DYNAMICS OF CORRELATIONS IN A SYSTEM OF HARD SPHERES

V. I. Gerasimenko*
Institute of mathematics of the NAS of Ukraine
Kyiv, Ukraine

I. V. Gapyak†
Taras Shevchenko National University of Kyiv,
Department of Mathematics and Mechanics
Kyiv, Ukraine

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Abstract

The possible ways to describe the states of a system of many hard spheres are considered, in particular by means of functions describing correlations of states. It is stated an approach to the description of the evolution based on the dynamics of correlations in a system of hard spheres. In addition, we consider another approach to describing the evolution of correlations in a system of many hard spheres, namely, in the framework of a one-particle distribution function (correlation function) governed by the non-Markovian Enskog kinetic equation.

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*E-mail address: gerasym@imath.kiev.ua
†E-mail address: gapjak@ukr.net
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1. Introduction

Recently, mainly in connection with the problem of the rigorous derivation of the kinetic equations \([1]-[5]\), a number of articles have discussed approaches to describing the evolution of states of a system of many hard spheres \([4]-[17]\).

As known, many-particle systems are described in terms of such notions as observables and states. The functional for the mean value of observables determines a duality of observables and states. As a consequence, there are two approaches to describing the evolution of a system from a finite number of particles, namely, in terms of observables, which are governed by the Liouville equation for the function of observables, or in terms of states governed by the dual Liouville equation for the probability distribution function, respectively \([17],[19]\).

An alternative approach to the description of states of a system of finitely many particles is to describe states using functions determined by the cluster expansions of the probability distribution function. They are cumulants (semi-invariants) of the probability distribution function and are interpreted as correlations of the state (correlation functions). The evolution of correlation functions is governed by the Liouville hierarchy (the von Neumann hierarchy for quantum many-particle systems \([20]-[24]\)).

One more approach to describing a state of many-particle systems is to describe a state by means of a sequence of so-called reduced distribution functions (marginal distribution functions) governed by the BBGKY (Bogolyubov–Born–Green–Kirkwood–Yvon) hierarchy \([1]-[4]\). An alternative approach to such a description of a state is based on sequences of functions determined by the cluster expansions of reduced distribution functions. These functions are interpreted as the reduced correlation functions that are governed by the hierarchy of nonlinear evolution equations (in papers \([25],[26]\) in the case of quantum many-particle systems). The mention approaches are allowed to describe the evolution of states of systems both with a finite and infinite number of particles.

In the paper, it is also developed an approach to the description of the evolution by means of both reduced distribution functions and reduced correlation functions which is based on the dynamics of correlations in a system of hard spheres. It should be emphasized that the structure of solution expansions of the corresponding hierarchies is induced by the structure of a solution expansion of the Liouville hierarchy for a sequence of correlation functions. We note the importance of the description of the processes of the creation and propagation of correlations \([27]\), in particular, it is related to the problem of the description of the entanglement of states in many-particle systems.

In addition in the paper, an approach to the description of the evolution of states of a hard sphere system by means of the state of a typical particle governed by the generalized Enskog equation \([13]\) is discussed, or in other words, the foundations of describing the evolution of states by kinetic equations are considered. We note that the conventional approach to the mentioned problem is based on the consideration of asymptotic behavior (the Boltzmann–Grad asymptotic behavior \([28]-[31]\)) of a solution of the BBGKY hierarchy for reduced distribution functions represented in the form of series expansions of the perturbation theory in case of initial states specified by a one-particle distribution function without correlation functions \([1]-[5],[32]\).

Thus, the paper deals with the mathematical problems of the description of the evolution of many hard spheres based on various ways of describing of the state, in particular by means of functions describing correlations of states. Moreover, the origin of different approaches to the description of states is discussed.
2. Dynamics of finitely many hard spheres

The system of many hard spheres is describing in terms of observables and states. The functional for the mean value of observables determines the duality of observables and states, and, as a consequence, there are two equivalent ways to describe the evolution of a system of finitely many hard spheres as the evolution of observables governed by the Liouville equation, and as the evolution of states governed by the dual Liouville equation (usually called the Liouville equation). An equivalent approach adapted to describing the evolution of observables and states of systems of both finite and infinite number of hard spheres is to describe a state by means of a sequence of so-called reduced distribution functions (marginals) governed by the BBGKY hierarchy of equations and of observables by means of sequences of so-called reduced functions of observables (marginal observables) that are governed by the dual BBGKY hierarchy of equations.

2.1. Observables and states

We consider a system of identical particles of a unit mass interacting as hard spheres with a diameter of $\sigma > 0$. Every particle is characterized by its phase coordinates $(q_i, p_i) \equiv x_i \in \mathbb{R}^3 \times \mathbb{R}^3$, $i \geq 1$. For configurations of such a system the following inequalities are satisfied: $|q_i - q_j| \geq \sigma, i \neq j \geq 1$, i.e. the set $\mathbb{W}_n \equiv \{(q_1, \ldots, q_n) \in \mathbb{R}^{3n} | q_i - q_j < \sigma$ for at least one pair $(i, j) : i \neq j \in (1, \ldots, n)\}, n > 1$, is the set of forbidden configurations.

Let $C_n$ be the space of sequences $b = (b_0, b_1, \ldots, b_n, \ldots)$ of bounded continuous functions $b_n \in C_n$ equipped with the norm: $\|b_n\|_{C_n} = \max_{n \geq 0} \frac{1}{n!} \|b_n\|_{C_n}$, and let $L^1_n \equiv L^1(\mathbb{R}^{3n} \times \mathbb{R}^{3n})$ be the space of integrable functions that are symmetric with respect to permutations of the arguments $x_1, \ldots, x_n$, equipped with the norm: $\|f_n\|_{L^1(\mathbb{R}^{3n} \times \mathbb{R}^{3n})} = \int dx_1 \ldots dx_n |f_n(x_1, \ldots, x_n)|$. Hereafter the subspace of continuously differentiable functions with compact supports we will denote by $L^1_{n,0} \subset L^1_n$ and the subspace of finite sequences of continuously differentiable functions with compact supports let be $L^1_{0} \subset L^1_{\alpha} = \oplus_{n=0}^{\infty} \alpha^n L^1_n$, where $\alpha > 1$ is a real number.

For a hard-sphere system of a non-fixed, i.e. arbitrary but finite average number of identical particles (nonequilibrium grand canonical ensemble) in the space $\mathbb{R}^3$, an observable describes by the sequence $A = (A_0, A_1(x_1), \ldots, A_n(x_1, \ldots, x_n), \ldots)$ of functions $A_n \in C_n$ defined on the phase spaces of the corresponding number $n$ of hard spheres.

Then the meaning of positive continuous linear functional on the space $C_n$ is determined the average value of an observable (the expected value or mean value of an observable). For a system of non-fixed number of hard spheres it is defined as follows [1]:

$$\langle A \rangle = (A, D) = (I, D)^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_1 \ldots dx_n A_n D_n,$$

where $D = (1, D_1, \ldots, D_{n}, \ldots)$ is a sequence of symmetric nonnegative functions $D_n = D_n(x_1, \ldots, x_n)$, $n \geq 1$, equal to zero on the set of forbidden configurations $\mathbb{W}_n$ and the normalizing factor $(I, D) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_1 \ldots dx_n D_n(x_1, \ldots, x_n)$ is a grand canonical partition function. The sequence of functions $D$ describes a state of a system of a non-fixed number of hard spheres.

For the sequences $A \in C_n$ and $D \in L^1_{\alpha}$ mean value functional [1] exists and it determines a duality between observables and states.

We note that in the particular case of a system of $N < \infty$ hard spheres the observables and states are described by the one-component sequences: $A^{(N)} = (0, \ldots, 0, A_N, 0, \ldots)$.
and $D^{(N)} = (0, \ldots, 0, D_N, 0, \ldots)$, respectively, and therefore, functional (1) has the following representation

$$
\langle A \rangle = \langle A, D \rangle = \frac{1}{N!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^N} dx_1 \ldots dx_N \ A_N \ D_N,
$$

where $(I, D) = \frac{1}{N!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^N} dx_1 \ldots dx_N D_N$ is the normalizing factor (canonical partition function), and it is usually assumed that the normalization condition $\int_{(\mathbb{R}^3 \times \mathbb{R}^3)^N} dx_1 \ldots dx_N D_N = 1$ holds.

The function $D_N(x_1, \ldots, x_N)$, which describes all possible states of a system of $N$ hard spheres, is called a probability distribution function, since the expression $(I, D)^{-1} D_N(x_1, \ldots, x_N) dx_1 \ldots dx_N$ is the probability of finding the phase states of the $1st, \ldots, Nth$ hard sphere in the phase volumes $dx_1, \ldots, dx_N$ centered at the phase points $x_1, \ldots, x_N$, respectively.

If at initial instant an observable specified by the sequence $A(0) = (A_0, A_0^0(x_1), \ldots, A_0^n(x_1, \ldots, x_n), \ldots)$, then the evolution of observables $A_n(t), n \geq 1$, i.e. the sequence $A(t) = (A_0, A_1(t, x_1), \ldots, A_n(t, x_1, \ldots, x_n), \ldots)$ is determined by the following the one-parameter mapping $S(t) = \oplus_{n=0}^{\infty} S_n(t)$:

$$
A(t) = S(t)A(0),
$$

which is defined on every the space $C_n = C(\mathbb{R}^{3n} \times (\mathbb{R}^{3n} \setminus \mathbb{W}_n))$ by means of the phase trajectories of a hard-sphere system, which are defined almost everywhere on the phase space $\mathbb{R}^{3n} \times (\mathbb{R}^{3n} \setminus \mathbb{W}_n)$, namely, beyond of the set $\mathbb{M}_n^0$ of the zero Lebesgue measure, as follows

$$
(S_n(t)b_n)(x_1, \ldots, x_n) = S_n(t, 1, \ldots, n)b_n(x_1, \ldots, x_n) \equiv \begin{cases} 
    b_n(X_1(t, x_1, \ldots, x_n), \ldots, X_n(t, x_1, \ldots, x_n)), & \text{if } (x_1, \ldots, x_n) \in (\mathbb{R}^{3n} \times (\mathbb{R}^{3n} \setminus \mathbb{W}_n)), \\
    0, & \text{if } (q_1, \ldots, q_n) \in \mathbb{W}_n,
\end{cases}
$$

where for $t \in \mathbb{R}$ the function $X_i(t)$ is a phase trajectory of $ith$ particle constructed in [1] and the set $\mathbb{M}_n^0$ consists of the phase space points which are specified such initial data that during the evolution generate multiple collisions, i.e. collisions of more than two particles, more than one two-particle collision at the same instant and infinite number of collisions within a finite time interval [11,31].

