1 Introduction.

Spinor fields and Dirac operators on Riemannian manifolds $M^n$ can be introduced by means of a reduction of the frame bundle from the structure group $O(n)$ to one of the groups $Spin(n)$, $Pin(n)$ or $Spin^C(n)$. In any case the existence of a corresponding reduction imposes topological restrictions on the manifold $M^n$. The aim of the present paper is the construction of spinor bundles of Cartan type over certain non-orientable manifolds. The bundles under consideration do not split into subbundles invariant under the action of the Clifford algebra and therefore they are not induced from $Pin^C$ structure of the manifold. Moreover, we study the case of the real projective space $\mathbb{R}P^n$ and its spinor bundle in more detail. These bundles over $\mathbb{R}P^n$ admit a suitable metric connection $\nabla^S$ and the corresponding Dirac operator. In particular it turns out that there are Killing spinors in the constructed twisted spinor bundle over $\mathbb{R}P^n$ for all dimensions.

Let us fix some notations. In case $n \equiv 0 \mod 2$ we denote by $\gamma : \text{Cliff}^C(n) \sim \text{End}(\Delta_n)$ the isomorphism between the Clifford algebra and the algebra of endomorphisms of the space $\Delta_n$ of all Dirac spinor. In this way we obtain the so called Dirac representation $\gamma$ of the Clifford algebra. In case $n \equiv 1 \mod 2$ we have the Pauli representation $\gamma : \text{Cliff}^C(n) \to \text{End}(\Delta_n)$. Denote by $\alpha : \text{Cliff}^C(n) \to \text{Cliff}^C(n)$ the canonical involution. Then

$$\gamma \oplus (\gamma \circ \alpha) : \text{Cliff}^C(n) \sim \text{End}(\Delta_n) \oplus \text{End}(\Delta_n)$$

is an isomorphism and the corresponding representation of the Clifford algebra in $\Delta_n \oplus \Delta_n$ is the so called Cartan representation (see [Fr2], [Tr]).

2 The construction of spinor bundles of Cartan type.

We consider odd-dimensional Riemannian manifolds $M^{2k+1}$ only. A spinor bundle of Cartan type is a $2^{k+1}$-dimensional complex Clifford bundle $S$ realizing at any point the Cartan representation. $\gamma \oplus (\gamma \circ \alpha) : \text{Cliff}^C(\mathbb{R}^{2k+1}) \sim \text{End}(\Delta_{2k+1}) \oplus \text{End}(\Delta_{2k+1})$ of the Clifford algebra (see [Tr]). Any spinor bundle of Cartan type over an orientable manifold.
manifold splits into two Clifford subbundles. Indeed, the volume form \(dM^{2k+1} = e_1 \cdots e_{2k+1} \in \text{Cliff}(T(M^{2k+1}))\) commutes with the Clifford multiplication. Since
\[
\alpha(e_1 \cdots e_{2k+1}) = -e_1 \cdots e_{2k+1}
\]
d\(M^{2k+1}\) acts on \(S\) with two different eigenvalues. Therefore \(dM^{2k+1}\) defines a splitting of the bundle \(S\) into two Clifford subbundles.

Consider a simply-connected, odd-dimensional Riemannian spin manifold \(M^{2k+1}\) and denote by \(S\) its spinor bundle. Any isometry \(\gamma : M^{2k+1} \to M^{2k+1}\) admits two lifts \(\hat{\gamma}_\pm : S \to S\) into the spinor bundle \(S\). We once again describe the construction of \(\hat{\gamma}_\pm\). The differential \(d\gamma\) (resp. \((-d\gamma)\) - \(M\) is odd-dimensional !) maps the frame bundle into itself if \(\gamma\) preserves the orientation (resp. does not preserve the orientation). Moreover, \(d\gamma\) (or \((-d\gamma)\)) lifts into the spin structure of \(M^{2k+1}\) and defines two lifts \(\hat{\gamma}_\pm\). The lifts \(\hat{\gamma}_\pm\) commute (anticommute) with the Clifford multiplication, i.e.
\[
\hat{\gamma}_\pm(X \cdot \varphi) = \deg(\gamma)d\gamma(X) \cdot \hat{\gamma}_\pm(\varphi)
\]
(see [Fr1]).

