Transient Solutions of an Isotropic Thermostatic Pervious Half Space Subjected to a Point Hot Fluid Injection

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Keywords: Point Hot Fluid Injection, Closed-form Solution, Half Space, Transient Solution

Abstract. Elastic transient deformation of a half space due to a point of hot fluid injection at constant rate is studied in this paper. The half space ground surface is modelled as a thermostatic pervious boundary. Biot’s three-dimensional consolidation theory with thermal effect is introduced to derive the analytical solutions of the transient consolidation deformation due to a point of hot fluid injection. The stratum is modelled as a saturated isotropic poroelastic half space. Using Laplace and Hankel integral transforms, closed-form solutions of the horizontal and vertical displacements are obtained. The maximum ground surface horizontal displacement can be exactly derived, and it is around 30% of the maximum ground surface vertical displacement. Moreover, the golden ratio appears in the maximum ground surface horizontal displacement and corresponding vertical displacement of a half space. The study concludes that golden ratio emerges in this phenomenon, and the horizontal displacement should be properly considered in the prediction of vertical displacement induced by a hot fluid injection. The study can provide better understanding of the hot fluid injection induced responses of an isotropic porous elastic half space.

1. Introduction

Half space responses due to hot fluid injection are an important geotechnical engineering issues. Considerable research has been devoted to analyze the mechanical, hydraulic, and thermal behavior of hot fluid injection for the impact on engineering safety. Usually, hydraulic and thermal impact result in a volumetric change of fluid and solid skeleton of the strata. The increasing excess pore fluid pressure can cause decrease in effective stress, which can result in a hydraulic-thermal failure in the strata due to the loss of shear resistance of solid skeleton. The physical simulation of these features is a complex task, and its validation is a major concern for the safety improvement of the hot fluid injection.

Biot’s general three-dimensional consolidation theory [1, 2] is usually regarded as the fundamental theory for modelling geotechnical issues. Based on Biot’s theory, Tarn and Lu [3] presented closed-form solutions of subsidence due to a point sink embedded in saturated elastic half space at a constant rate. Tarn and Lu found that groundwater withdrawal from an impervious half space induces a larger amount of consolidation settlement than from a pervious half space. Recently, the hot spring withdrawals experienced rapid growths in Jiao-Shi, Taiwan, and the estimation of hot spring recharge were considered by Lin et al. [4] and Chen et al. [5]. Sasaki et al. [6] investigated a system of gas production from methane hydrate layers involving hot water injection using dual horizontal wells. Al-Wahaibi et al. [7] presented experimentally the potential of enhancing oil recovery from Middle East heavy oil field via hot water injection. Rosenbrand et al. [8] studied seasonal energy storage achieved by hot water injection in geothermal sandstone aquifers. In the study of Gai et al. [9], the development characteristics of hot water flooding in different rhythm reservoir after steam drive by means of physical simulation were investigated. Nevertheless, transient closed-form solutions of the half space due to hot fluid injection were not obtained in the above studies.
The basic governing equations are based on porous mechanics with thermal effect. Therefore, the hot fluid injection point induced transient ground surface horizontal and vertical displacements are obtained by integral transforms. Figure 1 shows a point of hot fluid source injected into a stratum at depth \( h \) where the stratum is modelled as a saturated thermally isotropic elastic half space, and the ground surface of the half space is treated as a thermostatic pervious boundary. Using Laplace and Hankel transforms, the transient horizontal and vertical displacements of the ground surface due to a hot fluid injection are obtained. Results can provide better understanding of the time dependent ground surface displacements due to a point of hot fluid injection.

![Diagram of hot fluid injection](image)

Figure 1. Hot fluid injection with constant rate at a depth \( h \).

