Genus two Smale-Williams solenoid attractors in 3-manifolds

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Abstract: Using alternating Heegaard diagrams, we construct some 3-manifolds which admit diffeomorphisms such that the non-wandering sets of the diffeomorphisms are composed of Smale-Williams solenoid attractors and repellers, an interesting example is the truncated-cube space. In addition, we prove that if the nonwandering set of the diffeomorphism consists of genus two Smale-Williams solenoids, then the Heegaard genus of the closed manifold is at most two.

Keywords: 3-manifolds, Smale-Williams solenoid attractors, alternating Heegaard diagram.

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1. Introduction

For a diffeomorphism of a manifold $f : M \rightarrow M$, Smale introduced the notion of hyperbolic structure on the non-wandering set, $\Omega(f)$, of $f$. It is Smale’s long range program to classify a Baire set of these diffeomorphisms, and $\Omega(f)$ plays a crucial role in this program. He also introduced solenoid into dynamics in [S], in the literature this solenoid is called Smale solenoid or pure solenoid.

To carry out this program, Williams defined 1-dimensional solenoid in terms of 1-dimensional branched manifold, which is the generalization of Smale solenoid. There are two methods to define Smale-Williams solenoid: the inverse limit of an expanding map on branched manifold or the nested intersections of handlebodies.

Bothe studied the ambient structure of attractors in [B1], through this work, we can see the two definitions above are equivalent. Boju Jiang, Yi Ni and Shicheng Wang studied the global question in [JNW]. The question is, if a closed 3-manifold admit a diffeomorphism $f$ such that the non-wandering set of $f$ consists of two Smale solenoids, what we can say about the manifold
They proved that the manifold must be a lens space. Furthermore, for any lens space, they can construct such a diffeomorphism. Our previous paper [MY] also considered this question, we got all Smale solenoids realized in a given lens space through an inductive construction. Actually, part work of [JNW] and [MY] have been studied in [B2].

A manifold \( M \) admitting a diffeomorphism \( f \) such that \( \Omega(f) \) consists of two hyperbolic attractors presents a symmetry of the manifold with certain stability. In the paper [JNW], the authors noted that they believe many more 3-manifolds admit such symmetry if we replace the Smale solenoid by its generalization——Smale-Williams solenoid.

In this paper we consider this question, more precisely, we have the following problem: is there any closed orientable three manifold \( M \) which admits a diffeomorphism \( f \) such that the non-wandering set of \( f \), \( \Omega(f) \), is composed of Smale-Williams solenoid attractors and repellers?

In fact, for any 3-manifolds \( M \), there is a diffeomorphism \( f \) such that \((M, f)\) has a Smale-Williams solenoid as one attractor, so in the question, we must require that all of \( \Omega(f) \) consists of Smale-Williams solenoids, and using standard arguments in dynamics, in this case, there is exactly one attractor and one repeller.

Gibbons studied this question in \( S^3 \) in [G]. He constructed many such diffeomorphisms on \( S^3 \). Similar with the discussion in [JNW], we focus on the question that which manifold admit such a diffeomorphism. The main results of this paper are the following

**Theorem 3.4:** Let \( M \) be a closed 3-manifold, and there is \( f \in Diff(M) \) such that \( \Omega(f) \) consists of genus two Smale-Williams solenoids, then the Heegaard genus of \( M \), \( g(M) \leq 2 \).

**Theorem 4.5:** If a Heegaard splitting \( M = N_1 \cup N_2 \) of the closed orientable 3-manifolds \( M \) is a genus two alternating Heegaard splitting, then there is a diffeomorphism \( f \), such that \( \Omega(f) \) consists of two Smale-Williams solenoids.

In fact, we give the first example that genus two Smale-Williams solenoids can be realized globally in a Heegaard genus two closed 3-manifold. An interesting example is the rational homology sphere whose fundamental group is the extended triangle group of order 48, i.e, the truncated-cube space, see [M].
Some notions in 3-dimensional manifolds theory and in dynamics will be given in Section 2, for the definition of alternating Heegaard splitting, see Section 4.

