KK-mode contribution to the crossover scale for the brane-induced force

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Abstract

We discuss contributions of the KK modes to the crossover length scale $r_c$ for the brane-induced force when the brane is given by a solitonic background field. We work in a 5D scalar model with a domain-wall background that mimics the DGP model. In spite of the infinite number of the KK modes, the crossover scale remains finite due to the warping effect on the ambient space of the domain wall. The inclusion of the KK modes relaxes the hierarchy among the model parameters that is required to realize a phenomenologically viable size of $r_c$. We also discuss whether a nontrivial dilaton background enlarges $r_c$ or not.

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1 Introduction

The gravitational kinetic term is radiatively corrected by the quantum loop effect of the massive matters \[1\,2\,3\]. Thus, when matter fields are localized on a 3-brane in a higher-dimensional gravitational theory, the four-dimensional (4D) Einstein-Hilbert term is induced on the brane. If the induced 4D Planck mass \(M_4\) is much larger than the higher dimensional one \(M_*\), the gravity that acts between two sources on the brane behaves like the 4D one for shorter distances than the crossover length scale \(r_c\), which is defined by \[4\,5\]

\[
    r_c \equiv \begin{cases} 
        \frac{M_4^2}{M_*^2} & (D = 5) \\
        \frac{M_*^4}{M_4^3} & (D \geq 6) 
    \end{cases}, 
\]

(1.1)

where \(D\) is the dimension of the spacetime, while behaves like the higher-dimensional one for longer distances than \(r_c\). This is called the DGP mechanism \[4\]. The crossover scale \(r_c\) must be larger than the present Hubble size \(\sim 10^{26}\) m from the consistency with the observational data.

The DGP-type models have various interesting features when its cosmological time evolution is taken into account \[6\]. It has the self-accelerating solution, in which the late-time accelerated expansion of the three-dimensional space occurs even in the absence of the cosmological constant. It also has the degravitating solution, which is relevant to the solution of the cosmological constant problem \[7\,8\,9\].

However, the five-dimensional (5D) DGP model is found to be disfavored by the observations \[10\]. Thus we have to consider six- or higher- dimensional theories for phenomenologically viable models. When we work in such theories, the finite width of the brane has to be considered, otherwise the correlation functions diverge on the brane. So we consider a solitonic object in the field theory as the brane in this paper. The brane-localized modes are obtained as the low-lying Kaluza-Klein (KK) modes of a bulk field that couples to the soliton in such a case. Therefore, not only the low-lying KK modes but also higher-level KK modes that propagate into the bulk contribute to the induced gravity on the brane through the quantum loop effects.

As can be seen from \[11\], we have to realize a huge hierarchy between the bulk gravitational scale \(M_*\) and the brane-induced Planck scale \(M_4\) in order to obtain a phenomenologically viable value of \(r_c\). Naively, this seems possible since we have an infinite number of the KK modes that contribute to \(r_c\). However, this is not so trivial because higher KK
modes have larger KK masses and their couplings to the gravitational field are suppressed due to the rapid oscillating profiles in the extra dimensions so that their contributions to \( r_c \) are also suppressed.

In this paper, we estimate the crossover scale \( r_c \) for the brane-induced force when the brane is given by a solitonic field configuration, and discuss the possibility to realize a huge value of \( r_c \). Although the 5D DGP is observationally disfavored, it is instructive to understand the situation how the KK modes contribute to \( r_c \) in a 5D model. Hence we consider a 5D toy model with a domain-wall background in this paper. Besides, we neglect the tensor structure of the gravitational field and analyze a scalar field theory to simplify the situation.

The paper is organized as follows. In the next section, we briefly review how the brane-induced force is generated through the quantum loop effect of a brane-localized mode. In Sec. 3, we consider a scalar model that mimics the 5D gravitational theory with a domain wall background, perform the KK decomposition of the fields and derive explicit forms of the 5D propagators. In Sec. 4, we calculate the self-energy of the bulk field, expand the one-loop corrected propagator in terms of 4D momentum, and derive the expression of the crossover scale from 4D to 5D. In Sec. 5, we discuss the effect of a nontrivial dilaton background, which was proposed to enlarge the crossover scale. Sec. 6 is devoted to the summary. In Appendix A, we complement the details of the domain-wall sector. In Appendix B, we collect the definitions and various properties of the special functions that are used in the text.

## 2 Brane-induced force

Let us briefly review how the brane-induced force is generated by the quantum loop effect.

We consider a 5D scalar theory with a brane located at \( y = 0 \). The Lagrangian is given by

\[
\mathcal{L} = -\frac{M_5^3}{2} \partial^M \Phi_G \partial_M \Phi_G - \delta(y) \left\{ -\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{m^2}{2} \phi^2 - \lambda \Phi_G \phi^2 \right\}, \quad (2.1)
\]

where the real scalars \( \Phi_G(x^\mu, y) \) and \( \phi(x^\mu) \) are a 5D bulk field and a 4D brane field, respectively. In this model, \( \Phi_G \) mimics the gravitational field whose tensor structure is neglected. The positive constant \( M_5 \) is the 5D Planck mass. The 4D scalar \( \phi \) is a brane matter field whose “gravitational” interaction is parametrized by \( \lambda \).

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1 The indices \( M = 0, 1, 2, 3, 4 \) and \( \mu = 0, 1, 2, 3 \) are the 5D and 4D Lorentz indices, respectively.

2 The mass dimensions of the quantities are \( [\Phi_G] = 0, [\phi] = 1, [m] = 1 \) and \( [\lambda] = 2 \), respectively.
The (tree-level) 5D propagator of $\Phi_G$ is given by a solution of

$$(-p^2 + \partial_y^2) G_G(y, y'; p) = -\frac{1}{M_5^3} \delta(y - y'), \quad (2.2)$$

where $p^2 \equiv p^\mu p_\mu$, and has the following properties.

$$G_G(y, y'; p) = G_G(y', y; p), \quad \lim_{y \to \pm \infty} |G_G(y, y'; p)| < \infty. \quad (2.3)$$

We can easily solve the above equation, and obtain

$$G_G(y, y'; p) = \frac{1}{2M_5^3} e^{-p|y-y'|}, \quad (2.4)$$

where $p \equiv \sqrt{p^\mu p_\mu}$. This tree-level propagator receives the quantum correction induced by the brane field $\phi$. The quantum-corrected propagator $G_G(y, y'; p)$ is given by

$$G_G(y, y'; p) = G_G(y, y'; p) + G_G(0, 0; p) \Sigma(p) G_G(0, y'; p) + \cdots$$

$$= \frac{e^{-p|y-y'|} - e^{-p(|y|+|y'|)}}{2M_5^3 p} + \frac{e^{-p(|y|+|y'|)}}{2M_5^3 p} \left\{ 1 + \frac{\Sigma(p)}{2M_5^3 p} + \left( \frac{\Sigma(p)}{2M_5^3 p} \right)^2 + \cdots \right\}$$

$$= \frac{e^{-p|y-y'|} - e^{-p(|y|+|y'|)}}{2M_5^3 p} + \frac{e^{-p(|y|+|y'|)}}{2M_5^3 p - \Sigma(p)}, \quad (2.5)$$

where $\Sigma(p)$ is the self-energy of $\Phi_G$. The one-loop contribution to the self-energy of $\Phi_G$ is

$$\Sigma_G^{1\text{loop}}(p) = \lambda^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q + p)^2 + m^2} \frac{1}{q^2 + m^2}. \quad (2.6)$$

Here we expand $\Sigma_G^{1\text{loop}}(p)$ in terms of the external momentum $p$.

$$\Sigma_G^{1\text{loop}}(p) = \Sigma_G^{(0)} + \Sigma_G^{(1)} p + \Sigma_G^{(2)} p^2 + O(p^3). \quad (2.7)$$

Then, we find that $\Sigma_G^{(0)}$ is logarithmically divergent, $\Sigma_G^{(1)}$ vanishes, and

$$\Sigma_G^{(2)} = -\frac{\lambda^2}{96\pi^2 m^2}. \quad (2.8)$$

Since $\Phi_G$ mimics the gravitational field, we neglect $\Sigma_G^{(0)}$. In the genuine gravity theory, it will vanish due to the invariance under the general coordinate transformation [1]. Thus,

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3 We have moved to the momentum basis for the 4D coordinates [11, 12].
the one-loop-corrected propagator between two sources located at the brane is calculated as
\[ G_{G}^{\text{1loop}}(0, 0; p) = \frac{1}{2M_{5}^{2}p - \Sigma_{G}^{(2)}p^{2} + \mathcal{O}(p^{3})} \] (2.9)

From this expression, we find that the “gravitational force” mediated by $\Phi_{G}$ behaves like four-dimensional for shorter distances than the crossover scale $r_{c}$, which is defined by
\[ r_{c} \equiv \frac{\mid \Sigma_{G}^{(2)} \mid}{2M_{5}^{2}} = \frac{\lambda^{2}}{192\pi^{2}M_{5}^{3}m^{2}}. \] (2.10)

Note that the “gravitational coupling” $\lambda$ has mass-dimension 2.

If all the mass scales are comparable, the crossover scale cannot be much larger than the compton length $m^{-1}$ (or the 5D Planck length $M_{5}^{-1}$). One simple way to enlarge $r_{c}$ is to introduce vast number of matter fields on the brane [13]. If the model has $N$ copies of $\phi$ that equally interact with $\Phi_{G}$, the crossover scale is enhanced by the factor $N$.
\[ r_{c} = \frac{N\lambda^{2}}{192\pi^{2}M_{5}^{3}m^{2}}. \] (2.11)

In fact, there are an infinite number of KK modes or stringy modes in compactified extra-dimensional models or the string theory. However, in these theories, higher modes have very large masses and thus their contribution to $r_{c}$ is suppressed. Besides, the coupling of each mode to $\Phi_{G}$ depends on the KK level, and those of higher modes are suppressed due to the rapidly oscillating profile in the extra dimension. Therefore, it is nontrivial whether $r_{c}$ is finite or explicitly depends on the cutoff of the theory. In this paper, we will answer this question in a simple toy model.

