Dolbeault cohomology
of compact complex manifolds
with an action of a complex Lie group.

Nikita Klemyatin

Abstract: Let \( G \) be a complex Lie group acting on a compact complex Hermitian manifold \( M \) by holomorphic isometries. We prove that the induced action on the Dolbeault cohomology and on the Bott-Chern cohomology is trivial. We also apply this result to compute the Dolbeault cohomology of Vaisman manifolds.

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1 Introduction.

One of the main invariants of a compact complex manifold is its Dolbeault cohomology groups \( H^n_{\bar{\partial}}(M) \). They are defined as cohomology groups of the complex \( (\Omega^{*,*}(M), \bar{\partial}) \). Whenever \( M \) is compact Kähler, Dolbeault cohomology are isomorphic to de Rham cohomology groups \( H^n_{dR}(M; \mathbb{C}) \). More precisely, there is a Hodge decomposition for de Rham cohomology groups:
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\[ H^k_{dR}(M; \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_{\overline{\partial}}(M). \]

However, when \( M \) is non-Kähler, this decomposition generally fails.

Now let \( M \) be a manifold which is equipped with a smooth action of a connected Lie group \( G \). A simple computation with Cartan formula \( L_X = d\iota_X + \iota_X d \) shows that \( G \) acts trivially on \( H^*_{dR}(M) \). Hence, in the case of a compact Kähler \( M \) with holomorphic action of \( G \) the induced action of on \( H^{*,*}_{\overline{\partial}}(M) \) is again trivial. In contrary, the non-Kähler case provides examples of non-trivial action (see [Akh1], [Akh2]) on the Dolbeaut cohomology groups. So one could ask when the holomorphic action of a Lie group induces the trivial action on \( H^{*,*}_{\overline{\partial}}(M) \).

This question could be extended on the \textit{Bott-Chern} \( H^{*,*}_{BC}(M) \) and \textit{Aeppli} \( H^{*,*}_{A}(M) \) cohomology groups of a compact complex manifold \( M \) with holomorphic action of a group \( G \). These groups are defined as follows:

\[ H^{*,*}_{BC}(M) = \frac{\ker(d) \cap \ker(d^c)}{\text{im}(dd^c)} \]

and

\[ H^{*,*}_{A}(M) = \frac{\ker(dd^c)}{\text{im}(d) + \text{im}(d^c)}. \]

The induced action of \( G \) on these cohomology groups could be non-trivial. However, there are conditions on \( M \) and \( G \), which implies the triviality of the induced action on \( H^{*,*}_{\overline{\partial}}(M) \), \( H^{*,*}_{BC}(M) \) and \( H^{*,*}_{A}(M) \).

\textbf{Theorem 1.1:} Let \( G \) be a complex Lie group, which acts on a compact Hermitian manifold \( M = (M, h) \) by holomorphic isometries. Then \( G \) acts trivially on Dolbeaut, Bott-Chern and Aeppli cohomologies of \( M \).

As a corollary, we obtain that the Dolbeaut cohomology may be represented by invariant forms and any \( \Delta_{\overline{\partial}} \) -harmonic form is invariant.

As an application of \textbf{Theorem 1.1} we compute the Dolbeaut cohomology of compact Vaisman manifolds. Indeed, Dolbeaut cohomology groups of \( M \) are isomorphic to the Dolbeaut cohomology groups of invariant forms and they can be easily computed by this observation.

\textbf{Theorem 1.2:} The Dolbeaut cohomology groups of a Vaisman manifold \( M \) are organized as follows:
$H^{p,q}_\partial(M) = \begin{cases} H^{p,q}_\partial(M) \oplus \theta^{0,1} \wedge H^{p,q-1}_\partial(M), & p + q \leq \dim_\mathbb{C}(M) \\ \text{Ker}(L_{\omega_0}) H^{p,q}_\partial(M) \oplus \theta^{0,1} \wedge H^{p,q-1}_\partial(M), & p + q > \dim_\mathbb{C}(M) \end{cases}$

2 Preliminaries.

2.1 Basic properties of complex Lie groups.

We briefly introduce some notions about holomorphic group actions on complex manifolds.

**Definition 2.1:** A group $G$ is called a complex Lie group if $G$ is a complex manifold and the maps $G \times G \to G, (g_1; g_2) \mapsto g_1 g_2$ and $G \to G, g \mapsto g^{-1}$ are holomorphic.

A complex Lie group $G$ is compact if the underlying complex manifold is compact.

The following claim is well-known.

**Claim 2.2:** Let $G$ be a compact complex Lie group of dimension $n$. Then $G$ is a torus $\mathbb{C}^n / \mathbb{Z}^n$.

