BOUND STATES OF THE DIRAC EQUATION FOR A CLASS OF EFFECTIVE QUADRATIC PLUS INVERSELY QUADRATIC POTENTIALS

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Abstract

The Dirac equation is exactly solved for a pseudoscalar linear plus Coulomb-like potential in a two-dimensional world. This sort of potential gives rise to an effective quadratic plus inversely quadratic potential in a Sturm-Liouville problem, regardless the sign of the parameter of the linear potential, in sharp contrast with the Schrödinger case. The generalized Dirac oscillator already analyzed in a previous work is obtained as a particular case.
The Schrödinger equation with a quadratic plus inversely quadratic potential, known as singular oscillator, is an exactly solvable problem [1]-[6] which works for constructing solvable models of $N$ interacting bodies [7]-[8] as well as a basis for perturbative expansions and variational analyses for spiked harmonic oscillators [9]-[14]. Generalizations for finite-difference relativistic quantum mechanics [15] as well as for time-dependent parameters in the nonrelativistic version have also been considered [16]-[17]. In the present paper we approach the time-independent Dirac equation with a conserving-parity pseudoscalar potential given by a linear plus a Coulomb-like potential. This sort of potential gives rise to an effective quadratic plus inversely quadratic potential in a Sturm-Liouville problem, regardless the sign of the parameter of the linear potential. Therefore, this sort of problem is exactly solvable in the Dirac equation. The generalized Dirac oscillator [18] is obtained as a particular case. In addition to its importance as a new solution of the Dirac equation this problem might be relevant to studies of confinement of neutral fermions by a linear plus inversely linear electric field.

The two-dimensional Dirac equation can be obtained from the four-dimensional one with the mixture of spherically symmetric scalar, vector and anomalous magnetic-like (tensor) interactions. If we limit the fermion to move in the $x$-direction ($p_y = p_z = 0$) the four-dimensional Dirac equation decomposes into two equivalent two-dimensional equations with 2-component spinors and $2 \times 2$ matrices [19]. Then, there results that the scalar and vector interactions preserve their Lorentz structures whereas the anomalous magnetic interaction turns out to be a pseudoscalar interaction. Furthermore, in the 1+1 world there is no angular momentum so that the spin is absent. Therefore, the 1+1 dimensional Dirac equation allow us to explore the physical consequences of the negative-energy states in a mathematically simpler and more physically transparent way. The confinement of fermions by a pure conserving-parity pseudoscalar double-step potential [20] and their scattering by a pure nonconserving-parity pseudoscalar step potential [21] have already been analyzed in the literature providing the opportunity to find some quite interesting results. Indeed, the two-dimensional version of the anomalous magnetic-like interaction linear in the radial coordinate, christened by Moshinsky and Szczepaniak [22] as Dirac oscillator, has also received attention. Nogami and Toyama [23], Toyama et al. [24] and Toyama and Nogami [25] studied the behaviour of wave packets under the influence of such a conserving-parity potential, Szmytkowski and Gruchowski [26] proved the completeness of the eigenfunctions and Pacheco et al. [27] studied some
thermodynamics properties. More recently de Castro [18] analyzed the possibility of existence of confinement of fermions under the influence of pseudoscalar power-law potentials, including the case of potentials unbounded from below, and discussed in some detail the eigenvalues and eigenfunctions for conserving- and nonconserving-parity linear potentials.

In the presence of a time-independent pseudoscalar potential the 1+1 dimensional time-independent Dirac equation for a fermion of rest mass $m$ reads

$$\left( c\alpha p + \beta mc^2 + \beta\gamma^5 V_p \right) \psi = E\psi \tag{1}$$

where $E$ is the energy of the fermion, $c$ is the velocity of light and $p$ is the momentum operator. $\alpha$ and $\beta$ are Hermitian square matrices satisfying the relations $\alpha^2 = \beta^2 = 1$, $\{\alpha, \beta\} = 0$. From the last two relations it follows that both $\alpha$ and $\beta$ are traceless and have eigenvalues equal to $\pm 1$, so that one can conclude that $\alpha$ and $\beta$ are even-dimensional matrices. One can choose the $2 \times 2$ Pauli matrices satisfying the same algebra as $\alpha$ and $\beta$, resulting in a 2-component spinor $\psi$. The positive definite function $|\psi|^2 = \psi^\dagger \psi$, satisfying a continuity equation, is interpreted as a probability position density and its norm is a constant of motion. This interpretation is completely satisfactory for single-particle states [28]. It is worth to note that the Dirac equation is covariant under $x \to -x$ if $V(x)$ changes sign. This is because the parity operator $P = \exp(i\varepsilon)P_0\sigma_3$, where $\varepsilon$ is a constant phase and $P_0$ changes $x$ into $-x$, changes sign of $\alpha$ and $\beta\gamma^5$ but not of $\beta$. Using $\alpha = \sigma_1$ and $\beta = \sigma_3$, $\beta\gamma^5 = \sigma_2$ and provided that the spinor is written in terms of the upper and the lower components

