N=3 SUPERSYMMETRIC BORN-INFELD THEORY

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Abstract

We construct an off-shell $N=3$ supersymmetric extension of the abelian $D=4$ Born-Infeld action starting from the action of supersymmetric Maxwell theory in $N=3$ harmonic superspace. A crucial new feature of the $N=3$ super BI action is that its interaction part contains only terms of the order $4k$ in the $N=3$ superfield strengths. The correct component bosonic BI action arises as the result of elimination of auxiliary tensor field which is present in the off-shell $N=3$ vector multiplet in parallel with the gauge field strength. In this new Legendre-type representation, the bosonic BI action is fully specified by a real function of the single variable quartic in the auxiliary tensor field. The generic choice of this function amounts to a wide set of self-dual nonlinear extensions of the Maxwell action. All of them admit an off-shell $N=3$ supersymmetrization.

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1 Introduction

Supersymmetric extensions of the Born-Infeld (BI) and Dirac-Born-Infeld actions play the important role in modern string theory, being essential parts of the worldvolume actions of Dp-branes [1]. A manifestly $N = 1$ supersymmetric $D = 4$ BI action was constructed in Refs. [2]. Later on, it was rederived in [3] within the nonlinear realizations approach as the Goldstone-Maxwell superfield action describing one of possible patterns of the partial spontaneous breaking of $N = 2, D = 4$ supersymmetry down to $N = 1$ supersymmetry. It was interpreted as a manifestly worldvolume supersymmetric form of the static-gauge action of the “space-filling” super D3-brane.

An interesting and challenging problem is to construct off-shell super BI actions with manifest (linearly realized) extended supersymmetries. A direct $N = 2$ supersymmetrization of the $D = 4$ BI action in terms of the $N = 2$ Maxwell superfield strength was constructed in [5]. In [9], a modified action was proposed, such that it can be interpreted as the Goldstone-Maxwell superfield action for the partial supersymmetry breaking $N = 4 \to N = 2$ in $D = 4$ (and, respectively, as a static-gauge form of the super D3-brane action in $D = 6$, with two scalar fields of the vector $N = 2$ multiplet being the transverse brane coordinates). No super BI actions with manifest linearly realized higher $N$ supersymmetries were constructed so far. Only partial results related to the supersymmetrization of the quartic term in the $\alpha'$ expansion of the BI action (the so called Euler-Heisenberg (EH) action) were known. An $N = 4$ supersymmetric extension of this term was found in terms of $N = 1$ superfields [1] and $N = 2$ superfields in the projective [6] and harmonic [7] $N = 2$ superspaces. Accordingly, only $N = 1$ or $N = 2$ supersymmetries in these actions are realized linearly and off-shell. A manifestly $N = 3$ supersymmetric extension of the EH action in the light-cone superspace (lacking manifest Lorentz invariance) was presented in [8].

In this paper we construct an $N = 3$ superextension of the full BI action in $N = 3$ harmonic superspace (HSS). This superspace is a generalization of $N = 2$ HSS [10], and it was introduced in [11] to obtain an off-shell unconstrained superfield formulation of $N = 3$ gauge theory (amounting to $N = 4$ gauge theory on shell). The off-shell action of $N = 3$ gauge theory has an unusual form of superfield Chern-Simons-type term and exists entirely due to a few unique (almost miraculous) peculiarities of $N = 3$ HSS. The opportunity to construct the $N = 3$ BI action also amounts to one of such peculiarities.

The direct $N = 1$ and $N = 2$ superextensions of the BI action are collections of separate superfield terms which supersymmetrize each order of the expansion of the bosonic BI action in powers of the Maxwell field strength. In the $N = 3$ harmonic superfield formalism it is easy to construct the 4th order interaction term from the off-shell $N = 3$ superfield strengths defined by the Grassmann-analytic harmonic gauge potentials. However, the 6th order superfield term does not exist, though the terms of 8th and higher orders exist again. Thus, a naive $N = 3$ completion of the full variety of the bosonic nonlinear terms of the BI action seems impossible.

A surprising way around this difficulty is related to the following unusual feature of the off-shell $N = 3$ HSS formalism [11]. The Grassmann-analytic gauge potentials of $N = 3$
gauge theory contain, besides the physical fields including the standard gauge potential \(A_m\), also an infinite number of the auxiliary fields. Among them there is an independent bispinor field \(H_{\alpha\beta}\). The correct bilinear Maxwell term in the component action arises only after elimination of this field by its algebraic equation of motion. The \(N = 3\) superfield strength contains the combination \(V_{\alpha\beta} = \frac{1}{4}[H_{\alpha\beta} + F_{\alpha\beta}(A)]\) of the auxiliary field and the gauge field strength.

This off-shell structure of the \(N = 3\) harmonic superfields suggests the following way of solving the problem of \(N = 3\) supersymmetrization of the BI theory. In Ref. [13] it was noticed that the vector auxiliary components of the off-shell \(N = 2\) hypermultiplet in \(N = 2\) HSS are capable to generate higher-order terms in the bosonic part of some super \(p\)-brane action already from the 4th order superfield term. This observation indicates that in the presence of tensor auxiliary components the issue of supersymmetric generalization of given nonlinear bosonic action does not amount to a straightforward order-by-order reconstruction.

We use the auxiliary component \(V_{\alpha\beta}\) as the Legendre-type transform variable for the gauge field strength \(F_{\alpha\beta}(A)\). It turns out that this specific Legendre transform of the standard bosonic BI action is determined by a real function \(E\) of the single variable \(a = V^2\bar{V}^2\) where \(V^2 = V^{\alpha\beta}V_{\alpha\beta}\). The problem of \(N = 3\) supersymmetrization is then reduced to the construction of the superfield terms of the order 4\(k\) in the auxiliary field \(V_{\alpha\beta}\). All these terms can be constructed as the appropriate powers of the off-shell \(N = 3\) superfield strengths and their spinor derivatives in the framework of the analytic subspace of \(N = 3\) HSS. Thus an off-shell \(N = 3\) BI action proves to exist despite the absence of the 6th order self-interaction superfield terms. A generic function \(E(a)\) exhausts the complete set of the \(SO(2)\) self-dual nonlinear extensions of the Maxwell action, the BI one being a special representative of them. All such actions can be off-shell \(N = 3\) supersymmetrized.

## 2 \(N=3\) harmonic superspace

### 2.1 Constraints of \(N=3\) gauge theory in ordinary \(N=3\) superspace

We start by recapitulating basic facts about the formulation of \(N = 3\) supersymmetric gauge theory in the standard \(N = 3\), \(D = 4\) superspace \(R(4|12) = \{z^M\}\),

\[
z^M = (x^{\alpha\beta}, \theta^i, \bar{\theta}^{\dot{i}k}) .
\]  

(2.1)

Here \(i, k \ldots = 1, 2, 3\) are indices of the fundamental representations of the group \(SU(3)\), the \(R\)-symmetry group of the \(N = 3\), \(D = 4\) Poincaré superalgebra.

The algebra of spinor derivatives in \(R(4|12)\) has the form

\[
\{D^k_{\alpha}, D^l_{\beta}\} = 0 , \quad \{D^k_{\alpha}, \bar{D}^{\dot{i}}_{\alpha\dot{\beta}}\} = -4i\delta^k_l \partial_{\alpha\dot{\beta}} \text{ and c.c.} .
\]  

(2.2)

The superfield constraints defining the \(N = 3\) supersymmetric Maxwell theory are the following gauge-covariantized version of these relations:

\[
\{\nabla^i_{\alpha}, \nabla^k_{\alpha}\} = \epsilon_{\alpha\beta} \bar{W}^{ik} , \quad \{\nabla^k_{\alpha}, \nabla^l_{\dot{\beta}}\} = -4i\delta^k_l \nabla_{\alpha\dot{\beta}} \text{ and c.c.} ,
\]  

(2.3)
where
\[ \nabla^i_{\alpha} = D^i_{\alpha} + A^i_{\alpha}(z), \quad \nabla_{\dot{i}\dot{\alpha}} = \bar{D}_{\dot{i}\dot{\alpha}} + \bar{A}_{\dot{i}\dot{\alpha}}(z) \] (2.4)
and \( A^i_{\alpha}(z), \bar{A}_{\dot{i}\dot{\alpha}}(z) \) are the corresponding spinor gauge connections.

