Geometrical Mechanics on algebroids

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Abstract

A natural geometric framework is proposed, based on ideas of W. M. Tulczyjew, for constructions of dynamics on general algebroids. One obtains formalisms similar to the Lagrangian and the Hamiltonian ones. In contrast with recently studied concepts of Analytical Mechanics on Lie algebroids, this approach requires much less than the presence of a Lie algebroid structure on a vector bundle, but it still reproduces the main features of the Analytical Mechanics, like the Euler-Lagrange-type equations, the correspondence between the Lagrangian and Hamiltonian functions (Legendre transform) in the hyperregular cases, and a version of the Noether Theorem.

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1 Introduction

Lie algebroids have been introduced repeatedly into differential geometry since the early 1950s, and also into physics and algebra, under a wide variety of names. They have been also recognized as infinitesimal objects for Lie groupoids by J. Pradines [21]. We refer to [15] for basic definitions, examples, and an extensive list of publications in this area. The classical Cartan differential calculus on a manifold $M$, including the exterior derivative $d$, the Lie derivative $\mathcal{L}$, etc., can be viewed as being associated with the canonical Lie algebroid structure on $TM$ represented by the Lie bracket of vector fields, and therefore it has obvious generalizations to an arbitrary Lie algebroid.

The tangent bundle $TM$ is of course a canonical Lie algebroid. It is associated with the canonical Poisson tensor (symplectic form) on $T^*M$. Other canonical objects associated with $TM$ are: the canonical isomorphism

\[ \alpha_M : \mathcal{T}T^*M \to T^*T^*M \]

of double vector bundles, discovered by Tulczyjew [24], that is the dual to the well-known flip

\[ \kappa_M : \mathcal{T}TM \to \mathcal{T}TM, \]

and the tangent lift $d_T$ of tensor fields on $M$ to tensor fields on $TM$ (cf. [30, 19, 5]).

Being related to many areas of geometry, like connection theory, cohomology theory, invariants of foliations and pseudogroups, symplectic and Poisson geometry, etc., Lie algebroids became recently an object of extensive studies. In [28], A. Weinstein had posed the problem of finding a framework for Analytical Mechanics based on a general Lie algebroid. This task was undertaken in [13, 11, 16, 17] to get a Lie algebroid version of the geometric construction of the Euler-Lagrange equations due to J. Klein [9]. In this approach the E-L equation is a first order equation on the Lie algebroid bundle $E$.

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with some compatibility conditions. In the classical version \((E = TM)\), the Klein’s method is based on the vector bundle structure of \(TM\) and the existence of a vector-valued 1-form, so called the ‘soldering form’. Such a form does not exist for a general Lie algebroid and the conclusion is that the immediate analogy for the Klein’s approach does not exist as well. In a series of papers E. Martínez has proposed an interesting modified version of the Klein’s method, in which the bundles tangent to \(E\) are replaced by the prolongations (in the sense of Higgins and Mackenzie [8]) of \(E\) with respect to the vector bundle projections \(\tau: E \to M\) and \(\tau^*: E^* \to M\). A similar approach for structures more general than Lie algebroids has been proposed by M. Popescu and P. Popescu [20].

The ideas of J. Klein go back to 1962. Since then a lot of work has been done to get a better understanding of the geometric background for Analytical Mechanics. In the papers of Tulczyjew [25] and de León with Lacomba [12] we find another geometric constructions of the E-L equation. The starting point for the Tulczyjew’s construction is the dynamics of a system, i.e. a Lagrangian submanifold of \(TT^*M\), which is the inverse image of \(dL(M)\) with respect to the canonical diffeomorphism \(\alpha_M: TT^*M \to T^*TM\), where \(L\) is a function (Lagrangian) on \(M\). The diffeomorphism \(\alpha_M\), or its dual \(\kappa_M: TT^*M \to TT^*M\), represent, unlike the ‘soldering form’, the complete structure of the tangent bundle.

On the other hand, in a couple of papers [6, 7], two of us have developed an approach to Lie algebroids based on the analogue of these canonical diffeomorphisms, which form a part of the so-called Tulczyjew triple, and which appear to be morphisms of double vector bundles. This allowed us to introduce the notion of a (general, not necessarily Lie) algebroid as a morphism of certain double vector bundles or, equivalently, as a vector bundle equipped with a linear 2-contravariant tensor.

What we propose in this paper is to adopt the Tulczyjew approach [24] (cf. also [26]) to the case of a general algebroid. In particular, we obtain a geometric construction of an equation which was suggested by A. Weinstein as a Lie algebroid version of the Euler-Lagrange equation. The main difference with the papers like [13, 16, 17, 20] is not only that we deal with general algebroids but also that in this case we avoid objects like ‘soldering form’, symplectic algebroids, Poincaré-Cartan sections, Lie algebroid prolongations, etc.

We postpone to a separate publication the discussion of the calculus of variations for a general Lie algebroid. It requires more fundamental discussion on the virtual displacements and it is closely related to the problem of integrability of algebraic structures in the sense in which Lie algebroids can be integrated to Lie groupoids.

The paper is organized as follows. First we fix the notation. In the next two sections, the classical Tulczyjew triple and rudiments of the approach from [6] and [7] on viewing Lie algebroids as double bundle morphisms are presented. In Section 3 we recall the definition of the complete lift of tensor fields. In Section 4 we present the concepts of deriving dynamics on general algebroids \(E\), similar to that in the Lagrangian and Hamiltonian formalisms. The equations we discuss are equations for paths in \(E^*\) and \(E\). The latter are often regarded as algebroid generalizations of the Euler-Lagrange equations. A variant of the Noether Theorem is proved. Section 5 contains a couple of examples which generalize the geodesic and the Wong equations.

### 1.1 Notation

Let \(M\) be a smooth manifold. We denote by \(\tau_M: TM \to M\) the tangent vector bundle and by \(\pi_M: T^*M \to M\) the cotangent vector bundle.