2. Notions and facts in 3-dimensional manifolds theory and in dynamics

For fundamental facts about 3-manifolds see [H] and [J]. Let $G$ be a finite graph in $\mathbb{R}^3$, then a regular neighborhood $H$ of $G$ in $\mathbb{R}^3$ is called a handlebody, it is a 3-manifold with boundary, the genus of its boundary is called the genus of $H$, denoted by $g(H)$. Let $M$ be a closed orientable 3-manifold, if there is a closed orientable surface $S$ in $M$ which separates $M$ into two handlebodies $H_1$ and $H_2$, then we say $M = H_1 \cup_S H_2$ is a Heegaard splitting of $M$, $S$ is called a Heegaard surface. Any closed orientable 3-manifold has infinitely many Heegaard splittings, and the minimum of the genus of the Heegaard surfaces is called the Heegaard genus of the 3-manifold $M$.

A properly embedded 2-sided surface $F$ in a 3-manifold $M$ is called an incompressible surface if it is $\pi_1$-injective, otherwise, it is a compressible surface.

The following theorems will be used in the paper:

**Haken Finiteness Theorem.** Let $M$ be a compact orientable 3-manifold. Then the maximum number of pairwise disjoint, non-parallel closed connected incompressible surfaces in $M$, denoted by $h(M)$, is a finite integer $\geq 0$.

**Papakyriakopoulos Loop Theorem.** Let $M$ be a compact orientable 3-manifold and $S \subset M$ a closed orientable surface. If the homomorphism $i_* : \pi_1(S) \to \pi_1(M)$ induced by the embedding $i : S \to M$ is not injective, then there is an embedded disk $D \subset M$ such that $D \cap S = \partial D$ and $\partial D$ is an essential circle in $S$.

We recall some facts about Smale-Williams solenoid from the famous paper [W1].

**Definition 2.1:** A branched 1-manifold $L$, is just like a smooth 1-manifold, but there are two type of coordinate neighborhoods are allowed. These are the real line $R$ and $Y = \{(x, y) \in R^2 : y = 0$ or $y = \varphi(x)\}$. Here $\varphi : R \to R$ is a fixed $C^\infty$ function such that $\varphi(x) = 0$ for $x \leq 0$ and $\varphi(x) > 0$ for $x > 0$. The branch set $B$, of $L$, is the set of all points of $L$ corresponding to $(0, 0) \in Y$. 

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The first Betti number $\beta_1(L)$ is called the \textit{genus} of the branched 1-manifold, and which is just the genus of the handlebody that $L$ induced (See Example 2.6).

Note that a branched 1-manifold $L$ has a tangent bundle $T(L)$, and a differentiable map $f : L \to L'$ between branched 1-manifolds induces a map $Df : T(L) \to T(L')$ of their tangent bundles.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}

\textbf{Definition 2.2}: Let $L$ be a branched 1-manifold, a $C^r$ immersion $g : L \to L$ is called an \textit{expansion map}, if there are $c > 0, \lambda > 1$, such that

$$\|(Dg)^n(v)\| \geq c\lambda^n\|v\|$$

$\forall n \in \mathbb{N}, \forall v \in T(L)$.

\textbf{Definition 2.3}: Let $L$ be a branched 1-manifold with branch set $B$. We call $g : L \to L$ a \textit{Williams expansion map}, if:

Axiom 1. $g$ is an expansion map;

Axiom 2. $\Omega(g) = L$;

Axiom 3. Any point $p$ of $L$ has a neighborhood $U(p)$, such that $g(U(p))$ is an arc;

Axiom 4. There is a finite set $A \subset L$, such that, $g(A \cup B) \subset A$.

\textbf{Definition 2.4}: Let $\Sigma$ be the inverse limit of the sequence

$$L \xleftarrow{g} L \xleftarrow{g} L \xleftarrow{g} ....$$

where $L$ is a branched 1-manifold, $g$ is a Williams expansion map on $L$. For a point $a=(a_0, a_1, a_2, ...)$ $\in \Sigma$, let $h^{-1}(a) = (a_1, a_2, a_3, ...)$, then $h : \Sigma \to \Sigma$ is a homeomorphism. $\Sigma$ is called the \textit{Smale-}
**Williams solenoid** with shift map $h$, denoted it by $(\Sigma, h)$.

**Example 2.5**: See [W1], Figure 1 contains all of the branched 1-manifolds with two branched points. Only the first two allow immersions satisfying Axiom 2. Let $K$ be the first one and define the Williams expansion map $g : K \to K$ on its oriented 1-cells by:

\[
K_1 \to K_3^{-1}K_1K_3
\]

\[
K_2 \to K_3K_2K_3^{-1}
\]

\[
K_3 \to K_2K_3^{-1}K_1
\]

It is easy to check that $g$ satisfies Axiom 1, 3, 4 of Definition 2.3. For Axiom 2, for some semi-conjugacy reason we only need to check that the induced symbolic dynamical system matrix $X$ is irreducible, that is, for all $1 \leq i, j \leq \text{dim}(X)$, there exists $N(i, j) > 0$ such that the $ij^{th}$ entry of $X^{N(i,j)}$ is positive, see [BH]. The induced matrix $X$ of $g$ is

\[
X = \begin{pmatrix}
1 & 0 & 2 \\
0 & 1 & 2 \\
1 & 1 & 1
\end{pmatrix}
\]

obviously $X$ is irreducible.

