ON BOUNDARY VALUES FOR RECTIFIABLE CURVES OF A GENERALIZATION OF THE CAUCHY-TYPE INTEGRAL RELATED TO THE HELMHOLTZ OPERATOR IN $\mathbb{R}^2$

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Abstract. There are considered vector fields and quaternionic $\alpha$-hyperholomorphic functions in a domain of $\mathbb{R}^2$ which generalize the notion of solenoidal and irrotational vector fields. There are established sufficient conditions for the corresponding Cauchy-type integral along a closed Jordan rectifiable curve to be continuously extended onto the closure of a domain. The Sokhotski-Plemelj-type formulas are proved as well.

1. A generalization of holomorphy in $\mathbb{R}^2$.

1.1. Given a domain $\Omega \in \mathbb{R}^2$ consider a $\mathbb{C}^3$-valued function $f : \Omega \mapsto \mathbb{C}^3$. Let $i_1, i_2, i_3$ be a canonical basis in $\mathbb{C}^3$, then any $f$ is of the form $f = f_1i_1 + f_2i_2 + f_3i_3$; moreover, we let $\mathbb{R}^2$ be a real linear space with the basis $i_1, i_2$. Of course, here $f_1, f_2, f_3$ are complex-valued functions in $\Omega$ and $\|f\|^2 := |f_1|^2 + |f_2|^2 + |f_3|^2$.

1.2. In this paper we are interested in those vector-functions $f$ which are solutions of the following system:

$$
\begin{cases}
\text{div } f = 0, \\
\text{rot } f = -\alpha f,
\end{cases}
$$

where $\alpha$ is a given complex number and for $z := xi_1 + yi_2 \in \Omega$

$$
\text{div } f := \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y},
$$

$$
\text{rot } f := \frac{\partial f_3}{\partial y}i_1 - \frac{\partial f_3}{\partial x}i_2 + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)i_3.
$$

Solutions of the system (1) are natural generalizations of the notion of a solenoidal and irrotational vector field which corresponds to $\alpha = 0$. There are many papers about both the case $\alpha = 0$ and $\alpha \in \mathbb{C} \setminus \{0\}$, see for instance [GR], [KS], [RS]. Note that the system (1) can be naturally considered in a domain of $\mathbb{R}^3$ but since the two-dimensional case has its essential peculiarities, we present here the latter one, and the former will be done elsewhere.

Date: February 4, 2003.

1991 Mathematics Subject Classification. 30G35, 32V05.
1.3. But it appears that there are deep mathematical reasons to consider a more general system, namely, the following one:

\[
\begin{align*}
\text{div} \, f &= \alpha f_0, \\
\text{rot} \, f + \alpha f &= -\text{grad} \, f_0,
\end{align*}
\]

where \(f_0\) is a \(\mathbb{C}\)-valued function and \(\text{grad} \, f_0 := \frac{\partial f_0}{\partial x} i_1 + \frac{\partial f_0}{\partial y} i_2\). Thus we shall be working with pairs \(F := (f_0, f)\) with norm \(\|F\| := |f_0|^2 + \|f\|^2\) and satisfying (2) from where certain conclusions will be made for vector–functions satisfying (1).

1.4. Let \(H_n^{(p)}\) be the Hankel functions of the kind \(p \in \{1, 2\}\) and of order \(n \in \{0, 1, 2\}\) (see [GR]), introduce the following notation:

\[
K_{\alpha, 0}(z) := \begin{cases} 
(-1)^p \frac{i \alpha}{4} H_n^{(p)}(\alpha \|z\|), & \text{if } \alpha \neq 0, \\
0, & \text{if } \alpha = 0,
\end{cases}
\]

\[
K_\alpha(z) := \begin{cases} 
(-1)^p \frac{i \alpha}{4} H_n^{(p)}(\alpha \|z\|) \frac{z}{\|z\|^2}, & \text{if } \alpha \neq 0, \\
-\frac{z}{2\pi \|z\|^2}, & \text{if } \alpha = 0,
\end{cases}
\]

where \(z \in \mathbb{R}^2 \setminus \{(0; 0)\}\), \(i\) is the complex imaginary unit in \(\mathbb{C}\), and

\[
p = \begin{cases} 
1, & \text{if } \text{Im}(\alpha) > 0 \text{ or } \alpha > 0, \\
2, & \text{if } \text{Im}(\alpha) < 0 \text{ or } \alpha < 0.
\end{cases}
\]

Now the following pair:

\[
K_\alpha(z) := \left( K_{\alpha, 0}(z), K_\alpha(z) \right)
\]

will play the role, in a sense, of the Cauchy kernel for the system (2). It generates an analog of the Cauchy-type integral for a continuous pair \(F := (f_0, f)\), \(f_0 : \Gamma \mapsto \mathbb{C}\), \(f : \Gamma \mapsto \mathbb{C}^3\); by the formulas: if \(\zeta := \xi i_1 + \eta i_2\), \(\sigma := d\eta i_1 - d\xi i_2\), and if \(\Gamma\) is a closed rectifiable Jordan curve which is the boundary of a bounded domain \(\Omega^+ := \Omega\), and \(\Omega^- := \mathbb{R}^2 \setminus (\Omega^+ \cup \Gamma)\) is its complement; then we define the pair

\[
\Phi_\alpha[F](z) := \left( \Phi_{\alpha, 0}[F](z), \Phi_\alpha[F](z) \right), \quad z \in \mathbb{R}^2 \setminus \Gamma,
\]

with

\[
\Phi_{\alpha, 0}[F](z) := -\int_\Gamma \left( \langle K_\alpha(\zeta - z), \sigma \rangle f_0(\zeta) + \langle [K_\alpha(\zeta - z), \sigma] + K_{\alpha, 0}(\zeta - z) \cdot \sigma, f(\zeta) \rangle \right).
\]
\[
\Phi_\alpha[F](z) := \int_\Gamma \left( \left[ K_\alpha(\zeta - z), \sigma \right] + K_{\alpha,0}(\zeta - z) \cdot \sigma, f(\zeta) \right) - \\
- \left\langle K_\alpha(\zeta - z), \sigma \right\rangle f(\zeta) + \\
+ f_0(\zeta) \left( \left[ K_\alpha(\zeta - z), \sigma \right] + K_{\alpha,0}(\zeta - z) \cdot \sigma \right),
\]
(7)
where \(<, \cdot >\) and \([, , ]\) denote, respectively, the scalar and the vector products in \(\mathbb{C}^3\), i.e., for \(\{a, b\}, a = \sum_{k=1}^3 a_k i_k, b = \sum_{k=1}^3 b_k i_k, < a, b > = \sum_{k=1}^3 a_k b_k;\)
\[
[a, b] := \begin{vmatrix}
    i_1 & i_2 & i_3 \\
    a_1 & a_2 & a_3 \\
    b_1 & b_2 & b_3
\end{vmatrix}.
\]

We shall write \(\Phi^+_\alpha[f]\) and \(\Phi^-\alpha[f]\) for the respective restrictions of \(\Phi_\alpha[f]\) onto \(\Omega^+\) and \(\Omega^-\).

1.5. If, in particular, the scalar component of the pair \(F\) is identically zero, \(F = \{0; f\}\), this does not mean, in general, that \(\Phi_\alpha[F]\) is vector-valued, which follows directly from (6). Thus, thinking of an analog of the Cauchy-type integral for the system (1) as a purely vectorial object, we must exclude the scalar component of \(\Phi_\alpha[F]\) in the following sense. Let \(C(\Gamma; \mathbb{C}^3)\) be the complex linear space of all \(\mathbb{C}^3\)-valued continuous vector–functions \(f\) on \(\Gamma\), introduce
\[
\mathcal{M}(\Gamma; \mathbb{C}^3) := C(\Gamma; \mathbb{C}^3) \cap \left\{ f : \int_\Gamma \left( \left[ K_\alpha(\zeta - z), \sigma \right] + K_{\alpha,0}(\zeta - z) \cdot \sigma, f(\zeta) \right) = 0, z \notin \Gamma \right\}.
\]
Now, for \(f \in \mathcal{M}(\Gamma; \mathbb{C}^3)\), the analog of the Cauchy-type integral for the system (1) is given by the formula
\[
\Phi_\alpha[f](z) := \int_\Gamma \left( \left[ K_\alpha(\zeta - z), \sigma \right] + K_{\alpha,0}(\zeta - z) \cdot \sigma, f(\zeta) \right) - \\
- \left\langle K_\alpha(\zeta - z), \sigma \right\rangle f(\zeta).
\]
One may check up that now \(\Phi_\alpha[f]\) is a solution to (1).

