Asymptotic Form of Gopakumar–Vafa Invariants from Instanton Counting

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Abstract

We study the asymptotic form of the Gopakumar–Vafa invariants at all genera for Calabi–Yau toric threefolds which have the structure of fibration of the $A_n$ singularity over $\mathbb{P}^1$. We claim that the asymptotic form is the inverse Laplace transform of the corresponding instanton amplitude in the prepotential of $\mathcal{N} = 2$ $SU(n+1)$ gauge theory coupled to external graviphoton fields, which is given by the logarithm of the Nekrasov’s partition function.
1 Introduction

In this article we study the geometric engineering of the four-dimensional $\mathcal{N} = 2$ $SU(n+1)$ gauge theory without matter hypermultiplets. It is known that the gauge theory can be realized by compactifying the type IIA string on a certain Calabi–Yau threefold and taking a certain limit \cite{1}. Such a Calabi–Yau threefold must satisfy two conditions: it must have $A_n$-singularity so that the gauge group is $SU(n+1)$; it must be a fibration over the Riemann surface of genus zero (i.e. $\mathbb{P}^1$) so that the gauge theory has the asymptotic freedom. The limit is a double scaling limit such that the exceptional curves of the $A_n$ singularity shrink and the base $\mathbb{P}^1$ expands simultaneously. In this limit, the free energy of topological strings on the Calabi–Yau threefold becomes the prepotential of the gauge theory. The implication of this phenomenon is that the worldsheet instanton correction should reproduce the spacetime instanton correction in the gauge theory and the precise relation between them is what we derive in this article.

In the language of mathematics we study the asymptotic form of the Gopakumar–Vafa invariants at all genera for Calabi–Yau threefolds which are smooth toric varieties and possess the structure of the fibration of $A_n$-singularity over $\mathbb{P}^1$. Recently, Iqbal and Kashani-Poor showed that the topological string amplitude obtained by the method of the geometric transition \cite{2,3,4} agrees with Nekrasov’s partition function for instanton counting \cite{5,6} in the limit which they call field theory limit. This limit involves the limit of the string coupling (the genus expansion parameter) as well as the usual limit of the Kähler parameters. By taking the logarithm, one can see that the generating function of the Gromov–Witten invariants of the Calabi–Yau threefold agrees with the logarithm of Nekrasov’s partition function \cite{7} in this limit: the genus zero part of the generating function corresponds to the Seiberg–Witten prepotential \cite{8} and the higher genus parts correspond to the effects due to the external graviphoton fields since the genus expansion of the former matches the expansion in terms of $\hbar$ in the latter in the field theory limit. From this relation, we derive the asymptotic form of the Gopakumar–Vafa invariants of the $A_n$-fibration over $\mathbb{P}^1$ at all genera: it is obtained as the inverse Laplace transform of the corresponding term in the logarithm of Nekrasov’s partition function for $SU(n+1)$ instanton counting.

The asymptotic form of the Gopakumar–Vafa invariants was first studied in the quintic case \cite{9}. Other cases studied so far are: the canonical bundle of $\mathbb{P}^2$ and other one modulus local mirror systems with Picard-Fuchs equations given by Meijer’s equation \cite{10}; the canonical
bundle of Hirzebruch surface $F_0, F_1, F_2$ \[ \mathbb{P}^1 \]; the canonical bundle of $F_2$ blown up at 1,2,3-points \[ \mathbb{P}^1 \]. The last two cases are the results of the geometric engineering of $SU(2)$ gauge theory. All of these results concern Gopakumar–Vafa invariants at genus zero. To our knowledge, our result is the first for the higher genus cases.

The organization of the paper is as follows. In section 2 we review Nekrasov’s partition function \[ [7, 12] \]. Section 3 is devoted to the description of the $A_n$-fibration over $P^1$. In section 4 we review the results of Iqbal and Kashani-Poor \[ [5, 6] \]. In section 5 we derive the asymptotic form of the Gopakumar–Vafa invariants. In section 6 and in appendix A we will test our results in the case of $A_2$-fibration over $P^1$. The data of local B-model is included in appendix B.

## 2 Nekrasov’s Partition Function

The closed formula for the Seiberg–Witten prepotential of the $\mathcal{N} = 2$ $SU(n+1)$ gauge theory was derived by Nekrasov \[ [7] \] and its mathematical proof was given by Nakajima–Yoshioka \[ [12] \]. The instanton correction part of the prepotential is given by

\[
-\hbar^2 \log Z_{\text{Nekrasov}}^{A_n} \big|_{\hbar=0}.
\]

Here $Z_{\text{Nekrasov}}^{A_n}$ is Nekrasov’s partition function for instanton counting. The terms with higher order in $\hbar$ correspond to the effect of the external graviphoton fields. Before giving the form of $Z_{\text{Nekrasov}}^{A_n}$, let us explain notations. We use the letter $R, R_i$ for a partition (or a Young diagram) and $\mu_i, \mu_{i,j}$ for its parts: $R = (\mu_1, \mu_2, \ldots), R_i = (\mu_{i,1}, \mu_{i,2}, \ldots)$. $l(R)$ denotes the weight of $R$ (the number of boxes of the Young diagram) and $d(R)$ the length of $R$ (the number of rows of the Young diagram). $R^t$ denotes the conjugate partition of $R$ (the transposed Young diagram). $\kappa(R) := \sum_{i=1}^{d(R)} \mu_i (\mu_i - 2i + 1)$.

Nekrasov’s partition function is

\[
Z_{\text{Nekrasov}}^{A_n} = \sum_{R_1, \ldots, R_{n+1}} \Lambda^{l(R_1) + \cdots + l(R_{n+1})} \prod_{i,j=1}^{n+1} \prod_{k,l=1}^{\infty} \frac{a_{i,j} + \hbar (\mu_{i,k} - \mu_{j,l} + l - k)}{a_{i,j} + \hbar (l - k)}.
\]

The summation is over $n+1$ partitions $R_1, \ldots, R_{n+1}$. $\Lambda$ is the dynamical scale. $a_{i,j} = a_i - a_j$, where $a_1, \ldots, a_{n+1}$ are the vacuum expectation values of the complex scalar fields in the gauge multiplet of unbroken $U(1)^n \subset SU(n+1)$ of the $\mathcal{N} = 2$ $SU(n+1)$ gauge theory. From the mathematical viewpoint, Nekrasov’s partition function is the integration in the equivariant
cohomology over the moduli space of instantons on $\mathbb{R}^4$ and $a_1, \ldots, a_{n+1}$ are the generators of the symmetric algebra of the dual of $\bigoplus_n u(1) \subset su(n+1)$.

Note that the seemingly infinite product should read an abbreviated form of the finite product: for two partitions $R_i, R_j$,

$$\prod_{k,l=1}^{\infty} \frac{a_{i,j} + \hbar(\mu_{i,k} - \mu_{j,l} + l - k)}{a_{i,j} + \hbar(l - k)} = \prod_{(k,l) \in R_i} \frac{1}{a_{i,j} + \hbar(\mu_{i,k} - l + \mu_{j,l}^\vee - k + 1)} \prod_{(k,l) \in R_j} \frac{1}{a_{i,j} + \hbar(-\mu_{i,l}^\vee + k - \mu_{j,k} + l - 1)}.$$  \hspace{1cm} (3)

Here $\mu_{i,k}^\vee$ (resp. $\mu_{j,k}^\vee$) is the $k$-th part of $R_i^t$ (resp. $R_j^t$), i.e. $R_i^t = (\mu_{i,1}^\vee, \mu_{i,2}^\vee, \ldots)$. $(k, l) \in R$ means that there is a box in the Young diagram $R$ at the place of $k$-th row and $l$-th column. It becomes important later that all the factors appear only in the denominator, not in the numerator when we derive the asymptotic form of the Gopakumar–Vafa invariants.

