The Random Quadratic Assignment Problem

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Optimal assignment of classes to classrooms [1], design of DNA microarrays [2], cross species gene analysis [3], creation of hospital layouts [4], and assignment of components to locations on circuit boards [5] are a few of the many problems which have been formulated as a quadratic assignment problem (QAP). Originally formulated in 1957, the QAP is one of the most difficult of all combinatorial optimization problems. Here, we use statistical mechanical methods to study the asymptotic behavior of problems in which the entries of at least one of the two matrices that specify the problem are chosen from a random distribution $P$. Surprisingly, this case has not been studied before using statistical methods despite the fact that the QAP was first proposed over 50 years ago [6]. We find simple forms for $C_{\text{min}}$ and $C_{\text{max}}$, the costs of the minimal and maximum solutions respectively. Notable features of our results are the symmetry of the results for $C_{\text{min}}$ and $C_{\text{max}}$ and the dependence on $P$ only through its mean and standard deviation, independent of the details of $P$. After the asymptotic cost is determined for a given QAP problem, one can straightforwardly calculate the asymptotic cost of a QAP problem specified with a different random distribution $P$.

The quadratic assignment problem (QAP) is a combinatorial optimization problem first introduced by Koopmans and Beckmann [6]. It is NP-hard and is considered to be one of the most difficult problems to be solved optimally. The problem was defined in the following context: A set of $N$ facilities are to be located at $N$ locations. The quantity of materials which flow between facilities $i$ and $j$ is $A_{ij}$ and the distance between locations $i$ and $j$ is $B_{ij}$. The problem is to assign to each location a single facility so as to minimize (or maximize) the cost

$$C = \sum_{i=1}^{N} \sum_{j=1}^{N} A_{ij} B_{p(i)p(j)}, \quad (1)$$

where $p(i)$ represents the location to which $i$ is assigned.

In addition to being important in its own right, the QAP includes such other combinatorial optimization problems as the traveling salesman problem and graph partitioning as special cases. There is an extensive literature which addresses the QAP and is reviewed in [7–18]. With the exception of specially constructed cases, optimal algorithms have solved only relatively small instances with $N \leq 36$. Various heuristic approaches have been developed and applied to problems typically of size $N \approx 100$ or less.

Most work on the QAP has focused on solution techniques, bounds on optimal solutions, heuristics, and properties of problems with specially structured matrices. Previous work on asymptotic properties of random QAP instances has been limited to the case in which the elements of both matrices are drawn from random distributions. In this case it was shown by rigorous arguments [13,22] that almost surely as $N \to \infty$, $(C_{\text{max}} - C_{\text{min}})/C_{\text{max}} \to 0$; the minimum and maximum solutions approach the solution obtained by a random permutation of $p$.

Here we consider the properties of solutions to the QAP under the requirement that the elements of only one of the matrices need be drawn from random distribution $P$. Our approach makes use of the replica approach of statistical mechanics.

Without loss of generality, we will choose $A$ as a matrix the elements of which are chosen from the random distribution $P(A_{ij})$; the elements of $B$ are arbitrary. We find that in the asymptotic limit in which the size of the problem $N \to \infty$

$$C_{\text{min}} = \mu_A \mu_B N^2 - \sigma_A f(B) N^{3/2}, \quad (2)$$
$$C_{\text{max}} = \mu_A \mu_B N^2 + \sigma_A f(B) N^{3/2}. \quad (3)$$

Here $C_{\text{min}}$ and $C_{\text{max}}$ are the costs of the minimum and maximum solutions, respectively, and $\mu_A$ and $\sigma_A$ are the mean and standard deviations of the distribution $P(A)$; $\mu_B$ is the mean of the entries of $B$ and $b$ is a function of $B$ and $N$. Our goal is to argue for the form of equations (2) and (3). We do not attempt to determine the value of the functions $f(B)$.