Theorem 1.6 (Analogue of N. A. Davydov theorem (see [D]) for the system (2)). Let \(\Gamma\) be a closed rectifiable Jordan curve, \(f_0 : \Gamma \mapsto \mathbb{C}\) and \(f : \Gamma \mapsto \mathbb{C}^3\) be continuous functions, \(F := (f_0, f)\), and let the integral
\[
\Psi_\alpha[F](t) := \lim_{\delta \to 0} \int_{\Gamma \setminus \Gamma_{t,\delta}} \|K_\alpha(\zeta - t)\| \cdot \|\sigma\| \cdot \|F(\zeta) - F(t)\|, t \in \Gamma,
\]
(9)
where \(\Gamma_{t,\delta} := \{ \zeta \in \Gamma : \|\zeta - t\| \leq \delta \}\), exist uniformly with respect to \(t \in \Gamma\). Then there exists the pair of integrals
\[
F_\alpha[F](t) := \left( F_{\alpha,0}[F](t), F_\alpha[F](t) \right), t \in \Gamma,
\]
where

\begin{equation}
F_{\alpha,0}[\mathcal{F}](t) := -\lim_{\delta \to 0} \int_{\Gamma \setminus \Gamma_{t,\delta}} \left( \langle K_{\alpha}(\zeta - t), \sigma \rangle (f_0(\zeta) - f_0(t)) + \right.
\end{equation}

\begin{equation}
\left. + \langle [K_{\alpha}(\zeta - t), \sigma] + K_{\alpha,0}(\zeta - t) \cdot \sigma, (f(\zeta) - f(t)) \rangle \right) \right. \right.
\end{equation}

\begin{equation}
F_{\alpha}[\mathcal{F}](t) := \lim_{\delta \to 0} \int_{\Gamma \setminus \Gamma_{t,\delta}} \left( \left[ [K_{\alpha}(\zeta - t), \sigma] + K_{\alpha,0}(\zeta - t) \cdot \sigma, (f(\zeta) - f(t)) \right] - 
\end{equation}

\begin{equation}
- \langle K_{\alpha}(\zeta - t), \sigma \rangle (f_0(\zeta) - f_0(t)) + 
\end{equation}

\begin{equation}
+ (f_0(\zeta) - f_0(t)) \left( [K_{\alpha}(\zeta - t), \sigma] + K_{\alpha,0}(\zeta - t) \cdot \sigma \right) \right)
\end{equation}

moreover, the functions \( \Phi^+_\alpha[\mathcal{F}] \) extend continuously onto \( \Gamma \) and the following analogues of the Sokhotski-Plemelj formulas hold:

\begin{equation}
\Phi^+_{\alpha,0}[\mathcal{F}](t) = (I_{\alpha,\Gamma,0}(t) + 1)f_0(t) - \langle I_{\alpha,\Gamma}(t), f(t) \rangle + F_{\alpha,0}[\mathcal{F}](t), \quad t \in \Gamma,
\end{equation}

\begin{equation}
\Phi^+_{\alpha}[\mathcal{F}](t) = [I_{\alpha,\Gamma}(t), f(t)] + (I_{\alpha,\Gamma,0}(t) + 1)f(t) + f_0(t)I_{\alpha,\Gamma}(t) + F_{\alpha}[\mathcal{F}](t), \quad t \in \Gamma,
\end{equation}

where the following notation is used:

\begin{equation}
I_{\alpha,\Gamma,0}(t) := -\alpha \int_{\Omega^+} K_{\alpha,0}(\zeta - t)d\xi d\eta,
\end{equation}

\begin{equation}
I_{\alpha,\Gamma}(t) := -\alpha \int_{\Omega^+} K_{\alpha}(\zeta - t)d\xi d\eta,
\end{equation}

and \( \Phi^+_{\alpha,0}[\mathcal{F}](t) := \lim_{\Omega^+ \ni z \to t} \Phi_{\alpha}[\mathcal{F}](z) \).

The proof will be given after some preparatory work which is of an independent interest.

**Corollary 1.7** (Analogue of N. A. Davydov theorem for the system (1)). Let \( \Gamma \) be a closed rectifiable Jordan curve, \( f \in M(\Gamma; \mathbb{C}^d) \), and let the integral

\begin{equation}
\Psi_{\alpha}[f](t) := \lim_{\delta \to 0} \int_{\Gamma \setminus \Gamma_{t,\delta}} \left\| K_{\alpha}(\zeta - t) \right\| \cdot \left\| \sigma \right\| \cdot \left\| f(\zeta) - f(t) \right\|
\end{equation}

exist uniformly with respect to \( t \in \Gamma \). Then there exists the integral

\begin{equation}
F_{\alpha}[f](t) := \lim_{\delta \to 0} \int_{\Gamma \setminus \Gamma_{t,\delta}} \left( \left[ [K_{\alpha}(\zeta - t), \sigma] + K_{\alpha,0}(\zeta - t) \cdot \sigma, (f(\zeta) - f(t)) \right] - 
\end{equation}

\begin{equation}
- \langle K_{\alpha}(\zeta - t), \sigma \rangle (f_0(\zeta) - f_0(t)) + 
\end{equation}

\begin{equation}
+ (f_0(\zeta) - f_0(t)) \left( [K_{\alpha}(\zeta - t), \sigma] + K_{\alpha,0}(\zeta - t) \cdot \sigma \right) \right)
\end{equation}

moreover, the functions \( \Phi^+_\alpha[f](z) \) extend continuously onto \( \Gamma \) and the following analogues of the Sokhotski-Plemelj formulas hold:

\begin{equation}
\Phi^+_\alpha[f](t) = [I_{\alpha,\Gamma}(t), f(t)] + (I_{\alpha,\Gamma,0}(t) + 1)f(t) + F_{\alpha}[f](t), \quad t \in \Gamma,
\end{equation}

where the notation is used:

\begin{equation}
I_{\alpha,\Gamma,0}(t) := -\alpha \int_{\Omega^+} K_{\alpha,0}(\zeta - t)d\xi d\eta,
\end{equation}

\begin{equation}
I_{\alpha,\Gamma}(t) := -\alpha \int_{\Omega^+} K_{\alpha}(\zeta - t)d\xi d\eta,
\end{equation}

and \( \Phi^+_{\alpha,0}[f](t) := \lim_{\Omega^+ \ni z \to t} \Phi_{\alpha}[f](z) \).
Lemma 2.2. For complex quaternions the situation is different.

2. Quaternions and quaternion-valued $\alpha$-hyperholomorphic functions in $\mathbb{R}^2$

2.1. We shall denote, as usual, by $\mathbb{H} = \mathbb{H}(\mathbb{R})$ and $\mathbb{H}(\mathbb{C})$ the sets of real and complex quaternions, i.e., each quaternion is of the form

$$a = \sum_{k=0}^{3} a_k i_k$$

with $\{a_k\} \subset \mathbb{R}$ for real quaternions and $\{a_k\} \subset \mathbb{C}$ for complex quaternions; $i_0 = 1$ stands for the unit and $i_1, i_2, i_3$ stand for imaginary units; the complex numbers imaginary unit in $\mathbb{C}$ will be denoted by $i$. $\mathbb{H}$ has the structure of a real non-commutative, associative algebra without zero divisors. $\mathbb{H}(\mathbb{C})$ is a complex non-commutative, associative algebra with zero divisors.

