Nontrivial topological dynamics in Minkowskian Higgs model quantized by Dirac.

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Abstract

We study the nontrivial topological dynamics inherent in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac.

It comes to persistent collective solid rotations inside the physical BPS monopole vacuum, accompanied by never vanishing vacuum "electric" fields (vacuum monopoles) \( E \).

The enumerated rotary effects inside the physical BPS monopole vacuum suffered the Dirac fundamental quantization are the specific display of the Josephson effect, whose nature will be reveal in the present study.

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1 Introduction.

Continuing the investigations about the Dirac fundamental quantization [1] of the Minkowskian Higgs model with vacuum BPS monopole solutions started in the recent papers [2, 3], now we concentrate our efforts to the nontrivial topological dynamics, proves to be inherent in the enumerated model side by side with manifest superfluid properties [3].

The origin of that nontrivial topological dynamics (we reveal in the present study) lies in the Gauss-shell reduction of the Minkowskian Higgs model (with vacuum BPS monopole solutions).

At resolving the Yang-Mills (YM) Gauss law constraint

\[
\frac{\delta W}{\delta A^a_0} = 0 \iff [D^2(A)]^{ac} A_{0c} = D_i^{ac}(A) \partial_0 A^i_c
\]

with the covariant (Coulomb) gauge \[4, 5\]

\[
A^a || \sim [D_i^{ac}(\Phi^{(0)}) A^i_c] = 0|_{t=0}
\]

(in the fixed time instant \(t_0\) and in the [topologically trivial] YM BPS monopole background \(\Phi^{(0)}\)), the former turns into the second-order homogeneous differential equation

\[
[D_i^2(\Phi^{(0)})]^{ac} A_{0c} = 0,
\]

permitting the family of so-called zero mode solutions \[6, 7\]

\[
A^c_0(t, x) = \dot{N}(t) \Phi^{(0)}(x) \equiv Z^c,
\]

implicating the topological variable \(\dot{N}(t)\) and Higgs (topologically trivial) vacuum Higgs BPS monopole modes \(\Phi^{(0)}(x)\).

The appearance of \(Z^a\) solutions at the constraint-shell reduction of the Minkowskian Higgs model with vacuum BPS monopoles involves lot of important consequences for this model, we shall discuss in the present study.

First of all, knowing YM potentials \(Z^a\) (referring obviously to the BPS monopole vacuum), it is easy to write down \(F^a_{i0}\) components of the YM tension tensor, taking the shape of so-called vacuum "electric" monopoles \[4, 5\]

\[
F^a_{i0} = \dot{N}(t) D_i^{ac}(\Phi^{(0)}_k) \Phi_{0c}(x).
\]

Issuing from vacuum "electric" monopoles \(F^a_{i0}\), one can construct \[4, 5, 8, 9\] the action functional

\[
W_N = \int d^4x \frac{1}{2} (F^c_{i0})^2 = \int dt \frac{\dot{N}^2 I}{2}
\]

involving the rotary momentum \[8\]

\[
I = \int_V d^3x (D_i^{ac}(\Phi^{(0)}_k) \Phi_{0c})^2 = \frac{4\pi^2 \epsilon}{\alpha_s} = \frac{4\pi^2}{\alpha_s^2} \frac{1}{V < B^2 >}.
\]
The YM coupling constant
\[ \alpha_s = \frac{g^2}{4\pi(\hbar c)^2} \]
enters this expression for \( I \) together with the typical size \( \epsilon \) \cite{4,5} of BPS monopoles and the vacuum expectation value \( < B^2 > \) of the ”magnetic” field.

It will be shown that the action functional \( W_N \) describes correctly collective solid rotations of the BPS monopole vacuum (suffered the Dirac fundamental quantization \cite{1}) with constant angular velocities \( \dot{N}(t) \).

These constant angular velocities \( \dot{N}(t) \) determine the real energy-momentum spectrum
\[ P_{N} = \dot{N} I = 2\pi k + \theta; \quad \theta \in [-\pi, \pi]; \] (1.6)
of the free rotator \( W_N \), accompanied by the wave function
\[ \Psi_N \equiv < P_{N}|N> = \exp(iP_{N}N). \] (1.7)

The enumerated rotary effects inside the BPS monopole vacuum suffered the Dirac fundamental quantization \cite{1} may be explained good \cite{10} as a particular manifestation of the Josephson effect, coming to persistent circular motions of material points (in particular, quantum fields) without (outward) sources.

In the present study we consider in detail vacuum rotary effects in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac, associated, in the first place, with properties of the topological variable \( N(t) \), proving to be the noninteger degree of the map referring to the \( U(1) \rightarrow SU(2) \) embedding.

All this implies the following plan of the article, we now propose our readers.

Section 2, devoted to the nontrivial topological dynamics inherent in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac \cite{1}, consists of two subsections.

In Subsection 2.1 we study more in detail how the nontrivial topological dynamics arises in the Minkowskian Higgs model quantized by Dirac and involving vacuum BPS monopole solutions.

Repeating the arguments \cite{4,5,7}, we show that its origin in the constraint-shell reduction of this model in terms of topological Dirac variables \( \hat{A}_i^p \) \( (i = 1, 2) \) \cite{3,8}: gauge invariant and transverse functionals of YM fields, satisfying the covariant transverse gauge.

Although Subsection 2.1 contains, actually, no new information in comparison with Refs. \cite{4,5,7}, it will be helpful, nevertheless, as the preparatory material for Subsection 2.2, in which we study the properties of the topological variable \( N(t) \), the noninteger degree of the map referring to the \( U(1) \rightarrow SU(2) \) embedding.

The principal result will be got in Subsection 2.2 is that \cite{10}
\[ \dot{N}(t) = \text{const} = (n_{out} - n_{in})/T \equiv \nu/T, \] (1.8)
where \( n_{out}, n_{in} \in \mathbb{Z} \) refer to the fixed time instants \( t = \pm T/2 \), respectively; \( \nu \) is referred to as the Pontryagin index \([7, 10]\).

It will be argued herewith (due to general QFT reasoning) that it would be set \( T \to \infty \).

By that the claim to \( N(t) \) to take integers at \( t = \pm T/2 \to \infty \) is satisfied automatically when one represents this variable as \([10]\)

\[
N(t) = (n_+ - n_-)\frac{t}{T} + (n_+ + n_-)\frac{1}{2};
\]

\( n_- \equiv n_{in}; \quad n_+ \equiv n_{out}. \)

Section 3 will be devoted to the general discussion about the Josephson effect, in which the arguments \([10]\) will be repeated.

It will be shown that persistent circular motions of quantum fields (that is the essence of the Josephson effect) are characterized \([9, 10]\) by never vanishing (until \( \theta \neq 0 \)) momenta

\[
P = \hbar \frac{2\pi k + \theta}{L},
\]

with \( L \) being the length of the whole closed line along which the given quantum field moves.

Such momenta \( P \) attain their nonzero minima \( p = \hbar \theta / L \) as \( k = 0 \) and if \( \theta \neq 0 \).

Besides collective solid rotations inside the BPS monopole vacuum suffered the Dirac fundamental quantization \([1]\) (these rotations are characterized constant angular velocities \( \dot{N}(t) \)), the Josephson effect in the appropriate Minkowskian Higgs model comes to never vanishing (until \( \theta \neq 0 \)) vacuum ”electric” fields (”electric” monopoles)

\[
(E^a_i)_{\text{min}} = \theta \frac{\alpha_s}{4\pi^2 \epsilon} B^a_i; \quad -\pi \leq \theta \leq \pi.
\]

Such minimum value of the vacuum ”electric” field \( E \) corresponds to trivial topologies \( k = 0 \), while generally \([8]\),

\[
F^a_{i0} \equiv E^a_i = \dot{N}(t) (D_i(\Phi_k^{(0)})(\Phi_{(0)}), B^a_i(\Phi_{(0)}) = \frac{\alpha_S}{4\pi^2 \epsilon} \dot{B}_i(\Phi_{(0)}) = (2\pi k + \theta) \frac{\alpha_S}{4\pi^2 \epsilon} \dot{B}_i(\Phi_{(0)}).
\]
2 Nontrivial topological dynamics in Minkowskian Higgs model quantized by Dirac.

2.1 Collective vacuum modes arise at Gauss-shell reduction of Minkowskian Higgs model with BPS monopoles.

In Introduction we have already discussed the outlines appearing collective vacuum modes $Z^a$, (1.3), in the Minkowskian Higgs model with vacuum BPS monopole solutions quantized by Dirac [1].

In details, this looks as following.

In the Minkowskian Higgs model with vacuum BPS monopole solutions (including topologically trivial YM BPS monopole background $\Phi^{(0)}$), the YM Gauss law constraint (1.1) may be satisfied with the covariant (Coulomb) gauge [4, 5]

$A_i^\parallel \sim \left[D^{ac}_i(\Phi^{(0)})A^c_i(0)\right] = 0|_{t=0}.$

(2.1)

In particular, in the enumerated model, the YM Gauss law constraint (1.1) may be resolved in terms of topological Dirac variables $\hat{A}^D_i (i = 1, 2)$ [5] with the symbol $T$ standing for time ordering the matrices under the exponent sign.

As it was discusses in Ref. [3] (repeating the arguments [8, 12]), the topological Dirac variables (2.2), involving Gribov topological Gribov multipliers $v^{(n)}(x)$, are gauge invariant and transverse functionals of YM fields $A$.

Herewith for topologically trivial multipliers $v^{(0)}(x)$, functionals (2.2) satisfy the gauge (2.1), while at $n \neq 0$ they satisfy the Coulomb gauge [7]

$D^{ab}_{i}(\Phi^{(n)}_{k})A^{i(n)}_{b} = 0$

(2.3)

(implicating topologically nontrivial YM BPS monopole modes $\Phi^{(n)}_{k}$).

Upon fixing the Coulomb gauge (2.1), (2.3) for topological Dirac variables (2.2), the YM Gauss law constraint (1.1) turns into the homogeneous second-order differential equation

$$(D^2)^{ab}\Phi_{b(0)} = 0,$$

(2.4)

permitting $Z^a$ as its solutions.

1 According [3, 10],

$\hat{A}_{\mu} = \frac{gA_{\mu}^a\tau_a}{2i\hbar c}$.

2 The just described method resolving the YM Gauss law constraint was proposed at the first time by Polubarinov [13]. It consists, generally, in straight solving constraint equations: classical equations on field components having equal to zero canonical momenta.
Now let us return to Eq. (1.4), described collective solid rotations inside the Minkowskian
BPS monopole vacuum suffered the Dirac fundamental quantization [1].
To ground that it is correct, let us appeal to classical solid mechanics.
It is well known (see e.g. §32 in [14]) that the general solid Lagrangian has the look

\[ L_s = \frac{\mu V^2}{2} + \frac{1}{2} I_{ik} \Omega_i \Omega_k - U \]  

in a chosen rest reference frame.

Herewith \( \mu \) is the mass of the given solid:

\[ \mu = \sum_a m_a, \]

with \( m_a \) being the mass of the \( a \)th “particle”; \( V \) is the velocity of the c.m. translational
motion; \( \Omega \) is the angular velocity of its rotary motion; \( U \) is the potential energy of the solid.

In conclusion, the tensor \( I_{ik} \) is the tensor of inertia momenta (or simply, inertia tensor):

\[ I_{ik} = \sum_a m_a (x^2_{ai} \delta_{ik} - x_{ia} x_{ka}). \]

We can compare both Eqs.: (1.4) and (2.5) [14].

This comparison shows that the role of the inertia tensor \( I_{ik} \) is played, in the Minkowskian
Higgs model quantized by Dirac and involving vacuum BPS monopole solutions, by the
value \( I \) [8], (1.5), of the vacuum rotary momentum. Simultaneously, the role of the angular
velocity \( \Omega \) is played therein by \( \dot{N}(t) \).

The way deriving Eq. (1.5) for the vacuum "rotary momentum" \( I \) was outlined in
Ref. [8]:

\[ I \simeq \frac{4\pi^2}{\alpha_s} \int^R_\epsilon dr \frac{d}{dr} (r^2 \frac{d}{dr} f_{BP S}^0 (r)); \quad R \to \infty. \]

This Eq. implicates the "Higgs" BPS monopole ansatz [4, 5]

\[ f_{BP S}^0 (r) = \left[ \frac{1}{\epsilon \tanh(r/\epsilon)} - \frac{1}{r} \right]. \]

Herewith the typical size \( \epsilon \) of ("Higgs", "YM") BPS monopoles may be set via [4, 5]

\[ \frac{1}{\epsilon} \equiv \frac{gm}{\sqrt{\lambda}}, \]

with \( m \) and \( \lambda \) being, respectively, the Higgs mass and selfinteraction constant.