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \tag{2}$$

the Dirac equation decomposes into :

$$-\left( E - mc^2 \right) \psi_+ = i\hbar c \psi'_- + iV \psi_- \tag{3}$$

$$-\left( E + mc^2 \right) \psi_- = i\hbar c \psi'_+ - iV \psi_+$$

where the prime denotes differentiation with respect to $x$. In terms of $\psi_+$ and $\psi_-$ the spinor is normalized as $\int_{-\infty}^{+\infty} dx \left( |\psi_+|^2 + |\psi_-|^2 \right) = 1$, so that $\psi_+$ and $\psi_-$ are square integrable functions. It is clear from the pair of coupled first-order
differential equations \(\text{(3)}\) that \(\psi_+\) and \(\psi_-\) have definite and opposite parities if the Dirac equation is covariant under \(x \to -x\). In the nonrelativistic approximation (potential energies small compared to \(mc^2\) and \(E \approx mc^2\)) Eq. \(\text{(3)}\) loses all the matrix structure and becomes

\[
\psi_- = \frac{p}{2mc} \psi_+ \tag{4}
\]

\[
\left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^+_{\text{eff}}\right) \psi_+ = \left(E - mc^2\right) \psi_+ \tag{5}
\]

where \(V^+_{\text{eff}} = V^2/2mc^2 + \hbar/2mc V'\). Eq. \(\text{(4)}\) shows that \(\psi_-\) if of order \(v/c << 1\) relative to \(\psi_+\) and Eq. \(\text{(5)}\) shows that \(\psi_+\) obeys the Schr"odinger equation with the effective potential \(V^+_{\text{eff}}\). It is noticeable that this peculiar (pseudoscalar-) coupling results in the Schr"odinger equation with an effective potential in the nonrelativistic limit, and not with the original potential itself. The form in which the original potential appears in the effective potential, the \(V^2\) term, allows us to infer that even a potential unbounded from below could be a confining potential. This phenomenon is inconceivable if one starts with the original potential in the nonrelativistic equation.

The coupling between the upper and the lower components of the Dirac spinor can be formally eliminated when Eq. \(\text{(3)}\) is written as second-order differential equations:

\[
-\frac{\hbar^2}{2m} \psi''_\pm + V^\pm_{\text{eff}} \psi_\pm = E_{\text{eff}} \psi_\pm \tag{6}
\]

where

\[
E_{\text{eff}} = \frac{E^2 - m^2 c^4}{2mc^2} \tag{7}
\]

\[
V^\pm_{\text{eff}} = \frac{V^2}{2mc^2} \pm \frac{\hbar}{2mc} V' \tag{8}
\]

These last results show that the solution for this class of problem consists in searching for bounded solutions for two Schr"odinger equations. It should not be forgotten, though, that the equations for \(\psi_+\) or \(\psi_-\) are not indeed independent because the effective eigenvalue, \(E_{\text{eff}}\), appears in both equations. Therefore, one has to search for bound-state solutions for \(V^+_{\text{eff}}\) and \(V^-_{\text{eff}}\) with a common eigenvalue.
Now let us consider a pseudoscalar potential in the form

\[ V = m\omega cx + \frac{\hbar cg}{x} \]  

(9)

where \( \omega \) and \( g \) are real parameters. Then the effective potential becomes the singular harmonic oscillator

\[ V_{\text{eff}}^\pm = Ax^2 + \frac{B_\pm}{x^2} + C_\pm \]  

(10)

where

\[
A = \frac{1}{2} m \omega^2 \\
B_\pm = \frac{\hbar^2 g}{2m} (g \mp 1) \\
C_\pm = \hbar \omega \left( g \pm \frac{1}{2} \right)
\]

