A PROJECTIVE VARIETY WITH DISCRETE, NON-FINITELY GENERATED AUTOMORPHISM GROUP

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ABSTRACT. We construct a projective variety with discrete, non-finitely generated automorphism group.

1. Introduction

Suppose that $X$ is a projective variety over a field $K$. The set of automorphisms of $X$ can be given the structure of a scheme by realizing it as an open subset of $\text{Hom}(X, X)$. In general, $\text{Aut}(X)$ is locally of finite type, but it may have countably many components, arising from components of the Hilbert scheme. Write $\pi_0(\text{Aut}(X)) = (\text{Aut}(X)/\text{Aut}^0(X))_{\bar{K}}$ for the group of geometric components.

Examples.

1. Let $X = \mathbb{P}^r$. Then $\text{Aut}(X) \cong \text{Aut}^0(X) \cong \text{PGL}_{r+1}(K)$, and $\pi_0(\text{Aut}(X))$ is trivial.
2. Let $E$ be a general elliptic curve over $K$. Then $\pi_0(\text{Aut}(E \times E)) \cong \text{GL}_2(\mathbb{Z})$ is an infinite discrete group.
3. Let $X$ be a general hypersurface of type $(2, 2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then $X$ is a K3 surface, and the covering involutions associated to the three projections $X \to \mathbb{P}^1 \times \mathbb{P}^1$ generate a subgroup of $\pi_0(\text{Aut}(X))$ isomorphic to $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. 

According to a result of Brion [3], any connected algebraic group over a field of characteristic 0 can be realized as $\text{Aut}^0(X)$ for some smooth, projective variety. In contrast, very little seems to be known in general about the component group $\pi_0(\text{Aut}(X))$. In what follows, let $K$ be a field of characteristic 0, not necessarily algebraically closed, and let $\bar{K}$ be an algebraic closure. All varieties are defined over $K$, except where noted otherwise, and by a point we mean a $K$-point. Our main result is the following.

**Theorem 1.** There exists a smooth, geometrically simply connected, projective variety $X$ over $K$ for which $\pi_0(\text{Aut}(X))$ is not finitely generated.

The question of finite generation of $\pi_0(\text{Aut}(X))$ has been raised several times in various arithmetic [12,11] and geometric [3,6] contexts.

The component group $\pi_0(\text{Aut}(X))$ is an algebraic analog of the mapping class group $\pi_0(\text{Diff}(M))$ of a smooth manifold $M$. In general, the mapping class group is not finitely generated, with an example provided by tori in dimension at least five [9]. However, at least in high dimensions, the failure of $\pi_0(\text{Diff}(M))$ to be finitely generated is attributable to the fundamental group of $M$: according to a theorem of Sullivan [15], if $\dim M \geq 5$ and $\pi_1(M) = 0$, then $\pi_0(\text{Diff}(M))$ is finitely generated.

The related group $\text{Bir}(X)$ of birational automorphisms of $X$ can also be very complicated, even for simple varieties over $\mathbb{C}$. To begin with, the birational automorphism group does not admit a reasonable scheme structure in general [2]. If the canonical class $K_X$ has some
positivity, the situation is somewhat better. If $X$ is of general type, then $\text{Bir}(X)$ is finite. As long as $X$ is not uniruled, $\text{Bir}(X)$ admits the structure of a group scheme of locally finite type [8]. If the canonical class of $X$ is nef, then the Kawamata–Morrison cone conjecture places additional constraints on $\text{Aut}(X)$ and $\text{Bir}(X)$ [11]. In dimension 2, the automorphism group of a K3 surface is known to be finitely generated [14], but need not be commensurable with an arithmetic group [16]. Little seems to be known for Calabi–Yau varieties in higher dimensions.

Before giving the example, we sketch the technique. If $X$ is a variety and $Z$ is a closed subscheme of $X$, then the automorphisms of $X$ that lift to automorphisms of the blow-up $\text{Bl}_Z(X)$ are precisely those that map $Z$ to itself (not necessarily fixing $Z$ pointwise). Our approach, roughly speaking, is to find a variety $X$ with a subscheme $Z$ so that $\text{Stab}(Z) \subset \text{Aut}(X)$ is not finitely generated, and then to pass to the blow-up $\text{Bl}_Z(X)$ to obtain a variety realizing $\text{Stab}(Z)$ as an automorphism group. There are two main difficulties. The first is to find $X$ and $Z$ for which the stabilizer of $Z$ in $\text{Aut}(X)$ is not finitely generated. The second is to ensure that $\text{Bl}_Z(X)$ does not have any automorphisms other than those that lift from $X$.

