Scalar mesostatic field with regard for gravitational effects

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Abstract

It is considered the scalar mesostatic field of a point source with the regard for spacetime curvature caused by this field. For the field with $\chi = 0$ the exact solution of Einstein equations was obtained. It was demonstrated that at small distance from a source the gravitational effects are so large that they cause the significant changes in behavior of meson field. In particular, the total energy of static field diverges logarithmically.

1 See, for instance, O. Bergman and R. Leipnik, Phys. Rev. 107 (1957) 1157; H. A. Buchdahl, Phys. Rev. 111 (1959) 1417; M. Wyman, Phys. Rev. D 24 (1981) 839, and so on. [Comment by translator].
It is usually assumed that gravitational forces can always be neglected when considering elementary particles. In present paper the incorrectness of such a viewpoint is proved by virtue of calculation of scalar mesostatic field with the regard for the spacetime curving caused by it. It turns out, that in such a case at close distance to a point source the field completely differs from that desired.

Let us suppose $U$ to be a scalar satisfying with the de Broglie equation

$$\Box - \chi^2 U = 0. \quad (1)$$

We will be interested in its static spherically symmetrical (vanishing at infinity) solutions hence $U = U(r)$. Then from (1) we obtain

$$U(r) = Ge^{-\chi r}/r, \quad (2)$$

where the constant $G$ is the charge of a source. However, (2) was obtained in the assumption that spacetime remains to be flat even at small distances from a source that is evidently wrong. The consistent theory must take into account the gravitational effects which cannot be regarded as “small” as it will be shown below. The expression (2) is just the limit case for large $r$.

In the case we are interested $U$ always can be regarded as real-valued. Otherwise it always can be made real-valued by means of the transformation $U \rightarrow U e^{i\alpha}$.

1. In the most general case and for any coordinates the stress-energy tensor of scalar field can be written in the form:

$$T^k_i = (1/8\pi)\{2U_i U^k - \delta^k_i (U_i U^l - \chi^2 U^2)\}; \quad T_{ik} = T_{ki}, \quad (3)$$

where

$$U_i = \partial U/\partial x^i, \quad U^k = g^{ik} U_i. \quad (4)$$

Keeping in mind the static field $U = U(r)$ we will use the coordinates $x^1 = r, x^2 = \vartheta, x^3 = \varphi$ and $x^0 = t$, with the line element

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2 (d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2), \quad (5)$$

where $\nu = \nu(r)$ and $\lambda = \lambda(r)$. Supposing $U = U(r)$ and denoting by prime the differentiation with respect to $r$ we obtain from (3):

$$8\pi T^1_1 = -e^{-\lambda} U'^2 + \chi^2 U^2, \quad 8\pi T^2_2 = 8\pi T^3_3 = 8\pi T^0_0 = e^{-\lambda} U'^2 + \chi^2 U^2. \quad (6)$$

If there is no other field and masses then the Einstein equations,

$$R^k_i - 1/2 \delta^k_i R = -(8\pi k/c^4) T^k_i, \quad (7)$$

are written with $T^k_i$ from (6), and by virtue of the known expressions for left hand side (7) in the chosen coordinates [1] we obtain:

$$e^{-\lambda} \left( \frac{1}{r^2} + \frac{\nu'}{r} \right) - \frac{1}{r^2} = -\frac{k}{c^4} (e^{-\lambda} U'^2 + \chi^2 U^2),$$

$$\frac{1}{2} e^{-\lambda} \left( \nu'' + \frac{1}{2} \nu'^2 - \frac{1}{r} \nu' - \frac{1}{r} (\nu' - \lambda') \right) = -\frac{k}{c^4} (e^{-\lambda} U'^2 + \chi^2 U^2), \quad (8)$$

$$e^{-\lambda} \left( \frac{1}{r^2} - \frac{\lambda'}{r} \right) - \frac{1}{r^2} = -\frac{k}{c^4} (e^{-\lambda} U'^2 + \chi^2 U^2).$$

