AFFINENESS OF SOME QUOTIENT DUR SHEAVES OF
A SUPER AFFINE GROUP

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Dedicated to Professor Mitsuhiro Takeuchi in honor of his distinguished career

Abstract. We prove that given a super affine closed subgroup $H$ of a super affine group $G$ over a field $k$ of characteristic $k \neq 2$, the dur $k$-sheaf $\tilde{G}/\tilde{H}$ of right cosets is affine if the affine $k$-group $\Pi$ associated to $H$ is (a) reductive or (b) pro-finite. Especially when $G$ is algebraic, the result in Case (a) gives rise to a positive answer to Brundan’s question which was recently discussed by Zubkov [10].

Key Words: super affine group, super Hopf algebra, dur sheaf.

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1. Introduction

Throughout this paper we work over a fixed field $k$, whose characteristic $k$ is supposed to differ from 2 unless otherwise stated. Let $S$ denote the category of vector spaces graded by the group $\mathbb{Z}_2 = \{0, 1\}$. It is a $k$-linear tensor category given the canonical symmetry

$$V \otimes W \rightarrow W \otimes V, \quad v \otimes w \mapsto (-1)^{|v||w|} w \otimes v,$$

where $v(\in V), w(\in W)$ are supposed to be homogeneous elements of degree $|v|, |w|$. Algebraic systems in the vector space category are generalized to those in $S$. The generalized latter are called with super prefixed; for example, commutative (Hopf) algebras in $S$ are called super commutative (Hopf) algebras. A super $k$-functor (resp., super $k$-group functor) is a set-valued (resp., group-valued) functor on the category of super commutative algebras. Generalizing the subject worked out by Demazure and Gabriel [1], Zubkov [10] recently defined the notion of (dur) $k$-sheaves in the super context, and successfully associated a (dur) $k$-sheaf with universal property to every super $k$-functor.

A super $k$-(group) functor is said to be affine if it is representable. It is then represented by a super commutative (Hopf) algebra, say $A$, and the functor is denoted by $\text{Sp} A$ (denoted by $\text{SSp} A$ in [10]). An affine super $k$-group functor is called a super affine $k$-group. Let $G = \text{Sp} A$ be as such, where $A = A_0 \bigoplus A_1$ is a super commutative Hopf algebra. It is said to be algebraic (resp., finite) if $A$ is finitely generated as an algebra (resp., finite-dimensional). This $G$, restricted to the category of ordinary commutative algebras, gives rise to the affine $k$-group $\overline{G} = \text{Sp} \overline{A}$ which is represented by the largest quotient ordinary Hopf algebra

$$\overline{A} := A_0/A_1^2.$$
of $A$; see [11, p.298]. We call $\mathcal{G}$ the affine $k$-group associated to $G$.

Our main results of this paper are the following two.

**Theorem 1.1.** Let $G$ be a super affine $k$-group, and let $H$ is a super affine closed subgroup of $G$. Then the dur $k$-sheaf $\mathcal{G}/H$ associated to the super $k$-functor $G/H$ of right cosets is affine, if the affine $k$-group $\mathcal{H}$ associated to $H$ is (a) reductive or (b) pro-finite.

**Corollary 1.2.** Let $G, H$ be as above, and assume that $G$ is algebraic. Then the super $k$-sheaf $\mathcal{G}/H$ associated to $G/H$ is affine, and coincides with $\mathcal{G}/\mathcal{H}$, if $\mathcal{H}$ is (a) reductive or (b) finite.

The last result in Case (a) answers in the positive Brundan’s question discussed by Zubkov [10], who proved the same result as Corollary 1.2 in the restricted situation when (a) $\text{ch } k > 2$ and $\mathcal{H}$ is reductive or (b) $G$ is finite.

If $G$ is algebraic in the situation above, then $H$ and hence $\mathcal{H}$ are algebraic, in which case Condition (b) in Theorem 1.1, which is restated so as by (b) in Theorem 1.3 below, is equivalent to (b) in Corollary 1.2. Therefore, the corollary follows from Theorem 1.1 and the following.

**Proposition 1.3.** Let $G, H$ be as in Theorem 1.1. Assume that $G$ is algebraic. If $\mathcal{G}/H$ is affine, then $\mathcal{G}/\mathcal{H}$ is affine, and coincides with $\mathcal{G}/\mathcal{H}$.

Our three results above remain true if $G/H$ is replaced by the super $k$-functor $H\backslash G$ of left cosets; to see this, apply each of the results to the opposite group $G^{\text{op}}$.

