A REMARK ON DEFORMATIONS OF 1-CONVEX MANIFOLDS WITH EXCEPTIONAL CURVES

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ABSTRACT. A formula for the dimension of the smoothing component of a 3-dimensional isolated Cohen–Macaulay singularity is shown. We apply this formula for a 1-convex threefold with a connected exceptional curve which is blown down to a terminal Gorenstein singularity.

1. INTRODUCTION

An $n$-dimensional complex manifold $X$ is called a 1-convex (or strongly pseudoconvex) $n$-fold if there is a proper surjective morphism $\pi$ from $X$ onto a Stein space $V$ with $\pi_* \mathcal{O}_X \cong \mathcal{O}_V$ and a finite subset $\Sigma \subseteq V$ such that $X \setminus E \to V \setminus \Sigma$ is biholomorphic where $E = \pi^{-1}(\Sigma)$. We call $E$ the exceptional set and $\pi: (X, E) \to (V, \Sigma)$ the Remmert reduction.

In [17, Theorem 2], Laufer proved that if $X$ is a 1-convex manifold with a 1-dimensional exceptional set $E$ then it has (the germ of) the miniversal deformation space $\text{Def}(X)$. It is a natural question to ask for a formula for the dimension of $\text{Def}(X)$.

We may assume without loss of generality that the curve $E$ is connected, i.e., $\Sigma = \{p\}$ and thus the germ $(V, p)$ is an isolated singularity. In the case of surfaces, Stephen Yau gave a formula of $\dim \text{Def}(X)$ in [30, Theorem 3.1] for an isolated hypersurface singularity $(V, p)$ and Wahl in [27, Theorem 3.13 (c)] for a smoothable, normal surface singularity.

In the present note, we will generalize the results of [27, 30]. The main result of this note is Theorem 3.1 in Section 3 for a 3-dimensional isolated Cohen–Macaulay singularity which is smootheable. As an application, we derive a formula of $\dim \text{Def}(X)$ for a certain 1-convex threefold $X$:

**Theorem 1.1.** Let $X$ be a 1-convex threefold with a connected exceptional curve $E$, and $\pi: (X, E) \to (V, p)$ the Remmert reduction as above. If $K_X$ is $\pi$-trivial, then

$$\dim \text{Def}(X) = \tau - \frac{\mu + \sigma}{2}$$

where $\mu$ (resp. $\tau$) is the Milnor (resp. Tjurina) number of the singularity $(V, p)$ and $\sigma$ is the rank of the local divisor class group $\text{Cl}(\mathcal{O}_{V, p})$.
Here the canonical divisor $K_X$ is said to be $\pi$-trivial if the intersection number of $K_X$ with every irreducible component of $E$ is zero. In such a case, the exceptional curve $E$ blows down to a terminal Gorenstein singularity, so to an isolated cDV (compound Du Val) singularity [22]. We remark that in general $(V, p)$ need not even be Cohen–Macaulay (see for example [1, p.626 (1)-(4)] and [2, Example 3.2]).

We will relate $\dim \text{Def}(X)$ to Du Bois invariants $b^{r,s}(V, p)$ of the rational isolated hypersurface singularity $(V, p)$, which were introduced in [24] (see §2.1).

**Corollary 1.2.** Under the hypotheses of Theorem 1.1, the dimension of $\text{Def}(X)$ is equal to $b^{2,1}(V, p)$. Moreover, the following are equivalent:

1. The $(V, p)$ is an ordinary double point.
2. $\dim \text{Def}(V) = \dim \text{Def}(X) + 1$.
3. $b^{1,1}(V, p) = 0$.
4. $b^{2,1}(V, p) = 0$.

The conditions (2) and (3) occur in the deformation theory of singular Calabi–Yau threefolds (see [11, Lemma 3.6] and [20, Theorem 2.2]), and the condition (4) was studied in [25, Theorem 5.4, Remark 5.12]. There are similar results in [8, §3] (see Remark 3.4).

We close this introduction with a few remarks on the techniques used in this paper. We will compare the Euler characteristics on the smoothing and the resolution (see Proposition 2.8 and 2.12) by using a globalization property of smoothings, proved by Looijenga [19]. Then we derive a formula of $\dim \text{Def}(X)$ for a 1-convex $n$-fold with a connected exceptional curve (cf. Remark 2.9). It can be computed in principle by the Riemann–Roch theorem.

