Vibrations of a viscoelastic isotropic plate under periodic load without considering the tangential forces of inertia

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Abstract. A mathematical model of the problem of viscoelastic isotropic plate vibrations based on the Kirchhoff-Love hypothesis in a geometrically nonlinear formulation was presented. The mathematical model was built without considering the tangential forces of inertia. To describe the viscoelastic properties of the plate material, a weakly singular Koltunov-Rzhanitsyn kernel with three different rheological parameters was chosen. To solve the problem of parametric vibrations of a viscoelastic plate with a weakly singular relaxation kernel, a numerical method based on the use of quadrature formulas was applied. A discrete model of this problem was first constructed using the Bubnov-Galerkin method; i.e., a system of integro-differential equations with variable coefficients was obtained, and then, using a numerical method based on the elimination of a singularity of the kernel, the problem of parametric vibrations of viscoelastic rectangular plates was solved. The influence of the viscoelastic properties of the material and the variability of the plate thickness on the oscillatory process was shown.

1. Introduction

It is known that bridges, crossovers, overpasses are artificial structures on roads and railways. In total, there are more than 100 thousand bridges on the roads and railways in Russia alone. Many issues related to the design of bridges, crossovers, and overpasses lead to the calculation of plates of variable thickness since they can be interpreted as plates from the point of view of the theory of elasticity and structural mechanics. Bridges, crossovers, and overpasses are designed to operate under the influence of power loads (static and dynamic ones). In the material of artificial structures, under long-term loads, the creep phenomenon (viscoelasticity) may appear, which may lead to a significant decrease in the bearing capacity. Thus, in order to obtain a more realistic picture of the stress-strain state in the design of structural elements in the form of plates of variable thickness, it is necessary to conduct research in a geometrically nonlinear formulation of the problem with proper joint consideration of the viscoelastic properties and, in the general case, anisotropic properties of the material. It should be noted that there are practically no solutions to these problems in the literature, despite their practical and theoretical importance. Obviously, this is, to some extent, explained by significant difficulties (both mathematical and computational ones) of their solution [1, 2].

There are a number of studies in which nonlinear vibrations and dynamic stability of plates, panels, and shells of constant thickness are investigated. A review of such works for the period from 2003 to
2013 is presented in [3]. It gives the results of theoretical and experimental studies devoted to specific dynamic problems.

In [4], parametric vibrations of geometrically nonlinear cylindrical shells are investigated on the basis of Donnell’s theory.

The results of the study using the Bolotin method of the dynamic stability of composite plates under periodic loads are given in [5].

In [6], the problem of parametric vibrations of composite plates is solved on the basis of the finite element method. The regions of dynamic instability of the plates are constructed.

The study of the dynamic stability of composite plates under periodic loads is given in [7]. The problem is solved by the Bolotin method. The influence of its dimensions, different boundary conditions, and load intensity on the behavior of the plate is studied.

There are a number of articles devoted to the study of the stability of plates and shells of variable thickness under various compressive loads.

In [8], the vibrations of various types of composite plates and shells of variable thickness are investigated. The results obtained on the basis of the proposed method are compared with the results of other authors obtained analytically and numerically.

In [9,10], on the basis of the Bolotin method, the problem of dynamic instability of composite plates of variable thickness is solved. The influence of the geometric and physical-mechanical parameters of the plate on the region of dynamic instability is shown.

In [11], the finite element method is used to solve the problem of dynamic stability of composite rectangular panels of variable thickness under compressive loads.

The study in [12] is devoted to the vibrations of a shell of double curvature of variable thickness. Various types of thickness profiles are considered.

In [13], the bending of isotropic rectangular plates of variable thickness is investigated. The method proposed allows considering any laws of thickness variation.

Free vibrations of composite shells of variable thickness are considered in [14].

The study of parametric vibrations of an isotropic cylindrical shell of variable thickness under various boundary conditions is considered in [15].

In [16–18], parametric vibrations of orthotropic viscoelastic plates, panels, and shells of variable thickness are investigated.

In this paper, we investigate nonlinear parametric vibrations of an isotropic viscoelastic rectangular plate of variable thickness without considering tangential forces of inertia.

2. Materials and methods

Let us consider the problem of parametric vibrations of an isotropic viscoelastic rectangular plate of variable thickness \( h=h(x,y) \) with sides \( a \) and \( b \). Let the plate be subjected to dynamic loading on side \( a \) by a periodic load \( P(t)=P_0+P_1\cos\Theta t \) \( (P_0, P_1=\text{const}; \Theta \) is the frequency of external periodic load), provided that the plate has initial deflections.