Let \(\tau: E \to M\) be a vector bundle and let \(\pi: E^* \to M\) be the dual bundle. We use the following notation for tensor bundles:

\[
\tau^\otimes_k: E \otimes_M \cdots \otimes_M E = \otimes^k_M(E) \to M
\]

and the module of sections over \(\mathcal{C}^\infty(M)\):

\[
\otimes^k(\tau) = \Gamma(\otimes^k_M(E)).
\]

By \(\langle \cdot, \cdot \rangle\), we denote the canonical pairing between \(E\) and \(E^*\) as well as pairings between the corresponding tensor bundles, e.g.,

\[
\langle \cdot, \cdot \rangle: \otimes^k_M(E) \times_M \otimes^k_M(E^*) \to \mathbb{R}.
\]
and pairings of sections, e. g.,

\[ (\cdot, \cdot) : \otimes^k (\tau) \times \otimes^k (\pi) \longrightarrow C^\infty (M). \]

Let \( K \) be a section of the tensor bundle \( \otimes^k_M (E) \), \( K \in \otimes^k (\tau) \). We denote by \( \iota (K) \) the corresponding linear function on the dual bundle

\[ \iota (K) : \otimes^k_M (E^*) \rightarrow \mathbb{R} \]
\[ : a \mapsto \langle K (m), a \rangle, \quad m = \pi \otimes^k (a). \]

For a section \( X \) of \( \tau \) (\( X \in \otimes^1 (\tau) \)), we have the usual operator of insertion

\[ \iota_X : \otimes^{k+1} (\pi) \rightarrow \otimes^k (\pi) \\
: \mu_1 \otimes \cdots \otimes \mu_{k+1} \mapsto \langle X, \mu_1 \rangle \otimes \cdots \otimes \mu_{k+1}. \]

Let \( \Lambda \in \otimes^2 (\tau) \). We denote by \( \tilde{\Lambda} \) the mapping

\[ \tilde{\Lambda} : E^* \rightarrow E, \quad \tilde{\Lambda} \circ \mu = \iota_\mu \Lambda. \]

By \( \Delta_E \) we denote the Liouville (called also Euler) vector field on the vector bundle \( E \).

### 1.2 Local coordinates

Let \((x^a), a = 1, \ldots, n\), be a coordinate system in \( M \). We introduce the induced coordinate systems

\[
\begin{align*}
(x^a, \dot{x}^b) & \quad \text{in } TM, \\
(x^a, p_b) & \quad \text{in } T^* M.
\end{align*}
\]

Let \((e^1, \ldots, e^m)\) be a basis of local sections of \( \tau : E \rightarrow M \) and let \((e^1_*, \ldots, e^m_*)\) be the dual basis of local sections of \( \pi : E^* \rightarrow M \). We have the induced coordinate systems:

\[
\begin{align*}
(x^a, y^i), & \quad y^i = \iota (e^i_*) , \quad \text{in } E, \\
x^a, \xi_i), & \quad \xi_i = \iota (e^i) , \quad \text{in } E^*,
\end{align*}
\]

and

\[
\begin{align*}
(x^a, y^i, \dot{x}^b, \dot{y}^j) & \quad \text{in } TE, \\
(x^a, \xi_i, \dot{x}^b, \dot{\xi}_j) & \quad \text{in } TE^*, \\
(x^a, y^i, p_b, \pi_j) & \quad \text{in } T^* E, \\
(x^a, \xi_i, p_b, \varphi^j) & \quad \text{in } T^* E^*.
\end{align*}
\]

The adapted coordinates on the above bundles define canonical double vector bundle structures on them in the sense of J. Pradines ([22, 23], cf. also [2, 14]). The Liouville vector field has the form \( \Delta_E = y^i \partial_{y^i} \). We have the canonical symplectic forms:

\[ \omega_E^* = dp_a \wedge dx^a + dp_i \wedge d\xi_i \]

on \( T^* E^* \) and

\[ \omega_E = dp_a \wedge dx^a + d\pi_i \wedge dy^i \]

on \( T^* E \), and the corresponding Poisson tensors

\[
\begin{align*}
\Lambda_{E^*} = \partial_{p_a} \wedge \partial_{x^a} + \partial_{p_i} \wedge \partial_{\xi_i} & \quad \text{and } \Lambda_E = \partial_{p_a} \wedge \partial_{x^a} + \partial_{\pi_i} \wedge \partial_{y^i},
\end{align*}
\]

There is also a canonical isomorphism (cf. [4, 10, 7])

\[ \mathcal{R}_r : T^* E^* \longrightarrow T^* E \]
being an anti-symplectomorphism and also an isomorphism of double vector bundles:

\[
\begin{array}{ccc}
T^*E^* & \xrightarrow{\mathcal{R}_\tau} & T^*E \\
\pi_{E^*} & & \downarrow id \\
E & \xrightarrow{id} & E^* \\
\pi & \downarrow id & \pi \\
M & \xrightarrow{id} & M
\end{array}
\]

In local coordinates, \( \mathcal{R}_\tau \) is given by
\[
(x^a, y^i, p_b, \pi_j) \circ \mathcal{R}_\tau = (x^a, \varphi^i, -p_b, \xi_j).
\]

### 1.3 Classical Tulczyjew triple

Let \( M \) be the configuration manifold of a mechanical system. The cotangent bundle \( T^*M \) is the phase space of the system. Elements of the phase space are momenta. The commutative diagram

\[
\begin{array}{ccc}
T^*T^*M & \xrightarrow{\beta(T^*M, \omega_M)} & TT^*M \\
\pi_{T^*M} & \xrightarrow{id} & \pi_{T^*M} \\
\pi^*M & \xrightarrow{id} & \pi^*M \\
M & \xrightarrow{id} & M
\end{array}
\]

known as the Tulczyjew triple, contains the geometric structures used to formulate the dynamics of the system. The dynamics is a differential equation \( D \subset TT^*M \). A solution \( \gamma : I \to T^*M \) of this equation is a phase space trajectory of the system. Trajectories of the system in the configuration manifold \( M \) are solutions of the second-order Euler-Lagrange equation

\[
E_L = T^2\pi_M(PD),
\]

where the set
\[
PD = TD \cap T^2T^*M \subset T^2T^*M
\]
is a second order differential equation called the prolongation of \( D \). Note however that in general these trajectories in \( M \) do not determine \( D \).

