PAIRS OF DIAGONAL QUADRATIC FORMS AND LINEAR CORRELATIONS AMONG SUMS OF TWO SQUARES

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Abstract. For suitable pairs of diagonal quadratic forms in 8 variables we use the circle method to investigate the density of simultaneous integer solutions and relate this to the problem of estimating linear correlations among sums of two squares.

1. Introduction

Let \( Q_1, Q_2 \in \mathbb{Z}[x_1, \ldots, x_n] \) be quadratic forms, with \( Q_2 \) non-singular. Suppose, furthermore, that as a variety \( V \) in \( \mathbb{P}^{n-1} \), the intersection of quadrics \( Q_1 = Q_2 = 0 \) is also non-singular. In this paper we return to our recent investigation [3] into the arithmetic of the singular varieties \( X \subset \mathbb{P}^{n+1} \) defined by the pair of quadratic forms

\[
q_1(x_1, \ldots, x_{n+2}) = Q_1(x_1, \ldots, x_n) - x_{n+1}^2 - x_{n+2}^2,
\]
\[
q_2(x_1, \ldots, x_{n+2}) = Q_2(x_1, \ldots, x_n).
\]

Let \( r(M) \) be the function that counts the number of representations of an integer \( M \) as a sum of two squares and let \( W : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) be an infinitely differentiable bounded function of compact support. In [3, Theorem 1] we were able to prove the expected asymptotic formula for the associated counting function

\[
S(B) = \sum_{x \in \mathbb{Z}^n, 2|Q_1(x)} r(Q_1(x)) W \left( \frac{x}{B} \right), \quad (B \to \infty),
\]

under the assumption that \( n \geq 7 \). In particular this establishes the Hasse principle for \( X \) when \( n \geq 7 \), a fact previously attained in a much more general setting by Colliot-Thélène, Sansuc and Swinnerton-Dyer [4].

Our goal is to show that the sum \( S(B) \) can also be estimated asymptotically when \( n = 6 \), provided that \( Q_1 \) and \( Q_2 \) are taken to be diagonal. We will deal here only with forms of the shape

\[
Q_1(x) = \alpha(x_1^2 + x_2^2) + \alpha'(x_3^2 + x_4^2),
\]
\[
Q_2(x) = \beta(x_1^2 + x_2^2) + \beta'(x_3^2 + x_4^2) + \beta''(x_5^2 + x_6^2),
\]

where \( \alpha, \alpha', \beta, \beta', \beta'' \) are non-zero integers such that \( \alpha \beta' - \alpha' \beta \neq 0 \). Note that the common zero locus of these polynomials is no longer non-singular in \( \mathbb{P}^5 \).

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We will estimate $S(B)$ using the same version of the circle method that we used to handle $n \geq 7$, taking care to avoid duplicating unnecessary effort. We will arrive at the same exponential sums

$$S_{d,q}(m) = \sum_{a \pmod{q}}^* \sum_{\substack{k \pmod{dq} \quad Q_1(k) \equiv 0 \pmod{d} \\ Q_2(k) \equiv 0 \pmod{d}}} e_{dq}(aQ_2(k) + m.k),$$

(1.3)

for positive integers $d$ and $q$ and varying $m \in \mathbb{Z}^n$. When $Q_1$ and $Q_2$ are both diagonal it will be easier to analyse these sums explicitly. Nonetheless, the situation for $n = 6$ is more delicate, since we are no longer able to win sufficient cancellation solely through an analysis of the Dirichlet series

$$\sum_{q=1}^{\infty} \frac{S_{1,q}(m)}{q^s},$$

as in [3]. Instead we will attempt to profit from cancellation due to sign changes in the exponential sum $S_{d,1}(m)$. The latter sum is associated to a pair of quadratic forms, rather than a single form, and this raises significant technical obstacles. The following is our main result.

**Theorem 1.1.** Assume that $Q_1(x) \gg 1$ and $\nabla Q_1(x) \gg 1$, for some absolute implied constant, for every $x \in \text{supp}(W)$. Suppose that $X(\mathbb{R})$ and $X(\mathbb{Q}_p)$ are non-empty for each prime $p$. Then there exists a constant $c > 0$ such that

$$S(B) = cB^4 + O(B^{4-\delta}),$$

for any $\delta < \frac{1}{160}$. The implied constant is allowed to depend on $\alpha, \alpha', \beta, \beta', \beta''$ and $W$.

This result compares favourably with work of Cook [5], who is able to handle suitable pairs of diagonal quadratic forms in at least 9 variables, rather than the 8 variables that we deal with. The leading constant in Theorem 1.1 is an absolutely convergent product of local densities $c = \sigma_\infty \prod_p \sigma_p$, whose positivity is equivalent to the hypothesis that $X(\mathbb{R})$ and $X(\mathbb{Q}_p)$ are non-empty for each prime $p$.

A central problem in analytic number theory is to study the average order of arithmetic functions as they range over the values taken by polynomials. Let $L = (L_1, \ldots, L_4)$ be a collection of pairwise non-proportional binary linear forms defined over $\mathbb{Z}$, for which each $L_i(x,y)$ is congruent to $x$ modulo 4 as a polynomial. Our choice of forms (1.2) is largely motivated by their connection to the sums

$$T_\omega(B; L) = \sum_{(x,y) \in \mathbb{Z}^2 \atop 2|x} r(L_1(x,y)) \cdots r(L_4(x,y))\omega\left(\frac{x}{B}, \frac{y}{B}\right),$$

(1.4)

with $\omega : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ a suitable weight function. When $\omega = 1_{\mathcal{R}}$ is taken to be the characteristic function of an open, bounded and convex region $\mathcal{R} \subset \mathbb{R}^2$, with piecewise continuously differentiable boundary, it is possible to derive an asymptotic formula for the sum, as $B \to \infty$. This has been the focus of work by Heath-Brown [7], which in turn has been
improved in joint work of the first author with de la Bretèche [1]. Assume that $L_i(x, y) > 0$ for every $(x, y) \in R$. Then there exists a constant $c$ such that

$$T_{1\varphi}(B; L) = cB^2 + O\left(\frac{B^2}{(\log B)^\eta}\right),$$

(1.5)

for any $\eta < 0.08607$, where the implied constant is allowed to depend on $L_1, \ldots, L_4, R$ and $c$ can be interpreted as a product of local densities. This topic has also been addressed by Matthiesen [9] using recent developments in additive combinatorics. In this case a far-reaching generalisation of $T_{1\varphi}(B; L)$ is studied, which as a special case retrieves the asymptotic formula (1.5), but without an explicit error term.

Theorem 1.1 can be adapted to study $T_\omega(B; L)$ for other weights $\omega$. We make the choice

$$\omega(x, y) = w_1(L_1(x, y))w_0(L_2(x, y))w_0(L_3(x, y))w_1(L_4(x, y)),$$

where $w_0, w_1 : \mathbb{R} \to \mathbb{R}_{>0}$ are infinitely differentiable bounded functions of compact support, with $w_0$ supported away from 0. Suppose that $L_i(x, y) = a_ix + b_iy$, with $(a_i, b_i)$ congruent to (1, 0) modulo 4, for $1 \leq i \leq 4$. For each $1 \leq i < j \leq 4$ we write $\Delta_{i,j} = a_ib_j - a_jb_i$ for the non-zero resultant of $L_i$ and $L_j$. For simplicity we will assume that $\Delta_{1,2} = 1$, although the general case can be handled with more work. Opening up the $r$-functions we see that

$$T_\omega(B; L) = \sum_{(x,y)\in\mathbb{Z}^2} \sum_{s,t} \omega\left(\frac{x}{B}, \frac{y}{B}\right),$$

where the inner sum is over $(s, t) \in \mathbb{Z}^2$ for which $L_i(x, y) = s_i^2 + t_i^2$, for $1 \leq i \leq 4$. It is clear that the condition $2 \nmid x$ is equivalent to $2 \nmid s_4^2 + t_4^2$ since $L_4(x, y) \equiv x \pmod{4}$. Eliminating $x, y$ via the transformation

$$x = b_2(s_1^2 + t_1^2) - b_1(s_2^2 + t_2^2), \quad y = a_1(s_2^2 + t_2^2) - a_2(s_1^2 + t_1^2),$$

we see that the system of four equations is equivalent to the pair of quadratics

$$\Delta_{2,3}(s_1^2 + t_1^2) - \Delta_{1,3}(s_2^2 + t_2^2) + s_3^2 + t_3^2 = 0,$$

$$\Delta_{2,4}(s_1^2 + t_1^2) - \Delta_{1,4}(s_2^2 + t_2^2) + s_4^2 + t_4^2 = 0.$$

Since either equation involves a sum of two squares of variables not apparent in the other equation, this variety is clearly of the type that are central to the present investigation. Taking $W(x) = w_1(x_1^2 + x_2^2)w_0(x_3^2 + x_4^2)w_0(x_5^2 + x_6^2)w_1(Q_1(x))$, for $x = (x_1, \ldots, x_6)$, one sees that $T_\omega(B; L) = S(B^2)$, where $Q_1, Q_2$ are as in [1.2], with

$$(\alpha, \alpha', \beta, \beta', \beta'') = (-\Delta_{2,4}, \Delta_{1,4}, \Delta_{2,3}, -\Delta_{1,3}, 1).$$

The following result is now a trivial consequence of Theorem 1.1.

