Entanglement and quantum transport in integrable systems

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Understanding the entanglement structure of out-of-equilibrium many-body systems is a challenging yet revealing task. Here we investigate the entanglement dynamics after an inhomogeneous quench in integrable systems. This is the prototypical setup for studying quantum transport, and it consists in the sudden junction of two macroscopically different states. By exploiting the recently developed integrable hydrodynamic approach and the quasiparticle picture for the entanglement dynamics, we conjecture a formula for the entanglement production rate after joining two semi-infinite reservoirs, as well as the steady-state entanglement entropy of a finite subregion. We show that both quantities are determined by the quasiparticles created in the Non Equilibrium steady State (NESS) appearing at large times at the interface between the two reservoirs. Specifically, the steady-state entropy coincides with the thermodynamic entropy of the NESS, whereas the entropy production rate reflects its spreading into the bulk of the two reservoirs. Our results are numerically corroborated using tDMRG simulations in the paradigmatic XXZ spin-1/2 chain.

Introduction.— The quest for a complete understanding of entanglement spreading in out-of-equilibrium many-body system is a fruitful research theme in contemporary physics. Recently, this became experimentally relevant, as it is now possible to measure the entanglement dynamics with cold atoms [1, 2]. The best-known entanglement diagnostic tool is the von Neumann (entanglement) entropy $S \equiv -\text{Tr} \rho_A \ln \rho_A$, where $\rho_A$ is the reduced density matrix of a subsystem $A$ (see Figure 1 (a) for a one-dimensional setup).

A prominent out-of-equilibrium situation is that of the inhomogeneous quantum quench. The prototypical setup for spin chains is depicted in Figure 1 (a), and it consists in the sudden junction of two chains $A$ and $B$ that are prepared in two macroscopically different quantum states $|\Psi_A\rangle$ and $|\Psi_B\rangle$, respectively. Subsequently, the state $|\Psi_A\rangle \otimes |\Psi_B\rangle$ is evolved in real time using a quantum many-body Hamiltonian $H$. The main focus so far has been the study of transport of local quantities, such as the local energy and local magnetization. Several techniques have been used, such as Conformal Field Theory (CFT), free-fermion methods [9–17], field theory methods [18–20], integrability [21–25], and numerical techniques [26–30]. For integrable models a recent breakthrough [31, 32] allows for an analytic treatment of transport problems using Thermodynamic Bethe Ansatz (TBA) techniques [33–43].

Surprisingly, as of now the entanglement dynamics after inhomogeneous quenches remained unexplored. Notable exceptions are inhomogeneous systems that can be mapped to CFTs in curved spacetime [6–8]. In this case it is established that $S$ grows logarithmically after the quench [7]. Also, there is growing interest in studying inhomogeneous quenches using holography [44].

In contrast, for homogeneous quenches a well-known quasiparticle picture [45] allows one to understand the entanglement dynamics in terms of quasiparticles traveling ballistically, with opposite quasimomenta, through the system. The underlying idea is that only quasiparticles created at the same point in space are entangled. At a generic time $t$ the entanglement entropy $S(t)$ between a subregion $A$ and the rest is proportional to the number of quasiparticles emitted at the same point in space at $t = 0$, and being at time $t$ one in subsystem $A$ and the other in $B$. More quantitatively,

$$S(t) \propto 2t \int_{|v| < t} d\lambda |v(\lambda)| f(\lambda) + \ell \int_{|v| > t} d\lambda |v(\lambda)|,$$

where $f(\lambda)$ depends on the cross section for creating quasiparticles with quasimomentum $\lambda$, and $v(\lambda)$ is their velocity. Eq. (1) holds in the space-time scaling limit, i.e., $\ell, t \to \infty$ with fixed $t/\ell$. In many physical situations a maximum velocity $v_M$ exists, for instance, due to the Lieb-Robinson bound [46]. Then, Eq. (1) predicts a linear growth for $t \leq \ell/(2v_M)$, followed by a volume-law $S \propto \ell$ at longer times.
The validity of (1) has been proven rigorously only for free-fermion models [47–52], for which \( f(\lambda) \) and \( v(\lambda) \) can be determined analytically. The quasiparticle picture has been also confirmed in several numerical studies [53–55], and using holographic methods [56–63]. Remarkably, a framework to render (1) predictive for generic integrable systems has been developed in Ref. 64 (see Ref. 65 for an application to the Hubbard chain). Specifically, for integrable systems \( f(\lambda) \) coincides with the thermodynamic entropy associated with the post-quench steady state, while the quasiparticle velocities \( v(\lambda) \) are those of the low-lying excitations around it.

