Non-thermal fixed points and solitons in a one-dimensional Bose gas

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New Journal of Physics 14 (2012) 075005 (21pp)
Received 20 March 2012
Published 3 July 2012
Online at http://www.njp.org/
doi:10.1088/1367-2630/14/7/075005

Abstract. Single-particle momentum spectra for a dynamically evolving one-dimensional Bose gas are analysed in the semi-classical wave limit. Representing one of the simplest correlation functions, these provide information on a possible universal scaling behaviour. Motivated by the previously discovered connection between (quasi-) topological field configurations, strong wave turbulence and non-thermal fixed points of quantum field dynamics, soliton formation is studied with respect to the appearance of transient power-law spectra. A random-soliton model is developed for describing the spectra analytically, and the analogies and differences between the emerging power laws and those found in a field theory approach to strong wave turbulence are discussed. The results open a new perspective on solitary wave dynamics from the point of view of critical phenomena far from thermal equilibrium and the possibility of studying this dynamics by experiment without the need for detecting solitons in situ.

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1. Introduction

Many-body systems far away from thermal equilibrium can show a much wider range of characteristics than equilibrium systems. However, they buy this abundance for the price of transiency. Among the wealth of possible non-equilibrium many-body configurations, the most interesting candidates for theoretical and experimental study are those at which generic time evolutions get stuck for an extraordinarily long time. This is particularly possible at critical points where universal properties such as power-law behaviour of correlation functions appear, leading to slowing-down phenomena as the infrared (IR) modes become dominant. Out of equilibrium fluid turbulence is among the phenomena that were first described as a critical phenomenon [1–4].

In the realm of quantum physics, far-from-equilibrium many-body dynamics, from the formation of Bose–Einstein condensates in ultracold gases to quark–gluon plasmas produced in heavy-ion collisions and reheating after early-universe inflation, exhibits many interesting phenomena. In this context, increasing attention [5–18] is being given to wave turbulence phenomena [19–21]. Critical points far from equilibrium, the so-called non-thermal fixed points, were proposed [6] and related to strong wave turbulence [7–10] and the formation of quasi-topological defects [15–17]. Such defects play an important role in superfluid turbulence, also referred to as quantum turbulence (QT), which has been the subject of extensive studies in the context of helium [22, 23] and dilute Bose gases [24–27]. In contrast to eddies in classical fluids, vorticity in a superfluid is quantized [28, 29] and the creation and annihilation processes of quantized vortices are distinctly different [22, 23].

Considering a single vortex or vortex line, geometry imposes a power-law shape of the angle-averaged flow velocity away from the vortex core. This spatial power-law dependence implies the power-law momentum spectrum predicted to occur at the non-thermal fixed point [15, 17]. In this paper, we focus on solitary waves in one-dimensional (1D) quantum gases and show that these exhibit scaling behaviour in analogy to the universal properties of vortex ensembles in 2D and 3D systems. Self-similarity here is due to the absence of a scale in a soliton which has zero width and is randomly positioned in space. See [30, 31] for related studies in relativistic field theory.
Solitons are non-dispersive wave solutions which can arise in many nonlinear systems spanning a wide range from the Earth’s atmosphere [32] or water surface waves [33] to optics [34]. The characteristics of a single or a few solitons in ultracold Bose gases have been studied during the last decade regarding their movement and interaction in traps [35–39], their formation and creation [40–44] and their decay [45–47]; for a review, see [48]. Here, we propose detecting and characterizing soliton excitations in ultracold gases by measuring the single-particle momentum spectra for such systems. We emphasize that these can give a strong indication of the presence of solitons even in the cases where they cannot be observed in situ.

Superfluid turbulence plays an important role in the context of the kinetics of condensation and the development of long-range order in a dilute Bose gas. This, as well as turbulence in its acoustic excitations, has been discussed in [49–53] and more recently in [15, 17, 18, 54]. The dimensionality of the system under consideration plays an important role for the particular signature of turbulence. Hence, an interesting aspect is how do properties change as one approaches the crossover from, e.g., 1D to elongated 3D systems [55, 56]. A possible observation of QT in ultracold atomic gases at present poses an exciting challenge for experiments [57–59]. We stress that the experimental study of superfluid turbulence and, more generally, of non-thermal fixed points in ultracold Bose gases is central for understanding the processes important during the build-up of coherence and degeneracy, in particular in 1D systems [60–62].

In the following, we study the formation of soliton excitations in trapped 1D Bose gases by means of simulations in the classical-wave limit of the underlying quantum field theory. We analyse the momentum spectra of the time-evolving system and discuss them with respect to non-thermal fixed points discussed in field theory. In section 2, we develop a model of independent, randomly positioned grey solitons, locally being solutions to the GPE describing a degenerate, 1D dilute Bose gas. Single-particle momentum spectra are derived for such systems, both in a homogeneous system and within a trapping potential. A protocol for the formation of such soliton ensembles within a trapped gas is described in section 3, and the resulting states are analysed with respect to the predicted momentum spectra and power-law signatures of non-thermal fixed points. Our conclusions are presented in section 4.

2. Momentum spectra of soliton ensembles

In the classical-wave limit, a dilute ultracold Bose gas is well described by a positive definite Wigner phase-space distribution function $W[\phi, \phi^\ast]$ for the complex field $\phi$ at each point in position or momentum space [63, 64]. The dynamics of the gas is determined by the time evolution of this Wigner function according to the classical field equation:

$$i \partial_t \phi(x, t) = \left( -\frac{\partial_x^2}{2m} + V(x, t) + g_{1D} |\phi(x, t)|^2 \right) \phi(x, t),$$

which is identical in form to the Gross–Pitaevskii equation (GPE) for the field expectation value. Here, $m$ is the mass of the bosons, $V$ an external trapping potential and $g_{1D}$ the coupling constant in one spatial dimension (1D).