2 In the journal version the first from eqs. (8) contained a misprint so I have corrected it. [Comment by translator].
This is a complete system of equations for gravitational field coupled with the field \( U \). Indeed, from (7) it follows \((T^k_i)_k = 0\), where the index behind the brace means covariant differentiation, or in the explicit form:

\[
\frac{\partial}{\partial x^k}(\sqrt{-g} T^k_i) - \frac{1}{2}\sqrt{-g} \frac{\partial g_{lm}}{\partial x^i} T^{lm} = 0.
\]  

(9)

Substituting here \( T^k_i \) from (6) and \( g_{ik} \) from (5) we obtain:

\[
U'' + \left( \frac{2}{r} + \frac{1}{2}(\nu' - \lambda') \right) U' - \chi^2 e^\lambda U = 0.
\]  

(10)

This is the equation for field \( U \) in our case. At \( e^\lambda = 1 \) and \( e^\nu = c^2 \) it yields (2) as it was desired. Equation (10) can be obtained also immediately if one proceeds from the generalization of equation (1) for arbitrary coordinates.

In the case \( \chi = 0 \) the equations become simple, and it is possible to integrate them completely; (10) yields us right away:

\[
U' = -(cG/r^2)e^{1/2 (\lambda-\nu)},
\]  

(11)

where \( cG \) is a constant of integration. Substituting it into (8) (at \( \chi = 0 \)) it is easily to check that the one of equations appears to be a consequence of the two rest. We consider the more symmetric pair of equations as a basic one:

\[
e^{-\lambda}(1 + r\nu') - 1 = (kG^2/c^2r^2)e^{-\nu},
e^{-\lambda}(1 - r\lambda') - 1 = -(kG^2/c^2r^2)e^{-\nu}.
\]  

(12)

Summing up them we obtain:

\[
(r^2e^{\nu-\lambda})' = 2re^\nu.
\]  

(13)

Hence, if one designates:

\[
Z(r) = re^{1/2 (\nu-\lambda)},
\]  

(14)

then from (13), (11) and (14) we obtain:

\[
e^\nu = (1/r)ZZ'; \quad e^\lambda = rZ'/Z; \quad U' = -(cG/r)Z,
\]  

(15)

hence all the values of interest are expressed via the function \( Z(r) \), thus, the knowledge of it resolves the problem. Substituting (15) into (12) after simplification we obtain the equation for \( Z(r) \):

\[
Z^2Z'' = a^2Z'/r \quad \left( a^2 = \frac{kG^2}{c^2} \right).
\]  

(16)

According to (14) from the condition that the metric must tend to the Galilean one at infinity we obtain the condition for \( Z \):

\[
Z(r) \sim cr \quad \text{at} \quad r \to \infty.
\]  

(17)

First of all we consider the case \( \chi = 0 \) when a solution can be found exactly. Then we will consider the case of the field with \( \chi \neq 0 \) at small \( r \).

2. If one divides the equation (16) by \( Z^2 \) and multiplies by \( r \) that it can be written in the form:

\[
(rZ' - Z)' + (a^2/Z)' = 0,
\]  

(18)

that can be integrated right away:

\[
\frac{ZZ'}{Z^2 + (2km/c)Z - a^2} = \frac{1}{r},
\]  

(19)
where $2km/c$ is the constant of integration. The secondary integration yields
\[ (Z - Z_0)^{1-p}(Z + Z_1)^{1+p} = (Cr)^2, \]  
where $C$ is the constant of integration, and
\[ Z_0 = c^{-1}(\sqrt{(km)^2 + c^2a^2} - km); \quad Z_1 = c^{-1}(\sqrt{(km)^2 + c^2a^2} + km), \]  
\[ p = km[(km)^2 + (ca)^2]^{-1/2} < 1. \]  
The equation (20) can be rewritten in the form:
\[ \left( Z^2 + \frac{2km}{c}Z - a^2 \right) \left( \frac{Z + Z_1}{Z - Z_0} \right)^p = (Cr)^2. \]  
Equations (20) or (22) determine $Z$ as the function of $r$ depending on the constants $C$, $m$ and parameter $a^2$. At $Z \to \infty$ from (22) we obtain $Z \approx \pm Cr$ so that $Z$ has two branches. Comparing this with (17) we see that it is necessary to suppose $C = c$ to choose the positive branch $Z$. The curve $Z(r)$ is shown on Fig. 1 at the following values of parameters:
\[ c = 1; \quad km = 0.158; \quad a = 3.158; \quad p = 0.05. \]  
At $p \ll 1$ according to (22) the curve $Z(r)$ is close to the hyperbola:
\[ Z^2 + (2km/c)Z - a^2 = c^2r^2, \]  
which is shown on Fig. 2 as well.