To prove that our variety $X$ has non-finitely generated automorphism group, we will exhibit a smooth rational curve $C$ which is fixed by every automorphism of $X$. Restriction of automorphisms then determines a map $\rho : \text{Aut}(X) \rightarrow \text{Aut}(C) \cong \text{PGL}_2(K)$. We arrange that the image of $\rho$ is contained in an abelian subgroup of $\text{PGL}_2(K)$ and exhibit an explicit non-finitely generated subgroup of $\text{Im}(\rho)$. It follows that $\text{Aut}(X)$ is not finitely generated.

We turn now to the construction. Given a subvariety $V \subset X$, write

$$\text{Aut}(X; V) = \{ \phi \in \text{Aut}(X) : \phi(V) = V \}$$

There are three main steps. First, we describe a family of elliptic rational surfaces $S$ for which $\text{Aut}(S)$ is a large discrete group, and there is a rational curve $C$ on $S$ with $\text{Aut}(S; C)$ of finite index. Second, we specialize the surface $S$ in order to control the image of $\text{Aut}(S; C) \rightarrow \text{Aut}(C) \cong \text{PGL}_2(K)$. We show that there is a point $p$ on $C$ so that the subgroup $\text{Aut}_{\text{par}}(S; C, p)$ of automorphisms $\phi$ so that $\phi|_C$ is parabolic with fixed point $p$ is not finitely generated. At last, by some auxiliary constructions, we arrive at a six-dimensional variety $X$ whose automorphisms are precisely given by $\text{Aut}_{\text{par}}(S; C, p)$.

**Step 1: Automorphisms of surfaces with prescribed action on a curve**

If $z_1$, $z_2$, $z_3$, and $z_4$ are four distinct points in $\mathbb{P}^1$, there is a unique involution $\iota : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ with $\iota(z_1) = z_2$ and $\iota(z_3) = z_4$. Figure 1 shows how this map can be constructed geometrically when $\mathbb{P}^1$ is embedded as a conic in $\mathbb{P}^2$.

![Figure 1. Geometric construction of $\iota$](image-url)
Theorem 2. Suppose that $P = (p_1, p_2, p_3, p_4, p_5)$ of points in $\mathbb{P}^1$, let $\Gamma_P \subset \text{PGL}_2(K)$ be the subgroup generated by the involutions $\iota_{ij,kl} : \mathbb{P}^1 \to \mathbb{P}^1$ satisfying $\iota(p_i) = p_j$ and $\iota(p_k) = p_l$, where $i, j, k$ and $l$ are distinct indices. For a given configuration $P$, there are 15 such involutions for different choices of points.

(1) $\text{Aut}(S)$ is discrete;

(2) $\text{Aut}(S; C)$ has finite index in $\text{Aut}(S)$;

(3) The image of $\rho : \text{Aut}(S; C) \to \text{Aut}(C)$ contains $\Gamma_P$.

Proof. Let $L_0, \ldots, L_5$ be six lines in $\mathbb{P}^2$ intersecting at 15 distinct points, and let $S$ be the blow-up of $\mathbb{P}^2$ at these 15 points. Write $R$ for a partition of the six lines into two sets of three, with a distinguished line in each set. Given such a labelling, denote by $L_{R,0}, L_{R,1}, L_{R,2}$ and $L'_{R,0}, L'_{R,1}, L'_{R,2}$ the two triples, with $L_{R,0}$ and $L'_{R,0}$ the two distinguished lines. Let $O_R$ be the point of intersection of $L_{R,0}$ and $L'_{R,0}$.