For a complex quaternion $a = \sum_{k=0}^{3} a_k i_k$ its quaternionic conjugate is defined by

$$\overline{a} := a_0 - \sum_{k=1}^{3} a_k i_k.$$  

The module of a quaternion $a$ coincides with its Euclidean norm: $|a| = \|a\|_{\mathbb{R}^n}$. In particular, for $a \in \mathbb{H}$ we have $|a| = \|a\|_{\mathbb{R}^4}$ and besides $|a|^2 = a \cdot \overline{a} = \overline{a} \cdot a$ while for a complex quaternion $|a|^2 \neq a \cdot \overline{a}$. What is more, for $a, b$ from $\mathbb{H}$ there holds: $|a \cdot b| = |a| \cdot |b|$ which is extremely important working with real quaternions. For complex quaternions the situation is different.

**Lemma 2.2.** $|ab| \leq \sqrt{2} \cdot |a| \cdot |b|$ for all $a, b \in \mathbb{H}(\mathbb{C})$.

**Proof.** Let

$$a := \sum_{k=0}^{3} a_k i_k, \quad a_k := \alpha_k + i\lambda_k, \quad \alpha_k, \lambda_k \in \mathbb{R},$$

$$b := \sum_{k=0}^{3} b_k i_k, \quad b_k := \beta_k + i\gamma_k, \quad \beta_k, \gamma_k \in \mathbb{R}.$$  

We have $a = a' + ia''$ and $b = b' + ib''$, where $a', a'', b', b''$ are real quaternions.

Since

$$|a|^2 = \sum_{k=0}^{3} (\alpha_k^2 + \lambda_k^2) = |a'|^2 + |a''|^2, \quad |b|^2 = \sum_{k=0}^{3} (\beta_k^2 + \gamma_k^2) = |b'|^2 + |b''|^2,$$

then

$$|a| \cdot |b|^2 = |a'|^2 |b'|^2 + |a''|^2 |b''|^2 + |a'|^2 |b''|^2 + |a''|^2 |b'|^2 + |a''|^2 |b''|^2.$$  

Therefore

$$|ab|^2 = |a'b' - a''b'' + ia'(a''b' + a''b')|^2 = |a'b' - a''b''|^2 + |a'b' + a''b''|^2 =
= (a'b' - a''b'') \cdot (a'b' - a''b'') + (a'b'' + a''b') \cdot (a'b'' + a''b') =
= (a'b' - a''b'') \cdot (a'b' - a''b'') + (a'b'' + a''b') \cdot (a'b'' + a''b') = (|a| \cdot |b|)^2 + d,$$  

where $d$ is a real number.
where $\mathbb{H} \ni d = a b^\dagger a'' b' + a'' b' a'' b' - a'' b' a'' b' - a b^\dagger a' b'$, and
\begin{equation}
|d| \leq 2|a' b'| \cdot |a'' b'| + 2|a' b'||a'' b'| \leq (|a| \cdot |b|)^2. \tag{19}
\end{equation}
Combining (18) and (19), we obtain the assertion of Lemma. \hfill \Box

2.3. Let $\Omega$ be a domain in the plane $\mathbb{R}^2$, we consider $\mathbb{H}(\mathbb{C})$-valued functions defined in the domain $\Omega$. On the bi-$\mathbb{H}(\mathbb{C})$-module $C^2(\Omega; \mathbb{H}(\mathbb{C}))$ there introduced the two-dimensional Helmholtz operator with a wave number $\lambda \in \mathbb{C}$:
\[ \lambda \Delta := \Delta_{\mathbb{R}^2} + \lambda M, \]
where $\Delta_{\mathbb{R}^2} = \partial_1^2 + \partial_2^2$, $\partial_k = \dfrac{\partial}{\partial x_k}$ and for $a \in \mathbb{H}(\mathbb{C})$ we denote $\lambda M$ the operator of the multiplication by $\lambda$ on the left-hand side, analogously for $M^\lambda$. The operators
\[ \overline{\partial} := \partial_1 + i \partial_2 \quad \text{and} \quad \partial := \partial_1 - i \partial_2 \]
determine, respectively, classes of holomorphic and anti-holomorphic functions of the complex variable, and the following factorization holds:
\[ \partial \circ \overline{\partial} = \overline{\partial} \circ \partial = \Delta_{\mathbb{R}^2}. \]
Consider the following partial differential operators with quaternionic coefficients:
\[ s(t) := i_1 \cdot \partial_1 + i_2 \cdot \partial_2; \quad s(t) := i_1 \cdot \partial_1 + i_2 \cdot \partial_2; \]
\[ \partial_s := \partial_1 \circ M^{i_1} + \partial_2 \circ M^{i_2}; \quad \overline{\partial}_s := \partial_1 \circ M^{i_1} + \partial_2 \circ M^{i_2}. \]
The following equalities can be easily verified:
\[ \partial_s \circ \overline{\partial}_s = \overline{\partial}_s \circ \partial_s = \Delta_{\mathbb{R}^2} = s(t) \circ s(t) = s(t) \circ s(t), \]
which mean that
\[ s(t) \circ s(t)^2 = s(t)^2 = -\Delta_{\mathbb{R}^2}. \]
For $\alpha \in \mathbb{C}$ to be a complex square root of $\lambda \in \mathbb{C}$, i.e. $\alpha^2 = \lambda$, set
\[ \alpha \partial := \partial_s + \alpha^\alpha M; \quad \overline{\partial}_s := s(t) \circ \partial + M^\alpha. \]
Then we have the following factorizations of the Helmholtz operator:
\[ \lambda \Delta = -\partial_\alpha \circ \partial_\alpha = -\partial_\alpha \circ \partial_\alpha = -\alpha \partial \circ \partial_\alpha = -\alpha \partial \circ \partial_\alpha. \]
In analogy with the usual notion of a holomorphic function, consider the following definition of $\alpha$-hyperholomorphic functions in $\mathbb{R}^2$.

**Definition 2.4** ([ST1]). Let $f \in C^1(\Omega, \mathbb{H}(\mathbb{C}))$, $f$ is called $\alpha$-hyperholomorphic if $\alpha \partial f \equiv 0$ in $\Omega$.

Of course, more exactly such functions should be called, for instance, left-$\alpha$-hyperholomorphic because there is a “symmetric” definition for $\partial_{\alpha}$, as well as for $\alpha \overline{\partial}$ and $\overline{\partial}_s$. We shall deal with the above case only.

Such a definition for $\alpha$-hyperholomorphic functions was introduced in [ST1] both for complex and quaternionic values of $\alpha$, and some essential properties were established there. Main integral formulas for $\alpha$-hyperholomorphic functions were constructed in [ST2]. All proofs and details can be found in these papers, see also [KS, Appendix 4]. Some developments of the topic are presented in [RS] and [GSS]. One can find much more relevant bibliographical references in all these papers.
2.5. In what follows we shall be in need of some properties of the Hankel functions \(H_n^{(p)}(t), \ t \in \mathbb{C},\) (see [GR]) which we concentrate in this section for the reader’s convenience. The following equalities are valid:

\[
\frac{d}{dt} H_1^{(p)}(t) = \frac{1}{2} \left( H_0^{(p)}(t) - H_2^{(p)}(t) \right),
\]

\[
\frac{d}{dt} H_0^{(p)}(t) = -H_1^{(p)}(t),
\]

\[
t H_2^{(p)}(t) = 2H_1^{(p)}(t) - t H_0^{(p)}(t),
\]

and the following series expansions of the Hankel functions \(H_0^{(p)}(t)\) and \(H_1^{(p)}(t)\) hold:

\[
H_0^{(p)}(t) = \left( 1 - (-1)^p \frac{2i}{\pi} (\log \frac{t}{2} + C) \right) \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{2^{2k}(k!)^2} + \frac{2i}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k t^{2k}}{2^{2k}(k!)^2} \sum_{m=1}^{k} \frac{1}{m},
\]

\[
H_1^{(p)}(t) = \left( 1 - (-1)^p \frac{2i}{\pi} (\log \frac{t}{2} + C) \right) \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{2^{2k+1}(k+1)!} + (-1)^p \left( \frac{2i}{\pi t} + \frac{it}{2\pi} \right) + \frac{i}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{2^{2k+1}(k+1)!} \left( \sum_{m=1}^{k+1} \frac{1}{m} + \sum_{m=1}^{k} \frac{1}{m} \right),
\]

where \(C\) is the Euler constant.

