On the Quantum Quarter Plane and the Real Quantum Plane

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Abstract

Suppose that \( q \neq \pm 1 \) is a complex number of modulus one. Let \( \mathcal{O}(\mathbb{R}^2_q) \) be the \( \ast \)-algebra with two hermitean generators \( x \) and \( y \) satisfying the relation \( xy = qyx \). Using operator representations of the \( \ast \)-algebra \( \mathcal{O}(\mathbb{R}^2_q) \) on Hilbert space and the Weyl calculus of pseudodifferential operators we construct \( \ast \)-algebras of “functions” on the quantum quarter plane \( \mathbb{R}^+ + q \) and on the real quantum plane \( \mathbb{R}^2_q \) which are left module \( \ast \)-algebras for the Hopf \( \ast \)-algebra \( \mathcal{U}_q(gl_2(\mathbb{R})) \). We define a family \( h_k, k \in \mathbb{Z}^2 \), of covariant positive linear functionals on these \( \ast \)-algebras and study the actions of the \( \ast \)-algebras \( \mathcal{O}(\mathbb{R}^2_q) \) and \( \mathcal{U}_q(gl_2(\mathbb{R})) \) on the associated Hilbert spaces. Quantum analogs of the partial Fourier transforms and the Fourier transform are found. A differential calculus on the “function” \( \ast \)-algebras is also developed and investigated.

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0. Introduction

Suppose that \( q \neq \pm 1 \) is a complex number of modulus one. Let \( \mathcal{O}(\mathbb{R}^2_q) \) be the \( \ast \)-algebra with two hermitean generators \( x \) and \( y \) satisfying the relation

\[
xy = qyx.
\]

In quantum group theory \( \mathcal{O}(\mathbb{R}^2_q) \) is usually called the coordinate \( \ast \)-algebra of the real quantum plane. It is well-known that \( \mathcal{O}(\mathbb{R}^2_q) \) is a left module \( \ast \)-algebra of the Hopf \( \ast \)-algebra \( \mathcal{U}_q(gl_2(\mathbb{R})) \) with action given by formulas (75)–(77) below. However these structures are not sufficient in order to study analytic properties of the real quantum plane. In the undeformed case \( q = 1 \) the \( \ast \)-algebra \( \mathcal{O}(\mathbb{R}^2_q) \) is just the polynomial algebra \( \mathbb{C}[x, y] \) in two hermitean indeterminates \( x \) and \( y \) equipped with the usual action of the Lie algebra \( gl_2(\mathbb{R}) \). In this situation we can replace the polynomial algebra by the larger \( \ast \)-algebra \( \mathcal{A}(\mathbb{R}^2) := \mathbb{C}[x, y] + C_0^\infty(\mathbb{R}^2) \) of functions on \( \mathbb{R}^2 \) and extend the action of \( gl_2(\mathbb{R}) \) to \( \mathcal{A}(\mathbb{R}^2) \). On the algebra \( C_0^\infty(\mathbb{R}^2) \) we can study differential and integral calculus and so we can develop analysis on \( \mathbb{R}^2 \). Roughly speaking, for the real quantum plane we try to proceed in a similar way.

An interesting approach to the quantum space \( \mathbb{R}^d_q \) has been developed by M. Rieffel [R] in the framework of his theory of deformation quantization. This approach is essentially based on \( C^\ast \)-algebras.
In the undeformed case the points of \( \mathbb{R}^2 \) are in one-to-one correspondence to the well-behaved irreducible \(*\)-representations (see [S1]) of the polynomial algebra \( \mathbb{C}[x, y] \) in Hilbert space. Thus we are lead to look for well-behaved irreducible \(*\)-representations of the \(*\)-algebra \( \mathcal{O}(\mathbb{R}^2_q) \). This problem has been studied in [S2]. In this paper we consider four irreducible well-behaved \(*\)-representations of \( \mathcal{O}(\mathbb{R}^2_q) \). They are defined as follows. We fix a real number \( \gamma \) such that \( q = e^{2\pi i \gamma} \) and two reals \( \alpha \) and \( \beta \) such that \( \alpha \beta = \gamma \). Let \( \mathcal{P} \) and \( \mathcal{Q} \) be the self-adjoint operators on the Hilbert space \( L^2(\mathbb{R}) \) given by \((\mathcal{P}f)(x) = (2\pi i)^{-1} f'(x) \) and \( \mathcal{Q}f = xf(x) \). Then, for \( \epsilon, \epsilon' \in \{+, -\} \) there exists an irreducible \(*\)-representation \( \rho_{\epsilon \epsilon'} \) of the \(*\)-algebra \( \mathcal{O}(\mathbb{R}^2_q) \) on \( L^2(\mathbb{R}^2) \) such that

\[
\rho_{\epsilon \epsilon'}(x) = e^{2\pi i \epsilon \alpha Q}, \quad \rho_{\epsilon \epsilon'}(y) = e^{2\pi i \epsilon' \beta P}.
\]

Because of the spectra of operators \( \rho_{\epsilon \epsilon'}(x) \) and \( \rho_{\epsilon \epsilon'}(y) \), we think of the \(*\)-representation \( \rho_{\epsilon \epsilon'} \) as realization of the algebra \( \mathcal{O}(\mathbb{R}^2_q) \) on the quantum quarter plane \( \mathbb{R}^2_q \).

Let us sketch the main ideas of our investigations and begin with the \(*\)-representation \( \rho_{++} \) corresponding to the (open) quantum quarter plane \( \mathbb{R}^{++}_q \). We want to construct “functions on the quantum quarter plane which are vanishing at the boundary”. In order to do so, we define these “functions” as pseudodifferential operators by means of the Weyl calculus [Fo], [St], that is,

\[
\text{Op}(a) = \gamma \int \hat{a}(\alpha s, \beta t) e^{2\pi i (s\alpha Q + t\beta P)} ds dt.
\]

As symbol class we take the set of all functions \( a(x_1, x_2) \) on \( \mathbb{R}^2 \) which are in the intersection of domains of operators \( e^{2\pi c_1 Q_1} e^{2\pi d_1 P_1} \otimes e^{2\pi c_2 Q_2} e^{2\pi d_2 P_2} \), where \( c_1, c_2, d_1, d_2 \in \mathbb{R} \). This set is a \(*\)-algebra, denoted \( \mathfrak{A}(\mathbb{R}^2) \), with respect to the twisted product and involution of pseudodifferential operators. The first aim of our construction is to extend the algebraic structure and the left action of \( U_q(gl_2) \) to the direct sum

\[
\mathcal{A}(\mathbb{R}^{++}_q) := \mathcal{O}(\mathbb{R}^2_q) + \mathfrak{A}(\mathbb{R}^2)
\]

such that \( \mathcal{A}(\mathbb{R}^{++}_q) \) becomes a left \( U_q(gl_2(\mathbb{R})) \)-module \(*\)-algebra. We think of \( \mathcal{A}(\mathbb{R}^{++}_q) \) as counterpart of the \(*\)-algebra \( \mathcal{A}(\mathbb{R}^{++}) := \mathbb{C}[x, y] + C^\infty(\mathbb{R}^{++}) \) of functions on the quarter plane \( \mathbb{R}^{++} \) equipped with the action of Lie algebra \( gl_2(\mathbb{R}) \). For each \( k \in \mathbb{Z}^2 \) there exists a faithful linear functional \( h_k \) on the \(*\)-algebra \( \mathfrak{A}(\mathbb{R}^2) \) which is covariant with respect to the left action of \( U_q(gl_2(\mathbb{R})) \). These functionals \( h_k \) can be viewed as quantum analogs of the state on the \(*\)-algebra \( C^\infty_0(\mathbb{R}^{++}) \) given by the Lebesgue measure. Further, there are two \( U_q(gl_2) \)-covariant differential calculi on the algebra \( \mathcal{O}(\mathbb{R}^2_q) \) invented in [PW] and [WZ]. We extend one of these calculi to a differential calculus on the larger algebra \( \mathcal{A}(\mathbb{R}^{++}_q) \). Thus the key ingredients for a differential and integral calculus on the quantum quarter plane \( \mathbb{R}^{++}_q \) are developed. Similar considerations can be done for three other quantum quarter planes. We strongly believe that
this approach, with some technical modifications, could serve as a guide-line for the constructions of other non-compact quantum spaces as well.

The next main step is the construction of the function \(*\)-algebra for the real quantum plane \(\mathbb{R}_q^2\). The idea is to obtain the quantum plane by “gluing together the four quantum quarter planes at the coordinate axis”. In order to do so, it is natural to begin with the direct sum \(\mathfrak{A}_0(\mathbb{R}^2)_4\) of the four \(*\)-algebras \(\mathfrak{A}(\mathbb{R}^2)\) corresponding to the quantum quarter planes. The elements of \(\mathfrak{A}_0(\mathbb{R}^2)_4\) are interpreted as “functions on the quantum plane which are vanishing at the coordinate axis”. As in the case of the “ordinary” plane we consider \(\mathfrak{A}_0(\mathbb{R}^2)_4\) as subspace of the Hilbert space \(L^2(\mathbb{R}^2)\). In this manner, the symbol algebra \(\mathfrak{A}_0(\mathbb{R}^2)_4\) (the \(\mathfrak{A}(\mathbb{R}^2)\)-algebras \(\mathfrak{A}(\mathbb{R}^2)\)-module \(\mathfrak{A}(\mathbb{R}^2)\)) are described on the generators. The \(\mathfrak{A}(\mathbb{R}^2)\)-algebra \(\mathfrak{A}(\mathbb{R}^2)\) is a subspace of the Hilbert space obtained from a covariant positive linear functional. Then the generators \(E' := q^{1/2}(q^{-1} - q^{-1})E\) and \(F' := q^{-1/2}(q - q^{-1})F\) of \(U_q(gl_2(\mathbb{R}))\) act on \(\mathfrak{A}_0(\mathbb{R}^2)_4\) as symmetric operators which are not essentially self-adjoint. In order to remedy this defect, we extend \(\mathfrak{A}_0(\mathbb{R}^2)_4\) to a larger \(*\)-algebra \(\mathfrak{A}(\mathbb{R}^2)_4\) by allowing, roughly speaking, symbols having singularities. All hermitean generators \(E', F', q^{-1/4}K_1, q^{-1/4}K_2\) of \(U_q(gl_2(\mathbb{R}))\) and \(x, y\) of \(O(\mathbb{R}^2)\) act on the larger function \(*\)-algebra \(\mathfrak{A}(\mathbb{R}^2)_4\) by essentially self-adjoint operators. Because of the singularities of symbols, we do not get actions of the whole \(*\)-algebras \(U_q(gl_2(\mathbb{R}))\) and \(O(\mathbb{R}^2)\) on \(\mathfrak{A}(\mathbb{R}^2)_4\).

The aims and main steps of our construction are explained, at least to some extent, by the preceding discussion. However, the rigorous undertaking of this program requires a number of technical lemmas on unbounded operator theory, on quantum groups, and on the Weyl calculus. We have collected these results in a rather long preliminary Section 1. Moreover, notation and terminology are fixed in Section 1. The reader might start at Section 2.

Let us describe the content of this paper more in detail. In Section 2 we investigate the left \(U_q(gl_2(\mathbb{R}))\)-module \(*\)-algebra \(O(\mathbb{R}^2_q)\) and the covariant first order differential calculus on \(O(\mathbb{R}^2_q)\). In Section 3 the corresponding formulas and structures are extended to a larger auxiliary \(*\)-algebra \(W\). This \(*\)-algebra contains in an operator representation on the Hilbert space \(L^2(\mathbb{R}^2)\) the operators \(e^{2\pi i(\alpha \beta + t \gamma \rho)}\), \(s, t \in \mathbb{R}\). In Section 4 the left action of \(U_q(gl_2(\mathbb{R}))\) on these operators is used in order to derive a left action on operators \(Op(a)\) and so on symbols \(a \in \mathfrak{A}(\mathbb{R}^2)\). In this manner, the symbol algebra \(\mathfrak{A}(\mathbb{R}^2)\) and the direct sum \(\mathcal{A}(\mathbb{R}^2_q) = O(\mathbb{R}^2)_q + \mathfrak{A}(\mathbb{R}^2)\) of vector spaces become left \(U_q(gl_2(\mathbb{R}))\)-module \(*\)-algebras. This is the function algebra of the quantum quarter plane \(\mathbb{R}^q_+\) mentioned above. We also extend the differential calculus of \(O(\mathbb{R}^2_q)\) to the function algebra \(\mathcal{A}(\mathbb{R}^2_q)\). In Section 5 we define for each \(k \in \mathbb{Z}^2\) a \(U_q(gl_2(\mathbb{R}))\)-covariant faithful positive linear functional \(h_k\) on the \(*\)-algebra \(\mathfrak{A}(\mathbb{R}^2)\) by a weighted integral over the symbol. The \(*\)-representations \(\psi_k\) of the \(*\)-algebras \(O(\mathbb{R}^2)_q\) and \(U_q^{\text{cov}}(gl_2(\mathbb{R}))\) on the associated unitary space \(\mathfrak{A}_k := (\mathfrak{A}(\mathbb{R}^2), \langle \cdot, \cdot \rangle_k)\) are described on the generators. The \(*\)-representations \(\psi_k\) and product and involution of the \(*\)-algebra \(\mathcal{A}(\mathbb{R}^2_q)\) are transformed by a unitary transformation to the Hilbert space \(L^2(\mathbb{R}^2)\). These transformed structures are essentially used for the construction of the quantum plane in Section 6. In the last subsection of Section 5 a uniqueness theorem for the covariant functional
Section 6 is devoted to the construction of the real quantum plane from four quantum quarter planes. The function algebra of $\mathbb{R}^2$ can be thought as direct sum of function algebras of the four quarter planes with boundary conditions $f(+0,y) = f(-0,y)$ and $f(x,+0) = f(x,-0)$. We first give an equivalent formulation of this picture and use then the corresponding formulas as motivation for the definitions of structures of the real quantum plane. We also define three unitary operators $F^q_x, F^q_y$ and $F^q$ which interchange up to some powers of the generators $K_1$ and $K_2$ the coordinate operators $x, y$ and the corresponding $q$-deformed partial derivatives $D^q_x, D^q_y$, respectively. The unitaries $F^q_x, F^q_y$, and $F^q$ can be considered as quantum analogs of the partial Fourier transforms and the Fourier transform, respectively.

1. Preliminaries

1.1 Algebraic Preliminaries

All algebras in this paper are over the complex field. All the notions and facts on Hopf algebras and quantum groups used in this paper can be found in [Mo] and [KS], see also [FRT].

Let $\mathcal{U}$ be a Hopf algebra. We use the Sweedler notation $\Delta(f) = f^{(1)} \otimes f^{(2)}$ for the comultiplication $\Delta(f)$ of $f \in \mathcal{U}$. A left $\mathcal{U}$-module algebra $Z$ is an algebra (without unit in general) which is a left $\mathcal{U}$-module with left action $\triangleright$ such that

$$f \triangleright (zz') = (f^{(1)} \triangleright z)(f^{(2)} \triangleright z'), \quad z, z' \in Z, \quad f \in \mathcal{U}. \quad (1)$$

A dual pairing of two Hopf algebras $\mathcal{U}$ and $\mathcal{A}$ is a bilinear mapping $\langle \cdot, \cdot \rangle : \mathcal{U} \times \mathcal{A} \to \mathbb{C}$ such that

$$\langle \Delta(f), a_1 \otimes a_2 \rangle = \langle f, a_1 a_2 \rangle, \quad \langle f_1 f_2, a \rangle = \langle f_1 \otimes f_2, \Delta(a) \rangle,$$

$$\langle f, 1 \rangle = \varepsilon(f), \quad \langle 1, a \rangle = \varepsilon(a), \quad \langle S(f), a \rangle = \langle f, S(a) \rangle$$

for all $f, f_1, f_2 \in \mathcal{U}$ and $a, a_1, a_2 \in \mathcal{A}$. By a dual pairing of two Hopf $\ast$-algebras $\mathcal{U}$ and $\mathcal{A}$ we mean a dual pairing $\langle \cdot, \cdot \rangle$ of the Hopf algebras $\mathcal{U}$ and $\mathcal{A}$ which has the additional property that

$$\langle f^\ast, a \rangle = \overline{\langle f, S(a)^\ast \rangle} \quad \text{and} \quad \langle f, a^\ast \rangle = \overline{\langle S(f)^\ast, a \rangle}, \quad f \in \mathcal{U}, \quad a \in \mathcal{A}. \quad (2)$$

Let $\langle \cdot, \cdot \rangle$ be a dual pairing of Hopf algebras $\mathcal{U}$ and $\mathfrak{A}$. Any right $\mathfrak{A}$-comodule algebra $Z$ is a left $\mathcal{U}$-module algebra with left action $\triangleright$ defined by

$$f \triangleright z = \langle f, z_{(1)} \rangle z_{(0)}, \quad f \in \mathcal{U}, \quad z \in Z, \quad (3)$$

where $\phi(z) = z_{(0)} \otimes z_{(1)}$ is the Sweedler notation for the right coaction $\phi$.

Lemma 1. Suppose that $\langle \cdot, \cdot \rangle$ is a dual pairing of Hopf $\ast$-algebras $\mathcal{U}$ and $\mathfrak{A}$. If $Z$ is a right $\mathfrak{A}$-comodule $\ast$-algebra with right coaction $\phi : Z \to Z \otimes \mathfrak{A}$, then the associated left action of $\mathcal{U}$ on $Z$ satisfies the condition

$$(f \triangleright z)^\ast = (S(f)^\ast) \triangleright z^\ast, \quad f \in \mathcal{U}, \quad z \in Z. \quad (4)$$
**Proof.** Since \( \mathcal{Z} \) is a \( \mathfrak{A} \)-comodule \(*\)-algebra, the coaction \( \phi \) preserves the involution, so that
\[
\phi(z^*) \equiv (z^*)_{(0)} \otimes (z^*)_{(1)} = (z_{(0)})^* \otimes (z_{(1)})^* \equiv \phi(z)^*.
\]
Using this condition and the second relation of (3) we conclude that
\[
S(f)^* \circ z^* = \langle S(f)^*, (z^*)_{(1)} \rangle (z^*)_{(0)} = \langle f, (z_{(1)})^{**} \rangle (z_{(0)})^* = (f \circ z)^*.
\]

From now on we suppose that \( \mathcal{U} \) is a Hopf \(*\)-algebra. Equation (4) in Lemma 1 gives the motivation for the following definition: A \(*\)-algebra \( \mathcal{Z} \) is called a \textit{left} \( \mathcal{U} \)-\textit{module} \(*\)-\textit{algebra} if \( \mathcal{Z} \) is a left \( \mathcal{U} \)-module algebra with left action \( \circ \) such that equation (4) holds. Then Lemma 1 says any right \( \mathfrak{A} \)-comodule \(*\)-algebra \( \mathcal{Z} \) is a left \( \mathcal{U} \)-module \(*\)-algebra with respect to the associated left action.

Suppose that \( \mathcal{Z} \) is a left \( \mathcal{U} \)-module \(*\)-algebra with left action \( \circ \) and let \( \chi \) be a linear functional on the Hopf \(*\)-algebra \( \mathcal{U} \). We shall say that a linear functional \( h \) on \( \mathcal{Z} \) is \textit{covariant with respect to} \( \chi \) if
\[
h(f \circ z) = \chi(f) h(z), \quad f \in \mathcal{U}, z \in \mathcal{Z}.
\]
(5)

Suppose for a moment that \( h \not\equiv 0 \) is covariant with respect to \( \chi \). Then it follows from the conditions of a left action that \( \chi \) is a character, that is,
\[
\chi(fg) = \chi(f) \chi(g), \quad f, g \in \mathcal{U}, \quad \text{and} \quad \chi(1) = 1.
\]
(6)

If in addition \( h \) is hermitian (that is, \( h(z^*) = \overline{h(z)} \) for \( z \in \mathcal{Z} \)), then we conclude from (2) and (4) that
\[
\overline{\chi(f)} = \chi(S(f)^*), \quad f \in \mathcal{U}.
\]
(7)

Note that a linear function \( h \) on the \( \mathcal{U} \)-module algebra \( \mathcal{Z} \) is invariant if and only if \( h \) is covariant with respect to the counit \( \varepsilon \).

**Lemma 2.** Let \( h \) be a linear functional on the left \( \mathcal{U} \)-module \(*\)-algebra \( \mathcal{Z} \) and set
\[
\langle y, x \rangle := h(x^* y), \quad x, y \in \mathcal{Z}.
\]
(8)

Consider the following three conditions:
(i) \( h \) is covariant with respect to \( \chi \).
(ii) \( \chi(f) \langle y, x \rangle = \langle f_{(2)} \circ y, S(f_{(1)})^{*} \circ x \rangle \) for \( f \in \mathcal{U}, x, y \in \mathcal{Z} \).
(iii) \( \langle f \circ y, x \rangle = \chi(f_{(2)}) \langle y, f_{(1)}^{*} \circ x \rangle \) for \( f \in \mathcal{U}, x, y \in \mathcal{Z} \).

Then we have (i) \( \rightarrow \) (ii) \( \rightarrow \) (iii). If \( \mathcal{Z} \) has a unit, then (iii) \( \rightarrow \) (i) and so all three conditions are equivalent.

**Proof.** (i) \( \rightarrow \) (ii): Using the formulas (5), (6) and (7) and the fact that \( S \circ * \circ S \circ * = \text{id} \) in any Hopf \(*\)-algebra we get
\[
\langle f_{(2)} \circ y, S(f_{(1)})^{*} \circ x \rangle = h((S(f_{(1)})^{*} \circ x)^{(f_{(2)} \circ y)}) = h((S(f_{(1)})^{*} \circ x^{*})(f_{(2)} \circ y))
= h((f_{(1)} \circ x^{*})(f_{(2)} \circ y)) = h(f \circ x^{*} y) = \chi(f) \langle y, x \rangle.
\]
(ii)→(iii): Using once more the relation $S \circ \ast \circ S \circ \ast = \text{id}$ and condition (ii) we compute

$$\langle f \circ y, x \rangle = \langle f(\varepsilon(f_1)) x \rangle = \langle f(\varepsilon(f_1)) , S^{-1}(f(\varepsilon(f_1)) f(x) \rangle = \langle f(\varepsilon(f_1)) x \rangle.$$

(iii)→(i): Applying condition (iii) with $x = 1$ we obtain

$$h(f \circ y) = \langle f \circ y, 1 \rangle = \chi(f(\varepsilon(f_1)) \langle y, f(\varepsilon(f_1)) 1 \rangle = \chi(f(\varepsilon(f_1)) = \chi(f(\varepsilon(f_1)) \langle y, \varepsilon(f(\varepsilon(f_1)) 1 \rangle.$$

The special case where $\chi$ is the counit $\varepsilon$ and $\mathcal{Z}$ has a unit will be stated separately as

**Corollary 3.** A linear functional $h$ on the left $\mathcal{U}$-module $\ast$-algebra $\mathcal{Z}$ with unit is invariant (that is, $h(f \circ z) = \varepsilon(f) h(z)$ for $f \in \mathcal{U}$ and $z \in \mathcal{Z}$) if and only if $\langle f \circ y, x \rangle = \langle y, f \circ x \rangle$ for all $x, y \in \mathcal{Z}$ and $f \in \mathcal{U}$.

Suppose $h$ is an invariant linear functional on the left $\mathcal{U}$-module $\ast$-algebra $\mathcal{Z}$ such that the form (9) is a scalar product. Then, by the implication (i)→(iii) of Lemma 2, the left action of $\mathcal{U}$ on $\mathcal{Z}$ is a $\ast$-representation of the $\ast$-algebra on the unitary space $(\mathcal{Z}, \langle \cdot, \cdot \rangle)$.

Let $\sigma_1$ and $\sigma_2$ be automorphisms of an algebra $\mathcal{Z}$. Recall that a linear mapping $\mathcal{D}$ of $\mathcal{Z}$ is called at $(\sigma_1, \sigma_2)$-derivation if

$$\mathcal{D}(z_1 z_2) = \sigma_1(z_1) \mathcal{D}(z_2) + \mathcal{D}(z_1) \sigma_2(z_2), \quad z_1, z_2 \in \mathcal{Z}.$$ 

A first order differential calculus (briefly, a FODC) over an algebra $\mathcal{Z}$ is a $\mathcal{Z}$-bimodule $\Gamma$ equipped with a linear mapping $d: \mathcal{Z} \to \Gamma$ such that $\Gamma$ is the linear span of elements $z_1 d z_2, z_1, z_2 \in \mathcal{Z}$, and $d(z_1 z_2) = z_1 d z_2 + d z_1 z_2$ for $z_1, z_2 \in \mathcal{Z}$.

Next we recall the definitions of the Hopf algebras $\mathcal{U}_q(\mathfrak{g}l_2)$, $\mathcal{U}_q(\mathfrak{s}l_2)$ and $\mathcal{O}(GL_q(2))$ as used in what follows. Let $\mathcal{U}_q(\mathfrak{g}l_2)$ be the complex unital algebra with generators $E$, $F$, $K_1$, $K_2$, $K_1^{-1}$, $K_2^{-1}$ and defining relations

$$K_1 K_2 = K_2 K_1, \quad K_j K_j^{-1} = K_j^{-1} K_j = 1 \quad \text{for} \quad j = 1, 2,$$

$$K_1 E K_1^{-1} = q^{1/2} E, \quad K_2 E K_2^{-1} = q^{-1/2} E, \quad K_1 F K_1^{-1} = q^{-1/2} F, \quad K_2 F K_2^{-1} = q^{1/2} F,$$

$$EF - FE = \lambda^{-1}(K^2 - K^{-2}),$$

where we set

$$K := K_1 K_2^{-1} \quad \text{and} \quad \lambda := q - q^{-1}.$$ 

The algebra $\mathcal{U}_q(\mathfrak{g}l_2)$ is a Hopf algebra with structure maps given on the generators by

$$\Delta(K_j) = K_j \otimes K_j, \quad \Delta(E) = E \otimes K + K^{-1} \otimes E, \quad \Delta(F) = F \otimes K + K^{-1} \otimes F.$$
\( \varepsilon(K_j) = 1, \varepsilon(E) = \varepsilon(F) = 0, S(K_j) = K_j^{-1}, S(E) = -qE, S(F) = -q^{-1}F \)

for \( j = 1, 2 \). Note that the element \( L := K_1K_2 \) is group-like and central. Let \( \mathcal{U}_q({\text{sl}}_2) \) denote the subalgebra of the algebra \( \mathcal{U}(gl_2) \) generated by the elements \( E, F, K, K^{-1} \). Clearly, \( \mathcal{U}_q({\text{sl}}_2) \) is a Hopf subalgebra of \( \mathcal{U}(gl_2) \).

Let \( \mathcal{O}(GL_q(2)) \) be the coordinate Hopf algebras of the quantum group \( GL_q(2) \) and let \( u_{jk}, j, k = 1, 2 \), be the entries of the corresponding fundamental matrix. There exists a dual pairing \( \langle .., \rangle \) of the Hopf algebras \( \mathcal{U}_q(gl_2) \) and \( \mathcal{O}(GL_q(2)) \). It is determined by the values on the generators \( K_1, K_2, E, F \) and \( u_{11}, u_{12}, u_{21}, u_{22} \), respectively, which are given by

\[ \langle K_1, u_{11} \rangle = \langle K_2, u_{22} \rangle = q^{-1/2}, \langle K_1, u_{22} \rangle = \langle K_2, u_{11} \rangle = \langle E, u_{21} \rangle = \langle F, u_{12} \rangle = 1 \]  

and zero otherwise.

The algebra with generators \( x \) and \( y \) satisfying the relation \( xy = qyx \) is called the coordinate algebra \( \mathcal{O}(\mathbb{C}_q^2) \) of the quantum plane \( \mathbb{C}_q^2 \). It is a right \( \mathcal{O}(GL_q(2)) \)-comodule algebra with right coaction \( \varphi \) given by

\[ \varphi(x) = x \otimes u_{11} + y \otimes u_{21}, \varphi(y) = x \otimes u_{12} + y \otimes u_{22}. \]  

Let \( \mathcal{O}(\mathbb{C}_q^2) \) denote the algebra with genearors \( x, x^{-1}, y, y^{-1} \) and relations

\[ xy = qyx, \quad xx^{-1} = x^{-1}x = 1, \quad yy^{-1} = yy^{-1} = 1. \]

That is, \( \mathcal{O}(\mathbb{C}_q^2) \) is the localization of \( \mathcal{O}(\mathbb{C}_q^2) \) with respect to the elements \( x \) and \( y \) and the algebra \( \mathcal{O}(\mathbb{C}_q^2) \) can be considered as a subalgebra of \( \mathcal{O}(\mathbb{C}_q^2) \).

Assume now that the deformation parameter \( q \) is of modulus one. Then there exists an involution \( f \rightarrow f^* \) on the algebra \( \mathcal{U}_q(gl_2) \) determined by

\[ E^* := -qE, \quad F^* := -q^{-1}F, \quad K_1^* := K_1, \quad K_2^* := K_2. \]  

Equipped with this involution the Hopf algebra \( \mathcal{U}_q(gl_2) \) is a Hopf \( * \)-algebra denoted by \( \mathcal{U}_q(gl_2(\mathbb{R})) \). We often work with the hermitean elements

\[ E' := q^{1/2}E \text{ and } F' := q^{-1/2}F \]

of the \( * \)-algebra \( \mathcal{U}_q(gl_2(\mathbb{R})) \). Further, there is an involution \( f \rightarrow f^\dagger \) given by

\[ E^\dagger := -q^{-1}E, \quad F^\dagger := -qF, \quad K_1^\dagger := q^{-1/2}K_1, \quad K_2^\dagger := q^{-1/2}K_2 \]  

such that \( \mathcal{U}_q(gl_2) \) becomes a \( * \)-algebra. It will be denoted by \( \mathcal{U}_q^{w}(gl_2(\mathbb{R})) \). In Section 5 we study covariant linear functionals with respect to the character \( \chi \) on \( \mathcal{U}_q(gl_2(\mathbb{R})) \) defined by \( \chi(K_1) = \chi(K_2) = q^{1/2} \) and \( \chi(E) = \chi(F) = 0 \). Then, by Lemma 2, the corresponding left action of \( \mathcal{U}_q(gl_2) \) is a \( * \)-representation of the \( * \)-algebra \( \mathcal{U}_q^{w}(gl_2(\mathbb{R})) \).

The Hopf algebra \( \mathcal{O}(GL_q(2)) \) is a Hopf \( * \)-algebra, denoted \( \mathcal{O}(GL_q(2,\mathbb{R})) \), with involution determined on the generators by \( u_{jk}^* = u_{jk}, j, k = 1, 2 \). The
dual pairing $\langle \ldots \rangle$ of the Hopf algebras $U_q(gl_2)$ and $O(GL_q(2))$ given by (9) is also a dual pairing of the Hopf $*$-algebras $U_q(gl_2(\mathbb{R}))$ and $O(GL_q(2, \mathbb{R}))$.

Further, there exist an involution of the algebra $O(C_2^q)$ given by (13) such that this algebra is a $*$-algebra. It is denoted by $O(R^2_2)$ and called the coordinate $*$-algebra of the real quantum plane. From the preceding formulas we see at once that the right coaction $\varphi$ of $O(GL_q(2))$ on $O(C_2^q)$ preserves the corresponding involutions, that is, $O(R^2_2)$ is a right comodule $*$-algebra of the Hopf $*$-algebra $O(GL_q(2, \mathbb{R}))$. Hence, by Lemma 1, $O(R^2_2)$ is a left module $*$-algebra for the Hopf $*$-algebra $U(gl_2(\mathbb{R}))$.

Remark 1. In the literature the involution of $U_q(gl_2(\mathbb{R}))$ is often defined by the requirements $E^* = -E, F^* = -F, K_1^* = K_1, K_2^* = K_2$. The latter defines indeed an involution which makes $U_q(gl_2(\mathbb{R}))$ into a Hopf $*$-algebra. However, with respect to this involution the dual pairing with $O(GL_q(2, \mathbb{R}))$ is not a dual pairing of Hopf $*$-algebras and the $*$-algebra $O(R^2_2)$ is not a left module $*$-algebra.

