Condensation transition in zero-range processes with diffusion

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Abstract. Recent studies have indicated that the coarse grained dynamics of a large class of traffic models and driven-diffusive systems may be described by urn models. We consider a class of one-dimensional urn models whereby particles hop from an urn to its nearest neighbor by a rate which decays with the occupation number $k$ of the departure site as $(1 + b/k)$. In addition a diffusion process takes place, whereby all particles in an urn may hop to an adjacent one with some rate $\alpha$. Condensation transition which may take place in this model is studied and the $(b, \alpha)$ phase diagram is calculated within the mean field approximation and by numerical simulations. A driven-diffusive model whose coarse grained dynamics corresponds to this urn model is considered.

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1. Introduction

Ordering and condensation transitions in one-dimensional systems far from thermal equilibrium have been studied extensively in recent years [1, 2, 3]. It has been repeatedly demonstrated that unlike systems in thermal equilibrium, driven one-dimensional systems whose dynamics does not obey detailed-balance can be ordered even when the dynamics is local and noisy.

By making a correspondence between phase separation in one-dimensional systems and condensation in urn models, two mechanisms which allow for phase separation in driven systems have been suggested. Urn models are simple lattice models defined on a ring geometry, where each site can either be vacant or occupied by one or more particles. The first mechanism is described in terms of the Zero Range Process (ZRP) [2, 4]. In this model particles hop between nearest neighbor lattice sites with rates $\omega_k$ which depend only on the number of particles $k$ at the departure site. If the rates decay to zero in the large $k$ limit, or if the rates decay slowly enough to a finite value, a condensation transition takes place, whereby a single lattice site becomes macroscopically occupied as the density is increased beyond a critical value. It has been suggested that the coarse-grained dynamics in a broad class of one-dimensional driven models can be described
by a ZRP with rates which at large $k$ decay as $\omega_k = \omega_\infty (1 + b/k)$ \cite{5,6}. In this case phase separation takes place only if $b > 2$.

A second mechanism is described in terms of the of the Chipping Model (CM)\cite{7,8}. The dynamics of this model involves two processes: chipping, where a single particle hops to a nearest neighbor site at a constant rate $\omega$; and diffusion, where all particles in a site hop together to an adjacent site with rate $\alpha$. This model can be viewed as a ZRP with constant hopping rate ($b = 0$), extended to include diffusion processes as well. Mean-field analysis of this model indicates that this model exhibits a condensation transition at a critical density. This result remains valid beyond the mean-field approximation as long as the chipping process is symmetric \cite{7}. It has also been shown that if the chipping process is biased either to the left or to the right, no condensation transition takes place \cite{8}.

Recently it has been suggested that the coarse-grained dynamics of certain cellular-automata traffic models can be modeled by the asymmetric CM \cite{9}. Within this picture, the coarse-grained evolution of traffic models is described in terms of domain dynamics, which essentially involves asymmetric chipping and diffusion processes. It has thus been concluded that no phase separation transition should be expected in this class of models, and that jamming phenomena take place as a broad crossover process rather than via a sharp phase transition. On the other hand, there exist models which may be related to traffic, in which the rate of chipping a particle from a domain is not constant, but rather depends on the domain size. Such an example is the Bus Route model \cite{10}, where an approximate description assigns hopping rates to buses that decay as a function of the distance to the next bus ahead. It is thus of interest to combine the features of the chipping model and the ZRP processes and consider urn modes which exhibit chipping and diffusion processes with occupation dependent chipping rates.

In this paper we consider in details a class of urn models which incorporate both diffusion and chipping processes with occupation ($k$) dependent chipping rates of the form $\omega_k = 1 + b/k$. In Section 2 we introduce the generalized ZRP and analyze its behavior both within mean-field approximation and by numerical simulations. In Section 3 we consider a simple driven diffusive model, and show that its coarse grained dynamics is described by this generalized ZRP. Conclusions and summary are presented in Section 4.

