A phase-field approach to model evaporation in porous media

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\[ \dot{m} := \rho_\alpha (v_\alpha \cdot \mathbf{n} - v_n) \]
Introduction

Motivation

- Evaporation and drying in porous media occur in many environmental and industrial systems
- Moving fluid-fluid interface at the pore-scale
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- Evaporation and drying in porous media occur in many environmental and industrial systems
- Moving fluid-fluid interface at the pore-scale

Objectives

- Formulate a mathematical model of the relevant processes at the pore scale, including a better description of the evolving liquid-gas interface
- Derive effective models valid at the REV scale through upscaling the pore-scale processes
Sharp Interface Formulation

- Advantage: determines the location of the moving interface
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- Challenge: location not known a-priori, depends on the model unknowns
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- Question: how does the normal velocity of the interface $v_n$ behave?
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- Information: this velocity is directly linked to the mass transfer across the interface
Sharp Interface Formulation

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- Question: how does the normal velocity of the interface $v_n$ behave?

- Information: this velocity is directly linked to the mass transfer across the interface

- Requirement: some kinematic condition coupling the normal velocity to an evaporation rate
Sharp Interface Formulation

Mass conservation -

\[ \partial_t \rho_\alpha + \nabla \cdot (\rho_\alpha \mathbf{v}_\alpha) = 0 \quad \text{in} \quad \Omega_\alpha(t), \quad \alpha = l, g \] (1)

\[ (\rho_l \mathbf{v}_l - \rho_g \mathbf{v}_g) \cdot \mathbf{n} = \nu_n (\rho_l - \rho_g) \quad \text{on} \quad \Gamma_{lg}(t) \] (2)

\[ \partial_t (\rho_g \chi^\nu_g) + \nabla \cdot (\rho_g \chi^\nu_g \mathbf{v}_g) = \nabla \cdot (D^\nu_g \rho_g \nabla \chi^\nu_g) \quad \text{in} \quad \Omega_g(t) \] (3)

\[ (\rho_l \mathbf{v}_l - (\rho_g \chi^\nu_g \mathbf{v}_g - D^\nu_g \rho_g \nabla \chi^\nu_g)) \cdot \mathbf{n} = \nu_n (\rho_l - \rho_g \chi^\nu_g) \quad \text{on} \quad \Gamma_{lg}(t) \] (4)

Momentum balance -

\[ \partial_t \rho_\alpha + \nabla \cdot (\rho_\alpha \mathbf{v}_\alpha \otimes \mathbf{v}_\alpha) = -\nabla p_\alpha + \nabla \cdot T_\alpha + \rho_\alpha \mathbf{g} \quad \text{in} \quad \Omega_\alpha(t), \quad \alpha = l, g \] (5)

\[ -\left( (p_l - p_g) I + (T_l - T_g) \right) \cdot \mathbf{n} = \dot{m} (\mathbf{v}_l - \mathbf{v}_g) - \sigma \kappa \mathbf{n} \quad \text{on} \quad \Gamma_{lg}(t) \] (6)

\[ \mathbf{v}_l \cdot \mathbf{t}_i = \mathbf{v}_g \cdot \mathbf{t}_i \quad \text{on} \quad \Gamma_{lg}(t) \] (7)

where,

\[ T_\alpha = \mu_\alpha \nabla \mathbf{v}_\alpha + \nabla \mathbf{v}_\alpha^T + \xi_\alpha (\nabla \cdot \mathbf{v}_\alpha) I, \]

\[ \dot{m} = \rho_\alpha (\mathbf{v}_\alpha \cdot \mathbf{n} - \mathbf{v}_n), \]

\[ \sigma \text{- surface tension}, \quad \kappa = \nabla \Gamma \cdot \mathbf{n}, \]

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Sharp Interface Formulation

■ Mass conservation -

\[
\partial_t \rho_\alpha + \nabla \cdot (\rho_\alpha \mathbf{v}_\alpha) = 0 \quad \text{in } \Omega_\alpha(t), \ \alpha = l, g
\]  

