Abstract. We study birational maps among 1) the moduli space of semistable torsion sheaves of Hilbert polynomial $4m + 2$ on a smooth quadric surface, 2) the moduli space of semistable torsion sheaves of Hilbert polynomial $m^2 + 3m + 2$ on $\mathbb{P}^3$, 3) Kontsevich’s moduli space of genus zero stable maps of degree 2 to Grassmannian $Gr(2,4)$. A regular birational morphism from 1) to 2) is described in terms of Fourier-Mukai transform. The map from 3) to 2) is Kirwan’s partial desingularization. Also we investigate several geometric properties of 1) by using the variation of moduli spaces of stable pairs.

1. Introduction

In this paper, $V$ is a fixed complex vector space of dimension 4. Let $\{x, y, z, w\}$ be a basis of $V^*$. In $\mathbb{P}^3 = \mathbb{P}(V)$, $Q = Z(xy - zw)$ is a smooth quadric surface. Let $Gr(2,4)$ be the space of lines in $\mathbb{P}(V)$.

The aim of this paper is understanding birational maps between several moduli spaces. We list three main characters in this article.

- The space $M_2$ is the moduli space of semistable sheaves $F$ on $Q$ with Hilbert polynomial $4m + 2$ and the first Chen class $c_1(\text{Supp}(F)) = (2,2)$.
- The space $R$ is the moduli space of semistable sheaves $F$ on $\mathbb{P}^3$ with Hilbert polynomial $m^2 + 3m + 2$.
- The space $K$ is the moduli space of genus zero stable maps of degree 2 to $Gr(2,4)$.

By using the standard deformation theory on each moduli space, it is not difficult to show that all of those three moduli spaces are 9 dimensional varieties. However, it is not obvious that they are indeed birational; they are parametrizing certain geometric objects with different dimensions on different algebraic varieties.

1.1. Main results. In this paper, we prove that they are indeed birational. Moreover, we show that the birational morphisms can be described in terms of Fourier-Mukai (shortly, FM) transform and Kirwan’s partial desingularization.

Theorem 1.1. (Theorem 5.7) There exists a birational morphism $\Psi : M_2 \rightarrow R$.

The map $\Psi$ is a composition of two FM transforms $\Psi_1 : D^b(\text{Coh}(Q)) \rightarrow D^b(\text{Coh}(\mathbb{P}^3)^*))$ and $\Psi_2 : D^b(\text{Coh}(\mathbb{P}^3)^*)) \rightarrow D^b(\text{Coh}(\mathbb{P}^3))$. Furthermore, $\Psi$ is a smooth blow-up of two points on $R$.

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By a work of Le Potier in [23], it is well-known that \( R \) can be constructed as an elementary \( (\text{PGL}_2 \times \text{PGL}_2) \)-GIT quotient of a projective space \( \mathbb{P}(\mathbb{C}^2 \otimes V, \mathbb{C}^2) \). This GIT quotient has strictly semistable points and \( R \) has some bad singularities. Surprisingly, its partial resolution of singularities has very different moduli theoretic interpretation.

**Theorem 1.2.** (Theorem 6.1) There is a birational morphism

\[ \pi : K \to R, \]

which is Kirwan’s partial desingularization of \( R \). It is a composition of two blow-ups along the singular loci of \( R \).

In summary, we have a common contraction

\[
\begin{array}{ccc}
M_2 & \xleftarrow{\Psi} & K \\
& \searrow \nearrow & \\
& R. &
\end{array}
\]

Because these two maps are blow-ups along disjoint centers, on the log minimal model program (MMP) of \( M_2 \) or \( K \), the other space does not appear.

In a recent few years, there has been an explosion of the study of birational geometry of moduli spaces of semistable sheaves in the viewpoint of log MMP. Indeed many of such birational models are moduli spaces of Bridgeland stable objects in the derived category of coherent sheaves ([2, 4, 5, 6, 7, 10, 13]).

Our work arose from the study of the geometry of \( M_2 \) in the viewpoint of the Bridgeland wall crossing on moduli spaces of torsion sheaves on rational surfaces with Picard number \( \geq 2 \). During the study, the first author observed that by applying the work of Le Potier in [23, Section 4], one may describe the birational morphism between its birational models in terms of FM transforms. In [23], Le Potier applied a FM transform to obtain a birational map from a moduli space of torsion sheaves on \( \mathbb{P}^2 \) with Euler characteristic zero to a moduli space of torsion free sheaves on its dual plane \((\mathbb{P}^2)^*\). Theorem 1.1 tells us that FM transform may provide a birational map between two moduli spaces of sheaves on different projective varieties.

### 1.2. Rationality

The original motivation of this paper was to show the rationality of \( M_2 \), which was asked in [3, Conjecture 35]. The fact that \( M_2 \) is birational to \( K \) gives the rationality of \( M_2 \), because the rationality of \( K \), or more precisely, the space of conics in \( \text{Gr}(2,4) \) is well-known, for example in [28, Section 3.10] or in [8]. On the diagram below, \( H = \text{Hilb}^{2m+1}(\text{Gr}(2,4)) \) is the Hilbert scheme of conics in \( \text{Gr}(2,4) \) and \( C = \text{Chow}_{1,2}(\text{Gr}(2,4)) \) is the normalization of the irreducible component of Chow variety containing smooth conics. All maps are birational morphisms.

\[
\begin{array}{ccc}
K & \xleftarrow{-} & H \\
& \searrow \nearrow & \\
C & & \text{Gr}(3,6)
\end{array}
\]
1.3. **Desingularization of \( M_2 \).** Along the locus of strictly semistable sheaves, the moduli space \( M_2 \) has some singularities. Finding a desingularization of a singular moduli space is a problem with a very long history and interesting implications. For example, in the case of moduli spaces of vector bundles on a curve, several desingularizations with certain moduli theoretic interpretations are constructed in [31, 27]. Of course, finding a resolution of given moduli space is useful to study the geometry of the singular moduli space. For instance in [22], by using her desingularization method, Kirwan computed the rank of intersection cohomology groups of the moduli space of semistable vector bundles on a curve when it is singular. Furthermore, the desingularization can be a highly nontrivial example of a variety with desired geometric property. In [29], O’Grady constructed a new compact hyperkähler manifold by taking a desingularization of a certain singular moduli space of torsion-free sheaves on a K3 surface.

Two main theorems in this paper combined with [12, Theorem 1.2] provide an explicit desingularization of \( M_2 \). \( K \) is singular along the locus \( D \) of stable maps which are two to one maps to their images. By [12, Theorem 1.2], the blow-up of \( K \) along \( D \) is a smooth projective variety \( CC_2 := CC_2(Gr(2,4)) \), namely, the space of complete conics. On \( R \), two blow-up centers for \( M_2 \to R \) and \( K \to R \) are disjoint. Thus the fiber product \( \tilde{M}_2 := M_2 \times_R CC_2 \) is a desingularization of \( M_2 \).

It would be very interesting to find a moduli theoretic interpretation of \( \tilde{M}_2 \).

\[
\begin{array}{ccc}
\tilde{M}_2 & \to & CC_2 \\
\downarrow & & \downarrow \\
K & \to & R \\
\downarrow & & \downarrow \\
M_2 & \to & R
\end{array}
\]

**Question 1.3.** What is the moduli theoretic meaning of \( \tilde{M}_2 \)?

1.4. **Geometry of \( M_2 \) and its numerical invariants.** Besides the result described above, by using the moduli space \( M_2^\alpha \) of \( \alpha \)-stable pairs, we computed some topological invariants of \( M_2 \). In Section 3, we study 1) the variation of the moduli space of \( \alpha \)-stable pairs from \( \alpha = \infty \) to \( \alpha = 0^+ \) and 2) the fiber of the map \( \phi : M_2^{0^+} \to M_2 \). Indeed, when \( \alpha = 2 \), there is a single wall-crossing which is a composition of a smooth blow-up and down. In Section 3 we show that there is a diagram

\[
\begin{array}{cccc}
M_2^\infty & \leftarrow & \cdots & \leftarrow M_2^{0^+} \\
\pi & \downarrow & & \downarrow \phi \\
|O_Q(2,2)|| & \to & M_2 & M_2
\end{array}
\]

where \( \phi, \pi \) are algebraic fiber spaces and the other two maps are birational morphisms. As consequences, we compute 1) the singular locus, 2) the rank of Picard group, and 3) the virtual Poincaré polynomial of \( M_2 \). Also we obtain a classification of all free resolution types of objects in \( M_2 \), which is crucial in Section 5.
1.5. Stream of the paper. This paper is organized as the following. In Section 2, we give definitions of moduli spaces appearing in this paper and review some well-known properties. In Section 3, we study the geometry of the moduli space $M_2$ and compute its numerical invariants. Section 4 offers an elementary and classical argument to show the rationality of $M_2$. In the next two sections, we describe the birational map in Section 4 in terms of FM transform and partial desingularization. In Section 5, we prove Theorem 1.1. Finally, in Section 6, we show Theorem 1.2.

1.6. Notation and convention. We work over the complex number $\mathbb{C}$. We will denote the direct sum $F^m$ of $m$-copies of a coherent sheaf $F$ by $mF$ if there is no confusion. The boldface letters $R, M, K, N$ refer (coarse) moduli or parameter spaces.

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2. Preliminaries

In this section, we review definitions and several well-known properties of moduli spaces we will discuss.

2.1. Moduli space of semistable sheaves. Let $X$ be a smooth projective variety with fixed polarization $L$. For a coherent sheaf $F$ on $X$, the Hilbert polynomial $P(F)(m)$ is defined by $\chi(F \otimes L^m)$. If the support of $F$ has dimension $d$, $P(F)(m)$ has degree $d$ and it can be written as

$$P(F)(m) = \sum_{i=0}^{d} a_i \frac{m^d}{d!}.$$ 

The coefficient $r(F) := a_d$ is called the multiplicity. the reduced Hilbert polynomial is $p(F)(m) := P(F)(m)/r(F)$. A sheaf $F$ is semistable if

- $F$ is pure;
- for every nonzero proper subsheaf $F' \subset F$, $p(F')(m) \leq p(F)(m)$ for $m \gg 0$.

We say $F$ is stable if the inequality is strict. For each semistable sheaf $F$, there is a filtration (so called Jordan-Hölder filtration) $0 = F_0 \subset F_1 \subset \cdots \subset F_n = F$ such that $\text{gr}_i(F) := F_i/F_{i-1}$ is stable and $p(F)(m) = p(\text{gr}_i(F))(m)$ for all $i$. Finally, two semistable sheaves $F_1$ and $F_2$ are $S$-equivalent if $\text{gr}(F_1) \cong \text{gr}(F_2)$ where $\text{gr}(F) := \oplus_i \text{gr}_i(F)$.

In his monumental paper [32], Simpson proved that there is a projective coarse moduli space $M_1(X, P(m))$ of $S$-equivalent classes of semistable sheaves of fixed Hilbert polynomial $P(m)$. There is an open subset $M_1(X, P(m))$ parametrizing stable sheaves. For a great introduction and the details of proofs and related results, see [18].

In this paper, we will focus on two examples.

Let $Q \subset \mathbb{P}(V)$ be a smooth quadric surface. Since $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, we may denote a line bundle as $O_Q(a, b)$ for two integers $a, b$. Take $L = O_Q(1, 1)$ as a polarization.