It is useful to first consider the solution for which $p=p^*$ is a random permutation. Because the elements of $A$ are assigned randomly and since $p^*(i)$ and $p^*(j)$ are random, each $B_{ij}$ in the sum is multiplied by a random value of $A_{ij}$ the average of which is $\mu_A$. Hence, the cost of a random permutation is

$$C_{\text{rand}} = \mu_A \sum_{i,j=1}^{N} B_{p^*(i)p^*(j)} = \mu_A \mu_B N^2. \quad (4)$$

We now use the replica method of statistical mechanics to derive the form for $C_{\text{min}}$ and then derive the relationship of $C_{\text{max}}$ to $C_{\text{min}}$. Employing a Hamiltonian, $H$, defined as the QAP cost function our goal is to compute the partition function

$$Z = \sum_{\{p\}} \exp\left[\frac{H}{kT}\right] = \sum_{\{p\}} \exp\left[\frac{1}{kT} \sum_{i,j=1}^{N} A_{ij} B_{p(i)p(j)}\right]. \quad (5)$$

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and the free energy
\[- \frac{F}{kT} = \lim_{N \to \infty} \ln Z\]
where \(k\) and \(T\) are the Boltzmann constant and temperature respectively. Then,
\[C_{\text{min}} = F(T = 0).\]

Since the Hamiltonian includes a random matrix, \(A\), we want to calculate the value of the free energy \(F\) averaged over the disorder specified by the probability distribution \(P(A)\). However, averaging the log of the partition function is difficult. The replica method of statistical mechanics \([24]–[28]\) was introduced to make calculation of this average possible. The replica method has been used not just on models of physical systems (such as spin glasses \([24]–[26]\)) but also on such combinatorial optimization problems as graph partitioning and the traveling salesman problem \([26]–[28]\). The calculation of the average of the partition function is simplified using a mathematical identity known as the replica trick, \(\ln(x) = \lim_{n \to 0} (x^n - 1)/n\). Then equation (6) becomes
\[- \frac{F}{kT} = \lim_{N \to \infty} \lim_{n \to 0} \frac{1}{n} (Z^n - 1),\]
where
\[Z^n = \left( \sum_{\{p^1\}} \exp\left[ \frac{H(\{p^1\})}{kT} \right] \right) \ldots \left( \sum_{\{p^n\}} \exp\left[ \frac{H(\{p^n\})}{kT} \right] \right)\]
and \(Z^n = \int P(A) Z^n dA\) denotes \(Z^n\) averaged over the disorder. Here each Hamiltonian represents a replica of the original system and the sum over \(\{p^n\}\) now denotes the sum over all permutations in all replicas.

In order to achieve physically sensible results with \(f \equiv F/N\) intensive, we require the following dependence on the mean and standard deviation of \(P(A)\) to scale as (see \([25]–[27]\)):
\[
\mu_A = \bar{\mu}_A/N \\
\sigma_A = \bar{\sigma}_A/\sqrt{N}
\]
with \(\bar{\mu}_A\) and \(\bar{\sigma}_A\) independent of \(N\). In Appendix A we then find that
\[Z^n = \exp\left[ \frac{n \mu_A \mu_B N^2}{kT} \right] \times \sum_{\{p^n\}} \exp \left[ \frac{\sigma_A^2}{2(kT)^2} \sum_{i,j} \sum_{\alpha} B_{p^n(i)p^n(j)} \right] (11)\]

We can make the following observations based on equations (11) and (8):

- Consistent with equation (2), the dependence of \(F\) on \(A\) is only through \(\mu_A\) and \(\sigma_A\).
- If \(\sigma_A = 0\) and/or \(T \to \infty\), substituting equation (11) in equation (8) yields \(F = \mu_A \mu_B N^2\) which is the cost of the random solution, equation (3). This is reasonable because (i) physically for high temperatures we expect randomness and (ii) if \(\sigma_A = 0\), all entries in \(A\) are identical and all permutations yield the same costs.

We infer the form of \(F(T = 0)\) as follows:
- In equation (11) \(\sigma_A\) appears in the combination \(\sigma_A/T\). Thus, from equation (3) we see that in the \(T \to 0\) limit, only a term linear in \(\sigma_A\) can survive in \(F\).

