Distillability and Positivity of Partial Transposes in General Quantum Field Systems

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Abstract. Criteria for distillability, and the property of having a positive partial transpose, are introduced for states of general bipartite quantum systems. The framework is sufficiently general to include systems with an infinite number of degrees of freedom, including quantum fields. We show that a large number of states in relativistic quantum field theory, including the vacuum state and thermal equilibrium states, are distillable over subsystems separated by arbitrary spacelike distances. These results apply to any quantum field model. It will also be shown that these results can be generalized to quantum fields in curved spacetime, leading to the conclusion that there is a large number of quantum field states which are distillable over subsystems separated by an event horizon.

1 Introduction

In the present work we investigate entanglement criteria for quantum systems with infinitely many degrees of freedom, paying particular attention to relativistic quantum field theory.

The specification and characterization of entanglement in quantum systems is a primary issue in quantum information theory (see [34] for a recent review of quantum information theory). Entanglement frequently appears as a resource for typical quantum information tasks, in particular for teleportation [2], key distribution [18], and quantum computation [48]. Ideally these processes use bipartite entanglement in the form of maximally entangled states, such as the singlet state of two spin-1/2 particles. But less entangled sources can sometimes be converted to such maximally entangled ones by a some “distillation process” using only local quantum operations and classical communication [46, 3]. States for which this is possible are called “distillable”, and this property is the strongest entanglement property for generic states (as opposed to special parameterized families). Indeed, it is stronger than merely being entangled, where a state is called entangled if it cannot be written as a mixture of uncorrelated product states. The existence of non-distillable entangled states (also called “bound entangled states”) was first shown in [28].

For a given state it is often not easy to decide to which class it belongs. A very efficient criterion is obtained from studying the partial transpose of the density operator, and asking whether it is a positive operator. In this case the state is called a ppt state, and an npt state otherwise. Originally, the npt property was established by Peres [44] as a sufficient condition for entanglement, and was subsequently shown to be also sufficient for low dimensional systems [29, 66] and some highly symmetric systems [59].
It turns out that ppt states cannot be distilled, so the existence of bound entangled states shows that the ppt condition is a much tighter fit for non-distillability than for mere separability. In fact, it is one of the major open problems [47] to decide whether there is equivalence, i.e., whether all npt states are distillable. There have been indications that this conjecture might fail for bipartite quantum systems having finitely and sufficiently many discrete degrees of freedom [14, 16]. On the other hand, for bipartite quantum systems having finitely many continuous degrees of freedom (such as harmonic oscillators) it was found that Gaussian states which are npt are also distillable (about this and related results, cf. [23] and refs. cited there).

While this brief recapitulation of results documents that the distinction between entangled, npt and distillable states is a subtle business already in the case of quantum systems with finitely many degrees of freedom, we should now like to point out that the study of entanglement is also a longstanding issue in general quantum field theory. Already before the advent of quantum information theory, the extent to which Bell-inequalities are violated has been investigated in several articles by Landau [39, 40] and by Summers and Werner [55, 54, 56]. In fact, the studies [55, 54] motivated the modern concept of separable states (then called “classically correlated” [65]) and raised the question of the connection between separability and Bell’s inequalities. More recently, there has been a renewed interest [50, 26, 42, 1, 17, 45] in the connection between “locality” as used in quantum information theory on the one hand, and in quantum field theory on the other. However, for some of the relevant questions, like distillability, the usual framework of quantum information theory, mainly focussing on systems with finite dimensional Hilbert spaces, is just not rich enough. This lack, which is also serious for the connections between entanglement theory and statistical mechanics of infinite systems, is addressed in the first part of our paper.

In particular, we extend the notions of separability and distillability for the general bipartite situations found in systems which have infinitely many degrees of freedom, and which cannot be expressed in terms of the tensor product of Hilbert spaces. These generalizations are fairly straightforward. Less obvious is our generalization of the notion of states with positive partial transpose, since the operation “partial transposition” itself becomes meaningless. Of course, we also establish the usual implications between these generalized concepts.

It turns out that 1-distillability of a state follows from the Reeh-Schlieder property, which has been thoroughly investigated for quantum field theoretical systems. After establishing this connection, we can therefore bring to bear known results from quantum field theory to draw some new conclusions about the non-classical nature of vacuum fluctuations. In particular, the vacuum is 1-distillable, even when Alice and Bob operate in arbitrarily small spacetime regions, and arbitrarily far apart in a Minkowski spacetime. Such a form of distillability can then also be deduced to hold for a very large (in a suitable sense, dense) class of quantum field states, including thermal equilibrium states. We comment on related results in [26, 38] and [50] in the remarks following Thm. 7.2. Furthermore, we generalize the distillability result to free quantum fields on curved spacetimes. We also point out that this entails distillability of a large class of quantum field states over subsystems which may be separated by an event horizon in spacetime, inhibiting two-way classical communication between the system parts, and we will discuss what this means for the distillability concept.
2 General Bipartite Quantum Systems

The bipartite quantum systems arising in quantum field theory are systems of infinitely many degrees of freedom. In contrast, the typical descriptions of concepts and results of quantum information theory are for quantum systems described in finite dimensional Hilbert spaces. In this section we describe the basic mathematical structures needed to describe systems of infinitely many degrees of freedom and, in particular, bipartite systems in that context.

For the transition to infinitely many degrees of freedom it does not suffice to consider Hilbert spaces of infinite dimension: this level of complexity is already needed for a single harmonic oscillator. The key idea allowing the transition to infinitely many oscillators is to look at the observable algebra of the system, which is then no longer the algebra of all bounded operators on a Hilbert space, but a more general operator algebra. This operator algebraic approach to large quantum systems has proved useful in both quantum field theory and quantum statistical mechanics [4, 5, 19, 24, 52].

For many questions we discuss, it suffices to take the observable algebra \( \mathcal{R} \) as a general C*-algebra: this is defined as an algebra with an adjoint operation \( X \mapsto X^* \) on the algebra elements \( X \) and also with a norm with respect to which it is complete and which satisfies \(||X^*X|| = ||X||^2\). In practically all applications, \( \mathcal{R} \) is given in a Hilbert space representation, so that it is usually no restriction of generality to think of \( \mathcal{R} \) as a norm-closed and adjoint-closed subalgebra of the algebra \( B(\mathcal{H}) \) of all bounded linear operators on a Hilbert space \( \mathcal{H} \). We should emphasize, though, that \( \mathcal{R} \) is usually really a proper subalgebra of \( B(\mathcal{H}) \), and also its weak closure (in the sense of convergence of expectation values) \( \overline{\mathcal{R}} \) will typically be a proper subalgebra of \( B(\mathcal{H}) \). This is of particular importance in the present context when we consider C*-algebras of local observables in relativistic quantum field theories: These are proper subalgebras of some \( B(\mathcal{H}) \) which don’t contain any finite-dimensional projection (in technical terms, the von Neumann algebras arising as their weak closures are purely infinite, cf. Sec. 2.7 in [4]). Therefore, the properties of these algebras are fundamentally different from those of the full \( B(\mathcal{H}) \); in particular, arguments previously developed in quantum information theory for finite dimensional systems modelled on \( B(\mathbb{C}^m) \otimes B(\mathbb{C}^n) \) are typically based on the use of finite-dimensional projections and thus they can usually not simply be generalized to the quantum field theoretical case.

We only consider algebras with unit element \( 1 \). For some questions we will consider a special type of such algebras, called von Neumann algebras, about which we collect some basic facts later. In any case, the “observables” are specified as selfadjoint elements of the algebra, or, more generally as measures (POVMs) with values in the positive elements of \( \mathcal{R} \).

Discussions of entanglement always refer to distinguished subsystems of a given quantum system. Subsystems are specified as subalgebras of the total observable algebra. For a bipartite system we must specify two subsystems with the crucial property that every observable of one subsystem can be measured jointly with every observable of the other, which is equivalent to saying that the observable algebras commute elementwise. Hence we arrive at

**Definition 2.1.** A *(generalized)* **bipartite system**, usually denoted by \( (A, B) \subset \mathcal{R} \), is a pair of C*-subalgebras \( A, B \) of a larger C*-algebra \( \mathcal{R} \), called the ambient algebra of the system, such that the identity is contained in both algebras, and all elements of \( A \in A \) and \( B \in B \) commute.

Thinking of typical situations in quantum information theory, \( A \) corresponds to the observables controlled by ‘Alice’ and \( B \) to the observables controlled by ‘Bob’. The ambient algebra \( \mathcal{R} \) will not play an important role for the concepts we define.
For most purposes it is equivalent to choose $\mathcal{R}$ either “minimal”, i.e., as the smallest C*-subalgebra containing both $A$ and $B$, or else “maximal” as $B(\mathcal{H})$, the algebra of all bounded operators on the Hilbert space $\mathcal{H}$ on which all the operators under consideration are taken to operate.

The standard quantum mechanical example of a bipartite situation is given by the tensor product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ of two Hilbert-spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, with the observable algebras of Alice and Bob defined as $\mathcal{R} = B(\mathcal{H})$, $A = B(\mathcal{H}_A) \otimes 1$, $B = 1 \otimes B(\mathcal{H}_B)$.

Note that in this example both algebras $A, B$ are of the form $B(\tilde{\mathcal{H}})$ (for suitable Hilbert-space $\tilde{\mathcal{H}}$), and as mentioned above, this will not be the case any more when $A$ and $B$ correspond to algebras of local observables in quantum field theory. Furthermore, if we do not want to impose unnecessary algebraic restrictions on the subsystems, we must envisage more general compositions than of tensor product form, too. Such systems arise naturally in quantum field theory, for tangent space-subsystems, we must envisage more general compositions than of tensor product form, too. Such systems arise naturally in quantum field theory, for tangent space-time regions [54], but also if we want to describe a state of an infinite collection of singlet pairs, and other “infinitely entangled” situations [35].

A state on a C*-algebra $\mathcal{R}$ is a linear functional $\omega : \mathcal{R} \to \mathbb{C}$, which can be interpreted as an expectation value functional, i.e., which is positive ($\omega(A) \geq 0$ for $A \geq 0$), and normalized ($\omega(1) = 1$). When $\mathcal{R} \subset B(\mathcal{H})$, i.e., when we consider a particular representation of all algebras involved as algebras of operators, we can consider the special class of states of the form

$$\omega(A) = \text{Tr}(\rho_\omega A) \quad \text{for all } A \in \mathcal{R},$$

(2.1)

for some positive trace class operator $\rho_\omega$, called the density operator of $\rho$. Such states are called normal (with respect to the representation). As usual, for $A = A^*$ representing an observable, the value $\omega(A)$ is the expectation value of the observable $A$ in the state $\omega$. A bipartite state is simply a state on the ambient algebra of a bipartite system. Since every state on the minimal ambient algebra can be extended to a state on the maximal algebra, this notion does not intrinsically depend on the choice of ambient algebra.

A bipartite state $\omega$ is a product state if $\omega(AB) = \omega(A)\omega(B)$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Similarly, $\omega$ is called separable, if it is the weak limit$^1$ of states $\omega_\alpha$, each of which is a convex combination of product states.

