Risk-sensitive Necessary and Sufficient Optimality Conditions and Financial Applications: Fully Coupled Forward-Backward Stochastic Differential Equations with Jump diffusion

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Abstract
Throughout this paper, we focused our aim on the problem of optimal control under a risk-sensitive performance functional, where the system is given by a fully coupled forward-backward stochastic differential equation with jump. The risk neutral control system has been used as preliminary step, where the admissible controls are convex, and the optimal solution exists. The necessary as well as sufficient optimality conditions for risk-sensitive performance are proved. At the end of this work, we illustrate our main result by giving an example of mean-variance for risk sensitive control problem applied in cash flow market.

Key words: Fully Coupled Forward Backward Stochastic Differential Equation with Jump, Risk-sensitive, Necessary Optimality Conditions, Sufficient Optimality Conditions, Logarithmic Transformation, Mean variance, Cash flow.

1 Introduction
Maximum principle for controlled stochastic differential equations (SDE in short), whose objective is to obtain necessary as well as sufficient optimality conditions of controls, has been extensively investigated since 1970s. The initial work was done by Kushner [19]. The other fundamental advance was developed by Haussmann [17,18]. Versions of the stochastic maximum principle (SMP in short), in which the diffusion coefficient is allowed to depend explicitly on the control variable, have been derived by Arkin & Saksonov [2], Bensoussan [3], and Bismut
The results of [2] and [4, 5, 6] consider the case of random coefficients. Necessary and sufficient optimality conditions for linear systems with random coefficients, where no $L^p$-bounds are imposed on the controls, are established by Cadellinas and Karatzas [9]. The general case, where the control domain is not convex, and the diffusion coefficient depends explicitly on the variable control, was derived by Peng [23] by introducing two adjoint processes, and a variational inequality of the second order. Recently, by considering risk sensitive performance control with an exponential functional cost, Djehiche et al [14] generalized the previous results on the subject, and derive necessary optimality conditions, by adding the mean field process.

The initial works on optimal control of jump processes was first considered by Boel [7, 8], Rishel [25]. Later, many authors studied this kind of control problems including Situ [26], Cadellinas [10], and Framstad Øksendal & Sulem [16]. We note that in [10] and [16], some applications in finance are treated. The general case, where the control domain is not convex and the diffusion coefficient depends explicitly on the control variable, was derived by Tang and Li [31], by using the second order expansion, the results of [31] are given with two adjoint processes and a variational inequality of the second order. For more details on the controlled systems with jumps and their applications, see Øksendal and Sulem [22] and the references therein.

The purpose of this paper is to generalize the model governed with SDE and BSDE, before that we must give this motivation example which has taken from the thesis of Armerin [1].

Modeling and controlling cash flow processes of a firm or a project, such as pricing and managing an insurance contract, is a class of problems where forward backward stochastic differential equations (FBSDEs in short) provide a natural setup and a powerful tool. In this paper, we shall investigate an example of such a situation arising in the pricing of a simple insurance contract.

A policyholder at an insurance company has paid premiums that at time zero have accumulated to the sum $m_0$. The money is invested in an asset portfolio with wealth $(x_t)_{t \in [0,T]}$ managed by the insurance company under a time interval $[0,T]$. At each instant $t \in [0,T]$, the policyholder ought to receive an amount $c_t x_t$. The present value (price) of the cash stream $(c_s x_s)$, discounted to time $t$ with a discount factor (deflator) $\exp \left\{ - \int_0^t \lambda_s ds \right\}$, where $\lambda_t$ is assumed nonnegative, bounded, and deterministic, is given by

$$y_t = \mathbb{E} \left[ \int_t^T e^{-\int_0^r \lambda_s \, ds} c_s \lambda_s ds \mid \mathcal{F}_t \right].$$

(1.1)

Assume that the portfolio is invested in a simple Black-Scholes market model consisting of a risk-free asset (for example, a bond or a bank account) with a short interest rate $r_t$ assumed bounded and deterministic, and a risky asset evolving as a geometric Brownian motion with rate of return $\mu_t$ and volatility $\sigma_t$, both assumed to be bounded and deterministic functions of time, with $\sigma_t \geq 2$.\]
\(\varepsilon > 0\) for all \(t \in [0, T]\). In this market the wealth process \((x_t)_{t\in[0,T]}\) is governed by the dynamics given by

\[
\begin{align*}
\begin{cases} 
  dx_t & = (r_t x_t + \rho_t u_t) \, dt + \sigma_t u_t dW_t, \\
  x_0 & = m_0,
\end{cases}
\end{align*}
\tag{1.2}
\]

where \(u_t\) is the amount invested in the risky asset and \(\rho_t = \mu_t - r_t\) is the risk premium held for this investment.

The insurance company allocates the amounts \((u_t)\) in order to come close to the following target at time \(T\): Find the admissible strategies \((c,u)\) which maximize the policyholder’s preferences represented by the utility function \(F\) of the cash streams, under the condition that the total amount to be paid out is equal to the total premium \(m_0\):

\[
\max_{(c,u)} \frac{1}{\theta} \mathbb{E} \left[ F^\theta (x_T) \right].
\tag{1.3}
\]

By selecting an appropriate portfolio choice strategy \(u(\cdot)\), where the exponent \(\theta > 0\) is called the risk sensitive parameter. Assume that the policyholder’s utility function is of HARA (hyperbolic absolute risk aversion) type. That is, \(F(X) = \frac{X^\theta}{\theta}\), where \(\theta \in (0, 1)\). We can rewrite the expectation \(\mathbb{E} \left[ F^\theta (x_T) \right]\) in terms of an expected exponential of integral criterion, by applying Itô’s formula to \(\ln x_t^\theta = \theta \ln x_t\), we get

\[
\max_{(c,u)} m_0^\theta \mathbb{E} \left[\int_0^T \exp \theta \{ f(t,x_t,u_t) \} \, dt \right],
\]

where

\[
f(t,x_t,u_t) = (\theta - 1) \frac{\sigma^2}{2} u_t^2 + \left( \frac{1}{2} \sigma^2 + m - r_t - c_t x_t \right) u_t + r_t,
\]

and

\[
y_0 = \mathbb{E} \left[ \int_0^T e^{-\int_0^t \lambda_s \, ds} \, c_s x_s \, \, \, d\mathcal{F}_s \bigg| \mathcal{F}_0 \right],
\]

is the total value of the stream of cash flows discounted to time zero.

We need the following definition of admissible strategies suitable for our problem.

**Definition 1.1** An admissible strategy is a pair of \((\mathcal{F}_t)_{t\geq0}\)-adapted processes \((c,u)\) such that (1.2) has a strong solution \((x_t)_{t\in[0,T]}\) that satisfies

\[
\mathbb{E} \int_0^T |x_t| \, dt < \infty,
\]

and

\[
\mathbb{E} \left[ \int_0^T e^{-\int_0^t \lambda_s \, ds} \, c_t x_t \, dt \right]^2 < \infty.
\]
controls. The criteria to be minimized over $U$ with values in some set and $W$ is a Poisson martingale measure with characteristic ability space where $(z_t)_{t \geq 0}$ is $(\mathcal{F}_t)_{t \geq 0}$-adapted and square-integrable with respect to $dt \times d\mathbb{P}$ over $[0, T] \times \Omega$.

Hence, (1.2) and (1.4) satisfied by $(x, y, z)$ is a FBSDE, in the next step we want to improve this notion of cash flow problem into a system of fully coupled FBSDE with jump diffusion, as the best of our acknowledge, this is not a simple or trivial extension, because of we have a lot of work to do. Firstly the function minimize has the form an expected exponential, secondly the problem of control governed by a fully coupled FBSDE with jump diffusion as in system (3.3) is very hard to solve it especially if we want to derive the stochastic maximum principle (Lemma 3.3 ??, and ?? below).

Our aim in this paper is to derive necessary as well as sufficient optimality conditions for jump process, controlled diffusion and generator for the system driven by a fully coupled forward backward stochastic differential equation (FBSDE in short) under a risk sensitive performance. We give the results, in the form of global SMP, by using an auxiliary process as a preliminary step see the section 3 below.

In the risk sensitive performance case, the system is governed by a FBSDE with jump diffusion

$$
\begin{align*}
\left\{ \begin{array}{l}
    dx^v(t) = & b(t, x^v(t), y^v(t), z^v(t), r^v(t, \cdot), v(t)) \ dt \\
    & + \sigma(t, x^v(t), y^v(t), z^v(t), r^v(t, \cdot), v(t)) \ dW(t) \\
    & + \int_0^t \gamma(t, x(t^-), y(t^-), z(t^-), r(t^-), v(t), \lambda) \tilde{N}(dt, d\lambda) \\
    dy^v(t) = & -g(t, x^v(t), y^v(t), z^v(t), r^v(t, \cdot), v(t)) \ dt + z^v(t) \ dW(t) \\
    & + r^v(t, \lambda) \tilde{N}(dt, d\lambda) \\
    x^v(0) = & d, \\
    y(T) = & a,
    \end{array} \right.
\end{align*}
$$

where $b$, $\sigma$, $\gamma$ and $g$ are given functions, $d$ is the initial data, $a$ is terminal data, and $W = (W(t))_{t \geq 0}$ is a standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, satisfying the usual conditions, and $\tilde{N}(dt, d\lambda)$ is a Poisson martingale measure with characteristic $m(d\lambda) dt$.

The control variable $v = (v_t)$, called strict control, is an $\mathcal{F}_t$-adapted process with values in some set $U$ of $\mathbb{R}$. We denote by $\mathcal{U}$ the class of all strict admissible controls. The criteria to be minimized over $\mathcal{U}$ has the form

$$
J^0(v) = \mathbb{E} \left( \exp \theta \left[ \int_0^T f(t, x^v(t), y^v(t), z^v(t), r^v(t, \cdot), v(t)) \ dt + \Phi(x^v(T)) + \Psi(y^v(0)) \right] \right),
$$

Now, for each admissible strategy $(c, u)$, the $(\mathcal{F}_t)_{t \geq 0}$-adapted value process $(y_t)_{t \geq 0}$ in (1.1) satisfies the following BSDE:

$$
\left\{ \begin{array}{l}
    dy_t = (\lambda_t y_t - c_t x_t) \ dt + z_t dW_t, \\
    y_T = 0,
    \end{array} \right.
$$

1.4)

where $(z_t)_{t \geq 0}$ is $(\mathcal{F}_t)_{t \geq 0}$-adapted and square-integrable with respect to $dt \times d\mathbb{P}$ over $[0, T] \times \Omega$.
where $\Phi$, $\Psi$ and $f$ are given maps and $(x^v(t), y^v(t))$ is the trajectories controlled by $v$.

A control $u \in \mathcal{U}$ is called optimal if it satisfies

$$J(u) = \inf_{v \in \mathcal{U}} J(v).$$

To achieve the objective of this paper, and establish the necessary and sufficient optimality conditions, the existence and uniqueness of the optimal control which minimize the functional cost is proved, we proceed as follows. Firstly, we give the optimality conditions for risk neutral controls. The idea is to use the fact that the auxiliary state process $\xi^v(t)$ is the best intermediate step to translate the system of forward backward SDE into three equations see (3.8) in section 3. Secondly, we suggest a transformation of the adjoint equations $(p_1, q_1), (p_2, q_2), (p_3, q_3)$ and $\pi(\lambda)$ into following adjoint equations $(\tilde{p}_2, \tilde{q}_2), (\tilde{p}_3, \tilde{q}_3)$ and $\tilde{\pi}(\lambda)$ by applying the result obtained by both Yong [32] and Wu [34], but with some additional ideas, we use this transformation and virtue of the logarithm transformed introduced by El Karoui & Hamadene [15] to solve this problem and driven the necessary as well as sufficient optimality conditions of the type risk sensitive performance.

