Doubling of Scattering Phase Shifts
for Two Dimensional Strings

Antal Jevicki       Miao Li
Department of Physics
Brown University
Providence, RI 02912
antal@het.brown.edu    li@het.brown.edu

Tamiaki Yoneya
Institute of Physics
University of Tokyo
Komaba, Tokyo 153
yoneya@tansei.cc.u-tokyo.ac.jp

We discuss the origin of the leg factors appearing in 2D string theory. Computing in the world sheet framework we use the semiclassical method to study string amplitudes at high energy. We show that in the case of a simplest 2-point amplitude these factors correspond entirely to the time delay for reflection off the Liouville wall. Our semiclassical calculation reveals that the string longitudinal modes, although nonpropagating in 2D spacetime, have the effect of doubling the phase shift. Particular emphasis is put on comparison with the point particle (center of mass) case. A general method is then given for calculating an arbitrary amplitude semiclassically.

April 1995
1. Introduction

It is generally believed that the $c = 1$ matrix model describes a critical string theory in a two-dimensional target spacetime with coordinate dependent condensation of dilaton and tachyon fields $[1]$. The scaling properties of the matrix model observables with respect to the fermi energy agree precisely with the predictions of the string theory, when suitable correspondences are made with the physical variables of string theory. In particular, the string $S$-matrix coincides with the $S$-matrix of the collective-field quanta of the matrix model after making a momentum dependent renormalization of the asymptotic fields. The renormalization factor, often called the leg factor, is a pure phase, but cannot simply be discarded as unobservable since it contributes to a momentum dependent phase shift of the scattering amplitudes. The leg factors contain important physical information on the nature and content of two dimensional string theory. When continued to Euclidean metric, they exhibit an infinite number of poles whose positions coincide with the characteristic values of momenta of 2D strings. Namely, the poles occur at the momenta where energy-momentum conservation law is obeyed in the linear-dilaton background up to, in general, a finite number of insertions of the background tachyon fields $[2]$. If one first neglects the tachyon condensation, one can easily perform an ordinary free field calculation of the $S$-matrix for bulk scattering. Then, the poles can be interpreted as arising from the short distance singularities of the product of the vertex operators on the world sheet. The short-distance singularities on the world sheet correspond to large distance propagation of strings in the asymptotic region of the target spacetime. We expect that such bulk amplitudes are, in general, exponentially damped, because of the linear dilaton condensation. Ref. $[3]$ gives a spacetime interpretation of the bulk scattering, and shows that the first few singularities indeed account for the exponentially small effect of the string interactions occurring in the asymptotic region of the target spacetime.

From the viewpoint of bulk scattering, the poles reflect only the asymptotic properties of the target spacetime and do not reflect the presence of the tachyon condensation. However, the same leg factor simultaneously accounts for the amplitudes in scattering (so-called wall scattering) against the tachyon background. In particular, the two-point amplitude extracts the response of the string propagation upon the tachyon background. Here, the poles should be interpreted as arising solely from the resonance of strings with a coordinate dependent tachyon background.

It is not at all clear, however, why the same leg factor can explain both of these properties of 2D strings which seem to be independent to each other. Is it just a coincidence valid only for a particular background? Or, is there anything deeper, related to unknown universal properties of string theories? This is, we believe, one of crucial questions to be
clarified in seeking a nonperturbative and background independent formulation of string theories, based on a hint provided by the matrix models. The problem becomes serious especially when we consider possible matrix models corresponding to different string backgrounds, since the pole structure implied from the resonance with the background does not completely coincide the pole structure required for bulk scattering. For a discussion on this problem and a tentative proposal in the case of large-mass black hole background, see [4].

In view of this situation, the purpose of the present paper is to give a new direct derivation of the scattering phase shifts at high energy from the viewpoint of continuum string theory. Although our results are still too modest to answer the above questions, we hope that they enlighten some aspects that have not been fully recognized in the previous works.

To our knowledge, no string theoretic derivation of the wall scattering amplitude has been given except for the one based on the method of analytical continuation with respect to the number of insertions of the tachyon condensate operators [5]. Although such a derivation nicely gives a general n-point amplitude in closed form, it does not reveal the physical origin of the leg factors.

We shall consider the high energy limit of the wall scattering in order to probe inside the wall region of the target spacetime. Throughout the present paper, we remain in the lowest approximation with respect to the string coupling constant by assuming a strong tachyon condensation. We use a direct semi-classical approximation and explicitly compute the effect of higher string modes in the tachyon background. We then find that string fluctuations around the center of mass give an equal contribution to the phase shift as that coming from the center of mass motion itself. This result is consistent with the double pole structure of the two-point amplitude and suggests that the leg factor not only reflects the properties of the asymptotic propagation, but also embodies a property originated from the intrinsically extended nature of 2D string in a given background. Although there are no transverse oscillations, the extension in the longitudinal direction does give an observable effect. This should perhaps be interpreted as a manifestation of discrete physical states of 2D string theories.

