Boundary feedback stabilization by piecewise constant time delay for the wave equation

Martin Gugat
Lehrstuhl 2 für Angewandte Mathematik,
Martensstr. 3, 91058 Erlangen, Germany *

Marius Tucsnak
Institut Élie Cartan Nancy (Mathématiques)
Université Henri Poincar Nancy 1
B.P. 70239, F-54506 Vandoeuvre-les-Nancy Cedex
France †

Abstract
For vibrating systems, a delay in the application of a feedback control can destroy the stabilizing effect of the control. In this paper we consider a vibrating string that is fixed at one end and stabilized with a boundary feedback with delay at the other end.

We show that for certain feedback parameters the system is exponentially stable with constant delays of the form $4L/c$, $8L/c$, $12L/c$, ... Moreover, we show that the system is exponentially stable with piecewise constant delays that attain the values $4L/c$ and $8L/c$.

Key Words: hyperbolic pde, feedback stabilization of pdes, delay, boundary feedback, switching delay, wave equation, feedback with delay, time-dependent feedback-parameter, past observation, circular string.

AMS Subject Classification 49K20, 49K25, 34H05

1 Introduction

Datko et al. have described the following problem in the application of feedback laws: Some second-order vibrating systems cannot tolerate small time delays in their damping (see [7]). In other words: Delays can destabilize a system that is asymptotically stable in the absence of delays (see [6]). The problem of instability caused by small constant delays has also been considered in [18], where a systematic frequency domain treatment of this phenomenon has been given and examples for the instability created by small delays have been presented.

*gugat@am.uni-erlangen.de
†Marius.Tucsnak@loria.fr
In [14] a constant delay with the value $2L/c$ has been considered and it has been shown that with this delay, exponential damping is possible for feedback parameters with sufficiently small absolute value that have an opposite sign as the parameters that generate exponential damping in the case without delay.

In this paper we show that for a constant delay that is an integer multiple of $4L/c$, exponential damping with feedback parameters of sufficiently small absolute value is possible if the feedback parameters have the same sign as the parameters that work in the case without delay.

Moreover, we consider piecewise constant delays with values $4L/c$ or $8L/c$ and show that also for delays that switch between those two values in an arbitrary way, the energy decays exponentially for certain feedback parameters.

For the problem considered in this paper some progress has been made in [3] for the wave equation. In [4] the related problem for the Euler-Bernoulli beam has been considered. In most studies of feedback stabilization of second-order vibrating systems, no delays are considered: In [5], a vibrating string is considered and a feedback law is presented for which the energy vanishes in finite time. In [11] it is shown that the result from [5] is stable in the sense that also with moving boundaries, the energy is driven to zero in finite time. The problem of boundary control of the wave equation has also been studied in [20], [17], [15], [16], [2], [21] and the references therein.

This paper has the following structure: In Section 2 we define the considered system and in Section 3 we show that it is well-posed.

In Section 4 we show that the system is stable with piecewise constant delays that attain the values $4L/c$ and $8L/c$. To our knowledge, this is the first example of a system that is stabilized with a switching delay, where the switching occurs between the two delay values.

In the last section we show that our feedback law is stabilizing without delay and for a certain sequence of constant delays with appropriately chosen feedback parameters of the same sign. We show the exponential decay of the energy in the system.

2 The System

Let a string of length $L > 0$ and the corresponding wave speed $c > 0$ be given. Define the set $\Omega = (0, \infty) \times (0, L)$. Define the set of initial states

$$B = \{(y_0, y_1) \in H^1(0, L) \times L^2(0, L) : y_0(0) = 0\}.$$

Let a number $\iota \in \{0, 1, 2, \ldots\}$ be given. Assume that $\delta$ is a piecewise constant function with $\delta(t) \in [2L/c, 4L/c]$ for all $t \geq 0$.

For $(y_0, y_1) \in B$ we consider the system $S_1$:

$$v(0, x) = y_0(x), \quad (2.1)$$
$$v_t(0, x) = y_1(x), \quad x \in (0, L) \quad (2.2)$$
$$v_{tt}(t, x) = c^2 v_{xx}(t, x), \quad (t, x) \in \Omega \quad (2.3)$$
3 Well-posedness of the system $S_1$

In this section, we study the well-posedness of system $S_1$ that is (2.1)-(2.6).

**Theorem 1** Assume that $\delta$ is a piecewise constant function with $\delta(t) \in [2L, 4L]$ for all $t \geq 0$.

Let $(y_0, y_1) \in B$ be given. Define the function $\alpha$ recursively by

$$\alpha(x) = \begin{cases} -\frac{1}{2} y_0(-x) + \frac{1}{2c} \int_0^{-x} y_1(s) \, ds, & x \in [-L, 0), \\ \frac{1}{2} y_0(x) + \frac{1}{2c} \int_0^x y_1(s) \, ds, & x \in [0, L), \end{cases}$$

and for $k \in \{1, 2, ..., 2(t-1)\}$ and $x \in [L + 2kL, 3L + 2kL)$ by

$$\alpha'(x) = -\alpha'(x - 2L)$$

and for $k \in \{0, 1, 2, ...\}$ and $x \in [L + 4kL + 2kL, 3L + 4kL + 2kL)$ by

$$\alpha'(x) = -\alpha'(x - 2L) + f \alpha'(x - c\delta(x)) - f \alpha'(x - 2L - c\delta(x))$$

and the condition that $\alpha$ is continuous on the interval $[-L, \infty)$. Let

$$v(t, x) = \alpha(ct + x) - \alpha(ct - x), \quad (t, x) \in \Omega.$$  

For every finite interval $I \subset [-L, \infty)$ we have $\alpha' \in L^2(I)$. The function $v$ is continuous on $\Omega$ and $v_t, v_x \in L_{loc}^1(\Omega)$. Define the family of test functions $\mathcal{T}$ as

$$\mathcal{T} = \{ \varphi \in C^2(\Omega) : \text{There exists a set } Q = [t_1, t_2] \times [x_1, x_2] \subset \Omega \text{ such that the support of } \varphi \text{ is contained in the interior of } Q \}.$$