**Theorem 1.2.** Let $\delta < \frac{1}{320}$. Then there exists a constant $c$ such that

$$T_\omega(B; L) = cB^2 + O(B^{2-\delta}).$$
The constant $c$ appearing in Theorem 1.2 is a product of local densities. As in Theorem 1.1 one can ensure its positivity by determining whether or not the underlying variety has points everywhere locally. At the expense of additional labour it would be possible to work with a more general class of weight functions than the one we have chosen. In this way it seems feasible to substantially improve the error term in (1.5) by selecting a weight function that approximates the characteristic function of $\mathcal{R}$.

While interesting in their own right, the study of sums like (1.4) can play an important rôle in the Manin conjecture for rational surfaces. This arises from using descent to pass from counting rational points of bounded height on a surface $S$ to counting suitably constrained integral points on associated torsors $\mathcal{T} \to S$ above the surface. The asymptotic formula (1.5) can be interpreted as the density of integral points on a torsor above the Châtelet surface

$$y^2 + z^2 = f(x),$$

with $f$ a totally reducible separable polynomial of degree 3 or 4 defined over $\mathbb{Q}$. In joint work of the first author with de la Bretèche and Peyre [2], this is a crucial ingredient in the resolution of the Manin conjecture for this family of Châtelet surfaces. It seems likely that Theorem 1.2 could prove the basis of an improved error term in this work.

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## 2. Preliminaries

Our analysis of $S(B)$ in (1.1) is largely based on our previous work [3]. We shall follow the same conventions regarding notation that were introduced there. Recall the definition [3, Eq. (3.5)] of $I_{d,q}(m)$. We begin by recording a version of [3, Lemma 12], in which a partial derivative with respect to $d$ is taken.

**Lemma 2.1.** For $0 < |m| \leq dQB^{-1+\varepsilon} = \sqrt{d}B^\varepsilon$ and $q \ll Q = B/\sqrt{d}$, we have

$$\frac{\partial^i}{\partial d^i} I_{d,q}(m) \ll d^{-i} \left| \frac{Bm}{dq} \right|^{1-\frac{n}{2}} B^\varepsilon,$$

for any $i \in \{0,1\}$.

**Proof.** When $i = 0$ this is due to Heath-Brown [6, Lemma 22]. Let us suppose that $i = 1$. After a change of variables we have

$$I_{d,q}(m) = d^n \int_{\mathbb{R}^n} h \left( \frac{q\sqrt{d}}{B}, d^2Q_2(y) \right) W_{d,T} (dy) e_{4q}(-Bm.y)dy.$$

We proceed to take the derivative with respect to $d$. The right hand side is seen to be

$$\frac{n}{d} I_{d,q}(m) + d^n \int_{\mathbb{R}^n} g_d(y)e_{4q}(-Bm.y)dy,$$
where if \( h^{(1)}(x, y) = \frac{\partial}{\partial x} h(x, y) \) and \( h^{(2)}(x, y) = \frac{\partial}{\partial y} h(x, y) \), then

\[
g_d(y) = \frac{q}{2B \sqrt{d}} h^{(1)} \left( \frac{q \sqrt{d}}{B}, d^2 Q_2(y) \right) W_{d,T}(dy) \\
+ 2dQ_2(y) h^{(2)} \left( \frac{q \sqrt{d}}{B}, d^2 Q_2(y) \right) W_{d,T}(dy) + h \left( \frac{q \sqrt{d}}{B}, d^2 Q_2(y) \right) \frac{\partial}{\partial d} W_{d,T}(dy).
\]

Let \( W^{(1)}(y) = y. \nabla W(y) \). One finds that

\[
\frac{\partial}{\partial d} W_{d,T}(dy) = \frac{1}{d} W^{(1)}(dy) V_T(d) + W(dy) V'_T(d),
\]

if \( T \leq B \), and

\[
\frac{\partial}{\partial d} W_{d,T}(dy) = \frac{1}{d} W^{(1)}(dy) V_T \left( B^2 dQ_1(y) \right) + W(dy) V'_T \left( B^2 dQ_1(y) \right) B^2 Q_1(y) ,
\]

otherwise. Hence

\[
\frac{\partial}{\partial d} W_{d,T}(dy) = \frac{1}{d} \hat{W}_{d,T}(dy),
\]

where the new function \( \hat{W}_{d,T} \) has the same analytic behaviour as \( W_{d,T} \). Another change of variables now yields

\[
\frac{\partial}{\partial d} I_{d,q}(m) = \frac{n}{d} I_{d,q}(m) + \frac{1}{2d} \int_{\mathbb{R}^n} \frac{q \sqrt{d}}{B} h^{(1)} \left( \frac{q \sqrt{d}}{B}, Q_2(y) \right) W_{d,T}(y) e_{4dq}(-Bm.y) dy \\
+ \frac{2}{d} \int_{\mathbb{R}^n} h^{(2)} \left( \frac{q \sqrt{d}}{B}, Q_2(y) \right) \hat{W}_{d,T}(y) e_{4dq}(-Bm.y) dy \\
+ \frac{1}{d} \int_{\mathbb{R}^n} h \left( \frac{q \sqrt{d}}{B}, Q_2(y) \right) \hat{W}_{d,T}(y) e_{4dq}(-Bm.y) dy,
\]

where \( \hat{W}_{d,T}(y) = W_{d,T}(y) Q_2(y) \). The last three integrals can be compared with \( I_{d,q}(m) \), and the lemma now follows using the bounds in the statement of the lemma for \( i = 0 \). \( \square \)

Let \( \varrho(d) = S_{d,1}(0) \), in the notation of (1.3). In [3, §1] we defined “Hypothesis-\( \varrho \)” to be the hypothesis that \( \varrho(d) = O(d^{n-2+\varepsilon}) \), for any \( \varepsilon > 0 \). Our present investigation will be streamlined substantially by the convention adopted in [3] that any estimate concerning quadratic forms \( Q_1, Q_2 \in \mathbb{Z}[x_1, \ldots, x_n] \) was valid for arbitrary forms such that \( Q_2 \) is non-singular, with \( n \geq 5 \), for which the variety \( Q_1 = Q_2 = 0 \) defines a (possibly singular) geometrically integral complete intersection \( V \subset \mathbb{P}^{n-1} \). The quadratic forms \( Q_1, Q_2 \) in (1.2) clearly adhere to these constraints. Our next task is to verify Hypothesis-\( \varrho \) in the present setting.

**Lemma 2.2.** Hypothesis-\( \varrho \) holds if \( Q_1, Q_2 \) are given by (1.2).
It follows that

$$ \varrho(p^r) \leq (1 + r)^3 p^{3r} \# \{(u, v, w) \pmod{p^r} : p^r \mid \alpha u + \alpha' v, \ p^r \mid \beta u + \beta' v + \beta'' w \}. $$

Suppose \( p^k \parallel \beta'' \). We may clearly assume without loss of generality that \( r > k \). Then from the congruence \( p^r \mid \beta u + \beta' v + \beta'' w \) we get a congruence modulo \( p^{r-k} \) which gives a unique solution for \( w \) modulo \( p^{r-k} \). These lift to give us at most \( p^k \) possibilities for \( w \) modulo \( p^r \), for any given \( u, v \). Similarly the congruence \( p^r \mid \alpha u + \alpha' v \) gives at most \( p^d \) many \( u \) for any given \( v \), where \( p^d \parallel \alpha \). Hence \( \varrho(p^r) \ll (1 + r)^3 p^{4r} \), which is satisfactory for the lemma. \( \square \)

The exponential sum \( S_{d,q}(m) \) in (1.3) satisfies the multiplicativity property recorded in [3, Lemma 10]. This makes it natural to introduce the sums

$$ \mathcal{D}_q(m) = S_{1,q}(m), \quad \mathcal{D}_d(m) = S_{d,1}(m), \quad \mathcal{M}_{d,q}(m) = S_{d,q}(m), $$

the latter sum only being of interest when \( d \) and \( q \) exceed 1 and are constructed from the same set of primes. Since the variety \( V \) defined by the common zero locus of \( Q_1 \) and \( Q_2 \) is singular, we will need alternatives to the estimates obtained in [3, §5] for \( \mathcal{D}_d(m) \).