Consider $M_1(Q, 4m + 2)$. This moduli space has several connected components, which are parametrized by $\beta \in H_2(Q, \mathbb{Z})$ which represent the scheme theoretic
support of $F$. Thus we may write

$$M_L(Q, 4m + 2) = \bigsqcup_{\beta \in H_2(Q, \mathbb{Z})} M_{L}(Q, \beta, 4m + 2).$$

**Definition 2.1.** Let $M_2 := M_L(Q, c_1(O_Q(2, 2)), 4m + 2)$. So $M_2$ is the moduli space of $S$-equivalent classes of semistable torsion sheaves of multiplicity 4, of the support class $c_1(O_Q(2, 2))$ and $\chi = 2$. It is a $c_2^2 + 1 = 9$-dimensional variety (see [23, Proposition 2.3]).

**Definition 2.2.** Let $R := M_{\mathcal{O}_{P^3}(1)}(P^3, m^2 + 3m + 2)$. It is a 9-dimensional variety.

2.2. **Moduli space of stable pairs.** To analyze some geometric properties of $M_2$, we will use the moduli space of stable pairs. As in the previous section, let $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth quadric surface and let $L = O_Q(1, 1)$ be a polarization. A pair $(s, F)$ consists of a coherent sheaf $F$ on $Q$ and a nonzero section $O_Q \rightarrow F$. Fix a positive rational number $\alpha$. A pair $(s, F)$ is called $\alpha$-semistable if $F$ is pure and for any subsheaf $F' \subset F$, the inequality

$$\frac{P(F')(m) + \delta \cdot \alpha}{r(F')} \leq \frac{P(F)(m) + \alpha}{r(F)}$$

holds for $m \gg 0$. Here $\delta = 1$ if the section $s$ factors through $F'$ and $\delta = 0$ otherwise. When the strict inequality holds, $(s, F)$ is called as a $\alpha$-stable pair. As in the case of sheaves, we can define Jordan-Hölder filtration and $S$-equivalent classes of pairs.

There exists a projective scheme $M^\alpha_P(Q, P(m))$ parameterizing $S$-equivalence classes of $\alpha$-semistable pairs with Hilbert polynomial $P(m)$ ([24, Theorem 4.12]). As in the case of the moduli space of semistable sheaves, we have a decomposition of the moduli space

$$M^\alpha_P(Q, 4m + 2) = \bigsqcup_{\beta \in H_2(Q, \mathbb{Z})} M_{\alpha,L}(Q, \beta, 4m + 2).$$

We denote the moduli space $M^\alpha_P(Q, c_1(O_Q(2, 2)), 4m + 2)$ by $M^{\alpha,P}_2$. The extremal case that $\alpha$ is sufficiently large (resp. small) is denoted by $\alpha = \infty$ (resp. $\alpha = 0^+$). The deformation theory of pairs has been studied by many authors. For our purpose, see [17, Corollary 1.6 and Corollary 3.6].

2.3. **Kontsevich’s moduli space of stable maps.** For a smooth projective variety $X$, fix a Chow class $\beta \in A_1(X, \mathbb{Z})$. Let $(C, x_1, \cdots, x_n)$ be a projective connected reduced curve with $n$ marked points. A map $f : (C, x_1, \cdots, x_n) \rightarrow X$ is called stable if

- $C$ has at worst nodal singularities;
- $x_1, \cdots, x_n$ are $n$ distinct smooth points;
- $|\text{Aut}(f)| < \infty$, or equivalently, for a polarization $L$ on $X$, $\omega_C(\sum x_i) \otimes f^*L^{\otimes 3}$ is ample.

Let $\mathcal{M}_{g,n}(X, \beta)$ be the moduli stack of $n$-pointed stable maps with $g(C) = g$, $f_*[C] = \beta$. It is well-known that $\mathcal{M}_{g,n}(X, \beta)$ is a proper Deligne-Mumford stack and its coarse moduli space $M_{g,n}(X, \beta)$ is a projective scheme ([16, Theorem 1]).

In this paper, let $K := M_{0,0}(\text{Gr}(2, 4), 2L)$ and let $K_1 := M_{0,1}(\text{Gr}(2, 4), 2L)$ where $L$ be the effective generator of $A_1(\text{Gr}(2, 4), \mathbb{Z})$. By [20, Corollary 1], they are normal irreducible varieties. There is a natural forgetful map $\pi : K_1 \rightarrow K$ which forgets the marked point. One can compute the dimension of $K$ (resp. $K_1$), which is 9 (resp. 10), by using [16, Theorem 2].
3. Geometry of $M_2$

In this section, we compute some numerical invariants of $M_2$ by using the variation of moduli space of stable pairs.

3.1. Wall crossing and geometry of $M_2$. Recall that $M_2^\infty$ is the moduli space of $\alpha$-semistable pairs on $Q$ with Hilbert polynomial $4m + 2$ and support class $c_1(\mathcal{O}_Q(2,2))$.

In two extremal cases $\alpha = \infty$ and $\alpha = 0^+$, we have a well-known structure morphisms on $M_2^\infty$.

Lemma 3.1. (1) The moduli space $M_2^\infty$ is isomorphic to the relative Hilbert scheme

$$\text{Hilb}^2(\mathcal{C}/\mathcal{O}_Q(2,2))$$

of two points over the universal quartic curve $\mathcal{C} \to |\mathcal{O}_Q(2,2)|$ in $Q$. The latter space is a $\mathbb{P}^1$-bundle over the Hilbert scheme $\text{Hilb}^2(Q)$ of two points.

(2) There exists a forgetting map $\phi : M_2^{\alpha^+} \to M_2$ associates $F$ to the pair $(s,F)$.

Proof. The isomorphism in Item (1) is given by the following way (c.f., [30, Proposition B.8] and [17, §4.4]). For the pair $(Z,C) \in \text{Hilb}^2(\mathcal{C}/\mathcal{O}_Q(2,2))$, we associates a $\infty$-stable pair $(s,1^2_{Z,C} := \mathcal{E}xt^1_Q(1_{Z,C},\omega_Q)) \in M_2^\infty$ where $s$ arises as the following way. Let us take the dual $\mathcal{E}xt_Q(\omega_Q)$ in the short exact sequence $0 \to I_{Z,C} \to \mathcal{O}_C \to \mathcal{O}_Z \to 0$. Then we obtain a short exact sequence

$$0 \to \mathcal{E}xt^1_Q(\mathcal{O}_C,\omega_Q) \to 1^2_{Z,C} \to \mathcal{E}xt^2_Q(\mathcal{O}_Z,\omega_Q) \to 0,$$

where the first terms are zero because $\mathcal{O}_Z$ is supported on a zero dimensional scheme and the last one is zero because of the pureness of $\mathcal{O}_C$. Note that the length $l(\mathcal{E}xt^2_Q(\mathcal{O}_Z,\omega_Q)) = 2$. But one can easily see that $\mathcal{E}xt^1_Q(\mathcal{O}_C,\omega_Q) \cong \mathcal{O}_C$ by using the resolution of $\mathcal{O}_C$. As combining with the restriction map $\mathcal{O}_Q \to \mathcal{O}_C$ with the second map in equation (3.1), we have a canonical section $s : \mathcal{O}_Q \to 1^2_{Z,C}$.

The second part of Item (1) comes from the fact that $h^0(1_{[p,q]} \otimes \mathcal{O}_Q(2,2)) = 7$ for any two points $(p,q)$ on $Q$ (possibly $p = q$) (See [6, Example 6.1]).

Let us choose $\alpha < \frac{1}{1}$. Then one can easily check that if $(s,F)$ is a $\alpha$-stable pair, then $F$ is semistable. Hence the forgetful functor $(s,F) \mapsto F$ indeed defines a map between moduli spaces $\phi : M_2^{\alpha^+} \to M_2$. Thus we have (2). \hfill $\square$

Now let us study the wall crossing when the stability parameter $\alpha$ decreases. We denote the pair $(s,F)$ by $(1,F)$ if the section $s \in H^0(F)$ is non-zero and $(0,F)$ otherwise.

Proposition 3.2. There is a simple wall crossing among

$$M_2^\infty \leftarrow \cdots \leftarrow \cdots \leftarrow M_2^{\alpha^+}$$

at $\alpha = 2$. When $\alpha = 2$, the modified locus is given by the subvariety parametrizing $(1,F) = (1,\mathcal{O}_C) \oplus (0,\mathcal{O}_L)$ for a rational cubic curve $C$ of the class $\mathcal{O}_Q(2,1)$ (resp. $\mathcal{O}_Q(1,2)$) and a line $L$ of the class $\mathcal{O}_Q(0,1)$ (resp. $\mathcal{O}(1,0)$). Furthermore, the moduli space $M_2^{\alpha^+}$ is a smooth variety of dimension 10.
Let \((1,F) \in \mathbb{P}(\text{Ext}^1_Q((1,\mathcal{O}_C),(0,\mathcal{O}_L))) \cong \mathbb{P}^2\) be the \(\infty\)-stable pairs. That is, \((s,F)\) fits into the exact sequence \(0 \rightarrow (0,\mathcal{O}_L) \rightarrow (1,F) \rightarrow (1,\mathcal{O}_C) \rightarrow 0\). Geometrically, the pairs \((1,F)\) parameterizes the pair of subschemes \((Z,L \cup C)\) such that \(Z\) lies on the line \(L\). After the wall crossing, these extension changes the sub pairs and the quotients one. Hence we obtain \(0^+\)-stable pairs \((1,G) \in \mathbb{P}(\text{Ext}^1_Q((0,\mathcal{O}_L),(1,\mathcal{O}_C))) \cong \mathbb{P}^1\).

**Proof of Proposition 3.2.** Consider a pair \((s,F)\) on \(Q\) as that on \(\mathbb{P}^3\) by using the embedding \(Q \subset \mathbb{P}^3\). From an upper bound of the dimension of the space of global sections of semistable sheaves in \(\mathbb{P}^3\) ([9, Theorem 1.1]), one can see that the possible wall is one of the followings:

1. \((1,F_{4m+2}) = (1,F_{3m}) \oplus (0,F_{m+2})\) or \((2m)\)
2. \((1,F_{4m+2}) = (1,F_{3m+1}) \oplus (0,F_{m+1})\).

Here \(F_k\) means a semistable sheaf with Hilbert polynomial \(k\).

But the case (1) cannot occur because \(F_{3m} \cong \mathcal{O}_C\) for some plane cubic curve \(C\) and thus it is not contained in the quadric surface \(Q\). Hence the wall occurs if and only if \((1,F_{4m+2}) = (1,F_{3m+1}) \oplus (0,F_{m+1})\). By the classification of stable sheaves with small Hilbert polynomials in \(\mathbb{P}^3\) ([15]), we know that \(F_{3m+1} \cong \mathcal{O}_C\) and \(F_{m+1} = \mathcal{O}_L\) for some twisted cubic curve \(C\) and a line \(L\).

Let us prove that \(M_2^+\) is a smooth variety. Since the complement of the wall crossing locus in \(M_2^+\) is isomorphic to the corresponding locus in \(M_2\) and \(\chi(\text{Ext}^*Q((1,F),(1,F))) = 10\), it is enough to check that the obstruction space \(\text{Ext}^2(Q((1,F),(1,F)))\) vanishes for the pairs \((1,F)\) in the wall crossing locus of \(M_2^+\) ([17, Corollary 3.10, Theorem 3.12]). The pair \((1,F)\) fits into a nontrivial extension \(0 \rightarrow (1,\mathcal{O}_C) \rightarrow (1,F) \rightarrow (0,\mathcal{O}_L) \rightarrow 0\). This implies that \(H^1(F) = 0\) and \(F\) is a semistable sheaf on \(Q\). From [17, Corollary 1.6], the obstruction space appears in the exact sequence

\[
\text{Hom}(C,H^1(F)) \rightarrow \text{Ext}^2(Q((1,F),(1,F))) \rightarrow \text{Ext}^2(F,F) \cong \text{Ext}^0(F,F(-2,-2))^*\]

where the last isomorphism is given by the Serre duality. But \(\text{Ext}^0(F,F(-2,-2)) = 0\) because of the semistability of \(F\). \(\square\)

**Lemma 3.3.** Let \(F \in M_2 \setminus M_2^+\) be a polystable sheaf, that is, a semistable sheaf satisfying \(F = \text{gr}(F)\). Then \(F \cong \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2}\) for two conics \(C_1\) and \(C_2\) on \(Q\).