The function $v$ satisfies the wave equation (2.3) in the following weak sense:

$$\int_{\Omega} v_t(t, x) \varphi_t(t, x) \, d(t, x) = c^2 \int_{\Omega} v_x(t, x) \varphi_x(t, x) \, d(t, x) \text{ for all } \varphi \in \mathcal{T}. \quad (3.5)$$

The function $v$ satisfies (2.1) and (2.2) and (2.4)-(2.6). In this sense, $v$ is the solution of the system $S_1$ that is (2.1)-(2.6).
**Proof.** Since \( y'_0 \in L^2(0, L) \), the Sobolev imbedding Theorem implies that \( y_0 \) is continuous. Moreover, \( y_1 \) is in \( L^2(0, L) \), thus \( \alpha \) is well defined. Now we discuss the regularity of \( \alpha \). On the intervals \([-L, 0), [0, L) \) and \([L + 2kL, 3L + 2kL) \) \((k \in \{0, 1, 2, 3, \ldots \})\) the function \( \alpha \) is continuous. Due to the definition of the set \( B \) we have

\[
\lim_{x \to 0^-} \alpha(x) = -(1/2)y_0(0) = 0 = (1/2)y_0(0) = \lim_{x \to 0^+} \alpha(x),
\]

\[
\lim_{x \to L^-} \alpha(x) = \frac{1}{2}y_0(L) + \frac{1}{2c} \int_0^L y_1(s) \, ds
\]

\[
= \frac{1}{2} \int_0^L y_1(s) \, ds - \left( -\frac{1}{2}y_0(L) + \frac{1}{2c} \int_0^L y_1(s) \, ds \right)
\]

\[
= \frac{1}{2} \int_0^L y_1(s) \, ds - \alpha(-L) = \lim_{x \to L^+} \alpha(x),
\]

\[
\alpha(3L + 2kL) = (f - 1)\alpha(L + 2kL) - f\alpha(2kL - L) + C_k
\]

\[
= \lim_{x \to 3L + 2kL^-} \alpha(x)
\]

hence \( \alpha \) is continuous on the interval \([-L, \infty) \). The derivative \( \alpha' \) in the sense of distributions exists on the intervals \((-L, 0), (0, L), (L, 3L) \) and \((3L + 2kL, 5L + 2kL) \) as \( L^2 \)-function. Since \( \alpha \) is continuous, this implies that \( \alpha \) is absolutely continuous on \((-L, \infty) \). Hence \( \alpha' \in L^2_{\text{loc}}(-L, \infty) \). The continuity of \( v \) follows from the continuity of \( \alpha \). For \( t = 0 \) and \( x \in (0, L) \) we have

\[
v(0, x) = \alpha(x) - \alpha(-x) = y_0(x).
\]

For \((t, x) \in \Omega \) almost everywhere, we have

\[
v_t(t, x) = c[\alpha'(x + ct) - \alpha'(-x + ct)]. \quad (3.6)
\]

Thus the definition of \( \alpha \) implies the equation \( v_t(0, x) = y_1(x) \). Hence the initial conditions \((2.1)\) and \((2.2)\) are valid.

For \((t, x) \in \Omega \) almost everywhere, we have

\[
v_x(t, x) = \alpha'(x + ct) + \alpha'(-x + ct). \quad (3.7)
\]

By Tonelli’s Theorem (see e.g. [19]), \((3.7)\) implies \( v_x \in L^1_{\text{loc}}(\Omega) \) and \((3.6)\) implies \( v_t \in L^1_{\text{loc}}(\Omega) \).
For all $\varphi \in \mathcal{T}$, integration by parts, (3.7) and (3.6) yield

\[
\int_\Omega v_x(t,x)\varphi_x(t,x)\,d(t,x)
= \int_{x_1}^{x_2} \int_{t_1}^{t_2} \varphi_x(t,x)[\alpha'(x+ct) + \alpha'(-x+ct)]\,dt\,dx
= -\int_{x_1}^{x_2} \int_{t_1}^{t_2} \varphi_x(t,x)[\alpha(x+ct) + \alpha(-x+ct)]/c\,dt\,dx
= -\int_{t_1}^{t_2} \int_{x_1}^{x_2} \varphi_{tx}(t,x)[\alpha(x+ct) + \alpha(-x+ct)]/c\,dx\,dt
= \int_{t_1}^{t_2} \int_{x_1}^{x_2} \varphi_t(t,x)[\alpha'(x+ct) - \alpha'(x+ct)]/c\,dx\,dt
= \int_\Omega \varphi_t(t,x) v_t(t,x)/c^2\,d(x,t)
\]

hence (3.5) holds.

For $x = 0$ we have $v(t,0) = \alpha(ct) - \alpha(ct) = 0$, hence at $x = 0$ the boundary condition $v(t,0) = 0$ holds for all $t > 0$.

For $x = L$, (3.7) implies for $t \in (0, 4\iota L/c)$ the equation

\[
v_x(t,L) = \alpha'(L+ct) + \alpha'(ct-L) = -\alpha'(ct-L) + \alpha'(ct-L) = 0.
\]

Therefore, the boundary condition (2.5) holds for all $t \in (0, 4\iota L/c)$.

For $t > 4\iota L/c$, we have

\[
v_x(t,L) = \alpha'(ct+L) + \alpha'(ct-L)
= f\alpha'(ct-c\delta(t)) - f\alpha'(ct-2L-c\delta(t))
= f[\alpha'(L+ct-c\delta(t)) - \alpha'(-L+ct-c\delta(t))]
= \left(f/c\right) v_t(t-c\delta(t),L).
\]

Therefore, the boundary condition (2.6) holds for all $t > 4\iota L/c$.