The Riemann–Roch defect for isolated singularities has been studied in [19, §3]. However, the scissor relation (c) in the proof of [19, Theorem 3.3] only holds for certain singularities (see Remark 3.3). Fortunately, the theorem of Riemann–Roch on threefolds is easy to compute (see Proposition 3.2). Based on the method of Laufer, Wahl and Looijenga [19, 27], we will apply the Riemann–Roch formula to prove our results (see §3).

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2. Preliminaries

2.1. Du Bois invariants. We start by recalling the setup from [24]. Let \((Y, D) \to (V, p)\) be a good resolution of an isolated singularity \((V, p)\) of pure dimension \(n\), i.e., \(D\) is a divisor with simple normal crossings on \(Y\). We fix a representative \(Y \to V\) with \(V\) a contractible Stein space.

Steenbrink defined invariants \(b^{r,s}\) of \((V, p)\), called the Du Bois invariants (see [24, §2]), by

\[
b^{r,s}(V, p) := \dim H^s(Y, \Omega^r_Y(\log D)(-D))
\]

for \(r \geq 0\) and \(s \geq 1\). These invariants do not depend on the choice of the good resolution, as they can be defined in terms of the filtered de Rham complex ([5], [23, (3.5)]).

There are another invariants of \((V, p)\), defined in [24, §3]. By [23, Theorem 1.9], there is a mixed Hodge structure on \(H^k(V \setminus \{p\}) \cong H^{k+1}_{\{p\}}(V)\). The Hodge filtration \(F\) on the local cohomology \(H^{k+1}_{\{p\}}(V)\) arises from a spectral sequence

\[
E_1^{pq} = H^q(D, \Omega^p_Y(\log D) \otimes \mathcal{O}_D) \Rightarrow H^{p+q+1}_{\{p\}}(V, \mathbb{C})
\]

which degenerates at the \(E_1\)-term. Therefore its Hodge filtration defines invariants \(l^{r,s}\) of the singularity \((V, p)\) by \(l^{r,s} := \dim E_1^{r,s}\).

The following proposition follows from Lemma 2, Theorem 4 and 6 in [24]. Note that the isolated rational singularity \((V, p)\) is Du Bois, i.e., \(b^{0,s} = 0\) for \(0 < s < n\) (see [23, (3.7)]).

**Proposition 2.1.** Let \((V, p)\) be a 3-dimensional, rational isolated hypersurface singularity. Then

\[
\tau = b^{2,1} + b^{1,1} + l^{1,1} \quad \text{and} \quad \mu = 2b^{1,1} + l^{1,1}.
\]

We remark that the main results in [24] express the Tjurina and Milnor numbers of certain isolated singularities in terms of Du Bois invariants \(b^{r,s}\) and \(l^{r,s}\). We state the proposition above considering such hypersurfaces for simplicity, which is all we need in the proof of Corollary 1.2.

Let \(\sigma(V, p)\) denote the rank of the local divisor class group \(\text{Cl}(\mathcal{O}_{V,p})\), i.e.,

\[
\sigma(V, p) = \text{rank}(\text{Weil}(V, p)/\text{Cart}(V, p)),
\]

where \(\text{Weil}(V, p)\) is the Abelian group of Weil (resp. Cartier) divisors of the singularity \((V, p)\). It is a finite number if \((V, p)\) is a rational singularity (see [16, Lemma 1.12]).

**Proposition 2.2.** If \((V, p)\) is a 3-dimensional, rational isolated hypersurface singularity, then \(\sigma(V, p) = l^{1,1}(V, p)\).

**Proof.** Recall that \(l^{1,1}(V, p) = \dim \text{Gr}^1_{\{p\}} H^3_{\{p\}}(V, \mathbb{C})\) is defined by the spectral sequence (2.1). Since \((V, p)\) is rational, we have \(l^{0,i} = l^{i,0} = 0\) for all \(i\) by [24, Lemma 2] and in particular \(E_{\infty}^{02} = E_{\infty}^{20} = 0\). Hence we get \(H^3_{\{p\}}(V, \mathbb{C}) = \)
The proposition follows from the fact that
\[ \sigma(V, p) = \dim H^2(V \setminus \{p\}, \mathbb{C}) \] (see [6 (6.1)] or the proof of [20 Proposition 3.10]).

