Coupled Wire description of the Correlated Physics in Twisted Bilayer Graphene

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Since the discovery of superconductivity and correlated insulator at fractional electron fillings in the twisted bilayer graphene, most theoretical efforts have been focused on describing this system in terms of an effective extended Hubbard model. However, it was recognized that an exact tight binding model on the Moiré superlattice which captures all the subtleties of the bands can be exceedingly complicated. Here we pursue an alternative framework of coupled wires to describe the system based on the observation that the lattice relaxation effect is strong at small twist angle, which substantially enlarges the AB and BA stacking domains. Under an out-of-plane electric field which can have multiple origins, the low energy physics of the system is dominated by interconnected wires with (approximately) gapless 1d conducting quantum valley hall domain wall states. We demonstrate that the Coulomb interaction likely renders the wires a U(2)2 (1 + 1)d conformal field theory with a tunable Luttinger parameter for the charge U(1) sector. Spin triplet and singlet Cooper pair operator both have quasi-long range order in this CFT. The junction between the wires at the AA stacking islands can lead to either a two dimensional superconductor, or an insulator.

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Surprising correlated physics such as superconductivity and correlated insulator at fractional electron fillings away from charge neutrality has been discovered in different systems with Moiré superlattices [1–4], which motivated a series of active theoretical studies [5–37]. These systems have narrow electron bandwidth near charge neutrality [28–31], hence interaction effects are significantly enhanced. In several systems that are microscopically different, for example, (1) the heterostructure of trilayer graphene (TLG) and hexagonal boron nitride (hBN), and (2) twisted bilayer graphene (TBG), (3) twisted double bilayer graphene (TDBG) [32], insulating behavior was observed at commensurate fractional fillings away from the charge neutral point [3, 4, 32, 33]. These results suggest that an alternative theoretical framework to understand the observed correlated physics is highly desired for the TBG. Here we pursue a coupled wire network framework to describe the TBG with a small twisted angle. A similar description based on coupled wires, such as the Chalker-Coddington model [39, 40], has been used to describe states without local Wannier orbitals. But in TBG, the coupled wire network description is not just motivated by theoretical convenience, it is also physically realistic, based on the following observations:

(1) At small twisted angle, the lattice relaxation and deformation effect is expected to be strong, and lead to substantially enlarged AB and BA stacking domains [41, 42], and narrow 1d domain walls.

(2) A displacement field will drive an AB (or BA) stacking bilayer graphene into a “quantum valley Hall insulator” [43–49], and this displacement field can be turned on manually experimentally [4], or intrinsically exists in the system due to lack of \( \tilde{\sigma} \rightarrow -\tilde{\sigma} \) reflection symmetry (strongly asymmetric response to the displacement field was indeed observed in Ref. [4]), or even be generated spontaneously due to interaction [50]. Compared with a single layer graphene, in an AB (or BA) stacking bilayer graphene, interaction has much stronger effects due to the quadratic band touching at each valley [41, 55].

(3) Under a uniform displacement field (regardless of its origin), the AB and BA stacking domains are quantum valley Hall insulators with opposite valley Hall conductivities, and they are separated by domain walls with conducting 1d states. The long wavelength modulation of the entire system prohibits large momentum trans-
fer, hence the valley quantum number is approximately conserved, and the domain wall states are approximately gapless. These conducting wires (AB/BA domain walls) have been observed directly in numerics [56] and experiment on TBG [57, 58].

In fact, an effective network model has been proposed to describe the noninteracting physics of the system [59]. Along each 1d wire, there are four counter-propagating localized electron modes, which without interaction would constitute the $U(4)_1$ conformal field theory (CFT). The 1d fermions carry three quantum numbers: valley ($L, R$), spin ($\uparrow, \downarrow$), and channel (1, 2) index (Fig. 1):

$$ H = \int dx \sum_{c=1,2} \sum_{\alpha = \uparrow, \downarrow} i v (\psi^\dagger_{L,c,\alpha} \partial_x \psi_{L,c,\alpha} - \psi^\dagger_{R,c,\alpha} \partial_x \psi_{R,c,\alpha}). $$

The left and right moving modes come from two different valleys (which differ by a large momentum orthogonal to the wire), and each valley will contribute two channels (AB/BA domain wall). The left and right moving fermions differ by a large lattice momentum (orthogonal to the wires) which is the size of the Brillouin zone of the original honeycomb lattice.

in transition metals. Let us consider two electrons with the following two-body wave functions $\Psi_A(x_1, x_2)$ and $\Psi_B(x_1, x_2)$ ($x_1, x_2$ are 2d coordinates):

$$\Psi_A(x_1, x_2) \sim \varphi_{L,1}(x_1) \varphi_{R,2}(x_2) - \varphi_{L,2}(x_1) \varphi_{R,1}(x_2)$$

$$+ \varphi_{R,1}(x_1) \varphi_{L,2}(x_2) - \varphi_{R,2}(x_1) \varphi_{L,1}(x_2);$$

$$\Psi_B(x_1, x_2) \sim \varphi_{L,1}(x_1) \varphi_{R,2}(x_2) - \varphi_{L,2}(x_1) \varphi_{R,1}(x_2)$$

$$- \varphi_{R,1}(x_1) \varphi_{L,2}(x_2) + \varphi_{R,2}(x_1) \varphi_{L,1}(x_2).$$

Here $\varphi_{L,1}(x)$ represents the spatial wave function of the left-moving fermions (which comes from one of the two valleys) at channel 1. Both states $\Psi_A, B$ are “channel singlet” states (they are antisymmetric in the channel indices), while $\Psi_A$ is symmetric in the valley space, $\Psi_B$ is antisymmetric in the valley space. The spin space wave function was not written down but can be straightforwardly inferred. Both states cost low energy under Coulomb interaction, i.e. they have considerable lower energy compared with states that are symmetric in the channel space, and this energy difference is not suppressed by large momentum transfer (more detailed estimate will be given in the supplementary material). Thus the channel space is analogous to the gauged “color space” of spin chains [60, 61], which must form a color singlet state.

