Queuing Networks with Varying Topology – A Mean-Field Approach.

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November 18, 2013

Abstract

We consider the queuing networks, which are made from servers, exchanging their positions. The customers, using the network, try to reach their destinations, which is complicated by the movements of the servers, taking their customers with them, while they wait for the service. We develop the general theory of such networks, and we establish the convergence of the symmetrized version of the network to the Non-Linear Markov Process.

1 Introduction

In this paper we consider a model of queuing network containing servers moving on the set of nodes of some graph $G$, in such a way that at any time, a node harbors a single server. Customers enter the network at each entrance node. When it arrives to some node, a customer joins the queue currently harbored by this node. Customer $c$ also has a designated exit node $D(c)$,
which he needs to reach in order to exit the network. In order to reach its
destination, a customer visits a series of intermediate servers: when at server
$v$, a customer waits in the associated queue before being served. The waiting
time depends on the service discipline at $v$. Once a customer $c$ being served
by $v$ leaves, he is sent to the server $v'$, located at the adjacent node, which is
the closest to the destination node $D(c)$. Once it gets to $D(c)$, the customer
leaves the system.

The main feature of the network we are considering is that servers are
moving on the graph. So while customers are waiting to be served, two
adjacent servers can move by simultaneously swapping their locations. When
two servers swap, each one carries its queue with it. So if the server $v$,
currently containing $c$, moves, then the distance between $c$ and its destination
$D(c)$ might change, in spite of the fact that customer $c$ has not yet completed
its service at $v$.

In such networks with moving nodes new effects take place, which are
not encountered in the usual situations with stationary nodes. For example,
it can happen that ‘very nice’ networks – i.e. networks with fast servers
and low load – become unstable once the nodes start to move. Here the
instability means that the queues becomes longer and longer with time, for
a large network. In contrast, for the same parameters, the queues remain
finite in the network with stationary servers.

This instability, which appear as a result of the movement of the servers,
will be a subject of our forthcoming papers, [BRS]. In the present paper
we develop the general qualitative theory of such networks, and we focus
on the mean-field approach to them. The main result can be described as
follows. We start with the definition of the class of networks with jumping
nodes. The network can be finite or infinite. In order to be able to treat the
problem, we consider the mean-field version of it, which consists of $N$ copies
of the network, interconnected in a mean-field manner. We show that as $N$
increases, the limiting object becomes a Non-Linear Markov Process (NLMP)
in time. The ergodic properties of this process determine the stability or
instability of our network. They will be investigated in our forthcoming
papers, [BRS]. Here we will establish the existence of the NLMP and the
convergence to it.

We now recall what is meant by Non-Linear Markov Processes. We
do this for the simplest case of discrete time Markov chains, taking values in
a finite set $S$, $|S| = k$. For the general case see [RS], Sect. 2 and 3. In the
discrete case the set of states of our Markov chain is the simplex $\Delta_k$ of all
probability measures on $S$, $\Delta_k = \{\mu = (p_1, \ldots, p_k) : p_i \geq 0, p_1 + \ldots + p_k = 1\}$, while the Markov evolution defines a map $P : \Delta_k \to \Delta_k$. In the case of usual Markov chains the map $P$ is linear with $P$ coinciding with the matrix of transition probabilities. A non-linear Markov chain is defined by a family of transition probability matrices $P_\mu, \mu \in \Delta_k$, so that the matrix entry $P_\mu(i, j)$ is the probability of going from $i$ to $j$ in one step, starting in the state $\mu$. The (non-linear) map $P$ is then defined by $P(\mu) = \mu P_\mu$.

The ergodic properties of linear Markov chains are settled by the Perron-Frobenius theorem. In particular, if the linear map $P$ is such that the image $P(\Delta_k)$ belongs to the interior $\text{Int} (\Delta_k)$ of $\Delta_k$, then there is precisely one point $\mu \in \text{Int} (\Delta_k)$, such that $P(\mu) = \mu$, and for every $\nu \in \Delta_k$ we have the convergence $P^n(\nu) \to \mu$ as $n \to \infty$. In case $P$ is non-linear, we are dealing with a more or less arbitrary dynamical system on $\Delta_k$, and the question about the stationary states of the chain or about measures on $\Delta_k$ invariant under $P$ cannot be settled in general.

2 Formulation of the Main Result

2.1 Network Description and the Main Result in a Preliminary Form

We consider a queueing network with jumping nodes on a connected graph $G = [V(G), E(G)]$.

This means that at every node $v \in V$, at any time, there is one server with a queue $q_v$ of customers waiting there for service. Every customer $c$ at $v$ carries its destination, $D(c) \in V(G)$. The goal of the customer is to reach its destination. In order to get there, a customer completing its service at $v$ jumps along the edges of $G$ to one of the nodes of $G$ which is closest (in the graph distance) to $D(c)$ where he joins the queue of the server currently harbored by this node. Once the destination of a customer is reached, he leaves the network. In the meantime, the servers of our network can jump. More precisely, the two servers at $v, v'$, which are neighbors in $G$ can exchange their positions with rate $\beta_{vv'}$. The queues $q_v$ and $q_{v'}$ then exchange their positions as well. Of course, such an exchange may bring some customers of $q_v$ and $q_{v'}$ closer to their destinations, and some other customers further
away from their destinations. We suppose that the rates $\beta_{v'v}$ of such jumps are uniformly bounded by a constant:

$$|\beta_{v'v}| < \beta.$$  \hspace{1cm} (1)

Our goal is to study the behavior of such a network.

The graphs $G$ we are interested in can be finite or infinite. The case of finite graphs is easier, while the infinite case requires certain extra technical points. In particular, when we talk about the functions of the states of our network in the infinite graph case, we will always assume that they are either local or quasi-local, with dependence on far-away nodes decaying exponentially fast with the distance. This exponential decay property will be conserved by our dynamics.

We will suppose that the degrees of the vertices of $G$ are finite and uniformly bounded by some constant $D(G)$. Of course, this automatically holds in the finite graph case.

In order to make our network tractable, we will study its symmetrized, or mean-field, modification. This means that we pass from the graph $G$ to its mean-field version, the graph $G_N = G \times \{1, \ldots, N\}$, and eventually take the limit $N \to \infty$. By definition, the graph $G_N$ has the set of vertices $V(G_N) = V(G) \times \{1, \ldots, N\}$; two vertices $(v, k), (v', k') \in V(G_N)$ define an edge in $E(G_N)$ iff $(v, v') \in E(G)$. As we shall see, the restriction of our state process from $G \times \{1, \ldots, N\}$ to the subgraph $G \equiv G\times\{1\}$ goes, as $N \to \infty$, to a Non-Linear Markov Process on $G$, which is a central object of our study. The limiting network $K$ (which coincides, in a sense, with the above mentioned NLMP) will be the limit of the networks $K_N$ on $G \times \{1, \ldots, N\}$.

In our model the servers can exchange their positions (bringing all the customers queuing at them to the new locations). The above rate of exchange $\beta_{v'v}$ should be renormalized, as we pass from $G$ to $G_N$, in order for the limit to exist. So for a server located at $(v, k) \in V(G \times \{1, \ldots, N\})$, the rate of transposition with the server $(v', k')$, where the node $v'$ is a neighbor of the node $v$, is given by

$$\frac{\beta_{v'v}}{N}.$$  \hspace{1cm}

This implies that the server at $(v, k)$ will jump to the set of nodes $v' \times \{1, \ldots, N\}$ with the rate $\beta_{v'v}$, independent of $N$.

Every server has a queue of customers. Each customer has a class, $\kappa \in K$. These classes are letters of some finite alphabet $K$. If a customer of class
κ completes his service on the server at v and goes to server u, it then gets a new class \( \kappa' = T(\kappa; v, u) \). Once a server finishes serving a customer, it chooses another one from the queue, according to the class of the customers present in the queue and the service discipline. It can happen that the service of a customer is interrupted if a customer with higher priority comes, and then the interrupted service is resumed after the appropriate time.

The service time distribution \( \eta \) depends on the class of the customer and on the server v, \( \eta = \eta(\kappa, v) \). We do not suppose that \( \eta \) is exponential.

Every customer \( c \) in our network \( \mathcal{K}_N \) has its destination node, \( D(c) = w \in V(G) \). In spite of the fact that our servers do change their positions, this location \( D(c) \) does not change with time. The customer \( c \) tries to get to its destination node. In order to do so, if it is located at \( (v, k) \) and finishes its service there, then it goes to the server at \( (v', n) \), where \( v' \in G \) is the neighbor of \( v \) which is the closest to \( D(c) \). If there are several such \( v' \), one is chosen uniformly. The coordinate \( n \in \{1, ..., N\} \) is chosen uniformly as well. If at the moment of the end of the service it so happens that \( v \) is at distance 1 from \( D(c) \) or that \( v \) coincides with \( D(c) \), then the customer leaves the network. However, if the customer \( c \) is waiting for service at the server \( (v, k) \), then nothing happens with him even if \( v = D(c) \); it can be that at a later moment this server will drift away from the node \( D(c) \), and the distance between \( c \) and its destination \( D(c) = w \) will increase during this waiting time.

A formal definition of the Markov process describing the evolution of the network \( \mathcal{K}_N \) will be provided later, in Section 5.1. Our main result is the proof of the convergence of the network \( \mathcal{K}_N \) to the Non-Linear Markov Process – which is the limiting mean-field system. The formulation of our main theorem is given in Section 2.5.

