Vacuum energy in conical space with additional boundary conditions

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Abstract
The total vacuum energy of some quantized fields in conical space with additional boundary conditions is calculated. These conditions are imposed on a cylindrical surface which is coaxial with the symmetry axis of conical space. The explicit form of the matching conditions depends on the field under consideration. In the case of the electromagnetic field, the perfectly conducting or isorefractive matching conditions are imposed on the cylindrical surface. For a massless scalar field, the semi-transparent conditions (δ-potential) on the cylindrical shell are investigated. As a result, the total Casimir energy of the electromagnetic field and the scalar field, per unit length along the symmetry axis, proves to be finite, unlike the case of an infinitely thin cosmic string. In these studies, the spectral zeta functions are widely used. It is shown briefly how to apply this technique for obtaining the asymptotics of the relevant thermodynamical functions with a high-temperature limit.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Quantum matter fields on manifolds with conic singularity are of interest in black hole physics and especially in cosmic string theory. The status of cosmic strings has changed during their rather long history. Therefore, it is worth mentioning a few words about this topic.

In the course of time evolution and extension, the Universe should undergo a number of phase transitions. From a quite general point of view, these transitions may be accompanied by the emergence of topological defects³ such as monopoles, cosmic strings and domain walls [1]. When neglecting the internal structure of these objects, we are left with point-like, ³ The topological defects are the regions with higher dynamical symmetry surrounded by a space with a lower symmetry group.
one- and two-dimensional defects, respectively. For the cosmic scenarios, the most interesting were monopoles and cosmic strings. On this subject, there is a vast literature (see, for instance, [2] and references therein). In the 1980’s, the cosmic strings were considered as candidates for a mechanism of galaxy formation. However, recent observations on the cosmic microwave background have shown conclusively that this effect can at most account for a small fraction of the total power, up to 10% [3, 4]. Nevertheless, cosmic strings are still considered as plausible sources of detectable gravitation waves (see, for example, [5, 6] and references therein), gamma ray burst [7], high-energy cosmic rays [8] and due to their gravitational lensing effects [9]. Recently, cosmic strings attract a renewed interest in the framework of brane inflation [10] and as macroscopic fundamental strings [11–14]. Comprehensive reviews of cosmic string studies are presented in papers [15, 16].

The gravitational field, i.e. the geometry of the spacetime, connected with cosmic string, is determined by the solution of the Einstein equations with the energy–momentum tensor defined by the string. When an infinitely thin straight string with the linear mass density \( \mu \) is located along the \( z \) axis, then the space outside the string is isomorphic to the manifold \( \mathbb{R}^1 \times C_{2\pi - \phi} \), where \( \mathbb{R}^1 \) is an infinite line along the \( z \) axis, and instead of a plane perpendicular to this axis, one has a two-dimensional cone \( C_{2\pi - \phi} \) with the angle deficiency \( \phi = 8\pi \mu G \), \( G \) being the gravitational constant. The cone \( C_\beta \) of an angle \( \beta \) becomes a plane when \( \beta = 2\pi \).

The static straight thin cosmic string is thus represented by a locally flat spacetime with a conical singularity along an axis [17–20]. A string with a deficit angle has a positive mass density and it is stretched; a string with an excess angle has a negative mass density and is squeezed. Beside the empty spacetime with a single string, cosmic strings appear in a wide variety of solutions to Einstein equations. Any axially symmetric spacetime can be trivially modified to contain a string on the axis of symmetry. Keeping in mind the physical origin of a cosmic string, one should characterize it by the gauge flux parameter in addition to the deficit angle parameter. However, this point is beyond the scope of our consideration.

For physical applications, it is interesting to study different fields in the background of a cosmic string. As explained above, this problem is reduced to the consideration of the fields on the spacetime manifold \( \mathbb{T}^1 \times \mathbb{R}^1 \times C_{2\pi - \phi} \). It is assumed that the effect of the fields on the spacetime geometry can be neglected.

One of the difficulties in these studies is nonintegrable singularity at the string axes for the energy density of fields considered in the background of an infinitely thin cosmic string [21]. As a result, the total vacuum energy (per unit length of the string) is not defined. To overcome this drawback, the cosmic string of finite thickness can be considered. For this purpose one may use, for example, the Gott–Hiscock metric [18, 22] which is the solution to the Einstein equations when the energy density is a constant inside the string of a finite thickness and zero outside of it [23].

In this paper, we are going to draw the attention to the following interesting fact that is closely related to the problem stated above. Namely, in order to render the total field energy in a conical space to be finite, it is sufficient to impose some boundary (or matching) conditions for quantized fields on a cylindrical surface which is coaxial with the symmetry axes of the conical space. In a loose form, this observation can be interpreted as follows: isolation of a ‘naked’ conic singularity by introducing additional boundary conditions on a surface surrounding it results in ‘improving’ the physical consequences generated by this singularity.

When choosing the fields for calculating the vacuum energy, we take into consideration the following. The vacuum energy of massless fields is, as a rule, much greater than that
of massive fields. Therefore, we study the Casimir energy of electromagnetic and massless scalar fields in the problem in question. As regards the boundary or matching conditions on the cylindrical surface, there is a large freedom here. We impose such conditions which are usually considered in the analogous studies [24, 25]. In the case of electromagnetic field, the perfectly conducting boundary conditions or isorefractive matching conditions are widely used in the Casimir calculations. In our consideration, we follow in this way. For a massless scalar field, we impose the semi-transparent conditions on the cylindrical shell. The point is that the real boundaries cannot confine the physical fields exactly within a given region, i.e. real boundaries are always more or less transparent for fields. This property is simulated, for example, by the $\delta$-potential [25].

As a result, the total Casimir energy of the electromagnetic field and the scalar field, per unit length along the symmetry axis, in the conical space with additional boundary conditions proves to be finite unlike the case of an infinitely thin cosmic string. In our studies of vacuum energy, we use substantially the spectral zeta function technique. In addition, this approach enables us to calculate directly the asymptotics of the relevant thermodynamical functions with a high-temperature limit.

The layout of the paper is as follows. In section 2, the spectra of an electromagnetic field and a scalar massless field in the geometry under consideration are analysed. For an electromagnetic field, two types of boundary conditions are considered: perfectly conducting conditions and isorefractive ones. For a massless scalar field, semi-transparent matching conditions are applied, i.e. the $\delta$-potential at the point $r = a$ is introduced, both signs of the respective coupling constant being investigated in detail. In section 3, the global zeta functions for these spectra are constructed by implementing the analytical continuation needed. Proceeding from these results, the finite and unique values for the vacuum energy of the electromagnetic field and scalar massless field in the conical space with additional boundary conditions are derived. In section 4, the high-temperature asymptotics of the relevant thermodynamical functions in the problem under study are obtained by making use of the heat kernel coefficients calculated in our previous work [26] and in this paper. In section 5, we discuss briefly the obtained results and compare them with analogous studies.

2. The excitation spectrum of the fields in a conical space with additional boundary conditions

When the energy–momentum tensor is determined by the $\delta$-like mass distribution along the $z$-axis with the linear density $\mu$, the solution to the Einstein equations is given by the following metric [17–20]:

$$ds^2 = -dt^2 + dr^2 + (1 - 4\mu G)^2 r^2 d\phi^2 + dz^2, \quad 0 \leq \phi \leq 2\pi,$$  \hspace{1cm} (2.1)

where $G$ is the gravitational constant. The cylindrical coordinates $(r, \phi, z)$ are used with the $z$-axis coinciding with the thin straight infinite string. It is easily convinced that the space part of metric (2.1) describes the geometry on the surface of a cone with the deficit angle $\Phi = 8\pi \mu G$ or with the opening angle $2\pi - \Phi$.

When considering matter fields on the cone surface, one can obviously substitute the latter by a plane with a segment of the angle $\Phi$ cut out, provided the periodicity conditions in a new angular variable $\theta$

$$f(\theta) = f(\theta + 2\pi - \Phi), \quad 0 \leq \theta \leq 2\pi - \Phi$$  \hspace{1cm} (2.2)

are imposed.
In terms of the angular variable $\theta$, metric (2.1) looks exactly like the metric of the Minkowski spacetime
\[ ds^2 = -dt^2 + dr^2 + r^2\, d\theta^2 + dz^2 \] (2.3)
supplemented by conditions (2.2) (see for details [26]).

2.1. Electromagnetic field

In the case of an electromagnetic field in a conical space, we apply perfectly conducting conditions or isorefractive matching conditions on an auxiliary cylindrical surface. As a result, the problem under consideration is reduced to the calculation of the electromagnetic vacuum energy for a cylindrical shell by imposing on its lateral surface the boundary conditions (or matching conditions) mentioned above and by substituting the usual $2\pi$-periodicity by the periodicity condition (2.2).

The solutions to the Maxwell equations with boundary conditions on the circular cylindrical shell have been considered in many papers (see, for example, [21, 26, 27]). We shall take advantage of this analysis and allow for the periodicity condition (2.2). The latter requirement leads to the substitution of the integer index $n$ in the Bessel and Hankel functions by $np$, where
\[ \frac{1}{p} = 1 - \frac{\Phi}{2\pi}. \] (2.4)
According to the physics of cosmic string formation [2, 11, 15], $p$ is very close to 1 and $p > 1$.

Let us address now the spectrum of electromagnetic oscillations in the configuration under consideration. In view of the axial symmetry of the conical space, the eigenvalues of the spectral problem (eigenfrequencies) $\omega_q$ are 'numbered' by the following three indices:
\[ q \equiv \{n_r, n, k_z\}, \] (2.5)
which correspond to the coordinates $\{r, \theta, z\}$. The radial index $n_r$ can take discrete values or it can be continuous. The angle index $n$ corresponds to the compact angle variable $\theta$: $0 \leq \theta \leq 2\pi - \Phi = 2\pi/p$. The continuous variable $k_z$ in (2.5) is the wave vector of the plane waves propagating in the $z$-direction: $-\infty < k_z < \infty$.

The general solution to the Maxwell equations in metric (2.3) in unbounded spacetime is expressed in terms of two independent scalar functions that correspond to the TE-polarization and TM-polarization of the electromagnetic field [28–30]. In the conical space, these functions should satisfy the periodicity conditions (2.2).