The results of this paper generalize all the previous works on the subject, into FBSDE with jumps diffusion under the risk sensitive performance. we combine between two important results the first one was such of Djehiche et al [14], while the second was Chala [11, 12], for more details for the risk sensitive the readers can see the papers [29, 30] and references of therein.

The paper is organized as follows: In section 2, we give the precise problem formulations, and introduce the risk-sensitive model, and give the various assumptions used throughout this paper. In section 3, we shall study our system of fully coupled forward backward SDE, the new approach method transformation of the adjoint process is given and studied, SMP for risk-neutral is given, which will be the main result in next section, we give our first main result, the necessary optimality conditions for risk-sensitive control problem under an additional hypothesis is established. In section 4, The sufficient optimality conditions for risk-sensitive performance cost is our second main result, is obtained under the convexity of the Hamiltonian function. In section 5, we finished the paper by given an application, a financial model of mean variance with risk-sensitive performance functional is the best application for our problem. The conclusion and remarks is the last section (section 6).

2 Problem and settings

In all what follows, we will be worked on the classical probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \leq T}, \mathbb{P}\right)$, such that $\mathcal{F}_0$ contains all the $\mathbb{P}$–null sets, $\mathcal{F}_T = \mathcal{F}$ for an arbitrarily fixed time horizon $T$, and $(\mathcal{F}_t)_{t \leq T}$ satisfies the usual conditions. We assume that the filtration $(\mathcal{F}_t)_{t \leq T}$ is generated by the following two mutually independent processes
(i) \((W(t))_{t \geq 0}\) is a one-dimensional standard Brownian motion.

(ii) Poisson random measure \(N\) on \([0, T] \times \Gamma\), where \(\Gamma \subset \mathbb{R} - \{0\}\). We denote by \((\mathcal{F}_t^W)_{t \leq T}\) (resp. \((\mathcal{F}_t^N)_{t \leq T}\)) the \(\mathbb{P}\)-augmentation of the natural filtration of \(W\) (resp. \(N\)). Obviously, we have

\[
\mathcal{F}_t := \sigma \left[ \int_0^t \int_A N(d\lambda, dr); \ s \leq t, \ A \in \mathcal{B}(\Gamma) \right] \vee \sigma [W(s); \ s \leq t] \vee \mathcal{N},
\]

where \(\mathcal{N}\) contains all \(\mathbb{P}\)-null sets in \(\mathcal{F}\), and \(\sigma_1 \vee \sigma_2\) denotes the \(\sigma\)-field generated by \(\sigma_1 \cup \sigma_2\). We assume that the compensator of \(N\) has the form

\[
\mu(dt, d\lambda) = m(d\lambda) dt,
\]

for some positive and \(\mathcal{F}\)-finite Lévy measure \(m\) on \(\Gamma\), endowed with its Borel \(\sigma\)-field \(\mathcal{B}(\Gamma)\). We suppose that

\[
\int_{\Gamma} 1 \wedge |\lambda|^2 m(d\lambda) < \infty,
\]

and write \(\tilde{N} = N - mdt\) for the compensated jump martingale random measure of \(N\).

**Notation 2.1** We need to define some additional notations. Given \(s \leq t\), let us introduce the following spaces

\[
\mathcal{S}_2([0,T], \mathbb{R}) \text{ the set of } \mathbb{R}\text{-valued adapted cadlag processes } P \text{ such that }
\]

\[
\|P\|_{\mathcal{S}_2([0,T], \mathbb{R})} := \mathbb{E} \left[ \sup_{r \in [0,T]} |P(r)|^2 \right]^{1/2} < +\infty.
\]

\[
\mathcal{M}_2([0,T], \mathbb{R}) \text{ is the set of progressively measurable } \mathbb{R}\text{-valued processes } Q \text{ such that }
\]

\[
\|Q\|_{\mathcal{M}_2([0,T], \mathbb{R})} := \mathbb{E} \left[ \int_0^T |Q(r)|^2 dr \right]^{1/2} < +\infty.
\]

\[
\mathcal{L}_m^2([0,T], \mathbb{R}) \text{ is the set of } \mathcal{B}([0,T] \times \Omega) \otimes \mathcal{B}(\Gamma) \text{ measurable maps } R : [0,T] \times \Omega \times \Gamma \rightarrow \mathbb{R} \text{ such that }
\]

\[
\|R\|_{\mathcal{L}_m^2([0,T], \mathbb{R})} := \mathbb{E} \left[ \int_0^T \int_{\Gamma} |R(r)|^2 m(d\lambda) dr \right]^{1/2} < +\infty,
\]

we denote by \(\mathbb{E}\) the expectation with respect to \(\mathbb{P}\).

Let \(T\) be a strictly positive real number and \(U\) is a convex nonempty subset of \(\mathbb{R}\).

**Definition 2.1** Let \(U\) be a nonempty closed subset in \(\mathbb{R}\). An admissible control is a \(U\)-valued measurable \(\mathcal{F}_t\)-adapted process \(v\), such that \(\|v\|_{\mathcal{S}_2} < \infty\). We denote by \(\mathcal{U}\) the set of all admissible controls.
For all \( v \in \mathcal{U} \), we consider the following fully coupled forward-backward with jump system

\[
\begin{align*}
    dx(t) &= b(t, x(t), y(t), z(t), r(t, .), v(t)) dt \\
        &+ \sigma(t, x(t), y(t), z(t), r(t, .), v(t)) dW(t) \\
        &+ \int_{\Gamma} \gamma(t-, x(t-), y(t-), z(t-), r(t-, \lambda), v(t-), \lambda) \tilde{N}(dt, d\lambda) \\

dy(t) &= -g(t, x(t), y(t), z(t), r(t, .), v(t)) dt + z(t) dW(t) \\
        &+ \int_{\Gamma} r(t, \lambda) \tilde{N}(dt, d\lambda) \\
    x(0) &= d, \quad y(T) = a, \quad t \in [0, T]
\end{align*}
\tag{2.5}
\]

where \( b : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Gamma \times \mathcal{U} \to \mathbb{R}, \sigma : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Gamma \times \mathcal{U} \to \mathbb{R}, \) \( g : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Gamma \times \mathcal{U} \to \mathbb{R}, \) and \( \gamma : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Gamma \times \mathcal{U} \to \mathbb{R} \) are given maps. If \( (x(\cdot), y(\cdot), z(\cdot), r(\cdot, \cdot)) \) is the unique solution of (2.5) associated with \( v(\cdot) \in \mathcal{U} \).

The functional cost of the risk-sensitive type is given by

\[
J^\theta(v) = \mathbb{E}
\left[ \exp \theta \left( \int_0^T f(t, x(t), y(t), z(t), r(t, .), v(t)) dt + \Phi(x^v(T)) + \Psi(y^v(0)) \right) \right],
\tag{2.6}
\]

where \( \Phi : \mathbb{R} \to \mathbb{R}, \Psi : \mathbb{R} \to \mathbb{R}, f : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Gamma \times \mathcal{U} \to \mathbb{R} \) are given maps, and \( \theta > 0 \) is called the risk-sensitive parameter.

Our risk-sensitive stochastic optimal control problem is stated as follows: For given \( (t, x(t), y(t), z(t), r(t, .)) \in [0, T] \times \mathbb{R}^4 \), minimize (2.6) subject to (2.5) over \( \mathcal{U} \).

\[
\inf_{v \in \mathcal{U}} J^\theta(v) = J^\theta(u). \tag{2.7}
\]

A control that solves the problem \{2.5, 2.6, 2.7\} is called optimal. Our goal is to establish a necessary optimality conditions as well as a sufficient optimality conditions, satisfied by a given optimal control, in the form of stochastic maximum principle (SMP in short).

We give some notations \( \Upsilon = (x^v(t), y^v(t), z^v(t), r^v(t, .))^T \), where \( (\cdot)^T \) denotes the transport of the matrix,

and \( M(t, \Upsilon) = \begin{pmatrix} b \\ \sigma \\ -g \end{pmatrix}(t, \Upsilon) \).

We introduce the following assumptions.

**\( H_1 \):**

For each \( \Upsilon \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \), \( M(t, \Upsilon) \) is an \( \mathcal{F}_t \)-measurable process defined on \([0, T]\) with \( M(t, \Upsilon) \in \mathcal{M}^2([0, T]; \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Gamma) \).

**\( H_2 \):**

\( M(t, .) \) satisfies Lipschitz conditions: There exists a constant \( k > 0 \), such that

\[
|M(t, \Upsilon) - M(t, \Upsilon')| \leq k|\Upsilon - \Upsilon'| \forall \Upsilon, \Upsilon' \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Gamma, \forall t \in [0, T].
\]
The following monotonic conditions introduced in \[34\], are the main assumptions in this paper. 

\(H_3:\)

\[
\langle M(t,\mathcal{Y}) - M(t,\mathcal{Y}'), \mathcal{Y} - \mathcal{Y}' \rangle \leq \beta |\mathcal{Y} - \mathcal{Y}'|^2,
\]

for every \(\mathcal{Y} = (x, y, z, r)\) and \(\mathcal{Y}' = (x', y', z', r')\) \(\in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Gamma, \forall t \in [0, T]\), where \(\beta\) is a positive constant.

\(U\) is a convex subset of \(\mathbb{R}\).

**Proposition 2.1** For any given admissible control \(v(\cdot)\) and under the assumptions \((H_1), (H_2)\) and \((H_3)\), the fully coupled FBSDE with jump diffusion \((2.3)\) admits an unique solution

\[
(x^v(t), y^v(t), z^v(t), r^v(t, \cdot), v(t)) \in \left(\mathcal{M}^2([0, T]; \mathbb{R} \times \mathbb{R} \times \Gamma) \right)^2 \times \mathcal{S}^2([0, T]; \mathbb{R} \times \Gamma).
\]

**Proof.** The proof can be seen in \[34\]. \(\blacksquare\)

Next, we assume that

\(H_4:\)

i) \(b, \sigma, g, \Phi\) and \(\Psi\) are continuously differentiable with respect to \((x^v, y^v, z^v, r^v(\cdot))\).

ii) All the derivatives of \(b, \sigma, g\) and \(f\) are bounded by

\(C(1 + |x^v| + |y^v| + |z^v| + |r^v|)\).

iii) The derivatives of \(\Phi, \Psi\) are bounded by \(C(1 + |x^v|)\) and \(C(1 + |y^v|)\) respectively.

Under the above assumptions, for every \(v \in U\) equation \((2.3)\) has a unique strong solution and the function cost \(J^v\) is well defined from \(U\) into \(\mathbb{R}\).

