FACTORIZATION AND THE DRESSING METHOD
FOR THE GEL’FAND-DIKII HIERARCHY

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Abstract. The isospectral flows of an $n^{th}$ order linear scalar differential operator $L$ under the hypothesis that it possess a Baker-Akhiezer function were originally investigated by Segal and Wilson from the point of view of infinite dimensional Grassmanians, and the reduction of the KP hierarchy to the Gel’fand-Dikii hierarchy. The associated first order systems and their formal asymptotic solutions have a rich Lie algebraic structure which was investigated by Drinfeld and Sokolov. We investigate the matrix Riemann-Hilbert factorizations for these systems, and show that different factorizations lead respectively to the potential, modified, and ordinary Gel’fand-Dikii flows. Lie algebra decompositions (the Adler-Kostant-Symes method) are obtained for the modified and potential flows. For $n > 3$ the appropriate factorization for the Gel’fand-Dikii flows is not a group factorization, as would be expected; yet a modification of the dressing method still works. A direct proof, based on a Fredholm determinant associated with the factorization problem, is given that the potentials are meromorphic in $x$ and in the time variables. Potentials with Baker-Akhiezer functions include the multisoliton and rational solutions, as well as potentials in the scattering class with compactly supported scattering data. The latter are dense in the scattering class.

1. The Gel’fand-Dikii Hierarchies

Gel’fand and Dikii [GD] constructed a hierarchy of isospectral flows of the $n^{th}$ order scalar differential operator

$$L = \sum_{j=0}^{n} u_j(x) D^{n-j}; \quad D = \frac{1}{i} \frac{d}{dx}; \quad u_0 = 1, \quad u_1 = 0.$$ 

where $u_j = u_j(x), j > 2$, are elements of the Schwartz class $S(\mathbb{R})$. The flows are given by

$$\dot{L} = [L_{+}^{k/n}, L], \quad k = 1, 2, \ldots$$

(1.1)

where $k \neq 0 \mod n$, and $L_{+}^{k/n}$ denotes the differential part of $L^{k/n}$ considered as a pseudodifferential operator. We shall refer to the coefficients $u_j$ of $L$ as the potential. The coefficients of $L_{+}^{k/n}$ are obtained by solving formal recursion relations and are universal differential polynomials in the $\{u_j\}$.

The forward and inverse scattering problem for $L$ has been investigated for potentials in $S$, and the Gel’fand-Dikii flows for potentials in this class have been constructed by Beals [B], and Beals, Deift, and Tomei [BDT]. Action-angle variables for flows in $S$ were constructed in terms of the scattering data by Beals and

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Segal and Wilson [SW] investigated these flows under the ansatz that $L$ possess a wave function of Baker-Akhiezer type: $w(x, z) \exp\{ixz\}$, where $w$ is analytic at $z = \infty$ and takes the value 1 there. From now on, we denote this class of potentials by $B$. They analysed the flows from the point of view of infinite dimensional Grassmanians, and the reduction of the KP hierarchy to the Gel’fand-Dikii hierarchy.

The $n^{th}$ order scalar equations can be converted to $n \times n$ first order systems. These have a rich Lie algebraic structure (cf. Drinfeld and Sokolov [DS]), just as the Dirac equation rather than the Klein Gordon equation exhibits the internal symmetries of the electron. All reductions are gauge equivalent, but besides the Gel’fand-Dikii flows, one finds others, such as the modified and potential Gel’fand-Dikii flows. We analyze the matrix Riemann-Hilbert problems corresponding to the first order systems. For potentials in $B$, the formal asymptotic series in [DS] are actually convergent.

Given a Baker-Akhiezer function, it is easily seen that $L$ has a basis of such wave functions. On the other hand, there is always a basis of wave functions which are entire in $z$. Denoting the Wronskian of this basis by $\phi_-$ and that of the basis of Baker-Akhiezer functions by $\phi_+$ we form the quotient $g = \phi_+^{-1}\phi_-$. The matrices $\phi_\pm$ can be chosen in such a way that $g$ is a function only of $\lambda = z^n$. Conversely, appropriate Riemann-Hilbert factorizations of a given a matrix $g(\lambda)$ together with a standard argument (the so-called “dressing method”), yields a potential in $B$. Thus, while we cannot precisely characterize the class of potentials $B$, we nevertheless can construct them by solving a factorization problem.

All conversions of the $n^{th}$ order equation to a first order system are gauge equivalent, and the entries of $L$ are differential polynomials in the entries of the potential $q$ [DS]. The canonical conversion (by Wronskians) leads to the Gel’fand-Dikii flows themselves. The modified Gel’fand-Dikii flows are obtained by first factoring $L$ as a product of $n$ first order operators, and constructing a system of AKNS type. We investigate in particular the standard and modified Gel’fand-Dikii flows together with a third type we call the “potential” flows. All three are obtained by factoring exactly the same matrix $q$, but taking the factors in different infinite dimensional submanifolds. For the potential and modified flows, that submanifold is a subgroup and the components of the connections are obtained by Lie (or loop) algebra decompositions (à la Adler-Kostant-Symes). For the Gel’fand-Dikii flows themselves the submanifold is not a group for $n > 3$. A Lie algebra decomposition is not available, but the dressing argument need only be modified slightly.

Schilling [Sch] obtained a Lie algebra decomposition for the first order system of equations for $\phi_j = L_j^{(j-1)/n}v$, $j = 1, \ldots, n$, but did not consider the corresponding factorization problem and flows.

The hypothesis that $L$ possess a Baker-Akhiezer function is a very strong one, as observed in [SW]. If $q$ is rational in $x$ the wave functions are in general multiple valued in the punctured neighborhoods of the poles of $q$; but if $L$ has a Baker-Akhiezer function then the wave functions have trivial monodromy (see §2.3). Moreover, if $q$ is a priori defined only on some interval $I$ on the real line, and $L$ has a Baker-Akhiezer function for $x \in I$, then $q$ has a meromorphic extension to the entire complex plane (Corollary 3.1.7).

In the case $n = 2$, assuming, say, that $(1+|x|)u_2 \in L_1(\mathbb{R})$, the assumption implies that the reflection coefficient is compactly supported. In particular if $w$ is rational
in $z$, then the reflection coefficient vanishes, and we have a so-called “reflectionless potential.” Properly interpreted, the same holds for general $n$, so that in a sense the ansatz is a generalization of the notion of reflectionless potential. Segal and Wilson showed that the class $B$ includes the multi-soliton solutions, the rational solutions, and in general the “algebro-geometric” solutions of the KdV equation. We show in this paper that the class $B$ is actually somewhat larger, and in fact that $S_0 \cap B$ is dense in $S_0$ (Theorem 2.1.7). ($S_0$ is the class of potentials for which the scattering transform is well-defined in [BDT]).

We show, by arguments based on the Fredholm determinant of an operator of trace class associated with the factorization problem, that the potentials in $B$ are meromorphic in $x$ and in the time variables under the Gel’fand-Dikii flows. Such results were proved by Segal and Wilson [SW] using the theory of the $\tau$ function for the Kadomtsev-Petviashvili (KP) hierarchy, together with the reduction of the KP hierarchy to the Gel’fand-Dikii flows. Our proof is similar in spirit but requires considerably less machinery; furthermore, it can be extended to other systems in the Drinfeld-Sokolov hierarchy, as well as the $n \times n$ AKNS flows.

The non-vanishing of the Fredholm determinant for the system guarantees both a left and right factorization; whereas the non-vanishing of the $\tau$ function guarantees only a one-sided factorization.

The factorization technique used here provides an efficient computational algorithm for constructing the rational and multi-soliton solutions. We hope to take this up in a future paper.

The restricted class $B$ is distinct from the potentials for which the scattering transform of $L$ is well-defined. The “scattering class” of potentials is invariant under perturbations by functions in the Schwartz class $S$, whereas $B$ is not. On the other hand, $B$ includes the rational solutions, which, due to their slow decay, or to possible poles on the real axis, are not in the scattering class.

For generic potentials in the scattering class, the wave functions are meromorphic in sectors of the complex plane of angle $\pi/n$. The scattering class leads to a Riemann-Hilbert factorization problem on a set of rays $\Sigma$ running from the origin to infinity, [BDT] while potentials in $B$ are described by a factorization problem on a circle centered at the origin. This factorization problem is considerably simpler than that associated with the inverse scattering problem; moreover, it is closely tied to the infinite dimensional Lie Group/ Lie algebra structure associated with the first order systems [DS].

In §2 we develop the forward problem for potentials in $B$ by showing that the ansatz leads to a matrix factorization problem on a circle centered at the origin in the complex $\lambda$ plane.

In §3 we consider the inverse problem, formulated as a Riemann-Hilbert problem. The parameters $x$ and $t$ appear analytically in the factorization problem, and the solutions of the factorization problem are shown to be meromorphic in the parameters. The dressing method is used to construct a flat connection from the solutions of the factorization problem and show that the resulting connection potentials are meromorphic in $x$ and $t$. Similar considerations arise in the proof of the Painlevé property for solutions of isomonodromy deformation problems [Mal] (cf. also Miwa, who proved the Painlevé property using the $\tau$ function.)

In §4 we discuss the infinite dimensional Lie structure of the factorization problem, using gradings of the algebra introduced by Drinfeld and Sokolov. In §5 we prove that the three factorizations lead to the standard, modified, and potential
Gel’fand-Dikii flows.

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2. The forward problem

2.1 Reduction to a first order system. The scattering theory of \( L \) proceeds by converting the \( n \)th order scalar equation \( Lf = \lambda f \) to a first order system. Let \( \phi_j = D_j^{-1}f \) for \( j = 1, \ldots, n \). Then the column vector \( \phi = (\phi_1, \phi_2, \ldots, \phi_n)^\dagger \) satisfies the first order system

\[
(D - J_\lambda - q)\phi = 0, \tag{2.1.1}
\]

where

\[
J_\lambda = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
& \ddots & & & \\
0 & 0 & 0 & \ldots & 1 \\
\lambda & 0 & 0 & \ldots & 0
\end{pmatrix}, \quad q = -\begin{pmatrix}
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 \\
& & & & \\
0 & 0 & \ldots & 0 & 0 \\
u_n & u_{n-1} & \ldots & u_2 & 0
\end{pmatrix}.
\]

The linear space of all such \( q \) constitutes a special class of potentials, which we call the Gel’fand-Dikii potentials. For the rest of this section, \( q \) will denote a potential in this class. In general, we do not assume the entries of \( q \) are in the Schwartz class; but we say that \( q \in \mathcal{S}(\mathbb{R}) \) if each entry of \( q \) belongs to \( \mathcal{S} \).

Matrix solutions of (2.1.1) are necessarily Wronskians, the entries of the first row being scalar solutions of \( Lf = \lambda f \).

Let \( f(x, z) \) be a Baker-Akhiezer function for \( L \), \( \alpha_j, 1 \leq j \leq n \), the \( n \)th roots of unity, and let \( f_j(x, z) = f(x, \alpha_j z) \). The Wronskian matrix \( W = \|D^{k-1}f_j\|, 1 \leq k, j \leq n \), satisfies (2.1.1). From the asymptotic behavior of \( f \) we conclude

\[
W \sim \|D^{k-1}e^{ix\alpha_jz}\| = \Lambda_z e^{ixzJ_\alpha} = d(z)e^{ixzJ_\alpha}\Lambda_\alpha
\]

\[
z \rightarrow \infty
\]

where

\[
J_\alpha = \text{diag}(\alpha_1, \ldots, \alpha_n), \quad d(z) = \text{diag}(1, z, \ldots, z^{n-1}), \quad \Lambda_z = d(z)\Lambda_\alpha,
\]

and

\[
\Lambda_\alpha = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
\alpha_1 & \alpha_2 & \ldots & \alpha_n \\
\alpha_1^2 & \ldots & \alpha_n^2 \\
& \ddots & & & \\
\alpha_1^{n-1} & \ldots & \alpha_n^{n-1}
\end{pmatrix}.
\]
The matrix function \( \psi = d^{-1}W\Lambda^{-1}_\alpha \) satisfies

\[
D\psi = (zJ + q(z))\psi \quad \lim_{z \to \infty} \psi e^{-ixzJ} = 1 \quad (2.1.2)
\]

where \( q(z) = d^{-1}qd, \ d^{-1}J\lambda d = zJ \), and

\[
J = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ldots & 1 \\
1 & 0 & 0 & \ldots & 0
\end{pmatrix}.
\]

This proves:

**2.1.3 Theorem.** \( L \) has a Baker-Akhiezer wave function iff (2.1.2) has a fundamental solution of the form \( \psi = m(x, z)e^{ixzJ} \) where \( m(x, z) \) is analytic at \( z = \infty \) and tends to \( I \) there.

