = Na$ and five dimensional gauge coupling $g_5 = ag^2$. The fluctuations $U_{i,i+1}$ higgs gauge symmetry down to a diagonal subgroup and mass square spectral of gauge bosons are

$$M_k = 8g^2 f_s^2 \sin^2(\pi k/N)$$

where $k = 0, 1, \ldots, N - 1$, and diagonal gauge coupling is $g_4^2 = g_5^2/R$.

The above-mentioned observation also triggered our intuition to utilize noncommutative geometry (NCG) method to study scalar potential in ACG-model because the dynamics of Yang-Mills field is geometric and the (differential) geometry of lattices is a specification of NCG. In fact, the first attempt to adopt NCG in ACG-model was explored by M. Alishahiha whose concern is concentrated mainly on gravity.
2 Noncommutative Yang-Mills theory on cyclic groups

2.1 Noncommutative geometry on cyclic groups

A comprehensive introduction to NCG can be found in [4]. NCG on a cyclic group can be characterized by a generalized Dirac operator. Let $\mathbb{Z}_N$ be $N$-order cyclic group, then Dirac operator on $\mathbb{Z}_N$ is defined to be $D[\omega] = \bar{\omega}Tb^\dagger + T^\dagger b\omega$ in which $T$ is the induced translation acting on functions over $\mathbb{Z}_N$, $b, b^\dagger$ is a pair of fermionic annihilation/creation operators, and $\omega$ is so-called link variable being parametrized by $\omega = \rho e^{i\theta}$. Under a gauge transformation $U$, which is a unitary function on $\mathbb{Z}_N$, $\omega \rightarrow (TU)\omega U^\dagger$. Two gauge covariants, which are the candidates of generalized field strength, are defined by

$$F_1 := D[\omega]^2 = T^\dagger(\phi)bb^\dagger + \phi b^\dagger b, F_2 := \partial^\dagger(\phi)b \wedge b^\dagger =: D[\omega] \wedge D[\omega] \quad (3)$$

in which $\phi := \rho^2$, and the formal partial derivative $\partial := T - I d$.

2.2 Dynamics of Dirac operator

Thanks for ACG’s observation, the following identifications can be carried out. Firstly, the $N$-site lattice generated dynamically in ACG-model is identified with $\mathbb{Z}_N$; secondly, $U_{i,i+1}$ is identified with $\omega$; last and most importantly, scalar potential in action (1) is identified with Yang-Mills action on $\mathbb{Z}_N$. Below three types of Yang-Mills actions will be constructed.

$$V_I = Tr(F_1^2)$$

After a representation for $b, b^\dagger$ being specified, $V_I \sim \sum_{i \in \mathbb{Z}_N} \phi(i)^2$, whose equation of motion admits only zero solution obviously; consequently, no spontaneous symmetry broken happens under this construction.

$$V_{II} = Tr(F_2^2)$$

Introduce a lattice laplacian $\Delta = \partial \partial^\dagger$, then

$$V_{II} \sim \sum_{i \in \mathbb{Z}_N} \phi(i)(\Delta \phi)(i)$$

Equation of motion is read as

$$\rho \Delta \phi = 0$$

whose nontrivial solutions are constant functions. This fact can be understood schematically by the extremal value principle in commutative harmonic analysis. This class of solutions describes flat directions, because for any of them the potential energy equals to zero.
$$V_{III} = \frac{1}{2} Tr(F_2^2 + \frac{\alpha}{2} F_1^2 - \frac{\beta}{2} F_1)$$

where the coupling constants $\alpha \geq 0$. Note that the last term in $V_{III}$ is called Sitarz term, which vanishes in continuum limit if there exists such a limit. The equation of motion is given by

$$\rho(2t \phi - T(\phi) - T(\phi) - \beta) = 0$$

where $t := (2 + \alpha)/2 \geq 1$. In any circumstance, there exists at least a zero solution. It is easy to see that if $\beta < 0, \alpha \geq 0$ or $\beta = 0, \alpha > 0$, the zero solution is the minimum of $V_{III}$, which implies that no SSB happens. So below only case $\beta > 0, \alpha \geq 0$ will be considered.

Translation invariant solution (ACG phase) $\phi(i) = \frac{\beta}{\alpha}, \forall i \in \mathbb{Z}_N, \alpha > 0$; no solution exists for $\alpha = 0$ in this case. Energy is computed to be $V_{III} = -\frac{N \beta^2}{\alpha} =: V_{inv}^N$. Accordingly, the same ACG type of SSB pattern is found out as shown in Eq.(2).

