Quantum Mechanics as a Classical Theory I: 
Non-relativistic Theory

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march, 30, 1995

Abstract

The objective of this series of three papers is to axiomatically derive 
quantum mechanics from classical mechanics and two other basic axioms. 
In this first paper, Schrödinger’s equation for the density matrix is first 
obtained and from it Schrödinger’s equation for the wave functions is 
derived. The momentum and position operators acting upon the den-
sity matrix are defined and it is then demonstrated that they commute. 
Pauli’s equation for the density matrix is also obtained. A statistical 
potential formally identical to the quantum potential of Bohm’s hidden 
variable theory is introduced, and this quantum potential is reinterpreted 
through the formalism here proposed. It is shown that, for dispersion free 
ensembles, Schrödinger’s equation for the density matrix is equivalent 
to Newton’s equations. A general non-ambiguous procedure for the con-
struction of operators which act upon the density matrix is presented. It 
is also shown how these operators can be reduced to those which act upon 
the wave functions.

1 General Introduction

Ever since the establishment of the Copenhagen Interpretation and it’s purely 
epistemological point of view, quantum mechanics has been subject to criti-
cism, debates and of constant revisionary attempts trying to alter it in one way 
or another[1]-[7]. Other than the hidden variables theories, few attempts were 
made to reconstruct quantum theory on the lines of Realism. Even these the-
ories, according to Bell’s theorem[8]-[10], must posses non-local characteristics 
little acceptable to a realist such as Einstein[1].

Part of the problem resides in the fact that, unlike relativistic theories, it has 
been hitherto impossible to derive quantum mechanics from classical mechanics 
though the insertion of a few complementary postulates capable of identifying
the differences between both theories. The absence of this derivation has led most physicists to believe in the existence of an unbridgeable conceptual abyss separating the two theories [11]. Because of this, most of the ontologies proposed for quantum mechanics reject Realism a priori.

In this series of papers we intend to demonstrate that both relativistic and non-relativistic quantum mechanics can be derived from classical mechanics thorough the addition of two somewhat "natural" postulates which do not alter the classical character of the derivation. We intend to demonstrate that, based on a rather simple generalization of these postulates, it is possible to construct a relativistic general quantum theory for ensembles of single particle systems.

One of the most important results, at least where epistemology is concerned, is the mathematically backed negation of the ontological origin of Heisenberg’s uncertainty relations. This is the key to refuting the Copenhagen Interpretation as a whole together with it’s ontology [12]. Also in the relation to epistemology, we propose a quantum model in which the observer is included without its ad hoc postulation being necessary [13]. From there, we maintain that a general measurement theory in quantum mechanics (and in all of physics) is impossible, analyzing von Neumann’s [4] attempt as an example.

In the same manner, and independently of Bell’s argument [9, 15], which we reject, we demonstrate the errors in von Neumann’s theorem on the impossibility of hidden variable theories. Our counter example is a theory of local behavior. We show that Bell’s theorem does not necessarily entail in non-local quantum mechanics and that quantum mechanics’ hidden variable theory is, in fact, newtonian mechanics itself. We also take this opportunity to demonstrate that Bohm’s [7, 16, 17] theory is not truly a hidden variable theory and reinterpret its formalism.

In a totally formal perspective, we show that Schroedinger’s equation for the density matrix is fundamental, and not that for the probability amplitudes. Also, Dirac’s second order equation is shown to be the fundamental one (and not his linear equation) [18]. We also present a general technique for the construction of operators [19].

We therefore demonstrate that, starting from our basic axioms, it is possible to obtain all of quantum mechanics and much more, and that this theory is nothing more than a classical theory.

This series of papers is divided in the following manner:

In this first paper we develop non-relativistic formalism in it’s most fundamental results, obtaining Schroedinger’s equations for the density matrix (which we call the density function) and for the wave function (which we call the probability amplitude). We develop the concept of operators which act upon the density function and upon the probability amplitude, and demonstrate that the first commute according to different rules from the latter - in the appendix we present a general technique free from ambiguities for obtaining operators. Pauli’s equation is also obtained by including internal degrees of freedom associated to the intrinsic magnetic moment. The basic equations for Bohm’s hidden
variable theory are also found, but only shortly discussed. We also demonstrate how to introduce the observer into quantum formalism. The epistemological implications of our results are left for the third paper of this series.

In the second paper we derive Schroedinger’s relativistic equations for the density function and for the probability amplitude through small modifications in the axioms of the first paper, making them coherent with the special theory of relativity. The calculations of the first paper are repeated in order to find a relativistic theory for one particle. We also find a system of equations involving Einstein’s which take into account the effects of gravitation.

The third and final paper of this series discusses the epistemological implications of the results of the first two papers. Based upon the clarification of Heisenberg’s misconception interpretation of the uncertainty relations, we construct a Realistic Interpretation to contrast with Copenhagen’s. Von Neumann’s measurement theory is discussed in comparison to the results of our first paper through Realistic epistemology. We demonstrate that von Neumann’s theorem on the impossibility of a hidden variable theory is not correct but that, on the other hand, Bell’s argumentation is also unacceptable. To be more precise, we demonstrate that, taking the non-relativistic case as an example, newtonian theory is the theory of hidden variables of quantum mechanics. Bell’s theorem is analyzed and we demonstrate that it does not imply in a non-locality of quantum mechanics. We discuss Bohm’s theory and demonstrate that it cannot be a hidden variable theory, since it is not free of the dispersions associated with Heisenberg’s relations.

2 Introduction

In this paper, we obtain non-relativistic quantum theory’s mathematical formalism from newtonian mechanics and two additional postulates. For reasons of clarity, these results are presented in an axiomatic way.

In the third section, we present the basic formalism, deriving Schroedinger’s equations for the density function and for the probability amplitude. We introduce the operator concept for these two functions and demonstrate that they obey distinct commutation laws. From there we show that Heisenberg’s uncertainty relations are not ontological, but purely formal.

In the fourth section, Pauli’s equation for the density function is derived taking into consideration the particle’s internal magnetic moments.

In the fifth section, we introduce and interpret the idea of statistical potential, formally similar to Bohm’s quantum potential.

