Time dependent nonclassical properties of even and odd nonlinear coherent states

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Abstract

We construct even and odd nonlinear coherent states of a parametric oscillator and examine their nonclassical properties. It has been shown that these superpositions exhibit squeezing and photon antibunching which change with time.

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1 Introduction

Coherent states of various Lie algebras as well as various superpositions of coherent states have attracted considerable attention over the years. It has been shown that particular superpositions of harmonic oscillator coherent states known as the even and odd coherent states [1], exhibit nonclassical properties like squeezing and antibunching [2-4]. Recently another type of coherent states, called the f-coherent states [5] or the nonlinear coherent states [6] has been introduced. It has been shown [6] that nonlinear coherent states with a particular class of nonlinearities is useful in the description of a trapped ion and that such states have strong nonclassical properties. Subsequently various superpositions of nonlinear coherent states have been studied [7,8] and shown to possess nonclassical properties like squeezing, antibunching etc. Recently a scheme of generating even and odd nonlinear coherent states has also been suggested [9].

On the other hand time dependent quantum systems are of interest in different areas of physics including quantum optics. In particular time dependent harmonic oscillator has been widely studied and exact solutions of a class of parametric oscillator with time dependent frequency is known [10]. Using these solutions one can construct coherent states [11] as well as various superpositions of coherent states e.g., even and odd coherent states [12]. It has been shown that these cat states possess interesting nonclassical properties which depend on time [12]. In the present work our objective is to examine nonclassical properties of even and odd nonlinear coherent states of a parametric oscillator. More specifically we shall study squeezing and antibunching properties of these states and examine how these properties change with time.

2 Even and odd nonlinear coherent states of a parametric oscillator

Here we shall consider a time dependent oscillator with the Hamiltonian (we take $\hbar = 2m = \omega(0) = 1$)

$$H = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\omega^2(t)x^2$$

(1)

The time dependent integral of motion are then given by [10]
\[ A = \frac{i}{\sqrt{2}}[\varepsilon(t)p - \dot{\varepsilon}(t)x] \]  

(2)

where \( \varepsilon(t) \) satisfies the following conditions:

\[ \ddot{\varepsilon}(t) + \omega^2(t)\varepsilon(t) = 0, \quad \varepsilon(0) = 1, \quad \dot{\varepsilon}(0) = i \]  

(3)

It can be readily verified that the operators \( A \) and \( A^\dagger \) satisfy the commutation relation

\[ [A, A^\dagger] = 1 \]  

(4)

It can be shown [10] that the normalised eigenstates of the Schroedinger equation corresponding to (1) is given by

\[ \psi_n(x, t) = (\frac{\varepsilon^*(t)}{2\epsilon(t)})^{n} \frac{1}{\sqrt{(n!)}} \psi_0(x, t) H_n(\frac{x}{|\varepsilon(t)|}) \]  

(5)

where \( \psi_0(x, t) \) is given by

\[ \psi_0(x, t) = \pi^{-\frac{1}{2}}\epsilon(t)^{-\frac{1}{2}}exp\left(\frac{-i\dot{\varepsilon}(t)x^2}{2\epsilon(t)}\right) \]  

(6)

The coherent state corresponding to the parametric oscillator (1) can now be constructed as in the time independent case and it is given by

\[ \psi_{\beta}(x, t) = \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \psi_n(x, t) = \psi_0(x, t)exp\left(\frac{-1}{2} |\beta|^2 - \frac{\beta^2\varepsilon^*(t)}{2\epsilon(t)} + \frac{\sqrt{2}\beta x}{\epsilon(t)}\right) \]  

(7)

where \( \beta \) is an arbitrary complex number. It can now be easily verified that the coherent states (7) are also eigenstates of the operator \( A \):

\[ A\psi_{\beta}(x, t) = \beta\psi_{\beta}(x, t) \]  

(8)