The Bianchi identities following from (2.3) produce the constraints on the covariant superfield strengths
\[ D^i_{\alpha} \bar{W}^{kl} + D^k_{\alpha} \bar{W}^{il} = 0, \] (2.5)
\[ \bar{D}_{\dot{i}\dot{\alpha}} \bar{W}^{kl} = \frac{1}{2} (\delta^k_i \bar{D}_{\dot{j}\dot{\alpha}} \bar{W}^{jl} - \delta^l_i \bar{D}_{\dot{j}\dot{\alpha}} \bar{W}^{jk}) \text{ and c.c.} \] (2.6)

They can be shown to reduce the component fields content of the superfield strengths to the on-shell \( N = 3 \) vector multiplet [14].

2.2 Off-shell gauge superfields in \( N=3 \) harmonic superspace

The \( SU(3)/U(1) \times U(1) \) harmonic superspace has been introduced in [11] to construct an off-shell unconstrained superfield formulation of \( N = 3 \) gauge theories. We quote our conventions for coordinates and derivatives in \( N = 3 \) HSS in the Appendix A (they are in essence the same as in ref. [15]). Here we recall the basic elements of abelian \( N = 3 \) gauge theory in \( N = 3 \) HSS using this notation.

The fundamental objects of the abelian \( N = 3 \) gauge theory are three harmonic gauge potentials living as unconstrained superfields on the \( (4+6|8) \)-dimensional analytic subspace \( H(4+6|8) = \{ \zeta, u \} \) of \( N = 3 \) HSS
\[ V^1_2(\zeta, u) \equiv V^{(2,-1)}(\zeta, u), \quad V^1_3(\zeta, u) \equiv V^{(1,1)}(\zeta, u), \quad V^2_3 \equiv V^{(-1,2)}(\zeta, u), \]
\[ V^1_2 = -(\widetilde{V^2_3}), \quad V^1_3 = (\widetilde{V^1_3}). \] (2.7)

The definition of the generalized conjugation \( \sim \) preserving \( N = 3 \) Grassmann harmonic analyticity and the precise content of the analytic coordinate set \( \{ \zeta, u \} \) are given in Appendix A. We employed here the double notation in order to emphasize a contact with the original paper [11].

The potentials undergo abelian gauge transformations with a real analytic parameter \( \lambda(\zeta, u) \):
\[ \delta V^1_2 = iD^1_2 \lambda, \quad \delta V^1_3 = iD^1_3 \lambda, \quad \delta V^2_3 = iD^2_3 \lambda. \] (2.8)

The potential \( V^1_3 \) can be consistently expressed in terms of the two remaining ones by imposing the conventional constraint
\[ \hat{V}^1_3 \equiv D^1_2 V^2_3 - D^2_3 V^1_2. \] (2.9)

Below we shall use the off-shell formalism with such composite potential \( \hat{V}^1_3 \).

The free \( N = 3 \) gauge theory action has the following form:
\[ S_2(V^1_2, V^2_3) = -\frac{1}{4f^2} \int d\zeta^{(33)}(u) \left[ V^2_3 D^1_3 V^1_2 + \frac{1}{2}(D^1_2 V^2_3 - D^2_3 V^1_2)^2 \right], \] (2.10)

where the analytic superspace integration measure \( d\zeta^{(33)}(u) = d^4x d^8\theta^{(33)}(u) \) is defined in (A.13) and we have introduced the coupling constant \( f \) of dimension \( -2 \), so that \([V^1_2] = -2\) and the
gauge field strength is dimensionless. This convention will turn out useful when constructing nonlinear extensions of (2.10).

Besides an infinite number of gauge components accounted for by the gauge freedom (2.8), the gauge potentials possess an infinite number of the auxiliary field components. The latter disappear only on the mass shell defined by the free equations of motion following from (2.10):

$$V_{2}^{11} = D_{3}^{1}V_{2}^{1} - D_{2}^{1}V_{3}^{1} = 0 , \quad V_{3}^{12} = D_{3}^{2}V_{3}^{1} - D_{1}^{1}V_{3}^{2} = 0 . \quad (2.11)$$

These equations, being a sort of harmonic zero-curvature conditions, imply the on-shell harmonic connections through the real non-analytic bridge superfield $v$

$$V_{K}^{I}(v) = iD_{K}^{I}v . \quad (2.12)$$

This representation, together with the $N = 3$ Grassmann analyticity conditions for the potentials, can be used to demonstrate [11] that (2.11) are equivalent to the original $R(4|12)$ constraints (2.9), (2.6).

For our further purposes it will be important to know the full structure of the bosonic $SU(3)$ singlet sector in the component expansion of the off-shell analytic potentials $V_{2}^{1}$ and $V_{3}^{2}$ in the WZ gauge. A simple analysis yields

$$v_{2}^{1} = \theta_{2}^{\alpha} \bar{\theta}^{\dot{\alpha}}A_{\alpha \dot{\alpha}} + i(\theta_{2}^{1})^{2}\bar{\theta}^{1(\dot{\alpha} \bar{\theta}^{2})\dot{H}_{\dot{\alpha}}} + i(\theta_{2}^{2})(\bar{\theta}^{1\dot{\theta}^{2}}C , \quad (2.13)$$

$$v_{3}^{2} = \theta_{3}^{\alpha} \bar{\theta}^{2\dot{\alpha}}A_{\alpha \dot{\alpha}} - i(\theta_{2}^{1})^{2}\bar{\theta}^{(\dot{\alpha} \bar{\theta}^{2})2}H_{\dot{\alpha} \beta} - i(\theta_{2}^{1}(\dot{\theta}^{1\dot{\theta}^{2}}C , \quad (2.14)$$

$$\dot{v}_{3}^{1} = 2\theta_{2}^{\alpha} \bar{\theta}^{1\dot{\alpha}}A_{\alpha \dot{\alpha}} - 2i(\bar{\theta}^{1\dot{\theta}^{2}}\theta_{2}^{1(\dot{\alpha} \bar{\theta}^{2})2}(H - F)_{\alpha \dot{\alpha}} + 2i(\theta_{2}^{1}(\dot{\theta}^{1\dot{\theta}^{2}}(H - F)_{\dot{\alpha} \beta})$$

$$+ 4i(\theta_{2}^{1}(\dot{\theta}^{1\dot{\theta}^{2}}C + (\theta_{2}^{1})^{2}(\bar{\theta}^{1\dot{\theta}^{2}}\dot{\theta}^{2\dot{\alpha}}\dot{H}_{\dot{\alpha} \dot{\beta}} + \dot{\theta}^{2\dot{\alpha}}\dot{H}_{\dot{\alpha} \dot{\beta}} , \quad (2.15)$$

where $H_{\alpha \dot{\alpha}} = H_{\dot{\delta} \alpha}$, $C = C$ and the spinor representation for the gauge field strength was used

$$F_{\alpha \dot{\alpha}}(A) \equiv \partial_{\alpha}^{\dot{\alpha}}A_{\alpha \dot{\alpha}} , \quad F_{\dot{\alpha} \alpha}(A) \equiv \partial_{\dot{\alpha}}^{\alpha}A_{\alpha \dot{\alpha}} , \quad (2.16)$$

$$\partial_{\alpha}A_{\beta} + \partial_{\beta}A_{\alpha} = 0 . \quad (2.17)$$

We observe that the auxiliary dimensionless symmetric tensor and scalar fields $H_{\alpha \dot{\alpha}}$ and $C$ are present in the off-shell $SU(3)$ singlet sector in parallel with the gauge potential $A_{\alpha \dot{\alpha}}$ and its covariant field strength. The presence of these auxiliary fields is the pivotal difference of the off-shell vector $N = 3$ multiplet from the $N = 2$ one arising from the WZ gauge of the analytic harmonic $N = 2$ gauge connection [10]. As we shall see soon, the fields $H_{\alpha \dot{\alpha}}$, $H_{\dot{\alpha} \alpha}$ play a crucial role in constructing $N = 3$ supersymmetric BI action.