For further specification of the spectrum in the problem at hand, the boundary conditions on the cylindrical shell should be taken into account explicitly.

2.2. Perfectly conducting boundary conditions

On the surface of a perfect conductor, the tangential component of electric field ($E_\parallel$) and the normal component of the magnetic field ($H_\perp$) should vanish
\[ E_\parallel = 0, \quad H_\perp = 0. \] (2.6)
It is important to note that these conditions do not couple the polarizations of electromagnetic fields, as the Maxwell equations do, and furthermore the fields inside and outside of the shell are independent [31].

Inside the shell, the (unnormalized) eigenfunctions are [26]
\[ u_{nn_r}(r, \theta) = J_n(\lambda_{nn_r} r) \left( \frac{\sin np\theta}{\cos np\theta} \right), \quad n = 0, 1, 2, \ldots, \] (2.7)
where the parameter \( p \) was introduced in (2.4) and
\[
\lambda^2 = \frac{\omega^2}{c^2} - k_z^2. \tag{2.8}
\]

For simplicity, we drop the common multiplier \( \exp(-i\omega t + ik_z z) \) here. The solutions to the Maxwell equations are expressed in terms of the eigenfunctions (2.7) through the well-known formulas [27, 28].

For the TE-polarization, \( \lambda_{nnr} \), in (2.7) is the \( n_r \)th root of the equations
\[
J'_{np}(\lambda_{nnr} a) = 0, \quad n = 0, 1, 2, \ldots, \quad n_r = 1, 2, \ldots, \tag{2.9}
\]
and for the TM-polarization, we should take the roots of the equations
\[
J_{np}(\lambda_{nnr} a) = 0, \quad n = 0, 1, 2, \ldots, \quad n_r = 1, 2, \ldots. \tag{2.10}
\]

The explicit form of the eigenfunctions (2.7) implies, specifically, that at \( n \neq 0 \), all the eigenvalues in the spectral problem under consideration are twice degenerate (both the functions \( \sin np\theta \) and \( \cos np\theta \) work), and at \( n = 0 \) this degeneracy is eliminated because \( \sin n p \theta \) vanishes.

Summarizing, we can infer the following. At a fixed value of \( k_z \), the spectrum of the frequencies \( \omega(\text{TE}, k_z) \) inside the shell is discrete, and it is given by the formula
\[
\omega = c\sqrt{\lambda^2 + k_z^2}, \tag{2.11}
\]
where
\[
\omega = \omega_{nnr}(\sigma, k_z), \quad \sigma = \text{TE, TM},
\]
and for the TE-modes, \( \lambda \) in (2.11) should be substituted by the roots of equation (2.9), and in the case of the TM-modes the roots of equation (2.10) should be used.

Outside of the shell, the spectrum of the electromagnetic oscillations is continuous
\[
ck_z < \omega < \infty , \tag{2.12}
\]
because we are dealing here with a free unbounded configuration space. For the summation over such a spectrum to be accomplished the scattering formalism should be employed [32], i.e. one has to calculate the pertinent \( S \)-matrix. The regular solutions defining the Jost functions \( a^+_{n}(\lambda) \) in our model read
\[
f_n(r) = a^-_{n}(\lambda) H_{np}(\lambda r) - a^+_{n}(\lambda) H^*_{np}(\lambda r), \quad r > a, \tag{2.13}
\]
where \( \lambda \) is a continuous variable related to \( \omega \) and \( k_z \) by (2.8) and \( H^\pm_{np}(\lambda r) \) are the Hankel functions
\[
H^+_n(\lambda r) \equiv H^{(1)}_n(\lambda r), \quad H^-_n(\lambda r) \equiv H^{(2)}_n(\lambda r).
\]
The regular solutions (2.13) should satisfy the boundary conditions
\[
\left. \frac{d f_n^{\text{TE}}(r)}{dr}\right|_{r=a} = 0, \quad \left. f_n^{\text{TM}}(r)\right|_{r=a} = 0. \tag{2.14}
\]

By making use of the definition of the \( S \)-matrix
\[
S_n(\omega) = \frac{a^+_{n}(\lambda)}{a^-_{n}(\lambda)}
\]
and (2.13), (2.14), we obtain
\[
S_n^{\text{TE}}(\omega) = \frac{H_{np}^*(\lambda a)}{H_{np}(\lambda a)}, \quad S_n^{\text{TM}}(\omega) = \frac{H_{np}(\lambda a)}{H_{np}^*(\lambda a)}. \tag{2.15}
\]
The note concerning the different degeneracy of the states with $n = 0$ and $n \neq 0$ for oscillations inside the shell applies also to the scattering states outside of the shell. When $\omega^2 < c^2k_z^2$, there are solutions to the wave equation which decay in the radial direction:

$$H_{np}^+(\lambda r) \equiv H_{np}^{(1)} \left( i \frac{\ell}{c} \sqrt{c^2k_z^2 - \omega^2} \right) = \frac{2}{\pi} i^{n-1} K_{np} \left( \frac{\ell}{c} \sqrt{c^2k_z^2 - \omega^2} \right).$$

However, they do not meet the boundary conditions (2.14). Hence, in the problem at hand, there are no surface (or evanescent) waves.

### 2.3. Isorefractive (diaphanous) matching conditions

When employing these conditions, we assume that the electric and magnetic properties of the media inside ($\varepsilon_1, \mu_1$) the shell and outside of it ($\varepsilon_2, \mu_2$) are different, but the velocities of light in these regions are the same: $c_1 = c_2, c_i = c/\sqrt{\varepsilon_i \mu_i}, i = 1, 2$ ($c$ is the light velocity in vacuum). The fields $E$ and $H$ inside the shell and outside of it are coupled due to the continuity requirement for their tangential components

$$\text{discont} (E_i) = 0, \quad \text{discont} (H_i) = 0, \quad r = a. \quad (2.16)$$

The configuration space is noncompact, and as a consequence, the electromagnetic spectrum is continuous (2.12). Indeed, for a given $n$ and $k_z$, the scattering solutions to the Maxwell equations contain six amplitudes: the TE-solution has one internal amplitude and two amplitudes outside of the shell. The same holds for the TM-solution. The matching conditions (2.16) at the cylinder surface lead to four linear homogeneous equations for these amplitudes. Hence, no restrictions arise here for the spectral parameter $\omega^2/c^2$. The polarizations in this problem decouple in the equations determining the eigenfrequencies [33], but they do not decouple on the level of the scattering solutions. However, the scattering matrix in this case has a simple structure (more precisely, the Jost matrices have zero diagonal elements). In our paper [32], the $S$-matrix has been derived for the scattering of electromagnetic waves on an infinite circular material cylinder in the general case when $c_1 \neq c_2$. These formulae are considerably simplified if $c_1 = c_2$. As before, the $S$-matrix is the $(2 \times 2)$ matrix acting on polarizations (TE and TM) and it satisfies the matrix equation

$$K^+ S + K^- = 0, \quad (2.17)$$

where the matrices $K^\pm$ are now

$$K^\pm = \pm \begin{pmatrix} 0 & \beta^\pm \\ \gamma^\pm & 0 \end{pmatrix},$$

$$\beta^\pm_{np} = -i \frac{\alpha \lambda}{c J_{np}} \left( \mu_2 J_{np} H_{np}^{\pm} - \mu_1 J_{np} H_{np}^{\mp} \right),$$

$$\gamma^\pm_{np} = i \frac{\alpha \lambda}{c J_{np}} \left( \varepsilon_2 J_{np} H_{np}^{\pm} - \varepsilon_1 J_{np} H_{np}^{\mp} \right).$$

In these formulae, all the Bessel ($J_{np}$) and Hankel ($H_{np}^{\pm}$) functions have the same argument $a\lambda$, where $\lambda = \sqrt{(\omega/c)^2 - k_z^2}$. Ultimately, the determinant of the $S$-matrix

$$\det S = -\frac{\det K^-}{\det K^+} \quad (2.19)$$

---

5. This is also true for the natural modes of a compact cylinder with $c_1 = c_2$ (see (6) and (7) in section 9.15 of the book [31]). This set of equations does not split into equations separately for $a_e^n$ and $b_e^n$, or $a_m^n$ and $b_m^n$ when $c_1 = c_2$.

6. It is convenient to interchange the notations for the matrices $K^\pm$ in [32]: $K^+ \to K^-$, and take on the right-hand side of (20) in [32] the difference instead of the sum in order to comply with (2.13) in this paper.
is expressed in terms of the known multipliers $\Delta^{\text{TE}}$ and $\Delta^{\text{TM}}$ [27, 33, 34], these multipliers being constructed for both the Hankel functions of the first $(H_0^{(1)})$ and second $(H_0^{(2)})$ kinds\(^7\).

It is easy to show that in the problem under consideration, there are no surface modes. For $\omega^2 < c^2 k_z^2$, there are solutions to the wave equation which decay in both directions from the lateral surface of the cylindrical shell: $u_n(r) \sim I_{np}(r \sqrt{c^2 k_z^2 - \omega^2}/c)$, $r < a$ and $u_n(r) \sim K_{np}(r \sqrt{c^2 k_z^2 - \omega^2}/c)$, $r > a$. The respective frequency equations are

\[
\varepsilon_1 I'_{np}/I_{np} = \varepsilon_2 K'_{np}/K_{np}, \quad \mu_1 I'_{np}/I_{np} - \mu_2 K'_{np}/K_{np} = 0, \quad n = 0, 1, 2, \ldots \quad (2.20)
\]

Here all the modified Bessel functions have the same argument $\sqrt{c^2 k_z^2 - \omega^2}$. Since $I_{np}' > 0$ and $K_{np}' < 0$, the left-hand sides of equations (2.20) are strictly positive. Hence, these equations have no real roots in the interval $0 < \omega < c k_z$.

Summarizing this subsection, we infer that the spectrum of electromagnetic oscillations in the case of isorefractive matching conditions is pure continuous (see (2.12)).