In the next section, we begin by briefly reviewing 2D string theory with tachyon and linear-dilaton background. It will be emphasized that a simple reduction to the center of mass motion is not sufficient to account for the phase shift of the 2-point amplitude, especially in the high energy limit. In section 3, we will proceed to calculate the effect of higher string modes and compute the contribution of them to the phase shift in the high-energy limit. We first present a calculation based on a direct mode counting in
Minkowski metric. Next in section 4, we propose a general scheme for calculating arbitrary amplitudes in the semi-classical approximation in Euclidean space. The result for the two-point amplitude obtained in section 3 is reproduced by this scheme. In the concluding section, we will briefly discuss the spacetime meaning of the phase shift and try to give a qualitative interpretation of our result.

2. Leg Factors of the Scattering Amplitudes

In this section we present a brief review of 2d string theory, introduce the notation and describe some earlier attempts at understanding the origin and pole structure of its S-matrix.

In the world sheet description of the theory one has \( c_M = 1 \) matter \( X(z, \bar{z}) \) together with a \( c_L = 25 \) Liouville field \( \varphi(z, \bar{z}) \):

\[
S = \frac{1}{8\pi} \int d^2z \sqrt{\hat{g}} \left\{ \hat{g}^{ab} \partial_a X \partial_b X + \hat{g}^{ab} \partial_a \varphi \partial_b \varphi + Q \hat{R}^{(2)} \varphi + \mu e^{\alpha \varphi} \right\},
\]

with \( Q = -2\sqrt{2} \), \( \alpha = \alpha_- = -\sqrt{2} \). The vertex operators for massless tachyons read

\[
V(z, \bar{z}) = e^{i\sqrt{2}kX(z, \bar{z})} e^{-\sqrt{2}(1-|k|)\varphi(z, \bar{z})},
\]

where one only takes the (physical) + dressing of the Liouville field. In the above, we have the Euclidean theory, a transformation:

\[
iX \rightarrow X, \quad k \rightarrow \pm i\omega,
\]

(with \( \omega > 0 \)) brings us to the Minkowski time and the vertex operators

\[
T_{\pm \omega} = e^{-i\sqrt{2}\omega X(z, \bar{z})} e^{-\sqrt{2}(1+i|\omega|)\varphi(z, \bar{z})}
\]

describing respectively left and right moving waves.

The \( n \)-point tachyon S-matrix was found through a combination of continuum and matrix model techniques to have the general structure \[3\]

\[
S(k_1, k_2 \cdots k_n) = \prod_i \frac{\Gamma(-2|k_i|)}{\Gamma(2|k_i|)} \tilde{S}(k_1, k_2, \cdots k_n).
\]

It is characterized by a leg factor for each scattering particle and a residual S-matrix \( \tilde{S} \) described by the simple dynamics of a collective field. The origin and physical interpretation of the leg factors however is more mysterious. They contain poles at imaginary values
of momenta, these poles partially come from intermediate discrete states which are known to appear in the spectrum of 2d String Theory. But the poles also come from the nontrivial tachyon background described by the Liouville exponential potential \( \int \mu e^{-\sqrt{2}\varphi(z, \bar{z})} \). The latter is especially clear if one takes the simplest example of the 2-point \( S \)-matrix. In this case one has a single string reflecting of the Liouville wall with the amplitude

\[
S_2(\omega, -\omega) = \mu^{-2i\omega} \left( \frac{\Gamma(2i\omega)}{\Gamma(-2i\omega)} \right)^2. \tag{2.6}
\]

This amplitude exhibits double poles at \( 2i\omega_n = -n \). Since this is a one string process, these poles can not come from typical exchange states associated with intermediate Feynman propagators. They entirely reflect the property of the background and the fact that the object being scattered is a string.

In this connection one can consider a point particle which would correspond to the center of mass (or the zero mode) of the string. The scattering of the Liouville wall is given by the Klein-Gordon (Wheeler-de Witt) equation \[6\text{][7]}

\[
\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \varphi_0^2} + \mu e^{-\sqrt{2}\varphi_0} \right) \psi(t, \varphi_0) = 0,
\]

which is solved to give the scattering amplitude

\[
T(\omega) = \frac{\Gamma(2i\omega)}{\Gamma(-2i\omega)} \mu^{-2i\omega} = e^{i\delta(\omega)}. \tag{2.7}
\]

This amplitude is again a pure phase but exhibits single imaginary energy poles of \(-2i\omega = n\). These clearly come from the Liouville wall \( e^{-\sqrt{2}\varphi_0} \) itself. It has been suggested that the center of mass Wheeler-de Witt equation might be capable of giving an exact description of string theory. For that scenario to be true, one has to come up with a mechanism for generating the double poles of the string \( S \)-matrix. Ref. \[8\] has made a suggestion of using a modified tachyon background of the form

\[
T_0(\varphi_0) = (b_1 + b_2 \varphi) e^{-\sqrt{2}\varphi}. \tag{2.8}
\]