A $U(4)_1$ CFT can be decomposed as

$$U(4)_1 \sim U(1)_s^4 \oplus SU(2)_s^2 \oplus SU(2)_g^2,$$  (3)

where $SU(2)_s^2$ corresponds to the sector of the channel space. The interaction effect discussed in the previous paragraph contributes to the marginally relevant term $\lambda J^s_L \cdot J^s_R$ in the CFT, where $J^s_{L,R}$ are the left and right Kac-Moody currents of the channel space, and it will gap out the $SU(2)_s^2$ sector of the CFT. The residual degrees of freedom would form CFT

$$U(2)_2 \sim U(1)_s^4 \oplus SU(2)_s^2.$$  (4)

The $U(1)_s^4$ sector of the CFT corresponds to the charge degrees of freedom, and it can be represented by a pair of conjugate bosons $\theta$ and $\phi$ which satisfy $[\nabla_x \phi, \theta] = [\nabla_x \theta, \phi] = i$. The $SU(2)_s^2$ corresponds to the spin degrees of freedom, and as we discussed before, due to the prohibition of large momentum transfer, the left and right modes have approximately separate spin $SU(2)$ symmetries. The $SU(2)_s^2$ CFT can be represented by a $(1 + 1)d$ nonlinear sigma model whose order parameter is a $SU(2)$ matrix $g_{\alpha \beta}$, plus a Wess-Zumino-Witten term at level-2 [62]. The left and right spin symmetry acts on $g_{\alpha \beta}$ as the left and right $SU(2)$ transformations.

Physical operators can be represented as CFT fields. For example, a fermion mass operator (which is a backscattering term) can be written as [62]

$$\hat{M}_{\alpha \beta} = \sum_c \psi^\dagger_{L,c,\alpha} \psi_{R,c,\beta} \sim \exp (i \sqrt{\pi} \phi) g_{\alpha \beta}. $$  (5)
where $g_{\alpha\beta}$ is the spin $SU(2)$ matrix order parameter mentioned previously. Notice that the mass operator must be a channel singlet, because otherwise it must involve the $SU(2)_2$ sector, which as we argued is already gapped out.

Likewise, a Cooper pair operator can be written as

$$
\Delta_{\alpha\beta} = \epsilon_{\alpha\gamma} \epsilon_{\alpha\beta} \psi_L \epsilon_{\gamma\delta} \psi_R, \quad \sim \exp(i\sqrt{\pi}g) g_{\alpha\beta},
$$

(6)

where $\theta$ and $\phi$ are the pair of conjugate bosons that describe the charge sector of the CFT. The representation of the mass operator $M_{\alpha\beta}$ is given in Ref. [22]. The Cooper pair operator representation can be inferred by defining a new set of fermions: $\tilde{\psi}_L = \epsilon_0 \psi_L^\dagger$, $\tilde{\psi}_R = \psi_R$, where the two $\epsilon$ matrices act in the spin and channel indices respectively. The fermion operator $\tilde{\psi}_L$ transforms exactly the same as $\psi_L$ in the channel and spin space, but carries opposite charge. The Cooper pair operator in Eq. (6) becomes precisely the mass term (backscattering) between $\tilde{\psi}_L$ and $\tilde{\psi}_R$.

The Cooper pair operator $\Delta_{\alpha\beta}$ is a channel singlet pairing. The pairing matrix $\Delta_{\alpha\beta}$ can always be expanded as a four component vector ($\Delta^0$, $\tilde{\Delta}$):

$$
\Delta_{\alpha\beta} = \Delta^0 1_{2\times2} + i\tilde{\Delta} \cdot \sigma.
$$

(7)

Here $\Delta^0$ is a spin singlet pairing order parameter, while $\tilde{\Delta}$ is a spin triplet pairing order parameter. Together they form a four component vector representation under the $SO(4) \sim SU(2)_L \times SU(2)_R$ symmetry. Without a further Hund’s (or anti-Hund’s) coupling that favors either spin triplet or singlet pairing, these four components pairing order parameters are all degenerate. In the supplementary material, we discuss a different method to obtain the CFT field expressions Eq. (5) and Eq. (6) where the fermion mass and the Cooper pair operators are treated on equal footing.

The scaling dimensions of the fermion mass and Cooper pair operators are

$$
[M_{\alpha\beta}] = \frac{3}{8} + \frac{1}{4K}, \quad [\tilde{\Delta}_{\alpha\beta}] = \frac{3}{8} + \frac{K}{4},
$$

(8)

where $3/8$ comes from the scaling dimension of the $g$ matrix order parameter in the $SU(2)_2$ CFT, and $K$ is the Luttinger parameter in the $U(1)^c_2$ CFT. Soon we will see that these scaling dimensions will determine whether the system becomes superconductor or insulator due to wire junctions at the AA islands. Notice that both $M_{\alpha\beta}$ and $\tilde{\Delta}_{\alpha\beta}$ can simultaneously have lower scaling dimensions (which implies enhanced correlation) compared with non-interacting 1d fermion systems, where both operators have scaling dimensions 1. Thus the interaction which gaps out the $SU(2)^c_2$ channel space indeed enhances the system’s tendency to form superconductor and insulator.

