Generación de entrelazamiento y simulación relativista con osciladores paramétricos en cQED

Entanglement generation and relativistic simulation with cQED parametric oscillators

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DECLARACIÓN DE AUTORÍA Y ORIGINALIDAD DE LA TESIS 
PRESENTADA PARA OBTENER EL TÍTULO DE DOCTOR

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Resumen

Estamos viviendo una época en la que nuestra comprensión de la mecánica cuántica está avanzando a un ritmo excitante. Desde su concepción, siempre se le ha considerado el cimiento de nuestro entendimiento sobre una gran variedad de fenómenos físicos. Sin embargo, su papel solía consistir en ser un primer paso hacia la creación de teorías efectivas sobre, por ejemplo, las propiedades de semiconductores o la estadística de procesos que impliquen partículas fundamentales. Pero hoy en día, podemos explorar sus características más íntimas y sorprendentes con la más exquisita de las precisiones. O dicho de otra forma, se aproxima el tiempo en el que sea una tecnología.

El desarrollo de las tecnologías cuánticas está claramente motivado por sus aplicaciones. Las peculiaridades de la mecánica cuántica se han convertido en oportunidades para el almacenamiento, la transferencia y el procesamiento de información de formas imposibles para las ciencias clásicas. Por lo tanto, el diseño de computadores cuánticos capaces de otorgarnos esas ventajas es una tarea central para la investigación actual.

Esta intersección, entre curiosidad científica y posibilidad tecnológica, es donde esta tesis se desarrolla. Se prestará atención al estado del arte en tecnologías cuánticas, en particular a la electrodinámica cuántica de circuitos (cQED, por sus siglas en inglés). Este término engloba a cualquier circuito diseñado con material superconductor, normalmente incluyendo versátiles uniones Josephson. Se modelizarán circuitos existentes o se propondrán nuevos diseños que puedan responder preguntas actuales sobre mecánica cuántica. Esta tesis explora dos de esas preguntas.

Por un lado, se investiga la generación y detección de nuevos estados entrelazados no gaussianos de la radiación microondas. Estos estados son producidos en un nuevo oscilador paramétrico, construido recientemente en el ámbito de cQED, capaz de convertir un tono microondas en tres tonos diferentes simultáneamente. Esos nuevos tres fotones comparten entre sus magnitudes correlaciones cuánticas, en particular entrelazamiento genuino, que es considerado el recurso que permite el procesamiento cuántico de la información imposible a las ciencias clásicas. En el texto este entrelazamiento es denominado no gaussiano debido a que no se manifiesta en segundos momentos estadísticos y proponemos un criterio simple y práctico para la creación de testigos capaces de detectarlo: deben construirse con momentos estadísticos superiores, que cambien en el tiempo. Adicionalmente, se exploran las implicaciones teóricas que puede tener este criterio y se encuentran conexiones con otras clases de entrelazamiento, como la paradigmática inequivalencia entre los estados GHZ y W de tres qubits.
Resumen

Por otro lado, se explora otra de las posibles aplicaciones de las tecnologías cuánticas: la simulación de sistemas cuánticos. En la literatura anterior a la presente tesis abundan los circuitos que imitan sistemas en los que se deben considerar simultáneamente fenómenos cuánticos y relativistas, como pueden ser los efectos Casimir dinámico y Unruh. Estos últimos no han sido observados en sus formulaciones originales sino, alguno, en circuitos que crean fenómenos análogos. En este trabajo se explora la información que puede obtenerse de dichos análogos, proponiendo un circuito que permita explorar la dinámica interna de un espejo experimentando una trayectoria relativista, es decir, un espejo que produce el efecto Casimir dinámico.
Abstract

We are living in an age where our understanding of quantum mechanics is increasing at an exciting pace. Since its inception, we have always considered it at the foundation of a great variety of physical phenomena. However, it was often used as a first step in the creation of effective theories about, for instance, properties of semiconductors or statistics of processes involving fundamental particles. But nowadays, we are learning to harness the theory to its full extent. Nowadays, we can explore its most intimate and shocking features with the most exquisite control. It is close to become a technology.

The development of quantum technologies is clearly motivated by their applications. The oddities of quantum mechanics have become opportunities to store, transfer and process information in ways impossible to the classical sciences. Therefore, the task of designing a quantum computer capable of delivering said advantages is central to the research being done today.

This interface, between scientific curiosity and technological possibility, is where this thesis stands. Attention will be paid to the current state of the art in quantum technologies, mainly in circuit quantum electrodynamics (cQED). This acronym is an umbrella term that encompass any circuit designed to operate with superconducting material, most often composed of versatile Josephson junctions. Then, we will model existing circuits or propose new designs that may shed light on interesting topics in quantum mechanics. This thesis explores two specific topics:

On one hand, we study the generation and detection of new entangled non-gaussian states of microwave radiation. These states are produced in a new parametric oscillator, built recently within the field of cQED, capable of down-converting a microwave tone into three different tones at once. These new three photons share among their magnitudes quantum correlations, in particular genuine entanglement. This kind of entanglement is considered one of the resources that allow for the quantum processing of information impossible to the classical sciences. In this text we refer to it as non-Gaussian because of its manifestation on statistical moments higher than covariances and we propose a simple and practical criterion for the design of witnesses capable of detecting it: they must be built from higher statistical moments that change through time. Additionally, we speculate on the theoretical implications of the criterion and find suggestive connections to other entanglement classes, such as the paradigmatic nonequivalent GHZ and W three qubit states.
Abstract

On the other hand, we explore one of the possible applications of quantum technologies: the simulation of quantum systems. The literature prior to this thesis showcases multiple examples of superconducting circuits capable of mimicking systems in which one must consider both quantum and relativistic phenomena, such as the dynamical Casimir and Unruh effects. These effects have not been observed in their original setting, and some have only been observed in their analogue circuits. This work explores the information that can be obtained through analog simulation, proposing a circuit capable of featuring the internal dynamics of a mirror experiencing a relativistic trajectory, that is, a mirror producing the dynamical Casimir effect.
List of Publications

This thesis presents the results published in the following peer-reviewed articles

1. A. A. C., C. W. S. Chang, F. Quijandría, G. Johansson, C. M. Wilson and C. Sabín, Physical Review Letters 125 020502 (2020)

2. A. A. C., L. García Álvarez, E. Solano and C. Sabín, Physical Review A 103 062201 (2021)

3. A. A. C. and C. Sabín Physical Review A 105 022401 (2022).

Additionally, during the time the author of this thesis spent within the Universidad Complutense de Madrid Doctorate program, another publication took place

- P. Cordero Encinar, A. A. C. and C. Sabín Physical Review A 104 052609 (2021).
1 Introduction

The main purpose of chapter 1 is to present the reader an overview of the work contained in this thesis. There are three intentions behind such a summary. Firstly, we expect most of our readers to be interested only in pieces of the thesis, perhaps because they suspect our research might be connected to theirs. Therefore, the present overview intends to guide them to the pieces they are interested in, see section 1.2. Secondly, other group of readers will travel through the whole thing. For them, the overview provides a rationale, a thread of thoughts that encompass this thesis entirely, which is provided in section 1.1. Thirdly, this work builds upon the research of many others. We consider it is unfair to imagine the readers are familiar with all that previous research. Therefore, this chapter introduces some concepts on multipartite entanglement in section 1.3. Additionally, we provide references that we recommend to readers not familiar with circuit quantum electrodynamics (cQED) in section 1.4.

1.1 Objectives

Quantum theory has been a successful theory for around a century. As time passed, more and more phenomena were satisfactorily predicted, up to the point of considering it the most relevant foundation of our understanding of nature, together with general relativity. However, our attitude towards quantum effects has not always been the same. In its early years, its probabilistic and non-local features collided with our prejudices on the workings of nature. Quantum mechanics was considered a faulty tool that needed improvement, an instrument that needed getting rid of those features. With time we saw that was not the case, those striking features did not imply worrisome relations with consciousness, in the case of the measurement problem, or with faster than light actions, in the case of non-locality. Today, coherence and entanglement are considered the defining properties of quantum theories. Today, we understand the importance of exploring the consequences of uncertainty relations and non-locality.

It took more than theoretical work to change our opinions in this way. It took experimenting. The history of the XX and XXI centuries spans a chain of experiments that, ranging from the early frequency-discrete radiation of atoms to the surprising teleportation of quantum states, has discovered us increasingly intimate phenomena of quantum theory. The realization of those experiments and many others has granted us technologies that allow for the control and design of physical systems previously ignored. In order
1 Introduction

to exemplify this paradigm shift in greater detail, consider the text-book postulates of quantum mechanics: they specify what possible states a system may have (rays on any Hilbert space), what possible dynamics can take place (any Hamiltonian and measurements) and how to compose systems together (tensor products of states and operators); but they all rely on external theories to specify the actual content of the Universe. One of those theories is quantum electrodynamics: it tells us what is the particular set of possible states of the electromagnetic field and charged particles, the specific dynamics (the Hamiltonian) that will govern them and so on. Of course, those externals theories are, to our knowledge, correct and valuable all the way to the Standard Model of Fundamental Particles. But in the present day we can put the focus back on quantum mechanics and away from the external theories. Nowadays systems and hamiltonians are engineered from the more basic ingredients that the external theories postulate. As such, we design qubits with features of our choosing, instead of letting external theories specify which features physical systems have, or we design interactions between particular pairs or triplets of systems instead of figuring out how many nearest neighbours we must consider in our models. In this new paradigm we can explore the consequences of those postulates in a deeper sense, we can choose the content of our Universe to take quantum effects to its ultimate consequences.

Having described this mindset, this thesis is about using present-day quantum technologies to focus our theoretical work on quantum phenomena. With that context in mind, we explore systems that may contest our understanding of pieces of the theory. Lastly, we try to close the loop by proposing experiments that may prove or disprove the theory, as well as create new resources for further theoretical and practical developments.

1.2 Navigating this thesis

Following the lines of the last paragraph of section 1.1, we have taken concepts of a very versatile technology, circuit quantum electrodynamics (cQED), and used them to explore two different projects: genuinely tripartite entanglement and analog simulation of quantum-relativistic phenomena. Those two ideas led to three publications around which the chapters to come revolve.

Chapter 2 is about a proposal of an entanglement detection protocol. Said entanglement is thought to be produced by an already engineered three-body interaction in bosonic systems, that we denominate three-mode spontaneous parametric down conversion. Said interaction is known for generating non-gaussian states, so we will denominate its entanglement as non-gaussian throughout this work.

Chapter 3 is about the modification of a common Hamiltonian in cQED and other fields, the Rabi Hamiltonian, so that it may become the platform for the analog simulation of the microscopical basis of the dynamical Casimir effect. Said modification
1.3 Minimal introduction to tripartite entanglement

consists in the ability to tune the qubit-field coupling over time, which very much relates to the movement, real or simulated, of the qubit within the field modes. Then, we identify the qubit movement as if it were the movement of a mirror, providing the field with energy so that vacuum might be populated by photons.

Chapter 4 is about non-gaussian entanglement. In particular, we provide a precise definition for said entanglement and generalise the application of the concepts to systems beyond the bosonic modes of chapter 2. We find connections to the GHZ and W classes of pure 3 qubit states and we strengthen the results of the previous chapter.

Finally, we point out that chapters 2 to 4 share a similar structure. They revolve around a publication in a peer-reviewed journal. Therefore, we take the opportunity of this thesis to broaden the target audience of our work, which implies that those chapters contain accessible introductions to those publications, as well as providing greater detail in the derivation of their results. Additionally, we provide some materials that did not make it into the publications, as well as a discussion of the different connections between them. Note that throughout the text, $\hbar = 1$. Furthermore, we commit an abuse of notation and frequencies measured in hertz denote radians per second, when the correct definition of hertz is cycles per second.

1.3 Minimal introduction to tripartite entanglement

Approximately two thirds of this work deals with entanglement. We ignore the percentage of scientific articles produced today that deal with quantum information, but it is certainly high enough to make it impossible for one person to keep up with all of them. Therefore, we will use the present section to make a non-exhaustive introduction to entanglement. In fact, it only covers the concepts strictly required to understand the following chapters.

If the reader is familiar with any quantum technology, then they must have heard about so many different kinds of entanglement that it is impossible to keep a count of them: bipartite, multipartite, genuine, completely inseparable, $k$-producible, GHZ- or W-like, ... Behind this diversity, however, there is a fundamental reason: the definition of entanglement itself. So, let’s take a step back and briefly ask *What is entanglement?* The shortest definition we know of is *non-separability*. In order to give meaning to the definition, consider one of the text-book postulates of Quantum Mechanics

**Postulate 4:** The state space of a composite physical system is the tensor product of the state spaces of the component physical systems [...]

*From Quantum Information and Quantum Computation,*
M. A. Nielsen and I.L. Chuang.
It indicates how to build new, more complicated, systems from simpler ones. That is a very convenient postulate to have. With its aid, we can split the Universe into simple constituents, figure out their dynamics when isolated from the rest, and then join some of them together and try to solve the whole thing.

However, when the compound system evolves in time and correlations appear among its components, a tricky situation may happen: Even if we know everything there is to know about the compound system, this is, the global state $\psi$; it might make no sense to separate pieces of the global state and consider them the states of each of the components, as it happens in $|\psi\rangle = \frac{1}{\sqrt{2}} |\psi_1\rangle \otimes |\psi_2\rangle + \frac{1}{\sqrt{2}} |\phi_1\rangle \otimes |\phi_2\rangle$. Such a phenomenon is non-separability, and when it takes place we claim that the global state is entangled.

Please note the negation in its definition: There is entanglement when there is no separable description of at least one of the constituents. And if entanglement consists in the absence of a property... How can it not be astonishingly diverse? Think, for instance, about the variety of things that are not blue. Such a set contains elements as different as a robin and the Sahara desert, not to mention ridiculous elements to which the concept of color does not apply at all, such as the scent of wine, the dialectic process and many others. There is, however, a crucial difference between entanglement and the set of not-blue things. The definition of entanglement does have an affirmative ingredient, entanglement is (no negation there) a property of some quantum states, which is a much tamer set to begin with than the not-blue set. Having said that, the dimension of the global state space grows exponentially with the number of subsystems, so even if it is not as incomprehensible as the not blue set, it is still difficult to handle.

### 1.3.1 Multipartite entanglement

One of the first displays of the diversity of entangled states arises when considering tripartite systems. When the global system is made of three different constituents its state might be non-separable in more than one way

$$
\psi_{123} = \frac{1}{\sqrt{2}} \psi_1 \otimes \psi_2 \otimes \psi_3 + \frac{1}{\sqrt{2}} \phi_1 \otimes \phi_2 \otimes \phi_3
$$

$$
\psi_{12-3} = \left[ \frac{1}{\sqrt{2}} \psi_1 \otimes \psi_2 + \frac{1}{\sqrt{2}} \phi_1 \otimes \phi_2 \right] \otimes \psi_3
$$

The state $\psi_{123}$ is definitely non-separable, but the state $\psi_{12-3}$ has two separable pieces, one of which is, in turn, non-separable. Situations like this one motivated the distinction between bipartite, tripartite and, in general, $N$ multipartite entanglement. In section 1.3.3 we provide another criterion, based on whether two states can be converted one into another with only local operations and classical communication, that allows to further divide $N$ multipartite entangled states for a fixed $N$. 
1.3 Minimal introduction to tripartite entanglement

1.3.2 Genuine entanglement

The second insight into the variety of entangled states, relevant in the chapters to come, deals with our ignorance about the actual global state. Up until this point, our definition of entanglement relied on an assumption impossible to meet: *if we know everything that there is to know about the global system* is a ridiculous statement in an experimental science such as physics. In order to introduce a model of our ignorance, physicists argued that Born’s rule should be modified

**Born’s rule:** Given a system with state $\psi$ and an observable $O$, the expectation value of the latter is

$$\langle O \rangle = \langle \psi | O | \psi \rangle$$

Note that given a state, the rule returns the expectation value of an observable. If instead of being certain about the state of the system, we could only be certain of sampling the state from an ensemble of orthonormal (so they are distinguishable) states, then Born’s rule should be combined with classical probability theory

**generalized Born’s rule:** Given a system with state sampled from a ensemble of orthonormal states $\{\psi_i\}_{i=1}^N$ with probabilities $P_i$, as well as an observable $O$, its expectation value is

$$\langle O \rangle = \sum_{i=1}^R P_i \langle \psi_i | O | \psi_i \rangle$$

(1.1)

Since the sampled $\psi_i$s are orthonormal then they are a basis of the state space, or if that is not the case, the ensemble can be extended with more states with probability zero until it is a basis. In that case, the modified Born’s rule becomes taking the trace of a multiplication of two operators

$$\langle O \rangle = Tr\{\rho O\}$$

where $\rho = \sum_i P_i |\psi_i\rangle\langle \psi_i|$. Any reader with basic knowledge in quantum mechanics knows this operator as the density matrix, and it is customary to interpret it as a mixed state (even though, within this interpretation, it is not a state properly speaking) while the actual states the density matrix samples from are called pure states.

Now, within this formalism, what is quantum entanglement, again? We could take a naive route and define it directly as non-separability of the density matrix. In this case, a system, even when we do not know the global state, is entangled whenever the global density matrix is not a tensor product of density matrices of each elementary subsystem, that is, $\rho = \rho_1 \otimes \rho_2$. If we were to use that definition, the following density matrix should
be considered entangled and non-separable

$$\rho = \frac{1}{2} |00\rangle \langle 00| + \frac{1}{2} |11\rangle \langle 11|$$

But, if we take a look at the modified Born’s rule in Eq. (1.1) we can appreciate this density matrix can be interpreted as having the global state sampled from the two clearly separable possibilities |00⟩ or |11⟩, both with the same probability. Therefore, we should not claim this density matrix contains any entanglement, as it can be interpreted as a classically correlated distribution of separable pure states. We take this chance to define separable pure states as product states, since their density matrices are tensor products of the subsystem’s matrices.

Following with the example, we could be led to believe that the fruitful definition of separability does not pay attention to the separability of the whole matrix, but to the separability of the pure states it samples from. In that case our density matrix should be considered separable because it can be interpreted as sampling from separable pure states. However, many different probability distributions of pure states end up producing the same density matrix. In particular, the density matrix we are considering can be reinterpreted in terms of Bell pairs

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2}} |00\rangle \pm \frac{1}{\sqrt{2}} |11\rangle$$

and then it follows

$$\rho = \frac{1}{2} |00\rangle \langle 00| + \frac{1}{2} |11\rangle \langle 11| = \frac{1}{2} |\psi_-\rangle \langle \psi_-| + \frac{1}{2} |\psi_+\rangle \langle \psi_+|$$

Therefore, we conclude that given a density matrix, the particular ensemble of states producing it is ambiguous and inadequate for a definition of separability. But, on the other hand, not all density matrices display such extreme ambiguity. If we know the state is a Bell pair, there is only one density matrix for the system and it is clearly entangled. Thus, somewhere between these two extreme cases must lie the border between separable and non-separable states. The actual definition of separability is as follows: a density matrix is said to be separable if there is at least one ensemble of only product states that produces it. Conversely, the definition of non-separability follows: a density matrix is said to be non-separable if there is no ensemble of only product states that produces it.

$$\rho_{\text{non-separable}} \neq \sum_{i=1}^{R} P_i \rho_i^{(1)} \otimes \rho_i^{(2)}$$

Note that, just as it happened with pure states, the definition of entanglement is formulated as a negation, this is, as non-separability, and that opens the door to many different possible ways of being entangled. Following the steps we took with pure states,
1.3 Minimal introduction to tripartite entanglement

Tripartite systems display some of those different kinds of entanglement. Firstly, consider a separable density matrix

\[ \rho_{1-2-3} = \sum_{i=1}^{R} P_i \rho_i^{(1)} \otimes \rho_i^{(2)} \otimes \rho_i^{(3)} \]

Again, note that the decomposition in the right hand side of the equation is not unique, but as soon as one like this one exists, the left hand side is a separable tripartite density matrix. From here, we can try to define new non-separable density matrices. Take for example

\[ \rho_{12-3} = \sum_{i=1}^{R} P_i \rho_i^{(1,2)} \otimes \rho_i^{(3)} \text{ with } \rho_i^{(1,2)} \neq \sum_{j=1}^{R'} P_{ij} \rho_{ij}^{(1)} \otimes \rho_{ij}^{(2)} \]

This density matrix is entangled, according to our definition, but that entanglement is shared only between the subsystems 1 and 2. Therefore, we denominate its entanglement as bipartite. Additionally, we take the opportunity offered by this example to make a second definition: biseparability. Here, the third system is separable from the composite system made from the first and second ones, and all three together form the global system. Then, we claim the state is biseparable. Such concept becomes very convenient when making the jump to tripartite entanglement. Consider a new example in which a density matrix is not biseparable in any possible way

\[ \rho_{\text{fully inseparable}} \neq \sum_{j=1}^{R} P_{ij} \rho_{ij}^{(\alpha,\beta)} \otimes \rho_{ij}^{(\gamma)} \]

where the superindices \( \alpha, \beta \) and \( \gamma \) indicate the subsystem those operators act upon. They range from 1 to 3 without repetition and the order between \( \alpha \) and \( \beta \) does not matter, so Eq. (1.2) is a summarized way of imposing 3 equations on \( \rho_{\text{fully inseparable}} \). Is this tripartite entanglement? The answer is that it might, but does not have to be. We call situations like these \( \text{fully inseparable} \), but not tripartite entanglement. To clarify this distinction consider

\[ \rho_{\text{generalized biseparable}} = \frac{1}{3} \rho^{(1,2)} \otimes \rho^{(3)} + \frac{1}{3} \rho^{(1)} \otimes \rho^{(2,3)} + \frac{1}{3} \rho^{(1,3)} \otimes \rho^{(2)} \]

with \( \rho^{(\alpha,\beta)} \neq \sum_{i=1}^{R} P_i \rho_i^{(\alpha)} \otimes \rho_i^{(\beta)} \) (1.3)

This density matrix is fully inseparable because there is no bipartition that separates the matrix in two. But, if we interpret the global density matrix with the modified Born’s rule, we are modeling a system in which our knowledge about the state is enough to tell that there is some bipartite entanglement, but we do not have enough information to point out which pair of subsystems is entangled. Therefore, we claim the density matrix contains a kind generalized bipartite entanglement, not a tripartite one. Conversely,
when the density matrix can not be brought into such a decomposition, we characterize it as \textit{genuinely} entangled in a tripartite way.

\[
\rho_{\text{genuine}} = \sum_{i=1}^{R} P_i \rho_i^{(1,2,3)} \text{ with } \rho_i^{(1,2,3)} \neq \sum_{j=1..R', \alpha, \beta, \gamma=1,2,3} P_{ij}^{\alpha,\beta,\gamma} \rho_{ij}^{(\alpha)} \otimes \rho_{ij}^{(\beta,\gamma)} \tag{1.4}
\]

For these states, and following again the interpretation provided to us by the modified Born’s rule about density matrices, our knowledge of the state of the system is not absolute but it is sufficient to be certain about the tripartite nature of its entanglement, hence the name \textit{genuine tripartite entanglement}. Unfortunately, the name \textit{genuine entanglement} has been coined twice and for different concepts. Some authors use it to denote the entanglement of the \textit{GHZ} state as opposite to the entanglement of the \textit{W} state, a completely different notion than the one considered here. In our opinion, the name \textit{genuine} is sticking to the concept of non-generalized biseparability considered here; and to some extend, it has the right to be called genuine. Even more unfortunate is that both meanings of \textit{genuine entanglement} are relevant in the present thesis. Therefore, section 1.3.3 deals with that second meaning, that we will denominate \textit{GHZ}-like entanglement.