2.4. Semitransparent matching conditions for the scalar field

Now we address the consideration of a massless scalar field $\varphi(t, \mathbf{x})$ in the conical space with a cylindrical surface of radius $a$. On this surface, we impose on the field $\varphi(t, \mathbf{x})$ the matching conditions which may be interpreted as the scalar $\delta$-potential. The wave equations in this case read

\[
\left[ \frac{1}{r^2} \frac{\partial^2}{\partial r^2} - \Delta + \frac{g}{r} \delta(r - a) \right] \varphi(t, \mathbf{x}) = 0,
\]

where $g$ is a dimensionless constant specifying the strength of the potential and $\Delta$ is the Laplace operator for the spatial part of metric (2.3). Separating the variables

\[
\varphi(t, \mathbf{x}) \sim e^{-\text{int} \epsilon k_z z} f_n(r) \left( \frac{\sin np \theta}{\cos np \theta} \right), \quad n = 0, 1, 2, \ldots,
\]

we arrive at the equation for the radial function $f_n(r)$

\[
\frac{d^2 f_n(r)}{dr^2} + \frac{1}{r} \frac{df_n(r)}{dr} + \left[ \frac{\omega^2}{c^2} - k_z^2 - \frac{n^2 p^2}{r^2} - \frac{g}{r} \delta(r - a) \right] f_n(r) = 0. \quad (2.23)
\]

First we demand the function $f_n(r)$ to be continuous at the point $r = a$:

\[
f_n(a + 0) - f_n(a - 0) = 0,
\]

so that one could integrate the left-hand side of (2.23) over $rd\mathbf{r}$. Such an integration in the vicinity of the point $r = a$ yields

\[
\frac{d}{dr} f_n(a + 0) - f_n(a - 0) = \frac{g}{a} f_n(a).
\]

As usual [35], the wave equation with the $\delta$-potential (2.23) is substituted by a pertinent free equation (without potential)

\[
\frac{d^2 f_n(r)}{dr^2} + \frac{1}{r} \frac{df_n(r)}{dr} + \left( \frac{\omega^2}{c^2} - k_z^2 - \frac{n^2 p^2}{r^2} \right) f_n(r) = 0, \quad (2.26)
\]

the solutions of the latter, $f_n(r)$, being subjected to the matching conditions (2.24) and (2.25). Sometimes the matching conditions generated by the $\delta$-potential are referred to as the semitransparent ones.

\(^7\) In [34] (see endnote 18 there), it was proposed to do this ‘by hand’ without rigorous justification. In [33], rather complicated contours were used when going to the imaginary frequencies.
Finally, at a given \( k_z, \ -\infty < k_z < \infty \), we face the spectral problem specified by the differential equation (2.26), matching conditions (2.24), (2.25) and physical conditions at the origin \((r = 0)\) and at the infinity \((r \to \infty)\), the frequency \( \omega > 0 \) being the spectral parameter. It turns out that the structure of the spectrum in the problem at hand depends strongly on the sign of the constant \( g \), namely for positive values of \( g \) the spectrum is pure continuous, and when \( g < 0 \) we have, in addition to the continuous branch, surface modes (or bound states).

First we consider the continuous part of the spectrum, i.e. we construct the scattering matrix in this problem for both signs of \( g \). The regular solution is

\[
 f_n(r) = \begin{cases} 
 a_n^m(\lambda)J_{np}(\lambda r) & \text{for } r < a, \\
 a_n^m(\lambda)H_{np}^+(\lambda r) - a_n^m(\lambda)H_{np}^-(\lambda r) & \text{for } r > a, 
\end{cases} 
\]  
(2.27)

where

\[
 \lambda = \sqrt{(\omega/c)^2 - k_z^2}. 
\]  
(2.28)

The matching conditions (2.24) and (2.25) yield

\[
 a_n^m(\lambda)J_{np}(\lambda a) = a_n^m(\lambda)H_{np}^+(\lambda a) - a_n^m(\lambda)H_{np}^-(\lambda a), 
\]  
(2.29)

\[
 a_n^m(\lambda) \left[ \frac{g}{\lambda a}J_{np}(\lambda a) + J_{np}'(\lambda a) \right] = a_n^m(\lambda)H_{np}^+(\lambda a) - a_n^m(\lambda)H_{np}^-(\lambda a). 
\]  
(2.30)

Here and below, the prime in Bessel and Hankel functions denotes the differentiation with respect to their arguments, \( \lambda a \). Eliminating from (2.29) and (2.30) the amplitudes \( a_n^m(\lambda) \), we obtain the S-matrix

\[
 S_n(\omega) = \frac{gJ_{np}(\lambda a)H_{np}^+(\lambda a) + 2i/\pi}{gJ_{np}(\lambda a)H_{np}^+(\lambda a) - 2i/\pi}. 
\]  
(2.31)

Now we address the consideration of the surface modes in the problem under study. Such solutions to the boundary-value problem (2.26), (2.24), (2.25) exist only for the negative constant \( g \). Indeed, when

\[
 \frac{\omega^2}{c^2} - k_z^2 < 0, 
\]

the radial equation (2.26) has the solutions \( K_{np}(\kappa r) \) and \( I_{np}(\kappa r) \), which exponentially vanish outside the cylinder shell in both directions, for \( r > a \) and for \( r < a \), respectively. Here

\[
 \kappa = \sqrt{k_z^2 - \omega^2/c^2}. 
\]  
(2.32)

Thus, the complete solution describing the evanescent waves (surface modes) is given by

\[
 f_n(r) = \begin{cases} 
 a_n^m(\kappa)I_{np}(\kappa r) & \text{for } r < a, \\
 a_n^m(\kappa)K_{np}(\kappa r) & \text{for } r > a. 
\end{cases} 
\]  
(2.33)

The matching conditions (2.24) and (2.25) give

\[
 a_n^m(\kappa)I_{np}(\kappa a) = a_n^m(\kappa)K_{np}(\kappa a), 
\]  
(2.34)

\[
 a_n^m(\kappa)K'_{np}(\kappa a) - a_n^m(\kappa)I_{np}'(\kappa a) = \frac{g}{a\kappa}a_n^m(\kappa)I_{np}(\kappa a). 
\]  
(2.35)

Taking into account the relation [36]

\[
 I_n(x)K'_n(x) - I'_n(x)K_n(x) = -\frac{1}{x}, 
\]

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we obtain the equation determining the frequencies of the surface modes:

\[ 1 + g I_n(x) K_n(x) = 0, \]  

(2.36)

where

\[ x^2 = a^2 \kappa^2 = a^2 \left( k_z^2 - \frac{\omega^2}{c^2} \right). \]  

(2.37)

The product of the modified Bessel functions \( I_\nu(x)K_\nu(x) \) has the following properties:

\[ I_0(x)K_0(x) \sim \ln x, \quad x \to 0, \]

\[ I_\nu(0)K_\nu(0) = \frac{1}{2^\nu}, \quad \nu > 0, \]

\[ I_\nu(x)K_\nu(x) \sim \frac{1}{2x} \left[ 1 - \frac{4\nu^2 - 1}{(2x)^2} + \cdots \right], \quad x \to \infty, \]  

(2.38)

(see, for example, [36]). By numerical calculations,\(^8\) one can easily make sure that, for all \( \nu \), the product \( I_\nu(x)K_\nu(x) \) monotonically decreases in the interval \( 0 < x < \infty \) varying between the limiting values given in (2.38). Hence, the frequency equation (2.36) with \( g < 0 \) has always a root for \( n = 0 \), but for \( n \geq 1 \) this equation has the root, provided that the inequality

\[ |g|^{-1} < \frac{1}{2np} \]  

(2.39)

is satisfied. Obviously for a given \( |g| \), condition (2.39) can be true only for some first values of \( n \). As a consequence, merely for such \( n \), the frequency equation (2.36) can have solution.

Let \( x = a \kappa \) be a root of (2.36), then the frequency of the surface mode, \( \omega_{sm} \), is given by

\[ \omega_{sm} = c \sqrt{k_z^2 - \kappa^2} \]  

(2.40)

(see (2.37)). Obviously, the frequency \( \omega_{sm} \) should be a real quantity; thus, one has to add to (2.40) the restriction

\[ k_z > \kappa. \]  

(2.41)

Summarizing, we can claim that the spectral problem under consideration has the continuous spectrum in the domain (2.12) for both signs of the coupling constant \( g \). For negative \( g \) there appear, in addition, the surface modes with frequencies (2.40), (2.41).

Closing this section, it is worth mentioning solutions in the problem under consideration which are obtained from the surface modes when \( k_z < \kappa \). These solutions, as the surface modes, are located around \( r = a \) (see (2.33)), but they have pure imaginary frequencies

\[ \omega = \pm i \tilde{\omega} \quad \text{with} \quad \tilde{\omega} = c \sqrt{k_z^2 - k_\kappa^2}, \quad \kappa > k_z. \]  

(2.42)

As a result, such solutions exponentially rise with time \( \sim \exp(\pm \tilde{\omega} t) \). Hence, already at the first-quantized level, there is instability in the system at hand. However, in what follows, we disregard this instability. In our paper [37], an analogous solution was discussed briefly in the framework of the flat plasma sheet model with a negative parameter \( q \), and it was interpreted as a resonance solution. In reality, it is not the case and it is impossible to confront the considered solutions of the Klein–Gordon equation with any special solution of the Schrödinger equation.