Here by combining a recent hydrodynamic approach for integrable systems [31, 32] with the quasiparticle picture (1), we provide the first exact results for the entanglement dynamics after an inhomogeneous global quench in generic integrable systems. Specifically, we derive the steady-state entanglement entropy of a finite region \( A' \) (see Figure 1 (b) for the considered geometry), as well as the entanglement production rate when two semi-infinite reservoirs \( A \) and \( B \) are put in contact. Our main result is that both are determined by the physics of the Non-Equilibrium-Steady-State (NESS) (ray with \( \zeta = 0 \) in Figure 1 (a)) appearing at the interface between the two reservoirs. The steady-state entanglement entropy coincides with the thermodynamic entropy of the NESS, whereas the entanglement production rate reflects the spreading of the NESS from the interface into the bulk of the two reservoirs. To corroborate our results we present time-dependent Density Matrix Renormalization Group [66–69] (tDMRG) simulations [70].

**Entanglement via integrable hydrodynamics.**— The first key ingredient to derive our results is that the spectrum of integrable models exhibits families of stable quasiparticles. Typically, these are composite objects of elementary excitations. For spin chains, they correspond to bound states of magnon-like excitations. The possible set of quasimomenta (rapidity) \( \lambda \) that can be assigned to the quasiparticles are obtained by solving the so-called Bethe equations [71]. In the thermodynamic limit the rapidities form a continuum. Thermodynamic properties of integrable models are described by the rapidity densities \( \rho_n(\lambda) \) and the hole densities \( \rho_n^h(\lambda) \). The latter is the density of unoccupied rapidities. The total density is defined as \( \rho_n^{(t)}(\lambda) \equiv \rho_n + \rho_n^h \). Here the subscript \( n \) labels different quasiparticle families, and for spin chains is the size of the bound states. Importantly, for free models in terms of quasimomenta \( \rho_n^{(t)} = \text{const} \), reflecting that the quasimomenta are equally-spaced. The densities are the central objects in the Thermodynamic Bethe Ansatz (TBA) approach [71]. Every set of \( \rho_n, \rho_n^h \) is interpreted as a thermodynamic macrostate, which corresponds to an exponentially large number of microscopic eigenstates of the model. Their number is given as \( e^{S_{YY}} \), with \( S_{YY} \) the so-called Yang-Yang entropy [72]

\[
S_{YY} = s_{YY} L = L \sum_{n=1}^{\infty} \int d\lambda [\rho_n^{(t)} \ln \rho_n^{(t)} - \rho_n \ln \rho_n - (\rho_n^{(t)} - \rho_n) \ln \rho_n^h] \\
\equiv L \sum_n \int d\lambda s_n^{(t)}[\rho_n, \rho_n^h]. \tag{2}
\]

Similar to free models, \( S_{YY} \) counts the number of ways of assigning the different rapidities \( \lambda \) to the quasiparticles, compatibly with the densities \( \rho_n, \rho_n^h \).

The second key ingredient in our approach is the integrable hydrodynamics framework [31, 32]. Due to integrability, after the quench information spreads ballistically from the interface between \( A \) and \( B \). As a consequence, physical observables depend only on the combination \( \zeta \equiv x/t \) (see Figure 1 (a)), with \( x \) the distance from the interface between \( A \) and \( B \). For each fixed \( \zeta \), dynamical properties of local and quasilocabilities are described by a thermodynamic macrostate, i.e., a set of densities \( \rho_{\zeta,n} , \rho_{\zeta,n}^{(t)} \). Each macrostate identifies a different Generalized Gibbs Ensemble [45, 73–103]. The macrostate with \( \zeta = 0 \) is known as Non Equilibrium Steady State (NESS). The central result of Ref. 31 and 32 is that the densities \( \rho_{\zeta,n} \) satisfy the continuity equation à la Boltzmann

\[
(\zeta - v_{\zeta,n}(\lambda)) \partial_\lambda \rho_{\zeta,n}(\lambda) = 0, \tag{3}
\]

where \( \partial_{\zeta,n} \equiv \rho_{\zeta,n} / \rho_{\zeta,n}^{(t)} \), together with the standard TBA equations [71]

\[
\rho_{\zeta,n}^{(t)}(\lambda) = a_n(\lambda) + \sum_{n=1}^{\infty} (a_{n,m} \ast \rho_{\zeta,m}(\lambda)). \tag{4}
\]

Here \( a_n \) and \( a_{n,m} \) are known functions of \( \lambda \) (see Supplementary Material for their expression for the XXZ chain), and the star symbols denotes the convolution \( f \ast g \equiv \int d\mu f(\lambda - \mu)g(\mu) \). Crucially, in (3) \( v_{\zeta,n} \) are the velocities of the low-lying excitations around the macrostate \( \rho_{\zeta,n} \), and fully encode the interactions (scatterings) between quasiparticles [104]. They can be calculated using standard TBA techniques [104]. The solutions of (3) can be conveniently written as [32]

