THE BACKGROUND FIELD METHOD APPLIED TO COSMOLOGICAL PHASE TRANSITION

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Abstract. In this paper, the cosmological phase transition is investigated by background gauge field method. As a continuation of previous our work, some numerical results and graphic solutions at $T \neq 0$ are presented. Hence the mechanism of cosmological phase transition in the early Universe is considered. It is shown that the breaking of symmetry significantly depend on the nonzero temperature and chemical potential. Furthermore, it is the first order of phase transition. Non-restoration of symmetry in hot gauge theories for Cosmology.

I. INTRODUCTION

Phase transition is a complicate physical process, its nature is non-perturbative phenomenon [1] - [5]. Therefore, it is worth to mention that the finite temperature effective action basing on functional integral as a method which provides a general approximation beyond one loop and higher free energy density in the pertubative as well as non-pertubative sector. In particular, it plays an important role in the investigation of phase transition and non-equilibrium phenomena [6], [7], as dynamics of processes.

Recently there has been considerable interest in the symmetry in both hot scalar field theories and hot gauge theories for cosmology [8], [10]. It is shown that the phase transition in the early Universe could be described by non-Abelian gauge theories at high temperature [11]. Furthermore, the contribution to the free energy only by calculated by using non-pertubative methods. However, the effective potential of gauge theories may fail to be gauge because it generally does depend on the $\xi$ - gauge. Therefore the background field method have been applied to compute quantum effects without losing gauge invariance [12], [13].

Our main aim is to apply the background gauge field method at high temperature to investigation of cosmological phase transition. In this connection, it is possible to consider this paper as being complementary numerical computation to result of previous our work.

This paper is organized as follows. In section II, we present the formalism of effective action and background gauge field method. The one loop free energy density at nonzero temperature an chemical potential is obtained in section III. Section VI is devoted to investigation the cosmological phase transition from some numerical calculations and graphic solutions. Our conclusion is summarized in section V.
II. FORMALISM

We start from the Lagrangian density

\[
L_0 = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \bar{\Psi} (i\gamma^\mu D_\mu - G_i \Phi_i) \Psi \\
+ [(D_\mu - i\mu \delta_{\mu 0}) \Phi_i^+]^+ [(D^\mu - i\mu \delta^{\mu 0}) \Phi_i] - m^2 \Phi_i^+ \Phi_i - \lambda (\Phi_i^+ \Phi_i)^2 \\
- \frac{1}{2\xi} (\partial^\mu A^a_\mu)^2 - \partial_\mu \omega^a_\mu \partial^\mu \omega_a + f_{abc} (\partial_\mu \omega^a_\mu) A^b_\mu \omega^c.
\]

where \(\bar{\Psi}, \Psi\) are multiplet of fermion fields, \(\Phi_i (i = 1, 2 \ldots n)\) are components of scalar fields, \(A^a_\mu\) - gauge fields and \(\omega, \omega^*\) - ghost fields. Here \(\mu\) is chemical potential, \(G_i\) and \(\lambda\) are coupling constants, \(\lambda > 0\)

\[
D_\mu \equiv \partial_\mu - iT^a A^a_\mu, \\
F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f_{abc} A^b_\mu A^c_\nu,
\]

where \(T_a\) are group generators, \(f_{abc}\) are structure constants which satisfy Lie algebra

\[
f_{abc} f_{dce} = g^2 C_A \delta_{ad} \\
Tr(T_a T_b) = g^2 C_F \delta_{ab}
\]

with \(C_A\) is numerical constant of gauge group, \(C_A = N\) for \(SU(N)\), \(C_F\) is representation of this group.

The fields are shifted by

\[
A_\mu \rightarrow A_\mu + A'_\mu; \quad \langle 0 | A_\mu | 0 \rangle = const, \quad \langle 0 | A'_\mu | 0 \rangle = 0, \\
\Phi_i \rightarrow \Phi_i + \Phi'_i; \quad \langle 0 | \Phi_i | 0 \rangle = \phi_{0i}, \quad \langle 0 | \Phi'_i | 0 \rangle = 0, \\
\Psi \rightarrow \Psi + \Psi'; \quad \langle 0 | \Psi | 0 \rangle = \langle 0 | \Psi' | 0 \rangle = 0, \\
\omega_a \rightarrow \omega_a + \omega'_a; \quad \langle 0 | \omega_a | 0 \rangle = \langle 0 | \omega'_a | 0 \rangle = 0,
\]

where \(A^a_\mu, \Phi, \Psi, \omega_a\) are the background fields, and \(A'_\mu, \Phi'_i, \Psi', \omega'_a\) are the quantum fields, which are variables of integration in the functional integral.

