On the vanishing discount approximation for compactly supported perturbations of periodic Hamiltonians: the 1d case

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ABSTRACT

We study the asymptotic behavior of the viscosity solutions $u_k^G$ of the Hamilton-Jacobi (HJ) equation

$$\lambda u(x) + G(x, u') = c(G) \quad \text{in } \mathbb{R}$$

as the positive discount factor $\lambda$ tends to 0, where $G(x, p) := H(x, p) - V(x)$ is the perturbation of a Hamiltonian $H \in C(\mathbb{R} \times \mathbb{R})$, $\mathbb{Z}$–periodic in the space variable and convex and coercive in the momentum, by a compactly supported potential $V \in C_c(\mathbb{R})$. The constant $c(G)$ appearing above is defined as the infimum of values $a \in \mathbb{R}$ for which the HJ equation $G(x, u') = a$ in $\mathbb{R}$ admits bounded viscosity subsolutions. We prove that the functions $u_k^G$ locally uniformly converge, for $\lambda \to 0^+$, to a specific solution $u_0^G$ of the critical equation $G(x, u') = c(G)$ in $\mathbb{R}$.

We identify $u_0^G$ in terms of projected Mather measures for $G$ and of the limit $u_0^H$ to the unperturbed periodic problem. Our work also includes a qualitative analysis of the critical equation with a weak KAM theoretic flavor.

1. Introduction

In this paper we study the asymptotic behavior of the viscosity solutions $u_k^G$ of discounted Hamilton-Jacobi (HJ) equations of the form

$$\lambda u(x) + G(x, d_x u) = c(G) \quad \text{in } M$$

(1)

as the discount factor $\lambda$ tends to 0. Here we consider the case where $M$ is the 1-dimensional Euclidean space $\mathbb{R}$, and $G := H - V$ is the perturbation of a $\mathbb{Z}$–periodic (in the space variable) Hamiltonian $H \in C(\mathbb{R} \times \mathbb{R})$ via a compactly supported potential $V \in C_c(\mathbb{R})$. The Hamiltonian $H$ is furthermore assumed convex and coercive in the momentum variable $d_x$. The constant $c(G)$ appearing above is defined as the infimum of values $a \in \mathbb{R}$ for which the HJ equation $G(x, d_x u) = a$ in $M$ admits bounded viscosity subsolutions.

It turns out that the functions $u_k^\lambda$ are equi–Lipschitz and locally equi–bounded in $\mathbb{R}$, hence, by the Ascoli-Arzelà Theorem and the stability of the notion of viscosity solution, they converge, along subsequences as $\lambda$ goes to 0, to viscosity solutions of the critical equation...
\[ G(x, d_x u) = c(G) \quad \text{in } M. \] (2)

Since the above critical equation has infinitely many solutions, even up to additive constants in general, it is not clear at this level that the limit of the \( u_G^\lambda \) along different subsequences is the same. The main theorem we prove is the following.

**Theorem 1.** Let \( M := \mathbb{R} \) and let \( G(x, p) = H(x, p) - V(x) \) be as above. For every \( \lambda > 0 \), let us denote by \( u_G^\lambda \) the unique bounded viscosity solution of the discounted Hamilton-Jacobi Eq. (1). Then the whole family \( \{ u_G^\lambda \}_{\lambda > 0} \) converges, locally uniformly in \( \mathbb{R} \) as \( \lambda \to 0^+ \), to a distinguished solution \( u_0^G \) of the critical Eq. (2).

Since solutions to Eqs. (1) and (2) are equi-Lipschitz, the function \( H \) can be possibly modified for \( p \) outside a large ball without affecting the analysis. We shall therefore assume, without any loss in generality, the Hamiltonians to be superlinear in \( p \). In that case, we can introduce the conjugated Lagrangians via Fenchel's formula and make use of tools issued from weak KAM Theory, suitably adapted to the case at issue, to characterize the limit function \( u_0^G \). Our output can be summarized as follows. Let us denote by \( c_f(G) \) the free critical value, defined as the minimum value \( a \in \mathbb{R} \) for which the Hamilton-Jacobi equation \( G(x, d_x u) = a \) in \( M \) admits possibly unbounded viscosity subsolutions, cf. [1]. Let \( \Xi_{b-}(G) \) be the family of bounded subsolutions \( v : \mathbb{R} \to \mathbb{R} \) of the critical Eq. (2) such that

\[
\int_{\mathbb{R} \times \mathbb{R}} v(y) \, d\hat{\mu}(y, q) \, 1_0(c(G) - c_f(G)) \quad \text{for every } \mu \in \hat{\mathcal{M}}(G),
\] (3)

where \( \hat{\mathcal{M}}(G) \) denotes the set of Mather measures for \( G \) and \( 1_0 \) the indicator function of the set \( \{0\} \) in the sense of convex analysis, i.e. \( 1_0(t) = 0 \) if \( t = 0 \) and \( 1_0(t) = +\infty \) otherwise. The constraint (3) is meant to be empty whenever \( c(G) > c_f(G) \).

In what follows, we will denote by \( u_H^0 \) the uniform limit of the functions \( u_G^\lambda \) on \( \mathbb{R} \) as \( \lambda \to 0^+ \), which is well-defined according to the results in [2], and by

\[
p_H^-(x) := \min\{ p \in \mathbb{R} : H(x, p) \leq c(H) \},
\]

\[
p_H^+(x) := \max\{ p \in \mathbb{R} : H(x, p) \leq c(H) \}.
\]

We have the following characterization.

**Theorem 2.** The limit \( u_0^G \) of the discounted solutions \( u_G^\lambda \) is characterized as follows.

I. If \( c(G) > c(H) \), then \( c(G) = c_f(G) \) and

\[ u_G^0(x) := \sup_{v \in \Xi_{b-}(G)} v(x), \quad x \in \mathbb{R}. \]

Furthermore, \( u_G^0 \) is coercive on \( \mathbb{R} \).

II. If \( c(G) = c(H) \), then:

A. If \( \int_0^1 p_H^+(x)dx = 0 \), then for every \( x \in \mathbb{R} \)

\[ u_G^0(x) := \sup\{ v(x) : v \in \Xi_{b-}(G), v \leq u_H^0 \quad \text{in} \quad (-\infty, \gamma_v) \}, \]

where \( \gamma_v := \min(\text{supp}(V)) \);
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carried out in [2], where Theorem 1 is proved in the case

where \( \bar{y}_V := \max(\text{supp}(V)) \).

In either case, \( u^0_G \) is bounded on \( \mathbb{R} \).

III. if \( c(G) = c(H) = c_f(H) \), then \( c(G) = c_f(G) \) and

Furthermore, \( u^0_G \) is bounded on \( \mathbb{R} \).

The model example we have in mind is \( H(x, p) := |\theta + p| - U(x) \), where \( \theta \in \mathbb{R} \) and \( U \) is a 1-periodic function. Here, \( c_f(H) = -\min_{\mathbb{R}} U \), \( c_f(G) = -\min_{\mathbb{R}} (U + V) \) and clearly \( c_f(G) \geq c_f(H) \) (although this is a general fact, see Proposition 2.4). Furthermore, there exists \( \theta^- \leq 0 \leq \theta^+ \) such that

When \( c_f(G) = c_f(H) \), then \( \theta^- = \theta^+ = 0 \) and only cases (II) and (III) above occur. This happens, for instance, when the potential \( V \) is nonnegative. When \( c_f(G) > c_f(H) \), then \( (\theta^-, \theta^+) \) is nonempty and case (I) occurs when \( \theta \in (\theta^-, \theta^+) \), while case (II) occurs when \( \theta \in \mathbb{R} \setminus (\theta^-, \theta^+) \).

The interest to analyze the asymptotic behavior as \( \lambda \to 0^+ \) of solutions to Eq. (1) has roots, as it is well-known, in homogenization theory, see [3] and also [4, 5], as well as in ergodic optimal control, see for example [6–9].

The present work can be seen as an extension to a noncompact setting of the study carried out in [2], where Theorem 1 is proved in the case \( G = H \) and \( M := \mathbb{R}^d \) for any \( d \geq 1 \). In this instance, the constant \( c(H) \), as defined above, agrees with the one considered in [2], known as ergodic constant in the PDE literature, or Mane critical value in the context of weak KAM Theory. It can be characterized by the property of being the unique constant \( a \in \mathbb{R} \) for which the equation \( H(x, d_u) = a \) admits \( \mathbb{Z}^d \)-periodic viscosity solutions on \( \mathbb{R}^d \), or, equivalently, viscosity solutions on the torus \( \mathbb{T}^d \). The analysis in [2] is carried out by making use of tools from weak KAM Theory in the spirit of [10]. The breakthrough brought in by [2] consists in pointing out the crucial role played in the asymptotic study by Mather measures and by a distinguished family of probability measures associated with the discounted equations. These kinds of results have been subsequently generalized in many different directions, such as, for instance, HJ equations with a possibly degenerate second-order term [11–13], weakly coupled systems of HJ equations [14–16], mean field games [17], HJ equations with vanishing negative discount factor [18], contact-type HJ equations [19–21]. In all these papers, the base manifold \( M \) is assumed compact (typically, the \( d \)-dimensional flat torus \( \mathbb{T}^d \)), which is a crucial assumption in order to have pre-compactness of the family of measures introduced for the study.

Few results are available when \( M \) is noncompact. In this direction, the first convergence result appeared in literature is, to the best of our knowledge, the work [22]. Here, \( M := \mathbb{R}^d \) and \( G(x, p) := g(x) \cdot p - f(x) \), where \( f \) is a bounded, uniformly continuous function and \( g \) is a Lipschitz, bounded and strongly monotone vector field on \( \mathbb{R}^d \). This
latter condition yields that the dynamical system induced by $g$ has a unique fixed point $x_0$, which is an attractor for the dynamics. This means that the associated critical Eq. (2) has a unique solution up to additive constants. With the language of weak KAM Theory, which was formalized and introduced by A. Fathi around ten years later, we may say that $\{x_0\}$ is the Aubry set for (2) in this setting.

The vanishing discount problem for HJ equations in the whole $d$-dimensional Euclidean space has been recently considered in [23], where the authors study the asymptotic of solutions to an equation of the form (1) with $M := \mathbb{R}^d$ and where $c(G)$ is replaced by $c_f(G)$. The Hamiltonian $G$ is a continuous function of general form, which is only assumed convex and coercive in the momentum variable $p$, locally uniformly in $x$. The asymptotic convergence is proved under a condition (see (A3) or its weaker version (A3') in [23]) that implies the existence of a bounded subsolution to (2) which is uniformly strict outside some compact set of $\mathbb{R}^d$. In the basic example $G(x,p) := |p| - f(x)$, these conditions boil down to requiring that the infimum of $f$ is not attained at infinity. As in [2], the asymptotic behavior of discounted solutions is derived from weak convergence of suitably associated measures, but the way these are introduced is rather different. Condition (A3)/(A3') in [23] is used to prove tightness of these families of measures, thus recovering the necessary compactness which is no longer guaranteed by the ambient space.

Even though the setting and the point of view is not the same, this study is similar in spirit with case (I) here (cf. Theorem 2). Indeed, case (I) is characterized by the existence of a bounded subsolution to (2) which is uniformly strict outside some compact set containing the support of $V$. This implies tightness of the families of discounted probability measures introduced for the asymptotic analysis, which are defined, in accordance with [2], as occupational measures on optimal curves for the discounted solutions. We are again in a situation where, even though the ambient space is noncompact, there is no dispersion of mass at infinity, so the strategy followed in [2] can be implemented. It is worth pointing out that the dimension 1 of the ambient space plays no role for this part of the work. Indeed, the proofs of Theorem 1 and Theorem 2 in case (I) can be easily adapted to the case $M := \mathbb{R}^d$ for any $d \geq 1$. In this regard, we point out that the definitions of the objects we work with, most notably the critical values $c(G)$ and $c_f(G)$, the associated intrinsic semidistances and the free Aubry set, are given in dimension 1 for uniformity of notation, but they make sense in $\mathbb{R}^d$ for any $d \geq 1$. We have also taken care of writing proofs of the facts needed in this part of the analysis in such a way to be easily adapted to any space dimension.

We also remark that case (I) herein considered has some intersection with the work [23]. For instance, when $H(x,p) := |p| - U(x)$, we have $c(H) = c_f(H) = -\min_{\mathbb{R}} U$ and $c(G) = c_f(G) = -\min_{\mathbb{R}} (U + V)$, so case (I) occurs if and only if $\min_{\mathbb{R}} (U + V) < \min_{\mathbb{R}} U$, namely if and only if $G$ satisfies condition (A3)/(A3') in [23].

The novelty of our analysis consists in dealing with cases where a bounded subsolution to (2), uniformly strict outside some compact set, does not exist, cf. case (II) and (III) in Theorem 2 above and Theorem 2.5. In this situation, the probability measures $\tilde{\mu}_k^x$ associated with the discounted equations may and actually do present dispersion of mass at infinity as $\lambda \to 0^+$. We deal with this lack of compactness by showing that any such measure $\tilde{\mu}_k^x$ can be written as a convex combination of a probability measure $\tilde{\mu}_{x,\lambda}^x$. 

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whose support is contained in a ball large enough so to contain the support of $V$, and a probability measure $\bar{\mu}_{x,2}$ having support contained in the complementary set. By standard compactness, the measures $\bar{\mu}_{x,1}$ possibly converge to a Mather measure $\bar{\mu}_{x,1}$ associated with $G$ as $\lambda \to 0^+$, and this can happen only when $c(G) = c_f(G)$. On the other hand, the measures $\bar{\mu}_{x,2}$ do not see the perturbation of $H$ through the potential $V$, so we can exploit the compactness hidden in the model by showing that their projections on the tangent bundle of the 1-dimensional torus converge, as $\lambda \to 0^+$, to Mather measures $\bar{\mu}_{x,2}$ for $H$. Even though we believe this phenomenon should hold in any space dimension, we are able to prove it only in dimension 1. The crucial point consists in showing that the limit measures $\bar{\mu}_{x,1}$ and $\bar{\mu}_{x,2}$ are closed. For this, we use a purely 1-dimensional property, namely the fact that the optimal curves for the discounted solutions can be taken to be monotone, see Theorem 1.18. This implies that any such optimal curve cannot visit again a ball once it has left it. Dimension 1 comes also crucially into play in other parts of our analysis for cases (II) and (III) and hence it cannot be generalized as it is to higher dimensions. For the sake of completeness, we must specify that Theorems 1 and 2 are proved in case (III) under a condition on the asymptotic limit $u^0_H$ of the periodic problem, see condition (U) in Section 7.2, which is known to hold when $H$ is Tonelli. We have some partial results showing that condition (U) keeps holding when $H$ is merely continuous, but this is not a general fact, cf. Proposition 6.4 and Remark 6.5.

Our work also includes a qualitative study of the critical Eq. (2) with a weak KAM theoretic flavor, which serves as a preliminary step for the asymptotic analysis. The definition of the constant $c(G)$ is essentially borrowed from the literature on homogenization of HJ equations, but its use for the analysis herein addressed seems new. Indeed, equations of the form $G(x,d_xu) = c$ in $M$ with $M$ noncompact are usually studied in the weak KAM theoretic literature for $c = c_f(G)$. Notice that the value of $c_f(G)$ is not affected by shifts of $G$ in the momentum variable, while that of $c(G)$ is. Indeed, when homogenization occurs, for instance when $G = H$, the constant $c(G)$ is equal to the effective Hamiltonian associated with $G$ evaluated at $p = 0$.

When $c(G) > c(H)$, i.e. in case (I), our analysis reveals that Eq. (2) does not admit bounded solutions, see Theorem 2.5. Furthermore, we show that there exists a coercive solution to (2), see Corollary 4.6. This case is also characterized by the existence of a bounded subsolution, which is uniformly strict outside a compact set, cf. Theorems 2.5 and 4.2. This can be interpreted by saying that there is no Aubry set at infinity. These results are not specific of dimension 1, in fact they can be extended to any dimension with essentially the same proofs. We point out that, in this case, the existence of Mather measures for $G$ in any space dimension can be derived through Proposition 4.11 (rather than via Theorem 3.3, whose statement and proof are specifically 1d).

The case $c(G) = c(H)$, corresponding to cases (II) and (III), is characterized by the nonexistence of a bounded subsolution such to be uniformly strict outside a compact set, cf. Theorem 2.5. This feature is not specific of dimension 1 and can be interpreted by saying that there is an Aubry set hidden at infinity. Our study reveals that this Aubry set at infinity is intervening in the asymptotic problem. We believe it would be interesting per se extending this qualitative analysis to higher dimensions and better understanding the role played by the Aubry set at infinity, cf. [24, 25].
When this work was about to be posted on the ArXiv e-print repository, the authors learnt of a recent pre-print by M. Bardi and H. Kouhkouh [26] where the authors prove, among other things, Theorem 1 in the case

\[ M = \mathbb{R}^d, \quad G = \inf_{x \in \mathbb{R}^d} f(x) \]

and

\[ c(G) = -\inf_{x \in \mathbb{R}^d} f(x), \]

where \( f \) is a bounded, Lipschitz and semi-concave function on \( \mathbb{R}^d \) which attains its infimum on \( \mathbb{R}^d \). Differently from [23], the set of minimizers of \( f \) does not need to be compact. This study has some intersection with case (III) herein considered, even though it relies on completely different techniques.

The paper is organized as follows. Section 2 is devoted to fix the main notation and assumptions adopted in the paper and to collect some preliminary facts about critical and discounted HJ equations for a general convex and superlinear Hamiltonian \( G \in C(\mathbb{R}^d) \). In Section 3 we introduce the notion of critical value \( c(G) \) and compare it with the ergodic constant \( c(H) \) when \( G \) is the perturbation of a periodic Hamiltonian \( H \) by a compactly supported potential. Section 4.1 contains some known facts about Mather measures on a compact manifold \( M \), together with an extension of these tools to the 1-dimensional noncompact case \( M = \mathbb{R} \). Section 4.2 is devoted to the study of discounted measures, defined following [2]: we have collected here the new ideas and facts needed to study their asymptotic behavior in the noncompact setting at issue. The analysis of cases (I), (II) and (III) from Theorem 2 above is presented in Sections 5–7, respectively. Each of these sections is divided in two subsections: the first one is devoted to a preliminary analysis of the critical equations for \( G \) and \( H \), the second one contains the asymptotic analysis for the discounted equations. More precisely, Theorems 1 and 2 in cases (I), (II) and (III) correspond, respectively, to Theorem 4.1.3, Theorem 5.1.1 and Theorem 6.1.0.

2. Preliminaries

2.1. A few notations

A function \( g : \mathbb{R}_+ \to \mathbb{R} \) will be termed coercive if \( g(h) \to +\infty \) as \( h \to +\infty \); it will be termed superlinear if \( g(h)/|h| \) is coercive.

Given a metric space \( X \), we will write \( \varphi_n \rightharpoonup \varphi \) on \( X \) to mean that the sequence of functions \( (\varphi_n)_n \) uniformly converges to \( \varphi \) on compact subsets of \( X \). Furthermore, we will denote by \( \text{Lip}(X) \) the family of Lipschitz–continuous real functions defined on \( X \).

We will denote by \( B_r(x_0) \) and \( B_r \) the open balls in \( \mathbb{R} \) of radius \( R \) centered at \( x_0 \) and 0, respectively. The symbol \( |\cdot| \) stands for the Euclidean norm.

With the term curve, without any further specification, we refer to an absolutely continuous function from some given interval \([a, b]\) to \( \mathbb{R} \).

2.2. Viscosity solutions

We will consider Hamilton–Jacobi equations of the general form

\[ F(u(x), x, u') = 0 \quad \text{in} \ \mathbb{R}, \quad (1.1) \]

where \( F \in C(\mathbb{R} \times \mathbb{R} \times \mathbb{R}) \). The notion of solution, subsolution and supersolution of (1.1) adopted in this paper is the one in the viscosity sense, see [7, 27, 28].
subsolutions and supersolutions will be implicitly assumed continuous and the adjective viscosity will be often omitted, with no further specification.

Let us recall that a Lipschitz–continuous subsolution $u$ of (1.1) is also an almost everywhere subsolution, i.e. $F(u(x), x, u'(x)) \leq 0$ for a.e. $x \in \mathbb{R}$. The converse is true when $F$ is convex in $p$, see [7, 27–29].

We will also use the following standard results, see for instance [7, 27–29]:

**Proposition 1.1.** Assume $F \in C(\mathbb{R} \times \mathbb{R})$ is independent of $u$ and such that $F(x, \cdot)$ is convex in $\mathbb{R}$ for every $x \in \mathbb{R}$. Let $u \in C(\mathbb{R})$. The following properties hold:

(i) if $u$ is the pointwise supremum (respectively, infimum) of a family of subsolutions (resp., supersolutions) to (1.1), then $u$ is a subsolution (resp., supersolution) of (1.1);

(ii) if $u$ is the pointwise infimum of a family of equi-Lipschitz subsolutions to (1.1), then $u$ is a Lipschitz subsolution of (1.1);

(iii) if $u$ is a convex combination of a family of equi-Lipschitz subsolutions to (1.1), then $u$ is a Lipschitz subsolution of (1.1).

Note that items (ii) and (iii) above require the convexity of $F$ in the momentum, while item (i) is actually independent of this condition.

We conclude this section by a standard approximation result, see [1], Theorem 8.1, that we shall repeatedly use in our analysis.