The 1D GPE (1) possesses quasi-topological soliton solutions which may travel with a fixed velocity but are non-dispersive, i.e. stationary in shape [48]. For positive coupling constant, $g_{1D} > 0$ solitons are characterized by an exponentially localized density depression with respect to the surrounding bulk matter and a corresponding shift in the phase angle $\varphi$ of the complex
Figure 1. Snapshots of a single run of the nonlinear classical field equation, showing solitons that oscillate inside a trapped 1D ultracold Bose gas. The gas is initially non-interacting and thermalized, with $T = 360 \omega_{ho}$, in a trap with oscillator length $l_{ho} = 8.5$ (in grid units). At time $t = 0$ the interaction is switched to $g_{1D} = 7.3 \times 10^{-3}$, and a cooling period using a high-energy knife is applied; see section 3 for details of grid units, the chosen parameters and protocol. The panels show the 1D colour-encoded density distribution as a function of time. Top left panel: the initially imposed interaction quench causes strong breathing-like oscillations and the creation of many solitons. Top right panel: breathing oscillations have damped out, leaving a dipolar oscillation of the bulk distribution in the trap. Clearly distinct solitons have formed. Bottom left panel: at the end of the cooling period (here at $t = t_c = 9.1 \times 10^3$) only a few solitons are left oscillating within the oscillating bulk. Bottom right panel: a single soliton is left at late times.

field $\phi = |\phi| \exp[i\varphi]$. Depending on the depth of this depression, the soliton is called either grey or, for maximum depression, black. In the background of a homogeneous bulk density $n$ it is described by

$$
\phi_{v}(x, t) = \sqrt{n} \left[ \gamma^{-1} \tanh \left( \frac{x - x_s(t)}{\sqrt{2} \gamma \xi} \right) + iv \right],
$$

where $x_s(t) = x_0 + vt$ is the position of the soliton at time $t$. Here, $\xi = \left[ 2m n g_{1D} \right]^{-1/2}$ is the healing length, and we express time in units of the inverse speed of sound $c_s = \left[ n g_{1D} / m \right]^{1/2}$, i.e. $t = \tilde{t} / c_s$, and drop the overbars. $\gamma = 1 / \sqrt{1 - v^2}$ is the ‘Lorentz factor’ corresponding to the velocity $v$ of the grey soliton in units of the speed of sound, $v = v / c_s = |\phi_{v}(vt, t)| / \sqrt{n}$. Being related to the density minimum, $\nu$ is also called the ‘greyness’ of the soliton, ranging between 0 (black soliton, $|\phi_{v}(vt, t)| = v \sqrt{n} = 0$) and 1 (no soliton, $|\phi_{v}(vt, t)| = \sqrt{n}$).

In section 3, we will study the formation and evolution of soliton excitations in trapped 1D Bose gases by means of the GPE (1). An example of one run is shown in figure 1: a sudden initial quench of the coupling $g_{1D}$ causes strong oscillations of the bulk gas in the trap and lets solitons form out of the short-wave-length collective oscillations. These solitons are seen as black lines surviving within the oscillating gas for a long time.

In this paper, we are mainly interested in the characterization of the ensemble of solitons emerging in our simulations in terms of single-particle spectra in momentum space. Hence, see

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The single-particle momentum spectrum is determined by evaluating ensemble averages
\[ n(k, t) = \langle |\phi(k, t)|^2 \rangle_{\text{ensemble}} \] over a large number of runs. Figure 2 shows \( n(k, t) \) at three different times during the period where the initial breathing oscillations are still present. Note the double-logarithmic scale. At low momenta, the spectrum shows a plateau, whereas at high momenta it falls off exponentially. At the point of time corresponding to the red curve, the cloud is further expanded such that solitons are more separated from each other, which causes an intermediate power-law dependence \( n(k) \sim k^{-2} \) to appear, as indicated by the straight line. The plateau, the power law and the exponential decay form the characteristic signature for solitons that we discuss in more detail in the following.

2.1. The random-soliton model: a uniform gas

To obtain an analytical understanding of the possible spectra in the context of non-thermal fixed points, we discuss, in the following, the case of a dilute ensemble of well-separated solitons with random velocities and positions. The wave function of a single grey soliton in a homogeneous bulk background condensate is given in equation (2). Using this we write down an expression for
a set of $N_i$ uncorrelated solitons with density minima at $\{x_i, i = 1, \ldots, N_i\}$ and (dimensionless) velocities $v_i$, in a background of constant bulk density $n$:

$$
\phi^{(N_i)}(x, t) = \sqrt{n} \prod_{i=1}^{N_i} \left[ n^{-1/2} \phi_{v_i}(x - x_i) \right].
$$

(4)

Note that due to the neglect of correlations, this field does not, in general, represent a solution to the GPE in which the solitons remain non-dispersive.

We make use of the assumption that the ensemble is dilute, i.e. that the distance between each pair of neighbouring solitons is much larger than the healing length. This assumption, for grey solitons, is not valid as soon as two oppositely moving solitons encounter each other, but for simplicity we will assume that these collisional configurations can be neglected in view of a majority of well-separated solitons. Since, for any $i$, $\phi'_{v_i}(x - x_i) \equiv d\phi_{v_i}(x - x_i)/dx \simeq 0$ as $|x - x_i| \gg 1$, we can rewrite the spatial derivative of the field (4) as

$$
\phi^{(N_i)'}(x, t) = \sum_{i=1}^{N_i} \phi'_{v_i}(x - x_i, t) \prod_{j \neq i} \left[ n^{-1/2} \phi_{v_j}(x - x_j, t) \right]
\simeq \sum_{i=1}^{N_i} \left[ \delta(x - x_i(t)) \prod_{j \neq i} e^{i \beta_j \theta(x - x_j(t))} \right] \star \phi'_{v_i}(x, 0).
$$

(5)

Here $x_i(t) = x_i(0) - v_i t$, $\beta_j = 2 \arccos v_j$, $\star$ denotes the convolution over the spatial dependence on $x$, $\theta(x)$ the Heaviside function, and we have neglected an irrelevant overall phase. Note that the sign of $\beta_i$ indicates the direction of propagation of the $i$th soliton. The term in square brackets in the second line of equation (5) is proportional to the spatial derivative of the field describing an ensemble of $N_i$ infinitely thin solitons ($\xi \to 0$), at the positions $\{x_i\}$,

$$
\phi^{(N_i)'}(x, t) \simeq \sum_i \frac{i \gamma_i}{2n_s \sqrt{n}} \phi^{(N_i)'}_{\xi \to 0}(x_i, t) \delta(x - x_i) \star \phi'_{v_i}(x, 0),
$$

(6)

where $n_s$ is the number of solitons per unit length and the prefactor containing $\gamma_i$ takes into account that the phase jump by $\exp[i \beta_i \theta(x - x_i)]$ is itself proportional to a theta function $\gamma_i \theta(x - x_i)$. Note that although the derivative $\phi^{(N_i)'}_{\xi \to 0}(x_i, t)$ gives a sum of terms, each being proportional to a delta distribution, only one of these remains when evaluated at $x_i$, which gives the term in square brackets in equation (5). We take the Fourier transform of $\langle \phi^{(N_i)'}(x)^* \phi^{(N_i)'}(y) \rangle$ with respect to $x - y$, integrate over $R = (x + y)/2$, and divide by $k^2$,

$$
n(k, t) = \frac{1}{4n_s^2} \sum_{i, j=1}^{N_i} \gamma_i \gamma_j e^{ik(x_i - x_j)} \phi^{*}_{v_i}(k) \phi_{v_j}(k) \partial_x \partial_y e^{-n_s|x_i - x_j|} \left[ 1 - P(\beta) \exp[i \beta \text{sgn}(x_i - x_j)] \right].
$$