It is worthwhile to note at this point that the singularity at \( x = 0 \) never menaces the fermion to collapse to the center \( \Box \) because in any condition \( B_\pm \) is never less than the critical value \( B_c = -\hbar^2/8m \). Furthermore, \( \omega \) must be different from zero, otherwise there would be an everywhere repulsive effective potential as long as the condition \( B_+ < 0 \) and \( B_- < 0 \) is never satisfied simultaneously. Therefore, the parameters of the effective potential with \( \omega \neq 0 \) fulfill the key conditions to furnish spectra purely discrete with infinite sequence of eigenvalues, regardless the signs of \( \omega \) and \( g \). Note also that the parameters of the effective potential are related in such a manner that the change \( \omega \to -\omega \) induces the change \( V_{\text{eff}}^\pm \to V_{\text{eff}}^\pm - 2C_\pm \) (\( C_\pm \to -C_\pm \)) whereas under the change \( g \to -g \) one has \( V_{\text{eff}}^\pm \to V_{\text{eff}}^\pm - 2C_\mp \) (\( B_\pm \to B_\mp \) and \( C_\pm \to -C_\mp \)). The combined transformation \( \omega \to -\omega \) and \( g \to -g \) has as effect \( V_{\text{eff}}^\pm \to V_{\text{eff}}^\mp \), meaning that the effective potential for \( \psi_+ \) transforms into the effective potential for \( \psi_- \) and vice versa. In Figure 1 is plotted the effective potential for some illustrative values of \( g \). From this Figure one can see that in the case \( |g| > 1 \) both components of the Dirac spinor are subject to the singular harmonic oscillator, for \( |g| = 1 \) one component is subject to the harmonic oscillator while the other one is subject to the singular harmonic oscillator, for \( |g| < 1 \) one component is subject to a bottomless potential well.
and the other one is subject to the singular harmonic oscillator, in the case $g = 0$ both components are subject to harmonic oscillator potentials.

Defining

$$\xi = \sqrt{\frac{2mA}{\hbar}} x^2$$

(12)

and using (6)-(8) one obtains the equation

$$\xi \psi''_\pm + \frac{1}{2} \psi'_\pm + \left[ -\frac{\xi}{4} - \frac{mB_\pm}{2\hbar^2 \xi} + \frac{1}{4\hbar} \sqrt{\frac{2m}{A}} (E_{eff} - C_\pm) \right] \psi_\pm = 0$$

(13)

Now the prime denotes differentiation with respect to $\xi$. The normalizable asymptotic form of the solution as $\xi \to \infty$ is $e^{-\xi/2}$. As $\xi \to 0$, when the term $1/x^2$ dominates, the regular solution behaves as $\xi^{s/2}$, where $s$ is a nonnegative solution of the algebraic equation

$$s(s - 1) - 2mB_\pm/\hbar^2 = 0$$

(14)

viz.

$$s = \frac{1}{2} \left( 1 \pm \sqrt{1 + \frac{8mB_\pm}{\hbar^2}} \right) \geq 0$$

(15)

If $B_\pm > 0$ there is just one possible value for $s$ (that ones with the plus sign in front of the radical) and the same is true for $B_\pm = B_c$ when $s = 1/2$, but for $B_c < B_\pm < 0$ there are two possible values for $s$ in the interval $0 < s < 1$. If the singular potential is absent ($B_\pm = 0$) then $s = 0$ or $s = 1$. The solution for all $\xi$ can be expressed as $\psi_\pm(\xi) = \xi^{s/2} e^{-\xi/2} w(\xi)$, where $w$ is solution of the confluent hypergeometric equation [29]

$$\xi w'' + (b - \xi)w' - aw = 0$$

(16)

with

$$a = \frac{b}{2} - \frac{1}{4\hbar} \sqrt{\frac{2m}{A}} (E_{eff} - C_\pm)$$

(17)

$$b = s + 1/2$$
Then $w$ is expressed as $_1F_1(a, b, \xi)$ and in order to furnish normalizable $\psi_\pm$, the confluent hypergeometric function must be a polynomial. This demands that $a = -n$, where $n$ is a nonnegative integer in such a way that $_1F_1(a, b, \xi)$ is proportional to the associated Laguerre polynomial $L^{b-1}_n(\xi)$, a polynomial of degree $n$. This requirement, combined with the top line of (17), also implies into quantized effective eigenvalues:

$$E_{\text{eff}} = \left(2n + s + \frac{1}{2}\right) \hbar|\omega| + C_\pm, \quad n = 0, 1, 2, \ldots$$  \hspace{1cm} (18)

with eigenfunctions given by

$$\psi_\pm(x) = N_\pm \xi^{s/2} e^{-\xi/2} L^{s-1/2}_n(\xi)$$  \hspace{1cm} (19)