### 3 Necessary optimality conditions and auxiliary process

First of all, we may introduce an auxiliary state process \(\xi^v(t)\) which is solution of the following stochastic differential equation (SDE in short):

\[
d\xi^v(t) = f(t, x^v(t), y^v(t), z^v(t), r^v(t, \cdot), v(t)) \, dt, \quad \xi^v(0) = 0.
\]

From the above auxiliary process, the fully coupled forward-backward type control problem is equivalent to

\[
\inf_{u \in U} \mathbb{E} \left[ \exp \theta \{ \Phi(x^v(T)) + \Psi(y^v(0)) + \xi(T) \} \right],
\]

subject to

\[
\begin{align*}
\inf_{u \in U} d\xi^v(t) &= f(t, x^v(t), y^v(t), z^v(t), r^v(t, \cdot), v(t)) \, dt, \\
\begin{cases}
dx^v(t) = b(t, x^v(t), y^v(t), z^v(t), r^v(t, \cdot), v(t)) \, dt \\
+ \sigma(t, x^v(t), y^v(t), z^v(t), r^v(t, \cdot), v(t)) \, dW(t) \\
+ \int_{\Gamma} \gamma(t, x(t-), y(t-), z(t-), r(t-), \lambda, v(t-), \lambda) \, \tilde{N}(dt, d\lambda),
\end{cases} \\
gy^v(t) &= -g(t, x^v(t), y^v(t), z^v(t), r^v(t, \cdot), v(t)) \, dt + z^v(t) \, dW(t) \\
&+ \int_{\Gamma} r^v(t, \lambda) \, \tilde{N}(dt, d\lambda), \\
\xi^v(0) &= 0, \quad x^v(0) = d, \quad y^v(T) = a.
\end{align*}
\] (3.8)
We denote by
\[ A^\theta_T := \exp \left\{ \Phi (x^v (T)) + \Psi (y^v (0)) + \int_0^T f (t, x^v (t), y^v (t), z^v (t), r^v (t, \cdot), v (t)) \, dt \right\}, \]
and we can put also
\[ \Theta_T := \Phi (x^v (T)) + \Psi (y^v (0)) + \int_0^T f (t, x^v (t), y^v (t), z^v (t), r^v (t, \cdot), v (t)) \, dt, \]
the risk-sensitive loss functional is given by
\[
\Theta_\theta := \frac{1}{\theta} \log \mathbb{E} \left\{ \exp \left\{ \Phi (x^v (T)) + \Psi (y^v (0)) + \int_0^T f (t, x^v (t), y^v (t), z^v (t), r^v (t, \cdot), v (t)) \, dt \right\} \right\}
= \frac{1}{\theta} \log \mathbb{E} \left\{ \exp \{ \theta \Theta_T \} \right\}.
\]
When the risk-sensitive index \( \theta \) is small, the functional \( \Theta_\theta \) can be expanded as \( \mathbb{E} (\Theta_T) + \frac{1}{2} \text{Var} (\Theta_T) + O (\theta^2) \), where, \( \text{Var} (\Theta_T) \) denotes the variance of \( \Theta_T \).

If \( \theta < 0 \), the variance of \( \Theta_T \), as a measure of risk, improves the performance \( \Theta_\theta \), in which case the optimizer is called risk seeker. But, when \( \theta > 0 \), the variance of \( \Theta_T \) worsens the performance \( \Theta_\theta \), in which case the optimizer is called risk averse. The risk-neutral loss functional \( \mathbb{E} (\Theta_T) \) can be seen as a limit of risk-sensitive functional \( \Theta_\theta \) when \( \theta \to 0 \), for more details the reader can see the papers [13].

**Notation 3.1** We will use the following notation throughout this paper. For \( \phi \in \{ b, \sigma, f, g, H^\theta, \tilde{H}^\theta \} \), we define

\[
\begin{align*}
\phi (t) &= \phi (t, x^v (t), y^v (t), z^v (t), r^v (t, \cdot), v (t)), \\
\partial \phi (t) &= \partial \phi (t, x^v (t), y^v (t), z^v (t), r^v (t, \cdot), v (t)) \\
-\phi (t, x^v (t), y^v (t), z^v (t), r^v (t, \cdot), v (t)) &= \frac{\partial \phi (t, x^v (t), y^v (t), z^v (t), r^v (t, \cdot), v (t))}{\partial \zeta} (t, x^v (t), y^v (t), z^v (t), r^v (t, \cdot), v (t)) \quad \zeta = x, y, z, r (\cdot), \\
\phi (t, x^v (t), y^v (t), z^v (t), r^v (t, \cdot), v (t)) &= \phi (t, x^v (t), y^v (t), z^v (t), r^v (t, \cdot), v (t)) \\
\end{align*}
\]

and \( \gamma (t；\lambda) \) it means that the function \( \gamma \) is càdlàg.

Where \( v_0 \) in an admissible control from \( \mathcal{U} \).
We assume that \( (H_1), (H_2), (H_3) \) and \( (H_4) \) hold, we might apply the SMP for risk-neutral of fully coupled forward-backward type control from Yong [32], to augmented state dynamics \( (\xi, x, y, z, r) \) and derive the adjoint equation. There exist unique \( \mathcal{F}_T \)-adapted of processes \( (p_1, q_1, \pi_1), (p_2, q_2, \pi_2), (p_3, q_3, \pi_3) \),
which solve the following system matrix of backward SDEs

\[
\begin{aligned}
\begin{pmatrix}
\tilde{p}_1 (t) \\
\tilde{p}_2 (t) \\
\tilde{p}_3 (t)
\end{pmatrix}
&= \begin{pmatrix}
dp_1 (t) \\
dp_2 (t) \\
dp_3 (t)
\end{pmatrix} \\
&= - \begin{pmatrix}
0 & 0 & 0 \\
f_x (t) & b_x (t) & g_x (t) \\
f_y (t) & b_y (t) & g_y (t)
\end{pmatrix} \begin{pmatrix}
p_1 (t) \\
p_2 (t) \\
p_3 (t)
\end{pmatrix} dt \\
- \begin{pmatrix}
0 & 0 & 0 \\
0 & \sigma_x (t) & 0 \\
0 & \sigma_y (t) & 0
\end{pmatrix} \begin{pmatrix}
q_1 (t) \\
q_2 (t) \\
q_3 (t)
\end{pmatrix} dt \\
+ \int_{\Gamma} \begin{pmatrix}
0 & 0 & 0 \\
0 & \gamma_x (t-\lambda) & 0 \\
0 & \gamma_y (t-\lambda) & 0
\end{pmatrix} \begin{pmatrix}
\pi_1 (t, \lambda) \\
\pi_2 (t, \lambda) \\
\pi_3 (t, \lambda)
\end{pmatrix} m (d\lambda) dt \\
+ \begin{pmatrix}
q_1 (t) \\
q_2 (t) \\
q_3 (t)
\end{pmatrix} dW (t) + \int_{\Gamma} \begin{pmatrix}
\pi_1 (t, \lambda) \\
\pi_2 (t, \lambda) \\
\pi_3 (t, \lambda)
\end{pmatrix} \tilde{N} (dt, d\lambda)
\end{aligned}
\]

\[
\begin{pmatrix}
p_1 (T) \\
p_2 (T) \\
p_3 (0)
\end{pmatrix}
= \theta A_T \left( \frac{1}{\Phi_x (x_T^n)} \right)
\]

(3.9)

with \( \mathbb{E} \left[ \sum_{i=1}^{3} \sup_{0 \leq t \leq T} |p_i (t)|^2 + \sum_{i=1}^{2} \int_{0}^{T} |q_i (t)|^2 dt \right] < \infty \), and

\[
\begin{aligned}
q_3 (t) &= -Tr \left[ \begin{pmatrix}
f_x (t) & b_x (t) \\
\sigma_x (t) & g_x (t)
\end{pmatrix} \begin{pmatrix}
p_1 (t) \\
p_2 (t) \\
p_3 (t)
\end{pmatrix} \right] + \int_{\Gamma} \gamma_x (t-\lambda) \pi_2 (t, \lambda) m (d\lambda), \\
\pi_3 (t, \lambda) &= -Tr \left[ \begin{pmatrix}
f_y (t) & b_y (t) \\
\sigma_y (t) & g_y (t)
\end{pmatrix} \begin{pmatrix}
p_1 (t) \\
p_2 (t) \\
p_3 (t)
\end{pmatrix} \right] + \int_{\Gamma} \gamma_y (t-\lambda) \pi_2 (t, \lambda) m (d\lambda).
\end{aligned}
\]

To this end we may define (3.9) in the compact form as

\[
\begin{aligned}
\begin{pmatrix}
d\tilde{p}_1 (t) \\
d\tilde{p}_2 (t) \\
d\tilde{p}_3 (t)
\end{pmatrix}
&= -F (t) dt + \Sigma (t) dW (t) + \int_{\Gamma} R (t, \lambda) \tilde{N} (dt, d\lambda) \\
\begin{pmatrix}
p_1 (T) \\
p_2 (T) \\
p_3 (0)
\end{pmatrix}
&= \theta A_T \left( \frac{1}{\Phi_x (x_T^n)} \right), \text{ and } p_3 (0) = \theta \Phi_y (y^n (0)) A_T,
\end{aligned}
\]

where

\[
F (t) = \begin{pmatrix}
0 & 0 & 0 \\
f_x (t) & b_x (t) & g_x (t) \\
f_y (t) & b_y (t) & g_y (t)
\end{pmatrix} \begin{pmatrix}
p_1 (t) \\
p_2 (t) \\
p_3 (t)
\end{pmatrix} \\
+ \begin{pmatrix}
0 & 0 & 0 \\
0 & \sigma_x (t) & 0 \\
0 & \sigma_y (t) & 0
\end{pmatrix} \begin{pmatrix}
q_1 (t) \\
q_2 (t) \\
q_3 (t)
\end{pmatrix} \\
- \int_{\Gamma} \begin{pmatrix}
0 & 0 & 0 \\
0 & \gamma_x (t-\lambda) & 0 \\
0 & \gamma_y (t-\lambda) & 0
\end{pmatrix} \begin{pmatrix}
\pi_1 (t, \lambda) \\
\pi_2 (t, \lambda) \\
\pi_3 (t, \lambda)
\end{pmatrix} m (d\lambda),
\]
\[ \Sigma(t) = \begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix}, \]

and

\[ R(t, \cdot) = \begin{pmatrix} \pi_1(t, \cdot) \\ \pi_2(t, \cdot) \\ \pi_3(t, \cdot) \end{pmatrix}. \]

We suppose here that \( \bar{H}^\theta \) be the Hamiltonian associated with the optimal state dynamics \((\xi^u, x^u, y^u, z^u, r^u(.))\), and the triplet of adjoint processes \((\bar{p}^\theta(t), \bar{q}^\theta(t), \bar{r}^\theta(t, .))\) is given by

\[
\bar{H}^\theta(t, \xi^u(t), x^u(t), y^u(t), z^u(t), r(t, .), u(t), \bar{p}^\theta(t), \bar{q}^\theta(t), \bar{r}^\theta(t, .))
= \begin{pmatrix} f(t) \\ b(t) \\ g(t) \end{pmatrix} (\bar{p}^\theta(t))^\top + \begin{pmatrix} 0 \\ \sigma(t) \\ 0 \end{pmatrix} (\bar{q}^\theta(t))^\top
- \int_\Gamma \begin{pmatrix} \gamma(t-, \lambda) \\ 0 \\ 0 \end{pmatrix} (\bar{r}^\theta(t, \lambda))^\top m(d\lambda).
\]

**Theorem 3.1** Assume that \((H_1), (H_2), (H_3)\) and \((H_4)\) hold.

If \((\xi^u(\cdot), x^u(\cdot), y^u(\cdot), z^u(\cdot), r(\cdot, \cdot))\) is an optimal solution of the risk-neutral control problem \((3.3)\), then there exist \(\mathcal{F}_t^-\)-adapted processes

\((p_1, q_1, \pi_1), (p_2, q_2, \pi_2), (p_3, q_3, \pi_3)\) that satisfy \((3.9)\), such that

\[ \bar{H}^\theta_u(t) (u_t - v_t) \geq 0, \quad (3.10) \]

for all \(u \in \mathcal{U}\), almost every \(t\) and \(\mathbb{P}\)-almost surely, where \(\bar{H}^\theta_u(t)\) is defined in notation \((3.1)\).

**Proof.** For more details the reader can see paper \([32]\) with the result of paper \([27]\). \(\blacksquare\)

### 3.1 Expected Exponential Utility

The expected exponential utility can be transformed into quadratic BSDE, this Backward stochastic differential equation it permits us to find an other way to resoundre the problem of adjoint equation which play a good role in the component of the Hamiltonian function.

