A dynamical proof of the van der Corput inequality

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\textbf{ABSTRACT}

We provide a dynamical proof of the van der Corput inequality for sequences in Hilbert spaces that is based on the Furstenberg correspondence principle. This is done by reducing the inequality to the mean ergodic theorem for contractions on Hilbert spaces. The key difficulty therein is that the Furstenberg correspondence principle is, \textit{a priori}, limited to scalar-valued sequences. We, therefore, discuss how interpreting the Furstenberg correspondence principle via the Gelfand–Naimark–Segal construction for C\textsuperscript{*}-algebras allows to study not just scalar but general Hilbert space-valued sequences in terms of unitary operators. This yields a proof of the van der Corput inequality in the spirit of the Furstenberg correspondence principle and the flexibility of this method is discussed via new proofs for different variants of the inequality.

\textbf{ARTICLE HISTORY}

Received 21 March 2022
Accepted 30 June 2022

\textbf{KEYWORDS}

Ergodic theorems; Furstenberg correspondence principle; C\textsuperscript{*}-algebras; Hilbert modules

\textbf{2020 MATHEMATICS SUBJECT CLASSIFICATIONS}

Primary: 47A35; Secondary: 46L08; 46L55

The \textit{van der Corput inequality} (also known as \textit{van der Corput lemma} or \textit{van der Corput trick}) is a well-known and versatile tool in ergodic theory commonly used for complexity reduction since it allows to study the decay of norms in terms of the decay of correlations. It can be used to prove van der Corput’s difference theorem on equidistribution of sequences (see, e.g. \cite{17, Section 3} or \cite{26}), to show that every weakly mixing system is weakly mixing of all orders (see \cite[Theorem 9.31]{10}), or to obtain convergence to zero in the proofs of weighted, subsequential, polynomial and multiple ergodic theorems as well as Wiener–Wintner results (see, e.g. \cite{16}, \cite[Section 7.4]{8}, \cite[Chapter 21]{10} and \cite{9}). Here is a common formulation of the inequality (see, e.g. \cite[Lemma 21.5]{14}).

\textbf{Theorem 0.1:} For every bounded sequence \((u_n)_{n \in \mathbb{N}}\) in a Hilbert space \(H\) the inequality

\[
\limsup_{N \to \infty} \left\| \frac{1}{N} \sum_{n=1}^{N} u_n \right\|^2 \leq \limsup_{J \to \infty} \frac{1}{J} \sum_{j=1}^{J} \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \text{Re}(u_n \overline{u}_{n+j})
\]

holds.

The standard proof of the inequality rests on an application of the Cauchy–Schwarz inequality as well as some elementary computations (see, e.g. the proof of \cite[Lemma 21.5]{14}).
There also exist more conceptual approaches, for example by means of positive definite functions, dilation theory, and the mean ergodic theorem (see [2, Theorem 2.12] and the blog post [19]). This article provides a new perspective on the inequality in terms of the Furstenberg correspondence principle.

Given a bounded sequence of scalars, the Furstenberg correspondence principle provides a universal way to study the statistical properties of the sequence via dynamical systems. It does so by constructing Hilbert spaces and unitary operators that encode the statistical behaviour of the sequence in dynamical terms. Below we show that one can use the strategy laid out by the correspondence principle to readily derive the scalar van der Corput inequality. Thereby, the scalar van der Corput inequality reduces to the mean ergodic theorem and a simple application of the Cauchy–Schwarz inequality.

The interest of reparing such a well-known inequality with a less elementary proof is twofold: on the one hand, such a conceptual proof provides a simple way to derive and understand not only the van der Corput inequality but also similar inequalities. More importantly, however, only the scalar-valued case can be treated in this way since, classically, the Furstenberg correspondence principle only concerns sequences of numbers. Given the importance of the correspondence principle as a bridge between number theory and ergodic theory, this begs the question: Does there exist some version of the correspondence principle for Hilbert space-valued sequences? We address this question in the first two sections by showing that, also in the case of Hilbert space-valued sequences, it is possible to construct a Hilbert space and a unitary operator acting on it which encode properties of the sequence. We show that this allows, in complete analogy to the scalar case, to reduce the general Hilbert space-valued van der Corput inequality to the mean ergodic theorem and the Cauchy–Schwarz inequality. Since the key tool in the proof is the Gelfand–Naimark–Segal construction for C*-algebras, the remaining sections discuss how these techniques allow to derive several other van der Corput-type inequalities on C*-algebras and Hilbert-C*-modules. This leads to new proofs of different uniform versions of the van der Corput inequality, see Corollaries 5.2, 5.3 and 5.4.

Organization of the article. Section 1 is a short recall of the correspondence principle and explains the difficulty of extending it to the vector-valued setting. Section 2 then presents all key ideas by giving a short dynamical proof of the scalar van der Corput inequality in Theorem 2.1 and then extending these ideas to a proof of the general van der Corput inequality in Theorem 2.2. We generalize these methods to C*-algebras in Section 3 and to Hilbert modules in Section 4. In Section 5, we show how these extensions allow to recover different uniform versions of the van der Corput inequality. The article concludes with a discussion on extensions to Følner nets and more general groups in Section 6.

1. Motivation: the Furstenberg correspondence principle

Given a bounded sequence \((a_n)_n \in l^\infty(\mathbb{N})\) of scalars, the Furstenberg correspondence principle provides a way to encode the sequence in a dynamical system. Classically, this is done by considering the closure \(A := \{a_n \mid n \in \mathbb{N}\}\) of its range, forming the symbolic shift system \((A^\mathbb{N}, \tau)\) with \(\tau((x_n)_n) = (x_{n+1})_n\), and passing to the subshift \((K; \tau)\) generated by the sequence \((a_n)_n\), i.e. \(K := \{\tau^k((a_n)_n) \mid k \in \mathbb{N}_0\}\). (Note that \(A^\mathbb{N}\) and hence \(K\) is compact since \(A\) is a closed, bounded subset of \(\mathbb{C}\).) For the study of statistical properties of the sequence \((a_n)_n\), i.e. the asymptotic behaviour of averages, one can then pass from the
topological system \((K; \tau)\) to the measure-preserving system \((K, \mathcal{B}, \mu; \tau)\) where \(\mathcal{B}\) denotes the Borel \(\sigma\)-algebra of \(K\) and \(\mu\) is some \(\tau\)-invariant probability measure that represents, e.g. density properties of the original sequence \((a_n)_n\). Thus the problem of understanding statistical properties of the sequence \((a_n)_n\) can be translated into the study of the system \((K, \mathcal{B}, \mu; \tau)\) and the Koopman operator \(T_\tau : L^2(K, \mathcal{B}, \mu) \to L^2(K, \mathcal{B}, \mu)\), \(f \mapsto f \circ \tau\). This approach has proven to be a critical interface between number theory and ergodic theory with applications ranging from Furstenberg’s celebrated proof of Szemerédi’s theorem (see [13]) to countless further applications and developments such as [11, 21, 28] or [12].