The modification given by the second term is capable of converting a single into a double pole

\[
\langle k | \varphi_0 e^{-\sqrt{2}\varphi_0} | k \rangle = \frac{1}{(2ik + 1)^2} \tag{2.9}
\]

at least for the lowest case. The modified background was argued by Polchinski \[8\] as arising in the effective tachyon Lagrangian

\[
L_{eH} = \frac{1}{2g_{st}^2} \int d^2 x \sqrt{-G} e^{-2D} \left[ - (\nabla T)^2 + 4T^2 - V(T) + \{ R + 4 (\nabla D)^2 + 16 \} \right] + \cdots, \tag{2.10}
\]
where it corresponds to a static solution for the tachyon field $T_0(\varphi_0)$. The value of the coefficients $b_1$ and $b_2$ can in principle be specified by the nonlinear interaction term of the effective Lagrangian or the full string field theory. In the continuum, it has been suggested that a similarly modified Liouville term could be generated through the fact that for $c < 1$ one actually has two possible candidates

$$e^{\alpha_\pm} \varphi \quad \text{with} \quad \alpha_\pm = -\sqrt{2} \mp \epsilon$$

as $c \to 1$. Forming a linear combination and taking the limit $\epsilon \to 0$ we can then have:

$$\frac{1}{2\epsilon} \left[ e^{\alpha_+} - e^{\alpha_-} \right] \to \varphi(z, \bar{z}) e^{-\sqrt{2} \varphi(z, \bar{z})}.$$  \hspace{1cm} (2.12)

The problem with the above mechanism for generating double poles is that it appears to be at best relevant only for low momenta. It does not survive at high energy where it would require the addition of an infinite number of further terms. It is then more appropriate to concentrate on the high energy region in trying to discover the actual mechanism for doubling of poles in the string $S$-matrix. At high energy this means a doubling of the scattering phase shift and the time delay. As we have mentioned, a simple modification by an additional term of the form $\varphi e^{-\sqrt{2} \varphi}$ can not achieve this at high momenta as is easily seen in the classical-particle picture: Near the turning point $e^{-\sqrt{2} \varphi + \log \varphi} \sim e^{-\sqrt{2} \varphi}$ as $\varphi \sim -\ln \omega \to -\infty$ and the prefactor $\varphi$ does not play any leading role in the high-energy limit.

Typical high energy scattering can be studied by semiclassical techniques and that is what we do in the next section. In the mechanism that we suggest, a central role is played by the higher string modes. In an expansion

$$X_\mu(\sigma) = X_\mu^0 + \sum_{n \neq 0} e^{i n \sigma} X_\mu^{(n)},$$  \hspace{1cm} (2.13)

the non-zero string modes will be shown to double the contribution to the phase shift. Such effects of higher modes participating to renormalize the leading classical result were known to systematically appear in integrable two-dimensional soliton models [9]. Their presence in Liouville theory has been emphasized originally in [10] and [11]. Our discussion concerns the evaluation of a physical quantity (the string $S$-matrix) at high energies with a demonstration of a significant one loop effect doubling the classical contribution.
3. Semi-classical Calculation in Minkowski Space

It is argued in the previous section that the degree of the center of mass of a string alone does not account for the phase shift of the 2-point scattering amplitude. We now set out to show this is indeed true by an explicit computation. For the 2-point amplitude, it is more physical to work with a Minkowski spacetime, and for this matter to work with a Minkowski world sheet also. The amplitude is then expressed as a world sheet path integral with appropriate boundary conditions set by wave functions at \( t = \pm \infty \), the world sheet topology is that of a cylinder, as we consider closed string only. There is a systematic quantization procedure developed in \cite{12}. Unfortunately we have found it difficult to adapt that procedure to the calculation of string scattering amplitudes. Therefore, we shall perform only a semi-classical calculation, in which the zero mode of \( \phi \) is taken as the one satisfying the classical Liouville equation, and the one-loop calculation is done by expanding around this classical solution. It turns out that the semi-classical computation on the world sheet becomes very accurate at high energies.

The Minkowski world sheet action with a Minkowski target space is

\[
S(\varphi, X) = \frac{1}{8\pi} \int dt d\sigma \left( (\partial_t \varphi)^2 - (\partial \varphi)^2 - \mu e^{-\sqrt{2} \varphi} - \sqrt{2} \omega (\varphi(\infty) + \varphi(-\infty)) \right),
\]

where \( X \) is the real time of the target Minkowski space and \( t \) is the World sheet time. The period of world sheet space \( \sigma \) is \( 2\pi \). Consider the 2-point amplitude. The incoming wave is left-moving, so at \( t = -\infty \), we shall insert wave function \( T_{\omega} = \exp \left( -i \sqrt{2} \omega (X + \varphi) \right) \), this determines the boundary condition of the path integral at \( t = -\infty \). The outgoing wave is right-moving, therefore an insertion of the complex conjugate of \( T_{-\omega} = \exp \left( -i \sqrt{2} \omega (X - \varphi) \right) \) is used. Note that in the wave functions a factor \( \exp(-\sqrt{2} \varphi) \) is dropped out, since this factor is cancelled by appropriate boundary term in the action, whose origin is the background charge term. Now the 2-point amplitude is given by

\[
S_2(\omega, -\omega) = \int [dX d\varphi] T_{-\omega}(X(\infty), \varphi(\infty)) T_{\omega}(X(-\infty), \varphi(-\infty)) e^{iS}. \tag{3.2}
\]

The integral over \( X \) is Gaussian, thus can be easily performed. The insertions of wave functions only constrain \( \partial_t X(\pm \infty) \sim \omega \). Their effect is then removed by a shift \( X \to 2\sqrt{2} \omega t + X \).