**Proof:** Consider the adjoint representation $\text{Ad} : G \to \text{End}(\text{Lie}(G)) \cong \mathbb{C}^{n^2}$. This is a holomorphic map, hence it is constant by the maximum principle. Since $\text{Ad}(e) = 1_n$, we have $\text{Ad}(G) = 1_n$ (here $1_n$ is a $n \times n$ identity matrix). Hence $G$ is a commutative group and the exponential map $\exp : \text{Lie}(G) \to G$ is a homomorphism. Since $G$ is compact, $\exp$ has a nontrivial kernel, which is a finitely generated abelian group without torsion, i.e. $\ker(\exp) \cong \mathbb{Z}^p$. It is easy to see, that $p = 2n$ and $G \cong \mathbb{C}^n / \mathbb{Z}^{2n}$.

When $G$ is non-compact, it might be non-abelian. However, when $G$ acts on $M$ by holomorphic isometries, we can say something about the closure of $G$. First of all we need the following result about the isometry group of a compact Riemannian manifold.

**Theorem 2.3:** (see [Kob], Chapter II, Thm.1.2) Let $(M, g)$ be a compact Riemannian manifold. Then the group $\text{Isom}(M)$ isometries of $(M, g)$ is a compact Lie group.

We also need a following statement.
Proposition 2.4: Let $G$ be a complex group which acts by a holomorphic isometries on a compact Hermitian manifold $(M; h)$. Then the closure of $G$ in the group Isom($M$) still acts by holomorphic isometries on $M$.

Proof: Let $\{g_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $G$. Since each $g_n$ lies in Isom, the family $\{g_n\}_{n=1}^{\infty}$ is bounded. By the Montel’s theorem, the limit $g$ of $\{g_n\}_{n=1}^{\infty}$ is holomorphic. By the theorem of Cartan and von Neumann (see [Kob]), the closure is a Lie group. It will be compact by Theorem 2.3. ■

2.2 Holomorphic vector fields and Lie derivative.

Let $G$ be a Lie group and $M = (M, J)$ be a compact complex manifold. We assume that both $G$ and $M$ are connected.

Definition 2.5: A holomorphic action of complex Lie group $G$ on a complex manifold $M$ is a holomorphic map $\sigma : G \times M \to M$.

Definition 2.6: ([Gau]) A vector field $X$ is called holomorphic vector field if $L_X J = 0$.

Clearly holomorphic vector fields form a Lie subalgebra $\mathfrak{h}(M)$ in the algebra of all holomorphic vector fields on $M$. If $G$ acts holomorphically on $M$, then the action induces a homomorphism $d_c \sigma : \text{Lie}(G) \to \mathfrak{h}(M)$ of (real) Lie algebras. This homomorphism gives us the action of $\text{Lie}(G)$ on tensors on $M$ via the Lie derivative. Recall that Lie derivative acts on differential forms via Cartan formula:

$$L_X \alpha = d_L X \alpha + \iota_X d \alpha.$$ 

It also commutes with de Rham differential:

$$L_X d \alpha = d L_X \alpha.$$ 

Also the Lie derivative satisfies the Leibniz rule for the tensor product:

$$L_X (P \otimes Q) = (L_X P) \otimes Q + P \otimes (L_X Q).$$

In particular, the following identity holds:

$$L_X Y = [X; Y].$$
As mentioned above, all holomorphic vector fields form a Lie algebra. Moreover, on a complex manifold, the algebra of holomorphic vector fields is a complex Lie algebra and $JX$ is holomorphic whenever $X$ is holomorphic. Indeed, for a holomorphic vector field $X$ and for any vector field $Y$ one has

$$L_{JX}(JY) = [JX; JY] = J[JX; Y] + J[X; JY] + [X; Y] = J[JX; Y] - [X; Y] + [X; Y] = J[JX; Y] = JL_X Y.$$ 

Here in the second equality we use that Nijenhuis tensor is zero and in the third equality we use the definition of holomorphic vector fields.

Now, in the case of $G$-action on $M$ that the homomorphism $d_e\sigma : \text{Lie}(T) \to \mathfrak{h}(M)$ is actually a homomorphism of complex Lie algebras.

### 2.3 $d^c$-operator and Lie derivative.

We start from the following definition:

**Definition 2.7:** (see [Gau], sect. 1.11) Consider an operator $d^c$, defined as following:

$$d^c \alpha = Jd J^{-1} \alpha = (-1)^{|\alpha|} Jd J \alpha.$$ 

Here $\alpha$ is an arbitrary form of degree $|\alpha|$.

Note that $d^c$ is a real operator.

One can write down $d^c$ in terms of operators $\partial$ and $\overline{\partial}$ (see [Gau]):

$$d^c = i(\overline{\partial} - \partial)$$

Here $d = \partial + \overline{\partial}$ is the standard decomposition of the de Rham differential on complex manifolds.

We also have

$$\overline{\partial} = \frac{1}{2}(d + i d^c) \quad \partial = \frac{1}{2}(d - i d^c).$$

There is a simple Cartan-type formula, which connects the operator $d^c$ and the Lie derivative along $L_{JX}$.