We have recognized the presence of a canonical symplectic structure in \( TT^*M \) with the symplectic form \( d_T\omega_M \). In most cases of interest in physics the dynamics is a Lagrangian submanifold of \( (TT^*M, d_T\omega_M) \). Morphisms \( \alpha_M \) and \( \beta(T^*M, \omega_M) \) are canonical symplectomorphisms from \( (TT^*M, d_T\omega_M) \) to \( (T^*T^*M, \omega_{T^*M}) \) and to \( (T^*T^*M, \omega_{T^*M}) \). These symplectomorphisms with cotangent bundles create the possibility of generating the dynamics from (generalized) Lagrangians associated with \( TM \) or (generalized) Hamiltonians associated with \( T^*M \) (cf. [24, 26, 27]).

### 2 Algebroids as double vector bundle morphisms

It is well known that Lie algebroid structures on the vector bundle \( E \) correspond to linear Poisson structures on \( E^* \). A 2-contravariant tensor \( \Lambda \) on \( E^* \) is called linear if the corresponding mapping
$\Lambda: T^*E^* \to TE^*$ is a morphism of double vector bundles. This is the same as to say that the corresponding bracket of functions is closed on (fiber-wise) linear functions. The commutative diagram

\[ \begin{array}{ccc}
T^*E^* & \xrightarrow{\Lambda} & TE^* \\
\xrightarrow{\pi} & \xrightarrow{\varepsilon} & \\
T^*E & \xrightarrow{\tau} & TM \\
\end{array} \]

describes a one-to-one correspondence between linear 2-contravariant tensors $\Lambda$ on $E^*$ and homomorphisms of double vector bundles covering the identity on $E^*$ (cf. [10, 7]).

The core of a double vector bundle is the intersection of the kernels of the projections. It is obvious that the core of $T^*E$ (resp., $TE^*$) can be identified with $T^*M$ (resp., $E^*$). With these identifications the induced by $\varepsilon$ morphism of cores is a morphism

$\varepsilon_c: T^*M \to E^*$.

In local coordinates, every $\varepsilon$ as in (1) is of the form

$$
(x^a, \xi_i, x^b, \xi_j) \circ \varepsilon = (x^a, \pi_i, \rho_k(x)y^k, c_{ij}^k(x)y^i\pi_k + \sigma_j^k(x)p_a)
$$

and it corresponds to the linear tensor on $E^*$

$$
\Lambda_\varepsilon = c_{ij}^k(x)\partial_{\xi_i} \otimes \partial_{\xi_j} + \rho_i^b(x)\partial_{\xi_i} \otimes \partial_{x^b} - \sigma_j^a(x)\partial_{x^a} \otimes \partial_{\xi_j}.
$$

We have also

$$
(x^a, \dot{x}^b) \circ \varepsilon_r = (x^a, \rho_k^b(x)y^k),
$$

$$
(x^a, \xi_i) \circ \varepsilon_c = (x^a, \sigma_j^a(x)p_b).
$$

In [7] by algebroids we meant the morphisms (1) of double vector bundles covering the identity on $E^*$, while Lie algebroids were those algebroids for which the tensor $\Lambda_\varepsilon$ is a Poisson tensor. The relation to the canonical definition of Lie algebroid is given by the following theorem.

**Theorem 1** [6, 7] An algebroid structure $(E, \varepsilon)$ can be equivalently defined as a bilinear bracket $[\cdot, \cdot]_\varepsilon$ on sections of $\tau: E \to M$, together with vector bundle morphisms $a^\varepsilon_r, a^\varepsilon_c: E \to TM$ (left and right anchors), such that

$$
[fX, gY]_\varepsilon = f(a^\varepsilon_r \circ X)(g)Y - g(a^\varepsilon_c \circ Y)(f)X + fg[X, Y]_\varepsilon
$$

for $f, g \in C^\infty(M)$, $X, Y \in \otimes^1(\tau)$. The bracket and anchors are related to the 2-contravariant tensor $\Lambda_\varepsilon$ by the formulae

$$
\iota([X, Y]_\varepsilon) = \{\iota(X), \iota(Y)\}_\Lambda_\varepsilon,
$$

$$
\pi^*(a^\varepsilon_r \circ X(f)) = \{(\iota(X), \pi^*f\}_\Lambda_\varepsilon,
$$

$$
\pi^*(a^\varepsilon_c \circ X(f)) = \{\pi^*f, \iota(X)\}_\Lambda_\varepsilon.
$$

We have also $a^\varepsilon_r = \varepsilon_r$ and $a^\varepsilon_c = (\varepsilon_c)^*$. The algebroid $(E, \varepsilon)$ is a Lie algebroid if and only if the tensor $\Lambda_\varepsilon$ is a Poisson tensor.
To every algebroid \( \varepsilon \) there is the \textit{adjoint algebroid} \( \varepsilon^+ : T^*E \to TE^* \) which is the bundle morphism dual to \( \varepsilon \) with respect to the projections onto \( E^* \) and which corresponds to the transposition of the tensor \( \Lambda_\varepsilon \). The algebroid is \textit{skew-symmetric} if and only if \( \varepsilon^+ = -\varepsilon \). An algebroid we call a (left or right) \textit{connection} if one of the anchors (right or left, respectively) is trivial. In this case we have for the bracket \( \nabla^\varepsilon_X Y = [X, Y]_\varepsilon \) associated with, say, a left connection \( \varepsilon \), the standard properties of a linear connection: \( \nabla^\varepsilon_X Y = f \nabla^\varepsilon_X Y \) and \( \nabla^\varepsilon_X f Y = f \nabla^\varepsilon_X Y + (d_f \circ X)(f)Y \).