As it can be read from (1.5), \( \epsilon \sim V^{-1} \).
More exactly, this dependence may be given \([4, 5, 8]\) as

\[
\frac{1}{\epsilon} = \frac{g_m}{\sqrt{\lambda}} \sim \frac{g^2 < B^2 >}{4\pi}.
\]  

(2.9)

Latter Eq. comes from evaluating \([8]\) the vacuum "magnetic" energy referring to the BPS monopole configuration:

\[
\frac{1}{2} \int_\epsilon^\infty d^3x [B_i^a(\Phi_k)]^2 \equiv \frac{1}{2} V < B^2 > \sim \frac{1}{2\alpha_s} \int_\epsilon^\infty \frac{dr}{r^2} \sim \frac{1}{2\alpha_s} = \frac{2\pi}{g^2\sqrt{\lambda}} = \frac{2\pi}{g^2\epsilon}.
\]  

(2.10)

(at setting \(c = \hbar = 1\) in the definition of \(\alpha_s\)).

Draw our attention also to the \(< B^2 >^{-1}\) dependence of the rotary momentum \(I, (1.5)\).

This is the specific trace of manifest superfluid properties \([2, 8]\) of the Minkowskian Higgs model quantized by Dirac \([1]\).

More precisely, this dependence is stipulated by the Bogomol’nyi equation \([2, 4, 5]\)

\[
B = \pm D\Phi.
\]  

(2.11)

implicating the Higgs isomultiplet \(\Phi\) (having the shape of vacuum BPS monopole solutions), characterizing \([2]\) the superfluid properties of the Minkowskian BPS monopole vacuum.

In the infinite spatial volume limit \(V \to \infty\), the rotary momentum \(I, (1.5)\), disappears with disappearing the BPS monopoles size \(\epsilon\) via (2.9).

This implies also suppressing, in the infinite spatial volume limit, the action functional (1.4), describing collective solid rotations of the Minkowskian BPS monopole vacuum suffered the Dirac fundamental quantization \([1]\).

Due to the variation principle, the action functional (1.4) yields, as it is easy to see, the YM Gauss law constraint (1.2).

This confirms the rightfulness of the Gauss-shell reduction of the Minkowskian Higgs model.

Generally speaking, this Gauss-shell reduction of the Minkowskian Higgs model, involving collective zero mode solutions \(Z^a, (1.3)\), is one of the features of the Dirac fundamental quantization \([1]\) distinguishing it from the "heuristic" Faddev-Popov (FP) quantization \([15]\), implicating FP path integrals with gauge fixing.

As it was demonstrated in the recent paper \([9]\) (repeating the arguments \([16]\)), the presence of collective (vacuum) excitations in a gauge theory (collective zero mode solutions \(Z^a, (1.3)\), are the example of such collective excitations) involves violating the gauge equivalence theorem \([17, 18]\) between the Dirac \([1]\) and FP \([15]\) quantization approaches.

The both methods prove to be of equal worth when S-matrixes for quantum fields on-shell are in question.
But when composite fields are present in a gauge model (for instance, the collective excitations or bound states), the gauge equivalence theorem [17, 18] is violated [9, 16] and various spurious Feynman diagrams (SD) [7, 19] appear on the level of appropriate Green functions:

\[(\text{FR})^F + (\text{SD}) \equiv (\text{FR})^* \quad \text{for S–matrices with composite fields);}\]

\[(\text{FR})^F + (\text{SD}) \equiv (\text{FR})^* \quad \text{for Green functions),}\]

where the Feynman rules (FR)^F refer to the FP gauge fixing method [15], while the Feynman rules (FR)^* refer to the Dirac fundamental quantization approach [1]. The latter rules implicate (topological) Dirac variables.

As it was demonstrated in Refs. [7, 9, 19], spurious Feynman diagrams (SD) take account of the manifest relativistic covariance of (topological) Dirac variables [7, 9, 13, 19] (we refer our readers to Ref. [9] for details).

### 2.2 Properties of topological variable \(N(t)\).

The topic of this subsection will be investigating properties of the topological variable \(N(t)\) (with its time derivative \(\dot{N}(t)\), entering the expression (1.4) for the BPS monopole vacuum rotary action functional).

Herewith \(N(t)\) plays the crucial role in the (topologically nontrivial) dynamics inherent in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [1]. Therefore the present subsection is most important.

As it was demonstrated in Refs. [4, 5, 7], the topological variable \(N(t)\) may be specified via the relation

\[
\nu[A_0, \Phi(0)] = \frac{g^2}{16\pi^2} \int_{t_{\text{in}}}^{t_{\text{out}}} dt \int d^3x F_{\mu\nu}^a \tilde{F}^{a\mu\nu} = \frac{\alpha_s}{2\pi} \int d^3x F_{i0}^a B_i^a(\Phi(0))[N(t_{\text{out}}) - N(t_{\text{in}})]
\]

\[
= N(t_{\text{out}}) - N(t_{\text{in}}) = \int_{t_{\text{in}}}^{t_{\text{out}}} dt \dot{N}(t), \quad (2.12)
\]

taking account of the natural duality between the tensors \(F_{i0}^a\) and \(F_{ij}^a\).

Herewith \(\nu[A_0, \Phi(0)]\) is referred to as the vacuum Chern-Simons functional, implicating the asymptotical states “in” and “out” taking in the time instants \(t_{\text{in}}\) and \(t_{\text{out}}\), respectively. Herewith the mentioned time instants may be chosen to be \(t_{\text{in}} = T/2 = -t_{\text{out}}\) (with an arbitrary \(T\)).

In Ref. [7], the vacuum Chern-Simons functional (Pontryagin index) \(\nu[A]\) was defined as

\[
\nu[A] = \frac{g^2}{16\pi^2} \int_{t_{\text{in}}}^{t_{\text{out}}} dt \int d^3x F_{\mu\nu}^a \tilde{F}_a^{\mu\nu} = X[A_{\text{out}}] - X[A_{\text{in}}] = n(t_{\text{out}}) - n(t_{\text{in}}); \quad n_{\text{in, out}} \in \mathbb{Z};
\]

\[(2.13)\]
with
\[ X[A] = -\frac{1}{8\pi^2} \int_V d^3 x \epsilon^{ijk} \text{tr} \left[ \dot{A}_i \partial_j \dot{A}_k - \frac{2}{3} \dot{A}_i \dot{A}_j \dot{A}_k \right], \quad A_{\text{in, out}} = A(t_{\text{in, out}}, x), \quad (2.14) \]

being the topological winding number functional of gauge fields.

It follows from (2.12), (2.13) and (2.14) that \( N(t) \) may be treated as the noninteger degree of the map referring to the \( U(1) \to SU(2) \) embedding.\(^3\)

As we have discussed above, the topological dynamical variable \( N(t) \) may be represented in the shape (1.9)\(^{11}\).

In this case, as it is easy to see\(^{11}\), \( N(t) \) takes integers as \( t_{\text{in}} = T/2 = -t_{\text{out}} \):
\[ N(\pm T/2) = n_{\pm}, \quad n_{\pm} \in \mathbb{Z}. \quad (2.15) \]

Latter Eq. may be considered as an original ”boundary condition” imposed on \( N(t) \).

As a consequence of (1.9), one comes to Eq. (1.8) for \( \dot{N}(t) \).

Indeed, and this has a crucial importance, from the QFT point of view, it should be set \( T \to \infty \).

This setting has profound roots in QFT associated with the Haag theorem\(^{21}\).

It turns out that at considering asymptotical states for quantum fields in finite time instants \( t_{\text{in, out}} \) one encounters lot of problems and troubles.

The thing is that\(^{21}\) the interaction picture for quantum fields is based on assuming that, in QFT, generalized coordinates (quantum fields) interacted \( \phi \) and appropriate generalized momenta \( \pi \) (canonically conjugate to \( \phi \)) are related with an unitary transformation with ”free” variables \( \phi_0 \) and \( \pi_0 \) in the fixed time instants \( t \).

For instance,
\[ \dot{\phi}(t, x) = V(t) \phi_0(t, x) V^{-1}(t) \quad (2.16) \]
for generalized coordinates.

In this situation, as the Haag theorem shows\(^{3}\), the quested QFT becomes trivial if the claim for quantum fields to be relativistic invariant is added to the standard claims.

\(^3\)The Pontryagin degree of a map theory is stated enough good in the monograph\(^{20}\) (in §T21), and we recommend this monograph to our readers for studying the question.

\(^{11}\)Now note only that degrees of maps take integers according the Pontryagin theory.

\(^{21}\)Briefly, in the relativistic case, the Haag (-Hall) theorem may be formulated as follows\(^{21}\).

One assumes that, in the fixed time instants \( t \), the fields \( \phi(t, x) \), \( \dot{\phi}(t, x) \) and \( \phi_0(t, x) \), \( \dot{\phi}_0(t, x) \) (the existence of such fields is postulated) form two irreducible systems of quantum fields in the Hilbert spaces: respectively, \( \mathcal{H} \) and \( \mathcal{H}_{(0)} \), of quantum states.

Then, in addition to (2.16), it is assumed
\[ \dot{\phi}(t, x) = V(t) \dot{\phi}_0(t, x) V^{-1}(t). \]

In this case four first Wightman functions of the fields \( \phi(t, x) \) and \( \dot{\phi}_0(t, x) \) coincide in the both theories.

Moreover, if \( \phi_0(t, x) \) is a free field with the mass \( m \geq 0 \), then \( \phi(t, x) \) is also the free field with the mass \( m \) and the both theories coincide to within an unitary transformation \( V \).
to these fields in QFT \cite{21} (the so-called Wightman axioms). In other words, quantum fields prove to be free.

More exactly \cite{21}, in this case (at $t \neq \pm \infty$), there are no good specified matrix $V(t)$ for the abovementioned unitary transformation over the Hilbert space $\mathcal{H}$ of quantum states.

To ground the latter statement, let us analyse briefly the canonical Lagrangian formalism in QFT.

The canonical quantization scheme is based \cite{13} on the classical Lagrangian

$$ L = L(\phi_\alpha(x), \frac{\partial \phi_\alpha(x)}{\partial x^\nu}), $$

which is a function of (quantum) fields $\phi_\alpha(x)$ and their first derivatives.

The canonically conjugate "momentum" to the field $\phi_\alpha(x)$ in the time instant $t$ is given by Eq.

$$ \pi_\alpha(t, x) = \frac{\partial L}{\partial \frac{\partial \phi_\alpha(t, x)}{\partial t}}. $$

It is postulated that canonically conjugate fields $\phi_\alpha$ and $\pi_\alpha$ are the elements of an algebra determined by canonical commutation relations (CCR) in the case of bosonic fields\footnote{They would be replaced with anticommutation relations in the case of fermionic fields.}

$$ [\phi_\alpha(t, x), \phi_\beta(t, y)] = [\pi_\alpha(t, x), \pi_\beta(t, y)] = 0; $$

$$ [\phi_\alpha(t, x), \pi_\beta(t, y)] = i\delta_{\alpha\beta}\delta(x - y). \tag{2.17} $$

As a result, QFT may be formulated as quantum mechanics of a system with the infinite number of degrees of freedom (herewith $x$ and $y$ plays the role of the "number" of the given generalized coordinate and momentum).

It is convenient also to deal with a countable basis instead of a continuous.

For this purpose, in \cite{21} it was proposed to introduce an orthonormalized system of functions $h_\nu(x)$ in the three-dimensional space (for instance, there can be Hermit functions $N_\nu e^{-x^2/2}H_{\nu_1\nu_2\nu_3}(x)$) and to specify "coordinates" and "momenta" with Eqs.

$$ Q_n(t) = \int \phi_\alpha(t, x)h_\nu(x)d^3x, $$

$$ P_n(t) = \int \pi_\alpha(t, x)h_\nu(x)d^3x, \tag{2.18} $$

with $n$ being the composite discrete index: $n \equiv (\alpha, \nu)$.

It may be shown herewith \cite{21} that CCR (2.17) take the look

$$ [Q_n(t), Q_{n'}(t)] = [P_n(t), P_{n'}(t)] = 0; $$

$$ [Q_n(t), P_{n'}(t)] = i\delta_{nn'}. \tag{2.19} $$

Further, the abstract algebra of elements $Q_n$, $P_n$ is realized satisfying (2.19).
It is, indeed, an algebra of unrestricted operators in the Hilbert space $\mathcal{H}$ of state vectors.

In the case of a system with finite degrees of freedom, some two irreducible representations of CCR \((2.19)\) realized via selfconjugate (Hermitian) operators in the Hilbert space $\mathcal{H}$ are unitary equivalent \(^6\).

In particular, an unitary operator $V(t_2, t_1)$ exists associated the operators $P_n$ and $Q_n$ referring to different time instants:

$$Q_n(t_2) = V(t_2, t_1)Q_n(t_1)V^{-1}(t_2, t_1),$$

$$P_n(t_2) = V(t_2, t_1)P_n(t_1)V^{-1}(t_2, t_1). \tag{2.20}$$

But the said is not correct for systems with infinite numbers of degrees of freedom.