**Proposition 2.8:** Let $X$ be a holomorphic vector field on compact complex manifold $M$. Then $L_{JX} \alpha = -\{d^c; \iota_X\} = -(d^c \iota_X + \iota_X d^c) \alpha$ for any form $\alpha$. 

2.4 Elliptic operators on compact manifolds.

In this section we recall some facts about elliptic operators on compact manifolds.

Let $M$ be a compact manifold and $E$ a vector bundle on $M$. Throughout this section the symbol $\Gamma(E)$ denotes the space of smooth sections on $M$.

**Definition 2.9:** Let $E, F$ are complex vector bundles over compact manifold $M$. A linear differential operator $P$ of degree $k$ from $E$ to $F$ is a $C$-linear operator $P: \Gamma(E) \to \Gamma(F), \ s \mapsto Ps$ of the form

$$Ps(x) = \sum_{|j|=0}^{k} a_j(x) \frac{\partial^j}{\partial x_1^{j_1} \cdots \partial x_m^{j_m}} s(x). \quad (2.1)$$

The principal symbol of the operator $P$ is the morphism of vector bundles $\sigma_P(x, \xi) = \sum_{|j|=k} a_j(x)\xi^j, \ \sigma_P(x, \xi): E \to F$. Here $\xi \in T_x M$.

The operator $P$ is called elliptic, if the principal symbol $\sigma_P(x, \xi)$ of $P$ is injective for any nonzero $\xi$.

Fix a volume form $d\mu$ on $M$. Let $E$ be a Hermitian vector bundle, the operator $P : \Gamma(E) \to \Gamma(E)$ is called self-adjoint, if it is self-adjoint with respect to the $L^2$ scalar product

$$(s; t) := \int_M h(s, t)d\mu$$

for any two $s, t \in \Gamma(E)$.

Elliptic self-adjoint operators have very nice spectral properties.

**Proposition 2.10:** (see [Gil]) Let $P : \Gamma(E) \to \Gamma(E)$ be an elliptic self-adjoint operator. Then we can find a complete basis $\{s_j\}_{j=1}^{\infty}$ of $L^2(E)$ of eigensections of $P$. Each eigensection of $P$ is smooth and each eigenspace
has finite dimension. Moreover the set of eigenvalues is a discrete subset in $\mathbb{R}$.

This proposition has a very nice application for the case of $E = \Omega^*(M)$ on a compact Hermitian manifold $M$ with the holomorphic and isometric action of compact Lie group $G$.

**Proposition 2.11:** Let $G$ and $M$ are as above and $P : \Omega^*(M) \rightarrow \Omega^*(M)$ be a self-adjoint elliptic operator. Suppose that for any $g \in G$ and for any smooth form $\alpha$ we have $g^*P\alpha = Pg^*\alpha$. Then each eigenspace of $P$ is a direct sum of irreducible representations of $G$. Moreover, for any $g \in G$ there exist a complete orthogonal basis $\{\alpha_j\}_{j=0}^{\infty}$ such that $P\alpha_j = \lambda_j\alpha_j$ and $g^*\alpha_j = e^{ia_j}\alpha_j$, $a_j \in \mathbb{R}$.

**Proof:** Since $g^*P\alpha = Pg^*\alpha$, $g^*$ and $P$ commute on each eigenspace of $P$. Hence they have common eigenvectors on each eigenspace. For an arbitrary $g \in G$ the map $g^*$ is $L^2$-isometry and hence its restriction on each eigenspace is an unitary operator. Hence all eigenvalues of $g^*$ are equal to $e^{ia_j}$ for some real $a_j$. $lacksquare$

### 2.5 Various cohomology groups on compact complex manifolds.

Recall that Dolbeault cohomology groups are defined as follows:

$$H^{\omega,\ast}_{\overline{\partial}}(M) = \frac{\ker(\overline{\partial})}{\text{im}(\overline{\partial})}.$$  

In the case of Kähler manifolds there is a decomposition of de Rham cohomology groups on a direct sum of Dolbeault cohomology groups:

$$H^k_{dR}(M; \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_{\overline{\partial}}(M).$$

