SYMMETRY REDUCTION OF THE 3-BODY PROBLEM IN $\mathbb{R}^4$

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Dedicated to James Montaldi

Abstract. The 3-body problem in $\mathbb{R}^4$ has 24 dimensions and is invariant under translations and rotations. We do the full symplectic symmetry reduction and obtain a reduced Hamiltonian in local symplectic coordinates on a reduced phase space with 8 dimensions. The Hamiltonian depends on two parameters $\mu_1 > \mu_2 \geq 0$, related to the conserved angular momentum. The limit $\mu_2 \to 0$ corresponds to the 3-dimensional limit. We show that the reduced Hamiltonian has relative equilibria that are local minima and hence Lyapunov stable when $\mu_2$ is sufficiently small. This proves the existence of balls of initial conditions of full dimension that do not contain any orbits that are unbounded.

1. Introduction

Consider $N$ masses $m_i$ at positions $r_i \in \mathbb{R}^d$, $i = 1, \ldots, N$, moving under the influence of Newtonian attraction with potential

$$U = -\sum_{1 \leq i < j \leq N} \frac{m_im_j}{||r_i - r_j||},$$

so that Newton’s equations of motion are

$$m_i\ddot{r}_i = -\nabla_{r_i}U, \quad i = 1, \ldots, N.$$

Here $|| \cdot ||$ denotes the Euclidean norm on $\mathbb{R}^d$. These equations are invariant under translations and Galilein boosts $r_i \to r_i + c + vt$ for some constant vectors $c, v \in \mathbb{R}^d$ and under rotations $r_i \to Mr_i$ for some constant matrix $M \in SO(d)$. The corresponding conserved quantities are the total linear momentum $\sum m_i \dot{r}_i$ and the total angular momentum $\sum m_i r_i \wedge \dot{r}_i$. In addition there is a scaling symmetry $r_i \to sr_i$ and $t \to ts^{3/2}$ for constant scalar $s$.

The goal of this paper is to reduce these equations by translation and rotation symmetry, specifically for $N = 3$ and $d = 4$. The cases $d = 1, 2, 3$ have been studied extensively in the classical literature, see, e.g., [Whi37]. Larger $d$ have more recently been studied by Albouy and Chenciner [AC98] and Chenciner [Che13]. For $N = 3$ the case $d = 4$ is interesting because new dynamics appears compared to $d = 3$. In particular the balanced configurations introduced in [AC98] are relative equilibria that do not exist for $d = 3$. For $N = 3$, cases with $d > 4$ do not, by contrast, produce new dynamics compared to $d = 4$. The reduction we are going to present holds for an arbitrary potential that depends on the
distances $||r_1 - r_j||$ only. It is based on a novel approach to the well-known procedure of eliminating the nodes which dates back to Jacobi [Jac43] in the case $d = 3$.

With the fully reduced Hamiltonian function (Hamiltonian) at hand it is then straightforward to find new relative equilibria and analyse their stability. Our second main theorem shows that there is a family of relative equilibria that corresponds to minima of the Hamiltonian, and thus constitute solutions of the 3-body problem that are Lyapunov stable. A simple corollary is that M. Herman’s “Oldest problem in dynamical systems” on whether the set of unbounded solutions is dense for negative energy [Her98] can be answered for $d = 4$ by “No!”: There is a ball of full dimension that does not contain any unbounded solutions.

The realisation that the 3-body problem in $\mathbb{R}^4$ has Lyapunov stable relative equilibria was conceived in discussions with Alain Albouy, Rick Moeckel, James Montaldi and Alain Chenciner at the Observatory in Paris in 2015. Some of the results of these discussions are presented in the preprint [AD19]. In [AD19] is it shown that there is a global minimum of the Hamiltonian for generic angular momentum, and some properties of the families of relative equilibria are proved. By contrast, in the present paper we prove that all three families of relative equilibria are minima when the angular momentum is sufficiently close to the (non-generic) 3-dimensional case.

After this paper was finished a related preprint [SS19] appeared. In that preprint only the subgroup $SO(2) \times SO(2)$ of the full rotational symmetry group $SO(4)$ is considered in the reduction and hence somewhat different results are obtained.

2. Translation Reduction

Translation reduction is well known and can be achieved by introducing Jacobi vectors. Define vectors $x_i \in \mathbb{R}^d$ by

$$x_1 = r_2 - r_3, \quad x_2 = r_1 - \frac{m_2 r_2 + m_3 r_3}{m_2 + m_3}, \quad x_3 = \frac{m_1 r_2 + m_2 r_2 + m_3 r_3}{m_1 + m_2 + m_3}$$

and conjugate momenta $y_i \in \mathbb{R}^d$ by

$$y_1 = -\frac{m_3 \dot{r}_2 + m_2 \dot{r}_3}{m_2 + m_3}, \quad y_2 = \frac{(m_2 + m_3) \dot{r}_1 + m_1 \dot{r}_2 + m_1 \dot{r}_3}{m_1 + m_2 + m_3}, \quad y_3 = m_1 \dot{r}_1 + m_2 \dot{r}_2 + m_3 \dot{r}_3.$$

Clearly $x_3$ is the centre of mass and $y_3$ is the total linear momentum, both of which are set to zero from now on.

The mutual distances in these coordinates become

$$||r_2 - r_3|| = ||x_1||, \quad ||r_3 - r_1|| = ||a_2 x_1 + x_2||, \quad ||r_1 - r_2|| = ||a_3 x_1 - x_2||$$

with $a_i = m_i/(m_2 + m_3)$, $i = 2, 3$, so that the potential is a function of the scalar products $x_i \cdot x_j$, $i, j = 1, 2$ only.

Define the reduced masses

$$\nu_1 = \frac{m_2 m_3}{m_2 + m_3}, \quad \nu_2 = \frac{m_1 (m_2 + m_3)}{m_1 + m_2 + m_3}$$
so that the translation reduced Hamiltonian becomes

\[ H = \frac{1}{2\nu_1} ||y_1||^2 + \frac{1}{2\nu_2} ||y_2||^2 + V(||x_1||^2, ||x_2||^2, x_1 \cdot x_2) \]

with \(x_1, x_2, y_1, y_2 \in \mathbb{R}^d\).

The Hamiltonian \(H\) is invariant under rotations \((x_1, x_2, y_1, y_2) \rightarrow (Mx_1, Mx_2, My_1, My_2)\).

The corresponding angular momentum is given by the angular momentum

\[ L = x_1 \wedge y_1 + x_2 \wedge y_2 \in \mathfrak{so}(d) \]

which for \(d = 4\) has 6 independent components. The wedge product can be expressed as an anti-symmetric matrix using \(x \wedge y = xy^t - yx^t\). Since \(L\) is anti-symmetric, for \(d = 4\) its characteristic polynomial is even and has two invariants: The determinant of \(L\) which is a perfect square called the Pfaffian of \(L\), and the trace of \(L^2\). Denote the eigenvalues of \(L\) by \(\pm i\mu_1\) and \(\pm i\mu_2\), so that \(\text{Pf}(L) = \mu_1\mu_2\) and \(\text{tr}L^2 = -2(\mu_1^2 + \mu_2^2)\).

3. Rotation Reduction

Reduction by rotations does depend on the dimension \(d\). In order to generalize the reduction procedure due to Jacobi as, e.g., described in [Whi37], to the case \(d = 4\), we introduce a basis for the plane in \(\mathbb{R}^4\) spanned by the two vectors \(x_1\) and \(x_2\) through an orthogonal rotation matrix \(M\). In this basis we can write

\[ x_1 = Mq_{12}, \quad q_{12} = (q_1, q_2, 0, 0)^t, \quad x_2 = Mq_{34}, \quad q_{34} = (q_3, q_4, 0, 0)^t. \]

Since the potential is a function of the scalar products \(x_i \cdot x_j\), in the new coordinates the potential depends only on \(q_1^2 + q_2^2, q_3^2 + q_4^2, q_1q_3 + q_2q_4\). The main problem is to determine the form of the kinetic energy in the new coordinates.