On the space $C_n$ one-parameter mapping (3) is an isometric $*$-weak continuous group of operators, i.e. it is a $C_0^*$-group. For the group of evolution operators (3) the Duhamel equation holds

$$
S_n(t, 1, \ldots, n)b_n = \prod_{i=1}^{n} S_1(t, i)b_n + \int_0^t d\tau \prod_{i=1}^{n} S_1(t - \tau, i) \sum_{j_1 < j_2 = 1}^{n} \mathcal{L}_{int}(j_1, j_2) S_n(\tau, 1, \ldots, n)b_n = \prod_{i=1}^{n} S_1(t, i)b_n + \int_0^t d\tau S_n(t - \tau, 1, \ldots, n) \sum_{j_1 < j_2 = 1}^{n} \mathcal{L}_{int}(j_1, j_2) \prod_{i=1}^{n} S_1(\tau, i)b_n,
$$
where for \( t > 0 \) the operator \( \mathcal{L}_{\text{int}}(j_1, j_2) \) is defined by the formula

\[
\mathcal{L}_{\text{int}}(j_1, j_2)b_n \doteq \sigma^2 \int_{S^2_+} d\eta \langle \eta, (p_{j_1} - p_{j_2}) \rangle \left( b_n(x_1, \ldots, p_{j_1}^*, q_{j_1}, \ldots), \right)
\]

\[
p_{j_2}^* = b_n(x_1, \ldots, x_n) - b_n(x_1, \ldots, x_n) \delta(q_{j_1} - q_{j_2} + \sigma \eta).
\]

In definition (4) the symbol \( \langle \cdot, \cdot \rangle \) means a scalar product, the symbol \( \delta \) denotes the Dirac measure, \( S^2_+ \doteq \{ \eta \in \mathbb{R}^3 \mid \eta = 1, (\eta, (p_1 - p_2)) > 0 \} \) and the momenta \( p_i^*, p_j^* \) are determined by the equalities:

\[
p_i^* = p_i - \eta \langle \eta, (p_i - p_j) \rangle ,
\]

\[
p_j^* = p_j + \eta \langle \eta, (p_i - p_j) \rangle.
\]

If \( t < 0 \), the operator \( \mathcal{L}_{\text{int}}(j_1, j_2) \) is defined by the corresponding expression (1). Thus, the infinitesimal generator \( \mathcal{L}_n \) of the group of operators (3) has the structure

\[
\mathcal{L}_n b_n \doteq \sum_{j=1}^n \mathcal{L}(j)b_n + \sum_{j_1 < j_2=1} \mathcal{L}_{\text{int}}(j_1, j_2)b_n,
\]

where the Liouville operator of free motion \( \mathcal{L}(j) \doteq \langle p_j, \frac{\partial}{\partial q_j} \rangle \) defined on the set \( C_{n,0} \), we had denoted by the symbol \( \mathcal{L}(j) \).

If \( A(0) \in C_n \), the sequence (2) is a unique solution of the Cauchy problem for the sequence of the weak formulation of the Liouville equations

\[
\frac{\partial}{\partial t} A(t) = \mathcal{L} A(t),
\]

\[
A(t)|_{t=0} = A(0),
\]

where the operator \( \mathcal{L} = \oplus_{n=0}^\infty \mathcal{L}_n \) is defined by formula (5).

Taking into account the equality \( (I, D(0)) = (I, S^*(t)D(0)) \), and of the validity for functional (1) the following representations:

\[
(A(t), D(0)) = (I, D(0))^{-1} \sum_{n=0}^\infty \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_1 \ldots dx_n S_n(t)A^0_n D^0_n = \]

\[
(I, S^*(t)D(0))^{-1} \sum_{n=0}^\infty \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_1 \ldots dx_n A^0_n S_n^*(t)D^0_n \equiv \]

\[
(I, D(t))^{-1}(A(0), D(t)),
\]

where the adjoint group of operators \( S^*_n(t) \) to group of operators (3) is defined on the space of integrable functions \( L^1_n \)

\[
S^*_n(t) = S_n(-t),
\]

then, as a result, we can describe the evolution of many hard spheres within the evolution of states.

On the space \( L^1_n \) the one-parameter mapping defined by formula (9) is an isometric strong continuous group of operators. Indeed, \( \|S^*_n(t)\| = 1 \).
We note that the group of operators \( S_n^*(t) \) satisfies the Duhamel equation

\[
S_n^*(t, 1, \ldots, n) = \prod_{i=1}^{n} S_i^*(t, i) + \int_0^t d\tau \prod_{i=1}^{n} S_i^*(t - \tau, i) \sum_{j_1 < j_2 = 1}^{n} \mathcal{L}_{\text{int}}^*(j_1, j_2) S_n^*(\tau, 1, \ldots, n) = \prod_{i=1}^{n} S_i^*(t, i) + \int_0^t d\tau S_n^*(t - \tau, 1, \ldots, n) \sum_{j_1 < j_2 = 1}^{n} \mathcal{L}_{\text{int}}^*(j_1, j_2) \prod_{i=1}^{n} S_i^*(\tau, i),
\]

where for \( t > 0 \) the operator \( \mathcal{L}_{\text{int}}^*(j_1, j_2) \) is defined by the formula

\[
\mathcal{L}_{\text{int}}^*(j_1, j_2) f_n = \sigma^2 \int_{S^2_+} d\eta \langle \eta, (p_{j_1} - p_{j_2}) \rangle f_n(x_1, \ldots, p_{j_1}, q_{j_1}, \ldots, p_{j_2}, q_{j_2}, \ldots, x_n) \delta(q_{j_1} - q_{j_2} + \sigma \eta) - f_n(x_1, \ldots, x_n) \delta(q_{j_1} - q_{j_2} - \sigma \eta) \rangle.
\]

In formula (10) the notations similar to (4) are used.

Hence the infinitesimal generator \( \mathcal{L}_n^* \) of the group of operators \( S_n^*(t) \) has the structure

\[
\mathcal{L}_n^* f_n = \sum_{j=1}^{n} \mathcal{L}^*(j) f_n + \sum_{j_1 < j_2 = 1}^{n} \mathcal{L}_{\text{int}}^*(j_1, j_2) f_n,
\]

where the Liouville operator of free motion \( \mathcal{L}^*(j) \) is defined on the subspace \( L^1_{n,0} \subset L^1_n \), we had denoted by the symbol \( \mathcal{L}^*(j) \).

In view of the validity of equality (8) the evolution of all possible states, i.e. the sequence \( D(t) = (1, D_1(t), \ldots, D_n(t), \ldots) \in L^1_n \) of the probability distribution functions \( D_n(t), n \geq 1 \), is determined by the formula

\[
D(t) = S^*(t)D(0),
\]

where the one-parameter family of operators \( S^*(t) = \oplus_{n=0}^{\infty} S_n^*(t) \), is defined as above.

If \( D(0) \in L^1_{\alpha,0} \), the sequence of distribution functions defined by formula (12) is a unique solution of the Cauchy problem for the sequence of the weak formulation of the dual Liouville equation for states (known as the Liouville equation)

\[
\frac{\partial}{\partial t} D(t) = \mathcal{L}^* D(t),
\]

\[
D(t)|_{t=0} = D(0),
\]

where the generator \( \mathcal{L}^* = \oplus_{n=0}^{\infty} \mathcal{L}_n^* \) of the dual Liouville equations (13) is the adjoint operator to generator (5) of the Liouville equation (6) in the sense of functional (1) i.e. it is defined by formula (11).

2.2. Reduced functions of observables and states

For the description of a system of hard spheres of both finite and infinite number of particles another approach to describing observables and states is used, which is equivalent to the approach formulated above in the case of systems of finitely many hard spheres [11,52].
Indeed, for a system of finitely many particles mean value functional \((1)\) can be represented in one more form:

\[
\langle A \rangle = (I, D)^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_1 \ldots dx_n \, A_n \, D_n =
\]

\[
\sum_{s=0}^{\infty} \frac{1}{s!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^s} dx_1 \ldots dx_s \, B_s(x_1, \ldots, x_s) \, F_s(x_1, \ldots, x_s),
\]

where, for the description of observables and states, the sequence of so-called reduced functions of observables \(B = (B_0, B_1(x_1), \ldots, B_s(x_1, \ldots, x_s), \ldots)\) (other used terms: marginal or \(s\)-particle observable \((17)\)) was introduced and the sequence of so-called reduced distribution functions \(F = (1, F_1(x_1), \ldots, F_s(x_1, \ldots, x_s), \ldots)\) (other used terms: marginals \([4, 5]\), truncated or \(s\)-particle distribution function \([32]\)), respectively. Thus, the reduced functions of observables are defined by means functions of observables by the following expansions \([19]\):

\[
B_s(x_1, \ldots, x_s) = \sum_{n=0}^{s} \frac{(-1)^n}{n!} \sum_{j_1 \neq \ldots \neq j_n = 1} A_{s-n}((1, \ldots, s) \setminus (j_1, \ldots, j_n)), \quad s \geq 1,
\]

and the reduced distribution functions are defined by means of probability distribution functions as follows \([1]\):

\[
F_s(x_1, \ldots, x_s) = (I, D)^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_{s+1} \ldots dx_{s+n} \, D_{s+n}(x_1, \ldots, x_{s+n}), \quad s \geq 1.
\]

We emphasize that the possibility of describing states within the framework of reduced distribution functions naturally arises as a result of dividing the series in expression \([1]\) by the series of the normalization factor, i.e. in consequence of redefining of mean value functional \([15]\).

