Adiabatic matter effect with three generation neutrinos

and the solar neutrino problem

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Abstract
We find an exact analytic solution for the time evolution of a three Dirac neutrino system adiabatically oscillating in matter, constructing explicitly the relevant $3 \times 3$ mixing matrix in matter. Using this result we investigate the solar neutrino data in a scenario where the neutrino masses are such that $m_1 \lesssim m_2 \ll m_3$, taking into account several phenomenological constraints on neutrino mixing angles and masses. A solution of the solar neutrino problem for large values of the parameter $\delta m^2 = m_2^2 - m_1^2$ which are not usually associated with a resonance is found. This is an essentially three-generation effect.

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Most of the analyses of the neutrino oscillation hypothesis assume that this phenomenon involves only two generations of neutrinos. It is difficult, however, to justify why oscillations would not involve also the third family. From the conceptual point of view, mixing and mass parameters required for three generation oscillations are not different from mixing and mass parameters that appear in this same phenomenon involving only two generations of neutrinos. Why would such parameters related with the third family vanish and not those ones leading to two neutrino oscillations? In our opinion, the two generation analysis is just an indicative approach to the more realistic three generation case.

The larger number of mixing and mass parameters in three neutrino oscillations can be quoted as a difficulty to approach this scenario. There are three mixing angles, one phase and two independent mass parameters that are, in principle, free parameters. Due to this fact, when three generations are considered in the literature [1], some assumptions have been made to restrict this parameter space resulting that either only two family transitions are effective or the parameters are fixed arbitrarily.

In this letter we find an exact analytic solution for the time evolution of a three Dirac neutrino system adiabatically oscillating in matter, constructing the three dimensional mixing matrix in matter, to calculate the electron neutrino survival probability and compare it to the solar neutrino data.

Assuming the minimal extension of the standard electroweak model when only three right-handed neutrino singlets are introduced to generate Dirac neutrino masses, a mixing can then occur among the three lepton flavors. Therefore, neutrinos produced in weak processes are in general linear combinations of the mass eigenstates: \( \nu_\alpha = \sum_i V_{\alpha i} \nu_i \) (\( \alpha = e, \mu, \tau; \ i = 1, 2, 3 \)), where

\[
V = \begin{pmatrix}
c_\theta c_\beta & s_\theta c_\beta & s_\beta \\
-s_\theta c_\gamma - c_\theta s_\gamma s_\beta & c_\theta c_\gamma - s_\theta s_\gamma s_\beta & s_\gamma c_\beta \\
s_\theta s_\gamma - c_\theta c_\gamma s_\beta & -c_\theta s_\gamma s_\beta - s_\theta c_\gamma s_\beta & c_\gamma c_\beta
\end{pmatrix}.
\]

(1)

We have set to zero the CP violating phase in Eq. (1).
The matter effects\footnote{4} for the generalized case of three generations are described by the
time evolution equation\footnote{3}
\[
\begin{align*}
  \frac{d}{dt} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} &= \left[ \frac{E_1 + E_2}{2} + H \right] \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} \\
\end{align*}
\tag{2a}
\]
where
\[
H = V \begin{pmatrix} \frac{E_1 - E_2}{2} \\ -\frac{E_1 - E_2}{2} \\ E_3 - \frac{E_1 + E_2}{2} \end{pmatrix} V^{-1} + \begin{pmatrix} A \\ 0 \\ 0 \end{pmatrix}
\tag{2b}
\]
and \(A = \sqrt{2}G_FN_e(t)\), with \(G_F\) the Fermi constant, \(N_e(t)\) the electron number density in the region reached by the neutrino at the instant \(t\).