**Lemma 1.2.** Assume $F \in C(\mathbb{R} \times \mathbb{R})$ such that $F(x, \cdot)$ is convex in $\mathbb{R}$ for every $x \in \mathbb{R}$, and let $u$ be a Lipschitz subsolution of Eq. (1.1). Then, for all $\varepsilon > 0$, there exists a smooth function $u_\varepsilon : \mathbb{R} \to \mathbb{R}$ such that

$$||u - u_\varepsilon||_\infty \leq \varepsilon \quad \text{and} \quad F(x, u^\prime_\varepsilon(x)) \leq \varepsilon \quad \text{for all} \ x \in \mathbb{R}.$$ 

2.3. Hamilton–Jacobi equations and the free Aubry set

Throughout the paper, we will call Hamiltonian a continuous function $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. If not otherwise stated, we will always assume that $G$ satisfies the following assumptions:

(G1) (convexity) for every $x \in \mathbb{R}$, the map $p \mapsto G(x, p)$ is convex on $\mathbb{R}$.

(G2) (superlinearity) there exist two superlinear functions $\alpha, \beta : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\alpha(|p|) \leq G(x, p) \leq \beta(|p|) \quad \text{for all} \ (x, p) \in \mathbb{R} \times \mathbb{R}.$$ 

(G3) (uniform continuity) $G \in UC(\mathbb{R} \times B_R)$ for all $R > 0$.

The Hamiltonian $G$ will be termed Tonelli if it is furthermore of class of $C^2$ on $\mathbb{R} \times \mathbb{R}$ and satisfies $\partial^2 G/\partial p^2(x, p) > 0$ in $\mathbb{R} \times \mathbb{R}$.

**Remark 1.3.** Condition (G2) is equivalent to saying that $G$ is superlinear and locally bounded in $p$, uniformly with respect to $x$. We deduce from (G1)

$$|G(x, p_1) - G(x, p_2)| \leq M_R|p - q| \quad \text{for all} \ x \in \mathbb{R}, \ \text{and} \ p_1, p_2 \in B_R, \quad (1.2)$$

where $M_R = \sup \{|G(x, p)| : x \in \mathbb{R}, \ |p| \leq R + 2\}$, which is finite thanks to (G2).
To any such Hamiltonian, we can associate a Lagrangian function $L : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ via the Fenchel transform of $G$:

$$L(x, q) = G^*(x, q) := \sup_{p \in \mathbb{R}} \langle p, q \rangle - G(x, p).$$  \hspace{1cm} (1.3)

Such a function $L$ satisfies (G1), (G2) and (G3) as well, for a suitable pair of superlinear functions $\alpha, \beta$ in place of $\alpha, \beta$. In the sequel we shall often use the so-called Fenchel inequality, namely

$$L(x, q) \geq \langle p, q \rangle - G(x, p) \quad \text{for all } x \in \mathbb{R} \text{ and } p, q \in \mathbb{R}.$$

Next, we recall some preliminary facts about stationary Hamilton–Jacobi equations of the form

$$G(x, u') = a \quad \text{in } \mathbb{R}$$  \hspace{1cm} (1.4)

where $a \in \mathbb{R}$. Note that any given $C^1$ function $u : \mathbb{R} \to \mathbb{R}$ is a subsolution (resp. supersolution) of $G(x, u') = a$ provided $a \geq \max_{x \in \mathbb{R}} G(x, u')$ (resp. $a \leq \min_{x \in \mathbb{R}} G(x, u')$). Moreover, the coercivity and convexity of $G$ entails the following characterization of viscosity subsolutions of (1.4), see for instance [27, 28].

**Proposition 1.4.** Let $G \in C(\mathbb{R} \times \mathbb{R})$ satisfy (G1)-(G2) and let $a \in \mathbb{R}$. A continuous function $u$ is a viscosity subsolution $u$ of (1.4) if and only if it is Lipschitz continuous and satisfies

$$G(x, u'(x)) \leq a \quad \text{for a.e. } x \in \mathbb{R}.$$

Moreover, the set of viscosity subsolutions of (1.4) is equi-Lipschitz, with a common Lipschitz constant $\kappa_a$ given by

$$\kappa_a = \sup \{ |p| : G(x, p) \leq a \}.$$  \hspace{1cm} (1.5)

We define the free critical value $c_f(G)$ as

$$c_f(G) = \inf \{ a \in \mathbb{R} : \text{equation (1.4) admits viscosity subsolutions} \}.$$  \hspace{1cm} (1.6)

The term free refers to the fact that we are not prescribing a priori any constraint on the kind of growth of subsolutions at infinity. Of course, the subsolutions we are considering have at most linear growth due to their Lipschitz character. The existence of almost everywhere subsolutions for (1.4) readily implies that the set $Z(x, a) := \{ p \in \mathbb{R} : G(x, p) \leq a \}$ is non-empty for every $x \in \mathbb{R}$. In particular, $c_f(G)$ is greater than or equal to the level of equilibria $\sup \min \{ G(x, p) \}$. By the Ascoli–Arzelà Theorem and the stability of the notion of viscosity subsolution, it can be easily proved that the infimum in (1.6) is attained, meaning that there are subsolutions also at the free critical level.

We will refer to

$$G(x, u') = c_f(G) \quad \text{in } \mathbb{R}$$  \hspace{1cm} (1.7)

as the free critical equation.
Following [30], we carry out the study of properties of subsolutions of (1.4), for 

\[ a \geq c_f(G), \]

by means of the semidistances \( S_a \) defined on \( \mathbb{R} \times \mathbb{R} \) as follows:

\[
S_a(x, y) = \inf \left\{ \int_0^1 \sigma_a(\gamma(s), \dot{\gamma}(s)) \, ds \right\},
\]

where the infimum is taken over all absolutely continuous curves \( \gamma : [0, 1] \to \mathbb{R} \) with \( \gamma(0) = x \) and \( \gamma(1) = y \), and \( \sigma_a(x, q) \) is the support function of the \( a \)-sublevel \( Z(x, a) \) of \( G \), namely

\[
\sigma_a(x, q) := \sup \{ \langle q, p \rangle : p \in Z(x, a) \}.
\]

There exists a positive constant \( \kappa_a \) such that the function \( S_a \) satisfies the following properties, for all \( x, y, z \in \mathbb{R} \):

\[ S_a(x, y) \leq S_a(x, z) + S_a(z, y), \quad S_a(x, y) \leq \kappa_a |x - y| \]

Further properties of \( S_a \) which be useful in the sequel are summarized in the statement below, see [30].

**Proposition 1.5.** Let \( a \geq c_f(G) \).

(i) A function \( \phi \) is a viscosity subsolution of (1.4) if and only if

\[ \phi(x) - \phi(y) \leq S_a(y, x) \quad \text{for all } x, y \in \mathbb{R}. \]

In particular, all viscosity subsolutions of (1.4) are \( \kappa_a \)-Lipschitz continuous.

(ii) For any \( y \in \mathbb{R} \), the functions \( S_a(y, \cdot) \) and \( -S_a(\cdot, y) \) are both subsolutions of (1.4).

(iii) For any \( y \in \mathbb{R} \)

\[ S_a(y, x) = \sup \{ v(x) : v \text{ is a subsolution to (1.4) with } v(y) = 0 \}. \]

In particular, by maximality, \( S_a(y, \cdot) \) is a viscosity solution of (1.4) in \( \mathbb{R} \setminus \{ y \} \).

It is also well known that Eq. (1.4) admits solutions for every \( a \geq c_f(G) \), thanks to the noncompact character of the ambient space \( \mathbb{R} \).

Let us now go back to the free critical Eq. (1.7). We define the free Aubry set \( \mathcal{A}_f(G) \) as

\[ \mathcal{A}_f(G) := \{ y \in M : S_{c_f(G)}(y, \cdot) \text{ is a solution to (1.7)} \}. \]

The free Aubry set is (possibly empty) closed subset of \( \mathbb{R} \) characterized by the following property.

**Proposition 1.6.** A point \( y \in \mathbb{R} \) belongs to \( \mathcal{A}_f(G) \) if and only if every subsolution \( v \) to (1.7) satisfies the supersolution test at \( y \), i.e.

\[ G(y, p) \geq c_f(G) \quad \text{for every } p \in D^- v(y), \]

where \( D^- v(y) \) denotes the set of subdifferentials of \( v \) at \( y \).

---

1 Pick a diverging sequence \( (y_n)_n \). The functions \( v_n(\cdot) := S_{c_f(G)}(y_n, \cdot) - S_{c_f(G)}(y_n, 0) \) form a relatively compact sequence in \( C(\mathbb{R}) \) by the Ascoli-Arzela Theorem. Any cluster point \( u \) of \( (v_n)_n \) in \( C(\mathbb{R}) \) solves (1.4) in any fixed open ball in \( \mathbb{R} \) by property (iii) above and by stability of the notion of viscosity solution.
The free Aubry set $\mathcal{A}_f(G)$ can be also characterized in terms of strict subsolutions. We give a definition first.

**Definition 1.7.** A subsolution $v \in \text{Lip}(\mathbb{R})$ to (1.4) is said to be strict in an open set $U \subset \mathbb{R}$ if for every open set $V$ compactly contained in $U$ there exist a constant $\delta = \delta(V) > 0$ such that

$$G(x, v'(x)) < a - \delta \quad \text{for a.e. } x \in V.$$  

We will say that $v$ is uniformly strict in $U$ if the constant $\delta$ can be chosen independently of $V$. The existence of a subsolution to (1.7) which is strict in an open set $U$ implies $U \cap \mathcal{A}_f(G) = \emptyset$, in view of Proposition 1.6. The converse implication is guaranteed by the following Theorem, cf. [30, 31].

**Theorem 1.8.** There exist a subsolution $v$ to the free critical Eq. (1.7) which is strict and of class $C^\infty$ in $\mathbb{R} \setminus \mathcal{A}_f(G)$.

Proposition 1.6 and Theorem 1.8 yield altogether the following characterization: $\mathcal{A}_f(G)$ is the minimal closed subset of $\mathbb{R}$ for which there exists a strict subsolution outside of it. Otherwise stated, the free Aubry set $\mathcal{A}_f(G)$ is the set where the obstruction to the existence of strict subsolutions to (1.7) concentrates.

All the results presented above are actually true in $\mathbb{R}^d$ for any $d \geq 1$. The next results are instead specific of dimension 1.

Let us begin by pointing out some properties of the support function $\sigma_a$ that we shall need in the sequel. Let us denote by $\text{dom}(Z) := \{(x, a) \in \mathbb{R} \times \mathbb{R} : Z(x, a) \neq \emptyset\}$. This set is closed, as it can be easily checked. The following result holds.

**Lemma 1.9.** The function $(x, a, q) \mapsto \sigma_a(x, q)$ is continuous from $\text{dom}(Z) \times \mathbb{R}$ to $\mathbb{R}$. Furthermore, the map $a \mapsto \sigma_a(x, q)$ is strictly increasing and the map $q \mapsto \sigma_a(x, q)$ is convex and positively homogeneus.

**Proof.** To show that $(x, a, q) \mapsto \sigma_a(x, q)$ is continuous on $\text{dom}(Z) \times \mathbb{R}$, it suffices to notice that, for $(x, a) \in \text{dom}(Z)$, the set $Z(x, a)$ has either nonempty interior or is a singleton in view of (G1). The other assertions easily follows from the definition of $\sigma_a$ and from properties (G1)-(G2). \hfill \Box

**Remark 1.10.** For every $(x, a) \in \text{dom}(Z)$, the set $Z(x, a)$ is an interval of the form $Z(x, a) = [p^-(x, a), p^+(x, a)]$. The functions $p^\pm(x, a)$ are jointly continuous in $\text{dom}(Z)$. This is a consequence of the previous Lemma since $p^-(x, a) = -\sigma_a(x, -1)$ and $p^+(x, a) = \sigma_a(x, 1)$. Also notice that $Z(x, a) \neq \emptyset$ for every $a \geq c_f(G)$ and $x \in \mathbb{R}$, i.e. $\mathbb{R} \times [c_f(G), +\infty) \subseteq \text{dom}(Z)$.

In dimension 1, we have the following explicit characterization of the free critical value and of the free Aubry set associated with $G$.

**Theorem 1.11.** The following holds:

$$c_f(G) = \sup_{x \in \mathbb{R}} \min_{p \in \mathbb{R}} G(x, p). \quad (1.11)$$
Furthermore, there exists $v \in C^1(\mathbb{R})$ satisfying

$$G(x, v'(x)) \leq c_f(G) \quad \text{for all } x \in \mathbb{R}$$

which is strict outside the set $\mathcal{E}(G)$ of equilibria of $G$ defined as follows:

$$\mathcal{E}(G) := \left\{ y \in \mathbb{R} : \min_{p \in \mathbb{R}} G(y, p) = c_f(G) \right\}.$$ 

In particular, $\mathcal{A}_f(G) = \mathcal{E}(G)$.

We note that the set of equilibria is empty if and only if the supremum in (1.11) is not attained.

**Proof.** Let us temporarily denote by $c_0$ the right-hand side term in (1.11). We already know that $c_f(G) \geq c_0$. Let us write $Z(x, c_0) = \left[ p^-(x), p^+(x) \right]$. In view of Remark 1.10, we know that the functions $x \rightarrow p^\pm(x)$ are continuous. Furthermore, they satisfy $p^-(x) \leq p^+(x)$ for all $x \in \mathbb{R}$, with equality holding if and only if $x \in \mathcal{E}(G)$. Let us set

$$\nu(x) := \int_0^x \frac{p^-(z) + p^+(z)}{2} \, dz$$

It is easily seen that $\nu$ is a $C^1$ subsolution of Eq. (1.4) with $a = c_0$, which is furthermore strict in $\mathbb{R} \setminus \mathcal{E}(G)$. This implies $c_f(G) = c_0$ and $\mathcal{A}_f(G) \subseteq \mathcal{E}(G)$ in view of Proposition 1.6. The converse inclusion $\mathcal{E}(G) \subseteq \mathcal{A}_f(G)$ follows from the fact that (1.10) is authomatically satisfied at points $y \in \mathcal{E}(G)$.

### 2.4. Discounted Hamilton-Jacobi equations

We will be also interested in discounted Hamilton-Jacobi equations of the form

$$\lambda u(x) + G(x, u') = a \quad \text{in } \mathbb{R}, \quad (1.12)$$

where $a \in \mathbb{R}$ and $\lambda > 0$ is the discount factor. The following holds.

**Proposition 1.12.** Let $G \in C^{C(\mathbb{R} \times \mathbb{R})}$ be a Hamiltonian satisfying (G1)–(G2) and $\lambda > 0$. Then any subsolution $w$ of (1.12) is locally Lipschitz–continuous and satisfies

$$\lambda w(x) + G(x, w'(x)) \leq a \quad \text{for } \text{a.e. } x \in \mathbb{R}. \quad (1.13)$$

Furthermore, if $w$ is bounded from below on $\mathbb{R}$, then $w$ is Lipschitz–continuous in $\mathbb{R}$.

**Proof.** A subsolution $w$ of (1.12) satisfies

$$G(x, w') \leq a - \lambda w(x) \leq \|(a - \lambda w)^+\|_{L^\infty(U)} \quad \text{in } U$$

in the viscosity sense for every open and bounded subset $U$ of $\mathbb{R}$, where we have denoted by $(a - \lambda w)^+$ the positive part of the function $a - \lambda w$. We infer that $w$ is locally Lipschitz continuous by the coercivity of $G$, see [27]. In particular, it satisfies the inequality (1.13) at every differentiability point, i.e. almost everywhere by Rademacher’s theorem. If $w$ is bounded from below, we can take $U = \mathbb{R}$ above and derive that $w$ is globally Lipschitz on $\mathbb{R}$, with Lipschitz constant given by (1.5) with $\|(a - \lambda w)^+\|_{\infty}$ in place of $a$. \qed
The crucial difference between Eq. (1.4) and the discounted Eq. (1.12) with \( k > 0 \) is that the latter satisfies a strong comparison principle, see for instance [27, Théorème 4.3].

**Theorem 1.13.** Let \( G \in C(\mathbb{R} \times \mathbb{R}) \) be a Hamiltonian satisfying (G1)–(G3) and \( a \in \mathbb{R} \). Let \( w, v \in C(\mathbb{R}) \) be a bounded super and subsolution to (1.12), respectively. Then \( w \geq v \) in \( \mathbb{R} \).

This comparison principle and a standard application of Perron’s method yields the following result, see for instance [27, Théorème 2.12].

**Theorem 1.14.** Let \( G \in C(\mathbb{R} \times \mathbb{R}) \) be a Hamiltonian satisfying (G1)–(G3) and \( a \in \mathbb{R} \). Then there exists a unique solution \( u^k \) to (1.12) in the class of bounded continuous functions on \( \mathbb{R} \). Furthermore, \( ||u^k||_\infty \leq (M + |a|)/\lambda \) with \( M := \max\{|x(0)|, |\beta(0)|\} \), and \( u^k \) is \( \kappa_M \)-Lipschitz continuous, with \( \kappa_M \) given by (1.5).

By exploiting the fact that the solution \( u^k \) is globally Lipschitz on \( \mathbb{R} \) and by arguing as in the proof of [27, Théorème 4.3], we derive the following result, cf. proof of Proposition 2.6 in [2].

**Proposition 1.15.** Let \( G \in C(\mathbb{R} \times \mathbb{R}) \) be a Hamiltonian satisfying (G1)–(G3) and \( a \in \mathbb{R} \). Let \( v \leq 0, w \geq 0 \) be a sub and a supersolution to (1.4), respectively. Then \( v \leq u^k \leq w \) in \( \mathbb{R} \) for every \( \lambda > 0 \), where \( u^k \) is the solution to (1.12). In particular, the family of functions \( \{u^k : \lambda > 0\} \) is equi-Lipschitz and locally equi-bounded in \( \mathbb{R} \), hence pre-compact in \( C(\mathbb{R}) \).

In our analysis, we will exploit the following representation formula for the solution \( u^k \) of the discounted Eq. (1.12):

\[
    u^k(x) = \inf_{\gamma} \int_{-\infty}^{0} e^{st} [L(\gamma(s), \dot{\gamma}(s)) + a] \, ds \quad \text{for every } x \in \mathbb{R},
\]

where the infimum is taken over all absolutely continuous curves \( \gamma : (0,0] \to \mathbb{R} \), with \( \gamma(0) = x \), see [7] for more details. Moreover, we will need the following property concerning minimizing curves, see [2, Appendix 2] for a proof:

**Proposition 1.16.** Let \( G \in C(\mathbb{R} \times \mathbb{R}) \) be a Hamiltonian satisfying (G1)–(G3). Let \( \lambda > 0 \) and \( x \in \mathbb{R} \). Then there exists a Lipschitz curve \( \gamma^k_x : (-\infty, 0] \to \mathbb{R} \) with \( \gamma^k_x(0) = x \) such that

\[
    u^k(x) = e^{-\lambda t} u^k(\gamma^k_x(-t)) + \int_{-t}^{0} e^{\lambda s} \left[ L(\gamma^k_x(s), \dot{\gamma}^k_x(s)) + a \right] \, ds \quad \text{for every } t > 0.
\]

In particular

\[
    u^k(x) = \int_{-\infty}^{0} e^{\lambda s} \left[ L(\gamma^k_x(s), \dot{\gamma}^k_x(s)) + a \right] \, ds.
\]

Furthermore, there exists a constant \( \bar{\kappa} > 0 \), independent of \( \lambda \) and \( x \), such that \( ||\dot{\gamma}^k_x||_\infty \leq \bar{\kappa} \).

A curve \( \gamma : (-\infty, 0] \to \mathbb{R} \) will be termed **minimizing** or **optimal** for \( u^k(x) \) if \( \gamma(0) = x \) and it satisfies (1.15).
Remark 1.17. The constant $\bar{\kappa} > 0$ in Proposition 1.16 actually depends on the functions $\alpha, \beta$ appearing in condition (G2) only, cf. with the proof of Proposition 6.2 in [2] and with Theorem 1.14. If $\gamma_{x}^{c}$ is differentiable at $t \leq 0$ and $u^{c}$ if differentiable at $\gamma_{x}^{c}(t)$, we furthermore have

$$\dot{\gamma}_{x}^{c}(t) \in \partial_{p}G(\gamma_{x}^{c}(t), (u^{c})'(\gamma_{x}^{c}(t))) \quad \text{and} \quad (u^{c})'(\gamma_{x}^{c}(t)) \in \partial_{q}L(\gamma_{x}^{c}(t), \dot{\gamma}_{x}^{c}(t)), \quad (1.16)$$

see for instance the proof of Proposition 5 in [32]. In (1.16), the symbols $\partial_{p}G(y,p)$ and $\partial_{q}L(y,q)$ stand for the subdifferentials of the functions $G(y, \cdot)$ and $L(y, \cdot)$ in the sense of convex analysis, respectively. The above differential inclusion holds with an equality for every $t < 0$ whenever $G$ is Tonelli, see for instance Proposition 6 in [32].

We now proceed by proving a monotonicity property of minimizing curves for $u^{c}$ that will play a crucial role in the proofs of the asymptotic result.

Theorem 1.18. Let $G \in C(\mathbb{R} \times \mathbb{R})$ be a Hamiltonian satisfying (G1)–(G3) and let $u^{c}$ be the unique bounded solution to (1.12) with $\lambda > 0$. For every $x \in \mathbb{R}$, there exists a minimizing curve $\gamma : (-\infty, 0] \to \mathbb{R}$ for $u^{c}(x)$ such that $t \mapsto \gamma(t)$ is monotone.