**Proof.** Note that if \(F \in M_2\) is strictly semistable, then the destabilizing subsheaf must have the Hilbert polynomial \(2m+1\), because this is the only nontrivial integer valued polynomial dividing \(4m+2\). The only semistable sheaf with the Hilbert polynomial is \(\mathcal{O}_{C_1}\) for some conic \(C_1\) in \(Q\). The quotient sheaf also has the Hilbert polynomial \(2m+1\) and hence it is \(\mathcal{O}_{C_2}\) for some conic \(C_2\) in \(Q\). Therefore \(F \cong \text{gr}(F) \cong \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2}\). \(\square\)

**Corollary 3.4.** [3, Corollary 23] The strictly semistable locus \(M_2 \setminus M_2^+\) is isomorphic to \(\mathbb{P}^3 \times \mathbb{P}^3/\mathbb{Z}_2\).

In the next proposition we compute the fibers of the last forgetful map \(\phi : M_2^+ \rightarrow M_2\).

**Proposition 3.5.** Let \(\phi : M_2^+ \rightarrow M_2\) be the forgetful map \((s,F) \mapsto F\).

1. Over \(M_2^+\), \(\phi\) is a \(\mathbb{P}^1\)-bundle.
(2) Let $D \subset \mathbb{P}^3 \times \mathbb{P}^3/\mathbb{Z}_2$ be the diagonal. Over $(\mathbb{P}^3 \times \mathbb{P}^3/\mathbb{Z}_2 - D) \subset M_2 \setminus M_2^\circ$, $\phi$ is a $\mathbb{P}^2 \cup_{p=q} \mathbb{P}^2$-bundle over $\mathbb{P}^3 \times \mathbb{P}^3/\mathbb{Z}_2 - D$, where $\mathbb{P}^2 \cup_{p=q} \mathbb{P}^2$ is the reducible variety obtained by gluing two $\mathbb{P}^2$’s along two closed points $p$ and $q$. On $(\mathbb{P}^2 - (p)) \cup (\mathbb{P}^2 - (q))$, the fiber parametrizes extensions such that $(1, F) \in \mathbb{P} \operatorname{Ext}^1((1, \mathcal{O}_{C_1}), (0, \mathcal{O}_{C_2})) \cong \mathbb{P}^2$ for $C_1 \neq C_2 \in |\mathcal{O}_Q(1, 1)| = \mathbb{P}^3$. The gluing point $p = q$ parametrizes the unique pair $(s, \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2})$ for conics $C_1 \neq C_2$.

(3) Over $D \cong \mathbb{P}^3 = |\mathcal{O}_Q(1, 1)| \subset M_2 \setminus M_2^\circ$, $\phi$ is a $\mathbb{P}^2$-bundle parametrizing pairs $(1, F) \in \mathbb{P} \operatorname{Ext}^1((1, \mathcal{O}_C), (0, \mathcal{O}_C)) \cong \mathbb{P}^2$ for $C \in |\mathcal{O}_Q(1, 1)|$.

Proof. Over the stable locus $M_2^\circ$, there exists a universal family $F$ on $M_2^\circ \times Q$. Then the projectivized direct image sheaf $\mathbb{P}(p_* F)$ on $M_2^\circ$ parameterizes the $0^+\pi$-stable pairs $(s, F)$ where $p : M_2^\circ \times Q \to M_2^\circ$ is the first projection map. Hence, $\mathbb{P}(p_* F) \subset M_2^\circ$. This implies that, if $F$ is stable, then the fiber $\phi^{-1}(F)$ is $\mathbb{P}^1$ because $h^0(F) = 2$. This proves Item (1).

Suppose that $(s, F)$ is in the inverse image of the strictly semistable locus. We may assume that $\operatorname{gr}(F) = [\mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2}]$ for two conics $C_1, C_2 \subset |\mathcal{O}_Q(1, 1)|$. First assume that $C_1 \neq C_2$ and $F \not\cong \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2}$. Then by [17, Corollary 1.6], there exists a long exact sequence

$$\operatorname{Ext}^0(\mathcal{O}_{C_1}, \mathcal{O}_{C_2}) \to \operatorname{Hom}(\mathcal{C}, H^0(\mathcal{O}_{C_2})) \to \operatorname{Ext}^1((1, \mathcal{O}_{C_1}), (0, \mathcal{O}_{C_2})) \to \operatorname{Ext}^1(\mathcal{O}_{C_1}, \mathcal{O}_{C_2}) \to \operatorname{Hom}(\mathcal{C}, H^1(\mathcal{O}_{C_1})).$$

Since $C_1 \neq C_2$, we have $\operatorname{Ext}^0(\mathcal{O}_{C_1}, \mathcal{O}_{C_2}) = 0$. Also, $\operatorname{Hom}(\mathcal{C}, H^0(\mathcal{O}_{C_2})) \cong \mathbb{C}$ and $\operatorname{Hom}(\mathcal{C}, H^1(\mathcal{O}_{C_2})) = 0$. By using the standard resolution of conics, we can compute that $\operatorname{Ext}^1(\mathcal{O}_{C_1}, \mathcal{O}_{C_2}) \cong \mathbb{C}^2$. Thus we obtain a short exact sequence

$$0 \to \operatorname{Hom}(\mathcal{C}, H^0(\mathcal{O}_{C_2})) \cong \mathbb{C} \to \operatorname{Ext}^1((1, \mathcal{O}_{C_1}), (0, \mathcal{O}_{C_2})) \to \operatorname{Ext}^1(\mathcal{O}_{C_1}, \mathcal{O}_{C_2}) \cong \mathbb{C}^2 \to 0.$$

The extension classes $\mathbb{P}(\operatorname{Ext}^1((1, \mathcal{O}_{C_1}), (0, \mathcal{O}_{C_2}))) \cong \mathbb{P}^2$ except $\mathbb{P}(\operatorname{Hom}(\mathcal{C}, H^0(\mathcal{O}_{C_2}))) \cong \mathbb{P}(\mathcal{C}) = pt$ (which is the trivial extension of $F$) parameterizes stable pairs whose underlying sheaves are nontrivial extensions in the fiber $\phi^{-1}(F)$.

Finally, we prove (3). Let $\operatorname{gr}(F) = [\mathcal{O}_C \oplus \mathcal{O}_C]$ for a conic $C \in |\mathcal{O}_Q(1, 1)|$. Then by the same computation in the case (2), one can see that $\mathbb{P} \operatorname{Ext}^1((1, \mathcal{O}_C), (0, \mathcal{O}_C)) \cong \mathbb{P}^2$.

3.2. Some numerical invariants. By using the wall-crossing results in the previous section, we can compute some numerical invariants of $M_2$.

Proposition 3.6. The rank of the Picard group of $M_2$ is $3$.

Proof. For a smooth projective surface $X$ with $q(X) = 0$, it is well-known that $\operatorname{rank} \operatorname{Pic}(\operatorname{Hilb}^n(X)) = \operatorname{rank} \operatorname{Pic}(X) + 1$. So $\operatorname{rank} \operatorname{Pic}(\operatorname{Hilb}^2(Q)) = 3$. Then by Item (1) of Proposition 3.2, we know that $\operatorname{rank} \operatorname{Pic}(M_2^\circ) = 4$. Note that the space $M_2^\circ$ is isomorphic to $M_2^\circ$ up to codimension $2$. So $\operatorname{rank} \operatorname{Pic}(M_2^\circ) = 4$.

By Item (1) of Proposition 3.5, the restriction $M_2^\circ|_{\phi^{-1}(M_2^\circ)}$ is a $\mathbb{P}^1$-bundle over $M_2^\circ$. Also from the dimension counting using the explicit description of the base
and fibers in Items (2) and (3) of the same proposition, codim_{M_2}(M_2^2)^c ≥ 2 and codim_{M_2^0} φ^{-1}(M_2^2)^c ≥ 2. Thus we can conclude that

\[ \text{rank Pic}(M_2^0) = \text{rank Pic}(M_2^0 |_{φ^{-1}(M_2)}) = \text{rank Pic}(M_2^0) + 1 = \text{rank Pic}(M_2) + 1. \]

For a variety X, the virtual Poincaré polynomial of X is defined by

\[ P(X) = \sum (-1)^{i+j} \dim Q \text{gr}_W^i H^j_c(X, \mathbb{Q}) p^{i/2}, \]

where gr_W^i H^j_c(X, \mathbb{Q}) is the j-th weight-graded piece of the mixed Hodge structure on the i-th cohomology of X with compact supports. Since odd cohomology groups of moduli spaces of our interest always vanish, P(X) is a polynomial. For the motivic properties of the virtual Poincaré polynomial, see [26].

The classification of stable pairs in Proposition 3.5 enables us to compute the virtual Poincaré polynomial of the space M_2, because we have a stratification of the space M_2^0 in term of Zariski locally trivial fibrations over each base space.

**Corollary 3.7.** The virtual Poincaré polynomial of M_2 is given by

\[ P(M_2) = p^9 + 3p^8 + 4p^7 + 3p^6 + 3p^5 + 2p^4 + 3p^3 + 3p^2 + 3p + 1. \]

In particular, the virtual Euler number is e(M_2) = 26.

**Proof.** By Proposition 3.2,

\[ P(M_2^0) = P(M_2^0) + 2P(P^1) - 2P(P^2))P(P^1) = P(Hilb^2(Q)) · P(P^1) + 2P(P^1) - 2P(P^2))P(P^1). \]

But P(Hilb^2(Q)) = 1/2 (P(Q)^2 - P(Q)) + P(Q) · P(P^1). Hence

\[ P(M_2^0) = p^{10} + 4p^9 + 8p^8 + 9p^7 + 10p^6 + 10p^5 + 9p^4 + 8p^3 + 8p^2 + 4p + 1. \]

On the other hand, by the proof of Proposition 3.5, we have

\[ P(M_2^0) = P(P^1)P(M_2^0) + (2P(P^2) - 1)(P(Sym^2P^3) - P(P^3)) + P(P^2) · P(P^3) \]

and thus obtain the virtual Poincaré polynomial P(M_2^0). Finally,

\[ P(M_2) = P(M_2^0) + P(Sym^2P^3). \]

\[ \square \]

### 3.3. Free resolution of elements of M_2

Another application of the wall-crossing analysis is the classification of free resolutions of elements in M_2. We will use the result of this section in Section 5. Through the wall crossing, let us present the resolution of all of the semistable sheaves in M_2 which we will use later.