In this section, using [3], we shall establish the veracity of Theorem 1.1 subject to new bounds for the exponential sums \( \mathcal{D}_d(m) \) and \( \mathcal{M}_{d,q}(m) \), whose truth will be demonstrated in subsequent sections. We can be completely explicit about the analogue of the polynomial \( G(m) \) in [3, §5]. Define

$$ c_0 = c_0(m) = \alpha \alpha'(m_5^2 + m_6^2), $$
$$ c_1 = c_1(m) = \alpha' \beta''(m_1^2 + m_2^2) + \alpha \beta''(m_3^2 + m_4^2) + (\alpha \beta' + \alpha' \beta)(m_5^2 + m_6^2), $$
$$ c_2 = c_2(m) = \beta' \beta''(m_1^2 + m_2^2) + \beta \beta''(m_3^2 + m_4^2) + \beta \beta'(m_5^2 + m_6^2). \quad (2.1) $$

In particular \( c_2 = Q_2^*(m) \), where \( Q_2^* \) is the adjoint quadratic form. We will set

$$ \delta(m) = c_1^2 - 4c_0 c_2, \quad (2.2) $$
a quartic form in \( m \), and

$$ \sigma(m) = \alpha' \beta''(m_1^2 + m_2^2) + \alpha \beta''(m_3^2 + m_4^2) + (\alpha \beta' - \alpha' \beta)(m_5^2 + m_6^2). \quad (2.3) $$

The rôle of \( G \) is now played by the polynomial \( \delta(m)H(m) \), where

$$ H(m) = (m_1^2 + m_2^2)(m_3^2 + m_4^2)(m_5^2 + m_6^2). $$

We henceforth put \( \Delta_V = 2\alpha \alpha' \beta' \beta''(\alpha \beta' - \alpha' \beta) \neq 0 \).

Our proof of [3, Theorem 1] was based on a careful analysis of the sum \( U_{T,a}(B, D) \) in [3, Eq. (7.3)\], for \( D \geq 1 \). Rather than summing non-trivially over \( q \), as there, our course of action for Theorem 1.1 is based on summing non-trivially over \( d \). As before it suffices to consider the contribution to \( S_{T,a}(B) \) from \( m \) such that \( m = 0 \) or \( 0 < |m| \leq \sqrt{d}B^r \).
Dealing with the latter contribution leads us to study the expression

$$V_{T,a}(B, D) = B^4 \sum_{\delta \Delta \leq B^\varepsilon} \sum_{q \leq B/\sqrt{D}} \frac{1}{q^6} \sum_{\substack{d, q \mid \delta, d, q \Delta \leq \Xi}} \frac{\chi(d)}{d^5} T_{d,q}(m) I_{d,q}(m),$$

for $D \geq 1$, where $T_{d,q}(m)$ is given in [3, Lemma 8]. We will show that

$$V_{T,a}(B, D) = O\left(\frac{B^2}{\varepsilon^{\varepsilon}}\right),$$

for any $D \ll B$.

Define the non-zero integer

$$N = \begin{cases} \delta(m)H(m), & \text{if } \delta(m)H(m) \neq 0, \\ H(m), & \text{if } \delta(m) = 0 \text{ and } H(m) \neq 0, \\ \delta(m), & \text{if } \delta(m) \neq 0 \text{ and } H(m) = 0, \\ 1, & \text{otherwise.} \end{cases}$$

We split $d$ as $\delta d$ with $(d, q\Delta_V N) = 1$ and $\delta \mid (q\Delta_V N)$. Then

$$V_{T,a}(B, D) \leq B^4 \sum_{\delta \Delta \leq B^\varepsilon} \sum_{q \leq B/\sqrt{D}} \frac{1}{q^6} \sum_{\delta \leq D} |T_{\delta,q}(m)| \frac{\delta}{\delta \Delta} \sum_{d, q \Delta \leq \Xi} \frac{\chi(d)}{d^5} \mathcal{D}_d(m) I_{d,q}(m),$$

where $\sum'$ means that the sum is restricted to odd integers only. Applying partial summation we see that the inner sum over $d$ can be written

$$J = \Sigma(D/\delta) \cdot \frac{I_{\delta,q}(m)}{(D/\delta)^5} - \int_{D/(2\delta)}^{D/\delta} \Sigma(x) \frac{\partial}{\partial x} \left( \frac{I_{\delta,q}(m)}{x^5} \right) dx,$$

where

$$\Sigma(x) = \sum_{D/(2\delta) < d \leq x} \chi(d) \mathcal{D}_d(m).$$

We will establish the following result in [3].

**Lemma 2.3.** We have $\Sigma(x) \ll |m|^{\theta(m)+\varepsilon} x^{\frac{3}{2}} + \psi(m)+\varepsilon$, with

$$\psi(m) = \begin{cases} 0, & \text{if } \delta(m) \neq \square \text{ and } H(m) \neq 0, \\ \frac{1}{2}, & \text{if } \delta(m) = \square \text{ and } H(m) \neq 0, \\ \frac{1}{2}, & \text{if } \delta(m) \neq 0 \text{ and } H(m) = 0, \\ \frac{3}{2}, & \text{otherwise,} \end{cases}$$

and

$$\theta(m) = \begin{cases} \frac{7}{8}, & \text{if } \delta(m) \neq \square \text{ and } H(m) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$
Taking Lemma 2.3 on faith for the moment, and appealing to Lemma 2.1, we therefore deduce that

\[ J \ll B^\varepsilon \left| \frac{Bm}{Dq} \right|^{-2} |m|^{\theta(m)} \frac{\left( \frac{D}{\delta} \right) \delta^{\frac{\varepsilon}{2} + \psi(m)} \left( \frac{\delta}{D} \right)^5}{\delta^{\frac{5}{2} + \psi(m)}}. \]

Inserting this into our expression for \( V_{T,a}(B, D) \) gives

\[ V_{T,a}(B, D) \ll \frac{B^{2+\varepsilon}}{D^2} \sum_{0<|m|\leq \sqrt{DB^\varepsilon}} \frac{|m|^{\theta(m)}D^{\psi(m)}}{|m|^2} \sum_{q\leq B/\sqrt{D}} \frac{1}{q^4} \sum_{\delta \mid \delta, \Delta_V \leq \Xi} |T_{\delta,q}(m)| q^{\varepsilon}. \]

Combining [3] Lemma 9 with [3] Eq. (4.1)], we see that

\[ \frac{|T_{\delta,q}(m)|}{q^4} \ll \frac{|S_{\delta,q}(m)|}{q^{4}}, \]

where \( q' \) is the odd part of \( q \). Hence we may restrict attention to odd values of \( q \) in the above estimate without loss of generality. Let us write \( \delta = \delta_1 \delta_2 \) with \( \delta_1 \mid \Delta_V \) and \( (\delta_2, \Delta_V) = 1 \).

Similarly we write \( q = q_1 q_2 \) with \( q_1 \mid \Delta_V^\infty \) and \( (q_2, \Delta_V) = 1 \). In particular \( q_2 \) is odd and we have \( S_{\delta,q}(m) = S_{\delta_1,q_1}(m)S_{\delta_2,q_2}(m) \) by [3] Lemma 10. Combining Lemma 2.2 with [3] Eq. (4.1) and [3] Lemma 25 we see that \( S_{\delta_1,q_1}(m) \ll \delta_1^{1+\varepsilon} q_1^4 \), whence

\[ \frac{S_{\delta_1,q_1}(m)}{\delta_1^\frac{3}{2} q_1^4} \ll \Xi^2 B^\varepsilon. \]  

(2.7)

Note that there are \( O(B^\varepsilon) \) choices for \( \delta_1 \) and \( q_1 \) by [3] Eq. (1.3)]. Finally, appealing once more to [3] Lemma 10], we deduce there is a factorisation \( \delta_2 = \delta_2 \delta_2 \) and \( q_2 = q_2 q_2 \), with

\[ \delta_2 \mid N^\infty, \quad (\delta_2, q_2) = 1, \quad \delta_2 \mid q_2^\infty, \quad (\delta_2 q_2, \delta_2 q_2) = 1, \]

such that

\[ S_{\delta_2,q_2}(m) = \mathcal{D}_{\delta_2}(m) \mathcal{D}_{q_2}(m). \mathcal{M}_{\delta_2,q_2}(m). \]  

(2.8)

We have \((\delta_2 \delta_2 q_2 q_2, \Delta_V) = 1 \) here. We will need good upper bounds for these sums.