Define two essential quantities: The oriented area \(A\) of the triangle formed by two vectors \(x_1\) and \(x_2\) in configuration space and an angular momentum \(L_3\) as, respectively,

\[ A = \frac{1}{2}(q_1q_4 - q_2q_3), \quad L_3 = q_1p_2 - q_2p_1 + q_3p_4 - q_4p_3. \]

The orthogonal matrix \(M\) is a product of elementary rotation matrices. Since \((q_1, q_2, q_3, q_4)\) already give 4 degrees of freedom, the rotation \(M\) needs to have another 4 degrees of freedom, say \((\psi_1, \psi_2, \theta_1, \theta_2)\). Notice that not all of the 6 dimensions of \(\text{SO}(4)\) are used in this way.

Denote a basis of \(\text{so}(4)\) by \(B_{ij} = E_{ij} - E_{ji}\), where \(E_{ij}\) is the matrix with all entries equal to zero except for the \(ij\)-entry which is 1. Now define the rotation matrix \(M \in \text{SO}(4)\) by

\[ M = \exp(B_{12}\theta_1) \exp(B_{34}\theta_2) \exp(B_{13}\psi_1) \exp(B_{24}\psi_2) = M_\theta M_\psi. \]

Notice that the first two factors and the last two factors commute.

**Lemma 1.** A symplectic transformation from \((x_1, x_2, y_1, y_2) \in \mathbb{R}^{16}\) to new local coordinates \(q_{\text{new}} = (q_1, q_2, q_3, q_4, \psi_1, \psi_2, \theta_1, \theta_2)\) and momenta \(p_{\text{new}} = (p_1, p_2, p_3, p_4, p_\psi_1, p_\psi_2, p_\theta_1, p_\theta_2)\) is
given by

\[
(2) \quad x_1 = M \begin{pmatrix} q_1 \\ q_2 \\ 0 \\ 0 \end{pmatrix}, \quad x_2 = M \begin{pmatrix} q_3 \\ q_4 \\ 0 \\ 0 \end{pmatrix}, \quad y_1 = M \begin{pmatrix} p_1 \\ p_2 \\ \alpha_1 \\ \alpha_2 \end{pmatrix}, \quad y_2 = M \begin{pmatrix} p_3 \\ p_4 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}
\]

where \( \alpha_i \) are linear in all momenta and given by

\[
\alpha_1 = q_2 B - \frac{q_4 p_4}{2A}, \quad \alpha_2 = -q_4 C + \frac{q_3 p_4}{2A}, \quad \alpha_3 = -q_1 B + \frac{q_2 p_4}{2A}, \quad \alpha_4 = q_2 C - \frac{q_1 p_4}{2A}
\]

\[
B = \frac{L_3 \sin 2\psi_1 + 2(p_{\theta_1} \sin \psi_1 \cos \psi_2 + p_{\theta_2} \cos \psi_1 \sin \psi_2)}{2A(\cos 2\psi_1 - \cos 2\psi_2)}
\]

\[
C = \frac{L_3 \sin 2\psi_2 + 2(p_{\theta_1} \cos \psi_1 \sin \psi_2 + p_{\theta_2} \sin \psi_1 \cos \psi_2)}{2A(\cos 2\psi_1 - \cos 2\psi_2)}
\]

**Proof.** In configuration space, the old coordinates are expressed as functions of the new coordinates, \( q_{\text{old}} = F(q_{\text{new}}) \). This is extended to a canonical symplectic transformation by cotangent lift \( p_{\text{old}} = (DF)^{-t}p_{\text{new}} \). In our case there is a special structure because in front of the vectors on the right-hand sides of the relations in (2) we have an orthogonal matrix as a prefactor.

The derivative of \( x_1 \) is given by

\[
\frac{\partial x_1}{\partial q_{\text{new}}} = M \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & q_2 \cos \psi_1 \cos \psi_2 & q_2 \sin \psi_1 \sin \psi_2 \\ 0 & 1 & 0 & 0 & 0 & -q_4 \cos \psi_1 \cos \psi_2 & -q_4 \sin \psi_1 \sin \psi_2 \\ 0 & 0 & 0 & 0 & q_2 \cos \psi_1 \cos \psi_2 & -q_2 \cos \psi_1 \sin \psi_2 & q_1 \sin \psi_1 \cos \psi_2 \\ 0 & 0 & 0 & 0 & -q_2 & -q_1 \cos \psi_1 \sin \psi_2 & 0 \end{pmatrix} = MU_{12},
\]

and the derivative of \( x_2 \) is

\[
\frac{\partial x_2}{\partial q_{\text{new}}} = M \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & q_3 \cos \psi_1 \cos \psi_2 & q_3 \sin \psi_1 \sin \psi_2 \\ 0 & 0 & 0 & 1 & 0 & 0 & -q_3 \cos \psi_1 \cos \psi_2 & -q_3 \sin \psi_1 \sin \psi_2 \\ 0 & 0 & 0 & 0 & q_4 \cos \psi_1 \cos \psi_2 & -q_4 \cos \psi_1 \sin \psi_2 & q_1 \sin \psi_1 \cos \psi_2 \\ 0 & 0 & 0 & 0 & -q_4 & -q_3 \cos \psi_1 \sin \psi_2 & 0 \end{pmatrix} = MU_{34}.
\]

The non-trivial entries can be computed from \( M^t M \). For example the last column of \( U_{12} \) is given by \( M^t(\partial M/\partial \theta_2)q_{12} \). Forming the matrix \( U^t = (U_{12}^t, U_{34}^t) \) the hard work is to invert \( U \). The determinant of \( U \) is

\[
\det U = \frac{1}{2}A^2(\cos 2\psi_1 - \cos 2\psi_2).
\]

Now the cotangent lift is given by \( p_{\text{old}} = \text{diag}(M, M)U^{-t}p_{\text{new}} \) and this gives the formulas claimed. \( \square \)

**Lemma 2.** In the new coordinates the Hamiltonian becomes

\[
(3) \quad H = \frac{1}{2}(p_1^2 + p_2^2 + \tilde{f}(q_3, q_4)) + \frac{1}{2}(p_3^2 + p_4^2 + \tilde{f}(q_1, q_2)) + V(q_1^2 + q_2^2, q_3^2 + q_4^2, q_1 q_3 + q_2 q_4)
\]
where
\[ \tilde{f}(q_i, q_j) = \left( q_i B - \frac{q_j p_{\psi_1}}{2A} \right)^2 + \left( -q_j C + \frac{q_i p_{\psi_2}}{2A} \right)^2. \]

In particular, \( H \) is independent of \( \theta_1 \) and \( \theta_2 \).

Proof. This is a simple consequence of the previous Lemma. \( \square \)

Notice that \( H \) can be considered as a partially reduced Hamiltonian function with two parameters \( p_{\theta_1} = \mu_1 \) and \( p_{\theta_2} = \mu_2 \) and two cyclic angles \( \theta_1 \) and \( \theta_2 \). This reduced Hamiltonian has 6 degrees of freedom and a 12-dimensional phase space \( M^{12} \).

