CONFLUENT TERMINATING EXTENSIONAL LAMBDA-CALCULI WITH SURJECTIVE PAIRING AND TERMINAL TYPE

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Abstract. For the lambda-calculus with surjective pairing and terminal type, Curien and Di Cosmo were inspired by Knuth-Bendix completion, and introduced a confluent rewriting system that (1) extends the naive rewriting system, and (2) is stable under contexts. The rewriting system has (i) a rewrite rule “a term of a terminal type rewrites to a term constant ∗, unless the term is not ∗,” (ii) rewrite rules for the extensionality of function types and product types, and rewrite rules mediating (i) and (ii). Curien and Di Cosmo supposed that because of (iii), any reducibility method cannot prove the strong normalization (SN) of Curien-Di Cosmo’s rewriting system, and they left the SN open. By relativizing Girard’s reducibility method to the ∗-free terms, we prove SN of their rewriting, and SN of the extension by polymorphism. The relativization works because: for any SN term \(t\), and for any variable \(z\) of terminal type not occurring in \(t\), \(t\) with all the occurrences of ∗ of terminal type replaced by the variable \(z\) is SN. KEYWORDS: relativized reducibility method; strong normalization;

1. Introduction

Equational theories for terminal types, unit types, singleton types are useful in mathematics and computer science:

- Coherence problem of cartesian closed category [37, 38, 39, 36, 34].
- An extension LF\(_{\Sigma, \frac{1}{1}}\) of LF [22] by dependent sum types and the type 1 for the empty context.
- Useless code elimination [9, 32].
- Proof irrelevant types [4].
- Higher-order unification for a proof assistant system Agda [5]. Agda supports Σ-types in form of records with associated η-equality in its general form.

We study the extensional \(\lambda\)-calculus \(\lambda\beta\eta\pi\ast\) with surjective pairing and unit types. It is an equational theory useful to solve the coherence problem of cartesian closed category. The equational theory \(\lambda\beta\eta\pi\ast\) is decidable. As we see below, typical proofs of the decidability employ, more or less, the following two methods:

- Tait’s reducibility methods to prove the strong normalization (SN, for short) of rewriting relations; Here, SN states that there is no infinite sequence of the rewriting relation. Variants of Tait’s reducibility method include reducibility candidate method [19] and computability closure [10].
- Logical relation methods. For the historical account, see [25].
In both methods, by induction on types, we define a family \( \{ P_\varphi \}_{\varphi} \) of sets of terms, indexed by all types \( \varphi \). Here

\[
(t_1, \ldots, t_n) \in P_{\varphi \rightarrow \psi} : \iff \forall (s_1, \ldots, s_n) \in P_\varphi. \ ((t_1 s_1, \ldots, t_n s_n) \in P_\psi).
\]

Then we carry out an induction on terms to prove the target property. Logical relations more fit to semantical problems \[45\] of \( \lambda \)-calculi.

We list proofs of the decidability of the equational theory \( \lambda \beta \eta \pi^* \).

- **Type-directed expansions.** See \[37, 38, 21, 12, 1, 17, 27, 35\], to cite a few. The SN proof of the type-directed expansion in \[27\] is as follows: They first restricted the places of terms to be replaced, proved the SN of such restricted rewriting system by a reducibility method, and then derived the SN of the type-directed expansion.

- **Sarkar’s algorithm.** The extension \( \text{LF}^{\Sigma,1} \) corresponds to \( \lambda \beta \eta \pi^* \). Sarkar \[46\] studied \( \text{LF}^{\Sigma,1} \) by the standard techniques of \[24\]. For \( \text{LF}^{\Sigma,1} \), to give a type-checking algorithm, Sarkar \[46\] provided an decision algorithm of the definitional equality. For the decision algorithm, he proved the completeness for equality by a Kripke logical relation, the soundness of the algorithm and the existence of canonical forms in \( \text{LF}^{\Sigma,1} \).

- **Normalization-by-evaluation** (\[15, 7, 3\], to cite a few). From a given term \( t \), we obtain a normal form \( v \) judgmentally equal to \( t \), by evaluating \( t \) and then by reification it. \[7\] (\[3\], resp.) used Grothendieck logical relation (Kripke logical relation, resp.) between well-typed terms \( t \) and semantic objects \( d \), which for base types expresses that \( d \) reifies to a normal form \( v \) judgmentally equal to \( t \).

- **A translation that incorporates type-directed expansions by type-indexed functions on terms.** The translation reduces the decidability of the equational theory \( \lambda \beta \eta \pi^* \) to that of the corresponding intensional equational theory \[20, 49\]. It, however, turns out that this idea does not yield a decision procedure for the equational theory \( \lambda^2 \beta \eta \pi^* \), which is the polymorphic extension of \( \lambda \beta \eta \pi^* \).

In \[16\], the decidability of the equational theory \( \lambda \beta \eta \pi^* \) are proved, much more based on rewriting technique \[6\]. They first introduced a rewriting system that generates the equational theory \( \lambda \beta \eta \pi^* \), as follows: To the simply-typed \( \beta \eta \)-rewriting, we add the rewrite rule “a term of type \( \top \) rewrites to \( * \) unless the term is not \( * \),” and then keep adding rewriting rules, like from a term rewriting system we obtain a confluent term rewriting system through Knuth-Bendix completion \[0\]. For this extensional \( \lambda \)-calculus \( (\lambda \beta \eta \pi^*)^\prime \) with surjective pairing and terminal type, they proved the weak normalization of the rewriting system, and derived the confluence from it. This rewriting system so directly depends on the rewriting technique. The reducibility methods are not so flexible as rewriting rules. Curien and Di Cosmo suggested no direct application of reducibility method proves SN of \( (\lambda \beta \eta \pi^*)^\prime \).

We prove the SN of Curien-Di Cosmo’s rewriting system \( (\lambda \beta \eta \pi^*)^\prime \) by relativizing Girard’s reducibility method to the \( * \)-free terms. We introduce the reducibility predicates for \( (\lambda \beta \eta \pi^*)^\prime \), apply them only for the set of \( * \)-free terms, derive the SN of all \( * \)-free terms. To make our relativization argument handy, we introduce the non-Haussdorf Alexandrov topological space of terms for the rewriting, and interpret our argument.
The rest of paper is organized as follows: In the next section, we recall the definition of Curien-Di Cosmo’s rewriting system (Subsection 2.1), and explain how their rewriting system suggests relativization of reducibility method (Subsection 2.2), and uncover the essence of the relativization by using Alexandrov topological space \([28]\), in (Subsection 2.3). In Section 3 we prove SN of \((\lambda^2 \beta \eta \pi\)′). In Section 4 we prove SN of \((\lambda \beta \eta \pi\)′), the extension by the polymorphism. In Section A we comment type-directed expansions as related work for Curien-Di Cosmo’s rewriting.

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2. Preliminary

2.1. Curien and Di Cosmo’s rewriting system based on eta-reduction. We recall the equational theory \(\lambda \beta \eta \pi\) from \([10]\).

Types are built up from the distinguished type constant \(\top\), and type variables, by means of the product type \(\varphi \times \psi\) and the function type \(\varphi \rightarrow \psi\). Terms are built up from the distinguished term constant \(*^{\top}\) and term variables \(x^\varphi, y^\varphi, \ldots, x^\psi, y^\psi, \ldots\), by means of \(\lambda\)-abstraction \(\lambda x^\varphi. t^\psi\rightarrow^\varphi\), term application \((u^\varphi \rightarrow^\psi v^\psi)^\psi\), pairing \(\langle u^\varphi, v^\psi \rangle^{\varphi \times \psi}\), left-projection \((\pi_1 t^{\varphi \times \psi})^\psi\), and right-projection \((\pi_2 t^{\varphi \times \psi})^\psi\). The superscript represents the type. The superscript is often omitted. The set of free variables of a term \(t\) is denoted by \(\text{FV}(t)\). The equational theory \(\lambda \beta \eta \pi\) consists of the following axioms:

\[
\begin{align*}
(\beta) & \quad (\lambda x. u)v = u[x := v]. \\
(\pi_1) & \quad \pi_1(u, v) = u. \\
(\pi_2) & \quad \pi_2(u, v) = v. \\
(\eta) & \quad \lambda x. tx = t, \quad (x \in \text{FV}(t).) \\
(SP) & \quad \langle \pi_1 u, \pi_2 u \rangle = u. \\
(c) & \quad s^\top = *^\top.
\end{align*}
\]

By the last equality, the type \(\top\) corresponds to the singleton. The singleton does to the terminal object of a cartesian closed category (CCC for short). So \(\top\) is called the terminal type.

By orienting the equational axioms \((\beta), (\pi_1), (\pi_2), (\eta), (SP)\) left to right, we obtain rewrite rule schemata. Let \((T)\) be a rewrite rule schema \(s^\top \rightarrow *^\top\) \((s^\top \neq *^\top)\). Here for terms \(t\) and \(s\), we write \(t \equiv s\), provided that by renaming bound variables, \(t\) becomes identical to \(s\). Let \(\rightarrow\) be the closure of these rewrite rule schemata by contexts. By abuse of notation, we write \(\lambda \beta \eta \pi\) for a so-obtained rewriting system. The reverse of \(\rightarrow\) is denoted by \(\leftarrow\). \(\rightarrow\) is the reflexive, transitive closure of \(\rightarrow\). Let us abbreviate confluence by CR.