Unfortunately, this approach via symbolic dynamics must inevitably fail for problems involving sequences \((u_n)_n \in \ell^\infty(N; H)\) in infinite-dimensional Hilbert spaces \(H\), for lack of the compactness of the unit ball. To overcome this obstacle, we recall an alternative approach has proven to be a critical interface between number theory and ergodic theory with applications ranging from Furstenberg’s celebrated proof of Szemerédi’s theorem (see [13]) to countless further applications and developments such as [11, 21, 28] or [12].

To overcome this obstacle, we recall an alternative approach to the correspondence principle commonly used in Ramsey theory that Furstenberg himself was already aware of, see [6, p. 50]: Using the Stone–Čech compactification \(\beta N\) of the natural numbers and the identification

\[
\ell^\infty(N) \cong C(\beta N),
\]

every sequence \((a_n)_n\) corresponds uniquely to its continuous extension \(a \in C(\beta N)\). Since the map \(\tau : N \to N, x \mapsto x + 1\) extends continuously to \(\beta N\), the sequence \((a_n)_n\) is encoded in the topological dynamical system \((\beta N, \tau)\) via \(a_n = a(\tau^{n-1}(1))\). The invariant measures \(\mu\) of this system can be identified with the Banach limits on \(\ell^\infty(N) \cong C(\beta N)\), i.e. the shift-invariant positive linear forms, and one can again study the sequence \((a_n)_n\) in terms of the dynamical system \((\beta N, \mathcal{B}, \mu; \tau)\) and its Koopman operator \(T_\tau : L^2(\beta N, \mathcal{B}, \mu) \to L^2(\beta N, \mathcal{B}, \mu)\).

This approach does not immediately extend either to vector-valued sequences \((u_n)_n\) in some Hilbert space \(H\) since the isomorphism \(\ell^\infty(N; H) \cong C(\beta N; H)\) holds if and only if \(H\) is finite-dimensional. However, a conceptual view of \(L^2(\beta N, \mathcal{B}, \mu)\) is to regard it as the Hilbert space \(C(\beta N)^{(1)}\mu\) obtained from the \(C^*\)-algebra \(C(\beta N)\) by means of the Gelfand–Naimark–Segal construction for the state \(\mu \in C(\beta N)'\). Thus, by constructing Hilbert spaces \(\mathcal{H}_\mu\) out of \(\ell^\infty(N; H)\), we may achieve a generalization of the Hilbert spaces \(L^2(\beta N, \mathcal{B}, \mu) \cong \ell^\infty(N)^{(1)}\mu\) to the Hilbert space-valued setting. As we show below, this can be done in terms of a more general GNS construction for Hilbert modules instead of \(C^*\)-algebras. We carry these arguments out in fairly high generality to underline that the GNS construction is the key tool needed. However, we start with the most basic example in the next section and then develop the general ideas in the following sections.

2. Van der Corput via the correspondence principle

We start with a short proof of the scalar van der Corput inequality based on the correspondence principle, followed by a second proof in the general case that adapts this idea to the vector-valued setting.

**Theorem 2.1:** For every bounded sequence \((u_n)_{n \in \mathbb{N}}\) of complex numbers the inequality

\[
\limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} u_n \right|^2 \leq \liminf_{j \to \infty} \left( \frac{1}{j} \sum_{j=1}^{j} \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \Re(u_n u_{n+j}) \right).
\]
holds.

**Proof:** Let \((u_n)_n \in \ell^\infty(\mathbb{N})\) and \(p \in \beta \mathbb{N} \setminus \mathbb{N}\) be a nonprincipal ultrafilter such that

\[
\limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} u_n \right|^2 = \lim_{N \to p} \left| \frac{1}{N} \sum_{n=1}^{N} u_n \right|^2.
\]

Denote by \(u \in C(\beta \mathbb{N})\) the canonical extension of \((u_n)_n\) to \(\beta \mathbb{N}\) and for each \(n \in \mathbb{N}\) let \(\delta_n \in C(\beta \mathbb{N})'\) be the Dirac measure at \(n\). Then

\[
\lim_{N \to p} \frac{1}{N} \sum_{n=1}^{N} u_n = \lim_{N \to p} \left\langle u, \frac{1}{N} \sum_{n=1}^{N} \delta_n \right\rangle = \left\langle u, \lim_{N \to p} \frac{1}{N} \sum_{n=1}^{N} \delta_n \right\rangle = \int_{\beta \mathbb{N}} u \, d\mu.
\]

Here, the probability measure \(\mu \in C(\beta \mathbb{N})'\) exists as a weak* limit because any limit along an ultrafilter exists in a compact space. Now, let \(T_\tau : L^2(\beta \mathbb{N}, \mathcal{B}, \mu) \to L^2(\beta \mathbb{N}, \mathcal{B}, \mu)\) be the Koopman operator for the right shift \(\tau : \beta \mathbb{N} \to \beta \mathbb{N}\) and let \(P_\mu\) denote its mean ergodic projection. By the Cauchy–Schwarz inequality and since \(P_\mu\) is a conditional expectation, we conclude that

\[
\limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} u_n \right|^2 = \int_{\beta \mathbb{N}} |P_\mu u|^2 \, d\mu \leq \int_{\beta \mathbb{N}} P_\mu u \, d\mu = \int_{\beta \mathbb{N}} P_\mu u \, d\mu = \int_{\beta \mathbb{N}} u \, d\mu.
\]

To extend this idea to the general Hilbert space case, let \(\mu \in C(\beta \mathbb{N})'\) be a shift-invariant probability measure as in the proof above. As described in Section 1, \(\mu\) is a state on the C*-algebra \(C(\beta \mathbb{N})\), i.e., a positive linear form of norm 1. As such, \(\mu\) gives rise to a sesquilinear form \((\cdot|\cdot)_\mu\) which yields the Hilbert space \(L^2(\beta \mathbb{N}, \mathcal{B}, \mu)\) by means of the Gelfand–Naimark–Segal construction. Now, let \(H\) be a Hilbert space and consider the space \(\ell^\infty(\mathbb{N}, H)\) of bounded sequences in \(H\). This space is no longer a C*-algebra but still
a C*-Hilbert module by means of the $\ell^\infty(\mathbb{N})$-valued inner product

$$(\cdot|\cdot): \ell^\infty(\mathbb{N},H) \times \ell^\infty(\mathbb{N},H) \to \ell^\infty(\mathbb{N}), \quad ((u_n)_n, (v_n)_n) \mapsto ((u_n|v_n)_H)_n.$$ 

(We will introduce Hilbert modules formally in Section 4.) Next, we can consider the sesquilinear form

$$(\cdot|\cdot)_\mu: \ell^\infty(\mathbb{N},H) \times \ell^\infty(\mathbb{N},H) \to \mathbb{C}, \quad ((u_n)_n, (v_n)_n) \mapsto \mu((u_n|(v_n)_n)).$$

Since this sesquilinear form is positive semi-definite (which also implies that it is hermitian), it induces an inner product on the quotient

$$\ell^\infty(\mathbb{N},H)/\{x \in \ell^\infty(\mathbb{N},H) \mid (x|x)_\mu = 0\}.$$ 