In general, there is no such decomposition for de Rham cohomology. However, there is the Frölicher inequality which is obtained from the Frölicher spectral sequence (see [Dem]):

$$\dim_{\mathbb{C}} H^k_{dR}(M; \mathbb{C}) \leq \sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(M).$$
One can define another cohomology groups, namely Bott-Chern cohomology

$$H_{BC}^{p,q}(M) = \frac{\ker(d) \cap \ker(d^c)}{\text{im}(dd^c)}$$

and Aeppli cohomology

$$H_A^{p,q}(M) = \frac{\ker(dd^c)}{\text{im}(d) + \text{im}(d^c)}.$$ 

There is an analogue of harmonic decomposition for these cohomologies. Indeed, for any Hermitian metric $h$ on $M$ one can construct the following Laplacians for Bott-Chern and Aeppli cohomologies (see [Sch]):

$$\Delta_{BC} := (\partial \overline{\partial})(\partial \overline{\partial})^* + (\partial \overline{\partial})(\partial \overline{\partial})^* + (\partial \overline{\partial})(\partial \overline{\partial})^* + (\partial \overline{\partial})(\partial \overline{\partial})^* + \partial \partial + \partial^* \partial$$

and

$$\Delta_A = \overline{\partial} \overline{\partial} + \partial \partial^* + (\partial \overline{\partial})(\partial \overline{\partial})^* + (\partial \overline{\partial})(\partial \overline{\partial})^* + (\partial \overline{\partial})(\partial \overline{\partial})^* + (\partial \overline{\partial})(\partial \overline{\partial})^*.$$ 

These operators are self-adjoint and elliptic and their kernels are isomorphic to $H_{BC}^{p,q}(M)$ and $H_A^{p,q}(M)$ respectively.

Both Bott-Chern and Aeppli cohomologies are “dual” to each other in the following sense.

**Theorem 2.12:** (see [A], Theorem 2.5, and [Sch]) Let $M$ be a compact complex manifold of complex dimension $m$. Then there is an isomorphism between $H_{BC}^{p,q}(M)$ and $(H_A^{m-p,m-q}(M))^*$. The isomorphism is given by the nondegenerate pairing

$$\int_M : H_{BC}^{p,q}(M) \times H_A^{m-p,m-q}(M) \to \mathbb{C}.$$ 

All these cohomology groups are related in the following way (see [AT]):