Lemma 3. In the new coordinates the angular momentum \( L \) satisfies
\[ M^i \dot{L} M^j = -B_{12} p_{\theta_1} - B_{34} p_{\theta_2} - B_{13} p_{\psi_1} - B_{24} p_{\psi_2} + \frac{1}{2 \sin \delta \sin \sigma} (B_{23} (u_1 + u_2) + B_{14} (u_1 - u_2)) \]
where
\[ u_1 = L_3 - \Sigma \cos \delta, \quad u_2 = L_3 - \Delta \cos \sigma \]
and
\[ \sigma = \psi_1 + \psi_2, \quad \delta = \psi_1 - \psi_2, \quad \Sigma = p_{\theta_1} + p_{\theta_2}, \quad \Delta = p_{\theta_1} - p_{\theta_2} \]

Proof. Note that for orthogonal \( M \) we have
\[ M x \wedge M y = M x (M y)^t - M y (M x)^t = M (x^t y - y^t x) M^t, \]
which says that the momentum map \( L \) is equivariant with respect to the rotation given by \( M \). Using \( M = M^i \tilde{L} M^j \psi \) then gives
\[ M^i \dot{L} M^j = M^i \tilde{L} M^j \psi, \quad \tilde{L} = q_{12} \wedge p_{12} + q_{34} \wedge p_{34}, \]
where \( \tilde{L} \) is the angular momentum tensor in the body frame defined by \( M \). Here \( q_{ij} \) and \( p_{ij} \) refer to the vectors on the right-hand sides of the relations in (2). It is straightforward to compute
\[ \tilde{L} = B_{12} L_3 + B_{13} (q_1 \alpha_1 + q_3 \alpha_3) + B_{14} (q_1 \alpha_2 + q_3 \alpha_4) + B_{23} (q_2 \alpha_1 + q_4 \alpha_3) + B_{24} (q_2 \alpha_2 + q_4 \alpha_4). \]

Using the definitions of \( \alpha_i \) the coefficient of \( B_{13} \) reduces to \( -p_{\psi_1} \) and the coefficient of \( B_{24} \) reduces to \( -p_{\psi_2} \). Conjugating \( \tilde{L} \) with \( M^j \psi \) gives the result. \( \square \)

So far this is a partial reduction: We introduced two cyclic angles \( \theta_1 \) and \( \theta_2 \) with conjugate momenta that are now constants of motion. However, this is a reduction by two degrees of freedom only, i.e. to 6 degrees of freedom, but we would like to reduce by another two degrees of freedom (elimination of the nodes), so that the reduced system has 4 degrees of freedom. Here and subsequently, we assume the values \( \mu_1 \) and \( \mu_2 \) of the momenta \( p_{\theta_1} \) and \( p_{\theta_2} \), respectively, to be fixed and generic.

Symplectic symmetry reduction in the abstract is described by a fundamental theorem by Marsden and Weinstein [MW74]. According to that, one fixes a regular value \( \mu \) of a momentum map which is supposed to be equivariant with respect to the symmetry group, and then takes the quotient of the corresponding level set of the momentum map by the isotropy subgroup of \( \mu \). Provided that the isotropy subgroup acts freely and properly on that level set, the quotient defines the reduced symplectic manifold with a unique
symplectic form. For commutative groups both steps reduce by the same dimension. For the case of $SO(d)$ the isotropy group that fixes a generic element of $so(d)$ has dimension $[d/2]$, which is the dimension of the number of real invariant 2-dimensional eigenspaces of $L$, and corresponds, for $d = 4$ to our two cyclic angles $\theta_1$ and $\theta_2$. In the 3-body problem collinear configurations with zero angular momentum are fixed by a continuous group of rotations. Hence, the value of the momentum map is not regular there, and we expect the corresponding reduced space to have a singularity there. In fact, the symplectic coordinates which we have introduced in the present paper are valid near generic planar configurations only.

4. Restriction to an 8-dimensional invariant subset

We are now choosing a coordinate system in which $L$ has a particularly simple form, which is adapted to our choice of $M$. Let $L$ be equal to

$$L_0 = \begin{pmatrix} 0 & \mu_1 & 0 & 0 \\ -\mu_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_2 \\ 0 & 0 & -\mu_2 & 0 \end{pmatrix} = \mu_1 B_{12} + \mu_2 B_{34}$$

Notice that since $L_0$ is spanned by $B_{12}$ and $B_{34}$ and since these two matrices commute, we conclude that we have $M_\theta^2 L_0 M_\theta = L_0$. Thus the two-parameter subgroup of matrices $M_\theta$ is the isotropy group that fixes $L_0$. Our approach was inspired by the treatment of the case $d = 3$ in Whittaker [Whi37].

Lemma 4. The subset $I \subset M^{12}$ defined as the zero-level of the map $C : M^{12} \to \mathbb{R}^4$ with components $c_i$ defined by

$$c_1 = p_{\psi_1}, \quad c_2 = p_{\psi_2}, \quad c_3 = \Sigma \cos \delta - L_3, \quad c_4 = \Delta \cos \sigma - L_3$$

is an invariant set with respect to the Hamiltonian flow of the partially reduced Hamiltonian (3) for fixed values

$$p_{\theta_1} = \mu_1, \quad p_{\theta_2} = \mu_2.$$

Locally $I$ is an 8-dimensional manifold almost everywhere (near any regular point with respect to the map $C$).

Proof. Since $L$ is constant we can impose $L = L_0$. Combined with the previous Lemma on $L$ this implies that $p_{\theta_1} = \mu_1, p_{\theta_2} = \mu_2$, and four additional equations that together determine the six components of $L$. After a little bit of algebra we see that these imply $c_i = 0$ as stated in the Lemma. Of course it is also possible to check that the set $I$ is invariant by directly using the Hamiltonian vector field corresponding to the partially reduced Hamiltonian $H$ in (3).

We next show that it is possible to obtain the reduced Hamiltonian (near any regular point of the invariant set $I$) by simply restricting the Hamiltonian $H$ in (3) to $I$. This is a consequence of the following general Theorem.
Theorem 1. Let \((M, \omega)\) be a smooth, 2d-dimensional symplectic manifold equipped with some symplectic form \(\omega\), and let \(N = C^{-1}(c) \subset M\) be a smooth, \((2d - k)\)-dimensional submanifold \((k, d \in \mathbb{N}, 0 < k < d)\), where \(c \in \mathbb{R}^k\) is a regular value of the smooth map
\[
\mathcal{C} : M \rightarrow \mathbb{R}^k; \quad z \mapsto \begin{pmatrix} c_1(z) \\ \vdots \\ c_k(z) \end{pmatrix}.
\]
Furthermore, assume the square matrix
\[
\mathcal{A}(z) := \left( \omega(z)(X_{c_1}(z), X_{c_j}(z)) \right)_{i,j=1, \ldots, k}
\]
to be regular for all \(z \in N\). Here, for any smooth function \(g, X_g : M \rightarrow TM\) denotes the corresponding Hamiltonian vector field which is uniquely defined by
\[
\omega(z)(X_g(z), w) = dg(z)w
\]
for all \(z \in M, \ w \in T_zM\). Then \((N, \omega|_N)\) is a symplectic submanifold of \((M, \omega)\). Note, accordingly, \(k\) is even.

In addition, consider a Hamiltonian dynamical system
\[
\dot{x} = X_H(z), \quad z \in M
\]
corresponding to some smooth Hamiltonian \(H\), and suppose that \(N\) is invariant under the flow of that system, i.e.
\[
X_H(z) \in T_zN
\]
for all \(z \in N\). Then
\[
(X_H)|_N = X_{H|_N}
\]
is the Hamiltonian vector field of the reduced (restricted) system on \((N, \omega|_N)\).