As we said, Theorem \((3.1)\) is a good SMP for the risk-neutral of forward backward control problem. We follow the same approach used in \([11] [14]\), and suggest a transformation of the adjoint processes \((p_1, q_1, \pi_1(\cdot)), (p_2, q_2, \pi_2(\cdot)), (p_3, q_3, \pi_3(\cdot))\) in such a way to omit the first component \((p_1, q_1, \pi_1(\cdot))\) in \((3.9)\), and to obtain the SMP \((3.10)\) in terms of only the last two adjoint processes, that we denote them by \((\tilde{p}_2, \tilde{q}_2, \tilde{\pi}_2(\cdot)), (\tilde{p}_3, \tilde{q}_3, \tilde{\pi}_3(\cdot))\). Noting that \(dp_1(t) = \)
q_1(t) \, dW_t + \int \pi_1(t, \lambda) \, N(dt,d\lambda) \) and \( p_1(T) = \theta A^0_T \), the explicit solution of this backward SDE is

\[
p_1(t) = \theta \mathbb{E} \left[ A^0_T \mid \mathcal{F}_t \right] = \theta V^\theta(t), \quad (3.11)
\]

where

\[
V^\theta(t) := \mathbb{E} \left[ A^0_T \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T. \quad (3.12)
\]

As a good look of (3.11), it would be natural to choose a transformation of \((\tilde{p}, \tilde{q}, \tilde{r}(\_))\) instead of \((p, q, r(\_))\), where \( \tilde{p}_1(t) = \frac{1}{\theta V^\theta(t)} p_1(t) = 1 \).

We consider the following transform

\[
\tilde{p}(t) = \left( \begin{array}{c} \tilde{p}_1(t) \\ \tilde{p}_2(t) \\ \tilde{p}_3(t) \end{array} \right) := \frac{1}{\theta V^\theta(t)} \tilde{p}(t), \quad 0 \leq t \leq T. \quad (3.13)
\]

By using (3.9) and (3.13), we have

\[
\tilde{p}(T) := \left( \begin{array}{c} \tilde{p}_1(T) \\ \tilde{p}_2(T) \end{array} \right) = \left( \begin{array}{c} 1 \\ \Phi_x(x^u(T)) \end{array} \right), \quad \text{and} \quad \tilde{p}(0) = \Psi_y(y^u(0)).
\]

The following properties of the generic martingale \( V^\theta \) are essential in order to investigate the properties of these new processes \((\tilde{p}(t), \tilde{q}(t), \tilde{r}(t, \_))\).

In this part, we want to prove the relationship between the exponential utility and the backward quadratic stochastic equation. First of all, it’s very important to write the expected exponential utility under this form

\[
e^{\lambda t} = \mathbb{E} \left[ A_{t,T} \mid \mathcal{F}_t \right] = \mathbb{E} \left[ \exp \theta \left[ \int_t^T f(s, x(s), y(s), z(s), r(s, \_), v(s)) \, ds + \Phi(x^v(T)) + \Psi(y^v(0)) \right] \mid \mathcal{F}_t \right]. \quad (3.14)
\]

For more details about the Expected exponential utility optimization, the reader can visit the papers [?].

**Lemma 3.1** The necessary and sufficient condition for the expected exponential utility \((3.14)\) is the backward quadratic stochastic equation

\[
\exp \{ \theta A^\theta(t) \} \quad (3.15)
\]

\[
e^{\lambda t} = \mathbb{E} \left[ \exp \theta \left[ \int_t^T f(t, x(t), y(t), z(t), r(t, \_), v(t)) \, dt + \Phi(x^v(T)) + \Psi(y^v(0)) \right] \mid \mathcal{F}_t \right] \Leftrightarrow
\]

\[
\left\{ \begin{array}{l}
d A^\theta(t) = - \left\{ f(t) + \frac{\theta}{2} |l(t)|^2 + \frac{\theta}{2} \int_{\Gamma} |L(t, \lambda)|^2 m(d\lambda) \\
\quad + \int_{\Gamma} \left\{ \frac{\exp(\theta r(t, \lambda)) - 1}{\theta} - r(t, \lambda) \right\} m(d\lambda) \right\} dt + l(t) \, dW(t) \\
- \int_{\Gamma} \left\{ \frac{\exp(\theta r(t, \lambda)) - 1}{\theta} \right\} \tilde{N}(dt, d\lambda) + \int_{\Gamma} L(t, \lambda) \, \tilde{N}(dt, d\lambda),
\end{array} \right.
\]

\[
A^\theta(T) = \Phi_x(x^u(T)) + \Psi(y^u(0)).
\]
where
\[ E \left[ \int_0^T |l(t)|^2 \, dt + \int_0^T \int_T |L(t, \lambda)|^2 \, m(d\lambda) \, dt \right] < \infty. \]

**Proof.** We assume that (3.13) holds, the we get
\[
\exp \left\{ \theta \Lambda^\theta (t) + \theta \int_t^T f(s, x(s), y(s), z(s), r(s), v(s)) \, ds \right\}
= \mathbb{E} \left[ \exp \theta \mathbb{E} \left[ \int_t^T f(s, x(s), y(s), z(s), r(s), v(s)) \, ds + \right.ight.
\left. \int_0^t f(s, x(s), y(s), z(s), r(s), v(s)) \, ds + \Phi (x^v (T)) + \Psi (y^v(0)) | \mathcal{F}_t \right] \right] = \mathbb{E} \left[ \theta \Lambda^\theta_T | \mathcal{F}_t \right].
\]

By using of martingale representation theorem, there exist a process square integrable \( Z \) with respect to norm \( ||Q||_{L^2([0, T], \mathbb{R})} \), and the process \( r(t, \lambda) \) in the space \( L^2_m ([0, T], \mathbb{R}) \), putting \( \mathbb{E} [\Lambda^\theta_T] = \exp \{ \theta \Lambda^\theta (0) \} \), we get
\[
\exp \{ \theta \Lambda^\theta (t) \} - \exp \{ \theta \Lambda^\theta (0) \} = \theta \int_0^t Z(s) \, dW(s) + \int_0^t \int_\Gamma r(t, \lambda) \, \tilde{N} (ds, d\lambda).
\]

By applying Lévy-Ito’s formula to \( \left( \exp \left\{ \theta \Lambda^\theta (t) + \theta \int_0^T f(s) \, ds \right\} \right) \), we get
\[
\theta \left[ \theta \Lambda^\theta (t) + \theta \int_0^T f(s) \, ds \right] + \frac{\theta^2}{2} \langle d\Lambda^\theta, d\Lambda^\theta \rangle_t + \int_\Gamma \{ \exp (\theta r(t, \lambda) - 1) - \theta r(t, \lambda) \} \, m(d\lambda) \, dt
\]
\[
+ \int_\Gamma \{ \exp (\theta r(t, \lambda) - 1) \} \, \tilde{N} (ds, d\lambda)
\]
\[
= \theta Z(t) \exp \left\{ \theta \Lambda^\theta (t) + \theta \int_0^T f(s) \, ds \right\} \, dW_t + \theta \int_\Gamma r(t, \lambda) \exp \left\{ \theta \Lambda^\theta (t) + \theta \int_0^T f(s) \, ds \right\} \, \tilde{N} (ds, d\lambda).
\]

Hence,
\[
\langle d\Lambda^\theta, d\Lambda^\theta \rangle = \theta^2 \left[ Z(t) \exp \left\{ \theta \Lambda^\theta (t) + \theta \int_0^T f(s) \, ds \right\} \right]^2 \, dt
\]
\[
+ \theta^2 \int_\Gamma \left\{ r(t, \lambda) \exp \left\{ \theta \Lambda^\theta (t) + \theta \int_0^T f(s) \, ds \right\} \right\} \, m(d\lambda)
\]
\[
:= \theta^2 |l(t)|^2 \, dt + \theta^2 \int_\Gamma |L(t, \lambda)|^2 \, m(d\lambda) dt.
\]
Then, by replacing in (3.16), we have the backward quadratic as the following expression

\[
\begin{cases}
    d\Lambda^\theta (t) = -\left\{ f(t) + \frac{\theta}{2} l(t)^2 + \frac{\theta}{2} \int_{\Gamma} |L(t,\lambda)|^2 m(d\lambda) \\
    + \int_{\Gamma} \left\{ \frac{\exp(\theta r(t,\lambda))}{\theta} - r(t,\lambda) \right\} m(d\lambda) \right\} dt + l(t) dW(t) \\
    - \int_{\Gamma} \left\{ \frac{\exp(\theta r(t,\lambda))}{\theta} - r(t,\lambda) \right\} \tilde{N}(dt,d\lambda) + \int_{\Gamma} L(t,\lambda) \tilde{N}(dt,d\lambda),
\end{cases}
\]

\[\Lambda^\theta (T) = \Phi_x (x^u (T)) + \Psi (y^u (0)),\]

where,

\[l(t) =: Z(t) \exp \left( \theta \Lambda^\theta (t) + \theta \int_0^T f(s) ds \right) \]

\[L(t,.) =: r(t,.) \exp \left( \theta \Lambda^\theta (t) + \theta \int_0^T f(s) ds \right).\]

As is proved in lemma 3.1, the process \(\Lambda^\theta\) is the first component of the \(\mathcal{F}_t\)-adapted pair of processes \((\Lambda^\theta, l, L(.) )\) which is the unique solution to the quadratic backward SDE with jump diffusion (3.14).

**Lemma 3.2** Suppose that \((H_4)\) holds. Then

\[\mathbb{E} \left( \sup_{0 \leq t \leq T} |\Lambda^\theta (t)|^2 \right) \leq C_T,\]  

(3.17)

In particular, \(V^\theta\) solves the following linear backward SDE

\[dV^\theta (t) = \theta l (t) V^\theta (t) dW(t) + \theta V^\theta (t) \int_{\Gamma} L(t,\lambda) \tilde{N}(dt,d\lambda), \quad V^\theta (T) = A^\theta_T.\]  

(3.18)

Hence, the process defined on \(\left( \Omega, \mathcal{F}, \left( \mathcal{F}_t^{(W,N)} \right)_{t \geq 0}, \mathbb{P} \right)\) by

\[L^\theta_t := \frac{V^\theta (t)}{V^\theta (0)} = \exp \left( \int_0^t \theta l (s) dW(s) - \frac{\theta^2}{2} \int_0^t |l(s)|^2 ds + \int_0^t \int_{\Gamma} L(s,\lambda) \tilde{N}(ds,d\lambda) \right. \]

\[\left. - \int_{\Gamma} \left\{ \frac{\exp(\theta r(t,\lambda))}{\theta} - r(t,\lambda) \right\} \tilde{N}(dt,d\lambda) - \frac{\theta^2}{2} \int_0^t \int_{\Gamma} |L(s,\lambda)|^2 m(d\lambda) ds \right) , \quad 0 \leq t \leq T,\]  

(3.19)

is a uniformly bounded \(\mathcal{F}\)-martingale.

**Proof.** First we prove (3.17). We assume that \((H_4)\) holds, \(f, \Phi\) and \(\Psi\) are bounded by a constant \(C > 0\), we have

\[0 < e^{-(2+T)C\theta} \leq A^\theta_T \leq e^{(2+T)C\theta} .\]  

(3.20)
Therefore, $V^\theta$ is a uniformly bounded $\mathcal{F}_t$–martingale satisfying
\begin{equation}
0 < e^{-(2+T)C\theta} \leq V^\theta(t) \leq e^{(2+T)C\theta}, \quad 0 \leq t \leq T. \tag{3.21}
\end{equation}

The complete proof see the Lemma 3.1 page 405 [11].

In the next, we will state and prove the necessary optimality conditions for the system driven by fully coupled FBSDE with jumps diffusion with a risk sensitive performance functional type. To this end, let us summarize and prove some lemmas that will we use thereafter.