Moreover, linear canonical transformations (i.e. transformations of variables $P_n$ and $Q_n$ maintaining the look of CCR \((2.19)\)) don’t correspond, generally speaking, to an

\(^6\)In the latter statement is the substance of the so-called von Neumann theorem (see Theorem 6.14 in \[21\]).

Let us an irreducible Weyl system be given. It is set over the Hilbert space $\mathcal{H}$ and involves CCR represented in the exponential form for a quantum system with some $n$ degrees of freedom.

Any Weyl system always comes to the pair of Abelian $n$-parameter groups $U(a)$ and $V(b)$:

$$U(a) = e^{iap}; \quad V(b) = e^{ibq},$$

respectively.

Here $a, b \in \mathbb{R}^n$, and

$$ap \equiv \sum_{j=1}^{n} a_j p_j; \quad bp \equiv \sum_{j=1}^{n} b_j q_j$$

for a system of (generalized) coordinates $q \equiv (q_1, \ldots q_n)$ and (generalized) momenta $p \equiv (p_1, \ldots p_n)$ conjugate to former.

It is well known \[22\] that the operators $p$ and $q$ are selfconjugate (Hermitian).

Herewith also

$$U(a)U(a') = U(a + a'); \quad V(b)V(b') = V(b + b'),$$

and the commutation relations between $p$ and $q$ come to Eq. \[21\]

$$U(a)V(b) = e^{iab}V(b)U(a).$$

The claim for a Weyl system $\{U(a), V(b)\}$ to be irreducible comes to the claim that there are no nontrivial closed subspace in the Hilbert space $\mathcal{H}$ invariant with respect the operators $U(a), V(b)$.

On the other hand, if

$$p_i = -i \frac{\partial}{\partial q_i}$$

for any $i$, and if the vectors $p$ and $q$ are defined over the Lebesgue space $\mathcal{G}^2(\mathbb{R}^n)$, the operators $p$ and $q$ form the representation for CCR referred to as the Schrödinger representation \[21\].

In the above terms, the von Neumann theorem may be formulated as follows \[21\].

Any irreducible Weyl system with $n$ degrees of freedom is unitary equivalent to the Schrödinger representation in the Lebesgue space $\mathcal{G}^2(\mathbb{R}^n)$. Any reducible Weyl system (with $n$ degrees of freedom) is the direct sum of irreducible representations and thus it is a multiple of the Schrödinger representation.
unitary equivalence transformation.\footnote{To understand this idea, we recommend our readers to study the arguments given in Exercise 7.24 in Ref. \cite{21}.}

Among the infinite set of non-equivalent representations of CCR, one can pick out the Fock representation (see §8.4 in \cite{21}) in the theory of free fields.

This picking out may be achieved with the aid of the additional claim that there is the unique normalized relativistic invariant state (vacuum) $\Psi_0$ annihilated under acting the positive frequencies operators:

$$\tilde{\phi}^{(+)}(p)\Psi_0 = 0.$$ (2.21)

This claim is equivalent to existing the vacuum $\Psi_0$.

The Fock representation is specified in the unique way (to within the unitary equivalence).

On the face of it, it can be assumed (and it is assumed often) that one can pick out the Fock representation also in the theory of \textit{interacting} fields. Herewith that representation would involve a physical vacuum maintained in the time.

Indeed, the Haag theorem shows that it is not correct and that one would utilize another representations than the Fock ones for CCR (herewith there is the ”actual” infinite tower of ”bare” particles in each state in such representations).

The mentioned difficulties associated with the Haag theorem may be avoid at utilizing the asymptotical interaction picture in which interacting fields are equated in the time instants $t = \pm \infty$ instead their equating in a finite time instants $t$.

In this case, interacting, $\phi(x)$, and free, $\phi^{\text{in}}(x)$, fields are related as \cite{21}

$$\phi(x) = S^* T(S\phi^{\text{in}}(x)),$$ (2.22)

with $T$ standing for time ordering and $S$ being the appropriate scattering matrix.\footnote{The properties of S-matrices were described good in the monographs \cite{21,23}.}

In particular, it is worth to note that S-matrices are always Poincare invariant unitary operators in the Fock space $F$ of a system of free relativistic particles (see e.g. §7.3 in \cite{21}): for instance, this space may be identified with the space of fallen particles.

Herewith the irreducible system of (Wightman) free (”in”) fields $\phi^{\text{in}(\chi)}$ with CCR

$$[\phi^{\text{in}(\chi)}(x), \phi^{\text{in}(\chi')}(y)] = \frac{1}{i} D_{\nu\nu'}^{\chi\chi'}(x-y)$$

(where in the denotations \cite{21}, $l$, $l'$ being Lorenz indices and $\chi$, $\chi'$ being indices specifying types of fields; $D(x)$ is the appropriate Pauli-Jordan permutation function) at the normal spin-statistic connection (referring to the time instant $t = -\infty$) acts in $F$.

As it is well known, the S-matrix permits its (functional) expansion \cite{21}

$$S = \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 \cdots \int dx_n \sum_{\alpha_1}^{\alpha_n} S^{(n)}_{\alpha_1 \cdots \alpha_n}(x_1, \ldots x_n) \times \phi_{\alpha_1}(x_1) \cdots \phi_{\alpha_n}(x_n),$$

involving composite indices $\alpha_i$ ($i = 1, \ldots n$) consisting of $l$, $\chi$ and ”in” (”out”) indices \cite{21}.
Latter Eq. is equivalent to the Yang-Feldman equation \[\text{[21, 25]}\]

\[
\phi_t^{(\chi)}(x) = \phi_t^{\text{in}(\chi)}(x) + \sum_{\chi'\nu} \int D_{tt'}^{\text{ret}(\chi\chi')} (x - y) J^{(\chi')(\chi')'}(y) dy.
\] \tag{2.23}

This Eq. implicates the retarded Green function \[\text{[21, 23]}\]

\[
D^{\text{ret}}(x) = \int \frac{1}{m^2 - p^2 + i0\epsilon(p^0)} e^{ipx} d^4p,
\] \tag{2.24}

where \[\text{[23]}\]

\[
\epsilon(\alpha) = \frac{1}{\pi i} R \int_{-\infty}^{\infty} \frac{e^{i\alpha\tau} d\tau}{\tau} = \begin{cases} 1 & \text{at } \alpha > 0 \\ -1 & \text{at } \alpha < 0 \end{cases}.
\]

(with \(R\) denoting the main value).

For the field \(\phi_t^{\text{out}(\chi)}(x)\) (at \(t = +\infty\)), one would replace \(D^{\text{ret}}\) with the advanced Green function

\[
D^{\text{adv}}(x) = \int \frac{1}{m^2 - p^2 - i0\epsilon(p^0)} e^{ipx} d^4p
\] \tag{2.25}

in the Yang-Feldman equation \[\text{(2.23)}\].

Note also, that from the viewpoint of (relativistic) quantum mechanics, attributing free fields to the time instants \(t = \pm\infty\) corresponds just to the interaction picture in such its picture \[\text{[23]}\] when an interaction \(H_1\) is "switched on" adiabatically at \(t = -\infty\) and is "switched off" also adiabatically at \(t = +\infty\).

Herewith, at denoting as \(\Phi(-\infty)\) the amplitude of the initial ("in") state and as \(\Phi(+\infty)\) the amplitude of the final ("out") state (in the appropriate Fock space \(F\)), takes place the relation

\[
\Phi(+\infty) = S\Phi(-\infty)
\] \tag{2.26}

between these amplitudes, implicating the scattering matrix \(S\).

The interdependence between the relations \[\text{(2.26)}, \text{(2.22)}\] and the Yang-Feldman equations \[\text{(2.23)}\] is highly transparent.

The just describing procedure \[\text{[21]}\] referring free fields \(\phi^{\text{in}}(x)\) and \(\phi^{\text{out}}(x)\) to the time instants \(t = \pm\infty\), respectively (involving the Yang-Feldman equations \[\text{(2.23)}\]), results (as it may be demonstrated) deleting infrared divergences in quested QFT (although ultraviolet divergences can occur).

The mentioned infrared divergences are just associated with the Haag theorem. The origin of them is in the infinite spatial volume utilized at the proof of this theorem (for...
instance, in Eqs. (2.18) specifying the operators $Q_n(t)$ and $P_n(t)$; this just causes the bad definition for the operator $V(t_2, t_1)$ because of (2.20).

These divergences can be tamed at a compactification performed, but quested QFT may still be plagued by divergences of other types; for example, irreducible representations for a free scalar field and a $\phi^4$ field in the spatially compactified (2+1)-dim Minkowski space-time are known to belong to inequivalent representations because of ultraviolet divergences that occur at high energy-momentum.

Returning again to studying the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac, note that setting $T \to \infty$ in all the formulas about the topological dynamical variable $N(t)$ seems to be quite correct in the light of the said above about the role of such setting in QFT for free fields as the way to avoid various troubles associated with infrared divergences in quested QFT (in the first place, it is connected closely with the possibility to expand S-matrices by asymptotically free quantum fields at $t \to \pm \infty$ with integrable coefficients).

The consequence of setting $T \to \infty$ in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac is that $\dot{N}(t) \to 0$ due to (1.8). In other words, angular velocities of collective solid rotations inside the Minkowskian BPS monopole vacuum suffered the Dirac fundamental quantization approach zero in the limit $T \to \infty$.

And this is, indeed, a bad news. The rotary effects disappear in this limit $T \to \infty$. In other words, the Minkowskian BPS monopole vacuum will be motionless as $T \to \infty$

The only way to resolve a contradiction now arising, is to circumvent the Haag theorem. It turns out that there are odds to do this, but with numerous warnings.

For instance, the Haag theorem does not exclude the existence of the interaction (Dirac) picture at violating the translational invariance of QFT by introducing the spatial cut-off. When the interaction picture exists for the Hamiltonian cutting off in such a way, we are dealing with the so-called local Fock representation of CCR. However, a difficult mathematical problem arises in this case with the cut-off removal.

Such a ”local” interpretation of the interaction picture, bypassing the Haag theorem, was proposed, for instance, in the work [27].

In Ref. [9] there was assumed the so-called ”discrete vacuum geometry”

$$R_{YM} \equiv SU(2)/U(1) \simeq \mathbb{Z} \otimes G_0/U_0,$$

with

$$\pi_1(U_0) = \pi_1(G_0) = 0$$

and

$$SU(2) \equiv G; \quad U(1) \equiv U,$$

\[9\] The arguments by J. Earman and D. Fraser were utilized here (visit the site http://philsci-archive.pitt.edu/archive/00002673).
necessary to justify various rotary effects (in particular, the above described collective solid rotations) proper to the Minkowskian BPS monopole vacuum suffered the Dirac fundamental quantization.

This "discrete vacuum geometry" will be investigated in next papers, are planned.

Now we only should like point to the following circumstance just associated with angular velocities $\dot{N}(t)$ of collective solid rotations inside the Minkowskian BPS monopole vacuum.

As it was discussed in Ref. [9], typical widths of domain walls between different topological sectors inside the discrete space $R_{YM}$ is comparable with $\epsilon(r)$, the typical size [4, 5] of BPS monopoles.

Now let’s examine eq. (2.9) [4, 5, 8]. If the coupling constant $g$ is finite (although it is indeed large at the distance $r \sim 1$ fm, i.e. of the hadrone size order, from the chosen origin of coordinates; this corresponds to the quark confinement), (2.9) implies $\epsilon \to 0$ in the $V \to \infty$ limit (this limit is considered as fixed in the model we discuss now due to the necessary integration over the whole three-space in (2.10)).

Vice versa, in the asymptotical freedom limit $g \to 0$, $\epsilon$ can take any finite values (due to the $0 \times \infty$ uncertainty in that case of $V \to \infty$). This means that walls between topological domains inside $R_{YM}$ can be of finite widths $O(\epsilon(0)) \neq 0$, at the origin of coordinates.

Concluding this subsection, we should like discuss some important features of the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [1] that just associated with the properties of the topological dynamical variable $N(t)$.

First of all, as it follows from (1.8), angular velocities $\dot{N}(t)$ of collective solid rotations inside the Minkowskian BPS monopole vacuum suffered the Dirac fundamental quantization [1] are motion integrals [6]:

$$\ddot{N}(t) = 0. \quad (2.28)$$

Secondly, as it was shown in Ref. [29], introducing the noninteger degree of the map $N(t)$ (in the (2.12) wise) involves the gauge transformations

$$\mathcal{N}_1[\Phi^{(0)}, N(t)] = \sin[2\pi N(t)]/2\pi,$$

$$\mathcal{N}_2[N(t)] = \{N(t) - \sin[2\pi N(t)]/2\pi\}; \quad (2.29)$$

connected with gauge transformations

$$X[A^{(n)}] = X[A^{(0)}] + \mathcal{N}_1[\Phi^{(0)}, N(t)] + \mathcal{N}_2[N(t)] = X[A^{(0)}] + N(t). \quad (2.30)$$

for the winding number functional $X[A]$ [3, 7]. Herewith

$$\nu[A] = X[A_{out}] - X[A_{in}] = n(t_{out}) - n(t_{in}).$$
Furthermore [7], the "boundary" condition (2.15) for the topological dynamical variable \( N(t) \) is equivalent to shifts this variable onto integers:

\[
N \implies N + n \equiv \tilde{N}; \quad n = \pm 1, \pm 2, ...
\]  
(2.31)

(as \( T \to \infty \) and \( t = \pm T/2 \)).