Nekrasov’s partition function is invariant under the action of the Weyl group of $A_n$ (the symmetric group $S_{n+1}$). It is also invariant under the $\mathbb{Z}_2$ action, which is generated by $(a_1, 2, \ldots, a_{n,n+1}) \to (-a_1, 2, \ldots, -a_{n,n+1})$ (this $\mathbb{Z}_2$ action coincides with the Weyl group action in the case of $A_1$). Some of these symmetries will later appear in the result of the asymptotic form of the Gopakumar–Vafa invariants.

### 3 \textit{A}_n\textit{-fibration over }\mathbb{P}^1

In this section we describe the $A_n$-fibration over $\mathbb{P}^1$ ($n \geq 1$). By this term, we mean the smooth, Calabi–Yau (i.e., the canonical bundle of which is trivial), toric variety of complex three dimensions which has the structure of the fibration of the minimal resolution of the $A_n$-singularity over $\mathbb{P}^1$.

There exist $(n+2)$ different such Calabi–Yau toric threefolds. We label them by an integer $m$ ($-n+1 \leq m \leq 2$) and call it $X_{A_n}^m$. $X_{A_n}^m$ is specified by the polytope

$$\Delta_{A_n}^m = [(0,1), (-m,-1), (-1,0), (n,0)] \subset \mathbb{R}^2$$  \hspace{1cm} (4)

with the following triangulation; let $v_i$ ($1 \leq i \leq n+4$) be the integral points:

$$v_1 = (0,1), v_2 = (-m,-1), v_3 = (-1,0), v_4 = (0,0), \ldots, v_{n+4} = (n,0);$$  \hspace{1cm} (5)
then the triangulation is such that its 2-simplices are \([v_1, v_i, v_{i+1}]\) and \([v_2, v_i, v_{i+1}]\) \((3 \leq i \leq n+3)\). (see Figure \[\text{I}\] for \(n = 2\) example). Here \([v_j, \ldots, v_k]\) is the convex hull of the vectors \(v_j, \ldots, v_k\).

The fan for \(X^m_{A_n}\) is such that its section at the height 1 is the polytope \(\Delta^m_{A_n}\).

A basis of \(H_2(X^m_{A_n}; \mathbb{Z})\) consists of the homology classes of the base space \(\mathbb{P}^1 =: C_B\), and \(n\)-exceptional curves of the fiber space \(C_j\) \((1 \leq j \leq n)\). We can calculate \(H_2(X^m_{A_n}; \mathbb{Z})\) using the spectral sequence argument \([13]\) (chapter 2) and the result is as follows. Each interior point corresponds to a compact surface and such surfaces generate \(H_4(X^m_{A_n}; \mathbb{Z})\) freely. Each interior edge in \(\Delta^m_{A_n}\) corresponds to a torus invariant curve \(\mathbb{P}^1\) and such \(\mathbb{P}^1\)s generate \(H_2(X^m_{A_n}; \mathbb{Z})\). Let \(C_{ij}\) be the \(\mathbb{P}^1\) corresponding to the interior edge spanned by \(v_i\) and \(v_j\). Then there are two relations for each interior point \(v_i\):

\[
\sum_j (v_j - v_i) [C_{ij}] = 0 \quad (4 \leq i \leq n + 3).
\]

Here the summation is over \(j\) such that \(v_i\) and \(v_j\) span a 1-simplex. Therefore \([C_B]\) corresponds to the edge \([v_3, v_4]\) for \(m = 1, 2\), to both \([v_{r+3}, v_{r+4}]\) and \([v_{r+4}, v_{r+5}]\) for \(m = -2r\) \((0 \leq r \leq \left\lfloor \frac{m-1}{2} \right\rfloor)\), to \([v_{r+3}, v_{r+4}]\) for \(m = -2r + 1\) \((1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor)\); the exceptional curve \([C_i]\) corresponds to \([v_1, v_{i+3}]\) and \([v_2, v_{i+3}]\) \((1 \leq i \leq n)\) (more precisely, we should say that we define the order \(i\) of \(C_i\) as such).

The generating function of the Gromov–Witten invariants for \(A_n\)-fibration over \(\mathbb{P}^1\) denoted by \(X^m_{A_n}\), takes the following form:

\[
\mathcal{F}_{GW}(X^m_{A_n}) = \sum_{g=0}^{\infty} g^2 \mathcal{F}_{GW}^g(X^m_{A_n}),
\]

\[
\mathcal{F}_{GW}^g(X^m_{A_n}) = \sum_{d_B,d_1,\ldots,d_n \geq 0} N_{g,d_B,d_1,\ldots,d_n} q_B^{d_B} q_1^{d_1} \cdots q_n^{d_n}.
\]

\(N_{g,d_B,\ldots,d_n}\) is the genus \(g\), 0-pointed Gromov–Witten invariant of \(X^m_{A_n}\) for the homology class \(d_B[C_B] + d_1[C_1] + \cdots + d_n[C_n]\). \(t_B\) (resp. \(t_i\)) is the Kähler parameter for the 2-cycle \([C_B]\) (resp. \([C_i]\) \((1 \leq i \leq n)\)). \(g_s\) is the variable for the genus expansion, and is identified with the string coupling.

Gromov–Witten invariants are generically rational numbers, but the generating function can be expressed in terms of integral invariants called the Gopakumar–Vafa invariants \([14, 15]\).
In the case of the $A_n$-fibration over $\mathbf{P}^1$, the generating function is written as follows:

$$F_{GW}(X^m_{A_n}) = \sum_{g=0}^{\infty} \sum_{d_0, d_1, \ldots, d_n \geq 0} \sum_{k=1}^{\infty} \frac{n^g_{d_0, d_1, \ldots, d_n}}{k} \left(2 \sin \frac{kq_s}{2}\right)^{2g-2} \left(q_B \prod_{i=1}^{d_0} q_i \cdot \prod_{j=1}^{d_1} q_j \cdot \prod_{i+j \geq 2} q_i q_j\right)^k. \quad (8)$$

Here $n^g_{d_0, d_1, \ldots, d_n}$ is the Gopakumar–Vafa invariant of $X^m_{A_n}$ for the homology class $d_B[C_B] + d_1[C_1] + \cdots + d_n[C_n]$ and for genus $g$.

## 4 Topological String Amplitudes

In this section we briefly review some of Iqbal and Kashani-Poor’s results: the topological string amplitude for $A_n$-fibration over $\mathbf{P}^1$ and its field theory limit [5].

It is conjectured that the topological string amplitude obtained by the geometric transition and the Chern–Simons theory gives the generating function of Gromov–Witten invariants [16]. For the $A_n$-fibration over $\mathbf{P}^1$,

$$\log Z(X^m_{A_n})_{q=e^{ig_s}} = F_{GW}(X^m_{A_n}). \quad (9)$$

Here $q$ is a parameter which should be identified with $\exp\left(\frac{2\pi i}{N+k}\right)$ in the $SU(N)$ Chern–Simons theory but in this context it is just a formal variable. $i$ denotes the imaginary unit. $g_s$ is the parameter of the genus expansion appeared in [7][8].

