ON SOME TENSOR TUBAL-KRYLOV SUBSPACE METHODS VIA THE T-PRODUCT

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Abstract. In the present paper, we introduce new tensor Krylov subspace methods for solving linear tensor equations. The proposed methods use the well known T-product for tensors and tensor subspaces related to tube fibers. We introduce some new tensor products and the related algebraic properties. These new products will enable us to develop third-order the tensor tubal GMRES and the tensor tubal Golub Kahan methods. We give some properties related to these methods and propose some numerical experiments.

Keywords. Arnoldi, Krylov, GMRES, Krylov subspaces, Linear tensor equations, Tensors, T-products.

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1. Introduction. A tensor is a multidimensional array of data. The number of indices of a tensor is called modes or ways. Notice that a scalar can be regarded as a zero mode tensor, first mode tensors are vectors and matrices are second mode tensor. The order of a tensor is the dimensionality of the array needed to represent it, also known as ways or modes. For a given N-mode (or order-N) tensor \( X \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_N} \), the notation \( x_{i_1, \ldots, i_N} \) (with \( 1 \leq i_j \leq n_j \) and \( j = 1, \ldots, N \)) stands for the element \( (i_1, \ldots, i_N) \) of the tensor \( X \). The norm of a tensor \( A \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{\ell}} \) is specified by

\[
\|A\|_F^2 = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_{\ell}=1}^{n_{\ell}} a_{i_1, i_2, \ldots, i_{\ell}}^2.
\]

Corresponding to a given tensor \( A \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_N} \), the notation

\[
A_{\underbrace{\cdots,\cdots}_{(N-1)-time}}\kappa \quad \text{for} \quad \kappa = 1, 2, \ldots, n_N
\]

denotes a tensor in \( \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_{N-1}} \) which is obtained by fixing the last index and is called frontal slice. Fibers are the higher-order analogue of matrix rows and columns. A fiber is defined by fixing all the indexes except one. A matrix column is a mode-1 fiber and a matrix row is a mode-2 fiber. Third-order tensors have column, row and tube fibers/scalars. An element \( C \in \mathbb{R}^{1 \times 1 \times n} \) is called a fiber/scalar tube of length \( n \). More details are found in [8, 11, 10].

In this paper, we will develop some new tubal-Krylov subspace methods using projections onto special low dimensional tensor Krylov subspaces via the the T-product. To this end, we introduce some new tensor products with some new related algebraic properties. Then we propose the tubal-GMRES and the tubal-Golub Kahan algorithms to solve linear tensor equations.

The paper is organized as follows: In Section 2, we give notations and definitions related to the T-product. In Section 3, we develop some new tensor products and give some algebraic properties. After defining a tubal-QR factorisation algorithm we defined in Section 4, the tensor tubal-Global Arnoldi process that allows us to introduce the tubal-global GMRES method. Section 5 is devoted to the tensor tubal Golub Kahan method. Finally, some numerical tests are reported in Section 6.

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2. Definitions and notations. In this section we recall some definitions and properties of the discrete Fourier transformation and the T-product.

2.1. Discrete Fourier Transformation. The Discrete Fourier Transformation (DFT) plays a very important role in the definition of the T-product of tensors. The DFT on a vector \( v \in \mathbb{R}^n \) is defined by

\[
\hat{v} = F_n(v) \in \mathbb{C}^n,
\]

where \( F_n \) is the matrix defined as

\[
F_n(v) = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)}
\end{pmatrix} \in \mathbb{C}^{n \times n},
\]

where \( \omega = e^{-2\pi i / n} \) with \( i^2 = -1 \). It is not difficult to show that (see [5])

\[
F_n^* = F_n, \quad \text{and} \quad F_n^* F_n = F_n F_n^* = nI_n.
\]

Then \( F_n^{-1} = \frac{1}{n} F_n \) which show that \( \frac{1}{\sqrt{n}} F_n \) is a unitary matrix.

The cost of computing the vector \( \hat{v} \) directly from (2.1) is \( O(n^2) \). Using the Fast Fourier Transform (fft), it will cost only \( O(n \log n) \) and this makes the FFT very fast for large problems. It is known that

\[
F_n \circ(v) F_n^{-1} = \text{Diag}(\hat{v}),
\]

which is equivalent to

\[
F_n \circ(v) F_n^* = n \text{Diag}(\hat{v}),
\]

where

\[
\circ(v) = \begin{pmatrix}
v_1 & v_2 & \cdots & v_n \\
v_2 & v_1 & \cdots & v_{n-1} \\
\vdots & \vdots & \cdots & \vdots \\
v_n & v_{n-1} & \cdots & v_1
\end{pmatrix},
\]

and \( \text{Diag}(\hat{v}) \), is the diagonal matrix whose \( i \)-th diagonal element is \( \text{Diag}({\hat{v}}_i) \).

2.2. Definitions and properties of the T-product. In this part, we briefly review some concepts and notations related to the T-Product, see [2, 10, 9] for more details. Let \( A \in \mathbb{R}^{n_1 \times n_2 \times n_3} \) be a third-order tensor, then the operations \( \circ \), unfold and fold are defined by

\[
\circ(A) = \begin{pmatrix}
A_{11} & A_{n_1} & A_{n_1-1} & \cdots & A_2 \\
A_{21} & A_{n_2} & A_{n_2-1} & \cdots & A_3 \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
A_{n_1, n_2} & A_{n_1, n_3-1} & A_{n_1, n_3-2} & \cdots & A_2 \\
A_{11} & A_{n_1, n_3-1} & A_{n_1, n_3-2} & \cdots & A_2
\end{pmatrix} \in \mathbb{R}^{n_1 n_2 \times n_2 n_3}
\]
\begin{align*}
\text{unfold}(A) &= \begin{pmatrix}
A_1 \\
A_2 \\
\vdots \\
A_{n_3}
\end{pmatrix} \in \mathbb{R}^{n_2n_3 \times m_2}, \quad \text{fold(\text{unfold}(A))} = A.
\end{align*}

Let \( \tilde{A} \) be the tensor obtained by applying the DFT on all the 3-mode tubes of the tensor \( A \). With the Matlab command \texttt{fft}, we have

\[
\tilde{A} = \text{fft}(A, [], 3), \quad \text{and } \text{ifft}(\tilde{A}, [], 3) = A,
\]

where \( \text{ifft} \) denotes the Inverse Fast Fourier Transform.

The tensor \( \tilde{A} \) can be obtained using the 3-mode product \cite{11, 9, 10} as follows

\[
\tilde{A} = A \times_3 F_{n_3}, \quad A = \tilde{A} \times_3 F_{n_3}^{-1}
\]

(2.6)

where \( \times_3 \) is the 3-mode product defined in \cite{11}.

Let \( A \) be the matrix

\[
A = \text{Diag}(\tilde{A}) = \begin{pmatrix} A^{(1)} & & \\ & A^{(2)} & \\ & & \ddots \\ & & & A^{(n_3)} \end{pmatrix},
\]

(2.7)

and the matrices \( A^{(i)} \)'s are the frontal slices of the tensor \( \tilde{A} \). The block circulant matrix \( \text{bcirc}(A) \) can also be block diagonalized by using the DFT and this gives

\[
(F_{n_3} \otimes I_{n_1}) \text{bcirc}(A)(F_{n_3}^{-1} \otimes I_{n_2}) = A,
\]

(2.8)

As noticed in \cite{7, 8}, the diagonal blocks of the matrix \( A \) satisfy the following property

\[
\left\{
\begin{array}{l}
A^{(1)} \in \mathbb{R}^{n_1 \times n_2} \\
\text{conj}(A^{(i)}) = A^{(n_3-i+2)},
\end{array}
\right.
\]

(2.9)

where \( \text{conj}(A^{(i)}) \) is the complex conjugate of the matrix \( A^{(i)} \). Next we recall the definition of the T-product.