We denote its completion by $\mathcal{H}_\mu$. Note here that $\{x \in \ell^\infty(\mathbb{N},H) \mid (x|x)_\mu = 0\}$ is actually a subspace of $\ell^\infty(\mathbb{N},H)$ by the Cauchy–Schwarz inequality for positive semi-definite sesquilinear forms (see, e.g. [5, Inequality 1.4]). If $\mu = \lim_{N \to p} \frac{1}{N} \sum_{n=1}^{N} \delta_n$ for some ultrafilter $p \in \beta \mathbb{N} \setminus \mathbb{N}$, it can be shown that $\mathcal{H}_\mu$ is isomorphic to the quotient

$$\left\{ u: \mathbb{N} \to H \middle| \limsup_{N \to p} \frac{1}{N} \sum_{n=1}^{N} \|u_n\|^2 < +\infty \right\} / \left\{ u: \mathbb{N} \to H \middle| \limsup_{N \to p} \frac{1}{N} \sum_{n=1}^{N} \|u_n\|^2 = 0 \right\}$$

and that under this identification the scalar product takes the form

$$((u_n)_n|(v_n)_n)_\mu = \lim_{N \to p} \frac{1}{N} \sum_{n=1}^{N} (u_n|v_n)_H.$$ 

Related spaces also play an essential role in [21] to capture asymptotics of sequences along Følner sequences (see in particular Remark 3.2 for when their spaces are complete), but we will not need this representation here. The nontrivial part of proving this representation is to show that the above-defined space is complete; the reader can find the relevant ideas in [4, Section II.2]. The only thing we will need below is that the shift $T: \ell^\infty(\mathbb{N},H) \to \ell^\infty(\mathbb{N},H), \ (u_n)_n \mapsto (u_{n+1})_n$ induces a contraction on $\mathcal{H}_\mu$. To see this, denote by $S: \ell^\infty(\mathbb{N}) \to \ell^\infty(\mathbb{N})$ the shift $(u_n)_n \mapsto (u_{n+1})_n$ and observe that for $x, y \in \ell^\infty(\mathbb{N},H), \ (Tx|Ty) = S(x|y)$ and so $(Tx|Ty)_\mu = (x|y)_\mu$ by $S$-invariance of $\mu$. Hence, if $x \in \ell^\infty(\mathbb{N},H)$ is such that $(x|x)_\mu = 0$, then also $(Tx|Tx)_\mu = 0$. This shows that $T$ induces an isometry on $\mathcal{H}_\mu$ which we denote by $T_\mu$. With this set-up, we are able to prove the full van der Corput inequality. The only notable change is that we interpret the left-hand side in terms of an invariant sesquilinear form instead of an invariant linear form.

**Theorem 2.2:** For every bounded sequence $(u_n)_{n \in \mathbb{N}}$ in a Hilbert space $H$ the inequality

$$\limsup_{N \to \infty} \left\| \frac{1}{N} \sum_{n=1}^{N} u_n \right\|^2 \leq \liminf_{J \to \infty} \frac{1}{J} \sum_{j=1}^{J} \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \text{Re}(u_n|u_{n+j})$$

holds.
Proof: Let $u = (u_n)_n \in \ell^\infty(\mathbb{N}, H)$ be a bounded sequence and let $p \in \beta\mathbb{N} \setminus \mathbb{N}$ be a non-principal ultrafilter such that

$$
\limsup_{N \to \infty} \left\| \frac{1}{N} \sum_{n=1}^N u_n \right\|^2 = \lim_{N \to p} \left\| \frac{1}{N} \sum_{n=1}^N u_n \right\|^2
$$

and define a shift-invariant state $\mu$ on $\ell^\infty(\mathbb{N})$ via

$$
\mu := \lim_{N \to p} \frac{1}{N} \sum_{n=1}^N \delta_n.
$$

Consider the sesquilinear form

$$
B: \ell^\infty(\mathbb{N}, H) \times \ell^\infty(\mathbb{N}, H) \to \mathbb{C}, \quad (v, w) \mapsto \lim_{N \to p} \left( \frac{1}{N} \sum_{n=1}^N v_n \middle| \frac{1}{N} \sum_{n=1}^N w_n \right)
$$

and observe that

$$
B(u, u) = \limsup_{N \to \infty} \left\| \frac{1}{N} \sum_{n=1}^N u_n \right\|^2.
$$

We show that $B$ induces a contractive sesquilinear form on $\mathcal{H}_\mu$, i.e.

$$
|B(v, w)| \leq \sqrt{(v|v)_\mu \sqrt{(w|w)_\mu}} = \|v\|_{\mathcal{H}_\mu} \|w\|_{\mathcal{H}_\mu}.
$$

(1)

for all $v, w \in \ell^\infty(\mathbb{N}, H)$. Take $v, w \in \ell^\infty(\mathbb{N}, H)$ and note that for $N \in \mathbb{N}$ we can apply the Cauchy–Schwarz inequality both in $H$ and in $\mathbb{C}^N$ to obtain

$$
\left\| \left( \frac{1}{N} \sum_{n=1}^N v_n \right) \middle| \frac{1}{N} \sum_{n=1}^N w_n \right\| \leq \left\| \frac{1}{N} \sum_{n=1}^N v_n \right\| \left\| \frac{1}{N} \sum_{n=1}^N w_n \right\| \leq \frac{1}{N} \sum_{n=1}^N \|v_n\| \frac{1}{N} \sum_{n=1}^N \|w_n\|
$$

$$
\leq \left( \frac{1}{N} \sum_{n=1}^N \|v_n\|^2 \right)^{\frac{1}{2}} \left( \frac{1}{N} \sum_{n=1}^N \|w_n\|^2 \right)^{\frac{1}{2}}.
$$

Taking the limit as $N \to p$ proves (1). Therefore, $B$ induces a contractive sesquilinear form on $\mathcal{H}_\mu$ which we denote by $B_\mu$. Since for any $n, m \in \mathbb{N}$ and $v, w \in \ell^\infty(\mathbb{N}, H)$ one has $B(S^n v, S^m w) = B(v, w)$, $B_\mu$ satisfies an analogous invariance on $\mathcal{H}_\mu$. Denoting by $P_\mu$ the
mean ergodic projection of $T_\mu$ acting on $H_\mu$ and using this invariance, we conclude that

$$
\limsup_{N \to \infty} \left\| \frac{1}{N} \sum_{n=1}^{N} u_n \right\|^2 = B_\mu([u], [u]) = B_\mu(P_\mu [u], P_\mu [u]) \leq \|P_\mu [u]\|_{H_\mu} \|P_\mu [u]\|_{H_\mu}
$$

$$
= ([u]|P_\mu [u])_{H_\mu} = \lim_{J \to \infty} \frac{1}{J} \sum_{j=1}^{J} ([u]|S^j [u])_{H_\mu}
$$

$$
= \lim_{J \to \infty} \frac{1}{J} \sum_{j=1}^{J} \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} (u_n|u_{n+j})
$$

$$
\leq \liminf_{J \to \infty} \frac{1}{J} \sum_{j=1}^{J} \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \Re(u_n|u_{n+j}).
$$

Having seen the key ideas in these simple cases, the following sections exploit that only very little is actually used in the proofs, allowing to cover generalizations in different directions and several versions of the van der Corput inequality. We believe that the method presented here could also find applications to other problems but this is beyond the scope of this article.

### 3. Van der Corput on $C^*$-algebras

Since the above proof of the van der Corput inequality depends mostly on the GNS construction, we generalize it to arbitrary $C^*$-algebras in this section. To explain our approach, we start again with the scalar case of the original result.