### 3 Positivity of Partial Transpose (ppt)

Consider again the standard situation in quantum information theory, where all Hilbert spaces are finite dimensional, and a bipartite system with Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Then we can define the partial transpose of a state $\omega$, or equivalently, its density matrix with $\rho_\omega$ with $\omega(.) = \text{Tr}(\rho_\omega .)$, by introducing orthonormal bases $\{|e_k^{(A)}\rangle\}$ in $\mathcal{H}_A$ and $\{|e_l^{(B)}\rangle\}$ in $\mathcal{H}_B$ for each of the Hilbert spaces, and swapping the matrix indices belonging to one of the factors, say the first, so that

$$\langle e_k^{(A)} \otimes e_l^{(B)} | \rho_\omega^{T_A} | e_m^{(A)} \otimes e_n^{(B)} \rangle = \langle e_m^{(A)} \otimes e_l^{(B)} | \rho_\omega | e_k^{(A)} \otimes e_n^{(B)} \rangle .$$

(3.2)

Then it is easy to see that in general $\rho_\omega \geq 0$ does not imply $\rho_\omega^{T_A} \geq 0$, i.e., the partial transpose operation is not completely positive. On the other hand, if $\omega$ is separable, then $\rho_\omega^{T_A} \geq 0$. More generally, we say that $\omega$ is a ppt-state when this is the case. As we just noted, the ppt property is necessary for separability, and also sufficient in low dimensions (2 $\otimes$ 2 and 2 $\otimes$ 3), which is known as the Peres-Horodecki criterion for separability [44].

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$^1$This means that $\lim_\alpha \omega_\alpha(X) = \omega(X)$, for all $X \in \mathcal{R}$
It is important to note that while the definition of the partial transpose depends on the choice of bases, the ppt-condition does not: different partial transposes are linked by a unitary transformation and so have the same spectrum. In the more involved context of general bipartite systems, we will follow a similar approach by defining a ppt property without even introducing an object which one might call the ‘partial transpose’ of the given state, and which would in any case be highly dependent on further special choices.

**Definition 3.1.** We say that a state $\omega$ on a bipartite system $(A, B) \subset R(A)$ has the ppt property if for any choice of finitely many $A_1, \ldots, A_k \in A$, and $B_1, \ldots, B_k \in B$, one has
\[
\sum_{\alpha,\beta} \omega(A_\beta A_\alpha^* B_\alpha^* B_\beta) \geq 0.
\]

Clearly, this definition is independent of the choice of ambient algebra $R$, since only expectations of the form $\omega(AB)$ enter. It is also symmetrical with respect to the exchange of $A$ and $B$ (just exchange $A_\alpha$ and $B_\alpha^*$, with concomitant changes).

Our first task is to show that this notion of ppt coincides with that given by Peres [44] in the case of finite-dimensional Hilbert spaces. We show this by looking more generally at situations in which there is a candidate for the role of the “partial transpose of $\omega$”.

**Proposition 3.2.** Let $(A, B) \subset B(\mathcal{H})$ be a bipartite system, and let $\theta$ be an anti-unitary operator on $\mathcal{H}$ such that the algebra $\tilde{B} \equiv \theta B \theta^*$ commutes elementwise with $A$.

1. Suppose that $\tilde{\omega}$ is a state on $B(\mathcal{H})$ such that
\[
\tilde{\omega}(A \tilde{B}) = \omega(A \theta \tilde{B}^* \theta^*)
\]
for $A \in A$ and $\tilde{B} \in \tilde{B}$. Then $\omega$ is ppt.

2. In particular, if $A$, $B$ are finite dimensional matrix algebras, Definition 3.1 is equivalent to the positivity of the partial transpose in the sense of Eq. (3.2).

Note that the star on the right hand side of Eq. (3.3) is necessary so the whole equation becomes linear in $\tilde{B}$. When $\theta$ is complex conjugation in some basis, $X \mapsto \theta^* X^* \theta$ is exactly the matrix transpose in that basis. This proves the second part of the Proposition: if $A$, $B$ are matrix algebras, we can identify $B$ with the algebra of all transposed matrices $\theta^* B \theta$, and with this identification Eq. (3.3) defines a linear functional on $A \otimes B$, which is just the partial transpose of $\omega$. The only issue for the ppt property in both formulations is indeed whether this functional is positive, i.e., a state.

**Proof.** Let $A_1, \ldots, A_k \in A$, and $B_1, \ldots, B_k \in B$ be as in Definition 3.1, and introduce $\tilde{B}_\alpha = \theta^* B_\alpha^* \theta$, so that also $B_\alpha = \theta B_\alpha^* \theta^*$. Then
\[
\sum_{\alpha,\beta} \omega(A_\beta A_\alpha^* B_\alpha^* B_\beta) = \sum_{\alpha,\beta} \omega \left( A_\beta A_\alpha^* \theta \tilde{B}_\alpha^* \theta^* \theta \tilde{B}_\beta^* \theta^* \right) = \sum_{\alpha,\beta} \omega \left( A_\beta A_\alpha^* \theta \tilde{B}_\alpha^* \tilde{B}_\beta^* \theta^* \right) = \sum_{\alpha,\beta} \omega \left( A_\beta A_\alpha^* \theta (\tilde{B}_\beta^* \tilde{B}_\alpha^*)^* \theta^* \right) = \sum_{\alpha,\beta} \tilde{\omega} \left( A_\beta A_\alpha^* \tilde{B}_\beta \tilde{B}_\alpha^* \right) = \tilde{\omega}(XX^*),
\]

(3.4)
with \( X = \sum_{\alpha} A_{\alpha} \tilde{B}_{\alpha} \). Clearly, when \( \tilde{\omega} \) is a state, this is positive. \( \square \)

Another consistency check is the following.

**Lemma 3.3.** Also for general bipartite systems, separable states are ppt.

**Proof.** Obviously, the ppt property is preserved under weak limits and convex combinations. By definition, each separable state arises by such operations from product states. Hence it is enough to show that each product state on \( \mathcal{R} \) is ppt. If \( \omega(AB) = \omega(A)\omega(B) \) is a product state, and \( A_1, \ldots, A_k \in \mathcal{A} \), and \( B_1, \ldots, B_k \in \mathcal{B} \) we introduce the \( k \times k \)-matrices \( M_{\alpha\beta} = \omega(A_\alpha A_\beta^*) \) and \( N_{\alpha\beta} = \omega(B_\beta^* B_\beta) \). What we have to show according to Definition 3.1 is that \( \text{tr}(MN) \geq 0 \). But this is clear from the observation that \( M \) and \( N \) are obviously positive semi-definite. \( \square \)

Therefore the set of states which are not ppt across \( \mathcal{A} \) and \( \mathcal{B} \) (the “npt-states”) forms a subset of the class of entangled states. As is well-known already for low dimensional examples (larger than \( 3 \otimes 3 \)-dimensional systems) the converse of this Lemma fails.

We add another result, an apparent strengthening of the ppt condition, which will turn out to be useful in proving below that a ppt state fulfills the Bell inequalities. Again, the assumptions on \( \mathcal{A} \) and \( \mathcal{B} \) are of the generic type as stated at the beginning of the section.

**Lemma 3.4.** Let \( \omega \) be a ppt state on \( \mathcal{R} \) for the bipartite system \( (\mathcal{A}, \mathcal{B}) \subset \mathcal{R} \). Then for any choice of finitely many \( A_1, \ldots, A_k \in \mathcal{A} \) and \( B_1, \ldots, B_k \in \mathcal{B} \), it holds that

\[
|\omega(T)|^2 \leq \sum_{\alpha,\beta} \omega(A_\alpha A_\beta^* B_\alpha^* B_\beta)
\]

where \( T = \sum_{\alpha} A_{\alpha} B_{\alpha} \).

**Proof.** We add new elements \( A_0 = 1 \) and \( B_0 = \lambda I \) for \( \lambda \in \mathbb{C} \) to the families \( A_1, \ldots, A_k, B_1, \ldots, B_k \). The condition of ppt then applies also with the new families \( A_0, A_1, \ldots, A_k \in \mathcal{A}_1, B_0, B_1, \ldots, B_k \in \mathcal{A}_2 \), entailing that

\[
0 \leq \sum_{\alpha,\beta=0}^k \omega(A_\alpha A_\beta^* B_\alpha^* B_\beta) = \sum_{\alpha,\beta=1}^k \omega(A_\alpha A_\beta^* B_\alpha^* B_\beta) + \omega(\lambda T^*) + \omega(\overline{T}^*) + \omega(|\lambda|^2 I).
\]

Now insert \( \lambda = -\omega(T) \) and use that, since \( \omega \) is a state, it holds that \( \omega(T^*) = \overline{\omega(T)} \). This yields immediately the inequality claimed in Lemma 3.4. \( \square \)

In a similar spirit, we can apply the standard trick of *polarization*, i.e., of replacing the arguments in a positive definite quadratic form by linear combinations to get a condition on a bilinear form. The *polarized version* of the ppt-property is the following, and makes yet another connection to the ordinary matrix version of the ppt-property:

**Lemma 3.5.** Let \( \omega \) be a state on a bipartite system \( (\mathcal{A}, \mathcal{B}) \subset \mathcal{R} \). Then for any choice of elements \( A_1, \ldots, A_n \in \mathcal{A} \) and \( B_1, \ldots, B_m \in \mathcal{B} \), introduce the \( (nm) \times (nm) \)-matrix \( X \)

\[
\langle i\alpha|X|j\beta \rangle = \omega(A_i B_\alpha B_\beta^* A_j^*)
\]

where \( \alpha, \beta = 1, \ldots, n \) and \( \lambda, \mu = 1, \ldots, m \).

All such matrices are positive definite for any state \( \omega \). Moreover, they all have a positive partial transpose if and only if \( \omega \) is ppt.
Proof: The positivity for arbitrary states says that, for all complex $n \times m$-matrices $\Phi$, we have
\[
\sum_{i\alpha j\beta} \Phi_{i\alpha} \langle i\alpha | X | j\beta \rangle \Phi_{j\beta} = \omega(X^*X) \geq 0 , \tag{3.6}
\]
where $X = \sum_{i\alpha} \Phi_{i\alpha} B^*_\alpha A^*_i$. For the ppt-property, decompose an arbitrary $\Phi$ as
\[
\Phi_{i\alpha} = \sum_{\mu} u_{i\mu} v_{\alpha\mu},
\]
for suitable coefficient matrices $u, v$. For example, we can get $u$ and $v$ from the singular value decomposition of $\Phi$. Inserting this into the condition for the positivity of $X^T_2$, we find
\[
\sum_{i\alpha j\beta} \Phi_{i\alpha} \langle i\alpha | X^T_2 | j\beta \rangle \Phi_{j\beta} = \sum_{i\alpha j\beta \mu\nu} \Phi_{i\alpha} \langle i\nu | X | j\mu \rangle \Phi_{j\beta} = \omega(\tilde{A}_\mu \tilde{A}^*_\mu \tilde{B}_\nu \tilde{B}^*_\nu),
\]
with
\[
\tilde{A}_\mu = \sum_i u_{i\mu} A^*_i \quad \text{and} \quad \tilde{B}_\nu = \sum_\alpha v_{\alpha\nu} B^*_\alpha.
\]
The ppt-property demands that all these expressions are positive, and conversely, positivity of all these expressions entails that $\omega$ is ppt. \hfill \Box

This Lemma greatly helps to sort the big mess of indices which would otherwise clutter the proof of the following result. It contains as a special case the observation that the tensor product of ppt states is ppt, provided we consistently maintain the Alice/Bob distinction, which will be important for establishing the preservation of the ppt-property under general distillation protocols. In the standard case this is an easy property of the partial transposition operation. Since this is not available in general, we have to give a separate proof based on our definition.

