ON DIMENSION-FREE VARIATIONAL INEQUALITIES
FOR AVERAGING OPERATORS IN \( \mathbb{R}^d \)

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Abstract. We study dimension-free \( L^p \) inequalities for \( r \)-variations of the Hardy–Littlewood averaging operators defined over symmetric convex bodies in \( \mathbb{R}^d \).

1. Introduction

The purpose of this paper is to initiate the study of \( r \)-variational estimates in the setting of dimension-free bounds for averaging operators defined over symmetric convex bodies in \( \mathbb{R}^d \).

We will assume that \( G \) is a non-empty convex symmetric body in \( \mathbb{R}^d \), which simply means that \( G \) is a bounded convex open and symmetric subset of \( \mathbb{R}^d \). For every \( x \in \mathbb{R}^d \), \( t > 0 \) and \( f \in L^1_{\text{loc}}(\mathbb{R}^d) \) let

\[
M^G_t f(x) = \frac{1}{|G_t|} \int_{G_t} f(x-y)dy = \frac{1}{|G_t|} \mathbf{1}_{G_t} * f(x)
\]

be the Hardy–Littlewood averaging operator defined over the sets

\[
G_t = \{ y \in \mathbb{R}^d : t^{-1}y \in G \}.
\]

For \( r \in [1, \infty) \) the \( r \)-variation seminorm \( V_r \) of a complex-valued function \( (0, \infty) \times \mathbb{R}^d \ni (t, x) \mapsto a_t(x) \) is defined by setting

\[
V_r(a_t(x) : t \in Z) = \sup_{0 < t_0 < \ldots < t_J \in Z} \left( \sum_{j=0}^{J} |a_{t_{j+1}}(x) - a_{t_j}(x)|^r \right)^{1/r},
\]

where \( Z \) is a subset of \( (0, \infty) \) and the supremum is taken over all finite increasing sequences in \( Z \). Usually \( r \)-variation \( V_r \) defined over the dyadic set \( Z = \{ 2^n : n \in \mathbb{Z} \} \) is called the long \( r \)-variation seminorm. In order to avoid some problems with measurability of \( V_r(a_t(x) : t \in Z) \) we assume that \( (0, \infty) \ni t \mapsto a_t(x) \) is always a continuous function for every \( x \in \mathbb{R}^d \). The \( r \)-variational seminorm is an invaluable tool in pointwise convergence problems. If for some \( r \in [1, \infty) \) and \( x \in \mathbb{R}^d \) we have

\[
V_r(a_t(x) : t > 0) < \infty
\]

then the limits \( \lim_{t \to 0} a_t(x) \) and \( \lim_{t \to \infty} a_t(x) \) exist. So we do not need to establish pointwise convergence on a dense class, which in many cases is a challenging problem, see [11] and the references given there. Moreover, \( V_r \)'s control the supremum norm in the following sense, for any \( t_0 > 0 \) we have the pointwise estimate

\[
\sup_{t > 0} |a_t(x)| \leq |a_{t_0}(x)| + 2V_r(a_t(x) : t > 0).
\]

There is an extensive literature about estimates for \( r \)-variational seminorms. However for our purposes the most relevant will be [9], [10] and [11], see also the references given there.

One of the main results of this paper is the following theorem.

Theorem 1.1. Let \( p \in (3/2, 4) \) and \( r \in (2, \infty) \). Then there exists a constant \( C_{p, r} > 0 \) independent of the dimension \( d \in \mathbb{N} \) and such that for every symmetric convex body \( G \subset \mathbb{R}^d \) we have

\[
\|V_r(M^G_t f : t > 0)\|_{L^p} \leq C_{p, r}\|f\|_{L^p}
\]

for all \( f \in L^p(\mathbb{R}^d) \).
Theorem 1.2. Let \( p \in (1, \infty) \) and \( r \in (2, \infty) \). Then there exists a constant \( C_{p,r} > 0 \) independent of the dimension \( d \in \mathbb{N} \) and such that for every symmetric convex body \( G \subset \mathbb{R}^d \) we have
\[
\left\| V_r \left( M_{n}^{G} f : n \in \mathbb{Z} \right) \right\|_{L^p} \leq C_{p,r} \| f \|_{L^p}
\]
for all \( f \in L^p(\mathbb{R}^d) \).

If we restrict our attention to the balls induced by small \( \ell^q \) norms in \( \mathbb{R}^d \) then we can obtain the full range of \( p \)'s in Theorem 1.1. To be more precise for \( q \in [1, \infty) \) let us define these balls
\[
B_q = \left\{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d : |x|_q = \left( \sum_{1 \leq k \leq d} |x_k|^q \right)^{1/q} \leq 1 \right\}, \quad \text{and}
\]
\[
B_\infty = \left\{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d : |x|_\infty = \max_{1 \leq k \leq d} |x_k| \leq 1 \right\}.
\]

Then we have the following theorem.

Theorem 1.3. Suppose that \( G = B_q \) is one of the balls defined in (1.4) for some \( q \in [1, \infty] \). Let \( p \in (1, \infty) \) and \( r \in (2, \infty) \). Then there exists a constant \( C_{p,q,r} > 0 \) independent of the dimension \( d \in \mathbb{N} \) and such that
\[
\left\| V_r \left( M_{\ast}^{G} f : t > 0 \right) \right\|_{L^p} \leq C_{p,q,r} \| f \|_{L^p}
\]
holds for all \( f \in L^p(\mathbb{R}^d) \).

The range for parameter \( r \in (2, \infty) \) in Theorem 1.1, Theorem 1.2 and Theorem 1.3 is sharp, see [10].

Dimension dependent versions of Theorems 1.1 and 1.2 with sharp ranges of parameters \( p \in (1, \infty) \) and \( r \in (2, \infty) \), follow from [11, Theorem A.1]. Also related to our results is the paper of Jones, Seeger and Wright [10]. Especially, [10, Theorem 1.4], where \( L^p \) bounds for \( r \)-variations associated with the spherical averages have been established. Their estimates, however, depend on the dimension.

For a symmetric convex body \( G \subset \mathbb{R}^d \), \( x \in \mathbb{R}^d \) and \( f \in L^1_{\text{loc}}(\mathbb{R}^d) \) we set
\[
M^G_{\ast} f(x) = \sup_{t > 0} M^G_t f(x).
\]

The present paper may be thought of as a variational counterpart for a series of articles establishing dimension-free bounds on \( L^p(\mathbb{R}^d) \) for \( M^G_p \) and various symmetric convex bodies \( G \). The starting point of this line of research was the work of the third author [15], where he obtained the dimension-free bounds on \( L^p(\mathbb{R}^d) \), with \( p \in (1, \infty) \), for \( M^{B_2} \), where \( B_2 \) is the Euclidean ball. The fully corresponding result for \( r \)-variations is our Theorem 1.3 for \( G = B_2 \), see also Theorem A.1 in the Appendix. Then, the first author proved a dimension-free estimate on \( L^2(\mathbb{R}^d) \) for \( M^G \) when \( G \) is a general symmetric convex body, see [7].

This result has been independently extended by the first author and Carbery [6] to \( L^p(\mathbb{R}^d) \) for \( p \in (3/2, \infty] \). An \( r \)-variational counterpart of [11, 2, and 6] is our Theorem 1.3. Contrary to these papers we also have to restrict \( p \) from above by 4. Removing this restriction does not seem easy, as we do not have a natural endpoint estimate to interpolate. In [11, 2, and 6], such an estimate was the trivial bound for \( M^G \) on \( L^\infty(\mathbb{R}^d) \). Next, imposing a certain geometric constraint, Müller [14] enlarged the range of \( p \)'s, for which one has dimension-free bounds on \( L^p(\mathbb{R}^d) \) for \( M^G \), to all \( p \in (1, \infty) \). His result includes the cases when \( G = B_q \) with \( q \in [1, \infty) \). The most recent development in the study of dimension-free bounds for averaging operators is [3], in which the first author provided dimension-free estimates on \( L^p(\mathbb{R}^d) \), with \( p \in (1, \infty) \) for the cubes \( G = B_\infty \). Theorem 1.3 is a variational counterpart of the maximal results from [12] and [4] for \( G = B_2 \) with \( q \in [1, \infty) \).

Now, let us describe Müller’s result more precisely. Let \( G \) be a symmetric convex body in \( \mathbb{R}^d \). By the argument from [11], there exists an invertible linear transformation \( U \in \text{Gl}(\mathbb{R}^d) \) and a constant \( L(G) > 0 \) such that
\[
\text{Vol}_d U(G) = 1 \quad \text{and} \quad \int_{U(G)} (\xi \cdot x)^2 dx = L(G)^2
\]
for all unit vectors \( \xi \in S^{d-1} = \{ y \in \mathbb{R}^d : |y| = 1 \} \), where \( \xi \cdot x = (\xi, x) \) is the standard inner product in \( \mathbb{R}^d \) and \( |y| = |y|_2 \) is the Euclidean norm in \( \mathbb{R}^d \) as in (1.3) with \( q = 2 \). It is not difficult to note that \( L(G) \) is determined uniquely by (1.6), moreover \( U \) is determined uniquely up to multiplication from the left by an orthogonal transformation of \( \mathbb{R}^d \).
Similarly as in [1], for every $\xi \in S^{d-1}$ and $u \in \mathbb{R}$, we define
\begin{equation}
\varphi_\xi(u) = \text{Vol}_{d-1}\{x \in U(G) : (\xi \cdot x) = u\}.
\end{equation}
Moreover, we define the constants
\begin{align}
\sigma(G)^{-1} &= \max \{\varphi_\xi(0) : \xi \in S^{d-1}\}, \\
Q(G) &= \max \{\text{Vol}_{d-1}(\pi_\xi(U(G))) : \xi \in S^{d-1}\},
\end{align}
where $\pi_\xi : \mathbb{R}^d \to \mathbb{R}^\perp$ denotes the orthogonal projection of $\mathbb{R}^d$ onto the hyperplane perpendicular to $\xi$.

We note that $\sigma(V(G)) = \sigma(G)$ and $Q(V(G)) = Q(G)$ for any $V \in \text{Gl}(\mathbb{R}^d)$. Moreover, in [1] it was proven that there is a universal constant $a > 0$ such that $a^{-1}L(G) \leq \sigma(G) \leq aL(G)$.

Using these two linear invariants $\sigma(G)$ and $Q(G)$ Müller estimated $\|M^G_{\ast}\|_{L^p \to L^p}$. Namely, the main result of [12] states that for every $p \in (1, \infty]$ and for every symmetric convex body $G \subset \mathbb{R}^d$ there is a constant $C(p, \sigma(G), Q(G)) > 0$ independent of the dimension $d$ such that
\begin{equation}
\|M^G_{\ast}\|_{L^p \to L^p} \leq C(p, \sigma(G), Q(G)).
\end{equation}
In other words $\|M^G_{\ast}\|_{L^p \to L^p}$ may depend on $\sigma(G)$ and $Q(G)$, but not explicitly on the dimension $d$. In fact [2] and [6] show that for $p \in (3/2, \infty]$ the norm $\|M^G_{\ast}\|_{L^p \to L^p}$ can be even chosen independently of $\sigma(G)$ and $Q(G)$. Using (1.11) Müller showed that $\|M^G_{\ast}\|_{L^p \to L^p}$ is independent of the dimension for all $G = B_q$ with $q \in [1, \infty]$, since $\sigma(B_q)$ and $Q(B_q)$ can be explicitly computed and they are independent of $d$. However, for the cubes $G = B_\infty$ it turned out that $\sigma(B_\infty)$ is independent of the dimension, but $Q(B_\infty) = \sqrt{d}$. The question about the dimension-free bounds for the cubes in $\mathbb{R}^d$ for $p \in (1, 3/2]$ was left open until [4], where the first author significantly refined and extended the methods from [12]. Using the product nature of the cubes he showed that for every $p \in (1, \infty]$ there is a constant $C_p > 0$ independent of the dimension $d$ such that
\begin{equation}
\|M^G_{\ast}\|_{L^p \to L^p} \leq C_p.
\end{equation}
The question whether $\|M^G_{\ast}\|_{L^p \to L^p}$ with $p \in (1, 3/2]$ can be controlled by a constant independent of the dimension for general symmetric convex bodies remains still open. The situation is even more complicated for $r$-variational estimates. A question which we are unable to answer is what happens not only when $p \in (1, 3/2]$, but also for $p \in [4, \infty]$ in the case of general symmetric convex bodies. However the method of the proof of Theorem 1.3 in view of Müller’s result, allows us to deduce that for every $p \in (1, \infty)$, for every $r \in (2, \infty)$ and for every symmetric convex body $G \subset \mathbb{R}^d$ there is a constant $C(p, r, \sigma(G), Q(G)) > 0$ independent of $d$ such that
\begin{equation}
\sup_{\|f\|_{L^p} \leq 1} \|V_r(M^G_{\ast}f : t > 0)\|_{L^p} \leq C(p, r, \sigma(G), Q(G)).
\end{equation}

Theorem 1.1 and Theorem 1.3 give some partial evidence to support the conjecture which asserts that for all $p \in (1, \infty)$ and $r \in (2, \infty)$ and for every general symmetric convex body $G \subset \mathbb{R}^d$ the left-hand side of (1.11) can be controlled by a constant independent of the dimension and the linear invariants $\sigma(G)$ and $Q(G)$.

Let us briefly describe the strategy for proving our main results. The first step is splitting the consideration into long and short variations, see (2.3). The long variations are treated in Theorem 1.2 by appealing to known dimension-free estimates for $r$-variations of the Poisson semigroup (see [3] Theorem 3.3]) together with a square function estimate. The square function estimate is proved by using a dimension-free Littlewood–Paley theory, which allows us to prove very good estimates on $L^2$ and acceptable estimates on $L^p$. Then interpolation establishes Theorem 1.2. In view of Theorem 1.2 in order to prove Theorems 1.1 and 1.3 it is enough to consider short variations. To this end we also use a dimension-free Littlewood–Paley theory and interpolation between $L^2$ and $L^p$ bounds. The analysis of short variations breaks basically into two cases, whether $p \in [2, \infty]$ or $p \in (1, 2]$. In the first case we rely on the numerical inequality, Lemma 2.1 which reduces estimates for $r$-variations to the situation where the division intervals over which differences are taken are all of the same size. The case $r = 2$ of this is particularly suited to an application of the Fourier transform. On the other hand, when $p \in (1, 2]$ we use an orthogonality principle (an $r$-variation adaptation of an idea in [5]), together with an appropriate characterization of $r$-variation in terms of fractional derivatives, given in Proposition 2.1. Of course fractional derivatives had already occurred in [6] and [12], as well as the earlier proof of the spherical maximal theorem for $d \geq 3$. It should be pointed out that for $p \in (1, 2)$ it is essential that we can use $r$-variational estimates for $r \in (1, 2)$. 
1.1. Applications. The results from the previous paragraph have an ergodic theoretical interpretation. Namely, let \((X, \mathcal{B}, \mu)\) be a \(\sigma\)-finite measure space with families of commuting and measure-preserving transformations \((T_1^t : t \in \mathbb{R}), (T_2^t : t \in \mathbb{R}), \ldots, (T_d^t : t \in \mathbb{R})\), which map \(X\) to itself. For every symmetric convex body \(G \subset \mathbb{R}^d\) for every \(x \in X\) and \(f \in L^1(X, \mu)\) we define the ergodic Hardy–Littlewood averaging operator by setting
\[
\mathcal{A}_t^G f(x) = \frac{1}{|G_t \cap \mathbb{Z}^d|} \sum_{y \in G_t \cap \mathbb{Z}^d} f(T_1^{ty_1} \circ T_2^{ty_2} \circ \ldots \circ T_d^{ty_d} x).
\]

For this operator we also have dimension free \(r\)-variational estimates.