Let $G = \text{Sp } A$ be a super affine $k$-group. Every super affine closed subgroup $H$ of $G$ arises uniquely from a quotient super Hopf algebra $A \to D$ of $A$ so that $H = \text{Sp } D$. As a generalization of Takeuchi’s Theorem [8, Theorem 10], it is proved by Zubkov [10, Theorem 5.2] that $G/H$ (or equivalently, $G\backslash H$) is affine if and only if $A$ is faithfully coflat, regarded as a right or equivalently, left $D$-comodule along the quotient map $A \to D$. If this last condition is satisfied, we say that

\[(1.3)\quad A \to D \text{ is faithfully coflat};\]

see Proposition 2.2 below for some equivalent conditions. In virtue of the equivalence stated above, Theorem 1.1 is translated into Hopf-algebra language as follows.

**Theorem 1.4.** A quotient $A \to D$ of a super commutative Hopf algebra $A$ is faithfully coflat if the commutative Hopf algebra $\overline{D} = D_0/D_1^2$ is (a) cosemisimple or (b) a directed union of finite-dimensional Hopf subalgebras.

Generalizing [11 III, Sect.3, 7.2], the main theorem, Theorem 6.2, of [10] states that if a super affine closed subgroup $H$ of a super affine $k$-group $G$ is normal, then $G/H$ is affine and is a super affine $k$-group. A Hopf-algebraic counterpart of this result as well as of some others from [10] (published 2009) had been proved by the author in the article [4] published 2005. In the article [4] just cited, an important role was played by the Tensor Product
Decomposition Theorem, which will be reproduced in Section 4 as Theorem 4.1 and whose proof will be given there in a refined form because a part of the original proof was not quite well. That theorem plays an important role in this paper as well, to prove Theorem 1.4 above. The proof of Theorem 1.4 is given in the last Section 5, while Proposition 1.3 is proved in Section 3. The two results imply the remaining Theorem 1.1 and Corollary 1.2 as was already noted.

2. Some basic results on super (co)algebras

As in [4], we will often write $\mathbb{Z}_2$ to denote the group (Hopf) algebra $k\mathbb{Z}_2$. Given an ordinary (resp., super) algebra $R$, we let $R_{\text{M}}$, $M_R$ (resp., $R_{\text{S}}$, $S_R$) denote the categories of left and respectively, right $R$-modules (resp., those modules in $S$). If $R$ is super, it is naturally regarded as a right $\mathbb{Z}_2$-module algebra, which constitutes the algebra $\tilde{R} = \mathbb{Z}_2 \ltimes R$ of smash product.

Note that $R_{\text{S}} = R_{\text{M}}$, $S_R = M_{\tilde{R}}$.

Proposition 2.1. For a super algebra $R$, the following are equivalent:

(a) $R$ is a right Noetherian ring;
(b) $R$ is a Noetherian object in $S_R$, or in other words, super right ideals in $R$ satisfy the descending chain condition;
(c) $\tilde{R} = \mathbb{Z}_2 \ltimes R$ is a right Noetherian ring.

The parallel conditions with ‘right’ replaced by ‘left’ are equivalent to each other.

Proof. (a) $\Rightarrow$ (b): Obvious.
(b) $\Rightarrow$ (c): Let $e_i (i = 1, 2)$ denote the primitive idempotents in the group (Hopf) algebra $k\mathbb{Z}_2$. Suppose that $e_1$ corresponds to the counit $\varepsilon$, so that $\varepsilon(e_i) = \delta_{i,1}$. We see that $\tilde{R} = \bigoplus_{i=1}^{2} e_i \otimes R$ in $S_R$, $e_1 \otimes R \simeq R$ in $S_R$, and $e_2 \otimes R$ is a degree shift of $R$. Therefore, (b) implies that $e_i \otimes R$ ($i = 1, 2$) are both Noetherian in $S_R$, which in turn implies (c).

(c) $\Rightarrow$ (a): This is seen if one introduces to $\tilde{R}$, just as in [4] Page 304, lines 17–18, an alternative, but isomorphic structure in $M_{\tilde{R}}$.

We have worked above in the ‘right case’. In the ‘left’ case, discuss in parallel, using a mirror. □

Lemma 5.1 of [3] characterizes faithfully flat or projective modules over a super algebra. We will dualize the result as far as will be needed. Given an ordinary (resp., super) coalgebra $C$, we let $C_{\text{M}}$, $M^C$ (resp., $C_{\text{S}}$, $S^C$) denote the categories of left and respectively, right $C$-comodules (resp., those comodules in $S$). If $C$ is super, it constitutes the coalgebra $\tilde{C} = \mathbb{Z}_2 \bowtie C$ of smash coproduct. Note that $C^S = \tilde{C}^M$, $S^C = M^\tilde{C}$.