The sixth section proposes a quantum experiment arrangement where a physical system acting as an external observer is introduced and its influence can be formally taken into account.

In the last section, we present our conclusions.
The appendix A discusses some of Wigner-Moyal Infinitesimal Transformation’s properties of the state functions \( F(x, p; t) \). It is also demonstrated that Schrödinger’s equation for the density function can be reduced to Newton’s equations in the limit of dispersion free ensembles.

Appendix B brings a general free from ambiguities method for the construction of operators.

In appendix C the calculations made in the body of the work are generalized for three-dimensional systems composed of many particles.

Appendix D is concerned with the definition of mixed states and the introduction of the concept of density matrix. The relation between the calculations made in the main text and the trace operation upon the density matrix is also investigated.

In both this paper and the other two we will use the terms ”classical” and ”quantum” meaning a classical statistical mechanics built upon phase or configuration space, respectively (as will become clear ahead).

3 Axioms and Formalism

We begin our theory with ensembles described by probability density functions in classical phase space written as \( F(x, p; t) \). The variables which label this function represent the position and the momentum of the particles that make up the ensemble.

Let us now list the theory’s axioms:

(A1) Newtonian particle mechanics is valid for all particles which constitute the systems which compose the ensemble.

(A2) For an isolated system the joint probability density function is conserved:

\[
\frac{dF(x, p; t)}{dt} = 0
\]  

(A3) The Wigner-Moyal Infinitesimal Transformation defined as

\[
\rho \left( \frac{x + \delta x}{2}, \frac{x - \delta x}{2}; t \right) = \int F(x, p; t) \exp \left( \frac{ip\delta x}{\ell} \right) dp
\]

where \( \ell \) is a universal parameter (having the same value for all transformations) with dimensions of angular momentum, is adequate for the description of any non-relativistic quantum system (this definition differs from that usually made in the literature \[23\] in its infinitesimal nature just emphasized).

We now demonstrate that all non-relativistic quantum mechanics, and some additional results, can be obtained from these axioms.
Using equation (1) we have
\[
\frac{dF(x, p; t)}{dt} = \frac{\partial F}{\partial t} + \frac{dx}{dt} \frac{\partial F}{\partial x} + \frac{dp}{dt} \frac{\partial F}{\partial p} = 0. \tag{3}
\]
We can use Newton’s equations, axiom (A1), on (3)
\[
\frac{dx}{dt} = \frac{p}{m}; \quad \frac{dp}{dt} = f, \tag{4}
\]
supposing that the force \(f\) derives from a potential \(V(x)\):
\[
f = -\frac{\partial V}{\partial x}. \tag{5}
\]
Multiplying this equation by the exponential in (2) and integrating, we reach
\[
-\frac{\partial \rho}{\partial t} + \frac{i \ell}{m} \frac{\partial^2 \rho}{\partial x \partial (\delta x)} - \frac{i}{\ell} \delta V \rho = 0, \tag{6}
\]
where we use the infinitesimal character of \(\delta x\) to write
\[
\frac{\partial V}{\partial x} \delta x = \delta V(x) = V\left(x + \frac{\delta x}{2}\right) - V\left(x - \frac{\delta x}{2}\right), \tag{7}
\]
and also from the fact that
\[
\left[F(x, p; t) \exp\left(\frac{ip\delta x}{\ell}\right)\right]_{p=+\infty}^{p=-\infty} = 0, \tag{8}
\]
as is expected from a probability density function
Changing the variables
\[
y = x + \frac{\delta x}{2}; \quad y' = x - \frac{\delta x}{2}, \tag{9}
\]
we can rewrite equation (3) above as
\[
\left\{ \frac{\ell^2}{2m} \left[ \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial y'^2} \right] - [V(y) - V(y')] \right\} \rho(y, y'; t) = -i\hbar \frac{\partial}{\partial t} \rho(y, y'; t), \tag{10}
\]
which we call Schroedinger’s First Equation for the density function \(\rho(y, y'; t)\).
This equation is valid for all values of \(y\) and \(y'\), as long as these are infinitesimally close, as can be seen in (3). It must not be forgotten that this consideration is not inconvenient to our calculations, for reasons that will become clear ahead.
We have proven the following theorem:

(T1) The density function \(\rho(y, y'; t)\) satisfies Schroedinger’s First Equation (10) being \(\ell\) a parameter to be determined experimentally (we know that this parameter is Plank’s constant and thus we will henceforth write \(\hbar\) instead of \(\ell\)).

5
Let us now suppose that we can write
\[ \rho (y, y'; t) = \Psi^* (y'; t) \Psi (y; t), \]
where we will call \( \Psi (y; t) \) the probability amplitude. Since this is usually a complex function, we can write it as
\[ \Psi (y; t) = R (y; t) \exp \left( \frac{iS (y; t)}{\hbar} \right), \]
where \( R (y; t) \) and \( S (y; t) \) are real functions.

We can now expand the function \( \rho (y, y'; t) \), given in equation (11) in terms of \( x \) and \( \delta x \), to obtain
\[ \rho \left( x + \frac{\delta x}{2}, x - \frac{\delta x}{2}; t \right) = \]
\[ \left\{ R (x; t)^2 - \frac{\delta x^2}{2} \left[ \left( \frac{\partial R}{\partial x} \right)^2 - R (x; t) \frac{\partial^2 R}{\partial x^2} \right] \right\} \exp \left( \frac{i \delta x}{\hbar} \frac{\partial S}{\partial x} \right), \]
which, substituted in equation (6), yields
\[ \frac{\partial (R^2)}{\partial t} + \frac{\partial}{\partial x} \left( R \frac{\partial S}{\partial x} \right) = 0, \]
\[ \frac{\delta x}{\hbar} \frac{\partial}{\partial x} \left[ \frac{1}{2m} \left( \frac{\partial S}{\partial x} \right)^2 + V + \frac{\partial S}{\partial t} - \frac{\hbar^2}{2mR^2} \frac{\partial^2 R}{\partial x^2} \right] = 0. \]
Canceling the real and complex terms, we obtain the following equations
\[ \frac{\partial P}{\partial t} + \frac{\partial}{\partial x} \left( P \frac{\partial S}{\partial x} \right) = 0, \]
\[ \frac{\partial}{\partial x} \left[ \frac{1}{2m} \left( \frac{\partial S}{\partial x} \right)^2 + V + \frac{\partial S}{\partial t} - \frac{\hbar^2}{2mR^2} \frac{\partial^2 R}{\partial x^2} \right] = 0, \]
where
\[ P (x; t) = R (x; t)^2 = \lim_{\delta x \to 0} \rho \left( x + \frac{\delta x}{2}, x - \frac{\delta x}{2}; t \right) \]
is the probability density in configuration space, as becomes clear if we look at expression (2). This last equation justifies the comment made after (10); the mean values of the quantities will always be calculated within the limit given in (17) so that the calculation of the density function for infinitesimally close points can be done without the loss of generality. (for mixtures, see appendix D). This does not imply that only the element for which \( \delta x \) is equal to zero contributes. The kinematical evolution of the density function is governed by
equation (6) which mixes all the contributions. The above mentioned limit must be taken after this equation has been solved.