**Theorem 1.4.** Let \(p \in (3/2, 4)\) and \(r \in (2, \infty)\). Then there exists a constant \(C_{p,r} > 0\) independent of the dimension \(d \in \mathbb{N}\) and such that for every symmetric convex body \(G \subset \mathbb{R}^d\) we have
(1.13) \[
\|V_r(A_t^G f : t > 0)\|_{L^p(X, \mu)} \leq C_{p,r} \|f\|_{L^p(X, \mu)}
\]
for all \(f \in L^p(X, \mu)\). Moreover, if we consider only long variations, then (1.13) remains true for all \(p \in (1, \infty)\) and \(r \in (2, \infty)\) and we have
(1.14) \[
\|V_r(A_n^G f : n \in \mathbb{Z})\|_{L^p(X, \mu)} \leq C_{p,r} \|f\|_{L^p(X, \mu)}.
\]

Theorem (1.3) is an ergodic counterpart of Theorem 1.4 and Theorem 1.5. If \(G = B_q\) for some \(q \in [1, \infty]\) then we obtain sharp ranges of exponents with respect to the parameters \(p \) and \(r\) for the operator (1.2). Namely, we can prove an analogue of Theorem 1.3.

**Theorem 1.5.** Suppose that \(G = B_q\) is one of the balls defined in (1.4) for some \(q \in [1, \infty]\). Let \(p \in (1, \infty)\) and \(r \in (2, \infty)\). Then there exists a constant \(C_{p,r} > 0\) independent of the dimension \(d \in \mathbb{N}\) and such that
(1.15) \[
\|V_r(A_t^G f : t > 0)\|_{L^p(X, \mu)} \leq C_{p,q,r} \|f\|_{L^p(X, \mu)}
\]
holds for all \(f \in L^p(X, \mu)\).

In Section 5 we prove a transference principle (see Proposition 5.1), which allows us to deduce inequalities (1.13), (1.14) and (1.15) from the corresponding estimates in (1.2), (1.3), and (1.5), respectively.

The remarkable feature of Theorem 1.4 and Theorem 1.5 is that the implied bounds in (1.13), (1.14), and (1.15) are independent of the number of underlying commuting and measure-preserving transformations \(T_1^{ty_1}, \ldots, T_d^{ty_d}\). On the other hand, due to the properties of \(r\)-variational seminorm we immediately know that the limits
\[
\lim_{t \to 0^+} \mathcal{A}_t^G f(x), \quad \text{and} \quad \lim_{t \to \infty} \mathcal{A}_t^G f(x)
\]
exist almost everywhere on \(X\) for every \(f \in L^p(X, \mu)\) and \(p \in (1, \infty)\). It is worth emphasizing that although the pointwise convergence for \(M_t^G f\) as \(t \to 0\) (respectively \(t \to \infty\)) can be easily deduced from the maximal bounds, this is much harder for \(A_t^G f\). For \(M_t^G f\) there are many natural dense subspaces which could be used to establish pointwise convergence. However, for \(A_t^G f\), which is defined on an abstract measure space there is no obvious way how to even find such a candidate for a dense class. Fortunately, \(r\)-variational estimates allow us to obtain the desired conclusion directly.

At this stage a similar question concerning the discrete analogue of (1.12) arises. One can ask about \(r\)-variational estimates independent of the dimension for the following operator
\[
\mathcal{M}_t^G f(x) = \frac{1}{|G_t \cap \mathbb{Z}^d|} \sum_{y \in G_t \cap \mathbb{Z}^d} f(T_1^{ty_1} \circ T_2^{ty_2} \circ \ldots \circ T_d^{ty_d} x).
\]

This immediately lead us to the averaging operators on \(\mathbb{Z}^d\), i.e.
\[
\mathcal{M}_t^G f(x) = \frac{1}{|G_t \cap \mathbb{Z}^d|} \sum_{y \in G_t \cap \mathbb{Z}^d} f(x - y).
\]

However, nothing is known in the discrete setup, even the dimension free estimates for the maximal functions \(\mathcal{M}_t^G f(x) = \sup_{t > 0} |\mathcal{M}_t^G f(x)|\) are unknown. There is no easy way to derive these estimates from the corresponding dimension free estimates of the continuous analogue of \(\mathcal{M}_t^G f\). In the ongoing project [5] we initiated investigations in this direction and we are able to provide dimension free bounds for the discrete maximal functions \(\mathcal{M}_t^G f\) on \(L^p(\mathbb{Z}^d)\) in a certain range of parameters \(p\). We only handled the case of the Euclidean balls \(G = B_2\) and the cubes \(G = B_\infty\). In the case of the discrete cubes we can also provide \(r\)-variational estimates (in some range of \(p\) and \(r\)) and the results from this paper find applications there.
1.2. Notation and basic reductions. Let \( S(\mathbb{R}^d) \) denote the set of all Schwartz functions on \( \mathbb{R}^d \). To prove our results, by a simple density argument, it suffices to establish inequalities (1.12), (1.13) and (1.15) for all \( f \in S(\mathbb{R}^d) \). From now on, in the proofs presented in the paper \( f \) will be always a Schwartz function.

Let \( U \) be an invertible linear transformation of \( \mathbb{R}^d \) and let \( U_p \) be the isometry of \( L^p \) given by

\[
U_p f(x) = |\det U|^{-1/p} f(U^{-1}x) \quad \text{for any} \quad p \geq 1.
\]

Then we have

\[
U_p(\mathcal{V}_t(M_t^G f : t > 0)) = \mathcal{V}_t(U_p(M_t^G f) : t > 0) = \mathcal{V}_t(M_t^{U(G)}(U_p f) : t > 0),
\]

since

\[
U_p \circ M_t^G = M_t^{U(G)} \circ U_p.
\]

Therefore \( G \) in (1.11) can be freely replaced with any other symmetric convex body \( U(G) \) and \( L^p \) bounds remain unchanged. This is an important remark which allows us to assume that (1.7) always holds. From now on we will assume that \( U(G) = G \). More precisely we will assume that Vol\(_d\) \( G = |G| = 1 \) and that \( G \) is in the isotropic position, which means that there is an isotropic constant \( L = L(G) > 0 \) such that for every unit vector \( \xi \in \mathbb{S}^{d-1} \) we have

\[
(1.16) \quad \int_G (\xi \cdot x)^2 \, dx = L(G)^2.
\]

As in [1] the Fourier methods will be extensively exploited here to establish \( L^2 \) bounds in the aforementioned theorems. Let us define the Fourier transform \( \mathcal{F} \) of a function \( f \in S(\mathbb{R}^d) \) by setting

\[
\mathcal{F} f(\xi) = \mathcal{F}_{\mathbb{R}^d} f(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x} \, dx
\]

for any \( \xi \in \mathbb{R}^d \) and let \( \mathcal{F}^{-1} = \mathcal{F}_{\mathbb{R}^d}^{-1} \) denote the inverse Fourier transform on \( \mathbb{R}^d \).

Since \( |G| = 1 \) we see that the kernel of (1.11) satisfies

\[
|G_t|^{-1} \mathbb{1}_G(x) = t^{-d} K_G(t^{-1} x)
\]

for all \( t > 0 \), where \( K_G(x) = \mathbb{1}_G(x) \) and

\[
\mathcal{F}(|G_t|^{-1} \mathbb{1}_G)(\xi) = m^G(t \xi),
\]

where

\[
m^G(\xi) = \mathcal{F}(K_G)(\xi) = \mathcal{F}(\mathbb{1}_G)(\xi).
\]

The isotropic position of \( G \) allows us to provide dimension-free estimates for the multiplier \( m \).

**Proposition 1.1** ([1] eq. (10),(11),(12)). Given a symmetric convex body \( G \subset \mathbb{R}^d \) with volume one, which is in the isotropic position, there exists a constant \( C > 0 \) such that for every \( \xi \in \mathbb{R}^d \) we have

\[
|m^G(\xi)| \leq C |\xi|^{-1}, \quad |m^G(\xi) - 1| \leq C L |\xi|, \quad |\xi, \nabla m^G(\xi)| \leq C.
\]

The constant \( L = L(G) \) is defined in (1.16), while \( C \) is a universal constant which does not depend on \( d \).

Throughout the whole paper \( d \in \mathbb{N} \) denotes the dimension. Unless otherwise stated \( C > 0 \) will stand for an absolute constant whose value may vary from occurrence to occurrence and it will never depend on the dimension. We will use the convention that \( A \lesssim B \) (\( A \gtrsim B \)) to say that there is an absolute constant \( C_\delta > 0 \) (which possibly depends on \( \delta > 0 \)) such that \( A \leq C_\delta B \) (\( A \geq C_\delta B \)). We abbreviate \( A \lesssim B \) when the implicit constant is independent of \( \delta \). We will write \( A \asymp B \) when \( A \lesssim B \) and \( A \gtrsim B \) hold simultaneously.

For simplicity of the notation we will often abbreviate \( M_t = M_t^G \) and \( m = m^G \).

**Acknowledgements**

The authors are grateful to the referees for careful reading of the manuscript and useful remarks that led to the improvement of the presentation.

2. Useful tools

In this section we gather some general tools which will be used in the proofs of our main results.
2.1. Properties of r-variations. We begin with some simple properties of r-variation seminorms. For \( r \in [1, \infty) \) the r-variation seminorm \( V_r \) of a complex-valued function \((0, \infty) \ni t \mapsto a_t\) is defined by
\[
V_r(a_t : t \in \mathbb{Z}) = \sup_{J \in \mathbb{N}} V_r^J(a_t : t \in \mathbb{Z}),
\]
where \( \mathbb{Z} \subseteq (0, \infty) \) and
\[
V_r^J(a_t : t \in \mathbb{Z}) = \sup_{0 < t_0 < \ldots < t_J \in \mathbb{Z}} \left( \sum_{j=0}^{J} |a_{t_{j+1}} - a_{t_j}|^r \right)^{1/r}
\]
and the supremum is taken over all finite increasing sequences in \( \mathbb{Z} \) of length \( J + 1 \).

The function \( r \mapsto V_r(a_t : t \in \mathbb{Z}) \) is non-increasing and for any \( U \subseteq V \subseteq \mathbb{Z} \) satisfies
\[
V_r(a_t : t \in \mathbb{Z}) \leq V_r(a_t : t \in \mathbb{Z}).
\]
Moreover, for every \( t_0 \in \mathbb{Z} \) we have
\[
\sup_{t \in \mathbb{Z}} |a_t| \leq |a_{t_0}| + 2 V_r(a_t : t \in \mathbb{Z}).
\]
If \( \mathbb{Z} \) is a countable set, then
\[
V_r(a_t : t \in \mathbb{Z}) \leq 2 \left( \sum_{t \in \mathbb{Z}} |a_t|^r \right)^{1/r}.
\]
Finally, for every \( r \in [1, \infty) \) there exists \( C_r > 0 \) such that
\[
V_r(a_t : t \geq 0) \leq C_r V_r(a_{2^n} : n \in \mathbb{Z}) + C_r \left( \sum_{n \in \mathbb{Z}} \left( V_r((a_t - a_{2^n}) : t \in [2^n, 2^n + 1]) \right)^{1/r} \right).
\]

The first quantity on the right side in (2.3) is called the long variation seminorm, whereas the second is called the short variation seminorm. This is a very useful inequality which, in view of Theorem 1.2, will permit us to reduce the proofs of Theorem 1.1 and Theorem 1.4 to the estimates of short variations associated with \( M_1^2 \). In order to deal with the short variations efficiently we will prove an elementary inequality (2.4) which allows us to dominate each dyadic block in the short variations by suitable square functions, which are simpler objects to handle.

**Lemma 2.1.** Given \( r \in [1, \infty) \), \( n \in \mathbb{Z} \), and a continuous function \( a : [2^n, 2^{n+1}] \to \mathbb{C} \), we have
\[
V_r(a_t : t \in [2^n, 2^{n+1}]) \leq \left( \sum_{m \geq -n} \sum_{k=0}^{2^{m+n-1}} |a_{2^{m+n}+(k+1)/2m} - a_{2^{m+n}+k/2m}|^r \right)^{1/r}
\]
\[
= 2^{1-1/r} \sum_{l \geq 0} \left( \sum_{k=0}^{2^{l-1}} |a_{2^{n+2^{n-l-1}}(k+1)} - a_{2^{n+2^{n-l}}}|^r \right)^{1/r}.
\]

**Proof.** First of all we observe that any interval \([s, t]\) for some real numbers \( 2^n \leq s < t < 2^{n+1} \) can be written as a disjoint sum (possibly infinite sum) of dyadic intervals, i.e.
\[
[s, t] = \bigcup_{l \in \mathbb{Z}} [w_l, w_{l+1}]
\]
with the following properties:

- Each \([w_l, w_{l+1}]\) is in \( I_m \), for some \( m \geq -n \), where
  \[
  I_m = \left\{ \left[ 2^n + \frac{k}{2^m}, 2^n + \frac{k+1}{2^m} \right] : 0 \leq k \leq 2^{m+n} - 1 \right\}.
  \]
- There are at most two dyadic intervals on the right-hand side of (2.5) with the same length.

To prove (2.4) let us consider dyadic intervals of maximal length contained in \( A = [s, t] \). There are at most two such intervals. Let \( I_0 \) be the one which lies closer to the left endpoint \( s \). Then \([s, t] \setminus I_0\) is a sum of at most two intervals \( A_1, B_1 \). Without loss of generality we may assume that \( s \in A_1 \). Now let \( I_1 \) be a dyadic interval contained in \( A_1 \) with maximal length such that \( \text{dist}(I_1, I_0) = 0 \). Then by the maximality of \( I_0 \) we see that \(|I_1| \leq |I_0|/2\). Now we define \( A_2 = A_1 \setminus I_1 \). If \( A_2 \neq \emptyset \) then we proceed as in the previous step. Namely we take a dyadic interval \( I_2 \) contained in \( A_2 \) with maximal length such that \( \text{dist}(I_2, I_1) = 0 \). By the maximality of \( I_1 \) we see that \(|I_2| \leq |I_1|/2\). Now we define \( A_3 = A_2 \setminus I_2 \). If \( A_3 \neq \emptyset \) then we proceed likewise above. Then inductively we obtain a sequence of disjoint dyadic
intervals \((I_j : j \in \mathbb{N})\) such that \(|I_1| > |I_2| > \ldots\) and \(A_1 = \bigcup_{j \in \mathbb{N}} I_j\). If \(B_1\) is empty we are done. If not then we can repeat the argument as for \(A_1\) and obtain a sequence of disjoint dyadic intervals \((J_j : j \in \mathbb{N})\) contained in \(B_1\) such that \(|J_1| > |J_2| > \ldots\) and \(B_1 = \bigcup_{j \in \mathbb{N}} J_j\). Thus we have proven that

\[
[s, t) = \bigcup_{j \in \mathbb{N}} I_j \cup I_0 \cup \bigcup_{j \in \mathbb{N}} J_j.
\]

From the construction described above it is clear that there are at most two dyadic intervals in the sum which have the same length.

Having proven (2.5) we can show (2.4). Namely, let \(2^n \leq t_0 < t_1 < \ldots < t_J < 2^{n+1}\) be any increasing sequence. By (2.5) for any integer \(0 \leq j < J\) we write

\[
[t_j, t_{j+1}) = \bigcup_{l \in \mathbb{Z}} [w^l_j, w^l_{j+1}),
\]

where each \([w^l_j, w^l_{j+1})\) is a dyadic interval which belongs to \(I_m\) for some \(m \geq -n\). Thus

\[
|a_{t_{j+1}} - a_{t_j}| \leq \sum_{l \in \mathbb{Z}} |a_{w^l_{j+1}} - a_{w^l_j}| = \sum_{m \geq -n} \sum_{l \in \mathbb{Z}} |a_{w^l_{j+1}} - a_{w^l_j}|
\]

and \(|\{l \in \mathbb{Z} : [w^l_j, w^l_{j+1}) \in I_m\}| \leq 2\) for any \(m \geq -n\). Hence, we obtain

\[
\left(\sum_{j=0}^{J-1} |a_{t_{j+1}} - a_{t_j}|^r\right)^{1/r} \leq \left(\sum_{j=0}^{J-1} \left(\sum_{m \geq -n} \sum_{l \in \mathbb{Z}} |a_{w^l_{j+1}} - a_{w^l_j}|\right)^r\right)^{1/r}
\]

\[
\leq \sum_{m \geq -n} \left(\sum_{j=0}^{J-1} \sum_{l \in \mathbb{Z}} |a_{w^l_{j+1}} - a_{w^l_j}|\right)^{1/r} \quad \text{by triangle inequality}
\]

\[
\leq 2^{1-1/r} \sum_{m \geq -n} \left(\sum_{j=0}^{J-1} \sum_{l \in \mathbb{Z}} |a_{w^l_{j+1}} - a_{w^l_j}|\right)^{1/r} \quad \text{by Hölder’s inequality.}
\]

The last sum is dominated by the right-hand side of (2.4), since \([t_j, t_{j+1}) \cap [t_{j'}, t_{j'+1}) = \emptyset\) for \(j \neq j'\) and the proof is completed.