By means of (15), (19) and (22) we obtain:
\[ e^\nu = \frac{1}{r^2} \left( Z^2 + \frac{2km}{c}Z - a^2 \right) = c^2 \left( \frac{Z - Z_0}{Z + Z_1} \right)^p, \]  
\[ e^\lambda = \frac{1}{Z^2} \left( Z^2 + \frac{2km}{c}Z - a^2 \right) = \frac{c^2r^2}{Z^2} \left( \frac{Z - Z_0}{Z + Z_1} \right)^p. \]  
From here it is seen that for all $r \neq 0$ we have $e^\nu > 0$ and $e^\lambda > 0$. In the point $r = 0$ both functions tend to zero hence the gravitational radius is absent. The curves (25) are represented on Fig. 3 at the same values of parameters (23).
As for the root of the determinant of metric tensor that we find from (5) and (25):
\[
\sqrt{-g} = \frac{r \sin \vartheta}{Z} \left( Z^2 + \frac{2km}{c} Z - a^2 \right) = \frac{c^2 r^3 \sin \vartheta}{Z} \left( Z - Z_0 \right)^p.
\]  
(26)

Approximately for \( r \to 0 \) we find from (20): \( Z - Z_0 = \text{const} \cdot r^{2/(1-p)} + \ldots \), then from (25) we obtain
\[
e^\nu = \text{const} \cdot r^{2/(1-p)} + \ldots; \quad e^\lambda = \text{const} \cdot r^{2/(1-p)} + \ldots
\]  
(27)

For \( r \to \infty \) we find from (24)
\[
Z(r) = cr - (km/c) + (kG^2 + k^2m^2)/c^3r + \ldots
\]  
(28)

Then from (25) up to the terms of order \( 1/r^2 \) inclusive we obtain:
\[
e^\nu = c^2, \quad e^{-\lambda} = 1 - 2km/c^2 r + (kG^2 + 2k^2m^2)/c^4r^2 + \ldots
\]  
(29)

Thus, the metric has no Schwarzschild-like behavior in the sense that it is non-symmetric with respect to \( e^\nu \) and \( e^{-\lambda} \). At not very small scales the curving of spacetime reveals itself only in the spatial part of metric whereas time remains to be flat. It is seen from Fig. 2 as well. Finally, from (29) we find the justification for the designation \( 2km/c \) as an integration constant which was done in (19).

3. Let us consider now the expression for field \( U \). From (15), (19) and (25) we find:
\[
U'(r) = -\frac{cG}{rZ} = -cG \frac{Z'}{Z^2 + (2km/c)Z - a^2}.
\]  
(30)

From here we obtain
\[
U(r) = \frac{c^2 G}{2\sqrt{kG^2 + k^2m^2}} \ln \left( \frac{Z + Z_1}{Z - Z_0} \right),
\]  
(31)

where the constant of integration is chosen in such a way that \( U(\infty) = 0 \). At \( r \to 0 \) we hence have:
\[
U = \text{const} \cdot \ln (1/r) + \ldots,
\]  
(32)

i.e., field has only the logarithmic singularity at \( r = 0 \) unlike the ordinary result (3).