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\begin{align*}
H_{BC}^{p,q}(M) \quad &\quad H_A^{p,q}(M) \\
H_{\partial}^{p,q}(M) \quad &\quad H_{dR}^k(M; \mathbb{C}) \\
H_{\overline{\partial}}^{p,q}(M) \quad &\quad H_{dR}^k(M; \mathbb{C})
\end{align*}
```
Note that in the general case all arrows in this diagram are neither injective nor surjective. For instance, if $M$ is non-Kähler, then the map $H^p_{BC}(M) \to H^q_{dR}(M)$ may have a nontrivial kernel (see [OVV], Theorem 2.3).

3 Proof of the main theorem

Proposition 3.1: Suppose that complex connected Lie group $G$ acts by holomorphic isometries on compact complex non-Kähler Hermitian manifold $(M, h)$. Then $G$ acts trivially on $H^*_{BC}(M)$.

Proof: Consider the natural map $H^*_{BC}(M) \to H^*_{dR}(M; \mathbb{C})$. Since $G$ acts trivially on $H^*_{dR}(M; \mathbb{C})$, it is sufficient to prove that the action on $\text{Ker}(H^*_{BC}(M) \to H^*_{dR}(M; \mathbb{C}))$ is trivial.

Suppose now that $\eta = d\alpha$ for some $\alpha$. Let $X \in \text{Lie}(G)$. Each $\alpha$ can be written as $\alpha = \sum_{j=0}^\infty c_j \alpha_j$, there $\Delta_{BC}\alpha_j = \lambda_j \alpha_j$. We also have $L_X\alpha = \sum_{j=0}^\infty c_j L_X \alpha_j$. Hence, by Proposition 2.11 we can assume $L_X\alpha = a\alpha$ and $L_{JX}\alpha = b\alpha$ for some $a, b \in i\mathbb{R}$.

Since $d^c\eta = 0$, we have

\[
0 = \iota_X d^c d\alpha = -L_{JX} d\alpha - d^c \iota_X d\alpha = -L_{JX} d\alpha - d^c L_X \alpha + d^c d\iota_X \alpha = -(bd + ad^c)\alpha + d^c d\iota_X \alpha.
\]

Let $\delta := bd + ad^c$, where $a, b \in i\mathbb{R}$ are as above. From the equation above we can see that $0 = -\delta \alpha + d^c d\iota_X \alpha$.

Now we want to write down $L_X$ in the form $\{\delta, \iota_Y\}$ for some $Y = y_1 X + y_2 JX$:

\[
\{\delta, \iota_Y\} = by_1 \{d^c, \iota_X\} + ay_1 \{d; \iota_X\} + ay_2 \{\iota_X ; d\} + by_2 \{\iota_{JX} ; d^c\} = (ay_2 - by_1)L_{JX} + (ay_1 + by_2)L_X = L_X.
\]

Therefore we have the system of linear equations:

\[
\begin{cases}
ay_2 - by_1 = 0 \\
ay_1 + by_2 = 1.
\end{cases}
\]

The determinant of this system is equal to $-b^2 - a^2$ and it is nonzero whenever either $a$ or $b$ is nonzero. Hence this system has a solution, i.e. $L_X\alpha = \{\delta, \iota_Y\} = \delta_{tY}\alpha$. 
If $\eta = d\alpha$ then $L_X\eta = L_Xd\alpha = dL_X\alpha = d\delta_Y\alpha = dd^c\iota_Y\alpha = dd^c\beta$ for some $\beta$. Thus $L_X$ does not change the class of $\eta$ in $H^{*\ast}_{BC}(M)$.

**Corollary 3.2:** In the assumptions of Proposition 3.1, $G$ acts trivially on $H^{*\ast}_{A}(M)$.

**Proof:** This is a direct corollary of Proposition 3.1 and Theorem 2.12.

Now we can prove Theorem 1.1.

**Theorem 3.3:** (also Theorem 1.1) Let $G$ be a compact Lie group which acts on a compact Hermitian manifold $M = (M, h)$ by holomorphic isometries. Then $G$ acts trivially on Dolbeault, Bott-Chern and Aeppli cohomologies of $M$.

**Proof:**
Consider the exact sequence:

$$0 \rightarrow A^{p,q} \rightarrow B^{p,q} \rightarrow H^{p,q}_{\partial}(M) \rightarrow H^{p,q}_{A}(M) \rightarrow C^{p,q} \rightarrow 0$$

(see [AT]).

Here we define $A^{p,q}$, $B^{p,q}$ and $C^{p,q}$ as follows:

$$A^{p,q} = \frac{\text{im}(\partial) \cap \text{im}(\overline{\partial})}{\text{im}(\overline{\partial})}, \quad B^{p,q} = \frac{\text{im}(\partial) \cap \ker(\overline{\partial})}{\text{im}(\overline{\partial})}, \quad C^{p,q} = \frac{\ker(\partial\overline{\partial})}{\text{im}(\partial) + \text{im}(\overline{\partial})}.$$

Since $A^{p,q}$ and $B^{p,q}$ are subspaces of $H^{p,q}_{BC}(M)$, the action of $G$ on $A^{p,q}$ and $B^{p,q}$ is trivial. The $G$-action on groups $H^{p,q}_{A}(M)$ are also trivial by Corollary 3.2. The groups $C^{p,q}$ have trivial $G$-action because the map $H^{p,q}_{A}(M) \rightarrow C^{p,q}$ is surjective and commutes with action of $G$. Hence the induced $G$-action on each group $H^{p,q}_{A}(M)$ is trivial.