2. Mathematical model

2.1. Governing Equations

Figure 1 shows a point of hot fluid recharge into a saturated porous stratum at a depth \( h \). The soil mass is considered as a homogeneous isotropic porous medium, and the constitute law can be expressed as:

\[
\sigma_{ij} = 2Ge_{ij} + \frac{2G}{1-2\nu} \varepsilon_{kk} \delta_{ij} - \frac{2G(1+\nu)}{1-2\nu} \nu \delta_{ij} - p \delta_{ij} \quad i, j = 1, 2, 3, \tag{1}
\]

in which \( \sigma_{ij} \) are the total stress components; \( \vartheta \) is the temperature increment measured from the reference state; \( \varepsilon_{ij} \) are the strain components; \( \alpha_s \) is linear thermal expansion coefficient; \( \nu \) is Poisson’s ratio, and \( G \) is shear modulus of the stratum. The excess pore fluid pressure \( p \) is positive for compression, and \( \delta_{ij} \) is the Kronecker delta. The strains \( \varepsilon_{ij} \) and displacement components \( u_i \) are given by the linear law:

\[
\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad i, j = 1, 2, 3. \tag{2}
\]

The total stress \( \sigma_{ij} \) must satisfy the equilibrium equations:

\[
\sigma_{ij,j} + b_i = 0 \quad i, j = 1, 2, 3. \tag{3}
\]

Where \( b_i \) denote the body forces. Equations (1) and (2) are used in the equilibrium equations to express their forms in displacements \( u_i \), temperature increment \( \vartheta \) and excess pore fluid pressure \( p \) as follows:

\[
Gu_{i,j} + \frac{G}{1-2\nu} \varepsilon_{kk} \delta_{ij} - \frac{2G(1+\nu)}{1-2\nu} \nu \delta_{ij} - p \delta_{ij} = 0 \quad i = 1, 2, 3. \tag{4}
\]

Consider a point of hot fluid recharge at a constant rate with volume \( Q_f \) and thermal strength \( Q_t \) per unit time at point \((0, h)\). The uncoupled hydraulic and thermal governing equations in axially symmetric coordinates \((r, z)\) are derived from the conservation of mass and Darcy’s law, conservation of energy and heat conduction law as below:

\[
-\frac{k}{\gamma_f} p_{,jj} + n \beta \frac{\partial p}{\partial t} - \frac{Q_f}{2\pi h} \sigma_{\delta}(r) \delta(z-h) u(t) = 0, \tag{5a}
\]
\[-λ_\varphi β + c_r \frac{\partial \varphi}{\partial t} - \frac{Q_i}{2\pi r} \delta(r) \delta(z-h) u(t) = 0, \quad (5b)\]

Where \(k\) and \(n\) are the permeability and porosity of the porous medium, respectively; \(β\) is the compressibility of pore fluid; \(γ_f\) is the unit weight of pore fluid; \(λ_\varphi\) is the thermal conductivity, and \(c_r = ρ c\). The constants \(ρ\) and \(c\) define density and specific heat of the poroelastic medium, respectively; \(δ(x)\) is the Dirac delta, and \(u(t)\) is Heaviside unit step function. Equations (4), (5a) and (5b) constitute the basic governing equations of the time-dependent thermally poroelastic responses of a saturated porous medium.

2.2. Boundary Conditions and Initial Conditions

The pervious half space surface, \(z = 0\), is considered as traction-free, and it does not have temperature change for all times \(t ≥ 0\). Then the boundary conditions on half space ground surface \(z = 0\) are given by

\[\sigma_{ez}(r,0,t) = 0, \quad \sigma_{zz}(r,0,t) = 0, \quad \varphi(r,0,t) = 0 \quad \text{and} \quad p(r,0,t) = 0. \quad (6)\]

Assuming there are no initial change of displacement, seepage and temperature for the thermally poroelastic medium, the initial conditions at time \(t = 0\) due to a point of hot fluid recharge can be treated as

\[u_r(r,0,0) = 0, \quad u_z(r,0,0) = 0, \quad \varphi(r,0,0) = 0 \quad \text{and} \quad p(r,0,0) = 0. \quad (7)\]

The point of hot fluid injection source is assumed no effect at the remote boundary of \(z → \infty\) for all times \(t ≥ 0\). Hence

\[\lim_{z→∞} [u_r(r,z,t), u_z(r,z,t), \varphi(r,z,t), p(r,z,t)] = [0, 0, 0, 0]. \quad (8)\]

Equations (6), (7) and (8) formulate the boundary conditions and initial conditions of this study.

