EFFICIENT STORAGE OF PARETO POINTS IN BIOBJECTIVE MIXED INTEGER PROGRAMMING

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ABSTRACT. Biobjective mixed integer linear programs (BOMILP) are optimization problems where two linear objectives are optimized over a polyhedron while restricting some of the variables to be integer. Since many of the techniques for solving BOMILP (or approximating its solution set) are iterative processes which utilize data discovered during early iterations to aid in the discovery of improved data during later iterations, it is highly desirable to efficiently store the nondominated subset of a given set of data. This problem has not received considerable attention in the context of BOMILP; only naive methods have been implemented. We seek to bridge this gap by presenting a new data structure in the form of a modified binary tree that stores, updates, searches and returns nondominated solutions. This structure takes points and line segments in \( \mathbb{R}^2 \) as input and stores the nondominated subset of this input. We note that when used alongside an exact solution procedure, such as branch-and-bound (BB), at termination the data stored by this structure is precisely the set of Pareto optimal solutions. We perform two experiments. The first is designed to compare the utility of our structure for storing nondominated data to that of a dynamic list which updates via pairwise comparison. In the second we use our data structure alongside the biobjective BB techniques available in the literature and solve specific instances of BOMILP. The results of our first experiment suggest that the data structure performs reasonably well in handling input of up to \( 10^7 \) points or segments and does so much more efficiently than a dynamic list. The results of the second experiment show that when our structure is utilized alongside BB fathoming is enhanced and running times improve slightly.

1. INTRODUCTION

Biobjective mixed integer linear programs (BOMILP) have the following form,

\[
\begin{align*}
\min_{x,y} & \quad f(x,y) := [f_1(x,y) := c_1^\top x + d_1^\top y, \ f_2(x,y) := c_2^\top x + d_2^\top y] \\
\text{s.t.} & \quad (x,y) \in P_T := \{(x,y) \in \mathbb{R}^m \times \mathbb{Z}^n : Ax + By \leq b\} 
\end{align*}
\]

(1)

where \( P_T \) is a bounded set. Thus BOMILP encompasses both biobjective linear programs (BOLP) and biobjective integer programs (BOIP). Define \( \Omega := \{\omega \in \mathbb{R}^2 : \omega = f(x,y) \ \forall (x,y) \in P_T\} \), the collection of all points in \( \mathbb{R}^2 \) which can be obtained using the objective function values of feasible solutions to (1). We refer to the space \( \mathbb{R}^2 \) containing \( \Omega \) as the objective space.

Unlike single-objective programs, one cannot expect to find a single optimal solution to biobjective programs since the objective functions are oftentimes conflicting. Instead, a set of efficient solutions which offer an acceptable compromise between the objectives is sought. In order to determine what types of solutions are “acceptable,” we provide several notations and definitions. For any two vectors \( v^1, v^2 \in \mathbb{R}^2 \) we use the following notation: \( v^1 \leq v^2 \) if \( v_i^1 \leq v_i^2 \) for \( i = 1, 2 \); \( v^1 \leq v^2 \) if \( v^1 \leq v^2 \) and \( v^1 \neq v^2 \); and \( v^1 < v^2 \) if \( v_i^1 < v_i^2 \) for \( i = 1, 2 \). Given distinct \((\overline{x}, \overline{y}), (x', y')\) \( \in P_T \), we say that \( f(\overline{x}, \overline{y}) \) dominates \( f(x', y') \) if \( f(\overline{x}, \overline{y}) \leq f(x', y') \). This dominance is strong if \( f(\overline{x}, \overline{y}) < f(x', y') \); otherwise it is weak. A point \((\overline{x}, \overline{y}) \in P_T \) is (weakly) efficient if \( \not\exists \) \((x', y') \in P_T \) such that \( f(x', y') \) (strongly) dominates \( f(\overline{x}, \overline{y}) \). The set of all efficient solutions...
in $P_I$ is denoted by $X_E$. A point $\varpi = f(\overline{x}, \overline{y})$ is called Pareto optimal if $(\overline{x}, \overline{y}) \in X_E$. Given $\Omega' \subseteq \Omega$ we say that $\omega' \in \Omega'$ is nondominated in $\Omega'$ if for no $\omega'' \in \Omega'$ such that $\omega''$ dominates $\omega'$. Note that Pareto optimal points are nondominated in the set of Pareto optimal points $\Omega = \{\omega \in \mathbb{R}^2 : \omega = f(x, y) \forall (x, y) \in X_E\}$ is found. Note that the definitions given here extend to problems with more than two objectives, but we give them for biobjective problems since that is our focus for this paper.

It is well-known [cf. 4] that a BOLP can be solved by taking convex combinations of $f_1(\cdot)$ and $f_2(\cdot)$ and solving a finite number of LPs. Thus for BOLP, the set of Pareto points can be characterized as $\Omega_P = \{(f_1, f_2) \in \mathbb{R}^2 : f_2 = \psi(f_1)\}$ where $\psi(\cdot)$ is a continuous convex piecewise linear function obtained using extreme points of the dual feasible region. Similarly, for BOIP it is known that $\Omega_P$ is a finite set of discrete points in $\mathbb{R}^2$. Now consider the case of BOMILP. Let $Y = \text{Proj}_y P_I$ be the set of integer feasible subvectors to (1). Since $P_I$ is bounded, we have $Y = \{y^1, \ldots, y^k\}$ for some finite $k$. Then for each $y^i \in Y$ there is an associated BOLP, referred to as a slice problem and denoted $\mathbb{P}(y^i)$, obtained by fixing $y = y^i$ in (1).

$$\mathbb{P}(y^i) \quad \min_x \{f_1(x) = c_1^T x + d_1^T y^i, \quad f_2(x) = c_2^T x + d_2^T y^i\}$$

s.t. $Ax \leq b - By^i$ \quad (2)

Problem $\mathbb{P}(y^i)$ has a set of Pareto solutions $S_i := \{(f_1, f_2) \in \mathbb{R}^2 : f_2 = \psi_i(f_1)\}$, where $\psi_i(\cdot)$ is a continuous convex piecewise linear function as explained before. Then $\Omega_P \subseteq \bigcup_{i=1}^k S_i$ and this inclusion is strict in general. In particular, we have:

$$\Omega_P = \bigcup_{i=1}^k \left( S_i \setminus \bigcup_{j \neq i} \left( S_j + \mathbb{R}_+^2 \setminus \{0\} \right) \right) \quad (3)$$

Such union of sets is not, in general, represented by a convex piecewise linear function. Figure 1 shows an example with $k = 4$.

It should be noted that finding $\Omega_P$ is not a trivial task in general. In the worst case, $\Omega_P = \bigcup_{i=1}^k S_i$ and one may have to solve every slice problem to termination, which can have exponential complexity. For multiobjective IP’s (i.e. $m = 0$), De Loera et al. [3] prove that $\Omega_P$ can be enumerated in polynomial-time for fixed $n$, which extends the well known result that single-objective IP’s can be solved in polynomial-time for fixed $n$. We are unaware of any similar results for BOMILP.

Not many exact procedures have been presented for solving BOMILP with general integers. The works of Belotti et al. [1] and Boland et al. [2] are the only ones we know of, though Özpeynirci and Köksalan [12] give an exact method for finding supported solutions of BOMILP. Most other techniques in the literature have been devoted to specific cases. Vincent et al. [20] improved upon the method of Mavrotas and Diakoulaki [10] for mixed 0-1 problems. Stidsen et al. [17] propose a method for solving mixed 0-1 problems in which only one of the of the
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objectives contains continuous variables. We point out that the works of Belotti et al. [1], Mavrotas and Diakoulaki [10], Stidsen et al. [17] and Vincent et al. [20] are based on biobjective branch-and-bound (BB) procedures in which the Pareto set is determined by solving several BOLPs, while the works of Boland et al. [2] and Ö兹eynirci and Köksalan [12] utilize other techniques in which the Pareto set is determined by solving several MIPs. We also note that the pure integer case has been studied for binary variables [8], general integers [14] and specific classes of biobjective combinatorial problems [6, 13, 16].

In this paper we present a data structure for efficiently storing a nondominated set of feasible solutions to a BOMILP. This structure is useful alongside exact solution procedures as well as heuristics which aim to approximate the Pareto set. The structure we present is a modified version of a quad-tree. For a detailed description and background information on quad-trees, we suggest [15]. Although quad-trees have been used extensively for storing Pareto points in the past [18, 19], they have been used only in the pure integer case. Sun and Steuer [19] stored nondominated solutions using both quad-trees and dynamic lists which were updated via pairwise comparison. They showed that in the pure integer, biobjective case, dynamic lists were able to store nondominated solutions more efficiently than quad-trees. However, notice that in the pure integer case all nondominated solutions are singletons while in the mixed integer case nondominated solutions can consist of line segments as well as singletons. Therefore, since we consider the mixed integer case in this work, our data structure stores line segments as well as singletons. Data stored in a quad-tree is organized in such a way that it can be easily searched to find a desired subset, which is desirable in certain situations. The algorithms we use to implement this tree force it to remain balanced, which is significant because having a balanced tree reduces the time complexity required to access an individual node.