It should be emphasized that the above $SU(3)$ singlet fields are components of the infinite-dimensional off-shell $N = 3$ gauge supermultiplet. But all other bosonic fields in the WZ gauge have a non-trivial $SU(3)$ assignment. In what follows we shall be interested in the pure Maxwell part of the component form of the action (2.11), so for us it will be enough to know just the $SU(3)$ singlet sector (2.13) - (2.15) of the analytic gauge potentials.
It is easy to explicitly evaluate contributions of two terms in the action (2.10) to the \( SU(3) \) singlet part of the component form of this action (up to surface terms)

\[
-\frac{1}{4f^2} \int d\zeta \nabla^2 v_3^2 D_3^1 v_2^1 \Rightarrow -\frac{1}{8f^2} \int d^4 x (HF + \bar{H}\bar{F}) ,
\]

\[
-\frac{1}{8f^2} \int d\zeta \nabla^2 (\bar{v}_3^1)^2 \Rightarrow \frac{1}{16f^2} \int d^4 x \left[ H^2 + \bar{H}^2 - 4 (HF + \bar{H}\bar{F}) + F^2 + \bar{F}^2 + 8C^2 \right] ,
\]

where the definition (A.13) and the following notation

\[
F^2 = F^{\alpha\beta} F_{\alpha\beta} , \quad H^2 = H^{\alpha\beta} H_{\alpha\beta} , \quad FH = F^{\alpha\beta} H_{\alpha\beta} ,
\]

\[
\bar{F}^2 = \bar{F}^{\dot{\alpha}\dot{\beta}} \bar{F}_{\dot{\alpha}\dot{\beta}} , \quad \bar{H}^2 = \bar{H}^{\dot{\alpha}\dot{\beta}} \bar{H}_{\dot{\alpha}\dot{\beta}} , \quad \bar{F}\bar{H} = \bar{F}^{\dot{\alpha}\dot{\beta}} \bar{H}_{\dot{\alpha}\dot{\beta}}
\]

were used.

Thus the gauge field part of the off-shell super \( N = 3 \) Maxwell component Lagrangian is

\[
L_2(F, H, C) = \frac{1}{16f^2} \left[ H^2 + \bar{H}^2 - 6 (\bar{H}\bar{F} + HF) + F^2 + \bar{F}^2 + 8C^2 \right] .
\]

Eliminating the auxiliary fields \( H_{\alpha\beta} , \bar{H}_{\dot{\alpha}\dot{\beta}} , C \) by their algebraic equations of motion

\[
H_{\alpha\beta} = 3 F_{\alpha\beta} , \quad \bar{H}_{\dot{\alpha}\dot{\beta}} = 3 \bar{F}_{\dot{\alpha}\dot{\beta}} , \quad C = 0 ,
\]

we arrive at the standard Maxwell action

\[
L_2(F) = -\frac{1}{2f^2} (F^2 + \bar{F}^2) = -\frac{1}{4f^2} \mathcal{F}^{mn} \mathcal{F}_{mn} ,
\]

where \( \mathcal{F}_{mn} = \partial_m A_n - \partial_n A_m \) and the precise relation of the spinor and vector notations is given in Appendix B. We shall see soon that the appropriate starting point for the construction of \( N = 3 \) BI action is just the \( N = 3 \) supersymmetry-inspired form (2.22) of the Maxwell action, with the properly redefined tensor auxiliary fields.

To this end, let us construct the analytic superfield strengths for the \( N = 3 \) gauge theory. Like in the \( N = 2 \) gauge theory in \( N = 2 \) HSS \( ^{16} \), one firstly defines the non-analytic abelian connections via the harmonic zero-curvature equations

\[
D_2^1 V_1^2 - D_1^2 V_1^1 = 0 , \quad D_3^2 V_2^3 - D_2^3 V_3^2 = 0 ,
\]

where \( V_2^3 = -\bar{V}_1^2 , \Delta V_2^3 = iD_2^3 \lambda , \Delta V_1^2 = iD_1^2 \lambda \) and the explicit form of the harmonic derivatives is given in Appendix A. Then the mutually conjugated Grassmann-analytic off-shell superfield strengths of the \( N = 3 \) Maxwell theory are constructed as follows \( ^{16} \):

\[
W_{23} = \frac{1}{4} (\bar{D}_3)^2 V_2^3 , \quad \bar{W}^{12} = -\frac{1}{4} (D_1)^2 V_1^2 ,
\]

i.e., quite analogously to the construction of superfield strengths in \( N = 2 \) HSS.
These off-shell superfield strengths satisfy the following Grassmann analyticity conditions:
\[ \overline{D}_{2a} W_{23} = \overline{D}_{3a} W_{23} = D_a^1 W_{23} = 0, \quad D^1_a \overline{W}^{12} = D^2_a \overline{W}^{12} = \overline{D}_{3a} \overline{W}^{12} = 0 \quad (2.27) \]
and harmonic differential conditions
\[ D^2_3 W_{23} = 0, \quad D^1_2 W^{12} = 0. \quad (2.28) \]
They can be checked using the analyticity of the basic gauge potentials, the relations (2.25), and the properties like
\[ D^2_1 f_2 = 0 \Rightarrow f_2 = 0, \]
which is valid for any \( SU(3) \) harmonic function with at least one lower-case index 2 (such relations can be easily proved in the central basis, where harmonic derivatives are short). It is also straightforward to check gauge invariance of the superfield strengths (2.26).

Free equations of motion for the harmonic potentials (2.11) yield the on-shell harmonic-analyticity equations for the superfield strengths \[15\]
\[ V^I_K = i D^I_K v \Rightarrow D^2_1 W_{23} = 0, \quad D^2_3 \overline{W}^{12} = 0. \quad (2.29) \]
Together with (2.28), these equations imply that on shell and in the central basis
\[ W_{23} = u^k_2 u^l_3 W_{kl}(z^M), \quad \overline{W}^{12} = u^1_k u^2_l \overline{W}^{kl}(z^M), \quad (2.30) \]
where the superfield strengths \( W_{kl}, \overline{W}^{kl} \) satisfy the original constraints (2.3), (2.6). An important consequence of the dynamical harmonic constraints (2.29) are the following on-shell conditions:
\[ (D^2)^2 W_{23} = (D^3)^2 W_{23} = (D^1_1)^2 W_{23} = (D^3)^2 \overline{W}^{12} = (D^1_1)^2 \overline{W}^{12} = (D^2_3)^2 \overline{W}^{12} = 0 \quad (2.31) \]

We shall need the full off-shell \( SU(3) \) singlet component structure of \( W_{23}, \overline{W}^{12} \). In order to find it one should firstly solve the harmonic equations (2.23) for the corresponding parts of the non-analytic harmonic connections \((v_1^1, v_2^1)\), assuming the ansatz (2.13), (2.14) for the analytic connections \( V_2^1, V_3^1 \). Using the property \((D^1_2)^2 v_1^1 = 0\), one obtains the following exact representation for \( v_2^1 \) in terms of \( v_1^1 \):
\[ v_2^1 = \frac{1}{2} (D^2_1)^2 v_2^1 - \frac{1}{12} (D^1_1)^3 D^1_2 v_1^1. \quad (2.32) \]
The expression for \( v_3^1 \) can be obtained by the \( \sim \) conjugation of \( v_1^1 \). Substituting (2.32) into (2.26), we get the corresponding representation for \( \overline{w}^{12} \):
\[ \overline{w}^{12} = -\frac{1}{4} (D^1)^2 v_1^2 = -\frac{1}{4} (D^2)^2 v_2^1 + \frac{1}{8} (D^3)^2 D^1_2 D^1_2 v_1^1. \quad (2.33) \]
Once again, \( w_{23} \) can be recovered by the \( \sim \) conjugation.
The relevant explicit expressions are

\[
v_1^2 = -\theta_1^a \bar{\theta}^{\gamma \delta} A_{a \gamma} - i(\bar{\theta}^2)^a \theta_1^a (F_{\alpha \beta} + \epsilon_{\alpha \beta} C) + i(\bar{\theta}_1)^2 \bar{\theta}^{1 \alpha} \bar{\theta}^{2 \beta} (H_{\alpha \beta} + \bar{F}_{\alpha \beta}) \\
- (\bar{\theta}^2)^2(\theta_1)^2 \theta_2^a \bar{\theta}^{1 \beta} \bar{\theta}^2_{\alpha} (H_{\alpha \beta} + \bar{F}_{\alpha \beta}) , \quad (2.34)
\]