Remark 2.3. Let \( \pi : X \to V \) be a resolution of a 3-dimensional rational isolated singularity \((V, p)\). If the exceptional set \( E \) of \( \pi \) has dimension 1, then the number of irreducible components of \( E \) equals \( \sigma(V, p) \) (cf. [16 Lemma 3.4] and [28 Remark 2.8, 2.10]).

2.2. The Riemann–Roch defect. Let \((V, p)\) be an isolated normal singularity of pure dimension \( n \geq 2 \) which is smoothable.

Let \( f : V' \to \Delta \subseteq \mathbb{C} \) be a good representative of a smoothing of \( V \). According to a globalization theorem of Looijenga [19, Appendix], it follows that \( f \) is a restriction of a projective flat family \( F : Z \to \Delta \) which is smooth outside the point \( \{p\} \). We write \( Z \) for the fiber \( Z_0 \) and \( Z_t, t \neq 0 \), for non-singular fiber. Let \( S \) be the smoothing component on which the smoothing \( f : V' \to \Delta \) takes place and \( \beta_f := \dim S \) (cf. [27, (4.1)]).

The following lemma is known in [27, (3.8)], and we recall the argument for the convenience of the reader.

Lemma 2.4. With the above hypothesis and notation, for \( t \neq 0 \) we have
\[ \chi(\Theta_Z) = \chi(\Theta_{Z_t}) + \beta_f. \]

Proof. By a conjecture of Wahl proved by Greuel and Looijenga [10 (2.6)], we have
\[ \beta_f = \dim \text{Coker}(\Theta_{Z_t} \to \Theta_{Z,p}). \]
Note that the natural morphism is injective because the sheaf of relative derivations \( \Theta_{Z_t} \) has depth \( \geq 2 \). Since \( \Theta_{Z_t} \) is flat over \( \Delta \) and induces \( \Theta_{Z_t} \) for \( t \neq 0 \), the lemma follows from the additivity of the Euler characteristic and the fact that the function \( \chi(\Theta_{Z_t} \to \Theta_{Z,p}) \) is constant on \( t \in \Delta \) [3, III, Theorem 4.12].

Remark 2.5. If \((V, p)\) is an isolated complete intersection singularity, then it is unobstructed. Hence \( \beta_f \) is the Tjurina number \( \tau \) of \((V, p)\), which is independent of the smoothing \( f \).

Notation 2.6. Let \( \pi : X \to V \) be any resolution of the singularity \((V, p)\). By gluing the resolution \( \pi \) and the identity map of \( Z \setminus \{p\} \), we get a resolution \( \Pi : Z_{\hat{}} \to Z \).

The following lemma will be useful in the sequel.

Lemma 2.7. Let \( \hat{Z} \) and \( \Pi \) be as in Notation [26] For a coherent sheaf \( \mathcal{F} \) on \( Z_{\hat{}} \), we have
\[ \chi(\mathcal{F}) = \chi(\Pi_*\mathcal{F}) + \sum_{1 \leq i \leq n-1} (-1)^i \dim(R^i \Pi_*\mathcal{F})_p. \]

Proof. Notice that \( R^i \Pi_*\mathcal{F} \) is supported on the point \( \{p\} \) for \( i > 0 \). The lemma follows from the Leray spectral sequence for \( \Pi \) and \( \mathcal{F} \). \( \square \)
We recall the notion of equivariant resolutions (in the sense of Hironaka). A resolution \( \pi : X \to V \) is called equivariant if the natural injection \( \pi_* \Theta_X \to \Theta_V \) is an isomorphism. Note that such resolutions always exist (see [13] or [15, p.14]).

**Proposition 2.8.** With the above hypothesis and Notation 2.6, we assume that \( \pi \) is equivariant. Then, for \( t \neq 0 \) we have

\[
\chi(\Theta_{Z_t}) - \chi(\Theta_{\hat{Z}}) = \sum_{1 \leq i \leq n-1} (-1)^{i-1} h^i(\Theta_X) - \beta_f,
\]

and \( h^i(\Theta_X) = 0 \) for all \( i > r \) if the resolution \( \pi \) has relative dimension \( \leq r \).