Lemma 3.6. Let $(A_k, B_k) \subset \mathcal{R}_k$ be a finite collection of bipartite systems, all contained in a common ambient algebra $\mathcal{R}$ such that all algebras $\mathcal{R}_k$ commute. Let $\mathcal{A}$ (resp. $\mathcal{B}$) denote the $C^*$-algebra generated by all the $A_k$ (resp. $B_k$). Let $\omega$ be a state on $\mathcal{R}$, which is ppt for each subsystem, and which factorizes over the different $\mathcal{R}_k$. Then $\omega$ is ppt for $(\mathcal{A}, \mathcal{B}) \subset \mathcal{R}$.

Proof. We show the ppt property in polarized form. Since $\mathcal{A}$ is generated by the commuting algebras $A_k$, we can approximate each element by linear combinations of products $A = \prod_k A^{(k)}$. Since the polarized ppt-condition is continuous and linear in $A_i$, it suffices to prove it for choices $A_i = \prod_k A_i^{(k)}$, and similarly for $B_\alpha$. For such choices the factorization of $\omega$ implies that the $X$-matrix from the Lemma is the tensor product of the matrices $X_k$ obtained for the subsystems. The partial transposition of the whole matrix is done factor by factor, and since all the $X^T_2$ are positive, so is their tensor product $X^T_2$. \hfill \Box

We close this section by pointing out a mathematically more elegant way of expressing the ppt property. It employs the concept of the opposite algebra $\mathcal{A}^{\text{op}}$ of a given $*$-algebra $\mathcal{A}$. The opposite algebra is the $*$-algebra formed by $\mathcal{A}$ with
its original vector addition, scalar multiplication, and adjoint (and operator norm), but endowed with a new algebra product:

\[ A \bullet B = BA, \quad A, B \in A, \]

where on the right hand side we read the original algebra product of \( A \). There is a linear, \( * \)-preserving, one-to-one, onto map \( \theta : A^{\text{op}} \to A \) given by \( \theta(A) = A \), which is an anti-homomorphism (i.e., \( \theta(A \bullet B) = \theta(B)\theta(A) \) for all \( A, B \in A^{\text{op}} \). With its help one can define a linear, \( * \)-preserving map \( \theta \circ \text{id} : A^{\text{op}} \otimes B \to \mathbb{R} \) by

\[ (\theta \circ \text{id})(A \otimes B) = \theta(A)B, \]

where we have distinguished the “algebraic tensor product” \( \otimes \), i.e., the tensor product as defined in linear algebra, from the ordinary tensor product “\( \otimes \)” of \( C^* \)-algebras, which also contains norm limits of elements in \( A \otimes B \).

By definition, \( (\theta \circ \text{id}) \) has dense range, but is usually unbounded, and does not preserve positivity. Given any state \( \omega \) on \( \mathbb{R} \), it induces a linear functional \( \omega \circ (\theta \circ \text{id}) \) on \( A^{\text{op}} \otimes B \). Then it is not difficult to check that the functional \( \omega \circ (\theta \circ \text{id}) \) is positive (i.e. \( \omega \circ (\theta \circ \text{id})(C^*C) \geq 0 \) for all \( C \in A^{\text{op}} \otimes B \)) if and only if \( \omega \) is a ppt state.

It would be interesting to study “mild failures” of the ppt condition, i.e., cases in which \( \omega \circ (\theta \circ \text{id}) \), although not positive, is a bounded linear functional, or maybe even a normal linear functional on \( A^{\text{op}} \otimes B \).

### 4 Relation to the Bell-CHSH Inequalities

Now we study the connection of the ppt-property to Bell-inequalities in the CHSH form [11]. Again, we have to recall some terminology. A state \( \omega \) on a bipartite system \((A, B) \subset \mathbb{R} \) is said to satisfy the Bell-CHSH inequalities if

\[ |\omega(A(B' + B) + A'(B' - B))| \leq 2 \quad (4.1) \]

holds for all hermitean \( A, A' \in A \) and \( B, B' \in B \) whose operator norm is bounded by 1. A quantitative measure of the failure of a state to satisfy the Bell-CHSH inequalities is measured by the quantity

\[ \beta(\omega) = \sup_{A, A', B, B'} \omega(A(B' + B) + A'(B' - B)). \]

where the supremum is taken over all admissible \( A, A', B, B' \) as in (4.1). By Cirel’son’s inequality [10],

\[ \beta(\omega) \leq 2\sqrt{2}. \quad (4.2) \]

If equality holds here, we say that the bipartite state \( \omega \) violates the Bell-CHSH inequalities maximally. The proof of the following result is adapted from the finite dimensional case [63].

**Theorem 4.1.** If a bipartite state is ppt, then it satisfies the Bell-CHSH inequalities.

**Proof.** The right hand side in (4.1) is linear in each of the arguments \( A, A' \in A \) and \( B, B' \in B \). Hence we can search for the maximum of this expression taking each of these four variables as an extreme point of the admissible convex domain. The extreme points of the set hermitean \( X \) with \( ||X|| \leq 1 \) are those with \( X^2 = \mathbf{1} \). Hence it is sufficient to show that the bound (4.1) holds for all hermitean arguments
fulfilling $A^2 = A'^2 = B^2 = B'^2 = 1$. For such operators $A, A'$ and $B, B'$ we set, following [39],

$$C = A(B' + B) + A'(B' - B)$$

and obtain

$$|\omega(C)|^2 \leq \omega(C^2) = 4 + \omega([A, A'][B, B']) \quad (4.3)$$

where $[X, Y] = XY - YX$ denotes the commutator. On the other hand, if we set $A_1 = A, A_2 = A', B_1 = B' + B, B_2 = B' - B$, we get according to Lemma 3.4, since $\omega$ admits a ppt,

$$|\omega(C)|^2 \leq \sum_{\alpha, \beta = 1}^2 \omega(A^{\alpha}_\beta A^{\star}_\alpha B^{\star}_\beta B_\beta) = 4 - \omega([A, A'][B, B']) \quad (4.4)$$

Adding (4.3) and (4.4) yields $|\omega(C)|^2 \leq 4$ which is equivalent to (4.1).

\[\square\]

5 Distillability for General Quantum Systems

If entanglement is considered as a resource provided by some source of bipartite systems, it is natural to ask whether the particular states provided by the source can be used to achieve some tasks of quantum information processing, such as teleportation. Usually the pair systems provided by the source are not directly usable, so some form of preprocessing may be required. This upgrading of entanglement resources is known as distillation. The general picture here is that the source can be used several times, say $N$ times. The allowed processing steps are local quantum operations, augmented by classical communication between the two labs holding the subsystems (“LOCC operations” [3, 34], see also [2]), usually personified by the two physicists operating the labs, called Alice and Bob. That is, the decision which operation is applied by Bob can be based on measuring results previously obtained by Alice and conversely. The aim is to obtain, after several rounds of operations, some bipartite quantum systems in a state which is nearly maximally entangled. The number of these systems may be much lower than $N$, whence the name “distillation”.

The idea of distillation can be generalized to combinations of resources. For example, a bound entangled (i.e., not by itself distillable) state can sometimes be utilized to improve entanglement in another state [27].

The optimal rate of output particles per input particle is an important quantitative measure of entanglement in the state produced by the source. Distillation rates are very hard to compute because they involve an optimization over all distillation procedures, a set which is difficult to parameterize. A simpler question is to decide whether the rate is zero or positive. In the latter case the state is called distillable.

In this paper we will look at two types of results on distillability, ensuring either success or failure: We will show that many states in quantum field theory are distillable, by using an especially simple kind of distillation protocol. States for which this works are also called 1-distillable (see below).

On the other hand we will show that distillable states cannot be ppt. Note that this is a statement about all possible LOCC protocols, so we will need to define this class of operations more precisely in our general context. The desired implication will become stronger if we allow more operations as LOCC, so we should make only minimal technical assumptions about this class of operations. To begin with, LOCC operations are operations between different bipartite systems. So let
(A_i, B_i) \subset \mathcal{R}_i \text{ and } (A_2, B_2) \subset \mathcal{R}_2 \text{ be bipartite systems. An operation localized on the Alice side will be a completely positive map } T : A_1 \rightarrow A_2 \text{ with } T(1) = 1. \text{ Note that since we defined the operation in terms of observables, we are working in the Heisenberg picture, hence 1 labels the output system and 2 labels the input system. An operation also producing classical results is called an instrument in the terminology of Davies [12]. When there are only finitely many possible classical results, this is given by a collection } T_x \text{ of completely positive maps, labelled by the classical result } x, \text{ such that } \sum_x T_x(1) = 1. \text{ Similarly, an operation depending on a classical input } x \text{ is given by a collection of completely positive maps } S_x \text{ such that } S_x(1) = 1. \text{ Hence, whether the classical parameter } x \text{ is an input or an output is reflected only in the normalization conditions. A LOCC operation with information flow only from Alice to Bob is then given by a completely positive map } M : \mathcal{R}_1 \rightarrow \mathcal{R}_2 \text{ such that }

\[ M(AB) = \sum_x T_x(A)S_x(B) , \]  

where the sum is finite, and for each } x, T_x : A_1 \rightarrow A_2 \text{ and } S_x : B_1 \rightarrow B_2 \text{ are completely positive with the normalization conditions specified above. This will be the first round of a LOCC protocol. In the next round, the flow of information is usually reversed, and all operations are allowed to depend on the classical parameter } x \text{ measured in the first round. Iterating this will lead to a similar expression as (5.1), with } x \text{ replaced by the accumulated classical information obtained in all rounds together. The normalization conditions will depend in a rather complicated way on the information parameters of each round. However, as is easily seen by induction the overall normalization condition

\[ \sum_x T_x(1)S_x(1) = 1 \]  

will also hold for the compound operation. Fortunately, we only need this simple condition. An operator } M \text{ of the form (5.1), with completely positive } T_x, S_x, \text{ but with only the overall normalization condition (5.2), is called a separable superoperator, in analogy to the definition of separable states. More generally, we use this term also for limits of such operators } M_n, \text{ such that probabilities converge for all input states, and all output observables. By such limits we automatically also cover the case of continuous classical information parameters } x, \text{ in which the sums are replaced by appropriate integrals.

Then we can state the following implication:

\textbf{Proposition 5.1.} Let } M \text{ be a separable superoperator between bipartite systems } (A_i, B_i) \subset \mathcal{R}_i, \; (i = 1, 2), \text{ and let } \omega_2 \text{ be ppt. Then the output state } \omega_1(X) = \omega_2(M(X)) \text{ is also ppt. In particular, ppt states are not distillable with LOCC operations.}

\textit{Proof.} The ppt-preserving property is preserved under limits as described above, and also under sums, so it suffices to consider superoperators } M, \text{ in which the sum (5.1) has only a single term, i.e., } \omega_1(AB) = \omega_2(T(A)S(B)). \text{ Let } A_1, \ldots, A_k \in A_1, \text{ and } B_1, \ldots, B_k \in B_1. \text{ We have to show that } \sum_{\alpha, \beta} \omega_2(T(A_\beta A_\alpha^*)S(B_\alpha^* B_\beta)) \geq 0. \text{ Now because } T \text{ is completely positive, the matrix } T(A_\beta A_\alpha^*) \text{ is positive in the algebra of } A_2\text{-valued } k \times k\text{-matrices, and hence we can find elements } t_{\alpha\beta} \in A_2 \text{ (the matrix elements of the square root) such that}

\[ T(A_\beta A_\alpha^*) = \sum_n (t_{\alpha\beta})^* t_{n\alpha} . \]  

Of course, there is an analogous decomposition

\[ S(B_\beta B_\alpha^*) = \sum_m (s_{m\beta})^* s_{m\alpha} . \]
Hence, observing the changed order of the indices $\alpha, \beta$ in the $S$-term:

$$\sum_{\alpha, \beta} \omega_1 (T(A_\beta A_\alpha^*)S(B_\alpha B_\beta^*)) = \sum_{n,m} \sum_{\alpha, \beta} \omega_2 ((t_{n\beta})^*t_{n\alpha} (s_{m\alpha})^*s_{m\beta}) ,$$

which is positive, because the input state $\omega_2$ was assumed to be ppt.