2. Zero-Range Processes with diffusion

The generalized zero-range process is defined on a lattice of $M$ sites, with periodic boundary conditions, occupied by $N = \phi M$ particles. The dynamics is defined through the rates by which two nearest neighbor sites containing $k$ and $m$ particles, respectively, exchange particles:

\begin{align*}
\text{diffusion:} \quad (k, m) &\xrightarrow{\alpha} (k + m, 0) \\
\text{chipping:} \quad (k, m) &\xrightarrow{q \omega_m} (k + 1, m - 1) \quad , \quad (k, m) \xrightarrow{(1-q)\omega_k} (k - 1, m + 1).
\end{align*}
where the rate $\omega_k$ depends on the number of particles in the departure site,
\[ \omega_k = 1 + \frac{b}{k}. \] (1)
Thus the model is specified by three dynamical parameters, $\alpha, b$ and $q$, and by the average occupancy $\phi$. The limit $\alpha = 0$ corresponds to a ZRP with no diffusion; the limit $b = 0$ recovers the chipping model. The parameter $q$ controls the spatial bias, with $q = 1/2$ being the symmetric case.

2.1. Mean-Field analysis

We first analyze the model within mean-field approximation, where correlations between sites are neglected. Let $p_k$ be the probability of a given site to be occupied by $k$ particles. The mean-field evolution equations for $p_k$ are given by
\[
\frac{\partial p_0}{\partial t} = \alpha (1 - p_0)^2 + \omega_1 p_1 (1 - p_0) - (\lambda - \omega_1 p_1) p_0 \] (2)
\[
\frac{\partial p_n}{\partial t} = -2 \alpha p_n (1 - p_0) + \alpha \sum_{k=1}^{n-1} p_k p_{n-k} + [\lambda (p_{n-1} - p_n) + \omega_{n+1} p_{n+1} - \omega_n p_n],
\]
with $n \geq 1$ and $\lambda = \sum_{k=1}^{\infty} \omega_k p_k$.

To solve these equations in the steady state, where all time derivatives vanish, let us define a generating function $g(s) = \sum_{n=1}^{\infty} p_n s^n/n$. Multiplying the $n$th equation in (2) by $s^n$ and summing over $n \geq 1$ one obtains
\[
\alpha s^2 g'(s)^2 - 2 \alpha s (1 - p_0) g'(s) + (1 - s)(1 - \lambda s) g'(s) + b \frac{1-s}{s} g(s) - \lambda p_0 (1-s) + \alpha (1-p_0)^2 = 0.
\] (3)

It is assumed that at criticality $p_k$ has the asymptotic large $k$ behavior $p_k \sim 1/k^\tau$. One is interested in evaluating $\tau$ as a function of $\alpha$ and $b$. For the model to exhibit a condensation transition one needs $\tau > 2$. Within the mean-field approximation the model exhibits condensation transition already at $b = 0$. We therefore expect such transition to take place for any $b \geq 0$ (an assertion which has been checked numerically), and hence assume in the following that $\tau > 2$. This form of $p_k$ implies that $g(s)$ is well-defined for $|s| < 1$, and is singular at $s = 1$. Taking $s \equiv e^{-\epsilon}$, with $\epsilon > 0$, we find that for non-integer $\tau$ the singular part of $g(s)$ has the following behavior for $\epsilon \ll 1$,
\[
g(s) \sim \int_{1}^{\infty} \frac{e^{-\epsilon n}}{n^{\tau+1}} dn \sim \epsilon^\tau,
\] (4)
where for the purpose of extracting the leading singularity we replace the sum by an integral. For integer $\tau$ the singularity of $g(s)$ is of the form $\epsilon^\tau \log \epsilon$. In the following analysis we assume for convenience that $\tau$ is non-integer. The results derived in this analysis apply to the integer $\tau$ case as well. We thus make the anzats
\[
g(s) = (a_0 + a_1 \epsilon + a_2 \epsilon^2 + \ldots) + \epsilon^\tau (b_0 + b_1 \epsilon + b_2 \epsilon^2 + \ldots) + \ldots,
\] (5)
where the terms in the first brackets represent the regular part of $g(s)$, while the terms in the second brackets correspond to the singular part, with $\tau$ being its leading power.
Figure 1. The exponent $\tau$ as a function of $b$ for $\alpha = 0.2$. Solid line is given by the solution to the mean-field equations. Boxes are results of numerical simulations of the mean-field dynamics with $M = 1000, \phi = 3$. An estimation for $\tau$ is obtained by fitting $p_k$ to $k^{-\tau}$ over the range $10 < k < 100$. An example for such a fit in the case $b = 3$ is given in the inset.