**Theorem 3.1:** For every bounded sequence $(u_n)_{n \in \mathbb{N}}$ of complex numbers the inequality

$$
\limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} u_n \right|^2 \leq \liminf_{J \to \infty} \frac{1}{J} \sum_{j=1}^{J} \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \Re(u_n u_{n+j})
$$

holds.

To reformulate this in the language of $C^*$-algebras, we consider the $C^*$-algebra $\ell^\infty(\mathbb{N})$ of all bounded complex sequences and the shift operator $S(v_n)_{n \in \mathbb{N}} := (v_{n+1})_{n \in \mathbb{N}}$ for $(v_n)_{n \in \mathbb{N}} \in \ell^\infty(\mathbb{N})$. The shift $S \in \mathcal{L}(\ell^\infty(\mathbb{N}))$ is a Markov operator meaning that it is

(i) **unital**, i.e. $S1 = 1$, and

(ii) **positive**, i.e. $Sx \geq 0$ for every $x \in \ell^\infty(\mathbb{N})$ with $x \geq 0$. 
With the state defined by the point evaluation \( \mu \equiv \delta_1 : \ell^\infty(\mathbb{N}) \to \mathbb{C}, (v_n)_{n \in \mathbb{N}} \mapsto v_1 \) and \( x \equiv (u_n)_{n \in \mathbb{N}} \in \ell^\infty(\mathbb{N}) \) we can rewrite the left-hand side of the desired inequality as

\[
\limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} u_n \right|^2 = \limsup_{N \to \infty} \left( \frac{1}{N} \sum_{n=0}^{N-1} S^n x, \nu \right)^2.
\]

Likewise, we obtain

\[
\limsup_{J \to \infty} \frac{1}{J} \sum_{j=1}^{J} \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \text{Re} \langle u_{n+j} \rangle = \limsup_{J \to \infty} \frac{1}{J} \sum_{j=1}^{J} \limsup_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \text{Re} \langle S^n ((S^j x)^* x), \mu \rangle
\]

for the right-hand side.

With these considerations, we can obtain Theorem 2.1 by proving that for every Markov operator \( S \in \mathcal{L}(A) \) on a unital commutative C*-algebra \( A \) and every state \( \mu \in A' \) one has

\[
\limsup_{N \to \infty} \left( \frac{1}{N} \sum_{n=0}^{N-1} S^n x, \mu \right)^2 \leq \liminf_{J \to \infty} \frac{1}{J} \sum_{j=1}^{J} \limsup_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \text{Re} \langle S^n ((S^j x)^* x), \mu \rangle.
\]

In fact, our methods even allow to extend inequality (2) to non-commutative C*-algebras.

The following operators are the analogue of Markov operators in the non-commutative situation. Their only essential property for the following is that they define contractions on all induced GNS Hilbert spaces.

**Definition 3.2:** A bounded operator \( S \in \mathcal{L}(A) \) on a unital C*-algebra \( A \) is a **Markov–Schwarz operator** if

(i) it is unital, i.e. \( S1 = 1 \), and

(ii) it satisfies the **Schwarz inequality**, i.e. \((Sx)^* Sx \leq S(x^* x)\) for all \( x \in A \).

Every unital 2-positive operator is Markov–Schwarz. In particular, every unital *-homomorphism is a Markov–Schwarz operator. If \( A \) is commutative, then the concepts of Markov and Markov–Schwarz operators coincide. We refer to [25, Chapters 1 and 2] for an introduction to positive and Schwarz operators on C*-algebras.

We show the following version of inequality (2) for Markov–Schwarz operators by employing the GNS construction for C*-algebras and the mean ergodic theorem for contractions on Hilbert spaces. Here and in the following, we denote by \( S(A) \subseteq A' \) the set of states on a unital C*-algebra \( A \). This is a convex subset of \( A' \) and, equipped with the weak* topology, a compact space (see, e.g.[23, Section 3.2.1]).

**Theorem 3.3:** Let \( S \in \mathcal{L}(A) \) be a Markov–Schwarz operator on a unital C*-algebra \( A \). Moreover, let \( x \in A \), \((N_i)_{i \in I} \) a subnet of \( \mathbb{N} \) and \( \mu_i \in S(A) \) a state on \( A \) for every \( i \in I \). Then

\[
\limsup_{i} \left( \frac{1}{N_i} \sum_{n=0}^{N_i-1} S^n x, \mu_i \right)^2 \leq \liminf_{J \to \infty} \frac{1}{J} \sum_{j=1}^{J} \limsup_{i} \frac{1}{N_i} \sum_{n=0}^{N_i-1} \text{Re} \langle S^n ((S^j x)^* x), \mu_i \rangle.
\]
Proof: For the sake of convenience, we write $C_N := \frac{1}{N} \sum_{n=0}^{N-1} S^n \in \mathcal{L}(A)$ for every $N \in \mathbb{N}$. Passing to a subnet of $(C_{N_i}, \mu_{N_i})_{i \in I}$ we may assume that

$$\lim \sup_{i \in I} |\langle C_{N_i} x, \mu_{N_i} \rangle|^2 = \lim \sup_{i \in I} |\langle C_{N_i} x, \mu_{N_i} \rangle|^2 = \lim \sup_{i \in I} |\langle x, C_{N_i} \mu_{N_i} \rangle|^2$$

Using compactness and convexity of the state space $S(A)$, we may also assume that the weak* limit $\mu := \lim_{i \in I} C_{N_i} \mu_i \in S(A)$ exists. Thus we obtain

$$\lim \sup_{i} \left( \frac{1}{N_i} \sum_{n=0}^{N_i-1} S^n x, \mu_i \right)^2 = |\langle x, \mu \rangle|^2.$$ 

It is clear that the state $\mu$ is invariant, i.e. $S^* \mu = \mu$. As in the GNS construction (see, e.g. [23, Section 3.2]) the map

$$(\cdot | \cdot)_\mu : A \times A \rightarrow \mathbb{C}, \quad (y, z) \mapsto \langle z^* y, \mu \rangle$$ 

is a positive sesquilinear form yielding a Hilbert space $\mathcal{H}_\mu$: Take the subspace $A_\mu := \{ y \in A \mid \langle y | y \rangle_\mu = 0 \}$ and the completion $\mathcal{H}_\mu$ of the quotient $A/A_\mu$ with respect to the induced norm $\| \cdot \|_\mu$ given by $\| [y] \|_\mu = \langle y | y \rangle_\mu$ for $y \in A$. Since $\mu$ is invariant and $S$ satisfies the Schwarz inequality, $S$ induces a contraction $S_\mu \in \mathcal{L}(\mathcal{H}_\mu)$ with $S_\mu [y] = [Sy]$ for every $[y] \in A/A_\mu$. By the mean ergodic theorem the sequence $(\frac{1}{J} \sum_{j=1}^{J} (S_\mu)^j)_{j \in \mathbb{N}}$ converges strongly to the orthogonal projection $P_\mu \in \mathcal{L}(\mathcal{H}_\mu)$ onto the fixed space $\text{fix}(S_\mu)$ of the operator $S_\mu$ (see [10, Theorem 8.6]). This yields

$$|\langle x, \mu \rangle| = \lim_{J \to \infty} \left| \frac{1}{J} \sum_{j=1}^{J} S_j^* x, \mu \right| = \lim_{J \to \infty} \left| \left( \frac{1}{J} \sum_{j=1}^{J} S_j^* x \right) \langle [1] \rangle_\mu \right| = |\langle P_\mu [x] | [1] \rangle_\mu|.$$ 