\[
v_2^3 = -\theta_2^a \bar{\theta}^{\gamma \delta} A_{a \gamma} + i(\bar{\theta}_2)^a \theta_2^a \bar{\theta}^{1 \beta} \bar{\theta}^{2 \alpha} (F_{\gamma \delta} - \epsilon_{\gamma \delta} C) - i(\bar{\theta}_2)^2 \theta_2^a \theta_3^a \bar{\theta}^{1 \gamma} \bar{\theta}^{2 \delta} (H_{\gamma \delta} + F_{\gamma \delta}) \\
+ (\bar{\theta}^2)^2(\theta_2)^2 \bar{\theta}^{1 \beta} \bar{\theta}^{2 \alpha} \bar{\theta}_{\gamma} (H_{\gamma \delta} + F_{\gamma \delta}) , \quad (2.35)
\]

\[
w_{23} = i\theta_2^a \theta_3^b (H_{\alpha \beta} + F_{\alpha \beta} - \theta_2)^2 \theta_3^a \bar{\theta}^{1 \beta} \bar{\theta}^{2 \alpha} (H_{\alpha \beta} + F_{\alpha \beta}) , \quad (2.36)
\]

\[
\bar{w}^{12} = i\bar{\theta}_1^{\gamma} \bar{\theta}^{\gamma \delta} (\bar{H}_{\alpha \beta} + \bar{F}_{\alpha \beta}) + (\bar{\theta}^{1 \gamma} \bar{\theta}^{2 \alpha} \bar{\theta}_{\alpha} (H_{\alpha \beta} + F_{\alpha \beta}) . \quad (2.37)
\]

One can directly check that \( w_{23}, \bar{w}^{12} \) on their own obey the off-shell conditions \((2.27)\) and \((2.28)\).

One observes two distinguished features of \( W_{23} \) and \( W^{12} \). Firstly, they do not include the scalar auxiliary field \( C(x) \) which is present in the free \( N = 3 \) gauge theory action \((2.10), (2.22)\). Secondly, the gauge field strengths appear inside them only in a fixed combination with the tensor auxiliary fields \( H_{\alpha \beta}, \bar{H}_{\alpha \beta} \). Thus, the gauge field strengths can be fully removed from \( W_{23} \) and \( \bar{W}^{12} \) by redefining the auxiliary fields

\[
H_{\alpha \beta} \Rightarrow V_{\alpha \beta} = \frac{1}{4} (H_{\alpha \beta} + F_{\alpha \beta}) , \quad \bar{H}_{\alpha \beta} \Rightarrow \bar{V}_{\alpha \beta} = \frac{1}{4} (\bar{H}_{\alpha \beta} + \bar{F}_{\alpha \beta}) , \quad (2.38)
\]

\[
W_{23} = 4i \theta_2^a \theta_3^b V_{\alpha \beta} + \ldots , \quad \bar{W}^{12} = 4i \bar{\theta}_1^\alpha \bar{\theta}^{2 \beta} V_{\alpha \beta} + \ldots , \quad (2.39)
\]

\[
(W_{23})^2 = 4(\theta_2)^2(\theta_3)^2 V^2 + \ldots , \quad (\bar{W}^{12})^2 = 4(\bar{\theta}_1)^2(\bar{\theta}^2)^2 \bar{V}^2 + \ldots . \quad (2.40)
\]

The free Maxwell Lagrangian \((2.22)\) (with \( C = 0 \)), being rewritten through \( V_{\alpha \beta}, \bar{V}_{\alpha \beta} \), reads

\[
L_2(F, H, 0) \equiv B_2(F, V) = \frac{1}{f^2} [V^2 + \bar{V}^2 - 2 (VF + \bar{V} \bar{F}) + \frac{1}{2} (F^2 + \bar{F}^2)] . \quad (2.41)
\]

The algebraic equations of motion for \( V_{\alpha \beta}, \bar{V}_{\alpha \beta} \) giving rise to the standard Lagrangian \((2.24)\) are simply

\[
V_{\alpha \beta} = F_{\alpha \beta} , \quad \bar{V}_{\alpha \beta} = \bar{F}_{\alpha \beta} . \quad (2.42)
\]

From the above discussion one infers two important properties of the off-shell description of \( N = 3 \) gauge theory in \( N = 3 \) HSS having no direct analogs in the \( N = 1 \) and \( N = 2 \) cases. First, the free Maxwell component Lagrangian appears in the unusual forms \((2.22)\) or \((2.41)\), while its standard form is recovered only after eliminating the auxiliary fields \( V_{\alpha \beta}, \bar{V}_{\alpha \beta} \) by their linear algebraic equations of motion \((2.42)\). Secondly, the off-shell superfield strengths contain just these tensor auxiliary fields, but not the ordinary gauge field strengths \( F_{\alpha \beta}, \bar{F}_{\alpha \beta} \).

These surprising features suggest a non-standard approach to constructing nonlinear and non-polynomial superextensions of the off-shell \( N = 3 \) Maxwell theory. One should start by modifying \((2.22)\) by proper terms which are nonlinear (and/or non-polynomial) in the auxiliary fields \( V_{\alpha \beta}, \bar{V}_{\alpha \beta} \), such that nonlinearities in \( F_{\alpha \beta}, \bar{F}_{\alpha \beta} \) are regained as the result of

\[\footnote{This field enters some other superfield strengths which are of no relevance for our purpose of constructing a minimal \( N = 3 \) extension of the BI action.} \]
eliminating these auxiliary fields by their nonlinear equations of motion. Then one can hope to $N = 3$ supersymmetrize the terms nonlinear in $V_{\alpha\beta}, \bar{V}_{\dot{\alpha}\dot{\beta}}$ with the help of the above superfield strengths $W_{\alpha\beta}, \bar{W}_{\dot{\alpha}\dot{\beta}}$ which contain just these auxiliary fields. In the next Sections we shall show that in this way the BI action as well as a wide class of self-dual extensions of the Maxwell action can be $N = 3$ supersymmetrized.

3 New Legendre-type representation of the Born-Infeld action and self-dualities

Let us introduce the notation

$$\varphi = F^2, \quad \bar{\varphi} = \bar{F}^2, \quad X(\varphi, \bar{\varphi}) \equiv (\varphi + \bar{\varphi}) + (1/4)(\varphi - \bar{\varphi})^2.$$  \hspace{1cm} (3.1)

In terms of these variables the standard BI Lagrangian has the following form:

$$L_{BI}(F, \bar{F}) = \frac{1}{f^2} \left[ 1 - \sqrt{-\text{det}(\eta_{mn} + \mathcal{F}_{mn})} \right] \equiv \frac{1}{f^2} \left[ 1 - Q(\varphi, \bar{\varphi}) \right],$$  \hspace{1cm} (3.2)

$$Q(\varphi, \bar{\varphi}) = \sqrt{1 + X} = 1 + \frac{1}{2}X - \frac{1}{8}X^2 + \frac{1}{16}X^3 - \frac{5}{128}X^4 + \ldots$$  \hspace{1cm} (3.3)

$$= 1 + \frac{1}{2}(\varphi + \bar{\varphi}) - \frac{3}{4}(\varphi\bar{\varphi}/2 - \varphi\bar{\varphi}/2) + O(\varphi^5)$$

(one should make use of Eqs. (3.1) of the Appendix B).

