Global existence and stabilization in a diffusive predator-prey model with population flux by attractive transition

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Abstract
The diffusive Lotka–Volterra predator-prey model
\[
\begin{aligned}
\frac{u_t}{u} &= \nabla \cdot \left[ d_1 \nabla u + \chi v^2 \nabla \left( \frac{u}{v} \right) \right] + u (m_1 - u + av), & x \in \Omega, \ t > 0, \\
\frac{v_t}{v} &= d_2 \Delta v + v (m_2 - bu - v), & x \in \Omega, \ t > 0,
\end{aligned}
\]
is considered in a bounded domain \( \Omega \subset \mathbb{R}^n, n \in \{2, 3\} \), under Neumann boundary condition, where \( d_1, d_2, m_1, \chi, a, b \) are positive constants and \( m_2 \) is a real constant. The purpose of this paper is to establish global existence and boundedness of classical solutions in the case \( n = 2 \) and global existence of weak solutions in the case \( n = 3 \) as well as show long-time stabilization. More precisely, we prove that the solutions \((u(\cdot,t), v(\cdot,t))\) converge to the constant steady state \((u^*, v^*)\) as \( t \to \infty \), where \( u^*, v^* \) solves \( u^*(m_1 - u^* + av^*) = v^*(m_2 - bu^* - v^*) = 0 \) with \( u^* > 0 \) (covering both coexistence as well as prey-extinction cases).

Key words: large time behavior; predator-prey model; attractive transitional flux

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1 Introduction

Initiated by the modeling of *dictyostelium discoideum* slime mold using a system of coupled partial differential equations in the seminal paper by Keller and Segel in 1970 ([6]) and spurred on by the subsequent successful mathematical verification of its real world aggregation behavior in the model ([20]), in the decades since we have seen more and more applications of this same modeling approach to various system of biological agents whose movement is similarly affected by some outside influence. The versatility of this technique is further made evident by it not only generalizing to different dynamics in similar microscopic systems but even to macroscopic systems, such as the modeling of criminal behavior ([10]) as well as predator-prey models ([8]). Additionally from a mathematical perspective, these kinds of movement dynamics involving an external stimulus can have a significant destabilizing influence on the dynamics of the system as a whole, making even the question of existence of (potentially generalized) global model solutions as well as the nature of long-time behavior challenging and thus answering said questions an often worthwhile endeavor. Given that answers in this regard also help verify whether a proposed model agrees with reality, it is exactly this type of inquiry we will pursue in this paper for the following problem.

**Problem.** Motivated by Oeda–Kuto [8], we consider the diffusive Lotka–Volterra predator-prey model with population flux by attractive transition, as given by

\[
\begin{align*}
  u_t &= \nabla \cdot \left[ d_1 \nabla u + \chi v^2 \nabla \left( \frac{u}{v} \right) \right] + u(m_1 - u + av), \quad x \in \Omega, \ t > 0, \\
  v_t &= d_2 \Delta v + v(m_2 - bu - v), \quad x \in \Omega, \ t > 0,
\end{align*}
\]

which will be precisely studied as the following initial-boundary value problem with the equivalent first equation in this paper:

\[
\begin{align*}
  u_t &= \nabla \cdot \left[ (d_1 + \chi v) \nabla u \right] - \chi \nabla \cdot (u \nabla v) + u(m_1 - u + av), \quad x \in \Omega, \ t > 0, \\
  v_t &= d_2 \Delta v + v(m_2 - bu - v), \quad x \in \Omega, \ t > 0, \\
  \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, \ t > 0, \\
  u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega
\end{align*}
\]

in a bounded domain \( \Omega \subset \mathbb{R}^n, \ n \in \{2, 3\} \), with smooth boundary \( \partial \Omega \), where

\[
d_1, d_2, m_1, \chi, a, b > 0, \quad m_2 \in \mathbb{R}
\]

are constants, \( \frac{\partial}{\partial \nu} \) denotes differentiation with respect to the outward normal of \( \partial \Omega \), and the initial data \( u_0, v_0 \neq 0 \) are suitably regular and nonnegative. The problem (1.1) is a diffusive Lotka–Volterra predator-prey model in which unknown functions \( u = u(x, t) \) and \( v = v(x, t) \) represent the population densities of the predator and the prey at location \( x \in \Omega \) and time \( t \geq 0 \), respectively. In this model the drift-diffusion of the predator consists of the population flux \( -\left[ d_1 \nabla u + \chi v^2 \nabla \left( \frac{u}{v} \right) \right] \), where the first term shows the usual linear diffusion, while the nonlinear diffusion described by the second term

\[
-\chi \nabla \cdot \left[ v^2 \nabla \left( \frac{u}{v} \right) \right] = -\chi \nabla \cdot \left[ v \nabla u - u \nabla v \right]
\]

models an ecological situation, in which predators are prone to move towards higher concentration of the prey. As typical cases of drift-diffusion in biological models, the cross-diffusion in the Shigesada–Kawasaki–Teramoto model and the chemotaxis in the Keller–Segel model are well-known. As explained
by Okubo–Levin [9], the transition probabilities of each predator are determined by conditions at departure, arrival and the middle point. The so-called taxis $\chi u \nabla v$ is determined by the difference of the conditions between departure and arrival and is considered as the middle case, whereas $-\chi v^2 \nabla (u/v)$ in (1.1) shows that the transition probabilities are determined by the condition at arrival (9) and the latter case has not been mathematically studied much except for some recent work on the topic of stationary solutions (7, 8). In particular, there seems to be no work on global existence and asymptotic behavior of non-stationary solutions to the problem (1.1).

Related works. Before stating our results regarding exactly this matter, let us first briefly widen our perspective somewhat to examine the larger context of prior work in this area.

In particular for prey-taxis systems with a more standard taxis term, there has been some classical existence (and sometimes boundedness) theory established under the assumption that the taxis term either vanishes for large values of $u$ (11, 28) or the taxis sensitivity is sufficiently small (19, 26). See also 16 for a discussion of classical solution theory in an indirect prey-taxis model. Relaxing the necessary assumptions, there has also been some foray into the construction of weak solutions in similar settings (1, 19).

Often already established in tandem with the already mentioned existence theory, there has also been some exploration of long-time behavior in various diffusive predator-prey models. More precisely, it has been shown that in many scenarios the solutions tend (potentially exponentially fast) to their constant equilibria (5, 16, 26). In 19, the constructed weak solutions are even further shown to become classical as a consequence of stabilization.

Moving yet slightly further away from the model we are interested in but staying in the realm of predator-prey modeling and analysis, there have been some efforts made to analyze predator-taxis systems, which model prey fleeing from its predators as opposed to the predators chasing the prey (27). When this dynamic is combined with the already discussed prey-taxis, the resulting system seems to be rather challenging as there are to our knowledge thus far only some constructions of weak solutions available (15) considering the one dimensional case, 3 featuring nonlinear diffusion) and classical solutions seem to have only been constructed if both taxis terms are mediated by a regularizing indirection (25) or the initial data is already very close to the equilibria (2). There has also been some discussion of models featuring multiple predators attracted by the same prey (17).

Purpose. The purpose of this paper is

1) to establish global existence and boundedness of classical solutions in the case $n = 2$ and global existence of weak solutions in the case $n = 3$;

2) to show stabilization or more specifically to show that the solution $(u(\cdot, t), v(\cdot, t))$ converges to the constant steady state $(u_*, v_*)$ in some appropriate sense as $t \to \infty$, where $u_*, v_*$ are either the positive solutions to $u_*(m_1 - u_* + av_*) = v_*(m_2 - bu_* - v_*) = 0$ in the coexistence case ($m_2 - bm_1 > 0$) or equal to $m_1$ and $0$, respectively, in the prey-extinction case ($m_2 - bm_1 \leq 0$).

Main results. The main result for the first purpose concerning global existence reads as follows.

Theorem 1.1 Let $n \in \{2, 3\}$, $d_1, d_2, m_1, \chi, a, b > 0$, $m_2 \in \mathbb{R}$ and assume that $u_0, v_0 \in W^{1,\infty}(\Omega)$ are nonnegative with $u_0, v_0 \neq 0$. In addition, if $n = 3$, assume that $u_0 \in L \log L(\Omega)$ and $\sqrt{v_0} \in W^{1,2}(\Omega)$.
(i) If \( n = 2 \), then the problem (1.1) possesses a unique global classical solution \((u, v)\) such that
\[
\begin{align*}
&u \in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)), \\
v \in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty))
\end{align*}
\]
and such that \( u, v > 0 \) in \( \Omega \times (0, \infty) \). Moreover, this solution is bounded in the sense that there exists a constant \( C > 0 \) fulfilling
\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C \quad \text{for all } t > 0.
\]

(ii) If \( n = 3 \), then there exists at least one global weak solution \((u, v)\) of (1.1) in the sense that
\[
\begin{align*}
u \in L^2_{\text{loc}}([0, \infty); L^2(\Omega)), & \quad v \in L^\infty(\Omega \times (0, \infty))
\end{align*}
\]
and that \( u, v \geq 0 \) a.e. on \( \Omega \times (0, \infty) \),
\[
\begin{align*}
u \in L^2_{\text{loc}}([0, \infty); L^2(\Omega)), & \quad v \in L^\infty(\Omega \times (0, \infty))
\end{align*}
\]
as well as the identities
\[
\begin{align*}
- \int_0^\infty & \int_\Omega u \varphi_t - \int_\Omega u_0 \varphi(\cdot, 0) = - \int_0^\infty \int_\Omega (d_1 + \chi v) \nabla u \cdot \nabla \varphi + \chi \int_0^\infty \int_\Omega u \nabla v \cdot \nabla \varphi \\
+ \int_0^\infty & \int_\Omega u(m_1 - u + av)\varphi, \quad (1.2)
\end{align*}
\]
\[
\begin{align*}
- \int_0^\infty & \int_\Omega v \varphi_t - \int_\Omega v_0 \varphi(\cdot, 0) = - d_2 \int_0^\infty \int_\Omega \nabla v \cdot \nabla \varphi + \int_0^\infty \int_\Omega v(m_2 - bu - v)\varphi \quad (1.3)
\end{align*}
\]
hold for all \( \varphi \in C^0_0(\overline{\Omega} \times [0, \infty)). \)

Given that we have now properly formalized the first main result of this paper, we can now transition to stating our second main result concerning the stabilization behavior of the now established solutions to (1.1).

**Theorem 1.2** Let \( n \in \{2, 3\} \), \( d_1, d_2, m_1, \chi, a, b > 0 \), \( m_2 \in \mathbb{R} \) and assume that \( u_0, v_0 \in W^{1,\infty}(\Omega) \) are nonnegative with \( u_0, v_0 \neq 0 \). In addition, if \( n = 3 \), assume that \( u_0 \in L \log L(\Omega) \) and \( \sqrt{v_0} \in W^{1,2}(\Omega) \). Let
\[
\begin{align*}
&u_* := \begin{cases} 
\frac{m_1 + am_2}{ab+1} & \text{if } m_2 - bm_1 \geq 0 \\
\frac{m_1}{m_2} & \text{if } m_2 - bm_1 < 0
\end{cases}
&v_* := \begin{cases} 
\frac{m_2-bm_1}{ab+1} & \text{if } m_2 - bm_1 \geq 0 \\
0 & \text{if } m_2 - bm_1 < 0
\end{cases}
\end{align*}
\]
be the constant steady states. If
\[
\log(u_0) \in L^1(\Omega) \quad \text{and} \quad \log(v_0) \in L^1(\Omega) \text{ if } v_* > 0
\]
as well as
\[
\chi^2 < \frac{4d_1d_2}{bm_2+u_*} \left( \frac{av_*}{m_2} + \frac{4}{b} \right), \quad \text{where } m_2^+ := \max(0, m_2),
\]
then the solution \((u, v)\) constructed in Theorem 1.1 has one of the following stabilization properties depending on the space dimension \( n \):
(i) If $n = 2$, then
\[ u(\cdot, t) \to u_* \text{ in } L^p(\Omega) \quad \text{and} \quad v(\cdot, t) \to v_* \text{ in } W^{1,p}(\Omega) \] (1.7)
for all $p \in [1, \infty)$ as $t \to \infty$.