Thus, by the Cauchy–Schwarz inequality and the fact that $\mu$ is unital, we obtain

$$|\langle x, \mu \rangle|^2 \leq \| P_\mu [x] \|_\mu^2 \cdot \| [1] \|_\mu^2 = \| P_\mu [x] \|_\mu^2 = \langle [x] | P_\mu [x] \rangle_\mu = \lim_{J \to \infty} \frac{1}{J} \sum_{j=1}^{J} \langle x | S_j^* x \rangle_\mu$$

$$= \lim_{J \to \infty} \frac{1}{J} \sum_{j=1}^{J} \lim_{i \in I} \frac{1}{N_i} \sum_{n=0}^{N_i-1} \langle C_{N_i} \mu_i, ((S_i^j)^n x) \rangle_\mu = \lim_{J \to \infty} \frac{1}{J} \sum_{j=1}^{J} \lim_{i \in I} \frac{1}{N_i} \sum_{n=0}^{N_i-1} \langle S^n((S_i^j)^n x), \mu_i \rangle.$$

Therefore,

$$|\langle x, \mu \rangle|^2 \leq \lim \inf_{J \to \infty} \frac{1}{J} \sum_{j=1}^{J} \lim \sup_{i} \frac{1}{N_i} \sum_{n=0}^{N_i-1} \text{Re} \langle S^n((S_i^j)^n x), \mu_i \rangle.$$

Using the reformulation at the beginning of this section, we obtain the scalar van der Corput inequality Theorem 2.1 as a special case.
Corollary 3.4 (van der Corput for complex numbers): For every bounded sequence \((u_n)_{n \in \mathbb{N}}\) of complex numbers the inequality
\[
\limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} u_n \right|^2 \leq \liminf_{J \to \infty} \frac{1}{J} \sum_{j=1}^{J} \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \text{Re}(u_n \overline{u}_{n+j})
\]
holds.

Proof: Apply Theorem 3.3 with \(A = \mathcal{L}^\infty(\mathbb{N}), S \in \mathcal{L}(A)\) the shift given by \(S(v_n)_{n \in \mathbb{N}} := (v_{n+1})_{n \in \mathbb{N}}\) for \((v_n)_{n \in \mathbb{N}} \in \mathcal{L}^\infty(\mathbb{N})\), \((N_i)_{i \in I} = (N)_{n \in \mathbb{N}}, \mu_i = \delta_1: \mathcal{L}^\infty(\mathbb{N}) \to \mathbb{C}, (v_n)_{n \in \mathbb{N}} \mapsto v_1\) for every \(i \in \mathbb{N}\) and \(x = (u_n)_{n \in \mathbb{N}} \in \mathcal{L}^\infty(\mathbb{N})\).

However, Theorem 3.3 can also be applied to non-commutative C*-algebras, e.g. the C*-algebra \(\mathcal{L}(H)\) of all bounded operators on a Hilbert space \(H\).

Corollary 3.5 (van der Corput for operators): Let \(H\) be a Hilbert space, \(S \in \mathcal{L}(H)\) an isometry and \(\xi \in H\) with \(\|\xi\| = 1\). Then the inequality
\[
\limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=0}^{N-1} (TS^n \xi | S^n \xi) \right|^2 \leq \liminf_{J \to \infty} \frac{1}{J} \sum_{j=1}^{J} \limsup_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \text{Re}(S^j T S^n \xi | T S^{n+j} \xi)
\]
holds for every \(T \in \mathcal{L}(H)\).

Proof: We consider the implemented operator \(S\) acting on \(\mathcal{L}(H)\) via \(ST := S^* TS\) for \(T \in \mathcal{L}(H)\). Since \(S\) is an isometry, this is a Markov–Schwarz operator. Now take the vector state \(\mu\) on \(\mathcal{L}(H)\) given by \(\mu(T) := (T \xi | \xi)\) for all \(T \in \mathcal{L}(H)\) and set \(\mu_i := \mu\) for all \(i \in \mathbb{N}\). Theorem 3.3 applied to \(S\) yields the desired inequality.

4. Van der Corput on Hilbert modules

Similarly to our first proof of the scalar van der Corput inequality, also the proof of the full inequality Theorem 2.2 can be carried out in greater generality on so-called Hilbert modules. As we will see in Corollary 5.4, being able to consider Hilbert modules with other underlying algebras than \(\mathcal{L}^\infty(\mathbb{N})\), e.g. \(C_b(\mathbb{N} \times \mathbb{T})\), gives additional control that allows to establish uniform versions of the van der Corput inequality. These are often used for establishing Wiener–Wintner results.

Definition 4.1: Let \(A\) be a unital C*-algebra. A pre-Hilbert module \(E\) over \(A\) is a unital\(^3\) left module \(E\) over the algebra \(A\) together with a map
\[
(\cdot | \cdot)_A : E \to A, \quad (x, y) \mapsto (x | y)_A
\]
such that

(i) \((x | x)_A \geq 0\) and \((x | x)_A = 0\) if and only if \(x = 0\) for \(x \in E\).

(ii) \((x | y)^*_A = (y | x)_A\) for all \(x, y \in E\).
(iii) $(\cdot|y)_A: E \to A$ is an $A$-linear map for every $y \in E$.

An introduction to this concept can be found in [18].

We will henceforth assume that $A$ is commutative since in this case the Cauchy–Schwarz inequality

$$|(x|y)_A| \leq (x|x)_A^{\frac{1}{2}} \cdot (y|y)_A^{\frac{1}{2}}$$

holds for all $x, y \in E$, where modulus and square root are defined via continuous functional calculus (use the Gelfand–Naimark representation of $A$ and apply the Cauchy–Schwarz inequality for semi-inner products pointwise, see [7, page 49]). Setting

$$|x|_A := (x|x)_A^{\frac{1}{2}} \quad \text{for} \quad x \in E,$$

we then obtain a ‘vector-valued norm’ $|\cdot|: E \to A_+$. This also yields a real-valued norm $\|\cdot\|$ via $\|x\| := \|x|_A\|$ for $x \in E$. If $E$ is complete with respect to this norm, then $E$ is called a Hilbert module (or a Hilbert $C^*$-module) over $A$. However, completeness is not needed in the following.

**Example 4.2:** Let $H$ be a Hilbert space. By defining multiplication componentwise we turn $\ell^\infty(N, H)$ into a unitary module over $\ell^\infty(N)$. Setting $(u|v)_{\ell^\infty(N)} := ((u_n|v_n))_{n \in N}$ for $u = (u_n)_{n \in N}, v = (v_n)_{n \in N} \in \ell^\infty(N, H)$, we arrive at a Hilbert module over $\ell^\infty(N)$.