We now proceed to the construction of time dependent even and odd nonlinear coherent states. For the sake of convenience let us denote the time dependent coherent state by the ket \( |\beta, t> \) so that \( \langle x|\beta, t > \) i.e, the wave function representation of the coherent state is given by equation (7). First we note that the time dependent nonlinear coherent states are eigenstates of a generalised time dependent integral of motion B defined by
\[ B = f(A^\dagger A) \]

where \( f(A^\dagger A) \) is a real function and is called the nonlinearity function. Thus the generalised integrals of motion \( B \) and \( B^\dagger \) satisfy the following algebra:

\[ [B, B^\dagger] = f^2(A^\dagger A)(A^\dagger A + 1) - f^2(A^\dagger A - 1)A^\dagger A \]  \hspace{1cm} (10)

Thus in contrast to \( A \) and \( A^\dagger \) the operators \( B \) and \( B^\dagger \) satisfy a deformed Heisenberg algebra whose nature of deformation depends on on the choice of the nonlinearity function \( f \). Clearly if \( f = 1 \) the relation (10) becomes the same as (4). Now defining the time dependent nonlinear coherent state \( |\alpha, t >_{NL} \) as right eigenstate of the operator \( B \) we obtain using (9)

\[ |\alpha, t >_{NL} = C \sum_{n=0}^{\infty} d_n \alpha^n |n, t > \]  \hspace{1cm} (11)

where \( \alpha \) is a complex number and \( |n, t > \) denotes the ket corresponding to the wave function (5). The normalisation constant \( C \) and the coefficients \( d_n \) are given by

\[ C^2 = [\sum_{n=0}^{\infty} d_n^2 (\bar{\alpha} \alpha)^n]^{-1} \]
\[ d_0 = 1 \]
\[ d_n = [\sqrt{n!} f(n)!]^{-1} \]  \hspace{1cm} (12)

where \( f(n)! = f(1)f(2)....f(n) \).

Even nonlinear coherent states of a parametric oscillator are then defined by

\[ |\alpha, t >_{ENL} = N_+ (|\alpha, t >_{NL} + | - \alpha, t >_{NL}) = C_+ \sum_{n=0}^{\infty} d_{2n} |2n, t > \]  \hspace{1cm} (13)

where \( C_+ \) is a normalisation constant and is given by

\[ C_+ = [\sum_{n=0}^{\infty} d_{2n}^2 |\alpha|^{4n}]^{-1/2} \]  \hspace{1cm} (14)

Proceeding similarly the odd nonlinear coherent states are found to be

\[ |\alpha, t >_{ONL} = N_-(|\alpha, t >_{NL} - | - \alpha, t >_{NL}) = C_- \sum_{n=0}^{\infty} d_{2n+1} \alpha^{2n+1} |2n + 1, t > \]  \hspace{1cm} (15)
where $C_-$ is given by

$$C_- = \left[ \sum_{n=0}^{\infty} d_{2n+1}^2 |\alpha|^{4n+2} \right]^{-1/2} \tag{16}$$

### 3 Squeezing of the even nonlinear coherent states

Before discussing squeezing we note that the standard time independent harmonic oscillator creation and annihilation operators $a$ and $a^\dagger$ are connected to the time dependent one $A$ and $A^\dagger$ by the following relation:

$$\begin{pmatrix} a \\ a^\dagger \end{pmatrix} = \begin{pmatrix} u^* & -v \\ -v^* & u \end{pmatrix} \begin{pmatrix} A \\ A^\dagger \end{pmatrix} \tag{17}$$

where $u$ and $v$ are defined in terms of $\epsilon$ and $\dot{\epsilon}$:

$$u = \frac{1}{2}(\epsilon - i\dot{\epsilon}), \quad v = -\frac{1}{2}(\epsilon + i\dot{\epsilon}) \tag{18}$$

Thus the relations (17) enables us to determine the action of the operators $a$ and $a^\dagger$ on the states given by (13) and (15).