Additionally, in the literature about this topic one can find that \( N > 3 \) multipartite genuine entanglement is defined in two different ways: Firstly, it can mean that a system with \( N > 3 \) constituents is in a state that is not biseparable nor a convex sum of biseparable states on any possible bipartition (for instance [1]). Other authors consider that an \( N \)-partite system can have \( M \)-partite genuine entanglement (with \( N \geq M \geq 3 \)) by extending the definition recursively. That is, a system is genuinely \( M \)-partite entangled if its state can not be decomposed as a convex sum of \((M - 1)\)partite genuine entangled states (for instance [2]). Needless to say, the first definition is the same as the second one if \( M \) is 3.

Before concluding this section, please note that tripartite genuine entanglement, despite being of fundamental interest because of its experimental robustness, is quite difficult to handle. It has a convoluted definition that indicates a system is entangled whenever, from the many different decompositions of the density matrix, there is no generalized biseparable description of it. Therefore, there are few measures of genuine entanglement, a reasonable search on arXiv (this one) on January 2022 yields 15 results about genuine entanglement measures, the most relevant of them in our opinion being [3, 4, 5, 6, 7, 8, 1, 2]. Those measures are often difficult to compute, not to mention that many do not apply to continuous variables systems as the systems we will study in the following chapters. A very common approach in the literature is, then, to compute simpler lower bounds of those measures. That way, if the bound turns out to be positive, we know the system is genuinely entangled, but not how much exactly. If the bound turns out to be negative, that is inconclusive evidence because the actual measure might be positive. An even simpler approach consists in using entanglement witnesses. Witnesses are bounds that combinations of observables of a system with no entanglement (genuine or any kind considered) must follow. If after measuring the observables, the
bound is violated then the system must be entangled (in the way considered). If the
bound is followed, the evidence is again inconclusive. Witnesses are, therefore, very
similar to measure bounds, and they are usually easier to come up with from a theo-
retical perspective, and they are more accessible experimentally, as they are based on
expectation values of observables. Chapters 2 and 4 will deal with the construction of
these witnesses.

1.3.3 Nonequivalent tripartite entanglement classes in 3 qubit systems

The third insight into the variety of phenomena we denominate entanglement deals with
a broadly known result [9] about 3 qubit pure states. At the beginning of section 1.3
we stated that entanglement is non-separability. Then, we claimed that because of its
definition as the absence of a property, we are bound to have many different kinds of
entanglement. That definition is perfectly fine, but it ignores a crucial fact. Quantum
mechanical systems are often described effectively by Classical Mechanics if there is
no specific experimental effort to produce or protect their quantum features. Thus, an
ongoing effort in quantum research is to identify, from all the possible operations that
could be performed on a system, those that generate entanglement and those that do
not.

One of the first steps in that direction could be considered the class of Local Operations
and Classical Communication (LOCC) protocols, which was built to determine which
operations do not generate entanglement. Its proper definition requires paying attention
to some subtleties beyond the scope of this work, but in order to introduce it we indicate
that, if we consider a particular separable state, we can try to find all the operations
that map it to another separable state.

The first kind of LOCC operations we consider is the unitary evolution of an isolated
subsystem. If a particular subsystem is not allowed to interact with anything else, then
the time evolution of the whole system factors into a tensor product containing the
separable evolution of the isolated subsystem and the rest of systems. Then, we define a
local unitary as the evolution of the isolated subsystem. If its initial state was separable
from the rest of the system, it will remain separable. In fact, the initial state has to
come from the evolution of separable states in the past. Because of this reversibility,
local unitaries are known not to generate nor to destroy entanglement. If they did the
latter, their inverse would do the former.

The second kind of elementary operations that we consider in LOCC are measurements
on single subsystems. These, very much like local unitaries, map the set of separable
states to itself. But they are no longer reversible, and in fact, they map entangled states
to separable ones. Therefore, they are said to conserve or destroy entanglement, never
to create it.
1 Introduction

The final step to build the complete set of LOCC protocols is to consider the composition of both kinds of operations on the same subsystem or in different ones, as well as their conditional execution based on the results of previous measurements. In short, LOCC is the set of all operations that can be produced by local action on each subsystem and classical communication. There is much more to tell about LOCC protocols, but in the sake of brevity we only point out two properties. First, there are operations that map separable states to other separable states that are not in LOCC \cite{10}. Second, and despite the first point, LOCC protocols are often considered the complete set of non-entanglement-generating operations. Therefore, when given a general state $\rho$ we can find a LOCC protocol that generates another arbitrary state $\sigma$ from $\rho$, we say that $\sigma$ is less or equally entangled than $\rho$. If the LOCC protocol is reversible then $\sigma$ and $\rho$ are equally entangled. This argument lies at the core of the construction of every entanglement measure.

Now, we are in the position to introduce \cite{9}. It is a well known fact that Bell pairs represent the maximally entangled states of 2 qubit systems. That is, from any Bell pair we can obtain any other 2 qubit pure state by means of a LOCC operation \cite{11}. Furthermore, all 2 qubit states can be partially ordered from least to most entangled based on LOCC convertibility. But that is no longer the case with 3 qubit pure states. In \cite{9} it was shown that the following two states

$$|GHZ\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle)$$

$$|W\rangle = \frac{1}{\sqrt{3}} (|001\rangle + |010\rangle + |100\rangle)$$

can not be converted in any direction into each other by means of LOCC protocols with non-zero success probability. Even more so, any other tripartitely entangled pure state can be obtained with non-zero success probability from only one of those two states. Therefore, the authors concluded that there are two different tripartite entanglement classes, one with the $|GHZ\rangle$ state as its maximally entangled representative, and the other one lead by $|W\rangle$. As pointed out in section 1.3.2, it was unfortunate that some posterior literature coined the expression genuine entanglement to the class of $|GHZ\rangle$-like states. We believe that choice was made on the grounds of how the two different states react to single system measurements. The $|W\rangle$ state becomes a Bell pair between the unmeasured qubits, while the $|GHZ\rangle$ state leaves those qubits in a separable state. Under that scope, it appears the word genuine is applied to the $|GHZ\rangle$ entanglement because the entanglement contained in the $|W\rangle$ state is considered not genuinely tripartite, given that it becomes bipartite under single-qubit measurement. We find this consideration misleading, the $|W\rangle$ state is tripartitely inseparable, just as the $|GHZ\rangle$ is. In fact, we find a much more appropriate term would be to denominate the entanglement of the $|W\rangle$ as robust to particle loss, while the $|GHZ\rangle$ is fragile. Moreover, chapter 4 generalizes the concept of (non-)Gaussian introduced in chapter 2 to discrete variables systems and, there, we will show how the $|GHZ\rangle$ must be considered non-Gaussian while the $|W\rangle$ state is Gaussian.
1.4 Resources for learning cQED

An introduction to the broad field of circuit quantum electrodynamics (cQED) is beyond the expected contents of this thesis. Fortunately, its actual contents deal with a limited set of circuits. But if the reader finds out they are unfamiliar with the design of superconducting circuits, we provide some guidance on interesting references for the particular set of systems that we consider.

cQED deals with the modeling of nearly dissipativeless superconducting circuits. Quantum mechanics provides us with a microscopical explanation on why those materials contain dissipativeless currents, but their quantum properties reach much further than that. These circuits are modeled by macroscopic degrees of freedom that are quantum in nature. While a typical electronic circuit can not be in a superposition state such as \textit{an electron has gone through the device plus that same electron has not}, a superconducting circuit can. Such quantum behaviour is due to multiple factors: from the Bose-Einstein condensation of Cooper pairs to the experimental exploitation of the Josephson effect and flux quantization, which lead to the design of Josephson junctions and superconducting quantum interference devices (SQUIDs). A very accessible introduction to these topics is provided on the last chapter of the Feynman lectures [12].

Josephson junctions are of great importance to cQED because they behave as nonlinear inductors, which in turn produce anharmonic energy levels when they are included in circuits. If the reader is unfamiliar with how those circuits are modeled from its constituent Josephson junctions, capacitors and other elements, we recommend the course on quantum fluctuations in electrical circuits, given by Devoret at Les Houches in 1995 [13]. Having the ability to produce anharmonic energy levels is fundamental from a perspective of controlling the circuit’s state. In anharmonic systems, electromagnetic pulses of different frequencies address transitions between different levels. This fact allows for greater control of the state than if the levels were harmonic, as it would happen with no Josephson junctions. Such control led to the design of qubits. These systems are electrical circuits that can be cooled down to its ground state and operated coherently so that its dynamics remain bounded to a state subspace of dimension two. However, there are a variety of circuits that can behave that way, leading to the multitude of different qubits we have today: charge and flux qubits, transmons, xmons and so on. A modern review such as [14] covers the circuits that are most often used nowadays, as well as many of the successes of cQED: the strong light-matter coupling of qubits to the field on waveguides, the ability of modulate over time the circuit’s parameters when they include SQUIDs and many more. Ultimately, that review explains how these circuits are operated to realize a digital quantum computer. Albeit those results are of great importance, they are not relevant for this thesis. We often deal with analog simulators or with entanglement generation in systems prior to interpret them as quantum computers.

Up to this point we have discussed cQED in a broad sense. It is a very active area of
research, so keeping up with the entirety of the literature is a challenging task. Therefore, we provide now references that are tailored to this thesis, instead of trying to cover cQED completely. In the chapters to come, we often consider one-dimensional cavities terminated in a SQUID. The article [15] contains a good introduction to how to model that system, despite they only consider a single Josephson junction instead of a complete SQUID. Other systems that we consider are tunable-coupling transmons, which are introduced in [16].

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Towards the detection of genuine tripartite non-Gaussian entanglement

At the beginning of this PhD, experiments were taking place in the groups of C. M. Wilson (University of Waterloo) and G. Johansson (Chalmers University of Technology) with the theoretical collaboration of my PhD advisor, C. Sabín. They were running an experiment in which a superconducting circuit behaved as a parametric amplifier capable of down-converting a pump tone into three different tones at once [1], a process denominated three-mode spontaneous parametric down-conversion (3SPDC). Loosely speaking, the vacuum state of the three output tones was evolved into a superposition of that vacuum plus the state with a photon on each mode. This just happens to be a good approximation, in general many other photon number states are involved in that evolution, but those two are the main participants at short times. Such an approximate state is reminiscent of the GHZ state in discrete variables, and therefore we expected that the parametric oscillator, operated as this three-mode mixer, generated tripartite entanglement. Because we were modeling an experiment, we had to consider the system as being in a mixed state, and in those cases the entanglement we were expecting is denominated genuine. Unfortunately that term has been coined twice with different meanings. Here, we use it to describe that we are certain about the tripartite nature of the entanglement even when we are not absolutely certain about the system’s state, as we introduced in section 1.3. But other literature has considered it as the entanglement of the GHZ and related states, which can be confusing in this chapter, and even more so in chapter 4.

However, prior literature [2] had argued that this was not a fruitful scheme for generating genuine entanglement. In that work, a similar system in the context of nonlinear quantum optics was studied and they indicated that a pump tone that parametrically down-converted into three output tones produces a state that was not detected as genuinely entangled by a family of witnesses. They proceeded to study other pump tones and seeding schemes to successfully generate entanglement. Nevertheless, we must point out that witnesses are not entanglement measures, they are only sufficient (but not necessary) conditions to entanglement. Therefore, their argument did not rule out tripartite entanglement in the 3SPDC setting and we set out to find new witnesses families that were tailored for the experiment.

A first analysis of the system indicated that the witnesses used in [2] were not well suited for the situation at hand because they paid attention only to covariances of the
Towards the detection of genuine tripartite non-Gaussian entanglement quadratures. The 3SPDC system does not change the covariances of vacuum, its action becomes apparent in higher statistical moments. Witnesses that were sensitive to those moments were not found, so we built one and it detected genuine tripartite entanglement in our system.

Then, we realized that all the witnesses we had gathered followed a pattern: those that paid attention only to covariances could not detect entanglement encoded in higher statistical moments and, conversely, witnesses that paid attention to those higher moments did not detect entanglement encoded in covariances. A prime example of the latter entanglement is produced in this same or related systems when the pump consists of two tones that resonate with different pairs of the three output modes. That way, the initial vacuum state in the output nodes $|000\rangle$ is driven into a superposition in which a pair of modes is excited $|110\rangle$, plus another pair excited $|011\rangle$. We denominate that process double two-mode spontaneous parametric down-conversion (2-2SPDC) and it is known to generate genuine tripartite entanglement [3, 4, 5]. As pointed out above, the witness built to detect the entanglement produced by 3SPDC is oblivious to the entanglement produced in 2-2SPDC. Inspired by the field of quantum optics, we chose to coin the adjective Gaussian for the covariance-sensitive witnesses, as those moments determine completely a Gaussian; while we coined non-Gaussian for the witnesses composed of higher statistical moments. Since Gaussian witnesses reported entanglement only in the 2-2SPDC process, while non-Gaussian witnesses did only in the 3SPDC, we were lead to believe that these could be mutually exclusive entanglement classes, regardless of the witnesses considered, hence the expression non-Gaussian in the title of this chapter.

The following sections in this chapter meet three purposes: Firstly, they give an introduction to topics required to understand the results summarized above. Secondly, one of those sections is the actual publication these results lead to. And thirdly, some sections provide improvements over the results published in [6]. Therefore, section 2.1 introduces the design on the superconducting system that enables the 3SPDC process. Section 2.2 introduces our non-Gaussian witness. Section 2.3 is the publication [6], whose introduction covers in far lesser detail the topics discussed in the previous sections, but its main body discusses the application of our non-Gaussian witness to the 3SPDC system. There, numerical simulations can be found that identify what experimentally accessible parameters regimes are adequate for genuine tripartite non-Gaussian entanglement generation. After the publication, section 2.4 introduces an improvement of our witness that was found later on. Lastly, we provide some concluding remarks on section 2.5.

### 2.1 Three-mode spontaneous parametric down conversion in circuit Quantum Electrodynamics

Here we describe the experimental system that we studied in [6] and that was introduced in [1]. In fact, we gather some of the result of [1] here. In short, what we re-derive in
2.1 Three-mode spontaneous parametric down conversion in circuit Quantum Electrodynamics

this section is the combination of resonators and Josephson junctions that lead to an electromagnetic field that can be effectively described by the Hamiltonian

\[ H_{3SPDC} = \sum_{i=1}^{3} \omega_i a_i^\dagger a_i + g_0 a_1^\dagger a_2^\dagger a_3^\dagger + g_0 a_1 a_2 a_3, \]

that is, three normal modes with characteristic frequencies \( \omega_i \quad i = 1, 2, 3 \) that interact in a three-body fashion with coupling \( g_0 \). The \( i \)-th mode creation operator is \( a_i^\dagger \). Note that we set \( \hbar = 1 \).

2.1.1 Slightly asymmetric and weakly pumped SQUID

The system described in [1] is composed of a superconducting cavity and an asymmetric Superconducting QUantum Interference Device (SQUID) sitting at one of its edges. We start our model with the latter. As pointed out in section 1.4, a SQUID is a loop of superconducting material, except for two points (Josephson junctions, from now on JJ) that break the loop in two different islands that are so close together that the tunneling of Cooper pairs becomes a relevant phenomenon. Because of the Josephson effect and flux quantization through the loop, the inductance of the device depends on both the current and the external magnetic flux passing through it, enabling experimentalists to tune the device’s non-linear inductance in real time. In particular, the Lagrangian is

\[ L_{\text{SQUID}}(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2, t) = \frac{C_1}{2} \dot{\phi}_1^2 + E_{J1} \cos \left( \frac{\phi_1}{\phi_0} \right) + \frac{C_2}{2} \dot{\phi}_2^2 + E_{J2} \cos \left( \frac{\phi_2}{\phi_0} \right), \]

where \( \phi_1 \) is the phase difference between the Cooper pairs at each side of the first junction, whereas \( \phi_2 \) is the same at the second one. We denote time derivatives with the dot notation, for instance in \( \dot{\phi}_i \). The capacities of each junction are \( C_1 \) and \( C_2 \), while \( E_{J1} \) and \( E_{J2} \) are their respective Josephson energies. The reduced flux quantum is \( \phi_0 = 1/2e \), with \( 1 = \hbar \) the reduced Planck’s constant and \( e \) the (positive) electron charge. Note that the phase differences must follow the flux quantization condition

\[ \phi_2 - \phi_1 = \phi_{\text{ext}}, \]

where \( \phi_{\text{ext}} \) is the external magnetic flux through the loop and we suppose the system contains no flux quanta. Note that \( \phi_{\text{ext}} \) must be considered time dependent. In fact, it is the mechanism that introduces the pump tone into the system, so it must be regarded as oscillating with the pump frequency. For convenience, we define an independent flux \( \phi = \phi_1 + \phi_{\text{ext}}/2 = \phi_2 - \phi_{\text{ext}}/2 \) so that the Lagrangian might be rewritten as having only one dynamical variable. After some simple but convoluted algebra, the single-variable Lagrangian turns out to be equivalent to a single tunable Josephson junction

\[ L_{\text{SQUID}}(\phi, \dot{\phi}, t) = \frac{C_T}{2} \dot{\phi}^2 + E_J(\phi_{\text{ext}}) \cos \left( \frac{\phi}{\phi_0} - \alpha(\phi_{\text{ext}}) \right), \] (2.1)
where the effective parameters $E_J$, $\alpha$ and $C_T$ can be written in terms of both JJs characteristic parameters and the external flux

$$E_J(\phi_{\text{ext}}) = \sqrt{E_{J,1}^2 + E_{J,2}^2 + 2E_{J,1}E_{J,2}\cos\left(\frac{\phi_{\text{ext}}}{\phi_0}\right)}$$

$$\alpha(\phi_{\text{ext}}) = \tan^{-1}\left(\tan\left(\frac{\phi_{\text{ext}}}{2\phi_0}\right) \frac{E_{J,1} - E_{J,2}}{E_{J,1} + E_{J,2}}\right)$$

$$C_T = C_1 + C_2.$$ 

The dependence of the effective Josephson energy $E_J$ and phase offset $\alpha$ on the external flux is rather complicated, in order to simplify our analysis we consider three approximations. Firstly, the JJs in the SQUID are only slightly asymmetric

$$\Delta = \frac{E_{J,2} - E_{J,1}}{E_{J,1} + E_{J,2}},$$

that is, we neglect any quadratic term in $\Delta$ from the effective Josephson energy

$$E_J \approx 2E_{J,1}\sqrt{1 + 2\Delta}\left|\cos\left(\frac{\phi_{\text{ext}}}{\phi_0}\right)\right|,$$  

(2.2)

which now has a simpler trigonometric dependence on the external flux. The second approximation we perform involves the pump tone. The external magnetic flux oscillates around some mean value with the pump tone frequency

$$\phi_{\text{ext}}(t) = \phi_{0}^{\text{ext}} + \lambda\cos(\omega_{d}t),$$

where $\omega_{d}$ is the pump tone. We consider that the amplitude of the oscillations is small compared to the flux quantum, that is $\lambda \ll \phi_0$ and we neglect any quadratic or higher term in $\lambda/\phi_0$. This way the cosine dependence of the effective Josephson energy in Eq. (2.2) can be approximated by

$$\cos\left(\frac{\phi_{\text{ext}}}{\phi_0}\right) \approx \cos\left(\phi_{0}^{\text{ext}} / \phi_0\right) - \sin\left(\phi_{0}^{\text{ext}} / \phi_0\right) \frac{\lambda\cos\omega_{d}t}{\phi_0} + O\left(\frac{\lambda^2}{\phi_0^2}\right),$$

which yields

$$E_J \approx 2E_{J,1}\sqrt{1 + 2\Delta}\left|\cos\left(\frac{\phi_{0}^{\text{ext}}}{\phi_0}\right)\right|\left|1 - \tan\left(\frac{\phi_{0}^{\text{ext}}}{\phi_0}\right) \frac{\lambda}{\phi_0} \cos\omega_{d}t\right|,$$
Additionally we define some constants in order to clarify the dependence of the effective Josephson energy on the pump tone.