In Section 2 we describe the structure in detail, provide the algorithms necessary for its implementation, and discuss the complexity and correctness of each of these algorithms. Section 3 provides an example of utilizing the structure to determine the nondominated subset of a particular set of solutions. In Section 4 we present the results of two experiments. The results of the first experiment show that in the mixed integer case our data structure is able to store nondominated solutions more efficiently than a dynamic list and, in most cases, can handle up to $10^7$ inserted solutions in reasonable time. In the second experiment we utilize our structure alongside the BB procedure of Belotti et al. [11] to solve specific instances of BOMILP. The results show that the use of our structure leads to faster solution times for almost all solved instances of BOMILP.

2. Tree data structure

We begin this section by presenting the high-level idea of our data structure. Next we give a detailed description of the data structure and the algorithms we used to implement it. We finish by discussing some theoretical results including the complexity of each algorithm, and thus the overall structure. Throughout this discussion, when we refer to storing solutions we are referring to points in the objective space. Recall that we will be storing the nondominated subset of the union of several Pareto sets. One convenient way to store this subset is to store each of the individual points and line segments in $\mathbb{R}^2$ that it comprises.

2.1. Purpose and principle. Figure 2(a) shows an example of solutions that might be generated when solving an instance of BOMILP. We would like to store the nondominated portion of these points and segments, as shown in Figure 2(b). Our goal is to have a data structure $S$ which can take points and line segments as input, and store only the nondominated subset of the solutions regardless of the order in which they are inserted. Therefore, when a new solution is added to $S$, it needs to not only recognize whether or not it is dominated by solutions already in $S$, but it must also be able to determine whether or not the new solution dominates any currently stored solutions. Once these checks have been made, $S$ must be able to update itself and store only nondominated solutions. Consider the set of solutions depicted in Figures 2(a) and 2(b), and suppose that the segments connecting (1,17), (2,15), (4,14), and (9,13) are currently stored in $S$. When inserting the point (5,11) into $S$, it must recognize that the point dominates
a portion of the segment connecting (4,14) and (9,13), and thus this portion of the segment must be removed from $S$ before the point is added to $S$. Similarly, when the segment connecting (6,16) and (7,10) is inserted, it must recognize that a portion of this segment is dominated by (5,11) and therefore only allow the nondominated portion of the segment to be added. 

The data structure we use is a modified version of a quad-tree in which each node represents either a singleton or a line segment associated with a Pareto point or set of Pareto points of (1). Note that a quad-tree is a data structure specifically designed for storing data in $\mathbb{R}^2$. Each node $\pi$ in a quad-tree must contain at most four children, one for each each quadrant of the Cartesian plane. The four children of $\pi$ must lie within $\pi + \mathbb{R}_{++}$, $\pi + \mathbb{R}_{-+}$, $\pi + \mathbb{R}_{+-}$, and $\pi + \mathbb{R}_{--}$, respectively, where, for example, $\mathbb{R}_{++} := \{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$.

2.2. Operations and details. Due to the fact that dominated solutions are not stored in our structure, our modified quad-tree actually reduces to a modified binary tree. Let $\Pi$ be the set of nodes in the tree. For a given $\pi \in \Pi$, notice that if solutions are present in $\pi + \mathbb{R}_{++}$, they are dominated by $\pi$ and should not be stored in the tree. Similarly, if solutions are present in $\pi + \mathbb{R}_{--}$, they dominate $\pi$ and $\pi$ should be removed from the tree. 

Thus, for any node $\hat{\pi} \in \Pi$ the children of $\hat{\pi}$ associated with $\hat{\pi} + \mathbb{R}_{++}$ and $\hat{\pi} + \mathbb{R}_{--}$ are unnecessary. Hence, each $\hat{\pi} \in \Pi$ has only two children, and thus the tree reduces to a binary tree.

In order to present our structure in a clear, understandable manner, we define the following terms for each $\pi \in \Pi$:

1. $\pi$.type - Sgmt for $\pi$ representing line segment, and Pnt for $\pi$ representing a singleton.
2. $\pi.x_1, \pi.x_2, \pi.y_1,$ and $\pi.y_2$ - $\pi$ is identified by:
   (i) Point ($\pi.x_1, \pi.y_1$) if $\pi$.type = Pnt.
   (ii) Extreme points ($\pi.x_1, \pi.y_1$) and ($\pi.x_2, \pi.y_2$) if $\pi$.type = Sgmt.
   Note that if $\pi$.type = Pnt, we assume ($\pi.x_1, \pi.y_1$) = ($\pi.x_2, \pi.y_2$).
3. $\pi.p$ - parent node of $\pi$.
4. $\pi.l$ - left child node of $\pi$.
5. $\pi.r$ - right child node of $\pi$.
6. $\pi.size$ = total # of nodes contained in the sub-tree rooted at $\pi$.
7. $\pi.ideal.left = (\pi^{nw}.x_1, \pi^{nw}.y_1)$ where $\pi^{nw}$ is the north-west-most node in the subtree rooted at $\pi$.
8. $\pi.ideal.right = (\pi^{se}.x_2, \pi^{se}.y_2)$ where $\pi^{se}$ is the south-east-most node in the subtree rooted at $\pi$.

We say that $\pi$ is the root node if $\pi.p = \emptyset$ and $\pi$ is instead a leaf node if $\pi.l = \pi.r = \emptyset$. See Figure 3(a) for the details of $\pi.ideal.left$ and $\pi.ideal.right$.

Now, in order to further simplify the descriptions of the algorithms we use in implementing our data structure, we partition $\mathbb{R}^2$ into 4 regions relative to any node $\pi$. Figures 3(b) and 3(c) show the details of this partition for each type of node. We denote these regions by $R_\alpha(\pi)$.

Figure 2. Example of solutions generated when solving an instance of BOMILP.
Given distinct nodes $\pi$ and $\pi^*$, we use the notation $\pi^* \cap R_\alpha(\pi)$ to denote any portion of the point or segment associated with node $\pi^*$ that lies in region $R_\alpha(\pi)$. If no such portion exists, we say $\pi^* \cap R_\alpha(\pi) = \emptyset$. In order to ensure that these regions are disjoint let us assume that each region contains its lower and left boundaries, but not its upper or right boundaries. Also assume that $\pi$ itself is contained in $R_2(\pi)$ and not $R_i(\pi)$ for $i \in \{1, 3, 4\}$. Note that this convention is taken so that weakly dominated points will not be included in our structure. This convention is convenient since we are working with minimization problems, but if working with maximization problems one should include upper and right boundaries in each region and include $\pi$ in $R_3(\pi)$ rather than $R_2(\pi)$. Now, for an example, suppose $\pi \in \Pi$ is defined by the segment between (2,5) and (3,3). Further suppose that $\pi_1$ is the point (1,5), $\pi_2$ is the point (2,6) and both $\pi_1$ and $\pi_2$ are inserted into our structure. Observe Figure 3(d). The point associated with node $\pi_1$ weakly dominates the left-most point of the segment associated with $\pi$ and thus $\pi_1$ should be stored. However, the point associated with node $\pi_2$ is weakly dominated by the segment associated with $\pi$ and so $\pi_2$ should not be stored. Situations like this motivate our decision to include lower and left boundaries with a given region, but not upper or right boundaries. Such a convention further simplifies the descriptions of the algorithms we use in implementing our data structure.

This data structure has three main purposes: (i) it should be able to handle the insertion of several thousands (or even millions) of points and segments and update itself efficiently, (ii) the structure must be organized so that it can easily be searched and a desired subset can be obtained, and (iii) it must be able to return the current set of nondominated solutions. So, the main algorithms needed for the utilization of this data structure are functions for insertion of new solutions, deletion of dominated solutions, and rebalancing of the tree. We describe these algorithms next.

### 2.2.1. Insertion

Recall that $\Omega_P \subseteq \bigcup_{i=1}^k S_i$ and is hence a collection of points and segments. Thus only points or segments will be inserted into the structure. For this purpose we define the `insert` function which takes two inputs: a node $\pi^*$ which is being inserted and a node $\pi$ which is the root of the tree or subtree where $\pi^*$ is inserted. The point or segment associated with $\pi^*$ is compared against $\pi$. Consider the following four situations:

1. If $\pi^* \subseteq R_2(\pi)$ then $\pi \leq \pi^*$ and thus $\pi^*$ is discarded.
2. If $\pi \not\subseteq \pi^*$ but $\pi^* \cap R_2(\pi) \neq \emptyset$ then a portion of $\pi^*$ is either dominated by $\pi$ or is a repetition of a solution stored in $\pi$. We denote this situation by $\pi \preceq_p \pi^*$. In this case $\pi^* \cap R_2(\pi)$ is discarded.
3. If $\pi^* \subseteq \pi$ then $\pi$ is removed from the tree.
4. If $\pi^* \not\subseteq \pi$ but $\pi \cap R_2(\pi^*) \neq \emptyset$ then $\pi$ is reduced to $\pi \setminus R_2(\pi^*)$.