1.2. Operator-theoretic Preliminaries

First we fix some notation. Let $J(a, b)$ be the strip $\{z \in \mathbb{C} : a < \text{Im} z < b\}$. The Fourier transform $F$ and its inverse are used in the form

$$Ff(x) = \hat{f}(x) = \int e^{-2\pi itx} f(t) dt, \quad (F^{-1}f)(x) = \int e^{2\pi itx} f(t) dt. \quad (14)$$

Throughout, we denote the domain of an operator $T$ by $\mathcal{D}(T)$ and the scalar product of $L^2(\mathbb{R}^n)$ by $\langle \cdot, \cdot \rangle$. Let $P$ and $Q$ be the self-adjoint operators on the Hilbert space $L^2(\mathbb{R})$ defined by

$$(Pf)(x) = \frac{1}{2\pi} f'(x) \text{ and } (Qf)(x) = xf(x).$$

The operators $P$ and $Q$ are unitarily equivalent by the Fourier transform

$$FQF^{-1} = -P, \quad FPF^{-1} = Q. \quad (15)$$

The first assertion of the following lemma describes the domain and the action of the operators $e^{-2\pi \beta P}$ for real $\beta$. We formulate the result for $\beta > 0$; the case $\beta < 0$ is completely similar.

Lemma 4. (i): Suppose that $\beta > 0$. Let $g(z)$ be a holomorphic function on the strip $J(0, \beta)$ such that

$$\sup_{0 < y < \beta} \int_{-\infty}^{\infty} |g(x+iy)|^2 dx < \infty.$$
Then there exist functions \( g(x) \in L^2(\mathbb{R}) \) and \( g_\beta(x) \in L^2(\mathbb{R}) \) such that \( \lim_{y \downarrow 0} g_y = g \) and \( \lim_{y \uparrow \beta} g_y = g_\beta \) in \( L^2(\mathbb{R}) \), where \( g_y(x) := g(x+iy) \) for \( x \in \mathbb{R} \) and \( y \in (0, \beta) \).

Setting \( g(x+i\beta) := g_\beta(x), x \in \mathbb{R} \), we have

\[
\lim_{n \to \infty} g(x+n^{-2}i) = g(x) \quad \text{and} \quad \lim_{n \to \infty} g(x+(\beta-n^{-2})i) = g(x+\beta i) \quad \text{a.e. on } \mathbb{R}.
\]

The function \( g \) belongs to the domain \( \mathcal{D}(e^{-2\pi\beta P}) \) and we have

\[
(e^{-2\pi\beta P} g)(x) = g(x+\beta i).
\] (16)

Conversely, each function \( g \) in the domain \( \mathcal{D}(e^{-2\pi\beta P}) \) arises in this way.

(ii): For any \( \beta \in \mathbb{R} \) and \( \delta > 0 \), the vector space \( \mathcal{D}_\delta := \text{Lin}\{e^{-\delta x^2+cx}, c \in \mathbb{C}\} \) is a core for the self-adjoint operators \( e^{2\pi\beta P} \) and \( e^{2\pi\beta Q} \).

**Proof.** [S2], Lemma 1–3. \( \square \)

Throughout this paper we assume that the deformation parameter \( q \neq \pm 1 \) of modulus one and that \( \gamma \) is a fixed real number such that

\[ q = e^{2\pi i\gamma}. \]

Further, let \( \alpha \) and \( \beta \) denote real numbers such that \( \alpha \beta = \gamma \) and put

\[ X = e^{2\pi \alpha Q} \quad \text{and} \quad Y = e^{2\pi \beta P}. \] (17)

From (17) it follows that the operators \( X \) and \( Y \) defined by (17) satisfy the relation \( XY \eta = qYX \eta \) for each vector \( \eta \in \mathcal{D}(XY) \cap \mathcal{D}(YX) \). Therefore, for each \( \epsilon, \epsilon' \in \{+, -\} \), there exist a unique faithful \(*\)-representation \( \rho_{\epsilon \epsilon'} \) of the \(*\)-algebra \( \mathcal{O}(\mathbb{R}^2_q) \) on the domain \( \mathfrak{A}(\mathbb{R}) \) such that

\[ \rho_{\epsilon \epsilon'}(x) = \epsilon X |\mathfrak{A}(\mathbb{R}), \quad \rho_{\epsilon \epsilon'}(y) = \epsilon' Y |\mathfrak{A}(\mathbb{R}). \] (18)

Let \( \mathfrak{A}(\mathbb{R}) \) be the set of entire holomorphic functions \( a(x) \) on the complex plane satisfying

\[
\sup_{\delta_1 < y < \delta_2} \int_{-\infty}^{\infty} |a(x+iy)|^2 e^{2sx} dx < \infty
\] (19)

for all \( s, \delta_1, \delta_2 \in \mathbb{R}, \delta_1 < \delta_2 \). From Lemma 4 we easily derive that

\[ \mathfrak{A}(\mathbb{R}) = \bigcap_{n,m=-\infty}^{+\infty} \mathcal{D}(X^n Y^m) = \bigcap_{n,m=-\infty}^{+\infty} \mathcal{D}(Y^n X^m). \]

Clearly, \( \mathfrak{A}(\mathbb{R}) \) is invariant under the Fourier transform and its inverse.

Throughout, we denote by \( f_\alpha \) the function

\[ f_\alpha(x) = -2 \sinh \pi \beta (2x+\alpha i) \] (20)
and by $L_\alpha$ and $R_\alpha$ the operators on the Hilbert space $L^2(\mathbb{R})$ given by

$$L_\alpha = \overline{f_\alpha(P)} e^{-2\pi \alpha Q}, \quad R_\alpha = e^{-2\pi \alpha Q} f_\alpha(P).$$

(21)

Some properties of these operators are collected in the next lemma.

**Lemma 5.** (i) $L_\alpha$ is a closed symmetric operator.  
(ii) $R_\alpha$ is the adjoint operator of $L_\alpha$.  
(iii) $A(\mathbb{R})$ is a core for the operator $L_\alpha$.  
(iv) $f_\alpha(P)\overline{A(\mathbb{R})}$ is a core for the operator $R_\alpha$.

**Proof.** By formula (13), we can replace $P$ by $Q$ and $Q$ by $-P$. But then the assertions (i)–(iii) have been stated in [S2] and [S4].

It remains to prove assertion (iv). First note that $f_\alpha(P)^{-1}$ is a bounded normal operator on the Hilbert space $L^2(\mathbb{R})$, so $B_\alpha := f_\alpha(P)^{-1}A(\mathbb{R})$ is a dense linear subspace of $L^2(\mathbb{R})$. We show that

$$(R_\alpha | B_\alpha)^* \subseteq L_\alpha.$$ 

(22)

Indeed, suppose that $\zeta \in D((R_\alpha | B_\alpha)^*)$. Then there exists a vector $\xi \in L^2(\mathbb{R})$ such that $\langle R_\alpha \eta, \zeta \rangle = \langle \eta, \xi \rangle$ for all $\eta \in B_\alpha$. Writing $\eta$ as $\eta = f_\alpha(P)^{-1} \eta'$ with $\eta' \in A(\mathbb{R})$, we obtain $\langle e^{-2\pi \alpha Q} \eta', \zeta \rangle = \langle \eta', \overline{f_\alpha(P)}^{-1} \xi \rangle$. Since $A(\mathbb{R})$ is a core for $e^{-2\pi \alpha Q}$ by Lemma 4(ii), the latter implies that $\zeta \in D(e^{-2\pi \alpha Q})$ and $e^{-2\pi \alpha Q} \zeta = \overline{f_\alpha(P)}^{-1} \xi$. Thus, we have $\zeta \in D(\overline{f_\alpha(P)} e^{-2\pi \alpha Q}) = D(L_\alpha)$ which proves (22).

By the assertion of (i), (22) implies that $(R_\alpha | B_\alpha)^{**} \supseteq (L_\alpha)^* = R_\alpha$. But the latter means that $B_\alpha$ is a core for $R_\alpha$.

Next we essentially use some results obtained in [S4]. We restate them here using a slight different notation. For $\delta_1, \delta_2 \in \mathbb{R}$, $\delta_1 > \delta_2$, let $\mathcal{H}(\delta_1, \delta_2)$ denote the set of all holomorphic functions $f$ on the strip $J(\delta_1, \delta_2)$ satisfying

$$\sup_{\delta_1 < y < \delta_2} \int_{-\infty}^{\infty} |f(x + iy)|^2 e^{-sx^2} \, dx < \infty$$

for all $s > 0$. By Lemma 2 in [S4], each function $f \in \mathcal{H}(\delta_1, \delta_2)$ has a.e. boundary limits $f(x + i\delta_1)$ and $f(x + i\delta_2)$ on $\mathbb{R}$. For notational simplicity we assume that $\alpha > 0$. With some obvious modifications all results remain valid for $\alpha < 0$.

We apply Theorem 1 in [S4] to the function $f(x) := -2 \sinh 2\pi \beta x$ and with $\alpha$ replaced by $\alpha/2$. Note that $f(x - i\alpha/2) = f_{-\alpha}(x)$, where $f_\alpha$ is defined by (20). Then Theorem 1 in [S4] can be restated as follows.

**Lemma 6.** There exist holomorphic functions $w_\alpha \in \mathcal{H}(\alpha, 0)$ and $v_\alpha \in \mathcal{H}(\alpha, -\alpha)$ such that

$$|w_\alpha(x)| = |v_\alpha(x)| = 1 \quad a.e. \text{ on } \mathbb{R},$$

$$w_\alpha(x) = f_{-\alpha}(x)v_\alpha(x - \alpha i), \quad v_\alpha(x) = f_{-\alpha}(x)w_\alpha(x - \alpha i) \quad a.e. \text{ on } \mathbb{R}.$$ 

(23)

(24)
The functions $w_\alpha, v_\alpha$ are uniquely determined up to a constant factor of modulus one by these properties.

Let $W_\alpha$ and $A_\alpha$ denote the operator matrices

$$W_\alpha(\mathcal{P}) = \begin{pmatrix} w_\alpha(\mathcal{P}) & 0 \\ 0 & v_\alpha(\mathcal{P}) \end{pmatrix}, A_\alpha = \begin{pmatrix} 0 & L_\alpha \\ R_\alpha & 0 \end{pmatrix}, B_\alpha = \begin{pmatrix} e^{2\pi\alpha q} & 0 \\ 0 & e^{2\pi\alpha q} \end{pmatrix}. \quad (25)$$

Since $|w_\alpha| = |v_\alpha| = 1$ a.e. on $\mathbb{R}$ by (23) and $L_\alpha^* = R_\alpha$ by Lemma 5, $W_\alpha(\mathcal{P})$ is a unitary operator and $A_\alpha$ and $B_\alpha$ are self-adjoint operators on the Hilbert space $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$.

**Lemma 7.** $W_\alpha(\mathcal{P})^* A_\alpha W_\alpha(\mathcal{P}) = B_{-\alpha}$ and $W_\alpha(\mathcal{P})^* B_\alpha W_\alpha(\mathcal{P}) = A_{-\alpha}$.

**Proof.** By (14), we have $\mathcal{F}W_\alpha(\mathcal{P}) \mathcal{F}^{-1} = W_\alpha(\mathcal{Q})$, $\mathcal{F}e^{-2\pi\alpha Q} \mathcal{F}^{-1} = e^{2\pi\alpha P}$ and $\mathcal{F}L_\alpha \mathcal{F}^{-1} = -2 \sin \pi \beta (2x-\alpha i)e^{2\pi\alpha P}$, that is, $\mathcal{F}L_\alpha \mathcal{F}^{-1}$ is the operator $L_f$ with $f(x) = -2 \sinh 2\pi \beta x$ and $\alpha$ replaced by $\alpha/2$ in the notation of [S4]. Thus, under the Fourier transform the assertion $W_\alpha(\mathcal{P})^* A_\alpha W_\alpha(\mathcal{P}) = B_{-\alpha}$ is just equation (24) in [S4].

Next we prove that $W_\alpha(\mathcal{P})^* B_\alpha W_\alpha(\mathcal{P}) = A_{-\alpha}$. Since the self-adjoint operator $A_{-\alpha}$ has no proper self-adjoint extension in the same Hilbert space, it suffices to show that $W_\alpha(\mathcal{P})^* B_\alpha W_\alpha(\mathcal{P}) \supseteq A_{-\alpha}$ which means that

$$w_\alpha(\mathcal{P})^* e^{2\pi\alpha Q} v_\alpha(\mathcal{P}) \supseteq L_{-\alpha} \equiv \mathcal{F}_{-\alpha}(\mathcal{P}) e^{2\pi\alpha Q},$$

$$v_\alpha(\mathcal{P})^* e^{2\pi\alpha Q} w_\alpha(\mathcal{P}) \supseteq R_{-\alpha} \equiv e^{-2\pi\alpha Q} f_{-\alpha}(\mathcal{P}).$$

Note that $\mathcal{F}_{-\alpha}(x) = f_{\pm \alpha}(x)$. Applying the unitary transformation $\mathcal{F} \cdot \mathcal{F}^{-1}$ and using (15) it follows that the latter relations are equivalent to

$$w_\alpha(x)e^{2\pi\alpha P} v_\alpha(x) \supseteq f_{\pm \alpha}(x)e^{-2\pi\alpha P}, \quad (26)$$

$$v_\alpha(x)e^{-2\pi\alpha P} w_\alpha(x) \supseteq e^{-2\pi\alpha P} f_{\pm \alpha}(x). \quad (27)$$

Recall that $f(x \pm \frac{\alpha}{2} i) = f_{\pm \alpha}(x)$. Therefore, formula (24) can be rewritten as

$$f_{-\alpha}(x)e^{2\pi\alpha P} = w_\alpha e^{2\pi\alpha P} \overline{v_\alpha}, \quad (28)$$

$$e^{2\pi\alpha P} f_{\alpha}(x) = v_\alpha e^{2\pi\alpha P} \overline{w_\alpha}. \quad (29)$$

Let $\eta \in \mathfrak{B}(\mathbb{R})$. Since $\overline{\eta} \in \mathcal{D}(f_{-\alpha}(x)e^{2\pi\alpha P})$ and hence $\overline{v_\alpha} \overline{\eta} \in \mathcal{D}(e^{2\pi\alpha P})$ by (28), we have $v_\alpha \eta \in \mathcal{D}(e^{-2\pi\alpha P})$. The relations (26) combined with the facts that $w_\alpha \in \mathcal{H}(\alpha,0)$ and $v_\alpha \in \mathcal{H}(\alpha,-\alpha)$ imply that $v_\alpha(x+\alpha i) = f_{-\alpha}(x+\alpha i)w_\alpha(x)$ a.e. on $\mathbb{R}$ (see e.g. formula (6) in [S4]). Since $f_{-\alpha}(x+\alpha i) = f_{\alpha}(x)$, we obtain

$$w_\alpha(x)e^{-\pi\alpha P} v_\alpha(x) \eta = w_\alpha(x) v_\alpha(x+\alpha i)e^{-2\pi\alpha P} \eta = f_{\alpha}(x)e^{-2\pi\alpha P} \eta.$$

That is, the operators $\overline{w_\alpha} e^{-2\pi\alpha P} v_\alpha$ and $f_{\alpha} e^{-2\pi\alpha P}$ coincide on the domain $\mathfrak{B}(\mathbb{R})$. By Lemma 5, $\mathfrak{B}(\mathbb{R})$ is a core for the closed operator $L_{-\alpha}$ and so for $f_{\alpha}(x)e^{-2\pi\alpha P} = \mathcal{F}L_{-\alpha} \mathcal{F}^{-1}$. Thus we conclude that

$$\overline{w_\alpha} e^{-2\pi\alpha P} v_\alpha \supseteq f_{\alpha} e^{-2\pi\alpha P}. \quad (25)$$
Next we verify the second relation \((24)\). By \((23)\), \(\bar{w}_\alpha(x) f_\alpha(x)^{-1} \eta(x) \in \mathcal{D}(e^{2\pi \alpha P})\) and hence \(w_\alpha(x) f_\alpha(x)^{-1} \eta(x) \in \mathcal{D}(e^{-2\pi \alpha P})\). From \((26)\) we derive that \(w_\alpha(x+\alpha i) = f_\alpha(x+\alpha i) v_\alpha(x)\) (see formula \((3)\) in [S4]). Therefore, for \(\varphi(x) = f_\alpha(x)^{-1} \eta(x)\) with \(\eta \in \mathfrak{A}(\mathbb{R})\) we obtain

\[
\bar{v}_\alpha(x) e^{-2\pi \alpha P} w_\alpha(x) \varphi = \bar{v}_\alpha(x) e^{-2\pi \alpha P} x_\alpha(x) f_\alpha(x)^{-1} \eta \\
= \bar{v}_\alpha(x) w_\alpha(x+\alpha i) f_\alpha(x+\alpha i)^{-1} e^{-2\pi \alpha P} \eta \\
= \bar{v}_\alpha(x) v_\alpha(x) e^{-2\pi \alpha P} \eta = e^{-2\pi \alpha P} f_\alpha(x) \varphi.
\]

Thus, the operators \(\mathcal{T}_\alpha e^{-2\pi \alpha P} w_\alpha\) and \(e^{-2\pi \alpha P} f_\alpha\) coincide on the dense domain \(f_\alpha(x)^{-1} \mathfrak{A}(\mathbb{R})\). Since \(f_\alpha(\mathfrak{P})^{-1} \mathfrak{A}(\mathbb{R})\) is a core for \(R_\alpha\) by Lemma 5 and \(\mathcal{F} R_\alpha \mathcal{F}^{-1} = e^{-2\pi \alpha P} f_\alpha(x)\), it follows that \(\bar{v}_\alpha e^{-2\pi \alpha P} w_\alpha \supseteq e^{-2\pi \alpha P} f_\alpha(x)\). This proves \((24)\) and completes the proof of Lemma 7.

**Lemma 8.** Let \(c, d \in \mathbb{R}\) and \(\delta_0 > 0\). Suppose that \(8|dc| < 1\). Then the vector space \(\tilde{E}_{\delta_0} = \text{Lin}\{e_{t,\delta}(x) = e^{2\pi \alpha (ix - \delta x^2)}; t \in \mathbb{R}, 0 < \delta < \delta_0\}\) is dense in \(\mathfrak{A}(\mathbb{R})\) with respect to the norm \(\| \cdot \|_{c,d} = \|(e^{2\pi \alpha Q} + e^{-2\pi \alpha Q})(e^{2\pi \alpha P} + e^{-2\pi \alpha P})\|\).

**Proof.** Since both operators \(e^{2\pi \alpha Q} + e^{-2\pi \alpha Q}\) and \(e^{2\pi \alpha P} + e^{-2\pi \alpha P}\) on the Hilbert space \(L^2(\mathbb{R})\) are self-adjoint and greater than the identity, the operator domain \(\mathcal{E}_{c,d} := \mathcal{D}(e^{2\pi \alpha Q} + e^{-2\pi \alpha Q})(e^{2\pi \alpha P} + e^{-2\pi \alpha P})\) equipped with the norm \(\| \cdot \|_{c,d}\) is a Hilbert space. Assume to the contrary that the assertion of the lemma is not true. Then there exists a non-zero vector \(\psi_0 \in \mathcal{E}_{c,d}\) such that

\[
((e^{2\pi \alpha Q} + e^{-2\pi \alpha Q})(e^{2\pi \alpha P} + e^{-2\pi \alpha P})\psi_0, (e^{2\pi \alpha Q} + e^{-2\pi \alpha Q})(e^{2\pi \alpha P} + e^{-2\pi \alpha P})e_{t,\delta}) = 0
\]

for \(t \in \mathbb{R}\) and \(0 < \delta < \delta_0\). In order to write this relation in another form, we set

\[
\psi = (e^{2\pi \alpha Q} + e^{-2\pi \alpha Q})(e^{2\pi \alpha P} + e^{-2\pi \alpha P})\psi_0, \quad \xi_\delta = e^{-2\pi \delta x^2} (e^{2\pi \alpha x} + e^{-2\pi \alpha x})\psi.
\]

Since \(\psi \in L^2(\mathbb{R})\), it follows that \(e^{n|x|}\xi_\delta(x) \in L^1(\mathbb{R})\) for all \(n \in \mathbb{N}\). Further, we compute

\[
((e^{2\pi \alpha P} + e^{-2\pi \alpha P})e_{t,\delta}) (x) = e^{2\pi \delta (x^2 - \delta^2)}(e^{2\pi ix(t+2\delta d) + \pi \delta\xi} + e^{2\pi ix(t-2\delta d) - 2\pi \delta\xi}).
\]

Therefore, condition \((31)\) means that the Fourier transform \(\hat{\xi}_\delta\) of the \(L^1\)-function \(\xi_\delta\) satisfies the relation

\[
e^{2\pi \delta t} \hat{\xi}_\delta(t+2\delta d) + e^{-2\pi \delta t} \hat{\xi}_\delta(t-2\delta d) = 0, \quad t \in \mathbb{R}.
\]

Setting

\[
\eta_\delta(t) := e^{\pi(2\delta)^{-1} t^2 - \pi(4\delta d)^{-1} t i} \hat{\xi}_\delta(t),
\]

equation \((33)\) is obviously equivalent to the relation

\[
\eta_\delta(t+2\delta d) = \eta_\delta(t-2\delta d), \quad t \in \mathbb{R}.
\]
Since $e^{n|x|}\xi_\delta(x) \in L^1(\mathbb{R})$ for $n \in \mathbb{N}$, $\hat{\xi}_\delta$ is an entire holomorphic function on the complex plane and so is $\eta_\delta$ by (34). Therefore, by (33), the function $\eta_\delta$ is bounded on the real axis and hence $e^{n|x|}\hat{\xi}_\delta(t) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ for $n \in \mathbb{N}$ by (34). Consequently, $\xi_\delta(x)$ is an entire holomorphic function and $\xi_\delta(x) \in \mathcal{D}(e^{n|x|})$ for $n \in \mathbb{N}$. Hence, by (31),

$$
\zeta(x) := (e^{2\pi cx} + e^{-2\pi cx})\psi(x) = e^{2\pi \delta x^2} \xi_\delta(x)
$$

is also an entire function.

Computing $\xi_\delta(x)$ by the inverse Fourier transform from $\hat{\xi}_\delta(t)$ and using equation (33), we derive that

$$
\xi_\delta(x) + e^{-8\pi \delta d^2} e^{8\pi \delta x} \xi_\delta(x+2di) = 0, \ x \in \mathbb{R} .
$$

Inserting (36) into (37) we conclude that

$$
\zeta(x) + \zeta(x+2di) = 0 .
$$

Since $8|dc| < 1$ by assumption, the function $(e^{2\pi cx} + e^{-2\pi cx})^{-1}$ is holomorphic on the strip $J_{d,\varepsilon} := \{ x \in \mathbb{C} : |\text{Im} \ x| < 2d+\varepsilon \}$ for small $\varepsilon > 0$ and

$$
\inf \{ \| (e^{2\pi cx} + e^{-2\pi cx})^{-1} \| ; x \in J_{d,\varepsilon} \} > 0 .
$$

Therefore, since $\xi_\delta(x) \in \mathcal{D}(e^{-4\pi dP})$ as noted above, we conclude from Lemma 4 that the function $e^{-2\pi \delta x^2} \psi(x) = (e^{2\pi cx} + e^{-2\pi cx})^{-1} \xi_\delta(x)$ belongs to the domain $\mathcal{D}(e^{-4\pi dP})$ and

$$
e^{-4\pi dP} (e^{-2\pi \delta x^2} \psi(x)) = e^{-2\pi \delta(x-2di)^2} \psi(x+2di) .
$$

Note that $\psi(x) \in L^2(\mathbb{R})$ by construction and $\psi(x+2di) \in L^2(\mathbb{R})$ by (33), (38) and (39). Since $e^{-2\pi \delta x^2} \psi(x) \to \psi(x)$ and $e^{-4\pi dP} (e^{-2\pi \delta x^2} \psi) \to \psi(x+2di)$ as $\delta \to 0$ by (30) and the operator $e^{-4\pi dP}$ is closed, it follows that $\psi \in \mathcal{D}(e^{-4\pi dP})$ and $(e^{-4\pi dP} \psi)(x) = \psi(x+2di)$. Applying this fact and formula (38) we obtain

$$
(e^{-4\pi dP} \psi, \psi) = \int \psi(x+2di) \overline{\psi(x)} \, dx
$$

$$
= - \int (e^{2\pi bx} + e^{-2\pi bx})(e^{2\pi bx(x+2di)} + e^{-2\pi bx(x+2di)})^{-1}|\psi(x)|^2 \, dx .
$$

Because of the assumption $8|cd| < 1$, we have $\cos 4\pi cd > 0$. Hence the function under the integral sign in (30) is non-negative. On the other hand, since $\psi \neq 0$ by construction, we have $(e^{-4\pi dP} \psi, \psi) > 0$. Thus we arrived at a contradiction and the assertion of Lemma 8 is proved.

Next we turn to some facts on tensor products of certain operators. If $T_1$ and $T_2$ are closed operators on a Hilbert space $\mathcal{H}$, then the symbol $T_1 \otimes T_2$ means the closure of the linear operator on the domain $\mathcal{D}(T_1) \otimes \mathcal{D}(T_2)$ in the Hilbert space $\mathcal{H} \otimes \mathcal{H}$ defined by $(T_1 \otimes T_2)(\eta_1 \otimes \eta_2) = T_1 \eta_1 \otimes T_2 \eta_2$.
Let $\mathcal{P}_j$ and $\mathcal{Q}_j$, $j = 1, 2$, be the self-adjoint operators on the Hilbert space $L^2(\mathbb{R}^2)$ given by

$$(\mathcal{P}_j f)(x_1, x_2) = \frac{1}{2\pi} \frac{\partial}{\partial x_j} (x_1, x_2)$$

and

$$(\mathcal{Q}_j f)(x_1, x_2) = x_j f(x_1, x_2).$$

Let $\mathbb{R}^{++} := \{\mu = (\mu_1, \mu_2) \in \mathbb{R}^2 : \mu_1 > 0, \mu_2 > 0\}$. For $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$ we set

$$S(e^{2\pi \mu \mathcal{Q}}) := (e^{2\pi \mu_1 \mathcal{Q}_1} \otimes I + e^{-2\pi \mu_1 \mathcal{Q}_1} \otimes I)(I \otimes e^{2\pi \mu_2 \mathcal{Q}_2} + I \otimes e^{-2\pi \mu_2 \mathcal{Q}_2}),$$

$$e^{2\pi \mu \mathcal{Q}} := e^{2\pi \mu_1 \mathcal{Q}_1} \otimes e^{2\pi \mu_2 \mathcal{Q}_2}.$$  

The operators $S(e^{2\pi \mu \mathcal{P}}), e^{2\pi \mu \mathcal{P}}, e^{2\pi \mu |\mathcal{P}|}$ and $e^{2\pi \mu |\mathcal{Q}|}$ are defined in a similar manner. Then we have

$$\mathcal{D}_{\mu, \nu} := \mathcal{D}(S(e^{2\pi \mu \mathcal{Q}})S(e^{2\pi \nu \mathcal{P}})) = \bigcap_{\varepsilon, \delta \in \mathbb{Z}_2^2} \mathcal{D}(e^{2\pi \varepsilon \mathcal{Q}} e^{2\pi \delta \mathcal{P}}) = \bigcap_{\varepsilon, \delta \in \mathbb{Z}_2^2} \mathcal{D}(e^{2\pi \varepsilon \mathcal{Q}} e^{2\pi \delta \mathcal{Q}}).$$

where $\mathbb{Z}_2^2 = \{\varepsilon = (\varepsilon_1, \varepsilon_2) : \varepsilon_1, \varepsilon_2 \in \{1, -1\}\}$ and $2\pi \varepsilon \mu := (2\pi \varepsilon_1 \mu_1, 2\pi \varepsilon_2 \mu_2)$.

If $\nu \in \mathbb{R}^{++}$, then $\mathcal{D}_{\mu, \nu}$ is the vector space of all holomorphic functions on $\{(z_1, z_2) \in \mathbb{C}^2 : \Im z_j < |\nu_j|, j = 1, 2\}$ satisfying

$$\sup_{|\nu_j| < |\nu_j|} \iint |a(x_1+iy_1, x_2+iy_2)| e^{2\pi i(\mu_1 x_1 + |\mu_2 x_2|)} dx_1 dx_2 < \infty.$$  

The latter fact can be proved in a similar manner as Lemma 1.1 in [S2] using the Paley-Wiener Theorem.

**Lemma 9.** (i) Suppose that $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$ and $\nu = (\nu_1, \nu_2) \in \mathbb{R}^{++}$. If $f \in \mathcal{D}(S(e^{2\pi \nu \mathcal{P}})e^{2\pi \mu \mathcal{Q}}) \cap \mathcal{D}(e^{2\pi \mu \mathcal{Q}} S(e^{2\pi \nu \mathcal{P}}))$, then

$$|f(x_1+iy_1, x_2+iy_2)| \leq \frac{1}{2\pi} ((|\nu_1| - |y_1|)(|\nu_2| - |y_2|))^{-1/2} e^{-2\pi (\mu_1 x_1 + \mu_2 x_2)} \| e^{2\pi \nu |\mathcal{P}|} e^{2\pi \mu \mathcal{Q}} f \|$$

for $x_1, x_2, y_1, y_2 \in \mathbb{R}, |y_1| < \nu_1, |y_2| < \nu_2$.

(ii) Let $\mu = (\mu_1, \mu_2), \nu = (\nu_1, \nu_2) \in \mathbb{R}^{++}$. If $f \in \mathcal{D}_{\mu, \nu}$, then

$$|f(x_1+iy_1, x_2+iy_2)| \leq \frac{1}{2\pi} ((|\nu_1| - |y_1|)(|\nu_2| - |y_2|))^{-1/2} e^{-2\pi (\mu_1 x_1 + \mu_2 x_2)} \sum_{\varepsilon, \delta \in \mathbb{Z}_2^2} \| e^{2\pi \varepsilon \mathcal{Q}} e^{2\pi \delta \mathcal{Q}} f \|$$

for $x_1, x_2, y_1, y_2 \in \mathbb{R}, |y_1| < \nu_1, |y_2| < \nu_2$. The vector space $\mathcal{D}_{\mu, \nu}$ is contained in the Schwartz space $S(\mathbb{R}^2)$.

**Proof.** (i): Setting $g = \mathcal{F} f$ and $\varepsilon_j = \nu_j - |y_j|, j = 1, 2$, and using formulas (43),
which proves (44). Note that by the domain assumptions on \(f\) the function \(g = \mathcal{F}f\) belongs to the corresponding operator domains.

(ii): Since obviously \(\|e^{2\pi \nu | P|} e^{2\pi \mu Q} f\| \leq \sum_{\epsilon, \delta} \| e^{2\pi \epsilon \nu | P|} e^{2\pi \delta \mu Q} f\|\), inequality (45) follows at once from (44) applied with \(\epsilon\) replaced by \(\epsilon\mu\).

Finally, we prove that \(S(\mathbb{R}^2) \supseteq D_{\mu, \nu}\). Let \(a \in D_{\mu, \nu}\). By (12), we have \(a \in D(e^{\nu | Q|})\) which implies that \(a \in D(Q_1^m \otimes Q_2^m)\) for all \(n, m \in \mathbb{N}_0\). Similarly, since \(\mathcal{F}(a) \in D_{\nu, \mu}\) by (13), we have \(\mathcal{F}(a) \in D(Q_1^n \otimes Q_2^n)\) and hence \(a \in D(P_1^n \otimes P_2^n)\) for \(n, m \in \mathbb{N}_0\). Both conditions implies that \(a\) belongs to the Schwartz space \(S(\mathbb{R}^2)\) (see, for instance, Example 10.2.14 in [S1] for this apparently weaker characterization of the Schwartz space).

