When do Trajectories have Bounded Sensitivity to Cumulative Perturbations

Arsalan Sharifnassab and S. Jamaloddin Golestani

Abstract

A dynamical system is said to have bounded sensitivity if an additive disturbance leads to a change in the state trajectory that is bounded by a constant multiple of the size of the cumulative disturbance. It was shown in [1] that the (negative) (sub)gradient field of a piecewise constant and convex function with finitely many pieces has always bounded sensitivity. In this paper, we investigate sensitivity of some generalizations of this class of systems, as well as other classes of practical interest. In particular, we show that the result in [1] does not generalize to the (negative) gradient field of a convex function. More concretely, we provide examples of gradient field of a twice continuously differentiable convex function, and subgradient field of a piecewise constant convex function with infinitely many pieces, that have unbounded sensitivity. Moreover, we give a necessary and sufficient condition for a linear dynamical system to have bounded sensitivity, in terms of the spectrum of the underlying matrix.

The proofs and the development of our examples involve some intermediate results concerning transformations of dynamical systems, which are also of independent interest. More specifically, we study some transformations that when applied on a dynamical system with bounded sensitivity, preserve the bounded sensitivity property. These transformations include discretization of time and spreading of a system; a transformation that captures approximate solutions of the original system in a certain sense.

I. Introduction

We study a property of dynamical systems, that when satisfied, provides a bound on sensitivity of the state trajectory with respect to additive disturbances. Consider a dynamical system of the form

\[ \dot{x}(t) = f(x(t)), \]

and its perturbed counterpart

\[ \frac{d}{dt} \tilde{x}(t) = f(\tilde{x}(t)) + u(t). \]

The authors are with the Department of Electrical Engineering, Sharif University of Technology, Tehran, Iran; email: sharifnassab@ee.sharif.edu, golestani@sharif.edu
Here, $x(t)$ and $u(t)$ take values in $\mathbb{R}^n$. In order to motivate our results, let's temporarily assume that the system is nonexpansive, in the sense that for any solution $y(\cdot)$ of $\dot{y}(t) = f(y(t))$, and any pair of times $t_1$ and $t_2$ with $t_2 \geq t_1$,

$$\|y(t_2) - x(t_2)\| \leq \|y(t_1) - x(t_1)\|,$$

for a given norm $\|\cdot\|$. In this case, assuming the same initial conditions, $\tilde{x}(0) = x(0)$, a simple integration yields a bound of the form

$$\|\tilde{x}(t) - x(t)\| \leq \int_0^t \|u(s)\| \, ds. \quad (2)$$

However, our goal is to derive stronger bounds, of the form

$$\|\tilde{x}(t) - x(t)\| \leq C \sup_{\tau < t} \|\int_0^\tau u(s) \, ds\|, \quad (3)$$

for some constant $C > 0$ independent of $u(\cdot)$.

A bound of the form (3) is not valid in general. However, it is shown in [11] that a bound of type (3) is valid for the class of Finitely Piecewise Constant Subgradient (FPCS) systems. An FPCS system is, by definition, the (negative) gradient field of a piecewise linear and convex function with finitely many pieces. It is shown in [2] that FPCS systems actually contain the seemingly larger class of nonexpansive finite-partition systems. Finite-partition systems are dynamical systems that have a constant drift over each of the finitely many regions that form a partition of $\mathbb{R}^n$. Such systems are common in control, when dealing with hybrid systems with a finite set of control actions, that can be applied in certain parts of the state space. Examples include communication networks [3], [4], processing systems [5], manufacturing systems and inventory management [6], [7], etc.

A bound of type (3) is particularly useful in dealing with systems driven by stochastic noise. Under usual probabilistic assumptions, $\sup_{\tau < t} \|\int_0^\tau u(s) \, ds\|$ roughly grows as $\sqrt{t}$, whereas $\int_0^t \|u(\tau)\| \, d\tau$ grows at the rate of $t$, with high probability. See [8] for applications of this bound to the analysis of the celebrated Max-Weight policy for real-time job scheduling [3].

In this paper, we investigate the extent to which a bound of type (3) can or cannot generalize to some other classes of dynamical systems of practical interest. We consider linear systems and derive a necessary and sufficient condition for them to have bounded sensitivity. In particular, we show that a linear system admits a bound of the form (3) if and only if it is stable and has no closed orbit. More importantly, we show that the gradient field of a strictly convex function can have unbounded sensitivity. In the same spirit, we provide an example of the
gradient field of a piecewise linear convex function with infinitely many pieces, for which \( (3) \) does not hold. The two latter results are quite counter-intuitive: while the (negative) subgradient field of a piecewise constant convex function with finitely many pieces has bounded sensitivity, the (negative) subgradient field of some continuously differentiable (or even infinitely piecewise linear) convex functions can have unbounded sensitivity. These examples shed some light on the limitations of extending the sensitivity bound to generalizations of FPCS systems, and also on the inevitable complications of any proof for bounded sensitivity of these (FPCS) systems; cf. Section XI for a detailed discussion.

In the course of the proofs of our main results (i.e., unbounded sensitivity of some subgradient dynamical systems), we develop some machineries that are also of independent interest. More specifically, we study some transformations that when applied on a dynamical system with bounded sensitivity, preserve the bounded sensitivity property. In particular, we show for any continuous time dynamical system with bounded sensitivity that its analogous discrete time system also has bounded sensitivity. We establish a similar result when the dynamical system is convolved by a kernel, and when the system is spread, that is allowing for the trajectories to move along the drifts of nearby points.

A seemingly relevant literature is the input-to-state stability \([9], [10], [11], [12], [13], [14], [15]\). As discussed in Section 1 of \([1]\), for a system with additive disturbance, \( \dot{x}(t) = f(x(t)) + u(t) \), the input-to-state stability and a bound of the form \( (3) \) do not imply one another. Moreover, the Lyapunov method \([16]\) that underlies the proofs of input-to-state stability results is not powerful enough to establish sensitivity bounds of type \( (3) \).

The rest of the paper is organized as follows. We start with some definitions and preliminaries in Section II. We then present our main results in Sections III, IV, and V. In Section III we give necessary and sufficient conditions for bounded sensitivity of linear systems. In Section IV we investigates sensitivity of gradient fields of convex functions, and provide examples of differentiable (as well as piecewise constant) convex functions whose subgradient fields have unbounded sensitivity. In Section V we study transformations on dynamical systems that preserve boundedness of sensitivity, and set the stage and provide the required machinery for the proofs of the results of Section IV. We give the proofs of our main results in Sections VI, VII, VIII, IX, and X while relegating some of the details to the appendix, for improved readability. Finally, we discuss our results as well as several open problems and directions of future research in Section XI.
II. Preliminaries

As in [17], we identify a dynamical system with a set-valued function $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ and the associated differential inclusion $\dot{x}(t) \in F(x(t))$. We start with a formal definition, which allows for the presence of perturbations.

**Definition 1 (Perturbed Trajectories).** Consider a dynamical system $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$, and let $U : \mathbb{R} \to \mathbb{R}^n$ be a right-continuous function, which we refer to as the perturbation. Suppose that there exist measurable and integrable functions $\tilde{x}(\cdot)$ and $\zeta(\cdot)$ of time that satisfy

$$
\tilde{x}(t) = \int_0^t \zeta(\tau) \, d\tau + U(t), \quad \forall \ t \geq 0, \\
\zeta(t) \in F(\tilde{x}(t)), \quad \forall \ t \geq 0.
$$

(4)

We then call $U$ the perturbation, and such $\tilde{x}$ and $\zeta$ are called a perturbed trajectory and a perturbed drift, respectively. In the special case where $U$ is identically zero, we also refer to $\tilde{x}$ as an unperturbed trajectory.

We now define two notions of bounded sensitivity, the second of which implies the first.

**Definition 2 (Bounded Sensitivity).** A dynamical system $F(\cdot)$ is said to have bounded sensitivity if there exists a constant $C$ such that for any unperturbed trajectory $x(\cdot)$, any perturbation function $U(\cdot)$, and its corresponding perturbed trajectory $\tilde{x}(\cdot)$ initialized at $\tilde{x}(0) = x(0)$,

$$
\|\tilde{x}(t) - x(t)\| \leq C \sup_{\tau \leq t} \|U(\tau)\|, \quad \forall \ t \geq 0.
$$

(5)

Further, if for any pair $U_1(\cdot)$ and $U_2(\cdot)$ of perturbation functions and their corresponding perturbed trajectories $\tilde{x}_1(\cdot)$ and $\tilde{x}_2(\cdot)$, initialized at $\tilde{x}_1(0) = \tilde{x}_2(0)$,

$$
\|\tilde{x}_1(t) - \tilde{x}_2(t)\| \leq C \sup_{\tau \leq t} \|U_1(\tau) - U_2(\tau)\|, \quad \forall \ t \geq 0,
$$

(6)

then $F(\cdot)$ is said to have bounded sensitivity in strong sense.

A bound of type (6) implies the bound in (5), by simply letting one of the perturbation functions equal to zero.

Throughout the paper we often assume existence of a constant $\gamma$, for which

$$
\|F(x)\| \leq \gamma(1 + \|x\|_2), \quad \forall x \in \mathbb{R}^n
$$

(7)

This assumption is to prevent the solutions from blowing up in finite time.
We say that a dynamical system \( F(\cdot) \) is a subgradient dynamical system if there exists a convex function \( \Phi(\cdot) \), such that for any \( x \in \mathbb{R}^n \), \( F(x) = -\partial\Phi(x) \), where \( \partial\Phi(x) \) denotes the subdifferential of \( \Phi \) at \( x \). If further, \( \Phi(x) \) is of the form
\[
\Phi(x) = \max_i (-\mu_i^T x + b_i),
\]
for some \( \mu_i \in \mathbb{R}^n \), \( b_i \in \mathbb{R} \), and with \( i \) ranging over a finite set, we say that \( F \) is a Finitely Piecewise Constant Subgradient (FPCS, for short) system. It is shown in [1] that any FPCS system has bounded sensitivity.

**Theorem 1** (Theorem 1 of [1]). Every FPCS system has bounded sensitivity in the sense of (5).

In the rest of the section we briefly discuss quasi-convexity. A function \( f : \mathbb{R}^n \to \mathbb{R} \) is said to be quasi-convex if all of its sub-level-sets are convex sets. Equivalently, for any \( x, y \in \mathbb{R}^n \) and any \( \lambda \in (0, 1) \),
\[
f(\lambda x + (1 - \lambda)y) \leq \max (f(x), f(y)).
\]
(8)

It further, for any \( x, y \in \mathbb{R}^n \) and any \( \lambda \in (0, 1) \), (8) holds with strict inequality, then \( f \) is said to be strictly quasi-convex. Under some mild assumptions, one can convexify any strictly quasi-convex function [18]:

**Lemma 1** (Corollary 1 of [18]). For any continuously twice differentiable and strictly quasi-convex function \( f : \mathbb{R}^n \to \mathbb{R} \) with compact level-sets, there exists an increasing and continuously twice differentiable function \( h : \mathbb{R} \to \mathbb{R} \) such that \( h \circ f \) is strictly convex.

Finally, we denote by \( \mathbb{R}_+ \) and \( \mathbb{Z}_+ \) the sets of non-negative real numbers and non-negative integers, respectively.

**III. Sensitivity of Linear Systems**

In this section we present a necessary and sufficient condition for a linear dynamical system to have bounded sensitivity. A linear dynamical system is a system of the form \( \dot{x} = Ax \), defined in terms of a square matrix \( A \). Before going over our result for linear systems, we define a property for general dynamical systems.

**Definition 3** (Stable and Orbit-Free Systems). A dynamical system is said to be stable if every unperturbed trajectory stays in a bounded region, and it is orbit-free if no unperturbed trajectory is a periodic orbit. We use the shorthand SOF for a stable and orbit-free system.
The following lemma provides well-known facts on the stability of linear systems.

**Lemma 2.** A linear system $\dot{x} = Ax$ is SOF if and only if its eigenvalues are either zero or have negative real parts, and the multiplicity of the zero eigenvalue equals the dimension of the associated eigenspace.

The next theorem shows that a linear system has bounded sensitivity if and only if it is SOF.

**Theorem 2** (Sensitivity of Linear Systems). A linear dynamical system has bounded sensitivity in the (strong) sense of (6), if and only if it is SOF. Further, every non-SOF linear system has unbounded sensitivity in the sense of (5). Moreover, for an SOF dynamical system in which $A$ is diagonalizable (of the form $A = P\Lambda P^{-1}$), (6) is satisfied by the following constant,

$$C = 1 + \frac{\sigma_{\text{max}}}{\sigma_{\text{min}}} \sum_{i} \frac{|\lambda_i|}{|\text{Re}(\lambda_i)|},$$

(9)

where $\lambda_i$’s are eigenvalues of $A$, and $\sigma_{\text{max}}$ and $\sigma_{\text{min}}$ are the largest and smallest singular values of $P$, respectively. In the special case that $A$ is symmetric, (9) is simplified to $C = n + 1$.