**Lemma 3.3** The second and the third risk-sensitive adjoint equations of the solution $\tilde{p}_2(t), \tilde{q}_2(t), \tilde{\pi}_2(t, \lambda), \tilde{p}_3(t), \tilde{q}_3(t), \tilde{\pi}_3(t, \lambda)$ and $(V^\theta(t), l(t), L(t))$ become
\begin{equation}
\begin{cases}
\tilde{d}p_2(t) = -H^\theta(t) dt + (\tilde{q}_2(t) - \theta l(t)) \tilde{p}_2(t) dW_t^\theta + \int_\Gamma (\tilde{\pi}_2(t, \lambda) - \theta L(t, \lambda) \tilde{p}_2(t)) \tilde{N}^\theta(dt, d\lambda), \\
\tilde{d}p_3(t) = -H^\theta(t) dt - (H^\theta(t) - \theta l(t) \tilde{p}_3(t)) dW_t^\theta - \int_\Gamma (\nabla H_r(t) - \theta L(t, \lambda) \tilde{p}_3(t)) \tilde{N}^\theta(dt, d\lambda), \\
\tilde{d}V^\theta(t) = \theta V^\theta(t) l(t) dW_t + \theta V^\theta(t) \int_\Gamma L(t, \lambda) \tilde{N} dt, d\lambda, \\
V^\theta(T) = A^\theta(T), \\
\tilde{p}_2(T) = \Phi_x(x_T), \quad \tilde{p}_3(0) = \Psi_y(y(0)).
\end{cases} \tag{3.22}
\end{equation}

The solution $(\tilde{p}(t), \tilde{q}(t), \tilde{\pi}(t, \lambda), V^\theta(t), l(t), L(t))$ of the system \((3.22)\) is unique, such that
\begin{equation}
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |\tilde{p}(t)|^2 + \sup_{0 \leq t \leq T} |V^\theta(t)|^2 + \int_0^T \left( |\tilde{q}(t)|^2 + |l(t)|^2 \right) dt + \int_\Gamma \left( |\tilde{\pi}(t, \lambda)|^2 + |L(t, \lambda)|^2 \right) m(\lambda) d\lambda \right] < \infty, \tag{3.23}
\end{equation}

where
\begin{equation}
H^\theta(t, x(t), y(t), z(t), r(t), \tilde{p}(t), \tilde{q}(t), \tilde{\pi}(t, \lambda), V^\theta(t), l(t), L(t)) = f(t) + b(t) \tilde{p}_2 + \sigma(t) \tilde{q}_2 + \tilde{q}(t) - \theta z(t) l(t) \tilde{p}_3 + \int_\Gamma \left\{ \gamma (t^-), \lambda \tilde{\pi}_2(t, \lambda) - (g(t) - \theta r(t, \lambda) L(t, \lambda)) \tilde{p}_3 \right\} \lambda m(\lambda). \tag{3.24}
\end{equation}

**Proof.** We want to identify the processes $\tilde{\alpha}, \tilde{\beta}$ and $\tilde{\gamma}$ such that
\begin{equation}
\tilde{d}p(t) = -\tilde{\alpha}(t) dt + \tilde{\beta}(t) dW(t) + \int_\Gamma \tilde{\gamma}(t^-, \lambda) \tilde{N}(d\lambda, dt)
\end{equation}

By applying Itô’s formula to the process $\tilde{p}(t) = \theta V^\theta(t) \tilde{p}(t)$, and using the
expression of $V^\theta$ in [3.18], we obtain

\[
d\bar{p}(t) = -\left[ \frac{1}{\varphi V^\theta(t)} \begin{pmatrix} 0 & 0 & 0 \\ f_x(t) & b_x(t) & g_x(t) \\ f_y(t) & b_y(t) & g_y(t) \end{pmatrix} \begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{pmatrix} \right] \\
+ \frac{1}{\varphi V^\theta(t)} \left[ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_x(t) & 0 \\ 0 & \sigma_y(t) & 0 \end{pmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix} - \theta \begin{pmatrix} l_1(t) \\ l_2(t) \\ l_3(t) \end{pmatrix} \right] \bar{\beta}(t) \\
- \frac{1}{\varphi V^\theta(t)} \int_t^\infty \left[ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma_x(t-\lambda) & 0 \\ 0 & \gamma_y(t-\lambda) & 0 \end{pmatrix} \begin{pmatrix} \pi_1(\lambda) \\ \pi_2(\lambda) \\ \pi_3(\lambda) \end{pmatrix} \right] dt \\
+ \frac{1}{\varphi V^\theta(t)} \left[ \begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix} - \theta \begin{pmatrix} l_1(t) \\ l_2(t) \\ l_3(t) \end{pmatrix} \right] \bar{\rho}(t) dW(t) \\
+ \frac{1}{\varphi V^\theta(t)} \int_t^\infty \left[ \begin{pmatrix} \pi_1(\lambda) \\ \pi_2(\lambda) \\ \pi_3(\lambda) \end{pmatrix} - \theta \int_t^\infty \begin{pmatrix} L_1(\lambda) \\ L_2(\lambda) \\ L_3(\lambda) \end{pmatrix} \bar{\rho}(t) \right] \bar{N}(d\lambda, dt)
\]

By identifying the coefficients, and using the relation $\bar{p}(t) = \frac{1}{\varphi V^\theta(t)} \tilde{p}(t)$, the diffusion coefficient $\tilde{\beta}(t)$ will be

\[
\tilde{\beta}(t) = \begin{pmatrix} \tilde{q}_1(t) \\ \tilde{q}_2(t) \\ \tilde{q}_3(t) \end{pmatrix} - \theta \begin{pmatrix} l_1(t) \\ l_2(t) \\ l_3(t) \end{pmatrix} \tilde{\rho}(t),
\]

the drift term of the process $\tilde{\rho}(t)$

\[
\tilde{\alpha}(t) = \begin{pmatrix} 0 & 0 & 0 \\ f_x(t) & b_x(t) & g_x(t) \\ f_y(t) & b_y(t) & g_y(t) \end{pmatrix} \begin{pmatrix} \bar{p}_1(t) \\ \bar{p}_2(t) \\ \bar{p}_3(t) \end{pmatrix} \\
+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_x(t) & 0 \\ 0 & \sigma_y(t) & 0 \end{pmatrix} \begin{pmatrix} \bar{q}_1(t) \\ \bar{q}_2(t) \\ \bar{q}_3(t) \end{pmatrix} \\
+ \theta \begin{pmatrix} l_1(t) \\ l_2(t) \\ l_3(t) \end{pmatrix} \tilde{\beta}(t) - \int_t^\infty \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma_x(t-\lambda) & 0 \\ 0 & \gamma_y(t-\lambda) & 0 \end{pmatrix} \bar{\gamma}(t-\lambda) m(d\lambda).
\]

the jump diffusion gets the form

\[
\tilde{\gamma}(t^-) = \begin{pmatrix} \bar{\pi}_1(t^-) \\ \bar{\pi}_2(t^-) \\ \bar{\pi}_3(t^-) \end{pmatrix} - \theta \begin{pmatrix} L_1(t^-) \\ L_2(t^-) \\ L_3(t^-) \end{pmatrix} \bar{\rho}(t)
\]

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Finally, we obtain
\[
d\tilde{p}(t) = -\left[ \begin{array}{ccc}
0 & 0 & 0 \\
& (f_x(t) & b_x(t) & g_x(t) \\
& f_y(t) & b_y(t) & g_y(t) \\
& 0 & 0 & 0 \\
& 0 & \sigma_x(t) & 0 \\
& 0 & \sigma_y(t) & 0 \\
\end{array} \right] \left[ \begin{array}{c}
\tilde{q}_1(t) \\
\tilde{q}_2(t) \\
\tilde{q}_3(t) \\
\end{array} \right] - \theta \left[ \begin{array}{c}
\tilde{l}_1(t) \\
\tilde{l}_2(t) \\
\tilde{l}_3(t) \\
\end{array} \right] \tilde{\beta}(t)
\]
\[
-\int \Gamma \left[ \begin{array}{ccc}
0 & 0 & 0 \\
& L_1(t,\lambda) & \bar{q}_1(t) \\
& L_2(t,\lambda) & \bar{q}_2(t) \\
& L_3(t,\lambda) & \bar{q}_3(t) \\
\end{array} \right] \left[ \begin{array}{c}
\pi_1(t,\lambda) \\
\pi_2(t,\lambda) \\
\pi_3(t,\lambda) \\
\end{array} \right] \gamma(t,\lambda) m(d\lambda) dt
\]
\[
+ \int \Gamma \left[ \begin{array}{ccc}
\tilde{\pi}_1(t,\lambda) & \tilde{\pi}_2(t,\lambda) & \tilde{\pi}_3(t,\lambda) \\
\end{array} \right] - \theta \left[ \begin{array}{c}
\tilde{l}_1(t) \\
\tilde{l}_2(t) \\
\tilde{l}_3(t) \\
\end{array} \right] \tilde{p}(t) dW(t)
\]
\[
+ \int \Gamma \left[ \begin{array}{ccc}
\tilde{\pi}_1(t,\lambda) & \tilde{\pi}_2(t,\lambda) & \tilde{\pi}_3(t,\lambda) \\
\end{array} \right] - \theta \left[ \begin{array}{c}
\tilde{l}_1(t) \\
\tilde{l}_2(t) \\
\tilde{l}_3(t) \\
\end{array} \right] \tilde{p}(t) \tilde{N}(d\lambda, dt)
\]

It is easily verified that
\[
\left\{ \begin{array}{l}
d\tilde{p}_1(t) = \tilde{q}_1(t) [-\theta l_1(t) dt + dW(t)] + \int \tilde{\pi}_1(t,\lambda) [-\theta L_1(t,\lambda) m(d\lambda) dt + \tilde{N}(d\lambda, dt)] \\
\tilde{p}_1(T) = 1
\end{array} \right.
\]

In view of (3.19), we may use Girsanov’s Theorem to claim that
\[
\left\{ \begin{array}{l}
d\tilde{p}_1(t) = \tilde{q}_1(t) dW^\theta(t) + \int \tilde{\pi}_1(t,\lambda) \tilde{N}^\theta(d\lambda, dt) \\
\tilde{p}_1(T) = 1
\end{array} \right., \quad \mathbb{P}^\theta - a.s.,
\]
where,
\[
dW^\theta(t) = -\theta l(t) dt + dW(t)
\]
\[
\tilde{N}^\theta(d\lambda, dt) = -\theta L(t,\lambda) m(d\lambda) + \tilde{N}(d\lambda, dt),
\]

(3.25)

\(W^\theta(t)\) is a \(\mathbb{P}^\theta\)-Brownian motion and \(\tilde{N}^\theta(\lambda, t)\) is a \(\mathbb{P}^\theta\)-compensator Poisson measure, where,
\[
\frac{d\mathbb{P}^\theta}{d\mathbb{P}} |_{\mathcal{F}_t} := L_t^\theta = \exp \left( \int_0^t \theta l(s) dW(s) - \frac{\theta^2}{2} \int_0^t |l(s)|^2 ds + \int_0^t \int \tilde{N}(ds, d\lambda) \right)
\]
\[
- \int \Gamma \left( \frac{\exp(\theta r(t,\lambda)) - 1}{\theta} \right) \tilde{N}(dt, d\lambda) - \frac{\theta^2}{2} \int_0^t \int |L(s,\lambda)|^2 m(d\lambda) ds
\]
\[
- \int \Gamma \left( \frac{\exp(\theta r(t,\lambda)) - 1}{\theta} - r(t,\lambda) \right) m(d\lambda) \quad 0 \leq t \leq T.
\]
Therefore, the second and third components of \( \tilde{p}_2 \) and \( \tilde{p}_3 \) in (3.26), are given
by

\[
\begin{aligned}
    d\tilde{p}_2 (t) &= - \left\{ f_x (t) + b_x (t) \tilde{p}_2 (t) + g_x (t) \tilde{p}_3 (t) + \sigma_x (t) \tilde{q}_2 (t) + \int_\Gamma \gamma_x (t-\cdot, \lambda) \tilde{p}_2 (t, \lambda) m (d\lambda) \right\} dt \\
    &+ \left\{ \tilde{q}_2 (t) - \theta l_2 (t) \tilde{p}_2 (t) \right\} dW^\theta (t) + \int_\Gamma \left\{ \tilde{p}_2 (t, \lambda) - \theta L_2 (t, \lambda) \tilde{p}_2 (t) \right\} \tilde{N}^\theta (d\lambda, dt), \\
    \tilde{p}_2 (T) &= \Phi_x (x_T),
\end{aligned}
\]