\[
\partial_{\zeta,n}(\lambda) = \theta_H (v_{\zeta,n}(\lambda) - \zeta)(\rho_n^{h}(\lambda) - \zeta \rho_n^{h}(\lambda)) + \rho_n^{h}(\lambda), \tag{5}
\]
with $\theta_H(x)$ the Heaviside function, and $\varrho^{A(\ell)}_{n}$ the densities describing the homogeneous quenches with initial states $|\Psi_A\rangle$ and $|\Psi_B\rangle$, respectively. Although in general it is a formidable task, exact results for $\varrho^{A(\ell)}_{n}$ are available for several quenches [105–120]. We now present our main results. We start discussing the steady-state entanglement entropy of a finite region $A'$ of length $\ell$ embedded in part $A$ and placed next to the interface with $B$ (see Figure 1 (b)). The steady-state entropy is obtained in the limit $\ell/t \to 0$, which identifies the macrostate with $\zeta = 0$ (NESS). The spatial extension of the region described by this macrostate increases linearly with time, reflecting ballistic propagation of information. This implies that for $t \gg \ell$ the $\zeta = 0$ macrostate is expected to describe the entire subsystem $A'$. Similar to Ref. 64, it is natural to conjecture that the entanglement entropy density $s_{\infty}$ is calculated using the solutions of (3) or (4).

\[ s_{\infty} = \sum_{n=1}^{\infty} \int d\lambda |v_{\zeta=0,n}| \rho^{(n)}_{\zeta=0,n} \]  

where $s^{(n)}_{\zeta}$ is calculated using the solutions of (3) or (4).

We now turn to the entanglement production rate. For homogeneous quenches, this corresponds to the slope of the linear term in (1). Physically, Eq. (1) implies that $S(t)$ is determined by the total number of quasiparticles that at time $t$ crossed the interface between the two subsystems. It is natural to assume that the same applies to the inhomogeneous case. Again, quasiparticles crossing the interface are described by the macrostate with $\zeta = 0$. At generic time $t$ their number is proportional to the width of the associated lightcone spreading from the interface (see Figure 1 (b)). Equivalently, the entanglement growth at short times reflects the spreading of the NESS into the bulk of the two chains. Since the lightcone width increases linearly with time, reflecting the ballistic propagation of the quasiparticles, one should expect the linear behavior $S(t) \propto s't$, with $s' \equiv dS(t)/dt$ the entanglement production rate given as

\[ s' = \sum_{n=1}^{\infty} \int d\lambda |v_{\zeta=0,n}| \rho^{(n)}_{\zeta=0,n} \]  

Formally, Eq. (7) is the same as that for the homogeneous quench conjectured in Ref. 64. However, in contrast with the homogeneous case, the quasiparticle velocities are not odd under parity, i.e., $v_{\pm}(\lambda) \neq -v_{\pm}(\lambda)$, implying that the lightcone is not symmetric (in Figure 1 (b) $v^{(A)}_{\pm}$ and $v^{(B)}_{\pm}$ denote the different quasiparticle velocities in the two reservoirs). Similar to the homogeneous case the velocity of the entangling particles depends only on the local equilibrium state at large times [64] (here the NESS).

In the following we provide numerical evidence for (6) and (7) considering several inhomogeneous quenches in the anisotropic spin-1/2 Heisenberg chain (XXZ chain). The Hamiltonian reads

\[ H_{XXZ} = J \sum_{i=1}^{L} [S^x_i S^x_{i+1} + S^y_i S^y_{i+1} + \Delta s^z_i S^z_{i+1}] \]  

Here $S^{x,y,z}_{i} \equiv \sigma^{x,y,z}_{i} \otimes |0\rangle$ are spin-1/2 operators, $L$ the chain length and $\Delta$ the anisotropy parameter. We set $J = 1$ in (8), restricting ourselves to $\Delta > 1$. The XXZ chain is the paradigm of Bethe ansatz solvable models [71] (see the Supplementary Materials for details on the solution). In our tDMRG simulations we employ open boundary conditions.

Here we consider the inhomogeneous quenches in which parts $A$ or $B$ are prepared in the Néel state $|N\rangle \equiv (|\downarrow\uparrow\downarrow\cdots\rangle + |\downarrow\uparrow\downarrow\cdots\rangle)/\sqrt{2}$, the Majumdar-Ghosh or dimer state $|MG\rangle \equiv (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}$, and the ferromagnetic state $|F\rangle \equiv |\uparrow\uparrow\cdots\rangle$. For all these cases the bulk densities $\varrho^{A(\ell)}_{n}$ (cf. (5)) are known analytically (see the Supplementary Material).