The proof is given in Section VI. The proof relies on a closed form expression for the solutions of linear differential equations and the Jordan normal form of the underlying matrix $A$.

**IV. Sensitivity of Subgradient Dynamical Systems**

In this section, we present examples of subgradient dynamical systems that have unbounded sensitivity. In particular, we show that a bound of form (5) is not necessarily valid for a dynamical system driven by the (negative) gradient of a twice differentiable and convex function, as well as a dynamical system driven by subgradient of a piecewise constant and convex function with infinitely many pieces. These examples shed some light on the limits and challenges of generalizing Theorem I. More concretely, the examples suggest that Theorem I will no longer hold true if we remove/weaken any of its assumptions.

For the ease of presentation, throughout this section we will be working with polar and cylindrical coordinates. For a trajectory $x : \mathbb{R} \rightarrow \mathbb{R}^3$, and for a fixed $t \in \mathbb{R}_+$, we represent the location of $x(t)$ in the cylindrical coordinates by $(r, \phi, z)$, which equals $(r \cos \phi, r \sin \phi, z)$ in the Cartesian coordinates. At the same point, $x(t)$, we denote the vector $\dot{x}(t)$ in the local cylindrical coordinates by $\alpha \hat{r} + \phi \hat{\phi} + \gamma \hat{z}$, which equals $\alpha \cos \phi \hat{x} + \alpha \sin \phi \hat{y} + \gamma \hat{z}$ in the Cartesian coordinates.
In our first example, we consider a nonexpansive dynamical system and show that it has unbounded sensitivity. The example is simple by itself and is an immediate consequence of Theorem 2. Having said that, we include it here to build intuition and set the stage for the more elaborate examples that follow.

**Example 1.** (A nonexpansive system with unbounded sensitivity) Consider the two-dimensional linear dynamical system \( \dot{x} = F(x) = r \hat{\phi} \) in the polar coordinates. The system is nonexpansive, and its trajectories are circular orbits centered at the origin. Fig. 1 shows an illustration of the trajectories of this system. Consider an unperturbed trajectory \( x(t) = (1, t) \) in the polar coordinates. For any \( t \in \mathbb{R}_+ \), let \( \tilde{x}(t) = (t + 1, t) \) in the polar coordinates, and \( u(t) = 1 \hat{r} \) be the differential perturbation in the local polar coordinates at \( \tilde{x}(t) \). Then, for any \( t \in \mathbb{R}_+ \),

\[
\frac{d}{dt} \tilde{x}(t) = 1 \hat{r} + (t + 1) \hat{\phi} = F(\tilde{x}(t)) + u(t).
\]

Therefore, \( \tilde{x}(t) \) is a perturbed trajectory corresponding to perturbation \( U(t) = \int_0^t u(\tau) \, d\tau \). Moreover, \( \|U(t)\| = \| \int_0^t u(\tau) \, d\tau \| \leq \| \int_0^t \cos(\tau) \, d\tau \| + \| \int_0^t \sin(\tau) \, d\tau \| \leq 2 \), which is bounded. However, \( \|\tilde{x}(t) - x(t)\| = t \), which grows unbounded for large \( t \). Therefore, no constant \( C \) can satisfy (5), and the system has unbounded sensitivity.

In the next example we present the main result of this section, showing that the sensitivity of a subgradient field is not necessarily bounded.
Example 2 (Gradient field of a strictly convex function with unbounded sensitivity).

In cylindrical coordinates, consider the half cylinder

$$\Omega = \{(r, \phi, z) \mid r \leq 1/4, z \leq -1\},$$  \hspace{1cm} (10)

and let \(f(r, \phi, z)\) be a solution of the following equation over \(\Omega\):

$$f + z - \frac{1}{f} \ln \left( \cosh \left( fr \sin(f - \phi) \right) \right) - \frac{r^2}{1 + f} = 0.$$  \hspace{1cm} (11)

The following lemma shows that \(f\) is well-defined and is strictly quasi-convex.

**Lemma 3.**  
\(a)\) For any \((r, \phi, z) \in \Omega\), there is a unique \(f \geq 1\) that satisfies (11).

\(b)\) For any \(\alpha \geq 1\), the level-set \(f(r, \phi, z) = \alpha\) is a surface of the form

$$z(r, \phi) = -\alpha + \frac{1}{\alpha} \ln \left( \cosh \left( ar \sin(\alpha - \phi) \right) \right) + \frac{r^2}{1 + \alpha}.$$  \hspace{1cm} (12)

\(c)\) \(f\) is a smooth and strictly quasi-convex function, and its level-sets are compact.

The proof of the lemma is given in Appendix A.

For the intuition behind the definition of \(f\), note that for sufficiently large \(f\) (when \(z\) goes to \(-\infty\)), \(\ln \left( \cosh \left( fr \sin(f - \phi) \right) \right) / f \approx r \sin(f - \phi)\). Then, (12) implies that for sufficiently large \(\alpha\), the level set \(f = \alpha\) is very close to the surface \(z(r, \phi) = -\alpha + r |\sin(\alpha - \phi)|\). This surface has the shape of an opened book, and the books rotate as \(\alpha\) varies. An illustration of different level-sets of \(f\) is shown in Fig. 2 (a). In light of the rotating books analogy, we can show that the gradient field of \(f\) admits spring-shaped unperturbed trajectories and diverging spiral-shaped perturbed trajectories of the forms depicted in Fig. 2 (b). Having discussed the insight, we proceed to construct the desired convex function.

It follows from Lemma 3 (c) and Lemma 1 that there exists an increasing and twice continuously differentiable function \(h : \mathbb{R} \rightarrow \mathbb{R}\) such that \(h \circ f\) is a strictly convex function. Let \(\Phi \triangleq h \circ f\) and \(F\) be the gradient field of \(\Phi\). Then,

**Theorem 3.** \(F\) has unbounded sensitivity, even when its domain is restricted to

$$\Omega_{\zeta} \triangleq \{(r, \phi, z) \mid r \leq 1/6, z < \zeta\},$$  \hspace{1cm} (13)

for all \(\zeta \leq -1\).
Fig. 2. The dynamical system of Example 2 that has unbounded sensitivity. (a) shows the level sets of the quasi-convex function $f$ defined in (11) and (b) depicts a pair of perturbed (red) and unperturbed (black) trajectories (traversing the curves upwards).

The proof of the theorem is given in Section VIII and goes by showing that $F$ admits spiral trajectories of the form depicted in Fig. 2 (b). In the course of the proof we need the machinery that is developed later in Section V-A.

In the next example, we provide a counterpart of Example 2 for subgradient systems that are piecewise constant with infinitely many pieces.

**Example 3** (Gradient field of a piecewise linear convex function that has unbounded sensitivity). Consider the convex potential function $\Phi$ of Example 2 whose gradient field $F$ has unbounded sensitivity. We construct a piecewise constant approximation $\Psi$ of $\Phi$ with infinite number of pieces, such that the approximation error tends to zero as $z$ goes to $-\infty$. To do this, we consider a fine grid within the half-cylinder $\Omega$, defined in (10), with increasing resolution as $z$ goes to $-\infty$, and a corresponding triangulation of $\Omega$ with simplexes. We then let $\Psi(p) = \Phi(p)$ on the grid points $p$, and let $\Psi$ be the linear interpolation inside each simplex. The resulting $\Psi$ is a convex function. Let $H$ be the (sub)gradient field of $\Psi$. Since the resolution of grid points increases as $z$ goes to $-\infty$, for sufficiently small values of $z$, $H$ would give a good approximation of $F$, and we can use Theorem 3 to deduce unbounded sensitivity.
also for $H$. We now state the main result in the following theorem, and leave the detailed construction and the proof to Section IX.

**Theorem 4.** *There exists a piecewise linear convex function whose (sub)gradient field has unbounded sensitivity.*

Example 3 shows that a piecewise constant subgradient field with infinitely many pieces can have unbounded sensitivity. This is in contrast to Theorem 1, according to which the sensitivity of any piecewise constant subgradient field with a finite number of pieces is bounded. We conclude that the assumption of finiteness of the number of pieces is indeed necessary for Theorem 1 to be true.

V. **Transformations that Preserve bounded sensitivity**

In this section, we study transformations on a dynamical system that preserve bounded sensitivity, and provide the required machinery for the proof of Theorem 3. In particular, we show for any dynamical system with bounded sensitivity that discretization of time and spreading will preserve bounded sensitivity, up to an additive constant. The section comprises two subsections, each devoted to one of these transformations. The result of the first subsection on spreading a system (cf. Theorem 5) is used later in the proof of Theorem 3. There, we also give a corollary, on convolution of a system with a kernel, that is possibly of independent interest. In the second subsection, we prove that discrete time counterparts of continuous time systems inherit the bounded sensitivity property from the underlying continuous time system, showing the soundness of the concept.

A. **Spreading a Systems**

In this subsection, we consider the spreading of a dynamical system $F$, and will show that if $F$ has bounded sensitivity, then its spread systems also have bounded sensitivity in a weaker sense. The main result of this subsection not only provides insight into the sensitivity of approximate trajectories of a system, but also serves as a stepping-stone for many more results, including the proof of Theorem 3.

We start with a definition:
**Definition 4 (ϵ-Spread System).** Consider a dynamical system $F(\cdot)$, and an $\epsilon \geq 0$. For every point $x$ in the domain, let 

$$\tilde{F}_\epsilon(x) = \text{Conv}\{\xi \mid \xi \in F(y), y \in B_\epsilon(x)\},$$

where $B_\epsilon(x)$ is the closed Euclidean ball of radius $\epsilon$ centered at $x$, and for a subset $S$ of $\mathbb{R}^n$, Conv$(S)$ stands for the convex hull of $S$. Then, we refer to $\tilde{F}_\epsilon$ as the $\epsilon$-spread of $F$.

The definition of a spread-system allows for the trajectories to follow the drift of a neighbouring point. Such models find applications in control systems, where the control applied is chosen on the basis of noisy state measurements. Therefore, given an initial point $p$, the unperturbed trajectories of $\tilde{F}_\epsilon$ that emanate from $p$ are not typically unique. In this view, trajectories of $\tilde{F}_\epsilon$ can be perceived as approximate solutions of $F$.

There are several notions of a generalized solution of a differential equation, including weak solutions [19] and viscosity solutions [20], to name some. These are the solutions that satisfy the differential equation almost everywhere, while allowing for non-differentiability at some zero-measure set of times. In contrast, a solution of a spread system of $F$ may satisfy the differential equation $\dot{x} \in F(x)$ at no point of time whatsoever. In fact, the generalized solution of a differential equation are primarily developed to deal with non-differentiability of the solutions, while spread-systems allow for uncertainty about the current state, and lead to a notion of approximate solutions.

Despite the fact that several spread solutions can emerge form the same initial point, it turns that if a system has bounded sensitivity, then a weaker notion of bounded sensitivity still pertains to its spread systems.

**Theorem 5 (Sensitivity of Spread Systems).** Consider a dynamical system $F$ and an $\epsilon > 0$. Let $\bar{x}(\cdot)$ be a perturbed trajectory of $\epsilon$-spread system, $\tilde{F}_\epsilon$, of $F$, corresponding to perturbation $U(\cdot)$.

(a) Suppose that (5) is valid with constant $C$, and let $x(\cdot)$ be an unperturbed trajectory of $\tilde{F}_\epsilon$, initialized at $x(0) = \bar{x}(0)$. Then, for any $t \geq 0$,

$$\|\bar{x}(t) - x(t)\| \leq C \left( 2\epsilon + \sup_{\tau \leq t} \|U(\tau)\| \right) + 3\epsilon. \quad (15)$$

(b) Suppose that (6) is valid with constant $C$, and let $\bar{x}'(\cdot)$ be a perturbed trajectory of $\tilde{F}_\epsilon$, corresponding to perturbation $U'(\cdot)$, and initialized at $\bar{x}'(0) = \bar{x}(0)$. Then, for any $t \geq 0$,

$$\|\bar{x}(t) - \bar{x}'(t)\| \leq C \left( 2\epsilon + \sup_{\tau \leq t} \|U(\tau) - U'(\tau)\| \right) + 3\epsilon, \quad (16)$$
The proof is given in Section VII, and involves constructing a perturbation function, the spread trajectory of which is a corresponding perturbed trajectory of the initial system.

Conversely, if (15) holds for all spread systems of $F$, then by letting $\epsilon \to 0$, it follows that $F$ has bounded sensitivity in the sense of (5).