For stating a van der Corput inequality on such (pre-)Hilbert modules we now need two operators: a Markov operator on $S$ on the $C^*$-algebra $A$ and an operator $T \in L(E)$ on the pre-Hilbert module $E$ which is dominated by $S$ in the following sense.

**Definition 4.3:** Let $E$ be a pre-Hilbert module over a commutative unital $C^*$-algebra $A$ and $S \in L(A)$ a Markov operator. A bounded operator $T \in L(E)$ is $S$-dominated if $|Tx|_A^2 \leq S|x|_A^2$ for all $x \in E$.

**Example 4.4:** Let $H$ be a Hilbert space and consider the Hilbert module $\ell^\infty(N, H)$ over $\ell^\infty(N)$. Then the shift $T \in L(\ell^\infty(N, H))$ defined by $T(u_n)_{n \in N} := (u_{n+1})_{n \in N}$ for $(u_n)_{n \in N} \in \ell^\infty(N, H)$ is dominated by the shift $S \in L(\ell^\infty(N))$.

Using similar methods as in the previous section, we obtain the following version of the van der Corput inequality.

**Theorem 4.5:** Let $S \in L(A)$ be a Markov operator on a commutative unital $C^*$-algebra $A$ and let $T \in L(E)$ be an $S$-dominated operator on a pre-Hilbert module $E$ over $A$. Moreover, let $x \in E, (N_i)_{i \in I}$ a subnet of $\mathbb{N}$ and $\mu_i \in S(A)$ a state on $A$ for every $i \in I$. Then

$$\limsup_i \left( \left\| \frac{1}{N_i} \sum_{n=0}^{N_i-1} T^nx \right\|_A^2, \mu_i \right) \leq \liminf_{j \to \infty} \frac{1}{j} \sum_{j=1}^I \limsup_i \frac{1}{N_i} \sum_{n=0}^{N_i-1} \Re(S^n(x|^jx)_A, \mu_i).$$
Proof: We take the Cesàro means $C_N := \frac{1}{N} \sum_{n=0}^{N-1} T^n$ for every $N \in \mathbb{N}$. For each $i \in I$ we consider the positive semi-definite sesquilinear form

$$\varphi_i := \mu_i \circ (\cdot | \cdot)_A \circ (C_{N_i} \times C_{N_i}) : E \times E \to \mathbb{C},$$

$$(y, z) \mapsto \left\langle \left( \frac{1}{N_i} \sum_{n=0}^{N_i-1} T^n y, \frac{1}{N_i} \sum_{n=0}^{N_i-1} T^n z \right), \mu_i \right\rangle$$
on $E$. As above, we may then assume that

$$\limsup_i \left\langle \frac{1}{N_i} \sum_{n=0}^{N_i-1} T^n x, \mu_i \right\rangle = \limsup_i \varphi_i(x, x) = \lim_i \varphi_i(x, x).$$

Using that $T$ is $S$-dominated, we obtain that

$$|\varphi_i(y, y)| \leq \left\langle \left( \frac{1}{N_i} \sum_{n=0}^{N_i-1} |T^n y|_A \right)^2, \mu_i \right\rangle \leq \left\langle \frac{1}{N_i} \sum_{n=0}^{N_i-1} |T^n y|^2, \mu_i \right\rangle \leq \left\langle |y|_A^2, \frac{1}{N_i} \sum_{n=0}^{N_i-1} (S^n y)^\prime \mu_i \right\rangle$$

for $y \in E$ and consequently

$$|\varphi_i(y, z)| \leq \left( \left\langle |y|_A^2, \frac{1}{N_i} \sum_{n=0}^{N_i-1} (S^n y)^\prime \mu_i \right\rangle \right)^{\frac{1}{2}} \cdot \left( \left\langle |z|_A^2, \frac{1}{N_i} \sum_{n=0}^{N_i-1} (S^n y)^\prime \mu_i \right\rangle \right)^{\frac{1}{2}}$$

for all $y, z \in E$ and $i \in I$ by the Cauchy–Schwarz inequality. In particular, we obtain that $|\varphi_i(y, z)| \leq \|y\| \cdot \|z\|$ for all $y, z \in E$ and $i \in I$. We may therefore assume (by passing to a subnet using Tychonoff’s theorem) that there is a positive sesquilinear form $\varphi : E \times E \to \mathbb{C}$ with $\lim_i \varphi_i(y, z) = \varphi(y, z)$ for all $y, z \in E$. Thus the left-hand side of the desired inequality is given by $\varphi(x, x)$. Finally, again passing to a subnet, we can assume that (as in the proof of Theorem 3.3) the weak-* limit $\mu := \lim_i (\frac{1}{N_i} \sum_{n=0}^{N_i-1} (S^n y)^\prime) \mu_i$ exists in the state space $S(A)$. Inequality (3) then implies

(i) $|\varphi(y, z)| \leq ((|y|_A^2, \mu))^\frac{1}{2} \cdot ((|z|_A^2, \mu))^\frac{1}{2}$ for all $y, z \in E$.

We also note that $\varphi$ is invariant, i.e.

(ii) $\varphi \circ (T^n \times T^m) = \varphi$ for all $n, m \in \mathbb{N}_0$.

As in the proof of Theorem 3.3, we now construct a contraction on a Hilbert space. To do so, consider the positive semi-definite sesquilinear form

$$(\cdot | \cdot)_\mu : E \times E \to \mathbb{C}, \quad (y, z) \mapsto \langle (y|z)_A, \mu \rangle$$

and the subspace $E_\mu := \{x \in E | (x|x)_\mu = 0\}$. As in the discussion in Section 2, we write $H_\mu$ for the Hilbert space constructed as the completion of the quotient $E/E_\mu$ with respect to the induced norm $\|\cdot\|_\mu$. Since $T$ is $S$-dominated, it induces a contraction $T_\mu \in \mathcal{L}(H_\mu)$ with $T_\mu[y] = [Ty]$ for every $y \in E$. Denote the corresponding mean ergodic projection by
Moreover, by (i) \( \varphi \) defines a bounded positive semi-definite sesquilinear form \( \varphi_\mu \) on \( H_\mu \). Using (i) and (ii) we now obtain

\[
\varphi(x, x) = \varphi_\mu(P_\mu [x], P_\mu [x]) \leq ||P_\mu [x]||^2_\mu = ([x], P_\mu [x])_\mu = \lim_{J \to \infty} \frac{1}{J} \sum_{j=1}^{J} \langle x | T_j^J x \rangle_\mu.
\]

This implies the claim. 

**Remark 4.6:** Theorems 3.3 and 4.5 are related as follows: If \( S \in \mathcal{L}(A) \) is a Markov operator on a commutative unital C*-algebra \( A \), then, with \( T = S \) and \( (x | y)_A := y^* x \) for \( x, y \in A \) in Theorem 4.5, the right-hand sides of the two inequalities coincide. While Theorem 3.3 gives an estimate of

\[
\limsup_{N \to \infty} \left| \left| \frac{1}{N} \sum_{n=0}^{N-1} S^n x, \mu_i \right| \right|^2,
\]

Theorem 4.5 only provides an upper bound for

\[
\limsup_i \left| \left| \left( \frac{1}{N_i} \sum_{n=0}^{N_i-1} S^n x \right)^* \left( \frac{1}{N_i} \sum_{n=0}^{N_i-1} S^n x \right), \mu_i \right| \right|.
\]

Therefore, Theorem 4.5 is slightly weaker than Theorem 3.3 in the C*-algebra setting (however, if \( \mu_i \) is multiplicative for every \( i \in I \), then both estimates are the same).

