On critical normal sections for two-dimensional immersions in $\mathbb{R}^4$ and a Riemann-Hilbert problem

Steffen Fröhlich, Frank Müller

Abstract

For orthonormal normal sections of two-dimensional immersions in $\mathbb{R}^4$ we define torsion coefficients and a functional for the total torsion. We discuss normal sections which are critical for this functional. In particular, a global estimate for the torsion coefficients of a critical normal section in terms of the curvature of the normal bundle is provided.

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1 Introduction

Consider a two-dimensional, conformally parametrized immersion

$$X = X(u, v) = (x^1(u, v), x^2(u, v), x^3(u, v), x^4(u, v)) \in C^4(B, \mathbb{R}^4)$$ (1.1)

on the closed unit disc $B = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\} \subset \mathbb{R}^2$, together with an orthogonal moving 4-frame

$$\{X_u, X_v, N_1, N_2\},$$ (1.2)

which consists of the orthogonal tangent vectors $X_u, X_v$, and orthogonal unit normal vectors $N_1, N_2 \in C^3(B, \mathbb{R}^4)$:

$$X_u \cdot X'_u := g_{11} = W = g_{22} := X_v \cdot X'_v, \quad g_{12} := X_u \cdot X'_v = 0,$$

$$X_u \cdot N'_\sigma = 0 = X_v \cdot N'_\sigma \quad \text{for } \sigma = 1, 2,$$

$$|N_1| = 1 = |N_2|, \quad N_1 \cdot N'_2 = 0.$$ (1.3)

Here $W$ denotes the area element of $X$, and $X'$ means the transposed vector of $X$.

Finally, we set $\dot{B} = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 < 1\}$ for the open unit disc and $\partial B = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 = 1\}$ for its boundary.

Remarks.

1. Note the relation $W > 0$ in $B$.

2. For the introduction of conformal parameters into a Riemannian metric we refer to [4].

With the present notes we want to draw the reader’s attention to a definition of torsion coefficients $T_{\sigma,i}^\theta$ for an orthonormal normal section $\{N_1, N_2\}$, which is deduced from the theory of space curves. In addition, we introduce an associated functional of total torsion $T_X(N_1, N_2)$ and study its critical points.
The paper is organized as follows:

- In Chapter 2, we define torsion coefficients of orthonormal normal sections.
- In Chapter 3, we introduce the concept of the total torsion of an orthonormal normal section. We provide conditions for a normal section to be critical and optimal for the total torsion.
- Chapter 4 contains some aspects about generalized analytic functions and Riemann-Hilbert problems. These results are used to prove a global pointwise estimate for the torsion coefficients of critical normal sections.
- Finally, an example of a critical normal section for holomorphic graphs \((w, \Phi(w))\) will be discussed in Chapter 5.

2 Torsion coefficients and curvature of the normal bundle

According to the classical theory of curves in \(\mathbb{R}^3\), we introduce torsion coefficients as follows:

**Definition.** For an orthonormal normal section \(\{N_1, N_2\}\) we define

\[
T_{\sigma,i}^\vartheta := N_{\sigma,u^i} \cdot N_{\vartheta}^t, \quad i = 1, 2, \quad \sigma, \vartheta = 1, 2,
\]

setting \(u^1 \equiv u\) and \(u^2 \equiv v\).

**Remarks.**

1. Obviously, \(T_{\sigma,i}^\vartheta = -T_{\vartheta,i}^\sigma\) holds for any \(i = 1, 2, \quad \sigma, \vartheta = 1, 2\), and consequently \(T_{\sigma,i}^\sigma \equiv 0\).

2. The \(T_{\sigma,i}^\vartheta\) are exactly the coefficients of the normal connection. In their terms one defines the coefficients of the curvature tensor \(\mathcal{G}\) of the normal bundle (summation convention!)

\[
S_{\sigma,ij} := T_{\vartheta,j}^\sigma, u^i - T_{\vartheta,j}^\sigma, u^i + T_{\vartheta,i}^\vartheta, T_{\vartheta,j}^\vartheta - T_{\vartheta,j}^\vartheta, T_{\vartheta,i}^\vartheta, \quad i, j = 1, 2, \quad \sigma, \vartheta = 1, 2.
\]

In contrast to the case of codimension \(n \geq 3\), the quadratical terms in (2.5) vanish in \(\mathbb{R}^4\), and \(\mathcal{G}\) consists essentially of the single term

\[
S := S_{1,12}^2 = T_{1,1,v}^2 - T_{1,2,u}^2 = \text{div} (-T_{1,2,u}^2, T_{1,1})\). (2.6)

This is the reason why we concentrate on immersions in \(\mathbb{R}^4\).