Lemma 10. Let \(c = (c_1, c_2), d = (d_1, d_2) \in \mathbb{R}^2\), \(\delta_1 > 0\), \(\delta_2 > 0\). Suppose that \(8|c_jd_j| < 1\) for \(j = 1, 2\). Then the vector space \(\mathcal{L}_{\delta_1} \otimes \mathcal{L}_{\delta_2}\) is dense in \(\mathfrak{A}(\mathbb{R}^2)\) with respect to the norm

\[
\| \cdot \|_{c,d} := \| S(e^{2\pi c Q}) S(e^{2\pi d P}) \|.
\]

Proof. Assume the contrary. Then there exists a vector \(\psi \neq 0\) which is orthogonal in \(L^2(\mathbb{R}^2)\) to \(S(e^{2\pi c Q}) S(e^{2\pi d P})(\mathcal{L}_{\delta_1} \otimes \mathcal{L}_{\delta_2})\). From the assertion of Lemma 8 it follows that \((e^{2\pi c_j Q_j} + e^{-2\pi c_j Q_j})(e^{2\pi d_j P_j} + e^{-2\pi d_j P_j})\mathcal{L}_{\delta_j}\) is dense in \(L^2(\mathbb{R})\) for \(j = 1, 2\). But this in turn implies that \(\psi = 0\). 

Let \(\mathfrak{A}(\mathbb{R}^2)\) be the intersection of all domains \(D(S(e^{\nu Q}) S(e^{\nu P}))\), \(c, d \in \mathbb{R}^2\), or equivalently the vector space of all holomorphic functions on \(\mathbb{C}^2\) satisfying condition (44) for all \(\mu, \nu \in \mathbb{R}^2\). Let \(\tau\) denote the locally convex topology on \(\mathfrak{A}(\mathbb{R}^2)\) defined by the family of norms (44), \(c, d \in \mathbb{R}^2\). Since it obviously suffices to take a countable subfamily of such norms, the topology \(\tau\) is metrizable. Since \(\mathfrak{A}(\mathbb{R}^2)\) is the intersection of domains \(D(e^{2\pi c Q} e^{2\pi d P})\), \(\mathfrak{A}(\mathbb{R}^2)\) is complete.
with respect to this topology. Thus \( \mathfrak{A}(\mathbb{R}^2) \) is a Frechet space. The space \( \mathfrak{A}(\mathbb{R}^2) \) will play a crucial role as symbol algebra for the Weyl calculus.

1.3. The Weyl Calculus

In this subsection we shall be concerned with pseudodifferential operators on the Hilbert space \( L^2(\mathbb{R}) \) defined by means of the Weyl calculus. Our standard references in this matter are the books [Fo] and [St], see also [GV] and [H]. The Weyl correspondence assigns an operator \( Op(a) \) to any function \( a \) on \( \mathbb{R}^2 \) such that \( \hat{a} \in L^1(\mathbb{R}^2) \) by

\[
Op(a) = \gamma \int \int \hat{a}(\alpha s, \beta t) e^{2\pi i (s\alpha + t\beta)} \, ds \, dt.
\] (47)

Recall that \( \hat{a} \) is the Fourier transform \([14]\) of the function \( a \). \( \alpha \) and \( \beta \) are real numbers such that \( \alpha \beta = \gamma \) and \( q = e^{2\pi i \gamma} \). Since \( \hat{a} \in L^1(\mathbb{R}^2) \), the integral \((47)\) can be understood as a Bochner integral and it defines a bounded operator \( Op(a) \) on the Hilbert space \( L^2(\mathbb{R}) \).

Let us restate some well-known facts on the Weyl calculus (see [Fo], Chapter 2). The operator \( Op(a) \) acts by the formula

\[
(Op(a)f)(x) = \int \int a(\frac{1}{2}(x+y), t) e^{2\pi i (x-y)t} f(y) \, dy \, dt.
\] (48)

For the operator product \( Op(a)Op(b) \) and the adjoint operator \( Op(a)^* \) we have

\[
Op(a)Op(b) = Op(a\#b) \quad \text{and} \quad Op(a)^* = Op(a^*),
\] (49)

where the symbols \( a\#b \) and \( a^* \) are defined by

\[
(a\#b)(x_1, x_2) := 4 \int \int a(u_1, u_2)b(v_1, v_2)e^{4\pi i[(x_1-u_1)(x_2-v_2)-(x_1-v_1)(x_2-u_2)]} \, du_1 \, du_2 \, dv_1 \, dv_2,
\] (50)

\[
a^*(x_1, x_2) := a(x_1, x_2), \quad x_1, x_2 \in \mathbb{R}.
\] (51)

Lemma 11. Let \( \mu = (\mu_1, \mu_2), \nu = (\nu_1, \nu_2), \mu' = (\mu'_1, \mu'_2), \nu' = (\nu'_1, \nu'_2) \in \mathbb{R}^+, a \in \mathcal{D}_{\mu, \nu}, b \in \mathcal{D}_{\nu', \mu'} \). For \( t \in \mathbb{C} \), we have

\[
e^{2\pi tQ_1}(a\#b) = (e^{2\pi tQ_1}a)\#(e^{2\pi tP_2}b) \quad \text{if} \quad \text{Re} \, t < \mu_1, \text{Re} \, t < 2\nu_2,
\] (52)

\[
e^{2\pi tQ_1}(a\#b) = (e^{-\pi tP_2}a)\#(e^{2\pi tQ_1}b) \quad \text{if} \quad \text{Re} \, t < 2\nu_2, \text{Re} \, t < \mu_1,
\] (53)

\[
e^{2\pi tQ_2}(a\#b) = (e^{2\pi tQ_2}a)\#(e^{-\pi tP_1}b) \quad \text{if} \quad \text{Re} \, t < \mu_1, \text{Re} \, t < 2\nu'_1,
\] (54)

\[
e^{2\pi tQ_2}(a\#b) = (e^{\pi tP_1}a)\#(e^{2\pi tQ_2}b) \quad \text{if} \quad \text{Re} \, t < 2\nu_1, \text{Re} \, t < \mu_2,
\] (55)

\[
e^{2\pi tP_1}(a\#b) = (e^{\pi tP_1}a)\#(e^{2\pi tP_1}b) \quad \text{if} \quad \text{Re} \, t < \nu_1, \text{Re} \, t < \nu'_1.
\] (56)

\[
e^{2\pi tP_1}(a\#b) = (e^{4\pi tQ_2}a)\#(e^{-4\pi tQ_2}b) \quad \text{if} \quad \text{Re} \, t < \nu_2/2, \text{Re} \, t < \nu'_2/2.
\] (57)

\[
e^{2\pi tP_2}(a\#b) = (e^{2\pi tP_2}a)\#(e^{2\pi tP_2}b) \quad \text{if} \quad \text{Re} \, t < \nu_2, \text{Re} \, t < \nu'_2.
\] (58)

\[
e^{2\pi tP_2}(a\#b) = (e^{-4\pi tQ_1}a)\#(e^{4\pi tQ_1}b) \quad \text{if} \quad \text{Re} \, t < \nu_1/2, \text{Re} \, t < \nu'_1/2.
\] (59)
Proof. As samples, we carry out the proofs of formulas (52) and (57). The other equations are proved by a similar reasoning.

First we prove formula (52) for real $t$. It is well-known (see [Fo], p. 104) that the Fourier transform of the product $a \ast b$ is the twisted convolution of the Fourier transform $\mathcal{F}(a)$ and $\mathcal{F}(b)$, that is $\mathcal{F}(a \ast b) = \mathcal{F}(a) \ast_t \mathcal{F}(b)$, where

$$(c \ast_t d)(x_1, x_2) = \iint c(u_1, u_2)d(x_1-u_1, x_2-u_2)e^{\pi i(x_1u_2-x_2u_1)}du_1du_2.$$ 

Using the preceding fact and formula (15) we compute

$$\mathcal{F}(e^{2\pi tP_2}(a \ast b))(x_1, x_2) = \left(e^{2\pi tQ_1} \mathcal{F}(a \ast b)\right)(x_1, x_2)$$

$$= \iint \mathcal{F}(a)(u_1, u_2)\mathcal{F}(b)(x_1-u_1, x_2-u_2)e^{\pi i(x_1(u_2-2ti)-x_2u_1)}du_1du_2$$

$$= \iint \mathcal{F}(a)(u_1, u_2+2ti)\mathcal{F}(b)(x_1-u_1, x_2-u_2-2ti)e^{\pi i(x_1u_2-x_2u_1)}du_1du_2$$

$$= \iint \left(e^{-4\pi tP_2} \mathcal{F}(a)\right)(u_1, u_2)\left(e^{4\pi tP_2} \mathcal{F}(b)\right)(x_1-u_1, x_2-u_2)e^{\pi i(x_1u_2-x_2u_1)}du_1du_2$$

$$= \iint \left(e^{-4\pi tP_2} \mathcal{F}(a) \ast_t e^{4\pi tP_2} \mathcal{F}(b)\right)(x_1, x_2)$$

$$= \mathcal{F}(e^{4\pi tQ_2}a) \ast_t \mathcal{F}(e^{-4\pi tQ_2}b)(x_1, x_2)$$

$$= \mathcal{F}(e^4\pi tQ_2, a \ast e^{-4\pi tQ_2}b)(x_1, x_2)$$

which in turn implies (57). It remains to justify the fourth equality sign which follows by the formal substitution $u_2 \to u_2+2ti$. First we note that the assumptions $a \in \mathcal{D}_{\nu,\mu}$ and $b \in \mathcal{D}_{\nu',\mu'}$ imply that $\mathcal{F}(a) \in \mathcal{D}_{\nu,\mu}$ and $\mathcal{F}(b) \in \mathcal{D}_{\nu',\mu'}$, so that $\mathcal{F}(a) \in \mathcal{D}((e^{2\pi \nu_2P_2})$ and $\mathcal{F}(b) \in \mathcal{D}((e^{2\pi \nu_2P_2})$. Therefore, since $2|t| < \nu_2$ and $2|t| < \nu_2$, the function

$$\mathcal{F}(a)(u_1, u_2)\mathcal{F}(b)(x_1-u_1, x_2-u_2)e^{\pi i(x_1(u_2-2ti)-x_2u_1)}$$

of $u_2$ is holomorph on a strip $-\varepsilon < \text{Im } u_2 < 2|t|+\varepsilon$ of the complex $u_2$-plane for some small $\varepsilon > 0$. Hence the integral of this function along the boundary of the rectangle with corners $-R, R, R+2ti, -R+2ti$ vanishes. In order to justify the substitution $u_2 \to u_2+2ti$, it is sufficient to show that the corresponding integrals from $\pm R$ to $\pm R+2ti$ tend to zero as $R \to +\infty$. Using formula (15) we estimate

$$\left| \iint_{0 \to \infty} \mathcal{F}(a)(u_1, \pm R + si)\mathcal{F}(b)(x_1-u_1, x_2-(\pm R + si))e^{\pi i(x_1(\pm R+si-2ti)-x_2u_1)}dsdu_1 \right|$$

$$\leq C \left| \iint_{0 \to \infty} e^{-2\pi (\nu_2|u_1|+\nu_2 R)-2\pi (\nu'_2|x_1-u_1|+\nu'_2|x_2+R|)+\pi x_1(2t-s)}dsdu_1 \right|$$

$$\leq C x_1, x_2 e^{-2\pi \nu_2 R},$$
where $C$ and $C_{x_1,x_2}$ are not depending on $R$. Since $\nu_2 > 0$, the integral goes to zero if $R \to +\infty$. This proves formula \((57)\) for real $t$.

Next we prove \((52)\) for real $t$. From the definition \((54)\) of the product $\#$ we obtain

$$
(e^{2\pi t Q_1}(a \# b))(x_1, x_2) = \iint e^{2\pi tu_1} a(u_1, u_2) b(v_1, v_2) e^{4\pi i [(x_1-u_1)(x_2-v_2-\pi t/2)-(x_1-v_1)(x_2-u_2)]} du_1 du_2 dv_1 dv_2.
$$

Recall that $a, b \in D_{\mu, \nu}$ by assumption. Hence we have $a \in D(e^{\pm 2\pi \mu_1 Q_1})$ and $b \in D(e^{\pm 2\pi \nu_1 P_2})$. Since $|t| < \mu_1$ and $|t| < 2\nu_2$, the latter implies that $a \in D(e^{2\pi t Q_1})$ and $b \in D(e^{2\pi P_2})$. In fact, we even have that $e^{2\pi t Q_1} a, e^{2\pi P_2} a \in D_{\tilde{\mu}, \tilde{\nu}}$ for certain $\tilde{\mu}, \tilde{\nu} \in \mathbb{R}^{++}$. Equation \((52)\) follows from the preceding formula by the formal substitution $v_2 \to v_2 + it/2$. In order to show that this formal replacement is justified we integrate in the complex $v_2$-plane along the boundary of the rectangle with corners $-R, R, R + it/2, -R + it/2$, where $R > 0$. To complete the proof, it suffices to show that the integrals from $\pm R$ to $\pm R + it/2$ tend to zero as $R \to +\infty$. Indeed, using formula \((13)\) and the assumptions $|t| < \mu_1$ and $|t| < 2\nu_2$, we estimate

$$
\left| \int \iint ds du_1 du_2 dv_1 e^{2\pi tu_1} a(u_1, u_2) b(v_1, \pm R + si) e^{4\pi i [(x_1-u_1)(x_2-(\pm R + si)-\pi t/2)-(x_1-v_1)(x_2-u_2)]} \right|
\leq C \left| \int \iint e^{2\pi tu_1} e^{-2\pi (\mu_1 |u_1| + \mu_2 |u_2| + \nu_1 |v_1| + \nu_2 |v_2|) R} e^{4\pi (x_1-u_1)(t/2-s)} ds du_1 du_2 dv_1 \right|
\leq C' e^{-2\pi \mu'_2 R} \int \iint_{\infty}^{t/2} e^{2\pi (2u_1-1)|u_1|} du_1 ds
\leq C'' e^{-2\pi \mu'_2 R} \int_{-\infty}^{\infty} e^{2\pi |t| |u_1|} du_1 \leq C''' e^{-2\pi \mu'_2 R},
$$

where $C, C', C'', C'''$ are numbers not depending on $R$. Since $\mu'_2 > 0$, the integral goes to zero if $R \to +\infty$. This completes the proof of \((52)\) for real $t$.

For imaginary $t$ the above reasoning works as well. In this case we are lead to real translations of $u_2$ and $v_2$, respectively, which are possible by the translation invariance of the Lebesgue measure. The case of general $t \in \mathbb{C}$ follows by combining the real and the imaginary cases.

For $\mu, \nu \in \mathbb{R}^{++}$, let $D_{\mu, \nu}$ denote the intersection of domains $D_{\mu, \nu}$ (see \((13)\)), where $\mu', \nu' \in \mathbb{R}^{++}, \mu'_j < \mu_j, \nu'_j < \nu_j$ for $j = 1, 2$. 

\[\Box\]
Corollary 12. Let $\mu, \nu \in \mathbb{R}^+$. If $\mu_1 < 2\nu_2$ and $\mu_2 < 2\nu_1$, then $\mathcal{D}^{\mu,\nu}$ is a $\ast$-algebra with product $\#$ and involution $\ast$ defined $(49)$ and $(52)$, respectively. In particular, $\mathfrak{A}(\mathbb{R}^2)$ is a $\ast$-algebra.

Proof. Since $\mu_1 < 2\nu_2$ and $\mu_2 < 2\nu_1$, we conclude from formulas $(52)$, $(54)$, $(56)$, $(58)$ and $(59)$ that $a, b \in \mathcal{D}^{\mu,\nu}$ implies that $a\# b \in \mathcal{D}^{\mu,\nu}$. By $(52)$ it is obvious that $a^* \in \mathcal{D}^{\mu,\nu}$ for $a \in \mathcal{D}^{\mu,\nu}$. Thus $\mathcal{D}^{\mu,\nu}$ is a $\ast$-algebra. Since $\mathfrak{A}(\mathbb{R}^2)$ is the intersection of all domains $\mathcal{D}^{\mu,\nu}$, $\mathfrak{A}(\mathbb{R}^2)$ is a $\ast$-algebra as well.

Lemma 13. Suppose that $\mu, \nu \in \mathbb{R}^+$. Let $\| \cdot \|$ denote the norm of $L^2(\mathbb{R})$. If $a, b \in \mathcal{D}^{\mu,\nu} = \mathcal{D}(S(e^{2\pi\mu\xi})S(e^{2\pi\nu\xi}))$, then $a, b, b\# a \in \mathcal{S}(\mathbb{R})$ and we have

$$
\int\int a(x_1, x_2)b(x_1, x_2)dx_1dx_2 = \int\int (\bar{b}\# a)(x_1, x_2)dx_1dx_2,
$$

(60)

and we have

$$
\| a\# b \| \leq \| a \| \| b \|.
$$

(61)

Proof. By Lemma 9(ii) and Corollary 12, we have $a, b \in \mathfrak{A}(\mathbb{R}^2)$ and so $b\# a \in \mathcal{S}(\mathbb{R}^2)$. From Proposition 2 in [St], p. 555, it follows that $\text{Op}(a)$ and $\text{Op}(b)$ are Hilbert-Schmidt operators and

$$
\langle \text{Op}(a), \text{Op}(b) \rangle_{\text{HS}} = \text{Tr} \text{Op}(b)^{\ast} \text{Op}(a) = (a, b) \quad \text{and} \quad \| \text{Op}(a) \|_{\text{HS}} = \| a \|,
$$

(62)

where $\text{Tr}$ is the trace and $\langle \cdot, \cdot \rangle_{\text{HS}}$ and $\| \cdot \|_{\text{HS}}$ denote scalar product and Hilbert-Schmidt norm of Hilbert-Schmidt operators, respectively. Using formulas $(62)$ and $(63)$ and the submultiplicativity of the Hilbert-Schmidt norm we obtain

$$
\| a\# b \| = \| \text{Op}(a\# b) \|_{\text{HS}} = \| \text{Op}(a) \text{Op}(b) \|_{\text{HS}}
$$

$$
\leq \| \text{Op}(a) \|_{\text{HS}} \| \text{Op}(b) \|_{\text{HS}} = \| a \| \| b \|.
$$

This proves $(61)$.

Put $c := b\# a$. By $(63)$, the operator $\text{Op}(c)$ is an integral operator with kernel

$$
K_c(x_1, x_2) = \int c \left( \frac{1}{2}(x_1 + x_2), t \right) e^{2\pi i(x_1 - x_2)t}dt.
$$

(63)

Since $c \in \mathcal{S}(\mathbb{R}^2)$, the function $d$ defined by $d(y_1, y_2) = \int c(y_1, t) e^{2\pi iy_2t}dt$ is in $\mathcal{S}(\mathbb{R}^2)$ and so is the function $K_c(x_1, x_2) = d \left( \frac{1}{2}(x_1 + x_2), x_1 - x_2 \right)$. It is well-known that any integral operator with kernel in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ is a trace class operator on $L^2(\mathbb{R})$ and that its trace is given by the integral over the diagonal. Using this fact and formulas $(63)$ and $(64)$ we get

$$
\text{Tr} \text{Op}(b)^{\ast} \text{Op}(a) = \text{Tr} \text{Op}(b^\ast \# a) = \text{Tr} \text{Op}(c)
$$

$$
= \int K_c(x_1, x_1)dx_1 = \int\int c(x_1, x_2)dx_1dx_2
$$

$$
= \int\int a(x_1, x_2)\bar{b}(x_1, x_2)dx_1dx_2.
$$
Comparing the latter with (62), formula (61) follows.

Our next proposition says that $\mathcal{A}(\mathbb{R}^2)[\tau]$ is a Frechet $\ast$-algebra with approximate identity.

**Proposition 14.** (i) Provided with the product $\#$, the involution $\ast$ and the locally convex topology $\tau$, $\mathcal{A}(\mathbb{R}^2)$ is a Frechet topological $\ast$-algebra.

(ii) Set $f_\varepsilon(x_1, x_2) := e^{-\pi\varepsilon(x_1^2 + x_2^2)}$. For each $a \in \mathcal{A}(\mathbb{R}^2)$, we have

$$\lim_{\varepsilon \to +0} f_\varepsilon \# a = \lim_{\varepsilon \to +0} a \# f_\varepsilon = a \quad (64)$$

in the locally convex space $\mathcal{A}(\mathbb{R}^2)[\tau]$.

**Proof.** (i): Recall that by definition the topology $\tau$ is generated by the family of norms $\| e^{2\pi c Q} e^{2\pi d P} \|$, $c, d \in \mathbb{R}^2$, where $\| \cdot \|$ is the norm of $L^2(\mathbb{R}^2)$. Fix $c, d \in \mathbb{R}^2$ and put

$$\mu = (\mu_1, \mu_2), \mu_1 := d_1 + c_1/2, \mu_2 := d_2 - c_2/2. \quad (65)$$

By (53), (55), (56), (58) and (61), we obtain

$$\| e^{2\pi c Q} e^{2\pi d P} (a \# b) \| = \| (e^{2\pi c P} a) \# (e^{2\pi c Q} e^{2\pi c d P} b) \| \leq \| e^{2\pi c P} a \| \| e^{2\pi c Q} e^{2\pi c d P} b \| \quad (66)$$

for $a, b \in \mathcal{A}(\mathbb{R}^2)$. Since $\| e^{2\pi c Q} e^{2\pi d P} a \ast \| = \| e^{2\pi c Q} e^{2\pi d P} a \|$, product and involution are $\tau$-continuous, so $\mathcal{A}(\mathbb{R}^2)[\tau]$ is indeed a topological $\ast$-algebra. Since $\mathcal{A}(\mathbb{R}^2)[\tau]$ is a Frechet space as noted above, it is a Frechet topological $\ast$-algebra.

(ii): Let $b \in \mathcal{A}(\mathbb{R}^2)$ and $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$. Our aim is to prove by explicit estimations that

$$\lim_{\varepsilon \to +0} (e^{2\pi c P} f_\varepsilon) \# b = b \quad (67)$$

in $L^2(\mathbb{R}^2)$. Note first that from the well-known equation

$$\int e^{-\pi(s-c)^2 - 2\pi c t}s ds = \varepsilon^{-1/2} e^{-\pi\varepsilon^{-1} t^2} e^{-2\pi \varepsilon t c}, \quad c \in \mathbb{C}, \quad (68)$$

we obtain that

$$\mathcal{F}(e^{2\pi c P} f_\varepsilon)(x_1, x_2) = \varepsilon^{-1} e^{-\pi\varepsilon^{-1} (x_1^2 + x_2^2)} e^{2\pi(\mu_1 x_1 + \mu_2 x_2)}. \quad (69)$$

Using the latter formulas and the definition (51) of the product $\#$ we compute

$$|b(x_1, x_2) - ((e^{2\pi c P} f_\varepsilon) \# b)(x_1, x_2)|$$

$$= |b(x_1, x_2) - 4 \int \mathcal{F}(e^{2\pi c P} f_\varepsilon)(x_2 - 2v_2, 2v_1 - 2x_1)b(v_1, v_2) e^{4\pi i (v_1 x_2 - x_1 v_2)} dv_1 dv_2|$$

$$= \frac{4}{\varepsilon} \int \int (b(x_1, x_2) - b(v_1, v_2)) dv_1 dv_2 \quad (69)$$

$$\leq \frac{4}{\varepsilon} \int \int e^{-4\pi \varepsilon^{-1} (x_1 - v_1)^2 + 4\pi i (x_2 - v_2)(x_1 - v_1)} \left| b(x_1, x_2) - b(v_1, v_2) \right| dv_1 dv_2. \quad (70)$$
Fix a number $\delta > 0$. Since $b \in \mathfrak{A}(\mathbb{R}^2)$, (43) holds for arbitrary $\nu, \mu \in \mathbb{R}^+$. Hence there exists $M \in \mathbb{R}$ such that for $(x_1, x_2) \in \mathbb{R}^2$,

$$e^{4\pi|\mu_1|+1|x_1|+(4\pi|\mu_2|+1)x_2}|b(x_1, x_2)| \leq M.$$  \hfill (71)

Further, since $e^{x_1+|x_2|}b(x_1, x_2) \to 0$ as $|x_1| + |x_2| \to +\infty$ by (44), the function $c(x_1, x_2) := e^{x_1+|x_2|}b(x_1, x_2)$ is uniformly continuous on $\mathbb{R}^2$. Thus there exists $\delta_1$ such that $1 > \delta_1 > 0$ and

$$(1 + eM)e^{4\pi(\mu_1|+|\mu_2|)}|c(x_1, x_2) - c(v_1, v_2)| < \delta$$

for $(x_1 - v_1)^2 + (x_2 - v_2)^2 < \delta_1^2$. From this and (71) we easily derive that

$$e^{x_1+|x_2|}e^{4\pi((x_2-v_2)\mu_2-(x_1-v_1)\mu_1)}|b(x_1, x_2) - b(v_1, v_2)| < \delta$$

when $(x_1 - v_1)^2 + (x_2 - v_2)^2 < \delta_1^2$.

Next we turn to the domain where $(x_1 - v_1)^2 + (x_2 - v_2)^2 \geq \delta_1^2$. Obviously, there exists $K \in \mathbb{R}$ such that

$$-\pi\varepsilon^{-1}(t_1^2 + t_2^2) + 4\pi(|\mu_1| + 1)|t_1| + (\mu_2 + 1)|t_2| \leq K$$

for all $(t_1, t_2) \in \mathbb{R}^2$ and $1 > \varepsilon > 0$. If $(x_1 - v_1)^2 + (x_2 - v_2)^2 \geq \delta_1^2$, then by (71) and (73) we obtain that

$$e^{x_1+|x_2|}e^{-2\pi\varepsilon^{-1}((x_1-v_1)^2+(x_2-v_2)^2)}e^{4\pi((x_2-v_2)\mu_2-(x_1-v_1)\mu_1)}|b(x_1, x_2) - b(v_1, v_2)| \leq e^{-\pi\varepsilon^{-1}(x_1-v_1)^2+(x_2-v_2)^2}K^2M \leq e^{-\pi\varepsilon^{-1}\delta_1^2}K^2M < \delta$$

for sufficiently small $\varepsilon > 0$.

Using the relation

$$\frac{4}{\varepsilon}\int\int e^{-2\pi\varepsilon^{-1}((x_1-v_1)^2+(x_2-v_2)^2)}dv_1dv_2 = 2$$

by (68), it follows from estimates (69), (72) and (74) that

$$|b(x_1, x_2) - ((e^{2\pi\mu P}f_\varepsilon))#b)(x_1, x_2)| \leq 3\delta e^{-\varepsilon|x_1|-|x_2|}$$

and so $\|b - (e^{2\pi\mu P}f_\varepsilon)#b\| \leq 12\delta$ for small $\varepsilon > 0$. This proves (67).

Now let $a \in \mathfrak{A}(\mathbb{R}^2)$ and $c, d \in \mathbb{R}^2$. Let $\mu$ be as in (65). Applying (67) with $b = e^{2\pi\varepsilon Q}e^{2\pi dP}a$, we get

$$\|e^{2\pi\varepsilon Q}e^{2\pi dP}(f_\varepsilon#a - a)\| = \|e^{2\pi\mu P}f_\varepsilon#(e^{2\pi\varepsilon Q}e^{2\pi dP}a) - e^{2\pi\varepsilon Q}e^{2\pi dP}a\| \to 0$$

as $\varepsilon \to +0$. This proves that $\lim_{\varepsilon \to +0}f_\varepsilon#a = a$ in $\mathfrak{A}(\mathbb{R}^2)[\tau]$. Applying the involution we obtain the second equality in (64). \hfill \square

Remark 2. Upon scaling and multiplying by parameters, the operators $Op(f_\varepsilon)$, $\varepsilon > 0$, form the so-called Hermite semigroup $e^{-2\pi t(P^2+Q^2)}$, $t > 0$, acting on the Hilbert space $L^2(\mathbb{R}^2)$, see [Fo], pp. 236–238.
In this paper we shall mainly use the symbol algebra \( \mathfrak{A}(\mathbb{R}^2) \). However, for most considerations it suffices to work with the smaller symbol algebras

\[
\mathfrak{A}_{ex}(\mathbb{R}^2) := \text{Lin}\{ e^{-\varepsilon_1 x_1^2-\varepsilon_2 x_2^2+c_1 x_1+c_2 x_2}; \varepsilon_1 > 0, \varepsilon_2 > 0, c_1, c_2 \in \mathbb{C} \},
\]
\[
\mathfrak{A}_{pex}(\mathbb{R}^2) := \text{Lin}\{ x_1^{n_1} x_2^{n_2} e^{-\varepsilon_1 x_1^2-\varepsilon_2 x_2^2+c_1 x_1+c_2 x_2}; \varepsilon_j > 0, c_j \in \mathbb{C}, n_j \in \mathbb{N}_0 \}.
\]

Both \( \mathfrak{A}_{ex}(\mathbb{R}^2) \) and \( \mathfrak{A}_{pex}(\mathbb{R}^2) \) are \(*\)-algebras with multiplication \((50)\) and involution \((51)\). In order to prove this assertion it is sufficient to show that \( a \# b \) is in \( \mathfrak{A}_{ex}(\mathbb{R}^2) \) resp. \( \mathfrak{A}_{pex}(\mathbb{R}^2) \) when \( a \) and \( b \) are so. In the case of \( \mathfrak{A}_{ex}(\mathbb{R}^2) \) this can be verified by direct computation of the twisted product \( a \# b \) using formula \((68)\). From formula (3) in \([GV]\) it follows at once that \( a \# b \in \mathfrak{A}_{pex}(\mathbb{R}^2) \) for \( a, b \in \mathfrak{A}_{pex}(\mathbb{R}^2) \).

2. The Coordinate Algebra \( \mathcal{O}(\mathbb{C}_q^2) \) of the Quantum Plane

2.1 \( \mathcal{O}(\mathbb{C}_q^2) \) as a left module algebra of \( \mathcal{U}_q(\mathfrak{gl}_2) \)

Let \( \triangleright \) be the left action of \( \mathcal{U}_q(\mathfrak{gl}_2) \) on \( \mathcal{O}(\mathbb{C}_q^2) \) associated with the right coaction of \( \mathcal{O}(GL_q(2)) \) defined by \((10)\). From \((10)\) and \((3)\) we then obtain

\[
K_1 \triangleright x = q^{-1/2} x, \quad K_1 \triangleright y = y, \quad K_1 \triangleright x = x, \quad K_2 \triangleright y = q^{-1/2} y, \quad E \triangleright x = y, \quad E \triangleright y = 0, \quad F \triangleright x = 0, \quad F \triangleright y = x.
\]

Moreover, since \( \varepsilon(K_1) = \varepsilon(K_2) = 1 \) and \( \varepsilon(E) = \varepsilon(F) = 0 \), we also have

\[
K_1 \triangleright 1 = 1, \quad K_2 \triangleright 1 = 1, \quad E \triangleright 1 = 0, \quad F \triangleright 1 = 0.
\]

The following proposition derives the action of the generators \( K_1, K_2, E, F \) of \( \mathcal{U}_q(\mathfrak{gl}_2) \) on general elements of the algebra \( \mathcal{O}(\mathbb{C}_q^2) \). We set

\[
D_{q^{-2}}(f)(x) := \frac{f(x) - f(q^{-2} x)}{(1 - q^{-2}) x}.
\]

**Proposition 15.** If \( g \) and \( h \) are complex polynomials in a single variable, then we have

\[
K_1 \triangleright (g(x)h(y)) = g(q^{-1/2} x)h(y), \quad K_2 \triangleright (g(x)h(y)) = g(x)h(q^{-1/2} y), \quad (78)
\]
\[
E \triangleright (g(x)h(y)) = q^{-1/2} D_{q^{-2}}(g(q^{1/2} \cdot))(x)yh(q^{1/2} y), \quad (79)
\]
\[
F \triangleright (g(x)h(y)) = q^{-1/2} g(q^{1/2} x)xD_{q^{-2}}(h(q^{1/2} \cdot))(y). \quad (80)
\]

**Proof.** Since \( \mathcal{O}(\mathbb{C}_q^2) \) is a \( \mathcal{U}_q(\mathfrak{gl}_2) \)-comodule algebra, equation \((11)\) holds. The assertion follows from this equation combined with formulas \((73)\) and \((17)\). For the generators \( K_1 \) and \( K_2 \) this is obvious. We carry out the proof of formula \((80)\). The proof of formula \((73)\) is similar.