**Lemma 2.4.** Assume that \((d, \Delta_V) = 1 \). Then we have

\[ |\mathcal{D}_d(m)| \leq 4^{\omega(d)} \tau(d)\Gamma(d)(d, m, \delta(m)). \]

The proof of this result is deferred to [3]. It implies that \( \mathcal{D}_d(m) \ll d^{3+\varepsilon}(d, m) \), which recovers [3] Lemma 22 in the present setting. An application of [3] Lemma 25 yields \( \mathcal{M}_{d,q}(m) \ll d^{3+\varepsilon} q^4 \). This is not fit for purpose when \( m \) is generic, although it does suffice for non-generic \( m \). The following result will be established in [4].

**Lemma 2.5.** Assume that \((d, \Delta_V) = 1 \). Then we have

\[ \mathcal{M}_{d,q}(m) \ll d^{2+\varepsilon} q^{3+\varepsilon} (d, m)(q, m)^2 (d, \delta(m))(q, Q_2^*(m)). \]
Returning to (2.8), we are now ready to deduce the estimate
\[ S_{\delta_2,q_2}(m) \ll \delta_2^{3+\epsilon} q_2^{3+\epsilon}(\delta_2, m)(q_2, m)^2(\delta_2, \delta(m))(q_2, Q_2^{*}(m)), \]
if \( \delta(m)H(m)Q_2^{*}(m) \neq 0 \), and
\[ S_{\delta_2,q_2}(m) \ll \delta_2^{3+\epsilon} q_2^{4}(\delta_2, m), \]
in general. The latter follows easily from combining Lemma 2.4 with [3, Eq. (4.1)] and [3, Lemma 25]. For the former we act similarly but substitute Lemma 2.5 for [3, Lemma 25] in general. The latter follows easily from combining Lemma 2.4 with [3, Eq. (4.1)] and [3, Lemma 15] for [3, Eq. (4.1)].

We proceed to partition \( \mathbb{Z}^6 \) into a disjoint union of four sets. Let \( \mathcal{M}_1 \) denote the set of \( m \in \mathbb{Z}^6 \) such that \( \delta(m) \neq \square \) and \( H(m)Q_2^{*}(m) \neq 0 \). Likewise, let \( \mathcal{M}_2 \) denote the set of \( m \) for which \( \delta(m) = \square \) or \( Q_2^{*}(m) = 0 \) but \( H(m) \neq 0 \). Let \( \mathcal{M}_3 \) denote the set of \( m \) such that \( \delta(m)Q_2^{*}(m) \neq 0 \) and \( H(m) = 0 \). Finally, let \( \mathcal{M}_4 \) be the set of \( m \) such that \( \delta(m)Q_2^{*}(m) = H(m) = 0 \). We will need the following result.

**Lemma 2.6.** Let \( M \geq 1 \) and let \( \epsilon > 0 \). Then we have
\[ \#\{m \in \mathcal{M}_i : |m| \leq M\} = \begin{cases} O(M^{4+\epsilon}), & \text{if } i = 2 \text{ or } 3, \\ O(M^{2+\epsilon}), & \text{if } i = 4. \end{cases} \]
Furthermore, for any \( A \in \mathbb{Z} \), we have
\[ \#\{m \in \mathbb{Z}^6 : |m| \leq M, \ \delta(m) = A\} = O((1 + |A|)^\epsilon M^{2+\epsilon}). \]

**Proof.** Let us write \( R_i(M) \) for the quantity on the left hand side in the first displayed equation and \( R(A; M) \) for the quantity in the second displayed equation. Note that \( H(m) = 0 \) if and only if \( m_i = m_j = 0 \) for some \( (i, j) \in \{(1, 2), (3, 4), (5, 6)\} \). In particular the bound for \( R_3(M) \) is trivial. Likewise it is easy to see that there are \( O(M^{4+\epsilon}) \) choices of \( |m| \leq M \) for which \( Q_2^{*}(m) = 0 \) and \( O(M^{2+\epsilon}) \) choices of \( |m| \leq M \) for which \( Q_2^{*}(m) = H(m) = 0 \).

To handle the contributions \( \delta(m) \), it will be convenient to make the change of variables \( X = c_1(m), Y = c_0(m) \) and \( Z = c_2(m) \), in the notation of (2.1). In particular we have \( X, Y, Z \ll M^2 \). Using the familiar estimate \( r(m) = O(m^\epsilon) \) and recalling that \( \Delta_V \neq 0 \), we see that there are \( O(M^\epsilon) \) choices of \( m \) associated to a given triple \( X, Y, Z \). We begin by estimating
\[ R(A; M) \ll M^\epsilon \#\{X, Y, Z \ll M^2 : X^2 - 4YZ = A\}. \]
For a fixed \( X \) with \( X^2 - A \neq 0 \), the trivial estimate for the divisor function reveals that there are \( O((1 + |A|)^\epsilon M^\epsilon) \) choices for \( Y, Z \). For \( X \) with \( X^2 - A = 0 \) there are \( O(M^2) \) choices for \( Y, Z \) such that \( YZ = 0 \). Combining these contributions therefore shows that \( R(A; M) = O((1 + |A|)^\epsilon M^{2+\epsilon}) \), as claimed.

Turning to the estimation of \( R_2(M) \) and \( R_4(M) \), we first observe that
\[ R_2(M) \ll M^{4+\epsilon} + M^\epsilon \#\{W, X, Y, Z \ll M^2 : X^2 - 4YZ = W^2\} \ll M^{4+\epsilon}. \]
Finally, we have \( R_4(M) \ll M^{2+\epsilon} + M^\epsilon R(0, M) \ll M^{2+\epsilon} \), which completes the proof. \( \Box \)
Let us write $V_i$ for the overall contribution to $V_{T,a}(B, D)$ from $m \in \mathcal{M}$, for $1 \leq i \leq 4$. We relabel $\delta_2$ to be $\delta$ and $q_2$ to be $q$. Beginning with the contribution from generic $m$, an application of (2.7) and (2.9) yields

$$V_1 \ll \frac{\Xi^2 B^{2+\varepsilon}}{D^\frac{1}{2}} \sum_{m \in \mathcal{M}_1} \sum_{|m| \leq \sqrt{DB}^\varepsilon} \frac{(q, m)^2(q, Q_2^*(m))}{q} \sum_{\delta \leq D} \frac{(\delta, m)(\delta, \delta(m))}{\delta^\frac{1}{2}}.$$ 

The inner sum is

$$\sum_{\delta \leq D} \frac{(\delta, m)(\delta, \delta(m))}{\delta^\frac{1}{2}} \ll B^\varepsilon \max_{\delta \leq D} \left\{ (\delta, m)(\delta, \delta(m))^{\frac{1}{2}} \right\}.$$ 

Extracting the greatest common divisor $h$ of $m_1, \ldots, m_6$ and rearranging our expression, we therefore obtain

$$V_1 \ll \frac{\Xi^2 B^{2+\varepsilon}}{D^\frac{1}{2}} \max_{\delta \leq D} \sum_{h \leq \sqrt{DB}^\varepsilon} \frac{(\delta, h)}{h^\frac{1}{2}} \sum_{m \in \mathcal{M}_1} \frac{(\delta, \delta(hm))^{\frac{1}{2}}}{|m|^{\frac{1}{2}}} \sum_{|m| \leq h^{-1}\sqrt{DB}^\varepsilon} \frac{(q, h)^2(q, Q_2^*(hm))}{q}.$$ 

The inner sum over $q$ is clearly $O(h^2 B^\varepsilon)$, whence

$$V_1 \ll \frac{\Xi^2 B^{2+\varepsilon}}{D^\frac{1}{2}} \max_{\delta \leq D} \sum_{h \leq \sqrt{DB}^\varepsilon} \frac{h^2(\delta, h)^3}{h^\frac{2}{8}} \sum_{m \in \mathcal{M}_1} \frac{(\delta, \delta(m))^{\frac{1}{2}}}{|m|^{\frac{2}{8}}}.$$ 

since $(\delta, \delta(hm)) \leq (\delta, h)^4(\delta, \delta(m))$.