\[ E = 2E_{J1} \sqrt{1 + 2\Delta} \left| \cos \left( \frac{\phi_{\text{ext}}^0}{\phi_0} \right) \right| \]

\[ \delta E = \frac{E}{\phi_0} \tan \left( \frac{\phi_{\text{ext}}^0}{\phi_0} \right) \]

so that the final, conveniently approximated, effective Josephson energy is

\[ E_J \approx |E - \lambda \delta E \cos \omega_d t|, \quad (2.3) \]

where we have kept the dependence on the pump amplitude \( \lambda \) explicit, that is, outside of the definitions of \( E \) and \( \delta E \) because we are not done with the approximation \( \lambda^2/\phi_0^2 \ll 1 \). In particular, that approximation has further simplifying effects on the effective phase offset \( \alpha(\phi_{\text{ext}}) \). After some algebra and expanding a Taylor series on \( \lambda \), it becomes

\[ \alpha(\phi_{\text{ext}}) \approx \arctan \left( \Delta \tan \left( \frac{\phi_{\text{ext}}^0}{2\phi_0} \right) \right) + \frac{\sec^2(\phi_{\text{ext}}^0/2\phi_0) \Delta}{1 + \tan^2(\phi_{\text{ext}}^0/2\phi_0)} \lambda \cos \omega_d t + O \left( \frac{\lambda^2}{\phi_0^2} \right) \]

where the first time-independent factor can be ignored redefining the independent potential \( \phi \) as the addition of the previous potential plus that constant factor. The second time-dependent factor oscillates with the pump tone and a complicated amplitude which we encapsulate in the variable \( \delta \alpha \), that is equal to

\[ \delta \alpha = \frac{\sec^2(\phi_{\text{ext}}^0/2\phi_0) \Delta}{1 + \tan^2(\phi_{\text{ext}}^0/2\phi_0)} \Delta^2. \]

Note that, again, we do not include the pump amplitude \( \lambda \) within the definition of \( \delta \alpha \) because we are not finished taking the approximation \( \lambda^2/\phi_0^2 \ll 1 \) yet. Summarizing, the Lagrangian for the slightly asymmetric, low amplitude pump SQUID is

\[ L_{\text{SQUID}} \approx \frac{C_T \phi^2}{2} + |E - \lambda \delta E \cos \omega_d t| \cos \left( \frac{\phi}{\phi_0} - \lambda \delta \alpha \cos \omega_d t \right) \]

(2.4)

which has a simpler dependence on the pump tone than the full effective Lagrangian (2.1). Lastly, the third approximation that we perform consists in considering the internal flux of the SQUID, that is the dynamical variable \( \phi \) itself, small in units of the quantum flux. However, we take this approximation not as strongly as the others before, and we neglect terms in the Lagrangian that behave as \( O(\phi^6/\phi_0^6) \)

\[ L_{\text{SQUID}} \approx \frac{C_T \phi^2}{2} + |E - \lambda \delta E \cos \omega_d t| \times \left[ 1 - \frac{1}{2} \left( \frac{\phi}{\phi_0} - \lambda \delta \alpha \cos \omega_d t \right)^2 + \frac{1}{4!} \left( \frac{\phi}{\phi_0} - \lambda \delta \alpha \cos \omega_d t \right)^4 + O(\phi^6) \right] \]

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We ignore the zero-order term, which is constant in the dynamic variable $\phi$ and has no effect on the equations of motion. Additionally, we find the final terms that are proportional to $\lambda^2$ and regard them as very small.

$$L_{\text{SQUID}} \approx \frac{C_T}{2} \dot{\phi}^2 + \lambda E \delta \alpha \cos(\omega_d t) \frac{\phi}{\phi_0} - \frac{|E - \lambda \delta E \cos(\omega_d t)|}{2} \frac{\phi^2}{\phi_0^2}$$

$$- \frac{\lambda E \delta \alpha}{6} \cos(\omega_d t) \frac{\phi^3}{\phi_0^3} + \frac{|E - \lambda \delta E \cos(\omega_d t)|}{24} \frac{\phi^4}{\phi_0^4} \quad (2.5)$$

Note this Lagrangian can be split in a linear and a non-linear part in $\phi$ the dynamic variable

$$L_{\text{SQUID linear}} = \frac{C_T}{2} \dot{\phi}^2 + \lambda E \delta \alpha \cos(\omega_d t) \frac{\phi}{\phi_0} - \frac{|E - \lambda \delta E \cos(\omega_d t)|}{2} \frac{\phi^2}{\phi_0^2} \quad (2.6)$$

$$L_{\text{SQUID non-linear}} = - \frac{\lambda E \delta \alpha}{6} \cos(\omega_d t) \frac{\phi^3}{\phi_0^3} + \frac{|E - \lambda \delta E \cos(\omega_d t)|}{24} \frac{\phi^4}{\phi_0^4} \quad (2.7)$$

Summarizing, we have proven that a slightly asymmetric and weakly pumped SQUID can be modeled as a time-dependent non-linear inductor joined by a conventional capacitor. In the following section we will describe the effects that such an inductor can have on a cavity when it is built at one of its edges.

2.1.2 Effects of the slightly asymmetric and weakly pumped SQUID on a one-dimensional cavity

Now, we turn our attention to the complete system, which is composed of said SQUID at the edge of a one-dimensional cavity. This cavity is described by a magnetic flux field $\phi(x,t)$, with the coordinate $x$ spanning from $x = 0$ to $x = d$, being $d$ the cavity length. What we named $\phi$ as the SQUID’s internal magnetic flux is now the field at the edge, $\phi(x = d, t)$. However, before writing down the system’s Lagrangian, we must point out that the transition from a discrete description of the SQUID to a continuous field when considering the cavity must be paid some attention. In particular, we will begin with a discrete description of the cavity in terms of $N$ coupled and equally spaced $LC$-resonators

$$L_{\text{resonators}} = \sum_{i=1}^{N} \frac{c \Delta x o_i^2}{2} + \sum_{i=1}^{N-1} \frac{(\phi_{i+1} - \phi_i)^2}{2l\Delta x},$$

where $\Delta x = \frac{d}{N-1}$ is the spacing between the resonators and $c$ and $l$ are the capacitance and inductance of the cavity per unit of length. The magnetic flux at the $i$-th resonator is $\phi_i$ and it can be thought of as an approximation to the magnetic flux field at $\phi(x = i\Delta x, t)$. In fact, it converges to it when the jump to a field description is given, that is, $N \to \infty$ and $\Delta x \to 0$ while keeping $(N-1)\Delta x = d$ constant.
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![Circuit Diagram]

Figure 2.1: Circuit representing the discrete lumped element approximation model for a one-dimensional cavity ending on a slightly asymmetric weakly-pumped SQUID. The cavity is split into $N$ identical LC resonators with flux variables $\phi_i$, each of them accounting for a fractional piece $l\Delta x$ of the total inductance $ld$, and another fractional piece $c\Delta x$ of the total capacitance $cd$. Two Josephson junctions with characteristic energies $E_{J1}$ and $E_{J2}$ form the SQUID which introduces a parametric dependence on the external magnetic flux $\phi_{ext}$ going through its loop.

In Figure (2.1) we show a schematic of the lumped element circuit that approximates the cavity field and how we connect the SQUID to the system. In short, we connect it to the last node, $\phi_N$, which will become the field at the edge of the cavity $\phi(d, t)$ when we take the continuum limit. Then, the complete Lagrangian is

$$L_{\text{total}} = L_{\text{resonators}} + L_{\text{SQUID linear}} + L_{\text{SQUID non-linear}}$$

$$= \sum_{i=1}^{N} \frac{c\Delta x \dot{\phi}_i^2}{2} + \sum_{i=1}^{N-1} \frac{(\phi_{i+1} - \phi_i)^2}{2l\Delta x}$$

$$+ \lambda E\delta \alpha \cos(\omega_d t) \frac{\dot{\phi}}{\phi_0} - \frac{|E - \lambda \delta E \cos(\omega_d t)| \phi^2}{\phi_0^2}$$

$$- \frac{\lambda E\delta \alpha}{6} \cos(\omega_d t) \frac{\dot{\phi}^3}{\phi_0^3} + \frac{|E - \lambda \delta E \cos(\omega_d t)| \phi^4}{\phi_0^4}$$

(2.8)

where we have performed a fourth approximation: considering that the plasma frequency of the SQUID is much larger than any of the characteristic frequencies of the system. That way, the SQUID self-capacitance $C_T \dot{\phi}_N^2$ can be safely ignored.

Even after that approximation, the Lagrangian produces complicated equations of motion, in particular non-linear time-dependent ones. We could, in principle, ignore those equations, perform a Legendre transformation to get the Hamiltonian of the system and quantize its canonical variables in order to get a quantum description of the system. Then, we could try to solve a complicated quantum system to find out whether three-mode spontaneous parametric down-conversion takes place in it. However, there is a simpler procedure: we can solve the linear time-independent piece of the equations of motion and then quantize the system. This way, the eigen-modes obtained in this classical description become bosonic modes and we can focus our attention only on the non-linear and/or time-dependent terms at the quantum level. Therefore, it is
worthwhile to take a look at the equations of motion having ignored the non-linear terms and taken the time average of the time-dependent ones

\[
\ddot{\phi}_1 = -\frac{\phi_1 - \phi_2}{c\Delta x^2} \\
\ddot{\phi}_i = -\frac{2\phi_i - \phi_{i-1} - \phi_{i+1}}{c\Delta x^2} \\
\ddot{\phi}_N = -\frac{\phi_{N-1} - \phi_N}{c\Delta x^2} + E \frac{1}{2} \frac{\phi_N}{\phi_0^2}
\]

when the resonators’ fluxes are interpreted as values of the field, those equations become

\[
\ddot{\phi}(0,t) = -\frac{1}{c\Delta x} \left[ \frac{\phi(0,t) - \phi(\Delta x,t)}{\Delta x} \right]
\]

\[
\ddot{\phi}(x,t) = -\frac{1}{c\Delta x} \left[ \frac{2\phi(x,t) - \phi(x - \Delta x,t) - \phi(x + \Delta x,t)}{\Delta x^2} \right]
\]

\[
\ddot{\phi}(d,t) = \frac{1}{c\Delta x} \left[ -\frac{\phi(d - \Delta x,t) - \phi(d,t)}{\Delta x} + E \frac{\phi(d,t)}{\phi_0} \right]
\]

where \(x\) is not a continuous variable yet, but rather \(x = i\Delta x\). However, when the field limit is taken, Eq. (2.10) becomes the well known wave equation. That same limit exists for Eq. (2.9) only if the divergence of the factor \(\frac{1}{c\Delta x}\) is compensated by a null second factor, which gives the open boundary condition \(\partial_x \phi(0,t) = \phi'(0,t) = 0\). Once that condition is satisfied, the equation becomes again the wave equation. Performing the same analysis on Eq. (2.11), the boundary condition at \(x = d\) turns out to be

\[
\frac{\phi'(d,t)}{t} - \frac{E_j \phi(d,t)}{\phi_0} = 0
\]

Then, we can propose eigen-modes that implicitly follow the boundary condition at \(x = 0\), that is \(\phi_n(x) = A_n \cos(k_n x)\), which in turn transform the boundary condition at \(x = d\) into a transcendental equation yielding the wave numbers

\[
k_n d \tan(k_n d) = \frac{ldE}{2\phi_0^2}.
\]

The best we can do is numerically finding out the values of \(k_n\) given the systems parameters. Considering those wave numbers as known, the general solution to the wave equation \(\phi(x,t) = \sum \phi_n(t) \cos(k_n x)\) allows us by direct integration to write down the linear time-average Lagrangian as

\[
L_{\text{linear time-averaged}} = \sum_n \left[ c_n \dot{\phi}_n^2(t) + l_n^{-1} \ddot{\phi}_n(t) \right]
\]

with an effective capacitance and inductance per \(n\)-th mode that can be expressed as

\[
c_n = c \int_0^d \! dx \cos^2(k_n x) = \frac{cd}{2} \left( 1 + \frac{\sin(2k_n d)}{2k_n d} \right)
\]

\[
l_n^{-1} = l_n^{-1} k_n^2 \int_0^d \! dx \sin^2(k_n x) + \frac{E_j}{\phi_0} \cos(k_n d) = \frac{k_n^2 d^2}{2l} \left( 1 - \frac{\sin(2k_n d)}{2k_n d} \right)
\]
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Now, it is time to pay attention to the time-dependent and/or non-linear terms that we ignored in the total Lagrangian in Eq. (2.8). There is a linear time-dependent term that becomes:

\[ E \delta \alpha \cos(\omega_d t) \frac{\phi(d, t)}{\phi_0} = \lambda \cos(\omega_d t) \sum_n M_n^{(1)} \phi_n(t), \]

where we have grouped the constant coefficients into \( M_n^{(1)} \) for convenience, its definition is

\[ M_n^{(1)} = \frac{E \delta \alpha}{\phi_0} \cos(k_n d) \]  

(2.12)

If we interpret each mode \( \phi_n \) as a harmonic oscillator, this Lagrangian term represents a classical drive, that is, a force, acting on each mode individually. In the next section we will see that the pumping tone \( \omega_d \) will not resonate with any of the modes, and therefore this driving will be negligible. Back in Eq. (2.8) we see that we ignored another Lagrangian term that produced linear time-dependent terms in the equations of motion

\[ \frac{\lambda \delta E \cos(\omega_d t)}{2} \frac{\phi^2}{\phi_0^2} = \lambda \cos(\omega_d t) \sum_{n,m} M_{n,m}^{(2)} \phi_n(t)\phi_m(t) \]

which can be interpreted as a pair-wise interaction between the modes, driven with the same pump tone when the indices \( n \) and \( m \) are different. When they are equal, that term can be interpreted as a time-dependent Kerr non-linearity. Again, we have grouped together the constant coefficients into a new symbol \( M_{n,m}^{(2)} \) for convenience

\[ M_{n,m}^{(2)} = \frac{E}{2\phi_0} \cos(k_n d) \cos(k_m d) \]  

(2.13)

Now it is time to turn our attention to the non-linear terms in the total Lagrangian. We split them in three different terms: one time-dependent and proportional to \( \phi^3(d, t) \), another time-independent and proportional to \( \phi^4(d, t) \) and a final one time-dependent and proportional to \( \phi^4 \) as well (see Eq. (2.8)). Those terms become now

\[ \frac{\lambda E \delta \alpha}{6} \cos(\omega_d t) \frac{\phi^3}{\phi_0^3} = \lambda \cos(\omega_d t) \sum_{n,m,l} M_{n,m,o}^{(3)} \phi_n(t)\phi_m(t)\phi_o(t) \]

\[ E \frac{\phi^4}{24 \phi_0^4} = \sum_{n,m,o,p} N_{n,m,o,p}^{(4)} \phi_n(t)\phi_m(t)\phi_o(t)\phi_p(t) \]

\[ \frac{\lambda \delta E \cos(\omega_d t)}{24} \frac{\phi^4}{\phi_0^4} = \lambda \cos(\omega_d t) \sum_{n,m,o,p} M_{n,m,o,p}^{(4)} \phi_n(t)\phi_m(t)\phi_o(t)\phi_p(t) \]
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where, again, we have grouped together the constant coefficients in the new symbols $M_{n,m,o}^{(3)}$ and $M_{n,m,o,p}^{(4)}$, which are defined as

\begin{align}
M_{n,m,o}^{(3)} &= \frac{E\delta\alpha}{6\phi_0} \cos(k_n d) \cos(k_m d) \cos(k_o d) \quad (2.14) \\
N_{n,m,o,p}^{(4)} &= \frac{E}{24\phi_0} \cos(k_n d) \cos(k_m d) \cos(k_o d) \cos(k_p d) \quad (2.15) \\
M_{n,m,o,p}^{(4)} &= \delta E \frac{1}{24\phi_0} \cos(k_n d) \cos(k_m d) \cos(k_o d) \cos(k_p d) \quad (2.16)
\end{align}

Summing up we can write the total Lagrangian in terms of the classical modes as

\begin{equation}
L_{\text{total}} = \sum_n \left( \frac{c_n}{2} \dot{\phi}_n^2(t) - \frac{1}{2\ell_n} \phi_n^2(t) - \lambda \cos(\omega_d t) M_{n}^{(1)} \phi_n(t) \right) \\
- \lambda \cos(\omega_d t) \sum_{n,m} M_{n,m}^{(2)} \phi_n(t) \phi_m(t) \\
+ \lambda \cos(\omega_d t) \sum_{n,m,o} M_{n,m,o}^{(3)} \phi_n(t) \phi_m(t) \phi_o(t) \\
+ \sum_{n,m,o,p} N_{n,m,o,p}^{(4)} \phi_n(t) \phi_m(t) \phi_o(t) \phi_p(t) \\
- \lambda \cos(\omega_d t) \sum_{n,m,o,p} M_{n,m,o,p}^{(4)} \phi_n(t) \phi_m(t) \phi_o(t) \phi_p(t) \quad (2.17)
\end{equation}

Summarizing, a slightly asymmetric and weakly pumped SQUID built at the edge of an otherwise open one-dimensional cavity introduces a tunable classical driving, as well as tunable two-, three- and four-mode interactions. Additionally it introduces quadratic, cubic and quartic non-linearities on single modes as well as pairs and triplets of them. Although this complete Lagrangian seems very complicated, it is ready to be quantized and then, after an astute choice of the pump tone, it will greatly simplify into a three-mode spontaneous parametric down-conversion Hamiltonian.

### 2.1.3 Quantization and the Rotating Wave Approximation

Following the standard procedure, the conjugate momenta $\varphi_n(t)$ to the magnetic fluxes $\phi_n(t)$ are introduced by means of $\varphi_n(t) = \partial L_{\text{total}}/\partial \dot{\phi}_n = c_n \dot{\phi}_n$. Then, the Legendre
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The transform of the Lagrangian returns the Hamiltonian 

\[ H_{\text{total}} = \sum_n \left( \frac{\varphi_n^2(t)}{2c_n} + \frac{\phi_n^2(t)}{2l_n} + \lambda \cos(\omega_d t)M_n^{(1)}(t)\phi_n(t) \right) \]

\[ + \lambda \cos(\omega_d t) \sum_{n,m} M_n^{(2)}(t)\phi_n(t)\phi_m(t) \]

\[ - \lambda \cos(\omega_d t) \sum_{n,m,o} M_n^{(3)}(t)\phi_n(t)\phi_m(t)\phi_o(t) \]

\[ - \sum_{n,m,o,p} N_n^{(4)}(t)\phi_n(t)\phi_m(t)\phi_o(t)\phi_p(t) \]

\[ + \lambda \cos(\omega_d t) \sum_{n,m,o,p} M_n^{(4)}(t)\phi_n(t)\phi_m(t)\phi_o(t)\phi_p(t) \]

\[ (2.18) \]

Now, the conjugate variables \( \phi_n \) and \( \varphi_n \) can be quantized by turning them into operators on a Hilbert space fulfilling the canonical commutation relations \([\varphi_n, \phi_m] = i\hbar \delta_{n,m}\). In that quantum description we can introduce creation and annihilation bosonic operators

\[ \phi_n = \sqrt{\frac{1}{2}} \sqrt{\frac{l_n}{c_n}} (a_n^\dagger + a_n) \]

\[ (2.19) \]

so that \([a_n, a_m^\dagger] = \delta_{n,m} \) and \([a_n, a_m] = [a_n^\dagger, a_m^\dagger] = 0\). In terms of those ladder operators the Hamiltonian is

\[ H_{\text{total}} = \sum_n \left( \omega_n a_n^\dagger a_n + \lambda \cos(\omega_d t)\tilde{M}_n^{(1)}(t)(a_n^\dagger + a_n) \right) \]

\[ + \lambda \cos(\omega_d t) \sum_{n,m} \tilde{M}_n^{(2)}(t)(a_n^\dagger + a_n)(a_m^\dagger + a_m) \]

\[ - \lambda \cos(\omega_d t) \sum_{n,m,o} \tilde{M}_n^{(3)}(t)(a_n^\dagger + a_n)(a_m^\dagger + a_m)(a_o^\dagger + a_o) \]

\[ - \sum_{n,m,o,p} \tilde{N}_n^{(4)}(t)(a_n^\dagger + a_n)(a_m^\dagger + a_m)(a_o^\dagger + a_o)(a_p^\dagger + a_p) \]

\[ + \lambda \cos(\omega_d t) \sum_{n,m,o,p} \tilde{M}_n^{(4)}(t)(a_n^\dagger + a_n)(a_m^\dagger + a_m)(a_o^\dagger + a_o)(a_p^\dagger + a_p) \]

\[ (2.20) \]

where the tildes on the symbols \( \tilde{M}_n^{(1)} \), \( \tilde{M}_n^{(2)} \), ... mean that they include the constant coefficients that relate the creation-annihilation operators to the canonical variables in Eq. \( (2.19) \).