We wish to point that a similar phenomenon has been previously studied in the literature of input-to-state stability [14]. Given an external disturbance $u(\cdot)$ and an $\epsilon > 0$, it is shown in [14] that if the dynamical system $\dot{x} \in F(x,u)$ is input-to-state stable, then the system $\dot{x} \in \bigcup_{y \in B_\epsilon(x)} F(y,u)$ is also input-to-state stable, for sufficiently small values of $\epsilon$.

Leveraging Theorem 5, in the rest of this subsection we study a special type of transformation; and show that convolution with a kernel preserves bounded sensitivity.

**Definition 5** (Kernel and Convolution). For an $\epsilon > 0$, an $\epsilon$-kernel is any integrable function $h : B_\epsilon(0) \to \mathbb{R}_+$. Given a dynamical system $F$ and an $\epsilon$-kernel $h$, we define their convolution $F * h$ as

$$ (F * h)(x) = \int_{B_\epsilon(x)} \xi_x(y) h(x - y) \, dy, \quad \forall x \in \mathbb{R}^n; \quad (17) $$

where $\xi_x(y)$ is an arbitrary vector in $F(y)$.

**Corollary 1.** Consider a dynamical system $F$ for which (5) is valid, and let $h(\cdot)$ be an $\epsilon$-kernel, for some $\epsilon > 0$. Then, for any unperturbed trajectory $x(\cdot)$ of $F * h$, and any perturbed trajectory $\tilde{x}(\cdot)$ of $F * h$ corresponding to perturbation $U(\cdot)$,

$$ \|\tilde{x}(t) - x(t)\| \leq C \left( 2\epsilon + \sup_{\tau \leq t} \|U(\tau)\| \right) + 3\epsilon, \quad \forall t \in \mathbb{R}_+. \quad (18) $$

**Proof.** Without loss of generality, we assume that $\int_{B_\epsilon(0)} h(x) \, dx = 1$. Then, for any $x \in \mathbb{R}^n$,

$$ (F * h)(x) \in \text{Conv}\{\xi \mid \xi \in F(y), y \in B_\epsilon(x)\} = \tilde{F}_\epsilon(x), \quad (19) $$

where $\tilde{F}_\epsilon(x)$ is the $\epsilon$-spread of $F$. Therefore, every perturbed (respectively, unperturbed) trajectory of $(F * h)$ is also a perturbed (unperturbed) trajectory of $\tilde{F}_\epsilon(x)$. The theorem then follows from Theorem 5. \qed

A similar result is possible also for sensitivity bounds of the form (6).
B. Discretization of Time

A discrete time trajectory is attained by taking (small) steps along the drifts of a continuous time system. Formally,

\textbf{Definition 6} (Discrete Time Trajectories). Consider a dynamical system $F : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ and a function $V : \mathbb{Z}_+ \rightarrow \mathbb{R}^n$, which we refer to as discrete time perturbation. Suppose that there exist functions $z : \mathbb{Z}_+ \rightarrow \mathbb{R}^n$ and $\mu : \mathbb{Z}_+ \rightarrow \mathbb{R}^n$ such that
\[
z(t + 1) = \sum_{k \leq t} \mu(k) + V(t), \quad \forall t \in \mathbb{Z}_+,
\]
\[
\mu(t) \in F(z(t)), \quad \forall t \in \mathbb{Z}_+. \tag{20}
\]
We then call $z(\cdot)$ a \textit{discrete time trajectory} corresponding to the perturbation function $V(\cdot)$.

Discrete time trajectories correspond to systems that operate in slotted times. Examples include the queue lengths dynamics of job scheduling algorithms [4]. The following theorem shows that in any system whose continuous time trajectories have bounded sensitivity, a similar property also holds for its discrete time trajectories.

\textbf{Theorem 6} (Sensitivity in Discrete Time). Consider a dynamical system $F$, and let $z(\cdot)$ be a discrete time perturbed trajectory of $F$ corresponding to perturbation function $V(\cdot)$. For every $k \in \mathbb{Z}_+$, let $\mu_k \triangleq z(k + 1) - z(k) - V(k)$, which is an element of $F(z(k))$ (cf. (20)).

(a) Suppose that a bound of type (5) is valid with constant $C$. Let $x(\cdot)$ be the continuous time unperturbed trajectory of $F$ initialized at $x(0) = z(0)$. Then, for any $k \in \mathbb{Z}_+$,
\[
\|x(k) - z(k)\| \leq C \left( \max_{j < k} \|\mu_j\| + \max_{j < k} \|V(j)\| \right). \tag{21}
\]

(b) Suppose that a bound of type (6) is valid with constant $C$. Consider a discrete time perturbation function $V'(\cdot)$ and the corresponding discrete time perturbed trajectory $z'(\cdot)$ initialized at $z'(0) = z(0)$. For any $k \in \mathbb{Z}_+$, let $\mu'_k \triangleq z'(k + 1) - z'(k) - V'(k)$. Then, for any $k \in \mathbb{Z}_+$,
\[
\|z'(k) - z(k)\| \leq C \left( \max_{j < k} \|\mu_j - \mu'_j\| + \max_{j < k} \|V(i) - V'(i)\| \right). \tag{22}
\]

The proof is given in Section [X]. The high level idea is to simulate the discrete time perturbed trajectories by continuous time perturbed trajectories, and take advantage of the bounded sensitivity properties (5) and (6).
In Theorem 6, unlike its continuous time counterparts, the deviation bound also depends on the maximum jump size of the discrete time system, i.e., \( \|\mu_k\| \). This dependence is inevitable because the distance between continuous time and discrete time trajectories cannot go arbitrarily small, even if the perturbation term is zero.

The next corollary is a consequence of Theorems 1 and 6.

**Corollary 2.** Consider an FPCS system, a continuous time unperturbed trajectory \( x(\cdot) \) and a discrete time perturbed trajectory \( z(\cdot) \) corresponding to perturbation \( V(\cdot) \). Then, for any \( k \in \mathbb{Z}_+ \),

\[
\|x(k) - z(k)\| \leq C \left( \mu_{\max} + \max_{j<k} \|\sum_{i=0}^{j} V(i)\| \right),
\]

where \( C \) is the constant of Theorem 7 and \( \mu_{\max} \) is a constant independent of the trajectories.

**VI. Proof of Theorem 2**

We start with a well-known result on solvability of linear dynamical systems.

**Lemma 4 (Solution of a Linear System).** Given a measurable perturbation function \( U(\cdot) \), and an initial condition \( x(0) = x_0 \), the linear dynamical system \( \dot{x} = Ax \) has a unique perturbed trajectory, of the following form

\[
x(t) = e^{At}x(0) + U(t) + Ae^{At} \int_0^t e^{-A\tau} U(\tau) \, d\tau, \quad \forall t \geq 0.
\]

The proof is given in Appendix C.

Consider a pair \( U_1(\cdot) \) and \( U_2(\cdot) \) of perturbation functions and a pair of corresponding perturbed trajectories \( \tilde{x}_1(\cdot) \) and \( \tilde{x}_2(\cdot) \), with the same initial condition \( \tilde{x}_1(0) = \tilde{x}_2(0) \). Then, Lemma 4 implies that for any \( t \geq 0 \),

\[
\|\tilde{x}_1(t) - \tilde{x}_2(t)\| = \|U_1(t) - U_2(t)\| + \left\| \int_0^t e^{A\tau} \left( U_1(t-\tau) - U_2(t-\tau) \right) \, d\tau \right\|
\]

\[
\leq \|U_1(t) - U_2(t)\| + \left\| \int_0^t Ae^{A\tau} \left( U_1(t-\tau) - U_2(t-\tau) \right) \, d\tau \right\|.
\]

Consider the Jordan normal form of \( A \),

\[
A = PDP^{-1}, \quad D = \Lambda + B,
\]

where \( \Lambda \) is the diagonal matrix of eigenvalues of \( A \), and \( B \) is a matrix with some superdiagonal entries equal to one, and all other entries equal to zero. It follows that

\[
e^{At} = Pe^{Dt}P^{-1}.
\]
Fix a \( t > 0 \), and let
\[
\theta \triangleq \sup_{\tau \leq t} \|U_1(t - \tau) - U_2(t - \tau)\|. \tag{28}
\]
For any \( \tau \in [0, t] \), let
\[
V(\tau) \triangleq P^{-1} \left( U_1(t - \tau) - U_2(t - \tau) \right). \tag{29}
\]
Then, for any \( \tau \in [0, t] \),
\[
\|V(\tau)\| \leq \frac{\theta}{\sigma_{\text{min}}}, \tag{30}
\]
where \( \sigma_{\text{min}} \) is the smallest singular value of \( P \). Since \( P \) is invertible, \( \sigma_{\text{min}} > 0 \). It follows from (25) and (27) that
\[
\|\tilde{x}_1(t) - \tilde{x}_2(t)\| \leq \|U_1(t) - U_2(t)\| + \int_0^t A e^{A\tau} \left( U_1(t - \tau) - U_2(t - \tau) \right) d\tau = \|U_1(t) - U_2(t)\| + \int_0^t PDP^{-1} Pe^{D\tau} P^{-1} \left( U_1(t - \tau) - U_2(t - \tau) \right) d\tau = \|U_1(t) - U_2(t)\| + \|P\int_0^t e^{D\tau} V(\tau) d\tau\|. \tag{31}
\]
From the SOF assumption and Lemma 2 in the Jordan decomposition, every block associated with the zero eigenvalue has unit size. Equivalently, \( D \) has the following form
\[
D = \begin{bmatrix}
0 & 0 \\
0 & \tilde{D}
\end{bmatrix}, \tag{32}
\]
where \( \tilde{D} \) comprises the blocks of \( D \) corresponding to non-zero eigenvalues. Then, for any \( \tau \geq 0 \),
\[
e^{D\tau} = \begin{bmatrix}
I & 0 \\
0 & e^{\tilde{D}\tau}
\end{bmatrix}. \tag{33}
\]
For any \( \tau \geq 0 \), consider the decomposition
\[
V(\tau) = \begin{bmatrix}
V_0(\tau) \\
V_1(\tau)
\end{bmatrix}, \tag{34}
\]
where \( V_0 \) is a vector of length equal to the multiplicity of the zero eigenvalue in \( A \). Therefore, for any \( \tau \geq 0 \),
\[
\|De^{D\tau} V(\tau)\| = \left\| \begin{bmatrix}
0 \\
\tilde{D}e^{\tilde{D}\tau} V_1(\tau)
\end{bmatrix} \right\| = \|\tilde{D}e^{\tilde{D}\tau} V_1(\tau)\|. \tag{35}
\]
For a matrix $M$, we denote its Frobenius norm by $\|M\|$. We also let $\sigma_{\text{max}}$ be the largest singular value of $P$. It follows from (31) that

$$\|	ilde{x}_1(t) - \tilde{x}_2(t)\| \leq \|U_1(t) - U_2(t)\| + P \int_0^t D e^{D\tau} V(\tau) d\tau \leq \|U_1(t) - U_2(t)\| + \sigma_{\text{max}} \int_0^t \|D e^{D\tau} V(\tau)\| d\tau = \|U_1(t) - U_2(t)\| + \sigma_{\text{max}} \int_0^t \|\tilde{D} e^{\tilde{D}\tau} V_1(\tau)\| d\tau \leq \theta + \sigma_{\text{max}} \int_0^t \|\tilde{D} e^{\tilde{D}\tau}\| \|V_1(\tau)\| d\tau \leq \left(1 + \frac{\sigma_{\text{max}}}{\sigma_{\text{min}}} \int_0^\infty \|\tilde{D} e^{\tilde{D}\tau}\| d\tau\right) \theta,$$

(36)

where the equality is from (35) and the last inequality is due to (30). Since all eigenvalues of $\tilde{D}$ have negative real part, the integral in the right hand side of (36) is finite. Then, bounded sensitivity of the SOF linear system follows from (36).

For the second part, if the linear system is not SOF, then it is either unstable or has a periodic orbit. If the system is unstable, then a small perturbation at time zero can cause a perturbed trajectory $\tilde{x}_1(\cdot)$ with initial condition $\tilde{x}_1(0) = 0$ to have $\lim_{t \to \infty} \|\tilde{x}_1(t)\| = \infty$. On the other hand, $x(t) = 0$, for all $t \geq 0$, is an unperturbed trajectory. Then, the distance between $\tilde{x}_1(\cdot)$ and the unperturbed trajectory $x(\cdot)$ grows unbounded. In the second case, if the system is not orbit-free, consider a periodic orbit $x(t) = e^{At}x_0$, with $x(t_0) = x(0) = x_0$, for some $t_0 > 0$. Then, $u(t) = e^{At}x_0$ has a bounded integral. However, $\tilde{x}(t) = (t+1)e^{At}x_0$ satisfies the differential equation $\frac{d}{dt}\tilde{x}(t) = A\tilde{x}(t) + u(t)$, and hence is a perturbed trajectory whose deviation from $x(t)$ is unbounded as $t$ goes to infinity. We conclude that, in either case, if the linear system is not SOF, a bounded perturbation can cause unbounded changes in the trajectories, and there exists no constant $C$ under which (6) holds.