Equation (16) can be rewritten as

$$\frac{1}{2m} \left( \frac{\partial S}{\partial x} \right)^2 + V + \frac{\partial S}{\partial t} - \frac{\hbar^2}{2mR} \frac{\partial^2 R}{\partial x^2} = \text{const.}$$  \hspace{1cm} (18)$$

We can obtain the constant’s value considering the solution for a free particle. Calculations done, it is easy to demonstrate that this constant should cancel itself. With this constant equal to zero, equation (16) is equivalent to

$$\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} - V(x) \Psi = -i\hbar \frac{\partial \Psi}{\partial t}$$  \hspace{1cm} (19)$$

since, if we substitute the decomposition made in expression (12) in the equation above, we obtain (19) Schroedinger’s Second Equation for the probability amplitude. It is thus demonstrated that

(T2) If we can write the density function \(\rho\) as the product (11), than the probability amplitude \(\Psi (x; t)\) satisfies Schroedinger’s Second Equation (19) together with the Equation of Continuity (15). (We will justify the nomenclature given to equation (15) further on).

We introduce the operator concept through the formal identification (an apostrophe will always be put in order to distinguish operators which act upon the density function from those that act upon the probability amplitude)

$$\hat{p}' = -i\hbar \frac{\partial}{\partial (\delta x)} ; \quad \hat{x}' = x,$$  \hspace{1cm} (20)$$

based on the fact that

$$\overline{p} = \lim_{\delta x \to 0} -i\hbar \frac{\partial}{\partial (\delta x)} \int F(x, p; t) \exp \left( \frac{ip\delta x}{\hbar} \right) dx dp$$  \hspace{1cm} (21)$$

and

$$\overline{x} = \lim_{\delta x \to 0} \int x F(x, p; t) \exp \left( \frac{ip\delta x}{\hbar} \right) dx dp.$$  \hspace{1cm} (22)$$

Thus,

(T3) The result of the operation upon the density function \(\rho\) of the momentum and position operators, defined by the expressions in (20), represents, respectively, the mean values for position and momentum for the ensembles’ components.
Remember that, using the expansion of the density function given in (13), we have
\[
\mathcal{T} = \lim_{\delta x \to 0} -i\hbar \frac{\partial}{\partial (\delta x)} \int F(x, p; t) \exp \left( \frac{i p \delta x}{\hbar} \right) dx dp = \int R(x)^2 \left( \frac{\partial S}{\partial x} \right) dx,
\]
and this justifies our calling equation (13) the Continuity Equation.

In order to obtain the momentum operator action upon the probability amplitude we can rewrite the equation above as
\[
\lim_{\delta x \to 0} -i\hbar \int \frac{\partial}{\partial (\delta x)} \left[ \Psi^* \left( x - \frac{\delta x}{2}; t \right) \Psi \left( x + \frac{\delta x}{2}; t \right) \right] dx
\]
and reach, after some calculations, the result
\[
\text{Re} \left\{ \int \Psi^* (x; t) \left(-i\hbar \frac{\partial}{\partial x} \right) \Psi (x; t) dx \right\}.
\]
The same can be done for the position operator (it is worth stressing that the hermitian character is automatically established). This allows us to define the position and momentum operators
\[
\hat{p} \Psi (x; t) = -i\hbar \frac{\partial}{\partial x} \Psi (x; t); \quad \hat{x} \Psi (x; t) = x \Psi (x; t)
\]
as usual. Defining the Hamiltonian operator as
\[
\hat{H} \Psi (x; t) = \left[ \frac{p^2}{2m} + V(x) \right] \Psi (x; t) = i\hbar \frac{\partial}{\partial t} \Psi (x; t),
\]
we can rewrite Schroedinger’s Second Equation in operator terms such as
\[
\hat{H} \Psi (x; t) = i\hbar \frac{\partial}{\partial t} \Psi (x; t).
\]
If we now define the commutator of two operators in the usual form, it becomes clear that
\[
\left[ \hat{x}, \hat{p} \right] = i\hbar.
\]
From this result, it is easy to show that the relation
\[
\Delta x \Delta p \geq \frac{\hbar}{2}
\]
known as Heisenberg’s uncertainty relation, must be valid. Before this result is interpreted, it is necessary to stress that
\[
\left[ \hat{x}', \hat{p}' \right] = 0
\]
and, therefore, that we will have the following relation associated with these operators:

\[ \Delta x \Delta p \geq 0. \]  \hspace{1cm} (32)

This result was, in fact, expected since no hypothesis about the mean squared deviations associated to the (classical) function \( F(x, p; t) \) were made. It demonstrates that the relation \( (30) \) results from the recognition of the possibility of writing the density function \( \rho \) as the product represented in \( (11) \). In this manner, far from representing a fundamental property of nature, relation \( (30) \) represents a limitation of our descriptions according to equation \( (13) \). We put this as a theorem:

\[ (T4) \] Quantum mechanics, as developed according to equation \( (14) \), is only applicable to problems where the density function \( \rho \) can be decomposed according to \( (11) \). In these cases, the product of the mean quadratic deviations in the position and momentum of a system represented by the joint probability density function \( F(x, p; t) \) is such that

\[ \Delta x \Delta p \geq \hbar/2 \]

and has a lower limit.