**Definition 2.1.** The **T-product** \( (\ast) \) between two tensors \( A \in \mathbb{R}^{n_1 \times n_2 \times n_3} \) and \( B \in \mathbb{R}^{n_2 \times m \times n_3} \) is the \( n_1 \times m \times n_3 \) tensor given by:

\[
A \ast B = \text{fold}(\text{bcirc}(A)\text{unfold}(B)).
\]

Notice that from the relation \cite{2, 4}, we can show that the product \( C = A \ast B \) is equivalent to \( C = AB \). So, the efficient way to compute the T-product is to use Fast Fourier Transform (FFT). Using the relation \cite{2, 4}, the following algorithm allows us to compute in an efficient way the T-product of the tensors \( A \) and \( B \).
Algorithm 1 Computing the T-product via FFT

Inputs: $A \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ and $B \in \mathbb{R}^{n_2 \times m \times n_3}$
Output: $C = A \star B \in \mathbb{R}^{n_1 \times l \times n_3}$

1. Compute $\tilde{A} = \text{fft}(A, [], 3)$ and $\tilde{B} = \text{fft}(B, [], 3)$.
2. Compute each frontal slices of $\tilde{C}$ by
   \[
   C^{(i)} = \begin{cases} 
   A^{(i)}B^{(i)}, & i = 1, \ldots, \left\lfloor \frac{n_3+1}{2} \right\rfloor \\
   \text{conj}(C^{(n_3+i-2)}), & i = \left\lfloor \frac{n_3+1}{2} \right\rfloor + 1, \ldots, n_3.
   \end{cases}
   \]
3. Compute $C = \text{ifft}(\tilde{C}, [], 3)$.

For the T-product, we have the following definitions

**Definition 2.2.**

1. The identity tensor $I_{n_1, n_1, n_3}$ is the tensor whose first frontal slice is the identity matrix $I_{n_1, n_1}$ and the other frontal slices are all zeros.
2. An $n_1 \times n_1 \times n_3$ tensor $A$ is invertible, if there exists a tensor $B$ of order $n_1 \times n_1 \times n_3$ such that
   \[A \star B = I_{n_1, n_1, n_3} \quad \text{and} \quad B \star A = I_{n_1, n_1, n_3}.
   \]
   In that case, we set $B = A^{-1}$. It is clear that $A$ is invertible if and only if $\text{bcirc}(A)$ is invertible.
3. The transpose of $A$ is obtained by transposing each of the frontal slices and then reversing the order of transposed frontal slices 2 through $n_3$.
4. If $A$, $B$ and $C$ are tensors of appropriate order, then
   \[(A \star B) \star C = A \star (B \star C).
   \]
5. Suppose $A$ and $B$ are two tensors such $A \star B$ and $B^T \star A^T$ are defined. Then
   \[(A \star B)^T = B^T \star A^T.
   \]

**Definition 2.3.** Let $A$ and $B$ two tensors in $\mathbb{R}^{n_1 \times n_2 \times n_3}$. Then

1. The scalar inner product is defined by
   \[
   \langle A, B \rangle = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \sum_{i_3=1}^{n_3} a_{i_1i_2i_3}b_{i_1i_2i_3}.
   \]
2. The norm of $A$ is defined by
   \[
   \|A\|_F = \sqrt{\langle A, A \rangle}.
   \]

**Remark 2.1.** Another interesting way for computing the scalar product and the associated norm is as follows:

\[
\langle A, B \rangle = \frac{1}{n_3^{1/2}}\langle A, B \rangle; \quad \|A\|_F = \frac{1}{\sqrt{n_3^{1/2}}}\|A\|_F
\]

where the block diagonal matrix $A$ is defined by (2.7).

**Definition 2.4.**
1. An $n_1 \times n_1 \times n_3$ tensor $Q$ is orthogonal if
\[ Q^T \ast Q = Q \ast Q^T = I_{n_1,n_1,n_3}. \]

2. A tensor is called f-diagonal if its frontal slices are orthogonal matrices. It is called upper triangular if all its frontal slices are upper triangular.

Next, we introduce now the $T$-trace transformation.

**Definition 2.5.** Let $A$ be a tensor in $\mathbb{R}^{n_1 \times n_1 \times n_3}$. The tensor $T$-trace of $A$ is a fiber-tensor of $\mathbb{R}^{1 \times 1 \times n_3}$ defined such that its $i$-th frontal slice is the trace of $i$-th frontal slice of $\tilde{A}$, for $i = 1, \ldots , n_3$.

The $T$-trace($A$) can be computed by the following algorithm.

**Algorithm 2** Tensor T-trace

1. **Input.** $A \in \mathbb{R}^{n_1 \times n_1 \times n_3}$.
2. Set $\tilde{A} = \text{fft}(A,[] ,3)$
   (a) for $i = 1, \ldots , n_3$
   i. $z_{::i} = (\text{trace}(A^{(i)}))$ (trace matrix)
   (b) End
3. (T-trace($A$)) = $\text{ifft}(z,[] ,3)$
4. End

3. **New tensor products.** In this section, we introduce new tensor products, that will be used for simplifying the algebraic computations of the main results.

**Definition 3.1.** We have the following definitions:
1. Let $a \in \mathbb{R}^{1 \times 1 \times n_3}$ and $B = B(i,j,:) = b_{ij} \in \mathbb{R}^{m_1 \times m_2 \times n_3}$ for $i = 1, \ldots , m_1$, $j = 1, \ldots , m_2$. Then, the product $(a \ast B)$ is an $(m_1 \times m_2 \times n_3)$ tensor defined by
\[
\begin{pmatrix}
  a_b^{b_{11}} & \cdots & a_b^{b_{1m_2}} \\
  \vdots & \ddots & \vdots \\
  a_b^{b_{m_11}} & \cdots & a_b^{b_{m_1m_1}}
\end{pmatrix}
\]

**Remark 3.1.** The $\ast$ product is a generalisation of the product of a scalar with a matrix where the tubal-fiber plays the role of a scalar.

**3.0.1. The T-Kronecker and the T-inner products.** In the following we introduce the T-Kronecker product between two tensors as a generalisation of the classical Kronecker product for matrices.

**Definition 3.2.** (T-Kronecker product) Let $A \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ and $B \in \mathbb{R}^{m_1 \times m_2 \times n_3}$, the T-Kronecker product $A \ast B$ between $A$ and $B$ is the $n_1 m_1 \times n_2 m_2 \times n_3$ tensor given by:
\[
(A \ast B) = (A \ast B) \times_3 F_{n_3}^{-1}
\]

where the $i$-th frontal slice of $(A \ast B)$ is given by
\[
(A \ast B)^{(i)} = ((A \ast B) \times_3 F_{n_3})^{(i)} = (A \times_3 F_{n_3})^{(i)} \otimes (B \times_3 F_{n_3})^{(i)} = A^{(i)} \otimes B^{(i)}.
\]
which shows that $\langle A \otimes \langle R \rangle \rangle$ the Kronecker product between two matrices. The matrices $(A \times_3 F_{n_3})^{(i)}$ and $(B \times_3 F_{n_3})^{(i)}$ are the i-th frontal slices of $\mathcal{A}$ and $\mathcal{B}$ respectively. The T-Kronecker product of two tensors can be computed by the following algorithm.

**Algorithm 3 Tensor T-Kronecker**

1. **Input.** $A \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ and $B \in \mathbb{R}^{m_1 \times m_2 \times n_3}$
2. Set $\mathcal{A} = \text{fft}(A, [], 3)$ and $\mathcal{B} = \text{fft}(B, [], 3)$
   (a) for $i = 1, \ldots, n_3$
   i. $(A \otimes B)_{i, i, i} = (A^{(i)} \otimes B^{(i)})$
   (b) End
3. $(A \otimes B) = \text{ifft}((A \otimes B), [], 3)$
4. End
5. **Output.** $(A \otimes B)$ is the tensor of size $n_1 m_1 \times n_2 m_2 \times n_3$

**Proposition 3.3.** Let $A \in \mathbb{R}^{n_1 \times n_2 \times n_3}$, $B \in \mathbb{R}^{m_1 \times m_2 \times n_3}$, $C \in \mathbb{R}^{n_3 \times r_1 \times n_3}$ and $D \in \mathbb{R}^{m_3 \times r_2 \times n_3}$. Then we have the following properties
1. $(A \otimes B)^T = A^T \otimes B^T$
2. $(A \otimes B) \ast (C \otimes D) = (A \ast C) \otimes (B \ast D)$
3. If $A \in \mathbb{R}^{n \times n \times z}$ and $B \in \mathbb{R}^{p \times p \times z}$ are invertible then $(A \ast B)^{-1}$ is invertible and we have :
   
   $$(A \ast B)^{-1} = A^{-1} \ast B^{-1}$$

**Proof.** Obviously, the results stems directly from the properties of the matrix-Kronecker. In fact, for $i = 1, \ldots, n_3$, we have :

$$(A^T \otimes B^T)_{i, i, i} = ((A \times_3 F_{n_3})^{(i)} T \otimes (B \times_3 F_{n_3})^{(i)} T)$$

$$= ((A \times_3 F_{n_3})^{(i)} T \otimes (B \times_3 F_{n_3})^{(i)} T)$$

$$= ((A \otimes B)^T_{i, i, i})$$

which shows that $(A \otimes B)^T = (A^T \otimes B^T)$.