For a fixed integer $\delta \leq D$ and a choice of $M$ with $0 < M \leq h^{-1}\sqrt{DB}^\varepsilon$, we deduce from Lemma 2.6 that

$$\sum_{M < |m| \leq 2M} (\delta, \delta(m)) \leq \sum_{0 < |A| \leq M^4} (\delta, A)R(A; M)$$

$$\ll M^{2+\varepsilon} \sum_{0 < |A| \leq M^4} (\delta, A)$$

$$\ll \delta^\varepsilon M^{6+\varepsilon}.$$ 

Armed with this we conclude that

$$V_1 \ll \frac{\Xi^2 B^{2+\varepsilon}}{D^\frac{1}{2}} \max_{\delta \leq D} \sum_{h \leq \sqrt{DB}^\varepsilon} \frac{h^2(\delta, h)^3}{h^\frac{2}{8}} \left( \frac{\sqrt{D}}{h} \right)^{6-\frac{\varepsilon}{8}} \ll \Xi^2 B^{2+\varepsilon} D^{2-\frac{1}{16}+\varepsilon}.$$ 

This is $O(\Xi^2 B^{4-\frac{1}{16}+\varepsilon})$, since $D \ll B$, which is satisfactory for (2.4).
It is now time to consider the contribution from non-generic $m$. Invoking our estimate for $S_{\delta,q}(m)$ in (2.10) we find that

$$V_i \ll \frac{\Xi}{D} B^{3+\varepsilon} \max_{\delta \leq D} \sum_{m \in \mathcal{M}_i, 0<|m| \leq \sqrt{DB^\varepsilon}} |m|^{-2} \frac{D\psi(m) \delta^2(\delta, m)}{\delta\psi(m)},$$

for $2 \leq i \leq 4$. Suppose first that $i \in \{2,3\}$, so that $\psi(m) \leq \frac{1}{2}$. Then Lemma 2.6 implies that

$$V_i \ll \frac{\Xi}{D} B^{3+\varepsilon} \max_{\delta \leq D} \sum_{\ell \in \delta} \sum_{m \in \mathcal{M}_i, |m| \leq \sqrt{DB^\varepsilon}} |m|^{-2} \ll \Xi B^{3+\varepsilon},$$

since $D \ll B$. This is satisfactory for (2.4).

Finally, when $i = 4$, so that $\psi(m) \leq \frac{3}{2}$, Lemma 2.6 yields

$$V_4 \ll \frac{\Xi}{D} B^{3+\varepsilon} D^\frac{1}{2} \sum_{m \in \mathcal{M}_i, |m| \leq \sqrt{DB^\varepsilon}} |m|^{-2} \ll \Xi B^{4-\frac{1}{2}+\varepsilon},$$

since $D \ll B$. This too is satisfactory for (2.4) and thereby concludes our treatment of the contribution from non-zero $m$ to $S_{\gamma,a}(B)$.

Our analogue of [3, Eq. (8.1)] is now

$$S(B) = M^g(B) + O(\Xi^{-\frac{1}{2}} B^{4+\varepsilon} + \Xi B^{3+\varepsilon} + \Xi \frac{3}{2} B^{4-\frac{1}{2}+\varepsilon}),$$

where $M^g(B)$ is given by [3, Eq. (8.2)] with $n = 6$. To handle the remaining term we may invoke [3, Lemma 30]. Taking $\Xi = B^\frac{3}{10}$ we obtain the final error term $O(B^{4-\frac{1}{160}+\varepsilon})$ in our asymptotic formula for $S(B)$. This therefore completes the proof of Theorem 1.1, subject to the exponential sum estimates Lemmas 2.3–2.5.

3. Analysis of $\mathcal{D}_d(m)$

In this section we establish Lemmas 2.3 and 2.4 for which we will need to undertake a detailed analysis of the sum $\mathcal{D}_d(m)$ when $Q_1, Q_2$ are given by (1.2) and $(d, \Delta_V) = 1$, with $\Delta_V = 2\alpha\beta'\beta''(\alpha\beta' - \alpha'\beta)$. Before launching into this endeavour let us record a preliminary result concerning the quantity

$$\varrho_f(p^r) = \#\{n \pmod{p^r} : f(n) \equiv 0 \pmod{p^r}\},$$
for any quadratic polynomial $f$ defined over $\mathbb{Z}$ and any prime power $p^r$.

**Lemma 3.1.** Let $r \geq 1$ and let $f(x) = c_0 x^2 + c_1 x + c_2$ for $c_0, c_1, c_2 \in \mathbb{Z}$. Then we have
\[
g_f(p^r) \leq 2p^{v_2(c_1^2 - 4c_0c_2)}.
\]

*Proof.* Let $\mu = v_p(c_1^2 - 4c_0c_2)$ and let $p^i|_v(c_0, c_1, c_2)$. Writing $c_i = p^i c'_i$, for $0 \leq i \leq 2$, we denote by $f'$ the quadratic polynomial with coefficients $c'_0, c'_1, c'_2$. In particular we have $\mu' = v_p(c_1^2 - 4c'_0c'_2) = \mu - 2\ell$. If $r \leq \ell$ then it is clear that $g_f(p^r) = p^r \leq p^\frac{\mu}{2}$, which is satisfactory. Alternatively, if $r > \ell$, then $g_f(p^r) = p^r g_f(p^{r-\ell})$. The content of $f'$ is now coprime to $p$ and it therefore follows from work of Huxley [8] that $g_{f'}(p^{r-\ell}) \leq 2p^\frac{\mu'}{2}$. But then $g_f(p^r) \leq 2p^\frac{\mu}{2}$ when $r > \ell$, as required to complete the proof of the lemma. $\square$

We base our analysis of $\mathcal{D}_d(m)$ on the initial steps after the statement of [3, Lemma 20]. Let
\[
\mathcal{D}_d(m; b) = \sum_{k (\text{mod } d)} e_d(b_1 Q_1(k) + b_2 Q_2(k) + m k).
\]

(3.1)

Extracting the greatest common divisor between $d$ and $b$, as there, we conclude that
\[
\mathcal{D}_d(m) = \sum_{h | (d, m)} h^4 \cdot \sum_{(d', m')} \mathcal{D}_{d'}(m'; b) = \sum_{h | (d, m)} h^4 \cdot \mathcal{D}_{d'}^*(m'),
\]

(3.2)

say, with $d = h d'$ and $m = h m'$. By multiplicativity it suffices to analyse $\mathcal{D}_{d'}^*(m')$ when $d' = p^r$ for $r \geq 1$ and $p \nmid \Delta_V$. Let
\[
L(x, y) = L_1(x, y) = L_2(x, y) = \alpha x + \beta y,
\]

\[
L'(x, y) = L_3(x, y) = L_4(x, y) = \alpha' x + \beta' y,
\]

\[
L''(x, y) = L_5(x, y) = L_6(x, y) = \beta'' y.
\]

(3.3)

We will write $g(x, y) = L(x, y) L'(x, y) L''(x, y)$. Then we are led to consider
\[
\mathcal{D}_{p^r}(m'; b) = \prod_{i=1}^{6} \sum_{k (\text{mod } p^r)} e_{p^r}(L_i(b) k^2 + m' k).
\]

We would like to employ the explicit formulae for Gauss sums to evaluate these sums, which we proceed to recall. Let $p$ be a prime with $p \nmid 2a$. Then we have
\[
\sum_{k (\text{mod } p^r)} e_{p^r}(ak^2 + mk) = p^{\frac{\mu}{2}} e_{p^r}(-4am^2) \times \begin{cases} 1, & \text{if } r \text{ is even}, \\ \chi_p(a) \varepsilon(p), & \text{if } r \text{ is odd}. \end{cases}
\]

(3.4)

Here $\chi_p(\cdot) = \left( \frac{\cdot}{p} \right)$ is the Legendre symbol and $\varepsilon(p) = 1$ or $i$ according to whether $p$ is congruent to $1$ or $3$ modulo $4$, respectively. We will also need the following evaluation of the Ramanujan sum
\[
c_{p^r}(b) = \sum_{k (\text{mod } p^r)} e_{p^r}(bk) = \begin{cases} \varphi(p^r), & \text{if } p^r \mid b, \\ -p^{r-1}, & \text{if } p^{r-1} \mid b, \\ 0, & \text{otherwise}. \end{cases}
\]

(3.5)
Armed with these facts we be able to prove the following result.