For the third part, if $A$ is diagonalizable, then $\tilde{D}$ is a diagonal matrix with the non-zero eigenvalues of $A$ on its main diagonal. Therefore,

$$\int_0^\infty \|\tilde{D} e^{\tilde{D}\tau}\| d\tau \leq \int_0^\infty \sum_{i=1}^n |\lambda_i| e^{\Re(\lambda_i)\tau} d\tau = \sum_{i=1}^n \frac{|\lambda_i|}{\Re(\lambda_i)}.$$

(37)

Plugging (37) into (36) implies (9).

Finally, if $A$ is symmetric, then all eigenvalues are real and $P$ is orthonormal. Therefore, $|\lambda_i|/|\Re(\lambda_i)| = 1$ and $\sigma_{\text{max}} = \sigma_{\text{min}} = 1$. Then, (9) can be further simplified into $\|\tilde{x}_1(t) - \tilde{x}_2(t)\| \leq (n + 1)\theta$.  

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VII. Proof of Theorem [5]

We start with two lemmas, and then give the proof of Theorem [5].

Lemma 5. Consider a dynamical system $F$ and its $\epsilon$-spread system $\tilde{F}_\epsilon$, for some $\epsilon > 0$.

a) Let $\tilde{x}(\cdot)$ be a perturbed trajectory of $\tilde{F}_\epsilon$, corresponding to perturbation $U(\cdot)$. Then, for any $\delta > 0$, there exists a perturbation $U'(\cdot)$, and a corresponding perturbed trajectory $\tilde{y}(\cdot)$ of $F$, such that for any $t \geq 0$,

\[ \| \tilde{y}(t) - \tilde{x}(t) \| \leq \epsilon + \delta, \]  

(38)

\[ \sup_{\tau \leq t} \| U'(\tau) \| \leq \sup_{\tau \leq t} \| U(\tau) \| + \epsilon + \delta. \]  

(39)

b) For any pair $U_1(\cdot)$ and $U_2(\cdot)$ of perturbations and corresponding pair $\tilde{x}_1(\cdot)$ and $\tilde{x}_2(\cdot)$ of perturbed trajectories of $\tilde{F}_\epsilon$, and for any $\delta > 0$, there exists a pair $U'_1(\cdot)$ and $U'_2(\cdot)$ of perturbations and corresponding pair $\tilde{y}_1(\cdot)$ and $\tilde{y}_2(\cdot)$ of perturbed trajectories of $F$ such that

\[ \| \tilde{y}_i(t) - \tilde{x}_i(t) \| \leq \epsilon + \delta, \quad i = 1, 2, \]  

(40)

\[ \sup_{\tau \leq t} \| U'_1(\tau) - U'_2(\tau) \| \leq \sup_{\tau \leq t} \| U_1(\tau) - U_2(\tau) \| + 2\epsilon + \delta. \]  

(41)

The proof is elaborate and is given in Appendix [B]. The convex hull, $\text{Conv}(\cdot)$ in the definition of $\tilde{F}_\epsilon$ brings a tremendous amount of complication to the proof of Lemma [5]. Here, to see the main idea, we present and prove a simpler counterpart of Lemma [5].

Lemma 6. Consider a dynamical system $F$ and an $\epsilon > 0$. For any $x \in \mathbb{R}^n$, let

\[ \hat{F}(x) = \{ \xi \mid \xi \in F(y), \ y \in B_\epsilon(x) \}. \]  

(42)

Let $\tilde{x}(\cdot)$ be a perturbed trajectory of $\hat{F}$, corresponding to some perturbation $U(\cdot)$. Then, there exist a perturbation $U'(\cdot)$, and a corresponding perturbed trajectory $\tilde{y}(\cdot)$ of $F$ that satisfy (38) and (39) for $\delta = 0$.

Proof. By definition,

\[ \tilde{x}(t) = \tilde{x}(0) + \int_0^t \xi(\tau) \, d\tau + U(t), \quad \forall t \geq 0, \]  

(43)

where $\xi(\tau) \in \hat{F}(\tilde{x}(\tau))$. Therefore, for any $\tau$, there exists a $\tilde{y}(\tau)$ in the $\epsilon$-neighbourhood of $\tilde{x}(\tau)$, such that $\xi(\tau) \in F(\tilde{y}(\tau))$. Then, for any $t$,

\[ \tilde{y}(t) = \tilde{x}(t) + \left( \tilde{y}(t) - \tilde{x}(t) \right) = \tilde{x}(0) + \int_0^t \xi(\tau) \, d\tau + \left( U(t) + \tilde{y}(t) - \tilde{x}(t) \right). \]  

(44)
Hence, \( \tilde{y}(\cdot) \) is a perturbed trajectory of \( F \) associated with perturbation \( U'(t) = U(t) + \tilde{y}(t) - \tilde{x}(t) \). Since for any \( t \geq 0, \| \tilde{y}(t) - \tilde{x}(t) \| \leq \epsilon \), then \( \| U'(t) \| \leq \| U(t) \| + \epsilon \), and the lemma follows.

\[ \square \]

**Proof of Theorem 5.** For Part (a), let

\[ \delta = \frac{\epsilon}{2(C + 1)}. \]  

\[ (45) \]

From Lemma 5, there exists a pair \( U_1(\cdot) \) and \( U_2(\cdot) \) of perturbation functions and a corresponding pair \( \tilde{y}_1(\cdot) \) and \( \tilde{y}_2(\cdot) \) of perturbed trajectories of \( F \) such that for any \( t \geq 0 \),

\[ \| \tilde{y}_1(t) - \tilde{x}(t) \| \leq \epsilon + \delta, \]

\[ \| \tilde{y}_2(t) - x(t) \| \leq \epsilon + \delta, \]

\[ \| \sup_{\tau \leq t} U'_1(\tau) \| \leq \| \sup_{\tau \leq t} U(\tau) \| + \epsilon + \delta \]

\[ \| \sup_{\tau \leq t} U'_2(\tau) \| \leq \epsilon + \delta. \]  

\[ (46) \]

Let \( y(\cdot) \) be an unperturbed trajectory of \( F \) initialized at \( \tilde{y}(0) = x(0) \). Then,

\[ \| \tilde{x}(t) - x(t) \| \leq \| \tilde{x}(t) - \tilde{y}_2(t) \| + \| \tilde{y}_2(t) - y(t) \| + \| y(t) - \tilde{y}_1(t) \| + \| \tilde{y}_1(t) - x(t) \| \]

\[ \leq (\epsilon + \delta) + \| \tilde{y}_2(t) - y(t) \| + \| y(t) - \tilde{y}_1(t) \| + (\epsilon + \delta) \]

\[ \leq C \sup_{\tau \leq t} \| U'_2(\tau) \| + C \sup_{\tau \leq t} \| U'_1(\tau) \| + 2\epsilon + 2\delta \]

\[ \leq C \left( \sup_{\tau \leq t} \| U(\tau) \| + \epsilon + \delta \right) + C(\epsilon + \delta) + 2\epsilon + 2\delta \]

\[ = C \left( 2\epsilon + \sup_{\tau \leq t} \| U(\tau) \| \right) + 3\epsilon, \]

where the equations are due to the triangle inequality, \[ (46), \] again \[ (46), \] and \[ (45) \], respectively. This completes the proof of Part (a).

The proof of Part (b) is similar to the proof of Part (a). In view of Lemma 5 (b), consider a pair \( U_1(\cdot) \) and \( U_2(\cdot) \) of perturbations and a corresponding pair \( \tilde{y}_1(\cdot) \) and \( \tilde{y}_2(\cdot) \) of perturbed trajectories of \( F \) such that for any \( t \geq 0 \),

\[ \| \tilde{y}_i(t) - \tilde{x}_i(t) \| \leq \epsilon + \delta, \quad i = 1, 2, \]

\[ \sup_{\tau \leq t} \| (U'_1(\tau) - U'_2(\tau)) \| \leq \sup_{\tau \leq t} \| (U_1(\tau) - U_2(\tau)) \| + 2\epsilon + \delta. \]  

\[ (48) \]
Then,
\[
\|\tilde{x}(t) - x(t)\| \leq \|\tilde{x}(t) - \tilde{y}_2(t)\| + \|\tilde{y}_2(t) - \tilde{y}_1(t)\| + \|\tilde{y}_1(t) - \tilde{x}_1(t)\|
\]
\[
\leq \|\tilde{y}_2(t) - \tilde{y}_1(t)\| + 2(\epsilon + \delta)
\]
\[
\leq C \sup_{\tau \leq t} \|U'_{\tau} - U'_{\tau'}\| + 2(\epsilon + \delta)
\]
\[
\leq C \left( \sup_{\tau \leq t} \|U_2(\tau) - U_1(\tau)\| + 2(\epsilon + \delta) \right) + 2(\epsilon + \delta)
\]
\[
\leq C \left( 2\epsilon + \sup_{\tau \leq t} \|U_2(\tau) - U_1(\tau)\| \right) + 3\epsilon,
\]
where the relations are due to the triangle inequality, (48), (6), again (48), and (45), respectively. This completes the proof of Theorem 5.

VIII. Proof of Theorem 3

Here, we prove Theorem 3 by first showing that the spread systems of F admit spiral trajectories of the form depicted in Fig. 2 (b). We then conclude that the spread systems of F has unbounded sensitivity, in the sense that no constant C can satisfy (15). Finally, we use Theorem 5 to show that F has unbounded sensitivity.

Fix an \(\epsilon > 0\) and let \(\tilde{F}_\epsilon\) be the \(\epsilon\)-spread system of F.

Claim 1. There exists a constant \(z_\epsilon\) and a smooth function \(\alpha : \Omega \to \mathbb{R}_+\), such that for any \(r < 1/4\), any \(z < z_\epsilon\), and point \(p = (r, z, z)\) in the cylindrical coordinates,
\[
\frac{-2r}{1-z} \hat{r} + r \hat{\phi} + \hat{z} \in \frac{1}{\alpha(p)} \tilde{F}_\epsilon(p).
\]

Proof. Fix a sufficiently large \(n \in \mathbb{N}\), to be determined later. Consider the level-set \(f(r, \phi, z) = 2\pi n:\)
\[
z(r, \phi) = -2\pi n + \frac{1}{2\pi n} \ln \left( \cosh \left(2\pi nr \sin(\phi)\right) \right) + \frac{r^2}{1 + 2\pi n}.
\]
This level-set has the following representation in the Cartesian coordinates:
\[
z(x, y) = -2\pi n + \frac{1}{2\pi n} \ln \left( \cosh \left(2\pi ny\right) \right) + \frac{x^2 + y^2}{1 + 2\pi n}.
\]
Fix some \(x_0, y_0 \leq 1/4\), and consider the vector
\[
v = -\left( \frac{dz}{dx} \right)_{(x_0,y_0)} \hat{x} - \left( \frac{dz}{dy} \right)_{(x_0,y_0)} \hat{y} + \hat{z}.
\]
Then\(^1\) \(v\) is orthogonal to the surface \(z(x, y)\) at point \((x_0, y_0, z(x_0, y_0))\). Equivalently, \(v\) is a scaled normal vector for the level-set \(f(r, \phi, z) = 2\pi n\) at \((x_0, y_0, z(x_0, y_0))\). Then, there exists \(\hat{\alpha}(x, y, z) > 0\) such that

\[
\hat{\alpha}(x_0, y_0, z(x_0, y_0)) \hspace{10pt} v = \nabla f(x_0, y_0, z(x_0, y_0)).
\]  

(54)

Note that the \(z\) coordinate entry of \(v\) is unity and, form Lemma\(^3\) (c), \(\nabla f\) is smooth. Then, the function \(\hat{\alpha}\) is smooth, as well.

We proceed by elaborating on the partial derivatives in (53). We have

\[
- \frac{dz}{dx}\bigg|_{(x_0,y_0)} = -\frac{2x_0}{1 + 2\pi n}, \quad \text{(55)}
\]

and

\[
- \frac{dz}{dy}\bigg|_{(x_0,y_0)} = -\tanh\left(2\pi n y_0\right) - \frac{2y_0}{1 + 2\pi n} \quad \text{as } n \to \infty
\]

\[
\begin{cases}
-1 & \text{if } y_0 > 0, \\
1 & \text{if } y_0 < 0.
\end{cases}
\]  

(56)

Hence, for any \(\epsilon > 0\), there is a sufficiently large \(\tilde{n}_\epsilon\), such that for any \(n \geq \tilde{n}_\epsilon\) and any \(x_0 \in [-1/4, 1/4]\), there exists a \(y_0 < \epsilon/3\), such that

\[
- \frac{dz}{dy}\bigg|_{(x_0,y_0)} = x_0.
\]  

(57)

Let \(y_0(\cdot)\) be a function, such that for any \(x_0 \in [-1/4, 1/4]\), and the corresponding constant \(y_0\) that satisfies (57), \(y_0(x_0) = y_0\). From (56), \(y_0(\cdot)\) is the inverse of a smooth and Lipschitz function, and is thereby smooth.