Proof. To prove that \((N, \omega|_N)\) is a symplectic submanifold of \((M, \omega)\) we first show that \(\omega|_N\) is a symplectic form on \(N\). In fact,
\[
V_z := \text{span}\{X_{c_1}, \ldots, X_{c_k}\} \subset T_zM
\]
equipped with the symplectic form \(\omega(z)|_{V_z}\) defines a \(k\)-dimensional symplectic subspace of the symplectic vector space \((T_zM, \omega(z))\) for all \(z \in N\). This is a straightforward consequence of the regularity of the matrix \(\mathcal{A}(z)\). In turn, the \(\omega(z)\)-orthogonal, complementary subspace of \(V_z\) in \(T_zM\),
\[
V_z^\omega := \{v \in T_zM \mid \omega(z)(v, w) = 0 \ \text{for all} \ \ w \in V_z\}
\]
is symplectic, too; \(T_zM = V_z \oplus V_z^\omega\). But \(V_z \subset (T_zN)^\omega\), since by definition
\[
\omega(z)(X_{c_i}(z), w) = dc_i(z)w = 0
\]
for all \(i\) and all \(w \in T_zN\). So, \(T_zN \subset V_z^\omega\). Moreover,
\[
dim T_zN = \dim TM - k = \dim TM - \dim V_z = \dim V_z^\omega,
\]
i.e. \(T_zN = V_z^\omega\) is a symplectic subspace of \((T_zM, \omega(z))\) for all \(z \in N\). In particular, this implies that \(\omega|_{T_zN}\) is a symplectic form on \(T_zN\) for all \(z \in N\). Also, \(\omega|_N\) is a closed
differential form on the manifold $N$, since $d(\omega|_N) = (d\omega)|_N = 0$. Therefore, $\omega|_N$ satisfies all the axioms to be fulfilled for a symplectic form on $N$.

Now, we proceed to prove the second assertion of Theorem 1. Since $(N, \omega|_N)$ is a symplectic manifold, there exists a unique vector field $X_{H|_N}$ on $N$ such that

$$\omega(z)(X_{H|_N}(z), w) = d(H|_N)(z)w$$

for all $z \in N$ and all $w \in T_zN = V^*_z\omega$. But we also have

$$\omega(z)(X_H(z), w) = dH(z)w = d(H|_N)(z)w$$

for all $z \in N$ and all $w \in T_zN = V^*_z\omega \subset T_zM$, where $X_H(z) \in T_zN$ holds by the invariance property assumed to be satisfied for $N$. Therefore, uniqueness of the vector field $X_{H|_N}$ on $N$ implies $(X_H)|_N = X_{H|_N}$.

□

So, in order for the invariant set $\mathcal{I}$ described in Lemma 4 to be a symplectic submanifold almost everywhere, we need to check that the matrix $A(z)$ defined in the previous Theorem is non-singular. There are 4 non-zero entries. The determinant of the matrix is

$$\frac{(\Sigma + L_3 \cos \delta)(\Delta + L_3 \cos \sigma)}{\sin^4 \sigma \sin^4 \delta}$$

Restricted to the invariant set this simply becomes $\Delta^2 \Sigma^2 = (\mu_1^2 - \mu_2^2)^2$ and hence is non-vanishing as long as $|\mu_1| \neq |\mu_2|$. In fact, even though the matrix $A(z)$ is somewhat complicated, when restricted to the invariant set $\mathcal{I}$ the only non-zero entries are $\pm p\psi_i$.

Lemma 5. The functions $\tilde{f}$ in (3) restricted to the invariant set $\mathcal{I}$ defined in (4) are given by

$$\tilde{f}|_{\mathcal{I}} = f(q_i, q_j, L_3) = \frac{1}{16A^2} \left( (L_d + L_s)^2 q_i^2 + (L_d - L_s)^2 q_j^2 \right)$$

where $L_d$ and $L_s$ are functions of $L_3$ and the constants $\Delta = \mu_1 - \mu_2$ and $\Sigma = \mu_1 + \mu_2$:

$$L_d^2 = \Delta^2 - L_3^2, \quad L_s^2 = \Sigma^2 - L_3^2.$$

Proof. Setting $p\psi_i = 0$ and rewriting $f$ in terms of $\sigma$ and $\delta$ gives

$$B = -\frac{(L_3 \cos \sigma + \Delta)/ \sin \sigma + (L_3 \cos \delta + \Sigma)/ \sin \delta}{4A},$$

$$C = \frac{(L_3 \cos \sigma + \Delta)/ \sin \sigma - (L_3 \cos \delta + \Sigma)/ \sin \delta}{4A}.$$ 

Now inserting the definitions of the invariant set (4) reduces $\tilde{f}$ to $f$. □

This leads us to our first main result.
**Theorem 2.** The fully reduced Hamiltonian of the 3-body problem in 4-dimensional space with constant angular momentum matrix with eigenvalues $\pm i\mu_1, \pm i\mu_2$, $\mu_1 > \mu_2 \geq 0$ in local coordinates $q_i, p_i, i = 1, \ldots, 4$ assuming $A = \frac{1}{2}(q_1q_4 - q_2q_3) \neq 0$ is given by

\[
H = \frac{1}{2\nu_1}(p_1^2 + p_2^2 + f(q_3, q_4)) + \frac{1}{2\nu_2}(p_3^2 + p_4^2 + f(q_1, q_2)) + V(q_1^2 + q_2^2 + q_3^2 + q_4^2, q_1q_3 + q_2q_4)
\]

where $f$ is defined in Lemma 3.

Note that the old momenta are linear in the new momenta, and hence the kinetic energy is homogeneous of degree 2 in momenta. After restricting to the invariant set, however, the kinetic energy is not polynomial in the momenta, even though it still is homogeneous of degree 2 in momenta (including the constant momenta $p_\theta$).

We can introduce polar coordinates in the plane with vectors $(q_1, q_2)$ and $(q_3, q_4)$. The momentum conjugate to the corresponding angle will be the angular momentum $L_3$. However, the terms in $\tilde{f}$ are not rotationally symmetric, so this angle will not be cyclic. Introducing this angle would make explicit the separation into shape coordinates and orientation coordinates.

### 5. Effective Potential

According to Smale’s program [Sma70] finding relative equilibria is reduced to finding critical points of an effective potential after reduction. It is interesting to note that for $d = 2, 3$ this approach works nicely, since the Hamiltonian is quadratic in momenta, and additional terms can be considered to be part of the potential. Linear terms in momenta can be considered as effective magnetic fields. For $d = 4$ the reduced Hamiltonian is, however, not quadratic in momenta, and thus defining an effective potential in the usual way is not possible. However, we are interested in relative equilibria with vanishing $p$.

**Lemma 6.** For relative equilibria with $p_i = 0$, $i = 1, \ldots, 4$ define

\[
f(q_i, q_j, L_3) = c_0(q_i, q_j) + c_2(q_i, q_j)L_3^2 + O(L_3^4),
\]

such that the Hamiltonian to leading order in $p$ is

\[
H = K_{\text{eff}} + V_{\text{eff}} + O(p^4)
\]

where

\[
K_{\text{eff}} = \frac{1}{2\nu_1}(p_1^2 + p_2^2 + c_2(q_3, q_4)L_3^2) + \frac{1}{2\nu_2}(p_3^2 + p_4^2 + c_2(q_1, q_2)L_3^2),
\]

\[
V_{\text{eff}} = V(q_1^2 + q_2^2 + q_3^2 + q_4^2, q_1q_3 + q_2q_4) + \frac{1}{2\nu_1}c_0(q_3, q_4) + \frac{1}{2\nu_2}c_0(q_1, q_2).
\]

Then critical points of $V_{\text{eff}}$ are relative equilibria of $H$. If in addition $\frac{1}{2\nu_1}c_2(q_3, q_4) + \frac{1}{2\nu_2}c_2(q_1, q_2) > 0$ at the critical point, and the critical point is a minimum of $V_{\text{eff}}$, then it is a minimum of $H$. 
The function $c_0$ is the constant term of the Taylor expansion $c_0(q_i, q_j) = f(q_i, q_j, 0)$. Define the (effective) moments of inertia

$$I_1^{-1} = \frac{ q_1^2/\nu_2 + q_3^2/\nu_1}{ 4A^2}, \quad I_2^{-1} = \frac{ q_2^2/\nu_2 + q_3^2/\nu_1}{ 4A^2}. $$

Then the effective potential can be written as

$$V_{\text{eff}} = \frac{1}{2}(\mu_1^2 I_1^{-1} + \mu_2^2 I_2^{-1}) + V. $$

The additional terms in $K_{\text{eff}}$ proportional to $L_3^2$ are obtained from the Taylor series of $f$ as

$$\frac{L_3^2}{2(\mu_1^2 - \mu_2^2)}(-\mu_1^2 I_1^{-1} + \mu_2^2 I_2^{-1}).$$

It appears as if in the limit $\mu_2 \to 0$ we never have a positive definite $K_{\text{eff}}$. However, notice that for small $\mu_2$ the orders in $\mu_2$ of the various $q_i$ are different, in particular $q_3$ is of order 1 while $q_1$ is of order $\mu_2^2$, and $q_2$ and $q_3$ are negligibly small. Also notice that the condition stated in the Lemma is sufficient for the definiteness of $K_{\text{eff}}$, but not necessary.