Since \( \text{tr}(zJ + q) = 0 \) it follows that \( \det \psi \), and hence \( \det m \) is constant in \( x \). Since \( m \to I \) as \( z \to \infty \), \( \det m = 1 \) at infinity. We may therefore renormalize it so that \( \det m \) is identically 1 in a neighborhood of infinity. From now on we shall refer to \( m \) as a Baker-Akhiezer function for (2.1.2).

We now discuss, briefly, the scattering data of \( L \) when \( q \in \mathcal{S}[BDT] \). Let \( \Sigma = \{ \xi : \text{Re} \, ix(\alpha_k - \alpha_j) = 0, \ j, k = 1, \ldots, n \} \), and denote the \( 2n \) sectors which complement \( \Sigma \) by \( \Omega_{\nu} \), \( \nu = 1, 2, \ldots, 2n \). Solutions of (2.1.2) are constructed in each \( \Omega_{\nu} \) for which the corresponding \( m_{\nu} \) satisfies

\[
Dm = [zJ, m] + q(z)m, \quad (2.1.4)
\]

and the “radiation conditions"

\[
\lim_{x \to -\infty} m(x, z) = I, \quad \sup_x |m(x, z)| < +\infty, \quad (2.1.5)
\]

where \( |\cdot| \) denotes the sum of the absolute values of the entries of the matrix \( m \). The wave function \( m_{\nu} \) is uniquely determined, except possibly at a bounded discrete set \( Z \), where it has poles. The points of \( Z \) may cluster only along the rays of \( \Sigma \). Away from those cluster points, \( m_{\nu} \) is continuous up to the boundary of \( \Omega_{\nu} \). For a generic set of potentials, denoted by \( \mathcal{S}_0 \), \( Z \) consists of a finite set of points. For these \( q \) the boundary values of \( m \) exist everywhere on \( \Sigma \setminus \{0\} \).

The scattering data consists of the multiplicative jumps (continuous component) of \( m \) across the rays of \( \Sigma \), together with its principal part (discrete component) at points in \( Z \). The continuous component of the scattering data is algebraically related, in the case \( n = 2 \), to the reflection coefficient. The multi-soliton potentials of \( L = D^2 + u \) are examples of potentials in \( \mathcal{B} \cap \mathcal{S}_0 \) with a trivial multiplicative jump. These are typically referred to as “reflectionless potentials”. We show below that for potentials in \( \mathcal{B} \cap \mathcal{S}_0 \) the multiplicative jumps are compactly supported. Thus \( \mathcal{B} \) is an extension of the class of potentials with compactly supported scattering data, and in particular contains the reflectionless potentials.

Let \( J \) denote the kernel of \( \text{ad} \, J \), and let \( \mathcal{O} \) denote a neighborhood of infinity in the complex plane.
2.1.6 Lemma. Let $O$ be a neighborhood of infinity in the complex plane, let $m_1$ and $m_2$ be two solutions of (2.1.4) which are analytic in $O \cap \Omega_\nu$, bounded for $x \in \mathbb{R}$ and, for all real $x$, tend to $I$ as $z \to \infty$ in $\Omega_\nu$. Then $m_1 = m_2 \delta_\nu(z)$, where $\delta_\nu$ takes its values in $J$, is analytic in $O \cap \Omega_\nu$, and tends to $I$ as $z$ tends to infinity along rays in the interior of $\Omega_\nu$. If $m_1$ and $m_2$ satisfy (2.1.4) for $z \in O$, and take the value $I$ at infinity, then $m_1 = m_2 \delta(z)$ in $O$ where $\delta(z)$ takes its values in $J$, is analytic in $O$, and takes the value $I$ at infinity.

Proof. The matrix $w = m_1^{-1}m_2$ satisfies $Dw = [zJ, w]$, hence $w = e^{izJ} \delta(z)e^{-izJ}$ for some matrix valued function $\delta$ which is independent of $x$ and analytic in $O \cap \Omega_\nu$. Since $w$ is bounded for $-\infty < x < \infty$ and tends to $I$ as $z$ tends to infinity, $\delta$ must have the required properties. This proves the first statement.

In the second case, $w$ is analytic at infinity, so has a Laurent expansion $\Sigma_j w_j(z)z^{-j}$ where the coefficients $w_j$ satisfy the recursion relations $[J, w_{j+1}] = d w_j/dx$, with $w_0 = I$. It follows that $[J, w_1] = 0$, so that $w_1 \in J$. Since $\text{ad} J$ is semi-simple, its kernel and range intersect only at zero. Hence $[J, w_2] = d w_1/dx$ implies that $w_1$ is independent of $x$ and $w_2 \in J$. By induction, all the coefficients $w_j$ are independent of $x$ and lie in $J$. □

We are now ready to prove

2.1.7 Theorem. In the following, let $q \in S$.

i. If the continuous component of the scattering data is trivial, and $Z$ is finite, then $L$ has a Baker-Akhiezer function for which $m$ is rational in $z$ and is bounded for all real $x$ for regular values of $z$.

ii. Conversely if $L$ has a Baker Akhiezer function for which $m$ is rational in $z$ and bounded for all real $x$ for regular values of $z$, then the continuous component of the scattering data is trivial.

iii. If the scattering data of $q$ is compactly supported, then $q \in B$: and conversely, if $L$ has a Baker-Akhiezer function for which $m$ is bounded in $x$ for regular values of $z$ then the scattering data is compactly supported.

iv. $S_0 \cap B$ is dense in $S_0$.

Proof. To prove i., define a sectionally meromorphic function $m$ by $m(x, z) = m_\nu(x, z)$ for $z \in \Omega_\nu$, where $m_\nu$ are the wave functions constructed in the forward scattering problem. Since $m$ has no jumps across $\Sigma \setminus 0$ and $Z$ is a finite set $m$ is meromorphic in $\mathbb{C}\setminus 0$. (It takes the value $I$ at infinity.) By Theorem 8.15 in [BDT], $z^{n-1}d(z)m(x, z)d^{-1}(z)$ has a finite limit at $z = 0$. Consequently, $m$ is a rational function of $z$ which is bounded in $x$ for regular values of $z$.

Conversely, let $m$ and $m_\nu$ denote the fundamental solutions of (2.1.4) obtained respectively from the Baker-Akhiezer function and the forward scattering problem. By the first statement in Lemma 2.1.6, $m = m_\nu \delta_\nu(z)$ for $z \in \Omega_\nu$, where $\delta_\nu$ takes its values in $J$ and is analytic in the intersection of $\Omega_\nu$ with a neighborhood of infinity. Since $m_\nu$ is continuous up to the boundary of $\Omega_\nu$,

$$m_\nu^{-1}m_{\nu+1}(x, \xi) = \delta_\nu \delta_{\nu+1}^{-1}(\xi), \quad \xi \in \Sigma_\nu,$$

where $\Sigma_\nu = \Omega_\nu \cap \Omega_{\nu+1}$. Since $\text{ad} J$ is a derivation, $s_\nu(\xi) = \delta_\nu \delta_{\nu+1}^{-1} \in J$.

We wish to show that each $s_\nu$ is the identity matrix. We can just as well work in a representation in which $J$ is diagonal; hence $J = \text{diag}(\alpha, \ldots, \alpha)$, where $\alpha$ is an eigenvalue, and

$$s_\nu(\xi) = \delta_\nu \delta_{\nu+1}^{-1} \in J.$$
denote the roots of unity. In that case, $s_\nu$ is also diagonal, and the above equation may be written as

$$m_{\nu+1}(x, \xi) = m_\nu(x, \xi)s_\nu(\xi),$$

or, since $s_\nu$ commutes with $J$,

$$e^{ix\xi J}m_{\nu+1}e^{-ix\xi J} = e^{ix\xi J}m_\nu e^{-ix\xi J}s_\nu(\xi) \quad (2.1.8)$$

For $\xi \in \Sigma$, let $\Pi_\xi$ denote the projection

$$\Pi_\xi e_{jk} = \begin{cases} e_{jk} & \text{if } Re i\xi(\alpha_j - \alpha_k) = 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $e_{jk}$ denotes the matrix with a 1 in the $jk$ place and zeroes elsewhere. We prove below that

$$\lim_{x \to -\infty} \Pi_\xi e^{ix\xi J}m_\nu e^{-ix\xi J} = I + v_\nu, \quad \kappa = \nu, \nu + 1, \quad (2.1.9)$$

where $v_\nu, v_{\nu+1}$ lie in the range of $\Pi_\xi$ and are respectively strictly upper and lower triangular in an appropriate ordering of the roots $\alpha_j$. From the definition of $\Pi_\xi$, we see that $\Pi_\xi a = (\Pi_\xi a)s$ for a diagonal matrix $s$. Applying $\Pi_\xi$ to (2.1.8) and taking the limit as $x \to -\infty$, we obtain

$$I + v_{\nu+1} = (I + v_\nu)s_\nu.$$

By the structure of $v_\nu$ and $v_{\nu+1}$ we must have $s_\nu = I$ and $v_\nu = v_{\nu+1} = 0$.

We establish (2.1.9) by considering an integral equation for the boundary values of the wave functions. For $z \in \Omega_\nu \setminus Z$, $m_\nu$ satisfies the Fredholm integral equation [B]

$$m_\nu(x, z) = I + \int_{-\infty}^{x} (\Pi^-_z + \Pi_0) e^{iz(x-y)J}(qm_\nu)e^{-iz(x-y)J} dy$$

$$- \int_{x}^{\infty} \Pi^+_z e^{iz(x-y)J}(qm_\nu)e^{-iz(x-y)J} dy,$$

where $\Pi_0$ projects onto the diagonal matrices and $\Pi^\pm_z$ denote the projections

$$\Pi^\pm_z e_{jk} = \begin{cases} e_{jk} & \text{if } \pm Re iz(\alpha_j - \alpha_k) > 0, \\ 0 & \text{if } \pm Re iz(\alpha_j - \alpha_k) < 0. \end{cases}$$

An ordering of the $\alpha_j$ may be chosen so that, for $z \in \Omega_\nu$, $\Pi^\pm_z$ are the projections $U_\pm$ onto the strictly upper and lower triangular matrices. Letting $z$ tend to $\xi \in \Sigma_\nu$ from $\Omega_\nu$, we obtain the following integral equation for the boundary values of $m_\nu$:

$$m_\nu(x, \xi) = I + \int_{-\infty}^{x} (U_+ + \Pi_0) e^{i\xi(x-y)J}(qm_\nu)e^{-i\xi(x-y)J} dy$$

$$- \int_{x}^{\infty} U_- e^{i\xi(x-y)J}(qm_\nu)e^{-i\xi(x-y)J} dy. \quad (2.1.10)$$
Conjugating (2.1.10) by $e^{-ix\xi J}$, applying $\Pi_\xi$, and taking the limit as $x \to -\infty$, we obtain
\[
\lim_{x \to -\infty} \Pi_\xi e^{-ix\xi J} m_\nu e^{ix\xi J} = I - \int_{-\infty}^{\infty} \Pi_\xi U_+ e^{-i\xi y J} (qm_\nu) e^{i\xi y J} dy = I + v_\nu.
\]

For $z \in \Omega_{\nu+1}$, $\Pi_\xi^+\Pi_\xi^-$ no longer projects onto the upper triangular matrices; however, $\Pi_\xi \Pi_\xi^+ = \Pi_\xi U_-$. Hence, repeating the previous arguments, we obtain
\[
\lim_{x \to -\infty} \Pi_\xi e^{-ix\xi J} m_{\nu+1} e^{ix\xi J} = I - \int_{-\infty}^{\infty} \Pi_\xi U_- e^{-i\xi y J} (qm_{\nu+1}) e^{i\xi y J} dy = I + v_{\nu+1}.
\]

This establishes (2.1.9), hence ii.

If the scattering data is trivial outside some compact set, then there is some open neighborhood $O$ of infinity on which $m_\nu = m_{\nu+1}$ in $\Sigma_\nu \cap O$, and we obtain a Baker-Akhiezer function by setting $m = m_\nu$ for $z \in \Omega_\nu \cap O$. Hence $q \in \mathcal{B}$, and iii. is established.