(ii) If $n = 3$, then
\[ u(\cdot, t) \to u_* \text{ in } L^1(\Omega) \quad \text{and} \quad v(\cdot, t) \to v_* \text{ in } L^p(\Omega) \] (1.8)
for all $p \in [1, \infty)$ as $(0, \infty) \setminus N \ni t \to \infty$, where $N$ is a set of measure zero.

**Main ideas.** Before diving into the details of the proofs, we will first give a brief overlook over our main ideas as well as the organization of this paper.

In Section 2, we begin by establishing some basic local existence theory for not only (1.1), which we will directly investigate for the two-dimensional case, but also for a family of variants featuring a regularization of the taxis term, which we will use in the three-dimensional case to gain our desired weak solution as a limit of the thus gained approximate solutions. We further derive a set of uniform baseline a priori estimates for these local solutions.

Having established the necessary preliminaries, we devote Section 3 to the proof of Theorem 1.1. To this end, we begin by treating the two-dimensional case following the approach from [14] by essentially iteratively bootstrapping ourselves to sufficiently good a priori estimates to rule out finite-time blow-up in (1.1). To handle the three-dimensional case, we do not work directly with (1.1) because of the lack of regularity but instead with the family of similar systems seen in (3.6) only differing due to a slight regularization of the taxis term. Our aim is then to gain our desired solution as the limit of the thus gained family of approximate solutions. The key to establishing the necessary bounds to achieve this is the derivation of an energy-type inequality of the form
\[
\frac{d}{dt} \left[ \int_{\Omega} u \log(u) + \frac{X}{2b} \int_{\Omega} \frac{|\nabla v|^2}{v} \right] + \frac{1}{K} \left[ \int_{\Omega} \frac{d_1 + Xv}{u} |\nabla u|^2 + \int_{\Omega} u^2 \log(u) + \int_{\Omega} \frac{|D^2 v|^2}{v^3} + \int_{\Omega} |\nabla v|^4 v^3 \right] \leq K
\]
in Lemma 3.3. We then use the bounds resulting from the above inequality as well as some of their immediate consequences to both ensure that the approximate solutions are in fact global in Lemma 3.6 by employing similar arguments to the ones used in the two-dimensional case as well as to construct our actual solution candidates by using compactness arguments in Lemma 3.9. It then only remains to show that the solution properties of the approximate solutions survive the limit process in such a way as to make our newly constructed solution candidates actual weak solutions to (1.1).

Given that we have now firmly established the existence of our desired solutions, Section 4 focuses on the analysis of their long-time behavior and thus the proof of Theorem 1.2. Here, we will not only treat both the two-dimensional and three-dimensional cases at once but also deal with both the coexistence and prey-extinction cases in a way that tries to minimize the duplication of similar arguments. We do this by basing our arguments for all cases on another energy-type inequality, which has the form
\[
\frac{d}{dt} E(u, v) \leq -\delta G(u, v), \quad \text{where} \quad E(u, v) := \int_{\Omega} H_{u_*}(u) + \frac{a}{b} \int_{\Omega} H_{v_*}(v) + \frac{2}{b^2 m_2} \int_{\Omega} (v - v_*)^2
\] with $H_{\xi}(\eta) = \eta - \xi - \xi \log(\frac{\eta}{\xi})$ if $\xi > 0$ and $H_0(\eta) = \eta$, and
\[
G(u, v) := \int_{\Omega} \frac{|\nabla u|^2}{u^2} + \int_{\Omega} \frac{|\nabla v|^2}{v^2} + \int_{\Omega} (u - u_*)^2 + \int_{\Omega} (v - v_*)^2.
\]
We then show in Lemma 4.3 that, under the parameter condition (1.6), both the classical solution in the two-dimensional case as well as the approximate solutions in the three-dimensional case fulfill said energy-type inequality after some (uniform) waiting-time $T_E$ while the energy $E$ is (uniformly) bounded up to said same time. The need for this waiting time is owed to eliminating an otherwise necessary initial data condition by using Lemma 4.1. As a consequence of this energy-type inequality, we conclude that the energy $E$ is monotonically decreasing for almost all times $t > T_E$ in Lemma 4.6 by first deriving a space-time square integrability bound for $H_u(u)$ and $H_v(v)$ in Lemma 4.5, which is necessary to ensure that the monotonicity property survives the limit process used in our weak solution construction, using properties of the functions $H_1(\eta)$ established in e.g. [24]. Combining this same square integrability property with the second notable consequence of the energy-type inequality, that is, $\int_{T_E}^{\infty} G(u, v) < \infty$, we can construct a sequence of times along which $E(u, v)$ converges to zero, which combined with the already established monotonicity almost immediately yields a slightly weaker version of our desired result in Lemma 4.8 and after some slight refinement Theorem 1.2 itself.

2 Preliminaries. Local existence and basic estimates

In this section we give lemmas which present local existence and basic estimates for classical solutions of (1.1) as well as a more regularized version of the same system, which we will later use in our weak solution construction. For convenience we write down the first and second equations of the system in equation as follows:

\[
\begin{align*}
  u_t &= \nabla \cdot ((d_1 + \chi v)\nabla u) - \chi \nabla \cdot (F(u)\nabla v) + u(m_1 - u + av), \quad x \in \Omega, \ t > 0, \\
v_t &= d_2 \Delta v + v(m_2 - bu - v), \quad x \in \Omega, \ t > 0, \\
  \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, \ t > 0, \\
  u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x),
\end{align*}
\]

where $F$ is given by

\[ F(u) := \begin{cases} 
  u & \text{if } n = 2, \\
  \frac{u}{1 + \varepsilon u} & \text{if } n = 3.
\end{cases} \]

The following statement on local existence and uniqueness can be proved by using fixed point arguments as in the proof of [12, Lemma 2.1] or [23, Lemma 2.1].

Lemma 2.1 Let $n \in \{2, 3\}$, $d_1, d_2, m_1, \chi, a, b > 0$, $m_2 \in \mathbb{R}$ and assume that $u_0, v_0 \in W^{1,\infty}(\Omega)$ are nonnegative and $u_0, v_0 \neq 0$. Then there exist $T_{\max} \in (0, \infty]$ and a uniquely determined pair $(u, v)$ of functions

\[ u \in C^0([0, T_{\max}) \times [0, \Omega]) \cap C^{2,1}(\Omega \times (0, T_{\max})) \quad \text{and} \quad v \in C^0([0, T_{\max}) \times [0, \Omega]) \cap C^{2,1}(\Omega \times (0, T_{\max})) \]

such that $u, v > 0$ in $\Omega \times (0, T_{\max})$, such that $(u, v)$ solves (2.1) classically, and such that

\[ \text{if } T_{\max} < \infty, \quad \text{then } \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \to \infty \text{ as } t \nearrow T_{\max}. \]

(2.2)
We now establish some basic estimates that will not only provide the foundation for our arguments refuting blow-up of the now established local solutions but will also prove useful in our later discussion of long-time behavior. Importantly, at no point are the constants derived in the following two lemmas dependent on the exact structure of $F$ as the term containing $F$ in the first equation always vanishes as a result of integration by parts and the prescribed Neumann boundary conditions.

**Lemma 2.2** Under the assumption of Lemma 2.1, there exists a constant $C > 0$ such that
\[
\|v(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all} \ t \in (0, T_{\text{max}}) \tag{2.3}
\]
and
\[
\int_t^{t+\tau} \int_\Omega |\nabla v|^2 \leq C \quad \text{for all} \ t \in (0, T_{\text{max}} - \tau) \tag{2.4}
\]
with $\tau := \min\{1, \frac{1}{2} T_{\text{max}}\}$.

**Proof.** Multiplying the second equation in (2.1) by $v^{p-1}$ ($p > 1$) and integrating it over $\Omega$, we see from the H"older inequality that
\[
\frac{1}{p} \frac{d}{dt} \int_\Omega v^p + d_2(p-1) \int_\Omega v^{p-2} |\nabla v|^2 = m_2 \int_\Omega v^p - b \int_\Omega uv^p + \int_\Omega v^{p+1} \\
\leq m_2 \int_\Omega v^p - |\Omega|^{-\frac{1}{p}} \left( \int_\Omega v^p \right)^{1+\frac{1}{p}}. \tag{2.5}
\]
By dropping the second term on the left-hand side, an ODE comparison shows that
\[
\int_\Omega v^p(\cdot, t) \leq \max \left\{ \int_\Omega v_0^p, m_2^p |\Omega|^{\frac{1}{p}} \right\}, \quad \text{i.e.,} \quad \|v(\cdot, t)\|_{L^p(\Omega)} \leq \max\{\|v_0\|_{L^p(\Omega)}, m_2 |\Omega|^{\frac{1}{p}}\}.
\]
Letting $p \to \infty$ precisely warrants (2.3). Moreover, a straightforward integration of (2.5) with $p = 2$ in time yields (2.4). \qed

**Lemma 2.3** Under the assumption of Lemma 2.1, there exists a constant $C > 0$ such that
\[
\int_\Omega u(\cdot, t) \leq C \quad \text{for all} \ t \in (0, T_{\text{max}}) \tag{2.6}
\]
and
\[
\int_t^{t+\tau} \int_\Omega u^2 \leq C \quad \text{for all} \ t \in (0, T_{\text{max}} - \tau) \tag{2.7}
\]
with $\tau := \min\{1, \frac{1}{2} T_{\text{max}}\}$.

**Proof.** Integrating the first equation in (2.1) over $\Omega$, we immediately infer from (2.3) that
\[
\frac{d}{dt} \int_\Omega u = \int_\Omega (m_1 + av)u - \int_\Omega u^2 \leq (m_1 + ac_1) \int_\Omega u - \int_\Omega u^2 \tag{2.8}
\]
for some $c_1 > 0$. Thus the Cauchy–Schwarz inequality gives

$$\frac{d}{dt} \int_{\Omega} u \leq (m_1 + ac_1) \int_{\Omega} u - |\Omega|^{-1} \left( \int_{\Omega} u \right)^2,$$

which by an ODE comparison derives

$$\int_{\Omega} u(\cdot, t) \leq \max \left\{ \int_{\Omega} u_0, (m_1 + ac_1)|\Omega| \right\}.$$ 

This proves (2.6), and then (2.7) results from a straightforward integration of (2.8) in time. 

3 Global existence

In this section, following the arguments in [14], [21] and [22], we establish global existence in (1.1). The proof will be divided into the two cases that $n = 2$ in Section 3.1 and that $n = 3$ in Section 3.2.

3.1 Global existence and boundedness of classical solutions in the case $n = 2$

We will show that $T_{\text{max}} = \infty$ as in the proof of [14] while we will use an energy inequality approach to show the same for the approximate solutions used in Section 3.2 to handle the case $n = 3$.

Lemma 3.1 Under the assumption of Lemma 2.1, there exists a constant $C > 0$ such that

$$\int_{\Omega} |\nabla v(\cdot, t)|^2 \leq C \quad \text{for all } t \in (0, T_{\text{max}})$$

and

$$\int_{t}^{t+\tau} \int_{\Omega} |\Delta v|^2 \leq C \quad \text{for all } t \in (0, T_{\text{max}} - \tau)$$

with $\tau := \min\{1, \frac{1}{2} T_{\text{max}}\}$.