If initial state specified by the sequence of reduced distribution functions \(F(0) = (1, F_1^0(x_1), \ldots, F_n^0(x_1, \ldots, x_n), \ldots) \in L^2\), then the evolution of all possible states, i.e. the sequence \(F(t) = (1, F_1(t,x_1), \ldots, F_s(t,x_1, \ldots, x_s), \ldots)\) of the reduced distribution functions \(F_s(t), s \geq 1\), is determined by the following series expansions \([33]\):

\[
F_s(t, x_1, \ldots, x_s) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_{s+1} \ldots dx_{s+n} \, \mathcal{A}_{1+n}(t, \{1, \ldots, s\}),
\]

\[
s + 1, \ldots, s + n \rangle F_{s+n}^0(x_1, \ldots, x_{s+n}), \quad s \geq 1,
\]

where the generating operator

\[
\mathcal{A}_{1+n}(t, \{1, \ldots, s\}, s + 1, \ldots, s + n) = \sum_{P: \{(1, \ldots, s), s+1, \ldots, s+n\} = \bigcup X_i} (-1)^{|P|-1} (|P| - 1)! \prod_{X_i \subset P} S_{\theta(X_i)}(t, \theta(X_i))
\]

is the \((1+n)th\)-order cumulant of the groups of operators \([12, 33]\). In expansion \((19)\) the symbol \(\{1, \ldots, s\}\) is a set consisting of one element \((1, \ldots, s)\), i.e. \(|\{1, \ldots, s\}| = 1\). The symbol \(\Sigma_P\) means the sum over all possible partitions \(P\) of the set \((\{1, \ldots, s\}, s + 1, \ldots, s + n)\) into
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|P| nonempty mutually disjoint subsets $X_i \subset \{1, \ldots, s\}, s + 1, \ldots, s + n$ and $\theta$ is the declustering mapping: $\theta(\{1, \ldots, s\}, s + 1, \ldots, s + n) = (1, \ldots, s + n)$. The simplest examples of cumulants (19) of groups of operators (12) have the form

$$A_1(t, \{1, \ldots, s\}) = S^*_s(t, 1, \ldots, s),$$

$$A_{1+1}(t, \{1, \ldots, s\}, s + 1) = S^*_s(t, 1, \ldots, s + 1) - S^*_s(t, 1, \ldots, s)S^*_1(t, s + 1).$$

If $F(0) \in L^1_\alpha = \oplus_{n=0}^\infty \alpha^n L^1_n$, series (18) converges on the norm of the space $L^1_\alpha$ provided that $\alpha > e$. The parameter $\alpha$ can be interpreted as the magnitude inverse to the average number of hard spheres.

We note that the method of constructing the reduced distribution functions (18) is based on the application of cluster expansions to the generating operators (12) of series (17), as a result of which the generating operators of series (18) are the corresponding-order cumulants of the groups of operators $S^*(t)$ (11), (33).

If $F(0) \in \oplus_{n=0}^\infty \alpha^n L^1_n$ and $\alpha > e$, then for $t \in \mathbb{R}$ the sequence of reduced distribution functions (18) is a unique solution of the Cauchy problem of the BBGKY hierarchy (1), (32):

$$\frac{\partial}{\partial t} F_s(t, x_1, \ldots, x_s) = \mathcal{L}_s^* F_s(t, x_1, \ldots, x_s) +$$

$$\sum_{j=1}^s \int_{(\mathbb{R}^3 \times \mathbb{R}^3)} dx_{s+1} \mathcal{L}_{\text{int}}^* (j, s + 1) F_{s+1}(t, x_1, \ldots, x_{s+1}),$$

$$F_s(t, x_1, \ldots, x_s) |_{t=0} = F^0_s(x_1, \ldots, x_s), \quad s \geq 1,$$

where we used notations accepted in formula (11), i.e. for $t \geq 0$ the Liouville operator $\mathcal{L}_s^*$ is defined in (11) and the equality holds

$$\sum_{j=1}^s \int_{(\mathbb{R}^3 \times \mathbb{R}^3)} dx_{s+1} \mathcal{L}_{\text{int}}^* (j, s + 1) F_{s+1}(t) \doteq$$

$$\sigma^2 \sum_{i=1}^s \int_{\mathbb{R}^3 \times S^*_s} dp_{s+1} d\eta(\eta, (p_i - p_{s+1})) (F_{s+1}(t, x_1, \ldots, q_i, p_i^*), \ldots,$$

$$x_s, q_i - \sigma \eta, p_{s+1}^*) - F_{s+1}(t, x_1, \ldots, x_s, q_i + \sigma \eta, p_{s+1}^*),$$

and for $t \leq 0$, the generator of the BBGKY hierarchy (20) is defined by the corresponding expression (11). Sequences of functions from the space $L^1_\alpha$ describe the state of a finitely many-particle system, because in this case the average number of hard spheres $\langle N \rangle = \int_{(\mathbb{R}^3)^N} dx F_1(t, x)$ is finite.

We note that traditionally (11), (5), (32) the reduced distribution functions are represented by means of the perturbation theory series of the BBGKY hierarchy (20)

$$F_s(t, x_1, \ldots, x_s) =$$

$$\sum_{n=0}^\infty \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_{s+1} \ldots dx_{s+n} S^*_s(t - t_1) \sum_{j_1=1}^s \mathcal{L}_{\text{int}}^* (j_1, s + 1) \times$$

$$S^*_{s+1}(t_1 - t_2) \ldots S^*_{s+n-1}(t_{n-1} - t_n) \sum_{j_n=1}^{s+n-1} \mathcal{L}_{\text{int}}^* (j_n, s + n) S^*_{s+n}(t_n) F^0_{s+n}(x_1, \ldots, x_{s+n}), \quad s \geq 1,$$
where we used notations accepted in formula (10). The nonperturbative series expansion for reduced distribution functions (18) is represented in the form of the perturbation theory series for suitable interaction potentials and initial data as a result of the employment of analogs of the Duhamel equation to cumulants (19) of the groups of operators (12).

We remark that, if initial observable (16) specified by the sequence of reduced observables \( B(0) = (B_0, B_1(x_1), \ldots, B_s(x_1, \ldots, x_s), \ldots) \in \mathcal{C}_s \), then the evolution of observables, i.e. the sequence \( B(t) = (B_0, B_1(t, x_1), \ldots, B_s(t, x_1, \ldots, x_s), \ldots) \) of the reduced observables \( B_s(t), s \geq 1 \), is determined by the following series expansions (19):

\[
B_s(t, x_1, \ldots, x_s) = \sum_{n=0}^{s} \frac{1}{n!} \sum_{j_1 \neq \ldots \neq j_n} ^{s} \mathfrak{A}_{1+n}(t, \{ (1, \ldots, s) \setminus (j_1, \ldots, j_n) \}),
\]

\[
(j_1, \ldots, j_n) B_{s-n}^0(x_1, \ldots, x_{j_1-1}, x_{j_1+1}, \ldots, x_{j_n-1}, x_{j_n+1}, \ldots, x_s), \quad s \geq 1.
\]

The generating operators of expansions (22) is the \((1 + n)\text{th}\)-order cumulant of groups of operators (3) defined by the following expansion:

\[
\mathfrak{A}_{1+n}(t, \{ (1, \ldots, s) \setminus (j_1, \ldots, j_n) \}, (j_1, \ldots, j_n)) = \sum_{P: \{(1, \ldots, s) \setminus (j_1, \ldots, j_n) \}, (j_1, \ldots, j_n)) = \bigcup_i X_i} (-1)^{|P| - 1} (|P| - 1)! \prod_{X_i \subset P} \Lambda(\theta(X_i))(t, \theta(X_i)), \quad n \geq 0,
\]

where the symbol \( \sum_P \) means the sum over all possible partitions \( P \) of the set \( (1, \ldots, n) \) into \( |P| \) nonempty mutually disjoint subsets \( X_i \subset (1, \ldots, n) \). This sequence is a unique solution of the Cauchy problem of the weak formulation of the dual BBGKY hierarchy for hard spheres (18), (19):

\[
\frac{\partial}{\partial t} B_s(t, x_1, \ldots, x_s) = \left( \sum_{j=1}^{s} \mathcal{L}(j) + \sum_{j_1 < j_2=1}^{s} \mathcal{L}_{\text{int}}(j_1, j_2) \right) B_s(t, x_1, \ldots, x_s) + \]

\[
+ \sum_{j_1 \neq j_2=1}^{s} \mathcal{L}_{\text{int}}(j_1, j_2) B_{s-1}(t, x_1, \ldots, x_{j_1-1}, x_{j_1+1}, \ldots, x_s),
\]

\[
B_s(t, x_1, \ldots, x_s)|_{t=0} = B_s^0(x_1, \ldots, x_s), \quad s \geq 1,
\]

where it is used notations accepted in formula (5).

Thus, there exist two approaches to the description of the evolution of many hard spheres, namely, within the framework of observables that are governed by the dual BBGKY hierarchy (23) for reduced functions of observables, or in terms of states governed by the BBGKY hierarchy (20) for the reduced distribution functions, respectively. For a system of finitely many hard spheres, these hierarchies are equivalent to the Liouville equation for observables and to the Liouville equation for states (the dual Liouville equation), respectively.

3. Dynamics of correlations of a hard-sphere system

An alternative approach to the description of states of a hard-sphere system of finitely many particles is given by means of functions determined by the cluster expansions of the probability distribution functions. They are interpreted as correlation functions (cumulants of probability distribution functions).
3.1. Correlation functions

We introduce the sequence of correlation functions \( g(t) = (1, g_1(t, x_1), \ldots, g_s(t, x_1, \ldots, x_n), \ldots) \) by means of the cluster expansions of the probability distribution functions \( D(t) = (1, D_1(t, x_1), \ldots, D_n(t, x_1, \ldots, x_n), \ldots) \), defined on the set of allowed configurations \( \mathbb{R}^{3n} \setminus \mathbb{W}_n \) as follows:

\[
D_n(t, x_1, \ldots, x_n) = g_n(t, x_1, \ldots, x_n) + \sum_{P : (x_1, \ldots, x_n) = \bigcup_i X_i, |P| > 1} \prod_{X_i \subset P} g_{|X_i|}(t, X_i), \quad (25)
\]

\( n \geq 1, \)

where \( \sum_{P : (x_1, \ldots, x_n) = \bigcup_i X_i, |P| > 1} \) is the sum over all possible partitions \( P \) of the set of the arguments \( (x_1, \ldots, x_n) \) into \( |P| > 1 \) nonempty mutually disjoint subsets \( X_i \subset (x_1, \ldots, x_n) \).