As was already mentioned, our ultimate aim is to find a nonlinear extension of the free Maxwell Lagrangian in the $N = 3$ supersymmetry-inspired form $B_2(F, V)$, eq. (2.41), such that this extension becomes the BI Lagrangian (3.2) after eliminating the auxiliary fields $V_{\alpha\beta}, \bar{V}_{\dot{\alpha}\dot{\beta}}$ by their algebraic equations of motion. By Lorentz covariance, such a nonlinear Lagrangian should have the following general form:

$$B(F, V) = B_2(F, V) + \frac{1}{f^2}E(V^2, \bar{V}^2)$$

$$= \frac{1}{f^2}[\nu + \bar{\nu} - 2(VF + \bar{V}\bar{F}) + \frac{1}{2}(\varphi + \bar{\varphi}) + E(\nu, \bar{\nu})],$$  \hspace{1cm} (3.4)

where $\nu \equiv V^2, \bar{\nu} \equiv \bar{V}^2$ and $E(\nu, \bar{\nu})$ is a real function to be determined. The nonlinear generalization of the free equations (2.42) reads

$$\partial B(F, V)/\partial V_{\alpha\beta} = 0 \Rightarrow V_{\alpha\beta} = F_{\alpha\beta} \left[ 1 + \frac{1}{f^2}E_\nu(\nu, \bar{\nu})/\partial \nu \right] = F_{\alpha\beta}N(\nu, \bar{\nu})$$  \hspace{1cm} (3.5)

(with its conjugate). Further, (3.5) implies

$$\nu = \varphi \bar{N}^2(\nu, \bar{\nu}), \quad \bar{\nu} = \bar{\varphi} \bar{N}^2(\nu, \bar{\nu}) \Rightarrow \nu = \nu(\varphi, \bar{\varphi}), \quad \bar{\nu} = \bar{\nu}(\varphi, \bar{\varphi}),$$  \hspace{1cm} (3.6)

$$N(\nu(\varphi, \bar{\varphi}), \bar{\nu}(\varphi, \bar{\varphi})) \equiv G(\varphi, \bar{\varphi}).$$  \hspace{1cm} (3.7)

The function $G(\varphi, \bar{\varphi})$ can be found from the basic requirement that (3.4) coincides with (3.2) after elimination of $V_{\alpha\beta}, \bar{V}_{\dot{\alpha}\dot{\beta}}$

$$\nu + \bar{\nu} - 2(VF + \bar{V}\bar{F}) + \frac{1}{2}(\varphi + \bar{\varphi}) + E(\nu, \bar{\nu}) = f^2L_{BI}(\varphi, \bar{\varphi}) = 1 - Q(\varphi, \bar{\varphi}).$$  \hspace{1cm} (3.8)
After substituting the expression (3.7) and its conjugate for $V_{\alpha \beta}, \bar{V}_{\dot{\alpha} \dot{\beta}}$ and making use of the definition (3.7), this condition can be rewritten as

$$\frac{1}{2} (\varphi + \bar{\varphi}) - 2(\varphi G + \bar{\varphi} \bar{G}) + \nu + \bar{\nu} + E(\nu, \bar{\nu}) = f^2 L_{BI}(\varphi, \bar{\varphi}) = 1 - Q(\varphi, \bar{\varphi}) .$$  

(3.9)

Differentiating it with respect to $\varphi$ and using the relations

$$\frac{\partial \nu}{\partial \varphi} = G^2 + 2\varphi G \frac{\partial G}{\partial \varphi}, \quad \frac{\partial \bar{\nu}}{\partial \varphi} = 2\bar{\varphi} \bar{G} \frac{\partial \bar{G}}{\partial \varphi},$$

which follow from (3.6), one obtains the simple expression for $G(\varphi, \bar{\varphi})$:

$$G(\varphi, \bar{\varphi}) = \frac{1}{2} \left( 1 - \frac{1}{2} f^2 \frac{\partial L_{BI}}{\partial \varphi} \right) = \frac{1}{2} \left( 1 + \frac{1}{Q(\varphi, \bar{\varphi})} \left[ 1 + \frac{1}{2} (\varphi - \bar{\varphi}) \right] \right) .$$  

(3.10)

A useful corollary of this representation is

$$G + \bar{G} = 1 + \frac{1}{Q} .$$  

(3.11)

The relation inverse to (3.10) reads

$$\varphi = 2 \bar{G} \frac{1 - \bar{G}}{1 - (G + \bar{G})^2} .$$  

(3.12)

Our aim is to find $E$ as a function of the variables $\nu = V^2, \bar{\nu} = \bar{V}^2$. As the first step, one expresses $\nu, \bar{\nu}$ in terms of $G$ and $\bar{G}$, using (3.6) and (3.12)

$$\nu = \varphi G^2 = 2 \bar{G} G^2 \frac{1 - \bar{G}}{1 - (G + \bar{G})^2} .$$  

(3.13)

Introducing

$$t \equiv \frac{G \bar{G}}{1 - (G + \bar{G})^2} ,$$  

(3.14)

one finds that $t$, as a consequence of (3.13), satisfies the following quartic equation:

$$t^4 + t^3 - \frac{1}{4} \nu \bar{\nu} = 0 .$$  

(3.15)

It allows one to express $t$ in terms of $a \equiv \nu \bar{\nu}$

$$t(a) = -1 - a \frac{a^2}{4} + \frac{3a^2}{16} - \frac{15a^3}{64} + \ldots .$$  

(3.16)

Of course, one can write a closed expression for $t(a)$ as the proper solution of (3.15), but we do not present it here in view of its complexity. The next (and last) step is to find $E(\nu, \bar{\nu})$. Taking into account the explicit expressions (3.12), (3.13) and (3.11) and substituting all
this into (3.9), one finally finds a simple expression for $E(\nu, \bar{\nu})$ through the real variable $t$ (and much more involved expression in terms of $a = \nu \bar{\nu}$):

$$E(a) \equiv E[t(a)] = 2[2t^2(a) + 3t(a) + 1] = \frac{a}{2} - \frac{a^2}{8} + \frac{3a^3}{32} + \ldots .$$  \hspace{1cm} (3.17)

The remarkable property of $E$ which is most important for further consideration is that it is a function of the single real variable $a = V^2 \bar{V}^2$ which is quartic in the auxiliary fields. Thus only terms $\sim V^{2n} \bar{V}^{2n}$ can appear in the power expansion of $E(V^2, \bar{V}^2)$.

Using the representation (3.17) we can find

$$\frac{\partial E}{\partial \nu} = 2 (3 + 4t) \frac{\partial t}{\partial \nu} .$$

On the other hand, from eq. (3.15),

$$\frac{\partial t}{\partial \nu} = \frac{\bar{\nu}}{4 (3 + 4t)^2} = \frac{1}{2 (3 + 4t)G} ,$$

where we have used eqs. (3.13) and (3.14). Thus

$$\frac{\partial E}{\partial \nu} = G^{-1} - 1$$  \hspace{1cm} (3.18)

in agreement with (3.3) and (3.7).

Finally, let us show that the substitution of an arbitrary function of the variables $t$ or $a = V^2 \bar{V}^2$ for $E(V^2, \bar{V}^2)$ into (3.4) gives rise, upon eliminating $V_{\alpha\beta}, \bar{V}_{\dot{\alpha}\dot{\beta}}$, to a general set of self-dual nonlinear extensions of the Maxwell Lagrangian.

As the first step in proving this, let us note that for an arbitrary nonlinear extension $L(F, \bar{F})$ of the Maxwell action one can always pick up the appropriate function $E(V^2, \bar{V}^2)$ such that

$$B(F, V(P)) = L(F, \bar{F}) .$$  \hspace{1cm} (3.19)