Let \(\Gamma\) be a discrete subgroup of isometries and suppose that \(\Gamma\) acts freely on \(M^{2k+1}\). We consider the spinor bundle of Cartan type
\[
S^* = S \oplus S
\]
over \(M^{2k+1}\). The Clifford multiplication in \(S^*\) is given by the formula
\[
X \cdot \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} X \cdot \varphi \\ -X \cdot \psi \end{pmatrix} \quad X \in T(M^{2k+1})
\]
where \(X \cdot \varphi\) denotes the Clifford multiplication in the spinor bundle \(S\). Any isometry \(\gamma \in \Gamma\) admits the following 4 lifts into the bundle \(S^*\):

- \(\begin{pmatrix} \hat{\gamma}_\pm & 0 \\ 0 & \hat{\gamma}_\pm \end{pmatrix}\) in case \(\gamma\) preserves the orientation.

- \(\begin{pmatrix} 0 & \hat{\gamma}_\pm \\ \hat{\gamma}_\pm & 0 \end{pmatrix}\) in case \(\gamma\) reverses the orientation.

Any of these lifts commutes with the Clifford multiplication in the bundle \(S^*\).

We introduce now a family of automorphisms \(\Pi(\Theta_1, \Theta_2, \epsilon_1, \epsilon_2, \gamma)\) of the bundle \(S^*\) depending on four parameters \(0 \leq \Theta_1 \leq 2\pi, 0 \leq \Theta_2 \leq 2\pi\) and \(\epsilon_1, \epsilon_2 \in \{\pm 1\}\). If \(\gamma\) preserves the orientation of the manifold we define
\[
\Pi(\Theta_1, \Theta_2, \epsilon_1, \epsilon_2, \gamma) = \begin{pmatrix} e^{i\Theta_1 \hat{\gamma}_\epsilon_1} & 0 \\ 0 & e^{i\Theta_2 \hat{\gamma}_\epsilon_2} \end{pmatrix},
\]

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otherwise let us introduce the automorphism

\[
\Pi(\Theta_1, \Theta_2, \varepsilon_1, \varepsilon_2, \gamma) = \begin{pmatrix}
0 & e^{i \Theta_2 \hat{\gamma}_2} \\
e^{i \Theta_1 \hat{\gamma}_1} & 0
\end{pmatrix}.
\]

Suppose that \(\varepsilon : \Gamma \to \text{Aut}(S^*)\) is a homomorphism such that for any \(\gamma \in \Gamma\) the automorphism \(\varepsilon(\gamma)\) coincides with one of the lifts \(\Pi(\Theta_1, \Theta_2, \varepsilon_1, \varepsilon_2, \gamma)\) of \(\gamma\) into \(S^*\). Then

\[
\bar{S} = S^*/\varepsilon(\Gamma)
\]

is a Clifford bundle of Cartan type over the Riemannian manifold \(\bar{M}^{2k+1} = M^{2k+1}/\Gamma\). The bundle \(\bar{S}\) splits into two parts invariant under the action of the Clifford bundle \(\text{Cliff}(T(M^{2k+1}))\) if and only if \(\bar{M}^{2k+1}\) is orientable. For a non-orientable manifold \(\bar{M}^{2k+1}\) the non-splitting bundle \(\bar{S}\) is not induced from a \(\text{Pin}^C(2k+1)\) structure of \(\bar{M}^{2k+1}\).