Proof. Testing the second equation in (2.1) by $-\Delta v$, we see from (2.3) and Young’s inequality that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla v|^2 + d_2 \int_{\Omega} |\Delta v|^2 = m_2 \int_{\Omega} v(-\Delta v) + b \int_{\Omega} uv\Delta v - 2 \int_{\Omega} v|\nabla v|^2 \leq c_1 + c_2 \int_{\Omega} u^2 + \frac{d_2}{2} \int_{\Omega} |\Delta v|^2$$

for some $c_1, c_2 > 0$. In view of (2.4) we know that there exists a constant $c_3$ such that

$$\int_{t}^{t+\tau} \int_{\Omega} |\nabla v|^2 \leq c_3 \quad \text{for all } t \in (0, T_{\text{max}} - \tau),$$

and hence, given $t \in (0, T_{\text{max}})$, we can pick $t_0 \in (t - \tau, t) \cap [0, \infty)$ fulfilling

$$\int_{\Omega} |\nabla v(\cdot, t_0)|^2 \leq c_4 := \max \left\{ \int_{\Omega} |\nabla v_0|^2, \frac{c_3}{\tau} \right\}.$$ 

Integrating (3.3) over $(t_0, t)$ and invoking (2.7), we thus infer (3.1), whereupon another integration of (3.3) yields (3.2). \qed
Lemma 3.2 Under the assumption of Lemma 2.1 let $p \geq 2$, $L > 0$ and $\tau := \min\{1, \frac{1}{2}T_{\text{max}}\}$. Then there exists a constant $C = C(p, L) > 0$ such that if

$$\int_t^{t+\tau} \int_{\Omega} u^p \leq L \quad \text{for all } t \in (0, T_{\text{max}} - \tau),$$

then

$$\int_{\Omega} u^p(\cdot, t) \leq C \quad \text{for all } t \in (0, T_{\text{max}})$$

and

$$\int_t^{t+\tau} \int_{\Omega} u^{p+1} \leq C \quad \text{for all } t \in (0, T_{\text{max}} - \tau).$$

Proof. Testing the first equation in (2.1) by $u^{p-1}$, we see from integration by parts and (2.3) that

$$\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p + (p-1) \int_{\Omega} (d_1 + \chi_v) u^{p-2} |\nabla u|^2 = (p-1)\chi \int_{\Omega} u^{p-1} \nabla u \cdot \nabla v + \int_{\Omega} u^p (m_1 - u + av) \leq (p-1)\chi \int_{\Omega} u^{p-1} \nabla u \cdot \nabla v + c_1 \int_{\Omega} u^p - \int_{\Omega} u^{p+1}$$

for some $c_1 > 0$. This implies that the situation is the same as in [14, Proof of Lemma 3.8, (3.32)]. Therefore the same argument yields the lemma. \qed

Recalling the basic estimate for $u^2$ in Lemma 2.3, we can now obtain the following lemma immediately.

Lemma 3.3 Under the assumption of Lemma 2.1, for all $p \in (1, \infty)$ there exists a constant $C(p) > 0$ such that

$$\int_{\Omega} u^p(\cdot, t) \leq C(p) \quad \text{for all } t \in (0, T_{\text{max}}).$$

Proof. The lemma results from an induction via Lemma 3.2 with the starting estimate (2.7). \qed

We are now in a position to prove global existence in the case $n = 2$.

Proof of Theorem 1.1 (i). Let $(u, v)$ denote the classical solution of (2.1) in $\Omega \times (0, T_{\text{max}})$ provided by Lemma 2.1 According to Lemmas 2.2 and 3.3 we know that

$$\|v(\cdot, t)\|_{L^\infty} \leq c_1 \quad \text{for all } t \in (0, T_{\text{max}})$$

for some $c_1 > 0$ and, given any $p \in (1, \infty)$, we can pick $c_2(p) > 0$ fulfilling

$$\int_{\Omega} u^p(\cdot, t) \leq c_2(p) \quad \text{for all } t \in (0, T_{\text{max}}). \quad (3.4)$$

Hence, for any $q \in [1, \infty]$, by taking $p \in (1, \infty)$ such that $\frac{1}{p} - \frac{1}{q} < \frac{1}{2}$, we infer from the variation-of-constant formula and well-known smoothing estimates for the Neumann heat semigroup [18, Lemma
Lemma A.1] that
\[ \|v(t)\|_{L^\infty(\Omega)} \leq c_3 \|v_0\|_{L^\infty(\Omega)} + c_3 \int_0^t \left(1 + (t-s)^{-\frac{1}{2}} \left(\frac{1}{p} - \frac{1}{q}\right)\right) e^{-\lambda(t-s)} \|v(m_2 - bu - v)\|_{L^p(\Omega)} ds \]
\[ \leq c_4 \left(1 + c_1[(m_2 + c_1)\|\nabla v\|_p + bc_2(p)]\right) \int_0^t \left(1 + (t-s)^{-\frac{1}{2}} \left(\frac{1}{p} - \frac{1}{q}\right)\right) e^{-\lambda(t-s)} ds \]
\[ \leq c_5 \quad \text{for all } t \in (0, T_{\text{max}}) \tag{3.5} \]
with some constants $c_3, c_4, c_5 > 0$ and $\lambda > 0$. In light of (3.4) and (3.5), we can deduce from Lemma 2.1] that
\[ \|u(t)\|_{L^\infty} \leq c_6 \quad \text{for all } t \in (0, T_{\text{max}}) \]
for some $c_6 > 0$. Thus we have $T_{\text{max}} = \infty$ and the conclusion of Theorem 1.1 (i) holds. \qed

### 3.2 Global existence of weak solutions in the case $n = 3$

In the case $n = 3$, since we will not be able to obtain boundedness of $\|u(t)\|_{L^\infty(\Omega)}$ in time, we shift to global classical solvability of approximate problems (2.1) with $F(u) := \frac{u}{1+\delta u}$ and then pass to the limit $\varepsilon \searrow 0$ as in the arguments in [21] and [22]. More precisely, for $\varepsilon \in (0, 1)$ we consider
\[
\begin{cases}
(u_\varepsilon)_t = \nabla \cdot ((d_1 + \chi \varepsilon u_\varepsilon) \nabla u_\varepsilon) - \chi \nabla \cdot \left(u_\varepsilon \frac{\nabla v_\varepsilon}{1+\varepsilon u_\varepsilon}\right) + u_\varepsilon(m_1 - u_\varepsilon + av_\varepsilon), & x \in \Omega, \ t > 0, \\
(v_\varepsilon)_t = d_2 \Delta v_\varepsilon + v_\varepsilon(m_2 - bu_\varepsilon - v_\varepsilon), & x \in \Omega, \ t > 0, \\
\frac{\partial u_\varepsilon}{\partial \nu} = \frac{\partial v_\varepsilon}{\partial \nu} = 0, & x \in \partial \Omega, \ t > 0, \\
u_\varepsilon(x, 0) = u_0(x), \ v_\varepsilon(x, 0) = v_0(x). & x \in \Omega,
\end{cases}
\tag{3.6}
\]

As provided in Lemma 2.1, the approximate problem (3.6) admits a unique local-in-time classical solution $(u_\varepsilon, v_\varepsilon)$ up to a maximal time $T_{\text{max}, \varepsilon} \in (0, \infty)$. Before proving that $T_{\text{max}, \varepsilon} = \infty$ we shall establish an energy inequality through a combination of $\int_\Omega u_\varepsilon \log(u_\varepsilon)$ and $\int_\Omega \frac{1}{\varepsilon^3} \nabla v_\varepsilon$.

**Lemma 3.4** Under the assumption of Lemma 2.1, there exists a constant $K > 0$ such that the solution of (3.6) satisfies
\[
\frac{d}{dt} \int_\Omega u_\varepsilon \log(u_\varepsilon) + \frac{1}{2\theta} \int_\Omega \frac{1}{\varepsilon^3} |\nabla v_\varepsilon|^2 \leq K \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}).
\tag{3.7}
\]

**Proof.** From the first equation in (3.6) and integration by parts it follows that
\[
\frac{d}{dt} \int_\Omega u_\varepsilon \log(u_\varepsilon) = \int_\Omega (u_\varepsilon)_t \log(u_\varepsilon) + \int_\Omega (u_\varepsilon)_t = -\int \frac{d_1 + \chi \varepsilon u_\varepsilon}{\varepsilon^3} |\nabla u_\varepsilon|^2 \chi \int \frac{1}{1+\varepsilon u_\varepsilon} \nabla u_\varepsilon \cdot \nabla v_\varepsilon + \int u_\varepsilon(m_1 - u_\varepsilon + av_\varepsilon)(\log(u_\varepsilon) + 1).
\]
Using (2.3) and noting that \( \log(r) + 1 \leq r \ (r > 0) \), we observe that if \( \log(u_\epsilon) + 1 \geq 0 \) then
\[
\begin{align*}
\epsilon (m_1 - u_\epsilon + av_\epsilon)(\log(u_\epsilon) + 1) & \leq u_\epsilon (c_1 - u_\epsilon)(\log(u_\epsilon) + 1) \\
& \leq -\frac{1}{2} u_\epsilon^2 \log(u_\epsilon) + \frac{1}{2} c_1^2 u_\epsilon
\end{align*}
\]
for some \( c_1 > 1 \) and this holds even if \( \log(u_\epsilon) + 1 < 0 \) because
\[
\begin{align*}
\epsilon (m_1 - u_\epsilon + av_\epsilon)(\log(u_\epsilon) + 1) & \leq -u_\epsilon \cdot u_\epsilon \log(u_\epsilon) \leq \frac{1}{c_1} u_\epsilon.
\end{align*}
\]
Hence by virtue of (2.6) we have
\[
\frac{d}{dt} \int u_\epsilon \log(u_\epsilon) \leq -\int \frac{d_1 + \epsilon u_\epsilon}{u_\epsilon} |\nabla u_\epsilon|^2 + \epsilon \int \frac{1}{1 + \epsilon u_\epsilon} \nabla u_\epsilon \cdot \nabla u_\epsilon - \frac{1}{2} \int u_\epsilon^2 \log(u_\epsilon) + c_2 \quad (3.8)
\]
for some \( c_2 > 0 \). On the other hand, we infer from integration by parts and the second equation in (3.6) that
\[
\frac{d}{dt} \int \frac{|\nabla v_\epsilon|^2}{v_\epsilon} \quad (3.9)
\]
for all \( t \in (0, T_{\text{max}, \epsilon}) \). Here, thanks to [21, Lemma 3.1], we have
\[
- \int \frac{|\Delta v_\epsilon|^2}{v_\epsilon} = -\int \frac{|D^2 v_\epsilon|^2}{v_\epsilon} - \frac{3}{2} \int \frac{|\nabla v_\epsilon|^2}{v_\epsilon} \Delta v_\epsilon + \int \frac{|\nabla v_\epsilon|^4}{v_\epsilon} + \frac{1}{2} \int \partial_{\partial v} \frac{1}{v_\epsilon} \partial \nabla v_\epsilon^2
\]
and as in [21, Proof of Lemma 3.2] we can see from a direct computation and integration by parts that
\[
\int v_\epsilon |D^2 \log(v_\epsilon)|^2 = \int \frac{|D^2 v_\epsilon|^2}{v_\epsilon} + \int \frac{|\nabla v_\epsilon|^2}{v_\epsilon} \Delta v_\epsilon - \int \frac{|\nabla v_\epsilon|^4}{v_\epsilon},
\]
whence adding these two identities entail
\[
- \int \frac{|\Delta v_\epsilon|^2}{v_\epsilon} + \int v_\epsilon |D^2 \log(v_\epsilon)|^2 = -\frac{1}{2} \int \frac{|\nabla v_\epsilon|^2}{v_\epsilon} \Delta v_\epsilon + \int \frac{1}{2} \partial_{\partial v} \frac{1}{v_\epsilon} \partial v_\epsilon^2 \cdot |\nabla v_\epsilon|^2.
\]
Thus we obtain
\[
\frac{d}{dt} \int \frac{|\nabla v_\epsilon|^2}{v_\epsilon} \leq -2d_2 \int v_\epsilon |D^2 \log(v_\epsilon)|^2 + d_2 \int \frac{1}{\partial v} \partial v_\epsilon^2 |\nabla v_\epsilon|^2 - 2b \int v_\epsilon \nabla u_\epsilon \cdot \nabla v_\epsilon + 3m_2 \int \frac{|\nabla v_\epsilon|^2}{v_\epsilon} (3.9)
\]
for all \( t \in (0, T_{\text{max}, \varepsilon}) \). Therefore, adding (3.8) and (3.9) multiplied by \( \frac{\chi}{2b} \), we have that with some constants \( c_3, c_4 > 0 \),

\[
\frac{d}{dt} \left[ \int_{\Omega} u_{\varepsilon} \log(u_{\varepsilon}) + \frac{\chi}{2b} \int_{\Omega} \frac{|\nabla v_{\varepsilon}|^2}{v_{\varepsilon}} \right] + \int_{\Omega} \frac{d_1 + \chi v_{\varepsilon}}{u_{\varepsilon}} |\nabla u_{\varepsilon}|^2 + \frac{1}{2} \int_{\Omega} u_{\varepsilon}^2 \log(u_{\varepsilon}) + \frac{\chi d_2}{b} \int_{\Omega} v_{\varepsilon}|D^2 \log(v_{\varepsilon})|^2 \\
\leq c_2 + c_3 \int_{\partial \Omega} \frac{1}{v_{\varepsilon}} \frac{\partial}{\partial \nu} |\nabla v_{\varepsilon}|^2 + c_4 \int_{\Omega} \frac{|\nabla v_{\varepsilon}|^2}{v_{\varepsilon}} - \chi^\varepsilon \int_{\Omega} \frac{u_{\varepsilon}}{1 + \varepsilon u_{\varepsilon}} \nabla u_{\varepsilon} \cdot \nabla v_{\varepsilon}
\]