### 3 Thick-brane model

#### 3.1 Setup

We consider the following 5D model.
\[ \mathcal{L} = -\frac{M_{5}^{3}}{2} \partial^{M} \Phi_{G} \partial_{M} \Phi_{G} - \frac{1}{2} \partial^{M} \Phi_{b} \partial_{M} \Phi_{b} - \frac{W(y)}{2} \Phi_{b}^{2} - \lambda \Phi_{G} \Phi_{b}^{2}, \] (3.1)
where $\Phi_{G}$ and $\Phi_{b}$ are real scalar fields. As in the previous section, $\Phi_{G}$ mimics the gravitational field whose tensor structure is neglected, and the low-lying KK modes of $\Phi_{b}$
correspond to the brane-localized modes. Both of them are assumed to have vanishing backgrounds. The cubic coupling parametrized by $\lambda$ mimics the gravitational coupling.

The real function $W(y)$ represents the domain-wall background of some other scalar field. As a typical example, we assume that

$$W(y) = A^2 \tanh^2(By) + W_0,$$

where $A$, $B$ and $W_0$ are real constants, and we take $A$ and $B$ to be positive (see Appendix A). In this section, we neglect the warping of the spacetime induced by the domain wall. It will be considered in Sec. 4.4.

### 3.2 “Gravitational field” $\Phi_G$

The linearized equation of motion for $\Phi_G$ is

$$(\Box + \partial_y^2) \Phi_G = 0.$$

Since the extra dimension is not compactified, the KK expansion of $\Phi_G$ is

$$\Phi_G(x^\mu, y) = \int_0^\infty dm \, g_m(y) \chi_m(x^\mu),$$

where the mode function $g_m(y)$ is a solution of

$$\partial_y^2 g_m(y) = -m^2 g_m(y).$$

Namely, $g_m(y)$ is expressed by a linear combination of $\sin(my)$ and $\cos(my)$.

The (tree-level) 5D propagator $G_G(y, y'; p)$ satisfies the same equations as (2.2) with (2.3), and is given by (2.4).

### 3.3 “Brane field” $\Phi_b$

#### 3.3.1 KK expansion

The linearized equation of motion for $\Phi_b$ is

$$\partial^M \partial_M \Phi_b - W(y) \Phi_b = 0.$$ 

Thus, the KK expansion of $\Phi_b$ is

$$\Phi_b(x^\mu, y) = \sum_k f_k(y) \phi_k(x^\mu),$$

where $f_k(y)$ is a solution of

$$\partial_y^2 f_k(y) = -m^2 f_k(y).$$
where the mode functions $f_k(y)$ are solutions of the mode equations:

$$\{\partial_y^2 - W(y)\} f_k(y) = -m_k^2 f_k(y), \quad (3.8)$$

with the KK masses $m_k$.

With (3.2), the mode equation (3.8) is expressed as

$$\{\partial_y^2 + m_k^2 - A^2 \tanh^2(By) - W_0\} f_k(y) = 0. \quad (3.9)$$

Here we change the coordinate $y$ as

$$y \to s \equiv \tanh(By). \quad (3.10)$$

Then, the above equation becomes

$$\left[(1 - s^2)\partial_s^2 - 2s\partial_s + a(a + 1) - \frac{b^2}{1 - s^2}\right] f_k(s) = 0, \quad (3.11)$$

where $f_k(s) \equiv f_k(y)$, and

$$a \equiv -\frac{1}{2} + \sqrt{\frac{A^2}{B^2} + \frac{1}{4}}, \quad b \equiv \frac{\sqrt{A^2 + W_0 - m_k^2}}{B}. \quad (3.12)$$

This is the general Legendre equation, and thus its solution is expressed by a linear combination of the Legendre functions of the first and second kinds $P^b_a(s)$ and $Q^b_a(s)$. To describe the mode functions, it is more convenient to choose

$$u^b_a(s) \equiv (1 - s^2)^{b/2} F\left(\frac{a + b + 1}{2}, -\frac{a + b}{2}; \frac{1}{2}; s^2\right),$$

$$v^b_a(s) \equiv (1 - s^2)^{b/2} s F\left(\frac{a + b + 2}{2}, -\frac{a + b + 1}{2}; \frac{3}{2}; s^2\right), \quad (3.13)$$

as two independent solutions of (3.11), where $F(\alpha, \beta, \gamma; z) = _2F_1(\alpha, \beta; \gamma; z)$ is the hypergeometric function. The relations to $P^b_a(s)$ and $Q^b_a(s)$ are given by (B.12) in Appendix B.5.

Note that $u^b_a(s)$ and $v^b_a(s)$ are even and odd functions, respectively.

As shown in Appendix B.5, the functions $u^b_a(s)$ and $v^b_a(s)$ are even functions of the parameter $b$, and thus can be rewritten as

$$u^b_a(s) = (1 - s^2)^{-b/2} F\left(\frac{a - b + 1}{2}, -\frac{a - b}{2}; \frac{1}{2}; s^2\right), \quad (3.14)$$

$$v^b_a(s) = (1 - s^2)^{-b/2} s F\left(\frac{a - b + 2}{2}, -\frac{a - b + 1}{2}; \frac{3}{2}; s^2\right). \quad (3.15)$$

We should also note that the mode functions $f_k(y)$ should satisfy

$$\lim_{y \to \pm \infty} |f_k(y)| < \infty. \quad (3.16)$$
### 3.3.2 Localized modes

Let us first consider a case of $m_k^2 < A^2 + W_0$. In this case, both $a$ and $b$ are positive. Thus, from (3.14) and (3.15), the necessary condition to satisfy (3.16) is

$$0 = F\left(\frac{a - b + 1}{2}, \frac{-a - b}{2}, 1; \frac{1}{2}\right) = \frac{\sqrt{\pi} \Gamma(b)}{\Gamma(-a+b)\Gamma(\frac{a+b+1}{2})},$$

or

$$0 = F\left(\frac{a - b + 2}{2}, \frac{-a - b + 1}{2}, 3; \frac{1}{2}\right) = \frac{\sqrt{\pi} \Gamma(b)}{2\Gamma(-a+b+1)\Gamma(\frac{a+b+2}{2})}.\quad (3.17)$$

We have used (B.4) at the second equalities.

The condition (3.17) indicates that $(-a + b)/2$ is zero or a negative integer. In this case, $u_a^b(s)$ in (3.13) can be rewritten by using (B.14) as

$$u_a^b(s) = \frac{\sqrt{\pi} \Gamma(-b)}{\Gamma\left(\frac{-a+b}{2}\right)\Gamma\left(\frac{a-b+1}{2}\right)}(1 - s^2)^{b/2} F\left(\frac{a + b + 1}{2}, \frac{-a + b}{2}, b + 1; 1 - s^2\right).\quad (3.19)$$

We can see that $u_a^b(\pm 1)$ vanishes.

The condition (3.18) indicates that $(-a + b + 1)/2$ is zero or a negative integer. In this case, $v_a^b(s)$ is rewritten by using (B.15) as

$$v_a^b(s) = \frac{\sqrt{\pi} \Gamma(b)}{2\Gamma(-a+b+1)\Gamma\left(\frac{a-b+2}{2}\right)}(1 - s^2)^{b/2} s F\left(\frac{a + b + 2}{2}, \frac{-a + b + 1}{2}, b + 1; 1 - s^2\right).\quad (3.20)$$

Thus $v_a^b(\pm 1)$ vanishes.

For example, in the case of (A.6), the parameters become $a = 2$ and $b = \sqrt{4 - \hat{m}_k^2}$, where $\hat{m}_k \equiv m_k/B$. The allowed KK mass eigenvalues are

$$m_k^2 = k(4 - k)B^2,$$

where $k = 0, 1$. The corresponding mode functions are

$$f_0(y) = \mathcal{N}_0 u_2^2(\tanh(By)) = \frac{\mathcal{N}_0}{\cosh^2(By)},$$

$$f_1(y) = \mathcal{N}_1 v_2^1(\tanh(By)) = \frac{\mathcal{N}_1 \tanh(By)}{\cosh(By)},\quad (3.22)$$

where $\mathcal{N}_0 = \sqrt{3B}/2$ and $\mathcal{N}_1 = \sqrt{3B}/\sqrt{2}$ are the normalization constants. These are localized modes around the domain wall.

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4 When $b$ is a non-negative integer, $\frac{a+b}{2}$ is also a non-negative integer. Thus, $\Gamma(-b)/\Gamma(-\frac{a-b}{2})$ remains finite.
3.3.3 Bulk-propagating modes

Next we consider a case of \( m_k^2 \geq A^2 + W_0 \). In this case, \( b \equiv i\beta \) is pure imaginary. Since
\[
\left| (1 - s^2)^{-b/2} \right| = \left| e^{-\frac{b}{2} \ln(1-s^2)} \right| = 1,
\]
we find that
\[
\lim_{s^2 \to 1} \left| u^b_{a}(s) \right| = \left| F \left( \frac{a - i\beta + 1}{2}, \frac{-a - i\beta + 1}{2}; 1 \right) \right|,
\]
\[
\lim_{s^2 \to 1} \left| v^b_{a}(s) \right| = \left| F \left( \frac{a - i\beta + 2}{2}, \frac{-a - i\beta + 1}{2}; 1 \right) \right|,
\]
which are finite. Thus the condition (3.16) is always satisfied. This indicates that there is a continuous spectrum above \( m_k = \sqrt{A^2 + W_0} \). The mode functions are expressed as
\[
f_k(y) = C_{1k} u^b_{a}(\tanh(By)) + C_{2k} v^b_{a}(\tanh(By)) + \text{h.c.},
\]
where \( C_{1k} \) and \( C_{2k} \) are constants. Since \( \lim_{y \to \pm\infty} |f_k(y)| \neq 0 \), these modes are not localized around the domain wall. In fact, using (B.3), we can rewrite the above solutions in the form of
\[
\tilde{f}_k(s) = C_{k+}(s)(1 - s^2)^{b/2} + C_{k-}(s)(1 - s^2)^{-b/2} + \text{h.c.},
\]
where \( C_{k\pm}(s) \) depend on the parameters \( a \) and \( b \), and take finite values at \( s = \pm 1 \). Since
\[
(1 - s^2)^{\pm b/2} = \exp \left\{ \pm \frac{i\beta}{2} \ln(1 - s^2) \right\} = \exp \left\{ \mp i\beta \ln \cosh(By) \right\} \sim \exp \left\{ \mp i\beta B \text{sgn} \left( y \right) y \right\},
\]
for \( |y| \gg 1/B \), the above mode functions behave as plane-wave solutions at points far from the domain wall.