(\rho_l \mathbf{v}_l - \rho_g \mathbf{v}_g) \cdot \mathbf{n} = \mathbf{v}_n (\rho_l - \rho_g) \quad \text{on } \Gamma_{lg}(t)

\[
\partial_t (\rho_g \mathbf{v}_g) + \nabla \cdot (\rho_g \mathbf{v}_g \mathbf{v}_g) = \nabla \cdot (D_g \rho_g \nabla \mathbf{v}_g) \quad \text{in } \Omega_g(t)
\]

\[
(\rho_l \mathbf{v}_l - (\rho_g \mathbf{v}_g - D_g \rho_g \nabla \mathbf{v}_g)) \cdot \mathbf{n} = \mathbf{v}_n (\rho_l - \rho_g \mathbf{v}_g) \quad \text{on } \Gamma_{lg}(t)
\]

■ Momentum balance -

\[
\partial_t \rho_\alpha + \nabla \cdot (\rho_\alpha \mathbf{v}_\alpha \otimes \mathbf{v}_\alpha) = -\nabla \rho_\alpha + \nabla \cdot \mathbf{T}_\alpha + \rho \mathbf{g} \quad \text{in } \Omega_\alpha(t), \ \alpha = l, g
\]

\[
-(\rho_l - \rho_g) \mathbf{l} + (\mathbf{T}_l - \mathbf{T}_g) : \mathbf{n} = \dot{m}(\mathbf{v}_l - \mathbf{v}_g) - \sigma \kappa \mathbf{n} \quad \text{on } \Gamma_{lg}(t)
\]

\[
\mathbf{v}_l \cdot \mathbf{t}_i = \mathbf{v}_g \cdot \mathbf{t}_i \quad \text{on } \Gamma_{lg}(t)
\]

where, \( \mathbf{T}_\alpha := \mu_\alpha (\nabla \mathbf{v}_\alpha + \nabla \mathbf{v}_\alpha^T) + \xi_\alpha (\nabla \cdot \mathbf{v}_\alpha) \mathbf{l} \), \( \dot{m} := \rho_\alpha (\mathbf{v}_\alpha \cdot \mathbf{n} - \mathbf{v}_n) \), \( \sigma \) - surface tension, \( \kappa = \nabla \Gamma \cdot \mathbf{n} \) - curvature
Sharp Interface Formulation

■ Energy balance -

\[ \partial_t (\rho_\alpha u_\alpha) + \nabla \cdot (\rho_\alpha h_\alpha v_\alpha) = \nabla \cdot (k_\alpha \nabla T_\alpha) + v_\alpha \cdot \nabla p_\alpha + T_\alpha : \nabla v_\alpha \quad \text{in } \Omega_\alpha(t), \ \alpha = l, g \quad (8) \]

\[ (k_g \nabla T_g - k_l \nabla T_l) \cdot n = \dot{m} (h_g - h_l) = \dot{m} \mathcal{L} \quad \text{on } \Gamma_{lg}(t) \quad (9) \]

\[ T_l = T_g = T_{sat} \quad \text{on } \Gamma_{lg}(t) \quad (10) \]

where, \( u_\alpha \) - internal energy per unit mass of phase \( \alpha \), \( h_\alpha := u_\alpha + p_\alpha / \rho_\alpha \) - specific enthalpy of phase \( \alpha \), \( \mathcal{L} := h_g - h_l \) - latent heat of evaporation

\[ \partial_t (\rho_S C_p S T_S) = \nabla \cdot (k_S \nabla T_S) \quad \text{in } \Omega_S \quad (11) \]

\[ k_S \nabla T_S \cdot n_S = k_\alpha \nabla T_\alpha \cdot n_S \quad \text{on } \Gamma_{S\alpha}(t) \quad (12) \]

\[ T_S = T_\alpha \quad \text{on } \Gamma_{S\alpha}(t) \quad (13) \]
Sharp Interface Formulation