(7)

Here, $P$ is the probability of finding a soliton with greyness $\nu = \cos(\beta/2)$, and averaging over the random positions of all solitons other than those at $x_i$ and $x_j$ has been done in order to obtain the exponential decay of the coherence function. Combining equation (7) with the Fourier transform of $\phi_v(x, 0)$,

$$
\phi_v(k, t) = \frac{1}{L} \sqrt{\frac{2\pi n}{L}} \left[ 2\pi \nu \delta(k) + \frac{\sqrt{2\pi \xi}}{\sin \left( \pi \gamma k \xi / \sqrt{2} \right)} \right],
$$

(8)
one derives the single-particle momentum distribution for a set of \( N_s \) solitons defined by greyness and position, \{\( v_i, x_i \mid i = 1, \ldots, N_s \)\}, as

\[
n(k, t) \simeq \frac{n}{4n_s^2} \int \frac{dk'}{2\pi} \sum_{i,j=1}^{N_s} \gamma_i \gamma_j e^{i(k-k')(x_i-x_j)} \times 2\pi v_i v_j \delta(k) + \frac{2\delta(0)\pi^2 \xi^2}{\sinh(\pi \gamma_j k \xi / \sqrt{2}) \sinh(\pi \gamma_j k \xi / \sqrt{2})} \times 4k^2 n_s \Re \alpha \frac{4n_s n \Re \alpha}{4n_s^2 (\Re \alpha)^2 + (k' - 2n_s \Im \alpha)^2} e^{-ik'(v_i - v_j)t}. \tag{9}
\]

Here, the inverse volume \( \delta(0) = L^{-1} \) appears as we first choose, in equation (7), the arguments of \( \phi_{v_i}^* \) and \( \phi_{v_j} \) different and take the identity limit only at the end. \( \alpha \) is the average over all \( \alpha_i = (1 - \exp(i\beta_i))/2 \).

Assuming the dependence of \( \gamma / \sinh(\pi \gamma_j k \xi / \sqrt{2}) \) on \( v_i \) to be negligible, we obtain an approximate stationary distribution

\[
n(k) \simeq \frac{4n_s n \Re \alpha}{4n_s^2 (\Re \alpha)^2 + (k - 2n_s \Im \alpha)^2} \left( \frac{\pi \gamma k \xi}{\sqrt{2}} \right)^2 \sinh^2 \left( \frac{\pi \gamma k \xi}{\sqrt{2}} \right), \tag{10}
\]

with a yet to be determined parameter \( \gamma \). For black solitons (\( v_i \equiv 0 \)), one obtains the exact expression

\[
n(k)|_{v=0} = \frac{4n_s n}{4n_s^2 + k^2} \left( \frac{\pi k \xi}{\sqrt{2}} \right)^2 \frac{2}{\sinh^2 \left( \frac{\pi k \xi}{\sqrt{2}} \right)}. \tag{11}
\]

For an ensemble of grey solitons of identical \( |v_i| \equiv v \), travelling with probabilities \( P \) into the positive \( x \)-direction and \( Q = 1 - P \) into the negative direction, one finds that

\[
n(k) = \frac{4n_s n}{4n_s^2 \gamma^{-4} + [k + 2(1 - 2Q)n_s v \gamma^{-1}]^2} \frac{2}{\sinh^2 \left( \pi \gamma k \xi / \sqrt{2} \right)} \left( \frac{\pi k \xi}{\sqrt{2}} \right)^2. \tag{12}
\]

Here, \( \gamma = (1 - v^2)^{-1/2} \). To demonstrate the applicability of the above analytic expressions, we construct ensembles of phase-space distributions of spatially well-separated solitons in a box with periodic boundary conditions and compute the ensemble average (3). These simulations are done on a 1D grid of \( N = 2^{14} \) sites, generating \( 5 \times 10^3 \) configurations for taking ensemble averages. For this, we multiply \( N_s = 20 \) single-soliton solutions (2) with positions \( x_i \) and greyness \( v_i \) chosen randomly according to a given phase-space distribution. To make sure that their relative distance on average is much larger than their widths, we chose the phase-space distribution to allow for a maximum greyness, \(|v_i| < |v_{\max}| < 1\), such that the diluteness criterion requires an approximate minimum box length of \( L = 4(1 - v_{\max}^2)^{-1/2} N_s \).

Figure 3(a) shows the single-particle momentum spectrum \( n(k) \) on a double-logarithmic scale for an ensemble of \( 5 \times 10^3 \) configurations with \( N_s = 20 \) solitons each distributed according to a flat distribution across the phase-space defined by the positions in the box and the maximum greyness \( |v_{\max}| = 0.99 \). Solid (black) squares represent the results of the numerical ensemble average, while the solid line corresponds to the analytical formula (10), with fitted parameters \( \alpha = 0.7, \gamma = 1.05 \). Compare this to the analytical average \( \alpha = 2/3 \). For comparison, we give
the results for the same number of purely black solitons (red squares and line) as well as for a fixed greyness $|v| = 0.707$ (blue squares and line), choosing an equal number of right- and left-movers. The comparison validates the approximate expressions (10)–(12) which exhibit scaling behaviour in the regime of momenta $k_n \gamma^{-2} \ll k \ll k_\xi \gamma^{-1}$.

In figure 3(b), we show the single-particle momentum spectrum for an ensemble of $5 \times 10^3$ configurations with $N_s = 20$ solitons each distributed according to a phase-space distribution with an unequal weight for solitons with positive $v$ (right-movers) and negative $v$ (left-movers). We specifically restricted the greyness to the interval $0 = v_{\text{min}} \leq v \leq v_{\text{max}} = 0.99$, and in addition to that, we chose the same parameters as those for figure 3(a). Black squares and the line show the ensemble average and compare with equation (12) for $Q = 0.2$, $\gamma = 1.179$ and $\nu = 0.412$, shown up to $k_\xi \approx 1$ where a finite-size effect sets in due to the random set of solitons not matching periodic boundary conditions. For comparison, the results for $N_s = 20$ solitons with fixed greyness $\nu = 0.707$, i.e. $\gamma = 1.41$, (blue circles and dash-dotted line) are shown.