The canonical example of a mapping \( \varepsilon \) in the case of \( E = TM \) is given by \( \varepsilon = \varepsilon_M = \alpha_M^{-1} \) – the inverse to the Tulczyjew isomorphism \( \alpha_M \) that can be defined as the dual to the isomorphism of double vector bundles

\[ TTM \rightarrow \tau \rightarrow TM \]

In general, the algebroid structure map \( \varepsilon \) is not an isomorphism and, consequently, its dual \( \kappa^{-1} = \varepsilon^{*\tau} \) with respect to the right projection is a relation and not a mapping.

### 3 The algebroid lift \( d^\varepsilon_T \)

For a tensor field \( K \in \otimes^k(\tau) \), we can define the vertical lift \( v_\tau(K) \in \otimes^k(\tau_E) \) (cf. [5, 30]). In local coordinates,

\[ v_\tau(f^{i_1\ldots i_k} e_{i_1} \otimes \ldots \otimes e_{i_k}) = f^{i_1\ldots i_k} \partial y^{i_1} \otimes \ldots \otimes \partial y^{i_k}. \]  

(4)

A particular case of the vertical lift is the lift \( v_T(K) \) of a contravariant tensor field \( K \) on \( M \) into a contravariant tensor field on \( TM \). It is well known (see [30, 5]) that in the case of \( E = TM \) we have also the tangent lift \( d_T : \otimes (\tau_M) \to \otimes (\tau_{TM}) \) which is a \( \tau_T \)-derivation. It turns out that the presence of such a lift for a vector bundle is equivalent to the presence of an algebroid structure. Note first that we can extend \( \varepsilon \) naturally to mappings (cf. [6, 7])

\[ \varepsilon^\otimes r : \otimes^r E^* \to T \otimes^r_M E^*, \quad r \geq 0. \]

**Theorem 2** [6, 7] \( E, \varepsilon \) be an algebroid. For \( K \in \otimes^k(\tau), \ k \geq 0 \), the equality

\[ \iota(d_T^\varepsilon(K)) = d_T(\iota(K)) \circ \varepsilon^\otimes k \]  

(5)

defines the tensor field \( d_T^\varepsilon(K) \in \otimes^k(\tau_E) \) which is linear and the mapping

\[ d_T^\varepsilon : \otimes(\tau) \to \otimes(\tau_E) \]

is a \( v_\tau \)-derivation of degree 0. In local coordinates,

\[ d_T^\varepsilon(f(x)) = y^i \rho^i_\tau(x) \frac{\partial f}{\partial y^i}(x), \]

\[ d_T^\varepsilon(f^i(x)e_i) = f^i(x) \sigma^i_\tau(x) \partial_{x^i} + \left(y^j \rho^j_\tau(x) \frac{\partial f^i}{\partial y^j}(x) + c^i_{jk}(x) y^j f^k(x)\right) \partial y^k. \]  

(6)

Conversely, if \( D : \otimes(\tau) \to \otimes(\tau_E) \) is a \( v_\tau \)-derivation of degree 0 such that \( D(K) \) is linear for each \( K \in \otimes^1(\tau) \), then there is an algebroid structure \( \varepsilon \) on \( \tau : E \to M \) such that \( D = d_T^\varepsilon \).

**Theorem 3** [7] \( \varepsilon \) be an algebroid structure on \( \tau : E \to M \). The following properties of \( \varepsilon \) are equivalent:
(a) $\varepsilon$ is a Lie algebroid structure,
(b) $\Lambda_{\varepsilon}$ is a Poisson tensor,
(c) $\Lambda_{E}$ and $d_T \Lambda_{\varepsilon}$ are $\varepsilon$-related,
(d) $d_T([X, Y]) = [d_T(X), d_T(Y)]$ for all $X, Y \in \mathfrak{g}(\tau)$.

4 Lagrangian and Hamiltonian formalisms for general algebroids

The double vector bundle morphism (1) can be extended to the following algebroid analogue of the Tulczyjew triple

\[ \begin{array}{c}
\mathbb{T}^*E^* \xrightarrow{\lambda} \mathbb{T}E^* \xrightarrow{\varepsilon} \mathbb{T}^*E \\
\mathbb{T}E^* \xrightarrow{\varepsilon_{\tau}} \mathbb{T}M \xrightarrow{\varepsilon_{\tau}} \mathbb{T}^*E
\end{array} \] (7)

The left-hand side is Hamiltonian, the right-hand side is Lagrangian, and the 'dynamics' lives in the middle.

Any Lagrangian function $L : E \to \mathbb{R}$ defines a Lagrangian submanifold $N = (dL)(E)$ in $\mathbb{T}^*E$, being the image of the de Rham differential $dL$, i.e. the image of the section $dL : E \to \mathbb{T}^*E$. The further image $D = \varepsilon(N)$ can be understood as an implicit differential equation on $E^*$, solutions of which are 'phase trajectories' of the system. The Lagrangian defines also smooth maps: $L_{eg} : E \to E^*$ and $\bar{L}_{eg} : E \to \mathbb{T}E^*$ by

$$L_{eg} = \tau_{E^*} \circ \varepsilon \circ dL = \mathbb{T}^*\tau \circ dL$$

and $\bar{L}_{eg} = \varepsilon \circ dL$. The map $L_{eg}$ is de facto the vertical derivative of $L$ and is the analogue of the Legendre mapping, incorrectly interpreted by many authors as the Legendre transformation associated with $L$ (as the Legendre transformation is the passage from a Lagrangian to a Hamiltonian generating object as explained in [26]). The introduced ingredients produce implicit differential equations, this time for curves $\gamma : I \to E$.