Iqbal and Kashani-Poor derived the topological string amplitude using certain identities on the summation over partitions [5][6]. The proof of the identities appeared later in the paper by Zhou [17] (Theorem 8.1). The topological string amplitude is

$$Z(X^m_{A_n}) = Z_{d_B=0} \cdot Z_{d_B \geq 1},$$

$$Z_{d_B=0} := \prod_{1 \leq i < j \leq n+1} K(q_{i,j})^2, \quad (10)$$

$$Z_{d_B \geq 1} := \sum_{R_1, \ldots, R_{n+1}} \left[ (-1)^m \sum_{i=1}^{n+1} l(R_i) q_i^{\sum_{j=1}^{n+3} \frac{m+1}{2} \kappa(R_i)} q_B^{\sum_{j=1}^{n+3} l(R_i)} \times \prod_{1 \leq i \leq \frac{m+1}{2}} q_i^{\frac{-m+2-2i}{m+2i} \sum_{k=1}^{i} l(R_k)} \prod_{i \leq n} q_i^{2m+2i} \sum_{k=1}^{n+1} l(R_k)} \times \prod_{1 \leq i \leq n+1} W_{R_i} q_i^{2} \prod_{1 \leq i < j \leq n+1} g_{R_i, R_j} (q_{i,j}, q)^2. \right] \quad (11)$$

(11)
Here \( R_1, \ldots, R_{n+1} \) are partitions, \( q_{i,j} := \prod_{k=i}^{j-1} q_k. \) \( \kappa(R) = \sum_{i=1}^{d(R)} \mu_i(\mu_i - 2i + 1). \)

\[
K(x) := \exp \left[ \sum_{k=1}^{\infty} \frac{q^k}{k(q^k - 1)^2} x^k \right].
\]

\[
W_R(q) := q^{2d(R)} \prod_{1 \leq i < j \leq d(R)} \frac{[\mu_i - \mu_j + j - i]}{[j - i]} \prod_{i=1}^{d(R)} \prod_{j=1}^{d(R)} \frac{1}{[j - i + d(R)]}, \quad ([x] := q^x - q^{-x}).
\]

\[
g_{R_1, R_2}(x; q) = \prod_{(i,j) \in R_1} \frac{1}{(1 - xq^{\mu_i-j+\mu_j-i+1})} \prod_{(i,j) \in R_2} \frac{1}{(1 - xq^{-\mu_i-j+\mu_j-i+1})}.
\]

Note that we can deal with the Gopakumar–Vafa invariants with \( d_B = 0 \) and \( d_B \geq 1 \) separately. \( Z_{d_B=0} \) gives the Gopakumar–Vafa invariants with \( d_B = 0 \) while \( Z_{d_B \geq 1} \) gives those with \( d_B \geq 1 \). This is because \( Z_{d_B=0} \) does not depend on \( q_B \) and \( Z_{d_B \geq 1} = 1 + O(q_B) \). From \( Z_{d_B=0} \) one can easily read the Gopakumar–Vafa invariants for \( d_B = 0 \): \( n_{d_B=0,d_1,\ldots,d_n}^q = -2 \) when \( d_i = d_{i+1} = \cdots = d_j = 1 \) \( (1 \leq i \leq j \leq n) \) and other \( d_k \)'s are zero, and \( n_{d_B=0,d_1,\ldots,d_n}^q = 0 \) otherwise. In other words, the value of \( n_{d_B=0,d_1,\ldots,d_n}^q \) is nonzero only for a homology class corresponding to a positive root under the identification of \([C_1], \ldots, [C_n]\) with the simple roots of the Lie algebra \( A_n \).

\( Z_{d_B \geq 1} \) in (13) is also written in the following form:

\[
Z_{d_B \geq 1} = \sum_{R_1, \ldots, R_{n+1}} \left[ (-1)^{n+m+1} \prod_{i=1}^{n+1} l(R_i) \right] 2^{-2(n+1)} \prod_{i=1}^{n+1} l(R_i) q^{\frac{n+m-1}{2} \sum_{i=1}^{n+1} \kappa(R_i)} \prod_{1 \leq k \leq \left[ \frac{n+m}{2} \right]} q_k \prod_{l \leq k \leq n} q_k^{(-n-1+k)(l(R_1)+\cdots+l(R_k)+m-2+k)(l(R_{k+1})+\cdots+l(R_{n+1}))} \prod_{1 \leq i,j \leq n} \prod_{1 \leq k,l} \frac{\sinh^{\frac{\beta}{2}}(a_{i,j} + h(\mu_i-k + l-k))}{\sinh^{\frac{\beta}{2}}(a_{i,j} + h(l-k))}.
\]

Here \( \beta, a_i, h \) are introduced as

\[
q_i = e^{-\beta(a_i-a_{i+1})}, \quad q = e^{-\beta h},
\]

and \( a_{i,j} := a_i - a_j \). The last product in (15) is equal to the factor that appears in the Nekrasov’s complete string partition function \([7, 5]\). The product is the finite product in the same manner as we mentioned in section 2.

The field theory limit is the limit \( \beta \to 0 \) with

\[
q_B = (-1)^{n+m+1}(\beta \Lambda)^{2(n+1)}, \quad t_i = -\beta a_{i,i+1} (1 \leq i \leq n), \quad q = e^{-\beta h}.
\]
Taking the field theory limit, $Z_{d_B \geq 1}$ becomes

$$\lim_{\beta \to 0} Z_{d_B \geq 1} = Z_{\text{Nekrasov}}^A.$$  \hfill (18)

Therefore, the field theory limit is the limit where the amplitude of the four-dimensional theory is reproduced from the topological strings.

Let us summarize the correspondence between the parameters in the topological string amplitude and those in the four-dimensional gauge theory (Nekrasov’s partition function). The Kähler parameters $t_i (1 \leq i \leq n)$ of the fiber are proportional to the vacuum expectation value of the complex scalar $a_{i,i+1}$ in the gauge theory in four dimensions. The genus expansion parameter $g_s$ is proportional to the parameter $\hbar$. In the recent work of Eguchi and Kanno \[18\], the parameter $\beta$ is identified with the radius of the fifth-dimensional circle in the five-dimensional gauge theory.

The list below is the identification of the notation of Iqbal–Kashani-Poor \[5\] with ours.

| Iqbal–Kashani-Poor \[5\] | Here |
|-------------------------|------|
| $\prod_k (1 - q^k x)^{-C_k(R_1,R_2)}$ | $g_{R_1,R_2}(x,q)$ |
| $N$ | $n + 1$ |
| $m$ | $m + n - 1$ |
| $t_F(Q_F)$ | $t_{n+1-i}(q_{n+1-i})$ |
| $\{a_1, \ldots, a_{N-1}\}$ | $\{a_n, \ldots, a_1\}$ |
| $\{R_1, \ldots, R_N\}$ | $\{R_{n+1}, \ldots, R_1\}$ |

Remark: The relation between the three point vertex amplitudes in \[5\] and those appeared in the recent paper of Aganagic et al. \[3\] is $V_{R_1,R_2,R_3} = q^{\frac{\gamma(R_3)}{2}} C_{R_1,R_2,R_3}.$

5 Asymptotic Form of the Gopakumar–Vafa Invariants

In this section, we derive the asymptotic form $r^{(g)}_{d_B}(d_1, \ldots, d_n)$ of the Gopakumar–Vafa invariants $n_{d_B,d_1,\ldots,d_n}^g$ for $d_B \geq 1$.