**Lemma 3.2.** Let \( p \nmid \Delta_V \) and let \( m \in \mathbb{Z}^\delta \). Then we have

\[
\mathcal{D}_{p^r}(m) \leq 4(r + 1)p^{2r + \min\{r, \frac{\nu_p(d(m))}{2}\}}.
\]

Moreover, when \( r = 1 \) and \( p \nmid H(m) \), we have

\[
\mathcal{D}_{p}(m) = p^2 \chi_p(-\delta(m)) + O(p).
\]

Assuming this to be true for the moment we can establish Lemmas 2.3 and 2.4. Beginning with the latter, we deduce that \( |\mathcal{D}_{d}(m')| \leq 4^{\omega(d)} \tau(d')d'^2(d', \delta(m')) \). Once inserted into (3.2), we obtain

\[
|\mathcal{D}_{d}(m)| \leq 4^{\omega(d)} \tau(d)d^2 \sum_{h | (d, m)} h^2 \left( \frac{d}{h} \frac{\delta(m)}{h^4} \right) \leq 4^{\omega(d)} \tau(d)d^2(d, m)(d, \delta(m)),
\]

since \((d, \Delta_V) = 1\). This therefore establishes Lemma 2.4.

We now turn to the proof of Lemma 2.3 and recall the definition of \( \Sigma(x) \) in (2.6). Suppose first that \( \delta(m) \neq 0 \). Then it follows from Lemma 2.4 that \( \mathcal{D}_{d}(m) \leq d^{2+\varepsilon} \), since \( \gcd(d, \delta(m)) = \gcd(d, m) = 1 \). But then \( \Sigma(x) \ll x^{3+\varepsilon} \). If \( \delta(m) = 0 \) then Lemma 2.4 yields the trivial bound

\[
\Sigma(x) \ll \sum_{0 < d \leq x} d^{3+\varepsilon}(d, m) \ll |m|^{\varepsilon} x^{4+\varepsilon},
\]

since \( m \neq 0 \).

When \( \delta(m) H(m) \neq 0 \) we can do better using complex analysis. Recall the definition (2.3) of \( N \). Let \( M \) be any non-zero integer divisible by \( qN \). We will need to examine the Dirichlet series

\[
\eta_M(s; m) = \sum_{(d, M) = 1} \frac{\chi(d) \mathcal{D}_d(m)}{d^s} = \prod_{p | M} \left\{ \sum_{r = 0}^\infty \frac{\chi_{p^r}(-1) \mathcal{D}_{p^r}(m)}{p^{rs}} \right\},
\]

for \( s \in \mathbb{C} \). One deduces from (3.2) and Lemmas 2.3 and 3.2 that

\[
\sum_{r = 0}^\infty \frac{\chi_{p^r}(-1) \mathcal{D}_{p^r}(m)}{p^{rs}} = 1 + \frac{\chi_p(-1) \mathcal{D}_p(m)}{p^s} + O \left( \sum_{r = 2}^\infty 4(r + 1)^2 p^{2r - r\sigma} \right)
\]

\[
= 1 + \frac{\chi_p(\delta(m))}{p^{s-2}} + O \left( p^{1-\sigma} + p^{4-2\sigma} \right),
\]

for the primes under consideration, where \( \sigma = \Re(s) \). It follows that

\[
\eta_M(s; m) = L(s - 2, \psi_m) E_M(s),
\]

where \( L(s, \psi_m) \) is the Dirichlet \( L \)-function with Jacobi symbol \( \psi_m(\cdot) = (\frac{\delta(m)}{\cdot}) \) and \( E_M(s) \) is an Euler product which converges absolutely in the half plane \( \sigma > \frac{5}{2} \) and satisfies the bound \( E_M(s) = O(M^\varepsilon) \) there. One notes that \( \psi_m \) is non-trivial when \( \delta(m) \neq \square \) and has conductor \( O(|m|^4) \) since \( \delta \) is a quartic form.
Invoking the truncated Perron formula, as in [3, Eq. (4.5)] with $c = 3 + \varepsilon$, we easily arrive at the statement of Lemma 2.3 when $\delta(m) \neq 0$ by repeating the proof of [3, Lemma 18] and moving the line of integration back to $\sigma = \frac{5}{2} + \varepsilon$. When $\delta(m) = 0$, we see that $\eta_M(s; m)$ is absolutely convergent and bounded by $O(M^\varepsilon)$ in the half-plane $\sigma > 3$. We apply the truncated Perron formula, as in [3, Eq. (4.5)] , with $c = 3 + \varepsilon$. But then we immediately arrive at the second estimate in Lemma 2.3 without the need to move the line of integration.

Proof of Lemma 3.2. In order to apply (3.4) we must deal with the possibility that one of the linear forms $L_i(b)$ is divisible by $p$. Let us write

$$D_p^*(m) = \sum_{0 \leq j < r} D_p^{(j)}(m) + D_p^{(r)}(m),$$

where for $0 \leq j < r$ we denote by $D_p^{(j)}(m)$ the overall contribution to $D_p^*(m)$ from $b$ such that $p^j | g(b)$. Likewise $D_p^{(r)}(m)$ is the contribution from $b$ such that $p^r | g(b)$. For the first part of the lemma it will suffice to show that

$$D_p^{(j)}(m) \leq 4p^{2r+\min\{r, \frac{\varepsilon(p^r(m))}{2}\}},$$

for $0 \leq j \leq r$.

We begin with an analysis of $D_p^{(0)}(m)$, recalling the notation (3.1) for $D_p^*(m; b)$. Note that $p \nmid b_2$ in $D_p^*(m; b)$. Applying (3.4), we deduce that

$$D_p^*(m; b) = p^{3r} \chi_p (-g(b)^2) e_{p^r} \left( -4 \left\{ \sum_{i=1}^6 L_i(b) m_i^2 \right\} \right)$$

$$= p^{3r} \chi_p (-1) e_{p^r} \left( -4 b_2 \left\{ \sum_{i=1}^6 L_i(b_1 b_2, 1) m_i^2 \right\} \right).$$

Dividing by $p^{2r}$, introducing the sum over $b$ and making the change of variables $b = b_1 b_2^r$, we obtain

$$D_p^{(0)}(m) = p^r \chi_{p^r} (-1) \sum_{b (\text{mod } p^r)} c_{p^r} (q_m(b)),$$

where

$$q_m(b) = \beta^r L'(b, 1) (m_1^2 + m_2^2) + \beta^r L(b, 1) (m_3^2 + m_4^2) + L(b, 1) L'(b, 1) (m_5^2 + m_6^2)$$

$$= c_0 b^2 + c_1 b + c_2,$$

in the notation of (2.1). Recall from (2.2) that $\delta(m) = c_1^2 - 4c_0 c_2$. It follows from (3.5) and Lemma 3.1 that

$$|D_p^{(0)}(m)| \leq p^{2r} \left( g_{q_m}(p^r) + g_{q_m}(p^{r-1}) \right) \leq 4p^{2r+\min\{r, \frac{\varepsilon(p^r(m))}{2}\}}.$$ 

This is satisfactory for (3.7).

Next we require an estimate of similar strength for the sums $D_p^{(j)}(m)$, for $1 \leq j < r$. We begin by noting that if $p | g(b)$ in $D_p^*(m; b)$ then $p$ can divide precisely one of the
linear factors since $p \nmid \Delta_V b$. Consequently we may write $\mathcal{D}_{p'}^{(j)}(m) = D + D' + D''$ where $D$ denotes the overall contribution arising from $b$ such that $p^j \parallel L(b)$, and similarly for $D'$ and $D''$. We will deal here only with the sum $D$, which is typical. Since $p \nmid L'(b)L''(b)$ we deduce that $p \nmid b_2$ and (3.4) yields

$$\mathcal{D}_{p'}(m; b) = p^{2r} e_{p'} \left( -4b_2 \left\{ \sum_{i=3}^{6} L_i((b_1 b_2, 1)m_i^2) \right\} \right) \mathcal{G}_1 \mathcal{G}_2,$$

where

$$\mathcal{G}_i = \sum_{k \equiv i (\text{mod } p')} e_{p'}(L(b)k^2 + m_i k),$$

for $i = 1, 2$. Write $L((b_1 b_2, 1) = p^j u$, with $p \nmid u$. It is easily checked that $\mathcal{G}_i = 0$ unless $p^j \mid m_i$, for $i = 1, 2$. Writing $m_i = p^j m_i'$ for $i = 1, 2$, it follows from a further application of (3.4) that

$$\mathcal{G}_i = p^j \sum_{k \equiv p^j \text{mod } p'} e_{p^j} (b_2 u k^2 + m_i' k)$$

$$= \begin{cases} p^{r+j} e_{p^j} \left( \frac{4b_2 u m_i'^2}{p} \right), & \text{if } r - j \text{ is even}, \\ \varepsilon(p) \chi_p(b_2 u) p^{\frac{r+j}{2}} e_{p^j} \left( \frac{-4b_2 u m_i'^2}{p} \right), & \text{if } r - j \text{ is odd}. \end{cases}$$