The rewriting system \(\lambda \beta \eta \pi\) is not CR, as follows: In each line of the following, \(x\) and \(y\) are variables, and it is not the case that for the leftmost term \(t_1\) and the rightmost \(t_2\), there is a term \(t_0\) such that \(t_1 \rightarrow t_0 \rightarrow t_2\):

\[
\begin{align*}
y^\varphi \rightarrow^\top \leftarrow & \lambda x. (yx)^\top \rightarrow \lambda x. *^\top, \\
& \quad x \leftarrow \langle (\pi_1 x)^\top, (\pi_2 x)^\top \rangle \rightarrow \langle *^\top, *^\top \rangle, \\
\lambda x^\top. y^\top \leftarrow & \lambda x^\top. yx^\top \rightarrow y^\top \rightarrow^\varphi, \\
\langle \pi_1 x, * \rangle \leftarrow & \langle (\pi_1 x)^\varphi, (\pi_2 x)^\top \rangle \rightarrow x^\varphi \times^\top, \\
\langle *, \pi_2 x \rangle \leftarrow & \langle (\pi_1 x)^\top, (\pi_2 x)^\varphi \rangle \rightarrow x^\top \times^\varphi.
\end{align*}
\]
The behavior of the rewrite rule schemata \( g \) is not so simple as it looks like. The rewrite relation \( \rightarrow_{\beta\eta\pi_1\pi_2} \) is CR \[44\]. In the type-free setting, \( \rightarrow_{\beta SP} \) is not CR \[31\]. In dependent type theories such as Agda, the unit type (=terminal type) is important in relation to the record type, but in the presence of the unit type, the type-checking is rather difficult; Not all subterms has a type label as our terms. So, we should infer the type of the term before we apply the equational axiom (\( \varepsilon \)) to cope with a typing rule such as “M has a type A whenever M has a type B such that A is equal to B.”

For the equational theory \( \lambda\beta\eta\pi* \), Curien and Di Cosmo, inspired by completion of term rewriting systems, introduced a rewriting system \( (\lambda\beta\eta\pi*)' \) in \[16\]. First they inductively defined the types “isomorphic to” the terminal type \( \top \) and the canonical terms of such types.

**Definition 2.1** \((\lambda\beta\eta\pi*)'\).

- \( \top \) is “isomorphic to” \( \top \) and the canonical term of \( \top \) is \( \ast^\top \).
- Suppose \( \varphi \) is a type and \( \tau \) is a type “isomorphic to” \( \top \). Then the type \( \varphi \rightarrow \tau \) is “isomorphic to” \( \top \) and the canonical term \( \ast^\varphi \rightarrow^\tau \) of \( \varphi \rightarrow \tau \) is \( \lambda x \varphi. \ast^\tau \).
- If each type \( \tau_i \) is “isomorphic to” \( \top \) (\( i = 1, 2 \)), then the type \( \tau_1 \times \tau_2 \) is “isomorphic to” \( \top \) and the canonical term \( \ast^\tau_1 \times^\tau_2 \) of \( \tau_1 \times \tau_2 \) is \( (\ast^\tau_1, \ast^\tau_2) \).

The set of types “isomorphic to” \( \top \) is denoted by \( \text{Iso}(\top) \). Whenever we write \( \ast^\varphi \), we tacitly assume \( \varphi \in \text{Iso}(\top) \). The canonical terms are not directly related to ‘the canonical forms of \[46\] Sect. 8.1].

The rewrite relation \( \rightarrow \) of the rewriting system \( (\lambda\beta\eta\pi*)' \) is defined by the rewrite rule schemata obtained from the first five equational axioms \( (\beta), (\pi_1), (\pi_2), (\eta) \), and \( (SP) \) of \( \lambda\beta\eta\pi* \) by orienting left to right, and the following four rewrite rule schemata:

\[
\begin{align*}
(g) & \quad u^\tau \rightarrow^\top \ast^\tau, & (u \text{ is not canonical}) \\
(\eta_{\text{top}}) & \quad \lambda x^\tau. x^\ast^\tau \rightarrow t, & (x \notin \text{FV}(t).) \\
(SP_{\text{top}1}) & \quad \langle \pi_1 u, \ast^\tau \rangle \rightarrow u, & (u \text{ has type } \varphi \times \tau.) \\
(SP_{\text{top}2}) & \quad \langle \ast^\tau, \pi_2 u \rangle \rightarrow u, & (u \text{ has type } \tau \times \psi.).
\end{align*}
\]

The first rule \((g)\) schema that generates a canonical term \( \ast^\tau \) is called “gentop” in \[16\].

In \[16\], the rewriting system \( (\lambda\beta\eta\pi*)' \) is proved to be CR and weakly normalizing, by using an ingenious lemma for abstract reduction systems. \( (\lambda\beta\eta\pi*)' \) is non-left-linear and has a rewrite rule schema with side conditions. We cannot apply criteria for CR of left-linear (higher-order) term rewriting system based on closed condition of (parallel) critical pairs (e.g., \[45\] \[42\]). \( \beta\eta\pi_{\text{top}}g \)-reduction is the triangulation \[43\] of \( \beta\eta g \)-reduction, and thus CR by \[43\] Corollary 2.6. However, \( (\lambda\beta\eta\pi*)' \) is not a triangulation of the rewriting system \( \lambda\beta\eta\pi* \); As we see \[1\], \( g \)-rule schema rewrites the one-step reduct \( u^\top \rightarrow^\top \) of \( \langle \pi_1 u, \pi_2 u \rangle \) to the two-step reduct of \( \langle \pi_1 u, \pi_2 u \rangle \). This does not fit to the definition of the triangulation.

2.2. **Rewrite rule schema** \((\eta_{\text{top}})\), and relativized reducibility method to the \( \ast^\tau \)-terms. All variations (e.g., reducibility candidate method \[19\], computability closure \[10\]) of Tait’s reducibility method uses reducibility predicates. The reducibility predicates for \( (\lambda\beta\eta\pi*)' \) are as usual:
By an atomic type, we mean the distinguished type constant \( \top \) or a type variable.

**Definition 2.2.**

(a): A term of an atomic type is reducible, if the term is \( SN \).

(\( \times \)): A term \( t^\varphi \times \psi \) is reducible, if so are \( (\pi_1 t)^\varphi \) and \( (\pi_2 t)^\psi \).

(\( \rightarrow \)): A term \( t^\varphi \rightarrow \psi \) is reducible, if for any reducible term \( u^\varphi \), \( (tu)^\psi \) is reducible.

Let \( RED_\varphi := \{ t^\varphi \mid t^\varphi \text{is reducible} \} \). All variations of reducibility method require to show a key statement

\[
(2) \quad \forall u^\varphi \in RED_\varphi \quad (v^\psi [x^\varphi := u^\varphi] \in RED_\psi) \implies \lambda x. v \in RED_{\varphi \rightarrow \psi}.
\]

The rewrite rule schema \( (\eta_{top}) \), however, causes the difficulty to prove the key statement \([2]\), as follows \([16]\). In the reducibility candidate method \([19]\), an available auxiliary property is that, a term \( tu \) is reducible, as soon as \( s \) is reducible for all reducts \( s \) of \( tu \). So the proof of the key statement amounts to the proof that all reducts of \( (\lambda x. v) u \) are reducible. The rewrite rule schema \( (\eta_{top}) \) can rewrite \( (\lambda x. v) u \) to \( (v' u) \) which is not \( v' x := u \equiv v \). The standard argument indeed proves the following statement (Lemma 5.4 \([2]\)):

\[
(3) \quad \begin{cases}
\forall u \in RED_\varphi \quad (v^\psi [x := u] \in RED_\psi) \quad \text{and} \\
(v \equiv (v' + \top'), x \not\in \text{FV}(v')) \implies (v' \in RED_{\top \rightarrow \psi})
\end{cases}
\implies \lambda x. v \in RED_{\varphi \rightarrow \psi},
\]

This immediately implies

\[
(4) \quad v^\psi \in F \land \forall u \in RED_\varphi \quad (v^\psi [x^\varphi := u^\varphi] \in RED_\psi) \implies \lambda x. v \in RED_{\varphi \rightarrow \psi} \cap F,
\]

where

**Definition 2.3.** Let \( t \) be a term of \( (\lambda \beta \pi \ast') \). \( t \) is called \( \ast \)-free, if the term constant \( \ast' \) does not occur in \( t \). Let \( F \) be the set of \( \ast \)-free terms. Let \( T \) be the set of terms and \( SN \) be the set of \( SN \) terms.

The lemma \([4]\) suggests to split \( T \subseteq SN \) into two statements

\[
(5) \quad F \subseteq SN \implies T \subseteq SN,
\]

\[
(6) \quad F \subseteq SN,
\]

and to prove \( F \subseteq SN \) by employing \( \{ RED_\varphi \cap F \mid \varphi \text{is a type} \} \).

### 2.3. Essence of reducibility predicate relative to \( \ast \)-free terms.

To prove \( F \subseteq SN \), we prove the following relativization of the key statement \([2]\) to \( F \).

\[
(7) \quad v^\psi \in F \land \forall u \in (RED_\varphi \cap F) \quad (v[x := u] \in RED_\psi \cap F) \implies \lambda x. v \in RED_{\varphi \rightarrow \psi} \cap F.
\]

This follows from lemma \([4]\), if for a function \( f_{v^\psi,x^\varphi}(u^\varphi) := v^\psi [x^\varphi := u^\varphi] \),

\[
(8) \quad f_{v^\psi,x^\varphi} (RED_\varphi \cap F) \subseteq RED_\psi \cap F \implies f_{v^\psi,x^\varphi} (RED_\varphi) \subseteq RED_\psi
\]

So, relativizing reducibility method to \( F \) is introducing a topology to \( T \) such that

1. \( f_{v^\varphi,x^\varphi} : T \rightarrow T \) is continuous,
2. \( \overline{RED_\varphi \cap F} = RED_\varphi (\forall \varphi) \), where \( (\overline{\ }) \) is the closure operation, and
3. \( F \subseteq SN \implies T \subseteq SN \).
The pair of the first two implies $[9]$, because
\[
\text{RED}_\psi = \text{RED}_\psi \cap F \supseteq f_{u,x}(\text{RED}_\psi \cap F) \supseteq f_{u,x}(\text{RED}_\varphi \cap F) = f_{u,x}(\text{RED}_\varphi).
\]

A topological space $X$ is called Alexandrov, if there is a preorder $\leq$ such that the closed sets are exactly the upwardly closed sets. The Alexandrov topological space induced by a preorder $\leq$ is denoted by $T(\leq)$. Let $\leq$ and $\sqsubseteq$ be preorders. A function $f : T(\leq) \rightarrow T(\sqsubseteq)$ is continuous, if and only if $f$ preserves the preorders.