The nontrivial part of the computation concerns integration over \( \varphi \). Let us separate \( \varphi \) into a (center of mass) zero mode and oscillating modes \( \varphi = \varphi_0(t) + \varphi_{os} \). The total action for \( \varphi \), taking the boundary wave functions into account, is

\[
S(\varphi) = \frac{1}{8\pi} \int dt d\sigma \left( (\partial_t \varphi)^2 - (\partial \varphi)^2 - \mu e^{-\sqrt{2} \varphi} \right) - \sqrt{2} \omega (\varphi(\infty) + \varphi(-\infty)). \tag{3.3}
\]
It follows from variation of the above action that the Liouville zero mode satisfies the Liouville equation

$$\partial_t^2 \varphi_0 - \frac{\mu}{\sqrt{2}} e^{-\sqrt{2} \varphi_0} = 0$$

(3.4)

with the boundary conditions

$$\partial_t \varphi_0(\pm \infty) = \pm 2 \sqrt{2} \omega.$$

The solution is well-known [13]:

$$e^{-\sqrt{2} \varphi_0} = \frac{8 \omega^2}{\mu \cosh^2 2 \omega t}.$$ 

(3.5)

Plugging this solution into (3.3), we will get an infinite contribution. The divergence is proportional to the time lapse and therefore should be subtracted in order to obtain the true phase shift. Perhaps the simplest way to calculate the classical action is to vary the action with respect to \( \omega \), then integrate the result to get the zero mode (or center of mass) contribution to the phase shift. Vary the action with respect to \( \omega \) and make use of the Liouville equation and the boundary conditions

$$\delta S_{cl} = -\sqrt{2}(\varphi_0(\infty) + \varphi_0(-\infty)) \delta \omega$$

$$= 2 \ln\left(\frac{8 \omega^2}{\mu}\right) \delta \omega - 4 \ln(\cosh 2 \omega T) \delta \omega,$$

where a cut-off in the time lapse \( T \) was introduced. The last term in the second equality, depending on \( T \), should be dropped out. The second term, after integrating over \( \omega \), gives

$$S_{2cl}^\omega(\omega, -\omega) = e^{i S_{cl}} = \left(\frac{\mu}{2}\right)^{-2i \omega} e^{i(-4 \omega + 4 \omega \ln(2 \omega))}.$$ 

(3.6)

Comparing this result to that in (2.8), we see that they essentially agree. The difference gets smaller and smaller when \( \omega \) gets larger and larger. This computation shows that another factor \( \Gamma(2i \omega) / \Gamma(-2i \omega) \) in the full 2-point amplitude is missing, if one takes only the degree of center of mass into account as in [1].

We now show that most of the other gamma ratio can be recovered from the one-loop calculation. One-loop contribution will involve all oscillating modes in an essential way. This suggests that even though the longitudinal modes are nonpropagating in two dimensional spacetime, they are essential in the full stringy description of the scattering amplitudes (they might reflect the remnant modes, the discrete states). Expand the action around the above classical solution to the second order of the high modes

$$S(\varphi_0 + \varphi) = S_{cl} + \frac{1}{8 \pi} \int dt d\sigma \left( (\partial \varphi)^2 - \frac{8 \omega^2}{\cosh^2 2 \omega t} \varphi^2 \right).$$

(3.7)
There are two ways to proceed to calculate the determinant resulting from integration of high modes $\varphi$. The first method is the so-called phase shift method. This method has the advantage of showing clearly how high modes make a contribution $4i\omega \ln(2\omega)$. The second method is that of heat kernel. We shall present yet a third, the most general method in Euclidean space in the next section.

The phase shift method requires solving the eigen-value problem of the following equation

$$
\left(-\partial_t^2 + \partial_\sigma^2 - \frac{8\omega^2}{\cosh^2 2\omega t}\right) \varphi = \lambda^2 \varphi. \tag{3.8}
$$

The operator $\partial_\sigma^2$ can be replaced by $-m^2$, its eigen-value. It is easy to see that the equation

$$
\left(-\partial_t^2 - \frac{8\omega^2}{\cosh^2 2\omega t}\right) \varphi = \nu^2 \varphi
$$

has the solution

$$
\varphi = e^{i\nu t} \left(\tanh 2\omega t - \frac{i\nu}{2\omega}\right). \tag{3.9}
$$

The other solution is obtained by $\nu \to -\nu$. The eigen-value of the equation (3.8) is then $\nu^2 - m^2$, and the one-loop phase shift is given by

$$
\ln Z_1 = -\frac{1}{2} \sum_{\nu,m} \ln(\nu^2 - m^2 + i\epsilon). \tag{3.10}
$$