Proof. We shall prove the assertion in two steps.

Step 1. Let us assume $G$ to be Tonelli. Pick a minimizing curve $\gamma : (-\infty, 0] \to \mathbb{R}$ for $u^{c}(x)$. Clearly, we can assume $\gamma$ non-constant. Hence, there exists $B < 0$ such that $\gamma(B) \neq \gamma(0)$. Let us assume for definiteness $\gamma(B) < \gamma(0)$. We aim to show that $t \mapsto \gamma(t)$ is non-decreasing.

Claim I: $\gamma(t) \leq \gamma(B)$ for all $t \leq B$.

Indeed, if this were not the case, there would exists $A < B$ such that $\gamma(A) > \gamma(B)$. Since $\gamma$ is moving from the point $\gamma(A)$ to the point $\gamma(B)$ in the interval $[A,B]$, there must exist a point $t_{0} \in (A,B)$ such that

$$\gamma(B) < z := \gamma(t_{0}) < \min\{\gamma(A), \gamma(0)\} \quad \text{and} \quad \dot{\gamma}(t_{0}) < 0 \quad (1.17)$$

On the other hand, since the curve $\gamma$ is moving from $\gamma(B)$ to $\gamma(0)$ in the interval $[B,0]$, there must exist a point $t_{1} \in (B,0)$ such that $z = \gamma(t_{1})$ and $\dot{\gamma}(t_{1}) \not\geq 0$. Now we use the fact that $\gamma$ is optimal for $u^{c}$, $u^{c}$ is differentiable at $z = \gamma((\infty, 0))$ and

$$\dot{\gamma}(t_{0}) = \partial_{p}G(z, u'(z)) = \dot{\gamma}(t_{1})$$

to get a contradiction.

Claim II: $\gamma(a) \leq \gamma(b)$ for all $a < b \leq 0$.

If this were not the case, there would exists $a < b \leq 0$ such that $\gamma(b) < \gamma(a)$. If $\gamma(b) < \gamma(0)$, we can use the same argument used in Claim I with $b$ in place of $B$ and $a$ in place of $A$ to get a contradiction. Hence, in view of Claim I, we have

$$\gamma(a) > \gamma(b) \geq \gamma(0) > \gamma(B) \quad \text{and} \quad B < a < b.$$ 

Then there exists a point $t_{0}$ of $\gamma$ in $(a,b)$ such that

$$\gamma(b) < z := \gamma(t_{0}) < \gamma(a) \quad \text{and} \quad \dot{\gamma}(t_{0}) < 0.$$
On the other hand, since the curve \( \gamma \) is moving from \( \gamma(B) \) to \( \gamma(a) \) in the interval \([B,a]\), there must exist a point \( t_1 \in (B,a) \) such that \( z = \gamma(t_1) \) and \( \dot{\gamma}(t_1) \geq 0 \). Now we use the fact that \( \gamma \) is optimal for \( u^k \), \( u^k \) is differentiable at \( z \in \gamma((-,0)) \) and
\[
\dot{\gamma}(t_0) = \partial_p G(z,u'(z)) = \dot{\gamma}(t_1)
\]
to get a contradiction.

**Step 2.** Let us now go back to the case of a Hamiltonian \( G \) of general type. Let \((\rho_n)_n\) a sequence of standard regularizing kernels on \( \mathbb{R} \times \mathbb{R} \) and set \( G_n(x,p) := (\rho_n * G)(x,p) + |p|^2/n \). The Hamiltonians \( G_n \) are Tonelli and satisfy (G2) for a common pair of functions \( \alpha, \beta \). Let us denote by \( u^k_n \) the solutions of the discounted Eq. (1.12) with \( G := G_n \). The functions \( u^k_n \) are equi-bounded and equi-Lipschitz, in view of Theorem 1.14, hence they converge, up to extracting a subsequence (not relabeled), to a limit function \( u^k \). Since \( G_n = G \) in \( \mathbb{R} \times \mathbb{R} \), by the stability of the notion of viscosity solution we derive that \( u^k \) is the solution of the discounted Eq. (1.12). Let us fix \( x \in \mathbb{R} \) and denote by \( \gamma_n : (-\infty,0] \to \mathbb{R} \) a minimizing curve for \( u^k_n(x) \). In view of Step 1, these curves are monotone. Up to extracting a subsequence, that again we shall not relabel to ease notation, we can furthermore assume that these curve all have the same kind of monotonicity, let us say they are all non-decreasing, to fix ideas. In view of Proposition 1.16 and Remark 1.17, these curves \( \gamma_n \) are \( \tilde{k} \)-Lipschitz for some common constant \( \tilde{k} > 0 \), hence they converge, up to extracting a subsequence (not relabeled), to a limit curve \( \gamma : (-\infty,0] \to \mathbb{R} \) with \( \gamma(0) = x \). Clearly, the curve \( \gamma \) is nondecreasing. Let us show it is optimal for \( u^k(x) \). The Lagrangians \( L_n \) associated with \( G_n \) via (1.3) uniformly converge to the Lagrangian \( L \) associated with \( G \) on compact subsets of \( \mathbb{R} \times \mathbb{R} \). By standard semi-continuity results in the Calculus of Variations, see [33, Theorem 3.6], for every fixed \( t > 0 \) we get
\[
u^k(x) - e^{-\tilde{s}t} u^k(\gamma(-t)) = \lim_{n \to +\infty} u^k_n(x) - e^{-\tilde{s}t} u^k_n(\gamma_n(-t)) = \lim_{n \to +\infty} \int_{-t}^{0} e^{\tilde{s}s} [L_n(\gamma_n,\dot{\gamma}_n) + a] \, ds
\]
\[
\geq \int_{-t}^{0} e^{\tilde{s}s} [L(\gamma,\dot{\gamma}) + a] \, ds,
\]
thus proving the minimality of \( \gamma \).

**Remark 1.19.** Please note that, when \( G \) is a Tonelli Hamiltonian, we have actually shown that monotonicity holds for any minimizing curve \( \gamma : (-\infty,0] \to \mathbb{R} \) for \( u^k(x) \).

**3. Critical values**

Let \( G : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be a continuous Hamiltonian satisfying (G1)-(G2), and let us consider a family of Hamilton–Jacobi equations of the form
\[
G(x,u') = a \quad \text{in} \ \mathbb{R},
\]
with \( a \in \mathbb{R} \). We define the following critical value associated with \( G \) as follows:
\[ c(G) := \inf \{ a \in \mathbb{R} : \text{there exists a bounded subsolution of (2.1)} \}. \quad (2.2) \]

We also recall the definition of free critical value associated with \( G \), cf. [1, 31]:
\[ c_f(G) = \min \{ a \in \mathbb{R} : \text{there exists a subsolution of (2.1)} \}. \]

The following holds.

**Proposition 2.1.** Assume there exists a bounded supersolution \( w \in \text{Lip}(\mathbb{R}) \) to (2.1). Then \( a \leq c(G) \). In particular, if \( w \) is a solution of (2.1), then \( a = c(G) \).

**Proof.** Assume by contradiction that \( a > c(G) \). Then there exists a bounded and Lipschitz function \( \tilde{u} \) such that
\[ G(x, \tilde{u}') \leq a - 2\delta \quad \text{in } \mathbb{R}. \]

Set \( u(x) := \tilde{u}(x) - \varepsilon \sqrt{1 + |x|^2} \). For \( \varepsilon > 0 \) small enough
\[ G(x, u') < a - \delta \quad \text{in } \mathbb{R}. \]

Furthermore, since
\[ \lim_{|x| \to \infty} \frac{w(x) - u(x)}{|x|} = \varepsilon, \]
we infer that \( \lim_{|x| \to +\infty} (w(x) - u(x)) = +\infty \). In particular, there exists \( x_0 \in \mathbb{R} \) such that
\[ (w - u)(x_0) = \inf_{\mathbb{R}} (w - u), \]
i.e. \( u \) is a subtangent to \( w \) at \( x_0 \). Being \( w \) a supersolution of \( G \geq a \), by Proposition 4.3 in [34], there exists \( p_0 \) belonging to the Clarke generalized gradient \( \partial u(x_0) \) of \( u \) at \( x_0 \) such that \( G(x_0, p_0) \geq a \). On the other hand, \( G(x_0, p_0) \leq a - \delta \) being \( u \) a subsolution of \( G < a - \delta \), cf. [29, Proposition 3], yielding a contradiction. When \( w \) is a solution, the converse inequality \( a \geq c(G) \) holds as well by definition of \( c(G) \).

In the periodic case, the following characterization of the critical value holds:

**Proposition 2.2.** Let \( H \in C(\mathbb{R} \times \mathbb{R}) \) be Hamiltonian satisfying (G1)-(G2) which is \( \mathbb{Z} \)-periodic in \( x \). Then
\[ c(H) = \min \{ a \in \mathbb{R} : \text{there exist periodic subsolutions of (2.1)} \}. \quad (2.3) \]

**Proof.** Let us call \( c_p(H) \) the infimum of the set appearing at the right-hand side term of (2.3). It is well known that this infimum is attained, i.e. periodic subsolutions exist at level \( a = c_p(H) \) as well, see [3, 30]. Clearly, \( c(H) \leq c_p(H) \). Let \( a > c(H) \) and pick a bounded subsolution \( u \in \text{Lip}(\mathbb{R}) \) of (2.1) with \( G := H \). Set
\[ v(x) := \inf_{z \in \mathbb{Z}} u(x + z), \quad x \in \mathbb{R}. \]

It is easily seen that the function \( v \) is well defined, Lipschitz continuous and \( \mathbb{Z} \)-periodic. By \( \mathbb{Z} \)-periodicity of \( H \) in the state variable, the map \( x \to u(x + z) \) is a subsolution of (2.1) with \( G := H \) for each \( z \in \mathbb{Z} \). Since \( H \) is convex in \( p \), we infer from Proposition 1.1 that \( v \) is
itself a subsolution of (2.1) as an infimum of subsolutions. Hence \( a > c_p(H) \) and the claim follows since \( a > c(H) \) was arbitrarily chosen.

Let \( H \) be as in the statement of Proposition 2.2. For any fixed \( \theta \in \mathbb{R} \), let \( H_\theta \) be the Hamiltonian defined as \( H_\theta(x,p) := H(x,\theta + p) \) for all \((x,p) \in \mathbb{R} \times \mathbb{R} \). The effective Hamiltonian \( \bar{H} \) associated with \( H \) (also called \( \alpha \)-function in the context of weak KAM Theory) is the function defined as \( \bar{H}(\theta) := c(H_\theta) \) for every \( \theta \in \mathbb{R} \). It is well known that \( \bar{H} \) satisfies (G1)-(G2), i.e. is convex and superlinear. Furthermore, the following holds, see for instance [35, Sections 6 and 7].

**Theorem 2.3.** For every \( a \geq c_f(H) \), let

\[
Z^H(x,a) := \{ p \in \mathbb{R} : H(x,p) \leq a \} = [p^-_H(x,a),p^+_H(x,a)]
\]

for all \( x \in \mathbb{R} \).

Then

\[
\bar{H}(\theta) = \inf \left\{ a \geq c_f(H) : \theta \in \left[ \int_0^1 p^-_H(x,a)dx, \int_0^1 p^+_H(x,a)dx \right] \right\}
\]

for all \( \theta \in \mathbb{R} \).

Furthermore, \( c_f(H) = \min_a \bar{H} \).

Given \( H \) as in the statement of Proposition 2.2 and \( V \in C_c(\mathbb{R}) \), we define

\[
G(x,p) := H(x,p) - V(x)
\]

for every \((x,p) \in \mathbb{R} \times \mathbb{R} \).

The following holds.

**Proposition 2.4.** Let \( H \) and \( G \) be as above. Then \( c(G) \geq c(H) \) and \( c_f(G) \geq c_f(H) \).

**Proof.** Let us prove \( c(G) \geq c(H) \). Pick \( a > c(G) \) and let \( u \in \text{Lip}(\mathbb{R}) \) be a bounded subsolution to (2.1). Let \((z_n)_n\) be a diverging sequence in \( \mathbb{Z} \), i.e. \( \lim_n |z_n| = +\infty \), and set \( u_n(\cdot) := u(z_n + \cdot) - u(z_n) \). Each function \( u_n \) is a subsolution of

\[
G_n(x,u_n') \leq a \quad \text{in} \ \mathbb{R},
\]

with \( G_n(\cdot,\cdot) := G(z_n + \cdot,\cdot) \). Since \( u \) is bounded and Lipschitz on \( \mathbb{R} \), the family of functions \((u_n)_n\) is equi–bounded and equi–Lipschitz, in particular there exists a bounded function \( v \in \text{Lip}(\mathbb{R}) \) such that, up to subsequences, \( u_n \rightharpoonup v \) in \( \mathbb{R} \). Now, since \( G_n \rightharpoonup H \) in \( \mathbb{R} \times \mathbb{R} \), by the stability of the notion of viscosity subsolution we conclude that \( v \) satisfies

\[
H(x,v') \leq a \quad \text{in} \ \mathbb{R}.
\]

This proves that \( a \geq c(H) \) for every \( a > c(G) \), hence \( c(G) \geq c(H) \).

The proof of the inequality \( c_f(G) \geq c_f(H) \) goes along the same lines, the only difference being that the functions \((u_n)_n\) are locally equi–bounded in \( \mathbb{R} \) since they are equi–Lipschitz and satisfy \( u_n(0) = 0 \) for all \( n \in \mathbb{N} \). \( \square \)

The fact that the critical value \( c(G) \) is either equal or not equal to \( c(H) \) carries the following interesting information.

**Theorem 2.5.** Let us consider the critical equation

\[
G(x,u') = c(G) \quad \text{in} \ \mathbb{R}.
\]  

(2.4)
(i) If $c(G) > c(H)$, Eq. (2.4) does not admit bounded solutions.

(ii) If $c(G) = c(H)$, Eq. (2.4) does not admit a bounded subsolution which is uniformly strict outside some compact set $K$.

Proof. Let us prove (i). Assume by contradiction that there exists a bounded solution $u$ of (2.4). The same argument used in the proof of Proposition 2.4 shows that there exists a bounded solution $v$ of $H(x, v') = c(G)$ in $\mathbb{R}$. From Proposition 2.1 we infer that $c(H) = c(G)$, reaching a contradiction.

Let us prove (ii). Assume by contradiction that there exists a bounded subsolution $u$ of (2.4) and a constant $\delta > 0$ such that $G(x, u'(x)) < c(G) - \delta$ for a.e. $x \in \mathbb{R} \setminus K$. The same argument used in the proof of Proposition 2.4 shows that there exists a bounded subsolution $v$ of $H(x, v') < c(G) - \delta$ in $\mathbb{R}$, contradicting the definition of $c(H)$ since $c(H) = c(G)$.

4. Minimizing measures

4.1. Mather measures

Let $X$ be a metric separable space. A probability measure on $X$ is a nonnegative, countably additive set function $\mu$ defined on the $\sigma$-algebra $\mathcal{B}(X)$ of Borel subsets of $X$ such that $\mu(X) = 1$. In this paper, we deal with probability measures defined either on $M$, with $M = \mathbb{T}^1$ or $M = \mathbb{R}$, or on its tangent bundle $TM$. A measure on $TM$ is denoted by $\tilde{\mu}$, where the tilde on the top is to keep track of the fact that the measure is on the space $TM$. We say that a sequence $(\tilde{\mu}_n)_n$ of probability measures on $TM$ (narrowly) converges to a probability measure $\tilde{\mu}$ on $TM$, in symbols $\tilde{\mu}_n \rightharpoonup \tilde{\mu}$, if

$$\lim_{n \to +\infty} \int_{TM} f(x, q) \, d\tilde{\mu}_n(x, q) = \int_{TM} f(x, q) \, d\tilde{\mu}(x, q) \quad \text{for every} \ f \in C_b(TM), \quad (3.1)$$

where $C_b(TM)$ denotes the family of continuous and bounded real functions on $TM$. If $\tilde{\mu}$ is a probability measure on $TM$, we denote by $\mu$ its projection $\pi_1 \# \tilde{\mu}$ on $M$, i.e. the probability measure on $M$ defined as

$$\pi_1 \# \tilde{\mu}(B) := \tilde{\mu}(\pi_1^{-1}(B)) \quad \text{for every} \ B \in \mathcal{B}(M).$$

Note that

$$\int_{M} f(x) \, \pi_1 \# \tilde{\mu}(x) = \int_{TM} (f \circ \pi_1)(x, q) \, d\tilde{\mu}(x, q) \quad \text{for every} \ f \in C_b(M),$$

where $C_b(M)$ denotes the family of bounded continuous functions on $M$.

Let $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous Hamiltonian satisfying (G1)-(G2) and such that $H$ is $\mathbb{Z}$-periodic in $x$. Let us denote by $L_H$ the Lagrangian associated with $H$ via the Fenchel transform (1.3). Mather theory states that the constant $-c(H)$, where $c(H)$ is the critical value, can be also obtained by minimizing the integral of the Lagrangian over $TM$ with respect to closed probability measures on $TM$. The definition of closed measure is the following:
Definition 3.1. A probability measure $\tilde{\mu} \in \mathcal{P}(TM)$ is termed closed if it satisfies the following properties:

(i) $\int_M |q| \, d\tilde{\mu}(x, q) < +\infty$;
(ii) for every $\varphi \in C^1_b(M) \cap \operatorname{Lip}(M)$

$$\int_{TM} \varphi'(x) \, q \, d\tilde{\mu}(x, q) = 0.$$ 

A way to construct a closed measure is the following: if $\gamma : [a, b] \to M$ is an absolutely continuous curve, define the probability measure $\tilde{\mu}_\gamma$ on $TM$, by

$$\int_{TM} f(x, q) \, d\tilde{\mu}_\gamma(x, q) := \frac{1}{b-a} \int_a^b f(\gamma(t), \dot{\gamma}(t)) \, dt,$$ 

for every $f \in C_b(TM)$. It is easily seen that $\tilde{\mu}_\gamma$ is a closed measure whenever $\gamma$ is a loop.

The following holds, see for instance [2, Appendix 1] for a proof.

Theorem 3.2. Let $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous Hamiltonian satisfying (G1)-(G2) and $\mathbb{Z}$-periodic in $x$. Let us denote by $L_H$ the Lagrangian associated with $H$ via the Fenchel transform (1.3). Then

$$\min_{\tilde{\mu}} \int_{T^1 \times \mathbb{R}} L_H(x, v) \, d\tilde{\mu}(x, v) = -c(H)$$

where $\tilde{\mu}$ varies in the set of closed measures on $T^1 \times \mathbb{R}$.

A closed probability measure on $T^1 \times \mathbb{R}$ that solves the minimization problem (3.3) is called Mather measure for $H$. The set of Mather measures for $H$ will be denoted by $\mathfrak{M}(H)$. A projected Mather measure is a Borel probability measure in $\mu$ on $T^1$ of the form $\mu = \pi_1 \# \tilde{\mu}$ with $\tilde{\mu} \in \mathfrak{M}(H)$. The set of projected Mather measures will be denoted by $\mathfrak{M}(H)$.

Let us now consider the Hamiltonian $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ obtained by perturbing the periodic Hamiltonian $H$ via a potential $V \in C_0(\mathbb{R})$ as follows:

$$G(x, p) := H(x, p) - V(x) \quad \text{for all } (x, p) \in \mathbb{R} \times \mathbb{R}.$$ 

Let us denote by $L_G$ the Lagrangian associated with $G$ via the Fenchel transform (1.3). The following holds.

Theorem 3.3. Let $G(x, p) := H(x, p) - V(x)$ for all $(x, p) \in \mathbb{R} \times \mathbb{R}$, where $V \in C_0(\mathbb{R})$ and $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous Hamiltonian satisfying (G1)-(G2) and $\mathbb{Z}$-periodic in $x$. Let us denote by $L_G$ the Lagrangian associated with $G$ via the Fenchel transform (1.3). Then

$$\min_{\tilde{\mu}} \int_{\mathbb{R} \times \mathbb{R}} L_G(x, q) \, d\tilde{\mu}(x, q) = -c_f(G)$$

where $\tilde{\mu}$ varies in the set of closed measures on $\mathbb{R} \times \mathbb{R}$.

Proof. By Theorem 1.11, there exists a $C^1$ subsolution $v$ of the equation $G(x, v') = c_f(G)$ in $\mathbb{R}$ which is strict outside $\mathcal{E}(G)$. For any closed measure $\tilde{\mu}$ on $\mathbb{R} \times \mathbb{R}$ we have...
\[
\int_{\mathbb{R} \times \mathbb{R}} L_G(x, q) \, d\tilde{\mu}(x, q) \geq \int_{\mathbb{R} \times \mathbb{R}} v'(x)q \, d\bar{\mu}(x, q) - \int_{\mathbb{R} \times \mathbb{R}} G(x, v'(x)) \, d\bar{\mu}(x, q) \geq -c_f(G),
\]

where we have used Fenchel inequality and the fact that the measure \( \bar{\mu} \) is closed. On the other hand, it is easily seen that the set \( E(G) \) of equilibria is nonempty and that any measure of the form \( \tilde{\mu} := \delta_{(y, 0)} \) with \( y \in E(G) \) is a closed measure on \( \mathbb{R} \times \mathbb{R} \) that solves the minimization problem (3.4).