This linear dependence on \(\sigma_A\) as well as on \(\mu_A\) is consistent with the simple case in which all of the elements in \(A\) are scaled by a constant, \(z\), in which case \(\sigma_A \to z \sigma_A\) and \(\mu_A \to z \mu_A\). Clearly the optimal permutation is unchanged but the cost is also scaled by \(z\). Thus, in this simple case, for any permutation (including the optimal one) the linear dependence on \(\mu_A\) and \(\sigma_A\) must hold.
- Given that we obtained equation (11) by expanding in \(1/\sqrt{N}\), we expect the second term in the expressions for \(C_{\text{min}}\) to be proportional \(N^{3/2}\) since the leading term is proportional to \(N^2\).

Given the considerations, above the only possible expression for the second term in \(F\) is \(\sigma_A f(B) N^{3/2}\) where \(f\) is a function of \(B\) only.

The form of \(C_{\text{max}}\) follows directly as follows. Let \(C_{\text{min}}(A, B)\) and \(C_{\text{max}}(A, B)\) denote the optimal minimum and maximum costs respectively of the QAP problem with matrices \(A\) and \(B\) and let \(-A\) denote a matrix with elements \(-A_{ij}\). Since \(C_{\text{max}}(A, B) = C_{\text{min}}(-A, B)\) and since \(\mu_A = -\mu_A\) and \(\sigma_A = \sigma_A\), the form for \(C_{\text{max}}\) in equation (3) follows directly from the form for \(C_{\text{min}}\).

If the entries of \(B\) are also drawn from a random distribution, it is straightforward to show that
\[
C_{\text{min}} = \mu A^2 \mu B N^2 - c \sigma_A \sigma_B N^{3/2} \\
C_{\text{max}} = \mu A^2 \mu B N^2 + c \sigma_A \sigma_B N^{3/2}
\]
(12)
where \(c\) is a constant independent of \(A\) and \(B\) and \(\sigma_B\) is the standard deviation of the entries in \(B\).

Zdeborová et al. \([29]\) have conjectured that, in the large \(N\) limit, the minimal and maximal costs of partitioning random regular graphs into two equal sized subgraphs are related by \(C_{\text{max}} - |E| = \frac{|E| - C_{\text{min}}}{2}\) where \(|E|\) is the total number of edges in the random regular graph. Given that the graph partitioning problem can be represented as a QAP (see Appendix B) this relationship follows directly from equations (2) and (3) and supports this conjecture for the partitioning of random graphs into two subgraphs of any size. The more general relationship \(C_{\text{max}} - \frac{\mu A^2 \mu B N^2}{2} = -\mu A^2 \mu B^2 N^2 - C_{\text{min}}\) can be interpreted as an extension to weighted as well as unweighted graphs.

We are not aware of a method to proceed further with the replica calculation for the case in which the \(B\) matrix
In Fig. 2(a) we plot $C \leq C_N$ where $\sigma_A$ is equal for a given $\sigma$. This is consistent with equations (2) and (3).

A stronger test is achieved by studying instances specified by a matrix that represents a random graph of average degree $k$. In this case,

$$\sigma_A(k) = \sqrt{k(N - 1 - k)} \equiv \sqrt{(\frac{N - 1}{2})^2 - (k - \frac{N - 1}{2})^2}$$

which represents a circle with origin at $(\frac{(N - 1)}{2}, 0)$. In Fig. 2(a) we plot $C_{\text{min}}$, $C_{\text{rand}}$, and $C_{\text{max}}$ versus $k$, $0 \leq k \leq N - 1$, for an instance of type Random-Grid. In order to illustrate the behavior of $C_{\text{min}}$ and $C_{\text{max}}$ in more detail, in Fig. 2(b), we plot

$$\Delta C_{\text{min/max}} \equiv C_{\text{min/max}} - C_{\text{rand}}$$

The solid line is an ellipse of the form

$$C_{\text{max/min}}^{\text{theory}} = \pm \sigma_A(k) f(B) N^{3/2}$$

where $f(B)$ is chosen to best fit of equation (15) to the data. The fit is consistent with the theory, exhibiting both the expected linear dependence of the optimal costs on $\sigma_A(k)$ and the symmetry represented by equations (2) and (3). In Fig. 2 we show similar plots for other varied QAP instances. Of particular interest are the plots for instances representing graph partitioning; the plots illustrate the confirmation of the conjecture of Ref. 29 for both random and random regular graphs and also the validity for partitioning of graphs into unequal sized sets of vertices.