**Example 2.6**: Figure 2(a) contains a genus two handlebody $N$ with disk foliation and a self embedding $f$. In fact, we take $N$ as a "neighborhood" of $K$, and there is a natural projection $\pi$ from $N$ to $K$, and the embedding $f$ is induced by $g$ in Example 2.5. We define $\Lambda = \bigcap_{n=1}^{\infty} f^n(N)$, then $(\Lambda, f)$ is conjugate to $(\Sigma, h)$ via $T$ constructed below. We have the following commutative diagram,

\[
\begin{array}{ccc}
N & \xrightarrow{f} & N \\
\downarrow \pi & & \downarrow \pi \\
K & \xrightarrow{g} & K
\end{array}
\]

$\forall x \in \Lambda$, $T(x)$ is defined to be $(\pi(x), \pi(f^{-1}(x)), \ldots, \pi(f^{-n}(x)), \ldots)$. As we shall see in Proposition 2.7, $T$ is the conjugate map.
Example 2.6 shows the way to construct Smale-Williams solenoid locally in a geometrical way, the key is to deal with the branched points, we can get the following proposition:

**Proposition 2.7:** Let $L$ be a branched 1-manifold with Williams expansion map $g$ which induces Smale-Williams solenoid $\Sigma$ and shift map $h$. Then the dynamical system $(\Sigma, h)$ has local three dimensional model $(\Lambda, f)$, this means, there is a genus $\beta_1(L)$ handlebody $N$, an embedding $f : N \hookrightarrow N$, such that

1. There is a projection map $\pi : N \longrightarrow L$ which gives a disc foliation structure of $N$ by $\pi^{-1}(p)$, $\forall p \in L$. $f$ preserves the foliation structure, furthermore, the area satisfies

$$\frac{S_{f(\pi^{-1}(p))}}{S_{\pi^{-1}(p)}} \leq \epsilon$$
for some $\epsilon > 0$ small enough.

(2) Define $\Lambda = \bigcap_{m=1}^{\infty} f^n(N)$, we still call $f|_{\Lambda}$ $f$, such that, $(\Sigma, h)$ is conjugate to $(\Lambda, f)$.

Proof: This proof is based on the Example 2.6.

1. Construct $(N, \pi, f)$.

Step a. Take a "neighborhood" of $L$, which is a handlebody $N$, and we foliated $N$ by disks as in Figure 2(a), so there is a natural projection $\pi : N \rightarrow L$.

Step b. Obviously there is a local immersion map $i : L \rightarrow N$ such that $\pi \circ i = g$. Because of $\text{dim}(N) = 3 = 2 \text{dim}(L) + 1$, we can disturb $i(L)$ in $N$ such that we get an embedding map $i_1 : L \hookrightarrow N$ such that $\pi \circ i_1 = g$. Now we take a neighborhood of $i_1(L)$, say $N_1$, such that $N_1$ is embedded into $N$. Just like step a, it is easy to get an embedding map $f : N \hookrightarrow N$ such that, $f(N) = N_1$ and satisfies condition (1) of Proposition 2.7.

2. Conjugation.

Let $\Sigma$ be the Smale-Williams solenoid defined by $(L, g)$, and $h$ is the shift map. Define $T : \Lambda \rightarrow \Sigma$, $\forall x \in \Lambda$, $T(x) = (\pi(x), \pi(f^{-1}(x)), ..., \pi(f^{-n}(x)))$. Now it is easy to check that $T \circ f = h \circ T$, $T \circ f(x) = (\pi \circ f(x)), \pi(x), \pi(f^{-1}(x)), ..., h \circ T(x) = (g \circ \pi(x), \pi(x), \pi(f^{-1}(x)), ...)$. Since $\pi \circ f = g \circ \pi$, we get $T \circ f = h \circ T$. On the other hand, $T$ has inverse map $T^{-1} : (x_0, x_1, x_2, ...) \rightarrow \pi^{-1}(x_0) \cap f(\pi^{-1}(x_1)) \cap f(\pi^{-2}(x_2)) \cap ..., this map is well-defined by $S_{f(\pi^{-1}(p))}/S_{\pi^{-1}(p)} \leq \epsilon$, it is easy to check it is the inverse map of $T$. So $(\Sigma, h)$ is conjugate to $(\Lambda, f)$ by $T^{-1}$. Obviously, $(N, f)$ constructed above is an attractor model. Q.E.D.