In this quantum description, the equation of motion is the Schrödinger equation. In principle, finding its solution from Hamiltonian \( (2.20) \) is a challenging process, given the multipartite and time-dependent nature of the interactions. However, an astute choice of the pumping tone can greatly simplify our analysis. It is a well-known fact in the phenomenon of parametric down-conversion that the pump tone must be equal to the...
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sum of the output tones, so that energy is conserved. Therefore, if we are looking after three-mode parametric down-conversion processes, the pump tone $\omega_d$ must be equal to the sum of three of the mode frequencies $\omega_n$. For simplicity and convenience we can choose those modes to be the three in lowest energy, that is, $\omega_d = \sum_{i=1}^{3} \omega_n$. Under that assumption, the rotating wave approximation (RWA) indicates that most of the terms in Eq. (2.20) produce little to no dynamics, at least at short times. We remind the reader that the RWA indicates that the Hamiltonian terms that dominate the system’s dynamics are those that are constant in time in the interaction picture. To make the transition to that picture clearer, let us introduce a compact notation to indicate products of creation and annihilation operators

$$a_n^s = \begin{cases} a_n^\dagger & \text{if } s = +1 \\ a_n & \text{if } s = -1 \end{cases}$$

This way, the Hamiltonian can be conveniently rewritten in the interaction picture. Note that the effect this change of picture has on the Hamiltonian is adding a oscillating phase to each $a_n^s$ operator.

$$H_{\text{total}}^\text{int}(t) = \lambda \cos(\omega_d t) \sum_n \tilde{M}^{(1)}_{n} e^{i(\pm \omega_d + \omega_n) t} a_n^p$$

$$+ \lambda \sum_{n,m} \tilde{M}^{(2)}_{n,m} e^{i(\pm \omega_d + \omega_n + \omega_m) t} a_n^p a_m^q$$

$$- \lambda \sum_{n,m,l} \tilde{M}^{(3)}_{n,m,l} e^{i(\pm \omega_d + \omega_n + \omega_m + \omega_l) t} a_n^p a_m^q a_l^r$$

$$- \sum_{n,m,l,k} \tilde{N}^{(4)}_{n,m,l,k} e^{i(\pm \omega_d + \omega_n + \omega_m + \omega_l + \omega_k) t} a_n^p a_m^q a_l^r a_k^s$$

Thus, finding the constant terms is the same as finding the roots of the exponents, that is, which combination of indices $p, q, r, s$ makes each exponent zero, provided that the normal frequencies $\omega_n$ and the pump tone $\omega_d$ are known. Given the anharmonicity of the normal frequencies, we can assume that none of them are a multiple of each other. Therefore, the exponents that we find to have roots are those composed of subtracting pairs of equal frequencies, that is, degenerate solutions such as $\omega_1 - \omega_1$ and the like. Notice that we have chosen the driving to be $\omega_d = \omega_1 + \omega_2 + \omega_3$. Because of this pump tone, none of the terms containing $M^{(1)}, M^{(2)}$ or $M^{(4)}$ are constant: they always have an anharmonic non-zero frequency/ies left out in the exponent. But some terms containing $M^{(3)}$ and $N^{(4)}$ have zero exponent, which we consider to be the ones that dominate the
2.2 Entanglement witness construction

dynamics and compose the RWA Hamiltonian

$$H_{\text{RWA}}^{\text{int}} = -\frac{6\lambda}{2} \tilde{M}_{1,2,3}^{(3)} \left( a_1^\dagger a_2^\dagger a_3^\dagger + a_1 a_2 a_3 \right)$$

$$- 4 \sum_{n,m,p,q} \tilde{N}_{n,n,m,m}^{(4)} a_n^p (a_n^p)^\dagger a_n^q (a_n^q)^\dagger$$

(2.22)

where we indicate with $H_{\text{RWA}}^{\text{int}}$ that we are dealing with the Hamiltonian from Eq. (2.20) in the interaction picture and after performing the RWA. The first term in Eq. (2.22) is the Hamiltonian that we are looking for, one that produces three-mode spontaneous parametric down-conversion, as can be seen from the fact that it maps the vacuum state $|000\rangle$ to the three-photon state $|111\rangle$ on each mode. However, there are some fourth-order non-linearities, the terms containing $N^{(4)}$, that are relevant according to the RWA. Luckily, they add up to a constant, as the reader can easily prove by rewriting that summation in normal ordering. \footnote{That is, creation operators to the left, followed by annihilation operators to the right} Then, the system is described by

$$H_{\text{RWA}}^{\text{int}} = -\frac{6\lambda}{2} \tilde{M}_{1,2,3}^{(3)} \left( a_1^\dagger a_2^\dagger a_3^\dagger + a_1 a_2 a_3 \right)$$

(2.23)

Summarizing, we have proven that a system composed of a one-dimensional cavity with an open edge and a slightly asymmetric, weakly pumped SQUID at the other edge is effectively described by a three-mode spontaneous parametric down-conversion Hamiltonian if the pump is conveniently tuned to match the addition of the three lowest normal modes of the cavity. Note that the three-body interaction in Eq. (2.23) could, in principle, generate an entangled state from vacuum.

2.2 Entanglement witness construction

In the previous section we studied a system that is governed by a three-body interaction we denominated three-mode spontaneous parametric down-conversion (3SPDC). In this section we give a brief motivation about why such interaction should produce genuine tripartite entanglement and then proceed to build entanglement witnesses that are capable of reporting the presence of that kind of entanglement. It is advisable that the reader not familiar with genuine entanglement reads our introduction to the topic on section 1.3.2.

2.2.1 Genuine entanglement witness

In section 1.3.2 we defined what tripartite genuine entanglement is: the property of mixed states not decomposable into biseparable mixtures. Here, we will develop a witness tailored to detect the genuine entanglement produced by the 3SPDC interaction
Towards the detection of genuine tripartite non-Gaussian entanglement

described in section 2.1. As pointed out at the beginning of this chapter 2, a witness-based methodology was already used for an analogous system in quantum optics [2] and led to inconclusive results. Therefore, we had to improve that approach, and a first analysis of the system concluded that the non-Gaussian nature of the 3SPDC interaction must be taken into account. We remind the reader that in the field of quantum optics, and all the quantum technologies that have followed, a state is called Gaussian if its Wigner function is a Gaussian. In other words, a continuous variable state is considered Gaussian if the first and second statistical moments resulting from measuring its canonical variables are enough to determine it. In order to illustrate why the 3SPDC interaction makes non-Gaussian states, consider a first-order perturbative expansion on the Schrödinger equation with the Hamiltonian in Eq. (2.23) and taking the initial state as the vacuum

\[ \psi(t) \approx |000\rangle + g_0 t |111\rangle \]

(2.24)

at short times \( t \) and low couplings \( g_0 \), where \( |n\rangle \) is the static Hamiltonian eigenstate populated with \( n \) photons and \( |nnn\rangle = |n\rangle \otimes |n\rangle \otimes |n\rangle \). On one hand, this approximate state is genuinely entangled. To prove this, note that it is a pure state, so its associated density matrix has only one possible decomposition. Furthermore, that density matrix is clearly non-biseparable on every bipartition. On the other hand, a simple calculation shows that the covariances of the modes’ quadratures for the approximate state in Eq. (2.24) are zero

\[ \Delta^2 x_i x_j = \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle = 0 \]
\[ \Delta^2 p_i p_j = \langle p_i p_j \rangle - \langle p_i \rangle \langle p_j \rangle = 0 \]

with \( i \neq j \), so these are not variances but covariances. We consider this fact compelling evidence of the non-Gaussianity of the 3SPDC interaction. Now, consider the family of witnesses used in [2], which was introduced in [7] and takes the form

\[ S = + \min_{i \neq j \neq k \neq i} \left( |h_i g_i| + |h_j g_j + h_k g_k| \right) - \sum_{i,j=1}^{3} g_i g_j \Delta^2 x_i x_j - \sum_{i,j=1}^{3} h_i h_j \Delta^2 p_i p_j \]

where the six \( g_i \) and \( h_i \) are free real parameters. Whenever \( S > 0 \), then the state is genuinely entangled. But if the covariances are zero, then \( S \) will never be positive, regardless of the values taken by the free parameters. We conclude that this witness, which we denominate Gaussian because it is only sensitive to covariances, is not fit to detect the entanglement of the system if there is any.

Then, we set out to construct non-Gaussian witnesses. A promising candidate was introduced in [8]. That work proposed a family of witnesses tailored to detect \( N \) multipartite inseparability. In particular, given two operators \( O_1 \) and \( O_2 \) acting on two different subsystems, if the condition

\[ | \langle O_1 O_2 \rangle | > \sqrt{\langle O_1^2 \rangle \langle O_2^2 \rangle} \]

is satisfied.
2.2 Entanglement witness construction

is fulfilled, then the (probably mixed) state comprising the systems labeled 1 and 2 is not separable. This condition is well suited for our system because it allows us to pick the operators so that the expectation values are third order (and above) statistical moments of the canonical variables. That way, the condition becomes sensitive to the state’s non-Gaussianity. Consider that we set \( O_1 = a_1 \) and \( O_2 = a_2a_3 \), where \( a_i \) is the destruction operator on the \( i \)-th mode. In this case, if we define

\[
I_1 = | \langle a_1 a_2 a_3 \rangle | - \sqrt{\langle a_1^4 a_1 \rangle \langle a_2^4 a_3^4 \rangle} \tag{2.25}
\]

then whenever \( I_1 > 0 \) we know the first mode is not separable from the composite system comprising the second and third modes. If we go back to the approximated state in Eq. (2.24) and compute \( I_1 \) we get \( |g_0t| - g_0^2 t^2 \), which is bigger than zero at short times and low couplings, precisely the perturbative regime in which Eq. (2.24) works. Note that other combinations of destruction operators can be used to build \( I_2 \) and \( I_3 \), witnesses that prove the inseparability of the second and third modes from the rest, respectively.

Those witnesses and their perturbative values look promising, but they prove only the inseparability of each of the modes from the other two. In order to build a genuine entanglement witness we need to mirror the derivation of \( I_1 \) in Eq. (2.25) or any typical witness. Those derivations often start assuming the state does not possess the kind of entanglement we are looking for. Then, some bounds are derived that those unentangled states must necessarily follow. Because of elementary logic, if we take any other state and it turns out to violate the bounds, then it must be entangled. In other words, the violation of the bound is a sufficient condition to entanglement. Thus, what do non-genuine entangled states look like? If we look back to section 1.3.2, those states can range from completely separable to generalized biseparable states like in Eq. (1.3). All those possibilities can be summarized as the density matrices that allow at least one decomposition of the form

\[
\rho = P^{1-23} \sum_{i=1}^{\mathcal{R}} P_i^{1-23} (1) \otimes \rho_i^{(2,3)}
+ P^{2-13} \sum_{i=1}^{\mathcal{R}'} P_i^{2-13} (2) \otimes \rho_i^{(1,3)}
+ P^{3-12} \sum_{i=1}^{\mathcal{R}''} P_i^{3-12} (3) \otimes \rho_i^{(1,2)} \tag{2.26}
\]

where the density matrices \( \rho_i^{(\alpha,\beta)} \) might or might not be biseparable in return.

If we consider a state like Eq. (2.26) then the expectation value of \( a_1 a_2 a_3 \) in absolute value is bounded from above by the expectation value of each of the three possible
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Biseparable pieces of $\rho$, which we name

$$\rho^{1-23} = \sum_{i=1}^{R} P_i^{1-23} \rho_i^{(1)} \otimes \rho_i^{(2,3)}$$

$$\rho^{2-13} = \sum_{i=1}^{R} P_i^{2-13} \rho_i^{(2)} \otimes \rho_i^{(1,3)}$$

$$\rho^{3-12} = \sum_{i=1}^{R} P_i^{3-12} \rho_i^{(3)} \otimes \rho_i^{(1,2)}$$

so that the total density matrix is their convex sum, and the expectation value of $a_1 a_2 a_3$ follows

$$|\langle a_1 a_2 a_3 \rangle_\rho| \leq P^{1-23} |\langle a_1 a_2 a_3 \rangle_{\rho^{1-23}}| + P^{2-13} |\langle a_1 a_2 a_3 \rangle_{\rho^{2-13}}| + P^{3-12} |\langle a_1 a_2 a_3 \rangle_{\rho^{3-12}}|$$

(2.27)

because of the triangle inequality. After the publication of [6] we realized there is a tighter and simpler bound, which we introduce in section 2.4. In order to make a genuine non-Gaussian witness from this bound we need to take a couple more steps. Firstly, the density matrices $\rho^{\alpha-\beta\gamma}$ are at least biseparable by construction. Therefore, we know each of them make $I_\alpha \leq 0$, so that

$$\langle a_1 a_2 a_3 \rangle_{\rho^{\alpha-\beta\gamma}} \leq \sqrt{\langle a_\alpha a_\alpha \rangle_{\rho^{\alpha-\beta\gamma}} \langle a_\beta a_\beta a_\gamma a_\gamma \rangle_{\rho^{\alpha-\beta\gamma}}}$$

Secondly, we need to drop any dependence on the biseparable density matrices. Those are not the actual state of the system and checking any possible decomposition of a state looking for them is unpractical. Luckily, for any operator we have

$$P^{1-23} \langle O \rangle_{\rho^{1-23}} = \langle O \rangle_\rho - P^{2-13} \langle O \rangle_{\rho^{2-13}} - P^{3-12} \langle O \rangle_{\rho^{3-12}} \leq \langle O \rangle_\rho,$$

and similarly for the other permutations of the subsystems. This inequality points out that the expectation value over one of the biseparable matrices composing the total mixed state will never exceed the expectation value of the actual mixed state. That way we can drop any reference to the biseparable density matrices in the bound to $|\langle a_1 a_2 a_3 \rangle|$. Summarizing, any state ranging from completely separable to generalized biseparable, or in other words, not genuinely entangled, necessarily follows

$$|\langle a_1 a_2 a_3 \rangle| \leq \sqrt{\langle a_1 a_1 \rangle \langle a_2 a_3 a_3 \rangle} + \sqrt{\langle a_2 a_2 \rangle \langle a_1 a_3 a_3 \rangle} + \sqrt{\langle a_3 a_3 \rangle \langle a_1 a_1 a_2 a_2 \rangle}.$$

Therefore, we conclude that violating this bound is a sufficient condition to genuine entanglement. Grouping its terms together, we define our non-Gaussian genuine witness

$$G_1 = |\langle a_1 a_2 a_3 \rangle|$$

$$- \sqrt{\langle a_1 a_1 \rangle \langle a_2 a_3 a_3 \rangle} - \sqrt{\langle a_2 a_2 \rangle \langle a_1 a_3 a_3 \rangle} - \sqrt{\langle a_3 a_3 \rangle \langle a_1 a_1 a_2 a_2 \rangle}.$$  

(2.28)
2.3 Publication

So that when $G_1 > 0$, the state has to be genuinely entangled. Note that because of the triplet and quartets of annihilation operators, the witness is indeed sensitive to third and fourth order statistical moments of the canonical variables, that is, it is non-Gaussian.

2.3 Publication

In this section the article [6] is copied verbatim, as the main section of the present chapter. Please note it has its own page numbering as well as bibliography.
Tripartite Genuine Non-Gaussian Entanglement in Three-Mode Spontaneous Parametric Down-Conversion

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We show that the states generated by a three-mode spontaneous parametric down-conversion (SPDC) interaction Hamiltonian possess tripartite entanglement of a different nature to other paradigmatic three-mode entangled states generated by the combination of two-mode SPDC interactions. While two-mode SPDC generates Gaussian states whose entanglement can be characterized by standard criteria based on two-mode quantum correlations, these criteria fail to capture the entanglement generated by three-mode SPDC. We use criteria built from three-mode correlation functions to show that the class of states recently generated in a superconducting-circuit implementation of three-mode SPDC ideally have tripartite entanglement, contrary to recent claims in the literature. These criteria are suitable for triple SPDC but we show that they fail to detect tripartite entanglement in other states which are known to possess it, which illustrates the existence of two fundamentally different notions of tripartite entanglement in three-mode continuous-variable systems.

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Parametric amplification of the quantum vacuum in superconducting-circuit architectures [1] has proven to be a very fruitful paradigm for quantum technologies. For instance, the high-frequency modulation of a superconducting quantum interference device (SQUID) terminating a superconducting transmission line can generate pairs of photons out of the vacuum—a particular realization of the dynamical Casimir effect [2]—which exhibit entanglement and other forms of quantum correlations [3–6]. These correlations become resources that can be used in many applications of quantum technologies, for instance, entangling distant qubits [7,8] in distributed quantum computing architectures [9].

However, the use of these resources is limited by their bipartite nature, meaning the correlations only span two systems. Extending the entanglement to more modes would unlock access to a large number of new protocols including boson sampling [10], the generation of microwave cluster states [11], quantum state sharing [12], quantum secret sharing [13,14], and quantum teleportation networks [15].

One strategy to accomplish this is multitone modulation of the SQUID, with frequencies addressing multiple pairs of modes. Theory predicts that this approach can produce genuine multipartite entanglement [16,17] and it has recently been experimentally validated for three modes [18]. While the demonstrated entanglement was genuinely tripartite, it was generated by the simultaneous action of a pair of two-mode interactions and was detected purely through the measurement of second-order correlations [19].

It stands to reason that a single three-mode Hamiltonian might be better suited for the task of generating tripartite entanglement.

In fact, a three-mode spontaneous parametric down-conversion (SPDC) Hamiltonian can be engineered in superconducting circuits by suitably flux pumping an asymmetric SQUID terminating a coplanar waveguide resonator, as recently demonstrated experimentally in [20]. As this scheme includes a direct three-mode interaction, the relevant physical features cannot be captured by second-order correlations, making it necessary to include higher-order correlations in the characterization of the state [20]. The presence of independent higher-order correlations is often referred to as non-Gaussianity in the system. Common second-order criteria, including those previously mentioned [19], fail to detect multipartite entanglement in these states, as was noted in [21]. This has led to the impression that three-mode SPDC may not be a useful quantum resource. Nevertheless, in this Letter, we show that three-mode SPDC does produce entanglement, as well as the necessity to use higher-order correlations to detect the generated tripartite entanglement. Therefore, the claim in [21] that there is no entanglement in these states is overly broad. As we will prove below, the correct statement is that entanglement is, indeed, generated, but that it is non-Gaussian in nature.

In this work, we use entanglement criteria based on third- and fourth-order correlations to detect tripartite entanglement in the class of states produced experimentally by
three-mode SPDC [20]. We show that the class exhibits both full inseparability and genuine tripartite entanglement. We also show that the same criteria fail to detect tripartite entanglement in states produced by quadratic Hamiltonians, in which the multimode interaction is induced by the combination of two-mode interactions. Since we know the latter also include states with genuine tripartite entanglement, as shown experimentally in [18], our results clearly suggest that higher-order SPDC interactions generate a different kind of multipartite entanglement, distinguished from the Gaussian entanglement most commonly studied in continuous-variable systems. We will refer to this novel notion of entanglement as genuine non-Gaussian entanglement.

Let us now start with the description of our results. We analyze a system related to the experimental setup of [20], consisting of a superconducting resonator terminated by an asymmetric SQUID. We consider three field modes with frequencies \( \omega_i \) (i = a, b, c) and the corresponding creation and annihilation operators \( i, i^\dagger \) with standard bosonic commutation relations. We assume that initially each mode is in a weak thermal state \( \rho(i|n_{th}^i) \), characterized by the corresponding low average number of thermal photons according to its frequency and temperature, as given by \( \langle n_{th}^i \rangle = 1/(e^{\beta_0} - 1) \), where \( \beta_0 = \hbar \omega_i/(k_B T) \gg 1 \).

The system evolves under the interaction Hamiltonian:

\[
H_1 = \hbar g_0 \cos \omega_0 t (e^{i\theta_a} a + e^{-i\theta_a} a^\dagger)(e^{i\theta_b} b + e^{-i\theta_b} b^\dagger) (e^{i\theta_c} c + e^{-i\theta_c} c^\dagger), \tag{1}
\]

where \( \theta_i \) are locally controllable phases and \( g_0 \) is the coupling strength. Choosing the coupling modulation \( \omega_0 \) as

\[
\omega_0 = \omega_a + \omega_b + \omega_c \tag{2}
\]
gives rise to the effective Hamiltonian, with a derivation resembling a rotating-wave approximation (RWA), that has the form in the interaction picture of

\[
H_I = \frac{\hbar g_0}{2} (e^{i\theta} abc + e^{-i\theta} a^\dagger b^\dagger c^\dagger), \tag{3}
\]

where \( \theta = \theta_a + \theta_b + \theta_c \), and in the following we fix it to zero, as it plays no interesting role in entanglement generation. This is the three-mode SPDC Hamiltonian [22–24] required.

The standard criteria to detect tripartite entanglement, such as [19,25], are based on inequalities concerning expectation values and correlations which involve of course the three modes but in a pairwise fashion, such as \( \langle x_i x_j \rangle \) \( x_i, x_j \) being the position quadratures associated to the modes, e.g., \( x_i = (i + i^\dagger)/\sqrt{2} \). However, looking at the Hamiltonian (3) it seems natural to think that these criteria are not suitable in this case. Indeed it was shown in [21] that some of these criteria were not able to detect tripartite entanglement for these states. Perhaps the most compelling evidence proving this point is that the covariance matrix of an initial thermal state—including the vacuum—evolved under Hamiltonian (3) remains diagonal. That is, Hamiltonian (3) does not produce any second-order correlations. But we must stress that the criteria used in [21] are sufficient but not necessary conditions on entanglement, and as such they are inconclusive when they fail. Thus, what is needed is to look for higher-order criteria able to capture the pure three-mode nature of the states generated by the Hamiltonian (3). This nature has been demonstrated, both theoretically and experimentally, by the absence of second-order correlations together with the existence of third-order ones [20].

A typical approach to tripartite entanglement is to study correlations between all the possible bipartitions of the system. We recall that the definition of an entangled system is a system whose density matrix \( \rho \) is neither separable nor a mixture of separable states, that is \( \rho \neq \sum_i \rho_i \otimes \rho_i^2 \), where each \( \rho_i \) spans the first system and each \( \rho_i^2 \) the second. For instance, for each bipartition we can look at the inequalities developed in [26]. As usual, if the state is not entangled between the two subsystems, correlations between them are, by definition, classical and the inequalities hold. But if they are violated, we can conclude the state has to be entangled between those subsystems. If we define \( A^{(1)} \) and \( A^{(2)} \) as operators acting respectively on the Hilbert spaces of two subsystems in which the total system is split, by [26] if the total state is not entangled with respect to this partition, then

\[
|\langle A^{(1)} A^{(2)} \rangle| \leq \sqrt{\langle A^{(1)} \rangle \langle A^{(1)} \rangle \langle A^{(2)} \rangle \langle A^{(2)} \rangle}, \tag{4}
\]

Therefore, in our case, choosing the annihilation operator as the reference operator in all cases, we have that if condition

\[
|\langle abc \rangle| \leq \sqrt{\langle N_a \rangle \langle N_b \rangle \langle N_c \rangle}, \tag{5}
\]

is violated for all three possible \( i - jk \) bipartitions (namely \( a - bc, b - ac, c - ab \)) of the system then we know that the state is not biseparable with respect to any bipartition. In the above, \( N \) is the number operator. If the state is not biseparable for the three bipartitions, then the state is said to be fully inseparable. Defining \( I_i = |\langle abc \rangle|^2 - \sqrt{\langle N_a \rangle \langle N_b \rangle \langle N_c \rangle} \), we have that the state is fully inseparable if \( I_i > 0 \) for the three bipartitions.