On the set \( \mathbb{R}^{3n} \setminus \mathbb{W}_n \) solutions of recursion relations (25) are given by the following expansions:

\[
g_s(t, x_1, \ldots, x_s) = D_s(t, x_1, \ldots, x_s) + \sum_{P : (x_1, \ldots, x_s) = \bigcup_i X_i, |P| > 1} (-1)^{|P| - 1} (|P| - 1)! \prod_{X_i \subset P} D_{|X_i|}(t, X_i), \quad s \geq 1. \quad (26)
\]

The structure of expansions (26) is such that the correlation functions can be treated as cumulants (semi-invariants) of the probability distribution functions (12).

Thus, correlation functions (26) are to enable to describe of the evolution of states of finitely many hard spheres by the equivalent method in comparison with the probability distribution function, namely within the framework of dynamics of correlations [20], [21].

If initial state described by the sequence \( g(0) = (1, g_1^0(1), \ldots, g_n^0(x_1, \ldots, x_n), \ldots) \), of correlation functions \( g_n^0 \in L_n^1, n \geq 1 \), then the evolution of all possible states, i.e. the sequence \( g(t) = (1, g_1(t, x_1), \ldots, g_s(t, x_1, \ldots, x_s), \ldots) \) of the correlation functions \( g_s(t), s \geq 1 \), is determined by the following group of nonlinear operators [21]:

\[
g_s(t, x_1, \ldots, x_s) = \mathcal{G}(t; 1, \ldots, s \mid g(0)) = \sum_{P : (1, \ldots, s) = \bigcup_j X_j} \mathfrak{A}_{|P|}(t, \{X_1\}, \ldots, \{X_{|P|}\}) \prod_{X_j \subset P} g_{|X_j|}^0(X_j), \quad s \geq 1, \quad (27)
\]

where \( \sum_{P : (1, \ldots, s) = \bigcup_j X_j} \) is the sum over all possible partitions \( P \) of the set \( (1, \ldots, s) \) into \( |P| \) nonempty mutually disjoint subsets \( X_j \), the set \( (\{X_1\}, \ldots, \{X_{|P|}\}) \) consists from elements of which are subsets \( X_j \subset (1, \ldots, s) \), i.e. \( |(\{X_1\}, \ldots, \{X_{|P|}\})| = |P| \). The generating operator \( \mathfrak{A}_{|P|}(t) \) in expansion (27) is the \( |P|th \)-order cumulant of the groups of operators (12) which is defined by the expansion

\[
\mathfrak{A}_{|P|}(t, \{X_1\}, \ldots, \{X_{|P|}\}) = \sum_{P' : (X_1, \ldots, (X_{|P|}) = \bigcup_k Z_k} (-1)^{|P'| - 1} (|P'| - 1)! \prod_{Z_k \subset P'} S_{\theta(Z_k)}^0(t, \theta(Z_k)), \quad (28)
\]

where \( \theta \) is the declusteringization mapping: \( \theta(\{X_1\}, \ldots, \{X_{|P|}\}) = (1, \ldots, s) \). The simplest examples of correlation operators (27) are given by the following expansions:

\[
g_1(t, x_1) = \mathfrak{A}_1(t, 1) g_1^0(x_1), \]
\[
g_2(t, x_1, x_2) = \mathfrak{A}_1(t, \{1, 2\}) g_2^0(x_1, x_2) + \mathfrak{A}_2(t, 1, 2) g_1^0(x_1) g_1^0(x_2).
\]
Thus, the cumulant nature of correlation functions induces the cumulant structure of a one-parametric mapping (27).

In particular, in the absence of correlations between hard spheres at the initial moment, known as the initial states satisfying the chaos condition [1]-[3], the sequence of the initial correlation functions has the form $g^{(c)}(0) = (0, g^{(1)}_1(x_1), 0, \ldots, 0, \ldots)$ (in terms of a sequence of the probability distribution functions it means that $D^{(c)}(0) = (1, D^{(1)}_1(x_1), D^{(1)}_1(x_1)D^{(2)}_1(x_2)x_{(x_1,x_2)}^{\delta_{(x_1,x_2)}}, \ldots, \prod_{i_1=1}^{n} D^{(i_1)}_1(x_{i_1})x_{(x_1,\ldots,x_{i_1})}^{\delta_{(x_1,\ldots,x_{i_1})}}, \ldots)$, where $X_{(x_1,\ldots,x_{i_1})}^{\delta_{(x_1,\ldots,x_{i_1})}}$ is the Heaviside step function of allowed configurations of $n$ hard spheres). In this case for $(x_1, \ldots, x_s) \in \mathbb{R}^{3s} \times (\mathbb{R}^{3s} \setminus \mathbb{W}_n)$ expansions (27) are represented as follows:

$$g_s(t, x_1, \ldots, x_s) = \mathfrak{A}_s(t, 1, \ldots, s) \prod_{i=1}^{s} g^{(i)}_1(x_i), \quad s \geq 1,$$

where $\mathfrak{A}_s(t)$ is the $s$th-order cumulant of groups of operators defined by the expansion

$$\mathfrak{A}_s(t, 1, \ldots, s) = \sum_{P: (1, \ldots, s) = \bigcup_{i} X_i} (-1)^{|P| - 1}(|P| - 1)! \prod_{X_i \subset P} S^s_{X_i}(t, X_i),$$

and it was used notations accepted in formula (12). From the structure of series (29) it is clear that in case of absence of correlations at the initial instant the correlations generated by the dynamics of a system of hard spheres are completely determined by the cumulants of the groups of operators (30).

### 3.2. The Liouville hierarchy

If $g^{(s)}_s \in L^1_s$, $s \geq 1$, then for $t \in \mathbb{R}$ the sequence of correlation functions (27) is a unique solution of the Cauchy problem of the weak formulation of the Liouville hierarchy [20, 21]:

$$\frac{\partial}{\partial t} g_s(t, x_1, \ldots, x_s) = \mathcal{L}^s_\alpha g_s(t, x_1, \ldots, x_s) +$$

$$\sum_{P: (x_1, \ldots, x_s) = X_1 \cup X_2} \sum_{i_1 \in X_1} \sum_{i_2 \in X_2} \mathcal{L}^s_{\text{int}}(i_1, i_2) g_{|X_1|}(t, X_1) g_{|X_2|}(t, X_2),$$

$$g_s(t, x_1, \ldots, x_s)|_{t=0} = g^{(s)}_s(x_1, \ldots, x_s), \quad s \geq 1,$$

where $\sum_{P: (x_1, \ldots, x_s) = X_1 \cup X_2}$ is the sum over all possible partitions $P$ of the set $(x_1, \ldots, x_s)$ into two nonempty mutually disjoint subsets $X_1$ and $X_2$, and the operator $\mathcal{L}^s_\alpha$ is defined on the subspace $L^1_\delta \subset L^1_\alpha$ by formula (11). It should be noted that the Liouville hierarchy (31) is the evolution recurrence equations set.

For $t \geq 0$ we give a few examples of recurrence equations set (31) for a system of hard spheres:

$$\frac{\partial}{\partial t} g_1(t, x_1) = -\langle p_1, \frac{\partial}{\partial q_1} \rangle g_1(t, x_1),$$

$$\frac{\partial}{\partial t} g_2(t, x_1, x_2) = -\sum_{j=1}^{2} \langle p_j, \frac{\partial}{\partial q_j} \rangle g_2(t, x_1, x_2) + \sigma^2 \int_{S^2_+} d\eta \langle \eta, (p_1 - p_2) \rangle (g_2(t, q_1, p^*_1)$$

$$q_2, p^*_2) \delta(q_1 - q_2 + \sigma \eta) - g_2(t, x_1, x_2) \delta(q_1 - q_2 - \sigma \eta)) +$$

$$\sigma^2 \int_{S^2_+} d\eta \langle \eta, (p_1 - p_2) \rangle (g_1(t, q_1, p^*_1)g_1(t, q_2, p^*_2) \delta(q_1 - q_2 + \sigma \eta) -$$

$$g_1(t, x_1)g_1(t, x_2) \delta(q_1 - q_2 - \sigma \eta)), $$

$$\ldots$$
Correlations in a hard-sphere system

where it was used notations accepted above in definition (4).

We note that because the Liouville hierarchy (31) is the recurrence evolution equations set, we can construct a solution of the Cauchy problem (31), (32), integrating each equation of the hierarchy as the inhomogeneous Liouville equation. For example, as a result of the integration of the first two equations of the Liouville hierarchy (31), we obtain the following equalities:

\[
\begin{align*}
g_1(t, x_1) &= S_1^*(t, 1)g_1^0(x_1), \\
g_2(t, 1, 2) &= S_2^*(t, 1, 2)g_2^0(x_1, x_2) + \\
&\int_0^t dt_1 S_2^*(t - t_1, 1, 2)L_{int}^*(1, 2)S_1^*(t_1, 1)S_1^*(t_1, 2)g_1^0(x_1)g_1^0(x_2).
\end{align*}
\]

Then for the corresponding term on the right-hand side of the second equality, an analog of the Duhamel equation holds

\[
\begin{align*}
\int_0^t dt_1 S_2^*(t - t_1, 1, 2)L_{int}^*(1, 2)S_1^*(t_1, 1)S_1^*(t_1, 2)g_1^0(x_1)g_1^0(x_2) &= \\
= -\int_0^t dt_1 \frac{d}{dt_1}(S_2^*(t - t_1, 1, 2)S_1^*(t_1, 1)S_1^*(t_1, 2)) &= \\
= S_2^*(t, 1, 2) - S_1^*(t, 1)S_1^*(t, 2) = \mathcal{A}_2(t, 1, 2),
\end{align*}
\]

where \(\mathcal{A}_2(t)\) is the second-order cumulant of groups of operators (30). As a result of similar transformations for \(s > 2\), the solution of the Cauchy problem (31), (32), constructed by an iterative procedure, is represented in the form of expansions (27).