3.3.4 5D propagator

Here we derive the (tree-level) 5D propagator \( G_b(y, y'; p) \) for \( \Phi_b \) (see Appendix of Ref. [12] in the case of no domain wall background). It is defined as a solution of
\[
\left\{ -p^2 + \partial_y^2 - W(y) \right\} G_b(y, y'; p) = -\delta(y - y'),
\]
where \( p^2 = p^\mu p_\mu \), and \( p_\mu \) is the Euclidean 4D momentum, and has the following properties.
\[
G_b(y, y'; p) = G_b(y', y; p), \quad \lim_{y \to \pm\infty} G_b(y, y'; p) = 0.
\]

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5 Because \( \text{Re} \, b = 0 \) in this case, we cannot use the formula (B.4).

6 Note that when \( u^b_{a}(s) \) and \( v^b_{a}(s) \) are solutions of (3.11), so are \( \{ u^b_{a}(s) \}^\dagger \) and \( \{ v^b_{a}(s) \}^\dagger \).
In order to solve (3.27), it is convenient to decompose $G_b(y, y'; p)$ into the following two parts.

$$G_b(y, y'; p) = \partial(y - y') G_{b>}(y, y'; p) + \partial(y' - y) G_{b<}(y, y'; p).$$  

(3.29)

where $\partial(y)$ is the Heaviside step function, and

$$\{-p^2 + \partial_y^2 - W(y)\} G_{b\geq}(y, y'; p) = 0.$$  

(3.30)

Then, (3.28) is rewritten as

$$G_{b>}(y, y'; p) = G_{b<}(y, y'; p), \quad \lim_{y \to \infty} G_{b>}(y, y'; p) = \lim_{y \to -\infty} G_{b<}(y, y'; p) = 0.$$  

(3.31)

Besides, by integrating (3.27) for $y$ over the infinitesimal interval $[y' - \epsilon, y' + \epsilon]$, we obtain the following matching condition.

$$\{\partial_y G_{b>}(y, y'; p) - \partial_y G_{b<}(y, y'; p)\}|_{y'=y} = -1.$$  

(3.32)

Note that (3.30) is the same equation as (3.8) if we replace $-p^2$ with $m_k^2$. Thus, its solution can be expressed by a linear combination of $u_{a}^{b}(p) (\tanh(By))$ and $v_{a}^{b}(p) (\tanh(By))$, where

$$a \equiv -\frac{1}{2} + \sqrt{\frac{A^2}{B^2} + \frac{1}{4}}, \quad b(p) \equiv \sqrt{\frac{A^2 + W_0 + p^2}{B}}.$$  

(3.33)

Taking (3.31) into account, $G_{b\geq}(y, y'; p)$ are expressed as

$$G_{b>}(y, y'; p) = F_{a>}(y') F_{a>}^{(p)}(y),$$

$$G_{b<}(y, y'; p) = F_{a>}(y') F_{a<}^{(p)}(y).$$  

(3.34)

where $F_{a\geq}(s)$ is defined by

$$F_{a\geq}^{b}(y) \equiv \frac{R(a, b)}{\sqrt{2B}}u_{a}^{b}(\tanh(By)) + \frac{v_{a}^{b}(\tanh(By))}{\sqrt{2B}R(a, b)},$$

$$R(a, b) \equiv \left\{ \frac{\Gamma(a+b+1)}{2} \Gamma\left(\frac{a+b}{2}\right) \right\}^{1/2},$$  

(3.35)

so that they satisfy

$$\lim_{y \to \infty} F_{a>}(y) = \lim_{y \to -\infty} F_{a<}(y) = 0.$$  

(3.36)

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7 Since $u_{a}^{b}(s)$ and $v_{a}^{b}(s)$ are even and odd functions of $s$, we should note that $F_{a<}(y) = F_{a>}(y)$.

8 See (B.22) in Appendix B.5.
We can check that the condition (3.32) is satisfied by using the identity,
\[ u^b_a(s) \frac{du^b_a}{ds}(s) - v^b_a(s) \frac{dv^b_a}{ds}(s) = \frac{1}{1 - s^2}. \] (3.37)
As a result, the 5D propagator is expressed as
\[ G_b(y, y'; p) = \partial(y - y') \mathcal{F}_{a<}^{b(p)}(y') \mathcal{F}_{a>}^{b(p)}(y) + \partial(y' - y) \mathcal{F}_{a>}^{b(p)}(y') \mathcal{F}_{a<}^{b(p)}(y). \] (3.38)
From (3.8) and (3.27), the mode function \( f_k(y) \) is expressed as
\[ f_k(y) = \int_{-\infty}^{\infty} dy' \delta(y - y') f_k(y') \\
= -\int_{-\infty}^{\infty} dy' \left\{ -p^2 + \partial_y^2 - W(y') \right\} G_b(y, y'; p) f_k(y') \\
= -\int_{-\infty}^{\infty} dy' G_b(y, y'; p) \left\{ -p^2 + \partial_y^2 - W(y') \right\} f_k(y') \\
= (p^2 + m_k^2) \int_{-\infty}^{\infty} dy' G_b(y, y'; p) f_k(y'). \] (3.39)
We have performed the partial integral at the third equality. Thus, the 5D propagator is also expressed as
\[ G_b(y, y'; p) = \sum_k f_k(y) f_k(y'). \] (3.40)

4 Quantum correction to \( G_G \) and crossover scale

4.1 Quantum-corrected propagator

The quantum-corrected 5D propagator \( G_G(y, y'; p) \) is given by
\[ G_G(y, y'; p) = G_G(y, y'; p) + (G_G \cdot \Sigma_G \cdot G_G)(y, y'; p) \\
+ (G_G \cdot \Sigma_G \cdot G_G)(y, y'; p) + \cdots, \] (4.1)
where \( \Sigma_G(y, y'; p) \) is the self-energy of \( \Phi_G \), and the dot denotes the integral over the extra-dimensional coordinate. For example, the second term is explicitly written as
\[ (G_G \cdot \Sigma_G \cdot G_G)(y, y'; p) \equiv \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dy_2 G_G(y, y_1; p) \Sigma_G(y_1, y_2; p) G_G(y_2, y'; p). \] (4.2)

The 4D propagator between the sources on the brane is given by
\[ G_{4D}(y) = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dy' f_0(y) G_G(y, y'; p) f_0(y'). \] (4.3)
where \( f_0(y) \) denotes the mode function for the source localized on the domain-wall.

The one-loop contribution to \( \Sigma_G \) is expressed as

\[
\Sigma_G^{1\text{loop}}(y, y'; p) = \int \frac{d^4 q}{(2\pi)^4} \lambda^2 G_b(y, y'; q + p) G_b(y', y; q). \tag{4.4}
\]

Since the integrand only depends on \( q = \sqrt{q^2} \) and the angle \( \theta \) between the two vectors \( p^\mu \) and \( q^\mu \) and is independent of the other two angles, the integrals for the latter angles can be trivially performed. Hence (4.4) is rewritten as

\[
\Sigma_G^{1\text{loop}}(y, y'; p) = \lambda^2 \int_0^\infty dq \frac{q^3}{4\pi^3} \int_0^\pi d\theta \sin^2 \theta G_b(y, y'; q + p) G_b(y', y; q). \tag{4.5}
\]

### 4.2 4D momentum expansion of 5D propagator

Here we expand the 4D propagator \( G_{\text{4D}}(p) \) in terms of the magnitude of 4D (Euclidean) momentum \( p = \sqrt{p^\nu p_\nu} \).