The strategy to use (6) and (7) is to first solve the coupled systems of integral equations (3) and (4) for $\zeta = 0$. The obtained densities $\rho^{(n)}_{\zeta=0,n}$ are then substituted in (2) and (7). Our results are summarized in Figure 2. The Figure shows the steady-state entropy $s_{\infty}$ (dotted lines) and the entanglement production rate $s'$ as a function of $\Delta$. For the quench with initial state $|N\rangle \otimes |F\rangle$, both $s_{\infty}$ and $s'$ vanish for $\Delta = 0$, reflecting that the Néel state is the ground state of the XXZ chain in that limit. Interestingly, for both $|N\rangle \otimes |F\rangle$ and $|MG\rangle \otimes |F\rangle$, since the ferromagnet is an exact eigenstate of the XXZ chain at any $\Delta$, no quasiparticle production happens in subsystem $A$. As a consequence, $S(t)$ is fully determined by the quasiparticle transport from $B$ to $A$. However, $s_{\infty}$ and $s'(t)$ are smaller than the corresponding values for the homogeneous quench from the Néel state and the dimer state [64]. It is also interesting to observe that for these quenches only the quasiparticles with $n = 1$ contribute in (6) and (7), i.e., the bound states contribution to the entanglement vanishes (see the Supplementary Materials). This is not the case for the quench from $|MG\rangle \otimes |F\rangle$, for which all bound states contribute and quasiparticles are generated in both reservoirs. It is
FIG. 4. Entanglement dynamics after the inhomogeneous quench in the XXZ chain: Entanglement production rate. Panels (a,b,c) and (d,e,f) show tDMRG results for the initial states $|N⟩ \otimes |F⟩$ and $|MG⟩ \otimes |F⟩$, respectively. Different symbols are for different $\Delta$ and $\ell \leq 64$. $S/\ell$ is plotted versus the rescaled time $t/\ell$. The dash-dotted line is the theoretical result using Bethe ansatz (Figure 2) in the thermodynamic limit $\ell \rightarrow \infty$.

FIG. 5. The same as in Figure 4 for the initial state $|N⟩ \otimes |MG⟩$. Notice the smaller subsystem sizes ($\ell \leq 24$) as compared with Figure 4. The different panels are for different $\Delta$. The dash-dotted lines are the Bethe ansatz results.

interesting to observe that in this case both $s_\infty$ and $s'$ exhibit a rather weak dependence on $\Delta$ (see Figure 2).

**Numerical checks in the XXZ chain.** — We now turn to verify the theoretical predictions presented in Figure 2 using tDMRG simulations. We first focus on the steady-state entropy density $s_\infty$. We consider the setup in Figure 1 (b), with $A'$ being a finite block of length $\ell$ embedded in $A$. We consider only the quenches from the states $|N⟩ \otimes |F⟩$ and $|MG⟩ \otimes |F⟩$, as they are easier to simulate. In fact, we observe that tDMRG simulations give accurate results for the entanglement dynamics already with the comparatively small value of the bond dimension $\chi \approx 75$. This is even more dramatic for local observables. Specifically, we verified that for the quench from $|N⟩ \otimes |F⟩$, tDMRG simulations with $\chi \approx 20$ allow one to obtain very reliably the post-quench dynamics of the local conserved quantities and their currents, in the regime $-2 \leq \zeta \leq 2$ (see Supplemental Materials), provided that a space-time average is performed. This somehow confirms the recent results of Ref. 121, which show that transport-related quantities can be reliably extracted from tDMRG simulations with rather small bond dimension.

Our tDMRG results are reported in Figure 3 (a) and (b), plotting $s_\infty$ versus $1/\ell$. The data are in the regime $t \gg \ell$, i.e., in the steady state. To minimize the effects of oscillating (with the block size) scaling corrections we averaged the data for $t \gg \ell$ (see the Supplementary Material for the raw data). The results are for chains with $L = 40$ sites, several values of $\Delta$ (shown with different symbols) and $\ell \leq 10$. The star symbols are the Bethe ansatz results (6). The dash-dotted lines are fits to the behavior $S/\ell = s_\infty + a/\ell + b/\ell^2$, with $s_\infty$ the Bethe ansatz result in the thermodynamic limit and $a$, $b$ fitting parameters. Finite-size corrections are clearly visible for small values of $\ell$. However, for both initial states the numerical data are compatible in the thermodynamic limit with the Bethe ansatz results. Note that for the quench from $|MG⟩ \otimes |F⟩$ the dependence on $\Delta$ for large $\ell$ is not visible within the numerical precision, as expected from Figure 2.

We now focus on the entanglement production rate in Figure 4, again considering the quench from $|N⟩ \otimes |F⟩$ (panels (a)-(c)) and $|MG⟩ \otimes |F⟩$ (panels (d)-(e)), plotting $S/\ell$ versus $t/\ell$. The data are now for chains with $L \leq 128$. We always consider the half-chain entropy, i.e., $A' = A$ and $A$ half of the chain (see Figure 1). A crucial consequence is that only one boundary is present between the subsystem and the rest. The dash-dotted lines are the Bethe ansatz predictions (Figure 2 and (7)). After an initial transient, in all cases $S(t)$ exhibits linear behavior. The slope of this linear increase is in agreement with the Bethe ansatz results, already for $\ell \approx 16$. For the quench from $|MG⟩ \otimes |F⟩$, again, one should observe the weak dependence on $\Delta$. Finally, in Figure 5 we show the results for the initial state $|N⟩ \otimes |MG⟩$. The different panels are for different $\Delta$. Due to the larger amount of entanglement, we can only provide reliable tDMRG data for $\ell \leq 24$. The data are obtained from tDMRG simulations with $\chi = 400$. To highlight finite-size and finite-time corrections we also show data for $\ell = 8$, which show large deviations from the Bethe ansatz predictions. On the other hand, for $\ell = 24$ the data are clearly compatible with (7).