We now study the dependence of $\Delta C$ on $N$. We treat instances in which the $A$ matrix is random or random regular and consider different types of $B$ matrix. To compare results for instances of different sizes, we define the normalized quantities $\Delta C_{\text{norm}}$ and $k_{\text{norm}}$

$$\Delta C_{\text{norm}} = \frac{\Delta C}{\mu_B N^2}$$

$$k_{\text{norm}} \equiv \frac{k}{N - 1}.$$ (16)

With this normalization we expect

$$\Delta C_{\text{norm}} \sim \frac{\Delta C}{\mu_B N^{1/2}}.$$ (17)

In Fig. 3(a) we plot $\Delta C_{\text{norm}}$ for various values of $N$ for the Random-Grid instance. We confirm the $N^{-1/2}$ dependence by plotting

$$\Delta C_{\text{collapsed}} \equiv \Delta C_{\text{norm}} N^{1/2}$$

in Fig. 3(b). The collapse is consistent with equation (17). Additional plots for other instance types are shown in Fig. 5.

In summary, using the replica method of statistical analysis, we have found simple forms for the minimum and maximum costs of QAP problems in which at least one matrix is determined by a random distribution.

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FIG. 1: For an $N = 100$ QAP instance consisting of an $A$ matrix with elements from a Gaussian distribution and a $B$ matrix representing a two-dimensional grid, (from top to bottom), $C_{\text{max}}$, $C_{\text{rand}}$, and $C_{\text{min}}$ versus standard deviation $\sigma_A$. For a given $\sigma_A$, $C_{\text{max}}$ and $C_{\text{min}}$ values are equidistant from $C_{\text{rand}}$ value.
FIG. 2: For an $N = 100$ QAP instance consisting of an $A$ matrix representing a random graph and $B$ matrix representing a two-dimensional grid, (from top to bottom), (a) $C_{\text{max}}, C_{\text{rand}},$ and $C_{\text{min}}$ versus average degree $k$ and (b) $\Delta C_{\text{max}}$ and $\Delta C_{\text{min}}$ versus $k$. In this and all following figures, the upper and lower semi-circles are the $\Delta C_{\text{max}}$ and $\Delta C_{\text{min}}$ plots, respectively. The solid circular line represents the theoretical prediction.

FIG. 3: For a QAP instance of type Random-Grid, (a) normalized $\Delta C$ versus normalized $k$ for instance sizes $N=100$ (light gray); $N=225$ (medium gray), and $N=400$ (black); (b) corresponding collapsed plots (see equation (18)).
Appendix A: Integration over disorder

In the following we retain only terms which do not vanish in the $N \to \infty$ limit. This is equivalent to retaining terms only to second order in $A_{ij}$. Because we want to maintain the exponential form, we write

\[ \bar{Z}^n = \int P(A) \sum_{\{p^\alpha\}} \exp \left[ \frac{1}{kT} \sum_{\alpha=1}^{n} \sum_{i,j=1}^{N} A_{ij} B_{p^\alpha(i)p^\alpha(j)} \right] dA \]

\[ = \sum_{\{p^\alpha\}} \exp \left[ \ln \int P(A) e^{\frac{1}{kT} \sum_{i,j} \sum_{\alpha}^{n} A_{ij} B_{p^\alpha(i)p^\alpha(j)} } dA \right] \]

\[ = \sum_{\{p^\alpha\}} \exp \left[ \ln \prod_{i,j} \int P(A_{ij}) e^{\frac{1}{kT} \sum_{\alpha}^{n} A_{ij} B_{p^\alpha(i)p^\alpha(j)} } dA_{ij} \right] \]

\[ = \sum_{\{p^\alpha\}} \exp \left[ \sum_{i,j} \ln(1 + y_{ij}) \right] \quad (A1) \]

where $y_{ij} \equiv \int P(A_{ij}) \exp \left[ \frac{1}{kT} \sum_{\alpha}^{n} A_{ij} B_{p^\alpha(i)p^\alpha(j)} \right] dA_{ij} - 1$