In analogy with the periodic case, we shall call Mather measure for \( G \) a closed probability measure on \( \mathbb{R} \times \mathbb{R} \) that solves the minimization problem (3.4). We will denote by \( \tilde{\mathcal{M}}(G) \) the set of Mather measures for \( G \). A projected Mather measure for \( G \) is a Borel probability measure \( \mu \) on \( \mathbb{R} \) of the form \( \mu = \pi_1\#\bar{\mu} \) with \( \bar{\mu} \in \tilde{\mathcal{M}}(G) \). The set of projected Mather measures will be denoted by \( \mathcal{M}(G) \).

As a byproduct of the previous computation, we easily derive the following information.

**Corollary 3.4.** The set \( \mathcal{M}(G) \) of projected Mather measures for \( G \) coincides with the closed convex hull of \( \{ \delta_y : y \in E(G) \} \), i.e. the set of delta Dirac measures concentrated at points of \( E(G) \).

**Proof.** It is enough to show that any projected Mather measure for \( G \) has support contained in \( E(G) \). Let \( \mu = \pi_1\#\bar{\mu} \) for some Mather measure \( \bar{\mu} \). Since \( \bar{\mu} \) is minimizing, all inequalities in (3.5) are equalities. This means that \( L_G(x, q) = v'(x)q - G(x, v'(x)) \) and \( G(x, v'(x)) = c_f(G) \) for \( \bar{\mu} \)-a.e. \( (x, q) \in \text{supp}(\bar{\mu}) \), namely

\[
x \in E(G) \quad \text{and} \quad q \in \partial_p G(x, v'(x)) \quad \text{for} \quad \bar{\mu} \text{-a.e. } (x, q) \in \text{supp}(\bar{\mu}).
\]

We conclude that \( \text{supp}(\mu) \subseteq E(G) \). When \( G(x, \cdot) \) is differentiable at \( v'(x) \), we also get \( q = 0 \), since \( G(x, v'(x)) = \min_p G(x, p) \).

### 4.2. Discounted minimizing measures

In this section we introduce and study the properties of a class of measures associated with the discounted Eq. (1.12). We start by considering the case of a general continuous Hamiltonian \( G \) satisfying (G1)-(G2). For every \( \lambda > 0 \), we denote by \( u^\lambda \) the solution of Eq. (1.12). Let us fix \( x \in \mathbb{R} \) and choose a minimizer \( \gamma^\lambda_x : (-\infty, 0] \rightarrow \mathbb{R} \) for \( u^\lambda(x) \), i.e. a curve satisfying (1.15) with \( \gamma^\lambda_x(0) = x \), see Proposition 1.16. We define a measure \( \tilde{\mu}^\lambda_x = \tilde{\mu}^\lambda_x \) on \( \mathbb{R} \times \mathbb{R} \) by setting

\[
\int_{\mathbb{R} \times \mathbb{R}} f(y, q) \, d\tilde{\mu}^\lambda_x := \int_{-\infty}^0 (e^{\lambda s}) f(\gamma^\lambda_x(s), \dot{\gamma}^\lambda_x(s)) \, ds \quad \text{for every } f \in C_b(\mathbb{R} \times \mathbb{R}).
\]

Clearly, the measures \( \{ \tilde{\mu}^\lambda_x : \lambda > 0, x \in \mathbb{R} \} \) are probability measures whose supports are contained in \( \mathbb{R} \times \bar{B}_k \) for some constant \( k > 0 \), according to Proposition 1.16. The
projection of $\tilde{\mu}_x^t$ on the base manifold $\mathbb{R}$ via the map $\pi_1: \mathbb{R} \times \mathbb{R} \ni (x,p) \mapsto x \in \mathbb{R}$ is the measure $\mu_x^t$ defined as follows:
\[
\int_{\mathbb{R} \times \mathbb{R}} f(x) \, d\mu_x^t = \int_{-\infty}^{0} (e^{is} f(\gamma_x^s(t))) \, ds \quad \text{for every } f \in C_0(\mathbb{R}).
\] (3.7)

We record for later use the following result.

**Proposition 3.5.** Let $v \in \text{Lip}(\mathbb{R})$ be a subsolution of (1.4) such that $v \leq 0$ in $\mathbb{R}$ and
\[
G(x, v'(x)) < a - \delta \quad \text{for a.e. } x \in U,
\]
where $U$ is an open subset of $\mathbb{R}$ and $\delta > 0$ is a constant. Let $\mu_x^t$ be the probability measure defined in (3.7). Then
\[
\mu_x^t(U) \leq \frac{\lambda}{\delta} (u^t(x) - v(x)).
\]

**Proof.** Let us pick $r > 0$ small enough such that the set $U_r := \{x \in U : \text{dist}(x, \partial U) > r\}$ is nonempty. By Urysohn’s Lemma, there exists a continuous function $\psi: \mathbb{R} \to [a - \delta, a]$ such that $\psi \equiv a - \delta$ on $U_r$ and $\psi \equiv a$ on $\mathbb{R} \setminus U_r$. In particular, the subsolution $v$ satisfies $G(x, v'(x)) \leq \psi(x)$ for a.e. $x \in \mathbb{R}$. Fix $\varepsilon > 0$ and apply Lemma 1.2 with $u := v$ and $F(x, p) := G(x, p) - \psi(x)$ to obtain a function $v_\varepsilon \in \text{Lip}(\mathbb{R}) \cap C^\infty(\mathbb{R})$ such that $G(x, v_\varepsilon'(x)) \leq \psi(x) + \varepsilon$ for every $x \in \mathbb{R}$. In particular,
\[
G(x, v_\varepsilon'(x)) \leq a + \varepsilon \quad \text{in } \mathbb{R} \setminus U_r, \quad G(x, v_\varepsilon'(x)) \leq a - \delta + \varepsilon \quad \text{in } U_r. \tag{3.8}
\]

By exploiting Fenchel’s inequality and by taking into account (3.8) we have
\[
u^t(x) = \int_{-\infty}^{0} e^{is} \left[ L(\gamma_x^s(t), \gamma_x^s(t)) + a \right] \, ds
\]
\[
\geq \int_{-\infty}^{0} e^{is} \left[ v_\varepsilon'(\gamma_x^s(t)) \gamma_x^s(t) - G(\gamma_x^s(t), v_\varepsilon(\gamma_x^s(t)) + a \right] \, ds
\]
\[
\geq \int_{-\infty}^{0} e^{is} \left[ v_\varepsilon'(\gamma_x^s(t)) \right] \, ds + (\delta - \varepsilon) \int_{-\infty}^{0} e^{is} \psi_x(\gamma_x^s(t)) \, ds - \frac{\varepsilon}{\lambda}
\]
\[
\geq v_\varepsilon(x) - \int_{-\infty}^{0} (e^{is} v_\varepsilon(\gamma_x^s(t)) \, ds + \frac{(\delta - \varepsilon)}{\lambda} \mu_x^t(U_r) - \frac{\varepsilon}{\lambda},
\]
where the last inequality follows by an integration by parts and by taking into account that $v \leq 0$ in $\mathbb{R}$. By multiplying both sides by $\lambda > 0$, sending $\varepsilon \to 0$ and exploiting the fact that $v \leq 0$ we get
\[
\mu_x^t(U_r) \leq \frac{\lambda}{\delta} \left( u^t(x) - v(x) + \int_{\mathbb{R}} v(y) \, d\mu_x^t(y) \right) \leq \frac{\lambda}{\delta} (u^t(x) - v(x)).
\]

The assertion follows by sending $r \to 0^+$.

Let us now consider the case $G(x, p) := H(x, p) - V(x)$, where $V \in C_c(\mathbb{R})$ and $H$ is a Hamiltonian satisfying (G1)-(G2) which is $\mathbb{Z}$-periodic in $x$. Let us assume that $c(G) = c(H)$ and denote by $c$ this common critical constant. Let us denote by $u^t_G$ and $u^t_H$ the solutions of the discounted Eq. (1.12) with $a := c$, associated with with Hamiltonians $G$.
and $H$, respectively. For every fixed $x \in \mathbb{R}$ and $\lambda > 0$, let us denote by $\gamma_{x}^{\lambda} : (-\infty, 0] \to \mathbb{R}$ a monotone minimizer for the variational formula related to $u_{C}^{\lambda}(x)$, see (1.14), which exists in force of Proposition 1.16 and Theorem 1.18. Fix $r > 0$ big enough so that $\{x\} \cup [\tilde{y}_{V}, \tilde{y}_{V}] \subseteq B_{r}$, where $\tilde{y}_{V} := \min(\supp(V))$ and $\tilde{y}_{V} := \max(\supp(V))$. All objects we are about to introduce depend on $r$, yet explicit dependence on $r$ will be omitted to ease notation. Set

$$T_{x}^{\lambda} := \sup \{t > 0 : \gamma_{x}^{\lambda}(-t) \in \tilde{B}_{r} \}.$$  \hfill (3.9)

Since $t \to \gamma_{x}^{\lambda}(t)$ is monotone, we have $\gamma_{x}^{\lambda}((-T_{x}^{\lambda}, 0]) \subseteq \tilde{B}_{r}$. When $T_{x}^{\lambda} < +\infty$, we furthermore have $\gamma_{x}^{\lambda}(-T_{x}^{\lambda}) \in \partial \tilde{B}_{r}$, $\gamma_{x}^{\lambda}((-\infty, -T_{x}^{\lambda})) \subseteq \mathbb{R} \setminus \tilde{B}_{r}$. Set $\theta_{x}^{\lambda} := 1 - e^{-2T_{x}^{\lambda}} \in (0, 1]$ and define two probability measures $\check{\mu}_{x,1}^{\lambda}$, $\check{\mu}_{x,2}^{\lambda}$ on $\mathbb{R} \times \mathbb{R}$ by setting

$$\int_{\mathbb{R} \times \mathbb{R}} f(y, q) \, d\check{\mu}_{x,1}^{\lambda}(y, q) := \frac{1}{\theta_{x}^{\lambda}} \int_{-T_{x}^{\lambda}}^{0} (e^{k_{s}}) f(\gamma_{x}^{\lambda}(s), \gamma_{x}^{\lambda}(s)) \, ds,$$  \hfill (3.10)

and, when $T_{x}^{\lambda} < +\infty$,

$$\int_{\mathbb{R} \times \mathbb{R}} f(y, q) \, d\check{\mu}_{x,2}^{\lambda}(y, q) := \frac{1}{1 - \theta_{x}^{\lambda}} \int_{-T_{x}^{\lambda}}^{0} (e^{k_{s}}) f(\gamma_{x}^{\lambda}(s), \gamma_{x}^{\lambda}(s)) \, ds$$  \hfill (3.11)

for all $f \in C_{b}(\mathbb{R} \times \mathbb{R})$. In order to simplify some statements in what follows, it is convenient to assume the probability measure $\check{\mu}_{x,2}^{\lambda}$ to be defined also when $T_{x}^{\lambda} = +\infty$ (and $\theta_{x}^{\lambda} = 1$). Any choice will serve the purpose. Hence, we set $\check{\mu}_{x,2}^{\lambda} := \delta_{(x+r,0)}$ (the Dirac measure concentrated in the point $(x + r, 0)$) whenever $T_{x}^{\lambda} = +\infty$.

Let us regard $\mathbb{R}$ as the universal covering of $\mathbb{T}^1$ and denote by $\pi : \mathbb{R} \to \mathbb{T}^1$ the projection map. Given a probability measure on $\mathbb{R} \times \mathbb{R}$, we define $\tilde{\pi} \# \check{\mu}$ as the push-forward of $\check{\mu}$ via the map $\tilde{\pi} : \mathbb{R} \times \mathbb{R} \to \mathbb{T}^1 \times \mathbb{R}$ defined as $\tilde{\pi}(x, p) = (\pi(x), p)$. Such a measure $\tilde{\pi} \# \check{\mu}$ can be equivalently defined as follows:

$$\int_{\mathbb{T}^1 \times \mathbb{R}} f(x, q) \, d(\tilde{\pi} \# \check{\mu})(x, q) := \int_{\mathbb{R} \times \mathbb{R}} f(x, q) \, d\check{\mu}(x, q) \quad \text{for all } C_{b}(\mathbb{T}^1 \times \mathbb{R}),$$

where $C_{b}(\mathbb{T}^1 \times \mathbb{R})$ is also interpreted as the space of functions $f \in C_{b}(\mathbb{R} \times \mathbb{R})$ that are $\mathbb{Z}$-periodic in $x$. Any such measure belongs to the space $\mathcal{P}(\mathbb{T}^1 \times \mathbb{R})$ of probability measures on $\mathbb{T}^1 \times \mathbb{R}$.

The following holds:

**Proposition 3.6.** Let $\check{\mu}_{x,1}^{\lambda}$, $\check{\mu}_{x,2}^{\lambda}$ be the probability measures defined above for $x \in \mathbb{R}$ and $\lambda > 0$.

- (i) The family of probability measures $\{\check{\mu}_{x,1}^{\lambda} : \lambda > 0\}$ is pre-compact in $\mathcal{P}(\mathbb{R} \times \mathbb{R})$.
- (ii) The family of Borel measures $\{\tilde{\pi} \# \check{\mu}_{x,2}^{\lambda} : \lambda > 0\}$ is pre-compact in $\mathcal{P}(\mathbb{T}^1 \times \mathbb{R})$.

**Proof.** The curves $\{\gamma_{x}^{\lambda} : \lambda > 0\}$ are equi-Lipschitz, let us say $\tilde{\kappa}$-Lipschitz, in view of Proposition 1.16. By definition of $T_{x}^{\lambda}$, we have
Proposition 3.7. Let \( \bar{\mu}_{x,1}^n \), \( \bar{\mu}_{x,2}^n \) and \( \theta_x^n \) be, respectively, the probability measures and the constant defined above for \( x \in \mathbb{R} \) and \( \lambda > 0 \). Let \( \lambda_n \) be a sequence such that \( \lambda_n \searrow 0 \).

(i) Suppose that \( \bar{\mu}_{1,x}^n \to \bar{\mu}_{1,x} \) in \( \mathcal{P}(\mathbb{R} \times \mathbb{R}) \) and \( \theta_x^n \to 0 \in (0,1] \). Then \( \bar{\mu}_{1,x} \) is a closed measure in \( \mathcal{P}(\mathbb{R} \times \mathbb{R}) \). Furthermore, \( \bar{\mu}_{1,x} \) is a Mather measure in \( \mathcal{P}(\mathbb{R} \times \mathbb{R}) \) for \( L_G \).

(ii) Suppose that \( \bar{\pi}_x \mu_{2,x}^n \to \bar{\mu}_{2,x} \) in \( \mathcal{P}(\mathbb{T}^1 \times \mathbb{R}) \) and \( \theta_x^n \to 0 \in [0,1) \). Then \( \bar{\mu}_{2,x} \) is a closed measure in \( \mathcal{P}(\mathbb{T}^1 \times \mathbb{R}) \). Furthermore \( \bar{\mu}_{2,x} \) is a Mather measure in \( \mathcal{P}(\mathbb{T}^1 \times \mathbb{R}) \) for \( L_H \).

Proof. Let us first prove that \( \bar{\mu}_{1,x} \) is a closed measure in \( \mathcal{P}(\mathbb{R} \times \mathbb{R}) \). For \( \varphi \in C_b^1(\mathbb{R}) \cap \text{Lip}(\mathbb{R}) \) we have

\[
\int_{\mathbb{R} \times \mathbb{R}} \varphi'(y) d\bar{\mu}_{1,x}(y, q) = \lim_{n \to +\infty} \int_{\mathbb{R} \times \mathbb{R}} \varphi'(y) d\bar{\mu}_{1,x}^n(y, q)
\]

\[
= \lim_{n \to +\infty} \frac{1}{\theta_x^n} \int_{-T_x^n}^{0} \lambda_n e^{\lambda_n s} \varphi' \left( \gamma_x^n(s) \right) \dot{\gamma}_x^n(s) ds
\]

\[
= \lim_{n \to +\infty} \frac{\lambda_n}{\theta_x^n} \left( \varphi(y) - e^{-\lambda_n T_x^n} \varphi(\gamma_x^n(-T_x^n)) - \int_{-T_x^n}^{0} (e^{\lambda_n s})' \varphi(\gamma_x^n(s)) ds \right)
\]

where for the third equality we have used integration by parts. Now

\[
\left| \frac{\lambda_n}{\theta_x^n} \left( \varphi(y) - e^{-\lambda_n T_x^n} \varphi(\gamma_x^n(-T_x^n)) \right) \right| \leq 2 \frac{\lambda_n}{\theta_x^n} ||\varphi||_\infty \to 0,
\]

implying the assertion. The proof of the fact that \( \bar{\mu}_{2,x} \) is a closed measure in \( \mathcal{P}(\mathbb{T}^1 \times \mathbb{R}) \) goes along the same lines, with the only difference that \( \varphi \) needs to be chosen in \( C^1(\mathbb{T}^1) \).

To prove the rest of the assertion, we begin by noticing that, by the choice of \( \gamma_x^n \), the following holds:

\[
\lambda \mu_G(x) =
\]

\[
= \theta_x^n \int_{\mathbb{R} \times \mathbb{R}} (L_H(y, q) + V(y) + c) \ d\bar{\mu}_{1,x}^n(y, q) + (1 - \theta_x^n) \int_{\mathbb{R} \times \mathbb{R}} (L_G(y, q) + V(y) + c) \ d\bar{\mu}_{2,x}^n(y, q)
\]

\[
= \theta_x^n \int_{\mathbb{R} \times \mathbb{R}} (L_H(y, q) + V(y) + c) \ d\bar{\mu}_{1,x}(y, q) + (1 - \theta_x^n) \int_{\mathbb{R} \times \mathbb{R}} (L_H(y, q) + V(y) + c) \ d\bar{\mu}_{2,x}(y, q),
\]
where for the last equality we have taken into account that \((\text{supp}(V) \times \mathbb{R}) \cap \text{supp}(\tilde{\mu}_{2,x}) = \emptyset)\) together with the fact that \(L_H\) is \(\mathbb{Z}_{\tau}\)-periodic in the state variable. Let us set \(\lambda := \lambda_n\) and send \(n \to +\infty\) in the equality above to obtain

\[
0 = \theta \int_{\mathbb{R} \times \mathbb{R}} (L_H + V + c) \, d\tilde{\mu}_{1,x} + (1 - \theta) \int_{\mathbb{T}^1 \times \mathbb{R}} (L_H + c) \, d(\tilde{\pi} \# \tilde{\mu}_{2,x}).
\]

The assertion readily follows from this after noticing that

\[
\theta \int_{\mathbb{R} \times \mathbb{R}} (L_H + V + c) \, d\tilde{\mu}_{1,x} \geq 0, \quad (1 - \theta) \int_{\mathbb{T}^1 \times \mathbb{R}} (L_H + c) \, d(\tilde{\pi} \# \tilde{\mu}_{2,x}) \geq 0
\]

due to the fact that the measures \(\tilde{\mu}_{1,x}\) and \(\tilde{\mu}_{2,x}\) are closed when \(\theta \neq 0\) and \(\theta \neq 1\), respectively. □

The next proposition will play a crucial role in the proof of the asymptotic convergence.

**Proposition 3.8.** Let \(v\) be a bounded subsolution to

\[
G(x, v') \leq c \quad \text{in } \mathbb{R}.
\]

For every \(\lambda > 0\) and \(x \in \mathbb{R}\) we have

\[
u^\lambda_G(x) \geq v(x) - \left(\theta^\lambda \int_{\mathbb{R} \times \mathbb{R}} v(y) \, d\tilde{\mu}_{1,1}^\lambda(y, q) + (1 - \theta^\lambda) \int_{\mathbb{R} \times \mathbb{R}} v(y) \, d\tilde{\mu}_{1,2}^\lambda(y, q)\right).
\]

**Proof.** Fix \(\epsilon > 0\) and apply Lemma 1.2 with \(F := G - c\), \(u := v\) to obtain a function \(v_\epsilon \in \text{Lip} (\mathbb{R}) \cap C^\infty (\mathbb{R})\) such that

\[
||v - v_\epsilon||_\infty < \epsilon \quad \text{and} \quad G(x, v'_\epsilon(x)) < c + \epsilon \quad \text{for every } x \in \mathbb{R}.
\]

By making use of Fenchel’s inequality and integrating by parts we get

\[
u^\lambda_G(x) \geq \int_{-\infty}^0 e^{\lambda s} \left[ v'_\epsilon (\gamma^\lambda_x(s)) \left( \gamma^\lambda_x(s) - \epsilon \right) \right] \, ds = \int_{-\infty}^0 e^{\lambda s} \frac{d}{ds} v_\epsilon (\gamma^\lambda_x(s)) \, ds - \frac{\epsilon}{\lambda}
\]

\[
= v_\epsilon(x) - \int_{-\infty}^0 \left( e^{\lambda s} \right)' v_\epsilon (\gamma^\lambda_x(s)) \, ds - \frac{\epsilon}{\lambda}
\]

\[
= v_\epsilon(x) - \left( \int_{-T^\lambda_2}^0 \left( e^{\lambda s} \right)' v_\epsilon (\gamma^\lambda_x(s)) \, ds + \int_{-\infty}^{-T^\lambda_2} \left( e^{\lambda s} \right)' v_\epsilon (\gamma^\lambda_x(s)) \, ds \right) - \frac{\epsilon}{\lambda}.
\]

Now we send \(\epsilon \to 0\) and recall the definition of \(\tilde{\mu}_{1,1}^\lambda\), \(\tilde{\mu}_{1,2}^\lambda\) to get the assertion. □

### 5. The case \(c(G) > c(H)\)

In this section we shall prove the asymptotic convergence in the case \(c(G) > c(H)\). We start with a qualitative study of the critical equation associated with \(G\).