3. QUATERNIONIC GENERALIZATION OF THE CAUCHY-TYPE INTEGRAL

3.1. Given \(\alpha \in \mathbb{C}\) and real quaternions \(z := x i_1 + y i_2, \ \zeta := \xi i_1 + \eta i_2\) thinkable as points of the Euclidean space \(\mathbb{R}^2\) equipped with the additional structure of quaternionic multiplication; introduce the notation:

\[
\theta_\alpha(z) := \begin{cases} 
(-1)^p \frac{i\alpha}{4} H_0^{(p)}(\alpha|z|), & \text{if } \alpha \neq 0, \\
\frac{1}{2\pi} \log |z|, & \text{if } \alpha = 0,
\end{cases}
\]

where \(p\) depends on \(\alpha\) by the formula (3). It is a well known fact (see e.g. [V]) that the function \(\theta_\alpha\) is the fundamental solution of the Helmholtz operator \(\Delta_{\alpha^2} := \Delta_{\mathbb{R}^2} + \alpha^2 M,\) written down for all values of \(\alpha.\)

The \(\alpha\)-hyperholomorphic Cauchy kernel, i.e., the fundamental solution to operators \(\alpha \partial\) and \(\partial_\alpha,\) is defined as

\[
K_\alpha(z) := -\alpha \partial [\theta_\alpha](z) = -\partial [-\alpha] [\theta_\alpha](z).
\]

Hence one has explicitly:

\[
K_\alpha(z) = \begin{cases} 
(-1)^p \frac{i\alpha}{4} \left( \frac{H_1^{(p)}(\alpha|z|) z}{|z|} + H_0^{(p)}(\alpha|z|) \right), & \text{if } \alpha \neq 0, \\
-\frac{1}{2\pi |z|^2}, & \text{if } \alpha = 0.
\end{cases}
\]
Now, for a continuous function $f : \Gamma \to \mathbb{H}(\mathbb{C})$ and $\sigma := d\eta_1 - d\xi_2$ the Cauchy-type integral of $f$ is given by the formula
\begin{equation}
\Phi_\alpha[f](z) := \int_\Gamma K_\alpha(\zeta - z) \cdot \sigma \cdot f(\zeta), \quad z \in \mathbb{R}^2 \setminus \Gamma;
\end{equation}
as in the previous section here $\Gamma$ is a closed rectifiable Jordan curve.

**Theorem 3.2.** Let $\Gamma$ be a closed rectifiable Jordan curve, $f : \Gamma \to \mathbb{H}(\mathbb{C})$ be a continuous function, and let the integral
\begin{equation}
\Psi_\alpha[f](t) := \lim_{\delta \to 0} \int_{\Gamma \setminus \Gamma_{t,\delta}} |K_\alpha(\zeta - t)| \cdot |\sigma| \cdot |f(\zeta) - f(t)|, \quad t \in \Gamma,
\end{equation}
where $\Gamma_{t,\delta} := \left\{ \zeta \in \Gamma : |\zeta - t| \leq \delta \right\}$, exist uniformly with respect to $t \in \Gamma$. Then there exists the integral
\begin{equation}
F_\alpha[f](t) := \lim_{\delta \to 0} \int_{\Gamma \setminus \Gamma_{t,\delta}} K_\alpha(\zeta - t) \cdot \sigma \cdot (f(\zeta) - f(t)), \quad t \in \Gamma;
\end{equation}
moreover, the functions $\Phi_\alpha^\pm[f]$ extend continuously onto $\Gamma$, and the following analogues of the Sokhotski-Plemelj formulas hold:
\begin{align}
\Phi_\alpha^+[f](t) &= (I_{\alpha,\Gamma}(t) + 1)f(t) + F_\alpha[f](t), \quad t \in \Gamma, \\
\Phi_\alpha^-[f](t) &= I_{\alpha,\Gamma}(t)f(t) + F_\alpha[f](t), \quad t \in \Gamma,
\end{align}
where $\Phi_\alpha^+[f](t) := \lim_{\Omega \ni z \to t} \Phi_\alpha[f](z)$, and
\begin{equation*}
I_{\alpha,\Gamma}(t) := -\alpha \int_{\Omega^+} K_\alpha(\zeta - t) d\xi d\eta.
\end{equation*}

The proof is based on several lemmas which are of interest by themselves.

**Lemma 3.3.** The limit (28) exists uniformly with respect to $t \in \Gamma$ and $F_\alpha[f]$ is a continuous function on $\Gamma$.

**Lemma 3.4.**
\begin{equation}
\Phi_\alpha[1](z) = \begin{cases} 
I_{\alpha,\Gamma}(z) + 1, & z \in \Omega^+, \\
I_{\alpha,\Gamma}(z), & z \in \Omega^-.
\end{cases}
\end{equation}

**Lemma 3.5.** $I_{\alpha,\Gamma}$ is a continuous function in $\mathbb{R}^2$.

4. **Proof of the results of Section 3.**

4.1. **Proof of Lemma 3.3.** Denote
\begin{align*}
\Psi_\alpha(\delta, t) &:= \int_{\Gamma \setminus \Gamma_{t,\delta}} |K_\alpha(\zeta - t)| \cdot |\sigma| \cdot |f(\zeta) - f(t)|, \\
F_\alpha(\delta, t) &:= \int_{\Gamma \setminus \Gamma_{t,\delta}} K_\alpha(\zeta - t) \cdot \sigma \cdot (f(\zeta) - f(t)).
\end{align*}
We have $F_\alpha(\delta, t) = \sum_{k=0}^3 (F^{(1)}_{\alpha,k}(\delta, t) + iF^{(2)}_{\alpha,k}(\delta, t))\mathbf{i}_k$, where $F^{(1)}_{\alpha,k}$ and $F^{(2)}_{\alpha,k}$ are real-valued functions.
Under conditions of Theorem 3.2 the function $\Psi_\alpha(\delta, t)$ tends to the finite limit $\Psi_\alpha(t)$, when $\delta \to 0$, uniformly with respect to $t \in \Gamma$. Using the criterion of uniform convergence for the integral and Lemma 2.2, we get that for $\forall \varepsilon > 0 \exists \delta(\varepsilon) > 0 \forall t \in \Gamma$:

$$0 < \delta_1 < \delta_2 < \delta(\varepsilon) \Rightarrow$$

$$\Rightarrow \Psi_\alpha(\delta_1, t) - \Psi_\alpha(\delta_2, t) = \int_{\Gamma_{r_1}\setminus\Gamma_{r_1}} |K_\alpha(\zeta - t)| \cdot |\sigma| \cdot |f(\zeta) - f(t)| < \varepsilon \Rightarrow$$

$$\Rightarrow |F_\alpha(\delta_1, t) - F_\alpha(\delta_2, t)| = \left| \int_{\Gamma_{r_1}\setminus\Gamma_{r_1}} K_\alpha(\zeta - t) \cdot \sigma \cdot (f(\zeta) - f(t)) \right| \leq 2 \int_{\Gamma_{r_1}\setminus\Gamma_{r_1}} |K_\alpha(\zeta - t)| \cdot |\sigma| \cdot |f(\zeta) - f(t)| < 2\varepsilon \Rightarrow$$

$$(32) \quad \Rightarrow |F_{\alpha,k}^{(j)}(\delta_1, t) - F_{\alpha,k}^{(j)}(\delta_2, t)| < 2\varepsilon \quad (j = 1, 2; \ k = 0, \ldots, 3).$$