**Example:** \(\Gamma = \mathbb{Z}_p\).

Suppose \(\Gamma\) is generated by one involution \(\gamma : M^{2k+1} \to M^{2k+1}\) with the lifts \(\hat{\gamma}_\pm\). If \(\gamma\) preserves the orientation and \(\hat{\gamma}_\pm^2 = \text{Id}\) the manifold \(\bar{M}^{2k+1} = M^{2k+1}/\mathbb{Z}_2\) admits a spin structure. In case \(\hat{\gamma}_\pm^2 = -\text{Id}\) \(M^{2k+1}\) is orientable, but does not admit a spin structure. Nevertheless we can construct a spinor bundle using the homomorphism

\[
\varepsilon(\gamma) = \begin{pmatrix}
i \hat{\gamma}_+ & 0 \\
0 & i \hat{\gamma}_+
\end{pmatrix}.
\]

If \(\gamma\) does not preserve the orientation we define the homomorphism \(\varepsilon : \mathbb{Z}_2 \to \text{Aut}(S^*)\) by the formula

\[
\varepsilon(\gamma) = \begin{pmatrix}
0 & \hat{\gamma}_+ \\
\hat{\gamma}_+ & 0
\end{pmatrix} \quad \text{if } \hat{\gamma}_+^2 = \text{Id}
\]

\[
\varepsilon(\gamma) = \begin{pmatrix}
0 & i \hat{\gamma}_+ \\
i \hat{\gamma}_+ & 0
\end{pmatrix} \quad \text{if } \hat{\gamma}_+^2 = -\text{Id}.
\]

A similar construction for an arbitrary cyclic group \(\mathbb{Z}_p\) yields the result:

Let \(\bar{M}^{2k+1}\) be a Riemannian manifold with fundamental group \(\pi_1(\bar{M}^{2k+1}) = \mathbb{Z}_p\) and suppose that the universal covering \(M^{2k+1}\) admits a spin structure. Then over \(\bar{M}^{2k+1}\) there exists a spinor bundle \(\bar{S}\) of Cartan type. The bundle \(\bar{S}\) splits into two subbundles invariant under the action of the Clifford bundle \(\text{Cliff}(T(M^{2k+1}))\) if and only if \(\bar{M}^{2k+1}\) is orientable.

Let \(D : \Gamma(S) \to \Gamma(S)\) be the Dirac operator acting on sections of the spinor bundle \(S\) over \(M^{2k+1}\). The Dirac operator \(\mathcal{D}\) of the spinor bundle \(S^*\) of Cartan type is given by
\[ \mathcal{D} = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}. \]

Therefore the eigenspaces \( E_\lambda(\mathcal{D}) \) consists of pairs \( \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \) of eigenspinors of \( D \), i.e.

\[ E_\lambda(\mathcal{D}) = E_\lambda(D) \oplus E_{-\lambda}(D). \]

The group \( \Gamma \) acts on \( E_\lambda(\mathcal{D}) \) by the matrices \( \varepsilon(\gamma) \) and the eigenspace of the Dirac operator \( \mathcal{D} \) on \( \bar{M}^{2k+1} \) coincides with the subspace of \( \varepsilon(\Gamma) \)-invariant pairs in \( E_\lambda(\mathcal{D}) \).

In particular, in case \( \Gamma = \mathbb{Z}_2 \) and \( \bar{M}^{2k+1} \) is non-orientable we have only one transformation

\[ \varepsilon(\gamma) = \begin{pmatrix} 0 & \gamma_+ \\ \gamma_+ & 0 \end{pmatrix} \quad \text{or} \quad \varepsilon(\gamma) = \begin{pmatrix} 0 & i\gamma_+ \\ i\gamma_+ & 0 \end{pmatrix}. \]

The map \( E_\lambda(D) \ni \varphi \mapsto \begin{pmatrix} \varphi \\ \gamma_+(\varphi) \end{pmatrix} \in E_\lambda(\mathcal{D}) \) (resp. \( \varphi \mapsto \begin{pmatrix} \varphi \\ i\gamma_+(\varphi) \end{pmatrix} \)) defines an isomorphism between the corresponding eigenspaces. Therefore the spectrum of the operator \( (\mathcal{D}, \bar{S}, \bar{M}^{2k+1}) \) coincides with the spectrum of the operator \( (D, S; M^{2k+1}) \).