For our further discussion, we should recall the explicit look of Gribov stationary topological multipliers \( v^{(n)}(x) \) in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac.

This proves to be [3, 4, 5, 7]

\[
v^{(n)}(x) = \exp[n\hat{\Phi}_0(x)],
\]  
(2.32)

with \( \hat{\Phi}_0(x) \) being the Gribov phase, that is, indeed,

\[
\hat{\Phi}_0(r) = -i\pi \frac{\tau^a}{r} f^{BPS}_a(r), \quad f^{BPS}_a(r) = \frac{1}{\tanh(r/\epsilon)} - \frac{\epsilon}{r}.
\]  
(2.33)

(where \( \tau^a \) (\( a = 1, 2, 3 \)) are Pauli matrices).

Thus \( \hat{\Phi}_0(x) \) is an \( U(1) \subset SU(2) \) scalar constructed over (topologically trivial) Higgs vacuum BPS monopole solutions [4, 5].

It is easy to see that Gribov stationary topological multipliers \( v^{(n)}(x), (2.32) \), may be replaced with [7]

\[
\exp[N(t)\hat{\Phi}_0(x)],
\]  
(2.34)

that implies replacing

\[
\hat{A}^{(N)}_i = \exp[N(t)\hat{\Phi}_0(x)][\hat{A}^{(0)}_i + \partial_i] \exp[-\tilde{N}(t)\hat{\Phi}_0(x)]
\]  
(2.35)

for topological Dirac variables \( \hat{A}^D_i \) in the initial time instant \( t_0 \).

This replacing for topological Dirac variables is quite legitimate since exponential multipliers \( \exp(N(t)) \) can be always formally included into \( U(1) \subset SU(2) \) gauge matrices \( u(t, x) \).

In this case topological Dirac variables \( \hat{A}^D_i, (2.2) \), remain gauge invariant at the replacement (2.35) due to the transformation law [8, 12]

\[
U(t, x) \to U_u(t, x) = u^{-1}(t, x)U(t, x),
\]  
(2.36)

with

\[
U(t, x) \equiv T \exp \left\{ \int_{t_0}^t dt \hat{A}_0(t, x) \right\}.
\]

Thus \( U(t, x) = 0 \) as \( t = t_0 \).
Gribov topological multipliers (2.34) may be rewritten in the alternative form [7]

\[ U_Z = T \exp\left[ \int_{t_0}^{t} dt' \hat{Z}(t', x) \right] \big|_{\text{asymptotic}} = \exp[N(t)\hat{\Phi}_0(x)] \] (2.37)

with

\[ \hat{Z}(t, x) \big|_{\text{asymptotic}} = \dot{N}(t)\hat{\Phi}_0(x) + O(\frac{1}{r^{l+1}}); \quad l > 1. \] (2.38)

Since the topological variable \( N(t) \) satisfies the "boundary condition" (2.15) [10] in the initial time instant \( t = t_{\text{in}} \equiv t_0 = -T/2 \to -\infty \), in this time instant, Gribov topological multipliers \( U_Z \): (2.34), (2.37), turn into \( v^{(n)}(x) \) ones at \( t = t_0 \).

Thus one can assert that in the initial time instant \( t = t_0 \) there is a natural isomorphism between the sets \( \{U_Z\} \) and \( \{v^{(n)}(x)\} \) of Gribov topological multipliers (the same is correctly, of course, also at \( t = T/2 \)).

And moreover, Gribov topological multipliers \( U_Z(t_0, x) \) acts as authomorphisms on the set \( \{v^{(n)}(x)\} \).

The latter fact is in a good agreement with Eq. (2.31). More exactly, the shift (2.31) of integers \( n \in \mathbb{Z} \) onto \( N(t) \) involves in this case (because of the "boundary condition" (2.15)) mapping a one (say, "large" [7], with the fixed \( n \neq 0 \) \( U(1) \subset SU(2) \) orbit into another, while the alone \( U(1) \subset SU(2) \) gauge group remains immovable at this shift.

Now there are two important conclusions may be drawn concerning Gribov topological multipliers \( \{U_Z\} \): (2.34), (2.37).

Firstly, the topological degeneration of Dirac variables \( A_i^D \), (2.2), may be reduced, in the initial time instant \( t = t_0 \), to shifts (2.31) of the topological variable \( N(t) \) onto integers.

Secondly, the us said about Gribov topological multipliers \( U_Z(N(t), x) \) shows that these multipliers play the role equivalent the role of topological multipliers \( v^{(n)}(x) \) in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [1].

In particular, the constraint-shell Hamiltonian of the mentioned model may be expressed in terms of \( U_Z(N(t), x) \) as well as in terms of \( v^{(n)}(x) \), as it was demonstrated in Ref. [10].

In the papers planed this will be discussed in detail and also the properties of Gribov topological multipliers \( U_Z(N(t), x) \) will be analysed.

3 Josephson effect in Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac.

Collective solid rotations inside the Minkowskian Higgs BPS monopole vacuum suffered the Dirac fundamental quantization [11] imply the purely real energy-momentum spectrum \( P_N \), (1.6), accompanied by the wave function \( \Psi_N \), (1.7).

Herewith the topological momentum [5] \( P_N \), (1.6), is read easily from the action functional (1.3).
As it follows from (1.6), the topological momentum $P_N$ depends explicitly on the $\theta$-angle.

In the present section we attempt to demonstrate our readers that such dependence of $P_N$ on the $\theta$-angle is the particular display of the Josephson effect, coming to persistent circular motions of material points (quantum fields).

In Ref. [10] the Josephson effect was explained with the example of two superconductors joined in an electric circuit [10].

At a contact of two superconductors, the motion of electrons in the both is strongly correlated.

This means that all the electrons possess one and the same momentum and the common wave function [10]

$$
\Psi = \prod_{j=1}^{n} \Psi_j = \exp[ip \sum_{j}^{N} X_j / \hbar] = \exp[i(pN) \sum_{1}^{N} X_j / (N\hbar)] \equiv \exp[i\mathcal{P}\mathcal{X}/\hbar],
$$

(3.1)

with $\mathcal{X}$ being the coordinate of the centre of inertia of the system of electrons and $\mathcal{P}$ being their total momentum.

At a contact of two different superconductors: say, in a point $x_0$, a flip of the wave function’s phase occurs:

$$
\Psi(x_0 + \epsilon) = \exp[i\theta]\Psi(x_0 - \epsilon), \quad \epsilon \rightarrow 0.
$$

(3.2)

The origin of latter Eq. is following [10].

When a closed (one-dimensional) way is given and $L$ is the length of this closed curve, then points $X$ and $X + L$ coincide and are physically equivalent.

This means that the wave function at the point $X$ coincides, up to the phase $\exp[i\theta]$ ($|\theta| \leq \pi$), with the wave function at the point $X + L$:

$$
\Psi(X + L) = \exp[i\theta]\Psi(X).
$$

(3.3)

On the other hand, the fact [22] that any circular motion possesses a discrete momentum spectrum implies that any closed way can be topologically nontrivial (and it is easy to ascertain, applying definite methods of differential geometry, alike those stated in the monograph [31]: in Lectures 3, 4, 26).

But also in the analytical way, one can make sure in the nontrivial topological content of circular motions.

To do this, it is necessary to utilize the periodicity property of $\exp[i\theta]$ and to express explicitly the momentum $p$ of the considered circular motion in terms of the Planck constant $\hbar$ according to the Heisenberg uncertainty principle [10]:

$$
p = \hbar \frac{2\pi k + \theta}{L}, \quad k = \pm(0, 1, 2, \ldots).
$$

(3.4)

---

10 See the original paper [30] by Josephson. The investigation of B. D. Josephson about tunnelling effects in superconductors were honoured, in 1973y., by Nobel Prize in the sphere of physics.
Generally, a quantum object (a "quantum train" in the terminology [10]) moving along a closed way cannot stop at \( \theta \neq 0 \), as it follows from (3.4).

The state with the minimum energy at \( k = 0 \) (treated as a "vacuum") corresponds to the persistent motion of the considered "train" involving the momentum

\[ p = \hbar \theta / L. \tag{3.5} \]

This momentum \( p \) of the persistent motion along the closed way disappears in the classical limit \( \hbar \rightarrow 0 \).

This was, in effect, the general mathematical description [10] of the Josephson effect taking place in physical models involving circular motions.

Note that (gauge) physical theories may distinguish by trajectories of circular motions, and this may tell upon their specific.

So, for instance, in QED\(_{(1+1)} \) [32] one deals with circular motions along the circle \( S^1 \simeq U(1) \) of the infinite radius due to identifying points of the configuration space \( \{ A_1(x,t) \} \) at the spatial infinity.

More exactly [32], the "points"

\[ A^{(n)}(x,t) = \exp(i\Lambda^{(n)}(x))(A_1(x,t) + i\frac{\partial_1}{e}) \exp(-i\Lambda^{(n)}(x)), \quad n \in \mathbb{Z}, \tag{3.6} \]

in the QED\(_{(1+1)} \) configuration space \( \{ A_1(x,t) \} \) are physically identical.

Herewith under the physical identity (physical equivalence) of the points \( A^{(n)}(x,t) \), with \( n \) running about the set \( \mathbb{Z} \) of integers, in the configuration space \( \{ A_1(x,t) \} \), the identical probabilities distributions for gauge fields belonging to different topological domains in the model [32] as well as to the one fixed topological domain there may be understood.

11Here \( \Lambda^{(n)}(x) \) are the \( U(1) \) generators, entering explicitly Weyl base elements \( P_1^{(n)} \in U(1) \):

\[ P_1^{(n)}(x) = \exp(i\frac{\Lambda^{(n)}(x)}{\hbar}). \]

As it was explained in [32], Weyl base elements \( P_1^{(n)} \) would satisfy the spatial asymptotic

\[ \lim_{|x| \rightarrow \pm \infty} P_1^{(n)}(x) = 1, \]

that is equivalent to

\[ \Lambda^{(n)}(\infty) - \Lambda^{(n)}(-\infty) = 2\pi n \hbar, \]

in order to deal with transverse electric fields \( \partial_x \tilde{E}(x,t) = 0 \) in the model [32].

To satisfy the above spatial asymptotic, it is expediently to assume [10]

\[ \Lambda^{(n \pm)}(x) = \hbar 2\pi n \pm \frac{x}{R} \]

(with \( R \) standing for the spatial infinity) at arbitrary values of the spatial coordinate \( x \).

Here, following [10], it is convenient to pick out the subsets of positive, \( n_+ \), and negative, \( n_- \), numbers among integers \( n \in \mathbb{Z} \).
This affects immediately the QED\(_{(1+1)}\) wave function \(\Psi\):

\[
\Psi(A^{(n+1)}) = e^{i\theta} \Psi(A^{(n)}), \quad |\theta| \leq \pi.
\]  

(3.7)

Eq. (3.7) is the particular case of general Eq. (3.3) \([10]\) in the actual limit \(L \to \infty\) for the infinite large circle \(S^1\).

In superconductors the Josephson effect becomes possible only at their contact and when a closed electric circuit including these two superconductors is built \([12]\).

In a liquid helium specimen at rest such persistent motion arises at the spontaneous breakdown of the initial \(U(1)\) gauge symmetry of the Bogolubov helium Hamiltonian \([34]\) together with the superfluidity phenomenon \([35]\).

The specific of a liquid helium specimen at rest model \([34, 35]\) is such that superfluid potential motions (proceeding with velocities do not exceeding a crucial \(v_0\) one \([2, 35]\)) coexist there with vortices.

A good analysis of vortices in a liquid helium specimen (at rest) was performed in the monograph \([36]\), in §§30, 31.

It turns out herewith that that nontrivial topologies \(n \neq 0\) in a liquid helium specimen (at rest) just correspond to vortices therein, while the trivial topology \(n = 0\) corresponds to the superfluidity phenomenon \([2, 34, 35]\).

More exactly, there is the simple relation \([36]\) associated the tangential velocity \(v^{(n)}\) of a rectilinear vortex in a liquid helium (at rest) to the given topological number \(n \in \mathbb{Z}\):

\[
n = \frac{m}{2\pi\hbar} \oint_{\Gamma} v^{(n)} dl,
\]

(3.8)

with \(m\) being the mass of the helium atom; \(dl\) being the element of the length along the axis \(z\) of this (rectilinear) vortex.

It follows from Eq. (3.8) that this cyclic integral vanishes as \(n = 0\).