We remark that a steady solution of the Liouville hierarchy (31) is a sequence of the Ursell functions on the allowed configurations of a hard-sphere system, i.e. \(g^{eq} = (0, e^{-\beta p_2^2/2}, 0, \ldots, 0, \ldots)\), where \(\beta\) is a parameter inversely proportional to temperature.

We emphasize that the dynamics of correlations, that is, the fundamental equations (31) describing the evolution of correlations of states, can be used as a foundation for describing the evolution of states of a system of both a finite and an infinite number of hard spheres instead of the Liouville equation for states [17]-[27].

4. Processes of the propagation of correlations in a hard-sphere system

Another approach to the description of states of hard-sphere systems of both finite and infinite number of particles is can be formulated as in above by means of functions determined by the cluster expansions of the reduced distribution functions. Such functions are interpreted as reduced correlation functions of states (marginal or \(s\)-particle correlation functions [24]-[26], or cumulants of marginals [7],[8]). On a microscopic scale, the macroscopic characteristics of fluctuations of observables are directly determined by means of the reduced correlation functions.

The following also outlines the approach to the description of the evolution of states by means of both reduced distribution functions and reduced correlation functions which is
based on the dynamics of correlations in a system of hard spheres governed by the Liouville hierarchy of equations for a sequence of correlation functions.

4.1. Reduced correlation functions

Traditionally reduced correlation functions are introduced by means of the cluster expansions of the reduced distribution functions similar to the cluster expansions of the probability distribution functions \( 25 \) and on the set of allowed configurations \( \mathbb{R}^{3n} \setminus \mathbb{W}_n \) they have the form:

\[
F_s(t, x_1, \ldots, x_s) = \sum_{\mathcal{P} : (x_1, \ldots, x_s) = \bigcup_i X_i} \prod_{X_i \in \mathcal{P}} G_{|X_i|} (t, X_i), \quad s \geq 1, \tag{33}
\]

where \( \sum_{\mathcal{P} : (x_1, \ldots, x_s) = \bigcup_i X_i} \) is the sum over all possible partitions \( \mathcal{P} \) of the set \( (x_1, \ldots, x_s) \) into \( |\mathcal{P}| \) nonempty mutually disjoint subsets \( X_i \subset (x_1, \ldots, x_s) \). As a consequence of this, the solution of recurrence relations \( 33 \) represented through reduced distribution functions as follows:

\[
G_s(t, x_1, \ldots, x_s) = \sum_{\mathcal{P} : (x_1, \ldots, x_s) = \bigcup_i X_i} (-1)^{|\mathcal{P}| - 1} (|\mathcal{P}| - 1)! \prod_{X_i \in \mathcal{P}} F_{|X_i|} (t, X_i), \tag{34}
\]

are interpreted as the functions that describe the correlations of states in a hard-sphere system. The structure of expansions \( 34 \) is such that the reduced correlation functions can be treated as cumulants (semi-invariants) of the reduced distribution functions \( 18 \).

We note that the reduced correlation functions give an equivalent approach to the description of the evolution of states of many hard spheres along with the reduced distribution functions. Indeed, the macroscopic characteristics of fluctuations of observables are directly determined by the reduced correlation functions on the microscopic scale \( 25, 26 \), for example, the functional of the dispersion of an additive-type observable, i.e. the sequence \( A^{(1)} = (0, a_1(x_1), \ldots, \sum_{i_1=1}^{n} a_1(x_{i_1}), \ldots) \), is represented by the formula

\[
\langle (A^{(1)} - \langle A^{(1)} \rangle)^2 (t) \rangle = \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_1 (a_1^2(x_1) - \langle A^{(1)} \rangle^2 (t)) G_1 (t, x_1) + \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^2} dx_1 dx_2 a_1(x_1) a_1(x_2) G_2 (t, x_1, x_2),
\]

where \( \langle A^{(1)} \rangle (t) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_1 a_1(x_1) G_1 (t, x_1) \) is the mean value functional of an additive-type observable.

If \( G(0) = (1, G_1^0(x_1), \ldots, G_s^0(x_1, \ldots, x_s), \ldots) \) is a sequence of reduced correlation functions at initial instant, then the evolution of all possible states, i.e. the sequence \( G(t) = (1, G_1(t, x_1), \ldots, G_s(t, x_1, \ldots, x_s), \ldots) \) of the reduced correlation functions \( G_s(t), s \geq 1 \), is determined by the following series expansions \( 24 \):

\[
G_s(t, x_1, \ldots, x_s) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_{s+1} \ldots dx_{s+n} \mathfrak{A}_{1+n} (t; \{1, \ldots, s\}, s + 1, \ldots, s + n \mid G(0)), \tag{35}
\]

\[ s \geq 1, \]
where the generating operator \(\mathfrak{A}_{1+n}(t; \{1, \ldots, s\}, s + 1, \ldots, s + n \mid G(0))\) of this series is the \((1 + n)th\)-order cumulant of groups of nonlinear operators (27):

\[
\mathfrak{A}_{1+n}(t; \{1, \ldots, s\}, s + 1, \ldots, s + n \mid G(0)) = \sum_{P: ((1, \ldots, s), s + 1, \ldots, s + n) = \bigcup \mathcal{X}_k} (-1)^{|P| - 1} (|P| - 1)! G(t; \theta(X_1) \mid \ldots \mathcal{G}(t; \theta(X_{|P|})(0) \mid \ldots), \quad n \geq 0,
\]

and where the composition of mappings (27) of the corresponding noninteracting groups of particles was denoted by \(\mathcal{G}(t; \theta(X_1) \mid \ldots \mathcal{G}(t; \theta(X_{|P|})(0) \mid \ldots)\), for example,

\[
\mathcal{G}(t; 1 \mid \mathcal{G}(t; 2 \mid G(0))) = \mathfrak{A}_1(t, 1) \mathfrak{A}_1(t, 2) G_2^0(x_1, x_2),
\]

\[
\mathcal{G}(t; 1, 2 \mid \mathcal{G}(t; 3 \mid G(0))) = \mathfrak{A}_1(t, \{1, 2\}) \mathfrak{A}_1(t, 3) G_3^0(x_1, x_2, x_3) + \mathfrak{A}_2(t, 1, 2) \mathfrak{A}_1(t, 3) (G_1^0(x_1) G_2^0(x_2, x_3) + G_1^0(x_2) G_2^0(x_1, x_3)).
\]

We will adduce examples of expansions (36). The first order cumulant of the groups of nonlinear operators (27) is the group of these nonlinear operators

\[
\mathfrak{A}_1(t; \{1, \ldots, s\} \mid G(0)) = \mathcal{G}(t; 1, \ldots, s \mid G(0)).
\]

In case of \(s = 2\) the second order cumulant of nonlinear operators (27) has the structure

\[
\mathfrak{A}_{1+1}(t; \{1, 2\}, 3 \mid G(0)) = \mathcal{G}(t; 1, 2, 3 \mid G(0)) - \mathcal{G}(t; 1, 2 \mid G(t; 3 \mid G(0))) = \mathfrak{A}_{1+1}(t, \{1, 2\}, 3) G_3^0(1, 2, 3) +
\]

\[
(\mathfrak{A}_{1+1}(t, \{1, 2\}, 3) - \mathfrak{A}_2(t, 2, 3) \mathfrak{A}_1(t, 1)) G_1^0(x_1) G_2^0(x_2, x_3) +
\]

\[
(\mathfrak{A}_{1+1}(t, \{1, 2\}, 3) - \mathfrak{A}_2(t, 1, 3) \mathfrak{A}_1(t, 2)) G_1^0(x_2) G_2^0(x_1, x_3) +
\]

\[
\mathfrak{A}_{1+1}(t, \{1, 2\}, 3) G_1^0(x_3) G_2^0(x_1, x_2) + \mathfrak{A}_3(t, 1, 2, 3) G_1^0(x_1) G_1^0(x_2) G_1^0(x_3),
\]

where the operator

\[
\mathfrak{A}_3(t, 1, 2, 3) = \mathfrak{A}_{1+1}(t, \{1, 2\}, 3) - \mathfrak{A}_2(t, 2, 3) \mathfrak{A}_1(t, 1) - \mathfrak{A}_2(t, 1, 3) \mathfrak{A}_1(t, 2)
\]

is the third-order cumulant (30) of groups of operators (9) of a system of hard spheres.

In the case of the initial state specified by the sequence of reduced correlation functions \(G^{(c)} = (0, G_1^0, 0, \ldots, 0, \ldots)\), that is, in the absence of correlations between hard spheres at the initial moment of time [5,30], according to definition (36), on the allowed configurations reduced correlation functions (35) are represented by the following series expansions:

\[
G_s(t, x_1, \ldots, x_s) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_{s+1} \ldots dx_{s+n} \mathfrak{A}_{s+n}(t; 1, \ldots, s + n) \prod_{i=1}^{s+n} G_1^0(x_i) \chi_{\mathbb{R}^3}(s+n) \setminus \mathbb{W}_{s+n},
\]

where the generating operator \(\mathfrak{A}_{s+n}(t)\) is the \((s + n)th\)-order cumulant (30) of groups of operators (9).

If \(G(0) \in \oplus_{n=0}^{\infty} L_n^\infty\), then provided that \(\max_{n \geq 1} \|G_n^0\|_{L_n^\infty} < (2e^3)^{-1}\) [24], for \(t \in \mathbb{R}\) the sequence of reduced correlation functions (35) is a unique solution of the Cauchy problem.
of the hierarchy of evolution nonlinear equations for hard spheres (for quantum systems known as the nonlinear BBGKY hierarchy [25]):

\[
\frac{\partial}{\partial t} G_s(t, x_1, \ldots, x_s) = \mathcal{L}_s^\ast G_s(t, x_1, \ldots, x_s) + \sum_{P_1: (x_1, \ldots, x_s) = X_1 \cup X_2} \sum_{i_1 \in X_1} \sum_{i_2 \in X_2} \mathcal{L}_s^\ast(i_1, i_2) G_{|X_1|}'(t, X_1) G_{|X_2|}'(t, X_2)) + \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_{s+1} \sum_{i=1}^s \mathcal{L}_s^\ast(i, s+1) (G_{s+1}(t, x_1, \ldots, x_{s+1}) + \sum_{P: (1, \ldots, s+1) = X_1 \cup X_2, i \in X_1; s+1 \in X_2} G_{|X_1|}'(t, X_1) G_{|X_2|}'(t, X_2)) \),
\]

where the generators of these evolution equations are defined as in (11), and we used notations accepted in the Liouville hierarchy of equations (31).