We thus constructed non-splitting spinor bundles over non-orientable manifolds \( \bar{M}^{2k+1} \) via coverings. We mention that other situations give similarly bundles with the described properties. For example, any immersion of \( \bar{M}^{2k+1} \) into an oriented spin manifold \( N^{2k+2} \) also defines a spinor bundle of Cartan type over \( \bar{M}^{2k+1} \). Indeed, let \( f : \bar{M}^{2k+1} \rightarrow N^{2k+2} \) be the immersion and consider the Dirac spinor bundle \( \bar{S} \) of \( N^{2k+2} \). Then the induced bundle \( \bar{S} = f^*(S) \) inherits the structure of a spinor bundle of Cartan type over \( \bar{M}^{2k+1} \).

3 The Spinor Bundle over \( \mathbb{R}P^n \).

We start with the \( n \)-dimensional sphere

\[ S^n = \{ x \in \mathbb{R}^{n+1} : |x| = 1 \} \]

and we denote by \( S^* \) the \( 2\left[\frac{n+1}{2}\right] \)-dimensional bundle

\[ S^* = S^n \times \Delta_{n+1}. \]

The tangent bundle of the sphere

\[ T(S^n) = \{ (x, t) \in S^n \times \mathbb{R}^{n+1} : \langle x, t \rangle = 0 \} \]

acts by Clifford multiplication

\[ \mu((x, t) \otimes (x, \varphi)) := (x, t \cdot \varphi) \]
on $S^*$. Therefore $S^*$ is a Clifford bundle over $S^n$. Let $g : S^n \to S^n, g(x) = -x$ be the antipodal map. Its differential $dg : T(S^n) \to T(S^n)$ acts on a tangent vector $(x, t) \in T(S^n)$ by

$$dg(x, t) = (-x, -t).$$

Let us define a lift $\hat{g}$ of the antipodal map into the bundle $S^*$ by the formula

$$\hat{g}(x, \varphi) = (-x, x \cdot \varphi).$$

Then $\hat{g}$ is again an involution, $(\hat{g})^2 = \text{Id}$. Moreover, $\hat{g}$ respects the Clifford multiplication, i.e. the diagram

\[
\begin{array}{ccc}
T(S^n) \otimes S^* & \xrightarrow{\mu} & S^* \\
\downarrow{(dg) \otimes \hat{g}} & & \downarrow{\hat{g}} \\
T(S^n) \otimes S^* & \xrightarrow{\mu} & S^*
\end{array}
\]

commutes. Indeed, given $(x, t) \otimes (x, \varphi) \in T(S^n) \otimes S^*$ we have

$$\mu \circ (dg \otimes \hat{g})((x, t) \otimes (x, \varphi)) = \mu((-x, -t) \otimes (-x, x \cdot \varphi)) = (-x, -t \cdot x \cdot \varphi).$$

On the other hand,

$$\hat{g} \mu((x, t) \otimes (x, \varphi)) = \hat{g}((x, t \cdot \varphi)) = (-x, x \cdot t \cdot \varphi).$$

Since $x$ and $t$ are orthogonal vectors in $\mathbb{R}^{n+1}$ we have in the Clifford algebra the relation

$$x \cdot t + t \cdot x = 0,$$

i.e. the mentioned diagram commutes.

Over the real projective space $\mathbb{R}P^n = S^n / g$ we define the bundle

$$S = S^*/\hat{g}.$$  

The tangent bundle $T(\mathbb{R}P^n)$ can be identified with $T(\mathbb{R}P^n) = T(S^n)/dg$ and the above discussed property of the lift $\hat{g} : S^* \to S^*$ implies that we obtain a well-defined Clifford multiplication

$$\mu : T(\mathbb{R}P^n) \otimes S \to S.$$  

At any point of $\mathbb{R}P^n$ the bundle $S$ realizes the Dirac (in case $n \equiv 0 \text{ mod } 2$) or the Cartan (in case $n \equiv 1 \text{ mod } 2$) representation of the $n$-dimensional Clifford algebra.