\(^8\) The authors are indebted to M Bordag for such calculations.
3. Global zeta functions and Casimir energies

Now we are in a position to construct the spectral zeta functions
\[ \zeta(s) = \sum_{|q|} \left( \omega_{q}^{-2s} - \bar{\omega}_{q}^{-2s} \right) \]  
(3.1)

and to calculate, on this basis, the vacuum energy of quantum fields in question
\[ E = \frac{\hbar}{2} \zeta \left( s = -\frac{1}{2} \right). \]  
(3.2)

The summation in (3.1) should be done over the whole spectrum (discrete, continuous, and with allowance for the surface modes if such exist). The frequencies \( \bar{\omega}_{q} \) in (3.1) are obtained from \( \omega_{q} \) when \( a \to \infty \). The subtraction of \( \bar{\omega}_{q} \) in (3.1) corresponds to removing from the vacuum energy the contributions proportional to the volume of the manifold and to the manifold boundary. In this way, the renormalization of the vacuum energy is accomplished. The parameter \( s \) is considered at first to belong to the region of the complex plane \( s \) where the sum in (3.1) exists. After that the analytical continuation of (3.1) to the point \( s = -\frac{1}{2} \) and further for \( \text{Re} \ s < -\frac{1}{2} \) should be done. When summing over the continuous branch of the spectrum, the function of the spectral density shift has to be employed [32]. The rigorous mathematical theory of scattering gives the following expression for this function:
\[ \Delta \rho(\omega) \equiv \rho(\omega) - \rho_{0}(\omega) = \frac{1}{2\pi i} \frac{d}{d\omega} \det S(\omega), \]  
(3.3)

where \( S(\omega) \) is the \( S \)-matrix in the spectral problem at hand. Here \( \rho(k) \) is the density of states for a given potential (or for a given boundary condition in the case of compound media) and \( \rho_{0}(k) \) is the spectral density in the respective free spectral problem (for a vanishing potential or for a homogeneous unbounded space).

3.1. Perfectly conducting boundary conditions

As shown in section 2.2, the spectrum in this problem has two branches: discrete branch (electromagnetic oscillations inside the shell) and continuous branch (oscillations outside of the shell). In a complete form, (3.1) reads now
\[ \zeta(s) = 2 \sum_{\sigma} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sum_{n=0}^{\infty} \left[ \sum_{n_{r}} \omega_{nr}^{-2s}(\sigma, k) + \int_{ck_{r}}^{\infty} \omega^{-2s} \Delta \rho_{n}(\sigma, \omega, k) d\omega \right] - (a \to \infty), \]  
(3.4)

where \( \sigma = \text{TE}, \text{TM} \) and the prime over the sum sign means that the term with \( n = 0 \) is taken with the weight \( \text{1/2} \). The discrete frequencies \( \omega_{nr}(\sigma, k) \) in (3.4) are defined by (2.9) and (2.10), and the function of the spectral density shift is given by (3.3) and (2.15).

The further treatment of (3.4) is aimed at deriving a unique integral representation for the contributions of both the discrete branch and the continuous branch of the spectrum, the integration being done over imaginary frequencies \( \omega \to i\omega \). This is accomplished by transforming the sum \( \sum_{n_{r}} \) into the contour integral applying the argument principle theorem from the complex analysis (see figure 1(a)). After that, the initial contour \( C \) is continuously transformed into the contour \( C_{\Omega} \) with \( \Omega \to \infty \). The contribution of the continuous part of the spectrum is also expressed in terms of integrals along the contours \( C_{\Omega} \) shown in figure 1(b). The contour \( C_{\Omega} \) is used for transforming the integral \( \int_{ck}^{\infty} d\omega \omega^{-2s} \cdots \) in (3.4) containing \( H^{+}(H^{-}) \) and \( H^{+\prime}(H^{-\prime}) \) in accord with (2.15). In both the cases, the
integration is accomplished on the complex frequency plane $\omega$ with the cut connecting the points $ck_z$ and $-ck_z$.

It is worth noting here that the analytic properties of the scattering matrix (or the Jost function) in the Casimir studies are different in comparison with the standard theory of potential scattering [38]. In fact they are close to those for the Klein–Gordon equation, the role of mass squared in Casimir calculations being played by $k^2_z$. This implies in particular that the analytical properties of the scattering matrix in the Casimir calculations should be revealed by a direct analysis of its explicit form without referring to the nonrelativistic potential scattering.

Taking into account all this, we obtain the following formula for the spectral zeta function in the problem under consideration:

$$
\zeta_{\text{shell}}(s) = \frac{1}{\sqrt{\pi}} \frac{\alpha(s)}{\Gamma(\frac{3}{2} - s)} \left(\frac{a}{\alpha}\right)^{-2s} \sum_{n=0}^{\infty} \int_{0}^{\infty} dy \ y^{1-2s} \frac{d}{dy} \ln \left[1 - \mu_{np}^2(y)\right]. \tag{3.5}
$$

where

$$
\mu_{np}(y) = y[I_{np}(y)K_{np}(y)]'. \tag{3.6}
$$

It is worth comparing (3.5) with (3.11) in [34]. Further we will follow the well-elaborated procedure for the construction of the analytical continuation of the formulae like (3.5) into the left semi-plane of the complex variable $s$ (see, for example, [39]). This regular procedure substantially uses the uniform asymptotic expansion (UAE) of the modified Bessel functions. Remarkably, the result obtained in this way can be of arbitrary required accuracy. To obtain it, one should keep in UAE sufficiently many terms. As far as we know, there are no other methods to construct the analytical continuation needed.

Upon the change of the integration variable in (3.5) $y = npz$, $n = 1, 2, \ldots$, we can use UAE here. We content ourselves with the first two terms in UAE

$$
\ln \left\{1 - \left[\frac{y}{dz}(I_{np}(npz)K_{np}(npz))\right]^2\right\} = -\frac{z^4y^6}{4n^2p^2} \left[1 + \frac{y^2}{4n^2p^2} \left(3 - 30y^2 + 35y^4 + \frac{1}{2}z^4y^4\right) + O(n^{-4})\right], \quad t = (1 + z^2)^{-1/2}. \tag{3.7}
$$
Now we substitute (3.7) into all the terms in (3.5) with $n \neq 0$. The term with $n = 0$ in this sum will be treated by subtracting and adding to the logarithmic function the quantity \[ -\frac{1}{4} \frac{y^4}{(1 + y^2)^7}. \] (3.8)

As a result, the zeta function $\zeta_{\text{shell}}(s)$ can be presented now as the sum of three terms

$$
\zeta_{\text{shell}}(s) = Z_1(s) + Z_2(s) + Z_3(s),
$$

where

$$
Z_1(s) = \frac{2s - 1}{2\sqrt{\pi}a} \left( \frac{c}{a} \right)^{2s} \int_0^\infty dy \, y^{-2s} \left\{ \ln \left[ 1 - \mu_0^2(y) \right] + \frac{1}{4} y^4 f(y) \right\},
$$

$$
Z_2(s) = \left( \frac{c}{a} \right)^{-2s} \frac{(1 - 2s)(3 - 2s)}{64\sqrt{\pi}a} [2p^{-1-2s} \zeta_2(2s + 1) + 1] \frac{\Gamma \left( \frac{1}{2} + s \right)}{\Gamma(s)},
$$

$$
Z_3(s) = \left( \frac{c}{a} \right)^{-2s} \frac{(1 - 2s)(3 - 2s)(284s^2 - 104s - 235)}{64144\sqrt{\pi}a p^{3+2s}} \frac{\Gamma \left( \frac{3}{2} + s \right)}{\Gamma(s)} \zeta_2(3 + 2s).
$$

As shown in [34], $Z_1(s)$ is an analytic function of the complex variable $s$ in the region $-3/2 < \Re s < 1/2$. In (3.11) and (3.12), $\zeta_2(s)$ is the Riemann zeta function which accomplishes the analytical continuation of the sum $\sum_{n=1}^\infty n^{-2s}$ to the left semi-plane of $s$.

It is left now to take the limit $s \to -1/2$ in (3.9). The contribution $Z_1(-1/2)$ to the zeta function does not contain the parameter $p$, i.e. it remains the same as in the case of Minkowski spacetime. Hence, for $Z_1(-1/2)$, the value from [34] can be taken:

$$
Z_1(-1/2) = \frac{c}{2\pi a^2} (-0.6517).
$$

Special care should be taken when calculating the limit $s \to -1/2$ in $Z_2(s)$ (3.11) in view of the poles of the function $\Gamma(s + 1/2)$ at this point. Using the values

$$
\zeta_2(0) = -\frac{1}{2}, \quad \zeta_2'(0) = -\frac{1}{2} \ln 2\pi, \quad \Gamma(x) = \frac{1}{x} - \gamma + O(x),
$$

we derive [34, 40]

$$
\lim_{s \to -1/2} \left[ 2p^{-1-2s} \zeta_2(1 + 2s + 1) \Gamma \left( \frac{1}{2} + s \right) \right] = \lim_{s \to -1/2} \left[ 2[1 - (1 + 2s) \ln p + \cdots] \right]
$$

$$
\times \left[ \zeta_2(0) + \zeta_2'(0)(1 + 2s + \cdots) + 1 \right] \left( \frac{2}{1 + 2s} - \gamma + \cdots \right)
$$

$$
= -2 \ln \frac{2\pi}{p}.
$$

Thus, we have for $Z_2(-1/2)$

$$
Z_2(-1/2) = \frac{c}{8\pi a^2} \ln \frac{2\pi}{p}.
$$

The third term (3.12) gives

$$
Z_3(-1/2) = \frac{c}{2\pi a^2} \frac{7}{480} \frac{\pi^2}{6p^2}.
$$

Gathering together $Z_i(-1/2), \ i = 1, 2, 3$, we obtain the finite value for the vacuum energy of the electromagnetic field in the problem under consideration:

$$
E_{\text{shell}} = \frac{\hbar}{2} \zeta_{\text{shell}} \left( -\frac{1}{2} \right).
$$
where
\[ \zeta_{\text{shell}} \left( \frac{1}{2} \right) = \frac{c}{2\pi a^2} \left( -0.6517 + \frac{1}{4} \ln \frac{2\pi}{p} + \frac{7}{480} \frac{\pi^2}{p^2} \right). \] (3.19)

At \( p = 1 \), we reproduce (3.35) from [34].

The consideration presented here can be extended to the next terms in the UAE of the Bessel functions (3.7) in a straightforward way. Therefore, we shall not present here these rather cumbersome expressions.