Figure 3. (a) Single-particle momentum spectrum as a function of $k$ on a double-logarithmic scale for an ensemble of $5 \times 10^3$ configurations with $N_s = 20$ solitons each distributed according to a flat distribution across the phase space defined by the positions in the box and the maximum greyness $|v_{\text{max}}| = 0.99$. To avoid the set of solitons not matching the periodic boundary conditions, solitons are randomly chosen within one half of the phase space, with matching partners in the other half. We chose $\xi = 8a_s$. Solid (black) squares: numerical ensemble averages; solid line: equation (10) with $\alpha = 0.7$, $\gamma = 1.05$. For comparison, results for $N_s = 20$ purely black solitons (red triangles and dashed line) are shown as well as for $N_s = 20$ solitons with fixed greyness $|v| = 0.707$, i.e. $\gamma = 1.4$, (blue circles and the dash-dotted line) choosing an equal number of right- and left-movers, $P = Q = 1/2$. The comparison validates the approximate expressions (10)–(12) which exhibit scaling behaviour in the regime of momenta $k_n \gamma^{-2} \ll k \ll k_\xi \gamma^{-1}$. (b) The same as in (a), for an unequal weight of solitons with positive $v$ (right-movers) and negative $v$ (left-movers). The greyness is uniformly distributed within $0 = \nu_{\text{min}} \leq \nu \leq \nu_{\text{max}} = 0.99$. All other parameters are the same as in (a). Black squares and the solid line show the ensemble average and compare with equation (12) for $Q = 0.2$, $\gamma = 1.179$, and $\nu = 0.412$, shown up to $k_\xi \approx 1$ where a finite-size effect sets in due to the random set of solitons not matching periodic boundary conditions. For comparison, the results for $N_s = 20$ solitons with fixed greyness $\nu = 0.707$, i.e. $\gamma = 1.41$, (blue circles and dash-dotted line) are shown.

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2.2. Relation to non-thermal fixed points and vortical excitations in superfluids

Universal power-law behaviour in a many-body system far from equilibrium points to the appearance of turbulence phenomena. In the following, we summarize a few basic ideas of wave-turbulence theory for quantum gases and discuss the power-law spectra derived for soliton ensembles in this context.

A dilute, degenerate Bose gas is compressible such that collective sound-wave excitations can occur. This allows for the so-called weak wave turbulence which can occur in a regime where kinetic theory applies: the wave-kinetic equation for a Bose gas, which describes the evolution of the system under the collisional interactions between different wave modes, is known to have non-trivial stationary solutions. These solutions are non-thermal and exhibit a power-law dependence of mode occupations on momentum \([19–21]\). As in fluid turbulence, such solutions imply that the energy flows from, e.g., small to large momentum scales. In between these momentum scales of the source and the sink, the energy passes through the so-called inertial interval, where the distribution over momenta is stationary and follows a power law with a universal exponent predicted by weak-wave-turbulence theory \([19–21]\). The stationary solution is metastable, i.e. requires a constant and equal flow in and out of the inertial interval. We note that weak wave turbulence does not bear, in general, vortical excitations. It is defined by scaling and stationary transport between different scales and hence can also occur in one spatial dimension where vortices are absent.

Applying the above to a degenerate Bose gas one faces, however, the problem that the description in terms of wave-kinetic equations breaks down in the IR regime of long wavelengths where amplitudes, i.e. talking about a Bose gas, single-particle occupation numbers grow large and the description in terms of, e.g., elastic two-to-two collisions becomes unreliable \([19]\). As a consequence of this, the so-called strong wave turbulence is expected to occur in the IR regime. Recent developments presented in \([6–10, 13]\) allow one to set up a unifying description of scaling, both in the ultraviolet (UV) regime, where the wave-kinetic and quantum Boltzmann equations apply, and in the IR limit. Recall that at a critical point the IR modes dominate the system’s behaviour. In this IR regime, new scaling laws were found by analysing non-perturbative Kadanoff–Baym dynamic equations, as in the UV, with respect to non-thermal stationary power-law solutions \([6–8, 10]\). These solutions were termed non-thermal fixed points. Analogous predictions for dilute Bose gases were given in \([9]\), proposing what was termed as strong matter-wave turbulence in the regime of long-range excitations.

In \([15, 17]\), it was then shown that these non-thermal fixed points can also be understood, in two and three dimensions, in terms of vortex excitations of the superfluid: in the IR limit of large wave numbers the incompressible, superfluid component of the gas dominates, and the predicted IR power laws appear due to the algebraic radial decay of the flow velocity around the vortex cores. As a consequence, within a window in momentum space, which is limited by the inverse mean distance between different vortices and the inverse core size, the single-particle occupation number spectrum shows the power law predicted in \([9]\).

Before we discuss this in more detail let us first turn back to the soliton spectra derived in the previous section. Assuming an equal number of solitons travelling with positive and
negative velocities, \( P = Q = 1/2 \), i.e. assuming \( \text{Im} \alpha = 0 \), the single-particle spectrum (12) is characterized by a maximum of two scales. Consider the case \( n_s \ll \gamma / (\sqrt{2} \pi \xi) \). For momenta greater than the reduced soliton density but smaller than the reduced inverse healing length, \( k_n \gamma^{-2} \ll k \ll k_{\xi} \gamma^{-1} \), with \( k_n = 2n_s \) and \( k_{\xi} = \sqrt{2} / (\pi \xi) \), the momentum distribution exhibits a power-law behaviour, \( n(k) \sim k^{-2} \). This reflects, firstly, the random position of the kink-like phase jump across the centre of the soliton, and secondly, that these momenta cannot resolve the spatial width of the kink. In other words, looking within a spatial window of size between \( 1/k_{\xi} \) and \( \gamma k_{n_s}^{-1} \), the appearance of a single sharp solitonic phase jump inside the window is observed in a random manner. Hence, for any of these window sizes, the system looks identical; it appears self-similar. This self-similarity is at the base of the scaling momentum distribution. Already for a single soliton, the momentum distribution does not know anything about the position of the kink (this information appears as a phase in the momentum-space Bose field which is irrelevant for \( n(k) \)), and thus the single-soliton distribution is self-similar, too.