**Remark 1** Note that our system has a continuous state. Optimal boundary control problems for the wave equation with continuous states have been considered in [4]. The proof of Theorem 7 is similar to the proof of Theorem 4.1 in [10]. Theorem 7 is a generalization of Theorem 1 in [13], where the case $\delta(x) = 2L/c$ has been considered.
3.1 Transformation of the recursion to a vector recursion

Instead of the recursion (3.3) we can also use the following linear system to characterize the solution of $S_1$:

\[
\begin{pmatrix}
\alpha'(x) \\
\alpha'(x - 2L) \\
\alpha'(x - 4L) \\
\alpha'(x - 6L) \\
\alpha'(x - c\delta(\frac{3}{2}))
\end{pmatrix} =
\begin{pmatrix}
-1 & 0 & 0 & f & -f \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
\alpha'(x - 2L) \\
\alpha'(x - 4L) \\
\alpha'(x - 6L) \\
\alpha'(x - 8L) \\
\alpha'(x - 10L)
\end{pmatrix}.
\]

(3.8)

Let $B_2$ be the matrix in system (3.8). Let $\det(\lambda I - B_2) = p_f(\lambda)$ denote the characteristic polynomial of $B_2$. Then we have the equation

\[p_f(\lambda) = \lambda^5 + \lambda^4 - \lambda f + f.\]

If $\delta(x) = 4$, we can write (3.8) in the form of the linear system

\[
\begin{pmatrix}
\alpha'(x) \\
\alpha'(x - 2L) \\
\alpha'(x - 4L) \\
\alpha'(x - 6L) \\
\alpha'(x - 8L)
\end{pmatrix} =
\begin{pmatrix}
-1 & f & -f & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
\alpha'(x - 2L) \\
\alpha'(x - 4L) \\
\alpha'(x - 6L) \\
\alpha'(x - 8L) \\
\alpha'(x - 10L)
\end{pmatrix}.
\]

(3.9)

Let $B_1$ be the matrix in system (3.9). Let $\det(\lambda I - B_1) = d_f(\lambda)$ denote the characteristic polynomial of $B_1$. Then we have the equation

\[d_f(\lambda) = \lambda^2(\lambda^3 + \lambda^2 - \lambda f + f).\]

3.2 The characteristic Polynomial

Let $j$ be a natural number. For a real number $f$ we define the polynomial

\[p_f(t) = t^{2j+1} + t^{2j} - ft + f.\]

So for $f = 0$ we have $p_0(t) = t^{2j}(1+t)$ with the roots $(-1)$ (with multiplicity 1) and zero as the second root (with multiplicity $2j$).

**Lemma 1** There exists a number $\delta_j > 0$, such that for all $f \in (-\delta_j, 0)$, all roots of $p_f$ have a modulus that is strictly less than one.

For the proof of Lemma 1 we use an intermediate result. The following Lemma 2 states that outside a neighbourhood of $(-1)$, all roots of $p_f$ have a modulus that is strictly less than one.

**Lemma 2** Let $z$ be a root of $p_f$ with $|z - (-1)| > \frac{2|f|}{1+|f|}$. Then

\[|z| < 1,
\]

that is the modulus of $z$ is strictly less than one.
Proof. The equation $p_f(z) = 0$ implies $z^{2j}(z + 1) - f(z + 1) + 2f = 0$. Hence $(z^{2j} - f)(z + 1) = -2f$ which implies the inequality

$$|z^{2j} - f| = \frac{2|f|}{|z + 1|}.$$ 

We have

$$|z|^{2j} \leq |z^{2j} - f| + |f| = \frac{2|f|}{|z + 1|} + |f| < (1 - |f|) + |f| = 1.$$ 

Hence we have $|z| < 1$ and the assertion follows.

Note that if $f < 4j + 1$, for all $t \geq 1$ we have $p_f'(t) = t^{2j-1}((2j+1)t+2j) - f > 0$, and since $p_f(1) = 2$ this implies that $p_f$ does not have a root in $[1, \infty)$.

Now we come to the proof of Lemma 1.

For $t \leq -1$, we have $p_f'(t) = t^{2j-1}((2j+1)t+2j) - f > 0$, so $p_f$ is strictly increasing on $(-\infty, -1)$ and since $p_f(-1) = 2f < 0$ this implies that $p_f$ does not have a root in $(-\infty, -1]$. Since $p_f(1) = 2 > 0$, there exists a real root of $p_f$ in $(-1, 1)$ and all real roots of $p_f$ are contained in this interval.

For $f = 0$, $z_0 = -1$ is a single root of $p_f$. Hence there exist numbers $\epsilon > 0$, $\delta > 0$ and a neighbourhood $U(-1) = \{z \in C : |z + 1| \leq \epsilon\}$ such that for all $f \in (-\delta, 0)$ there exists exactly one root of $p_f$ in $U(-1)$. Since the complex roots appear in conjugate pairs, this must be a real root, hence it is in $(-1, 1)$.

In particular, its absolute value is strictly less than one.

The other $2j$ roots of $p_f$ are all outside of $U(-1)$. If $|f|$ is sufficiently small, for all $z \not\in U(-1)$ we have $|z - (-1)| > \epsilon > \frac{2|f|}{|1+f|}$ and thus Lemma 2 implies $|z| < 1$ which finishes the proof of Lemma 1.

The following Lemma implies a necessary condition that must hold if all roots of $p_f$ have a modulus that is strictly less than one.

**Lemma 3** If $f < 0$ and $p_f(-f) \geq 0$, there exists a real root of $p_f$ with absolute value greater than or equal to one.

**Proof.** We have $p_f(0) = f < 0$. Since $p(-f) \geq 0$, there exists a real root $t_* \in (0, -f]$ with $p_f(t_*) = 0$. Hence $\frac{1}{t_*} \geq 1$. Let $z_1, \ldots, z_{2j}$ denote the other roots. Then we have $|z_1z_2 \cdots z_{2j}| = \frac{1}{t_*} \geq 1$. Thus the assertion follows.

**Lemma 4** If $f < 0$ and $\lambda \in (3 - 2\sqrt{2}, 3 + \sqrt{2})$, we have $p_f((\lambda|f|)^{\frac{1}{2j+1}}) > 0$. 