Expanding $y_{ij}$ to second order in $A_{ij}$ we have:

\[ y_{ij} \sim \int P(A_{ij}) \left[ 1 + \frac{1}{kT} \sum_{\alpha}^{n} A_{ij} B_{p^\alpha(i)p^\alpha(j)} + \frac{\left( \sum_{\alpha}^{n} A_{ij} B_{p^\alpha(i)p^\alpha(j)} \right)^2}{2(kT)^2} \right] dA_{ij} - 1 \]

\[ = \frac{\mu_A}{kT} \sum_{\alpha}^{n} B_{p^\alpha(i)p^\alpha(j)} + \frac{\mu_{2A}}{(kT)^2} (\sum_{\alpha}^{n} B_{p^\alpha(i)p^\alpha(j)})^2 \]

(A2)

where $\mu_{A2}$ is the second moment (around zero) of $P$. Expanding $\ln(1 + y_{ij})$ to the second order in $y_{ij}$ and substituting equation (A2), we have

\[ \ln(1 + y_{ij}) \sim \frac{\mu_A}{kT} \sum_{\alpha}^{n} B_{p^\alpha(i)p^\alpha(j)} + \frac{\mu_{2A}}{(kT)^2} (\sum_{\alpha}^{n} B_{p^\alpha(i)p^\alpha(j)})^2 - \frac{1}{2} \frac{\mu_A}{kT} \sum_{\alpha}^{n} B_{p^\alpha(i)p^\alpha(j)}^2 \]

\[ = \frac{\mu_A}{kT} \sum_{\alpha}^{n} B_{p^\alpha(i)p^\alpha(j)} + \frac{\sigma_{A}^2}{(kT)^2} (\sum_{\alpha}^{n} B_{p^\alpha(i)p^\alpha(j)})^2 \]

(A3)

where we have only retained terms to $O(A_{ij}^2)$. Finally we have

\[ \bar{Z}^n = \sum_{\{p^\alpha\}} \exp \left[ \frac{\mu_A}{kT} \sum_{i,j}^{n} B_{p^\alpha(i)p^\alpha(j)} + \frac{\sigma_{A}^2}{(kT)^2} (\sum_{\alpha}^{n} B_{p^\alpha(i)p^\alpha(j)})^2 \right] \]

\[ = \sum_{\{p^\alpha\}} \exp \left[ \frac{\mu_A \beta n N^2}{kT} + \frac{\sigma_{A}^2}{(kT)^2} \sum_{i,j}^{n} B_{p^\alpha(i)p^\alpha(j)}^2 \right] \quad (A4) \]

where we use the fact that $\sum_{i,j} B_{p^\alpha(i)p^\alpha(j)}$ is independent of permutation.

Appendix B: Relationship to Graph Partitioning

The problem of partitioning a graph into two subgraphs of size $rN$ and $(1-r)N$ with the minimum number
of edges between the two subgraphs can be represented as a QAP as follows: One matrix, $A$, is the adjacency matrix of the graph to be partitioned. The other matrix, $B$, the graph partitioning matrix, is the adjacency matrix for a bipartite graph in which edges are present between two sets of vertices; one set contains $rN$ vertices and the second set contains $(1-r)N$ vertices. The QAP cost function is the cost of partitioning the graph represented by $A$.

Appendix C: Specific B Matrix

Here we treat the case in which the matrix elements $B_{ij}$ can be represented as $b_i b_j$. We follow the spin-glass calculation of Ref. [25].

Let

$$Z'_n = \sum_{\{p^n\}} \exp \left[ \frac{\sigma_A^2 N}{2(kT)^2} N \sum_{i,j=1}^n B_{p^\alpha(i)p^\beta(j)} \right]$$

so

$$Z' = \exp \left[ \frac{\mu_A \mu_B n N^2}{kT} \right] Z'_n. \quad \text{(C1)}$$

Using

$$\sum_{i,j=1}^n \left( \sum_{\alpha=1}^n b_{p^\alpha(i)b_{p^\beta(j)}} \right)^2$$

$$= \sum_{i=1}^n \left( \sum_{\alpha=1}^n b_{p^\alpha(i)b_{p^\beta(i)}} \right)^2$$

$$= nN^2 \mu_{2b} + \sum_{(\alpha\neq\beta)} \left( \sum_{i=1}^n b_{p^\alpha(i)b_{p^\beta(i)}} \right)^2, \quad \text{(C3)}$$