Let \(n_\epsilon = \max(\tilde{n}_\epsilon, 1/\epsilon)\). Then, for \((x_0, y_0(x_0))\), plugging (55) and (57) into (53), we get

\[
v = -\frac{2x_0}{1 + 2\pi n} \hat{x} + x_0 \hat{y} + \hat{z}.
\]  

(58)

Let \(p = (x_0, 0, -2\pi n)\). Then,

\[
\| (x_0, y_0(x_0), z(x_0, y_0(x_0))) - p \| \leq |y_0(x_0)| + |z(x_0, y_0(x_0)) + 2\pi n|,
\]

\[
= |y_0(x_0)| + \frac{1}{2\pi n} \ln \left( \cosh \left(2\pi n y_0(x_0)\right) \right) + \frac{x_0^2 + y_0(x_0)^2}{1 + 2\pi n},
\]

\[
\leq |y_0(x_0)| + \frac{y_0(x_0)}{1 + 2\pi n} + \frac{x_0^2 + y_0(x_0)^2}{1 + 2\pi n},
\]

\[
< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{(1/4)^2 + (1/4)^2}{1 + 2\pi / \epsilon} < \epsilon,
\]

(59)

\(^1\)Note that for any surface \(z(x, y)\), (53) gives an orthogonal vector to that surface.
where the first inequality is a triangle inequality, the equality is from the definition of $z(x_0, y_0)$ in (52), the second inequality is because $\ln (\cosh(x)) \leq |x|$, for all $x \in \mathbb{R}$, and the third inequality is due to the assumptions $y_0(x_0) < \epsilon/3$, $x_0, y_0(x_0) < 1/4$, and $n \geq n_\epsilon \geq 1/\epsilon$. Combining (54) and (59), and letting $\alpha(p) = \tilde{\alpha}(x_0, y_0(x_0), z(x_0, y_0(x_0)))$ we obtain
\[
\alpha(p) v \in \tilde{F}_\epsilon(p). \tag{60}
\]

Since $\tilde{\alpha}(\cdot), y_0(\cdot), \text{and } z(\cdot)$ are smooth, so is $\alpha(\cdot)$.

Back to the cylindrical coordinates, letting $r_0 = x_0$, we have $p = (r_0, -2\pi n, -2\pi n)$, and (58) turns into $v = -2r_0/(1 + 2\pi n)\tilde{r} + r_0\tilde{\phi} + \tilde{z}$. Then, (60) implies that
\[
-\frac{2r_0}{1 + 2\pi n}\tilde{r} + r_0\tilde{\phi} + \tilde{z} \in \frac{1}{\alpha(p)}\tilde{F}_\epsilon(p). \tag{61}
\]

By rotation of the coordinates around the $z$ axis, we can make a similar argument for every $z \leq -2\pi n_\epsilon$, which is not necessarily an integer multiple of $2\pi$. Then, letting $z_\epsilon = -2\pi n_\epsilon$, the claim follows from (61).

For any point $x_0$ on the $z$ axis, there is an unperturbed trajectory $x(\cdot)$ of $F$, which is also an unperturbed trajectory of $\tilde{F}_\epsilon$, initialized at $x(0) = x_0$, that always stays on the $z$-axis. In what follows, we will use Claim 1 to show that there is a perturbed trajectory $\tilde{x}(\cdot)$ of $\tilde{F}_\epsilon$ corresponding to a perturbation of size $\epsilon$, that is initialized on the $z$ axis and whose distance from the $z$ axis grows larger than $1/6$ at some positive time.

Consider an auxiliary dynamical system $\dot{x} = G(x)$, with
\[
G(r, \phi, z) = \frac{-2r}{1 - z}\tilde{r} + r\tilde{\phi} + \tilde{z}, \tag{62}
\]
over the half-cylinder $\Omega$. Let
\[
z_0 = z_\epsilon + \zeta - 2/\epsilon, \tag{63}
\]
where $z_\epsilon$ is the constant in the statement of Claim 1 and $\zeta$ is the constant in the statement of the lemma. Let $r(0) = 0$, and let $r : \mathbb{R}_+ \to \mathbb{R}$ be the solution of
\[
\dot{r}(t) = \epsilon - \frac{2r}{1 - z_0 - t}. \tag{64}
\]
Then, for any $t \in [0, 1/(2\epsilon)]$, we have $\dot{r}(t) < \epsilon$, and as a result,
\[
r(t) \leq \epsilon t \leq 1/2. \tag{65}
\]
Moreover, for any \( t \in [0, 1/(2 \epsilon)] \),
\[
\dot{r}(t) = \epsilon - \frac{2r}{1 - z_0 - t} \\
\geq \epsilon - \frac{2 \times (1/2)}{1 - z_\epsilon - \zeta + 2/\epsilon - 1/(2 \epsilon)} \\
> \epsilon - \frac{1}{3/(2 \epsilon)} \\
= \frac{\epsilon}{3},
\]
where the first inequality is due to (65) and the definition of \( z_0 \) in (63). Therefore, \( r\left(1/(2 \epsilon)\right) > 1/6 \). Then, there exists a \( t_0 \in (0, 1/(2 \epsilon)) \) at which
\[
r(t_0) = 1/6,
\]
\[
0 \leq r(t) \leq 1/6, \quad \forall t \in [0, t_0].
\]
We fix this \( t_0 \) for the rest of the proof.

In cylindrical coordinates, for any \( t \in [0, t_0] \) let
\[
p(t) = (r(t), z_0 + t, z_0 + t).
\]
Also let \( u(t) = \epsilon \hat{r} \) in the local cylindrical coordinates at \( p(t) \), and \( \bar{U}(t) = \int_0^t u(\tau) \, d\tau \). Then,
\[
\| \bar{U}(t) \| = \| \int_0^t u(\tau) \, d\tau \| \\
\leq \| \int_0^t \epsilon \cos(z_0 + \tau) \, d\tau \| + \| \int_0^t \epsilon \sin(z_0 + \tau) \, d\tau \| \\
\leq 2 \epsilon.
\]
Moreover, for any \( t \in [0, t_0] \),
\[
\dot{p}(t) = \dot{r}(t) \hat{r} + r(t) \hat{\phi} + \hat{z} \\
= \epsilon \hat{r} - \frac{2r}{1 - z_0 - t} \hat{r} + r(t) \hat{\phi} + \hat{z} \\
= u(t) + G(p(t)).
\]
where the equalities are due to (68), (64), and (62), respectively. Then, \( p(\cdot) \) is a perturbed trajectory of \( G(\cdot) \) corresponding to perturbation function \( \bar{U}(\cdot) \).

Consider the function \( \alpha(\cdot) : \Omega \to \mathbb{R}_+ \) defined in Claim \( \Box \) and let \( \beta(\cdot) \) be a solution of
\[
\dot{\beta}(t) = \alpha\left(p(\beta(t))\right).
\]
Since $\alpha(\cdot)$ and $p(\cdot)$ are smooth and locally Lipschitz functions, $\alpha \circ p(\cdot)$ is also locally Lipschitz, and (71) has a solution [21].

Let $t_1 = \beta^{-1}(t_0)$, and let $\tilde{x}(t) = p(\beta(t))$, for all $t \in [0, t_1]$. Then,

$$\tilde{x}(t_1) = p(t_0).$$

(72)

For $t \in [0, t_1]$, let $\tilde{x}_z(t)$ and $p_z(\beta(t))$ be the $z$ coordinates of $\tilde{x}(t)$ and $p(\beta(t))$, respectively. Then, for any $t \in [0, t_1]$,

$$\tilde{x}_z(t) = p_z(\beta(t))
= z_0 + \beta(t)
\leq z_0 + \beta(t_1)
= z_0 + t_0$$

(73)

where the first inequality is because $\dot{\beta} > 0$ and the second inequality is because $t_0 \leq 1/(2\epsilon)$.

For any $t \in [0, t_1]$, let $U(t) = \tilde{U}(\beta(t))$. We now show that $\tilde{x}(\cdot)$ is a perturbed trajectory of $\tilde{F}_\epsilon$, corresponding to the perturbation function $U(\cdot)$. We have for any $t \in [0, t_1]$,

$$\frac{d}{dt} \tilde{x}(t) = \frac{d}{dt} p(\beta(t))
= \dot{\beta}(t) \frac{d}{d\beta(t)} p(\beta(t))
= \dot{\beta}(t) G\left(p(\beta(t))\right) + \dot{\beta}(t) u(\beta(t))
= \alpha\left(p(\beta(t))\right) G(\tilde{x}(t)) + \frac{d}{dt} \tilde{U}(\beta(t))$$

(74)

where the last three equalities are due to (70), (71), and the definition of $\tilde{x}(\cdot)$, respectively. Moreover, it follows from Claim 1, definition of $G(\cdot)$ in (62), and (73), that $\alpha(\tilde{x}(t)) G(\tilde{x}(t)) \in \tilde{F}_\epsilon(\tilde{x}(t))$, for all $t \in [0, t_1]$. Then, (74) implies that for any $t \in [0, t_1]$,

$$\frac{d}{dt} \tilde{x}(t) \in \tilde{F}_\epsilon(\tilde{x}(t)) + \frac{d}{dt} U(t).$$

(75)

Therefore, $\tilde{x}(\cdot)$ is a perturbed trajectory of $\tilde{F}_\epsilon$, corresponding to the perturbation function $U(t)$. 

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Finally, let $x(\cdot)$ be an unperturbed trajectory of $\tilde{F}$, initialized at $x(0) = \tilde{x}(0) = (0, z_0, z_0)$, that always stays on the $z$ axis. Then, (72) and (67) imply that
\[ \|\tilde{x}(t_1) - x(t_1)\| = \|p(t_0) - x(t_1)\| \geq r(t_0) = 1/6. \] (76)
Moreover, it follows from (69) that
\[ \sup_{t \leq t_1} \|U(t)\| = \sup_{t \leq t_1} \|\tilde{U}(\beta(t))\| \leq 2\epsilon. \] (77)
Since for any $\epsilon > 0$, there exists such pair $x(\cdot)$ and $\tilde{x}(\cdot)$ of trajectories and perturbation $U(\cdot)$ for which (76) and (77) hold, no constant $C$ can satisfy (15). Then, Theorem 5 implies that $F$ has unbounded sensitivity, and the theorem follows.

IX. Proof of Theorem 4

To prove Theorem 4, we follow the high level idea discussed in Example 3. Consider the convex potential function $\Phi$ of Example 2 over the half-cylinder $\Omega$ (defined in (10)), whose gradient field $F$ has unbounded sensitivity. It was shown in Theorem 3 that for any $\zeta \leq -1$, $F$ has unbounded sensitivity over $\Omega_{\zeta}$ (defined in (13)). We now construct a sequence $\zeta_i, i = 1, 2, \ldots$, of negative numbers as follows. Let $\zeta_0 = -1$. For any $i \in \mathbb{Z}_+$, Theorem 3 implies that there exist an $\epsilon_i > 0$, a time $t_i > 0$, an unperturbed trajectory $x_i(\cdot)$ of $F$, and a perturbed trajectory $\tilde{x}_i(\cdot)$ of $F$ corresponding to a perturbation function $U_i(\cdot)$ and initialized at $\tilde{x}_i(0) = x_i(0)$, such that
\[ x_i(t), \tilde{x}_i(t) \in \Omega_{\zeta_i}, \quad \forall t \in [0, t_i], \] (78)
\[ \|U_i(t)\| \leq \epsilon_i, \quad \forall t \in [0, t_i], \] (79)
and
\[ \|\tilde{x}_i(t_i) - x_i(t_i)\| \geq i \epsilon_i. \] (80)
We then let for any $i \in \mathbb{Z}_+$,
\[ \zeta_{i+1} = -\left(1 + \sup_{t \in [0, t_i]} \|\tilde{x}_i(t_i)\| + \sup_{t \in [0, t_i]} \|x_i(t_i)\|\right). \] (81)
Then, it follows from (78) that $\zeta_i, i = 0, 1, \ldots$, is a decreasing sequence. Therefore, $\Omega_{\zeta_{i+1}} \subset \Omega_{\zeta_i}$, for all $i \in \mathbb{Z}_+$. For any $i \in \mathbb{Z}_+$, let
\[ \Gamma_i = \Omega_{\zeta_i} \setminus \Omega_{\zeta_{i+1}} = \left\{(r, \phi, z) \mid r \leq 1/6, z \in [\zeta_{i+1}, \zeta_i]\right\}. \] (82)
Then, each $\Gamma_i$ is bounded. Moreover, for any $t \in [0, t_i]$, we have $x_i(t), \tilde{x}_i(t) \in \Gamma_i$.