Proof. We have
\[ p_f((\lambda |f|^{2j+1})) = \lambda|f| + \lambda^{2j+1} \frac{t^{2j+1}}{2j+1} - \lambda^{2j+1} f|f|^{2j+1} + f \]
\[ = (\lambda - 1)|f| + \lambda^{2j+1} \frac{t^{2j+1}}{2j+1} + \lambda^{2j+1} f|f|^{2j+1} \]
\[ = |f|^{2j+1} \left[ (\lambda - 1)|f|^{2j+1} + \lambda^{2j+1} + \lambda^{2j+1} f|f|^{2j+1} \right] \]
\[ = |f|^{2j+1} \cdot h(|f|^{2j+1}), \]
where
\[ h(t) = \lambda^{2j+1} t^2 + (\lambda - 1)t + \lambda^{2j+1}. \]
Since
\[ \Delta = (\lambda - 1)^2 - 4\lambda = \lambda^2 - 6\lambda + 1 < 0 \]
due to our choice of \( \lambda \), we have \( h(t) > 0 \) for all \( t \in (-\infty, \infty) \) and the assertion follows.

Lemma 5 If \( f < 0 \) and
\[ f > -\frac{1}{2j} \left( \frac{2j-1}{2j+1} \right)^{2j}, \quad (3.10) \]
we have
\[ p_f(-\frac{2j-1}{2j+1}) = \frac{1}{2j+1} \left[ 4jf + 2 \left( \frac{2j-1}{2j+1} \right)^{2j} \right] > 0, \quad (3.11) \]
\[ p_f(-2j|f|^{\frac{1}{2j}}) > 0, \quad (3.12) \]
\[ p_f(-|f|^{\frac{1}{2j}}) \leq 0, \quad (3.13) \]
\[ p_f((\frac{1}{2j}|f|^{\frac{1}{2j}}) < 0 \quad (3.14) \]
and there exist three real roots of \( p_f \), one of them in the interval \((-1, -\frac{2j-1}{2j+1})\), another in the interval \((-2j|f|^{\frac{1}{2j}}, -|f|^{\frac{1}{2j}})\) and the third in \((\frac{1}{2j}|f|^{\frac{1}{2j}}, 1)\).

For \( f = -\frac{1}{2j} \left( \frac{2j-1}{2j+1} \right)^{2j} \), we have \( p_f(-\frac{2j-1}{2j+1}) = 0 \).

Proof. Let \( t = -\frac{2j-1}{2j+1} \). We have
\[ p_f(t) = t^{2j}(1 + t) + f(1 - t) \]
\[ = f(1 + \frac{2j-1}{2j+1}) + t^{2j}(1 - \frac{2j-1}{2j+1}) \]
\[ = f \frac{4j}{2j+1} + t^{2j} \frac{2}{2j+1} \]
\[ = \frac{1}{2j+1} \left[ 4jf + 2t^{2j} \right] \]
\[ > 0 \]
which implies (3.11).

For $\lambda \in (1, \infty)$, we have $p_f(-\lambda f(\lambda f)^{\frac{1}{\lambda}}) = |f|[-\lambda f^{\frac{1}{\lambda}} |f|^{\frac{1}{\lambda}} + \lambda - 1] > 0$ if

$$|f| < \left(\frac{\lambda - 1}{\lambda + 1}\right)^{\frac{1}{\lambda}}$$

and (3.12) follows with the choice $\lambda = 2j$.

We have $p_f(-|f|^{\frac{1}{\lambda}}) = |f||-2|f|^{\frac{1}{\lambda}}| \leq 0$ and (3.13) follows.

For $\lambda \in (0, 1)$, we have $p_f((\lambda f)^{\frac{1}{\lambda}}) = |f|[(\lambda + 1)\lambda (f^{\frac{1}{\lambda}} + \lambda - 1] < 0$ if

$$|f| < \left(\frac{\lambda - 1}{1 + \lambda}\right)^{\frac{1}{\lambda}}$$

and (3.14) follows with the choice $\lambda = \frac{1}{2j}$.

Since $p_f(-1) = 2f < 0$ and $p_f(0) = f < 0$ and $p_f(1) = 2 > 0$ the assertion follows.

Lemma 3 implies that we only need to consider values of $f < 0$ with $p_f(-f) < 0$.

For $j = 1$ this yields the sharper result given in Lemma 6.

**Lemma 6** Let $j = 1$. Then for all $f \in (1 - \sqrt{2}, 0)$ we have $p_f(-f) < 0$ and $p_f((-f)/(5)^{1/3}) > 0$. Hence there exists a root of $p_f$ in the interval $(-f, (-f)/(5)^{1/3})$. The other two roots we have a modulus that is strictly less than one.

**Proof.** We have $p(-f) = -f^3 + 2f^2 + f = -f[(f-1)^2-2]$. Since $f \in (1 - \sqrt{2}, 0)$ we have $1 < (1-f)^2 < 2$ which implies $p_f(-f) < 0$. Lemma 4 with $\lambda = 1/5$ implies $p_f((-f)/(5)^{1/3}) > 0$. Hence there exists a root $t_*$ of $p_f$ in the interval $(-f, (-f)/(5)^{1/3})$.

If the other roots are complex conjugate, we call them $z$ and $\bar{z}$ and have $|z|^2 = z\bar{z} = -f/t_*$.

Now we consider the case that the other roots are real. Note that for $t > 0$, we have $p'(t) > -f > 0$ so there exists nor root that is greater than $t_*$. On the other hand, for $t < -1$ we have $p'(t) > 1 - f > 0$. Since $p(-1) = 2f < 0$, this implies that there is no root in $(-\infty, -1]$ hence also in this case the absolute value of all three roots is strictly less than one. Hence the assertion follows.

For the case $j = 2$ where $p_f$ is a polynomial of degree five we only have the result given in Lemma 7.

**Lemma 7** Let $j = 2$. Then for all $f \in (-81/2500, 0)$ we have $p_f(-f) < 0$ and the roots of $p_f$ have a modulus that is strictly less than one.

