SOLUTIONS OF THE DIFFERENTIAL INEQUALITY WITH A NULL LAGRANGIAN: REGULARITY AND REMOVABILITY OF SINGULARITIES

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Abstract. We prove a theorem on self-improving regularity for derivatives of solutions of the inequality $F(v'(x)) \leq KG(v'(x))$ constructed by means of a quasiconvex function $F$ and a null Lagrangian $G$. We apply this theorem to improve the stability and Hölder regularity results of [15] and to establish a theorem on removability of singularities for solutions of this inequality.

1. Introduction

In the present paper, which is a sequel to [15], we study properties of solutions of the following inequality

$$F(v'(x)) \leq KG(v'(x)) \quad \text{a.e. } V$$

constructed by means of a quasiconvex function $F$ and a null Lagrangian $G$. The results on closer of sets of such solutions with respect to the local convergence in the Lebesgue space, their Hölder regularity, and precompactness of these sets with respect to the locally uniform convergence [15, Theorems 7 and 8 and Corollary 1] are applied to obtaining the stability theorems [15, Theorems 1 and 3-6] for the class of solutions to the equation

$$F(u'(x)) = G(u'(x)) \quad \text{a.e. } V.$$

The main result below is the theorem on self-improving regularity for derivatives of solutions of (1) (Theorem 3.1). We apply this result to improve the above-mentioned Hölder regularity and stability theorems (see Theorems 3.2 and 3.4). Also we prove the theorem on removability of singularities for solutions of (1) (Theorem 3.5).

Observe that if for a mapping $v: V \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$, $n \geq 2$, we define $F(v'(x)) = |v'(x)|^n$ and $G(v'(x)) = \det v'(x)$ then inequality (1) is the dilatation inequality

$$|v'(x)|^n \leq K \det v'(x) \quad \text{a.e. } V.$$

We remind that a solution of the class $W^{1,n}(V; \mathbb{R}^n)$ of the dilatation inequality is called a mapping with $K$-bounded distortion or a $K$-quasiregular mapping. The theory of mappings with bounded distortion is the key part of the geometric function theory which has many diverse applications (for example, see monographs [17, 18, 22, 32, 33, 34] and the bibliography therein).

A remarkable feature of the class of conformal mappings (mappings with 1-bounded distortion) is the stability phenomenon. The first results on stability of

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classes of plane and spatial conformal mappings were obtained by M. A. Lavrent’ev while studying quasiconformal mappings (homeomorphic mappings with bounded distortion) [27, 28]. Later, the theory of stability of conformal mappings which appeared in the framework of the theory of quasiconformal mappings was developed mainly by M. A. Lavrent’ev himself as well as P. P. Belinskii and Yu. G. Reshetnyak (for example, see the monographs [4, 22, 32, 33, 34] and bibliography therein).

One of the main results of this theory is the following assertion (for example, see [4, 22, 32, 33, 34, 35]): For a ball \( B(x,r) \subset \mathbb{R}^n, n \geq 2 \), each \( K \)-quasiconformal mapping \( v: B(x,r) \to \mathbb{R}^n \) with coefficient \( K \) close to 1 deviates little in the \( C \)-norm from conformal mappings on each subball \( B(x,\rho r) \), \( 0 < \rho < 1 \); moreover, the deviation vanishes as \( K \to 1 \). The stability property of conformal mappings is applied to obtaining important theorems both in the theory of quasiconformal mappings and its applications; therefore, finding other classes of mappings possessing the stability properties represents an interesting problem. Starting from the stability theory for conformal mappings, A. P. Kopylov [21] (also see [22]) proposed the general conception of stability in the \( C \)-norm for classes of mappings, while he himself named \( \xi \)-stability. This conception agrees properly with the theory of stability of conformal mappings (see [21, 22]). Indeed, the above result is equivalent to the theorem on \( \xi \)-stability of the class of conformal mappings in the class of quasiconformal mappings (see [22, Chapter 1, § 1.3]). In the \( \xi \)-stability framework various stability theorems were obtained for classes of multidimensional holomorphic mappings, classes of solutions to elliptic systems of linear partial differential equations, classes of homeotheties, and a series of other mapping classes (for example, see the articles by Kopylov [21, 22, 23], Dairbekov [9, 10], Sokolova [35, 37], and the bibliography therein). Most of the above-mentioned mapping classes can be considered as classes of solutions to equations of the form (2). In [15] we obtained a theorem on \( \xi \)-stability of classes of solutions to (2) (see [15, Theorem 1]).

Some notes on the history of results on the self-improving regularity and on the removability of singularities for mappings with bounded distortion can be found in the book of T. Iwaniec and G. Martin [19] (see also [8, 16, 18, 17]). We would like to point out that removability problems and regularity theory under minimal hypothesis are of crucial interest in PDE’s. The recent article of A. P. Kopylov [24] contains an exposition of new results on stability and regularity of solutions to elliptic systems of linear partial differential equations. As in [12, 13, 14, 15] we develop the approaches and methods used for investigations of mappings with bounded distortion to study properties of solutions of (1). In particular, we apply the Hodge decomposition theory developed by T. Iwaniec and G. Martin [18, 17, 19] and used by them, for instance, for obtaining the theorems on self-improving regularity and removability of singularities for mappings with bounded distortion (for example, see [19, Theorem 14.4.1 and 17.3.1]).

We now describe the structure of the article. In § 2 we give the basic notation and terms. In § 3 we state the main results. In § 4 we expose the preliminary results. The proof of Theorem 3.1 is presented in § 5. In § 6 we give the proof of Theorem 3.5.

2. Notation and Terminology

Let \( A \) be a set in \( \mathbb{R}^n \). The topological boundary of \( A \) is denoted by \( \partial A \). The diameter of \( A \) is defined as \( \text{diam} A := \sup\{|x - y| : x, y \in A\} \). The outer Lebesgue
measure of $A$ is denoted by $|A|$. We use the symbol $\dim_H A$ for the Hausdorff dimension of $A$.