However, even if the state has full inseparability, there is still the possibility that

\[
\rho = P_1 \rho_1^{(a)} \otimes \rho_1^{(bc)} + P_2 \rho_2^{(b)} \otimes \rho_2^{(ac)} + P_3 \rho_3^{(c)} \otimes \rho_3^{(ab)} \tag{6}
\]

or in other words, the state is a mixture of biseparable states (which implies \( P_1 + P_2 + P_3 = 1 \)). In this particular state,
it is as if the tripartite correlations were classical and, hence, we do not refer to full inseparability as a form of tripartite entanglement. Then it becomes immediate to define genuine tripartite entanglement as the correlations of fully inseparable states that cannot be written as (6) [19,27,28]. Note that we are in an analogous situation as before, looking for a condition every state like (6) must follow, and concluding that any state violating it has to be genuinely entangled. Note too that the difference between full inseparability and genuine entanglement is only relevant for mixed states.

We have not found in the literature a condition for genuine tripartite entanglement involving correlations of more than two modes. However, we can derive an inequality involving $|\langle abc \rangle|$ that every state of the form (6) follows. Using (6) and the triangle inequality, it is straightforward to write

$$|\langle abc \rangle| \leq P_1|\langle abc \rangle_{\rho_1}| + P_2|\langle abc \rangle_{\rho_2}| + P_3|\langle abc \rangle_{\rho_3}|,$$

where in the lhs the expectation value refers to the total state $\rho$ while in the rhs refer to the different elements of the convex sum, that we are denoting $\rho_1$, $\rho_2$, $\rho_3$, namely

$$\rho_1 = \rho_1^{(a)} \otimes \rho_1^{(bc)}, \quad \rho_2 = \rho_2^{(b)} \otimes \rho_2^{(ac)},$$

$$\rho_3 = \rho_3^{(c)} \otimes \rho_3^{(ab)}.$$

Now we know that, by construction, $\rho_1$, $\rho_2$, and $\rho_3$ are biseparable and therefore they must follow the inequalities (5). Therefore we have

$$|\langle abc \rangle| \leq P_1\sqrt{|\langle N_{a} \rangle_{\rho_1}|^2|\langle N_{b} N_{c} \rangle_{\rho_1}|} + P_2\sqrt{|\langle N_{b} \rangle_{\rho_2}|^2|\langle N_{a} N_{c} \rangle_{\rho_2}|} + P_3\sqrt{|\langle N_{c} \rangle_{\rho_3}|^2|\langle N_{a} N_{b} \rangle_{\rho_3}|},$$

Finally, using again (6) we have that, for instance,

$$P_1 \langle N_{a} \rangle_{\rho_1} = \langle N_{a} \rangle_{\rho} - P_2 \langle N_{a} \rangle_{\rho} - P_3 \langle N_{a} \rangle_{\rho} \leq \langle N_{a} \rangle_{\rho},$$

and similarly with all the expectation values in (9). Putting everything together, we find that if the state is of the form (6) then

$$|\langle abc \rangle| \leq \sqrt{|\langle N_{a} \rangle| |\langle N_{b} N_{c} \rangle|} + \sqrt{|\langle N_{b} \rangle| |\langle N_{a} N_{c} \rangle|} + \sqrt{|\langle N_{c} \rangle| |\langle N_{a} N_{b} \rangle|},$$

where we have let the subindex $\rho$ drop since it would be the same for all expectation values. Therefore, we conclude that if a state violates (11), then it possesses genuine tripartite entanglement. If it does not violate the inequality (11) but violates (5) for the three bipartitions, then it is just fully inseparable. We define $G = |\langle abc \rangle| - \sqrt{|\langle N_{a} \rangle| |\langle N_{b} N_{c} \rangle|} - \sqrt{|\langle N_{b} \rangle| |\langle N_{a} N_{c} \rangle|} - \sqrt{|\langle N_{c} \rangle| |\langle N_{a} N_{b} \rangle|}$, and then $G > 0$ is the condition for genuine tripartite entanglement.

With these witnesses $I_1$ and $G$, we can begin to study the entanglement generated by three-mode SPDC. If we consider the initial temperature negligible, then the initial state is the vacuum. If in addition to this we suppose that we are in a perturbative regime where the system evolves to a pure state containing only the vacuum and a triplet with small probability amplitude $\alpha$, then $\langle abc \rangle \approx \alpha$, while the $\langle N_i \rangle$, $\langle N_i N_i \rangle$ are of order $|\alpha|^2$. Therefore, for very low temperatures and coupling strengths, the conditions for entanglement are expected to be satisified.

In order to confirm and generalize this analytical intuition, we present now numerical results for the above inequalities for the states generated by the evolution under the interaction Hamiltonian (3) for three modes $\omega_a = \omega$, $\omega_b = 2\omega$, $\omega_c = 3\omega$ in the parameter regime $\beta_i > 1$, $g_0/\omega_0 \ll 1$ (low temperature and low coupling).

We find that the system is fully inseparable in all the explored parameter regime, except for very low coupling and high temperature ($g_0 < 0.002\omega_0$ and $\beta_0 \omega_0 < 1.2$). This can be seen in Fig. 1, where we present the maximum value of $I_1$ over the time interval (0,50/$\omega_0$) as a function of coupling and temperature. This proves $a - b c$ inseparability. The values of $I_2$ and $I_1$ are not shown, but, in fact, they generally exceed $I_1$. We expect this behavior since the single modes in their bipartitions have higher frequencies which leads to fewer thermal photons and a better parametric amplification of vacuum by the Hamiltonian. Genuine entanglement is detected for temperatures lower than $\beta_0 \omega_0 = 1.6$, as reported in Fig. 2. Summarizing, almost
everywhere in the explored parameter regime (that is, \( g_0 < 0.1\omega_a \) and \( \beta < 1 \)) the system contains tripartite correlations, but only for temperatures below \( \beta\omega_a = 1.6 \) (blue or darker upper region) while the witness fails to capture the system’s entanglement, if any, for higher temperatures comparable to level spacing (red or darker lower region).

The role of the coupling is richer than the temperature, as higher coupling can require higher-order corrections in perturbation theory, as well as break the rotating-wave approximation, leading to a discrepancy between the full Hamiltonian in Eq. (1) and the effective one in Eq. (3). Intuitively, the coupling controls the rate of evolution of the interacting system. Figure 3 shows how the system evolves from a slightly thermal state under the RWA Hamiltonian (3), developing an ever increasing value of \( G \) with a rate determined by the coupling.

However, we may expect that the Hamiltonian (3) will stop being an effective description of the full Hamiltonian (1) at high couplings and long times. That is, the RWA may break down. In fact, we observe this breakdown as shown in Fig. 4, which plots the same information but evolving the system under the full Hamiltonian (1). In this new scenario, the behavior of \( G \) is the same as Fig. (3) for short times (in units of the coupling), indicating genuine entanglement. However, \( G \) becomes negative after some time. This behavior is expected, since the RWA neglects terms that do not create or annihilate photon triplets, e.g., \( a^\dagger b c, a^\dagger b^\dagger c \). Therefore, when their effects become relevant, we expect weaker third-order correlations. In the experimental setup of [20], the reported correlations follow from the evolution of the system under their RWA Hamiltonian, therefore the relevant parameter regime in this work has to be contained in the region with \( G > 0 \) in Fig. 4. In addition to this, the timescale during which the RWA is valid is short compared to dissipative ones, so we conclude that decoherence in the system will not produce a relevant change in \( G \) before the full Hamiltonian (1) spoils it.

We have seen, then, that the states generated by the action of a three-mode SPDC Hamiltonian such as (1) or (3) evolving from an initial weakly thermal state possess tripartite entanglement—contrary to the claim in [21]—which can be detected by our three-mode criteria. Now we compare these results with the case of double two-mode
SPDC (that is, a Hamiltonian of the form $ab + ac + a^\dagger b^\dagger + a^\dagger c^\dagger$). In [18], it was shown that the resulting state possesses not only full inseparability, but also genuine entanglement by means of entanglement tests based on second-order correlations. We report that $G$ fails at detecting any entanglement at any coupling or temperature in the regime in which it would be expected to (that is, $\beta > 1$ and $g_0/\omega_0 < 0.1$).

Therefore, we are now in the opposite scenario: tripartite entanglement is detected by two-mode criteria but not by our three-mode criteria. This suggests that $G$ and the conditions explored in [18,21] detect two different kinds of entanglement. Hence, we label the entanglement signaled by $G > 0$ as tripartite genuine non-Gaussian entanglement, given the clear non-Gaussian nature of states evolved under either Eqs. (1) or (3).

Summarizing, we have shown that three-mode SPDC interaction Hamiltonians generate states with both full inseparability and genuine tripartite entanglement when acting upon a weak thermal state, contrary to previous claims in the literature. The type of tripartite entanglement displayed by these states is different from other paradigmatic three-mode states, and therefore needs to be captured by different entanglement criteria. We introduce entanglement criteria based on three-mode correlations and show that our three-mode criteria. This suggests that double-SPDC entanglement is detected by two-mode criteria but not by our three-mode criteria. This points to two different classes of continuous-variable tripartite entanglement in three-mode systems.

Our results pave the way for multipartite entanglement tests in the experimental setup of [20] and could be a guide for the characterization and measurement of entanglement in three-mode SPDC in other platforms [29,30].

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2.4 Non-Gaussian witness improvement

The reader at this point might be wondering why the sections before the article [6] derived the non-Gaussian witness $G$ just to have that same derivation repeated in the article itself. The reason is that the redundancy simplifies the discussion of an improvement we found on that witness, as we have pointed out a couple times previously. The key insight that made possible the improvement is a rather trivial fact: a convex sum follows the triangle inequality, yes, but it also follows the generally tighter bound that it can not be bigger than the largest of its individual terms. In other words,

$$\left| \sum_{i=1}^{R} P_i v_i \right| \leq \sum_{i=1}^{R} P_i |v_i|$$

is true, but a better bound is

$$\left| \sum_{i=1}^{R} P_i v_i \right| \leq \max_{i=1...R} |v_i|$$

Therefore, this can be applied to the construction $G_1$ back in section 2.2.1. There, we studied a chain of bounds for non genuine states of the shape in Eq. (2.26). Those states are in general convex sums of biseparable density matrices, and we used the triangle inequality to bound the expectation value of the operator $a_1a_2a_3$ in Eq. (2.27).

If we use the better bound for convex sums, we have

$$|\langle a_1a_2a_3 \rangle_\rho| \leq \max_{\alpha,\beta,\gamma=1,2,3, \alpha \neq \beta \neq \gamma \neq \alpha} |\langle a_1a_2a_3 \rangle_{\rho^{\alpha-\beta-\gamma}}|$$  \hspace{1cm} (2.29)

and then the same steps as in section 2.2.1 build the following witness

$$G_2 = |\langle a_1a_2a_3 \rangle| - \max_{\alpha,\beta,\gamma=1,2,3, \alpha \neq \beta \neq \gamma \neq \alpha} \sqrt{\langle N_\alpha \rangle \langle N_\beta N_\gamma \rangle}$$ \hspace{1cm} (2.30)

We claim this witness is an improvement because it is clear from the argument above that there might be states that are reported as genuinely entangled by $G_2$ but not from $G_1$.

2.5 Conclusions

In this chapter we have described our research on the feasibility of the detection of genuine entanglement in a 3SPDC process. In particular, we have given a detailed derivation of the results published in [6], in the hope of increasing its accessibility to a broader audience than the article itself. We conclude that a non-Gaussian witness
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is capable of detecting genuine entanglement in experimentally accessible parameter regimes, and we raise to the readers attention the fact that Gaussian witnesses appear to be better suited for the Gaussian states produced by 2-2SPDC processes, while the non-Gaussian witness is better suited for the non-Gaussian states produced by the 3SPDC process.

The reader might be wondering how far that mutual exclusion between Gaussian and non-Gaussian witnesses can be taken from a theoretical perspective. Do Gaussian and non-Gaussian genuinely entangled states define some notion of different entanglement classes? During the development of the research showcased in the Thesis we became interested in that question, which lead to a different publication that is the center of chapter 4. Thus, we refer the reader to that chapter for further information.

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3 Qubit Motion as a Microscopic Model for the Dynamical Casimir Effect

One of the heralded applications of quantum technologies is a radical improvement on the simulation of quantum systems. As the pioneers of quantum computing noted [1, 2, 3], the very same fact that makes quantum simulation difficult, that is, the exponential growth of memory and operations required to store and compute the time evolution of a quantum system of increasing size; is the reason why we should consider simulators governed by quantum mechanics. As more physical elements are added to the simulator, its accessible resources grow exponentially too, so they compensate the growth in the computing costs of a bigger system to simulate. Once this idea was formalized [4], the proposal, design and demonstration of quantum simulators became a very active research area. However, as quantum technologies grew more capable, multiple paradigms for performing quantum simulations have risen.

Universal fault-tolerant quantum computers, provided one is built, are capable of efficient quantum simulation. When those computers are designed within the circuit formalism we say they execute digital simulations. We have engaged in the design of some of those digital simulations, see [5] for further details. Needless to say, devices designed today are Noisy Intermediate Scale Quantum (NISQ) computers [6], so digital simulations are far from being executed on fault-tolerant machines. It remains an open question whether there is an error mitigation scheme that will allow a NISQ device to outperform classical supercomputers on simulation problems, while there are claims of having obtained that quantum advantage on sampling tasks [7, 8].

Alternatively, and inspired by the history of classical computing, devices that are not Turing complete, but rather mimic the physical processes of the systems to be simulated, are denominated analog simulators. In these devices, there is a map (from now on, the analogy) between the magnitudes of interest in the simulated system and the actual experimental magnitudes of the simulator. Then, the device is designed so that the simulator’s magnitudes have, in principle, the same dynamics as the magnitudes of the simulated system. This strategy places less requirements on the device than universal fault-tolerant computation, which makes it an attractive research area today. Making any claims on quantum advantage delves into details of complexity theory that are beyond the scope of this thesis, but it is undeniable that many analog simulators have produced insightful results in many-body systems [9] or even high energy physics [10], to name a few.
Additionally, we believe it is worthwhile to point out two other benefits of analog simulators beyond quantum exponential growth of available resources. When a system is expected to exhibit some interesting phenomenon, but it is not experimentally accessible, a common strategy is to consider analog simulators that reproduce that same phenomenon on a different setting. Therefore, fundamental knowledge is gained both about the phenomenon, despite the inaccessibility of the simulated system, and about the physics of the simulator, which now displays the phenomenon of interest. This idea is further developed in more concrete examples in section 3.1.

On this chapter we deal with the design of an analog simulator for relativistic and quantum effects. The system that we explore contributes to present day quantum technologies in both ways: On one hand, it exploits the accessibility of the strong coupling regime in circuit QED to simulate relativistic effects. Note that these effects, in their original formulation in free space, often suffer from weak coupling and small signals. In particular, we propose a simulator for the internal degrees of freedom of a mirror undergoing relativistic motion, which in turn is expected to produce photons, a phenomenon denominated dynamical Casimir effect. Surprisingly, we find that a simulator expected to reproduce another relativistic phenomenon, the Unruh effect, is capable of such task. Both effects are briefly introduced in section 3.2, and the publication containing the results in section 3.3. On the other hand, this simulator makes use of experimental proposals for qubits with time-dependent couplings. Some of these qubits were built with the intention of reducing crosstalk in digital quantum computers, so finding applications for them in analog simulators increases their interest. Other of these systems have been proposed for introducing time-dependent couplings driven by mechanical means. In the Appendix to the publication in section 3.3 we summarize the benefits and weaknesses of each proposal. Then, in section 3.4, we present the conclusions and possible future directions of research.

### 3.1 Analog simulators

A characteristic that sets analog simulators apart from digital computers, and very relevant to understand this chapter, is the varying transparency the analogy can have between the simulator and the simulated system. Sometimes, the system to be simulated is expected to exhibit some phenomenon and, when the simulator reproduces the same phenomenon with high fidelity, we claim the phenomenon has happened on the simulator instead.

To speak in less abstract terms, allow us to consider an uncontroversial example that any undergraduate student has heard of: It is a well known fact that a spring, manipulated in the parameter regime where Hooke’s law stands, is a mechanical harmonic oscillator. Likewise, an LC resonator operated within the lumped-element approximation is an electrical harmonic oscillator. Therefore, there is an analogy between the length of
3.1 Analog simulators

the spring and the voltage across the circuit, as well as relations between their parameters. This example is enlightening when we consider the discovery of the phenomenon of resonance. It is beyond the point of this thesis to make this example historically accurate, but it stands to reason that, at some point in History, someone was the first person to both observe and to have a theoretical model of the phenomenon of resonance. Let us speculate that the system considered was mechanical. Then, the system was driven with a force oscillating at the same frequency as the oscillations, which in turn rapidly grew in amplitude. Later on, the theory of electrical circuits is developed and LC resonators are designed. When driven with an oscillating power source the system undergoes an analogue growth in amplitude. The point of this example is: can we claim that we are dealing with two distinct phenomena, mechanical and electrical resonance, or we must claim they are the same phenomenon?

There might be no definitive answer to that question. An interesting perspective on it pays attention to how we use the word simulator. The difference between a simulator and its simulated system often implies that the former is more accessible than the later. At the beginning, mechanical systems were more common, so they could be considered simulators for the more rare electrical circuits. Nowadays, on the other hand, assembling LC resonators from premanufactured elements is simpler for students than dealing with the fragility of mechanical oscillators. So it is understandable to build, for example, coupled LC resonators to simulate toy models of solids, instead of the mechanical experiment. Given the fact that the preferred system is a matter of convenience beyond any physical criterion, we are inclined to claim that resonance is a phenomenon common to both systems. But, what if one of the pair of analogous systems is much more difficult to build?

This is the case for the Dynamical Casimir effect (DCE). It will be briefly introduced in section 3.2.2, but for now consider that its original formulation described the generation of photons when a mirror is undergoing relativistic velocities and accelerations. Needless to say, actually building that system is a difficult technological feat, that has not been achieved at time of writing. However, one can understand that the mirror is relevant to the electromagnetic field because it provides a time-dependent boundary condition. There are many other different systems that can undergo time-dependent boundary conditions. In particular, C. M. Wilson et al [11] built, within the context of circuit QED, a one-dimensional cavity with a SQUID sitting at one of its edges so that it acts as a time-dependent inductor. In that system, the relativistic movement of the mirror is replaced by the fast tuning of the inductor, creating an analog simulator of the DCE. Photon generation was found and the DCE was claimed to be observed. Or should we say electrical DCE? We believe we find ourselves in the same situation as in the example above and, given that some research groups are after optomechanical experiments producing the DCE [12], then the main distinction between the electrical or mechanical DCE could be which equipment a particular laboratory has, provided the mechanical DCE is observed at some point. Therefore, the reader must be aware that when we discuss analog simulations, we make no distinction between the simulated and
simulator’s effects. In fact, depending on context, we refer to the effect as some sort of equivalence class between all analogous systems, or to the particular experiment under consideration.

Summarizing, the design of analog simulators widens our understanding of the simulated phenomena and increases the applications current technologies have. This benefit adds up to the exponential growth in resources that set the field of quantum simulation in motion, and is the one this chapter pays most attention to.

3.2 Relativistic effects in Quantum Field Theories

Up to this point we have discussed quantum analog simulations from a perspective centered on the simulator. It is understandable to have such a bias nowadays, as building those simulators is an exciting new area of research, to which the article at section 3.3 belongs. In this section we will balance that bias and briefly explore the perspective taken by the theoretical physicists that explore relativistic effects on quantum field theories for their own sake.

Quantum field theories are one of the most successful formalisms we have so far. They are the primary ingredient of many, if not all, theories that need to take into consideration both quantum and relativistic features, and succeed in making many experimentally contrasted predictions. Take, for instance, the magnetic moment of the electron, which is claimed to be the most accurately known property of an elementary particle [13]. That magnitude is in agreement with its theoretical prediction all the way to the thirteenth decimal place. Thus, because of this experiment and many others, we are very confident about the soundness of standard model of particle physics.

There are, however, predictions that have not been measured yet, mostly due to the difficulty of designing experiments providing a signal strong enough to be measured, or observations sensitive enough to detect them. The most famous example could be considered the Hawking radiation, with its associated difficulty of taking place at the event horizon of a black hole. Given that requirement, it is accessible only to observations and not to experiments, not to mention the difficulty of those observations. Furthermore, such phenomenon bears special interest because takes current quantum field theories to parameter regimes near to where they are expected to fail. That is, in intense gravitational fields where general relativity must be taken into account. We conclude that somewhere in between this extreme example and the tamer experiments cited above, new physics could be found.

Because of the equivalence principle in general relativity, we know that gravity is locally equivalent to acceleration. During the 1970s, Fulling [14], Davies [15] and Unruh [16] considered an accelerating observer running quantum experiments in otherwise flat
and empty space-time. They concluded that a detector accelerating with the observer would detect photons, even though the state of the field, as seen from a resting detector, was vacuum. This phenomenon was coined the Unruh effect, and it is further discussed in 3.2.1. Alternatively, there are other mechanisms that can introduce acceleration into quantum systems. That is the case of an accelerating mirror, which was studied by G. Moore [17] and later coined as the Dynamical Casimir effect (DCE). It is further discussed in 3.2.2.

### 3.2.1 Unruh effect

As explained above, Davies [15] and Unruh [16] noted that a detector experiencing constant proper acceleration through vacuum should obtain a reading equivalent to being at rest in the presence of a thermal field. Prior to that, Fulling [14] had noted that the situation was more intricate than what the previous statement leads us to believe. When one considers the quantization of a field beyond Minkowski space-time, be it on actually curved space-time or on an accelerated coordinate system, there is some ambiguity when it comes to define particle number operators. Therefore, the authors cited so far concluded that what should be interpreted as the number operator for an inertial observer is not the same that for an accelerating one. Then, for the same field state the inertial and accelerated observers can detect different number of particles.