We can now use the Gaussian integral identity

$$e^{\lambda z^2} = \frac{1}{\sqrt{2\pi}} \int \left( -\frac{1}{2} x^2 + (2\lambda)^{1/2} x \right) dx$$

with $dx \to \left( \frac{\sigma_A^2 N}{2(kT)^2} \right)^{1/2} dQ_{\alpha\beta}$ and find

$$Z'_n = \exp \left[ \frac{\sigma_A^2 N}{2(kT)^2} (n \mu_{2b}) \right] \int \prod_{\alpha\beta} \left( \frac{N \sigma_A^2}{2\pi(kT)^2} \right)^{1/2} dQ_{\alpha\beta} \exp \left[ -N D[Q_{\alpha\beta}] \right] \quad \text{(C4)}$$

where

$$D[Q_{\alpha\beta}] = \frac{\sigma_A^2}{2(kT)^2} \sum_{(\alpha\neq\beta)=1}^n Q_{\alpha\beta}^2 - \ln \sum_{\{p^n\}} \exp \left[ \frac{1}{2} \frac{\sigma_A^2}{kT} \right] Q \sum_{(\alpha\neq\beta)=1}^n \sum_{i=1}^N b_{p^\alpha(i)b_{p^\beta(i)}} / N. \quad \text{(C5)}$$

The form of equation (C4) suggests that the integrals be evaluated with the method of steepest descent with the value of the integral determined by the maxim value of $D$. Assuming no replica symmetry breaking, at the maximum all values of $Q_{\alpha\beta}$ are equal [26]; we denote this maximum value as $Q$ and