The first equation, which will be denoted by $(E^1_L)$, is represented by the inverse image

$$E^1_L = \mathbb{T}(L_{eg})^{-1}(D)$$ (8)

of $D$ with respect to the derivative $\mathbb{T}(L_{eg}) : \mathbb{T}E \to \mathbb{T}E^*$ of $L_{eg} : E \to E^*$. This simply means that $\gamma : I \to E$ is a solution $(E^1_L)$ if and only if $L_{eg} \circ \gamma$ is a solution of $(D)$, i.e. $\mathbb{T}(L_{eg} \circ \gamma) \subset D$. This construction corresponds to that of de Léon and Lacomba [12].

The second equation, which will be denoted by $(E^2_L)$, is defined by

$$E^2_L = \{ v \in \mathbb{T}E : \varepsilon \circ dL(v) \in \mathbb{T}^2E^*\},$$ (9)

where $\mathbb{T}^2E^* \subset \mathbb{T}\mathbb{T}E^*$ is the subset of holonomic vectors, i.e. such that $\tau_{\mathbb{T}E^*}(w) = \mathbb{T}\tau_{E^*}(w)$. It follows from the commutativity of the diagram (7) that

$$E^2_L = \{ v \in \mathbb{T}E : \varepsilon \circ \tau_{E^*}(\mathbb{T}dL(v)) = \mathbb{T}(\mathbb{T}^\#\tau)(\mathbb{T}dL(v))\}.$$
It is clear that this definition can be extended to any subset of $T^*E$. The solutions of the second equation, are such paths $\gamma : I \to E$ that the tangent prolongation $T(L_{eg} \circ \gamma)$ of $L_{eg} \circ \gamma$ is exactly $\tilde{L}_{eg} \circ \gamma$. Since the path $\tilde{L}_{eg} \circ \gamma$ belongs to $D$ by definition, $(E^2_L) \subset (E^1_L)$. It is easy to see that the equation $(E^2_L)$ is represented by the inverse image

$$E^2_L = \mathcal{T}(\tilde{L}_{eg})^{-1}(T^2E^*)$$

of the subbundle $T^2E^*$ of holonomic vectors in $\mathcal{T}T^*E^*$ with respect to the derivative $\mathcal{T}(\tilde{L}_{eg}) : TE \to \mathcal{T}T^*E^*$. In local coordinates, $D$ has the parametrization by $(x^a, y^k)$ in the form (cf. (2))

$$\tilde{L}_{eg}(x^a, y^k) = (x^a, \frac{\partial L}{\partial y^i}(x, y), \rho^a_k(x)y^k, c^k_j(x)y^j, \frac{\partial L}{\partial y^k}(x, y) + \sigma^a_j(x)\frac{\partial L}{\partial x^a}(x, y))$$

and the equation $(E^2_L)$, for $\gamma(t) = (x^a(t), y^i(t))$, reads

$$(E^2_L) : \begin{align*}
\frac{dx^a}{dt} &= \rho^a_k(x)y^k, \\
\frac{d}{dt} \left( \frac{\partial L}{\partial y^i}(x, y) \right) &= \frac{\partial L}{\partial y^j}(x, y) + \sigma^a_j(x)\frac{\partial L}{\partial x^a}(x, y),
\end{align*}$$

in the full agreement with [13, 16, 17, 28], if only one takes into account that, for Lie algebroids, $\sigma^a_j = \delta^a_j$. As one can see from (11), the solutions are automatically admissible curves in $E$, i.e. the velocity $\frac{d}{dt}(\tau \circ \gamma)(t)$ is $\varepsilon_\tau(\gamma(t))$.

The equation $(E^1_L)$ is weaker and reads

$$(E^1_L) : \begin{align*}
\frac{dx^a}{dt} &= \rho^a_k(x)y^k, \\
\frac{d}{dt} \left( \frac{\partial L}{\partial y^i}(x, y) \right) &= \frac{\partial L}{\partial y^j}(x, y) + \sigma^a_j(x)\frac{\partial L}{\partial x^a}(x, y),
\end{align*}$$

for certain choice of $(x, y_0) \in E$ (associated with $(x, y)$) satisfying $L_{eg}(x, y_0) = L_{eg}(x, y)$, i.e. for $(x, y_0)$ such that

$$\frac{\partial L}{\partial y^i}(x, y) = \frac{\partial L}{\partial y^j}(x, y_0).$$

Of course, when the Lagrangian is hyperregular, i.e. when $L_{eg} : E \to E^*$ is a diffeomorphism, the equations $(E^1_L)$ and $(E^2_L)$ coincide.

We have also the following variant of the Noether Theorem.

**Theorem 4** If $X$ is a section of $E$ and $f$ is a function on $M$, then

$$\mathcal{L} \delta^\tau_X(f) = \delta^\tau_X(f)$$

if and only if the function $(\iota(X) - \nu_\tau(f)) \circ L_{eg}$ on $E$ is a first integral of the equation $(E^2_L)$. In particular, if (13) is satisfied, then $(\iota(X) - \nu_\tau(f)) \circ L_{eg}$ is a first integral of the equation $(E^1_L)$.