Finally let us remark that up to this point we have not assumed any particular form of the potential, other than that it depends on $x_i \cdot x_j$ only. From now (with the exception of Lemma 10) we will treat the Newtonian case only.

6. Equilibria of the Reduced Hamiltonian for Two Equal Masses

Before treating the case of arbitrary masses we now discuss the case of two equal masses $m_2 = m_3 = m$. This case is technically simpler since the equilibrium conditions for one of the equilibria can be solved explicitly. In the general case, instead we only have a series solution near $\mu_2 = 0$.

Theorem 3. For $m_2 = m_3 = m$ an isosceles triangle is a relative equilibrium of the 3-body problem in $\mathbb{R}^4$ for any momenta $\mu_1 > \mu_2 > 0$. The relative equilibrium is given by $q_2 = q_3 = 0$ and $p_1 = p_2 = p_3 = p_4 = 0$, and two additional equations that relate $q_1, q_4$ to $\mu_1, \mu_2$:

$$\frac{m^2}{q_1^2} + \frac{4mm_1q_1}{(q_1^2 + 4q_2^2)^{3/2}} - \frac{\mu_2^2}{\nu_1q_1^2} = 0, \quad \frac{16mm_1q_4}{(q_1^2 + 4q_4^2)^{3/2}} - \frac{\mu_1^2}{\nu_2q_4^2} = 0. $$

For $\mu_2$ sufficiently small the corresponding critical point of the reduced Hamiltonian is a minimum.
Figure 1. Scaled energy-momentum diagram of the isosceles family of relative equilibria (or balanced configuration) in the 3-body problem in dimension 4 for two different mass ratios. These relative equilibria are minima of the Hamiltonian for sufficiently large negative scaled energy $h$, which occurs for small $b$ corresponding to small $\mu_2$.

Proof. In the isosceles case $a_1 = a_2 = \frac{1}{2}$, $\nu_1 = m/2$, and $\nu_2 = 2mm_1/(2m + m_1)$. The derivative of the function $f$ with respect to $p_i$ at vanishing momenta vanishes. The claim that the critical point is a minimum is proved in the following Lemmas.

Denote the mass ratio as $n = m_1/m$ and the length ratio as $\rho = q_1/q_4$. To rationalise the square root use $\rho = 4t/(1 - t^2)$ where $t \in (0, 1)$ and $t = 2 - \sqrt{3}$ corresponds to the equilateral triangle. With this parametrisation the equilibrium condition determines $\mu_2^2$ as rational functions of $n$ and $t$ (up to scaling with $m^3q_4$). One can check that by the implicit function theorem this is always possible instead of eliminating $q_1, q_4$.

A family of equilibria is best described in an energy-momentum diagram, see Figure 1. Define the scaled energy $h$ and dimensionless momentum $b$ as

$$(h, b) = \left( (\mu_1 + \mu_2)^2 H_{\text{eq}}, \frac{\mu_1 \mu_2}{(\mu_1 + \mu_2)^2} \right).$$

In Fig. 1 instead $(-1/h, b)$ is plotted since we are interested in the limit $\mu_2 \to 0$ where $h \to -\infty$ and $b \to 0$. The parameter along the curve is the shape parameter $\rho = q_1/q_4$ ranging from $\rho = 0$ (collision, left endpoint with $b = 0$) through $\rho = 2/\sqrt{3}$ (equilateral) to $\rho = \infty$ (collinear, right endpoint with $b = 0$). Equivalently, the parameter $t \in (0, 1)$ can be used.

In the limit $\mu_2 \to 0$ and hence $b \to 0$, while $\mu_1$ remains finite, the equilibrium condition of Theorem 3 can be written as ($n = \frac{m_1}{m}$)

$$q_1 = \frac{2\mu_1^2}{m^3} b^2 \left( 1 + 8b + O(b^2) \right), \quad q_4 = \frac{\mu_1^2}{4m^3n^2(2 + n)} \left( (2 + n)^2 + 24b^4 + O(b^5) \right).$$
The distance between the equal mass particles goes to zero with \( b^2 \), while the distance to the third particle remains finite in this limit.

**Lemma 7.** The Hessian of the reduced Hamiltonian at the isosceles equilibrium is block-diagonal, with three non-trivial \( 2 \times 2 \) blocks and two explicit eigenvalues given by \( 1/\nu_i \). In the following expressions for these \( 2 \times 2 \) blocks the relationship between \( q_1, q_4 \) and \( \mu_1, \mu_2 \) has not been used.

\[
\frac{\partial^2 H}{\partial^2 (q_2, q_3)} \bigg|_{eq} = \left( \begin{array}{cc}
\frac{\mu_1^2}{\nu_2 q_1 q_4} + \frac{m_1^2}{q_1} + \frac{4 m m_1 (q_1^2 - 8 q_4^2)}{(q_1^2 + 4 q_4^2)^{3/2}} & \frac{\mu_1^2}{\nu_2 q_1 q_4} + \frac{\mu_2^2}{\nu_4 q_1 q_4} - \frac{48 m m_1 q_1 q_4}{(q_1^2 + 4 q_4^2)^{3/2}} \\
\frac{\mu_2^2}{\nu_4 q_1 q_4} - \frac{48 m m_1 q_1 q_4}{(q_1^2 + 4 q_4^2)^{3/2}} & \frac{\mu_3^2}{\nu_4 q_1 q_4} - \frac{32 m m_1 (q_1^2 - 2 q_4^2)}{(q_1^2 + 4 q_4^2)^{3/2}}
\end{array} \right)
\]

\[
\frac{\partial^2 H}{\partial^2 (q_1, q_4)} \bigg|_{eq} = \left( \begin{array}{cc}
\frac{3 \mu_1^2}{\nu_2 q_1^2} - 2 \frac{m_1^2}{q_1} - \frac{8 m_1 (q_1^2 - 2 q_4^2) m}{(q_1^2 + 4 q_4^2)^{3/2}} & \frac{48 m m_1 q_1 q_4}{(q_1^2 + 4 q_4^2)^{3/2}} \\
\frac{48 m m_1 q_1 q_4}{(q_1^2 + 4 q_4^2)^{3/2}} & \frac{3 \mu_2^2}{\nu_2 q_1^2} + \frac{16 m m_1 (q_1^2 - 8 q_4^2)}{(q_1^2 + 4 q_4^2)^{3/2}}
\end{array} \right)
\]

\[
\frac{\partial^2 H}{\partial^2 (p_2, p_3)} \bigg|_{eq} = \frac{1}{\mu_1^2 - \mu_2^2} \left( \begin{array}{ccc}
\mu_1^2 & \frac{1}{\nu_1} - \frac{q_1^2}{\nu_2 q_1 q_4} & \frac{\mu_1^2 q_1}{\nu_2 q_4} - \frac{\mu_2^2 q_1}{\nu_1 q_4} \\
\frac{\mu_2^2}{\nu_2 q_1 q_4} - \frac{\mu_2^2 q_4}{\nu_1 q_4} & \mu_2^2 & \frac{q_4^2}{\nu_1 q_1} - \frac{1}{\nu_2}
\end{array} \right)
\]

**Proof.** The blocks are found by differentiating \( V_{eff} \) and \( K_{eff} \), evaluated at \( p_1 = p_2 = p_3 = p_4 = 0, q_2 = q_3 = 0 \).