Let us consider the even-dimensional case, $n = 2k$. The Dirac representation of the Clifford algebra decomposes into two irreducible representations with respect to the action of the even part $\text{Cliff}_e(n)$ of the Clifford algebra. Therefore the bundle $S$ decomposes at any fixed point as a Clifford module too. However, globally the
bundle $S$ over $\mathbb{R}P^n$ does not split as a $\text{Cliff}_o(T(\mathbb{R}P^n))$-bundle. Indeed suppose that there exists a decomposition $S = S_1 \oplus S_2$ invariant under the action of the Clifford bundle. Then we obtain a corresponding decomposition $S^* = S_1^* \oplus S_2^*$ of the bundle $S^*$ over $S^2$. The spin module $\Delta_{n+1}$ decomposes in a unique way as a $\text{Cliff}_o(T_xS^n)$-module

$$\Delta_{n+1} = \Delta^+_{n+1}(x) \oplus \Delta^-_{n+1}(x).$$

We describe this decomposition in an explicit way. Let $e_1, \ldots, e_n$ an orthonormal basis of the tangent space $T_x(S^n)$ and consider the element

$$f = i^{n/2} e_1 \cdot \ldots \cdot e_n : \Delta_{n+1} \to \Delta_{n+1}.$$

Then $f^2 = \text{Id}$ and $\Delta_{n+1}$ decomposes into the eigenspaces of $f, \Delta_{n+1} = \Delta^+_{n+1}(x) \oplus \Delta^-_{n+1}$ with

$$\Delta^\pm_{n+1}(x) = \{ \varphi \in \Delta_{n+1} : f(\varphi) = \pm \varphi \}.$$

Now the volume form $e_1 \cdot \ldots \cdot e_n \cdot x$ of $\mathbb{R}P^n$ acts on $\Delta_{n+1}$ by multiplication

$$e_1 \cdot \ldots \cdot e_n \cdot x|_{\Delta_{n+1}} = \alpha_{n+1}.$$

Then $f = i^{n/2} e_1 \cdot \ldots \cdot e_n = -i^{n/2} \alpha_{n+1}x$ and the subspaces $\Delta^\pm_{n+1}(x)$ are given by

$$\Delta^\pm_{n+1}(x) = \{ \varphi \in \Delta_{n+1} : \pm x\varphi = i^{n/2} \alpha_{n+1} \varphi \}.$$

The bundles $S_1^*$ and $S_2^*$ coincide therefore with the subbundles $\Delta^\pm_{n+1}(x)$. Since $S_i^*(i = 1, 2)$ are induced bundles by the covering $S^n \to \mathbb{R}P^n = S^n/g$ they are invariant under the lift $\tilde{g} : S^* \to S^*$. However, the lift $\tilde{g}$ defined by the Clifford multiplication does not preserve the decomposition $\Delta_{n+1} = \Delta^+_{n+1}(x) \oplus \Delta^-_{n+1}(x)$, a contradiction.

We discuss now the odd-dimensional case, $n = 2k + 1$. Again we start with an orthonormal basis $e_1, \ldots, e_n \in T_x(S^n)$ and we introduce the automorphism

$$f = e_1 \cdot \ldots \cdot e_n : \Delta_{n+1} \to \Delta_{n+1}.$$

Then we have

$$f^2 = (-1)^{k+1}, \quad f \cdot x = -x \cdot f$$

and

$$f \cdot t = t \cdot f \quad \text{for all tangent vectors } t \in T_x(S^n).$$

We decompose $\Delta_{n+1}$ into the eigenspaces of $f$

$$\Delta_{n+1} = \Delta^+_{n+1}(x) \oplus \Delta^-_{n+1}(x), \quad \Delta^\pm_{n+1}(x) = \{ \varphi \in \Delta_{n+1} : e_1 \cdot \ldots \cdot e_n \varphi = \pm i^{k+1} \varphi \}.$$

Since $g$ commutes with all tangent vectors $t \in T_x(S^n)$ the subspaces $\Delta^\pm_{n+1}(x)$ are invariant under the action of the Clifford algebra $\text{Cliff}(T_x(S^n))$. The lift $\tilde{g}$ preserves
now \((n = 2k + 1)\) the decomposition \(\Delta_{n+1} = \Delta^+_{n+1}(x) \oplus \Delta^-_{n+1}(x)\), i.e. the Clifford bundle \(S\) over \(\mathbb{RP}^n\) decomposes into two Clifford subbundles.