Note that the second possibility above may result in $\pi^*$ being split into two pieces. Similarly, the final possibility may result in $\pi$ being split into two nodes. If none of the above four scenarios occur, then neither $\pi$ nor $\pi^*$ dominates the other and they can therefore coexist in the tree.
If $\pi^*$ is not discarded while being compared with $\pi$, then if $\pi^* \cap R_1(\pi) \neq \emptyset$, $\pi^* \cap R_1(\pi)$ will need to be inserted at $\pi.l$. Similarly, if $\pi^* \cap R_4(\pi) \neq \emptyset$, $\pi^* \cap R_4(\pi)$ will need to be inserted at $\pi.r$. For this reason, the INSERT function is recursive. Notice that it may be the case that $\pi.l = \emptyset$ or $\pi.r = \emptyset$. A node $\pi^*$ is added to the tree if and only if it is inserted at an empty node. Thus, the typical use of the INSERT function is to insert a new node $\pi^*$ at the root node, $\pi_0$. Then $\pi^*$ is either discarded or $\pi^* \cap R_1(\pi)$ and $\pi^* \cap R_4(\pi)$ are inserted at $\pi.l$ and $\pi.r$, respectively. This process repeats recursively until either (i) $\pi^*$ has been fully discarded, or (ii) all nondominated portions of $\pi^*$ have been added to the tree as new nodes. Throughout the remainder of this paper we will use the notation REPLACE($\pi', \tilde{\pi}$) to denote the process of replacing the point or segment associated with $\pi' \in \Pi$ with the point or segment associated with $\tilde{\pi} \in \Pi$ and leaving the tree structure otherwise unchanged. We use the notation $\pi' \leftarrow \tilde{\pi}$ to denote the process of replacing $\pi'$ and its entire subtree with $\tilde{\pi}$ and its entire subtree. Algorithm 1 describes the INSERT procedure.

**Algorithm 1** Inserting a new point or segment, $\pi^*$, into the data structure at node $\pi$

1: function INSERT($\pi^*, \pi$)
2: if $\pi^* = \emptyset$ then Return
3: if $\pi = \pi_0$ and $\pi_0 \neq \emptyset$ then REBALANCE($\pi$) $\triangleright$ $\pi_0$ represents the root node
4: if $\pi = \emptyset$ then REPLACE($\pi, \pi^*$), $\pi$.size $\leftarrow 1$, UPDATE($\pi$)
5: else REPLACE($\pi, \pi \setminus cl(R_2(\pi^*))$)
6: if $\pi = \emptyset$ then
7: if $\pi$.ideal_left $\cap R_2(\pi^*) \neq \emptyset$ then $\pi.l \leftarrow \emptyset$
8: if $\pi$.ideal_right $\cap R_2(\pi^*) \neq \emptyset$ then $\pi.r \leftarrow \emptyset$
9: REMOVE_NODE($\pi$)
10: INSERT($\pi^*, \pi$)
11: else
12: if $\exists \pi_1, \pi_2$ s.t. $\pi = \pi_1 \cup \pi_2$ & $cl(\pi_1) \cap cl(\pi_2) = \emptyset$ then
13: $\pi_1.l \leftarrow \pi.l$, $\pi_2.r \leftarrow \pi.r$
14: $\pi \leftarrow \pi_1$, $\pi.r \leftarrow \pi_2$
15: UPDATE($\pi$)
16: INSERT($\pi^* \cap R_1(\pi), \pi.l$)
17: INSERT($\pi^* \cap R_4(\pi), \pi.r$)

In Algorithm 1 the functions REMOVE_NODE and REBALANCE refer to the processes of deleting nodes from the tree and rebalancing the tree, respectively. These algorithms will be discussed further in Sections 2.2.2 and 2.2.3 respectively. The recursive UPDATE function has a node $\pi$ as input and traverses up the tree from $\pi$ until reaching the root node. UPDATE then performs two actions: (i) ensures that $\pi$.size $= (\pi.l).size + (\pi.r).size + 1$ where $(\pi').size = 0$ if and only if $\pi' = \emptyset$, and (ii) ensures that $\pi$.ideal_left and $\pi$.ideal_right are updated appropriately. After this has been done, if $\pi.p \neq \emptyset$ then UPDATE($\pi, \pi$) is called.

We now introduce a property that is maintained throughout all operations on the tree as described in the remainder of the paper.

**Property 1.** Given an arbitrary node in the tree $\pi$, all nodes in the subtree of $\pi.l$ are located completely within $R_1(\pi)$ and all nodes in the subtree of $\pi.r$ are located completely within $R_4(\pi)$.

2.2.2. Deletion. Removing a dominated node from the tree is the next task that frequently needs to be performed. Notice that when a node is deleted, in order for the tree structure to be retained, another node must replace it. This is precisely where the difficulty lies. Usually, when a node is deleted from a quad-tree structure, all nodes contained in the subtree of the deleted node are reinserted in order to maintain proper organization of the tree [13, 19]. Since our quad-tree simplifies to a binary tree, however, we propose something much simpler. Notice that in order for our tree to maintain the appropriate structure, Property 1 must be met. For any node $\pi$ that needs to be removed and replaced, there are precisely two nodes that may replace it and satisfy Property 1. They are the right-most node in the subtree of $\pi.l$ and left-most node in the subtree of $\pi.r$. The procedure for REMOVE_NODE is in Algorithm 2.
Algorithm 2 Remove a node that has been shown to be dominated.

1: function RemoveNode(π)
2: if π.size = 1 then π → ∅
3: else Define ˜π = ∅
4: if (π.l).size > (π.r).size then ˜π → FindRightmostNode(π.l)
5: else ˜π → FindLeftmostNode(π.r)
6: Replace(π, ˜π)
7: RemoveNode(˜π)
8: if π.p ̸= ∅ then Update(π.p)

2.2.3. Rebalancing. The final task to perform in maintaining our structure is rebalancing. To maintain balance we use the following strategy of Overmars and van Leeuwen [11]: for each non-leaf node π, the subtrees of π.l and π.r must contain no more than \( \frac{1}{2 - \delta} \) nodes, where \( k \) is the number of nodes in π’s subtree and \( \delta \) is a pre-selected value in the open interval \((0, 1)\). Enforcing this requirement causes the depth of the tree to be at most \( \log_{2-\delta} t \) where \( t \) is the number of nodes in the tree. Now, based on this requirement we develop two rebalancing methods, RebalanceLeft1 and RebalanceLeft2 (and similarly RebalanceRight1 and RebalanceRight2) each of which take a node π as input. In RebalanceLeft2, the left-most node of the subtree of π.r is found and is used to replace π. Then π is moved to right-most position of the subtree of π.l. Notice that RebalanceLeft2 moves a single node from one side of a tree to the other. In certain situations it may be more beneficial to move several nodes from one side of the tree to the other in a single operation. RebalanceLeft1 is designed for this purpose. In RebalanceLeft1, the nodes of the tree are shifted in the following fashion: (i) π.r and its right subtree shift up and left to take the place of π and its right subtree, (ii) π and its left subtree shift down and left to become the new left subtree of π.r, and (iii) the original left subtree of π.r is then placed as the new right subtree of π. RebalanceLeft1 and RebalanceLeft2 are illustrated in Figure 4.

Algorithm 3 is used to determine which rebalancing procedure to apply in order to balance the tree. Its correctness is shown in Proposition 5, which is presented in the next section.

2.3. Performance Guarantees. We now present results about the correctness and complexity of the insertion, deletion, and rebalancing procedures. In this section we will use the notation π ∈ Subtree(ˆπ) to denote the case in which π is a node contained in the subtree of ˆπ which is rooted at ˆπ. We assume ˆπ ∈ Subtree(π).

Proposition 1. Insert removes any portion of a currently stored node π which is dominated by an inserted node π*.