Here and subsequently, \( h^i(F) \) denotes the dimension of \( H^i(X, F) \) for a coherent sheaf \( F \) on \( X \).

**Proof.** We apply Lemma 2.7 to \( F = \Theta_{\hat{Z}} \), and replace \( (R^i \Pi_* \Theta_{\hat{Z}})_p \) by \( h^i(\Theta_X) \) because \( V \) is Stein. The first assertion follows from the fact that \( \Pi \) is equivariant and Lemma 2.4 and the second assertion from the theorem on formal functions [3, III, Theorem 3.1].

**Remark 2.9.** Let \( X \) be a 1-convex \( n \)-fold with 1-dimensional exceptional set. By [17, Theorem 2], the miniversal deformation space \( \text{Def}(X) \) is smooth of dimension \( h^1(\Theta_X) \). It is also known that the Remmert reduction \( \pi : X \to V \) is equivariant (cf. [7, (3.1)]). If \( V \) has a smoothing \( f : V' \to \Delta \), then by Proposition 2.8 we get

\[
\dim \text{Def}(X) = \chi(\Theta_{Z_t}) - \chi(\Theta_{\hat{Z}}) + \beta_f.
\]

For \( n = 2 \), this was given in [27, (3.10.3)].

Let \( F \) be the Milnor fiber of the smoothing \( f : V' \to \Delta \), i.e., it is the fiber of the topological fiber bundle \( f : V' \setminus V \to \Delta \setminus \{0\} \). The middle Betti number \( b_n(F) \) of \( F \) of the smoothing \( f \) is called the Milnor number \( \mu_f \) of \( f \). If we write \( E \) for the exceptional fiber \( \Pi^{-1}(p) \), then the difference of topological Euler characteristics \( \chi_{\text{top}} \) of \( E \) and \( F \) equals the global topological defect (see [27, (3.5.3)]),

\[
\chi_{\text{top}}(F) - \chi_{\text{top}}(E) = \chi_{\text{top}}(Z_t) - \chi_{\text{top}}(\hat{Z}).
\]

(2.3)

We remark that if the isolated singularity \((V, p)\) is a complete intersection then \( \mu_f \) depends only on the singularity. In this case, we denote it by \( \mu(V, p) \). Moreover, the Milnor fiber \( F \) is \((n-1)\)-connected (see, e.g., [18 (5.8)]) and there are only two non-vanishing Betti numbers \( b_0(F) = 1 \) and \( b_n(F) \), and thus

\[
\chi_{\text{top}}(F) = 1 + (-1)^n \mu(V, p).
\]

(2.4)
2.3. Dual of dualizing sheaves. Keep the notation as Section 2.2. We further assume that \((V, p)\) is Cohen–Macaulay. Then there is a relative dualizing sheaf \(\omega_{\mathcal{X}/\Delta}\) for the globalization \(\mathcal{X} \to \Delta\) of the smoothing \(f\). It is flat over the 1-dimensional disk \(\Delta\). The dual \(\omega_{\mathcal{X}/\Delta}'\) is still torsion-free and thus flat over \(\Delta\), and induces \(\omega_{\mathcal{X}'}\) for \(t \neq 0\) and an inclusion \(\omega_{\mathcal{X}/\Delta}' \otimes \mathcal{O}_Z \hookrightarrow \omega_{\mathcal{X}'/\mathcal{D}}\).

Wahl introduced the notion of \(\omega_{\mathcal{X}'}\)-constant deformations [26 (1.4)], to which the dual of the dualizing differentials lifts, and an invariant of the smoothing \(f\) [27 (3.7)],

\[
\alpha_f := \dim_C \text{Coker}(\omega_{\mathcal{X}/\Delta}' \otimes \mathcal{O}_{Z,p} \hookrightarrow \omega_{\mathcal{X}',p}).
\]

Note that the smoothing \(f\) is \(\omega_{\mathcal{X}'}\)-constant if and only if \(\alpha_f = 0\) (automatic if \((V, p)\) is Gorenstein). By semicontinuity [3 III, Theorem 4.12], we get for \(t \neq 0\),

\[
\chi(\omega_{\mathcal{X}'}') = \chi(\omega_{\mathcal{X}/\Delta}' \otimes \mathcal{O}_Z) = \chi(\omega_{\mathcal{X}'}') - \alpha_f.
\]

Next we treat the case of resolutions.