### 5. From operators to sequences

We now apply our abstract operator theoretic inequality to derive more concrete versions of the van der Corput lemma. We start with the Hilbert space version of Corollary 3.4 which is a direct consequence of Theorem 4.5.

**Corollary 5.1 (van der Corput for Hilbert spaces):** For every bounded sequence \( (u_n)_{n \in \mathbb{N}} \) in a Hilbert space \( H \) the inequality

\[
\limsup_{N \to \infty} \left| \left| \frac{1}{N} \sum_{n=1}^{N} u_n \right| \right|^2 \leq \liminf_{J \to \infty} \frac{1}{J} \sum_{j=1}^{J} \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \text{Re}(u_n | u_{n+j})
\]

holds.

**Proof:** As in the proof of Corollary 3.4 we take \( A = \ell^\infty(\mathbb{N}) \), \( S \in \mathcal{L}(A) \), \( (N_i)_{i \in I} = (N)_{N \in \mathbb{N}} \) and \( (\mu_i)_{i \in I} = (\delta_1)_{n \in \mathbb{N}} \). Moreover, we consider the Hilbert module \( E = \ell^\infty(\mathbb{N}, H) \) over \( \ell^\infty(\mathbb{N}) \) (see Example 4.2) and let \( T \) be the shift on \( E \) which is \( S \)-dominated (see Example 4.4). Then Theorem 4.5 yields the claim. 

\[ \blacksquare \]
By considering nets \((\mu_i)_{i \in I}\) of different states \(\mu_i\) in Theorem 4.5 we also obtain the following versions of the van der Corput inequality (compare with [15, Lemma 3.18], [10, Exercise 20.8] and [20, Proposition 6], respectively).

**Corollary 5.2:** For every bounded sequence \((u_n)_{n \in \mathbb{N}}\) in a Hilbert space \(H\) the inequality

\[
\limsup_{N \to \infty} \sup_{M \in \mathbb{N}} \left\| \frac{1}{N} \sum_{n=M+1}^{M+N} u_n \right\|^2 \leq \liminf_{j \to \infty} \left\{ \frac{1}{j} \sum_{j=1}^{N} \limsup_{N \to \infty} \frac{1}{N} \sum_{n=M+1}^{N+M} \Re(u_n | u_{n+j}) \right\}
\]

holds.

**Proof:** We find a sequence \((M_N)_{N \in \mathbb{N}}\) with

\[
\limsup_{N \to \infty} \sup_{M \in \mathbb{N}} \left\| \frac{1}{N} \sum_{n=M+1}^{M+N} u_n \right\|^2 = \limsup_{N \to \infty} \frac{1}{N} \sum_{n=M+1}^{N+M} u_n
\]

Consider the states \(\mu_i := \delta_{M_i} : \ell^\infty(\mathbb{N}) \to \mathbb{C}\), \((v_n)_{n \in \mathbb{N}} \mapsto v_{M_i}\) and proceed as in the proof of Corollary 5.1. This yields the desired inequality. ■

**Corollary 5.3:** For every bounded sequence \((u_n)_{n \in \mathbb{N}}\) in a Hilbert space \(H\) the inequality

\[
\limsup_{N,M \to \infty} \left\| \frac{1}{N} \sum_{n=M+1}^{M+N} u_n \right\|^2 \leq \liminf_{j \to \infty} \left\{ \frac{1}{j} \sum_{j=1}^{N} \limsup_{N,M \to \infty} \frac{1}{N} \sum_{n=M+1}^{N+M} \Re(u_n | u_{n+j}) \right\}
\]

holds.

**Proof:** We find subsequences \((N_i)_{i \in \mathbb{N}}\) and \((M_i)_{i \in \mathbb{N}}\) of \(\mathbb{N}\) such that

\[
\limsup_{N,M \to \infty} \left\| \frac{1}{N} \sum_{n=M+1}^{M+N} u_n \right\|^2 = \limsup_{i \to \infty} \left\| \frac{1}{N_i} \sum_{n=M_i+1}^{M_i+N_i} u_n \right\|^2.
\]

We then again proceed as in the proof of Corollary 5.1 and apply Theorem 4.5 with \((N_i)_{i \in \mathbb{N}}\) and \((\mu_i)_{i \in \mathbb{N}}\) given by \(\mu_i := \delta_{M_i} : \ell^\infty(\mathbb{N}) \to \mathbb{C}\), \((v_n)_{n \in \mathbb{N}} \mapsto v_{M_i}\) for \(i \in \mathbb{N}\). ■

**Corollary 5.4:** For every bounded sequence \((u_n)_{n \in \mathbb{N}}\) in a Hilbert space \(H\) the inequality

\[
\limsup_{N \to \infty} \sup_{|\lambda|=1} \left( \left\| \frac{1}{N} \sum_{n=1}^{N} \lambda^n u_n \right\|^2 \right) \leq \liminf_{j \to \infty} \left\{ \frac{1}{j} \sum_{j=1}^{N} \limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} (u_n | u_{n+j}) \right| \right\}
\]

holds.

**Proof:** We write \(\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}\), consider the \(\mathbb{C}^*\)-algebra \(A := \mathcal{C}_b(\mathbb{T} \times \mathbb{N})\) of bounded continuous functions on \(\mathbb{T} \times \mathbb{N}\) and the shift operator \(S \in \mathcal{L}(A)\) given by \(Sf(\lambda, n) \mapsto f(\lambda, n + 1)\) for \((\lambda, n) \in \mathbb{T} \times \mathbb{N}\) and \(f \in \mathcal{C}_b(\mathbb{T} \times \mathbb{N})\). Similar to the proof of
Corollary 5.1 the space of bounded vector-valued continuous functions \( E := C_b(\mathbb{T} \times \mathbb{N}, H) \) is canonically a Hilbert module over \( A \). The shift \( T \in \mathcal{L}(E) \) given by \( Ty(\lambda, n) := y(\lambda, n + 1) \) for \((\lambda, n) \in \mathbb{T} \times \mathbb{N}\) and \( y \in C_b(\mathbb{T} \times \mathbb{N})\) is \( S \)-dominated. We set \( x(\lambda, n) := \lambda^n u_n \) for \((\lambda, n) \in \mathbb{T} \times \mathbb{N}\). Finally, we pick a sequence \((\lambda_N)_{N \in \mathbb{N}}\) in \( \mathbb{T} \) with

\[
\limsup_{N \to \infty} \sup_{|\lambda| = 1} \left\| \frac{1}{N} \sum_{n=1}^{N} \lambda^n u_n \right\|^2 = \limsup_{N \to \infty} \left\| \frac{1}{N} \sum_{n=1}^{N} \lambda_N^n u_n \right\|^2
\]

and set \( \mu_i(f) := f(\lambda_i, 1) \) for \( f \in C_b(\mathbb{T} \times \mathbb{N}) \) and \( i \in \mathbb{N} \). Then by Theorem 4.5

\[
\limsup_{N \to \infty} \sup_{|\lambda| = 1} \left\| \frac{1}{N} \sum_{n=1}^{N} \lambda^n u_n \right\|^2 = \limsup_{N \to \infty} \left\langle \left| \sum_{n=0}^{N-1} T^n x \right|^2, \mu_N \right\rangle
\]

\[
\leq \liminf_{j \to \infty} \frac{1}{j} \sum_{j=1}^{j} \limsup_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \text{Re} \langle S^n(x|T^jx)_A, \mu_N \rangle.
\]

However, for every \( j \in \mathbb{N} \), we have

\[
\text{Re} \frac{1}{N} \sum_{n=0}^{N-1} \langle S^n(x|T^jx)_A, \mu_N \rangle \leq \frac{1}{N} \sum_{n=0}^{N-1} \left| (\lambda_N^n u_n | \lambda_N^{n+j} u_{n+j}) \right| = \frac{1}{N} \sum_{n=0}^{N-1} \left| (u_n | u_{n+j}) \right|
\]

which yields the claim. 