We finish this section with an approximate characterization of the \(r\)-variation seminorm, which is interesting in its own right. In the proposition stated below we will be dealing with functions \(F\) defined on \(\mathbb{R}\) that are compactly supported and belong to \(L^r(\mathbb{R})\) for some \(r \in [1, \infty]\). For such a function \(F\) we will write a fractional derivative \(D^\alpha F\) defined as a tempered distribution by the formula

\[
\mathcal{F}_\mathbb{R}(D^\alpha F)(\xi) = \left(2\pi |\xi|\right)^{\alpha} \mathcal{F}_\mathbb{R} F(\xi),
\]

for every \(\xi \in \mathbb{R}\) and \(\alpha \in [0, 1]\). Here \(\mathcal{F}_\mathbb{R}\) stands for the Fourier transform on \(\mathbb{R}\) given by

\[
\mathcal{F}_\mathbb{R} f(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i \xi x} dx.
\]

**Proposition 2.1.** Let \(F\) be a complex-valued function with a compact support in \(\mathbb{R}\).

(i) Suppose that \(F\) and \(D^\alpha F\) are in \(L^r(\mathbb{R})\) for some \(r \in [1, \infty]\) and \(\alpha > 1/r\). Then there is \(C_r > 0\) such that

\[
V_r(F(t) : t \in \mathbb{R}) \leq C_r \left(\|F\|_{L^r(\mathbb{R})} + \|D^\alpha F\|_{L^r(\mathbb{R})}\right).
\]

(ii) Conversely, assume that \(F \in L^r(\mathbb{R})\) and \(V_r(F(t) : t \in \mathbb{R}) < \infty\). Then for every \(\beta < 1/r\) there is \(C_{\beta, r} > 0\) such that

\[
\|D^\beta F\|_{L^r(\mathbb{R})} \leq C_{\beta, r} \left(\|F\|_{L^r(\mathbb{R})} + V_r(F(t) : t \in \mathbb{R})\right).
\]

**Proof.** We begin with demonstrating part (i). Let \(J_\alpha\) be the Bessel potential operator defined for any \(h \in L^r(\mathbb{R})\) by

\[
J_\alpha(h)(x) = G_\alpha \ast h(x),
\]

where

\[
\mathcal{F}_\mathbb{R} G_\alpha(\xi) = (1 + 4\pi |\xi|^2)^{-\alpha/2}.
\]
It is known from [13] Chapter 5, Section 3.2 that there are finite measures \( \nu_\alpha \) and \( \lambda_\alpha \) such that
\[
(1 + 4\pi^2\|\xi\|^2)^{\alpha/2} = \hat{\nu}_\alpha(\xi) + (2\pi|\xi|)^\alpha \hat{\lambda}_\alpha(\xi).
\]
Thus one can represent \( F \) in terms of \( J_\alpha \). Namely, \( F = J_\alpha(f) \), where
\[
f = F \ast_R \nu_\alpha + (D^\alpha F) \ast_R \lambda_\alpha,
\]
and by our assumptions \( f \in L^r(\mathbb{R}) \). Moreover, for every \( t \in \mathbb{R} \) we have
\[
F(t) = \int_{\mathbb{R}} G_\alpha(u)f(t - u)\,du.
\]
For the Bessel kernel \( G_\alpha \) one has the following estimates (see [13] Chapter 5, Section 3.1), here \( n = 1 \)
\[
\tag{2.8}
G_\alpha(u) = O(|u|^{-1+\alpha}) \quad \text{and} \quad G_\alpha(u) \text{ is rapidly decreasing at infinity},
\]
and
\[
\tag{2.9}
\frac{d}{du} G_\alpha(u) = O(|u|^{-2+\alpha}) \quad \text{and} \quad \frac{d}{du} G_\alpha(u) \text{ is rapidly decreasing at infinity}.
\]
Note that this representation of \( F \), and the properties of \( G_\alpha \) stated immediately above, show that \( F \) may be taken to be continuous on \( \mathbb{R} \) and not merely defined almost-everywhere, and hence \( V_1(F(t): t \in \mathbb{R}) \) is well-defined. Now for any \( s = \sigma + i\tau \) consider \( F_s \) defined by
\[
F_s(t) = e^{(s-1+\alpha/2)\tau} \int_{\mathbb{R}} G_\alpha(u)|u|^{1-s-1/\tau} f(t - u)\,du.
\]
When \( s = 1 - 1/r \), then \( F_s(t) = F(t) \). Next, we show that
\[
\tag{2.10}
V_1(F_s(t): t \in \mathbb{R}) \leq C_0\|f\|_{L^1(\mathbb{R})}, \quad \text{if } \text{Re}(s) = 0.
\]
Indeed, when \( \text{Re}(s) = 0 \), then the derivative of \( F_s \) (more precisely, here we mean the weak derivative) can be estimated by
\[
\left| \frac{d}{dt} F_s(t) \right| \leq ce^{-\tau^2} \int_{\mathbb{R}} \left| \frac{d}{du} G_\alpha(u)|u|^{1-s-1/\tau} \right| |f(t - u)|\,du.
\]
Thus taking into account the properties of \( G_\alpha \) stated in (2.8) and (2.9) and the fact that \( \alpha > 1/r \) (which imply that \( F_s \in L^\infty(\mathbb{R}) \) and both \( F_s \) and \( \frac{d}{dt} F_s \) are in \( L^1(\mathbb{R}) \)), we obtain
\[
V_1(F_s(t): t \in \mathbb{R}) \leq \int_{\mathbb{R}} \left| \frac{d}{dt} F_s(t) \right| \,dt \leq C_0\|f\|_{L^1(\mathbb{R})},
\]
where
\[
C_0 := c'(1 + |\tau|)e^{-\tau^2} \left( \int_{|u|\leq 1} |u|^{-1+\alpha-1/\tau}\,du + \int_{|u|> 1} |u|^{-2}\,du \right) < \infty.
\]
On the other hand, we show
\[
\tag{2.11}
V_\infty(F_s(t): t \in \mathbb{R}) \leq 2C_1\|f\|_{L^\infty(\mathbb{R})}, \quad \text{if } \text{Re}(s) = 1.
\]
Indeed, when \( \text{Re}(s) = 1 \), then
\[
\sup_{t \in \mathbb{R}} |F_s(t)| \leq e^{1/\tau^2} \int_{\mathbb{R}} |G_\alpha(u)||u|^{1-1/r}\,du \|f\|_{L^\infty(\mathbb{R})},
\]
since
\[
C_1 := e^{1/\tau^2} \int_{\mathbb{R}} |G_\alpha(u)||u|^{1-1/r}\,du \leq c \left( \int_{|u|\leq 1} |u|^{-1+\alpha-1/\tau}\,du + \int_{|u|> 1} |u|^{-2}\,du \right) < \infty.
\]
Now the mappings \( f \mapsto F_s \) can be rephrased as an analytic family of operators as follows. Choose any sequence \( t_0 < t_1 < \cdots < t_N \) and define \( T_s(f) \) to be the sequence
\[
(F_s(t_k) - F_s(t_{k-1})): k \in \mathbb{Z}_N,
\]
where \( \mathbb{Z}_N = \{1, 2, \ldots, N\} \). Observe now that (2.10) and (2.11) imply that
\[
\|T_s(f)\|_{l^1(\mathbb{Z}_N)} \leq C_0\|f\|_{L^1(\mathbb{R})}, \quad \text{if } \text{Re}(s) = 0
\]
and
\[
\|T_s(f)\|_{l^\infty(\mathbb{Z}_N)} \leq 2C_1\|f\|_{L^\infty(\mathbb{R})}, \quad \text{if } \text{Re}(s) = 1.
\]
Now the complex interpolation theorem, see [17] Chapter 5, Section 4, shows that for \( s = 1 - 1/r \),
\[
\|T_s(f)\|_{l^\infty(\mathbb{Z}_N)} \leq C\|f\|_{L^r(\mathbb{R})}.
\]
As a result, since $F_s = F$ for $s = 1 - 1/r$, we obtain
\[
\left( \sum_{k=1}^{N} |F(t_k) - F(t_{k-1})|^r \right)^{1/r} \leq C \|f\|_{L^r(\mathbb{R})}.
\]
Since the constant $C$ depends only on the constants $C_0$, $C_1$, and $r$, therefore $C$ is independent of the choice of $(t_k : 0 \leq k \leq N)$ and $N$. This gives
\[
V_r(F(t) : t \in \mathbb{R}) \leq C \|f\|_{L^r(\mathbb{R})}
\]
and combined with
\[
\|f\|_{L^r(\mathbb{R})} \leq c(\|f\|_{L^r(\mathbb{R})} + \|D^\alpha F\|_{L^r(\mathbb{R})})
\]
proves part (i) and (2.6).

We now focus on part (ii). We can assume that $r < \infty$ for otherwise there is nothing to prove. Let $h \geq 0$ and $A = V_r(F(t) : t \in \mathbb{R}) < \infty$, then
\[
(2.12) \quad \sum_{k \in \mathbb{Z}} |F(v + (k + 1)h) - F(v + kh)| \leq A^r,
\]
Now integrate this inequality for $v$ in the interval $[0, h]$. Since
\[
\int_0^h |F(v + (k + 1)h) - F(v + kh)|^r \, dv = \int_{kh}^{(k+1)h} |F(v + h) - F(v)|^r \, dv,
\]
inserting this in (2.12) gives the modulus of continuity inequality
\[
\|F(\cdot + h) - F(\cdot)\|_{L^r(\mathbb{R})} \leq Ah^{1/r}, \quad \text{for } h \geq 0,
\]
from which the result for negative $h$ also follows yielding
\[
(2.13) \quad \|F(\cdot + h) - F(\cdot)\|_{L^r(\mathbb{R})} \leq V_r(F(t) : t \in \mathbb{R}) |h|^{1/r}, \quad \text{for } h \in \mathbb{R}.
\]
We now invoke the fact that for any $0 < \beta < 1$
\[
D^\beta F(t) = c_\beta \lim_{\varepsilon \to 0} \int_{|h| \geq \varepsilon} \frac{F(t + h) - F(t)}{|h|^{1+\beta}} \, dh,
\]
for a suitable constant, where the limit is taken in the sense of distributions (see [14] Chapter 5, 6.10 and the references given there). Writing
\[
D^\beta F(t) = c_\beta \lim_{\varepsilon \to 0} \int_{|h| \geq \varepsilon} \frac{F(t + h) - F(t)}{|h|^{1+\beta}} \, dh + c_\beta \int_{|h| > 1} \frac{F(t + h) - F(t)}{|h|^{1+\beta}} \, dh,
\]
we see that both terms belong to $L^r(\mathbb{R})$. To estimate the first term we apply Minkowski’s integral inequality and (2.13) and obtain
\[
\left( \int_\mathbb{R} \left| \int_{|h| \geq \varepsilon} \frac{F(t + h) - F(t)}{|h|^{1+\beta}} \, dh \right|^r \, dt \right)^{1/r} \leq V_r(F(t) : t \in \mathbb{R}) \int_{|h| \geq \varepsilon} \left| \frac{F(t + h) - F(t)}{|h|^{1+\beta}} \right| ^r \, dh
\]
\[
\leq C_\beta V_r(F(t) : t \in \mathbb{R}),
\]
since $\beta < 1/r$. The second term is controlled by
\[
\left( \int_\mathbb{R} \left| \int_{|h| \geq 1} \frac{F(t + h) - F(t)}{|h|^{1+\beta}} \, dh \right|^r \, dt \right)^{1/r} \leq 2 \|F\|_{L^r(\mathbb{R})} \int_{|h| \geq 1} \left| \frac{F(t + h) - F(t)}{|h|^{1+\beta}} \right| \, dh \leq C_\beta \|F\|_{L^r(\mathbb{R})},
\]
because $\beta > 0$ and $\|F(t + h)\|_{L^r(\mathbb{R})} = \|F\|_{L^r(\mathbb{R})}$. This proves part (ii) and (2.8). \hfill \square

2.2. An almost orthogonality principle. Now we prove an $r$-variational counterpart of an almost orthogonality principle from [6]. Proposition 2.2 will be a key ingredient in the proofs of Theorem 1.1 and Theorem 1.3 for $p < 2$.

**Proposition 2.2.** Suppose that $(X, \mathcal{B}, \mu)$ is a $\sigma$-finite measure space and let $(T_t : t \in Z)$ be a family of positive linear operator defined on $\bigcup_{1 \leq p \leq \infty} L^p(X, \mathcal{B}, \mu)$ for some index set $Z \subseteq (0, \infty)$. Moreover, for a given $p_0 \in [1, 2)$ the following conditions are satisfied:

\footnote{We say that a linear operator $T$ is positive, if $Tf \geq 0$ for every function $f \geq 0$.}
• There is a family of linear operators \((S_n : n \in \mathbb{Z})\) with the property that \(f = \sum_{n \in \mathbb{Z}} S_n f\), for any \(f \in L^2(X, \mathcal{B}, \mu)\). Moreover, for every \(p \in (1, 2]\) there exists \(C_{1,p} > 0\) for which for every \(f \in L^p(X, \mathcal{B}, \mu)\) we have

\[
\left\| \left( \sum_{n \in \mathbb{Z}} |S_n f|^2 \right)^{1/2} \right\|_{L^p} \leq C_{1,p} \|f\|_{L^p}.
\]

(2.14)

• For every \(p \in (p_0, 2]\) there exists \(C_{2,p} > 0\) such that for every \(f \in L^p(X, \mathcal{B}, \mu)\) we have

\[
\left\| \sup_{t \in \mathbb{Z}} |T_t f| \right\|_{L^p} \leq C_{2,p} \|f\|_{L^p}.
\]

(2.15)

• For every \(p \in (p_0, 2]\) there exists \(C_{3,p} > 0\) such that for every \(f \in L^p(X, \mathcal{B}, \mu)\) we have

\[
\sup_{n \in \mathbb{Z}} \left\| V_p (T_t f : t \in Z_n) \right\|_{L^p} \leq C_{3,p} \|f\|_{L^p},
\]

where \(Z_n = [2^n, 2^{n+1}) \cap \mathbb{Z}\).

• There exists a sequence \((a_j : j \in \mathbb{Z})\) of positive numbers such that \(\sum_{j \in \mathbb{Z}} a_j^p = A_p < \infty\) for every \(\rho > 0\) and there exists \(C_4 > 0\) such that for every \(j \in \mathbb{Z}\) and for every \(f \in L^2(X, \mathcal{B}, \mu)\) we have

\[
\left\| \left( \sum_{n \in \mathbb{Z}} V_2 (T_t S_{j+n} f : t \in Z_n) \right)^{1/2} \right\|_{L^2} \leq C_4 a_j \|f\|_{L^2}.
\]

(2.17)

Then for every \(p \in (p_0, 2]\), there exists \(C_p > 0\) such that for every \(f \in L^p(X, \mathcal{B}, \mu)\) we have

\[
\left\| \left( \sum_{n \in \mathbb{Z}} V_2 (T_t f : t \in Z_n) \right)^{1/2} \right\|_{L^p} \leq C_p \|f\|_{L^p}.
\]

(2.18)

Proof. Let us fix \(p \in (p_0, 2]\). We will show that there is \(C_p > 0\) such that for every \(L, N \in \mathbb{N}\) we have

\[
\left\| \left( \sum_{|n| \leq N} V_2^L (T_t f : t \in Z_n) \right)^{1/2} \right\|_{L^p} \leq C_p \|f\|_{L^p}.
\]

(2.19)

Letting \(N \to \infty\) and \(L \to \infty\) and invoking the monotone convergence theorem we obtain (2.13).