It is well known, the background field method allows one to fix a gauge, thereby compute quantum effects without losing explicit gauge invariance [14].

Now we consider the effective action in background field for which \(A^a_\mu\) and \(\Phi_i\) are constant, and \(\bar{\Psi} = \Psi = \omega = \omega^* = 0\). The effective action is calculated from the part of the action that is quadratic in quantum fields \(A'_\mu, \Phi'_i, \Psi'\) and \(\omega'_a, \omega'^*\) over which one integrated

\[
I_{quad} = \int dx L_{quad} = \int dx \left[ -\frac{1}{4} \left( \bar{D}_\mu A'_\mu - \bar{D}_\mu A'^a_\mu \right)^2 - \frac{1}{4} F^a_{\mu\nu} f_{abc} A'^b_\mu A'^c_\nu \right] \\
- \int dx \bar{\Psi}' (\gamma^\mu D_\mu + \mu \gamma^0 + M + g_s \phi_i') \Psi' \\
+ \int dx \left[ \frac{1}{2} (D_\mu \phi_i' D^\mu \phi_i' - M^2 \phi_i'^2) - \frac{\lambda}{4} \phi_i'^4 \right] \\
- \int dx \left[ \frac{1}{2\xi} \left( \bar{D}_\mu A'_\mu \right)^2 + \left( \bar{D}_\mu \omega'_a \right) \left( \bar{D}_\mu \omega'_a \right) \right].
\]
\[ I_{\text{quad}} = \frac{1}{2} \int dxdy A'_\mu(x) \mathcal{D}^{ab}_{\mu\nu}(x, y) A'_\nu(y) - \int dxdy \Psi'(x) \mathcal{D}_{ab}(x, y) \Psi'(y) \]

(8)

By using the Fourier transformation \( \mathcal{D}(k) = \int dx e^{ik(x-y)} \mathcal{D}(x-y) \) the matrices in (8) are given by

\[
\mathcal{D}_{\mu\nu}^{ab}(k) = g_{\mu\nu} \left[ (-ik_\rho \delta_{ca} + f_{cda} A^d_\rho) (-ik_\delta \delta_{cb} + f_{ceb} A^e_\rho) 
- (-ik_\rho \delta_{ca} + f_{cda} A^d_\rho) (-ik_\mu \delta_{cb} + f_{ceb} A^e_\mu) + F_{\mu\nu} f_{cab} \right] + g_{\mu\nu} \delta_{ij} \Phi_i(k) \Phi_j(k) T^a T^b + \epsilon \text{ terms,}
\]

with \( F_{\mu\nu} = f_{abc} A^b_\mu A^c_\nu \).

\[
\mathcal{D}(k) = (-ik - iT_a A^a_\mu + M + \mu \gamma_0) + \epsilon \text{ terms,}
\]

(10)

\[
\mathcal{D}_{ab}(k) = \left[ (-ik_\rho \delta_{ca} + f_{cda} A^d_\rho) (-ik_\mu \delta_{cb} + f_{ceb} A^e_\mu) + \epsilon \text{ terms,}
\]

(11)

\[
\mathcal{D}_{ij}(k) = (-ik_\mu - i\mu - iT_a A^a_\mu)_{ij} (ik_\mu + i\mu + iT_a A^a_\mu)_{ij} - m^2_{ij} - \frac{\lambda}{2} \phi_i \phi_j + \epsilon \text{ terms,}
\]

(12)

In momentum representation, the effective action takes the general form

\[
\Gamma_\beta [\psi, \bar{\psi}, \phi, A_\mu, \omega, \omega^*] = I [\psi, \bar{\psi}, \phi, A_\mu, \omega, \omega^*] - \frac{i}{2} Tr \ln G^{ab}_{\mu\nu}(k) 
+ iTr \ln S(k) - \frac{i}{2} Tr \ln \Delta_{ij}(k) + iTr \ln D_{ab}(k) + \sum_{n=2}^{\infty} n \text{ loops } 1PI.
\]

(13)

Here the action \( I [\psi, \bar{\psi}, \phi, A_\mu, \omega, \omega^*] \) is given in (8) - (12). The Trace, the logarithm are taken in functional sense, and the free propagators are given by

\[
S^{-1}(k) = \mathcal{D} + M - i\epsilon; \quad M = g_{\mu\nu},
\]