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\[ \partial_t (\rho_\alpha u_\alpha) + \nabla \cdot (\rho_\alpha h_\alpha \mathbf{v}_\alpha) = \nabla \cdot (k_\alpha \nabla T_\alpha) + \mathbf{v}_\alpha \cdot \nabla \rho_\alpha + T_\alpha : \nabla \mathbf{v}_\alpha \quad \text{in } \Omega_\alpha(t), \quad \alpha = l, g \quad (8) \]

\[ (k_g \nabla T_g - k_l \nabla T_l) \cdot \mathbf{n} = \dot{m} (h_g - h_l) = \dot{m}\mathcal{L} \quad \text{on } \Gamma_{lg}(t) \quad (9) \]

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■ Reaction/Evaporation rate -

\[ \frac{(\rho_l + \rho_g)}{2} (\mathbf{v}_n - \mathbf{v}_g \cdot \mathbf{n}) = -f(\chi^v_g) = -R \left\{ \left( \frac{\chi^v_g}{\chi^{v sat}} \right)^2 - 1 \right\} \quad (14) \]

where, \( f \) is the resulting evaporation rate, and \( R \) is a reaction constant of dimension kg m\(^{-2}\) s\(^{-1}\)
Phase-field Formulation

- $\phi = 1$ (Fluid I), $\phi = 0$ (Fluid II)
- Approximate sharp interface by a smooth phase field $\phi$
- Moving interface replaced by a thin, diffuse layer
- All equations solved in a fixed domain

▷ Redeker, M., Rohde, C., Pop., I.S., Upscaling of a tri-phase phase-field model for precipitation in porous media, *IMA J Appl Math*, 81, 898–939, 2016.
▷ Bringedal, C., Von Wolff, L., Pop, I.S., Phase field modeling of precipitation and dissolution processes in porous media: upscaling and numerical experiments, *Multiscale Model. Simul.*, 18(2), 1076–1112, 2020.
Diffuse Interface Model: Our Take

- **Phase-field equation** -

\[
\rho \left\{ \partial_t \phi + \nabla \cdot (\phi \mathbf{v}) \right\} = \frac{\gamma}{\nu} \left\{ \nabla^2 \phi - \lambda^{-2} P'(\phi) \right\} - \sqrt{2} \lambda^{-1} \phi(1 - \phi)f(\chi^\nu_g) \quad \text{in } \Omega_F \quad (15)
\]

where, \( P(\phi) = \phi^2(1 - \phi)^2 \), \( \rho = \phi \rho_l + (1 - \phi) \rho_g \), \( \mathbf{v} \) - velocity of the mixture, \( \gamma \) and \( \nu \) are parameters having dimensions kg s\(^{-2}\) and m s\(^{-1}\)

\[
\nabla \phi \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_S \quad (16)
\]

- **Mass conservation equation** -

\[
\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \text{in } \Omega_F \quad (17)
\]

\[
\partial_t \left( \phi \rho_l + (1 - \phi) \rho_g \chi^\nu_g \right) + \nabla \cdot \left\{ (\phi \rho_l + (1 - \phi) \rho_g \chi^\nu_g) \mathbf{v} \right\} = \nabla \cdot \left( D^\nu_g \rho_g (1 - \phi) \nabla \chi^\nu_g \right) \quad \text{in } \Omega_F \quad (18)
\]

\[
D^\nu_g \rho_g (1 - \phi) \nabla \chi^\nu_g \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_S \quad (19)
\]
Diffuse Interface Model: Our Take

- **Momentum balance equation** -

\[
\partial_t (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) = -\nabla p + \nabla \cdot \mathbf{T} - \nabla \cdot (\lambda \sigma \nabla \phi \otimes \nabla \phi) + \rho \mathbf{g} \quad \text{in } \Omega_F
\]  

where, \( p \) - pressure of the mixture and \( \mathbf{T} = \phi \mathbf{T}_l + (1 - \phi) \mathbf{T}_g \)

\[
\mathbf{v} = 0 \quad \text{on } \Gamma_S
\]  