In the following \( \mu_1, \mu_2 \) have been eliminated using the equilibrium condition, parametrised by \( t \). The eigenvalues then depend (up to an overall scaling) on the essential parameters \( n = m_1/m \) and \( t \).

**Lemma 8.** For \( t \to 0 \) all eigenvalues of the Hessian are positive. The \( m^2/q_1^3 \)-scaled eigenvalues of the \( (q_2, q_3) \)-block for \( t \to 0 \) are

\[
\frac{n^2}{(2n + 4)^2} - \frac{1}{4 t} - \frac{2 (7 n^2 + 2 n)}{n + 2} + O(t), \quad \frac{1}{64 t^3} + \frac{17}{64} + \frac{1}{8 n} + \frac{11 n^2 + 6 n}{n + 2} + O(t).
\]

The \( m^2/q_1^3 \)-scaled eigenvalues of the \( (q_1, q_4) \)-block for \( t \to 0 \) are

\[
\frac{1}{64 t^3} - \frac{3}{64 t} + 2 n + O(t), \quad 2 n + 12 n t^2 + O(t^3).
\]

The \( m \)-scaled eigenvalues of the \( (p_2, p_3) \)-block for \( t \to 0 \) are

\[
2 - \frac{8 (3 n - 2) t^2}{n} + O(t^3), \quad \frac{n + 2}{16 n t^2} + \frac{(n + 2)^2}{32 n^4} + O(t).
\]

**Proof.** Because of the block-diagonal structure of the Hessian these can be obtained by solving quadratic equations and expanding the roots in the small parameter \( t \).

Instead of using series expansion we can check conditions for the Hessian to be positive definite. This can be done globally, for all \( t \in (0, 1) \). Definiteness is lost when the determinants of the blocks go through zero or infinity. The expression \( \mu_1^2 - \mu_2^2 \) appears in the
Figure 2. Parameter space $n = m_1/m > 0$ and shape parameter $t \in (0, 1)$ of the isosceles equilibrium. The curves divide the positive quadrant into 6 regions. The horizontal line $t = 2 - \sqrt{3}$ corresponds to the equilateral triangles. The parabola-shaped curve $P_1(n, t) = 0$ indicates a vanishing of the determinant of the $(q_2, q_3)$-block. The curve $P_2(n, t) = 0$ starting at the origin indicates a vanishing of the determinant of the $(q_2, q_3)$-block and an infinity in the determinant of the $(p_2, p_3)$-block. In the region adjacent to the $n$-axis all eigenvalues are positive and the isosceles solution is a minimum of the 3-body problem in $\mathbb{R}^4$.

Lemma 9. The eigenvalues of the $(q_2, q_3)$-block are positive if $(\mu_1^2 - \mu_2^2)P_1(n, t) > 0$.

The eigenvalues of the $(q_1, q_4)$-block are always positive.

The eigenvalues of the $(p_2, p_3)$-block are positive if $(2 - \sqrt{3}) - t)(\mu_1^2 - \mu_2^2) > 0$.

Proof. The determinant of the $(q_2, q_3)$-block vanishes when $\mu_1^2 = \mu_2^2$ and when the polynomial
\[
P_1(n, t) = 32t^3 ((3n(t^4 - 6t^2 + 1) + 2(t^4 - 10t^2 + 1)) - (t^2 + 1)^5
\]
vanishes.
The determinant of the \((q_1, q_4)\)-block is
\[
m^4 n (t^2 - 1)^6 \left(128 n t^3 + t^6 + 15 t^4 + 15 t^2 + 1\right)
\]
\[
32 q_4^3 (t^2 + 1)^6
\]
which is positive for positive \(n\) and \(t\).

The determinant of the \((p_2, p_3)\)-block (without the prefactor \(\mu_1^2 - \mu_2^2\)) is
\[
\frac{(1 - t^2) (t^2 - 4 t + 1) (t^4 + 4 t^3 + 18 t^2 + 4 t + 1)}{2 t (t^2 + 1)^3},
\]
where only the middle factor in the numerator changes sign at \(t = 2 - \sqrt{3}\) (equilateral triangle). The prefactor itself vanishes when \(\mu_1^2 = \mu_2^2\), which implies \(P_2(t) = 0\).

The curves \(P_1(t) = 0\) and \(P_2(t) = 0\) together with \(t = 2 - \sqrt{3}\) are shown in Figure 2.

The frequencies \(\omega_i\) for rotation in the eigenplanes are determined by differentiating the Hamiltonian with respect to \(\mu_1\) and \(\mu_2\). At the equilibrium the only contribution comes from \(V_{\text{eff}}\) such that \(\omega_i = \mu_i I_i^{-1}\). This gives
\[
\omega_1 = \frac{\mu_1}{\nu_2 q_4^2}, \quad \omega_2 = \frac{\mu_2}{\nu_1 q_4^2}
\]
and hence
\[
\omega_1 = \sqrt{\frac{m (2 + n)}{q_4^3}} \sqrt{\frac{(1 - t^2)^3}{(1 + t^2)^3}}, \quad \omega_2 = \sqrt{\frac{2 m}{q_3^3} \frac{1 + 32 n t^3}{(1 + t^2)^3}}
\]
such that for \(t \to 0\) the frequencies of rotation are given by Kepler’s third law. Frequency \(\omega_2\) is determined by masses \(m_2\) and \(m_3\) orbiting around each other with distance \(q_1\), ignoring \(m_1\), while frequency \(\omega_1\) is determined by mass \(m_1\) orbiting around the combined mass \(m_2 + m_3\) at distance \(q_3\), and hence behaves like \(\sqrt{M/q_4^3}\). Note that \(\omega_2\) diverges like \(t^{-3/2}\), while \(\omega_1\) remains finite. The frequency ratio simply is \((1 + t^2)^2 + 32 n t^3)/(32 (2 + n) t^3)\), which in general is irrational so that the relative equilibrium is a quasiperiodic motion with two incommensurate frequencies.

7. General masses

Denote by \(\mu\) the ratio \(\mu = \mu_2/\mu_1\) and by \(M = m_1 + m_2 + m_3\). We are now giving a series expansion of the coordinates of the equilibrium in the limit \(\mu_2 \to 0\) (and hence \(\mu \to 0\)).

**Lemma 10.** The equilibrium condition \(D_q V_{\text{eff}} = 0\) implies the solvability condition \(q_1 q_2 \nu_1 + q_3 q_4 \nu_2 = 0\). Simplifying the equilibrium condition using the solvability condition gives
\[
\begin{align*}
\frac{I_1 \mu_2^2 q_4}{8 A^3 \nu_1 \nu_2} &= 2 q_1 V_1 + q_3 V_3, \quad -\frac{I_2 \mu_1^2 q_3}{8 A^3 \nu_1 \nu_2} = 2 q_2 V_1 + q_4 V_3, \\
-\frac{I_1 \mu_2^2 q_2}{8 A^3 \nu_1 \nu_2} &= 2 q_3 V_2 + q_1 V_3, \quad \frac{I_2 \mu_1^2 q_1}{8 A^3 \nu_1 \nu_2} = 2 q_4 V_2 + q_2 V_3.
\end{align*}
\]
Proof. After reduction the potential is a function of the form $V(q_1^2 + q_2^2, q_3^2 + q_4^2, q_1 q_3 + q_2 q_4)$. Thus the gradient with respect to $q$ is $V_i = (2q_1 V_1 + q_3 V_3, 2q_2 V_1 + q_4 V_3, 2q_3 V_1 + q_1 V_3, 2q_4 V_1 + q_2 V_3)$, where $V_i$ denotes the derivative of $V$ with respect to its $i$th argument. Reading the right hand side as a linear equation in $V_i$, $i = 1, 2, 3$ the solvability condition is that the left hand side is orthogonal to the kernel of the transpose of the matrix of the linear system. The kernel is given by $(-q_2, q_1, -q_4, q_3)^t$ and is equal to the derivative of $L_3$ with respect to $p$. The solvability condition is $(\mu_1^2 - \mu_3^2)(q_1 q_2 \nu_1 + q_2 q_4 \nu_2)/(4 A^2 \nu_1 \nu_2) = 0$, which proves the stated solvability condition. Using the solvability condition the moments of inertia simplify to

$$I_1 = \nu_2 q_1^2 + \nu_1 q_2^2, \quad I_2 = \nu_1 q_1^2 + \nu_2 q_3^2,$$

and this leads to the stated left hand side of the equilibrium condition.