(14)

\[
\Delta_{ij}^{-1}(k) = \delta_{ij} k^2 - M_{ij}^2 - i\epsilon; \quad M_{ij}^2 = (\mu^2 - m^2) \delta_{ij} - \frac{\lambda}{2} \phi_i \phi_j,
\]

(15)

\[
\left[ G_{0ab}(k) \right]_{\mu\nu}^{-1} = (M_{ab}^2 - k^2 \delta_{ab}) \left[ \frac{k_\mu k_\nu}{k^2} - g_{\mu\nu} \right] + \left[ \delta_{ab} \frac{k^2}{\xi} - M_{ab}^2 \right] \frac{k_\mu k_\nu}{k^2},
\]

(16)

\[
D_{0ab}(k) = \delta_{ab} (k^2 - i\epsilon), \quad M_{ab} = \frac{1}{2} \delta_{ab} g_{\mu\nu}.
\]

(17)

III. ONE LOOP THERMAL FREE ENERGY DENSITY

Next we consider the theory at finite temperature by "imagine time" formalism. For \( \langle A_\mu \rangle_\beta = \delta_{0\mu} A^0_\mu, \langle \Phi \rangle = \phi_0 \) the effective potential is defined by

\[
V_\beta = -\frac{\Gamma_\beta}{\beta} \int dx
\]

(18)

where \( \beta = T^{-1} \) (we set Boltzmann constant k = 1). It is just thermal free energy density, which concerns with the phase transition at \( T = T_c \).
In $d = 4 - 2\epsilon$ dimension, the divergent integrals can be regularized by using the "imagine time" formalism, where nonzero chemical potential $\mu$ is added to the fermionic Matsubara frequencies, i.e.

$$i \int \frac{d^4k}{(2\pi)^4} f(k) \to T \sum_k \frac{d^4k}{(2\pi)^4} f(i\omega_n, \bar{k}),$$

where $i\omega_n = 2\pi n T$ for bosons, $\omega_n = (2n + 1)\pi T$ for fermions and $i\omega_n \to i\omega_n + \mu$.

From (8) - (13) and (18) we arrived at the expression for the effective potential

$$V_\beta = V_{cl} - \sum_k \left[ \ln(k^2 + M^2) - \ln(k^2 + M_{ij}^2) - \ln(k^2 + M_{ab}^2) \right]$$

$$+ ig^2 \left( \frac{11}{12} N - \frac{1}{6} N_F + \frac{1}{12} N_B \right) \sum_k \frac{1}{(k^2)^2} \int dx F_{\mu\nu}^a F^{a\mu\nu}$$

$$+ \frac{i}{8} g^2 \sum_k \frac{1}{(k^2 + m^2)^2}$$

$$+ \frac{i}{2} g_i \sum_k \sum_p \frac{1}{(p^2 + M_1^2)(k^2 + M_2^2)} \left[ (k + p)^2 + M_{ab}^2 \right].$$

Finally, the one loop thermal free energy density is given

$$V_\beta = -\frac{1}{2} (\mu^2 - m^2) \phi^2 + \frac{\lambda}{4} \phi^4 - \frac{1}{2} \delta_{ab} M_{ab}^2 A_{\beta\mu}^2 - \frac{\pi^2 T^4}{90} \left( N_B + \frac{7}{8} N_F \right)$$

$$+ \frac{T^2}{24} \left\{ (\mu^2 - m^2 - \frac{\lambda}{2} \phi^2) + 3 Tr M_{ab}^2 + \frac{1}{2} Tr [\gamma_0 (M + \mu \gamma_0) \gamma_0 (M + \mu \gamma_0)] \right\}$$

$$- \frac{T}{12\pi} (2\Re^2 + \delta_{ab} M_{ab}^2) - \frac{g^2 T^2}{48 \times 4\pi} (2\Re + \delta_{ab} M_{ab} + 2M)$$

$$+ \frac{g^2}{(4\pi)^2} \left( \frac{11}{12} N - \frac{1}{6} N_F + \frac{1}{12} N_B \right) \left( \frac{1}{\epsilon} - 2 \ln \frac{\bar{\eta}}{4\pi T} + 2\gamma_E \right) \int dx F_{\mu\nu}^a F^{a\mu\nu}. \tag{20}$$

where $\Re, M_{ab}, M$ are thermal masses, e.g the squared scalar mass is

$$\Re^2 = (\mu^2 - m^2) + \frac{\lambda}{24} T^2 - \frac{\lambda}{2} \phi^2. \tag{21}$$