- **Energy balance equation** -

\[
\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho h \mathbf{v}) = \nabla \cdot (k \nabla T) + \mathbf{v} \cdot \nabla p + \mathbf{T} : \nabla \mathbf{v} + \mathbf{v} \cdot \{ \nabla \cdot (\lambda \sigma \nabla \phi \otimes \nabla \phi) \} \quad \text{in } \Omega_F
\]  

where, \( \mathbf{u}, h, k \) and \( T \) are the internal energy, specific enthalpy, heat conductivity and Temperature of the mixture, respectively

\[
k \nabla T \cdot \mathbf{n} = k_s \nabla T_S \cdot \mathbf{n} \quad \text{on } \Gamma_S
\]

\[
T = T_S \quad \text{on } \Gamma_S
\]
Decreasing Energy

The energy associated with the above phase-field model is given by

\[
E = \int_{\Omega_F} \left( \frac{1}{2} \rho \nabla v^2 + \gamma \lambda^{-1} P(\phi) + \frac{1}{2} \gamma \lambda | \nabla \phi |^2 + \rho u + \rho F(\rho, \phi) \right) dx
\]  

(25)

Here the density energy \( \rho F(\rho, \phi) \) is defined as follows

\[
\partial_\phi (\rho F(\rho, \phi)) = \sqrt{2} \nu \phi (1 - \phi) f(\chi^V_g)
\]  

(26)

Then one can compute

\[
\frac{d}{dt} E(t) = \int_{\Omega_F} \left( \nabla \cdot (\rho \nabla \phi) - \frac{\eta^2}{\rho \lambda \nu} - \partial_\rho (\rho F(\rho, \phi)) \nabla \cdot (\rho \mathbf{v}) \right) dx
\]  

(27)

where, \( \eta = \gamma \lambda^{-1} P' (\phi) - \gamma \lambda \nabla^2 \phi + \sqrt{2} \nu \phi (1 - \phi) f(\chi^V_g) \).

\( \Rightarrow \) Decreasing energy for zero velocities, or low enough divergence.

▷ Ghosh, T., Bringedal, C., A phase-field approach to model evaporation in porous media: Upscaling from pore to Darcy scale, arXiv preprint, 2021.

https://arxiv.org/abs/2112.13104
Sharp Interface Limit

- Phase field/diffuse interface model can be seen as an approximation of the sharp interface model

- Introduce the dimensionless parameter $\zeta = \lambda / L$ related to thickness of the diffuse interface region

- Investigate the behavior of the solution as $\zeta \to 0$ : We recover the sharp interface formulation!
Sharp Interface Limit

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- Introduce the dimensionless parameter $\zeta = \lambda / L$ related to thickness of the diffuse interface region

- Investigate the behavior of the solution as $\zeta \to 0$ : We recover the sharp interface formulation!

- Matched asymptotic expansion: away from the diffuse interface (outer expansion), close to it (inner expansion) and applying matching condition at the transition region

- Outer expansion: gives the governing equations in the corresponding phases

- Inner expansion: provides the interface conditions between the two phases
- Scale separation: $\varepsilon = \frac{l}{L} \ll 1$ and $y = \varepsilon^{-1} x$
- Write the unknowns as a series expansion in terms of the scale separation $\varepsilon$
- Diffusion dominated regime, i.e. $\text{Pe} = \mathcal{O}(\varepsilon)$
Upscaling: Periodic Homogenization

**Upscaled equations: Summary**

\[
\frac{\partial \overline{\rho}_0}{\partial t} + \varepsilon \nabla_x \cdot (\overline{\rho}_0 \overline{v}_0) = \mathcal{O}(\varepsilon) \quad \text{in } \mathbb{D} \tag{28}
\]

\[
\frac{\partial}{\partial t} \left( \overline{\rho}_0 \chi \overline{v}_0 \right) + \varepsilon \nabla_x \cdot \left\{ \left( \overline{\rho}_0 \chi \overline{v}_0 \right) \overline{v}_0 \right\} = \nabla_x \cdot (D \nabla_x \chi \overline{v}_0) + \mathcal{O}(\varepsilon) \quad \text{in } \mathbb{D} \tag{29}
\]