**Corollary 3.4:** Let $h$ be a $G$-invariant Hermitian metric on $M$ and $\alpha \in H^{p,q}_{\overline{\partial}}(M)$ be a harmonic form (with respect to $h$) on $M$. Then $\alpha$ is $G$-invariant.
**Proof:** Let \( X \) be a holomorphic vector field from \( \text{Lie}(G) \). Since \( h \) is \( G \)-invariant, \( L_X \) commutes with \( \Delta_{\overline{\omega}} \) and hence \( G \) acts on \( \Delta_{\overline{\omega}} \)-harmonic forms. However, the space of \( \Delta_{\overline{\omega}} \)-harmonic forms is isomorphic to the Dolbeault cohomology (see [Dem]). Since \( g^*\alpha \) is again \( \Delta_{\overline{\omega}} \)-harmonic and has the same Dolbeault cohomology class as \( \alpha \), we have \( g^*\alpha = \alpha \). \( \square \)

**Corollary 3.5:** Let \( M \) and \( G \) be as above. Then the closure \( \overline{G} \) of \( G \) in \( \text{Isom}(M) \) acts trivially on \( H^{p,q}_{\overline{\omega}}(M) \).

**Proof:** By Proposition 2.4 the group \( \overline{G} \) acts on the space of \( \Delta_{\overline{\omega}} \)-harmonic forms. The group \( G \) acts trivially on harmonic forms because \( G \) acts trivially on it. \( \square \)

4 The application to Vaisman manifolds.

4.1 Sasakian and Vaisman manifolds.

In this section we recall some facts about Vaisman and Sasakian manifolds.

**Definition 4.1:** An odd-dimensional Riemann manifold \((S, g)\) is called a Sasakian manifold if it cone \( C(S) := S \times \mathbb{R}_{>0} \) with the metric \( \tilde{g} := t^2 g + dt^2 \) is Kähler and the natural action of \( \mathbb{R}_{>0} \) on \( C(S) \) is holomorphic.

**Definition 4.2:** A compact complex Hermitian manifold \((M, J, \omega)\) of \( \dim_{\mathbb{C}} > 1 \) is called a locally conformally Kähler (LCK for short), if it admits a Kähler covering \((\tilde{M}, \tilde{J}, \tilde{\omega})\), such that covering group acts by holomorphic homoteties on \( \tilde{M} \).

The LCK property is equivalent to existence of a closed form \( \theta \) such that \( d\omega = \theta \wedge \omega \). The form \( \theta \) is called the Lee form. It is obviously closed. When \( \theta \) is exact, an LCK manifold can be equipped a Kähler metric. Indeed, if \( \theta = d\varphi \), then \( e^{-\varphi} \omega \) is closed.

A very important example of LCK manifolds are Vaisman manifolds.

**Definition 4.3:** A Vaisman manifold \((M, J, \omega, \theta)\) is an LCK manifold such that the Lee form \( \theta \) is parallel with respect to the Levi-Civita connection which is associated to the Hermitian metric.
The typical examples of Vaisman manifolds are Hopf varieties $H_A := (\mathbb{C}^n \setminus 0)/\langle A \rangle$, where $A = \text{diag}(\lambda_i)$ with $|\lambda_i| < 1$ (see [OV2]).

The Vaisman manifold has a foliation $\Sigma$ which is called the canonical (or fundamental) foliation. It is generated by $X = \theta^2 := g^{-1}(\theta, \cdot)$ and $JX = J\theta^2$. It is well-known that $X$ and $JX$ acts holomorphically on $M$. Moreover, there is a transversely Kähler metric on $M$. It is given by the following formula:

$$2\omega_0 = d\theta^c = d(J\theta) = \omega - \theta \wedge \theta^c.$$  

(see [V1]). The local structure of compact Vaisman manifolds is well-known and it is described by the following theorem.

**Theorem 4.4:** (The local structure theorem for compact Vaisman manifolds, see [OV1]) Let $(M, J, \omega, \theta)$ be a Vaisman manifold. Denote by $X$ the vector field dual to $\theta$. Then $L_X J = 0$ and $M$ is locally isomorphic to the Kähler cone of a Sasakian manifold. Moreover, $X$ acts on Kähler covering $\tilde{M}$ by holomorphic homotheties of Kähler metric.

The following proposition is well-known.

**Proposition 4.5:** The vector field $JX$ is a Killing.

**Proof:** This is a local statement.

By Theorem 4.4 we can assume that locally $M = S \times \mathbb{R}$ with product metric $g_S + dt^2$. Moreover, we can assume that $\theta = dt$ and $X = \frac{d}{dt}$. Hence the metric $\tilde{g} = e^{-t}(g_S + dt^2)$ is Kähler. Denote by $\tilde{\omega}$ a Kähler form of $\tilde{g}$.

Since $JX$ is orthogonal to $X$, it is tangent to $S$. Since $JX$ is holomorphic, we have $L_{JX}\tilde{g} = L_{JX}\tilde{\omega} = dt_{JX}\tilde{\omega} = d\theta = 0$. ■

Since $\nabla X = \nabla \theta = 0$, $X$ is Killing as well and we have the following statement

**Claim 4.6:** The group generated by $e^{tX}$ and $e^{tJX}$ acts by holomorphic isometries on $M$.

### 4.2 Dolbeault cohomology of Vaisman manifolds.

Let $(M, J, \omega, \theta)$ be a Vaisman manifold. We start from the corollary of Theorem 1.1.