#### 5.1. The critical equation

Let us consider the critical equation

\[
G(x, u') = c(G) \quad \text{in } \mathbb{R}.
\] (4.1)
By definition, we infer that \( c(G) \geq c_f(G) \). Let us denote by \( S_G \) the critical semi-distance associated with \( G \) via (1.8) with \( a = c(G) \). The set \( \{ S_G(y, \cdot) : y \in \mathbb{R} \} \), with \( S_G \) defined according to (1.8) with \( a := c(G) \), is a family of subsolutions to (4.1). We recall that such subsolutions are \( \kappa \)-Lipschitz, with \( \kappa \) given by (1.5) with \( a := c(G) \), but they are not a priori bounded on \( \mathbb{R} \). In fact, the following holds.

**Proposition 4.1.** Let \( \bar{K} \) be a compact subset of \( \mathbb{R} \). Then there exists a constant \( C = C(\bar{K}) \) such that

\[
S_G(y, \cdot) \geq C|x| - \frac{1}{C} \quad \text{on} \quad \mathbb{R} \quad \text{for every} \quad y \in \bar{K}.
\]

**Proof.** Let \( B_r \) be an open ball centered in 0 with radius \( r \) large enough so that \( \bar{K} \cup \text{supp}(V) \subset B_r \). By the triangular inequality we get \( S_G(y, x) \geq S_G(0, x) - S_G(0, y) \geq -\kappa|x - y| \) for all \( x \in \mathbb{R} \), in particular

\[
S_G(y, x) \geq -2kr \quad \text{for all} \quad x \in B_r \quad \text{and} \quad y \in \bar{K}.
\]

Let us pick \( x \in \mathbb{R} \setminus B_r \) and let \( \gamma \in \text{Lip}_{y,x}([0, 1]; \mathbb{R}) \) such that

\[
S_G(y, x) + 1 > \int_0^1 \sigma^G_{c(G)}(\gamma, \dot{\gamma}) \, ds.
\]

Let us denote by \( T := \max \{ s \in [0, 1] : \gamma(s) \in \bar{B}_r \} \) and by \( z := \gamma(T) \). Let us denote by \( u \) a periodic solution to \( H(x, u') = c(H) \) in \( \mathbb{R} \). By the coercivity of \( H \) in \( p \), see condition (H2), we get that there exists an \( R > 0 \) such that

\[
Z^H_{c(G)} := \{ p \in \mathbb{R} : H(x, p) \leq c(G) \} \subseteq B_R \quad \text{for every} \quad x \in \mathbb{R}.
\]

By taking into account Remark 1.3, we get

\[
c(G) - c(H) = H(x, p_1) - H(x, p_2) \leq M_R |p_1 - p_2|
\]

for every \( x \in \mathbb{R} \), \( p_1 \in \partial Z^H_{c(G)} \), \( p_2 \in \partial Z^H_{c(H)} \). We infer that

\[
\sigma^H_{c(G)}(x, q) \geq \sigma^H_{c(H)}(x, q) + \delta|q| \quad \text{for every} \quad (x, q) \in \mathbb{R} \times \mathbb{R},
\]

with \( \delta := (c(G) - c(H))/M_R > 0 \). By the choice of \( r \) and \( T \) we have

\[
\sigma^G_{c(G)}(\gamma, \dot{\gamma}) = \sigma^H_{c(G)}(\gamma, \dot{\gamma}) \geq \sigma^H_{c(H)}(\gamma, \dot{\gamma}) + \delta|\dot{\gamma}| \quad \text{in} \quad [T, 1].
\]

We infer

\[
S_G(y, x) + 1 > \int_0^T \sigma^G_{c(G)}(\gamma, \dot{\gamma}) \, ds + \int_T^1 \sigma^H_{c(G)}(\gamma, \dot{\gamma}) \, ds
\]

\[
\geq S_G(y, z) + \delta \int_T^1 |\dot{\gamma}| \, ds + u(x) - u(z) \geq \delta|x - z| - 2(Kr + ||u||_\infty)
\]

\[
\geq \delta|x| - (\delta r + 2K r + 2||u||_\infty),
\]

proving the assertion for \( C > 0 \) small enough.

We proceed by showing the following
**Theorem 4.2.** There exist a compact set $K$ in $\mathbb{R}$ with $\text{supp}(V) \subseteq \text{int}(K)$ and a bounded subsolution $v_G$ to (4.1) which is uniformly strict outside $K$, i.e.

$$G(x, v'_G(x)) \leq c(G) - \delta \quad \text{for a.e. } x \in \mathbb{R} \setminus K$$

for some constant $\delta > 0$. In particular, up to an additive constant, we can assume $v_G \leq 0$ in $\mathbb{R}$.

In view of Proposition 1.6, we infer

**Corollary 4.3.** Let $K$ be the compact set given by Theorem 4.2. Then $A_f(G) \subseteq K$.

For the proof of Theorem 4.2, we will need the following preliminary fact.

**Proposition 4.4.** Let $a \in (c(H), c(G)]$. Then there exist $u_a, v_a \in \text{Lip}(\mathbb{R})$ such that

(i) $u_a$ is a solution to

$$H(x, u'_a) = a \quad \text{in } \mathbb{R},$$

with $u_a(x) \leq -C|x| + 1/C$ in $\mathbb{R}$ for some constant $C > 0$;

(ii) $v_a$ is subsolution of (4.2) with $v_a(x) \geq C|x| - 1/C$ in $\mathbb{R}$ for some constant $C > 0$.

**Proof.** Let $\bar{H} : \mathbb{R} \to \mathbb{R}$ be the effective Hamiltonian associated with $H$. Since $\bar{H}(0) = c(H) < c(G)$, we infer that $0 \in \text{int}\{\bar{H} \leq a\}$. In particular, for each $\xi \in \{1, -1\}$, there exists a unique $P_\xi \in \{t\xi : t > 0\}$ such that $\bar{H}(P_\xi) = a$. Let $\tilde{w}_\xi$ be a periodic solution of

$$H(x, P_\xi + \tilde{w}'_\xi) = a \quad \text{in } \mathbb{R}$$

satisfying $\tilde{w}_\xi(0) = 0$ and set $w_\xi(x) := \tilde{w}_\xi(x) + P_\xi x$ for every $x \in \mathbb{R}$. Being $w_1, w_{-1}$ Lipschitz and locally bounded solutions of (4.2), we infer that the functions

$$u_a(x) := \min\{w_1(x), w_{-1}(x)\}, \quad x \in \mathbb{R}$$

$$v_a(x) := \max\{w_1(x), w_{-1}(x)\}, \quad x \in \mathbb{R}$$

are a solution and a subsolution of (4.2), respectively, cf. Proposition 1.1. Moreover, since $\min_{\xi \in \{1, -1\}} |P_\xi| > 0$, it is easily seen that there exists a constant $C > 0$ such that

$$u_a(x) \leq -C|x| + \frac{1}{C}, \quad v_a(x) \geq C|x| - \frac{1}{C} \quad \text{for every } x \in \mathbb{R}.$$  

This proves the assertion. $\square$

**Proof of Theorem 4.2.** Let take $w(\cdot) := S_G(0, \cdot)$. By Proposition 4.1, we know that $w$ is a subsolution to (4.1) satisfying $w(x) \to +\infty$ as $|x| \to +\infty$. Pick a periodic solution $u_H$ to $H(x, u') = c(H)$ in $\mathbb{R}$ and set

$$v_G(x) := \min\{w(x), u_H(x) + k\} \quad x \in \mathbb{R},$$

where $k > 0$ is chosen large enough so that $v_G \equiv w$ in a open neighborhood $U$ of $\text{supp}(V)$. It is easily seen that $v_G$ is a bounded subsolution to (4.1). Furthermore, as $u_H$ is bounded while $w$ is coercive, there exists a compact set $K$ such that $v_G \equiv u_H + k$ in
\[ \mathbb{R} \setminus K. \] Note that this implies \( \text{supp}(V) \subseteq U \subseteq K. \) The assertion follows by setting \( \delta := c(G) - c(H). \)

The following holds:

**Theorem 4.5.** We have that \( c(G) = c_f(G) \) and \( \mathcal{A}_f(G) \neq \emptyset. \)

**Proof.** Suppose either \( c(G) > c_f(G) \) or \( c(G) = c_f(G) \) and \( \mathcal{A}_f(G) = \emptyset. \) We claim that there exists \( v \in \text{Lip}(\mathbb{R}) \) which is a strict subsolution to (4.1). In the first case, pick an \( a \in (c_f(G), c(G)) \) and take a subsolution \( v \in \text{Lip}(\mathbb{R}) \) of \( G(x, u') = a \) in \( \mathbb{R}, \) which exists by definition of \( c_f(G). \) In the second case, the claim follows as a direct application of Theorem 1.8.

Since \( v \) is a priori neither bounded nor uniformly strict in \( \mathbb{R}, \) we are going to modify it in order to produce a uniformly strict and bounded subsolution to (4.1). To this aim, choose \( a \in (c(H), c(G)) \), take \( v_a \) as in the statement of Proposition 4.4–(ii) and set

\[ w(x) := \max\{v(x), v_a(x) - k\}, \quad x \in \mathbb{R}, \]

with \( k > 0 \) large enough so that \( w \equiv v \) in an open neighborhood \( U \) of \( \text{supp}(V). \) The function \( w \) is a coercive subsolution to (4.1), which is uniformly strict in every bounded open subset of \( \mathbb{R}. \) Now take a periodic solution \( u \) to \( H(x, u') = c(H) \) in \( \mathbb{R} \) and set

\[ \tilde{w}(x) := \min\{w(x), u(x) + k\}, \quad x \in \mathbb{R}, \]

with \( k > 0 \) large enough so that \( \tilde{w} \equiv w \equiv v \) in the open neighborhood \( U \) of \( \text{supp}(V). \) It is easily seen that \( \tilde{w} \) is a bounded subsolution to (4.1), uniformly strict in every open and bounded subset of \( \mathbb{R}. \) Since \( w \) is coercive while \( u \) is bounded, \( \tilde{w} \) agrees with \( u + k \) outside a compact set \( K \) containing \( \text{supp}(V). \) Since \( H(x, \cdot) \equiv G(x, \cdot) \) for \( x \in \mathbb{R} \setminus \text{supp}(V) \) and \( c(H) < c(G) \), we infer that there exist a \( \delta > 0 \) such that

\[ G(x, \tilde{w}'(x)) \leq c(G) - \delta \quad \text{for a.e. } x \in \mathbb{R}, \]

thus contradicting the minimality of \( c(G). \)

The results gathered so far yield the following simple consequence:

**Corollary 4.6.** There exists a solution \( u_G \) to (4.1) such that

\[ u_G(x) \geq C|x| \quad \text{on } \mathbb{R} \]

for some constant \( C > 0. \) In particular, \( u_G \geq 0 \) in \( \mathbb{R}. \)

**Proof.** Pick \( y \in \mathcal{A}(G) \) and set \( u_G(\cdot) := S_G(y, \cdot) + 1/C, \) where \( C \) is the constant appearing in the statement of Proposition 4.1.

**5.2. Asymptotic analysis**

Let us denote by \( u_G^\dagger \) the (unique) bounded solution of the following discounted equation:

\[ \lambda u + G(x, u') = c(G) \quad \text{in } \mathbb{R} \]

(4.3)

We recall that the critical Eq. (4.1) possesses a coercive solution \( u_G \geq 0, \) see Corollary 4.6, and a bounded subsolution \( v_G \leq 0 \) such that
\[ G(x, v_G'(x)) \leq c(G) - \delta \quad \text{for a.e. } x \in \mathbb{R} \setminus K, \]

for some \( \delta > 0 \) and some compact set \( K \supseteq \text{supp}(V) \), see Theorem 4.2. In view of Proposition 1.15, we infer that the functions \( \{ u_G^\lambda : \lambda > 0 \} \) are equi-Lipschitz and locally equi-bounded on \( \mathbb{R} \), and satisfy

\[ v_G \leq u_G^\lambda \leq u_G \quad \text{in } \mathbb{R}. \quad (4.4) \]

Let us denote by

\[ \omega(\{ u_G^\lambda \}) := \{ u \in \text{Lip}(\mathbb{R}) : u_G^\lambda \Rightarrow u \text{ in } \mathbb{R} \text{ for some sequence } \lambda_k \to 0 \}. \]

We aim at showing that the whole family \( \{ u_G^\lambda : \lambda > 0 \} \) converges to a distinguished function \( u_G^0 \) as \( \lambda \to 0^+ \), namely that \( \omega(\{ u_G^\lambda \}) = \{ u_G^0 \} \). The first step consists in identifying a good candidate \( u_G^0 \) for the limit of the solutions \( u_G^\lambda \) of the discounted equations. To this aim, we consider the family \( \mathcal{E}_{b-}(G) \) of bounded subsolutions \( w : \mathbb{R} \to \mathbb{R} \) of the critical Eq. (4.1) satisfying the following condition

\[ \int_{\mathbb{R}} w(y) \, d\mu(y) \leq 1_0(c(G) - c_f(G)) \quad \text{for every } \mu \in \mathcal{M}(G), \quad (4.5) \]

where \( \mathcal{M}(G) \) denotes the set of projected Mather measures for \( G \) and \( 1_0 \) the indicator function of the set \( \{0\} \) in the sense of convex analysis, i.e. \( 1_0(t) = 0 \) if \( t = 0 \) and \( 1_0(t) = +\infty \) otherwise. In the case considered in this section, the right-hand side term of (4.5) is 0 since \( c(G) = c_f(G) \).

**Lemma 4.7.** The following holds:

\[ \text{supp} (\mu) \subseteq K \quad \text{for all } \mu \in \mathcal{M}(G). \quad (4.6) \]

**Proof.** Let us set \( U := \mathbb{R} \setminus K \) and choose \( r > 0 \) small enough so that the set \( U_r := \{ x \in U : \text{dist}(x, \partial U) > r \} \) is nonempty. By arguing as in the proof of Proposition 3.5 with \( v_G \) in place of \( v \), we infer that, for every fixed \( \varepsilon > 0 \), there exists a function \( v_\varepsilon \in \text{Lip}(\mathbb{R}) \cap C^\infty(\mathbb{R}) \) such that

\[ G(x, v_\varepsilon'(x)) \leq c(G) + \varepsilon \quad \text{in } \mathbb{R} \setminus U_r, \quad G(x, v_\varepsilon'(x)) \leq c(G) - \delta + \varepsilon \quad \text{in } U_r. \quad (4.7) \]

Pick a Mather measure \( \tilde{\mu} \in \mathcal{M}(G) \) and let \( \mu = \pi_1 # \tilde{\mu} \). By exploiting Fenchel’s inequality, the closed character of \( \tilde{\mu} \) and (4.7) we have

\[ 0 = \int_{\mathbb{R} \times \mathbb{R}} (L(x, q) + c(G)) \, d\tilde{\mu}(x, q) \geq \int_{\mathbb{R} \times \mathbb{R}} (v_\varepsilon'(x)q + c(G) - G(x, v_\varepsilon'(x))) \, d\tilde{\mu}(x, q) \]

\[ \geq -\varepsilon + (\delta - \varepsilon) \mu(U_r). \]

By sending first \( \varepsilon \to 0^+ \) and then \( r \to 0^+ \), we conclude that \( \mu(U) = 0 \), as it was to be proved. \( \square \)

From the previous lemma, we infer that the integral in (4.5) is well defined for any continuous function \( w \). Let us also note that the set \( \mathcal{E}_{b-}(G) \) is not empty, since it contains the bounded and nonpositive subsolution \( v_G \) given by Theorem 4.2. Furthermore, the following holds.
Lemma 4.8. The family $\mathcal{S}_b(G)$ is locally uniformly bounded from above in $\mathbb{R}$, i.e.

$$\sup \{ w(x) : u \in \mathcal{S}_b(G) \} < +\infty \quad \text{for all } x \in \mathbb{R}.$$ 

Proof. The family of critical subsolutions is equi–Lipschitz. Call $\kappa$ a common Lipschitz constant. Pick a projected Mather measure $\mu \in \mathcal{W}(G)$. In view of (4.6), for every $w \in \mathcal{S}_b(G)$ we have $\min_K w = \int_{\mathbb{R}} \min_K w \, d\mu \leq \int_{\mathbb{R}} w \, d\mu \leq 0$. Hence, $\max_K w \leq \max_K w - \min_K w \leq \kappa \text{diam}(K) < +\infty$. By using again the fact that $w$ is $\kappa$–Lipschitz, we conclude that $w(x) \leq \kappa \text{dist}(x,K) + \max_K w \leq \kappa (\text{dist}(x,K) + \text{diam}(K))$ for every $w \in \mathcal{S}_b(G)$.

Therefore we can define $u^0_G : \mathbb{R} \to \mathbb{R}$ by

$$u^0_G(x) := \sup_{w \in \mathcal{S}_b(G)} w(x), \quad x \in \mathbb{R}. \quad (4.8)$$

As the supremum of a family of viscosity subsolutions to (4.1), we know that $u^0_G$ is itself a critical subsolution of the same equation. We will obtain later that $u^0_G$ is a solution, see Theorem 4.13 below. Next, let us show that, in the definition of $u^0_G$, we can remove the constraint that critical subsolutions are bounded.

Lemma 4.9. Let $w \in \text{Lip}(\mathbb{R})$ be a subsolution of (4.1). Then there exists a diverging sequence $(r_n)_n$ in $(0, +\infty)$ and a sequence $(w_n)_n \subset \text{Lip}(\mathbb{R}^d)$ of bounded subsolutions to (4.1) such that $B_{r_n} \subseteq \{ w_n = w \}$ for every $n \in \mathbb{N}$. In particular,

$$u^0_G(x) = \sup_{w \in \mathcal{S}_b(G)} w(x) \quad x \in \mathbb{R}, \quad (4.9)$$

where $\mathcal{S}_b(G)$ denotes the family of subsolutions to (4.1) satisfying (4.5).

Proof. Let $w \in \text{Lip}(\mathbb{R})$ be a subsolution to (4.1) and set

$$w_n(x) = \min \{ \max \{ w(x), v_G(x) - n \}, v_G(x) + n \}, \quad x \in \mathbb{R},$$

where $v_G$ is the bounded subsolution to (4.1) provided by Theorem 4.2. As a maximum and minimum of critical subsolutions, $w_n$ is a critical subsolution by Proposition 1.1. Moreover, $w_n$ is bounded and agrees with $w$ on larger and larger balls as $n \to +\infty$. In particular, for $n$ large enough, $w_n \equiv w$ on $K$, therefore such a $w_n$ satisfy (4.5) whenever $w \in \mathcal{S}_b(G)$ in view of (4.6). The remainder of the assertion easily follows from this.

Let $u_G$ be the coercive critical solution obtained according to Corollary 4.6. Since $u_G - \|u_G\|_{L^\infty(K)} \in \mathcal{S}_b(G)$, from the previous Lemma we infer that

$$u^0_G(x) \geq C|x| - 1/C \quad \text{on } \mathbb{R} \text{ for some positive constant } C > 0. \quad (4.10)$$

We now start to study the asymptotic behavior of the discounted value functions $u^\lambda_G$ as $\lambda \to 0^+$ and the relation with $u^0_G$. We begin with the following result:
Proposition 4.10. Let $\lambda > 0$. Then, for every $\mu \in \mathcal{M}(G)$, we have
\[
\int_{\mathbb{R}} u_{\lambda}^{G}(x) \, d\mu(x) \leq 0.
\]

In particular, $u \leq u_{\lambda}^{0}$ on $\mathbb{R}$ for every $u \in \omega(\{u_{\lambda}^{G}\})$.

Proof. By applying Lemma 1.2 with $F(x, p) := \lambda u_{\lambda}^{G}(x) + G(x, p) - c(G)$, we infer that there exists a sequence $(w_n)_n$ of functions in $C^1(\mathbb{R}) \cap \text{Lip}(\mathbb{R})$ such that $\|u_{\lambda}^{G} - w_n\|_{\infty} \leq 1/n$ and
\[
\lambda u_{\lambda}^{G}(x) + G(x, w_n'(x)) \leq c(G) + 1/n \quad \text{for every } x \in \mathbb{R}.
\]

By the Fenchel inequality
\[
L_G(x, q) + G(x, w_n'(x)) \geq w_n'(x) \quad \text{for every } (x, q) \in \mathbb{R} \times \mathbb{R},
\]

Combining these two inequalities we infer
\[
\lambda u_{\lambda}^{G}(x) + w_n'(x) \leq L_G(x, q) + c(G) + \frac{1}{n} \quad \text{for every } (x, q) \in \mathbb{R} \times \mathbb{R}.
\]

Pick $\bar{\mu} \in \mathcal{M}(G)$, and set $\mu = \pi_{\#} \bar{\mu} \in \mathcal{M}(G)$. Since $\bar{\mu}$ is closed and minimizing, we have $\int_{\mathbb{R} \times \mathbb{R}} w_n'(x, q) \, d\bar{\mu}(x, q) = 0$, and $\int_{\mathbb{R} \times \mathbb{R}} L_G(x, q) \, d\bar{\mu}(x, q) = -c(G)$. Therefore, if we integrate (4.11), we obtain
\[
\lambda \int_{\mathbb{R}} u_{\lambda}^{G}(x) \, d\mu(x) \leq \frac{1}{n}.
\]

Since $\lambda > 0$, letting $n \to \infty$ we obtain $\int_{\mathbb{R}} u_{\lambda}^{G}(x) \, d\mu(x) \leq 0$. If $u$ is the uniform limit of $(u_{\lambda_n}^{G})_n$ for some $\lambda_n \to 0$, we know that it is a solution of the critical Eq. (4.1). Moreover, it also has to satisfy $\int_{\mathbb{R}} u(x) \, d\mu(x) \leq 0$ for every projected Mather measure $\mu$. Therefore $u \in \mathcal{S}(G)$ and $u \leq u_{\lambda}^{0}$. $\square$

The next (and final) step consists in showing that $u \geq u_{\lambda}^{0}$ in $\mathbb{R}$ whenever $u \in \omega(\{u_{\lambda}^{G}\})$. For this, we will exploit the representation formula (1.14) for $u_{\lambda}^{G}$ and Proposition 1.16.