Translation non-invariant solution (nontrivial phase) $\phi = \beta v$,

$$v(i) = \sum_{j=0}^{N-2-i} f_{i,i+j}(t) \psi_{i+j}(t), i = 0, 1, ..., N - 2; v(N - 1) = 0$$

in which $f_{i,j}(t) = \frac{U_i(t)}{U_j(t)}$, $\psi_n = \sum_{n=0}^n U_j(t)$, $\Sigma_n(t) = \sum_{j=0}^n U_j(t)$. $U_n(t)$ are Chebyshev's polynomials of the second kind whose roots are all real and lie in $(-1, 1)$; for all $n$, $U_n(t) > 0$ if $t \geq 1$.

Energy of low energy phenomenology of nontrivial phase, $\alpha > 0$. For $N = 3$, $V_{III} = -\frac{3 \alpha^3 + 11 \alpha^2 + 23 \alpha + 5}{\alpha^2 + 4 \alpha + 2} \beta^2 \left\{ \begin{array}{ll}
> V_{inv}^N, & \alpha \in (0, (\sqrt{73} - 5)/3) \\
= V_{inv}^N, & \alpha = (\sqrt{73} - 5)/3 \\
< V_{inv}^N, & \alpha > (\sqrt{73} - 5)/3
\end{array}\right.$

Gauge boson mass square matrix is given by $M^2 \sim$

$$\begin{pmatrix}
\phi(0) & -\phi(0) & 0 & 0 & ... & 0 & 0 \\
-\phi(0) & \phi(0) + \phi(1) & -\phi(1) & 0 & ... & 0 & 0 \\
0 & -\phi(1) & \phi(1) + \phi(2) & -\phi(2) & ... & 0 & 0 \\
... & ... & ... & ... & ... & ... & ... \\
0 & 0 & 0 & 0 & ... & \phi(N - 3) + \phi(N - 2) - \phi(N - 2) \\
0 & 0 & 0 & 0 & ... & -\phi(N - 2) & \phi(N - 2)
\end{pmatrix}$$

which is semi-positive-definite and has only one zero mode. For $N = 3$, the eigenvalues are $m^2 \sim (0, 1, 3) \frac{\beta}{1+\alpha}$ vs $(0, 3, 3)\beta/\alpha$ for ACG phase according to
Eq. (3): for $N = 4$, $m^2 \sim (0, 2, 1 + r + \sqrt{1 + r^2}, 1 + r - \sqrt{1 + r^2})\phi(0)$ where $\phi(0) = \beta(\alpha + 3)/(\alpha^4 + 4\alpha^2 + 2)$, $r = 1 + 1/(3 + \alpha)$, vs $(0, 2, 2, 4)\beta/\alpha$ for ACG phase.

3 Conclusion

In general, different actions from NCG give rise to different low energy phenomena. With $V_I$, no SSB happens; with $V_{II}$, there appears a random balance; for ACG phase from $V_{III}$, scenario of ACG is recovered; nontrivial phase from $V_{III}$ is a distinctively novel pattern, which will be the real vacuum in strong coupling range of $\alpha$ when $N = 4$.

Acknowledgments

This work was supported by Climb-Up (Pan Deng) Project of Department of Science and Technology in China, Chinese National Science Foundation and Doctoral Programme Foundation of Institution of Higher Education in China.

References

1. T. Kaluza, *Sitz. Preuss. Akad. Wiss. K* 1, 966 (1921); O. Klein, *Z. Phys.* 37, 895 (1926); N. S. Manton, *Nucl. Phys.* B 158, 141 (1979); A. Connes and J. Lott, *Nucl. Phys. Proc. Suppl.* B 18, 29 (1990); N. Arkani-Hamed *et al., Phys. Lett.* B 429, 263 (1998); *ibid, Phys. Rev.* D 59, 086004 (1999); I. Antoniadis *et al., Phys. Lett.* B 436, 257 (1998); L. Randall and R. Sundrum, *Phys. Rev.* D 83, 3370 (1999); *ibid, Phys. Rev. Lett.* 83, 4690 (1999).
2. N. Arkani-Hamed *et al., Phys. Rev. Lett.* 86, 4757 (2001).
3. M. Alishahiha, *Phys. Lett.* B 517, 406 (2001).
4. J. Madore, *An Introduction to Noncommutative Differential Geometry and its Physical Applications*, (University Press, Cambridge, 1995); J. M. Gracia-Bondía *et al, Elements of Noncommutative Geometry*, (Birkhäuser, Boston, 2001).
5. L-S. Jiang *et al, Lectures on Mathematical Physics Equations, 2nd Edition*, (Higher Education Press, Beijing, 1996).
6. A. Sitarz, *Phys. Lett.* B 308, 311 (1993).
7. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, (Academic Press, New York, 1980).Commun. Theo. Phys., gr-qc/0102028.