Gliders and Ether in Rule 54

Markus Redeker

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This is a study of the one-dimensional elementary cellular automaton rule 54 in the new formalism of “flexible time”. We derive algebraic expressions for groups of several cells and their evolution in time. With them we can describe the behaviour of simple periodic patterns like the ether and gliders in an efficient way. We use that to look into their behaviour in detail and find general formulas that characterise them.

1 Introduction

This is a case study of one specific cellular automaton, Rule 54, with the methods developed in [3]. They were developed to allow the study of cellular automata with the methods of theoretical mathematics and without the need for computer simulations. While the previous paper concentrates on the development of the theory, here the ideas are presented in a less formal way and used to work with larger structures.

Section 2 of this paper introduces the formalism in a less formal way than in [3] and shows how the transition function of the cellular automaton can be expressed in it. The resulting formulas still describe only the behaviour of a small number of cells at a time. Therefore in Section 3 rules for larger groups of cells are found. We use them in Section 4 to study the behaviour of four simple periodic structures that occur under Rule 54: the
ether and three types of gliders (Figure 1). We find formulas for them and general expressions for gliders and ethers and look into their behaviour.

2 Local Interactions

2.1 Rule 54

“Rule 54” is the common name – following the naming convention of Stephen Wolfram [4] – of a one-dimensional cellular automaton with two states and a three-cell neighbourhood.

At every time it consists of an infinite line of cells. The state of each cell is an element of the set \( \Sigma = \{0, 1\} \), and the behaviour of the automaton is given by its local transitions function

\[
\varphi : \Sigma^3 \rightarrow \Sigma. \tag{1}
\]

It is applied to every three-cell subsequence of the infinite cell line, and the resulting value is the state of the cell in the middle at the next time step. Rule 54 has

\[
\varphi(s) = \begin{cases} 
1 & \text{for } s \in \{001, 100, 010, 101\}, \\
0 & \text{otherwise.}
\end{cases} \tag{2}
\]

Sequences of elements of \( \Sigma \) – like 001 – stand here and later for elements of \( \Sigma^* = \bigcup_{k \geq 0} \Sigma^k \). Note that \( \varphi \) is symmetric under the interchange of left and right.

2.2 Situations

The formalism of Flexible Time [3] is motivated by the idea that it is easier to find patterns in the evolution of cellular automata if one considers structures that involve cells at different times.

These structures are here called situations. They are a generalisation of the sequences of cell states (like 001) considered before. These sequences give the states of neighbouring cells at a certain unspecified time. Thus the sequence 001 describes the states of three cells, possibly at the positions \( x = 0, 1, 2 \), and tells us that the cells at \( x = 0 \) and \( x = 1 \) are in state 0 and the cell at \( x = 2 \) is in state 1. The position information is implicit in the ordering of the symbols: When a symbol in the sequence stands for the state of a certain cell, its right neighbour in the sequence gives the state of its right neighbour cell, and so on.

Situations are then cell sequences that extend over space and time. To write them down we need additional symbols for a change of time. The symbols we actually use stand for a displacement in time and also in space, because they harmonise then better with the way a cellular automaton evolves.

Under Rule 54, situations are written as sequences of the symbols 0, 1, \( \ominus \) and \( \oplus \). The intended interpretation can most easily be described in terms of instructions to write symbols on a grid. The fields of the grid are labelled by pairs \( (t, x) \in \mathbb{Z}^2 \); \( x \) is the position of a cell and \( t \) a time in its evolution. The writing rules are then

- At the beginning the writing position is at \( (0, 0) \).
• If the next symbol is 0 or 1, write it down and move the writing positions one step forward; if it was \((t, x)\) it is now \((t, x + 1)\).

• If the next symbol is \(\ominus\), move the writing position from \((t, x)\) to \((t - 1, x - 1)\).

• If the next symbol is \(\oplus\), move the writing position from \((t, x)\) to \((t + 1, x - 1)\).

• **No overwriting:** One cannot write different symbols at the same field. (This concerns expressions like \(01 \oplus \ominus 1\): After writing \(01 \oplus \ominus\) one is again at position \((0, 1)\) and tries to write a 1 in a field that contains already a 0. So \(01 \oplus \ominus 1\) is not a valid situation, but \(01 \oplus \ominus 0\) is.)