The choice of a labelling $R$ determines two completely reducible cubics $C = L_{R,0} \cup L_{R,1} \cup L_{R,2}$ and $C' = L'_{R,0} \cup L'_{R,1} \cup L'_{R,2}$, which span a pencil in $\mathbb{P}^2$. The base locus of the pencil is the nine points $L_{R,i} \cap L'_{R,j}$. Let $\pi_R : S_R \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at these points, so that the pencil gives rise to an elliptic fibration $\gamma_R : S_R \to \mathbb{P}^1$. Note that the fibration $\gamma_R$ must be relatively minimal (i.e. there are no $(-1)$-curves contained in the fibers): a general fiber is linearly equivalent to $-K_{S_R}$, and so a $(-1)$-curve on $S_R$ must have intersection 1 with every fiber.

The exceptional divisor of $\pi_R$ above the point $O_R$ provides a section $E$ of $\gamma_R$. Let $\iota_R : S_R \dashrightarrow S_R$ be the birational involution induced by the $\gamma_R$-fiberwise action of $x \mapsto -x$ on the smooth fibers, with the section $E$ as the identity. Since $\gamma_R$ is a relatively minimal fibration, $\iota_R$ extends to a regular map $\iota_R : S_R \to S_R$ on the entire surface $S_R$ (see e.g. [10 II.10, Theorem 1]). Such a map necessarily permutes the three nodes on each of the fibers $C$ and $C'$, and so lifts to a biregular involution on the fifteen point blow-up $S$.

There is a simple geometric description of $\iota_R$ as a rational map of $\mathbb{P}^2$, and in particular of its action on $L_{R,0}$. Suppose that $\ell$ is a line in $\mathbb{P}^2$ passing through the point $O_R$, and that $x$ is a point on $\ell$ lying on a smooth fiber $C_x$ of $\gamma_R$. Then $\ell$ meets $C_x$ at $x$, $O_R$, and the third point $\iota_R(x)$. This description remains valid on components of the singular fibers not containing $\ell$, and so $\iota_R$ acts on $\ell$ so that the two points $L_{R,1} \cap \ell$ and $L_{R,2} \cap \ell$ are exchanged, as are $L'_{R,1} \cap \ell$ and $L'_{R,2} \cap \ell$. This uniquely determines the map: if $\ell$ is any line through $O_R$ for which the four points $L_{R,1} \cap \ell$, $L_{R,2} \cap \ell$, $L'_{R,1} \cap \ell$ and $L'_{R,2} \cap \ell$ are distinct, including $L = L_{R,0}$, then $\iota_R$ restricts to $\ell$ as the unique involution exchanging these two pairs of points.

The rational surface $S$ claimed by the theorem can now be constructed by choosing the lines in special position. Fix a line $C \subset \mathbb{P}^2$, and choose five other lines $L_1, \ldots, L_5$ so that $L_i \cap C = p_i$, where the $p_i$ are the points of the configuration $P$. Since the field $K$ is infinite, for general choices of the $L_i$, the fifteen points of intersection are distinct. The involution of $C$ exchanging $p_i$ with $p_j$ and $p_k$ with $p_l$ is realized as the restriction of $\iota_R : S \to S$ for a suitable labelling $R$: let $m$ be the unique index which does not appear among $i, j, k$, and $l$, and take $L_{R,0} = C$, $L_{R,1} = L_i$, $L_{R,2} = L_j$, $L'_{R,0} = L_m$, $L'_{R,1} = L_k$, and $L'_{R,2} = L_l$. Thus each involution $\iota_{ij,kl}$ on $C$ is the restriction of an automorphism of $\iota_R : S \to S$ fixing $C$, as claimed.

A blow-up $X$ of $\mathbb{P}^2$ at four points with no three collinear satisfies $H^0(X, TX) = 0$, and so $\text{Aut}^0(S)$ is trivial since $S \to \mathbb{P}^2$ factors through such a blow-up. It remains only to check
that the subgroup \( \text{Aut}(S; C) \) has finite index in \( \text{Aut}(S) \). This is a consequence of the fact that \( S \) is a Coble rational surface [7, 5]: the linear system \( |-2K_S| \) has a unique element, the union of the strict transforms of the six lines \( L_i \). Indeed, each line satisfies \( -2K_S \cdot L_i = -4 \), and so must be contained in the base locus of \( |-2K_S| \). An automorphism preserves the anticanonical class, so the six lines are permuted by any element of \( \text{Aut}(S) \), giving rise to a map \( \text{Aut}(S) \to S_6 \). The subgroup \( \text{Aut}(S; C) \) is the preimage of the subgroup of permutations fixing \( C \), and thus of finite index.

\[ \square \]

**Figure 2.** Construction of \( \iota_R : S \to S \)

Figure 2 illustrates the geometry of the map \( \iota_R \). The restriction of \( \iota_R \) to the line \( \ell \) is the unique involution exchanging the two points marked “\( \blacklozenge \)” and the two points marked “\( \blacksquare \)”.