(3.10)

for all \( t \in (0, T_{\text{max}, \varepsilon}) \). Invoking [22] (3.9), we can estimate the last term on the left-hand side of (3.10) as

\[
\frac{\chi d_2}{b} \int_{\Omega} v_{\varepsilon}|D^2 \log(v_{\varepsilon})|^2 \geq c_5 \int_{\Omega} \frac{|D^2 v_{\varepsilon}|^2}{v_{\varepsilon}} + c_5 \int_{\Omega} \frac{|\nabla v_{\varepsilon}|^4}{v_{\varepsilon}^3}
\]

for some \( c_5 > 0 \), while Young’s inequality and (2.3) give

\[
c_4 \int_{\Omega} \frac{|\nabla v_{\varepsilon}|^2}{v_{\varepsilon}} \leq c_5 \frac{8}{\varepsilon} \int_{\Omega} \frac{|\nabla v_{\varepsilon}|^4}{v_{\varepsilon}^3} + c_6
\]

(3.11)

as well as

\[-\chi^\varepsilon \int_{\Omega} \frac{u_{\varepsilon}}{1 + \varepsilon u_{\varepsilon}} \nabla u_{\varepsilon} \cdot \nabla v_{\varepsilon} = \int_{\Omega} \frac{u_{\varepsilon}}{v_{\varepsilon}} \nabla v_{\varepsilon} \cdot \nabla v_{\varepsilon} - \chi^\varepsilon \frac{4}{4} \frac{1}{v_{\varepsilon}^2} \int_{\Omega} \frac{v_{\varepsilon} |\nabla u_{\varepsilon}|^2}{u_{\varepsilon}} \leq \frac{d_1}{2} \int_{\Omega} \frac{|\nabla u_{\varepsilon}|^2}{u_{\varepsilon}} + c_5 \int_{\Omega} \frac{|\nabla v_{\varepsilon}|^4}{v_{\varepsilon}^3} + c_7 \int_{\Omega} u_{\varepsilon}^2
\]

for some \( \varepsilon \)-independent constants \( c_6, c_7 > 0 \), all of which imply that

\[
\frac{d}{dt} \left[ \int_{\Omega} u_{\varepsilon} \log(u_{\varepsilon}) + \frac{\chi}{2b} \int_{\Omega} \frac{|\nabla v_{\varepsilon}|^2}{v_{\varepsilon}} \right] + \int_{\Omega} \frac{d_1 + \chi v_{\varepsilon}}{u_{\varepsilon}} |\nabla u_{\varepsilon}|^2 + \frac{1}{2} \int_{\Omega} u_{\varepsilon}^2 \log(u_{\varepsilon}) + \frac{\chi d_2}{b} \int_{\Omega} v_{\varepsilon}|D^2 \log(v_{\varepsilon})|^2 \\
\leq c_2 + c_3 \int_{\partial \Omega} \frac{1}{v_{\varepsilon}} \frac{\partial}{\partial \nu} |\nabla v_{\varepsilon}|^2 + c_6 + c_7 \int_{\Omega} u_{\varepsilon}^2
\]

(3.12)

for all \( t \in (0, T_{\text{max}, \varepsilon}) \). In the same way as in the proof of [4] (3.10), noting that \( \frac{\partial}{\partial \nu} |\nabla v_{\varepsilon}|^2 \leq c_8 |\nabla v_{\varepsilon}|^2 \) for some \( c_8 > 0 \) and combining the Sobolev compact embedding \( W^{r+\frac{1}{2}-2}(\Omega) \hookrightarrow L^2(\partial \Omega) \) \((r \in (0, \frac{1}{2}))\) with the fractional Gagliardo-Nirenberg inequality, we can obtain

\[
\int_{\partial \Omega} \frac{1}{v_{\varepsilon}} \frac{\partial}{\partial \nu} |\nabla v_{\varepsilon}|^2 \leq 4c_8 \| \nabla \sqrt{v_{\varepsilon}} \|^2_{L^2(\partial \Omega)} \leq c_9 \| D^2 \sqrt{v_{\varepsilon}} \|_{L^2(\Omega)}^{2(1-\alpha)} + c_9 \| \nabla \sqrt{v_{\varepsilon}} \|^2_{L^2(\Omega)}
\]

for some \( c_9 > 0 \) and \( \alpha \in (0, 1) \). Here a direct computation gives

\[
\| D^2 \sqrt{v_{\varepsilon}} \|_{L^2(\Omega)} \leq c_{10} \left( \int_{\Omega} \frac{|D^2 v_{\varepsilon}|^2}{v_{\varepsilon}} + \int_{\Omega} \frac{|\nabla v_{\varepsilon}|^4}{v_{\varepsilon}^3} \right)
\]
with some $c_{10} > 0$, and like in (3.11) for any small $\delta > 0$ we can take $c_{11}(\delta) > 0$ such that
\[
\|\nabla \sqrt{v}\|_{L^2}\leq \frac{1}{4} \int_{\Omega} \frac{|\nabla v|^2}{v} \leq \delta \int_{\Omega} \frac{|\nabla v|^4}{v^3} + c_{11}(\delta).
\]
Therefore the Young inequality yields
\[
c_3 \int_{\partial \Omega} \frac{1}{v} \frac{\partial}{\partial n} |\nabla v|^2 \leq \frac{c_5}{2} \int_{\Omega} \frac{|D^2 v|^2}{v} + \frac{c_5}{4} \int_{\Omega} \frac{|\nabla v|^4}{v^3} + c_{12}
\]
for some $c_{12} > 0$. Plugging this inequality into (3.12) and employing the inequality $c_7 r^2 \leq \frac{1}{4} r^2 \log(r) + c_{13} (r \geq 0)$ with some $c_{13} > 0$ in the last summand in (3.12) imply the desired inequality (3.7). □

As a consequence of Lemma 3.4 we can draw the following a priori estimates.

**Lemma 3.5** Under the assumption of Lemma 2.1, assume further that $u_0 \in L \log L(\Omega)$ and $\sqrt{v_0} \in W^{1,2}(\Omega)$. Then there exists $C > 0$ such that for all $\varepsilon \in (0,1)$ the solution of (3.6) satisfies
\[
\int_{\Omega} u_\varepsilon(\cdot, t) \log(u_\varepsilon(\cdot, t)) + \chi \frac{b}{2b} \int_{\Omega} |\nabla v_\varepsilon(\cdot, t)|^2 \leq C \quad \text{for all } t \in (0, T_{\text{max, } \varepsilon})
\]
and
\[
\int_{0}^{T} \int_{\Omega} \frac{d_1 + \chi v_\varepsilon}{u_\varepsilon} |\nabla u_\varepsilon|^2 + \int_{0}^{T} \int_{\Omega} u_\varepsilon^2 \log(u_\varepsilon) + \int_{0}^{T} \int_{\Omega} \frac{|D^2 v_\varepsilon|^2}{v_\varepsilon} + \int_{0}^{T} \int_{\Omega} \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \leq C \cdot (T + 1)
\]
for all $T \in (0, T_{\text{max, } \varepsilon})$.

**Proof.** We first make use of the trivial inequality $r \log(r) \leq r^2 \log(r)$ ($r \geq 0$) to obtain
\[
\int_{\Omega} u_\varepsilon \log(u_\varepsilon) \leq \int_{\Omega} u_\varepsilon^2 \log(u_\varepsilon).
\]
On the other hand, we next invoke the Young inequality and (2.3) to infer
\[
\frac{\chi}{2b} \int_{\Omega} \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} \leq \int_{\Omega} \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + c_1
\]
for some $c_1 > 0$. Using these two inequality in (3.7), we observe that
\[
y_\varepsilon(t) := \int_{\Omega} u_\varepsilon(\cdot, t) \log(u_\varepsilon(\cdot, t)) + \frac{\chi}{2b} \int_{\Omega} \frac{|\nabla v_\varepsilon(\cdot, t)|^2}{v_\varepsilon(\cdot, t)}
\]
satisfies
\[
y_\varepsilon'(t) + \frac{1}{K} y_\varepsilon(t) \leq c_2 := K + \frac{c_1}{K} \quad \text{for all } t \in (0, T_{\text{max, } \varepsilon}).
\]
This provides
\[
y_\varepsilon(t) \leq c_3 := \max \left\{ \int_{\Omega} u_0 \log(u_0) + \frac{\chi}{2b} \int_{\Omega} \frac{|\nabla v_0|^2}{v_0}, \ K c_2 \right\}
\]
for all $t \in (0, T_{\text{max, } \varepsilon})$, which precisely warrants (3.13). Integrating (3.7), we arrive at (3.14). □
Lemma 3.6 Under the assumption of Lemma 3.5, for all \( \varepsilon \in (0,1) \), we have \( T_{\max, \varepsilon} = \infty \); that is, the solution of (3.6) is global in time.

Proof. To facilitate a proof by contradiction, we assume that \( T_{\max, \varepsilon} < \infty \). Then, multiplying the first equation in (3.6) by \( u_\varepsilon^3 \) and noting that \( \frac{u_\varepsilon}{1 + u_\varepsilon} \leq \frac{1}{\varepsilon} \), we see from integration by parts, (2.3) and Young’s inequality that

\[
\frac{1}{4} \frac{d}{dt} \int _\Omega u_\varepsilon^4 = -3 \int _\Omega (d_1 + \chi u_\varepsilon) u_\varepsilon^2 |\nabla u_\varepsilon|^2 + 3\chi \int _\Omega \frac{u_\varepsilon^3}{1 + \varepsilon u_\varepsilon} \nabla u_\varepsilon \cdot \nabla v_\varepsilon + \int _\Omega u_\varepsilon^4 (m_1 - u_\varepsilon + av_\varepsilon)
\]

\[
\leq -3d_1 \int _\Omega u_\varepsilon^2 |\nabla u_\varepsilon|^2 + c_1 \int _\Omega u_\varepsilon |\nabla u_\varepsilon| \cdot u_\varepsilon \cdot \frac{|\nabla v_\varepsilon|}{\varepsilon^2} + c_2 \int _\Omega u_\varepsilon^4
\]

for some \( c_1, c_2, c_3, c_4 > 0 \). In view of (3.14) this enables us to find \( c_5 > 0 \) fulfilling

\[
\int _\Omega u_\varepsilon^4 (\cdot, t) \leq c_5 \quad \text{for all } t \in (0, T_{\max, \varepsilon}). \tag{3.16}
\]