The two other properties are shown in the same way. $\blacksquare$

Next, we define the new Tubal-inner product.

**Definition 3.4.** (Tubal-inner product) For $X, Y$ two tensors in $\mathbb{R}^{n_1 \times s \times n_3}$, we define the T-inner product $\langle \cdot, \cdot \rangle_T$ is defined by:

$$\begin{align*}
\mathbb{R}^{n_1 \times s \times n_3} \times \mathbb{R}^{n_1 \times s \times n_3} & \longrightarrow \mathbb{R}^{1 \times l \times n_3} \\
(X, Y) & \longmapsto \langle X, Y \rangle_T = T\text{-trace}(X^T \ast Y).
\end{align*}
$$

Let $X_1, \ldots, X_\ell$ be a collection of $\ell$ third tensors in $\mathbb{R}^{n_1 \times 1 \times n_3}$, if

$$\langle X_i, X_j \rangle_T = \begin{cases} 
\alpha_i \delta_{i,j} & i = j \\
0 & i \neq j.
\end{cases}$$
Then the T-diamond product $A$ to be a T-orthogonal collection of tensors. The collection is called T-orthonormal if $\alpha_i = 1$, $i = 1, \ldots, \ell$.

Notice that the T-trace of $(X^T \star y)$ can be expressed via the 3-mode product as follows:

$$(\text{T-trace}(X^T \star y))_{3}^{n_3} = \text{trace}((X_{3} \times_3 F_{n_3})^{(i)}(y_{3} \times_3 F_{n_3})^{(i)}) \; , i = 1, \ldots, n_3$$

**Proposition 3.5.** Let $A, B$ and $C$ be tensors of $\mathbb{R}^{n_1 \times s \times n_2}$ and $a \in \mathbb{R}^{1 \times 1 \times n_3}$. Then the T-inner product satisfies the following properties

1. $\langle A, B + C \rangle_T = \langle A, B \rangle_T + \langle A, C \rangle_T$,
2. $\langle A, a \star B \rangle_T = a \star \langle A, B \rangle_T$,
3. $\langle A, X \star B \rangle_T = \langle X^T \star A, B \rangle_T$, for $X \in \mathbb{R}^{n_1 \times n_1 \times n_3}$.

**Proof.** For $i = 1, \ldots, n_3$, we have

$$(\text{T-trace}(A^T \star (B + C))_{3}^{n_3})^{(i)} = \text{trace} \left( (A_{3} \times_3 F_{n_3})^{(i)} (B_{3} \times_3 F_{n_3})^{(i)} + (C_{3} \times_3 F_{n_3})^{(i)} \right)$$

$$= \text{trace} \left( (A_{3} \times_3 F_{n_3})^{(i)} (B_{3} \times_3 F_{n_3})^{(i)} + (A_{3} \times_3 F_{n_3})^{(i)} (C_{3} \times_3 F_{n_3})^{(i)} \right)$$

$$= (\text{T-trace}(A^T \star B + A^T \star C)_{3}^{n_3})^{(i)}$$

which shows the first property. The other properties could be easily shown in the same way. \)

3.0.2. T-Diamond product of third order tensor format. Now, we introduce the T-Diamond product between two tensors and give some algebraic properties.

**Definition 3.6.** Let

$$A = [A_1, \ldots, A_p]; \quad A_i : n_1 \times s \times n_3, i = 1, \ldots, p$$

$$B = [B_1, \ldots, B_l]; \quad B_j : n_1 \times s \times n_3, j = 1, \ldots, l.$$

Then the T-diamond product $A^T \diamond B$ is the tensor of size $p \times \ell \times n_3$ given by :

$$(A \diamond B) = (\stackrel{\_}{A} \diamond \stackrel{\_}{B})_{3}^{F_{n_3}^{-1}}$$

where the $i$-th frontal slice of $(\stackrel{\_}{A} \diamond \stackrel{\_}{B})$ is given by

$$(\stackrel{\_}{A} \diamond \stackrel{\_}{B})^{(i)}_{3}^{n_3} = (A_{3} \times_3 F_{n_3})^{(i)} \diamond (B_{3} \times_3 F_{n_3})^{(i)} = A^{(i)T} \diamond B^{(i)}$$

for $i = 1, \ldots, n_3$, where $\diamond$ is the diamond product between two matrices; for more details about the diamond product between two matrices, see \cite{1}. The T-diamond product can be computed by the following algorithm.
**Algorithm 4 Tensor T-Diamond product**

1. **Input.** \( \mathcal{A} \in \mathbb{R}^{n_1 \times p \times n_3} \) and \( \mathcal{B} \in \mathbb{R}^{n_1 \times q \times n_3} \)
2. Set \( \mathcal{A} = \text{fft}(\mathcal{A},[],3) \) and \( \mathcal{B} = \text{fft}(\mathcal{B},[],3) \)
   
   (a) for \( i = 1, \ldots, n_3 \)
      
      i. \( (\mathcal{A}^T \diamond \mathcal{B})_{:,i} = (\mathcal{A}^T \circ \mathcal{B}^T)_{:,i} \)
   
   (b) End
3. \( (\mathcal{A}^T \diamond \mathcal{B}) = \text{ifft}((\widehat{\mathcal{A}^T \diamond \mathcal{B}}),[],3) \)
4. End
5. **Output.** \( \mathcal{A}^T \diamond \mathcal{B} \) is the tensor of size \( p \times \ell \times n_3 \)

The next proposition gives some algebraic properties of the T-diamond product.

**Proposition 3.7.** Let \( \mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{R}^{n_1 \times p \times n_3} \), \( \mathcal{D} \in \mathbb{R}^{n_1 \times n_1 \times n_3} \) and \( \mathcal{L} \in \mathbb{R}^{p \times p \times n_3} \). We have the following proposals:

1. \( (\mathcal{A} + \mathcal{B})^T \diamond \mathcal{C} = \mathcal{A}^T \diamond \mathcal{C} + \mathcal{B}^T \diamond \mathcal{C} \)
2. \( \mathcal{A}^T \diamond (\mathcal{B} + \mathcal{C}) = \mathcal{A}^T \diamond \mathcal{B} + \mathcal{A}^T \diamond \mathcal{C} \)
3. \( (\mathcal{A}^T \diamond \mathcal{B})^T = \mathcal{B}^T \diamond \mathcal{A} \)
4. \( (\mathcal{D} \star \mathcal{A})^T \diamond \mathcal{B} = \mathcal{A}^T \diamond (\mathcal{D}^T \star \mathcal{B}) \)
5. \( \mathcal{A}^T \diamond (\mathcal{B} \star (\mathcal{L} \odot \mathcal{J}_{s,s,n_3})) = (\mathcal{A}^T \diamond \mathcal{B}) \star \mathcal{L} \)

**Proof.** Obviously, the results stems directly from the properties of the matrix-\( \odot \) product. Indeed, we have For \( i = 1, \ldots, n_3 \) we have

\[
(\mathcal{A}^T \diamond (\mathcal{B} \star (\mathcal{L} \odot \mathcal{J}_{s,s,n_3})))_{\times 3} F_{n_3}^{(i)} = (\mathcal{A} \times F_{n_3}^{(i)} T \circ ((\mathcal{B} \times F_{n_3}^{(i)} T) \odot (\mathcal{L} \odot F_{n_3}^{(i)} T)))
\]

\[
= \left[ ((\mathcal{A} \times F_{n_3}^{(i)} T \circ (\mathcal{B} \times F_{n_3}^{(i)} T)) \odot (\mathcal{L} \times F_{n_3}^{(i)} T)) \right]
\]

\[
= ((\mathcal{A}^T \diamond \mathcal{B}) \star \mathcal{L}) \times F_{n_3}^{(i)}
\]

Finally we get : \( \mathcal{A}^T \diamond (\mathcal{B} \star (\mathcal{L} \odot \mathcal{J}_{s,s,n_3})) = (\mathcal{A}^T \diamond \mathcal{B}) \star \mathcal{L} \). The other results are obtained by following the same manner. \( \square \)

4. **The tensor tubal global GMRES method.**

4.1. **Tubal global QR factorization.** Next, we present the Tubal-Global Gram—Schmidt process.