The set $\mathbb{R}^{m \times n} := \{ \zeta = (\zeta_{\mu\nu})_{\mu=1,\ldots,m, \nu=1,\ldots,n} : \zeta_{\mu\nu} \in \mathbb{R}, \mu = 1, \ldots, m, \nu = 1, \ldots, n \}$ consists of all real $(m \times n)$-matrices. We identify a matrix $\zeta = (\zeta_{\mu\nu})_{\mu=1,\ldots,m, \nu=1,\ldots,n} \in \mathbb{R}^{m \times n}$ with the linear mapping $(\zeta_1, \ldots, \zeta_n) : \mathbb{R}^n \to \mathbb{R}^m$, where $\zeta_{\mu}(x) := \sum_{\nu=1}^n \zeta_{\mu\nu} x_\nu$, $\mu = 1, \ldots, m, x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. The operator norm in $\mathbb{R}^{m \times n}$ is defined as $|\zeta| := \sup \{ |\zeta(x)| : x \in \mathbb{R}^n, |x| < 1 \}$; and the Hilbert–Schmidt norm is defined as $\|\zeta\| := \left( \sum_{\mu=1}^m \sum_{\nu=1}^n \zeta_{\mu\nu}^2 \right)^{1/2}$. The number of $k$-tuples of ordered indices $\Gamma^k_n := \{I = (i_1, \ldots, i_k) : 1 \leq i_1 < \cdots < i_k \leq n, i_\kappa \in \{1, \ldots, n\}, \kappa = 1, \ldots, k \}$ equals the binomial coefficient $\binom{n}{k} := \frac{n!}{k!(n-k)!}$. Given $x \in \mathbb{R}^n$ and $I \in \Gamma^k_n$, we put $x_I := (x_{i_1}, \ldots, x_{i_k}) \in \mathbb{R}^k$. For $I \in \Gamma^k_n$ we denote $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. We use a convention that $dx_I = 1$ if $k = 0$. The entries of the $k$th associated matrix $M_k(\zeta) := (\det J_I \zeta)_{J \in \Gamma^k_m, I \in \Gamma^k_n} \in \mathbb{R}^{n \times n}$ for the matrix $\zeta \in \mathbb{R}^{m \times n}$ are the $k \times k$-minors $\det J_I \zeta := \det \begin{pmatrix} \zeta_{i_1j_1} & \cdots & \zeta_{i_1j_k} \\ \vdots & \ddots & \vdots \\ \zeta_{i_kj_1} & \cdots & \zeta_{i_kj_k} \end{pmatrix}$. Here and in the sequel we enumerate the entries of $\Upsilon \in \mathbb{R}^{n \times n} \times (\zeta^\times_k)$ by lexicographically ordered $k$-tuples $I \in \Gamma^k_n$ and $J \in \Gamma^k_m$, i.e. $\Upsilon = (\gamma_{IJ})_{J \in \Gamma^k_m, I \in \Gamma^k_n}$. We identify $M_1(\zeta)$ with $\zeta$.

The Jacobian matrix of $u = (u_1, \ldots, u_m) : U \subset \mathbb{R}^n \to \mathbb{R}^m$ at a point $x \in U$ is the matrix $u'(x) := \begin{pmatrix} \frac{\partial u_\mu}{\partial x_\nu}(x) \end{pmatrix}_{\nu=1,\ldots,n}$. If $I \in \Gamma^k_n$ and $J \in \Gamma^k_m$ then $\frac{\partial u_I}{\partial x_J}(x) = \frac{\partial (u_{i_1} \cdots u_{i_k})}{\partial (x_{j_1} \cdots x_{j_k})}(x) = \det J_I u'(x)$.

Let $\mathcal{V}$ be a real vector space. We say that a function $\Phi : \mathcal{V} \to \mathbb{R}$ is positively homogeneous of degree $p$ if $\Phi(tx) = t^p \Phi(x)$ for all $t > 0$ and $x \in \mathcal{V} \setminus \{0\}$. Following Ch. B. Morrey [30], we say that a continuous function $F : \mathbb{R}^{m \times n} \to \mathbb{R}$ is quasiconvex, if

$$|B(0,1)|F(\zeta) \leq \int_{B(0,1)} F(\zeta + \varphi(x)) \, dx$$

for all $\varphi \in C_0^\infty(B(0,1) ; \mathbb{R}^m)$ and $\zeta \in \mathbb{R}^{m \times n}$. Let $p \geq 1$. Following M. A. Sychev [35], we say that a quasiconvex function $F$ is strictly $p$-quasiconvex if, for $\zeta \in \mathbb{R}^{m \times n}$ and $\varepsilon, C > 0$, there is $\delta = \delta(\zeta, \varepsilon, C) > 0$ such that, for each mapping $\varphi \in C_0^\infty(B(0,1) ; \mathbb{R}^m)$ satisfying $\|\varphi\|_{L^p(B(0,1) ; \mathbb{R}^{m \times n})} \leq C|B(0,1)|^{1/p}$, the condition $\int_{B(0,1)} F(\zeta + \varphi'(x)) \, dx \leq |B(0,1)|F(\zeta + \delta)$ implies $\{x \in B(0,1) : |\varphi'(x)| \geq \varepsilon\} \leq \varepsilon|B(0,1)|$. Observe that in the mathematical literature the term strictly quasiconvexity is also used for another property (which is close but nonequivalent to ours) consisting in the fact that the strict inequality in the definition of quasiconvexity (4) is valid for nonzero mappings $\varphi$ (for example, see [20]). In this article we use the term in the sense of M. A. Sychev’s definition [35]. In the case $p > 1$ the notion of strictly $p$-quasiconvexity for functions $F$ of this article is equivalent to the notion of strictly closed $p$-quasiconvexity from J. Kristensen’s article [25] which is defined in terms of the theory of gradient Young measures (see [25, Proposition 3.4]). Observe that we can replace the ball $B(0,1)$ in the definitions of quasiconvexity and strictly $p$-quasiconvexity by an arbitrary bounded domain $U$ with $|\partial U| = 0$ (for example, see [31]). A function $G : \mathbb{R}^{m \times n} \to \mathbb{R}$ is a null Lagrangian if both functions $G$ and $-G$ are quasiconvex. The term “null Lagrangian” appeared due to
the following fact: The Euler–Lagrange equation corresponding to the variational integral \( \int_U G(u'(x)) \, dx \) with null Lagrangian \( G \) holds identically for all admissible deformations \( u: U \subset \mathbb{R}^n \to \mathbb{R}^m \) (see [2] and also [19, 3, 6, 7, 31]). The only the affine combinations of minors (called \textit{quasiaffine functions}) are null Lagrangians [11, 26] (also see [1, 2, 3, 6, 7, 19, 30, 31]): i.e.,

\begin{equation}
G(\zeta) = \gamma_0 + \sum_{k=1}^{\min\{m,n\}} \sum_{J \in \Gamma^k_n, J \in \Gamma^k_n} \gamma_{JJ} \det_{JJ} \zeta, \quad \zeta \in \mathbb{R}^{m \times n},
\end{equation}

for some \( \gamma_0, \gamma_{JJ} \in \mathbb{R} \).