Further, from (6), (25), (26) and (30) we find for the tensor density \( \sqrt{-g} T^k_i \) the following expressions:
\[
T^0_0 \sqrt{-g} = \frac{(c^2 G^2/8\pi)(\sin \vartheta/rZ)}{Z^2 + (2km/c)Z - a^2},
\]  
(33)

\[
T^0_0 \sqrt{-g} = T^2_2 \sqrt{-g} = T^3_3 \sqrt{-g} = -T^1_1 \sqrt{-g},
\]  

which also differ completely from the ordinary result. Hence for the total energy of static field we obtain:
\[
W = \frac{1}{c} \int T^0_0 \sqrt{-g} dx^1 dx^2 dx^3 = (cG^2/2) \int dr/rZ(r).
\]  
(34)
The integral here is analogous to that in (30), (31), and if we take 0 and $\infty$ as limits that on the lower limit we obtain the divergence of order $W \sim \ln (1/r)$ at $r \to 0$, i.e., the total energy of the field of a point source diverges logarithmically. It is interesting to point out that despite this circumstance the “classical charge radius” remains to be the same as in the ordinary theory, $r_0 = C^2/2mc$, as it can easily be made sure. It is explained simply by the fact that the domain, where the curvature of spacetime is sufficient, is much smaller than the “classical radius”.

Omitting elementary but bulk calculations we point out just that by virtue of the substitution

\[ Z = c\rho - (km/c) + (kC^2 + k^2m^2)/4c^3\rho, \tag{35} \]

the metric (5), (25) can be reduced to “isotropic” coordinates:

\[ ds^2 = e^\nu dt^2 - e^\mu(d\rho^2 + \rho^2d\vartheta^2 + \rho^2\sin^2\vartheta d\varphi^2) = \]

\[ = e^{\nu'}dt^2 - e^{\mu'}(dx^2 + dy^2 + dz^2), \tag{36} \]

where $\nu = \nu(\rho)$, $\mu = \mu(\rho)$ and $\rho = \sqrt{x^2 + y^2 + z^2}$. By means of the latter expression for $ds^2$ one can calculate, also in an elementary but bulk way, the energy-momentum pseudo-tensor of gravitational field $t^k_i$ and then the energy of total field

\[ P_0 = c^{-1}\int (T^0_0 + t^0_0)\sqrt{-g} \, dx \, dy \, dz. \tag{37} \]

As it is known just $P_0$ rather than $W$ has physical sense. We just point out that these calculations lead again to the logarithmic divergence of total energy and to the same order of “classical radius” value as those above.

4. In the case of scalar field with $\chi \neq 0$ the system (8) cannot be resolved so simply as at $\chi = 0$. However we are interested in the solution in the neighborhood of the point $r = 0$, and by direct substitution (27) and (32) into (8) one can make sure that the solutions for the cases $\chi \neq 0$ and $\chi = 0$ coincide at $r \to 0$. It is quite analogous to the result (2) of the ordinary theory. The same result can be obtained without reference to the explicit form of the solution for the case $\chi = 0$. Following from the invariance of system (8) with respect to the transformation

\[ r \to \sigma r, \quad \chi \to \chi/\sigma, \quad (\sigma = \text{const}), \tag{38} \]

it can be shown that the solution of this system has the form (if then it is supposed $\sigma = \chi$):

\[ U = U_0V(r\chi), \quad e^\nu = e^{\nu_0}\varphi(r\chi), \quad e^\lambda = e^{\lambda_0}\psi(r\chi), \tag{39} \]

where $U_0$, $e^{\nu_0}$, $e^{\lambda_0}$ are the solutions for the case $\chi = 0$, and $V$, $\varphi$, $\psi$ are functions of a single variable ($r\chi$); these three functions $\to 1$ at ($r\chi$) $\to 0$.

Thus, we see that in the case of scalar mesostatic field (with $\chi \neq 0$ and $\chi = 0$) the considering of gravitational effects at small distances from a source leads to the results completely distinguished from the ordinary theory, and these effects cannot be regarded as small.

References

[1] L. Landau and E. Lifshitz, Field Theory (Litzdat, Moskow, 1941).