In order to evaluate this sum, we put equation (3.8) into a box ($-T, T$). Without the background, $\nu_n$ would be $\pi n/T$. With the background, $\nu_n = \pi n/T - \delta_n/(2T)$. $\delta_n$ is called the phase shift (hence the name phase shift method), and is determined by requiring $\varphi(T) = \varphi(-T)$. Using (3.9) we find

$$
e^{i\delta_n} = \frac{\nu_n + 2i\omega}{\nu_n - 2i\omega},$$

or $\delta(\nu) = 2 \arctan(2\omega/\nu)$. The one-loop phase shift (3.10) is written as

$$
-\frac{T}{2\pi} \sum_m \int d\nu (1 + \frac{\delta'(\nu)}{2T}) \ln(\nu^2 - m^2 + i\epsilon) =
$$

$$
-\frac{T}{2\pi} \sum_m \int d\nu \ln(\nu^2 - m^2 + i\epsilon) - \frac{1}{4\pi} \sum_m \int d\nu \delta'(\nu) \ln(\nu^2 - m^2 + i\epsilon).
$$

The first term is independent of the background, therefore of $\omega$. It should be dropped out. The second term after integration by parts becomes

$$
\frac{1}{2\pi} \int d\nu \delta(\nu) \sum_m \frac{\nu}{\nu^2 - m^2 + i\epsilon} = -\frac{i\omega}{2} \int d\nu \epsilon(\nu) \delta(\nu),
$$

8
where $\epsilon(\nu)$ is the step function. The above equality is valid only when $\omega >> 1$. Use the previous result for $\delta(\nu)$, the above integral over $\nu$ is logarithmically divergent. Imposing cut-off $\Lambda$ for $\nu$ we finally obtain the result

$$\ln Z_1 = -4i\omega (1 - \ln(2\omega/\Lambda)).$$

The finite part is $-4i\omega (1 - \ln(2\omega))$, just the asymptotic value of the logarithmic of the missing ratio $\Gamma(2i\omega)/\Gamma(-2i\omega)$. The cut-off part $-4i\omega \ln \Lambda$ can be combined with the classical part $-2i\omega \ln \mu$ to give a renormalized “cosmological constant” $\mu\Lambda^2$. Note that this one-loop calculation is valid for large $\omega$. An Euclidean calculation presented in the next section will give an exact one-loop result.

The above calculation tells us the following points. First, it is not necessary to use the operator $\phi \exp(-\sqrt{2}\phi)$ in the world sheet formalism as the tachyon condensate. This tachyon condensate can be viewed at best as an effective condensate at low energies, and perhaps incapable of accounting for the high energy behavior. Second, high modes play an important role even in the 2-point amplitude. If one trusted in a naive gauge-fixing procedure, one would have concluded that the higher modes are irrelevant. Third, the logarithmic behavior of the amplitude is associated with the logarithmic ultraviolet divergence of the one-loop contribution. This shows that there is no further logarithmic and power behaved terms from higher loops, since higher loops are not divergent in two-dimensions. Thus, one-loop calculation gives an exact result in the high-energy limit.

The next method employed to calculate the determinant is the heat kernel method. This method is efficient when one tries to calculate the variation of the determinant with $\omega$ first

$$\delta \ln Z_1 = -\frac{1}{2} \delta \text{tr} \ln \left( -\partial_t^2 + \partial_\sigma^2 - \frac{8\omega^2}{\cosh^2 2\omega t} \right) = 4 \int dt d\sigma K(t\sigma, t\sigma) \delta\left( \frac{\omega^2}{\cosh^2 2\omega t} \right),$$

where $K$ is the inverse of the operator $-\partial_t^2 + \partial_\sigma^2 - 8\omega^2/(\cosh^2 2\omega t)$. A general method of computing $K$ was given in the last reference in [9]. We shall not give the derivation except the lengthy result:

$$K(t\sigma, t\sigma) = \frac{i}{4\pi} \sum_{m \neq 0} \frac{1}{m(a_m^2 - 1)} (1 + \frac{m^2}{4\omega^2})[(1 + a_m^2)(\tanh^2 2\omega t + \frac{m^2}{4\omega^2})]$$

$$- a_m \left( (\tanh 2\omega t - \frac{im}{2\omega})^2 + c.c \right),$$

$$a_m = e^{2imT} \left( 1 - \frac{im/2\omega}{1 + i\omega/2\omega} \right),$$

again we have put the system in a box ($-T, T$). After a straightforward but tedious calculation, we obtain the same result as in (3.11).

This second method will be most efficient in the case of semi-classical calculation of a general amplitude, where one has to deal with Euclidean spacetime and Euclidean world sheet. This we shall do in the next section.
4. Semi-classical Calculation in Euclidean Space

Amplitudes are calculated on the Euclidean world sheet by inserting vertex operators and integrating over positions of these operators. The Euclidean world sheet action is given in eq.(2.1). In complex coordinates it reads

\[ S = \frac{1}{2\pi} \int d^2 z \left( \partial X \bar{\partial} X + \partial \varphi \bar{\partial} \varphi + \mu e^{-\sqrt{2} \varphi} \right), \]  

(4.1)

where the background charged term drops out due to the flatness of the world sheet metric. A general amplitude is given by

\[ S_n(\omega_i) = \int \prod_i d^2 z_i \int [dX d\varphi] e^{-S} \prod_i V_{\omega_i}(z_i), \]  

(4.2)

with

\[ V_{\omega_i} = e^{i \sqrt{2} \omega_i X - \sqrt{2} (1 - |\omega_i|) \varphi}. \]

Again the \( X \) part of the path integral can be easily performed, leaving a standard correlation function together with a \( \delta \) function enforcing the conservation of energy.