Further, after adding a Lagrange multiplier term to such general $B(F, V)$,

$$B(F, V) \Rightarrow \tilde{B}(F, V, P) = B(F, V) + \frac{i}{f^2} \left( P_{\alpha\beta} F^{\alpha\beta} - \bar{P}_{\dot{\alpha}\dot{\beta}} \bar{F}^{\dot{\alpha}\dot{\beta}} \right) ,$$  \hspace{1cm} (3.20)

$$P_{\alpha\beta}(B) \equiv \partial_{(\alpha} B_{\beta)\dot{\beta}} , \quad \bar{P}_{\dot{\alpha}\dot{\beta}}(B) \equiv \partial_{(\dot{\alpha}} B_{\dot{\beta}\beta)} ,$$  \hspace{1cm} (3.21)

it becomes possible to reproduce the Bianchi identity (2.17) for the field strengths $F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}}$ and, hence, their standard representation through the Maxwell potential $A_{\alpha\dot{\alpha}}$ (Eqs. (2.16)), by varying (3.20) with respect to the unconstrained Lagrange multiplier $B_{a\dot{a}}$ (the dual gauge potential). On the other hand, since $F_{\alpha\beta}, \bar{F}_{\dot{\alpha}\dot{\beta}}$ are now off-shell unconstrained, one can trade them for the dual gauge field strengths $P_{\alpha\beta}, \bar{P}_{\dot{\alpha}\dot{\beta}}$ using their algebraic equations of motion:

$$F_{\alpha\beta} = 2V_{\alpha\beta} - iP_{\alpha\beta} \quad \text{and c.c .}$$

It is easy to show that after substituting this expression back into $\tilde{B}(F, VP)$ the latter becomes

$$\tilde{B}(F, V, P) \Rightarrow B_2(P, -iV) + \frac{1}{f^2} E(V^2, \bar{V}^2) \equiv B_2(P, \bar{V}) + \frac{1}{f^2} E(-\bar{V}^2, -\bar{V}^2) .$$  \hspace{1cm} (3.22)
The self-duality means that the final Lagrangian (after elimination of $V_{\alpha\beta}$ and $\bar{V}_{\dot{\alpha}\dot{\beta}}$) has the same form in terms of $P_{\alpha\beta}$ and $\bar{P}_{\dot{\alpha}\dot{\beta}}$ as the original one \((3.19)\) in terms of $F_{\alpha\beta}$ and $\bar{F}_{\dot{\alpha}\dot{\beta}}$. From \((3.22)\) it is clear that the necessary and sufficient condition for such a self-duality is that the function $E$ is even with respect to its both arguments,

$$E(V^2, \bar{V}^2) = E(-V^2, -\bar{V}^2). \quad (3.23)$$

Obviously, this is valid for an arbitrary function of $a = V^2 \bar{V}^2$, which proves the self-duality of the corresponding class of nonlinear actions, including the BI action.

The “discrete” self-duality just discussed is sometimes called “self-duality by Legendre transformation” \([19]\). There exists another type of self-duality which can be called $SO(2)$-duality. It holds essentially on shell and can be formulated as the property of covariance with respect to $SO(2)$ transformations mixing up the Bianchi identities for $F_{\alpha\beta}$, $\bar{F}_{\dot{\alpha}\dot{\beta}}$ with the equations of motion associated with the Lagrangian $L(F, \bar{F})$. The differential condition which singles out the Lagrangians $L(F, \bar{F})$ revealing such a type of self-duality is as follows (for details, see \([18, 19]\)):

$$F^2 - \bar{F}^2 + P^2 - \bar{P}^2 = 0, \quad (3.24)$$

where now

$$P_{\alpha\beta} \equiv i f^2 \frac{\partial L(F, \bar{F})}{\partial F_{\alpha\beta}}, \quad P^2 \equiv P_{\alpha\beta} P_{\alpha\beta}, \quad \bar{P}^2 \equiv \bar{P}_{\dot{\alpha}\dot{\beta}} \bar{P}_{\dot{\alpha}\dot{\beta}}. \quad (3.25)$$

In order to find the restrictions which this kind of self-duality imposes on the function $E(V^2, \bar{V}^2)$, let us again start from the representation \((3.19)\), with $L(F, \bar{F})$ being general and unspecified for the moment. Differentiating this identity with respect to $F_{\alpha\beta}$ and taking account of the relations \((3.8), (3.9)\) with $L_{\text{BI}}$ instead of $L_{\text{BI}}$, as well as of the general relation \((3.18)\), one obtains

$$P_{\alpha\beta}(F) = -2i V_{\alpha\beta}(F) + i F_{\alpha\beta}. \quad (3.26)$$

Substituting this back into \((3.24)\) brings the latter into the form

$$V^2 - \bar{V}^2 - (VF - \bar{V} \bar{F}) = 0. \quad (3.27)$$

Using the same general relations once again, after some algebra one finds that the latter condition is reduced to the following linear differential constraint on the function $E(\nu, \bar{\nu})$:

$$\nu \frac{\partial E(\nu, \bar{\nu})}{\partial \nu} = \bar{\nu} \frac{\partial E(\nu, \bar{\nu})}{\partial \bar{\nu}}. \quad (3.28)$$

After passing to the variables $a = \nu \bar{\nu}$, $b = \nu + \bar{\nu}$, this condition becomes

$$\frac{\partial E(a, b)}{\partial b} = 0 \quad \Rightarrow \quad E = E(a) = E(\nu \bar{\nu}). \quad (3.29)$$

Thus we come to the surprising result that the whole class of nonlinear extensions of the Maxwell action admitting the $SO(2)$ self-duality is parametrized by an arbitrary real function of one argument $E(\nu \bar{\nu})$ (the only natural restriction is $E(0) = 0$, which implies the standard Maxwell action in the limit of vanishing self-interaction).
A similar conclusion has been made in [20] in a different context. The authors of [20] have reduced the $SO(2)$ self-duality equation (3.24) to a nonlinear differential equation for $L(F, \bar{F})$ and have found that its perturbative solution is specified by some arbitrary function of the single variable $\sqrt{F^2 \bar{F}^2}$ (in our notation). Our consideration clearly demonstrates that the class of actions which reveal the “discrete” self-duality is wider than that of the $SO(2)$ self-dual ones: the functions (3.29) form a subclass of (3.23). The BI action obviously respects both types of self-duality.

4 N=3 Born-Infeld action and its generalizations

The problem of constructing a manifestly $N = 3$ supersymmetric superfield action which would yield, in the bosonic sector, the previously elaborated $F, V$ form of the BI action amounts to setting up a collection of superfield monomials which extend the appropriate terms in the power expansion of the function $E(V^2 \bar{V}^2)$ defined in (3.17). This procedure is in a sense analogous to the construction of the $N = 1$ and $N = 2$ superfield BI actions [2, 4] (see also [1]). An essential difference is, however, that in our case we are led to supersymmetrize the powers of the auxiliary fields $V_{\alpha \beta}, \bar{V}_{\dot{\alpha} \dot{\beta}}$, while in the $N = 1$ and $N = 2$ cases the powers of $F_{\alpha \beta}, \bar{F}_{\dot{\alpha} \dot{\beta}}$ in the expansion of the standard form (3.2) of the BI action are supersymmetrized.

The rescaled $N = 3$ superfield strengths have the following dimension:

$$[W_{IK}] = [\bar{W}^{IK}] = -1.$$  (4.1)

The 4th order superfield invariant lives in the same $N = 3$ analytic superspace as the free term (2.10):

$$S_4 = \frac{1}{32f^2} \int dud\zeta (33)_{11}(W_{23})^2(\bar{W}^{12})^2$$

$$= \frac{1}{2f^2} \int d^4x V^2 \bar{V}^2 + \ldots ,$$  (4.2)

where we omitted the fermionic and scalar field terms.

Given the function $E(V^2 \bar{V}^2)$ defined by Eq. (3.17), let us introduce the new function $\hat{E}(V^2 \bar{V}^2)$ by

$$E(V^2 \bar{V}^2) = \frac{1}{2} V^2 \bar{V}^2 \hat{E}(V^2 \bar{V}^2) ,$$  (4.3)

with $\hat{E}(a) = 1 - a/4 + O(a^2)$. Then the whole sequence of higher order terms in the $N = 3$ generalization of the BI-action, including the previous 4th order term, can be written as a closed expression in the analytic superspace,

$$S_E = \frac{1}{32f^2} \int dud\zeta (33)_{11}(W_{23})^2(\bar{W}^{12})^2 \hat{E}(A) .$$  (4.4)

Here $A$ is the following real analytic superfield:

$$A = \frac{1}{24} (D^1)^2(D_3)^2 [D^{2\alpha} W_{12} D^{2\beta} W_{12} \bar{D}_2 \bar{W}^{23} \bar{D}_2 \bar{W}^{23}] = V^2 \bar{V}^2 + \ldots ,$$  (4.5)

$$W_{12} = D_1^3 W_{23} = -4i \theta_1^{(\alpha} \theta_2^{\beta)} V_{\alpha \beta} + \ldots , \quad \bar{W}^{23} = -D_1^3 \bar{W}^{12} .$$  (4.6)
Thus, we have obtained an $N = 3$ generalization of the Born-Infeld action using the off-shell Grassmann-analytic potentials $V_2^1$ and $V_3^2$

$$S_{BI}^{N=3} = S_2 + S_E.$$

The substitution of generic function $\hat{E}(A) = 1 + O(A)$ into (4.4) yields $N = 3$ superextensions of the self-dual nonlinear deformations of Maxwell theory discussed in the previous Section.