We now construct a fine grid within the half-cylinder $\Omega$, defined in (10), with increasing resolution as $z \to -\infty$, and consider a corresponding triangulation of $\Omega$ with simplexes vertexed at these grid points. We then consider a piecewise constant approximation $\Psi$ of $\Phi$, by letting $\Psi(x) = \Phi(x)$ on the grid points, and $\Psi$ be a linear interpolation inside each simplex. Since $\Phi$ is continuously twice differentiable and $\Gamma_i$ is bounded, for all $i \in \mathbb{Z}_+$, we can choose the grid points so that $\Psi$ gives an arbitrarily accurate approximation of $\Phi$ inside $\Gamma_i$, in the following sense: for any $i \in \mathbb{Z}_+$, any point $p \in \Gamma_i$, and any $\xi \in \partial \Psi(p)$,

$$
\|\xi - \nabla \Phi(p)\| \leq \frac{\epsilon_i}{t_i}.
$$

(83)

Capitalizing on (83), we now show that $\tilde{x}_i(\cdot)$ and $x_i(\cdot)$ are perturbed trajectories of the gradient field of $\Psi$, corresponding to perturbations of size no larger than $2\epsilon_i$.

Let $H$ be the (sub)gradient field of $\Psi$, and fix an $i \in \mathbb{Z}_+$. For any $t \in [0, t_i]$,

$$
x_i(t) = x_i(0) + \int_0^t \nabla \Phi(x_i(\tau)) \, d\tau
$$

(84)

$$
= x_i(0) + \int_0^t \xi(x_i(\tau)) \, d\tau + \int_0^t \left( \nabla \Phi(x_i(\tau)) - \xi(x_i(\tau)) \right) \, d\tau,
$$

where $\xi(x_i(\tau))$ is an arbitrary subgradient of $\Psi$ at $x_i(\tau)$. In the same vein, for any $t \in [0, t_i]$,

$$
\tilde{x}_i(t) = \tilde{x}_i(0) + \int_0^t \nabla \Phi(\tilde{x}_i(\tau)) \, d\tau + U(t)
$$

(85)

$$
= \tilde{x}_i(0) + \int_0^t \tilde{\xi}(\tilde{x}_i(\tau)) \, d\tau + \left[ U(t) + \int_0^t \left( \nabla \Phi(\tilde{x}_i(\tau)) - \tilde{\xi}(\tilde{x}_i(\tau)) \right) \, d\tau \right],
$$

where $\tilde{\xi}(\tilde{x}_i(\tau))$ is an arbitrary subgradient of $\Psi$ at $\tilde{x}_i(\tau)$. For $t \in [0, t_i]$, let

$$
V_i(t) = \int_0^t \left( \nabla \Phi(x_i(\tau)) - \xi(x_i(\tau)) \right) \, d\tau,
$$

(86)

and

$$
\tilde{U}_i(t) = U_i(t) + \int_0^t \left( \nabla \Phi(\tilde{x}_i(\tau)) - \tilde{\xi}(\tilde{x}_i(\tau)) \right) \, d\tau.
$$

(87)
It then follows from (84) and (85) that \( x_i(\cdot) \) and \( \tilde{x}_i(\cdot) \) are perturbed trajectories of \( H \) corresponding to perturbations \( V_i(\cdot) \) and \( \tilde{U}_i(\cdot) \), respectively. Moreover, for any \( t \in [0, t_i] \),

\[
\|V_i(t)\| = \| \int_0^t \left( \nabla \Phi(x_i(\tau)) - \xi(x_i(\tau)) \right) d\tau \|
\leq \int_0^t \| \nabla \Phi(x_i(\tau)) - \xi(x_i(\tau)) \| d\tau
\leq \int_0^t \frac{\epsilon_i}{t_i} d\tau
= \frac{t \epsilon_i}{t_i}
< 2\epsilon_i,
\]

where the second inequality is due to (83). In the same vein,

\[
\|\tilde{U}_i(t)\| = \| U_i(t) + \int_0^t \left( \nabla \Phi(\tilde{x}_i(\tau)) - \xi(\tilde{x}_i(\tau)) \right) d\tau \|
\leq \|U_i(t)\| + \int_0^t \| \nabla \Phi(\tilde{x}_i(\tau)) - \xi(\tilde{x}_i(\tau)) \| d\tau
\leq \epsilon_i + \int_0^t \frac{\epsilon_i}{t_i} d\tau
= \epsilon_i + \frac{t \epsilon_i}{t_i}
\leq 2\epsilon_i.
\]

Let \( y_i(\cdot) \) be an unperturbed trajectory of \( H \) initialized at \( y_i(0) = x_i(0) \). Then,

\[
\|\tilde{x}_i(t_i) - y_i(t_i)\| + \| x_i(t_i) - y_i(t_i) \| \geq \| \tilde{x}_i(t_i) - x_i(t_i) \| \geq i \epsilon_i.
\]

Therefore, either \( \|\tilde{x}_i(t_i) - y_i(t_i)\| \geq i\epsilon_i/2 \) or \( \| x_i(t_i) - y_i(t_i) \| \geq i\epsilon_i/2 \). Hence, for any \( i \in \mathbb{Z}_+ \), there is a perturbation function of size no more than \( 2\epsilon_i \), and a corresponding pair of perturbed and unperturbed trajectories of \( H \), with the same initial conditions, whose distance grows larger than \( i\epsilon_i/2 \) in time \( t_i \). This implies that \( H \) has unbounded sensitivity and completes the proof of the theorem.

X. Proof of Theorem 6

The high level idea is to simulate a discrete time perturbed trajectory with a continuous time perturbed trajectory, and then take advantage of the bounded sensitivity property of the continuous time system.
**Lemma 7** (Simulation of Discrete Time Trajectories with Continuous Time Trajectories). Consider a dynamical system $F$, a discrete time perturbation $V(\cdot)$, a corresponding discrete time trajectory $z(\cdot)$, and a $\mu(\cdot)$ that satisfies (20). Let $U(\cdot)$ be a continuous time perturbation, 

$$U(t) = V(\lfloor t \rfloor) - (t - \lfloor t \rfloor)\mu(\lfloor t \rfloor), \quad \forall t \in \mathbb{R}^n. \quad (88)$$

Then, there exists a corresponding continuous time perturbed trajectory $\tilde{x}(\cdot)$ such that 

$$\tilde{x}(k) = z(k), \quad k = 0, 1, \ldots. \quad (89)$$

**Proof.** For any $t \in \mathbb{R}^+$, let 

$$\tilde{x}(t) = z(\lfloor t \rfloor) \quad (90)$$

We show that $\tilde{x}$ is a perturbed trajectory corresponding to perturbation $U$. For any $t \in \mathbb{R}^+$, let $\xi(t) = \mu(\lfloor t \rfloor)$. Then, $\xi(t) \in F(z(\lfloor t \rfloor)) = F(\tilde{x}(t))$. Moreover, for any $t \in \mathbb{R}^+$,

$$\tilde{x}(t) = z(\lfloor t \rfloor) = \sum_{k \leq t} \mu(k) + V(\lfloor t \rfloor)$$

$$= \sum_{k \leq t} \int_{k-1}^{k} \xi(\tau) d\tau + \left[ U(t) + (t - \lfloor t \rfloor)\mu(\lfloor t \rfloor) \right]$$

$$= \int_{0}^{\lfloor t \rfloor} \xi(\tau) d\tau + U(t) + \int_{\lfloor t \rfloor}^{t} \xi(t)$$

$$= \int_{0}^{t} \xi(\tau) d\tau + U(t).$$

Therefore, $\tilde{x}$ is a perturbed trajectory corresponding to perturbation $U$, and the lemma follows. 

**Proof of Theorem 6.** For Part (a), it follows from Lemma 7 that there exists a perturbation function $U(\cdot)$ with corresponding continuous time perturbed trajectory $\tilde{x}(\cdot)$ that satisfy (88) and (89). Then, for any $k \in \mathbb{Z}^+$,

$$\|x(k) - z(k)\| = \|x(k) - \tilde{x}(k)\|$$

$$\leq C \sup_{t \leq k} \|U(\tau)\|$$

$$\leq C \left( \max_{j < k} \|\mu(j)\| + \max_{j < k} \|V(j)\| \right), \quad (92)$$
where the relations are due to (89), (5), and (88), respectively. This completes the proof of Part (a).

For Part (b), consider, from Lemma 7, a pair $U_1(\cdot)$ and $U_2(\cdot)$ of perturbations and a corresponding pair $\tilde{x}_1(\cdot)$ and $\tilde{x}_2(\cdot)$ of perturbed trajectories such that for $i = 1, 2$,

$$\tilde{x}_i(k) = z_i(k), \quad \forall k \in \mathbb{Z}_+,$$

$$U_i(t) = V_i(\lfloor t \rfloor) - (t - \lfloor t \rfloor)\mu_i(\lfloor t \rfloor), \quad \forall t \in \mathbb{R}^n.$$  (93)

Then, for any $k \in \mathbb{Z}_+$,

$$\|z_1(k) - z_2(k)\| = \|\tilde{x}_1(k) - \tilde{x}_2(k)\|$$

$$\leq C \sup_{t \leq k} \|U_1(t) - U_2(t)\|$$

$$\leq C \left( \max_{j < k} \|\mu_1(j) - \mu_2(j)\| + \max_{j < k} \|V_1(j) - V_2(j)\| \right),$$

(95)

where the relations are due to (93), (6), and (94), respectively. This completes the proof of the theorem.

\[\square\]

XI. Discussion

We studied boundedness of sensitivity to cumulative perturbations for some classes of dynamical systems, a property that, when holds, provides strong conclusions and tools for systems driven by stochastic noise. We derived a necessary and sufficient condition for bounded sensitivity of a linear dynamical system, in terms of its spectrum. More specifically, we showed that a linear system has bounded sensitivity if and only if it is stable and has no periodic orbits.

Moreover, we gave examples of subgradient fields of strictly convex as well as piecewise linear convex functions with unbounded sensitivity. The latter results is particularly important because it certifies the necessity of “finiteness” assumption in a former result (cf. Theorem 1), according to which the subgradient field of a piecewise linear convex function with finitely many pieces has bounded sensitivity.

We also studied several transformations of a dynamical system that preserve a bounded sensitivity property. In particular, we showed for a dynamical system with bounded sensitivity that a similar property holds for its induced discrete time systems, spread systems, and the systems obtained via convolution with a kernel.

In the rest of this section, we point some open problems and directions for future research.
• Sensitivity of new classes of dynamical systems: A first direction for future research is extending the sensitivity bounds to other classes of dynamical systems. A potential candidate is the class of nonexpansive piecewise linear dynamical systems (with all pieces being SOF). This class generalizes the class of FPCS systems as well as the class of SOF linear systems.

• Combination of two systems: Besides fining more transformations of a system that preserve bounded sensitivity, an interesting research direction concerns bounded sensitivity of combination of two or several systems. For example, does the pointwise sum of two systems that have bounded sensitivity still have bounded sensitivity? The answer is already known to be negative. However, the pointwise sum of systems with specific structures might yield bounded sensitivity. A prominent example is the sum of an FPCS system and a linear system. These combinations include the systems that underlie the gradient flows of the LASSO cost function, i.e., \( \|x\|_1 + \|Ax - b\|_2 \).

Another direction involves preservation of bounded sensitivity under other means of combination of two dynamical systems, more general than a pointwise sum. An interesting combination involves dynamical systems over disjoint domains, that are glued/cast together in a nonexpansive manner.

• Strong vs weak bounded sensitivity: As mentioned before, a sensitivity bound of type (6) implies the sensitivity bound of type (5). The reverse however is not still fully understood. It remains open to obtain the conditions under which a dynamical system with a bound of type (5) has bounded sensitivity also in the sense of (6). As a concrete example, Theorem 1 shows that FPCS systems have bounded sensitivity in the sense of (5). It would be interesting if one could prove or disprove analogues results in the strong sensitivity sense of (6).

• Bounded domain: So far, we have dealt with dynamical systems defined over the entire \( \mathbb{R}^n \). This however is not the case in many applications such as queueing networks where the state space (i.e., queue lengths) is restricted to the positive orthant. Boundedness of the domain gives rise to boundary conditions like projecting the “escaping trajectories” back onto the domain, which typically further complicate the dynamics, and which need to be addressed in future research. In particular, one can investigate what type of boundary conditions will preserve bounded sensitivity, once the domain is restricted.

• Convolution by a kernel: We showed in Corollary 1 that a system with bounded sensitivity, when convolved with a kernel, still pertains a weaker notion of bounded sensitivity that incorporates additive penalties. However, it remains open that under what conditions on
the kernel, the latter system would have bounded sensitivity in the sense of (5), with no additive penalties.