Since \( \Delta(F) = F \otimes K + K^{-1} \otimes F \), it follows from \((11)\) that

\[
F \triangleright (zz') = (F \triangleright z)(K \triangleright z') + (K^{-1} \triangleright z)(F \triangleright z'). \quad (81)
\]
Recall that $F\triangleright x = 0$ and $F\triangleright 1 = 0$ by (73) and (77). Using these facts it follows from (81) by induction on $n$ that $F\triangleright x^n = 0$ for $n \in \mathbb{N}_0$. Thus we have $F\triangleright g(x) = 0$.

Since $K^{-1}\triangleright g(x) = K^{-1}_1K_2\triangleright g(x) = K^{-1}_1\triangleright g(x) = g(q^{1/2}x)$, we derive from (81), applied to $z = g(x)$ and $z' = h(y)$, that

$$F\triangleright (g(x)h(y)) = g(q^{1/2}x)(F\triangleright h(y)).$$

(82)

Therefore, in order to prove (83) it suffices to show that

$$F\triangleright y^n = q^{-1/2}q^{n/2}x D_q^{-2}(y^n), \quad n \in \mathbb{N}. \quad (83)$$

We prove (83) by induction on $n$. If $n = 1$, then (83) is true by (73). If (83) is valid for $a$, then it follows from (81) and (75) that

$$D_q^a(y^n) = q^{-1/2}(1 - q^{-2a})^{-1}q^{n/2}y^n.$$ (84)

which proves (83) in the case of $n + 1$. □

For $z \in O(C_q^2)$, we define

$$D^q_x(z) = Ky^{-1}E'\triangleright z, \quad D^q_y(z) = Kx^{-1}F'\triangleright z,$$

(84)

where the elements $y^{-1}$ and $x^{-1}$ of $\hat{O}(C_q^2)$ act by left multiplication on $O(C_q^2)$. From (73) and (74) we obtain

$$D^q_x(g(x)h(y)) = q^{1/2}\lambda D_q^{-2}(g)(qx)h(qy), \quad (85)$$

$$D^q_y(g(x)h(y)) = q^{-1/2}\lambda g(x)D_q^{-2}(h)(qy). \quad (86)$$

for polynomials $g$ and $h$.

For $r, s \in \mathbb{N}_0$, let $\sigma_{rs}$ denote the automorphism of the algebra $O(C_q^2)$ defined by $\sigma_{rs}(z) = K_1^rK_2^s z, z \in O(C_q^2)$. From formulas (73)-(74) or (75) we easily derive that $D^q_x$ is a $(\sigma_{-2,0}, \sigma_{2,-2})$-derivation and $D^q_y$ is a $(\sigma_{0,-2}, \sigma_{2,2})$-derivation of the algebra $O(C_q^2)$, that is, for $z_1, z_2 \in O(C_q^2)$ we have

$$D^q_x(z_1z_2) = (K_1^{-2\triangleright z_1})D^q_x(z_2) + D^q_x(z_1)(K_1^2K_2^{-2\triangleright z_2}),$$

$$D^q_y(z_1z_2) = (K_2^{-2\triangleright z_1})D^q_y(z_2) + D^q_y(z_1)(K_1^2K_2^{-2\triangleright z_2}).$$

In the limit $q \to 1$ the preceding equations go into the Leibniz rule. We shall consider the linear mappings $D^q_x$ and $D^q_y$ as $q$-deformed partial derivatives of the algebra $O(C_q^2)$. 
2.2 Covariant Differential Calculus on $\mathcal{O}(\mathbb{C}_q^2)$

As shown in [PW] and [WZ], there are two distinguished first order differential calculi $\Gamma_+$ and $\Gamma_-$ on $\mathcal{O}(\mathbb{C}_q^2)$. For both calculi, the set of differentials $\{dx, dy\}$ is a basis for the right (and for the left) $\mathcal{O}(\mathbb{C}_q^2)$-module of first order forms. Therefore, for any $z \in \mathcal{O}(\mathbb{C}_q^2)$ there exist uniquely determined elements $\partial_x(z)$ and $\partial_y(z)$ of $\mathcal{O}(\mathbb{C}_q^2)$, called partial derivatives of $z$, such that

$$dz = dx \partial_x(z) + dy \partial_y(z). \quad (87)$$

The bimodule structures of the calculi $\Gamma_+$ and $\Gamma_-$ are described by the following commutation relations:

\begin{align*}
\Gamma_+ : \quad & xd\!y = qdy x + (q^2 - 1) dx \cdot y, \quad ydx = qdx y, \quad (88) \\
& xdx = q^2 dx \cdot x, \quad ydy = q^2 dy \cdot y. \quad (89)
\end{align*}

\begin{align*}
\Gamma_- : \quad & ydx = q^{-1} dx \cdot y + (q^{-2} - 1) dy \cdot x, \quad xdy = q^{-1} dy \cdot x, \quad (90) \\
& xdx = q^{-2} dx \cdot x, \quad ydy = q^{-2} dy \cdot y. \quad (91)
\end{align*}

From these relations we see that $\eta_+ := q^{-2}xdx$ and $\eta_- := x^{-2}ydy$ are non-zero central elements of the bimodules $\Gamma_+$ and $\Gamma_-$, respectively. Recall that an element $\eta$ of a bimodule over an algebra $Z$ is called central if $\eta z = z \eta$ for all $z \in Z$.

Note that the relations for $\Gamma_+$ go into the relations of $\Gamma_-$ if we interchange the coordinates $x$ and $y$ and the numbers $q$ and $q^{-1}$. The partial derivatives $\partial_x$ and $\partial_y$, considered as linear mappings of $\mathcal{O}(\mathbb{C}_q^2)$, and the coordinate functions $x$ and $y$, acting on $\mathcal{O}(\mathbb{C}_q^2)$ by left multiplication, satisfy the relations:

\begin{align*}
\Gamma_+ : \quad & \partial_x y = qy \partial_x, \quad \partial_y x = qx \partial_y, \\
& \partial_x x - q^2 x \partial_x = 1 + (q^2 - 1) y \partial_y, \quad \partial_y y - q^2 y \partial_y = 1.
\end{align*}

\begin{align*}
\Gamma_- : \quad & \partial_y x = q^{-1} y \partial_x, \quad \partial_y x = q^{-1} x \partial_y, \\
& \partial_x x - q^{-2} x \partial_x = 1, \quad \partial_y y - q^{-2} y \partial_y = 1 + (q^{-2} - 1) x \partial_x.
\end{align*}

From these formulas one derives by induction the expressions for the actions of $\partial_x$ and $\partial_y$ on general elements of $\mathcal{O}(\mathbb{C}_q^2)$. If $g$ and $h$ are complex polynomials in a single variable, then we have:

\begin{align*}
\Gamma_+ : \quad & \partial_x (g(y) h(x)) = g(qy) D_{q^{-2}}(h)(x), \quad \partial_y (g(y) h(x)) = D_{q^{-2}}(g)(y) h(x), \quad (92) \\
\Gamma_- : \quad & \partial_x (g(x) h(y)) = D_{q^{-2}}(g)(x) h(y), \quad \partial_y (g(x) h(y)) = g(q^{-1} x) D_{q^{-2}}(h)(y). \quad (93)
\end{align*}

All these facts and formulas are well-known. We now give another description of these calculi. Let $\Omega$ be the free bimodule of the localization algebra $\hat{\mathcal{O}}(\mathbb{C}_q^2)$ generated by a central vector space $V$. That is, $\Omega$ is the vector space $\hat{\mathcal{O}}(\mathbb{C}_q^2) \otimes \mathbb{C}_q^2$ with bimodule structure given by

$$u \left( \sum_j z_j \otimes e_j \right) v := \sum_j u z_j v \otimes e_j, \quad (94)$$
where $u, v, z_j \in \mathcal{O}(\mathbb{C}_q^2)$, $e_j \in V$. For notational simplicity, we write $ze$ instead of $z \otimes e$, where $z \in \mathcal{O}(\mathbb{C}_q^2)$ and $e \in E$. Fix two elements $e_1, e_2 \in V$ and put
\[
\omega_+ := q^{-2}y^2x^{-2}e_1 + x^{-2}e_2, \quad \omega_- = q^2x^2y^{-2}e_1 + y^{-2}e_2, \\
d_+ z := \omega_+ z - \omega_+, \quad d_- z := \omega_- z - \omega_-, \quad z \in \mathcal{O}(\mathbb{C}_q^2).
\]

Let us abbreviate $\mathcal{Z} := \mathcal{O}(\mathbb{C}_q^2)$. Obviously, $\tilde{\Gamma}_\varepsilon := \mathcal{Z} \cdot d_\varepsilon \mathcal{Z} \cdot \mathcal{Z}$ is a $\mathcal{Z}$-bimodule and the mapping $d_\varepsilon : \mathcal{Z} \to \tilde{\Gamma}_\varepsilon$ satisfies the Leibniz rule for $\varepsilon = +, -$. Thus, the pair $(\tilde{\Gamma}_\varepsilon, d_\varepsilon)$ is a first order differential calculus over the algebra $\mathcal{Z} = \mathcal{O}(\mathbb{C}_q^2)$.

For the differentials of the coordinate functions we obtain
\[
d_+ x = (q^{-2} - 1)y^2x^{-1}e_1, \quad d_+ y = (q^{-2} - 1)q^{-2}y^3x^{-2}e_1 + (q^{-2} - 1)yx^{-2}e_2, \\
d_- x = (q^2 - 1)q^2x^3y^{-2}e_1 + (q^2 - 1)xy^{-2}e_2, \quad d_- y = (q^2 - 1)x^2y^{-1}e_1. \tag{96}
\]

**Lemma 16.** Suppose that the elements $e_1$ and $e_2$ are linearly independent. Then the first order differential calculi $\Gamma_\varepsilon$ and $\tilde{\Gamma}_\varepsilon, \varepsilon = +, -$, are isomorphic.

**Proof.** Since $\{dx, dy\}$ is a free left $\mathcal{O}(\mathbb{C}_q^2)$-module basis of $\Gamma_\varepsilon$, there is a well-defined left $\mathcal{O}(\mathbb{C}_q^2)$-module homomorphism $\psi_\varepsilon : \Gamma_\varepsilon \to \tilde{\Gamma}_\varepsilon$ such that
\[
\psi_\varepsilon(udx + vdy) = ud_\varepsilon x + v d_\varepsilon y, \quad u, v \in \mathcal{O}(\mathbb{C}_q^2).
\]

In order to prove that $\psi_\varepsilon$ is an $\mathcal{O}(\mathbb{C}_q^2)$-bimodule homomorphism, it suffices to show that the relations (88) and (89) resp. (90) and (91) hold also in $\Omega_+$ and $\Omega_-$. As a sample, we verify the first relation of (90). The other relations follow by similar straightforward computations. Using formulas (96) and the commutation rules in the algebra $\mathcal{O}(\mathbb{C}_q^2)$, we obtain
\[
q^{-1}d_- x \cdot y + (q^{-2} - 1)d_- y \cdot x = \\
(q^2 - 1)(qx^3y^{-2}e_1 y + q^{-1}xy^{-2}e_2 y + (q^{-2} - 1)x^2y^{-1}e_1 x) = \\
(q^2 - 1)((qx^3y^{-1} + (q^{-2} - 1)qx^3y^{-1})e_1 + yxy^{-2}e_2) = yd_- x.
\]

From the construction it is clear that $\psi_\varepsilon$ is a surjective FODC homomorphism. We show that $\psi_-$ is injective and suppose that $ud_- x + vd_- y = 0$ for some elements $u, v \in \mathcal{O}(\mathbb{C}_q^2)$. Inserting the expressions from (96) and using the assumption that $e_1$ and $e_2$ are linearly independent, we get
\[
u q^2x^3y^{-2} + v x^2y^{-1} = 0, vxy^{-2} = 0
\]
which in turn implies that $u = v = 0$. The proof for $\psi_+$ is similar. \hfill \Box

We shall identify the isomorphic calculi $\Gamma_\varepsilon$ and $\tilde{\Gamma}_\varepsilon$. The above approach to the calculi $\tilde{\Gamma}_\varepsilon$ is convenient for many purposes. Among others, it allows us easily to extend these calculi to larger algebras.
The partial derivatives $\partial_x$ and $\partial_y$ can be also expressed in terms of the action of the generators of $\mathcal{U}_q(gl_2)$. Combining the formulas (79), (80) and (92) we obtain for the calculus $\Gamma_-$ the relations

$$
\partial_x(z) = q^{\frac{3}{2}}y^{-1}EK_1^3K_2\varphi z, \quad \partial_y(z) = q^{\frac{3}{2}}x^{-1}FK_3^1K_2\varphi z
$$

for $z \in \mathcal{O}(\mathbb{C}_q^2)$, where the action of $E, F, K_1, K_2$ is given by Proposition 15 and the elements $y^{-1}$ and $x^{-1}$ act by left multiplication on $\mathcal{O}(\mathbb{C}_q^2)$.

3. An Auxiliary $*$-algebra $\mathcal{W}$

3.1 The $\mathcal{U}_q(gl_2(\mathbb{R}))$-module $*$-algebra $\mathcal{W}$

Let $\mathcal{W}$ denote the $*$-algebra generated by the operators

$$
W(s, t) := e^{2\pi i(s\alpha Q + t\beta P)}, \quad s, t \in \mathbb{C}.
$$

These operators satisfy the relations

$$
W(s_1, t_1)W(s_2, t_2) = e^{\pi i(s_1t_1 - s_1t_2)}W(s_1 + s_2, t_1 + t_2), \quad (98)
$$

$$
W(s, t)^* = W(-\overline{s}, -\overline{t}) \quad (99)
$$

for $s_1, t_1, s_2, t_2, s, t \in \mathbb{C}$. By Lemma 4, the operator $W(s, t)$ acts as

$$
(W(s, t)f)(x) = e^{2\pi is\alpha x + \pi ist\gamma}f(x + t). \quad (100)
$$

Equations (98) and (100) hold for vectors contained in the corresponding operator domains. For instance, they hold on the domains $\mathcal{D}_\delta$, where $\delta > 0$, and $\mathfrak{A}(\mathbb{R}^2)$ in the Hilbert space $L^2(\mathbb{R}^2)$. Each of the dense subspaces is an invariant dense core for all operators $W(s, t)$. From (17) and (100) we see that

$$
W(-i, 0) = X, W(0, -i) = Y, W(s, 0) = X^{is}, W(0, t) = Y^{it}, \quad s \in \mathbb{C}. \quad (101)
$$

Our next aim is to define a left action of the Hopf algebra $\mathcal{U}_q(gl_2)$ on $\mathcal{W}$. Let us identify the generators $x$ with $X = W(-i, 0)$ and $y$ with $Y = W(0, -i)$. Then $\mathcal{O}(\mathbb{R}^2_q)$ becomes a $*$-subalgebra of $\mathcal{W}$. We now use formulas (78)-(80) (which have been proved only for polynomials $g$ and $h$!) as a motivation and extend them formally to the functions $g(X) = X^{is}$ and $h(Y) = Y^{it}$, $s, t \in \mathbb{C}$, of the positive self-adjoint operators $X$ and $Y$ defined by (17). Throughout we interpret expressions $(q^{k/2}X)^{is}$ and $(q^{k/2}Y)^{it}$ as $e^{-\pi k\gamma s}e^{2\pi is\alpha Q}$ and $e^{-\pi k\gamma t}e^{2\pi it\beta P}$, respectively, for $k = 0, \pm 1, 3$ and $s, t \in \mathbb{C}$. Recall that

$$
W(s, t) = e^{\pi i\gamma st}W(s, 0)W(0, t) = e^{\pi i\gamma st}X^{is}Y^{it} \quad (102)
$$
by (97). Applying now (73) and (80) formally (!) and using (102) we derive
\[
\lambda F^t W(s, t) = \lambda e^{\pi i \gamma s t} ((F^t X^i s) (K \triangleright Y^{it}) + (K^{-1} \triangleright X^i s)(F^t Y X^{it})
\]
\[
= \lambda e^{\pi i \gamma s t} (0 + (q^{1/2} X^i s q^{-1/2} X D_{q^{-2}}((q^{1/2} Y)^{it}))
\]
\[
= q^{1/2} (1 - q^{-2}) e^{\pi i \gamma s t} e^{-\pi \gamma s} X^i s X \left( \frac{(q^{1/2} Y)^{it} - (q^{1/2} q^{-2} Y)^{it}}{(1 - q^{-2}) Y} \right)
\]
\[
= q^{1/2} e^{-\pi \gamma s} e^{\pi i \gamma s t} X^i s X (e^{-\pi \gamma t} - e^{3\pi \gamma t}) Y^{it} Y^{-1}
\]
\[
= q^{1/2} e^{-\pi \gamma s} (e^{-\pi \gamma t} - e^{3\pi \gamma t}) e^{\pi i \gamma s t} X^i s + 1 Y^{it - 1}
\]
\[
= q^{1/2} e^{-\pi \gamma s} (e^{-\pi \gamma t} - e^{3\pi \gamma t}) e^{\pi i \gamma s t} W(s - i, 0) W(0, t + i)
\]
\[
= q^{1/2} e^{-\pi \gamma s} (e^{-\pi \gamma t} - e^{3\pi \gamma t}) e^{\pi i \gamma s t} e^{-\pi \gamma(s - i)(t + i)} W(s - i, t + i)
\]
\[
= (e^{-2\pi \gamma t} - e^{2\pi \gamma t}) W(s - i, t + i).
\]

The formulas for the actions of the other generators $E, K_1$ and $K_2$ are derived by a similar formal reasoning. Replacing (80) by (73) and (78) we obtain
\[
\lambda E^t W(s, t) = (e^{-2\pi \gamma s} - e^{2\pi \gamma s}) W(s + i, t - i),
\]
\[
K_1^t W(s, t) = e^{\pi \gamma s} W(s, t), \quad K_2^t W(s, t) = e^{\pi \gamma t} W(s, t).
\]

We now take the above formulas which have been obtained by formal algebraic manipulations as the starting point for the rigorous definition of a left action of $\mathcal{U}_q(gl_2)$ on the $*$-algebra $\mathcal{W}$. That is, for $s, t \in \mathbb{C}$ we define
\[
E \triangleright W(s, t) = \lambda^{-1} (e^{-2\pi \gamma s} - e^{2\pi \gamma s}) W(s + i, t - i), \quad (103)
\]
\[
F \triangleright W(s, t) = \lambda^{-1} (e^{-2\pi \gamma t} - e^{2\pi \gamma t}) W(s - i, t + i), \quad (104)
\]
\[
K_1 \triangleright W(s, t) = e^{\pi \gamma s} W(s, t), \quad K_2 \triangleright W(s, t) = e^{\pi \gamma t} W(s, t). \quad (105)
\]

**Proposition 17.** With definitions (103)–(105), $\mathcal{W}$ is a left $\mathcal{U}_q(gl_2(\mathbb{R}))$-module $*$-algebra.

**Proof.** Since the set of operators $W(s, t), s, t \in \mathbb{C}$, is linearly independent as easily shown, the preceding definitions extend uniquely to well-defined linear mappings of $\mathcal{W}$ into itself. It is straightforward to check that the terms $K_1 E - q^{1/2} E K_1, K_2 E - q^{-1/2} E K_2, K_1 F - q^{-1/2} F K_1, K_2 F - q^{1/2} F K_2$ and $\lambda E F - \lambda F E - K^2 + K^{-2}$ applied to an arbitrary basis element $W(s, t)$ of $\mathcal{W}$ vanish. Thus, formulas (103)–(105) define indeed a left action of the algebra $\mathcal{U}_q(gl_2)$ on $\mathcal{W}$.

That the left module $\mathcal{W}$ is a $\mathcal{U}_q(gl_2)$-module algebra means that (1) is satisfied. It suffices to check this condition for the generators $f = E, F, K_1, K_2, K_1^{-1}, K_2^{-1}$ and $z = W(s, t), z' = W(s', t'), s, t, s', t' \in \mathbb{C}$. As a sample, we carry out
this for the generator $f = E$. Using (103), (105) and (100) we compute

\[
\lambda(E^\flat W(s, t))(K W(s', t')) + \lambda(K^{-1} \circ W(s, t))(E^\flat W(s', t'))
\]
\[
= (e^{-2\pi\gamma s} - e^{2\pi\gamma s})e^{\pi i(s'-t')}W(s+i, t-i)W(s', t')
+ e^{\pi i(t-s)}(e^{-2\pi\gamma s'} - e^{2\pi\gamma s'})W(s, t)W(s'+it'-i)
\]
\[
= (e^{-2\pi\gamma s} - e^{2\pi\gamma s})e^{\pi i(s'-t')}e^{\pi i(y(s'+i)-(s+i)t')}W(s + s'+i, t + t'-i)
+ e^{\pi i(t-s)}(e^{-2\pi\gamma s'} - e^{2\pi\gamma s'})e^{\pi i((s'+i)t-s(t'-i))}W(s + s'+i, t + t'-i)
\]
\[
= \lambda e^{\pi i(s'-st')}E^\flat W(s+s', t+t')
\]
\[
= \lambda E^\flat W(s, t)W(s', t').
\]

This proves (4) in the case $f = E$.

Finally, it remains to check that (4) holds. Since $W$ is a left $U_q(gl_2)$-module algebra as just shown, it suffices to do this for the generators $f$ of $U_q(gl_2)$.

Again we restrict ourselves to the case $f = \lambda E, z = W(s, t)$. Since $S(\lambda E)^* = \lambda(-q)E^* = -\lambda E$ by (11) and $W(s, t)^* = W(-\bar{s}, -\bar{t})$, we have

\[
(\lambda E^\flat W(s, t))^* = (e^{-2\pi\gamma s} - e^{2\pi\gamma s})W(s+\bar{i}, t-\bar{i})
\]
\[
= (e^{-2\pi\gamma -\bar{s}} - e^{2\pi\gamma -\bar{s}})W(-\bar{s} + \bar{i}, -\bar{t} - \bar{i})
= -\lambda E^\flat W(-\bar{s}, -\bar{t}).
\]

The $*$-algebra $W$ consists of Hilbert space operators and formulas (103)–(105) have been derived by using formal operator calculus. However, the content of Proposition 17 is purely algebraic: It is obvious that the complex vector space $W$ with basis $W(s, t), s, t \in \mathbb{C}$ is a $*$-algebra with multiplication and involution defined by (12) and (13). Proposition 17 says that $W$ is a $U_q(gl_2(\mathbb{R}))$-module $*$-algebra with respect to the left action defined by (103)–(105).

Recall that $\mathcal{O}(\mathbb{R}^2_q)$ is a $U_q(gl_2(\mathbb{R}))$-module $*$-subalgebra of $W$. By definition, the products $xw$ and $yw$ for $w \in W$ are the operator products $Xw$ and $Yw$, respectively, in the Hilbert space $L^2(\mathbb{R})$. From (97) and (101) we obtain

\[
xW(s, t) = e^{-\pi t}W(s-i, t), \quad yW(s, t) = e^{\pi s}W(s, t-i), \quad (106)
\]
\[
W(s, t)x = e^{\pi t}W(s-i, t), \quad W(s, t)y = e^{-\pi s}W(s, t-i). \quad (107)
\]

3.2 Covariant differential calculus on $W$

In this subsection we extend the differential calculus $\Gamma_-$ of $\mathcal{O}(\mathbb{R}^2_q)$ to $W$. In order to do so, we use the approach given in 2.2 with $Z = W$ and write $\Gamma, d, \omega$ for $\Gamma_-, d_-, \omega_-$, respectively.

As in 2.2, we set $\omega = q^2x^2y^{-2}e_1 + y^{-2}e_2$ and define

\[
dz = \omega z - z\omega, \quad z \in W.
\]

Obviously, $\Gamma := W \cdot dW \cdot W$ is a first order differential calculus over $W$ with differentiation $d$ such that the differentials $dx, dy$ form a free left $W$-module
basis of $\Gamma$. Because of this property, the partial derivatives $\partial_s(z)$ and $\partial_t(z)$ are well-defined by (87). In order to compute the latter for $z = W(s, t)$, we use the commutation rules $xW(s, t) = e^{-2\pi\gamma_1 t}W(s, t)x$ and $yW(s, t) = e^{2\pi\gamma_2 s}W(s, t)y$ (by (103) and (107)) and the expressions (91) for $dx$ and $dy$. Comparing coefficients in (87), we obtain for $s, t \in \mathbb{C}$,

$$\partial_s(W(s, t)) = \frac{1 - e^{4\pi\gamma_2 s}}{1 - q^{-2}} e^{\pi\gamma_1 t}W(s + i, t), \quad \partial_t(W(s, t)) = \frac{1 - e^{4\pi\gamma_2 t}}{1 - q^{-2}} e^{3\pi\gamma_2 s}W(s, t + i).$$

4. The $\mathcal{U}_q(\mathfrak{gl}_2(\mathbb{R}))$-module $*$-algebra $\mathcal{A}(\mathbb{R}_q^{++})$

4.1 In the preceding section we extended the action of the Hopf $*$-algebra $\mathcal{U}_q(\mathfrak{gl}_2(\mathbb{R}))$ on $\mathcal{O}(\mathbb{R}_q^2)$ to the larger $*$-algebra $\mathcal{W}$ such that $\mathcal{W}$ is a left module $*$-algebra of $\mathcal{U}_q(\mathfrak{gl}_2(\mathbb{R}))$. We now go one step further and make the $*$-algebra $\mathcal{A}(\mathbb{R}^2)$ into a left $\mathcal{U}_q(\mathfrak{gl}_2(\mathbb{R}))$-module $*$-algebra. In order to do so we use the formulas (103)–(105) in order to derive the corresponding formulas for the action of the generators $E, F, K_1, K_2$ on $Op(a)$. Suppose that $a \in \mathcal{A}(\mathbb{R}^2)$. For the generator $E$ we obtain

$$\lambda E \ast Op(a) = \gamma \int \int \hat{a}(\alpha s, \beta t)(\lambda E \ast W(s, t))dsdt$$

$$= \gamma \int \int \hat{a}(\alpha s, \beta t)(e^{-2\pi\gamma_2 s}e^{2\pi\gamma_2 s})W(s + i, t - i)dsdt$$

$$= \gamma \int \int \hat{a}(\alpha(s - i), \beta(t + i))(e^{-2\pi\gamma(s - i)}e^{2\pi\gamma(s - i)})W(s, t)dsdt.$$  

Let us explain the steps of this computation. The first equality is only a formal interchanging of integrals and left action, while the second follows from formula (103). The third equality is obtained by the formal replacements $s \to s + i$ and $t \to t - i$. These substitutions are justified by a standard argument from complex analysis which has been used already in the proof of Lemma 11: The integral of the holomorphic operator-valued function

$$s \to \hat{a}(\alpha s, \beta t)(e^{-2\pi\gamma_2 s}e^{2\pi\gamma_2 s})W(s + i, t - i)$$

along the boundary of the rectangle $-R, R, R - i, -R - i$ for fixed $t \in \mathbb{C}$ and $R > 0$ is zero. By Lemma 9, the integrals from $R$ to $R - i$ and from $-R - i$ to $-R$ tend to zero as $R \to +\infty$. Arguing similarly for the variable $t$, the third equality is obtained. In order to complete this reasoning, we note that the function

$$\hat{a}(\alpha(s - i), \beta(t + i))(e^{-2\pi\gamma(s - i)}e^{2\pi\gamma(s - i)})$$

is the Fourier transform of the function $a_E \in \mathcal{A}(\mathbb{R}^2)$ defined by

$$a_E(x_1, x_2) := e^{2\pi(\beta x_2 - \alpha x_1)}(a(x_1 + \beta i, x_2) - a(x_1 - \beta i, x_2)).$$  \ (108)
Thus, we have seen that \( \lambda E \circ Op(a) = Op(a)E \). Using (104) and (103) instead of (103) a similar reasoning shows that \( \lambda F \circ Op(a) = Op(a)F \) and \( K_j \circ Op(a) = Op(a)K_j \), where the symbol \( a_F, a_{K_j} \in \mathfrak{A}(\mathbb{R}^2) \) are given by

\[
\begin{align*}
    a_F(x_1, x_2) &= e^{2\pi i (\alpha x_1 - \beta x_2)}(a(x_1, x_2 + \alpha i)) - a(x_1, x_2 - \alpha i)), \\
    a_{K_1}(x_1, x_2) &= a(x_1 - \frac{\beta}{2} i, x_2), \\
    a_{K_2}(x_1, x_2) &= a(x_1, x_2 - \frac{\alpha}{2} i).
\end{align*}
\]

Summarizing, in terms of the symbol we have derived the following formulas for the actions of the generators \( E, F, K_1, K_2 \) of \( U_q(gl_2(\mathbb{R})) \):

\[
\begin{align*}
    (E \circ a)(x_1, x_2) &= \lambda^{-1} e^{2\pi i (\beta x_2 - \alpha x_1)}(a(x_1 + \beta i, x_2)) - a(x_1 - \beta i, x_2), \quad (109) \\
    (F \circ a)(x_1, x_2) &= \lambda^{-1} e^{2\pi i (\alpha x_1 - \beta x_2)}(a(x_1, x_2 + \alpha i)) - a(x_1, x_2 - \alpha i)), \quad (110) \\
    (K_1 \circ a)(x_1, x_2) &= a(x_1 - \frac{\beta}{2} i, x_2), \quad (K_2 \circ a)(x_1, x_2) = a(x_1, x_2 - \frac{\alpha}{2} i). \quad (111)
\end{align*}
\]

The derivation of these formulas is rigorous except for the justification of the interchanging of integrals and actions. This could be made rigorous by introducing appropriate locally convex topologies. We shall not proceed this way, because we shall use formulas (109)–(111) only as definitions of the action of \( U_q(gl_2(\mathbb{R})) \) on \( \mathfrak{A}(\mathbb{R}^2) \) and prove the corresponding properties directly in 4.3.

Note that formulas (109)–(111) and also formulas (112) and (113) below are meaningful for larger classes of symbols rather than \( \mathfrak{A}(\mathbb{R}^2) \). For instance, for the function \( a(x_1, x_2) = e^{2\pi i (\alpha x_1 + \beta x_2)} \) (which is of course not in \( \mathfrak{A}(\mathbb{R}^2) \)) we have \( Op(a) = e^{2\pi i (\alpha Q + \beta P)} = W(s, t) \). In this case formulas (109)–(111) reduces to the equations (103)–(103) derived in the preceding section. If we allow the symbols to be distributions, then we recover also formulas (78)–(80).

In a similar manner the product of the operators \( Op(a) \) with operators \( X, Y \) can be computed by using formulas (106) and (107) (or (48)). We then obtain \( XOp(a) = Op(x_{a}), Op(a)X = Op(a_{x}), YOp(a) = Op(y_{a}) \) and \( Op(a)Y = Op(a_{y}) \), where the symbol \( x_{a}, a, y_{a} \in \mathfrak{A}(\mathbb{R}^2) \) are given by

\[
\begin{align*}
    x_{a}(x_1, x_2) &= e^{2\pi i \alpha x_1}a(x_1, x_2 + \frac{\alpha}{2} i), \quad a_{x}(x_1, x_2) = e^{2\pi i \alpha x_1}a(x_1, x_2 - \frac{\alpha}{2} i), \\
    y_{a}(x_1, x_2) &= e^{2\pi i \beta x_2}a(x_1 - \frac{\beta}{2} i, x_2), \quad a_{y}(x_1, x_2) = e^{2\pi i \beta x_2}a(x_1 + \frac{\beta}{2} i, x_2).
\end{align*}
\]

Let \( \mathfrak{A}(\mathbb{R}^2) \) denote the direct sum of vector spaces \( \mathcal{O}(\mathbb{R}^2) \) and \( \mathfrak{A}(\mathbb{R}^2) \).