**Definition 4.1.** Let \( \mathbf{z} \in \mathbb{R}^{1 \times n_3} \), the tubal rank of \( \mathbf{z} \) is the number of its non-zero Fourier coefficients. If the tubal-rank of \( \mathbf{z} \) is equal to \( n_3 \), we say that it is invertible and we denote by \( (\mathbf{z})^{-1} \) the inverse of \( \mathbf{z} \) if: \( \mathbf{z} \star (\mathbf{z})^{-1} = (\mathbf{z})^{-1} \star \mathbf{z} = \mathbf{e} \).

First, we need to introduce a normalization algorithm. This means that given a non-zero \( \mathcal{A} \in \mathbb{R}^{n_1 \times s \times n_3} \) we need to be able to write:

\[
\mathcal{A} = \mathbf{a} \star \mathcal{Q} = \mathcal{Q} \star (\mathbf{a} \odot \mathbf{J}_{s,n_3})
\]

where \( \mathbf{a} \) is invertible and \( \langle \mathcal{Q}, \mathcal{Q} \rangle_T = \mathbf{e} \), where \( \mathbf{e} \) is the tubal-fiber such that MatVec(\( \mathbf{e} \)) = \((1, 0, 0, \ldots, 0)^T\).

We consider the following normalization algorithm.

---

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Algorithm 5 A normalization algorithm (Normalization($A$))

1. **Input.** $A \in \mathbb{R}^{n_1 \times s \times n_3}$ and tol.
2. Set $\tilde{Q} = \text{fft}(A, [], 3)$
   (a) for $j = 1, \ldots, n_3$
      i. $a_j = \text{trace}(\tilde{Q}_j^T \tilde{Q}_j) = ||\tilde{Q}_j||_F$
      ii. if $a_j < \text{tol}$, stop
      iii. else $\tilde{Q}_j = \frac{\tilde{Q}_j}{a_j}$
   (b) End for
3. $Q = \text{ifft}(\tilde{Q}, [], 3)$, $a = \text{ifft}(a, [], 3)$
4. End

Note that $a_i$ is a scalar (the $i$-th frontal slice of the tube scalar $a \in \mathbb{R}^{1 \times 1 \times n_3}$) and $\tilde{Q}_i$ is the $i$-th frontal slice of $\tilde{Q} \in \mathbb{R}^{n_1 \times s \times n_3}$.

The following algorithm summarizes the different steps defining a Tubal-Global QR factorization of a tensor.

Algorithm 6 Tensor Tubal-Global QR decomposition

1. **Input.** $A \in \mathbb{R}^{n \times s \times n_3}$ s < n.
2. Set $[Q_1, r_{1,1,:}] = \text{Normalization}(A_1)$
3. For $j = 1, \ldots, k$
   (a) $W = Q_j$
   (b) for $i = 1, \ldots, j - 1$
      i. $r_{i,j,:} = \langle Q_i, W \rangle_T$
      ii. $W = W - r_{i,j,:} \ast R_i$
   (c) End for
   (d) $[Q_j, r_{j,j,:}] = \text{Normalization}(W)$: using Algorithm 5
4. End

**Proposition 4.2.** Let $Z = [Z_1, \ldots, Z_k]$ be an $n \times ks \times n_3$ tensor where $Z_j$ is an $n \times s \times n_3$ tensor, for $j = 1, \ldots, k$. Then from Algorithm 6 $Z$ can be factored as

$$Z = Q \ast (R \odot I_{s,s,n_3}),$$

where $Q = [Q_1, \ldots, Q_k]$ is an $(n \times ks \times n_3)$ T-orthonormal tensor satisfying $Q^T \odot Q = I_{k,k,n_3}$ and $R$ is an upper triangular $(k \times k \times n_3)$ tensor (each frontal slice of $R$ is an upper triangular matrix of size $k \times k$).

**Proof.** This will be shown by induction on $k$. For $k = 1$, we have from line 2 of Algorithm 6 $\langle Q_1, Q_1 \rangle_T = e$. Assume now that the result is true for some $k$. Using the results of Proposition 3.3 we obtain

$$(r_{k+1,k+1,:}) \ast \langle Q_j, Q_{k+1} \rangle = \langle Q_j, W - \sum_{i=1}^{k-1} r_{i,k,:} \ast Q_i \rangle$$

$$= \left( \langle Q_j, W \rangle - \sum_{i=1}^{k-1} r_{i,k,:} \ast \langle Q_j, Q_i \rangle \right)$$

$$= (r_{j,k,:} - r_{j,k,:}) = 0, j = 1, \ldots, k.$$
Then we get $Q^T \odot Q = I_{k \times n_3}$. Let $\mathcal{Z} = [Z_1, \ldots, Z_k]$ be an $n \times ks \times n_3$ tensor with $Z_j$ an $n \times s \times n_3$ tensor, for $j = 1, \ldots, k$. Then from Algorithm 6 we have $Z_j = \sum_{i=1}^j r_{i,j} \ast Q_i$ and the $j$-th lateral slice of $(Z)$ is given by:

$$(Z)_j = Z_j = \sum_{i=1}^j r_{i,j} \ast Q_i$$

$$= \sum_{i=1}^j Q_i \ast (r_{i,j}) \ast \mathcal{I}_{s,s,n_3}$$

$$= [Q_1, \ldots, Q_j] \ast \begin{bmatrix} r_{1,j} \\ \vdots \\ r_{j,j} \\ 0 \end{bmatrix} \ast \mathcal{I}_{s,s,n_3}.$$

If we set $R_j = \begin{bmatrix} r_{1,j} \\ \vdots \\ r_{j,j} \\ 0 \end{bmatrix} \in \mathbb{R}^{k \times 1 \times n_3}$ the $j$-th lateral slice of the $(k \times k \times n_3)$ $\mathcal{R} = [R_1, \ldots, R_k]$ tensor. Then we have the decomposition

$$Z_j = [Q_1, \ldots, Q_j] \ast (R_j \ast \mathcal{I}_{s,s,n_3}) \quad j = 1, \ldots, k$$

Therefore, $\mathcal{Z}$ be factored as $\mathcal{Z} = \mathcal{Q} \ast (\mathcal{R} \ast \mathcal{I}_{s,s,n_3})$ where $[Q_1, \ldots, Q_k]$ is an $(n \times ks \times n_3)$ T-orthonormal tensor $Q^T \odot Q = I_{k \times k \times n_3}$ and $\mathcal{R}$ is an upper triangular $(k \times k \times n_3)$ tensor. Note that $Q^T \odot \mathcal{Z} = Q^T \odot (\mathcal{Q} \ast (\mathcal{R} \ast \mathcal{I}_{s,s,n_3}))$, by using the results 5 in Proposition 3.7, we get $Q^T \odot \mathcal{Z} = (Q^T \odot \mathcal{Q}) \ast \mathcal{R} = \mathcal{R}$. 