Proposition 2.2. For an object $V$ in $S^C$, the following are equivalent:

(a) $V$ is (an) injective (cogenerator) in $M^C$;
(b) $V$ is (faithfully) coflat in $M^C$;
(c) The cotensor product functor $V \square_C : C^S \rightarrow S$ is (faithfully) exact.

A parallel result holds true for every object in $C^S$. 

Proof. (a) ⇔ (b): This is due to Takeuchi [7, Proposition A.2.1].

(b) ⇔ (c): Dualize the proof of [4, Lemma 5.1(1)]. In fact, (b) ⇒ (c) is obvious. For the converse, let \( \tilde{V} = \mathbb{Z}_2 \otimes V \) denote the tensor product in \( S \); this is an object in \( S^C \), given the right \( C \)-comodule structure arising from that on \( V \). If \( W \in C S \), we see that \( \varepsilon \otimes \text{id} \otimes \text{id} : \mathbb{Z}_2 \otimes V \otimes W \to V \otimes W \) induces a natural isomorphism \( \tilde{V} \square_C W \xrightarrow{\sim} V \square_C W \). Therefore, (c) implies that \( \tilde{V} \) is (faithfully) coflat in \( M^C \), which in turn implies (b), since \( \tilde{C} \) is faithfully coflat in \( M^C \).

Next, we give a supplementary result to [4]; see [9] for a more substantial, supplementary result to [4], which concerns structure of super cocommutative Hopf algebras. Let \( R \) be a super algebra. Let \( R^o \) denote the dual super coalgebra of \( R \) as defined in [4, p.290, line 16]; this consists of those elements in the dual vector space \( R^* \) which annihilate some super ideal in \( R \) of cofinite dimension.

**Lemma 2.3.** This \( R^o \) coincides with the dual coalgebra of \( R \) as defined in [6, p.109] in non-super context.

**Proof.** The latter is defined to be the pullback of \( R^* \otimes R^* \) along the dual \( R^* \to (R \otimes R)^* \) of the product map, which is therefore \( \mathbb{Z}_2 \)-graded. Each homogeneous element in it annihilates some super ideal in \( R \) of cofinite dimension, as is seen from the proof of [6, Proposition 6.0.3], 3) ⇒ 4) ⇒ 1). This proves the lemma.

**Remark 2.4.** The three results above are generalized as follows, with \( \mathbb{Z}_2 \) replaced by more general Hopf algebras. Here the characteristic \( ch \) may be arbitrary.

1. Let \( J \) be a finite-dimensional semisimple commutative Hopf algebra, and let \( R \) be a right, say, \( J \)-module algebra, which constitutes the algebra \( J \ltimes R \) of smash product. Proposition 2.1 holds for this \( R \), with \( \tilde{R}, S_R \) replaced by \( J \ltimes R, M_{J \ltimes R} \).

2. Let \( J \) be a Hopf algebra with bijective antipode, and let \( C \) be a right, say, \( J \)-comodule coalgebra, which constitutes the coalgebra \( J \bowtie_C \) of smash coproduct. Proposition 2.2 holds in the generalized situation that \( C_S, S^C \) are replaced by the categories \( C(M^J), (M^J)^C (= M^J \bowtie C) \) of left and respectively, right \( C \)-comodules in the tensor category \( M^J \).

3. Let \( R \) be an algebra graded by a finite group \( G \). Lemma 2.3 is generalized so that if an element in \( R^* \) annihilates some ideal in \( R \) of cofinite dimension, it necessarily annihilates some \( G \)-graded ideal of cofinite dimension.

3. **Proof of Proposition 1.3**

A super commutative algebra \( R \) is said to be Noetherian if super ideals in \( R \) satisfy the descending chain condition. By Proposition 2.1, this last condition is equivalent to that \( R \) is left or equivalently, right Noetherian ring.

**Lemma 3.1.** Let \( R \to A \) be a map of super commutative algebras, with which \( A \) is regarded as a super algebra over \( R \). Assume that \( A \) is finitely
generated over \( R \), and \( R \) is Noetherian. Then, \( A \) is Noetherian, and is 

clasically presented over \( R \) in the super sense as defined in [10, Page 721, line 4–6].