(3.27)

and

\[
\begin{aligned}
    d\tilde{p}_3 (t) &= - \left\{ f_y (t) + b_y (t) \tilde{p}_2 (t) + g_y (t) \tilde{p}_3 (t) + \sigma_y (t) \tilde{q}_2 (t) + \theta l_3 (t) \tilde{q}_3 (t) - \int_\Gamma \gamma_y (t-\cdot, \lambda) \tilde{p}_2 (t, \lambda) m (d\lambda) \right\} dt \\
    &+ \left\{ \tilde{q}_3 (t) - \theta l_3 (t) \tilde{p}_3 (t) + \int_\Gamma \gamma_\tilde{z} (t-\cdot, \lambda) \tilde{p}_2 (t, \lambda) m (d\lambda) + \theta l_3 (t) \tilde{p}_3 (t) \right\} dW^\theta (t) \\
    &- \int_\Gamma \left\{ f_\tilde{r} (t) + b_\tilde{r} (t) \tilde{p}_2 (t) + g_\tilde{r} (t) \tilde{p}_3 (t) + \int_\Gamma \gamma_\tilde{r} (t-\cdot, \lambda) \tilde{p}_2 (t, \lambda) m (d\lambda) \right\} \tilde{N}^\theta (d\lambda, dt), \\
    \tilde{p}_3 (0) &= \Psi_y (y (0)),
\end{aligned}
\]

(3.28)

or in equivalent expression the adjoint equations for \((\tilde{p}_2, \tilde{q}_2), (\tilde{p}_3, \tilde{q}_3), (\tilde{\pi}_2, \tilde{\pi}_3)\) and \((V^\theta, l, L)\) become

\[
\begin{aligned}
    d\tilde{p}_2 (t) &= - H^\theta_x (t) dt + (\tilde{q}_2 (t) - \theta l_2 (t) \tilde{p}_2 (t)) dW^\theta (t) + \int_\Gamma \left\{ \tilde{p}_2 (t, \lambda) - \theta L_2 (t, \lambda) \tilde{p}_2 (t) \right\} \tilde{N}^\theta (d\lambda, dt), \\
    d\tilde{p}_3 (t) &= - H^\theta_y (t) dt - H^\theta_\tilde{r} (t) dW^\theta (t) - \int_\Gamma \nabla H^\theta_\tilde{r} (t) \tilde{N}^\theta (d\lambda, dt), \\
    dV^\theta (t) &= \theta l (t) V^\theta (t) dW (t) + \theta V^\theta (t) \int_\Gamma L (t, \lambda) \tilde{N} (d\lambda, dt), \\
    V^\theta (T) &= A^\theta (T), \\
    \tilde{p}_2 (T) &= \Phi_x (x (T)), \quad \tilde{p}_3 (0) = \Psi_y (y (0)).
\end{aligned}
\]

The solution \((\tilde{p}, \tilde{q}, \tilde{\pi}, V^\theta, l, L)\) of the system (3.22) is unique, such that

\[
E \left[ \sup_{0 \leq t \leq T} |\tilde{p} (t)|^2 + \sup_{0 \leq t \leq T} |V^\theta (t)|^2 + \int_0^T \left( |\tilde{q} (t)|^2 + |l (t)|^2 \right) m (d\lambda) dt \right] < \infty,
\]

where

\[
H^\theta (t) := H^\theta_x (t, x (t), y (t), z (t), \gamma^u (t, \lambda), \tilde{p}_2 (t), \tilde{q}_2 (t), \tilde{p}_3 (t), \tilde{p}_2 (t, \lambda), V^\theta (t), l (t), L (t, \lambda))
\]

\[
= f (t) + b (t) \tilde{p}_2 + \sigma (t) \tilde{q}_2 + (g (t) + z (t) \theta l (t)) \tilde{p}_3 - \int_\Gamma \left\{ \gamma (t-\cdot, \lambda) \tilde{p}_2 (t, \lambda) - (g (t) + r (t, \lambda) L (t, \lambda)) \tilde{p}_3 \right\} m (d\lambda).
\]

The proof is completed. ■
**Theorem 3.2 (Risk-Sensitive necessary optimality conditions):** We assume that \( (H_1) \) holds, if \( (x^u, y^u, z^u, r^u, u) \) is an optimal solution of the risk-sensitive control problem \( \{2.3, 2.6, 2.7\} \), then there exist \( \mathcal{F}_t \)-adapted processes \( (V^0(t), l(t), L(t, \lambda)) \), and \( (\tilde{p}_2(t), \tilde{q}_2(t)), (\tilde{p}_3(t), \tilde{q}_3(t)) \) such that \( \partial H^0(t) \leq 0 \),

for all \( u \in \mathcal{U} \), almost every \( 0 \leq t \leq T \) and \( \mathbb{P} \)-almost surely.

**Proof.** The Hamiltonian \( \tilde{H}^0 \) associated with \( \{2.3\} \), is given by

\[
\tilde{H}^0(t, \xi^u(t), x^u(t), y^u(t), z^u(t), r^u(t, \cdot), \tilde{q}^u(t), \tilde{\pi}^u(t, \cdot)) = \{ \theta V^0(t) \} \sum_{\tilde{p}_2(t), \tilde{q}_2(t), (\tilde{p}_3, \tilde{q}_3) \in U} (V^0(t), l_2(t), l_3(t), L_2(t, \cdot), L_3(t, \cdot)) ,
\]

and \( H^0 \) is the risk-sensitive Hamiltonian given by \( \{2.21\} \). To arrive at a risk-sensitive stochastic maximum principle expressed in terms of the adjoint processes \( (\tilde{p}_2, \tilde{q}_2), (\tilde{p}_3, \tilde{q}_3), (\tilde{\pi}_2, \tilde{\pi}_3) \) and \( (V^0, l, L) \), which solve \( \{2.22\} \). Hence, since \( V^0 > 0 \), the variational inequality \( \{3.10\} \) translates into \( \partial H^0(t) \leq 0 \), for all \( u \in \mathcal{U} \), almost every \( 0 \leq t \leq T \) and \( \mathbb{P} \)-almost surely. □

### 4 Risk sensitive sufficient optimality conditions

This section is concerned with a study of the necessary condition of optimality \( \{3.10\} \) when it becomes sufficient.

**Theorem 4.1 (Risk sensitive sufficient optimality conditions):** Assume that \( \Phi(.) \) and \( \Psi(.) \) are convex and for all \( (x, y, z, r, v) \in \mathbb{R} \times \mathbb{R} \times \prod \mathcal{U} \times \mathcal{U} \) the function \( H(t, x, y, z, r, v, p, q, \pi) \) is convex, and for any \( v \in \mathcal{U} \) such that \( \mathbb{E}[v]^2 < \infty \). Then, \( u \) is an optimal control of the problem \( \{2.25, 2.26, 2.27\} \), if it satisfies \( \{4.10\} \).

**Proof.** Let \( u \) be an admissible control (candidate to be optimal) for any \( v \in \mathcal{U} \), we have

\[
J^0_v - J^0_u = \mathbb{E}[\exp\{\theta \Psi(y^v(0)) + \theta \Phi(x^v(T)) + \theta \xi^v(T)\}] - \mathbb{E}[\exp\{\theta \Psi(y^u(0)) + \theta \Phi(x^u(T)) + \theta \xi^u(T)\}].
\]

Since \( \Psi \) and \( \Phi \) are convex, and applying Taylor’s expansion, we get

\[
J^0_v - J^0_u \geq \mathbb{E}[\theta A_T(x^v(T) - x^u(T))] + \mathbb{E}[\theta \Phi_x(x^u(T)) A_T(x^v(T) - x^u(T))] + \mathbb{E}[\theta \Psi(y^u(0)) A_T(y^v(T) - y^u(T))].
\]

According to \( \{3.10\} \), we remark that \( p_1(T) = \theta A_T \), \( p_2(T) = \theta \Phi_x(x^u(T)) A_T \), \( p_3(T) = \theta \Psi(y^u(0)) A_T \), and \( p_3(0) = \theta \Psi_y(y^u(0)) A_T \), then

\[
J^0_v - J^0_u \geq \mathbb{E}[p_1(T)(\xi^v_T - \xi^u_T)] + \mathbb{E}[p_2(T)(x^v(T) - x^u(T))] + \mathbb{E}[p_3(0)(y^v(0) - y^u(0))].
\]

(4.29)
We apply Itô’s formula to \( p_1(t) (\xi^v(t) - \xi^u(t)) \),
\[
d(p_1(t) (\xi^v(t) - \xi^u(t))) = (\xi^v(t) - \xi^u(t)) dp_1(t) + p_1(t) d(\xi^v(t) - \xi^u(t))
+ \langle (\xi^v - \xi^u), p_1 \rangle_t dt + \int_T \langle (\xi^v - \xi^u), p_1 \rangle_t m(d\lambda) dt
\]
then
\[
\int_0^T (p_1(t) (\xi^v(t) - \xi^u(t))) dt = \int_0^T (\xi^v(t) - \xi^u(t)) dp_1(t) + \int_0^T p_1(t) d(\xi^v(t) - \xi^u(t))
+ \int_T \langle (\xi^v - \xi^u), p_1 \rangle_t dt + \int_0^T \int_T \langle (\xi^v - \xi^u), p_1 \rangle_t m(d\lambda) dt
= \int_0^T (f(t, x^v(t), y^v(t), z^v(t), r^v(t,.), v_\lambda) - f(t, x^u(t), y^u(t), z^u(t), r^u(t,.), u_\lambda)) p_1(t) dW_t
+ \int_0^T \int_T \int_T (f(t, x^v(t), y^v(t), z^v(t), r^v(t,.), v_\lambda) - f(t, x^u(t), y^u(t), z^u(t), r^u(t,.), u_\lambda)) p_1(t) \pi_\lambda \tilde{N}(d\lambda, dt)
+ \int_0^T (f(t, x^v(t), y^v(t), z^v(t), r^v(t,.), v_\lambda) - f(t, x^u(t), y^u(t), z^u(t), r^u(t,.), u_\lambda)) p_1(t) dt
\]
We apply expectation, we get
\[
\mathbb{E}[p_1(T) (\xi^v(T) - \xi^u(T))] = \mathbb{E}
\left[\int_0^T (f(t, x^v(t), y^v(t), z^v(t), r^v(t,.), v_\lambda) - f(t, x^u(t), y^u(t), z^u(t), r^u(t,.), u_\lambda)) p_1(t) dt\right]
\tag{4.30}
\]
And we apply also Itô’s formula to \( p_2(t) (x^v(t) - x^u(t)) \)
\[
d(p_2(t) (x^v(t) - x^u(t))) = (x^v(t) - x^u(t)) dp_2(t) + p_2(t) d(x^v(t) - x^u(t))
+ \langle x^v - x^u, p_2 \rangle_t dt + \int_T \langle x^v - x^u, p_2 \rangle_t m(d\lambda) dt
\]
then

\[
\int_0^T d\left( p_2(t)(x^v(t) - x^u(t)) \right) = \int_0^T (b(t, x^v(t), y^v(t), z^v(t), r^v(t, \cdot), v_t) - b(t, x^u(t), y^u(t), z^u(t), r^u(t, \cdot), u_t)) p_2(t) dt \\
+ \int_0^T (\sigma(t, x^v(t), y^v(t), z^v(t), r^v(t, \cdot), v_t) - \sigma(t, x^u(t), y^u(t), z^u(t), r^u(t, \cdot), u_t)) p_2(t) dt \\
+ \int_0^T \int_\Gamma (\gamma(t, x^v(t), y^v(t), z^v(t), r^v(t, \cdot), v_t) - \gamma(t, x^u(t), y^u(t), z^u(t), r^u(t, \cdot), u_t)) p_2(t) dW_t \\
+ \int_0^T g_x(t) q_3 + \int_\Gamma \gamma_x(t-, \lambda) \pi_2(\lambda, t) m(\lambda) \left( x^v_t - x^u_t \right) dt \\
+ \int_0^T g_y(t) q_2(t) (x^v_t - x^u_t) dB_t + \int_0^T \int_\Gamma \pi_2(\lambda, t) (x^v_t - x^u_t) \tilde{N}(d\lambda, dt) \\
+ \int_0^T \int_\Gamma (\gamma(t, x^v(t), y^v(t), z^v(t), r^v(t, \cdot), v_t) - \gamma(t, x^u(t), y^u(t), z^u(t), r^u(t, \cdot), u_t)) \pi(t, \lambda) m(d\lambda) dt
\]