The previous Lemma is valid for an arbitrary potential depending on $x_i \cdot x_j$ only. From now on all statements are about the Newtonian case.

Lemma 11. A critical point with $p_i = 0$, $i = 1, \ldots, 4$ of the reduced Hamiltonian (6) for general masses has a power series expansion for small $\mu$ given by

$$\frac{q_1}{\kappa \mu_1^2} = u^2 - \frac{m_1}{m_2 + m_3} u^8 + O(u^{12})$$

$$\frac{q_2}{\kappa \mu_1^2} = \frac{3 u^{10} m_1 (m_2 - m_3) (1 - u^4 (5 m_2^2 + 24 m_3 m_2 + 5 m_3^2))}{2 (m_2 + m_3)^2} + O(u^{16})$$

$$\frac{q_3}{\kappa \mu_1^2} = -\frac{3 u^{12} M m_2 m_3 (m_2 - m_3)}{2 (m_2 + m_3)^4} \left(1 - \frac{5 u^4 (m_2^2 + 6 m_3 m_2 + m_3^2)}{4(m_2 + m_3)^2}\right) + O(u^{18})$$

$$\frac{q_4}{\kappa \mu_1^2} = 1 + \frac{3 u^4 m_2 m_3}{2 (m_2 + m_3)^2} + O(u^8)$$

where $\kappa = \frac{M}{m_1^2 (m_2 + m_3)^2} = \frac{\nu_1}{\nu_2 m_1 m_2 m_3}$ and $u = \mu/(m_2 m_3 \sqrt{\kappa/(m_2 + m_3)})$.

Proof. The equilibrium condition $\partial V_{\text{eff}}/\partial q_i = 0$ in the limit $\mu \to 0$ has only $q_4$ with a non-vanishing limit. The leading orders of $q_1, q_2, q_3$ in $\mu$ are 2, 10, 12, respectively. This balances the leading order of 3 of the 4 equilibrium conditions of Lemma 10. However, the condition $\partial V_{\text{eff}}/\partial q_2 = 0$ is not balanced at leading order but determines two higher order coefficients. The remaining higher order coefficients of the power series solutions are then determined order by order. A natural dimensionless expansion parameter is $u$ as determined by the leading order coefficients of $q_1$ and $q_4$. □

The surprisingly high powers of the leading order in $\mu$ (or $u$) for $q_2$ and $q_3$ can be interpreted as saying that in the collision limit $\mu_2 \to 0$ the configuration is approximately isosceles. Of course for $m_3 = m_2$ the solution is exactly isosceles and $q_2 = q_3 = 0$. The distances between the particles are

$$((|r_2 - r_3|, |r_3 - r_1|, |r_1 - r_2|)) = \kappa \mu_1^2 \left(u^2, 1 + u^4 m_2 m_3 + 2 m_2 + m_3^2, 1 + u^4 m_3 + 3 m_2 + m_3^2\right) + O(u^8)$$
and the scalar products are
\[ \langle |x_1|^2, |x_2|^2, x_1 \cdot x_2 \rangle = \kappa^2 \mu_1^4 \left( u^4 + O(u^{10}), 1 + O(u^4), \frac{3m_1(m_2 - m_3)}{2(m_2 + m_3)^2} u^{10} + O(u^{14}) \right). \]

The area behaves like \[ A = \frac{1}{2} u^2 \kappa^2 \mu_1^4 + O(u^6). \]

**Theorem 4.** The relative equilibrium of the 3-body problem in 4-dimensional space given in Lemma 11 is a minimum of the reduced Hamiltonian \( \tilde{H} \).

**Proof.** The reduced Hamiltonian is the sum of kinetic and potential energy. The effective potential \( \tilde{V} \) has a minimum at this equilibrium, as is shown in the next Lemma. We now show that the effective kinetic energy \( K_{\text{eff}} \) is positive definite for sufficiently small \( \mu \). The coefficient of the correction term proportional to \( L^2 \) in \( K_{\text{eff}} \) is
\[ -\frac{\mu_1^2}{I_1} + \frac{\mu_2^2}{I_2} = -\frac{\mu_1^2}{\nu_2 q_1^2 + \nu_1 q_2^2} + \frac{\mu_2^2}{\nu_1 q_2^2 + \nu_2 q_3^2} = -\frac{\mu_1^2}{\nu_2 q_1^2} + \frac{\mu_2^2}{\nu_1 q_1^2} + O(\mu_2^{10}) \]
and since \( q_1 = O(\mu_2^2) \) the second term dominates for \( \mu_2 \to 0 \) while the first (negative) term is \( O(1) \) and so the coefficient is positive for sufficiently small \( \mu_2 \). By Lemma 6 the Hessian of the Hamiltonian with respect to \( (p_1, p_2, p_3, p_4) \) is thus positive definite for sufficiently small \( \mu_2 \). Together with the following Lemma on the positivity of the Hessian of \( V_{\text{eff}} \) this implies that the critical point is a minimum of \( \tilde{H} \). \( \square \)

We remark that \( K_{\text{eff}} \) ceases to be positive definite for \( \mu_2 \to \mu_1 \). We also remark that the moments of inertia are the non-zero eigenvalues of the original inertia tensor, which is similar to the inertia tensor \( q_1 q_2^2 \nu_1 + q_3 q_4^2 \nu_2 \) and using the identity \( q_1 q_2 \nu_1 + q_3 q_4 \nu_2 = 0 \) gives the above moments of inertia.

**Lemma 12.** The scaled eigenvalues of the Hessian of the effective potential \( \tilde{V} \) evaluated at the equilibrium of Lemma 11 have Laurent expansions given by
\[
\frac{m_1(m_2 + m_3)}{m_2 m_3} + O(u^4)
\]
\[
\frac{m_1^2(m_2 + m_3)^3}{m_2^2 m_3^2 M u^4} - \frac{1}{u^2} + O(u^0)
\]
\[
\frac{1}{u^6} + \left( 1 + \frac{11m_2 m_3}{2(m_2 + m_3)^2} + \frac{m_2 m_3}{m_1(m_2 + m_3)} \right) \frac{1}{u^2} + O(u^0)
\]
\[
\frac{1}{u^6} + \frac{9m_2 m_3}{2(m_2 + m_3)^2 u^2} + \frac{7m_1}{m_2 + m_3} + O(u^1)
\]
with an overall scaling factor of \( m_2 m_3/q_3^3 \) removed.