**Conclusions.** — We investigated the entanglement dynamics after inhomogeneous quenches in integrable models. Specifically, we conjectured an analytic formula for the entanglement production rate after joining two semi-infinite reservoirs as well as for the steady-state entropy of a finite subregion (6) and (7), respectively. Our work opens several promising new directions. First, although we provided some
of its ingredients, we were not able to provide a quasiparticle description for the full dynamics of a finite region. Although this is possible in principle, it requires to solve the kinematic problem for the quasiparticles moving inside the subsystem, which could depend on macrostates with $\zeta \neq 0$. This remains an open task. Second, it would be interesting to consider the inhomogeneous quench in which an integrability-breaking term acting at the interface between $A$ and $B$ is added to the Hamiltonian. This should lead to thermal behavior for the steady-state entanglement entropy. It would be interesting to observe this effect. Finally, it would be interesting to consider a time-dependent bipartition in which subsystem $A'$ moves away from the boundary. By appropriately tuning the speed at which $A'$ changes its position, it should be possible to change the thermodynamic macrostate, i.e., the ray $\zeta$, governing the entanglement production.

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In this Supplementary Materials we provide some additional information on our results. Specifically:
1) We provide details on the Bethe ansatz solution of the XXZ chain.
2) We provide some results on the Bethe ansatz treatment of homogeneous quantum quenches in the XXZ chain.
3) We discuss how to determine the velocities of the low-lying excitations around a TBA thermodynamic macrostate.
4) We discuss the post-quench dynamics of the local conserved quantities and their currents in the XXZ chain at $\Delta > 1$ using the integrable hydrodynamics approach.
5) We detail the Bethe ansatz treatment of the inhomogenous quench consisting in the expansion of the Néel state in the ferromagnet.
6) We present the full tDMRG data for the entanglement entropy of a finite region after the quench from the state $|N\rangle \otimes |F\rangle$.

**BETHE ANSATZ SOLUTION OF THE XXZ CHAIN**

Due to the conservation of the total magnetization $S_z$, the eigenstates of the XXZ can be classified according to the total magnetization $S_T = \sum_i S_i^z$. Equivalently, one can use the total number $M$ of down spins as a good quantum number for the eigenstates.

In the Bethe ansatz [71] solution of the XXZ chain, the eigenstates in the sector with $M$ down spins (particles) are identified by a set of $M$ rapidities $\lambda_j$, which are solutions of the so-called Bethe equations [71]

\[
\left[ \frac{\sin(\lambda_j + i\frac{\eta}{2})}{\sin(\lambda_j - i\frac{\eta}{2})} \right]^L = -\prod_{k=1}^{M} \frac{\sin(\lambda_j - \lambda_k + i\eta)}{\sin(\lambda_j - \lambda_k - i\eta)},
\]

(S.1)

where $\eta \equiv \text{arccosh}(\Delta)$. Here we are interested in the thermodynamic limit $L, M \rightarrow \infty$ with $M/L$ fixed. Then, the solutions of the Bethe equations (S.1) form string patterns in the complex plane. Rapidities forming a $n$-string can be written as

\[
\lambda_{n,\gamma}^j = \lambda_{n,\gamma} + i\frac{\eta}{2}(n + 1 - 2j) + \delta_{n,\gamma}^j,
\]

(S.2)

where $j = 1, \ldots, n$ denotes the different string components, $\lambda_{n,\gamma}$ is the "string center", and $\delta_{n,\gamma}^j$ are the string deviations. For the majority of the eigenstates of the XXZ chain, the string deviations are exponentially small, i.e., $\delta_{n,\gamma}^j = \mathcal{O}(e^{-L})$ (string hypothesis). The $n$-strings describe bound states of $n$ down spins. The string centers $\lambda_{n,\gamma}$ are obtained by solving the Bethe-Gaudin-Takahashi (BGT) equations [71]

\[
L\theta_n(\lambda_{n,\alpha}) = 2\pi I_{n,\alpha} + \sum_{(n,\alpha) \neq (m,\beta)} \Theta_{n,m}(\lambda_{n,\alpha} - \lambda_{m,\beta}).
\]

(S.3)
For $\Delta > 1$, one has $\lambda_{n,\gamma} \in [-\pi/2, \pi/2]$. Here we used that $\theta_n(\lambda) \equiv 2\arctan[\tan(\lambda)/\tanh(n\eta/2)]$. The scattering phases $\Theta_{n,m}(\lambda)$ are given as

$$
\Theta_{n,m}(\lambda) \equiv (1 - \delta_{m,n})\theta_{n-m}(\lambda) + 2\theta_{n-m+2}(\lambda) + \cdots + \theta_{n+m-2}(\lambda) + \theta_{n+m}(\lambda).
$$