For black solitons, the self-similar scaling region is limited by the scales \( k_n \) and \( k_{\xi} \). Below \( k_n \), the distribution is constant because wave numbers that are too low cannot resolve the kink structure. This corresponds to the first-order coherence function decaying exponentially in space, with the decay scale set by the soliton distance \( 1/n_s \),

\[
(\phi^*(x)\phi(y)) \sim \exp[-2n_s|x-y|].
\]

Above \( k_{\xi} \), the momentum spectrum resolves the finite width of the soliton density dip which results in an exponential suppression of the mode occupations. We recall that also \( k_{\xi} \sim k_{n_s} \) is the local tangential unit vector and \( \phi \) the phase angle of the complex Bose field. The \( r \)-dependence implies a \( k^{-1} \) scaling of \( |\phi(k)| \) and thus a \( k^{-2} \) scaling of the kinetic energy \( E(k) \sim k^2 n(k) \sim v(k)^2 \), i.e. \( n(k) \sim k^{-4} \) [17, 65]. Similar arguments led to \( n(k) \sim k^{-5} \) for the radial momentum distribution in the presence of a vortex line in three dimensions [17]. Extending these arguments, the scaling \( n(k) \sim k^{-d-2} \) was shown to appear for ensembles of randomly positioned vortices/vortex lines in a range of momenta \( l^{-1}_v \lesssim k \lesssim \xi^{-1} \) between the inverse of the inter-vortex distance \( l_v \) and the inverse of the healing length \( \xi \), which is a measure for the core width.

As pointed out above, the power-law spectrum \( n(k) \sim k^{-d-2} \), in turn, had been predicted by the use of non-perturbative field-theory methods in [6, 9] where it resulted for a strong-wave-turbulence cascade in the IR, characterizing the scaling behaviour at a non-thermal fixed
point. This cascade was shown in [17] to be caused by particles being transported towards the IR, where they build up high mode occupations and thus coherence in the sample; see also [49–54]. Note that in the picture of the evolving Bose field this momentum-space transport corresponds to the mutual annihilation of vortices and antivortices in the system, which results in an increase of the intervortex distance and thus of the range over which phase coherence is established.

Having recalled all this, we note that there is a discrepancy between the predicted scaling \( \sim k^{-d-2} \), which was found to be consistent with the vortex picture for \( d = 2 \) and \( 3 \), and the scaling \( \sim k^{-2} \) obtained here for the solitons in \( d = 1 \). To gain more insight into this issue we consider the spatial decay of the phase coherence for a system in two dimensions over distances considerably larger than the intervortex spacing. In analogy to the soliton ensembles discussed in section 2.3, we consider randomly positioned and well-separated vortices. One finds that the decay of the phase coherence follows the same exponential law,

\[
\langle \phi^\dagger(x)\phi(y) \rangle \sim \exp[-2n_{v,1}|x - y|],
\]

where \( n_{v,1} \) is the 1D uniform vortex density along the straight line through \( x \) and \( y \). Fourier transforming (14) with respect to \( x - y \) results in a momentum spectrum \( n_{v}(k) \sim (4n_{v,1}^2 + k^2)^{-3/2} \) which scales as \( k^{-3} \) for momenta considerably larger than the inverse vortex distance \( n_{v,1} = 1/l_v \). Analogously, one finds \( n_c(k) \sim k^{-4} \) for vortex line tangles in three dimensions. We use the subscript ‘c’ to distinguish these spectra from the \( n(k) \) discussed above.

The apparent contradiction between the respective scalings of \( n(k) \) and \( n_c(k) \) whose exponents differ by 1 is resolved by observing the following: while the exponential decay (14) is valid over large distances \( |x - y| \gg l_v \), it is the algebraic dependence of the flow field \( \mathbf{v} \) as a function of distance from the vortex core which matters on length scales below the inter-vortex distance, causing the steeper power law \( n(k) \sim k^{-d-2} \) at momenta \( k \gg 1/l_v \). Hence, we find an important qualitative difference between the physical properties underlying the universal scaling \( n(k) \sim k^{-2} \) in one dimension and the scalings \( n(k) \sim k^{-d-2} \) in \( d = 2 \) and \( 3 \) dimensions: while black solitons in one dimension are at rest and no particle flow can occur, in higher dimensions, transverse flow circling around the vortex cores gives rise to an additional contribution to the kinetic energy. As far as this transverse flow dominates over a possible additional longitudinal flow component for which \( \mathbf{v} \cdot \nabla \mathbf{v} \neq 0 \), see [17], this comparison also holds when allowing for grey solitons, which are moving opposite to the (longitudinal) particle flow across the soliton dip.

As can be seen from the above predictions for the IR scaling laws at a non-thermal fixed point, the dimensionality of the system under consideration plays an important role for the particular signature of superfluid turbulence. Hence, an interesting aspect is how do properties change as one approaches the crossover from, e.g., 1D to elongated 3D systems [55, 56] where the snake instability is known to become relevant. A detailed study of this crossover will be the subject of a future publication.

In summary, there is a principal difference between the scalings of the momentum spectra in one and higher dimensions, giving rise to a deviation of the scaling \( \sim k^{-2} \) in \( d = 1 \) from the field-theory prediction \( k^{-d-2} \) which, in turn, is valid in \( d = 2 \) and \( 3 \) dimensions. To recover this discrepancy within a field-theory approach to strong wave turbulence is beyond the scope of this paper.
2.3. The random-soliton model: a trapped gas

In our dynamical simulations, we will consider soliton formation in a harmonic trapping potential rather than in a homogeneous system; see section 3. We therefore need to take into account, in the random-soliton model, the inhomogeneous bulk distribution of the gas.

Assuming a sufficiently shallow harmonic potential, \( l_{\text{ho}}/\xi = (m \omega_{\text{ho}} \xi^2)^{-1/2} \approx n_s^{-1} \), we can describe the Bose field in local-density approximation with respect to a bulk density distribution given in Thomas–Fermi approximation, \( n_{\text{TF}}(x) \approx n_0 [1 - (x/R)^2] \), with \( R = \sqrt{2} \zeta / (\omega_{\text{ho}} \xi) = 2 g_{1D} \eta / \omega_{\text{ho}} \) being the Thomas–Fermi radius in units of \( \xi \). We take the maximum density \( n_0 \) large enough to ensure \( k_\xi \gg k_n \) for solitons not too close to the edge of the cloud. In such a bulk density distribution, single solitons oscillate harmonically between classical turning points where the solitons ‘touch ground’, i.e. momentarily turn black [46]. Their oscillation frequency is by a factor of \(~1/\sqrt{2}~\) smaller than the trap frequency, \( \omega_\xi \approx \omega_{\text{ho}} / \sqrt{2} \).

