Pion–nucleon scattering in chiral perturbation theory II: 
Fourth order calculation

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\textbf{Abstract}

We analyze elastic–pion nucleon scattering to fourth order in heavy baryon chiral perturbation theory, extending the third order study published in Nucl. Phys. A640 (1998) 199. We use various partial wave analyses to pin down the low–energy constants from data in the \textit{physical} region. The S–wave scattering lengths are consistent with recent determinations from pionic hydrogen and deuterium. We find an improved description of the P–waves. We also discuss the pion–nucleon sigma term and problems related to the prediction of the subthreshold parameters.

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1 Introduction and summary

Effective field theory allows one to analyze the chiral structure of Quantum Chromodynamics in the low energy domain, which is not accessible to a perturbative expansion in the strong coupling constant. The spontaneous violation of the chiral symmetry of QCD entails the existence of Goldstone bosons, the pions (we consider here the two flavor case of the light up and down quarks). The interactions of the Goldstone bosons among themselves and with matter vanish as the momentum transfer goes to zero. This is a consequence of Goldstone’s theorem. Consequently, such interactions can be analyzed in a perturbative expansion, where all momenta and masses are small compared to the typical scale of hadronic interactions, say the mass of the rho meson. This is the so-called chiral expansion. A systematic investigation of processes involving pions allows therefore to understand in a precise and quantitative manner how the symmetry violation takes place and also to pin down the ratios of the light quark masses. One such process is elastic pion–nucleon scattering. In ref.[1] (called (I) from here on), we considered this reaction in the framework of heavy baryon chiral perturbation theory (HBCHPT) to third order in the chiral expansion. At that order, the first contributions from pion loop graphs, which perturbatively restore unitarity, appear. Besides pion loop diagrams, there are tree graphs. Some of these have fixed coefficients, others are accompanied by coupling constants not fixed by chiral symmetry. These so-called low energy constants (LECs) must be determined by a comparison to data or can be estimated using models. As has been shown in refs.[2, 3], these LECs encode the information from higher mass states not present explicitly in the effective field theory. There are three important reasons to extend the calculations of (I) to fourth order: First, only at that order one has a complete one-loop representation. Second, it is known that these fourth order corrections can be large (for a review see ref.[4] and an update is given in ref.[5]). Third, only after having obtained an accurate representation of the isospin–symmetric amplitude, as done here, one can attack the more subtle problem of isospin symmetry violation [6, 7] in low energy pion–nucleon scattering. First steps in the framework of HBCHPT have been reported in refs.[8, 9, 10].

There have been some interesting new developments since (I) appeared: First, a manifestly Lorentz invariant form of baryon chiral perturbation theory was proposed in ref.[11] and some implications for the nucleon mass and the scalar form factor to fourth order were worked out. The same group is also investigating pion–nucleon scattering in that framework [12]. By construction, their approach leads to the correct analytical structure of the pion–nucleon scattering amplitude, whereas in the heavy baryon approach special care has to be taken in certain regions of the complex plane. Second, a different determination of the dimension two and three LECs by fitting the HBCHPT amplitude to the dispersive representation (based on the Karlsruhe partial wave analysis) inside the Mandelstam triangle was performed in ref.[13]. While that method has the a priori advantage that the chiral expansion is expected to converge best in this special region of the Mandelstam plane, it is difficult to work out the theoretical uncertainties. Another drawback of this procedure is that only three dimension two LECs could be determined with sufficient precision (if one uses the third order HBCHPT amplitude as input). This is related to the fact that close to the point $\nu = t = 0$ the contribution from the third order terms is accompanied by very small kinematical prefactors.

Here, we will follow the approach used in (I), namely to fit to the phase shifts provided by three different partial wave analyses for pion laboratory momenta between 40 and 100 MeV. This not only allows for a discussion of the uncertainties due to the input but also gives us the possibility to predict the phase shifts at higher and lower momenta, in particular the scattering lengths and the range parameters. Furthermore, we are then able to directly compare to the third order calculation and draw conclusions on the convergence of the chiral expansion. Of course, at the appropriate places we will discuss the relation to the work reported in refs.[11, 13].
The pertinent results of the present investigation can be summarized as follows:

(i) We have constructed the complete one-loop amplitude for elastic pion–nucleon scattering in heavy baryon chiral perturbation theory, including all terms of order $q^4$. It contains 13 low–energy constants plus one related to fixing the pion–nucleon coupling constant through the Goldberger–Treiman discrepancy. Their values can be determined by fitting to the two S– and four P–wave partial wave amplitudes for three different sets of available pion–nucleon phase shifts in the physical region at low energies (typically in the range of 40 to 100 MeV pion momentum in the laboratory frame).

(ii) We have performed two types of fits. In the first one, we fit four combinations of the dimension two and four LECs, together with five LECs from the third and five from the fourth order. This means that the dimension two LECs are subject to quark mass renormalizations from certain fourth order terms. Most fitted LECs are of “natural” size. In the second approach, we fix the dimension two LECs as determined from the third order fit and determine the corresponding dimension four LEC combinations separately. We have studied the convergence of the chiral expansion by comparing the best fits based on the second, third and fourth order representation of the scattering amplitudes. The fourth order corrections are in general not large, but they improve the description of most partial waves. This indicates convergence of the chiral expansion.

(iii) We can predict the phases at lower and at higher energies, in particular the threshold parameters (scattering lengths and effective ranges). The results are not very different from the third order study in (I), but the description of the P–waves is improved, in particular the scattering length in the delta channel and the energy dependence of the small P–waves. The errors on the S-wave scattering lengths are as in (I) since they are due to the differences in the partial wave analyses used as input. Our theoretical predictions are consistent with recent determinations from pionic hydrogen and deuterium [21].

(iv) The pion–nucleon sigma term (at zero momentum transfer) can not be predicted without further input since at fourth order a new combination of LECs appears, that is not pinned down by the scattering data. Therefore, we have analyzed the sigma term at the Cheng–Dashen point. Using a family of sum rules which relate this quantity to threshold parameters and known dispersive integrals, we find results consistent with other determinations using the various partial wave analyses.

(v) We do not find any improvement of the chiral description of the so–called subthreshold parameters as reported in (I). In some cases, the fourth order prediction is worse than the third order one. Since our amplitude is pinned down in the physical region, we do not expect the extrapolation to the subthreshold region to be very precise. To circumvent this problem, it is mandatory to combine the chiral representation obtained here with dispersion relations, see e.g. ref.[12], or by fitting directly inside the Mandelstam triangle [13].

The manuscript is organized as follows. In section 2, we briefly discuss the effective Lagrangian underlying our calculation. All details were given in (I), so here we only spell out the new terms at fourth order. Section 3 contains the HBCHPT results for the pion–nucleon scattering amplitudes $g^\pm, h^\pm$ to fourth order. The fitting procedure together with the results for the phase shifts and threshold parameters are presented in section 4. We also spell out the remaining problems related to the sigma term and the subthreshold parameters. The appendix contains the analytical expressions for the various threshold parameters.
2 Effective Lagrangian

The starting point of our approach is the most general chiral invariant Lagrangian built from pions, nucleons and external scalar sources (to account for the explicit chiral symmetry breaking). The Goldstone bosons are collected in a $2 \times 2$ matrix-valued field $U(x) = u^2(x)$. We use the so-called sigma model parametrisation (gauge). We work in the framework of heavy baryon chiral perturbation theory, thus the nucleons are described by structureless non-relativistic spin-$\frac{1}{2}$ particles, denoted by $N(x)$. The effective theory admits a low energy expansion, i.e. the corresponding effective Lagrangian can be written as (for more details and references, see e.g. [4])

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\pi \pi}^{(2)} + \mathcal{L}_{\pi \pi}^{(4)} + \mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_{\pi N}^{(2)} + \mathcal{L}_{\pi N}^{(3)} + \mathcal{L}_{\pi N}^{(4)} \ldots,$$

(2.1)

where the ellipsis denotes terms of higher order. For the explicit form of the meson Lagrangian and the dimension one, two and three pion–nucleon terms, we refer to (I). The complete fourth order Lagrangian is given in ref.[14] and the renormalization is discussed in ref.[15]. For completeness, we display here the finite terms from $\mathcal{L}_{\pi N}^{(4)}$ which contribute to elastic $\pi N$ scattering

$$\mathcal{L}_{\pi N}^{(4)} = \mathcal{N} \left\{ \left( \bar{e}_{14} - \frac{1}{16 m^2 c_2} \langle h_{\mu \nu} h^{\mu \nu} \rangle + \left( \bar{e}_{15} - \frac{1}{256 m^3 g_A^2} - \frac{d_{14} - d_{15}}{16 m} \right) v^{\mu \nu} \langle h_{\mu \rho} h_{\nu}^{\rho} \rangle \right) + \left( \bar{e}_{16} + \frac{3}{256 m^3 g_A^2} \langle h_{\mu \rho} h_{\nu}^{\rho} \rangle + \bar{e}_{17} [S^\mu, S^\nu] \right) \left[ h_{\mu \rho} h_{\nu}^{\rho} \right] + \left( \bar{e}_{18} - \frac{1}{128 m^3} (2 - g_A^2) - \frac{1}{16 m^2 c_4} [S^\mu, S^\nu] \langle h_{\mu \rho} h_{\nu}^{\rho} \rangle + \bar{e}_{19} \langle \chi_+ \rangle \langle u \cdot u \rangle \right) + \left( \bar{e}_{20} - \frac{1}{32 m^2 g_A^2} - \frac{1}{8 m g_A d_{16}} \langle \chi_+ \rangle \langle (v \cdot u)^2 \rangle + \left( \bar{e}_{21} + \frac{1}{16 m^2 c_1} [S^\mu, S^\nu] \langle \chi_+ \rangle \right) \langle u \cdot u \rangle \right) + \bar{e}_{22} [D^\mu, [D^\mu, \langle \chi_+ \rangle]] + \bar{e}_{35} v^{\mu \nu} \langle \chi_- h_{\mu \nu} \rangle + \bar{e}_{36} \langle u \mu [D^\mu, \chi_-] \rangle + \bar{e}_{37} [S^\mu, S^\nu] \langle u \mu [D^\mu, \chi_-] \rangle + \bar{e}_{38} \langle \chi_+ \rangle \langle \chi_+ \rangle + \bar{e}_{39} \langle \chi_+ \rangle \langle \chi_+ \rangle + \bar{e}_{40} (\det \chi + \det \chi^\dagger) \right\}$$