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APPENDIX A

Proof of Lemma 3

For Part (a), consider a function $h : \mathbb{R}^4 \rightarrow \mathbb{R}$,
\[
h(f, r, \phi, z) = f + z - \frac{1}{f} \ln \left( \cosh \left( fr \sin(f - \phi) \right) \right) - \frac{r^2}{1 + f}.
\] (96)
Then, $f(r, \phi, z)$ in (11) is a solution of the implicit equation $h(f, r, \phi, z) = 0$. For any $(r, \phi, z) \in \Omega$,
\[
h(1, r, \phi, z) = 1 + z - \ln \left( \cosh \left( r \sin(1 - \phi) \right) \right) - r^2 \leq 1 + z \leq 0
\]
and
\[
\lim_{f \to \infty} h(f, r, \phi, z) = z + \lim_{f \to \infty} \left( f - \frac{1}{f} \ln \left( \cosh \left( fr \sin(f - \phi) \right) \right) \right)
\]
\[
\geq z + \lim_{f \to \infty} \left( f - \frac{fr \sin(f - \phi)}{f} \right)
\]
\[
= \infty,
\]
where the last inequality is because $\ln \left( \cosh(x) \right) \leq |x|$, for all $x \in \mathbb{R}$. Then, for any $(r, \phi, z) \in \Omega$, there is an $f \geq 1$ for which $h(f, r, \phi, z) = 0$. We now prove the uniqueness of this $f$, by showing that for any fixed $(r, \phi, z) \in \Omega$, $h(f, r, \phi, z)$ is a strictly increasing function in $f$.

We have
\[
\frac{\partial}{\partial f} h(f, r, \phi, z) = 1 + \frac{1}{f^2} \ln \left( \cosh \left( fr \sin(f - \phi) \right) \right)
\]
\[
- \frac{r \sin(f - \phi) + fr \cos(f - \phi)}{f} \tanh \left( fr \sin(f - \phi) \right) + \frac{r^2}{(1 + f)^2}
\]
\[
\geq 1 - \frac{r + fr}{f} \tanh \left( fr \right)
\]
\[
= 1 - \frac{r \tanh \left( fr \right)}{f} - r \tanh \left( fr \right)
\]
\[
\geq 1 - r^2 - r
\]
\[
> 0.
\] (97)
where the the first inequality is by removing the positive terms and the trigonometric functions, the second inequality is because $\tanh(x) \leq x$ and $\tanh(x) \leq 1$, for $x > 0$, and the last inequality is from $(r, \phi, z) \in \Omega$. Then, $h$ is a strictly increasing function in its first argument, and for any fixed $(r, \phi, z) \in \Omega$, there is a unique $f$ that satisfies $h(f, r, \phi, z) = 0$. This completes the proof of Part (a).
Part (b) is immediate from the definition of $f$ in (11). For Part (c), note that $\frac{d^2}{dx^2} \ln \cos(x) = 1/\cosh(x)^2 \geq 0$, and $\ln \cosh(\cdot)$ is thereby a convex function. Then, the surface of each level-set of $f$, given in (12), is the sum of a convex function, $z(r, \phi) = -f + \ln \left( \cosh \left( fr \sin(f - \phi) \right) \right) / f$, and a strictly convex function, $z(r, \phi) = r^2 / (1 + f)$. Hence, the surface, $z$, of each level-set is a strictly convex function. Then, for any pair of points $p_1, p_2 \in \Omega$ with $f(p_1) = f(p_2) = a$, the line segment connecting $p_1$ and $p_2$ lies above the level set $f = a$. Equivalently, for any $\alpha \in (0, 1)$, $f(\alpha p_1 + (1 - \alpha)p_2) > a$. Thus, $f$ is strictly quasi-convex. Moreover, each level-set of $f$ is the intersection of $\Omega$ with a surface of type (12), and is thereby compact. For smoothness, note that $f$ is a solution of the implicit equation $h(f, r, \phi, z) = 0$, where $h$ is smooth and, from (97), $\partial h/\partial f > 0$. Then, it follows from the “implicit function theorem” for smooth functions [22] (Theorem 12 of Appendix B) that $f(\cdot)$ is smooth.

APPENDIX B

Proof of Lemma 5

By the definition of a perturbed trajectory,
\[
\bar{x}(t) = \int_0^t \xi(\tau) \, d\tau + U(t), \quad \forall t \geq 0, \tag{98}
\]
\[
\xi(t) \in \bar{F}_\epsilon(\bar{x}(t)), \quad \forall t \geq 0.
\]
The term $\int_0^t \xi(\tau) \, d\tau$ in the equality, is a continuous function of $t$ and $U(\cdot)$ is a right-continuous function. Then, $\bar{x}(\cdot)$ is right continuous.

In the proof that follows, we use a transfinite recursion [23] to partition $\mathbb{R}$ into a number of time intervals $[t_i, t_{i+1})$. Let $\text{Ord}$ be the collection of all ordinal numbers [23]. Consider the sequence $t_i$ defined by the following transfinite recursion:

- Base case: $t_0 = 0$.
- Successor case: For any successor ordinal $\alpha$, let
\[
t_\alpha = \min \left( t_{\alpha-1} + \frac{\delta}{\gamma \left( \|\bar{x}(t_{\alpha-1})\| + \epsilon + \delta \right) + 1} \right), \tag{99}
\]
where $\gamma$ is the constant in (7).
- Limit case: For any limit ordinal $\alpha$, let
\[
t_\alpha = \sup \left\{ t \in \mathbb{R} \mid \bar{x}(\tau) \in \mathcal{B}_\delta(\bar{x}(t_{\alpha-1})), \forall \tau \in [t_{\alpha-1}, t) \right\}, \tag{100}
\]
• Termination: If $t_\alpha = \infty$, halt and let $\alpha^* = \alpha$.

Claim 2. The ordinal $\alpha^*$ exists and is a limit ordinal. Moreover, the intervals $[t_\alpha, t_{\alpha+1})$, $\alpha < \alpha^*$, cover $\mathbb{R}_+$. 

Proof of Claim 2. It follows from (99) and the right-continuity of $\tilde{x}(\cdot)$ that for any successor ordinal $\alpha$, $t_\alpha > t_{\alpha-1}$. Together with (100), this implies that for any ordinal $\alpha < \alpha^*$, and any ordinal $\beta < \alpha$, $t_\beta < t_\alpha$. Then, all values of $t_\alpha$ are distinct, and the length of the sequence $t_\alpha$, $\alpha \leq \alpha^*$, can be no larger than the cardinality of $\mathbb{R}$, i.e., $2^{\aleph_0}$.

Assuming the “axiom of choice”, the “Von-Neumann’s cardinal assignment” \cite{24} implies that $2^{\aleph_0}$ equals some ordinal number $\beta$. Then, since $2^{\aleph_0} < 2^{2^{\aleph_0}} = \beta$, the transfinite recursion defining $t_\alpha$ must terminate for some value of $\alpha < \beta$. Hence, $\alpha^*$ exists and is less than $\beta$. Moreover, $\alpha^*$ cannot be a successor ordinal, because in that case, $t_{\alpha^*-1} < \infty$ and (99) would have implied that $t_{\alpha^*} < \infty$.

For the second part of the claim, note that
\[
\bigcup_{\alpha < \alpha^*} [t_\alpha, t_{\alpha+1}) = \sup_{\alpha < \alpha^*} [t_0, t_\alpha) = [t_0, t_{\alpha^*}) = [0, \infty) = \mathbb{R}_+,
\]
and the claim follows. \hfill \square

We continue by defining the perturbation $U'(\cdot)$ and its corresponding perturbed trajectory $\tilde{y}(\cdot)$. Fix some $\alpha < \alpha^*$. It follows from (99) that for any $t \in [t_\alpha, t_{\alpha+1})$, $\|\tilde{x}(t) - \tilde{x}(t_\alpha)\| \leq \delta$. Then, (98) implies that for any $t \in [t_\alpha, t_{\alpha+1})$,
\[
\xi(t) \in \tilde{F}_\epsilon(\tilde{x}(t)) \subseteq \tilde{F}_{\epsilon+\delta}(\tilde{x}(t_\alpha)).
\]
Therefore,
\[
\frac{1}{t_{\alpha+1} - t_\alpha} \int_{t_\alpha}^{t_{\alpha+1}} \xi(t) \, dt \in \text{Conv}\left(\tilde{F}_{\epsilon+\delta}(\tilde{x}(t_\alpha))\right)
\]
\[
= \tilde{F}_{\epsilon+\delta}(\tilde{x}(t_\alpha))
\]
\[
= \text{Conv}\{\xi \mid \xi \in F(y), y \in B_{\epsilon+\delta}(x)\}.
\]
Then, from the “Carathéodory’s theorem” \cite{25}, there exist $n+1$ number, $\xi_1^\alpha, \ldots, \xi_{n+1}^\alpha$, of vectors in $\tilde{F}_{\epsilon+\delta}(\tilde{x}(t_\alpha))$, and non-negative constants $\theta_1^\alpha, \ldots, \theta_{n+1}^\alpha$ with $\theta_1^\alpha + \cdots + \theta_{n+1}^\alpha = 1$ such that
\[
\frac{1}{t_{\alpha+1} - t_\alpha} \int_{t_\alpha}^{t_{\alpha+1}} \xi(t) \, dt = \sum_{i=1}^{n+1} \theta_i^\alpha \xi_i^\alpha.
\]
For any $i \leq n + 1$, let $z_i^\alpha$ be a point in $\mathcal{B}_{\epsilon + \delta}(\bar{x}(t_\alpha))$ such that $\xi_i^\alpha \in F(z_i^\alpha)$. Also let $T_0^\alpha = t_\alpha$, and for any $i \leq n + 1$,

$$t_i^\alpha = (t_{i+1} - t_i) \sum_{j=1}^i \theta_i^\alpha. \tag{105}$$

We now define the functions $U', \tilde{y}', \xi': \mathbb{R}_+ \to \mathbb{R}^n$ as follows. For any $\alpha \leq \alpha^*$, let

$$U'(t_\alpha) = U(t_\alpha),$$

$$\tilde{y}(t_\alpha) = \bar{x}(t_\alpha),$$

$$\xi'(t_\alpha) = \xi(t_\alpha). \tag{106}$$

For any $\alpha \leq \alpha^*$, any $i \leq n$, and any $t \in [T_{i-1}^\alpha, T_i^\alpha)$ excluding $t = t_\alpha$, let

$$\tilde{y}(t) = z_i^\alpha,$$

$$\xi'(t) = \xi_i^\alpha,$$

$$U'(t) = U(t_\alpha) + (z_i^\alpha - \bar{x}(t_\alpha)) - \int_{t_\alpha}^t \xi'(\tau) d\tau. \tag{107}$$

Then, for any $t \geq 0$,

$$\xi'(t) \in F(\tilde{y}(t)). \tag{108}$$

In the reset of the proof, we will show that (38) and (39) hold, and that $\tilde{y}(\cdot)$ is a perturbed trajectory of $F(\cdot)$ corresponding to perturbation $U'(\cdot)$.

Since $z_i^\alpha \in \mathcal{B}_{\epsilon + \delta}(\bar{x}(t_\alpha))$, for all $\alpha < \alpha^*$ and all $i \leq n + 1$, it follows that for any $t \in [T_{i-1}^\alpha, T_i^\alpha)$,

$$\|\tilde{y}(t) - \bar{x}(t)\| = \|z_i^\alpha - \bar{x}(t)\| \leq \|z_i^\alpha - \bar{x}(t_\alpha)\| + \|\bar{x}(t_\alpha) - \bar{x}(t)\| \leq (\epsilon + \delta) + \delta, \tag{109}$$

and (38) is satisfies by a proper choice of $\delta$. Moreover, it follows from (39) and (7) that for any $\alpha < \alpha^*$ and any $i \leq n + 1$,

$$\|(t_{i+1} - t_\alpha) \xi_i^\alpha\| \leq \frac{\delta}{\gamma (\|\bar{x}(t_\alpha)\| + \epsilon + \delta) + 1} \|\xi_i^\alpha\|$$

$$\leq \frac{\delta}{\gamma (\|\bar{x}(t_\alpha)\| + \epsilon + \delta) + 1} \|z_i^\alpha\|$$

$$\leq \frac{\delta}{\gamma (\|\bar{x}(t_\alpha)\| + \epsilon + \delta) + 1} \gamma (\|\bar{x}(t_\alpha)\| + \epsilon + \delta)$$

$$< \delta. \tag{110}$$

Then, for any $\alpha < \alpha^*$, and $i \leq n + 1$, and any $t \in [t_\alpha, t_{i+1})$,

$$\|U'(t)\| \leq \|U(t_\alpha)\| + \|z_i^\alpha - \bar{x}(t_\alpha)\| + \|\int_{t_\alpha}^t \xi'(\tau) d\tau\|$$

$$\leq \|U(t_\alpha)\| + (\epsilon + \delta) + \delta. \tag{111}$$
Therefore,
\[
\sup_{\tau \leq t} \|U'(\tau)\| \leq \sup_{\tau \leq t} \|U(\tau)\| + \epsilon + 2\delta, \tag{112}
\]
and (41) follows by a proper choice of \(\delta\).