Note that these formulas reduce to the isosceles case \( m_3 = m_2 \) using the relationship
\[ u^2 = 4t + 28t^3 + 128nt^4 + O(t^5). \]
As in the isosceles case, three of the eigenvalues diverge to positive infinity in the limit.
Proof. The Hessian of the effective potential evaluated at the equilibrium condition does not block-diagonalise as in the isosceles case. The Hessian can be simplified using \( q_1 q_2 \mu_1 + q_3 q_4 \nu_2 = 0 \) and the result is

\[
\frac{\mu_1^2 \nu_1^3 q_4^4}{\nu_2^2 I_{12}^2 q_4^4} \begin{pmatrix}
\frac{q_2^2}{\nu_2} & -q_3 q_4 & q_2 q_4 & q_2 q_3 \\
-q_3 q_4 & 3 q_2^2 & q_1 q_4 & -3 q_1 q_3 \\
q_2 q_4 & q_1 q_4 & \frac{q_2^2}{\nu_2} & -q_1 q_2 \\
-q_3 q_4 & -q_1 q_2 & 3 q_2^2 & q_2 q_3
\end{pmatrix}
+ \frac{\mu_2^2 \nu_2^3 q_4^4}{\nu_1 I_{12}^2 q_1^4} \begin{pmatrix}
3 q_4^2 & -q_3 q_4 & -3 q_3 q_4 & q_2 q_3 \\
-q_3 q_4 & \frac{q_1 q_4}{\nu_2} & q_1 q_4 & q_1 q_3 \\
-3 q_2 q_4 & q_1 q_4 & 3 q_2^2 & -q_1 q_2 \\
q_2 q_3 & q_1 q_3 & -q_1 q_2 & q_1 q_3
\end{pmatrix} + D^2 q \frac{V}{\nu_2}
\]

The prefactors of the first two matrices are of order \(-4\) and \(-6\) in \( \mu \), respectively. Considering all terms that do not involves \( q_2 \) or \( q_3 \) in the first two terms gives the terms denoted by \( a_{ij}^k \) (except \( a_{22}^{-3} \)) of order \( 2k \) in \( \mu \) where \( k \) ranges from \(-3\) to \( 0 \). All other terms that involves \( q_2 \) or \( q_3 \) are of order at least \( 4 \) in \( \mu \). The Hessian \( D^2 q V \) is diagonally dominant for small \( \mu \) with terms of order \(-6\) and \( 0 \) in \( \mu \) in the diagonal, and these terms are also included in \( a_{ij}^k \). All off-diagonal terms in \( D^2 q V \) are at least of order \( \mu^2 \). Thus the Hessian can be decomposed as

\[
D = D_a + D_b = \begin{pmatrix}
a_{11}^{-3} & 0 & 0 & 0 \\
0 & a_{22}^{-3} & a_{22}^{-2} & 0 \\
0 & a_{22}^{-2} & a_{33}^{-3} & 0 \\
0 & 0 & 0 & a_{44}^{-0}
\end{pmatrix}
+ \begin{pmatrix}
0 & b_{12}^1 & b_{13}^2 & b_{14}^3 \\
b_{12}^1 & 0 & 0 & b_{24}^3 \\
b_{13}^2 & 0 & 0 & b_{34}^4 \\
b_{14}^3 & b_{24}^3 & b_{34}^4 & 0
\end{pmatrix}
\]

where \( a_{ij}^k, b_{ij}^k \) denote entries of leading order \( 2k \) in \( \mu \). Extracting the overall leading order \( \mu^{-6} \) from \( D \) makes all entries power series in \( \mu \). So the symmetric \( \mu^6 D \) is a perturbation of the symmetric \( \mu^6 D_a \), and by standard theory, see, e.g. [Kat13] chapter II, §2.3, the eigenvalues of \( \mu^6 D_a \) only change at order \( \mu^8 \), the leading order of \( \mu^6 D_b \).

The entries \( a_{11}^{-3} \) and \( a_{44}^{-0} \) give two eigenvalues up to order \( 6 \) in \( \mu \). Expanding the eigenvalues of the middle \( 2 \times 2 \) block of \( D_a \) gives the leading order terms of the other two eigenvalues as stated in the Lemma. This shows that the eigenvalues of the Hessian of the effective potential are positive for small \( \mu \). \[\square\]

The frequencies of rotation in the cyclic angles \( \theta_i \) are obtained by differentiating the reduced Hamiltonian with respect to \( \mu_i \). At the equilibrium the only contribution comes from \( V_{\text{eff}} \) and hence

\[
\omega_1 = \frac{\mu_1}{\nu_2 q_4^2 + \nu_1 q_2^2} \approx \frac{\mu_1}{\nu_2 q_4^2}, \quad \omega_2 = \frac{\mu_2}{\nu_1 q_4^2 + \nu_2 q_3^2} \approx \frac{\mu_2}{\nu_1 q_3^2}.
\]

Expanding gives

\[
\omega_1 = \sqrt{\frac{M}{q_4^2}} \left( 1 - \frac{3 m_2 m_3}{4 (m_2 + m_3)^2} u^4 + O(u^8) \right)
\]

\[
\omega_2 = \sqrt{\frac{m_2 + m_3}{q_4^3}} \left( 1 + \frac{m_1}{2 (m_2 + m_3)} u^6 + O(u^{10}) \right)
\]
These formulas allow for the following nice interpretation of the three-dimensional limit $u, \mu, \mu_2 \to 0$: The mass $m_1$ encircles the colliding binary pair with frequency $\omega_1$ at distance $q_4$ such that $\omega_1^3 q_4^3 = M + O(u^4)$ is constant at leading order, which is Keplers 3rd law for $m_1$ encircling $m_2 + m_3$. The binary pair of masses $m_2, m_3$ has diverging frequency $\omega_2$ and vanishing distance $q_1$ such that $\omega_2^3 q_1^3 = m_2 + m_3 + O(u^6)$ is constant at leading order, which is again Keplers 3rd law for $m_2$ and $m_3$ encircling each other at distance $q_1$. In general the two frequencies are incommensurate, and hence this is a quasi-periodic relative equilibrium.

The value $h$ of the scaled value of the Hamiltonian as a function of the dimensionless angular momentum $b$ is

$$h = -\frac{m_2^3 m_3^3}{2(m_2 + m_3)b^2} \left( 1 - 2b + b^2 - \frac{2b^2 m_1^3 (m_2 + m_3)^4}{M m_2^3 m_3^3} + O(b^3) \right)$$

The quadratic behaviour of $-1/h$ as a function of $b$ near the origin can clearly be seen in Fig. 1 near the origin.

There are two additional similar such solutions obtained by exchanging the masses. The formulas for the critical point and its eigenvalues are symmetric in $m_2$ and $m_3$ (except for $q_2$ and $q_3$, which flip their signs). The fact that $m_2, m_3$ are singled out is a result of the choice of Jacobi coordinates in the translation reduction. The two alternative choices defining $x_1$ as either $r_3 - r_1$ or as $r_1 - r_2$ leads to two more solutions that are related by permuting the masses. All three solutions limit to collision solutions, but their precise asymptotic behaviour is different depending on the values of the masses. When all masses are equal there is only a single family of such solutions. When two masses are equal there are two families, one of which is shown in Fig. 1. For distinct masses there are three families with different limiting values for $hb^2$ given by $(m_i + m_j)/(m_i m_j)^3$ for each pair of indices $i, j$.

It is interesting to note that these limiting values of $hb^2$ are exactly the critical values at which a bifurcation of the energy surface takes place at infinity, see [Alb93]. In fact, there are some remarks in Albouy’s paper about higher spatial dimensions [Alb93, section B4]. Considering $hb^2$ in the limit $\mu_2 \to 0$ has the same order as $H\mu_2^2$, and hence is the correct scaling invariant combination in the 3-dimensional limit. In our analysis we found $q_4$ finite and $q_1 \to 0$. By rescaling, instead one can consider $q_4 \to \infty$ and $q_1$ non-zero. This corresponds to the bifurcation at infinity. It would be interesting to analyse bifurcations at infinity in the $n$-body problem in general from the point of view of higher spatial dimensions.

8. Acknowledgements

HRD would like to thank Jürgen Scheurle and the Fakultät für Mathematik at the Technische Universität München for their hospitality during his sabbatical in 2018. HRD would like to thank Alain Albouy, Alain Chenciner, Rick Moeckel and James Montaldi for extensive discussions at the Observatory in Paris in 2015 when the existence of minima of the 3-body problem in $\mathbb{R}^4$ was conceived. The preprint [AD19] describes some of the results of these discussions.
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