$$D[Q] = -\left( \frac{\sigma_A}{2kT} \right)^2 n(n-1)Q^2 - \ln \sum_{\{p^n\}} \exp \left[ \frac{1}{2} \frac{\sigma_A}{kT} \right] Q \sum_{(\alpha\neq\beta)=1}^n \sum_{i=1}^N b_{p^\alpha(i)b_{p^\beta(i)}} / N \quad \text{(C6)}$$
Using
\[
\sum_{(\alpha \neq \beta) = 1}^{n} \sum_{i=1}^{N} b_{p^\alpha(i)} b_{p^\beta(i)}
\]
\[
= \sum_{i=1}^{N} [\sum_{\alpha}^{n} (b_{p^\alpha(i)})^2 - \sum_{\alpha}^{n} b_{p^\alpha(i)}^2]
\]
\[
= \sum_{i=1}^{N} [\sum_{\alpha}^{n} (b_{p^\alpha(i)})^2 - nN\mu_{2b}]
\]
\[
(C7)
\]
to uncouple the replicas, we have
\[
D[Q] = -\left(\frac{\tilde{\sigma}_A}{2kT}\right)^2(n(n-1)Q^2 + 2n\mu_{2b}Q) - \ln \sum_{\{p^\alpha\}} \exp\left[\frac{\tilde{\sigma}_A}{kT} \sum_{\alpha}^{n} (\sum_{\alpha}^{n} b_{p^\alpha(i)})^2Q/N\right]
\]
\[
(C8)
\]
In the spin glass calculation the sum over \(i\) can be carried out here and the factor of \(1/N\) drops out. Since we cannot perform the sum over \(i\) here, the \(1/N\) factor is carried through to the end of the calculation as a normalization factor. Using the Gaussian integral identity again we can write
\[
E[Q] = \ln \sum_{\{p^\alpha\}} \exp\left[\frac{\tilde{\sigma}_A}{kT} \sum_{\alpha}^{n} (\sum_{\alpha}^{n} b_{p^\alpha(i)})^2Q/N\right]
\]
\[
= \ln \prod_{i=1}^{N} \int \exp\left[-\frac{z_i^2}{2} + \tilde{\sigma}_A \sqrt{Q/N} \sum_{\alpha}^{n} b_{p^\alpha(i)} z_i\right] \frac{dz_i}{\sqrt{2\pi}}
\]
\[
(C9)
\]
Now considering just the terms in the exponent dependent on \(b_{p^\alpha(i)}\), we find
\[
\sum_{\{p^\alpha\}} \exp\left[\frac{\tilde{\sigma}_A}{kT} \sqrt{Q/N} \sum_{\alpha}^{n} b_{p^\alpha(i)} z_i\right] = \prod_{\alpha} \left(\sum_{\{p^\alpha\}} \exp\left[\frac{\tilde{\sigma}_A}{kT} \sqrt{Q/N} \sum_{\alpha}^{n} b_{p^\alpha(i)} z_i\right]\right)^n
\]
\[
(C10)
\]
This is the key step which makes the \(n\) dependence explicit and allows non-ambiguous analytic continuation of \(n \to 0\). Keeping only terms to first order in \(n\), and using \((1/2\pi) \int_{-\infty}^{\infty} e^{-z^2/2} dz = 1\) we can then write
\[
E[Q] = \prod_{i=1}^{N} \int e^{-z_i^2/2} \left(\sum_{\alpha} \exp\left[\frac{\tilde{\sigma}_A}{kT} \sqrt{Q/N} b_{p(i)} z_i\right]\right)^n \frac{dz_i}{\sqrt{2\pi}}
\]
\[
= \prod_{i=1}^{N} \int e^{-z_i^2/2} \ln \left(\sum_{\alpha} \exp\left[\frac{\tilde{\sigma}_A}{kT} \sqrt{Q/N} b_{p(i)} z_i\right]\right) \frac{dz_i}{\sqrt{2\pi}}
\]
\[
= \ln(1 + n \prod_{i=1}^{N} \int e^{-z_i^2/2} \ln \left(\sum_{\alpha} \exp\left[\frac{\tilde{\sigma}_A}{kT} \sqrt{Q/N} b_{p(i)} z_i\right]\right) \frac{dz_i}{\sqrt{2\pi}})
\]
\[
= n \prod_{i=1}^{N} \int e^{-z_i^2/2} \ln \left(\sum_{\alpha} \exp\left[\frac{\tilde{\sigma}_A}{kT} \sqrt{Q/N} b_{p(i)} z_i\right]\right) \frac{dz_i}{\sqrt{2\pi}}
\]
\[
(C11)
\]
For small \(T\), as shown in Appendix D
\[
E[q] = n \frac{\tilde{\sigma}_A}{kT} \sqrt{Q/N} \Psi(b),
\]
\[
(C12)
\]
where \(\Psi(b)\) depends only on the values of \(b_i\) normalized by \(\sqrt{N}\). We can then write
\[
D[Q] = -\left(\frac{\tilde{\sigma}_A}{2kT}\right)^2(n(n-1)Q^2 + 2n\mu_{2b}Q) - n \frac{\tilde{\sigma}_A}{kT} \sqrt{Q} \Psi(b)
\]
\[
(C13)
\]
To find the \( Q \) which maximizes \( Z' \) we differentiate equation (C13) with respect to \( Q \) and solve for \( Q \). In the limit \( n \to 0 \) we find

\[
Q = \mu_{2b} - \frac{kT}{\sigma_A} \Psi. \tag{C14}
\]

Substituting for \( D[Q] \) and evaluating equation (C14), using the method of steepest descent we find, to first order in \( n \),

\[
Z' = \exp[-n\left(\frac{\tilde{\sigma}_A}{2kT}\right)^2(-\mu_{2b}^2n + n(n-1)Q^2 + 2n\mu_{2b}Q) - n\frac{\tilde{\sigma}_A}{kT}\sqrt{Q}\Psi(b)]
\]

\[
= \exp[nN((\frac{\tilde{\sigma}_A}{2kT})^2((\mu_{2b} - Q)^2 - \frac{\tilde{\sigma}_A}{kT}\sqrt{Q}\Psi(b)))] \tag{C15}
\]

and thus from equation (C2)

\[
F(T = 0) = -kT \lim_{n \to 0} \frac{1}{n} (Z'_n - 1)
\]

\[
= -kT[\mu_{A\mu B}N^2 + (\frac{\tilde{\sigma}_A}{2kT})^2N((\mu_{2b} - Q)^2 - \frac{\tilde{\sigma}_A}{kT}\sqrt{Q}\Psi(b))] \tag{C16}
\]

Using equation (C14), in the limit \( T \to 0 \), and defining \( f(b) = \sqrt{2a}\Psi(b) \)

\[
F(T = 0) = \mu_{A\mu B}N^2 - \tilde{\sigma}_A N f(b)
= \mu_{A\mu B}N^2 - \sigma_A N^{3/2} f(b) \tag{C17}
\]