Now to examine squeezing behaviour of the even and odd nonlinear coherent states we introduce the following quadratures:

$$X_1 = \frac{a + a^\dagger}{\sqrt{2}}, \quad X_2 = \frac{a - a^\dagger}{\sqrt{2i}} \tag{19}$$

Then it follows that the operators $X_1$ and $X_2$ satisfy the following uncertainty relation:

$$<\Delta X_1^2> <\Delta X_2^2> \geq \frac{1}{4} <[X_1, X_2]^2> \tag{20}$$

Where $<\Delta X_i^2>$ is defined as

$$<\Delta X_i^2> = <X_i^2> - <X_i>^2 \tag{21}$$

Thus it is clear that a state is squeezed if either of the following inequalties hold:

$$<\Delta X_1^2> < \frac{1}{2} |<[X_1, X_2]>| \quad \text{or} \quad <\Delta X_2^2> < \frac{1}{2} |<[X_1, X_2]>| \tag{22}$$
Now to examine whether or not the squeezing conditions (22) hold we need to evaluate several expectation values like $< A^\dagger A >$, $< A^2 >$, $< A^2 >$ etc. This in turn requires a specific choice of the parameter $\epsilon(t)$. For a given frequency $\omega(t)$ this can be obtained from the solution of equation (3). Here we take the oscillator frequency $\omega(t)$ and the corresponding parameter $\epsilon(t)$ as [12,13]

$$\omega(t) = \frac{1 + \kappa \cos(2t)}{1 + \kappa}$$

$$\epsilon(t) = \cosh(\frac{1}{4}\kappa t)e^{it} + isinh(\frac{1}{4}\kappa t)e^{-it}, \quad \kappa << 1$$ (23)

It may be pointed out that with this choice of $\epsilon(t)$, the functions $u$ and $v$ defined in (18) become determined too. Finally we need to specify the nonlinearity function $f$. Obviously for each choice of the nonlinearity function we get a different nonlinear coherent state. In the present case we take as nonlinearity function the one considered in ref [6] to describe the motion of a trapped ion:

$$f(n) = L_{1}^{1}(\eta^{2})[(n + 1)L_{0}^{0}(\eta^{2})]^{-1}$$ (24)

where $L_{n}^{m}(x)$ are generalised Laguerre polynomials and $\eta$ is known as the Lamb-Dicke parameter. Clearly $f(n) = 1$ when $\eta = 0$ and in this case the even and odd nonlinear coherent states become standard even and odd coherent states. However as $\eta$ increases the nonlinearity develops and it is reflected by the structure of the phase probability distribution [14] of the even and odd nonlinear coherent states. It has been shown that for a reasonably large value of the Lamb-Dicke parameter $\eta/\alpha$ the phase probability distributions have multiple peaks due self splitting (which is a consequence of nonlinearity). For very large values of the Lamb-Dicke parameter $\eta/\alpha$ the number of peaks also increases and the structure becomes fairly complicated. So in what follows we shall confine ourselves to reasonable values of $\eta$ and $\alpha$ such that all essential features associated with nonlinearity are present without being too complicated.

It may be noted that with this choice of the nonlinearity function the algebra (10) becomes a nonpolynomial deformation of the Heisenberg algebra. We now proceed to examine the inequalities in (22). After some calculations using (17) and (19) as well as (23) and (24) it can be shown that the uncertainty relation (20) and the squeezing conditions (22) can be expressed respectively as
\[ F(\alpha, \eta, \kappa, t) \cdot G(\alpha, \eta, \kappa, t) \geq \frac{1}{4} \] (25)

and

\[ F(\alpha, \eta, \kappa, t) < \frac{1}{2} \text{ or } G(\alpha, \eta, \kappa, t) < \frac{1}{2} \] (26)

where \( F(\alpha, \eta, \kappa, t) \) and \( G(\alpha, \eta, \kappa, t) \) are given by

\[ F(\alpha, \eta, \kappa, t) = \frac{2}{(\kappa \cos(2t) + 4)} (\epsilon^2 < A \dagger^2 > + \epsilon^* \epsilon < A^\dagger A > + \epsilon \epsilon^* ) \]
\[ G(\alpha, \eta, \kappa, t) = \frac{2}{(\kappa \cos(2t) + 4)} (\dot{\epsilon}^2 < A \dagger^2 > + \dot{\epsilon}^* \dot{\epsilon} < A^\dagger A > -2 \dot{\epsilon} \dot{\epsilon}^* < A^\dagger A > - \dot{\epsilon} \dot{\epsilon} ) \] (27)

and in obtaining the above expressions we have neglected terms of higher order in \( \kappa \).