### 4.2. The tensor tubal-global Arnoldi process

In this section, we define the Tubal-Global Arnoldi process that could be considered as a generalisation of the global Arnoldi process defined in 6 for matrices. In 4 Jbilou & all introduced the T-global Arnoldi process. The main difference between the Tubal-Global Arnoldi and the T-Global Arnoldi is that for T-Global Arnoldi process the tensor Krylov subspace $\mathcal{T}_K(A, \mathcal{V})$ associated to the T-product, defined for the pair $(A, \mathcal{V})$ is as follows

$$\mathcal{T}_K(A, \mathcal{V}) = \text{Tspan} \{\mathcal{V}, A \ast \mathcal{V}, \ldots, A^{m-1} \ast \mathcal{V}\} = \left\{ \mathcal{Z} \in \mathbb{R}^{n \times s \times n_3} : \mathcal{Z} = \sum_{i=1}^m \alpha_i (A^{i-1} \ast \mathcal{V}) \right\} \quad (4.1)$$

where $\alpha_i \in \mathbb{R}, i = 1, \ldots, m$, $A \in \mathbb{R}^{n \times s \times n_3}$ and $\mathcal{V} \in \mathbb{R}^{n \times s \times n_3}$ ($s << n$). In the case of the Tubal Global Arnoldi process, the tensor Tubal global Krylov subspace of order $m$ generated by $A$ and $\mathcal{V}$ and denoted by $\mathcal{T}_K^m(A, \mathcal{V}) \subset \mathbb{R}^{n \times s \times n_3}$ is defined by:

$$\mathcal{T}_K^m(A, \mathcal{V}) = \text{TSpan} \{\mathcal{V}, A \ast \mathcal{V}, A^2 \ast \mathcal{V}, \ldots, A^{m-1} \ast \mathcal{V}\}$$

$$= \left\{ \mathcal{Z} \in \mathbb{R}^{n \times s \times n_3} : \mathcal{Z} = \sum_{i=1}^m a_i (A^{i-1} \ast \mathcal{V}) \right\} \quad (4.2)$$

$$= \left\{ \mathcal{Z} \in \mathbb{R}^{n \times s \times n_3} : \mathcal{Z} = \sum_{i=1}^m a_i (A^{i-1} \ast \mathcal{V}) \right\} \quad (4.3)$$
where $a_i \in \mathbb{R}^{1 \times 1 \times n_3}$, $A_i^{i-1} \ast V = A_i^{i-2} \ast A \ast V$, $i = 2, \ldots, m$ and $A^0$ is the identity tensor. The following Tubal-Global Arnoldi process produces a T-orthonormal basis of $\mathcal{T}K^2_m(A, V)$. The algorithm is described as follows.

**Algorithm 7** The Tensor Tubal-Global Arnoldi

1. **Input.** $A \in \mathbb{R}^{n \times n \times n_3}$, $V \in \mathbb{R}^{n \times s \times n_3}$ and the positive integer $m$.
2. Set $[V_1, r_{1,1:}] = \text{Normalization}(V)$
3. For $j = 1, \ldots, m$
   a. $W = A \ast V_j$
   b. for $i = 1, \ldots, j$
      i. $h_{i,j,:} = \langle V_i, W \rangle$
      ii. $W = W - h_{i,j,:} \ast V_i$
   c. End for
   d. $[V_{j+1}, h_{j+1,j,:}] = \text{Normalization}(W)$
4. End

**Proposition 4.3.** Suppose that $m$ steps of Algorithm 7 have been run. Then, the tensors $V_1, \ldots, V_m$, form a T-orthonormal basis of the Tubal-global Krylov subspace $\mathcal{T}K^2_m(A, V)$.

**Proof.** This will be shown by induction on $m$. For $m = 1$, we have from line 2 in Algorithm 6: $\langle V_1, V_1 \rangle_T = e$. Assume now that the result is true for some $m$, then from Algorithm 6 and by using the results of Proposition 3.5, we get

$$
(h_{m+1,m,:}) \ast \langle V_j, V_{m+1} \rangle = \langle V_j, (h_{m+1,m,:}) \ast V_{m+1} \rangle = V_j, (W - \sum_{i=1}^m h_{i,m,:} \ast V_i) = \left( \langle V_j, W \rangle - \sum_{i=1}^m h_{i,m,:} \ast \langle V_j, V_i \rangle \right) = (h_{j,m,:} - h_{j,m,:}) = 0, = 1, \ldots, m.
$$

Furthermore, from line 3(d) of Algorithm 7, we immediately have $\langle V_{m+1}, V_{m+1} \rangle = e$. Therefore, the result is true for $m + 1$ which completes the proof.

Let $V_m$ be the $(n_1 \times sm \times n_3)$ tensor whose frontal slices are $V_1, \ldots, V_m$ and let $\tilde{H}_m$ the $(m+1) \times m \times n_3$ Hessenberg tensor defined by (TTGA) (Hessenberg tensor mean that every frontal slice of $\tilde{H}_m$ is a Hesemberg matrix) and by $\tilde{H}_m$ the tensor obtained from $\tilde{H}_m$ by deleting its last horizontal slice. $A \ast V_m$ is the $(n_1 \times (sm) \times n_3)$ tensor whose frontal slices are $A \ast V_1, \ldots, A \ast V_m$, respectively. We can set

$$V_m := [V_1, \ldots, V_m], \text{ and } A \ast V_m := [A \ast V_1, \ldots, A \ast V_m]$$

We can now state the following algebraic properties

**Proposition 4.4.** Suppose that $m$ steps of Algorithm 7 have been run. Then, the
following statements hold:

\[ A \ast V_m = V_m \ast (H_m \odot I_{s,s,n}) + V_{m+1} \ast (h_{m+1,m} \ast (e_{1,m} \odot I_{s,s,n})) \],

\[ V^T_m \ast A \ast V_m = H_m, \]

\[ A \ast V_m = V_{m+1} \ast (\tilde{H}_m \odot I_{s,s,n}) \],

\[ V^T_{m+1} \ast A \ast V_m = \tilde{H}_m, \]

\[ V^T_m \ast V_m = I_{m,m,n} \],

where \(e_{1,m} : \in \mathbb{R}^{1 \times m \times n} \) with 1 in the \((1,m,1)\) position and zeros in the other positions and \(h_{m+1,m} : \) is the fibre tube in the \((m+1,m,:)\) position of \(\tilde{H}\).

**Proof.** We give a proof only for the third relation, the other relations could be obtained in the same way. From Algorithm \(\mathcal{A} \ast V_j = \sum_{i=1}^{j+1} h_{i,j,:} \ast V_i \). Using the fact that \(A \ast V_m = [A \ast V_1, \ldots, A \ast V_m]\), the j-th frontal slice of \(A \ast V_m\) is given by:

\[
(A \ast V_m)_j = A \ast V_j = \sum_{i=1}^{j+1} h_{i,j,:} \ast V_i
\]

\[
= \sum_{i=1}^{j+1} V_i \ast ((h_{i,j,:}) \odot I_{s,s,n})
\]

\[
= [V_1, \ldots, V_{j+1}] \ast \left( \begin{bmatrix} h_{1,j,:} \\ \vdots \\ h_{j+1,j,:} \end{bmatrix} \odot I_{s,s,n} \right).
\]

If we set \(H_j = \begin{bmatrix} h_{1,j,:} \\ \vdots \\ h_{j+1,j,:} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{m+1 \times 1 \times n} \) the j-th lateral slice of of the Hessemberg tensor \(\tilde{H} = [H_1, \ldots, H_m] \). We have

\[
(A \ast V_m)_j = [V_1, \ldots, V_{j+1}] \ast (H_j \odot I_{s,s,n}) \quad j = 1, \ldots, m
\]

Now, we may conclude the result immediately. \(\square\)

4.3. The tensor Tubal-Global GMRES method. We briefly describe how the well-known global GMRES can be extended in the tensor form via the T-product. The tubal-global GMRES method is based on tubal-global Arnoldi process to build an orthonormal basis of the tensor tensor tubal Krylov subspace \(\mathcal{H}_{s,s,n}^{(t)}\). First, we need to introduce a new T-\(\ell_2\) norm defined in the space \(Y \in \mathbb{R}^{m \times 1 \times n} \).
DEFINITION 4.5. *(T-ℓ₂-inner product)* Let \(X \in \mathbb{R}^{m \times 1 \times n_3}, \ y \in \mathbb{R}^{m \times 1 \times n_3}\). Then the T-ℓ₂ inner scalar product of \(X\) and \(y\) is defined by

\[
\langle X, y \rangle_{T_2} = \frac{1}{\sqrt{n_3}} \sum_{i=1}^{n_3} (X \times_3 F_{n_3})^{i}(y \times_3 F_{n_3})^{i}.
\]

(4.4)

The associated T-ℓ₂ norm is defined by Let \(y \in \mathbb{R}^{m \times 1 \times n_3}\), the ℓ₂-T-norm noted by \(||.||_{T_2}\) is defined as follow:

\[
||y||_{T_2} = \frac{1}{\sqrt{n_3}} \left( \sum_{i=1}^{n_3} (||y \times_3 F_{n_3})^{i}||_{F}^2 \right)^{\frac{1}{2}}
\]

(4.5)

where \(||.||_{F}\) denote the usual vector ℓ₂ norm.