Proof. Assume π* \( \leq_p \) π ∈ II. Since the case in which π = π0 is trivial, we assume π ≠ π0. Assume WLOG that π ∈ Subtree(π0,l) and consider the insertion of π* at π0. Notice that if π* \( \leq \) π0 then in line 5 of Algorithm 1 π is replaced by π \( \setminus cl(R2(π*)) \) = ∅. Thus, one of the following occurs: (i) π* \( \leq \) π0,ideal_left (i.e., π.ideal_left \( \cap R2(π*) \) ≠ ∅) and all nodes contained in
Subtree($\pi_0.l$) are removed from $\Pi$, or (ii) $\pi^* \not\subset \pi_0.ideal.left$ and so REMOVE_NODE replaces $\pi_0$ with a node $\hat{\pi} \in \text{Subtree}(\pi_0)$ and the insert procedure recurses. Let $\pi' = \pi^* \cap R_3(\pi)$ which is the portion of $\pi^*$ that dominates $\pi$. Notice that if $\pi^* \not\subset \pi_0$ then $\pi^* \cap cl(R_1(\pi_0))$, which contains $\pi'$, is inserted at $\pi_0.l$.

The above arguments can be repeated to show that for every $\hat{\pi} \in \Pi$ such that $\pi \in \text{Subtree}(\hat{\pi})$, either: (i) all nodes in $\text{Subtree}(\hat{\pi})$ are removed from $\Pi$, or (ii) a portion of $\pi^*$ containing $\pi'$ is inserted at $\hat{\pi}$. Notice that if a node $\pi''$ containing $\pi'$ is inserted at $\pi$, $\pi$ will be reduced to $\pi \setminus cl(R_2(\pi'')) = \emptyset$.

**Proposition 2.** INSERT adds a portion of an inserted node $\pi^*$ to the tree if and only if it is not dominated by any node currently stored in the tree.

**Proof.** Notice that the reverse direction is trivial because if $\hat{\pi}$ is a portion of $\pi^*$ not dominated by any $\pi \in \Pi$, then $\hat{\pi}$ will be inserted at one of the children of every node it is compared against. Thus, since there are a finite number of nodes in the tree, $\hat{\pi}$ must eventually be inserted at an empty node and added to the tree.

Now, for the forward direction we show the contrapositive. Suppose there is $\pi \in \Pi$ such that $\pi \not\leq_p \pi^*$. Let $\pi' = \pi^* \cap cl(R_2(\pi))$, (i.e., the portion of $\pi^*$ that is dominated by $\pi$). Assume WLOG that $\pi \in \text{Subtree}(\pi_0.l)$ and consider the insertion of $\pi^*$ at $\pi_0$. Notice that by Property 1 $\pi' \subset R_1(\pi_0) \cup R_2(\pi_0)$. If $\pi' \subset R_2(\pi_0)$ then $\pi'$ will not be added to the tree since only $\pi^* \cap R_1(\pi)$ and $\pi^* \cap R_3(\pi)$ are inserted to the children of $\pi_0$. On the other hand, if $\pi' \not\subset R_2(\pi_0)$ then $\pi' \cap R_1(\pi_0) \subset \pi^* \cap R_1(\pi_0)$ and the latter is inserted at $\pi_0.l$.

The above arguments can be repeated to show that for every $\hat{\pi} \in \Pi$ such that $\pi \in \text{Subtree}(\hat{\pi})$, either: (i) $\pi'$ is thrown out when inserted at a parent or grandparent of $\hat{\pi}$, or (ii) a portion of $\pi^*$ containing a subset of $\pi'$ is inserted at $\hat{\pi}$. Notice that if a node $\pi''$ containing a subset of $\pi'$ is inserted at $\pi$, neither $\pi'' \cap R_1(\pi)$ nor $\pi'' \cap R_3(\pi)$ will contain any portion of $\pi'$. Thus, no portion of $\pi'$ can be added to the tree.

**Proposition 3.** Use of the REMOVE_NODE procedure does not violate Property 1.

**Proof.** Suppose some $\pi \in \Pi$ has been shown to be dominated and therefore needs removed from $\Pi$. The case in which $\pi$ is a leaf node is trivial, so we assume that $\pi$ has at least one child. By Property 1, if $\pi.l \neq \emptyset$ then $\pi \subset R_4(\pi.l)$ and if $\pi.r \neq \emptyset$ then $\pi \subset R_1(\pi.r)$. Thus, if $\pi.l \neq \emptyset$ and $\pi'$ is the right-most node in $\text{Subtree}(\pi.l)$, then $\pi'$ is the unique node in $\text{Subtree}(\pi.l)$ such that $\hat{\pi} \subset R_1(\pi')$ for all $\hat{\pi} \in \text{Subtree}(\pi.l) \setminus \pi'$. Similarly, if $\pi.r \neq \emptyset$ and $\pi''$ is the left-most node in $\text{Subtree}(\pi.r)$, then $\pi''$ is the unique node in $\text{Subtree}(\pi.r)$ such that $\hat{\pi} \subset R_4(\pi'')$ for all $\hat{\pi} \in \text{Subtree}(\pi.r) \setminus \pi''$. Therefore, replacing $\pi$ with either the right-most node in $\text{Subtree}(\pi.l)$ or the left-most node in $\text{Subtree}(\pi.r)$ satisfies Property 1.

**Proposition 4.** Use of the REBALANCE procedure does not violate Property 1.

**Proof.** To show that the proposition holds, we must show that neither REBALANCE_LEFT1 nor REBALANCE_LEFT2 violates Property 1. First consider REBALANCE_LEFT1($\pi$):

Note that after this procedure is carried out, $\pi.r$ becomes the root node of the subtree that was once rooted at $\pi$. All nodes that were in the subtree of $\pi.r.l$ remain in their original
positions relative to π.r. Now notice that π becomes the left child of π.r, which does not violate Property 1 since π is completely within R₁(π.r). Finally, the entire subtree of π.r.l becomes the right subtree of π. Since π is now the left child of π.r, all of these nodes are still located in the left subtree of π.r. Furthermore, since these nodes were originally located in π’s right subtree, Property 1 is still satisfied.

Now consider RebalanceLeft2(π):

In this procedure π is replaced by the left-most node in the subtree of π.r. We proved that this would not violate Property 1 in the proof of Proposition 4. After this, π is placed as the right child of the node that was previously the right-most node in the subtree of π.l. We can see that this placement also does not violate Property 1 since all nodes originally within the subtree of π.l are completely within R₁(π).

Proposition 5. After one call of Rebalance(π) the balance criterion is satisfied at π.

Proof. WLOG assume that (π.r).size > π.size. Now, if (π.r.r).size < \( \frac{(1-\delta)\pi\text{size}}{2-\delta} - 1 \) then the proposition is trivially satisfied since in this case RebalanceLeft2 is repeated until (π.r).size = \( \frac{\pi\text{size}}{2-\delta} \). Thus, we focus on the case in which (π.r.r).size ≥ \( \frac{(1-\delta)\pi\text{size}}{2-\delta} - 1 \). Notice that by the construction of the Rebalance procedure, the subtrees of π.l and π.r are balanced before that of π. Thus (π.r.r).size ≤ \( \frac{\pi\text{size}}{2-\delta} \) because otherwise (π.r.r).size > \( \frac{\pi\text{size}}{2-\delta} \) which contradicts the fact that the subtree of π.r is balanced. Now, suppose that after calling RebalanceLeft1(π), π’ is the new root node of the subtree originally rooted at π. Then the subtree of π’.r will be the original subtree of π.r.r. Thus, since (π.r.r).size ≤ \( \frac{\pi\text{size}}{2-\delta} \), the balance criterion will be satisfied for π’.r. Also notice that RebalanceLeft1(π) is only called if (π.r.r).size ≥ \( \frac{(1-\delta)\pi\text{size}}{2-\delta} - 1 \)

\[ \Rightarrow (π.r.r).size ≥ \left( \frac{1-\delta}{2-\delta} \right) \pi\text{size} + \pi\text{size} - \frac{\pi\text{size}}{2-\delta} - 1 \]

\[ \Rightarrow \frac{\pi\text{size}}{2-\delta} ≥ \pi\text{size} - (π.r.r).size - 1. \]

After the procedure is completed, it will be the case that (π’.l).size = \( \pi\text{size} - (π.r.r).size - 1 \) where \( \pi\text{size} \) is the size of the original subtree rooted at π. Thus, the balance criterion will be satisfied for π’.l.

Proposition 6. If \( t \) is the number of nodes stored in the tree, the worst case complexities of Insert, RemoveNode and Rebalance are \( O(t) \), \( O(\log t) \) and \( O(t^2 \log t) \), respectively.

Proof. Insert:

It is clear that for any comparison between an inserted node π* and some π ∈ Π, it is possible that π* ∩ R₁(π) ≠ ∅ and π* ∩ R₃(π) ≠ ∅. Thus it is possible for a portion of π* to be compared with every node in a subtree, implying \( O(t) \) complexity.