3. Note that \(S\) does not depend on the choice of the orthonormal section \(\{N_1, N_2\}\), compare Subsection 3.1.

4. Distinguish our definition from that of the normal torsion of a surface (see i.e. \[1\]): It can be defined as the torsion of the one-dimensional normal section (as a curve on the surface), which arises from a suitable intersection of \(X\) with a three-dimensional hyperplane.

3 Total torsion and optimal normal sections

To an orthonormal section \(\{N_1, N_2\}\) of the normal bundle we assign the total torsion

\[
T_X(N_1, N_2) := \sum_{\sigma, \vartheta=1}^2 \iint_B g^{ij} T_{\sigma,i}^\vartheta T_{\sigma,j}^\vartheta W \, dudv = 2 \iint_B \left\{ (T_{1,1}^2)^2 + (T_{1,2}^2)^2 \right\} \, dudv,
\]

where \(g^{ij} g^{jk} = \delta_{i}^{k}\), and \(\delta_{i}^{k}\) is the Kronecker symbol (see the conformity relations in (1.3)).
3.1 Critical orthonormal normal sections

The total torsion depends on the chosen orthonormal section \( \{N_1, N_2\} \), and it can be controlled by means of a rotation angle \( \varphi = \varphi(u,v) \), depending smoothly on \((u,v) \in B\). Indeed, starting with the section \( \{\tilde{N}_1, \tilde{N}_2\} \), we write

\[
\tilde{N}_1 = \cos \varphi N_1 + \sin \varphi N_2, \quad \tilde{N}_2 = -\sin \varphi N_1 + \cos \varphi N_2 \tag{3.2}
\]

for the rotated normal section \( \{\tilde{N}_1, \tilde{N}_2\} \). Then, the new torsion coefficients are given by

\[
\tilde{T}_{1,1}^2 = T_{1,1}^2 + \varphi_u, \quad \tilde{T}_{1,2}^2 = T_{1,2}^2 + \varphi_v. \tag{3.3}
\]

Due to (3.1), the difference between new and old total torsion now computes to

\[
T_X(\tilde{N}_1, \tilde{N}_2) - T_X(N_1, N_2) = 2 \iint_B |\nabla \varphi|^2 \, dudv + 4 \iint_B (T_{1,1}^2 \varphi_u + T_{1,2}^2 \varphi_v) \, dudv \tag{3.4}
\]

In general, the right hand side does not vanish.

**Proposition.** Let \( \{N_1, N_2\} \) be critical for \( T_X \). Then the torsion coefficients satisfy

\[
\text{div} \left( T_{1,1}^2, T_{1,2}^2 \right) = 0 \quad \text{in} \ B, \quad (T_{1,1}^2, T_{1,2}^2) \cdot \nu^t = 0 \quad \text{on} \ \partial B. \tag{3.5}
\]

3.2 Construction of critical orthonormal normal sections

How can we construct a critical section \( \{N_1, N_2\} \) from a given section \( \{\tilde{N}_1, \tilde{N}_2\} \)?

If \( \{N_1, N_2\} \) is critical, then we have

\[
0 = \text{div} \left( T_{1,1}^2, T_{1,2}^2 \right) = \text{div} \left( \tilde{T}_{1,1}^2 - \varphi_u, \tilde{T}_{1,2}^2 - \varphi_v \right) \quad \text{in} \ B,
\]

\[
0 = (T_{1,1}^2, T_{1,2}^2) \cdot \nu^t = (\tilde{T}_{1,1}^2 - \varphi_u, \tilde{T}_{1,2}^2 - \varphi_v) \cdot \nu^t \quad \text{on} \ \partial B, \tag{3.6}
\]

by virtue of (3.3), (3.5). This implies our next result:

**Proposition.** The given section \( \{\tilde{N}_1, \tilde{N}_2\} \) transforms into a critical section by means of (3.2), iff

\[
\Delta \varphi = \text{div} \left( \tilde{T}_{1,1}^2, \tilde{T}_{1,2}^2 \right) \quad \text{in} \ B,
\]

\[
\frac{\partial \varphi}{\partial \nu} = (\tilde{T}_{1,1}^2, \tilde{T}_{1,2}^2) \cdot \nu^t \quad \text{on} \ \partial B \tag{3.7}
\]

holds for the rotation angle \( \varphi = \varphi(u,v) \).