**Proof.** Let \( P = k[x_1, ..., x_n] \) denote a polynomial algebra in finite indeterminates, and let \( T = \wedge(V) \) denote the exterior algebra of a finite-dimensional vector space \( V \). It suffices to prove that the super commutative algebra \( R \otimes P \otimes T \) is Noetherian, since \( A \) is a homomorphic image of such a super algebra over \( R \). By (the proof of) Hilbert’s Basis Theorem, \( R \otimes P \) is Noetherian. This implies each sub-quotient \( R \otimes P \otimes \wedge^i(V) \) of \( R \otimes P \otimes T \) is Noetherian, which in turn implies the desired Noetherian property. \( \square \)

Let \( A \) be a super commutative Hopf algebra, whose coalgebra structure 

maps will be denoted by

\[
\Delta : A \to A \otimes A, \quad \Delta(a) = \sum a_1 \otimes a_2; \quad \varepsilon : A \to k.
\]

A super left or right coideal subalgebra \( B \subset A \) is said to be **faithfully flat** if \( A \) is faithfully flat as a left or equivalently, right \( B \)-module; see [4, Corollary 5.5] also for further equivalent conditions. Recall from [4, Proposition 5.6] the following.

**Proposition 3.2.** Fix a super commutative Hopf algebra \( A \). Then the faithfully flat super left coideal subalgebras \( B \subset A \) and the faithfully coflat quotient super Hopf algebras \( A \to D \) (see (1.3)) are in one-to-one correspondence, by

\[
B \mapsto A/B^+ A, \quad D \mapsto A^{\co D}.
\]

Here, \( B^+ = B \cap \ker \varepsilon \). Given a quotient \( A \to D, \ a \mapsto \bar{a} \), \( A^{\co D} \) is defined by

\[
A^{\co D} = \{ a \in A \mid \sum a_1 \otimes \bar{a}_2 = a \otimes 1 \},
\]

the right \( D \)-coinvariants in \( A \).

**Proof of Proposition 1.3.** Let \( A \to D \) be a faithfully coflat quotient of a super commutative Hopf algebra \( A \). This gives rise to a closed embedding

\[
H := \text{Sp} D \hookrightarrow G := \text{Sp} A \text{ of super affine } k\text{-groups with } G/H \text{ affine; see [10, Theorem 5.2].}
\]

Set \( B = A^{\co D} \). By Proposition 3.2 \( B \subset A \) is faithfully flat and \( A/B^+ A = D \). It follows by [4, Proposition 1.3] that \( N \mapsto N \otimes_B A \) gives a category equivalence

\[
S_B \xrightarrow{\cong} S_A^D,
\]

where \( S_A^D \) denotes the category of \((D, A)\)-Hopf modules in \( S \).

Assume that \( G \) is algebraic, or \( A \) is finitely generated. Since \( A \) is then Noetherian by Lemma 3.1, it is a Noetherian object in \( S_A^D \), whence the corresponding \( B \) in \( S_B \) is Noetherian. This, combined with Lemma 3.1 and [10, Proposition 5.2], implies the desired result. \( \square \)
4. TENSOR PRODUCT DECOMPOSITION THEOREM, REVISITED

Throughout this section we let $A = A_0 \oplus A_1$ denote a super commutative Hopf algebra. Recall from [12] the definition of the quotient ordinary Hopf algebra $\overline{A}$ of $A$. As in [3], we define

\begin{equation}
W^A := A_1 / A_0^+ A_1,
\end{equation}

where $A_0^+ = A_0 \cap \text{Ker} \, \varepsilon$; this is the odd part of the cotangent space of the super affine $k$-group Sp $A$ at unity. The assignments $A \mapsto \overline{A}$, $A \mapsto W^A$ are functorial. Recall that for any vector space $V$, the exterior algebra $\bigwedge (V)$ forms a super commutative and cocommutative Hopf algebra in which every element in $V$ is an odd primitive. We reproduce from [4] the following.

**Theorem 4.1.** ([4, Theorem 4.5, Remark 4.8])

1. There is a counit-preserving isomorphism $A \xrightarrow{\sim} \overline{A} \otimes \bigwedge (W^A)$ of super left $A$-comodule algebras.