The tree-level 5D propagator (2.4) is expanded as

\[
G_G(y, y'; p) = \frac{1}{2M_G^2 p} \left\{ G_G^{(0)}(y, y') + G_G^{(1)}(y, y') p + \mathcal{O}(p^2) \right\}, \tag{4.6}
\]

where

\[
G_G^{(0)}(y, y') \equiv 1, \quad G_G^{(1)}(y, y') \equiv -|y - y'|. \tag{4.7}
\]

The self-energy \( \Sigma_G^{1\text{loop}} \) in (4.5) is expanded as

\[
\Sigma_G^{1\text{loop}}(y, y'; p) = \lambda^2 \int_0^\infty dq \frac{q^3}{4\pi^3} \int_0^\pi d\theta \sin^2 \theta \times G_b(y, y'; q) + 2q^2 \partial_q G_b(y, y'; q) \cdot p \cos \theta \\
+ \left\{ \partial_q^2 G_b(y, y'; q) + 2q^2 \partial_q^2 G_b(y, y'; q) \cos^2 \theta \right\} p^2 + \mathcal{O}(p^3) \} G_b(y', y; q) \\
= \Sigma_G^{(0)}(y, y') + \Sigma_G^{(2)}(y, y') p^2 + \mathcal{O}(p^4), \tag{4.8}
\]

where the odd-power terms in \( p \) vanish after the \( \theta \)-integration, and

\[
\Sigma_G^{(0)}(y, y') \equiv \frac{\lambda^2}{8\pi^2} \int_0^\infty dq q^3 G_b^2(y, y'; q), \\
\Sigma_G^{(2)}(y, y') \equiv \frac{\lambda^2}{16\pi^2} \int_0^\infty dq G_b(y', y; q) \left\{ 2q^3 \partial_q^2 G_b(y, y'; q) + q^5 \partial_q^2 G_b(y, y'; q) \right\}. \tag{4.9}
\]

As we did in Sec. 2, we will neglect \( \Sigma_G^{(0)} \) in the following. In the genuine gravity theory, the \( p \)-independent term vanishes due to the invariance under the general coordinate transformation.
Therefore, the 4D propagator (4.3) is expanded as

\[ G_{4D}(p) = f_0 \cdot (G_G + G_G \cdot \Sigma_G \cdot G_G + \cdots) \cdot f_0 = \frac{f_0 \cdot G_G^{(0)} \cdot f_0}{2M_5^2 p} + f_0 \cdot \left( \frac{G_G^{(1)}}{2M_5^3} + \frac{G_G^{(0)} \cdot \Sigma_G^{(2)} \cdot G_G^{(0)}}{4M_5^6} \right) \cdot f_0 + \mathcal{O}(p), \]  

(4.10)

where the dot denotes the \( y \)-integral like (4.2).

### 4.3 Crossover scale

The crossover scale \( r_c \) is read off as the ratio of the expansion coefficients of the first two terms in (4.10). (See (2.5) and (2.10).)

\[ r_c = \left| \frac{1}{f_0 \cdot G_G^{(0)} \cdot f_0} \times f_0 \cdot \left( \frac{G_G^{(1)}}{2M_5^3} + \frac{G_G^{(0)} \cdot \Sigma_G^{(2)} \cdot G_G^{(0)}}{2M_5^3} \right) \cdot f_0 \right|. \]  

(4.11)

Since \( \Sigma_G^{(2)}(y, y') \) is induced by a quantum correction induced by the \( \Phi_b \)-loop, we are interested in the case that the second term in the parentheses dominates. Namely, \( r_c \) becomes

\[ r_c \simeq \left| \frac{f_0 \cdot G_G^{(0)} \cdot \Sigma_G^{(2)} \cdot G_G^{(0)} \cdot f_0}{2M_5^3 \left( f_0 \cdot G_G^{(0)} \cdot f_0 \right)} \right|. \]  

(4.12)

In the following, we assume that the zero-mode function \( f_0(y) \) is given by that in (3.22). Then, we can calculate

\[ f_0 \cdot G_G^{(0)} \cdot f_0 = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dy' \ f_0(y)G_G^{(0)}(y, y')f_0(y') = \left\{ \int_{-\infty}^{\infty} dy \ f_0(y) \right\}^2 = \frac{4N_0^2}{B^2}. \]  

(4.13)

From (3.38), the integrands in (4.9) are expressed by

\[ I_b^{(1)}(y, y'; q) \equiv G_b(y', y; q)\partial_q G_b(y, y'; q) \]

\[ = \vartheta(y - y') \left\{ A^{(0)}_<(y)A^{(1)}_>(y') + A^{(1)}_<(y)A^{(0)}_>(y') \right\} + \vartheta(y' - y) \left\{ A^{(0)}_<(y)A^{(1)}_>(y') + A^{(1)}_<(y)A^{(0)}_>(y') \right\}, \]

\[ I_b^{(2)}(y, y'; q) \equiv G_b(y', y; q)\partial_q^2 G_b(y, y'; q) \]

\[ = \vartheta(y - y') \left\{ A^{(0)}_<(y)A^{(2)}_>(y') + 2A^{(1)}_<(y)A^{(1)}_>(y') + A^{(2)}_<(y)A^{(0)}_>(y') \right\} + \vartheta(y' - y) \left\{ A^{(0)}_<(y)A^{(2)}_>(y') + 2A^{(1)}_<(y)A^{(1)}_>(y') + A^{(2)}_<(y)A^{(0)}_>(y') \right\}, \]

(4.14)
The integrand in the right-hand-side is calculated as

\[ A_{\geq}^{(n)}(y) \equiv f^{b(q)}(y) \partial_{q^2}f^{b(q)}(y). \quad (n = 0, 1, 2) \]  (4.15)

Then, \( \Sigma_{G}^{(2)}(y, y') \) in \((4.9)\) is expressed as

\[ \Sigma_{G}^{(2)}(y, y') = \frac{\lambda^2}{16\pi^2} \int_0^\infty dq \left\{ 2q^3 I_b^{(1)}(y, y'; q) + q^5 I_b^{(2)}(y, y'; q) \right\}. \]  (4.16)

Thus, the numerator in \((4.12)\) is

\[ f_0 \cdot G_G^{(0)} \cdot \Sigma_{G}^{(2)} \cdot G_G^{(0)} \cdot f_0 = \frac{\lambda^2}{16\pi^2} \int_0^\infty dq \left\{ f_0 \cdot G_G^{(0)} \cdot \big( 2q^3 I_b^{(1)} + q^5 I_b^{(2)} \big) \right\} \cdot G_G^{(0)} \cdot f_0 , \]  (4.17)

The integrand in the right-hand-side is calculated as

\[
\left( f_0 \cdot G_G^{(0)} \cdot I_b^{(1)} \cdot G_G^{(0)} \cdot f_0 \right)(q) = \left( \int_{-\infty}^\infty dy f_0(y) \right)^2 \int_{-\infty}^\infty \int_{-\infty}^\infty dy' dy'' I_b^{(1)}(y', y''; q) \\
= \frac{8N_0^2}{B^2} \left\{ J_b^{(0,1)}(q) + J_b^{(1,0)}(q) \right\}, \\
\left( f_0 \cdot G_G^{(0)} \cdot I_b^{(2)} \cdot G_G^{(0)} \cdot f_0 \right)(q) = \frac{8N_0^2}{B^2} \left\{ J_b^{(0,2)}(q) + 2J_b^{(1,1)}(q) + J_b^{(2,0)}(q) \right\},
\]

where

\[ J_b^{(n,l)}(q) \equiv \int_{-\infty}^\infty dy A_{\geq}^{(n)}(y) \int_{-\infty}^y dy' A_{\leq}^{(l)}(y'). \]  (4.19)

We have used that \( \int_{-\infty}^\infty dy \int_{y}^\infty dy' = \int_{-\infty}^\infty dy' \int_{-\infty}^y dy \). Since \( A_{\geq}^{(n)}(y') = A_{\leq}^{(n)}(-y') \) and \( \int_{-\infty}^\infty dy \int_{-\infty}^y dy' = \int_{-\infty}^\infty d(-y') \int_{-\infty}^{-y'} d(-y) \), we can see that

\[ J_b^{(n,l)}(q) = J_b^{(l,n)}(q). \]  (4.20)

Therefore, \((4.18)\) can be simplified as

\[ f_0 \cdot G_G^{(0)} \cdot \Sigma_{G}^{(2)} \cdot G_G^{(0)} \cdot f_0 = \frac{\lambda^2 N_0^2}{\pi^2 B^2} \int_0^\infty dq \left[ 2q^3 J_b^{(0,1)}(q) + q^5 \left\{ J_b^{(0,2)}(q) + J_b^{(1,1)}(q) \right\} \right]. \]  (4.21)

As a result, the crossover scale is expressed as

\[ r_c \simeq \frac{\lambda^2}{8\pi^2 M_5^2} \left| \int_0^\infty dq \left[ 2q^3 J_b^{(0,1)}(q) + q^5 \left\{ J_b^{(0,2)}(q) + J_b^{(1,1)}(q) \right\} \right] \right|. \]  (4.22)

In order to see whether this quantity is finite or not, the asymptotic behavior of the integrand for large \( q \) is important. From \((3.29)\), we find that

\[ A_{\geq}^{(n)}(y) \simeq \left( \frac{-y}{2q} \right)^n e^{-2qy} \frac{e^{2qy}}{2q}, \]  (4.23)

\[ \text{Notice that all the } q\text{-dependence of } I_b^{(1,2)} \text{ comes from } b(q), \text{ which is a function of } q^2. \]
for $|q| \gg B$. This indicates that each $J_b^{(n,l)}(q)$ diverges. This divergence comes from the fact that higher KK modes propagate in the bulk just like plane waves and the extra dimension has an infinite volume. However, a domain wall generically warps the ambient space, and its volume can be finite if we work in a gravitational theory. In the next subsection, we will see that $J_b^{(n,l)}(q)$ become finite in the warped geometry.