We here note that manuals commonly assume that the Second Schroedinger’s Equation for the probability amplitude is the fundamental one to be obtained and that, writing the density function as in \( (11) \), the First Schroedinger’s Equation is derived. We demonstrate here that this sequence is unjustifiable. While the equation for the amplitudes presents us with a dispersion relation such as in \( (30) \), the equation for the density function is dispersion free (see appendix D).

The results obtained above are the foundation of the whole non-relativistic quantum mechanical formalism, which we will not derive again.

Let us pass on to the derivation of Pauli’s equation

4 Pauli’s Equation

Up to this point we have discussed only systems constituted of particles with no internal degree of freedom. Consider now the case in which the system’s components possess an intrinsic magnetic moment capable of coupling to an external magnetic field. We can expand this field around the region occupied by the particle

\[ H(x) = H(x_0) + (x - x_0) \cdot \nabla H(x_0) + ..., \]  \hspace{1cm} (33)

where \( x_0 \) is the particle’s position. The coupling between the particle’s intrinsic magnetic moment and the magnetic field is

\[ F_m = -\nabla (m \cdot H) + ... \]  \hspace{1cm} (34)
Newton’s equation for this system can be written as

\[
\frac{dp}{dt} = f_{mec} + F_m, \tag{35}
\]

where \( f_{mec} \) represents general mechanical forces (derivable from a potential \( V(x) \)). Following similar steps to those used in the previous section and in appendix C, we reach the following equation for the density function

\[
\left\{ \frac{\hbar^2}{2m} [\nabla_y^2 - \nabla^2_{y'}] - [V(y) - V(y')] - [m \cdot H(y) - m \cdot H(y')] \right\} \rho = -i\hbar \frac{\partial}{\partial t} \rho \tag{36}
\]

which we call Pauli’s Equation for the density function[23].

The extra degrees of freedom are represented by quantities \( m_i \), which we do not know. To obtain information about these quantities, we can consider the precession equation which they should obey

\[
\frac{dm_i}{dt} = \epsilon_{ijk}m_k H_j, \tag{37}
\]

where \( \epsilon_{ijk} \) is the totally anti-symmetric tensor. Comparing these equations with those that we obtained when we wrote \( m_i \) as operators, we obtain the following commutation relation:

\[
\frac{1}{i\hbar} [\hat{m}_i, \hat{m}_j] = \epsilon_{ijk} \hat{m}_k \tag{38}
\]

from which we can construct an appropriate matrix representation.

We can still write

\[
\hat{m} = g \frac{e}{2mc} \hat{S} \quad ; \quad \hat{S} = \frac{\hbar}{2} \hat{\sigma}, \tag{39}
\]

where \( g \) is the Landé factor. Obviously, we can not know the value of this factor until we study the relativistic problem.

We have obtained the following result:

\[\text{(T5)}\] A body with internal magnetic moment capable of coupling to an external magnetic field obeys Pauli’s Equation [23].

In the second paper of this series we approach the general and special relativistic quantum mechanical problem of \textit{ensembles} composed of single particle systems. We will also derive Dirac’s and Klein Gordon’s Equations in addition to a general relativistic quantum equation.

## 5 The Statistical Potential

We can return to equation [18]

\[
\frac{1}{2m} \left( \frac{\partial S}{\partial x} \right)^2 + V + \frac{\partial S}{\partial t} - \frac{\hbar^2}{2mR} \frac{\partial^2 R}{\partial x^2} = 0 \tag{40}
\]
and note that it may be considered a Hamilton-Jacobi equation for one particle subjected to an effective potential

\[ V_{\text{eff}}(x) = V(x) - \frac{\hbar^2}{2mR} \frac{\partial^2 R}{\partial x^2}. \]  

Thus, we can formally write

\[ \frac{dp}{dt} = -\frac{\partial}{\partial x} V_{\text{eff}}(x) = -\frac{\partial}{\partial x} \left[ V(x) - \frac{\hbar^2}{2mR} \frac{\partial^2 R}{\partial x^2} \right], \]

along with the initial condition

\[ p = \frac{\partial S}{\partial x}. \]

The integration of the system \((42), (43)\) will give us a series of trajectories which will be equivalent to the "force" lines associated to the effective potential \((41)\). The resolution method for this system of equations is as follows: first Schrödinger’s equation must be solved in order to obtain the probability amplitudes referent to the ensemble. Once these amplitudes have been obtained, the effective potential, which will act as a statistical field for the ensemble, is built. One rather instructive example of this calculation is its application to the double slit experiment\([24]\) (an example of a time dependent problem can be found in the literature\([25]\).

It must be stressed that the equations for the individual constituents of the system are Newton’s equations. Thus, the potential \((41)\) must not be considered a real potential, but a fictitious potential which acts as a field in reproducing, through "trajectories", the statistical results of the original equation \((40)\). We see this potential as a statistical potential.

The discussion of Bohm’s hidden variable theory\([7]\) along this reinterpretation of the potential \((41)\) will be undertaken in the last paper of this series.

6 The Observer

With the advent of quantum mechanics, the question of the observer began to occupy a prominent position in physics. Treated with mathematical rigor for the first time by von Neumann\([14]\), and later by a series of authors\([4, 5, 17, 26, 27]\), the observer acquired elevated epistemological status through quantum theory, even though his function within this theory’s formalism is rather disputable\([13]\), since the variables to him associated never occur within the formalism itself (this will be further discussed in the third paper of this series).

The various measurement theories which have been proposed have profound epistemological and philosophical implications. One example is the Copenhagen Interpretation’s inevitable conclusion that the observer’s consciousness is necessary in the measuring process - as it is responsible for the collapse of the state
vector when an observation is made. We shall present a model in which the observer, seen as a physical system, may be introduced into quantum theory. We are not suggesting a general measurement theory, as did von Neumann, but analyzing a specific physical system. Nevertheless, the model here proposed will give us conditions to discuss the philosophical framework of measurement theories proposed for quantum theory.