\[ = \mathcal{N} \left\{ \left( \bar{e}_{14} - \frac{1}{16 m^2 c_2} \langle h_{\mu \nu} h^{\mu \nu} \rangle + \left( \bar{e}_{15} - \frac{1}{256 m^3 g_A^2} - \frac{d_{14} - d_{15}}{16 m} \right) v^{\mu \nu} \langle h_{\mu \rho} h_{\nu}^{\rho} \rangle \right) + \left( \bar{e}_{16} + \frac{3}{256 m^3 g_A^2} \langle h_{\mu \rho} h_{\nu}^{\rho} \rangle + \bar{e}_{17} [S^\mu, S^\nu] \right) \left[ h_{\mu \rho} h_{\nu}^{\rho} \right] + \left( \bar{e}_{18} - \frac{1}{128 m^3} (2 - g_A^2) - \frac{1}{16 m^2 c_4} [S^\mu, S^\nu] \langle h_{\mu \rho} h_{\nu}^{\rho} \rangle + \bar{e}_{19} \langle \chi_+ \rangle \langle u \cdot u \rangle \right) + \left( \bar{e}_{20} - \frac{1}{32 m^2 g_A^2} - \frac{1}{8 m g_A d_{16}} \langle \chi_+ \rangle \langle (v \cdot u)^2 \rangle + \left( \bar{e}_{21} + \frac{1}{16 m^2 c_1} [S^\mu, S^\nu] \langle \chi_+ \rangle \right) \langle u \cdot u \rangle \right) + \bar{e}_{22} [D^\mu, [D^\mu, \langle \chi_+ \rangle]] + \bar{e}_{35} v^{\mu \nu} \langle \chi_- h_{\mu \nu} \rangle + \bar{e}_{36} \langle u \mu [D^\mu, \chi_-] \rangle + \bar{e}_{37} [S^\mu, S^\nu] \langle u \mu [D^\mu, \chi_-] \rangle + \bar{e}_{38} \langle \chi_+ \rangle \langle \chi_+ \rangle + \bar{e}_{39} \langle \chi_+ \rangle \langle \chi_+ \rangle + \bar{e}_{40} (\det \chi + \det \chi^\dagger) \right\}

\]

\[ = \mathcal{N} \left\{ \left( \bar{e}_{14} - \frac{1}{16 m^2 c_2} \langle h_{\mu \nu} h^{\mu \nu} \rangle + \left( \bar{e}_{15} - \frac{1}{256 m^3 g_A^2} - \frac{d_{14} - d_{15}}{16 m} \right) v^{\mu \nu} \langle h_{\mu \rho} h_{\nu}^{\rho} \rangle \right) + \left( \bar{e}_{16} + \frac{3}{256 m^3 g_A^2} \langle h_{\mu \rho} h_{\nu}^{\rho} \rangle + \bar{e}_{17} [S^\mu, S^\nu] \right) \left[ h_{\mu \rho} h_{\nu}^{\rho} \right] + \left( \bar{e}_{18} - \frac{1}{128 m^3} (2 - g_A^2) - \frac{1}{16 m^2 c_4} [S^\mu, S^\nu] \langle h_{\mu \rho} h_{\nu}^{\rho} \rangle + \bar{e}_{19} \langle \chi_+ \rangle \langle u \cdot u \rangle \right) + \left( \bar{e}_{20} - \frac{1}{32 m^2 g_A^2} - \frac{1}{8 m g_A d_{16}} \langle \chi_+ \rangle \langle (v \cdot u)^2 \rangle + \left( \bar{e}_{21} + \frac{1}{16 m^2 c_1} [S^\mu, S^\nu] \langle \chi_+ \rangle \right) \langle u \cdot u \rangle \right) + \bar{e}_{22} [D^\mu, [D^\mu, \langle \chi_+ \rangle]] + \bar{e}_{35} v^{\mu \nu} \langle \chi_- h_{\mu \nu} \rangle + \bar{e}_{36} \langle u \mu [D^\mu, \chi_-] \rangle + \bar{e}_{37} [S^\mu, S^\nu] \langle u \mu [D^\mu, \chi_-] \rangle + \bar{e}_{38} \langle \chi_+ \rangle \langle \chi_+ \rangle + \bar{e}_{39} \langle \chi_+ \rangle \langle \chi_+ \rangle + \bar{e}_{40} (\det \chi + \det \chi^\dagger) \right\}

\]
\begin{align*}
&+ \frac{1}{4m} \left( \bar{d}_{14} - \bar{d}_{15} \right) \left( [S^\mu, S^\nu] \langle h^\lambda_{\mu} u^\nu \rangle D_\lambda + \text{h.c.} \right) \\
&- \left( \frac{1}{64m^3} g_A^2 + \frac{1}{8m^2} c_2 - \frac{1}{4m} \left( \bar{d}_{14} - \bar{d}_{15} \right) \right) \left( [S^\lambda, S^\mu] v^\nu \langle h_{\mu\nu}, v \cdot u \rangle D_\lambda + \text{h.c.} \right) \\
&+ \left( \frac{1}{64m^3} g_A^2 - \frac{1}{4m} \left( \bar{d}_{14} - \bar{d}_{15} \right) \right) \left( [S^\lambda, S^\mu] v^\nu v^\rho \langle u^\mu, h^\nu_{\rho} \rangle D_\lambda + \text{h.c.} \right) \\
&+ \frac{1}{32m^3} g_A \left( i S^\lambda \left[ v \cdot D, h^\mu_{\mu} \right] D_\lambda + \text{h.c.} \right) + \frac{1}{16m^3} g_A \left( i \left[ S \cdot D, h^\mu_{\mu} \right] v \cdot D + \text{h.c.} \right) \\
&- \frac{1}{32m^3} g_A \left( i S^\nu [v \cdot D, h^\mu_{\mu}] v \cdot D + \text{h.c.} \right) - \frac{1}{32m^3} g_A \left( i S^\nu [v \cdot D, h^\mu_{\mu}] v \cdot D + \text{h.c.} \right) \\
&+ \frac{1}{32m^3} g_A \left( i S^\nu [v \cdot D, h^\mu_{\mu}] v \cdot D + \text{h.c.} \right) - \frac{1}{32m^3} g_A \left( i S^\nu [v \cdot D, h^\mu_{\mu}] v \cdot D + \text{h.c.} \right) \\
&- \frac{1}{2m^2} c_2 D_\mu \langle u^\mu u^\nu \rangle D_\nu + \left( \frac{1}{64m^3} g_A^2 - \frac{1}{8m^2} c_2 \right) D_\mu \langle u \cdot u \rangle D_\mu \\
&- \left( \frac{1}{64m^3} g_A^2 + \frac{1}{8m^2} c_2 \right) D_\mu \langle (v \cdot u)^2 \rangle D_\nu - \left( \frac{1}{32m^3} g_A^2 + \frac{1}{8m^2} c_4 \right) D_\mu \langle S^\rho, S^\tau \rangle [u^\rho, u^\tau] D_\mu \\
&+ \left( \frac{1}{16m^3} g_A^2 + \frac{1}{4m^2} c_4 \right) \left( D_\mu [S^\rho, S^\tau] [v^\rho, u_\tau] D_\nu + \text{h.c.} \right) - \frac{1}{4m^2} c_1 D_\mu \langle \chi_+ \rangle D_\mu \\
&- \frac{1}{4m^3} g_A \left( i u \cdot D S \cdot Dv \cdot D + \text{h.c.} \right) + \frac{1}{8m^3} g_A \left( i S \cdot u D^2 v \cdot D + \text{h.c.} \right) \\
&+ \frac{3}{8m^3} g_A \left( i v \cdot u S \cdot Dv \cdot D + \text{h.c.} \right) - \frac{1}{8m^3} g_A \left( i S \cdot w v \cdot Dv \cdot D + \text{h.c.} \right) \right) N ,
\end{align*}

with \( u^\mu \) the nucleon’s four–velocity, \( S^\mu \) the covariant spin–vector, \( D_\mu \) the chiral covariant derivative, \( u^\mu = i (\partial_\mu u^\nu + u^\nu \partial_\mu u) \), \( h^\mu_{\nu} = [D^\mu_{\nu}, u^\nu] + [D^\nu_{\mu}, u^\nu] \) and \( \chi_\pm = u^\uparrow \chi u^\dagger \pm u^\dagger \chi u \) encoding the explicit chiral symmetry breaking. Traces in flavor space are denoted by \( \langle \ldots \rangle \) and \( \tilde{\chi}_\mp = \chi_\mp - \langle \chi_\mp \rangle /2 \). We remark that the various parameters like \( g_A, m_s, \ldots \) appearing in the effective Lagrangian have to be taken in the chiral SU(2) limit \( (m_u = m_d = 0, m_s \text{ fixed}) \) and should be denoted as \( g_A, m_s, \ldots \). Throughout this manuscript, we will not specify this but it should be kept in mind. It is also worth mentioning the particular role of the terms \( \sim e_{115,116} \). These operators have no pion matrix elements but are simply contact interactions of the external scalar source with the matter fields and thus contribute to the nucleon mass and the scalar form factor. This will be of importance later on. Having constructed the effective pion–nucleon Lagrangian to order \( q^4 \), we now turn to use it in order to describe elastic pion–nucleon scattering. To account for isospin breaking, one has to extend this Lagrangian to include virtual photons. This has already been done in [8, 10] and we refer the reader to these papers. For a systematic study of isospin violation in the elastic and charge exchange channels, one first has to find out to what accuracy the low energy \( \pi N \) phase shifts can be described in the isospin symmetric framework.\footnote{An investigation of isospin violation in the threshold amplitudes to third order was reported in ref.[9].} This is the question which will be addressed in the remaining sections of this investigation.
3 Pion–nucleon scattering