**Proposition 3.8.** Let F ∈ M_2. Then F has one of the following free resolutions:

1. \( 0 \to 2O_Q(-1, -1) \to 2O_Q \to F \to 0; \)
2. \( 0 \to O_Q(-1, -2) \to O_Q(1, 0) \to F \to 0; \)
3. \( 0 \to O_Q(-2, -1) \to O_Q(0, 1) \to F \to 0. \)
Proof. We describe all possible resolutions of underlying sheaves, for stable pairs $(s,F) \in M^3_2$. If a pair $(s,F)$ does not belong to the wall crossing locus in $M^3_2$, then $F \cong \oplus D[p,q]$, for some $p,q \in \mathbb{C}$ ([30, Proposition B.8] and [17, §4.4]). Here $D$ denotes the dual $G^D := \mathcal{E}xt^1_Q(G, \omega_Q)$. If the line $L = \langle p,q \rangle$ generated by $p$ and $q$ is not one of two rulings of $Q$, then $\langle p,q \rangle$ is a complete intersection of two conics in $Q$. Hence we have a short exact sequence $0 \to \mathcal{O}(2,1) \to 2\mathcal{Q}(0,1) \to I_{(p,q),Q} \to 0$. By performing the mapping cone operation to $0 \to I_{C,Q} \cong \mathcal{O}(2,1) \to I_{(p,q),Q} \to I_{(p,q),C} \to 0$, we obtain a free resolution $0 \to 2\mathcal{O}(2,1) \to 2\mathcal{O}(0,1) \to I_{(p,q),C} \to 0$.

By taking the dual $D$, we can obtain the case (1). If the line $L = \langle p,q \rangle$ is a ruling of $Q$, then $I_{(p,q),C} \cong \mathcal{O}(1,0)$ or $\mathcal{O}(1,0)$. By taking the dual operation again, we get the cases (2) or (3).

Now assume that the $0^+$-stable pair $(1,F)$ is contained in the wall crossing locus. Then $(1,F)$ fits into the short exact sequence

$$0 \to (1,\mathcal{O}_C) \to (1,F) \to (0,\mathcal{O}_L) \to 0$$

for a cubic curve $C$ and a line $L$ in $Q$. If $F$ is strictly semistable, then $gr(F) = \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2}$ for two conics $C_1$ and by the horseshoe lemma type argument, $F$ has the resolution of the form (1). Now assume that $F$ is stable. Then every pair $(s,F)$ with a nonzero section $s \in H^0(F) = \mathbb{C}^2$ is $0^+$-stable. Also, the pair $(1,F)$ in our consideration is exactly given by the unique (up to scalar) nonzero section $1 \in H^0(\mathcal{O}_C) = \mathbb{C} \subset H^0(F)$. We can construct a flat family of pairs such that the underlying sheaf is constant but the section does vary. Let

$$0 \to (0,\mathcal{O}_C) \to (s,F) \to (s_t,\mathcal{O}_L) \to 0$$

be the non-split extension such that $\lim_{t \to 0}(s_t,F) = (1,F)$ for $s_t \in H^0(F) \setminus H^0(\mathcal{O}_C)$. Then obviously $(s_t,F)$ for all $t \neq 0$ is contained in the complement of the wall crossing locus. Hence as we discussed before, the possible resolutions of the underlying sheaf $F$ is given by the list. This proves the claim.

**Definition 3.9.** Let $D_{1,0}$ (resp. $D_{0,1}$) be the locus of semistable sheaves $F \in M_2$ having the resolution of type (2) (resp. type (3)) of Proposition 3.8. Both of them are isomorphic to the complete linear space $|\mathcal{O}(2,2)| \cong \mathbb{P}^8$ because the isomorphism class of sheaves completely determined by the support. Note that $D_{1,0} \cup D_{0,1} \subset M^3_2$.

**Remark 3.10.** By using the language of derived categories, a point in the loci $D_{1,0}$ or $D_{0,1}$ can be described by the extension classes. If $F \in D_{1,0}$ (similarly for $D_{0,1}$), then $F$ fits into the exact triangle

$$0 \to \mathcal{O}(1,0) \to F \to \mathcal{O}(-1,2)[1] \to 0$$

Conversely, every non-trivial triangle of such form is in the locus $D_{1,0}$ because $\text{Ext}^1(\mathcal{O}(-1,2)[1], \mathcal{O}(1,0)) \cong \text{Ext}^1(\mathcal{O}(-1,2), \mathcal{O}(1,0)) = H^0(\mathcal{O}(2,2)) = \mathbb{C}^9$. As assuming that these one forms one of the walls in the sense of Bridgeland, after the wall crossing, we have a unique non-trivial triangle

$$0 \to \mathcal{O}(-1,2)[1] \to G \to \mathcal{O}(1,0) \to 0$$

because $\text{Ext}^1(\mathcal{O}(1,0), \mathcal{O}(-1,2)[1]) \cong \text{Ext}^2(\mathcal{O}(1,0), \mathcal{O}(-1,2)) = \mathbb{C}$. Compare the $\mathbb{P}^2$ case in [7, §6] and [25, §6]. In Section 5, we will describe such a phenomenon in terms of FM transforms ([23, §4]).
4. Rationality of $M_2$

In this short section, we answer the rationality question. In Sections 5 and 6, we will provide a more theoretical description of the birational map. But in this section, we leave a very elementary proof of the rationality of $M_2$ for the readers who are interested in the description of the birational map in classical terms.

**Proposition 4.1.** Two spaces $K$ and $M_2$ are birationally equivalent.

**Proof.** It is enough to construct an injective map on some Zariski dense open subset of the space $K$ since both varieties have the same dimension and irreducible. We begin with $K_1 := M_{0,1}(Gr(2,4), 2\mathcal{L})$, the moduli space of 1-pointed stable maps. Let $K_0$ be the open subset parametrizing smooth 1-pointed conics in $Gr(2,4)$ such that 1) it generates a smooth quadric in $\mathbb{P}(V) = \mathbb{P}^3$ and 2) the smooth quadric is not equal to $Q$. And let $K^0 := \pi(K_0)$ for the forgetful map $\pi$. For each stable map $[f: (\mathbb{P}^3, x) \rightarrow Gr(2,4)] \in K^0$, we have a pair $(Q', \ell)$ of a quadric surface in $\mathbb{P}^3$ and a line $\ell \subset Q'$ obtained by a pointed conic in $Gr(2,4)$. Clearly, it is well-defined on families.

Define $C := Q' \cap Q$. Then $C$ is a quartic curve since $Q' = Q$. Let $[p, q] = \ell \cap Q$ be the set of intersection points. Then the line bundle $\mathcal{L} := \mathcal{O}_C(p + q)$ on $C$ is a stable sheaf on $Q$. Therefore we have a map $\psi : K^0 \rightarrow M_2$.

A different choice of a marked point $x \in \mathbb{P}^3$ defines a different line $\ell$ in the same ruling class. But the line bundle $\mathcal{L}$ does not depend on the choice of $x$, because the natural map $\mathbb{P}^3 \rightarrow Pic^2(C)$ is constant since the image is a smooth elliptic curve. Therefore the map descends to a map $\tilde{\psi} : K^0 \rightarrow M_2$.

Now we show that $\tilde{\psi}$ is injective. Let $F_1 := \mathcal{O}_{C_1}(p_1 + q_1) \equiv \mathcal{O}_{C_2}(p_2 + q_2) : \mathcal{F}_2$ where $C_1 = Q \cap Q_1$ and $\{p_i, q_i\} = Q \cap Q_i$ for $i = 1, 2$. Also let $\ell_i = \langle p_i, q_i \rangle$. Obviously $C_1 = C_2$, so let $C := C_1 = C_2$. Then $\langle Q, Q_1, Q_2 \rangle \subset H^0(\mathcal{I}_C(2)) = \mathbb{C}^2$. So the set $\{Q, Q_1, Q_2\}$ is linearly dependent. Without loss of generality, we may assume that

\[ Q_1 = aQ_2 + bQ. \]

Note that $a \neq 0$ because $Q_1 \neq Q$. If $b \neq 0$, we have $\{p_1, q_1\} \subset Q_2$. If $\ell_1 = \langle p_1, q_1 \rangle \subset Q_2$, then $\ell_1 \subset Q$ since $\ell_1 \subset Q_1$ and (4.1). This is a contradiction to the fact that $\ell_1 \cap Q = \{p_1, q_1\}$. So $\ell_1 \cap Q_2 = \{p_1, q_1\}$ and $\ell_1$ is not a ruling of $Q_2$. Now, by using their different resolutions of the sheaves $F_1$ on $Q_2$ (see Proposition 3.8), one can check that $F_1 \not\cong F_2$, which makes a contradiction.

Thus $b = 0$ and $Q_1 = Q_2$. By Proposition 3.8 again, if $\ell_1$ and $\ell_2$ are on the different ruling classes, then $F_1 \not\cong F_2$. Thus $\ell_1$ and $\ell_2$ are on the same ruling class and $[Q_1, \ell_1]$ and $[Q_2, \ell_2]$ come from the same stable map. \hfill $\square$

As a consequence, we obtain the following result, which gives an affirmative answer for [3, Conjecture 35].

**Corollary 4.2.** The moduli space $M_2$ is rational.

**Proof.** The moduli space $K$ is birational to $H := Hilb^{2m+1}(Gr(2,4))$, the Hilbert scheme of conics in $Gr(2,4)$. The latter space is birational to $Gr(3,6)$ ([28, Section 3.10]), which is a rational variety. \hfill $\square$
5. Birational map between $M_2$ and $R$ via Fourier-Mukai transforms

5.1. The geometry of the moduli space $R$. Recall that $R$ is the moduli space of semistable sheaves in $\mathbb{P}^3$ with Hilbert polynomial $m^2 + 3m + 2$. Here we collect the well-known properties of $R$ intensively studied in [24, Section 3].

**Proposition 5.1.** [24, Proposition 3.6, Remark 3.8]

1. Every semistable sheaf $F \in R$ has a free resolution
   \[
   0 \rightarrow 2\mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow 2\mathcal{O}_{\mathbb{P}^3} \rightarrow F \rightarrow 0.
   \]

   For the converse, see Lemma 5.2.

2. (a) If $F \in R^s$, then $F \cong I_{L,S}(1)$ for an irreducible quadric surface $S$ and a line $L \subset S$.
   (b) If $F \in R \setminus R^s$, then $\text{gr}(F) = |\mathcal{O}_H \oplus \mathcal{O}_{H'}|$ for planes $H$ and $H'$ in $\mathbb{P}^3$.

3. The map $\pi : R \rightarrow |\mathcal{O}_{\mathbb{P}^3}(2)| \cong \mathbb{P}^9$ defined by the Fitting ideal of $F$ is a double covering ramified along the discriminant divisor $\Delta$. Here, $\Delta$ denotes the locus of singular quadric surfaces in $\mathbb{P}^3$, which is a degree 4 singular hypersurface.

We believe that the following result is observed by Le Potier. But since we could not find the precise statement and its proof, we leave it as a lemma.

**Lemma 5.2.** If $F \in \text{Coh}(\mathbb{P}^3)$ has a resolution
\[
0 \rightarrow 2\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{M} 2\mathcal{O}_{\mathbb{P}^3} \rightarrow F \rightarrow 0,
\]
then $F$ is semistable.

**Proof.** Choose an injective homomorphism $\mathcal{O}_{\mathbb{P}^3}(-1) \subset 2\mathcal{O}_{\mathbb{P}^3}(-1)$. By composing with $M$, we have an injective homomorphism
\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow 2\mathcal{O}_{\mathbb{P}^3}
\]
such that the cokernel is isomorphic to a twisted ideal sheaf $I_{L,\mathbb{P}^3}(1)$ for a line $L$ or $\mathcal{O}_H \oplus \mathcal{O}_{\mathbb{P}^3}$ for a plane $H$.