To construct a mathematical model of the problem in a geometrically nonlinear formulation according to the Kirchhoff-Love hypothesis, we take the physical relationship between stresses \( \sigma_x, \sigma_y, \tau_{xy} \) and strains \( \varepsilon_x, \varepsilon_y, \gamma_{xy} \) in the following form [2]:

\[
\sigma_x = \frac{E}{1-\mu^2} \left( I - \Gamma^* \right) \varepsilon_x + \mu \varepsilon_y, \quad \tau_{xy} = \frac{E}{2(1+\mu)} \left( I - \Gamma^* \right) \gamma_{xy}, \quad (x \leftrightarrow y),
\]

where \( \Gamma^* \) is the integral operator with relaxation kernel \( \Gamma^* : \Gamma^* \varphi = \int_0^1 \Gamma(t-\tau)\varphi(\tau)d\tau \), \( \mu \) - the Poisson's ratio, \( E \) is the elastic modulus; hereinafter, the symbol \( (x \leftrightarrow y) \) indicates that the remaining relations are obtained by circular substitution of indices.
The relationship between strains $\varepsilon_x, \varepsilon_y, \gamma_{xy}$ in the middle surface and displacements $u, v, w$ in directions $x, y, z$, taking into account the initial irregularities, are taken in the following form [2]:

$$
\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left[ \left( \frac{\partial w_0}{\partial x} \right)^2 - \left( \frac{\partial w_0}{\partial y} \right)^2 \right],
\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2} \left[ \left( \frac{\partial w_0}{\partial y} \right)^2 - \left( \frac{\partial w_0}{\partial x} \right)^2 \right],
\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial w_0}{\partial y} - \frac{\partial w_0}{\partial x},
$$

(2)

where $w_0 = w_0(x, y)$ is the initial deflection of the plate.

The bending and torques of the plate element are as follows:

$$
M_x = -D(1 - \mu^2) \left[ \frac{\partial^2 (w-w_0)}{\partial x^2} + \mu \frac{\partial^2 (w-w_0)}{\partial y^2} \right] (x \leftrightarrow y),
H = -D(1 - \mu^2) \frac{\partial^2 (w-w_0)}{\partial x \partial y},
$$

(3)

where $D = \frac{Eh^3 (x, y)}{12(1 - \mu^2)}$ is the cylindrical stiffness of the plate.

When deriving the equation of motion for the element of an isotropic viscoelastic rectangular plate, we proceed from the following equation [2]:

$$
q + \frac{1}{h} \left[ \frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 H}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} \right] + \frac{\partial}{\partial x} \left( \frac{\partial w_0}{\partial x} + \gamma_{xy} \frac{\partial w_0}{\partial y} \right) + \frac{\partial}{\partial y} \left( \frac{\partial w_0}{\partial y} + \gamma_{xy} \frac{\partial w_0}{\partial x} \right) - \rho \frac{\partial^2 w}{\partial t^2} = 0.
$$

(4)

Substituting (1) and (3) in (4), we obtain:

$$
\left(1 - \mu^2 \right) \left[ \frac{\partial \varepsilon_x}{\partial x} + \frac{\partial \varepsilon_y}{\partial y} - \frac{\rho \partial^2 u}{\partial t^2} = 0, \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} - \rho \frac{\partial^2 \gamma_{xy}}{\partial t^2} = 0 \right],
$$

$$
\left(1 - \mu^2 \right) \left[ \frac{\partial \varepsilon_x}{\partial x} + \frac{\partial \varepsilon_y}{\partial y} + \frac{1 - \mu}{2} \frac{\partial \gamma_{xy}}{\partial y} \right] + \frac{\partial \varepsilon_x}{\partial x} \left( \varepsilon_x + \mu \varepsilon_y \right) + \frac{\partial \varepsilon_y}{\partial y} \left( \varepsilon_y + \mu \varepsilon_x \right) = 0,
$$

$$
\left(1 - \mu^2 \right) \left[ \frac{\partial \varepsilon_x}{\partial x} + \frac{\partial \varepsilon_y}{\partial y} + \frac{1 - \mu}{2} \frac{\partial \gamma_{xy}}{\partial y} \right] + \frac{\partial \varepsilon_x}{\partial x} \left( \varepsilon_x + \mu \varepsilon_y \right) + \frac{\partial \varepsilon_y}{\partial y} \left( \varepsilon_y + \mu \varepsilon_x \right) = 0,
$$

$$
\left(1 - \mu^2 \right) \left[ \frac{\partial \varepsilon_x}{\partial x} + \frac{\partial \varepsilon_y}{\partial y} + \frac{1 - \mu}{2} \frac{\partial \gamma_{xy}}{\partial y} \right] + \frac{\partial \varepsilon_x}{\partial x} \left( \varepsilon_x + \mu \varepsilon_y \right) + \frac{\partial \varepsilon_y}{\partial y} \left( \varepsilon_y + \mu \varepsilon_x \right) = 0,
$$