3.1 Basic definitions

In this section, we only give a few basic definitions pertinent to elastic pion–nucleon scattering. For a more detailed discussion, we refer to (I). In the center-of-mass system (cms), the amplitude for the process \( \pi^a(q_1) + N(p_1) \to \pi^b(q_2) + N(p_2) \) takes the following form (in the isospin basis):

\[
T_{\pi N}^{ba} = \left(\frac{E + m}{2m}\right) \left\{ \delta_{ba} \left[ g^+(\omega, t) + i\vec{\sigma} \cdot (\vec{q}_2 \times \vec{q}_1) h^+(\omega, t) \right] \\
+ i\epsilon^{bac} \left[ g^-(\omega, t) + i\vec{\sigma} \cdot (\vec{q}_2 \times \vec{q}_1) h^-(\omega, t) \right] \right\}
\]  

(3.1)

with \( \omega = v \cdot q_1 = v \cdot q_2 \) the pion cms energy, \( E_1 = E_2 \equiv E = (\vec{q}^2 + m^2)^{1/2} \) the nucleon energy and \( \vec{q}^2 = \vec{q}_2^2 \equiv \vec{q}^2 = ((s - M^2 - m^2)/4m^2M^2)/(4s) \). \( t = (q_1 - q_2)^2 \) is the invariant momentum transfer squared and \( s \) denotes the total cms energy squared. Furthermore, \( g^\pm(\omega, t) \) refers to the isoscalar/isovector non-spin-flip amplitude and \( h^\pm(\omega, t) \) to the isoscalar/isovector spin-flip amplitude. This form is most suitable for the HBCHPT calculation since it is already defined in a two–component framework.

3.2 Chiral expansion of the amplitudes

What we are after is the chiral expansion of the various amplitudes \( g^\pm, h^\pm \). These consist of essentially three pieces, which are the tree and counterterm parts of polynomial type as well as the unitarity corrections due to the pion loops. To be precise, we have

\[
X = X^{\text{tree}} + X^{\text{ct}} + X^{\text{loop}}, \quad X = g^\pm, h^\pm, 
\]  

(3.2)

where the tree contribution subsumes all Born terms with fixed coefficients, the counterterm amplitude the ones proportional to the dimension two, three and four LECs. The last term in Eq.(3.2) is the complete one–loop amplitude consisting of terms of order \( q^3 \) and \( q^4 \). The latter is a complex–valued function and restores unitarity in the perturbative sense. Its various terms are all proportional to \( 1/F^4 \). We remark that the topologies of the new loop graphs are not different from the ones already present at third order. The loops of fourth order have exactly one insertion from the dimension two Lagrangian. Note that we treat the chiral symmetry breaking scale \( \Lambda_\chi \simeq 1 \text{ GeV} \) on the same footing as the nucleon mass. In principle, one could also organize the loop expansion, which proceeds in powers of \( 1/F^2 \), and the \( 1/m \) expansion independently of each other (with some prescription for the mixed terms). These amplitudes are functions of two kinematical variables, which we choose to be the pion energy and the invariant momentum transfer squared, i.e. \( X = X(\omega, t) \). In what follows, we mostly suppress these arguments. The full one–loop amplitude to order \( q^4 \) is obtained after mass and coupling constant renormalization,

\[
(\tilde{g}_A, \tilde{m}, F, M) \to (g_A, m, F_\pi, M_\pi). 
\]  

(3.3)
To fourth order, these read (we also give the corresponding Z–factors for the pion and the nucleon)

\[
Z_\pi = 1 - \frac{2M^2}{F^2} \ell_4 - \frac{\Delta_\pi}{F^2},
\]

(3.4)

\[
Z_N = 1 - \frac{g_A^2}{F^2} \frac{3M^2}{16\pi^2} + \frac{g_A^2}{F^2} \frac{9M^3}{64\pi m},
\]

(3.5)

\[
M^2_\pi = M^2 \left\{ 1 + \frac{2M^2}{F^2} \ell_3 + \frac{\Delta_\pi}{2F^2} \right\},
\]

(3.6)

\[
m = \bar{m} - 4M^2 c_1 - \frac{3g_A^2 M^3}{32\pi F^2}
\]

\[
- M^4 (16\ell_{38} + 2\ell_{115} + \frac{1}{2}\ell_{116}) + \frac{3M^4 c_2}{128\pi^2 F^2} - \frac{3g_A^2 M^4}{64\pi^2 m F^2},
\]

(3.7)

\[
F_\pi = F \left\{ 1 + \frac{M^2}{F^2} \ell_4 - \frac{\Delta_\pi}{F^2} \right\},
\]

(3.8)

\[
g_A \frac{\Delta_\pi}{F^2} = \left\{ \frac{M^2}{F^2} \ell_4 + \frac{4M^2}{g_A} d_{16}(\lambda) + \frac{g_A^2}{4F^2} \left( \Delta_\pi - \frac{M^2}{4\pi^2} \right) 
\]

\[
- \frac{M^3}{6\pi F^2} \left( - \frac{1}{8m} + c_3 - \left( 2c_4 + \frac{1}{2m} \right) - \frac{3g_A^2}{4m} \right) \right\},
\]

(3.9)

with

\[
\Delta_\pi = 2M^2_\pi \left( L + \frac{1}{16\pi^2} \ln \frac{M_\pi}{\lambda} \right) + O(d - 4)
\]

(3.10)

and

\[
L = \frac{\lambda^{d-4}}{16\pi^2} \left( \frac{1}{d - 4} + \frac{1}{2} (\gamma_E - 1 - \ln 4\pi) \right)
\]

(3.11)

where Euler’s constant \( \gamma_E = 0.557215 \) has been used, \( \lambda \) is the scale of dimensional regularization and \( d \) the number of space–time dimensions.

After these preliminaries, we give the final expressions for the tree, counterterm and loop graphs at fourth order in terms of the renormalized quantities:

**Tree and counterterm graphs:**

\[
m^3 F_\pi^2 g^{+}(\omega, t) = -\frac{g_A^2}{64\omega^4} \left[ -32\omega^4 M^4_\pi t^4 + 45M^4_\pi t^2 - 11M^2_\pi t^3 - 80M^6_\pi t + 52M^8_\pi 
\]

\[
+ 7\omega^2 t^3 + 110\omega^2 M^4_\pi t - 49\omega^2 M^2_\pi t^2 - 76\omega^2 M^6_\pi + 11\omega^4 t^2 
\]

\[
+ t^4 - 4\omega^4 t + 28\omega^4 M^4 \right]
\]

\[
+ \frac{1}{2} M^2_\pi (2\omega^2 - 2M^2_\pi + t)mc_1 + 8M^4_\pi m^3 c_1 \frac{\ell_3}{F^2_\pi}
\]

\[
+ \frac{1}{4} (-22\omega^2 M^4_\pi + 8M^4_\pi + 3\omega^2 t + 14\omega^4 - 4M^2_\pi t)mc_2
\]

\[
+ \frac{1}{8} (-4\omega^2 M^2_\pi + 2\omega^2 t + 4M^4_\pi - 4M^2_\pi t + t^2)mc_3 - 16M^2_\pi \omega^2 m^2 c_1 c_2
\]
\[ m^3 F_{\pi}^2 h^+(\omega, t) = \frac{g_A^2}{32\omega^4} \left[ 5\omega^2 t^2 + 27M_{\pi}^4 t - 9M_{\pi}^2 t^2 - 25\omega^2 M_{\pi}^2 t - 28M_{\pi}^6 + t^3 \right. \\
+30\omega^2 M_{\pi}^4 - 4\omega^4 M_{\pi}^2 + 3\omega^4 t \right] \\
+M_{\pi}^2 mc_1 - \frac{1}{2}\omega^2 mc_2 + \frac{1}{4}(-2M_{\pi}^2 + t)mc_3 \\
+\frac{1}{2}(8\omega^2 - 4M_{\pi}^2 + t)m^2(\bar{d}_{14} - \bar{d}_{15}) - \frac{g_A}{2\omega^2} M_{\pi}^2(-4M_{\pi}^2 + t)m^2 \bar{d}_{18} \right], \quad (3.12) \\