3. Statement of the Main Results

Fix a number \( k \in \mathbb{N}, 2 \leq k \leq \min\{n, m\} \). Below we assume that continuous functions \( F: \mathbb{R}^{m \times n} \to \mathbb{R} \) and \( G: \mathbb{R}^{m \times n} \to \mathbb{R} \) satisfy the following conditions:

(H1) \( F \) is a quasiconvex function;

(H2) \( G \) is a null Lagrangian;

(H3) \( F \) and \( G \) are positively homogeneous of degree \( k \);

(H4) \( \sup \{ K \geq 0 : F(\zeta) \geq KG(\zeta), \zeta \in \mathbb{R}^{m \times n} \} = 1 \);

(H5) \( c_F := \inf \{ F(\zeta) : \zeta \in \mathbb{R}^{m \times n}, |\zeta| = 1 \} > 0 \);

(H6) \( d_G := \sup \{ \sum_{J \in \Gamma^k_n, J \in \Gamma^k_n} |\gamma_{JJ}|x|^2 : x \in \mathbb{R}^n, |x| = 1 \} < k c_F / (n - k) \) in the case \( k < n \).

Here the coefficients \( \gamma_{JJ} \) are taken from (5) for the null Lagrangian \( G \). By (H3), the representation (5) for the null Lagrangian \( G \) consists only of \( (k \times k) \)-minors; i.e.,

\begin{equation}
G(\zeta) = \sum_{J \in \Gamma^k_n, J \in \Gamma^k_n} \gamma_{JJ} \det_{JJ} \zeta, \quad \zeta \in \mathbb{R}^{m \times n}.
\end{equation}

Since \( F \) is continuous, (H3) implies the inequalities

\begin{equation}
c_F |\zeta|^k \leq F(\zeta) \leq C_F |\zeta|^k, \quad \zeta \in \mathbb{R}^{m \times n},
\end{equation}

with the constants \( c_F \) from (H5) and \( C_F := \sup \{ F(\zeta) : \zeta \in \mathbb{R}^{m \times n}, |\zeta| = 1 \} < \infty \).

\textbf{Theorem 3.1} (Self-improving regularity). \textit{Suppose that \( F \) and \( G \) satisfy (H2)--(H5). Let \( K \geq 1 \). Then there exist two numbers \( q(F, G, K) \) and \( p(F, G, K) \) with \( 1 < q(F, G, K) < k < p(F, G, K) \) such that for a given exponent \( p > q(F, G, K) \) every mapping \( v \in W^{1,p}_{\text{loc}}(V; \mathbb{R}^m) \), which is defined on an open set \( V \subset \mathbb{R}^n \) and satisfies inequality (1), actually lies in \( W^{1,p}_{\text{loc}}(V; \mathbb{R}^m) \) for all \( s \in (q(F, G, K), p(F, G, K)) \). Moreover, for each test function \( \varphi \in C^\infty_0(V) \) we have the Caccioppoli-type inequality

\begin{equation}
\|\varphi v'\|_{L^s(V; \mathbb{R}^m)} \leq C(F, G, K, s) \|v \otimes \varphi'\|_{L^p(V; \mathbb{R}^m)}
\end{equation}

for some constant \( C(F, G, K, s) > 0 \).

The following \textbf{Theorems 3.2, 3.3} are a straightforward consequence of \textbf{Theorems 3.1} and \textbf{13 Theorem 8, 4, and 6}.

\textbf{Theorem 3.2} (Hölder regularity). \textit{Let \( F \) and \( G \) be functions satisfying (H2)--(H6). Put \( K_0 = \infty \) for \( k = n \) and \( K_0 = \frac{k c_F}{(n-k) d_G} \) for \( k < n \). Suppose that \( K \in [1, K_0) \) and \( \delta \in (0, 1) \) satisfy the inequality

\begin{equation}
\frac{Kd_G}{k c_F} \leq \frac{1}{n - k + k \delta}.
\end{equation}

Let $V$ be an open set in $\mathbb{R}^n$. Then each solution $v \in W^{1,p}_{\text{loc}}(V; \mathbb{R}^m)$ of inequality (1) satisfies the Hölder condition with exponent $\delta$ on each compact subset in $V$.

**Theorem 3.3** (Stability in the $C$-norm). Suppose that $F$ and $G$ satisfy (H1)–(H6). Let $K \geq 1$, and let $q(F,G,K)$ denote the exponent from Theorem 3.1. Let $V$ be a domain in $\mathbb{R}^n$, and let $U$ be a compact subset in $V$. Then there is a function $\alpha(K) = \alpha_{F,G,V,U}(K)$ defined for $1 \leq K < K_0$ and such that $\lim_{K \to 1} \alpha(K) = \alpha(1) = 0$ and, for each mapping $v \in W^{1,p}_{\text{loc}}(V; \mathbb{R}^m)$, $p > q(F,G,K)$, which satisfies inequality (1) there is a mapping $u \in W^{1,k}_{\text{loc}}(V; \mathbb{R}^m)$ which is a solution to (2) such that

$$
\|v - u\|_{C(U; \mathbb{R}^m)} \leq \alpha(K) \text{diam } V.
$$

The next theorem improves Theorems 3.3 in the case when the function $F$ satisfies the following condition:

(H1') $F$ is strictly $k$-quasiconvex.

Note that condition (H1') is stronger than (H1). In this case, in addition to the estimate (10) of proximity (in the $C$-norm) of solutions of inequality (1) to solutions to equation (2), we obtain proximity estimates (in the $L^k$-norm) for the derivatives of these mappings.

**Theorem 3.4** (Stability in the Sobolev norm). Suppose that $F$ and $G$ satisfy (H1') and (H2)–(H6). Then the conclusion of Theorem 3.3 is valid together with (10) and the following inequality:

$$
\|v' - u'\|_{W^{1,1}_{\text{loc}}(U; \mathbb{R}^m)} \leq \alpha(K) \text{diam } V.
$$

**Theorem 3.5** (Removability of singularities). Suppose that $F$ and $G$ satisfy (H2)–(H6). Let $K \geq 1$, and let $q(F,G,K)$ denote the exponent from Theorem 3.1. Consider a domain $V \subset \mathbb{R}^n$. Then for a closed subset $E$ of $V$ with the Hausdorff dimension $\text{dim}_H(E) < n - q(F,G,K)$ every bounded mapping $v \in W^{1,k}_{\text{loc}}(V \setminus E; \mathbb{R}^m)$ which satisfies inequality (1) can be extended to a mapping of the class $W^{1,k}_{\text{loc}}(V; \mathbb{R}^m)$ which is defined over the whole domain $V$ and also satisfies inequality (1).