$$
\left(1 - \mu^2 \right) \left[ \frac{\partial \varepsilon_x}{\partial x} + \frac{\partial \varepsilon_y}{\partial y} + \frac{1 - \mu}{2} \frac{\partial \gamma_{xy}}{\partial y} \right] + \frac{\partial \varepsilon_x}{\partial x} \left( \varepsilon_x + \mu \varepsilon_y \right) + \frac{\partial \varepsilon_y}{\partial y} \left( \varepsilon_y + \mu \varepsilon_x \right) = 0,
$$

$$
\left(1 - \mu^2 \right) \left[ \frac{\partial \varepsilon_x}{\partial x} + \frac{\partial \varepsilon_y}{\partial y} + \frac{1 - \mu}{2} \frac{\partial \gamma_{xy}}{\partial y} \right] + \frac{\partial \varepsilon_x}{\partial x} \left( \varepsilon_x + \mu \varepsilon_y \right) + \frac{\partial \varepsilon_y}{\partial y} \left( \varepsilon_y + \mu \varepsilon_x \right) = 0,
$$

$$
\left(1 - \mu^2 \right) \left[ \frac{\partial \varepsilon_x}{\partial x} + \frac{\partial \varepsilon_y}{\partial y} + \frac{1 - \mu}{2} \frac{\partial \gamma_{xy}}{\partial y} \right] + \frac{\partial \varepsilon_x}{\partial x} \left( \varepsilon_x + \mu \varepsilon_y \right) + \frac{\partial \varepsilon_y}{\partial y} \left( \varepsilon_y + \mu \varepsilon_x \right) = 0,
$$

(5)

where $\varepsilon_x, \varepsilon_y, \gamma_{xy}$ are determined from (2). This is a system of nonlinear integro-differential equations of motion of an isotropic viscoelastic rectangular plate with respect to displacements $u, v$ and $w$. 


Equation (5) is simplified when considering the process without tangential forces of inertia. It becomes possible to discard the inertial terms in the first two equations. These two equations are satisfied (in the absence of \( p_x \) and \( p_y \)), by introducing the stress function \( \Phi \) in the middle surface according to formulas [2]

\[
\sigma_x = \frac{N_x}{h} = \frac{\partial^2 \Phi}{\partial y^2}, \quad \sigma_y = \frac{N_y}{h} = \frac{\partial^2 \Phi}{\partial x^2}, \quad \tau_{xy} = \frac{N_{xy}}{h} = -\frac{\partial^2 \Phi}{\partial x \partial y}
\]

In this case, instead of three equations (5), we obtain the following two equations of the Karman type

\[
\left(1 - \Gamma^*\right)\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + 3\left(1 - \Gamma^*\right)\left[2h\left(\frac{\partial^2 h}{\partial x^2} + h^2 \frac{\partial^2 h}{\partial y^2}\right) + \partial^2 w + \mu \frac{\partial^2 w}{\partial y^2}\right] + 6\left(1 - \Gamma^*\right)h^2 \frac{\partial^2 h}{\partial y^2} \left[\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}\right]
\]

\[
+ 3\left(1 - \Gamma^*\right)\left[2h\frac{\partial^2 h}{\partial y^2} + h^2 \frac{\partial^2 h}{\partial x^2}\right] + 6\left(1 - \mu \right)^2 \left[2h\frac{\partial^2 h}{\partial x \partial y^2} + 2h \frac{\partial^2 h}{\partial x^2 \partial y} \frac{\partial^2 w}{\partial y^2}\right] = 12\left(1 - \mu^2\right)\frac{E}{h} \frac{\partial^2 w}{\partial t^2} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) - \frac{12\left(1 - \mu^2\right)}{E} \rho \frac{\partial^2 w}{\partial t^2} = 0,
\]

Deflection \( w(x, y, t) \) and stress function \( \Phi \) in the resulting system are approximated by

\[
w(x, y, t) = \sum_{n=1}^{N} \sum_{m=1}^{M} w_{nm}(t) \psi_{nm}(x, y), \quad \Phi(x, y, t) = \sum_{n=1}^{N} \sum_{m=1}^{M} \Phi_{nm}(t) \chi_{nm}(x, y)
\]

where \( w_{nm} = \frac{w_{nm}}{h_0} \) and \( \Phi_{nm} = \frac{\Phi_{nm}}{h_0} \) are the sought functions of time; \( \psi_{nm}(x, y) \), \( \chi_{nm}(x, y) \); \( n = 1, 2, \ldots, N; \quad m = 1, 2, \ldots, M \) - are the coordinate functions that satisfy the given boundary conditions of the problem.