Case 1. The cokernel is $I_{L,\mathbb{P}^3}(1)$.

By using the snake lemma to
\[
\begin{CD}
0 @>>> \mathcal{O}_{\mathbb{P}^3}(-1) @>>> 2\mathcal{O}_{\mathbb{P}^3} @>>> 0 @>>> 0 \\
@. @VVV @VVV @. \\
0 @>>> 2\mathcal{O}_{\mathbb{P}^3}(-1) @>>> \mathcal{O}_{\mathbb{P}^3} @>>> I_{L,\mathbb{P}^3}(1) @>>> 0,
\end{CD}
\]
we have an exact sequence
\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow I_{L,\mathbb{P}^3}(1) \rightarrow F \rightarrow 0.
\]
From $0 \rightarrow I_{L,\mathbb{P}^3}(1) \rightarrow \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0$, by using the snake lemma again, one can show that $F$ is the kernel of the canonical surjection $\mathcal{O}_{\mathbb{P}^3}(1) \rightarrow \mathcal{O}_{\mathbb{P}^3}(1)$ for a quadric surface $Q'$ determined by the inclusion $\mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow I_{L,\mathbb{P}^3}(1) \rightarrow \mathcal{O}_{\mathbb{P}^3}(1)$.
Therefore $F = I_{L,Q'}(1)$. By Proposition 5.1, $F$ is semistable.

Case 2. The cokernel is $\mathcal{O}_H \oplus \mathcal{O}_{\mathbb{P}^3}$.

By a similar diagram chasing, we have a short exact sequence
\[
0 \rightarrow \mathcal{O}_H \rightarrow F \rightarrow \mathcal{O}_H' \rightarrow 0
\]
for two planes $H$ and $H'$. Hence, $F$ is semistable because $\text{gr}(F) = \mathcal{O}_H \oplus \mathcal{O}_H'$.

\[\square\]
So we have the following result.

**Proposition 5.3.** [24, Corollary 3.7] The space $R$ is isomorphic to the space $N(4; 2, 2)$ of Kronecker modules of type $(4; 2, 2)$.

By definition, the space $N(4; 2, 2)$ is constructed as the GIT quotient
$$\mathbb{P} \left( \text{Hom} \left( O_{\mathbb{P}^3}(-1)^2, O_{\mathbb{P}^3}^2 \right) \right) / G \cong \mathbb{P} \left( \text{Hom} \left( \mathbb{C}^2 \otimes V, \mathbb{C}^2 \right) \right) / G \cong \mathbb{P} \left( V^* \otimes gl_2 \right) / G,$$
where $G = \text{PGL}_2 \times \text{PGL}_2$ acts as $(A, B) \cdot M = AMB^{-1}$ ([24, Corollary 3.7]). Therefore $R \cong N(4; 2, 2)$ is a normal variety. For this action, there are strictly semistable points and the GIT quotient is singular. In the next section, we will discuss a systematic (partial) resolution of this singularities, so called Kirwan’s paritial desingularization ([21]).

As a simple corollary of Proposition 5.1, we have:

**Corollary 5.4.** The canonical divisor $K_R$ is $-8\pi^* O_{\mathbb{P}^9}(1)$.

**Proof.** The singular locus $\text{Sing}(\Delta)$ of $\Delta$ is isomorphic to $\mathbb{P}^3 \times \mathbb{P}^3 / \mathbb{Z}_2$. Thus the codimension of $\text{Sing}(\Delta)$ in $\Delta$ is two. Hence, if we apply the covering formula for the map $\pi$, then we have
$$K_R = \pi^* K_{\mathbb{P}^9} + \frac{1}{2} [\Delta].$$
Since $[\Delta] = 4\pi^* O_{\mathbb{P}^9}(1)$, we have the result. \qed

5.2. **A divisorial contraction of $M_2$.** In this section, by applying the method developed in [23, Section 4], we construct a birational morphism $\Psi : M_2 \rightarrow R$ by using FM transforms.

Let $\Delta_1$ be the universal conic in $Q \subset \mathbb{P}(V) = \mathbb{P}^3$. Let $\Delta_2$ be the space of the universal planes in $(\mathbb{P}^3)^*$. We have two diagrams:

$$\Delta_1 \subset (\mathbb{P}^3)^* \times Q \xrightarrow{p} Q$$
$$\downarrow q$$
$$([\mathbb{P}^3]^* \cong |O_Q(1, 1)|$$

and

$$\Delta_2 \subset \mathbb{P}^3 \times (\mathbb{P}^3)^* \xrightarrow{r} (\mathbb{P}^3)^*$$
$$\downarrow s$$
$$\mathbb{P}^3,$$
where $p, q, r, s$ indicate the projection maps onto each factor.

**Definition 5.5.** Let
$$\Psi_1(F) := q_*(O_{\Delta_1} \otimes p^*(F \otimes O_Q(1, 1))) =: F'$$
and
$$\Psi_2(F') := R^2s_*(O_{\Delta_2} \otimes r^*(F' \otimes O_{\mathbb{P}^3}(-3))).$$

**Remark 5.6.** As we will see below in the the proof of Theorem 5.7, for those two transforms the other higher direct image sheaves vanish and thus the $\Psi_i$’s can be regraded as the FM transforms with the kernel $O_{\Delta_1}$ on derived categories.

Now we prove the first main theorem of this paper.
Theorem 5.7. There exists a birational morphism

$$
\Psi : M_2 \to R.
$$

given by $F \mapsto \Psi(F) := \Psi_2(\Psi_1(F))$. The map $\Psi$ contracts two divisors $D_{1,0}$ and $D_{0,1}$ (Definition 3.9) to two points $\Psi(D_{1,0}) = ([\mathcal{O}_Q(1,0)])$, $\Psi(D_{0,1}) = ([\mathcal{O}_Q(0,1)])$. Furthermore, $\Psi$ is a smooth blow-up of two points $[\mathcal{O}_Q(1,0)]$ and $[\mathcal{O}_Q(0,1)]$.

Proof. Step 1. For any $F \in M_2$ with the resolution of type (1) of Proposition 3.8, $\Psi(F) \in R$.

If $F$ has the resolution of type (1) of Proposition 3.8, by tensoring $\mathcal{O}_Q(1,1)$, we have

$$
0 \to 2\mathcal{O}_Q \to 2\mathcal{O}_Q(1,1) \to F(1,1) \to 0.
$$

First we compute the direct image sheaves $R^i\mathcal{q}_* (\mathcal{O}_{\Delta_i} \otimes p^* \mathcal{O}_Q)$ and $R^i\mathcal{q}_* (\mathcal{O}_{\Delta_i} \otimes p^* \mathcal{O}_Q(1,1))$. The resolution of $\mathcal{O}_{\Delta_i}$ is given by

$$
0 \to \mathcal{O}_{(P^3)_i} \times \mathcal{O}(-1, (1, 1)) \to \mathcal{O}_{(P^3)_i} \times \mathcal{O} \to \mathcal{O}_{\Delta_i} \to 0.
$$

By tensoring $\mathcal{q}^* \mathcal{O}_{(P^3)_i}$ and taking the direct image functor $R^i\mathcal{p}_*$, we obtain a long exact sequence

$$
0 \to \mathcal{q}_* \mathcal{O}_{(P^3)_i} \times \mathcal{O}(-1, (1, 1)) \to \mathcal{q}_* \mathcal{O}_{(P^3)_i} \times \mathcal{O} \to \mathcal{q}_* \mathcal{O}_{\Delta_i} \otimes p^* \mathcal{O}_Q(1,1) \to 0.
$$

But $H^i(\mathcal{O}_Q(-1, (1,1))) = H^i(\mathcal{O}_Q) = 0$ for all $i$ and $j \geq 1$. Also, $H^0(\mathcal{O}_Q) = \mathbb{C}$. Thus we obtain that $R^i\mathcal{q}_* \mathcal{O}_{(P^3)_i} \times \mathcal{O}(-1, (1, 1)) = 0$ for all $i$ and $R^i\mathcal{q}_* \mathcal{O}_{(P^3)_i} \times \mathcal{O} = 0$ for all $i$. Also $\mathcal{q}_* \mathcal{O}_{(P^3)_i} \times \mathcal{O} = 0$ for all $i \geq 1$.

One can similarly compute $R^i\mathcal{q}_* (\mathcal{O}_{\Delta_i} \otimes p^* \mathcal{O}_Q(1,1))$. From the resolution (5.3) of $\mathcal{O}_{\Delta_i}$, we have

$$
0 \to \mathcal{q}_* \mathcal{O}_{(P^3)_i} \times \mathcal{O}(-1, (0,0)) \to \mathcal{q}_* \mathcal{O}_{(P^3)_i} \times \mathcal{O}(0, (1,1)) \to \mathcal{q}_* \mathcal{O}_{\Delta_i} \otimes p^* \mathcal{O}_Q(1,1) \to 0.
$$

But $H^i(\mathcal{O}_Q) = \mathbb{C}$ for $i = 0$ and $0$ for otherwise. Also, $H^j(\mathcal{O}_Q(1,1)) = \mathbb{C}^4$ for $j = 0$ and $0$ for otherwise. Hence $R^i\mathcal{q}_* \mathcal{O}_{(P^3)_i} \times \mathcal{O}(-1, (0,0)) = \mathcal{O}_{(P^3)_i}(-1)$ for $i = 0$ and $0$ for otherwise. Also, $\mathcal{q}_* \mathcal{O}_{(P^3)_i} \times \mathcal{O}(0, (1,1)) = 4\mathcal{O}_{(P^3)_i}$, for $j = 0$ and $0$ for otherwise.

By applying these two computations to (5.2), we have a resolution

$$
0 \to \mathcal{O}_{(P^3)_i}(-1) \to 4\mathcal{O}_{(P^3)_i} \to \mathcal{q}_*(\mathcal{O}_{\Delta_i} \otimes p^* \mathcal{O}_Q(1,1)) = Q \to 0.
$$

Therefore, by taking the FM-transform $\Psi_1$, we obtain a short exact sequence

$$
0 \to 2\mathcal{O}_{(P^3)_i}(-1) \to 2\mathcal{O}_{(P^3)_i} \to \Psi_1(F) \to 0.
$$

Note that $R^i\mathcal{q}_* (\mathcal{O}_{\Delta_i} \otimes p^* F(1,1)) = 0$ for all $i \geq 1$.

Take the second FM-transform for $F' := \Psi_1(F)$. By a similar computation using the resolution of $\mathcal{O}_{\Delta_2}$ and the Serre duality,

$$
\Psi_2(\mathcal{O}_{(P^3)_i}) \cong \mathcal{O}_{(P^3)_i}(-1), \quad \Psi_2(Q) \cong \mathcal{O}_{P^3}.
$$

Furthermore, $R^j\mathcal{s}_* (\mathcal{O}_{\Delta_2} \otimes r^*(F'(-3))) = 0$ for $j \neq 2$. Hence from the above short exact sequence, we obtain a resolution

$$
0 \to 2\mathcal{O}_{P^3}(-1) \to 2\mathcal{O}_{P^3} \to \Psi_2(F') \to 0.
$$
But by Lemma 5.2, the sheaf $\Psi_2(F')$ is semistable with Hilbert polynomial $m^2 + 3m + 2$.