**Corollary 4.7:** The Dolbeault cohomology groups of \( M \) are the cohomology groups of complex \( (\Lambda^*(M)^{\text{inv}}, \bar{\partial}) \) of invariant forms on \( M \).

**Proof:** The group, generated by \( e^{tX} \) and \( e^{tJX} \) acts by holomorphic isometries on \( M \) (Claim 4.6). Hence we can apply Theorem 1.1 and Corollary 3.5. Each \( \bar{\partial} \)-closed invariant form lies in \( \Lambda^*(M)^{\Sigma} \) and it is an element of cohomology group of the complex \( (\Lambda^*(M)^{\text{inv}}, \bar{\partial}) \). □

Recall the important definition.

**Definition 4.8:** Let \( M \) be a manifold with foliation \( \Sigma \). A form \( \alpha \) is basic, if \( \iota_X \alpha = \iota_X d\alpha = 0 \) for any vector field \( X \) tangent to \( \Sigma \).

**Proposition 4.9:** Let \( \eta \) be an invariant form on \( M \). Then \( \Lambda^*(M)^{\text{inv}} = (\pi^*\Lambda_B^*)^{\text{inv}} \otimes \Lambda^* (F) \).

**Proof:** A form \( \alpha \) on \( M \) is \( G \)-invariant iff it is invariant under the induced action of \( \text{Lie}(G) \). Hence this is a purely local statement. Denote by \( F \) the fiber of foliation \( \Sigma \) on \( M \) and by \( B \) the leaf space.

We know that locally

\[ \Lambda^*(M) = \pi^*\Lambda_B^* \otimes \Lambda^*(F). \]

Hence, the following equality holds for invariant forms

\[ \Lambda^*(M)^{\text{inv}} = (\pi^*\Lambda_B^*)^{\text{inv}} \otimes \Lambda^* (F)^{\text{inv}}. \]

But the \( (\pi^*\Lambda_B^*)^{\text{inv}} \) are just basic forms and \( \Lambda^* (F)^{\text{inv}} \) is the exterior algebra generated by \( \theta \) and \( \theta^c \). □

Recall some important definition from homological algebra.

**Definition 4.10:** Suppose \( (K^*, d_K) \) and \( (L^*, d_L) \) are complexes and \( f : K^* \to L^* \) be a morphism of these complexes. Define a complex \( (C(f), d_f) \) as follows: \( C(f)_i = K_{i+1} \oplus L_i \) and \( d_f = (d_K, f - d_L) \). This complex is called the cone of \( f \).

For each cone of a morphism we can construct the long exact sequence of cohomology. Indeed, we have the short exact sequence of complexes:
0 \rightarrow L^* \rightarrow C(f) \rightarrow K^*[1] \rightarrow 0.

There is a well-known way to construct a long sequence in cohomology from a short sequence of complexes:

\[ \ldots \rightarrow H^i(L^*) \rightarrow H^i(C(f)) \rightarrow H^{i+1}(K^*) \rightarrow H^{i+1}(K^*) \rightarrow \ldots \]

(see [GM] for the details).

Now we can compute the Dolbeault cohomology groups for a Vaisman manifold \( M \).

Consider the subcomplex \( \Lambda^{*,*}_{B, \theta^0, 1} := (\pi^* \Lambda^*_B)^{\text{inv}} \oplus \theta^{0, 1} \wedge (\pi^* \Lambda^*_B)^{\text{inv}} \) of the complex \( \Lambda^*(M)^{\text{inv}} \). Denote by \( L_{\omega_0} \) the operator of multiplication by \( \omega_0 \). Clearly, \( \Lambda^*(M)^{\text{inv}} = \Lambda^{*,*}_{B, \theta^0, 1} \oplus \theta^{1,0} \wedge \Lambda^{*,*}_{B, \theta^0, 1} \) and \( L_{\omega_0} \) is a morphism \( \Lambda^{*,*}_{B, \theta^0, 1} \rightarrow \Lambda^{*,*}_{B, \theta^0, 1} \).

**Proposition 4.11:** ([OV3] in the case of de Rham cohomology) The complex \( \Lambda^*(M)^{\text{inv}} \) is isomorphic to the cone \( C(L_{\omega_0}) \) of the morphism \( \Lambda^{*,*}_{B, \theta^0, 1} \xrightarrow{L_{\omega_0}} \Lambda^{*,*}_{B, \theta^0, 1} \).

**Proof:** We have

\[ \Lambda^*(M)^{\text{inv}} = \Lambda^{*,*}_{B, \theta^0, 1} \oplus \theta^{1,0} \wedge \Lambda^{*,*}_{B, \theta^0, 1} = \Lambda^{*,*}_{B, \theta^0, 1} \oplus \Lambda^{*,*}_{B, \theta^0, 1}[-1]. \]

The Dolbeault differential \( \overline{\partial} \) on \( \Lambda^*(M)^{\text{inv}} \) acts in the following way: it is the ordinary \( \overline{\partial} \) on \( \Lambda^{*,*}_{B, \theta^0, 1} \). On the other hand, we have \( \overline{\partial} \theta^{1,0} = \omega_0 \). Hence the Dolbeault differential acts on \( \Lambda^{*,*}_{B, \theta^0, 1}[-1] \) as \( \omega_0 - \overline{\partial} \).

**Theorem 4.12:** (also Theorem 1.2) The Dolbeault cohomology groups of a Vaisman manifold \( M \) are organized as follows:

\[ H^p_q(M) = \begin{cases} H^p_q(M) \oplus \epsilon^{0,1} \wedge H^{p,q-1}(M), & p + q \leq \dim_C(M) \\ \text{Im}(L_{\omega_0}) \big|_{H^p_q(M) \oplus \epsilon^{0,1} \wedge H^{p,q-1}(M)} & p + q > \dim_C(M) \end{cases} \]