The rest of the paper is organized as follows. In Subsection 2.2 we present the description of the state space of our NLMP. In Subsection 2.5 we then describe its possible jumps, write down its evolution equation and finally formulate our main result about the existence of the NLMP and the convergence of the networks \( \mathcal{K}_N \) to it, as \( N \to \infty \). In Section 3 we prove the existence theorem for the NLMP. Next Section 4 is devoted to various compactification arguments. We use these arguments in Section 5 to check the applicability of the Trotter-Kurtz theorem, thus proving the convergence part of our main result.
2.2 The State Space of the Mean-Field Limit

We describe below the state space of the mean-field limit, which will be referred to as the *Comb*.

At any given time, at each node \( v \in G \), we have a finite *ordered* queue \( q_v \) of customers, \( q_v = \{ c_i \} \equiv \{ c^v_1, ..., c^v_{l(q_v)} \} \), where \( l(q_v) \) is the length of the queue \( q_v \). The customers are ordered according to their arrival times to \( v \). The information each customer carries consists of

1. Its class \( \kappa_i \equiv \kappa_i(c_i) \in K, (|K| < \infty) \) and

2. The final address \( v_i = D(c_i) \in V(G) \) which the customer wants to reach. The class \( \kappa \) of the customer can change as a result of the service process. We will denote by \( \bar{\kappa}(c) \) the class of the customer \( c \) once its service at the current server is over. In what follows we consider only conservative disciplines. This means that the server cannot be idle if the queue is not empty.

3. We will denote by \( C(q_v) \) the customer of queue \( q_v \) which is being served, and we denote by \( \tau(C(q_v)) \) the amount of time this customer already spent being served. We need to keep track of it since our service times are not exponential in general. It can happen that in the queue \( q_v \) there are customers of lower priority than \( C(q_v) \), which already received some service, but whose service is postponed due to the arrival of higher priority customers.

4. Let \( i^*(q_v) \) be the location of the customer \( C(q_v) \) in the queue \( q_v \), i.e. \( C(q_v) \equiv c^v_{i^*(q_v)} \). The service discipline is the rule \( R_v \) to choose the location \( i^*(q_v) \) of the customer which has to be served. In what follows we will suppose that the rule \( R_v \) is some function of the sequence of classes of our customers, \( \kappa_1, ..., \kappa_{l(q_v)} \) and of the sequence of their destinations, \( D(c^v_1), ..., D(c^v_{l(q_v)}) \), so that

\[
i^*(q_v) = R_v \left[ \{ \kappa_1, ..., \kappa_{l(q_v)} \}, \{ D(c^v_1), ..., D(c^v_{l(q_v)}) \} \right].
\]

We assume that the function \( R_v \) depends on \( \{ D(c^v_1), ..., D(c^v_{l(q_v)}) \} \) only through the relative distances \( \text{dist}(D(c^v_i), v) \).
5. The amount of service already acquired by the customers \( c_v^1, \ldots, c_v^l \) will be denoted by \( \tau_1, \ldots, \tau_{l(q_v)} \). At the given time, the only \( \tau \) variable which is growing is \( \tau_{i^*(q_v)} \equiv \tau [C(q_v)] \equiv \tau \left[ c_{i^*(q_v)}^v \right] \). Sometime we will write \( c_v \equiv c_v(z, \tau, v') \) for a customer at \( v \) of class \( z \), who has already received \( \tau \) units of service and whose destination is \( v' \).

The space of possible queue states at \( v \) is denoted by \( M(v) \). The ‘coordinates’ in \( M(v) \) are listed in the five items above. So \( M(v) \) is a countable union of finite-dimensional positive orthants; it will be referred to as the Comb.

Let \( M = \prod_{v \in G} M(v) \). The measure \( \mu \) of the NLMP is defined on this product space \( M \), which is hence a product of Combs. It turns out that we will encounter only the product measures on \( M \). We will discuss this point below; see also [PRS], where we prove a simple extension of the de Finetti’s theorem.

2.3 Possible State Jumps of the Mean-Field Limit and their Rates

We give here a brief summary of the jumps of the NLMP on the Comb and their rates.

2.3.1 The Arrival of a New (external) Customer

A customer \( c^{v'} \) of class \( z \in K \) arrives at node \( v \), with destination \( D(v) = v' \) with rate \( \lambda = \lambda(z, v, v') \). We suppose that

\[
\sum_{z, v'} \lambda(z, v, v') < C,
\]

uniformly in \( v \). As far as the notation is concerned, we will say that the queue state \( q = \{q_u, u \in G\} \) changes to \( q' = \{q_u, u \in G\} \oplus c^{v'} \). The associated jump rate \( \sigma_e(q, q') \) is

\[
\sigma_e(q, q') = \lambda(z, v, v').
\] (2)

2.3.2 Service Completion

It is easy to see that the customer in service at node \( v \), who received the amount \( \tau \) of service time, finishes his service at \( v \) with the rate \( \frac{F'_e(C(q_v)), \tau)}{1 - F_e(C(q_v)), \tau)} \).
where $F_{\kappa,v}$ denotes the distribution function of the service time. For future use we suppose that this rate has a limit as $\tau \to \infty$. We also suppose that it is uniformly bounded by a constant $F < \infty$. The queue state $q = \{q_u, u \in G\}$ changes to $q' = \{q_u, u \in G\} \ominus C(q_v)$, so we denote this rate by

$$\sigma_f(q, q') = \frac{F'_v(C(q_u)), v \tau}{1 - F_v(C(q_u)), v \tau} \leq F.$$  (3)

2.3.3 Servers Jumping

Let the server at $v$ jump and exchange with the one at $v'$. As a result, the queue $q_v$ is replaced by a (random) queue $Q$ distributed according to the distribution law $\mu_{v'}(dQ)$. So the state changes from $q = \{q_u, u \in G\}$ to $q' = \{Q, q_u, u \neq v \in G\}$ with rate

$$\sigma_{ex}(q, q') = \beta_{vv'}\mu_{v'}(dQ).$$  (4)

2.3.4 The Arrival of the Transit Customers

Suppose we are at node $v'$, and that a customer $c^v$ of class $\kappa$ located in the server of the neighboring node $v$ completes his service. What are the chances that this customer joins node $v' \in \mathcal{N}(v)$ to be served there? Here we denote by $\mathcal{N}(v)$ the set of all vertices of the graph $G$, which are neighbors of $v$. For this to happen it is necessary that

$$\text{dist}(v, D(c^v)) = \text{dist}(v', D(c^v)) + 1, \text{ and dist}(v', D(c^v)) > 0.$$  

If there are several such nodes in $\mathcal{N}(v)$, then all of them have the same chance. In the case $\text{dist}(v, D(c^v)) = 1$, the customer $c^v$ goes to the node $D(c^v)$ and leaves the network immediately. Let $E(v, D(c^v))$ be the number of such nodes:

$$E(v, D(c^v)) = \# \{w \in \mathcal{N}(v) : \text{dist}(v, D(c^v)) = \text{dist}(w, D(c^v)) + 1\}.$$  

\footnote{Queuing theorists might be surprised by these departures without arrivals whereas customers do not necessarily leave the network. As we shall see, in the mean-field limit, any single departure from $v$ to $v'$ has no effect on the state of the queues of $v'$ because of the uniform routing to the mean-field copies. However, the sum of the departure processes from all copies of the servers at $v$ leads to a positive arrival rate from $v$ to $v'$ which is evaluated in subsection 2.3.4 below.}
Thus, for every pair \( v, D \in G \) of sites with \( \text{dist}(v, D) > 1 \), we define the function \( e_{v,D} \) on the sites \( w \in G \):

\[
e_{v,D}(w) = \begin{cases} \frac{1}{E(v,D)} & \text{if } w \in \mathcal{N}(v) : \text{dist}(v, D) = \text{dist}(w, D) + 1 \\ 0 & \text{otherwise} \end{cases}.
\]

Then, in the state \( \mu \), the rate of the transit customers of class \( \kappa \) arriving to node \( v' \) is given by

\[
\sigma_{tr} \left( q, q \oplus c^{v', \kappa} \right) \equiv \sigma_{tr}^\mu \left( q, q \oplus c^{v', \kappa} \right) = \sum_{v \in \mathcal{N}(v')} \int d\mu(q_v) e_{v,D(C(q_v))}(v') \frac{\mathcal{F}_{v,v'}(\tau(C(q_v)))}{1 - \mathcal{F}_{v,v'}(\tau(C(q_v)))} \delta(\bar{\kappa}(C(q_v)), \kappa). \tag{6}
\]

Here \( \bar{\kappa} \) is the class the customer \( C(q_v) \) gets after his service is completed at \( v \). Such a customer \( c^{v', \kappa} \) just arrived to \( v' \), if generated by the customer \( C(q_v) \), has the site \( D(C(q_v)) \) as its destination.

Note that the two rates (5), (6) do depend on the measure \( \mu \), which is the source of the non-linearity of our process.

### 2.4 Evolution Equations of the Non-Linear Markov Process

We start with a more precise description of the state space of the initial network.

Denote by \( v \in V(G) \) a vertex of \( G \). The state of the server \( q \) at \( v \) is

- \( w \), the sequence of customers present in its queue, ordered according to the times of their arrivals (we recall that customers belong to classes, so that we need the sequence of customer classes to represent the state of the queue);

- the vector describing the amount of service already obtained by these customers; the dimension of this vector is the length \( \|w\| \) of \( w \).

The \( i \)-th coordinate of this vector will be denoted by \( \tau_i \). For example, for the FIFO discipline only the first coordinate can be non-zero. For the LIFO discipline all coordinates are positive in general.
Thus, $q_v$ is a point in $M(v)$, which is the disjoint union of all positive orthants $\mathbb{R}^+_w$, with $w$ ranging over the set of all finite sequences of customer classes. For $w$ the empty sequence, the corresponding orthant $\mathbb{R}^+_\emptyset$ is a single point.