\[
\overline{v}_0 = -K \nabla_x \rho_0 - G \quad \text{in } \mathbb{D} \tag{30}
\]

\[
\frac{\partial (\rho_0 u_0)}{\partial t} + \frac{\partial (\rho_S C_p, s T_S)}{\partial t} + \varepsilon \nabla_x \cdot \left( \rho_0 h_0 \overline{v}_0 \right) = \nabla_x \cdot (A \nabla_x T_0) + \mathcal{O}(\varepsilon) \quad \text{in } \mathbb{D} \tag{31}
\]

The phase field \( \phi_0(t, x, y) \) is updated locally in each pore by solving

\[
\rho_0 \left\{ \frac{\partial \phi_0}{\partial t} + \nabla_y \cdot (\phi_0 \overline{v}_0) \right\} = \frac{\gamma}{\nu} \left\{ \nabla^2_y \phi_0 - \lambda^{-2} P'(\phi_0) \right\} - \frac{\sqrt{2}}{\lambda} \phi_0 (1 - \phi_0) f(\chi \overline{v}_0) \quad \text{in } P
\]

\[
\nabla_y \phi_0 \cdot n_P = 0 \quad \text{on } \Gamma_P \quad \text{(32)}
\]

Periodicity in \( y \) across \( \partial Y \).
Upscaling: Periodic Homogenization

- The effective diffusion matrix $\mathcal{D}(t, x)$:

$$d_{ij}(t, x) = \int_P D^v_{gR_0}(1 - \phi_0) \left( \delta_{ij} + \partial_{y_i} \kappa^j \right) \, dy, \quad \text{with } i, j \in \{1, 2, \cdots, d\} \quad (33)$$

where

$$\nabla_y \cdot \{ D^v_{gR_0}(1 - \phi_0)(e_j + \nabla_y \kappa^j) \} = 0 \quad \text{in } P$$

$$D^v_{gR_0}(1 - \phi_0)(e_j + \nabla_y \kappa^j) \cdot n_P = 0 \quad \text{on } \Gamma_P \quad (34)$$

Periodicity in $y$ across $\partial Y$

$$\partial_t \left( \frac{\rho \chi V}{g_0} \right) + \varepsilon \nabla_x \cdot \left( \frac{\rho \chi V}{g_0} \mathbf{V}_0 \right) = \nabla_x \cdot (\mathcal{D} \nabla_x \chi g_0) + \mathcal{O}(\varepsilon)$$
Upscaling: Periodic Homogenization

- The effective "permeability" matrix $\mathcal{K}(t, \mathbf{x})$:

$$k_{ij}(t, \mathbf{x}) = \int_P w_i^j \, dy, \quad \text{with } i, j \in \{1, 2, \cdots, d\}$$

(35)

where

$$(\mathbf{e}_j + \nabla_y \Pi^j) + \nabla_y \cdot \left\{ \mu_0 \left( \nabla_y w^j + \nabla_y w^T \right) \right\} + \nabla_y \cdot \left\{ \xi_0 \left( \nabla_y \cdot w^j \right) \mathbf{I} \right\} = 0$$

in $P$

$$\nabla_y \cdot (\rho_0 w^j) = 0$$

in $P$

$$w^j = 0$$

on $\Gamma_P$

(36)

Periodicity in $y$ across $\partial Y$

$$\overline{v}_0 = -\mathcal{K} \nabla_x p_0 - \mathcal{G}$$
Upscaling: Periodic Homogenization