Hence, similarly in (3.5), we have

\[
\| \nabla v (\cdot, t) \| _{L^\infty (\Omega)} \leq c_6 \| \nabla v_0 \| _{L^\infty (\Omega)} + c_6 \int _0^t \left( 1 + (t-s)^{- \frac{1}{2} - \frac{3}{2} \varepsilon} \right) e^{-\lambda(t-s)} \| v(m_2 - bu - v) \| _{L^4 (\Omega)} \, ds
\]

\[
\leq c_7 \quad \text{for all } t \in (0, T_{\max, \varepsilon}) \tag{3.17}
\]

with some constants \( c_6, c_7 > 0 \) and \( \lambda > 0 \). Therefore, for all \( p \in (1, \infty) \), again by multiplying the first equation in (3.6) by \( u_\varepsilon^{p-1} \), we obtain

\[
\frac{1}{p} \frac{d}{dt} \int _\Omega u_\varepsilon^p \leq -(p-1)d_1 \int _\Omega u_\varepsilon^{p-2} |\nabla u_\varepsilon|^2 + (p-1)\chi \int _\Omega \frac{u_\varepsilon^{p-1}}{1 + \varepsilon u_\varepsilon} \nabla u_\varepsilon \cdot \nabla v_\varepsilon + \int _\Omega u_\varepsilon^p (m_1 - u_\varepsilon + av_\varepsilon)
\]

\[
\leq -(p-1)d_1 \int _\Omega u_\varepsilon^{p-2} |\nabla u_\varepsilon|^2 + c_8 \int _\Omega u_\varepsilon^{p-1} |\nabla u_\varepsilon| + c_9 \int _\Omega u_\varepsilon^p
\]

\[
\leq c_{10} \int _\Omega u_\varepsilon^p
\]

for some \( c_8, c_9, c_{10} > 0 \). This implies that for all \( p \in (1, \infty) \) there is \( c_{11}(p) > 0 \) such that

\[
\int _\Omega u_\varepsilon^p (\cdot, t) \leq c_{11}(p) \quad \text{for all } t \in (0, T_{\max, \varepsilon}),
\]

so that an iteration argument as in [13, Lemma A.1] yields the existence of \( c_{12} > 0 \) satisfying

\[
\| u_\varepsilon (\cdot, t) \| _{L^\infty (\Omega)} \leq c_{12} \quad \text{for all } t \in (0, T_{\max, \varepsilon}).
\]

Combined with (3.17), this means that \( T_{\max, \varepsilon} = \infty \) as our initial assumption combined with the aforementioned bounds leads to a contradiction according to the extensibility criterion (2.2). \( \square \)

The following further estimates for spatio-temporal integrals are immediate consequences of Lemmas 2.3 and 3.5.
Corollary 3.7 Under the assumption of Lemma 3.5 there exists \( C > 0 \) such that for all \( \varepsilon \in (0, 1) \) the solution of (3.6) satisfies

\[
\int_0^T \int_\Omega u_\varepsilon^2 \leq C \cdot (T + 1) \quad \text{for all } T > 0
\] (3.18)

and

\[
\int_0^T \int_\Omega |\nabla u_\varepsilon|^\frac{4}{3} \leq C \cdot (T + 1) \quad \text{for all } T > 0
\] (3.19)

as well as

\[
\int_0^T \int_\Omega |\nabla v_\varepsilon|^4 \leq C \cdot (T + 1) \quad \text{for all } T > 0.
\] (3.20)

Proof. In view of Lemma 2.3 we find \( c_1 > 0 \) fulfilling

\[
\int_0^T \int_\Omega u_\varepsilon^2 \leq c_1 \cdot (T + 1) \quad \text{for all } T > 0,
\]

which means (3.18). According to Lemma 3.5 there exists \( c_2 > 0 \) such that

\[
\int_0^T \int_\Omega |\nabla u_\varepsilon|^2 u_\varepsilon + \int_0^T \int_\Omega |\nabla v_\varepsilon|^4 v_\varepsilon^3 \leq c_2 \cdot (T + 1) \quad \text{for all } T > 0.
\] (3.21)

Thus, the Hölder inequality gives

\[
\int_0^T \int_\Omega |\nabla u_\varepsilon|^\frac{4}{3} = \int_0^T \int_\Omega u_\varepsilon^\frac{4}{3} \cdot |\nabla u_\varepsilon|^\frac{4}{3} \leq \left( \int_0^T \int_\Omega u_\varepsilon^2 \right)^{\frac{2}{3}} \left( \int_0^T \int_\Omega |\nabla u_\varepsilon|^2 \right)^{\frac{2}{3}} \leq c_1^{\frac{2}{3}} c_2^{\frac{2}{3}} (T + 1) \quad \text{for all } T > 0,
\]

which shows (3.19). Finally, (3.21) together with (2.3) implies (3.20).

In preparation for the shortly following compactness argument used as the source for our weak solutions, we now derive some additional uniform estimates for the time derivatives of our approximate solutions.

Lemma 3.8 Under the assumption of Lemma 3.5 there exists \( C > 0 \) such that for all \( \varepsilon \in (0, 1) \) the solution of (3.6) satisfies

\[
\int_0^T \|u_{\varepsilon t}(\cdot, t)\|_{(W^{1,4}(\Omega))^*} \ dt \leq C \cdot (T + 1) \quad \text{for all } T > 0
\] (3.22)

and

\[
\int_0^T \|v_{\varepsilon t}(\cdot, t)\|_{(W^{1,4}(\Omega))^*} \ dt \leq C \cdot (T + 1) \quad \text{for all } T > 0.
\] (3.23)
Proof. Let $\varphi$ be a function in $W^{1,4}(\Omega) \subset L^\infty(\Omega)$. We then test the first equation in (3.6) with $\varphi$ and apply integration by parts as well as the Young and Hölder inequalities to see that there exists $c_1 > 0$ such that
\[
\int_\Omega u_\varepsilon t \varphi \leq \int_\Omega (d_1 + \chi_{\varepsilon})(\nabla u_\varepsilon \nabla \varphi) + \chi \int_\Omega \frac{u_\varepsilon}{1 + \varepsilon u_\varepsilon} |\nabla v_\varepsilon| |\nabla \varphi| + \int_\Omega |\varphi| u_\varepsilon (m_1 + u_\varepsilon + av_\varepsilon)
\]
\[
\leq (d_1 + \chi \|v_\varepsilon\|_L^\infty(\Omega)) \|\nabla u_\varepsilon\|_L^\frac{4}{3}(\Omega) \|\nabla \varphi\|_L^4(\Omega) + \chi \|u_\varepsilon\|_L^2(\Omega) \|\nabla v_\varepsilon\|_L^2(\Omega) \|\nabla \varphi\|_L^4(\Omega)
\]
\[
+ \|\varphi\|_L^\infty(\Omega) \int_\Omega u_\varepsilon (m_1 + u_\varepsilon + av_\varepsilon)
\]
\[
\leq c_1 \left( \int_\Omega |\nabla u_\varepsilon|^4 + \int_\Omega u_\varepsilon^2 + \int_\Omega |\nabla v_\varepsilon|^4 + \int_\Omega v_\varepsilon^2 + \|v_\varepsilon\|_L^\infty(\Omega)^2 + 1 \right) \left( \|\nabla \varphi\|_L^4(\Omega) + \|\varphi\|_L^\infty(\Omega) \right)
\]
for all $t \in (0, \infty)$. Due to the bounds established in Corollary 3.7 and Lemma 2.2 as well as the fact that in three dimensions $W^{1,4}(\Omega)$ is continuously embedded into $L^\infty(\Omega)$, we immediately gain the bound (3.22) from the above. By a similar testing procedure applied to the second equation in (3.6), we gain $c_2 > 0$ such that
\[
\int_\Omega v_\varepsilon t \varphi \leq d_2 \int_\Omega |\nabla v_\varepsilon| |\nabla \varphi| + \int_\Omega |\varphi| v_\varepsilon |m_2| + bu_\varepsilon + v_\varepsilon
\]
\[
\leq c_2 \left( \int_\Omega u_\varepsilon^2 + \int_\Omega |\nabla v_\varepsilon|^4 + \int_\Omega v_\varepsilon^2 + 1 \right) \left( \|\nabla \varphi\|_L^\frac{4}{3}(\Omega) + \|\varphi\|_L^\infty(\Omega) \right)
\]
for all $t \in (0, \infty)$. Again by application of Corollary 3.7 and Lemma 2.2, this implies the second bound (3.23) and therefore completes the proof.

We now construct candidates for our weak solutions as limits of our approximate solutions along a suitable null sequence $(\varepsilon_j)_{j \in \mathbb{N}}$ by applying the Aubin–Lions compact embedding theorem as well as the weak compactness properties of bounded sets in Sobolev spaces.

**Lemma 3.9** Under the assumption of Lemma 3.3 there exist a null sequence $(\varepsilon_j)_{j \in \mathbb{N}}$ and a.e. non-negative functions $u, v : \Omega \times [0, \infty) \rightarrow \mathbb{R}$ such that $v \in L^\infty(\Omega \times (0, \infty))$ as well as
\[
\begin{align*}
u_\varepsilon &\to u \quad \text{in} \quad L^2_{\text{loc}}((0, \infty); L^2(\Omega)) \quad \text{and a.e. in} \quad \Omega \times (0, \infty), & (3.24) \\
u_\varepsilon &\to u \quad \text{in} \quad L^\frac{4}{3}_{\text{loc}}((0, \infty); W^{1,\frac{4}{3}}(\Omega)), & (3.25) \\
v_\varepsilon &\to v \quad \text{in} \quad L^2_{\text{loc}}((0, \infty); L^4(\Omega)) \quad \text{and a.e. in} \quad \Omega \times (0, \infty) \quad \text{and} & (3.26) \\
v_\varepsilon &\to v \quad \text{in} \quad L^\frac{4}{3}_{\text{loc}}((0, \infty); W^{1,\frac{4}{3}}(\Omega)) & (3.27)
\end{align*}
\]
as $\varepsilon = \varepsilon_j \searrow 0$.

Proof. According to Corollary 3.7, the family $(u_\varepsilon)_{\varepsilon \in (0,1)}$ is bounded in $L^\frac{4}{3}((0,T); W^{1,\frac{4}{3}}(\Omega))$ for all $T > 0$ and, according to Lemma 3.8, the family $(u_\varepsilon t)_{\varepsilon \in (0,1)}$ is bounded in $L^1((0,T); (W^{1,4}(\Omega))^*)$ for all $T > 0$. Thus applying the Aubin–Lions lemma to the triple of spaces $W^{1,\frac{4}{3}}(\Omega) \subset L^\frac{4}{3}(\Omega) \subset (W^{1,4}(\Omega))^*$ as well as the weak compactness property of bounded sets in Sobolev spaces for all $T \in \mathbb{N}$ combined with a diagonal sequence argument yields a null sequence $(\varepsilon_j)_{j \in \mathbb{N}}$ and $u : \Omega \times [0, \infty) \rightarrow \mathbb{R}$
such that (3.25) holds and such that \( u_\varepsilon \to u \) in \( L^\frac{4}{3}_{\text{loc}}((0, \infty); L^\frac{4}{3}(\Omega)) \) as well as a.e. on \( \Omega \times (0, \infty) \) as \( \varepsilon = \varepsilon_j \downarrow 0 \). As further Lemma 3.3 provides \( c > 0 \) such that

\[
\int_0^T \int_\Omega u_\varepsilon^2 \log(u_\varepsilon) \leq c \cdot (T + 1) \quad \text{for all } T > 0,
\]
a combination of the De La Vallée Poussin criterion for uniform integrability and the Vitali convergence theorem yields the convergence property (3.24).

Similarly according to Lemma 2.2 and Corollary 3.7 the family \((v_\varepsilon)_{\varepsilon \in (0,1)}\) is bounded in the space \( L^1((0,T); W^{1,4}(\Omega)) \) for all \( T > 0 \) and, according to Lemma 3.8 the family \((v_\varepsilon t)_{\varepsilon \in (0,1)}\) is bounded in the space \( L^1((0,T); (W^{1,4}(\Omega))^*) \) for all \( T > 0 \). Given this, we can again apply the Aubin–Lions lemma to the triple of spaces \( W^{1,4}(\Omega) \subset L^4(\Omega) \subset (W^{1,4}(\Omega))^* \) and use the same weak compactness properties of bounded sets in Sobolev spaces to immediately gain a function \( v : \Omega \times [0, \infty) \to \mathbb{R} \) such that both (3.26) and (3.27) hold (after potentially switching to a subsequence).