### 3.2. Diaphanous matching conditions

In this case, we have a continuous spectrum of electromagnetic oscillations (2.12). As a result, only the second term in (3.4) survives. When going to the imaginary frequencies, the contours \( C_{\pm}/Omega_1 \) should be used (see figure 1(b)). We again obtain the integral representation (3.5) for the spectral zeta function, but now \( \mu_{np}(y) \) should be multiplied by the parameter \( \xi^2 \):

\[ \xi^2 = \frac{(\xi_1 - \xi_2)^2}{(\xi_1 + \xi_2)^2} = \frac{(\mu_1 - \mu_2)^2}{(\mu_1 + \mu_2)^2} \leq 1. \] (3.20)

Performing in the same way as for perfectly conducting boundary conditions, we arrive at the result

\[ \zeta_{\text{lin}} \left( \frac{1}{2} \right) = \frac{c}{2\pi a^2} \left( -0.490878 + \frac{1}{4} \ln \frac{2\pi}{p} + \frac{\pi^2}{288} \frac{7}{360} \frac{1}{p^2} \right). \] (3.21)

We are considering only the contribution to the zeta function which is linear in \( \xi^2 \). When \( p \to 1 \), the derived value for \( \zeta_{\text{lin}}(-1/2) \) vanishes [33, 41, 42]. As noted earlier, in cosmic string physics the parameter \( p \) is close to 1. Hence, the vacuum energy of the electromagnetic field for these boundary conditions

\[ E = \frac{\hbar}{2} \zeta_{\text{lin}} \left( -\frac{1}{2} \right) \] (3.22)

will also be small.

### 3.3. Semitransparent matching conditions

At first, we assume that \( g > 0 \) and, as a consequence, the spectrum of excitations of the field \( \phi(t, x) \) is pure continuous (see section 2.3). When constructing the spectral zeta function, we carry out the integration over the continuous spectrum (2.12) by making use of (3.4) keeping there the second integral term alone. Further there is no need to subtract here the contribution of Minkowski spacetime (the term obtained in the limit \( a \to \infty \)). The point is that such a subtraction has already been done in (3.3). By making use of the exact expression for the \( S \)-matrix (2.31) and the integration contours shown in figure 1(b), we obtain the following representation for the zeta function:

\[ \zeta_{s-1}(s) = C(s) \sum_{n=0}^{\infty} \int_{0}^{\infty} dy y^{1-2s} \frac{d}{dy} \ln \left[ 1 + g I_{\nu}(y) K_{\nu}(y) \right], \] (3.23)

where \( \nu = np \) and

\[ C(s) = \frac{c^{-2s}}{a^{1-2s} \sqrt{\pi} \Gamma(s) \Gamma(3/2 - s)}, \quad C(-1/2) = -\frac{c}{2\pi a^2}. \] (3.24)
Further we shall consider the zeta function (3.23) in the linear approximation with respect to $g$:

$$\zeta_{s-t}(s) = g \, C(s) \sum_{n=0}^{\infty} \int_{0}^{\infty} dy \, y^{1-2s} \frac{d}{dy} [I_{\nu}(y)K_{\nu}(y)].$$

(3.25)

Analytical continuation of this function to the left semi-plane of the complex variable $s$ will be accomplished in the standard way (see, for example, [39]): we add and subtract to the integrand in (3.25) for $n \geq 1$ the first few terms of the UAE for the product of the modified Bessel functions $I_{\nu}(y)K_{\nu}(y)$:

$$I_{\nu}(y)K_{\nu}(y) \approx \sum_{k=0}^{\infty} \frac{P_k(t)}{y^{2k+1}}, \quad t = \frac{1}{\sqrt{1+z^2}},$$

(3.26)

where

$$P_0(t) = \frac{t^2}{2}, \quad P_1(t) = \frac{t^3}{16} (1 - 6t^2 + 5t^4),$$

(3.27a)

$$P_2(t) = \frac{t^5}{256} (27 - 580t^2 + 2170t^4 - 2772t^6 + 1155t^8),$$

(3.27b)

$$P_3(t) = \frac{t^7}{2048} (t^2 - 1)(425425t^{10} - 1106105t^8 + 1014442t^6)$$

$$- 383570t^4 + 51445t^2 - 1125),$$

(3.27c)

and so on [36].

Without high accuracy, we take in UAE (3.26) for subtracting and adding only two terms

$$I_{\nu}(y)K_{\nu}(y) \approx \sum_{k=0}^{1} \frac{P_k(t)}{y^{2k+1}} = \frac{t}{2\nu} \left[ 1 + \frac{t^2(1 - 6t^2 + 5t^4)}{8\nu^2} + \cdots \right].$$

(3.28)

In the case of the product $I_0(y)K_0(y)$, we add and subtract the usual asymptotic of this expression when $y \rightarrow \infty$

$$1 \quad \frac{2\sqrt{1+2y^2}}{2\sqrt{1+2y^2}}$$

(3.29)

As a result, the zeta function (3.25) can be represented now as the sum of four terms:

$$\zeta_{s-t}(s) = g \, C(s) \sum_{i=1}^{4} Z_i(s),$$

(3.30)

where

$$Z_1(s) = \frac{1}{2} \int_{0}^{\infty} dy \, y^{1-2s} \frac{d}{dy} \left[ I_0(y)K_0(y) - \frac{1}{2\sqrt{1+2y^2}} \right],$$

(3.31)

$$Z_2(s) = \frac{1}{2} \left( \sum_{n=1}^{\infty} v^{-2s} \frac{1}{2} \int_{0}^{\infty} dz \, z^{1-2s} \frac{d}{dz} d \right),$$

(3.32)

$$Z_3(s) = \frac{1}{16} \sum_{n=1}^{\infty} v^{-2-2s} \int_{0}^{\infty} dz \, z^{1-2s} \frac{d}{dz} t^3 (1 - 6t^2 + 5t^4),$$

(3.33)
\[ Z_4(s) = \sum_{n=1}^{\infty} v^{1-2s} \int_0^\infty dz z^{1-2s} \frac{d}{dz} \left\{ I_\nu(vz)K_\nu(vz) - \frac{t}{2\nu} \left[ 1 + \frac{t^2(1 - 6t^2 + 5t^4)}{8v^2} \right] \right\} \]. (3.34)

The obtained formulae (see (3.31), (3.32) and (3.33)) show clearly the basic idea of making use of the UAE for analytic continuation: this method allows one to factorize the divergences which are originated in the sum over \( n \) and in integration over \( y \) or \( z \). In what follows, we express the result of the summation in terms of the Riemannian zeta function and the results of integration through the gamma function. It is this that provides us with explicit analytic continuation needed. The last term, \( Z_4(s) \), can be evaluated only numerically upon fixing the parameter \( \nu = np \) and the value of the variable \( s \), for example, \( s = -1/2 \). As a rule, this term gives a small correction \([33, 40]\). Fortunately, in the problem under consideration, this term does not seem to contribute to the Casimir energy for arbitrary values of \( p \) (see below).

Taking into account the behaviour of \( \frac{d}{dy} [I_\nu(y)K_\nu(y)] \) at the origin and at infinity

\[
\frac{d}{dy} [I_\nu(y)K_\nu(y)] \simeq \begin{cases} 
-\frac{1}{y}, & y \to 0, \\
-\frac{1}{2y^2} - \frac{3}{16y^4}, & y \to \infty,
\end{cases}
\] (3.35)

we infer that the function \( Z_1(s) \) defined by (3.31) is an analytic function in the region

\[-1 < \Re s < 1/2.\] (3.36)
The upper bound in (3.36), which is caused by the behaviour of the integrand in (3.31), can be removed by introducing the infrared cutoff or the mass of the field \( \phi(t, x) \) (see, for example, \([40, p 4524]\)). At the end of calculations, this cutoff (or mass) should be set to zero. At the point \( s = -1/2 \), we obtain numerically

\[ g C(-1/2)Z_1(-1/2) = \frac{cg}{4\pi a^2} \cdot 0.999 \ldots .\] (3.37)

The integral in (3.32) exists originally in the region

\[ 0 < \Re s < 3/2,\] (3.38)

and the sum over \( n \) in this formula is finite in the domain

\[ \Re s > 1/2.\] (3.39)

Thus, in the region

\[ 1/2 < \Re s < 3/2,\] (3.40)
equation (3.32) defines an analytic function of the variable \( s \). We can analytically continue this function to the whole complex plane \( s \), except for simple poles at some isolated points. For this purpose, we have to put

\[ \sum_{n=1}^{\infty} v^{-2t} = p^{-2t} \Gamma(2s),\] (3.41)

\[ \int_0^\infty dz z^{1-2s} \frac{d}{dz} t = -\int_0^\infty dz z^{2-2s} t^3 = -\frac{1}{2} \frac{\Gamma(s)\Gamma(3/2 - s)}{\Gamma(3/2)}.\] (3.42)

We have used here the table integral \([43]\)

\[ \int_0^\infty z^{a-1}r^\beta dz = \frac{1}{2} \frac{\Gamma\left(\frac{a+\beta}{2}\right)\Gamma\left(\frac{-\beta}{2}\right)}{\Gamma\left(\frac{a}{2}\right)} .\] (3.43)
The right-hand side of (3.43) has no singularities (poles) in the region
\[ \text{Re } \alpha < 0, \quad \text{Re } (\alpha + \beta) > 0. \] (3.44)
Admitting the appearance of poles, we define the left-hand side of (3.42) in the whole complex plane \( s \) by the right-hand side in this formula.

Finally, we have
\[ g \mathcal{C}(s)Z_2(s) = -\frac{g}{8} \mathcal{C}(s)[2p^{-2s} \xi_R(2s) + 1] \frac{\Gamma(s) \Gamma(3/2 - s)}{\Gamma(3/2)} \]
\[ = -\frac{ge^{-2s}}{2\pi a^{1/2}} \left[ p^{-2s} \xi_R(2s) + \frac{1}{2} \right]. \] (3.45)
At the point \( s = -1/2 \), it gives
\[ g \mathcal{C}(-1/2)Z_2(-1/2) = -\frac{c g}{2\pi a^2} \left[ p \xi_R(-1) + \frac{1}{2} \right] = -\frac{c g}{4\pi a^2} \left( 1 - \frac{p}{6} \right). \] (3.46)

In the same way, we can construct the analytic continuation of the function \( Z_3(s) \), which is defined originally by (3.33) in the region
\[ 0 < \text{Re } s < 3/2. \] (3.47)
In the whole complex plane \( s \), we have
\[ g \mathcal{C}(s)Z_3(s) = -\frac{g e^{-2s}}{2\pi a^{1/2}} \xi_R(2 + 2s)(2s + 1)(2s - 1). \] (3.48)
At the point \( s = -1/2 \), we find
\[ g \mathcal{C}(-1/2)Z_3(-1/2) = -\frac{c g}{24\pi a^2} \frac{p - 1}{p}. \] (3.49)

The functions \( Z_i(s) \), \( i = 1, 2, 3 \), are defined by the initial formulae (3.31)–(3.33) in the domains of the complex plane \( s \) which have a common strip, namely
\[ 1/2 < \text{Re } s < 3/2 \] (3.50)
(see (3.36) with the note following (3.40) and (3.47)). This implies that the analytic continuations of the individual functions \( Z_i(s) \), \( i = 1, 2, 3 \), constructed above, applies also to the sum \( \sum_{i=1}^{3} Z_i(s) \).