- The effective gravity vector $G(t, x)$:

$$ g_i(t, x) = \int_P w_0^i \, dy, \quad \text{with } i \in \{1, 2, \cdots, d\} \tag{37} $$

where

$$ \nabla_y \Pi^0 = \nabla_y \cdot \left\{ \mu_0 \left( \nabla_y w^0 + \nabla_y w^0 \top \right) + \xi_0 \left( \nabla \cdot w^0 \right) I \right\} - \nabla_y \cdot (\lambda \sigma \nabla \phi_0 \otimes \nabla \phi_0) + \rho_0 g \quad \text{in } P $$

$$ \nabla_y \cdot (\rho_0 w^0) = \partial_t \rho_0 \quad \text{in } P $$

$$ w^0 = 0 \quad \text{on } \Gamma_P \tag{38} $$

$$ \bar{v}_0 = -\kappa \nabla_x p_0 - G $$
Upscaling: Periodic Homogenization

• The effective heat conductivity matrix $\mathcal{A}(t, \mathbf{x})$:

$$a_{ij}(t, \mathbf{x}) = \int_{P} \left\{ k_0 \left( \delta_{ij} + \partial_{y_i} \psi^j \right) + k_S \left( \delta_{ij} + \partial_{y_i} \eta^j \right) \right\} dy,$$

with $i, j \in \{1, 2, \cdots, d\}$ (39)

where

$$\nabla_y \cdot \left\{ k_0 \left( e_j + \nabla_y \psi^j \right) \right\} = 0 \quad \text{in } P$$

$$\nabla_y \cdot \left\{ k_S \left( e_j + \nabla_y \eta^j \right) \right\} = 0 \quad \text{in } S$$

$$\psi^j = \eta^j \quad \text{on } \Gamma_P$$

$$k_0 \left( e_j + \nabla_y \psi^j \right) \cdot \mathbf{n}_P + k_S \left( e_j + \nabla_y \eta^j \right) \cdot \mathbf{n}_P = 0 \quad \text{on } \Gamma_P$$

Periodicity in $y$ across $\partial Y$

$$\partial_t (\rho_0 \mathbf{u}_0) + \partial_t (\rho_S C_{p,S} T_{S0}) + \varepsilon \nabla_\mathbf{x} \cdot (\rho_0 h_0 \mathbf{v}_0) = \nabla_\mathbf{x} \cdot (\mathcal{A} \nabla_\mathbf{x} T_0) + \mathcal{O}(\varepsilon)$$
Summary

- Derived a new phase field model to describe evaporation on the pore scale

- Proposed model follows a decreasing free energy only for the diffusion dominated case

- Sharp interface limit of the phase field formulation recovers the sharp interface formulation (i.e., governing equations and boundary conditions)

- Derived an upscaled (REV-scale) model taking into account the pore-scale information

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Thank you!

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Outer Expansion: Leading Order

- Phase field equation: \( P'(\phi_0^{out}) = 0 \implies \phi_0^{out} = 0, 1/2, 1. \)

- Governing equations in \( \Omega^l_0(t) \):

\[
\begin{align*}
\partial_t \rho_{l,0}^{out} + \nabla \cdot (\rho_{l,0}^{out} \mathbf{v}_{l,0}^{out}) &= 0 \\
\partial_t \left( \rho_{l,0}^{out} \mathbf{v}_{l,0}^{out} \right) + \nabla \cdot \left( \rho_{l,0}^{out} \mathbf{v}_{l,0}^{out} \otimes \mathbf{v}_{l,0}^{out} \right) &= -\nabla p_{l,0}^{out} + \nabla \cdot \mathcal{T}_{l,0}^{out} + \rho_{l,0}^{out} \mathbf{g} \\
\partial_t \left( \rho_{l,0}^{out} \mathbf{u}_{l,0}^{out} \right) + \nabla \cdot \left( \rho_{l,0}^{out} \mathbf{h}_{l,0}^{out} \mathbf{v}_{l,0}^{out} \right) &= \nabla \cdot \left( k_l \nabla T_{l,0}^{out} \right) + \mathbf{v}_{l,0}^{out} \cdot \nabla p_{l,0}^{out} + \mathcal{T}_{l,0}^{out} : \nabla \mathbf{v}_{l,0}^{out}
\end{align*}
\]