The $U(1)^c_2$ CFT deserves some clarifications. It can always be written as a free boson theory with the Hamiltonian:

$$
H = \int dx \left( \frac{1}{2K} (\nabla_x \theta)^2 + \frac{K}{2} (\nabla_x \phi)^2 \right).
$$

(9)

where $\theta$ and $\phi$ are a pair of conjugate bosons. We can fermionize this theory through standard procedure, and define new fermion operators as

$$
C_{L,R} \sim \eta L \exp(i\sqrt{\pi} \theta \pm i\sqrt{\pi} \phi),
$$

(10)

where $\eta_{L,R}$ are the Klein factors. Then the Cooper pair and the mass term of the new fermion $C_{L,R}$ should be represented as $\exp(i\sqrt{\pi} \theta)$, and $\exp(i\sqrt{\pi} \phi)$. But these Cooper pairs should correspond to the charge−2e bound state of the electrons, and the mass term should correspond to a two electron backscattering. This is because under the assumption of separate left and right spin $SU(2)$ symmetries, a charge−2e Cooper pair, or a singlet electron back scattering term, cannot be invariant under the $SU(2)_L \times SU(2)_R$ spin symmetry. Later we will show that the charge−4e $U(1)^c_2$ sector may become relevant to the finite temperature physics of the system.

The 1d CFTs will intersect at the AA stacking islands, and due to the lattice relaxation and deformation, the size of the AA stacking islands has shrunk [41]. Let us first look at a single AA island which is a junction between CFTs along three directions. At this junction, the Cooper pairs can tunnel between 1d CFTs along different wires. This Josephson tunnelling between CFTs can be described by a $(0+1)d$ action at the junction

$$
S = \int d\tau \sum_{I,J} u_0 \Delta^0_{\epsilon_I} \Delta^0_{\epsilon_J} + u_1 \Delta^\dagger_{\epsilon_I} \cdot \Delta_{\epsilon_J},
$$

(11)

$\epsilon_I$ with $I = 1, 2, 3$ represent wires along three directions that meet at this junction. The scaling dimension of $u_0$ and $u_1$ are both $[u_0] = [u_1] = 1/4 - K/2$, where $K$ is the Luttinger parameter in Eq. (8) thus when $K < 1/2$ even a single junction Josephson Cooper pair tunnelling becomes relevant, and we expect this Josephson tunnelling to drive the entire system into a superconductor. If we take into account of the tunnelling between parallel wires, which happens along the entire 1d wires rather than one junction, then this parallel tunnelling will be relevant and the entire system becomes a superconductor for $K < 5/2$.

Here we allow $u_0$ and $u_1$ to be different, which breaks the two separate $SU(2)$ spin symmetries to its diagonal spin $SU(2)$ symmetry. The AA island has shrunk substantially due to lattice relaxation, thus the potential modulates at a shorter length scale compared with other regions of the system, which enhances the large momentum transfer and leads to the mixing between the left and right $SU(2)$ symmetries. If $u_0$ dominate $u_1$, the system would favor to form a global spin singlet pairing. Now the global structure of the system can be mapped to the following classical XY model:

$$
H \sim \sum_{\vec{r}} -V \sum_{I=1}^3 \cos(\theta^I_{\vec{r}} - \theta^I_{\vec{r} + a_{\vec{e}_I}}).
$$
When one assumes that $\vec{a}$ spin triplet superconductor. As an example let us consider a singlet of spin, valley, and channel indices.

The SC order parameter along different wires, with $u_0 < 0$ and $u_0 > 0$ in Eq. (12) Table I refers to the wires in Fig. 1.

| wires | $u_0 < 0$, s-wave pairing | $u_0 > 0$, d + id or d − id pairing |
|-------|--------------------------|-----------------------------------|
| $I = 1$ | $\Delta$ | $\Delta$ |
| $I = 2$ | $\Delta$ | $\Delta e^{\pm i \frac{\pi}{3}}$ |
| $I = 3$ | $\Delta$ | $\Delta e^{\pm i \frac{\pi}{3}}$ |

TABLE I: The SC order parameter along different wires, with $u_0 < 0$ and $u_0 > 0$ in Eq. (12) The index $I$ refers to the wires in Fig. 1.

$$+ u_0 \sum_{I,J=1}^{3} \cos(\theta^I_r - \theta^J_r) + \cdots$$ (12)

Here $\vec{r}$ denotes the AA stacking islands of the lattice, and $e_1$ with $I = 1, 2, 3$ are unit vectors along the wires (Fig. 1). $\hat{r}$ is the distance between two AA stacking islands, and $\theta^I_r$ is the phase angle of the spin singlet Cooper pair of wire along direction $e_1$. The ellipsis in Eq. (12) represent other weaker terms allowed by symmetry in the system.

Here naturally $V > 0$, which reflects the fact that along each wire the superconductor order parameter has a quasi long range order and prefers the Cooper pair to have a uniform pairing phase along the wire. Then when $u_0 < 0$, the Josephson couplings between different wires are “unfrustrated”, hence the entire system should form a spin singlet $s$-wave pairing with a uniform pairing phase; while when $u_0 > 0$, the Josephson coupling between wires along three directions is “frustrated”. The two terms in Eq. (12) demands a uniform $\theta^I$ along direction $e_1$, while wires that intersect each other at one island will have Cooper pair phases which differ from each other by ±120 degrees. Then the pairing symmetry of the entire system is identical to the $d + id$ (or $d - id$) pairing, as under a spatial 60 degree rotation (a cyclic permutation between wires along three directions), the pairing phase angle changes by ±120 degrees. This $d + id$ pairing superconductor is a singlet of spin, valley, and channel indices.