RemoveNode:

Here we assume that the tree is balanced prior to calling RemoveNode. Recall that when a node π is removed it is replaced with either the left-most node in the subtree of π.r or the right-most node in the subtree of π.l. Since the tree is balanced, finding such a node is clearly an \( O(\log t) \) process. If π’ is the node replacing π and π’ is not a leaf node, then its original position must then be filled using the same process. Note though that in finding the replacement for π’ a path through the tree is traversed which begins precisely where the path traversed in finding the replacement for π ended. Thus, even though multiple nodes may need replaced in order for π to be removed, the overall process must result in the traversal of only one path through the tree, resulting in an \( O(\log t) \) procedure.

Rebalance:

As currently implemented, rebalancing requires checking the balance criterion at every node of the tree. Ensuring that the balance criterion is met at one of these nodes could require repeating the strategy RebalanceLeft2 up to \( \frac{t}{2} \) times. Thus, since RebalanceLeft2 calls FindLeftmostNode, FindRightmostNode, and Update, which are \( O(\log t) \) procedures, the complexity of rebalancing is \( O(t^2 \log t) \). \( \square \)
3. ILLUSTRATIVE EXAMPLE

Recall the points and segments specified in Figure 2(a). We use these points and segments as input to our data structure and show a few of the nontrivial steps of developing our tree. Assume that the solutions shown in the figure are obtained from five separate slice problems and that the Pareto sets of these slice problems, listed in respective order, are: (i) the singleton (1,19), (ii) the piecewise linear curve connecting (1,17) and (9,13), (iii) the piecewise linear curve connecting (6,16) and (11,4), (iv) the singleton (5,11), and (v) the piecewise linear curve connecting (8,7) and (17,2). The points and segments which define these Pareto sets will be inserted into our structure in the order of (iii), (iv), (ii), (v), (i). Piecewise linear curves will be inserted as individual line segments from left to right.

The reader is encouraged to review the pseudocode given previously (particularly Algorithm 1). To begin we let \( \pi^* \leftarrow (6,16) \) to \( (7,10) \) and call Insert\((\pi^*, \pi_0)\). Since \( \pi_0 = \emptyset \) we replace \( \pi_0 \) with \( \pi^* \). Clearly the current tree structure is now a single node. Next we let \( \pi^* \leftarrow (7,10) \) to \( (10,5) \) and call Insert\((\pi^*, \pi_0)\). Notice that \( \pi^* \subset R_4(\pi_0) \) and should be inserted at \( \pi_0.r \). Since \( \pi_0.r = \emptyset \) this insertion results in \( \pi^* \) being added to the tree. Therefore the tree now contains the root node which has one child to its right. The insertion of the segment connecting \( (10,5) \) to \( (11,4) \) is analogous. Next consider Pareto set (iv). Let \( \pi^* \leftarrow (5,11) \) and call Insert\((\pi^*, \pi_0)\). Observe Figure 5(a). Clearly we can see that \( \pi^* \preceq_p \pi_0 \) and thus we remove the dominated portion of \( \pi_0 \) by letting \( \pi_0 = \pi_0 \setminus R_2(\pi^*) \). After this has been done, notice that \( \pi^* \subset R_1(\pi_0) \). Therefore, since \( \pi_0.l = \emptyset \), \( \pi^* \) becomes the left child of \( \pi_0 \). Figure 5(b) shows the tree structure after \( \pi^* \) has been inserted. We leave it to the reader to consider Pareto set (ii). Note, though, that after processing this set the subtree rooted at \( \pi_0.l \) needs to be rebalanced. The resulting tree is shown in Figure 5(c).

Next we consider the insertion of Pareto set (v). Let \( \pi^* \leftarrow (8,7) \) to \( (14,3) \) and call Insert\((\pi^*, \pi_0)\). Clearly \( \pi^* \subset R_4(\pi_0) \) and will therefore be inserted to \( \pi_0.r \). Observe from Figure 5(b) that \( \pi^* \preceq_p \pi_0.r \). This time, though, the portion of \( \pi_0.r \) which is dominated is the center section of the segment. This means that \( \pi_0.r \) must be split into two nodes \( \pi_1 \) and \( \pi_2 \). Node \( \pi_1 \) takes the place in the tree where \( \pi_0.r \) originally was, and the left subtree of \( \pi_0.r \) becomes the
left subtree of \( \pi_1 \). Node \( \pi_2 \) becomes the right child of \( \pi_1 \) and the right subtree of \( \pi_0.r \) becomes the right subtree of \( \pi_2 \). Now, after this process has been completed, observe that \( \pi^* \subseteq R_4(\pi_1) \) and thus \( \pi^* \) will be inserted to \( \pi_2 \) (which is now \( \pi_0.r.r \)). Notice that \( \pi_0.r.r \leq \pi^* \) and that it is the center portion of \( \pi^* \) that is dominated. Thus the calls to \( \text{INSERT}(\pi^* \cap R_1(\pi_0.r), \pi_0.r.r.l) \) and \( \text{INSERT}(\pi^* \cap R_4(\pi_0.r.r), \pi_0.r.r.r) \) will each cause a portion of \( \pi^* \) to be inserted at \( \pi_0.r.r.l \) and \( \pi_0.r.r.r \) respectively. Since \( \pi_0.r.r.l = \emptyset , \pi^* \cap R_4(\pi_0.r.r) \) will become \( \pi.r.r.l \). Since \( \pi_0.r.r.r \) is the segment \((10,5)\) to \((11,4)\), it is clear that another portion of \( \pi^* \) will need to be removed, and then the remainder of \( \pi^* \) will become \( \pi_0.r.r.r.r \).

We end our example now because the remaining insertions result in scenarios which are analogous to those that we have now observed. Note that if we were to continue, one more rebalance would be required and the final tree structure would be that found in in Figure 6(c). Note that this tree structure is dependent on the order of insertion.

4. Computational Experiments

We implemented our data structure in the C programming language and performed two tests. The first was designed to test the number of solutions our structure can effectively store and how quickly they can be processed. The second was designed to test the utility of our data structure when used alongside the BB algorithm of Belotti et al. [1]. Both tests were run using Clemson University’s Palmetto Cluster. Specifically, an HP SL250s server node with a single Intel E5-2665 CPU core with 16GB of RAM running Scientific Linux 6.4 was used.

In both of these experiments we compare the performance of our structure with that of a dynamic list (L). Like our structure, this list also takes points and segments in \( \mathbb{R}^2 \) as input and stores only the nondominated subset of all input. When a point or segment is inserted, it is compared with every other stored point or segment. Then, during each comparison dominated solutions are discarded. Such lists have been used for storing nondominated solutions in both the pure integer [19] and mixed-integer cases [11] [20].

4.1. Implementation of Rebalance. Recall from Proposition 6 that maintaining a balanced tree is the most costly of the three operations needed to create our structure. It is also the one operation that is unnecessary in order to ensure that we store the correct solutions. For this reason we decided to further consider the rebalancing operations in hopes of finding an alternative implementation that is less computationally costly, but still performs well in practice.

Note that Overmars and van Leeuwen [11] suggest rebalancing by traversing the path travelled by an inserted solution in the reverse order and checking whether or not the balance criterion is satisfied at each of these nodes. This saves one from having to check the balance criterion at every node in the tree since the only places where it could have been altered are at nodes along this path. In our case, though, when a line segment is inserted into our structure, it often does not remain intact, but is separated into many smaller segments, each traversing its own path through the tree before finally being added. For this reason, rearranging the tree after the insertion of a segment into the tree is troublesome. Hence, we propose a few alternative approaches:

A1 - Before allowing a point or segment to be inserted at the root node, check the balance criterion at every node in the tree and rebalance where necessary. Using this approach one is able to guarantee that the balance of the tree is maintained, but its complexity is clearly very high. (Notice that this approach is the implementation used as presented in Algorithms 1-5.)

A2 - Performing the previous procedure, but periodically rather than before every insertion at the root node. For example, we could determine to check the entire tree for balance every time 100 new solutions are added to the tree. This approach significantly decreases the complexity of rebalancing, but eliminates the balance guarantee.

A3 - Another approach could be to check the balance criterion at any node that is currently being inserted at. This approach has a much lower complexity, and would cause balance to be maintained at the root node, and along any frequently travelled paths in the tree. However, again the guarantee of balance is lost.
A4 - A final approach is to combine approaches 2 and 3. Using approach 3 may allow one to maintain a fairly well balanced tree by applying approach 2 much more infrequently than if using approach 2 alone. Clearly this has a higher complexity than approach 3, but it may be less than that of approach 2 and allow for a more balanced tree.

We implemented each of these approaches in our first experiment, described in Section 4.2. We utilize approach A2 when performing our second experiment, which is described in Section 4.3, because it provided the fastest running times in the majority of the tests we ran in our first experiment.

4.2. Experiment 1 - Random Data.

4.2.1. Setup. This test has two main purposes: (i) to compare the efficiency of our data structure with that of a dynamic list (which updates via pairwise comparison) when storing nondominated solutions, and (ii) to determine the number of solutions our structure can take as input and process in a reasonable amount of time.