**Proof:** $(\iota(X) - \nu_\tau(f)) \circ L_{eg}$ is a first integral of $(E^2_L)$ if $(\iota(X) - \nu_\tau(f))$ is a first integral of the implicit differential equation on $E^*$ defined by $D \iff d\tau(\iota(X) - \nu_\tau(f))$ vanishes on $D \iff d\tau(\iota(X) - \nu_\tau(f)) \circ \varepsilon$ vanishes on $N$. Writing $\nu_\tau(f)$ as $\iota(f)$ we have, according to (5),

$$d\tau(\iota(X) - \nu_\tau(f)) \circ \varepsilon = \iota(d^\tau_F(X)) - \iota(d^\tau_F(f))$$

and

$$(\iota(d^\tau_F(X)) - \iota(d^\tau_F(f))) (dL(e)) = \mathcal{L} \delta^\tau_X(f) L(e) - d^\tau_F(f)(e).$$

$$\blacksquare$$

Note that the tensor $\Lambda_\tau$ gives rise also to kind of a Hamiltonian formalism (cf. [18]). In [7] and [18] one refers to a 2-contravariant tensor as to a Leibniz structure, that however may cause some confusion with the Leibniz algebra in the sense of J.-L. Loday as a non-skew-symmetric analog of a Lie algebra.
Anyhow, in the presence of $\Lambda_\varepsilon$, by the hamiltonian vector field associated with a function $H$ on $E^*$ we understand the contraction $i_H \Lambda_\varepsilon$. Thus the question of the Hamiltonian description of the dynamics $D$ is the question if $D$ is the image of a Hamiltonian vector field. (Of course, one can also try to extend such a Hamiltonian formalism to more general generating objects like Morse families.) Every such a function $H$ we call a Hamiltonian associated with the Lagrangian $L$. However, it should be stressed that, since $\varepsilon$ and $\Lambda_\varepsilon$ can be degenerated, we have much more freedom in choosing generating objects (Lagrangian and Hamiltonian) than in the symplectic case. For instance, the Hamiltonian is defined not up to a constant but up to a Casimir function of the tensor $\Lambda_\varepsilon$ and for the choice of the Lagrangian we have a similar freedom. However, in the case of a hyperregular Lagrangian we recover the standard correspondence between Lagrangians and Hamiltonians. Let us start with the following lemma which can be easily proved exactly like in the classical case and which reflects the fact that this correspondence is, in principle, independent on the algebroid structure on $E$ but which comes directly from the isomorphism $\mathcal{R}_\varepsilon$.

**Lemma 1** If the Lagrangian $L$ is hyperregular, then the Lagrange submanifold $N = dL(M)$ in $\mathbb{T}^*E$ corresponds under the canonical isomorphism $\mathcal{R}_\varepsilon$ to the Lagrange submanifold $dH(M)$ in $\mathbb{T}^*E^*$, where $H = (\Delta_\varepsilon(L) - L) \circ L^{-1}_{\varepsilon g}$.

**Collorary 1** If the Lagrangian $L$ is hyperregular, then the function $H = (\Delta_\varepsilon(L) - L) \circ L^{-1}_{\varepsilon g}$ is a Hamiltonian associated with $L$.

**Proof:** Since $\varepsilon = \tilde{\Lambda}_\varepsilon \circ \mathcal{R}_\varepsilon^{-1}$, in view of the above Lemma, $\varepsilon(N) = \tilde{\Lambda}_\varepsilon(dH(M))$ which means exactly that $\varepsilon(N)$ is the image of the hamiltonian vector field associated with $H$.

All the above shows that the presented Lagrangian and Hamiltonian formalisms work well for all algebroids and not only for Lie algebroids. The fact that the tensor $\Lambda_\varepsilon$ is Poisson played no role in the above considerations and Lie brackets have not been explicitly used.

## 5 Examples

a) **Generalized geodesics.** The simplest hyperregular Lagrangian on $E$ is given by a symmetric positive-definite metric $g$ on $E$. The metric induces an isomorphism of the vector bundles $\tilde{g} : E \to E^*$ which, in turn, induces an isomorphism of corresponding tensor bundles. With respect to this isomorphism the metric $g$ corresponds to a ‘contravariant metric’ $G$. In local coordinates:

$$
g = g_{ij}(x)e^i \otimes e^j, \quad G = g^{ij}(x)e_i \otimes e_j, \quad g_{ij}g^{jk} = \delta^i_k, \quad and \quad L(x, y) = \frac{1}{2}g_{ij}(x)y^iy^j.
$$

The Legendre map is given by $L_{\varepsilon g}(x, y) = (x^a, g_{ij}(x)y^i)$ and its inverse is $L^{-1}_{\varepsilon g}(x, \xi) = (x^a, g^{ij}(x)\xi_j)$. It is easy to see that the Hamiltonian dynamics $D$ on $E^*$ is represented by the vector field

$$
D(x, \xi) = \rho^i \partial_{x^i} + \sigma^{ij} \partial_{\xi^i} \partial_{\xi^j}.
$$

This is the Hamiltonian vector field of $\Lambda_\varepsilon$ with a Hamiltonian

$$
H(x, \xi) = \frac{1}{2}g^{ij}(x)\xi_i\xi_j.
$$

The equations $(E^1_L)$ and $(E^2_L)$ coincide and read

$$
\dot{x}^a = \rho^a \xi^k,
$$

$$
\frac{d}{dt} \left( g_{ik}y^i \right) = \left( c^i_{ik}g_{sj} + \frac{1}{2}\sigma^i_k \frac{\partial g_{ij}}{\partial x^a} \right) y^j y^j.
$$

The last equation can be rewritten in the form

$$
y^i + \Gamma^i_{1j}(x)y^j y^j = 0,
$$

where $\Gamma^i_{1j}(x) = \frac{1}{2}\sigma^i_k g^{jk}$. The metric $g$ is positive-definite and the tensor $\Lambda_\varepsilon$ corresponds to a ‘contravariant metric’ $G$. This is the Hamiltonian vector field of $\Lambda_\varepsilon$ with a Hamiltonian

$$
H(x, \xi) = \frac{1}{2}g^{ij}(x)\xi_i\xi_j.
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The equations $(E^1_L)$ and $(E^2_L)$ coincide and read