Every orthant $\mathbb{R}^+_w$ with $|w| = n > 0$ is equipped with a vector field $r = \{ r(x), x \in \mathbb{R}^+_n \}$. The coordinate $r_i(x)$ of $r(x)$ represents the fraction of the processor power spent on customer $c_i$, $i = 1, \ldots, n$. This fraction is a function of the current state $x$ of the queue. We have $\sum_i r_i(x) \equiv 1$. In what follows we will consider only disciplines where exactly one coordinate of the vector $r$ is 1, while the rest of them are 0. The vectors $r(x)$ are defined by the service discipline. For example, for FIFO $r_1(x) = 1$, $r_i(x) = 0$ for $i > 1$.

In our general notation $r_i(x) = 1$ iff $i = i^*(x)$.

There are two natural maps between the spaces $\mathbb{R}^+_w$. One is the embedding $\chi: \mathbb{R}^+_w \to \mathbb{R}^+_w \cup \mathbb{R}^+\emptyset$, corresponding to the arrival of the new customer $c$; it is given by $\chi(x) = (x, 0)$. The other one is the projection, $\psi: \mathbb{R}^+_w \to \mathbb{R}^+_w \setminus c_{i^*(x)}$, corresponding to the completion of the service of the customer $c_{i^*(x)}$, currently served. It is given by $\psi(x) = (x_1, \ldots, x_{i^*(x)-1}, x_{i^*(x)+1}, \ldots, x_{|w|})$. For $|w| = 0$ the space $\mathbb{R}^+\emptyset$ is a point, and the map $\psi: \mathbb{R}^+_\emptyset \to \mathbb{R}^+_\emptyset$ is the identity.

The third natural map $\zeta_{vw'}: M(v) \times M(v') \to M(v) \times M(v')$ is defined for every ordered pair $v, v'$ of neighboring nodes. It corresponds to the jump of a customer, who has completed his service at $v$, to $v'$, where he is going to be served next. It is defined as follows: if the destination $D(C(q_v))$ of the attended customer $C(q_v)$ of the queue $q_v$ is different from $v'$, then

$$\zeta_{vw'}(q_v, q_v') = (q_v \ominus C(q_v), q_{v'} \oplus c(C(q_v))),$$

where the customer $c(C(q_v))$ has the properties:

1. $D(c(C(q_v))) = D(C(q_v))$,
2. $\kappa(c(C(q_v))) = T(\kappa(C(q_v)); v, v')$,
3. $\tau(c(C(q_v))) = 0$. 

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If $D(C(q_v)) = v'$ or if $q_v = \varnothing$, then we put $\zeta_{vv'}(q_v, q_{v'}) = (q_v, q_{v'})$.

As already explained, in order to make our problem tractable, we have to pass from the graph $G$ studied above to the mean-field graphs $G_N$, mentioned earlier. They are obtained from $G$ by taking $N$ disjoint copies of $G$ and by interconnecting them in a mean-field manner. Let $\Omega_N$ be the infinitesimal operator of the corresponding continuous time Markov process. We want to pass to the limit $N \to \infty$, in the hope that in the limit, the nature of the process will become simpler. The key observation is that if this limit exists, then the arrivals to each server at every time should be a Poisson point process (with time dependent rate function). Indeed, the flow to every server is the sum of $N$ flows of rates $\sim \frac{1}{N}$. Since the probability that a customer, served at a given node revisits this node, goes to zero as $N \to \infty$, the arrivals to a given server in disjoint intervals are asymptotically independent in this limit.

In order to check that the limit $N \to \infty$ exists, we will formally write down the limiting infinitesimal generator $\Omega$. We will then show that it defines a (non-linear) Markov process. Finally, we will check that the convergence $\Omega_N \to \Omega$ is such that the Trotter-Kurtz theorem applies.

Let $M = \Pi_v M(v)$. We first write the evolution of the measure $\mu$ on $M$ when $\mu$ is a product measure, $\mu = \Pi_v \mu_v$. For a warm-up we first consider now the case when the measure $\mu$ has a density. The general situation will be treated below, see Proposition 3.

For $q \in M(v)$ denote by $e(q)$ the last customer in the queue $q$, by $l(q)$ length of the queue. Then the last customer $e(q)$ in the queue can be also denoted by $c_l(q)$ and the quantity $\tau(e(q))$ denotes the time this customer already was served at $v$.

We then have

$$\frac{d}{dt} \mu_v(q_v, t) = A + B + C + D + E \quad (10)$$

with

$$A = -\frac{d}{dr_i(q_v)} \mu_v(q_v, t) \quad (11)$$

the derivative along the direction $r(q_v)$;

$$B = \delta(0, \tau(e(q_v))) \mu_v(q_v \ominus e(q_v), t) [\sigma_{tr}(q_v \ominus e(q_v), q_v) + \sigma_e(q_v \ominus e(q_v), q_v)] \quad (12)$$
where \( q_v \) is created from \( q_v \ominus e(q_v) \) by the arrival of \( e(q_v) \) from \( v' \), and \( \delta(0, \tau(e(q_v))) \) takes into account the fact that if the last customer \( e(q_v) \) has already received some amount of service, then he cannot arrive from the outside (see (6) and (2));

\[
C = -\mu_v(q_v, t) \sum_{q_v'} [\sigma_{tr}(q_v, q_v') + \sigma_e(q_v, q_v')] ,
\]

which corresponds to changes in queue \( q_v \) due to customers arriving from the outside and from other servers;

\[
D = \int_{q_v': q_v' \ominus C(q_v')} d\mu_v(q_v', t) \sigma_f(q_v', q_v' \ominus C(q_v')) - \mu_v(q_v, t) \sigma_f(q_v, q_v \ominus C(q_v)) ,
\]

where the first term describes the situation where the queue state arises \( q_v \) after a customer was served in a queue \( q_v' \) (longer by one customer), such that \( q_v' \ominus C(q_v') = q_v \), while the second term describes the completion of service of a customer in \( q_v \);

\[
E = \sum_{v' \text{n.n.v}} \beta_{vv'} \left[ \mu_{v'}(q_v, t) - \mu_v(q_v, t) \right] ,
\]

where the \( \beta \)-s are the rates of exchange of the servers.

### 2.5 Main Result

Before stating the main result, we need some observations on the states of the network. To compare the networks \( K_N \) and the limiting network \( K \), it is desirable that their states are described by probability distributions on the same space. This is in fact easily achievable, due to the permutation symmetry of the networks \( K_N \). Indeed, if we assume that the initial state of \( K_N \) is \( \prod_{v \in G} S_{v,N} - \)invariant – where each permutation group \( S_{v,N} \) permutes the \( N \) servers at the node \( v \) – then, evidently, so is the state at every later time. After the factorization by the permutation group \( (S_N)^G \) the configuration at any vertex \( v \in G \) can be conveniently described by an atomic probability measure \( \Delta_N^v \) on \( M(v) \), of the form \( \sum_{k=1}^{N} \frac{1}{N} \delta(q_{v,k}, \tau) \), where \( \tau \) is the vector of already received services in queue \( q_{v,k} \). We put \( \Delta_N = \{\Delta_N^v\} \).

We study the limit of the networks \( K_N \) as \( N \to \infty \). For this limit to exist, we need to choose the initial states of the networks \( K_N \) appropriately.
So we suppose that the initial states \( \nu_N \) of our networks \( \mathcal{K}_N \) – which are atomic measures on \( M \) with atom weight \( 1/N \) – converge to the state \( \nu \) of the limiting network \( \mathcal{K} \).

We are now in a position to state the main result:

**Theorem 1** Let \( S_{N,t} \) be the semigroup \( \exp\{t\Omega_N\} \) defined by the generator \( \Omega_N \) of the network \( \mathcal{K}_N \), described in Section 2.1 (the operator \( \Omega_N \) is formally defined in \((20-26)\)). Let \( S_t \) be the semigroup \( \exp\{t\Omega\} \), defined by the generator \( \Omega \) of the network \( \mathcal{K} \) (the operator \( \Omega \) is defined in \((10-15)\) for ‘nice’ states, and in \((27-31)\) for the general case).

1. The semigroup \( S_t \) is well-defined, i.e., for every measure \( \nu \) on \( M \), the trajectory \( S_t(\nu) \) exists and is unique.
2. Suppose that the initial states \( \nu_N \) of the networks \( \mathcal{K}_N \) converge to the state \( \nu \) of the limiting network \( \mathcal{K} \). Then for every \( t > 0 \) \( S_{N,t}(\nu_N) \to S_t(\nu) \).

### 3 The Non-Linear Markov Process: Existence

In this section we prove Part 1 of Theorem 1.

Our Non-Linear Markovian evolution is a jump process on \( M = \Pi_v M(v) \). Between the jumps, the point \( q \in M(v) \) moves with unit speed in its orthant along the field \( r(q) \); this movement is deterministic. The point \( q \in M(v) \) can also perform various jumps, as described above.

**Theorem 2** For every initial state \( \mu(0) = \Pi_v \mu(v)(0) \) Equation \((10-15)\) has a solution, which is unique.

**Proof.** The idea of the proof is the following. Let us introduce an auxiliary system on the same set of servers with the same initial condition \( \mu(0) \). Instead of the internal Poisson flows of the initial system with rates \( \lambda_{v^v}(t) \) (which are (hypothetically) determined uniquely by \( \mu(0) \)), we consider, for every node \( v \), an arrival Poisson flow of customers with arbitrary rate function \( \lambda_{v^v}(t) \). The result of the service at \( v \) will then be a collection of (individually non-Poisson) departure flows to certain nodes \( v'' \). The flow from \( v \) to \( v'' \) is non-Poisson in general. Consider its rate function, \( b_{v,v''}(t) \):

\[
b_{v,v''}(t) = \lim_{\Delta t \to 0} \frac{\mathbb{E}\text{ (number of cust. arrived from } v \text{ to } v'' \text{ in } [t, t + \Delta t])}{\Delta t}.
\]
Thus, we have an operator $\psi_{\mu(0)}$, which transforms the collection $\{\lambda_{v'}(t)\}$ to $\{b_{v'}(t)\}$. Our theorem about the existence and uniqueness will follow from the fact that the map $\psi_{\mu(0)}$ has a unique fixed point, $\bar{\lambda}$. Note, that the ratefunctions $\lambda$ and $b$ depend not only on the nodes $v, v'$, but also on the class of customers. Below we often omit some coordinates of these vectors, but we always keep them in mind.