The result in iv) is a consequence of the fact that potentials with compactly supported scattering data are dense in $S_0$ in the Schwartz topology. This fact is stated without proof in [BDT] (cf. p. 154). A rough sketch of the proof runs like this. Let $q \in S_0$, and denote its scattering data by $v$. (We also include the discrete data in $v$.) On $\Sigma$ each entry of $v - I$ belongs to $S(\Sigma)$ ([BDT], Theorem 13.1). In particular, each entry of $v - I$ is rapidly decreasing at infinity and has an asymptotic expansion in positive powers of $z$ as $z \to 0$ along each ray $\Sigma_\nu$ of $\Sigma$. We approximate $v$ by smooth data $v_n$ such that $v_n = v$ in $|z| \leq n$, and $v_n - I$ vanishes identically in the region $|z| \geq n + 1$. Since the inverse transform is defined in a sufficiently small neighborhood of $v$, it is defined for each $v_n$ for sufficiently large $n$; and the corresponding potentials $q_n$ belong to $\mathcal{B}$. The inverse transform, as constructed in [BDT], is the composition of two continuous processes: a rational approximation of the scattering data; and the inversion of a Fredholm operator close to the identity. Hence the inverse transform is (locally) continuous from $L_2(\Sigma)$ to $L_2(\mathbb{R})$. Using the estimates in [BDT], one could also prove, with some effort, that the corresponding potentials $q_n$ converge to $q$ in the topology of $S$. □

For potentials in $S$ the $m_\nu$ are uniquely determined by their normalization as $x \to -\infty$. No such unique normalization is possible for the Baker-Akhiezer wave functions; but we do obtain:

2.1.11 Theorem. If $q \in \mathcal{B}$ then a fundamental solution of (2.1.2) can be constructed which satisfies the symmetry condition $\psi(x, \alpha z) = d^{-1}(\alpha) \psi(x, z) d(\alpha)$, where $\alpha$ is a primitive $n^{th}$ root of unity.

Remark. This symmetry condition does not uniquely determine the wave function.

Proof. First note that (2.1.4) is invariant under right multiplication by holomorphic $\delta(z) \in \mathcal{J}$ which tend to $I$ as $z$ tends to $\infty$. These matrices have the representation
\[
\delta(z) = \sum_{j=0}^{n-1} \delta_j(z) J^j, \quad \delta_j(z) \to \delta_{j0} \text{ as } z \to \infty.
\]
Since \(d^{-1}(\alpha) z J d(\alpha) = \alpha z J\) and \(d^{-1}(\alpha) q(z) d(\alpha) = q(\alpha z),\) (2.1.4) is also satisfied by \(d(\alpha) m(x, \alpha z) d^{-1}(\alpha).\) By the second statement of Lemma 2.1.6 there is a \(\delta(z)\) with values in \(J\) such that

\[d(\alpha) m(x, \alpha z) d^{-1}(\alpha) = m(x, \alpha z) \delta(z).\]

Let \(q\) be regular at some point \(x_0,\) and let \(a(\alpha) = m(x_0, \alpha z).\) Then

\[a(\alpha z) = d^{-1}(\alpha) a(z) \delta(z) d(\alpha).\]  

(2.1.12)

The function \(m\) is uniquely determined by its values at \(x_0.\) Moreover, \(m\mu\) is also a solution of (2.1.4) for any holomorphic matrix \(\mu(z)\) with values in \(J.\) For the proof of Theorem 2.1.11 it therefore suffices to find a matrix \(\mu(z)\) with values in \(J\) such that

\[d(\alpha) a(\alpha z) \mu(\alpha z) d^{-1}(\alpha) = a(z) \mu(z);\]

for it then follows that \(m(x, z) \mu\) satisfies the required symmetry condition. Using (2.1.12) we may write this condition as

\[\delta(z) = \mu(z) d(\alpha) \mu^{-1}(\alpha z) d^{-1}(\alpha)\]  

(2.1.13)

where

\[\delta, \mu \in J, \quad \delta(\infty) = I, \quad \det \delta = 1.\]

We represent \(\delta\) and \(\mu\) in the form

\[\delta(z) = \exp\{\sum_{j=1}^{n-1} \eta_j(z) J^j\}, \quad \mu(z) = \exp\{\sum_{j=1}^{n-1} \theta_j(z) J^j\}.\]

Iterating (2.1.13) \(n\) times, we obtain

\[I = \delta(z) d(\alpha) \delta(\alpha z) \ldots \delta(\alpha^{n-1} z) d(\alpha).\]

Bringing each of the \(n\) matrices \(d(\alpha)\) all the way to the left in the above expression (and noting that \(J d(\alpha) = d(\alpha) \alpha J\)), we find

\[I = \exp\{\sum_{j=1}^{n-1} \sum_{k=0}^{n-1} (\eta_j(\alpha^k z) \alpha^{-k}) J^j\},\]

and therefore

\[\sum_{k=0}^{n-1} \eta_j(\alpha^k z) \alpha^{-k} = 0, \quad j = 1, \ldots, n - 1\]

These equations represent a set of constraints on the \(\eta_j,\) which are given functions. From (2.1.13) we find by similar arguments,

\[\eta_j(z) = \theta_j(z) - \alpha^{-j} \theta_j(\alpha z).\]  

(2.1.14)

We seek a solution of (2.1.14) in the form

\[\theta_j(z) = \sum_{k=0}^{n-1} v_k \eta_j(\alpha^k z), \quad j = 1, \ldots, n - 1,\]
where the coefficients $v_k$ (more precisely $v_{j,k}$) are to be determined. The three preceding equations reduce to the linear system

\[
\begin{pmatrix}
1 & 0 & \ldots & \alpha^{-j} \\
-\alpha^{-j} & 1 & 0 & \ldots & \alpha^{-(j+1)} \\
0 & -\alpha^{-j} & 1 & 0 & \ldots & \alpha^{-(j+2)} \\
0 & \ldots & -\alpha^{-j} & 1 & +\alpha^{-(j-2)} & \ldots & 1
\end{pmatrix}
\begin{pmatrix}
v_0 \\
v_1 \\
\vdots \\
v_{n-2} \\
v_{n-1}
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix}.
\]

The coefficient matrix on the left is invertible, as can be seen by reducing it to row echelon form. Adding the first row to the second multiplied by $\alpha^j$, etc. one arrives at a matrix in row echelon form whose diagonal is $1, \alpha^j, \alpha^{2j}, \ldots, -\alpha^{-(j-1)}(1-\alpha^{-j})$. The last entry is non-zero, since $\alpha$ is a primitive $n^{th}$ root of unity. □

### 2.2 Properties of the wave function.

We next analyze the structure of the wave function $m$ constructed in theorem 2.1.11. From the symmetry condition

\[\psi(x,\alpha z) = d^{-1}(\alpha)\psi(x,z)d(\alpha)\] (2.2.1)

and the expansion

\[m(x,z) = \sum_{j=0}^{\infty} \frac{m_j}{z^j}, \quad m_0 = I\] (2.2.2)

we deduce

\[d^{-1}(\alpha)m_jd(\alpha) = \alpha^{-j}m_j, \quad j \geq 0.\] (2.2.3)

We claim that

\[m_j(x) = E_j(x)J^{-j}, \quad E_j \text{ diagonal, } \quad E_0 = I.\] (2.2.4)

First, from the identity $e_{ij} = e_{ii}J^{j-i}$ we see that any $A \in M_n(\mathbb{C})$, the space of $n \times n$ matrices with complex entries, can be represented as

\[A = \sum_{k=0}^{n-1} E_k J^k, \quad E_k \text{ diagonal.}\] (2.2.5)

Furthermore,

\[d^{-1}(\alpha)J^kd(\alpha) = (\alpha J)^k, \quad k \in \mathbb{Z}.\] (2.2.6)

Expanding $m_j$ in the form (2.2.5) and using (2.2.6) we see that $E_k = 0$ for $j \neq k \mod n$. This establishes (2.2.4).

From now on we often suppress the $x$ dependence of the $E_j$. Further restrictions are placed on them by the requirement that $m$ satisfy (2.1.4). First we note that

\[q(z) = \sum_{j=1}^{n-1} Q_{j+1}(zJ)^{-j}, \quad Q_j = \text{diag}(0,0,\ldots,-u_j).\] (2.2.7)

Substituting (2.2.2) into (2.1.4) we obtain a sequence of recursion relations for the $m_j = E_j J^{-j}$. Straightforward calculations lead to

\[(\sigma - I)E_1 = 0,\] (2.2.8)

\[(\sigma - I)E_{j+1} = DE_j - \sum_{k=1}^{j} Q_{k+1} \sigma^{-k}(E_{j-k}), \quad j \geq 1\]

where we set $Q_k = 0$ for $k > n$ and $\sigma(E_j) = JE_j J^{-1}$. We shall need the following simple facts about $\sigma$.
2.2.9 Lemma. Let $\mathcal{D}$ denote the algebra of diagonal $n \times n$ matrices and let $\mathcal{D}_0$ be the subspace of traceless such matrices. Then

i) $\sigma \in \text{Aut} \mathcal{D}$

ii) $\sigma^n = I$

iii) $\text{Range} (\sigma - I) = \mathcal{D}_0, \quad \text{Ker}(\sigma - I) = \mathbb{C} I$

iv) On $\mathcal{D}_0$ and for $n > 2$,

$$(\sigma - I)^{-1} = \frac{1}{n}(\sigma^2 + 2\sigma^3 + \cdots + (n - 2)\sigma^{n-1} - I).$$

Proof. Items (i-iii) are immediate from the definition of $\sigma$, while (iv) follows from the identity

$I + \sigma + \sigma^2 + \cdots + \sigma^{n-1} = 0$ on $\mathcal{D}_0$.

2.2.10 Theorem. Let $D_1$ denote the matrix diag $(1, 1, \ldots, 1, 1 - n)$. Then

$$E_0 = I, \quad E_1 = \omega_1 I,$$

$$E_j = \omega_j I + \sum_{k=1}^{j-1} (D^k \omega_{j-k})(\sigma - I)^{-k} D_1, \quad j \geq 2.$$

The $\omega_j$ are determined from the coefficients $u_j$ of $L$ by a sequence of recursion relations

$$u_j = -nD\omega_{j-1} + h_j \quad 2 \leq j \leq n$$

where $h_2 = 0$ and $h_j$ is a polynomial in $\omega_1, \ldots, \omega_{j-2}$ and their derivatives for $j > 2$.

Proof. By (2.2.8) $E_1$ is a scalar multiple of the identity, so $E_1 = \omega_1 I$. By the second relation in (2.2.8)

$$(\sigma - I)E_2 = DE_1 - Q_2 I.$$ 

This equation is solvable provided that $\text{tr} (DE_1 - Q_2) = nD\omega_1 + u_2 = 0$; in that case

$$DE_1 - Q_2 = D\omega_1 I - nD\omega_1 e_{nn} = D\omega_1 (I - ne_{nn}) = (D\omega_1)D_1,$$

and

$$E_2 = \omega_2 I + (D\omega_1)(\sigma - I)^{-1} D_1.$$ 

Assume the result holds for $j \geq 2$. Since $(\sigma - I)^{-1}$ maps $\mathcal{D}_0$ to itself, $\text{tr}DE_j = \text{tr}D\omega_j I = nD\omega_j$, hence the recursion relation (2.2.8) for $E_{j+1}$ is solvable provided

$$nD\omega_j - \text{tr} \sum_{k=1}^{j} Q_{k+1}\sigma^{-k}(E_{j-k}) = 0. \quad (2.2.11)$$

It follows from (2.2.11) that

$$\sum_{k=1}^{j} Q_{k+1}\sigma^{-k}(E_{j-k}) = e_{nn} \text{tr} \sum_{k=1}^{j} Q_{k+1}\sigma^{-k}(E_{j-k}) = n(D\omega_j)e_{nn},$$
so that (2.2.8) may be written as

\[(\sigma - I)E_{j+1} = D\omega_j I + \sum_{k=1}^{j-1} D^{k+1} \omega_{j-k}(\sigma - I)^{-k} D_1 - \sum_{k=1}^{j} Q_{k+1} \sigma^{-k}(E_{j-k})\]

\[= (D\omega_j) D_1 + \sum_{k=1}^{j-1} D^{k+1} \omega_{j-k}(\sigma - I)^{-k} D_1\]

\[= \sum_{k=1}^{j} D^k \omega_{j+1-k}(\sigma - I)^{-k+1} D_1.\]

Therefore

\[E_{j+1} = \omega_{j+1} I + \sum_{k=1}^{j} D^k \omega_{j+1-k}(\sigma - I)^{-k} D_1.\]

For \(2 \leq j \leq n-1\) (2.2.11) implies

\[n D\omega_j + \sum_{k=1}^{j} u_{k+1} \text{Tr}[e_{nn} \sigma^{-k}(E_{j-k})] = n D\omega_j + u_{j+1} + \sum_{k=1}^{j-1} u_{k+1} \text{Tr}[e_{nn} \sigma^{-k}(E_{j-k})] = 0.\]

This yields the appropriate recursion relation for \(\omega_j\) and the induction step is completed. \(\square\)

Now we are ready to state the main theorem of this section.