4. Preliminary Results

Let $l \in \mathbb{Z}$ with $0 \leq l \leq n$, and let $p \geq 1$. Denote by $L^p(\mathbb{R}^n; \Lambda^l)$ the space of differential $l$-forms on $\mathbb{R}^n$ with coefficients in $L^p(\mathbb{R}^n)$.

The following theorem is a modification of the result of T. Iwaniec and G. Martin on integral estimates concerning wedge products of closed differential forms [19, Theorem 13.6.1].

**Theorem 4.1** (Estimates beyond the natural exponent). Let $n, k \in \mathbb{N}$ with $2 \leq k \leq n$. Consider $p_1, \ldots, p_k, \varepsilon_1, \ldots, \varepsilon_k \in \mathbb{R}$ and $l_1, \ldots, l_k \in \mathbb{N}$ such that $1 < p_\infty < \infty$, \( \frac{1}{p_1} + \cdots + \frac{1}{p_k} = 1 \), $-1 \leq 2\varepsilon_k \leq \frac{p_\infty - 1}{p_\infty}$, and $l := n - l_1 - \cdots - l_k \geq 0$. Let $\hat{l} = (n_1, \ldots, n_l) \in \Gamma^l_n$. Suppose that $\{(\varphi_1, \ldots, \varphi_k)\}$ be $k$-tuple of closed differential forms with $\varphi_\hat{l} \in L^{1-\varepsilon_k}p_\infty(\mathbb{R}^n; \Lambda^l)$. Then

$$
\int \frac{\varphi_1 \wedge \cdots \wedge \varphi_k}{|\varphi_1|^{\varepsilon_1} \cdots |\varphi_k|^{\varepsilon_k}} \wedge dx_j \leq C(p_1, \ldots, p_k) \max(|\varepsilon_1|, \ldots, |\varepsilon_k|) \||\varphi_1|^{1-\varepsilon_1}L^{1-\varepsilon_1}_{p_1}(\mathbb{R}^n; \Lambda^l) \cdots |\varphi_k|^{1-\varepsilon_k}L^{1-\varepsilon_k}_{p_k}(\mathbb{R}^n; \Lambda^l),
$$
Remark 4.2. For the case $\hat{t} = 0$, i.e. $dx_{\hat{t}} = 1$, the estimate (12) was established in [19, Theorem 13.6.1]. In the proof of Theorem 4.1 we use the technique of Hodge decompositions developed in [19] (see also [17, 18]) and applied for proving of [19, Theorem 13.6.1].

Proof of Theorem 4.1. Observe that $(1 - \varepsilon, p)_{\infty} \geq \frac{p+1}{2} > 1$, $\varepsilon = 1, \ldots, k$. We have $\frac{\varphi}{|\varphi|_{\infty}^{\varepsilon}} \in L^{p_{\infty}}(\mathbb{R}^{n}; \Lambda^{l} \wedge \varepsilon)$. Denote by $W^{1,p}(\mathbb{R}^{n}; \Lambda^{l})$, $0 \leq l \leq n$, $p \geq 0$, the space of differential $l$-forms on $\mathbb{R}^{n}$ with coefficients in $W^{1,p}(\mathbb{R}^{n})$. We can consider the following Hodge decomposition in $L^{p_{\infty}}(\mathbb{R}^{n}; \Lambda^{l} \wedge \varepsilon)$ (see [18, Theorem 6.1], see also [19, § 10.6]):

\[
\frac{\varphi}{|\varphi|_{\infty}^{\varepsilon}} = d\alpha_{\varepsilon} + d^*\beta_{\varepsilon}
\]

with some $\alpha_{\varepsilon} \in W^{1,p_{\infty}}(\mathbb{R}^{n}; \Lambda^{l-1} \wedge \varepsilon)$ and $\beta_{\varepsilon} \in W^{1,p_{\infty}}(\mathbb{R}^{n}; \Lambda^{l+1} \wedge \varepsilon)$. Here $d$ is the exterior derivative, and $d^*$ is its formal adjoint, the coexterior derivative. The forms $d\alpha_{\varepsilon}$ and $d^*\beta_{\varepsilon}$, $\varepsilon = 1, \ldots, k$, are uniquely determined and can be expressed by means of the Hodge projection operators

\[
E : L^p(\mathbb{R}^{n}; \Lambda^l) \to dW^{1,p}(\mathbb{R}^{n}; \Lambda^{l-1}) \quad \text{and} \quad E^* : L^p(\mathbb{R}^{n}; \Lambda^l) \to d^*W^{1,p}(\mathbb{R}^{n}; \Lambda^{l+1})
\]

defined by [19, § 10.6, formulas (10.71) and (10.72)] for $1 < p < \infty$ and $1 \leq l \leq n-1$. Namely we have

\[
d\alpha_{\varepsilon} = E\left(\frac{\varphi}{|\varphi|_{\infty}^{\varepsilon}}\right) \quad \text{and} \quad d^*\beta_{\varepsilon} = E^*\left(\frac{\varphi}{|\varphi|_{\infty}^{\varepsilon}}\right).
\]

Applying [18, Theorem 6.1], we get the following bound for exact term:

\[
\|d\alpha_{\varepsilon}\|_{L^{p_{\infty}}(\mathbb{R}^{n}; \Lambda^{l})} \leq C_1(p_{\infty})\|\varphi_{\varepsilon}\|_{L^{(1-\varepsilon)p_{\infty}}(\mathbb{R}^{n}; \Lambda^{l})}.
\]

By [19, § 10.6, formulas (10.73) and (10.74)] we have $\text{Ker} E = \{\varphi \in L^p(\mathbb{R}^{n}; \Lambda^l) : d^*\varphi = 0\}$ and $\text{Ker} E^* = \{\varphi \in L^p(\mathbb{R}^{n}; \Lambda^l) : d\varphi = 0\}$ for $1 < p < \infty$ and $1 \leq l \leq n-1$. Then $E^*(\varphi_{\varepsilon}) = 0$. Therefore we can write $d^*\beta_{\varepsilon}$ as a commutator

\[
d^*\beta_{\varepsilon} = E^*\left(\frac{\varphi_{\varepsilon}}{|\varphi_{\varepsilon}|_{\infty}^{\varepsilon}}\right) - E^*\left(|\varphi_{\varepsilon}|_{\infty}^{\varepsilon}\right).
\]