### 4.4 Warped geometry

In the presence of the domain wall, $\Phi_{\text{wall}}(y) = v \tanh(By)$, the spacetime is warped with the metric

$$ds^2 = a_w^2(y) \eta_{\mu\nu} dx^\mu dx^\nu + dy^2,$$

where the warp factor $a_w(y)$ is given by \[14, 15\]

$$a_w(y) = \cosh^{-2\beta}(By) \exp \left\{ -\frac{\beta}{2} \tanh^2(By) \right\}, \quad \beta \equiv \frac{v^2}{9M_5^3}.$$ \hspace{1cm} (4.26)

We are interested in the asymptotic behaviors of the 5D propagator $G_b(y, y'; q)$ for $q \gg B$ in a region far from the domain wall. In such a region, the above background geometry is approximated as the Randall-Sundrum spacetime \[16\] with the warp factor

$$a_w^{\text{RS}}(y) = e^{-k|y|}, \quad k \equiv 2\beta B.$$ \hspace{1cm} (4.27)

We have rescaled the 4D coordinates $x^\mu$ so that $a_w^{\text{RS}}(0) = 1$. Then, (3.27) is modified as \[12\]

$$\{ -e^{2k|y|} p^2 + \partial_y^2 - 4k \text{sgn}(y) \partial_y - W_\infty \} G_b(y, y'; p) = -e^{4k|y|} \delta(y - y'),$$

where $W_\infty \equiv A^2 + W_0$. With the conditions in (3.28), this is solved as

$$G_b(y, y'; p) = \vartheta(y - y') F_<(y'; p) F_>(y; p) + \vartheta(y' - y) F_>(y'; p) F_<(y; p),$$

where

$$F_>(y; p) \equiv \begin{cases} \frac{e^{2ky}}{\sqrt{H(\tilde{p})}} K_\nu(\tilde{p}e^{-ky}) & (y > 0) \\ \sqrt{kH(\tilde{p})} e^{-2ky} \left[ I_\nu(\tilde{p}e^{-ky}) - \frac{1}{K_\nu(\tilde{p})} \left\{ K_\nu(\tilde{p}) H(\tilde{p}) + I_\nu(\tilde{p}) \right\} K_\nu(\tilde{p}e^{-ky}) \right] & (y < 0) \end{cases},$$

$$F_<(y; p) \equiv F_>(-y; p),$$

$$H(\tilde{p}) \equiv -2K^2_\nu(\tilde{p}) - 2\tilde{p}K'_\nu(\tilde{p})K_\nu(\tilde{p}) = -2K^2_\nu(\tilde{p}) + \tilde{p}K_\nu(\tilde{p}) \{ K_{\nu+1}(\tilde{p}) + K_{\nu-1}(\tilde{p}) \}.$$ 

(4.30)
Here, \( \nu \) and \( \tilde{p} \) are defined by
\[
\nu \equiv \sqrt{4 + \frac{W_\infty}{k}}, \quad \tilde{p} \equiv \frac{p}{k},
\]
(4.31)
and \( I_\nu(z) \) and \( K_\nu(z) \) are the modified Bessel functions of the first and second kinds, respectively.

Now we focus on the case that \( p \gg k \). Using (B.8), we find that \( F_\nu(y; p) \) behaves as
\[
F_\nu(y; p) \simeq \begin{cases} 
\frac{1}{\sqrt{2p}} \exp \left\{ \frac{\tilde{p}(1 - e^{ky}) + \frac{3}{2} ky}{2} \right\} & (y > 0) \\
\frac{1}{\sqrt{2p}} \exp \left\{ -\frac{\tilde{p}(1 - e^{-ky}) - \frac{3}{2} ky}{2} \right\} & (y < 0)
\end{cases}.
\]
(4.32)
In the flat limit \( k \to 0 \), this is reduced to
\[
F_\nu(y; p) \simeq \frac{e^{-py}}{\sqrt{2p}},
\]
(4.33)
which agrees with (B.28) that leads to (4.23).

Here we define the new coordinate \( Y \) as
\[
Y \equiv \text{sgn}(y) \frac{e^{k|y|} - 1}{k}.
\]
(4.34)
Then, (4.32) is rewritten as
\[
\tilde{F}_\nu(Y; p) \equiv F_\nu(y; p) \simeq \frac{e^{-pY}}{\sqrt{2p}} (k |Y| + 1)^{3/2}.
\]
(4.35)
This leads to
\[
\tilde{A}_\nu^{(0)}(Y; q) \equiv \tilde{F}_\nu^2(Y; q) \simeq \frac{e^{-2qY}}{2q} (k |Y| + 1)^3,
\]
\[
\tilde{A}_\nu^{(1)}(Y; q) \equiv \tilde{F}_\nu(Y; q) \partial_q \tilde{F}_\nu(Y; q) \simeq \left( -\frac{Y}{2q} - \frac{1}{4q^2} \right) \frac{e^{-2qY}}{2q} (k |Y| + 1)^3,
\]
\[
\tilde{A}_\nu^{(2)}(Y; q) \equiv \tilde{F}_\nu(Y; q) \partial_q^2 \tilde{F}_\nu(Y; q) \simeq \left( \frac{Y^2}{4q^2} + \frac{Y}{2q^3} + \frac{5}{16q^4} \right) \frac{e^{-2qY}}{2q} (k |Y| + 1)^3.
\]
(4.36)
Using these quantities, \( J_b^{(n,l)}(q) \) defined in (4.19) is now modified as
\[
J_b^{(n,l)}(q) = \int_{-\infty}^{\infty} dy \sqrt{-G} \tilde{A}_\nu^{(n)}(y) \int_{-\infty}^{y} dy' \sqrt{-G} \tilde{A}_\nu^{(l)}(y')
\]
\[
\simeq \int_{-\infty}^{\infty} \frac{dY}{(k |Y| + 1)^5} \tilde{A}_\nu^{(n)}(Y; q) \int_{-\infty}^{Y} \frac{dY'}{(k |Y'| + 1)^5} \tilde{A}_\nu^{(l)}(Y'; q),
\]
(4.37)
where $\sqrt{-G} = a_w^4(y) \simeq e^{-4k|y|} = (k|Y| + 1)^{-4}$. Using (B.10), these integrals are approximated as

\begin{align*}
J_b^{(1,0)}(q) &= J_b^{(0,1)}(q) \simeq -\frac{1}{32kq^5} + \mathcal{O}(\tilde{q}^{-7}), \\
J_b^{(2,0)}(q) &= J_b^{(0,2)}(q) \simeq \frac{1}{48k^3q^5} + \frac{19}{480kq^7} + \mathcal{O}(\tilde{q}^{-8}), \\
J_b^{(1,1)}(q) &\simeq -\frac{1}{48k^3q^5} + \frac{1}{80kq^7} + \mathcal{O}(\tilde{q}^{-8}).
\end{align*}

Namely, we have

\begin{align*}
2q^2J_b^{(0,1)}(q) + q^5 \left\{ J_b^{(0,2)}(q) + J_b^{(1,1)}(q) \right\} = -\frac{1}{96kq^2} + \mathcal{O}(\tilde{q}^{-3}).
\end{align*}

As a result, the integral in (4.22) now converges. Since the approximate expression (4.32) is valid when $q \gg k$, and the integral for that region is dominant when $\beta$ in (4.31) is small enough, the crossover scale is estimated as

\begin{align*}
r_c &\sim \frac{\lambda^2}{8\pi^2 M_5^3} \int_k^\infty dq \frac{1}{96kq^2} = \frac{\lambda^2}{768\pi^2 M_5^3 k^2} = \frac{27M_5^3 \lambda^2}{1024\pi^2 v^4 B^2}.
\end{align*}

In the flat geometry limit ($k \to 0$), this diverges as we saw in the previous subsection. However, when we take into account the warping effect of the geometry by the domain wall, we have a finite value of $r_c$. Namely, we have to allow a hierarchy among the parameters in order to obtain the crossover scale $r_c$ that is larger than the present Hubble radius $\sim 10^{26}$ m, even after summing up contributions from an infinite number of KK modes. For example, in the case that the bulk gravitational scale is $M_5 \sim \sqrt{\lambda} \sim 10^{11}$ GeV, the domain wall scale, which is characterized by $v^{2/3}$ and $B$, has to be less than $10^4$ GeV.

5 Nontrivial dilaton background

In Refs. [17, 18], the authors proposed a mechanism to enlarge the crossover scale $r_c$. The idea is to allow the coefficient of the kinetic term for $\Phi_G$ in (3.1) to have a nontrivial $y$-dependence.

\begin{align*}
\mathcal{L} = -\frac{K(y)}{2} \partial^M \Phi_G \partial_M \Phi_G + \cdots,
\end{align*}

where $K(y)$ is a real function that satisfies

\begin{align*}
\lim_{y \to \pm \infty} K(y) = M_5^3, \quad K(0) \ll M_5^3.
\end{align*}
The function \( K(y) \) is understood as a background field configuration of the dilaton.

In Ref. \[18\], the thin-wall limit (i.e., \( B \to \infty \)) is considered, and \( K(y) \) is assumed to take a tiny (positive) value \( \epsilon M_5^3 \) (\( \epsilon \ll 1 \)) on the brane. After the canonical normalization of \( \Phi_G \), its coupling to \( \Phi_b \) is rescaled as \( \lambda / \sqrt{K(y)} \). Since the quantum-induced 4D Planck mass is proportional to \( \lambda / \sqrt{K(0)} \), the crossover scale \( r_c \) is enhanced by a factor \( 1 / \epsilon \).

However, we have to be more careful to discuss this enhancement because they did not consider the internal structure of the brane and the profile of the “gravitational field” \( \Phi_G \) there. To illustrate the situation, we assume that

\[
K(y) = M_5^3 \tanh(B_K y),
\]

where \( B_K \) is a positive constant.

Since the behavior of the propagator around the brane is essential in this mechanism, in this section, we neglect the warping of the spacetime, which mainly affects the behavior at positions far from the brane. On the nontrivial dilaton background, \((2.2)\) is modified as

\[
\left( -p^2 + \partial_y^2 - \frac{K''(y)}{2K(y)} + \frac{K'^2(y)}{4K^2(y)} \right) \tilde{G}_G(y, y'; p) = -\frac{1}{M_5^3} \delta(y - y'),
\]

where \( \tilde{G}_G(y, y'; p) \) is the 5D propagator of the canonically normalized field \( \tilde{\Phi}_G \equiv \sqrt{K(y)} \Phi_G \). We can solve this equation by the technique that we did in Sec. \[3.3.4\] and find that

\[
\tilde{G}_G(y, y'; p) = \vartheta(y - y') \tilde{G}_G^\vartheta(y) \tilde{G}_G^\vartheta(y') + \vartheta(y' - y) \tilde{G}_G^\vartheta(y') \tilde{G}_G^\vartheta(y),
\]

where \( \hat{p} \equiv p/B_K \), and

\[
\tilde{G}_G^\vartheta(y) \equiv \frac{R(1, \hat{p})}{\sqrt{2B_K M_5^3}} u_1^\vartheta(\tanh(B_K y)) \mp \frac{v_1^\vartheta(\tanh(B_K y))}{\sqrt{2B_K M_5^3} R(1, \hat{p})},
\]

which is similar to \( F_{a\vartheta}^b(y) \) defined by \((3.35)\).