Note that the functions $b$ are continuous and uniformly bounded. Moreover, without loss of generality we can suppose that they are Lipschitz, with a Lipschitz constant $L$, which depends only on $\gamma$ – the supremum of the service rates. Since we look for the fixed point, we can assume that the functions $\lambda$ are bounded as well, and also that they are integrable and Lipschitz, with the same Lipschitz constant. So we restrict the functions $\lambda$ to be in this class $L^1[0, T]$. In case the graph $G$ is finite, we put the $L_1$ metric on our functions,

$$
\int_0^T |\lambda_1^1(\tau) - \lambda_2^2(\tau)| \, d\tau \equiv \sum_{v'v} \int_0^T |\lambda_{v'}^1(\tau) - \lambda_{v'}^2(\tau)| \, d\tau.
$$

Note that this metric turns $L^1[0, T]$ into complete compact metric space, by Arzelà–Ascoli. For the infinite $G$ we choose an arbitrary vertex $v_0 \in G$ as a ‘root’, and we define likewise

$$
\int_0^T |\lambda_1(\tau) - \lambda_2(\tau)| \, d\tau = \sum_{v'v} \exp\left\{-2D(G) \left[\text{dist}(v_0, v') + \text{dist}(v_0, v)\right]\right\} \int_0^T |\lambda_{v'}^1(\tau) - \lambda_{v'}^2(\tau)| \, d\tau.
$$

We recall that $D(G)$ is the maximal degree of the vertex of $G$. The topology on $L^1[0, T]$ thus defined is equivalent to the Tikhonov topology; in particular, $L^1[0, T]$ is again a complete compact metric space.

We will now show that for every $\mu(0)$, the map $\psi_{\mu(0)}$ is a contraction on $L^1[0, T]$; by the Banach theorem this will imply the existence and the uniqueness of the fixed point for $\psi_{\mu(0)}$. Without loss of generality we can assume that $T$ is small.

Let $\{\lambda_1(t), \lambda_2(t) : t \in [0, T], v \in G\}$ be the rates of two collections of Poisson inflows to our servers; assume that for all $v$

$$
\int_0^T |\lambda_1^1(\tau) - \lambda_2^2(\tau)| \, d\tau < \Lambda
$$

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uniformly in \( v \). We want to estimate the difference \( b^1_0 (t) - b^2_0 (t) \) of the rates of the departure flows at \( 0 \in G \); clearly, this will be sufficient. The point is that the rates \( b^i_0 (\cdot) \) depend also on the rates \( \lambda^i_v (\cdot) \) for \( v \neq 0 \), due to the possibility of servers jumps, after which the state at the node 0 is replaced by that at the neighboring node.

Let \( \tau_1 < \tau_2 < \ldots < \tau_k \in [0, T] \), \( k = 0, 1, 2, \ldots \) be the (random) moments when the state at 0 is replaced by the state at a neighbor node, due to the ‘server jumps’. We will obtain an estimate on \( \int_0^T |b^1_0 (t) - b^2_0 (t)| \, dt \) under the condition that the number \( k \) and the moments \( \tau_1 < \tau_2 < \ldots < \tau_k \) are fixed; since our estimate is uniform in the conditioning, this is sufficient. Note that the probability to have \( k \) jumps during the time \( T \) is bounded from above by \( (T \beta)^k \) (see (1)).

Informally, the contraction takes place because the departure rates \( b (t) \) for \( t \in [0, T] \) with \( T \) small depend mainly on the initial state \( \mu (0) \): the new customers, arriving during the time \( [0, T] \) have little chance to be served before \( T \), if there were customers already waiting. Therefore the ‘worst’ case for us is when in the initial states all the servers are empty, i.e. the measures \( \mu_v (0) \) are all equal to the measure \( \delta_0 \), having a unit atom at the empty queues \( \emptyset \).

A. Let us start the proof by imposing the condition that the jumps to the server at \( v = 0 \) do not happen, i.e. \( k = 0 \). So let \( \lambda^1 (t) \), \( \lambda^2 (t) \), \( t \in [0, T] \) be the rates of two collections of the Poisson inflows to the empty server, and \( \gamma \) be the supremum of the service rates. We want to estimate the difference \( \int_0^T |b^1(t) - b^2(t)| \, dt \) of the rates of the departure flows. For that we will use the representation of integrals as expectations of random variables.

Consider first the integral

\[
I^\lambda_T = \sum_{v'} \int_0^T \left| \lambda^1_{v'v} (t) - \lambda^2_{v'v} (t) \right| \, dt.
\]

For every value of the index \( v' \) let us consider the region between the graphs of the functions

\[
H_{v'} (t) = \max \{ \lambda^1_{v'v} (t) , \lambda^2_{v'v} (t) \} \quad \text{and} \quad h_{v'} (t) = \min \{ \lambda^1_{v'v} (t) , \lambda^2_{v'v} (t) \} .
\]

Then the integral \( \int_0^T (\lambda^1_{v'v} (t) - \lambda^2_{v'v} (t)) \, dt \) is the expectation of the number of the rate 1 Poisson points \( \omega_{v'} \in \mathbb{R}^2 \), falling between \( H_{v'} \) and \( h_{v'} \). Let us denote the corresponding random variables by \( \alpha_{v'} \). Let us declare the customers
that correspond to the points falling below \( h_v' = \min \{ \lambda^1_{v,v} (t), \lambda^2_{v,v} (t) \} \) as colorless; let us color the points falling between \( H = h_v' \) and \( h_v' \) in the following way: the customers from the first flow, falling between \( \lambda^1_{v,v} (t) \) and \( h_v' (t) \), get the red color, while the customers from the second flow, falling between \( \lambda^2_{v,v} (t) \) and \( h_v' (t) \), get the blue one. The variable \( \alpha_v' \) is thus the number of colored customers. This defines the coupling between the two input flows.

Note that, on the event that each of the departure flows is not affected by the colored customers, the two departure flows are identical. Therefore the contribution of this event to the integral \( \int_0^T |b_{vv'}^1 (t) - b_{vv'}^2 (t)| \, dt \) is zero.

Let \( \mathcal{C} \) be the complement to this event. Note that the probability of arrival of colored customer is equal to \( h_v' \) of the (unconditional) probability of \( \mathcal{C} \) that correspond to the points falling below \( h_v' \). The contribution of this event to the integral by the colored customers, the two departure flows are identical. Therefore this defines the coupling between the two input flows.

B. Consider now the case when we also allow the jumping of servers. Then the operator \( \psi_{\mu_0} \) from the collection \( \{ \lambda_e (t), e \in E (G) \} \) to \( \{ b_e (t), e \in E (G) \} \), is still defined, once the initial condition \( \mu_0 = \{ \mu_v (0) \} \) is fixed. We argue that the contribution of the event of having \( k \) server jumps at the node \( v = 0 \) to \( \int_0^T |b_{vv'}^1 (t) - b_{vv'}^2 (t)| \, dt \) is of the order of \( T^{k+1} \).

Consider the case \( k = 1 \), and let us condition on the event that \( \tau_1 = \tau < T \) and the jump to \( v = 0 \) is made from the nearest neighbor node (n.n.) \( w \). So we need to compare the evolution at \( v = 0 \) defined by the Poisson inputs with rates \( \lambda^1_{v,0} (t), \lambda^1_{w,v} (t) \) with that defined by the rates \( \lambda^2_{v,0} (t), \lambda^2_{w,v} (t) \). We will use the same couplings between the pairs of flows. Before time \( \tau \), the server at \( v = 0 \) behaves as described above in Section A. At the moment \( \tau \) its state \( \mu^i_0 (\tau) \) is replaced by that drawn independently from \( \mu^i_w (\tau) \), which then evolves according to the in-flow defined by \( \lambda^i_{w,0} (t) \), \( i = 1, 2 \). A moment thought shows that all the arguments of Part A still apply, so under all the conditions imposed we have that \( \int_0^T |b_{0v'}^1 (t) - b_{0v'}^2 (t)| \, dt \) is again bounded from above by \( \text{const} \cdot T \mathcal{F} \max (I_{1,v}^{\ast v}, I_{2,v}^{\ast w}) \) which is \( \ll \Lambda \) uniformly.
The averaging over $\tau$ brings the extra factor $T\beta$, so the corresponding contribution to $\int_0^T |b_{0v'}^1 (t) - b_{0v'}^2 (t)| \, dt$ is $\text{const} \cdot T^2 \beta \mathcal{F} \max \{ I^\lambda_{0v}, I^\lambda_{v'} \} \ll T \Lambda$.

The case of general values of $k$ follows immediately.

C. Thus far we have shown that the initial condition $\mu (0) = \{ \mu_v (0) \}$ defines all the in-flow rates $\lambda_{v'v} (t), \ t \in [0, T]$ uniquely. If the graph $G$ is finite, that implies the uniqueness of the evolving measure $\mu (t)$. For the infinite graph it can happen in principle that different ‘boundary conditions’—i.e. different evolutions of $\mu$ ‘at infinity’—might still be a source of non-uniqueness. However, the argument of Part B shows that the influence on the origin $0 \in G$ from the nodes at distance $R$ during the time $T$ is of the order of $T^R$, provided the degree of $G$ is bounded. Therefore the uniqueness holds for the infinite graphs as well. ■

Proposition 3 The semigroup, defined by Equations (10–15) is Feller.