For distillability we have to consider tensor powers of the given state and try to obtain a good approximation of a singlet state of two qubits by some LOCC operation. However, since the final state is clearly not ppt, and the input tensor power is ppt by Lemma 3.6, the statement just proved shows that this impossible.

For positive distillability results it is helpful to reduce the vast complexity of all LOCC operations, applied to arbitrary tensor powers, and to look for specific simple protocols for the case at hand. Since we are not concerned with rates, but only with the yes/no question of distillability, some major simplifications are possible. The first simplification is to restrict the kind of classical communication. Suppose that the local operations are such that every time they also produce a classical signal “operation successful” or “operation failed”. Then we can agree to use only those pairs in which the operation was successful on both sides. In all other cases we just try again. Note that this requires two-way classical communication, since Alice and Bob both have to give their ok for including a particular pair in the ensemble. However, in the simplest case no further communication between Alice and Bob is used. To state this slightly more formally, let $T$ denote the distillation operation in such a step, written in the Heisenberg picture. This is a selective operation in the sense that $T(1)I \leq 1$, and $\omega(T(I))$ is the probability for successfully obtaining a pair. Then by the law of large numbers we can build from this a sequence of non-selective distillation operations on many such pairs, which produce systems in the state

$$\omega^{[T]}(A) = \left( \frac{\omega(T(A))}{\omega(T(I))} \right),$$

(5.5)

with rate close to the probability $\omega(T(I))$. If we are only interested in the yes/no question of distillability and not in the rate, then obviously selective operations are just as good as non-selective ones. Moreover, it is sufficient for distillability that $\omega^{[T]}$ be distillable for some such $T$. It is also convenient to restrict the type of output systems: it suffices to produce a pair of qubits (2-level systems) in a distillable state, because from a sufficient number of such pairs any entangled state can be generated by LOCC operations. Any target state which has non-positive partial transpose will do, because for qubits ppt and non-distillability are equivalent. Finally, we look at situations where the criterion can be applied without going to higher tensor powers.

In the simplest case only one pair prepared in the original state $\omega$ is needed to obtain a distillable qubit pair with positive probability.

**Definition 5.2.** A state $\omega$ on a bipartite system $(A, B) \subset \mathcal{R}$ is called 1-distillable, if there are completely positive maps $T : B(C^2) \rightarrow A$ and $S : B(C^2) \rightarrow B$ such that the functional $\omega_2(X \otimes Y) = \omega(T(X)S(Y))$, $X \otimes Y \in B(C^2 \otimes C^2)$, on the two-qubit system is not ppt.

Then according to the discussion just given, 1-distillable states are distillable. If the maps $T, S$ are normalized such that $||T(I)|| = ||S(I)|| = 1$, and $\omega_2$ is close to a multiple of a singlet state, a rough estimate of the distillation rate achievable from $\omega$ is the normalization constant $\omega_2(I)$. In the field theoretical applications below this rate will be very small.
Note that specifying a completely positive map $T : B(\mathbb{C}^2) \to A$ is equivalent to specifying the four elements $T_{kl} = T(|k\rangle\langle l|) \in A$ or, in other words, an $A$-valued $2 \times 2$-matrix, called the Choi matrix of $T$. It turns out that $T$ is completely positive iff the Choi matrix is positive in the algebra of such matrices (isomorphic to $A \otimes B(\mathbb{C}^2)$). This allows a partial converse of the implication “distillable $\Rightarrow$ npt”:

**Lemma 5.3.** Let $\omega$ be a state on a bipartite system $(A, B) \subset \mathcal{R}$, and suppose that the ppt condition in Definition 3.1 fails already for $k = 2$. Then $\omega$ is 1-distillable in the sense of Definition 3.2.

**Proof:** Let $A_1, A_2, B_1, B_2$ be as in Definition 3.1. Then we can take the matrix $A_\alpha A_\beta^*$ as the Choi matrix of $T$, i.e., with a similar definition for $S$:

$$T(M) = \sum_{\alpha, \beta} A_{\beta} \langle \beta | M | \alpha \rangle A_{*\alpha}^*$$

$$S(N) = \sum_{\alpha', \beta'} B_{\alpha'}^* \langle \alpha' | N | \beta' \rangle B_{\beta'}.$$

Inserting this into Definition 5.2, we find

$$\omega_2(Z) = \sum_{\alpha, \beta, \alpha', \beta'} \langle \beta \alpha' | Z | \alpha \beta' \rangle \omega \left( A_{\beta} A_{\alpha}^* B_{\alpha'}^* B_{\beta'} \right), \quad Z \in B(\mathbb{C}^2 \otimes \mathbb{C}^2). \quad (5.6)$$

In particular, when $Z$ is equal to the transposition operator $Z | \alpha \beta' \rangle = | \beta' \alpha \rangle$, this expectation is equal to the sum in Definition 3.1, hence negative by assumption. On the other hand, $Z$ has a positive partial transpose (proportional to the projection onto a maximally entangled vector), hence $\omega_2$ cannot be positive. \qed

## 6 The Reeh-Schlieder property

In this section we will establish a criterion for 1-distillability which will be useful in quantum field theoretical applications. We prove it in an abstract form, which for the time being makes no use of spacetime structure. We will assume that all observable algebras are given as operator algebras, i.e., we look at bipartite systems of the kind $(A, B) \subset B(\mathcal{H})$. This is no restriction of generality, since every C*-algebra (here the ambient algebra of the bipartite system) may be isomorphically realized as an algebra of operators. The non-trivial information contained in any such representation is about a special class of states, namely the normal ones (see (2.1)). Any state of a C*-algebra becomes normal in a suitable representation, so the choice of representation is mainly the choice of a class of states of interest. In particular, we have the vector states on $B(\mathcal{H})$, which are states of the form

$$\omega_\psi(R) = \langle \psi | R | \psi \rangle, \quad R \in B(\mathcal{H}), \quad (6.1)$$

with $\psi \in \mathcal{H}$ a unit vector. Again, this is not a loss of generality, since every bipartite system can be written in this way, by forming the GNS-representation [4] of the ambient algebra\(^2\). However, in this language the key condition of this section is more easily stated. It has two formulations: one emphasizing the operational content from the physical point of view, and one which is somewhat simpler mathematically. We state their equivalence in the following Lemma (whose proof is entirely trivial).

\(^2\)For a state $\omega$ on a C*-algebra $\mathcal{R}$, there is always a triple $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ where: (1) $\pi_\omega$ is a $*$-preserving representation of $\mathcal{R}$ by bounded linear operators on the Hilbert-space $\mathcal{H}_\omega$. (2) $\Omega_\omega$ is a unit vector in $\mathcal{H}_\omega$ so that $\pi_\omega(\mathcal{R}) \Omega_\omega$ is dense in $\mathcal{H}_\omega$. (3) $\omega(R) = \langle \Omega_\omega | \pi_\omega(R) | \Omega_\omega \rangle$ for all $R \in \mathcal{R}$. $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ is called the GNS representation of $\omega$; see, e.g., [4] for its construction.
Lemma 6.1. Let $A \subset B(\mathcal{H})$ be a C*-algebra, and $\psi \in \mathcal{H}$ a unit vector. Then the following are equivalent:

1. $\psi$ has the **Reeh-Schlieder property** with respect to $A$, i.e., for each unit vector $\chi \in \mathcal{H}$ and each $\varepsilon > 0$, there is some $A \in A$, so that

$$|\omega_\chi(R) - \omega_\psi(A^*RA)/\omega_\psi(A^*A)| < \varepsilon ||R||$$

holds for all $R \in B(\mathcal{H})$.

2. $\psi$ is cyclic for $A$, i.e., the set $A\psi = \{A\psi : A \in A\}$ is dense in $\mathcal{H}$.

We also remark that a vector $\psi$ in $\mathcal{H}$ is called *separating* for $A$ if for each $A \in A$, the relation $A\psi = 0$ implies that $A = 0$. It is a standard result in the theory of operator algebras that $\psi$ is cyclic for a von Neumann algebra $A$ if and only if $\psi$ is separating for its commutant $A'$ (see, e.g., [4]). Note that $A$, a subset of $B(\mathcal{H})$, is a von Neumann algebra if it coincides with its bicommutant $A''$, where for $\mathcal{B} \subset B(\mathcal{H})$, its bicommutant is the von Neumann algebra $\mathcal{B}' = \{R \in B(\mathcal{H}) : RB = BR \forall B \in \mathcal{B}\}$.

The physical meaning of the Reeh-Schlieder property is that any vector state on $B(\mathcal{H})$ can be obtained from $\omega_\psi$ by selecting according to the results of a measurement on the subsystem $A$. Let us denote by $A_1$ a multiple of the $A$ from the Lemma, normalized so that $||A_1|| \leq 1$, and set $A_0 = (1 - A_1^*A_1)^{1/2}$. Then the operation elements $T_i(R) = A_i^*RA_i$ ($i = 0, 1$) together define an instrument. The operation without selecting according to results is $T(R) = T_0(R) + T_1(R)$. This instrument is **localized** in $A$ in the sense that $T_i(A) \subset A$, and that for any $B$ commuting with $A$, in particular for all observables of the second subsystem of a bipartite system, we get $T(B) = B$. That is, no effect of the operation is felt for observables outside the subsystem $A$. Of course, $T_i(B) \neq B$, but this only expresses the state change by selection in the presence of correlations. The state appearing in the Reeh-Schlieder property is just a selected state, obtained by running the instrument on systems prepared according to $\omega_\psi$, and keeping only the systems with a 1-response. By taking convex combinations of operations, one can easily see that also every convex combination of vector states, and hence any normal state can be approximately obtained from $\omega_\psi$.

Our next result connects these properties with distillability.

Theorem 6.2. Let $(A,B) \subset B(\mathcal{H})$ be a bipartite system, with both $A,B$ non-abelian. Suppose $\psi \in \mathcal{H}$ is a unit vector which has the Reeh-Schlieder property with respect to $A$. Then $\omega_\psi$ is 1-distillable.

The proof of this statement takes up ideas of Landau, and utilizes Lemma 5.5 in [56]. To keep our paper self-contained, we nevertheless give a full proof here.

**Proof, step 1:** We first treat the special case in which $A$ and $B$ are von Neumann algebras, i.e., of algebras also closed in the weak operator topology. Then a theorem due to M. Takesaki [57] asserts that there are non-vanishing *-homomorphisms $\tau : B(C^2) \to A$ and $\sigma : B(C^2) \to B$, which may, however, fail to preserve the identity. Consider the map $\pi : B(C^2) \otimes B(C^2) \to B(\mathcal{H})$, given by

$$\pi(X \otimes Y) = \tau(X)\sigma(Y).$$

(6.2)

One easily checks that, because the ranges of $\tau$ and $\sigma$ commute, $\pi$ is a *-homomorphism. But as a C*-algebra $B(C^2) \otimes B(C^2) \cong B(C^4)$ is a full matrix algebra. Since this has no ideals, $\pi$ is either an isomorphism or zero.