The result, in mathematical terms, is a function from a subset of \(\mathbb{Z}^2\) to \(\Sigma\) together with an element of \(\mathbb{Z}^2\) (the final writing position). The function, which is called \(\pi_s\) for a situation \(s\), describes the states of some cells at some times, while the element of \(\mathbb{Z}^2\), written \(\delta(s)\), will be important when parts of situations are substituted for others. The whole situation is then the pair \(s = (\pi_s, \delta(s))\). We will also need an empty situation, which is written\(\lambda\).

\[
\begin{align*}
\varepsilon_- = \ominus 00: & \quad \varepsilon_- = \ominus 1: & \quad e_- = \ominus \ominus 1: \\
w_- = \varepsilon_- 1: & \quad e_+ = 00 \ominus: & \quad e_+ = 1 \varepsilon_+: \\
\varepsilon_+ = 00 \oplus: & \quad w_+ = 1 \varepsilon_+: & \quad e_+ = 1 \oplus \varepsilon_+: \\
g_0 = 0 \varepsilon_+ 0 \varepsilon_- e_-: & \quad g_o = \varepsilon_+ \varepsilon_+ 0 \varepsilon_- e_-: & \\
g_o = \varepsilon_+ \varepsilon_+ 0 \varepsilon_- e_-: & \quad g_o = \varepsilon_+ \varepsilon_+ 0 \varepsilon_- e_-: \\
\end{align*}
\]

**Figure 2:** Useful situations in Rule 54.

In Figure \[2\] you can see diagrams for some situations that will become useful later. Cells in the states 0 and 1 appear as \(\square\) and \(\blacksquare\), while the initial and final writing position are marked by small triangles: \(<\) stands left of the start position, \(\triangleright\) at the end position. The diagram for \(g_o\) becomes less surprising if one notices that \(\varepsilon_\varepsilon_+ = 00 \ominus \ominus 00\) has the diagram \(<\square\triangleright\): a first case of overwriting.

I have also treated there the situations as normal algebraic expressions, like elements of a semigroup. Product and exponentiation are defined in the usual way: \(\varepsilon^2\) is the result of writing \(\varepsilon\) twice, and so on. However, due to the restrictions on overwriting, not all products of situations exist.
2.3 Reactions

The evolution of cellular automata is described by reactions, expressions of the form \( a \rightarrow b \) with two situations \( a \) and \( b \). The situation \( b \) represents a “partially later” state of the cellular automaton than \( a \), with the states of some cells at a later time than \( a \).

To make this notion more precise, let us consider functions of the form \( \pi : E \rightarrow \Sigma \). They are called cellular processes in [3]. If a cellular process fulfills the condition

\[
(\text{If } (t, x - 1), (t, x), (t, x + 1) \in E \text{ then } (t + 1, x) \in E \\
\quad \text{and } \pi(t + 1, x) = \varphi(\pi(t, x - 1)\pi(t, x)\pi(t, x + 1)),
\]

then it describes a part of the evolution of a cellular automaton under the rule \( \varphi \).

With this notion we can define “\( \rightarrow \)” as a binary relation on the set of situations: \( a \rightarrow b \) is true if \( \delta(a) = \delta(b) \) and for all cellular processes \( \pi \) that fulfill (3) we have: If \( \pi \supseteq \pi_a \) then \( \pi \supseteq \pi_b \).

One can see that if \( xay \) and \( xby \) are situations and there is a reaction \( a \rightarrow b \), then \( xay \rightarrow xby \) is a reaction too. This is called the application of \( a \rightarrow b \) on \( xay \). We can use that and describe the behaviour of a cellular automaton by a small set of generator reactions between a small number of cells. All the others follow from them by application on larger situations and by chaining the reactions. Table 1 shows a set of generator reactions for Rule 54. It is derived from (2) but contains some shortcuts.

To derive Table 1 we start with the rule that

\[
\varphi(\alpha, \beta, \gamma) = \sigma \quad \text{becomes} \quad \ominus\alpha, \beta, \gamma \rightarrow \sigma \ominus\beta, \gamma \quad \text{and} \quad \alpha, \beta, \gamma \ominus \rightarrow \alpha, \beta \ominus \sigma,
\]

because then \( \sigma \) is placed correctly one time step later than \( \beta \). The first two lines of Table 1 are found this way. Other reactions, like \( \ominus 10 \rightarrow 1 \ominus 0 \), are the result of a unification: There would be both \( \ominus 100 \rightarrow 1 \ominus 00 \) and \( \ominus 101 \rightarrow 1 \ominus 01 \), but the state of the rightmost cell has no influence on the result and is therefore left out at both sides of the reaction. These new, shorter reactions can now be applied on the results of some others:
\( \ominus 010 \rightarrow 1 \ominus 10 \), a reaction that one gets from \( \mathcal{H} \), is then extended by \( 1 \ominus 10 \rightarrow 11 \ominus 0 \) to \( \ominus 010 \rightarrow 11 \ominus 0 \). With these methods the top block of Table 1 is derived.