It must be noted that the fundamental equation used, Liouville's equation in phase space, is valid only for a closed system. Such a system may have its behavior described by Schrödinger's First equation, but then there is no collapse of the state vector (no observer at all). In this manner, the state vector's collapse must always be postulated as a consequence of the intervention of a ghost external observer, as it is by von Neumann, for example. And more, since this observer could be considered a member of a larger system, we are forced, in traditional analysis, to an infinite regression which stops only when an observer with a consciousness capable of such a reduction is postulated.

Yet let us see how we can treat a particular case in the perspective of Realism. Consider a two-slit experiment as shown in figure 1. In this experiment the source sends particles through one of the first sources slits. These particles have distribution \( F_0(x, p; t) \) when they leave source and, without the influence of external factors, would have a \( F_1(x, p; t) \) distribution measured by the detectors of the second screen. But let us say that we desire to measure this distribution in a position previous to the second screen (a typically interphenomenon measure). In this case we position source, capable of emitting particles, as is shown in figure 1. These particles exit source with the distribution \( F_0^2(x, p; t) \) and if it was not for the particles they would hit detectors with the distribution \( F_2(x, p; t) \). According to the hypothesis of Realism, we can expect that the alteration of these two distributions will be the product of collisions between both system’s particles (considered distinct here for simplicity). So, after the collisions take place we expect to find system and system represented by the distributions \( F_1^1(x, p; t) \) and \( F_1^2(x, p; t) \) respectively.

We expect, as we do for Boltzmann’s Equation, that the equation satisfied by the new distribution be now given as

\[
\frac{dF_1^1}{dt} = \frac{\partial F_1^1}{\partial t} + \frac{p_1}{m_1} \cdot \nabla_x F_1^1 + \frac{d\mathbf{p}_1}{dt} \cdot \nabla_p F_1^1 = D_CF_1^1,
\]

where \( D_CF_1 \) represents the change in distribution \( F_1 \) due to the collisions. We can divide this change according to two sources: one caused by the collisions which fling the particles into the phase space element, which we call \( D_C^{(+)}F_1dx_1d\mathbf{p}_1 \), and one that flings them out of this volume, which we call \( D_C^{(-)}F_1dx_1d\mathbf{p}_1 \). It is clear that

\[
D_CF_1 = D_C^{(+)}F_1 - D_C^{(-)}F_1.
\]

The term \( D_C^{(-)}F_1dx_1d\mathbf{p}_1 \) can be calculated once considered that, within
the volume element, the probability of a collision sending particles outside this volume is

\[ \sigma (p_1, p_2 \rightarrow p_1', p_2') \, d^3p_1' \, d^3p_2', \]  

(46)

where \( \sigma (p_1, p_2 \rightarrow p_1', p_2') \) is the cross section for the collisions in which the particles, initially with momenta \( p_1 \) and \( p_2 \), begin to have \( p_1' \) and \( p_2' \) momenta, respectively. If we multiply this number by the flux of particle existing within this volume

\[ F_1(x, p_1; t) \left| \frac{p_1}{m_1} - \frac{p_2}{m_2} \right| \, d^3p_1 \]  

(47)

and by the number of type 2 particles which can bring forth such a collision

\[ F_2(x, p_2; t) \, d^3x \, d^3p_2, \]  

(48)

we obtain

\[ D_C^{-} F_1(x, p_1; t) = \int_{p_1'} \int_{p_2'} \int_{p_2} \left| \frac{p_1}{m_1} - \frac{p_2}{m_2} \right| F_1(1) F_2(2) \sigma (p_1, p_2 \rightarrow p_1', p_2') \, d^3p_1' \, d^3p_2' \, d^3p_2, \]  

(49)

where \( F_1(1) \) and \( F_2(2) \) represent \( F_1(x, p_1; t) \) and \( F_2(x, p_2; t) \), respectively. In the same manner, using the inverse scattering arrangement \( \Theta \), we get for \( D_C^{(+)} F_1 \):

\[ D_C^{(+)} F_1(x, p_1; t) = \int_{p_1'} \int_{p_2'} \int_{p_2} \left| \frac{p_1}{m_1} - \frac{p_2}{m_2} \right| F_1^{(1')} F_2^{(2')} \sigma (p_1', p_2' \rightarrow p_1, p_2) \, d^3p_1' \, d^3p_2' \, d^3p_2, \]  

(50)

where \( F_1^{(1')} \) and \( F_2^{(2')} \) represent \( F_1^{(1)}(x, p_1'; t) \) and \( F_2^{(1)}(x, p_2'; t) \), respectively. Using the fact that

\[ \sigma (p_1, p_2 \rightarrow p_1', p_2') = \sigma (p_1', p_2' \rightarrow p_1, p_2) \]  

(51)

and that, for elastic collisions,

\[ \left| \frac{p_1}{m_1} - \frac{p_2}{m_2} \right| = \left| \frac{p_1'}{m_1} - \frac{p_2'}{m_2} \right| = \vartheta, \]  

(52)

we finally have the factor

\[ D_C F_1(1) = \int_{p_1'} \int_{p_2'} \int_{p_2} \left| \frac{p_1}{m_1} - \frac{p_2}{m_2} \right| (F_1(1) F_2(2) - F_1^{(1')} F_2^{(2')}) \cdot \sigma (p_1', p_2' \rightarrow p_1, p_2) \, d^3p_1' \, d^3p_2' \, d^3p_2. \]  

(53)

Taking this result to equation \([\square]\), we have

\[ \frac{\partial F_1}{\partial t} + \frac{p_1}{m_1} \cdot \nabla_x F_1 + \frac{d p_1}{d t} \cdot \nabla_p F_1 = \]
\[ \int_{p_1} \int_{p_2} \int_{p_1'} \int_{p_2'} (F_1 (1) F_2 (2) - F_1^1 (1') F_2^1 (2')) \vartheta \sigma d^3 p_1 d^3 p_1' d^3 p_2 d^3 p_2' \]  

(54)

and, after applying the Wigner-Moyal Infinitesimal Transformation, we obtain

\[ \left\{ \frac{\hbar^2}{2m_1} \left[ \nabla^2_{y_1} - \nabla^2_{y_1'} \right] - [V (y_1) - V (y_1')] \right\} \rho (1) + i \hbar \frac{\partial}{\partial t} \rho (1) = \]

\[ = \int_{p_1} \int_{p_2} \int_{p_1'} \int_{p_2'} (F_1 (1) F_2 (2) - F_1^1 (1') F_2^1 (2')) \vartheta \sigma (p_1', p_2' \rightarrow p_1, p_2) \cdot \exp \left[ \frac{i}{\hbar} p_1' \cdot (y_1 - y_1') \right] d^3 p_1 d^3 p_1' d^3 p_2 d^3 p_2' , \]

where, as usual, we did

\[ y_1 = x_1 + \frac{\delta x_1}{2} ; \quad y_1' = x_1 - \frac{\delta x_1}{2} \]

and \( \rho (1) = \rho (y_1', y_1) \) refers, naturally, to system1.