(S.4)

Each choice of the so-called BGT quantum numbers $I_{n,\alpha} \in \frac{1}{2} \mathbb{Z}$ corresponds to a different eigenstate of the chain. The corresponding eigenstate energy $E$ and total momentum $P$ are obtained by summing over all the rapidities as $E = \sum_{n,\alpha} \epsilon_n(\lambda_{n,\alpha})$, and $P = \sum_{n,\alpha} z_n(\lambda_{n,\alpha})$ with

$$
\epsilon_n(\lambda) \equiv -\frac{\sinh(\eta) \sinh(n\eta)}{\cosh(n\eta) - \cos(2\lambda)},
$$

$$
z_n(\lambda_{n,\alpha}) = \frac{2\pi I_{n,\alpha}}{L}.
$$

(S.5)

In the thermodynamic limit, one works with the rapidity densities. The root densities $\rho_{n,p}$ are formally defined as

$$
\rho_{n,p}(\lambda) \equiv \lim_{L \to \infty} \frac{1}{L} \sum_{\lambda_{n,\alpha} + 1} \lambda_n(\lambda_{n,\alpha} - \lambda_{n,\alpha+1}).
$$

(S.6)

It is also convenient to define the associated hole densities $\rho_{n,h}$, i.e., the density of unoccupied rapidities, and the total densities $\rho_{n,t} = \rho_{n} + \rho_{n,h}$. The BGT equations (S.3) in the thermodynamic limit become a system of integral equations

$$
\rho_{n,t}(\lambda) = a_n(\lambda) + \sum_{m=1}^{\infty} (a_{n,m} \ast \rho_{m})(\lambda),
$$

(S.7)

where we defined

$$
a_{nm}(\lambda) = (1 - \delta_{nm})a_{n-m}(\lambda) + 2a_{n-m}(\lambda) + \cdots + 2a_{n+m-2}(\lambda) + a_{n+m}(\lambda),
$$

(S.8)

with

$$
a_n(\lambda) = \frac{1}{\pi} \frac{\sinh(\eta\lambda)}{\cosh(n\eta) - \cos(2\lambda)}.
$$

(S.9)

MACROSTATES FOR HOMOGENEOUS QUENCHES

Here we report the analytical results for the root densities $\varphi_n^{(A)}(\lambda)$ and $\varphi_n^{(B)}(\lambda)$ appearing in (3). We consider the homogeneous quenches from the Néel state and the Majumdar-Ghosh state. These densities characterize the thermodynamic macrostate in the bulk of the two systems $A$ and $B$ (see Figure 1), i.e., for $|\lambda| \to \infty$. We first define the densities $\eta_n \equiv \rho_{n,h}/\rho_n$. In terms of $\eta_n$ one has $\eta_0 = 1/(1 + \eta_n)$.

For both the Néel state and the Majumdar-Ghosh state, the $\eta_n$ obey the recursive equation [91, 115]

$$
\eta_n(\lambda) = \frac{\eta_{n-1}(\lambda + i\eta/2)\eta_{n-1}(\lambda - i\eta/2)}{1 + \eta_{n-2}(\lambda)} - 1,
$$

(S.10)

$$
\rho_{n,h}(\lambda) = \rho_{n,t}(\lambda + \frac{i\eta}{2}) + \rho_{n,t}(\lambda) - \rho_{n-1,h}(\lambda),
$$

(S.11)

where $\eta_0 = 0$ and $\rho_{0,h} = 0$.

For the Néel state one has [115]

$$
\eta_1 = \frac{2[2\cosh(\eta) + 2\cosh(3\eta) - 3\cosh(2\lambda)\sin(2\lambda)]}{[\cosh(\eta) - \cosh(2\lambda)][\cosh(4\eta) - \cosh(4\lambda)]},
$$

(S.12)

$$
\rho_{1,h} = \rho_1 + 1 - \frac{\cosh^2(\eta/2)}{2},
$$

(S.13)

For the Majumdar-Ghosh one has [107, 114]

$$
\eta_1 = \frac{\cos(4\lambda) - \cosh(2\eta)}{\cos^2(2\lambda) - \cosh(2\eta)} - 1,
$$

(S.14)

and

$$
\rho_{1,h} = \rho_1 + \frac{1}{2\pi} \left(\omega(\lambda - i\eta/2) + \omega(\lambda + i\eta/2)\right),
$$

(S.15)