**Step 2: Specializing the configuration \( P \)**

We now exhibit a configuration \( P = (p_1, p_2, p_3, p_4, p_5) \) for which the group \( \Gamma_P \) contains parabolic and hyperbolic elements with a common fixed point. Fix coordinates on \( \mathbb{P}^1 \).

**Lemma 3.** For the configuration

\[ P = (0, 1, 2, 3, 6) \]

the group \( \Gamma_P \) contains the two elements

\[ \sigma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}. \]

**Proof.** We claim that \( \sigma = \iota_{12,34} \circ \iota_{13,45} \circ \iota_{13,25} \) and \( \tau = \iota_{13,45} \circ \iota_{24,35} \circ \iota_{15,23} \). Indeed,

\[
\begin{align*}
(t_{12,34} \circ t_{13,45} \circ t_{13,25})(p_1) &= (t_{12,34} \circ t_{13,45})(p_3) = t_{12,34}(p_1) = p_2, \\
(t_{12,34} \circ t_{13,45} \circ t_{13,25})(p_2) &= (t_{12,34} \circ t_{13,45})(p_5) = t_{12,34}(p_4) = p_3, \\
(t_{12,34} \circ t_{13,45} \circ t_{13,25})(p_3) &= (t_{12,34} \circ t_{13,45})(p_1) = t_{12,34}(p_3) = p_4.
\end{align*}
\]

For the configuration \( P \), this yields \((t_{12,34} \circ t_{13,45} \circ t_{13,25})(0) = 1\), \((t_{12,34} \circ t_{13,45} \circ t_{13,25})(1) = 2\), and \((t_{12,34} \circ t_{13,45} \circ t_{13,25})(2) = 3\), so the composition must be the automorphism \( \sigma \) given by
Furthermore, φ is an automorphism fixing \( p \) if and only if \( \rho(\phi) \) lies in the subgroup \( B \subset \text{PGL}_2(K) \) comprising matrices of the form

\[
\begin{pmatrix}
1 & c \\
0 & 1
\end{pmatrix},
\]

which correspond to parabolic M"obius transformations \( z \mapsto z + c \). The subgroup \( B \) is abelian, isomorphic to \( \mathbb{G}_a \); since \( \rho(\text{Aut}_{\text{par}}(S; C, p)) \) is contained in \( B \), this group is abelian as well. For any integer \( n \), the transformation

\[
\tau^{-n} \circ \sigma \circ \tau^n = \begin{pmatrix} 1 & \frac{1}{3^n} \\ 0 & 1 \end{pmatrix}
\]

is contained in \( B \). Since \( \sigma \) and \( \tau \) both lie in \( \text{Im}(\rho) \) by the construction of Theorem 2, the elements \( \tau^{-n} \circ \sigma \circ \tau^n \) all lie in \( \rho(\text{Aut}_{\text{par}}(S; C, p)) \), and so \( \rho(\text{Aut}_{\text{par}}(S; C, p)) \) has a subgroup isomorphic to \( \mathbb{Z}[\frac{1}{3}] \). Since \( \rho(\text{Aut}_{\text{par}}(S; C, p)) \) is abelian and has a non-finitely generated subgroup, it is not finitely generated. A quotient of a finitely generated group is finitely generated, and we conclude that \( \text{Aut}_{\text{par}}(S; C, p) \) itself is not finitely generated.

The following geometric characterization of elements of \( \text{Aut}_{\text{par}}(S; C, p) \) will prove useful. Let \( \Delta_S : S \to S \times S \) be the diagonal map.

**Lemma 5.** Suppose that \( \phi : S \to S \) is an automorphism fixing \( p \). Then \( \phi \) fixes \( C \) as well. Furthermore, \( \phi \) lies in \( \text{Aut}_{\text{par}}(S; C, p) \) if and only if \( \text{id}_S \times \phi : S \times S \to S \times S \) fixes the tangent direction \( T_{\Delta_S(p)}(\Delta_S(C)) \).