**Step 2.** For $F \in M_2$ with a resolution of the type (2) or (3) in Proposition 3.8, $\Psi(F) \in R$.

If $F \in D_{1,0}$, then $F \cong O_C(1,0)$ for some quartic curve $C$. From the resolution

$$0 \to O_Q(0,-1) \to O_Q(2,1) \to F(1,1) \to 0,$$

it is straightforward to check that

$$\Psi_1(F) = q_*(O_{\Delta_1} \otimes p^*O_Q(2,1)).$$

by using a similar computation in the previous step. Note that the image $\Psi_1(F)$ does not depend on the choice of the quartic curve $C$.

Let $i : Q \hookrightarrow \mathbb{P}^3$ be the inclusion and let $j : (\mathbb{P}^3)^* \times Q \to \mathbb{P}^3 \times (\mathbb{P}^3)^*$ be the map defined by $j(x,y) = (y,i(x))$. Then

$$\Psi_2(\Psi_1(F)) = R^2s_*(O_{\Delta_2} \otimes r^*(\Psi_1(F) \otimes O_{(\mathbb{P}^3)^*}(-3)))$$

$$\cong R^2s_*(O_{\Delta_2} \otimes r^*(q_*O_{\Delta_1} \otimes p^*O_Q(2,1)) \otimes O_{(\mathbb{P}^3)^*}(-3)))$$

$$\cong R^2s_*(O_{\Delta_2} \otimes r^*(r_*O_{\Delta_2} \otimes j_*p^*O_Q(2,1)) \otimes O_{(\mathbb{P}^3)^*}(-3)))$$

The third isomorphism comes from $q \circ j$ and the fifth isomorphism comes from $s \circ j = i \circ p$. This can be regarded as a composition of two FM-transforms of $i_*O_Q(2,1)$ with the same kernel $O_{\Delta_2}$.

Let

$$0 \to 2O_{\mathbb{P}^3} \xrightarrow{M} 2O_{\mathbb{P}^3}(1) \to i_*O_Q(2,1) \to 0 \tag{5.4}$$

be a resolution of $i_*O_Q(2,1)$ on $\mathbb{P}^3$. By using the resolution of $O_{\Delta_2}$, we obtain a short exact sequence on $\mathbb{P}^3$*

$$0 \to 2O_{\mathbb{P}^3} \to 2Q'' \to r_*(O_{\Delta_2} \otimes s^*(i_*O_Q(2,1))) \to 0,$$

where $Q'' := \text{coker}(O_{(\mathbb{P}^3)^*}(-1) \to 4O_{(\mathbb{P}^3)^*})$. By twisting $O_{(\mathbb{P}^3)^*}(-3)$ and taking the direct image functor $R^q$, we have a short exact sequence

$$0 \to 2O_{\mathbb{P}^3}(-1) \to 2O_{\mathbb{P}^3} \to G \to 0.$$

By Lemma 5.2, $G \in R$. One can also check that $R^q s_*(-)$ for $i \neq 2$ vanish.

Furthermore, we can show that $G \cong O_Q(1,0)$. Indeed, the map $M$ in (5.4) is completely recovered (up to a twisting) after taking two FM-transforms and thus the cokernel also is. This implies that $G(1) = i_*O_Q(2,1)$. Hence $\Psi_2(\Psi_1(F)) \cong O_{\mathbb{P}^3}(-1) \otimes i_*O_Q(2,1) = i_*O_Q(1,0)$.

The case of $F \in D_{0,1}$ can be computed in a similar way.

**Step 3.** The map $\Psi$ is birational.

The inverse map $\Psi_1^{-1} \circ \Psi_2^{-1} : R - [O_Q(1,0), [O_Q(0,1)] \to M_2 - D_{1,0} \cup D_{0,1}$ can be defined by inverse FM-transforms

$$\Psi_1^{-1}(F') := R^2p_*(O_{\Delta_1} \otimes q^*(F'(-3)))$$

$$\Psi_2^{-1}(F'') := r_*(O_{\Delta_2} \otimes s^*(F''(1))).$$

By using the standard resolution of $F'' \in R$ in Proposition 5.1, by the same computation, one can show that $\Psi_2^{-1}(F'') = F'$ and $\Psi_1^{-1}(F') = F$ by the same argument.
in the proof of the claim before. Hence $\Psi$ is injective and thus $\Psi$ is a birational morphism.

**Step 4.** The map $\Psi$ is a smooth blow-up.

Finally, we show that the map $\Psi$ is a smooth blow-up. From the standard deformation theory on the moduli space of sheaves, at the point $[\mathcal{O}_Q(1, 0)] \in R$, the tangent space of $R$ is canonically isomorphic to $\text{Ext}_q^1(\mathcal{O}_Q(1, 0), \mathcal{O}_Q(1, 0))$. We show that $T_{[\mathcal{O}_Q(1, 0)], R} \cong H^0(\mathcal{O}(2, 2)) = \mathbb{C}^9$ by the following argument. Consider the long exact sequence ([11, Lemma 13]):

\[
0 \rightarrow \text{Ext}_q^1(\mathcal{O}_Q(1, 0), \mathcal{O}_Q(1, 0)) \rightarrow \text{Ext}_q^1(\mathcal{O}_Q(1, 0), \mathcal{O}_Q(1, 0)) \rightarrow \\
\text{Ext}_q^0(\mathcal{O}_Q(1, 0), \mathcal{O}_Q(1, 0)) \rightarrow \text{Ext}_q^0(\mathcal{O}_Q(1, 0), \mathcal{O}_Q(1, 0)) \rightarrow \cdots
\]

But $\text{Ext}_q^0(\mathcal{O}_Q(1, 0), \mathcal{O}_Q(1, 0)) = 0$ for $i \geq 1$. So

\[
\text{Ext}_q^0(\mathcal{O}_Q(1, 0), \mathcal{O}_Q(1, 0)) \cong \text{Ext}_q^0(\mathcal{O}_Q(1, 0), \mathcal{O}_Q(1, 0) \otimes \mathcal{N}_{Q/P^2}),
\]

where the latter space is isomorphic to $\text{Ext}_q^0(\mathcal{O}_Q(1, 0), \mathcal{O}_Q(1, 0) \otimes \mathcal{N}_{Q/P^2}) \cong H^0(\mathcal{O}_Q(2, 2))$.

On the other hand, by definition (see Proposition 3.8 and Definition 3.9), the sheaves on the divisor $D_{1,0}$ can be parametrized by

\[
i : \mathbb{P}(\text{Hom}(\mathcal{O}_Q(-1, -2), \mathcal{O}_Q(1, 0))) = \mathbb{P}(T_{[\mathcal{O}_Q(1, 0)], R}) \hookrightarrow M_2.
\]

Hence, we obtain a commutative diagram

\[
\begin{array}{ccc}
D_{1,0} & \xrightarrow{i} & M_2 \\
\downarrow & & \downarrow \Psi \\
[\mathcal{O}_Q(1, 0)] & \xrightarrow{c} & R.
\end{array}
\]

and thus the morphism $\Psi$ is a smooth blow-up morphism of $R$ at the point $[\mathcal{O}_Q(1, 0)]$. The case of $[\mathcal{O}_Q(0, 1)]$ is identical. \hfill \Box

**Corollary 5.8.** The canonical divisor $K_{M_2}$ is given by $\Psi^*K_R + 8(D_{1,0} + D_{0,1})$.

**Remark 5.9.** Recall that the virtual Euler number $e(M_2)$ is 26 (Corollary 3.7). By Theorem 5.7 and the blow-up formula of the virtual Poincaré polynomial, one can easily see that $e(R) = 10$. This can be explained by the following different way. The virtual Euler number of the partial stable part $R^s$ is $e(R^s) = 0$ ([33, Corollary 6.14]). But by Proposition 5.1, the complement $R \setminus R^s$ is isomorphic to the space $\text{Sym}^2(\mathbb{P}^3)$ parameterizing two planes in $\mathbb{P}^3$. Thus $e(R) = e(R^s) + e(\text{Sym}^2(\mathbb{P}^3)) = 10$.

6. **Birational map between $R$ and $K$ via partial desingularization**

Recall that the space $R$ can be constructed as the GIT quotient

\[
\mathbb{P}(\text{Hom}(\mathbb{C}^2 \otimes V, \mathbb{C}^2))/G,
\]

where $G = \text{PGL}_2 \times \text{PGL}_2$ acts as $(A, B) \cdot M = AMB^{-1}$. For this action, there are strictly semistable points and the GIT quotient $R$ is very singular. In this section, we study Kirwan’s partial desingularization ([21]) of $R$, which is a systematic procedure to resolve these singularities $G$-equivariant way.

The next result is the second main theorem of this paper.

**Theorem 6.1.** The partial desingularization of $R$ is isomorphic to $K$. Kontsevich’s moduli space of degree 2 stable maps to the Grassmannian $\text{Gr}(2, 4)$. 
6.1. **A description of the birational map.** The birational map \( \mathbf{R} \to \mathbf{K} \) can be described in the following way. Let

\[
\mathcal{O}_{S \times \mathbb{P}^3}(-1)^2 \stackrel{M}{\to} \mathcal{O}_{S \times \mathbb{P}^3}^2
\]

be a family of Kronecker modules over \( S \). Consider the projectivization \( \mathbb{P}(\mathcal{O}_{S \times \mathbb{P}^3}(-1)^2) \cong \mathbb{P}(\mathcal{O}_{S \times \mathbb{P}^3}^2) \cong S \times \mathbb{P}^3 \times \mathbb{P}^1 \) and let \( p : S \times \mathbb{P}^3 \times \mathbb{P}^1 \to S \times \mathbb{P}^1 \) and \( q : S \times \mathbb{P}^3 \times \mathbb{P}^1 \to S \times \mathbb{P}^3 \) be two projections. By taking the pull-back, we have

\[
\mathcal{O}_{S \times \mathbb{P}^3 \times \mathbb{P}^1}(-1,0)^2 \cong \mathcal{O}_{S \times \mathbb{P}^3 \times \mathbb{P}^1}^2.
\]

Let \( \iota : \mathcal{O}_{S \times \mathbb{P}^3 \times \mathbb{P}^1}(-1,-1) \to \mathcal{O}_{S \times \mathbb{P}^3 \times \mathbb{P}^1}(-1,0)^2 \) be the tautological subbundle. By taking the dual, we have

\[
\mathcal{O}_{S \times \mathbb{P}^3 \times \mathbb{P}^1}^2 \cong \mathcal{O}_{S \times \mathbb{P}^3 \times \mathbb{P}^1}(1,1).
\]

By taking the push-forward \( p_* \) and \( \otimes \mathcal{O}_{\mathbb{P}^3}(1) \), we obtain

\[
\mathcal{O}_{S \times \mathbb{P}^3}(1,-1)^2 \cong \mathcal{O}_{S \times \mathbb{P}^3}(1,1).
\]

Finally, take its dual again:

\[
\mathcal{O}_{S \times \mathbb{P}^3}(1,-1)^2 \cong \mathcal{O}_{S \times \mathbb{P}^3}(1,1).
\]

For a general fiber, it is an epimorphism to a rank 2, degree 2 bundle, so we have a degree 2 map \( \mathbb{P}^1 \to \text{Gr}(2,4) \). Hence we have a rational family of stable maps

\[
\Phi(M) : S \to \mathbf{K},
\]

which is regular on the locus that \( p_*(q^*M \otimes \iota)^* \otimes \mathcal{O}_{\mathbb{P}^1}(-1)^* \) is surjective.