The a.e. nonnegativity of both \( u \) and \( v \) as well as the global boundedness of \( v \) then follows from the positivity of the approximate solutions and the boundedness properties laid out in Lemma 2.2 combined with the a.e. convergence already established in (3.24) and (3.26).

As we have now constructed our solution candidates, it remains only to be shown that they in fact fulfill our desired weak solution property and thus prove the second half of Theorem 1.1.

**Proof** of Theorem 1.1 (ii). Let \( u \) and \( v \) as well as the null sequence \((\varepsilon_j)_{j \in \mathbb{N}}\) be as constructed in Lemma 3.9. Notably given the convergence, nonnegativity and boundedness properties established in said lemma, we immediately gain the necessary regularity properties for our weak solution definition. We thus only need to further show that \( u \) and \( v \) fulfill (1.2) and (1.3). To this end, we first note that each \( u_\varepsilon \) already fulfills a slightly modified version of (1.2) corresponding to the first equation in (3.6) and each \( v_\varepsilon \) already fulfills (1.3) exactly. As such, it now only remains to show that all terms in the weak solution equation solved by our approximate solutions converge to their counterparts without \( \varepsilon \) as \( \varepsilon = \varepsilon_j \downarrow 0 \) as well as handle the slightly different taxis term. The former can be easily seen by application of the convergence properties from Lemma 3.9 while we will now discuss the latter in a little more detail. To facilitate this, we first fix \( \varphi \in C_0^\infty(\Omega \times [0, \infty)) \). Due to the dominated convergence theorem and the a.e. pointwise convergence given in (3.24) we can conclude that \( \frac{\nabla \varphi}{1+\varepsilon u_\varepsilon} \) converges to \( \nabla \varphi \) in \( L^4(\Omega \times (0, \infty)) \) as \( \varepsilon = \varepsilon_j \downarrow 0 \). Combining this with (3.24) then yields that \( \frac{u_\varepsilon \nabla v \cdot \nabla \varphi}{1+\varepsilon u_\varepsilon} \) converges to \( u \nabla v \cdot \nabla \varphi \) as \( \varepsilon = \varepsilon_j \downarrow 0 \) and therefore completes the proof.

**4 Stabilization**

As we have now proven the first of our two main results, we shift our focus from the aforementioned existence result to our analysis of long-time behavior. Similarly to the previous section, we begin by deriving some a priori bounds for the solutions to (2.1). Importantly, we again take great care to ensure that all the constants involved in this process are independent of \( F \) to enable us to not only use said bounds in the two-dimensional case but also in the three-dimensional case, where the solution construction relies on approximation.
In an effort to eliminate an otherwise necessary initial data condition on \( v_0 \), we first show that after an appropriate waiting time the second component of solutions to (2.1) approaches \( m_2^+ = \max(0, m_2) \) uniformly from above.

**Lemma 4.1** Under the assumption of Lemma 2.1 and for each \( m > m_2^+ = \max(0, m_2) \), there exists \( T \equiv T(m) > 0 \) such that

\[
v(x, t) \leq m
\]

for all \( x \in \overline{\Omega} \) and \( t \in (T, \infty) \).

**Proof.** By comparison with the solution to the initial value problem \( y' = y(m_2 - y) \), \( y(0) = \|v_0\|_{L^\infty(\Omega)} \), we gain that

\[
v(\cdot, t) \leq \begin{cases} 
\left( \frac{1}{m_2} + \left( \frac{1}{\|v_0\|_{L^\infty(\Omega)}} - \frac{1}{m_2} \right) e^{-m_2 t} \right)^{-1} & \text{if } m_2 \neq 0 \\
\left( \frac{1}{\|v_0\|_{L^\infty(\Omega)}} + t \right)^{-1} & \text{if } m_2 = 0
\end{cases}
\]

for all \( t \in (0, \infty) \). Our desired result then follows directly from this as the right-hand side converges to \( m_2 \) if \( m_2 > 0 \) and to 0 if \( m_2 \leq 0 \) as \( t \to \infty \). \( \square \)

### 4.1 An eventual uniform energy-type inequality for solutions to (2.1)

As is not uncommon (see e.g. [24]), our argument for the stabilization behavior laid out in Theorem 1.2 chiefly rests on the derivation of an eventual (uniform) energy-type inequality for solutions of (2.1), which in the two-dimensional case can then be directly applied to the classical solutions we have constructed in the previous section or whose most important consequences survive the limit process used in our construction of weak solutions in the three-dimensional case. Notably, the energy functional involved is different and in a sense weaker from the one employed in Section 3.2 as our long-time stabilization argument in turn necessitates a stronger type of energy inequality than the aforementioned existence result.

To establish an important building block for the functionals involved in said energy-type inequality, we now define the family of nonnegative functions

\[
H_\xi(\eta) := \begin{cases} 
\eta - \xi - \xi \log \left( \frac{\eta}{\xi} \right) & \text{if } \xi > 0, \\
\eta & \text{if } \xi = 0
\end{cases}
\]

for all \( \xi \geq 0, \eta \geq 0 \) and recall the following functional inequalities involving the aforementioned functions as e.g. established in [24].

**Lemma 4.2** Let \( \Omega \) be a smooth bounded domain and \( \xi \geq 0 \). Then

\[
\int_\Omega |\varphi - \xi| \leq \frac{1}{1 - \log(2)} \int_\Omega H_\xi(\varphi) + \sqrt{8\xi |\Omega|} \left\{ \int_\Omega H_\xi(\varphi) \right\}^{\frac{1}{2}} \tag{4.1}
\]

for all measurable \( \varphi : \Omega \to (0, \infty) \) and there exists \( C > 0 \) such that

\[
\int_\Omega H^2_\xi(\varphi) \leq \int_\Omega \varphi^2 + C \xi^2 \left( \int_\Omega \frac{\nabla \varphi^2}{\varphi^2} + \left\{ \int_\Omega |\log \varphi| \right\}^2 + 1 \right) \tag{4.2}
\]

for all positive \( \varphi \in L^2(\Omega) \) with \( \log \varphi \in W^{1,2}(\Omega) \).
Proof. The case $\xi = 0$ is trivial while the remaining case of $\xi > 0$ follows from [23] Lemma 3.1, Lemma 3.2].

Using the now defined functions $H_\xi(\eta)$ and assuming that (1.6) holds, we can define the energy-type functional

$$E(u, v) := \int_\Omega H_{u*}(u) + \frac{a}{b} \int_\Omega H_{v*}(v) + \frac{2}{b^2 \hat{m}_2} \int_\Omega (v - v_*)^2$$

for all measurable functions $u, v : \Omega \to [0, \infty]$ as well as the functional

$$G(u, v) := \int_\Omega \frac{\nabla u}{u^2} + \int_\Omega \frac{\nabla v}{v^2} + \int_\Omega (u - u_*)^2 + \int_\Omega (v - v_*)^2$$

for all $u, v \in L^2(\Omega)$ with $\log(u), \log(v) \in W^{1,2}(\Omega)$. Here $u_*$ and $v_*$ are defined as in (1.4) and $\hat{m}_2 > m_{2+} = \max(0, m_2)$ is chosen in such a way that

$$\chi^2 < \frac{4d_1d_2}{b\hat{m}_2u_*} \left( \frac{av_*}{\hat{m}_2} + \frac{4}{b} \right). \quad (4.3)$$

Having introduced the central functionals of this section, we can now derive our desired eventual energy inequality as well as ensure that the energy $E$ remains uniformly bounded in the time interval before the energy inequality provides us with even stronger guarantees. The latter is especially important for the three-dimensional case. Note that, here as well as later in this section, we will make extensive use of the fact that $u_*$ and $v_*$ are chosen to exactly fulfill $(m_1 - u_*) + av_* = 0$ and $(m_2 - bu_* - v_*) = 0$ if $v_* > 0$ or $(m_2 - bu_* - v_*) \leq 0$ if $v_* = 0$.

Lemma 4.3 Under the assumption of Lemma 2.1 and if (1.5) as well as (1.6) hold, there exist $C > 0$, $T_E > 0$ and $\delta > 0$ such that

$$E(u(\cdot, t), v(\cdot, t)) \leq C \quad (4.4)$$

for all $t \in [0, T_E]$ and

$$\frac{d}{dt} E(u(\cdot, t), v(\cdot, t)) \leq -\delta G(u(\cdot, t), v(\cdot, t)) \quad (4.5)$$

for all $t \in (T_E, \infty)$.

Proof. We begin by fixing $\delta > 0$ such that

$$\delta \leq \frac{a}{b} \quad (4.6)$$

and

$$(1 - \delta)\hat{m}_2 > m_{2+} = \max(0, m_2). \quad (4.7)$$

as well as

$$\chi^2 < \frac{4(d_1 - \frac{\delta}{u_*})d_2}{b\hat{m}_2u_*} \left( \frac{av_*}{\hat{m}_2} + \frac{4}{b} - \delta \frac{b}{d_2\hat{m}_2} \right)$$

while ensuring that all factors on the right-hand side stay positive individually, which is possible due to (4.3). This then implies

$$\left( \frac{\chi^2u_*}{4(d_1 - \frac{\delta}{u_*})} - \frac{4d_2}{b^2 \hat{m}_2} \right) \hat{m}_2^2 - d_2v_\ast \frac{a}{b} < -\delta \quad (4.8)$$
after some slight rearrangement. As $\delta$ fulfills (4.7), we can use Lemma 4.1 to fix $T_E > 0$ such that

$$\|v(\cdot, t)\|_{L^\infty(\Omega)} \leq (1 - \delta)\hat{m}_2 \leq \hat{m}_2$$

(4.9)

for all $t \in (T_E, \infty)$.

Now integrating the first equation in (2.1) over $\Omega$ yields

$$\frac{d}{dt} \int_\Omega u = \int_\Omega u(m_1 - u + av)$$

for all $t \in (0, \infty)$ due to the Neumann boundary conditions. Testing the same equation with $-\frac{1}{u}$ further gives us

$$-\frac{d}{dt} \int_\Omega \log(u) = -\int_\Omega \frac{u_t}{u} = -\int_\Omega \nabla \cdot \left( (d_1 + \chi v)\nabla u \right) + \chi \int_\Omega \frac{1}{u} \nabla \cdot (F(u)\nabla v) - \int_\Omega (m_1 - u + av)$$

$$\leq -\int_\Omega (d_1 + \chi v) \frac{\nabla u^2}{u^2} + \chi \int_\Omega \frac{F(u)}{u^2} \nabla u \cdot \nabla v - \int_\Omega (m_1 - u + av)$$

$$\leq -\frac{\delta}{u_*} \int_\Omega \nabla u^2 + \frac{\chi^2 u_*}{4(d_1 - \frac{\delta}{u_*})} \int_\Omega |\nabla v|^2 - \int_\Omega (m_1 - u + av)$$

for all $t \in (0, \infty)$ by integration by parts, an application of Young’s inequality as well as the estimate $F(y) \leq y$ for all $y \geq 0$. Combining the above two inequalities then allows us to estimate

$$\frac{d}{dt} \int_\Omega H_{u_*}(u) = \frac{d}{dt} \int_\Omega u - u_* - u_* \log \left( \frac{u}{u_*} \right)$$

$$\leq -\delta \int_\Omega \frac{\nabla u^2}{u^2} + \frac{\chi^2 u_*}{4(d_1 - \frac{\delta}{u_*})} \int_\Omega |\nabla v|^2 + \int_\Omega (u - u_*)(m_1 - u + av)$$

$$= -\delta \int_\Omega \frac{\nabla u^2}{u^2} + \frac{\chi^2 u_*}{4(d_1 - \frac{\delta}{u_*})} \int_\Omega |\nabla v|^2 - \int_\Omega (u - u_*)^2 + a \int_\Omega (u - u_*)(v - v_*)$$

(4.10)

for all $t \in (0, \infty)$ as $(m_1 - u_* + av_*) = 0$. If $v_* = 0$, integrating the second equation in (2.1) over $\Omega$ similarly yields

$$\frac{d}{dt} \int_\Omega H_{v_*}(v) = \frac{d}{dt} \int_\Omega v = \int_\Omega v(m_2 - bu - v)$$

$$= -\int_\Omega v(v - v_*) - b \int_\Omega v(u - u_*) + \int_\Omega v(m_2 - bu_*) - v_*$$

$$\leq -\int_\Omega v(v - v_*) - b \int_\Omega v(u - u_*)$$

$$= -d_2 v_* \int_\Omega \frac{|\nabla v|^2}{v^2} - \int_\Omega (v - v_*)^2 - b \int_\Omega (v - v_*)(u - u_*)$$

(4.11)
for all \( t \in (0, \infty) \) as \((m_2 - bu_* - v_*) \leq 0\). The same inequality can be achieved for the case \( v_* > 0 \) by essentially the same procedure as the one applied to \( u \) to gain (4.10) because in this case \((m_2 - bu_* - v_*) = 0\).