When $u_0$ dominates $u_0$ in Eq. (11) the system will form a spin triplet superconductor. As an example let us assume that $\Delta_{\tilde{e}_1}(\vec{r}) = \exp(i\theta^I_r)\xi^I_r$ (the real and imaginary parts of the spin triplet Cooper pair are parallel with each other), which is similar to the so-called “polar state” of Bose-Einstein condensate (BEC) of the spin-1 spinor cold atoms [63, 65]. Then the effective Hamiltonian of the coupled Josephson wires reads

$$H \sim \sum_{\vec{r}} - V \sum_{I=1}^{3} \xi^I_r \cdot \xi^I_{\vec{r} + a\tilde{e}_1} \cos(\theta^I_r - \theta^I_{\vec{r} + a\tilde{e}_1})$$

$$+ u_1 \sum_{I,J=1}^{3} \xi^I_r \cdot \xi^J_r \cos(\theta^I_r - \theta^J_r) + \cdots$$ (13)

When $u_1 < 0$, the system forms a uniform $s$-wave spin triplet pairing. When $u_1 > 0$, again the Josephson coupling on every AA island is frustrated, then the system either forms a uniform state of $\theta$, with a 120 degree “antiferromagnetic” pattern of $\tilde{a}$, or forms a $d + id$ pattern of $\theta$, with a “ferromagnetic” state of $\tilde{a}$. Other symmetry allowed terms, or quantum fluctuation may lift the degeneracy of the two scenarios described above.

There is a $Z_2$ gauge transformation shared between $\exp(i\theta^I_r)$ and $\xi^I_r$, i.e. the spin triplet pairing order parameter is invariant under $\xi^I_r \rightarrow -\xi^I_r$ and $\theta^I_r \rightarrow \theta^I_r + \pi$. At any finite temperature, the vectors $\xi^I_r$ will be disordered due to thermal fluctuation because this system is purely two dimensional, then as was predicted in Ref. [10], the superconductor vortex at finite temperature will carry magnetic flux quantized as $n\Phi_0/(4\pi)$. This means that the charge sector will form an effective charge–$4e$ superconductor with algebraic correlation of charge–$4e$ order parameters. This charge–$4e$ superconductor is qualitatively the same as the Cooper pair of the fermions $C_{L,R}$ defined before. The same logic led to fractionalized vortices of the polar state of spin-1 BEC, which was confirmed numerically in Ref. [65].

At the AA islands, symmetry also allows charge backscattering within each wire. The charge sector of the system is described by the $C$ fermions defined in Eq. (10) $C_L$ and $C_R$ come from two different valleys in the bulk, which project to the same momentum (Dirac crossing) along the $id$ domain wall. Upon doping away from charge neutrality, the $C_{L,R}$ fermion will acquire a fermion wave vector $\pm 2\delta k_f$ away from the Dirac crossing, thus a backscattering involves a momentum transfer of $2\delta k_f$. The backscattering of the $C$ fermion is described by

$$S = \int d\vec{r} d\vec{x} uU(x) \left( C_L^\dagger C_R e^{2i\delta k_f x} + H.c. \right)$$ (14)

where $U(x)$ is the periodic potential along the wire due to the AA stacking islands. If the integral along the entire wire $\int dx U(x)e^{2i\delta k_f x}$ is nonzero, then this implies that $2\delta k_f = \pm 2\pi/a$, where $a$ is the lattice constant of the Moiré superlattice, or the distance between two AA stacking islands. This implies that there must be an integer multiple of $\pm 2e$ charges between two AA islands on each wire (one $C$ fermion carries charge $2e$). And if wires along two directions acquire $+2e$ between every two neighboring AA islands, and the wires along the third direction acquire $-2e$ between AA islands, the entire system becomes an insulator at half-filling away from charge neutrality with $+2e$ charge per unit cell on the superlattice. The insulator observed at the 1/4 filling should correspond to two particle backscattering, which is a much weaker effect. The backscattering will be more relevant with larger Luttinger parameter $K$.

We also notice that in experiment the resistivity at the same charge density can strongly depend on the displacement field [4]. This is a natural phenomenon in our formalism, because a stronger displacement field would lead to a larger gap in the quantum valley Hall insula-
tor, and hence stronger localization of the electron wave function at the wires. Stronger localization of the domain wall states would lead to a stronger effective particle density-density interaction in the $(1+1)d$ CFT, and hence a larger Luttinger parameter $K$ based on the standard bosonization formalism. A larger $K$ would render the backscattering at the AA islands more relevant. This means that the Luttinger parameter $K$ is tunable by the displacement field, and the field can potentially lead to a metal-insulator transition.

**Summary:** We study the correlated physics of the TBG based on a coupled wire framework. The low energy physics of the system is dominated by the conducting wires which are the domain walls between the AB/BA domains. These domains are enlarged due to lattice relaxation, and are driven into the quantum valley Hall insulators under a displacement field which can have multiple origins. The observed superconductivity and the correlated insulator of the system are interpreted as consequences of the Josephson tunnelling and also backscattering at the AA stacking islands, which are the junctions where the wires along three directions meet. One puzzle from the experiment is the weakness of the insulators at fractional fillings. In our description, the insulating behavior is due to the backscattering at the AA islands, which is still suppressed due to large momentum transfer (large momentum transfer orthogonal to the wire, which is still approximately defined due to the smoothness of the background potential), thus it will at most lead to a weak correlated insulator. In our formalism a displacement field can tune the Luttinger parameter of the CFT, and hence affect the relevance of backscattering and also charge transport, as was observed experimentally.