Clearly non-leading singularities with powers $\tau’ > \tau$ can exist in this expansion as well. In the following we proceed by inserting this anzats into (3), and systematically solving this equation order by order in $\epsilon$. From the equations for terms of order 0, 1, 2 and $\tau$ one readily derives expressions for both the critical occupancy $\phi_c$ and $p_0$ in terms of $\lambda$:

\[ \phi_c = \frac{1 - \lambda}{2\alpha}, \quad p_0 = 1 - \frac{\lambda}{b} + \frac{(1 - \lambda)^2}{4\alpha b}. \]  

(6)

Note that non-leading singular terms cannot contribute to terms of these orders, and they thus need not be considered.

In order to determine $\lambda$ and $\tau$ one needs to distinguish between two cases:

(i) $\tau > 3$ — One proceeds by considering the orders 3, 4 and $\tau + 1$ terms in (3). Since $\tau > 3$, non-leading singularities do not contribute to these terms as well. A straightforward calculation yields

\[ \lambda = \frac{2\alpha + b + 1 - \sqrt{\gamma}}{3}, \]  

(7)

where $\gamma = 4\alpha (\alpha + b + 1) + (b - 2)^2$. Inserting (7) into (3) one gets the critical occupancy $\phi_c$ in terms of the model parameters. This solution holds for values of $b$ larger than $b^*$, defined by $\tau(b^*) = 3$. Note that in the limit $b \to \infty$ one has $\tau \to b$.

This is a result of the fact that at large values of $b$ diffusion becomes rare and the ZRP result is approached.

(ii) $2 < \tau < 3$ — In this interval we find that for $b < b^*$ (for which $\tau < 3$) $\tau$ becomes independent of $b$, with $\tau = 5/2$. To analyze this case we consider the $2\tau - 2$ term
in \( \tau \). This term is given by \( \alpha \tau^2 b_0^2 \epsilon^{2\tau-2} \), which does not vanish since by definition \( b_0 \neq 0 \). In this \( \tau \) interval non-leading singular terms in \( g(s) \) cannot contribute to this term. To see this, let \( \tau' > \tau \) correspond to a such sub-leading term. Its leading contribution to \( \epsilon \) is of the order \( \epsilon^{\tau'+1} \), which is of higher order than \( \epsilon^{2\tau-2} \). This analysis implies that \( 2\tau - 2 \) has to be an integer. Within this \( \tau \) interval, the only possibility is \( \tau = 5/2 \).

The parameter \( \lambda \), and thus the critical occupancy, cannot be determined perturbatively in this \( \tau \) interval. It is determined by the two boundary conditions imposed on \( \epsilon \), namely \( g(0) = 0 \) and \( g(1) = (\lambda - 1 + p_0)/b \), with \( p_0 \) given by \( \epsilon \).

Integrating \( \epsilon \) numerically, one identifies the value of \( \lambda \) which satisfies the boundary conditions.

In summary, we find that within the mean-field approximation \( \tau = 5/2 \) for \( b < b^*(\alpha) \), while \( \tau \) is a continuous function of \( b \) for \( b > b^*(\alpha) \). The exponent \( \tau \) exhibits a discontinuity at \( b^*(\alpha) \) (see figure 1). The critical occupancy \( \phi_c \) is given as a function of \( b \) in figure 2. For \( b > b^*(\alpha) \) the curve is deduced from \( \phi_c \) and \( \epsilon \), while for \( b < b^*(\alpha) \) it is obtained numerically by finding the value of \( \lambda \) for which the solution of \( \epsilon \) satisfies the appropriate boundary conditions. One observes that although the exponent \( \tau \) exhibits a discontinuity at \( b^* \), the critical occupancy is a continuous monotonically decreasing function of \( b \). Note that increasing either \( \alpha \) or \( b \) results in decreasing the critical occupancy. This is due to the fact that condensation is favored by both the diffusion process and by the size-dependency of the chipping. The value of \( b^* \) as a function of \( \alpha \) is depicted in the inset of this figure.
Figure 3. The exponent $\tau$ as a function of $b$, obtained from numerical simulations. Solid lines are $\tau = 2$ corresponding to asymmetric CM ($b = 0$), and $\tau = b$ corresponding to ZRP ($\alpha = 0$). Simulations were performed on systems of size $M = 1000$ and occupancy $\phi = 3$.