This type of attractors have important meaning in the study of attractors, see Williams [W2].

3. Restriction on the Heegaard genus

We term the first two branched 1-manifolds type $I$ and type $II$ respectively, and discuss their Williams expansion map using their geometrical model introduced above. We call this type of attractor type $I$ (type $II$) Smale-Williams solenoid attractor.

Definition 3.1: Let $N$ be a genus two handlebody in a 3-manifold $M$, if there is $f \in Diff(M)$ such that $f|_N$ is conjugate to a local model of type $I$ (type $II$) Smale-Williams solenoid attractor,
then we call \((M, f)\) has type \(I\) (type \(II\)) Smale-Williams solenoid attractor.

**Lemma 3.2:** Let \((N, f)\) be a local model of a genus two Smale-Williams solenoid attractor, then there is no properly embedded disk \(D\) in \(N\) such that \(f(N) \cap D = \emptyset\).

**Proof:** We prove the lemma only for type \(I\) Smale-Williams solenoid attractor, and the proof for the case of type \(II\) is similar.

If there is an essential disk \(D\) such that \(f(N) \cap D = \emptyset\), then there is a solid torus \(V \subset N\) such that \(\text{Im}(f) \subset V\), this induces \(\text{Im}(f_*) \subset \pi_1(V) \cong \mathbb{Z}\), so \(\text{Im}(f_*)\) is abelian.

Let \(K\) be a type \(I\) branched 1-manifold as Figure 1 shows and \(g\) be the induced map of \(f\) on \(K\), which is a train track map, see [BH].

We choose a base point \(P\) for \(\pi_1(K)\), \(x = [K_1]\), \(y = [K_3K_2K_3^{-1}]\), \(\pi_1(K) = \langle x \rangle \ast \langle y \rangle\) is a free group of rank two(see Figure 1). Since \(\text{Im}(g_*)\) is an abelian group, we have that \(g_*(xyx^{-1}y^{-1}) = 1\), \(xyx^{-1}y^{-1} = [K_3K_1K_2K_3^{-1}]K_1^{-1}K_3^{-1}K_2^{-1}K_3^{-1}\). Since \(K_3K_1K_2K_3^{-1}K_1^{-1}K_3^{-1}K_2^{-1}K_3^{-1}\) is a legal path in \(K\) and \(g\) is a train track map, we have \(g(K_3K_1K_2K_3^{-1}K_1^{-1}K_3^{-1}K_2^{-1}K_3^{-1})\) is also a legal path, so it can not be a homotopic trivial path in \(K\)(See [BH]). This means \(g_*(xyx^{-1}y^{-1}) \neq 1\), it is a contradiction. Q.E.D.

**Proposition 3.3:** Let \(M\) be a closed 3-manifold, and \(N\) is a genus two handlebody in \(M\). If there is \(f \in \text{Diff}(M)\) such that \((N, f|_N)\) is conjugate to a local genus two Smale-Williams solenoid, then \(\partial N\) is compressible in \(\overline{M-N}\).

**Proof:** Suppose \(\partial N\) is incompressible in \(\overline{M-N}\). Let \(m\) be the Haken number of \(\overline{M-N}\), denoted by \(h(\overline{M-N}) = m\), \(S_1, S_2, ..., S_{m-1}\) are mutually disjoint nonparallel incompressible surfaces in \(\overline{M-N}\). Since \(f\) is a diffeomorphism from \(\overline{M-N}\) to \(\overline{M-f(N)}\), so \(h(\overline{M-f(N)}) = m\), and \(\partial f(N)\) is incompressible in \(\overline{M-f(N)}\). If \(\partial N\) is compressible in \(\overline{M-f(N)}\), then \(\partial N\) is compressible in \(\overline{N-f(N)}\), which contradicts to Lemma 3.2. Then, by standard arguments in 3-manifold topology, \(S_1, S_2, ..., S_{m-1}\) are incompressible surfaces in \(\overline{M-f(N)}\). And then, \(S_1, S_2, ..., S_{m-1}, \partial N, \partial f(N)\) are mutually disjoint nonparallel incompressible surfaces in \(\overline{M-f(N)}\), so \(h(\overline{M-f(N)}) \geq m + 1\), which contradicts to \(h(\overline{M-f(N)}) = m\). Q.E.D.
Theorem 3.4: Let $M$ be a closed 3-manifold, and there is $f \in \text{Diff}(M)$ such that $\Omega(f)$ consists of genus two Smale-Williams solenoids, then the Heegaard genus of $M$, $g(M) \leq 2$.