For Curien-Di Cosmo's rewriting $\rightarrow$, we consider the Alexandrov topological space $T = T(\rightarrow)$ of terms. For every $A \subseteq T(\rightarrow)$, the closure $\overline{A}$ is $\{t' \mid \exists t \in A. (t \rightarrow t')\}$. This topology satisfies the above-mentioned three conditions:
1. $f_{u,x}$ is continuous, because $u \rightarrow u' \implies v[x := u] \rightarrow v[x := u']$.
2. $\text{RED}_\varphi \cap F = \text{RED}_\varphi (\forall \varphi)$, because
3. A variable $z^\top$ does not occur in $t \in SN \implies t[\ast^\top := z^\top] \in SN$,
as we will see in the proof of Lemma 4.6 (2).

The Alexandrov topology uncover the essence of the reducibility method relativized to the $\ast$-terms, that is, the property $[9]$.

The property $[10]$ is also the essence of SN proof of the polymorphic extension $(\lambda^2 \beta \eta \pi \ast)'$ of $(\lambda \beta \eta \pi \ast)'$. Girard proved the polymorphic $\lambda$-calculus $\lambda^2$ by employing the candidates of reducibility for all types $[19]$. We will prove the SN of the Curien-Di Cosmo-style polymorphic $\lambda$-calculus $(\lambda^2 \beta \eta \pi \ast)'$ by relativizing the reducibility candidate method to the $\ast$-terms, as we did SN of $(\lambda \beta \eta \pi \ast)'$ by relativizing the reducibility method to the $\ast$-terms. For the SN proof of $(\lambda^2 \beta \eta \pi \ast)'$, in Definition 4.3 we additionally require the following property to each reducibility candidate $\mathcal{R}$ of each type $\varphi$:

**Property 2.4.** (1) If $\ast^\varphi$ is defined, then $\ast^\varphi \in \mathcal{R}$; and

(2) For any $t \in \mathcal{R}$, and for any variable $z^\top$ not occurring in $t$, $t[\ast^\top := z^\top] \in \mathcal{R}$.

Then $\mathcal{R} \cap F = \mathcal{R}$.

3. SN PROOF BY RELATIVIZED REDUCIBILITY METHOD

In our SN proofs, we will use a well-founded induction on a well-founded relation. A well-founded relation is, by definition, $A = (A, \succ)$ such that $\emptyset \neq \succ \subseteq A \times A$ and there is no infinite chain $a \succ a' \succ a'' \succ \cdots$. The well-founded induction on a well-founded relation $A = (A, \succ)$ is, by definition,

\[
\text{WFI}(A) : \forall P \subseteq A \left[ \forall x \in A (\forall x'(x \succ x' \Rightarrow x' \in P) \implies x \in P) \implies \forall x \in A (x \in P) \right].
\]

We call the subformula $\forall x'(x \succ x' \Rightarrow x' \in P)$ the WF induction hypothesis. For $n \geq 1$ well-founded relations $A_i = (A_i, \succ_i)$ ($i = 1, \ldots, n$), we define a binary relation

$A_1 \# \cdots \# A_n = (A_1 \times \cdots \times A_n, \succ_1 \# \cdots \# \succ_n)$

by: $(x_1, \ldots, x_n) \succ_1 \# \cdots \# \succ_n (y_1, \ldots, y_n)$, if there exists $i$ such that $x_i \succ_i y_i$ but $x_j = y_j$ ($j \neq i$). Then $A_1 \# \cdots \# A_n$ is a well-founded relation.

If the redex of $t \rightarrow t'$ is $\Delta$, we write $t \overset{\Delta}{\rightarrow} t'$. Below, “$\sqsubseteq$” reads “is a subterm occurrence of.”

Following $[19]$, we consider:
Definition 3.1 (Neutral). A term is called neutral if it is not of the form \langle u, v \rangle or \lambda x. v.

We state and prove four properties (CR0), (CR1), (CR2) and (CR3) of the reducibility (Definition 2.2). Girard verified the last three to prove the SN of \( \beta\pi_1\pi_2 \)-reduction in [19].

Lemma 3.2. (CR0): If \( \ast \varphi \) is defined, then \( \ast \varphi \) is reducible.

(CR1): If \( t \varphi \) is reducible, then \( t \varphi \) is SN.

(CR2): if \( t \varphi \) is reducible and \( t \varphi \rightarrow t' \varphi \), then \( t' \varphi \) is reducible.

(CR3): if \( t \varphi \) is neutral, and \( t' \) is reducible whenever \( t \varphi \rightarrow t' \varphi \), then \( t \varphi \) is reducible.

To prove Lemma 3.2, we first note the following:

Lemma 3.3. By (CR0) and (CR3), we have

(CR4): If \( t \varphi \) is a variable, then \( t \varphi \) is reducible.

Proof. Let \( t \rightarrow t' \). Then \( t' \) is canonical, since \( t \) is variable. By (CR0), \( t' \) is reducible.

Proof of Lemma 3.2. By induction on \( \varphi \).

\( \varphi \) is atomic.: (CR0) \( \ast \varphi \) is \( \ast \top \), and SN. So \( \ast \varphi \) is reducible.

(CR1) is clear. (CR2) As \( t \) is SN, so is every reduct \( t' \) of \( t \). (CR3) If all reducts of \( t \) are SN, then \( t \) is SN.

\( \varphi = \varphi_1 \times \varphi_2 \): (CR0) As \( \ast \varphi_1 \times \varphi_2 \) is a normal form \( \langle \ast \varphi_1, \ast \varphi_2 \rangle \), the reduct of \( \pi_1 \ast \varphi_1 \times \varphi_2 \) is \( \ast \varphi_i \), which is reducible by induction hypothesis (CR0). By induction hypothesis (CR3) for \( \varphi_i \), \( \pi_1 \ast \varphi_1 \times \varphi_2 \) is reducible. Hence \( \ast \varphi_1 \times \varphi_2 \) is reducible.

(CR1) Suppose that \( t \) is reducible. Then \( \pi_1 t \) is reducible. By induction hypothesis (CR1) for \( \varphi_i \), \( \pi_1 t \) is SN. So \( t \) is SN.

(CR2) If \( t \rightarrow t' \), then \( \pi_1 t \rightarrow \pi_1 t' \). As \( t \) is reducible by hypothesis, so are \( \pi_1 t \). By induction hypothesis (CR2) for \( \varphi_i \), \( \pi_1 t' \) is reducible, and so \( t' \) is reducible.

(CR3) Let \( \pi_1 t \rightarrow s \). We have two cases.

1. \( \Delta \equiv \pi_1 t \) and \( s \equiv \ast \varphi_i \): By induction hypothesis (CR0) for \( \varphi_i \), \( s \) is reducible.

2. Otherwise, \( s \equiv \pi_1 t' \) for some \( t' \) such that \( t \rightarrow t' \). \( s \) is reducible, because \( t' \) is reducible by the hypothesis. \( \pi_1 t \) is neutral, and all the terms \( s \) with \( \pi_1 t \rightarrow s \) are reducible. By induction hypothesis (CR3) for \( \varphi_i \), \( \pi_1 t \) is reducible. Hence \( t \) is reducible.

\( \varphi = \varphi_1 \rightarrow \varphi_2 \): (CR0) Let \( u \) be a reducible term of type \( \varphi_1 \). By induction hypothesis (CR1) for \( \varphi_1 \), \( u \) is SN. So WF1(((\{\ast \varphi_1 \mid \ast \varphi_1 \text{ is reducible} \}, \rightarrow)) is available, where \( \rightarrow \) is the rewrite relation. We will verify that \( \ast \varphi_1 \rightarrow \varphi_2 u \) is reducible. Suppose \( \ast \varphi_1 \rightarrow \varphi_2 u \rightarrow s \). As \( \ast \varphi_1 \rightarrow \varphi_2 \) is in normal form, we have two cases.

1. \( \Delta \equiv \ast \varphi_1 \rightarrow \varphi_2 u \): Then \( s \equiv \ast \varphi_2 \) is reducible by induction hypothesis (CR0) for \( \varphi_2 \).
(2) Otherwise, \( s \equiv \ast \rightarrow \rightarrow \rightarrow \rightarrow u' \) with \( u \rightarrow u' \). Then \( u' \) is reducible by induction hypothesis (CR2) for \( \varphi_1 \). So, by the WF induction hypothesis, \( s \equiv \ast \rightarrow \rightarrow \rightarrow \rightarrow u' \) is reducible.

In any case, the neutral term \( \ast \rightarrow \rightarrow \rightarrow \rightarrow u \) rewrits to reducible terms only. By induction hypothesis (CR3) for \( \varphi_2 \), \( * \rightarrow \rightarrow \rightarrow \rightarrow u \) is reducible. So \( * \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow 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Lemma 3.6. 

(1) \( \Delta \equiv \pi_1(u, v) \) is a redex of the rewrite rule \((\pi_1)\) and \(s \equiv u\): Then \(s\) is reducible by the hypothesis \((1a)\).

(2) \( \Delta \equiv (u, v) \) is a redex of \((SP)\) and \(s \equiv \pi_1(\ast^\varphi \times \psi)\): Then \(\ast^\varphi \times \psi\) is reducible by \((CR0)\).

(3) By the definition of the reducibility for the product type, \(s \equiv \pi_1(\ast^\varphi \times \psi)\) is reducible.

(4) \( \Delta \equiv (u, v) \) is a redex of \((SP)\) and \(s \equiv \pi_1w:\) Then \(u \equiv \pi_1w\) and \(v \equiv \pi_2w\).

(5) \( s \equiv \pi_1w \) is reducible by the hypothesis \((1a)\).

(6) \( s \equiv \pi_1w \) is reducible by the hypothesis \((1b)\).

(7) \( \Delta \subseteq u: \) Then \(s \equiv \pi_1(u', v)\) with \(u \rightarrow u'\). \(u'\) is reducible by \((1a)\) and \((CR2)\).

(8) \( \Delta \subseteq v: \) Then \(s \equiv \pi_1(u, v')\) with \(v \rightarrow v'\). \(v'\) is reducible by \((1a)\) and \((CR2)\).

In every case, the neutral term \(\pi_1(u, v)\) rewrites to reducible terms only, and by \((CR3)\), \(\pi_1(u, v)\) is reducible. We can similarly prove that \(\pi_2(u, v)\) is reducible. So \(\langle u, v \rangle\) is reducible.