4.2. On the description of states governed by the dynamics of correlations

A definition equivalent to the definition (17) of reduced distribution functions can be formulated on the basis of the correlation functions (27) of systems of a finite number of hard spheres, namely (see Appendix)

\[
F_s(t, x_1, \ldots, x_s) = \sum_{n=0}^\infty \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_{s+1} \ldots dx_{s+n} g_{1+n}(t, \{x_1, \ldots, x_s\}, x_{s+1}, \ldots, x_{s+n}), \quad s \geq 1,
\]

where the correlation functions of clusters of hard spheres \(g_{1+n}(t), n \geq 0\), are defined by the expansions:

\[
g_{1+n}(t, \{x_1, \ldots, x_s\}, x_{s+1}, \ldots, x_{s+n}) = \sum_{P: (1, \ldots, s, s+1, \ldots, s+n) = \bigcup_i X_i} \mathcal{A}_{|P|}'(t, \{\theta(X_1)\}, \ldots, \{\theta(X_{|P|})\}) \prod_{X_i \subseteq P} g_{|X_i|}'(X_i), \quad n \geq 0,
\]

and \(\mathcal{A}_{|P|}'(t)\) is the \(|P|\)th-order cumulant (28) of the groups of operators (12). The possibility of redefining the reduced distribution functions naturally arises as a result of dividing the series in expression (17) by the series of the normalization factor (21).

Since the correlation functions \(g_{1+n}(t), n \geq 0\), are governed by the corresponding Liouville hierarchy for clusters of hard spheres, the reduced distribution functions (40) are governed by the BBGKY hierarchy (20).

We note that correlation functions of hard-sphere clusters expressed through correlation functions of hard spheres (27) by the following relations:

\[
g_{1+n}(t, \{x_1, \ldots, x_s\}, x_{s+1}, \ldots, x_{s+n}) = \sum_{P: \{x_1, \ldots, x_s\}, x_{s+1}, \ldots, x_{s+n} = \bigcup_i X_i} (-1)^{|P|-1} (|P| - 1)! \times \prod_{X_i \subseteq P} \sum_{P': \theta(X_i) = \bigcup_j Z_j, Z_j \subseteq P'} \prod_{X_i \subseteq P} g_{|Z_i|}'(t, Z_i), \quad n \geq 0.
\]
In particular case $n = 0$, i.e. the correlation function of a cluster of the $s$ hard spheres, these relations take the form

$$g_{1+0}(t, \{x_1, \ldots, x_s\}) = \sum_{P: (x_1, \ldots, x_s) = \bigcup_i X_i \subset P} \prod_i g_{|X_i|}(t, X_i).$$

Assuming as a basis an alternative approach to the description of the evolution of states of a hard-sphere system within the framework of correlation functions (27), then the reduced correlation functions are defined by means of a solution of the Cauchy problem of the Liouville hierarchy (31),(32) as follows [26],[24]:

$$G_s(t, x_1, \ldots, x_s) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_{s+1} \ldots dx_{s+n} g_{s+n}(t, x_1, \ldots, x_{s+n}), \quad (42)$$

where the generating function $g_{s+n}(t, x_1, \ldots, x_{s+n})$ is defined by expansion (26). Such a representation is derived as a result of the fact that the reduced correlation functions are cumulants (34) of reduced distribution functions (40).

Since the correlation functions $g_{s+n}(t), n \geq 0$, are governed by the Liouville hierarchy for hard spheres (31), the reduced correlation functions defined as (42) are governed by the nonlinear BBGKY hierarchy (38).

We emphasize that $nth$ term of expansions (42) of the reduced correlation functions are determined by the $(s + n)th$-particle correlation function (27) as contrasted to the expansions of reduced distribution functions (40) which are determined by the $(1 + n)th$-particle correlation function of clusters of hard spheres (41).

In the absence of correlations of the states of hard spheres at the initial moment of time on allowed configurations, the reduced correlation functions (35) and the reduced distribution functions are represented by expansions in the series (37) and (18), respectively. Consequently, the generator of these series expansions differs only in the order of the cumulants of the groups of operators of hard spheres. As a result, the process of creating correlations in a system of hard spheres is described by means of such reduced distribution functions or reduced correlation functions.

Thus, as follows from the above, the cumulant structure of correlation function expansions (41) or (27) induces the cumulant structure of series expansions for reduced distribution functions (18) and reduced correlation functions (35), respectively, or other words, the evolution of the state of a system of an infinite number of hard spheres is governed by the dynamics of correlations.

5. **On the description of correlations by means of the kinetic equations**

Further, an approach to the description of states by means of the state of a typical particle of a system of many hard spheres is discussed, or in other words, foundations are overviewed of describing the evolution of states by kinetic equations.

We shall consider systems which the initial state specified by a one-particle reduced correlation (distribution) function, namely, the initial state specified by a sequence of reduced correlation functions satisfying a chaos property stated above, i.e. by the sequence
$G^{(c)} = (0, G_{1}^{0}, 0, \ldots, 0, \ldots)$. We remark that such an assumption about initial states is intrinsic in kinetic theory of many-particle systems [1]-[5].

Since the initial data $G^{(c)}$ is completely specified by the one-particle correlation (distribution) function, the Cauchy problem of the nonlinear BBGKY hierarchy (38), (39) is not completely well-defined the Cauchy problem, because the initial data is not independent for every unknown function of the hierarchy of evolution equations. Therefore, the opportunity takes place to reformulate such a Cauchy problem as a new Cauchy problem for the one-particle correlation function, with the independent initial data and explicitly determined functionals of the solution of this Cauchy problem. We formulate such a restated Cauchy problem and state functionals.

The following statement is true. In the case of the initial state specified by a one-particle correlation (distribution) function, the Cauchy problem of the nonlinear BBGKY hierarchy (38), (39) is not completely well-defined the Cauchy problem, because the initial data is not independent for every unknown function of the hierarchy of evolution equations. Therefore, the opportunity takes place to reformulate such a Cauchy problem as a new Cauchy problem for the one-particle correlation function, with the independent initial data and explicitly determined functionals of the solution of this Cauchy problem. We formulate such a restated Cauchy problem and state functionals.

The following statement is true. In the case under consideration the reduced correlation functionals $G_{s}(t | G_{1}(t))$, $s \geq 2$, are represented with respect to the one-particle correlation function

$$G_{1}(t, x_{1}) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^{3} \times \mathbb{R}^{3})^{n}} dx_{2} \ldots dx_{1+n} \mathfrak{A}_{1+n}(t, 1, \ldots, n+1) \prod_{i=1}^{n+1} G_{1}^{0}(x_{i}) \mathcal{V}_{\mathbb{R}^{3} \setminus \mathbb{W}_{n+1}}^{(n+1)},$$

where the generating operator $\mathfrak{A}_{1+n}(t)$ of this series is the $(1 + n)th$-order cumulant (30) of the groups of operators (12), by the following series:

$$G_{s}(t, x_{1}, \ldots, x_{s} | G_{1}(t)) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^{3} \times \mathbb{R}^{3})^{n}} dx_{s+1} \ldots dx_{s+n} \mathfrak{V}_{s+n}(t, 1, \ldots, s+n) \prod_{i=1}^{s+n} G_{1}(t, x_{i}),$$

$s \geq 2$.

The generating operator $\mathfrak{V}_{s+n}(t)$, $n \geq 0$, of the $(s+n)th$-order of this series is determined by the following expansion [13]:

$$\mathfrak{V}_{s+n}(t, 1, \ldots, s, s+1, \ldots, s+n) = \prod_{j=1}^{n} \sum_{|D_{j}| \leq s+n-s_{2}-\cdots-n_{j}} \frac{1}{|D_{j}|!} \mathfrak{A}_{s+n-s_{2}-\cdots-n_{j}}(t, 1, \ldots, s+n-s_{2}-\cdots-n_{j}) \times$$

$$\prod_{i_{1} \neq \ldots \neq i_{|D_{j}|}}^{\prod_{i_{j} \in C_{D_{j}}} X_{i_{j}}} \prod_{|D_{j}| \leq s+n-s_{2}-\cdots-n_{j}}^{\prod_{i_{j} \in C_{D_{j}}} X_{i_{j}}} \mathfrak{A}_{1+|X_{i_{j}}|}(t, i_{j}, X_{i_{j}}).$$
where \( \sum_{D_j ; Z_j = \bigcup_j x_{ij}} \) is the sum over all possible dissections of the linearly ordered set \( Z_j \equiv ( s + n - n_1 - \ldots - n_j + 1 , \ldots , s + n - n_1 - \ldots - n_{j-1} ) \) on no more than \( s + n - n_1 - \ldots - n_j \) linearly ordered subsets, the \((s + n)th\) order scattering cumulant is defined by the formula

\[
\hat{G}_{s+n}(t,1,\ldots, s+n) = G_{s+n}(t,1,\ldots, s+n) \mathcal{X}_{\mathbb{R}^{3(s+n)} \setminus \mathbb{W}_{s+n}} \prod_{i=1}^{s+n} \mathcal{A}_{1}^{-1}(t,i),
\]

and notations accepted above were used. A method of the construction of reduced correlation functionals \((44)\) is based on the application of the so-called kinetic cluster expansions \((43)\) to the generating operators \((30)\) of series \((37)\). If \( \|G_1(t)\|_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)} < e^{-(3s+2)} \), series \((44)\) converges in the norm of the space \(L^1_s\) for arbitrary \( t \in \mathbb{R} \) \([13]\).