Let us fix $x \in \mathbb{R}$. For every $\lambda > 0$, we choose $\gamma_{\lambda}^{x} : (-\infty, 0] \to \mathbb{R}$ with $\gamma_{\lambda}^{x}(0) = x$ to be an optimal curve for $u_{\lambda}^{G}$, cf. Proposition 1.16, and we define a measure $\tilde{\mu}_{\lambda}^{x} = \tilde{\mu}_{\lambda}^{x}$ on $\mathbb{R} \times \mathbb{R}$ via (3.6). The following holds:

Proposition 4.11. Let $x \in \mathbb{R}$ and $(\lambda_n)_n$ be an infinitesimal sequence. Then the set of measures $(\tilde{\mu}_{\lambda}^{x})_n$ defined above is a tight family of probability measures, whose supports are all contained in $\mathbb{R} \times B_{\tilde{k}}$ for some $\tilde{k} > 0$. In particular, they are relatively compact in the space of probability measures on $\mathbb{R} \times \mathbb{R}$ with respect to the narrow convergence. Furthermore, if $(\tilde{\mu}_{\lambda}^{x})_n$ is narrowly converging to $\tilde{\mu}$, then $\tilde{\mu}$ is a (closed) Mather measure.

Proof. Call $\tilde{k}$ a common Lipschitz constant for the family of curves $\{\gamma_{\lambda}^{x} : \lambda > 0\}$, according to Proposition 1.16. Then the measures $\tilde{\mu}_{\lambda}^{x}$ are all probability measures and all have support contained in the compact set $\mathbb{R} \times B_{\tilde{k}}$, as it can be easily checked by their definition. In order to prove that the set of probability measures $(\tilde{\mu}_{\lambda}^{x})_n$ is tight, it
is enough to prove that the set of measures \((\mu^\lambda_n)_n\) is tight, where each measure \(\mu^\lambda_n := \pi_1\#\tilde{\mu}^\lambda_n\) is defined as the push-forward of the measure \(\tilde{\mu}^\lambda_n\) via the projection map \(\pi_1 : \mathbb{R} \times \mathbb{R} \ni (x, p) \mapsto x \in \mathbb{R}\). Moreover, since any finite family of probability measure on a Polish space is tight, it suffices to show the following:

**Claim:** there exists a constant \(C > 0\) such that \(\mu^\lambda_n(\mathbb{R} \setminus K) < C\lambda\) for every \(\lambda > 0\).

But this follows as a direct application of Proposition 3.5 with \(v := v_G\) and \(U := \mathbb{R} \setminus K\). By Prohorov’s Theorem, see for instance [36, Theorem 5.1], we infer that the set of probability measures \((\tilde{\mu}^\lambda_n)_n\) is relatively compact in \(\mathcal{P}(\mathbb{R} \times \mathbb{R})\) with respect to narrow convergence.

Let now assume that \((\tilde{\mu}^\lambda_n)_n\) is narrowly converging to \(\tilde{\mu}\) for some \(\lambda_n \to 0\). Arguing as in the proof of Proposition 3.7-(i), we get that \(\tilde{\mu}\) is a closed probability measure. It remains to show that \(\int_{\mathbb{R} \times \mathbb{R}} L_G(x, q) \, d\tilde{\mu}(x, q) = -c(G)\). We have

\[
\int_{\mathbb{R} \times \mathbb{R}} (L_G(x, q) + c(G)) \, d\tilde{\mu}(x, q) = \lim_{n \to \infty} \int_{\mathbb{R} \times \mathbb{R}} (L_G(x, q) + c(G)) \, d\tilde{\mu}^\lambda_n(x, q)
\]

\[
= \lim_{n \to \infty} \int_{-\infty}^0 (e^{\lambda_0 u_G}(s))^{\frac{\partial u_G^{\lambda_0} - \lambda_0 u_G}{\partial q} + \lambda_0 u_G} (s) + c(G)) \, ds = \lim_{n \to \infty} \lambda_n u_G^{\lambda_n}(x) = 0,
\]

where the last equality follows from the fact that \(\lambda u_G^{\lambda_n}(x) \to 0\) in view of (4.4).

The following lemma will be crucial for the proof of our main result, see Theorem 4.13 below.

**Lemma 4.12.** Let \(w\) be any bounded critical subsolution. For every \(\lambda > 0\) and \(x \in \mathbb{R}\)

\[
u_G^{\lambda}(x) \geq w(x) - \int_{\mathbb{R} \times \mathbb{R}} w(y) \, d\tilde{\mu}^\lambda_n(y, q).
\]

**Proof.** It suffices to remark that the measure \(\tilde{\mu}^\lambda_n\) agrees with the measure \(\tilde{\mu}^\lambda_{n, 1}\) defined via (3.10) where we have taken \(r = + \infty\). The assertion follows from Proposition 3.8.

We are now ready to prove our main theorem:

**Theorem 4.13.** The functions \(u_G^{\lambda}\) uniformly converge to the function \(u_G^0\) given by (4.8) locally on \(\mathbb{R}\) as \(\lambda \to 0^+\). In particular, as an accumulation point of \(u_G^{\lambda}\) as \(\lambda \to 0^+\), the function \(u_G^0\) is a viscosity solution of (4.1).

**Proof.** By Theorem 1.14 and Proposition 1.15, we know that the functions \(u_G^{\lambda}\) are equi-Lipschitz and locally qu–bounded on \(\mathbb{R}\), hence it is enough, by the Ascoli–Arzelà theorem, to prove that any converging subsequence has \(u_G^0\) as limit.

Let \(\lambda_n \to 0\) be such that \(u_G^{\lambda_n}\) locally uniformly converge to some \(u \in C(\mathbb{R})\). We have seen in Proposition 4.10 that

\[u(x) \leq u_G^{\lambda_n}(x) \quad \text{for every } x \in \mathbb{R}.
\]

To prove the opposite inequality, let us fix \(x \in \mathbb{R}\). Let \(w\) be a bounded critical subsolution. By Lemma 4.12, we have
By Proposition 4.11, extracting a further subsequence, we can assume that \( \tilde{\mu}_x^\alpha \) converges narrowly to a Mather measure \( \tilde{\mu} \) whose projection on \( \mathbb{R} \) is denoted by \( \mu \). Passing to the limit in the last inequality, we get

\[
\mu(x) \geq w(x) - \int_{\mathbb{R}} w(y) \, d\tilde{\mu}_x^\alpha(y, q).
\]

If we furthermore assume that \( w \in \mathcal{Z}_{b-} \), the set of bounded subsolutions satisfying (4.5), we obtain \( \int_{\mathbb{R}} w(y) \, d\mu(y) \leq 0 \), hence \( u \geq w \). We conclude that \( u \geq u^0_G = \sup_{w \in \mathcal{Z}_{b-}} w \) in view of Lemma 4.9.

6. The case \( c(G) = c(H) > c_f(H) \)

In this section we shall prove the asymptotic convergence in the case \( c(G) = c(H) > c_f(H) \). Throughout this section, we shall denote by \( c \) this common critical constant for notational simplicity. We start with some remarks on the critical equations associated with \( H \) and \( G \).

6.1. Critical equations

Let us consider the critical equations

\[
G(x, u') = c \quad \text{in } \mathbb{R}, \quad (5.1)
\]
\[
H(x, u') = c \quad \text{in } \mathbb{R}. \quad (5.2)
\]

We introduce a piece of notation first. Let us set \( \bar{y}_V := \max(\text{supp}(V)) \), \( \underline{y}_V := \min(\text{supp}(V)) \), and, for every \( x \in \mathbb{R} \),

\[
Z_H(x) := \{ p \in \mathbb{R} : H(x, p) \leq c \} = [p^-_H(x), p^+_H(x)],
\]
\[
Z_G(x) := \{ p \in \mathbb{R} : G(x, p) \leq c \} = [p^-_G(x), p^+_G(x)].
\]

From the fact that \( c > c_f(H) \) we infer \( \rho := \min_{x \in \mathbb{R}} (p^+_H(x) - p^-_H(x)) > 0 \). Furthermore, one of the following circumstances occurs, cf. Theorem 2.3 (with \( \theta = 0 \))²:

A. \( \int_0^1 p^+_H(x) \, dx = 0 > -\rho > \int_0^1 p^-_H(x) \, dx \);
B. \( \int_0^1 p^+_H(x) \, dx > \rho > 0 = \int_0^1 p^-_H(x) \, dx \).

The following holds:

**Theorem 5.1.** There exists a bounded solution \( u_G \) to (5.1).

**Proof.** Let us assume we are in case (A). Set \( u_G(x) := \int_0^x p^+_G(z) \, dz \) for all \( x \in \mathbb{R} \). It is clear that \( u_G \) is a Lipschitz solution to (5.1). To see that it is bounded, simply notice that

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²We recall that \( c(H) = h(0) \), cf. Section 2.
\(u_G(y' - n) = u_G(y), \quad u_G(y' + n) = u_G(y)\) for all \(n \in \mathbb{N}\), since \(p^+_G = p^+_H\) in \(\mathbb{R} \setminus \text{supp}(V)\) and the mean of the 1-periodic function \(p^+_H\) is 0. The proof in case (B) is analogous.

Furthermore, we have the following additional information:

**Proposition 5.2.** Let \(v\) be a bounded solution to (5.1). Then:

(i) in case (A),
\[
v'(x) = p^+_H(x) \quad \text{for all } x \geq \bar{y}_V := \max(\text{supp}(V)). \tag{5.3}
\]

(ii) in case (B),
\[
v'(x) = p^-_H(x) \quad \text{for all } x \leq \underline{y}_V := \min(\text{supp}(V)). \tag{5.4}
\]

**Proof.** Let us prove (i). Let us assume by contradiction that (5.3) does not hold. Being \(v\) a viscosity solution to (5.1), its derivative \(v'\) can jump only downwards. Since
\[
p^+_G(x) - p^-_G(x) = p^+_H(x) - p^-_H(x) \geq \rho > 0 \quad \text{for all } x \in \mathbb{R} \setminus \text{supp}(V),
\]
there exists a point \(y \geq \bar{y}_V\) such that \(v'(x) = p^-_G(x)\) for all \(x > y\). Hence, for all \(n \in \mathbb{N}\) we have
\[
v(y + n) = v(y) + \int_y^{y+n} p^-_G(z)dz = v(y) + \int_y^{y+n} p^+_H(z)dz
\]
\[
\quad = v(y) - n \int_0^1 (p^+_H(z) - p^-_H(z))dz \leq v(y) - n\rho,
\]
in contrast with the fact that \(v\) is bounded. The proof of item (ii) is analogous. \(\square\)

**Remark 5.3.** Proposition 5.2 holds in particular in the case \(G := H\) (i.e. when \(V \equiv 0\)) with equality in (5.3) and in (5.4) holding for every \(x \in \mathbb{R}\), by periodicity of the solution \(v\).

### 6.2. Asymptotic convergence

Let us denote by \(u^\lambda_G, u^\lambda_H\) the solutions of the discounted equations
\[
\lambda u + G(x, u) = c \quad \text{in } \mathbb{R}, \tag{5.5}
\]
\[
\lambda u + H(x, u) = c \quad \text{in } \mathbb{R}. \tag{5.6}
\]

We want to prove that the solutions \(u^\lambda_G\) converge, as \(\lambda \to 0^+\), to a specific solution of \(u^0_G\) the critical Eq. (5.1). In view of [2], we know that this is true for the solutions of (5.6), i.e. \(u^\lambda_H \Rightarrow u^0_H\) in \(\mathbb{R}\) where \(u^0_H\) is a solution to the critical Eq. (5.6), and this convergence is actually uniform in \(\mathbb{R}\) since all these functions are \(\mathbb{Z}\)-periodic. Furthermore, \(u^0_H\) is identified as the maximal (sub-)solution to (5.2) such that
where \( \mathcal{M}(H) \) denotes the set of projected Mather measures for \( H \).

Let \( u_G \) be the bounded solution of (5.1) given by Theorem 5.1. Since \( u_G^- := u_G - ||u_G||_\infty \) and \( u_G^+ := u_G + ||u_G||_\infty \) are, respectively, a bounded negative and positive solution of (5.1), we derive from Proposition 1.15 that the functions \( \{u_G^\lambda : \lambda > 0\} \) are equi-bounded and equi-Lipschitz on \( \mathbb{R} \), and satisfy

\[
-2||u_G||_\infty \leq u_G^\lambda(x) \leq u_G^\lambda(x) \leq u_G^\lambda(x) \leq 2||u_G||_\infty \quad \text{for all } x \in \mathbb{R}. \tag{5.8}
\]

Let us denote by

\[
\omega\left(\left\{u_G^\lambda\right\}\right) := \left\{u \in \text{Lip}(\mathbb{R}) : u_G^\lambda \Rightarrow u \text{ in } \mathbb{R} \text{ for some sequence } \lambda_k \to 0\right\}.
\]

Of course, our aim is to show that this set reduces to \( \{u_G^0\} \). We first have to identify a good candidate for \( u_G^0 \). To this aim, we start with some preliminary remarks. For every \( \lambda > 0 \), let us set

\[
Z_H^\lambda(x) := \left\{p \in \mathbb{R} : H(x,p) \leq c - \lambda u_G^\lambda(x) \right\} = \left[p_H^{-\lambda}(x), p_H^{+\lambda}(x)\right] \quad \text{for all } x \in \mathbb{R},
\]

\[
Z_G^\lambda(x) := \left\{p \in \mathbb{R} : G(x,p) \leq c - \lambda u_G^\lambda(x) \right\} = \left[p_G^{-\lambda}(x), p_G^{+\lambda}(x)\right] \quad \text{for all } x \in \mathbb{R}.
\]

We shall need the following technical lemma.

**Lemma 5.4.** There exists a modulus of continuity \( \omega \) such that, for every \( \lambda \in (0,1) \),

\[
(i) \quad ||p_H^{-\lambda} - p_G^{-\lambda}||_\infty \leq \omega(\lambda), \quad ||p_H^{+\lambda} - p_G^{+\lambda}||_\infty \leq \omega(\lambda);
\]

\[
(ii) \quad ||p_H^{-\lambda} - p_H^{-\lambda}||_\infty \leq \omega(\lambda), \quad ||p_H^{+\lambda} - p_H^{+\lambda}||_\infty \leq \omega(\lambda).
\]

**Proof.** Let \( C > 0 \) be such that

\[
||u_H^\lambda||_\infty + ||u_G^\lambda||_\infty < C \quad \text{for all } \lambda \in (0,1).
\]

Within this proof, we shall use the following temporary notation:

\[
Z_H(x,a) := \left\{p \in \mathbb{R} : H(x,p) \leq a \right\} = \left[p_H(x,a), p_H^{+}(x,a)\right],
\]

\[
Z_G(x,a) := \left\{p \in \mathbb{R} : G(x,p) \leq a \right\} = \left[p_G(x,a), p_G^{+}(x,a)\right].
\]

Furthermore, we shall denote by \( \text{dom}(Z_H) \) (respectively, \( \text{dom}(Z_G) \)) the set of points \( (x,a) \in \mathbb{R} \times \mathbb{R} \) such that \( Z_H(x,a) \) (respectively, \( Z_G(x,a) \)) is nonempty. Notice that

\[
\text{dom}(Z_H) \setminus (\text{supp}(V) \times \mathbb{R}) = \text{dom}(Z_G) \setminus (\text{supp}(V) \times \mathbb{R}).
\]

The functions \((x,a) \to p_H^{+}(x,a)\) and \((x,a) \to p_G^{+}(x,a)\) are continuous in \( \text{dom}(Z_H) \) and \( \text{dom}(Z_G) \), respectively, in view of Lemma 1.9 and Remark 1.10. Furthermore, the functions \( p_H^{+}(x,a) \) are 1-periodic in \( x \) and \( p_H^{+}(x,a) = p_H^{+}(x,a) \), \( p_G^{+}(x,a) = p_G^{+}(x,a) \) for every \( (x,a) \in \text{dom}(Z_G) \setminus (\text{supp}(V) \times \mathbb{R}) \). We derive that there exists a modulus of continuity \( \tilde{\omega} \) such that

\[
|p_G(x,a) - p_G(x,b)| \leq \tilde{\omega}(|a - b|), \quad |p_G^{+}(x,a) - p_G^{+}(x,b)| \leq \tilde{\omega}(|a - b|) \tag{5.9}
\]
for all \((x,a),(x,b)\) \in \text{dom}(Z_G).\) Now notice that \(p^+_G(x) = p^+_G(x,c)\) and \(p^\pm_G(x) = p^\pm_G(x,c - \lambda u^\pm_G(x))\) for all \(x \in \mathbb{R}^d.\) Furthermore, it is clear that \((x, c)\) and \((x, c - \lambda u^\pm_G(x))\) both belong to \(\text{dom}(Z_G)\) for every \(x \in \mathbb{R}.\) By applying (5.9) with \(a := c\) and \(b := c + \lambda u^\pm_G(x),\) we get assertion (i) with \(\omega(h) := \tilde{\omega}(Ch)\) for every \(h \geq 0.\) The same argument with \(H\) in place of \(G\) gives assertion (ii). \(\square\)

We proceed by showing the following result.

**Proposition 5.5.** There exist \(\lambda_0 > 0\) and \(N_0 \in \mathbb{N}\) such that for every \(\lambda \in (0, \lambda_0)\) the following holds:

(i) in case (A),
\[
(u^\pm_G)'(x) = p^\pm_G(x) \quad \text{for all } x \in (-\infty, y_V - N_0) \cup (y_V, +\infty)
\]
\[
(u_H^\pm)'(x) = p^\pm_H(x) \quad \text{for all } x \in \mathbb{R};
\]

(ii) in case (B),
\[
(u^\pm_G)'(x) = p^\pm_G(x) \quad \text{for all } x \in (-\infty, y_V) \cup (y_V + N_0, +\infty)
\]
\[
(u_H^\pm)'(x) = p^\pm_H(x) \quad \text{for all } x \in \mathbb{R}.
\]

**Proof.** We only prove (i), being the proof of (ii) similar. Let \(C > 0\) be such that
\[
\|u^\pm_H\|_{\infty} + \|u^\pm_G\|_{\infty} < C \quad \text{for all } \lambda \in (0, 1).
\]

For every fixed \(\varepsilon > 0,\) choose \(\lambda(\varepsilon) > 0\) small enough so that \(\lambda C < \varepsilon\) for all \(\lambda \in (0, \lambda(\varepsilon)).\) Now we choose \(\tilde{\varepsilon} > 0\) small enough so that \(c - \tilde{\varepsilon} > c_f(H)\) and, thanks to Lemma 5.4,
\[
\|p^\pm_G - p^\pm_H\|_{\infty} < \frac{\rho}{4}, \quad \|p^\pm_G - p^\pm_H\|_{\infty} < \frac{\rho}{4} \quad \text{for all } \lambda \in (0, \lambda(\tilde{\varepsilon})),
\]
where \(\rho := \min_{x \in \mathbb{R}}(p^+_G(x) - p^+_H(x)) > 0.\) Set \(\lambda_0 := \lambda(\tilde{\varepsilon})\) and fix \(\lambda \in (0, \lambda_0).\) Due to the fact that \(p^\pm_G \equiv p^\pm_H\) on \(\mathbb{R} \setminus \text{supp}(V),\) we infer
\[
p^\pm_G(x) - p^\pm_G(x) \geq p^+_G(x) - p^-_G(x) - \frac{\rho}{2} \geq \frac{\rho}{2} \quad \text{for all } x \in \mathbb{R} \setminus \text{supp}(V)
\]
and
\[
\int_y^{y+1} p^\pm_G(z)dz \leq \frac{\rho}{4} + \int_y^{y+1} p^-_G(z)dz \quad \text{for all } y \in \mathbb{R},
\]
in particular
\[
\int_y^{y+1} p^\pm_G(z)dz \leq -\frac{3}{4}\rho \quad \text{whenever } (y, y+1) \cap \text{supp}(V) = \emptyset
\]
due to (A). Let us assume that (5.10) does not hold. Being \(u^\pm_G\) a viscosity solution of (5.5), its derivative can jump only downwards. In view of (5.12), two possibilities may occur:
a. \((u_G^\epsilon)'(x) = p_G^{\epsilon^-}(x)\) for all \(x > y\) for some \(y \geq \bar{y}_V\);
b. there exists \(y \leq \underline{y}_V\) such that
\[(u_G^\epsilon)'(x) = p_G^{\epsilon^+}(x) \quad \text{in } (-\infty, y) \quad \text{and} \quad (u_G^\epsilon)'(x) = p_G^{\epsilon^-}(x) \quad \text{in } (y, \underline{y}_V).\]

Let us proceed to show that instance (a) never occurs. Otherwise, we would have
\[u_G^\epsilon(y + n) - u_G^\epsilon(y) = \int_y^{y+n} p_G^{\epsilon^-}(z) \, dz \leq -\frac{3}{4}n\rho \quad \text{for all } n \in \mathbb{N},\]
in contradiction with the fact that \(u_G^\epsilon\) is bounded. In case (b), let us denote by \(N\) the integer part of \(\underline{y}_V - y\). Then
\[u_G^\epsilon(\underline{y}_V) - u_G^\epsilon(\underline{y}_V - N) = \int_{\underline{y}_V}^{\underline{y}_V - N} N^{\epsilon^+} p_G^{\epsilon^-}(z) \, dz \leq -\frac{3}{4}N\rho,\]
yielding
\[N \leq \frac{8||u_G^\epsilon||_\infty}{3\rho} \leq \frac{8C}{3\rho}.
\]
in view of (5.11). Assertion (5.10) follows by setting \(N_0 := 1 + [8C/(3\rho)]\) (where the symbol \([h]\) stands for the integer part of \(h\)). The same argument with \(V \equiv 0\) proves the second assertion in (i) with same constant \(\lambda_0 > 0\).