(2) By \((CR4)\), \(\ast^\varphi\) is reducible. So \(v^\varphi\) is, by the premise \((2a)\). Let \(u^\varphi\) be a reducible, possibly non-\(\ast\)-free term. By \((CR1)\), both of \(u, v\) are SN. By the well-founded induction \((10)\), we will verify that \((\lambda x.v)u\) is reducible. Assume \((\lambda x.v)u \rightarrow^\Delta s\). We will exhaust the positions of the redex \(\Delta\) in \((\lambda x.v)u\) from left to right, and the rewrite rule schemata of \(\rightarrow\). Then we have seven cases:

(1) \( \Delta \equiv (\lambda x.v)u \) is a redex and \(s \equiv \ast^\varphi\): Then \(s\) is reducible by \((CR0)\).

(2) \( \Delta \equiv (\lambda x.v)u \) is a redex of \((\beta)\) and \(s \equiv \ast^\varphi v[x := u]\): Then \(s\) is reducible by hypothesis \((2a)\).

(3) \( \Delta \equiv \lambda x.v \) is a redex and \(s \equiv \ast^\varphi \rightarrow^\psi u\): As \(\ast^\varphi \rightarrow^\psi\) is reducible by \((CR0)\), so is \(s\).

(4) \( \Delta \equiv \lambda x.v \) is a redex of \((\eta)\) and \(s \equiv v[x := u]\): Then, this case is case 2.

(5) \( \Delta \equiv \lambda x.v \) is a redex of \((\eta_{top})\) and \(s \equiv wu\) with \(v \equiv w^\ast \varphi\) and \(x \notin \text{FV}(w)\):

Then, since \(w\) is reducible by hypothesis \((2b)\), \(s \equiv wu\) is reducible.

(6) \( \Delta \subseteq v\) and \(s \equiv (\lambda x.v')u\) with \(v \rightarrow v'\): Then, by \((CR2)\), \(v'\) is reducible. By the WF induction hypothesis, \(s \equiv (\lambda x.v')u\) is reducible.

(7) \( \Delta \subseteq u\) and \(s \equiv (\lambda x.v')u\) with \(u \rightarrow u'\): Then, by \((CR2)\), \(u'\) is reducible. By the WF induction hypothesis, \(s \equiv (\lambda x.v')u\) is reducible.

In every case, the neutral term \((\lambda x.v)u\) reduces to reducible terms only. So, by \((CR3)\), \((\lambda x.v)u\) is reducible. Hence \(\lambda x.v\) is reducible.

Corollary 3.5. If \(u^\varphi, v^\psi\) are reducible and \(\ast\)-free, then so is \(\langle u, v \rangle\).

Lemma 3.6. 

(1) A \(\ast\)-free term with the variables substituted by \(\ast\)-free terms is \(\ast\)-free.

(2) Suppose that \(t\) is a term and \(z^\top\) is a variable not occurring in \(t\). Then

(a) if \(t\) is reducible, so is \(t[z^\top := z^\top]\).

(b) \(t[\ast^\top := z^\top] \rightarrow^* t\).

Proof. 

(1) Trivial.

(2) \((2a)\) is trivial.

(2a) By induction on the type \(\varphi\) of \(t\).
• $\varphi$ is atomic:
  Assume that $t[x^* := z^*] \neq \perp$ is not reducible. By the definition, $t$ is SN but there are $t[x^* := z^*] \equiv s_0, s_1, s_2, \ldots$ such that $s_i \rightarrow s_{i+1}$. Then $s_i[z^* := *^*] \equiv s_{i+1}[z^* := *^*]$ or $s_i[z^* := *^*] \rightarrow s_{i+1}[z^* := *^*]$. The former happens if $s_i \rightarrow g_s, \ldots$ with the redex being $z^*$. If $\{s_i[z^* := *^*]\}_{i} \neq 0$ is finite, then for any $i$ but finitely many, $s_i \rightarrow g_s, \ldots$. However, $\rightarrow g_s$ reduces the length of terms or the number of non-$*$ variables. Since $z$ is a fresh variable, $t \equiv s_0[z := *]$. Hence, $t$ is not SN. This contradicts the reducibility of $t$.

• $\varphi = \varphi_1 \cdot \varphi_2$:
  As $t[x^* := \varphi_1 \cdot \varphi_2]$ is reducible, $(t u^*) \varphi_2$ is so for every reducible $u^*$. $z^*$ does not occur in $t u^*$. So, by induction hypothesis on $\varphi_2$, $(t u^*)[x^* := z^*] \equiv t[x^* := z^*] u^*$ is reducible. By (2b), $u^*[x^* := z^*] \neq u$. So, $(t u^*)[x^* := z^*] \rightarrow t[x^* := z^*] u^*$. By (CR2), $t[x^* := z^*] u^*$ is reducible. Hence $t[x^* := z^*]$ is reducible.

• $\varphi = \varphi_1 \times \varphi_2$:
  As $t[x^* := \varphi_1 \times \varphi_2]$ is reducible, $\pi_i t[x^* := \varphi_1 \times \varphi_2]$ is so for each $i = 1, 2$. $z$ does not occur in any of $\pi_i$. So, by induction hypothesis, $(\pi_i(t[x^* := z^*]) \equiv \pi_i(t[x^* := z^*])$ is reducible. Hence $t[x^* := z^*]$ is reducible.

(2b) Just contract each occurrence of $z^*$ to $*^*$. \hfill \Box

Lemma 3.7. Given a $*$-free $v^\varphi$. If $v[x^\varphi := u^\varphi]$ is reducible for every reducible $*$-free $u^\varphi$, then $\lambda x^\varphi. v^\varphi$ is reducible and $*$-free.

Proof. Let $u^\varphi$ be a reducible term. By Lemma 3.6 [2], there is a $*$-free reducible term $u$ such that $u^* \rightarrow w$. By the premise, $v[x := u]$ is reducible. Because of $v[x := u] \rightarrow v[x := w]$, (CR2) implies that $v[x := w]$ is reducible. By Lemma 3.4 [2], $\lambda x^\varphi. v^\varphi$ is reducible.

In the following two theorems, we use Lemma 3.6

Theorem 3.8 (Relativized Reducibility). Assume that

(1) $t$ is a $*$-free term;

(2) a sequence of distinct variables $x^t_1, \ldots, x^t_n$ contains all free variables of $t$; and

(3) $u^t_i$ is reducible and $*$-free ($i = 1, \ldots, n$).

Then $t[x^t_1, \ldots, x^t_n := u^t_1, \ldots, u^t_n]$ is reducible.

Proof. By induction on $t$. By the premise (1), $t$ is not the constant $*^*$. So, we have five cases.

(1) $t \equiv x_i$: Then $t[x := u_i] \equiv u_i$ is reducible by the premise (3).

(2) $t \equiv \pi_i u$ ($i = 1, 2$): Then by induction hypothesis, $w[x := u_i] \equiv u_i$ is reducible. So each $\pi_i(w[x := u_i])$. This term is ($\pi_i(w[x := u_i]) \equiv t[x := u_i]$).

(3) $t \equiv (u, v)$: Then $t[x := u] \equiv (u[x := u], v[x := u])$. By induction hypotheses, both $u[x := u]$ and $v[x := u]$ are reducible. By Lemma 3.6 (1), $u[x := u]$ and $v[x := u]$ are $*$-free. By Lemma 3.4 (1), $t[x := u]$, that is, $u[x := u], v[x := u]$, is reducible.

(4) $t \equiv uv$: Then by induction hypotheses $w[x := u]$ and $v[x := u]$ are reducible, and so (by definition) is $w[x := u] (v[x := u])$; but this term is $t[x := u]$.
(5) \( t \equiv \lambda y^\varphi. w^\psi \) with \( y \) not free in any \( \vec{x}, \vec{u} \). By induction hypothesis, for every reducible \( \ast \)-free \( u^\varphi \), we have a reducible term \( w[\vec{x}, y^\varphi := \vec{u}, u^\varphi] \equiv w[\vec{x} := \vec{u}] \). By Lemma 3.7, \( \lambda y^\varphi. w[\vec{x} := \vec{u}] \equiv t[\vec{x} := \vec{u}] \) is reducible.

Hence we have established the relativized reducibility theorem. □

Theorem 3.9. All terms of \((\lambda \beta \eta \pi \ast)’\) are reducible.

Proof. Let \( t \) be a term. Som variable \( z^{\top} \) does not occurring in \( t \). By Lemma 3.6 (2b), there is a \( \ast \)-free term \( \tilde{t} \) such that \( \tilde{t} \ast \to t \). \( \tilde{t} \) is reducible by (CR4) and by Theorem 3.8 with \( u_i := x_i \), the identity substitution. As \( \tilde{t} \ast \to t \), (CR2) implies the reducibility of \( t \). □

Corollary 3.10. \((\lambda \beta \eta \pi \ast)’\) satisfies SN.

Proof. By (CR1) and Theorem 3.9, every term of \((\lambda \beta \eta \pi \ast)’\) is SN. □

We can define the extension of the equational theory \((\lambda \beta \eta \pi \ast)’\) by weakly extensional sum types, and the extension of the Curien-Di Cosmo style rewriting system, and prove the SN by a relativized reducibility method [2, Appendix].

Remark 3.11. In [26] ([47], resp.), ordinal numbers are assigned to typed \( \lambda \)-terms (typed combinators, resp.) in order to prove SN of typed \( \beta \)-reduction (typed combinatory reduction, resp.). In [6], cut-elimination procedure of a deduction system is used to give an optimal upper bound of typed \( \beta \eta \)-reduction. But these two proofs seem not to generalize for SN of \((\lambda \beta \eta \pi \ast)’\). In these two proofs, it is not the case that (1) the ordinal number of \( r \ast \tau \) is greater than that of \( r \) and (2) the ordinal number of the left-hand side \( \lambda x^\tau. t^\tau (x \notin \text{FV}(t)) \) of the rewrite rule schema \((\eta_{\text{top}})\) is greater than the ordinal number of the right-hand side \( t \).