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The equations $(E^1_L)$ and $(E^2_L)$ coincide and read

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$$

$$
\frac{d}{dt} \left( g_{ik}y^i \right) = \left( c^i_{ik}g_{sj} + \frac{1}{2}\sigma^i_k \frac{\partial g_{ij}}{\partial x^a} \right) y^j y^j.
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The last equation can be rewritten in the form

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where $\Gamma^i_{1j}(x) = \frac{1}{2}\sigma^i_k g^{jk}$. The metric $g$ is positive-definite and the tensor $\Lambda_\varepsilon$ corresponds to a ‘contravariant metric’ $G$. This is the Hamiltonian vector field of $\Lambda_\varepsilon$ with a Hamiltonian

$$
H(x, \xi) = \frac{1}{2}g^{ij}(x)\xi_i\xi_j.
$$

The equations $(E^1_L)$ and $(E^2_L)$ coincide and read

$$
\dot{x}^a = \rho^a \xi^k,
$$

$$
\frac{d}{dt} \left( g_{ik}y^i \right) = \left( c^i_{ik}g_{sj} + \frac{1}{2}\sigma^i_k \frac{\partial g_{ij}}{\partial x^a} \right) y^j y^j.
$$

The last equation can be rewritten in the form

$$
y^i + \Gamma^i_{1j}(x)y^j y^j = 0,
where
\[ \Gamma^l_{ij} = \frac{1}{2}g^{kl} \left( \rho^b_j \frac{\partial g_{ik}}{\partial x^b} + \rho^b_i \frac{\partial g_{jk}}{\partial x^b} - \sigma^b_{ik} \frac{\partial g_{bj}}{\partial x^b} - c^b_{ij} g_{bs} - c^b_{jk} g_{si} \right). \]  
(15)

The equations
\[ \dot{x}^a = \rho^a_i(x) \dot{y}^i, \quad \dot{y}^i + \Gamma^l_{ij}(x) y^i y^j = 0, \]  
(16)

with \( \Gamma^l_{ij} \) as in (15), are generalized geodesic equations, since for the case \( E = TM \) in adapted coordinates we have \( c^l_{ij} = 0, \) \( \rho^a_i = \sigma^a_i = \delta^a_i, \) so they reduce to the standard geodesic equation with \( \Gamma^l_{ij} \) being the Christoffel symbols of the Levi-Civita connection associated with the metric \( g. \) In the general case of an algebroid (except for skew-symmetric ones) the notion of Levi-Civita connection is unclear, since the concept of the torsion is unclear. The ‘Christoffel symbols’ \( \Gamma^l_{ij} \) can be understood however as the structure constants of the algebroid lift of the contravariant metric \( G \) pushed forward to \( E^* \) by the diffeomorphism \( \tilde{g}. \) To be more precise, let us observe first that the algebroid lift \( d_T^+ G \) of \( G \) with respect to the adjoint algebroid structure \( \varepsilon^+ \) is a symmetric contravariant tensor on \( E \) which reads
\[ d_T^+ G = \tilde{c}^a_{ij} y^a \partial_{y^i} \otimes \partial_{y^j} + \tilde{\rho}^a_i (\partial_{y^i} \otimes \partial_{x^a} + \partial_{x^a} \otimes \partial_{y^i}), \]
where
\[ \tilde{\rho}^a_i \overset{=}{=} -g^{ij} \rho^a_j; \]
\[ \tilde{c}^a_{ij} \overset{=}{=} g^{kj} c^j_{ks} + g^{ki} c^j_{ks} - \frac{\partial g^{ij}}{\partial x^a} g^a_s. \]

Its push-forward to \( E^* \) is a symmetric contravariant tensor which turns out to be
\[ \tilde{G} = g_*(d_T^+ G) = -2\Gamma^l_{ij} \xi_i \partial_{\xi_j} \otimes \partial_{\xi_j} - \rho^a_i (\partial_{\xi_i} \otimes \partial_{x^a} + \partial_{x^a} \otimes \partial_{\xi_i}), \]
with \( \Gamma^l_{ij} \) as in (15). One can produce, say, a left connection \( \nabla \) out of \( \tilde{G} \) by putting \( \nabla = \frac{1}{2} (A_\varepsilon - \tilde{g}_* (d_T^+ G)), \) i.e. by
\[ \nabla_X Y = \frac{1}{2} ([X, Y]_\varepsilon - \tilde{G}(X, Y)). \]
(17)

In the case of a skew-symmetric algebroid \( \varepsilon, \) the connection (17) is the Levi-Civita connection of the metric \( g \) (uniquely determined like in the standard case, cf. [7]) and the symmetrization of its Christoffel symbols gives exactly the symbols \( \Gamma^l_{ij} \).

b) Generalized Wong equations.

Consider an algebroid \( E \) which is the direct product \( E = TM \times g \) of the canonical Lie algebroid \( TM \) over a manifold \( M \) of dimension \( m \) and an arbitrary \( \mathbb{R} \)-algebra \( g \) of dimension \( n \) (the setting could be more general, based on a short exact sequence of algebroids, but we have chosen this one for simplicity).