That ambiguity in the particle number led to some authors to believe that the reading on the detector was an artifact, instead of representing the state of the field [18]. Others claim that the field is, indeed, populated by photons following a thermal distribution but not on the thermal state, rather on a state produced from spontaneous emission by the excited detector [19]. We conclude that our understanding of the effect is incomplete, so building an analog simulator could be a good way of gaining insight into these questions. We try to keep a broad enough definition of the Unruh effect until said unknowns are solved. For instance, we do not require the field to be in a thermal state. Firstly, because the founding articles on the topic only show that the response of the accelerated detector is as if the field was thermal when they are at rest, and the authors do not prove that the field is actually on the thermal state. Secondly, even in those pioneering formalisms different detector trajectories are bound to produce a different system evolution in which the reading of the detector and the actual state of the field become something completely different from thermal.

Following a similar line of thought, Scully et al. [19] considered enhancing the signal of the Unruh effect by accelerating two-level detectors into and through a high $Q$ cavity. In that setting, the signal is several orders of magnitude larger and the field state evolves further away from vacuum, which led them to believe this could be a step towards the detection of the effect. A key insight of [19] is that, both in free space and in the cavity, detector and field become excited because of the dynamics produced by counter rotating terms in the Hamiltonian. We remind the reader that the rotating terms are defined as
3 Qubit Motion as a Microscopic Model for the Dynamical Casimir Effect

those that are constant in the interaction picture for time independent Hamiltonians, and they are counter rotating otherwise. An example will be useful throughout this chapter. Consider the Rabi Hamiltonian

\[ H_{\text{Rabi}} = \omega a^\dagger a + \frac{\Omega}{2} \sigma_z + g \sigma_x (a^\dagger + a) \]

where \( \omega \) is the characteristic frequency of an oscillator with creation-annihilation operators \( a \) and \( a^\dagger \), respectively. The qubit’s characteristic frequency is \( \Omega \), while \( \sigma_x \) and \( \sigma_z \) are its first and third Pauli matrices. The coupling between the qubit and the oscillator is \( g \). When this Hamiltonian is written in the interaction picture, its dependency on time can be explicitly expressed if one considers the qubit’s ladder operators \( \sigma^+ + \sigma^- \) = \( \sigma_x \)

\[ H_{\text{Rabi}}^{\text{int}} = g \left[ \sigma^- a^\dagger + \sigma^+ a^\dagger e^{-i(\omega+\Omega)t} + \sigma^- a e^{-i(\omega-\Omega)t} + \sigma^+ a \right] \]

We bring to the reader’s attention that the term \( \sigma^+ a^\dagger \) is counter rotating, and it contributes to the excitation of both the two-level detector and the field. When the qubit is accelerating through space-time, we must consider the coupling \( g \) as dependent on the qubit’s position and its proper time, in order to account for the different eigen-mode amplitudes it will find throughout its trajectory. Then, the insight of [19] consists in studying how the time-modulation of the coupling can break the Rotating Wave Approximation (RWA) so that the dynamics produced by \( \sigma^+ a^\dagger \) become dominant. Scully et al. claim that this hint applies beyond the cavity-enhanced effect.

We have concluded the introduction about the Unruh effect. The introduction is minimal, in the sense that it covers the works required to understand the article at 3.3, and it overlooks the rest of the literature. This is intentional, since the scope of this thesis deals with analog simulators, and our discussion of the subtleties of the effect must end at some point. Please allow us to abandon the perspective of a physicist working on relativistic phenomena, and let us take the one centered on analogue simulators.

As stated above, the Rabi Hamiltonian models the interaction of a two-level atom interacting with a single field mode. When the atom moves, the coupling must be regarded as time-dependent. Therefore, if a superconducting circuit is capable of changing the qubit-field coupling without actually moving the qubit, we can interpret it as a simulator. This approach was taken in [20] by Felicetti et al., including my PhD advisor, and it studied a system similar to the one we will discuss in 3.3

3.2.2 Dynamical Casimir effect

The Dynamical Casimir effect (DCE) is closely related to the Unruh effect. When one studies the quantization of a field under the presence of a moving mirror, a similar ambiguity in the particle number operator appears as the one observed by Fulling in the Unruh effect. Such a task was pioneered by Moore [17], and later on developed
3.2 Relativistic effects in Quantum Field Theories

by Fulling and Davies [21]. Moore concluded that, after solving said ambiguity, if the initial field state was the vacuum, the movement of the mirror would change the notion of particle number operators to a point were the system would appear as populated by photons. Needless to say, this is a relativistic phenomenon, so non-relativistic mirror velocities produce an amount of photons of no experimental interest. Accelerating a massive mirror to relativistic trajectories was considered a difficult experimental feat, but some proposals exist [12].

However, if one looks at the works cited so far, the mirror is modeled by a boundary condition. In fact, Fulling and Davies [21] propose the boundary condition to be an event horizon of a black hole in some circumstances. Therefore, from the perspective of quantum technologies, we could consider many different mechanisms to impose time-dependent boundary conditions in the context of quantum technologies. This approach was explored in [22, 23, 11], leading to a very similar system to the one studied in chapter 2: a SQUID sitting at the edge of a cavity imposes a time-dependent boundary condition and becomes an electrical analogue of the DCE. Another successful approach considered using a metamaterial with tunable refractive index so that the optical length of the cavity is modulated over time [24].

We have seen that if the DCE is considered as the result of time-dependent boundary conditions, and not just relativistic massive mirrors, it becomes a versatile phenomenon that explains the behavior of different parametric systems. But from the relativistic point of view, its original formulation presents a number of inconveniences. Firstly, there are no rigid solids in relativistic theories. However, the mirror appears to be one. Therefore, the original formulation of the DCE should be considered a limiting and unattainable case [25]. In other words, it could be of fundamental interest to consider mirrors made of some matter, so that the absence of rigidity can be studied. Secondly, the matter composing the mirror does not get entangled with the field. If it did, the time-evolution of the DCE could be non-unitary. Thirdly, if we are to measure the mechanical DCE at some point, it is beneficial to know which matter does not take energy from the field. If it did, DCE photon detection would be more difficult.

All the concerns exposed in the previous paragraph could be addressed with the creation of a microscopic model of the DCE. We use the word microscopic after the so-called microscopic models of Ewald and Oseen [26, 27] for refraction and transmission in classical optics. In those models, the constitutive relations for dielectrics are derived from the Maxwell equations in vacuum and considering the dielectric material as a lattice of electric dipoles. In other words, it was shown that a plane wave incident on the lattice will make the dipoles absorb and emit radiation so that they produce two plane waves that propagate in the bulk of the lattice. One of those waves interferes destructively with the incident wave, while the other is interpreted as the transmitted wave, which travels though the bulk at a lower speed giving birth to the refractive index of a material from microscopic considerations. This suggestive process led to de Melo e Souza et al. [28] to propose a microscopic model for the DCE. In their article, a single atom moving
through open space replaces the time-dependent boundary condition as the representative of the mirror. Despite their system not answering all the concerns expressed above, we consider it is a very relevant step forward. With the work presented at section 3.3 we aim at creating a microscopic model that sheds light on some of those concerns in the context of the DCE within a cavity, as well as proposing analog electrical and mechanical simulators for it.

3.3 Publication

In this section the article [29] is copied verbatim, as the main section of the present chapter. Please note it has its own page numbering as well as bibliography.
Qubit motion as a microscopic model for the dynamical Casimir effect

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The generation of photons from the vacuum by means of the movement of a mirror is known as the dynamical Casimir effect (DCE). In general, this phenomenon is effectively described by a field with time-dependent boundary conditions. Alternatively, we introduce a microscopic model of the DCE capable of capturing the essential features of the effect with no time-dependent boundary conditions. Besides the field, such a model comprises a subsystem representing the mirror’s internal structure. In this work, we study one of the most straightforward mirror systems: a qubit moving in a cavity and coupled to one of the bosonic modes. We find that under certain conditions on the qubit’s movement that do not depend on its physical properties, a large number of photons may be generated without changing the qubit state, as should be expected for a microscopic model of the mirror.

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I. INTRODUCTION

In his seminal paper of 1970, Moore [1] discovered that relativistic movement of perfectly conducting mirrors could produce radiation even if the state of the electromagnetic field before the mirrors’ movement were the vacuum. In the next years, the phenomenon was subsequently studied [2–4] until the name dynamical Casimir effect (DCE) was coined [5], joining the broad family of quantum vacuum fluctuation effects that includes, among others, the Lamb shift [6], the Casimir-Polder effect [7,8], and the Unruh [9–11] and Hawking’s radiations [12], to name a few [13]. For a long time, the realization of the DCE and most of these effects remained out of reach due to the experimental requirements to access the quantum and relativistic parameter regime needed for a measurable signal [14]. This hurdle was overcome in this century with the advent of circuit quantum electrodynamics, as it allows experiments in the strong light-matter coupling regime [15] and has led to a number of proposals for experimental observations of the DCE [16,17]. In 2011, Wilson et al. [18] carried out an experiment in which the relativistically moving mirror was reproduced using a modulated magnetic flux threading a superconducting quantum interference device, leading to a time-dependent boundary condition in a microwave waveguide and a nonclassical DCE photon production [19]. The DCE was also observed in a Josephson metamaterial capable of modulating its refractive index [20], leading to an equivalent setting in which the effective length of a cavity changes over time.

All the works cited so far model the moving mirrors as time-dependent boundary conditions. However, this boundary condition is an effective description that reproduces the effects of a more complex system, disregarding its microscopic features. In this work, we are interested in formulating a model that captures some of those microscopical features and reproduces the DCE with no time-dependent boundary conditions. The earliest application of this idea of an underlying microscopic model dates back to the Ewald-Oseen extinction theorem [21–24]. According to the theorem, transmission and reflection of a plane wave at the interface between dielectric media can be understood as the collective response of the media’s dipoles. This approach was applied by de Souza et al. [25] to model moving mirrors in a quantum field theory leading to the DCE.

Following those steps, the goal of this work is to find a microscopic model for a moving mirror that reproduces the DCE and employs the ever-growing tool set of present-day quantum technologies. We study the most straightforward system that may accommodate this phenomenon: a discrete mirror corresponding to a qubit moving in a cavity and interacting with one of its bosonic modes, as depicted in Fig. 1. The interaction between a qubit and a cavity has been studied extensively in the literature, especially in the case of a static qubit with the well-known Rabi model [26,27], and in the rotating wave approximation (RWA) regime with the Jaynes-Cummings (JC) model [28]. Prior work has addressed the case of a moving qubit [29], and in particular, the photon and qubit excitation due to the so-called cavity-enhanced Unruh effect.
FIG. 1. (Upper panel) Diagram of the system proposed as a microscopic model for the DCE. A qubit with frequency $\Omega$ and in its ground state moves back and forth inside a cavity of length $L$, producing photons in the fundamental mode of frequency $\omega$ if the qubit speed is close to the speed of light in the medium while staying in its ground state. Said photons are not produced by time-dependent boundary conditions; the walls (dashed dark blocks) are static. (Lower panel) Qubit-mode coupling $g$ as a function of qubit position for the fundamental mode, as defined in Eqs. (1) and (2). Then the time-dependent coupling is due to its composition with the trajectory and a small abuse in notation $g(t) = g[x(t)]$.

[30–33]. However, the parameter regime found to reproduce the DCE, where both rotating and counter-rotating terms are relevant, has not been explored before, to the authors’ knowledge. In this article, we show that the qubit motion generates photons without changing its internal state for speeds close to the speed of light in the medium. This behavior provides a microscopic model for the DCE.

The structure of this article is as follows: First, we recall the conditions of a microscopic model for the DCE and analyze the qubit-cavity system that fulfills the requirements. Second, we numerically explore the system’s parameter space, finding both the JC model and the cavity-Unruh regime, as well as the novel microscopic DCE regime. Then we present a perturbative analytical justification for this latter regime, leading to the characterization of the microscopic DCE as a second-order perturbation theory effect. Finally, we conclude with a summary of the main results and our remarks about possible experimental implementations.

II. MICROSCOPIC DCE WITH A QUBIT-CAVITY SYSTEM

The DCE consists in the generation of photons through the relativistic movement of a perfect mirror. To identify a system as a microscopic model of the time-dependent boundary producing the DCE, we first need to consider what properties are characteristic of the DCE. First, the movement of the mirror triggers the generation of photons, and there is no generation if the mirror is static. Second, a perfect mirror does not take energy from the field; if anything the former will return any energy to the latter in a short amount of time. Third, the predicted evolution of the DCE field is unitary [1], and because of this, the global unitary must factor into two unitaries for the field and the mirror, producing no entanglement. Last, we find that, in all the DCE settings explored so far, few if not none make assumptions on the internal structure of the mirror, that is, its static Hamiltonian’s spectrum. Regardless of how it compares to the field eigenenergies all settings expect the mirror to behave as an inert boundary condition.

In order to propose a microscopic system that reproduces all these characteristics we require the said model to follow three requirements: (1) its movement must trigger the generation of photons, (2) it must stay in its ground state so that it does not take energy from the field and does not get entangled with it, and (3) its static Hamiltonian’s spectrum must play no role in the effect. In previous work [25], an oscillating atom moving nonrelativistically in free space was proposed as a microscopic model for the DCE. Such a model fulfills our first two requirements but not the third one. In the case of Ref. 25, the atom’s energy gap must be large compared to the photon and movement frequencies. Other models treat the mirror as a system rather than a boundary condition but do not meet the above requirements [34–37]. These are valuable generalizations of the DCE to new regimes, although they do not fit our definition of a microscopic model.

The discrete mirror that we find to follow these requirements is a pointlike electric dipole qubit coupled to one bosonic field mode. It is described by the Hamiltonian

$$H_{\text{total}} = H_0 + H_{\text{int}},$$

$$H_0 = \frac{\Omega}{2} \sigma_z + \omega a^+ a,$$

$$H_{\text{int}} = g(t) \sigma_z (a^+ + a),$$

(1)

with $\Omega$ and $\omega$ the qubit and mode frequencies, $\sigma_x, \sigma_z$ the first and third Pauli matrices, $a^+$ and $a$ the creation and annihilation operators for the bosonic mode, and $g$ the time-dependent coupling that varies due to the classical motion of the discrete mirror qubit. The coupling to the fundamental bosonic mode of a cavity with perfectly conducting and static edges takes the form

$$g(t) = g_0 \cos[k x(t)],$$

$$k = \frac{\pi}{L},$$

(2)

where $L$ is the length of the cavity and $x$ the trajectory of the discrete mirror qubit. The coupling’s cosine dependence with the qubit’s position $x$ results from the cavity’s fundamental mode field amplitude at different locations. Following the tradition of DCE models, we will consider the movement of the mirror $x(t)$ as externally prescribed and not a dynamical variable. This way the system must be regarded as open and driven from the outside so that energy conservation does not inhibit photon production. In the context of superconducting circuits, the coupling intensity $g_0$, is typically one or two orders of magnitude smaller than the circuit frequencies [15,38]. Also, coupling modulation usually lies in utilizing superconducting quantum interference devices, instead of physically moving circuit components. Following that approach, the Hamiltonian of Eq. (1) can be engineered using tunable-coupling qubits [38]. Alternatively, a promising candidate consists of film bulk acoustic resonators, which could behave as an actual moving discrete mirror [39,40].

Given the linearity of the Schrödinger equation, we expect a coupling modulated with a cosine shape of constant frequency to be the most appropriate for analytical calculations.
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FIG. 2. Number of photons \( \langle N \rangle \) as a function of time \( t \) in units of the mode frequency \( \omega \), and driving frequency \( \omega_d \) in units of the same mode frequency \( \omega \), or equivalently, qubit velocity \( v \) in units of the speed of light in the medium \( c_n \). The driving frequency was produced by a qubit moving back and forth within the cavity, with constant velocity \( v = L/\pi \omega d \) in one direction, and \(-v\) after bouncing in the opposite direction. The qubit frequency is given by \( \Omega = \omega \), and the coupling intensity is \( g_0 = 0.1 \omega \).

Such cosine coupling modulation is produced by qubit trajectories \( x = vt \) with constant velocity \( v \), for which the coupling oscillates with a driving frequency \( \omega_d = \pi v/L \). To bound the qubit trajectory to the cavity while keeping this pure cosine coupling, we invert the direction of the qubit’s (otherwise constant) velocity every time it reaches the cavity edges. In order to relate the velocity with the rest of parameters of the system, we will pay attention to it in adimensional units \( v/c_n = \omega d/\omega \), with \( c_n \) the speed of light in the medium. Moreover, we remind the reader that typical values of \( c_n \) in superconducting circuit setups are \( c_n/c \approx 0.4 \) [18].

III. PARAMETER SPACE AND PHYSICAL REGIMES OF THE SYSTEM

We numerically explore the parameter space of the Hamiltonian of Eq. (1) describing the qubit-cavity system and identify three physical regimes: the JC regime, the cavity-enhanced Unruh regime, and the microscopic DCE regime. For the sake of simplicity, we consider the resonant case with equal mode and qubit frequencies, \( \omega = \Omega \), respectively, in the following simulations. We show the average cavity photon number \( \langle N \rangle \) and the mean value of the qubit excited-state population \( P_e = \langle \sigma^+ \sigma^- \rangle \) as a function of time and qubit velocity in Figs. 2 and 3, respectively. The qubit-cavity system evolves from its ground state, for which \( \langle N \rangle = 0 \) and \( \langle \sigma^+ \sigma^- \rangle = 0 \), First, the case of a static qubit corresponds to the JC model. Indeed, when the qubit velocity is zero \( \omega d = 0 \), the coupling to the cavity is constant, \( g(t) = g_0 \), and for a coupling intensity \( g_0 < 0.1 \omega \), the RWA and the JC model hold. The counter-rotating terms \( a \sigma^+ \) and \( a \sigma^- \) produce no dynamics given their fast phase in the interaction picture. The rotating terms \( a \sigma^+ \) and \( a \sigma^- \) that compose the JC model do not generate dynamics either, given the initial ground state of the system, which remains unchanged as shown in Figs. 2 and 3.

Alternatively, the \emph{anti-RWA} holds when the driving frequency is the sum of the qubit and bosonic mode frequencies \( \omega_d = \omega + \Omega \). For the resonant case \( \omega = \Omega \) studied, it corresponds to a qubit velocity twice the speed of light in the medium. In this case, we neglect the now fast-oscillating rotating terms, leaving the constant counter-rotating ones to generate the cavity-enhanced Unruh effect. The qubit is not undergoing a uniform acceleration motion as in the canonical Unruh effect [9–11]. Therefore, the radiation is not thermal, as it happens with more general trajectories [41,42]. The common feature to these phenomena is dominant counter-rotating terms in the dynamics, as Scully \emph{et al.} noticed and exploited when developing the cavity-enhanced Unruh effect [30]. This relativistic effect has been studied previously for a qubit-cavity system [29]. There the ground state of the system evolves towards the qubit excited state and one cavity photon for this qubit velocity, \( v = 2c_n \), in a Rabi-like oscillation. Consequently, both the photon number \( \langle N \rangle \) (Fig. 2) and the qubit excited-state population \( P_e \) (Fig. 3) increase up to one for \( \omega d/\omega = 2 \) in our numerical simulations.

The third regime corresponds to the proposed \emph{microscopic} dynamic Casimir effect. We observe a photon generation in the cavity (Fig. 2), while the qubit remains in its ground state (Fig. 3) for \( \Omega = \omega = \omega_d \). This nonoscillatory monotonic photon production without qubit excitation occurs when the qubit moves at the speed of light in the medium, \( \omega_d/\omega = v/c_n = 1 \). As a final note on the numerical results, if the system is evolved further in time, the increasing number of photons requires a bigger subspace of the Hilbert space to be considered in the simulations. See Appendix A for proof that the said subspace was large enough.

IV. PERTURBATIVE ANALYSIS OF THE MICROSCOPIC DCE MODEL

A perturbative approach can explain the \emph{microscopic} DCE regime parameters, as it happens with the Unruh effect or the more widely known JC model. In order to make clear our discussion let us briefly fix some notation. Let the state be

\begin{align*}
\langle \sigma^+ \sigma^- \rangle = P_e = \frac{1}{2P} = \frac{1}{2}.
\end{align*}
written as a power series on the coupling:

\[
\psi(t; g_0) = \psi(0) + \frac{1}{2} \psi^{(2)}(t) g_0^2 + \cdots.
\]  

Each of the terms \(\psi^{(n)}(t)\) corresponds to the \(n\)th partial derivative of the state with respect to \(g_0\), evaluated at \(g_0 = 0\),

\[
\psi^{(n)}(t) = \frac{\partial^n}{\partial g_0^n} \psi(t; g_0 = 0).
\]

Although the functional dependence of \(\psi(t; g_0)\) is hardly ever known, time-dependent perturbation theory enables us to compute its derivatives recursively as

\[
\psi^{(n+1)}(t) = \int_0^t dt' \frac{H_T^2(t')}{g_0} \psi^{(n)}(t'),
\]

where \(H_T^2\) is the interaction Hamiltonian \(H_{\text{int}}\) in the interaction picture with respect to the static Hamiltonian term \(H_0\).

Following this method, we approximate the state of the system considering the Hamiltonian of Eq. (1) and the initial ground state when computing the corrections from Eq. (5). We remind the reader that for the static qubit case, the JC model is a good approximation because the rotating terms \(a^\dagger \sigma^- + \sigma^+ a\), that the JC model shares with Hamiltonian Eq. (1), do not oscillate over time. Those terms produce perturbative corrections linear in time when integrated in Eq. (5).

We define a resonance as those integrals that result in linear contributions. Alternatively, a superluminal qubit’s speed of \(|v| = (1 + \Omega/\omega)_c\) makes the counter-rotating terms resonate and the anti-RWA becomes a good approximation, leading to the cavity-enhanced Unruh effect.