Since we are considering high energy scattering, we will denote \(|\omega_i| - 1\) simply by \( \omega_i \), assuming that all \( \omega \)'s are positive. The semi-classical computation of the \( \varphi \) path integral is divided into two steps, the same as in the previous section. The first step is to solve the classical Liouville equation with sources provided by insertion of vertex operators

\[ \partial \bar{\partial} \varphi_0 + \frac{\mu}{\sqrt{2}} e^{-\sqrt{2} \varphi_0} - \sqrt{2} \pi \sum_i \omega_i \delta^2(z - z_i) = 0. \]  

(4.3)

This equation in principle can be solved for an arbitrary number of sources, see [14]. For a classical calculation, it is enough to know the behavior of the solution near each source. Near \( z_i \)

\[ \varphi_0(z) = \sqrt{2} \omega_i \ln |z - z_i|^2 + \frac{1}{\sqrt{2}} \ln \mu + \Delta_i(\omega), \]  

(4.4)

where \( \Delta_i(\omega) \) as a constant is a function of \( \omega \)'s, its precise form is to be determined by the exact solution. Following [14], we call it the time delay at \( z_i \).

Again it is easy to compute the classical action including source terms by first considering its variation with respect to \( \omega \)'s, as we did in the previous section. Making use of the Liouville equation, one finds

\[ \delta S_{cl}(\omega) = \sqrt{2} \sum_i \varphi_0(z_i) \delta \omega_i. \]  

(4.5)

1 This time delay should not be confused with the actual time delay discussed in the next section.
Once again this is divergent since $\varphi_0(z_i)$ is divergent. The divergence, similar to the one in the Minkowski calculation which depends on the time lapse, is regularized by introducing a short-distance cut-off on the world sheet and absorbed into a renormalization of the vertex operators. This being done, we obtain a finite result

$$S_{cl}(\omega) = \ln \mu \sum \omega_i + \sqrt{2} \int \Delta_i(\omega)d\omega_i. \quad (4.6)$$

In order for the last integral to make sense, the time delays $\Delta_i$ should satisfy the integrable conditions $\partial_{\omega_i} \Delta_j = \partial_{\omega_j} \Delta_i$. This formula for the classical part of the amplitude expresses it as integration of the time delays is due to Bilal and Gervais [14]. We now address the question of quantum correction.

The one-loop calculation boils down to a calculation of the determinant of operator $-\partial \bar{\partial} + \mu \exp(-\sqrt{2} \varphi_0)$. At the first sight, this appears a formidable task, since for an arbitrary number of sources, the exact solution $\varphi_0$ is rather complicated. We will see that as far as one can calculate $\varphi_0$, the calculation of the determinant is straightforward. Again we adopt the heat kernel method. The variation of the logarithmic of the determinant is given by

$$\delta S_1 = \frac{1}{2} \delta \ln(-\partial \bar{\partial} + \mu e^{-\sqrt{2} \varphi_0}) = \frac{\mu}{2} \int d^2 z K(z, z) \delta(e^{-\sqrt{2} \varphi_0}). \quad (4.7)$$

The heat kernel satisfies

$$(-\partial \bar{\partial} + \mu e^{-\sqrt{2} \varphi_0}) K(z, w) = \delta^2(z - w).$$

Suppose we know how to solve the Liouville equation (4.3) with an additional source $\omega$ at $w$, and call this solution $\varphi_\omega(z, w)$. It is easy to see that the following function

$$- \frac{1}{\sqrt{2\pi}} \frac{\partial \varphi_\omega(z, w)}{\partial \omega}|_{\omega=0} \quad (4.8)$$

satisfies the heat kernel equation. Therefore it is to be identified with $K(z, w)$ up to a function annihilated by the differential operator. It follows from this observation that

$$\frac{\partial \varphi_0(z_i)}{\partial \omega_j} = -\sqrt{2\pi} K(z_i, z_j),$$

and since $K(z_i, z_j)$ is symmetric in $z_i$ and $z_j$, we have $\partial_{\omega_i} \Delta_j = \partial_{\omega_j} \Delta_i$. Note that the above identification of the derivative of $\varphi_0(z_i)$ with the heat kernel is correct only up to a function which is annihilated by the Liouville differential operator. That the integrable conditions of time delays follow from this identification suggests that it is correct, and the function annihilated by the Liouville differential operator is zero.
The above considerations are general. Now we demonstrate this procedure by an explicit calculation of the 2-point function. In this case $\omega_1 = \omega_2 = \omega$. Placing one source at $z = 0$ and another at $z = \infty$, the classical solution is

$$e^{-\sqrt{2}\varphi_0} = \frac{8\omega^2|z|^{4\omega}}{\mu(1 + |z|^{4\omega})^2}. \quad (4.9)$$

Apply (4.9),

$$S_{cl} = 2\omega \ln(\mu/2) + 4\omega - 4\omega \ln(2\omega), \quad (4.10)$$

agrees with the Minkowski calculation.