- Governing equations in \( \Omega^g_0(t) \):

\[
\begin{align*}
\partial_t \rho_{g,0}^{out} + \nabla \cdot \left( \rho_{g,0}^{out} \mathbf{v}_{g,0}^{out} \right) &= 0, \\
\partial_t \left( \rho_{g,0}^{out} \mathbf{v}_{g,0}^{out} \right) + \nabla \cdot \left( \rho_{g,0}^{out} \mathbf{v}_{g,0}^{out} \otimes \mathbf{v}_{g,0}^{out} \right) &= -\nabla p_{g,0}^{out} + \nabla \cdot \mathcal{T}_{g,0}^{out} + \rho_{g,0}^{out} \mathbf{g} \\
\partial_t \left( \rho_{g,0}^{out} \mathbf{u}_{g,0}^{out} \right) + \nabla \cdot \left( \rho_{g,0}^{out} \mathbf{h}_{g,0}^{out} \mathbf{v}_{g,0}^{out} \right) &= \nabla \cdot \left( k_g \nabla T_{g,0}^{out} \right) + \mathbf{v}_{g,0}^{out} \cdot \nabla p_{g,0}^{out} + \mathcal{T}_{g,0}^{out} : \nabla \mathbf{v}_{g,0}^{out}
\end{align*}
\]
Inner Expansion

- Phase field equation:

\[ O(\zeta^{-2}) : \quad \phi_0^\text{in}(t, z, s) = \phi_0^\text{in}(z) = \frac{1}{2} \left(1 - \tanh \left( \frac{z}{\sqrt{2L}} \right) \right) \]

\[ O(\zeta^{-1}) : \quad \left( \frac{\rho_{i,0}^\text{out} + \rho_{g,0}^\text{out}}{2} \right) \left( \mathbf{v}_{n,0} - \mathbf{v}_{g,0} \cdot \mathbf{n}_0 \right) = - \left( \frac{\alpha}{\nu} \kappa_0 + f(\chi_{g,0}^\text{out}) \right) \quad \text{on } \Gamma_{lg}(t) \]

- Mass conservation equations:

\[ O(\zeta^{-1}) : \quad \mathbf{v}_{n,0} \left( \rho_{g,0}^\text{out} - \rho_{i,0}^\text{out} \right) = \left( \rho_{g,0}^\text{out} \mathbf{v}_{g,0}^\text{out} - \rho_{i,0}^\text{out} \mathbf{v}_{l,0}^\text{out} \right) \cdot \mathbf{n}_0 \quad \text{on } \Gamma_{lg}(t) \]

\[ O(\zeta^{-2}) : \quad \mathbf{v}_{n,0} \left( \rho_{g,0}^\text{out} \chi_{g,0}^\text{out} - \rho_{i,0} \right) = \left( \left( \rho_{g,0}^\text{out} \chi_{g,0}^\text{out} \mathbf{v}_{g,0}^\text{out} - D_g \rho_{g,0}^\text{out} \nabla \chi_{g,0}^\text{out} \right) - \rho_{i,0}^\text{out} \mathbf{v}_{l,0}^\text{out} \right) \cdot \mathbf{n}_0 \quad \text{on } \Gamma_{lg}(t) \]
Inner Expansion

- Momentum balance equation:
  \[ \mathcal{O}(\zeta^{-2}) : \quad v_{g,0} \cdot t_0 = v_{l,0} \cdot t_0 \quad \text{on} \quad \Gamma_{lg}(t) \]
  \[ \mathcal{O}(\zeta^{-1}) : \quad \text{Provides normal velocity condition on} \quad \Gamma_{lg}(t) \]

- Energy balance equation:
  \[ \mathcal{O}(\zeta^{-2}) : \quad T_{g,0}^{out} = T_{l,0}^{out} \quad \text{on} \quad \Gamma_{lg}(t) \]
  \[ \mathcal{O}(\zeta^{-1}) : \quad (k_g \nabla T_{g,0}^{out} - k_l \nabla T_{l,0}^{out}) \cdot n_0 = \dot{m}_0 \left( u_{g,0}^{out} - u_{l,0}^{out} \right) = \dot{m}_0 \mathcal{L} \quad \text{on} \quad \Gamma_{lg}(t) \]