To prove (2.19) we reduce the problem to study a certain family of linear operators. For this purpose, for every \(n \in \mathbb{Z}\), let \(\mathcal{T}^N_n\) be a family of all sequences \(t_L^n = (t_L^l : 0 \leq l \leq L)\) such that each component \(t_L^l : \mathbb{R}^2 \to Z_n\) is a measurable function and \(t_L^n(x) < t_L^0(x) < \ldots < t_L^N(x)\) for any \(x \in \mathbb{R}^2\). We will prove that there is \(C_p > 0\) such that for every \(L, N \in \mathbb{N}\) we have

\[
\sup_{t_L^n \in \mathcal{T}^N_n} \ldots \sup_{t_L^1 \in \mathcal{T}^N_1} \left\| \left( \sum_{|n| \leq N} \sum_{i=0}^{L-1} |(T_{t_L^{i+1}} - T_{t_L^i}) f|^2 \right)^{1/2} \right\|_{L^p} \leq C_p \|f\|_{L^p}.
\]

(2.20)

Suppose for the moment that (2.20) is proven. We will show how it implies (2.19). For every integer \(|n| \leq N\), let \(t_L^n = (t_L^l : 0 \leq l \leq L)\) be a sequence of measurable functions with the properties as above and additionally satisfying

\[
V_2^L (T_t f(x) : t \in Z_n) = \sum_{l=0}^{L-1} |(T_{t_L^{i+1}} - T_{t_L^i}) f(x)|^2.
\]

Then, assuming (2.20), we obtain

\[
\left\| \left( \sum_{|n| \leq N} V_2^L (T_t f : t \in Z_n) \right)^{1/2} \right\|_{L^p} = \left\| \left( \sum_{|n| \leq N} \sum_{i=0}^{L-1} |(T_{t_L^{i+1}} - T_{t_L^i}) f|^2 \right)^{1/2} \right\|_{L^p} \leq \sup_{t_L^n \in \mathcal{T}^N_n} \ldots \sup_{t_L^1 \in \mathcal{T}^N_1} \left\| \left( \sum_{|n| \leq N} \sum_{i=0}^{L-1} |(T_{t_L^{i+1}} - T_{t_L^i}) f|^2 \right)^{1/2} \right\|_{L^p} \leq C_p \|f\|_{L^p},
\]

and (2.19) follows. Thus it suffices to prove (2.20).

Due to (2.11) we see that (2.20) holds with the constant \((2N + 1)C_{3,p}\). Our task now will be to show that (2.20) holds with the constant which is independent of \(L, N \in \mathbb{N}\). For this purpose fix
Thus

\[ (2.21) \quad \left( \sum_{|n| \leq N} \left( \frac{1}{|T_{T_{n+1}} - T_{T_{l}}|} \right)^{s/r} \right)^{1/s} \leq A_N(p, r, s) \left( \sum_{|n| \leq N} |g_n|^r \right)^{1/s} \subseteq L^p. \]

Let \( q \) be a real number such that \( p_0 < q < p < 2 \) and define \( \theta \in (0, 1) \) by setting
\[ \frac{1}{q} = \frac{1 - \theta}{q} + \frac{\theta}{\infty}. \]

This implies that \( \theta = 1 - q/2 \) and consequently determines \( u \in (q, p) \) such that
\[ \frac{1}{u} = \frac{1 - \theta}{q} + \frac{\theta}{p}. \]

Therefore, interpolation between \( L^q(\ell^q(n)) \) and \( L^p(\ell^p(n)) \) ensures that
\[ A_N(u, 2, 2) \leq A_N(q, q, q)^{1-\theta} A_N(p, \infty, \infty)^{\theta}. \]

Invoking (2.16) we have \( A_N(q, q, q) \leq C_{3,q} \), since

\[ \left( \sum_{|n| \leq N} \left( \frac{1}{|T_{T_{n+1}} - T_{T_{l}}|} \right)^{1/q} \right)^{1/q} \subseteq L^q. \]

Let \( g = \sup_{|n| \leq N} |g_n| \) and observe that \( A_N(p, \infty, \infty) \leq 2C_{2,p} \), since by (2.15), we obtain
\[ \sup_{|n| \leq N} \sup_{0 \leq l \leq L-1} |(T_{T_{n+1}} - T_{T_{l}})g_n| \subseteq L^p, \sup_{e \in Z} g \subseteq 2C_{2,p} g \subseteq L^p. \]

Thus
\[ (2.22) \quad A_N(u, 2, 2) \leq A_N(q, q, q)^{1-\theta} A_N(p, \infty, \infty)^{\theta} \leq C_{3,q}^{1-\theta}(2C_{2,p})^\theta. \]

By (2.21), (2.14) and (2.22) we get
\[ (2.23) \quad \left( \sum_{|n| \leq N} \left( \frac{1}{|T_{T_{n+1}} - T_{T_{l}}|} \right)^{1/2} \right)^{1/2} \subseteq L^2. \]

By (2.17) we get
\[ (2.24) \quad \left( \sum_{|n| \leq N} \left( \frac{1}{|T_{T_{n+1}} - T_{T_{l}}|} \right)^{1/2} \right)^{1/2} \subseteq L^2. \]

Now let us introduce \( \rho \in (0, 1) \) obeying
\[ \frac{1}{p} = \frac{1 - \rho}{u} + \frac{\rho}{2}. \]

Interpolating (2.23) with (2.24) we have
\[ (2.25) \quad \left( \sum_{|n| \leq N} \left( \frac{1}{|T_{T_{n+1}} - T_{T_{l}}|} \right)^{1/2} \right)^{1/2} \subseteq L^p. \]

Summing (2.25) over \( j \in Z \) we obtain
\[ \left( \sum_{|n| \leq N} \left( \frac{1}{|T_{T_{n+1}} - T_{T_{l}}|} \right)^{1/2} \right)^{1/2} \subseteq L^p. \]

which proves (2.20) and completes the proof of Proposition 2.2.

\[ \square \]
2.3. The Poisson semigroup. As in [11] we shall exploit dimension-free bounds for the Poisson semigroup \( P_t \) which, for every \( x \in \mathbb{R}^d \), satisfies

\[
\mathcal{F}(P_t f)(\xi) = p_t(\xi) \mathcal{F}(f)(\xi),
\]

where

\[
p_t(\xi) = e^{-2\pi t|\xi|^2}.
\]

For every \( x \in \mathbb{R}^d \) let us introduce the maximal function

\[
P_\ast f(x) = \sup_{t > 0} P_t[f](x)
\]

and the square function

\[
g(f)(x) = \left( \int_0^\infty t \left| \frac{d}{dt} P_t f(x) \right|^2 \, dt \right)^{1/2}
\]

associated with the Poisson semigroup. We know from [13] that for every \( p \in (1, \infty) \) there exists a constant \( C_p > 0 \) independent of the dimension such that for every \( f \in L^p(\mathbb{R}^d) \) we have

\[
\|P_\ast f\|_{L^p} \leq C_p \|f\|_{L^p}
\]

and

\[
\|g(f)\|_{L^p} \leq C_p \|f\|_{L^p}.
\]

We will also need some variant of the Littlewood–Paley inequality. Namely, we define for every \( n \in \mathbb{Z} \) the projections \( S_n = P_{t_{2^n}} - P_{t_{2^{n-1}}} \) corresponding to the Poisson semigroup, where \( L = L(G) \) is the constant defined in [1,13]. With this definition, we clearly have, for every \( f \in L^2(\mathbb{R}^d) \), that

\[
f = \sum_{n \in \mathbb{Z}} S_n f.
\]

Moreover, for each \( n \in \mathbb{Z} \) and \( x \in \mathbb{Z}^d \) we see

\[
S_n f(x) = \int_{L_{2^n-1}}^{L_{2^n}} \frac{d}{dt} P_t f(x) \, dt.
\]

Hence, by the Cauchy–Schwarz inequality we obtain

\[
|S_n f(x)|^2 \leq \left( \int_{L_{2^n-1}}^{L_{2^n}} \left| \frac{d}{dt} P_t f(x) \right| \, dt \right)^2 \leq L^{2^n-1} \int_{L_{2^n-1}}^{L_{2^n}} \left| \frac{d}{dt} P_t f(x) \right|^2 \, dt \leq \int_{L_{2^n-1}}^{L_{2^n}} \left| \frac{d}{dt} P_t f(x) \right|^2 \, dt.
\]

Now summing over \( n \in \mathbb{Z} \) and appealing to (2.27) we obtain that the following dimension-free Littlewood–Paley inequality

\[
\left\| \left( \sum_{n \in \mathbb{Z}} |S_n f|^2 \right)^{1/2} \right\|_{L^p} \leq \|f\|_{L^p} \leq C_p \|f\|_{L^p},
\]

for every \( f \in L^p(\mathbb{R}^d) \).

Finally, since the Poisson semigroup \( P_t \) is Markovian, [3, Theorem 3.3] states that for every \( p \in (1, \infty) \) and every \( r \in (2, \infty) \) there is a constant \( C_{p,r} > 0 \) independent of the dimension such that for every \( f \in L^p(\mathbb{R}^d) \) we have

\[
\|V_r (P_t f : t > 0)\|_{L^p} \leq C_{p,r} \|f\|_{L^p}.
\]

We include the proof of (2.30) for completeness. It suffices to prove that there is a constant \( C_{p,r} > 0 \) such that for every \( M \in \mathbb{N} \) we have

\[
\|V_r (P_t f : t \in \mathcal{D}_M)\|_{L^p} \leq C_{p,r} \|f\|_{L^p},
\]

where \( \mathcal{D}_M = \{n/2^M : n \geq 0\} \). Then letting \( M \to \infty \) in (2.31), by the monotone convergence theorem, we obtain (2.30). To prove (2.31) we recall Rota’s theorem from [13].

**Theorem 2.1** (Rota’s theorem). Assume that \((X, \mathcal{B}, \mu)\) is a \( \sigma \)-finite measure space and let \( Q \) be a linear operator defined on \( \bigcup_{1 \leq p \leq \infty} L^p(X, \mathcal{B}, \mu) \) satisfying the following conditions:

- \( \|Q\|_{L^p \to L^p} \leq 1 \) for every \( p \in [1, \infty] \),
- \( Q = Q^* \) in \( L^2(X, \mathcal{B}, \mu) \),
- \( Qf \geq 0 \) for every \( f \geq 0 \),
- \( Q1 = 1 \).
Thus the operator $Q_R$ all Lebesgue measurable sets on $(2.32)$

As there is no 'dimension' in the proof of Lépingle’s inequality (it is purely probabilistic)

where the last inequality follows from Lépingle’s inequality for martingales, see [10] and the references

the following inequality holds

(3.1)

To prove (2.31) we apply Rota’s theorem with $Q = P_{1/2^{r+1}}$ and with $X = \mathbb{R}^d$ and the $\sigma$-algebra $\mathcal{B}$ of all Lebesgue measurable sets on $\mathbb{R}^d$ and the Lebesgue measure $\mu = |\cdot|$, on $\mathbb{R}^d$. Indeed,

$$
\|V_r(P_tf: t \in \mathcal{D}_M)\|_{L^p} = \|V_r(P_{n/2^rf}: n \geq 0)\|_{L^p} = \|V_r(Q^{2n}f: n \geq 0)\|_{L^p} \leq \|\mathbb{E}[V_r(E[F|F_n]|\mathcal{F})]\|_{L^p(\Omega,\mathcal{F},\mu)} \\
\leq C_p,f \|f\|_{L^p(\mathcal{F},\mathbb{R})} = C_p,f \|f\|_{L^p(\mathbb{R})}
$$

where the last inequality follows from Lépingle’s inequality for martingales, see [10] and the references
given there. As there is no 'dimension' in the proof of Lépingle’s inequality (it is purely probabilistic)
the implied constant $C_{p,f}$ in (2.30) is independent of the dimension.

3. Estimates for long variations: proof of Theorem 1.2

In this section we prove Theorem 1.2. Fix $\rho \in (2, \infty)$ and a non-empty convex symmetric body $G$ in $\mathbb{R}^d$. Recall that $M_t = M_t^G$ and $m = m^G$, where $m^G(\xi) = F(1_G)(\xi)$. The main ingredients in the proof will be the $r$-variational estimate (2.30) for the Poisson semigroup $P_t$, the estimates for the multiplier $m$ associated with $M_t$ from Proposition 1.1, and the lacunary maximal result from [2] or [6], which states
that for every $p \in (1, \infty)$ there exists $C_{p,\infty} > 0$ such that every $d \in \mathbb{N}$ and for every convex body $G \subset \mathbb{R}^d$
the following inequality holds

(3.1)

$$
\sup_{n \in \mathbb{Z}} |M_{2^n} f|_{L^p} \leq C_{p,\infty} \|f\|_{L^p}
$$

for all $f \in L^p(\mathbb{R}^d)$.

Proof of Theorem 1.2 For every $f \in L^p(\mathbb{R}^d)$ we obtain

(3.2)

$$
\|V_r(M_{2^n} f: n \in \mathbb{Z})\|_{L^p} \leq \|V_r(P_{2^n} f: n \in \mathbb{Z})\|_{L^p} + \left( \sum_{n \in \mathbb{Z}} |M_{2^n} f - P_{2^n} f|^2 \right)^{1/2} \|f\|_{L^p}.
$$

The first term in (3.2) is bounded on $L^p$ by (2.30). Therefore, it remains to obtain $L^p$ bounds for the square function in (3.2). For this purpose we will use (2.31). Indeed, observe that

$$
\left( \sum_{n \in \mathbb{Z}} |M_{2^n} f - P_{2^n} f|^2 \right)^{1/2} \leq \left( \sum_{n \in \mathbb{Z}} |M_{2^n} S_{j+n} f - P_{2^n} S_{j+n} f|^2 \right)^{1/2} \leq \sum_{j \in \mathbb{Z}} \left( \sum_{n \in \mathbb{Z}} |M_{2^n} S_{j+n} f - P_{2^n} S_{j+n} f|^2 \right)^{1/2} \|f\|_{L^p} \lesssim 2^{-d_{ij}j} \|f\|_{L^2} \lesssim \|f\|_{L^p}.
$$

In order to justify the last but one inequality in (3.3) we shall prove, for each $j \in \mathbb{Z}$, that

(3.4)

$$
\left( \sum_{n \in \mathbb{Z}} |M_{2^n} S_{j+n} f|^2 \right)^{1/2} \leq \left( \sum_{n \in \mathbb{Z}} |P_{2^n} S_{j+n} f|^2 \right)^{1/2} \leq \|f\|_{L^p}.
$$

and

(3.5)

$$
\left( \sum_{n \in \mathbb{Z}} |M_{2^n} S_{j+n} f - P_{2^n} S_{j+n} f|^2 \right)^{1/2} \lesssim 2^{-d_{ij}j/2} \|f\|_{L^2}.
$$
To prove (3.4) we first show the following dimension-free vector-valued bounds
\begin{align}
(3.6) & \left\| \left( \sum_{n \in \mathbb{Z}} |M_{2^n} g_n|^2 \right)^{1/2} \right\|_{L^p} \lesssim \left\| \left( \sum_{n \in \mathbb{Z}} |g_n|^2 \right)^{1/2} \right\|_{L^p} \\
(3.7) & \left\| \left( \sum_{n \in \mathbb{Z}} |P_{L^{2^n}} g_n|^2 \right)^{1/2} \right\|_{L^p} \lesssim \left\| \left( \sum_{n \in \mathbb{Z}} |g_n|^2 \right)^{1/2} \right\|_{L^p}
\end{align}
for all \( p \in (1, \infty) \). Then in view of (3.6), (3.7) and (2.20) we conclude
\[ \left\| \left( \sum_{n \in \mathbb{Z}} |M_{2^n} S_{j+n} f|^2 \right)^{1/2} \right\|_{L^p} + \left\| \left( \sum_{n \in \mathbb{Z}} |P_{L^{2^n}} S_{j+n} f|^2 \right)^{1/2} \right\|_{L^p} \lesssim \left\| \left( \sum_{n \in \mathbb{Z}} |S_{j+n} f|^2 \right)^{1/2} \right\|_{L^p} \lesssim \|f\|_{L^p}, \]
which proves (3.1).