**Lemma 2.10.** Under the above hypotheses, for the resolution \(\Pi: \mathcal{Z} \to \mathcal{Z}\) as in Notation 2.6, we have a natural inclusion \(\Pi_*(\omega_{\mathcal{Z}'}/\mathcal{D}) \hookrightarrow \omega_{\mathcal{Z}}\). Furthermore, it is an isomorphism if the exception set of \(\Pi\) has codimension \(\geq 2\).

**Proof.** Let \(E = \Pi^{-1}(p)\), and let \(\tilde{j}: \mathcal{Z} \setminus E \to \mathcal{Z}\) be the natural inclusion. Consider the exact sequence of local cohomology sheaves [12 Corollary 1.9]

\[
0 \to \mathcal{H}_E^0(\omega_{\mathcal{Z}}') \to \omega_{\mathcal{Z}}' \to \tilde{j}_*\tilde{i}^*\omega_{\mathcal{Z}}' \to \mathcal{H}_E^1(\omega_{\mathcal{Z}}') \to 0
\]

and note that \(\mathcal{H}_E^0(\omega_{\mathcal{Z}}') = 0\) by a depth argument (cf. [12 Theorem 3.8]). Furthermore, \(\mathcal{H}_E^1(\omega_{\mathcal{Z}}') = 0\) if \(E\) has codimension \(\geq 2\).

By \(\mathcal{Z}/\mathcal{D}\), we have \(\Pi_*(\omega_{\mathcal{Z}'}) \hookrightarrow \Pi_*\tilde{j}_*\tilde{i}^*\omega_{\mathcal{Z}'}\). Let \(U := \mathcal{Z} \setminus \{p\}\) be the smooth locus and \(j: U \hookrightarrow \mathcal{Z}\) the inclusion. Then, since depth \(\omega_{\mathcal{Z}',p} \geq 2\), we get \(\omega_{\mathcal{Z}'} \sim j_*(\omega_U)\) by \(\mathcal{H}_i^i(\omega_{\mathcal{Z}}') = 0\) for \(i = 0, 1\). Thus the proposition follows from \(\Pi_*\tilde{j}_*\tilde{i}^*\omega_{\mathcal{Z}'} = j_\Pi_*\tilde{j}_*\tilde{i}^*\omega_{\mathcal{Z}'} = j_*\Pi_*(\omega_{\mathcal{Z}'})\). \(\square\)

**Remark 2.11.** If \((V, p)\) has dimension two and \(\Pi: \mathcal{Z} \to \mathcal{Z}\) is the minimal resolution, then one always has \(\Pi_*(\omega_{\mathcal{Z}}') \sim \omega_{\mathcal{Z}}\) by [26 (3.5)].

**Proposition 2.12.** Let \(X, \mathcal{Z}, \pi\) and \(\Pi\) be as in Notation 2.6. Then we have

\[
\chi(\omega_{\mathcal{Z}'}) - \chi(\omega_{\mathcal{Z}}') = \sum_{1 \leq i \leq n-1} (-1)^{i-1}h^i(\omega_X^i) - \alpha_f + \gamma_{\pi}
\]

with \(\gamma_{\pi} = \dim_C \text{Coker}(\Pi_*(\omega_{\mathcal{Z}}')_p \hookrightarrow \omega_{\mathcal{Z}',p})\).

**Proof.** By Lemma 2.10 and the additivity of the Euler characteristic, we get \(\chi(\omega_{\mathcal{Z}'}') = \chi(\Pi_*(\omega_{\mathcal{Z}'})) + \gamma_{\pi}\). Then the corollary follows from Lemma 2.7 (2.6) and that \(V\) is Stein. \(\square\)
Remark 2.13. Suppose that \((V, p)\) is a rational Gorenstein singularity, i.e., it is a canonical singularity of index 1 (cf. [4] (6.4), (6.8) or [11] (3.3)). In the case \(\omega_X \cong \pi^* \omega_V\), we have \(h^i(\omega^\pi_X) = 0\) for \(i > 0\). Such \(\pi\) is called a crepant resolution.