In order to write the neutrino survival probabilities when matter effects are present, we
would like to have the exact solution of the three coupled differential equations given by Eq. (2). This has proven to be very difficult to obtain and only approximate solutions\footnote{3} has been achieved up to now. In the following we will obtain the explicit form of the relevant
3 × 3 mixing matrix in matter. After that we will be able to construct an exact solution
for the three neutrino time evolution equations assuming that the neutrino propagation is
adiabatic everywhere. In this case the problem is reduced to diagonalize the matrix \(H\) in
Eq. (2). We obtain the characteristic polynomial of \(H\)
\[
\lambda^3 + 3a\lambda^2 + 3b\lambda + c = 0,
\tag{3}
\]
where
\[
3a = -Tr \ H, \quad 3b = H_{ee}^m + H_{\mu\mu}^m + H_{\tau\tau}^m, \quad c = -det \ H
\tag{4}
\]
and \(H_{\alpha\alpha}^m\) denotes the minor of the \(H_{\alpha\alpha}\) elements. We will not write explicitly the elements
\(H_{\alpha\alpha'}\), \(\alpha, \alpha' = e, \mu, \tau\) since they can easily be obtained from Eqs. (1) and (2). The eigenvalues are
\[ \lambda_n = 2\sqrt{-h \cos \left( \frac{\Theta + 2n\pi}{3} \right) - a}, \quad n = 0, 1, 2; \]  
\[ h = b - a^2, \quad g = c - 3ab + 2a^3, \quad \cos \Theta = -\frac{g}{2\sqrt{-h^3}}. \]  

The respective eigenvectors are

\[ \tilde{V}_{\alpha i} = \frac{\delta_{\alpha i}}{\delta_i}, \quad \alpha = e, \mu, \tau; \quad i = 1, 2, 3, \]  
where

\begin{align*}
\delta_{e1} &= H_{ep}H_{\mu\tau} - H_{e\tau}(H_{\mu\mu} - \lambda_1), \quad \delta_{\mu1} = H_{ep}H_{e\tau} - H_{\mu\tau}(H_{ee} - \lambda_1), \\
\delta_{\tau1} &= (H_{ee} - \lambda_0)(H_{\mu\mu} - \lambda_1) - H_{e\mu}^2, \quad \delta_1 = \left[ (\delta_{e1})^2 + (\delta_{\mu1})^2 + (\delta_{\tau1})^2 \right]^\frac{1}{2}, \\
\delta_{e2} &= H_{e\mu}\delta_{\tau1} - H_{e\tau}\delta_{\mu1}, \quad \delta_{\mu2} = H_{e\tau}\delta_{e1} - (H_{ee} - \lambda_2)\delta_{\tau1}, \\
\delta_{\tau2} &= (H_{ee} - \lambda_2)\delta_{\mu1} - H_{e\mu}\delta_{e1}, \quad \delta_2 = \left[ (\delta_{e2})^2 + (\delta_{\mu2})^2 + (\delta_{\tau2})^2 \right]^\frac{1}{2}, \\
\delta_{e3} &= \delta_{\tau1}\delta_{\mu2} - \delta_{\tau2}\delta_{\mu1}, \quad \delta_{\mu3} = \delta_{\tau2}\delta_{e1} - \delta_{e2}\delta_{\tau1}, \\
\delta_{\tau3} &= \delta_{\mu1}\delta_{e2} - \delta_{\mu2}\delta_{e1}, \quad \delta_3 = \left[ (\delta_{e3})^2 + (\delta_{\mu3})^2 + (\delta_{\tau3})^2 \right]^\frac{1}{2}. 
\end{align*}

Therefore the phenomenological eigenstates can be written in terms of the matter eigenstates \( \nu_\alpha = \sum_i \tilde{V}_{\alpha i}\tilde{\nu}_i, \) \((i = 1, 2, 3),\) where the matrix \( \tilde{V} \) can be read from Eqs. (6) and (7) and parametrize in terms of the mixing angles in matter as the matrix in Eq. (1) but with \( \theta \rightarrow \tilde{\theta}, \gamma \rightarrow \tilde{\gamma} \) and \( \beta \rightarrow \tilde{\beta}. \) It is trivial now to write down the averaged adiabatic survival probability of finding a \( \nu_\alpha \) produced at the point \( x_0 \) inside the sun and detected at the point \( x \) in the Earth’s surface

\[ P_{\nu_\alpha \rightarrow \nu_\alpha} = \sum_i |\tilde{V}_{\alpha i}(x_0)|^2|\tilde{V}_{\alpha i}(x)|^2. \]  

The solar neutrino problem has been confirmed by many experiments. In the following we will consider experimental data from Homestake(H), Kamiokande(K) and Gallex(G) experiments [4,5].