Both anchors coincide with the projection \( pr_1 \) on the first factor, i.e. on \( TM. \) For local coordinates \( (x^a) \) in \( M \) and a basis \( (v_i) \) of \( g \) we have the adapted coordinates in \( E: \) \((x^a, x^b, \bar{v}^i), \) where \((\bar{v}^i)\) is the basis in \( g^* \) dual to \((v_i)\). Of course, here we understand \( TM \) and \( M \times g \) as subbundles of \( E \) according to the natural immersions \( I_1 \) and \( I_2. \) Let us assume additionally that we have a Riemannian metric \( g \) on \( M \) and a metric \( h \) on \( g. \) With every connection \( A: TM \to E, \) i.e. with every vector bundle morphism \( A: TM \to E \) over the identity such that \( pr_1 \circ A = id_{TM}, \) we can associate a metric \( g_A \) on \( E \) by
\[ g_A(X, Y) = g(pr_1 X, pr_1 Y) + h(X - A(pr_1 X), Y - A(pr_1 Y)). \]
(18)

In other words, this is the product of metrics \( g \) and \( h \) with respect to the identification of \( E \) with \( TM \times g \) via
\[ A: TM \times g \to TM \times g, \quad A = A \circ pr_1 + I_2 \circ pr_2. \]
The metric induces a hyperregular quadratic Lagrangian \( L_A \) on \( E \) as above and thus the corresponding Euler-Lagrange equations associated with the product algebroid structure \( \varepsilon. \)
An equivalent approach is to consider, instead of (18), just the product metric on \( T^*M \times g \) with the corresponding Lagrangian of the form
\[
L(x, \dot{x}, \dot{v}) = \frac{1}{2} \left( \sum_{i,j \leq n} h_{ij} \dot{v}^i \dot{v}^j + \sum_{a,b \leq m} g_{ab}(x) \dot{x}^a \dot{x}^b \right),
\]
(19)
together with a deformed algebroid structure \( \varepsilon_A \) on \( E = T^*M \times g \) obtained via the isomorphism \( \bar{A} : T^*M \times g \to T^*M \times g \). Let us fix this new algebroid structure on \( E \) by fixing a connection \( A \). In local coordinates, \( A(\partial_{x^a}) = \partial_{x^a} + \sum_{i \leq n} A^i_a(x) v_i \). The contravariant tensor \( \Lambda_{\varepsilon_A} \) on \( E^* \simeq T^*M \times g^* \) reads then
\[
\Lambda_{\varepsilon_A} = C^i_{ij} v_i \partial_{x^a} \otimes \partial_{v_j} + A^i_a(x) C^i_{ij} v_i \partial_{p^a} \otimes \partial_{v_j} + A^i_a(x) C^i_{ij} v_i \partial_{p^a} \otimes \partial_{p^a} + F^i_{ab}(x) \partial_{p^a} \otimes \partial_{p^b} + \partial_{p^a} \wedge \partial_{x^a},
\]
where \( C^i_{ij} \) are the structure constants of the algebra \( g \) with respect to the basis \( v_i \), and \( F^i_{ab} \) are the coordinates of the curvature \( F \) of the connection \( A \) with respect to the product algebroid structure \( \varepsilon \) defined by \( F(X, Y) = [A(X), A(Y)]_x - A([X, Y]_x). \) In local coordinates,
\[
F^i_{ab}(x) = \frac{\partial A^i_a(x)}{\partial x^b} - \frac{\partial A^i_b(x)}{\partial x^a} + A^i_a(x) A^j_b(x) C^i_{ij}.
\]
It is a matter of standard calculations to show that in this case the Hamiltonian dynamics \( D \) on \( T^*M \times g^* \) is represented by the equations
\[
\begin{align*}
\frac{d}{dt}(x^a) &= g^{ab}(x)p_b, \\
\frac{d}{dt}(p_a) &= F^i_{ba}(x) g^{bc}(x)p_c v_i - \frac{1}{2} \frac{\partial g^{bc}}{\partial x^a}(x)p_b p_c + A^i_a(x) C^i_{ij} h^{ij} v_i v_j, \\
\frac{d}{dt}(v_i) &= A^i_a(x) C^i_{ij} g^{ab}(x)p_a v_i + C^i_{ij} h^{js} v_i v_j.
\end{align*}
\]
(20)
Since the Lagrangian (19) is hyperregular, this phase dynamics is Hamiltonian for the tensor \( \Lambda_{\varepsilon_A} \) and a clear Hamiltonian function reads
\[
H = \frac{1}{2}(g^{ab} p_a p_b + h^{ij} v_i v_j).
\]
If \( g \) is a Lie algebra and the metric \( h \) is invariant, then \( C^i_{ij} h^{js} v_i v_j = 0 \) and the above equations reduce to the celebrated Wong equations [29], usually written in the form
\[
\begin{align*}
\frac{d}{dt}(v_i) &= A^i_a(x) C^i_{ij} h^{js} v_i, \\
\frac{d}{dt}(p_a) &= F^i_{ba}(x) \dot{x}^b v_i - \frac{1}{2} \frac{\partial g^{bc}}{\partial x^a}(x)p_b p_c.
\end{align*}
\]
(21)
(22)
Possible differences in signs in various references are effects of differences in conventions. The equation (21) is usually called the first Wong equation and the equation (22) – the second Wong equation. Note that these equations have an alternative Hamiltonian description [1]. The corresponding ‘Euler-Lagrange equation’ in our general case can be easily derived from (20):
\[
\begin{align*}
\frac{d}{dt}(x^a) &= \dot{x}^a, \\
\frac{d}{dt}(\dot{x}^d) &= g^{da} \left( F^i_{ba} h_{ls} \dot{x}^b \dot{v}^s + A^i_a C^i_{js} h_{ls} \dot{v}^s \dot{v}^j + \frac{1}{2} \left( \frac{\partial g^{ac}}{\partial x^b} + \frac{\partial g^{bc}}{\partial x^a} - \frac{\partial g^{ab}}{\partial x^c} \right) \dot{x}^b \dot{x}^c \right), \\
\frac{d}{dt}(\dot{v}^k) &= h^{lk} \left( A^i_a C^i_{jk} h_{ls} \dot{x}^a \dot{v}^s + C^i_{jk} h_{ls} \dot{v}^s \right).
\end{align*}
\]
(23)
The constants \( C^i_{ij} \) can be arbitrary in our approach. Of course, the generalized Wong equations (23) form a particular case of the generalized geodesic equations (16). Note that Lie algebroid versions of these equations are known and can be found e.g. in [3] and [13].
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