In this case, introducing the following dimensionless quantities

\[
\bar{w} = \frac{w}{h_0}; \quad \bar{w}_0 = \frac{\overline{w}_0}{h_0}; \quad \bar{x} = \frac{x}{a}; \quad \bar{y} = \frac{y}{b}; \quad \bar{h} = \frac{h}{h_0}; \quad \lambda = \frac{a}{b}; \quad \delta = \frac{b}{h_0}; \quad q = \frac{\rho}{E}; \quad \Theta = \frac{\sigma}{\bar{w}}; \quad \Gamma(t) = \frac{\Gamma(t)}{\omega}
\]

and keeping the previous notation, to determine unknowns \( w_{nm} = \frac{w_{nm}}{h_0(t)} \), we obtain the following system of nonlinear integro-differential equations:

\[
\sum_{n=1}^{N} \sum_{m=1}^{M} c_{klmn} \tilde{w}_{nm} + \eta_3 \sum_{n=1}^{N} \sum_{m=1}^{M} k_{lmn} \left(1 - 2\mu k_{lmn} \cos \theta \right) w_{nm} - \eta_3 \Gamma^* \sum_{n=1}^{N} \sum_{m=1}^{M} f_{klmn} w_{nm} = 12\left(1 - \mu^2\right)\bar{k}^2 \bar{q} \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) - \frac{12\left(1 - \mu^2\right)}{E} \rho \frac{\partial^2 \Phi}{\partial t^2} = 0,
\]

where \( \eta_3 \) is the external load parameter.

The system of nonlinear integro-differential equations (8) with corresponding initial conditions describes the problem of parametric vibrations of an isotropic viscoelastic rectangular plate of variable
thickness under initial deflections without considering tangential inertial forces.

A numerical method [19] based on the use of quadrature formulas is used to solve the system of equations. In this case, a weakly singular Koltunov-Rzhanitsyn kernel [20] of the following form is used as the relaxation kernel:

$$\Gamma(t) = Ae^{-\beta t}.t^{\alpha-1}, \quad A > 0, \quad \beta > 0, \quad 0 < \alpha < 1$$

3. Results and discussion
The results of calculations with various physical and geometric parameters of an isotropic viscoelastic rectangular plate of variable thickness are plotted in figures 1 to 3. The dependence of the thickness variation is taken as $h = 1 - \alpha^* x$.

Here, unless otherwise specified, the following initial data in the calculations are taken: $A=0.05; \alpha=0.25; \beta=0.05; \mu=0.3; \delta=25; w_0=0.01; \lambda=1; \alpha^*=0.5; \delta_0=0.3; \delta_1=0.5; \Theta=1.1$.

The influence of the viscoelastic properties of the material on the plate behavior is studied. Figure 1 shows the graphs of the deflection function for various values of rheological parameter $\alpha$. The results obtained show that an increase in the value of this parameter leads to a decrease in the vibration amplitude.

![Figure 1](image.png)

**Figure 1.** Dependence of the deflection on time at $\alpha=0.1$ (1); 0.25 (2); 0.5(3).

Note that in all cases a steady oscillatory process is observed.

Further, the behavior of the viscoelastic plate was studied at various values of external load $q$ (figure 2). It is seen that an increase in the external load leads to an increase in the vibration amplitude.

Figure 3 shows the results of a study of the influence of the thickness variation parameter $\alpha^*$ on the behavior of a viscoelastic plate.
It is clearly seen from the figure that the amplitude of the oscillations increases with an increase in this parameter. Note that at the beginning of the oscillation process, the amplitude values slightly differ from the amplitude of plates of constant thickness.

The following types of dependence of the thickness variations in one direction are considered:

\[ h(x) = \frac{1}{2} h_0 \left[ 1 - \alpha^* x \right] \]

Here \( h_0 = h(0) = const \), \( \alpha^* \) – is the parameter characterizing the thickness variability; \( h(x) = \frac{1}{2} h_0 \left[ 1 + \varepsilon \sin(2\pi r - 1) \alpha x \right] \); \( h(x) = \frac{1}{2} h_0 \left( ax^2 + bx + c \right) \) is the quadratic law.

4. Conclusion

A mathematical model, a solution method, an algorithm and a program for solving the problem of parametric vibrations of isotropic viscoelastic rectangular plates of variable thickness without regard to tangential inertia forces were developed.

Parametric vibrations of isotropic viscoelastic rectangular plates of variable thickness are investigated on the basis of a polynomial approximation of deflections.

The influence of various physical and geometric parameters of the plate on the amplitude-time characteristics of the plate is studied for various laws of thickness variation.
The proposed method and solution algorithm can be applied in the study of parametric vibrations of other types of thin-walled structures of variable thickness.

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