Let us state the result first: the asymptotic form is given by

$$r^{(g)}_{d_B}(d_1, \ldots, d_n) = (-1)^{(n+m+1)d_B} L_{d_1,a_{1,2}}^{-1} \circ \cdots \circ L_{d_n,a_{n,n+1}}^{-1} F^{(g)}_{d_B}(a_{1,2}, \ldots, a_{n,n+1}).$$  \hfill (19)

This formula holds in the region

$$d_B \geq 1, \quad d_1, \ldots, d_n \gg d_B(g+1).$$  \hfill (20)
Here \( L_{(x,s)} \) is the Laplace transform, i.e. \( L_{(x,s)}(f(x)) = \int_0^\infty dx \ e^{-sx}f(x) \) and \( L_{(x,s)}^{-1} \) is its inverse. \( F_k^{(g)}(a_1, \ldots, a_{n,n+1}) \) is defined from the Nekrasov’s partition function as follows:

\[
\log Z_{\text{Nekrasov}}^{A_n} = \sum_{g=0}^{\infty} (\eta h)^{2g-2} \sum_{k=1}^{\infty} A^{2(n+1)k} F_k^{(g)}(a_1, \ldots, a_{n,n+1}). \tag{21}
\]

It should be regarded as the function in \( n \) variables \( a_1, \ldots, a_{n,n+1} \).

There are several remarks.

1. \( F_k^{(g)}(a_1, \ldots, a_{n,n+1}) \) takes the following form

\[
F_k^{(g)}(a_1, \ldots, a_{n,n+1}) = \sum_{\sum k_{ij} = 2(n+1)k + 2g^{-2}} A_{\{k_{ij}\}} \prod_{1 \leq i \leq j \leq n} \frac{1}{a_{i,j}^{k_{ij}}} \tag{22}
\]

where \( A_{\{k_{ij}\}} \) is a constant prefactor. The counting of the degree \(-2(n+1)k - 2g + 2\) in \( a_{i,j} \)'s in the right-hand side is as follows. Assign the degree 1 to both \( a_{i,j} \)'s and \( h \) in Nekrasov’s partition function \[2\]. Then the term with \( l(R_1) + \cdots + l(R_{n+1}) = k \) should have the degree \(-2(n+1)k\). Subtracting the degree \( 2g - 2 \) of \( h \), we obtain the degree \(-2(n+1)k - 2g + 2\). Note that we can perform the inverse Laplace transform because \( a_{i,j} \)'s do not appear in the numerator in \[22\].

2. The appearance of the inverse Laplace transform is a very natural consequence. If we just rewrite \( q_i \) as \( e^{-t_i} \) \( (1 \leq i \leq n) \) in the generating function of the Gromov–Witten invariants \[7\], it becomes

\[
\sum_{d_B \geq 1} q_{d_B}^{d_B} \sum_{g=0}^{\infty} g^s^{2g-2} \sum_{d_1, \ldots, d_n \geq 0} N_{g,d_B,d_1,\ldots,d_n} e^{-t_1 d_1} \cdots e^{-t_n d_n}. \tag{23}
\]

We have shown only the relevant part for \( N_{g,d_B,d_1,\ldots,d_n} \) with \( d_B \geq 1 \) here. Then if we assume that the summation over \( d_1, \ldots, d_n \) can be replaced with the integration, we can regard the integral as the Laplace transform of \( N_{g,d_B,d_1,\ldots,d_n} \) from the variables \( d_1, \ldots, d_n \) to the variables \( t_1, \ldots, t_n \). Given that \( t_i \) is proportional to \( a_{i,i+1} \) \[17\] and that the inverse Laplace transform exists as mentioned above, the asymptotic form of the Gromov–Witten invariants is given by the inverse Laplace transform.

3. \( r_{d_B}^{(g)}(d_1, \ldots, d_n) \) is the homogeneous polynomial in \( d_1, \ldots, d_n \) of degree

\[
2(n+1)d_B + 2g - 2 - n. \tag{24}
\]

Note that the degree grows as \( d_B \) and \( g \) grow. \( (\cdot :) \) Recall that performing the inverse Laplace transform once reduces the degree by 1 (for example, \( L_{x,s}^{-1}(\frac{1}{x^n}) \) is a polynomial of degree \( n - 1 \) in \( x \)). Therefore the degree of the result of the inverse Laplace transform for \( A_n \) is smaller by
than the total degree $2(n + 1)k + 2g - 2$ of $a_{i,j}$'s in $\mathcal{F}_k^{(g)}(a_{1,2}, \ldots, a_{n,n+1})$.  

4. The condition $d_1, \ldots, d_n \gg 1$ is necessary so that we can replace the summation with the integration. The condition $d_1, \ldots, d_n \gg d_B(g + 1)$ is further required so that we can neglect the contribution from multiple covering and the bubbling.

5. $n_{d_B}^{(g)}(d_1, \ldots, d_n)$ is invariant under $(d_1, d_2, \ldots, d_n) \mapsto (d_n, d_{n-1}, \ldots, d_1)$. It is because $Z_{d_B \geq 1}$ is invariant under 

\begin{equation}
(q_1, q_2, \ldots, q_n) \mapsto (q_n, q_{n-1}, \ldots, q_1).
\end{equation}

It is possible that the Weyl invariance manifests itself in the asymptotic form in other manners. For $A_2$ cases, see the remark in section 6.

(Proof)

We consider the asymptotic form of the Gromov–Witten invariants $N_{g,d_B,d_1,\ldots,d_n}$ first and that of the Gopakumar–Vafa invariants $n_{d_B}^{(g)}$ next. As it turned out, the two are the same, because we can neglect the bubbling effect and the multicovering effect when $d_1, \ldots, d_n \gg d_B(g + 1)$.

We use the notations $a := (a_{1,2}, \ldots, a_{n,n+1})$, $d := (d_1, \ldots, d_n)$, $t := (t_1, \ldots, t_n)$ in the rest of this section.

Recall that the part $Z_{d_B \geq 1}$ becomes equal to Nekrasov’s partition function in the field theory limit. Let us consider the logarithm of the equation. Then its right-hand side is written as  

\begin{equation}
\lim_{\beta \to 0} \log Z_{d_B \geq 1} = \sum_{g=0}^{\infty} \left((i\hbar \beta)^{2g-2} \sum_{d_B=1}^{\infty} ((-1)^{n+m+1}(\beta \Lambda)^{2(n+1)})^{d_B} \times \sum_{d} N_{g,d_B,d} e^{-d t} |_{t=\beta a} \right).
\end{equation}

Comparing the right-hand side and the left-hand side as the formal power series in $\hbar$ and $\Lambda$, the following holds up to the lowest order in $\beta$:

\begin{equation}
\mathcal{F}_k^{(g)}(a) = (-1)^{(n+m+1)k} \beta^{2(n+1)k+2g-2} \sum_{d} N_{g,d_B=k,d} e^{-d t} |_{t=\beta a}.
\end{equation}