We apply expectation, we get

\[
\mathbb{E} \left[ p_2(T)(x^v(T) - x^u(T)) \right] = \\
\mathbb{E} \left[ \int_0^T - (f_x(t) p_1 + b_x(t) p_2 + \sigma_x(t) q_2) \\
+ g_x(t) q_3 + \int_\Gamma \gamma_x(t-, \lambda) \pi_2(\lambda, t) m(\lambda) \left( x^v_t - x^u_t \right) dt \right] \\
+ \mathbb{E} \left[ \int_0^T (b(t, x^v(t), y^v(t), z^v(t), r^v(t, \cdot), v_t) - b(t, x^u(t), y^u(t), z^u(t), r^u(t, \cdot), u_t)) p_2(t) dt \right] \\
+ \mathbb{E} \left[ \int_0^T (\sigma(t, x^v(t), y^v(t), z^v(t), r^v(t, \cdot), v_t) - \sigma(t, x^u(t), y^u(t), z^u(t), r^u(t, \cdot), u_t)) q_2(t) dt \right] \\
+ \mathbb{E} \left[ \int_0^T \int_\Gamma (\gamma(t, x^v(t), y^v(t), z^v(t), r^v(t, \cdot), v_t) - \gamma(t, x^u(t), y^u(t), z^u(t), r^u(t, \cdot), u_t)) \pi(t, \lambda) m(d\lambda) dt \right]
\]

We apply also Itô’s formula to \( p_3(t)(y^v(t) - y^u(t)) \)

\[
d(p_3(t)(y^v(t) - y^u(t))) = (y^v(t) - y^u(t)) dp_3(t) + p_3(t) d(y^v(t) - y^u(t)) \\
+ \langle y^v - y^u, p_3 \rangle dt + \int_\Gamma \langle y^v - y^u, p_3 \rangle m(d\lambda) dt
\]
then

\[
\int_0^T \{ p_3 (t) (y^v (t) - y^u (t)) \} = \int_0^T \{ g (t, x^v (t), y^v (t), z^v (t), r^v (t, \cdot), v_t) - g (t, x^u (t), y^u (t), z^u (t), r^u (t, \cdot), u_t) \} p_3 (t) \, dt
\]

\[
\int_0^T (z^v (t) - z^u (t)) \, p_3 (t) \, dW_t
\]

\[
+ \int_0^T \int_{\Gamma} (r^v (t, \lambda) - r^u (t, \lambda)) \, p_3 (t) \, \tilde{N} \, (dt, d\lambda)
\]

\[
+ \int_0^T (- (f_y (t) p_1 + b_y (t) p_2 + \sigma_y (t) q_2
\]

\[
+ g_y (t) p_3 + \int_{\Gamma} \gamma_y (t-, \lambda) \pi_2 (\lambda, t) \, m \, (d\lambda) \} \, (y^v_t - y^u_t) \, dt
\]

\[
+ \int_0^T (- (f_z (t) p_1 + b_z (t) p_2 + \sigma_z (t) q_2 + g_z (t) p_3
\]

\[
+ \int_{\Gamma} \gamma_z (t-, \lambda) \pi_2 (\lambda, t) \, m \, (d\lambda) \} \, (z^v_t - z^u_t) \, dW_t
\]

\[
+ \int_0^T \int_{\Gamma} (- (f_r (t) p_1 + b_r (t) p_2 + \sigma_r (t) q_2 + g_r (t) p_3
\]

\[
+ \int_{\Gamma} \gamma_r (t-, \lambda) \pi_2 (\lambda, t) \, m \, (d\lambda) \} \, (r^v_t (\lambda) - r^u_t (\lambda)) \, m \, (d\lambda) \, dt
\]

We apply expectation, We get:

\[
\mathbb{E} [p_3 (0) (y^v (0) - y^u (0))] =
\]

\[
\mathbb{E} \left[ \int_0^T \{ g (t, x^v (t), y^v (t), z^v (t), r^v (t, \cdot), v_t) - g (t, x^u (t), y^u (t), z^u (t), r^u (t, \cdot), u_t) \} p_3 (t) \, dt \right]
\]

\[
- \mathbb{E} \left[ \int_0^T (f_y (t) p_1 (t) + b_y (t) p_2 (t) + g_y (t) p_3 (t) + \sigma_y (t) q_2 (t)
\]

\[
+ \int_{\Gamma} \gamma_y (t-, \lambda) \pi_2 (\lambda, t) \, m \, (d\lambda) \} \, (y^v_t - y^u_t) \, dt \right] \tag{4.32}
\]

\[
- \mathbb{E} \left[ \int_0^T (f_z (t) p_1 (t) + b_z (t) p_2 (t) + g_z (t) p_3 (t) + \sigma_z (t) q_2 (t)
\]

\[
+ \int_{\Gamma} \gamma_z (t-, \lambda) \pi_2 (\lambda, t) \, m \, (d\lambda) \} \, (z^v_t - z^u_t) \, dt \right]
\]

\[
- \mathbb{E} \left[ \int_0^T \int_{\Gamma} (f_r (t) p_1 (t) + b_r (t) p_2 (t) + g_r (t) p_3 (t) + \sigma_r (t) q_2 (t)
\]

\[
+ \gamma_r (t-, \lambda) \pi_2 (\lambda, t) \, (r^v_t (\lambda) - r^u_t (\lambda)) \, m \, (d\lambda) \, dt \right]
\]
implies that
\[ E \geq \int_0^T \left( H^\theta (t, x^u(t), y^u(t), z^u(t), r^u(t, \cdot), v_t, p^u(t), q^u(t), \pi(t, \cdot)) \right) dt \]

By replacing (4.30), (4.31) and (4.32) into (4.29), we have

\[
J^\theta (v) - J^\theta (u)
\geq \mathbb{E} \left[ \int_0^T \left( H^\theta (t, x^v(t), y^v(t), z^v(t), r^v(t, \cdot), v_t, p^v(t), q^v(t), \pi(t, \cdot)) \right) dt \right]
- \mathbb{E} \left[ \int_0^T \left( H^\theta (t, x^u(t), y^u(t), z^u(t), r^u(t, \cdot), v_t, p^u(t), q^u(t), \pi(t, \cdot)) \right) dt \right]
- \mathbb{E} \left[ \int_0^T \left( H^\theta (t, x^u(t), y^u(t), z^u(t), r^u(t, \cdot), v_t, p^u(t), q^u(t, \cdot)) \right) dt \right]
- \mathbb{E} \left[ \int_0^T \nabla H^\theta (t, x^u(t), y^u(t), z^u(t), r^u(t, \cdot), v_t, p^u(t), q^u(t, \cdot)) \right) dt]

Since the Hamiltonian \( H \) is concave with respect to \((x, y, z, r, v)\), we have

\[
\mathbb{E} \left[ \int_0^T H^\theta (t, x^v(t), y^v(t), z^v(t), r^v(t, \cdot), v_t, p^v(t), q^v(t), \pi(t, \cdot)) dt \right]
\leq \mathbb{E} \left[ \int_0^T \left( H^\theta (t, x^u(t), y^u(t), z^u(t), r^u(t, \cdot), v_t, p^u(t), q^u(t, \cdot)) \right) dt \right]
+ \mathbb{E} \left[ \int_0^T \nabla H^\theta (t, x^u(t), y^u(t), z^u(t), r^u(t, \cdot), v_t, p^u(t), q^u(t, \cdot)) dt \right]
+ \mathbb{E} \left[ \int_0^T \nabla H^\theta (t, x^u(t), y^u(t), z^u(t), r^u(t, \cdot), v_t, p^u(t), q^u(t, \cdot)) dt \right]
+ \mathbb{E} \left[ \int_0^T \nabla H^\theta (t, x^u(t), y^u(t), z^u(t), r^u(t, \cdot), v_t, p^u(t), q^u(t, \cdot)) dt \right]

Then

\[
J^\theta (v) - J^\theta (u)
\geq \mathbb{E} \left[ \int_0^T \left( H^\theta (t, x^v(t), y^v(t), z^v(t), r^v(t, \cdot), v_t, p^v(t), q^v(t, \cdot)) \right) dt \right] .
\]

In virtue of the necessary condition of optimality (3.10), the last inequality implies that \( J^\theta (v) - J^\theta (u) \geq 0 \). Then, the theorem is improved. ■
5 Example: Mean-Variance (Cash-flow):

Now we return to the problem of optimal portfolio stated in the motivating example, and apply the risk sensitive necessary optimality condition (Theorem 3.2).

Our state dynamics is

\[
\begin{align*}
\frac{dx(t)}{dt} &= (\rho v(t) - cx(t)) dt + \sigma v(t) dW(t) + \int_{\Gamma} v(t) (1 + r(t, \lambda)) \tilde{N}(d\lambda, dt), \\
x(0) &= m_0 = d,
\end{align*}
\]

and

\[
\begin{align*}
\frac{dy(t)}{dt} &= (\rho v(t) - cx(t) + \lambda y(t)) dt + z(t) dW(t) + \int_{\Gamma} r(t, \lambda) \tilde{N}(d\lambda, dt), \\
y(T) &= 0 = a.
\end{align*}
\]

The cost functional is

\[ J^\theta(v(.)) = \exp \left\{ \theta J^0(v(.)) \right\}, \]

where \( J \) is the neutral cost functional given by the following expected with an exponential form see section 1.2.3

\[ \tilde{J}^0(v(.)) = \frac{\theta}{2} \mathbb{E}(\Psi_T - a)^2 + \mathbb{E}(\Psi_T) + o(\theta^2), \]  

(5.35)

Where \( \Psi_T = (x_T + y_0) \). The investor wants to minimize (5.35) subject to (5.33) and (5.34) by taking \( v(.) \) over \( \mathcal{U} \), the mean–variance portfolio selection problem is to find \( u(t) \) which minimize

\[ \text{Var}(\Psi_T) = \mathbb{E}(x_T + y_0 - a)^2 \]