PROPOSITION 4.6. Let \(A, B, C \in \mathbb{R}^{m \times 1 \times n_3}\) and \(\alpha \in \mathbb{R}\), the T-ℓ₂-inner product satisfies the following properties

1. \(\langle A, B + C \rangle_{T_2} = \langle A, B \rangle_{T_2} + \langle A, C \rangle_{T_2}\).
2. \(\langle A, \alpha B \rangle_{T_2} = \alpha \langle A, B \rangle_{T_2}\).
3. \(\langle A, X \odot B \rangle_{T_2} = \langle X^{T} \odot A, B \rangle_{T_2}\), for \(X, B \in \mathbb{R}^{n_1 \times 1 \times n_3}\).

Proof. The proofs come directly from the definitions of the T-ℓ₂ inner scalar product and the operator \(\varphi\).

PROPOSITION 4.7. Let \(y \in \mathbb{R}^{m \times 1 \times n_3}\) and \(V \in \mathbb{R}^{n \times m \times n_3}\) such that \(V^{T} \odot V = J_{mn_3}\). Then

\[
||V \odot (y \odot J_{s,s,n_3})||_{F} = ||y||_{T_2}.
\]

Proof. We have:

\[
||V \odot (y \odot J_{s,s,n_3})||_{F}^2 = \frac{1}{n_3} \sum_{i=1}^{n_3} ||(V \times_3 F_{n_3})^{i} \left( (y \times_3 F_{n_3})^{i} \odot (J_{s,s,n_3} \times_3 F_{n_3})^{i} \right) ||_{F}^2
\]

\[
= \frac{1}{n_3} \sum_{i=1}^{n_3} (||y \times_3 F_{n_3})^{i}||_{F}^2
\]

\[
= ||y||_{T_2}^2.
\]

Next, we will see how to define the Tubal-Global GMRES. Consider the linear system of tensor equations

\[
A \star X = B
\]

(4.6)

where \(A \in \mathbb{R}^{n \times n \times n_3}\) assumed to be nonsingular, \(B, X \in \mathbb{R}^{n \times s \times n_3}\), with \(1 \leq s \ll n, n_3\). The problem \((4.6)\) is a generalisation of multiple linear systems of equations corresponding to \(n_3 = 1\), to the tensor case. Let \(X_0 \in \mathbb{R}^{n \times s \times n_3}\) be an arbitrary initial guess with the associated residual tensor \(R_0 = B - A \star X_0\). The aim of the tensor Tubal-Global GMRES method is to find, at some step \(m\), an approximation \(X_m\) of the solution \(X^\star\) of the problem \((4.6)\) as follows

\[
X_m - X_0 \in \mathbb{R}^{n_3}_m (A, R_0),
\]

(4.7)
with
\[ \| \mathcal{R}_m \|_F = \min_{\mathcal{X}} \{ \| \mathcal{B} - \mathcal{A} \ast \mathcal{X} \|_F; \mathcal{X} - \mathcal{X}_0 \in \mathcal{X}_m^\mathcal{R}(\mathcal{A}, \mathcal{R}_0) \} \] (4.8)

Let \( \mathcal{X}_m = \mathcal{X}_0 + \mathcal{V}_m \ast (y \otimes J_{s,s,n_3}) \) with \( y \in \mathbb{R}^{m \times 1 \times n_3} \). Then the problem (4.8) is equivalent to
\[ \| \mathcal{R}_m \|_F = \min_{y \in \mathbb{R}^{m \times 1 \times n_3}} \| \mathcal{R}_0 - \mathcal{A} \ast \mathcal{V}_m \ast (y \otimes J_{s,s,n_3}) \|_F \] (4.9)

Using the Proposition 4.7 and Proposition 3.3 and Step 2 of Algorithm (7), we get the following
\[ \| \mathcal{R}_0 - \mathcal{A} \ast \mathcal{V}_m \ast (y \otimes J_{s,s,n_3}) \|_F = \| (e_{1,1} \ast r_{1,1,1} - (\mathcal{K}_m \ast y)) \|_{T_2} \]

where \( e_{1,1} \in \mathbb{R}^{m+1 \times 1 \times n_3} \) with 1 in the (1,1,1) position and zero in the other positions. Therefore, the tensor \( y_m \) solving the minimisation problem (4.10) is given by
\[ y_m = \arg \min_{y \in \mathbb{R}^{m \times 1 \times n_3}} \| (e_{1,1} \ast r_{1,1,1} - (\mathcal{K}_m \ast y)) \|_{T_2}. \] (4.10)

The approximate solution is given as
\[ \mathcal{X}_m = \mathcal{X}_0 + \mathcal{V}_m \ast (y_m \otimes J_{s,s,n_3}) \] (4.11)

### 4.4. Implementation of tensor Tubal-Global GMRES method

For the implementation of Algorithm 8, we use the tensor QR decomposition to solve the minimization (4.10). First, we consider the algorithm Tensor QR decomposition which is based on the application of the QR matrix decomposition to each sub-block of the obtained block diagonal matrix in the Fourier domain. In other word, for \( \mathcal{A} \in \mathbb{R}^{n \times m \times n_3} \) we have \( \mathcal{A} = \mathcal{Q} \ast \mathcal{R} \) where \( \mathcal{Q} \) is an \( n \times m \times n_3 \) orthogonal tensor \( \mathcal{Q}^T \ast \mathcal{Q} = \mathcal{I}_{mnn_3} \) and \( \mathcal{R} \in \mathbb{R}^{m \times m \times n_3} \) triangular tensor.

**Proposition 4.8.** Let \( y \in \mathbb{R}^{m \times 1 \times n_3} \) and \( \mathcal{Q} \in \mathbb{R}^{n \times m \times n_3} \) such that \( \mathcal{Q}^T \ast \mathcal{Q} = \mathcal{I}_{mnn_3} \). Then
\[ \| \mathcal{Q} \ast y \|_{T_2} = \| y \|_{T_2} \]

**Proof.** The proof is a direct application of the T-\( \ell_2 \) norm and the properties of the application \( \mathcal{Q} \).

Now, we apply the T-QR decomposition to \( \mathcal{K}_m \), and by using Proposition 4.8, we get
\[ \| (e_{1,1} \ast r_{1,1,1} - (\mathcal{K}_m \ast y)) \|_{T_2} = \| \mathcal{Q}^T \ast (e_{1,1} \ast r_{1,1,1} - (\mathcal{K}_m \ast y)) \|_{T_2} \]
\[ = \| \mathcal{Q}^T \ast (e_{1,1} \ast r_{1,1,1}) - \mathcal{Q}^T \ast (\mathcal{K}_m \ast y) \|_{T_2} \]
\[ = \| \hat{\mathcal{Y}}_m - \mathcal{K}_m \ast y \|_{T_2} \]
Algorithm 8 Tensor T-QR decomposition

1. **Input.** \( A \in \mathbb{R}^{n \times m \times n_3} \)
2. Set \( \hat{A} = \text{fft}(A, [], 3) \)
   (a) for \( i = 1, \ldots, n_3 \)
      i. \( \tilde{Q}_{:,i}, \tilde{R}_{:,i} = \text{QR}(A^i) \) (matrix QR decomposition)
   (b) End
3. \( (\tilde{Q}) = \text{ifft}(\tilde{Q}, [], 3), (\tilde{R}) = \text{ifft}(\tilde{R}, [], 3) \)
4. End
5. **Output.** \( Q \in \mathbb{R}^{n \times m \times n_3}, R \in \mathbb{R}^{m \times m \times n_3} \).

where \( \tilde{G}_m = \Omega^T \ast (e_{1,1,} \ast r_{1,1,}) \) and \( \tilde{R}_m = \Omega^T \ast \tilde{H}_m. \)

**Proposition 4.9.** Let \( \tilde{G} \) and \( \tilde{R}_m \) given as
\[
\tilde{R}_m = \Omega^T \ast \tilde{H}_m, \text{ and } \tilde{G}_m = \Omega^T \ast (e_{1,1,} \ast r_{1,1,}) = \begin{bmatrix}
g_1 \\
\vdots \\
g_{m+1}
\end{bmatrix} \in \mathbb{R}^{m+1 \times 1 \times n_3}. \tag{4.12}
\]