As shown in Appendix \[B.5\] \( u_1^\vartheta(s) \) and \( v_1^\vartheta(s) \) are even functions of \( \hat{p} \). Thus, they are expanded as

\[
u_1^\vartheta(\tanh(B_K y)) = (1 - \tanh^2(B_K y))^{\hat{p}/2} F \left( \frac{2 + \hat{p}}{2}, \frac{-1 + \hat{p}}{2}, \frac{1}{2}; \tanh^2(B_K y) \right)
= 1 - B_K y \tanh(B_K y) + \mathcal{O} (\hat{p}^2),
\]

\[
v_1^\vartheta(\tanh(B_K y)) = (1 - \tanh^2(B_K y))^{\hat{p}/2} \tanh(B_K y) F \left( \frac{3 + \hat{p}}{2}, \frac{\hat{p}}{2}, \frac{3}{2}; \tanh^2(B_K y) \right)
= \tanh(B_K y) + \mathcal{O} (\hat{p}^2).
\]
Here we have used that

\[
F\left(1, -\frac{1}{2}; \frac{1}{2}; \tanh(B_K y)\right) = 1 - B_K y \tanh(B_K y),
\]

\[
F\left(\frac{3}{2}, 0; \frac{3}{2}; \tanh(B_K y)\right) = 1.
\]

(5.8)

Since

\[
R(1, \hat{p}) = \left\{ \frac{\Gamma(\frac{2+\hat{p}}{2})\Gamma(\frac{-1+\hat{p}}{2})}{2\Gamma(\frac{3+\hat{p}}{2})\Gamma(\frac{\hat{p}}{2})} \right\}^{1/2} = \frac{i\sqrt{\hat{p}}}{\sqrt{2B_K M_5^2}} \left\{ 1 + \mathcal{O}(\hat{p}^2) \right\},
\]

(5.9)

the function \( G_{\hat{p}}(y) \) is expanded as

\[
G_{\hat{p}}(y) = \left\lfloor \frac{i\sqrt{\hat{p}}}{\sqrt{2B_K M_5^2}} \left\{ 1 - B_K y \tanh(B_K y) \right\} - \frac{\tanh(B_K y)}{i\sqrt{2B_K M_5^2}} \right\} \left\{ 1 + \mathcal{O}(\hat{p}^2) \right\}. (5.10)
\]

Thus, the 5D propagator \( \tilde{G}_G(y, y'; p) \) is expanded as

\[
\tilde{G}_G(y, y'; p) = \frac{1}{2M_5^2 p} \left\{ \tilde{G}_G^{(0)}(y, y') + \tilde{G}_G^{(1)}(y, y') p + \mathcal{O}(p^2) \right\}, (5.11)
\]

where

\[
\tilde{G}_G^{(0)}(y, y') \equiv \tanh(B_K y') \tanh(B_K y),
\]

\[
\tilde{G}_G^{(1)}(y, y') \equiv - \frac{\left| \tanh(B_K y) - \tanh(B_K y') \right|}{B_K} - |y - y'| \tanh(B_K y') \tanh(B_K y).
\]

(5.12)

The propagator of the “brane field” \( \Phi_b \), \( G_b(y, y'; p) \), is unchanged from that in the previous section. Hence, the one-loop self-energy \( \tilde{\Sigma}_G(y, y'; p) \) of \( \Phi_G \) is now given by

\[
\tilde{\Sigma}_G(y, y'; p) = \int \frac{d^4 q}{(2\pi)^4} \frac{\lambda}{\sqrt{K(y)}} \frac{\lambda}{\sqrt{K(y')}} G_b(y, y'; q + p) G_b(y', y; q) = \tilde{\Sigma}_G^{(0)}(y, y') + \tilde{\Sigma}_G^{(2)}(y, y') p^2 + \mathcal{O}(p^4), (5.13)
\]

where

\[
\tilde{\Sigma}_G^{(0)}(y, y') \equiv \frac{\lambda^2}{8\pi^2 \sqrt{K(y)K(y')}} \int_0^\infty dq q^3 G_b^2(y, y'; q),
\]

\[
\tilde{\Sigma}_G^{(2)}(y, y') \equiv \frac{\lambda^2}{16\pi^2 \sqrt{K(y)K(y')}} \int_0^\infty dq G_b(y', y; q) \left\{ 2q^3 \partial_q^2 G_b(y, y'; q) + q^5 \partial^2_q G_b(y, y'; q) \right\}.
\]

(5.14)
Note that these quantities are enhanced around the brane $y = 0$, as expected. Again, we drop the contribution of $\tilde{\Sigma}_G^{(0)}(y, y')$. Then, the crossover scale $r_c$ is obtained just in the same way as (4.12).

$$r_c \simeq \left| \frac{f_0 \cdot \tilde{G}_G^{(0)} \cdot \tilde{\Sigma}_G^{(2)} \cdot \tilde{G}_G^{(0)} \cdot f_0}{2M_5^2 (f_0 \cdot \tilde{G}_G^{(0)} \cdot f_0)} \right|. \quad (5.15)$$

Using (5.12) and (5.14), we have

$$f_0 \cdot \tilde{G}_G^{(0)} \cdot f_0 = \left\{ \int_{-\infty}^{\infty} dy \ f_0(y) \tanh(B_K y) \right\}^2,$$

$$f_0 \cdot \tilde{G}_G^{(0)} \cdot \tilde{\Sigma}_G^{(2)} \cdot \tilde{G}_G^{(0)} \cdot f_0 = \left\{ \int_{-\infty}^{\infty} dy \ f_0(y) \tanh(B_K y) \right\}^2 \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dy_1dy_2 \ \tanh(B_K y_1) \tilde{\Sigma}_G^{(2)}(y_1, y_2) \tanh(B_K y_2). \quad (5.16)$$

Note that the enhancement factor $1/\sqrt{K(y)}$ contained in $\tilde{\Sigma}_G^{(2)}$ is exactly cancelled by $\tanh(B_K y)$ coming from $\tilde{G}_G^{(0)}$. Therefore, the resultant expression of $r_c$ is the same as (4.22). Namely, this enhancement mechanism does not work.

In fact, this result is independent of the detailed function form of $K(y)$. Since

$$\left( \partial_y^2 - \frac{K''}{2K} + \frac{K'^2}{4K^2} \right) \sqrt{K(y)} = 0, \quad (5.17)$$

the leading term of the expansion (5.11) has the form of

$$\tilde{G}_G^{(0)}(y, y') = \frac{\sqrt{K(y)K(y')}}{M_5^3}.$$  \hspace{1cm} (5.18)

Thus, the enhancement factor $1/\sqrt{K(y)}$ in $\tilde{\Sigma}_G^{(2)}(y, y')$ is always cancelled when we calculate the crossover scale $r_c$.

### 6 Summary

We have discussed the contribution of an infinite number of the KK modes to the brane-induced force when the brane is given by a domain wall in an uncompactified 5D theory. In particular, we estimated the crossover scale $r_c$ from 4D to 5D for the induced force, and clarified whether $r_c$ is enhanced to be of the order of the Hubble radius by that contribution.
Taking into account the warping of the fifth-dimensional space by the domain wall, \( r_c \) becomes finite. So we cannot realize a large value of \( r_c \) without introducing a hierarchy among the model parameters.

The parameters in our model can be categorized into the “gravitational” ones \( \{ \lambda, M_5 \} \) and the ones relevant to the brane physics \( \{ m, A, B, v, B_K \} \). If we assume that the parameters in each class are of the same order of magnitude, the crossover scale is roughly estimated as

\[
 r_c \simeq \begin{cases} 
 10^{-3} \times \left( \frac{M_5}{m_b} \right)^2 \times l_5 & \text{(single brane mode)} \\
 10^{-3} \times \left( \frac{M_5}{m_b} \right)^8 \times l_5 & \text{(including KK modes)}
\end{cases} 
\]

where \( l_5 \equiv 1/M_5 \) is the 5D Planck length, and \( m_b \) denotes the mass scale of the brane physics. Thus, if we admit some hierarchy between \( M_5 \) and \( m_b \), the contribution of all the KK modes makes it easier to realize a phenomenologically viable size of \( r_c \).

The authors of Refs. [17, 18] proposed a mechanism to enlarge \( r_c \) by introducing a nontrivial dilaton background. After the canonical normalization of the “gravitational field” \( \Phi_G \), the “gravitational couplings” to the brane modes are enhanced in this case. However, this nontrivial background also affects the propagator of \( \Phi_G \), and in fact, this effect exactly cancels the enhancement of the above couplings. Hence this mechanism does not work in the 5D theory.

In higher-dimensional theories, the situation may be different. Since we have more KK modes and less warping effects on the ambient geometry, we expect that \( r_c \) explicitly depends on the cutoff scale of the theory. Besides, the nontrivial dilaton background may enlarge \( r_c \) because the behavior of the propagator near the brane is quite different from the 5D case in such a higher-dimensional theory, and thus the cancellation of the enhancement factor will not occur. In order to clarify these points, we need to extend our study to higher dimensions. We will discuss these issues in separate papers.