In the remainder of this section, \(\tilde{\epsilon}\) will denote a positive parameter chosen small enough so that \(c - \tilde{\epsilon} > c_f(H)\), and \(\lambda_0 = \lambda(\tilde{\epsilon})\) will denote the positive parameter chosen as in the proof of Proposition 5.5.

**Proposition 5.6.** Let \(\lambda \in (0, \lambda_0)\), \(x \in \mathbb{R}\) and \(\gamma_x^\epsilon : (-\infty, 0] \to \mathbb{R}\) be a minimizing curve for \(u_G^\epsilon(x)\). The following holds:

(i) in case (A), \(\gamma_x^\epsilon(t) \leq \max\{x, \bar{y}_V\}\) for all \(t \leq 0\);
(ii) in case (B), \(\gamma_x^\epsilon(t) \geq \min\{x, \underline{y}_V\}\) for all \(t \leq 0\).

**Proof.** We only prove (i), being the proof of item (ii) analogous. Let us assume by contradiction that there exists \(a < 0\) such that \(\gamma_x^\epsilon(a) > \max\{x, \bar{y}_V\}\). Let us set
\[b := \sup\left\{\beta \in (a, 0) : \gamma_x^\epsilon(t) > \max\{x, \bar{y}_V\} \quad \text{for all } t \in (a, \beta)\right\}.
\]
Since \(\gamma_x^\epsilon\) is optimal for \(u_G^\epsilon(x)\), in view of Remark 1.17 and of Proposition 5.5 we have
\[\gamma_x^\epsilon(t) \in \partial_p G(\gamma_x^\epsilon(t), (u_G^\epsilon)'(\gamma_x^\epsilon(t))) = \partial_p H(\gamma_x^\epsilon(t), p_G^{\epsilon^+}(\gamma_x^\epsilon(t))) \quad \text{for } \text{a.e. } t \in (a, b),\]
hence \(\gamma_x^\epsilon(t) > 0\) for a.e. \(t \in (a, b)\) due to the fact that (since we are in case (A))
\[H(\gamma_x^\epsilon(t), p_G^{\epsilon^+}(\gamma_x^\epsilon(t))) = G(\gamma_x^\epsilon(t), (u_G^\epsilon)'(\gamma_x^\epsilon(t))) = c - \lambda u_G^\epsilon(\gamma_x^\epsilon(t)) > c - \tilde{\epsilon} > c_f(H)\]
for every \(t \in (a, b)\) and \(c_f(H) \geq \min_p H(\gamma_x^\epsilon(t), p)\). We derive \(\gamma_x^\epsilon(b) > \max\{x, \bar{y}_V\}\).

This gives the sought contradiction when \(b < 0\), by maximality of \(b\), and also when \(b = 0\), since \(\gamma_x^\epsilon(0) = x\).

The previous argument in the case \(V \equiv 0\) gives the following result.
Corollary 5.7. Let $\lambda \in (0, \lambda_0)$, $x \in \mathbb{R}$ and $\gamma^x_\lambda : (-\infty, 0] \to \mathbb{R}$ be a minimizing curve for $u^\lambda_H(x)$. In case (A), $\gamma^x_\lambda(t) < x$ for all $t < 0$. In case (B), $\gamma^x_\lambda(t) > x$ for all $t < 0$.

As a consequence of the information gathered, we obtain the following relation between the solutions of the discounted Eqs. (5.5) and (5.6).

**Corollary 5.8.** Let $\lambda \in (0, \lambda_0)$. The following holds:

(i) in case (A), $u^\lambda_G(x) \leq u^\lambda_H(x)$ for all $x \in (-\infty, y_V)$;
(ii) in case (B), $u^\lambda_G(x) \leq u^\lambda_H(x)$ for all $x \in (y_V, +\infty)$.

**Proof.** Let us prove (i). Let $x < y_V$ and $\gamma^x_\lambda : (-\infty, 0] \to \mathbb{R}$ be a minimizing curve for $u^\lambda_H(x)$. According to Corollary 5.7, the support of $\gamma^x_\lambda$ does not intersect $\text{supp}(V)$. We derive

$$u^\lambda_H(x) = \int_{-\infty}^{0} e^{\lambda s}(L_H(\gamma^x_\lambda \dot{\gamma}^x_\lambda) + c) \, ds = \int_{-\infty}^{0} e^{\lambda s}(L_G(\gamma^x_\lambda \dot{\gamma}^x_\lambda) + c) \, ds \geq u^\lambda_G(x).$$

It is time to make our guess about the expected limit $u^0_G$ of the solutions $u^\lambda_G$ of the discounted Eq. (5.5) as $\lambda \to 0^+$. We introduce a piece of notation first: let us denote by $\mathcal{S}_{b-}(G)$ the family of bounded subsolutions $v : \mathbb{R} \to \mathbb{R}$ of the critical Eq. (5.1) satisfying

$$\int_{\mathbb{R}} v(y) \, d\mu(y) \leq 1_0(c(G) - c_f(G)) \quad \text{for every } \mu \in \mathcal{M}(G),$$

where we have denoted by $\mathcal{M}(G)$ the set of projected Mather measures for $G$ and by $1_0$ the indicator function of the set $\{0\}$ in the sense of convex analysis, i.e. $1_0(t) = 0$ if $t = 0$ and $1_0(t) = +\infty$ otherwise. Note that constraint (5.14) is empty whenever $c(G) > c_f(G)$. Motivated by the information gathered so far, we make the following guess:

(a) in case (A), for every $x \in \mathbb{R}$ we set

$$u^0_G(x) := \sup \left\{ v(x) : v \in \mathcal{S}_{b-}(G), v \leq u^0_H \text{ in } (-\infty, y_V) \right\};$$

(b) in case (B), for every $x \in \mathbb{R}$ we set

$$u^0_G(x) := \sup \left\{ v(x) : v \in \mathcal{S}_{b-}(G), v \leq u^0_H \text{ in } (y_V, +\infty) \right\}.$$

Our next proposition shows in particular that the sets appearing at the right-hand side of formulae (5.15) and (5.16) are nonempty.

**Proposition 5.9.** Let $u \in \omega(\{u^\lambda_G\})$. Then

$$\int_{\mathbb{R}} u \, d\mu \leq 0 \quad \text{for all } \mu \in \mathcal{M}(G).$$

Furthermore, $u \leq u^0_H$ in $(-\infty, y_V)$ in case (A), while $u \leq u^0_H$ in $(y_V, +\infty)$ in case (B). In particular, $u \leq u^0_G$ in $\mathbb{R}$. 

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Proof. By arguing as in the proof of Proposition 4.10 we infer that $u_G^\lambda$ satisfies (5.17) for every $\lambda > 0$, hence the same holds for $u \in \omega(\{u_G^\lambda\})$. Furthermore, in view of Corollary 5.8, we also know that $u \leq u_H^0$ in $(-\infty, y_V)$ if we are in case (A), and $u \leq u_H^0$ in $(y_V, +\infty)$, if we are in case (B). In either case, $u \leq u_G^0$ in $\mathbb{R}$. \qed

Next, we show that $u_G^0$ is well-defined.

Proposition 5.10. The function $u_G^0$ is a bounded viscosity subsolution of Eq. (5.1).

Proof. As a supremum of a nonempty and equi-Lipschitz family of subsolutions to (5.1), the function $u_G^0$ is a subsolution to the same equation provided it is finite-valued. Let us then prove that $u_G^0$ is bounded on $\mathbb{R}$. In view of Proposition 5.9, it is enough to show that $u_G^0$ is bounded from above. Let us assume for definiteness that we are in case (A). Let us set

$$\tilde{u}(x) := u_H^0(y_V) + \int_{y_V}^x p_G^+(z) \, dz \quad \text{for all } x \in \mathbb{R}.$$ 

It is easily seen that $\tilde{u}'(x) = p_G^+(x)$ for every $x \in \mathbb{R}$, in particular $\tilde{u}$ is a (classical) solution to (5.1) such that $\tilde{u} \equiv u_H^0$ on $(-\infty, y_V]$, cf. Remark 5.3. Furthermore,

$$\tilde{u}(x) - \tilde{u}(y_V) = \int_{y_V}^x p_G^+(z) \, dz = \int_{y_V}^x p_H^+(z) \, dz = u_H^0(x) - u_H^0(y_V) \quad \text{for all } x \geq y_V,$$

thus showing that $\tilde{u}$ is bounded. Let $v$ be a bounded subsolution to (5.1) such that $v \leq u_H^0$ in $(-\infty, y_V)$. For every $x > y_V$ we have

$$v(x) = v(y_V) + \int_{y_V}^x v'(z) \, dz \leq u_H^0(y_V) + \int_{y_V}^x p_G^+(z) \, dz = \tilde{u}(x).$$

This shows that $u_G^0(x) \leq \tilde{u}(x) \leq ||\tilde{u}||_\infty$ for every $x \in \mathbb{R}$. The proof in case (B) is analogous. \qed

We are now in position to prove the asymptotic convergence.

Theorem 5.11. The functions $u_G^\lambda$ uniformly converge to the function $u_G^0$ given either by (5.15) or by (5.16) locally on $\mathbb{R}$ as $\lambda \to 0^+$. In particular, as an accumulation point of $u_G^\lambda$ as $\lambda \to 0^+$, the function $u_G^0$ is a viscosity solution of (5.1).

Proof. By Proposition 1.15 we know that the functions $u_G^\lambda$ are equi-Lipschitz and equi-bounded, hence it is enough, by the Ascoli–Arzelà theorem, to prove that any converging subsequence has $u_G^0$ as limit.

Let $\lambda_n \to 0$ be such that $u_G^{\lambda_n}$ locally uniformly converge to some $u \in C(\mathbb{R})$. We have seen in Proposition 5.9 that

$$u(x) \leq u_G^0(x) \quad \text{for every } x \in \mathbb{R}.$$ 

Let us prove the opposite inequality. Let us assume, for definiteness, that we are in case (A). Fix $x \in \mathbb{R}$ and pick $v \in \mathcal{E}_{b-}(G)$ such that $v \leq u_H^0$ in $(-\infty, y_V]$. For every $\lambda > 0$, let $\gamma^\lambda_x : (-\infty, 0] \to \mathbb{R}$ with $\gamma^\lambda_x(0) = x$ to be a monotone minimizer for the variational
formula related to $u^G_C(x)$, see Proposition 1.16 and Theorem 1.18. Choose $r > 0$ big enough so that $\{x\} \cup \{y_V, \tilde{y}_V\} \subseteq B_r$ and let $\tilde{\mu}_{x,1}^2, \tilde{\mu}_{x,2}^2$ be the probability measures on $\mathbb{R} \times \mathbb{R}$ defined via (3.10) and (3.11), respectively. When $T^2_x < +\infty$, we infer from Proposition 5.6 that $\gamma^2_T(((-\infty, -T^2_x)) \subseteq (\infty, -r)$, consequently $\text{supp}(\tilde{\mu}_x^2) \subseteq (-\infty, -r) \times \mathbb{R}$. In view of Proposition 3.8 and of the fact that $v \leq u^0_H$ in $(-\infty, -r)$, we get, for every $\lambda > 0$,

$$u^G_C(x) \geq v(x) - \left( \theta^2_x \int_{\mathbb{R} \times \mathbb{R}} v(y) \, d\tilde{\mu}_{x,1}^2(y, q) + (1 - \theta^2_x) \int_{\mathbb{R} \times \mathbb{R}} u^0_H(y) \, d\tilde{\mu}_{x,2}^2(y, q) \right).$$

Now set $\lambda := \lambda_n$ and send $n \to +\infty$ in the inequality above. In view of Proposition 3.6, there exist measures $\tilde{\mu}_{1,x} \in \mathcal{P}(\mathbb{R} \times \mathbb{R})$ and $\tilde{\mu}_{2,x} \in \mathcal{P}(\mathbb{T}^1 \times \mathbb{R})$ and $\theta \in [0, 1]$ such that

$$u(x) \geq v(x) - \theta \int_{\mathbb{R} \times \mathbb{R}} v(y) \, d\tilde{\mu}_{x,1}(y, q) - (1 - \theta) \int_{\mathbb{T}^1 \times \mathbb{R}} u^0_H(y) \, d\tilde{\mu}_{x,2}(y, q). \tag{5.18}$$

Furthermore, in view of Propositions 3.7,

$$\int_{\mathbb{R} \times \mathbb{R}} v(y) \, d\tilde{\mu}_{x,1}(y, q) \leq 0 \quad \text{if } \theta \neq 0, \quad \int_{\mathbb{T}^1 \times \mathbb{R}} u^0_H(y) \, d\tilde{\mu}_{x,2}(y, q) \leq 0 \quad \text{if } \theta \neq 1.$$

By exploiting this information in (5.18), we get $u(x) \geq v(x)$. Hence

$$u(x) \geq \sup \left\{ v(x) : v \in \mathcal{V}_{b_+}(G), v \leq u^0_H \quad \text{in } (-\infty, y_V) \right\} = u^G_C(x)$$

for all $x \in \mathbb{R}$, as it was to be shown. The proof for the case (B) is analogous. \hfill \Box

### 7. The case $c(G) = c(H) = c_f(H)$

In this section we shall prove the asymptotic convergence when $c(G) = c(H) = c_f(H)$. Since $c(G) \geq c_f(G) \geq c_f(H)$, in this case we furthermore have $c(G) = c(H) = c_f(G) = c_f(H)$. We will denote by $c$ this common constant, for notational simplicity. We start with some remarks on the critical equations associated with $H$ and $G$.

#### 7.1. Critical equations

Let us therefore consider the critical equations

$$G(x, u') = c \quad \text{in } \mathbb{R}, \tag{6.1}$$

$$H(x, u') = c \quad \text{in } \mathbb{R}. \tag{6.2}$$

As in Section 6, we set $\bar{y}_V := \max(\text{supp}(V)), y_V := \min(\text{supp}(V))$, and, for every $x \in \mathbb{R},$

$$Z_H(x) := \left\{ p \in \mathbb{R} : H(x, p) \leq c \right\} = \left[ p_{H^-}(x), p_{H^+}(x) \right],$$

$$Z_G(x) := \left\{ p \in \mathbb{R} : G(x, p) \leq c \right\} = \left[ p_{G^-}(x), p_{G^+}(x) \right].$$
In view of Theorem 2.3 (with \( \theta = 0 \))\(^3\), the following holds:

\[
P_H^- := \int_0^1 p_H^-(z) \, dz \leq 0 \leq \int_0^1 p_H^+(z) \, dz =: P_H^+.
\]  

(6.3)

Let us denote by \( S_H \) and \( S_G \) the critical semi-distance associated with \( H \) and \( G \) via (1.8) with \( a = c \), respectively. A direct computation shows that

\[
S_H(y, x) = \int_y^x p_H^+(z) \, dz, \quad S_G(y, x) = \int_y^x p_G^+(z) \, dz \quad \text{if} \quad y \leq x
\]

\[
S_H(y, x) = \int_x^y - p_H^-(z) \, dz, \quad S_G(y, x) = \int_x^y - p_G^-(z) \, dz \quad \text{if} \quad y > x
\]

As a consequence, we derive the following result:

**Proposition 6.1.** For every \( y \in \mathbb{R} \) and \( n \in \mathbb{N} \), we have

\[
S_H(y, y + n) = nP_H^+, \quad S_H(y, y - n) = n(-P_H^-).
\]

Furthermore, there exists positive constants \( C \) and \( \kappa \), such that

\[-C \leq S_G(y, x) \leq \kappa|x - y| \quad \text{for all} \ x, y \in \mathbb{R}.
\]

**Proof.** The first assertion is apparent from the formulae for \( S_H \) provided above and the periodicity of \( p_H^+ \). Let us consider the second assertion. The lower bound for \( S_G \) follows via the same argument used in the proof of Proposition 4.1, with the only difference that here \( \delta = 0 \) since \( c(G) = c(H) \). The upper bound is a general fact already remarked in Section 2.3. \( \square \)

We record here for later use the following lemma.

**Lemma 6.2.** For every \( y \in \mathbb{R} \), we have

\[
\inf\{S_H(\xi, x), \quad \xi \in y + \mathbb{Z} \} = \min\{S_H(\xi, x), \quad \xi \in (y + \mathbb{Z}) \cap [x - 1, x + 1]\} \quad \text{for all} \ x \in \mathbb{R}.
\]

**Proof.** It is clearly enough to prove the following assertion:

**Claim:** let \( \xi \in y + \mathbb{Z} \). If \( |\xi - x| > 1 \), there exists \( \eta \in y + \mathbb{Z} \) such that

\[
|\eta - x| \leq |\xi - x| - 1 \quad \text{and} \quad S_H(\eta, x) \leq S_H(\xi, x).
\]

Let us divide the proof in two cases.

If \( x - \xi > 1 \), i.e. \( \xi < x - 1 \), we have

\[
S_H(\xi, x) = S_H(\xi, \xi + 1) + S_H(\xi + 1, x) = P_H^+ + S_H(\xi + 1, x) \geq S_H(\xi + 1, x),
\]

and the claim holds true with \( \eta := \xi + 1 \).

If \( \xi - x > 1 \), i.e. \( \xi > x + 1 \), we have

\[
S_H(\xi, x) = S_H(\xi, \xi - 1) + S_H(\xi - 1, x) = (-P_H^-) + S_H(\xi - 1, x) \geq S_H(\xi - 1, x),
\]

and the claim holds true with \( \eta := \xi - 1 \). \( \square \)

\(^3\)We recall that \( c(H) = \tilde{h}(0) \), cf. Section 2.
Let us denote by $\mathcal{E}(H)$ and $\mathcal{E}(G)$ the sets of equilibria associated with $H$ and $G$, respectively. Since $G = H$ in $\mathbb{R} \setminus \text{supp}(V)$, we have
\[ \mathcal{E}(H) \setminus \text{supp}(V) = \mathcal{E}(G) \setminus \text{supp}(V). \] (6.4)

The following holds:

**Theorem 6.3.** There exists a bounded solution $u_G$ to (6.1).

**Proof.** Let us set $u_G(x) := \inf_{y \in \mathcal{E}(G)} S_G(y, x)$. From (6.4), the fact that $\mathcal{E}(H)$ is $\mathbb{Z}$-periodic and of Proposition 6.1, we derive that $u_G$ is a well defined bounded, Lipschitz function. Furthermore, as an infimum of viscosity solutions of (6.1), we get that $u_G$ is a viscosity solution as well in view of Proposition 1.1. \qed

### 7.2. Asymptotic convergence

Let us denote by $u_G^\epsilon$, $u_H^\epsilon$ the solutions of the discounted equations
\[ \lambda u + G(x, u') = c \quad \text{in} \ \mathbb{R}, \] (6.5)
\[ \lambda u + H(x, u') = c \quad \text{in} \ \mathbb{R}. \] (6.6)

We want to prove that the solutions $u_G^\epsilon$ converge, as $\lambda \to 0^+$, to a specific solution of $u_G^0$ the critical Eq. (6.1). In view of [2], we know that this is true for the solutions of (6.6), i.e. $u_H^\epsilon \Rightarrow u_H^0$ in $\mathbb{R}$ where $u_H^0$ is a solution to the critical Eq. (6.6), and this convergence is actually uniform in $\mathbb{R}$ since all these functions are $\mathbb{Z}$–periodic. Furthermore, $u_H^0$ is identified as the maximal (sub-)solution $v$ to (6.2) such that
\[ \int_{\mathbb{T} \times \mathbb{R}} v(y) \ d\mu(y) \leq 0 \quad \text{for all} \ \mu \in \mathcal{M}(H), \] (6.7)

where $\mathcal{M}(H)$ denotes the set of projected Mather measures for $H$.

The asymptotic convergence of the solutions $u_G^\epsilon$ will be established under the following assumption:

$u_H^0(x) = \sup \{ v(x) : v \text{ is a } \mathbb{Z} – \text{periodic subsolution of (6.2) with } v \leq 0 \text{ on } \mathcal{E}(H) \}$.

This is motivated by the following fact.

**Proposition 6.4.** Condition (U) is fulfilled in either one of the following cases:

(i) \quad \text{when } \int_0^1 p_H^{-}(x)dx < 0 < \int_0^1 p_H^{+}(x)dx;

(ii) \quad \text{when } (\int_0^1 p_H^{-}(x)dx)(\int_0^1 p_H^{+}(x)dx) = 0 \text{ and } H \text{ is Tonelli}.