It only remains to show that \(\tilde{y}(\cdot)\) is a perturbed trajectory of \(F(\cdot)\) corresponding to perturbation \(U'(\cdot)\).

**Claim 3.**

a) For any \(\alpha < \alpha^*\),
\[
\tilde{y}(t_\alpha) = \tilde{y}(0) + \int_0^{t_\alpha} \xi'(\tau) \, d\tau + U'(t_\alpha). \tag{113}
\]

b) For any \(\alpha < \alpha^*\), and any \(t \in [t_\alpha, t_{\alpha+1})\),
\[
\tilde{y}(t) = \tilde{y}(t_\alpha) + \int_{t_\alpha}^{t} \xi'(\tau) \, d\tau + U'(t) - U'(t_\alpha). \tag{114}
\]

**Proof of Claim 3.** We first show that \(\xi'(\cdot)\) is Lebesgue integrable. Since \(\xi(\cdot)\) is a right-continuous function of time, it is measurable. Moreover, (7) implies that for any \(t \geq 0\),
\[
\sup_{\tau \in [0,T]} \|\xi'(\tau)\| \leq \sup_{\tau \in [0,T]} \gamma \|\tilde{y}(\tau)\| \leq \sup_{\tau \in [0,T]} \gamma (\|\tilde{x}(\tau)\| + \epsilon + 2\delta) \leq \gamma (e^{\epsilon \tau}\|0\| + \epsilon + 2\delta) < \infty. \tag{115}
\]

Therefore, \(\xi'(\cdot)\) has finite integral over every bounded interval. On the other hand, for any \(\alpha < \alpha^*\),
\[
\int_{t_\alpha}^{t_{\alpha+1}} \xi'(\tau) \, d\tau = (t_{\alpha+1} - t_\alpha) \sum_{i=1}^{n+1} \theta_i \xi_i = \int_{t_\alpha}^{t_{\alpha+1}} \xi(\tau) \, d\tau, \tag{116}
\]
where the equalities are due to the definition of \(\xi'(\cdot)\) and (104), respectively.

We now prove (113) via a transfinite induction \cite{23} on \(\alpha \in \text{Ord}\).

- **Base case:** \(\tilde{y}(0) = \tilde{y}(0) + U'(0)\).
- **Induction step for the successor case:** Consider a successor ordinal number \(\alpha < \alpha^*\), and suppose that (113) holds for \(\alpha - 1\). Then,
\[
\tilde{y}(0) + \int_0^{t_\alpha} \xi'(\tau) \, d\tau + U'(t_\alpha) = \left[\tilde{y}(0) + \int_0^{t_{\alpha-1}} \xi'(\tau) \, d\tau + U'(t_{\alpha-1})\right] + \int_{t_{\alpha-1}}^{t_\alpha} \xi'(\tau) \, d\tau + \left[U'(t_\alpha) - U'(t_{\alpha-1})\right]
\]
\[
= \tilde{y}(t_{\alpha-1}) + \int_{t_{\alpha-1}}^{t_\alpha} \xi'(\tau) \, d\tau + \left[U'(t_\alpha) - U'(t_{\alpha-1})\right] \tag{117}
\]
\[
= \tilde{x}(t_{\alpha-1}) + \int_{t_{\alpha-1}}^{t_\alpha} \xi(\tau) \, d\tau + \left[U(t_\alpha) - U(t_{\alpha-1})\right]
\]
\[
= \tilde{x}(t_\alpha)
\]
\[
= \tilde{y}(t_\alpha).
\]
where the second equality is due to the induction hypothesis, the third equality is from (106) and (116), the fourth equality is because $\tilde{x}(\cdot)$ is a perturbed trajectory corresponding to $U(\cdot)$, and the last equality is again from (106).

- Induction step for the limit case: Consider a limit ordinal number $\alpha < \alpha^*$, and suppose that (113) holds for all ordinals $\beta < \alpha$. Then, for any $\beta < \alpha$,

  \[
  \tilde{y}(0) + \int_0^{t_\alpha} \xi'(\tau) \, d\tau + U'(t_\alpha) = \left[ \tilde{y}(0) + \int_0^{t_\beta} \xi'(\tau) \, d\tau + U'(t_\beta) \right] \\
  + \int_{t_\beta}^{t_\alpha} \xi'(\tau) \, d\tau + [U'(t_\alpha) - U'(t_\beta)] \\
  = \tilde{y}(t_\beta) + \int_{t_\beta}^{t_\alpha} \xi'(\tau) \, d\tau + [U'(t_\beta) - U'(t_\beta)] \\
  = \tilde{x}(t_\beta) + \int_{t_\beta}^{t_\alpha} \xi(\tau) \, d\tau + [U(t_\alpha) - U(t_\beta)] \\
  + \int_{t_\beta}^{t_\alpha} (\xi'(\tau) - \xi(\tau)) \, d\tau \\
  = \tilde{x}(t_\alpha) + \int_{t_\beta}^{t_\alpha} (\xi'(\tau) - \xi(\tau)) \, d\tau \\
  = \tilde{y}(t_\alpha) + \int_{t_\beta}^{t_\alpha} (\xi'(\tau) - \xi(\tau)) \, d\tau.
  \]  

(118)

where the second equality is due to the induction hypothesis, the third equality is from (106) and (116), the fourth equality is because $\tilde{x}(\cdot)$ is a perturbed trajectory corresponding to $U(\cdot)$, and the last equality is again from (106). As a result, \( \int_{t_\beta}^{t_\alpha} (\xi'(\tau) - \xi(\tau)) \, d\tau \) is independent of choice of $\beta$.

It follows from (115) that $\|\xi'(t) - \xi(t)\|$ is bounded for $t \leq t_\alpha$. Moreover, since $\alpha$ is a limit ordinal, from the definition (100), the sequence $t_\beta$, for $\beta < \alpha$, converges to $t_\alpha$ from below. Then, $\| \int_{t_\beta}^{t_\alpha} (\xi'(\tau) - \xi(\tau)) \, d\tau \|$ becomes arbitrarily small for proper values of $\beta < \alpha$. Therefore, $\int_{t_\beta}^{t_\alpha} (\xi'(\tau) - \xi(\tau)) \, d\tau = 0$, and (113) follows from (118).

This completes the proof of Part (a).

For Part (b), recall the constants \( t_i^n, i = 1, \ldots, n+1 \), defined in (105). Then, for any $i \leq n+1$
and any \( t \in [t^\alpha_{i-1}, t^\alpha_i) \),
\[
\bar{y}(t^\alpha_i) + \int_{t^\alpha_i}^{t} \xi'(\tau) \, d\tau + U'(t) - U'(t^\alpha_i) = \bar{y}(t^\alpha_i) + \int_{t^\alpha_i}^{t} \xi'(\tau) \, d\tau + (z^\alpha_i - \bar{x}(t^\alpha_i)) - \int_{t^\alpha_i}^{t} \xi'(\tau) \, d\tau
\]
\[
= z^\alpha_i
\]
\[
= \bar{y}(t),
\]
(119)

where the first and the last equalities are due to the definition (107). This completes the proof of the claim.

Then, it follows from Parts (a) and (b) of claim 3 that for any \( t \geq 0 \),
\[
\bar{y}(t) = \bar{y}(0) + \int_{0}^{t} \xi'(\tau) \, d\tau + U'(t).
\]
(120)

Together with (108), this implies that \( \bar{y}(\cdot) \) is a perturbed trajectory of \( F \) corresponding to perturbation \( U' \).

We finally note that \( U' \) is right continuous everywhere, except for the times \( t^\alpha_i \). To satisfy right continuity also at times \( t^\alpha_i \), we modify the definitions of \( U' \), \( \bar{y} \), and \( \xi' \), by eliminating (106) and considering (107) also at \( t = t^\alpha_i \). It is straightforward to see that this modification does not impact any of the integrals, and (112) and (120) would still be valid. This completes the proof of Part (a) of the Lemma.

The proof of Part (b) is similar to the proof of Part (a). The only difference is the choice of \( t^\alpha_i \) in for successor ordinals \( \alpha \) in the transfinite recursion. Here, we replace (99) with
\[
t^\alpha_i = \min \left( t^\alpha_{i-1} + \frac{\delta}{\gamma(\|\bar{x}_1(t^\alpha_{i-1})\| + \|\bar{x}_2(t^\alpha_{i-1})\| + \epsilon + \delta) + 1}, \sup \left\{ t \in \mathbb{R} \mid \bar{x}_i(\tau) \in B^{\delta}(\bar{x}_{i}(t^\alpha_{i-1})), \forall \tau \in [t^\alpha_{i-1}, t), i = 1, 2 \right\} \right).
\]
(121)

The only point here is that we use the same sequence \( t^\alpha_i \) for both trajectories. Then, instead of (111), we can write
\[
\|U'(t) - U'_1(t)\| \leq \|U_1(t) - U_2(t)\| + \|z^\alpha_{i,1} - \bar{x}(t)\| + \|z^\alpha_{i,2} - \bar{x}(t)\|
\]
\[
+ \| \int_{t^\alpha_i}^{t} \xi'_1(\tau) \, d\tau \| + \| \int_{t^\alpha_i}^{t} \xi'_2(\tau) \, d\tau \|,
\]
\[
\leq \|U_1(t) - U_2(t)\| + 2(\epsilon + \delta) + 2\delta
\]
\[
= \|U_1(t) - U_2(t)\| + 2\epsilon + 4\delta.
\]
(122)

This implies (41) for a proper choice of \( \delta \), and completes the proof of Lemma 5.
Appendix C

Proof of Lemma 4

First note that, by definition, if \( y(\cdot) \) is a perturbed trajectory, then for any \( t \geq 0 \)

\[
y(t) - \int_0^t Ay(\tau) \, d\tau = y(0) + U(t).
\]

(123)

By multiplying both sides with \( e^{-A\tau} \) and integration, we get

\[
\int_0^t e^{-A\tau} y(\tau) \, d\tau - \int_0^t Ae^{-A\tau} \int_0^\tau y(s) \, ds \, d\tau = y(0) \int_0^t e^{-A\tau} \, d\tau + \int_0^t e^{-A\tau} U(\tau) \, d\tau.
\]

(124)

From integration by parts (e.g., Theorem 12.5 in [26]),

\[
\int_0^t e^{-A\tau} y(\tau) \, d\tau - \int_0^t Ae^{-A\tau} \int_0^\tau y(s) \, ds \, d\tau = e^{-At} \int_0^t y(\tau) \, d\tau.
\]

Pluging this into the left hand side of (124), we obtain

\[
\int_0^t y(\tau) \, d\tau = y(0) e^{At} \int_0^t e^{-A\tau} \, d\tau + \int_0^t e^{-A\tau} U(\tau) \, d\tau.
\]

(125)

We now show that \( x(\cdot) \) defined in (24) is a solution of (125). We have

\[
\int_0^t x(\tau) \, d\tau = x(0) e^{At} \int_0^t e^{-A\tau} \, d\tau + \int_0^t U(\tau) \, d\tau + \int_0^t Ae^{At} \int_0^\tau e^{-As} U(s) \, ds \, d\tau
\]

\[
= x(0) e^{At} \int_0^t e^{-A\tau} \, d\tau + \int_0^t U(\tau) \, d\tau + \int_0^t \left( \int_0^t Ae^{At} \, d\tau \right) e^{-As} U(s) \, ds
\]

\[
= x(0) e^{At} \int_0^t e^{-A\tau} \, d\tau + \int_0^t U(\tau) \, d\tau + \int_0^t \left( e^{At} - e^{As} \right) e^{-As} U(s) \, ds
\]

\[
= x(0) e^{At} \int_0^t e^{-A\tau} \, d\tau + e^{At} \int_0^t e^{-A\tau} U(\tau) \, d\tau.
\]

(126)

Then, \( x(\cdot) \) satisfies (125). Moreover, for any other solution \( x'(\cdot) \) of (125), \( \int_0^t (x'(\tau) - x(\tau)) = 0 \), for all \( t \geq 0 \). Then, \( x'(\cdot) \) equals \( x(\cdot) \), almost everywhere. Therefore, for any solution \( y(\cdot) \) of (123),

\[
y(t) = \int_0^t Ay(\tau) \, d\tau + y(0) + U(t)
\]

\[
= \int_0^t Ax(\tau) \, d\tau + y(0) + U(t)
\]

(127)

\[
= x(t).
\]

Therefore, \( x(\cdot) \) is the unique perturbed trajectory corresponding to the perturbation function \( U(\cdot) \), and the lemma follows.