Applying [18, Theorem 12.2.1] (see also [17, Theorems 8.1 and 8.2]), we obtain

\[
\|d\beta_{\varepsilon}\|_{L^{p_{\infty}}(\mathbb{R}^{n}; \Lambda^{l})} \leq C_2(p_{\infty})\|\varphi_{\varepsilon}\|_{L^{(1-\varepsilon)p_{\infty}}(\mathbb{R}^{n}; \Lambda^{l})}.
\]

Using (13), we have

\[
\int \frac{\varphi_1 \wedge \cdots \wedge \varphi_k \wedge dx_{\hat{t}}}{|\varphi_1|_{\infty}^{\varepsilon} \cdots |\varphi_k|_{\infty}^{\varepsilon}} = \int (d\alpha_1 + d^*\beta_1) \wedge \cdots \wedge (d\alpha_k + d^*\beta_k) \wedge dx_{\hat{t}} = \int d\alpha_1 \wedge \cdots \wedge d\alpha_k \wedge dx_{\hat{t}} + \int B.
\]

Since $p_1, \ldots, p_k$ represents a Hölder conjugate tuple, by Stokes’ formula via an approximation argument we obtain

\[
\int d\alpha_1 \wedge \cdots \wedge d\alpha_k \wedge dx_{\hat{t}} = 0.
\]
The integrand $B$ is a sum of wedge products of the type $\psi_1 \wedge \cdots \wedge \psi_k \wedge dx_I$, where $\psi_\kappa$ is either $d\alpha_\kappa$ or $d^*\beta_\kappa$ and at least one $d^*\beta_\kappa$ is always present, with at most $2^k - 1$ terms. Combining Hölder inequality with (15) and (16), we get

$$
\int \psi_1 \wedge \cdots \wedge \psi_k \wedge dx_I \leq C_3(k)\|\psi_1\|_{L^p(\mathbb{R}^n;\Lambda^1)} \cdots \|\psi_k\|_{L^p(\mathbb{R}^n;\Lambda^k)}
$$

$$
\leq C_4(p_1, \ldots, p_k)\varepsilon\|\varphi_1\|_{L^{(1-\varepsilon)p_1}(\mathbb{R}^n;\Lambda^1)} \cdots \|\varphi_k\|_{L^{(1-\varepsilon)p_k}(\mathbb{R}^n;\Lambda^k)}
$$

with $\varepsilon := \max(|\varepsilon_1|, \ldots, |\varepsilon_k|)$. This with (17) and (18) yields (12).

The following theorem is a modification of the results of T. Iwaniec and G. Martin on integral estimates for Jacobians [19, Theorems 7.8.1 and 13.7.1] and is a consequence of Theorem 4.1.

**Theorem 4.3** (Fundamental inequality for subdeterminants). Let $n, m, k \in \mathbb{N}$ with $2 \leq k \leq \min(m, n)$. Then there exists a constant $C(k) \geq 1$ such that for every distribution $v = (v_1, \ldots, v_m) \in \mathcal{D}'(\mathbb{R}^n;\mathbb{R}^m)$ with $v' \in L^p(\mathbb{R}^n;\mathbb{R}^{m \times n})$, $1 \leq p < \infty$, and for every $I = (i_1, \ldots, i_k) \in \Gamma^F_n$, $J = (j_1, \ldots, j_k) \in \Gamma^F_n$ we have the inequality

$$
\left| \int |v'|^{p-\varepsilon} \frac{\partial v_I}{\partial x_I} \right| \leq C(k) \left| 1 - \frac{p}{k} \right| \int |v'|^p.
$$

**Remark 4.4.** For the case $k = n = m$ the estimate (19) was established in [19, Theorems 7.8.1 and 13.7.1].

**Proof of Theorem 4.3.** Let $p_\kappa := k$, $\varepsilon_\kappa := \varepsilon := 1 - \frac{p}{k}$, and $l_\kappa := 1$ for $\kappa = 1, \ldots, k$. Then $1 < p_\kappa < \infty$, $1/l_1 + \cdots + 1/l_k = 1$, $I := n - k = n - l_1 - \cdots - l_k \geq 0$, $1 - p_\kappa = p$, and $\max(|\varepsilon_1|, \ldots, |\varepsilon_k|) = |\varepsilon| = 1 - \frac{p}{k}$. Let $\varphi_\kappa := dv_{j_\kappa} \in L^{(1-\varepsilon)p_\kappa}(\mathbb{R}^n;\Lambda^{l_\kappa})$. Let $I = (i_1, \ldots, i_l) \in \Gamma^F_n$ be the ordered $l$-tuple such that $\{i_1, \ldots, i_l\} = \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$. We chose the sign $\text{sgn} I$ such that $\text{sgn} I dx_I \wedge dx_I = dx_1 \wedge \cdots \wedge dx_n$.

When $p$ lies outside the interval $(\frac{2k+1}{k-1}, \frac{3k}{k-1})$ the estimate is clear as (19) always holds with 1 in place $C(k) \left| 1 - \frac{p}{k} \right|$. In this case $\left| 1 - \frac{p}{k} \right| \geq \frac{k-1}{2k}$ and inequality (19) holds with $C(k) = \frac{2k}{k-1}$.