**Proof.** We aim at showing that condition (6.7) is equivalent to $v \leq 0$ on $\mathcal{E}(H)$. We first remark that any measure of the kind $\delta_{(y, 0)}$ with $y \in \mathcal{E}$ is a Mather measure, i.e. $\delta_y$ is a projected Mather measure for every $y \in \mathcal{E}(H)$. In order to conclude, it suffices to show that any projected Mather measure has support contained in $\mathcal{E}(H)$. Let $\mu := \pi_1 \# \tilde{\mu}$ for some Mather measure $\tilde{\mu} \in \tilde{\mathcal{M}}(H)$. In case (i), pick $\lambda \in (0, 1)$ in such a way that the
function

\[ w(x) := \int_0^x (\lambda p_H^+(z) + (1 - \lambda)p_H^-(z)) \, dz, \quad x \in \mathbb{R} \]

is 1-periodic. It is easily seen that \( w \) is a \( C^1 \)-subsolution of the critical Eq. (6.2) and it is strict in \( \mathbb{R} \setminus \mathcal{E}(H) \). By using Fenchel inequality and the fact that \( \tilde{\mu} \) is closed, we have

\[-c = \int_{\mathbb{T}^1 \times \mathbb{R}} L_H(x, q) \, d\tilde{\mu}(x, q) \geq \int_{\mathbb{T}^1 \times \mathbb{R}} w'(x)q \, d\tilde{\mu}(x, q) - \int_{\mathbb{T}^1} H(x, w'(x)) \, d\mu(x) \geq -c,\]

hence all inequalities are equalities. We infer that \( H(x, w'(x)) = c \) for \( \mu \)-a.e. \( x \in \mathbb{T}^1 \), i.e. \( \text{supp}(\mu) \subseteq \mathcal{E}(H) \).

In case (ii), it is well known, cf. [31, Theorem 1.6] that \( \tilde{\mu} \) is invariant under the Lagrangian flow. Since the set \( C := \mathcal{E}(H) \times \{0\} \) is the maximal closed invariant set under the Lagrangian flow, the support of \( \tilde{\mu} \) needs to be contained in \( C \). We conclude that \( \text{supp}(\mu) \subseteq \mathcal{E}(H) \). \( \square \)

**Remark 6.5.** We refer the reader to the Appendix for an example of a non-Tonelli Hamiltonian for which condition (U) is violated.

**Theorem 6.6.** Assume condition (U) holds. We have

\[ u^0_{G}(x) = \inf \{ S_H(y, x) : y \in \mathcal{E}(H) \} \quad \text{for all } x \in \mathbb{R}. \]  

**Proof.** Let us call \( u(x) \) the infimum appearing at the right-hand side of (6.8). The fact that \( u \) is a bounded solution to (6.2) can be proved as in Theorem 6.3. The function \( u \) is also 1-periodic due to the fact that \( z + \mathcal{E}(H) = \mathcal{E}(H) \) and \( S_H(y + z, x + z) = S_H(y, x) \) for every \( z \in \mathbb{Z} \) and \( x, y \in \mathbb{R} \). Furthermore, since \( S_H(y, y) = 0 \) for every \( y \in \mathcal{E}(G) \), it is apparent from its very definition that \( u \leq 0 \) on \( \mathcal{E}(G) \). On the other hand, if \( v \) is a 1-periodic subsolution to (6.2) with \( v \leq 0 \) on \( \mathcal{E}(G) \), in view of Proposition 1.5 we have

\[ v(x) \leq v(y) + S_G(y, x) \leq S_G(y, x) \quad \text{for all } x \in \mathbb{R} \text{ and } y \in \mathcal{E}(G), \]

hence, by taking the infimum of the right-hand side term with respect to \( y \in \mathcal{E}(G) \), we get \( v(x) \leq u(x) \) for all \( x \in \mathbb{R} \). \( \square \)

Let \( u_G^- \) be the bounded solution of the critical Eq. (6.1). Since \( u_G^- := u_G - ||u_G||_{\infty} \) and \( u_G^+ := u_G + ||u_G||_{\infty} \) are, respectively, a bounded negative and positive solution of (6.1), from Proposition 1.15 we derive that the functions \( \{ u^\lambda_G : \lambda > 0 \} \) are equi-bounded and equi-Lipschitz on \( \mathbb{R} \), and satisfy

\[-2||u_G||_{\infty} \leq u_G^- (x) \leq u_G^+ (x) \leq u_G^+ (x) - 2||u_G||_{\infty} \quad \text{for all } x \in \mathbb{R}. \]  

Let us denote by

\[ \omega\left( \{ u^\lambda_G \} \right) := \left\{ u \in \text{Lip}(\mathbb{R}) : u^\lambda_G \Rightarrow u \text{ in } \mathbb{R} \right\} \text{ for some sequence } \lambda_k \rightarrow 0 \].
Of course, our aim is to show that this set reduces to \( \{ u_G^0 \} \). We first have to identify a good candidate for \( u_G^0 \). Our guess is the following:

\[
\begin{align*}
  u_G^0(x) := \sup \{ v(x) : v \in \mathcal{B}_{b_-(G)} \} & \quad x \in \mathbb{R}, \\
  \mathcal{B}_{b_-(G)} & \text{ denotes the family of bounded subsolutions } v : \mathbb{R} \to \mathbb{R} \text{ of the critical Eq. (6.1) satisfying } \\
  \int_{\mathbb{R}} v(y) \, d\mu(y) \leq 1_0(c(G) - c_f(G)) \quad \text{for every } \mu \in \mathcal{M}(G),
\end{align*}
\]

where \( \mathcal{M}(G) \) denotes the set of projected Mather measures for \( G \) and \( 1_0 \) is the indicator function of the set \( \{0\} \) in the sense of convex analysis, i.e. \( 1_0(t) = 0 \) if \( t = 0 \) and \( 1_0(t) = +\infty \) otherwise. Note that in this case the right-hand side of (6.11) is equal to 0 since \( c(G) = c_f(G) \).

**Theorem 6.7.** The following holds:

\[
\begin{align*}
  u_G^0(x) = \sup \{ v(x) : v \text{ subsolution of (6.1) such that } v \leq 0 \text{ on } \mathcal{E}(G) \} & \quad \text{for all } x \in \mathbb{R}.
\end{align*}
\]

In particular, \( u_G^0(x) = \inf_{y \in \mathcal{E}(G)} S_G(y, x) \) for all \( x \in \mathbb{R} \), and \( u_G^0 \) is a bounded solution to (6.1).

**Proof.** The first assertion is a straightforward consequence of Corollary 3.4. The second assertion can be proved arguing as in the proof of Theorem 6.6.

Let us begin to study the asymptotic behavior of the solutions \( u_G^\lambda \) of the discounted Eq. (6.5).

**Proposition 6.8.** Let \( \lambda > 0 \). Then

\[
\begin{align*}
  u_G^\lambda(y) \leq 0 & \quad \text{for all } y \in \mathcal{E}(G).
\end{align*}
\]

In particular, \( u_\leq u_G^0 \text{ in } \mathbb{R} \text{ for every } u \in \omega(\{u_G^0\}) \).

**Proof.** Let \( y \in \mathcal{E}(G) \). Then the stationary curve \( \tilde{\xi}(t) := y \) for all \( t \leq 0 \) satisfies

\[
L_G(\tilde{\xi}(t), \dot{\tilde{\xi}}(t)) + c = L_G(y, 0) + c = -\min_{p} G(y, p) + c = 0 \quad \text{for all } t \leq 0.
\]

By using \( \tilde{\xi} \) as a competitor curve in formula (1.14) for \( u_G^\lambda(y) \), we obtain \( u_G^\lambda(y) \leq 0 \).

We derive that any fixed \( u \in \omega(\{u_G^0\}) \) is a bounded solution of (6.1) satisfying \( u \leq 0 \) in \( \mathcal{E}(G) \), hence \( u \leq u_G^0 \text{ in } \mathbb{R} \text{ by the maximality property stated in Theorem 6.7} \).

In order to show that \( \omega(\{u_G^0\}) = \{u_G^0\} \), we need to show the converse inequality, i.e.

\[
\begin{align*}
  u \geq u_G^0 & \quad \text{for every } u \in \omega(\{u_G^0\}).
\end{align*}
\]

Let us show the following preliminary fact first.

**Proposition 6.9.** Assume condition (U) holds. Then

\[
\begin{align*}
  u_H^0(x) \geq u_G^0(x) & \quad \text{for all } x \in \mathbb{R} \setminus [\bar{y}_V - 1, \bar{y}_V + 1].
\end{align*}
\]
Proof. Let $x \in \mathbb{R} \setminus [y^V - 1, y^V + 1]$. According to Lemma 6.2,
\[
\begin{align*}
u_H^0(x) &= \min \{ S_H(y, x) : \ y \in \mathcal{E}(H) \cap [x - 1, x + 1] \}. \\
\end{align*}
\]
Since $[x - 1, x + 1] \cap \text{supp}(V) = \emptyset$, we have that $\mathcal{E}(G) \cap [x - 1, x + 1] = \mathcal{E}(H) \cap [x - 1, x + 1]$ and $S_H(y, x) = S_G(y, x)$ for all $y \in [x - 1, x + 1]$. We conclude that
\[
\begin{align*}
u_H^0(x) &= \min \{ S_H(y, x) : \ y \in \mathcal{E}(H) \cap [x - 1, x + 1] \} \\
&= \min \{ S_G(y, x) : \ y \in \mathcal{E}(G) \} = \nu_G^0(x).
\end{align*}
\]

We are now in position to prove the asymptotic convergence.

Theorem 6.10. Assume condition (U) holds. Then the functions $\nu_G^0$ uniformly converge to the function $\nu_H^0$ given by (6.10) locally on $\mathbb{R}$ as $\lambda \to 0^+$.

Proof. Let $\lambda_n \to 0$ be such that $\nu_G^0$ locally uniformly converge to some $u \in C(\mathbb{R})$. We have seen in Proposition 6.8 that
\[
u(x) \leq \nu_G^0(x) \quad \text{for every } x \in \mathbb{R}.
\]

To prove the opposite inequality, let us fix $x \in \mathbb{R}$. According to Theorem 1.18, there exists a monotone curve $\gamma_x : (-\infty, 0] \to \mathbb{R}$ with $\gamma_x(0) = x$ which is optimal for $\nu_G^0(x)$. Pick $r > 0$ big enough so that $\{x\} \cup [y^V - 1, y^V + 1] \subseteq B_r$ and let $\tilde{\mu}_{x,1}^\lambda$, $\tilde{\mu}_{x,2}^\lambda$ be the probability measures on $\mathbb{R} \times \mathbb{R}$ defined via (3.10) and (3.11), respectively. Let us apply Proposition 3.8 with $\nu := \nu_G^0$. By taking into account that $\text{supp}(\tilde{\mu}_{x,2}^\lambda) \subseteq (\mathbb{R} \setminus [y^V - 1, y^V + 1]) \times \mathbb{R}$ and $\nu_G^0 \leq \nu_H^0$ in $\mathbb{R} \setminus [y^V - 1, y^V + 1]$, we get
\[
u_G^0(x) \geq \nu_G^0(x) - \left( \theta_x^2 \int_{\mathbb{R} \times \mathbb{R}} \nu_G^0(y) \ d\tilde{\mu}_{x,1}^\lambda(y, q) + (1 - \theta_x^2) \int_{\mathbb{R} \times \mathbb{R}} \nu_H^0(y) \ d\tilde{\mu}_{x,2}^\lambda(y, q) \right)
\]

Now set $\hat{\lambda} := \lambda_n$ and send $n \to +\infty$ in the above inequality. In view of Propositions 3.6 and 3.7, there exist measures $\tilde{\mu}_{1,x} \in \mathcal{P}(\mathbb{R} \times \mathbb{R})$ and $\tilde{\mu}_{2,x} \in \mathcal{P}(\mathbb{T}^1 \times \mathbb{R})$ and $\theta \in [0, 1]$ such that
\[
u(x) \geq \nu_G^0(x) - \theta \int_{\mathbb{R} \times \mathbb{R}} \nu_G^0 \ d\tilde{\mu}_{x,1} - (1 - \theta) \int_{\mathbb{T}^1 \times \mathbb{R}} \nu_H^0 \ d\tilde{\mu}_{x,2}.
\]

Furthermore, in view of Propositions 3.7,
\[
\int_{\mathbb{R} \times \mathbb{R}} \nu_G^0 \ d\tilde{\mu}_{x,1} \leq 0 \quad \text{if } \theta \neq 0, \quad \int_{\mathbb{T}^1 \times \mathbb{R}} \nu_H^0 \ d\tilde{\mu}_{x,2} \leq 0 \quad \text{if } \theta \neq 1.
\]

By exploiting this information in (6.13), we get $\nu(x) \geq \nu_G^0(x)$, as it was to be shown. The proof is complete.

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Appendix A

In this section we give an example of a continuous Hamiltonian $H$ on $\mathbb{R} \times \mathbb{R}$, 1-periodic in the $x$-variable, for which condition (U) does not hold. The Hamiltonian $H$ is defined on $[0, 1] \times \mathbb{R}$ as follows:
\[ H(x, p) := (p - p_1(x))(p - p_2(x)) + c(x)|p - p_1(x)| \quad \text{for every } x \in [0, 1] \text{ and } p \in \mathbb{R}, \]

where \( p_1(x) := -\cos(2\pi x) \) and \( p_2(\cdot) \) is chosen in such a way that \( p_2(x) < p_1(x) \) for every \( x \in (0, 1) \) and

\[
p_2(x) = \begin{cases} -1 & \text{for } x \in \left[ 0, \frac{1}{2} + \frac{2}{100} \right] \cup \left[ 1 - \frac{1}{100}, 1 \right], \\ p_1(x) - \varepsilon_1 & \text{for } x \in \left[ \frac{1}{2} + \frac{3}{100}, 1 - \frac{3}{100} \right], \end{cases}
\]

with \( \varepsilon_1 > 0 \) small to be chosen later. The function \( \varepsilon(\cdot) \) is the piecewise affine function such that

\[
\varepsilon(0) = \varepsilon(1) = \frac{1}{100} \quad \text{and} \quad \varepsilon(x) = 0 \quad \text{for all } x \in \left[ \frac{1}{100}, 1 - \frac{1}{100} \right].
\]

We remark for later use that there exist \( \rho > 0 \), independent of \( \varepsilon_1 \), and \( \varepsilon_2 \in (0, \varepsilon_1] \) such that

\[
\varepsilon(x) + (p_1 - p_2)(x) \geq \rho \quad \text{for all } x \in \left[ 0, \frac{1}{2} + \frac{2}{100} \right] \cup \left[ 1 - \frac{1}{100}, 1 \right], \tag{A.1}
\]

\[
\varepsilon(x) + (p_1 - p_2)(x) \geq \varepsilon_2 \quad \text{for all } x \in [0, 1].
\]

Let us show that \( c_f(H) = c(H) = 0 \). To this aim, first note that the function \( u(x) := \int_0^x p_1(z) \, dz \) is a 1-periodic solution to the equation

\[
H(x, u') = 0 \quad \text{in } \mathbb{R}. \tag{A.2}
\]

This shows that \( c_f(H) \leq c(H) = 0 \). The converse inequality can be obtained as follows:

\[
c_f(H) = \max_x \min_p H(x, p) \geq \min_p H(0, p) = \min_p \left( (p + 1)^2 + \frac{1}{100} |p + 1| \right) = 0.
\]

Furthermore, since \( \min_p H(x, p) = \min_p (p - p_1(x))(p - p_2(x)) < 0 \quad \text{for all } x \in [1/100, 1 - 1/100] \), we derive that the set of equilibria \( \mathcal{E}(H) \) is contained in \([-1/100, 1/100] + \mathbb{Z} \).

Let us now show that we are in case (ii) of Proposition 6.4. Indeed, the set

\[
Z_H(x) := \{ p \in \mathbb{R} : H(x, p) \leq 0 \} \quad \text{for all } x \in \mathbb{R}
\]

is a closed interval of the form \([p_{\min}^H(x), p_{\max}^H(x)]\). It is not hard to see that \( p_{\max}^H(x) = p_1(x) \) for all \( x \in \mathbb{R} \). We derive \( \int_0^1 p_{\max}^H(x) \, dx = \int_0^1 p_1(x) \, dx = 0 \). In this case, any 1-periodic subsolution \( v \) to the critical Eq. (A.2) satisfies \( v'(x) = p_1(x) \) for every \( x \in \mathbb{R} \) (otherwise \( v \) would not be 1-periodic). In particular, 1-periodic subsolutions to (A.2) are unique up to additive constants.

Next, let \( \gamma : [0, T] \to \mathbb{R} \) be a curve satisfying \( \gamma(0) = 0, \gamma(T) = 1 \) and

\[
\dot{\gamma}(s) \in c(\gamma(s)) + (p_1 - p_2)(\gamma(s)) \in \partial_p H(\gamma(s), p_1(\gamma(s))) \quad \text{for all } s \in [0, T].
\]

The measure \( \tilde{\mu} \in \mathcal{M}(\mathbb{T}^1 \times \mathbb{R}) \) defined as

\[
\int_{\mathbb{R} \times \mathbb{R}} f(y, q) \, d\tilde{\mu} := \frac{1}{T} \int_0^T f(\gamma(s), \dot{\gamma}(s)) \, ds \quad \text{for every } f \in C_b(\mathbb{T}^1 \times \mathbb{R}) \tag{A.3}
\]

is a Mather measure. Indeed, it is closed, in the sense of Definition 3.1 with \( M := \mathbb{T}^1 \).

Furthermore,

\[
L(\gamma(t), \dot{\gamma}(t)) = p_1(\gamma(t))\dot{\gamma}(t) - H(\gamma(t), p_1(t)) = (v \circ \gamma)'(t) \quad \text{for every } t \in [0, T]
\]

and for any 1-periodic subsolution \( v \) to (A.2), since \( v'(x) = p_1(x) \) for all \( x \in \mathbb{R} \), as remarked above. By integrating this equality over \([0, T]\), we get that \( \tilde{\mu} \) is a minimizing Mather measure, cf. Theorem 3.2. Let us denote by \( \mu \) the projection of \( \tilde{\mu} \) on \( \mathbb{T}^1 \) and by \( u_{\gamma} \), for every fixed \( y \in \mathbb{R} \), the unique 1-periodic solution to (A.2) such that \( u_{\gamma}(y) = 0 \), i.e., the function \( u_{\gamma}(x) := \int_y^x p_1(z) \, dz \). We claim that
**Claim:** there exists $\varepsilon_1 > 0$ small enough so that

$$\int_{\Gamma^1} u_y(x) \, d\mu(x) > 0 \quad \text{for every } y \in \left[ -\frac{1}{100}, \frac{1}{100} \right].$$

In particular, for such a choice of $\varepsilon_1$, condition (U) does not hold. Indeed, the maximal 1-periodic subsolution $\tilde{u}^0$ to (A.2) satisfying $\tilde{u}^0 \leq 0$ on $E(H)$ will be of the form $u_y$ for some $\tilde{y} \in E(H)$.

Let us proceed to prove the claim. We first notice that $u_y(x) \geq 0$ for every $y \in \left[ -\frac{1}{100}, \frac{1}{100} \right]$ and $x \in \left[ \frac{1}{2} + \frac{1}{100}, 1 - \frac{1}{100} \right]$.

Furthermore, there exists $\delta > 0$ small enough such that $u_y(x) \geq \delta$ for every $y \in \left[ -\frac{1}{100}, \frac{1}{100} \right]$ and $x \in \left[ \frac{1}{2} + \frac{3}{100}, 1 - \frac{3}{100} \right]$.

We stress that such a $\delta > 0$ is independent of $\varepsilon_1 > 0$. Let $0 < A < a < b < B < 1$ such that $\gamma(A) = \frac{1}{2} + \frac{2}{100}$, $\gamma(a) = \frac{1}{2} + \frac{3}{100}$, $\gamma(b) = 1 - \frac{3}{100}$, $\gamma(B) = 1 - \frac{1}{100}$.

From the definitions of $\gamma(\cdot)$ and $p_1(\cdot)$ and from (A.1) we get $\gamma(A) - \gamma(0) \geq \rho A$, $\gamma(b) - \gamma(a) = \varepsilon_1(b - a)$, $\gamma(1) - \gamma(B) \geq \rho(1 - B)$, yielding in particular $A < 1/\rho$, $1 - B < 1/\rho$ and $b - a > 1/(4\varepsilon_1)$. Let us fix $y \in [-1/100, 1/100]$. We have

$$T\int_{\Gamma^1} u_y \, d\mu = \int_0^A u_y(\gamma(t)) \, dt + \int_A^B u_y(\gamma(t)) \, dt + \int_B^1 u_y(\gamma(t)) \, dt \geq -2 ||u_y||_{\infty}(A + 1 - B) + \int_a^b u_y(\gamma(t)) \, dt \geq \frac{-4}{\rho} + \frac{\delta}{4\varepsilon_1},$$

where we have used the fact that $u_y(\gamma(t)) \geq 0$ for all $t \in [A, B]$ and $||u_y||_{\infty} \leq 1$. The claim readily follows from this, by recalling that $\rho > 0$ and $\delta > 0$ do not depend on the choice of $y \in [-1/100, 1/100]$ and $\varepsilon_1 > 0$. 