In leading order in \( \epsilon \sim \bar{x}_s / \bar{x}_s \), the field of a single soliton can locally be written in the simple form given in equation (2), with \( v \) and thus \( \gamma \) replaced by local quantities,

\[
v \rightarrow v(x_s) = v_{r,\text{max}} \left[ 1 - \left( \frac{x_s}{x_{s,\text{max}}} \right)^2 \right],
\]

\[
\gamma \rightarrow \gamma(x_s) = [1 - v(x_s)^2]^{-1/2},
\]

evaluated at the position \( x_s = x_s(t) \) of the soliton [46]. \( v_{s,\text{max}} \) is the maximum greyness the soliton acquired in the centre of the trap, and \( x_{s,\text{max}} = v_{s,\text{max}} R \) is the distance of the soliton’s turning point from the trap centre. Only solitons whose velocity does not exceed the Landau critical velocity, i.e. for which \( v_{s,\text{max}} \leq 1 \), can oscillate in the trap for more than a quarter of the period \( T_s = 2 \pi / \omega_s \). This limits the maximum greyness at a distance \( x \) from the trap centre to a range between 0 and \( v_{\text{max}}(x) = 1 - (x/R)^2 \).

At a given time \( t \), we assume a particular set \( \{x_i(t)\} \) of \( N_s \) well-separated solitons across the trapped gas. The single-particle momentum spectrum corresponding to an ensemble of such sets depends on the distribution of the solitons over the greyness for each position in the trap. This distribution is best visualized in phase space which is parameterized by the \( (x, v) \), or equivalently and in dimensionless form, by \( (x/R, v) \), with both \( x/R \) and \( v \) ranging between \(-1\) and \(1\). In this space, the trajectory of a single soliton is a circle with radius \( v_{s,\text{max}} \) which is traced out with constant angular velocity \( \omega_s \). Hence, a stationary distribution of the \( N_s \) solitons is given by a circularly symmetric distribution in phase space, i.e. a distribution over the different possible maximum greynesses \( v_{s,\text{max}} \) or turning points \( x_{s,\text{max}} \). The simplest assumption would be that of a uniform distribution of the solitons in phase space, which amounts to a uniform distribution over the different possible \( v \) at each distance \( x \) from the trap centre and an integrated soliton density distribution \( n_s(x) \propto 1 - (x/R)^2 \).

Following the above considerations, we can obtain approximate expressions for the momentum spectrum. For instance, for a uniform density \( P[\bar{x}, \bar{v}] \equiv n_{s,0} / \bar{R}_s \) of solitons within a radius \( \bar{R}_s = R_s / R \) in phase space, i.e. for all \( (\bar{x}, \bar{v}) \) with \( \sqrt{x^2 + v^2} \leq \bar{R}_s \leq 1 \), with \( n_{s,0} = N_s / (\bar{R}_s \pi) \), the first-order coherence function for thin solitons becomes

\[
\langle \phi^{(N_s)}(\bar{x}) \star \phi^{(N_s)}(\bar{y}) \rangle_{\xi \rightarrow 0} = \sqrt{n_{\text{TF}}(\bar{x}) n_{\text{TF}}(\bar{y})} \exp \left\{ -2 n_{s,0} \int_{\bar{y}} \bar{z} d\bar{z} \alpha(\bar{z}) \right\}
\]

(16)
with a local average dephasing of

\[ \alpha(\bar{x}) = \frac{1}{2\bar{R}_s} \int \frac{\sqrt{\bar{R}_s^2 - \bar{x}^2}}{\sqrt{\bar{R}_s^2 - \bar{x}^2}} \, dv \left( 1 - v^2 \pm i v \sqrt{1 - v^2} \right) \]

\[ = \sqrt{1 - \bar{x}^2} \left( 1 - \frac{\bar{R}_s^2 - \bar{x}^2}{3} \right). \]  

(17)

Using this, the integral over \( \alpha \) to linear order in \( \bar{x} \) reads

\[ \int_{\bar{x} = 0}^{\bar{x}} d\bar{z} \, \alpha(\bar{z}) = \alpha^{(1)}(\bar{x}) + \mathcal{O}(\bar{x}^3), \]

\[ \alpha^{(1)} = 1 - \bar{R}_s^2 / 3. \]

This approximation is best for \( \bar{R}_s \rightarrow 1 \), in which limit we obtain

\[ \langle \phi^{(N)}(\bar{x}) \phi^{(N)}(\bar{y}) \rangle_{\xi} \rightarrow 0 \approx \sqrt{n_{TF}(\bar{x}) n_{TF}(\bar{y})} e^{-4N_s |\bar{x} - \bar{y}| / (3\pi \bar{R}_s)}. \]  

(18)

Analogously, one calculates the exponential dephasing factor for more complicated soliton phase-space distributions. From the above results taken together, one derives the momentum distribution in local-density approximation as the convolution of the spectrum for a homogeneous distribution of thin solitons with the Fourier transform of the bulk density, multiplied by the momentum spectrum of a single soliton:

\[ n(k) \approx \left( n_{TF}(k) \ast \frac{4n_{x,0}^2 \alpha^{(1)}}{4n_{x,0}^2 \alpha^{(1)} + k^2} \right) \frac{(\pi \gamma k_\xi)^2 / 2}{\sinh^2 \left( \pi \gamma k_\xi / \sqrt{2} \right)}, \]  

(19)

where the parameter \( \gamma \) is to be determined and \( \ast \) denotes the convolution with respect to \( k \).

In the case when single-soliton distributions contributing to the ensemble are not uniform throughout the phase space, the ensemble-averaged \( n(k) \) would rather be a sum of \( (k \rightarrow -k) \)-asymmetric distributions such that on average the momentum distribution can have local maxima at finite \( |k| > 0 \).

To study the quality of the above analytic expressions, we construct ensembles of phase-space distributions of spatially well-separated solitons inside a harmonic trap and compute the ensemble average (3). For this we multiply \( N_s \) single-soliton solutions (2) with positions \( x_i \) and greyness \( \nu_i \) chosen according to a given phase-space probability distribution and ensure that their relative distance on average is much larger than their widths.

Figure 4 shows the single-particle momentum spectrum \( n(k) \) on a double-logarithmic scale for an ensemble of \( 5 \times 10^3 \) configurations with \( N_s = 20 \) solitons each distributed according to a flat distribution in phase space \( \{x/R, \nu\} \), circularly symmetric around \( (x = 0, \nu = 0) \) with radius \( \bar{R}_s = 1 \). Solid (red) squares represent the results of the numerical ensemble average, whereas the solid line corresponds to the analytical formula (19), with \( \gamma = 1.1 \). For comparison, we give corresponding results for the same number of black solitons distributed randomly across the trap (blue squares and line).