**Proof.** Case 1: Suppose that $p_f$ had five real roots. Then they would all be in the interval $(-1, 1)$.

Case 2: Now we consider the case that $p_f$ has two complex conjugate roots $z$ and $\bar{z}$. Since $f \in (-81/(4 * 625), 0)$, Lemma 5 implies that we have three real roots $t_1, t_2, t_3$ such that $-1 < t_1 < -\frac{3}{5} < t_2 < -|f|^{1/4} < \frac{1}{|f|^{1/4}} < t_3$. 


We have

\[ |z|^2 = z\bar{z} = -\frac{f}{t_1 t_2 t_3} < \frac{5}{3} \frac{|f|}{\sqrt{|f|/2}} = \frac{5\sqrt{2}}{3} \sqrt{|f|} < 1. \]

Hence all roots have a modulus that is strictly less than one and the first part of the assertion follows with Lemma 3.

The following Lemma gives a construction of values of \( f \) for which a pair of complex conjugate roots of \( p_f \) for \( j = 2 \) is known. If \( |f| \) is sufficiently small and \( f < 0 \), the remaining three roots are all real, so they can be easily approximated to arbitrary precision.

**Lemma 8**  Let \( j = 2 \). Let \( a \geq 0 \) be given. Define the numbers

\[ q = a + 4a^2 + 2a^3, \]

\[ R = q + \sqrt{q^2 - 4a^3} \]

and

\[ f = (8a^3 + 4a^2)R - (1 + 4a)R^2. \]

Let \( b = 1 + 2a \), \( c = 2ab - R \) and \( d = 2ac - Rb \). Then we have

\[ p_f(z) = (z^2 - 2az + R) (z^3 + bz^2 + cz + d). \]

In particular, \( p_f \) has the roots \( z_1 = a + \sqrt{R-a^2}i \) and \( z_2 = a - \sqrt{R-a^2}i \).

**Proof.** We have

\[ (z - z_1)(z - z_2) = z^2 - 2az + R. \]

Hence

\[ (z - z_1)(z - z_2) (z^3 + bz^2 + cz + d) = (z^2 - 2az + R) (z^3 + bz^2 + cz + d) \]

\[ = z^5 + (b - 2a)z^4 + (c - 2ab + R)z^3 + (d - 2ac + Rb)z^2 + (Rc - 2ad)z + Rd \]

\[ = z^5 + z^4 + (Rc - 2ad)z + Rd. \]

Using the definition of \( f \) we obtain the equation

\[ Rd = R[4a^2(1 + 2a) - R(1 + 4a)] = f. \]

From the definition of \( R \) we have \( R^2 - 2qR + 4a^3 = 0 \). Hence

\[ 0 = (1 + 2a)R^2 - 2(2a^3 + 4a^2 + a)R + 4a^3(1 + 2a). \]

This is equivalent to the equation

\[ 0 = (R - 2a)d + Rc. \]

Hence we have \(-2ad + Rc = -Rd = -f\), thus

\[ (z^2 - 2az + R) (z^3 + bz^2 + cz + d) = z^5 + z^4 - fz + f = p_f(z) \]

and the assertion follows.
4 Exponential stability of system $S_1$ with piece-wise constant delay

We define the energy

$$E(t) = \frac{1}{2} \int_0^L \left( \partial_x v(t, x) \right)^2 + \frac{1}{c^2} \left( \partial_t v(t, x) \right)^2 \, dx$$  \hspace{1cm} (4.1)$$

and the energy $E_1$ by the equation

$$E_1(t) = \sum_{j=0}^4 \left( t + 2j \frac{L}{c} \right).$$  \hspace{1cm} (4.2)$$

Note that $E(t) \leq E_1(t)$.

To show the exponential stability of $S_1$, we use the following result:

**Lemma 9** Let $\lambda > 0$ and the function $E : [0, \infty) \to [0, \infty)$ be given. Then the following two statements are equivalent:

1. $E$ decays exponentially in the sense that there exist real numbers $C_1, \mu \in (0, \infty)$ such that

$$E(t) \leq C_1 E(0) \exp(-\mu t)$$

for all $t \in [0, \infty)$.

2. There exist real numbers $C_2 > 0$ and $f \in (0, 1)$ such that the inequality

$$E(t + j\lambda) \leq f^j C_2 E(0)$$

holds for all $t \in [0, \lambda)$ and for all $j \in \{0, 1, 2, \ldots\}$.

**Proof.** First we show that 1. implies 2. Assume that 1. holds. Then for all $t \in [0, \lambda)$ and all $j \in \{0, 1, 2, \ldots\}$ we have the inequality

$$E(t + j\lambda) \leq C_1 E(0) \exp(-\mu(t + j\lambda))$$

$$= C_1 E(0) \exp(-\mu t) \exp(-\mu j)$$

$$\leq C_1 E(0) \exp(-\mu j)$$

$$= C_1 E(0) \exp(-\lambda \mu)^j$$

$$= f^j C_2 E(0)$$

with $C_2 = C_1$ and $f = \exp(-\lambda \mu)$.

Now we show that 2. implies 1. Assume that 2. holds. For $j \in \{0, 1, 2, \ldots\}$ define $t_j = j\lambda$. For all $t \in [\lambda, \infty)$ there exists $j \in \{0, 1, 2, \ldots\}$ such that $t \in [t_j, t_{j+1})$. Hence we can write $t = t_j + s$, with $s \in [0, \lambda)$.

Define

$$\mu = -\frac{\ln(f)}{\lambda}.$$
Then $\ln(f) = -\lambda \mu$. Let $C_1 = C_2 \exp(\lambda \mu)$. Then (2) implies the inequality

$$E(t) = E(s + t_j) \leq f^j C_2 E(0) = \exp(j \ln(f)) C_2 E(0) = \exp(-j \lambda \mu) C_1 \exp(-\lambda \mu) E(0) = C_1 \exp(-\mu t_j) \exp(-\mu \lambda) E(0) \leq C_1 E(0) \exp(-\mu(t_j + s)) = C_1 E(0) \exp(-\mu t)$$

and the assertion follows.