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[1] G. Chen, L. Jiang, S. Wu, B. Lv, H. Li, K. Watanabe, T. Taniguchi, Z. Shi, Y. Zhang, and F. Wang, arXiv:1803.01985 (2018).
[2] Y. Cao, V. Fatemi, A. Demir, S. Fang, S. L. Tomarken, J. Y. Luo, J. D. Sanchez-Yamagishi, K. Watanabe, T. Taniguchi, E. Kaxiras, et al., Nature 556, 80 (2018).
[3] Y. Cao, V. Fatemi, S. Fang, K. Watanabe, T. Taniguchi, E. Kaxiras, and P. Jarillo-Herrero, Nature 556, 43 (2018).
[4] M. Yankowitz, S. Chen, H. Polsyny, Y. Zhang, K. Watanabe, T. Taniguchi, D. Graf, A. F. Young, and C. R. Dean, Science 363, 1059 (2019), ISSN 0036-8075, http://science.sciencemag.org/content/363/6431/1059.full.pdf, URL http://science.sciencemag.org/content/363/6431/1059.
[5] C. Xu and L. Balents, Phys. Rev. Lett. 121, 087001 (2018), URL https://link.aps.org/doi/10.1103/PhysRevLett.121.087001.
[6] H. C. Po, L. Zou, A. Vishwanath, and T. Senthil, Phys. Rev. X 8, 031089 (2018), URL https://link.aps.org/doi/10.1103/PhysRevX.8.031089.
[7] J. F. Dodaro, S. A. Kivelson, Y. Schattner, X. Q. Sun, and C. Wang, Phys. Rev. B 98, 075154 (2018), URL https://link.aps.org/doi/10.1103/PhysRevB.98.075154.
[8] N. F. Q. Yuan and L. Fu, Phys. Rev. B 98, 045103 (2018), URL https://link.aps.org/doi/10.1103/PhysRevB.98.045103.
[9] J. Kang and O. Välik, Phys. Rev. X 8, 031088 (2018), URL https://link.aps.org/doi/10.1103/PhysRevX.8.031088.
[10] B. Padhi, C. Setty, and P. W. Phillips, Nano Letters 18, 6175 (2018), PMID: 30185049, https://doi.org/10.1021/acs.nanolett.8b02033, URL https://doi.org/10.1021/acs.nanolett.8b02033.
[11] B. Padhi and P. Phillips, arXiv:1810.00884 (2018).
[12] G. Baskaran, arXiv:1804.00627 (2018).
[13] C.-C. Liu, L.-D. Zhang, W.-Q. Chen, and F. Yang, Phys. Rev. Lett. 121, 217001 (2018), URL https://link.aps.org/doi/10.1103/PhysRevLett.121.217001.
[14] L. Rademaker and P. Mellado, Phys. Rev. B 98, 235158 (2018), URL https://link.aps.org/doi/10.1103/PhysRevB.98.235158.
[15] H. Isobe, N. F. Q. Yuan, and L. Fu, Phys. Rev. X 8, 041041 (2018), URL https://link.aps.org/doi/10.1103/PhysRevX.8.041041.
[16] M. Koshino, N. F. Q. Yuan, T. Koretsune, M. Ochi, K. Kuroki, and L. Fu, Phys. Rev. X 8, 031087 (2018), URL https://link.aps.org/doi/10.1103/PhysRevX.8.031087.
[17] Y.-Z. You and A. Vishwanathan, arXiv:1805.06867 (2018).
[18] F. Wu, A. H. MacDonald, and I. Martin, Phys. Rev. Lett. 121, 257001 (2018), URL https://link.aps.org/doi/10.1103/PhysRevLett.121.257001.
[19] G.-Y. Zhu, T. Xiang, and G.-M. Zhang, arXiv:1806.07535 (2018).
[20] B. Lian, Z. Wang, and B. A. Bernevig, arXiv:1807.04382 (2018).
[21] H. Guo, X. Zhu, S. Feng, and R. T. Scalettar, Phys. Rev. B 97, 235453 (2018), URL https://link.aps.org/doi/10.1103/PhysRevB.97.235453.
[22] Y.-H. Zhang and T. Senthil, arXiv:1809.05110 (2018).
[23] H. C. Po, L. Zou, T. Senthil, and A. Vishwanath, arXiv:1808.02482 (2018).
[24] Q. K. Tang, L. Yang, D. Wang, F. C. Zhang, and Q. H. Wang, arXiv:1809.06772 (2018).
[25] A. Thomson, S. Chatterjee, S. Sachdev, and M. S. Scheurer, Phys. Rev. B 98, 075109 (2018), URL https://link.aps.org/doi/10.1103/PhysRevB.98.075109.
[26] K. Hejazi, C. Liu, H. Shapourian, X. Chen, and L. Balents, Phys. Rev. B 99, 035111 (2019), URL https://link.aps.org/doi/10.1103/PhysRevB.99.035111.
[27] J. Liu, J. Liu, and X. Dai, arXiv:1810.03103 (2018).
[28] R. Bistritzer and A. H. MacDonald, Proceedings of the National Academy of Sciences 108, 12233 (2011), ISSN 0027-8424, http://www.pnas.org/content/108/30/12233.full.pdf, URL http://www.pnas.org/content/108/30/12233.
[29] E. Suárez Morell, J. D. Correa, P. Vargas, M. Pacheco, and Z. Barticevic, Phys. Rev. B 82, 121407 (2010), URL https://link.aps.org/doi/10.1103/PhysRevB.82.121407.
wave functions in more detail in this appendix. The wave