Therefore there is a rational map

\[
\Phi : \mathbb{P}(\text{Hom}(C^2 \otimes V, C^2))^{ss} \to \mathbf{K}.
\]

It is straightforward to see that \( \Phi \) is \( G \)-invariant, because the \( G \)-action simply changes 1) the coordinate of the fiber \( \mathbb{P}^1 \) and the automorphism of \( \mathcal{O}_{\mathbb{P}^1}(-1)^2 \) \( \iota \). Therefore we have the quotient map

\[
\overline{\Phi} : \mathbf{R} \to \mathbf{K}.
\]

6.2. **GIT stability of the moduli space of Kronecker modules.** The GIT (semi)stability of the moduli space of Kronecker modules is already well-known. For the proof, consult [14, Proposition 15], [1, Remark 4.9], and [32, Proposition 1.14].

**Theorem 6.2.** A closed point \( M \in \mathbb{P}(A \otimes V,B) \) is (semi)stable with respect to the \( G \)-action if and only if for every nonzero proper \( A' \subset A \),

\[
\dim A' (\leq) < \dim \text{im} M(A' \otimes V).
\]

In our special case, we have:

**Corollary 6.3.** Let \( X = \mathbb{P}(C^2 \otimes V,C^2) \cong \mathbb{P}(V \otimes \mathfrak{gl}_2) \) with prescribed linearized \( G \)-action. \( M \in X \) is (semi)stable if and only if for every one dimensional subspace \( A' \subset C^2 \), \( \dim \text{im} \text{F}(A' \otimes G)(\geq) > 1 \).
This stability condition can be described in down-to-earth terms. We can describe \( M \in X \) as a \( 2 \times 2 \) matrix of linear polynomials with four variables \( x, y, z, w \). If \( F \) is semistable, then even after performing row/column operations, there is no zero row nor column. If \( F \) is stable, then for even after any row/column operations, \( F \) has no zero entry. Therefore we have the following results.

**Lemma 6.4.** Let \( M \in X \) be a closed point.

1. If \( M \) is unstable, then

\[
M \in G \cdot \begin{bmatrix} g & h \\ 0 & 0 \end{bmatrix}
\]

for some \( g, h \in V^* \).

2. If \( M \in X^{ss} - X^s \), then

\[
M \in G \cdot \begin{bmatrix} g & k \\ 0 & h \end{bmatrix}
\]

for some \( g, h, k \in V^* \).

3. If \( M \in X^{ss} - X^s \) and \( M \) has a closed orbit in \( X^{ss} \), then \( k = 0 \) and

\[
M \in G \cdot \begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix}
\]

for some \( g, h \in V^* \). If \( g = h \), then \( \text{Aut } M \cong \text{PGL}_2 \). If not, \( \text{Aut } M \cong \mathbb{C}^* \).

In particular, we obtain:

**Lemma 6.5.** The rational map \( \Phi \) in (6.2) is regular on \( X^s \).

**Proof.** By the above construction, for each \( M \in X^s \), \( \Phi(F) \) is given by a rank 2 quotient bundle

\[
\Phi(M) : O_{P^1}(1) \to O_{P^1}(2)
\]

such that for each \( p \in P^1 \), the map is given by a linear combination of coefficients of two rows of \( M \). Since \( M \in X^s \), every linear combination has full rank. Therefore \( \Phi(F) \) is a surjection and it defines a stable map. \( \square \)

### 6.3. Stratification on \( X^{ss} \)

Now we define a stratification

\[
X^{ss} = Y_0 \sqcup Z_0 \sqcup Y_1 \sqcup Z_1 \sqcup X^s
\]

as the following. The last (and general) stratum \( X^s \) is the stable locus.

Let \( Y_0' \) be the image of

\[
\rho_0 : \mathbb{P}(gl_2) \times \mathbb{P}(V^*) \to X, \quad (A, g) \mapsto A \times \begin{bmatrix} g & 0 \\ 0 & g \end{bmatrix}
\]

and let \( Y_0 := Y_0' \cap X^{ss} \). Then \( Y_0 = \rho_0(\text{PGL}_2 \times \mathbb{P}(V^*)) \) and on this locus \( \rho_0 \) is an embedding. So \( Y_0 \) is a smooth closed subvariety of dimension 6. Finally, at each closed point \( M \in Y_0 \), the normal bundle \( N_{Y_0/X^{ss}}|M \) is naturally isomorphic to \( H \otimes \mathfrak{sl}_2 \), where \( H \) is a 3-dimensional quotient space of \( V^* \).

Let \( Z_0' \) be the image of

\[
\tau_0 : G \times \mathbb{P}(V^* \oplus V^*) \times \mathbb{P}^2 \to X, \quad ([A, B], (g, h), [s : t : u]) \mapsto A \begin{bmatrix} sg & uh \\ 0 & tg \end{bmatrix} B^{-1}.
\]

Let \( Z_0 := (Z_0' \cap X^{ss}) \setminus Y_0 \) and let \( Z_0 := Z_0 \sqcup Y_0 \), the closure of \( Z_0 \) in \( X^{ss} \). By computing the dimension of general fiber, one can see that \( Z_0 \) is a 10-dimensional...
irreducible $G$-invariant variety. The normal cone $C_{Y_0}/Z_0$ is an analytic locally trivial bundle, whose fiber at $M \in Y_0$ is isomorphic to $\text{Aut } M \cdot (H \otimes (e)) = \text{Aut } M \cdot (H \otimes (f))$ where $(h, e, f)$ is the standard basis of $sl_2$. In the projectivized normal space $P(H \otimes sl_2)$, $P(C_{Y_0}/Z_0|_M) \cong P \times PGL_2 \cdot P(e) \cong P \times P^1$. Thus $C_{Y_0}/Z_0$ is a degree 2 bundle of dimension 4. Let $Y'_1$ be the image of

$$\rho_1 : G \times P(V^* \oplus V^*) \to X, \quad ((A, B), (g, h)) \mapsto A \begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix} B^{-1}.$$  

Similarly, let $Y_1 := (Y'_1 \cap X^{ss}) \setminus Y_0$ and let $\overline{Y}_1 := Y_0 \cup Y_1$, the closure of $Y_1$ in $X^{ss}$. The general fiber of $\rho_1$ has dimension 2, so $Y_1$ has dimension 11. By computing its general fiber, we can conclude that $\rho_1|_{\rho_1^{-1}(Y_1)}$ is smooth and thus $Y_1$ is smooth, too. Indeed $\overline{Y}_1$ is singular along $Y_0$. For $M \in Y_1$, the fiber of the normal bundle $N_{Y_1/X^{ss}}|_M$ is naturally isomorphic to $K \otimes (e, f)$, where $K$ is a 2-dimensional quotient space of $V^*$. Finally, the normal cone $C_{Y_0}/Y_1$ is a cone over a smooth 4-dimensional variety in $N_{Y_0/X^{ss}}$. Indeed, at $M \in Y_0$, $C_{Y_0/X^{ss}}|_M \cong \text{Aut } M \cdot (H \otimes (h)) \subset H \otimes sl_2 \cong N_{Y_0/X^{ss}}|_M$ and $P(C_{Y_0}/X^{ss}) \cong P \times PGL_2 \cdot P(h) \cong P \times P^2 \subset P(H \otimes sl_2)$.

Finally, let $Z_1$ be the image of

$$\tau_1 : G \times P((V^*)^3) \to X, \quad ((A, B), (g, h, k)) \mapsto A \begin{bmatrix} g & k \\ 0 & h \end{bmatrix} B^{-1}$$

and let $Z_1 := (Z'_1 \cap X^{ss}) \setminus (Y_0 \cup Z_0 \cup Z_1)$. And let $Z_1 = Z_0 \cup Z_1 \cup Z_0 \cup Y_1$, the closure of $Z_1$ in $X^{ss}$. By checking the dimension of a general fiber, we can see that $Z_1$ is a codimension two irreducible $G$-invariant subvariety of $X^{ss}$. The normal cone $C_{Y_1/Z_1}$ is a fiber bundle, analytic locally the union of two transversal rank 2 bundles of the normal bundle $N_{Y_1/X^{ss}}$. For $M \in Y_1$, $C_{Y_1/Z_1}|_M \cong K \otimes (e) \cup K \otimes (f) \subset K \otimes (e, f) \cong N_{Y_1/X^{ss}}|_M$.

### 6.4. Partial desingularization - an outline.

In our situation, Kirwan’s partial desingularization is obtained as the following. For the general statement and its proof, see [21]. Set $X^0 := X^{ss}$. First of all, take the blow-up $X^{1'}$ of $X^0$ along $Y_0$, the deepest stratum with the largest stabilizer. Then $X^{1'}$ is a smooth variety with $G$-action since $Y_0$ is a smooth $G$-invariant subvariety. Let $\tau_1 : X^{1'} \to X_0$ be the blow-up morphism. Let $Y_1^1$ be the exceptional divisor and let $Y_1^1, Z_i$ be the proper transforms of $Y_1$, $Z_i$ respectively. Note that $Y_1^1$ is a smooth subvariety of $X^{1'}$, since the normal cone $C_{Y_0}/Y_1$ was a cone over a smooth variety.

For a linearized $Q$-ample line bundle $L_0$ on $X^0$ (this is unique up to scaling, since rank $\text{Pic}(X) = 1$), let $L_1 := \pi_1^* L_0 \otimes \mathcal{O}(-\epsilon_1 Y_0^1)$ for sufficiently small $\epsilon_1 > 0$. Then $L_1$ is an ample $Q$-line bundle and induces a linearized $G$-action. With respect to this $G$-linearized action on $X^{1'}$, $Z_0^1 \cong Z_0$ becomes unstable. Let $X_1 := X^{1'} \setminus Z_0^1$

Let $\tau_2 : X^{2'} \to X^1$ be the blow-up along $Y_1^1$. Then $X^{2'}$ is smooth variety with $G$-action since $Y_1^1$ is a $G$-invariant smooth subvariety. Let $Y_1^2$ be the exceptional divisor and let $Y_0^2$ (resp. $Z_1^2$) be the proper transform of $Y_0$ (resp. $Z_1$). Let $L_2 := \pi_2^* L_1 \otimes \mathcal{O}(-\epsilon_2 Y_1^2)$ for sufficiently small $0 < \epsilon_2 < \epsilon_1$. Since $Y_1^1$ is $G$-invariant, there is a well-defined $G$-linearized action on $L_2$. With respect to this linearized action on $X^{2'}$, $Z_1^2$ is unstable. Let $X^2 := X^{2'} \setminus Z_1^2$. 
Now \(X^2 = (X^2)^{ss} = (X^2)^s\) because there is no semistable point with a positive dimensional stabilizer. The partial desingularization of \(X//G\) is defined by \(X^2//G\). If we denote \(\pi_1 \circ \pi_2\) by \(\pi\), then since
\[
\pi^{-1}(X^3) \subset (X^2)^s \subset (X^2)^{ss} \subset \pi^{-1}(X^{ss})
\]
in general, there are quotient maps \(\pi_i : X^i//G \to X^{1-i}//G\) for \(i = 1, 2\). In summary, we have:
\[
\begin{array}{ccc}
X^2 & \xrightarrow{\pi_2} & X^1 \\
\downarrow{\pi_1} & & \downarrow{\pi_0} \\
X^0 & & \end{array}
\]
\[
\begin{array}{ccc}
X^2//G & \xrightarrow{\pi_2} & X^1//G \\
\downarrow{\pi_1} & & \downarrow{\pi_0} \\
X^{0}//G & & \end{array}
\]
Note that every point \(M \in X^2\) has only finite stabilizer with respect to the \(G\)-action, \(X^2//G\) has orbifold singularities only.