One may be curious about whether the higher-order recursive path ordering (HORPO for short) [29] or the General Schema [11], could be extended with surjective pairing and hence be used for proving SN of \((\lambda \beta \eta \pi \ast)’\). If there is a convenient translation of the rewrite rule schemata \((g), (\eta_{\text{top}}), \) and \((SP_{\text{top}})\) with type-abstraction to an infinite simply-typed system, such that the translation can also put all the rules of \((\lambda \beta \eta \pi \ast)’\) in the right kind of format, it is possible that a HORPO-variant (with minimal symbol \( \ast \)) may handle \((\lambda \beta \eta \pi \ast)’\). However, we need a new HORPO variant, since the conventional ones are troubled with the non-left-linear \((SP)\)-rule \( \text{pair}(p_1(x), p_2(x)) \to x \). There is no type ordering that allows for the extraction of \( x \) from terms of smaller type in general. The top rule \((g)\): \( u^\tau \to \ast^\tau \) (\( \tau \in \text{Iso}(\top), u \notin \text{FV}(t) \)) is also problematic for most HORPO-variants. It could be handled by using a variation of HORPO with minimal symbols, such as the one used in WANDA [33]. Here, WANDA is one of the most powerful automatic termination provers for higher-order rewriting.

4. SN proof by relativized reducibility candidate method

In [16], an extension \((\lambda^2 \beta \eta \pi \ast)’\) of \((\lambda \beta \eta \pi \ast)’\) by polymorphism is introduced.

Types are generated from type variables \( X, Y, \ldots \) and the distinguished type constant \( \top \) by means of the product type \( \varphi \times \psi \), the function type \( \varphi \to \psi \), and \( \Pi X. \varphi \).

Terms are built up similarly as the terms of \( \lambda \beta \eta \pi \ast \), but we also consider the following two clauses:
• **universal abstraction**: if \( \nu^x \) is a term, then so is \( (\Lambda X. \nu^x)^{\Pi X. \varphi} \), whenever the type variable \( X \) is not free in the type of a free variable of \( \nu^x \); and

• **universal application**: if \( t^{\Pi X. \varphi} \) and \( \psi \) is a type, then so is \( (t^{\Pi X. \varphi})^\varphi_{[X:=\psi]} \).

An occurrence of a type variable \( X \) is called **bounded**, if it is within the scope of \( \Lambda X \ldots \) or \( \Pi X \ldots \). An occurrence of a type variable which is not bounded is called **free**. The set of free type variables of a term \( t \) is denoted by \( \text{FTV}(t) \).

The superscript representing the type is often omitted.

**Definition 4.1** (Terminal types of \( \lambda^2 \beta \eta\pi* \)).

1. \( \top \in \text{Iso}(\top) \).
2. \( \tau \in \text{Iso}(\top) \implies \varphi \to \tau \in \text{Iso}(\top) \).
3. \( \tau_1, \tau_2 \in \text{Iso}(\top) \implies \tau_1 \times \tau_2 \in \text{Iso}(\top) \).
4. \( \tau \in \text{Iso}(\top) \implies \Pi X. \tau \in \text{Iso}(\top) \).

**Definition 4.2** (Stars of \( \lambda^2 \beta \eta\pi* \)).

1. \( \tau \in \text{Iso}(\top) \implies *^\varphi \to \tau \equiv \lambda x . \varphi . ^* \).
2. \( \tau_1, \tau_2 \in \text{Iso}(\top) \implies *^{\tau_1 \times \tau_2} \equiv (x^* \tau_1, x^* \tau_2) \).
3. \( \tau \in \text{Iso}(\top) \implies *^{\Pi \tau} \equiv \Lambda X . ^* \tau \).

The rewrite rule schemata of the rewriting system \( \lambda^2 \beta \eta\pi* \) are the rewrite rules \( (\beta), (\pi_1), (\pi_2), (\eta), (SP) \) of \( \lambda \beta \eta \pi \ast \),

\[
\begin{align*}
(g) & \quad u^\tau \to *^\tau, \quad (u \neq *^\tau, \tau \in \text{Iso}(\top),) \\
(\eta_{\text{top}}) & \quad \lambda x^\tau . t_*^\tau \to t, \quad (x \notin \text{FV}(t), \tau \in \text{Iso}(\top),) \\
(S P_{\text{top}1}) & \quad (\tau_1 u, *^\tau) \to u, \quad (u \text{ has type } \varphi \to \tau, \tau \in \text{Iso}(\top),) \\
(S P_{\text{top}2}) & \quad (\ast^\tau, \tau_2 u) \to u, \quad (u \text{ has type } \tau \times \psi, \tau \in \text{Iso}(\top),) \\
\end{align*}
\]

and the following two:

\[
\begin{align*}
(\beta^2) & \quad (\Lambda X. t) \varphi \to t[X := \varphi], \quad (\eta^2) \quad \Lambda X. sX \to s, \quad (X \notin \text{FTV}(s)).
\end{align*}
\]

This completes the definition of \( \lambda^2 \beta \eta\pi* \).

In [10], to show SN of the rewriting system \( \lambda^2 \beta \eta\pi* \), they tried to prove that every term of \( \lambda^2 \beta \eta\pi* \) in the g-normal form is SN. But they observed that the set of g-normal form is not closed under \( \beta^2 \)-reduction; \( (\Lambda X. \lambda x . \lambda y . x \to \gamma . y x x) \top \) is in g-normal form, but its reduct \( u \equiv \lambda x^\tau . \lambda y^\tau \to \gamma . y x x \) is not, as \( u \to_g \lambda x^\tau . \lambda y^\tau \to \gamma . y . * \).

We will prove SN of \( \lambda^2 \beta \eta\pi* \).

Following [19], we consider:

**Definition 4.3** (Neutral). A term is neutral if it is not of the form \( \langle u, v \rangle, \lambda x. v \), or \( \Lambda X. u \).

**Definition 4.4**. (1) We say a term of \( \lambda^2 \beta \eta\pi* \) is star-free, if it has no sub-term \(*^\tau \) with \( \tau \in \text{Iso}(\top) \).

(2) A set \( A \) of terms is called variant-closed, provided for any \( t \in A \), and for any variable \( z^\top \) not occurring in \( t \), \( t[z^\top := z^\top] \in A \).

As in \( \lambda \beta \eta \pi \ast \), we consider (CR0) of Lemma 3.2 and variant-closedness to define a reducibility candidate [19].

**Definition 4.5**. A reducibility candidate (RC for short) of type \( \varphi \) is a set \( \mathcal{R} \) of terms of type \( \varphi \) such that:

1. **(CR0)**: If \( *^\varphi \) is defined, then \( *^\varphi \in \mathcal{R} \). Moreover \( \mathcal{R} \) is variant-closed.
2. **(CR1)**: If \( t^\varphi \in \mathcal{R} \), then \( t^* \) is SN.
Lemma 4.6. (CR0) and (CR3) implies

(CR4): If \( t^\varphi \) is a variable, then \( t \) is in \( \mathcal{R} \).

Proof. The proof is exactly the same as the proof of Lemma 3.3.

Definition 4.7.

(1) Let \( \mathcal{SN}^\psi \) be the set of \( \mathcal{SN} \) terms of type \( \psi \).

(2) For an RC \( \mathcal{R}_i \) of type \( \varphi_i \) of type \( \psi_i \) (\( i = 1, 2 \)),

\[
\mathcal{R}_1 \times \mathcal{R}_2 = \{ t^{\varphi_1 \times \varphi_2} \mid \pi t \in \mathcal{R}_i \ (i = 1, 2) \},
\]

\[
\mathcal{R}_1 \to \mathcal{R}_2 = \{ t^{\varphi_1 \to \varphi_2} \mid \forall u (u \in \mathcal{R}_1 \implies tu \in \mathcal{R}_2) \}.
\]

Lemma 4.8.

(1) For any type \( \psi \), \( \mathcal{SN}^\psi \) is an RC.

(2) If \( \mathcal{R}_i \) is an RC of type \( \varphi_i \) (\( i = 1, 2 \)),

(a) \( \mathcal{R}_1 \times \mathcal{R}_2 \) is an RC of type \( \varphi_1 \times \varphi_2 \), and

(b) \( \mathcal{R}_1 \to \mathcal{R}_2 \) is an RC of type \( \varphi_1 \to \varphi_2 \).

Proof. [1] (CR0): The variant-closedness is essentially the proof of Lemma 3.6.(2) for \( \varphi \) being atomic. (CR1): By the definition of \( \mathcal{SN}^{\varphi_2} \). (CR2): If \( t \in \mathcal{SN}^{\varphi_2} \) and \( t \to t' \), then \( t' \in \mathcal{SN}^{\varphi_2} \). (CR3): Let \( t \) be a neutral term of type \( \varphi_2 \) such that any reduct \( t' \) of \( t \) is in \( \mathcal{SN}^{\varphi_2} \). Then \( t \) is in \( \mathcal{SN}^{\varphi_2} \).

[2] The proof of (CR0), . . . , (CR3) is the proof of Lemma 3.6 for \( \varphi \) being a function type or a product type. But ‘by induction hypothesis (CRk) on \( \varphi_i \)’ should be replaced by ‘by (CRk) of RC \( \mathcal{R}_i \).’ The variant-closedness is the proof of Lemma 3.6(2) for corresponding \( \varphi \). But ‘reducible’ should be replaced by ‘in \( \mathcal{R} \)’ or ‘in \( \mathcal{S} \).’

For a type \( \varphi \), a sequence \( \vec{X} \) of distinct type variables \( X_1, \ldots, X_m \), and a sequence \( \vec{\psi} \) of types \( \psi_1, \ldots, \psi_m \), let \( \varphi[\vec{X} := \vec{\psi}] \) be the simultaneous substitution.

Definition 4.9 (Parametric Reducibility). Suppose that

(1) \( \varphi \) is a type;

(2) a sequence \( \vec{X} \) of distinct type variables \( X_1, \ldots, X_m \) contains all free type variables of \( \varphi \);

(3) \( \vec{\psi} \) is a sequence of types \( \psi_1, \ldots, \psi_m \); and

(4) \( \vec{R} \) is a sequence of RCs \( \mathcal{R}_1, \ldots, \mathcal{R}_m \) of corresponding types \( \vec{\psi} \).