Proof. Note that the trajectories $\mu_t$ are continuous in time. Indeed, they are defined uniquely by the inflow rates $\{ \tilde{\lambda}_{v'v} (t) \}$, which in turn are defined by the external flows and by the initial state $\mu_0$. Since for the NLMP they coincide with the departure flow rates, $\{ \tilde{b}_{v'v} (t) \}$, which are continuous in time, they are continuous themselves, and so the trajectories $\mu_t$ are continuous as well. Moreover, the dependence of the rates $\tilde{\lambda}_{v'v} (t)$ on the initial state $\mu_0$ is continuous, since the departure rates $\{ \tilde{b}_{v'v} (t) \}$ are continuous in $\mu_0$. Therefore the map $\mathcal{P} \rightarrow \mathcal{P}$, defined by $\mu_0 \mapsto \mu_t$, is continuous as well. ■

4 Compactifications

In order to study the convergence of our mean-field type networks $\mathcal{K}_N$ to the limiting Non-Linear Markov Process, network $\mathcal{K}$, we want the latter to be defined on a compact state space. This means that we have to add to the graph $G$ the sites $G_\infty$ lying at infinity, obtaining the extended graph $\bar{G} = G \cup G_\infty$, and to allow infinite queues at each node $v \in \bar{G}$. We then have to extend our dynamics to this bigger system. The way it is chosen among several natural options is of small importance, since, as we will show, if the initial state of our network assigns zero probability to various infinities, then the same holds for all times.

The compactification plays here a technical role. It allows us to use some standard theorems of convergence of Markov processes, and probably can be
avoided. The benefit it brings is that certain observables can be continuously extended to a larger space.

4.1 Compactification $\bar{G}$ of the Graph $G$

When the graph $G$ is infinite, we need to have its compactification. The compactification we are going to define does depend on our network. Namely, it will use the following feature of our network discipline: if a customer $c$ is located at $v$ and its destination $D(c) = w$, then the path $c$ takes to get from $v$ to $w$ is obtained using the random greedy algorithm. According to it, $c$ chooses uniformly among all the n.n. sites, which bring $c$ one unit closer to its goal. In principle, different disciplines on the same graph $G$ might lead to different compactifications.

To define it we proceed as follows. Let $\gamma = \{\gamma_n \in G\}$ be a n.n. path on $G$. We want to define a notion of existence of the limit $L(\gamma) = \lim_{n \to \infty} \gamma_n$. If the sequence $\gamma_n$ stabilizes, i.e. if $\gamma_n \equiv g \in G$ for all $n$ large enough, we define $L(\gamma) = g$. To proceed, for any $v \in G$ we define the Markov chain $P^v$ on $G$. It is a n.n. random walk, such that at each step a walker makes its distance to $v$ to decrease by 1. If he has several such choices, he choose one of them uniformly. Therefore the transition probabilities $P^v(u,w)$ are given by the function $e_{u,v}(w)$, defined in (5). If $T$ is some integer time moment, and $u$ and $v$ are at the distance bigger than $T$, then the $T$-step probability distribution $P^{v,T}_u$ on the trajectories starting at $u$ and heading towards $v$ is defined.

Let now the path $\gamma$ be given, with $\gamma_n \to \infty$. We say that the limit $L(\gamma) = \lim_{n \to \infty} \gamma_n$ exists, if for any $u \in G$ and any $T$, the limit $\lim_{n \to \infty} P^{u,T}_{\gamma_n}$ exists. For two paths $\gamma'$, $\gamma''$ we say that $L(\gamma') = L(\gamma'')$ iff both limits exist and moreover for any $u \in G$ and any $T$ the measures $P^{u,T}_{\gamma'}$ and $P^{u,T}_{\gamma''}$ coincide for all $n \geq n(u,T,\gamma',\gamma'')$ large enough.

Consider the union $\bar{G} = G \cup G^\infty \equiv G \cup \{L(\gamma) : \gamma$ is a n.n. path on $G\}$. It is easy to see that a natural topology on $\bar{G}$ makes it into a compact. For example, consider the case $G = \mathbb{Z}^2$. Let $f : \mathbb{Z}^2 \to \mathbb{R}^2$ be the following embedding: $f(n,m) = \left(\text{sign}(n)\left(1 - \frac{1}{|n|}\right), \text{sign}(m)\left(1 - \frac{1}{|m|}\right)\right)$. Then the closure $\bar{G}$ will be the closure of the image of $f$ in $\mathbb{R}^2$.

We can make $\bar{G}$ into a graph. To do this we need to specify pairs of vertices which are connected by an edge. If $v', v''$ both belong to $G$, then they are connected in $\bar{G}$ iff they are connected in $G$. If $v'$ is in $G$, while $v'' \in G^\infty$, then
they are never connected. Finally, if \( v', v'' \in G^\infty \), then they are connected iff one can find a pair of paths \( \gamma', \gamma'' \to \infty \) such that \( L(\gamma') = v' \), \( L(\gamma'') = v'' \), and the sites \( \gamma'_n, \gamma''_n \in G \) are n.n. In particular, every vertex \( v \in G^\infty \) has a loop attached. Note that the graph \( \bar{G} \) is not connected.

4.2 Extension of the Network to \( \bar{G} \)

This is done in a natural straightforward way. Now we have servers and queues also at ‘infinite’ sites. Note that the customers at infinity cannot get to the finite part \( G \) of \( \bar{G} \). Also, the customers from \( G \) cannot get to \( G^\infty \) in finite time.

4.3 Compactification of \( M(v) \)

In order to make the manifold \( M(v), v \in V(G) \) compact we have to add to it various ‘infinite objects’: infinite words, infinite waiting times and infinite destinations. Since the destinations of the customers in our network are vertices of the underlying graph, the compactification of \( M(v) \) will depend on it. For the last question, we use the graph \( \bar{G} \), which is already compact. The manifold of possible queues at \( v \) is a disjoint union of the positive orthants \( \mathbb{R}^+_w \), where \( w \) is a finite word, describing the queue \( q_v = \{c_i\} \equiv \{c_i^v\} \) at \( v \). The letters of the word \( w \) are classes \( \kappa_i \in K \) of the customers plus the final addresses \( v_i = D(c_i) \in \bar{G} \) of them. So the set of the values of each letter is compact. We have to compactify the set \( W \) of the finite ordered words \( w \).

To do it we denote by \( O(w) \) the reordering of \( w \), which corresponds to the order of service of the queue \( w \). This is just a permutation of \( w \). We say that a sequence \( w_i \) of finite words converge as \( i \to \infty \), iff the sequence of words \( O(w_i) \) converge coordinate-wise. If this is the case we denote by \( \bar{O} = \lim_{i \to \infty} O(w_i) \), and we say that \( \lim_{i \to \infty} w_i = \bar{O} \). We denote by \( \bar{W} \) the set of all finite words, \( w \equiv (w, O(w)) \), supplemented by all possible limit points \( \bar{O} \). We define the topology on \( \bar{W} \) by saying that a sequence \( w_i \in \bar{W} \) is converging iff the sequence \( O(w_i) \) converge coordinate-wise. In other words, we put on \( \bar{W} \) the Tikhonov topology. Since \( K \) is finite, \( \bar{W} \) is compact in the topology of the pointwise convergence.

According to what was said in the Section 2.2, our service discipline (≡ i.e. the function \( O \)) has the following property. Let the sequence of finite words \( w_i \in W \) converge in the above sense. Let \( c \) be a customer, and consider the new sequence \( w_i \cup c \in W \), where customer \( c \) is the last arrived. Then we
have the implication
\[
\lim_{i \to \infty} O(w_i) \text{ exists } \Rightarrow \lim_{i \to \infty} O(w_i \cup c) \text{ exists.}
\]

The continuity of the transition probabilities in this topology on the set of queues is easy to see; indeed, the closeness of the two queues \( q \) and \( q' \) means that the first \( k \) customers served in both of them are the same. But then the transition probabilities \( P_T(q, \cdot) \) and \( P_T(q', \cdot) \) differ by \( o(T^k) \). So the extended process is Feller, as well as the initial one.

The compactifications \( \bar{\mathbb{R}}^+ \) of the orthants \( \mathbb{R}^+ \) are defined in the obvious way: they are the products of \( |w| \) copies of the compactifications \( \bar{\mathbb{R}}^+ = \mathbb{R}^+ \cup \infty \). For the infinite words \( \bar{O} \) we consider the infinite products, in the Tikhonov topology. The notion of the convergence in the union \( \cup_{w \in \bar{W}} \bar{\mathbb{R}}^+_w \) is that of the coordinate-wise convergence.

The properties of the service times, formulated in Sect. 2.3.2, allow us to extend the relevant rates in a continuous way to a function on \( \tau \in \bar{\mathbb{R}}^+ \). Moreover, the analog of Proposition 3 holds.

5 The proof of Convergence

Let \( \Omega_N, \Omega : X \to X \) are (unbounded) operators on the Banach space \( X \). We are looking for the conditions of convergence of the semigroups \( \exp \{ t\Omega_N \} \to \exp \{ t\Omega \} \) on \( X \) as \( N \to \infty \).