**Remark.** It is well known that the solvability of the Neumann problem

\[
\Delta \varphi = f \quad \text{in} \ B, \quad \frac{\partial \varphi}{\partial \nu} = g \quad \text{on} \ \partial B \tag{3.8}
\]

depends on the integrability condition

\[
\iint_B f \, dudv = \int_{\partial B} g \, ds, \tag{3.9}
\]

which is fulfilled in our proposition.
3.3 Minimality of critical orthonormal normal sections

Let \( \{N_1, N_2\} \) be a critical section. Then, we conclude

\[
T_X(\tilde{N}_1, \tilde{N}_2) = T_X(N_1, N_2) + 2 \iint_B |\nabla \varphi|^2 \, du \, dv,
\]

(3.10)

taking (3.4) and (3.5) into account. This proves the following

**Proposition.** A critical orthonormal normal section \( \{N_1, N_2\} \) minimizes the total torsion, i.e. we have

\[
T_X(N_1, N_2) \leq T_X(\tilde{N}_1, \tilde{N}_2)
\]

(3.11)

for all smooth orthonormal normal sections \( \{\tilde{N}_1, \tilde{N}_2\} \). The equality occurs iff \( \varphi \equiv \text{const} \).

3.4 Flat normal bundles

For a critical normal section, the vector-field \((-T^2_{1,2}, T^2_{1,1})\) is parallel to \(\nu\) along \(\partial B\). Applying the Gaussian integral theorem to (2.6), we infer

\[
\iint_B S \, du \, dv = \int_{\partial B} (\neg T^2_{1,2}, T^2_{1,1}) \cdot \nu^i \, ds = \pm \int_{\partial B} \sqrt{(T^2_{1,1})^2 + (T^2_{1,2})^2} \, ds.
\]

(3.12)

In particular, if \( S \equiv 0 \), that is, the normal bundle is flat, then we find

\[
T^\vartheta_{\sigma,i} \equiv 0 \quad \text{on } \partial B
\]

(3.13)

for \( i = 1, 2 \) and \( \sigma, \vartheta = 1, 2 \).

Differentiating (2.6) and (3.5), we further obtain

\[
\Delta T^2_{1,1} = \frac{\partial}{\partial v} S = 0, \quad \Delta T^2_{1,2} = -\frac{\partial}{\partial u} S = 0 \quad \text{in } B
\]

(3.14)

for flat normal bundles. Therefore,

\[
T^\vartheta_{\sigma,i} \equiv 0 \quad \text{in } B \quad (i = 1, 2, \ \sigma, \vartheta = 1, 2)
\]

(3.15)

follows by the maximum principle.

**Remark.** Immersions of prescribed mean curvature with flat normal bundles are extensively studied in the literature; see e.g. \([6]\), \([3]\) for higher dimensional surfaces. Special results for two-dimensional immersions without curvature conditions on the normal bundle can be found in \([2]\).

In the following, we investigate the inhomogeneous case of non-flat normal bundles to extend the relation (3.15) appropriately.
4 Estimates for the torsion coefficients

4.1 A Riemann-Hilbert problem

Once again, let us consider (2.6) and (3.5) for critical sections:

\[ \frac{\partial}{\partial u} T_{1,1}^2 + \frac{\partial}{\partial v} T_{1,2}^2 = 0, \quad \frac{\partial}{\partial v} T_{1,1}^2 - \frac{\partial}{\partial u} T_{1,2}^2 = S \quad \text{in } B. \]  

(4.1)

The complex-valued torsion \( \Psi = T_{1,1}^2 - iT_{1,2}^2 \) solves the non-homogeneous Cauchy-Riemann equation

\[ \frac{\partial}{\partial w} \Psi(w) = \Psi_{\overline{w}}(w) := \frac{i}{2} (\Psi_u + i\Psi_v) = \frac{i}{2} S, \quad w = u + iv \in \hat{B}. \]  

(4.2)

In addition, we write the boundary condition in (3.5) as

\[ \Re \{ w \Psi(w) \} = 0, \quad w \in \partial B. \]  

(4.3)

The relations (4.2) and (4.3) form a linear Riemann-Hilbert problem for \( \Psi \).