\[ A. Vaezi, Y. Liang, D. H. Ngai, L. Yang, and E.-A. Kim, \]
\[ Z. Qiao, J. Jung, Q. Niu, and A. H. MacDonald, \]
\[ Nature 520, 650 (2015). \]

\[ E. C. Marino, L. O. Nascimento, V. S. Alves, and C. M. Smith, \]
\[ Phys. Rev. X 5, 011040 (2015), URL https://link.aps.org/doi/10.1103/PhysRevX.5.011040. \]

\[ H. Min, G. Borghi, M. Polini, and A. H. MacDonald, \]
\[ Phys. Rev. B 77, 041407 (2008), URL https://link.aps.org/doi/10.1103/PhysRevB.77.041407. \]

\[ R. Nandkishore and L. Levitov, Phys. Rev. B 82, 115124 (2010), URL https://link.aps.org/doi/10. \]
\[ 1103/PhysRevB.82.115124. \]

\[ R. Nandkishore and L. Levitov, Physica Scripta 2012, 041011 (2012), URL http://stacks.iop.org/10. \]
\[ 1050-2947/2012/i=T44/a=041011. \]

\[ M. Koshino, K. Sugisawa, and E. McCann, Phys. Rev. B 95, 235311 (2017), URL https://link.aps.org/doi/ \]
\[ 10.1103/PhysRevB.95.235311. \]

\[ K. Sun, H. Yao, E. Fradkin, and S. A. Kivelson, Phys. Rev. Lett. 103, 046811 (2009), URL https://link.aps.org/ \]
\[ doi/10.1103/PhysRevLett.103.046811. \]

\[ F. San-Jose and E. Prada, Phys. Rev. B 88, 121408 (2013), URL https://link.aps.org/doi/10. \]
\[ 1103/PhysRevB.88.121408. \]

\[ S. S. Sunku, G. X. Ni, B. Y. Jiang, H. Yoo, A. Sternbach, A. S. McLeod, T. Stauber, L. Xiong, T. Taniguchi, K. Watanabe, et al., Science 362, 1153 (2018), ISSN 0036-8075, \]
\[ URL http://science.sciencemag.org/content/362/6419/1153.full.pdf, \]
\[ URL http://science.sciencemag.org/content/362/6419/1153. \]

\[ P. Rickhaus, J. Wallbank, S. Slizovskiy, R. Pisoni, H. Overweg, Y. Lee, M. Eich, M.-H. Liu, K. Watanabe, T. Taniguchi, et al., Nano Letters 18, 6725 (2018), pMID: 30336041, \]
\[ https://doi.org/10.1021/acs.nanolett.8b02387. \]

\[ D. K. Efimkin and A. H. MacDonald, Phys. Rev. B 98, 035404 (2018), URL https://link.aps.org/doi/10. \]
\[ 1103/PhysRevB.98.035404. \]

\[ I. Affleck and F. D. M. Haldane, Phys. Rev. B 36, 5291 (1987), URL https://link.aps.org/doi/10. \]
\[ 1103/PhysRevB.36.5291. \]

\[ I. Affleck, Nuclear Physics B 265, 448 (1986), ISSN 0550-3213, \]
\[ URL http://www.sciencedirect.com/science/article/pii/0550321386901689. \]

\[ I. Affleck, Nucl. Phys. B 265, 409 (1986). \]

\[ T.-L. Ho, Phys. Rev. Lett. 81, 742 (1998), URL https:// \]
\[ link.aps.org/doi/10.1103/PhysRevLett.81.742. \]

\[ T. Ohmi and K. Machida, Journal of the Physical Society of Japan 67, 1822 (1998), \]
\[ https://doi.org/10.1143/JPSJ.67.1822. \]

\[ S. Mukerjee, C. Xu, and J. E. Moore, Phys. Rev. Lett. 97, 120406 (2006), URL https://link.aps.org/doi/10. \]
\[ 1103/PhysRevLett.97.120406. \]

\[ E. Witten, Comm. Math. Phys. 92, 455 (1984), URL https://projecteuclid.org:443/euclid.cmp/ \]
\[ 1103/PhysRevLett.103.020401. \]

A: Exchange energy of two-particle wave functions

Let us evaluate the exchange energy of two-particle wave functions in more detail in this appendix. The wave
function $\Psi_A(x_1, x_2)$ considered in the main text has the interaction energy
\[
E_{\text{int}} \sim \int dx_1 dx_2 \, \Psi_A^*(x_1, x_2)V_{x_1, x_2}\Psi_A(x_1, x_2) = E_0 + E_{\text{ex}};
\]
where $V_{x_1, x_2}$ is the (screened) Coulomb interaction. Both integrals $\int dx_1$, $\int dx_2$ are performed in the $2d$ space.

\[
E_0 = \int dx_1 dx_2 |\varphi_{L,1}(x_1)|^2 |\varphi_{R,2}(x_2)|^2 V_{x_1, x_2} + \cdots
\]