Note that one can modify the superfield $A$ in (4.5) by terms containing $(D^2)^2 W_{12}$ or $\bar{(D_2)^2} \bar{W}^{23}$ which vanish on the free mass shell (recall (2.31)). These extra terms do not influence the $SU(3)$ singlet sector of the bosonic action, but can prove to be relevant for implementing additional spontaneously broken symmetries (see [9] for the role of similar terms in the $N = 2$ BI action).

5 Concluding remarks

In this paper we have constructed a minimal $N = 3$ superextension of Born-Infeld theory as the novel non-trivial example of off-shell self-interacting gauge theory in $N = 3$ HSS. Like the standard $N = 3$ gauge theory action, the off-shell $N = 3$ BI action can be written as an integral over the $(4+6|8)$-dimensional analytic subspace of the full $N = 3$ HSS. It yields the bosonic BI action in a new unusual form involving tensor auxiliary fields which are present in the off-shell $N = 3$ gauge multiplet. This $N = 3$ supersymmetry-inspired form of the BI action can be generalized to encompass a wide class of self-dual deformations of the Maxwell action. All such nonlinear actions admit $N = 3$ supersymmetrization.

We conclude with a few remarks and conjectures.

Our nonlinear terms in the $N = 3$ BI-action (4.4) generate higher-order corrections to the free equations (2.11) or (2.29), so in the nonlinear $N = 3$ BI theory one cannot use the standard abelian representations $V_I^I = iD_I^i v$ or (2.30) even on shell. It is an interesting problem to explore how the standard superfield constraints (2.5), (2.6) describing the free on-shell $N = 3$ gauge theory can be generalized to the BI deformation of the latter constructed here.

A closely related problem is as follows. In [9], an $N = 4$ superfield form of the equations of motion of the $N = 4$ super BI theory with the second non-linearly realized $N = 4$ supersymmetry was given. It was derived in the framework of nonlinear realization of the properly central-charge extended $N = 8$ supersymmetry with the unbroken $N = 4$ subgroup. The basic Goldstone $N = 4$ superfield is associated with the central charge generator and it is a generalization of the standard $N = 4$ gauge superfield strength. The $N = 4$ BI equations are a covariantization, with respect to the non-linearly realized $N = 8$ supersymmetry, of the standard superfield constraints of $N = 4$ Maxwell theory and are expected to describe a type II super D3-brane in $D = 10$ in a static gauge. It seems that this approach could be directly extended to the $N = 3$ case. One should start from a nonlinear realization of $N = 6$ supersymmetry, such that $N = 3$ supersymmetry is unbroken and a complex central charge $Z^{ik}$ is present in the anticommutator of the broken and unbroken $N = 3$ supercharges. Then one introduces the Goldstone-Maxwell superfields $W_{ik}(z), \bar{W}^{ik}(z)$ as the coset parameters associated with $Z^{ik}, \bar{Z}_{ik}$ and replaces the derivatives in the on-shell $N = 3$ superfield constraints (2.3), (2.4) by those covariantized with respect to the nonlinear realization of $N = 6$
supersymmetry. The resulting equations, like in the $N = 8 \rightarrow N = 4$ case, can be expected to contain a “disguised” form of the $D = 4$ BI equations \[9\] (along with the Nambu-Goto type equations for 6 physical scalars). Just like the constraints \(2.5\), \(2.6\) are equivalent to the standard $N = 3$ Grassmann analyticity conditions for the harmonic projections \(2.30\) of the superfield strengths, the BI deformation of these constraints can be equivalent to some nonlinear version of the Grassmann analyticity conditions. One might expect that the latter, like in the non-deformed case, can be transformed to a sort of nonlinear harmonic equations for the analytic harmonic potentials, and that for these equations a proper off-shell action exists. It should be a modification of the minimal $N = 3$ BI action constructed here, such that it reveals a second hidden non-linearly realized $N = 3$ supersymmetry. It is an intriguing open question how to find the explicit relation between the $N = 6 \rightarrow N = 3$ coset superfield variables and the off-shell $N = 3$ Grassmann-analytic superfield strengths.

One can approach the same problem from the opposite side and try to find hidden spontaneously broken supersymmetries in the above off-shell $N = 3$ BI action, or in its proper modifications. Using the general formula for the variation of the action $S_2(V_1^2, V_3^2)$,

\[
\delta S_2(V_2^1, V_3^2) \sim \int d\zeta \left( 33 \right) du \left( V_1^1 [(D_3^1 - D_3^2 D_3^1) \delta V_3^1 + (D_3^2)^2 \delta V_1^1] + \text{h.c.} \right) = 0 ,
\]

(5.1)

it is straightforward to show that $S_2(V_1^2, V_3^2)$ is off-shell invariant with respect to the following Goldstone-type transformation of the harmonic gauge potentials:

\[
\delta G V_2^1 = -(\delta_G V_3^2)^{\dagger} = [(\theta_2)^2 u_k^k - (\theta_2 \theta_3) u_2^k] \bar{c}_k + (\theta_2)^2 (e \theta_3) + 2u_1^k (\eta_k \theta_2)(\bar{\theta}^1)^2 \\
+ (\theta_2)^2 [(\bar{\eta}^k \bar{\theta}^1) u_k^1 - (\bar{\eta}^k \bar{\theta}^2) u_k^1] ,
\]

(5.2)

where $\epsilon^k, \bar{c}_k, \eta_k^\alpha, e^{\alpha}, \bar{\eta}^{k\dot{\alpha}}$ and $\bar{e}^{\dot{\alpha}}$ are additional 6 bosonic and 16 Grassmann parameters. These transformations provide shifts of the corresponding physical fields in the $N = 3$ vector multiplet and so can be treated as the lowest-order part of the non-linearly realized symmetries. For the time being we do not know whether a nonlinear generalization of the transformations \(5.2\) and the appropriate modification of the minimal $N = 3$ BI action do exist. It is worth emphasizing that there are 16 shifting fermionic symmetries. This can be viewed as an indication that the hypothetical $N = 3$ BI action with spontaneously broken symmetries actually reveals the $N = 8 \rightarrow N = 4$ coset structure, with one extra pair of broken and unbroken on-shell $N = 1$ supersymmetries (and with all 16 physical fermions being Goldstone ones). Thus it could provide a manifestly $N = 3$ supersymmetric form of the action of the $N = 8 \rightarrow N = 4$ BI theory. This conjecture is supported by the fact that the ordinary $N = 3$ gauge theory action coincides on shell with the action of $N = 4$ gauge theory and so should show up one additional on-shell supersymmetry. A technical problem tightly related to the issue of hidden nonlinear symmetries and brane interpretation consists in examining the 6 physical bosons sector of the $N = 3$ BI action and comparing it with the Nambu-Goto action of 3-brane in $D = 10$.