We now proceed to describe the figures. In figs 1 and 2 we describe the plots of \( F(\alpha, \eta, \kappa, t) \) and \( G(\alpha, \eta, \kappa, t) \) against time \( t \) for the even nonlinear coherent state. From figure 1 we find that the even nonlinear coherent state exhibits squeezing in both quadratures at different times. From figure 1 we also find that at some instant of time \( F(\alpha, \eta, \kappa, t) = G(\alpha, \eta, \kappa, t) = .5 \) so that the uncertainty relation (25) is saturated. This implies that even nonlinear coherent states can also be instantaneous intelligent states. In fig 2 we plot the same quantities as in fig 1 but for different parameter values. From fig 2 we find that squeezing behaviour remains the same as before. However comparing fig 1 and fig 2 it is seen that squeezing increases as \( \alpha \) increases.

Now a word about squeezing behaviour of the odd nonlinear coherent states. We have evaluated \( F(\alpha, \eta, \kappa, t) \) and \( G(\alpha, \eta, \kappa, t) \) over a wide range of values of \( \alpha \) and \( \eta \) but it was found that they never satisfy the inequalities in (24). Thus we conclude that the odd nonlinear coherent state does not exhibit quadrature squeezing.

### 4 Antibunching of the odd nonlinear coherent states

In order to study antibunching property we have to evaluate the second order correlation function \( g^2(0) \) and the condition for antibunching is given by

\[ g^2(0) = \frac{< a^\dagger^2 a^2 >}{< a^\dagger a >^2} < 1 \] (28)

Now using equation (17) we obtain
\[
\langle a^{\dagger} a^2 \rangle = |u|^2|v|^2 + 4|u|^2|v|^2 < A^{\dagger} A > + 4|u|^2|v|^2 < (A^{\dagger} A)^2 > + |u|^4 < A^{\dagger} A^2 > + |v|^4 < A^2 A^{\dagger} 2 > - \left[ (u^2 u^* v < A^{\dagger} v^2 > + u^* v v^* < A^2 > + 2 u v u^* v^2 < A^1 A^3 > + 2 u^* v v^* < A^2 A^{\dagger} > - u^2 v^2 < A^{\dagger} > + c.c \right]
\]

\[
\langle a^{\dagger} a > = (|u|^2 + |v|^2) < A^{\dagger} A > - |v|^2 - u^* v^* < A^2 > - u v < A^{\dagger} A >
\]

(29)

We have evaluated \( g^2(0) \) for the odd nonlinear coherent state and the result is plotted in figure 3 against time \( t \). From figure 3 we find that while \( g^2(0) \) has an increasing trend it is less than 1 for a considerable period of time. This implies that the odd nonlinear coherent states exhibit antibunching. The plot of \( g^2(0) \) shows that it has an increasing trend and eventually for large values of time it may become more than 1.

Finally we note that for the even nonlinear coherent state we have evaluated \( g^2(0) \) for a large number of values of the parameters but it was found to very much larger than unity. We thus conclude that the even nonlinear coherent state does not exhibit antibunching at any time.

**Conclusion**

In this article we have studied squeezing and antibunching properties of the even and the odd nonlinear coherent states of a parametric oscillator. It has been shown that while the squeezing of the even nonlinear coherent states increases with time, antibunching of the odd nonlinear coherent states decreases with time.
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Fig 1. Variation of squeezing of even nonlinear coherent state for \( \eta = .75, \alpha = .2 \) and \( \kappa = .2 \)
Fig 2. Variation of squeezing of even nonlinear coherent state for $\eta = .75, \alpha = .35$ and $\kappa = .2$
Fig 3. Time variation of $g^{(2)}(0)$ of odd nonlinear coherent state for $\eta = .8, \alpha = \kappa = .2$