Appendix D: \( E[Q] \) calculation

Here we discuss the evaluation of \( E[Q] \). We want to evaluate multiple integrals of the form

\[
E[Q] = \prod_i^{N} \int e^{-\frac{z_i^2}{2}} \ln \sum_p \exp[\frac{\tilde{\sigma}_A}{kT}\sqrt{Q/Nb_p(i)z_i}] \frac{dz_i}{\sqrt{2\pi}} \tag{D1}
\]

To see how the evaluation would proceed, consider the case of \( N=2 \). Then

\[
E[Q] = \int e^{-\frac{z_1^2}{2}} \left( \int e^{-\frac{z_2^2}{2}} \ln[e^{ab_1z_1+zb_2z_2} + e^{ab_2z_1+b_1z_2}] \frac{dz_1}{\sqrt{2\pi}} \frac{dz_2}{\sqrt{2\pi}} \right) \tag{D2}
\]

where \( a \sim 1/T \). For small \( T \), for different regions of integration, one exponential in the sum of exponentials dominates. Specifically, assuming without loss of generality that \( b_1 > b_2 \),

\[
\ln[e^{ab_1z_1+zb_2z_2} + e^{ab_2z_1+b_1z_2}] \sim ab_1z_1 + \ln[e^{ab_2z_2}] \quad (z_1 > z_2)
\]

\[
\sim ab_2z_1 + \ln[e^{ab_1z_1}] \quad (z_1 < z_2). \tag{D3}
\]

Then

\[
E[Q] = \int e^{-\frac{z_1^2}{2}} \left( \int_{z_2}^{\infty} e^{-\frac{z_2^2}{2}} ab_1z_1 + \ln[e^{ab_2z_2}] \frac{dz_1}{\sqrt{2\pi}} \frac{dz_2}{\sqrt{2\pi}} \right) \tag{D4}
\]

These integrals can all be solved exactly, with intermediate results in terms of the error function, \( erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz \), and we find that for this case of \( N = 2 \), \( E[Q] = a(b_1 - b_2)/\sqrt{\pi} \). This approach can be extended to any \( N \) and the result is \( a \) times a linear combination of the constants \( b_i \). We can then write

\[
E[Q] = n\frac{\tilde{\sigma}_A}{kT}\sqrt{Q}\Psi(b), \tag{D5}
\]

where \( \Psi(b) \) is the linear combination of the \( b_i \) divided by \( \sqrt{N} \) which normalizes the expression.

Appendix E: Matrix Types

- Uniform - the matrix elements are chosen from a

We employ matrices of the following types:
uniform distribution on the interval $[0, 100]$.

- Gaussian - matrix elements are chosen from a Gaussian distribution with zero mean and standard deviation $\sigma$.

- Half-Gaussian - matrix elements are chosen from a Gaussian distribution as above but only elements with value greater or equal to zero are used.

- Random (graph) - the matrix is the adjacency matrix of a random graph with edges present with probability $p$. The average degree of the graph is $k = pN$.

- Random Regular (graph) - the matrix is the adjacency matrix of a random regular graph for which all vertices are degree $k$.

- Grid - the matrix elements are the Euclidean distances between points in a two-dimensional square grid. The distances between adjacent points along the $x$ and $y$ axes are 100.

- Graph Partitioning - the matrix is the graph partitioning matrix described in Appendix B.

All matrices are symmetrical with zero diagonal. For the Random and Random Regular matrices that represent graphs, we study cases of the graph degree $k$ ranging from 0 to $N - 1$.

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FIG. 4: For various $N = 100$ QAP instance, $\Delta C_{\text{max}}$ and $\Delta C_{\text{min}}$ versus $k$. The solid circular line represents the theoretical prediction.
FIG. 5: (a),(c),(e) Normalized $\Delta C$ versus normalized $k$ for instance sizes $N=100$ (light gray); $N=225$ for (a) and 200 for (c) and (e) (medium gray); and $N=400$ (black). The right hand column contains the corresponding collapsed plots. Panels (a) and (c) also appear in the main paper.