2. Given a map $f : A \rightarrow D$ of super commutative Hopf algebras, isomorphisms $A \xrightarrow{\sim} A \otimes \bigwedge (W^A)$, $D \xrightarrow{\sim} D \otimes \bigwedge (W^D)$ such as claimed above can be chosen so that they make the following diagram commute:

\begin{equation}
\begin{array}{ccc}
A & \xrightarrow{\sim} & \overline{A} \otimes \bigwedge (W^A) \\
\downarrow f & & \downarrow f \otimes \bigwedge (W^f) \\
D & \xrightarrow{\sim} & D \otimes \bigwedge (W^D)
\end{array}
\end{equation}

The original proof given in [4] was divided into 0-affine case and general case. As the proof in the latter case was not quite well, we will refine it below. To start we need to recall some argument in 0-affine case.

The dual super coalgebra $A^\circ$ of $A$ (see Lemma 2.3) is a super cocommutative Hopf algebra. Let

\begin{equation}
V_{A^\circ} := P(A^\circ)_1
\end{equation}

denote the vector space of all odd primitives in $A^\circ$.

Assume that $A$ is 0-affine [4, Definition 4.2] in the sense that $A$ is finitely generated over the central subalgebra $A_0$. This implies $\dim W^A < \infty$; see [4, Proposition 4.4]. In this case, $V_{A^\circ}$ and $W^A$ are naturally dual to each other. Moreover, the finite-dimensional super Hopf algebras $\bigwedge (V_{A^\circ})$ and $\bigwedge (W^A)$ are dual to each other; see [4, Proposition 4.3(2) and Page 291, lines 1–3]. Choose arbitrarily a totally ordered basis $X$ of $V_{A^\circ}$, and let $\iota_X : \bigwedge (V_{A^\circ}) \rightarrow A^\circ$ denote the unit-preserving super coalgebra map defined by

\begin{equation}
\iota_X (x_\lambda \wedge x_\mu \wedge ... \wedge x_\nu) = x_\lambda x_\mu ... x_\nu,
\end{equation}

where $x_\lambda < x_\mu < ... < x_\nu$ in $X$. Let $\rho_X : A \rightarrow \bigwedge (W^A)$ denote the composite

\begin{equation}
A \rightarrow (A^\circ)^* \xrightarrow{\iota_X^*} \bigwedge (V_{A^\circ})^* \xrightarrow{\sim} \bigwedge (W^A),
\end{equation}

in which the first arrow is the canonical map. It is proved in [4, Proof in 0-affine case, pp. 300–301] that

\begin{equation}
\psi_X : A \rightarrow \overline{A} \otimes \bigwedge (W^A), \quad \psi_X (a) = \sum a_i \otimes \rho_X (a_2)
\end{equation}
is necessarily such an isomorphism as claimed by Part 1 of the theorem above; as in [4], an isomorphism of this form will be said to be admissible. We will identify \( X \) with the dual basis of \( W^A \) given the corresponding total order.

Given a map \( f : A \to D \) to another 0-affine super commutative Hopf algebra \( D \), choose totally ordered bases \( X \) of \( W^A \), and \( Y \) of \( W^D \) so that \( W^f : W^A \to W^D \) restricts to a map \( X \to Y \cup \{0\} \) which strictly preserves the order on \( X \setminus (W^f)^{-1}(0) \). (Note that the dual map \( V_Y^C : V_{D^0} \to V_{A^0} \) has the same property.) One then sees that the admissible isomorphisms \( \psi_X, \psi_Y \) make the diagram (\[4.2\]) commute. It follows that if \( f \) is an inclusion, in particular, then \( f : A \to D \), \( W^f : W^A \to W^D \) are both injections, through which we will regard so as

\[
(4.7) \quad \overline{A} \subset \overline{D}, \quad W^A \subset W^D, \ X \subset Y \ (\text{as ordered sets}).
\]

In this case we write

\[
(4.8) \quad (A, X) \subset (D, Y).
\]

**Proof in general case.** Let \( A \) be in general.

(1) Let \( \mathcal{F} = \mathcal{F}_A \) denote the set of those pairs \( (B, X) \) in which \( B \) is a 0-affine super Hopf subalgebra of \( A \), and \( X \) is a totally ordered basis of \( W^B \).

This set is ordered with respect to the \( \subset \) defined in (\[1.3\]). Given a directed subset \( \{(B_\alpha, X_\alpha)\}_\alpha \) of \( \mathcal{F} \), the directed union \( B := \bigcup_\alpha B_\alpha \) is a super Hopf subalgebra of \( A \) such that \( \overline{B} = \bigcup_\alpha \overline{B_\alpha}, \ W^B = \bigcup_\alpha W^{B_\alpha}; \) see (\[4.7\]) and [4, Proposition 4.3(3)]. This implies that the inductive limit \( \varinjlim_\alpha \psi_{X_\alpha} \) gives an isomorphism on \( B \) such as claimed by the theorem.