\[ m^3 F_{\pi}^2 g^-(\omega, t) = \frac{-g_A^2}{64\omega^4} \left[ -32\omega^4 M_{\pi}^4 t + 45M_{\pi}^4 t^2 - 11M_{\pi}^2 t^3 - 82M_{\pi}^6 t + 56M_{\pi}^8 \\
+7\omega^2 t^3 + 110\omega^2 M_{\pi}^4 t - 49\omega^2 M_{\pi}^2 t^2 - 80\omega^2 M_{\pi}^6 + 11\omega^4 t^2 \\
+8\omega^8 + t^4 - 2\omega^6 t - 8\omega^6 M_{\pi}^2 + 24\omega^4 M_{\pi}^4 \right] \\
+\frac{1}{32\omega^4} \left[ -4\omega^8 + 8\omega^6 M_{\pi}^2 - 4\omega^4 M_{\pi}^4 - \omega^6 t + \omega^4 M_{\pi}^2 t \right] \\
+\frac{1}{16}t(4\omega^2 - 4M_{\pi}^2 + t)mc_4 \\
-\frac{1}{2}(8\omega^2 M_{\pi}^2 + 4\omega^2 t + 8M_{\pi}^4 + t^2 - 6M_{\pi}^2 t)m^2(\bar{d}_1 + \bar{d}_2) \\
+3\omega^2(4\omega^2 - 4M_{\pi}^2 + t)m^2 \bar{d}_3 + 2M_{\pi}^2(4\omega^2 - 4M_{\pi}^2 + t)m^2 \bar{d}_5 \\
-\frac{g_A}{4\omega^2} M_{\pi}^2(t^2 + 2\omega^2 t - 8\omega^4 + 8M_{\pi}^4 - 6M_{\pi}^2 t)m^2 \bar{d}_{18} \right], \quad (3.13) \\

\[ m^3 F_{\pi}^2 h^-(\omega, t) = \frac{-g_A^2}{32\omega^4} \left[ 5\omega^2 t^2 + 27M_{\pi}^4 t - 9M_{\pi}^2 t^2 - 25\omega^2 M_{\pi}^2 t - 26M_{\pi}^6 + t^3 \right. \\
+30\omega^2 M_{\pi}^4 - 4\omega^4 M_{\pi}^2 + 3\omega^4 t - 2\omega^6 \right] - \frac{\omega^4(\omega^2 - M_{\pi}^2)}{16\omega^4} \\
-M_{\pi}^2 mc_1 + \frac{1}{8}(2\omega^2 - 2M_{\pi}^2 + t)mc_4 + \frac{g_A}{2\omega^2} M_{\pi}^2(2\omega^2 - 2M_{\pi}^2 + t)m^2 \bar{d}_{18} \\
-4(-2M_{\pi}^2 + t)m^2 \bar{e}_{17} + 8\omega^2 M_{\pi}^2 \bar{e}_{18} + 4M_{\pi}^2 m^3(2\bar{e}_{21} - \bar{e}_{37}) \right]. \quad (3.14) \\

\textbf{Loop graphs:}

\[ mF_{\pi}^4 g^+(\omega, t) = -\frac{1}{12\omega^3} J_0(\omega) \left[ 2M_{\pi}^2 g_A^4 \left. (t M_{\pi}^2 - 2\omega^4 + 4\omega^2 M_{\pi}^2 - t \omega^2 - 2M_{\pi}^4) \right] \\
+\omega^2 g_A^2 \left. (-12\omega^2 M_{\pi}^2 - M_{\pi}^2 t + \omega^2 t + 12\omega^4) \right. + 6\omega^4 M_{\pi}^2 \right] \\
+\frac{1}{12\omega^3} J_0(-\omega) \left[ 2g_A^2 \left. (6M_{\pi}^6 + 8\omega^6 - 10\omega^4 M_{\pi}^2 + 3t\omega^2 M_{\pi}^2 + M_{\pi} t^2 + 2t\omega^4 \right] \\
-4\omega^2 M_{\pi}^4 - 5t M_{\pi}^2 \right. - \omega^2 g_A^2 \left. (M_{\pi}^2 t + 4\omega^2 M_{\pi}^2 + \omega^2 t - 4\omega^4) \right. + 6\omega^4 \left. (-3M_{\pi}^2 + t + 4\omega^2) \right] \]
\[-\frac{1}{12\omega^2} \frac{\partial J_0}{\partial \omega}(\omega) \left[ g_A^2 (8\omega^6 + \omega^2 t^2 - 8M_\pi^6 - M_\pi^2 t^2 + 24\omega^2 M_\pi^4 + 6t\omega^4 + 6tM_\pi^4 \\
- 12t\omega^2 M_\pi^2 - 24\omega^4 M_\pi^2 ) \\
+ 2\omega^2 g_A^2 (4\omega^4 - M_\pi^2 t - 8\omega^2 M_\pi^2 + \omega^2 t + 4 M_\pi^4 ) \\
+ 3\omega^4 (4\omega^2 - 4M_\pi^2 + t) \right] \]

\[+ \frac{g_A^2}{32} \frac{\partial K_0}{\partial \omega}(0,t)(12M_\pi^2 t - 4M_\pi^6 - 9M_\pi^2 t^2 + 2t^3 ) \]

\[-\frac{1}{24} J_0(t) \left[ 3g_A^2 t(2t - M_\pi^2) - 48M_\pi^2 m c_1(M_\pi^2 - 2t) \\
+ 2mc_2(2t^2 + 4M_\pi^4 - 9tM_\pi^2) + 12c_3(-5tM_\pi^2 + 2t^2 + 2M_\pi^4) \right] \]

\[- \frac{1}{1152\pi^2\omega^3} \left[ 2g_A^4 (-52\omega^5 M_\pi^2 + 24\omega^7 + \omega^3 t^2 + 10t\omega^5 - 96\pi M_\pi^7 + 96\pi M_\pi^5 \omega^2 \\
+ 72\pi M_\pi^4 - 12\pi M_\pi^2 t^2 - 12\pi M_\pi^2 t - 12\omega M_\pi^4 + 6\omega M_\pi^2 t^2 \\
+ 11\omega^3 M_\pi^2 t + 4M_\pi^4 \omega^3 ) \\
+ g_A^2 (-18\omega^3 t^2 - 288\omega^5 M_\pi^2 + 60\omega^3 M_\pi^4 + 33\omega^3 t M_\pi^2 + 192\omega^7 \\
+ 64\omega^5 t) + 4\omega^3 M c_2(2t^2 + 6M_\pi^4 - 13M_\pi^2 t) \right] , \quad (3.16) \]

\[m F_\pi^4 h^+(\omega, t) = \frac{g_A^2}{12\omega^3} J_0(\omega)(\omega^2 - M_\pi^2) \left[ g_A^2 M_\pi^2 - 2\omega^2 + 8m\omega^2(c_3 - c_4) \right] \]

\[- \frac{g_A^2}{12\omega^3} J_0(-\omega) \left[ g_A^2 (M_\pi^2 \omega^2 + M_\pi^2 t - 3M_\pi^4 + 2\omega^4) \\
+ (\omega^2 - M_\pi^2)(2\omega^2 - 8\omega^2 m(c_3 - c_4)) \right] \]

\[+ \frac{g_A^4}{24\omega^2} \frac{\partial J_0}{\partial \omega}(-\omega)(4\omega^4 - M_\pi^2 t - 8\omega^2 M_\pi^2 + \omega^2 t + 4M_\pi^4 ) \]

\[- \frac{g_A^4}{32} \frac{\partial K_0}{\partial \omega}(0,t)(-9M_\pi^2 t + 4M_\pi^4 + 2t^2) - \frac{g_A^2}{8} J_0(t)(2t - M_\pi^2) \]

\[- \frac{g_A^4}{1152\omega^3\pi^2} \left[ 2g_A^2 (-12\pi M_\pi^5 - 6\pi M_\pi^3 \omega^2 + 6\pi M_\pi^3 t + 32\omega^5 + 4\omega^3 t \\
- 16\omega^3 M_\pi^2 - 12\omega M_\pi^4 - 3\omega t M_\pi^2 ) \\
+ 3\omega^2(3\omega M_\pi^2 - 6t - 16\pi M_\pi^4) + 192\pi M_\pi^2 \omega^2 m(c_3 - c_4) \right] , \quad (3.17) \]

\[m F_\pi^4 g^-(\omega, t) = \frac{1}{24\omega^3} J_0(\omega) \left[ g_A^4 (t\omega^2 M_\pi^2 + 2\omega^4 M_\pi^2 - 4\omega^2 M_\pi^4 - t M_\pi^4 + 2M_\pi^6 ) \\
+ g_A^2 (-6\omega^2 M_\pi^4 + 12\omega^4 M_\pi^2 + 3\omega^2 M_\pi^2 t - 3\omega^4 t - 12\omega^6 \\
+ 8\omega^2 m(c_3 - c_4)(2\omega^4 - 4M_\pi^2 \omega^2 + 2M_\pi^4 - t M_\pi^2 + \omega^2 t)) \\
+ 6\omega^4 (-M_\pi^2 - 16M_\pi^2 m c_1 + 8\omega^2 m(c_2 + c_3)) \right] \]

\[- \frac{1}{24\omega^3} J_0(-\omega) \left[ g_A^4 (-5t M_\pi^4 - 10\omega^4 M_\pi^2 - 4\omega^2 M_\pi^4 + 3t \omega^2 M_\pi^2 + 6M_\pi^6 \\
+ 8\omega^6 + M_\pi^2 t^2 + 2t\omega^4 ) \\
+ \frac{g_A^2}{4} (6M_\pi^4 - 4\omega^2 M_\pi^2 + \omega^2 t - 3M_\pi^2 t + 4\omega^4 \\
+ 8m (c_3 - c_4)(-2M_\pi^4 + t M_\pi^2 - \omega^2 t - 2\omega^4 + 4\omega^2 M_\pi^2)) \right] \]
\[ m F^4 \bar{h}^{-}(\omega, t) = \frac{1}{1152 \omega^3 \pi^2} \left[ g_A^4 (96 \pi M^5_\pi - 64 \omega^3 M^2_\pi - 12 \omega t M^2_\pi + 8 \omega^3 t - 24 \pi M^3_\pi t \right. \]

\[ \left. - 48 \pi M^3_\pi \omega^2 + 64 \omega^3) \right) + g_A^2 (-18 \omega^3 M^2_\pi + 48 \omega^5 + 36 \pi \omega^4 M_\pi + 11 \omega^3 t + 16 \omega^3 m c_4 (5 M^2_\pi - 4 \omega^2) \right) \]

\[ + 2 \omega^3 (6 M^2_\pi - t + 4 m c_4 (6 M^2_\pi - t)) \right]. \]
We have used the loop functions of ref.[16]. The $e_i$ are scale-independent (using the same procedure to eliminate the chiral logarithms as detailed in (I) for the $d_i$). It is important to stress that we have obtained a more precise representation of the imaginary parts as compared to (I) since in that paper, only the leading terms were included. Here, we also have the next-to-leading order corrections of the unitarity corrections. Clearly, unitarity is perturbatively fulfilled, i.e. $\text{Im} T^{(4)} \sim (\text{Re} T^{(2)})^2$ in a highly symbolic notation, where $T^{(n)}$ refers to the chiral representation of the $\pi N$ amplitude to $n^{th}$ order. It goes without saying that we also expect the corresponding real parts to be given more accurately.