VELOCITIES OF ENTANGLING QUASIPARTICLES

A crucial ingredient in the integrable hydrodynamic approach [31, 32] and in the derivation of our results (6)(7) is the velocity of the low-lying excitations around the TBA macrostates $\rho_n, \rho_n^{(\alpha)}$. Here we outline its derivation following the approach of Ref. 104. Given a generic thermodynamic macrostate identified by some densities $\rho_n, \rho_n^{(\alpha)}$, one can imagine of choosing among the eigenstates of (8) a representative of the macrostate. This would be identified by some BGT quantum numbers $I_{n,\alpha}$. Low-lying excitations around it can be constructed as particle-hole excitations. In each $n$-string sector, these correspond to the change $I_{n,\alpha} \to I_{n,\beta}$, where $I_{n,\beta}(I_{n,\alpha})$ is the BGT number of the added particle (hole). Since the model is interacting, this local change in quantum numbers affects all the rapidities obtained by solving the new set of Bethe equations. The excess energy of the particle-hole excitation can be written as [104]

$$
\delta E_n = e_n(\lambda_{n,p}^*) - e_n(\lambda_{n,h}^*). \tag{S.16}
$$

Note that (S.16) is the same as for free models apart from the dressing of the single particle energy. The change in the total momentum is obtained from (S.5) as

$$
\delta P = z_n(\lambda_{n,p}^*) - z_n(\lambda_{n,h}^*). \tag{S.17}
$$

The group velocity associated with the particle-hole excitations is by definition

$$
\nu_n^*(\lambda) \equiv \frac{\partial e_n}{\partial z_n} = \frac{e_n'(\lambda)}{2\pi \rho_n(1 + \eta_n(\lambda))}, \tag{S.18}
$$
where we used that \[ d\nu_n(\lambda)/d\lambda = 2\pi\rho_{n,t}, \] and we defined \( e_\nu'(\lambda) = de(\lambda)/d\lambda \). Here \( e_\nu'(\lambda) \) is obtained by solving an infinite system of Fredholm integral equations of the second kind \[ (S.23) \]

\[
e_\nu'(\lambda) + \frac{1}{2\pi} \sum_{m=1}^{\infty} \int d\mu e_\mu'(\mu) \frac{\Theta'_{m,n}(\mu - \lambda)}{1 + \eta_m(\mu)} = e_\nu'(\lambda),
\]

where \( \Theta'_{m,n}(\lambda) \equiv d\Theta_{m,n}(\lambda)/d\lambda \) and \( e_\nu'(\lambda) \equiv de(\lambda)/d\lambda \) (cf. also (S.4) and (S.5)).

### INTEGRABLE HYDRODYNAMICS: CONSERVED CHARGES AND CURRENTS

In this section we provide some numerical checks of the integrable hydrodynamics approach \cite{31, 32}, focusing on the XXZ chain in the region with \( \Delta > 1 \). We consider the quenches with initial states \( |N\rangle \otimes |F\rangle \) and \( |MG\rangle \otimes |F\rangle \). Using tDMRG simulations we investigate the dynamics of the local magnetization \( S^z \), the local energy density \( E \), and the energy current \( J_E \). These are defined as \[ (S.24) \]

\[
S^z_i \equiv S^z_i,
\]

\[
E = S^z_i S^z_{i+1} + S^y_i S^y_{i+1} + \Delta S^z_i S^z_{i+1} - \frac{\Delta}{4},
\]

\[
J_E \equiv S^z_{i-1} S^z_{i+1} S^y_i - S^y_{i-1} S^y_{i+1} S^z_i + \Delta S^z_{i-1} S^z_{i+1} S^y_i + \Delta S^z_{i-1} S^z_{i+1} S^y_{i+1} - \Delta S^z_{i-1} S^z_{i+1} S^y_i + \Delta S^y_{i-1} S^y_{i+1} S^z_i - \Delta S^y_{i-1} S^y_{i+1} S^z_{i+1}.
\]

In the framework of the Bethe ansatz, these are written in terms of the root densities \( \rho_{\zeta,n} \) as

\[
S_z = \sum_n \int d\lambda \rho_{\zeta,n},
\]

\[
E = \sum_n \int d\lambda \rho_{\zeta,n} \frac{\sinh \eta \sinh(\eta n)}{\cosh(2\lambda) - \cosh(\eta n)},
\]

\[
J_E = \sum_n \int d\lambda \rho_{\zeta,n} \frac{\sin(2\lambda) \sinh^{\eta} \sinh(\eta n)}{(\cosh(2\lambda) - \cosh(\eta n))^2}.
\]

Note that all the observables are functions of \( \zeta \equiv x/t \). The dependence on \( \zeta \) is encoded in the densities \( \rho_{\zeta,n} \), which are solutions of (3) and (4).