Now we replace the sum over $d_1, \ldots, d_n$ with the integration in the right-hand side. Then, the integral is nothing but the Laplace transform of $N_{g,d_B,d}$ from the variables $(d_1, \ldots, d_n)$ to $(t_1, \ldots, t_n)$:

\begin{equation}
\mathcal{F}_{d_B}^{(g)}(a) \sim (-1)^{(n+m+1)d_B} \beta^{2(n+1)d_B+2g-2} \mathcal{L}(d_1, t_1) \circ \cdots \circ \mathcal{L}(d_n, t_n)(N_{g,d_B,d}).
\end{equation}
Therefore the asymptotic form of the Gromov–Witten invariant \( N_{g,d_B,d} \) for given \( g \) and \( d_B \) is obtained by the inverse Laplace transform of \( \mathcal{F}_{d_B}^{(g)}(a) \):

\[
N_{g,d_B,d} \sim (-1)^{(n+m+1)d_B} \beta^{2(n+1)d_B + 2g - 2} \mathcal{L}_{(d_1,t_1)}^{-1} \circ \cdots \circ \mathcal{L}_{(d_n,t_n)}^{-1} \mathcal{F}_{d_B}^{(g)}(a) \big|_{a=t/\beta} \tag{29}
\]

Since \( \mathcal{F}_{d_B}^{(g)}(a) \) is homogeneous in \( 1/a_i \)'s with degree \( 2(n+1)d_B + 2g - 2 \), \( \mathcal{F}_{d_B}^{(g)}(a) \big|_{a=t/\beta} = \beta^{-2(n+1)d_B - 2g + 2} \mathcal{F}_{d_B}^{(g)}(t) \). Hence the powers of \( \beta \) in (29) vanish and the result is finite at \( \beta \to 0 \). Thus, just rewriting \( t \) with \( a \), we obtain

\[
N_{g,d_B,d} \sim (-1)^{(n+m+1)d_B} \mathcal{L}_{(a_1,T_1)}^{-1} \circ \cdots \circ \mathcal{L}_{(a_n,T_n)}^{-1} \mathcal{F}_{d_B}^{(g)}(a) =: r_{d_B}^{(g)}(d). \tag{30}
\]

When we have replaced the summation with the integration in (27), we have assumed that the contribution from the Gromov–Witten invariants with large values of \( d_1, \ldots , d_n \) is dominant. The assumption could be justified by this result (\( \because (24) \)).

Next we consider the asymptotic form of the Gopakumar–Vafa invariants \( n_{d_B,d}^{g} \). The Gopakumar–Vafa invariants are written in terms the Gromov–Witten invariants as follows [19]:

\[
n_{d_B,d}^{g} = \sum_{g' = 0}^{g} \sum_{k(\mid d_B,d)} \mu(k)k^{2g' - 3} \alpha_{g,g'} N_{g',d_B,k,d/k} \tag{31}
\]

where \( \alpha_{g,g'} \) is the coefficient of \( r^{g-g'} \) in the series

\[
\left( \frac{\arcsin(\sqrt{r}/2)}{\sqrt{r}/2} \right)^{2g'-2}.
\]

Note that the degree of \( r_{d_B}^{(g)}(d) \) is \( 2(n+1)d_B + 2g - 2 - n \) [21] and grows with \( g \) and \( d_B \). Therefore \( N_{g',d_B,k,d/k} \) with \( k \geq 2 \) or \( g' < g \) is sufficiently smaller when \( d_1, \ldots , d_n \gg d_B \):

\[
N_{g',d_B,k,d/k} \ll N_{g,d_B,d} \quad \text{if} \quad g' < g \text{ or } k > 1.
\]

And the number of such terms in the right-hand side of (31) is at most \( d_B(g + 1) \). Thus, the contribution to the Gopakumar–Vafa invariant from the Gromov–Witten invariants with lower genera (bubbling effect) and lower degrees (multiple covering) can be neglected if \( d_1, \ldots , d_n \gg d_B(g + 1) \). When this condition is satisfied,

\[
N_{g,d_B,d} \sim n_{d_B,d}^{g} \sim r_{d_B}^{(g)}(d). \tag{32}
\]
Figure 1: The section $\Delta^m_{A_2}$ of the fan at the height 1 (top) and the web diagram $\Gamma$ (bottom) for $A_2$-fibration over $\mathbb{P}^1$.

6 Example: $A_2$-fibration over $\mathbb{P}^1$

The asymptotic form for the $A_1$ case at genus zero was studied in [1, 11]. In this section we study the case $n = 2$. There are four Calabi–Yau toric threefolds which has the structure of the fibration of $A_2$-singularity over $\mathbb{P}^1$ (figure 1). We look into $m = -1$ and $g = 0$, $d_B = 1, 2$ in detail for the illustrative purpose. For more thorough results, see appendix A. First we calculate the asymptotic form $r_{d_B}^{(g)}(d_1, d_2)$ (19) from Nekrasov’s partition function. Then we calculate the Gopakumar–Vafa invariants from the topological string amplitude of Iqbal–Kashani-Poor [5, 11]. (We have checked that the Gopakumar–Vafa invariants from [11] agree with those obtained from the local B-model calculation for all four $A_2$-fibration over $\mathbb{P}^1$, up to $d_B \leq 2$, $d_1, d_2 \leq 21$. The data of the local B-model is presented in the appendix B.) Finally we see that the ratio approaches to one when $d_1, d_2$ are large (figure 2).

Asymptotic form

In section 5 we have derived the asymptotic form $r_{d_B}^{(g)}(d_1, d_2)$ of the Gopakumar–Vafa in-
variants \( n_{d_B,d_1,d_2}^g \).

\[
Z_{\text{Nekrasov}}^{A_2} = \sum_{R_1,R_2,R_3} \Lambda^{l(R_1)+l(R_2)+l(R_3)} \prod_{i,j=1}^3 \prod_{k,l=1}^\infty \frac{a_{i,j} + \hbar(\mu_{i,k} - \mu_{j,l} + l - k)}{a_{i,j} + \hbar(l - k)}.
\]

\[
\log Z_{\text{Nekrasov}}^{A_2} = \sum_{g=0}^{\infty} (\sqrt{-1}\hbar)^{2g-2} \sum_{k=1}^{\infty} \mathcal{F}_k^{(g)}(a_{1,2}, a_{2,3}) \Lambda^{6k}.
\]

\[
r_{d_B}^{(g)}(d_1, d_2) = (-1)^{(n+m+1)d_B} \mathcal{L}_{d_1,a_{1,2}}^{-1} \circ \mathcal{L}_{d_2,a_{2,3}}^{-1} \mathcal{F}_k^{(g)}(a_{1,2}, a_{2,3}).
\]

For instance, an explicit expression for first few \( \mathcal{F}_k^{(g)} \)'s are as follows:

\[
\mathcal{F}_1^{(0)} = \frac{1}{a_{1,2}^2a_{1,3}^2} + \frac{1}{a_{1,2}^2a_{2,3}^2} + \frac{1}{a_{1,3}^2a_{2,3}^2},
\]

\[
\mathcal{F}_2^{(0)} = \frac{3}{2a_{1,2}^4a_{1,3}^6} + \frac{3}{2a_{1,2}^6a_{1,3}^4} + \frac{3}{2a_{1,2}^4a_{2,3}^6} + \frac{3}{2a_{1,3}^4a_{2,3}^6} + \frac{3}{2a_{1,2}^6a_{2,3}^4} + \frac{3}{2a_{1,3}^6a_{2,3}^4}
\]

\[
+ \frac{2}{a_{1,2}^5a_{1,3}^5} + \frac{2}{a_{1,2}^5a_{2,3}^5} + \frac{2}{a_{1,3}^5a_{2,3}^5}
\]

\[
+ \frac{2}{a_{1,2}^2a_{1,3}^2a_{2,3}^2} + \frac{2}{a_{1,2}^2a_{1,3}^2a_{2,3}^2} + \frac{2}{a_{1,2}^2a_{1,3}^2a_{2,3}^2}.
\]