**Step 2:** We have to show that $\tau$ can be chosen so that $\pi$ is non-zero. In many situations of interest this would follow automatically because both $\tau(1)$ and $\sigma(1)$ are non-zero: Often $A$ and $B$ also have the so-called Schlieder property [51] (an independence property [56]), which means that $A \in A$, $B \in B$, $A,B \neq 0$ imply $AB \neq 0$. (There seems to be an oversight in [56, Lemma 5.5] concerning this
assumption.) However, we do not assume this property, and instead rely once again on the Reeh-Schlieder property of $\mathcal{A}$.

Let us take $\sigma$ as guaranteed by Takesaki’s Theorem, and set $p = \sigma(\mathbf{1})$. Then if $p\mathcal{A}p$ is non-abelian, we can apply Takesaki’s result to this algebra, and find a homomorphism $\tau$ with $\pi(\mathbf{1}) = \pi(\mathbf{1})p = \pi(\mathbf{1}) \neq 0$. So we only need to exclude the possibility that $p\mathcal{A}p$ is abelian.

In other words, we have to exclude the possibility that in some Hilbert space $\mathcal{H}(p) \equiv p\mathcal{H}$ there is an abelian von Neumann algebra $\mathcal{A}(p) \equiv p\mathcal{A}$ with a cyclic vector $\psi(p) = p\psi$, so that $\mathcal{A}(p)$ commutes with a copy $\mathcal{B}(p) \equiv \sigma(B(\mathbb{C}^2))$ of the $2 \times 2$-matrices. The latter property entails that $(\mathcal{A}(p))'$ is non-abelian.

We will exclude this possibility by adopting it as a hypothesis and showing that this leads to a contradiction. Let $q$ denote the projection onto the subspace of $\mathcal{H}(p)$ generated by $(\mathcal{A}(p))'\psi(p)$. This projection is contained in $(\mathcal{A}(p))'' = \mathcal{A}(p)$. Let $\mathcal{H}(qp) = q\mathcal{H}(p)$, then $\psi(qp) = q\psi(p) = \psi(p) \in \mathcal{H}(qp)$ is both a cyclic and separating vector for the von Neumann algebra $(\mathcal{A}(p))'\psi(p) = (\mathcal{A}(p))'q$, and since $\psi(p)$ is separating for $(\mathcal{A}(p))'(\mathbf{1})$ (owing to the assumed cyclicity of $\psi(p)$ for $\mathcal{A}(p)$), $(\mathcal{A}(p))'\psi(p)$ is non-abelian since so is $(\mathcal{A}(p))'\psi(p)$ by hypothesis. On the other hand, abelianess of $\mathcal{A}(p)$ entails that $\mathcal{A}(qp) = (\mathcal{A}(p))'(\mathbf{1}) = q(\mathcal{A}(p))q$ is a von Neumann algebra in $B(\mathcal{H}(qp))$ for which $(\mathcal{A}(p))'\psi(p) = (\mathcal{A}(qp))'$, where the second commutant is taken in $B(\mathcal{H}(qp))$. Clearly, $\mathcal{A}(qp)$ is again abelian. However, since $\psi(qp)$ is cyclic and separating for $(\mathcal{A}(p))'(\mathbf{1}) = (\mathcal{A}(qp))'$, it follows by the Tomita-Takesaki theorem [4] that $\mathcal{A}(qp)$ is anti-linearly isomorphic to $(\mathcal{A}(qp))'$, which is a contradiction in view of the abelianess of $\mathcal{A}(qp)$ and non-abelianess of $(\mathcal{A}(qp))'$.

To summarize, we have shown that with a suitable $\tau$, the representation $\pi$ in (6.2) is an isomorphism.

**Step 3:** Now consider the singlet vector $\Omega = (|+\rangle - |\rangle - \rangle) / \sqrt{2} \in (\mathbb{C}^2 \otimes \mathbb{C}^2)$. Since $\pi$ has trivial kernel, the projection $Q = \pi(|\Omega\rangle\langle\Omega|)$ is non-zero, and hence there is a vector $\chi$ in the range of this projection. Obviously, $\omega_{\chi}(\pi(Z)) = \omega_{\chi}(Q\pi(Z)Q) = \omega_{\chi}(\pi(|\Omega\rangle\langle\Omega|Z|\Omega\rangle\langle\Omega|)) = \langle\Omega|Z|\Omega\rangle\omega_{\chi}(Q) = \langle\Omega|Z|\Omega\rangle$ holds for all $Z \in B(\mathbb{C}^2 \otimes \mathbb{C}^2)$. Now we introduce the distillation maps $T, S$ of Definition 5.2. On Bob’s side $S(Y) = \sigma(Y)$ is good enough. For Alice we take $T(X) = A^*\tau(X)A$, where $A \in \mathcal{A}$ is the operator from the Reeh-Schlieder property for some small $\varepsilon > 0$. The functional distilled from this is

$$
\omega_2(X \otimes Y) = \omega_{\psi}(T(X)S(Y)) = \omega_{\psi}(A^*\tau(X)A\sigma(Y)) \\
= \omega_{\psi}(A^*\tau(X)\sigma(Y)A) = \omega_{\psi}(A^*\pi(X \otimes Y)A).
$$

Now the Reeh-Schlieder property, applied to the operator $R = \pi(Z) \in B(\mathcal{H})$, asserts that $\omega_2(Z)/\omega_2(1)$ is close to $\omega_{\chi}(\pi(Z)) = \langle\Omega|Z|\Omega\rangle$, $Z \in B(\mathbb{C}^2 \otimes \mathbb{C}^2)$. Hence, up to normalization, $\omega_2$ is close to a singlet state, and therefore is not ppt. This proves the Theorem in the case that $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras.

**Step 4:** When the $C^*$-algebras $\mathcal{A}, \mathcal{B}$ satisfy the assumptions of the Theorem, we do their weak closures, the von Neumann $\mathcal{A}''$, $\mathcal{B}''$: since $\mathcal{A} \subset \mathcal{A}''$ these algebras are both non-abelian, and by taking commutants of the inclusion $\mathcal{B} \subset \mathcal{A}'$, we get the commutation property $\mathcal{A}'' \subset \mathcal{B}'$ of the von Neumann algebras. Of course, if $\mathcal{A}''$ is dense in $\mathcal{F}$, so is the larger set $\mathcal{A}''\psi$.

Now let $T' : B(\mathbb{C}^2) \to \mathcal{A}''$ and $S' : B(\mathbb{C}^2) \to \mathcal{B}''$ be the distillation maps, whose existence we have just proved. We have to find maps $T : B(\mathbb{C}^2) \to \mathcal{A}$, and $S : B(\mathbb{C}^2) \to \mathcal{B}$ with smaller ranges, which do nearly as well. This is the content of the following

**Lemma 6.3.** Let $\mathcal{A} \subset B(\mathcal{H})$ be a $C^*$-algebra, and let $k \in \mathbb{N}$. Consider a completely positive map $T : B(\mathbb{C}^k) \to \mathcal{A}''$. Then for any finite collection of vectors $\phi_1, \ldots, \phi_n$, ...
and $\varepsilon > 0$ we can find a completely positive map $\tilde{T} : B(\mathbb{C}^k) \to A$ such that, for all $X \in B(\mathbb{C}^k)$, and all $j$, we have $||\langle T(X) - \tilde{T}(X) \rangle \phi_j || \leq \varepsilon ||X||$.

Obviously, with such approximations (for just the single vector $\phi_1 = \psi$), we get a distilled state $\omega_2$ arbitrarily close to what we could get from the distillation in the von Neumann algebra setting. This concludes the proof of the Theorem, apart from the proof of the Lemma.

Proof of the Lemma: Note that the version of the Lemma with $k = 1$ just states that the positive cone of $A$ is strongly dense in the positive cone of $A''$, which is a direct consequence of Kaplansky’s Density Theorem [57, Theorem 4.8]. We will reduce the general case to this by parameterizing all completely positive maps $T_i : B(\mathbb{C}^k) \to A''$ by their Choi-matrices

$$t_i = \sum_{\alpha\beta=1}^k T_i(\langle \alpha \rangle \langle \beta \rangle) \otimes |\alpha\rangle \langle \beta | \in A'' \otimes B(\mathbb{C}^k), \quad (6.3)$$

where “subscript $i$” equals “tilde” or “no tilde”. Note that $A'' \otimes B(\mathbb{C}^k)$ is the von Neumann algebra closure of $A \otimes B(\mathbb{C}^k)$, so via Kaplansky’s Density Theorem we obtain, for the given positive element $t \in A'' \otimes B(\mathbb{C}^k)$, and any finite collection of vectors in $\mathcal{H} \otimes \mathbb{C}^k$, a positive approximant $\tilde{t} \in A \otimes B(\mathbb{C}^k)$. As the collection vectors we take the given $\phi_i$, tensored with the basis vectors of $\mathbb{C}^k$, which implies the desired approximation for all $X$, which are matrix units $|\alpha\rangle \langle \beta |$. However, because $k$ is finite, and all norms are equivalent on a finite dimensional vector space, we can achieve a bound as required in the Lemma.

We remarked in the beginning of this section that assuming the given bipartite state to be a vector state in some representation is not a restriction of generality. Therefore there should be a version of the Theorem, which does not require a representation. Indeed, we can go to the GNS-representation of the algebra generated by $A$ and $B$ in the given state, and just restate the conditions of the Theorem as statements about expectations in the given state. This leads to the following

**Corollary 6.4.** Let $\omega$ be a state on a bipartite system $(A, B) \subset \mathcal{R}$, and suppose that

1. For some $A \in A$, and $B_1, \ldots, B_4 \in B$, $\omega(AB_1[B_2,B_3]B_4) \neq 0$, and a similar condition holds with $A$ and $B$ interchanged.

2. For all $B \in B$ and $\varepsilon > 0$ there is an $A \in A$ such that

$$\omega((A-B)^*(A-B)) \leq \varepsilon.$$

Then $\omega$ is 1-distillable.

This opens an interesting connection with the theory of maximally entangled states on bipartite systems. These are generalizations of the EPR state, and have the property that for every (projection valued) measurement on Alice’s side there is a “double” on Bob’s side such that if the two are measured together the results agree with probability one [35]. The equation which has to be satisfied by Alice’s observable $A$, and the double $B$ looks very much like condition 2 in the Corollary with $\varepsilon = 0$, except that in addition one requires $\omega((A-B)(A-B)^*) = 0$.

Before going to the context of quantum field theory, let us summarize the implications we have established for a state $\omega$ on a general bipartite system $(A, B) \subset \mathcal{R}$:
7 Distillability in Quantum Field Theory

The generic occurrence of distillable states in quantum field theory can be deduced from the fact that the Reeh-Schlieder property, and non-abelianess, are generic features of von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ describing observables localized in spacelike disjoint regions $O_A$ and $O_B$ in relativistic quantum field theory. To see this more precisely, we have to provide a brief description of the basic elements of quantum field theory in the operator algebraic framework. The reader is referred to the book by R. Haag [24] for more details and discussion.

The starting point in the operator algebraic approach to quantum field theory is that each system is described in terms of a so-called “net of local observable algebras” $\{\mathcal{A}(O)\}_{O \subset \mathbb{R}^4}$. This is a family of $C^*$-algebras indexed by the open, bounded regions $O$ in $\mathbb{R}^4$, the latter being identified with Minkowski spacetime. In other words, to each open bounded region $O$ in Minkowski spacetime one assigns a $C^*$-algebra $\mathcal{A}(O)$, and it is required that the following assumptions hold:

(I) Isotony: $O_1 \subset O_2 \Rightarrow \mathcal{A}(O_2) \subset \mathcal{A}(O_1)$,

(II) Locality: If the region $O$ is spacelike to the region $O'$, then $AA' = A'A$ for all $A \in \mathcal{A}(O)$ and all $A' \in \mathcal{A}(O')$.