The solution \( y_m = \min_{y \in \mathbb{R}^{m \times 1 \times n_3}} \| (e_{1,1,} \ast r_{1,1,} - (\tilde{H}_m \ast y)) \|_{T_{r_2}} \) is given by:
\[
R_m \ast y_m = g_m
\]
where \( R_m = \tilde{R}_m(1:m, :) \) and \( g_m = \begin{bmatrix}
g_1 \\
\vdots \\
g_m
\end{bmatrix} \in \mathbb{R}^{m \times 1 \times n_3}. \)

**Proof.**
\[
\| (e_{1,1,} \ast r_{1,1,} - (\tilde{H}_m \ast y)) \|_{T_{r_2}}^2 = \| \Omega^T \ast (e_{1,1,} \ast r_{1,1,} - (\tilde{H}_m \ast y)) \|_{T_{r_2}}^2
= \| \Omega^T \ast (e_{1,1,} \ast r_{1,1,}) - \Omega^T \ast (\tilde{H}_m \ast y) \|_{T_{r_2}}^2
= \| \tilde{G}_m - R_m \ast y \|_{T_{r_2}}^2
= \| g_{m+1} \|_{T_{r_2}}^2 + \| g_m - R_m \ast y \|_{T_{r_2}}^2.
\]
Assuming that \( R_m \) is invertible, \( y_m \) as the solution of the triangular tensor problem
\[
R_m \ast y_m = g_m \tag{4.13}
\]

**Remark 4.1.** Notice that for solving the equation (4.13), we use the tubal-back substitution method, which is based on the back substitution matrix method applied with tube-fibers.

5. Tensor tubal-global Golub Kahan algorithm. Instead of using the tensor global Arnoldi to generate a basis for the projection subspace, we can use the tensor global Lanczos process as was first introduced in [7]. Here, we will use the tensor Golub Kahan algorithm related to the T-product. We notice here that we already defined in [3] another version of the
Algorithm 9 Implementation of Tensor Tubal-Global GMRES (m)

1. **Input.** $A \in \mathbb{R}^{n \times n \times n}$, $V, B, X_0 \in \mathbb{R}^{n \times s \times n}$, the maximum number of iteration $\text{Iter}_{\text{max}}$ the positive integer $m$ of global krylov subspace and $\text{tol} > 0$.
2. Compute $X_0 = B - A \star X_0$.
3. $k = 1, \ldots, \text{Iter}_{\text{max}}$
   (a) Apply Algorithm 7 to calculate $V_m$ and $\tilde{H}_m$.
   (b) Compute the T-QR decomposition of $\tilde{H}_m$ using Algorithm 8.
   (c) Compute $R_m$ and $G_m$ using the relations (4.12).
   (d) Solve the system (4.13).
   (e) Compute $X_m = X_0 + V_m \star (Y_m \odot I_{s \times s \times n})$
4. If $||R_m||_F < \text{tol}$: Output the approximation $X_m$
5. else $X_0 = X_m$
6. End if
7. End for
8. End
9. **Output.** $X_m \in \mathbb{R}^{n \times s \times n}$ approximation of the solution of the equation (4.6).

tensor Golub Kahan algorithm by using the $m$-mode product with applications to color image restoration.

Let $A \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ and $U \in \mathbb{R}^{n_1 \times s \times n_3}$ be two tensors. Then, the Tensor Global T-Golub Kahan bidiagonalization algorithm is defined as follows

Algorithm 10 The Tensor Tubal-global Golub Kahan algorithm

1. **Inputs** The tensors $A$, $B$, and an integer $m$.
2. Set $V_0 = 0 \in \mathbb{R}^{n_1 \times s \times n_3}$ and $[U_1, a_1] = \text{Normalization}(U)$
3. for $j = 1, \ldots, m$
   (a) $V = A^T \star U_j - a_j \star V_{j-1}$
   (b) Set $[V_j, b_j] = \text{Normalization}(V)$
   (c) $U = A \star V_j - b_j \star U_j$
   (d) $[U_{j+1}, a_{j+1}] = \text{Normalization}(U)$
4. End for
5. End

Let $\tilde{C}_m$ be the upper bidiagonal $(m + 1) \times m \times n_3$ tensor

$$\tilde{C}_m = \begin{bmatrix}
    b_1 \\
    a_2 & b_2 & \ddots \\
    \vdots & \ddots & \ddots \\
    a_m & b_m \\
    a_{m+1}
\end{bmatrix}$$

and let $C_m$ be the $(m \times m \times n_3)$ tensor obtain by deleting the last horizontal slice of $\tilde{C}_m$. Then we have the following results.
Proposition 5.1. The tensors \( V_m = [V_1, \ldots, V_m] \in \mathbb{R}^{n_2 \times m \times n_3} \) and \( U_m = [u_1, \ldots, u_m, u_{m+1}] \in \mathbb{R}^{n_1 \times (m+1) \times n_3} \) given by Algorithm TTG-GK have orthogonal lateral slices tensors \( V_i \in \mathbb{R}^{n_2 \times m \times n_3} \) for \( i = 1, \ldots, m \), respectively; i.e.,

\[
\langle V_i, V_j \rangle_T = (U_i, U_j)_T = \begin{cases} e & i = j \\ 0 & i \neq j \end{cases}.
\]

Proof. This will be shown by induction on \( j \). For \( j = 1 \), Algorithm II shows that \( \langle V_1, V_1 \rangle_T = e \), \( (U_1, U_1)_T = e \). Assume now that the result is true for some \( j \). Then, from Algorithm II and using the results in Proposition 3.5, we conclude that \( \langle V_i, V_j \rangle_T = (U_i, U_j)_T = \begin{cases} e & i = j \\ 0 & i \neq j \end{cases} \). For \( j = 1, \ldots, m \),

Proposition 5.2. The tensors produced by the tensor global Golub-Kahan algorithm satisfy the following relations

\[
A \ast V_m = U_{m+1} \ast (\tilde{C}_m \otimes J_{s,s,n_3}),
\]

(5.1)

\[
A^T \ast U_m = V_{m+1} \ast (\tilde{C}_m \otimes J_{s,s,n_3}) + U_m \ast (a_{m+1} \ast (e_{1,m} \otimes J_{s,s,n_3})), \quad \text{and}
\]

(5.2)

\[
A \ast V_m = U_{m+1} \ast (\tilde{C}_m \otimes J_{s,s,n_3}),
\]

(5.3)

\[
B = U_{m+1} \ast ((a_1 \ast e_{1,1,:}) \otimes J_{s,s,n_3}).
\]

(5.4)

Proof. Using \( A \ast V_m = [A \ast V_1, \ldots, A \ast V_m] \), the \((j-1)\)-th lateral slice of \((A \ast U_m)\) is given by:

\[
A \ast V_{j-1} = b_{j-1} \ast U_{j-1} + a_j \ast U_j
\]

Furthermore, for \( j = 1, \ldots, m-1 \), \( U_m \ast C_{j,:} = b_{j-1} \ast U_{j-1} + a_j \ast V_j \) and for \( j = m \), \( V_m \ast C_{m,:} = A \ast V_m + a_{m+1} \ast U_{m+1} \). The result follows from these relations.

Proposition 5.3. Let \( X_m = V_m \ast (y \otimes J_{s,s,n_3}) \in \mathbb{R}^{n_2 \times m \times n_3} \) with \( y \in \mathbb{R}^{m \times 1 \times n_3} \) where \( V_m \) is obtained from Algorithm II be an approximation of \( X^* \) the solution of the tensor linear system (4.6), we have:

\[
||B - A \ast X_m||_F = ||(a_1 \ast e_{1,1,:} - (\tilde{C}_m \ast y))||_{T_{n_2}}
\]

(5.5)

where \( e_{1,1,:} \in \mathbb{R}^{m+1 \times 1 \times n_3} \) with 1 in \((1,1,1)\) position and zero in the rest.