The test consists of repeating the following procedure until $N$ insertions have been made into our structure or the dynamic list. First, generate a random integer $i \in \{1, 6\}$ and a random number $r_1 \in (0, 10)$. Then, if $i > 1$, for each $j \in \{2, \ldots, i\}$ a random number $c_j \in (0, 1)$ is generated and we define $r_j = r_1 + \sum_{\ell=2}^{j} c_{\ell}$. Next, for each $j \in \{1, \ldots, i\}$ the following are computed: (i) $y_j = \frac{(10.5 - r_j)^2}{5} - k$, and (ii) $x_j = r_j + (5 - k)$. Here $k$ is a dynamic value which is defined as 1 at the start of the test and increases by $\frac{\mu}{N}$ each time the above process is repeated. Here $\mu \in \mathbb{R}$ is a parameter that allows us to determine how much the solutions should “improve” over the course of the test. If $i = 1$, the singleton $(x_1, y_1)$ is inserted into the structure, otherwise the points $(x_1, y_1), \ldots, (x_i, y_i)$ are arranged in order of increasing $x$ values and then the line segments connecting each adjacent pair of points are inserted into the structure. We performed this test 100 times for each combination of the values $N = 10^4, 10^5, 10^6$ and $10^7$ and $\mu = 0, 0.001, 0.01, 0.1, 1$ and 10. We used various values for $\delta$ and found that the results were quite similar, but determined to use a value of $\delta = 0.3$. For each test we recorded the time it took to insert all solutions into our structure, the time it took to insert all solutions into the dynamic list, the final depth of our tree, the final number of nodes stored in our tree, and the final number of nodes stored in the dynamic list.

We now explain the significance of $\mu$. Many procedures for determining or approximating the Pareto set of a BOMILP are iterative procedures which attempt to use solutions generated during early iterations to generate better solutions (i.e., solutions which are closer to being Pareto optimal) in later iterations. Such procedures include BB and most heuristic algorithms. Selecting values for $\mu$ which are close to zero is intended to replicate generating solutions during one of these procedures in which there is little or no separation between early generated solutions and later generated ones, and thus both early and later generated solutions are likely to be Pareto. Alternatively, selecting large values of $\mu$ is intended to replicate generating solutions during one of these procedures in which there is significant separation between early generated solutions and later generated ones, and in which solutions generated later are much more likely to be Pareto than those generated early. We expect to find that our structure performs better for large values of $\mu$ since there should be more domination of solutions, therefore requiring less storage. As a visual aid, we include Figure 7 which shows an example of solutions generated during this experiment for $\mu = 0.1, 1$ and 10 and for $N = 100$. The solutions shown in red are those that are stored by our structure at the end of the test.

4.2.2. Implementation Details. First, recall that as presented, the implementation of our structure performs a check in order to determine whether or not an entire subtree is dominated. If a subtree is found to be dominated, the entire subtree is removed. We found that in practice, however, this implementation does not outperform the implementation in which no check for dominated subtrees is performed, rather dominated nodes are removed one at a time. We feel
that there are two drawbacks to the former implementation which are most likely the reasons for this: (i) more information (i.e., an ideal point for each subtree) is stored in each node, and (ii) when new solutions are added to the tree, the Update function must ensure that these ideal points are updated appropriately, which can be a costly procedure. Also notice that the worst case complexity of RemoveNode remains the same for both implementations. For this reason, we used the latter implementation when performing our tests.

4.2.3. Numerical Results. We present the results obtained from our randomized tests when implementing the 4 rebalancing approaches discussed in section 4.1.2. When using approach A2, we chose to perform an initial rebalance after 100 new solutions had been added to the structure and then again each time there was a 101% increase in the number of stored solutions. When using approach A4 we again performed an initial rebalance after 100 new solutions had been added to the structure, but this time we did not rebalance again until the number of solutions increased by 800%. The minimum, maximum, and average elapsed times and final depths of the tree resulting from running experiment 1 can be found in Table [1]. The average numbers of nodes stored while running these tests are given in Table [2]. Note that all averages are reported as geometric means. The symbols T and L indicate runs in which our tree structure and the dynamic list were used for storing solutions, respectively. Also, entries in Tables [1] and [2] which contain dashes are those for which no results are available due to the fact that individual runs took over 12 hours to complete and were therefore terminated. The symbol ⊙, on the other hand, indicates results for which, due to the large amount of time taken for each individual run, we were unable to perform the test 100 times. For these results, each test was instead run 5 times.
| N Type | Min | Avg | Max | Min | Avg | Max | N Type | Min | Avg | Max | Min | Avg | Max |
|--------|-----|-----|-----|-----|-----|-----|--------|-----|-----|-----|-----|-----|-----|
| 0      | 16.4| 17  | 17  | 16.4| 17  | 17  | 0.01   | 16.4| 17  | 17  | 16.4| 17  | 17  |
| 1      | 19.1| 19  | 20  | 19  | 19  | 20  | 0.02   | 19  | 19  | 20  | 19  | 19  | 20  |
| 2      | 22.9| 23  | 23.3| 22.9| 23  | 23.3| 0.03   | 22  | 22  | 22  | 22  | 22  | 22  |
| 3      | 26.8| 27  | 27.3| 26.8| 27  | 27.3| 0.04   | 26  | 26  | 26  | 26  | 26  | 26  |
| 4      | 31.7| 32  | 32.3| 31.7| 32  | 32.3| 0.05   | 31  | 31  | 31  | 31  | 31  | 31  |
| 5      | 37.7| 38  | 38.3| 37.7| 38  | 38.3| 0.06   | 37  | 37  | 37  | 37  | 37  | 37  |
| 6      | 44.7| 45  | 45.3| 44.7| 45  | 45.3| 0.07   | 45  | 45  | 45  | 45  | 45  | 45  |
| 7      | 52.7| 53  | 53.3| 52.7| 53  | 53.3| 0.08   | 53  | 53  | 53  | 53  | 53  | 53  |
| 8      | 61.7| 62  | 62.3| 61.7| 62  | 62.3| 0.09   | 62  | 62  | 62  | 62  | 62  | 62  |
| 9      | 72.7| 73  | 73.3| 72.7| 73  | 73.3| 0.10   | 73  | 73  | 73  | 73  | 73  | 73  |
| 10     | 85.7| 86  | 86.3| 85.7| 86  | 86.3| 0.11   | 86  | 86  | 86  | 86  | 86  | 86  |

**Table 1. Time and depth of the tree for Experiment 1**
Table 2. Number of nodes stored (averaged over all rebalancing approaches) for Experiment 1

| \( \mu \) | \( N \) | Tree/List | Avg # of Nodes | \( \mu \) | \( N \) | Tree/List | Avg # of Nodes |
|-------|--------|-----------|---------------|-------|--------|-----------|---------------|
| 0     | \( 10^4 \) | T         | 25,737        | 0.1   | \( 10^4 \) | T         | 285          |
|       | L      |           | 25,170        |       | L      |           | 285          |
| 10^5  | T      | 198,544   |               | 10^5  | T      | 766       |
|       | L      | –         |               |       | L      | 765       |
| 10^6  | T      | 864,145   |               | 10^6  | T      | 2,366     |
|       | L      | –         |               |       | L      | 2,363     |
| 10^7  | T      | 2,154,322 |               | 10^7  | T      | 7,211     |
|       | L      | –         |               |       | L      | \( \ast \) \( 7,166 \) |
| 0.001 | \( 10^4 \) | T         | 2,368         | 1     | \( 10^4 \) | T         | 188          |
|       | L      |           | 2,364         |       | L      |           | 188          |
| 10^5  | T      | 7,239     |               | 10^5  | T      | 283       |
|       | L      |           | 7,200         |       | L      |           | 283          |
| 10^6  | T      | 22,358    |               | 10^6  | T      | 761       |
|       | L      | \( \ast \) | 21,849        |       | L      |           | 761          |
| 10^7  | T      | 74,165    |               | 10^7  | T      | 2,371     |
|       | L      | –         |               |       | L      |           | 2,368        |
| 0.01  | \( 10^4 \) | T         | 767           | 10    | \( 10^4 \) | T         | 133          |
|       | L      |           | 766           |       | L      |           | 133          |
| 10^5  | T      | 2,369     |               | 10^5  | T      | 189       |
|       | L      |           | 2,365         |       | L      |           | 189          |
| 10^6  | T      | 7,210     |               | 10^6  | T      | 284       |
|       | L      |           | 7,171         |       | L      |           | 284          |
| 10^7  | T      | 22,397    |               | 10^7  | T      | 764       |
|       | L      | –         |               |       | L      |           | 764          |

There are several things to notice from Tables 1 and 2. First, notice that in all cases our data structure is able to process inserted solutions much more quickly than the dynamic list. Next, notice that for fixed values of \( N \) and \( \mu \), A2 typically performs the best in terms of running time but A3 and A4 typically perform the best in terms of maintaining a tree of minimum depth. Also notice that for each fixed value of \( N \), the time taken to process inserted solutions decreases as the value of \( \mu \) increases. Furthermore, the larger the value of \( \mu \), the closer the time needed for the dynamic list to process the input solutions becomes to the time needed for our tree to process the solutions. By comparing 1 and 2 it is easy to see the correlation between the time taken to process solutions and the number of nodes stored, for both our structure and the dynamic list.