By Zorn’s Lemma we have a directed subset \( \mathcal{G} = \{(B_\alpha, X_\alpha)\}_\alpha \) of \( \mathcal{F} \) which is maximal with respect to inclusion. Set \( B = \bigcup_\alpha B_\alpha \). The argument in the preceding paragraph shows that it suffices to prove \( B = A \). Assume \( B \subset A \) on the contrary. As in the original proof we have a pair \( (B', X') \) in \( \mathcal{F} \) and a 0-affine super Hopf subalgebra \( C \subset A \) such that \( B' = B \cap C, \ C \not\subset B \).

Extend the basis \( X' \) of \( W^{B'} \) to a basis \( X' \cup Y \) (disjoint union) of \( W^C \). For each \( \alpha \) such that \( (B_\alpha, X_\alpha) \subset (B', X') \), consider the pair

\[
(4.9) \quad (B_\alpha C, X_\alpha \cup Y).
\]

This \( B_\alpha C \) is a 0-affine super Hopf subalgebra of \( A \). Since \( B_\alpha \cap C = B' \), it follows from the proof of [4, Proposition 4.7] that \( W^{B_\alpha C} = W^{B_\alpha} + W^C \), \( W^{B_\alpha} \cap W^C = W^{B'} \). Therefore, the \( X_\alpha \cup Y \) in (\[1.3\]) is a basis of \( W^{B_\alpha C} \).

Choose arbitrarily a total order on \( Y \), and extend the orders on \( X_\alpha, Y \) to a total order on \( X_\alpha \cup Y \), uniquely so that \( X_\alpha < Y \) (that is, \( x < y \) if \( x \in X_\alpha, y \in Y \)). Then the pair is in \( \mathcal{F} \). The subset \( \mathcal{G} \) joined with all pairs given in (\[4.9\]) forms a directed subset of \( \mathcal{F} \) which properly includes, contradicting the maximality of \( \mathcal{G} \). Therefore, we must have \( B = A \), as desired.

(2) Let \( f : A \to D \) be as in the theorem. Let \( \mathcal{F}_A \) denote the subset of \( \mathcal{F}_A \) consisting of those pairs \( (B, X) \) in which \( X \) decomposes as \( X = X' \cup X'' \) so that \( W^f : W^A \to W^D \) is injective on the subspace spanned by \( X' \), and vanishes on \( X'' \). Let \( \{(B_\alpha, X_\alpha)\}_\alpha \) be a maximal directed subset of \( \mathcal{F}_A \), which exists by Zorn’s Lemma. By modifying the proof of (1) above, it follows that \( A = \bigcup_\alpha B_\alpha \). We see that \( \mathcal{E} := \{(f(B_\alpha), f(X_\alpha))\}_\alpha \) is a directed subset of...
\( F_D \), where we suppose \( X_\alpha = X'_\alpha \cup X''_\alpha \) as above, and \( f(X'_\alpha) \) has the total order inherited from \( X'_\alpha \). Moreover, the isomorphisms \( \lim_\alpha \psi_{X_\alpha} \) on \( A \), and \( \lim_\alpha \psi_{f(X'_\alpha)} \) on \( f(A) = \bigcup_\alpha f(B_\alpha) \) make the diagram (4.2) with \( D \) replaced by \( f(A) \) commute. Again as above, we can choose a maximal directed subset \( \{(E_\beta, Y_\beta)\}_\beta \) of \( F_D \) which includes \( E \), and then necessarily have \( \bigcup_\beta E_\beta = D \). The proof completes if we replace the last isomorphism on \( f(A) \) with the isomorphism \( \lim_\beta \psi_{Y_\beta} \) on \( D \). \( \square \)

5. Proof of Theorem 1.4

**Lemma 5.1.** Let \( A \to D \) be a quotient of a super commutative Hopf algebra \( A \). Assume that \( A = \bigcup_\alpha A_\alpha \) is a directed union of super Hopf subalgebras \( A_\alpha \). If the induced quotients \( A_\alpha \to D_\alpha \) are all faithfully coflat, then \( A \to D \) is as well.