**Lemma 18.** There is a unique structure of a \( * \)-algebra on \( \mathfrak{A}(\mathbb{R}^2) \) such that \( \mathcal{O}(\mathbb{R}^2) \) and \( \mathfrak{A}(\mathbb{R}^2) \) are \( * \)-subalgebras of \( \mathfrak{A}(\mathbb{R}^2) \) and the products of the generators \( x, y \) of \( \mathcal{O}(\mathbb{R}^2) \) and symbols \( a \in \mathfrak{A}(\mathbb{R}^2) \) are given by

\[
\begin{align*}
    xa(x_1, x_2) &= e^{2\pi i \alpha x_1}a(x_1, x_2 + \frac{\alpha}{2} i), \quad ax(x_1, x_2) = e^{2\pi i \alpha x_1}a(x_1, x_2 - \frac{\alpha}{2} i), \quad (112) \\
    ya(x_1, x_2) &= e^{2\pi i \beta x_2}a(x_1 - \frac{\beta}{2} i, x_2), \quad ay(x_1, x_2) = e^{2\pi i \beta x_2}a(x_1 + \frac{\beta}{2} i, x_2). \quad (113)
\end{align*}
\]

**Proof.** We first note that the maps \( z \to \rho_{++}(z) \) (see (18)) and \( a \to Op(a) \) (see (13)) are faithful \( * \)-representations of the \( * \)-algebras \( \mathcal{O}(\mathbb{R}^2) \) and \( \mathfrak{A}(\mathbb{R}^2) \) on
that the sum $\rho$ and $\operatorname{Op}$ the domain $A$.

In this subsection we introduce two useful algebra homomorphisms in order to understand the algebraic content behind formulas (109)–(111). Let $\rho$ denote the complex unital algebra with generators and defining relations $A$. Thus, $\rho$ and $\operatorname{Op}$ are the multiples of the identity operator and that no operator in $\operatorname{Op}(A(\mathbb{R}^2))$ is a multiple of the identity. Thus, $\rho(\mathcal{O}(\mathbb{R}^2)) \cap \operatorname{Op}(A(\mathbb{R}^2)) = \{0\}$. Hence the map $\mathcal{J} : (z, a) \mapsto \rho(z) + \operatorname{Op}(a)$ of $A(\mathbb{R}^2)$ to $\rho(\mathcal{O}(\mathbb{R}^2)) + \operatorname{Op}(A(\mathbb{R}^2))$ is bijective. The unique $\ast$-algebra structure on $A(\mathbb{R}^2)$ for which $\mathcal{J}$ is a $\ast$-homomorphism has obviously the desired properties.

We shall show by Theorem 21 below that $A(\mathbb{R}^2) = \mathcal{O}(\mathbb{R}^2) + A(\mathbb{R}^2)$ is even a left $\mathcal{U}(gl_2(\mathbb{R}))$-module $\ast$-algebra. We call this left $\mathcal{U}(gl_2(\mathbb{R}))$-module $\ast$-algebra $A(\mathbb{R}^2)$ the $\ast$-algebra of functions on the quantum quarter plane. Obviously, the $\ast$-subalgebra $\mathcal{O}(\mathbb{R}^2)$ is considered as the algebra generated by the two coordinate functions $x$ and $y$ of the quantum quarter plane. The elements of $A(\mathbb{R}^2)$ can be interpreted as “functions on the quantum quarter plane which go rapidly to zero at the boundary of the quantum quarter plane”. Note that $A(\mathbb{R}^2)$ is a two-sided $\ast$-ideal of the $\ast$-algebra $A(\mathbb{R}^2)$.

4.2 In this subsection we introduce two useful algebra homomorphisms in order to understand the algebraic content behind formulas (109)–(111). Let $B_q$ denote the complex unital algebra with generators $x_1, x_{-1}^1, y_1, y_{-1}^1, x_2, x_{-2}^1, y_2, y_{-2}^1$ and defining relations

$$x_jy_j = q^{1/8}y_jx_j, x_jx_j^{-1} = x_j^{-1}x_j = 1, y_jy_j^{-1} = y_j^{-1}y_j = 1 	ext{ for } j = 1, 2,$$  \hspace{1cm} (114)

$$x_1x_2 = x_2x_1, y_1y_2 = y_2y_1, x_1y_2 = y_2x_1, x_2y_1 = y_1x_2,$$  \hspace{1cm} (115)

where we set $q^{1/8} : = e^{\pi i/4}$. The subalgebra $B_j, j = 1, 2$, generated by $x_j, x_j^{-1}, y_j, y_j^{-1}$ is nothing but the localization of the algebra $\mathcal{O}(\mathbb{C}^2_q)$ at the elements $x_j$ and $y_j$, and $B_q$ is just the tensor product of the algebras $B_1$ and $B_2$.

**Lemma 19.** There are injective algebra homomorphisms $\psi : \mathcal{U}_q(gl_2) \to B_q$ and $\psi : \mathcal{O}(\mathbb{R}^2_q) \to B_q$ such that

$$\psi(E) = \lambda^{-1}x_2^2x_1^{-2}(y_1^{-4} - y_1^4),$$  \hspace{1cm} (116)

$$\psi(F) = \lambda^{-1}x_1^2x_2^{-2}(y_2^{-4} - y_2^4),$$  \hspace{1cm} (117)

$$\psi(K_1) = y_1^2, \quad \psi(K_2) = y_2^2,$$  \hspace{1cm} (118)

$$\psi(x) = x_1^2x_2^{-2}, \quad \psi(y) = x_2^2y_1^2.$$  \hspace{1cm} (119)

**Proof.** In order to prove the assertion for $\mathcal{U}_q(gl_2)$ it suffices to check that the operators $\psi(E), \psi(F), \psi(K_1)$ and $\psi(K_2)$ satisfy the defining relations of the algebra $\mathcal{U}_q(gl_2)$. Using the relations (114)–(115) of the algebra $B_q$ we obtain

$$\lambda\psi(E)\psi(K_1) = x_2^2x_1^{-2}(y_1^{-4} - y_1^4)y_1^2 = x_2^2(x_1^2y_1^2)(y_1^{-4} - y_1^4)$$

$$= x_2^2(q^{1/4})^{-2}y_1^{-2}x_1^{-2}(y_1^{-4} - y_1^4) = q^{-1/2}y_1^2x_2^2x_1^{-2}(y_1^{-4} - y_1^4)$$

$$= q^{-1/2}\lambda\psi(K_1)\psi(E).$$
The relations \( \psi(E)\psi(K_2) = q^{1/2}\psi(K_2)\psi(E) \), \( \psi(F)\psi(K_1) = q^{1/2}\psi(K_1)\psi(F) \), \( \psi(F)\psi(K_2) = q^{-1/2}\psi(K_2)\psi(F) \) and \( \lambda(\psi(E)\psi(F) - \psi(F)\psi(E)) = \psi(K)^2 - \psi(K)^{-2} \) are verified by similar computations. Obviously we have \( \psi(x)\psi(y) = q\psi(y)\psi(x) \). Hence the above formulas define indeed algebra homomorphisms of \( \mathcal{U}_q(gl_2) \) and \( \mathcal{O}(\mathbb{R}^2) \) into \( \mathcal{B}_q \). Since the sets \( \{E^kK_1^nK_2^mF^l; k, l \in \mathbb{N}_0, n, m \in \mathbb{Z}\} \), \( \{x^ky^n; k, n \in \mathbb{N}_0\} \) and \( \{x_1^ky_1^ny_2^m; k, l, n, m \in \mathbb{N}_0\} \) are vector space bases of \( \mathcal{U}_q(gl_2) \), \( \mathcal{O}(\mathbb{R}^2) \) and \( \mathcal{B}_q \), respectively, it follows easily from formulas (109)-(119) that the mappings \( \psi: \mathcal{U}_q(gl_2) \rightarrow \mathcal{B}_q \) and \( \psi: \mathcal{O}(\mathbb{R}^2) \rightarrow \mathcal{B}_q \) are injective. \( \square \)

Since \( |q| = 1 \), \( \mathcal{B}_q \) is a *-algebra with involution determined by \( x_j^* := x_j \) and \( y_j^* := y_j \), \( j = 1, 2 \). The algebra homomorphism \( \psi: \mathcal{U}_q^w(gl_2(\mathbb{R})) \rightarrow \mathcal{B}_q \) does not preserve the involution. The next lemma shows that \( \psi \) is similar to a *-homomorphism.

**Lemma 20.** For \( z \in \mathcal{U}_q(gl_2(\mathbb{R})) \) and \( z \in \mathcal{O}(\mathbb{R}^2) \), define

\[
\varphi(z) = x_1x_2y_1^{-1}y_2\psi(z)(x_1x_2y_1^{-1}y_2)^{-1}.
\]

Then \( \varphi: \mathcal{U}_q^w(gl_2(\mathbb{R})) \rightarrow \mathcal{B}_q \) and \( \varphi: \mathcal{O}(\mathbb{R}^2) \rightarrow \mathcal{B}_q \) are injective *-homomorphisms of the corresponding *-algebras. In fact, we have

\[
\begin{align*}
\varphi(E') &= x_1^2x_1^{-1}(y_1^{-4} - y_1^4)x_1^{-1} = x_1^2x_1^{-2}(q^{-1/2}y_1^{-4} - q^{1/2}y_1^4) \\
&= x_1^2(q^{1/2}y_1^{-4} - q^{-1/2}y_1^4)x_1^{-2}, \\
\varphi(F') &= x_1^2x_2^{-1}(y_2^{-4} - y_2^4)x_2^{-1} = x_1^2x_2^{-2}(q^{-1/2}y_2^{-4} - q^{1/2}y_2^4), \\
&= x_2^2(q^{1/2}y_2^{-4} - q^{-1/2}y_2^4)x_2^{-2}, \\
\varphi(q^{-1/4}K_1) &= \psi(K_1) = y_1^2, \\
\varphi(q^{-1/4}K_2) &= \psi(K_2) = y_2^2, \\
\varphi(x) &= \psi(x) = x_1^2y_1^2, \\
\varphi(y) &= \psi(y) = x_2^2y_2^2.
\end{align*}
\]

**Proof.** Clearly, \( \varphi: \mathcal{U}_q(gl_2(\mathbb{R})) \rightarrow \mathcal{B}_q \) and \( \varphi: \mathcal{O}(\mathbb{R}^2) \rightarrow \mathcal{B}_q \) are injective homomorphisms, because \( \psi \) are by Lemma 19. Therefore, it is sufficient to prove that \( \varphi(z^*) = \varphi(z)^* \) for the four generators \( z = E', F', q^{-1/4}K_1, q^{-1/4}K_2 \) of \( \mathcal{U}_q^w(gl_2(\mathbb{R})) \) and the two generators \( z = x, y \) of \( \mathcal{O}(\mathbb{R}^2) \). Since all these generators \( z \) and their images \( \varphi(z) \) are hermitean, it suffices to check formulas (120)-(123). The latter formulas follow by straightforward computations from (109)-(119) combined with the relations (114)-(117) of the algebra \( \mathcal{B}_q \). As a sample we verify (120) and compute

\[
\begin{align*}
\varphi(\lambda E) &= x_1x_2y_1^{-1}y_2\psi(\lambda E)(x_1x_2y_1^{-1}y_2)^{-1} \\
&= x_1x_2y_1^{-1}y_2x_1^{-1}y_1^{-1}(y_1^{-4} - y_1^4)y_2^{-1}y_1x_1^{-1}x_1^{-1} \\
&= x_1x_2q^{-1/4}x_1^{-1}y_1^{-1}q^{-1/4}x_2y_2(y_1^{-4} - y_1^4)y_2^{-1}y_1x_1^{-1}x_1^{-1} \\
&= q^{-1/2}x_2^2x_1^{-1}(y_1^{-4} - y_1^4)x_1^{-1}
\end{align*}
\]

which gives the first formula of (120). The second and third formulas of (120) follow by applying once more the commutation rules (114) and (115). \( \square \)
The $*$-homomorphisms $\varphi$ of $\mathcal{U}_q^{tw}(gl_2(\mathbb{R}))$ and $\mathcal{O}(\mathbb{R}_q^2)$ are crucial in what follows.

4.3 Let us return to the left action of the Hopf $*$-algebra $\mathcal{U}(gl_2(\mathbb{R}))$ on $\mathfrak{A}(\mathbb{R}^2)$ given by the formulas (109)–(111). We define a $*$-representation $\rho_0$ of the $*$-algebra $\mathcal{B}_q$ on the invariant dense domain $\mathfrak{A}(\mathbb{R}^2)$ of the Hilbert space $L^2(\mathbb{R}^2)$ by

$$
\rho_0(x_1) = e^{\pi Q_1}, \rho_0(y_1) = e^{\pi P_1}, \rho_0(x_2) = e^{\pi Q_2}, \rho_0(y_2) = e^{\pi P_2},
$$

where $q^{1/4} = e^{\pi i/2}$ and as always $\alpha \beta = \gamma$. It is obvious that these operators satisfy the relations of the $*$-algebra $\mathcal{B}_q$, so (124) defines indeed a $*$-representation of $\mathcal{B}_q$. Inserting (124) into (116)–(118) we see that equations (109)–(111) can be expressed as

$$
f \triangleright a = \rho_0(\psi(f))a, \quad a \in \mathfrak{A}(\mathbb{R}^2),
$$

for the generators $f = E, F, K_1, K_2$ of $\mathcal{U}_q(gl_2)$. We now take this equation as a definition for arbitrary elements $f \in \mathcal{U}_q(gl_2)$. Since $\rho_0 \circ \psi$ is an algebra homomorphism, (125) gives a well-defined left action of the algebra $\mathcal{U}_q(gl_2)$ on $\mathfrak{A}(\mathbb{R}^2)$. Recall from 2.1 that we have also a left action $\triangleright$ of $\mathcal{U}_q(gl_2)$ on $\mathcal{O}(\mathbb{R}_q^2)$. Hence the equation

$$
f \triangleright (z + a) := f \triangleright z + f \triangleright a, \quad f \in \mathcal{U}_q(gl_2), \quad z \in \mathcal{O}(\mathbb{R}_q^2), \quad a \in \mathfrak{A}(\mathbb{R}^2),
$$

defines a left action of $\mathcal{U}_q(gl_2)$ on the direct sum $\mathcal{A}(\mathbb{R}_q^{++}) = \mathcal{O}(\mathbb{R}_q^2) + \mathfrak{A}(\mathbb{R}^2)$. In terms of the $*$-representations $\rho_0$ formulas (112) and (113) can be written as

$$
xa = \rho_0(\psi(x))a = \rho_0(x_2^2 y_2^{-2})a, \quad ax = \rho_0(x_1^2 y_2^{-2})a, \quad ay = \rho_0(x_2^2 y_1^{-2})a.
$$

(127)

The main result of this section is the following theorem.

**Theorem 21.** With the preceding definitions, the $*$-algebra $\mathcal{A}(\mathbb{R}_q^{++})$ of functions on the quantum quarter plane is a left $\mathcal{U}_q(gl_2(\mathbb{R}))$-module $*$-algebra.

**Proof.** We already noticed that $\triangleright$ is a left action of the algebra $\mathcal{U}_q(gl_2)$ on $\mathcal{A}(\mathbb{R}_q^{++})$. It remains to show that conditions (1) and (4) are fulfilled for arbitrary elements $z, z' \in \mathcal{A}(\mathbb{R}_q^{++})$ and $f \in \mathcal{U}_q(gl_2)$.

We first prove that $\mathfrak{A}(\mathbb{R}^2)$ is a $\mathcal{U}_q(gl_2)$-left module algebra. Since $\triangleright$ is a left action of $\mathcal{U}_q(gl_2)$, it suffices to prove (1) for the generators $f = \lambda E, \lambda F, K_1, K_2$. These verifications are lengthy but straightforward. We restrict ourselves to...
the case \( f = \lambda E \). Then we compute

\[
(\lambda E^\circ a)\# (K^\circ b)(x_1, x_2) + (K^{-1} v a)\# (\lambda E^\circ b)(x_1, x_2)
\]

\[
= 4 \int \int \int \int du_1 du_2 dv_1 dv_2 e^{4\pi i [(x_1 - u_1)(x_2 - v_2) - (x_1 - v_1)(x_2 - v_2)]}
\]

\[
\{ e^{2\pi (\beta u_2 - \alpha u_1)} (a(u_1 + \beta i, u_2) - a(u_1 - \beta i, u_2)) b(v_1 - \frac{\beta}{2} i, v_2) + \alpha (u_1 + \frac{\beta}{2} i, u_2 - \frac{\alpha}{2} i) e^{2\pi (\beta v_2 - \alpha v_1)} (b(v_1 + \beta i, v_2) - b(v_1 - \beta i, v_2)) \}
\]

\[
= 4 \int \int \int \int du_1 du_2 dv_1 dv_2 a(u_1, u_2) b(v_1, v_2)
\]

\[
\{ -e^{2\pi (\beta x_2 - \alpha x_1)} (a(x_1 - u_1)(x_2 - v_2) - (x_1 - v_1)(x_2 - u_2)) b(v_1) + e^{2\pi (\beta x_2 - \alpha x_1)} (a(x_1 - v_1)(x_2 - v_2) - (x_1 - v_1)(x_2 - v_2)) b(v_1) \}
\]

\[
= (\lambda E^\circ (a \# b))(x_1, x_2).
\]

The first equality is obtained by inserting the formulas (109) and (111) for the actions of \( E \) and \( K \) and (50) for the product \( \# \) of the algebra \( \mathcal{A}(\mathbb{R}^2) \). The second equality follows by the substitution \( u_1 \to u_1 + \beta i, v_1 \to v_1 - \frac{\beta}{2} i, v_2 \to v_2 + \frac{\beta}{2} i \) of the first summand and similar replacements of the other three summands. As noted in the considerations preceding (11), these substitutions are justified because of Lemma 9. Next let us consider the expressions in the four exponentials after the second equality sign. By regrouping these terms we see that the first and the fourth exponentials cancel, while the second and third ones can be reexpressed as the exponentials after the third equality sign. The fourth equality follows by applying once more formulas (50) and (109). By a similar reasoning condition (1) can be checked for the other generators \( f = \lambda F, K_1, K_2 \). Thus, \( \mathcal{A}(\mathbb{R}^2) \) is a \( \mathcal{U}_q(gl_2) \)-left module algebra.

Recall from 2.1 that \( \mathcal{O}(\mathbb{R}^2_1) \) is also a \( \mathcal{U}_q(gl_2) \)-left module algebra. Therefore, in order to prove that the sum \( \mathcal{A}(\mathbb{R}^2_1) = \mathcal{O}(\mathbb{R}^2_1) + \mathcal{A}(\mathbb{R}^2) \) is a \( \mathcal{U}_q(gl_2) \)-left module algebra, it remains to show that

\[
f^\circ (w a) = (f_1^\circ w)(f_2^\circ a),
\]

\[
f^\circ (a w) = (f_1^\circ a)(f_2^\circ w)
\]

for \( f \in \mathcal{U}_q(gl_2), w \in \mathcal{O}(\mathbb{R}^2_1) \) and \( a \in \mathcal{A}(\mathbb{R}^2) \). It is easily seen that equation (129) holds for the product \( fg \) and arbitrary \( w \) and \( a \) provided that (129) holds for \( f \).
and arbitrary \( w \) and \( a \) and also for \( g \) and arbitrary \( w \) and \( a \). Hence it suffices to check condition (129) for elements \( f \) from a set \( M \) of generators of the algebra \( \mathcal{U}_q(g_{l2}) \) and for arbitrary \( w \) and \( a \). Suppose in addition that \( M \) is a vector space such that \( \Delta(M) \subseteq \mathcal{U}_q(g_{l2}) \otimes M \). Let \( w, w' \in \mathcal{O}(\mathbb{R}^2_q) \) such that (129) holds for \( w \) and all \( f \in M \) and \( a \) and also for \( w' \) and all \( f \in M \) and \( a \). We show that then (129) holds for the product \( ww' \) and arbitrary \( f \in M \) and \( a \) by computing

\[
(f(1)^{\lambda E}(w_1))f(2)^{\lambda E}(a) = (f(1)^{\lambda E}w)(f(2)^{\lambda E}w')(f(3)^{\lambda E}a) = (f(1)^{\lambda E}w)(f(2)^{\lambda E}w') = f(4)(ww'a).
\]

Note that for the second equality we used that \( \Delta(f) \subseteq \mathcal{U}_q(g_{l2}) \otimes M \) by assumption and so (129) is valid for the elements in the second tensor factor of \( \Delta(f) \). Applying the preceding with \( M = \text{Lin}\{E, F, K_1, K_2\} \) we conclude that condition (129) is fulfilled provided that it holds for \( f = E, F, K_1, K_2, w = x, y \) and arbitrary \( a \in \mathfrak{A}(\mathbb{R}^2) \). Arguing in a similar manner with condition (130) it follows that it is sufficient to verify (130) for the generators \( f = E, F, K_1, K_2 \) and \( w = x, y \). As a sample, we prove equation (129) for \( f = E \) and \( w = x \). Using the formulas (15), (160), (168) and (127) we obtain

\[
(\lambda E^{\lambda x})(K^{x a}) = (K^{-1}^{\lambda x})(\lambda E^{\lambda a})
\]

\[
= \lambda y(K^{x a}) + q^{1/2} x(\lambda E^{\lambda a})
\]

\[
= \lambda y\rho_0(x_2^2y_1^2)\rho_0(y_1^2y_2^2)a + q^{1/2} \rho_0(x_1^2y_2^2)\rho_0(x_1^2x_2^2(y_1^4 - y_4^4))a
\]

\[
= \rho_0(x_1^4x_2^2y_2^2 - q^{1/2} y_2^2 x_2^2(y_1^4 - y_4^4))a
\]

\[
= \rho_0((y_1^4 - y_4^4)x_2^2y_2^2)a
\]

\[
= \rho_0(x_1^2x_2^2(y_1^4 - y_4^4))\rho_0(x_2^2y_2^2)a
\]

\[
= \lambda E^{\lambda x}(xa).
\]

The other verifications are carried out in a similar manner. Thus we have shown that \( \mathfrak{A}(\mathbb{R}^2_{q^+}) \) is a right \( \mathcal{U}_q(g_{l2}) \)-module algebra.

Finally, we turn to condition (4). Because \( \mathfrak{A}(\mathbb{R}^2_{q^+}) \) is a left \( \mathcal{U}_q(g_{l2}) \)-module algebra, it is enough to prove (4) for the generators \( f = \lambda E, \lambda F, K_1, K_2 \). We verify (4) for \( f = \lambda E \) and \( z = a \in \mathfrak{A}(\mathbb{R}^2) \). Since \( S(\lambda E)^* = -\lambda E \), we obtain

\[
(\lambda E^{\lambda x})(x_1, x_2) = e^{2\pi i(\beta x_2 - \alpha x_1)}(a(x_1 + \beta i, x_2) - a(x_1 - \beta i, x_2))
\]

\[
= e^{2\pi i(\alpha x_2 - \beta x_1)}(\pi(x_1 + \beta i, x_2) - \pi(x_1 - \beta i, x_2))
\]

\[
= (S(\lambda E)^* \pi)(x_1, x_2).
\]

By similar computations we check that condition (4) is satisfied for \( f = \lambda F, K_1, K_2 \). Hence it follows that (4) holds for \( f \in \mathcal{U}_q(g_{l2}(\mathbb{R})) \) and \( z = a \in \mathfrak{A}(\mathbb{R}^2) \). Since \( \mathcal{O}(\mathbb{R}^2_{q^+}) \) is a left \( \mathcal{U}_q(g_{l2}(\mathbb{R})) \)-module \(*\)-algebra, (4) holds also for \( z = a \in \mathcal{O}(\mathbb{R}^2_{q^+}) \). Thus it remains to show that (4) is satisfied for products of the form \( z = wa \) and \( z = aw \), where \( a \in \mathfrak{A}(\mathbb{R}^2) \) and \( w \in \mathcal{O}(\mathbb{R}^2_{q^+}) \). We carry out this for \( z = wa \) and compute

\[
S(f)^* S(\lambda E^{\lambda x}) = S(f)^* S(\lambda E^{\lambda x})
\]

\[
= (f(1)^{\lambda E}a)^* (f(1)^{\lambda E}w)^* = ((f(1)^{\lambda E}w)(f(1)^{\lambda E}a))^* = (f^{\lambda E}(wa))^*.
\]
This completes the proof of Theorem 21. □

We close this subsection by collecting some additional useful relations. From the above formulas and the defining relations of the algebra $B_q$ we derive the following cross commutation relations for elements $\psi(E), \psi(F), \psi(K)$ and $\psi(x), \psi(y)$ in the algebra $B_q$:

\[
\begin{align*}
\psi(E)\psi(x) - q^{1/2}\psi(x)\psi(E) &= \psi(y)\psi(K), \\
\psi(E)\psi(y) &= q^{-1/2}\psi(y)\psi(E), \\
\psi(F)\psi(x) &= q^{1/2}\psi(x)\psi(F), \\
\psi(F)\psi(y) - q^{-1/2}\psi(y)\psi(F) &= \psi(x)\psi(K), \\
\psi(K)\psi(x) &= q^{-1/2}\psi(x)\psi(K), \\
\psi(K)\psi(y) &= q^{1/2}\psi(y)\psi(K).
\end{align*}
\]

The images of the algebras $U_q(sl_2(\mathbb{R}))$ and $O(R^2_q)$ under the algebra homomorphism $\psi$ do not generate the whole algebra $B_q$. But they are large enough such that fourth powers of the generators $x_1, x_2, y_1^{-1}, y_2$ can be expressed as

\[
\begin{align*}
y_1^{-4} &= q^{-1} + q^{1/2}\lambda\psi(EK^{-1})\psi(x)\psi(y)^{-1}, \\
x_1^4 &= \psi(x^4)(q - q^{-1/2}\lambda\psi(FK^{-1})\psi(y)\psi(x)^{-1}), \\
x_2^4 &= \psi(y^4)(q^{-1} + q^{1/2}\lambda\psi(EK^{-1})\psi(x)\psi(y)^{-1}).
\end{align*}
\]

4.4. Covariant differential calculus on the quantum quarter plane

In this subsection we extend the differential calculus $\Gamma = \Gamma_{\ldots}$ of $O(R^2_q)$ to the larger algebra $A(R^{++}_q)$. As in 3.2, we use the approach developed in 2.2, but now with the algebra $Z = A(R^{++}_q)$. We briefly repeat the construction from 2.2. Let $V$ be a two-dimensional vector space with basis $\{e_1, e_2\}$. The vector space $\Omega = A(R^{++}_q) \otimes V$ becomes a $A(R^{++}_q)$-bimodule with bimodule structure defined by (94). The differentiation $d$ is defined by the commutator with the element $\omega = q^x x^2 y^{-2} e_1 + y^{-2} e_2$ of the $A(R^{++}_q)$-bimodule $\Omega$, that is,

\[
dz = \omega z - z\omega, \quad z \in A(R^{++}_q).
\]

It is clear that $\Gamma := Z \cdot dZ \cdot Z$ is a first order differential calculus over $Z = A(R^{++}_q)$ with differentiation $d$ such that $\{dx, dy\}$ is a free left $Z$-module basis of $\Gamma$.

Let us compute the partial derivatives $\partial_x(a)$ and $\partial_y(a)$ for $a \in A(R^2)$. Using formulas (112), (113) and (124) we obtain from the definition of $d$ that

\[
da = q^2 x^2 y^{-2} e_1 a + y^{-2} e_2 a - q^2 ax^2 y^{-2} e_1 - ay^{-2} e_2 = q^2 x^2 y^{-2} \rho_0(1 - y_1 y_2) a e_1 + y^{-2} \rho_0(1 - y_1) a e_2. \tag{131}
\]

On the other hand, by (94) and (87) we have

\[
da = (q^2 - 1)(q^2 x^3 y^{-2}\partial_x(a) e_1 + xy^{-2}\partial_x(a) e_2 + x^2 y^{-1}\partial_y(a) e_1). \tag{132}
\]

Comparing the coefficient of $e_1$ and $e_2$ in (131) and (132) we derive

\[
\partial_x(a) = (1 - q^{-2})^{-1} x^{-1} \rho_0(1 - y_1) a, \quad \partial_y(a) = (1 - q^{-2})^{-1} y^{-1} \rho_0(y_1^8(1 - y_2^8)) a
\]
or equivalently
\[
\begin{align*}
\partial_x(a) &= (1 - q^{-2})^{-1}e^{-2\pi\alpha x_1}(a(x_1, x_2 - \frac{\alpha}{2}i) - a(x_1 - 2\beta i, x_2 - \frac{\alpha}{2}i)), \\
\partial_y(a) &= (1 - q^{-2})^{-1}e^{-2\pi\beta x_2}(a(x_1 - \frac{\beta}{2}i, x_2) - a(x_1 - 2\beta i, x_2 - 2\alpha i)).
\end{align*}
\]
In terms of the \(\ast\)-representation \(\rho_0\) defined by (124) and the actions of generators \(E, F, K_1, K_2, x, y\) given by (109)–(111), these formulas can be written as
\[
\begin{align*}
\partial_x(a) &= (1 - q^{-2})^{-1}\rho_0(x_1^{-2}y_2^2(1 - y_1^8))a = q^2y^{-1}EK_1^3K_2^\ast a, \\
\partial_y(a) &= (1 - q^{-2})^{-1}\rho_0(x_2^{-2}y_1^6(1 - y_2^8))a = q^2x^{-1}FK_1^3K_2^\ast a
\end{align*}
\]
for \(a \in \mathfrak{A}(\mathbb{R}^2)\). In 2.2 we have shown that the two latter expressions of \(\partial_x(a)\) and \(\partial_y(a)\) hold also for elements of \(\mathcal{O}(\mathbb{R}^2)\). Therefore, we have proved that
\[
\partial_x(a) = q^{3/2}y^{-1}EK_1^3K_2^\ast a, \quad \partial_y(a) = q^{1/2}x^{-1}FK_1^3K_2^\ast a
\]
for all elements \(a\) in the \(\ast\)-algebra \(\mathfrak{A}(\mathbb{R}^2) = \mathcal{O}(\mathbb{R}^2) + \mathfrak{A}(\mathbb{R}^2)\).

5. Covariant linear functionals on the quantum quarter plane

5.1 In the first subsection we construct for any \(k = (k_1, k_2) \in \mathbb{Z}^2\) a \(\mathcal{U}_q(gl_2(\mathbb{R}))\)-covariant linear functional \(h_k\) on \(\mathfrak{A}(\mathbb{R}^2)\) and show that it defines a scalar product \(\langle a, b \rangle_k\) on the \(\ast\)-algebra \(\mathfrak{A}(\mathbb{R}^2)\). This functional and the associated scalar product can be considered as \(q\)-analogs of the state given by Lebesgue measure and the \(L^2\)-scalar product on the classical quarter plane.

From the defining relations of the algebra \(\mathcal{U}_q(gl_2(\mathbb{R}))\) it is clear that there exists a unique character \(\tau\) on \(\mathcal{U}_q(gl_2(\mathbb{R}))\) such that
\[
\chi(K_1) = \chi(K_2) = q^{1/2} \text{ and } \chi(E) = \chi(F) = 0.
\]
Since the restriction of \(\chi\) to the Hopf subalgebra \(\mathcal{U}_q(sl_2(\mathbb{R}))\) is the counit, any \(\mathcal{U}_q(sl_2(\mathbb{R}))\)-covariant linear functional with respect to \(\chi\) is \(\mathcal{U}_q(sl_2(\mathbb{R}))\)-invariant.