It is obvious that we can invert the problem and interpret system1 as the observer and system2 as the observed. In this case, we would have for system2 an equation similar to (55)

\[ \left\{ \frac{\hbar^2}{2m_2} \left[ \nabla^2_{y_2} - \nabla^2_{y_2'} \right] - [V' (y_2) - V' (y_2')] \right\} \rho (2) + i \hbar \frac{\partial}{\partial t} \rho (2) = \]

\[ = \int_{p_2} \int_{p_1'} \int_{p_1} (F_1 (1) F_2 (2) - F_1^1 (1') F_2^1 (2')) \vartheta \sigma (p_1', p_2' \rightarrow p_1, p_2) \cdot \exp \left[ \frac{i}{\hbar} p_2' \cdot (y_2 - y_2') \right] d^3 p_2 d^3 p_2' d^3 p_1 d^3 p_1' , \]

where \( \rho (2) = \rho (y_2', y_2) \) refers to system2 subject to a potential \( V' (x) \), possibly distinct from \( V (x) \) (we are supposing a purely contact based interaction between the different particles and that there are no multiple collisions between the particles).

This "symmetry" between observer and observable is extremely important for the special theory of relativity's point of view. It is also important to observe that it is no longer possible to obtain an equation such as Schrödinger's Second Equation from (55) or (56) (of course, for a weak interaction between the two systems, such an equation can be approximated). The "wave" properties associated to these particles should, depending on the intensity of the interaction between both system, disappear. We here note that this property, as all of this problem's treatment, is quite distinct from the state vector's collapse.

We will discuss the epistemological implications of these and other properties of equations (55) and (56) in the third paper of this series.
7 Conclusion

From no more than three axioms and within a classical and coherent with Realism view of nature, it was possible to derive all of quantum mechanical formalism. It was also possible to derive the important issue of Heisenberg’s dispersion relations, one of the most fundamental result of the Copenhagen Interpretation[12].

In the second paper of this series, we will undertake relativistic treatment. In the final paper, we will discuss the epistemological implications of the results that we have obtained.

A Mathematical Properties of the Transformation

The Wigner-Moyal Infinitesimal Transformation is defined as the following Fourier Transform

$$\rho \left( x + \frac{\delta x}{2}, x - \frac{\delta x}{2}; t \right) = \int F(x, p; t) \exp \left( i \frac{p\delta x}{\hbar} \right) dp.$$  \hspace{1cm} (57)

One could think, formally and at first sight, that the function $F(x, p; t)$ could be obtained through the inverse Fourier Transform

$$F(x, p; t) = \int \rho \left( x + \frac{\delta x}{2}, x - \frac{\delta x}{2}; t \right) \exp \left( -i \frac{p\delta x}{\hbar} \right) d(\delta x).$$  \hspace{1cm} (58)

Yet it is known[21, 22, 29] that the function defined in (58) is not positive-defined, and thus, the argument runs, it can not be considered a true probability density.

Nevertheless, we should stress that it was necessary to consider $\delta x$ as being infinitesimal, in order for the density function to satisfy Schroedinger’s First Equation. In this manner, the transformation (58) can not be performed. This means that, even possessing the solution to the quantum problem, given by the density function, we can not obtain the probability density $F(x, p; t)$ in phase space. On the other hand, it is interesting to note that, if we have the solution in phase space (usually called classical), we can obtain the density function (usually called quantum) through the application of (57).

Even in view of the impossibility of the inverse transformation (58), we can easily demonstrate that

$$F_p(p; t) = \int F(x, p; t) dx \geq 0$$  \hspace{1cm} (59)

and, therefore, $F_p(p; t)$ serves as a probability density in momentum space (however we must stress that this is presupposed in the present formalism). In the
same manner, it can be shown that

$$F_x(x; t) = \int F(x, p; t) \, dp \geq 0$$

(60)

and, therefore, $F_x(x; t)$ serve as a probability density in real space.

### A.1 Correspondence

Let us suppose that we have a dispersion free ensemble of a single particle system. In this case, the joint probability function is given as

$$F(x, p; t) = \delta(x - x_0(t)) \delta(p - p_0(t)),$$

(61)

meaning that, given the same initial conditions, the trajectory in phase space followed by the particle will always be the same, given by $x_0(t)$ and $p_0(t)$. The density function for this problem is, using (57)

$$\rho\left(x + \frac{\Delta x}{2}, x - \frac{\Delta x}{2}; t\right) = \delta(x - x_0(t)) \exp\left(\frac{i}{\hbar}p_0(t) \Delta x\right),$$

(62)

where we use $\Delta x$ for infinitesimal dislocations so as to avoid confusion with Dirac’s delta distributions. Note that we can not write the density function as the product of amplitudes. This is not unexpected, as this function was derived from a probability density which did not satisfy dispersion relations. Substituting this expression in equation (6), we obtain

$$i\hbar\left[\frac{p_0(t)}{m} \frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right] \delta(x - x_0(t)) +$$

$$+ \Delta x \left[\frac{dp_0(t)}{dt} + \left(\frac{\partial V}{\partial x}\right)_{x=x_0}\right] \delta(x - x_0(t)) = 0,$$

(63)

which is a decomposition formally similar to that done in equation (14), in the appropriate limit, with the first term representing the Continuity Equation and the last one, Schroedinger’s Second Equation.