2.2.12 Theorem. Let \(\tilde{m} = d(z) md^{-1}(z)\). Then

i) \[\tilde{m} = \sum_{j=0}^{\infty} E_j J_{\lambda}^{-j},\]

where the \(E_j\)’s are given in Theorem 2.2.10;

ii) \(\tilde{m}(x, \alpha z) = \tilde{m}(x, z)\), hence \(\tilde{m} = \tilde{m}(x, \lambda)\);

iii) \(\tilde{m}\) is regular at \(\lambda = \infty\) and

\[\lim_{\lambda \to \infty} \tilde{m}(x, \lambda) = I + \ell,\]

where \(\ell\) is a strictly lower triangular matrix.

iv) \(\det \tilde{m} = 1\)

Proof. Observe that \(d(z) z J d^{-1}(z) = J_{\lambda}\) and \(E_j\) diagonal imply that \(d(z) E_j(z J)^{-j} d^{-1}(z) = E_j J_{\lambda}^{-j}\). This proves i) and ii) follows immediately. By the definition

\[\tilde{m} = dmd^{-1} = \sum_{j=0}^{\infty} E_j J_{\lambda}^{-j} = \sum_{j=0}^{n-1} E_j J_{\lambda}^{-j} + O(|z|^{-n}).\]
Note however that
\[
\lim_{z \to \infty} J^{-1}_\lambda = \begin{pmatrix} 0 & 0 & \ldots & 0 \\
1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & 1 \\
\end{pmatrix} \equiv K.
\]
Thus
\[
\tilde{m}(x, \infty) = \sum_{j=0}^{n-1} E_j K^j = I + \ell
\]
where \(\ell\) is strictly lower triangular. Finally \(\det \tilde{m} = \det m = 1\). \(\square\)

2.3 Factorization for the Gelfand-Dikii flows..

The existence of a Baker-Akhiezer function for \(L\) leads to a factorization problem on a circle centered at the origin in the \(\lambda\) plane; specifically:

2.3.1 Theorem. Let \(q \in B\). Then there exist wave functions \(\phi_\pm(x, \lambda)\) of (2.1.1) such that \(\phi_+\) is entire and \(\phi_- = \tilde{m}e^{ixJ\lambda}\), where \(\tilde{m}(x, \infty) = I + \ell\), \(\ell\) strictly lower triangular. \(\phi_\pm\) satisfy the factorization problem
\[
g(\lambda) = \phi_-^{-1}(x, \lambda)\phi_+(x, \lambda),
\]
where \(g\) is analytic in \(\rho < |\lambda| < +\infty\) for some \(\rho > 0\). By a suitable rescaling of the potentials, we may assume that \(\rho < 1\).

Proof. The wave function
\[
\phi_- = d(z)\psi_-^{-1}(z) = d(z)m_-(x, z)d^{-1}(s)e^{ixJ\lambda} = \tilde{m}(x, \lambda)e^{ixJ\lambda}
\]
satisfies (2.1.1) and, by Theorem 2.2.12, has all the required properties. Since (2.1.1) depends analytically on \(\lambda\), there is a unique solution \(\phi_+(x, \lambda)\), which is entire in \(\lambda\) and satisfies the initial value condition \(\phi_+(0, \lambda) = I\). (We assume, for convenience, that \(q\) is regular at \(x = 0\).) Since \(\phi_\pm\) both satisfy (2.1.1), the matrix \(\phi_-^{-1}\phi_+\) is independent of \(x\).

Given a Baker-Akhiezer function for the potentials \(u_j(x)\) which is analytic in the exterior of some circle in the \(\lambda\)-plane, we define rescaled potentials by \(u_j^s(x) = s^j u_j(sx), \ s > 0\). If \(q^s(x)\) denotes the matrix of rescaled potentials, then \(\phi^s = d(s)\phi(x, s^{-n}\lambda)d^{-1}(s)\) satisfies \((D - J\lambda - q^s(x))\phi^s = 0\). Replacing the original wave functions by the rescaled ones, we obtain a factorization problem on any circle in the \(\lambda\) plane we choose. In particular, we may assume (2.3.2) holds for some \(\rho < 1\). \(\square\)

The isospectral flows of \(L\) are compatibility conditions for the pair of scalar equations
\[
Lf = \lambda f, \quad D_t f = (L_+^{k/n})f, \quad D_t = -i\partial / \partial t.
\]
By (2.1.1), all \(x\)-derivatives of \(f\) can be written as linear combinations of the first \(n - 1\) derivatives of \(f\) with coefficients which are differential polynomials in the \(u_j\). The equation for the time evolution of \(f\) is therefore equivalent to a first order system of the form \((D_t - v)W = 0\), where \(W = ||D^{k-1}f_j||\) is the Wronskian introduced in \(\S 2.1\). Since the coefficients of \(L_+^{k/n}\) are differential polynomials in
the coefficients of $L$, $v$ also possesses such a structure. When $q = 0$ we find that $v = J^k_\lambda$, so the Gel’fand-Dikii flows are zero curvature conditions for the connection $D - J_\lambda - q$, $D_t - J^k_\lambda - B_{k,n}$, $B_{k,n}$ denoting a matrix valued differential polynomial in $u_j$ of degree $k - 1$ in $\lambda$. Let $q(x, t) \in B$ evolve according to the $(n, k)$ Gel’fand-Dikii flow. The wave functions $\phi_\pm$ of Theorem 2.3.1 satisfy

$$(D - J_\lambda - q)\phi_\pm = 0, \quad (D_t - J^k_\lambda - B_{k,n})\phi_\pm = 0.$$  

They may be written as

$$\phi_\pm = \tilde{m}_\pm(x, t, \lambda)e^{i(xJ_\lambda + tJ^k_\lambda)},$$

where $\tilde{m}_\pm$ is an entire function of $\lambda$, and $\tilde{m}_-$ satisfies the conditions of Theorem 2.1.12. From Theorem 2.3.1 we have:

**2.3.3 Corollary.** The functions $\tilde{m}_\pm$ satisfy the factorization problem

$$e^{i(xJ_\lambda + tJ^k_\lambda)}g(\lambda)e^{-i(xJ_\lambda + tJ^k_\lambda)} = \tilde{m}_-^{-1}\tilde{m}_+,$$  

(2.3.4)

where again $g$ is analytic in an exterior domain containing the unit circle, except possibly at $\lambda = \infty$.

Later, in Theorem 4.3.4, we shall see that the factorization given in (2.3.4) is unique. Using this, we may prove:

**2.3.4 Theorem.** Suppose $q$ is rational in $x$ in a domain $U$ and in addition is a Baker-Akhiezer potential. Then the monodromy of the wave functions is trivial.

**Proof.** Suppose

$$e^{xJ_\lambda}g e^{-xJ_\lambda} = m_-^{-1}m_+$$

and suppose $\tilde{m}_\pm$ denote the analytic continuations of $m_\pm$ around a pole of $q$. Since the process of analytic continuation is continuous, we must have

$$e^{xJ_\lambda}g e^{-xJ_\lambda} = \tilde{m}_-^{-1}\tilde{m}_+.$$  

By the uniqueness of the decomposition (Theorem 4.3.4 below), we must have $\tilde{m}_\pm = m_\pm$. □

### 3. The Inverse Problem

#### 3.1 Factorization on the circle.

In §2 we derived the factorization problem (2.3.4) on the unit circle, beginning with $L$ and its isospectral flows for potentials in $B$. The inverse problem consists of solving a factorization problem and constructing the flat connection from the factors $\tilde{m}_\pm$. We begin with a brief summary of the solution of matrix factorization problems on the unit circle. (cf. [GK].)

Let $G$ denote the loop group

$$G = \{ g : g = g(\lambda) \in M_n(C), \det g = 1, g \text{ analytic in } \rho < |\lambda| < +\infty \}.$$
together with its subgroups (under standard matrix multiplication)

\[ G_+ = \{ g \in G : g = \sum_{k \geq 0} g_k \lambda^k \}; \]

\[ G_- = \{ g \in G : g = \sum_{k \leq 0} g_k \lambda^k, \quad g_0 = I \} \]

We always assume \( \rho < 1 \), so that the domain of holomorphy of \( g \) contains the unit circle. In practice, we use \( g \) to signify either the function \( g(\lambda) \) or the sequence of Fourier coefficients \( \{ g_k \} \). For \( g \in G \)

\[ \sum_k |g_k|s^k < +\infty \]

for all \( s > \rho \), where \( |g_k| \) denotes the sum of the absolute values of the entries of \( g_k \). Therefore for any \( s > \rho \), \( G \) is a subset of the Hilbert space \( H^s \) defined by

\[ H^s = \{ g : ||g||^2 = \frac{1}{2\pi} \int_0^{2\pi} \text{tr} [g(se^{i\theta})(g(se^{i\theta})^*)] \, d\theta = \frac{1}{2\pi} \sum_k s^{2k} \text{tr} \ g_k g_k^* < +\infty \}. \]

We define

\[ H^s_+ = \{ g \in H^s : g_k = 0 \text{ for } k < 0 \}, \quad H^s_- = \{ g \in H^s : g_k = 0 \text{ for } k \geq 0 \}. \]

Then \( H^s_\pm \) are the Hilbert spaces of boundary values of matrix-valued functions holomorphic in the interior and exterior of the unit circle. From now on we sometimes drop the superscript \( s \). Moreover, if \( \phi_- \in H^s_- \) for all \( s > \rho \) and \( \det \phi_- = 1 \) then \( \phi_- \in G_- \).

We associate a Fredholm determinant with the factorization problem. (Segal and Wilson obtain the \( \tau \) function as a determinant associated with a Riemann-Hilbert problem.) The Fredholm determinant is defined for operators of the form \( I + A \) where \( A \) is of trace class on a Hilbert space \( H \) [Si]. The operators of trace class on \( H \) form a complex Banach space \( I_1 \) with norm

\[ ||A||_1 = \text{tr} \sqrt{AA^*}. \]

The nonlinear functional \( \det(I + A) \) is well defined and differentiable on \( I_1 \). Under the identification of \( I_1^* \) with the space of bounded linear operators on \( H \), its Fréchet derivative is

\[ (I + A)^{-1} \det(I + A). \]

The operator

\[ D(A) = -A(I + A)^{-1} \det(I + A) \]

is the first Fredholm minor, and

\[ (I + A)^{-1} = I + \frac{D(A)}{\det(I + A)}. \]

Since \( \det(I + A) \) is differentiable on the complex Banach space \( I_1 \) it is analytic there, and therefore so is \( D(A) \).
Theorem 3.1.1. If \( A(x) \in \mathcal{J}_1 \) is holomorphic in a complex variable \( x \), and \( \det(I + A(x)) \neq 0 \) for at least one \( x \neq x_0 \), then \( (I + A(x))^{-1} \) is meromorphic in \( x \), with poles at the zeroes of \( \det(I + A(x)) \).

In the course of the solution of the factorization problem we need to consider operators on \( H_\pm \) of the following type:

\[
C_g^+ f = \sum_{k \geq 0} f_k g_{j-k}, \quad j \geq 0; \quad C_g^- f = \sum_{k < 0} f_k g_{j-k}, \quad j < 0.
\]

3.1.2. Theorem. For \( g \in G \), \( C_g^\pm \) are Fredholm operators on \( H_\pm \).

Proof. Let \( K_g \) denote the operation defined on \( H \) by \( K_g f = fg \), where \( g \in G \) and \( f \in H \). The equation \( K_g f = h \) can be written as the infinite system of equations

\[
\sum_k f_k g_{j-k} = h_j.
\]

This system is invertible for \( g \in G \) since \( g \) is bounded and \( \det g(\lambda) = 1 \) on the unit circle, and \( K_g \) has a bounded inverse on \( H \).

We rewrite (3.1.3) as

\[
\sum_{k \geq 0} f_k^+ g_{j-k} + \sum_{k > 0} f_k^- g_{j+k} = h_j^+, \quad j \geq 0,
\]

\[
\sum_{k \geq 0} f_k^+ g_{j-k} - \sum_{k > 0} f_k^- g_{j-k} = h_j^-, \quad j > 0
\]

where \( f_k^+ = f_k, \quad k \geq 0; \quad f_k^- = f_{-k}, \quad k > 0; \) etc.