The Hamiltonian function (5.24) gets the form

\[
H^\theta(t) := H^\theta(t, x(t), y(t), z(t), r(t, \lambda), \tilde{p}_2(t), \tilde{q}_2(t), \tilde{p}_3(t), \tilde{q}_3(t), l(t), L(t,.), v_1) = f(t) + b(t) \tilde{p}_2(t) + \sigma(t) \tilde{q}_2(t) + \{g(t) - \theta l(t) z(t)\} \tilde{p}_3(t) \\
+ \int_{\Gamma} \{\gamma(t, -\lambda) \tilde{q}_3(t, \lambda) - (g(t) - \theta L(t, \lambda) r(t, \lambda)) \tilde{p}_3(t)\} m(d\lambda) \\
= (\rho v(t) - cx(t)) \tilde{p}_2(t) + \sigma v(t) \tilde{q}_2(t) + \{\rho v(t) - cx(t) + \lambda y(t)) - \theta l(t) z(t)\} \tilde{p}_3(t) \\
- \int_{\Gamma} \{v(t) (1 + r(t, \lambda)) \tilde{q}_3(t, \lambda) - (\rho v(t) - cx(t) + \lambda y(t)) - \theta L(t, \lambda) r(t, \lambda)) \} \tilde{p}_3(t) \} m(d\lambda).
\]

Then, to get the optimal control, the derivative of the above Hamiltonian with respect to the control process gives us
\[
H^\theta_u (t) := H^\theta_u (t, x (t), y (t), z (t), r (t, \cdot), \bar{\theta}_2 (t), \bar{\theta}_3 (t), \bar{\pi}_2 (t), \bar{\pi}_3 (t), \bar{l} (t), \bar{L} (t, \cdot), v_t)
= \tilde{p} \bar{\theta}_2 (t) + \sigma \bar{\theta}_2 (t) + \int_{\Gamma} (1 + r (t, \lambda)) \bar{\pi}_2(t, \lambda) m (d\lambda)
= 0
\]

(5.36)

Let \((x^u (t), u (t))\) be an optimal pair, the adjoint equation (5.27), is given by

\[
\begin{align*}
\dot{p}^u_2 (t) &= c (p^u_2 (t) + c \bar{\theta}_2 (t)) dt + (\bar{\theta}_2 (t) \bar{\theta}_2 (t)) dW^\theta (t) \\
&+ \int_{\Gamma} (\bar{\pi}_2(t, \lambda) - \theta L_2 (t, \lambda) \bar{p}^u_2 (t)) \bar{N} (d\lambda, dt), \\
\bar{p}^u_2 (T) &= 1 + \theta (x_T - y_0 - a).
\end{align*}
\]

By using of (3.25), we get

\[
\begin{align*}
\dot{p}^u_2 (t) &= c (p^u_2 (t) + c \bar{\theta}_2 (t)) dt + (\bar{\theta}_2 (t) \bar{\theta}_2 (t)) dW^\theta (t) \\
&+ \int_{\Gamma} (\bar{\pi}_2(t, \lambda) - \theta L_2 (t, \lambda) \bar{p}^u_2 (t)) \bar{N} (d\lambda, dt), \\
\bar{p}^u_2 (T) &= 1 + \theta (x_T - y_0 - a).
\end{align*}
\]

(5.37)

Therefore, an optimal solution \((x^u_n, \bar{p}^u_2 (t), u_t)\) can be obtained by solving the system FBSDE with jumps diffusion (5.33) and (5.34), unfortunately, in such system is difficult to find the explicit solution, to this end we use the similar technique as in (33) see also (32), we conjecture the solution to (5.33) and (5.37) is related by

\[
\bar{p}^u_2 (t) = A (t) x^u (t) + B (t),
\]

(5.38)

for some deterministic differentiable functions \(A (t)\) and \(B (t)\). Applying Itô’s formula to (5.38), we get

\[
\begin{align*}
\dot{p}^u_2 (t) &= \left[ A (t) x^u (t) + A (t) (\rho u_t - c x^u (t)) + B (t) \right] dt + A (t) \sigma u_t dW (t) \\
&+ \int_{\Gamma} A(t) (1 + r (t, \lambda)) u_t \bar{N} (d\lambda, dt), \\
\bar{p}^u_2 (T) &= A (T) x^u (T) + B (T).
\end{align*}
\]

(5.39)

On the other hand, by substituting (5.38) into (5.37), and denote by

\[
\begin{align*}
\bar{\theta}_3 (t) &= \theta L_2 (t, \lambda) \bar{p}^u_2 (t) - \bar{\theta}_2 (t), \\
\bar{\pi}_3 (t, \cdot) &= \bar{\pi}_2 (t, \cdot) - \theta L_2 (t, \cdot) \bar{p}^u_2 (t).
\end{align*}
\]

(5.40)

By using the Girsanov’s transformation in (5.39), as in section 2 lemma (3.2),
we obtain
\[
\begin{cases}
\frac{d\bar{p}_2^u}{dt} = \left\{ (c + \theta^2 \rho^2 + t + \int I \theta^2 L^2 (t, \lambda) m (d\lambda)) \bar{p}_2^u - \theta \bar{q}_3^u - \theta \bar{q}_3^u \right\} dt \\
\bar{p}_2^u (T) = 1 + \theta (xT - y_0 - a).
\end{cases}
\] (5.41)

By equating the coefficients and the final conditions of (5.41) with (5.39), we have
\[
\begin{align*}
\bar{\pi}_3 (t, \lambda) &= A(t) (1 + r (t, \cdot)) u_t, \\
\bar{q}_3^u (t) &= \sigma u_t A(t), \\
A(T) &= \theta, \\
B(T) &= 1 - \theta (y_0 + a).
\end{align*}
\] (5.42)

By identifying (5.40) with (5.42), we can rewrite
\[
\bar{q}_2^v (t) = \theta \bar{L}_2 (t, \cdot) (A(t) x^u (t) + B(t)) + r (t, \cdot) u_t A(t),
\]
and
\[
\bar{\pi}_2^v (t, \cdot) = \theta L_2 (t, \cdot) (A(t) x^u (t) + B(t)) + r (t, \cdot) u_t A(t),
\]
then replacing the both equations (5.42), and the last equations of \( \bar{q}_2^v (t) \) and \( \bar{\pi}_2^v (t, \cdot) \) into (5.36), we have,
\[
\rho (A(t) x^u (t) + B(t)) + \rho \bar{p}_3^u (t) + \sigma \theta l (A(t) x^u (t) + B(t)) + \sigma^2 A(t) u_t \\
+ \int G \left\{ (1 + r (t, \lambda)) \theta L (t, \lambda) (A(t) x^u (t) + B(t)) + (1 + r (t, \lambda))^2 A(t) u_t - \rho \bar{p}_3^u \right\} m (d\lambda)
= 0,
\]
then we get,
\[
u (t, x_t) = -\frac{\left( \rho + \sigma \theta l (t) + \int G (1 + r (t, \lambda)) \theta L (t, \lambda) m (d\lambda) \right) (A(t) x^u (t) + B(t)) + \rho \bar{p}_3^u}{A(t) G(t)},
\] (5.43)

where \( G(t) = \sigma^2 - \int (1 + r (t, \lambda))^2 m (d\lambda) .\).

In the other side, we have from (5.39) and (5.41). Then
\[
u_t = -\frac{\dot{A}(t) x^u (t) - 2c A(t) x^u (t) - c B(t) + \dot{B}(t) - c \bar{p}_3^u}{A(t) (\rho + \sigma \theta l + \int G (1 + r (t, \lambda)) \theta L (t, \lambda) m (d\lambda))}.
\] (5.44)

From (5.43) and (5.44), we have
\[
\begin{cases}
\dot{A}(t) = \left\{ 2c + \frac{\rho + \sigma \theta l + \int G (1 + r (t, \lambda)) \theta L (t, \lambda) m (d\lambda))^2}{A(t)} \right\} A(t), \\
A(T) = \theta.
\end{cases}
\] (5.45)
and

$$
\begin{cases}
\dot{B}(t) = \left\{ \frac{c + \left( \rho + \sigma \theta l(t) + \int_{\Gamma} (1 + r(t, \lambda)) \theta L(t, \lambda) m(\lambda) \right)^2}{G(t)} \right\} B(t) + \sigma \theta l(t), \\
B(T) = 1 - \theta(y_0 + a).
\end{cases}
$$

Then the explicit solutions of (5.45), and (5.46) have the form

$$
\begin{cases}
A(t) = \theta \exp \int_{t}^{T} \left\{ 2c + \frac{\rho + \sigma \theta l(s) + \int_{\Gamma} (1 + r(s, \lambda)) \theta L(s, \lambda) m(\lambda) \right)^2}{G(s)} \right\} ds, \\
B(t) = (1 - \theta(y_0 + a)) \exp \int_{t}^{T} \left\{ \frac{c + \left( \rho + \sigma \theta l(s) + \int_{\Gamma} (1 + r(s, \lambda)) \theta L(s, \lambda) m(\lambda) \right)^2}{G(s)} \right\} ds, \\
B(s) + \sigma \theta l(s) \left\{ 1 - \theta(y_0 + a) \right\}
\end{cases}
$$

(5.47)

Remark 5.1 It’s very important to remark that the solution of the function $B(t)$ in the form (5.47) is depend to the solution of $\tilde{p}_3(t)$. If we put $\tilde{p}_3(t) = \psi(t) y(t) + \varphi(t)$, for smooth deterministic functions $\psi$, and $\varphi$, by using the similar technique as an optimal solution in the last paragraph, to the triplet $(y^u(t), \tilde{p}_3(t), u(t))$. Then the solutions of $\psi$, and $\varphi$ yield respectively the equations

$$
\begin{cases}
\dot{\psi}(t) = \rho^2 \psi^2(t) - (2\lambda \sigma^2 A(t) - \theta^2 l^2(t)) \psi(t), \\
\dot{\varphi}(t) = (\rho \psi(t) + \theta^2 l^2(t) - \lambda) \varphi(t) + K(t), \\
\psi(0) = \theta, \text{ and } \varphi(0) = 1 - \theta(y_0 - a).
\end{cases}
$$

(5.48)

The main result in this section, can be given in the form of maximum principle of mean variance problem with risk sensitive performance.

Theorem 5.1 We assume that the pair $(A(t), B(t))$ has unique solution given by (5.47), the pair $(\varphi(t), \psi(t))$ has also the explicit solution of the system (5.48). Then the optimal control of the problem (5.33), (5.34) and (5.35) has the state feedback form

$$
u(t, x, y, r, \lambda) = - \frac{\left( \rho + \sigma \theta l(t) + \int_{\Gamma} (1 + r(t, \lambda)) \theta L(t, \lambda) m(\lambda) \right) \left( A(t)x^u(t) + B(t) \right) + \rho \left( \psi(t) y^u(t) + \varphi(t) \right)}{A(t)G(t)}.$$

6 Conclusion and Remarks:

This paper contains two main results. The first one, Theorem 5.1, establishes the necessary optimality conditions for the system of fully coupled FBSDE with risk sensitive performance, using an almost similar scheme as in Chala [11, 14]. The second main result, Theorem 6.1, suggests sufficient optimality conditions of fully coupled FBSDE given in form of risk sensitive performance. Here that our paper is the second extension of result of Chala [12]. The proof is
based on the convexity conditions of the Hamiltonian function, the initial and
terminal terms of the performance function. It should be noted that the risk
sensitive control problems studied by Lim and Zhou in [20] are different from
ours. Our results can be compared with maximum principle obtained by Shi
and Wu [29], but we have to be able to discuss the general case -if we add the
jumps diffusion term to our system-. This result it will be discussed in our next
paper. On the other hand, in the case where the system is governed by mean
field type we may take the existing paper established by Djechiche et al [14].
We have generalized this last result into the fully coupled stochastic differential
equation which is motivated by an optimal portfolio choice problem in financial
market specially the model of control cash flow of a firm or project for example
we can setting the model of pricing and managing an insurance contract, this
counterpart without mean field term as in [14]. A problem to be thoroughly
addressed in our future paper, where the system is governed by fully coupled
stochastic differential equation of mean field type, and will be compared with
[21]. Remarkably, the maximum principle of risk-neutral obtained by Wu [34],
and Yong [32] is quite similar to our theorem 3.

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