Noting that the real and complex parts should be equal to zero separately, we obtain

$$\frac{dp_0(t)}{dt} = -\left(\frac{\partial V}{\partial x}\right)_{x=x_0},$$

(64)

for the real term, and

$$\frac{p_0(t)}{m} = \left(\frac{dx}{dt}\right)_{x=x_0},$$

(65)

for the complex one. These are nothing more than Newton’s equations satisfied by each one of the ensemble’s components.
It must be noted that the result obtained above does not depend on Planck’s constant. It is well stated in the existing literature that the "classical limit" is not always obtained when we make Planck’s constant tend to zero. Also note that, in the perspective of the present work, the limit \( \bar{\hbar} \to 0 \) would not make sense due to the Wigner-Moyal Infinitesimal Transformation.

## B Operator Construction

One of the problems found within the usual formulation of quantum mechanics concerns the construction of operators which represent a certain function in the phase space. This problem is basically caused by the fact that we usually work with operators which act upon the probability amplitude and do not commute with each other. We have seen, in the third section of this paper, that we can define position and momentum operators which act upon the density function and commute with each other. Therefore we hope to be capable of, given a function \( f(x, p; t) \), constructing an operator to represent it when acting upon the density function. In fact, let \( f(x, p; t) \) be a function whose mean value we desire to calculate. In this case we have

\[
\mathcal{F}(x, p; t) = \int \int f(x, p; t) F(x, p; t) \, d^3x \, d^3p =
\]

\[
= \lim_{\delta x \to 0} \int \int f(x, p; t) F(x, p; t) \exp\left(\frac{i}{\hbar} \frac{p \cdot \delta x}{\hbar}\right) \, d^3x \, d^3p =
\]

\[
= \lim_{\delta x \to 0} \int \int O_p(x, \delta x; t) \rho \left( x + \frac{\delta x}{2}, x - \frac{\delta x}{2}; t \right) \, d^3x,
\]

and can say that \( O_p(x, \delta x; t) \) is the operator associated to \( f(x, p; t) \), with the process of limit understood.

It is thus easy to demonstrate as an example that, for angular momentum,

\[
\mathbf{L} = \mathbf{x} \times \mathbf{p}
\]

we get for the components

\[
\hat{L}_i' = -i \hbar \epsilon_{ijk} \frac{\partial}{\partial (\delta x_k)}.
\]

where we place an apostrophe on \( L_i' \) to mark that this operator acts upon a density function and not upon the probability amplitude. We will have, in general, the following correspondence rule

\[
O_p(g(x, p)) = g(x, -i \hbar \frac{\partial}{\partial (\delta x)}).
\]
The greatest difficulty associated to the methods of operator construction in usual quantum mechanics refers to a certain ambiguity which they present or their incompatibility to the one to one correspondence between operators and observables postulated in quantum mechanics. Among the methods proposed in the literature, we can cite: von Neumann’s rules, Weyl’s rules, Revier’s rules, etc.

The ambiguity problem has, nevertheless, prevailed. In fact, for the function \( p^2 x^2 \), von Neumann’s rules give, for example:

\[
O \left( p^2 x^2 \right) = \hat{x} \hat{p} - 2i\hbar \hat{x} \hat{p} - \frac{1}{4} \hbar^2 - 2i \hbar \hat{x} \hat{p} - \hbar^2,
\]

where \( O \left( p^2 x^2 \right) \) represents the operator associated to the calculation of the mean value of his argument. According to the present theory we have, naturally,

\[
O' \left( p^2 x^2 \right) = -\hbar^2 x^2 \frac{\partial^2}{\partial (\delta x_k)^2},
\]

which, expanding the density function according to (11) and performing the calculations, reduces to the following operator for the probability amplitude

\[
O \left( p^2 x^2 \right) = \hat{x} \hat{p} - i\hbar \hat{x} \hat{p},
\]

which does not present any ambiguity.

\section{Three-dimensional Formalism}

For a problem involving \( N \) particles the state of the ensemble is represented by the function \( F(x_1, p_1, \ldots, x_N, p_N; t) \). For this function we will have

\[
\frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_i \frac{dx_i}{dt} \cdot \nabla_{x_i} F + \sum_i \frac{dp_i}{dt} \cdot \nabla_{p_i} F = 0.
\]

We can use

\[
\frac{dx_i}{dt} = \frac{p_i}{m} ; \quad \frac{dp_i}{dt} = f^i_i + f^i_k,
\]

where \( f^i_k \) are the internal forces represented by

\[
f^i_k = \sum_{l \neq k} f^i_{l \rightarrow k} (x_{kl}) ; \quad x_{kl} = x_k - x_l,
\]

with \( f^i_{l \rightarrow k} \) representing the internal force exercised by particle \( l \) upon particle \( k \), depending only on the relative position, and \( f^e_k \) are the external forces acting on particle \( k \).
Using the potentials
\[ f_k^i = -\nabla x_k V^i ; \quad V^i = \frac{1}{2} \sum_{l \neq k} V_{kl}^i (x_{kl}) \]  

(76)

and
\[ f_k^c = -\nabla x_k V_c^i, \]  

(77)

jointly with the Wigner-Moyal Infinitesimal Transformation
\[ \rho (x_1 + \frac{\delta x_1}{2}, x_1 - \frac{\delta x_1}{2}, ..., x_N + \frac{\delta x_N}{2}, x_N - \frac{\delta x_N}{2}; t) = \]
\[ = \int .. \int F(x_1, p_1, ..., x_N, p_N; t) \exp \left[ \frac{i}{\hbar} (p_1 \cdot \delta x_1 + .. + p_N \cdot \delta x_N) \right] d^3 p_1 .. d^3 p_N, \]  

(78)

we reach the equation
\[ \left\{ \frac{\hbar^2}{2m} \left[ \nabla^2_{y_k} - \nabla^2_{y'_k} \right] - \sum_i \left[ V^i (y_{kl}) - V^i (y'_{kl}) \right] \right\} \rho - \]
\[ - \sum_i \left[ V_c^i (y_{kl}) - V_c^i (y'_{kl}) \right] \rho = -i\hbar \frac{\partial}{\partial t} \rho, \]  

(79)

where we have made the usual variable transformations. Equation (79) is the Schroedinger’s First Equation for the density function for an ensemble built of \( N \) particle systems. Schroedinger’s Second Equation can be obtained from (79) with calculations similar to those realized for the one dimensional problem.