**Theorem 2** Let

$$f_0 = \frac{-519801 - 761\sqrt{467857}}{303170688} = 0.00343\ldots$$

Assume that the delay $\delta$ is piecewise constant and that for all $t \geq 0$ we have $\delta(t) \in \{4L/c, 8L/c\}$.

Then there exists a neighbourhood $U$ of $f_0$ such that for all $f \in U$ System $S_1$ with $\iota = 2$ is exponentially stable in the sense that the energy decays exponentially. In fact there exists a constant $C_0 > 0$ that is independend of the initial state $(y_0, y_1)$ and a constant $L < 1$ such that for all $j \in \{0, 1, 2, \ldots\}$ and for all $t \in [0, 2L/c)$ we have the inequality

$$E_1(t + 2j \frac{L}{c}) \leq L^j C_0 E_1(0).$$

**Proof.** Theorem 1 states that system $S_1$ has a solution for which we can compute the corresponding energy defined in (4.1) as

$$E(t) = \int_0^L \alpha'(x + ct)^2 + \alpha'(-x + ct)^2 dx = \int_{-L}^L \alpha'(x + ct)^2 dx.$$ 

Let $h = 2L$. Let $\lambda_i, i \in \{1, 2, 3, 4, 5\}$ denote the eigenvalues of the matrix $B_2$ from system (3.8). Assume that we have $|\lambda_5| \leq |\lambda_4| \leq |\lambda_3| \leq |\lambda_2| \leq |\lambda_1|$.

Note that for $a = 1/36$, $q = \frac{a^2 + 2a^2}{1+2a}$ and $R = q + \sqrt{q^2 - 4a^3}$ we have $f_0 = 4(2a^3 + a^2)R - (1 + 4a)R^2$. Hence for $f = f_0$, due to Lemma 3 we have the eigenvalues $a \pm \sqrt{R - a^2 i}$. Due to Lemma 5 the other three eigenvalues are real and can be approximated as the roots of the polynomial of degree that is given in Lemma 8, namely

$$z^3 + \frac{19}{18}z^2 + \frac{723 - \sqrt{467857}}{24624}z - \frac{3244 + 5\sqrt{467857}}{110808}.$$
Define the corresponding eigenvectors

\[ s_i = \frac{1}{\sqrt{1 + \lambda_i^2 + \lambda_i^3 + \lambda_i^4 + \lambda_i^5}} \begin{pmatrix} \lambda_i^4 \\ \lambda_i^3 \\ \lambda_i^2 \\ \lambda_i \\ 1 \end{pmatrix} \]

and the matrix

\[ V_2 = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 \end{pmatrix} \]

Choose the functions \( c_1(s), c_2(s), c_3(s), c_4(s), c_5(s) \) such that for \( s \in (-L, 9L) \) almost everywhere we have

\[
\begin{pmatrix}
\alpha'(s + 4h) \\
\alpha'(s + 3h) \\
\alpha'(s + 2h) \\
\alpha'(s + h) \\
\alpha'(s)
\end{pmatrix} = V_2
\begin{pmatrix}
c_1(s) \\
c_2(s) \\
c_3(s) \\
c_4(s) \\
c_5(s)
\end{pmatrix}.
\]

Since the matrix is invertible and \( \alpha' \in L^2_{\text{loc}}(-L, \infty) \) this implies \( c_1, c_2, c_3, c_4, c_5 \) in \( L^2_{\text{loc}}(-L, 9L) \). The functions \( c_i \) are the coefficients of the representation as a linear combination of the eigenvectors of the matrix \( B_2 \). Then for all natural numbers \( j \in \{0, 1, 2, \ldots\} \) due to (3.8) and (3.9) we have the representation

\[
\begin{pmatrix}
\alpha'(s + 4h + jh) \\
\alpha'(s + 3h + jh) \\
\alpha'(s + 2h + jh) \\
\alpha'(s + h + jh) \\
\alpha'(s + jh)
\end{pmatrix} = \sum_{i=1}^{5} \gamma_{i,j}(s)c_i(s)s_i,
\]

where

\[
\begin{pmatrix}
\gamma_{1,j+1}(s) \\
\gamma_{2,j+1}(s) \\
\gamma_{3,j+1}(s) \\
\gamma_{4,j+1}(s) \\
\gamma_{5,j+1}(s)
\end{pmatrix} = M(s)
\begin{pmatrix}
\gamma_{1,j}(s) \\
\gamma_{2,j}(s) \\
\gamma_{3,j}(s) \\
\gamma_{4,j}(s) \\
\gamma_{5,j}(s)
\end{pmatrix},
\]

with the matrix

\[
M(s) = \begin{cases} 
V_2^{-1}B_2V_2 & \text{if } \delta(s/c) = 8L/c, \\
V_2^{-1}B_1V_2 & \text{if } \delta(s/c) = 4L/c.
\end{cases}
\]

By our construction, the matrix \( V_2^{-1}B_2V_2 = D_2 \) is a diagonal matrix that contains the numbers \( \lambda_i \) as diagonal elements. Due to Lemma 7 this implies that \( \|D_2\|_1 < 1 \). (In fact, we have \( \|D_2\|_1 < 0.994 \).)