The proof of (3.6) and (3.7) follows respectively from (3.1) and (2.20) and a vector-valued interpolation. We only estimate (3.6), the estimate in (3.7) will be obtained similarly. Indeed, for \( p \in (1, \infty) \) and \( s \in [1, \infty] \), let \( A(p, s) \) be the best constant in the following inequality
\[ \left\| \left( \sum_{n \in \mathbb{Z}} |M_{2^n} g_n|^s \right)^{1/s} \right\|_{L^p} \leq A(p, s) \left\| \left( \sum_{n \in \mathbb{Z}} |g_n|^s \right)^{1/s} \right\|_{L^p}. \]

Then interpolation, duality \((A(p, s) = A(p', s'))\), and (3.1) yield (3.6), since
\[ A(p, 2) \leq A(p, 1)^{1/2} A(p, \infty)^{1/2} = A(p', \infty)^{1/2} A(p, \infty)^{1/2} \lesssim C_{p', \infty} C_{p, \infty}^{1/2}. \]

To prove (3.4) let us introduce \( k(\xi) = m(\xi) - p_L(\xi) = m(\xi) - e^{-2\pi L |\xi|} \) which is the multiplier associated with the operator \( M_1 - P_L \). Observe that by Proposition 1.4 and the properties of \( p_L(\xi) \) there exists a constant \( C > 0 \) independent of the dimension such that
\[ |k(\xi)| \leq |m(\xi) - 1| + |p_L(\xi) - 1| \leq C |\xi|, \quad |k(\xi)| \leq C |\xi|^{-1}, \quad |\langle \xi, \nabla p_L(\xi) \rangle| \leq C, \]
since \( \langle \xi, \nabla p_L(\xi) \rangle = -2\pi L |\xi| e^{-\pi L |\xi|} \). Therefore, by (3.8) and Plancherel’s theorem we get
\begin{align}
(3.9) & \left\| \left( \sum_{n \in \mathbb{Z}} |M_{2^n} S_{j+n} f - P_{L^{2^n}} S_{j+n} f|^2 \right)^{1/2} \right\|_{L^2} \\
& = \left( \int_{\mathbb{R}^d} \sum_{n \in \mathbb{Z}} |k(2^n \xi) (e^{-\pi 2^n L |\xi|} - e^{-\pi 2^n L^{j+n} |\xi|})|^2 |\mathcal{F} f(\xi)|^2 d\xi \right)^{1/2} \\
& \lesssim \left( \int_{\mathbb{R}^d} \sum_{n \in \mathbb{Z}} \min \{2^n L |\xi|, (2^n L |\xi|)^{-1} \} \left( |e^{-\pi 2^n L |\xi|} - e^{-\pi 2^n L^{j+n} |\xi|}|\right)^2 |\mathcal{F} f(\xi)|^2 d\xi \right)^{1/2} \\
& \lesssim 2^{-|j|/2} \left( \int_{\mathbb{R}^d} \sum_{n \in \mathbb{Z}} \min \{2^n L |\xi|, (2^n L |\xi|)^{-1} \} \left| \mathcal{F} f(\xi) \right|^2 d\xi \right)^{1/2} \lesssim 2^{-|j|/2} \|f\|_{L^2},
\end{align}
where we have used
\begin{align}
(3.10) & \min \{2^n L |\xi|, (2^n L |\xi|)^{-1} \} \left( |e^{-\pi 2^n L |\xi|} - e^{-\pi 2^n L^{j+n} |\xi|}|\right) \lesssim 2^{-|j|} \quad \text{and} \\
(3.11) & \sum_{n \in \mathbb{Z}} \min \{2^n L |\xi|, (2^n L |\xi|)^{-1} \} \lesssim 1.
\end{align}

The proof of Theorem 1.2 is completed. \( \square \)

4. ESTIMATES FOR SHORT VARIATIONS: PROOFS OF THEOREM 1.1 AND THEOREM 1.3

This section is devoted to prove Theorem 1.1 and Theorem 1.3. All these theorems will follow, in view of (2.38) and Theorem 1.2 which has been proven in the previous section, if we show that for appropriate parameters \( p \) the short variation seminorm is bounded, i.e. for every function \( f \in L^p(\mathbb{R}^d) \) the following inequality
\[ \left\| \left( \sum_{n \in \mathbb{Z}} V_2(M_t f : t \in [2^n, 2^{n+1}]) \right)^{1/2} \right\|_{L^p} \leq C_p \|f\|_{L^p}. \]
holds with a constant $C_p > 0$ which does not depend on the dimension, where $M_t = M^G_t$ as in the previous section.

One of the key tools in all of the proofs will be inequality (4.3) from Lemma 1.1. This inequality and the Littlewood–Paley decomposition (2.28) will result in (4.11) for $p \geq 2$. The proof of (4.10) for $p \leq 2$ will additionally require an almost orthogonality principle from Proposition (2.2). Due to (2.28) and the maximal results from [2], [6], [12] and [4] we will have to only verify conditions (4.10) and (4.17) from Proposition (2.2). The bound (2.17) will follow from inequality (2.4), the Littlewood–Paley decomposition (2.28), and properties of the multiplier $m^G$ from Proposition (1.1). The bounds for (4.10) will require a more sophisticated argument, which is subsumed in Lemma (1.1).

Now we need to recall some facts from [6], [12], and [4]. For $\alpha \in (0,1)$ let $D^\alpha$ be the fractional derivative

$$D^\alpha f(t) = D_t^\alpha f(t) = D^\alpha_t F(u)\big|_{u=t} = \mathcal{F}(\frac{2\pi \xi)^\alpha}{\Gamma(1-\alpha)} \mathcal{F}^{-1}(F)(\xi) (t)$$

for every $t \in \mathbb{R}$. This formula gives a well defined tempered distribution on $\mathbb{R}$. Note the resemblance of the fractional derivative $D^\alpha$ with its variant $D^\alpha_t$ used in Proposition (2.1). In fact, the version of that proposition, with $D^\alpha_t$ in place of $D^\alpha$, holds as well. Since we do not explicitly need it, we will forgo a proof of this fact.

Simple computations show that for $t > 0$ we have

$$D_t^\alpha m(t\xi) = \int_{\mathbb{R}^d} (2\pi ix \cdot \xi)x^\alpha K(x)e^{-2\pi itx}dx,$$

where $m = m^G = \mathcal{F}(K_G)$, and $K(x) = K_G(x) = 1_G(x)$. Moreover, [7] Lemma 6.6 guarantees that

$$D_t^\alpha m(t\xi) = -\frac{1}{\Gamma(1-\alpha)} \int_{t}^{\infty} (u-t)^{-\alpha} \frac{d}{du} m(u\xi)du.$$  

If $P^\alpha_{\nu}$ is the operator associated with the multiplier $p^\alpha_{\nu}(\xi) = u^{\alpha+1}D^\alpha_{\nu} \left( \frac{m(u\xi)}{u} \right) |_{\nu=u}$ for $\xi \in \mathbb{R}^d$, then one can see that

$$M_t f(x) = \mathcal{F}^{-1}(m(t\xi)\mathcal{F} f)(x) = \frac{1}{\Gamma(\alpha)} \int_{t}^{\infty} \frac{u}{u} \left( 1 - \frac{t}{u} \right)^{\alpha-1} P^\alpha_{\nu} f(x) du.$$

Carbery [8] showed that for general symmetric convex bodies one has

$$\|T_{(\xi,\nabla)^\alpha} m f\|_{L^p} \leq \|f\|_{L^p} + \|T_{(\xi,\nabla)^\alpha} m f\|_{L^p}.$$

where $T_{(\xi,\nabla)^\alpha} m f$ is the multiplier operator associated with the symbol $(\xi \cdot \nabla)^\alpha m(\xi) = D_t^\alpha m(t\xi)|_{t=1}.$

The estimate from (4.5) immediately implies that

$$\sup_{\alpha > 0} \|P^\alpha_{\nu}\|_{L^p \rightarrow L^p} \lesssim 1 + \|T_{(\xi,\nabla)^\alpha} m\|_{L^p \rightarrow L^p},$$

since the multipliers $P^\alpha_{\nu}$ are dilations of $P^\alpha_1$, i.e. $P^\alpha_1 (\lambda \xi) = P^\alpha_{\lambda
u}(\xi)$ for any $\lambda > 0$. Using identity (4.1) it follows from [6] that for a general symmetric convex body $G$, every $p \in (3/2, \infty]$, and for every $f \in L^p(\mathbb{R}^d)$ we have

$$\|\sup_{\alpha > 0} |M_t f|\|_{L^p} \lesssim p \|f\|_{L^p},$$

with the implicit constant independent of the dimension and the underlying body. The proof of (4.7) consists of two steps. In the first step it was proven that the $L^p$ boundedness in (4.4) can be deduced from (4.7) provided that $\alpha > 1/p$. Then using complex interpolation Carbery showed that there is $C_{\alpha,p} > 0$ such that

$$\|T_{(\xi,\nabla)^\alpha} m\|_{L^p \rightarrow L^p} \lesssim 1 + \|T_{(\xi,\nabla)^\alpha} m\|_{L^p \rightarrow L^p},$$

for $\alpha = 2 - 2/p$. Combining these two facts we obtain $1/p < 2 - 2/p$ which is equivalent to $p > 3/2$ and gives the desired range in (4.4).

When $G = B_q$ is as in (4.1) with $q \in [1, \infty)$, in [12] it has been proven that for every $\alpha \in (1/2, 1)$ and every $p \in (1, \infty)$ there is a constant $C_{\alpha,p,q} > 0$ independent of $d$ such that

$$\|T_{(\xi,\nabla)^\alpha} m\|_{L^p \rightarrow L^p} \leq C_{\alpha,p,q}.$$
The same estimate for $G = B_{\infty}$ is justified in \[4\]. In fact, in both \[12\] and \[4\] the estimate \[4.9\] boils down to controlling the operator $T_{\sqrt{|m|}}$ associated with the multiplier $|m|$. In view of \[1.8\] and \[4.9\], for the bodies studied in Theorem \[1.3\], we thus have for all $p \in (1, \infty)$ and all $f \in L^p(\mathbb{R}^d)$ that

\[ \sup_{u > 0} \|D^\alpha_f\|_{L^p} \lesssim_{\alpha, p} \|f\|_{L^p}. \]

Let $\eta$ be a smooth function on $\mathbb{R}$ such that $0 \leq \eta(t) \leq 1$ and

\[ \eta(t) = \begin{cases} 1, & \text{if } t \in [1, 2], \\ 0, & \text{if } t \not\in (1/2, 3). \end{cases} \]

**Lemma 4.1.** Let $\eta$ be a smooth function as in \[4.11\]. Then for any $p \in (1, 2)$ and any $\alpha \in (1/p, 1)$ there is $C_{\alpha, p} > 0$ such that for every Schwartz function $f \in S(\mathbb{R}^d)$ we have

\[ \|V_p((\eta(t)M_t f) : t \in \mathbb{R})\|_{L^p(\mathbb{R}^d)} \leq C_{\alpha, p}(\|f\|_{L^p} + \|\eta(t)\|_{F^{-1}}(\|D^\alpha_t m(t)\|_{F}(\xi))\|_{L^p(\mathbb{R}^d)}). \]

Moreover, we have

\[ \|V_p((\eta(t)M_t f) : t \in \mathbb{R})\|_{L^p(\mathbb{R}^d)} \leq C_{\alpha, p}(\|f\|_{L^p} + \sup_{t > 0} \|\eta(t)\|_{F^{-1}}((\|\xi\| \cdot \|\eta(t)\|_{F}(\xi)))\|_{L^p}). \]

**Proof.** In the proof we abbreviate $F(t, x) = \eta(t)M_t f(x)$. Note that Proposition \[2.1\] and Fubini’s theorem give, for $\alpha > 1/p$, that

\[ \|V_p(F(t, \cdot) : t \in \mathbb{R})\|_{L^p} \lesssim \|F\|_{L^p(\mathbb{R}^2)} + \|D^\alpha_t F\|_{L^p(\mathbb{R}^2)}. \]

For every $\alpha \in (0, 1)$ and $s \in \mathbb{R}$ we have

\[ (2\pi is)^\alpha = |2\pi s|^\alpha e^{it2\pi \text{sgn}(s)}, \quad \text{or, equivalently,} \quad |2\pi s|^\alpha = (2\pi is)^\alpha e^{-it2\pi \text{sgn}(s)}. \]

Taking into account this identity and the $L^p(\mathbb{R})$ boundedness of the projections

\[ \Pi_{\pm}(h) = \mathcal{F}_{\mathbb{R}}(1_{(0, \infty)}(\pm \xi)F^{-1}_{\mathbb{R}}h(\xi)) \]

we note that

\[ \|D^\alpha_t h\|_{L^p(\mathbb{R})} = \|e^{it\hat{\xi}} \Pi_+ D^\alpha_t h + e^{-it\hat{\xi}} \Pi_- D^\alpha_t h\|_{L^p(\mathbb{R})} \lesssim \|D^\alpha_t h\|_{L^p(\mathbb{R})}. \]

Thus, by \[4.14\] and Fubini’s theorem we obtain

\[ \|V_p(F(t, \cdot) : t \in \mathbb{R})\|_{L^p} \lesssim \|F\|_{L^p(\mathbb{R}^2)} + \|D^\alpha_t F\|_{L^p(\mathbb{R}^2)}. \]

We claim that

\[ \|D^\alpha_t F\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p} + \|\eta(t)\|_{F^{-1}}(\|D^\alpha_t M_t f\|_{L^p(\mathbb{R}^2)}). \]

To prove \[4.15\] we have to establish \[4.13\]. Suppose that $h$ is a function in $C^2(\mathbb{R})$ such that

\[ \left| \frac{d^j}{dt^j} h(t) \right| \lesssim (1 + |t|)^{-j-1}, \]

for $j = 0, 1, 2$, and

\[ \sup_{s \in \mathbb{R}} |(1 + s^2)F^{-1}_{\mathbb{R}}h(s)| \lesssim 1. \]

Then by \[7\], Lemma 6.6] when $\alpha \in (0, 1)$ we get

\[ D^\alpha_t h(t) = -\frac{1}{\Gamma(1 - \alpha)} \int_0^\infty u^{\alpha-1} h(t + u) du. \]

So an integration by parts yields

\[ D^\alpha_t h(t) = -\frac{\alpha}{\Gamma(1 - \alpha)} \lim_{\varepsilon \to 0} \int_x^\infty u^{\alpha-1} (h(t + u) - h(t)) du. \]

We fix $x \in \mathbb{R}^d$ and take alternatively $h(t) = \eta(t)M_t f(x)$, which is a Schwartz function or $h(t) = M_t f(x)$ which is a function in $C^\infty(\mathbb{R})$, since we have assumed that $f \in S(\mathbb{R}^d)$. To be able to apply formula \[4.15\] with these functions we have to only verify \[4.16\] and \[4.17\]. Indeed, for $h(t) = \eta(t)M_t f(x)$ or $h(t) = M_t f(x)$ we get

\[ \left| \frac{d^j}{dt^j} h(t) \right| \lesssim C(f, x, d) (1 + |t|)^{-j-1}. \]
We also have
\[
\sup_{s \in \mathbb{R}} |(1 + s^2)F_\eta h(s)| \leq C(f, x, d).
\]
Although the last two inequalities have bounds which depend on the dimension, it does not affect our result. These inequalities are only used to check that we are allowed to apply \[\text{(4.15)}\] to establish \(\text{(4.18)}\), which itself is independent of the dimension.