## A Domain-wall sector

Here we provide specific examples of the “brane field” \( \Phi_b \), and show their couplings have the form of (3.1) with (3.2).
A.1 Domain-wall background

The simplest model that has a domain-wall background is

\[ \mathcal{L}_X = -\frac{1}{2} \partial^M X \partial_M X + \frac{m_X^2}{2} X^2 - \frac{\lambda_X}{4} X^4, \tag{A.1} \]

where \( X \) is a real scalar field, and \( m_X, \lambda_X > 0 \). The equation of motion is

\[ \partial^M \partial_M X + m_X^2 X - \lambda_X X^3 = 0. \tag{A.2} \]

and thus, there are two degenerate vacua \( \langle X \rangle = \pm m_X / \sqrt{\lambda_X} \). When we take the boundary conditions \( \lim_{y \to \pm \infty} \langle X \rangle = \pm m_X / \sqrt{\lambda_X} \), this model has the domain wall solution

\[ X_{bg}(y) = \frac{m_X}{\sqrt{\lambda_X}} \tanh \left( \frac{m_X}{\sqrt{2}} y \right). \tag{A.3} \]

A.2 Fluctuation modes around the domain wall

Around this background, the scalar field \( X \) is divided as

\[ X = X_{bg} + \tilde{X}, \tag{A.4} \]

where \( \tilde{X} \) is the fluctuation part. Plugging this into (A.1), we obtain

\[ \mathcal{L}_X = -\frac{1}{2} \partial^M \tilde{X} \partial_M \tilde{X} + \left( \frac{m_X^2}{2} - \frac{3\lambda_X}{2} X_{bg}^2 \right) \tilde{X}^2 - \lambda_X X_{bg} \tilde{X}^3 - \frac{\lambda_X}{4} \tilde{X}^4 \]

\[ = -\frac{1}{2} \partial^M \tilde{X} \partial_M \tilde{X} - \frac{1}{2} \left\{ -m_X^2 + 3m_X^2 \tanh^2 \left( \frac{m_X}{\sqrt{2}} y \right) \right\} \tilde{X}^2 + \mathcal{O}(\tilde{X}^3), \tag{A.5} \]

where we have dropped the \( \tilde{X} \)-independent term and the total derivative term. Thus, at the quadratic order, the Lagrangian for \( \tilde{X} \) has the same form as \( \Phi_b \) in (3.1) if we identify \( A \) and \( B \) in (3.2) as

\[ A = \sqrt{3} m_X, \quad B = \frac{m_X}{\sqrt{2}}, \quad W_0 = -m_X^2. \tag{A.6} \]

A.3 Matter field coupled to the domain wall

Next we consider a matter field coupled to the domain-wall field \( X \) as

\[ \mathcal{L}_Q = -\frac{1}{2} \partial^M Q \partial_M Q - \frac{m_Q^2}{2} Q^2 - \frac{\lambda_X Q^2}{2} X^2, \tag{A.7} \]

\[ \]
where the matter field $Q$ is a real scalar, and $m_Q, \lambda x > 0$. Plugging (A.4) into this, we obtain

$$\mathcal{L}_Q = -\frac{1}{2} \partial^M Q \partial_M Q - \frac{1}{2} \left\{ m_Q^2 + \frac{\lambda x}{\lambda} m_X^2 \tanh^2 \left( \frac{m_X}{\sqrt{2}} y \right) \right\} Q^2, \quad (A.8)$$

at the quadratic order in $Q$. This has the same form as (3.1) if we identify $\Phi_b = Q$ and the constants in (3.2) as

$$A = \sqrt{\frac{\lambda x}{\lambda}} m_X, \quad B = \frac{m_X}{\sqrt{2}}, \quad W_0 = m_Q^2. \quad (A.9)$$

\section*{B Various properties of special functions}

\subsection*{B.1 General properties of hypergeometric functions}

The hypergeometric function $F(\alpha, \beta; \gamma; z) = \ _2F_1(\alpha, \beta; \gamma; z)$ is defined by

$$F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + n)\Gamma(\beta + n)}{\Gamma(\gamma + n)} z^n$$

$$= 1 + \frac{\alpha \beta}{\gamma} z + \frac{\alpha(\alpha + 1)\beta(\beta + 1)}{2\gamma(\gamma + 1)} z^2 + \mathcal{O}(z^3), \quad (B.1)$$

where $\Gamma(z)$ is the gamma function, and satisfies

$$[z(1 - z) \partial_z^2 + \{\gamma - (\alpha + \beta + 1)z\} \partial_z - \alpha \beta] F(\alpha, \beta; \gamma; z) = 0. \quad (B.2)$$

This function can be rewritten as

$$F(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)} (1 - z)^{\gamma - \alpha - \beta} F(\gamma - \alpha, \gamma - \beta, \gamma - \alpha - \beta + 1; 1 - z)$$

$$+ \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} F(\alpha, \beta; \alpha + \beta - \gamma + 1; 1 - z). \quad (B.3)$$

We also have the formula:

$$F(\alpha, \beta, \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}, \quad (B.4)$$

for $\text{Re}\, \gamma > 0$ and $\text{Re}\, (\gamma - \alpha - \beta) > 0$.

The derivative of $F(\alpha, \beta, \gamma; z)$ in terms of $z$ is given by

$$\partial_z F(\alpha, \beta, \gamma; z) = \frac{\alpha \beta}{\gamma} F(\alpha + 1, \beta + 1, \gamma + 1; z). \quad (B.5)$$
B.2 Gamma function

The gamma function $\Gamma(z)$ satisfies that

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\pi\alpha)},$$
$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma(z + \frac{1}{2}),$$
$$\lim_{z \to \infty} \frac{\Gamma(z + \alpha)}{z^\alpha \Gamma(z)} = 1. \quad (B.6)$$

B.3 Modified Bessel functions

The modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$ satisfy the relation,

$$I_\nu(z)K'_\nu(z) - I'_\nu(z)K_\nu(z) = -\frac{1}{z}. \quad (B.7)$$

For $z \gg 1$, they are approximated as

$$I_\nu(z) \approx \frac{e^z}{\sqrt{2\pi z}}, \quad K_\nu(z) \approx \sqrt{\frac{\pi}{2z}} e^{-z}. \quad (B.8)$$

B.4 Exponential integral

The exponential integral $E_1(x)$ is defined by

$$E_1(x) \equiv \int_x^\infty dt \frac{e^{-t}}{t}. \quad (B.9)$$

For $|x| \gg 1$, this can be approximated by

$$E_1(x) = \frac{e^{-x}}{x} \left\{ 1 - \frac{1!}{x} + \frac{2!}{x^2} + O(3!x^{-3}) \right\}. \quad (B.10)$$

For example, the following integral is expressed by $E_1(x)$.

$$\int_{-\infty}^Y \frac{e^{2qY}}{2q(-kY + 1)^2} = \left[ \frac{e^{2qY}}{2qk(1-kY')} - \frac{e^{2q}}{k^2} E_1(2q(1-kY')) \right]_{-\infty}^Y$$

$$= \frac{e^{2qY}}{2qk(1-kY)} - \frac{e^{2q}}{k^2} E_1(2q(1-kY))$$

$$= \frac{e^{2qY}}{4q^2(1-kY)^2} - \frac{k e^{2qY}}{4q^3(1-kY)^3} + \frac{3k^2 e^{2qY}}{8q^4(1-kY)^4} + O(q^{-5}). \quad (B.11)$$

We have used (B.10) at the last equality.
B.5 Properties of the mode functions

The Legendre functions $P_a^b(s)$ and $Q_a^b(s)$ are expressed in terms of $u_a^b(s)$ and $v_a^b(s)$ in (3.13) as

\[
P_a^b(s) = \frac{2^b}{\sqrt{\pi}} \left\{ \cos \left( \frac{a + b}{2} \pi \right) \dot{u}_a^b(s) + \sin \left( \frac{a + b}{2} \pi \right) \dot{v}_a^b(s) \right\},
\]

\[
Q_a^b(s) = 2^{b-1} \sqrt{\pi} \left\{ -\sin \left( \frac{a + b}{2} \pi \right) \dot{u}_a^b(s) + \cos \left( \frac{a + b}{2} \pi \right) \dot{v}_a^b(s) \right\},
\]

where

\[
\dot{u}_a^b(s) \equiv \frac{\Gamma(a+b+1)}{\Gamma(a-b+1)} u_a^b(s), \quad \dot{v}_a^b(s) \equiv \frac{2\Gamma(a+b+2)}{\Gamma(a-b+1)} v_a^b(s).
\]

We can show that $u_a^b(s)$ and $v_a^b(s)$ are even functions of the parameter $b$. By using (B.3), we find that

\[
F \left( \frac{a + b + 1}{2}, \frac{-a + b + 1}{2}; \frac{1}{2}; s^2 \right)
\]

\[
= \frac{\sqrt{\pi} \Gamma(b)}{\Gamma(a+b+1) \Gamma(-a+b)} (1 - s^2)^{-b} F \left( \frac{-a - b}{2}, \frac{a - b + 1}{2}, -b + 1; 1 - s^2 \right)
\]

\[
+ \frac{\sqrt{\pi} \Gamma(-b)}{\Gamma(-a-b) \Gamma(a+b+1)} F \left( \frac{a + b + 1}{2}, \frac{-a + b}{2}, b + 1; 1 - s^2 \right),
\]

and

\[
F \left( \frac{a + b + 2}{2}, \frac{-a + b + 1}{2}; \frac{3}{2}; s^2 \right)
\]

\[
= \frac{\sqrt{\pi} \Gamma(b)}{2 \Gamma(a+b+1) \Gamma(-a+b)} (1 - s^2)^{-b} F \left( \frac{-a - b + 1}{2}, \frac{a - b + 2}{2}, -b + 1; 1 - s^2 \right)
\]

\[
+ \frac{\sqrt{\pi} \Gamma(-b)}{2 \Gamma(-a-b) \Gamma(a+b+1)} F \left( \frac{a + b + 2}{2}, \frac{-a + b + 1}{2}, b + 1; 1 - s^2 \right).
\]