6. Generalizations to Følner nets

Our results can easily be generalized from Cesàro means to other ergodic nets. Recall the following definition.

**Definition 6.1**: Let \( S \) be a semigroup (with composition now denoted multiplicatively). A net \((F_i)_{i \in I}\) of non-empty finite subsets of \( S \) is a right Følner net if

\[
\lim_{s \to \infty} \frac{|F_i s \Delta F_i|}{|F_i|} = 0
\]

for every \( s \in S \).

**Remark 6.2**: Note that every abelian semigroup has a (right) Følner net (see [1, Theorem 4]) and every semigroup having a right Følner net is necessarily right amenable, i.e. has a right invariant mean (however, the converse does not hold, see [22, Section 4.22]).

If the semigroup \( S \) is right cancellative (see, e.g. [3, Definition 1.16]), i.e. \( rt = st \) implies \( r = s \) for \( r, s, t \in S \), then we readily obtain that right Følner nets induce ‘right ergodic operators nets’ in the following way (cf. [24, Definition 1.1]).

**Lemma 6.3**: Let \((F_i)_{i \in I}\) be a right Følner net in a right cancellative semigroup \( S \). If \( S : S \to \mathcal{L}(E) \) is a bounded semigroup representation on a Banach space \( E \), then the operators \( C_i := \)
\[ \frac{1}{|F_i|} \sum_{t \in F_i} S_t \in \mathcal{L}(E) \text{ for } i \in I \text{ satisfy} \]
\[ \lim_i \| C_i S_t - C_i \| = 0 \]
for every \( t \in S \).

Using the same arguments – mutatis mutandis – as in Sections 2 and 3 with the more general version of the mean ergodic theorem for right ergodic operator nets, see [10, Theorem 8.32] and [24, Theorem 1.7], we obtain the following analogues of Theorems 3.3 and 4.5.

**Theorem 6.4:** Let \( S \) be a right cancellative semigroup. Moreover, let

(i) \((F_i)_{i \in I}\) and \((G_j)_{j \in J}\) be right Følner nets for \( S \),
(ii) \( A \) be a unital C*-algebra,
(iii) \( S: S \to \mathcal{L}(A) \) a representation as Markov–Schwarz operators, and
(iv) \( \mu_i \in S(A) \) be a state for every \( i \in I \).

Then
\[
\limsup_i \left\| \frac{1}{|F_i|} \sum_{t \in F_i} S_t x, \mu_i \right\|_A^2 \leq \liminf_j \left\| \frac{1}{|G_j|} \sum_{s \in G_j} \limsup_i \frac{1}{|F_i|} \sum_{t \in F_i} \text{Re} \langle S_t ((S_s x)^* x), \mu_i \rangle \right\|
\]
for every \( x \in A \).

**Theorem 6.5:** Let \( S \) be a right cancellative semigroup. Moreover, let

(i) \((F_i)_{i \in I}\) and \((G_j)_{j \in J}\) be right Følner nets for \( S \),
(ii) \( E \) be a pre-Hilbert module over a unital commutative C*-algebra \( A \),
(iii) \( S: S \to \mathcal{L}(A) \) a representation as Markov operators,
(iv) \( T: S \to \mathcal{L}(A) \) a representation such that \( T_t \) is \( S_t \)-dominated for every \( t \in S \), and
(v) \( \mu_i \in S(A) \) a state for every \( i \in I \).

Then
\[
\limsup_i \left\| \frac{1}{|F_i|} \sum_{t \in F_i} T_t x, \mu_i \right\|_A^2 \leq \liminf_j \left\| \frac{1}{|G_j|} \sum_{s \in G_j} \limsup_i \frac{1}{|F_i|} \sum_{t \in F_i} \text{Re} \langle S_t (T_s x), \mu_i \rangle \right\|
\]
for every \( x \in E \).

As in Section 5, these results imply several van der Corput inequalities, e.g. the following one.

**Corollary 6.6 (van der Corput for semigroups):** Let \( S \) be a right cancellative semigroup with right Følner nets \((F_i)_{i \in I}\) and \((G_j)_{j \in J}\). For every bounded map \( u: S \to H \), \( t \mapsto u_t \) into a
Hilbert space $H$ and every $r \in S$ the inequality

$$\limsup_i \left\| \frac{1}{|F_i|} \sum_{t \in F_i} u_{rt} \right\|^2 \leq \liminf_j \frac{1}{|G_j|} \sum_{s \in G_j} \limsup_i \frac{1}{|F_i|} \sum_{t \in F_i} \Re(u_{rt}u_{rts})$$

holds.

**Remark 6.7:** It would be interesting to also apply our approach to closed subsemigroups $S$ of a locally compact group $G$ with right Haar measure $m$ having a topological right Følner net $(F_i)_{i \in I}$ (e.g. $S = \mathbb{R}_{\geq 0}$ and $G = \mathbb{R}$), i.e. $F_i \subset S$ is compact with positive measure for every $i \in I$ and

$$\lim_i \frac{m(F_i \Delta Fs)}{m(F_i)} = 0.$$ 

for every $s \in S$ (cf. [24, Examples 1.2 (e)]). In particular, this could lead to a new proof of [2, Theorem 2.12].

**Notes**

1. Related but differing approaches are also present in the blog posts [27] by Tao and [20] by Moreira in the context of the van der Corput lemma.
2. In fact, it is not hard to show that the opposite shift also induces an isometry on $\mathcal{H}_\mu$ that is inverse to $T_\mu$. Hence, $T_\mu : \mathcal{H}_\mu \to \mathcal{H}_\mu$ is in fact unitary.
3. A left module $E$ over $A$ is unital if $1 \cdot x = x$ for every $x \in E$.
4. By the Gelfand-Naimark theorem one can (and should) think of $A = C(K)$ for a compact space $K$.

**Acknowledgements**

The authors thank Bálint Farkas, Ulrich Groh and Marco Peruzzetto for ideas and inspiring discussions. They are also grateful to the referee for their valuable comments.

**Disclosure statement**

No potential conflict of interest was reported by the authors.

**Funding**

The second author was supported by a scholarship of the Friedrich-Ebert-Stiftung while working on this article. He also acknowledges the financial support from the DFG (Deutsche Forschungsgemeinschaft) (project number 451698284).

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