Indeed, we may assume that \(\omega_V \cong \mathcal{O}_V\) for the Gorenstein singularity \((V, p)\). Then \((R^i\pi_* \omega^\pi_X)_p \cong (R^i\pi_* \mathcal{O}_X)_p = 0\) for \(i > 0\) since \((V, p)\) is rational.

3. Main result

Suppose from now on that \((V, p)\) is a 3-dimensional isolated normal Cohen–Macaulay singularity with a smoothing \(f \colon \mathcal{V} \to \Delta\). Given a resolution \(\pi \colon X \to V\), one can define the geometric genus \(p_g(V, p)\) of \((V, p)\) by \(h^2(\mathcal{O}_X)\). It is well known that the number is independent of the resolution.

In the previous section, we have seen the invariants \(\alpha_f\) and \(\beta_f\) associated the smoothing \(f\), defined as in (2.5) and (2.2) respectively. If the resolution \(\pi\) is equivariant, i.e., \(\pi_* \mathcal{O}_X \cong \mathcal{O}_V\), then the following main result are going to relate these invariants \(\alpha_f\) and \(\beta_f\) with certain numbers induced by \(\pi\). This is a generalization of [27, Theorem 3.13 (c)].

Theorem 3.1. With the above hypothesis and notation, we assume that \(\pi\) is equivariant. Then we have

\[
\beta_f - \alpha_f = h^1(\mathcal{O}_X) - h^2(\mathcal{O}_X) - 22p_g(V, p) - \frac{1}{2}(\chi_{\text{top}}(F) - \chi_{\text{top}}(E)) \\
- (h^1(\omega^\pi_X) - h^2(\omega^\pi_X)) - \gamma_\pi
\]

where \(E\) is the exceptional set of the resolution \(\pi\), \(F\) the Milnor fiber of the smoothing \(f\) and \(\gamma_\pi = h^0(\omega^\pi_Y/\pi_* (\omega^\pi_X))\).

Before stating the result to be proved, we need the following proposition, which follows from Hirzebruch–Riemann–Roch theorem and the observation that \(\omega^\pi_Y = \det \mathcal{O}_M\) and \(\text{td}_3(\mathcal{O}_M) = (1/24) c_1(\mathcal{O}_M) c_2(\mathcal{O}_M)\) (cf. [9] Example 15.2.5).

Proposition 3.2. Let \(M\) be a 3-dimensional compact manifold. Then we have

\[
\chi(\mathcal{O}_M) = \chi(\omega^\pi_Y) - 22\chi(\mathcal{O}_M) + (1/2)\chi_{\text{top}}(M).
\]

We now obtain our main result:

Proof of Theorem 3.1 We shall use the same notation as in Section 2.2. By Proposition 2.8 for \(t \neq 0\), we have

\[
\chi(\mathcal{O}_{Z_t}) - \chi(\mathcal{O}_Z) = h^1(\mathcal{O}_X) - h^2(\mathcal{O}_X) - \beta_f.
\]

Applying Proposition 3.2 to the left hand side gives

\[
\chi(\mathcal{O}_{Z_t}) - \chi(\mathcal{O}_Z) = \chi(\omega^\pi_{Z_t}) - \chi(\omega^\pi_Z) - 22(\chi(\mathcal{O}_{Z_t}) - \chi(\mathcal{O}_Z)) \\
+ (1/2)(\chi_{\text{top}}(Z_t) - \chi_{\text{top}}(Z)).
\]

By (2.3) and Proposition 2.12, it suffices to show that

\[
(3.1) \quad \chi(\mathcal{O}_{Z_t}) - \chi(\mathcal{O}_Z) = -p_g(V, p).
\]
To do this, we notice that only $h^2(O_X)$ can be non-zero for the normal Cohen–Macaulay singularity $(V,p)$, and $\chi(O_Z) = \chi(O_Z)$ by semicontinuity [3, III, Theorem 4.12]. Then the equality (3.1) follows from Lemma 2.7, and the proof is complete.