Let \( H_1 = V_2^{-1}B_1V_2 \). Then numerical computations show that \( \|H_1\|_1 < 1 \). (In fact, we have \( \|H_1\|_1 < 0.997 \).)
Define $L_0 = \max\{\|D_2\|_1, \|H_1\|_1\} < 1$. Then we have the inequality

$$
\left\| \begin{pmatrix} \gamma_{1,j}(s) \\ \gamma_{2,j}(s) \\ \gamma_{3,j}(s) \\ \gamma_{4,j}(s) \\ \gamma_{5,j}(s) \end{pmatrix} \right\|_1 \leq L_0^j \left\| \begin{pmatrix} \gamma_{1,0}(s) \\ \gamma_{2,0}(s) \\ \gamma_{3,0}(s) \\ \gamma_{4,0}(s) \\ \gamma_{5,0}(s) \end{pmatrix} \right\|_1 = 5L_0^j. \quad (4.3)
$$

This implies the inequality

\[
\left\| \begin{pmatrix} \alpha'(s + 4h + jh) \\ \alpha'(s + 3h + jh) \\ \alpha'(s + 2h + jh) \\ \alpha'(s + h + jh) \\ \alpha'(s + jh) \end{pmatrix} \right\|_2 \leq \left\| \sum_{i=1}^5 \gamma_{i,j}(s) c_i(s) s_i \right\|_2 = V_2 \left( \begin{pmatrix} \gamma_{1,j}(s) c_1(s) \\ \gamma_{2,j}(s) c_2(s) \\ \gamma_{3,j}(s) c_3(s) \\ \gamma_{4,j}(s) c_4(s) \\ \gamma_{5,j}(s) c_5(s) \end{pmatrix} \right) \leq \left\| V_2 \right\|_2 \left( \begin{pmatrix} c_1(s) \\ c_2(s) \\ c_3(s) \\ c_4(s) \\ c_5(s) \end{pmatrix} \right) \leq \left\| V_2 \right\|_2 \left( \begin{pmatrix} c_1(s) \\ c_2(s) \\ c_3(s) \\ c_4(s) \\ c_5(s) \end{pmatrix} \right) \leq \left\| V_2 \right\|_2 \left( \begin{pmatrix} c_1(s) \\ c_2(s) \\ c_3(s) \\ c_4(s) \\ c_5(s) \end{pmatrix} \right) \leq \sqrt{5} \left\| V_2 \right\|_2 L_0^{j/2} \left( \begin{pmatrix} c_1(s) \\ c_2(s) \\ c_3(s) \\ c_4(s) \\ c_5(s) \end{pmatrix} \right). \]
Let \( t \in [0, 10L/c] \). For the energy \( E_1 \) we have the equation

\[
E_1(t + 2jL/c) = \int_{-L}^L \alpha'(x + ct + jh)^2 dx
\]

\[
\leq \int_{-L}^L \norm{\begin{pmatrix}
\alpha'(s + ct + 4h + jh) \\
\alpha'(s + ct + 3h + jh) \\
\alpha'(s + ct + 2h + jh) \\
\alpha'(s + ct + h + jh) \\
\alpha'(s + ct + jh)
\end{pmatrix}}_2^2 ds
\]

\[
\leq 5 \|V_2\|^2 \|L_0\| \int_{-L}^L \norm{\begin{pmatrix}
c_1(s) \\
c_2(s) \\
c_3(s) \\
c_4(s) \\
c_5(s)
\end{pmatrix}}_2 ds
\]

\[
\leq L_0^2 C_0 E_1(0)
\]

which implies the exponential decay for \( f = f_0 \) due to Lemma 9. Due to continuity, we find a neighbourhood \( U \) of \( f_0 \) such that for all \( f \in U \) we have \( \|D_2(f)\|_1 < 1 \) and \( \|H_1(f)\|_1 < 1 \) and this yields the assertion.

5 Exponential stability of system \( S_1 \) with constant delay

Theorem 3 For all \( \iota \in \{0, 1, 2, \ldots\} \) there exists a number \( \delta_\iota > 0 \) such that for all \( f \in (-\delta_\iota, 0) \) System \( S_1 \) with the constant delay \( \delta(t) = 4\iota L/c \) is exponentially stable in the sense that the energy decays exponentially. In fact, there exists a constant \( C_0 > 0 \) that only depends on the initial state \((y_0, y_1)\) and \( f \) such that for all \( j \in \{0, 1, 2, \ldots\} \) and for all \( t \in [0, 2L/c] \) we have the inequality

\[
E(t + 2jL/c) \leq f^j C_0 E(0).
\]

Remark 2 Note that for the corresponding feedback law without delay

\[
v_{2\iota}(t, L) = f v_{\iota}(t, L), \ t > 0
\]

(5.1)

with \( f = -1 \), the energy is controlled to zero in finite time.

5.1 Proof of Theorem 3

Let \( \iota \in \{0, 1, 2, \ldots\} \) be given. Define the characteristic polynomial \( p_f(t) \) as in Section 3.2 with \( j = \iota \). Lemma 4 states that there exists a number \( \delta_\iota > 0 \), such that for all \( f \in (-\delta_\iota, 0) \), all roots of \( p_f \) have a modulus that is strictly less than one. The proof uses the fact that from (3.3) we get an explicit representation of \( \alpha' \). Let \( z_1, \ldots, z_{2\iota+1} \) denote the roots of \( p_f \).
Theorem 1 states that system $S_1$ has a solution for which we can compute the corresponding energy defined in (4.1) as

$$E(t) = \int_{-L}^{L} \alpha'(x + ct)^2 \, dx.$$  

Let $h = 2L$. For $x \geq (4t + 1)L$ equation (3.3) yields the equation

$$\alpha'(s) + \alpha'(s - h) - f\alpha'(s - 2th) + f\alpha'(s - (2t + 1)h) = 0. \quad (5.2)$$

Using the usual method for linear difference equations we obtain an explicit representation of the solution $\alpha' \in L^2_{loc}(-L, \infty)$ of (5.2). Choose the functions $c_1(s), c_2(s), \ldots, c_{2L+1}(s)$ such that for $s \in (-L, (2t + 1)L)$ almost everywhere we have

$$\begin{pmatrix} \alpha'(s) \\ \alpha'(s + h) \\ \alpha'(s + 2h) \\ \vdots \\ \alpha'(s + (2L-1)h) \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_{2L+1} \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{2L} & z_2^{2L} & \cdots & z_{2L+1}^{2L} \end{pmatrix} \begin{pmatrix} c_1(s) \\ c_2(s) \\ \vdots \\ c_{2L+1}(s) \end{pmatrix}.$$  