**Proof.** Any automorphism of \( S \) permutes the components of \( |−2K_S| \), which are the strict transforms of the six lines \( L_i \). The only component containing \( p \) is \( C \) itself, and so \( C \) must be invariant under \( \phi \).

An automorphism fixing \( C \) and \( p \) lies in \( \text{Aut}_{\text{par}}(S; C, p) \) if and only if \( p \) is a fixed point of \( \phi|_C \) with multiplicity 2, which is the case if and only if \( \text{id}_S \times \phi : S \times S \to S \times S \) fixes \( \Delta_S(p) \) and the tangent direction \( T_{\Delta_S(p)}(\Delta_S(C)) \), so that \( (\text{id}_S \times \phi)(\Delta_S(C)) \) is tangent to the diagonal at \( \Delta_S(p) \).
Remark 1. Let $\bar{\sigma}$ and $\bar{\tau}$ be automorphisms of $S$ which restrict to $C$ as $\sigma$ and $\tau$, as constructed in Theorem 2. Although the restrictions to $C$ of the automorphisms $\mu_n = \bar{\tau}^{-n} \circ \bar{\sigma} \circ \bar{\tau}^n$ commute and satisfy $\mu_{n-1}|_C \circ \mu_n |_C^{-3} = \text{id}_C$, these maps do not commute as automorphisms of $S$, and the map $\text{Aut}(S;C) \to \text{Aut}(C)$ is not injective. For example, the commutator $[\mu_0, \mu_1]$ is an automorphism of $S$ which restricts to $C$ as the identity, but a straightforward if somewhat tedious computation of the action of the involutions $\iota_R$ on $\text{NS}(S)$ shows that the induced map $[\mu_0, \mu_1] : \mathbb{P}^2 -\to \mathbb{P}^2$ is a Cremona transformation of degree 1,944,353. It seems conceivable that $\text{Aut}_\text{par}(S; C; p)$ is a free group on the countably many generators $\mu_n$, though this is difficult to prove.

Remark 2. The kernel $G$ of $\text{Aut}(S; C) \to \text{Aut}(C)$ is also of interest: this is the subgroup of automorphisms which fix $C$ pointwise, including the maps $[\mu_n, \mu_n]$ of the remark above. It seems likely that $G$ is not finitely generated; if this is the case, then by choosing a very general point $q$ on $C$, we might obtain a rational surface $S' = \text{Bl}_q S$ such that $\text{Aut}(S')$ is isomorphic to $G$ and is not finitely generated. However, it is not clear how to prove either that $G$ is not finitely generated, or that the blow-up does not admit automorphisms other than those lifted from $S$.

Step 3: A variety with non-finitely generated $\text{Aut}(X)$

We now construct a higher-dimensional variety $X$ realizing $\text{Aut}_\text{par}(S; C; p)$ as $\text{Aut}(X)$. Although $\text{Aut}_\text{par}(S; C; p)$ is not the stabilizer of any closed subscheme of $S$, it is the stabilizer of a closed subscheme of $S \times S$ under the group of automorphisms of $S \times S$ of the form $\iota_R \times \phi$: an automorphism $\phi$ lies in $\text{Aut}_\text{par}(S; C; p)$ and only if $\iota_R \times \phi$ fixes both $\Delta_S(p)$ and the tangent direction $T_{\Delta_S(p)}(\Delta_S(C))$. Our variety $X$ will be realized as a blow-up of $S \times S \times T$, where $T$ is a surface of general type; taking the product with $T$ makes it simpler to control automorphisms of blow-ups.

We begin with a lemma enabling us to show that a blow-up $\text{Bl}_V X$ has no automorphisms except those that lift from $T$. Say that a variety $X$ is $\mathbb{P}^r$-averse if every $\overline{K}$-morphism $h : \mathbb{P}^r_K \to X_K$ is constant. Note that if $X$ is $\mathbb{P}^r$-averse, it is also $\mathbb{P}$-averse for any $s > r$.