Define a set \( \text{RED}_\varphi[\vec{X} := \vec{R}] \) of terms of type \( \varphi[\vec{X} := \vec{\psi}] \) as follows:

(1) If \( \varphi = \top \), \( \text{RED}_\varphi[\vec{X} := \vec{R}] = \mathcal{SN}^{\top} \);

(2) If \( \varphi = X_i \), \( \text{RED}_\varphi[\vec{X} := \vec{R}] = \mathcal{R}_i \);

(3) If \( \varphi \equiv \varphi' \circ \varphi'' \), \( \text{RED}_\varphi[\vec{X} := \vec{R}] = \text{RED}_{\varphi'}[\vec{X} := \vec{R}] \circ \text{RED}_{\varphi''}[\vec{X} := \vec{R}] \) (\( \circ \to, \times \) are defined in Definition 4.4);

(4) If \( \varphi \equiv \Pi Y. \varphi' \) and \( Y \neq X_i \) \( (i = 1, \ldots, m) \), then \( \text{RED}_\varphi[\vec{X} := \vec{R}] \) is the set of terms \( t^{\Pi Y. \varphi'}[\vec{X} := \vec{\psi}] \) such that for any type \( \psi \)

and any RC \( \mathcal{S} \) of type \( \psi \), \( \psi[\vec{X} := \vec{\psi}, \vec{Y}] \in \text{RED}_{\varphi'}[\vec{X}, \vec{Y} := \vec{R}, \mathcal{S}] \).

Lemma 4.10. Under the conditions of Definition 4.9, \( \text{RED}_\varphi[\vec{X} := \vec{R}] \) is an RC of type \( \varphi[\vec{X} := \vec{\psi}] \).
Proof. By induction on \( \varphi \). First consider the case \( \varphi \equiv \Pi Y, \varphi' \). Without loss of generality, \( Y \) does not occur free in \( \psi \). Let \( S \) be an RC of type \( \varphi'' \). By induction hypothesis,

\[
(11) \quad T := \text{RED}_{\varphi'}[\vec{X}, Y := \vec{R}, S], \text{ is an RC.}
\]

(CR0): Let \( \ast_{\Pi Y, \varphi'}[\vec{X} := \vec{S}] \varphi'' \to s \) where \( Y \) is not free in \( \psi \). We will verify \( s \in T \).

Then \( s \equiv s^{[\vec{X}, Y := \vec{S}]} \). By (11), \( s \in T \).

Thus \( \ast_{\Pi Y, \varphi'}[\vec{X} := \vec{S}] \varphi'' \in T \) by (CR3). So \( \ast_{\Pi Y, \varphi'}[\vec{X} := \vec{S}] \) ∈ RED_{\Pi Y, \varphi'}[\vec{X} := \vec{R}] (11).

(CR1): Let \( t \in \text{RED}_{\varphi'}[\vec{X} := \vec{R}] \). Then \( t \varphi'' \in T \) by Definition 4.11 By (11) and (CR1) of \( T \), \( t \varphi'' \) is SN. So \( t \) is SN.

(CR2): Let \( t \in \text{RED}_{\varphi'}[\vec{X} := \vec{R}] \). Then \( t \varphi'' \in T \) by Definition 4.11 Assume \( t \to t' \). Then \( t \varphi'' \to t' \varphi'' \). By (11) and (CR2) of \( T \), \( t' \varphi'' \in T \). So \( t' \in \text{RED}_{\varphi'}[\vec{X} := \vec{R}] \).

(CR3): Suppose that \( t \) is neutral and that \( t' \in \text{RED}_{\varphi'}[\vec{X} := \vec{R}] \) whenever \( t \to t' \). Let \( t \varphi'' \to s \). As \( t \) is neutral, we have two cases:

(1) \( \Delta \equiv t \varphi'' \). Then \( s \equiv s^{[\vec{X}, Y := \vec{S}]} \), because \( t \) is neutral. By (11) and (CR0) of \( T \), \( s \in T \).

(2) Otherwise, \( s \equiv t' \varphi'' \) with \( t \to t' \). \( s \in T \) by \( t' \in \text{RED}_{\varphi'}[\vec{X} := \vec{R}] \).

By (11) and (CR3) of \( T \), \( t \varphi'' \in T \). So \( t \in \text{RED}_{\varphi'}[\vec{X} := \vec{R}] \).

To prove the variant-closedness of RED_{\Pi Y, \varphi'}[\vec{X} := \vec{R}], take an arbitrary \( t \) from the set and a variable \( z \) not occurring in \( t \). Then \( t \varphi'' \in \text{RED}_{\varphi'}[\vec{X}, Y := \vec{S}, S] \) for every RC \( S \) of type \( \varphi'' \). \( z \) does not occur in \( t \varphi'' \). By induction on \( \varphi' \), RED_{\varphi'}[\vec{X}, Y := \vec{S}, S] \) is variant-closed. So, \( (t \varphi'')[z := z^{\top}] \equiv t[z := z^{\top}] \varphi'' \) is in RED_{\varphi'}[\vec{X}, Y := \vec{S}, S]. Thus \( t \in \text{RED}_{\Pi Y, \varphi'}[\vec{X} := \vec{R}] \). To sum up, RED_{\Pi Y, \varphi'}[\vec{X} := \vec{R}] \) is variant-closed.

When \( \varphi \not\equiv \Pi Y, \varphi' \), we can prove (CR0), (CR1), (CR2), and (CR3) of RED_{\varphi'}[\vec{X} := \vec{R}], by induction hypotheses on \( \varphi \) and Lemma 4.8.

\[ \square \]

Lemma 4.11. Suppose that

1. \( \varphi, \psi \) are types, \( Y \) is a type variable;
2. a sequence \( \vec{X} \) of distinct type variables \( X_1, \ldots, X_m \) contains all free type variables of \( \varphi[Y := \psi] \) and those of \( \psi \);
3. \( X_i \neq Y \) (\( i = 1, \ldots, m \)); and
4. \( \vec{R} \) is a sequence of RCs \( \vec{R}_1, \ldots, \vec{R}_m \).

Then

\[
\text{RED}_{\varphi[Y := \psi]}[\vec{X} := \vec{R}] = \text{RED}_{\varphi'}[\vec{X}, Y := \vec{R}, \text{RED}_{\psi}[\vec{X} := \vec{R}]].
\]

Proof. By induction on \( \varphi \).

\[ \square \]

Lemma 4.12. Let \( \vec{R}, S \) be RCs of type \( \varphi, \psi \).

1. Let \( u^\varphi, v^\psi \) be any terms. \( (u^\varphi, v^\psi) \in \vec{R} \times S \), provided that
   a. \( u \in \vec{R} \) and \( v \in S \);
   b. if \( u \equiv \pi_1 w \) and \( v \equiv \ast^\psi \), then \( w \in \vec{R} \times S \); and
   c. if \( v \equiv \pi_2 w \) and \( u \equiv \ast^\varphi \), then \( w \in \vec{R} \times S \).
Lemma 4.14 (Universal application). Let $v^\psi$ be any term. $\lambda x^\varphi.v^\psi \in R \to S$, provided that

1. $v^\psi[x^\varphi := u^\varphi] \in S$ for every possibly non-star-free term $u^\varphi \in R$;
2. if $v \equiv w^{\varphi \to \psi}^{\varphi}$ and $x^\varphi \notin \text{FV}(w^{\varphi \to \psi})$, then $w^{\varphi \to \psi} \in R \to S$.

Proof. The proof is similar to the proof of Lemma 3.3.

Lemma 4.13 (Universal abstraction). Suppose that

1. $\varphi$ is a type;
2. a sequence $\bar{X}$ of distinct type variables $X_1, \ldots, X_m$ contains all free type variables of $\Pi Y.\varphi$;
3. $X_i \neq Y (i = 1, \ldots, m)$, $\bar{\psi}$ is a sequence of types $\psi_1, \ldots, \psi_m$;
4. $\bar{R}$ is a sequence of RCs $\mathcal{R}_1, \ldots, \mathcal{R}_m$ of types $\bar{\psi}$;
5. $Y$ does not occur free in $\bar{\psi}$;
6. $w^\varphi[\bar{X} := \bar{\psi}]$ is a term; and
7. for any type $\psi$ and any RC $S$ of type $\psi$,

$$(w[Y := \psi])^\varphi[\bar{X}, Y := \bar{\psi}] \in \text{RED}_{\Pi Y.\varphi}[\bar{X}, Y := \bar{R}, S].$$

Then $\Lambda Y. w \in \text{RED}_{\Pi Y.\varphi}[\bar{X} := \bar{R}]$.

Proof. $\mathcal{S}^{\mathcal{N}Y}$ is an RC, by Lemma 4.8 [1]. By assumption 7,

$$(12) \quad w \in \text{RED}_{\varphi}[\bar{X}, Y := \bar{R}, \mathcal{S}^{\mathcal{N}Y}].$$

By (CR1) of this RC, $w$ is SN. By Definition 4.9 [4], we have only to verify:

$$(13) \quad (\Lambda Y. w) \psi \in \text{RED}_{\varphi}[\bar{X}, Y := \bar{R}, S], \text{ for every type } \psi \text{ and RC } S \text{ of type } \psi.$$ 

The proof is by WFI $\left(\left\{ w^\varphi[\bar{X} := \bar{\psi}] \mid \text{(12) holds} \right\}, \rightarrow \right)$ where $\rightarrow$ is the rewrite relation. Let $(\Lambda Y. w) \psi \xrightarrow{\Delta} s$. We have five cases. We verify $s \in \mathcal{T} := \text{RED}_{\varphi}[\bar{X}, Y := \bar{R}, S]$.

1. $\Delta \equiv (\Lambda Y. w) \psi$ is a redex of $(g)$: Then $s \equiv s[\bar{X}, Y := \bar{\psi}, \psi]$. By (CR0) of $\mathcal{T}$,
2. $\Delta \equiv (\Lambda Y. w) \psi$ is a redex of $(\beta^2)$: Then $s \equiv w[Y := \psi]$. By assumption 7.
3. $\Delta \equiv (\Lambda Y. w) \psi$ is a redex of $(g)$: Then $s \equiv s[\Pi Y.\varphi[\bar{X} := \bar{\psi}] \psi]$. By (CR0),
4. $\bar{R} \in \text{RED}_{\Pi Y.\varphi}[\bar{X} := \bar{R}]$. Hence $s \in \mathcal{T}$ by Definition 4.9.
5. Otherwise, for some $w'$, $s \equiv (\Lambda Y. w') \psi$ and $w \to w'$. By the WF induction hypothesis.