We will use the following version of the Trotter-Kurtz theorem, which is Theorem 6.1 from Chapter 1 of [EK], with the core characterization taken from Proposition 3.3 of the same Chapter 1, where a core of \( \Omega \) is a dense subspace \( \bar{X} \subset X \) such that \( \forall \psi \in \bar{X} \)
\[
\exp \{ t\Omega \} (\psi) \in \bar{X}.
\] (16)

**Theorem 4** Let \( X^1 \subset X^2 \subset \ldots \subset X \) be a sequence of subspaces of Banach space \( X \). We suppose that we have projectors \( \pi_N : X \to X^N \), such that for every \( f \in X \) we have \( f^N = \pi_N (f) \to f \), as \( N \to \infty \). Suppose we have (strongly continuous contraction) semigroups \( \exp \{ t\Omega_N \} \) on \( X^N \), and we want a condition ensuring that for every \( f \)
\[
\exp \{ t\Omega_N \} f^N \to \exp \{ t\Omega \} f.
\] (17)

For this it is sufficient that for every \( f \) in the core \( \bar{X} \) of \( \Omega \) we have
\[
\| \Omega_N (f^N) - \Omega (f^N) \| \to 0.
\] (18)
We are going to apply it to our situation. The main ideas of this application were developed in the paper [KR].

The strong continuity of the semigroups $\exp \{ t\Omega_N \}$ is straightforward, while that for the semigroup $\exp \{ t\Omega \}$ follows from the fact that the trajectories $S_t\mu$ are continuous in $t$. After compactification, the set of probability measures $\mu \in \mathcal{P}$ becomes a compact, $\overline{\mathcal{P}}$, and so the family $S_t\mu$ is equicontinuous. This implies the strong continuity.

It is clear that the operation $\Omega_N (f^N)$ consists of computing finite differences for the function $f$ at some atomic measures, while $\Omega (f^N)$ is the operation of computing the derivatives at the same atomic measures. We will have convergence (18) in the situation when the derivatives exists and can be approximated by the finite differences. Therefore we have to choose differentiable functions for our space $\bar{X}$. We will check (18) in the Section 5.1.

We then need to check that this space of differentiable functions is preserved by the semigroup $\exp \{ t\Omega \}$ (core property). The function $\exp \{ t\Omega \} f (\mu)$ is just $f (S_t\mu)$, so if $f$ is differentiable in $\mu$, then the differentiability of $f (S_t\mu)$ follows from that of $S_t\mu$.

We take for our function space $X$ the space $\mathcal{C}_u (\mathcal{P})$, with $\mathcal{P}$ being the compactified version of $\mathcal{P}$, the space of product probability measures on $\Pi M (v)$. So we consider continuous functions on $\mathcal{P}$, which are bounded and uniformly continuous. $\mathcal{P}$ in turn is a subspace in the space $\Pi \mathcal{C}^* (M (v))$ of linear functionals on $\Pi \mathcal{C} (M (v))$ (with weak convergence topology), while $\mathcal{C}_u (M (v))$ is the space of bounded uniformly continuous functions on the combs $M (v)$. However, we will use a different norm on $\mathcal{P}$. Namely, we consider the space $\mathcal{C}_1 (M (v))$ of functions $f$ which are continuous and have continuous derivatives $f'$, and we put $||f||_1 = ||f|| + ||f'||$. The space $\mathcal{P}$ belongs to the dual space $\Pi \mathcal{C}^*_1 (M (v))$, and we will define the norm $||\cdot||_1$ on $\mathcal{P}$ to be the restriction of the natural norm on $\Pi \mathcal{C}^*_1 (M (v))$.

To estimate the norm of the Frechet differential we have to consider a starting measure $\mu_1 = \Pi \mu_v (t = 0)$, its perturbation, $\mu^2 = \Pi (\mu_v + h_v)$, with $h = \{ h_v \}$ non-trivial for finitely many $v$-s (say), the perturbation having norm $\approx \sum_v ||h_v||$, then to take the difference $\mu^2 (T) - \mu_1 (T)$ and to write it as $\Phi (T, \mu^1) h + O (||h||^2)$, and finally to show that the norm of the operator $\Phi (T, \mu^1)$ is finite.

Note first that it is enough to prove this for $T$ small, the smallness being uniform in all relevant parameters.

Now, if the increment $h$ is small (even only in the (weaker) $||\cdot||_1$ sense,
i.e. only from the point of view of the smooth functions – like, for example, a small shift of a $\delta$-measure), then all the flows in our network, started from $\mu^1$ and $\mu^2$ differ only a little, during a short time, its shortness being a function of the service time distributions only. The amplitude of the flow difference is of the order of $D \times F \times ||h||_1$, where $D$ is the maximal degree of the graph $G$, and $F$ is defined by (3). In fact, it can be smaller; it is attained for the situations when a server, being empty in the state $\mu^1$, becomes non-empty in the $h-$perturbed state. The order is computed in the stronger $||\cdot||_0$ norm. Therefore, after time $T$ the norm of the difference is such that

$$||\mu^2(T) - \mu^1(T)|| \leq ||h|| + T \times D \times F \times ||h||,$$

which explains our claim about the operator $\Phi(T, \mu^1)$. We will give a formal proof in Section 5.2.

5.1 Generator Comparison

We start with the finite MF-type network, made from $N$ copies of the initial network.

We first describe the process as a process of the network containing $N |G|$ servers, and then do the factorization by the product of $|G|$ permutation groups $S_N$.

The former one will be described only briefly. At each of the $N |G|$ servers, there is a queue of customers. Some of the queues can be empty. As time goes, the queues evolve – due to (1) the arrivals of external customers; (2) the end of the service of a customer, which then leaves the network; (3) the end of the service of a customer, which then jumps to the next server; (4) the interchange of two servers. Each of these events leads to a jump of our process. If none of them happens, the process evolves continuously, with the help of the time-shift semigroup.

After the factorization by the permutation group $(S_N)^G$ the configuration at any vertex $v \in G$ can be conveniently described by an atomic probability measure $\Delta^v_N$ on $M(v)$ of the form $\sum_{k=1}^{N} \frac{1}{N} \delta(q_{v,k}, \tau)$, where $\tau$ is the vector of time durations some of the customers from the queue $q_{v,k}$ (in particular the customer $C(q_{v,k})$) were already under the service; the case of empty queue is included.

We have a semigroup $S_N$ and its generator $\Omega_N$; the existence of them is straightforward. $\Omega_N$ acts on functions $F$ on measures $\mu_N \in \mathcal{P}_N$, which are atomic measures with atoms of weight $\frac{1}{N}$. 

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Our goal is now the following. Let \( \mu_N \to \mu \), and \( F \) be a smooth function on measures. Let us look at the limit \( \Omega_N (F)(\mu_N) \) and the value \( \Omega (F)(\mu) \). Since \( F \) is smooth, we are able replace certain differences by derivatives; after this, we will see the convergence \( \Omega_N (F)(\mu_N) \to \Omega (F)(\mu) \) in a transparent way.

For this, we write the multiline formula (20—26) for the operator \( \Omega_N \), applied to a function \( F \). On each \( M (v) \), we have to take a probability measure \( \Delta_N^v \) of the form \( \sum_{k=1}^N \frac{1}{N} \delta (q_{v,k}, \tau) \), where \( \tau \) is the amount of service already received by customer \( C(q_{v,k}) \) of \( q_{v,k} \) (at the position \( i^* (q_{v,k}) \) in queue \( q_{v,k} \)). Let \( \Delta_N = \{ \Delta_N^v \} \). Then

\[
\left( \Omega_N (F) \right) (\Delta_N)
\]

\[
= \sum_v \sum_k \frac{\partial F}{\partial r (q_{v,k}, \tau (C(q_{v,k})))} (\Delta_N) 
\]

\[
+ \sum_v \sum_k \sum_{v' \neq v} \sum_k \sum_{k'} \frac{1}{N} e_{v,D(C(q_{v,k}))} (v') \sigma_f (q_{v,k}, q_{v,k} \cap C(q_{v,k})) \times 
\]

\[
\times [F(J_{v,v';k,k'} (\Delta_N)) - F(\Delta_N)]
\]

where, for a directed edge \( v, v' \) and for a pair of queues \( q_{v,k}, q_{v',k'} \), we denote by \( J_{v,v';k,k'} (\Delta_N) \) a new atomic measure, which is a result of the completion of the service of a customer \( C(q_{v,k}) \) in a queue \( q_{v,k} \) at the server \( v \) and its subsequent jump into the queue \( q_{v',k'} \), increasing thereby the length of the queue \( q_{v',k'} \) at \( v' \) by one;

\[
+ \sum_v \sum_k \sum_{v' \neq v} \sum_\xi \lambda (\xi, v, v') \times 
\]

\[
\times [F(\Delta_N - \frac{1}{N} \delta (q_{v,k}) + \frac{1}{N} \delta (q_{v,k} \oplus e^v (\xi, 0, v'))) - F(\Delta_N)]
\]

– here we see the arrival of a new customer of class \( \xi \) with a destination \( v' \);

\[
+ \sum_v \sum_k \sum_{v : \text{dist}(v, v') \leq 1} \delta (D(C(q_{v,k})) = v') \sigma_f (q_{v,k}, q_{v,k} \cap C(q_{v,k})) \times 
\]

\[
\times [F(J_{v,k} (\Delta_N)) - F(\Delta_N)]
\]

– here we account for the customers that leave the network; the operator \( J_{v,k} (\Delta_N) \) denotes the new atomic measure, which is a result of the completion of the service of a customer \( C(q_{v,k}) \) in a queue \( q_{v,k} \) at the server.
$v$ and its subsequent exit from the system: $J_{v,k} (\Delta_N) = \Delta_N - \frac{1}{N} \delta (q_v, k) + \frac{1}{N} \delta (q_v, q_v \ominus C (q_v, k))$;

$$+ \sum_{v} \sum_{k} \sum_{v'} \sum_{n.n.v} \sum_{k'} \frac{1}{N} \beta_{v,v'} [F (T_{v,k,v',k'} \Delta_N) - F (\Delta_N)]$$

(26)

- here the operator $T_{v,k,v',k'}$ acts on the measure $\Delta_N$ by exchanging the atoms $\frac{1}{N} \delta (q_v, k)$ and $\frac{1}{N} \delta (q_v', k')$.