**Proposition.** The problem (4.2)-(4.3) possesses at most one solution \( \Psi \in C^1(\hat{B}) \cap C^0(B) \).

**Proof.** Let \( \Psi_1, \Psi_2 \) be two such solutions. Then we set \( \Phi(w) := w[\Psi_1(w) - \Psi_2(w)] \) and note

\[ \Phi_{\overline{w}} = 0 \quad \text{in } \hat{B}, \quad \Re \Phi = 0 \quad \text{on } \partial B. \]  

(4.4)

Consequently, \( \Phi \equiv ic \) holds true in \( B \) with some constant \( c \in \mathbb{R} \), and the continuity of \( \Psi_1, \Psi_2 \) implies \( c = 0 \). \( \square \)

4.2 Some facts about generalized analytic functions

As general references for this subsection we name [7], [5].

For arbitrary \( f \in C^1(B, \mathbb{C}) \) we define

\[ T_B[f](w) := -\frac{1}{\pi} \iint_B \frac{f(\zeta)}{\zeta - w} d\xi d\eta, \quad w \in \mathbb{C}, \]  

(4.5)

using the notation \( \zeta = \xi + i\eta \). Then, there hold \( g := T_B[f] \in C^1(\mathbb{C} \setminus \partial B) \cap C^0(\mathbb{C}) \) as well as

\[ \frac{\partial}{\partial w} T_B[f](w) = \begin{cases} f(w), & w \in \hat{B} \\ 0, & w \in \mathbb{C} \setminus B \end{cases}, \]  

(4.6)

cf. [7] Kapitel I, §5. Next, we set

\[ P_B[f](w) := -\frac{1}{\pi} \iint_B \left\{ \frac{f(\zeta)}{\zeta - w} + \frac{\overline{\zeta f(\zeta)}}{1 - w\zeta} \right\} d\xi d\eta \]  

\[ = T_B[f](w) + \frac{1}{w} T_B[wf](\frac{1}{w}). \]  

(4.7)
We obtain $h := P_B[f] \in C^1(\hat{B}) \cap C^0(B)$, and (4.6) yields

$$\frac{\partial}{\partial w} P_B[f](w) = f(w), \quad w \in \hat{B}. \quad (4.8)$$

Finally, we note the relation

$$P_B[f](w) = T_C[f_s](w), \quad w \in B. \quad (4.9)$$

Here, $T_C$ is defined as $T_B$ but with integration over $C$, and we have abbreviated

$$f_s(w) := \begin{cases} f(w), & w \in B \\ \frac{1}{|w|^4} f\left(\frac{1}{w}\right), & w \in \mathbb{C} \setminus B \end{cases}. \quad (4.10)$$

Observe that $f_s(w)$ is not continuous in $\mathbb{C}$, but it belongs to the class $L_p, 2(B)$ for any $p \in [1, +\infty]$, that means, $f_s(w)$ as well as $|w|^{-2} f_s\left(\frac{1}{w}\right)$ belong to $L_p(B)$; compare [7] p. 12.

Consequently, Satz 1.24 in [7] yields the following

**Proposition.** With the definitions above, we have the uniform estimate

$$|P_B[f](w)| = |T_C[f_s](w)| \leq c(p) \|f\|_{L_p(B)}, \quad w \in B, \quad (4.11)$$

where $p \in (2, +\infty]$, and $c(p)$ is a positive constant dependent on $p$.

### 4.3 A global pointwise estimate for the torsion coefficients

**Theorem.** Consider a conformally parametrized immersion $X \in C^4(B, \mathbb{R}^4)$ and write

$$s_p := \|S\|_{L_p(B)}, \quad p \in (2, +\infty]. \quad (4.12)$$

Then, the complex-valued torsion $\Psi = T_{1,1}^2 - iT_{1,2}^2$ of a critical orthonormal section $\{N_1, N_2\}$ satisfies

$$|\Psi(w)| \leq c(p)s_p \quad \text{for all } w \in B, \quad (4.13)$$

with some positive constant $c(p)$.

**Remark.**

1. For a flat normal bundle, i.e. $s_p = 0$, we recover (3.15).
2. The general estimate (4.13) shall be useful, e.g., for proving curvature estimates for immersions with non-flat normal bundle.