The Seiberg–Witten prepotential for the \( A_2 \) case was also studied in [20] by using the Seiberg–Witten curve, and the result agree with that of Nekrasov’s formula if we rescale \( \Lambda^6 \) to \( \frac{\Lambda^6}{4} \). The inverse Laplace transform of each term is as follows. Recall that \( a_{1,3} = a_{1,2} + a_{2,3} \). For \( \alpha, \beta, \gamma \in \mathbb{Z}_{\geq 0} \),

\[
\mathcal{L}_{(d_1,a_{1,2})}^{-1} \circ \mathcal{L}_{(d_2,a_{2,3})}^{-1} \left( \frac{1}{a_{1,2}^\alpha a_{2,3}^\beta a_{1,3}^\gamma} \right)
\]

\[
= \theta(d_2 - d_1) \sum_{j=0}^{\beta-1} d_1^{\alpha+j} d_2^{\beta-1-j} \frac{(-1)^j (\gamma)_j}{j! \Gamma(\beta - j) \Gamma(\alpha + \gamma + j)} + (d_1, \alpha \leftrightarrow d_2, \beta)
\]

(35)

When \( \alpha = 0 \) (resp. \( \beta = 0 \)), the second (resp. first) term is just zero. \( \theta(x) \) is the Heaviside step function. Therefore from (34), we obtain

\[
r_1^{(0)} = (-1)^{m+1}(d_2(2d_1 - d_2)\theta(d_1 - d_2) + d_1(2d_2 - d_1)\theta(d_2 - d_1)),
\]

\[
r_2^{(0)} = -\frac{1}{2 \cdot 6!} d_2^3 (5d_1^4 - 10d_1^3 d_2 + 9d_1^2 d_2^2 - 4d_1 d_2^3 + 2d_2^4)(2d_1 - d_2)\theta(d_1 - d_2) + (d_1 \leftrightarrow d_2).
\]

(36)

Remark: Weyl invariance.

The asymptotic forms are symmetric with respect to \( d_1 \leftrightarrow d_2 \) and have the factor \( 2d_1 - d_2 \) (resp. \( 2d_2 - d_1 \)) when \( d_1 \geq d_2 \) (resp. \( d_2 \geq d_1 \)). These are the results of the Weyl \( \times \mathbb{Z}_2 \) invariance.
of $\mathcal{F}^{(g)}_{d_B}(a_{1,2}, a_{2,3})$. The former is due to the symmetry $(a_{1,2}, a_{2,3}) \leftrightarrow (a_{2,3}, a_{1,2})$, which is the composition of the exchange $1 \leftrightarrow 3$ and the multiplication by $-1$. The latter is due to the exchange $(1, 2) \leftrightarrow (2, 1)$ (resp. $(2, 3) \leftrightarrow (3, 2)$); when one sums up the terms in the same orbit of this action, its inverse Laplace transform turns out to have the factor $2d_1 - d_2$ (resp. $2d_2 - d_1$).

**Gopakumar–Vafa invariants**

Let us define the generating function of the Gromov–Witten invariants and the Gopakumar–Vafa invariants for given $d_B$ by

\[
F^{(g)}_{d_B}(q_1, q_2) := \sum_{d_1, d_2 = 0}^{\infty} N_{g, d_B, d_1, d_2} q_1^{d_1} q_2^{d_2},
\]

\[
G^{(g)}_{d_B}(q_1, q_2) := \sum_{d_1, d_2 = 0}^{\infty} n_{d_B, d_1, d_2}^{(g)} q_1^{d_1} q_2^{d_2}.
\]

One can immediately calculate $F^{(g)}_{d_B}(q_1, q_2)$ by simply expanding the logarithm of (11) as

\[
\log \mathcal{Z}(X_{A_n}^m)|_{q = i q_s} = \mathcal{F}_{GW}(X_{A_n}^m) = \sum_{g=0}^{\infty} g_s^{2g-2} \sum_{d_B=0}^{\infty} q_B^{d_B} F^{(g)}_{d_B}(q_1, q_2).
\]

And one can calculate $G^{(g)}_{d_B}(q_1, q_2)$ from $F^{(g)}_{d_B}(q_1, q_2)$ as follows (compare (7) and (8)):

\[
G^{(0)}_1(q_1, q_2) = F^{(0)}_1(q_1, q_2),
\]

\[
G^{(0)}_2(q_1, q_2) = F^{(0)}_2(q_1, q_2) - \frac{1}{2q_s} F^{(0)}_1(q_1^2, q_2^2),
\]

\[
G^{(1)}_1(q_1, q_2) = F^{(1)}_1(q_1, q_2) - \frac{1}{12} G^{(0)}_1(q_1, q_2),
\]

\[
G^{(1)}_2(q_1, q_2) = F^{(1)}_2(q_1, q_2) - \frac{1}{2} F^{(1)}_1(q_1^2, q_2^2) - \frac{1}{12} G^{(0)}_2(q_1, q_2),
\]

\[
G^{(2)}_1(q_1, q_2) = F^{(2)}_1(q_1, q_2) - \frac{1}{240} G^{(0)}_1(q_1, q_2),
\]

\[
G^{(2)}_2(q_1, q_2) = F^{(2)}_2(q_1, q_2) - 2 F^{(2)}_1(q_1^2, q_2^2) - \frac{1}{240} G^{(0)}_2(q_1, q_2).
\]

Therefore from (11), (37), (38), (39), one can calculate the generating function of the Gopakumar–Vafa invariants. For instance, for $m = -1$,

\[
G^{(0)}_1 = \frac{1 + q_1 q_2 - 6 q_1 q_2 + q_1^2 q_2 + q_1 q_2^2 + q_1^2 q_2^2}{(1 - q_1)^2 (1 - q_2)^2 (1 - q_1 q_2)^2},
\]

\[
G^{(0)}_2 = \frac{-2 \sum_{i, j = 0}^{12} C^{(0)}_{ij} q_1^i q_2^j}{(1 - q_1)^4 (1 - q_2)^4 (1 - q_1 q_2)^4 (1 - q_1^2)^2 (1 - q_2^2)^2 (1 - q_1^2 q_2^2)^2}.
\]
The coefficients \( \{C_{ij}^{(0)}\} \) are given by

\[
C_{ij}^{(0)} = \begin{bmatrix}
0 & 0 & 3 & 4 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 3 & 4 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 3 & 0 & 15 & 15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 4 & 54 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 3 & -13 & 2 & 0 & -36 & -74 & -74 & 0 & -2 & -13 & 3 \\
5 & 0 & -7 & 15 & 0 & -74 & 26 & 26 & -74 & 0 & 15 & -7 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 0 & -7 & 15 & 0 & -74 & 26 & 26 & -74 & 0 & 15 & -7 \\
8 & 0 & 0 & 0 & -13 & -2 & 0 & -36 & -74 & -36 & 0 & 2 & 13 \\
9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Then the Gopakumar–Vafa invariants are obtained by the series expansion of \( G_{d_B}^{(g)} \):

\[
d_B = 1
\]

\[
d_B = 2
\]

We remark here that if we plug \( q_i = -\beta a_{i,i+1}(i = 1, 2) \) into \( G_{d_B}^{(g)}(q_1, q_2) \), the terms with the lowest degree in \( \beta \) reproduce the gauge theory result \( \beta^{-2(n+1)}2_2^{2+2+n}F_{d_B}^{(g)}(a_{1,2}, a_{2,3}) \). For example in \( G_1^{(0)} \), the numerator is \( \beta \) the denominator is \( \beta \) and we can check that the lowest degree part in \( G_1^{(0)} \) is equal to \( \beta^{-4}F_1^{(0)} \) where \( F_1^{(0)} \) has been calculated in \text{[33]}.