$E_{\text{ex}}$ is the exchange energy, and it involves six integrals:
\[
\begin{align*}
I_{\text{ex,1}} &= -\int dx_1 dx_2 \varphi_{L,1}^*(x_1) \varphi_{L,2}(x_1)V_{x_1, x_2} \varphi_{R,2}^*(x_2) \varphi_{R,1}(x_2) + c.c; \\
I_{\text{ex,2}} &= +\int dx_1 dx_2 \varphi_{L,1}^*(x_1) \varphi_{R,1}(x_1)V_{x_1, x_2} \varphi_{R,2}^*(x_2) \varphi_{L,2}(x_2) + c.c; \\
I_{\text{ex,3}} &= -\int dx_1 dx_2 \varphi_{L,1}^*(x_1) \varphi_{R,2}(x_1)V_{x_1, x_2} \varphi_{R,1}^*(x_2) \varphi_{L,2}(x_2) + c.c; \\
I_{\text{ex,4}} &= -\int dx_1 dx_2 \varphi_{L,2}^*(x_1) \varphi_{R,1}(x_1)V_{x_1, x_2} \varphi_{R,2}^*(x_2) \varphi_{L,1}(x_2) + c.c; \\
I_{\text{ex,5}} &= +\int dx_1 dx_2 \varphi_{L,2}^*(x_1) \varphi_{R,2}(x_1)V_{x_1, x_2} \varphi_{R,1}^*(x_2) \varphi_{L,1}(x_2) + c.c; \\
I_{\text{ex,6}} &= -\int dx_1 dx_2 \varphi_{R,1}^*(x_1) \varphi_{R,2}(x_1)V_{x_1, x_2} \varphi_{R,2}^*(x_2) \varphi_{R,1}(x_2) + c.c;
\end{align*}
\]

\[
E_{\text{ex}} = \sum_{i=1}^{6} I_{\text{ex,i}},
\]

(15)

The single-particle wave functions are roughly (for example) $\varphi_{L,1}(x) \sim \exp(iK_1 x)F_{L,1}(y)$, etc. where $F_{L,1}(y)$ is an envelope function of the coordinate $y$ orthogonal to the wire, and localized at the wire. $F_{L,1}(y)$ should carry an approximately conserved large momentum, which inherits from the crystal momentum of one of the two valleys, assuming the domain wall is smooth enough compared with the lattice scale. In all these exchange energy integrals, $I_{\text{ex,2-5}}$ are expected to be considerably smaller than $I_{\text{ex,1}}$ and $I_{\text{ex,6}}$, because they involve large momentum transfer, i.e., integrals like $\int dx_1 \varphi_{L,1}^*(x_1) \varphi_{R,1}(x_1)$. These integrals are highly suppressed because $\varphi_{L,1}(x_1)$ and $\varphi_{R,1}(x_1)$ come from two valleys in the original honeycomb lattice, the two valleys have very large momentum difference.

$I_{\text{ex,1}} + I_{\text{ex,6}}$ is the main exchange energy gained by $\Psi_A$, both integrals do not involve large momentum transfer, and they both conserve the total momentum along the wire (time-reversal symmetry guarantees that $K_1 = -K'_2$, $K_2 = -K'_1$), assuming we focus on a single wire without junction. With the Coulomb interaction, or the standard form of screened Coulomb interaction, $I_{\text{ex,1}} + I_{\text{ex,6}}$ is negative. The exchange energy of $\Psi_B$ is very similar, and both wave functions are “channel” singlet states.

One can also run the same test on other two-particle wave functions which are symmetric in the channel space, such as
\[
\begin{align*}
\Psi_C(x_1, x_2) &\sim \varphi_{L,1}(x_1) \varphi_{R,2}(x_2) + \varphi_{L,2}(x_1) \varphi_{R,1}(x_2) \\
&\quad - \varphi_{R,1}(x_1) \varphi_{L,2}(x_2) - \varphi_{R,2}(x_1) \varphi_{L,1}(x_2); \\
\Psi_D(x_1, x_2) &\sim \varphi_{L,1}(x_1) \varphi_{R,1}(x_2) - \varphi_{L,2}(x_1) \varphi_{R,2}(x_2) \\
&\quad + \varphi_{R,1}(x_1) \varphi_{L,1}(x_2) - \varphi_{R,2}(x_1) \varphi_{L,2}(x_2);
\end{align*}
\]

(16)

None of these wave functions gain as much exchange energy compared with $\Psi_A$ and $\Psi_B$, because their exchange energy integrals either involve large momentum transfer, or violate total momentum conservation along the wire. For example, for $\Psi_D(x_1, x_2)$, its exchange energy contains terms like
\[
- \int dx_1 dx_2 \varphi_{L,1}^*(x_1) \varphi_{L,2}(x_1)V_{x_1, x_2} \varphi_{R,1}^*(x_2) \varphi_{R,2}(x_2)
\]

(17)

This integral represents the physical process of moving two particles at momenta $K_2$ and $K'_2$ to momenta $K_1$ and $K'_1$ (Fig. 2), which is suppressed because in general it violates total momentum conservation along the wire. This total momentum conservation can be viewed as a $U(1)$ symmetry in the channel space, i.e., $N_{L,1} + N_{R,1} - N_{L,2} - N_{R,2}$ must be a conserved quantity, where (for example) $N_{L,1}$ is the number of left moving particles at channel 1.
B: Fermion Bilinears as CFT fields

In the main text, we obtain the CFT field expressions of the fermion mass operator Eq. 5 using the non-Abelian bosonization of $U(4)_1$ and the decomposition $U(4)_1 \sim U(1)_A \otimes SU(2)^2_1 \otimes SU(2)^2_2$. For the Cooper pair operator Eq. 6, we first define a new basis of fermions such that the Cooper pair operator acts as a fermion mass operator in the new basis. Then, we conduct a similar non-Abelian bosonization and the decomposition of $U(4)_1$ to obtain its CFT field expression. In this section, we study a different method to obtain the CFT field expressions of the fermion mass operator and the Cooper pair operator while treating them in equal footing.