Therefore for each fixed $t \in \Gamma$ there exist limits

$$F_{\alpha,k}^{(j)}(t) := \lim_{\delta \to 0} F_{\alpha,k}^{(j)}(\delta, t) \quad (j = 1, 2; \ k = 0, \ldots, 3)$$

and consequently there exists

$$(33) \quad F_\alpha[f](t) = \lim_{\delta \to 0} F_\alpha(\delta, t).$$

Proceeding to the limit as $\delta_1 \to 0$ in inequality (32) we obtain that for $\forall \varepsilon > 0 \exists \delta(\varepsilon) > 0 \forall t \in \Gamma$:

$$0 < \delta < \delta(\varepsilon) \Rightarrow |F_{\alpha,k}^{(j)}(\delta, t) - F_{\alpha,k}^{(j)}(t)| \leq 2\varepsilon \quad (j = 1, 2; \ k = 0, \ldots, 3) \Rightarrow$$

$$\Rightarrow |F_\alpha(\delta, t) - F_\alpha[f](t)| \leq 4\sqrt{2}\varepsilon,$$

and this is all. \qed

4.2. Proof of Lemma 3.4. Let $\alpha \neq 0$. We have

$$(34) \quad \Phi_\alpha[1](z) = \int_{\Gamma} K_\alpha(\zeta - z) \cdot \sigma = (-1)^{p-1} i^p \frac{i}{4} \alpha(i_1 + i_2 i_3 - I_3),$$

where

$$I_1 = \int_{\Gamma} H_1^{(p)}(\alpha|\zeta - z|) \frac{((\xi - x)d\eta - (\eta - y)d\xi)}{|\zeta - z|},$$

$$I_2 = \int_{\Gamma} H_1^{(p)}(\alpha|\zeta - z|) \frac{((\xi - x)d\xi + (\eta - y)d\eta)}{|\zeta - z|},$$

$$I_3 = \int_{\Gamma} H_0^{(p)}(\alpha|\zeta - z|) (d\eta i_1 - d\xi i_2).$$

Let $z \in \Omega^+$, let $\rho > 0$ be such that $B(z, \rho) := \{\zeta \in \mathbb{C} : |\zeta - z| \leq \rho\}$ is contained in $\Omega^+$, and let $\gamma_\rho$ be the boundary of $B(z, \rho)$.
Using the Green formula and the equalities (20), (22) we get:
\[
\int_{\Gamma - \gamma_\rho} \frac{H_1^{(p)}(\alpha|\zeta - z|)}{|\zeta - z|} ((\xi - x)d\eta - (\eta - y)d\xi) = \\
= \int_{\Omega^+ \setminus B(z, \rho)} \left( \frac{\partial}{\partial \xi} \left( \frac{H_1^{(p)}(\alpha|\zeta - z|)}{|\zeta - z|} (\xi - x) \right) + \frac{\partial}{\partial \eta} \left( \frac{H_1^{(p)}(\alpha|\zeta - z|)}{|\zeta - z|} (\eta - y) \right) \right) d\xi d\eta = \\
= \frac{1}{2} \int_{\Omega^+ \setminus B(z, \rho)} \left( \alpha H_0^{(p)}(\alpha|\zeta - z|) - \alpha H_2^{(p)}(\alpha|\zeta - z|) + 2 \frac{H_1^{(p)}(\alpha|\zeta - z|)}{|\zeta - z|} \right) d\xi d\eta = \\
= \int_{\Omega^+ \setminus B(z, \rho)} \alpha H_0^{(p)}(\alpha|\zeta - z|) d\xi d\eta,
\]

Using the Green formula and the equalities (20), (22) we get:
\[
= \int_{\gamma_\rho} \frac{H_1^{(p)}(\alpha|\zeta - z|)}{|\zeta - z|} ((\xi - x)d\eta - (\eta - y)d\xi) = 2\pi \rho H_1^{(p)}(\alpha \rho) = \\
= (-1)^p \frac{4i}{\alpha} + o(1) \text{ as } \rho \to 0.
\]

Hence
\[
I_1 = \lim_{\rho \to 0} \left( \int_{\Gamma - \gamma_\rho} + \int_{\gamma_\rho} \right) \frac{H_1^{(p)}(\alpha|\zeta - z|)}{|\zeta - z|} ((\xi - x)d\eta - (\eta - y)d\xi) = \\
= \int_{\Omega^+} \alpha H_0^{(p)}(\alpha|\zeta - z|) d\xi d\eta + (-1)^p \frac{4i}{\alpha}.
\]

Furthermore,
\[
\int_{\Gamma - \gamma_\rho} \frac{H_1^{(p)}(\alpha|\zeta - z|)}{|\zeta - z|} ((\xi - x)d\xi + (\eta - y)d\eta) = \\
= \int_{\Omega^+ \setminus B(z, \rho)} \left( \frac{\partial}{\partial \xi} \left( \frac{H_1^{(p)}(\alpha|\zeta - z|)}{|\zeta - z|} (\eta - y) \right) - \frac{\partial}{\partial \eta} \left( \frac{H_1^{(p)}(\alpha|\zeta - z|)}{|\zeta - z|} (\xi - x) \right) \right) d\xi d\eta = \\
= \int_{\Omega^+ \setminus B(z, \rho)} \frac{\partial}{\partial |\zeta - z|} \left( \frac{H_1^{(p)}(\alpha|\zeta - z|)}{|\zeta - z|} \right) \left( \frac{\partial |\zeta - z|}{\partial \xi} (\eta - y) - \frac{\partial |\zeta - z|}{\partial \eta} (\xi - x) \right) d\xi d\eta = \\
= 0,
\]

\[
\int_{\gamma_\rho} \frac{H_1^{(p)}(\alpha|\zeta - z|)}{|\zeta - z|} ((\xi - x)d\xi + (\eta - y)d\eta) = 0
\]

and, consequently,
\[
(36) \quad I_2 = 0.
\]
Analogously, using the equality (21), we have

\[
\int_{\Gamma-\gamma_\rho} H_0^{(p)}(\alpha|\zeta-z|)(d\eta_1 - d\xi_2) =
\int_{\Omega^+} \left( \frac{\partial H_0^{(p)}(\alpha|\zeta-z|)}{\partial \xi} \right)_1 + \frac{\partial H_0^{(p)}(\alpha|\zeta-z|)}{\partial \eta} \right)_2 \ d\xi \ d\eta =
\int_{\Omega^+} \frac{\alpha H_1^{(p)}(\alpha|\zeta-z|)}{|\zeta-z|} \ (\zeta-z) \ d\xi \ d\eta,
\]

and

\[
\int_{\gamma_\rho} H_0^{(p)}(\alpha|\zeta-z|)(d\eta_1 - d\xi_2) = 0
\]

and

\[
I_3 = - \int_{\Omega^+} \frac{\alpha H_1^{(p)}(\alpha|\zeta-z|)}{|\zeta-z|} \ (\zeta-z) \ d\xi \ d\eta.
\]

Thus, from (34) – (37) we have

\[
\Phi_\alpha[1](z) = 1 + (-1)^{p-1} \frac{i\alpha^2}{4} \int_{\Omega^+} \left( H_0^{(p)}(\alpha|\zeta-z|) + \frac{H_1^{(p)}(\alpha|\zeta-z|)}{|\zeta-z|} \ (\zeta-z) \right) \ d\xi \ d\eta = I_{\alpha,\Gamma}(z) + 1.
\]

Let now \( \alpha = 0 \). We have

\[
\Phi_0[1](z) = \int_{\Gamma} K_0(\zeta-z) \cdot \sigma = \frac{1}{2\pi} \int_{\Gamma} \frac{\zeta-z}{|\zeta-z|^2} \cdot \sigma =
\left[ \frac{1}{2\pi} \int_{\Gamma} \frac{(\xi-x) d\eta - (\eta-y) d\xi}{|\zeta-z|^2} + \frac{1}{2\pi} \int_{\Gamma} \frac{(\xi-x) d\xi + (\eta-y) d\eta}{|\zeta-z|^2} \right] i_3.
\]

Going on along the same way as in the computation of the integrals \( I_1 \) and \( I_2 \), and using the Green formula, we obtain that \( \Phi_0[1](z) = 1 \).