Suppose that $k+1 \leq 2p \leq 3k$. Then $-1 \leq 2\varepsilon_\kappa \leq \frac{p_\kappa - 1}{p_\kappa}$ and $|\varepsilon| \leq 1/2$. Applying Theorem 4.3.1 we obtain

$$
\left( \left| \frac{\partial v_I}{\partial x_I} \right| \right)^{1-\varepsilon} \left( \left| \frac{\partial v_I}{\partial x_I} \right| \right)^{p}
$$

$$
= \frac{|v'|^{p-\varepsilon}}{|v'|^p} \left( \left| \frac{\partial v_I}{\partial x_I} \right| \right) \left( \left| \frac{\partial v_I}{\partial x_I} \right| \right)^{1-\varepsilon} \leq |\varepsilon||v'|^p.
$$

Using the elementary inequalities $|v'|^a \leq |v_j| \cdots |v_j| \alpha$ and $|a - a^{1-\varepsilon}| \leq |\varepsilon|$ for $0 \leq a \leq 1$ and $-1 < \varepsilon < 1$, we have

$$
\left| \frac{\partial v_I}{\partial x_I} \right| \left| \frac{\partial v_I}{\partial x_I} \right| \left( \left| \frac{\partial v_I}{\partial x_I} \right| \right) \left( \left| \frac{\partial v_I}{\partial x_I} \right| \right)^{1-\varepsilon} \leq |\varepsilon||v'|^p.
$$
Combining this with (20), we obtain
\[
\left| \int |v'|^{p-k} \frac{\partial v_j}{\partial x_j} \right| \leq \left( |v'|^p \frac{\partial v_j}{\partial x_j} \right) \right| - |dv_{j_1}| \ldots |dv_{j_k}| \right| = \left( C_1(k) + 1 \right) \varepsilon I \left| |v'|^p. \right.
\]

In the proof of Theorem 3.1 we use the following version of Gehring’s lemma (see, for example, [19, Corollary 14.3.1]):

Lemma 4.5 (Gehring’s Lemma). Suppose \( f \) and \( g \) are non-negative functions of class \( L^q(\mathbb{R}^n) \), \( 1 < q < \infty \), and satisfy
\[
\left( \frac{1}{|B(a, R)|} \int_Q f^q \right)^{1/q} \leq \frac{A}{|B(a, 2R)|} \int_{B(a, 2R)} f + \left( \frac{1}{|B(a, 2R)|} \int_{B(a, 2R)} g^q \right)^{1/q}
\]
for all balls \( B(a, R) \subset \mathbb{R}^n \) and some constant \( A > 0 \). Then there exists a new exponent \( q' = q' (n, q, A) > p \) and a constant \( C = C(n, q, A) > 0 \) such that
\[
\int f^{q'} \leq C \int g^{q'}.
\]

5. Proof of the Self-improving Regularity Theorem

We are now in a position to prove Theorem 3.1 given in Section 3.

Proof of Theorem 3.1. Let \( p > 1 \). Obviously, we may assume that \( \varphi \geq 0 \) as otherwise we could consider \( |\varphi| \) which has no effect on inequality (8). Consider the auxiliary mapping \( h := \varphi v \in W^{1,p}(\mathbb{R}^n; \mathbb{R}^m) \). We have \( h' = \varphi' v + v \otimes \varphi' \). Using (7) and (1), we deduce
\[
|h'|^k \leq (|\varphi' v| + |v \otimes \varphi'|)^k = \varphi^k |v'|^k + \sum_{k=0}^{k-1} \binom{k}{k-\varepsilon} |\varphi'|^{k-\varepsilon} |v \otimes \varphi'|^{k-\varepsilon}
\]
\[
\leq c_p^{-1} \varphi^k F(v') + \sum_{k=0}^{k-1} \binom{k}{k-\varepsilon} (|h'| + |v \otimes \varphi'|)^{k-\varepsilon} |v \otimes \varphi'|^{k-\varepsilon}
\]
\[
\leq c_p^{-1} \varphi^k F(v') + \sum_{k=0}^{k-1} \binom{k}{k-\varepsilon} (|h'| + |v \otimes \varphi'|)^{k-1} |v \otimes \varphi'|^{k-1}
\]
\[
= c_p^{-1} KG(h' - v \otimes \varphi') + C_1(k) (|h'| + |v \otimes \varphi'|)^{k-1} |v \otimes \varphi'|^{k-1}
\]
\[
\leq c_p^{-1} KG(h') + C_2(F, G) K(|h'| + |v \otimes \varphi'|)^{k-1} |v \otimes \varphi'|^{k-1}
\]
Multiplying this inequality by \( |h'|^{p-k} \), after a little manipulation we obtain
\[
|h'|^p \leq c_p^{-1} K |h'|^{p-k} G(h') + C_3(F, G, p) K(|h'| + |v \otimes \varphi'|)^{p-1} |v \otimes \varphi'|.
\]
We observe here that clearly \( (|h'| + |v \otimes \varphi'|)^p - |v \otimes \varphi'| \) enjoys higher integrability than \( |h'|^p \). Using (6), we have \( |h'|^{p-k} G(h') = \sum_{j \in \Gamma_m, \Omega \in \Gamma_n} \gamma_{ji} |h'|^{p-k} \det_{ji} v' \). Applying
Theorem 2.3 we obtain $\int |h'|^{p-k} G(h') \leq C_4(G) |1 - \frac{k}{p}| \int |h'|^p$. Combining this with (22), we get

$$\int |h'|^p \leq \frac{C_4(G) K}{C_{ef}} \left[ 1 - \frac{p}{k} \right] \int \left( |h'|^p + C_5(F, G, p) K \int (|h'| + |v \otimes \varphi'|)^{p-1} |v \otimes \varphi'| \right).$$

Put $q(F, G, K) = k \left( 1 - \frac{c_p}{C_{ef}(G) K} \right)$ and $p(F, G, K) = k \left( 1 + \frac{c_p}{C_{ef}(G) K} \right)$. Suppose now that $p \in (q(F, G, K), p(F, G, K))$. Then $\frac{C_4(G) K}{c_p} |1 - \frac{k}{p}| < 1$. In this case inequality (23) can be expressed as

$$\int |h'|^p \leq \frac{C_5(F, G, p) K}{C_{ef}} \left[ 1 - \frac{p}{k} \right] \int (|h'| + |v \otimes \varphi'|)^{p-1} |v \otimes \varphi'|.$$

We have

$$\int (|h'| + |v \otimes \varphi'|)^p \leq 2^{p-1} \int (|h'|^p + |v \otimes \varphi'|^p)$$

$$\leq 2^{p-1} \left( \frac{C_5(F, G, p) K}{C_{ef}} \left[ 1 - \frac{p}{k} \right] \int (|h'| + |v \otimes \varphi'|)^{p-1} |v \otimes \varphi'| + \int |v \otimes \varphi'|^p \right)$$

$$\leq C(F, G, K, p) \int (|h'| + |v \otimes \varphi'|)^{p-1} |v \otimes \varphi'|$$

$$\leq C(F, G, K, p) \left[ \int (|h'| + |v \otimes \varphi'|)^p \right]^{\frac{1}{p}} \left[ \int |v \otimes \varphi'|^p \right]^{\frac{1}{p}}.$$