Using \(\text{(4.18)}\) we see that
\[
\mathcal{D}_t^\alpha F(t, x) - \eta(t) \mathcal{D}_t^\alpha M_t f(x) = E(t, x) = \int_0^\infty K(t, u) M_{t+u} f(x) du,
\]
with
\[
K(t, u) = \frac{\alpha}{\Gamma(1 - \alpha)} \frac{\eta(t + u) - \eta(t)}{u^\alpha + 1}.
\]
Now for \(K(t, u)\) we have the following two immediate estimates
\[
\sup_{t > 0} \|K(t, u)\| \leq \min \{u^{-\alpha}, u^{-\alpha - 1}\}.
\]
Observe now that
\[
\|E\|_{L^p(\mathbb{R} \times \mathbb{R}^d)} \leq \|E\|_{L^p(\mathbb{R} \times (-\infty, 3) \times \mathbb{R}^d)} + \|E\|_{L^p((3, \infty) \times \mathbb{R}^d)} + \|E\|_{L^p([-3, 3] \times \mathbb{R}^d)}.
\]
We estimate each term separately. For the first one, if \(t < -3\) then \(\eta(t) = 0\) and \(\eta(t + u) \neq 0\) only when \(1/2 < t + u < 3\). Thus by Minkowski’s integral inequality we obtain
\[
\|E\|_{L^p((\mathbb{R} \times (-\infty, 3) \times \mathbb{R}^d)} \leq \left( \int_{-\infty}^{-3} \left( \int_0^\infty |K(t, u)| \sup_{t > 0} \|M_t f\|_{L^p} du \right)^p dt \right)^{1/p}
\]
\[
\leq \sup_{t > 0} \|M_t f\|_{L^p} \left( \int_{-\infty}^{-3} \left( \int_0^\infty u^{-\alpha - 1} du \right)^p dt \right)^{1/p}
\]
\[
\lesssim \|f\|_{L^p} \left( \int_{-\infty}^{-3} |t|^{-p(\alpha + 1)} dt \right)^{1/p} \lesssim \|f\|_{L^p}.
\]
For the second one we have \(\|E\|_{L^p((3, \infty) \times \mathbb{R}^d)} = 0\), since \(\eta(t + u) = \eta(t) = 0\) for \(t > 3\). Finally, by \(\text{(4.20)}\) and Minkowski’s integral inequality we have
\[
\|E\|_{L^p([-3, 3] \times \mathbb{R}^d)} \lesssim \left( \int_{-3}^3 \left( \int_0^\infty \min \{u^{-\alpha}, u^{-\alpha - 1}\} \sup_{t > 0} \|M_t f\|_{L^p} du \right)^p dt \right)^{1/p}
\]
\[
\lesssim \sup_{t > 0} \|M_t f\|_{L^p} \lesssim \|f\|_{L^p}.
\]
Therefore, we have proven that
\[
\|E\|_{L^p(\mathbb{R} \times \mathbb{R}^d)} \lesssim \|f\|_{L^p},
\]
which in view of \(\text{(4.19)}\) yields \(\text{(4.15)}\).

Now, by Fubini’s theorem applied twice we have
\[
\mathcal{D}_t^\alpha M_t f(x) = \frac{1}{\Gamma(1 - \alpha)} \int_t^\infty (u - t)^{-\alpha} \frac{d}{du} M_u f(x) du
\]
\[
= \frac{1}{\Gamma(1 - \alpha)} \int_t^\infty (u - t)^{-\alpha} \mathcal{F}^{-1}\left( \frac{d}{du} m(u\xi) \mathcal{F} f(\xi) \right) (x) du
\]
\[
= \mathcal{F}^{-1}\left( \frac{1}{\Gamma(1 - \alpha)} \int_t^\infty (u - t)^{-\alpha} \frac{d}{du} m(u\xi) du \right) \mathcal{F} f(\xi) (x)
\]
\[
= \mathcal{F}^{-1}(\mathcal{D}_t^\alpha m(t\xi) \mathcal{F} f(\xi))(x).
\]
This combined with \(\text{(4.15)}\) gives \(\text{(1.12)}\). Finally by formula \(\text{(4.3)}\) we obtain for any \(\xi \in \mathbb{R}^d\) and \(\lambda > 0\) that
\[
\mathcal{D}_s^\alpha m(s\lambda\xi)|_{s=t} = \lambda^\alpha \mathcal{D}_s^\alpha m(s\xi)|_{s=t},
\]
thus
\[
\mathcal{D}_s^\alpha m(s\xi)|_{s=t} = t^{-\alpha} \mathcal{D}_s^\alpha m(s(t\xi))|_{s=1} = t^{-\alpha} ((t\xi) \cdot \nabla)^\alpha m(t\xi).
\]
This identity combined with \(\text{(1.12)}\) implies
\[
\|\eta(t) \mathcal{F}^{-1}(\mathcal{D}_t^\alpha m(t\xi) \mathcal{F} f(\xi))(x)\|_{L^p(\mathbb{R} \times \mathbb{R}^d)} \leq \sup_{t > 0} \|\mathcal{F}^{-1}((t\xi) \cdot \nabla)^\alpha m(t\xi) \mathcal{F} f(\xi))\|_{L^p},
\]
which proves (4.23) and completes the proof of the lemma.

4.1. Proof of Theorem 1.1. Fix a non-empty convex symmetric body $G$ in $\mathbb{R}^d$. In view of Theorem 1.2 we are left with proving (4.13) for all $p \in (3/2, 4)$. For simplicity of the notation as in the previous section we will write $M_t = M_{tf}$ and $m = M_G$. In the proof we will use the maximal result from [2] or [6], which states that for every $p \in (3/2, \infty]$ there exists $C_{p, \infty} > 0$ such that every $d \in \mathbb{N}$ and for every convex body $G \subset \mathbb{R}^d$ the following inequality holds

\begin{equation}
\left\| \sup_{t > 0} |M_t f| \right\|_{L^p} \leq C_{p, \infty} \|f\|_{L^p}
\end{equation}

for all $f \in L^p(\mathbb{R}^d)$. We will also appeal to the lacunary maximal inequality from [8].

The proof will be split according to whether $p \in (2, 4)$ or $p \in (3/2, 2]$. In both cases we shall exploit the $L^2$ inequality

\begin{equation}
\left\| \left( \sum_{n \in \mathbb{Z}} \sum_{k=0}^{d-1} |M_{2^n+2^{n-1}(k+1)} S_{j+n} f - M_{2^n+2^{n-1}k} S_{j+n} f|^2 \right)^{1/2} \right\|_{L^2} \lesssim 2^{-(1-\varepsilon)/2} 2^{j/4} \|f\|_{L^2},
\end{equation}

valid for all $\varepsilon \in (0, 1)$, $j \in \mathbb{Z}$ and $l \geq 0$, with the implicit constant which does not depend on $j$ and $l$.

Proof of the estimate (4.22). We will need some preparatory estimates. First of all we observe that for every $\varepsilon \in [0, 1)$ we have

\begin{equation}
\sum_{k=0}^{2^j-1} |m((2^n + 2^{n-l}(k + 1)) \xi) - m((2^n + 2^{n-l}k) \xi)|^{2-\varepsilon} \lesssim \sum_{k=0}^{2^j-1} \left( \int_{2^n+2^{n-l}k}^{2^n+2^{n-l}(k+1)} |\xi, \nabla m(s \xi)| ds \right)^{2-\varepsilon} \lesssim \sum_{k=0}^{2^j-1} \left( \log \left( \frac{2^n + 2^{n-l}(k + 1)}{2^n + 2^{n-l}k} \right) \right)^{2-\varepsilon} \lesssim 2^{j/2} \left( \frac{1}{2^{j/2} + 2^{j/2}} \right)^{2-\varepsilon} \lesssim 2^{j/2} \left( \frac{1}{2^{j/2}} \right)^{2-\varepsilon}.
\end{equation}

where in the second inequality we have used (4.17).

Secondly, for every $0 \leq k \leq 2^j$, we have $(2^n + 2^{n-l}k) \approx 2^n$, which guarantees

\begin{equation}
|m((2^n + 2^{n-l}(k + 1)) \xi) - m((2^n + 2^{n-l}k) \xi)|^{2-\varepsilon} \lesssim \min \left\{ |L2^n \xi|, |L2^n \xi|^{-1} \right\} \left| e^{-2\pi L2^{l+n}\xi} - e^{-2\pi L2^{l+n-1}\xi} \right|^2 \lesssim 2^{-c(j)/2} \min \left\{ |L2^n \xi|, |L2^n \xi|^{-1} \right\}^{2-\varepsilon}.
\end{equation}

Combining (4.24) with (4.25) with $\varepsilon \in (0, 1)$ we immediately obtain

\begin{equation}
\sum_{k=0}^{2^j-1} \left| m((2^n + 2^{n-l}(k + 1)) \xi) - m((2^n + 2^{n-l}k) \xi) \right|^2 |F S_{j+n}(\xi)|^2 \lesssim 2^{-c(j)/2} 2^{-(1-\varepsilon)} \min \left\{ |L2^n \xi|, |L2^n \xi|^{-1} \right\}^{2-\varepsilon}.
\end{equation}

By the Plancherel theorem and (4.23) we get

\begin{equation}
\left\| \left( \sum_{n \in \mathbb{Z}} \sum_{k=0}^{d-1} |M_{2^n+2^{n-1}(k+1)} S_{j+n} f - M_{2^n+2^{n-1}k} S_{j+n} f|^2 \right)^{1/2} \right\|_{L^2} \lesssim 2^{-c(j)/2} 2^{-(1-\varepsilon)} \int_{\mathbb{R}^d} \sum_{n \in \mathbb{Z}} \min \left\{ |L2^n \xi|, |L2^n \xi|^{-1} \right\} \left| F f(\xi) \right|^2 d\xi \lesssim 2^{-c(j)/2} 2^{-(1-\varepsilon)} \|f\|_{L^2}^2.
\end{equation}

This completes the proof of (4.23).

We now pass to the case when $p > 2$. 

□
Proof of inequality (4.1) for \( p \in (2,4) \) in the settings of Theorem 1.7. Note that by formulas (2.4) and (4.23) we obtain

\[
(4.27) \quad \left\| \left( \sum_{n \in \mathbb{Z}} V_2(M_t f : t \in [2^n, 2^{n+1})^2 \right) \right\|_{L^p}^{1/2} \leq \sum_j \sum_{l \geq 0} \left\| \sum_{n \leq j} \sum_{k = 0}^{2^{l-1}} |M_{2^n+2^{n-l}(k+1)}S_{j+n} f - M_{2^n+2^{n-l}k}S_{j+n} f|^2 \right\|_{L^p}^{1/2} \leq \| f \|_{L^p}.
\]

To establish the last inequality in (4.27) we have to show that for every \( p \in [2,4) \), there are \( \delta_p, \varepsilon_p > 0 \) such that

\[
(4.28) \quad \left\| \left( \sum_{n \leq j} \sum_{k = 0}^{2^{l-1}} |M_{2^n+2^{n-l}(k+1)}S_{j+n} f - M_{2^n+2^{n-l}k}S_{j+n} f|^2 \right) \right\|_{L^p}^{1/2} \leq 2^{-\delta_p (2^{-\varepsilon_p j})} | f \|_{L^p}.
\]

holds for all \( f \in L^p(\mathbb{R}^d) \) uniformly in \( j \in \mathbb{Z} \) and \( l \geq 0 \).

To this end, we show that, for \( p \in [2, \infty) \), we have

\[
(4.29) \quad \left\| \left( \sum_{n \leq j} \sum_{k = 0}^{2^{l-1}} |M_{2^n+2^{n-l}(k+1)}S_{j+n} f - M_{2^n+2^{n-l}k}S_{j+n} f|^2 \right) \right\|_{L^p}^{1/2} \leq 2^{l/2} | f \|_{L^p}.
\]

Then interpolation of (4.29) with (4.23) does the job and we obtain (4.28) for all \( p \in [2,4) \).

Thus we focus on proving (4.29). Since \( p \geq 2 \) we estimate

\[
\left\| \left( \sum_{n \in \mathbb{Z}} \sum_{k = 0}^{2^{l-1}} |M_{2^n+2^{n-l}(k+1)}S_{j+n} f - M_{2^n+2^{n-l}k}S_{j+n} f|^2 \right) \right\|_{L^p}^{1/2} \leq \sum_{l \geq 0} \max_{0 \leq k \leq 2^{l-1}} \left\| \sum_{n \leq j} |M_{2^n+2^{n-l}(k+1)}S_{j+n} f - M_{2^n+2^{n-l}k}S_{j+n} f|^2 \right\|_{L^{p/2}} \lesssim 2^l \max_{0 \leq k \leq 2^l} \left\| \left( \sum_{n \leq j} |M_{2^n+2^{n-l}k}S_{j+n} f|^2 \right) \right\|_{L^p}^{1/2} \lesssim \| f \|_{L^p}^2,
\]

where the last inequality follows from (2.29) and

\[
(4.30) \quad \sup_{l \geq 0} \max_{0 \leq k \leq 2^l} \left\| \sum_{n \leq j} |M_{2^n+2^{n-l}k}g| |f| \left\|_{L^p} \lesssim \| \sum_{n \leq j} |g|^2 \right\|_{L^p}^{1/2},
\]

which holds for all \( p \in (1,\infty) \) and the implicit constant independent of the dimension. To prove (4.30) we follow the argument used to justify (3.6). This is feasible, since for every \( p \in (1,\infty) \) and every \( f \in L^p(\mathbb{R}^d) \) we have the following lacunary estimate

\[
\sup_{l \geq 0} \max_{0 \leq k \leq 2^l} \left\| \sum_{n \leq j} |M_{2^n+2^{n-l}k}f| \right\|_{L^p} \lesssim C_{p, \infty} \| f \|_{L^p}
\]

with the same constant \( C_{p, \infty} \) as in (3.7). The last inequality can be established by appealing to (3.1) with a new convex body \((1+2^{-l}k)G\), since

\[
M_{2^n+2^{n-l}k}f = M_{2^n+2^{n-l}(k+1)}f = M_{2^n+2^{n-l}(k+1)}f = M_{2^n+2^{n-l}(k+1)}f.
\]

Hence, (4.29) follows and the proof of (4.1) for \( p \in (2,4) \) is completed.

We now pass to the case when \( p < 2 \).

Proof of inequality (4.1) for \( p \in (3/2,2) \) in the settings of Theorem 1.7. Here we apply the almostORTHOGONAL PRINCIPLE from Proposition 2.2 with \( p_0 = 3/2, \mathbb{Z} = [0,\infty) \) and with \( X = \mathbb{R}^d \) endowed with the \( \sigma \)-algebra \( \mathcal{B} \) of all Lebesgue measurable sets on \( \mathbb{R}^d \) and the Lebesgue measure \( \mu = | \cdot | \) on \( \mathbb{R}^d \). Moreover, \( T_0 = M_t, S_n = P_{2^n} - P_{2^{n-1}} \) is the Poisson projection and \( a_j = 2^{-\varepsilon j} \) for some \( \varepsilon \in (0,1) \). Note that by (2.4) and (4.23) the inequality from (4.23) implies

\[
(4.31) \quad \left\| \left( \sum_{n \in \mathbb{Z}} V_2(M_t S_{j+n} : t \in [2^n, 2^{n+1})) \right) \right\|_{L^2}^{1/2} \lesssim 2^{-\varepsilon j/4} \| f \|_{L^2}.
\]
The above inequality together with (2.29) and (4.22) show that in order to apply Proposition 2.2 it remains to verify the estimate
\[(4.32)\]
\[\sup_{n \in \mathbb{Z}} \| V_p(M_t f : t \in [2^n, 2^{n+1})] \|_{L^p} \lesssim \| f \|_{L^p},\]
for \(p \in (3/2, 2]\). First note that by rescaling it suffices to prove that
\[\| V_p(M_t f : t \in [1, 2)) \|_{L^p} \lesssim \| f \|_{L^p}.\]
Lemma 4.1 gives for any \(\alpha \in (1/p, 1)\) that
\[\| V_p(M_t f : t \in [1, 2)) \|_{L^p} \leq \| V_p(\eta(t) M_t f : t \in \mathbb{R}) \|_{L^p} \lesssim \| f \|_{L^p} + \sup_{t > 0} \| T((t \xi)^\alpha m(t \xi)) f \|_{L^p},\]
where \(\eta\) is as in (4.1). Due to (4.3) we obtain
\[\sup_{t > 0} \| T((t \xi)^\alpha m(t \xi)) f \|_{L^p} \leq \sup_{t > 0} \| T((t \xi)^\alpha m) f \|_{L^p} \lesssim \| f \|_{L^p}\]
for \(\alpha = 2 - 2/p\) which combined with \(\alpha > 1/p\) gives \(p > 3/2\) and proves (4.32). Hence we are allowed to apply Proposition 2.2 and the inequality in (4.1) is proven.