In the case of \( z \in \Omega^- \) the proof of (31) is simplified because of the continuity of the kernel \( K_\alpha \) on \( \Omega^+ \).

4.3. Proof of Lemma 3.5. Making use of the series expansions of the Hankel functions (23), (24) we obtain:

\[
I_{\alpha,\Gamma}(z) = \frac{i\alpha}{8} \left( (-1)^{p-1} I_{\alpha,\Gamma}^{(1)}(z) + \frac{2i}{\pi} I_{\alpha,\Gamma}^{(2)}(z) + (-1)^p \frac{4i}{\pi} I_{\alpha,\Gamma}^{(3)}(z) \right),
\]

\[
I_{\alpha,\Gamma}(z) = \frac{i\alpha}{8} \left( (-1)^{p-1} I_{\alpha,\Gamma}^{(1)}(z) + \frac{2i}{\pi} I_{\alpha,\Gamma}^{(2)}(z) + (-1)^p \frac{4i}{\pi} I_{\alpha,\Gamma}^{(3)}(z) \right),
\]
where
\[ I_{a,1}^{(1)}(z) := \int_{\Omega^+} \sum_{k=0}^{\infty} \alpha^{2k+1} |z|^2 |a_k + b_k \alpha (z - \zeta)| d\xi d\eta, \]
\[ I_{a,1}^{(2)}(z) := \int_{\Omega^+} \log |z - \zeta| \sum_{k=0}^{\infty} (-1)^k \frac{\alpha^{2k+1} |z|^2 |(2k+1 + \alpha(z - \zeta))| d\xi d\eta, \]
\[ I_{a,1}^{(3)}(z) := \int_{\Omega^+} \frac{\zeta - z}{|\zeta - z|^2} d\xi d\eta, \]

and \( a_{k,p}, b_{k,p} \) are complex coefficients.

The continuity of \( I_{a,1}^{(1)} \) follows from the continuity of the integrand.

Let us prove the continuity of \( I_{a,1}^{(3)} \). For an arbitrary \( z \in \mathbb{C} \) and a measurable \( E \subset \mathbb{C} \) set
\[ I_E(z) := \int_{E} \frac{\zeta - z}{|\zeta - z|^2} d\xi d\eta. \]

Let us fix any point \( z_0 \in \mathbb{C} \). For an arbitrary \( z \in \mathbb{C} \) we have
\[ I_{a,1}^{(3)}(z_0) - I_{a,1}^{(3)}(z) = I_{\Omega^+ \cap B(z_0, \rho)}(z_0) + I_{\Omega^+ \setminus (B(z_0, \rho) \cup B(z, \rho))}(z_0) + I_{\Omega^+ \setminus (B(z_0, \rho) \cup B(z, \rho))}(z_0) - I_{\Omega^+ \setminus (B(z_0, \rho) \cup B(z, \rho))}(z). \]

Fix an arbitrary \( \varepsilon > 0 \). Since \( |I_{E \cap B(z_1, \rho)}(z_2)| \leq 16 \rho \) for an arbitrary \( \rho > 0 \), \( z_1 \in \mathbb{C}, z_2 \in \mathbb{C}, E \subset \mathbb{C} \), there exists \( \rho(\varepsilon) > 0 \) such that \( |I_{E \cap B(z_1, \rho)}(z_2)| \leq \frac{\varepsilon}{6} \).

Therefore
\[ \left| I_{a,1}^{(3)}(z_0) - I_{a,1}^{(3)}(z) \right| \leq \frac{2\varepsilon}{3} + \int_{\Omega^+ \setminus (B(z_0, \rho) \cup B(z, \rho))} d\xi d\eta = \frac{2\varepsilon}{3} + \frac{4}{\pi} |z_0 - z| \int_{\Omega^+ \setminus (B(z_0, \rho) \cup B(z, \rho))} \frac{d\xi d\eta}{|\zeta - z_0||\zeta - z|}. \]

Under the condition \( |z_0 - z| < \frac{\rho(\varepsilon)}{2} \) we get:
\[ \int_{\Omega^+ \setminus (B(z_0, \rho) \cup B(z, \rho))} \frac{d\xi d\eta}{|\zeta - z_0||\zeta - z|} \leq 4\pi \log \frac{d}{\rho(\varepsilon)}, \]

where \( d = \max_{t \in \Omega^+} |z_0 - t| \). By choosing \( |z_0 - z| < \min \left\{ \frac{\rho(\varepsilon)}{2}; \varepsilon \left( 48 \log \frac{d}{\rho(\varepsilon)} \right)^{-1} \right\} \) we obtain
\[ \left| I_{a,1}^{(3)}(z_0) - I_{a,1}^{(3)}(z) \right| < \varepsilon. \]

To prove the continuity of \( I_{a,1}^{(2)}(z) \) fix any point \( z_0 \in \mathbb{C} \). For any \( z \in \mathbb{C} \) denote \( \delta := |z - z_0|, \Omega^+_1 := B(z_0, 3\delta) \cap \Omega^+, \Omega^+_2 := \Omega^+ \setminus \Omega^+_1. \)

Denote
\[ R(\zeta, z) := \log |\zeta - z| \sum_{k=0}^{\infty} (-1)^k \frac{\alpha^{2k+1} |\zeta - z|^2 |(2k+1 + \alpha(z - \zeta))| d\xi d\eta, \]