Gathering together the contributions (3.37), (3.46) and (3.49) we obtain
\[ \zeta_{s-1}(-1/2) = -\frac{c g}{24\pi a^2} \left( 1 - \frac{p}{p} \right). \] (3.51)

It seems that this answer is exact (in the linear in \( g \) approximation). The point is that the last term, \( Z_4(s) \), in the sum (3.30) does not contribute to the vacuum energy for arbitrary \( p \). This assertion is based on the following fact noted, for the first time, in the paper [44, appendix A]: upon substituting into (3.34) the product \( I_{\nu}(\nu z)K_{\nu}(\nu z) \) by the UAE (3.26) and setting here \( s = -1/2 \), we obtain
\[ Z_4(-1/2) = \sum_{k=2}^{\infty} \int_0^{\infty} dz z^{2k-1} \frac{d}{dz} P_k(t) \sum_{n=1}^{\infty} \frac{1}{(np)^{2k-1}} \]
\[ = \sum_{k=2}^{\infty} p^{1-2k} \xi_R(2k - 1) \int_0^{\infty} dz z^{2k-1} \frac{d}{dz} P_k(t). \] (3.52)
Analytical integration in (3.52) gives zero:
\[ \int_0^{\infty} dz z^2 \frac{d}{dz} P_k(t) = 0. \] (3.53)
Figure 2. Contours in the complex $\omega$ plane which should be used when going on to the integration over imaginary frequencies in the case of spectrum having continuous branch and surface modes. There are two cuts along the real axes: $-\infty < \omega < 0$ and $ck_z < \omega < \infty$.

We have checked this relation for $k = 2, 3$ by making use of the table integral [43]
\[
\int_0^\infty dz \, z^{1-s} \frac{d}{dz} \Gamma(2\rho-1) = \left(1 - \rho\right) \frac{\Gamma\left(\frac{1-s}{2}\right) \Gamma\left(\rho - \frac{3-s}{2}\right)}{\Gamma(\rho)}, \quad 3 - 2 \text{Re} \, \rho < \text{Re} \, s < 3.
\]
(3.54)

Certainly, it is important to bring to light the origin of these null results (see also [25, 44]).

Proceeding from all this, we conclude that the exact value (in the linear in $g$ approximation) of the Casimir energy in the problem at hand is
\[
E_{st} = \frac{\hbar}{2} \zeta_{s-t}(-1/2) = \frac{ch g}{48\pi a^2} \left( p - \frac{1}{p} \right), \quad g > 0.
\]
(3.55)

The vacuum energy (3.55) is positive because we have assumed that the constant $g$ is positive. In the limit $p = 1$, (3.55) reproduces the vanishing vacuum energy of the cylindric delta-potential [25, 44].

Now we address the consideration of a negative constant $g$, when the spectrum of the field $\varphi(t, x)$ has a continuous part (2.12) and surface modes with frequencies (2.40). In conformity with this, we represent the zeta-regularized vacuum energy $E(s)$ as the sum of two terms
\[
E(s) = E_{sm}(s) + E_{cont}(s)
\]
(3.56)
(see (3.4)). Transition to the imaginary frequencies in $E_{cont}(s)$ is accomplished by making use of the integration along the contours $C^+_\Omega$ and $C^-_\Omega$ shown in figure 2. One can see from this figure that $E_{cont}(s)$ involves, with an opposite sign, the contributions of the surface modes $\omega_{sm}$ ((2.40) with $|k_z| > k$) and the contribution of pure imaginary frequencies $\pm i\tilde{\omega}$ (see (2.42)). The contribution of $\omega_{sm}$ to $E_{cont}(s)$ is cancelled with $E_{sm}(s)$ (integration along the contour $C_s$, the contributions of the imaginary frequencies $\pm i\tilde{\omega}$ are mutually cancelled at $s = -1/2$.

As a result, we again arrive at formula (3.23) with an alone change, namely the integral over imaginary frequencies$^9$ $y$ should be treated as the principal value integral
\[
E(s) = \frac{1}{2} C(s) \sum_{n=0}^{\infty} \text{P.V.} \int_0^\infty dy \, y^{1-2s} \frac{d}{dy} \ln \left[1 - |g| I_n(y) K_n(y)\right],
\]
(3.57)

$^9$ The variable $y$ is not exactly equal to $i\omega$; more precisely, it is related to $i\lambda$ in (2.8).
where $C(s)$ is defined, as before, in (3.24). Obviously, in the linear in $-|g|$ approximation, we do not ‘feel’ the P.V. prescription. Hence, in this approximation, we have the result (3.55) with the negative sign:

$$
E_s = -\frac{\hbar |g|}{48\pi a^2} \left( p - \frac{1}{p} \right), \quad g < 0.
$$

(3.58)

In [25, 44] dealing with a massless scalar field in the background of a semitransparent cylindrical shell ($p = 1$), the negative values of $g$ were not considered at all with the aim ‘to avoid the appearance of negative eigenfrequencies’ [25, p 29]. The vacuum energy of a massive scalar field in the background of a cylindrical $\delta$-potential was considered in [45, 46] for both repulsive and attractive potentials. The possibility of bound states was not investigated there, and it was noted that for $g < 0$ the renormalized Casimir energy acquires complex values.

4. High-temperature expansions

Quantum field theory predicts a specific temperature dependence for the energy of radiation connected with a heated body. One cannot anticipate here the Planck spectrum and the Boltzmann law. These laws hold for the photons being in unbounded space and nevertheless possessing nonzero temperature. For such photons, a notion of black body radiation has been introduced as far back as in pre-quantum physics time. It is clear that in the general case, the spectrum of a heat radiation and its temperature behaviour should depend on the shape of a heated body. By the way, this dependence is important for much radio engineering equipment (thermal noise of antennae, waveguides and so on).

For obtaining the asymptotic behaviour of thermodynamical functions, a powerful method of the zeta function technique and the heat kernel expansion [47] can be used. It is important that for these calculations, it is sufficient to find the zeta function and the heat kernel coefficients not for a complete operator, determining the quadratic action of the quantum field under consideration, but only for the space part of this operator, i.e. one has to calculate first the zeta function that is used for the Casimir calculations at zero temperature. This is an essential merit of this approach. It is these spectral zeta functions that have been constructed in the preceding sections. Obviously, these results can be used for obtaining the high-temperature expansions for the thermodynamic potentials in the problems under consideration.

The high-temperature asymptotics for the Helmholtz free energy has the form [39, 47]

$$
F(T) \simeq -\frac{T}{2} \zeta(0) + B_0 \frac{T^4}{\hbar^3} - \frac{B_{1/2}}{3} \frac{T^3}{4\pi^3/2h^2} \zeta_K(3) - \frac{B_1}{24} \frac{T^2}{\hbar} + \frac{B_{3/2}}{(4\pi)^{3/2}} T \ln \frac{\hbar}{T} - \frac{B_2}{16\pi^2} \hbar \ln \left( \frac{T}{4\pi T} \right) + \gamma - \frac{B_{5/2}}{(4\pi)^{3/2}} \frac{h^2}{24T} - T \sum_{n \geq 3} \frac{B_n}{(4\pi)^{3/2}} \left( \frac{h}{2\pi T} \right)^{2n-3} \Gamma(n-3/2) \zeta_K(2n-3), \quad T \to \infty.
$$

(4.1)

Here $\gamma$ is the Euler constant and $B_{n/2}$, $n = 0, 1, 2, \ldots$, are the heat kernel coefficients which can be calculated by making use of the spectral zeta function in the problem at hand

$$
\frac{B_{n/2}}{(4\pi)^{3/2}} = \lim_{s \to 3/2} \left( s + \frac{n - 3}{2} \right) \Gamma(s) \xi(s), \quad n = 0, 1, 2, \ldots.
$$

(4.2)

In order to use this formula, the function $\xi(s)$ should be known in the vicinity of the following points:

$$
s = \frac{3}{2}, 1, \frac{5}{2}, 0, \ldots.
$$

(4.3)
The heat kernel coefficient $B_{n/2}$ is different from zero only when the product $\Gamma(s)\zeta(s)$ has a (simple) pole at the point $s = (3 - n)/2$, $n = 0, 1, 2, \ldots$.

The asymptotic expansions for the internal energy $U(T)$ and the entropy $S(T)$ are deduced from (4.1) through the thermodynamic relations

$$U(T) = -T^2 \frac{\partial}{\partial T} \left( T^{-1} F(T) \right),$$

$$S(T) = T^{-1} \left( U(T) - F(T) \right) = -\frac{\partial F}{\partial T}. \tag{4.5}$$

For compact manifolds, $B_0 \sim V$ and $B_{1/2} \sim S$, where $V$ is the volume of the manifold and $S$ is the area of its boundary. In view of this, the second term on the right-hand side of (4.1) gives the Stefan–Boltzmann law ($U(T) \sim VT^4$).

In the case of noncompact manifolds, the relative spectral functions (zeta function and heat kernel) are employed (see, for example, [48, 49]). When defining these functions, the contribution due to free space is subtracted. We have noted earlier that the scattering formalism implements this automatically. As a result, in our approach $B_0/V \to 0$, when $V \to \infty$, and $B_{1/2}/S \to 0$, when $S \to \infty$ (see for details [26, p 446]).

Summarizing, we may infer that the approach applied gives in fact the corrections to the energy density of electromagnetic energy. The area of practical employment of these thermodynamical asymptotics is the calculation of the temperature dependence of respective Casimir forces.