The isotony assumption implies that there is a smallest $C^*$-algebra containing all the $\mathcal{A}(O)$; this will be denoted by $\mathcal{A}(\mathbb{R}^4)$. It is also assumed that there exists a unit element $1$ in $\mathcal{A}(\mathbb{R}^4)$ which is contained in all the local algebras $\mathcal{A}(O)$. Suggested by the assumptions (I) and (II), the hermitean elements in $\mathcal{A}(O)$ should be viewed as the observables of the quantum system which can be measured at times and locations within the spacetime region $O$. The locality (or microcausality) assumption then says that there are no uncertainty relations between measurements carried out at spacetime events that are spacelike with respect to each other, or that the corresponding observables are “jointly measurable”. In this way, the relativistic requirement of finite propagation speed of all effects is built into the description of a system. (See also [9] for a very recent discussion of locality aspects in quantum field theory.)

Nevertheless, there is usually in quantum field theory an abundance of states which are “non-local” in the sense that there are correlations between measurements carried out in spacelike separated regions on these states which are of quantum nature, i.e. there is entanglement over spacelike separations for such states.

Given a state $\omega$ on $\mathcal{A}(\mathbb{R}^4)$, one can associate with it a net of “local von Neumann algebras” $\{\mathcal{R}_\omega(O)\}_{O \subset \mathbb{R}^4}$ in the GNS-representation by setting

$$\mathcal{R}_\omega(O) = \pi_\omega(\mathcal{A}(O))''',$$

where $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ is the GNS representation of $\omega$ (cf. footnote in Sec. 6). On the right hand side we read the von Neumann algebra generated by the set of operators $\pi_\omega(\mathcal{A}(O)) \subset B(\mathcal{H}_\omega)$.

At this point we ought to address a point which often causes confusion. Although in the GNS-representation the state $\omega$ is given by a vector state, it need not hold that $\omega$ is a pure state for the simple reason that $\mathcal{R}_\omega(\mathbb{R}^4)$ need not coincide with

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**Figure 1.** Implications valid for any bipartite state.
\(B(\mathcal{H}_\omega)\), and in that case \(\omega\) corresponds to the vector state \((\Omega_\omega, \Omega_\omega)\) restricted to \(\mathcal{R}_\omega(\mathbb{R}^4)\). However, restrictions of vector states onto proper subalgebras of \(B(\mathcal{H}_\omega)\) are in general mixed states.

It is very convenient to distinguish certain states by properties of their GNS-representations. We call a state \(\text{covariant}\) if there exists a \(\text{(strongly continuous)}\) unitary group \(\{U_\omega(a)\}_{a \in \mathbb{R}^4}\) which in the GNS-representation acts like the translation group:

\[
U_\omega(a)A_\omega(O)U_\omega(a)^{-1} = \mathcal{R}_\omega(O + a),
\]

for all \(a \in \mathbb{R}^4\) and all bounded open regions \(O\). Among the class of covariant states there are two particularly important subclasses:

**Vacuum states:** \(\omega\) is called a vacuum state if \(U_\omega(a)\Omega_\omega = \Omega_\omega\) (the state is translation-invariant) and the joint spectrum of the selfadjoint generators \(p_\mu, \mu = 0, 1, 2, 3\), of \(U_\omega(a) = e^{i\sum_\omega a^\mu p_\mu}\) is contained in the closed forward lightcone \(V_+ = \{x = (x^\mu) \in \mathbb{R}^4 : x^0 \geq 0, (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \geq 0\}\). In other words, the energy is positive in any inertial Lorentz frame.

**Thermal equilibrium states:** \(\omega\) is called a thermal equilibrium state at inverse temperature \(\beta > 0\) (corresponding to the temperature \(T = 1/k\beta\) where \(k\) denotes Boltzmann’s constant) if there exists a time-like unit vector \(e\) \(\in \mathbb{R}^4\), playing the role of a distinguished time axis, so that \(U_\omega(t \cdot e)\Omega_\omega = \Omega_\omega\) and

\[
\langle \Omega_\omega | A e^{-\beta H_B} B | \Omega_\omega \rangle = \langle \Omega_\omega | B A | \Omega_\omega \rangle
\]

holds for \((\text{a suitable dense subset of})\) \(\mathbb{R}^4\), where the selfadjoint operator \(H_\beta\) is the generator of the time-translations in the time-direction determined by \(e\), i.e. \(U_\omega(t \cdot e) = e^{iH_\beta t}, t \in \mathbb{R}\).

We should note that (7.4) is a slightly sloppy way of expressing the condition of thermal equilibrium at inverse temperature \(\beta\) which in a mathematically more precise form would be given in terms of the so-called “KMS boundary condition” that refers to analyticity conditions of the functions \(t \mapsto \langle \Omega_\omega | A U_\omega(t \cdot e) B | \Omega_\omega \rangle\) (see any of the references [5, 19, 24, 25, 52] for a precise statement of the KMS boundary condition). That way of characterizing thermal equilibrium states has the advantage of circumventing the difficulty that \(e^{-\beta H_\beta}\) will usually be unbounded since the “thermal Hamiltonian” \(H_\beta\) in the GNS-representation of a thermal state has a symmetric spectrum (much in contrast to the Hamiltonians in a vacuum-state representation). We will not enter into further details here and refer the reader to [5, 19, 52] for discussion of these matters. There is, however, a point which is worth focussing attention on. The condition of thermal equilibrium makes reference to a single direction of time, and it is known that if a state is a thermal equilibrium state with respect to a certain time axis \(e\), then in general it won’t be a thermal equilibrium state (at any inverse temperature) with respect to another time-direction \(e'\) [41, 43]. Nevertheless, it has been shown by J. Bros and D. Buchholz that in a relativistic quantum field theory, the correlation functions of a thermal equilibrium state \(\omega\) (with respect to an arbitrarily given time-direction) possess, under very general conditions, a certain analyticity property which is Lorentz-covariant, and stronger than the thermal equilibrium condition with respect to the given time-direction itself [6]. This analyticity condition is called “relativistic KMS-condition”.

Let us state the relativistic spectrum condition of [6] in precise terms (mainly for the sake of completeness; we won’t make use of it in the following):

A state \(\omega\) on \(\mathfrak{A}(\mathbb{R}^4)\) is said to fulfill the relativistic KMS condition at inverse temperature \(\beta > 0\) if \(\omega\) is covariant and if there exists a timelike vector \(e\) in \(V_+\) (the open interior of \(\overline{V}_+\)) having Minkowskian length, so that for each pair of operators \(A, B \in \pi_\omega(\mathfrak{A}(\mathbb{R}^4))\) there is a function \(F = F_{AB}\) which is analytic in the domain
We will give an indication of the nature of those general conditions leading to the relativistic KMS-condition since that gives us opportunity of also introducing the lacking bits of terminology for eventually formulating our result.

Let us start with a vacuum state $\omega = \omega_{\text{vac}}$, and denote the corresponding GNS-representation by $(\pi_{\text{vac}}, \mathcal{H}_{\text{vac}}, \Omega_{\text{vac}})$ and the local von Neumann algebras in the vacuum representation by $\mathcal{R}_{\text{vac}}(O)$. When one deals with quantum fields $\phi$ of the Wightman type, then $\mathcal{R}_{\text{vac}}(O)$ is generated by quantum field operators $\phi(f)$ smeared with test-functions $f$ having support in $O$. More precisely, $\mathcal{R}_{\text{vac}}(O) = \{e^{i\phi(f)}, \, \text{supp} f \subset O\}''$. This is the typical way how local algebras of observables arise in quantum field theory. We note that in this case, the net $\{\mathcal{R}_{\text{vac}}(O)\}_{O \subset \mathbb{R}^4}$ of von Neumann algebras fulfills the condition of additivity which requires that $\mathcal{R}_{\text{vac}}(O)$ is contained in $\{\mathcal{R}_{\text{vac}}(O_n)\}_{n \in \mathbb{N}}$ whenever the sequence of regions $\{O_n\}_{n \in \mathbb{N}}$ covers $O$, i.e. $O \subset \bigcup_{n} O_n$. The additivity requirement can therefore be taken for granted in quantum field theory.

Now it is clear that the vacuum state $\omega_{\text{vac}}$ (like any state) determines a further class of states $\omega'$ on $\mathfrak{A}(\mathbb{R}^4)$, namely those states which arise via density matrices in its GNS-representation:

$$\omega'(A) = \text{Tr}(\rho' \pi_{\text{vac}}(A)) \quad \forall A \in \mathfrak{A}(\mathbb{R}^4)$$

for some density matrix $\rho'$ on $\mathcal{H}_{\text{vac}}$. These states are called normal states (in the vacuum representation, in this case), and they correspond in an obvious manner to normal states on $\mathcal{R}_{\text{vac}}(\mathbb{R}^4)$. Such normal states in the vacuum representation may be regarded as states with a finite number of particles.

For quantum systems with a finite number of degrees of freedom one would write a thermal equilibrium state $\omega_\beta$ as a Gibbs state

$$\omega_\beta(A) = \text{Tr}(e^{-\beta H_{\text{vac}}} \pi_{\text{vac}}(A)),$$

but for a system situated in the unboundedly extended Minkowski spacetime, $e^{-\beta H_{\text{vac}}}$ won't be a density matrix since the spectrum of the vacuum Hamiltonian $H_{\text{vac}}$ will usually be continuous. So a thermal equilibrium state is not a normal state in the vacuum representation. What one can however do is to approximate $\omega_\beta$ by a sequence of “local Gibbs states”

$$\omega_\beta^{(N)}(A) = \text{Tr}(e^{-\beta H_{\text{vac}}^{(N)} \pi_{\text{vac}}(A)}, \quad A \in \mathfrak{A}(O_N),$$

which are restricted to bounded spacetime regions $O_N$ with suitable local Hamiltonians $H_{\text{vac}}^{(N)}$. Now one lets $O_N \nearrow \mathbb{R}^4$ as $N \nearrow \infty$, and under fairly general assumptions on the behaviour of the theory in the vacuum representation that are expected to hold for all physically relevant quantum fields, it can be shown that in the limit one gets a thermal equilibrium state $\omega_\beta$ (this is a long known result due to Haag, Hugenholtz and Winnink [25]) and that, moreover, remnants of the spectrum condition in the vacuum representation survive the limit to the effect that the limiting state $\omega_\beta$ satisfies the relativistic KMS-condition [6].

The relativistic KMS condition has proved useful in establishing the Reeh-Schlieder theorem for thermal equilibrium states. We shall, for the sake of completeness, quote the relevant results in the form of a theorem.

**Theorem 7.1.** [49, 36, 15] Let $\omega$ be either a vacuum state on $\mathfrak{A}(\mathbb{R}^4)$, or a thermal equilibrium state on $\mathfrak{A}(\mathbb{R}^4)$ satisfying the relativistic KMS-condition. Assume also that the net $\{\mathcal{R}_{\omega}(O)\}_{O \subset \mathbb{R}^4}$ fulfills additivity and that $\mathcal{H}_{\omega}$ is separable. Then it holds that:
(a) The set $\mathcal{R}_\omega(O)\Omega_\omega$ is dense in $\mathcal{H}_\omega$, i.e., the Reeh-Schlieder property holds for $\omega = \langle \Omega_\omega \mid \cdot \rangle \Omega_\omega$ with respect to $\mathcal{R}_\omega(O)$, whenever $O$ is an open region\(^3\).