(2(k + 1) + \alpha(z - \zeta)).
Then
\[|I_{\alpha,1}^{(2)}(z) - I_{\alpha,1}^{(2)}(z_0)| \leq \left| \iint_{\Gamma^{+}} R(\zeta, z_0) d\xi d\eta \right| + \left| \iint_{\Omega^{+}_2} R(\zeta, z) d\xi d\eta \right| + \left| \iint_{\Omega^{+}_2} (R(\zeta, z) - R(\zeta, z_0)) d\xi d\eta \right| =: I_4 + I_5 + I_6,\]
and setting $4\delta < 1$ we get
\[
I_4 \leq \iint_{\Omega^{+}_2} |\log |\zeta - z_0|| \sum_{k=0}^{\infty} \frac{|\alpha|^{2k+1} |\zeta - z_0|^{2k}}{2^{2k} k! (k + 1)!} (2(k + 1) + |\alpha| \cdot |\zeta - z_0|) d\xi d\eta \leq \sum_{k=0}^{\infty} \frac{|\alpha|^{2k+1} |\delta|^{2k}}{2^{2k} k! (k + 1)!} (2(k + 1) + |\alpha| \cdot |\zeta - z_0|) d\xi d\eta \leq \sum_{k=0}^{\infty} \frac{|\alpha|^{2k+1} |4\delta|^{2k}}{2^{2k} k! (k + 1)!} (2(k + 1) + |\alpha| \cdot |\zeta - z_0|) d\xi d\eta \leq \sum_{k=0}^{\infty} \frac{|\alpha|^{2k+1} |4\delta|^{2k}}{2^{2k} k! (k + 1)!} (2(k + 1) + |\alpha| \cdot |\zeta - z_0|) d\xi d\eta \leq \sum_{k=0}^{\infty} \frac{|\alpha|^{2k+1} |\delta|^{2k}}{2^{2k} k! (k + 1)!} (2(k + 1) + |\alpha| \cdot |\zeta - z_0|) d\xi d\eta \leq \sum_{k=0}^{\infty} \frac{|\alpha|^{2k+1} |\delta|^{2k}}{2^{2k} k! (k + 1)!} (2(k + 1) + |\alpha| \cdot |\zeta - z_0|) d\xi d\eta = o(1) \text{ as } \delta \to 0,
\]
\[
I_5 \leq \iint_{\Omega^{+}_2} |\log |\zeta - z|| \sum_{k=0}^{\infty} \frac{|\alpha|^{2k+1} |\zeta - z|^{2k}}{2^{2k} k! (k + 1)!} (2(k + 1) + |\alpha| \cdot |\zeta - z_0|) d\xi d\eta \leq \sum_{k=0}^{\infty} \frac{|\alpha|^{2k+1} |\delta|^{2k}}{2^{2k} k! (k + 1)!} (2(k + 1) + |\alpha| \cdot |\zeta - z_0|) d\xi d\eta \leq \sum_{k=0}^{\infty} \frac{|\alpha|^{2k+1} |4\delta|^{2k}}{2^{2k} k! (k + 1)!} (2(k + 1) + |\alpha| \cdot |\zeta - z_0|) d\xi d\eta \leq \sum_{k=0}^{\infty} \frac{|\alpha|^{2k+1} |\delta|^{2k}}{2^{2k} k! (k + 1)!} (2(k + 1) + |\alpha| \cdot |\zeta - z_0|) d\xi d\eta \leq \sum_{k=0}^{\infty} \frac{|\alpha|^{2k+1} |\delta|^{2k}}{2^{2k} k! (k + 1)!} (2(k + 1) + |\alpha| \cdot |\zeta - z_0|) d\xi d\eta = o(1) \text{ as } \delta \to 0.
\]
\[
I_6 \leq \iint_{\Omega^{+}_2} |\log |\zeta - z| - \log |\zeta - z_0|| \sum_{k=0}^{\infty} \frac{|\alpha|^{2k+1} |\zeta - z|^{2k}}{2^{2k} k! (k + 1)!} \times (2(k + 1) + |\alpha| \cdot |\zeta - z_0|) d\xi d\eta + \iint_{\Omega^{+}_2} |\log |\zeta - z_0|| \sum_{k=0}^{\infty} \frac{|\alpha|^{2k+1} |\zeta - z_0|^{2k}}{2^{2k} k! (k + 1)!} (|\zeta - z|^{2k} - |\zeta - z_0|^{2k}) \times (2(k + 1) + \alpha(z - z_0)) d\xi d\eta + \iint_{\Omega^{+}_2} |\log |\zeta - z_0|| \sum_{k=0}^{\infty} \frac{|\alpha|^{2k+1} |\zeta - z_0|^{2k}}{2^{2k} k! (k + 1)!} (|\zeta - z_0|^{2k} - |\zeta - z|^{2k}) \cdot |\alpha| \cdot |z_0 - z| d\xi d\eta.
\]
Using the inequalities
\[
\left| \log \frac{|\zeta - z|}{|\zeta - z_0|} \right| < \frac{2\delta}{|\zeta - z_0|}, \quad \zeta \in \Omega^{+}_2, \quad |\zeta - z| \leq 2k (C(\Gamma, z_0))^{2k-1} \delta, \quad \zeta \in \Omega^{+}_2,
\]
we have
\[
I_6 \leq C(\Gamma, z_0, \alpha)\delta,
\]
where $C(\cdot)$ denotes a constant depending only on the parameters in the parenthesis.
\[\square\]
4.4. Proof of Theorem 3.2. Let us prove (29) (the relation (30) is proved similarly). We consider a sequence \( z_n \in \Omega^+ \), \( z_n \to t \in \Gamma \), and denote by \( \zeta_n \) the nearest to \( z_n \) point of the curve \( \Gamma \).

Applying formula (31) we have that
\[
|\Phi_\alpha[f](z_n) - (I_{\alpha,1}(t) + 1)f(t) - F_\alpha[f](t)| =
\]
\[
|\Phi_\alpha[f](z_n) - \Phi_\alpha[f(\zeta_n)](z_n) + \Phi_\alpha[f(\zeta_n)](z_n) - F_\alpha[f](\zeta_n) + F_\alpha[f](\zeta_n) - (I_{\alpha,1}(t) + 1)f(t) - F_\alpha[f](t)| \leq M_1 + M_2,
\]
where
\[
M_1 = |\Phi_\alpha[f - f(\zeta_n)](z_n) - F_\alpha[f](\zeta_n)|,
\]
\[
M_2 = |(I_{\alpha,1}(z_n) + 1)f(\zeta_n) + F_\alpha[f](\zeta_n) - (I_{\alpha,1}(t) + 1)f(t) - F_\alpha[f](t)|.
\]

Let \( \alpha \neq 0 \). On the basis of relations (23) – (25) we have the representation
\[
K_\alpha(z) = S_\alpha(z) + \varphi_\alpha(z),
\]
where \( \varphi_\alpha(z) \) is a continuous function in \( \mathbb{C} \) and
\[
S_\alpha(z) := -\frac{1}{2\pi} \left( \frac{z}{|z|^2} + \alpha \log |z| \right).
\]
Then
\[
M_1 \leq \int_{\Gamma} S_\alpha(\zeta - z_n) \cdot \sigma \cdot (f(\zeta) - f(\zeta_n)) - \int_{\Gamma} S_\alpha(\zeta - \zeta_n) \cdot \sigma \cdot (f(\zeta) - f(\zeta_n)) + \\
+ \int_{\Gamma} \varphi_\alpha(\zeta - z_n) \cdot \sigma \cdot (f(\zeta) - f(\zeta_n)) - \int_{\Gamma} \varphi_\alpha(\zeta - \zeta_n) \cdot \sigma \cdot (f(\zeta) - f(\zeta_n)) =: M_3 + M_4.
\]

By virtue of continuity of the functions \( F_\alpha[f] \) (Lemma 3.3), \( I_{\alpha,1} \) (Lemma 3.5), \( \varphi_\alpha \) and \( f \) we get that \( M_2 \to 0 \) and \( M_4 \to 0 \), when \( z_n \to t \).

Let us fix an arbitrary \( \varepsilon > 0 \). For any given \( \delta > 0 \) we have
\[
M_3 \leq M_5 + M_6 + M_7,
\]
where
\[
M_5 = \left| \int_{\Gamma_{\zeta_n,\delta}} S_\alpha(\zeta - \zeta_n) \cdot \sigma \cdot (f(\zeta) - f(\zeta_n)) \right|,
\]
\[
M_6 = \left| \int_{\Gamma_{\zeta_n,\delta}} S_\alpha(\zeta - z_n) \cdot \sigma \cdot (f(\zeta) - f(\zeta_n)) \right|,
\]
\[
M_7 = \left| \int_{\Gamma \setminus \Gamma_{\zeta_n,\delta}} (S_\alpha(\zeta - z_n) - S_\alpha(\zeta - \zeta_n)) \cdot \sigma \cdot (f(\zeta) - f(\zeta_n)) \right|.
\]
By virtue of the equality (40) it follows from the uniform existence of $F_n[f]$ (Lemma 3.3) that for all sufficiently small $\delta$ and for all $\zeta_n \in \Gamma$ the inequality $M_5 < \frac{\delta}{3}$ is valid.

Let us estimate $M_6$. For any $\delta > 0$ let us take $z_n$ near to $t$ so that $|\zeta_n - z_n| < \frac{\delta}{3}$. We have

$$M_6 \leq \left| \int_{\Gamma_{\zeta_n,3}|\zeta_n - z_n|} S_\alpha(\zeta - z_n) \cdot \sigma \cdot (f(\zeta) - f(\zeta_n)) \right| +$$

$$+ \left| \int_{\Gamma_{\zeta_n,\delta}\setminus\Gamma_{\zeta_n,3}|\zeta_n - z_n|} S_\alpha(\zeta - z_n) \cdot \sigma \cdot (f(\zeta) - f(\zeta_n)) \right| =: M_8 + M_9.$$

Let us estimate $M_8$. Using the inequalities $|\zeta - \zeta_n| \leq 3|\zeta - z_n| < 4\delta$, we obtain for sufficiently small $\delta < \frac{\delta}{3}$:

$$|S_\alpha(\zeta - z_n)| = \left| \frac{\zeta - z_n}{|\zeta - z_n|^2} + \alpha \log |\zeta - z_n| \right| \leq$$

$$\leq \frac{3}{|\zeta - z_n|^2} + |\alpha| \cdot |\log |\zeta - \zeta_n|| + |\alpha| \log 3 \leq 4,$$

(43) $|S_\alpha(\zeta - z_n)| = |S_\alpha(\zeta - \zeta_n)| \cdot \frac{S_\alpha(\zeta - z_n)}{|S_\alpha(\zeta - \zeta_n)|} \leq 4 \cdot |S_\alpha(\zeta - \zeta_n)|$.