On the other hand \([2, 35]\), it is mathematically equivalent to

\[
\text{rot } v^{(0)} = 0,
\]

with the velocity vector \(v^{(0)}\) chosen to be directed along the vector \(v_0\) of the superfluid motion in the liquid helium.

Thus the trivial topology \(n = 0\) corresponds really to the superfluid potential motion, without friction forces between the liquid helium specimen and walls of the (rested) vessel where it is contained.

Moreover, to the same result one comes by setting the (semi)classical limit \(\hbar \to 0\) in (3.8):

\[
m \oint_{\Gamma} v^{(n)} dl = 2\pi n 0 = 0.
\]

(3.9)

\(^{12}\text{We recommend our readers the monograph } [33], \S 3.6, \text{ for a detailed acquaintance with this question.}\)
On the other hand, according to the Landau two-component phenomenological model of liquid helium II [35] and its "quantum version" [34], created by N. N. Bogolubov with co-authors, superfluidity in a helium is possible only in the limit \( p \to 0 \) of small transferred momenta, i.e., when the momentum \( mv_0 \) of the superfluid motion in a liquid helium II specimen is comparable by its absolute value with momenta \( p_{ph} \to 0 \) of phonons, belonging to the excitations spectrum in this specimen.

In turn [36], the part of the energy spectrum in a liquid helium corresponding to phonons is set by Eq.

\[ \epsilon \sim c_s p_{ph}, \]

with \( c_s \) being the sound velocity.

In this case of small transferred momenta, the semi-classical limit \( \hbar \to 0 \) would correspond to large sizes (may be assumed to be infinite) of the vessel where the liquid helium specimen is contained in order for this specimen to possess the manifest superfluidity: it is true due to the Heisenberg uncertainty principle.

Thus one can assert that the Josephson effect [10] in a liquid helium II specimen comes to (rectilinear) vortices (3.8) interspersing the topologically trivial superfluid component in the liquid helium and bearing traces of nontrivial topologies \( n \in \mathbb{Z} \).

Geometrical spaces of these (rectilinear) vortices in a liquid helium [36] are, indeed, infinite narrow cylinders.

The important specific of the Josephson effect in a liquid helium II specimen is that the \( \theta \)-angle can and would attain its zero in the liquid helium II at rest theory [36] simultaneously with \( n = 0 \).

It is associated again with the superfluidity claims [34, 35] would be imposed undoubtfully onto the liquid helium II at rest theory [36].

In this case rotary effects disappear in the zero topological sector of the model [36] together with circular integrals of rotary velocities \( v^{(n)} (n \in \mathbb{Z}) \); this just corresponds to setting \( \theta = 0 \) for this sector [10].

Considering the examples of QED\((1+1)\) [32] and the liquid helium II at rest theory [2, 34, 35, 36] with their specifics of the Josephson effect, now we can proceed to studying the features of this effect in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [1].

First of all, it is obvious that Eq. (1.6) for the topological momentum \( P_N \) of the Minkowskian BPS monopole vacuum (suffered the Dirac fundamental quantization) is also the particular case of general Eq. (3.4) [10].

Herewith one would set (conditionally) \( L = 1 \) in (3.4) in order to get (1.6).

Further, Eq. (3.3) [10] for the wave function \( \Psi \) is transformed in the mentioned Minkowskian Higgs model into Eq. [8]

\[ \Psi_N(N + 1) = e^{\theta} \Psi_N(N). \]  

\[ \text{It may be argued that the coexistence of superfluid motions and (rectilinear) vortices in a liquid helium II specimen testifies in favour of the first-order phase transition occurring there.} \]
This Eq. is analogous to Eq. (3.7) in (Minkowskian) QED\((1+1)\)\footnote{\textsuperscript{32}}.

Finally, as it was already discussed in Introduction, the Josephson effect in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac may be reduced \footnote{\textsuperscript{8}} to the presence of vacuum "electric" monopoles \(E_i^a\), (1.10).

Herewith in the zero topological sector of this Minkowskian Higgs model any vacuum "electric" monopole \(E_i^a\) achieves its nonzero (until \(\theta \neq 0\)) minimum (1.11) \footnote{\textsuperscript{14}}.

Indeed, Eq. (1.11) is the particular case of Eq. (3.5) \footnote{\textsuperscript{10}} (at setting \(L = 1\)).

It is associated with the natural interpretation \footnote{\textsuperscript{37}} of various "electric" fields (if nonzero) in all kinds of non-Abelian theories as canonical momenta conjugate to temporal components \(A_0\) of gauge fields.

In particular, a vacuum "electric" field \(E_i^a\), (1.11), plays the role of the canonical momentum conjugate to a vacuum potential \(Z^a\), (1.3).

Eq. (1.10) allows to assert \footnote{\textsuperscript{8}} that "persistent field motions" around the "cylinder" of the diameter \(\sim \epsilon\), with its symmetry axis coinciding with the axis \(z\) of the chosen (rest) reference frame, take place, recognized as the Josephson effect \footnote{\textsuperscript{10}} occurring in the Minkowskian physical non-Abelian BPS monopole vacuum suffered the Dirac fundamental quantization \footnote{\textsuperscript{1}}.

These "persistent field motions" around the infinitely narrow "cylinder" of the diameter \(\sim \epsilon\) come to solid rotations inside the Minkowskian physical non-Abelian BPS monopole vacuum.

And moreover, Eq. (3.10) (as the particular case of general Eq. (3.3) \footnote{\textsuperscript{10}}) for the wave function \(\Psi_N\) of the Minkowskian physical non-Abelian BPS monopole vacuum suffered the Dirac fundamental quantization describes correctly the Josephson effect in that vacuum with the above discussed "cylinder" topology (in the terminology \footnote{\textsuperscript{10}}).

Thus \footnote{\textsuperscript{8, 10}} a field theoretical analogy of the Josephson effect is available in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac \footnote{\textsuperscript{1}}: a circular current without sources \footnote{\textsuperscript{15}}.

Note that in (Minkowskian) QED\((1+1)\)\footnote{\textsuperscript{32}}, there is an analogue of vacuum "electric" monopoles (1.11) inherent in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac.

It is the electric field \footnote{\textsuperscript{32, 38, 39}},

\[
E \equiv G_{10} = \hat{N}[A](t) \frac{2\pi}{\epsilon} = e\left(\frac{\theta}{2\pi} + k\right).
\]

In the zero topological sector of QED\((1+1)\)\footnote{\textsuperscript{32}}, i.e. when \(k = 0\) \footnote{\textsuperscript{10}},

\[
E_{\text{min}} = \frac{e\theta}{2\pi}.
\]
Herewith the topological dynamical variable $N[A](t)$ inherent in QED\textsubscript{(1+1)} [32] possesses the properties similar to these $N(t)$ possesses in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [1].

For instance, $N[A](t)$ may be represented as [10]

$$N[A](t) = (n_+ - n_-) \frac{t}{T} + (n_+ + n_-) \frac{1}{2},$$

(3.13)

by analogy with (1.9) (with that distinction that in QED\textsubscript{(1+1)} it was assumed [10] $n_- < 0$ and $n_+ > 0$, as we have mentioned above).

Then again the "boundary condition"

$$N[A](\pm T/2) \to n_{\pm}|_{x \to R/2}.$$  

(3.14)

(where $T \to \infty$ and $R \to \infty$ is set [10, 32]) is satisfied by the topological dynamical variable $N[A](t)$.

Just the topological dynamical variable $N[A](t)$ determine in QED\textsubscript{(1+1)} [32] circular motions around the circle $S^1 \simeq U(1)$ of the infinite radius, that is the essence of the Josephson effect [10] in this model.

These circular motions may be described with the aid of the action functional [10]

$$S(R, T, \nu) = \int_{-T/2}^{T/2} L(t)dt, \quad L(t) = \frac{1}{2V}(\frac{2\pi}{e})^2 \dot{N}[A]^2(t) \equiv \frac{1}{2}M \dot{N}[A]^2,$$

(3.15)

with $V$ being the spatial volume,

$$V \equiv \int_{-R/2}^{R/2} dx = R,$$

and

$$\dot{N}[A](t) = \nu/T$$

(3.16)

due to (3.14) (where again $\nu = n_+ - n_-$).

4 Conclusion

In the present study, we have discussed the nontrivial topological dynamics inherent in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [1].

This dynamics, coming to the specific Josephson effect [10] in the enumerated model, i.e. to the existence of collective solid rotations inside the physical BPS monopole vacuum.

These solid rotations are described correctly by the free rotator action functional [14] [8], implicating the topological dynamical variable $N(t)$ (given via Eq. (2.12) [7]), vacuum
electric monopoles $F_{i0}^a$ (given via Eq. (1.11) \[8\]) and the purely real energy momentum spectrum (1.6).

They also don’t vanish at nonzero values $\theta \neq 0$ of the $\theta$-angle, while ”geometrically”, there are, indeed, rotations around the infinitely narrow cylinder of the effective diameter $\sim \epsilon$ (with $\epsilon$ being the typical size of BPS monopoles) along the axis $z$ of the chosen (rest) reference frame.

These results about the nontrivial topological dynamics inherent in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac are got at the Gauss-shell reduction of that model in terms of topological Dirac variables $A_i^D$ $(i = 1, 2)$, (2.2), gauge invariant and transverse functionals of YM fields.

As a result of this constraint-shell reduction, the family $Z^a$, (1.3), exist satisfying the Gauss law constraint (1.2): the second-order homogeneous differential equation in partial derivatives.

This picture of the collective solid rotations inside the BPS monopole vacuum (suffered the Dirac fundamental quantization \[1\]) seems to be correct at least at the absolute zero temperature $T = 0$, when these rotations proceed in the ”non-stop” regime \[10\] and ”friction forces” between this BPS monopole vacuum and its surroundings are absent.

In Ref. \[9\], it was pointed out that grounding the above described nontrivial topological dynamics in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac lies in assuming the ”discrete” vacuum geometry (2.27) for the appropriate vacuum manifold (degeneration space) $\mathcal{R}_{YM}$.

This idea will be us developed in the further study.

In particular, series of interesting properties of the discrete vacuum manifold $\mathcal{R}_{YM}$, (2.27), will be revealed and discussed.

It will be argued (continuing the job begun in \[9\]) the presence of three kinds of topological defects inside this manifold: thread and point hedgehog topological defects and also walls between different topological domains of $\mathcal{R}_{YM}$.

We shall see that just thread topological defects inside the discrete vacuum manifold $\mathcal{R}_{YM}$, (2.27), are the cause of various rotary phenomena in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac \[1\].

It is, in fact, the same mechanism that causes (rectilinear) vortices \[36\] in a liquid helium II specimen.

Herewith (in Subsection 2.2 we have touched upon this topic) it is very important also that the typical widths of domain walls, which are determined by the natural mass scale $m/\sqrt{\lambda}$ in the discussed model, are inversely proportional to this scale \[40\], i.e. that these widths are comparable with the typical size $\epsilon(V) \sim V^{-1}$ of BPS monopoles inherent in the vacuum manifold $\mathcal{R}_{YM}$. As a result, in the asymptotical freedom limit $g \rightarrow 0$ domain walls inside $\mathcal{R}_{YM}$ acquire finite sizes in the spatial region near the origin of coordinates \[16\].

\[16\]At studying the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac, it should be distinguished the values \[8\] $\epsilon (0) \neq 0$ and $\epsilon (\infty) \rightarrow 0$ of $\epsilon$, very important for understanding this model.
Vice versa, in the spatial region $|\mathbf{x}| \to \infty (V \to \infty)$, domain walls become infinitely thin due to Eq. (2.9).

This promotes, in Minkowskian constraint-shell QCD, the infrared topological confinement of Gribov large multiplies $v^{(n)}(\mathbf{x})$, in the spirit \[12\] (see also \[9\]), in (quark and gluonic) Green functions in all the orders of the perturbation theory.

The existence of thread topological defects inside the vacuum manifold $R_{YM}$, (2.27), is associated with the presence therein of rectilinear threads around which collective solid rotations (as a specific kind of vortices) proceed inside the BPS monopole vacuum suffered the Dirac fundamental quantization \[11\].

In the study planned, it will be shown, repeating the arguments \[20\], that there are YM fields

$$A_\theta(\rho, \theta, z) = A_\mu \partial x^n / \partial \theta$$

These fields may be always represented as \[20\]

$$A_\theta(\rho, \theta, z) = \exp(iM\theta)A_\theta(\rho) \exp(-iM\theta),$$

with $M$ being the generator of the group $G_1$ of global rotations compensating changes in the vacuum (Higgs-YM) configuration $(\Phi^a, A^a_\mu)$ at rotations around the axis $z$ of the chosen (rest) reference frame.

The elements of $G_1$ may be set as \[20\]

$$g_{\theta} = \exp(iM\theta).$$

YM fields $A_\theta$ are manifestly invariant with respect to shifts along the axis $z$.