3.3 Counterterm combinations

In the counterterm and loop contributions given above, a set of LECs appears. Most of these only enter in certain combinations and some only lead to quark mass renormalizations of the dimension two LECs. Therefore, it is instructive to work out how many independent local contact terms can contribute to $\pi N$ scattering to fourth order. This can be most easily done based on a dispersive analysis by counting the number of possible subtractions. For that, it is most appropriate to describe the pertinent $T$-matrix in terms of the standard invariant amplitudes $A$ and $B$,

$$T^{\pm}_{\pi N} = A^{\pm} + q^2 B^{\pm},$$

in a highly symbolic notation. The invariant amplitudes are functions of two variables, which one can choose to be $\nu$ and $t$; these count as $\mathcal{O}(q)$ and $\mathcal{O}(q^2)$, respectively. The most general polynomial for the four amplitudes $A^{\pm}, B^{\pm}$ to fourth order commensurate with crossing and the other symmetries thus takes the form

$$\begin{align*}
A_{\text{pol}}^+ &= a_1^+ + a_2^+ t + a_3^+ \nu^2 + a_4^+ t^2 + a_5^+ t \nu^2 + a_6^+ \nu^4, \\
A_{\text{pol}}^- &= \nu (a_1^- + a_2^- t + a_3^- \nu^2), \\
B_{\text{pol}}^+ &= b_1^+ \nu, \\
B_{\text{pol}}^- &= b_1^- + b_2^- t + b_3^- \nu^2. 
\end{align*}$$

(3.21)

Certain combinations of dimension two, three and four LECs are related to the subtraction constants ($a_1^+, \ldots, b_3^-$). We refrain from giving the precise relationship here since it is not needed in what follows. Therefore, in total we have 14 LECs since at third order there is one more related to the Goldberger–Treiman discrepancy, i.e. a local term with a LEC which allows to express the axial–vector coupling $g_A$ in terms of the pion–nucleon coupling $g_{\pi N}$, i.e.

$$g_{\pi N} = \frac{g_A m}{F_\pi} \left( 1 - \frac{2M_\pi^2 d_{18}}{g_A} \right).$$

(3.22)

This term is important if one wants to properly account for the Born terms expressed as a function of the pion–nucleon coupling constant. If one then calculates to orders $q^2, q^3$ and $q^4$, one has to pin down 4, 9 and 14 LECs, respectively. This pattern is quite different from the total number of terms in the Lagrangian allowed at the various orders (7, 23, and 118, respectively); it is a general rule that simple processes do not involve an exorbitant number of LECs. Indeed,
the terms proportional to the dimension four LECs $\tilde{e}_i$ ($i = 19, 20, 21, 22, 35, 36, 37, 38$) only amount to quark mass renormalizations of the dimension two LECs $c_i$ ($i = 1, 2, 3, 4$) via

\begin{align*}
\tilde{c}_1 &= c_1 - 2M^2(\tilde{e}_{22} - 4\tilde{e}_{38}) , \\
\tilde{c}_2 &= c_2 + 8M^2\left(\tilde{e}_{20} + \tilde{e}_{35} - \frac{g_A d_{16}}{8m}\right) , \\
\tilde{c}_3 &= c_3 + 4M^2(2\tilde{e}_{19} - \tilde{e}_{22} - \tilde{e}_{36}) , \\
\tilde{c}_4 &= c_4 + 4M^2(2\tilde{e}_{21} - \tilde{e}_{37}) .
\end{align*}

(3.23)

We have also used these parameters in the one–loop graphs of order $q^4$, although this induces some higher order contributions. This is a very general phenomenon of CHPT calculations at higher orders (for a discussion, see e.g. ref.[17]). There are different ways of determining the LECs. As in (I), we use data from the physical region for doing so. Our first strategy is to fit the renormalized $c_i$, called $\tilde{c}_i$ here, together with the four (combinations of) dimension three LECs $d_1 + d_2, d_3, d_{14} - d_{15}$ and $d_{18}$ and the genuine dimension four LECs $\tilde{e}_{14}, \tilde{e}_{15}, \tilde{e}_{16}, \tilde{e}_{17}, \tilde{e}_{18}$. As enumerated before, we thus have 14 free parameters. In such a fit, we cannot disentangle the $\tilde{c}_i$ into their quark mass dependent and independent pieces without further information from other processes. This defines our best fit. To study the convergence compared to the lower order calculations, we also show the best fit from (I) and a best second order fit based on tree diagrams with the dimension two insertions $\tilde{c}_i$. One can argue that the second order contribution is given by the amplitude up to second order, with the $c$'s taking on their values as given by the best fit at that order. By including the amplitude at third order, the values of the $c$'s will change, but these changes are considered to be effects of third order. (The same is valid of course for the $d$'s, when going from third to fourth order.) One might also be interested in how big the contribution from genuine second and third order terms are (terms really proportional to $q^2$, respectively to $q^3$). In order to address this question, we consider an alternative strategy, in which we fix the $c_i$ ($i = 1, 2, 3, 4$) to their values determined from the best fit up to third order and use the four combinations of dimension four LECs appearing in eq.(3.23) as fit parameters. Of course, this leads to the same number of free parameters, but this second method allows for a clean separation of the contributions from the various orders. Clearly, physical observables do not depend on this reshuffling of fit parameters (modulo higher order corrections effectively included when using the $\tilde{c}_i$ in the loop graphs).

4 Results

4.1 The fitting procedure

There are various possibilities to fix the LECs, a general discussion is given in (I). We proceed here along the same lines as in (I), namely we fit to the phase shifts given by three different partial wave analyses in the low energy region. As input we use the phase shifts of the Karlsruhe (KA85) group [18], from the analysis of Matsinos [19] (EM98) and the solution called SP98 from the VPI/GW group [20]. In contrast to what was done in (I), we do not assign a common error of 3% to the Karlsruhe and VPI phases, but rather mimic the uncertainties.

In the meantime, novel solutions like SM99 have appeared. Since these are not very different from SP98 and we want to have a direct comparison with the results of (I), we use SP98 here. We come back to this later.
of the Matsinos analysis in all cases, which is 1.5% for $S_{31}$, 0.5% for $S_{11}$, 1% for $P_{33}$ and 3.5% for the other $P$–waves. This assignment gives more weight to the better determined larger partial waves and is more natural than one common global error. The LEC $d_{18}$ is fixed by means of the Goldberger–Treiman discrepancy, i.e. by the value for the pion–nucleon coupling constant extracted in the various analyses. The actual values of $g_{\pi N}$ are $g_{\pi N} = 13.4 \pm 0.1$, $13.18 \pm 0.12$, $13.13 \pm 0.03$, for KA85, EM98 and SP98. Throughout, we use $g_A = 1.26$, $F_\pi = 92.4$ MeV, $m = 938.27$ MeV and $M_\pi = 139.57$ MeV. Finally, we remark that we do not use the value of the pion–nucleon $\sigma$–term in the fitting procedure. This has two reasons: First, as noted before, we only want to use information from the physical region to pin down the LECs and second, it is known that the convergence of the chiral series for this quantity is slow [17]. Before presenting the results of the actual fits, we already anticipate that the EM98 data basis will lead to the smallest $\chi^2$ for the following reasons. First, this data base is only covering the low–energy region of pion–nucleon scattering. Also, the representation is available on a denser grid of points in momentum transfer. In contrast, the KA85 and SP98 analyses span a much larger range of energy and thus uncertainties also from higher energies will play a role in the energy range considered here. Furthermore, in (I) we had already discussed that the extraction of the threshold parameters from the SP98 analysis is not unproblematic. Note, however, that the model underlying the EM98 analysis should not be used in the unphysical region, quite in contrast to the dispersion theoretical approach on which the KA85 phase shifts are based.