We adduce simplest examples of generating operators \((45)\):

\[
G_s(t,1,\ldots,s) = G_s(t,1,\ldots,s) \mathcal{X}_{\mathbb{R}^{3s} \setminus \mathbb{W}_s} \prod_{i=1}^{s} \mathcal{A}_{1}^{-1}(t,i),
\]

\[
G_{s+1}(t,1,\ldots,s,s+1) = G_{s+1}(t,1,\ldots,s+1) \mathcal{X}_{\mathbb{R}^{3(s+1)} \setminus \mathbb{W}_{s+1}} \prod_{i=1}^{s+1} \mathcal{A}_{1}^{-1}(t,i) - \sum_{j=1}^{s} \mathcal{A}_2(t,j,s+1) \mathcal{X}_{\mathbb{R}^{6} \setminus \mathbb{W}_2} \mathcal{A}_{1}^{-1}(t,j) \mathcal{A}_{1}^{-1}(t,s+1).
\]

We note that reduced correlation functionals \((44)\) describe all possible correlations generated by the dynamics of many hard spheres in terms of a one-particle correlation function.

If \( G^0_1 \in L^1_s \), then for arbitrary \( t \in \mathbb{R} \) one-particle correlation function \((43)\) is a weak solution of the Cauchy problem of the generalized Enskog kinetic equation \([13]\)

\[
\frac{\partial}{\partial t} G_1(t,x_1) = \mathcal{L}^*(1) G_1(t,x_1) + \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_2 \mathcal{L}^*_\text{int}(1,2) G_1(t,x_1) G_1(t,x_2) + \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_2 \mathcal{L}^*_\text{int}(1,2) G_2(t,x_1,x_2 | G_1(t)), \tag{46}
\]

\[
G_1(t,x_1)|_{t=0} = G^0_1(x_1), \tag{47}
\]

where the first part of the collision integral in equation \((46)\) has the Boltzmann–Enskog structure, and the second part of the collision integral is determined in terms of the two-particle correlation functional represented by series expansion \((44)\) and it describes all possible correlations which are created by hard-sphere dynamics and by the propagation of initial correlations related to the forbidden configurations.

Indeed, by virtue of definitions \((10),(11)\) of the generator of the generalized Enskog equation \((46)\), for \( t > 0 \) the kinetic equation get the following explicit form

\[
\frac{\partial}{\partial t} G_1(t,x_1) = -\langle p_1, \frac{\partial}{\partial q_1} \rangle G_1(t,x_1) + \sigma^2 \int_{\mathbb{R}^3 \times \mathbb{S}_+^2} dp_2 dq \langle \eta, (p_1 - p_2) \rangle (G_1(t,p^*_1,q_1)G_1(t,p^*_2,q_1 - \sigma \eta, \ldots) -
\]

Correlations in a hard-sphere system
Thus, for the initial state specified by a one-particle correlation function, then all possible states of a system of hard spheres can be described without any approximations within the framework of a one-particle correlation function governed by non-Markovian kinetic equation (46), and of explicitly defined functionals (44) of its solution (43).

6. On the low-density approximation of reduced correlation functions

The conventional philosophy of the description of the kinetic evolution consists of the following. If the initial state specified by a one-particle distribution function, then the evolution of states can be effectively described by means of a one-particle distribution function governed by the nonlinear kinetic equation in a suitable scaling limit [28], [32].

In the last decade, the Boltzmann–Grad limit (low-density scaling limit) [28], [29] of the reduced distribution functions constructed by means of the theory of perturbations were rigorously established in numerous papers, for example, in papers [7], [11], [17] and references therein.

Further, we consider a scheme for constructing the scaling asymptotic behavior of reduced correlation functions (37) in the particular case of the Boltzmann–Grad limit in the case of the above-mentioned initial state, which is specified by the scaled one-particle correlation function $G_{1}^{0,\epsilon}$, satisfying the condition:

$$|G_{1}^{0,\epsilon}(x_{1})| \leq ce^{-\frac{\beta}{2}p_{1}^{2}},$$

where $\epsilon > 0$ is a scaling parameter (the ratio of the diameter $\sigma > 0$ to the mean free path of hard spheres), $\beta > 0$ is a parameter and $c < \infty$ is some constant, and for $t \geq 0$ the operator $L_{\text{int}}^{*}$ in the dimensionless hierarchy of equations (38) is scaled in such a way that

$$L_{\text{int}}^{*}(j_{1}, j_{2})f_{n} = \epsilon^{2} \int_{S_{+}^{3}} d\eta \langle \eta, (p_{j_{1}} - p_{j_{2}}) \rangle f_{n}(x_{1}, \ldots, p_{j_{1}}^{*}, q_{j_{1}}, \ldots, p_{j_{2}}^{*}, q_{j_{2}}, \ldots, x_{n}) \delta(q_{j_{1}} - q_{j_{2}} + \epsilon \eta) - f_{n}(x_{1}, \ldots, x_{n}) \delta(q_{j_{1}} - q_{j_{2}} - \epsilon \eta),$$

where the notations similar to (4) are used.

We emphasize that the states of a system of infinitely many hard spheres are described by sequences of functions bounded with respect to the configuration variables [1] as it assumed above.

We will assume the existence of such Boltzmann–Grad limit of the reduced correlation function $G_{1}^{0,\epsilon}$ in the sense of weak convergence

$$w - \lim_{\epsilon \to 0} \{\epsilon^{2}G_{1}^{0,\epsilon}(x_{1}) - g_{1}^{0}(x_{1})\} = 0.$$ (48)

Since the $n$th term of series (37) for the $s$-particle correlation function is determined by the $(s + n)$th-order cumulant of asymptotically perturbed groups of operators (9), then
on the finite time interval in the Boltzmann–Grad limit limit the property of the propagation of initial chaos holds in the following sense:

\[ w - \lim_{\epsilon \to 0} \epsilon^{2s} G_s(t, x_1, \ldots, x_s) = 0, \quad s \geq 2. \quad (49) \]

The equality (49) is derived by the following assertions.

If \(|f_s| \leq c_0 \epsilon^{-\beta} \sum_{i=1}^{s} p_i^2\), then for arbitrary finite time interval for asymptotically perturbed first-order cumulant (50) of the groups of operators (9), i.e. for the strongly continuous group (9) the following equality takes place \([1, 31]\)

\[ w - \lim_{\epsilon \to 0} \left( S^*_s(t, 1, \ldots, s) f_s - \prod_{j=1}^{s} S^*_1(t, j) f_s \right) = 0. \]

Therefore, for the \((s+n)th\)-order cumulant of asymptotically perturbed groups of operators (9) the following equalities true:

\[ w - \lim_{\epsilon \to 0} \frac{1}{\epsilon^{2n}} \mathfrak{A}_{s+n}(t, 1, \ldots, s + n) f_{s+n} = 0, \quad s \geq 2. \]

If equality (48) holds for the initial one-particle correlation operator, then in the case of \(s = 1\) for the series expansion (37) the following equality is true

\[ w - \lim_{\epsilon \to 0} (\epsilon^2 G_1(t, x_1) - g_1(t, x_1)) = 0, \]

where for arbitrary finite time interval the limit one-particle correlation function \(g_1(t, x_1)\) is represented by the series

\[ g_1(t, x_1) = \sum_{n=0}^{\infty} \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \int_{[R^3 \times R^3]^n} dx_{2} \cdots dx_{1+n} S^*_1(t - t_1, 1) \times \]

\[ \mathcal{L}^{0, *}_{\text{int}}(1, 2) \prod_{j=1}^{2} S^*_1(t_1 - t_2, j_1) \cdots \prod_{i_1=1}^{n} S^*_1(t_n - t_i, i_n) \times \]

\[ \sum_{k_n=1}^{n} \mathcal{L}^{0, *}_{\text{int}}(k_n, n+1) \prod_{j_n=1}^{n+1} S^*_1(t_n, j_n) \prod_{i=1}^{n+1} g_1^0(x_i). \quad (50) \]

In this series expansion for \(t \geq 0\) the operator \(\mathcal{L}^{0, *}_{\text{int}}(j_1, j_2)\) is defined by the formula

\[ \mathcal{L}^{0, *}_{\text{int}}(j_1, j_2) f_n = \int_{S^2_+} d\eta(\eta, (p_{j_1} - p_{j_2})) \left( f_n(x_1, \ldots, p_{j_1}, q_{j_1}, \ldots, p_{j_2}, q_{j_2}, \ldots, x_n) - f_n(x_1, \ldots, x_n) \right) \delta(q_{j_1} - q_{j_2}), \]

where notations accepted in formula (10) are used.

Thus, we conclude that the limit one-particle correlation function (50) is a weak solution of the Cauchy problem of the Boltzmann kinetic equation:

\[ \frac{\partial}{\partial t} g_1(t, x_1) = \mathcal{L}^*(1) g_1(t, x_1) + \int_{[R^3 \times R^3]} dx_2 \mathcal{L}^{0, *}_{\text{int}}(1, 2) g_1(t, x_1) g_1(t, x_2), \]

\[ g_1(t, x_1)|_{t=0} = g_1^0(x_1), \]
or, if \( t \geq 0 \), for a system of hard spheres the Boltzmann equation has the following explicit form

\[
\frac{\partial}{\partial t} g_1(t, x_1) = -\langle p_1, \frac{\partial}{\partial q_1} \rangle g_1(t, x_1) + \int_{\mathbb{R}^3 \times S^2} dp_2 d\eta \langle \eta, (p_1 - p_2) \rangle (g_1(t, q_1, p_1^*) g_1(t, q_1, p_2^*) - g_1(t, x_1) g_1(t, q_1, p_2)) .
\]

We remark that some other approaches to the derivation of kinetic equations, in particular, for a system of many hard spheres with initial correlations, were developed in the works [13]-[16]. In [19], a hierarchy of kinetic equations describing the evolution of the observables of a hard-sphere system in the low-density limit is constructed.

### 7. Conclusion

This article dealt with a hard-sphere system of a non-fixed, i.e. arbitrary but finite average number of identical hard spheres. The possible approaches to describing the evolution of the states of a system of hard spheres using various modifications of probability distribution functions were considered. One of these approaches allows one to describe the evolution of both a finite and an infinite average number of hard spheres using reduced distribution functions (18) or reduced correlation functions (35), which are governed by the dynamics of correlations (27).