## D Density Matrix

Until now we have only dealt with ensembles which can be represented by pure states. In this appendix we will proceed to generalize the theory for mixed states. We will also show that the procedure usually followed in the literature involves an additional assumption. Indeed, when developing the density matrix theory one usually begins with the amplitudes which are solutions of the Second Schroedinger’s Equation \( |\Psi\rangle \) and then define the density function as the product
\[ |\Psi (y)\rangle \langle \Psi (y')| \]  

(80)

where \( y \) and \( y' \) run independently. We will suppose throughout this appendix that it is always possible to write the density function as the product (11) and will show that (80) is related to this additional assumption.

It was seen for pure states (see appendix B) that, when the decomposition given by expression (11) is possible, we can always write an operator acting upon
the density function $\hat{Q}$ as another one acting upon the probability amplitudes $\hat{Q}$, mathematically:

$$\langle Q \rangle = \lim_{\delta x \to 0} \int \hat{Q} \rho \left( x - \frac{\delta x}{2}, x + \frac{\delta x}{2}; t \right) dx = \int \Psi^\ast(x; t) \hat{Q} \Psi(x; t) dx,$$

where

$$\rho \left( x - \frac{\delta x}{2}, x + \frac{\delta x}{2}; t \right) = \Psi^\ast \left( x - \frac{\delta x}{2}; t \right) \Psi \left( x + \frac{\delta x}{2}; t \right)$$

is the density function representing the pure state defined by $\Psi(x; t)$.

The generalization for mixed states can be done writing the density function as

$$\rho \left( x - \frac{\delta x}{2}, x + \frac{\delta x}{2}; t \right) = \sum_n W_n \Psi_n^\ast \left( x - \frac{\delta x}{2}; t \right) \Psi_n \left( x + \frac{\delta x}{2}; t \right),$$

where the $W_n$ are the statistical weights. It is easy to see that

$$\langle Q \rangle = \lim_{\delta x \to 0} \int \hat{Q} \rho \left( x - \frac{\delta x}{2}, x + \frac{\delta x}{2}; t \right) dx = \sum_n W_n \int \Psi_n^\ast(x; t) \hat{Q} \Psi_n(x; t) dx.$$  \hspace{1cm} (84)

We can now choose a convenient representation for our amplitudes using some set of orthonormal basis states $\{\phi_i\}$, $i = 1, 2, ..$ for which

$$\Psi_n(x; t) = |\Psi_n\rangle = \sum_m a_m^{(n)} |\phi_m\rangle,$$

where we used Dirac’s representation of Bras and Kets to simplify the calculations. With expression (85), the mean value of the operator $Q$ can be written as

$$\langle Q \rangle = \sum_{n,m',m} W_n a_{m'}^{(n)} a_m^{(n)} \langle \phi_{m'} | Q | \phi_m \rangle.$$  \hspace{1cm} (86)

Now we can define our density matrix, in the representation given by $\{\phi_i\}$, as the matrix which elements are given by

$$\rho \left( m', m \right) = \sum_n W_n a_{m'}^{(n)} a_m^{(n)}.$$  \hspace{1cm} (87)

Since, by means of (83) and orthonormality, we have

$$a_m^{(n)} = \langle \phi_m | \Psi_n \rangle \quad \text{and} \quad a_{m'}^{(n)} = \langle \Psi_n | \phi_{m'} \rangle,$$

we get

$$a_m^{(n)} = \langle \phi_m | \Psi_n \rangle \quad \text{and} \quad a_{m'}^{(n)} = \langle \Psi_n | \phi_{m'} \rangle.$$  \hspace{1cm} (88)
it is possible to write
\[ \rho_{m,m'} = \sum_n W_n \langle \phi_m | \Psi_n \rangle \langle \Psi_n | \phi_{m'} \rangle, \tag{89} \]
by which the density function follows
\[ \rho (y', y) = \sum_n W_n |\Psi_n (y') \rangle \langle \Psi_n (y) |, \tag{90} \]
where \( y \) and \( y' \) should be independent variables for the dot product implied by \( 89 \) to be correct. Using results \( 86-90 \) we can see that we have
\[ \langle Q \rangle = Tr \left( \rho \cdot Q \right). \tag{91} \]

In the above derivation, we begin with the density function and an operator acting upon it and then, supposing decomposition \( 82 \), turned into a formalism with probability amplitudes and the related modified operator. When making such calculations it was necessary to take the limit \( \delta x \to 0 \). This limit implies changing from a commutative formalism into a non–commutative one, as is stated by theorem (T4). Expression \( 84 \) and \( 91 \) are equivalent only if it is possible to consider \( y \) and \( y' \) as independent variables (not necessarily infinitesimally separated). That such a supposition is made when we use decomposition \( 82 \) will now be demonstrated.

When decomposition \( 82 \) is assumed, it is easy to show that the density function equation can be written as
\[
\left\{ \frac{1}{\Psi} \left[ H\Psi - i\hbar \frac{\partial \Psi}{\partial t} \right] \right\} \left( x + \frac{\delta x}{2} \right) - \left\{ \frac{1}{\Psi} \left[ H\Psi - i\hbar \frac{\partial \Psi}{\partial t} \right] \right\} \left( x - \frac{\delta x}{2} \right) = \\
= \delta x \frac{\partial}{\partial x} \left\{ \frac{1}{\Psi} \left[ H\Psi - i\hbar \frac{\partial \Psi}{\partial t} \right] \right\} (x) = 0, \tag{92} \]
where \( H \) is the Hamiltonian defined in \( 27 \). But if we say that
\[ H\Psi (x) - i\hbar \frac{\partial \Psi (x)}{\partial t} = 0, \forall x \in R, \tag{93} \]
then equation \( 92 \) is satisfied independently of the infinitesimal parameter. This ends the demonstration.

So we conclude that, only when it is possible to write the density function as in \( 82 \), the mean values of the operators \( Q \) can be calculated using formulae \( 85-91 \), as is usually done in the literature.
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Figure 1: Experimental environment for the introduction of the observer into quantum mechanical formalism.