### 4.2 Phase shifts and threshold parameters

After the remarks of the preceding paragraph, we can now present results. For the KA85 case, we have fitted the data up to 100 MeV pion lab momentum (i.e. 4 points per partial wave at $q_\pi = 40, 60, 79, 97$ MeV). For the analysis of Matsinos, we use 17 points for each partial wave in the range of $q_\pi = 41.4 – 96.3$ MeV. For the VPI SP98 solution, we use the 5 data points in the range between 60 and 100 MeV, which give a stable fit. Of course, we could now extend the fits to higher energies than it was done in (I), but for a better comparison we do not show the results of these extended fits here. As discussed in section 3.3, we have two options for pinning down the LECs. Using strategy one, i.e. working with the $\hat{c}_i$, we call the fits corresponding to the Karlsruhe, Matsinos and VPI analysis, fit 1, 2 and 3, in order. The resulting LECs are given in table 1. Note that the error on the LECs is purely the one given by the fitting routine and is certainly underestimated. We remark that the $\hat{c}_i$ and $\hat{d}_i$ (or combinations thereof) are mostly of natural size, whereas some of the $\hat{e}_i$ come out fairly large. Also, there is some sizeable variation in the actual values of most LECs among the different fits. The resulting S– and P–wave phase shifts are shown in figs. 1 (fit 1), 2 (fit 2) and 3 (fit 3), in order. The corresponding $\chi^2$/dof is 0.40, 0.008 and 0.44 for fits 1, 2 and 3, respectively. The description of the phase shifts is excellent for fit 2. For fits 1 and 3, the $S_{31}$ and $P_{11}$ partial waves at pion momenta above 150 MeV are somewhat off. In all cases, the description of the $P_{33}$ partial wave is improved as compared to the third order calculation. Since we do not fit data below $q_\pi = 40, 41, 60$ MeV (fit 1, 2, 3), the threshold parameters are now predictions. These are shown for the various fits.

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#7 Of course, this might induce some mismatch in the sense that real errors associated to the KA and VPI/GW phases are different from the ones of the Matsinos analysis. We believe, however, that this procedure is preferable to the one using common global errors.
Table 2: Values of the S– and P–wave threshold parameters for the various fits as described in the text in comparison to the respective data. Note that we have extracted $b_{0+}^+$ from the Matsinos phase shifts and thus no uncertainty is given. Units are appropriate inverse powers of the pion mass times 10$^{-2}$.

| Obs. | Fit 1       | Fit 2       | Fit 3       | KA85       | EM98       | SP98       |
|------|-------------|-------------|-------------|------------|------------|------------|
| $a_{0+}^+$ | −0.96 0.45 | 0.27 0.83 | 0.41 ± 0.09 | 0.0 ± 0.1 |            |            |
| $b_{0+}^+$ | −5.31 −4.82 | −7.16 −4.40 | −4.46 −4.46 | −4.83 ± 0.10 |            |            |
| $a_{0+}$   | 9.03 7.71 | 8.67 9.17 | 7.73 ± 0.06 | 8.83 ± 0.07 |            |            |
| $b_{0+}$   | 1.50 1.78 | 1.15 0.77 | 1.56 0.07 | 0.07 ± 0.07 |            |            |
| $a_{1-}$   | −5.66 −5.87 | −4.81 −5.53 | −5.46 ± 0.10 | −5.33 ± 0.17 |            |            |
| $a_{1+}$   | 13.15 13.04 | 13.18 13.27 | 13.13 ± 0.13 | 13.6 ± 0.1 |            |            |
| $a_{1-}$   | −1.25 −1.17 | −0.79 −1.13 | −1.19 ± 0.08 | −1.00 ± 0.10 |            |            |
| $a_{1+}$   | −7.99 −8.21 | −7.54 −8.13 | −8.22 ± 0.07 | −7.47 ± 0.13 |            |            |

The agreement with the threshold parameters based on the chiral amplitude with the ones
based on the approaches underlying the various partial wave analyses is best for fit 2, slightly worse for fit 1 and clearly problematic for some of the parameters of fit 3. The reason for this was already spelled out in (I). The bands for the S–wave scattering lengths $a_i^{0+}$ and $a_i^{-+}$ are as in (I) since the uncertainties extracted there are mostly due to the input and not to the theory. They agree with the recent determinations from the shift and width of pionic hydrogen and deuterium, cf. Fig.2 in ref.[21]. For comparison, we translate our bands on the isoscalar and isovector scattering lengths into the physical ones,

\begin{align}
  a_{\pi^0 p \to \pi^0 n} &= -0.131 \ldots -0.117 \, M^{-1}_\pi \left[ ( -0.128 \pm 0.006 ) M^{-1}_\pi \right], \\
  a_{\pi^- p \to \pi^- p} &= 0.073 \ldots 0.093 \, M^{-1}_\pi \left[ ( 0.0883 \pm 0.0008 ) M^{-1}_\pi \right],
\end{align}

where the experimental numbers (in the square brackets) are taken from ref.[21]. Note, however, that recent progress in calculating $\pi^- p$ atoms in effective field theory lets one expect that the uncertainty due to electromagnetic corrections for the band derived from the hydrogen shift has been underestimated, see e.g. ref.[23]. Furthermore, only recently deuteron wave functions have been obtained precisely enough in an EFT approach to readdress the question of the deuteron shift constraining the elementary $\pi N$ amplitudes. It would also be worthwhile to repeat the EFT calculation of pion–deuteron scattering [22] using our fourth order $\pi N$ amplitudes as input. As argued before, we can study the convergence of the chiral expansion. In figs.1–3, the dot–dashed, dotted, dashed and solid lines refer to the best fits up to first, second, third and fourth order, respectively. Since we used the errors of the Matsinos analysis, it is best to consider fit 2 shown in fig. 2. In most cases, the fourth order corrections are smaller than the third order ones, indicating convergence. This could not be concluded from the third order calculation, compare the discussion in (I) and ref.[16]. Note also that in some partial waves the second order result is close to the data. The resulting values of the $c_i$ are very different from the ones given in table 4. The second order best fit based on the (KA85, EM98, SP98) analysis leads to $c_1 = (-0.81, -0.77, -1.06)$, $c_2 = (2.47, 2.69, 2.36)$, $c_3 = (-3.78, -3.96, -4.04)$ and $c_4 = (2.49, 2.64, 2.35)$ (all in GeV$^{-1}$).

\#8 That these values are very different from the ones based on a one–loop third order amplitude fit was already pointed out in ref.[24]. It is of particular interest to study the convergence of the S–wave scattering lengths, which has been already discussed in ref.[25, 26] estimating LECs from resonance saturation. Our results are summarized in table 3. Although it was already shown in ref.[26] that there are no fourth order corrections to $a_{0+}$, the readjustment of the LECs when going from third to fourth order leads to a small difference. That this difference is so small is also in agreement with ref.[26], where it was argued that the dominant correction to the Weinberg–Tomozawa low–energy theorem is a pion loop effect. For fits 1 and 2, the fourth order correction to the isoscalar S–wave scattering length is fairly small, even for fit 3 the dominant correction is the one from second to third order.

The second option is to keep the dimension two LECs fixed to their value determined from the third order fits and fit the additional four dimension four combinations. This allows for a clean discussion of the various contributions to the chiral expansion. The results for the LECs are shown in table 4.

\#9 The slight differences for the values of the $c_i$ as compared to the ones given in (I) stem from the fact that we use different error bars for the KA85 and SP98 partial waves as explained before.
additional combinations of dimension four LECs are fairly large and vary considerably for the various fits. The resulting S- and P-wave phase shifts are shown in figs. 4 (fit 1*), 5 (fit 2*) and 6 (fit 3*), in order. The corresponding $\chi^2$/dof is 0.50, 0.14 and 0.58 for fits 1*,2* and 3*, respectively. In these plots a different way of looking at the convergence properties of the amplitude is adopted: all four curves are based on the same fit and are thus obtained with the same set of LECs. The dot–dashed, dotted, dashed and solid lines show the contributions from the amplitude up to first, second, third and fourth order, respectively. We remark that the fourth order contributions are mostly small, with the exception of the $P_{11}$ and $P_{13}$ partial waves. The threshold parameters determined from these fits come out very close to the values given before and we thus refrain from adding another table here. Similar remarks hold for the convergence of the S-wave scattering lengths, only that in this way of fixing the LECs there is indeed no contribution to the isovector one from fourth order.

4.3 Sigma term, subthreshold parameters and further comments

In this section, we briefly discuss the so-called subthreshold parameters and the sigma term. Already in (I) we noted that the representation of the chiral amplitude, when pinned down by scattering data, is not very precise in the unphysical region. In particular, the small isoscalar amplitudes are obtained from various contributions, which are individually much larger than their sum. Consequently, this fine balance which is enforced through the fit in the physical region down to the scattering lengths is disturbed because the strict $1/m$ expansion performed here does not properly account for all cuts appearing in the $\pi N$ amplitude. In fact, using our fourth order representation, we do not find an improvement of the subthreshold parameters as given in (I), in some cases even a clear disimprovement. This problem could e.g. be circumvented in the formulation of ref.[11]. Another option is to pin down the LECs inside the Mandelstam triangle [13], which will lead to an improved representation in the unphysical region. This is also reflected in the prediction for the sigma term, which came out rather large in the fits shown in (I), but was considerably different (and consistent with the result from dispersion theory) based on the method used in [13].