4.2. Proof of Theorem 1.3
Now \(G\) is a ball induced by a small \(\ell^q\) norm given in (1.3). Note that in this case we have
\[(4.33)\]
\[\| \sup_{t > 0} | M_t f | \|_{L^p} \leq C_{p,q,\infty} \| f \|_{L^p}\]
for all \(f \in L^p(\mathbb{R}^d)\) with the constant \(C_{p,q,\infty} > 0\) independent of the dimension. When \(q \in [1, \infty)\) the bound (4.33) is due to Müller [12], while for \(q = \infty\) it was proved by the first author [3].

Our aim will be to show (4.33) for all \(p \in (1, \infty)\). This is enough by Theorem 1.2 As before for simplicity of notation we will write \(M_t = M_{t}^G\) and \(m = m^G\).

The representation (4.1) and the bound (4.10) give a useful estimate for the \(L^p\) norm of the difference of \(M_{t+h}f\) and \(M_t f\) for the convex bodies considered in Theorem 1.3.}

**Lemma 4.2.** Fix \(p \in (1, \infty)\) and \(\alpha \in (1/2, 1)\). Then, there exists a constant \(C_{p,\alpha} > 0\) such that for every \(t, h > 0\), and for every \(f \in L^p(\mathbb{R}^d)\) we have
\[(4.34)\]
\[\| M_{t+h} f - M_t f \|_{L^p} \lesssim C_{p,\alpha} \left( \frac{h}{t} \right)^\alpha \| f \|_{L^p}.\]
The same estimate remains true when the operator \(M_t\) is replaced with its adjoint \(M_t^*\).

**Proof.** It suffices to consider the case \(h < t\). After rescaling we may assume that \(t = 1\) and \(0 < h < 1\). Now, by (4.3) we have,
\[M_t f(x) = \int_0^\infty A(t, u) P_u^\alpha f(x)\,du,\]
where we have set
\[A(t, u) = \begin{cases} \frac{1}{(\alpha) u^{\alpha} (1 - \frac{u}{t})^{\alpha-1}} & \text{if } u \geq t, \\ 0 & \text{if } u \leq t. \end{cases}\]
Denote
\[X(u, h) = |A(1 + h, u) - A(1, u)|.\]
Then,
\[\| M_{1+h} f - M_1 f \|_{L^p} \lesssim \int_1^\infty X(u, h) \| P_u^\alpha f(x) \|\,du.\]
In view of Minkowski’s integral inequality and (4.10) we obtain
\[(4.35)\]
\[\| M_{1+h} f - M_1 f \|_{L^p} \leq \sup_{u > 1} \| P_u^\alpha f \|_{L^p} \int_1^\infty X(u, h)\,du \leq \| f \|_{L^p} \int_1^\infty X(u, h)\,du.\]
Then simple calculations show that
\[X(u, h) \lesssim \begin{cases} \frac{|u - 1|^{\alpha-1}}{u^{\alpha-1}} & \text{if } 1 \leq u \leq 1 + h, \\ \frac{|u - 1|^{\alpha-1}}{u^{\alpha-1}} & \text{if } 1 + h \leq u \leq 1 + 2h, \\ \frac{|u - 1|^{\alpha-1}}{u^{\alpha-1}} & \text{if } 1 + 2h \leq u. \end{cases}\]
Then by the duality it will suffice to prove, for every $p \in (2, \infty)$

$$
\int_1^\infty X(u, h)du \lesssim \int_1^{1+h} |u - 1|^{-1}du + \int_1^{1+2h} |u - h - 1|^{-1}du + h \int_1^{1+2h} |u - 1|^{-2}du \lesssim h^\alpha.
$$

Hence, using (4.35) we finish the proof of the lemma. \qed

The proof of (1.1) is divided according to whether $p \in (2, \infty)$ or $p \in (1, 2]$. We consider first the case when $p > 2$.

**Proof of (1.1) for $p \in (2, \infty)$ in the settings of Theorem 1.3.** Here we would like to show that for every $\alpha \in (1/2, 1)$ and $p \in (2, \infty)$ there is a constant $C_{\alpha, p} > 0$ such that

$$
\left( \sum_{n \geq \mathbb{Z}} \left( \sum_{k=0}^{2^l-1} |M_{2^n+2n^\ast-(k+1)}S_{j+n}f - M_{2^n+2n^\ast-k}S_{j+n}f|^2 \right)^{1/2} \right)_{L^p} \lesssim C_{\alpha, p} 2^{\frac{1}{2} - \frac{1}{2\alpha}} \|f\|_{L^p}.
$$

Then, taking $\alpha$ sufficiently close to 1 and interpolating with (1.21) we complete the proof. Observe that

$$
\left( \sum_{n \geq \mathbb{Z}} \left( \sum_{k=0}^{2^l-1} |M_{2^n+2n^\ast-(k+1)}S_{j+n}f - M_{2^n+2n^\ast-k}S_{j+n}f|^2 \right)^{1/2} \right)_{L^p} \leq 2^{l/2} \max_{0 \leq k \leq 2^l-1} \left( \sum_{n \geq \mathbb{Z}} |M_{2^n+2n^\ast-(k+1)}S_{j+n}f - M_{2^n+2n^\ast-k}S_{j+n}f|^2 \right)^{1/2} \left( \sum_{n \geq \mathbb{Z}} |g_n|^2 \right)^{1/2} \left( \sum_{n \geq \mathbb{Z}} |g_n|^2 \right)^{1/2}.
$$

Since $p > 2$. Take any sequence $(g_n : n \in \mathbb{Z}) \in L^{p'}(L^2)$ such that

$$
\left( \sum_{n \geq \mathbb{Z}} |g_n|^2 \right)^{1/2} \left( \sum_{n \geq \mathbb{Z}} |g_n|^2 \right)^{1/2} \leq 1.
$$

Then by the duality it will suffice to prove, for every $p' \in (1, 2)$, that

$$
\max_{0 \leq k \leq 2^l-1} \left( \sum_{n \geq \mathbb{Z}} |M_{2^n+2n^\ast-k}g_n|^2 \right)^{1/2} \lesssim 2^{-\frac{l}{2\alpha}} \left( \sum_{n \geq \mathbb{Z}} |g_n|^2 \right)^{1/2}.
$$

Lemma 4.12 will be critical in the proof of (4.35). We shall prove that for every $q \in (1, 2)$ we have

$$
\left( \sum_{n \geq \mathbb{Z}} |M_{2^n+2n^\ast-(k+1)}g_n - M_{2^n+2n^\ast-k}g_n|^2 \right)^{1/2} \lesssim 2^{-\frac{q}{2\alpha}} \left( \sum_{n \geq \mathbb{Z}} |g_n|^2 \right)^{1/2}.
$$

Taking $q = p'$ in (4.39) we obtain (4.35). To prove the estimate (4.39) we will use a vector-valued interpolation between $L^q(L^q)$ and $L^q(L^\infty)$ for $q \in (1, 2)$. Take $\theta \in (0, 1)$ satisfying

$$
\frac{1}{2} = \frac{\theta}{q} + \frac{1 - \theta}{\infty},
$$

then $q = q/2$. For $L^q(L^q)$ we have

$$
\left( \sum_{n \geq \mathbb{Z}} |M_{2^n+2n^\ast-k}g_n|^q \right)^{1/q} \lesssim 2^{-\alpha l} \left( \sum_{n \geq \mathbb{Z}} |g_n|^q \right)^{1/q}.
$$

since, by (4.34) it holds

$$
\left( |M_{2^n+2n^\ast-(k+1)+1}g_n|^q \right)^{1/q} \lesssim 2^{-\alpha l} \left( \sum_{n \geq \mathbb{Z}} |g_n|^q \right)^{1/q}.
$$

The $L^q(L^\infty)$ endpoint is estimated using (4.35) as

$$
\sup_{n \geq \mathbb{Z}} |M_{2^n+2n^\ast-(k+1)+1}g_n - M_{2^n+2n^\ast-k}g_n| \lesssim 2^{-\alpha l} \|f\|_{L^q}.
$$

where $g(x) = \sup_{n \geq \mathbb{Z}} |g_n(x)|$. Now, invoking interpolation we obtain (4.39). \qed

Now we pass to the case when $p < 2$. 

Published on 2020-05-01
Proof of (1.1) for \( p \in (1, 2) \) in the settings of Theorem (1.3). Now our aim will be to prove that for every \( p \in (1, 2) \) there is a constant \( C_p > 0 \) independent of \( d \) such that

\[
(4.42) \quad \left( \sum_{n \in \mathbb{Z}} V_2(M_t f : t \in [2^n, 2^{n+1}))^2 \right)^{1/2} \leq C_p \|f\|_{L^p}
\]

for all \( f \in L^p(\mathbb{R}^d) \). For this purpose we will again apply our almost orthogonality principle for \( r \)-variations Proposition (2.3) with \( p_0 = 1 \), \( Z = (0, \infty) \), and with \( X = \mathbb{R}^d \) endowed with the \( \sigma \)-algebra \( \mathcal{B} \) of all Lebesgue measurable sets on \( \mathbb{R}^d \) and the Lebesgue measure \( \mu = |\cdot| \) on \( \mathbb{R}^d \). Moreover, we take \( T_t = M_t \), \( S_n = P_{L^2} - P_{L^{2^n-1}} \), and \( a_j = 2^{-ji/4} \) for some \( e \in (0, 1) \). Condition (2.14) was already justified in (4.31). It only remains to verify condition (2.10), which in our case says that for every \( p \in (1, 2) \) there is a constant \( C_p > 0 \) independent of \( d \) such that

\[
(4.43) \quad \sup_{n \in \mathbb{Z}} \left\| V_p(M_t f : t \in [2^n, 2^{n+1})) \right\|_{L^p} \leq C_p \|f\|_{L^p}
\]

holds for all \( f \in L^p(\mathbb{R}^d) \). Once (4.43) is proven then Proposition (2.2) applies and completes the proof of Theorem (1.3). By rescaling it suffices to prove that

\[
\left\| V_p(M_t f : t \in [1, 2)) \right\|_{L^p} \lesssim \|f\|_{L^p}.
\]

By Lemma (1.1) we obtain for any \( \alpha \in (1/p, 1) \)

\[
\left\| V_p(M_t f : t \in [1, 2)) \right\|_{L^p} \leq \left\| V_p(\eta(t)M_t f : t \in \mathbb{R}) \right\|_{L^p} \lesssim \|f\|_{L^p} + \sup_{t>0} \left\| T_{(t\xi \eta^{m(t)})} f \right\|_{L^p},
\]

where \( \eta \) is as in (1.11). Due to (1.13) for \( \alpha \in (1/2, 1) \) we obtain

\[
\sup_{t>0} \left\| T_{(t\xi \eta^{m(t)})} f \right\|_{L^p} \leq \sup_{t>0} \left\| T(t\xi \eta^{m(t)}) \right\|_{L^p} \|f\|_{L^p} \lesssim \|f\|_{L^p}.
\]

If \( p \in (1, 2) \) is close to 1 we can always take an \( \alpha \) such that \( \alpha > 1/p \) and the proof of (4.43) is completed. \( \square \)

5. Transference principle

In this section we prove the transference principle, which will allow us to deduce estimates for \( r \)-variations on \( L^p(X, \mu) \) for the operator \( A_G^F \) from the corresponding bounds for \( M_G^F \) on \( L^p(\mathbb{R}^d) \). Specifically, in view of Proposition 5.1, the inequalities from (1.12), (1.14) and (1.15) will follow respectively from (1.2), (1.3) and (1.5).

Proposition 5.1. Suppose that for some \( p \in (1, \infty) \) and \( r \in (2, \infty] \) there is a constant \( C_{p,r} > 0 \) such that for every \( d \in \mathbb{N} \) and for every symmetric convex body \( G \subset \mathbb{R}^d \) the following estimate

\[
(5.1) \quad \left\| V_r(M_G^F h : t \in Z) \right\|_{L^p(\mathbb{R}^d)} \leq C_{p,r} \|h\|_{L^p(\mathbb{R}^d)}
\]

holds for all \( h \in L^p(\mathbb{R}^d) \), where \( Z \subset (0, \infty) \). Let \( A_G^F \) be the ergodic counterpart of \( M_G^F \) defined in (1.12) for a given \( \sigma \)-finite measure space \( (X, \mathcal{B}, \mu) \) with families of commuting and measure-preserving transformations \( (T^1_t : t \in \mathbb{R}), \ldots, (T^n_t : t \in \mathbb{R}) \), which map \( X \) to itself.

Then for every \( f \in L^p(X, \mu) \) the inequality

\[
(5.2) \quad \left\| V_r(A_G^F f : t \in Z) \right\|_{L^p(X, \mu)} \leq C_{p,r} \|f\|_{L^p(X, \mu)}
\]

holds with the parameters \( p, r \), and the constant \( C_{p,r} \) as in (5.1).

Proof. We fix \( f \in L^p(X, \mu) \), \( \varepsilon > 0 \), and \( R > 0 \), and define for every \( x \in X \) the auxiliary function

\[
\phi_x(y) = \begin{cases} f(T_1^{y_1} \circ \cdots \circ T_d^{y_d} x), & \text{if } y \in G_{R(1+\varepsilon/d)}, \\ 0, & \text{otherwise.} \end{cases}
\]

Then, for every \( z \in G_R \) and \( t < R\varepsilon/d \), we have

\[
(5.3) \quad A_G^F f (T_1^{y_1} \circ \cdots \circ T_d^{y_d} x) = \frac{1}{|G_t|} \int_{G_t} f(T_1^{y_1} \circ \cdots \circ T_d^{y_d} x) dy_1 \cdots dy_d
\]

\[
= \frac{1}{|G_t|} \int_{G_t} \phi_x(z - y) dy = M_G^F \phi_x(z).
\]
Indeed, since $G$ is a symmetric convex body we have that $z - y \in G_{R(1 + \epsilon/d)}$, whenever $z \in G_R$ and $y \in G$, Hence, by (5.3) and (5.4) we get
\[
\int_{G_R} |V_r(A_1^G f(T_{\frac{z}{d}} z \circ \ldots \circ T_{\frac{z}{d}} z x)) : t \in Z \cap (0, R\epsilon/d)|^d dz_1 \ldots dz_d
\]
(5.4)
\[
\leq \int_{G_R} |V_r(M^G \phi_x(z) : t \in Z \cap (0, R\epsilon/d))|^d dz_1 \ldots dz_d
\]
\[
\leq \|V_r(M^G \phi_x : t \in Z)\|_{L^p(R^d)}^p
\]
\[
\leq C_{p,r} \|\phi_x\|_{L^p(R^d)}^p.
\]
Averaging (5.4) over $x \in X$ we obtain
\[
\int_{G_R} \|V_r(A_1^G f(T_{\frac{z}{d}} z \circ \ldots \circ T_{\frac{z}{d}} z x)) : t \in Z \cap (0, R\epsilon/d))\|_{L^p(X,\mu)}^p dz_1 \ldots dz_d
\]
(5.5)
\[
\leq \|C_{p,r} \int_{G_R(1+\epsilon/d)} \|f(T_{\frac{z}{d}} z \circ \ldots \circ T_{\frac{z}{d}} z x))\|_{L^p(X,\mu)}^p dz_1 \ldots dz_d,
\]
by definition of $\phi_x$, which is supported on $G_{R(1 + \epsilon/d)}$. Inequality (5.5) guarantees that
\[
|G_R| \cdot \|V_r(A_1^G f : t \in Z \cap (0, R\epsilon/d))\|_{L^p(X,\mu)}^p \leq C_{p,r} \cdot |G_R(1+\epsilon/d)| \cdot \|f\|_{L^p(X,\mu)}^p,
\]

for all $T_{\frac{1}{d}}, \ldots, T_{\frac{z}{d}}$ preserve the measure $\mu$ on $X$. Dividing both sides by $|G_R|$ we obtain that
\[
\|V_r(A_1^G f : t \in Z \cap (0, R\epsilon/d))\|_{L^p(X,\mu)}^p \leq C_{p,r}(1 + \epsilon/d)^d \|f\|_{L^p(X,\mu)}^p \leq C_{p,r}^\epsilon \epsilon^\epsilon \|f\|_{L^p(X,\mu)}^p.
\]
This is the place where the parameter $\epsilon > 0$ is helpful. Namely, taking $R \to \infty$ and invoking the monotone convergence theorem we conclude that
\[
\|V_r(A_1^G f : t \in Z))\|_{L^p(X,\mu)} \leq C_{p,r} \epsilon^\epsilon \|f\|_{L^p(X,\mu)},
\]
with arbitrary $\epsilon > 0$. Therefore, letting $\epsilon \to 0^+$ we obtain (5.2) and complete the proof of the proposition.

\[ \Box \]

Appendix A. Dimension-free bounds for the ball averages

We give a different proof of the dimension-free estimate for the averages over the Euclidean balls in the full range of $p \in (1, \infty)$ and $r \in (2, \infty)$. The proof is in spirit of the original proof for the maximal function from [15]. Here by $M_t$ we always mean $M^G_t$ with $G$ being the Euclidean ball in $\mathbb{R}^d$. The main result of this section reads as follows.