6.5. Analysis on fibers. Here we will take a look into the change in a fiber of two exceptional divisors. For \(M \in Y_0\), \(\pi_1^{-1}(M) \cong \mathbb{P}(H \otimes \mathfrak{s}_{l2})\) where \(H\) is a 3-dimensional subspace of \(V^*\). On \(\pi_1^{-1}(M)\), \(\text{Aut } M \cong \text{PGL}_{2}\) action is induced by a trivial action on \(V\) and the standard \(SL_2\)-adjoint action on \(\mathfrak{s}_{l2}\). With respect to this \(\text{PGL}_{2}\)-action, the unstable locus is isomorphic to \(\mathbb{P}H \times \mathbb{P}^1\), which is precisely \(\mathbb{P}(C_{Y_0} \otimes V, |M|)\). Thus in \(X^1\), the inverse image of \(M\) is \(\mathbb{P}(V \oplus \mathfrak{s}_{l2})^{ss}\). If we denote the image of \(M\) in \(X//G\) by \(\overline{M}\), then
\[
\pi_1^{-1}(\overline{M}) \cong \mathbb{P}(V \oplus \mathfrak{s}_{l2})//\text{Aut } M \cong \mathbb{P}(V \oplus \mathfrak{s}_{l2})//\text{PGL}_{2}.
\]
The strictly semistable locus is isomorphic to \(\mathbb{P}(V \oplus \mathfrak{s}_{l2}) \setminus (\mathbb{P}H \times \mathbb{P}^1)\), which is the projectivized normal cone \(\mathbb{P}(C_{Y_0} \otimes V, |M|)\). Therefore over the fiber of \(\overline{M}\), the second blow-up \(\tilde{\pi}_2 : X^2//G \to X^1//G\) is precisely the partial desingularization of the fiber \(\mathbb{P}(H \otimes \mathfrak{s}_{l2})//\text{PGL}_{2}\). This resolution is well-studied in [19, Theorem 4.1]. The blow-up fiber is isomorphic to \(\mathbb{M}_{2,0}(\mathbb{P}^2, 2)\), the moduli space of degree 2 stable maps in \(\mathbb{P}^2\). This is isomorphic to the blow-up of \(\mathbb{P}^3 = \mathbb{P}(H \otimes \mathfrak{s}_{l2})//\text{PGL}_{2}\) along the degree 2 Veronese embedding of \(\mathbb{P}^2\).

For \(F \in Y_1\), \(\tilde{\pi}_2^{-1}(FM = \mathbb{P}(W \otimes \langle e, f \rangle)\). With respect to the \(\text{Aut } M \cong \mathbb{C}^*\) action, the unstable locus is \(\mathbb{P}(W \otimes \langle e \rangle) \cup \mathbb{P}(W \otimes \langle f \rangle)\). So on \(X^2\), \(\tilde{\pi}_2^{-1}(M) = \mathbb{P}(W \otimes \langle e, f \rangle)^s\). Also in \(X^2//G\), \(\pi_2^{-1}(\overline{M}) = \mathbb{P}(W \otimes \langle e, f \rangle)\mathbb{C}^* \cong \mathbb{P}^1 \times \mathbb{P}^1\).

6.6. Elementary modification of maps. The two steps on the partial desingularization can be understood as two steps of the modification of a family of maps. In this section, we prove Theorem 6.1.

Proof of Theorem 6.1. Let \(X^0 := X^{ss} = \mathbb{P}(\text{Hom}(C^2 \otimes V, C^2))^{ss}\) as in previous sections. By (6.1) (applied to \(S = X^0\)), we have a rational map
\[
f_0 : X^0 \times \mathbb{P}^1 \dashrightarrow \text{Gr}(2, 4)
\]
and by Lemma 6.5, \(f_0\) is regular on \(X^s \times \mathbb{P}^1\).

By the Plücker embedding, \(\text{Gr}(2, 4)\) is embedded into \(\mathbb{P}^5\) as a quadric hypersurface. By composing with this embedding, we may regard \(f_0\) as a family of maps to \(\mathbb{P}^5\). Indeed, this map is given by a bundle morphism
\[
6\mathcal{O}_{X^0 \times \mathbb{P}^1} = \Lambda^2 4\mathcal{O}_{X^0 \times \mathbb{P}^1} \xrightarrow{\Lambda^2(p_*(q^*M_{l1^*}) \otimes \mathcal{O}_{\mathbb{P}^1}(-1)^*)} \Lambda^2 2\mathcal{O}_{X^0 \times \mathbb{P}^1}(1) = \mathcal{O}_{X^0 \times \mathbb{P}^1}(2).
\]
Let $F_0 := \Lambda^2(p\ast(q^*\mathcal{M} \circ t)^* \otimes \mathcal{O}$_{\mathbb{P}^1}(-1))^*$. Consider the pull-back map $\pi_1 \times \text{id} : X^1 \times \mathbb{P}^1 \rightarrow X^0 \times \mathbb{P}^1$. Then by taking the pull-back, we have a morphism of sheaves

$$6\mathcal{O}_{X^1 \times \mathbb{P}^1}(\pi_1 \times \text{id})^* F_0 \rightarrow 6\mathcal{O}_{X^1 \times \mathbb{P}^1}(2).$$

This map factors through

$$(6.4) \quad 6\mathcal{O}_{X^1 \times \mathbb{P}^1} \rightarrow \mathcal{O}_{X^1 \times \mathbb{P}^1}(2)(-Y^1_2),$$

since $F_0|_{\mathcal{Y}_0 \times \mathbb{P}^1}$ is a zero map and so $(\pi_1 \times \text{id})^* F_0|_{\mathcal{Y}_1 \times \mathbb{P}^1}$ is zero, too. Let $F_1$ be the map in (6.4). Then on $Y^1_0 \times \mathbb{P}^1 \setminus Y^1_1 \times \mathbb{P}^1$, $F_1$ is regular. Thus we have an extension $f_1$ of the map $f_0 \circ (\pi_1 \times \text{id})$ such that the diagram

$$\begin{array}{ccc}
X^1 \times \mathbb{P}^1 & \rightarrow & Y^1_1 \times \mathbb{P}^1 \\
\downarrow \pi_1 \times \text{id} & \searrow f_1 & \\
X^0 \times \mathbb{P}^1 & \rightarrow & \text{Gr}(2, 4) \rightarrow \mathbb{P}^5
\end{array}$$

commutes.

Let $\pi_2 \times \text{id} : X^2 \times \mathbb{P}^1 \rightarrow X^1 \times \mathbb{P}^1$ be the blow-up map. Then the base locus $B$ of $(\pi_2 \times \text{id})^* f_1$ is a two to one étale cover of $Y^2_1$, since on each fiber $|M| \times \mathbb{P}^1$ for $M \in \mathcal{Y}_1^2$, the undefined locus of $f_1|_{|M| \times \mathbb{P}^1}$ is the union of two distinct points. In particular, it is a codimension two smooth subvariety. Let $\sigma : \Gamma \rightarrow X^2 \times \mathbb{P}^2$ be the blow-up along $B_1$ and let $B_2$ be the exceptional divisor. Then $s : \Gamma \rightarrow X^2 \times \mathbb{P}^2 \rightarrow X^2$ is a flat family of (possibly nodal) rational curves. Furthermore, the pull-back morphism

$$6\mathcal{O}_\Gamma \rightarrow (\pi_2 \times \text{id})^* F_1 \rightarrow 6\mathcal{O}_\Gamma \rightarrow 6\mathcal{O}_{X^2 \times \mathbb{P}^1}(2)(-Y^2_0)$$

factors through

$$(6.5) \quad 6\mathcal{O}_\Gamma \rightarrow \sigma^* \mathcal{O}_{X^2 \times \mathbb{P}^1}(2)(-Y^2_0 - B_2).$$

Let $F_2$ be the map in (6.5). Then it is an epimorphism and we obtain a regular map $f_2 : \Gamma \rightarrow \mathbb{P}^5$ as below:

$$\begin{array}{ccc}
\Gamma & \rightarrow & \mathbb{P}^5 \\
\downarrow \sigma & & \\
X^2 \times \mathbb{P}^1 & \searrow f_2 & \\
\downarrow \pi_2 \times \text{id} & & \\
X^1 \times \mathbb{P}^1 & \rightarrow & \text{Gr}(2, 4) \rightarrow \mathbb{P}^5 \\
\downarrow \pi_1 \times \text{id} & \searrow f_1 & \\
X^0 \times \mathbb{P}^1 & \rightarrow & \mathbb{P}^5
\end{array}$$

Thus $(s : \Gamma \rightarrow X^2, f_2 : \Gamma \rightarrow \mathbb{P}^5)$ defines a flat family of maps of degree 2 over $X^2$. By applying the standard stabilization of maps using relative log MMP, we can obtain a family of stable maps $(\tilde{s} : \tilde{\Gamma} \rightarrow X^2, \tilde{f}_2 : \tilde{\Gamma} \rightarrow \mathbb{P}^5)$. Moreover, $\tilde{f}_2$ factors through $\text{Gr}(2, 4)$, since on an open dense subset $X^4$, the image is in a closed subvariety $\text{Gr}(2, 4)$. Hence we obtain a morphism $\Phi^2 : X^2 \rightarrow K$. 


Finally, it is straightforward to see that this map is $G$-invariant, since $G$ acts as a change of coordinates of the domain curve. Thus we have a quotient map

$$\overline{\Phi}^2 : X^2/G \to K.$$ 

This is a birational morphism between two normal projective varieties. Moreover, both $X^2/G$ and $\overline{M}(\text{Gr}(2,4),2)$ are $\mathbb{Q}$-factorial since they have only finite quotient singularities. They have the same Picard numbers, so $\overline{\Phi}^2$ is an isomorphism. □

Although the singularity is very mild, the space $K$ is still singular. Let $D \subset K$ be the locus of stable maps which are 2:1 maps to their images. This is precisely the locus of stable maps with nontrivial stabilizer groups. Along this locus, $K$ has $\mathbb{Z}_2$-quotient singularities.

In [12, Theorem 1.2], it is shown that the blow-up of $K$ along $D$ is a smooth projective variety $\mathbb{C}C_2$, so called the space of complete conics.

**Corollary 6.6.** The fiber product $\tilde{M}_2 := M_2 \times_{\mathbb{R}} CC_2$ is a desingularization of $M_2$.

**Proof.** The blow-up centers for $M_2 \to \mathbb{R}$ and $CC_2 \to K \to \mathbb{R}$ are disjoint. Therefore $\tilde{M}_2$ is a blow-up of $CC_2$ at two smooth points.

$$\begin{array}{ccc}
\tilde{M}_2 & \to & CC_2 \\
\left\downarrow\right. & & \left\downarrow\right. \\
\left\downarrow\right. & & \left\downarrow\right. \\
M_2 & \to & R
\end{array}$$

□

It would be very interesting to find a moduli theoretic meaning of $\tilde{M}_2$.

**Remark 6.7.** The log MMP of $K$ is studied in [8]. The authors computed the stable base locus decomposition of the effective cone of $K$, and computed some of modular birational models of $K$. In [8], some birational models are simply described as divisorial contractions of some loci and their moduli theoretic interpretations were not described. Two birational models $R$ and $X^1/G$, the first step of the partial desingularization are indeed their “modular” interpretations. Indeed, $R$ is the image of $\phi_{H_{\sigma^2}}$ in [8, Proposition 3.7]. For $D \in c(H_{\sigma^2}T)$, the image of $\phi_D$ is $X^1/G$ (see [8, Theorem 3.8] for the notation).

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