The $\tau(b)$ curve found above corresponds to the mean-field approximation in the thermodynamic limit. It is of interest to examine how this behavior is manifested in finite systems. This could give a useful insight for the model with nearest-neighbor exchange, where no analytic results are available, and where one has to resort only to numerical simulations of finite systems. To this end we carried out numerical simulations of the model, with a modified dynamics for which the mean-field equations yield the correct steady state. This modified dynamics is defined by the same rates as those of the original model, except that the two sites exchanging particles in each dynamical move are chosen at random, and are not necessarily nearest neighbors. The results obtained for $\tau$ from a simulation of a system of size $M = 1000$ are presented in figure III. It is observed that the qualitative behavior derived in the previous section is recovered. Namely, $\tau$ assumes the value $5/2$ for small $b$, while it continuously varies with $b$ at large $b$. In the vicinity of the critical parameter $b^*$, large crossover effects dominate the dynamics, and the numerical curve seems to deviate from the analytical one. This is a finite size effect, and one needs far larger systems in order to recover the true asymptotic exponent $\tau$ in this region, as given by the exact solution found above. We carried out similar calculations for larger systems and found that indeed the trend is to move towards the analytical curve.

2.2. Numerical simulations

In the case of nearest-neighbor exchange no analytical solution for the steady-state distribution function is available, and one has to resort to numerical simulations. The exponent $\tau(b)$ for the totally asymmetric case ($q = 0$), as extracted from studies of systems of size $M = 1000$, is given in figure III. It is readily seen that the numerical
results are consistent with $\tau = 2$ at small $b$ and with a continuously varying $\tau$ at large $b$. This indicates that as in the CM ($b = 0$) the generalized model does not exhibit a condensation transition for small $b > 0$. Condensation is obtained only above a certain threshold of the parameter $b$. The precise nature of the curve in hard to determine numerically, due to the finite size effects which dominate the dynamics in the vicinity of $b^*$, as for the mean-field dynamics. Thus, for example, accurate determination of $b^*$ is not possible, although it seems close, or equal, to 2. It is also not clear from the simulations whether or not $\tau(b)$ is discontinues at $b^*$.

We also carried out numerical studies of the symmetric case, $q = \frac{1}{2}$. For $b = 0$ it is known that $\tau = 5/2$, as in mean-field [7]. We find that $\tau$ remains locked at $5/2$ for sufficiently small $b$, while $\tau$ varies continuously above this value for large $b$.

3. Corresponding one-dimensional driven diffusive model

We now demonstrate that the generalized zero-range process considered in the previous section could be relevant to the coarse-grained dynamics of certain driven-diffusive models. To this end we introduce a simple driven-diffusive model, and analyze it using the generalized ZRP model discussed above. This model is related to a class of models whose coarse-grained dynamics has been described in terms of the ZRP [5, 6]. In contrast to those models, the model presented here exhibits also domain diffusion. The model evolves in discrete time (namely through a parallel update scheme), and is therefore related to cellular automata models introduced to study traffic flow [11, 12].

The model is defined on a lattice of $L$ sites. Each site can either be vacant (0), or occupied by positive (+) or negative (−) particle. The dynamical moves are carried out in parallel by two consecutive update steps.

Step 1: $x0x \rightarrow 0xx$

Step 2:

$\begin{align*}
+00 & \rightarrow a \quad 0 + 0 \\
+0000 & \rightarrow a \quad 0 + 0000 \\
+0000 & \rightarrow u \quad 00 + 0000 \\
+0000 & \rightarrow a \quad 00 + 00 \\
0000 & \rightarrow a \quad 0000 - 00 - 00 \\
0000 & \rightarrow u \quad 00 - 00 - 00 .
\end{align*}$

Here $x$ is a particle of either type and $u + a \leq 1$. The model allows for single and double site jumps, and it exhibits features which do not exist in models with only single-site jumps. Note that while all transition probabilities in step 2 are symmetric under particle-type exchange and left-right reflection, this symmetry is broken by the transitions of step 1. At high density the system has states in which all vacancies are isolated, and are thus deterministic. In this case ergodicity is broken.