The purpose of the equations and reactions at the bottom of Table 1 is to create and destroy \( \ominus \) and \( \oplus \) symbols. The destruction reactions at the right remove also cell states that cannot be used in another reaction.\(^1\)

Together the reactions of Table 1 define a reaction system \( \Phi \). It consists of a set of situations and the reactions between them. We use a common convention and write \( s \in \Phi \) if \( s \) is an element of the set of reactions of \( \Phi \).

### 3 A Reaction System with Triangles

#### 3.1 Triangles

Now we need rules for larger structures. If their behaviour is understood, we can find reaction that simulate them in one step. At the present stage these structures will be periodic sequences of cells, and we start with the simplest of them, the sequences in which all cells are in the same state.

![Figure 3: Triangles and triangle reactions](image)

There are only two of them, and we evolve them first for only one time step.

\[
0^k \rightarrow 0^2 \ominus 0^{k-2} \ominus 0^2 \quad k \geq 0 \tag{5}
\]

\[
1^k \rightarrow 1 \ominus 0^k \ominus 1 \quad k \geq 0 \tag{6}
\]

We can see that \( 0^k \) is a persistent pattern that reappears in the next time step, while \( 1^k \) is instantaneous and exists only for one time step. Our guiding principle for a new, faster reaction system will be that evolution should never stop when a persistent pattern is reached.

So both reactions should be continued. The result for (5) – and therefore also for (6) – depends on the parity of \( k \) and is best expressed as

\[
0^{2k+\iota} \rightarrow (0^2 \ominus)^k 0^\iota (\ominus 0^2)^k \quad k \geq 1, \iota \in \{0,1\}, \tag{7}
\]

\[
1^{2k+\iota} \rightarrow 1 \ominus (0^2 \ominus)^k 0^\iota (\ominus 0^2)^k \ominus 1 \quad k \geq 1, \iota \in \{0,1\}. \tag{8}
\]

They are both examples of triangle reactions, that are reactions of the general form

\[
a_– x^k b_+ \rightarrow a_+ y^k c_+ b_– \quad k \geq 0, \tag{9}
\]

\(^1\)In \([3]\), which uses a slightly other definition of situations, the equations would have to be written as reactions. The destruction reactions, which are chosen somewhat ad hoc, are also different from the result of the result of the rules given there.
States: \(0, 1, \varepsilon_-, \varepsilon_+, e_-, e_+\).

Situations: No subsequences \(\varepsilon_0, 0\varepsilon_+, e_-, 1e_+\).

Triangles:

\[0^{2k+i} \rightarrow \varepsilon_+^k 0^i \varepsilon_+^k, \quad k \geq 1, \ i \in \{0, 1\}\]

\[1^{2k+i} \rightarrow e_+ \varepsilon_-^{k-1} 10^i \varepsilon_-^{k-1} e_-, \quad k \geq 1, \ i \in \{0, 1\}\]

Boundary terms:

\[\varepsilon_- (10)^k 1 \varepsilon_+ \rightarrow 1^{2k+3} \varepsilon_+, \quad k \geq 0\]

\[\varepsilon_- (10)^k e_+ \rightarrow 1^{2k+1} \varepsilon_+, \quad k \geq 0\]

\[e_- (01)^k \varepsilon_+ \rightarrow 1^{2k+1} e_+, \quad k \geq 0\]

\[\varepsilon_- e_+ \rightarrow \varepsilon_+ \varepsilon_-\]

\[e_- e_+ \rightarrow \varepsilon_+ e_-\]

Table 2: Rule 54 in triangle form

which trace the boundaries of a space-time triangle. Figure 3 shows an example.