Above it was established that the notion of cumulants (28) of the groups of operators (12) underlies non-perturbative expansions of solutions for the fundamental evolution equations describing the evolution of the state of a hard-sphere system, namely of the Liouville hierarchy (31) for correlation functions, of the BBGKY hierarchy (20) for reduced distribution functions and of the nonlinear BBGKY hierarchy (38) for reduced correlation functions, as well as it underlies the kinetic description of infinitely many hard spheres (44).

We emphasize that the structure of expansions for correlation functions (41), in which the generating operators are the cumulants of the corresponding order (28) of the groups of operators (12) of hard spheres, induces the cumulant structure of series expansions for reduced distribution functions (18), reduced correlation functions (35) and marginal correlation functionals (44). Thus, in fact, the dynamics of systems of infinitely many hard spheres is generated by the dynamics of correlations.

The origin of the microscopic description of the collective behavior of a hard-sphere system by a one-particle correlation (distribution) function that is governed by the generalized Enskog kinetic equation (46) was also considered. One of the advantages of such an approach to the derivation of kinetic equations from underlying dynamics consists of an opportunity to construct the kinetic equations with initial correlations, which makes it possible to describe the propagation of initial correlations in the scaling limits (17), (34). Another advantage of this approach is related to the problem of a rigorous derivation of the non-Markovian-type kinetic equations on the basis of the hard-sphere dynamics, which make it possible to describe the memory effects in many-particle systems with collisional dynamics.

In addition, it was established that in the particular case of a low-density approximation for initial states specified by a one-particle correlation function the asymptotic behavior of the constructed reduced correlation functions (37) is governed by the Boltzmann kinetic equation with hard-sphere collisions.
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Appendix

The possibility of the description of the evolution of states based on the dynamics of correlations \( \text{(40)} \) or \( \text{(42)} \) occurs naturally in consequence of dividing the series in expression \( \text{(1)} \) by the series of the normalizing factor, or other words, as a result of redefining of mean value functional \( \text{(1)} \).

To provide evidence of this statement, we will introduce the necessary concepts and prove the validity of some equalities. On sequences of functions \( f, \tilde{f} \in L_{\alpha}^{1} \) we define the *-product
\[
(f * \tilde{f})|_{Y}(Y) = \sum_{Z \subset Y} f|_{Z}(Z) \tilde{f}|_{Y \setminus Z}(Y \setminus Z),
\]
where \( \sum_{Z \subset Y} \) is the sum over all subsets \( Z \) of the set \( Y = (x_{1}, \ldots, x_{s}) \). Using the definition of the *-product \( \text{(A.1)} \), we introduce the mapping \( \text{Exp}_{s} \) and the inverse mapping \( \text{Ln}_{s} \) on sequences \( h = (0, h_{1}(x_{1}), \ldots, h_{n}(x_{1}, \ldots, x_{n}), \ldots) \) of functions \( h_{n} \in L_{\alpha}^{1} \) by the expansions
\[
(\text{Exp}_{s}h)|_{Y}(Y) = \left( \mathbb{1} + \sum_{n=1}^{\infty} \frac{h^{sn}}{n!} \right)|_{Y}(Y) = \delta|_{Y}|.0 + \sum_{P:Y = \bigcup_{i}X_{i}} \prod_{X_{i} \subset P} h|_{X_{i}}(X_{i}),
\]
where we used the notations accepted in formula \( \text{(A.1)} \), \( \delta|_{Y}|.0 \) is the Kronecker symbol, \( \mathbb{1} = (1, 0, \ldots, 0, \ldots) \), and respectively,
\[
(\text{Ln}_{s}(\mathbb{1} + h))|_{Y}(Y) = \left( \sum_{n=1}^{\infty} (-1)^{n-1} \frac{h^{sn}}{n} \right)|_{Y}(Y) = \sum_{P:Y = \bigcup_{i}X_{i}} (-1)^{|P| - 1}(|P| - 1)! \prod_{X_{i} \subset P} h|_{X_{i}}(X_{i}).
\]
Therefore in terms of sequences of operators recursion relations \( \text{(25)} \) are rewritten in the form
\[
D(t) = \text{Exp}_{s} g(t),
\]
where \( D(t) = \mathbb{1} + (0, D_{1}(t, x_{1}), \ldots, D_{n}(t, x_{1}, \ldots, x_{n}), \ldots) \). As a result, we get
\[
g(t) = \text{Ln}_{s} D(t).
\]

Thus, according to definition \( \text{(A.1)} \) of the *-product and mapping \( \text{(A.3)} \), in the component-wise form solutions of recursion relations \( \text{(25)} \) are represented by expansions \( \text{(26)} \).

For arbitrary \( f = (f_{0}, f_{1}, \ldots, f_{n}, \ldots) \in L_{\alpha}^{1} \) and \( Y = (x_{1}, \ldots, x_{s}) \) we will define the linear mapping \( \partial_{Y} : f \rightarrow \partial_{Y} f \), by the formula
\[
(\partial_{Y} f)|_{n}(x_{1}, \ldots, x_{n}) = f|_{Y+n}(Y, x_{1}|Y|+1, \ldots, x_{|Y|+n}), \quad n \geq 0.
\]
For the set \( \{Y\} \) consisting of the one element \( Y = (x_{1}, \ldots, x_{s}) \), we have, respectively
\[
(\partial_{\{Y\}} f)|_{n}(x_{1}, \ldots, x_{n}) = f_{1+n}(\{Y\}, x_{s+1}, \ldots, x_{s+n}), \quad n \geq 0.
\]
On sequences $\partial_Y f$ and $\partial_Y \tilde{f}$ we introduce the $*$-product
\[
(\partial_Y f * \partial_Y \tilde{f})|_{X}(X) \doteq \sum_{Z \subset X} f|_{|Z| + |Y|}(Y, Z) \tilde{f}|_{X \setminus Z + |Y'|}(Y', X \setminus Z),
\]
where $X, Y, Y'$ are the sets, which terms characterize clusters of hard spheres, and $\sum_{Z \subset X}$ is the sum over all subsets $Z$ of the set $X$. In particular case $Y = \emptyset, Y' = \emptyset$, this definition reduces to definition (A.1).

For $f = (0, f_1, \ldots, f_n, \ldots)$, $f_n \in L^1_\alpha$, according to definitions of mappings (A.2) and (A.5), the following equality holds
\[
\partial_{\{Y\}} \text{Exp}_* f = \text{Exp}_* f * \partial_{\{Y\}} f,
\]
and for mapping (A.4) respectively
\[
\partial_Y \text{Exp}_* f = \text{Exp}_* f * \sum_{P: Y = \bigcup_i X_i} \partial_{X_i} f * \ldots * \partial_{X_{|P|}} f,
\]
where $\sum_{P: Y = \bigcup_i X_i}$ is the sum over all possible partitions $P$ of the set $Y \equiv (x_1, \ldots, x_s)$ into $|P|$ nonempty mutually disjoint subsets $X_i \subset Y$.

According to the definition
\[
(I, f) \doteq \sum_{n=0}^\infty \frac{1}{n!} \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_1 \ldots dx_n f_n(x_1, \ldots, x_n),
\]
where $I = (1, \ldots, 1, \ldots)$, for sequences $f, \tilde{f} \in L^1_\alpha$, the following equality holds
\[
(I, f * \tilde{f}) = (I, f)(I, \tilde{f}).
\]

In terms of mappings (A.4) and (A.5) the generalized cluster expansions of solutions (12) of a sequence of the Liouville equations
\[
D_{s+n}(t, Y, X \setminus Y) = \sum_{P: \{Y\}, X \setminus Y = \bigcup_i X_i} \prod_{X_i \subset P} g|_{X_i}(t, X_i), \quad s \geq 1,
\]
where $X \setminus Y \equiv (x_{s+1}, \ldots, x_{s+n})$, take the form
\[
\partial_Y D(t) = \partial_{\{Y\}} \text{Exp}_* g(t).
\]

Now let us prove the equivalence of the definition (17) of the reduced distribution functions and the definition of (40) in the framework of the correlation dynamics.

In terms of mapping (A.4) the definition of reduced distribution functions (17) is written as follows:
\[
F_a(t, Y) = (I, D(t))^{-1}(I, \partial_Y D(t)).
\]

Using generalized cluster expansions (A.8), and as a consequence of equalities (A.6),(A.7), we find
\[
(I, \partial_Y D(t)) = (I, \partial_{\{Y\}} \text{Exp}_* g(t)) =
(I, \text{Exp}_* g(t) * \partial_{\{Y\}} g(t)) = (I, \text{Exp}_* g(t))(I, \partial_{\{Y\}} g(t)).
\]
Taking into account that, according to the particular case \( Y = \emptyset \), of cluster expansions (A.8), the equality holds

\[
(I, \text{Exp}_s g(t)) = (I, D(t)),
\]

and as a result, we establish the following representation for the reduced distribution functions:

\[
F_s(t, Y) = (I, d_{\{Y\}} g(t)).
\]

Therefore, in componentwise-form, we obtain relation (40).

We remind that the correlation functions of particle clusters in series (40), i.e. the functions \( g_{1+n}(t, \{Y\}, X \setminus Y), n \geq 0 \), are defined as solutions of generalized cluster expansions (A.8), namely

\[
g_{1+n}(t, \{Y\}, X \setminus Y) = \sum_{P: \{Y\}, X \setminus Y = \bigcup_i X_i} (-1)^{|P|-1} (|P| - 1)! \prod_{X_i \in P} D(t, X_i),
\]

\( s \geq 1, n \geq 0, \)

where the probability distribution function \( D(t, X_i) \) is solution (12) of the Liouville equation (13).

Thus, we have established relation (40) between the reduced distribution functions and correlation functions. In a similar way, the validity of relation (42) between the reduced correlation functions defined by the cumulant expansions: \( G(t) = \ln_s F(t) \), and correlation functions, i.e. \( G_s(t, Y) = (I, d_Y g(t)) \), can be justified.