We now consider the sigma term, which is the matrix element of the explicit chiral symmetry breaking part of the QCD Lagrangian sandwiched between proton states at zero momentum.
Table 4: Values of the LECs in GeV⁻¹, GeV⁻² and GeV⁻³ for the $c_i$, $\bar{d}_i$ and $\bar{e}_i$, respectively, for the various fits based on the second procedure as described in the text. The * denotes an input quantity.

| LEC   | Fit 1* | Fit 2* | Fit 3* |
|-------|--------|--------|--------|
| $c_1$ | -1.21* | -1.42* | -1.47* |
| $c_2$ | 3.29*  | 3.13*  | 3.26*  |
| $c_3$ | -5.91* | -5.85* | -6.14* |
| $c_4$ | 3.47*  | 3.50*  | 3.50*  |
| $\bar{d}_1 + \bar{d}_2$ | 5.32 ± 0.08 | 5.26 ± 0.05 | 4.90 ± 0.05 |
| $\bar{d}_3$ | -4.37 ± 0.09 | -3.61 ± 0.05 | -4.19 ± 0.07 |
| $\bar{d}_5$ | -0.13 ± 0.06 | -1.03 ± 0.03 | -0.16 ± 0.05 |
| $\bar{d}_{14} - \bar{d}_{15}$ | -9.31 ± 0.18 | -8.70 ± 0.11 | -9.31 ± 0.10 |
| $\bar{d}_{18}$ | -1.47 ± 0.16 | -1.49 ± 0.10 | -0.84 ± 0.06 |
| $\bar{e}_{14}$ | 2.33 ± 0.27 | 2.64 ± 0.14 | 4.19 ± 0.23 |
| $\bar{e}_{15}$ | -2.21 ± 0.30 | -3.33 ± 0.15 | 4.54 ± 0.25 |
| $\bar{e}_{16}$ | 5.69 ± 0.28 | 4.02 ± 0.14 | 2.74 ± 0.24 |
| $\bar{e}_{17}$ | 6.18 ± 0.98 | 5.14 ± 0.53 | 7.20 ± 0.64 |
| $\bar{e}_{18}$ | -1.27 ± 0.98 | -2.56 ± 0.53 | -3.36 ± 0.64 |
| $\bar{e}_{22} - 4\bar{e}_{38}$ | 15.06 ± 0.79 | 7.38 ± 0.40 | 27.72 ± 0.74 |
| $\bar{e}_{20} + \bar{e}_{35} - g_A d_{16}/(8m)$ | -15.28 ± 0.43 | -10.49 ± 0.21 | -17.35 ± 0.36 |
| $2\bar{e}_{19} - \bar{e}_{22} - \bar{e}_{36}$ | -3.58 ± 0.83 | -1.49 ± 0.40 | -25.12 ± 0.69 |
| $2\bar{e}_{21} - \bar{e}_{37}$ | -7.12 ± 1.96 | -1.66 ± 1.10 | -5.00 ± 1.43 |

transfer. While at third order we can directly give the sigma term, $\sigma(0)$, in terms of the LEC $c_1$, this can no longer be done at fourth order due to the appearance of the LEC combination $2e_{115} + e_{116}/2$. These operators contribute to the nucleon mass shift and the sigma term (scalar form factor) as noted before. These contact interactions have no pion matrix–elements and therefore can not appear in the scattering amplitude, even not in higher order loop graphs. We therefore use a more indirect method to determine the sigma term. For that, we consider $\Sigma = F_π^2 \bar{D}^+(\nu = 0, t = 2M_π^2)$ which can be related to $\sigma(0)$ by the venerable low–energy theorem of ref.[27]. There exists a whole family of relations between $\Sigma$ and certain combinations of threshold parameters, as detailed in ref.[28]. These relations have been worked out to third order and should be generalized to fourth order. We will use here the version given in ref.[29],

$$\Sigma = \pi F_π^2 [(4 + 2\mu + \mu^2)a_{0+}^+ - 4M_π^2 b_{0+}^+ + 12\mu M_π^2 a_{1+}^+] + \Sigma_0 $$, \hspace{1cm} (4.2)

with $\Sigma_0 = -12.6$ MeV and $\mu = M_π/m \approx 1/7$. Using the pertinent threshold parameters from the fourth (third) order representation, we find $\Sigma = 65 (62)$ MeV, 73 (79) MeV and 90 (82) MeV
for fits 1, 2 and 3, respectively. A special variant, which also contains some fourth order pieces, has recently been given by Olsson [30],

\[
\Sigma = \left[ F_\pi^2 F(2M_\pi^2) \right],
\]

\[
F(2M_\pi^2) = 14.5 a_{0+}^{1/2} - 5.06 (a_{0+}^{1/2})^2 - 10.13 (a_{0+}^{3/2})^2 - 16.65 b_{0+}^+ - 0.06 a_{1-}^+ + 5.70 a_{1+}^+ - 0.05,
\]

with the quantities on the right–hand–side being given in units of the pion mass. This leads to \( \Sigma = 73 (69) \) MeV, \( 85 (91) \) MeV and \( 104 (93) \) MeV for fits 1, 2 and 3, respectively. We consider the differences between the results based on eqs.(4.2) and (4.3) (and also using the fourth order results for the threshold parameters in the third order representation, eq.(4.2)) as an indication of the size of the fourth order terms. We note that the values we find for the Karlsruhe analysis are consistent with the direct determination based on hyperbolic dispersion relations [31], whereas the results based on the SP98 partial waves lead to a sizeably larger value than advocated by the VPI/GW group [32].

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A Threshold parameters

In this appendix, we give the analytical expressions for the threshold parameters up to fourth order. These read:

\[
a_{0+}^+ = \frac{M_\pi^2 [-g_A^2 + 8m(-2c_1 + c_2 + c_3)]}{16\pi(m + M_\pi)F_\pi^2} + \frac{3g_A^2 m M_\pi^4}{256\pi^2(m + M_\pi) F_\pi^4} \frac{g_A^2 M_\pi^4}{64\pi(m + M_\pi) m^2 F_\pi^2} - \frac{4M_\pi^4 c_1 c_2}{\pi(m + M_\pi) F_\pi^2} + \frac{2m M_\pi^4 c_1 \ell_3}{\pi(m + M_\pi) F_\pi^4} - \frac{g_A M_\pi^4(2\bar{d}_{16} - \bar{d}_{18})}{4\pi(m + M_\pi) F_\pi^2} + \frac{2 M_\pi^4 m(2\bar{e}_{14} + 2\bar{e}_{15} + 2\bar{e}_{16} + 2\bar{e}_{19} + 2\bar{e}_{20} + 2\bar{e}_{35} - \bar{e}_{36} - 4\bar{e}_{38})}{\pi(m + M_\pi) F_\pi^2} \]

\[
- \frac{M_\pi^2 [8 - 3g_A^2 + 2g_A^4 + 4m(2c_1 - c_3)]}{256\pi^3(m + M_\pi) F_\pi^4},
\]

\[
a_{0+}^- = \frac{m M_\pi}{8\pi(m + M_\pi) F_\pi^2},
\]

17
\[
\begin{align*}
\mathcal{A}_0^+ & = \frac{M_\pi^3 (g_A^2 + 32m^2 (\bar{d}_1 + \bar{d}_2 + \bar{d}_3 + 2\bar{d}_5))}{32\pi m (m + M_\pi) F_\pi^2} + \frac{M_\pi^3 m}{64\pi^3 (m + M_\pi) F_\pi^4}, \\
\mathcal{A}_0^+ & = \frac{g_A^2 (4m^2 + 2mM_\pi - M_\pi^2)}{64\pi m^2 (m + M_\pi) F_\pi^2} + \frac{2c_1 (2mM_\pi - M_\pi^2) + (c_2 + c_3) (4m^2 - 2mM_\pi + M_\pi^2)}{8\pi m (m + M_\pi) F_\pi^2} \\
& + \frac{M_\pi (g_A^2 + 8mc_2)}{16\pi m (m + M_\pi) F_\pi^2} + \frac{g_A^2 M_\pi (154m^2 - 18mM_\pi + 9M_\pi^2)}{3072\pi^2 m (m + M_\pi) F_\pi^4} \\
& - \frac{g_A^2 M_\pi (-16m^2 - 2mM_\pi + M_\pi^2)}{256\pi (m + M_\pi) m^4 F_\pi^2} - \frac{2\pi (m + M_\pi) m F_\pi^2}{M_\pi^2 c_1 c_2 (4m^2 - 2mM_\pi + M_\pi^2)} \\
& - \frac{\pi (m + M_\pi) m^2 F_\pi^2}{M_\pi^2 [8m^2 (\bar{d}_{14} - \bar{d}_{15}) - g_A \bar{d}_{18} (M_\pi^2 - 2mM_\pi + 4m^2)]} \\
& - \frac{M_\pi^2 (\bar{e}_{14} + \bar{e}_{15} + \bar{e}_{16}) (-8m^2 + 2mM_\pi - M_\pi^2)}{\pi (m + M_\pi) m^2 F_\pi^2} - \frac{M_\pi^2 (2mM_\pi - M_\pi^2) (\bar{e}_{22} - 4\bar{e}_{38})}{2\pi (m + M_\pi) m F_\pi^2} \\
& - \frac{M_\pi^2 [2(\bar{e}_{20} + \bar{e}_{35} - \frac{g_A m}{4m}) + (2\bar{e}_{19} - \bar{e}_{22} - \bar{e}_{36})] (-4m^2 + 2mM_\pi - M_\pi^2)}{2\pi (m + M_\pi) m F_\pi^2} \\
& - \frac{M_\pi^2}{9216\pi^3 (m + M_\pi) m F_\pi^2} [-432m^2 - 144mM_\pi + 72M_\pi^2 + g_A^2 (44m^2 + 54mM_\pi - 27M_\pi^2) + g_A^2 m^2 (-88 + 192\pi) - 36mM_\pi + 18M_\pi^2 + mc_1 (1248m^2 - 144mM_\pi + 72M_\pi^2) - 24m^2 c_2 + mc_3 (-768m^2 + 72mM_\pi - 36M_\pi^2)] ,
\end{align*}
\]