Hence $||h'| + |v \otimes \varphi'||_{L^p(\mathbb{R}^n)} \leq C(F, G, K, p) ||v \otimes \varphi'||_{L^p(\mathbb{R}^n; \mathbb{R}^{m \times n})}$. Then, in view of the simple fact that $|\varphi| \leq |h'| + |v \otimes \varphi'|$, we obtain the Caccioppoli-type estimate

$$\|\varphi v'\|_{L^p(\mathbb{R}^n; \mathbb{R}^{m \times n})} \leq C(F, G, K, p) \|v \otimes \varphi'\|_{L^p(\mathbb{R}^n; \mathbb{R}^{m \times n})}.$$

Of course now we observe that this inequality holds with $p$ replaced by $s$ for any $s \in (q(F, G, K), p(F, G, K))$, provided we know a priori that $v \in W^{1,s}_{\text{loc}}(V; \mathbb{R}^m)$. Let $S = \{ s \in (q(F, G, K), p(F, G, K)) : v \in W^{1,s}_{\text{loc}}(V; \mathbb{R}^m) \}$. We have $p \in S$. Therefore, $S \neq \emptyset$. For $s \in S$ we have (3): the constant $C(F, G, K, p)$ which depends continuously on $s$ is finite in the range $q(F, G, K) < s < p(F, G, K)$ but may blow up at the endpoints. This shows that $S$ is relatively closed in $(q(F, G, K), p(F, G, K))$. The theorem will be proved if we can show that $S$ is open. Certainly, if $s \in S$, then $(q(F, G, K), s) \subset (q(F, G, K), p(F, G, K))$. We are therefore left only with the task of showing higher integrability of the differential. It is at this point that Gehring’s lemma comes to the rescue. We easily derive from (21) reverse Hölder inequality for $h'$. Let $B_R := B(a, R) \subset B(a, 2R) =: B_{2R}$ be a concentric balls in $V$ and let $0 \leq \eta \leq 1$ be a function in $C_0^\infty(B_{2R})$ which is equal to 1 on $B_{R/2}$ and has $|\eta'| \leq C(n)$. Now we repeat the above calculations with some modifications to obtain the Caccioppoli-type estimate for $h - h_{B_{2R}}$ and $\eta$, where $h_{B_{2R}} := \frac{1}{|B_{2R}|} \int_{B_{2R}} |h|$. Consider the mapping $H := \eta(h - h_{B_{2R}})$. We have $H' = \eta h' + (h - h_{B_{2R}}) \otimes \eta'$. 
Using (21), we deduce

\[ |H'|^k \leq (|\eta h' + (h - h_{B_{2R}}) \otimes \eta'|)^k \]

\[ = \eta^k |H'|^k + \sum_{k-1}^{k-1} \left( \begin{array}{c} k \\ k-x \end{array} \right) |\eta h'|^x (h - h_{B_{2R}}) \otimes \eta'|^{k-x} \]

\[ \leq c_F^{-1} K G(\eta h') + C_2(F, G) K (|\eta h'| + |v \otimes \varphi'|)^{k-1} \eta|v \otimes \varphi'| + C_2(F, G) K (|H'| + |(h - h_{B_{2R}}) \otimes \eta'| + |v \otimes \varphi'|)^{k-1} \eta|v \otimes \varphi'| \]

\[ + \sum_{k+1}^{k-1} \left( \begin{array}{c} k \\ k-x \end{array} \right) (|\eta h'|)^x (h - h_{B_{2R}}) \otimes \eta'|^{k-x} \leq c_F^{-1} K G(H' - (h - h_{B_{2R}}) \otimes \eta') \]

\[ + C_2(F, G) K (|H'| + |(h - h_{B_{2R}}) \otimes \eta'| + |v \otimes \varphi'|)^{k-1} \eta|v \otimes \varphi'| \]

Multiplying this inequality by \(|H'|^{p-n} G(H')\), after a little manipulation we obtain

\[ |H'|^p \leq c_F^{-1} K |H'|^{p-n} G(H') \]

\[ + C_6(F, G, p) K (|H'| + |(h - h_{B_{2R}}) \otimes \eta'| + |v \otimes \varphi'|)^{p-1} \eta|v \otimes \varphi'|). \]

Using again (6) and Theorem 1.3 we obtain

\[ \int |H'|^p \leq \frac{C_6(F, G, p) K}{1 - \frac{C_4(G) K}{c_p} |1 - \frac{p}{k}|} \]

\[ \times \int (|H'| + |(h - h_{B_{2R}}) \otimes \eta'| + |v \otimes \varphi'|)^{p-1} \eta|v \otimes \varphi'|). \]

We have

\[ \int (|H'| + |(h - h_{B_{2R}}) \otimes \eta'| + |v \otimes \varphi'|)^p \]

\[ \leq 2^{p-1} \int (|H'|^p + |(h - h_{B_{2R}}) \otimes \eta'| + |v \otimes \varphi'|)^p \leq 2^{p-1} \left( \frac{C_6(F, G, p) K}{1 - \frac{C_4(G) K}{c_p} |1 - \frac{p}{k}|} \times \right) \]

\[ \times \int (|H'| + |(h - h_{B_{2R}}) \otimes \eta'| + |v \otimes \varphi'|)^{p-1} \eta|v \otimes \varphi'| + \int (|(h - h_{B_{2R}}) \otimes \eta'| + |v \otimes \varphi'|)^p \]

\[ \leq C_7(F, G, K, p) \int (|H'| + |(h - h_{B_{2R}}) \otimes \eta'| + |v \otimes \varphi'|)^{p-1} \eta|v \otimes \varphi'|) \]

\[ \leq C_7(F, G, K, p) \left[ \int (|H'| + |(h - h_{B_{2R}}) \otimes \eta'| + |v \otimes \varphi'|)^p \right]^{\frac{p-1}{p}} \times \]

\[ \times \left[ \int (|(h - h_{B_{2R}}) \otimes \eta'| + |v \otimes \varphi'|)^p \right]^\frac{1}{p}. \]
Hence
\[ \|H'\| + \|(h - h_{B_{2R}}) \otimes \eta'\| + \eta|v \otimes \varphi'| \|_{L^p(\mathbb{R}^n)} \leq C_7(F, G, K, \rho)\|(h - h_{B_{2R}}) \otimes \eta'\| + \eta|v \otimes \varphi'| \|_{L^p(\mathbb{R}^n)}. \]