Vector fields \(V\) on the projective space \(\mathbb{RP}^n\) can be identified with maps \(V : S \to \mathbb{R}^{n+1}\) such that
\[
V(-x) = -V(x) \quad , \quad (V(x), x) = 0 \quad x \in S^n.
\]
The bundle \(S\) over \(\mathbb{RP}^n\) arises from \(S^* = S^n \times \Delta_{n+1}\) and the identification \(\hat{g}(x, \varphi) = (-x, x \cdot \varphi)\). Therefore a section \(\Phi \in \Gamma(S; \mathbb{RP}^n)\) is a map \(\Phi : S^n \to \Delta_{n+1}\) with the property
\[
\Phi(-x) = x \cdot \Phi(x) . \quad (\ast)
\]
The formula
\[
(\nabla^S_V \Phi)(x) = d\Phi(V)(x) + \frac{1}{2} V(x) \cdot x \cdot \Phi(x)
\]
defines a covariant derivative
\[
\nabla^S : \Gamma(T(\mathbb{RP}^n)) \times \Gamma(S) \to \Gamma(S).
\]
Indeed, a simple calculation shows that \(\nabla^S_V \Phi\) has the invariance property \((\ast)\) in case \(\Phi\) and \(V\) satisfy the corresponding transformation rules. Moreover, the covariant derivative \(\nabla^S\) is compatible with the Clifford multiplication, i.e.
\[
\nabla^S_V (W \cdot \Phi) = (\nabla^S_V W) \cdot \Phi + W \cdot (\nabla^S_V \Phi),
\]
and \(\nabla^S\) preserves the hermitian metric of the bundle \(S\). To sum up, we see that the triple \((S, \nabla^S, \langle, \rangle)\) is a Dirac bundle over \(\mathbb{RP}^n\) (see [LM] for the general definition of a Dirac bundle). A direct calculation yields the formula
\[
R^S(V, W) \Phi = \frac{1}{2} (W \cdot V + \langle V, W \rangle) \cdot \Phi
\]
for the curvature tensor \(R^S\) of the connection \(\nabla^S\).

**Remark:** Let us compare the above defined covariant derivative \(\nabla^S\) with the usual covariant derivative \(\nabla\) in the usual spin bundle of a hypersurface in \(\mathbb{R}^{n+1}\). The formula for \(\nabla\) is
\[
\nabla_V \Phi = d\Phi(V) + \frac{1}{2} \Pi(V) \cdot \vec{N} \cdot \Phi
\]
i.e. over the sphere \(S^n\) the introduced Dirac bundle coincides with the usual spinor bundle of the sphere.

The Dirac operator \(D = \sum_{i=1}^n e_i \nabla^S_{e_i}\) acting on sections \(\Gamma(S; \mathbb{RP}^n)\) is related to the Laplace operator \(\Delta = -\sum_{i=1}^n \nabla^S_{e_i} \nabla^S_{e_i} - \sum_{i=1}^n \text{div}(e_i) \nabla^S_{e_i}\) by the well-known formula
\[
D^2 = \Delta + R
\]
for the curvature tensor \(R\) of the connection \(\nabla^S\).
where the endomorphism $R$ (in case of a general Dirac bundle) is given by

$$R = \frac{1}{2} \sum_{j,k} e_j e_k R^S(e_j, e_k).$$

In our situation we simply obtain

$$D^2 = \Delta + \frac{1}{4} n(n - 1) = \Delta + \frac{\tau}{4}$$

where $\tau$ is the scalar curvature of $\mathbb{R}P^n$. The bundle $S$ over $\mathbb{R}P^n$ admits Killing spinors, i.e. spinor fields $\Phi$ satisfying the differential equation