**Remark 3.3.** Let the hypotheses be as in Theorem 3.1 and $S$ the smoothing component on which the smoothing $f$ takes place. Our result shows that $\dim S + (1/2)\chi_{\text{top}}(F) - \alpha_f$ is independent of $S$ (and depend only on $\pi: X \to V$). In particular, if one smoothing $f$ is $\omega^\vee$-constant (i.e., $\alpha_f = 0$), so are all smoothings on the same irreducible component $S$ (cf. [27, (3.14.2)]). As a consequence, the formula in [19, 4.4] is not correct in general.

Theorem 3.1 allows us to treat the case of certain 1-convex threefolds.

**Proof of Theorem 3.1.** Recall that the exceptional set $E$ of the Remmert reduction $\pi: X \to V$ has dimension 1 and $K_X$ is $\pi$-trivial. Then $(V, p)$ is a 3-dimensional Gorenstein terminal singularity (see [22] or [4, (16.2)]). In particular, it is a rational hypersurface singularity, and $p_g(V, p) = 0$ (cf. [14, Corollary 4.2]). According to (2.5), Lemma 2.10 and Remark 2.5, it follows that $\alpha_f = \gamma_\pi = 0$ and $\beta_f$ equals the Tjurina number $\tau$ of $(V, p)$ for any fixed smoothing $f$.

We note that $\pi$ is crepant since the exceptional set $E$ contains no divisors. By Remark 2.9, 2.13 and Theorem 3.1, we find that

$$\dim \text{Def}(X) = \tau + (1/2)(\chi_{\text{top}}(F) - \chi_{\text{top}}(E)).$$

It remains to prove that $\chi_{\text{top}}(F) - \chi_{\text{top}}(E)$ is equal to $-(\mu + \sigma)$ where $\mu$ is the Milnor number of $(V, p)$ and $\sigma$ is the rank of $\text{Cl}(O_{V, p})$. Indeed, the irreducible components of $E$ are smooth rational curves, meeting transversally with no cycles [21, Proposition 1]. By using the Mayer–Vietoris sequence, we get the number of the irreducible components of $E$ equals $\chi_{\text{top}}(E) - 1$. Hence the desired formula follows from (2.4) and Remark 2.3.

**Proof of Corollary 1.2.** First, under the hypotheses of Theorem 1.1, we have seen that $(V, p)$ is an isolated cDV hypersurface singularity, and also have $\sigma > 0$ by Remark 2.3. Observe that it is an ordinary double point if and only if $\mu = 1$; in particular, $\tau = \sigma = 1$.

By Theorem 1.1 and Propositions 2.1 and 2.2, we get that the dimension of $\text{Def}(X)$ equals the Du Bois invariant $b^{2,1}$, and the condition (1) clearly implies (2), (3) and (4). Conversely, we can rewrite (2) as $\mu + \sigma = 2$ and thus this implies (1). The condition (3) implies (1) by [20, Theorem 2.2] (see also [24, p.1374]).

Now suppose that $(V, p)$ is not an ordinary double point. Notice that $V$ can be considered as the total space of a deformation of a Du Val surface singularity. One can find a nontrivial small deformation of $X$ under which $E$ splits up into a finite disjoint union of smooth copies of $\mathbb{P}^1$ with normal
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bundle $O_p1(-1)^{\oplus 2}$ (see [29, Proposition 1.1] or [7, p.679]). Hence we get $\dim \text{Def}(X) > 0$ and complete the proof of the equivalence of (1) and (4). □

Remark 3.4. In [8, §3], Friedman and Laza proved similar results as Theorem 1.1 and Corollary 1.2 with a different approach.