Denoting \( \{f^+, f^-\} \) as well by \( f \), we introduce the operators

\[
B_g f = \left( \sum_{k=0}^{\infty} f_k^+ g_{j-k}, \quad \sum_{k>0} f_k^- g_{j-k} \right), \quad T_g f = \left( \sum_{k>0} f_k^- g_{j+k}, \quad \sum_{k=0}^{\infty} f_k^+ g_{(k+j)} \right).
\]

By the decomposition (3.1.4), \( K_g = T_g + B_g \), and \( B_g = K_g - T_g = K_g(I - K_g^{-1}T_g) \). The operators \( C_g^\pm \) are precisely the components of \( B_g \). Since \( K_g \) is invertible, it suffices to show that \( T_g \) is compact.

In fact, each of its components is of trace class. Consider for example, the operator on \( H_+ \) defined by \( T f = \sum_k f_k g_{k+j} \). Now an infinite matrix \( A = ||a_{jk}|| \) is of trace class if

\[
\sum_{j,k \geq 0} |a_{jk}| < +\infty.
\]

In fact, (cf. [Si]) there is a partial isometry \( U \) such that \( |A| = U^* A \), where \( |A| \) is the positive definite operator \((AA^*)^{1/2}\). Since \( ||U^* \psi|| = ||\psi|| \), all entries of \( U^* \) are bounded by one in absolute value. Consequently

\[
||A||_1 = tr|A| = \sum U_{jk}^* a_{kj} \leq \sum |a_{jk}|.
\]
Thus it suffices to show that \( \Sigma_{j,k} |g_{j+k}| < +\infty \) where \( |g_k| \) denotes the sum of the absolute values of \( g_k \). We have
\[
\sum_{j,k \geq 0} |g_{j+k}| = \sum_{n=0}^{\infty} (n+1)|g_n|;
\]
and the sum on the right converges since \( g \) is analytic in \(|\lambda| > 1\).

The second component of \( T_g \) is also of trace class. A similar argument leads to consideration of the series
\[
\sum_{n \leq 0} |g_n|(1 + |n|);
\]
This sum converges since \( g \) is analytic for \( \rho < |\lambda| < +\infty \).

□

Since \( T_g \) is of trace class, the following is immediate:

**3.1.5. Corollary.** The operators \( C_g^\pm \) are invertible iff \( \det (I - K_g^{-1}T_g) \neq 0 \). If \( g \) is holomorphic in \( x \) then \( (C_g^\pm)^{-1} \) are meromorphic in \( x \).

Let
\[
G^t = \{g : g = g(\lambda) \in M_n(\mathbb{C}), \det g = 1, g \text{ analytic in } t < |\lambda| < +\infty\};
\]
and define \( H^t \), \( H^t_\pm \) and \( G^- \) accordingly. We are now ready to prove:

**3.1.6. Theorem.** Let \( g \in G^\rho \). The factorization problem
\[
\phi_- g = \phi_+, \quad g \in G, \quad \phi_\pm \in G^t_\pm, \quad t > \rho,
\]
has a unique solution provided \( \det(I - K_g^{-1}T_g) \neq 0 \). Moreover, if \( g \) is analytic in a parameter \( x \), the factors \( \phi_\pm \) are meromorphic in \( x \), provided that for at least one value of \( x \), \( \det(I - K_g^{-1}T_g) \neq 0 \).

**Proof.** Writing
\[
\phi_- = I + \sum_{k<0} \phi_k \lambda^k, \quad \phi_+ = \sum_{k \geq 0} \phi_k \lambda^k,
\]
and substituting these expressions into our factorization problem, we get
\[
\sum_k g_k \lambda^k + \sum_k \left( \sum_{j<0} \phi_j g_{k-j} \right) \lambda^k = \sum_{k \geq 0} \phi_k \lambda^k.
\]
For \( k < 0 \) we get the system of equations
\[
g_k + \sum_{j<0} \phi_j g_{k-j} = 0, \quad k < 0.
\]
By Corollary 3.1.5 this system has a unique solution in \( H^t_- \), denoted by \( \phi_- \), provided that \( \det(I - K_g^{-1}T_g) \) does not vanish. Setting
\[
\phi_k = g_k + \sum \phi_j g_{k-j}, \quad k \geq 0,
\]
we obtain the required factorization provided we can show that \( \phi_{\pm} \) belong to \( G^t_{\pm} \) respectively. Since \( \phi_- \) and \( g \) are analytic in \( |\lambda| > t \), so is \( \phi_+ = \phi_- g \). The factorization shows that there are no negative powers of \( \lambda \) in the Laurent expansion for \( \phi_+ \). Therefore \( \phi_+ \) is entire.

Note that \( \det \phi_+ = \det \phi_- \) on \( |\lambda| = 1 \) so the sectionally holomorphic function given by

\[
\begin{align*}
\det \phi_- & , \quad |\lambda| > 1 \\
\det \phi_+ & , \quad |\lambda| < 1
\end{align*}
\]

is analytic on the extended complex plane. Therefore it is a constant. Since \( \phi_- (\infty) = I \), it is identically 1, and \( \det \phi_+ \equiv 1 \). Hence \( \phi_+ \) is entire and \( \phi_{\pm} \in G^t_{\pm} \). Since \( \phi_- \) does not depend on \( t \) it belongs to \( G^t_- \) for all \( t > \rho \), hence \( \phi_- \in G_- \).

If \( g \) depends analytically on \( x \), then the \( K^{-1}T_g \) is an analytic operator valued function of \( x \), and the solution \( \gamma \) of (3.1.7) is meromorphic in \( x \). If \( g \) is entire, then the poles of \( \phi_- \) do not cluster in the finite complex \( x \) plane. \( \square \)

### 3.2 The dressing method.

Once the factorization of \( g \in G \) has been obtained, a flat connection is reconstructed from the factors by a procedure often referred to as the “dressing method” [ZS]. We first summarize the method for the AKNS hierarchies with potentials in \( B \) since these are somewhat less complicated.

Given \( g \in G \), let \( m_{\pm} \) be the solutions of the factorization problem

\[
e^{xzJ+tz^k_\mu}g(z)e^{-xzJ-tz^k_\mu} = m_-^{-1}m_+.
\]

(3.2.1)

where \( J \) and \( \mu \) are diagonal matrices. It is immediate from (3.2.1) that

\[
D(m_-^{-1}m_+) = z[J,m_-^{-1}m_+], \quad D_t(m_-^{-1}m_+) = z^k[\mu,m_-^{-1}m_+],
\]

(3.2.2)

where for the present \( D = d/dx \) and \( D_t = d/dt \).

Denote by \( m \) the sectionally holomorphic function

\[
m = \begin{cases} m_+ & |z| < 1; \\ m_- & |z| > 1. \end{cases}
\]

A simple calculation based on (3.2.2) shows that the sectionally holomorphic functions

\[
(Dm)m^{-1} + m(zJ)m^{-1}, \quad (D_t m)m^{-1} + m(z^k\mu)m^{-1}
\]

have no jumps across the unit circle, and are consequently entire functions of \( z \). Since \( m_- \) tends to \( I \) as \( z \) tends to infinity, their highest order terms in \( z \) are respectively \( zJ \) and \( z^k\mu \); hence, with \( m = m_- \),

\[
(Dm)m^{-1} + m(zJ)m^{-1} = u(x,z) \equiv zJ + q(x,t) \quad (3.2.3a)
\]

\[
(D_t m)m^{-1} + m(z^k\mu)m^{-1} = v_k(x,z) \equiv z^k\mu + B_k \quad (3.2.3b)
\]

where \( B_k \) denotes a polynomial of degree \( k - 1 \) in \( z \).

Letting ( )\(_+\) denote the projection onto the non-negative powers of \( z \), we obtain

\[
zJ + q = (m_- zJm_-^{-1})_+ = zJ + [m_1, J] \quad (3.2.4)
\]

where \( m_1 \) is the coefficient of \( z^{-1} \) in the expansion of \( m_- \), and

\[
v_k(x,z) = (m_- z^k\mu m_-^{-1})_+ \quad (3.2.5)
\]

\[\text{□} \]
3.2.6 Theorem. \( v_k \) is a universal polynomial in \( q \) and its derivatives with zero constant term. Specifically,

\[
v_k = \sum_{j=0}^{k} F_j z^{k-j}
\]

where the matrix valued functions \( F_j \) satisfy the recursion relations

\[
[J, F_{j+1}] = DF_j - [q, F_j], \quad F_0 = \mu.
\]

The connection \( D - u, D_t - v_k \) is flat, and \( q \) satisfies the nonlinear partial differential equation

\[
q_t = [J, F_{k+1}].
\]

Solutions of (3.2.8) which are obtained from the factorization (3.2.1) are meromorphic in \( x \) and \( t \).

Likewise, the coefficients of \( mJ^k \mu m^{-1} \), where \( me^{xJ} \) satisfies (2.1.1), are differential polynomials in \( q \) with zero constant term.

Proof. We sketch a proof of the first statement based on arguments in [BS2]. The wave function \( m_- \) is uniquely determined up to right multiplication by a diagonal matrix, so \( F = m_- z^k \mu m_{-1} \) is independent of the choice of \( m_- \). For fixed \( x_0 \), construct a formal series \( m_- \) by requiring the diagonal entries in \( m_j \), \( j > 0 \) to vanish at \( x_0 \). It is clear that the coefficients of \( m_- \) are then uniquely determined polynomials in \( q \) and its derivatives at \( x_0 \), so the same is true of the \( F_j \). Since \( x_0 \) is arbitrary, the result holds for all \( x \). When \( q \) is zero, \( m_- \) is necessarily diagonal and \( F = z^k \mu \); hence these polynomials have zero constant term. The same argument holds for the Gel’fand-Dikii case, where \( m_- \) satisfies \( m_x = [J_{\lambda}, m] + q m \) and \( F = mJ^k \mu m^{-1} \).

To prove the second statement, rewrite (3.2.3) in the form \( (D - u)m = m(D - zJ), \ (D_t - v)m = m(D_t - z^k \mu) \). Then

\[
[D - u, D_t - v]m = m[D - zJ, \ D_t - z^k \mu] = 0.
\]

Since \( m \) is invertible, \( [D - u, D_t - v] = 0 \) and the connection is flat. The evolution equation (3.2.8) follows in the standard way from this fact and the recursion relations (3.2.7).

Since \( x \) and \( t \) appear analytically in (3.2.1), the factors \( m_{\pm} \) are meromorphic in \( x \) and \( t \) by Theorem 3.1.6. The meromorphic behavior of \( q \) follows from (3.2.4). \( \square \)

The wave function \( m \) is said to “dress” the bare connection \( D - zJ, D_t - z^k \mu \) to the “dressed connection” \( D - u, D_t - v \).

Flat connections for the Gel’fand-Dikii and associated hierarchies are constructed from solutions of the factorization problem (2.3.4), but require a number of algebraic considerations. Namely, one must decide how one is going to split the constant factors in the Lie group \( G \). By altering the choice of \( G_{\pm} \) one obtains not only the Gel’fand-Dikii flows, but their associated flows as well, such as the modified Gel’fand-Dikii flows, etc. Moreover, in the case of the Gel’fand-Dikii flows themselves, the factor \( G_- \) in the Riemann-Hilbert splitting is not a group, so the dressing method must be modified. This will be done in §5.1.

The analogs of (3.2.3a,b) are

\[
(D\hat{m})\tilde{m}^{-1} + \hat{m}J\chi\tilde{m}^{-1} = J\chi + q
\]

\[
(D_t\tilde{m})\hat{m}^{-1} + \hat{m}J^k\chi\hat{m}^{-1} = J^k + B_{n,k}.
\]
3.2.10 Corollary. Let \( q \) be defined for \( x \in I \subset \mathbb{R} \) and suppose \( L \) has a Baker-Akhiezer function for \( x \in I \). Then \( q \) has a meromorphic extension to the complex plane.

Proof. For \( x \in I \) the wave functions satisfy the factorization problem (2.3.4) (with \( t = 0 \)). Since the left side is entire in \( x \), the factors \( m_{\pm} \) have meromorphic extensions to the complex plane, by Theorem 3.1.6. The meromorphic extension of \( q \) is then given by (3.2.9a). \( \square \)

This result was obtained in [SW] using the \( \tau \) function; the method here applies to the AKNS and Drinfeld-Sokolov hierarchies as well.

4. Algebraic structure

4.1 Loop groups and algebras. We begin by reviewing the basic ideas of loop groups and algebras [K], [PS]. We continue to denote by \( G \) the loop group introduced in §3.1. Let \( |\cdot| \) denote a norm on \( M_n(\mathbb{C}) \). We may assume that the automorphism \( \sigma \) on diagonal matrices \( D \), which we introduced in §2.2, leaves \( |\cdot| \) invariant: \( |\sigma(D)| = |D| \).