The heat kernel coefficients needed in (4.1) were calculated in our previous article [26] and in appendices A and B of this paper. The value of $\zeta'(0)$ can be obtained by the technique employed for this purpose, for example, in [47]. By making use of all this, we obtain the thermodynamical asymptotics in the geometry under consideration.

For an electromagnetic field with perfectly conducting boundary conditions, these asymptotics are

$$F(T) \simeq -\frac{T}{2} \zeta'(0) + \frac{3}{64p} \frac{T}{a} \ln \frac{\hbar}{T} - \frac{153}{2048} \frac{1}{4\pi} \frac{\hbar^2 c^2}{24T a^3}, \tag{4.6}$$

$$U(T) \simeq \frac{3}{16} \frac{T}{4\pi a} - \frac{153}{2048} \frac{1}{4\pi} \frac{\hbar^2 c^2}{12T a^3}, \tag{4.7}$$

$$S(T) \simeq \frac{1}{2} \zeta'(0) + \frac{3}{64pT} \left( 1 - \ln \frac{\hbar}{T} \right) - \frac{153}{2048} \frac{1}{4\pi} \frac{\hbar^2 c^2}{24T a^3}, \tag{4.8}$$

where

$$\zeta'(0) = \frac{1}{a} \left[ 0.53490 + \frac{1}{32p} \left( 3 \ln \frac{a}{2pc} + 3\gamma - 4 - \frac{47}{256p^2} \zeta_R(3) \right) \right]. \tag{4.9}$$

For isorefractive matching conditions, we have

$$F(T) \simeq -\frac{T}{2} \zeta'(0) + \frac{2}{45} \frac{T^3 a^2 \xi^2 \pi^3}{c^3} + \frac{3}{16} \frac{T}{pa} T \ln \frac{\hbar}{T} - \frac{45}{1024} \frac{\xi^2 \hbar^2 c^2}{pa^3 96T}, \tag{4.10}$$

$$U(T) \simeq -\frac{T^4}{\hbar^4} \frac{2\pi^3}{15} \frac{a^2 \xi^2}{c^3} + \frac{3}{64p} \frac{T}{a} - \frac{45}{1024} \frac{\xi^2 \hbar^2 c^2}{48pa^3}, \tag{4.11}$$

where
\[ S(T) \simeq \frac{1}{2} \zeta'(0) - \frac{8\pi^3}{45} \frac{a^2 \xi^2}{c^3} \frac{T^3}{\hbar^3} + \frac{3}{16} \frac{\xi^2}{pa} \left( 1 - \ln \frac{\hbar}{T} \right) - \frac{45}{1024} \frac{\xi^2 c^2}{pa^3} \frac{\hbar^2}{94T^2}, \quad (4.12) \]

where

\[ \zeta'(0) = \frac{\xi^2}{a} \left[ 0.28428 + \frac{1}{32p} \left( 3 \ln \frac{a}{2pc} + 3\gamma - 4 - \frac{27}{128p^2} \zeta_R(3) \right) \right]. \quad (4.13) \]

And finally for a massless scalar field with the \( \delta \)-potential (\( g > 0 \)), we obtain

\[ F(T) \simeq -\frac{T}{2} \zeta'(0) - \frac{T^4}{\hbar^4} \frac{\pi^3 ga^2}{90c^3} - \frac{\pi g T^2}{12hc} \left( 1 - \frac{1}{p} \right), \quad (4.14) \]

\[ U(T) \simeq \frac{T^4}{\hbar^4} \frac{\pi^3}{30^c^3} \frac{g^2 a^2}{12hc} \left( 1 - \frac{1}{p} \right). \quad (4.15) \]

\[ S(T) \simeq \frac{1}{2} \zeta'(0) + \frac{T^3}{\hbar^3} \frac{2\pi^3 ga^2}{45c^3} + \frac{\pi g T}{6hc} \left( 1 - \frac{1}{p} \right). \quad (4.16) \]

where

\[ \zeta'(0) = \frac{g}{\pi a} \left( -0.9818 + \frac{\pi^2}{144p^2} + \frac{1}{2} \ln \frac{2\pi}{p} \right). \quad (4.17) \]

5. Conclusions

We have demonstrated that enclosing the conical singularity with a cylindrical surface and imposing here appropriate boundary conditions on quantized fields render the total vacuum energy of these fields finite, at least, in the zeta renormalization technique.

It is worth noting here that usually conical singularity is considered as the singularity of the curvature tensor [20, 50]. We have treated this singularity as a non-smoothness of the boundary [26].

Calculation of the Casimir energy for a configuration close to those considered in our paper has attracted much attention in recent studies. Ellingsen, Brevik, and Milton [51–55] calculated the vacuum energy for wedge and cylinder geometry which includes, as a special case, our cone configuration. The periodic boundary conditions in angular variable should be used for this choice. Only this case has rigorous justification for the calculations conducted (vanishing of the heat kernel coefficient \( B_2 \)).

The local energy density for a wedge with a coaxial cylindrical shell was considered in [56–60]. The singular behaviour of this quantity at \( r = 0 \) was noted in these studies. This singularity forbade the calculation of the respective total vacuum energy.

As a byproduct, we have presented here a complete analysis of the spectral problem for a scalar massless field in the background of the \( \delta \)-potential located at a cylinder lateral surface, both the \( \delta \)-wall and the \( \delta \)-well being considered.

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Appendix A. The heat kernel coefficients for isorefractive boundary conditions

Here we calculate the heat kernel coefficients for an electromagnetic field obeying the isorefractive matching conditions on the cylindrical surface of radius $a$ in the conical space. We shall use relation (4.2) between the heat kernel coefficients and the relative spectral zeta function $\zeta(s)$. As noted in section 3.2, the spectral function in the problem at hand is given by (3.5), where $\mu_{np}(y)$ should be multiplied now by the parameter $\xi^2$ given in (3.20). Thus, we have

$$\zeta(s) = C(s) \sum_{n=0}^{\infty} \int_{0}^{\infty} dy \, y^{1-2s} \frac{d}{dy} \ln \left[ 1 - \xi^2 \mu_{n}^2(y) \right],$$  \hspace{1cm} (A.1)

where $\nu = np$, $\mu_{n}(y)$ is explained in (3.6) and $C(s)$ is defined in (3.24). Further we separate in (A.1) the contribution due to the term with $n = 0$ denoting the rest by $\zeta(s)$:

$$\zeta(s) = \zeta_0(s) + \zeta(s).$$  \hspace{1cm} (A.2)

In the $\xi^2$-approximation, $\zeta_0(s)$ reads

$$\zeta_0(s) = -\frac{\xi^2}{2} C(s) \int_{0}^{\infty} dy \, y^{1-2s} \frac{d}{dy} \mu_0^2(y).$$  \hspace{1cm} (A.3)

The integral over $y$ exists in the strip $-1/2 < \text{Re} \, s < 3/2$, (A.4) because

$$\frac{d\mu_0^2}{dy} = 4 \left( \ln \frac{y}{2} + \gamma + \frac{1}{2} \right) y + O(y^3), \quad y \to 0;$$

$$\frac{d\mu_0^2}{dy} = -\frac{1}{2y} - \frac{3}{4y^3} + O(y^{-7}), \quad y \to \infty.$$  \hspace{1cm} (A.5)

In view of (4.2), it implies that $\zeta_0(s)$ does not contribute to the heat kernel coefficients $B_k$ with $k = 1/2, 1, 3/2$.

The analytic continuation of $\zeta_0(s)$ given in (A.3) to the right of the strip (A.4), i.e. to the region $\text{Re} \, s > 3/2$, is accomplished by the addition and subtraction of the $y \to 0$ asymptote

$$\zeta_0(s) = -\frac{\xi^2}{2} C(s) \int_{0}^{\infty} dy \, y^{1-2s} \frac{d}{dy} \mu_0^2(y)$$

$$+ \int_{0}^{1} dy \, y^{1-2s} \left[ \frac{d}{dy} \mu_0^2(y) - 4 \left( \ln \frac{y}{2} + \gamma + \frac{1}{2} \right) y \right] + \zeta_{0,\to}^{\text{sing}}(s),$$  \hspace{1cm} (A.6)

$$\zeta_{0,\to}^{\text{sing}}(s) = 2\xi^2 C(s) \left[ \frac{1}{(3-2s)^2} + \left( \ln 2 - \gamma - \frac{1}{2} \right) \frac{1}{3-2s} \right].$$  \hspace{1cm} (A.7)

By making use of this result, we derive the contribution to the heat kernel coefficient $B_0$ generated by $\zeta_{0,\to}^{\text{sing}}(s)$:

$$(4\pi)^{3/2} \left. \text{res}_{s=3/2} \Gamma(s) \zeta_{0,\to}^{\text{sing}}(s) \right) = -4\pi \xi^2 a^2 c.$$  \hspace{1cm} (A.8)

It should be noted that this contribution\(^{10}\) is due to the infrared singularity of the spectral density in the problem at hand.