Due to the uniform existence of the integral (27) and the equality (40) it follows from (43) that for all sufficiently small $\delta$ and for all $|\zeta_n - z_n| < \frac{\delta}{3}$

$$M_8 \leq 4 \int_{\Gamma_{\zeta_n,3}|\zeta_n - z_n|} |S_\alpha(\zeta - \zeta_n)| \cdot |\sigma| \cdot |f(\zeta) - f(\zeta_n)| < \frac{\varepsilon}{6}.$$

Let us estimate $M_9$. We get

$$|S_\alpha(\zeta - z_n)| \leq |S_\alpha(\zeta - \zeta_n)| + |S_\alpha(\zeta - z_n) - S_\alpha(\zeta - \zeta_n)|.$$

As long as $|z_n - \zeta_n| \leq \frac{1}{2} |\zeta - z_n|$, $3|\zeta_n - z_n| < |\zeta - \zeta_n| \leq \delta < \frac{\delta}{2}$ and $\frac{1}{2\pi |S_\alpha(\zeta - \zeta_n)|} \leq 2\pi |S_\alpha(\zeta - z_n) - S_\alpha(\zeta - \zeta_n)|$ we have

$$|S_\alpha(\zeta - z_n)| \leq$$

$$\leq \frac{1}{2\pi} \left| \frac{\zeta - z_n}{|\zeta - z_n|^2} - \frac{\zeta - \zeta_n}{|\zeta - \zeta_n|^2} \right| + \frac{|\alpha|}{2\pi} \left| \log \frac{|\zeta - z_n|}{|\zeta - \zeta_n|} \right| = \frac{|z_n - \zeta_n|}{2\pi |\zeta - z_n| \cdot |\zeta - \zeta_n|} + \frac{|\alpha|}{2\pi} \left| \log \frac{|\zeta - z_n|}{|\zeta - \zeta_n|} \right| \leq$$

$$\leq \frac{1}{4\pi |\zeta - \zeta_n|} + \frac{|\alpha|}{2\pi} \log \frac{3}{2} \leq \frac{1}{4\pi |\zeta - \zeta_n|} \leq \frac{1 + |\alpha|}{2} |S_\alpha(\zeta - \zeta_n)|.$$

From (45), (46) we get

$$M_9 \leq \frac{3 + |\alpha|}{2} \int_{\Gamma_{\zeta_n,\delta}\setminus\Gamma_{\zeta_n,3}|\zeta_n - z_n|} |S_\alpha(\zeta - z_n)| \cdot |\sigma| \cdot |f(\zeta) - f(\zeta_n)| < \frac{\varepsilon}{6}.$$
for sufficiently small \( \delta \) and for \(|\zeta_n - z_n| < \frac{\delta}{3}\).

So we have from (42), (44), (47) that \( M_6 < \frac{\delta}{3}\).

In order to estimate \( M_7 \) fix any \( \delta \) satisfying all the conditions stated above and take \( z_n \) near \( t \) so that \(|\zeta_n - z_n| \leq \frac{\delta}{3}\).

We have \( \delta < |\zeta - \zeta_n|, \quad 2\delta < |\zeta - z_n| \). Therefore

\[
(48) \quad \frac{|z_n - \zeta_n|}{|\zeta - z_n| \cdot |\zeta - \zeta_n|} \leq \frac{3}{2\delta^2} |z_n - \zeta_n|,
\]

and by Lagrange’s theorem

\[
|\log \frac{|\zeta - z_n|}{|\zeta - \zeta_n|}| = \frac{1}{\mu} ||\zeta - z_n| - |\zeta - \zeta_n|| \leq \frac{3}{2\delta} |z_n - \zeta_n|,
\]

where \( \mu \) lies between \(|\zeta - z_n|\) and \(|\zeta - \zeta_n|\). That is why using the relation (46), we get

\[
|S_\alpha(\zeta - z_n) - S_\alpha(\zeta - \zeta_n)| \leq 3\left(1 + |a|\frac{\delta}{4\pi\delta^2}\right)|z_n - \zeta_n|,
\]

and taking into account the boundedness of the function \( f \), we obtain for the above fixed \( \delta \) and for \( z_n \) sufficiently near to \( t \)

\[
M_7 \leq \int_{\Gamma \setminus C_{\zeta_n, \delta}} |S_\alpha(\zeta - z_n) - S_\alpha(\zeta - \zeta_n)| \cdot |\sigma| \cdot |f(\zeta) - f(\zeta_n)| d\Gamma \leq \frac{1 + |a|\frac{\delta}{4\pi\delta^2}}{2\delta^2} l(\Gamma) \max_{t \in \Gamma} |f(t)| |\zeta_n - z_n| < \frac{\varepsilon}{3},
\]

where \( l(\Gamma) \) denotes the length of \( \Gamma \).

Thus, we have \( M_3 < \varepsilon \) and, consequently, the relation (29) is proved.

The continuity of \( \Phi^n_{\alpha} [f] \) on \( \Gamma \) follows now from Lemmas 3.3 and 3.5. This completes the proof of Theorem 3.2. \( \square \)

5. Proof of main results

5.1. We identify a complex quaternion \( a = \sum_{k=0}^{3} a_k i_k \) with the scalar-vector pair \((a_0, a)\), where \( a = \sum_{k=1}^{3} a_k i_k \) is a vector of the complex linear space \( \mathbb{C}^3 \) with the canonical basis \( i_1, i_2, i_3 \). Then a quaternionic function \( f = \sum_{k=0}^{3} f_k i_k \) is interpretable as a pair \( \mathcal{F} = (f_0, f) \), operator \( \partial_\alpha \) as a pair \((M^\alpha, \sigma^f)\), where \( \sigma^f := \partial_{i_1} t_1 + \partial_{i_2} t_2 \). Using the vectorial representation of the multiplication of any complex quaternions \( a = (a_0, a) \) and \( b = (b_0, b) \) (see [KS], p. 24):

\[
(49) \quad ab = (a_0 b_0 - \langle a, b \rangle, [a, b] + a_0 b + b_0 a),
\]

we obtain

\[
\partial_\alpha f = (\alpha f_0 - div f, rot f + \alpha f + grad f_0),
\]

reducing to the system (2) as the vector form of Definition 2.4 of an \( \alpha \)-hyperholomorphic function.

5.2. Proof of Theorem 1.6. The representation (4) follows from the formula (25) and we obtain (5) from (26) by using the equality (49). Combining the vector form of the functions \( F_\alpha, I_{n, \Gamma}, \Phi^n_{\alpha} \) in Theorem 3.2 with the equality (49), we arrive at Theorem 1.6 as a vector reformulation of Theorem 3.2. \( \square \)
5.3. Proof of Corollary 1.7. Applying Theorem 1.6 to the pair \( \mathcal{F} = (0, f) \), we obtain the desired conclusion. Because of the condition \( f \in M(\Gamma; \mathbb{C}^3) \) the Cauchy-type integral \( \Phi_{\alpha}[\mathcal{F}] \) is purely vectorial and therefore its boundary values \( \Phi_{\alpha}^\pm[\mathcal{F}] \) are also purely vectorial. □

Acknowledgments

The first-named author was supported in part by INTAS-99-00089 and by CONACYT project.

The second-named author was partially supported by CONACYT projects as well as by Insituto Politécnico Nacional in the framework of COFAA and CGPI programs.

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