In our main case of study, the microscopic DCE, the qubit moves at a relativistic velocity \(v \approx c_\text{n}\), leading to a driving of \(\omega_d \approx \omega\). Then both rotating and counter-rotating terms oscillate and cannot dominate the dynamics. Therefore, it is not straightforward to find a simplified Hamiltonian that behaves as the complete Hamiltonian of Eq. (1). If we consider a qubit moving at exactly the speed of light in the medium \(v = c_\text{n}\) and apply twice Eq. (5) we find the second-order correction

\[
\psi^{(2)}(t) = \int_0^t dt' \int_0^{t'} dt'' \frac{H_T^2(t')}{g_0} \frac{H_T^2(t'')}{g_0} |g, 0\rangle
\]

\[
= \int_0^t dt' \int_0^{t'} dt'' e^{-i\omega t''} \sigma^- a^\dagger e^{-i\omega t'} e^{i\omega t''} \sigma^+ a |g, 0\rangle + O(t^3),
\]

where \(|g\rangle\) is the qubit ground state and \(|n\rangle\) is the \(n\)-photon state in the cavity fundamental mode. With \(O(t^3)\), we indicate that we neglect any term bounded by a constant for long enough times. In our case, we disregard constant terms and exponentials with imaginary arguments. The oscillatory coupling leads to the \(e^{-i\omega t''}\) terms, and the time-dependent counter-rotating term \(\sigma^+ a^\dagger\) gives the \(e^{i\omega t''}\) term. After integrating, it results in

\[
\langle g, 2|\psi^{(2)}(t)\rangle = \frac{g_0^2 \sqrt{2} t}{4\omega} + O(t^3).
\]

The \(\langle g, 2|\psi\rangle\) state component grows linearly with time, which indicates that the integral of Eq. (6) contains resonant terms. Moreover, this resonance increases the \(\langle g, 2|\psi\rangle\) state component, but not the amplitudes associated with the qubit excited state, \(\langle e, n|\psi\rangle\). This result is compatible with the numerical results of Figs. 2 and 3, where the photon production takes place without qubit excitation. We find higher order resonances related to qubit ground state amplitudes \(\langle g, 2m|\psi^{(2m)}\rangle\) in even-order perturbation terms, whereas the odd orders do not show new resonances (see Appendix B).

These formulas illustrate why there is an ever-increasing photon generation without appreciable qubit excitation. In fact, there is always a pairwise photon production, which is considered the spectral signature of the DCE [13,43].

Additionally, we consider the case where the system’s frequencies are no longer resonant, but the qubit frequency is detuned by \(\delta\) from the cavity mode frequency, \(\Omega = \omega + \delta\). Following perturbation theory, one gets

\[
\psi^{(2)}(t) = \int_0^t dt' \int_0^{t'} dt'' e^{-i(\omega_d + \delta)t''} e^{-i\omega t'} \sigma^- a^\dagger e^{i(\omega_d + \delta)t''}
\]

\[
\times e^{i(\omega + \delta + \delta)t''} \sigma^+ a |g, 0\rangle.
\]

Here the first integral over \(t''\) results in a term \(e^{i(\omega_d + 2\omega + 4\delta)t''}\) that cancels the \(\delta\) dependence for the second integral over \(t'\). Hence, we conclude that the detuning is irrelevant in the microscopic DCE photon generation. In contrast, the critical parameter relation is the resonance between the driving frequency and the mode frequency, \(\omega_d = \omega\). In terms of the qubit velocity, the DCE is produced when the qubit’s speed approaches the speed of light in the medium, \(\omega_d/\omega = v/c_\text{n} \approx 1\).

The photon generation independence on detuning is observed in Fig. 4. There we depict the maximum number of photons over the time interval \(t \in [0, 200/\omega]\) for different values of the qubit and driving frequencies, \(\Omega\) and \(\omega_d\), respectively. We consider a fixed mode frequency \(\omega\) and a coupling intensity of \(g_0 = 0.1\omega\). We note that regardless of the qubit frequency, the DCE photon generation occurs for parameter regimes near \(\omega_d/\omega = v/c_\text{n} \approx 1\) as expected. Like in the
original DCE, there is a transition between a nonrelativistic mirror movement regime, with no photon production, and a relativistic one featuring the effect. Moreover, increasing the qubit frequency reduces the photon generation as expected from analytical calculations, since $\Omega$ appears in the denominator of the perturbative corrections. Finally, the cavity-enhanced Unruh effect appears in the superluminal regime, shown with the diagonal line $\omega_d = \Omega + \omega$ of Fig. 4. In this region of the parameter space, the photon number never exceeds one, like in a Rabi oscillation. See Appendix A for tests on the accuracy of the simulations.

V. EXPERIMENTAL POSSIBILITIES AND CONCLUSIONS

Regarding the experimental implementation, we remark that it does not require any additional sophistication compared to the measurement of acceleration radiation or the cavity-enhanced Unruh effect, either by modulating the coupling to mimic the qubit motion [29] or by actual mechanical oscillation [39,44]. See Appendix C for further discussion on realistic experimental parameters that may accommodate both effects and dissipation.

Summarizing, we have found that a discrete mirror composed of a moving qubit reproduces features of the DCE, such as photon generation from the vacuum. This photon generation takes place regardless of the qubit’s internal structure and without changing its initial ground state, which supports the hypothesis that the qubit captures the essential features of a microscopic description for a moving mirror. This effect is different from the already known cavity-enhanced Unruh effect, where the excitation of the qubit always accompanies the photon production. The microscopic DCE explored here also differs from a more idealized proposal consisting of an atom oscillating in free space [25]. In that case, the oscillation frequency must match the sum of the frequencies of two electromagnetic modes that, in turn, must be very small compared to the atom’s internal frequency. If we translate those requirements into our system, we find the scenario of a largely detuned qubit oscillating at twice the frequency of the cavity, with constant velocity $v = L/\pi \omega_d$ in one direction, and $-v$ after bouncing in the opposite direction. See Appendix C for tests on the accuracy of the simulations.

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APPENDIX A: NUMERICAL SIMULATION DETAILS

We consider a system composed of a qubit of fixed frequency $\Omega$ moving within a cavity of length $L$ and coupled to its fundamental mode of frequency $\omega$ and wave number $k = \pi/L$. Due to the said movement, the coupling oscillates in time as $g(t) = g_0 \cos(kx(t))$, with $x(t)$ the trajectory of the qubit. The system is described by the Hamiltonian of Eq. (1) that we rewrite here for convenience,

$$H_{\text{total}} = H_0 + H_{\text{int}},$$
$$H_0 = \frac{\Omega}{2} \sigma_+ \sigma_- + \omega a^\dagger a,$$
$$H_{\text{int}} = g(t) \sigma_+ (a^\dagger + a). \quad \text{(A1)}$$

We simulate the dynamics generated by the previous time-dependent Hamiltonian with the QUTIP library (version 4.4.1) in PYTHON [47]. We consider an idealized two-level qubit (that is, we neglect higher energy level excitations) coupled to a cavity fundamental mode, represented by a Hilbert space truncated to dimension 8 that comprises the vacuum photon number states up to $7$. Given the nature of the dynamical Casimir effect, we expect a monotonic and unbounded parametric generation of photons [48]. Thus, first, we ensure that the truncated state space used in our calculations is large enough to describe the system’s dynamics for the analyzed time interval. The Hamiltonian of Eq. (A1) produces one photon per perturbation order (see details in Appendix B), and the vacuum state cannot evolve to a high photon number state directly. We limit the evolution time in the simulations such that the system does not reach the cutoff photon number state $7$ from the initial low-energy states with one-photon.

FIG. 5. Expectation value of the projector on the cutoff cavity excited state, $\langle 7 \rangle$, as a function of time $t$ in units of the cavity frequency $\omega$, for different qubit velocities related to the driving frequency $\omega_d$, given as well in units of the cavity frequency $\omega$. We consider the same parameter domain as in Figs. 2 and 3. The qubit frequency is given by $\Omega = \omega$, and the coupling intensity is $g_0 = 0.1\omega$. The qubit moves back and forth within the cavity, with constant velocity $v = L/\pi \omega_d$ in one direction, and $-v$ after bouncing in the opposite direction.
FIG. 6. Maximum value over the time period \(0 < t < 200/\omega\) of the cutoff cavity excited state expectation value, \(\max_{\omega \in [0,200/\omega]} \langle 1 \mid \hat{\sigma}_z \otimes |7(7)\rangle\), as a function of \(\Omega/\omega\), that is, only one order of magnitude less than the photon frequency. 

The presence of this resonance does not compromise the conclusions of the main text, as it is compatible with a total state composed of a relaxed qubit and an increasing number of photons.

In Eqs. (B1)–(B3), \(|02\rangle\) and \(|12\rangle\) are the qubits ground and excited state and \(|n\rangle\) is a photon number state with \(n\) photons. Notice that there is a resonance different to the one described in Eq. (7) that we have not discussed. It enlarges the projection of the state onto the ground state \(|02\rangle \otimes |0\rangle\). The presence of this resonance does not compromise the results in the regime perturbative regime \(\omega t < 10\). Moreover, we observe the monotonic unbounded nature of DCE photon generation beyond the perturbative regime for our ideal model without dissipation. Further notes on decoherence and experimental requirements can be found in Appendix C.

**APPENDIX B: COMPLETE AND HIGHER ORDER PERTURBATIVE CORRECTIONS**

In the main text, we characterize the bosonic mode population and conditions under which the dynamical Casimir population takes place by means of typical time-dependent perturbation theory. However, given the extension of the second-order corrections in the coupling’s magnitude \(g_0\), we considered only those terms that become relevant to the DCE. In the following we prove that no other terms produce noticeable dynamics, even those of third order, by giving the full expressions of the corrections in the series expansion of Eq. (3) in the main text up to third order, for a system described by the Hamiltonian of Eq. (1). We consider the case of resident qubit and mode frequencies \(\omega = \Omega\), and the qubit moving back and forth in the cavity with a speed \(|v| = \omega L/\pi\), which leads to a DCE resonant driving \(\omega_d = \omega\). Explicitly, the perturbative terms of Eq. (3) are given by Eqs. (B1)–(B3):

\[
\frac{g_0^2}{2} \psi^{(1)}(t) = \left( \frac{2g_0}{3\omega} + \frac{g_0 e^{i\omega t}}{6\omega} + \frac{g_0^2 e^{i\omega t}}{48\omega^3} + \frac{g_0^3 e^{-3i\omega t}}{18\omega^3} + \frac{g_0^4 e^{-3i\omega t}}{6\omega^5} + \frac{ig_0^5 e^{i\omega t}}{6\omega^7} \right) |02\rangle \otimes |0\rangle, 
\]

\[
\frac{g_0^3}{6} \psi^{(2)}(t) = \left( \frac{7g_0^3 e^{i\omega t}}{192\omega^3} - \frac{5g_0^3 e^{-3i\omega t}}{144\omega^3} - \frac{g_0^4 e^{-3i\omega t}}{1728\omega^3} + \frac{g_0^5 e^{-3i\omega t}}{4032\omega^3} + \frac{g_0^6 e^{-3i\omega t}}{320\omega^5} \right) |12\rangle \otimes |3\rangle + \left( \frac{49g_0^3 e^{i\omega t}}{432\omega^3} - \frac{7g_0^3 e^{-3i\omega t}}{216\omega} - \frac{g_0^4 e^{-3i\omega t}}{72\omega^3} - \frac{g_0^5 e^{-3i\omega t}}{648\omega^5} \right) |12\rangle \otimes |1\rangle.
\]

**APPENDIX C: EXPERIMENTAL REQUIREMENTS**

A detailed experimental proposal for the realization of the DCE lies beyond the scope of this paper. Nevertheless, we will briefly discuss experimental parameter regimes for which an implementation of the microscopic DCE model studied in the main text may be possible. We require a tunable coupling between the qubit and the cavity of magnitude \(g_0 = 0.1\omega\), that is, only one order of magnitude less than the photon frequency.
Moreover, we assume that the coupling can be modulated in time with a frequency \( \omega_\text{d} \) comparable to the cavity frequency \( \omega_0 \). We analyze the parameter regimes achievable with microwave-frequency superconducting circuits. Within this technology, we consider two candidates: analog simulators and FBAR-driven circuits.

First, we propose a modified superconducting qubit coupled to a microwave cavity. If the photons have a typical frequency of, for example, \( \omega_0 = 5 \text{ GHz} \) then in order to produce the microscopic DCE one would have to design a qubit with frequency preferably lower, as Fig. 4 shows that photons production is larger in that case. To continue with the example we propose \( \Omega = 2 \text{ GHz} \), and coupling intensity \( g_0 = 500 \text{ MHz} \) [38]. Then the system would have to evolve for 40 ns, a short enough time to make dissipation irrelevant since photon lifetimes are typically in the hundreds of nanoseconds [15]. On the other hand, if the coupling intensity is lower, for example, \( g_0 = 50 \text{ MHz} \), the time required to produce photons increases by one order of magnitude, making dissipation relevant. That will be the case for our second experimental proposal, and so we will discuss the effects of dissipation then. For now, we discuss that the qubit will not actually move in the cavity; instead, it will simulate its movement. As Eq. (1) shows, the only effect movement has on the Hamiltonian is changing the value of the coupling in time. Thus one could argue that as long as an experiment manages to produce that same Hamiltonian, the same phenomena will take place, even if the qubit is static. In the latter case, the qubit could produce the time-dependent coupling if its dipolar moment changed over time. We remind the reader that this analog simulator approach was taken to observe the DCE in [18, 20]. We recall that the qubit-field interaction Hamiltonian comes from

\[
H_{\text{int}} = \hat{a} \cdot \hat{E}(x_{\text{qubit}}(t), t) \propto (\sigma^+ + \sigma^-)(a^+ + a),
\]

where \( \hat{a} \) is the qubit’s dipolar moment operator and \( \hat{E}(x, t) \) is the electric field amplitude operator throughout the cavity. On one hand, if the qubit is actually moving, its coupling will change due to the different field amplitudes it will find during its trajectory. Note that the trajectory may take the qubit over length scales comparable to the field’s wavelength and the dipolar approximation will still hold, as long as the qubit charge distribution can be regarded as pointlike and, when added together, neutral [49, Sec. AIV.1.b]. Superconducting qubits in the microwave regime follow those premises: they are neutrally charged circuits several orders of magnitude smaller than microwave wavelengths [15]. On the other hand, if the qubit is static but it can change its dipolar moment,

\[
H_{\text{int}} = \hat{d}(t) \cdot \hat{E}(x_0, t),
\]

the same Hamiltonian can be produced with an appropriate \( \hat{d}(t) \). Proposals and experiments with such qubits already exist [38] which prove their effectiveness and feasibility of the experimental parameters mentioned before.

However, a few caveats may make this experiment challenging. We have been able to pinpoint four:

1. Modulating coupling with no qubit frequency modulation
2. Modulating longitudinal coupling with no transversal coupling
3. Populating other cavity modes
4. Populating the third or higher levels on the physical system that models the qubit.

The first two points are related, so we discuss them jointly. The most common way of introducing externally controlled parameters in the system is by means of superconducting interference devices (SQUIDs), which behave as nonlinear inductors that can be tuned with the external magnetic flux that passes through them. However, in our case, the circuit must be designed such that those external parameters will modify only the dipolar moment and not the two lowest levels energy gap defining the qubit (first point). Moreover, the interaction Hamiltonian must be designed to forbid transitions between global states with the same qubit state. If the latter condition is not met, a different two-level interaction Hamiltonian should be taken into account (second point), as we explain in the following. Suppose that the circuit is described by a static Hamiltonian \( H_{\text{circuit}} \) plus an interaction part of the form \( H_{\text{int}} = \eta O(a^+ + a) \). Both operators \( H_{\text{circuit}} \) and \( O \) act on the degrees of freedom of the circuit, and \( a^+ \) and \( a \) act on the cavity mode state. The circuit-mode coupling \( \eta \) is proportional to \( g_0 \). When the circuit is operated as a qubit the state has to belong to the span of the two lowest eigenvectors \( \langle \eta | H_{\text{circuit}} | g \rangle = 0 \) and \( \langle \eta | H_{\text{circuit}} | e \rangle = \Omega | e \rangle \). Then one can consider a reduced, two-level interaction Hamiltonian \( H_{\text{int,2}\times2} \) given by the matrix elements \( \langle \eta | H_{\text{int}} | g \rangle, \langle e | H_{\text{int}} | e \rangle \) and \( \langle g | H_{\text{int}} | e \rangle \), which will produce the same dynamics as long as the circuit is operated as a qubit. By expanding the reduced interaction Hamiltonian in the Pauli basis one has

\[
H_{\text{int,2}\times2} = \eta \left( | g \rangle \langle O | e \rangle | \sigma_x \rangle + \langle e | O | e \rangle | \sigma_z \rangle (a^+ + a) \right),
\]

where the energies have been rescaled so that \( | | g \rangle O | g \rangle | \sigma_z \rangle \) does not appear and the Pauli basis has been rotated to conveniently remove \( \sigma_x \). Then the time dependence of the coupling comes from the time dependence of the eigenvectors \( | g \rangle, | e \rangle \). In addition, it is now clear why the interaction must be engineered so that no transitions between global states with the same qubit state are allowed. If that were not the case, \( | g | O | e \rangle \) would not be zero and a longitudinal coupling would appear with operator \( \sigma_z (a^+ + a) \).

For example, the tunable coupling transmon designed in [38] addresses satisfactorily the first point, that is, it can change the coupling intensity keeping static the qubit’s frequency, but not the second. In other words, an experiment using the said transmon would have to take into account a longitudinal coupling \( g_z(t) \sigma_z (a^+ + a) \). However, it stands to reason that the said transmon is a step in the right direction, and that simple modifications to its design could eliminate that piece in the Hamiltonian. As we have seen, the parameter space of the qubit comprises three parameters: frequency \( \Omega \), transversal coupling \( g_x \), and longitudinal coupling \( g_z \). The transmon [38] takes as external parameters only two magnetic fluxes, and so it can explore only a two-dimensional manifold of its three-dimensional parameter space. Thus we conclude that a similar circuit with three SQUIDs could, in principle, independently tune every parameter.

The third point, populating other cavity modes, is not a relevant issue for the microscopic DCE, but it certainly is for the cavity-enhanced Unruh effect. If the mode structure is composed of equidistant modes, then producing the
The microscopic DCE for the fundamental mode would require a driving frequency of $\omega_d = \omega_0$. We define $\omega_0$ as the frequency of the fundamental mode, referred to as $\omega$ in the main text for simplicity. Then the higher modes have frequencies $\omega_n = (n+1)\omega = (n+1)\omega_d$ which do not resonate with the driving. However, producing the cavity-enhanced Unruh effect in the fundamental mode would require a driving $\omega_d = \omega_0 + \Omega$, with $\Omega$ the frequency of the qubit. If in addition to this $\Omega \approx \omega_0$, then that same driving would produce the DCE on the next mode if frequency $\omega_1 = 2\omega \approx \Omega + \omega = \omega_d$, and both phenomena would combine in a nontrivial way. This problem can be addressed by detuning the qubit frequency $\Omega$. In fact, we find advantageous to reduce the said frequency, as the velocity of the qubit required to produce the Unruh effect is $v/c_n = 1 + \Omega/\omega$. In other words, that velocity is always superluminal in the medium, but is closer to the speed of light the smaller the qubit frequency $\Omega$ is.

The last caveat of the first experiment we propose is populating higher levels of the qubit system. If the qubit’s complete level structure is nearly harmonic the DCE will be combined with a resonance at second order, in which two-photon and higher-level qubit excitations take place with a magnitude comparable to the DCE. We conclude the qubit’s complete level structure must be anharmonic, at least with regard to the third level, or the said level will have to be taken into account.

The second experiment we consider makes use of a film bulk acoustic resonator (FBAR) in order to relate the modulation of the coupling to an actual moving piece in the system. Some recent literature has considered this experiment with small variations; we encourage the reader to read Wang et al. [39]. In that work the authors propose coupling two transmission lines by overlapping them over some of their lengths. In that way they form a capacitor which, in turn, is filled with dielectric material and cooled to its ground mechanically oscillating level of frequency in the $1$–$10$ GHz regime, as reported in [40]. The capacitor can be actuated upon by an external piezo leading to the movement of the capacitor plates. The point of [39] was to interpret one of the transmission lines as a moving detector that would become excited as the other transmission line would be populated by photons by an analog Unruh effect.

![FIG. 7. Number of photons $\langle N \rangle$ as a function of both time $t$ in units of the mode’s frequency $\omega$ and driving frequency $\omega_d$ in units of the same mode’s frequency $\omega$, or equivalently, qubit speed $v$ in units of the speed of light in the medium $c_n$. The qubit frequency is given by $\Omega = \omega$, and the coupling intensity is $g_0 = 0.025\omega$, to mimic the parameter regime that the experimental proposal with an actual mechanical oscillation would impose on the system. The driving frequency was produced by a qubit moving back and forth within the cavity, with constant velocity $v = L/\pi\omega_d$ in one direction, and $-v$ after bouncing in the opposite direction. We consider the collapse operators $0.025\omega_d$ and $0.025\omega_d^-$ in the Lindblad master equation.

![FIG. 8. Population of the qubit’s excited state $P_e$ that is $|\sigma^+\sigma^-$), as a function of both time $t$ in units of the mode’s frequency $\omega$ and driving frequency $\omega_d$ in units of the same mode’s frequency $\omega$. The qubit frequency is given by $\Omega = \omega$, and the coupling intensity is $g_0 = 0.025\omega$, to mimic the parameter regime that the experimental proposal with an actual mechanical oscillation would impose on the system. The driving frequency was produced by a qubit moving back and forth within the cavity, with constant velocity $v = L/\pi\omega_d$ in one direction, and $-v$ after bouncing in the opposite direction. We consider the collapse operators $0.025\omega_d$ and $0.025\omega_d^-$ in the Lindblad master equation.