Consider now the coarse-grained dynamics of this model. Let us define a domain in this model as a sequence of particles, interrupted only by isolated vacancies. These vacancies move deterministically to the left. The evolution of a domain can be described in terms of two processes: (a) a chipping process, in which a positive (negative) particle
leaves a domain of size $k$ to the right (left) with rate $w_k^+$ ($w_k^-$); and (b) a diffusion process, in which a vacancy penetrates the domain from its right (with probability $a$) and advances deterministically to its left, thus shifting its center of mass one unit to the right. In figure 4 a space-time configuration of the model is given. One readily observes the evolution of domains through chipping and diffusion, and the coalescence of domains upon contact.

We now consider the current which flows through a domain. The size $k$ of a domain is defined as the number of particles it contains, ignoring the vacancies in it. Since vacancies move deterministically through a domain, one may project the vacancies out of the domain dynamics, and consider only the dynamics of the charges. This dynamics is in fact described by the totally asymmetric exclusion process (TASEP)\cite{13}. Typically, a domain is asymmetric, composed of an unequal + and − charge densities. In its maximal current state, obtained for large $a$ and $u$, the current through such a domain takes the form $j_k \sim 1 + b/k$ with $b = 3/2$\cite{14,15}. We thus conclude that $w_k^\pm = w_\infty^\pm (1 + b/k)$, with $w_\infty^+ \neq w_\infty^-$ some constants, and $b = 3/2$.

The coarse-grained domain dynamics of this model is thus related to the generalized ZRP picture of Section 2 with $q \neq 1/2$ and $b = 3/2$. Since our numerical data (figure 3) suggests that $b^* > 3/2$, we expect the domain size distribution $p_k$ to take the form $p_k \sim k^{-\tau}$, with $\tau = 2$. Within the domain length accessible in numerical studies one could not get an accurate estimate for the exponent $\tau$. However, numerical simulations of the model indicate that $\tau$ is larger than its zero-diffusion limit (3/2), and is close or equal to 2. Note that the correspondence between this model and the zero-range process can only be made in the region of the model parameters where the domains are in the maximal-current state, and where the absorbing states are not reached.

Before concluding let us note that nearest-neighbor attractive interaction between particle of the same species could result in larger values of $b$, possibly reaching $b > b^* \simeq 2$. In this case the dynamics within a domain becomes identical to that of the KLS model\cite{16,17}. Here $w_k^\pm = w_\infty^\pm (1 + b(\eta)/k)$, where $\eta$ is the strength of the interaction\cite{6}. For
large values of $\eta$ one expects the exponent $\tau$ to increase beyond $\tau = 2$. This should be accompanied by a phase separation at high densities.

4. Summary and Discussion

In this work we generalize a class of zero-range processes with hopping rates of the form $\omega (1 + b/k)$ to include diffusion. In particular the exponent $\tau$ associated with the occupation distribution $p_k \sim k^{-\tau}$, and its dependence on the parameter $b$, is studied. Both mean-field approximation and numerical analysis show that for sufficiently small $b$ the exponent $\tau$ is $b$ independent, and assumes its $b = 0$ value. In particular, in the mean-field approximation $\tau = 5/2$ while for asymmetric nearest-neighbor chipping numerical simulations indicate that $\tau = 2$. Above a critical value of $b$ the exponent $\tau$ becomes a continuously increasing function of $b$.

This model could be relevant for describing the coarse-grained dynamics of certain driven diffusive models, and possibly traffic models. A particular example of a driven diffusive system for which this is the case is presented.

The fact that $\tau = 2$ in the asymmetric chipping-model has been interpreted as an indication that jamming in traffic models do not take place via a sharp phase transition [9]. The analysis presented in this paper indicates that this result is rather robust, and could hold even for a more general class of traffic models, for which the chipping-like process exhibits a weak dependence on the length of the domain ($b < b^*$). Such dependence may be of relevance for traffic models which allow overtaking.

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