**Proof.** Set \( B = A^{\co D} \), \( B_\alpha = A^{\co A_\alpha} \); see (3.4). Note that \( B = \bigcup_\alpha B_\alpha \), \( D = \bigcup_\alpha D_\alpha \), directed unions. If \( A_\alpha \to D_\alpha \) are all faithfully coflat, it follows by Proposition 3.2 that \( B_\alpha \to A_\alpha \) are all faithfully flat, and \( A_\alpha/B_\alpha D_\alpha = D_\alpha \). This implies that \( B = \bigcup_\alpha B_\alpha \to \bigcup_\alpha A_\alpha = A \) is faithfully flat, and \( A/B^+A = \lim_\alpha k \otimes_{B_\alpha} A_\alpha = \lim_\alpha D_\alpha = D \). Again by Proposition 3.2, \( A \to D \) is faithfully coflat. \( \square \)

In what follows we prove Theorem 1.4. Let \( A \to D \) be a quotient of a super commutative Hopf algebra \( A \). Note that \( A \) is presented as a directed union \( A = \bigcup_\alpha A_\alpha \) of finitely generated super Hopf subalgebras \( A_\alpha \). Let \( A_\alpha \to D_\alpha \) be the induced quotients. As was seen in (4.7), each \( D_\alpha \) is a Hopf subalgebra of \( \overline{\mathcal{D}}_\alpha \). Therefore, if \( \overline{\mathcal{D}} \) satisfies Condition (a) or (b) in the theorem, each \( D_\alpha \) is

\[
\text{(5.1) (a) cosimisimple or (b) finite-dimensional,}
\]

respectively. In virtue of Lemma 5.1, replacing \( A \to D \) with \( A_\alpha \to D_\alpha \), we may suppose that \( A \) is finitely generated, and \( \overline{\mathcal{D}} \) is in each case of (5.1).

**Proof in Case (b).** By bozonization we have a surjection \( \tilde{A} := \mathbb{Z}_2 \bowtie A \to \tilde{D} := \mathbb{Z}_2 \bowtie D \) of ordinary Hopf algebras. Since \( A \) is assumed to be finitely generated and we are in case (b), Theorem 4.1 implies that \( \tilde{A} \) and hence \( \tilde{D} \) have polynomial growth, and \( \tilde{D} \) is finite-dimensional. It follows by the main theorem of [2] (see also [4, Theorem 5.4]) that \( \tilde{A} \) is cofree as a right \( \tilde{D} \)-comodule, which implies that \( A \) is an injective cogenerator (indeed, cofree) as a right \( D \)-comodule, as desired; see Proposition 2.1. \( \square \)

For our proof in Case (a), we need the following lemma, which immediately follows from Theorem 4.1.

**Lemma 5.2.** Let \( A \) be a super commutative Hopf algebra. Let \( \co A \) denote the super right coideal subalgebra of \( A \) which consists of all left \( A \)-coinvariants; this is defined as in (3.4), but on the opposite side, and has the restricted \( \varepsilon \) as counit.

1. There is a counit-preserving isomorphism \( \co A \to \wedge(W^A) \) of super algebras.
(2) Every quotient map \( A \to D \) restricts to a counit-preserving surjection \( co^\pi A \to co^\pi D \) of super algebras.

**Proof in Case (a).** Set \( R = co^\pi A \), \( T = co^\pi D \). By Theorem 4.1 (applied to \( D \)), \( T \subset D \) is faithfully flat and \( D/DT^+ = TD \). By [3, Proposition 1.1], we have a category equivalence,

\[
S^D T \xrightarrow{\sim} S^D T, \quad V \mapsto V \Box D
\]

where \( S^D T \) denotes the category of \((D,T)\)-Hopf modules in \( S \).

Assume that we are in Case (a). Then, \( Z_2 \otimes D \) is cosemisimple. Therefore, the category \( S^D T \), identified with \( M^Z_2 \otimes D \), is semisimple. By (5.2), \( S^D T \) is semisimple. It follows that each object \( M \) in \( S^D T \) is \( D \)-injective since the structure map \( M \to M \otimes D \) splits in \( S^D T \).