Thus $s \in \mathcal{T}$. So the statement (13) follows from (CR3) of $\mathcal{T}$.

Lemma 4.14 (Universal application). Suppose that

1. $\varphi, \psi$ are types, $Y$ is a type variable;
2. a sequence $\bar{X}$ of distinct type variables $X_1, \ldots, X_m$ contains all free type variables of $\varphi[Y := \psi]$ and those of $\psi$;
3. $X_i \neq Y (i = 1, \ldots, m)$, $\bar{\psi}$ is a sequence of types $\psi_1, \ldots, \psi_m$; and
4. $\bar{R}$ is a sequence of RCs $\mathcal{R}_1, \ldots, \mathcal{R}_m$ of types $\bar{\psi}$.
Proof. By induction on $t$. The proof proceeds by cases according to the form of $t$.

(1) Suppose that $\psi$ contains all free variables of $\phi$. By the premise of Lemma 4.16.

Then

$$w \in \text{RED}_{\text{HY} \cdot \phi}[X := \vec{R}] \implies w \left( \psi[X := \vec{R}] \right) \in \text{RED}_{\phi}[Y := \psi][X := \vec{R}].$$

Proof. By Lemma 4.16, $\text{RED}_{\phi}[X := \vec{R}]$ is an RC of type $\psi[X := \vec{R}]$. By the premise and Definition 4.9 (1) with $\psi := \psi[X := \vec{R}]$,

$$w \left( \psi[X := \vec{R}] \right) \in \text{RED}_{\phi}[X, Y := \vec{R}, \text{RED}_{\phi}[X := \vec{R}]].$$

So Lemma 4.16 implies the conclusion.

□

Lemma 4.15. Let $\tau \in \text{Iso}(\mathcal{T})$. Then

(1) $\tau$ is not of the form $\cdots \rightarrow \cdots \rightarrow \varphi$ where $\varphi \notin \text{Iso}(\mathcal{T})$.

(2) $*^\tau$ is defined and $\text{FV}(*^\tau) = \emptyset$.

Proof. By induction on $\tau$. □

The following is the counterpart of Lemma 3.6.

Lemma 4.16. In $(\lambda^2 \beta \eta \pi^*)'$, for every star-free term $t$,

(1) a term $t[X_1, \ldots, X_m := \psi_1, \ldots, \psi_m]$ is star-free for all distinct type variables $X_1, \ldots, X_m$ and for all types $\psi_1, \ldots, \psi_m$; and

(2) a term $t[x_1^{\varphi_1}, \ldots, x_n^{\varphi_n} := u_1^{\varphi_1}, \ldots, u_n^{\varphi_n}]$ is star-free for all distinct variables $x_1^{\varphi_1}, \ldots, x_n^{\varphi_n}$ and for all star-free terms $u_1^{\varphi_1}, \ldots, u_n^{\varphi_n}$.

Proof. By induction on $t$. Let

$$\Theta = [X_1, \ldots, X_m := \psi_1, \ldots, \psi_m]$$

$$\theta = [x_1^{\varphi_1}, \ldots, x_n^{\varphi_n} := u_1^{\varphi_1}, \ldots, u_n^{\varphi_n}].$$

The proof proceeds by cases according to the form of $t$. By the definition of $(\lambda^2 \beta \eta \pi^*)'$, $t$ is not a term constant, because otherwise $t$ is $*^\tau$.

- $t$ is a variable: Then (1) is by Lemma 4.15 (2). (2) is clear.

- $t$ is an abstraction, or an application: By induction hypotheses.

- $t \equiv \Lambda Y.w$ such that $X_i \neq Y$ and $Y$ does not occur free in any $\psi_i$: By induction hypothesis, $w\Theta$ and $w\theta$ are star-free. So, none of $t\Theta \equiv \Lambda Y.w\Theta$ and $t\theta \equiv \Lambda Y.w\theta$ is a star term.

- $t \equiv w\psi$: Then, by induction hypothesis, $w\Theta$ and $w\theta$ are star-free. Hence, none of $t\Theta \equiv w\Theta(\psi\Theta)$ and $t\theta \equiv (w\theta)\psi$ is a star-term.

This completes the proof of Lemma 4.16. □

Lemma 4.17. Suppose that

(1) $v^\psi$ is a star-free;

(2) a sequence $X$ of distinct type variables $X_1, \ldots, X_m$ contains all free type variables of $v^\psi$; and

(3) $\vec{R}$ is a sequence of RCs $R_1, \ldots, R_m$ of types $\psi \equiv \psi_1, \ldots, \psi_m$.

If $v[x^{\varphi} := u^\psi] \in \text{RED}_{\psi}[X := \vec{R}]$ for every star-free $u^\varphi \in \text{RED}_{\phi}[X := \vec{R}]$, then $\lambda x^\varphi, v^\psi \in \text{RED}_{\phi \rightarrow \psi}[X := \vec{R}]$ and star-free.

---

119 Lemma 14.2.3] corresponding to this lemma has a typo: “$tV$” should be “$t(V[U/X])$.”
Proof. Let \( \psi \in \text{RED}_\varphi[\vec{X} := \vec{R}] \). Because \( \text{RED}_\varphi[\vec{X} := \vec{R}] \) is variant-closed, there is a star-free term \( u \in \text{RED}_\varphi[\vec{X} := \vec{R}] \) such that \( u \rightarrow^* \psi \). We have \( v[x := u] \rightarrow v[x := \psi] \). So, by the premise \( v[x := u] \in \text{RED}_\varphi[\vec{X} := \vec{R}] \) and (CR2), we have \( v[x := \psi] \in \text{RED}_\varphi[\vec{X} := \vec{R}] \). Because of the premise (1), Lemma 4.12 (2) implies \( \lambda x \varphi, \psi \in \text{RED}_\varphi[\vec{X} := \vec{R}] \), while \( \lambda x, \psi \) is star-free.

**Theorem 4.18 (Relativized Reducibility).** Suppose that

1. \( \tau \psi \) is a star-free term;
2. a sequence of distinct variables \( x_1^\varphi, \ldots, x_n^\varphi \) contains all free variables of \( \tau \psi \);
3. a sequence \( \vec{X} \) of distinct type variables \( X_1, \ldots, X_m \) contains all free type variables of \( \tau \psi \);
4. \( \vec{R} \) is a sequence of RCs \( \vec{R}_1, \ldots, \vec{R}_m \) of types \( \vec{\psi} \equiv \psi_1, \ldots, \psi_m \);
5. \( u_i^\varphi[\vec{X} := \vec{\psi}] \) is in \( \text{RED}_\varphi[\vec{X} := \vec{R}] \) and is star-free \( (i = 1, \ldots, n) \); and
6. \( t[\vec{X} := \vec{\psi}][\vec{x} := \vec{u}] \) is the term obtained from \( t[\vec{X} := \vec{\psi}] \) by simultaneously substitution of \( u_1^\varphi[\vec{X} := \vec{\psi}], \ldots, u_n^\varphi[\vec{X} := \vec{\psi}] \) into \( x_1^\varphi[\vec{X} := \vec{\psi}], \ldots, x_n^\varphi[\vec{X} := \vec{\psi}] \).

Then \( t[\vec{X} := \vec{\psi}][\vec{x} := \vec{u}] \) is in \( \text{RED}_\varphi[\vec{X} := \vec{R}] \).

Proof. By the premise (1), the premise (5) and Lemma 4.16

(14) \( t[\vec{X} := \vec{\psi}][\vec{x} := \vec{u}] \) is star-free.

By induction on \( t \). The proof proceeds by cases according to the form of \( t \). By the premise (1), \( t \) is not a star term. Then we have five cases.

1. \( t \) is a variable \( x_i \): Then \( t[\vec{X} := \vec{\psi}][\vec{x} := \vec{u}] \equiv u_i^\varphi[\vec{X} := \vec{\psi}] \) is in \( \vec{R}_i \) by the premise (5).
2. \( t \) is a pairing: We can prove this case, similarly as in the proof of Theorem 3.8, by using (14) and Lemma 4.12 (1).
3. \( t \) is a \( \lambda \)-abstraction: We can prove this case, similarly as in the proof of Theorem 3.8 by using (14) and Lemma 4.17.
4. \( t \equiv w_1^\varphi w_2^\varphi \) where \( \sigma_1 \equiv \sigma_2 \rightarrow \varphi \): If a free type variable occur in \( w_i \) \((i = 1, 2) \), then it does so in \( w_1 w_2 \). So, by induction hypotheses, \( w_i[\vec{X} := \vec{\psi}][\vec{x} := \vec{u}] \in \text{RED}_\sigma[\vec{X} := \vec{R}] \). By \( \sigma_1 \equiv \sigma_2 \rightarrow \varphi \), Definition 4.9 and Definition 4.17 we have \( t[\vec{X} := \vec{\psi}][\vec{x} := \vec{u}] \in \text{RED}_\varphi[\vec{X} := \vec{R}] \).
5. \( t \equiv (\Lambda Y. w)^{\text{HY}, \sigma} \) where \( X_i \neq Y \) and \( Y \) does not occur free in any \( \psi_1[\vec{X} := \vec{\psi}] \):

Then by the induction hypothesis, for any type \( \psi \) and any RC \( S \) of \( \psi \), \( w[\vec{X}, Y := \vec{\psi}, \vec{\psi}][\vec{x} := \vec{u}] \) is in \( \text{RED}_\sigma[\vec{X}, Y := \vec{R}, S] \). Since \( Y \) occurs in no \( \vec{u} \) without loss of generality, we have \( w[\vec{X}, Y := \vec{\psi}][\vec{x} := \vec{u}] \in \text{RED}_\sigma[\vec{X}, Y := \vec{R}, S] \). By Lemma 4.13 (\( \Lambda Y. w)[\vec{X} := \vec{\psi}][\vec{x} := \vec{u}] \) is in \( \text{RED}_\text{HY, \sigma}[\vec{X} := \vec{R}] \).
6. \( t \equiv w^{\text{HY, \sigma}} \psi \): Then by the induction hypotheses, \( w[\vec{X} := \vec{\psi}][\vec{x} := \vec{u}] \in \text{RED}_\text{HY, \sigma}[\vec{X} := \vec{R}] \). By Lemma 4.14 \( w[\vec{X} := \vec{\psi}][\vec{x} := \vec{u}] (\psi[\vec{X} := \vec{\psi}] \) \( \in \text{RED}_\sigma[\psi := \psi][\vec{X} := \vec{R}] \). This term is just \( (w \psi)[\vec{X} := \vec{\psi}][\vec{x} := \vec{u}] \).