Theorem A.1. For $p \in (1, \infty)$ and $r \in (2, \infty)$ there is $C_{p,r} > 0$ independent of $d$ such that the inequality
\[
\|V_r(M_t f : t > 0)\|_{L^p} \leq C_{p,r} \|f\|_{L^p},
\]
holds for all $f \in L^p(\mathbb{R}^d)$.

From Section 3 we know that the estimate holds for long variations. Thus, it is enough to provide the bound for short variations
\[
\left\| \left( \sum_{n \in \mathbb{Z}} V_2(M_t f : t \in [2^n, 2^{n+1}]) \right)^{1/2} \right\|_{L^p} \leq C_{p,r} \|f\|_{L^p}.
\]
The estimates of short variations in Theorem A.1 will be based on (4.3) from the next lemma.

Lemma A.1. Let $u < v$ be real numbers and $a : [u, v] \to [0, \infty)$ be a differentiable function. For every $r \in [2, \infty)$ we have
\[
V_r(a_t : t \in [u, v]) \lesssim \left( \int_u^v |a_t|^2 dt \right)^{1/4} \cdot \left( \int_u^v \left| \frac{d}{dt} a_t \right|^2 dt \right)^{1/4}.
\]
(4.1)
If additionally $u = 2^l$ and $v = 2^{l+1}$ for some $l \in \mathbb{Z}$ then
\[
V_r(a_t : t \in [2^l, 2^{l+1}]) \lesssim \left( \int_{2^l}^{2^{l+1}} \left| \frac{d}{dt} a_t \right|^2 dt \right)^{1/2}.
\]
(4.2)
Thus, setting
\[ (\sum_{n \in \mathbb{Z}} V_n(a_t : t \in [2^n, 2^{n+1}]))^{1/2} \leq \left( \int_0^\infty \left| \frac{d}{dt} a_t \right|^2 \frac{dt}{t} \right)^{1/2} \]

Proof. For the proof of (A.1), let \( u \leq t_0 < t_1 < \ldots < t_M < v \) be an arbitrary increasing sequence. By the mean value theorem and the Cauchy–Schwarz inequality we obtain
\[
|a_{t_{j+1}} - a_{t_j}|^2 \leq |a_{t_{j+1}} - a_{t_j}^2| = \left| \int_{t_j}^{t_{j+1}} 2a_t \frac{d}{dt} a_t \frac{dt}{t} \right| \leq 2 \left( \int_{t_j}^{t_{j+1}} |a_t|^2 \frac{dt}{t} \right)^{1/2} \cdot \left( \int_{t_j}^{t_{j+1}} \left| \frac{d}{dt} a_t \right|^2 \frac{dt}{t} \right)^{1/2}.
\]

Therefore, by the Cauchy–Schwarz inequality, we see
\[
\sum_{j=0}^{M-1} |a_{t_{j+1}} - a_{t_j}|^2 \leq 2 \sum_{j=0}^{M-1} \left( \int_{t_j}^{t_{j+1}} |a_t|^2 \frac{dt}{t} \right)^{1/2} \cdot \left( \int_{t_j}^{t_{j+1}} \left| \frac{d}{dt} a_t \right|^2 \frac{dt}{t} \right)^{1/2}
\]

as desired. To prove (A.2) observe that by the mean value theorem and the Cauchy–Schwarz inequality we get
\[
|a_{t_{j+1}} - a_{t_j}|^2 = \left| \int_{t_j}^{t_{j+1}} \frac{d}{dt} a_t \frac{dt}{t} \right|^2 \leq (t_{j+1} - t_j) \int_{t_j}^{t_{j+1}} \left| \frac{d}{dt} a_t \right|^2 \frac{dt}{t^2} \leq \int_{t_j}^{t_{j+1}} \left| \frac{d}{dt} a_t \right|^2 \frac{dt}{t}
\]

for any sequence \( 2^l \leq t_0 < t_1 < \ldots < t_M < 2^{l+1} \), since \( t_{j+1} - t_j \leq 2^l \). Thus
\[
\sum_{j=0}^{M-1} |a_{t_{j+1}} - a_{t_j}|^2 \leq \int_{2^l}^{2^{l+1}} \left| \frac{d}{dt} a_t \right|^2 \frac{dt}{t}
\]

and the proof is completed. \( \square \)

Observe now that in view of (A.3) the estimate for short variations reduces to
\[
\left\| \left( \int_0^\infty \left| \frac{d}{dt} M_t f \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p} \leq C_p \| f \|_{L^p}.
\]

Let
\[
S_t f(x) = \int_{S^{d-1}} f(x - ty) d\sigma(y),
\]

where \( d\sigma \) denotes the normalized spherical measure on the unit sphere \( S^{d-1} \) of \( \mathbb{R}^d \). The estimate (A.4) will be deduced from an analogous statement for the spherical averages, namely
\[
\left\| \left( \int_0^\infty \left| \frac{d}{dt} S_t f \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p} \leq C_p \| f \|_{L^p}.
\]

To see that (A.5) does imply (A.4) we note that
\[
M_t f(x) = d \int_0^1 u^{d-1} S_{tu} f(x) du.
\]

Thus, setting \( F(t, x) = S_t f(x) \) we have
\[
\frac{d}{dt} M_t f(x) = d \int_0^1 u^{d-1} u(\partial_1 F)(ut, x) du,
\]

and, consequently, by Minkowski’s inequality
\[
\left\| \left( \int_0^\infty \left| \frac{d}{dt} M_t f \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p} \leq d \int_0^1 u^{d-1} \left\| \left( \int_0^\infty |u(\partial_1 F)(ut, x)|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p} du
\]

\[
= \left\| \left( \int_0^\infty \left| \frac{d}{dt} S_t f \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p}.
\]
From now on we focus on (A.5). More precisely, we shall prove the following proposition.

**Proposition A.1.** For each \( p \in (1, \infty) \) there exists \( d_0(p) \in \mathbb{N} \) such that for \( d \geq d_0(p) \) we have

\[
\left\| \left( \int_0^\infty \left| t \frac{d}{dt} S_{t} f \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p} \leq C_p \| f \|_{L^p}
\]

for every \( f \in L^p(\mathbb{R}^d) \), where \( C_p > 0 \) is independent of \( d \).

To see that Proposition A.1 is enough we note that it implies (A.4), hence, also Theorem A.1 for dimensions \( d > d_0(p) \). For the finite number of smaller dimensions \( d < d_0(p) \) we just use the known fact that Theorem A.1 holds with some \( C_{r,p,d} > 0 \).

We pass to the proof of the proposition.

**Proof of Proposition A.1.** Set

\[
K^\alpha(x) = \begin{cases} 
\frac{1}{(1 - |x|^2)^{\alpha - 1}}, & \text{for } |x| < 1, \\
0 & \text{for } |x| \geq 1,
\end{cases}
\]

and let \( m^\alpha(\xi) = \mathcal{F}(K^\alpha)(\xi) \). Then \( K^\alpha \), as a function of \( \alpha \in \mathbb{C} \), is analytic.

For \( t > 0 \) we define \( K^\alpha_t(x) = t^{-d}K^\alpha(x/t) \), thus \( \mathcal{F}(K^\alpha_t)(\xi) = m(t\xi) \). It is well known that \( m^\alpha(\xi) = \pi^{-\alpha + 1}|\xi|^{-d/2 + \alpha + 1}J_{d/2 + \alpha - 1}(2\pi|\xi|) \), where \( J_{\nu} \) is the Bessel function of order \( \nu \). Therefore

\[
(A.6) \quad |m^\alpha(\xi)| + |\nabla m^\alpha(\xi)| \leq C_{d,\text{Re}(\alpha)} \min(1, |\xi|^{-d/2+1/2-\text{Re}(\alpha)}).
\]

Set \( S^\alpha_t f = f \ast K^\alpha_t \) so that \( S^\alpha_0 = S_t \). Using (A.6) and a straightforward computation we obtain

\[
(A.7) \quad \left\| \left( \int_0^\infty \left| t \frac{d}{dt} S^\alpha_t f \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^2} \leq C_{d,\text{Re}(\alpha)} \| f \|_{L^2}.
\]

We claim that, if \( \text{Re}(\alpha) = 3 \), then for all \( 1 < p < \infty \) it holds

\[
(A.8) \quad \left\| \left( \int_0^\infty \left| t \frac{d}{dt} S^\alpha_t f \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p} \leq C_{d,p,\Gamma(\alpha)} \| f \|_{L^p}.
\]

To prove (A.8) we use the vector-valued Calderón–Zygmund theory. Namely, we consider the operator

\[
T f = f \ast \mathcal{K}, \quad \text{where} \quad \mathcal{K}(x) = \left( t \frac{d}{dt} K^\alpha_t(x) : t > 0 \right)
\]

as a mapping with values in the Hilbert space \( H = L^2((0, \infty), \nu^2) \). Then (A.8) is equivalent to proving that \( T \) maps \( L^p \) into \( L^p(H) \). Note that (A.7) implies that this is true for \( p = 2 \). Thus, the vector-valued Calderón–Zygmund theory reduces our task to checking that

\[
\| \mathcal{K}(x) \|_H \leq C_d \frac{1}{|\Gamma(\alpha)|} |x|^{-d}, \quad \| \nabla \mathcal{K}(x) \|_H \leq C_d \frac{1}{|\Gamma(\alpha)|} |x|^{-d-1}.
\]

When \( \text{Re}(\alpha) = 3 \) both of these estimates (which actually are equalities) follow from the definition of \( K^\alpha_t \). Hence, (A.8) is proved.

In the next step we fix \( 1 < p < \infty \). We will show the existence of \( d_0 = d_0(p) \) such that

\[
(A.9) \quad \left\| \left( \int_0^\infty \left| t \frac{d}{dt} S^\alpha_t f \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\mathbb{R}^{d_0})} \leq C_p \| f \|_{L^p(\mathbb{R}^{d_0})}.
\]

Assume first that \( 1 < p < 2 \). Take \( 1 < q < 2 \), where \( q \) is close to 1 and write \( 1/p = (1-\theta)/2 + \theta/q \), where \( \theta \in (0,1) \). We also let \( \varepsilon \) be a small real number. The quantities \( q, \theta, \) and \( \varepsilon \) will be determined in a moment. We use complex interpolation for the analytic family of operators

\[
S^\alpha f = \left( e^{\varepsilon \alpha} t \frac{d}{dt} S^\alpha_t f : t > 0 \right).
\]
Namely, by (A.7) and (A.8) we have
\begin{align}
\|S^\alpha\|_{L^2(\mathbb{R}^d; H)} & \leq C_{d, \epsilon} \quad \text{for } \text{Re}(\alpha) = (3 - d)/2 + \epsilon \\
\|S^\alpha\|_{L^p(\mathbb{R}^d; H)} & \leq C_{d, q} \quad \text{for } \text{Re}(\alpha) = 3.
\end{align}
(A.10)

We want to take \( \epsilon \) positive and such that \( 0 = (1 - \theta)((3 - d)/2 + \epsilon) + \theta \). Then
\begin{align}
p = ((1 - \theta)/2 + \theta/q)^{-1} > \frac{(d + 3)}{(d - 3)/q + 3}.
\end{align}
(A.11)

Based on the above considerations we claim that (A.9) holds for any \( d_0 = d_0(p) > 3/(p - 1) \). Indeed, then \( p > \frac{d_0(p) + 3}{d_0(p)} \), so that there is a \( 1 < q < p \) small enough and such that (A.11) holds for \( q \) and \( d_0(p) \). But then (A.10) holds with a positive \( \epsilon \), hence by complex interpolation in (A.10), for \( 1 < p < 2 \) we obtain the bound
\begin{align}
\left\| \left( \int_0^\infty \left| t \frac{d}{dt} S_t f \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} & \leq C_{d, p} \|f\|_{L^p(\mathbb{R}^d)}.
\end{align}

If \( p \geq 2 \), then a similar interpolation argument shows that (A.9) holds for any \( d_0(p) > 3/(p' - 1) \). Therefore, (A.9) is proved for \( d_0 = d_0(p) = \lceil \max\{3/(p - 1), 3/(p' - 1)\} \rceil \).

In what follows we fix \( d_0 \) such that (A.9) holds. Let \( d \geq d_0 \) and, for \( x = (x_1, x_2) \in \mathbb{R}^d = \mathbb{R}^{d_0} \times \mathbb{R}^{d - d_0} \)
define
\begin{align}
S'_t f(x_1, x_2) = \int_{\mathbb{S}^{d_0 - 1}} f(x_1 - ty_1, x_2) \, d\sigma(y_1).
\end{align}
(A.12)

Since \( S'_t \) acts only on \( x_1 \) from (A.9) we obtain
\begin{align}
\left\| \left( \int_0^\infty \left| t \frac{d}{dt} S'_t f \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} & \leq C_p \|f\|_{L^p(\mathbb{R}^d)}.
\end{align}
(A.13)

Note that the constant \( C_p \) from (A.9) is preserved. Let \( O(d) \) be the group of orthogonal transformations on \( \mathbb{R}^d \). For \( \rho \in O(d) \) we set
\begin{align}
S^\rho_t = \rho \circ S'_t \circ \rho^{-1}, \quad \text{where} \quad (\rho \circ g)(x) = g(\rho x).
\end{align}

Then \( \frac{d}{dt} (S^\rho_t) = \rho \circ \left( \frac{d}{dt} S'_t \right) \circ \rho^{-1} \), and (A.12) gives, for \( \rho \in O(d) \), the bound
\begin{align}
\left\| \left( \int_0^\infty \left| t \frac{d}{dt} S^\rho_t f \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} & \leq C_p \|f\|_{L^p(\mathbb{R}^d)}.
\end{align}

To complete the proof we note that
\begin{align}
S_t f(x) = \int_{O(d)} S^\rho_t f(x) \, d\rho,
\end{align}

where \( d\rho \) denotes the probabilistic Haar measure on \( O(d) \). Therefore
\begin{align}
t \frac{d}{dt} S_t f(x) = \int_{O(d)} t \frac{d}{dt} S^\rho_t f(x) \, d\rho.
\end{align}

Hence, an application of Minkowski’s integral inequality for the \( L^p(H) \) norm in question together with (A.13) leads to
\begin{align}
\left\| \left( \int_0^\infty \left| t \frac{d}{dt} S_t f \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p} & \leq C_p \|f\|_{L^p}.
\end{align}

This finishes the proof of the proposition.
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