From the identity
\[
\lambda^m e_{jk} = e_{jj}J_{\lambda}^{k-j+nm},
\]

it follows that an equivalent representation of \( G \) is given by
\[
G = \{ g : g = \sum_{k=-\infty}^{\infty} d_k J_{\lambda}^k; \, \det g = 1, \, d_k \in D \}
\]

where it is assumed that the series converges absolutely and uniformly on \( S^1 \). Given \( g = \sum_k d_k J_{\lambda}^k \), and \( h = \sum_k h_k J_{\lambda}^k \) we have
\[
gh = \sum_j \left( \sum_k d_k \sigma^k(h_{j-k}) \right) J_{\lambda}^j.
\]

Moreover, \( G \) is a Banach Lie group with norm
\[
|g| = \sum_k |d_k|,
\]
and Banach Lie algebra
\[
\mathfrak{g} = \{ h : h = \sum_k h_k J_{\lambda}^k; \, |h| < +\infty, \, \text{tr} \, h_k = 0, \, \text{for} \, k = 0 \mod n \}.
\]

An integer valued grading of \( \mathfrak{g} \) is a direct sum decomposition of \( \mathfrak{g} \) into subspaces \( \mathfrak{g}_k \) such that \([\mathfrak{g}_j, \mathfrak{g}_k] \subset \mathfrak{g}_{j+k}\). In our case, \( \mathfrak{g} \) has two natural gradings, induced by the two bases:
\[
\deg_1 (g_k \lambda^k) = k, \quad \deg_2 (d_k J_{\lambda}^k) = k.
\]

The homogeneous subspaces are different for the first and second gradings; let us denote them by \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \). Then,
4.1.1 Lemma. The complex dimension of $g_1^k$ is $n^2 - 1$ while

$$\dim g_2^k = \begin{cases} n & \text{if } k \not\equiv 0 \mod n \\ n - 1 & \text{otherwise.} \end{cases}$$

4.2 Lie algebra decompositions. Let

$$V_n = \bigoplus_{j=-n+1}^{n-1} g_2^j = \{ a : a = \sum_{k=-n+1}^{n-1} a_k J_{\lambda}^k, \ a_k \in D \}.$$

It is easily seen that $\dim V_n = n(n - 1)$ and that

$$V_n = U_- \oplus U_+ \lambda^{-1},$$

where $U_{\pm}$ are the subspaces of strictly upper and lower triangular matrices.

$G$ and $g$ are contained in the linear space

$$B = \{ a : a = \sum_k a_k J_{\lambda}^k, \ a_k \in D \} = \{ a : a = \sum_k g_k \lambda^k, \ g_k \in M_n(\mathbb{C}) \}.$$

This space also has gradings relative to the two bases, as above. We define projections $P_1$ and $P_2$ (denoted respectively by $P_+$ and $P^+$ in [DS]) onto the subspaces of non-negative degree relative to these gradings by

$$P_1 a = \sum_{k \geq 0} g_k \lambda^k, \quad P_2 a = \sum_{k \geq 0} a_k J_{\lambda}^k,$$

where

$$a = \sum_k g_k \lambda^k = \sum_k a_k J_{\lambda}^k.$$

We define a third projection on $B$ as follows. Let $D_1$ be as in Theorem 2.2.10.

4.2.1 Theorem. There is a projection $P_3$ whose image coincides with that of $P_1$ and whose kernel is given by

$$\ker P_3 = \{ g : g = \sum_{k > 0} e_k J_{-\lambda}^{-k}, \ e_k \in E_k \}$$

where

$$E_k = \begin{cases} \text{span} \left[ I, (\sigma - I)^{-1} D_1, \ldots, (\sigma - I)^{-(k-1)} D_1 \right], & k \leq n - 1; \\ \mathcal{D} & k \geq n. \end{cases}$$

Proof. We must show that every $a \in B$ can be written uniquely as $a = h + v$, where $v \in \text{Im} P_1$ and $h$ belongs to the subspace $\ker P_3$ defined above. We may assume $a$ is homogeneous with respect to the second grading. If $\deg_2 a \geq 0$, then $a \in \text{Im} P_1$, and $a = v$. On the other hand, if $\deg_2 a \leq -n$, then $a \in \ker P_3$ and $a = h$.

That leaves elements in $V_n$. If $\deg a = -j$ then $a = a_j J_{\lambda}^{-j}$ for some $a_j \in \mathcal{D}$. We must show that

$$a_j = e + f, \quad e \in E, \ f \in F \quad (4.2.2)$$

where $E$ and $F$ are the subspaces of upper and lower triangular matrices, respectively.
where
\[ F_j = \{ f : f = \text{diag}(f_1, \ldots, f_n), \; f_1 = f_2 = \cdots = f_j = 0 \}. \]

It is clear that \( \dim F_j = n - j \). We first prove that \( \dim \mathcal{E}_j = j \) and so it suffices to prove that \( \mathcal{E}_j \cap F_j = \{ 0 \} \). Suppose that
\[ e = \omega_0 I + \sum_{r=1}^{j-1} \omega_r (\sigma - I)^{-r} D_1 = 0. \]

Then
\[ (\sigma - I)^{(j-1)} e = \omega_1 (\sigma - I)^{j-2} D_1 + \cdots + \omega_{j-1} D_1 = 0. \]

It is easily checked that \((\sigma - I)^k D_1 \in F_{n-k-1}\) for \( k < n \); and this implies that \( \omega_{j-1} = 0 \). Applying \((\sigma - I)^{j-2}\) to \( e \) we find next that \( \omega_{j-2} = 0 \) and so forth. Hence all the coefficients are zero, and \( \dim \mathcal{E}_j = j \).

Now let \( e \) be as given above and suppose that \( e \in \mathcal{E}_j \cap F_j \). We repeat the above argument. Applying \((\sigma - I)^{j-1}\) to \( e \) we obtain
\[ (\sigma - I)^{(j-1)} e = \omega_1 (\sigma - I)^{j-2} D_1 + \cdots + \omega_{j-1} D_1 \in (\sigma - I)^{j-1} F_j \subset F_1. \]

Hence \( \omega_{j-1} = 0 \). Then we apply \((\sigma - I)^{j-2}\) to \( e \), and find that \( \omega_{j-2} = 0 \) and so forth. Hence \( e = 0 \), and \( \mathcal{E}_j \cap F_j = \{ 0 \} \). \( \square \)

The decomposition (4.2.2) can be obtained by multiplying on the right by \( e_{kk} \) with \( k < j \) and taking the trace. In that case we obtain the system of equations
\[ \text{tr}[e_{kk}a] = \omega_0 + \sum_{r=1}^{j-1} \omega_r \text{tr}[e_{kk}(\sigma - I)^{-r} D_1], \quad 1 \leq k \leq j, \]

(4.2.3)

for the coefficients \( \omega_0, \ldots, \omega_{j-1} \). For future reference we denote the matrix of coefficients of (4.2.3) by \( W \).

The projection \( P_3 \) is obtained by splitting \( V_n \subset \ker P_2 \) into the direct sum of two subspaces. The dimension of \( \ker P_3 \cap V_n \) is \( n(n-1)/2 \).

Let us set
\[ g_j^- = \ker P_j \cap g, \quad g_j^+ = \text{im} P_j \cap g, \quad j = 1, 2, 3. \]

Then

\begin{enumerate}
  \item \( g = g_j^- \oplus g_j^+ \) for \( j = 1, 2, 3 \).
  \item \( g_1^-, g_2^-, g_1^+, g_2^+ \), and \( g_3^+ \) are each Lie subalgebras of \( g \) for all \( n \).
  \item \( g_3^- \) is a subalgebra of \( g \) for \( n = 2, 3 \) but not for \( n > 3 \).
\end{enumerate}

**Proof.** Statement i) follows from the fact that each \( P_j \) is a projection. That the kernel and range of \( P_3 \) are subalgebras is immediate from the definition. To prove the corresponding fact for \( P_2 \), note first that \( J_\lambda b J_\lambda^{-1} = \sigma(b) \) for any diagonal matrix \( b \). Then, for \( a \) and \( b \) diagonal,
\[ [a, B_{b, k}] = (a \sigma I(b) - b \sigma^k(a)) H^{j+k}. \]
It follows that the kernel and range of $P_2$ are subalgebras, since they are the subspaces of elements spanned respectively by negative and positive powers of $J_\lambda$.

To prove iii) it suffices to consider the subspace

$$g_3^n \cap V_n = sp\{e_k J^{-k}_\lambda, \ k = 0, \ldots, n-1, \ e_k \in \mathcal{E}_k\}$$

For $n = 2$, this subspace is spanned by $J^{-1}_\lambda$. Since $[J^{-1}_\lambda, J^{-1}_\lambda] = 0$, it is trivial that $g_3^n$ is a subalgebra. For $n = 3$ one must also check the commutator

$$[J^{-1}_\lambda, (\sigma - I)^{-1} D_1 J^{-2}_\lambda] = -\sigma^{-1}(D_1)J^{-3}_\lambda.$$

The element on the right has degree -3 in the second grading. When $n = 3$ it automatically belongs to $g_3^n$, since in that case no further constraint is placed on the diagonal matrix multiplying $J^{-3}_\lambda$.

For $n \geq 4$, however, the condition that it belong to $g_3^n$ is

$$\sigma^{-1}D_1 = \alpha I + \beta(\sigma - I)^{-1}D_1 + \gamma(\sigma - I)^{-2}D_1$$

for some complex constants $\alpha, \beta, \gamma$. Operating on this equation by $\sigma(\sigma - I)^2$, we find

$$(\sigma^2(1 - \beta) + (\beta - 2 - \gamma)\sigma + 1)D_1 = 0.$$ 

It is not hard to see, from the definition of $D_1$, that these equations are inconsistent. Thus $g_3^n$ is not a subalgebra when $n \geq 4$. □

4.3 Lie group decompositions. The above Lie algebra decompositions have their counterparts on the group level. Let

$$G^+_j = \{g : g \in G, \ g \in \text{Im}P_j\}, \quad (4.3.1)$$

$$G^-_j = \{g : g \in G, \ g - I \in \text{Ker}P_j\}. \quad (4.3.2)$$

Then:

$$G^+_1 = \{g : \det g = 1, \ g = \sum_{j \leq 0} g_j \lambda^j, \ g_0 = I\}$$

$$G^-_1 = \{g : \det g = 1, \ g = \sum_{j \geq 0} g_j \lambda^j\}$$

$$G^+_2 = \{g : \det g = 1, \ g = \sum_{j \leq 0} g_j J^j_\lambda, \ g_j \in \mathcal{D}, \ g_0 = I\}$$

$$G^-_2 = \{g : \det g = 1, \ g = \sum_{j \geq 0} g_j J^j_\lambda, \ g_j \in \mathcal{D}\},$$

and

$$G^+_3 = \{g : \det g = 1, \ g = \sum_{j \leq 0} g_j J^j_\lambda, \ g_j \in \mathcal{E}_j, \ g_0 = I\}$$

$$G^-_3 = G^+_1$$
4.3.3 Theorem.
i) \( G^+_3 = G^+_1 \).
ii) \( G^+_2 \) is a Lie subgroup of \( G^+_1 \) and \( G^-_1 \) is a Lie subgroup of \( G^-_2 \).
iii) \( G^-_j \) and \( G^+_j \), \( j = 1, 2 \) are Banach Lie groups with Lie algebras \( g^-_j \) and \( g^+_j \).
iv) \( G^-_3 \) is a group only for \( n = 2, 3 \).

This theorem is proved in the same manner as Theorem 4.2.5; the demonstration is left to the reader.

We are now ready to state the main theorem of this section:

4.3.4 Theorem. For each of the three projections \( P_j \) there is an open dense subset \( \Omega_j \subset G \) such that each \( g \in \Omega_j \) has a unique factorization

\[ g = (\phi^-_j)^{-1}\phi^+_j, \quad \phi^\pm_j \in G^\pm_j. \]

Proof. For \( j = 1 \) this is the well-known Riemann-Hilbert factorization whose proof is given in Theorem 3.1.6. Now suppose \( g \in \Omega_1 \) admits such a factorization with \( \phi^\pm \in G^\pm_1 \). We write

\[ \phi^+(\lambda) = v\hat{\phi}^+(\lambda), \quad \hat{\phi}^+(0) = I, \quad v = \phi_+(0). \]

Let \( SL^\circ(n) \) denote the subset of \( v \) in \( SL(n) \) which have the factorization \( v = \ell^{-1}u \) where \( \ell = I + \text{strictly lower} \) and \( u \) is an upper triangular matrix. As is well known, \( SL^\circ(n) \) is a dense open subset of \( SL(n) \). When \( v \in SL^\circ(n) \), \( g \) factors as

\[ g = (\ell\phi^-)^{-1}(u\hat{\phi}^+), \]

where \( \ell\phi^- \in G^-_2 \) and \( u\hat{\phi}^+ \in G^+_2 \).