This completes the proof of Theorem 4.18.
Definition 4.19. A term $t^\varphi$ is called reducible, if for some sequence of distinct type variables $X_1, \ldots, X_m$ containing the free type variables of a type $\varphi$, 

\[ t^\varphi \in \text{RED}_{\varphi}[X_1, \ldots, X_m := SN^{X_1}, \ldots, SN^{X_m}] \]

Theorem 4.20. Any term $t^\varphi$ is in $\text{RED}_{\varphi}[X_1, \ldots, X_m := SN^{X_1}, \ldots, SN^{X_m}]$.

Proof. Any star-free term is reducible, by (CR4) and by Theorem 4.18 with $u^\varphi_i := x_i^\varphi$, $\psi_j := X_j$ and $R_j := SN^{X_j}$. Hence a star-free term $t[*^T := z^T]$ is reducible for some variable $z^T$. Because $t[*^T := z^T] \rightarrow t$. $t$ is reducible, by (CR2). \hfill $\square$

Corollary 4.21. $(\lambda^2 \beta \eta \pi^*)'$ satisfies SN and CR.

Proof. SN follows from (CR1) and Theorem 4.20. $(\lambda^2 \beta \eta \pi^*)'$ is weakly confluent by [16, Proposition 2.5]. So, Newman’s lemma [40] implies CR of $(\lambda^2 \beta \eta \pi^*)'$. \hfill $\square$

4.1. Parametric terminal types. According to [23], in the parametric polymorphism, a type $\Pi X.(X \to X)$ is a terminal type. We will add the following clauses to the definition of $\text{Iso}(\top)$ and $\ast^\varphi$ of $(\lambda^2 \beta \eta \pi^*)'$.

\[
\begin{cases}
\Pi X.(X \to X) \in \text{Iso}(\top) \\
\ast \Pi X.(X \to X) := \Lambda X. \lambda x^X.x^X
\end{cases}
\]

Then, for a suitable condition,

\[
(\Lambda X. t^X \to X) \varphi \rightarrow^\beta t^\varphi \to^\varphi
\]

\[
(\Lambda X. \lambda x^X.x^X) \varphi \rightarrow^\beta \lambda x^\varphi.x^\varphi.
\]

If $\varphi \notin \text{Iso}(\top)$, then it may not be the case $t^\varphi \to^\varphi \rightarrow^\beta \lambda x^\varphi.x^\varphi$. So, $(\lambda^2 \beta \eta \pi^*)' + (15)$ may be not confluent.

The following rewrite rule schema resolves the confluence problem (16):

\[(g_{aux}) \quad t^\varphi \rightarrow^\varphi \rightarrow \lambda x^\varphi.x^\varphi,\]

provided

$t$ is of the form $s^X \to X[X := \varphi],\n
\varphi \notin \text{Iso}(\top),\n
X$ does not occur free in the type of any free term variable of $s$, and

$t \neq \lambda x^\varphi.x^\varphi).

The rewrite rule schema $(g_{aux})$ can be regarded as an ‘instance’ of a rewrite rule $(g)$ $\mu \Pi X.(X \to X) \rightarrow \ast \Pi X.(X \to X)$ (the left-hand side is not the right-hand side).

If we attempt to prove the SN of $(\lambda^2 \beta \eta \pi^*)' + (15) + (g_{aux})$ by a relativized reducibility candidate method of Section 4, we require

\[(17) \quad \text{If } \mathcal{R} \text{ is an RC of type } \varphi \to \varphi, \text{ then } \lambda x^\varphi.x^\varphi \in \mathcal{R}.
\]

It is because $(g_{aux})$ will cause, at least, the following new cases in the proof of Lemma 4.12:

- “Case $\Delta \equiv \pi_1\{u, v\}$ is a redex of $(g_{aux})$, $s \equiv \lambda x^\theta.x^\theta$ and $\varphi_1 = \theta \to \theta$ for some type $\theta.$”
• “Case $\Delta \equiv (\lambda x. v)u$ is a redex of $(g_{aux})$, $s \equiv \lambda x^\theta. x^\theta$ and $\varphi_2 = \theta \rightarrow \theta$ for some type $\theta$.”

If we add the property (17) in the definition of RC, then we cannot prove “If $R, S$ are RCs of type $\varphi$, then $R \rightarrow S$ is an RC of type $\varphi \rightarrow \varphi$.” It is because $R \subseteq S$ is not always available.

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APPENDIX A. Type-directed expansions

For the typed λ-calculus, let a binary relation \( \Rightarrow_\eta (\Rightarrow_{SP}) \) replace a neutral subterm occurrence in a non-elimination context with the η (SP)-expansion [38]. Neither \( \Rightarrow_\eta \) nor \( \Rightarrow_{SP} \) is stable under contexts. We call the relation \( \Rightarrow := \Rightarrow_\beta \pi_1 \pi_2 T \Rightarrow_{\eta SP} \) **Mints’ reduction**, as Mints introduced it in [37, 38]. Mints’ reduction generates the equational theory \( \lambda \beta \eta \pi^* \), and is SN+CR ([1, 27] to cite a few). In [1], the author presented a divide-and-conquer lemma to infer SN+CR property of a reduction system from that property of its subsystems.

**Lemma A.1** ([1]). If two binary relations \( \Rightarrow_R \) and \( \Rightarrow_S \) on a set \( U \neq \emptyset \) have SN+CR property, then so does \( \Rightarrow_{SR} \), provided that we have

\[
\forall u, v \in U \ (u \Rightarrow_S v \implies u^R \overset{+}{\Rightarrow} S v^R),
\]

where \( u^R \) and \( v^R \) are the \( \Rightarrow_R \)-normal forms of \( u \) and \( v \) respectively, and \( \overset{+}{\Rightarrow} S \) is the transitive closure of \( \Rightarrow_S \).

By inductive arguments, the author proved that

\[
(18) \quad t \Rightarrow_\beta \pi_1 \pi_2 T s \implies t^\eta_{SP} \Rightarrow_\beta \pi_1 \pi_2 T s^\eta_{SP}
\]

where \( u^\eta_{SP} \) is \( \Rightarrow_\eta_{SP} \)-normal form of \( u \). By Lemma A.1 SN+CR of Mints’ reduction follows. The SN of Mints’ reduction implies \( \Rightarrow_\eta = \leftarrow_\eta \ \Leftarrow_\beta \) and \( \Rightarrow_{SP} = \leftarrow_{SP} \ \Leftarrow_{\pi_1} \ \Leftarrow_{\pi_2}. \)

Čubrić proved the weak normalization of Mints’ reduction in [12] and then published the proof of SN in his thesis [13]. His SN proof is showing (18) by proving the commutativity [13] Proposition 3.29] of \( \Rightarrow_\beta \pi_1 \pi_2 T \) and \( \Rightarrow_{\eta_{SP}} \) and the preservation [13] Proposition 3.40] of \( \Rightarrow_{\beta \pi_1 \pi_2 T} \)-normal form by \( \Rightarrow_{\eta_{SP}} \). The authors proved (18) by mostly inductive argument. This was part of his generalization of Friedman’s theorem for CCCs [14]. According to Phil Scott, the work of Čubrić was motivated by the fact that Mints’ expansionary rewrites contained mistakes and Mints’ results were wrong! He resolved the issue with a detailed analysis of η-expansion. The issue was extending the Friedman Set-interpretation from a free CCC into the
category of sets into the free CCC $C(X)$ with an infinite set of indeterminates $X$ adjoined. It was the problem of the faithfulness of the embedding $C \to C(X)$ which led to Čubrić finding the mistakes in Mints’ work.

Čubrić, Dybjer, and Scott employed normalization-by-evaluation (NBE) techniques to prove directly the decision problem for the free CCC, without needing any Church-Rosser, SN, or even rewriting at all. But they did a computability argument at the end to in fact show that, from a traditional viewpoint, they are actually constructing long $\beta\eta$-normal forms. They added appendix in the proof of paper (mostly by Čubrić) where for each typed lambda calculi generated by (i) a graph, (ii) a category, (iii) a cartesian category, he tried to prove that the NBE decision procedure makes sense, and reduces the problem of the higher order structure roughly down to the decision problem of the underlying theory. For this purpose, he attempted to prove the transitivity rule of the equality is admissible in a formalized equational theory, by a similar proof technique of cut-elimination theorem of proof theory.

Although $\to_{\eta SP}$ is not stable under contexts, the finite development-like argument based on $\leftarrow_{\eta SP}$ proves CR of $\to$ [30] pointed out that.

In [17], Di Cosmo and Kesner proved CR+SN of a reduction system $\to_\beta \cup \to_\Pi \cup \to_{\pi_1} \cup \to_{\pi_2} \cup \to_{\eta SP} \cup \to_g$ union the $\beta$-like reductions of sum types. By showing how substitution and the reduction interact with the context-sensitive rules, they proved the WCR. They simulated expansions without expansions, to reduce SN of the reduction to SN for the underlying calculus without expansions, provable by the standard reducibility method.

The rewriting system $(\lambda\beta\eta\pi^*)'$ of Curien and Di Cosmo is stable under contexts (i.e., $t \to t' \implies t \cdots \to \cdots \to t'$. . . ) Mints’ reduction decides the equational theory $\lambda\beta\eta\pi^*$. Mints’ reduction is not stable under contexts.

Mints’ reduction fits to semantic treatments such as NBE [7]. See [3] in the context of type-checking of dependent type theories. However, because of the complication of Mints’ reduction, in his book [39] on selected papers of proof theory, Mints replaced his reduction with the $\beta\eta$-reduction modulo equivalence relation on terms. His purpose is to give a simple proof of difficult theorems of category theory with typed $\lambda$-calculus and proof theory by using the correspondence objects = types = propositions and arrows = terms = proofs. Mac Lane is interested in his ambition [36].

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