On the other hand, $s = 0$ and $s = 1$ for the case $B_\pm = 0$ and the associated Laguerre polynomial $L^{-1/2}_n(\xi)$ and $L^{+1/2}_n(\xi)$ are proportional to $H_{2n} \left(\sqrt{\xi}\right)$ and $\xi^{-1/2} H_{2n+1} \left(\sqrt{\xi}\right)$, respectively [29]. Therefore, the solution for the harmonic oscillator can be succinctly written in the customary form in terms of Hermite polynomials:

$$E_{\text{eff}} = \left(n + \frac{1}{2}\right) \hbar|\omega| + C_\pm, \quad n = 0, 1, 2, \ldots$$  \hspace{1cm} (20)

$$\psi_\pm(x) = N_\pm e^{-\xi/2} H_n \left(\sqrt{\xi}\right)$$  \hspace{1cm} (21)

Note that the behaviour of $\psi_\pm$ at very small $\xi$ implies into the Dirichlet boundary condition ($\psi_\pm(0) = 0$) for $s \neq 0$. This boundary condition is essential whenever $B_\pm \neq 0$, nevertheless it also develops for $B_\pm = 0$ when $s = 1$ but not for $s = 0$. Since $V_{\text{eff}}^\pm$ is invariant under reflection through the origin ($x \rightarrow -x$), eigenfunctions with well-defined parities can be found. However, the eigenfunctions expressed in terms of associated Laguerre polynomials, due to the presence of the term $x^s$, are restricted to the half-line $x > 0$ in the event of $s$ is not an integer number. This drawback can be circumvented by taking $x$ by $|x|$. Then, there still an even eigenfunction such as in the case when $s$ is an even number. New eigenfunctions can be found by taking symmetric and antisymmetric linear combinations of the eigenfunctions expressed in terms of eigenfunctions defined on the positive side of the $x$-axis. These new eigenfunctions possess the same effective eigenvalue so that there is a two-fold degeneracy. This degeneracy in a one-dimensional quantum-mechanical
problem is due to the fact that the eigenfunctions, whether they are either even or odd, disappear at the origin. These questions do not occur when the eigenfunctions are expressed in terms of Hermite polynomials because they are well-behaved on the entire \( x \)-axis and have both even and odd parities from the beginning.

The necessary conditions for confining fermions in the Dirac equation with the potential \( V \) have been put forward. The formal analytical solutions have also been obtained. Now we move on to consider a survey for distinct cases in order to match the common effective eigenvalue. As we will see this survey leads to additional restrictions on the solutions, including constraints involving the nodal structure of the upper and lower components of the Dirac spinor. The effective potentials for a few specific cases are already plotted in Figure 1 and all the others can be obtained by making the transformation \( \omega \to -\omega \) and \( g \to -g \), as mentioned before.

**Case A** \( -|g| > 1 \)

This is the situation where \( B_{\pm} > 0 \) and for \( g > 1 \) \( s = |g| \) for \( V_{\text{eff}}^+ (V_{\text{eff}}^-) \), and \( s = 1 + |g| \) for \( V_{\text{eff}}^- (V_{\text{eff}}^+) \). The upper and lower components of the Dirac spinor share the same eigenvalue if the quantum numbers satisfy the relation

\[
n_- = n + \frac{\varepsilon(\omega) - \varepsilon(g)}{2}
\]  

(22)

where \( \varepsilon(\omega) \) and \( \varepsilon(g) \) stand for the sign function and \( n_+ = n \) \( (n_-) \) is related to \( \psi_+ \) \( (\psi_-) \). The solutions are

\[
\psi_{\pm \varepsilon(g)} = N_{\pm \varepsilon(g)} \xi^{\varepsilon(g)/2} e^{-\xi/2} L_n^{|g|-1/2} (\xi) \\
\psi_{- \varepsilon(g)} = N_{- \varepsilon(g)} \xi^{(1+|g|)/2} e^{-\xi/2} L_{n + \frac{\varepsilon(\omega) - \varepsilon(g)}{2}} (\xi)
\]  

(23)

\[
E_{\text{eff}} = \left[ 2n + 1 - \frac{\varepsilon(g)}{2} + |g| + \varepsilon(\omega) \left( g + \frac{1}{2} \right) \right] \hbar |\omega|
\]

with

\[
n \geq \frac{\varepsilon(g) - \varepsilon(\omega)}{2}
\]  

(24)

7
and the further proviso that \( n \geq 0 \) if \( \varepsilon(\omega) > \varepsilon(g) \). Note that the subscripts \( \pm \varepsilon(g) \) in \( \psi \) is a sequel of the transformation \( V_{eff}^{\pm} \rightarrow V_{eff}^{\mp} - 2C_\pm \) under \( g \rightarrow -g \). Note also that the constraint (22) asserts that the number of nodes of \( \psi_+ \) and \( \psi_- \) just differ by \( \pm 1 \).