**Remark.** If our graph is infinite, the sums in (20–26) are infinite. However, they make sense for *local* functions $F$, as well as for the *quasilocal* ones, which depend of far-away nodes (exponentially) weakly.

Next, let us pass in the above formula to a formal limit, obtaining thus the (formal) expression for the limiting operator $\Omega$. It acts on functions $F$ on all probability measures on $\cup_v M (v)$. Let $\Delta = \lim_{N \to \infty} \Delta_N$. We will need the differentiability of $F$ at $\Delta$, i.e. the existence of the Frechet differential $F'$ at $\Delta$. This differential will be denoted also by $F'_\Delta (\cdot)$; it is a linear functional on the space of tangent vectors to $P$ at $\Delta \in P$. Assuming the existence of this differential, we have another multiline expression (27–31):

$$(\Omega (F)) (\Delta) = (\hat{\sigma} (F)) (\Delta)$$

$$+ \sum_{v} \sum_{v'} F'_\Delta e_{v,D(C(\cdot))} (v') \sigma_f \times (\zeta_{v,v'} (\Delta) - \Delta)$$

(27)

- where $\zeta_{v,v'} (\Delta) - \Delta$ is a (signed) measure (see (24)), and where $e_{v,D(C(\cdot))} (v') \sigma_f \times (\zeta_{v,v'} (\Delta) - \Delta)$ denotes the measure having density $e_{v,D(C(q))} (v') \sigma_f (q_v, q_v \ominus C (q_v))$ with respect to $(\zeta_{v,v'} (\Delta) - \Delta) (dq)$;

$$+ \sum_{v} \sum_{v'} \sum_{\kappa} \lambda (\kappa, v, v') \sigma_f (\chi_{v,v';\kappa} (\Delta)) F'_\Delta (\chi_{v,v';\kappa} (\Delta) - \Delta)$$

(28)

- here $\chi_{v,v';\kappa} : M (v) \to M (v)$ is the embedding, corresponding to the arrival to $v$ of the external customer of class $\kappa$ and destination $v'$, see (7);

$$+ \sum_{v} \sigma_f (\psi_{v}^n (\Delta))$$

(29)

- here

$$\psi_{v}^n (q) = \begin{cases} \psi_{v}^n (q) & \text{for } q \text{ with } \text{dist} (v, D (C (q))) \leq 1 \\ q & \text{for } q \text{ with } \text{dist} (v, D (C (q))) > 1 \end{cases}$$

(30)
while $\psi^v : M(v) \to M(v)$ is the projection, see (8), and the term $\sigma_f \times (\psi_{nn}^v(\Delta) - \Delta)$ is the (signed-)measure, having density $\sigma_f(q)$ with respect to the measure $(\psi_{nn}^v(\Delta) - \Delta)(dq)$;

$$+ \sum_v \sum_{v'^{\text{n.n.n}}} \beta_{v'v} F'_\Delta((T_{v'v}\Delta) - \Delta); \quad (31)$$

– here the operator $T_{v'v}$ acts on the measure $\Delta$ in the following way: it replaces the component $\Delta_v$ of the measure $\Delta$ by the measure $\Delta_{v'}$ (via identification between $M(v)$ and $M(v')$).

We now check that the limiting operator $\Omega$ is the same one we were dealing with in our study of the non-linear Markov process, (10–15).

**Proposition 5** The formula (27–31) can be written in the form

$$(\Omega(F))(\mu) = (\hat{\sigma}(F))(\mu) + (F'(\mu))(g(\mu)); \quad (32)$$

where $\hat{\sigma}$ is the generator of the time-shift semigroup, acting on our manifold, $F'$ is the Frechet differential of $F$, and the (signed) measure $g(\mu)$ is given by the r.h.s. of (10–15).

**Proof.** For the convenience of the reader we repeat here the equation (10–15).

$$\frac{d}{dt} \mu_v(q_v, t) = - \frac{d}{d \tau_q(q_v)} \mu_v(q_v, t)$$

$$+ \delta (0, \tau(e(q_v))) \mu_v(q_v \ominus e(q_v)) [\sigma_{tr} (q_v \ominus e(q_v), q_v) + \sigma_e(q_v \ominus e(q_v), q_v)]$$

$$- \mu_v(q_v, t) \sum_{q_v'} [\sigma_{tr} (q_v, q_v') + \sigma_e(q_v, q_v')]$$

$$+ \left[ \int_{q_v' : q_v' \ominus C(q_v') = q_v} d\mu_v(q_v') \sigma_f(q_v', q_v' \ominus C(q_v')) \right] - \mu_v(q_v) \sigma_f(q_v, q_v \ominus C(q_v))$$

$$+ \sum_{v'^{\text{n.n.n}}} \beta_{v'v} [\mu_{v'}(q_v) - \mu_v(q_v)].$$

The term $(\hat{\sigma}(F))(\Delta)$ evidently corresponds to $- \frac{d}{d \tau_q(q_v)} \mu_v(q_v, t)$, and the term

$$\sum_v \sum_{v'^{\text{n.n.n}}} \beta_{v'v} F'_\Delta((T_{v'v}\Delta) - \Delta)$$
to \( \sum_{v',n.n.v} \beta_{v'v} [\mu_{v'} (q_v) - \mu_v (q_v)] \). The ‘external customer arrival’ term

\[
\sum_v \sum_{v'} \sum_{\kappa} \lambda (\kappa, v, v') F'_\Delta (\chi_{v,v',\kappa} (\Delta) - \Delta)
\]

matches the terms

\[
\delta (0, \tau (e (q_v))) \mu_v (q_v \ominus e (q_v)) \sigma_e (q_v \ominus e (q_v), q_v) - \mu_v (q_v, t) \sum_{q_v'} \sigma_e (q_v, q_v').
\]

The ‘intermediate service completion’ term

\[
\sum_v \sum_{v',n.n.v} F'_\Delta \left( e_{v,D(C(i))} (v') \sigma_f \times (\zeta_{v,v'} (\Delta) - \Delta) \right)
\]

matches the terms

\[
\delta (0, \tau (e (q_v))) \mu_v (q_v \ominus e (q_v)) \sigma_{tr} (q_v \ominus e (q_v), q_v) - \mu_v (q_v, t) \sum_{q_v'} \sigma_{tr} (q_v, q_v').
\]

Finally, the ‘final service completion’ term \( \sum_v F'_\Delta (\sigma_f \times (\psi_{nn} (\Delta) - \Delta)) \) matches

\[
\left[ \int_{\tau_{q_v} \ominus C(q_{q_v}) = q_v} d\mu_v (q_{q_v}') \sigma_f (q_{q_v}', q_v \ominus C(q_{q_v}')) \right] - \mu_v (q_v) \sigma_f (q_v, q_v \ominus C(q_v)).
\]

Let us check that indeed we have the norm-convergence of the operators \( \Omega_N \) to \( \Omega \), the one needed in the convergence statement, \([18]\). The norm we use here is again \( \| \cdot \|_1 \).

The precise statement we need is the following:

**Proposition 6** Let \( f \) be a function on \( P \), with \( \| f \|_1 \) finite. We can restrict \( f \) on each subspace \( P_N \), and then apply the operator \( \Omega_N \), thus getting a function \( \Omega_N f \) on \( P_N \). We can also restrict the function \( \Omega f \) from \( P \) to \( P_N \). Then

\[
\| \Omega_N f - \Omega f \|_1^{P_N} \leq C_N \| f \|_1,
\]

with \( C_N \to 0 \), where \( \| \cdot \|_1^{P_N} \) is the restriction of the norm \( \| \cdot \|_1 \) to the subspace of functions of the measures \( P_N \).
Proof. We have to compare the operators given by (20 − 26) and (27 − 31) term by term. For example, compare the term (22)

\[
\sum_v \sum_k \sum_v' \sum_{\kappa} \lambda(\kappa, v, v') \left[ F\left(\Delta_N - \frac{1}{N} \delta(q_{v,k}) + \frac{1}{N} \delta(q_{v,k} \oplus c^\nu(\kappa, 0, v')) \right) - F(\Delta_N) \right],
\]

which corresponds to the arrival of a new customer of class \(\kappa\) with a destination \(v'\); and the term (28)

\[
\sum_v \sum_v' \sum_{\kappa} \lambda(\kappa, v, v') F'_{\Delta_N}(\chi_{v,v';\kappa}(\Delta_N) - \Delta_N),
\]

where \(\chi_{v,v';\kappa} : M(v) \to M(v)\) is the embedding, corresponding to the arrival to \(v\) of the external customer of class \(\kappa\) and destination \(v'\). Due to the locality properties of \(F\) it is sufficient to establish the convergence:

\[
\sum_{k=1}^N \left[ F\left(\Delta_N - \frac{1}{N} \delta(q_{v,k}) + \frac{1}{N} \delta(q_{v,k} \oplus c^\nu(\kappa, 0, v')) \right) - F(\Delta_N) \right] \to F'_{\Delta_N}(\chi_{v,v';\kappa}(\Delta_N) - \Delta_N) \quad \text{(33)}
\]

as \(N \to \infty\).