This result is similar to Theorem 3.2 from [Ts].
Proof: We have a long exact sequence:

\[
\cdots \rightarrow H_{B,\theta^0}^{p,q}(M) \xrightarrow{L_{\omega_0}} H_{B,\theta^0}^{p+1,q+1}(M) \rightarrow H^p_{B,\theta^0}(M) \rightarrow \cdots
\]

The cohomology groups \(H^*_{B,\theta^0}(M)\) of complex \((\Lambda^*_B,\Lambda^*_\theta)\) are equal to \(H^*_{B}(M) \oplus \theta^0 \wedge H^{*+1}_{B}(M)\). Since \(H^*_{B}(M)\) admits a Lefshetz \(SL(2)\)-action (see [EK] and [EKH]), there is an analog of such action for \(H^*_{B,\theta^0}(M)\). Since \(L_{\omega_0}\) is injective on \(H_{B}^{p,q}(M)\) for \(p+q \leq \dim_{\mathbb{C}}(M)\), it is injective on \(H_{B,\theta^0}^{p,q}(M)\) with the same \(p, q\). Hence we obtain the short exact sequence:

\[
0 \rightarrow H_{B,\theta^0}^{p-1,q-1}(M) \xrightarrow{L_{\omega_0}} H_{B,\theta^0}^{p,q}(M) \rightarrow H_{B}^{p,q}(M) \rightarrow 0
\]

For the case \(p + q > \dim_{\mathbb{C}}(M)\) we have another short exact sequence:

\[
0 \rightarrow H_{B}^{p,q}(M) \rightarrow H_{B,\theta^0}^{p,q}(M) \xrightarrow{L_{\omega_0}} H_{B}^{p+1,q+1}(M) \rightarrow 0
\]

The statement of the theorem directly follows from these two sequences.

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References

[Akh1] D. Akhiezer, Group actions on the Dolbeault cohomology of homogeneous manifolds, Mathematische Zeitschrift 226(4):607-621, 1997. (Cited on page 2.)

[Akh2] D. Akhiezer, Sur les représentations de groupes de Lie dans les espaces de cohomologie de Dolbeault, C R Acad. Sci. Paris 321, (1995), 1583-1586. (Cited on page 2.)

[AT] D. Angella, A. Tomassini, On the \(\partial \bar{\partial}\)-Lemma and Bott-Chern cohomology, arxiv:1402.1954v1 (Cited on pages 8 and 10.)

[A] D. Angella, Cohomological Aspects in Complex Non-Kähler Geometry, Lecture Notes in Mathematics 2095, Springer, 2014. (Cited on page 8.)

[Dem] Jean-Pierre Demailly, Complex Analytic and Differential Geometry, https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf (Cited on pages 7 and 11.)
[EK] El Kacimi-Alaoui, Aziz, *Opérateurs transversalement elliptiques sur un feuilletage riemannien et applications*, Compositio Math. 73 (1990), no. 1, 571-06. (Cited on page 15.)

[EKH] El Kacimi-Alaoui, Aziz; Hector, Gilbert, *Décomposition de Hodge basique pour un feuilletage riemannien*, Ann. Inst. Fourier (Grenoble) 36 (1986), no. 3, 207-227. (Cited on page 15.)

[Gau] P. Gauduchon, *Calabi’s extremal Kähler metrics: an elementary introduction*, http://germanio.math.unifi.it/wp-content/uploads/2015/03/dercalabi.pdf (Cited on pages 4 and 5.)

[GM] Sergei I. Gelfand, Yuri I. Manin, *Methods of Homological Algebra*, Springer Monographs in Mathematics, Springer-Verlag Berlin Heidelberg, 2003. (Cited on page 14.)

[Gil] Peter B. Gilkey, *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*, Mathematics lecture series 11, Publish or Perish Inc., 1996. (Cited on page 6.)

[Kob] S. Kobayashi, *Transformation Groups in Differential Geometry*, Springer, 1995. (Cited on pages 3 and 4.)

[OV1] L. Ornea, M. Verbitsky, *Structure theorem for compact Vaisman manifolds*, Math. Res. Lett. 10 (2003), pp.799-805. (Cited on page 12.)

[OV2] L. Ornea, M. Verbitsky, *Locally conformally Kähler metrics obtained from pseudoconvex shells*, https://arxiv.org/abs/1210.2080 (Cited on page 12.)

[OV3] L. Ornea, M. Verbitsky, *Supersymmetry and Hodge theory of Sasakian and Vaisman manifolds*, in preparation (Cited on page 14.)

[OVV] Liviu Ornea, Victor Vuletescu, Misha Verbitsky, *Classification of non-Kähler surfaces and locally conformally Kähler geometry*, https://arxiv.org/abs/1810.05768 (Cited on page 9.)

[Sch] M. Schweitzer, Autour de la cohomologie de Bott-Chern, arXiv:0709.3528 (Cited on page 8.)

[Ts] K. Tsukada, *Holomorphic forms and holomorphic vector fields on compact generalized Hopf manifolds*, Compositio Mathematica, Volume 93 (1994) no. 1, p. 1-22. (Cited on page 14.)

[V1] M. Verbitsky, *Vanishing theorems for locally conformal Hyperkähler manifolds*, https://arxiv.org/abs/math/0302219v4 (Cited on page 12.)
NIKITA KLEMYATIN
NATIONAL RESEARCH UNIVERSITY HSE,
DEPARTMENT OF MATHEMATICS, 6 USACHEVA STR. MOSCOW, RUSSIA
ALSO:
SKOLKOVO INSTITUTE OF SCIENCE AND TECHNOLOGY
BOLSHOY BOULEVARD 30, BLD. 1. MOSCOW, RUSSIA
nklemyat@yandex.ru