Neutrinos produced in different reactions have different energies. While \(^7\text{Be} \) neutrinos are almost monochromatic [6], neutrinos produced in other source-reactions have different
energy spectra which have to be considered since, as we will see in the following, the survival probability of the solar neutrinos is sensitive to their energy $E$ or their momentum $p$. It is necessary also to take into account the value of the solar matter density at the neutrino creation point $x_0$. We use the solar matter distribution calculated through the Standard Solar Model which is tabulated in Ref. [7]. Notice that in Eq. (8), since neutrinos are detected at the Earth’s surface, the matrix elements at the point $x$ are essentially those of the vacuum mixing matrix.

We can compare the theoretical neutrino flux ($\phi_{th}$) calculated from the Standard Solar Model [7] with the observed flux ($\phi_{exp}$) measured by each experiment. The ratios $R = \phi_{exp}/\phi_{th}$ are given by $R(H) = 0.28 \pm 0.04$, $R(K) = 0.49 \pm 0.12$, and $R(G) = 0.66 \pm 0.12$ [4].

Considering the neutrino oscillations, these ratios can be calculated for each experiment. We take into account only the main source reactions of solar neutrinos which are sensible to each specific experiment. For Homestake we have

$$R(H) = 0.78P^H(8B) + 0.14P^H(7Be) + 0.04P^H(15O).$$

The neutrino flux measured by Kamiokande facilities is not merely the electron neutrino one since detector electrons will interact with other neutrino flavors via neutral currents. For energies involved in the solar neutrino experiments, the $\nu_e$-electron scattering cross section is about seven times larger than other neutrino flavor ($\nu_\mu$-electron and $\nu_\tau$-electron) cross sections. Hence, for Kamiokande, taking into account these neutral current effects we have

$$R(K) = P^K(8B) + \frac{1}{7}[1 - P^K(8B)].$$

Finally, for Gallex

$$R(G) = 0.26P^G(8B) + 0.11P^G(7Be) + 0.05P^G(15O) + 0.54P^G(pp).$$

In Eqs. (8) we use the notation

$$P^J(X) = \sum_{E>E_{thres}} f^X(E)P_{\nu_e \rightarrow \nu_e}(E, \delta m^2, \theta, x, x_0).$$
\( J = H, K \) and \( G \) for Homestake, Kamiokande and Gallex; \( X \) denotes the particular source reaction of solar neutrinos and \( P_{\nu_\alpha \rightarrow \nu_\alpha} \) is given in Eq. (8). The threshold energy for each one of these experiments and the energy spectrum of neutrinos produced in reaction \( X \) are denoted by \( E_{\text{thr}} \) and \( f^X(E) \), respectively. The spectra function \( f^X(E) \) are given in Ref. [8].

As we said before, in Eq. (8) there are still too many free parameters: three vacuum mixing angles and three masses. Hence, it is necessary to take into account the constraints from other physical processes to fix some of the neutrino parameters before taking into account the solar neutrino data. The neutrino masses and mixing angles for the case of \( m_1 \lesssim m_2 \ll m_3 \) have been determined in Ref. [9] using \( \tau \) leptonic decays, pion decays, \( Z^0 \) invisible width and end-point data from \( \tau \) decay into five pions and assuming world average data for the ratio \( G_\tau/G_\mu \). Assuming the above mass hierarchy, the lower masses \( m_1, m_2 \) and one angle \( \theta \) remain undetermined, but \( m_3 \sim 165 \text{ MeV}, 11.54^\circ < \beta < 12.82^\circ \) and \( \gamma < 4.05^\circ \). Thus, we have one mixing angle \( \theta \) and two lightest neutrino mass difference \( \delta m^2 = m_2^2 - m_1^2 \) to be determined in neutrino oscillation processes.