The domain \( \Omega_2 \) consists of all \( g \in \Omega_1 \) for which \( v = \phi^+(0) \in SL^\circ(n) \). Since \( \Omega_2 \) is the inverse image of \( SL^\circ(n) \) under the continuous mapping \( g \to v \), it is open. To show that \( \Omega_2 \) is dense in \( \Omega_1 \), consider \( g \in \Omega_1 \setminus \Omega_2 \). From (3.1.8),

\[ v = g_0 + \sum_{j<0} \phi_j g_{-j}. \]

Let \( \dot{g} \) denote a tangent vector at \( g \) to a curve in \( G \). For those curves for which \( \dot{g} \) is independent of \( \lambda \), \( \dot{v} = \dot{g}_0 \). The set of such tangent vectors is transversal to \( SL(n) \setminus SL^\circ(n) \); for example, it contains all strictly triangular matrices. So every neighborhood of \( g \) contains points in \( \Omega_2 \).

The construction of the third factorization is more involved. We begin by proving:

4.3.5 Lemma. Given \( a \) in \( G^-_1 \) or \( G^-_2 \) there is a unique constant lower triangular matrix \( \ell \) with 1’s on the diagonal such that \( \ell a \in G^-_3 \).

Proof. It suffices to prove the lemma for \( a \in G^-_2 \), since \( G^-_1 \) is a subgroup of \( G^-_2 \). Let

\[ a = \sum_{k=0}^\infty a_k J^{-k}_\lambda, \quad a_k \in D, \quad a_0 = I. \]
Any constant lower triangular $\ell$ with 1’s on the diagonal can be written as

$$\ell = \sum_{j=0}^{n-1} f_j J^j_\lambda, \quad f_0 = 1, \quad f_j \in \mathcal{F}_j.$$  

It follows that

$$\ell a = \sum_{k=0}^{\infty} \left( \sum_{j=0}^k f_j \sigma^{-j}(a_{k-j}) \right) J^{-k}_\lambda$$

where we set $f_j = 0$ for $j \geq n$. Thus $\ell a \in G^3$ if and only if

$$\sum_{j=0}^k f_j \sigma^{-j}(a_{k-j}) \in \mathcal{E}_k, \quad k = 0, \ldots, n - 1.$$  

Writing these equations out we get

$$f_1 + a_1 = e_1$$

$$f_2 + f_1 \sigma^{-1}(a_1) + a_2 = e_2$$

$$\vdots$$

$$f_{n-1} + \sum_{k=1}^{n-2} f_k \sigma^{-k}(a_{n-k-1}) + a_{n-1} = e_{n-1}.$$  

But these decompositions are precisely those obtained in (4.2.2). Given $a_1$ the first equation may be solved uniquely for $f_1$ and $e_1$. Then $f_2$ and $e_2$ are uniquely determined from $a_1$ and $a_2$ from the second equation, and so forth. \( \square \)

We now return to the proof of Theorem 4.3.4. We set $\Omega_3 = \Omega_1$. Given $g \in \Omega_1$, let $g = \phi_1^- \phi_1^+$. By the preceding lemma there exists a unique lower triangular matrix $\ell$ with 1’s on the diagonal such that $\ell \phi_1^- \in G^3_3$. Note that left multiplication by a constant matrix of determinant 1 leaves $G^+_1 = G^+_3$ invariant; and therefore

$$g = (\ell \phi_1^-)^{-1}(\ell \phi_1^+), \quad \ell \phi_1^+ \in G^+_3$$

is the desired factorization.

For $j = 1, 2$, the uniqueness of the factorizations is a simple consequence of the fact that $G^-_j$ and $G^+_j$ are groups and $G^-_j \cap G^+_j = \{I\}$. For $j = 3$, however, $G^-_3$ is not a group for $n > 3$, and we have to argue differently.

Let us assume therefore that $g = (\phi_3^-)^{-1}\phi_3^+ = (\tilde{\phi}_3^-)^{-1}\tilde{\phi}_3^+$. Let $\ell$, respectively $\tilde{\ell}$ denote the values of $\phi_3^-$ and $\tilde{\phi}_3^-$ at infinity; $\ell$ and $\tilde{\ell}$ are lower triangular matrices with 1’s on the diagonal. Writing $\phi_3^- = \ell v_3$ and $\tilde{\phi}_3^- = \tilde{\ell} \tilde{v}_3$, we have

$$g = v_3^{-1}(\ell^{-1} \phi_3^+) = \tilde{v}_3^{-1}(\tilde{\ell}^{-1} \tilde{\phi}_3^+),$$  

(4.3.6)

where now

$$v_3, \tilde{v}_3 \in G^-_1, \quad \ell^{-1} \phi_3^+, \tilde{\ell}^{-1} \tilde{\phi}_3^+ \in G^+_1.$$  

The factorization (4.3.6) of $g$ is of type 1, and is therefore unique, by what we have already proved, so $v_3 = \tilde{v}_3$. By Lemma 4.3.5 there is a unique matrix $\ell$ such that such that $\ell v_3 = \ell \tilde{v}_3 \in G^-_3$. Thus $\ell = \tilde{\ell}$, $\phi_3^- = \tilde{\phi}_3^-$, and $\phi_3^+ = \tilde{\phi}_3^+$. \( \square \)
5. Nonlinear Flows

5.1 The Gel’fand-Dikiǐ flows. In the preceding section we introduced three different factorizations of elements of $G$. These three factorizations lead to the Gel’fand-Dikiǐ, modified and “potential” Gel’fand-Dikiǐ flows. In this section we discuss the construction of these different flows from the different factorizations. We begin with the Gel’fand-Dikiǐ flows themselves.

5.1.1 Theorem. Let $g$ belong to $G$ and let $m_{\pm}$ be the solutions of the factorization problem

$$e^{i(xJ_\lambda + tJ_\lambda^k)}g(\lambda)e^{-i(xJ_\lambda + tJ_\lambda^k)} = m_{\pm}^{-1}(x, t, \lambda)m_{\pm}(x, t, \lambda)$$

(5.1.2)

where $m_{\pm} \in G^\pm_3$. Then

$$(Dm)m_{\pm}^{-1} + mJ_\lambda m^{-1} = J_\lambda + q, \quad m = m_{\pm},$$

(5.1.3)

where $q = q(x, t) \in B$ satisfies the Gel’fand-Dikiǐ equations for $(k, n)$

Proof. By the results of §3 and Theorem 4.3.4 the factors $m_{\pm}$ of (5.1.2) are meromorphic functions of $x$ for fixed $t$ and of $t$ for fixed $x$. By the arguments of §3.2, $(Dm)m_{\pm}^{-1} + mJ_\lambda m_{\pm}^{-1}$, where $m$ is the sectionally holomorphic function $m_{\pm}$, is an entire function of $\lambda$.

Since $m_- \in G^-_3$,

$$m_- = \sum_{j \geq 0} m_j J^{-j}_\lambda, \quad m_j \in \mathcal{E}_j.$$  

In particular, $m_- \sim I + \ell$, as $\lambda \to \infty$, where $\ell$ is strictly lower triangular. Since $\ell$ depends on $x$, $(Dm_-)m_-^{-1}$ is bounded as $\lambda \to \infty$ while

$$m_- J_\lambda m_-^{-1} - J_\lambda = [m_-, J_\lambda]m_-^{-1}$$

$$= \left( \sum_{j \geq 0} (m_j - \sigma(m_j)) J^{-j}_\lambda \right) m_-^{-1}$$

$$= \left( \sum_{j \geq 2} (m_j - \sigma(m_j)) J^{-j}_\lambda \right) m_-^{-1}.$$  

The above expression is bounded as $\lambda \to \infty$, and therefore so is

$$(Dm_-)m_-^{-1} + m_- J_\lambda m_-^{-1} - J_\lambda.$$  

It follows by Liouville’s theorem that this expression is independent of $\lambda$.

Let us call it $q(x, t)$. From the expansion for $m_-$ we conclude that $q$ may be expanded as

$$q = \sum_{k=0}^{n-1} q_k J^{-k}_\lambda,$$

where $q_k \in \mathcal{F}$, since $q$ is independent of $\lambda$. 

Writing (5.1.3) in the form $Dm + mJ_\lambda = (J_\lambda + q)m$ and expanding $q$ and $m$ in powers of $J_\lambda$, we obtain a sequence of recursion relations for $q_k$ (noting that $m_0 = I$):

$$Dm_k + (I - \sigma)(m_{k+1}) = \sum_{j=0}^{k} q_j \sigma^{-j}(m_{k-j}) = q_0m_k + \cdots + q_k, \quad (5.1.4)$$

for $0 \leq k \leq n - 1$.

Since $m_- \in G_3$, $m_1 = \omega_10I$, and $q_0 = (I - \sigma)(m_1) = 0$. From (5.1.4) we get for $k = 1$

$$q_1 = m_1 + (I - \sigma)(m_2) = D\omega_{10}I - (\sigma - I)(\omega_{20}I + \omega_{21}(\sigma - I)^{-1}D_1 = D\omega_{10}I - \omega_{21}D_1.$$ 

Since $q_1 \in F_1$ its first entry vanishes, so $\omega_{21} = D\omega_{10}$, and

$$q_1 = nD\omega_{10}e_{nn}.$$ 

We now prove, by induction, that in fact $q_j$ is a scalar multiple of $e_{nn}$ for all $1 \leq j \leq n - 1$. Assume this is true for all integers less than $j$. Since the $m_s$ are diagonal matrices, $q_1m_{j-1} + \cdots + q_{j-1}m_1$ on the right side of (5.1.4) is a scalar multiple of $e_{nn}$, so we need to prove that

$$Dm_j + (I - \sigma)(m_{j+1}) = d_j e_{nn} \quad (5.1.5)$$

for some constant $d_j$. Since $q_j \in F_j$ the right side of (5.1.4) belongs to $F_j$, for $k = j$, and so

$$\text{tr } e_{ss}[Dm_j + (I - \sigma)(m_{j+1})] = 0, \quad s \leq j. \quad (5.1.6)$$

Writing

$$m_j = \omega_{j0}I + \sum_{k=1}^{j-1} \omega_{jk}(\sigma - I)^{-k}D_1,$$

$$m_{j+1} = \omega_{j+1,0}I + \sum_{k=1}^{j} \omega_{j+1,k}(\sigma - I)^{-k}D_1,$$

we find, after some calculations, that (5.1.6) reduces to the linear system of equations

$$\text{tr } e_{ss}(D\omega_{j0}I - \omega_{j+1,1}D_1) + \sum_{k=1}^{j-1} (D\omega_{jk} - \omega_{j+1,k+1}tr e_{ss}(\sigma - I)^{-k}D_1 = 0, \quad s \leq j.$$ 

This is the homogeneous system of equations corresponding to (4.2.3); and as we saw in the proof of Theorem 4.2.1, that system has a unique solution. Therefore $D\omega_{j0} = \omega_{j+1,1}$,

$$Dm_j + (I - \sigma)(m_{j+1}) = nD_{j0}e_{nn}.$$
and (5.1.5) is proved.

We now prove that \( q \) satisfies the \((n, k)\) Gel’fand-Dikii equations. Since \( G^3_\lambda \) is not a group, we cannot use the Lie algebra decomposition that worked for the AKNS hierarchy. Instead, we argue as follows. The solution of the factorization problem satisfies the simultaneous equations (3.2.9a,b). Therefore we have

\[
(D - J_\lambda - q)\psi = 0, \quad (D_t - G_k)\psi = 0, \quad (5.1.7)
\]

where \( G_k = J^k_\lambda + B_{n,k} \) is the right side of (3.2.9b), and \( \psi = m \exp\{xJ_\lambda + tJ^k_\lambda\} \). From the first equation we see that \( \psi \) is a Wronskian: \( \psi = ||D^{-1}v_{i-1}||, i, j = 1, \ldots, n \), and that the entries of the first row of \( \psi \) satisfy \( Lv = \lambda v \). If \( x_0 \) is a regular point of \( q \), then these equations are also satisfied by \( \psi = \psi(x, \lambda)\psi^{-1}(x_0, \lambda) \), and \( \psi(x_0, \lambda) = I \).

We shall henceforth assume this is the case and drop the tilde.