Lemma 6. Suppose that $X$ is a $\mathbb{P}^{r-1}$-averse variety of dimension $n$, and $V \subset X$ is a smooth, geometrically connected subvariety of codimension $r$. Write $\pi : \text{Bl}_V X \to X$ for the blow-up, with exceptional divisor $E$. Then every automorphism of $\text{Bl}_V X$ descends to an automorphism of $X$, and the induced map $\text{Aut}(\text{Bl}_V X) \to \text{Aut}(X)$ is an isomorphism onto $\text{Stab}(V)$.

Proof. We first observe that any nonconstant morphism $h : \mathbb{P}^{r-1}_K \to \text{Bl}_V X_K$ must have image contained in a geometric fiber of $\pi|_{E_K}$. Indeed, $\pi \circ h : \mathbb{P}^{r-1}_K \to X_K$ must be constant since $X$ is $\mathbb{P}^{r-1}$-averse, and so the image of $h$ is contained in a geometric fiber.

Suppose that $\phi : \text{Bl}_V X \to \text{Bl}_V X$ is an automorphism, and let $h : \mathbb{P}^{r-1}_K \to \text{Bl}_V X_K$ be the inclusion of a geometric fiber of $\pi|_{E_K}$. Then $\phi \circ h$ is an inclusion from $\mathbb{P}^{r-1}_K \to \text{Bl}_V X_K$, and so must be the inclusion of some fiber of $\pi|_{E_K}$. Thus $\phi$ permutes the fibers of $\pi|_{E_K}$, and so descends to an automorphism of $X$ fixing $\pi(E) = V$. \hfill \Box

Lemma 7.

(1) Suppose that $X_1$ and $X_2$ are $\mathbb{P}^r$-averse. Then $X_1 \times X_2$ is $\mathbb{P}^r$-averse.

(2) Suppose that $X$ is $\mathbb{P}^r$-averse and $V \subset X$ is a smooth, geometrically connected subvariety of codimension $s \leq r$. Then $\text{Bl}_V X$ is $\mathbb{P}^r$-averse.
Proof. For (1), suppose that $h : \mathbb{P}^r_K \to X_{1,K} \times X_{2,K}$ is a morphism. Then the projections $p_1 \circ h : \mathbb{P}^r_K \to X_{1,K}$ and $p_2 \circ h : \mathbb{P}^r_K \to X_{2,K}$ must both be constant, so that $h$ is constant. For (2), the map $\pi \circ h$ must be constant, and so if $h$ is nonconstant, its image is contained in a fiber of $\pi|_{E_K}$. These fibers are isomorphic to $\mathbb{P}^{r-1}_E$, and since $s - 1 < r$, the map $h$ must be constant. \hfill \Box

We require one more simple lemma before proceeding to the construction.

Lemma 8. Suppose that $X$ is a smooth projective variety with $\text{Aut}(X)$ discrete. There exists a divisor $G \subset X$ for which $\text{Aut}(X;G)$ is trivial.

Proof. Choose a very ample linear system $\mathcal{G} \cong \mathbb{P}^N$ on $X$. By the Lieberman–Fujiki theorem, the subgroup $\text{Aut}(X;\mathcal{G})$ of automorphisms fixing $\mathcal{G}$ is of finite type, and hence finite since $\text{Aut}(X)$ is assumed discrete. If $\phi$ is any member of $\text{Aut}(X,\mathcal{G})$ other than the identity, it can not act trivially on $\mathcal{G}$. Indeed, suppose that $\phi$ fixes every element of $\mathcal{G}$. If $x$ is any point of $X$, then $x = \bigcap_{G \ni G} G$, and so $x$ is fixed by $\phi$. It follows that $\phi$ is the identity map. As a consequence, a general element of $\mathcal{G}$ is not fixed by any automorphisms. \hfill \Box

Let $T$ be a smooth, geometrically simply connected surface over $K$ for which $\text{Aut}(T)$ is trivial, $T$ is not generically finite, and there is at least one $K$-point $t$ on $T$; according to [13], we can take $T$ to be the hypersurface in $\mathbb{P}^3$ defined by $x_0^5 + x_0 x_1^4 + x_1 x_2^4 + x_2 x_3^4 + x_3^5$, which has the point $[0, 1, 0, 0]$. (Note that if we work over $K = \mathbb{C}$ or any other uncountable field, then any very general hypersurface in $\mathbb{P}^3$ of degree at least 4 suffices.)