**Proposition 22.** For \(k = (k_1, k_2) \in \mathbb{Z}^2\) and \(a \in \mathfrak{A}(\mathbb{R}^2)\) we define
\[
h_k(a) = \int \int e^{2\pi(\alpha_k x_1 + \beta_k x_2)}a(x_1, x_2)dx_1dx_2, \tag{133}
\]
where
\[
\alpha_k := \alpha + 2\beta^{-1}k_1, \quad \beta_k := \beta + 2\alpha^{-1}k_2.
\]
(i) The linear functional \(h_k\) on the \(\mathcal{U}_q(gl_2(\mathbb{R}))\)-module \(\ast\)-algebra \(\mathfrak{A}(\mathbb{R}^2)\) is covariant with respect to the character \(\chi\).
(ii) For \(s, t \in \mathbb{R}\) and \(a \in \mathfrak{A}(\mathbb{R}^2)\),
\[
h_k(a(x_1 + \beta s, x_2 + \alpha t)) = e^{-2\pi\gamma(s + t) - 4\pi(k_1 s + k_2 t)}h_k(a). \tag{134}
\]
(iii) \(h_k\) is continuous on the Frechet space \(\mathfrak{A}(\mathbb{R}^2)[\tau]\). More precisely, for any \(\varepsilon = (\varepsilon_1, \varepsilon_2), \varepsilon_1 > 0, \varepsilon_2 > 0,\) we have
\[
|h_k(a)| \leq \frac{1}{2\pi \sqrt{\varepsilon_1 \varepsilon_2}} \| S(e^{\varepsilon_1 Q_1}e^{2\pi(\alpha_k Q_1 + \beta_k Q_2)}a) \|, \quad a \in \mathfrak{A}(\mathbb{R}^2).
\]
Proof. (i): It suffices to verify condition \([3]\) for the generators \(f = K_1, K_2, E, F\).
That is, we have to show that

\[ h_k(K^a) = h_k(K^b) = q^{1/2}h_k(a) \text{ and } h_k(E^a) = h_k(F^a) = 0 \]

for \(a \in \mathfrak{A}(\mathbb{R}^2)\). By formulas \([109]-[111]\), the latter conditions are equivalent to the relations

\[
\begin{align*}
\int \int e^{2\pi(\alpha_k x_1 + \beta_k x_2)} a(x_1 - i\beta/2, x_2) dx_1 dx_2 &= \int \int e^{2\pi(\alpha_k x_1 + \beta_k x_2)} a(x_1, x_2 - i\alpha/2) dx_1 dx_2 \\
&= e^{\pi\alpha\beta i} \int \int e^{4\pi(\beta^{-1}k_1 x_1 + \alpha^{-1}k_2 x_2)} a(x_1, x_2) dx_1 dx_2,
\end{align*}
\]

\[
\begin{align*}
\int \int e^{2\pi(\alpha_k x_1 + \beta_k x_2)} a(x_1 + \beta i, x_2) dx_1 dx_2 &= \int \int e^{4\pi(\beta^{-1}k_1 x_1 + (\alpha^{-1}k_2 + \beta) x_2)} a(x_1 - \beta i, x_2) dx_1 dx_2, \\
\end{align*}
\]

\[
\begin{align*}
\int \int e^{4\pi((\alpha + \beta^{-1}k_1)x_1 + \alpha^{-1}k_2 x_2)} a(x_1, x_2 + \alpha i) dx_1 dx_2 &= \int \int e^{4\pi((\alpha + \beta^{-1}k_1)x_1 + \alpha^{-1}k_2 x_2)} a(x_1, x_2 - \alpha i) dx_1 dx_2.
\end{align*}
\]

These identities follow by the formal replacements \((x_1, x_2) \rightarrow (x_1 + i\beta/2, x_2), (x_1, x_2) \rightarrow (x_1, x_2 - i\alpha/2)\) and \((x_1, x_2) \rightarrow (x_1, x_2 - 2\alpha i)\), respectively. Similarly as above, these substitutions are justified by integrating in the complex plane and using the asymptotic estimate of Lemma 9.

(ii): The formula follows by the substitution \((x_1, x_2) \rightarrow (x_1 - \beta s, x_2 - \alpha t)\).

(iii) follows from \([133]\) and the Cauchy-Schwarz inequality. \(\square\)

Let \(\langle \cdot, \cdot \rangle_k\) be the sesquilinear form defined by means of the functional \(h_k\) on the \(\ast\)-algebra \(\mathfrak{A}(\mathbb{R}^2)\), that is,

\[ \langle a, b \rangle_k = h_k(b^\ast \# a), \ a, b \in \mathfrak{A}(\mathbb{R}^2). \] (135)

Recall that \(\langle \cdot, \cdot \rangle\) denotes the scalar product of the Hilbert space \(L^2(\mathbb{R}^2)\). For \(k \in \mathbb{Z}^2\), we abbreviate

\[ T_k := e^{\pi(\alpha_k Q_1 - \beta_k P_1)} \otimes e^{\pi(\beta_k Q_2 + \alpha_k P_2)} \equiv e^{\pi\alpha_k Q_1} e^{-\frac{\pi}{2}\beta_k P_1} \otimes e^{\pi\beta_k Q_2} e^{\frac{\pi}{2}\alpha_k P_2}. \] (136)

**Proposition 23.** The sesquilinear form \(\langle \cdot, \cdot \rangle_k\) is a scalar product and \(T_k\) is an isometric linear isomorphism of the unitary space \((\mathfrak{A}(\mathbb{R}^2), \langle \cdot, \cdot \rangle_k)\) on the unitary space \((\mathfrak{A}(\mathbb{R}^2), (\cdot, \cdot))\). For \(a, b \in \mathfrak{A}(\mathbb{R}^2)\), we have

\[ \langle a, b \rangle_k = \int \int e^{2\pi(\alpha_k x_1 + \beta_k x_2)} a(x_1 + i\beta_k/4, x_2 - i\alpha_k/4) \overline{b(x_1 - i\beta_k/4, x_2 + i\alpha_k/4)} dx_1 dx_2. \] (137)
5.2 In this subsection we investigate the left actions of the algebras $A(16)$. Since the operator $T$ and $\langle \cdot, \cdot \rangle$ from formula (138) that $T$ is indeed a scalar product on the vector space $A(R^2)$. The expression in (137) is obtained from (138) by inserting the actions of the operator $T_k$. Further, it follows from (138) that $T_k$ is an isometric linear isomorphism of the unitary space $A_k := (A(R^2), \langle \cdot, \cdot \rangle)$ onto the unitary space $A(\cdot, \cdot) := (A(R^2), \langle \cdot, \cdot \rangle)$.

**Proof.** For $a, b \in A(R^2)$, we compute

$$
\langle a, b \rangle_k = \int \int e^{2\pi i (\alpha_2 x_1 + \beta_2 x_2)} (b^* a)(x_1, x_2) dx_1 dx_2 \\
= \int \int (e^{2\pi i (\alpha_1 Q_1 + \beta_1 Q_2)} b)^* (e^{\pi i (\alpha_2 P_2 - \beta_2 P_1)} a)(x_1, x_2) dx_1 dx_2 \\
= \int \int (e^{2\pi i (\alpha_1 Q_1 + \beta_1 Q_2)} b)^* (x_1, x_2) \cdot (e^{\pi i (\alpha_2 P_2 - \beta_2 P_1)} a)(x_1, x_2) dx_1 dx_2 \\
= (e^{\pi i (\alpha_2 - \beta_2)} a, e^{2\pi i (\alpha_1 Q_1 + \beta_1 Q_2)} b) \\
= (T_k a, T_k b). 
$$

(138)

Here the first equality combines the definitions (138) and (135) of $h_k$ and $\langle \cdot, \cdot \rangle_k$, respectively. The second equality follows from (12) and (54), while the third one follows from formula (60). The fourth equality is just the definition of the scalar product $\langle \cdot, \cdot \rangle$, and the fifth and the sixth are easily derived from (10). Since the operator $T_k$ is a bijective linear mapping of $A(R^2)$, we conclude from formula (133) that $\langle \cdot, \cdot \rangle_k$ is indeed a scalar product on the vector space $A(R^2)$. The expression in (137) is obtained from (138) by inserting the actions of the operator $T_k$. Further, it follows from (138) that $T_k$ is an isometric linear isomorphism of the unitary space $A_k := (A(R^2), \langle \cdot, \cdot \rangle_k)$ onto the unitary space $A(\cdot, \cdot) := (A(R^2), \langle \cdot, \cdot \rangle)$. 

5.2 In this subsection we investigate the left actions of the algebras $U_q(gl_2)$ and $O(R^2)$ on the unitary space $A_k := (A(R^2), \langle \cdot, \cdot \rangle_k)$. Among others, we shall transform these actions to the domain $A(R^2)$ in the Hilbert space $L^2(R^2)$ by means of the unitary operator $T_k$.

Recall from 4.2 that the map $\rho_0 \circ \psi$ defines left actions of the algebras $U_q(gl_2)$ and $O(R^2)$ on $A(R^2)$. For the generators $E, F, K_1, K_2$ of $U_q(gl_2)$ this action has been also given by formulas (10), (11), see also (125). For the algebra $O(R^2)$ the action $\rho_0 \circ \psi$ is just the left multiplication in the larger algebra $A(R^2)$, see Lemma 18 and formulas (127) and (128). Let $\psi_k$ denote the action $\rho_0 \circ \psi$ of $U_q(gl_2)$ and $O(R^2)$ considered as representation on the unitary space $A_k$. Since $T_k$ is a unitary transformation of $A_k \equiv (A(R^2), \langle \cdot, \cdot \rangle_k)$ on $(A(R^2), \langle \cdot, \cdot \rangle)$ by Proposition 23, $\psi_k$ is unitarily equivalent to the representation

$$
\Psi_k(\cdot) := T_k \psi_k(\cdot) T_k^{-1}
$$

(139)
on the domain $A(R^2)$ in the Hilbert space $L^2(R^2)$. Further, the compositions $\Phi := \rho_0 \circ \phi$ of the $*$-homomorphisms $\phi$ (defined in Lemma 20) and the $*$-representation $\rho_0$ of $B_q$ (defined by (124)) are also $*$-representations of the $*$-algebras $U_q^{\tau}(gl_2(R))$ and $O(R^2)$, respectively, on the domain $A(R^2)$ in $L^2(R^2)$.

Let $T$ denote the operator $T_k$ defined by (138) for $k = (0, 0)$, that is,

$$
T = e^{\pi \alpha Q_1} e^{-\frac{\pi}{2} \beta P_1} \otimes e^{\pi \beta Q_2} e^{\frac{\pi}{2} \alpha P_2} = e^{\pi \alpha Q_1 - \frac{\pi}{2} \beta P_1} \otimes e^{\pi \beta Q_2 + \frac{\pi}{2} \alpha P_2}.
$$

(140)
Using the operator

\[ C_k := e^{2\pi k_1 \beta^{-1} Q_1} e^{-\pi k_2 \alpha^{-1} P_1} \otimes e^{2\pi k_2 \alpha^{-1} Q_2} e^{\pi k_1 \beta^{-1} P_2} \]

(141)

acting on the Hilbert space \( L^2(\mathbb{R}^2) \), we can write the operator \( T_k \), as

\[ T_k = \imath^{k_1-k_2} C_k T. \]

(142)

Comparing formulas (140) and (124) we see that \( T = \rho_0(x_1 x_2 y_1^{-1} y_2) \). Therefore, by Lemma 13, we get

\[ \Psi_0(z) = T \psi_0(z) T_0^{-1} = \rho_0((x_1 x_2 y_1^{-1} y_2) \psi(z)(x_1 x_2 y_1^{-1} y_2)^{-1}) = \rho_0 \circ \varphi(z) = \Phi(z) \]

for \( z \in \mathcal{U}_q(gl_2) \) and \( z \in \mathcal{O}(\mathbb{R}^2) \). That is, we have \( \Psi_0 = \Phi \) for both \( * \)-algebras \( \mathcal{U}_q(gl_2(\mathbb{R})) \) and \( \mathcal{O}(\mathbb{R}^2) \).

Next we relate the representations \( \Psi_k \) and \( \Psi_0 = \Phi \). Using the definitions of the operator \( C_k \) and \( \Phi(f) = \rho_0 \circ \varphi(f), f = E', F', K, K_1, K_2, x, y \), we compute

\[ C_k \Phi(f) C_k^{-1} = (-1)^{k_1+k_2} \Phi(f) \text{ for } f = E', F', K, \]

(143)

\[ C_k \Phi(K_j) C_k^{-1} = (-1)^{k_j} \Phi(K_j) \text{ for } j = 1, 2, \]

(144)

\[ C_k \Phi(x) C_k^{-1} = \Phi(x), \ C_k \Phi(y) C_k^{-1} = \Phi(y) \]

(145)

Because of the formulas (142) and (139) we therefore have

\[ \Psi_k(f) = (-1)^{k_1+k_2} \Phi(f) \text{ for } f = E', F', K, \]

(146)

\[ \Psi_k(K_j) = (-1)^{k_j} \Phi(K_j) \text{ for } j = 1, 2, \]

(147)

\[ \Psi_k(x) = \Phi(x), \ \Psi_k(y) = \Phi(y). \]

(148)

In order to complete the picture we collect the formulas for the operators \( \Phi(f) \), where \( f = E', F', K, K_1, K_2, x, y \). Recall from (24) and (24) that \( L_\alpha \) denotes the operator \( L_\alpha = \frac{f_0(\mathcal{P})}{f_0(\mathcal{P})} e^{-2\pi \alpha Q} \), where \( f_0(x) = -2 \sinh \pi \beta (2x + \alpha i) \). Combining (124)-(123) and (124) we obtain

\[ \Phi(E') = L_\alpha \otimes e^{2\pi \beta Q_2}, \Phi(F') = e^{2\pi \alpha Q_1} \otimes L_\beta, \]

(149)

\[ \Phi(q^{-1/4}K_1) = e^{\pi \beta P_1} \otimes I, \ \Phi(q^{-1/4}K_2) = I \otimes e^{\pi \alpha P_2}, \ \Phi(K) = e^{\pi \beta P_1} \otimes e^{-\pi \alpha P_2}, \]

(150)

\[ \Phi(x) = e^{2\pi \alpha Q_1} \otimes e^{-\pi \alpha P_2}, \Phi(y) = e^{\pi \beta P_1} \otimes e^{2\pi \beta Q_2}. \]

(151)

Let us briefly discuss the outcome of these considerations. Since the functional \( h_k \) on the left \( \mathcal{U}_q(gl_2(\mathbb{R})) \)-module \( * \)-algebra \( \mathfrak{A}(\mathbb{R}^2) \) is covariant with respect to \( \chi \), it follows from Lemma 2,(i)\( \rightarrow \)(ii), and the definition of the involution of \( \mathcal{U}_q^w(gl_2(\mathbb{R})) \) that \( \psi_k = \rho \circ \psi \) is a \( * \)-representation of the \( * \)-algebra \( \mathcal{U}_q^w(gl_2(\mathbb{R})) \) on the unitary space \( \mathfrak{A}_k = (\mathfrak{A}(\mathbb{R}^2), \langle \cdot, \cdot \rangle_k) \). The \( * \)-representation \( \psi_k \) is unitarily
equivalent to the ∗-representation Ψ_k of \( U_q^{tw}(gl_2(\mathbb{R})) \) on the domain \( \mathfrak{A}(\mathbb{R}^2) \) in the Hilbert space \( L^2(\mathbb{R}^2) \). The actions of the operators \( \Psi_k(f) \) for the generators \( f = E', F', K_1, K_2 \) are explicitly given by the formulas (140)–(147) and (148)–(151). Note that the dependence of the operators \( \Psi_k(f) \) on \( k \in \mathbb{Z}^2 \) appears only in the signs in (146)–(147). In particular, if \( k_1 \) and \( k_2 \) are both even, then the ∗-representation \( \Psi_k \) of \( U_q^{tw}(gl_2(\mathbb{R})) \) on \( \mathfrak{A}_k \) is unitarily equivalent to the fixed ∗-representation \( \Phi \) on the domain \( \mathfrak{A}(\mathbb{R}^2) \) in the Hilbert space \( L^2(\mathbb{R}^2) \). From (148) and (150) we see that \( \Psi_k \) is a ∗-representation of \( \mathcal{O}(\mathbb{R}^2_q) \) on the unitary space \( \mathfrak{A}_k \). For any \( k \in \mathbb{Z}^2 \), the ∗-representation \( \Psi_k \) of \( \mathcal{O}(\mathbb{R}^2_q) \) on \( \mathfrak{A}_k \) is unitarily equivalent to the ∗-representation \( \Phi \) of \( \mathcal{O}(\mathbb{R}^2_q) \) on the domain \( \mathfrak{A}(\mathbb{R}^2) \) in \( L^2(\mathbb{R}^2) \).

Remark 3. The preceding derivation shows the reason for the non-uniqueness of covariant functionals on the left \( U_q(gl_2(\mathbb{R})) \)-module ∗-algebra \( \mathfrak{A}(\mathbb{R}^2) \) from the technical side: For even numbers \( k_1 \) and \( k_2 \) the unbounded positive self-adjoint operator \( C_k \) commutes with all representation operator \( \Phi(z), z \in U_q^{tw}(gl_2(\mathbb{R})) \), so that the unbounded commutant of \( \Phi(U_q(gl_2)) \) is non-trivial. However, it can be shown that the ∗-representation \( \Phi \) of \( U_q^{tw}(gl_2(\mathbb{R})) \) on \( L^2(\mathbb{R}^2) \) is irreducible.

By the preceding we have expressed the actions \( \psi_k \) of the algebras \( U_q(gl_2(\mathbb{R})) \) and \( \mathcal{O}(\mathbb{R}^2_q) \) on the unitary space \( \mathfrak{A}_k = (\mathfrak{A}(\mathbb{R}^2), \langle \cdot , \cdot \rangle_k) \) by means of the ∗-representations \( \Phi \) on the domain \( \mathfrak{A}(\mathbb{R}^2) \) in \( L^2(\mathbb{R}^2) \). Since the representation \( \Phi = T\psi_0 T^{-1} \) is obtained from \( \psi_0 \) by the unitary operator \( T \), it is natural to transform also the structure of the ∗-algebra \( \mathcal{A}(\mathbb{R}^2_{q+}) \) and the covariant functional \( h := h_0 \) under the bijective linear mapping \( T \) of \( \mathfrak{A}(\mathbb{R}^2) \). That is, for \( f, g \in \mathcal{A}(\mathbb{R}^2_{q+}) \) and \( a \in \mathfrak{A}(\mathbb{R}^2) \) we define

\[
f^a g = T(T^{-1} f \cdot T^{-1} g), \quad f^* = T(T^{-1} f^*), \quad \tilde{h}(a) = h(T^{-1} a).
\]  

(152)

Since \( \mathcal{A}(\mathbb{R}^2_{q+}) \) is a ∗-algebra with product \( \cdot \) and involution \( f \to f^* \), the vector space \( \mathcal{A}(\mathbb{R}^2_{q+}) \) is a ∗-algebra, denoted \( \hat{\mathcal{A}}(\mathbb{R}^2_{q+}) \), with product \( \hat{\cdot} \) and involution \( f \to f^* \). Further, since \( \mathcal{A}(\mathbb{R}^2_{q+}) \) is a left \( U_q(gl_2(\mathbb{R})) \)-module ∗-algebra (by Theorem 21) and \( h \) is covariant with respect to the left action \( \psi_0 \) (by Proposition 22), it is clear that \( \hat{\mathcal{A}}(\mathbb{R}^2_{q+}) \) is a left \( U_q(gl_2(\mathbb{R})) \)-module ∗-algebra and the linear functional \( \hat{h} \) is covariant with respect to the left action \( \Phi = T\psi_0 T^{-1} \).

Let us make the transformed structures more explicit. Suppose that \( f = z + a \) and \( g = w + b \), where \( z, w \in \mathcal{O}(\mathbb{R}^2_q) \) and \( a, b \in \mathfrak{A}(\mathbb{R}^2) \). From (112), (113) and (140) it follows that \( xt\phi = Txc \) and \( \phi Tc = Tyic \) and so \( \phi Tc = Tyic \) for all \( v \in \mathcal{O}(\mathbb{R}^2_q) \) and \( c \in \mathfrak{A}(\mathbb{R}^2) \). Hence we get

\[(z + a)\hat{a}(w + b) = zw + zb + aw + a\hat{z}b \quad \text{and} \quad (z + a)^* = z^* + a^*.
\]

That is, product and involution of the ∗-algebra \( \mathcal{O}(\mathbb{R}^2_q) \) remain unchanged and also the products of elements of \( \mathcal{O}(\mathbb{R}^2_q) \) and \( \mathfrak{A}(\mathbb{R}^2) \). It remains to describe
the transformed product and involution of the \emph{*-subalgebra} \( \mathfrak{A}(\mathbb{R}^2) \). From the definition \textbf{(140)} of the operator \( T \) and the formulas \textbf{(153)}–\textbf{(155)} we obtain
\[
\begin{align*}
  a^*_2 b &= a\#(e^{-\frac{\alpha}{2}\beta}p_1 e^{-\pi\alpha q_1} \otimes e^{\frac{\alpha}{2}\beta}p_2 e^{-\pi\beta q_2} b) = (e^{\frac{\alpha}{2}\beta}p_1 e^{-\pi\alpha q_1} \otimes e^{\frac{\alpha}{2}\beta}p_2 e^{-\pi\beta q_2} a)\# b \\
  = (e^{\frac{\alpha}{2}\beta}p_1 e^{-\frac{\alpha}{2}\beta}q_1 \otimes e^{\frac{\alpha}{2}\beta}p_2 e^{-\frac{\alpha}{2}\beta}q_2) a\#(e^{\frac{\alpha}{2}\beta}p_1 e^{-\frac{\alpha}{2}\beta}q_1 \otimes e^{\frac{\alpha}{2}\beta}p_2 e^{-\frac{\alpha}{2}\beta}q_2) b, \\
  a^*(x_1, x_2) &= (e^{\pi(\beta - \alpha)q_2} a)(x_1, x_2) = \bar{a}(x_1 - \frac{\beta}{4} i, x_2 + \frac{\alpha}{4} i).
\end{align*}
\]
\textbf{(153)}

The vector space \( \mathfrak{A}(\mathbb{R}^2) \) equipped with this transformed \emph{*-algebra} structure is denoted by \( \tilde{\mathfrak{A}}(\mathbb{R}^2) \). Inserting the definition \textbf{(133)} of the functional \( h = h_0 \) and substituting \( (x_1, x_2) \to (x_1 + \frac{\beta}{4} i, x_2 - \frac{\alpha}{4} i) \) we get
\[
\tilde{h}(a) = \int \int e^{\pi(\alpha x_1 + \beta x_2)} a(x_1, x_2) dx_1 dx_2, \quad a \in \mathfrak{A}(\mathbb{R}^2).
\]
\textbf{(155)}

Summarizing, we have transformed all structures obtained so far of the left \( U_q(gl_2(\mathbb{R})) \)-module \emph{*-algebra} \( \mathcal{A}(\mathbb{R}^2_+^\tau) \) of “functions on the quantum quarter plane” under the the mapping \( T \). The main advantage of this new picture is that the scalar product
\[
(a, b)_{\tilde{h}} := \tilde{h}(b^* a), \quad a, b \in \mathfrak{A}(\mathbb{R}^2),
\]
derived from the transformed covariant functional \( \tilde{h} \) is just the \( L^2 \)-scalar product \((\cdot, \cdot)\) on \( \mathbb{R}^2 \). (This follows at once from the construction. It can be also verified directly by using formulas \textbf{(153)} and \textbf{(155)}.) The latter fact will be crucial when we are looking for self-adjoint extensions of the representation operators \( \Phi(E^0) \) and \( \Phi(F^0) \) in a larger Hilbert space.

5.3 In this last subsection of Section 5 we prove two uniqueness result for covariant linear functionals. They say that under some technical assumptions equation \textbf{(134)} essentially characterizes the covariant functional \( h_k \).

\textbf{Proposition 24.} Let \( k = (k_1, k_2) \in \mathbb{Z}^2 \) and let \( h \) be a faithful positive linear functional on the \emph{*-algebra} \( \mathfrak{A}_{\text{prex}}(\mathbb{R}^2) \) which satisfies relation \textbf{(134)} for all \( s, t \in \mathbb{R} \) and \( a \in \mathfrak{A}_{\text{prex}}(\mathbb{R}^2) \). Suppose that \( h \) is continuous relative to the topology \( \tau \). Then there exists a positive number \( \nu \) such that \( h(a) = \nu h_k(a) \) for all \( a \in \mathfrak{A}_{\text{prex}}(\mathbb{R}^2) \).

\textbf{Proof.} The proof mimics some of the preceding considerations in reversed order. Since \( h \) is a faithful positive linear functional and \( T_k \) is a bijective linear mapping of \( \mathfrak{A}_{\text{prex}}(\mathbb{R}^2) \), the equation
\[
(a, b)_h = h(T_k^{-1} b)^\#(T_k^{-1} a), \quad a, b \in \mathfrak{A}_{\text{prex}}(\mathbb{R}^2),
\]
\textbf{(156)}
defines a scalar product on \( \mathfrak{A}_{\text{prex}}(\mathbb{R}^2) \). (In the case when \( h = h_k \) this scalar product is just the \( L^2 \)-scalar product.) Consider the following four one-parameter groups of operators acting on the unitary space \( (\mathfrak{A}_{\text{prex}}(\mathbb{R}^2), (\cdot, \cdot)_h) \):
\[
(U_1(t)f)(x_1, x_2) = e^{2 \pi i t x_1} f(x_1, x_2), \quad (U_2(t)f)(x_1, x_2) = e^{2 \pi i t x_2} f(x_1, x_2),
\]
\[
(V_1(t)f)(x_1, x_2) = f(x_1 + \beta t, x_2), \quad (V_2(t)f)(x_1, x_2) = f(x_1, x_2 + \alpha t).
\]
It is straightforward to verify the commutation relations

\[ T_k^{-1}U_1(t) = e^{\frac{i}{2}t\gamma + \pi tk_2}U_1(t)T_k^{-1}, \quad T_k^{-1}U_2(t) = e^{-\frac{i}{2}t\gamma - \pi tk_1}U_2(t)T_k^{-1}, \]
\[ T_k^{-1}V_1(t) = e^{\pi t\gamma + 2\pi tk_1}V_1(t)T_k^{-1}, \quad T_k^{-1}V_2(t) = e^{\pi t\gamma + 2\pi tk_2}V_2(t)T_k^{-1} \]

for arbitrary \( t \in \mathbb{R} \). From these commutation rules and the assumption (134) we derive that the operators \( U_1(t), U_2(t), V_1(t), V_2(t) \) preserve the scalar product \((\cdot, \cdot)_h\) for real \( t \). Let us carry out this computation for \( U_1(t) \). Using the definition of the scalar product \((\cdot, \cdot)_h\), the above relation for \( T_k^{-1}U_1(t) \), formula (130) and finally the assumption (134), we obtain

\[ (U_1(t)a, U_1(t)b)_h = h((T_k^{-1}U_1(t)b)^* (T_k^{-1}U_1(t)a)) \]
\[ = e^{\pi t\gamma + 2\pi tk_2}h((e^{2\pi it\alpha}T_k^{-1}b)^* (e^{2\pi it\alpha}T_k^{-1}a)) \]
\[ = e^{\pi t\gamma + 2\pi tk_1}h((e^{-2\pi it\alpha}T_k^{-1}b)^* (e^{-2\pi it\alpha}T_k^{-1}a)) \]
\[ = e^{\pi t\gamma + 2\pi tk_1}e^{-\pi t\gamma - 2\pi tk_2}h((T_k^{-1}b)^* (T_k^{-1}a)) \]
\[ = (a, b)_h . \] (157)

The corresponding proofs for the other operators are similar. The continuous extensions of the unitary operators \( U_j(t) \) and \( V_j(t) \) to the Hilbert space completion \( \mathcal{G} \) of the unitary space \( (\mathfrak{A}_{\text{pex}}, (\cdot, \cdot)_h) \) are denoted by the same symbols.

Before we continue let us note an estimate for the Hilbert space norm \( \| \cdot \|_h \).

Since the functional \( h \) is \( \tau \)-continuous by assumption, there is a \( \tau \)-continuous seminorm \( \tau \) on \( \mathfrak{A}_{\text{pex}}(\mathbb{R}^2) \) such that

\[ \| a \|_h^2 = h((T_k^{-1}b)^* (T_k^{-1}a)) \leq \tau((T_k^{-1}b)^* (T_k^{-1}a)), \quad a \in \mathfrak{A}_{\text{pex}}(\mathbb{R}^2) . \]

By the definition of the topology \( \tau \), we can take \( \tau \) to be a sum of norms

\[ \| \cdot \|_{c,d} := \| e^{2\pi i(c_1p_1+c_2p_2)}e^{2\pi (d_1Q_1+d_2Q_2)} \cdot \| , \] (158)

where \( c = (c_1, c_2), d = (d_1, d_2) \in \mathbb{R}^2 \). Using the formulas (62) – (69) we conclude that the Hilbert space norm \( \| \cdot \|_h \) is \( \tau \)-continuous on \( \mathfrak{A}_{\text{pex}}(\mathbb{R}^2) \).

Now let \( s \in \mathbb{R} \) and let \( a \in \mathfrak{A}_{\text{pex}}(\mathbb{R}^2) \). For \( c, d \in \mathbb{R}^2 \), we then have

\[ \| (U_1(t) - I - s2\pi i\alpha Q_1) a \|_{c,d} \]
\[ = \| e^{2\pi i(c_1p_1+c_2p_2)}e^{2\pi (d_1Q_1+d_2Q_2)}(U_1(t) - I - s2\pi i\alpha Q_1) a \| \]
\[ = \| e^{2\pi i\alpha c_1}U_1(t) - I - s2\pi i\alpha Q_1 - s2\pi at)e^{2\pi (c_1p_1+c_2p_2)}e^{2\pi (d_1Q_1+d_2Q_2)}a \| \]

Since the norm \( \| \cdot \|_h \) can be estimated by sums of norms \( \| \cdot \|_{c,d}, c, d \in \mathbb{R}^2 \), it follows from the preceding equality, applied with \( s = 0 \), that \( U_1(t)a \to a \) in the Hilbert space \( \mathcal{G} \) as \( t \to 0 \). Therefore, the one-parameter unitary group \( t \to U_1(t) \) on the Hilbert space \( \mathcal{G} \) is strongly continuous. Next we set \( s = t \). Then we conclude from the preceding that \( t^{-1}(U_1(t) - I)a \to 2\pi i\alpha Q_1 \) in the
Hilbert space \( G \) as \( t \to 0 \). (These facts are obvious in case of the Hilbert space \( L^2(\mathbb{R}^2) \), but we do not yet know that \((\cdot, \cdot)_h\) is a multiple of the \( L^2 \)-scalar product \((\cdot, \cdot)\).) The corresponding assertions for the other unitary groups follow by a similar reasoning.

It is obvious that the operators \( U_j(t) \) and \( V_j(t) \) satisfy the commutation relations

\[
V_j(t)U_j(s) = e^{2\pi i j t s} U_j(s) V_j(t), \quad V_j(t)U_2(s) = U_2(s) V_j(t), \quad V_j(t)U_1(s) = U_1(s) V_j(t), \quad U_1(t)U_2(s) = U_2(s) U_1(t), \quad V_1(t)V_2(s) = V_2(s) V_1(t)
\]

for \( j = 1, 2 \) and \( s, t \in \mathbb{R} \). That is, the unitary one-parameter groups \( U_j(t) \) and \( V_j(t) \), \( j = 1, 2 \) and \( t \in \mathbb{R} \), fulfill the Weyl relation. Therefore, by the Stone-von Neumann uniqueness theorem [Pu] that there exist a Hilbert space \( \mathcal{K} \) and a unitary transformation \( \mathcal{J} \) of \( \mathcal{G} \) on the Hilbert space \( \mathcal{H} := L^2(\mathbb{R}^2) \otimes \mathcal{K} \) such that

\[
\mathcal{J} V_j(t) U_l(s) \mathcal{J}^{-1} = \tilde{V}_j(t) \tilde{U}_l(s) \quad \text{for} \quad j, l = 1, 2, s, t \in \mathbb{R},
\]

(159)

where \( \tilde{V}_j(t) \) and \( \tilde{U}_l(t) \) denote the unitary groups on the Hilbert space \( \mathcal{H} \) defined by the same formulas as \( V_j(t) \) and \( U_l(t) \), respectively.