3. Analytic solutions

Applying Laplace and Hankel integral transformations \([10, 11]\), the transient horizontal displacement and vertical displacement of the ground surface \(z = 0\) due to a point of hot fluid injection at constant rate in axially symmetric coordinates \((r, z)\) are obtained as follows:

\[u_r(r,0,t) = \frac{(1 - 2ν)Q_1 γ_f}{2πGk} \left\{ \frac{c_{rtr}}{(h^2 + r^2)^{3/2}} - \int_0^1 \frac{c_{rtr}}{(h^2 + r^2)^{3/2}} \exp \left\{ -r^2 - 2h^2 + \frac{8}{r^2} \right\} \left[ I_0 \left( \frac{r^2}{8σ} \right) - I_1 \left( \frac{r^2}{8σ} \right) \right] dσ \right\} + \frac{(1 + ν)Q_1 α}{πλ_1} \left\{ \frac{c_{rtr}}{(h^2 + r^2)^{3/2}} \right. \left( \int_0^1 \exp \left\{ -r^2 - 2h^2 + \frac{8}{r^2} \right\} \left[ I_0 \left( \frac{r^2}{8σ} \right) - I_1 \left( \frac{r^2}{8σ} \right) \right] dσ \right\}, \quad (9a)\]

\[u_z(r,0,t) = \frac{(1 - 2ν)Q_1 γ_f}{2πGk} \left\{ \frac{c_{zth}}{(h^2 + r^2)^{3/2}} \left( \frac{h^2 + r^2}{2c_{zth}} \right) \right\} \]

\[\left. - \frac{h}{h^2 + r^2} \sqrt{\frac{c_{zth}}{\pi}} \exp \left\{ -\frac{h^2 + r^2}{4c_{zth}} \right\} + \frac{h}{2\sqrt{h^2 + r^2}} \text{erfc} \left( \frac{\sqrt{h^2 + r^2}}{2c_{zth}} \right) \right\} + \frac{(1 + ν)Q_1 α}{πλ_1} \left\{ \frac{c_{zth}}{(h^2 + r^2)^{3/2}} \text{erf} \left( \frac{\sqrt{h^2 + r^2}}{2c_{zth}} \right) \right\}

\[\left. - \frac{h}{h^2 + r^2} \sqrt{\frac{c_{zth}}{\pi}} \exp \left\{ -\frac{h^2 + r^2}{4c_{zth}} \right\} + \frac{h}{2\sqrt{h^2 + r^2}} \text{erfc} \left( \frac{\sqrt{h^2 + r^2}}{2c_{zth}} \right) \right\}, \quad (9b)\]

Where \(c_f = k/\beta γ_f\) and \(c_i = λ/ρ c\). The special functions \(\text{erf}(x)\) and \(\text{erfc}(x)\) denote the error function and complementary error function, respectively; and \(I_n(x)\) is known as the modified Bessel
function of the first kind of order $n$. The long-term ground surface horizontal and vertical displacements can be obtained by letting $t \to \infty$ in equations (9a) and (9b):

$$\begin{align*}
u_r(r,0,\infty) &= \frac{1}{4\pi Gk} \left[ (1-2\nu)Q_f \gamma_f + (1+\nu)Q_s \alpha_s \right] h + \frac{1}{2\pi \lambda_t} \sqrt{h^2 + r^2}, & (10a) \\
u_z(r,0,\infty) &= \frac{1}{4\pi Gk} \left[ (1-2\nu)Q_f \gamma_f + (1+\nu)Q_s \alpha_s \right] \frac{h}{2\pi \lambda_t} + \frac{1}{\sqrt{h^2 + r^2}}. & (10b)
\end{align*}$$