3. Soliton spectra in dynamical simulations

3.1. Soliton formation and tracking in position space

In the following, we study the formation of soliton ensembles by the use of semiclassical simulations, with Gaussian noise for the initial field modes. Moments of the phase-space probability distribution at a later time are determined by sampling the initial distribution, propagating each realization according to the GPE, and averaging over many such
trajectories [63, 64]. At the initial time, we take the gas to be non-interacting and thermalized and impose an interaction quench. To allow the emerging collective excitations to form solitons at a desired density we furthermore apply evaporative cooling by opening the trapping potential at the edges in a controlled fashion. During the first, cooling period, \( t \leq t_c \), the potential is given by the inverted Gaussian \( V(x, t) = m_0 \omega_0^2 U(t) \left[ 1 - \exp \left\{ -x^2 / 2U(t) \right\} \right] \) with its maximum being ramped down by sweeping \( U(t) = U_0 + (U_c - U_0) t / t_c \) linearly in time from \( U_0 \) to \( U_c \). At the same time, highly energetic particles near the edge of the potential are removed by adding a loss term \( i \Gamma(x, t) / 2 = i \Gamma_\infty [V(x, t) / U(t)] \gamma / 2 \) to the trapping potential. Thereafter, during the interval \( t_c \leq t \leq t_{\max} \), the loss is switched off and the potential is ramped up again to harmonic shape across the extension of the gas, \( U(t) = U_c + (U_{\max} - U_c) (t - t_c) / (t_{\max} - t_c) \). We choose \( r = 10, U_0 = 2.75, \Gamma_\infty = 0.1 U_0, U_c = U_0 / 3 \) and \( U_{\max} = 10 U_0 \). The times \( t_c \) and \( t_{\max} \) vary and are given in the following. This protocol corresponds to the one used in [66]. Different cooling schemes have been used in experiments; see, e.g., [60, 62], but as we are primarily interested in the 1D dynamics, we here restrict ourselves to purely 1D calculations.

For the simulations, we map the system onto a grid of \( N = 1024 \) lattice sites with a lattice constant \( a_s \). Unless stated otherwise, quantities are given in grid units based on \( a_s \), and the parameters chosen are a dimensionless coupling constant \( g_{1D} = 2 m a_s g_{1D} = 7.3 \times 10^{-3} \) s. a cooling time \( t_c = t_c / (2 m a_s^2) = 9.1 \times 10^3 \), and a harmonic oscillator length \( \bar{l}_{\hbar} = l_{\hbar} / a = 8.5 \). Lattice momenta are \( \bar{k} = 2 \sin (\pi n / N), n \in \{-N/2, ..., N/2 - 1\} \). We drop overbars in the following.

Three stages of the induced dynamical evolution can be observed; see also figure 1:

**Initial oscillations.** See the top left panel of figure 1. Following the interaction quench, potential energy is transformed into kinetic energy. One observes strong breathing-like
Figure 5. Snapshots of a single run of the nonlinear classical field equation, showing solitons which oscillate inside the trap, thereby showing signs of mutual scattering and passing through each other (examples marked by letters). Parameters are chosen as in figure 1.

oscillations of the gas. These oscillations decay on a timescale of $t \approx 2 \times 10^3$, leaving a dipolar oscillation of the bulk in the harmonic trap. Solitons are formed in the wake of the decaying breathing oscillations.

**Solitonic regime.** See the top right panel of figure 1. The initial collective oscillations have largely decayed except for an overall dipole mode, and many solitons appear. The bottom left graph shows the evolution around $t = t_c = 9.1 \times 10^3$ when only a very few solitons have survived. The solitons oscillate in the trap, being nearly black at the edges and grey in the centre of the trap corresponding to a non-zero velocity. Figure 5 magnifies a short period of the evolution. On mutual encounters, the solitons get phase-shifted, such that collisions show signs of scattering or passing through each other. Collisions with different such shifts are marked by letters A and B in figure 5.

**Final stage:** At times $t \gg t_c$, a soliton is still visible; see the bottom right panel of figure 1. Comparing runs we find different numbers of solitons remaining during the late stage.

The smallest timescale is the oscillation period in the trap $T_{ho} \approx 230$, which leads to an initial collective oscillation with period $T_{oscillation} \approx 300$ (cf figure 1). The collective breathing motion dies out after $\tau \approx 2000$. The oscillation period of a soliton in the Thomas–Fermi bulk is $T_s \approx \sqrt{2} T_{ho} \approx 320$. The longest timescale in our setup is the cooling time $t_c = 9100$. Comparing these timescales to the total time of the simulation, at the end of which solitons are still present, we see that the solitons are quasi-stationary in the system. They emerge soon after the initial quench and remain throughout the whole evolution, while thermalization of the high-momentum modes is proceeding as we will see in the following.

3.2. The number of solitons after cooling ends

In order to study the statistics of the solitons emerging during the evolution, we have set up an efficient tracking algorithm that identifies the trajectories of the solitons oscillating in the gas. The algorithm scans the wave function for density minima coinciding with a phase jump around them. Figure 6(a) shows the evolution of the mean number of solitons, for an ensemble of 200 runs. The three stages described above can be identified. The strong initial oscillations give an oscillating number of solitons until $t \approx 10^3$. The number of solitons decreases while the gas is
Figure 6. (a) Time evolution of the mean number $N_s$ of solitons, for an ensemble of 100 runs. Strong initial oscillations occur during the breathing-like bulk oscillations after the interaction quench, until $t \simeq 10^3$. The number of solitons decreases while the gas is evaporatively cooled. After the end of the cooling, at $t = t_c$, the decay is considerably slowed down and the number of solitons remains largely stable. Three different cooling times and two ramp speeds are shown, with $t_c = 3.9 \times 10^3$ (red), $t_c = 4.55 \times 10^3$ (blue) and $t_c = 9.1 \times 10^3$ (black). (b) The number of solitons $N_s(t_c)$ at the end of the cooling period, $t = t_c$, as a function of $t_c$, for ensembles of 200 runs each (black dots and error bars). The function $f_0 \exp\{-\gamma t_c\}$, with $f_0 = 9.7$ and $\gamma = 9.77 \times 10^{-5}$, was fitted with $\chi^2 = 0.006$ (black line). The function $g_0 t_c^{-1}$, with $g_0 = 1.55 \times 10^5$, was fitted with $\chi^2 = 1.28$ (red dashed line). Hence, other than for the Kibble–Zurek scheme of [66], cooling after the initial quench results in an exponential dependence of $N_s(t_c)$ on $t_c$.

evaporatively cooled. After the end of the cooling, at $t = t_c$, the decay is considerably slowed down and the number of solitons remains largely stable. Three different cooling times and two ramp speeds are shown, with $t_c = 3.9 \times 10^3$ (red), $t_c = 4.55 \times 10^3$ (blue) and $t_c = 9.1 \times 10^3$ (black), where the same speed is chosen to obtain the blue and black data.