Case B – \( |g| = 1 \)

In this situation \( B_+ = 0 \) \((B_- = 0)\) and \( s = 0 \) or 1 for \( V_{eff}^{+, -} \) \((V_{eff}^{-, +})\), and \( B_- > 0 \) \((B_+ > 0)\) and \( s = 2 \) for \( V_{eff}^{-, +} \) \((V_{eff}^{+, -})\) when \( g = +1 \) \((g = -1)\). It follows from (18) and (20) that the quantum numbers satisfy

\[
\begin{align*}
    n^{\pm \varepsilon(g)} & = \frac{n^{\varepsilon(g)} - 2 - \varepsilon(g)\varepsilon(\omega)}{2}
\end{align*}
\]

and the same remarks for the Case A involving \( \psi^{\pm \varepsilon(g)} \), as well as the nodal structure of the eigenfunctions, apply in Case B.

Case C – \( 0 < |g| < 1 \)

Here the solutions break in two sets regarding the two distinct possibilities of \( s \) for \( B_+ < 0 \) \((B_- < 0)\) for \( V_{eff}^{+, -} \) \((V_{eff}^{-, +})\) when \( 0 < g < 1 \) \((-1 < g < 0)\), viz. \( s = |g| \) or \( 1 - |g| \). On the other side, \( B_- > 0 \) \((B_+ > 0)\) for \( V_{eff}^{-, +} \) \((V_{eff}^{+, -})\) and \( s = 1 + |g| \). For the first set, \( s = |g| \) for \( V_{eff}^{+, -} \) \((V_{eff}^{-, +})\), and the constraint involving the quantum numbers as well as the solutions for the effective eigenvalues and eigenfunctions are the very same as those ones presented in the Case A. For the second set, \( s = 1 - |g| \) for \( V_{eff}^{+, -} \) \((V_{eff}^{-, +})\), and the quantum numbers satisfy \( n^{-\varepsilon(g)} = n^{\varepsilon(g)} - |g| + \varepsilon(g)\varepsilon(\omega)/2 \). This second set provides \( n \) equal to an integer number on the condition that \( |g| = 1/2 \) and this very particular case is already built in the first set.
**Case D** \(- g = 0\)

This is the situation where \(B_\pm = 0\) and \(s = 0\) or 1. The quantum numbers satisfy

\[
n_- = n + \varepsilon(\omega)
\]

and the solutions are

\[
\psi_+ = N_+ e^{-\xi/2} H_n \left( \sqrt{\xi} \right)
\]

\[
\psi_- = N_- e^{-\xi/2} H_{n+\varepsilon(\omega)} \left( \sqrt{\xi} \right)
\]

\[
E_{eff} = \left[ n + \frac{1 + \varepsilon(\omega)}{2} \right] \hbar |\omega|
\]

with

\[
n \geq -\varepsilon(\omega)
\]

and the further proviso that \(n \geq 0\) if \(\varepsilon(\omega) = +1\).

The preceding analyses shows that the effective eigenvalues are equally spaced with a step given by \(2\hbar |\omega|\) when at least one the effective potentials is singular at the origin. It is remarkable that the level stepping is independent of the sign and intensity of the parameter responsible for the singularity of the potential. We also note that there is a continuous transition from the Case A to the Case C, despite the appearance of the Hermite polynomial in the Case B. It should be remembered, though, that Hermite polynomials can be seen as particular cases of associated Laguerre polynomials. When both effective potentials become nonsingular \((g = 0)\) the step switches abruptly to \(\hbar |\omega|\). There is a clear phase transition when \(g \to 0\) due the disappearance of the singularity for both components of the Dirac spinor. In the limit as \(g \to 0\) the Neumann boundary condition, in addition to the Dirichlet boundary condition always present for \(g \neq 0\), comes to the scene. This occurrence permits the appearance of even Hermite polynomials and their related eigenvalues, which intercalate among the pre-existent eigenvalues related to odd Hermite polynomials. The appearance of even Hermite polynomials makes
ψ(0) ≠ 0 and this boundary condition is never permitted when the singular potential is present, even though g can be small. You might also understand the lack of such a smooth transition by starting from a nonsingular potential (g = 0), when the solution of the problem involves even and odd Hermite polynomials, and then turning on the singular potential as a perturbation of the g = 0 potential. Now the “perturbative singular potential” by nature demands, if is either attractive or repulsive, that ψ(0) = 0 so that it naturally kills the solution involving even Hermite polynomials. Furthermore, there is no degeneracy in the spectrum for g = 0.