For \( \delta = (\delta_1, \delta_2) \in \mathbb{R}^2_{++} \) and \( c = (c_1, c_2) \in \mathbb{C}^2 \), let \( e_{c, \delta} \) denote the function \( e^{2\pi i (c_1 x_1 + c_2 x_2 - \delta_1 x_1^2 - \delta_2 x_2^2)} \) on \( \mathbb{R}^2 \). Let

\[
A_\delta := (\mathcal{P}_1 + 2i \delta_1 \mathcal{Q}_1)(\mathcal{P}_1 - 2i \delta_1 \mathcal{Q}_1) + (\mathcal{P}_2 + 2i \delta_2 \mathcal{Q}_2)(\mathcal{P}_2 - 2i \delta_2 \mathcal{Q}_2)
\]

be the operator on unitary space \( (\mathcal{A}_{\text{pez}}(\mathbb{R}^2), (\cdot, \cdot)_h) \) and let \( \tilde{A}_\delta \) denote the operator on \( \mathcal{H} \) given by the same formula. Since \((\mathcal{P}_j - 2i \delta_j \mathcal{Q}_j)e_{0, \delta} = 0 \) for \( j = 1, 2 \), it is clear that \( A_\delta e_{0, \delta} = 0 \). From (158) it follows that the unitary transformation \( \mathcal{J} \) intertwines the generators of the unitary groups \( V_j(t), U_l(t) \) and \( \tilde{V}_j(t), \tilde{U}_l(t) \), respectively. Hence we also have \( \tilde{A}_\delta \mathcal{J} (e_{0, \delta}) = 0 \). It is not difficult to show that ker \( A_\delta = e_{0, \delta} \otimes \mathcal{K} \). Hence we conclude that for arbitrary \( \delta \in \mathbb{R}^2_{++} \) there exist a vector \( x_\delta \in \mathcal{K} \) such that \( \mathcal{J} (e_{0, \delta}) = e_{0, \delta} \otimes x_\delta \).

We next show that the vector \( x_\delta \) does not depend on \( \delta \in \mathbb{R}^2_{++} \). For \( \varepsilon \in \mathbb{R} \), let \( \varepsilon_1 := (\varepsilon, 0) \) and \( \varepsilon_2 := (0, \varepsilon) \). It is not difficult to verify that

\[
\varepsilon^{-1}(e_{0, \delta + \varepsilon_j} - e_{0, \delta}) - 2\pi \mathcal{Q}_j^2 e_{0, \delta} \to 0
\]

(160)

as \( \varepsilon \to 0 \) in the topology \( \tau \) on \( \mathcal{A}_{\text{pez}}(\mathbb{R}^2) \). Since the norm \( \| \cdot \|_h \) is \( \tau \)-continuous as noted above, (160) holds also in the Hilbert space \( \mathcal{G} \). Therefore, the image under \( \mathcal{J} \) of the expression in (160) tends to zero in the Hilbert space \( \mathcal{H} \) as \( \varepsilon \to 0 \). This means that

\[
[e^{-1}(e_{0, \delta + \varepsilon_j} - e_{0, \delta}) - 2\pi \mathcal{Q}_j^2 e_{0, \delta}] \otimes x_\delta + e_{0, \delta + \varepsilon_j} \otimes \varepsilon^{-1}(x_{\delta + \varepsilon_j} - x_\delta) \to 0
\]

as \( \varepsilon \to 0 \) in \( \mathcal{H} \). Using once more the fact that (160) holds in \( L^2(\mathbb{R}^2) \), we conclude that \( e^{-1}(x_{\delta + \varepsilon_j} - x_\delta) \to 0 \) in \( \mathcal{K} \). That is, the partial derivatives of the \( \mathcal{K} \)-valued function \( \delta \to x_\delta \) vanish on \( \mathbb{R}^2_{++} \). Hence this function is constant with respect to \( \delta \in \mathbb{R}^2_{++} \), say \( x_\delta = x \).
Recall that $\mathfrak{A}_{\text{pex}}(\mathbb{R}^2)$ is defined as the linear span of function $x_1^nx_2^mc_{c,d}$, where $n, m \in \mathbb{N}_0, c \in \mathbb{C}^2$ and $\delta \in \mathbb{R}^2_{++}$. From (160) and the fact that $J(e_{0,\delta}) = e_{0,\delta} \otimes x$ it follows that $J(a) = a \otimes x$ for all $a \in \mathfrak{A}_{\text{pex}}(\mathbb{R}^2)$. Since $J$ is unitary, we obtain

$$(a, b)_h = (J(a), J(b))_{\bar{R}} = (a, b)_{L(\mathbb{R}^2)}(x, x)_{\bar{K}}.$$ 

Thus the scalar product $(\cdot, \cdot)_h$ is a positive multiple of the $L^2$-scalar product $(\cdot, \cdot)$ on $\mathfrak{A}_{\text{pex}}(\mathbb{R}^2)$, that is, $(\cdot, \cdot)_h = \nu(\cdot, \cdot)$, where $\nu := (x, x)$.

By the latter and the definition ([58]) of the scalar product $(\cdot, \cdot)_h$, we have

$$h(b^* \# a) = (T_k a, T_k b)_h = \nu(T_k a, T_k b) = \nu(T_k^2 a, b) = \nu(e^{2\pi \alpha_k Q_1 - \pi \beta_k P_1} \otimes e^{2\pi \beta_k Q_2 + \pi \alpha_k P_2} a, b)$$ 

for $a, b \in \mathfrak{A}(\mathbb{R}^2)$. Now we set $b = f_\varepsilon$, where $f_\varepsilon$ is the approximate identity from Proposition 14. Taking the limit $\varepsilon \to +0$ in the topology $\tau$ and using (54), we obtain

$$h(a) = \nu \int \int e^{2\pi \alpha_k x_1 + 2\pi \beta_k x_2} a(x_1 + \frac{\beta_k}{2}i, x_2 - \frac{\alpha_k}{2}i) dx_1 dx_2 .$$

Arguing as in the proof of Proposition 22, we substitute $(x_1, x_2)(x_1 - i\beta_k/2, x_2 + i\alpha_k/2)$ in the latter formula and obtain $h(a) = \nu h_k(a), a \in \mathfrak{A}_{\text{pex}}(\mathbb{R}^2)$. 

**Theorem 25.** Let $k = (k_1, k_2) \in \mathbb{Z}^2$ and let $c = (c_1, c_2), d = (d_1, d_2) \in \mathbb{R}^2$ be such that $8|c_1d_1| < 1$ and $8|c_2d_2| < 1$. Suppose that $h$ is a faithful positive linear functional on the $\ast$-algebra $\mathfrak{A}(\mathbb{R}^2)$ such that

$$h(a(x_1 + \beta s, x_2 + \alpha s)) = e^{-2\pi \gamma(s+t) - 4\pi(k_1s + k_2t)} h(a),$$

$$|h(a)| \leq C\|S(e^{\epsilon Q})S(e^{\epsilon P})a\|$$

(161)

for some positive constant $C$ and for all $s, t \in \mathbb{R}$ and $a \in \mathfrak{A}(\mathbb{R}^2)$.

Then there is a positive constant $\nu$ such that $h = \nu h_k$.

**Proof.** By Proposition 24, we have $h(a) = \nu h_k(a)$ for all $a \in \mathfrak{A}_{\text{pex}}(\mathbb{R}^2)$. By Lemma 10, $\mathfrak{A}_{\text{pex}}(\mathbb{R}^2)$ is dense in $\mathfrak{A}(\mathbb{R}^2)$ relative to the norm $\|S(e^{\epsilon Q})S(e^{\epsilon P})\|$. Since $h$ and $h_k$ are both continuous with respect to this norm, it follows that $h = \nu h_k$ on $\mathfrak{A}(\mathbb{R}^2)$. 

Remark 4. It is likely that the assertion of Theorem 25 remains valid if we assume only the $\tau$-continuity of the functional $h$ instead of inequality (161) with $8|c_jd_j| < 1, j = 1, 2$. For this it would be sufficient to know that $\mathfrak{A}_{\text{pex}}(\mathbb{R}^2)$ is $\tau$-dense in $\mathfrak{A}(\mathbb{R}^2)$. 

45
6. The real quantum plane

6.1 In order to motivate the definitions given below, we first recall the following well-known representation of the Lie algebra $gl_2(\mathbb{R})$ on the $C^\infty$-functions of $\mathbb{R}^2$:

The action of the generators $e, f, h_1, h_2$ of $gl_2(\mathbb{R})$ satisfying the relations

$$[h_1, e] = e, [h_1, f] = -f, [h_2, e] = -e, [h_2, f] = f, [e, f] = h_1 - h_2$$

is given by the formulas

$$e = -y \frac{\partial}{\partial x}, f = -x \frac{\partial}{\partial y}, h_1 = -x \frac{\partial}{\partial x}, h_2 = -y \frac{\partial}{\partial y}. \quad (162)$$

The groups of relations (78)–(80) and (109)–(111) can be interpreted as quantum versions of the formulas (162). Likewise, we see that the linear mappings $\mathcal{D}_x^q$ and $\mathcal{D}_y^q$, $q \geq 0$, give $q$-versions of the partial derivatives $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, respectively. We shall not make this more explicit because we shall not need these details.

Let us explain the underlying idea of our construction of the real quantum plane in the classical situation. We consider the plane $\mathbb{R}^2$ as the direct sum of the four quarter planes $\mathbb{R}^{++},\mathbb{R}^{-+},\mathbb{R}^{+-},\mathbb{R}^{--}$ that are glued together along the two coordinate axes. On the level of functions this means that a continuous function $f$ on $\mathbb{R}^2\setminus\{(0,0)\}$ is given by a 4-tuple $(f_{++}, f_{---})$ of continuous functions on the quarter planes satisfying the boundary conditions

$$f_{++}(+0, y) = f_{--}(-0, y), \quad f_{---}(+0, y) = f_{---}(-0, y),$$

$$f_{++}(x, +0) = f_{--}(x, -0), \quad f_{--}(x, +0) = f_{--}(x, -0).$$

We now turn to the quantum case and consider the direct sum

$$\mathcal{B}(\mathbb{R}^2)_4 := \tilde{\mathfrak{A}}(\mathbb{R}^2) \oplus \tilde{\mathfrak{A}}(\mathbb{R}^2) \oplus \tilde{\mathfrak{A}}(\mathbb{R}^2) \oplus \tilde{\mathfrak{A}}(\mathbb{R}^2) \quad (163)$$

of the four $*$-algebras $\tilde{\mathfrak{A}}(\mathbb{R}^2)$ corresponding to the four quarter planes. Recall that product and involution of the $*$-algebra $\tilde{\mathfrak{A}}(\mathbb{R}^2)$ are given by formulas (153) and (154). We could have also taken here the $*$-algebra $\mathfrak{A}(\mathbb{R}^2)$. The reason why we prefer to work with $\tilde{\mathfrak{A}}(\mathbb{R}^2)$ is that on direct sums of Hilbert spaces $L^2(\mathbb{R}^2)$ it is easier to describe self-adjoint extensions of the operators $\Phi(E')$ and $\Phi(F')$. Since the elements of $\tilde{\mathfrak{A}}(\mathbb{R}^2)$ correspond to “functions on the quantum quarter plane which vanish at the boundaries”, no boundary condition occurs and we can just take the direct sum of the four $*$-algebras.

Next we want to make the direct sum $\mathcal{B}(\mathbb{R}^2) = \mathcal{O}(\mathbb{R}^2) + \mathcal{B}(\mathbb{R}^2)_4$ of the vector spaces $\mathcal{O}(\mathbb{R}^2)$ and $\mathcal{B}(\mathbb{R}^2)_4$ into a left $\mathcal{U}_q(gl_2(\mathbb{R}))$-module $*$-algebra. This means that we have to define the products of the generators $x$ and $y$ of $\mathcal{O}(\mathbb{R}^2)$ by a 4-tuple $a = (a_1, a_2, a_3, a_4), a_j \in \tilde{\mathfrak{A}}(\mathbb{R}^2)$, and left actions of the generators $\tilde{E}, \tilde{F}, \tilde{K}_1, \tilde{K}_2$ of $\mathcal{U}_q(gl_2(\mathbb{R}))$ on $a$. Let us look for a moment at the classical case and consider a quarter plane $\mathbb{R}^{\epsilon\epsilon'}$, where $\epsilon, \epsilon' \in \{+, -\}$. From the formulas (162)
we see that if we pass from \( \mathbb{R}^+ \) to \( \mathbb{R}'^+ \) then \( x \) has to be replaced by \( \epsilon x \), \( y \) by \( \epsilon y \), \( e \) by \( \epsilon e' \) and \( f \) by \( \epsilon e' f \), while \( h_1 \) and \( h_2 \) remain unchanged. This suggests to take the following definitions in the quantum case:

\[
\begin{align*}
\Phi(E') a &= (\Phi'(E') a_1, -\Phi'(E') a_2, -\Phi'(E') a_3, \Phi'(E') a_4), \\
\Phi(F') a &= (\Phi'(F') a_1, -\Phi'(F') a_2, -\Phi'(F') a_3, \Phi'(F') a_4), \\
\Phi(K_j) a &= (\Phi(K_j) a_1, \Phi(K_j) a_2, \Phi(K_j) a_3, \Phi(K_j) a_4), \quad j = 1, 2,
\end{align*}
\]

where \( a = (a_1, a_2, a_3, a_4) \), \( a_j \in \mathfrak{A}(\mathbb{R}^2) \). Recall that the action of the operators \( x \equiv \Phi(x), y \equiv \Phi(y) \) and \( f, f' \), \( E', F', K_1, K_2 \), are described by formulas (149) - (151). Since \( \mathcal{A}(\mathbb{R}^2) = \mathcal{O}(\mathbb{R}^2) + \mathfrak{A}(\mathbb{R}^2) \) is a left \( \mathcal{U}_q(gl_2(\mathbb{R})) \)-module *-algebra with left action \( \Phi \), it is easily verified that the preceding formulas define indeed a unique *-algebra structure on \( \mathcal{B}(\mathbb{R}^2)_4 \) and that \( \mathcal{B}(\mathbb{R}^2)_4 \) becomes a left \( \mathcal{U}_q(gl_2(\mathbb{R})) \)-module *-algebra in this manner.

From the preceding it is clear that the functional \( \tilde{h} \) on \( \mathcal{B}(\mathbb{R}^2)_4 \) defined by

\[
\tilde{h}(a) = \tilde{h}(a_1) + \tilde{h}(a_2) + \tilde{h}(a_3) + \tilde{h}(a_4), \quad a = (a_1, a_2, a_3, a_4) \in \mathcal{B}(\mathbb{R}^2)_4,
\]

is \( \mathcal{U}_q(gl_2(\mathbb{R})) \)-covariant with respect to \( \chi \), where \( \tilde{h}(a_j) \) is given by (155).

### 6.2 The vector space \( \mathcal{B}(\mathbb{R}^2)_4 \)

The vector space \( \mathcal{B}(\mathbb{R}^2)_4 \) is also a dense domain of the Hilbert space

\[
\mathcal{H} = \mathcal{L}^2(\mathbb{R}^2) \oplus \mathcal{L}^2(\mathbb{R}^2) \oplus \mathcal{L}^2(\mathbb{R}^2) \oplus \mathcal{L}^2(\mathbb{R}^2).
\]

From the formulas (151) and (152) it is clear that the operators \( x, y \) and \( \Phi(q^{-1/4} K_j), j = 1, 2 \), are essentially self-adjoint on the domain \( \mathcal{B}(\mathbb{R}^2)_4 \). From (149) and Lemma 5(ii) it follows that the operators \( \Phi(E') \) and \( \Phi(F') \) are symmetric but not essentially self-adjoint on the domain \( \mathcal{B}(\mathbb{R}^2)_4 \) in \( \mathcal{H} \). Indeed, by Lemma 5(ii) the adjoint of \( L_\alpha \) is the operator \( R_\alpha \) and \( R_\alpha \) is easily seen to be a proper extension of \( L_\alpha \). That the operators \( \Phi(E') \) and \( \Phi(F') \) are not essentially self-adjoint on the domain \( \mathcal{B}(\mathbb{R}^2)_4 \) is not surprising, because \( \mathcal{B}(\mathbb{R}^2)_4 \) contains only “functions which vanish at the boundaries of the four quantum quarter planes”. In the classical case the corresponding symmetric operators \( i e = -y i \frac{\partial}{\partial x} \) and \( i f = -x i \frac{\partial}{\partial y} \) are also not essentially self-adjoint when the functions in the domain have boundary values zero at the \( x \)- and \( y \)-axis. Our next step is to “glue together the four quarter quantum planes” and to obtain self-adjoint extensions of the symmetric operators \( \Phi(E') \) and \( \Phi(F') \) in this manner.

In order to explain the glueing procedure let us return to the classical situation and consider the direct sum of four quarter planes. Since the corresponding functions have the boundary values zero at the \( x \)- and \( y \)-axis, the operators \( i e = -y i \frac{\partial}{\partial x} \) and \( i f = -x i \frac{\partial}{\partial y} \) are not essentially self-adjoint. The particular self-adjoint extensions of the symmetric operators \( i e \) and \( i f \) we are interested in are determined by the boundary conditions

\[
f(+0, y) = f(-0, y) \quad \text{and} \quad f(x, +0) = f(x, -0).
\]

(166)
For simplicity, let us first look how the two upper quarter planes $\mathbb{R}^{-+}$ and $\mathbb{R}^{++}$ are glued together along the positive $y$-axis. We identify a function $f_{-+}$ on $\mathbb{R}^{-+}$ with the function $f_{++}$ on $\mathbb{R}^{++}$ given by $f_{++}(x, y) := f_{-+}(-x, y)$, $y > 0, x \in \mathbb{R}$. Let $D_x$ denote the symmetric operator $-i\frac{\partial}{\partial x}$ on $L^2(\mathbb{R}^{++})$ with boundary condition $f(+0, y) = 0$, $y > 0$. Then the operator $T_0 = -i\frac{\partial}{\partial x}$ on $\mathbb{R}^{-+} \cup \mathbb{R}^{++}$ with boundary condition zero at the positive $y$-axis acts on the Hilbert space $L^2(\mathbb{R}^{++}) \oplus L^2(\mathbb{R}^{++})$ and has the form

$$T_0 = \begin{pmatrix} D_x & 0 \\ 0 & -D_x \end{pmatrix}.$$ 

Let $T$ denote the self-adjoint extension of $T_0$ with boundary condition $f(+0, y) = f(-0, y)$, $y > 0$, and let $J_0$ be the symmetry (that is, self-adjoint unitary)

$$J_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$ 

One easily verifies that

$$J_0 T J_0 = \begin{pmatrix} 0 & D_x \\ D_x^* & 0 \end{pmatrix},$$

where $D_x^*$ is the adjoint of the closed symmetric operator $D_x$ on the Hilbert space $L^2(\mathbb{R}^{++})$.

In a similar manner we proceed with the four quarter planes $\mathbb{R}^{++}$, $\mathbb{R}^{-+}$, $\mathbb{R}^{+-}$, $\mathbb{R}^{-}$ As above, we identify functions on $\mathbb{R}^{-+}$, $\mathbb{R}^{+-}$, $\mathbb{R}^{-}$ with the corresponding functions on $\mathbb{R}^{++}$, and let $D_y = -i\frac{\partial}{\partial y}$ on $L^2(\mathbb{R}^{++})$ with boundary condition $f(x, +0) = 0$, $x > 0$. Further, let $T_{10} = -i\frac{\partial}{\partial x}$ and $T_{20} = -i\frac{\partial}{\partial y}$ be the symmetric operators with boundary values zero at the positive $y$- resp. $x$-axis acting on the Hilbert space

$$\mathcal{G} = L^2(\mathbb{R}^{++}) \oplus L^2(\mathbb{R}^{++}) \oplus L^2(\mathbb{R}^{++}) \oplus L^2(\mathbb{R}^{++}).$$

Consider the symmetries

$$J_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad J_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

on the Hilbert space $\mathcal{G}$. Since $J_1$ and $J_2$ commute, their product

$$J := J_1 J_2 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$ (167)

is again a symmetry. We shall transform all structures by means of the symmetry $J$.  

48
In order to save space, let us introduce abbreviations for some operator matrices. If \( z_1, z_2 \) and \( z \) is an operator on a Hilbert space \( \mathcal{K}_0 \), we set

\[
\sigma_1(z_1, z_2) = \begin{pmatrix} z_1 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 \\ 0 & 0 & z_1 & 0 \\ 0 & 0 & 0 & z_2 \end{pmatrix}, \quad \sigma_2(z_1, z_2) = \begin{pmatrix} z_1 & 0 & 0 & 0 \\ 0 & z_1 & 0 & 0 \\ 0 & 0 & z_2 & 0 \\ 0 & 0 & 0 & z_2 \end{pmatrix},
\]

\[
\theta_1(z) = \begin{pmatrix} 0 & z & 0 & 0 \\ z^* & 0 & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & z^* & 0 \end{pmatrix}, \quad \theta_2(z) = \begin{pmatrix} 0 & 0 & z & 0 \\ 0 & 0 & 0 & z \\ z^* & 0 & 0 & 0 \\ 0 & z^* & 0 & 0 \end{pmatrix},
\]

\[
\kappa_1(z) = \begin{pmatrix} 0 & 0 & 0 & z \\ 0 & 0 & z^* & 0 \\ 0 & z & 0 & 0 \\ z^* & 0 & 0 & 0 \end{pmatrix}, \quad \kappa_2(z) = \begin{pmatrix} 0 & 0 & 0 & z \\ 0 & 0 & z & 0 \\ 0 & z^* & 0 & 0 \\ z^* & 0 & 0 & 0 \end{pmatrix}
\]

and \( \sigma_j(z) := \sigma_j(z, -z), j = 1, 2 \). The matrices \( \sigma_j(z_1, z_2), \theta_j(z) \) and \( \kappa_j(z) \) act as operators on the Hilbert space \( \mathcal{K}_0 \oplus \mathcal{K}_0 \oplus \mathcal{K}_0 \oplus \mathcal{K}_0 \).

Let us now continue the above discussion. In terms of the preceding notation we have \( T_{10} = \sigma_1(D_x) \) and \( T_{20} = \sigma_2(D_y) \). It is not difficult to check that the self-adjoint extensions \( T_1 \) and \( T_2 \) of \( T_{10} \) and \( T_{20} \), respectively, with boundary condition (166) satisfy the relations

\[
JT_1 = J_2 \kappa_1(D_x) J_2 = \theta_1(D_x), \quad JT_2 = J_1 \kappa_2(D_y) J_1 = \theta_2(D_y).
\]

That is, the particular self-adjoint extensions \( T_1 \) and \( T_2 \) are characterized in a simple manner by means of the symmetry transformation \( J \).

Let \( x \) and \( y \) denote the multiplication operators by the coordinate functions \( x \) and \( y \), respectively, on \( L^2(\mathbb{R}^+) \). Then the multiplication operators \( M_x \) and \( M_y \) by the coordinate functions on \( L^2(\mathbb{R}^2) \) can be expressed as

\[
JM_x J = J \sigma_1(x) J = \theta_1(x), \quad JM_y J = J \sigma_2(y) J = \theta_2(y).
\]  

(168)

Further, the self-adjoint operators \( i\hat{e} = -iy\frac{\partial}{\partial x} \) and \( i\hat{f} = -ix\frac{\partial}{\partial y} \) with boundary condition (166) can be written as

\[
Ji\hat{e} J = JM_x T_2 J = \kappa_2(x D_y), \quad Ji\hat{f} J = JM_y T_1 J = \kappa_1(y D_x).
\]  

(169)

Finally, we also transform the structure of the \(*\)-algebra of functions on \( \mathbb{R}^2 \) under the symmetry \( J \). If we consider functions \( f, g \) on \( \mathbb{R}^2 \) as 4-tuples of functions on the quarter planes, then the product \( f \cdot g \) is transformed under \( J \) as

\[
f \circ g := J(J(f) \cdot J(g)).
\]  

(170)
More explicitly, for \( f = (f_1, f_2, f_3, f_4) \) and \( g = (g_1, g_2, g_3, g_4) \) we compute

\[
\begin{align*}
fg &= 1/4(f_1g_1 + f_2g_2 + f_3g_3 + f_4g_4, f_1g_1 + f_2g_2 + f_3g_3, \\
f_f &= f_1g_1 + f_2g_2 + f_3g_3 + f_4g_4, f_1g_1 + f_2g_2 + f_3g_3, \\
fg &= f_1g_1 + f_2g_2 + f_3g_3 + f_4g_4, f_1g_1 + f_2g_2 + f_3g_3, \\
f_1g_3 + f_2g_4 + f_3g_1 + f_4g_2, f_1g_4 + f_2g_3 + f_3g_2 + f_4g_1),
\end{align*}
\]

where \( f_j g_k \) and \( g_k f_j \) mean the usual pointwise products of functions on the quarter plane. Obviously, the involution of functions is invariant under \( J \), that is, we have \( J(f^*) = J(f)^* \).

Thus, we have described the operators and the algebra of functions on the plane by means of the symmetry \( J \) and the corresponding operators and algebras of the quarter planes. The advantage of this approach is that it gives a convenient algebraic form for the particular self-adjoint extensions of the first order differential operators. There is the disadvantage that the algebra product has to be changed too. The preceding formulas and considerations will serve as guiding motivation for the construction of the real quantum plane in the next subsection.

6.3 In this subsection we develop the basics of the construction of the real quantum plane. Our starting point is the description of the quantum quarter plane by means of the left \( U_q(gl_2(\mathbb{R})) \)-module \(*\)-algebra \( \tilde{A}(\mathbb{R}^{++}) \) developed in 4.2. In what follows we assume that

\[
0 < |\gamma| < 1/3. 
\]

Let us begin by defining the left action of \( U_q(gl_2) \). Remembering the formulas (169), (149) and (150), we consider the self-adjoint operators on the Hilbert space \( \mathcal{H} \) (see (165)) defined by the 4×4-operator matrices

\[
E := \kappa_1(L_\alpha \otimes e^{2\pi \beta Q_2}), \quad F := \kappa_2(e^{2\pi \alpha Q_1} \otimes L_\beta), \quad K_1 := I(e^{\pi \beta P_1}), \quad K_2 := I(e^{\pi \alpha P_2}).
\]

Here \( I(z) \) denotes the diagonal matrix with diagonal entries equal to \( z \). Note that the entries of these matrices are just the operators occurring in formulas (149)–(150). Let \( \mathfrak{A}(\mathbb{R}^2)_4 \) be the domain

\[
\mathfrak{A}(\mathbb{R}^2)_4 := \mathfrak{A}_{12}(\mathbb{R}^2) \oplus \mathfrak{A}_{2}(\mathbb{R}^2) \oplus \mathfrak{A}_1(\mathbb{R}^2) \oplus \mathfrak{A}(\mathbb{R}^2)
\]

in the Hilbert space \( \mathcal{H} \), where

\[
\begin{align*}
\mathfrak{A}_{1}(\mathbb{R}^2) &= f_\alpha(P_1)^{-1}\mathfrak{A}(\mathbb{R}^2), \quad \mathfrak{A}_{2}(\mathbb{R}^2) = f_\beta(P_2)^{-1}\mathfrak{A}(\mathbb{R}^2), \quad \\
\mathfrak{A}_{12}(\mathbb{R}^2) &= f_\alpha(P_1)^{-1}\mathfrak{A}(\mathbb{R}^2) + f_\beta(P_2)^{-1}\mathfrak{A}(\mathbb{R}^2).
\end{align*}
\]

It is clear that \( \mathfrak{A}(\mathbb{R}^2) \subseteq \mathfrak{A}_{1}(\mathbb{R}^2) \cup \mathfrak{A}_{2}(\mathbb{R}^2) \subseteq \mathfrak{A}_{12}(\mathbb{R}^2) \). Since \( f_\alpha(P_j)\mathfrak{A}(\mathbb{R}^2) \subseteq \mathfrak{A}(\mathbb{R}^2) \) for \( j = 1, 2 \), \( \mathfrak{A}(\mathbb{R}^2)_4 \) contains the domain \( \mathfrak{B}(\mathbb{R}^2)_q \) defined by (163). In particular, \( \mathfrak{A}(\mathbb{R}^2)_4 \) is dense in \( \mathcal{H} \). Using the fact that the operator \( L_\alpha \) in \( L^2(\mathbb{R}) \) has the adjoint operator \( R_\alpha = e^{-2\pi \alpha Q}f_\alpha(P) \) by Lemma 5(ii), it is clear that \( \mathfrak{A}(\mathbb{R}^2)_4 \) is contained in the domains of the operators \( E, F, K_j, j = 1, 2 \). Define

\[
\Phi(E') = E|\mathfrak{A}(\mathbb{R}^2)_4, \quad \Phi(F') = F|\mathfrak{A}(\mathbb{R}^2)_4, \quad \Phi(q^{-1/4}K_j) = K_j|\mathfrak{A}(\mathbb{R}^2)_4.
\]

(174)
Proposition 26. (i) $\mathfrak{A}(\mathbb{R}^2)_4$ is contained in the domains of the operator products $\Phi(E')\Phi(F')$, $\Phi(F')\Phi(E')$, $\Phi(E')\Phi(K_j)$, $\Phi(K_j)\Phi(E')$, $\Phi(F')\Phi(K_j)$, and $\Phi(K_j)\Phi(F')$ for $j = 1, 2$.

(ii) The operators $\Phi(E')$, $\Phi(F')$ and $\Phi(K_j)$ satisfy the defining relations in terms of the generators $E'$, $F'$ and $K_j$ of the algebra $\mathcal{U}_q(gl_2)$.

(iii) The operators $\Phi(E')$, $\Phi(F')$ and $\Phi(q^{-1/4}K_j)$ are essentially self-adjoint on the domain $\mathfrak{A}(\mathbb{R}^2)_4$.

Proof. (i): We show that $\mathfrak{A}(\mathbb{R}^2)_4$ is contained in the domain of the product $\Phi(E')\Phi(F')$. By the definition of operators $\Phi(E')$ and $\Phi(F')$ this means that

$A_{12}(\mathbb{R}^2)\subseteq \mathcal{D}((L_\alpha \otimes e^{2\pi \beta \omega_2})(e^{2\pi \alpha \omega_1} \otimes R_\beta)), \quad A_{23}(\mathbb{R}^2)\subseteq \mathcal{D}((R_\alpha \otimes e^{2\pi \beta \omega_1})(e^{2\pi \alpha \omega_1} \otimes R_\beta)), \quad A_{34}(\mathbb{R}^2)\subseteq \mathcal{D}((L_\alpha \otimes e^{2\pi \beta \omega_1})(e^{2\pi \alpha \omega_1} \otimes L_\beta)), \quad A_{41}(\mathbb{R}^2)\subseteq \mathcal{D}((R_\alpha \otimes e^{2\pi \beta \omega_1})(e^{2\pi \alpha \omega_1} \otimes L_\beta)).$

The last inclusion for the fourth component is obvious. Let us verify the assertion for the third component of $\Phi(E')\Phi(F')$. That is, we have to show that each vector $\eta = f_\alpha(P_1)^{-1} \zeta \in \mathfrak{A}_1(\mathbb{R}^2)$, where $\zeta \in \mathfrak{A}(\mathbb{R}^2)$, belongs to the domain of the product $(L_\alpha \otimes e^{2\pi \beta \omega_1})(e^{2\pi \alpha \omega_1} \otimes L_\beta)$. In order to prove this, it suffices to check that for all $\xi \in \mathfrak{A}(\mathbb{R})$ the vector $f_\alpha(P_1)^{-1} \zeta$ belongs to the domain of the operator $e^{2\pi \alpha \omega_1}$. Applying the Fourier transform we see that the latter is equivalent to the fact that for arbitrary $\xi \in \mathfrak{A}(\mathbb{R})$ the function $f_\alpha(x)^{-1} \xi(x)$ belongs to the domain of the operator $e^{-2\pi \alpha \omega_1}$. The assumption $0 < |\gamma| < 1/3$ implies that the infimum of the holomorphic function $f_\alpha(x)$ on the strip $0 < \Im x < \alpha$ when $\alpha > 0$ resp. $0 > \Im x > -\alpha$ when $\alpha < 0$ is positive. It follows from the characterization of the operator $e^{-2\pi \alpha \omega_1}$ given in Lemma 4 that the function $f_\alpha(x)^{-1} \xi$ belongs to the domain of this operator. The corresponding proofs for the first and the second components of $\Phi(E')\Phi(F')$ and for the other operator products are similar.