We can see the ratio between the Gopakumar–Vafa invariants and the asymptotic form \text{[36]} approaches 1 when \( d_1, d_2 \) become large (figure \text{[2]}).

\textsuperscript{1}The suffix (0) means \( g = 0 \).
7 Conclusion

In this article, we have derived the asymptotic form of the Gopakumar–Vafa invariants of the toric Calabi–Yau threefold which is the $A_n$-fibration over $\mathbb{P}^1$. The asymptotic form is, as it turned out, obtained as the inverse Laplace transform of the corresponding term in the logarithm of Nekrasov’s partition function for instanton counting.

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A Asymptotic Forms: $A_2$ case

\[
\begin{align*}
\mathcal{R}_1^{(0)} &= (-1)^{m+1}(-d_2(2d_1 - d_2)\theta(d_1 - d_2) - d_1(2d_2 - d_1)\theta(d_2 - d_1)), \\
\mathcal{R}_2^{(0)} &= -\frac{1}{2 \cdot 6!} d_2^3(5d_1^4 - 10d_1^3d_2 + 9d_1^2d_2^2 - 4d_1d_2^3 + 2d_2^4)(2d_1 - d_2)\theta(d_1 - d_2) \\
&\quad + (d_1 \leftrightarrow d_2), \\
\mathcal{R}_3^{(0)} &= (-1)^{m+1}\left[-\frac{2}{12!} d_2^5(132d_1^8 - 528d_1^7d_2 + 1012d_1^6d_2^2 - 1188d_1^5d_2^3 + 1045d_1^4d_2^4 \\
&\quad - 726d_1^3d_2^5 + 383d_1^2d_2^6 - 130d_1d_2^7 + 26d_2^8)(2d_1 - d_2)\theta(d_1 - d_2) \\
&\quad + (d_1 \leftrightarrow d_2)\right], \\
\mathcal{R}_4^{(0)} &= -\frac{1}{18!} d_2^7(74919d_1^{12} - 449514d_1^{11}d_2 + 1315171d_1^{10}d_2^2 - 2455310d_1^9d_2^3 \\
&\quad + 3375333d_1^8d_2^4 - 3714126d_1^7d_2^5 + 3402567d_1^6d_2^6 - 2575908d_1^5d_2^7 \\
&\quad + 1577430d_1^4d_2^8 - 756534d_1^3d_2^9 + 270468d_1^2d_2^{10} - 64496d_1d_2^{11} \\
&\quad + 8062d_2^{12})(2d_1 - d_2)\theta(d_1 - d_2) \\
&\quad + (d_1 \leftrightarrow d_2), \\
\mathcal{R}_5^{(0)} &= (-1)^{m+1}\left[-\frac{2}{5 \cdot 24!} d_2^9(85968388d_1^{16} - 687747104d_1^{15}d_2 + 2703858840d_1^{14}d_2^2 \\
&\quad - 6891437560d_1^{13}d_2^3 + 12945418880d_1^{12}d_2^4 - 19376317800d_1^{11}d_2^5 \\
&\quad + 24280022404d_1^{10}d_2^6 - 26073264140d_1^9d_2^7 + 24117002613d_1^8d_2^8 \\
&\quad - 19125226340d_1^7d_2^9 + 12883936450d_1^6d_2^{10} - 7261935912d_1^5d_2^{11} \\
&\quad + 3345086558d_1^4d_2^{12} - 1213991290d_1^3d_2^{13} + 327303495d_1^2d_2^{14} \\
&\quad - 58677410d_1d_2^{15} + 5334310d_2^{16})(2d_1 - d_2)\theta(d_1 - d_2) \\
&\quad + (d_1 \leftrightarrow d_2)\right].
\end{align*}
\]
\[ r_1^{(1)} = 0, \]
\[ r_2^{(1)} = \frac{3}{5 \cdot 9!} d_2^3 (80d_1^6 - 240d_1^5 d_2 + 363d_1^4 d_2^2 - 326d_1^3 d_2^3 \]
\[ + 177d_1^2 d_2^4 - 54d_1 d_2^5 + 18d_2^6)(2d_1 - d_2)\theta(d_1 - d_2) \]
\[ + (d_1 \leftrightarrow d_2), \]
\[ r_3^{(1)} = (-1)^{m+1}\frac{1}{2 \cdot 15!} d_2^5 (93184d_1^{10} - 465920d_1^9 d_2 + 1151280d_1^8 d_2^2 - 1809600d_1^7 d_2^3 \]
\[ + 2026700d_1^6 d_2^4 - 1703364d_1^5 d_2^5 + 1127386d_1^4 d_2^6 - 595192d_1^3 d_2^7 \]
\[ + 240704d_1^2 d_2^8 - 65178d_1 d_2^9 + 10863d_2^{10})(2d_1 - d_2)\theta(d_1 - d_2) \]
\[ + (d_1 \leftrightarrow d_2) \],
\[ r_4^{(1)} = \frac{3}{21!} d_2^7 (17023392d_1^{14} - 119163744d_1^{13} d_2 + 412141863d_1^{12} d_2^2 - 923722506d_1^{11} d_2^3 \]
\[ + 1511937483d_1^{10} d_2^4 - 1932300342d_1^9 d_2^5 + 2015543579d_1^8 d_2^6 \]
\[ - 1760629718d_1^7 d_2^7 + 1297849853d_1^6 d_2^8 - 798184034d_1^5 d_2^9 \]
\[ + 400904978d_1^4 d_2^{10} - 159563216d_1^3 d_2^{11} + 48024252d_1^2 d_2^{12} \]
\[ - 9861840d_1 d_2^{13} + 1095760d_2^{14})(2d_1 - d_2)\theta(d_1 - d_2) \]
\[ + (d_1 \leftrightarrow d_2), \]
\[ r_5^{(1)} = (-1)^{m+1}\frac{2}{7 \cdot 26!} d_2^9 (8030422400d_1^{18} - 72273801600d_1^{17} d_2 + 321891762920d_1^{16} d_2^2 \]
\[ - 936927933760d_1^{15} d_2^3 + 2008087382840d_1^{14} d_2^4 - 3394094432680d_1^{13} d_2^5 \]
\[ + 4734232437620d_1^{12} d_2^6 - 5615707782160d_1^{11} d_2^7 + 5764569968010d_1^{10} d_2^8 \]
\[ - 5153404272590d_1^9 d_2^9 + 4005220160655d_1^8 d_2^{10} - 2687880271720d_1^7 d_2^{11} \]
\[ + 1541655660080d_1^6 d_2^{12} - 744490921270d_1^5 d_2^{13} + 296060372105d_1^4 d_2^{14} \]
\[ - 93685509760d_1^3 d_2^{15} + 22307121734d_1^2 d_2^{16} - 3590362824d_1 d_2^{17} \]
\[ + 299196902d_2^{18})(2d_1 - d_2)\theta(d_1 - d_2) \]
\[ + (d_1 \leftrightarrow d_2) \].
\[ r_1^{(2)} = 0, \]
\[ r_2^{(2)} = -\frac{1}{6 \cdot 10!} d_2^3 (55d_1^6 - 165d_1^5 d_2 + 246d_1^4 d_2^2 - 217d_1^3 d_2^3 \\
+ 105d_1^2 d_2^4 - 24d_1 d_2^5 + 8d_2^6) (2d_1 - d_2)(d_1^2 - d_1 d_2 + d_2^2) \theta(d_1 - d_2) \\
+ (d_1 \leftrightarrow d_2), \]
\[ r_3^{(2)} = (-1)^{m+1} \left[ -\frac{4}{3 \cdot 16!} d_2^5 (19656d_1^{10} - 98280d_1^9 d_2 + 238524d_1^8 d_2^2 - 364416d_1^7 d_2^3 \\
+ 388505d_1^6 d_2^4 - 302835d_1^5 d_2^5 + 182286d_1^4 d_2^6 - 88439d_1^3 d_2^7 + 33435d_1^2 d_2^8 \\
- 8436d_1 d_2^9 + 1406d_2^{10})(2d_1 - d_2)(d_1^2 - d_1 d_2 + d_2^2) \theta(d_1 - d_2) \\
+ (d_1 \leftrightarrow d_2) \right], \]
\[ r_4^{(2)} = -\frac{1}{2 \cdot 22!} d_2^7 (107343027d_1^{14} - 751401189d_1^{13} d_2 + 2568281452d_1^{12} d_2^2 \\
- 5641473255d_1^{11} d_2^3 + 8957835684d_1^{10} d_2^4 - 10984068587d_1^9 d_2^5 \\
+ 10884203580d_1^8 d_2^6 - 8988502881d_1^7 d_2^7 + 6266334159d_1^6 d_2^8 \\
- 3651235662d_1^5 d_2^9 + 1737697440d_1^4 d_2^{10} - 656195438d_1^3 d_2^{11} \\
+ 188620176d_1^2 d_2^{12} - 37438506d_1 d_2^{13} \\
+ 4159834d_2^{14})(2d_1 - d_2)(d_1^2 - d_1 d_2 + d_2^2) \theta(d_1 - d_2) \\
+ (d_1 \leftrightarrow d_2), \]
\[ r_5^{(2)} = (-1)^{m+1} \left[ -\frac{2}{28!} d_2^9 (50517013976d_1^{18} - 454653125784d_1^{17} d_2 + 2008600020180d_1^{16} d_2^2 \\
- 5763329310336d_1^{15} d_2^3 + 12097712620980d_1^{14} d_2^4 - 19894873394844d_2^5 \theta(d_1^{13}) \\
+ 26838758944032d_2^6 d_1^{12} - 30659562087156d_2^7 d_1^{11} + 30245624065647d_2^8 d_1^{10} \\
- 25967123885015d_2^9 d_1^9 + 19370347312179d_2^{10} d_1^8 - 12464676224742d_1^{11} d_2^{12} \\
+ 6849482740752d_2^{12} d_1^6 - 3169505142660d_2^{13} d_1^5 + 1209745627491d_2^{14} d_1^4 \\
- 368804218845d_2^{15} d_1^3 + 85189452681d_2^{16} d_1^2 - 13450408536d_2^{17} d_1^1 \\
+ 1120867378d_2^{18})(2d_1 - d_2)(d_1^2 - d_1 d_2 + d_2^2) \theta(d_1 - d_2) \\
+ (d_1 \leftrightarrow d_2) \right]. \]