Note that rectilinear threads $A_\theta$ don’t coincide with vacuum YM BPS monopole solutions, and, on the contrary, there are, indeed, gaps between directions of vectors $\mathbf{B}_1$:

$$|\mathbf{B}_1| \sim \partial_\rho A_\theta(\rho, \theta, z),$$

and $\mathbf{B}$, given by the Bogomol’nyi equation (2.11).

These gaps testify in favour the first-order phase transition occurring in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac \[1\], where the ”magnetic” field $\mathbf{B}$, given via the Bogomol’nyi equation (2.11) \[2, 4, 5, 20\], may be treated as the order parameter in the enumerated model.

As it was demonstrated in the original papers \[41\], this ”magnetic” field $\mathbf{B}$ diverges as $r^{-2}$ at the origin of coordinates, and it is always a sign of a phase transition (either of the first or the second order) occurring in the Minkowskian Higgs model with vacuum BPS monopoles.

Unlike YM modes, suffered gaps near the axis $z$, Higgs vacuum BPS monopole modes may be continued in the (smooth) wise in this spatial region. In detail, the situation is following.

As it was demonstrated in Ref. \[29\] (see also \[3\]), the Higgs BPS ansatz

$$f_{01}^{BPS}(r) = \left[ \frac{1}{\tanh(r/\epsilon)} - \frac{\epsilon}{r} \right]$$

26
has the asymptotic
\[ f_{01}^{BPS}(0) = 0; \quad f_{01}^{BPS}(\infty) = 1. \]

On the other hand, as it was shown in the monograph [20], there exist z-invariant (vacuum) Higgs solutions in a (small) neighbourhood of the origin of coordinates:
\[ \Phi^{(n)}(\rho, \theta, z) = \exp(M\theta) \phi(\rho) \quad (n \in \mathbb{Z}), \]
can join vacuum Higgs BPS monopoles, belonging to the same topology \( n \) and disappearing [29] at the origin of coordinates, in a (smooth) wise.

Herewith, speaking ”in a smooth wise”, we imply that the covariant derivative \( D\Phi \) of any vacuum Higgs field \( \Phi^{(n)} \) merges with the covariant derivative of such a vacuum Higgs BPS monopole solution.

This requirement for vacuum Higgs fields \( \Phi^{(n)} \) to be smooth is quite natural if the goal is pursued, in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [1], to justify various rotary effects inherent in this model.

In particular, vacuum ”electric” monopoles (1.11) are directly proportional to \( D_i(\Phi^{(0)}_k) \Phi^{(0)} \).

These vacuum ”electric” monopoles, in turn, enter explicitly the action functional (1.4), describing, in the Dirac fundamental quantization scheme [1], collective solid rotations inside the Minkowskian BPS monopole vacuum.

Moreover, as it will be discussed in the next study, such (smooth) sawing together appropriate vacuum Higgs modes \( \Phi^{(n)} \) and BPS monopoles is intended to remove the following problem.

As it was discussed recently in the papers [2, 3], manifest superfluid properties of the Minkowskian BPS monopole vacuum (suffered the Dirac fundamental quantization [1]) are set by the Bogomolnyi, (2.11), and Gribov ambiguity,
\[ [D^2_i(\Phi^{(0)}_k)]^{ab}\Phi^{(0)}_{(a)b} = 0, \]
equations.

Mathematically, latter Eq. is the consequence of the Bogomolnyi equation (2.11) and the Bianchi identity \( D B = 0 \).

The both above Eqs. describe correctly [2, 3] the Minkowskian BPS monopole vacuum suffered the Dirac fundamental quantization as a superfluid potential (incompressible) liquid.

Thus one can assert [6] that
\[ D B \sim D E = 0 \]
for vacuum ”magnetic” and ”electric” tensions in the quested Minkowskian Higgs model, i.e. that these tensions are, indeed, ”transverse” vectors collinear each other.

Note that Eq. (1.11) [8] just reflects this collinearity.

This implies, on the face of it, a contradiction between collective solid rotations inside the Minkowskian BPS monopole vacuum suffered the Dirac fundamental quantization [1] and its manifest superfluid and potential nature [2, 3], excluding formally any rotations.
(due to the same reasoning as for the superfluid component in a liquid helium II specimen [34, 35]).

Going out from this contradiction seems to be just in locating (topologically nontrivial) threads in the infinitely narrow cylinder of the effective diameter \( \epsilon \) around the axis \( z \) and in joining (in a smooth wise) vacuum Higgs fields \( \Phi^{(n)}_a \) and BPS monopole solutions.

In this case collective solid rotations (vortices) inside the Minkowskian BPS monopole vacuum, occurring actually in that spatial region around the axis \( z \) and described correctly by the action functional (1.4), become quite ”legitimate”, and simultaneously, the Gauss law constraint (1.2) is satisfied outward this region with smooth vacuum ”electric” monopoles \( E^a_{\mu} \) [8], (1.11).

The said indicates the coexistence of two thermodynamic phases inside the Minkowskian BPS monopole vacuum suffered the Dirac fundamental quantization [1], i.e. the first-order phase transition occurring therein (additional to the second-order one associated with the spontaneous breakdown of the initial \( SU(2) \) gauge symmetry down to the \( U(1) \) one).

There are the thermodynamic phases of collective solid rotations and superfluid potential motions inside that physical vacuum.

Herewith the enough clear-cut picture can be observed how the enumerated thermodynamic phases are distributed inside the discrete vacuum manifold \( R_{YM} \), (2.27).

Thread topological defects (vortices), associated with rectilinear threads \( A_\theta \), are located intimately near the axis \( z \) of the chosen (rest) reference frame. Actually, they refer to the cylinder of the effective diameter \( \epsilon \) with \( z \) serving its symmetry axis.

Simultaneously, superfluid potential motions refer to the spatial region out of this cylinder, including the spatial region \( |x| \to \infty \) (corresponding to the infrared region of the momentum space).

The question about the boundary between these phases (and surface effects associated with this boundary) is, however, not simply one. This question (we leave for later studies) is complicated, for instance, by the discrete geometry of the vacuum manifold \( R_{YM} \).

The important consequence (will be us studied in one of works what follow) of the presence of rectilinear threads \( A_\theta \) in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac and involving the ”discrete” vacuum geometry (2.27) is the effect [20] of annihilating two equal magnetic charges \( m_1 = m_2 = m(n) \neq 0 (n \in \mathbb{Z}) \) colliding at crossing a rectilinear topologically nontrivial thread \( A_\theta(n) \).

Note that this effect takes place actually within a fixed topological domain inside the discrete vacuum manifold \( R_{YM} \), (2.27), possessing the topological number \( n \).

Colliding magnetic charges with different topological numbers can be suppressed in the spatial region near the axis \( z \) (of the chosen rest reference frame) by nonzero widths of domain walls in this spatial region.

The said means the possible annihilation of all the topologically nontrivial YM vacuum BPS monopole modes and excitations over this BPS monopole vacuum (suffered the Dirac fundamental quantization [1]) during a definite time.

As a consequence of such possible annihilation, Higgs (BPS monopole) modes should be free electric fields: their electric charges \( e \) which are dual (due to the Dirac quantization
of electric and magnetic charges) to zero magnetic charges can only survive upon the above described annihilation.

Such situation when Higgs modes possess arbitrary electric charges $e$, while magnetic charges $m \neq 0$ are confined is referred to as the Higgs phase in modern physical literature (see e.g. [44]).

If quarks are incorporated in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [1], disappearing topologically nontrivial YM modes via the "colliding" mechanism [20] can cause the possibility to observe free "coloured" quarks in the spatial region along the axis $z$ and wherein near the origin of coordinates (in light of the said above).

This can serve as a (perhaps, enough rough) representation for the asymptotical freedom of quarks in that model.

A few remarks are appropriate here. It becomes obvious from our discussion in Subsection 2.1 that the asymptotical freedom limit $g \to 0$ and the typical size of domain walls $\epsilon(0) \neq 0$ at $r \to 0$ are two values which should be in agreement in order to obtain a consistent theory.

A good step toward this direction is to suppose the location of discrete topological domains inside $R_{YM}$ in the spheroidal region with an effective radius $r_1 \ll 1$ fm (since the region near the $S^1$ boundary of the 1 fm diameter ball is the region of quarks/ gluons confinement inside of a hadron/meson). Also it is quite constructively (in line with the presence of rectilinear threads $A_{\theta}$ inside the discrete vacuum manifold $R_{YM}$, (2.27)) to presume that topological domains inside this vacuum manifold are located parallel the axis $z$ at short distances $O(r_1)$. They possess finite sizes and are separated by domain walls also of finite sizes.

Vice verse, in the spatial region $|x| \to \infty$, where walls between different topological

---

17Herewith it is understood customary (for instance, in Ref. [44]) that all magnetic charges are confined by (narrow) Meisner flux tubes, similar to ones in a superconductor [28].

In turn, this involves the linearly increasing "Mandelstam" potential $O(Kr)$ (with $K$ being the string tension) between YM monopole and antimonopole.

In the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac, that is the subject of our discussion, the role of the "Mandelstam" mechanism [28] annihilating magnetic charges via their confinement by Meisner flux tubes diminish (perhaps, only at finite temperatures $T \neq 0$, a contribution from the linearly increasing "Mandelstam" potential $O(Kr)$ in that model is possible, with the specific shape $O[(g^2T)^{-1}]$ [43]).

This temperature-depending "Mandelstam" potential can and would join a linear combination of the Coulomb $r^{-1}$ and "golden section" potentials [4, 5, 7, 8, 9] proper to the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac.

On the other hand, at $T \to 0$, the role of the above described mechanism [20] annihilating magnetic charges $m \neq 0$ via their colliding with topologically nontrivial threads $A_{\theta}$ increases in forming the Higgs phase inside the BPS monopole vacuum, supplanting actually the "Mandelstam" annihilation mechanism [28].

It is, obviously the merit of assuming the "discrete" geometry [227] for the vacuum manifold $R_{YM}$, calling to justify the Dirac fundamental quantization of the Minkowskian Higgs model with vacuum BPS monopoles.
domains inside $R_{YM}$ become highly thin, the infrared topological and physical confinement of quarks (in the spirit [7, 12]) takes place. Herewith the domains inside $R_{YM}$ cease to be parallel each other and, conversely, merge with each other.

The important gain we got in the present study is also that, at the fixed spatial volume $V$, the 'effective' mass $m/\sqrt{\lambda}$ is directly proportional to the coupling constant $g$, i.e., together with $g$, obeys the Callan – Symanzik equation [45].

A Appendix 1. Specific of asymptotical states in Euclidian instanton theory.

The specific of the Euclidian instanton YM theory [11] is associated, in the first place, with the Wick rotation $t = -x_4$ of the time axis.

In this case the finite action functional [7, 28]

$$S_{\text{Eucl}}(A) = \frac{8\pi^2}{g^2}\nu,$$  

(A.1)

where now the Pontryagin index $\nu$ is expressed through instantons $A$ in the standard way (2.13), (2.14) [7] with fixing the Weyl gauge $A_0 = 0$ [28] for temporal components of these instantons.

It is assumed (and Eqs. (2.13), (2.14) indicate this fact) the topological degeneration of the Euclidian instanton vacuum [11], consisting herewith of purely gauge stationary configurations [7, 28]

$$\hat{A}_i \Rightarrow L_i^n \equiv v^{(n)}(x)\partial_i v^{(n)}(x)^{-1} \quad \text{as} \quad |x| \to \infty; \quad v^{(n)}(x) \in SU(2).$$  

(A.2)

This conception of the topologically degenerated Euclidian instanton vacuum was discussed in Refs. [11] and [16] [10].

When the Euclidian time $x_4$ runs from $x_4 = -\infty$ to $x_4 = +\infty$, an instanton $A$ interpolates between the classical vacua (A.2) with the topological numbers $n_+$ and $n_-$ (as it was suggested for the first time in some papers by V. N. Gribov). Herewith this instanton $A$ possesses the topological number $\nu = n_+ - n_-.$

It is well known that the mentioned transitions between classical vacua with different topological numbers (they refer to $|x| \to \infty$, in the good agreement with our discussion in Subsection 2.2 about the crucial role in QFT of asymptotical states at $t \to \pm \infty$) proceed with the complete energy $\epsilon = 0$.

This corresponds (see e.g. the analysis of the Euclidian instanton YM model [11] in Refs. [28, 47]) to the asymptotical condition imposed onto the YM tension tensor $F_{\mu\nu}$. Such confinement may be provided, for instance, by surviving, upon colliding [20] at threads $A_\theta$, topologically trivial and, perhaps, topologically nontrivial, YM modes.

[19] Generally speaking, the topological degeneration is proper to vacua in gauge theories independently on the space: either the Euclidian or the Minkowski one, in which these gauge theories are considered.
\[ F_{\mu\nu}(x) \to 0, \quad |x| \to \infty, \]  
(A.3)

that is mathematically equivalent to the asymptotic

\[ |B| = |E| \to 0, \quad |x| \to \infty \]  
(A.4)

for the "magnetic" tension \( B \) and the "electric" tension \( E \).