(b) Moreover, there is a dense set of vectors $\chi \in \mathcal{H}_\omega$ so that, for each such $\chi$, $\mathcal{R}_\omega(O)\chi$ is dense in $\mathcal{H}_\omega$ for all open regions $O$.

The proof of (a) in the vacuum case has been given in [49]. For the case of thermal equilibrium states, a proof of this property was only recently established by C.D. Jäkel in [36]. Statement (b) is implied by (a), as has been shown in [15]. We should also like to point out that the Schlieder property mentioned in the proof of Thm. 6.2 holds for the state $\omega$, cf. [51, 37].

These quoted results in combination with Thm. 6.2 now yield:

**Theorem 7.2.** Let $A = \mathcal{R}_\omega(O_A)$ and $B = \mathcal{R}_\omega(O_B)$ be a pair of local von Neumann algebras of a quantum field theory\(^4\) in the representation of a state $\omega$ which is either a vacuum state, or a thermal equilibrium state satisfying the relativistic KMS-condition (with $\mathcal{H}_\omega$ separable).

If the open regions $O_A$ and $O_B$ are spacelike separated by a non-zero spacelike distance, then the state $\omega = \langle \Omega_\omega \mid \cdot \rangle \Omega_\omega$ is 1-distillable on the bipartite system $(A, B)$. Moreover, there is a dense set $X \subseteq \mathcal{H}_\omega$ so that the vector states $\langle \chi \mid \cdot \rangle$ are 1-distillable on $(A, B)$ for all $\chi \in X$, $\|\chi\| = 1$. Also, $X$ may be chosen independently of $O_A$ and $O_B$. Consequently, the set of vector states on $\mathcal{R} = (A \cup B)^\omega$ which are 1-distillable on $(A, B)$ is strongly dense in the set of all vector states.

**Remarks.** (i) Actually, the statement of Thm. 7.2 shows distillability not only for a dense set of vector states on $\mathcal{R}$ but even for a dense set of normal states (i.e., density matrix states) on $\mathcal{R}$. To see this note that, owing to the assumption that the spacetime regions $O_A$ and $O_B$ are spacelike separated by a finite distance, there is for $\mathcal{R}$ a separating vector in $\mathcal{H}_\omega$, since $\Omega_\omega$ has just this property: There is an open region $O$ lying spacelike to $O_A$ and $O_B$. By the Reeh-Schlieder property, $\mathcal{R}_\omega(O)\Omega_\omega$ is dense in $\mathcal{H}_\omega$, and hence, $\Omega_\omega$ is a separating vector for $\mathcal{R} \subset \mathcal{R}_\omega(O)'$. This implies by Thm. 7.3.8 of [31] that, whenever $\tilde{\omega}$ is a density matrix state on $\mathcal{R}$, there is a unit vector $\chi \in \mathcal{H}_\omega$ so that $\tilde{\omega} = \omega_\chi|\mathcal{R}$. In other words, under the given assumptions every normal state on $\mathcal{R}$ coincides with the restriction of a suitable vector state.

(ii) It should also be noted that, under very general conditions, vacuum representations and also thermal equilibrium representations of quantum field theories fulfill the so-called “split property” (an independence property, cf. [24, 56, 64]), which implies (under the conditions of Thm. 7.2) that there exists an abundance of normal states which are separable and even ppt on $(A, B)$ for bounded, spacelike separated regions $O_A$ and $O_B$.

(iii) The second part of the statement, asserting that in the GNS representation of $\omega$ there is a dense set of normal states which are distillable over causally separated regions, is closely related to a result by Clifton and Halvorson [26] who show (for a vacuum state $\omega$; see [38] for a generalization of the argument to states satisfying the relativistic KMS condition) that there is a dense set of normal states in the GNS representation of $\omega$ which are Bell-correlated over spacelike separated regions. However, they cannot deduce that Bell-correlations over spacelike separated regions are present for the state $\omega$ itself (or for a specific class of states, like those having the Reeh-Schlieder property, which can often be constructed out of other states). It is here where our result provides some additional information.

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\(^3\)Here and in the following, we always assume that the open set $O$ is non-void.

\(^4\)The quantum field theory is supposed to be non-trivial in the sense that its local observable algebras are non-abelian, and this is also to hold for the local von Neumann algebras in the representations considered. This is the generic case in quantum field theory and holds for all investigated quantum field models.
(iv) In an interesting recent paper, Reznik, Retzker and Silman [50] propose a different method towards qualifying the degree of entanglement of a (free) quantum field vacuum state over spacelike separated regions. Their idea is to couple each local algebra \( A = \mathcal{R}(O_A) \) and \( B = \mathcal{R}(O_B) \) to an “external” algebra \( B(\mathbb{C}^2) \). They introduce a time-dependent coupling between the quantum field degrees of freedom in \( O_A \) and \( O_B \) and the corresponding “external” algebras, which are hence supposed to represent detection devices for quantum field excitations. It is then shown in [50] that this dynamical coupling, turned on for a finite amount of time during which the quantum field degrees of freedom remain causally separated, yields an entangled partial state for the pair of detector systems from an initially uncorrelated state coupled to the quantum field vacuum. Further local filtering operations are then used to distill that partial detector state to an approximate singlet state. It should, however, be remarked that the authors of [50] do not demonstrate the existence of Bell-correlations in the vacuum state over arbitrarily spacelike separated and arbitrarily small spacetime regions in the sense of [39, 40, 54, 26], i.e. in the sense of proving a violation of the CHSH inequalities by the quantum field observables themselves. Nevertheless, the approach of [50], while apparently less general than the one presented here, has some interesting aspects since potentially it may allow a more quantitative description of distillability in quantum field systems.

8 Distillability Beyond Spacetime Horizons

It is worth pointing out that in Thm. 7.2 the spacelike separated regions \( O_A \) and \( O_B \) are the localization regions of the operations that Alice and Bob can apply to a given, shared state. The spacetime pattern of any form of classical communication between Alice and Bob that might be necessary to “post-select” a sub-ensemble of higher entanglement (i.e. to normalize the state \( \omega^{[T]} \)) from a given shared ensemble (on which local operations have been applied) is not represented in the criterion of distillability. Put differently, the distillability criterion merely tests if there are sufficiently “non-classical” long-range correlations in the shared state \( \omega \) which can be provoked by local operations. It does not require that the post-selection is actually carried out via classical communication realizable between Alice and Bob in spacetime. Such a stronger demand would have to make reference to the causal structure of the spacetime into which Alice and Bob are placed.

We will illustrate this in the present section, and we begin by noting that Thm. 7.2 can actually be generalized to curved spacetime. Thus, we assume that \( M \) is a four-dimensional smooth spacetime manifold, endowed with a Lorentzian metric \( g \). To avoid any causal pathologies, we will henceforth assume that \((M, g)\) is globally hyperbolic (cf. [60]). In this case, it is possible to construct nets of local observable \( (C^*) \) algebras \( \{\mathfrak{A}(O)\}_{O \subset M} \) for quantized free fields, like the scalar Klein-Gordon, Dirac and free electromagnetic fields [13, 61]. Let us focus, for simplicity, on the free quantized Klein-Gordon field on \((M, g)\), and denote by \( \{\mathfrak{A}(O)\}_{O \subset M} \) the corresponding net of local observable algebras, fulfilling the conditions of isotony and locality, which can be naturally formulated also in curved spacetimes.

Let us briefly indicate how the local \( C^* \)-algebras \( \mathfrak{A}(O) \) are constructed in the case of the free scalar Klein-Gordon field; for full details, see [13, 33, 61]. The Klein-Gordon operator on \((M, g)\) is \((\nabla^\alpha\nabla^\mu + m^2)\) where \( \nabla \) denotes the covariant derivative of the spacetime metric \( g \) and \( m \geq 0 \) is some constant. Owing to global hyperbolicity of the underlying spacetime \((M, g)\), the Klein-Gordon operator possesses uniquely determined advanced and retarded fundamental solutions (Green’s functions), \( G_+ \) and \( G_- \), which can be viewed as distributions on \( C_0^\infty(M \times M, \mathbb{R}) \). Their difference \( G = G_+ - G_- \) is called the causal propagator. One can construct a \( * \)-algebra \( \mathfrak{A}(M) \) generated by symbols \( W(f) \), \( f \in C_0^\infty(M, \mathbb{R}) \), fulfilling the relations \( W(f_1)W(f_2) = \)
This algebra possesses a unit element and admits a unique $C^*$-norm. We identify $\mathfrak{A}(M)$ with the $C^*$-algebra generated by all the $W(f)$. Then $\mathfrak{A}(O)$ is defined as the $C^*$-subalgebra generated by all $W(f)$ where $f \in C_0^\infty(O, \mathbb{R})$.

Now, unless $(M, g)$ possesses time-symmetries, there are no obvious criteria to single out vacuum states or thermal equilibrium states on $\mathfrak{A}(M)$. Nevertheless, there is a class of preferred states on $\mathfrak{A}(M)$ which serve, for most purposes, as replacements for vacua or thermal equilibrium states. The states in this class are called quasifree Hadamard states. Given such a state, $\omega$, one has $\pi_\omega(W(f)) = e^{i\Phi_\omega(f)}$ in the GNS representation of $\omega$ with selfadjoint quantum field operators $\Phi_\omega(f)$ in $\mathfrak{H}_\omega$ depending linearly on $f$ and fulfilling $\Phi_\omega((\nabla^\mu \nabla_\mu + m^2)f) = 0$ and the canonical commutation relations in the form $[\Phi_\omega(f_1), \Phi_\omega(f_2)] = ig(f_1, f_2)1$. The Hadamard condition on the two-point distribution $\langle \Omega_\omega | \Phi_\omega(x)\Phi_\omega(y) | \Omega_\omega \rangle$ of $\omega$ (symbolically written as integral kernel with $x, y \in M$) and demands, essentially, that this has a leading singularity of the type “$1/(\text{squared geodesic distance between } x \text{ and } y)$”.

Quasifree Hadamard states are a very well investigated class of the free scalar field in curved spacetime. The reasons why they are considered as replacements for vacuum states or thermal equilibrium states are discussed, e.g., in the refs. [22, 33, 61, 21].

The Hadamard condition on the two-point distribution of a (quasifree) state $\omega$ can equivalently be expressed by requiring that the $C^\infty$-wavefront set of the Hilbert-space valued distribution $C_0^\infty(M) \ni f \mapsto \Phi_\omega(f)\Omega_\omega$ is confined to the set of future-pointing causal covectors on $M$ (cf. [53] and also refs. cited there). If $\omega$ satisfies this latter condition, one says that it fulfills the microlocal spectrum condition ($\mu SC$). If the latter condition holds even with the analytic wavefront set in place of the $C^\infty$-wavefront set, then one says that $\omega$ fulfills the analytic microlocal spectrum condition ($a\mu SC$) [53]. (For $a\mu SC$, it is also required that the spacetime $(M, g)$ be real analytic.) While the definitions of $C^\infty$-wavefront set and analytic wavefront set are a bit involved so that we do not present them here and refer to [53] and refs. given there for full details, we put on record that for any quasifree state $\omega$ on the obervable algebra $\mathfrak{A}(M)$ of the scalar Klein-Gordon field one has

$$\omega \text{ fulfills } a\mu SC \Rightarrow \omega \text{ fulfills } \mu SC \Leftrightarrow \omega \text{ Hadamard.}$$

Moreover, on a stationary, real analytic, globally hyperbolic spacetime $(M, g)$, the quasifree ground states or quasifree thermal equilibrium states on $\mathfrak{A}(M)$, which are known to exist under a wide range of conditions, fulfill $a\mu SC$ [53]. It is also known that there exist very many quasifree Hadamard states on $\mathfrak{A}(M)$ for any globally hyperbolic spacetime $(M, g)$.