Hence

$$|D| \leq \frac{1}{p^{2r}} \cdot p^{3r+j} \left| \sum_{b \equiv 0 (\text{mod } p')} e_{p'} \left( -4b_2 \left\{ \sum_{i=3}^{6} L_i((b_1 b_2, 1)m_i^2) \right\} \right) e_{p^j} \left( \frac{-4b_2 u (m_1'^2 + m_2'^2)}{p} \right) \right|$$

if $p^j \mid (m_1, m_2)$ and $D = 0$ otherwise. We proceed under the assumption that $p^j \mid (m_1, m_2)$. We may view the sum over $b_2$ as a Ramanujan sum, leading to the inequality

$$|D| \leq p^{r+j} \sum_{b \equiv 0 (\text{mod } p')} e_{p'} \left( r_m(b) \right),$$

where $r_m(b) = p^{-j} q_m(b)$, in the notation of (3.9).

We handle the remaining sum by writing $b = b_0 + p^j b'$ where $b_0 \equiv -\alpha/3 (\text{mod } p^j)$ and $b'$ runs modulo $p^r - j$. Let

$$r'_m(x) = r_m(b_0 + p^j x) = p^{-j} q_m(b_0 + p^j x) = p^j c_0 x^2 + (2b_0 c_0 + c_1) x + p^{-j} q_m(b_0),$$

and note that $r'_m$ has discriminant $c_1^2 - 4c_0 c_2 = \delta(m)$. We have

$$|D| \leq p^{r+j} \sum_{b' \text{ (mod } p^r - j)} |c_{p'}(r'_m(b'))|. $$
The summand is zero unless \( p^{r-1} \mid r_m' (b') \). In particular we may assume that \( p^j \mid r_m' (b') \), whence \( |c_{p^r} (r_m' (b'))| = p^j |c_{p^{r-j}} (p^{-j} r_m' (b'))| \). Hence it follows from (3.5) and Lemma 3.1 that
\[
|D| \leq p^{2r+j} (g_{p^{r-j} r_m} (p^{-j}) + g_{p^{r-j} r_m} (p^{r-j-1})) \leq 4p^{2r+\min\{r, \frac{\nu_p (d(m))}{2}\}}.
\]
The same bound holds for \( D', D'' \) and so also for \( \mathcal{D}_p^{(r)} (m) \), as required for (3.7).

Finally we consider \( \mathcal{D}_p^{(r)} (m) \). We adapt the preceding argument, dealing with the case \( p^r \mid L(b) \) and \( p \nmid L'(b) L''(b) \), corresponding to \( D \), say. Tracing through the argument one deduces that \( \mathcal{G}_1 \mathcal{G}_2 = 0 \) unless \( p^r \mid (m_1, m_2) \), which we henceforth assume, in which case \( \mathcal{G}_1 \mathcal{G}_2 = p^{2r} \). Hence
\[
D = p^{2r} \sum_{b (m_1, m_2)} c_{p^r} \left( \beta''(m_3^2 + m_4^2) + L'(b, 1)(m_5^2 + m_6^2) \right)
\]
\[
= p^{2r} c_{p^r} \left( \alpha \beta''(m_3^2 + m_4^2) + (\alpha \beta' - \beta' \beta')(m_5^2 + m_6^2) \right)
\]
\[
= p^{2r} c_{p^r} \left( \sigma(m) \right),
\]
in the notation of (2.3). We claim that
\[
|c_{p^r} (\sigma(m))| \leq p^{\min\{r, \frac{\nu_p (d(m))}{2}\}}.
\]
Once achieved, and coupled with companion estimates for \( D' \) and \( D'' \), this will suffice for (3.7). Suppose first that \( p^r \mid \sigma(m) \), so that \( c_{p^r} (\sigma(m)) = \varphi(p^r) \). It will be convenient to write \( \xi_i = m_i^2 + m_{i+1}^2 \), for \( i \in \{1, 3, 5\} \). Recalling that \( p^{2r} \mid \xi_1 \) and noting from (2.1) and (2.3) that \( \sigma(m) = c_1 - 2\alpha \beta \xi_5 \), we see that
\[
\delta(m) \equiv (\sigma(m) + 2\alpha \beta \xi_5)^2 - 4\alpha \alpha' \beta \xi_5 (\beta'' \xi_3 + \beta' \xi_5) (\mod p^{2r})
\]
\[
\equiv 4\alpha \alpha' \beta \xi_5 (\sigma(m) + (\alpha' \beta - \alpha \beta') \xi_5 - \alpha \beta'' \xi_3) (\mod p^{2r})
\]
\[
\equiv 0 (\mod p^{2r}).
\]
Hence \( \nu_p (\delta(m)) \geq 2r \), as required. The case in which \( p^{r-1} \parallel \sigma(m) \) is similar, since then \( c_{p^r} (\sigma(m)) = -p^{r-1} \) and the same argument shows that \( \nu_p (\delta(m)) \geq 2(r - 1) \).

It remains to analyse the case \( r = 1 \) when \( p \nmid H(m) \). In this case
\[
\mathcal{D}_p^* (m) = \mathcal{D}_p^{(0)} (m) + \mathcal{D}_p^{(1)} (m),
\]
by (3.6). Moreover, \( \mathcal{D}_p^{(1)} (m) = 0 \) since \( p \nmid H(m) \). Taking \( r = 1 \) in (3.8) we deduce from (3.5) that
\[
\mathcal{D}_p^* (m) = p \chi_p (-1) \left( p \sum_{b (\mod p)} \sum_{b (\mod p)} 1 - \sum_{b (\mod p)} 1 \right).
\]
The second sum in the brackets is $p + O(1)$ and the first sum can be written
\[ \varrho_{qm}(p) - \#\{(b \pmod{p}) : p \mid g(b, 1), p \mid qm(b)\} = \varrho_{qm}(p), \]
since $p \nmid H(m)$. Substituting the identity $\varrho_{qm}(p) = 1 + \chi_{H}(d)$, we easily arrive at the statement of the lemma.

\[ \square \]

4. Analysis of $\mathcal{M}_{d,q}(m)$

In this section we establish the estimate in Lemma 2.5 for the mixed sum. Let $m \in \mathbb{Z}^6$ and let $d, q \in \mathbb{N}$ with $d \mid q^\infty$ and $q \mid d^\infty$, such that $(d, \Delta_{\nu}) = 1$. We claim that

\[ \mathcal{M}_{d,q}(m) = \frac{q^6}{d^2} \sum_{h \mid d} h^6 \sum_{r \mid q} \mu(q/r) r^6 \sum_{u \mid r} \frac{u^6}{m} \sum_{b_1 \equiv (dr) \pmod{b_2}} \frac{b_2}{b_2 (mod d') \equiv (b_1, ub_2d') = (b_2, r') = 1} \mathcal{D}_{dr'}(m''; r'b_1, b_2), \tag{4.1} \]

where
\[ d' = d/h, \quad m'' = (uhq/r)^{-1}m \in \mathbb{Z}^6, \quad r' = r/u. \]

To see this we treat the sum over $a$ as a Ramanujan sum in [13], finding that

\[ \mathcal{M}_{d,q}(m) = \sum_{k \equiv dq \pmod{d}} e_{dq} (m, k) \sum_{a \equiv dq \pmod{q}} e_{dq} (aQ_2(k)) \]

\[ = \sum_{r \mid q} \left( \frac{q}{r} \right) r \sum_{k \equiv dq \pmod{d'}} e_{dq} (m, k). \]