Using Eqs. (9) we have investigated the compatibility regions of the three solar neutrino experiments. We have considered the region \( 10^{-7} \text{ eV}^2 \leq \delta m^2 \leq 10^9 \text{ eV}^2 \). For \( 10^{-7} - 10^6 \text{ eV}^2 \) we have solution for either Homestake and Kamiokande or Gallex and Kamiokande but not for all of them at the same time. The only two regions fitting the three experiments are shown in Fig. 1 at 95 % C.L., we have restricted the range of \( \theta \) to be bellow \( \pi \) as the survival probability is symmetric in \( \pi - \theta \). They correspond to: \( 3.5 \times 10^6 \text{ eV}^2 \leq \delta m^2 \leq 5 \times 10^7 \text{ eV}^2 \) with \( 0.9 \leq \theta \leq 1.2 \) radians and \( 2 \times 10^6 \text{ eV}^2 \leq \delta m^2 \leq 2 \times 10^7 \text{ eV}^2 \) with \( 1.4 \leq \theta \leq 2.2 \) radians. In fact we have found small regions of allowed values even at 68 % C.L..

Some remarks are in order. With the parametrization of the mixing matrix used in this work, the resulting survival probability of the electron neutrino is not sensible to the angle \( \gamma \). Setting \( \beta, m_3, \) and \( A \) equal to zero, we recover the vacuum solution in two generation case [10]. On the other hand keeping \( A \neq 0 \) we recover the MSW solution in two generations. In this case we have solution for \( \delta m^2 \approx 10^{-6} \text{ eV}^2 \) and \( \sin^2 2\theta \approx 1 \), but only for two experiments at a time, what is consistent with the fact that we are considering only the
adiabatic solutions \[11\].

General analytical descriptions of three generation neutrino oscillations are far from transparent. We have found a solution to the solar neutrino problem in a region of the relevant parameter space which is not usually associated with resonances in the three neutrino evolution equations \[3\]. This is a completely new feature compared with the two neutrino MSW solution where only in the vicinity of a resonance we can expect to find a dependence of the flavor survival probability on the neutrino momenta. In that case, if we do not have a resonance for the values of the oscillating parameters, all neutrinos undergo the same survival probability and it is impossible to conciliate all solar neutrino data.

Here we have looked for such dependence of the flavor survival probability on the neutrino momenta investigating the behavior of the relevant mixing angles in matter considering the momentum range of neutrinos produced in the sun. While pp-neutrinos have momenta not larger than 0.44 MeV, \(^8B\)-neutrinos can present larger values of momenta up to 15 MeV. In Fig. 2 we show the results of this analysis. We observed that for values of the mass difference \(\delta m^2\) and the vacuum mixing angle \(\theta\) found to be relevant for the compatibility of all solar neutrino data (see Fig. 1), a strong dependence of the mixing angle in matter \(\tilde{\theta}\) on the neutrino momentum \(p\) is present.

Summarizing. We have found an exact analytical solution for the adiabatic transition probability in the three generation neutrino case. Applying our solution to an example where one of the masses is rather large, of the order of several MeV, we have shown that the mixing angles in matter strongly depend on the neutrino momentum when we consider the range of momenta physically interesting for the solar neutrinos. Therefore a solution to the solar neutrino problem can be achieved.

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FIGURES

FIG. 1. The compatibility region of the three solar neutrino experiments in the $\delta m^2$-$\theta$ plane at 95% C.L.

FIG. 2. $\sin^2 \tilde{\theta}$ ($s_{2\tilde{\theta}}$) as a function of neutrino momentum $p$ for $\delta m^2 = 1 \times 10^7$ eV$^2$ and several values of $\theta$. 
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