**B Local B-model Calculation: A₂ case**

We computed the Gopakumar–Vafa invariants of $A_2$-fibration over $\mathbb{P}^1$, for $g = 0$ and $d_B \leq 2$, $d_1, d_2 \leq 21$ by the local B-model calculation [21]. The results agree with the results from the
partition function of Iqbal and Kashani-Poor \cite{Iqbal} and the results in \cite{21} (section 6.4) for \(m = -1\) and \(m = 2\). In this section, we list the relevant data.

"Charge vectors":

\[
\begin{bmatrix}
1 & 1 & -m & -2 + m & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 \\
1 & 1 & 0 & -1 & -1 & 0 \\
0 & 0 & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 1 & -2 & 1
\end{bmatrix}
\]

\((m = 0, 1, 2)\)

Gröbner basis of the toric ideal \(I_A\):

\[l^{(0)}, l^{(1)}, l^{(2)}, l^{(1)} + l^{(2)}\].

Solutions to GKZ-system \(H_A(\beta)\) with \(\beta = (0, 0, 0)\):

\[f, \quad -t_B = \partial_{\rho_0} f, \quad -t_1 = \partial_{\rho_1} f, \quad -t_2 = \partial_{\rho_2} f, \quad I_1, I_2.\]

with

\[m = -1 : \quad I_1 = (2\partial_{\rho_0}\partial_{\rho_1} + \partial_{\rho_1}^2)f, \quad I_2 = (2\partial_{\rho_0}\partial_{\rho_2} + \partial_{\rho_2}^2)f.\]

\[m = 0 : \quad I_1 = \partial_{\rho_0}\partial_{\rho_1} f, \quad I_2 = (\partial_{\rho_0}\partial_{\rho_2} + \partial_{\rho_2}^2)f.\]

\[m = 1 : \quad I_1 = (2\partial_{\rho_0}\partial_{\rho_1} + \partial_{\rho_1}^2)f, \quad I_2 = (2\partial_{\rho_2}\partial_{\rho_0} + 2\partial_{\rho_0}\partial_{\rho_2} + 3\partial_{\rho_2}^2)f.\]

\[m = 2 : \quad I_1 = (\partial_{\rho_0}\partial_{\rho_1} + \partial_{\rho_1}^2)f, \quad I_2 = (\partial_{\rho_0}\partial_{\rho_2} + 2\partial_{\rho_0}\partial_{\rho_2} + 2\partial_{\rho_2}^2)f.\]

Here

\[f = \sum_{n_0, n_1, n_2 \geq 0} \prod_{i=1}^{6} \frac{\Gamma\left(\sum_{j=0}^{2} \rho_j (l^{(j)})_i + 1\right)}{\Gamma\left(\sum_{j=0}^{2} (\rho_j + n_j) (l^{(j)})_i + 1\right)} \cdot z_0^{n_0+\rho_0} z_1^{n_1+\rho_1} z_2^{n_2+\rho_2}\]

Identification with the prepotential (here, \(\mathcal{F} = \mathcal{F}_{GW}^{g=0}(X_{A_n})\)):

\[m = -1 : \quad I_1 = 2(-\partial_{t_B} - 2\partial_{t_1} + \partial_{t_2})\mathcal{F}, \quad I_2 = 2(-\partial_{t_B} + \partial_{t_1} - 2\partial_{t_2})\mathcal{F}\]

\[m = 0 : \quad I_1 = (-2\partial_{t_B} - 2\partial_{t_1} + \partial_{t_2})\mathcal{F}, \quad I_2 = (\partial_{t_1} - 2\partial_{t_2})\mathcal{F}\]

\[m = 1 : \quad I_1 = 2(-\partial_{t_B} - 2\partial_{t_1} + \partial_{t_2})\mathcal{F}, \quad I_2 = 2(\partial_{t_1} - 2\partial_{t_2})\mathcal{F}\]

\[m = 2 : \quad I_1 = (-2\partial_{t_1} + \partial_{t_2})\mathcal{F}, \quad I_2 = (\partial_{t_1} - 2\partial_{t_2})\mathcal{F}\]

We have determined the overall normalization so that \(n_{1,1,0} = n_{1,0,1} = n_{1,1,1} = -2\).
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