Thus, we have

\[
u_a^b(s) = \mathcal{U}_a^b(s^2) + \mathcal{V}_a^{-b}(s^2),
\]

\[
v_a^b(s) = s \left\{ \mathcal{V}_a^b(s^2) + \mathcal{U}_a^{-b}(s^2) \right\},
\]

where

\[
\mathcal{U}_a^b(z) \equiv \frac{\sqrt{\pi} \Gamma(-b)}{\Gamma(-a-b) \Gamma(a+b+1)} (1 - z)^{b/2} F \left( \frac{a + b + 1}{2}, \frac{-a + b}{2}, b + 1; 1 - z \right)
\]

\[
= C_u(b) (1 - z)^{b/2} H_1(a, b; 1 - z),
\]

\[
\mathcal{V}_a^b(z) \equiv \frac{\sqrt{\pi} \Gamma(-b)}{2 \Gamma(-a+b+1) \Gamma(a+b+2)} (1 - z)^{b/2} F \left( \frac{a + b + 2}{2}, \frac{-a + b + 1}{2}, b + 1; 1 - z \right)
\]

\[
= C_v(b) (1 - z)^{b/2} H_2(a, b; 1 - z).
\]
Here \( H_1(a, b; w) \equiv F\left(\frac{a+b+1}{2}, \frac{-a+b}{2}, b+1; w\right) \) and \( H_2(a, b; w) \equiv F\left(\frac{a+b+2}{2}, \frac{-a+b+1}{2}, b+1; w\right) \). The functions \( C_u(b) \) and \( C_v(b) \) are defined as

\[
C_u(b) = \frac{\sqrt{\pi} \Gamma(-b)}{\Gamma\left(-\frac{a-b}{2}\right) \Gamma\left(-\frac{a-b+1}{2}\right)}, \quad C_v(b) = \frac{\sqrt{\pi} \Gamma(-b)}{2 \Gamma\left(-\frac{a-b+1}{2}\right) \Gamma\left(-\frac{a-b+2}{2}\right)}. \tag{B.18}
\]

From the expression (B.16), we can see that \( u^b_a(s) \) and \( v^b_a(s) \) are even functions of \( b \).

Next we see the behaviors of the functions \( F_{a>}^b(y) \) defined in (3.35) near infinity \( y = \pm \infty \). We should note that

\[
R(a, b) = \sqrt{\frac{C_v(-b)}{C_u(-b)}}. \tag{B.19}
\]

Thus, for \( y \geq 0 \) (\( s \geq 0 \)), \( F_{a>}^b(y) \) is expressed as

\[
F_{a>}^b(y) = \frac{R(a, b)}{\sqrt{2B}} u^b_a(s) - \frac{v^b_a(s)}{\sqrt{2BR(a, b)}}
= \frac{1}{\sqrt{2BC_u(-b)C_v(-b)}} \left\{ C_v(-b) u^b_a(s) - C_u(-b) v^b_a(s) \right\}
= \frac{(1 - s^2)^{b/2}}{2b\sqrt{2BC_u(-b)C_v(-b)}} F\left(\frac{a+b+1}{2}, \frac{-a+b}{2}, b+1; 1 - s^2\right), \tag{B.20}
\]

where \( s = \tanh(By) \). At the last equality, we have used (3.13) and

\[
F\left(\frac{a+b+1}{2}, \frac{-a+b}{2}, b+1; 1 - s^2\right) = \frac{\Gamma(b+1)\Gamma\left(-\frac{1}{2}\right)}{\Gamma\left(\frac{a+b+1}{2}\right)\Gamma\left(\frac{a+b}{2}\right)} |s| F\left(\frac{-a+b+1}{2}, \frac{a+b+2}{2}, \frac{3}{2}; s^2\right)
+ \frac{\Gamma(b+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-\frac{a+b+1}{2}\right)\Gamma\left(-\frac{a+b+2}{2}\right)} F\left(\frac{a+b+1}{2}, -a+b \frac{1}{2}; s^2\right), \tag{B.21}
\]

which follows from (B.3). Since \( F(\alpha, \beta, \gamma; 0) = 1 \), (B.20) indicates that

\[
F_{a>}^b(y) \approx \frac{(1 - s^2)^{b/2}}{2b\sqrt{2BC_u(-b)C_v(-b)}} \cosh^{-b}(By)
\approx \frac{2^{b-1} e^{-By}}{b\sqrt{2BC_u(-b)C_v(-b)}}, \tag{B.22}
\]

for \( y \gg 1/B \).

For \( y \ll -1/B \) (i.e., \( s \simeq -1 \)), we find that

\[
F_{a>}^b(y) \approx \sqrt{\frac{2C_u(-b)C_v(-b)}{B}} (1 - s^2)^{-b/2} \approx \sqrt{\frac{2C_u(-b)C_v(-b)}{B}} \cosh^b(By)
\approx \frac{1}{2^b} \sqrt{\frac{2C_u(-b)C_v(-b)}{B}} e^{-By}. \tag{B.23}
\]
We have used (B.16) and that $H_1(a,-b,1-s^2) \simeq H_2(a,-b,1-s^2) \simeq 1$.

Using the second relation in (B.6), we have

$$C_u(-b)C_v(-b) = \frac{\pi \Gamma^2(b)}{2 \Gamma(-a+b) \Gamma(a+b+1) \Gamma(-a+b+1) \Gamma(a+b+2)} \frac{2^{2b-2} \Gamma^2(b)}{\Gamma(-a+b) \Gamma(a+b+1)}$$

Therefore, the asymptotic behaviors of $F_{a>}(y)$ are

$$F_{a>}(y) \simeq \begin{cases} \sqrt{\Gamma(-a+b) \Gamma(a+b+1)} e^{-bBy} & \left( y \gg \frac{1}{B} \right) \\ \sqrt{2Bb \Gamma(b)} \Gamma(b) \sqrt{2B} \Gamma(-a+b) \Gamma(a+b+1) e^{-bBy} & \left( y \ll -\frac{1}{B} \right) \end{cases}$$

We can obtain the asymptotic behaviors of $F_{a<}(y)$ by using the relation $F_{a<}(y) = F_{a>}(-y)$.

When the loop momentum $q$ is large enough, the parameter $b(q)$ defined in (3.33) is approximated as

$$b(q) \simeq \frac{q}{B} \equiv \tilde{q} \gg 1.$$ (B.26)

Thus, since

$$\Gamma(b+c) \simeq b^c \Gamma(b),$$ (B.27)

for $b \gg c$, (B.25) is simplified as

$$F_{a>}(y) \simeq \frac{e^{-qy}}{\sqrt{2Bq}} = \frac{e^{-qy}}{\sqrt{2q}}.$$ (B.28)

for $|y| \gg 1/B$. Therefore, we have

$$A^{(0)}_>(y) \simeq \frac{e^{-2qy}}{2q}, \quad A^{(1)}_>(y) \simeq -\frac{y}{4q^2} e^{-2qy}, \quad A^{(2)}_>(y) \simeq \frac{y^2}{8q^3} e^{-2qy}.$$ (B.29)

References

[1] D. Capper, Nuovo Cim. A 25 (1975), 29.

[2] S. L. Adler, Phys. Rev. Lett. 44 (1980), 1567.

[3] A. Zee, Phys. Rev. Lett. 48 (1982), 295.

[4] G. R. Dvali, G. Gabadadze and M. Porrati, “4-D gravity on a brane in 5-D Minkowski space,” Phys. Lett. B 485 (2000) 208 [hep-th/0005016].
[5] S. Dubovsky and V. Rubakov, Phys. Rev. D 67 (2003), 104014 [arXiv:hep-th/0212222 [hep-th]].

[6] C. Deffayet, Phys. Lett. B 502 (2001), 199-208 [arXiv:hep-th/0010186 [hep-th]].

[7] G. Dvali, G. Gabadadze and M. Shifman, Phys. Rev. D 67 (2003), 044020 [arXiv:hep-th/0202174 [hep-th]].

[8] N. Arkani-Hamed, S. Dimopoulos, G. Dvali and G. Gabadadze, [arXiv:hep-th/0209227 [hep-th]].

[9] G. Dvali, S. Hofmann and J. Khoury, Phys. Rev. D 76 (2007), 084006 [arXiv:hep-th/0703027 [hep-th]].

[10] J. Q. Xia, Phys. Rev. D 79 (2009), 103527 [arXiv:0907.4860 [astro-ph.CO]].

[11] S. B. Giddings, E. Katz and L. Randall, JHEP 03 (2000), 023 [arXiv:hep-th/0002091 [hep-th]].

[12] T. Gherghetta and A. Pomarol, Nucl. Phys. B 602 (2001) 3 [hep-ph/0012378].

[13] G. Dvali, G. Gabadadze, M. Kolanovic and F. Nitti, Phys. Rev. D 65 (2002), 024031 [arXiv:hep-th/0106058 [hep-th]].

[14] A. Kehagias and K. Tamvakis, Phys. Lett. B 504 (2001), 38-46 [arXiv:hep-th/0010112 [hep-th]].

[15] M. Shaposhnikov, P. Tinyakov and K. Zuleta, JHEP 09 (2005), 062 [arXiv:hep-th/0508102 [hep-th]].

[16] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999), 4690-4693 [arXiv:hep-th/9906064 [hep-th]].

[17] G. R. Dvali, G. Gabadadze, M. Kolanovic and F. Nitti, Phys. Rev. D 64 (2001) 084004 [hep-ph/0102216].

[18] G. Gabadadze, Nucl. Phys. Proc. Suppl. 171 (2007) 88 [arXiv:0705.1929 [hep-th]].