Proof: If the nonwandering set $\Omega(f)$ consists of genus two Smale-Williams solenoids, then standard arguments in dynamics theory shows $\Omega(f) = \Lambda_1 \cup \Lambda_2$ where $\Lambda_1$ is an attractor, $\Lambda_2$ is a repeller (See [JNW]). In addition, $\Lambda_1(\Lambda_2)$ is realized by genus two handlebody $N_1(N_2) \subset M$, $\Lambda_1 = \cap_{n \geq 0} f^n(N_1)$ and $\Lambda_2 = \cap_{n \leq 0} f^n(N_2)$. There are $m, k \in \mathbb{Z}^+$, such that $\partial f^m(N_2) \subset \overline{N_1 - f^k(N_1)} \subset N_1$, so if necessary, we let $f^m(N_2)$ be the new $N_2$, then we have $\partial N_2 \subset \overline{N_1 - f^k(N_1)} \subset N_1$. By Proposition 3.3, $\partial N_2$ is compressible in $\overline{M - \overline{N_2}} = \overline{N_1 - N_2}$. Let $c$ be an essential simple closed curve in $\partial N_2$ which bounds a disk $D$ in $\overline{N_1 - N_2}$. Adding a neighborhood of $D$ in $N_1 - N_2$ to $N_2$ (2-handle addition along $c$), and denote the resulting manifold by $N_2^*$:

*Case 1.* If $c$ is non-separating in $\partial N_2$, $\partial N_2^* = S$ is a torus. In this case $N_1$ is divided into two parts $W_1$ and $W_2$ by $S$, $\partial W_2 = S \cup \partial N_1$ and $\partial W_1 = S$:

Subcase 1.1. $S$ is compressible in $W_1$, so $W_1$ is a solid torus, and $\overline{N_1 - N_2}$ is a genus two handlebody. Hence $g(M) \leq 2$.

Subcase 1.2. $S$ is incompressible in $W_1$, since $N_1$ is a handlebody, $S$ must be compressible in $W_2$. Compressing $S$ in $W_2$, we get a 2-sphere $P$ in $W_2$, so $P$ is also in $N_2^*$. $P$ separates $W_2$ into $A$ and $B$, where $S \subset \partial A$ and $\partial N_1 \subset \partial B$. And we have $N_2^* = A^* \sharp B^*$, where $A^*$ is obtained from $A$ by capping off $P$, and $B^*$ is obtained from $B \cup \overline{\overline{N_2 - N_1}}$ by capping off $P$. $W_1 \cup A$ is in the handlebody $N_1$, so $W_1 \cup A$ is a 3-ball, $\pi_1(A^*)$ is nontrivial. And the rank of $\pi_1(N_2^*)$ is at most two, we get that the rank of $\pi_1(B^*)$ is at most one. So $B^*$ has a genus one Heegaard splitting by the fact that the Heegaard genus of $N_2^*$ is at most two. So $M = B^*$ has Heegaard genus at most one.

Case 2. If $c$ is separating in $\partial N_2$, $\partial N_2^* = S$ is composed of two tori $S_1$ and $S_2$, so there are two manifolds $W_1$ and $W_2$ in $\overline{N_1 - N_2}$ with $\partial W_i = S_i$:

Subcase 2.1. One of $W_1$ and $W_2$ is a solid torus. Then there is a nonseparating simple closed curve in $\partial N_2$ which bound a disk in $\overline{N_1 - N_2}$, and turn to Case 1.