The one-loop calculation is considerably more involved. Here to obtain the heat kernel, one has to solve the Liouville equation (4.3) with three sources. Two sources are equal, and are the original sources placed at $z = 0$ and $z = \infty$. The third source is placed at $w$, and is taken to zero after the derivative in (4.8) is taken. The Liouville solution with three sources can be expressed in terms of hyper-geometric functions, we refer to [14] for details. Here it is sufficient for us to write down the formula for $K(z, z)$, which is obtained after a lengthy calculation

$$K(z, z) = -2 \ln |z|^2 + 2(\psi(2\omega) + 2\psi(-2\omega) - 2\psi(1)), \quad (4.11)$$

where $\psi(x)$ is the function $\Gamma'(x)/\Gamma(x)$. We remark that the above finite result is obtained also upon certain regularization. Plugging (4.11) into formula (4.7), we obtain the one-loop result

$$S_1 = \ln(\Gamma(-2\omega)/\Gamma(2\omega)) + 4\psi(1)\omega. \quad (4.12)$$

The asymptotic behavior of the first term at large $\omega$ is as desired. The last term may be interpreted as a finite renormalization of the cosmological constant, which is regularization scheme dependent.

We conclude that calculations in both the Minkowski world sheet and the Euclidean world sheet correctly account for the high energy behavior of the 2-point amplitude. While the Minkowski calculation is more physical, and shows clearly how higher string modes modify the zero mode contribution, the Euclidean calculation is more powerful if one wishes to calculate high point amplitudes.

5. Discussions: The Time Delay and String Extension

In this section, we briefly discuss the spacetime meaning of the leg factor in the high-energy limit and try to interpret our result qualitatively. Consider the in(+) and out(−) string-wave packets which are localized with respect to their center of mass,

$$\psi_{\pm}(X \pm \varphi) = \int \frac{d\omega}{2\pi\omega} f_{\pm}(\omega)e^{-i\sqrt{2}\omega(X \pm \varphi)}. \quad (5.1)$$
If the modulus $|f_\pm(\omega)|$ of the function $f_\pm(\omega)$ is sharply peaked at $\omega = \omega_0$, the trajectory of the wave packet is given, using the phase $\phi_\pm(\omega)$ of the function $f_\pm(\omega)$, as

$$X = \mp \varphi + \delta_\pm(\omega_0), \quad (5.2)$$

where

$$\delta_\pm(\omega_0) \equiv \left. \frac{1}{\sqrt{2}} \frac{\partial \phi_\pm(\omega)}{\partial \omega} \right|_{\omega=\omega_0}. \quad (5.3)$$

The general transition matrix element for the wave packet states is

$$S_{nm} = \left( \prod_{i=1}^{n} \int \frac{d\omega_i}{2\pi \omega_i} \right) \left( \prod_{j=n+1}^{n+m} \int \frac{d\omega_j}{2\pi \omega_j} \right) \delta(\sum_{i=1}^{n} \omega_i - \sum_{j=1}^{m} \omega_{n+j})$$

$$\times \left( \prod_{i=1}^{n} f_i(\omega_i) \right) S(\omega_1, \omega_2, \ldots, \omega_n; -\omega_{n+1}, \ldots, -\omega_{n+m})$$

$$\times \left( \prod_{j=1}^{m} f_j(\omega_{n+j}) \right). \quad (5.4)$$

The trajectories of the scattered wave packets are determined by the condition that the phase of the integrand in eq. (5.4) is stationary with respect to the variation of independent energies $\omega_i$'s. Since the matrix model $S$-matrix element $\bar{S}(\omega_i)$ has no phase factor, this means that the trajectories are determined by the leg factor. In the high energy limit studied here, they are thus obtained by solving

$$\frac{\partial}{\partial \tilde{\omega}_a} \left\{ \sum_{i=1}^{n} (-\phi_{i-}(\omega_i) + \gamma(\omega_i)) + \sum_{j=1}^{n+j} (\phi_{j+}(\omega_{j+n}) + \gamma(\omega_{n+j})) \right\} = 0, \quad (5.5)$$

where $\{\tilde{\omega}_a : a = 1, 2, \ldots, n + m - 1\}$ denotes the independent set of energies, $\phi_{i-}, \phi_{j+}$ the phases of the functions $f_{i-}, f_{j+}$, respectively, and

$$\gamma(\omega) = 8\omega(\ln 2\omega - 1) - 2\omega \ln \mu. \quad (5.6)$$

Here and what follows, we drop the indices 0 for the peaked values of energies.