Take $X_0 = S \times S \times T$. The variety $X$ will be constructed by a sequence of four blow-ups of $X_0$. In each case, the blowup satisfies the hypotheses of Lemma 6, so we may identify its automorphism group with a subgroup of $\text{Aut}(X_0)$.

Lemma 9. Let $X_0 = S \times S \times T$. Fix a point $s$ on $S$ and a divisor $G$ on $S$ with $\text{Aut}(S;G)$ trivial, as in Lemma 8. Choose three distinct smooth, geometrically connected curves $\Gamma_1$, $\Gamma_2$, and $\Gamma_3$ in $T$, and a point $t$ on $\Gamma_3$ which does not lie on $\Gamma_1$ or $\Gamma_2$.

(1) The variety $X_0$ is $\mathbb{P}^r$-averse for any $r \geq 2$. The automorphisms of $X_0$ are of the form $\text{Aut}(S \times S) \times \text{id}_{T}$.

(2) Let $\pi_1 : X_1 \to X_0$ be the blow-up of $X_0$ along $s \times S \times \Gamma_1$. The variety $X_1$ is $\mathbb{P}^r$-averse for any $r \geq 3$. The automorphisms of $X_1$ are all lifts of $\text{Aut}(S;S) \times \text{Aut}(S) \times \text{id}_T$.

(3) Let $\pi_2 : X_2 \to X_1$ be the blow-up along the strict transform of $G \times S \times \Gamma_2$. The variety $X_2$ is $\mathbb{P}^r$-averse for $r \geq 4$. The automorphisms of $X_2$ are given by $\text{id}_S \times \text{Aut}(S;p) \times \text{id}_T$.

(4) Let $\pi_3 : X_3 \to X_2$ be the blow-up along the strict transform of $p \times S \times \Gamma_3$. Then $X_3$ is $\mathbb{P}^3$-averse for $r \geq 5$, and the automorphisms of $X_3$ are of the form $\text{id}_S \times \text{Aut}(S;p) \times \text{id}_T$.

(5) Let $E_3$ be the exceptional divisor of $\pi_3 : X_3 \to X_2$. Then $\Delta_S(C) \times t$ meets $E_3$ at a single point $u$. Let $\pi_4 : X_4 \to X_3$ be the blow-up at $u$. The automorphism group of $X_4$ is isomorphic to $\text{id}_S \times \text{Aut}_{par}(S;C,p) \times \text{id}_T$.

Proof. We treat the blow-ups in order.

(1) To show that $X_0$ is $\mathbb{P}^r$-averse for $r \geq 2$, it suffices to check that $S$ and $T$ are both $\mathbb{P}^2$-averse, according to the first part of Lemma 7. For $T$ this follows since $T$ is not uniruled, while for $S$ we note that a nonconstant morphism $h : \mathbb{P}^2_K \to S_K$ must be generically finite, and so induce an injection $h^* : \text{Pic}(S_K) \to \text{Pic}(\mathbb{P}^2_K)$, which is impossible.

Suppose that $\phi : X_0 \to X_0$ is an automorphism. Let $p_3 : X_0 \to T$ be the third projection. We first claim that $\phi$ must permute the geometric fibers of $p_3$. If $p_3 \circ \phi$ contracts any geometric
fiber of $p_3$, it must contract every geometric fiber by the rigidity lemma. So if $\phi$ does not permute the fibers of $p_3$, then every fiber of $p_3$ has image in $T$ of dimension at least 1. Since these fibers are isomorphic to $S \times S$, the image of every geometric fiber is uniruled, which implies that $T$ must be geometrically uniruled, contradicting the choice of $T$.

Consequently every automorphism of $X_3$ is of the form $\phi \times \psi$, where $\phi$ is an automorphism of $S \times S$ and $\psi$ is an automorphism of $T$. Since $\text{Aut}(T)$ is trivial, the group $\text{Aut}(X_0)$ can be identified with $\text{Aut}(S \times S) \times \text{id}_T$.