Combining (4.10) with an appropriately scaled version of (4.11) then results in the estimate
\[
\frac{d}{dt} \left[ \int_{\Omega} H_u(u) + \frac{a}{b} \int_{\Omega} H_{v_*}(v) \right] \\
\leq -\delta \int_{\Omega} \frac{|\nabla u|^2}{u^2} + \frac{\chi^2 v_*}{4(d_1 - \frac{\delta}{u_*})} \int_{\Omega} |\nabla v|^2 - d_2 v_*^2 \int_{\Omega} \frac{|\nabla v|^2}{v^2} - \int_{\Omega} (u - u_*)^2 - \frac{a}{b} \int_{\Omega} (v - v_*)^2
\]
for all \( t \in (0, \infty) \). By now further testing the second equation in (2.1) with \((v - v_*)\), we gain
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} (v - v_*)^2
\]
\[
= -d_2 \int_{\Omega} |\nabla v|^2 - \int_{\Omega} (v - v_*)v(m_2 - bu - v)
\]
\[
= -d_2 \int_{\Omega} |\nabla v|^2 - \int_{\Omega} (v - v_*)^2v - b \int_{\Omega} (v - v_*)v(u - u_*) + \int_{\Omega} (v - v_*)v(m_2 - bu_* - v_*)
\]
\[
\leq -d_2 \int_{\Omega} |\nabla v|^2 - \int_{\Omega} (v - v_*)^2v - b \int_{\Omega} (v - v_*)v(u - u_*)
\]
\[
\leq -d_2 \int_{\Omega} |\nabla v|^2 + \frac{b^2}{4} \|v\|_{L^\infty(\Omega)} \int_{\Omega} (u - u_*)^2
\]
for all \( t \in (0, \infty) \) because \((m_2 - bu_* - v_*) \leq 0\) for \( v_* = 0 \) and \((m_2 - bu_* - v_*) = 0\) for \( v_* > 0\). Therefore, combining this with the inequality directly preceding it yields
\[
\frac{d}{dt} \int_{\Omega} E(u, v) \leq -\delta \int_{\Omega} \frac{|\nabla u|^2}{u^2} + \left( \frac{\chi^2 v_*}{4(d_1 - \frac{\delta}{u_*})} - \frac{4d_2}{b^2 m_2} \right) \int_{\Omega} |\nabla v|^2 - d_2 v_*^2 \int_{\Omega} \frac{|\nabla v|^2}{v^2} + \left( \frac{\|v\|_{L^\infty(\Omega)}}{m_2} - 1 \right) \int_{\Omega} (u - u_*)^2 - \frac{a}{b} \int_{\Omega} (v - v_*)^2
\]
(4.12)
for all \( t \in (0, \infty) \).

Notably, the above implies that
\[
\frac{d}{dt} \int_{\Omega} E(u, v) \leq \frac{\chi^2 v_*}{4(d_1 - \frac{\delta}{u_*})} \int_{\Omega} |\nabla v|^2 + \frac{\|v\|_{L^\infty(\Omega)}}{m_2} \int_{\Omega} (u - u_*)^2
\]
for all \( t \in (0, \infty) \). By application of Lemma 2.2 and Lemma 2.3 as well as the initial data condition (1.5), which ensures that \( E(u_0, v_0) < \infty \), this directly gives us uniform boundedness of the energy functional on \([0, T_E]\) and thus the first half of our result by time integration.

If we now consider the inequality (4.12) only on the time interval \((T_E, \infty)\), we can use (4.9) followed
by (4.6)–(4.8) to further estimate that
\[
\frac{d}{dt} \int_{\Omega} E(u, v) \leq -\delta \int_{\Omega} \frac{\nabla u^2}{u^2} + \left[ \frac{\chi^2 u_\ast}{4(d_1 \frac{\Delta}{u_\ast})} - \frac{4d_2}{b^2 \tilde{m}_2} \right] \frac{\tilde{m}_2^2}{v^2} - \frac{d_2 v_\ast}{b} \int_{\Omega} (v - v_\ast)^2 \\
+ \left( \frac{\|v\|_{L^\infty(\Omega)}}{\tilde{m}_2} - 1 \right) \int_{\Omega} (u - u_\ast)^2 - \frac{a}{b} \int_{\Omega} (v - v_\ast)^2 \\
\leq -\delta \int_{\Omega} \frac{\nabla u^2}{u^2} - \delta \int_{\Omega} \frac{\nabla v^2}{v^2} - \delta \int_{\Omega} (u - u_\ast)^2 - \delta \int_{\Omega} (v - v_\ast)^2 = -\delta G(u, v)
\]
for all \( t \in (T_E, \infty) \), which completes the proof. \( \square \)

### 4.2 Monotonicity of the energy-type functional

Having now established our energy inequality, the first key consequence we will derive from it is that after the time \( T_E \) our energy functional is in fact always monotonically decreasing. We do this to later gain an eventual smallness property for \( E \) by just constructing an increasing sequence of times along which \( E \) becomes small, which combined with monotonicity is of course sufficient.

While the aforementioned monotonicity property is a trivial consequence of Lemma 4.3 in the two-dimensional case, in the three-dimensional case we still need to argue that the same monotonicity property survives the limit process used in our weak solution construction. In fact, given that the constructed weak solutions are merely nonnegative, it is not even immediately clear that \( E(u, v) \) is finite for almost all times \( t > T_E \) if \( (u, v) \) is such a weak solution. As such, we will now first prove what is effectively a uniform lower bound for \( \int_{\Omega} \log(u) \) and \( \int_{\Omega} \log(v) \).

**Lemma 4.4** Under the assumption of Lemma 4.3, there exists \( C > 0 \) such that
\[
u_\ast \int_{\Omega} |\log(u(\cdot, t))| \leq C \quad \text{and} \quad v_\ast \int_{\Omega} |\log(v(\cdot, t))| \leq C
\]
for all \( t \in (T_E, \infty) \) with \( T_E \) as in Lemma 4.3.

**Proof.** We fix \( T_E > 0, C > 0 \) as provided by Lemma 4.3. Then using the energy inequality from the same lemma, we can conclude that
\[
-u_\ast \int_{\Omega} \log(u(\cdot, t))
\]
\[
= -u_\ast \int_{\Omega} \log \left( \frac{u(\cdot, t)}{u_\ast} \right) - u_\ast \int_{\Omega} \log(u_\ast) - E(u(\cdot, t), v(\cdot, t)) + E(u(\cdot, t), v(\cdot, t))
\]
\[
= -u_\ast \int_{\Omega} \log(u_\ast) - \int_{\Omega} (u(\cdot, t) - u_\ast) - \frac{a}{b} \int_{\Omega} H_{v_\ast}(v(\cdot, t)) - \frac{2}{b^2 \tilde{m}_2} \int_{\Omega} (v(\cdot, t) - v_\ast)^2 + E(u(\cdot, t), v(\cdot, t))
\]
\[
\leq -u_\ast \int_{\Omega} \log(u_\ast) + \int_{\Omega} u_\ast + E(u(\cdot, T_E), v(\cdot, T_E))
\]
\[
\leq u_\ast (1 + |\log(u_\ast)|) |\Omega| + E(u(\cdot, T_E), v(\cdot, T_E)) \leq u_\ast (1 + |\log(u_\ast)|) |\Omega| + C
\]

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for all \( t \in (T_E, \infty) \). As further

\[
u_* \int_\Omega |\log(u(\cdot, t))| = -u_* \int_{\{u \leq 1\}} \log(u(\cdot, t)) + u_* \int_{\{u > 1\}} \log(u(\cdot, t)) \leq -u_* \int_\Omega \log(u(\cdot, t)) + 2u_* \int_{\{u > 1\}} u(\cdot, t)
\]

for all \( t \in (T_E, \infty) \), our first bound follows directly due to Lemma 2.3. The remaining bound for \( v \) can be achieved in essentially the same way if \( v_* > 0 \) and is trivial if \( v_* = 0 \).

We further derive a uniform square integrability property for \( H_{u_*}(u) \) and \( H_{v_*}(v) \) based on the above result as well as Lemma 4.3.

**Lemma 4.5** Under the assumption of Lemma 4.3, there exists \( C > 0 \) such that

\[
\int_t^{t+1} \int_\Omega H_{u_*}^2(u) \leq C \quad \text{and} \quad \int_t^{t+1} \int_\Omega H_{v_*}^2(v) \leq C
\]

for all \( t \in (T_E, \infty) \) with \( T_E \) as in Lemma 4.3.

**Proof.** Using Lemma 4.3 as well as Lemma 4.4, we can fix \( T_E > 0, c_1 > 0 \) and \( \delta > 0 \) such that

\[
\int_{T_E}^{\infty} \int_\Omega \frac{\nabla u^2}{u^2} + \int_{T_E}^{\infty} \int_\Omega \frac{\nabla v^2}{v^2} + \int_{T_E}^{\infty} \int_\Omega (u - u_*)^2 + \int_{T_E}^{\infty} \int_\Omega (v - v_*)^2
\]

\[
= \int_{T_E}^{\infty} G(u, v) \leq \frac{1}{\delta} E(u(\cdot, T_E), v(\cdot, T_E)) \leq \frac{C}{\delta}
\]

as well as

\[
u_* \int_\Omega \log(u(\cdot, t)) \leq c_1 \quad \text{and} \quad v_* \int_\Omega \log(v(\cdot, t)) \leq c_1
\]

for all \( t \in (T_E, \infty) \). An application of (4.2) from Lemma 4.2 combined with the above bounds then yields \( c_2 > 0 \) such that

\[
\int_t^{t+1} \int_\Omega H_{u_*}^2(u)
\]

\[
\leq 2 \int_t^{t+1} \int_\Omega (u - u_*)^2 + 2|\Omega| u_*^2 + c_2 \left( u_*^2 \int_t^{t+1} \int_\Omega \frac{\nabla u^2}{u^2} + u_*^2 \int_t^{t+1} \int_\Omega \left\{ \int_\Omega |\log(u)| \right\}^2 + u_*^2 \right)
\]

\[
\leq 2 \frac{c_1}{\delta} + 2|\Omega| u_*^2 + c_2 \left( u_*^2 + c_1 + u_*^2 \right)
\]

for all \( t \in (T_E, \infty) \). Naturally, the same estimation applies to \( \int_t^{t+1} \int_\Omega H_{v_*}^2(v) \), completing the proof. \( \square \)

Given this, we can now derive our desired (almost everywhere) monotonicity property for \( E(u, v) \) in both two and three dimensions.
Lemma 4.6 Let \((u, v)\) be the solution constructed in Theorem 1.1. If (1.3) as well as (1.6) hold, then there exists a null set \(N \subset (T_E, \infty)\) such that

\[
E(u(\cdot, t_1), v(\cdot, t_1)) \leq E(u(\cdot, t_0), v(\cdot, t_0))
\]

for all \(t_0, t_1 \in (T_E, \infty) \setminus N\) with \(t_1 > t_0\) and \(T_E\) as in Lemma 4.3. If \(n = 2\), then \(N = \emptyset\).