Then for all natural numbers $j \in 1, 2, \ldots$ we have the representation

$$\alpha'(s + jh) = c_1(s)z_1^{j} + c_2(s)z_2^{j} + \cdots + c_{2L+1}(s)z_{2L+1}^{j}.$$

Let $M_0 = \max\{|z_1|, |z_2|, \ldots, |z_{2L+1}|\}$. Then we have the inequality

$$|\alpha'(s + jh)| \leq M_0^j(|c_1(s)| + |c_2(s)| + \cdots + |c_{2L+1}(s)|). \quad (5.3)$$

Let $t \in [0, 2L/c)$. For the energy we obtain the inequality

$$E(t + 2j\frac{L}{c}) = \int_{-L}^{L} \alpha'(x + jh + ct)^2 \, dx$$

$$= \int_{-L}^{-L-ct} \alpha'(x + ct + jh)^2 \, dx$$

$$+ \int_{-L-ct}^{L} \alpha'(x + ct - 2L + (j + 1)h)^2 \, dx$$

$$= \int_{-L}^{L} \left| c_1(x + ct)z_1^{j} + \cdots + c_{2L+1}(x + ct)z_{2L+1}^{j} \right|^2 \, dx$$

$$+ \int_{-L-ct}^{L} \left| c_1(x + ct - 2L)z_1^{j+1} + \cdots + c_{2L+1}(x + ct - 2L)z_{2L+1}^{j+1} \right|^2 \, dx$$

$$\leq 2M_0^j \int_{-L}^{L} \left| c_1(s) + |c_2(s)| + \cdots + |c_{2L+1}(s)| \right|^2 \, ds$$

which implies the exponential decay due to Lemma 9.
6 Conclusion

In this paper we have considered feedback laws that use observations from the past for the boundary control of the considered systems that are governed by the wave equation. In this way, there is enough time for the processing of the feedback law in practice.

We have shown that if the feedback parameters are chosen appropriately the feedback laws with constant delay lead to exponential decay of the energy of the vibrating systems if the delay is an integer multiple of \(4L/c\).

Moreover, we have shown that if the delay is piecewise constant with values in \(4L/c, 8L/c\), the system also decays exponentially if the feedback parameter is chosen appropriately.

Acknowledgement This paper was supported by the PROCOPE program of DAAD, D/0811409. This paper took benefit from discussions during the meeting Partial differential equations, optimal design and numerics 2009 at the BENASQUE Center for Science Pedro Pascual.

References

[1] K. Ammari and M. Jellouli and M. Khenissi (2005) Stabilization of Generic Trees of Strings. Journal of Dynamical and Control Systems 11, 177-193.

[2] S. A. Avdonin and S. A. Ivanov (1995) Families of Exponentials. Cambridge University Press.

[3] B.Z. Guo and C.Z. Xu (2008) Boundary output feedback stabilization of a one-dimensional wave equation system with time delay. Proc. 17th IFAC World Congress, 8755-8760.

[4] B.Z. Guo and K.Y. Yang (2009) Danamic stabilization of an Euler-Bernoulli beam equation with time delay in boundary observation. Automatica, 45 1468-1475.

[5] S. Cox and E. Zuazua (1995) The rate at which energy decays in a string damped at one end. Indiana Univ. Math. J., 44, No.2, 545–573.

[6] R. Datko, J. Lagnese, M.P. Polis (1986) An example of the effect of time delays in boundary feedback stabilization of wave equations. SIAM Journal on Control and Optimization 24, pp. 152-156.

[7] R. Datko, Y.C. You (1991) Some second-order vibrating systems cannot tolerate small time delays in their damping. Journal of Optimization Theory and Applications 20, pp. 521-537.

[8] R. Dáger and E. Zuazua (2006) Wave propagation, observation and control in 1-d flexible multi-structures. Mathématiques & Applications (Berlin) 50. Berlin: Springer.
[9] M. Gugat (2006) *Optimal boundary control of a string to rest in finite time with continuous state.* ZAMM 86, 134-150.

[10] M. Gugat (2007) *Optimal energy control in finite time by varying the length of the string.* SIAM Journal on Control and Optimization 46, 1705-1725.

[11] M. Gugat (2008) *Optimal boundary feedback stabilization of a string with moving boundary.* IMA Journal of Mathematical Control and Information 25, 111-121.

[12] M. Gugat (2008) *Optimal switching boundary control of a string to rest in finite time.* ZAMM 88, 283-305.

[13] M. Gugat, M. Herty (2009) *Existence of classical solutions and feedback stabilization for the flow in gas networks.* ESAIM: COCV, DOI: 10.1051/cocv/2009035.

[14] M. Gugat (2010) *Boundary feedback stabilization by time delay for one-dimensional wave equations.* IMA Journal of Mathematical Control and Information.

[15] W. Krabs (1982) *Optimal control of processes governed by partial differential equations part ii: Vibrations.* Zeitschrift fuer Operations Research, 26:63–86.

[16] W. Krabs. *On moment theory and controllability of one-dimensional vibrating systems and heating processes* (1992) Lecture Notes in Control and Information Science 173, Springer–Verlag, Heidelberg.

[17] J. L. Lions (1988) *Exact controllability, stabilization and perturbations of distributed systems.* SIAM Review, 30: 1–68.

[18] H. Logemann, R. Rebarber and G. Weiss (1996) *Conditions for Robustness and Nonrobustness of the Stability of Feedback Systems with Respect to Small Delays in the Feedback Loop* SIAM J. Control Optim., 34: 572-600.

[19] G. K. Pedersen (1989) *Analysis Now,* Springer–Verlag, New York, 1989.

[20] D. L. Russell (1967) *Nonharmonic Fourier Series in the Control Theory of Distributed Parameter Systems.* Journal of Mathematical Analysis and Applications, 18, 542–560.

[21] M. Tucsnak and G. Weiss (2009) *Observation and Control for Operator Semigroups.* Birkhäuser Advanced Texts, Basel.