Several properties of the local von Neumann algebras $\mathcal{R}_\omega(O)$ are known for quasifree Hadamard states $\omega$, and we collect those of interest for the present discussion in the following Proposition.

**Proposition 8.1.** Let $(M, g)$ be a globally hyperbolic spacetime, and let $\omega$ be a quasifree Hadamard state on $\mathfrak{A}(M)$, the algebra of observables of the Klein-Gordon field on $(M, g)$. Write $\mathcal{R}_\omega(O) = \pi_\omega(\mathfrak{A}(O))''$, $O \subset M$, for the local von Neumann algebras in the GNS representation of $\omega$. Then the following statements hold.

(a) $\mathcal{R}_\omega(O)$ is non-abelian whenever $O$ is open.

(b) There is a dense set of vectors $\chi \in \mathfrak{H}_\omega$ so that, for each such $\chi$, $\mathcal{R}_\omega(O)\chi$ is dense in $\mathfrak{H}_\omega$, for all open $O \subset M$.

(c) If $(M, g)$ is real analytic and if $\omega$ satisfies the $a\mu SC$, then the Reeh-Schlieder property holds for $\omega = \langle \Omega_\omega |, | \Omega_\omega \rangle$ with respect to $\mathcal{R}_\omega(O)$, whenever $O \subset M$ is open.
Proof. Statement (a) is clear from the fact that the canonical commutation relations hold for the field operators $\Phi_\omega(f)$. Statement (c) is a direct consequence of Thm. 5.4 in [53]. For statement (b), one can argue as follows. For a globally hyperbolic $(M, g)$, there is a countable neighbourhood base $\{O_n\}_{n \in \mathbb{N}}$ for the topology of $M$ where each $O_n$ has a special shape (called “regular diamond” in [58]; we assume here also that each $O_n$ has a non-void causal complement), which allows the conclusion that each $\mathcal{R}_\omega(O_n)$ is a type III$_1$ factor (cf. Thm. 3.6 in [58]). Since $\mathcal{H}_\omega$ is separable (cf. again Thm. 3.6 in [58]), one can make use of Cor. 2 and Prop. 3 of [15] which leads to the conclusion that there is a dense set $X \subset \mathcal{H}_\omega$ so that each $\chi \in X$ is cyclic for all $\mathcal{R}_\omega(O_n)$, $n \in \mathbb{N}$. Since $\{O_n\}_{n \in \mathbb{N}}$ is a neighbourhood base for the topology of $M$, each open set $O \subset M$ has $O_n \subset O$ for some $n$, and hence each $\chi \in X$ is cyclic for $\mathcal{R}_\omega(O)$ whenever $O$ is an open subset of $M$. \hfill \Box

As in the previous section, we can conclude distillability from the just asserted Reeh-Schlieder properties.

**Theorem 8.2.** Let $(M, g)$ be globally hyperbolic spacetime, and let $\omega$ be a quasifree state on the observable algebra $\mathfrak{A}(M)$ of the quantized scalar Klein-Gordon field on $(M, g)$. Let $O_A$ and $O_B$ be two open subsets of $M$ whose closures are causally separated (i.e., they cannot be connected by any causal curve), and let $A = \mathcal{R}_\omega(O_A)$, $B = \mathcal{R}_\omega(O_B)$. The following statements hold:

(a) If $(M, g)$ is real analytic and $\omega$ satisfies the $a\mu SC$, then the state $\omega = \langle \Omega_\omega | \cdot | \Omega_\omega \rangle$ is 1-distillable on $(A, B)$.

(b) There is a dense set $X \subset \mathcal{H}_\omega$ so that the vector states $\langle \chi | \cdot | \chi \rangle$ are 1-distillable on $(A, B)$ for all $\chi \in X$, $||\chi|| = 1$. Also, $X$ may be chosen independently of $O_A$ and $O_B$. Consequently, the set of normal states on $\mathcal{R} = (A \cup B)'''$ which are 1-distillable on $(A, B)$ is strongly dense in the set of normal states on $\mathcal{R}$.

The proof of this Theorem is a straightforward combination of the statements of Prop. 8.1 with Thm. 6.2. For part (b), we have already made use of the observation of Remark (i) following Thm. 7.2.

Again, as noted in Remark (iii) following Thm. 7.2, part (b) of the last Theorem is related to a similar statement by Clifton and Halvorson [26] which refers to the existence of a dense set of normal states which are Bell-correlated over causally separated spacetime regions. Also here, our comments of Remark (iii) apply.

In Thm. 8.2 the localization regions $O_A$ and $O_B$ of the system parts controlled by Alice and Bob, respectively, could also be separated by spacetime horizons. Let us give a concrete example and take $(M, g)$ to be Schwarzschild-Kruskal spacetime, i.e. the maximal analytic extension of Schwarzschild spacetime. This is a globally hyperbolic spacetime which is real analytic, and it has two subregions, denoted by I and II, that model the interior and exterior spacetime parts of an eternal black hole, respectively (see Sec. 6.4 in [60]). These two regions are separated by the black hole horizon, so that no classical signal can be sent from the interior region I to an observer situated in the exterior region II. The situation is depicted in Figure 2 below.

For the quantized scalar Klein-Gordon field on the Schwarzschild-Kruskal spacetime, there is a preferred quasifree state, the so-called Hartle-Hawking state, which is in a sense the best candidate for the physical “vacuum” state on this spacetime (cf. [32, 61]). It is generally believed that this state fulfills the $a\mu SC$ on all of $M$. (The arguments of [53] can be used to show that $a\mu SC$ is fulfilled in region II and its “opposite” region, which makes it plausible that this holds actually on all of $M$, although there is as yet no complete proof.) Anticipating that this is the case, we can choose the localization region $O_A$ inside the interior region I and
$O_B$ in the exterior region $\Pi$ (cf. Fig. 2). Then, by our last theorem, we find that the Hartle-Hawking state $\omega$ of the quantized Klein-Gordon field is distillable on the bipartite system $(A, B)$ with $A = \mathcal{R}_\omega(O_A)$ and $B = \mathcal{R}_\omega(O_B)$. Furthermore, there is a dense set of normal states in the GNS-representation of the Hartle-Hawking state with respect to which this distillability holds. (At any rate, since the existence of quasifree Hadamard states for the Klein-Gordon field on the Schwarzschild-Kruskal spacetime is guaranteed, part (b) of Thm. 8.2 always ensures the existence of an abundance of states which are distillable on $(A, B)$).

A similar example for regions $O_A$ and $O_B$ separated by a spacetime horizon (an event horizon) can be given for de Sitter spacetime; the de Sitter “vacuum state” for the quantized Klein-Gordon field actually has all the required properties for the distillability statement of Thm. 8.2, cf. [7].

This shows that distillability of quantum field states beyond spacetime horizons (event horizons) can be expected quite generally.

A similar situation occurs also in the standard Friedmann-Robertson-Walker cosmological models with an initial spacetime singularity. In this scenario, spacetime regions sufficiently far apart from each other are causally separated for a finite amount of time by their cosmological horizons [60]. However, also in this situation, a quantum field state fulfilling the $a\mu SC$ on any Friedmann-Robertson-Walker spacetime would be distillable on a bipartite system $(A, B)$ of the form $A = \mathcal{R}_\omega(O_A)$ and $B = \mathcal{R}_\omega(O_B)$ for spacetime regions $O_A$ and $O_B$ separated by a cosmological horizon. Again, there is at any rate a large class of states where such a distillability is found. In passing we should like to note that quantum field correlations, whose appearance is precisely expressed by the Reeh-Schlieder property, have already been considered in connection with the question if (potentially, very strong) quantum field fluctuations in the early universe could account for the structure of its later development [62].

**Figure 2.** This figure shows the interior region $\text{I}$ and exterior region $\text{II}$ of the conformal diagram of Schwarzschild-Kruskal spacetime, which is a model of a static black hole spacetime (at large times after collapse of a star to a black hole). The event horizon, represented by the double lines, separates region $\text{I}$ from region $\text{II}$ such that no signal can be sent from $\text{I}$ to $\text{II}$ across the horizon. A quantum field state which satisfies the Reeh-Schlieder property (as e.g. implied by the analytic microlocal spectrum condition) is distillable over the shaded spacetime regions $O_A$ (wherein ‘Alice’ conducts her experiments on the state) and $O_B$ (wherein ‘Bob’ conducts his experiments on the state). The dashed line represents the black hole singularity.
9 Discussion: Classical Communication in Spacetime?

Distillation was introduced as the process of taking imperfectly entangled systems, and turning them into a useful entanglement resource. Any such process requires classical communication, even though for realizing 1-distillability only a single step of post-selection is required. It is suggestive to describe the classical communication steps also as causal communication processes in spacetime.

This immediately raises a problem: if the laboratories of Alice and Bob are separated by an event horizon, they will never be able to exchange the required signals, so in this case the above results of the previous section might appear to be totally useless. Several comments to this idea are in order.

1. Event horizons are global features of a spacetime. Hence if we are interested in what can be gained from the local state between Alice and Bob, the future development of the universe remains yet unknown. Since the gravitational background is taken as “external” at this level of the theory, the adopted framework, using only spacetime structure up until the time the quantum laboratories close, never allows a decision on whether or not postselection will be causally possible.

2. The attempt to include the distillation process in the spacetime description meets the following characteristic difficulty: It becomes very hard to distinguish between classical and quantum communication. Obviously, a quantum operation disturbs the quantum field in its future light cone, but it is very hard to assert that this disturbance leaves alone the spacetime region where the negotiations for postselection take place. In other words: we cannot distinguish LOCC operations from exchanging quantum particles, and this would completely trivialize the distinction between distillable and separable states.

3. This difficulty is akin to the problem of realizing statistical experiments in spacetime. On the one hand, the statistical interpretation of quantum mechanics (and hence of quantum field theory) is based on independent repetitions of “the same” experiment. But in a dynamic space time it is clear that strictly speaking no repetition is possible, and the above disturbance argument casts additional doubt on the possibility of independent repetitions. Carrying this argument still further, into the domain of quantum cosmology, it has been debated [20] whether quantum theory may ever apply to the universe as a whole. Whether this can be resolved by showing that for typical (small) experimental setups statistical behavior can be shown to hold with probability 1 in any ensemble of universes admitted by the theory is a question far beyond the present paper.

To summarize: we have adopted here the most “local” approach to distillability, where it is strictly taken as a property of a state $\omega$ of a general bipartite quantum system $(A,B) \subset \mathcal{R}$, independent of the “surroundings” of that quantum system and the global structure of the spacetime into which it is placed. Still, it would be quite interesting to see if distillability criteria taking into account the realizability of distillation protocols in spacetime can be developed in a satisfactory manner (e.g. reconcilable with ideas like general covariance [8], and with the difficulties related to independence of measurements alluded to above). We should finally note that the difference between these two points of view is insignificant for present day laboratory physics where it can always be safely assumed that spacetime is Minkowskian.

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