\[
\mathcal{A}_0^+ = \frac{2m^2 - 2mM_\pi + M_\pi^2}{32\pi m M_\pi (m + M_\pi) F_\pi^2} \left( 1 - \frac{2g_A^2}{16\pi (m + M_\pi) F_\pi^2} \right) + \frac{M_\pi g_A^2 (-10m^2 - 2mM_\pi + M_\pi^2)}{128\pi m^3 (m + M_\pi) F_\pi^2} - \frac{M_\pi c_4}{4\pi (m + M_\pi) F_\pi^2} \\
+ \frac{M_\pi [(\bar{d}_1 + \bar{d}_2 + \bar{d}_3) (6M_\pi^2 - 2mM_\pi + M_\pi^2) + \bar{d}_5 (4m^2 - 4mM_\pi + 2M_\pi^2)]}{4\pi m (m + M_\pi) F_\pi^2} \\
- \frac{M_\pi (4m^2 + 6mM_\pi - 3M_\pi^2 - 14g_A^2 m^2)}{768\pi^3 m (m + M_\pi) F_\pi^4} \\
- \frac{3g_A^2 M_\pi^2 (\bar{d}_1 + \bar{d}_2 + 3\bar{d}_3 + 2\bar{d}_5 + g_A \bar{d}_{18})}{64\pi (m + M_\pi) m^2 F_\pi^2} + \frac{M_\pi^2 (18 + 69\pi g_A^2 + (4 - 12\pi) g_A^4)}{2304\pi^3 (m + M_\pi) F_\pi^4} ,
\]

\[
\mathcal{A}^- = -\frac{g_A^2 m}{24\pi M_\pi (m + M_\pi) F_\pi^2} + \frac{3 - 2g_A^2 + 8mc_4}{48\pi (m + M_\pi) F_\pi^2}.
\]
\[
\begin{align*}
&+ \frac{M_\pi [3 - 3g_A^2 + 24mc_4 - 32m^2(\bar{d}_1 + \bar{d}_2) + 16m^2g_A\bar{d}_{18}]}{96\pi (m + M_\pi)F^2_\pi} \\
&- \frac{M_\pi m^2 [3 + g_A^2(21 + 24\pi) + g_A^4(2 + 24\pi)]}{3456\pi^3 (m + M_\pi)F^4_\pi} \\
&- \frac{g_A^2M^2_\pi}{192\pi (m + M_\pi)m^2F^2_\pi} - \frac{M^2_\pi c_1}{6\pi (m + M_\pi)mF^2_\pi} + \frac{M^2_\pi (\bar{d}_1 + \bar{d}_2 + 3\bar{d}_3 + 2\bar{d}_5 + g_A\bar{d}_{18})}{6\pi (m + M_\pi)F^2_\pi} \\
&+ \frac{2mM^2_\pi (2\bar{e}_{17} + 2\bar{e}_{18} + 2\bar{e}_{21} - \bar{e}_{37})}{3\pi (m + M_\pi)F^2_\pi} \\
&- \frac{M^2_\pi}{6912\pi^3 (m + M_\pi)F^4_\pi} [18 + g_A^2(72 - 33\pi) + g_A^4(4 + 30\pi) + 96\pi g_A^2mc_3] \\
&+ (176 - 96\pi)g_A^2mc_4], \\
&= - \frac{g_A^2m}{12\pi M_\pi (m + M_\pi)F^2_\pi} \\
&- \frac{g_A^2}{12\pi (m + M_\pi)F^2_\pi} \\
&+ \frac{M_\pi [c_2 + 2m(\bar{d}_{14} - \bar{d}_{15} + g_A\bar{d}_{18})]}{6\pi (m + M_\pi)F^2_\pi} - \frac{mM_\pi g_A^2(231\pi + g_A^2(112 + 96\pi))}{13824\pi^3 (m + M_\pi)F^4_\pi} \\
&+ \frac{g_A^2M^2_\pi}{192\pi (m + M_\pi)m^2F^2_\pi} + \frac{M^2_\pi (2c_1 - c_2 - c_3)}{8\pi (m + M_\pi)mF^2_\pi} + \frac{M^2_\pi (3(\bar{d}_{14} - \bar{d}_{15}) + 2g_A\bar{d}_{18})}{6\pi (m + M_\pi)F^2_\pi} \\
&+ \frac{2mM^2_\pi (-4\bar{e}_{14} - 2\bar{e}_{15} - 2\bar{e}_{19} + \bar{e}_{22} + \bar{e}_{36})}{3\pi (m + M_\pi)F^2_\pi} \\
&- \frac{M^2_\pi}{6912\pi^3 (m + M_\pi)F^4_\pi} [-36 + g_A^2(133 - 48\pi) + g_A^4(74 + 12\pi)] \\
&- 6m(52c_1 - c_2 - 32c_3(1 + g_A^2\pi) + 32c_4)], \\
&= - \frac{mg_A^2}{24\pi M_\pi (m + M_\pi)F^2_\pi} \\
&- \frac{g_A^2}{24\pi (m + M_\pi)F^2_\pi} \\
&- \frac{M_\pi [3g_A^2 + 32m^2(\bar{d}_1 + \bar{d}_2) - 16m^2g_A\bar{d}_{18}]}{96\pi m (m + M_\pi)F^2_\pi} - \frac{M_\pi m [3 + g_A^2(21 - 18\pi) + g_A^4(2 - 12\pi)]}{3456\pi^3 (m + M_\pi)F^4_\pi} \\
&- \frac{g_A^2M^2_\pi}{48\pi (m + M_\pi)m^2F^2_\pi} + \frac{M^2_\pi c_1}{12\pi (m + M_\pi)mF^2_\pi} + \frac{M^2_\pi (\bar{d}_1 + \bar{d}_2 + 3\bar{d}_3 + 2\bar{d}_5 + g_A\bar{d}_{18})}{6\pi (m + M_\pi)F^2_\pi} \\
&+ \frac{mM^2_\pi (-2\bar{e}_{17} - 2\bar{e}_{18} - 3\bar{e}_{21} + \bar{e}_{37})}{3\pi (m + M_\pi)F^2_\pi} \\
&+ \frac{M^2_\pi}{6912\pi^3 (m + M_\pi)F^4_\pi} [-18 + g_A^2(36 + 141\pi) + g_A^4(-4 + 42\pi)] \\
&+ 8mg_A^2(-12c_3 + (12\pi + 11)c_4)] , \\
&= \frac{g_A^2m}{24\pi M_\pi (m + M_\pi)F^2_\pi}
\end{align*}
\]
\[
\begin{align*}
&+ \frac{g_A^2 - 4mc_3}{24\pi(m + M_\pi)F_\pi^2} \\
&+ \frac{M_\pi[3g_A^2 + 16mc_2 - 16m^2(\bar{d}_{14} - \bar{d}_{15} + g_A\bar{d}_{18})]}{96\pi m(m + M_\pi)F_\pi^2} - \frac{M_\pi mg_A^2(231\pi + g_A^2(-56 + 96\pi))}{13824\pi^2(m + M_\pi)F_\pi^4} \\
&+ \frac{g_A^2 M_\pi^2}{48\pi(m + M_\pi) m^2 F_\pi^2} - \frac{g_A M_\pi^2 \bar{d}_{18}}{6\pi (m + M_\pi) F_\pi^2} + \frac{2mM_\pi^2(-4\bar{e}_{14} - 2\bar{e}_{15} - 2\bar{e}_{19} + \bar{e}_{22} + \bar{e}_{36})}{3\pi(m + M_\pi) F_\pi^2} \\
&- \frac{M_\pi}{6912\pi^3(m + M_\pi) F_\pi^4}[-36 + g_A^2(133 + 24\pi) + g_A^4(-10 + 66\pi)] \\
&+ 6m(-52c_1 + c_2 + (32 - 16\pi g_A^2)c_3 + 16\pi g_A^2 c_4)] .
\end{align*}
\] (A.8)

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Figures

Figure 1: Fits and predictions for the KA85 phase shifts as a function of the pion laboratory momentum $q_\pi$ using strategy one as explained in the text. Fitted in each partial wave are the data between 40 and 97 MeV (filled circles). For higher and lower energies, the phases are predicted as shown by the solid lines. The other lines refer to the best fits at the various orders as explained in the text.
Figure 2: Fits and predictions for the EM98 phase shifts as a function of $q_{\pi}$ using strategy one as explained in the text. Fitted in each partial wave are the data between 41 and 97 MeV (filled circles). For higher and lower energies, the phases are predicted as shown by the solid lines. The other lines refer to the best fits at the various orders as explained in the text.
Figure 3: Fits and predictions for the SP98 phase shifts as a function of the pion laboratory momentum $q_\pi$ using strategy one as explained in the text. Fitted in each partial wave are the data between 60 and 100 MeV (filled circles). For higher and lower energies, the phases are predicted as shown by the solid lines. The other lines refer to the best fits at the various orders as explained in the text.
Figure 4: Fits and predictions for the KA85 phase shifts as a function of the pion laboratory momentum $q_\pi$ using strategy two as explained in the text. Fitted in each partial wave are the data between 40 and 97 MeV (filled circles). For higher and lower energies, the phases are predicted. The various lines refer to the contributions from the various orders as explained in the text.
Figure 5: Fits and predictions for the EM98 phase shifts as a function of $q_\pi$ using strategy two as explained in the text. Fitted in each partial wave are the data between 41 and 97 MeV (filled circles). For higher and lower energies, the phases are predicted as shown by the solid lines. The various lines refer to the contributions from the various orders as explained in the text.
Figure 6: Fits and predictions for the SP98 phase shifts as a function of the pion laboratory momentum $q_\pi$ using strategy two as explained in the text. Fitted in each partial wave are the data between 60 and 100 MeV (filled circles). For higher and lower energies, the phases are predicted as shown by the solid lines. The various lines refer to the contributions from the various orders as explained in the text.