We first rewrite the left/right-moving complex fermions $\psi_{L,c,\alpha}$, $\psi^\dagger_{L,c,\alpha}$, $\psi_{R,c,\alpha}$, and $\psi^\dagger_{R,c,\alpha}$ in a Majorana fermion basis $\chi_{L,R}$ where each of $\chi_L$ and $\chi_R$ is an 8-component Majorana spinor. The Majorana basis is chosen such that the generators of the symmetries $U(1)^c$, $SU(2)^s$ and $SU(2)^c$ are given by

$$U(1)^c : \sigma^0,$$

$$SU(2)^s : \sigma^{xy}, \sigma^{0z}, \sigma^{0y},$$

$$SU(2)^c : \sigma^{0x}, \sigma^{0y}, \sigma^{0z},$$

where $\sigma^{x,y,z}$ are the Pauli matrices and $\sigma^0$ is the $2 \times 2$ identity matrix. Here, we’ve adopted the notation $\sigma^{abc...} = \sigma^a \otimes \sigma^b \otimes \sigma^c \otimes ...$. The left and right-moving Majorana fermions can be described by the $O(8)_1$ CFT. More precisely, we can bosonize these Majorana fermions and describe them using a non-linear sigma model with the group $O(8)$ and with a Wess-Zumino-Witten term at level 1. Following the non-Abelian bosonization procedure given by Ref. 66, we can identify the fermion bilinears $\chi_L \chi_R^T$ with the field $h \in O(8)$ of the non-linear sigma model. The fermion mass operator in Eq. 5 and the Cooper pair operator Eq. 6 are included in $\chi_L \chi_R^T$ and hence can be expressed in terms of $h \in O(8)$ when bosonized. In the following, we will study the specific form of field $h \in O(8)$ which represents the fermion mass and the Cooper pair operators.

First of all, both of the fermion mass and the Cooper pair operators are $SU(2)^c$ singlets. Hence, we focus only on the field $h \in O(8)$ such that $h$ commutes with the $SU(2)^c$ generators given in Eq. 20. The field $h$ that satisfy this condition takes the general form

$$h = W(\tilde{h} \otimes \sigma^0)W^\dagger$$

where $W = \frac{1}{\sqrt{2}}(1 + i\sigma^{0y})$ and $\tilde{h}$ is a $4 \times 4$ matrix. Since $h \in O(8)$, $h$ has to be a real matrix, which implies

$$\sigma^{0y} \tilde{h} \sigma^{0y} = \tilde{h}^\ast.$$  \hspace{1cm} (22)

This condition implies that $\tilde{h}$ decompose into a linear superposition of the following basis matrices with real coefficients:

$$\sigma^{00}, \sigma^{0x}, \sigma^{0y}, \sigma^{0z},$$

$$\sigma^{y0}, \sigma^{yx}, \sigma^{yy}, \sigma^{yz},$$

$$\sigma^{z0}, \sigma^{zx}, \sigma^{zy}, \sigma^{zz}.$$ \hspace{1cm} (23)

Both the fermion mass and the Cooper pair operators transform non-trivially under the left and right $U(1)^c$ and $SU(2)^s$ actions. For the field $h$, the left and right $U(1)^c$ and $SU(2)^s$ actions are given by the left and right multiplication of $U(1)^c$ and $SU(2)^s$ matrices generated the generators given in Eq. 20. Hence, we should organize the basis of $\tilde{h}$ such that $h$ transforms properly under the left and right $U(1)^c$ and $SU(2)^s$ actions:

$$\hat{h} = \alpha(\cos \phi + i \sin \phi \sigma^y) \otimes g$$

$$+ \beta(\cos \theta \sigma^z + \sin \theta \sigma^z) \otimes g'',$$ \hspace{1cm} (24)

where $\alpha, \beta$ are real number, $\phi$ and $\theta$ are two angular variables, and $g, g' \in SU(2)$ are $2 \times 2$ $SU(2)$ matrices. Note that $\hat{h}$ contains two terms. Their transformations under the left and right $U(1)^c$ symmetries allow us to identify them as the fermion mass operator and the Cooper pair operators respectively. The angular variables $\phi$ and $\theta$ are then naturally identify with the $\phi$ and $\theta$ fields of the $U(1)^c_2$ CFT fields discussed in the main text. Finally, we need to consider the constrain of $\hat{h}^\ast \hat{h} = 1$ on $\hat{h}$:

$$\alpha^2 + \beta^2 = 1,$$

$$gg' = g'g.$$ \hspace{1cm} (25)

To treat the fermion mass operator and the Cooper pair operator in equal footing, we should choose $\alpha = \beta = \frac{1}{\sqrt{2}}$. The second equation is naturally satisfy by setting $g = g' \in SU(2)$. Now, we can conclude that the most generic form of $\hat{h}$ that captures the fermion mass operators and Cooper pair operators in equal footing is given by

$$\tilde{h} = \frac{1}{\sqrt{2}}(\cos \phi + i \sin \phi \sigma^y) \otimes g$$

$$+ \frac{1}{\sqrt{2}}(\cos \theta \sigma^z + \sin \theta \sigma^z) \otimes g'.$$ \hspace{1cm} (26)

Using this form of $\tilde{h}$, we can obtain the expression of $h$. We can furthermore transform the basis from $\chi_{L,R}$ back to the complex fermions $\psi_{L,c,\alpha}, \psi^\dagger_{L,c,\alpha}, \psi_{R,c,\alpha}$. After the basis transformation, we see that the two terms in $\tilde{h}$ (that comes from the two terms in $\tilde{h}$) agree respectively with the CFT field expressions of the fermion mass operator Eq. 5 and of the Cooper pair operator Eq. 6 in the main text.