Moreover, going over to the Euclidian space \( E_4 \) from the Minkowski one in the instanton theory \[ 11 \], implicating the time coordinate \( x_4 \equiv \tau = it \), involves also semi-classical paths between classical vacua \( (A.2) \) "in" and "out" (respectively, at \( x = \mp \infty \)).

In this case transition amplitudes between the mentioned "in" and "out" vacua turn out having the look \[ 28, 48 \]

\[ T \sim e^{-S_{\text{Eucl}}/\hbar}[1 + O(\hbar)]. \]  
(A.5)

On the other hand, going over to the Minkowski space in the instanton non-Abelian model is not desirable.

In that case definite problems arise.

For instance, instantons turn into complex fields in the Minkowski space, although they are real in the Euclidian space \( E_4 \).

It is necessary to recall herewith that in the Euclidian space \( E_4 \), instantons \[ 11 \] satisfy the duality conditions \[ 28 \]

\[ F_{\mu\nu} = \pm \tilde{F}_{\mu\nu}; \quad \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}; \]  
(A.6)

that is equivalent to \[ 6, 16 \]

\[ B = \pm E \]  
(A.7)

(where \( B \) is the absolute value of the "magnetic" field and \( E \) is the absolute value of the "electric" field) for the YM tension tensor \( F_{\mu\nu} \).

Indeed, when the duality conditions \( (A.6) \) are satisfied, the appropriate (Euclidian) action functional

\[ S_{\text{Eucl}} = \int d^4x \frac{1}{2g^2} \text{tr} (F_{\mu\nu} F^{\mu\nu}) \]  
(A.8)

achieves its minimum, as it was demonstrated in \[ 28 \].

The mathematical equivalence between Eqs. \( (A.8) \) and \( (A.1) \) for the Euclidian action functional may be checked easy repeating the arguments \[ 28 \], however we omit this checking here.

Herewith the alone Hodge duality operation *, specified in the \( (A.6) \) way, takes only two eigenvalues,

\[ *^2 = 1, \quad * = \pm 1, \]  
(A.9)

in the Euclidian space \( E_4 \).
But in the Minkowski space the Hodge duality operation $\ast$ is specified badly (as it may be shown, repeating the arguments \[31, 49\]). This just serves the cause why instantons turn into complex fields in the Minkowski space.

The topological degeneration of the Euclidian instanton (purely gauge) vacuum and the existence of semi-classical paths (tunnelling transitions) of the shape \[A.5\] \[28, 48\] between asymptotical ”in” and ”out” states with the total energy $\epsilon = 0$ allow to introduce the so-called $\theta$-vacuum as a specific superposition of classical vacua \[A.2\].

Following \[28\], such superposition may be written down as

$$\mid \theta > = \sum_{n} e^{-i\theta} \mid n >.$$ \hspace{1cm} (A.10)

Herewith the parameter $\theta$ is referred to as the $\theta$-angle (or as the Bloch pseudomomentum) in the modern physical literature.

It is relevant to consider now the wave function $\Psi_{0}[A]$ that is indeed the eigenfunction of the Hamiltonian \[6\]

$$H(E, A) = \int d^3 x \frac{1}{2}[(E^a_i)^2 + (B^a_i)^2],$$ \hspace{1cm} (A.11)

$$E^a_i = \partial_0 A^a_i, \quad B^a_i = \frac{1}{2} \epsilon_{ijk}(\partial^j A^a_k - \partial^k A^a_j + \frac{g}{2} \epsilon^{abc} A^j_b A^k_c),$$ \hspace{1cm} (A.12)

disappearing over the remote surface $|x| \to \infty$ because of the asymptotic \[A.4\] \[47\].

Thus on the quantum level, the Schrödinger equation \[6\]

$$\hat{H}(i\delta/\delta A, A)\Psi_{0}[A] = 0; \quad \hat{H} = \int d^3 x \frac{1}{2}[E^2 + B^2]; \quad E = \frac{\delta}{i\delta A};$$ \hspace{1cm} (A.13)

takes place.

The action of the $\theta$-vacuum \[A.10\] onto the wave function $\Psi_{0}[A]$ comes to \[28\]

$$T_1 \Psi_{0}[A] = e^{i\theta} \Psi_{0}.$$ \hspace{1cm} (A.14)

Herewith the operator $T_1$, called the raising operator \[28\], involves the winding number 1 as its eigenvalue:

$$T_1 \mid n > = \mid n + 1 >,$$ \hspace{1cm} (A.15)

or, in terms of the winding number functional $X[A]$, \[2.14\], as \[5\]

$$T_1 X[A] = X[A] + 1.$$ \hspace{1cm} (A.16)

In the series of physical literature, e.g. \[6, 16, 50\] (see also Ref. \[9\]), the analysis of the instanton Euclidian $\theta$-vacuum was performed and additional properties of this vacuum were discovered.

In particular, the properties of the instanton Euclidian $\theta$-vacuum at arbitrary values $\epsilon$ of the energy-momentum spectrum were investigated.
This proved to be very helpful, although it is enough to consider only the \( \epsilon = 0 \) value of this spectrum, corresponding to the boundary conditions \((A.3), (A.4)\) imposed onto instanton YM fields.

For an arbitrary \( \epsilon \), the (instanton) Euclidian \( \theta \)-vacuum may be specified, following Ref. \([51]\), as the ground state satisfied the equations \([6, 16, 50]\)

\[
\hat{H}(i\delta/\delta A, A)\Psi_\epsilon[A] = \epsilon \Psi_\epsilon[A], \tag{A.17}
\]

\[
\nabla_i^{ab}(A)(\frac{\delta}{i\delta A_{ib}}\Psi_\epsilon[A]) = 0; \quad \nabla_i^{ab}(A) = \delta^{ab}\partial_i - g\epsilon^{abc}A_{ic}; \tag{A.18}
\]

and

\[
T_1\Psi_\epsilon[A] = e^{i1^\theta}\Psi_\epsilon[A]. \tag{A.19}
\]

Herewith the Schrödinger equation \((A.17)\) generalizes the one \((A.13)\), while Eq. \((A.19)\) generalizes Eq. \((A.14)\).

Eq. \((A.18)\) reflects \([6]\) the invariance of the wave function \(\Psi_\epsilon[A]\) (the eigenfunction of the \(\theta\)-vacuum) with respect to "small" SU(2) gauge transformations belonging to the trivial topology \(n = 0\).

Eq. \((A.18)\) may be "translated" to the claim \([16]\) that the "electric" field \(E\) is transverse (formally):

\[
\nabla_i E^i\Psi_\epsilon[A] = 0. \tag{A.20}
\]

Indeed, the vector \(E\) disappears over an infinitely remote surface \(|x| \to \infty\) due to the boundary conditions \((A.3), (A.4)\) imposed onto instanton YM fields.

Unlike \((A.18)\), associated with "small" SU(2) gauge transformations, Eq. \((A.19)\) is associated with "large" SU(2) gauge transformations belonging to nontrivial topologies \(n \neq 0\).

Herewith the eigenfunction \(\Psi_\epsilon[A]\) of the \(\theta\)-vacuum proves to be manifestly covariant with respect to "large" SU(2) gauge transformations, as Eq. \((A.19)\) runs.

Discussing Eq. \((A.19)\), note that the raising operator \(T_1\) \([28], (A.15)\), can be represented explicitly in the shape \([6]\)

\[
T_1 = \exp\left(\frac{d}{dX[A]}\right) = \exp\left\{\left[\int d^3xB^2\frac{g^2}{16\pi^2}\right]^{-1}\int d^3xB_i^a\frac{\delta}{\delta A_i^a}\right\}. \tag{A.21}
\]

This look for \(T_1\) follows immediately from the interpretation \([28]\) of the \(\theta\)-angle as a pseudomomentum operator having the common system of eigenfunctions \(\{\Psi_\epsilon[A]\}\) with the momentum operator \(\nabla_i^{ab}(A)\).

Then the winding number functional \(X[A]\) may be interpreted as the (generalized) coordinate canonically conjugate to the (pseudo)momentum \(\theta\) \([6]\):

\[
\left[\frac{d}{dX[A]}, X[A]\right] = 1. \tag{A.22}
\]

In particular, at \(X[A] = 1\), one comes to Eq. \((A.19)\).
The case of zero eigenvalues $\epsilon = 0$ of Hamilton operators $\hat{H}$ is, indeed, an especial case in quantum mechanics.

In this case the existence of physical solutions is rather an exception to the rule than a rule in the Euclidian instanton model [11] where the raising operator $T_1$ and the Hamilton operator $\hat{H}$ do not commute indeed [6, 16]:

$$[\hat{H}, T_1] \neq 0, \quad [\hat{H}, [\hat{H}, T_1]] \neq 0$$

and so on [20].

As it was discussed in Refs. [6, 16, 50] (and also in the later papers [4, 5, 7, 9]), at $\epsilon = 0$, the functional of the plane wave type exists,

$$\Psi_0[A] = \exp\{i(2\pi k + \theta)X[A]\} \quad (k \in \mathbb{Z}), \quad (A.23)$$

satisfying simultaneously the conditions (A.18), (A.19) and the Schrödinger equation (A.13) for $\epsilon = 0$ and the “complex” momentum:

$$2\pi k + \theta = 2\pi k \pm i8\pi^2/g^2. \quad (A.24)$$

The expression

$$P_N \equiv 2\pi k + \theta \quad (A.25)$$

is called the momentum of the topological motion (e.g. in the terminology of the paper [5]) or simply the topological momentum [21].

Thus the topological momentum

$$P_N = \pm i8\pi^2/g^2 \quad (A.26)$$

(at $k = 0$) indeed proves to be purely imaginary.

Additionally, the wave function $\Psi_0[A], (A.23)$, satisfies the duality equation (A.7) (as it was noted in Refs. [6, 16]).

In the operator shape, the duality equation (A.7) has the look [6]

$$E \Psi_0[A] = \pm B \Psi_0[A]. \quad (A.27)$$

Moreover, the wave function $\Psi_0[A]$ is specified badly at the minus sign before $P_N$ in (A.23). Just this implies disappearing the Hamilton operator $\hat{H}, (A.17)$ (i.e. $\hat{H} = \epsilon = 0$), due to the asymptotic conditions (A.3).

The said implies that it is impossible to give the correct probability description of the instanton $\theta$-vacuum since the Hilbert space of (topologically degenerated) instanton states becomes non-separable in this case.

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20 Thus the raising operator $T_1$ and therefore the winding number functional $X[A]$, due to (A.21), are, indeed, cyclic variables [14].

21 Graphically (on the complex plane), the set of “complex” momenta $P_N, (A.24)$, is a discrete set (more exactly, it is swept by the discrete “real” part $2\pi k$ and continious “imaginary” part $i8\pi^2/g^2$.)
From the quantum-mechanical point of view, the existence of purely imaginary values (A.26) of the topological momentum \( P_N \) in the Euclidian instanton model [11] means the possibility of semi-classical paths between ”in” and ”out” vacua (A.2) implicating these purely imaginary values of the topological momentum \( P_N \).

Herewith, as it was discussed in Refs. [4, 5, 7], appropriate semi-classical transition amplitudes involve the quantum analogue of an instanton:

\[
\exp(iW[A_{\text{instanton}}] = \Psi_0[A = L_{\text{out}}] \times \Psi_0^*[A = L_{\text{in}}] = \exp(-\frac{8\pi^2}{g^2}[n_{\text{out}} - n_{\text{in}}]). \quad (A.28)
\]

It is easy to understand that these ”motions” of instantons between ”in” and ”out” vacua implicating purely imaginary values (A.26) of the topological momentum \( P_N \) cannot be referred to the class of physical motions (and rather to the class of “ghost” modes), and below we shall consider an important argument confirming this fact.

On the other hand, the conditions (A.18), (A.19) and the Schrödinger equation (A.13) for \( \epsilon = 0 \) can be satisfied also at real values of the \( \theta \)-angle, as it was discussed in Ref. [50].

Really, at acting by the operator \( \hat{H}(i\delta/\delta A, A) \) on the wave function \( \Psi_0[A] \), the factor \( 2\pi k + \theta \) is not important at achieving the result (A.13).

Thus real values of the \( \theta \)-angle as well as imaginary ones prove to be relevant for describing the \( \theta \)-vacuum in the spirit [51]. In particular, following [50], one can consider the real topological momentum

\[
P^R_N = \pm \frac{8\pi^2}{g^2} \quad (A.29)
\]

at \( k = 0 \) by analogy with (A.26).

The said allows to introduce the complex \( \theta \)-angle:

\[
\theta = \theta_1 + i\theta_2, \quad (A.30)
\]

which imaginary part \( \theta_2 \) just corresponds to the purely imaginary topological momenta \( P_N \) [5, 6, 7, 16], (A.26), just at \( k = 0 \). Its real part \( \theta_1 \) may be assumed to vary in the interval \([-\pi, \pi]\) [6].