From these results we can see that our data structure can handle the insertion of large sets of solutions, thus we suspect that it can do so without posing a significant overhead on a solution procedure such as BB or a heuristic method.

4.3. Experiment 2 - Fathoming in BB.

4.3.1. Setup. We performed tests in which we solved a variety of instances of BOMILP using the BB technique of Belotti et al. [1]. Each instance was solved three times, once using our structure in order to generate the upper bound set at each iteration of the BB, once using a dynamic list in order to generate these sets, and once using a predetermined subset of \( \Omega_P \) to generate a single upper bound set which was used for fathoming throughout the BB. Note that more details on each of these types of sets are provided in the following section.

The instances of BOMILP that we solved were taken from [1] and [2]. We present results on all instances that took between 10 seconds and 8 hours to solve. We note that the instances of [2] were divided into two types of problems, but during preliminary tests the BB code ran
into numerical issues with the second type and so we did not experiment with them. In the following section we provide a background on BB procedures for both the single objective and biobjective cases.

4.3.2. Background on BB. Many of the prevalent techniques for solving BOMILP are based on the branch-and-bound (BB) method, which has been well established in the single-objective case. During an iteration of a typical BB procedure for MILP (and by extension for BOMILP), one solves an LP subproblem at a node \( \eta \) selected from a list \( \mathcal{L} \) of open subproblems in the BB enumeration tree. Then if some \( y_i \) takes a fractional value \( \gamma_i \) in the LP optimal solution, a process referred to as branching is performed. During branching two new subproblems are created by adding the constraints \( y_i \geq \lceil \gamma_i \rceil \) and \( y_i \leq \lfloor \gamma_i \rfloor \), respectively, to the LP subproblem at \( \eta \). These two subproblems are then added to \( \mathcal{L} \) and a new iteration is begun by selecting a new node \( \eta' \in \mathcal{L} \) to explore and subdivide, if necessary. By continually selecting subproblems to explore and subdivide, a BB tree of subproblems is formed. In the single objective case, a fractional solution to the LP at node \( \eta \) provides a valid lower bound for all subproblems of \( \eta \), while the objective function value associated with any integer feasible solution is an upper bound for every subproblem of the BB tree. These BB methods are effective because by comparing this global upper bound with the lower bound of a particular subtree, one is often able to prove that the subtree cannot provide a better integer feasible solution. Once this has been done, it is said that this subtree has been pruned or fathomed. Thus, in most cases the entire BB tree does not need to be explored and the best integer feasible solution can be found relatively quickly.

In the biobjective case, the bound sets used for fathoming are no longer singletons in \( \mathbb{R} \), instead they are subsets of \( \mathbb{R}^2 \) formed by taking unions of finitely many continuous convex piecewise linear functions [5]. During each iteration \( s \) of BB, a node \( \eta_s \) of the BB tree is considered. The set of Pareto solutions to the LP relaxation of (1) associated with \( \eta_s \) provides a lower bound set \( \mathcal{L}_s \). The upper bound set \( \mathcal{U}_G \), on the other hand, is globally valid to all nodes of the BB tree, although it is generally not known in its entirety until completion of the BB. Therefore, at each iteration \( s \) of BB, since \( \mathcal{U}_G \) is unknown and cannot be used for fathoming, another set \( \mathcal{U}_s = \vartheta(\mathcal{N}_s) \) is used instead, where \( \mathcal{N}_s \subset \Omega \) is a set containing no dominated points at iteration \( s \). Now, in order to describe the mapping \( \vartheta(\cdot) \), we introduce several definitions. A point \((\kappa_1, \kappa_2) \in \mathcal{N}_s \subset \Omega \) is said to be isolated in \( S \) if \( \exists \epsilon > 0 \) for which \( B_\epsilon(\kappa) := \{ \kappa \in S : ||\kappa - \kappa||_2 < \epsilon \} \) is empty. Given distinct \( \pi, \kappa' \in S \) such that \( \pi_1 < \kappa_1' \) and \( \pi_2 > \kappa_2' \), the point \( \kappa^n = (\kappa_1', \pi_2) \) is called the local nadir point with respect to \( \pi \) and \( \kappa' \) (note that the above inequalities are strict so that \( \kappa^n \neq \kappa' \) and \( \kappa^n \neq \pi \)). Given a line segment containing points in \( S \), the segment itself is referred to as a local nadir set. We now describe the mapping \( \vartheta(\cdot) \), which can be used to construct \( \mathcal{U}_s \) given \( \mathcal{N}_s \). Notice that if a line segment contains no dominated points, then it must have a negative slope. We use the notation \([\kappa^{nw}, \kappa^{se}]\) to denote any such segment, where \( \kappa^{nw} \) and \( \kappa^{se} \) are the segment’s north-west and south-east endpoints, respectively. Now, since at any iteration \( s \) of BB \( \mathcal{N}_s \subset \Omega \) contains no dominated points, each of its elements must be either an isolated point or a line segment with a negative slope. For each point \( \kappa \in \mathcal{N}_s \) consider \( \kappa_{s1} \), and for each segment \( [\kappa^{nw}, \kappa^{se}] \in \mathcal{N} \) consider \( \kappa_{s1}^{nw} \). Arrange the elements of \( \mathcal{N}_s \) in increasing order of these values. Then for each pair of adjacent elements \( (\varepsilon_1, \varepsilon_2) \in \mathcal{N}_s \), if the south-east-most point of \( \varepsilon_1 \) is not equal to the north-west-most point of \( \varepsilon_2 \), calculate the local nadir point with respect to these two points and add it to a set \( \mathcal{N}_s' \). Note that if \( \varepsilon_i \) for \( i \in \{1, 2\} \) is a point and not a segment, then its north-west-most and south-west-most points are simply \( \varepsilon_i \) itself. Now let \( \mathcal{N}_s'' \) be the set of local nadir sets in \( \mathcal{N}_s \). Then \( \vartheta(\mathcal{N}_s) := \mathcal{N}_s' \cup \mathcal{N}_s'' \) and thus \( \mathcal{U}_s = \mathcal{N}_s' \cup \mathcal{N}_s'' \). Figure 8(a) illustrates the relationship between \( \mathcal{N}_s \) and \( \mathcal{U}_s \).

One of the fathoming rules presented by Belotti et al. [1] states that at iteration \( s \) of BB a node \( \eta_s \) can be fathomed if \( \mathcal{L}_s \) is separable from \( \mathcal{U}_s \), i.e., \( \mathcal{L}_s \cap (\mathcal{U}_s - \mathbb{R}^2) = \emptyset \). This is essentially the extension of the well known “fathoming by bound dominance” rule for single-objective problems to the biobjective case. Figure 8(b) shows examples of lower bound sets \( \mathcal{L}_{s1} \) and \( \mathcal{L}_{s2} \). Notice that the locations of these sets show that \( \eta_{s1} \) cannot be fathomed but \( \eta_{s2} \) can. Clearly,
efficient fathoming depends on the choice of \( N_s \) used to construct \( U_s \) since good approximations of \( U_G \) at each iteration of BB can help fathom a large number of nodes.

At iteration \( s \) of BB, let \( F_s \) be the set of all \( \omega \in \Omega \) discovered during iterations 1, \ldots, \( s-1 \) of BB. Then at iteration \( s \), the best choice for \( N_s \) is the nondominated subset of \( F_s \). Finding this set can be cumbersome though. Until now, there seem to have been only two approaches used:

**Dynamic List:** Each time \( \omega \in \Omega \) is found, store it in a list and then remove dominated points and segments by performing a pairwise comparison between all stored solutions. After completion of the pairwise comparison the set of stored solutions is precisely \( N_s \), and \( U_s \) can be constructed as \( \vartheta(\mathcal{N}_s) \). Mavrotas and Diakoulaki [10] used a set of this type, although, their work also considered problems with more than two objectives. Vincent et al. [20] also used such bound sets and this work focused on biobjective problems.