**Proof of the theorem.** Let us write $f := \frac{i}{2} S \in C^1(B)$. We claim that $\Psi$ possesses the integral representation

$$\Psi(w) = P_B[f](w) = -\frac{1}{\pi} \iint_B \left\{ \frac{f(\zeta)}{\zeta - w} + \frac{\overline{f(\zeta)}}{1 - w\zeta}\right\} d\xi d\eta, \quad w \in B. \quad (4.14)$$

Then, (4.13) follows at once from the proposition in Subsection 4.2.
An elementary calculation proves

\[ wP_B[f](w) = \frac{1}{\pi} \int_B \int f(\zeta) \, d\xi \, d\eta + T_B[wf](w) - T_B[wf]\left(\frac{1}{w}\right). \]  

(4.15)

Taking \( f = \frac{i}{2}S \) into account, we infer

\[ \text{Re}\{wP_B[f](w)\} = 0, \quad w \in \partial B. \]  

(4.16)

Consequently – remember (4.8) –, \( P_B[f](w) \) solves the Riemann-Hilbert problem (4.2)-(4.3). Now the uniqueness result of the proposition in Subsection 4.1 yields the identity (4.14).

**Remark.** We point out that the representation (4.14) relies crucially on the fact that the right hand side \( f = \frac{i}{2}S \) in (4.2) is purely imaginary. In general, a Riemann-Hilbert problem as in (4.2)-(4.3) is solvable, iff the integral of the right hand side \( f \) over \( B \) has vanishing real part, cf. (4.15). For details we refer to [7] Kapitel IV, §7.

**5 Example: Holomorphic graphs on \( B \)**

Let us consider graphs \( X(w) = (w, \Phi(w)), \ w = u + iv \in B \). If \( \Phi(w) = \varphi(w) + i\psi(w) \) is holomorphic on \( B \), then the vectors

\[ N_1 = \frac{1}{\sqrt{W}} (-\varphi_u, -\varphi_v, 1, 0), \quad N_2 = \frac{1}{\sqrt{W}} (-\psi_u, -\psi_v, 0, 1) \]  

(5.1)

form an orthonormal normal section, where \( W = 1 + |\nabla \varphi|^2 = 1 + |\Phi'|^2 \) is the area element.

**Remark.** Due to \( \varphi_u = \psi_v, \varphi_v = -\psi_u \) and thus \( \Delta \varphi = \Delta \psi = 0 \), the immersion \( X \) represents a conformally parametrized minimal graph.

For the torsion coefficients we compute

\[ T_{1,1}^2 = \frac{1}{W} (-\varphi_u \varphi_v + \varphi_w \varphi_u) = \frac{1}{2W} \frac{\partial}{\partial v}(|\nabla \varphi|^2), \quad T_{1,2}^2 = -\frac{1}{2W} \frac{\partial}{\partial u}(|\nabla \varphi|^2), \]  

(5.2)

Consequently, the relation

\[ \text{div} (T_{1,1}^2, T_{1,2}^2) = 0 \quad \text{in} \; B \]  

(5.3)

is satisfied. In order to check the boundary condition in (3.5), we introduce polar coordinates \( u = r \cos \alpha, v = r \sin \alpha \) and note \( \frac{1}{r} \frac{\partial}{\partial \alpha} = u \frac{\partial}{\partial v} - v \frac{\partial}{\partial u} \). According to (5.2), we then obtain

\[ (T_{1,1}^2, T_{1,2}^2) \cdot \nu' = \frac{1}{2W} \left( u \frac{\partial}{\partial v} - v \frac{\partial}{\partial u} \right)(|\nabla \varphi|^2) = \frac{1}{2W} \frac{\partial}{\partial \alpha}(|\Phi'|^2) \quad \text{on} \; \partial B. \]  

(5.4)

**Proposition.** Consider the graph \( (w, \Phi(w)), \ w \in B \), with a holomorphic function \( \Phi(w) = \varphi(w) + i\psi(w) \). Then the normal section \( \{N_1, N_2\} \) defined in (5.1) is critical, that is, it satisfies (3.5), iff \( |\Phi'| \) is constant on \( \partial B \).

**Remark.** As an example, we mention the graph \( X(w) = (w, w^n), \ w \in B \), for arbitrary \( n \in \mathbb{N} \).
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Steffen Fröhlich
Freie Universität Berlin
Fachbereich Mathematik und Informatik
Arnimallee 2-6
D-14195 Berlin
Germany

e-mail: sfroehli@mi.fu-berlin.de

Frank Müller
Brandenburgische Technische Universität Cottbus
Mathematisches Institut
Konrad-Zuse-Straße 1
D-03044 Cottbus
Germany

e-mail: mueller@math.tu-cottbus.de