PROOF. If \(n = 2\), this is a trivial consequence of Lemma 4.3. As such, we will focus our efforts here on the case \(n = 3\) where Lemma 4.3 only implies that there exists a uniform \(T_E > 0\) such that

\[
E(u_\varepsilon(\cdot, t_1), v_\varepsilon(\cdot, t_1)) \leq E(u_\varepsilon(\cdot, t_0), v_\varepsilon(\cdot, t_0))
\]  

(4.13)

for all \(t_1 > t_0 > T_E\) and \(\varepsilon \in (0, 1)\) with \((u_\varepsilon, v_\varepsilon)\) being the approximate solutions used in Section 3.2. Therefore, we still need to argue that this property in fact translates to their limit functions for it to apply to the weak solutions constructed in the same section. We first note that due to the convergence properties already laid out in Lemma 3.9, we can ensure that there exists a null sequence \((\varepsilon_j)_{j \in \mathbb{N}}\) such that

\[
\int_\Omega (v_\varepsilon(\cdot, t) - v_\ast)^2 \to \int_\Omega (v(\cdot, t) - v_\ast)^2 \quad \text{for a.e. } t \in (T_E, \infty)
\]

(4.14)

and thus

\[
\int_\Omega (v_\varepsilon(\cdot, t) - v_\ast)^2 \to \int_\Omega (v(\cdot, t) - v_\ast)^2 \quad \text{for a.e. } t \in (T_E, \infty)
\]

as well as \(u_\varepsilon \to u\) and \(v_\varepsilon \to v\) almost everywhere in \(\Omega \times (T_E, \infty)\).

To now treat the remaining parts of the energy functional \(E\), we first note that Lemma 4.4 gives us \(c_1 > 0\) such that

\[
\int_\Omega |\log(u_\varepsilon(\cdot, t))| \leq c_1
\]

for all \(t \in (T_E, \infty)\) and \(\varepsilon \in (0, 1)\) as \(u_\ast > 0\). Due to Fatou’s lemma this implies

\[
\int_\Omega |\log(u(\cdot, t))| \leq c_1
\]

for all \(t \in (T_E, \infty)\) and thus \(u\) must be positive for almost all \((x, t) \in \Omega \times (T_E, \infty)\). As such and due to the fact that \(H_{u_\ast}\) is continuous outside of 0, we know that \(H_{u_\ast}(u_\varepsilon) \to H_{u_\ast}(u)\) almost everywhere in \(\Omega \times (T_E, \infty)\) as \(\varepsilon = \varepsilon_j \searrow 0\) due to the already established almost everywhere convergence for \(u_\varepsilon\) itself. Given that Lemma 4.5 grants us a constant \(c_2 > 0\) with

\[
\int_t^{t+1} \int_\Omega H_{u_\ast}^2(u_\varepsilon) \leq c_2
\]

for all \(t \in (T_E, \infty)\) and \(\varepsilon \in (0, 1)\), the Vitali convergence theorem gives us that further

\[
\int_t^{t+1} \int_\Omega H_{u_\ast}(u_\varepsilon) \to \int_t^{t+1} \int_\Omega H_{u_\ast}(u)
\]
as \( \varepsilon = \varepsilon_j \searrow 0 \) for all \( t \in (T_E, \infty) \). But this directly implies

\[
\int_{\Omega} H_{u_\varepsilon}(u_\varepsilon(\cdot, t)) \rightarrow \int_{\Omega} H_u(u(\cdot, t))
\]

for almost all \( t \in (T_E, \infty) \). If \( v_\ast > 0 \), the argument for \( \int_{\Omega} H_v(v) \) is essentially the same while for \( v_\ast = 0 \) it is a straightforward consequence of (4.14). As such, the property (4.13) translates to the limit functions outside of a set \( N \subset (T_E, \infty) \) with measure zero by simply taking the limit \( \varepsilon = \varepsilon_j \searrow 0 \).

\[\Box\]

### 4.3 Eventual smallness of the energy functional and proof of Theorem 1.2

As already alluded to in the previous section, we will now construct a sequence of times along which \( E(u, v) \) becomes small. To do this, we now first translate some important space-time integral bounds stemming from Lemma 4.3 and Lemma 4.5 to the solutions constructed in Section 3 using Fatou’s lemma.

**Lemma 4.7** Let \((u, v)\) be the solution constructed in Theorem 1.1. If (1.5) as well as (1.6) hold, then there exists \( C > 0 \) such that

\[
\int_{T_E}^{\infty} \int_{\Omega} (u - u_\ast)^2 \leq C \quad \text{and} \quad \int_{T_E}^{\infty} \int_{\Omega} (v - v_\ast)^2 \leq C
\]

as well as

\[
\int_{t}^{t+1} \int_{\Omega} H_{u_\ast}^2(u) \leq C \quad \text{and} \quad \int_{t}^{t+1} \int_{\Omega} H_{v_\ast}^2(v) \leq C
\]

for all \( t \in (T_E, \infty) \) with \( T_E \) as in Lemma 4.3.

**Proof.** If \( n = 2 \), this is a trivial consequence of Lemma 4.3 and Lemma 4.5. If \( n = 3 \), Fatou’s lemma combined with the same lemmata yields the desired result. \[\Box\]

Due to the above combined with the Vitali convergence theorem, it now follows that the energy \( E \) becomes small along a sequence of increasing times, which combined with the monotonicity property of the previous section and Lemma 4.2 leads to the following stabilization properties.

**Lemma 4.8** Let \((u, v)\) be the solution constructed in Theorem 1.1. If (1.5) as well as (1.6) hold, then there exists a null set \( N \subset (0, \infty) \) such that

\[
u(\cdot, t) \rightarrow v_\ast \quad \text{in} \quad L^1(\Omega)
\]

and

\[
u(\cdot, t) \rightarrow v_\ast \quad \text{in} \quad L^p(\Omega) \quad \text{for all} \quad p \in [1, \infty)
\]

as \((0, \infty) \setminus N \ni t \rightarrow \infty\). If \( n = 2 \), then \( N = \emptyset \).
**Proof.** As a direct consequence of Lemma 4.7, we can fix a sequence of times \((t_k)_{k \in \mathbb{N}} \subset (T_E, \infty) \setminus N\) with \(t_k \nearrow \infty\) as \(k \to \infty\), \(T_E\) as in Lemma 4.3 and \(N\) as in Lemma 4.6 as well as a constant \(c > 0\) such that
\[
\int_\Omega (u(\cdot, t_k) - u_*)^2 \to 0 \quad \text{and} \quad \int_\Omega (v(\cdot, t_k) - v_*)^2 \to 0 \quad (4.15)
\]
as \(k \to \infty\) and
\[
\int_\Omega H_{u_*}^2(u(\cdot, t_k)) \leq c \quad \text{and} \quad \int_\Omega H_{v_*}^2(v(\cdot, t_k)) \leq c \quad (4.16)
\]
for all \(k \in \mathbb{N}\). By potentially switching to a subsequence, we can further ensure that \(u(\cdot, t_k) \to u_*\) and \(v(\cdot, t_k) \to v_*\) almost everywhere as \(k \to \infty\) as a direct consequence of the \(L^2(\Omega)\) convergence established in (4.15). As notably \(H_\xi\) is continuous in \(\xi\) for all \(\xi \geq 0\), this further implies that
\[H_{u_*}(u(\cdot, t_k)) \to H_{u_*}(u_*) = 0 \quad \text{and} \quad H_{v_*}(v(\cdot, t_k)) \to H_{v_*}(v_*) = 0\]
almost everywhere as \(k \to \infty\). Because we have already established the bound seen in (4.16), we can now apply the Vitali convergence theorem to conclude that
\[
\int_\Omega H_{u_*}(u(\cdot, t_k)) \to 0 \quad \text{and} \quad \int_\Omega H_{v_*}(v(\cdot, t_k)) \to 0
\]
as \(k \to \infty\) as well. Combined with (4.15), this implies that
\[E(u(\cdot, t_k), v(\cdot, t_k)) \to 0\]
as \(k \to \infty\), which due to the monotonicity property laid out in Lemma 4.6 gives us that
\[E(u(\cdot, t), v(\cdot, t)) \to 0\]
as \((0, \infty) \setminus N \ni t \to \infty\). Then an application of inequality (4.11) from Lemma 1.2 yields that \(u(\cdot, t) \to u_*\) and \(v(\cdot, t) \to v_*\) in \(L^1(\Omega)\) as \((0, \infty) \setminus N \ni t \to \infty\), which combined with the fact that \(v\) is globally bounded in \(L^\infty(\Omega)\) due to Theorem 1.1 completes the proof. \(\square\)

To complete this section, we will now use the regularity properties already established in Theorem 1.1 combined with a brief testing based argument to improve upon the strength of the above stabilization properties in the two-dimensional case and thus prove our second central theorem.

**Proof of Theorem 1.2** For \(n = 3\), Theorem 1.2 is an immediate consequence of Lemma 4.8. As such, we will now focus on the case \(n = 2\). Here, we first note that as consequence of the properties derived in Lemma 1.8 we gain
\[u(\cdot, t) \to u_* \quad \text{in} \quad L^p(\Omega) \quad \text{for all} \quad p \in [1, \infty)\]
as \(t \to \infty\) due to the \(L^\infty(\Omega)\) bound for \(u\) established in Theorem 1.1 if \(n = 2\). Similar to Lemma 3.1 we now test the second equation in (1.1) with \(-\Delta v\) and employ integration by parts as well as
Young’s inequality to gain

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla v|^2 + d_2 \int_{\Omega} |\Delta v|^2 = - \int_{\Omega} v(m_2 - bu - v) \Delta v
\]

\[
= b \int_{\Omega} v(u - u_*) \Delta v + \int_{\Omega} v(v - v_*) \Delta v - \int_{\Omega} v(m_2 - bu_* - v_*) \Delta v
\]

\[
= b \int_{\Omega} v(u - u_*) \Delta v + \int_{\Omega} v(v - v_*) \Delta v + \int_{\Omega} (m_2 - bu_* - v_*) |\nabla v|^2
\]

\[
\leq \frac{d_2}{2} \int_{\Omega} |\Delta v|^2 + \frac{b^2}{d_2} \|v\|_{L^\infty(\Omega)}^2 \int_{\Omega} (u - u_* )^2 + \frac{1}{d_2} \|v\|_{L^\infty(\Omega)}^2 \int_{\Omega} (v - v_*)^2
\]

for all \( t \in (0, \infty) \) as \((m_2 - bu_* - v_*) \leq 0 \). Given that we have already established that \( u(\cdot, t) \) and \( v(\cdot, t) \) converge to \( u_* \) and \( v_* \) in \( L^2(\Omega) \) as \( t \to \infty \) and \( v \) is globally bounded in \( L^\infty(\Omega) \) as well as by application of a well-known consequence of the Poincaré inequality when considering functions with Neumann boundary conditions, it immediately follows that there exists a constant \( c > 0 \) and, for each \( \delta > 0 \), a time \( t_0 = t_0(\delta) > T_E \) with \( T_E \) as in Lemma 4.3 such that

\[
\frac{d}{dt} \int_{\Omega} |\nabla v|^2 + c \int_{\Omega} |\nabla v|^2 \leq \frac{\delta}{2} c
\]

for all \( t \in (t_0, \infty) \). By a standard comparison argument this implies that

\[
\int_{\Omega} |\nabla v(\cdot, t)|^2 \leq \left( \int_{\Omega} |\nabla v(\cdot, t_0)|^2 - \frac{\delta}{2} \right) e^{-c(t-t_0)} + \frac{\delta}{2}
\]

for all \( t \in (t_0, \infty) \) and therefore after an appropriate waiting time \( t_1 = t_1(\delta) > t_0 \) it follows that

\[
\int_{\Omega} |\nabla v(\cdot, t)|^2 \leq \delta
\]

for all \( t \in (t_1, \infty) \). As such, we have gained that

\[
v(\cdot, t) \to v_* \quad \text{in} \quad W^{1,2}(\Omega)
\]

as \( t \to \infty \), which combined with the \( W^{1,\infty}(\Omega) \) bound for \( v \) from Theorem 11 completes the proof. \( \square \)

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