By the second equation each \( v_j \) satisfies

\[
\frac{\partial v}{\partial t} = \sum_{j=1}^{n} G_{1j} D^{j-1}v
\]

where the \( G_{1j} \) denote the entries of the first row of \( G_k \). Since \( Lv_j = \lambda v_j \), we can replace \( \lambda^*v \) by \( L^*v \). The result is that each \( v_j \) satisfies an equation of the form \( \dot{v}_j = P_kv_j \) where \( P_k = D^k + \ldots \) is a differential operator of order \( k \) and is independent of \( \lambda \). The simultaneous equations \( Lv_j = \lambda v_j \), and \( \dot{v}_j = P_kv_j \) imply that \( (\dot{L} - [P_k, L])v_j = 0 \); and moreover, \( D^m v_j(x_0, z) = \delta_{jm} \) for \( 0 \leq j, m \leq n - 1 \).

Let

\[
[P_k, L] = \sum_{j=0}^{k} r_j(x)D^j.
\]

Then

\[
0 = (\dot{L} - [P_k, L])v_j(x_0, z) = \dot{u}_j(x_0) - r_j(x_0) - \sum_{m=n}^{k} r_m(x_0)D^m v_j(x_0, z),
\]

where the sum over \( m \geq n \) is vacuous if \( k < n \). Since derivatives of \( v \) of order \( n \) or greater can be replaced by \( \lambda \), the summation is a polynomial in \( \lambda \) of degree at least one. Since the above equation holds identically in \( \lambda \), we conclude that the summation vanishes identically, \([P_k, L]\) is of order less than \( n \), and \( \dot{L} - [P_k, L] \) vanishes at \( x_0 \). Since \( x_0 \) was arbitrary, it vanishes at all regular points of \( q \).

This implies that \([P_k, L]\) is of order \( n - 2 \). It follows (cf. Proposition 2.3 [DS]) that the coefficients of \( P_k \) are differential polynomials of the coefficients of \( L \) and that \( P_k \) is a linear combination of the operators \( L^{j/n}_+ \) for \( j \leq k \):

\[
P_k = \sum_{j=1}^{k} c_j(L^{j/n}_+),
\]

for some constants \( c_j \).

To show that \( P_k \) is precisely \( L^{k/n}_+ \), it suffices to prove that all the \( c_j, j < k \), are zero. We do this by showing that for \( j < k \) the coefficient of \( D^j \) in \( P_k \) is a differential polynomial in the entries of \( q \) with zero constant term. The coefficients of \( P_k \) come from the first row of \( G_k \), and this matrix is in turn given by (3.2.9b). From the lower triangular structure of the leading term of \( m \in G^3_\lambda \), the first row of \( G_k \) is identical to that of \( P_1(mJ^k_\lambda m^{-1}) \). But the entries of \( mJ^k_\lambda m^{-1} \) are differential polynomials in \( q \) with zero constant terms by Theorem 3.2.6.
5.2 The Modified Gel'fand-Dikii flows. The so-called modified Gel'fand-Dikii flows are a special case of the AKNS hierarchy, and are defined as follows. Writing 
\[ L = (D + u_n)(D + u_{n-1}) \cdots (D + u_1), \]
the equation \( Lv = \lambda v \) can be written as a first order system
\[ (D - J_\lambda + q)\psi = 0 \]
where \( q \) is the diagonal matrix with entries \( u_1, \ldots, u_n \). (\( \text{tr} q = 0 \) since the coefficient of \( D^{n-1} \) vanishes.) \n
Recall (§2.1) that \( \Lambda^{-1}_z J_\lambda \Lambda_z = \tilde{z} J_\alpha \), where \( J_\alpha \) is diagonal. Since \( q \) is diagonal, \( \Lambda^{-1}_z q \Lambda_z = \tilde{q} \) is independent of \( z \). Hence we can write the isospectral problem as
\[ (D - zJ_\alpha + \tilde{q})\tilde{\psi} = 0. \]

This is the isospectral problem for an AKNS system, whose flows we constructed in §3.2.

Working in the original basis, we obtain the \((n, k)\) modified Gel'fand-Dikii flow as 
\[ [D_x, D_t] = 0, \]
where
\[ D_x = \frac{\partial}{\partial x} - J_\lambda + q, \quad D_t = \frac{\partial}{\partial t} - [F^k]_+, \]
\[ F = mJ_\lambda m^{-1}, \quad [\cdot]_+ \text{ denotes the projection } P_2, \text{ i.e. the projection onto non-negative powers of } J_\lambda. \]

The modified Gel'fand Dikii flows are obtained from the second factorization:

5.2.1 Theorem. Let \( g \in G \) and let \( m_\pm \) be the holomorphic solutions to the factorization problem
\[ e^{i(xJ_\lambda + tJ_k^\pm)} g e^{-i(xJ_\lambda + tJ_k^\pm)} = m_-^{-1} m_+, \quad m_\pm \in G_2^\pm. \]

Then
\[ q(x, t) = (Dm)m^{-1} + mJ_\lambda m^{-1} - J_\lambda \]
is a diagonal matrix of trace zero and satisfies the modified Gel'fand-Dikii equations for \((k, n)\).

Proof. We repeat the first part of the proof of Theorem 5.1.1. Again we may conclude that
\[ (Dm)m^{-1} + mJ_\lambda m^{-1} - J_\lambda \quad (5.2.2) \]
is an entire function of \( \lambda \), where \( m \) is the sectionally meromorphic function defined by \( m_+, m_- \) in the regions \( |\lambda| < \rho, |\lambda| > \rho \) respectively. This time
\[ m_- = \sum_{j \geq 0} m_j J_\lambda^{-j}, \quad m_0 = I, \quad m_j \in D. \quad (5.2.3) \]

Again \( m_- = I + \ell + O(\lambda^{-1}) \) as \( \lambda \to \infty \) and so the expression in (5.2.2) is bounded as \( \lambda \) tends to infinity. As before, we denote it by \( q \), where \( q \) depends only on \( x \) and \( t \). Inserting (5.2.3) into (5.2.2), we obtain \( q \) as the constant term in
\[ m_- J_\lambda m_-^{-1} \]

\[ = (I + m_1 J_\lambda^{-1} + \ldots)J_\lambda (I - m_1 J_\lambda^{-1} + \ldots) - J_\lambda \]
\[ = m_- I - \sigma(m_-) J_\lambda \]
\[ \sigma(m_-) = \sum_{j \geq 0} m_j J_\lambda^{-j} \]
where the dots denote negative powers of $J_\lambda$. Thus

$$q = (I - \sigma)(m_1) \in D_0.$$  

To show that $q$ satisfies the modified Gel’fand-Dikii $(n,k)$ flow, we consider the connection $D - J_\lambda - q$, $D_t - v_k$, where $v_k$ is given by the right side of (3.2.9b). As in the proof of Theorem 3.2.6, this connection is flat. This time $G^+_2$ is a group, so we obtain the Lie-algebra decomposition

$$v_k = P_2 \left[ (D_t \tilde{m})\tilde{m}^{-1} + \tilde{m}J^k_\lambda \tilde{m}^{-1} \right]$$

$$= P_2 (\tilde{m}J^k_\lambda \tilde{m}^{-1})$$

$$= \sum_{j=0}^k F_j J^{k-j}_\lambda.$$  

By conjugating by the constant matrix $\Lambda_\alpha$ everything is expanded in powers of $zJ_\alpha$, $(F_j$ is diagonal, so it commutes with $d(z)$) and we see that this is precisely the $k^{th}$ AKNS flow. \hfill \Box

5.3 The “potential” Gel’fand-Dikii flows. Factorizations in $G^+_1$ lead to a third structure for the potential. This time $m \in G^-_1$ and so has the expansion

$$m = \sum_{j=0}^\infty \frac{m_j}{\lambda^j}, \quad m_0 = I.$$  

The arguments of the previous section are repeated, this time using the projection $P_1$. From (5.2.2) we obtain

$$P_1((Dm)m^{-1} + mJ_\lambda m^{-1}) = P_1(mJ_\lambda m^{-1}) = J_\lambda + [m_1, e_{n1}]$$

and one finds that

$$q = \begin{pmatrix}
m_{1n} & 0 & \ldots & 0 \\
m_{2n} & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
m_{nn} - m_{11} & -m_{12} & \ldots & -m_{1n}
\end{pmatrix},$$

where $m_{jk}$ are the entries of the coefficient matrix $m_1$. The dressing argument goes through as in the previous section. The following is related to the general theory of Drinfeld and Sokolov. As they show, all reductions of the $n^{th}$ order equation $Lv = \lambda v$ to a first order system are related by a gauge transformation, and the entries of $L$ are differential polynomials in the coefficients the potential.

5.3.1 Theorem. The entries of $q$ are differential polynomials of precisely $n - 1$ independent functions.

We first need to prove the following lemma.
5.3.2 Lemma. For \( m \in G^-_1 \) we have the representation

\[
m = \sum_{j=0}^{\infty} m_j J^{-j}_\lambda, \quad m_0 = I, \ m_j \in \mathcal{D}
\]

(5.3.3)

where the diagonal matrices \( m_1, \ldots, m_n \) satisfy the additional constraints

\[
\text{tr} m_j e_{kk} = 0 \quad j = 1, \ldots, n-1; \ k \geq j+1 \quad (5.3.4)
\]

\[
\text{tr} m_j = 0 \text{ for } j = 0 \mod n. \quad (5.3.5)
\]

Proof. From the form of \( J^{-1}_\lambda \), namely

\[
J^{-1}_\lambda = \begin{pmatrix}
0 & \ldots & \lambda^{-1} \\
1 & 0 & \ldots \\
0 & 1 & 0 & \ldots \\
\ddots & & & & \\
1 & 0
\end{pmatrix},
\]

we see that the constraints (5.3.4) are necessary and sufficient that the constant term in \( m \) be the identity matrix. The constraint \( \text{tr} \ m_n = 0 \) is a consequence of the condition \( \det \ m = 1 \). □

Now write \( q \) in the form

\[
q = \sum_{j=0}^{n-1} q_j J^{-j}_\lambda
\]

where

\[
q_0 = u_0(e_{11} - e_{nn}), \quad q_j = u_j(e_{jj} - e_{nn}) + v_j(e_{jj} + e_{nn}), \quad q_{n-1} = v_{n-1} e_{nn}.
\]

Substituting this form of \( q \), together with the representation (5.3.3) into the equation \( Dm = [J_\lambda, \ m] +qm \), we obtain the recursion relations

\[
m_1 - \sigma(m_1) = q_0,
\]

\[
Dm_1 + m_2 - \sigma(m_2) = q_0 m_1 + q_1,
\]

\[
\vdots
\]

\[
Dm_j + m_{j+1} - \sigma(m_{j+1}) = \sum_{k=0}^{n-1} q_k \sigma^{-k}(m_{j-k}).
\]

We solve these equations recursively. From the first we get \( m_1 = u_0 e_{11} \) and \( q_0 = u_0(e_{11} - e_{nn}) \). Taking the trace of the second equation we find \( tr \ q_1 = 2v_1 = Du_0 - u_0^2 \). With this choice of \( v_1 \) the diagonal matrix \( m_2 \) is uniquely determined by the second equation and the constraint (5.3.4). This process continues up to \( j = n-2 \). At each stage the parameters \( v_j \) are determined in terms of the preceding \( u_0, \ldots, u_{j-1} \) and the \( m_j \) is uniquely determined.
The equation at \( j = n - 1 \) is

\[
D_{m_{n-1}} + m_n - \sigma(m_n) = q_0 m_{n-1} + \cdots + q_{n-1}.
\]  

(5.3.5)

The first \( n - 1 \) rows of (5.3.5) plus the trace condition \( \text{tr} m_n = 0 \) uniquely determine the entries of \( m_n \) as differential polynomials in \( u_j, v_j \) for \( j = 0, \ldots, n - 2 \) The last row of (5.3.5) then determines \( v_{n-1} \)

We thus see that the entries of \( q \) are differential polynomials of the functions \( u_0, \ldots, u_{n-1} \). □

For \( n = 2, 3 \) we get

\[
q = \begin{pmatrix} u & 0 \\ Du - u^2 & -u \end{pmatrix}, \quad \begin{pmatrix} u_0 & 0 & 0 \\ \frac{u_1 + (D - u_0)u_0}{2} & 0 & 0 \\ \frac{u_1 - u_0}{2} & -u_1 + (D - u_0)u_0 & -u_0 \end{pmatrix}.
\]

This class of potentials leads to a third class of flows, the “potential Gel’fand-Dikii” flows. These flows may be obtained as a reduction of the potential Kadomtsev-Petviashvili hierarchy [Sz].

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