Breaking the inner sum into residue classes modulo $dr$ we easily deduce that $m$ must be divisible by $q/r$, whence

\[ \mathcal{M}_{d,q}(m) = \sum_{r \mid q} \left( \frac{q}{r} \right) \left( \frac{q}{r} \right)^6 \sum_{k \equiv dq \pmod{dr}} e_{dr} (m', k), \]

where $m = (q/r)m'$. Using characters to detect the congruences involving $Q_1$ and $Q_2$, we may write the sum over $k$ as

\[ \frac{1}{d^2r} \sum_{b_1 \equiv (dr) \pmod{b_2}} \mathcal{D}_{dr}(m'; rb_1, b_2), \]

in the notation of (3.1). Next we extract the greatest common divisor $h$ of $b$ and $d$, writing $d = hd'$ and $b = h' b'$, with $(b', d') = 1$. Breaking the sum into congruence classes modulo $d' r$ we then see that

\[ \mathcal{D}_{dr}(m'; rb_1, b_2) = \sum_{k' \equiv dq \pmod{d'r}} \sum_{k'' \equiv (mod h)} e_{dr'} (rb_1' Q_2(k') + b_2' Q_2(k') + h^{-1} m', k') e_h (m', k''). \]
This is \( h^6 \mathcal{D}_{d',r}(h^{-1}m'; rb_1, b_2) \) if \( h \mid m' \) and 0 otherwise. Our work so far has shown that

\[
\mathcal{M}_{d,q}(m) = \frac{q^6}{d^2} \sum_{h \mid d} \frac{h^6}{r^6} \sum_{b_1 (\text{mod } d')} \sum_{b_2 (\text{mod } d')} \mathcal{D}_{d',r}(h^{-1}m'; rb_1, b_2). \tag{4.2}
\]

To complete the proof of (4.1), we will need to extract the greatest common divisor \( u \) of \( b_2 \) and \( r \), writing \( b_2 = ub_2' \) and \( r = ur' \) with \( (b_2', r') = 1 \). Breaking the sum in \( \mathcal{D}_{d',r}(m'; rb_1, b_2) \) into residue classes modulo \( d'r' \), as before, we conclude that the inner sum over \( b \) can be written

\[
\sum_{u \mid r} u^6 \sum_{u \mid h^{-1}m'} \sum_{b_1 (\text{mod } d')} \sum_{b_2' (\text{mod } d')} \mathcal{D}_{d',r'}(m''; rb_1, b_2'),
\]

where \( m'' = u^{-1}m' \). This concludes the proof of (4.1).

Returning to (4.1), we denote by \( D(d', r') \) the inner sum over \( b = (b_1, b_2) \). We will establish the following estimate

**Lemma 4.1.** Let \( d', r' \in \mathbb{N} \), with \( d' \mid r'^\infty \) and \( (d', \Delta_V) = 1 \). Then we have

\[
D(d', r') \ll d^{4+\epsilon} r^{3+\epsilon} (d', \delta(m''))(r', Q_2^*(m'')).
\]

Inserting this estimate into (4.1), we see that the contribution to \( \mathcal{M}_{d,q}(m) \) from \( h \neq d \) is

\[
\ll (dq)^\epsilon \sum_{h \mid d} \frac{q^6}{d^2} \sum_{r \mid h} h^5 \sum_{r \mid q} \frac{1}{r^6} \sum_{u \mid r} u^5 \left( \frac{d}{h} \right)^4 \left( \frac{r}{u} \right)^3 (d, \delta(m))(r, Q_2^*((q/r)^{-1}m))
\]

\[
\ll d^{2+\epsilon} q^{3+\epsilon} \sum_{h \mid d} \frac{1}{h^6} \sum_{r \mid q} \left( \frac{q}{r} \right)^2 \sum_{u \mid r \mid (uhq/r|m)} u^2 (d, \delta(m))(q, Q_2^*(m))
\]

\[
\ll d^{2+\epsilon} q^{3+\epsilon} (d, m)(q, m)^2 (d, \delta(m))(q, Q_2^*(m)).
\]

This is satisfactory for Lemma 2.5.

Turning to the contribution to \( \mathcal{M}_{d,q}(m) \) from the terms with \( h = d \), we see from (4.2) that this is equal to

\[
d^4 q^6 \sum_{r \mid q \mid (dq/r)m} \sum_{b_2 (\text{mod } r)} \mathcal{D}_{r}(m'; 0, b_2),
\]

with \( m' = (dq/r)^{-1}m \in \mathbb{Z}^6 \). A little thought reveals that

\[
\sum_{b_2 (\text{mod } r)} \mathcal{D}_{r}(m'; 0, b_2) = \sum_{h \mid r \cap \frac{h}{m'}} h^6 \mathcal{D}_{r'}(m''),
\]

This completes the proof of Lemma 4.1.
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with \( r = h r' \) and \( \mathbf{m}' = h \mathbf{m}'' \). But then it follows from [3, Lemma 15] that

\[
\sum_{b_2 \pmod{r}} \mathcal{D}_r(\mathbf{m}'; 0, b_2) \ll \sum_{h | r, h | \mathbf{m}'} h^6 r'^2 \left( r', Q_2^*(\mathbf{m}'') \right)
\]

\[
\ll r^{3+\varepsilon} \left( r, \mathbf{m}' \right)^2 \left( r, Q_2^*(\mathbf{m}') \right).
\]

Since \( d \mid \mathbf{m} \) and \( \delta(\mathbf{m}) \) is homogeneous, we clearly have \( d^2 \leq (d, \mathbf{m})(d, \delta(\mathbf{m})) \). Hence this case contributes

\[
\ll d^2 q^{3+\varepsilon} (d, \mathbf{m})(q, \mathbf{m})^2 (d, \delta(\mathbf{m}))(q, Q_2^*(\mathbf{m}))
\]

to \( M_{d,q}(\mathbf{m}) \). This too is satisfactory and so completes the proof of Lemma 2.5 subject to the verification of Lemma 4.1.

Proof of Lemma 4.1. We recall that

\[
D(d', r') = \sum_{b_1 \pmod{d'}} \mathcal{D}_{d'}(\mathbf{m}''; r'b_1, b_2),
\]

This sum satisfies a basic multiplicativity property, meaning that it will suffice to analyse the case in which \( d' = p^k \) and \( r' = p^\ell \) for integers \( k \geq 1 \) and \( \ell \geq 0 \), with \( p \nmid \Delta_V \). Let us write \( u = p^j \).

Suppose first that \( \ell = 0 \) and \( k \geq 1 \). Then

\[
D(d', r') = D(p^k, 1) = \sum_{b_1 \pmod{d'}} \mathcal{D}_{d'}(\mathbf{m}''; b_1, b_2),
\]

with \( d' = p^k \). This is equal to a modified version of \( d'^2 \mathcal{D}_{d'}(\mathbf{m}'') \), in the notation of (3.2), wherein one is only interested in \( b \) for which \( (b_1, ub_2, d') = (b,r') = 1 \). Obviously this precisely coincides with \( d'^2 \mathcal{D}'_{d'}(\mathbf{m}'') \) when \( j = 0 \). The proof of Lemma 3.2 therefore shows that

\[
|D(p^k, 1)| \leq 4(k + 1)p^{4k + \min\{k, \frac{v_p(\delta(\mathbf{m}''))}{2}\}},
\]

where \( \delta(\mathbf{m}'') \) is given by (2.2).

Next suppose that \( \ell \geq 1 \) and \( k \geq 1 \). In this case

\[
D(d', r') = D(p^k, p^\ell) = \sum_{b_1 \pmod{p^k}} \mathcal{D}_{p^\ell}(\mathbf{m}''; p^\ell b_1, b_2).
\]

Recall the notation (3.3) for \( L, L', L'' \) and the subsequent definition of \( g \). It is clear that \( p \nmid g(p^\ell b_1, b_2) \), since \( p \nmid b_2 \). We may now trace through the analysis leading to (3.8), finding
that
\[ |D(p^k, p^\ell)| \leq p^{3(k+\ell)} \sum_{b \pmod{p^k}} |c_{p^{k+\ell}}(q_{m^\nu}(p^\ell b))| \]
\[ \leq p^{4(k+\ell)} \left( q_{r_{m^\nu}}(p^k) + q_{r_{m^\nu}}(p^{k-1}) \right), \]
via (3.5), where \( r_{m^\nu}(x) = p^{-\ell}q_{m^\nu}(p^\ell x) \). Here we have observed that the right hand side is empty unless \( p^\ell | c_2 = Q_2^*(m'' \nu) \). But the discriminant of \( r_{m^\nu} \) is equal to \( \delta(m'') \). Hence Lemma 3.1 yields
\[ |D(p^k, p^\ell)| \leq 4p^{4(k+\ell)+\min\{k, \frac{\nu\delta(m'')}{2}\}}. \]
Combining this with (4.3), we readily arrive at the statement of Lemma 4.1 \[ \square \]

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