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2 In the early-time limit discussed in [3], the trajectories are determined by the behavior of the phase at the first pole on the imaginary axis. Note that the early-time limit is necessarily a low-energy approximation. In the present paper, we are interested in the high-energy limit and the phase is governed by the collective effect of all poles.
For example, in the simplest $(1 \to 1)$ scattering $n = m = 1$ with $\omega_1 = \omega_2 = \omega$, we have actual time delay

$$\delta_{1-} - \delta_{1+} = \frac{d\gamma(\omega)}{d\omega} = 8 \ln 2\omega - 2 \ln \mu.$$  \hspace{1cm} (5.7)

As we have emphasized in previous sections, the first term is twice that of ordinary point particle in Liouville potential.

For $(1 \to 2)$ scattering with $\omega_3 = \omega = \omega_1 + \omega_2$, the conditions (5.5) are

$$\frac{\omega_1}{\omega} \delta_{1-} + \frac{\omega_2}{\omega} \delta_{2-} - \delta_{3+} = \frac{d\gamma(\omega)}{d\omega} + \sum_{i=1}^{2} \omega_i \frac{d\gamma(\omega_i)}{d\omega_i},$$  \hspace{1cm} (5.8)

$$\delta_{1-} - \delta_{2-} = \frac{d\gamma(\omega_1)}{d\omega_1} - \frac{d\gamma(\omega_2)}{d\omega_2}.\hspace{1cm} (5.9)$$

Similarly, for $(2 \to 1)$ scattering with $\omega_1 = \omega = \omega_2 + \omega_3$, we have

$$\delta_{1-} - (\frac{\omega_1}{\omega} \delta_{1+} + \frac{\omega_2}{\omega} \delta_{2+}) = \frac{d\gamma(\omega)}{d\omega} + \sum_{i=2}^{3} \omega_i \frac{d\gamma(\omega_i)}{d\omega_i},$$  \hspace{1cm} (5.10)

$$\delta_{1+} - \delta_{2+} = -\frac{d\gamma(\omega_2)}{d\omega_2} + \frac{d\gamma(\omega_3)}{d\omega_3}.\hspace{1cm} (5.11)$$

The trajectory of each wave packet is obtained by plugging $\delta_{\pm}(\omega_i)$ into (7.2). From (5.8) and (5.10), we see, in the extreme high-energy limit $\omega \to \infty$ where we have $\frac{d\gamma(\omega)}{d\omega} \sim 8 \ln \omega$, that the average time delay between initial and final wave packets is always given by the result of the two-point scattering. In this qualitative sense, doubling of phase shift is universal for arbitrary scattering of 2D strings in tachyon background.

The computations of previous sections clearly show that the additional contribution to the time delay arises from the extension of strings. This is natural because the duration of interaction of a string with a given potential would in general be longer, owing to string extension, than for a point particle of the same velocity. Let us try to qualitatively estimate the time delay arising from string extension. In the case of ordinary critical strings, it has been argued by several authors [15][16][17] that the string extension $\Delta \ell$ increases in proportion to its energy $\omega$ in the high energy limit. We can interpret this as follows. As the energy of a string increases, more and more energy can be exchanged in the process of interaction between the center of mass and string excitation. A large fluctuation in excitation energy implies a large string extension, which is estimated to be proportional to the energy [17]. In our two-dimensional case, there are no transverse excitations in the usual sense. However, there still appears an infinite sequence of discrete excited states at
discrete imaginary momenta. In the high-energy limit, we naturally suppose that all these
discrete states can collectively participate in the fluctuation of energies. If the fluctuation
of the excitation energy and hence of the center of mass energy is of order $\Delta E \sim \omega$ during
interaction, it is expected, by converting it to uncertainty with respect to the $\varphi$ coordinate
in the Liouville potential $e^{-\sqrt{2}\varphi}$, to give an additional contribution to the time delay of
order $\ln \Delta E \sim \ln \omega \left(\sim \ln \Delta \ell \right)$.

To summarize, We have given a new direct derivation of the phase shift for 2-point
wall scattering amplitude in the high-energy limit and suggested a general scheme for
extending our calculation to higher-point amplitudes. The result indicates that the high-
energy behavior of the leg factor reflects the effect of string extension in a given background.
We have suggested that the additional contribution to the time delay is due to the large
energy fluctuation occurring between the center of mass and discrete excited states in the
high-energy limit.

Next immediate problem would be to extend the result to the case of black hole
background. If our interpretation is correct, the high-energy limit of the leg factor in
this case should also exhibit properties which are qualitatively the same as in the tachyon
background. We are planning to report about this in a separate publication.

Acknowledgments

The present collaboration was made possible by the US-Japan collaborative program
in Science conducted by NSF and JSPS. We would like to thank NSF and JSPS for their
financial support. The research of A. J. and M. L. was supported by DOE Contract No.
DE-FG02-91ER40688-Task A. A.J. would like to thank the Institute of Physics of the
University of Tokyo at Komaba campus for its kind hospitality. T. Y. would like to thank
the Physics Department of Brown University for its kind hospitality. The research of T. Y.
was partially supported by the Grant-in-Aid for Scientific Research (No. 06640378) and
Grant-in-Aid for Priority Area (No. 06221211) from the Ministry of Education, Science,
and Culture.
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