Since the “boundary terms” of the triangles will occur often, we will use abbreviations for them,

\[\varepsilon_- = \ominus 0^2, \quad \varepsilon_+ = 0^2 \oplus, \quad e_- = \ominus 0^2 \ominus 1, \quad e_+ = 1 \oplus 0^2 \oplus. \quad (10)\]

The definitions for \(e_-\) and \(e_+\) have been chosen with the benefit of hindsight – instead of choosing abbreviations for \(\ominus 1\) and \(1 \ominus\) – because these terms will be important later. With them (11) and (12) become

\[0^{2k+i} \rightarrow \varepsilon_+^k 0^i \varepsilon_+^k, \quad k \geq 1, \ i \in \{0, 1\}\]

\[1^{2k+i} \rightarrow e_+ \varepsilon_-^{k-1} 10^i \varepsilon_-^{k-1} e_-, \quad k \geq 1, \ i \in \{0, 1\}\]

3.2 Destruction of Boundary Terms

We must now extend these reactions to a full reaction system. Since (11) and (12) create the boundary terms \(\varepsilon_-, \varepsilon_+, e_-\) and \(e_+\), the new reactions should destroy them. To keep the number of new reactions small, we require that the triangle reactions are always used efficiently and never applied to only a part of a cell sequence. (A reaction like \(0^3 \rightarrow \varepsilon_+ \varepsilon_- 0\) will be forbidden then.) We may express that by the requirement that the situations may never contain the terms \(\varepsilon_- 0, 0\varepsilon_+, e_- 1\) or \(1e_+\): they would be the result of such an incomplete application.

It will be enough for a working system to consider reactions that start from terms of the form \(b_- s b_+\), with \(b_- \in \{\varepsilon_-, e_-\}, \ b_+ \in \{\varepsilon_+, e_+\}\), \(s \in \Sigma^*\), to which no other reactions are applicable. The last requirement means that \(s\) must consist of cells in states 0 and 1 in alternating order: Two cells in the same state are already the starting point of a triangle reaction. It turns out that there are only six types of reactions that satisfy this requirement and that of the forbidden subconfigurations in the previous paragraph.

\(^2\text{We can bring reaction (12) in that form by setting } a_- = 1^2 x, \ x = 1^2, \ a_+ = \lambda, \ y_+ = 0^2 \ominus y_+ = \ominus 0^2\text{ and } c = 0^2.\)

6
Table 3: Simple Reactions that are useful in Section 4. Most of them are special cases of Table 2 or derived from them.

Here they are, together with reactions that start from them:

\[
\begin{align*}
\varepsilon_+e_+ & \rightarrow 1\varepsilon_+ & \varepsilon_+e_+ & \rightarrow w_+ \\
\varepsilon_+e_+ & \rightarrow \varepsilon_+1 & \varepsilon_+e_+ & \rightarrow w_+ \\
\varepsilon_+1\varepsilon_+ & \rightarrow e_+0e_- & w_-e_+ & \rightarrow e_+0e_- \\
e_-0e_+ & \rightarrow e + 0e_- & e_-w_+ & \rightarrow e_+0e_- \\
e_+10e_+ & \rightarrow 1^3\varepsilon_+ & w_-0e_+ & \rightarrow 1^3\varepsilon_+ \\
e_+10\varepsilon_+ & \rightarrow 1^3\varepsilon_+ & e_-0w_+ & \rightarrow 1^3\varepsilon_+ \\
e_+101\varepsilon_+ & \rightarrow 1^5 & w_-w_+ & \rightarrow 1^5.
\end{align*}
\]

The first four reactions have been chosen minimally such that the cell states of \( b \) in \( b \rightarrow b+1 \) are replaced with states that are exactly one time step later, such as in (5) and (6). The last two reactions cover the situations with \( s = \lambda \) that are not special cases of the previous four reactions. The resulting reactions system is listed in Table 2.

4 Ether and Gliders

4.1 The Ether

Now we will use the new reaction system to look at some phenomena that occur under Rule 54. The first of them is the \textit{ether}, a robust background pattern. It consists at alternating time steps of either the cell sequence \( 01^3 \) or \( 10^3 \) infinitely repeated. (To verify the reactions in this section Table 3 may be helpful.)

In the reaction system a formula for the ether can be derived from the \( 01^3 \) generation: We have

\[
01^3 \rightarrow 0e_+0e_-
\]

and (see Figure 4)

\[
0e_-0e_+ \rightarrow 0e_+0e_-,
\]

therefore

\[
(01^3)^k \rightarrow (0e_+e_-)^k \quad k \geq 0,
\]
a very simple triangle reaction. This is in contrast to the other possible starting point, 10^3, where one gets

\[(10^3)^k \to 1\varepsilon_+(0e_+)^k(0e_-)^k\varepsilon_1 \quad k \geq 1, \tag{22}\]

a more complicated triangle reaction, in which also the components of the other ether phase, \(e_-\) and \(e_+\), reappear. The reaction system selects thus one of the phases of the ether as more natural than the other, which is a helpful simplification.