**Predetermined subset of \( \Omega \):** Before beginning BB a preprocessing phase is used to generate a set \( N \subset \Omega \). Then at every iteration \( s \) of BB, let \( N_s = N \). Therefore a single set \( \mathcal{U} = \vartheta(\mathcal{N}) \) is used for fathoming throughout the BB. Note that one way to generate \( \mathcal{N} \) is to use the \( \epsilon \)-constraint method, i.e., solve the MILP \[ \min_{x,y} \begin{cases} c^T_1 x + d^T_1 y : c^T_2 x + d^T_2 y \leq \epsilon, (x, y) \in P \end{cases} \] for various values of \( \epsilon \). Then, for each value of \( \epsilon \) such that this problem is feasible, its solution \( (x_\epsilon, y_\epsilon) \in X_E \) and thus corresponds to a point in \( \Omega \). \( \mathcal{N} \) is then the union of all Pareto points found this way. Although this option eliminates the need for updating via pairwise comparison, its effectiveness is highly dependent on the number of initial points generated. Clearly, there is a tradeoff between computational time versus quality of the upper bound set. This method was used by Belotti et al. [1].

### 4.3.3. Implementation Details.

First we point out that when utilizing the predetermined subset of \( \Omega \), the \( \epsilon \)-constraint method was used to generate \( M \leq M^* \) points from \( \Omega \) before beginning the BB, where \( M^* \) is a user-selected upper bound on the number of these points that are generated. Notice, though, that these \( M \) points can still be useful in the cases when either our structure or a dynamic list is being used alongside the BB. By inserting these points into either structure at the start of the BB, the procedure can be “warm-started,” increasing the frequency and efficiency of fathoming.

Initially we solved several instances using our structure both with and without warm-starting. However, the results we obtained without warm-starting were very poor, and are therefore not reported. Notice that warm-starting allows solutions which are “far” from the set of Pareto-optimal solutions to be discarded early in the BB, and therefore fewer nodes of the BB tree are explored. As a visual aid, observe Figure 9 which shows solutions generated during the BB procedure when warm-starting is and is not used.

Notice that warm-starting provides \( M \) points which are in most cases well dispersed throughout \( \Omega \). Therefore we attempted an implementation in which these points \( M \) were inserted into our data structure in such a way as to begin BB with a tree that was perfectly balanced. We felt that this may allow us to turn off the rebalancing procedures and in turn solve each instance more quickly. The results we obtained from this implementation did not provide any increased
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(a) Using warm-start with $M^* = 100$

(b) Not using warm-start

Figure 9. This figure is taken from the solution of an instance from [1] with 80 variables and 80 constraints, and using $M^* = 100$ when warm-starting. The $M$ points generated during the warm-start are shown as cyan stars, all solutions generated during BB are shown in blue, and the final set of Pareto-optimal solutions is shown in red.

efficiency, however, and so results are reported from the original implementation, in which we used rebalancing strategy A2.

We solved each instance using various values for $M^*$, ranging from 10 on small instances to 3000 on large ones. “Good” choices for the value of $M^*$ seem to be highly dependent on the size and difficulty of the instance being solved.

4.3.4. Numerical Results. We now present the results we obtained from Experiment 2, which are explicitly given in Tables 3 and 4 and are summarized in the performance profile of CPU time shown in Figure [10]. We use P, T, and L to represent the implementations of the Prede-
termined set of $\Omega_P$, our tree structure, and the dynamic list, respectively. When we refer to a “tree” in these results, we mean our data structure, not the BB tree. From [1], there were 30 instances available for each problem size and from [2], there were 5 instances available for each problem size. However, we were unable to solve instances from [2] which had 320 variables and constraints, because they took longer than 8 hours to solve. Each instance was solved once. The data presented in Table 3 is calculated as a geometric average over all available instances of each size from [1]. The results in Table 4 are from [2] and are not averaged. Note that problem sizes are reported as the number of variables, which in all cases also equalled the number of constraints.

As it is difficult to know ahead of time what value of $M^*$ is most appropriate for solving a given BOMILP, we ran each of these instances for a large number of values for $M^*$, but for the sake of space only report results for 3 values of $M^*$ for each problem size. For each instance size we report results for one relatively large value of $M^*$, one medium value of $M^*$ and one relatively small value of $M^*$. In Figure 10 we use L, M, and H to denote low, medium, and high values for $M^*$, respectively.

As can be seen in Tables 3 and 4, for low values of $M^*$ computation time is significantly reduced when using our structure or the dynamic list in place of the predetermined subset of $\Omega_P$. As $M^*$ increases, though, the computation time resulting from using the predetermined subset of $\Omega_P$ approaches the time resulting from using our tree or the list. In some cases, when $M^*$ is high enough, the computation time resulting from using the predetermined subset of $\Omega_P$ is actually less than the time resulting from using our tree or the list. The reason for this is that larger values of $M^*$ result in initial bound sets which more closely approximate the Pareto set of a given instance. Thus, the additions made to this set throughout BB by our tree or the list do not aid in fathoming as much as they do for low values of $M^*$. We point out, however, that the Pareto set is not readily available upon termination when BB is implemented using the predetermined subset of $\Omega_P$. Instead, the set of all integer feasible solutions is stored and a post-processing phase is needed in order to determine the Pareto set. The implementations using
Table 3. Timing, fathoming, and storage results for instances taken from [1] (reported as geometric averages).

| Size | M  | P  | T  | L  | BB Nodes | # of Fathomed | Final # of Insertions | Final Nodes Stored | Final Depth of Tree |
|------|----|----|----|----|----------|----------------|----------------------|-------------------|--------------------|
| 60   | 10 | 25.0 | 15.7 | 15.6 | 516 | 342 | 342 | 1,246 | 1,246 | 75.0 | 76.3 | 12.8 |
| 25   | 14.7 | 12.7 | 12.7 | 309 | 271 | 271 | 1,421 | 1,421 | 75.8 | 78.4 | 13.4 |
| 50   | 13.9 | 12.8 | 12.8 | 273 | 254 | 254 | 1,264 | 1,264 | 76.0 | 79.6 | 13.4 |
| 80   | 10 | 54.2 | 35.0 | 35.4 | 705 | 492 | 492 | 2,743 | 2,743 | 89.0 | 90.1 | 9.6 |
| 25   | 37.1 | 31.2 | 31.4 | 488 | 428 | 428 | 2,102 | 2,102 | 89.5 | 91.6 | 10.1 |
| 50   | 34.2 | 31.4 | 31.6 | 430 | 403 | 403 | 1,869 | 1,869 | 89.6 | 93.3 | 9.7 |

Table 4. Timing, fathoming, and storage results for instances taken from [2].
our tree and the list data structure, on the other hand, do have the Pareto set readily available upon termination. Thus, we do not feel as though the cases in which the predetermined subset of $\Omega_P$ outperformed the tree and the list is evidence against the utility of our data structure. We also note that the number of nodes fathomed from the BB tree is generally lower when using our structure or the dynamic list as opposed to the predetermined subset of $\Omega_P$. This is because nodes of the BB are being fathomed earlier, or higher, in the tree, therefore causing fewer nodes to be explored.

When comparing the results of this experiment to those of our first experiment one may wonder why our tree structure does not significantly outperform the list in all cases. On inspecting the number of solutions inserted to our structure versus the final number stored, we found that a value of $\mu \approx 100$ (cf. §4.2.1 and Figure 7) could be associated with most of the solved instances. This value of $\mu$ indicates that there is a high level of separation amongst the solutions generated during BB and therefore a large fraction of generated solutions ends up being dominated and hence not stored. Thus for BB experiments, there is not a significant difference between our structure and the list in terms of the time needed to process the data. In fact, the time for either structure to process all inserted solutions was approximately one order of magnitude less than the time for BB to solve the instance. Thus, the BB had a much larger impact on overall running time than the data structure used.

5. Conclusion

In this work we have introduced a new data structure, in the form of a modified binary tree, that is able to efficiently store sets of nondominated solutions of BOMILPs. Until now similar structures have only been used in the pure integer case. We provide an extension for the more difficult mixed-integer case. We showed this structure performs with a worst case guarantee of $O(t^2 \log t)$ where $t$ is the number of stored nodes. We tested the practical value of our data structure with two experiments. The results show that our structure provides a more efficient method for storing solutions to BOMILP than other current techniques. They also show that our structure is also a very useful tool when used alongside BB methods for solving BOMILPs.

We recognize that there may be ways to extend our data structure and increase its efficiency. Recall that each node of our structure may store either a point or a line segment. It is possible that in certain cases our structure stores several segments that all belong to a single piecewise linear curve. Therefore it may be beneficial to extend the functionality of our structure so that entire piecewise linear curves can be stored in a single node. Notice that in some cases this
may allow for a significant reduction of the size of the tree and thus allow the structure to be populated and maintained more quickly. The reason that we did not implement our structure in this fashion is that for the BOMILP solution techniques we are familiar with, segments are generated one at a time and in general connecting segments are not generated sequentially. Also, for the specific instances we solved, it was not often that a significant number of connected line segments generated from the same slice problem were Pareto optimal.

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