(ii): We carry out the proof for the relation $E'F' - F'E' = \lambda(K_2^2 - K_1^2)$ of the algebra $\mathcal{U}_q(gl_2)$. For the other defining relations these verifications are straightforward and will be omitted. Computing the commutator of the operator matrices $E$ and $F$, we obtain a diagonal operator having the operators

$A_1 := (L_\alpha \otimes e^{2\pi \beta \omega_2})(e^{2\pi \alpha \omega_1} \otimes R_\beta) - (e^{2\pi \alpha \omega_1} \otimes L_\beta)(R_\alpha \otimes e^{2\pi \beta \omega_2}),$

$A_2 := (R_\alpha \otimes e^{2\pi \beta \omega_2})(e^{2\pi \alpha \omega_1} \otimes R_\beta) - (e^{2\pi \alpha \omega_1} \otimes L_\beta)(L_\alpha \otimes e^{2\pi \beta \omega_2}),$

$A_3 := (L_\alpha \otimes e^{2\pi \beta \omega_2})(e^{2\pi \alpha \omega_1} \otimes L_\beta) - (e^{2\pi \alpha \omega_1} \otimes R_\beta)(R_\alpha \otimes e^{2\pi \beta \omega_2}),$

$A_4 := (R_\alpha \otimes e^{2\pi \beta \omega_2})(e^{2\pi \alpha \omega_1} \otimes L_\beta) - (e^{2\pi \alpha \omega_1} \otimes R_\beta)(L_\alpha \otimes e^{2\pi \beta \omega_2})$

as diagonal entries. By (i), for any vector $\eta = (\eta_1, \eta_2, \eta_3, \eta_4) \in \mathfrak{A}(\mathbb{R}^2)$ the $j$-th component $\eta_j$ belongs to the domain of each of the two summands of the operator $A_j$, $j = 1, 2, 3, 4$. Using this fact we compute the terms $A_j \eta_j$ and obtain

$A_j \eta_j = \lambda(e^{2\pi \beta \omega_1} \otimes e^{-2\pi \alpha \omega_2} - e^{-2\pi \beta \omega_1} \otimes e^{2\pi \alpha \omega_2}) \eta_j.$

Thus, $\Phi(E')\Phi(F') \eta - \Phi(F')\Phi(E') \eta = \lambda(\Phi(K_1)^2 - \Phi(K_2)^{-2} - \Phi(K_1)^{-2} \Phi(K_2)^2) \eta$. 

51
(iii): By Lemma 4(ii), the domain \( D_\delta \) is a core for the self-adjoint operator \( e^P, c \in \mathbb{R} \). Since \( D_\delta \otimes D_\delta \subseteq \mathfrak{A}(\mathbb{R}^2) \), the operator \( \Phi(K_f) \) is essentially self-adjoint even on the smaller domain \( \mathfrak{B}(\mathbb{R}^2) \). For the operators \( \Phi(E') \) and \( \Phi(F') \) the assertion follows from statements (iii) and (iv) of Lemma 5.

It is easily seen that the domain \( \mathfrak{A}(\mathbb{R}^2)_4 \) is not invariant under the operators \( \Phi(E') \) and \( \Phi(F') \). Therefore, we do not get a \( * \)-representation of the whole \( * \)-algebra \( U_q^{tw}(gl_2(\mathbb{R})) \) on the domain \( \mathfrak{A}(\mathbb{R}^2)_4 \) of the Hilbert space \( \mathcal{H} \). However, the actions of the generators \( f = E', F', K_1, K_2 \) satisfy the defining relations and they have the hermiticity properties of the \( * \)-algebra \( U_q^{tw}(gl_2(\mathbb{R})) \).

Our next step is to define a \( * \)-algebra structure on \( \mathfrak{A}(\mathbb{R}^2)_4 \). Recall that we used the \( * \)-algebra \( \mathfrak{A}(\mathbb{R}^2_4) \) with product \( \sharp \) and involution \( * \) (see 4.2) as \( * \)-algebra of functions on the quantum half plane. As in the case of functions on \( \mathbb{R}^2 \) we transform the componentwise product of \( 4 \)-tuples under the symmetry \( J \) (see (170) and (171)). That is, for \( a = (a_1, a_2, a_3, a_4), b = (b_1, b_2, b_3, b_4) \in \mathfrak{A}(\mathbb{R}^2)_4 \) we define

\[
a_0 b = 1/4 (a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4, a_1 b_2 + a_2 b_1 + a_3 b_4 + a_4 b_3, a_1 b_4 + a_2 b_3 + a_3 b_2 + a_4 b_1, a_1 b_3 + a_2 b_4 + a_3 b_1 + a_4 b_2),
\]

\[
a^* = (a_1^*, a_2^*, a_3^*, a_4^*).
\]

For the product \( a_j \sharp b_k \) we take the symmetrized version in formula (153):

\[
a_j \sharp b_k = (e^{\pi \beta P_1} e^{-\pi 2 \alpha Q_1} \otimes e^{\pi \alpha P_2} e^{-\pi 2 \beta Q_2}) a_j \# (e^{\pi \beta P_1} e^{-\pi 2 \alpha Q_1} \otimes e^{\pi \alpha P_2} e^{-\pi 2 \beta Q_2}) b_k.
\]

The involution \( a_j^* \) of a \( a_j \) is defined as in (154) by

\[
a_j^*(x_1, x_2) = (e^{\pi (\beta \alpha P_1 - \alpha P_2)} \bar{a}_j)(x_1, x_2).
\]

**Proposition 27.** The vector space \( \mathfrak{A}(\mathbb{R}^2)_4 \) is a \( * \)-algebra with product \( \circ \) and involution \( * \) given by the formulas (173), (177), (176) and (178).

**Proof.** By arguing as in the proof of assertion (i) of Proposition 26 it follows from assumption (172) that the components \( a_j, b_k \) in formula (177) are in the corresponding operator domains and in a domain \( D_{\mu, \nu} \) for certain \( \mu, \nu \in \mathbb{R}^2 \).

Thus, the product (177) is indeed well-defined. That is the reason why we have chosen the symmetrized version in (153) rather than the two other formulas in (153). Note that for \( a, b \in \mathfrak{A}(\mathbb{R}^2) \) all three formulas in (153) coincide, but we dealing now with larger classes of symbols.

Now we prove that for \( a, b \in \mathfrak{A}(\mathbb{R}^2)_4 \) the components of the product \( a \sharp b \) belong again to the corresponding component space in (173). As a sample, we show that \( \mathfrak{A}_{12}(\mathbb{R}^2) \sharp \mathfrak{A}_{12}(\mathbb{R}^2) \subseteq \mathfrak{A}_{12}(\mathbb{R}^2) \). Let \( a, b \in f_\alpha(P_1)^{-1} \mathfrak{A}(\mathbb{R}^2) \). From formulas (56) and (177) we get

\[
f_\alpha(P_1)(a \sharp b) = q^{1/2} (f_\alpha(P_1)a) \sharp e^{2\pi \beta P_1} b + q^{-1/2} e^{-2\pi \beta P_1} a \sharp f_\alpha(P_1)b.
\]
Since \( f_\alpha(\mathcal{P}_1)a \) and \( f_\alpha(\mathcal{P}_1)b \) are in \( \mathfrak{A}_2(\mathbb{R}^2) \), it follows from formulas \( 52 - 54 \) that \( f_\alpha(\mathcal{P}_1)(a\mathbb{Z}b) \) is in the domain of all operators \( e^{2\pi i Q_2}e^{-2\pi i P_2} \), \( c, d \in \mathbb{R}^2 \). Thus, we have \( f_\alpha(e\mathcal{P}_1)(a\mathbb{Z}b) \in \mathfrak{A}_2(\mathbb{R}^2) \). If \( a, b \in f_\beta(\mathcal{P}_2)^{-1}\mathfrak{A}(\mathbb{R}^2) \), then we use formula \( 58 \) rather than \( 56 \) and obtain the identity

\[
 f_\beta(\mathcal{P}_2)(a\mathbb{Z}b) = q^{1/2}(f_\beta(\mathcal{P}_2)a)\mathbb{Z}e^{2\pi i P_2}b + q^{-1/2}e^{-2\pi i P_2}a\mathbb{Z}f_\beta(\mathcal{P}_2)b
\]

which implies that \( f_\beta(\mathcal{P}_2)(a\mathbb{Z}b) \in \mathfrak{A}_2(\mathbb{R}^2) \). Similar verifications can be done for the other cases and components. Thus, we have shown that \( a\mathbb{Z}b \in \mathfrak{A}(\mathbb{R}^2)_4 \) for \( a, b \in \mathfrak{A}(\mathbb{R}^2)_4 \).

Recall that by construction the product \( \circ \) and the involution \( \ast \) have been transformed from the products \( \# \) and the involution \( \ast \), respectively, under the bijective mapping \( J \). Hence \( \mathfrak{A}(\mathbb{R}^2)_4 \) is a \(-\)algebra, because the products \( \# \) and the involution \( \ast \) satisfy all axioms of a \(-\)algebra.

Next we define the product of elements of the coordinate algebra \( \mathcal{O}(\mathbb{R}^2) \) and the algebra \( \mathfrak{A}(\mathbb{R}^2)_4 \). Because of \( 168 \) and \( 151 \), we consider the self-adjoint operators \( x \) and \( y \) on the Hilbert space \( \mathcal{G} \) given by the the operator matrices

\[
 x = \theta_1(e^{2\pi i Q_2} \otimes e^{-\pi i P_2}), \quad y = \theta_1(e^{\pi i P_1} \otimes e^{2\pi i Q_2}).
\]

and define

\[
x \circ a \equiv \Phi(x)a = xa, \quad y \circ a \equiv \Phi(y)a = ya
\] (180)

for \( a \in \mathfrak{A}(\mathbb{R}^2)_4 \). Using once more assumption \( 172 \) it follows that each \( a \in \mathfrak{A}(\mathbb{R}^2)_4 \) is contained in the domains of the self-adjoint operators \( x \) and \( y \), so that \( 180 \) is well-defined.

An arbitrary element \( a \in \mathfrak{A}(\mathbb{R}^2)_4 \) is in general not in the domains of the powers \( x^n \) and \( y^n \) for \( n \in \mathbb{N} \) and the expressions \( x \circ a \) and \( y \circ a \) defined by \( 180 \) are in general not in \( \mathfrak{A}(\mathbb{R}^2)_4 \). Hence the direct sum \( \mathfrak{A}(\mathbb{R}^2) = \mathcal{O}(\mathbb{R}^2) + \mathfrak{A}(\mathbb{R}^2)_4 \) of \(-\)algebras \( \mathcal{O}(\mathbb{R}^2) \) and \( \mathfrak{A}(\mathbb{R}^2)_4 \) is not an algebra with respect to the product \( 180 \). However, for certain elements \( z \in \mathcal{O}(\mathbb{R}^2) \) and \( a \in \mathfrak{A}(\mathbb{R}^2)_4 \) the above definition \( 180 \) gives indeed a well-defined product \( z \circ a \) and \( \mathfrak{A}(\mathbb{R}^2) \) becomes a partial \(-\)algebra with product \( 180 \) in this manner. We shall not pursue this further.

We now replace \( \mathfrak{A}(\mathbb{R}^2)_4 \) by its \(-\)subalgebra

\[
 \mathfrak{A}_0(\mathbb{R}^2)_4 := \mathfrak{A}(\mathbb{R}^2) \oplus \mathfrak{A}(\mathbb{R}^2) \oplus \mathfrak{A}(\mathbb{R}^2) \oplus \mathfrak{A}(\mathbb{R}^2).
\]

According to our general picture, the elements of this subalgebra can be considered as functions on the real quantum plane which are vanishing on the coordinate axis. For \( a \in \mathfrak{A}_0(\mathbb{R}^2)_4 \), the elements \( \Phi(x)a \) and \( \Phi(y)a \) are obviously again in \( \mathfrak{A}_0(\mathbb{R}^2)_4 \), so equation \( 180 \) defines a \(-\)representation \( \Phi \) of the \(-\)algebra \( \mathcal{O}(\mathbb{R}^2)_4 \) on the invariant dense domain \( \mathfrak{A}_0(\mathbb{R}^2)_4 \) of the Hilbert space \( \mathcal{H} \).

By Lemma \( 4(ii) \), the operators \( \Phi(x^n) \) and \( \Phi(y^n) \) are essentially self-adjoint on \( \mathfrak{A}_0(\mathbb{R}^2)_4 \). For \( z \in \mathcal{O}(\mathbb{R}^2) \) and \( a \in \mathfrak{A}_0(\mathbb{R}^2)_4 \) we define

\[
 z \circ a := \Phi(z)a.
\]
Then the direct sum \( A_0(\mathbb{R}^2) := \mathcal{O}(\mathbb{R}^2) + \mathfrak{a}_0(\mathbb{R}^2)_4 \) becomes a \(*\)-algebra with product (180) such that \( \mathcal{O}(\mathbb{R}^2) \) and \( \mathfrak{a}_0(\mathbb{R}^2)_4 \) are \(*\)-subalgebras. Indeed, \( A_0(\mathbb{R}^2) \) is merely the transformation of the left \( \mathcal{U}_q(gl_2(\mathbb{R})) \)-module \(*\)-algebra \( \mathfrak{B}(\mathbb{R}^2)_4 \) defined in 6.1 (see (163)) under the symmetry \( J \). Therefore, \( A_0(\mathbb{R}^2) \) is a \(*\)-algebra with product (180) and there is a left action of \( \mathcal{U}_q(gl_2) \) given by formulas (174) such that \( A_0(\mathbb{R}^2) \) is a left \( \mathcal{U}_q(gl_2(\mathbb{R})) \)-module \(*\)-algebra.

For the study of the coordinate functions \( x \) and \( y \) the \(*\)-algebra \( A_0(\mathbb{R}^2)_4 \) is “large enough”, since the operators \( \Phi(x) \) and \( \Phi(y) \) are essentially self-adjoint on \( A_0(\mathbb{R}^2)_4 \). However, for the action of the generators of \( \mathcal{U}_q(gl_2(\mathbb{R})) \) it is not, because \( A_0(\mathbb{R}^2)_4 \) is not a core for the essentially self-adjoint operators \( \Phi(E') \) and \( \Phi(F') \). The larger \(*\)-algebra \( A(\mathbb{R}^2)_4 \) is needed in this case. Since we have only an action of the generators and not of the whole algebra \( \mathcal{U}_q(gl_2) \), \( A(\mathbb{R}^2)_4 \) cannot be a left \( \mathcal{U}_q(gl_2(\mathbb{R})) \)-module algebra. But for the generators \( f = E', F', K_1, K_2 \) the two conditions [3] and [4] are valid on the \(*\)-algebra \( A(\mathbb{R}^2)_4 \).

The proof of the latter assertion requires a number of computations. We carry out this verification only for the generator \( f = E' \) and for elements of the form \( a = (a_1, 0, 0, 0), b = (b_1, 0, 0, 0) \in A(\mathbb{R}^2)_4 \), where \( a_1, b_1 \in f_\alpha(P_1)^{-1}A(\mathbb{R}^2) \). First we note that from formulas (52)–(55) and (177) we derive that for arbitrary elements \( c, d \in f_\alpha(P_1)^{-1}A(\mathbb{R}^2) \), using the definitions of the product \( \circ \) and of the operators \( \Phi(E') \) and \( \Phi(K) \) and formulas (179), (181) and (182) we compute

\[
(\Phi(E')(a\sharp b))_1 = R_\alpha \otimes e^{2\pi \alpha Q_1}(a_1\sharp b_1) \\
= e^{-2\pi \alpha Q_1} \otimes e^{2\pi \alpha Q_2}(a_1\sharp b_1) \\
= q^{-1/4}e^{-2\pi \alpha Q_1} \otimes e^{2\pi \beta Q_2}((f_\alpha(P_1)a_1)\sharp e^{2\pi \beta P_2}b_1) \\
+ q^{-1/4}e^{-2\pi \alpha Q_1} \otimes e^{2\pi \beta Q_2}(e^{-2\pi \beta P_1}a_1\sharp f_\alpha(P_1)b_1) \\
= (e^{-2\pi \alpha Q_1} \otimes e^{2\pi \beta Q_2}f_\alpha(P_1)a_1)\sharp (e^{2\pi \beta P_1} \otimes e^{-2\pi \alpha P_2}b_1) \\
+ (e^{-2\pi \alpha Q_1} \otimes e^{2\pi \beta Q_2}f_\alpha(P_1)a_1)\sharp (e^{-2\pi \alpha Q_1} \otimes e^{2\pi \beta Q_2}f_\alpha(P_1)b_1) \\
= (\Phi(E')a_1)\sharp (\Phi(K)b_1) + (\Phi(K^{-1})a_1)\sharp (\Phi(E')b_1),
\]

where the lower index 1 always refers to the first component. This proves condition (1) for \( f = E' \) and the particular elements \( a \) and \( b \). The other cases can be treated in a similar manner.

Next we want to have a counterpart on \( A(\mathbb{R}^2)_4 \) of the covariant linear functional \( h \equiv h_0 \) on \( A(\mathbb{R}^2) \). Recall that this counterpart on the \(*\)-algebra \( \mathfrak{B}(\mathbb{R}^2)_4 \) is the functional \( \tilde{h} \) defined by (143). We have to transform this functional \( \tilde{h} \) under the symmetry \( J \) by setting \( \tilde{h}(a) := \tilde{h}(Ja) \). By the definitions (167) of \( J \) and (164) of \( \tilde{h} \) we obtain \( \tilde{h}(a) = 2\tilde{h}(a_1) \), where \( \tilde{h}(a_1) \) is given by (156).
Inserting formula (155) we are lead to define

\[ h(a) = 2 \int \int e^{\pi(\alpha x_1 + \beta x_2)} a_1(x_1, x_2) dx_1 dx_2, \quad a = (a_1, a_2, a_3, a_4) \in \mathcal{A}(\mathbb{R}^2)_4. \] (183)

Let us check first that \( h(a) \) is well-defined for arbitrary elements \( a \in \mathcal{A}(\mathbb{R}^2)_4 \).
Indeed, if \( a \in \mathcal{A}(\mathbb{R}^2)_4 \), then we have \( a_1 \in f_\alpha(P_1)^{-1}\mathcal{A}(\mathbb{R}^2) + f_\beta(P_2)^{-1}\mathcal{A}(\mathbb{R}^2) \).
From assumption (172) we conclude that \( a_1 \) is in the domain \( D(e^{\pi \alpha Q_1} \otimes e^{\pi \beta Q_2}) \) and that \( (e^{\pi \alpha Q_1} \otimes e^{\pi \beta Q_2})a_1 \in D_{\nu, \mu} \) for some \( \nu, \mu \in \mathbb{R}^{++} \). By Lemma 9(ii), the latter implies that the function \( e^{\pi(\alpha x_1 + \beta x_2)}a_1(x_1, x_2) \) is in the Schwartz space \( S(\mathbb{R}^2) \). Hence the integral in (183) exists.

From the construction it follows that the sesquilinear form \( \langle \cdot, \cdot \rangle_h \) associated with \( h \) by (18) is the scalar product of the Hilbert space \( \mathcal{H} \) (see (163)). This can be also verified directly by using formulas (173), (176), (153), (154) and (172)-(18). That is, we have
\[
\langle a, b \rangle_h = h(b^*a) = \int \int_{\mathbb{R}^2} (a_1 \overline{b_1} + a_2 \overline{b_2} + a_3 \overline{b_3} + a_4 \overline{b_4}) dx_1 dx_2
\] (184)

for \( a = (a_1, a_2, a_3, a_4), b = (b_1, b_2, b_3, b_4) \in \mathcal{A}(\mathbb{R}^2)_4 \). Thus we have reduced the scalar product \( \langle \cdot, \cdot \rangle_h \) to \( L^2 \)-scalar products on \( \mathbb{R}^2 \). To achieve formula (184) was the main aim of our considerations and it is the reason why we have transformed all structures by means of the operator \( T \) and the symmetry \( J \).

Finally, let us turn to the differential calculus on the quantum plane. The construction from 3.2 carries over almost verbatim to the algebra \( \mathcal{A}_0(\mathbb{R}_q^2) := \mathcal{O}(\mathbb{R}_q^2) + \mathcal{A}_0(\mathbb{R}^2)_4 \) and yields a first order differential calculus \( \Gamma \) over the algebra \( \mathcal{A}_0(\mathbb{R}_q^2) \) such that \( \{dx, dy\} \) is a free left module basis of \( \Gamma \). The corresponding partial derivatives \( \partial_x \) and \( \partial_y \) are of the form

\[
\partial_x(a) = \begin{pmatrix}
0 & \partial_x(a_1) & 0 & 0 \\
\partial_x(a_2) & 0 & 0 & 0 \\
0 & 0 & 0 & \partial_x(a_3) \\
0 & 0 & \partial_x(a_4) & 0 \\
\end{pmatrix}
\]

\[
\partial_y(a) = \begin{pmatrix}
0 & 0 & \partial_y(a_1) & 0 \\
0 & 0 & 0 & \partial_y(a_2) \\
\partial_y(a_3) & 0 & 0 & 0 \\
0 & \partial_y(a_4) & 0 & 0 \\
\end{pmatrix}
\]

for \( a = (a_1, a_2, a_3, a_4) \in \mathcal{A}_0(\mathbb{R}^2)_4 \), where \( \partial_x(a_j) \) and \( \partial_y(a_j) \), \( j = 1, 2, 3, 4 \), are as in 3.2.

Recall that \( \mathcal{A}(\mathbb{R}_q^2) = \mathcal{O}(\mathbb{R}_q^2) + \mathcal{A}(\mathbb{R}^2)_4 \) is not an algebra, because not all \( a \in \mathcal{A}(\mathbb{R}^2)_4 \) are in the domains of \( \Phi(x)^n\Phi(y)^m \). Likewise, the differential \( da = \omega a - \omega \) and the partial derivatives \( \partial_x(a) \) and \( \partial_y(a) \) are well-defined only for those elements \( a \in \mathcal{A}(\mathbb{R}^2)_4 \) belonging to the corresponding operator domains.
We close this subsection by listing the formulas of the operator matrices for the generators $E', F', K_1, K_2, x$ and $y$. Recall that

\[
L_\alpha = \overline{f_\alpha(P)} e^{-2\pi \alpha Q}, \quad R_\alpha = e^{-2\pi \alpha Q} f_\alpha(P),
\]

\[
f_\alpha(P) = -2 \sinh \pi \beta (2P + \alpha i) = -q^{1/2} e^{2\pi \beta P} + q^{-1/2} e^{-2\pi \beta P}.
\]

\[
\Phi(E') = E = \begin{pmatrix}
0 & 0 & 0 & L_\alpha \otimes e^{2\pi \beta Q_2} \\
0 & 0 & R_\alpha \otimes e^{2\pi \beta Q_2} & 0 \\
0 & L_\alpha \otimes e^{2\pi \beta Q_2} & 0 & 0 \\
R_\alpha \otimes e^{2\pi \beta Q_2} & 0 & 0 & 0
\end{pmatrix},
\]

\[
\Phi(F') = F = \begin{pmatrix}
0 & 0 & 0 & e^{2\pi \alpha Q_1} \otimes L_\beta \\
0 & 0 & e^{2\pi \alpha Q_1} \otimes R_\beta & 0 \\
e^{2\pi \alpha Q_1} \otimes L_\beta & 0 & 0 & 0 \\
e^{2\pi \alpha Q_1} \otimes R_\beta & 0 & 0 & 0
\end{pmatrix},
\]

\[
\Phi(q^{-1/4} K_1) = K_1 = \begin{pmatrix}
\begin{pmatrix} e^{2\pi \beta P_1} \otimes I & 0 & 0 & 0 \\
0 & e^{2\pi \beta P_1} \otimes I & 0 & 0 \\
0 & 0 & e^{2\pi \beta P_1} \otimes I & 0 \\
0 & 0 & 0 & e^{2\pi \beta P_1} \otimes I
\end{pmatrix},
\end{pmatrix}
\]

\[
\Phi(q^{-1/4} K_2) = K_2 = \begin{pmatrix}
I \otimes e^{2\pi \alpha P_2} & 0 & 0 & 0 \\
0 & I \otimes e^{2\pi \alpha P_2} & 0 & 0 \\
0 & 0 & I \otimes e^{2\pi \alpha P_2} & 0 \\
0 & 0 & 0 & I \otimes e^{2\pi \alpha P_2}
\end{pmatrix},
\]

\[
\Phi(x) = x = \begin{pmatrix}
\begin{pmatrix} e^{2\pi \alpha Q_1} \otimes e^{-2\pi \alpha P_2} & 0 & 0 & 0 \\
0 & e^{2\pi \alpha Q_1} \otimes e^{-2\pi \alpha P_2} & 0 & 0 \\
0 & 0 & e^{2\pi \alpha Q_1} \otimes e^{-2\pi \alpha P_2} & 0 \\
0 & 0 & 0 & e^{2\pi \alpha Q_1} \otimes e^{-2\pi \alpha P_2}
\end{pmatrix},
\end{pmatrix}
\]

\[
\Phi(y) = y = \begin{pmatrix}
\begin{pmatrix} e^{2\pi \beta P_1} \otimes e^{2\pi \beta Q_2} & 0 & 0 & 0 \\
0 & e^{2\pi \beta P_1} \otimes e^{2\pi \beta Q_2} & 0 & 0 \\
e^{2\pi \beta P_1} \otimes e^{2\pi \beta Q_2} & 0 & 0 & 0 \\
0 & e^{2\pi \beta P_1} \otimes e^{2\pi \beta Q_2} & 0 & 0
\end{pmatrix},
\end{pmatrix}
\]

6.4 In this last subsection we introduce three unitary operators $F^q_x, F^q_y$ and $F^q$ on the Hilbert space $H$ which can be considered as quantum analogs of the partial Fourier transforms and the Fourier transform on $\mathbb{R}^2$, respectively.
First let us note that the counterparts of the \(q\)-deformed partial derivatives \(D^q_x\) and \(D^q_y\) (see (34)) on the algebra \(A(\mathbb{R}^2_q)\) are defined by

\[
D^q_x := Ky^{-1}E = \begin{pmatrix}
0 & L_\alpha \otimes e^{-\pi_\alpha P_2} & 0 & 0 \\
R_\alpha \otimes e^{-\pi_\alpha P_2} & 0 & 0 & 0 \\
0 & 0 & 0 & L_\alpha \otimes e^{-\pi_\alpha P_2} \\
0 & 0 & R_\alpha \otimes e^{-\pi_\alpha P_2} & 0
\end{pmatrix},
\]
\[
D^q_y := Kx^{-1}F = \begin{pmatrix}
0 & e^{\pi_\beta P_1} \otimes L_\beta & 0 & 0 \\
e^{\pi_\beta P_1} \otimes R_\beta & 0 & 0 & 0 \\
e^{\pi_\beta P_1} \otimes R_\beta & 0 & 0 & 0 \\
0 & 0 & 0 & e^{\pi_\beta P_1} \otimes L_\beta
\end{pmatrix}.
\]

Clearly, \(D^q_x\) and \(D^q_y\) are self-adjoint operators on the Hilbert space \(\mathcal{H}\).

Let \(u\) be the unitary operator on \(L^2(\mathbb{R})\) given by \((uf)(x) = f(-x)\) and let \(w_\alpha, v_\alpha\) resp. \(w_\beta, v_\beta\) be the holomorphic functions from Lemma 6. Then \(F^q_x := \sigma_1 (u \overline{w}_\alpha (P_1) \otimes I, w_\alpha (P_1) \otimes I)\), \(F^q_y := \sigma_2 (I \otimes u \overline{w}_\beta (P_2), I \otimes u \overline{w}_\beta (P_2))\) are commuting unitaries on the Hilbert space \(\mathcal{H}\). Set \(F^q := F^q_x F^q_y\), that is,

\[
F^q = \begin{pmatrix}
(u \overline{w}_\alpha (P_1) \otimes u \overline{w}_\beta (P_2)) 0 & 0 & 0 \\
0 & (u \overline{w}_\alpha (P_1) \otimes u \overline{w}_\beta (P_2)) & 0 \\
0 & 0 & (u \overline{w}_\alpha (P_1) \otimes u \overline{w}_\beta (P_2))
\end{pmatrix}.
\]

We call the unitaries \(F^q_x\) and \(F^q_y\) quantum partial Fourier transforms and \(F^q\) quantum Fourier transform of the real quantum plane. The reason for this terminology stems from the fact that, roughly speaking, these unitaries interchange the coordinate functions \(x, y\) and the \(q\)-deformed partial derivatives \(D^q_x, D^q_y\), respectively. More precisely, we have the following relations.

**Proposition 27.** (i) \(F^q_x (F^q_x)^{-1} = -D^q_x, F^q_x D^q_x (F^q_x)^{-1} = x, F^q_y (F^q_y)^{-1} = K_1^{-2} y, F^q_D_y (F^q_y)^{-1} = K_1^{-2} D_y\).

(ii) \(F^q_y (F^q_y)^{-1} = -D^q_y, F^q_y D^q_y (F^q_y)^{-1} = y, F^q_y x (F^q_y)^{-1} K_2^2 x, F^q_y D^q_y (F^q_y)^{-1} = K_2^2 D_y\).

(iii) \(F^q_x (F^q_x)^{-1} = -K_2^2 D^q_x, F^q_y (F^q_y)^{-1} = -K_1^{-2} D^q_y, F^q_y D^q_y (F^q_y)^{-1} = K_2^2 x, F^q_D^q_y (F^q_y)^{-1} = K_2^{-2} y\).

**Proof.** (i): By Lemma 7, written in terms of matrix entries, we have

\[
u \overline{w}_\alpha (P) L_\alpha v_\alpha (P) u = e^{2\pi_\alpha Q}, u \overline{w}_\alpha (P) R_\alpha w_\alpha (P) u = e^{2\pi_\alpha Q},
\]

\[
u \overline{w}_\alpha (P) e^{2\pi_\alpha Q} v_\alpha (P) u = u L_{-\alpha} u = -L_\alpha, u \overline{w}_\alpha (P) e^{2\pi_\alpha Q} w_\alpha (P) u = u R_{-\alpha} u = -R_\alpha.
\]

Because of the two relations (183), the matrix entries of \(F^q_x D^q_x (F^q_x)^{-1}\) and \(x\) coincide, while (184) implies that \(F^q_y x (F^q_x)^{-1}\) and \(-D^q_x\) have the same matrix.
entries. This proves the first two relations of (i). The two other relations follow at once from the corresponding definitions combined with the fact that 
\[ue^{\pi\beta P}u = e^{-\pi\beta P}.\]

The proof of (ii) is similar to the proof of (i). The relations of (iii) follow easily from (i) and (ii).

\[\blacksquare\]

Remark 5. The holomorphic functions \(w_{-\alpha}\) and \(v_{-\alpha}\) coincide with the functions \(w_1\) and \(w_2\) in [S4], where a closely related quantum Fourier transform of a \(q\)-deformed Heisenberg algebra appeared. Holomorphic functions of similar kind have been used by S.L. Woronowicz [W] in another context as quantum exponential functions for the quantum \(ax + b\)-group.

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