Kibble and Zurek have predicted that the number of defects created in the near-adiabatic crossing of a phase transition scales with the crossing rate according to a power law which depends on the universal properties of the transition [67, 68]. This was studied numerically in [66] using the cooling protocol described above. While the interacting gas was chosen to be in thermal equilibrium initially, with a temperature well above the critical point, we start our simulations, motivated by earlier work on vortex dynamics [15, 17], with an interaction quench driving the system strongly out of equilibrium. To compare the dynamics induced in this way with the results of [66] we show, in figure 6(b), the dependence of the number of solitons created on the cooling ramp time $t_c$. We find that, within the error bars which indicate the variance over 200 runs, the data are fitted by an exponential dependence $N_s(t_c) = f_0 \exp\{-\gamma t_c\}$ rather than by a power law $N_s(t_c) = g_0 / t_c$ as predicted in [40]. We emphasize, however, that in our system, solitons mainly form during the initial stage following the interaction quench.
3.3. Time evolution of single-particle spectra

We finally discuss the relaxation dynamics with respect to the evolution of the respective single-particle momentum spectra (3). The initial state chosen in the simulations is given by a thermal canonical ensemble of distributions over the single-particle eigenstates of the trap. In figure 7(a), we show the momentum spectrum at time $t = 5 \times 10^3$. Solitons have formed at high density such that the scales $k_{n_s}$ and $k_\xi$ are close to each other, as indicated in the graph. The solid line represents a fit of the analytical model spectrum (19), with $n_{s,0} = 0.076$, $\gamma = 1$, $\sigma^{-1} = 0.036$ where the Gaussian of width $\sigma$ was used to describe the bulk distribution in position space. A Rayleigh–Jeans tail is absent as the cooling is still on. Due to the proximity of $k_{n_s}$ and $k_\xi$, no $k^{-2}$ power law is seen in between the low-energy plateau and the high-energy exponential fall-off.

In section 2, figure 2, we showed the spectrum for a wider trap with $l_{ho} = 17$ which allows the solitons to be diluted more across the trap and results in $k_{n_s} \simeq 0.15$ and $k_\xi \simeq 0.65$. This allows for a $k^{-2}$ scaling to appear clearly, indicating a self-similar random distribution of the solitons. For $t > t_c$, the gas enters the final stage: the number of particles and the energy are now conserved and the healing length is ‘frozen out’. Figure 7(b) shows the development of $n(k, t)$ from $t \gtrsim t_c$ to late times. Once the cooling, and thus the removal of particles with high energy, is
Figure 8. Time evolution of the spectrum shown on a linear colour scale at late times $4.95 < 10^{-5} t < 5.0$, for a single run. The oscillatory pattern appears despite a large number of runs contributing to the statistical ensemble. A small number of runs with a few oscillating solitons dominate the averaged spectrum. Terminated, a transport process from low to high momenta starts and thermalization takes place. The influence of solitons is still present with two effects: firstly, there are still solitons in the gas for the times displayed in figure 7(b) which contribute with their spectral profile to the total spectrum and broaden the plateau at low momenta up to $k_x \simeq 0.1$. Secondly, the momentum distribution starts to oscillate between situations with a stronger weight on the positive and on the negative side. Figure 8 shows how the remaining solitons, which oscillate in the trap, influence the spectrum with their own momentum appearing as the oscillating maxima in the spectrum.

4. Conclusions

We have studied the formation of dark solitary waves in 1D Bose–Einstein condensates as well as their relaxation dynamics towards the equilibrium. The corresponding single-particle momentum spectra were predicted in the framework of an instantaneous model of well-separated grey solitons, whose width is considerably smaller than their mutual distances in the bulk. For comparison with these predictions, semiclassical simulations of the relaxation dynamics of 1D Bose gases after an initial interaction quench and a cooling period were used to determine the respective spectra numerically. The thereby found spectra compared well with the analytical predictions, giving insight into the many-body dynamics from the point of view of universal properties and critical physics far from equilibrium. We emphasize that the particular protocol used to produce the solitons is irrelevant in that the properties characterizing the fixed point do not depend on how it is reached.

We have discussed the power-law behaviour of the momentum spectra, which appears in a range of momenta between the inverse of the inter-soliton distance and the inverse healing length, with regard to the universal scaling laws predicted in non-perturbative field-theory approaches to strong wave turbulence. In the 1D case studied here, the derived power-law exponent $\zeta = 2$ differs by 1 from the exponent $\zeta = d + 2$ predicted for a system in $d$ spatial dimensions and previously recovered in the scaling due to vortex excitations in a $d = 2$ and 3 dimensional Bose gas. We can trace this discrepancy back to the different flow patterns possible in $d > 1$ versus $d = 1$ dimension: while in the 1D gas, particle flow cannot choose its orientation except for a sign, vortical excitations in two and three dimensions are characterized by flow circling around the vortex cores. This flow is transverse, i.e. it changes its strength...
perpendicular to its direction. It dominates, at sufficiently low energies and momenta, over any additional longitudinal flow caused by compressible sound-wave excitations. Moreover, for geometric reasons, the transverse-flow velocity field changes algebraically in space and causes the single-particle momentum spectra to scale as predicted by strong-wave-turbulence theory.

We point out that the single-particle momentum spectra discussed here could be used in experiment to study solitary-wave dynamics in 1D Bose gases without the necessity to detect solitons in situ. Studying universal properties during the relaxation dynamics from a nonequilibrium initial state or under a constant driving force in this way opens a new point of view on strong wave turbulence and non-thermal fixed points.

Acknowledgments

The authors thank J Berges, R Bücker, L Carr, M J Davis, M Karl, G Nikoghosyan, M K Oberthaler, J M Pawlowski, J Schmiedmayer and J Schole for useful discussions. They acknowledge support from the Deutsche Forschungsgemeinschaft (GA 677/7,8), from the University of Heidelberg (FRONTIER, Excellence Initiative, Center for Quantum Dynamics) and the Helmholtz Association (HA216/EMMI).

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