(2) The center of the blow-up $\pi_1$ has codimension 3, so it follows from Lemma 7 that $X_1$ is $\mathbb{P}^r$-averse for $r \geq 3$. According to Lemma 6, since $X_0$ is $\mathbb{P}^2$-averse, $\text{Aut}(X_1)$ is given by the stabilizer of $s \times S \times \Gamma_1$ in $\text{Aut}(X_0)$, which is isomorphic to the stabilizer of $s \times S$ in $\text{Aut}(S \times S)$.

We claim that an element $\phi$ of $\text{Aut}(S \times S)$ fixes $s \times S$ only if it is of the form $\phi_1 \times \phi_2$, where $\phi_1$ is in $\text{Aut}(S; s)$ and $\phi_2$ is in $\text{Aut}(S)$. Indeed, if $\phi$ fixes one fiber of $p_1 : S \times S \to S$, it must permute the fibers, and so induces an automorphism $\phi_1 : S \to S$ on the base with $p_1 \circ \phi = \phi_1 \circ p_1$. Then $(\text{id}_S \times \phi_1^{-1}) \circ \phi$ is an automorphism of $S \times S$ defined over $p_1$. This must be given by a map $\text{id}_S \times \phi_2 : S \times S \to S \times S$, since $\text{Aut}(S)$ is discrete, and so $\phi$ is of the form $\phi_1 \times \phi_2$, where $\phi_1$ fixes $s$.

(3) Since $X_1$ is $\mathbb{P}^r$-averse for $r \geq 3$ and $X_2$ is the blow-up of $X_1$ at a center of codimension 4, it follows that $X_2$ is $\mathbb{P}^r$-averse for $r \geq 4$. Since the center of $\pi_2$ has codimension 4 and $X_1$ is $\mathbb{P}^3$-averse, the automorphisms of $X_2$ are given by isomorphisms of $X_1$ that fix $G \times p \times t_2$. The automorphisms of $X_1$ are all of the form $\phi_1 \times \phi_2 \times \text{id}_T$, and so this stabilizer is exactly $\text{id}_S \times \text{Aut}(S; p) \times \text{id}_T$.

(4) We have seen that $X_2$ is $\mathbb{P}^4$-averse, and $X_3$ is the blow-up of $X_2$ at a center of codimension 5. It follows that $X_3$ is $\mathbb{P}^r$-averse for $r \geq 5$, and the automorphisms of $X_3$ are lifts of automorphisms of $X_2$ that fix $p \times p \times \Gamma_3$. Every automorphism of $X_2$ fixes $p \times p \times \Gamma_3$, and so the automorphisms of $X_3$ are again given by $\text{id}_S \times \text{Aut}(S; p) \times \text{id}_T$.

(5) The centers of the blow-ups $\pi_1$ and $\pi_2$ are both disjoint from the fiber $S \times S \times t$, since $t$ lies on neither $\Gamma_1$ nor $\Gamma_2$, while the center of the blow-up $\pi_3$ meets $S \times S \times t$ at the single point $p \times p \times t$. As a result, $\Delta_S(C) \times t$ meets $E_3$ at one point $u$, as claimed. The restriction of $\pi_3 \circ \pi_2 \circ \pi_1$ to the strict transform of $S \times S \times t$ is the blow-up at the point $p \times p \times t$.

Since $X_3$ is $\mathbb{P}^5$-averse and the center of $\pi_3$ has codimension 6, $\text{Aut}(X_3)$ is isomorphic to the stabilizer of $u$ in $\text{Aut}(X_3)$. These are exactly the automorphisms $\text{id}_S \times \phi \times \text{id}_T$ of $X_3$ that fix the tangent direction $T_{\Delta(p)}(\Delta_S(C)) \times t$. According to Lemma 3, these are exactly the lifts of automorphisms of the form $\text{id}_S \times \text{Aut}_{\text{par}}(S; C, p) \times \text{id}_T$.

This completes the construction.

**Proof of Theorem** Let $X = X_4$ be as in Lemma 9. The variety $X$ is smooth, projective and geometrically simply connected, since it is a blow-up of $S \times S \times T$ where $S$ is a rational surface and $T$ is smooth and geometrically simply connected. The group $\text{Aut}(X)$ is isomorphic to $\text{Aut}_{\text{par}}(S; C, p)$, which is not finitely generated according to Lemma 4. □

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