Finally, it is worthwhile to mention that our $N = 3$ BI action admits a straightforward nonabelian extension along the same lines as in the $N = 1$ and $N = 2$ cases \[21\].
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Appendix A. SU(3)/U(1) × U(1) harmonics

The SU(3)/U(1) × U(1) harmonics \[ \{1, \bar{1}\} \] form an SU(3) matrix \( u_i^j \) and are defined modulo the group U(1) × U(1) which acts on the index \( I \)

\[
u_i^1 = u_i^{(1,0)}, \quad u_i^2 = u_i^{(-1,1)}, \quad u_i^3 = u_i^{(0,-1)}. \tag{A.1}
\]

Here \( i \) is the index of the fundamental representation of SU(3). The complex conjugated harmonics \( \overline{u}_i^j = \overline{u}_i^j \) have the opposite U(1) × U(1) charges:

\[
u_i^1 = u_i^{(-1,0)}, \quad u_i^2 = u_i^{(1,-1)}, \quad u_i^3 = u_i^{(0,1)}. \tag{A.2}
\]

The harmonics satisfy the following relations:

\[
u_i^j \nu_j^k = \delta_i^j, \quad \nu_i^j \nu_j^k = \delta_i^k, \quad \varepsilon^{ikl} u_i^{(1,0)} u_k^{(-1,1)} u_l^{(0,-1)} = 1. \tag{A.3}
\]

The SU(3)-covariant harmonic derivatives act on the harmonics according to the rule

\[
\partial_j^I u^K = \delta^K_j u_I, \quad \partial_j^I u^K = -\delta^K_j u_I. \tag{A.4}
\]

The special SU(3) conjugation \( \sim \) of the harmonics is defined by

\[
\tilde{u}_i^1 = u_3^i, \quad \tilde{u}_3^1 = u_i^1, \quad \tilde{u}_2^i = -u_2^i. \tag{A.5}
\]

On the harmonic projections of spinor coordinates

\[
\theta_i^\alpha = u_i^\alpha \theta_i^\alpha, \quad \bar{\theta}^{\dot{\alpha}I} = u_\alpha^I \bar{\theta}^{\dot{\alpha}I} \tag{A.6}
\]

the \( \sim \) conjugation acts in the following way:

\[
\theta_i^\alpha \leftrightarrow \bar{\theta}^{\dot{\alpha}i}, \quad \theta_2^\alpha \leftrightarrow -\bar{\theta}^{\dot{\alpha}i}, \quad \theta_3^\alpha \leftrightarrow \bar{\theta}^{\dot{\alpha}i}. \tag{A.7}
\]

The conjugation rules of harmonic derivatives are as follows:

\[
\tilde{D}_3 f = -D_3 f, \quad \tilde{D}_2 f = D_2 f. \tag{A.8}
\]

The analytic superspace \( H(4+6|8) \) is parametrized by the coordinates \( \{\zeta, u\} \), where

\[
\zeta^M = \{x_\lambda^{a\dot{\alpha}}, x_\lambda^{a\dot{\alpha}} + 4i(\bar{\theta}_3^{\dot{\alpha}j} - \theta_2^{\dot{\alpha}j}), \theta_2^a, \theta_3^a, \bar{\theta}^{1\dot{\alpha}}, \bar{\theta}^{2\dot{\alpha}}\}, \tag{A.9}
\]

\[
\delta x_\lambda^{a\dot{\alpha}} = 4i(\theta_2^a u_k^2 + 2i\theta_3^a u_k^3)\delta^a_k - 4i\delta^a_k (2\bar{\theta}^{1\dot{\alpha}} u_1^k + \bar{\theta}^{2\dot{\alpha}} u_2^k), \tag{A.10}
\]
and it is closed (i.e., real) under the generalized conjugation. In these coordinates the spinor and harmonic derivatives have the following explicit form:

\[ D^1_{\alpha} = \partial^1_{\alpha} , \quad \bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} , \]
\[ \bar{D}^1_{1\alpha} = -\bar{\partial}_{1\alpha} - 4i\theta^1_{\alpha} \partial_{\beta} , \quad D^3_{\alpha} = \partial^3_{\alpha} + 4i\bar{\theta}^3_{\alpha} \partial_{\dot{\alpha}} , \]
\[ D^2_{\alpha} = \partial^2_{\alpha} + 2i\bar{\theta}^2_{\alpha} \partial_{\dot{\alpha}} , \quad \bar{D}^1_{2\alpha} = -\bar{\partial}^1_{2\alpha} - 2i\theta^2_{\alpha} \partial_{\beta} , \]
\[ D^2_{\alpha} = \bar{\partial}^2_{\alpha} + 2i\theta^2_{\alpha} \partial_{\dot{\alpha}} - \theta^2_{\alpha} \partial^1_{\alpha} + \bar{\theta}^1_{\alpha} \bar{\partial}_{\dot{\alpha}} , \]
\[ D^3_{\alpha} = \bar{\partial}^3_{\alpha} + 4i\theta^3_{\alpha} \partial_{\dot{\alpha}} - \theta^3_{\alpha} \partial^1_{\alpha} + \bar{\theta}^1_{\alpha} \bar{\partial}_{\dot{\alpha}} , \]
\[ D^3_{\alpha} = \bar{\partial}^3_{\alpha} + 4i\bar{\theta}^3_{\alpha} \partial_{\dot{\alpha}} - \bar{\theta}^3_{\alpha} \partial^1_{\alpha} + \theta^1_{\alpha} \partial_{\dot{\alpha}} . \] (A.11)

where \( \partial_{\dot{\alpha}} = \partial / \partial x^\alpha_{\dot{\alpha}} \).

The Grassmann and harmonic measures of integration over the \( N = 3 \) analytic harmonic superspace are normalized so that

\[ \int d^8\theta^{(3)}(\theta_1)^2(\theta_2)^2(\bar{\theta})^2 = 1 , \quad \int du = 1 . \] (A.13)

**Appendix B. Relation between spinor and vector representations**

\[ x^{\alpha\dot{\beta}} = (\tilde{\sigma}_m)^{\dot{\beta}m} x^m , \quad \partial_{\dot{\alpha}} = \partial / \partial x^{\alpha\dot{\beta}} = \frac{1}{2}(\sigma^m)_{\alpha\dot{\beta}} \partial / \partial x^m , \quad A_{\alpha\dot{\beta}} = (\sigma^m)_{\alpha\dot{\beta}} A_m , \]
\[ F_{\alpha\dot{\beta}} = \frac{1}{2} \left( \partial^\alpha_{\dot{\alpha}} A_{\beta\dot{\beta}} + \partial^\beta_{\dot{\beta}} A_{\alpha\dot{\alpha}} \right) , \quad \bar{F}_{\dot{\alpha}\dot{\beta}} = \frac{1}{2} \left( \partial^\dot{\beta}_{\dot{\alpha}} A_{\beta\dot{\beta}} + \partial^\dot{\alpha}_{\dot{\alpha}} A_{\alpha\dot{\beta}} \right) , \]
\[ \mathcal{F}_{mn} = \partial_m A_n - \partial_n A_m = i F_{\alpha\beta}(\sigma_{mn})_{\alpha\beta} - i \frac{1}{2} F^{\alpha\dot{\beta}}(\tilde{\sigma}_{mn})_{\dot{\alpha}\dot{\beta}} , \]
\[ (\mathcal{F}_{mn})^2 = 2 \left( F^2 + \bar{F}^2 \right) , \quad \frac{1}{2} \varepsilon^{mnpq} \mathcal{F}_{mn} \mathcal{F}_{pq} = -2i \left( F^2 - \bar{F}^2 \right) . \] (B.1)

We use \( \eta_{mn} = \text{diag}(1, -1, -1, -1) \) and the standard conventions of the two-component spinor formalism

\[ (\sigma^m)_{\alpha\dot{\beta}} = (1, \tilde{\sigma})_{\alpha\dot{\beta}} , \quad (\tilde{\sigma}_m)^{\dot{\beta}m} = \varepsilon^{\alpha\beta} \varepsilon^{\dot{\beta}\dot{\alpha}}(\sigma^m)_{\beta\dot{\alpha}} , \]
\[ (\sigma_{mn})_{\alpha\beta} = i \frac{1}{2} (\sigma_m \sigma_n - \sigma_n \sigma_m)_{\alpha\beta} , \quad (\tilde{\sigma}_{mn})_{\dot{\alpha}\dot{\beta}} = i \frac{1}{2} (\bar{\sigma}_m \sigma_n - \bar{\sigma}_n \sigma_m)_{\dot{\alpha}\dot{\beta}} , \]
\[ \varepsilon_{12} = \varepsilon_{\dot{1}\dot{2}} = -\varepsilon^{12} = -\varepsilon^{\dot{1}\dot{2}} = 1 , \quad \varepsilon^{\alpha\beta} \varepsilon_{\beta\gamma} = \delta^\alpha_\gamma , \quad \varepsilon^{\dot{\alpha}\dot{\beta}} \varepsilon_{\dot{\beta}\dot{\gamma}} = \delta_{\dot{\gamma}} . \] (B.2)
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