Subcase 2.2. $S_1$ is incompressible in $W_i$, so $S_i$ must be compressible in $\overline{N_1 - W_i}$. Let $E$ be a compressible disk of $S_1$ in $\overline{N_1 - W_i}$, we can assume $E \cap D = \emptyset$, $E \cap S_2$ is a set of simple closed
curves, and \(|E \cap S_2|\) is minimal along all such compressible disk of \(S_1\). If \(E \cap S_2 \neq \emptyset\), take an innermost disk, say \(\Delta\), of \(E\), which is a compressible disk of \(S_2\) in \(\overline{N_1 - W_2}\), which is also disjoint from \(D\). Compressing \(S_2\) along \(\Delta\), we get a 2-sphere \(P_2\), which bounds a 3-ball \(B_2\) in \(N_1\), and \(W_2 \subset B_2\). From this 3-ball, and the compressibility of \(S_1\) in \(\overline{N_1 - W_1}\), we can get a 3-ball \(B_1\), which is disjoint from \(D\) and \(B_2\), and contains \(W_1\), \(\partial B_1 = P_1\). Connecting \(P_1\) and \(P_2\) by \(D\), we get a separating 2-sphere \(P\) in \(\overline{N_1 \cap N_2}\), which separates \(M\) into two components, each component is a \(B^3\) since it is contained in a handlebody. So \(M = S^3\). Q.E.D

4. Globally geometric realization of Smale-Williams solenoid type attractors in 3-manifolds

**Definition 4.1:** Let \(l_1, l_2\) be two subarcs of a branched 1-manifold \(L\), \(e\) an arc such that \(e \cap (l_1 \cup l_2) = \partial e\), \(E = I \times I = [-1, 1] \times [-1, 1]\) is a band with the core \(\{0\} \times [-1, 1] = e\), \(E \cap (l_1 \cup l_2) = [-1, 1] \times \{-1, 1\}\). Then along \(e\), we get two new subarcs \(l_3, l_4\) of a new branched 1-manifold \(L'\), \(l_3\) and \(l_4\) has just one crossing, this process is said to be a *band move* along \(E\). And note that there are two band moves along \(E\), see Figure 3.

![Figure 3](image_url)

**Figure 3**

**Definition 4.2(Alternating Heegaard splitting of type I):** As in Figure 2(a), let \(N\) be a handlebody, \(K\) be a branched 1-manifold of type I, \(\pi : N \to K\) be the natural projection, then there are three disks in \(N\), say \(E_1, E_2\) and \(E_3\), which separate \(N\) into two 3-balls, and \(E_3\) is separating in \(N\), the image of \(E_i\) in \(K_i\) is an interior point in \(K_i\). Let \(c\) be a simple closed curve in \(\partial N\), we say \(c\) is alternating with respect to \((E_1, E_2, E_3)\), if the intersection points of \(E_i\) occur in \(c\) alternatively about \(E_3\), that is, along \(c\), we see \(E_3, E_{i_1}, E_3, E_{i_2}, E_3, E_{i_3}, E_3, \ldots\), where \(i_j = 1\) or 2 and \(i_j \cdot i_{j+1} = 2\) to all \(j\).
Let \( M = N_1 \cup N_2 \) be a genus two Heegaard splitting, if there are three disks \( D_1, D_2, D_3 \) separate \( N_1 \) into two 3-ball, where \( D_3 \) is separating in \( N_1 \), and there are three disks \( C_1, C_2, C_3 \) separate \( N_2 \) into two 3-ball, where \( C_3 \) is separating in \( N_2 \), moreover \( \partial C_i \) is alternating with respect to \((D_1, D_2, D_3)\), and \( \partial D_i \) is alternating with respect to \((C_1, C_2, C_3)\), then we say the Heegaard splitting is alternating of type I.

**Definition 4.3 (Alternating Heegaard splitting of type II):** As in Figure 2(b), let \( N \) be a handlebody, \( L \) be a branched 1-manifold of type II, \( \pi : N \to L \) be the natural projection, then there are three disks in \( N \), say \( E_1, E_2, E_3 \), which separate \( N \) into two 3-balls, none of \( E_i \) is separating in \( N \), the image of \( E_i \) in \( L_i \) is an interior point in \( L_i \). Let \( c \) be a simple closed curve in \( \partial N \), we say \( c \) is alternating with respect to \((E_1, E_2, E_3)\), if the intersection points occur in \( c \) alternatively, that is, along \( c \), we see \( E_3, E_1, E_3, E_{i_2}, E_3, E_{i_3}, E_3, ... \), where \( i_j = 1 \) or \( 2 \).

Let \( M = N_1 \cup N_2 \) be a genus two Heegaard splitting, if there are three disks \( D_1, D_2, D_3 \) separate \( N_1 \) into two 3-balls, none of \( D_i \) is separating in \( N_1 \), and there are three disks \( C_1, C_2, C_3 \) separate \( N_2 \) into two 3-balls, none of \( C_i \) is separating in \( N_2 \). Moreover \( \partial C_i \) is alternating with respect to \((D_1, D_2, D_3)\), and \( \partial D_i \) is alternating with respect to \((C_1, C_2, C_3)\), then we say the Heegaard splitting is alternating of type II.