The maximum ground surface horizontal displacement $u_{r,\text{max}}$ and vertical displacement $u_{z,\text{max}}$ of the half space due to a point of hot fluid injection are derived from equations (10a) and (10b) by letting $r = \sqrt{\phi}h = 1.272h$ and $r = 0$, respectively:

$$\begin{align*}
u_{r,\text{max}} &= \nu_r(\sqrt{\phi}h,0,\infty) = \frac{1}{\sqrt{1+\phi(1+\phi+1)}} \left[ (1-2\nu)Q_f \gamma_f + (1+\nu)Q_s \alpha_s \right] \\
&= \frac{1}{\sqrt{\phi^2}} \left[ (1-2\nu)Q_f \gamma_f + (1+\nu)Q_s \alpha_s \right] \\
&= 0.3003 \left[ (1-2\nu)Q_f \gamma_f + (1+\nu)Q_s \alpha_s \right]. & (11a)
\end{align*}$$

$$\begin{align*}
u_{z,\text{max}} &= \nu_z(0,0,\infty) = \frac{1}{4\pi Gk} \left[ (1-2\nu)Q_f \gamma_f + (1+\nu)Q_s \alpha_s \right] \frac{h}{2\pi \lambda_t}. & (11b)
\end{align*}$$

in which $\phi = (1+\sqrt{5})/2 = 1.618$ is known as the golden ratio. The maximum ground surface horizontal and vertical displacements can be exactly derived as shown in equations (11a) and (11b), and $u_{r,\text{max}}$ is around 30% of $u_{z,\text{max}}$. The value $r = \sqrt{\phi}h$ is derived when $du_r(r,0,\infty)/dr$ is equal to zero, i.e.,

$$\begin{align*}
du_r(r,0,\infty)/dr &= \left[ (1-2\nu)Q_f \gamma_f + (1+\nu)Q_s \alpha_s \right] \frac{h \sqrt{h^2 + r^2}}{2\pi \lambda_t} \left( h^2 + r^2 \right)^{1/2} = 0. & (12)
\end{align*}$$

This leads to solutions of $r = \pm \sqrt{(1+\sqrt{5})/2h}$ and $r = \pm \sqrt{(1-\sqrt{5})/2h}$. However, only $r = \sqrt{(1+\sqrt{5})/2h}$ is realistic for $r \in [0,\infty)$.

It is interesting to find that the golden ratio $\phi$ is not only appeared in the point of hot fluid injection induced maximum ground surface horizontal displacement but also on the corresponding vertical displacement by letting $r = \sqrt{\phi}h$ in Eq. (10b). Thus, we have:

$$\begin{align*}
u_z(\sqrt{\phi}h,0,\infty) &= \left[ (1-2\nu)Q_f \gamma_f + (1+\nu)Q_s \alpha_s \right] \frac{h}{2\pi \lambda_t} \frac{1}{\sqrt{1+\phi}} = \frac{u_{z,\text{max}}}{\phi} = 0.618u_{z,\text{max}}. & (13)
\end{align*}$$

This shows that the ground surface vertical displacement is around 61.8% of the maximum ground surface vertical displacement $u_{z,\text{max}}$ at $r = \sqrt{\phi}h$, where the maximum ground surface horizontal displacement $u_{r,\text{max}}$ occurred. Besides, the equations (11a) and (11b) show that the maximum ground surface horizontal and vertical displacements are not directly dependent on the depth $h$ of the hot fluid injection point. The study concludes that golden ratio emerges in this phenomenon, and the horizontal displacement should be properly considered in the prediction of vertical displacement induced by a hot fluid injection.

4. Conclusions
Using Laplace and Hankel transformations, transient closed-form solutions of the consolidation due to a hot fluid injection into the thermostatic pervious elastic half space were obtained. The ground
surface horizontal displacement and vertical displacement are presented in equations (9a) and (9b). The maximum ground surface vertical displacement $u_{c, \text{max}}$ is derived as shown in equation (11b). The results can provide better understanding of the hot fluid injection induced responses of an isotropic poroelastic half space. The results show:

1. The maximum ground surface horizontal displacement is around 30% of the maximum surface vertical displacement at $r = \sqrt{\phi} = 1.272h$ where $\phi = (1+\sqrt{5})/2 = 1.618$ is known as the golden ratio. Besides, the ground surface vertical displacement is around 61.8% of the maximum ground surface vertical displacement at the same location $r = \sqrt{\phi}h$.
2. The magnitude of maximum ground surface horizontal displacement and vertical displacement are independent of the depth $h$ of the hot fluid injection point.

Acknowledgements

This work is supported by the Ministry of Science and Technology of Taiwan, Republic of China, through grants NSC89-2211-E-216-003 and MOST105-2625-M-415-001.

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