Thus the complex \( \theta \)-angle (A.30) may be considered as a ”compromise variant” united real \( \theta \)-angles \( \theta_1 \) [28, 50] and purely imaginary \( \theta_2 \) ones [6], given via (A.24), in the Euclidian instanton YM model [11].

Nevertheless, imaginary values \( \theta_2 \) of the \( \theta \)-angle and appropriate imaginary values (A.26) (at \( k = 0 \)) of the topological momentum \( P_N \), referring to the unphysical spectrum, present always in the Euclidian instanton model [11].

In the paper [16] this statement was referred to as the so-called no-go theorem.

The results got can be also treated [6] as the presence of unphysical solutions to the Schrödinger equation (A.13) at the application of ordinary quantization methods to a topologically nontrivial theory.

At studying (A.29) it becomes obvious that such real topological momentum \( P^R_N \) vanishes (i.e. the instanton YM configuration stops) in the limit \( g \to \infty \) for the YM coupling
constant $g$. It is just the infrared QCD confinement limit as it is understood in modern physics. In the terminology [6] this case is referred to as **infrared catastrophe**.

Note that purely imaginary (and thus space-like and unphysical) values (A.24) have no relation to the infrared catastrophe.

There may be shown, repeating the arguments [28], that tunnelling transition amplitudes of the (A.5) type may be rewritten in terms of the $\theta$-vacuum [6, 16, 50, 51] as

\[ < \theta' | e^{-iHt} | \theta > _J = \delta(\theta' - \theta) I_J(\theta) \]  

(A.31)

with an unknown function $I_J(\theta)$.

Because of (A.15), (A.10), we can write down:

\[ < \theta' | e^{-iHt} | \theta > _J = \sum_{m,n} e^{im\theta'} e^{-im\theta} < m | e^{-iHt} | n > J \]

\[ = \sum_{m,n} e^{-i(n-m)\theta} e^{im(\theta' - \theta)} \times \]

\[ \times \int [dA]_{n-m} \exp\{-i \int (L + JA) d^4x\}, \]  

(A.32)

with $J$ being the appropriate non-Abelian current.

One can cast (A.32) in the standard form by substituting $n - m \to \nu$; then upon doing the $m$-summation one gets

\[ I_J(\theta) = \sum_{\nu} e^{-i\nu\theta} \int [dA]_{\nu} \exp\{-i \int (L + JA) d^4x\} \]

\[ = \sum_{\nu} \int [dA]_{\nu} \exp\{-i \int (L_{\text{eff}} + JA) d^4x\}, \]  

(A.33)

where, utilising the expression (2.13) for the Chern-Simon function $a_l$, we introduce the effective YM Lagrangian:

\[ L_{\text{eff}} = L + \frac{g^2\theta}{16\pi^2} \text{tr} (F_{\mu\nu} \tilde{F}^{\mu\nu}). \]  

(A.34)

The well-known $\theta$-term appears in the latter expression.

It determines the contribution of the natural $SU(2)$ topology

\[ \pi_3(SU(2)) = \pi_3S^3 = \mathbb{Z} \]  

(A.35)

and instanton $\theta$-vacuum to the Euclidian YM Lagrangian formalism.

It may be treated as an effective YM selfinteraction with the coupling constant $g^2\theta$.

The principal shortcoming of Eq. (A.34) is its actual dependence on the topological momentum $P_N$: (A.25), (A.26). This violates the P and CP symmetries in the instanton YM theory [11], the so-called instanton CP-problem arises.

The manifest CP-covariance of the effective Lagrangian (A.34) is accompanied by the bad ultraviolet behaviour of the Euclidian instanton theory [11] (such behaviour always testifies about a "bad physics").
Moreover (and this fact was pointed out, for instance, in Ref. [50]), the effective instanton Lagrangian (A.34) is not a Lorentz invariant since the $\theta$-term entering this Lagrangian contains, indeed, the $BE$ product (satisfying the duality condition (A.27)) that is manifestly Lorentz covariant.

This implies automatically the Poincare covariance of the Lagrangian (A.34) due to the natural embedding of the (general) Lorentz group in the Poincare one.

Herewith the Lorentz covariance [50] of the effective instanton Lagrangian (A.34) supplements its CP-covariance [28].

Simultaneously, the $\theta$-term in (A.34) is manifestly gauge invariant as that can be represented in the shape [20]

$$\sim \theta \int d^4x < F_{\mu\nu}, F_{\alpha\beta} > \epsilon^{\mu\nu\alpha\beta}.$$  

Meanwhile, the any product $< F_{\mu\nu}, F_{\alpha\beta} >$ is manifestly gauge invariant [28].

The definite analogy between the $\theta$-term in the effective instanton Lagrangian (A.34), involving imaginary values (A.26) of the topological momentum $P_N$ (that corresponds to the imaginary part $\theta_2$ of the $\theta$-angle), and the action functional (1.4), describing correctly collective solid rotations inside the Minkowskian BPS monopole vacuum suffered the Dirac fundamental quantization [1] can be observed.

Really, at identifying the ends of discussed semi-classical paths 22, one gets closed (three-dimensional) trajectories of infinite "radiuses".

In this case semi-classical paths between "in" and "out" instanton vacua (A.2) come to circular motions in $S^4$ some of which proceed with imaginary (topological) momenta (A.26).

Just these ("real" and "imaginary") closed trajectories are described correctly by the $\theta$-term in the effective instanton Lagrangian (A.34).

On the other hand, as it follows from (A.34) [28], at $\epsilon = 0$ (that corresponds to instanton field configurations with $E = B = 0$ at the spatial infinity [28, 47]), only the $\theta$-term is actual in (A.34).

On the face of it, the $\theta$-term in (A.34) would give additional nonzero contributions in the energy integral of the instanton model [11] (in comparison with $\epsilon = 0$ coming indeed from $\mathcal{L}$).

But two essential objections may be brought at once against the latter assertion.

Firstly, some of abovementioned circular three-dimensional motions (that are, geometrically, the spheres $S^3$ of infinite diameters) are accompanied by negative kinetic energies [50] $\propto P_N^2$ (with $P_N$ now being purely imaginary).

Secondly, the circular three-dimensional motions with positive kinetic energies (corresponding to real values of $P_N$ and the real part $\theta_1$ of the $\theta$-angle) are, indeed, fictive since

22As it was discussed in Ref. [31], such identifying may be performed by going over to the compact Euclidian space $\mathbb{R}^4 \cup \{\infty\}$ from the $E_4$ one.

In this case the asymptotic (A.3), (A.4) [28, 47] for the YM tension tensor $F_{\mu\nu}^a$ is mathematically equivalent to continuing this tensor with its zero value in the point $\infty$.

On the other hand, $\mathbb{R}^4 \cup \{\infty\} \simeq S^4$ (via the stereographical projection has been performed [31]).
the $\theta$-term in the instanton YM effective Lagrangian \[ A.34 \] does not alter the YM equations of motions \[ A.35 \]
\[ D_\mu F^{\mu \nu} = 0 \]
(this fact was pointed out, for example, in Ref. \[ 20 \]).

The important conclusion may be drawn from our discussion.

It is out of the question the physical Josephson effect \[ 10 \] in the Euclidian instanton YM model \[ 11 \] (unlike that taking place in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac \[ 1 \], or QED \[ W(1 + 1) \] or in the liquid helium II at best theory Leite-Halatnikov).

In three mentioned models the Josephson effect \[ 10 \] taking place is just a physical effect.

For instance, in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac, the appropriate action functional \[ 1.4 \] \[ 4, 5, 7, 8 \] and the purely real energy-momentum spectrum $P_N$, \[ 1.6 \], reading immediately from the action functional \[ 1.4 \], determine the physical nature of the Josephson effect taking place.

In this case \[ 10 \], the Hamilton operator

\[ \hat{H}_1 = \frac{P_N^2}{2I} \]

(A.36)
corresponds to the free rotator action functional \[ 1.4 \], describing collective solid rotations inside the Minkowskian Higgs BPS monopole vacuum suffered the Dirac fundamental quantization \[ 1 \].

Then, instead of the Schrödinger equation \[ A.17 \] inherent in the Euclidian instanton YM model \[ 11 \], the Schrödinger equation

\[ \hat{H}_1 \Psi(N) = \epsilon_\theta \Psi(N), \]

(A.37)

with $\epsilon_\theta$ being the kinetic (rotary) energy, don’t vanishing until $\theta \neq 0$, corresponding to the free rotator action functional \[ 1.4 \] represented in the shape \[ 7, 10, 16 \]

\[ W_{\text{rot}} = \int dt \frac{P_N^2(t)}{2I} \]

(A.38)
takes place in the mentioned Minkowskian Higgs model.

And moreover, as it is easy to see, herewith the topological momentum $P_N$, \[ 1.6 \], and the topological dynamical variable $N(t)$, given via Eq. \[ 2.12 \], prove to be canonically conjugated values \[ 10 \]:

\[ i[P_N, N(t)] = \hbar. \]

(A.39)

Further, unlike the Euclidian instanton YM model \[ 11 \], where the appropriate Hamilton operator $\hat{H}_1$, \[ A.13 \], does not commute with the raising operator $T_1$ given via \[ A.21 \] (this

\[ 23 \] These equations of motions follow immediately from the action functional $S_{\text{Eucl}}$ \[ A.8 \], of the instanton YM model \[ 11 \].
is equivalent to nonzero commutators also between $\hat{H}$ and the topological momentum $P_N$, now the Hamilton operator $\hat{H}_1$, (A.36), contains explicitly the topological momentum $P_N$, (1.6). Thus the values $P_N$ and $\hat{H}_1$ commute:

$$[P_N, \hat{H}_1] = 0.$$  (A.40)

In this case it is necessary to replace the raising operator $T_1$: (A.19), (A.21), proper to the Euclidian instanton YM model [11], with the one [16]

$$T_{G1} = \exp(iP_N) = \exp\left(\frac{d}{dN(t)}\right)$$  (A.41)

in order that $[T_{G1}, \hat{H}_1] = 0$.

The commutation relation (A.40) between $P_N$ and $\hat{H}_1$ proper to the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [1] may be treated also as the (sufficient and necessary) condition that the energy-momentum spectrum (1.6) is real in that model (this follows from the well-known fact [22] that Hamilton operators are Hermitian and involve real eigenvalues).

The fact that the Hamilton operator $H_1$, (A.36), is squared by the topological momentum $P_N$ is also remarkable.

This indicates the manifest relativistic (Poincare) invariance of this Hamilton operator (unlike the relativistic covariant $\theta$-term in the effective Lagrangian $L_{\text{eff}}$ [28], (A.34), inherent in the Euclidian instanton YM model [11]).

This seems to be the way to avoid the CP-problem in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [1].

Note also that introducing the complex $\theta$-angle, (A.30), in the Euclidian instanton YM model [11] implicates singularities in semi-classical transitions amplitudes (A.32) between asymptotical "in" and "out" states.

Now we shall attempt to demonstrate this fact. The arguments stated in Ref. [52] will help us to do this.

The account of the $\theta$-term in the effective instanton Lagrangian (A.34) allows to represent any transitions amplitude (A.32) between vacuum "in" and "out" states in the shape [11, 52]

$$\Gamma \propto \exp(-8\pi^2/g^2) \cos \theta$$  (A.42)

For complex $\theta$-s, as it is well known [53],

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$  (A.43)

\[24\]The result below was got in Ref. [52] for the minimal nontrivial topologies $n = \pm 1$ (or $N = |n| = 1$), while instantons with $N \geq 2$ give subdominant contributions in the Euclidian instanton model [11].

Moreover, in Ref. [52], the $\theta$-term was assumed to be $-iN\theta$. Herewith the presence of the $i$ multipliers does not change the equations of motion in the model [11] and maintains the gauge invariance of the $\theta$-term.

In this case $\cos \theta$ multipliers appear in transitions amplitudes between vacuum "in" and "out" states, as it follows from general Eq. (A.30) [28] for Euclidian semi-classical transitions amplitudes.
As it was argued in [53], this function of $\theta_1 + i\theta_2$ acquires values can be any amount large if $\theta_1 = \text{Re} \theta = \pm \pi$, i.e., indeed, it would be assumed that $\theta_1 \in ]-\pi, \pi[$, instead of $\theta_1 \in [-\pi, \pi]$ as it was done in Ref. [6].

The pointed singularities of the ”complex” $\cos \theta$ influence also the energy contribution $\epsilon \neq 0$ from the $\theta$-term in the effective instanton Lagrangian $L_{\text{eff}}$ [28] (A.34).

Indeed, this contribution [10]

$$\epsilon_\theta \sim V \cos \theta_1; \quad V = 4\pi R^3/3$$

(with $\theta = \theta_1 + i\theta_2$ given via (A.30)), even without that infinite in the limit $V = R^3 \to \infty$, becomes additionally singular at $\theta_1 = \pm \pi$ [53].

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