The measure \(\Delta_N\) is a collection of \(N\) atoms, corresponding to queues \(q_{v,k}\), \(k = 1, \ldots, N\). In the first expression, we change just one of the \(N\) atoms, adding a new customer \(c^\nu(\kappa, 0, v')\) to each of the queues \(q_{v,k}\), and then take a sum of the corresponding increments over \(k\). In the second expression, we change all atoms simultaneously, obtaining the measure \(\chi_{v,v';\kappa}(\Delta_N)\), plus instead of taking the increment \(F(\chi_{v,v';\kappa}(\Delta_N)) - F(\Delta_N)\) we take the differential \(F'_{\Delta_N}\) of the measure \(\chi_{v,v';\kappa}(\Delta_N) - \Delta_N\). To see the norm convergence in (33) let us rewrite the increments

\[
F\left(\Delta_N - \frac{1}{N} \delta(q_{v,k}) + \frac{1}{N} \delta(q_{v,k} \oplus c^\nu(\kappa, 0, v')) \right) - F(\Delta_N) = F'_{Q_{k,N}}\left(- \frac{1}{N} \delta(q_{v,k}) + \frac{1}{N} \delta(q_{v,k} \oplus c^\nu(\kappa, 0, v')) \right),
\]

by the intermediate value theorem. Here the points \(Q_{k,N}\) are some points on the segments

\[
[\Delta_N, \Delta_N - \frac{1}{N} \delta(q_{v,k}) + \frac{1}{N} \delta(q_{v,k} \oplus c^\nu(\kappa, 0, v'))].
\]
Note that the norms $\|Q_{k,N} - \Delta_N\|$ evidently go to zero as $N \to \infty$, and so $\left\|F_{Q_{k,N}}' - F'_{\Delta_N}\right\| \to 0$ as well. Thus

$$\sum_{k=1}^{N} F_{Q_{k,N}}' \left( -\frac{1}{N} \delta (q_{v,k}) + \frac{1}{N} \delta (q_{v,k} \oplus c^v (\nu, 0, v')) \right)$$

$$= F'_{\Delta_N} \left[ \sum_{k=1}^{N} \left( -\frac{1}{N} \delta (q_{v,k}) + \frac{1}{N} \delta (q_{v,k} \oplus c^v (\nu, 0, v')) \right) \right]$$

$$+ \sum_{k=1}^{N} \left( F_{Q_{k,N}}' - F'_{\Delta_N} \right) \left( -\frac{1}{N} \delta (q_{v,k}) + \frac{1}{N} \delta (q_{v,k} \oplus c^v (\nu, 0, v')) \right),$$

with the second term is uniformly small as $N \to \infty$. By definition,

$$\chi_{v,v';\nu} (\Delta_N) - \Delta_N = \sum_{k=1}^{N} \left( -\frac{1}{N} \delta (q_{v,k}) + \frac{1}{N} \delta (q_{v,k} \oplus c^v (\nu, 0, v')) \right),$$

and this proves the convergence needed. The other terms are compared in the same manner.

This completes the checking of the relation (18) of the Convergence Theorem.

5.2 Frechet Differential Properties

Proposition 7 The semigroup is uniformly differentiable in $t$. In the notations of Sect. 5 (see (19)) this means that for all $t < T$

$$\left\| \mu^2 (t) - \mu^1 (t) - [\mathcal{D} \mu^1 (t)] (h) \right\|_1 \leq ||h||_1 \circ (||h||_1),$$

where the function $\circ (||h||_1)$ is small uniformly in $t \leq T$ and $\mu^1$, provided $T$ is small enough.

Proof. To write the equation for the Frechet differential $\mathcal{D} \mu^1 (t) (h)$ of the map $\mu (t)$ at the point $\mu = \mu (0)$ in the direction $h$ we have to compare the evolving measures $\mu^1 (t)$ and $\mu^2 (t)$, which are solutions of the equation (10)-(15) with initial conditions $\mu$ and $\mu + h$, and keep the terms linear in $h$. In what follows we use the notation $\sigma_{tr}$, where the superscript refers to the state in which the rate $\sigma_{tr}$ is computed, see (5).
We have

\[
\frac{d}{dt} h_v (q_v, t) =
\]

\[
= - \frac{d}{dr_v (q_v)} h_v (q_v, t) +
\]

(derivative along the direction \( r (q_v) \))

\[
+ \delta (0, \tau (e (q_v))) h_v (q_v \ominus e (q_v)) \left[ \sigma^\mu (q_v \ominus e (q_v), q_v) + \sigma_e (q_v \ominus e (q_v), q_v) \right]
\]

\[
+ \delta (0, \tau (e (q_v))) \mu_v (q_v \ominus e (q_v)) \left[ \sigma^h (q_v \ominus e (q_v), q_v) \right] -
\]

\( q_v \) is created from \( q_v \setminus e (q_v) \) by the arrival of \( e (q_v) \) from \( v' \), \( \delta (0, \tau (e (q_v))) \) accounts for the fact that if the last customer \( e (q_v) \) was already served for some time, than he cannot arrive from the outside, see (6) and \([2]\);

\[
- h_v (q_v, t) \sum_{q'_v} [\sigma^\mu (q_v, q'_v) + \sigma_e (q_v, q'_v)] - \mu_v (q_v, t) \sum_{q'_v} [\sigma^h (q_v, q'_v)] +
\]

(the queue \( q_v \) is changing due to customers arriving from the outside and from other servers)

\[
+ \left[ \int_{q_v \setminus e (q_v) \ominus C (q'_v) = q_v} dh_v (q'_v) \sigma_f (q'_v, q'_v \ominus C (q'_v)) \right] - h_v (q_v) \sigma_f (q_v, q_v \ominus C (q_v)) +
\]

(here the first term describes the creation of the queue \( q_v \) after a customer was served in a queue \( q'_v \) (longer by one customer), such that \( q'_v \ominus C (q'_v) = q_v \), while the second term describes the completion of service of a customer in \( q_v \))

\[
+ \sum_{v' \neq v} \beta_{vv'} [h_v (q_v) - h_v (q_v)]
\]

(36)

(the \( \beta \)-s are the rates of exchange of the servers).

The existence of the solution to the (linear) equation \([35 - 36]\) follows by the Peano theorem, while the uniqueness of the solution is implied by the estimate

\[
\| h (t) \| \leq \| h (0) \| e^{ct},
\]

which follows from the Gronwall estimate.

Finally we want to estimate the remainder,

\[
\zeta (t) = [\mu + h] (t) - \mu (t) - [D \mu (t)] (h).
\]

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Here $\mu + h \equiv \mu (0) + h (0) \equiv [\mu + h] (0)$ is the small perturbation of $\mu$, $[\mu + h] (t)$ is its evolution, and $[\mathcal{D} \mu (t)] (h)$ is the application of the Frechet differential of the map $\nu (0) \sim \nu (t)$, computed at the point $\mu$ and applied to the increment $h$. Note that $\zeta (0) = 0$ and it satisfies the equation

$$\frac{d}{dt} \zeta_v (q_v, t) = - \frac{d}{dr_{\tau' (q_v')}} (\zeta_v (q_v, t)) +$$

$$+ \delta (0, \tau (v (q_v))) \zeta_v (q_v \ominus e (q_v)) \sigma_e (q_v \ominus e (q_v), q_v) -$$

$$+ \delta (0, \tau (v (q_v))) \zeta_v (q_v \ominus e (q_v)) \sigma^{[\mu + h]}_{tr} (q_v \ominus e (q_v), q_v) +$$

$$+ \delta (0, \tau (v (q_v))) [\mu + h] (q_v \ominus e (q_v)) \sigma^h_{tr} (q_v \ominus e (q_v), q_v) -$$

$$- \zeta_v (q_v, t) \sum_{q_v'} \left[ \sigma^{[\mu + h]}_{tr} (q_v, q_v') + \sigma_e (q_v, q_v') \right] -$$

$$- h_v (q_v, t) \sum_{q_v'} \sigma^h_{tr} (q_v, q_v') - [\mu + h] (q_v, t) \sum_{q_v'} \sigma^\epsilon_{tr} (q_v, q_v') +$$

$$+ \left[ \int_{q_v' : q_v' \ominus C (q_v')} d \zeta_{\nu'} (q_v') \sigma_f (q_v', q_v' \ominus C (q_v')) \right] - \zeta_v (q_v) \sigma_f (q_v, q_v \ominus C (q_v)) +$$

$$+ \sum_{v'' \text{n.n.v}} \beta_{v v'} \left[ \zeta_{v'} (q_v) - \zeta_v (q_v) \right].$$

Note that the initial condition for the last equation is $[\zeta]_v (q, 0) = 0$. The terms in the r.h.s. which do not contain $\zeta$ are

$$\delta (0, \tau (v (q_v))) h_v (q_v \ominus e (q_v)) \sigma^h_{tr} (q_v \ominus e (q_v), q_v) - h_v (q_v, t) \sum_{q_v'} \sigma^h_{tr} (q_v, q_v'),$$

which are of the order of $||h||^2$. Therefore by Gronwall inequality the same bound holds uniformly for the function $[\zeta]_v (q, t)$, provided $t \leq T$ with $T$ small enough. ■

**Proposition 8** The set of uniformly differentiable functions is a core of the generator of our semigroup.
Proof. Follows from the previous Proposition, by the chain rule, and the Stone-Weierstrass Theorem. ■

This implies that the second condition (16) of the Theorem [4] holds as well, so in our case it is indeed applicable.

6 Conclusion

In this paper we have established the convergence of the mean-field version of the spatially extended network with jumping servers to a Non-Linear Markov Process. The configuration of the \( N \)-component mean-field network is described by the (atomic) measure \( \mu_N(t) \), which randomly evolves in time. We have shown that in the limit \( N \to \infty \) the measures \( \mu_N(t) \to \mu(t) \), where the evolution \( \mu(t) \) is already non-random. In a sense, this result can be viewed as a functional law of large numbers.

Our results can easily be generalized to the situation when instead of the underlying (infinite) graph \( G \) we take a sequence of finite graphs \( H_n \), such that \( H_n \to G \), consider the \( N \)-fold mean-field type networks \( H_{n,N} \), and take the limit as \( n,N \to \infty \).

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