Then, in view of the simple facts that
\[ |\eta h'| \leq |H'| + \|(h - h_{B_{2R}}) \otimes \eta'\| \leq |H'| + \|(h - h_{B_{2R}}) \otimes \eta'\| + \eta|v \otimes \varphi'| \]
and
\[ \|(h - h_{B_{2R}}) \otimes \eta'\| + \eta|v \otimes \varphi'| \| \leq \|(h - h_{B_{2R}}) \otimes \eta'\| + \eta|v \otimes \varphi'| \|, \]
we obtain the Caccioppoli-type estimate
\[ \int |\eta h'|^p \leq C_8(F, G, K, \rho) \int \|(h - h_{B_{2R}}) \otimes \eta'\|^p + C_8(F, G, K, \rho) \int \eta|v \otimes \varphi'|^p. \]
Using the properties of the test function \( \eta \), we get
\[ \int_{B_R} |h'|^p \leq C_9(F, G, K, \rho)R^{-p} \int_{B_R} \|(h - h_{B_{2R}})\|^p + C_9(F, G, K, \rho) \int_{B_R} |v \otimes \varphi'|^p. \]
Combining this with the Poincaré–Sobolev inequality (see, for example, [19, Theorem 4.10.3]), we obtain
\[ \frac{1}{|B_R|} \int_{B_R} |h'|^p \leq C_{10}(F, G, K, \rho) \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} |h'| \frac{np}{n+p} \right)^{\frac{n+p}{n}} + \frac{C_{10}(F, G, K, \rho)}{|B_{2R}|} \int_{B_{2R}} |v \otimes \varphi'|^p. \]

Hence
\[ \left( \frac{1}{|B_R|} \int_{B_R} |h'|^p \right)^{\frac{n}{n+p}} \leq \frac{C_{11}(F, G, K, \rho)}{|B_{2R}|} \int_{B_{2R}} |h'| \frac{n}{n+p} \]
\[ + \left( \frac{1}{|B_{2R}|} \int_{B_{2R}} (C_12(F, G, K, \rho)|v \otimes \varphi'|)^p \right)^{\frac{1}{n+p}}. \]
Put \( q = \frac{n+p}{n} > 1, f = |h'| \frac{n}{n+p}, \) and \( g = |v \otimes \varphi'| \frac{n}{n+p}. \) By Lemma 4.3 we conclude that \( f \) is integrable with a power slightly larger than \( q \). This in turn means that \( h' \) is integrable with a slightly higher power then \( p \) and so \( v \in W^{1,p'}_{\text{loc}}(V; \mathbb{R}^m) \) for some \( p' > p \).

6. Proof of the Removability Theorem

As in the proof of [19, Theorem 17.3.1] there are two key components in the proof of Theorem 3.3. Firstly, the assumption on the size of set \( E \) implies that \( E \) has zero \( s \)-capacity for an appropriate value of \( s \). Secondly, the Caccioppoli estimate (3) holds for this particular value of \( s \).

Proof of Theorem 3.3. We have \( q(F, G, K) < n - \text{dim}_H(E) \). Let
\[ s \in (q(F, G, K), n - \text{dim}_H(E)). \]
From [19, Theorem 17.2.1] (see also [5, 29, 35, 39]), we obtain that the set \( E \) has zero \( s \)-capacity. It is clear that \( |E| = 0 \). Further, Theorem 3.1 gives the Caccioppoli estimate
\[ \|\varphi'\|_{L^q(V; \mathbb{R}^{m \times n})} \leq C\|v \otimes \varphi\|_{L^p(V; \mathbb{R}^{m \times n})} \]
(27)
for every $\varphi \in C_0^\infty (V \setminus E)$, where the constant $C = C(F,G,K,s)$ does not depend on the test function $\varphi$ or the function $\nu$.

Let $\chi \in C_0^\infty (V)$ and $E' := E \cap \text{supp} \chi$. Then $E'$ has zero $s$-capacity. Therefore there exists a sequence of functions $(\eta_j \in C_0^\infty (V))_{j \in \mathbb{N}}$ such that $0 \leq \eta_j \leq 1$; $\eta_j = 1$ on some neighbourhood of $E'$; $\lim_{j \to \infty} \eta_j = 0$ almost everywhere in $\mathbb{R}^n$, and $\lim_{j \to \infty} |\eta_j'| = 0$. Put $\varphi_j := (1 - \eta_j)\chi \in C_0^\infty (V \setminus E)$ and $v_j := \varphi_j \nu \in W_{0}^{1,s}(V; \mathbb{R}^m)$. Then the mappings $v_j$ are bounded in $L^\infty (V; \mathbb{R}^m)$ converge to $\nu$ almost everywhere. We have $v_j' = \varphi_j \nu' + \nu \otimes \varphi_j'$ and $\varphi_j' = -\chi \eta_j' + (1 - \eta_j)\chi'$. Using (27), we obtain

\[
\| v_j' \|_{L^\infty (V; \mathbb{R}^{m \times n})} \leq \| \varphi_j \nu' \|_{L^\infty (V; \mathbb{R}^{m \times n})} + \| \nu \otimes \varphi_j' \|_{L^\infty (V; \mathbb{R}^{m \times n})} \\
\leq (1 + C) \| \nu \|_{L^\infty (V)} \| \nu' \|_{L^\infty (V; \mathbb{R}^n)} + \| (1 - \eta_j)\nu \otimes \chi' \|_{L^\infty (V; \mathbb{R}^{m \times n})}.
\]

Passing to the limit over $j$, we get

\[
\limsup_{j \to \infty} \| v_j' \|_{L^\infty (V; \mathbb{R}^{m \times n})} \leq (1 + C) \| \nu \otimes \chi' \|_{L^\infty (V; \mathbb{R}^{m \times n})}.
\]

Therefore the sequence $(v_j)_{j \in \mathbb{N}}$ is bounded in $W^{1,s}(V; \mathbb{R}^m)$. Hence there exists its subsequence $(v_{j_s})_{s \in \mathbb{N}}$ converges weakly in $W^{1,s}(V; \mathbb{R}^m)$ to a mapping in this Sobolev space. Clearly, this limit coincides with $\nu$ almost everywhere in $V$.

Therefore $\nu \in W_{0}^{1,s}(V; \mathbb{R}^m)$ for all test functions $\chi \in C_0^\infty (V)$. This yields $v \in W_{0}^{1,k}(V; \mathbb{R}^m)$. Since $v$ is a solution of inequality (11) almost everywhere in $V$, Theorem 3.1 yields $v \in W_{0}^{1,k}(V; \mathbb{R}^m)$. \hfill \square

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