\(^{10}\) In [61], the calculations analogous to (A.6) and (A.7) were performed incorrectly. This resulted in an erroneous conclusion drawn in [47] that $B_0 = 0$ for a material cylinder with $c_1 = c_2$.}

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The analytic continuation of the integral in (A.3) to the left of the strip (A.4), i.e. to the region \( \text{Re} s < -1/2 \), is accomplished by the addition and subtraction of the \( y \to \infty \) asymptote (A.5):

\[
\zeta_0(s) = -\frac{\xi^2}{2} C(s) \left[ \int_0^1 dy y^{1-2s} \frac{d}{dy} \mu^2_0(y) + \int_1^\infty dy y^{1-2s} \left( \frac{d}{dy} \mu^2_0(y) + \frac{1}{2y^3} + \frac{3}{4y^5} \right) \right] + \zeta_0^\text{sing}(s),
\]

(A.9)

\[
\zeta_0^\text{sing}(s) = \frac{\xi^2}{2} C(s) \left( \frac{1}{2} \frac{1}{1 + 2s} + \frac{3}{4} \frac{1}{2s + 3} \right).
\]

(A.10)

From (A.10), we derive the respective contributions to the heat kernel coefficients \( B_2, B_{3/2}, B_3 \):

\[
\pi \xi^2 \frac{c}{a^2}, \quad 0, \quad \frac{3}{4} \pi \xi^2 \frac{c^3}{a^4}.
\]

(A.11)

Now we turn to \( \zeta(s) \). To draw out the singularities in \( \zeta(s) \), we apply the UAE of the modified Bessel functions. In the \( \xi^2 \)-approximation, it gives

\[
\ln[1 - \xi^2 \mu^2_c(vz)] = -\xi^2 \frac{\zeta(s)}{4v^2} \left[ 1 + \frac{j^2}{4v^2} (3 - 30v^2 + 35v^4)
\right.
\]

\[
\left. + \frac{t^4}{4v^4} (9 - 256t^2 + 1290t^4 - 2037t^6 + 1015t^8) \right] + O(\xi^4).
\]

(A.12)

The integrals over \( z \) converge in the range \( (5 - j)/2 < \text{Re} s < 5/2 \), where \( j \) is the power of \( t \) in (A.12). After the integration over \( z \) in the divergent parts of the zeta function \( \zeta(s) \), we arrive at the result

\[
\zeta^\text{div}(s) = -\frac{\xi^2 g^{2j-1} \rho^{1-2i-j}}{2\sqrt{\pi} e^{\pi/2} \Gamma(s) \Gamma(3/2 - s)} \xi_0(2s - 1 + i) A_i(s), \quad i = 1, 2, \ldots,
\]

\[
A_2(s) = -\frac{1}{64} \frac{\pi}{\cos(\pi s)} (-1 + 2s)^2 (-3 + 2s),
\]

\[
A_4(s) = -\frac{1}{24576} \frac{\pi}{\cos(\pi s)} (-3 + 2s) (27 - 46s - 44s^2 + 56s^3) (-1 + 4s^2),
\]

\[
A_6(s) = -\frac{1}{47185920} \frac{\pi}{\cos(\pi s)} (-9 + 4s^2) (-1 + 4s^2)
\]

\[
(9586s + 4640s^3 - 18000s^3 + 15272s^2 - 2576s^2 - 6345),
\]

\[
A_{2i+1} = 0, \quad i = 1, 2, \ldots.
\]

For obtaining (A.13), the integration was carried out by making use of formula (3.43).

The contributions of \( \zeta(s) \) to the heat kernel coefficients \( B_k, k = 0, 1/2, 1, 3/2, 2, 5/2, 3 \), respectively, are

\[
0, \quad 0, \quad 0, \quad \frac{3\pi^3/2}{8} \frac{\xi^2}{pa^2}, \quad \frac{\pi c \xi^2}{512} \frac{c^2 \xi^2}{pa}, \quad -\frac{3\xi^2 c^3}{4a^4}.
\]

(A.14)

Summing up the results for \( \zeta_0(s) \) and \( \zeta(s) \), (A.8), (A.11), (A.14), one obtains

\[
B_0 = -4\pi \frac{a^2}{c^2} \xi^2, \quad B_{3/2} = \frac{3\pi^3/2}{8} \frac{\xi^2}{pa}, \quad B_{5/2} = \frac{45\pi^3/2}{512} \frac{c^2 \xi^2}{pa},
\]

\[
B_{1/2} = B_1 = B_2 = B_3 = 0.
\]

(A.15)
Table B1. The contributions to the heat kernel coefficients $B_k$ coming from $\zeta_0(s)$ and $\tilde{\zeta}(s)$. The results are obtained in the linear in $g$ approximation.

| $B_k$       | $\zeta_0(s)$ | $\tilde{\zeta}(s)$ | Total          |
|-------------|--------------|---------------------|----------------|
| $B_0$       | $\frac{3}{2}$ $\pi g a^2/c^3$ | 0                    | $\pi g a^2/c^3$ |
| $B_{1/2}$   | 1            | 0                   | 0              |
| $B_1$       | $\frac{1}{2}$ $2\pi g/c$      | $-2\pi g/pc$        | $2\pi g/c (1 - 1/p)$ |
| $B_{3/2}$   | 0            | $-2\sqrt{\pi} g/a$ | 0              |
| $B_2$       | $\frac{1}{2}$ 0                           | 0              |
| $B_{5/2}$   | $-1$         | $-g \sqrt{\pi} c^2/2a^3$ | $g \sqrt{\pi} c^2/2a^3$ |

Appendix B. The heat kernel coefficients for a cone with the $\delta$-potential

In the zeta function (3.23), we again pick out the term with $n = 0$:

$$\zeta_0(s) = \frac{1}{2} C(s) \int_0^\infty dy \, y^{1-2s} \frac{d}{dy} \ln[1 + g I_0(y) K_0(y)],$$

(B.1)

$$\tilde{\zeta}(s) = C(s) \sum_{n=1}^\infty (np)^{1-2s} \int_0^\infty dz \, z^{1-2s} \frac{d}{dz} \ln[1 + g I_0(\nu z) K_0(\nu z)], \quad \nu = np.$$  

In $\tilde{\zeta}(s)$, we use the UAE of the modified Bessel functions

$$\ln[1 + g I_0(\nu z) K_0(\nu z)] = \sum_{i=1}^\infty K_i(g, t) \nu^i, \quad t = \frac{1}{\sqrt{1 + z^2}}.$$  

(B.2)

$$K_1 = g \frac{t}{2}, \quad K_2 = -g^2 \frac{t^2}{8}, \quad K_3 = g \left( \frac{t^3}{16} - \frac{3t^5}{8} + \frac{5t^7}{16} \right) + g^3 \frac{t^3}{24} \cdots .$$  

With (B.2), the integrals over $z$ converge in the strip $(5 - j)/2 < \text{Re} s < 5/2$, where $j$ is the power of $t$, and

$$\tilde{\zeta}^{\text{div}}(s) = \sum_{i=1}^\infty Z_i(s),$$

(B.3)

$$Z_1(s) = -a^{2s-1} \frac{p^{-2s} g \zeta_0(2s)}{\pi c^{2s}},$$

$$Z_2(s) = a^{2s-1} \frac{2 \pi c^{2s} \Gamma(s) p^{-2s+1} g^2 \zeta_0(2s + 1)}{\Gamma(s)} \Gamma(s + 1/2) \Gamma(s + 1) - \frac{4 g \Gamma(s + 3)}{3},$$

We calculate the contributions from $\tilde{\zeta}^{\text{div}}(s)$ to the heat kernel coefficients in the linear in $g$ approximation by making use of (4.2). These results are presented in the fourth column of the table B1.

In the same approximation, $\zeta_0(0)$ is given by

$$\zeta_0(s) = \frac{g}{2} C(s) \int_0^\infty \frac{d}{dy} [I_0(y) K_0(y)].$$

(B.4)
The integral (B.4) converges in the region $0 < \text{Re} \, s < 1/2$, because
\[
\frac{d}{dy} [I_0(y)K_0(y)] = -\frac{1}{y} - \left(\ln \frac{y}{2} + \gamma\right) y + \cdots, \quad y \to 0;
\]
\[
\frac{d}{dy} [I_0(y)K_0(y)] = -\frac{1}{2y^2} - \frac{3}{16} \frac{1}{y^4} + \cdots, \quad y \to \infty. \tag{B.5}
\]
To perform the analytic continuation of (B.4) to the domain $\text{Re} \, s > 1/2$, one has to add and subtract from the integrand in (B.4) several terms of its small $y$ asymptote
\[
\zeta_0(s) = \frac{g}{2} C(s) \left\{ \int_0^1 dy \, y^{1-2s} \left[ \frac{d}{dy} (I_0(y)K_0(y)) + \frac{1}{y} + \left(\ln \frac{y}{2} + \gamma\right) y \right] + \int_1^\infty dy \, y^{1-2s} \frac{d}{dy} I_0(y)K_0(y) \right\} + \zeta_0^{\text{sing}}(s), \tag{B.6}
\]
where the singular part
\[
\zeta_0^{\text{sing}}(s) = \frac{ga^{2s-1}}{2\sqrt{\pi} \, c^{2s} \Gamma(s) \Gamma(3/2 - s)} \left[ -\frac{1}{1 - 2s} - (\gamma - \ln 2) \frac{1}{3 - 2s} + \frac{1}{(3 - 2s)^2} \right] \tag{B.7}
\]
has simple poles at the points $s = 3/2$ and $s = 1/2$. We use (B.7) to calculate the contribution of $\zeta_0(s)$ to the heat kernel coefficients $B_0$, $B_{1/2}$, and $B_{1}$. These contributions are generated by the infrared singularities of the spectral density in the problem under consideration.

Similarly, the $y \to \infty$ asymptote (B.5) should be added and subtracted to continue the integral (B.4) to the region $\text{Re} \, s \leq 0$. In this way, we obtain
\[
\zeta_0(s) = \frac{g}{2} C(s) \left\{ \int_0^1 dy \, y^{1-2s} \left[ \frac{d}{dy} (I_0(y)K_0(y)) + \frac{1}{2y^2} + \frac{3}{16} \frac{1}{y^4} \right] + \int_1^\infty dy \, y^{1-2s} \frac{d}{dy} I_0(y)K_0(y) \right\} + \zeta_0^{\text{sing}}_{\text{asymp}}(s), \tag{B.8}
\]
where
\[
\zeta_0^{\text{sing}}_{\text{asymp}}(s) = \frac{ga^{2s-1}}{2\sqrt{\pi} \, c^{2s} \Gamma(s) \Gamma(3/2 - s)} \left( -\frac{1}{4s} - \frac{3}{16} \frac{1}{2s + 2} \right). \tag{B.9}
\]
According to (4.2), the singular part (B.7) gives contribution to the heat kernel coefficients $B_0$, $B_{1/2}$ and $B_{1}$, while the singular part (B.9) contributes to the heat kernel coefficients $B_{3/2}$, $B_2$ and $B_{3/2}$. These are given in the second column of table B1.

Summing the contributions from $\zeta_0$ and $\zeta$, one obtains the heat kernel coefficients for a scalar massless field considered on a cone with semitransparent boundary conditions. These results are presented in the last column of table B1.

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