**Proposition 4.3:** If there is a genus two handlebody \( N \) in \( M \), and an alternating simple closed curve \( c \) in \( \partial N \) which bounds a disk in \( M - N \), then there is \( f \in Diff(M) \), such that \( \Omega(f) \) contains a genus two Smale-Williams solenoid attractor.

**Proof:** The proof is an explicit construction, we construct the diffeomorphism in the type I case, the type II case is similar.

We choose three parallel curves \( c_1, c_2, c_3 \) in \( \partial N \) which are parallel to \( c \). The branched 1-manifold \( J \) is a spine of \( N \), which is composed of oriented 1-cells: \( J_1, J_2, J_3 \) as Figure 4 shows. Note that \( J \) induces a disk foliation of \( N \), so \( \pi : N \to J \) is the projection map. We do the following operations to \( J \):

**Operation 1.** As indicated in Figure(4-1) → Figure(4-2). We take a subarc \( J_{1,1} \) of \( J_1 \), half-twist and move it toward \( \partial N \) and identify it with a subarc \( c_{1,1} \) of \( c_1 \), just like Figure (4-2) shows, this process is a band move in Definition 4.1. Since \( c_1 \) bounds a disk in \( M - N \), we can push \( c_{1,1} \) across...
the disk to the arc of the subarc $c_{1,2}$, where $c_{1,2} = \overline{c_1 - c_{1,1}}$. We do the same surgeries to $J_2, J_3$. In the end, we get a branched 1-manifold $J^*$ which is isotopic to $J$ in $M$.

![Figure 4-1](Image)

$J_2$ $J_3$

$J^*$

$J^{**}$

![Figure 4-2](Image)

![Figure 4-3](Image)

![Figure 4](Image)

Figure 5

Operation 2. Note that the composite map $J \rightarrow J^* \rightarrow J$ is not a Williams expansion map, where $J^* \rightarrow J$ is induced by $\pi$. So as indicated in Figure(4-2)$\rightarrow$ Figure(4-3). Slide one end of $J^*_2$ along a subarc of $J^*_1$, also one end of $J^*_3$ along another subarc of $J^*_1$, locally as Figure 5 shows. We get a new branched 1-manifold $J^{**}$ which is isotopic to $J$ in $M$ obviously.

So we can choose an $f \in Diff(M)$ which is isotopic to the identity such that:

1. $f : N \leftrightarrow N$;
2. $f(J) = J^{**}$;
(3) $f$ preserve the disk foliation structure of $N$. For every leaf $D$ of the foliation, the area satisfies $S_{f(D)}/S_D \leq \epsilon$ for some $\epsilon > 0$ small enough.

(4) Let $g = \pi \circ f$, we get the following diagram,

$$
\begin{array}{ccc}
N & \xrightarrow{f} & N \\
\downarrow \pi & & \downarrow \pi \\
J & \xrightarrow{g} & J
\end{array}
$$

$g$ is linear on every edge of $J$.

Claim: $g$ is a Williams expansion map of $J$.

Proof of the Claim. We check Axiom 1, ..., Axiom 4 of Definition 2.3 one by one.

Axiom 1. From the construction of $f : N \rightarrow N$, for example in Figure 4, we know, $g$:

$J_1 \rightarrow J_2 J_1^{-1} J_3$, $J_3 \rightarrow J_1^{-1} J_3 J_1 J_2 J_1^{-1} J_3$, and $J_2 \rightarrow J_1 J_2 J_1^{-1} J_3 J_1 J_2 J_1^{-1}$. Since $f$ expands wholly and $c$ has the alternating property then $g$ is smooth immersion for every local smooth arc of $J$. Since there are two branched points, $g$ is a Williams expansion map.

Axiom 2. Since the matrix of the symbol dynamical system induced by $g : J \rightarrow J$ is irreducible, actually nowhere is zero in the matrix through the check of Axiom 1. Thus any point of $J$ is a nonwandering point of $(J, g)$.

Axiom 3 and Axiom 4. These are obviously. So the Claim follows.

By the Claim, we know $f \in Diff(M)$ and $\Omega(f)$ contains a genus two Smale-Williams solenoid attractor. Q.E.D.