REFERENCES

[1] T. Ando. On the normal bundle of an exceptional curve in a higher-dimensional algebraic manifold. Math. Ann., 306(4):625–645, 1996.
[2] T. Ando. Some examples of simple small singularities. Comm. Algebra, 41(6):2193–2204, 2013.
[3] C. Bănică and O. Stănășilă. Algebraic methods in the global theory of complex spaces. Editura Academiei, Bucharest; John Wiley & Sons, London-New York-Sydney, 1976. Translated from the Romanian.
[4] H. Clemens, J. Kollár, and S. Mori. Higher-dimensional complex geometry. Astérisque, (166):144 pp., 1989, 1988.
[5] P. Du Bois. Complexe de de Rham filtré d’une variété singulière. Bull. Soc. Math. France, 109(1):41–81, 1981.
[6] H. Flenner. Divisorenklassengruppen quasihomogener Singularitäten. J. Reine Angew. Math., 328:128–160, 1981.
[7] R. Friedman. Simultaneous resolution of threefold double points. Math. Ann., 274(4):671–689, 1986.
[8] R. Friedman and R. Laza. Deformations of some local Calabi-Yau manifolds. arXiv:2203.11738, 2022.
[9] W. Fulton. Intersection theory, volume 2 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, second edition, 1998.
[10] G.-M. Greuel and E. Looijenga. The dimension of smoothing components. Duke Math J., 52(1):263–272, 1985.
[11] M. Gross. Deforming Calabi-Yau threefolds. Math. Ann., 308(2):187–220, 1997.
[12] R. Hartshorne. Local cohomology, volume 1961 of A seminar given by A. Grothendieck, Harvard University, Fall. Springer-Verlag, Berlin-New York, 1967.
[13] H. Hironaka. Bimeromorphic smoothing of a complex-analytic space. Acta Math. Vietnam., 2(2):103–168, 1977.
[14] U. Karras. Local cohomology along exceptional sets. Math. Ann., 275(4):673–682, 1986.
[15] Y. Kawamata. Minimal models and the Kodaira dimension of algebraic fiber spaces. J. Reine Angew. Math., 363:1–46, 1985.
[16] Y. Kawamata. Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces. Ann. of Math. (2), 127(1):93–163, 1988.
[17] H. Laufer. Versal deformations for two-dimensional pseudoconvex manifolds. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 7(3):511–521, 1980.
[18] E. Looijenga. Isolated singular points on complete intersections, volume 77 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1984.
[19] E. Looijenga. Riemann-Roch and smoothings of singularities. Topology, 25(3):293–302, 1986.
[20] Y. Namikawa and J.H.M. Steenbrink. Global smoothing of Calabi-Yau threefolds. Invent. Math., 122(2):403–419, 1995.
[21] H. Pinkham. Factorization of birational maps in dimension 3. In Singularities, Part 2 (Arcata, Calif., 1981), volume 40 of Proc. Sympos. Pure Math., pages 343–371. Amer. Math. Soc., Providence, RI, 1983.
[22] M. Reid. Minimal models of canonical 3-folds. In Algebraic varieties and analytic varieties (Tokyo, 1981), volume 1 of Adv. Stud. Pure Math., pages 131–180. North-Holland, Amsterdam, 1983.

[23] J.H.M. Steenbrink. Mixed Hodge structures associated with isolated singularities. In Singularities, Part 2 (Arcata, Calif., 1981), volume 40 of Proc. Sympos. Pure Math., pages 513–536. Amer. Math. Soc., Providence, RI, 1983.

[24] J.H.M. Steenbrink. Du Bois invariants of isolated complete intersection singularities. Ann. Inst. Fourier (Grenoble), 47(5):1367–1377, 1997.

[25] J.H.M. Steenbrink. Adjunction conditions for one-forms on surfaces in projective three-space. In Singularities and computer algebra, volume 324 of London Math. Soc. Lecture Note Ser., pages 301–314. Cambridge Univ. Press, Cambridge, 2006.

[26] J. Wahl. Elliptic deformations of minimally elliptic singularities. Math. Ann., 253(3):241–262, 1980.

[27] J. Wahl. Smoothings of normal surface singularities. Topology, 20(3):219–246, 1981.

[28] Sz-Sheng Wang. A note on nodal determinantal hypersurfaces. Geom. Dedicata, 208:97–111, 2020.

[29] P. M. H. Wilson. Symplectic deformations of Calabi-Yau threefolds. J. Differential Geom., 45(3):611–637, 1997.

[30] S. S.-T. Yau. Deformations and equitopological deformations of strongly pseudoconvex manifolds. Nagoya Math. J., 82:113–129, 1981.

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