$$\nabla^S_V \Phi = \lambda V \cdot \Phi$$

for some $\lambda \in \mathbb{R}$. Indeed, suppose that $\Phi$ is a Killing spinor in $\Gamma(S; \mathbb{R}P^n)$. Then we obtain

$$R^S(V, W)\Phi = 2\lambda^2 (WV + \langle V, W \rangle) \cdot \Phi$$

using the differential equation. On the other hand, the formula for the curvature tensor implies

$$R^S(V, W)\Phi = \frac{1}{2} (WV + \langle V, W \rangle) \cdot \Phi,$$

i.e. $\lambda = \pm \frac{1}{2}$. Let us consider the spinor field

$$\Phi(x) = (1 - x)\Phi_0$$

where $\Phi_0 \in \Delta_{n+1}$ is constant. Then

$$\Phi(-x) = (1 + x)\Phi_0 = x(1 - x)\Phi_0 = x\Phi(x)$$

and therefore $\Phi \in \Gamma(S; \mathbb{R}P^n)$ is a section in the spinor bundle over $\mathbb{R}P^n$. We calculate the covariant derivative $\nabla^S_V \Phi$:

$$(\nabla^S_V \Phi)(x) = -V(x) \cdot \Phi_0 + \frac{1}{2} V(x) \cdot x \cdot (1 - x)\Phi_0 =$$

$$= -V(x) \cdot \Phi_0 + \frac{1}{2} V(x) \cdot (1 + x)\Phi_0 =$$

$$= \frac{1}{2} V(x) \cdot (-1 + x) \cdot \Phi_0 = -\frac{1}{2} V(x) \cdot \Phi(x),$$

i.e. $\Phi(x)$ is a Killing spinor with Killing number $\lambda = -\frac{1}{2}$.

**Remark:** Killing spinors with Killing number $\lambda = +\frac{1}{2}$ on $\mathbb{R}P^n$ appear if we use the identification $\hat{g}_- : S^* \to S^*$ defined by the formula

$$\hat{g}(x, \varphi) = (-x, -x \cdot \varphi).$$

$\hat{g}_-$ defines in a similar way a Clifford bundle $S_- = S^*/\hat{g}_-$ whose sections are maps $\Phi : S^n \to \Delta_{n+1}$ with the property
\[ \Phi(-x) = -x \Phi(x). \]

The spinor fields
\[ \Phi(x) = (1 + x) \Phi_0 \]
are Killing spinors in the bundle \( S_- \) over \( \mathbb{RP}^n \) with Killing number \( \lambda = +\frac{1}{2} \).

Killing spinors on compact Riemannian spin manifolds with positive scalar curvature correspond to eigenspinors of the Dirac operator related to the smallest eigenvalue. In case \( \mathbb{RP}^{4k+3} \) the first eigenvalue of the Dirac operator on usual spinors has been calculated \( \left( \lambda_1 = \pm \frac{1}{2} \sqrt{\frac{n}{n-1}} \tau \right) \) and there are Killing spinors (see [F1]). More general, we can calculate the spectrum of the Dirac operator acting on sections \( \Phi \in \Gamma(S; \mathbb{RP}^n) \) using the computation of the eigenvalues of \( D \) on the sphere. Let \( \Phi : S^n \to \Delta_{n+1} \) be a section in the bundle \( S^* \). We decompose \( \Phi \) into \( \Phi = \Phi_+ + \Phi_- \) with
\[
\Phi_+(x) = \frac{\Phi(x) - x \cdot \Phi(-x)}{2}, \quad \Phi_-(x) = \frac{\Phi(x) + x \cdot \Phi(-x)}{2}.
\]
The sections \( \Gamma(S; \mathbb{RP}^n) \) are described by the condition \( \Phi_- \equiv 0 \). Now we apply the calculation of the spectrum of the Dirac operator on spheres as well as the realization of eigenspinors by polynomials (see [Su], [Tr]). Imposing the additional condition \( \Phi_- \equiv 0 \) we obtain the spectrum of the Dirac operator acting on the sections \( \Gamma(S; \mathbb{RP}^n) \) for arbitrary \( n \).

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