Example 4.4($RP^3$): Figure 6 is an alternating genus 2 Heegaard diagram of $M = RP^3 = N_1 \cup N_2$. The left figure depicts the diagram seen from out of $N_1$, $d_i$ bounds a disk $D_i$ in $N_1$, $c_i$ bounds a disk $C_i$ in $N_2$. The right figure depicts the diagram seen from out of $N_2$, it comes from the left diagram by Dehn twists $D_{e_1} D_{e_2} D_{e_2} D_{e_1}$ in Figure 7 and then follows by a mirror symmetry.
And now, by the diagram, we construct a self-diffeomorphism \( f \) of \( RP^3 \) such that \( \Omega(f) \) consists of two type I Smale-Williams solenoids.

Let \( J \) be the natural spine of \( N_1 \) with respect to the three disks \( D_1, D_2 \) and \( D_3 \) bounded by \( d_1, d_2 \) and \( d_3 \), and \( L \) be the spine of \( N_2 \) with respect to the three disks \( C_1, C_2 \) and \( C_3 \) bounded by \( c_1, c_2 \) and \( c_3 \).

Let \( J \cap D_i = P_i, L \cap C_i = O_i \) and we fix a point \( Q_i \in c_i \cap d_i \). We connect \( P_i \) with \( Q_i \) by an arc \( v_i \) in \( D_i \), and connect \( Q_i \) with \( O_i \) by an arc \( w_i \) in \( C_i \).
We isotopy $J$ to $J^*$ in $M$ by performing three band moves along $v_1$, $v_2$ and $v_3$, similarly we can isotopy $L$ to $L^*$ by performing three band moves along $w_1$, $w_2$ and $w_3$. By Figure 8, we can see that $L^* \sqcup J$ is isotopic to $J^* \sqcup L$: after three local half twist surgeries and pushing moves to $J^* \sqcup L$ (see Figure 8-2), we get $L^* \sqcup J$ (see Figure 8-3), all the surgery can be regarded as appearing in three mutually disjoint 3-balls $N(v_i \sqcup w_i)$ in $M$ (see Figure 9), so this progress is an isotopy move, hence $L^* \sqcup J$ is isotopic to $J^* \sqcup L$. This process is similar with [JNW].

Figure 8-1  Figure 8-2  Figure 8-3

Figure 9

Slide
Now, we construct $f$. As Figures 10 and 11 show, we get $J^{**} (L^{**})$ from $J^* (L^*)$ by slide operations. Since $J^{**} \sqcup L$ is isotopic to $L^{**} \sqcup J$ and as in the proof of Proposition 4.3, we can construct $f \in Diff(M)$ which is isotopic to identity, such that $f(J) = J^{**}$, $f(L^{**}) = L$ and $\Omega(f)$ is composed of type I Smale-Williams solenoids. Note that here the alternating condition is used to show that the induced matrix is irreducible, so Axiom 2 of Smale-Williams solenoid follows.

**Theorem 4.5:** If a Heegaard splitting $M = N_1 \cup N_2$ of the closed orientable 3-manifolds $M$ is a genus two alternating Heegaard splitting, then there is a diffeomorphism $f$, such that $\Omega(f)$ consists of two Smale-Williams solenoids.

**Proof:** It is the same as Example 4.4. Q.E.D.

**Example 4.6(The Truncated-Cube Space):** Figure 12 is an alternating Heegaard diagram of a closed 3-manifold $M = N_1 \cup N_2$ see from outside of $N_1$.

Its fundamental group, $\pi_1(M) = \langle x_1, x_2; x_1x_2x_1x_2^{-1}x_1^{-1}x_2^{-1}, x_1x_2x_1^{-1}x_2x_1x_2^{-1} \rangle$. Let $a = x_2$ and $b = x_2x_1$, we have $\pi_1(M) = \langle a, b; a^4 = b^3 = (ab)^2 \rangle$, it is the extended triangle group, $|\pi_1(M)| = 48$, so $M$ is a genus two Seifert manifold with base surface $S^2$ and three singular fibers, $M$ is called the truncated-cube space, see [M] and [T].
Example 4.7 ($S^3$ with type II Smale-Williams solenoids): Figure 13 is an alternating Heegaard splitting of $S^3$, so type II Smale-Williams solenoids can be realized in $S^3$.

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\[ f(N) \]

Diagram showing labeled parts:
- \( K_1 \)
- \( K_2 \)
- \( K_3 \)
- \( E_1 \)
- \( E_2 \)
- \( E_3 \)
- \( f(N) \)