If one now looks back at (17) and compares it with (20), one sees that they follow a common pattern. Both are background reactions of the form

\[b_-b_+ \to b_+b_. \quad \tag{23}\]

This reaction can easily be iterated to \(b_+^k b_-^k \to b_+^k b_-^k\), which describes the evolution of a large piece of a periodic background pattern.

Their involvement in the ether is the reason why \(e_-\) and \(e_+\) got their names in (10).

4.2 Gliders

There are three kinds of long-lived structures that are described in [1] in some detail. There they are called particles, now usually gliders. There is one moving particle \(w\), which appears as \(\overrightarrow{w}\) and \(\overleftarrow{w}\), depending on the direction in which it moves, and the “odd” and “even gutter” \(g_o\) and \(g_e\), which are immobile.

The \(w\) particle “may be generated by three 0’s followed by three 1’s or the converse” [1 p. 870]. We try this now and get

\[0^31^3 \to \varepsilon_+0\varepsilon_-e_+0e_- \to \varepsilon_+01\varepsilon_+0e_-\cdots. \quad \tag{24}\]

In it we can recognise \(0e_-\) as a part of the ether and \(\varepsilon_-0\) as a part of the ether in the wrong phase (as in (20) and (22)), so the rest must be the \(w\) particle. Therefore we define

\[w_- = \varepsilon_-1, \quad w_+ = 1\varepsilon_+. \quad \tag{25}\]

These definitions must be verified: We must show that \(w\) actually moves through the ether. But we have

\[w_-0e_+0 = \varepsilon_-10e_+0 \to 1^3\varepsilon_+0 \to e_+0\varepsilon_-10e_+0 \to e_+0\varepsilon_-10 = e_+0w_-0. \quad \tag{26}\]
Figure 5: A $\vec{w}$ glider moving on an ether background. The $w_-$ part is emphasised.

which shows how $w_-$ is destroyed and reappears at the right of its previous position (Figure 5). $w_-$ is therefore stable and corresponds to the right-moving glider $\vec{w}$ of [1].

The two immobile gliders, $g_e$ and $g_o$, are in fact small triangles, as can be seen from the pictures in [1]. It turns out that the right definitions for them are

$$g_o = e_+ \varepsilon_+ 0 e_- , \quad g_e = e_+ \varepsilon_+ \varepsilon_- e_- .$$

The verification that they actually behave like gliders is straightforward (Figure 6),

$$e_- 0 g_o 0 e_+ 0 = e_- 0 e_+ \varepsilon_+ 0 e_- 0 e_+ 0 \quad \rightarrow \quad e_+ 0 \quad e_- 0 e_+ 0 e_- 0 e_+ 0 \quad \rightarrow \quad e_+ 0 \quad w_- 0 w_+ 0 e_- 0 \quad \rightarrow \quad e_+ 0 \quad 1^0 0 \quad e_- 0 \quad \rightarrow \quad e_+ 0 \quad e_+ \varepsilon_+ 0 e_- 0 e_- 0 e_+ 0 \quad \rightarrow \quad e_+ 0 \quad e_+ \varepsilon_+ \varepsilon_- e_- 0 e_- 0 e_+ 0 = e_+ 0 g_o 0 e_- 0 ,$$

but the appearance of the $w$ gliders in the process is a bit surprising. It suggests the interpretation that the gliders $g_o$ and $g_e$ decay into two $w$ gliders, which then collide and create its next incarnation. With flexible time the gliders suddenly have an internal structure.

The three glider reactions (26), (28) and (29) have again a common structure, which can be described by the glider condition

$$b^k g b^\ell_+ \rightarrow b^\ell_+ g b^k .$$

Here $b_-$ and $b_+$ form a background pattern as in (23) and $g$ is the glider. The number $(\ell - k)/(\ell + k)$ is a measure for the speed of the glider.

We have now already touched the creation of other gliders by the $w$ gliders. Of the two syntheses found in the behaviour of the $g$ particles, the first one,

$$w_- 0 w_+ \rightarrow g_o ,$$

(31)
$e_0g_00e_0 \rightarrow e_0w_0w_0e_0 \rightarrow e_0g_00e_0$.

$e_0g_00e_0 \rightarrow e_0w_0w_0e_0 \rightarrow e_0g_00e_0$.

Figure 6: Evolution of the $g_0$ and $g_e$ gliders, together with the intermediate states where the $w$ gliders appear.

is more important because here the $w$ gliders are at the right distance to have been part of the ether before. Such a glider synthesis has been already noticed in [1], but here it occurs as a corollary of a previous analysis.

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