The renormalized coupling is given by

$$g_R = g \left[ 1 + g^2 \left( \frac{11}{12} N - \frac{1}{6} N_F + \frac{1}{12} N_B \right) \left( \frac{1}{\epsilon} + 2 \ln \frac{\bar{\eta}}{4\pi T} + 2\gamma_E \right) \right] + 0(g^4) \tag{22}$$

i.e the physical coupling $g_R$ increases due to quantum corrections in the non-Abelian theory.
IV. SOME NUMERICAL RESULTS AT NONZERO TEMPERATURE AND CHEMICAL POTENTIAL

Let us consider the minimum of the effective potential from which we can determine the breaking or the restoration of symmetry.

In the case of $\mu \leq m$, the free energy density is minimum at $\Phi_0 = 0$ at any temperature $T$, i.e. the symmetry is not broken (Fig. 1).

In Fig. 2 it is easily seen that the high temperature and nonzero chemical potential significantly affect on the spontaneous breaking of symmetry. The importance is that it leads to the cosmological phase transition.

In Fig. 3, we can see that the effective squared scalar mass is change from negative to positive value when the temperature is high enough. It is due to the contribution of the $M^2 T^2$ term in the effective potential. This just is phenomena of breaking of symmetry, when it occurs the cosmological phase transition is manifested.

In Fig. 4, the free energy is represented as a continuous function of $T$ and $\mu$ when $\mu \leq m$, but in Fig. 5, it has a discontinuity when $\mu \geq m$. It is shown by the discontinuous buffer between the symmetry and its breaking part. This is just the cosmological phase transition. Furthermore, the restoration of symmetry does appear that means after symmetry was spontaneously broken, the Universe is asymmetry.

In Fig. 6, the effective squared scalar mass depend on the chemical potential $\mu$ at fixed temperature, which expresses the first order phase transition.
Fig. 2. The effective potential $V_\beta$ as a function of $T$ and $\Phi$ in the case $\mu/m = 1.5$.

a) $\Phi = -100 \div 100\,\text{MeV}$ with $T = 0, 100, 200, 400\,\text{MeV}$, respectively

b) $T = 0 \div 400\,\text{MeV}$ and $\Phi = -300 \div 300\,\text{MeV}$.

Fig. 3. The squared scalar mass $M^2$ as a function of temperature $T$.

a) $T = -400 \div 400\,\text{MeV}$ with $\Phi = 100\,\text{MeV}$ in cases $\mu = m, \mu = 1.5m, \mu = 0.5m$.

b,c) $T = -400 \div 400\,\text{MeV}$ and $\Phi = -300 \div 300\,\text{MeV}$ in cases $\mu = 1, 2m, \mu = 0, 7m$, respectively.
**Fig. 4.** The effective potential $V_\beta$ as a function of $\mu$ and $\Phi$ at $T = 400\text{MeV}$.

a) $\mu = -500 \div 500\text{MeV}$ with $\Phi = 0, 100, 200, 400\text{MeV}$, respectively.

b) $\mu = 0 \div 400\text{ MeV}$ and $\Phi = -300 \div 300\text{ MeV}$.

**Fig. 5.** The effective mass as a function of chemical potential. It expresses the first order phase transition. The discontinuity is phase transition region (white region).

Plot for $\mu = 0 \div 500\text{ MeV}$ and $\Phi = -300 \div 300\text{ MeV}$.

**Fig. 6.** Dependence of the effective squared scalar mass on $\mu$ at fixed temperature.

It expresses the first order phase transition.

Plot for $2\pi^2$ with $\mu = 0 \div 1200\text{ MeV}$.
V. DISCUSSION AND CONCLUSION

We have studied the free energy as a function of temperature and non-zero chemical potential by background gauge field method in frame of Abelian theories. Hence the mechanism of cosmological phase transition is investigated. The graphic solutions have shown that in the early Universe it is first order phase transition. Furthermore, after the spontaneous symmetric breaking, one also can see non-restoration of symmetry in hot gauge field theories for Cosmology, i.e the Universe is asymmetry.

We should mention that one can study the quantum effects in phase transition and find exactly the cosmological critical temperature if a suitable parameter set is given.

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