The Bethe ansatz results (S.23)(S.24)(S.25) are compared with tDMRG results in Figure S1. The upper and lower panels show tDMRG data for the quench with initial states \( |N\rangle \otimes |F\rangle \) and \( |MG\rangle \otimes |F\rangle \), respectively. The data are for the XXZ chain with \( L = 300 \) sites, up to times \( t \lesssim 100 \). The results in panels (a-c) are obtained using bond dimension \( \chi \approx 20 \). For the quench from the state \( |MG\rangle \otimes |F\rangle \) (panels (e-f)) we used \( \chi \approx 75 \). In all panels the different symbols correspond to different values of \( \Delta \). In order to remove spatial and temporal oscillations, we performed a spatio-temporal average. Specifically, for each fixed \( \zeta \) the results in the Figure are obtained by averaging the data in a window \( \zeta \pm \epsilon \) with \( \epsilon \approx 0.1 \). The dashed-dotted lines are the theoretical results (S.23)(S.24)(S.25). For
$|\psi| \gg 1$ all the observables become $\zeta$ independent. This happens in spatio-temporal regions with $|x| \gg v_M t$, with $v_M$ the maximum velocity in the system. Remarkably, for both quenches, and for all values of $\Delta$, the tDMRG data are in very good agreement with the Bethe ansatz. Note, however, that the theoretical predictions exhibits some cusp-like behaviors (see for instance the arrow in panel (b)). These are not well reproduced by the tDMRG results. It is natural to expect that these deviations should be attributed to the finite $\chi$ and to the finite-time and finite-size effects.

**NEEL-FERRO QUENCH: BETHE ANSatz RESULTS**

In this section we provide some details on the Bethe ansatz results for the inhomogeneous quench from the state $|N\rangle \otimes |F\rangle$. Our results are summarized in Figure S2.

Specifically, panels (a) and (b) show the densities $\vartheta_{\zeta,n}$ and $\rho_{\zeta,n}$ (cf. (3)), respectively. The curves are obtained by numerically solving the system (3)(4). The results are for $\zeta = 0$, which identifies the relevant macrostate to describe the entanglement dynamics after the quench (the subscript $\zeta$ in $\vartheta_{\zeta,n}$ and $\rho_{\zeta,n}$ is omitted in the Figure). First, it is interesting to observe that only $\vartheta_{\zeta,1}$ and $\rho_{\zeta,1}$ are nonzero. Moreover, $\vartheta_{1}$ is nonzero only for $\lambda > 0$. Similar behavior is observed for $\rho_{1}$ (see panel (b)).

The group velocities of the low-lying excitations around the macrostate with $\zeta = 0$ are reported in Figure S2 (c). These are obtained by numerically solving (S.19) and using (S.18). As anticipated in the main text, one has $v_n(\lambda) \neq -v_n(-\lambda)$. This is in stark contrast with homogeneous quenches, where one has [64] $v_n(\lambda) = -v_n(-\lambda)$. As a consequence the lightcone of the entangling quasiparticles is not symmetric (see Figure 2 (b)). It is also interesting to observe that for any $\lambda, v_n < 0$ for $n > 1$, which implies that there is no transport of bound states from $B$ to $A$ (see Figure 1) after the quench.

Finally, in Figure S2 (e) we show the contributions to the Yang-Yang entropy density $s_{YY}^{(n)}$ (cf. (2)) of the quasiparticles. Clearly, only quasiparticles with $n = 1$ and $\lambda > 0$ contribute to the entropy. Together with the results in panel (c), this implies that the entanglement between the two subsystems is generated by the transport of particles from $B$ to $A$.

**STEADY-STATE ENTROPY: DMRG DATA**

In this section we discuss in more detail the numerical data for the steady-state entanglement entropy presented in Figure 3, i.e., for the quench from the state $|N\rangle \otimes |F\rangle$. We consider the entanglement entropy of a subsystem $A'$ of length $\ell$ placed next to the boundary between $A$ and $B$. To avoid boundary effects $A'$ is embedded in $A$ (see Figure 1), which is chosen larger than $A'$.

Our tDMRG results are presented in Figure S3. The data are obtained using standard tDMRG simulations with bond dimension $\chi \sim 300$ for the XXZ chain with $L = 40$ sites and $t \sim 16$. To calculate the von Neumann entropy we employed the techniques described in Ref. 122. The different panels correspond to different values of $1.5 \leq \Delta \leq 10$. The figure shows $S/\ell$ plotted versus the time after the quench. In each panel the different symbols are for subsystems of different length $1 \leq \ell \leq 9$. For each $\ell$ a linear increase, followed by a saturation, is observed. The saturation value decreases upon increasing $\Delta$, in agreement with the theoretical predictions in Figure 2. For $t \to \infty$, in the limit $\ell \to \infty$, the entropy density $S/\ell$ should converge to the Bethe ansatz predictions in Figure 2. However, strong finite-size effects are visible in
the Figure, due to the small $\ell$ considered. Moreover, it is interesting to observe that upon increasing $\Delta$, the data exhibit strong oscillations with time. The data presented in Figure 3 correspond to the time average of the tDMRG data in the time window $13 \leq t \leq 16$. Finally, we should mention that similar results are obtained for the quench from the state $|MG\rangle \otimes |F\rangle$. 