Proof. Using the representation (5.1) and \( B = U_{m+1} \ast ((a_1 \ast e_{1,1,:}) \otimes J_{s,s,n_3}) \) we have:

\[
||B - A \ast X_m||_F = ||(U_{m+1} \ast ((a_1 \ast e_{1,1,:}) \otimes J_{s,s,n_3}) - U_{m+1} \ast (\tilde{C}_m \otimes J_{s,s,n_3}) \ast (y \otimes J_{s,s,n_3}))||_F
\]

\[
= ||(a_1 \ast e_{1,1,:} - (\tilde{C}_m \ast y)) \otimes J_{s,s,n_3}||_F
\]

\[
= ||(a_1 \ast e_{1,1,:} - (\tilde{C}_m \ast y))||_{T_{n_2}}
\]

The following algorithm summarizes the main steps to solve the linear tensor problem (4.6) using Tensor Tubal-Global Golub Kahan.
Algorithm 11 Tensor Tubal-Global Golub Kahan (m)

1. \textbf{Input.} \(A \in \mathbb{R}^{n \times n \times n^3}, \mathcal{V}, B, X_0 \in \mathbb{R}^{n \times s \times n^3}\), the maximum number of iteration \(\text{Iter}_{\max}\) the positive integer \(m\) of Tubal-global krylov subspace and \(\text{tol} > 0\).
2. Compute \(\mathcal{R}_0 = B - A \ast X_0\).

\[k = 1, \ldots, \text{Iter}_{\max}\]

(a) Apply Algorithm 7 to calculate \(V_m, \tilde{H}_m\).
(b) Compute \(Y_m = \arg \min_{Y \in \mathbb{R}^{m \times 1 \times n^3}} ||(a_1 \ast e_{1,1,:} - (\tilde{e}_m \ast y))||_{\ell_2}\)
(c) Compute \(X_m = X_0 + V_m \ast (Y_m \ast X, s, n^3)\)

4. If \(||\mathcal{R}_m||_{\ell_2} < \text{tol}\): Output the approximation \(X_m\)
5. else \(X_0 = X_m\)
6. End if
7. End for
8. End

9. \textbf{Output.} \(X_m \in \mathbb{R}^{n \times s \times n^3}\) approximate solution of the equation (4.6).

6. Numerical experiments. This section performs some numerical tests for the Tensor Tubal-Global GMRES and Tensor Tubal-Global Golub Kahan methods when solving the linear tensor problem (4.6). All computations were carried out using the MATLAB R2018b environment with an Intel(R) Core i7-8550U CPU @1.80 GHz and processor 8 GB. The stopping criterion was

\[\frac{||\mathcal{R}_m||_{\ell_2}}{||\mathcal{R}_0||_{\ell_2}} < \epsilon,\]

where \(\epsilon = 10^{-12}\) is a chosen tolerance and \(\mathcal{R}_m\) the \(m\)-th residual associated to the approximate solution \(X_m\). In all the presented tables, we reported the obtained residual norms to achieve the desired convergence, the iteration number and the corresponding cpu-time.

6.1. Example 1. The tensor \(A\) is constructed using \(n_3\) frontal slices. In this example, the frontal slices of size \(n \times n\) are given by the matrices taken from [13, Example 35.1] as follows

\[A_i = \text{eye}(n) + \frac{i}{2\sqrt{n}} \text{rand}(n) \quad i = 1, \ldots, n_3\]

In this example, the right-hand side tensor \(B\) is constructed such that the exact solution \(X^*\) of the equation (4.6) is given by \(X^* = \text{ones}(n, s, n^3)\).

6.2. Example 2. For this experiment, we used matrices from the Harwell Boeing collection. The tensor \(A\) was constructed from three frontal slices \(A_i\), \(i = 1, 2, 3\), each of size \(900 \times 900\), and given as \(A_1 = \text{PDE900}; A_2 = \text{GR3030}, A_3 = \text{gallery}(	ext{tridiag}, n, -1 + \frac{10}{n^2+2}, 2, -1 + \frac{10}{n^2+2})\). The right hand is chosen such that \(B = A \ast X^*\) where the exact solution \(X^*\) of equation (4.6) is \(X^* = \text{rand}(n, s, n_3)\). The obtained results are reported in Table 6.2.

6.3. Example 3. In this example, the tensor \(A\) of size \(m_0^2 \times m_0^2 \times n_3\) is the Laplacian tensor constructed by using the 7-point discretization of the three-dimensional Poisson equation (6.1). The tensor \(B\) was constructed such that \(B = A \ast X^*\) where \(X^* = \text{ones}(m_0^2, s, n_3)\).
Table 6.1

Results for Example 1. \( \epsilon = 10^{-12} \ n_3 = 4 \)

| Method       | n  | s  | m  | iteration | \( \frac{|x_m||T_2^2|}{|x_0||T_2^2|} \) | cpu-time in seconds |
|--------------|----|----|----|-----------|----------------|--------------------|
| Algorithm 9  | 500| 5  | 10 | 3         | \( 1.06 \times 10^{-12} \) | 0.55               |
| Algorithm 11 | 500| 5  | 10 | 3         | \( 1.67 \times 10^{-12} \) | 0.79               |
| Algorithm 9  | 1000| 5 | 10 | 3         | \( 1.26 \times 10^{-12} \) | 1.53               |
| Algorithm 11 | 1000| 5 | 10 | 3         | \( 3.03 \times 10^{-13} \) | 2.69               |
| Algorithm 9  | 1500| 5 | 10 | 3         | \( 3.91 \times 10^{-16} \) | 3.60               |
| Algorithm 11 | 1500| 5 | 10 | 3         | \( 3.97 \times 10^{-13} \) | 6.60               |

Table 6.2

Results for Example 3. \( \epsilon = 10^{-12} \ n_3 = 3 \)

| method      | s  | m  | iteration | \( \frac{|x_m||T_2^2|}{|x_0||T_2^2|} \) | CPU-time (second) |
|-------------|----|----|-----------|----------------|-------------------|
| Algorithm 9 | 3  | 10 | 23        | \( 1.59 \times 10^{-12} \) | 12.47             |

The three-dimensional Poisson equation is given by

\[
\begin{cases}
-\nabla^2 \nu = f, & \Omega = \{(x, y, z) = 0 < x, y, z < 1\}, \\
\nu = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( f \) is an given function, and

\[
\nabla^2 \nu = \frac{\partial^2 \nu}{\partial x^2} + \frac{\partial^2 \nu}{\partial y^2} + \frac{\partial^2 \nu}{\partial z^2}.
\]

The mesh step size is given by : \( \Delta x = \Delta y = \Delta z = \frac{1}{n_3} \) where \( \Delta x, \Delta y, \Delta z \) the step sizes in the x-direction, y-direction and z-direction, respectively. The difference formula obtain by the standard central difference approximation is given by

\[
6\nu_{ijk} - \nu_{i-1,jk} - \nu_{i+1,jk} - \nu_{ij-1,k} - \nu_{ij+1,k} - \nu_{ijk-1} - \nu_{ijk+1} = h^3 f_{ijk}
\]

The Laplacian tensor \( A \) can be obtained from the central difference approximations in several forms. Here, \( A \) can be expressed as a third order tensor as follow

\[
\text{unfold}(A) = \begin{pmatrix}
A_1 \\
A_2 \\
\vdots \\
A_{n_3}
\end{pmatrix}
\]
where the frontal slices $A_i$, $i = 2, \ldots, n_3 - 1$, of size $m_0^2 \times m_0^2$ are given by

$$
A_{i-1} = A_{i+1} = \left( \frac{-1}{m_0^2} \right) \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & 0 & 0 \\
0 & \cdots & 0 & 0 & 0
\end{pmatrix}
$$

$$
A_i = \left( \frac{-1}{m_0^2} \right) \begin{pmatrix}
0 & -1 & \cdots & 0 & 0 \\
-1 & 6 & -1 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & -1 & 6 & -1 \\
0 & \cdots & 0 & -1 & 0
\end{pmatrix}
$$

Table 6.3 reports on the obtained relative residual norms and the corresponding cpu-time to obtain the desired convergence. As can be seen from this table, the Tensor Tubal-Global GMRES method gives good results within small cpu-times.

**Conclusion.** In this paper, we presented some new Krylov subspace methods using the T-product and some new other tensor products. We gave new algebraic properties of these products that allowed us to build new tensor Krylov-subspace based algorithms for solving general linear tensor equations. We focussed on the tubal-global GMRES and the tubal-global Golub-Kahan algorithms. Some numerical test on simple examples are also reported.

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