In all the circumstances, at least one of the effective potentials has a well structure and the highest well governs the value of the zero-point energy. Matching the formal solutions for a common effective eigenvalue has imposed additional restrictions on the allowed eigenvalues and relations between the number of nodes of the upper and lower components of the Dirac spinor have been obtained. A sharp limitation occurred in the Case B (|g| = 1) when all the solutions involving even Hermite polynomials have been suppressed. In the Case D (g = 0), a case which cannot be obtained as a limit of the preceding ones, the solutions of the generalized Dirac oscillator [18] were obtained. The Dirac eigenvalues are obtained by inserting the effective eigenvalues in Eq. One should realize that the Dirac energy levels are symmetrical about E = 0. It means that the potential couples to the positive-energy component of the spinor in the same way it couples to the negative-energy component. In other words, this sort of potential couples to the mass of the fermion instead of its charge so that there is no atmosphere for the spontaneous production of particle-antiparticle pairs. No matter the intensity and sign of the coupling parameters, the positive- and the negative-energy solutions never meet. There is always an energy gap greater or equal to 2mc², thus there is no room for transitions from positive- to negative-energy solutions. This all means that Klein’s paradox never comes to the scenario.

Figures 2-5 illustrate the behaviour of |ψ⁺|², |ψ⁻|² and |ψ|² = |ψ⁺|² + |ψ⁻|² for the positive-energy solutions of the ground-state contemplating all the distinct classes of effective potentials plotted in Figure 1. The relative normalization of ψ⁺ and ψ⁻ is obtained by substituting the solutions directly into the original first-order coupled equations (3). Comparison of these Figures shows that |ψ⁺| is larger than |ψ⁻| (for E > 0) and that the fermion tends to avoid the origin more and more as |g| increases. A numerical calculation of the uncertainty in the position (with m = ω = c = ℏ = 1) furnishes
0.844, 1.130, 1.346 and 1.528 for $g$ equal to 0, 1/2, 1 and 3/2, respectively.

In conclusion, we have succeed in searching for Dirac bounded solutions for the pseudoscalar potential $V = m\omega cx + \hbar cg/x$. The satisfactory completion of this task has been possible because the methodology of effective potentials has transmuted the question into Sturm-Liouville problems with effective quadratic plus inversely quadratic potentials for both components of the Dirac spinor. As stated in the first paragraph of this work, the anomalous magnetic-like interaction in the four-dimensional world turns into a pseudoscalar interaction in the two-dimensional world. The anomalous magnetic interaction has the form $-i\mu \beta \vec{\alpha}.\vec{\nabla}\phi(r)$, where $\mu$ is the anomalous magnetic moment in units of the Bohr magneton and $\phi$ is the electric potential, i.e., the time component of a vector potential. Therefore, besides its importance as new analytical solutions of a fundamental equation in physics, the solutions obtained in this paper might be of relevance to the confinement of neutral fermions in a four-dimensional world.

Acknowledgments

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Figure captions

Figure 1 – Effective potentials for $V = m\omega cx + \hbar cg/x$ with $\omega > 0$, $g \geq 0$. The solid lines are for $V_{\text{eff}}^+$, the dashed lines are for $V_{\text{eff}}^-$. a) $g = 3/2$ ;  b) $g = 1$ ;  c) $g = 1/2$ ;  d) $g = 0$ (m = $\omega$ = c = $\hbar$ = 1).

Figure 2 – $|\psi_+|^2$ (full thin line), $|\psi_-|^2$ (dashed line), $|\psi_+|^2 + |\psi_-|^2$ (full thick line), corresponding to positive-ground-state energy for the potential $V = m\omega cx + \hbar cg/x$ with $g = 3/2$ and $m = \omega = c = \hbar = 1$.

Figure 3 – The same as in Figure 2 with $g = 1$.

Figure 4 – The same as in Figure 2 with $g = 1/2$.

Figure 5 – The same as in Figure 2 with $g = 0$. 