Recall that we may suppose that \( A \) is finitely generated, whence \( \dim W^A < \infty \). Then it follows by Lemma 5.2(2) that \( A \to D \) restricts to a counit-preserving surjection \( R \to T \) with nilpotent kernel, say \( I \). Suppose \( I^r = 0 \) with \( r > 0 \). For each \( 0 \leq i < r \), \( AI^i \) is naturally regarded as a \((D,R)\)-Hopf module, or in notation, \( AI^i \in S^D R \). Therefore, \( AI^i/AI^{i+1} \in S^D T \), whence it is \( D \)-injective. This implies that the short exact sequence

\[
0 \to AI^i/AI^{i+1} \to A/ AI^{i+1} \to A/ AI^i \to 0
\]

in \( S^D R \) splits \( D \)-colinearly. It follows that \( A \) decomposes so as

\[
A \simeq \bigoplus_{i=0}^{r-1} AI^i/ AI^{i+1}
\]

into direct summands of injective right \( D \)-comodules. Since the surjection \( A/ AI \to D \) induced from \( A \to D \) splits in \( S^D T \), \( A/ AI \) is a cogenerator as well in \( M^D \), whence \( A \) is an injective cogenerator in \( M^D \), as desired. \( \square \)

**6. Notes added after the first submission**

6.1. **Afterthought.** After I posted to the arXiv the original version of this paper on January 27, 2010, I found that Theorems 1.1 and 1.4 can be generalized to a large extent, as follows, by slightly modifying the original proofs.

Let \( G, H \) be as in Theorem 1.1. Let \( \bar{G}, \bar{H} \) denote the affine \( k \)-groups associated to \( G, H \), respectively. Recall that \( \bar{H} \) is an affine closed subgroup of \( \bar{G} \).

**Theorem 6.1.** The \( k \)-dur sheaf \( \bar{G}/ \bar{H} \) of right cosets is affine if and only if \( \bar{G}/ \bar{H} \) is affine.

In virtue of the affineness criteria for quotient dur sheaves due to Takeuchi [8, Theorem 10] and Zubkov [10, Theorem 5.2], the theorem just formulated is translated into Hopf-algebra language so as (a) \( \iff \) (c) of the next theorem.

Let \( A \to D \) be as in Theorem 1.1. Set \( \bar{A} = A_0/ A_1^1, \quad \bar{D} = D_0/ D_1^1 \); see (1.2). Then there arises a natural surjection \( \bar{A} \to \bar{D} \) of commutative Hopf algebras.

**Theorem 6.2.** The following are equivalent:

(a) \( A \to D \) is faithfully coflat;
(b) $A \to D$ is coflat;
(c) $\overline{A} \to \overline{D}$ is faithfully coflat;
(d) $\overline{A} \to \overline{D}$ is coflat;

**Proof.** (a) $\Rightarrow$ (b). Trivial.
(b) $\Rightarrow$ (d). This follows since by Theorem 4.1, $A \to \overline{A}$ is faithfully coflat, and $D \to \overline{D}$ is coflat.
(c) $\Leftrightarrow$ (d). This is due to Doi [3, Remark on Page 247].
(c) $\Rightarrow$ (a). Assume (c). As is seen from the proof of Theorem 1.4 in Case (a), it suffices to prove the following two:
(1) For each $0 \leq i < r$, the $\overline{A}i/\overline{A}i+1$ in (5.4) is an injective object in $SDT$.
(2) The surjection $A/\overline{A} \to D$ given in two lines below (5.4) splits in $SDT$.

Notice from [4, Proposition 1.1] that through the category equivalence (5.2), the last surjection $A/\overline{A} \to D$ corresponds to $\overline{A} \to \overline{D}$ in $SDT$, which splits by (c). This proves (2).

To prove (1), fix $i$, and set $M = \overline{A}i/\overline{A}i+1$. Since $A$ is an algebra in $SD$, we have the category $SA$ of $(D,A)$-Hopf modules in $S$. Similarly we have $SDA$. We see that $M$ has a natural structure in $SDA$; it induces the structure of $M$ in $SDT$ which was used in the proof of Theorem 1.4 in Case (a). Notice again that through (5.2), $M$ corresponds to

$$M/MT^+ = M/MR^+ (= \overline{A}i/\overline{A}iR^+)$$

in $SDT$. This last object indeed has the structure in $SDA$ which arises from the structure of $M$ in $SDA$. Since by (c), there exists a unit-preserving right $\overline{D}colinear map \overline{D} \to \overline{A}$, it follows by Doi’s Theorem [3, Theorem 1] that every ordinary $(\overline{D},A)$-Hopf module is injective as a right $\overline{D}$-comodule. It follows that $M/MT^+$ is an injective object in $SDT$; this proves (1). \qed

6.2. **Erratum.** In Section 1 the term ‘reductive’ is confused with ‘linearly reductive’, as was pointed out by Zubkov. Condition (a) in Theorem 1.1 and Corollary 1.2 should be replaced by (a) *linearly reductive*. Therefore, Corollary 1.2 in Case (a) does not generalize Zubkov’s result cited just below the corollary.

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