![FIG. 9. Expectation value of $I_{2,2} \otimes |7\rangle \langle 7|$ as a function of both time $t$ in units of the mode’s frequency $\omega$ and driving frequency $\omega_d$ in units of the same mode’s frequency $\omega$. The qubit frequency is given by $\Omega = \omega$, and the coupling intensity is $g_0 = 0.025\omega$, to mimic the parameter regime that the experimental proposal with an actual mechanical oscillation would impose on the system. The driving frequency was produced by a qubit moving back and forth within the cavity, with constant velocity $v = L/\pi\omega_d$ in one direction, and $-v$ after bouncing in the opposite direction. We consider the collapse operators $0.025\omega_d$ and $0.025\omega_d^-$ in the Lindblad master equation.]
That experiment could be used for the implementation of the microscopic DCE too. In that case, one of the transmission lines should be substituted or reinterpreted as a qubit, that instead of moving back and forth in the cavity would hover up and down on top of it. The Hamiltonian of the system is very similar to Eq. (1), with the difference being a coupling directly proportional to the qubit position instead of the cosine of its position. In that way the trajectory would not be one with very similar to Eq. (1), with the difference being a coupling observable or not. Figures 7–9 show that the DCE photon production could be observed for the same simplified system of previous simulations evolving under a Lindbladian composed of Hamiltonian in Eq. (1) with resonant mode’s, qubit’s, and driving frequencies’ \( \omega = \Omega \) with weak coupling \( g_0 = 0.025\omega \) plus the collapse operators \( 0.025\omega a \) and \( 0.025\omega \sigma^- \), with \( a \) the photon annihilation operator and \( \sigma^- \) the qubit relaxation operator. Notice that a dissipation as intense as the coupling puts the system in a parameter regime between the strong and weak coupling regime. Such dissipation would be an overestimation, as the quantum technologies we consider have been designed to operate in the strong coupling regime since 2004 [15].

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3.4 Conclusion

In this section we briefly add to the conclusions already presented in the article at 3.3. Namely, we believe it is worthwhile to interpret some of the results obtained outside of the analog interpretation.

We have proven that a two-level system moving through a cavity at the speed of light in the medium or around that velocity, produces a monotonically increasing amount of pairs of photons while staying at its ground state and not getting entangled with the field. That is a very satisfactory microscopic model of the DCE as it follows the requisites indicated at the beginning of the article and at the end of section 3.2.2. Then we proposed a number of mechanisms that could be analog simulators for the effect. Here, we wish to add that, as with any analog simulator, it is worthwhile to interpret the phenomena at both sides of the analogy.

From a purely electrical point of view, and abandoning any interpretation of the system as a moving mirror, the results obtained can be summarized as follows: A qubit with the ability of tuning its coupling to a waveguide is capable of being operated on four interesting regimes. Three of them were already known in previous literature. Firstly, the qubit can keep a constant strong coupling with the field, allowing for typical quantum information processing. Secondly, the qubit can decouple from the field, so that its state is protected from leaking to the waveguide. Switching between these two regimes was the original motivation behind their design [30]. Thirdly, the qubit can modulate its coupling so that the counter-rotating dynamics become relevant, which can be considered as a primitive to generate qubit-field entanglement. Lastly, in this work we present that a qubit can modulate its coupling to make both rotating and counter rotating dynamics relevant, leading to a large photon production while keeping the qubit at its initial state.

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4 Non-Gaussian entanglement swapping between continuous and discrete variables systems

In chapter 2 we developed an entanglement witness tailored to the states produced by vacuum amplification from three-mode spontaneous parametric down-conversion (3SPDC). There, we found a suggestive criterion to classify entanglement in continuous variables (CV) systems: if the covariances change in time and the state remains Gaussian, then a Gaussian witness will detect the entanglement if there is any [1]. If the state stops being Gaussian, considering non-Gaussian witnesses is often fruitful. Moreover, those two scenarios seem mutually exclusive. But, why should we stop at CV systems? The notions of (non-)Gaussianity discussed in the introduction of chapter 2 are perfectly suitable for discrete variables (DV) systems. In fact, they can be applied to hybrid systems too. Furthermore, since these entanglement notions can be applied to mixed states, we find them very flexible, as they make sense on most if not all quantum systems.

Because of their general applicability, in this chapter 4 we pay theoretical attention to the differences and similarities of entangled states detected with (non-)Gaussian witnesses. Here, we upgrade the notions of (non-)Gaussianity in order to talk, not about witnesses nor states, but about entanglement itself. We define Gaussian entanglement as the property contained in systems that can be detected as entangled with Gaussian witnesses. Conversely, we define non-Gaussian entanglement as the property of systems which require to be detected with non-Gaussian witnesses. Then, we ask the question of whether there are states with both Gaussian and non-Gaussian entanglement. We conclude that 3SPDC processes produce only non-Gaussian entanglement, strengthening our claims from chapter 2.

In order to take advantage of the generality of the (non-)Gaussian criterion we study DV systems too. When considering multipartite DV systems there is a broadly known result that comes to mind. There are two classes of 3 qubit pure states that are both tripartitely entangled, and yet they are not convertible one in the other by means of stochastic local operations and classical communication (SLOCC) [2], see introduction on the topic at section 1.3.3 for further details and definitions. We find that the maximally entangled states of those classes, namely the GHZ and W states, contain non-Gaussian and Gaussian entanglement, respectively. This parallelism between (non-)Gaussianity and SLOCC (in-)convertibility suggest that the latter, more mathematically-oriented,
classification can be interpreted in terms of the former, more experimentally-accessible, criterion.

To put that parallelism further to the test, we study a hybrid system, that is, a system composed of both CV and DV elements. Our interest on hybrid systems is motivated by the flexibility of (non-)Gaussian entanglement and, additionally, the technological significance of these systems. We remind the reader that most qubits are in fact CV systems designed so that a two-dimensional state subspace contains their dynamics. Then, necessarily, the degrees of freedom interacting with the qubit are still CV. These CV systems can be considered the qubits’ environment, or they can be engineered to couple qubits together. Thus, we find interest in modifying the original 3SPDC system. When the 3SPDC process is driven between the lowest frequency modes of three different resonators, instead of modes on the same resonator, it produces the same non-Gaussian entanglement as in chapter 2 and, additionally, qubits can be coupled to each resonator. The absorption of the 3SPDC radiation by the qubits leads to tripartite entangled states in those qubits that we prove to be non-Gaussian in nature. Therefore, the non-Gaussian entanglement is swapped from CV to DV systems. Moreover, the qubits are reported as entangled in a wider parameter regime. Both of these facts could enhance the technological relevance of non-Gaussian entanglement.

Summarizing, the present chapter revolves around the results published in [3], which is presented in section 4.1. The purpose of that article is two-fold. On one hand, its section II introduces the theoretical concept of (non-)Gaussian entanglement and equips us with tools for proving the non-Gaussianity of some states. In particular, we use those tools to prove the non-Gaussianity of the GHZ state, and the Gaussianity of the W state was already proved in the literature [4], despite not being interpreted this way. On the other hand, we propose the design of a hybrid superconducting circuit with three pairs of bosonic modes and qubits that puts the newly introduced theoretical concepts to the test, in order to prove that they are adequate for actual experiments. In section III we prove the non-Gaussianity of the field’s states produced from vacuum and 3SPDC amplification. Then, in section IV we prove the same non-Gaussianity on the qubit’s reduced state, introducing a witness that is the DV counterpart of the witness presented in chapter 2. Then, the article’s Conclusions and Appendices follow, which are advisable to read before continuing with the sections outside the article.

### 4.1 Publication

In this section the article [3] is copied verbatim, as the main section of the present chapter. Please note it has its own page numbering as well as bibliography.
Non-Gaussian entanglement swapping between three-mode spontaneous parametric down-conversion and three qubits

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In this work we study the production and swapping of non-Gaussian multipartite entanglement in a setup containing a parametric amplifier which generates three photons in different modes coupled to three qubits. We prove that the entanglement generated in this setup is of non-Gaussian nature. We introduce witnesses of genuine tripartite non-Gaussian entanglement, valid for both mode and qubit entanglement. Moreover, those witnesses show that the entanglement generated among the photons can be swapped to the qubits, and indeed the qubits display non-Gaussian genuine tripartite entanglement over a wider parameter regime, suggesting that our setup could be a useful tool to extract entanglement generated in higher-order parametric amplification for quantum metrology or quantum computing applications.

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I. INTRODUCTION

Entanglement is the key ingredient to most quantum technologies being designed today, ranging from teleportation [1–3] to boson sampling [4] and in general any quantum computational scheme. Therefore, plenty of present-day literature deals with how to generate entanglement, and a very fruitful paradigm at that is parametric amplification. Take, for example, its role as a primitive ingredient in the recent claim on boson sampling quantum advantage [5].

The first instances of quantum parametric amplifiers date back to the 1980s [6,7] in the setting of outperforming quantum measurements with single-mode squeezing. Then, in that same decade, it was discovered that parametric amplification could pump energy in two modes at once, leading to the generation of two-mode squeezing [8], perhaps the simplest form of continuous-variable (CV) entanglement [9]. During the past five years, some of us have predicted that such two-mode squeezing can be used to entangle three modes in a genuinely tripartite way by applying the process to two pairs at once [10,11], a prediction that has been experimentally validated [12]. We denominate this process double two-mode spontaneous parametric down-conversion (2-2SPDC). In a recent work [13], we predicted that a similar process experimentally demonstrated in [14], capable of generating three photons on different modes at once, 3SPDC, produces genuine tripartite entanglement too. In order to experimentally detect 2-2SPDC entanglement, inspection of the covariances of the field quadratures was enough, whereas the 3SPDC entanglement requires inspecting higher statistical moments.

As entanglement generation has become a well-established technology, produced in countless laboratories around the globe, still interesting theoretical questions remain open.

Take, for example, the inequivalent entanglement of the three-qubit W and Greenberger-Horne-Zeilinger (GHZ) states [15]. Those states are entangled in a tripartite way and yet they cannot be converted into each other by means of stochastic local operations and classical communication (SLOCC). A generalization of this result to general discrete-variable (DV) d-level systems has been proposed recently [16], and the generalization to n qubits is still incomplete, although we know that there have to be infinitely many SLOCC classes for N > 3 [15], which therefore have to be gathered into some finite number of entanglement families (which proves to be a formidable task even for N=4 [17–20]) whose physical meaning is not always transparent. Furthermore, extensions of the above results to mixed states (even for three qubits) or to continuous variables beyond Gaussian states remain as open problems. A physically meaningful criterion to classify quantum entanglement, valid in principle both for CV and DV systems and for pure and mixed states, might be the distinction between gaussian and non-Gaussian entanglement. Besides the theoretical interest, non-Gaussian entanglement provides also technological advantages, for instance, in quantum metrology [21,22] or quantum computing applications [23].

In [13] we found that the states generated by 2-2SPDC and 3SPDC processes have different types of entanglement, suggesting some sort of continuous-variable equivalence with the three-qubit W and GHZ classes. In this work we formalize this insight as well as analyze the swapping of entanglement from 3SPDC to three qubits. In particular, we provide formal definitions to Gaussian and non-Gaussian entanglement and prove both the Gaussianity of 2-2SPDC entanglement and the non-Gaussianity of the 3SPDC entanglement, finding similarities and differences between GHZ and W classes.

Moreover, we propose an experimental setup in which 3SPDC non-Gaussian entanglement can be swapped to three qubits. An asymmetric superconducting quantum interference
device (SQUID) generating 3SPDC is coupled to three separate resonators, each containing a coupled superconducting qubit. We show that the entanglement generated among the qubits is also of non-Gaussian nature, by using a natural extension of our CV entanglement witness which accommodates DV systems. Interestingly, we detect non-Gaussian qubit entanglement in a wider parameter regime (as compared to mode entanglement) which suggests that the swapping to qubits could be an efficient way of extending the technological usefulness of 3SPDC entanglement.

The structure of this work is as follows. First we introduce the notions of Gaussian and non-Gaussian entanglement in such a way that they may be applied to both CV and DV systems and pure and mixed states. Then we relate these notions to the widely known W and GHZ states. After that, we present arguments that can be used to prove the non-Gaussianity of the entanglement contained in a state and we will apply them to our three-mode 3SPDC system in the presence of three qubits, each interacting with a bosonic mode. We will obtain proof of the tripartite non-Gaussianity of the field’s state as well of the qubits’. Finally, some concluding remarks and future research directions will be presented.

II. NON-GAUSSIAN ENTANGLEMENT

We start with a description of non-Gaussian entanglement. The term is coined after the Gaussian states of quantum optics, those states represented by Wigner functions that happen to be Gaussians of the canonical variables.

Detecting entanglement in an experiment often involves measuring some witness, namely, a combination of expectation values of observables that is bounded by some constant for states that do not possess the kind of entanglement considered. An entanglement witness is Gaussian if its algebraic expression contains only linear and quadratic contributions of the canonical variables. That way, the witness is only sensitive to the means and (co)variances of a multipartite wave function or Wigner quasidistribution. If higher powers of the canonical variables appear in the witness or the witness cannot be brought into an algebraic formula of the canonical variables, then it is non-Gaussian.

The characterization of the entanglement of Gaussian states is well known [24]. Any entanglement in a Gaussian state will be detected by a Gaussian witness; thus a Gaussian state can only contain Gaussian entanglement. However, a non-Gaussian state might have the same mean and covariances of the canonical variables as some separable Gaussian state [13]. Then its entanglement would not be detected by a Gaussian witness and so it would be non-Gaussian entanglement. Finally, we can extend the concept of Gaussianity to DV systems, by replacing any reference to canonical variables with spin variables.

Interestingly, the concepts of Gaussian and non-Gaussian entanglement can be related with the two main representatives of tripartite qubit entanglement, the W and GHZ states. The W entanglement is Gaussian, since it can be detected by a Gaussian witness [25], while GHZ entanglement is non-Gaussian, since we can, for instance, find a state that contains no entanglement and yet has the same means and covariances on the spin variables as the GHZ state:

\( p_{\text{GHZ}} \) mimics GHZ

\[
\begin{align*}
|\psi\rangle & = \frac{1}{\sqrt{2}} (|01\rangle + |11\rangle) + |10\rangle \otimes |00\rangle + |11\rangle \otimes |10\rangle + |11\rangle \otimes |11\rangle \\
& + \frac{1}{\sqrt{2}} (|01\rangle + |11\rangle) \otimes |00\rangle + |11\rangle \otimes |10\rangle + |11\rangle \otimes |11\rangle
\end{align*}
\]

where \(|0\rangle\) is the ground state of the \(i\)th qubit and \(|1\rangle\) its excited state. Both the GHZ state and \(p_{\text{GHZ}}\) have the same first and second statistical moments of the spin variables

\[
\begin{align*}
\langle S_i^x \rangle & = 0, \quad \langle S_i^y \rangle = 0, \quad \langle S_i^z \rangle = 0, \\
\Delta^2 S_i^x S_i^x & = 0, \quad \Delta^2 S_i^y S_i^y = 0, \quad \Delta^2 S_i^z S_i^z = \frac{1}{4},
\end{align*}
\]

where the spin variables are defined by \(S_i^x = |0\rangle \langle 1| - |1\rangle \langle 0|\) and \(S_i^z = |1\rangle \langle 1| - |0\rangle \langle 0|\) and the angular momentum algebra.

III. NON-GAUSSIANITY OF ENTANGLEMENT IN 3SPDC RADIATION

The 3SPDC process studied in [14] takes place in a system composed of three bosonic modes subject to time-dependent boundary conditions, implemented by means of a SQUID, which behaves as a tunable nonlinear inductor at the edge of a superconducting waveguide. The SQUIDs inductance is modulated with the sum of the characteristic frequencies of the three modes, producing an effective three-mode interaction described by

\[
H_{3\text{SPDC\:RWA}} = i \sum_{j=1}^{3} \omega_j a_j^\dagger a_j + \frac{g_0}{2} \cos \omega \Omega_j (a_j^\dagger a_j a_{j+1}^\dagger a_{j+1} + a_j^\dagger a_j a_{j+2}^\dagger a_{j+2}),
\]

where \(\omega_1, \omega_2, \text{ and } \omega_3\) are the modes’ characteristic frequencies, \(a_i^\dagger \text{ and } a_i\) are the creation and annihilation operators on the \(i\)th mode, respectively, \(g_0\) is the intensity of the coupling between the modes, and \(\Omega_j\) is the driving of the SQUID, which is equal to \(\sum \omega_j\). Note that the rotating-wave approximation (RWA) was performed in order to illustrate the main process induced by this Hamiltonian: parametric creation or destruction of triplets of photons, one on each mode. The Hamiltonian is actually an approximation of a more general Hamiltonian

\[
H_{3\text{SPDC}} = i \sum_{j=1}^{3} \omega_j a_j^\dagger a_j + \frac{g_0}{2} \cos \omega \Omega_j (a_j^\dagger a_j (a_{j+1}^\dagger a_{j+1}) (a_j^\dagger a_j + a_j^\dagger a_j + a_{j+2}^\dagger a_{j+2}) (a_j^\dagger a_j + a_j^\dagger a_j + a_{j+3}^\dagger a_{j+3}),
\]

which will be the one that we will study throughout the text. We use this Hamiltonian for the sake of completeness, although the RWA Hamiltonian above would suffice to obtain the main results of this work and is generally valid under experimental conditions. However, using the general Hamiltonian allows us not to be concerned with the regime of validity of the RWA. Before we begin proving the non-Gaussian nature of the entanglement produced among the three modes, we will extend the system with three qubits, each one interacting with one mode. This modification is of interest because it paves the way to experimental production of non-Gaussian entanglement in both CV systems (the reduced state of the three modes) and DV systems (the qubits).
FIG. 1. Illustration of the system composed of three transmission lines (depicted as solid meandering lines) that meet at an asymmetric SQUID (loop with boxes, that is, Josephson junctions, at the sides). Each one of those transmission lines interacts with a transmon qubit (colored zippers, not to scale). Control lines have been omitted. If pumped with the appropriate tone, the asymmetric SQUID will drive three-mode spontaneous parametric down-conversion among the three fundamental modes of the transmission lines. Non-Gaussian tripartite entanglement will be produced between the modes as proved in Eq. (5) and Fig. 2 for some parameter regimes. Additionally, non-Gaussian tripartite entanglement will be swapped to the qubits, as proved in the text and Fig. 3. We show labels for the parameters that appear in the Hamiltonian (1) for reference.

Such a technological platform could ground our theory on experimental data and additionally find technical applications as the primitive for generation of tripartite entanglement between CV or DV systems.

When the three qubits are taken into account, the total Hamiltonian becomes

\[
H_{\text{3SPDC+3qubits}} = \sum_{i=1}^{3} \omega_i a_i^\dagger a_i + \frac{\Omega_i}{2} \sigma_{x,i} + g_i \sigma_{x,i} (a_i^\dagger + a_i) + g_0 \cos \omega_{0f} (a_1^\dagger + a_1)(a_2^\dagger + a_2)(a_3^\dagger + a_3),
\]

(1)

where \(\sigma_{x,1,2,3}\) are the Pauli matrices for the \(i\)th qubit and \(g_i\) is the intensity of its coupling to the respective mode. Note that the qubit-mode interaction takes the form of the Rabi interaction. An experimental setup that could be effectively modeled with Eq. (1) is described in Fig. 1. It is composed of three superconducting cavities joined together from one of their edges [26–28]. At that meeting point lies an asymmetric SQUID driven with a single tone of frequency \(\omega_{0f}\) or \(\omega_0\).

In order to prove the non-Gaussianity of the entanglement produced by the Hamiltonian in Eq. (1) when evolving the initial vacuum state \(|0\rangle|0\rangle|0\rangle\), where \(|0\rangle\) is the mode vacuum state and \(|g\rangle\) is the qubit ground state, we will examine the time derivatives of the quadratures and spin covariances, by making use of the condition

\[
\text{i}\hbar \Delta^2 O_i O_j = 0
\]

\[
\Leftrightarrow \langle [O_i, H] [O_j, H] \rangle = \langle O_i [O_j, H] \rangle + \langle [O_i, H] O_j \rangle,
\]

(2)

where \(O_i\) and \(O_j\) are canonical or spin variables, \(H\) is the Hamiltonian of the system and \(\Delta^2 O_i O_j\) is the covariance between the measurements of \(O_i\) and \(O_j\), that is, \(\langle O_i O_j \rangle\) – \(\langle O_i \rangle \langle O_j \rangle\). Equation (2) is easily derived from the Heisenberg equation of motion. See Appendix A for further notes on its derivation. Using the Hamiltonian in Eqs. (1) and (2), we have

\[
\partial_t \Delta^2 x_i x_j = \left\{ \frac{x_i p_j}{m_j} + \frac{x_j p_i}{m_i} - (x_i) \frac{p_j}{m_j} - (x_j) \frac{p_i}{m_i} \right\},
\]

\[
\partial_t \Delta^2 p_i p_j = \left\{ -m_i \omega_i^2 p_i x_j + m_j \omega_j^2 p_j x_i \right\}
\]

\[
- \text{i} \hbar (g_i \sigma_{x,i} p_j + g_j \sigma_{x,j} p_i)
\]

\[
- \text{i} \hbar \left( \frac{g_i}{\Omega_i} p_i x_i x_j + \frac{g_j}{\Omega_j} p_j x_j x_i \right)
\]

\[
+ \{ m_i \omega_i^2 x_i + g_i \sigma_{x,i} + \text{i} \hbar (\Omega_i/\alpha) x_i \} \langle p_i \rangle
\]

\[
+ \{ m_j \omega_j^2 x_j + g_j \sigma_{x,j} + \text{i} \hbar (\Omega_j/\alpha) x_j \} \langle p_j \rangle,
\]

\[
\partial_t \Delta^2 S_i S_j = \Omega_j \langle [\sigma_{x,i}^\dagger \sigma_{x,j}^\dagger] + [\sigma_{z,i}^\dagger \sigma_{z,j}^\dagger] \rangle
\]

\[
- \text{i} \hbar \left( \frac{g_i}{\Omega_i} p_i x_i x_j + \frac{g_j}{\Omega_j} p_j x_j x_i \right) \langle p_i \rangle
\]

\[
+ \{ m_i \omega_i^2 x_i + g_i \sigma_{x,i} + \text{i} \hbar (\Omega_i/\alpha) x_i \} \langle p_i \rangle
\]

\[
+ \{ m_j \omega_j^2 x_j + g_j \sigma_{x,j} + \text{i} \hbar (\Omega_j/\alpha) x_j \} \langle p_j \rangle,
\]

(3)

where \(x_i\) and \(p_i\) are the quadratures of the \(i\)th mode and \(S_i, S_j, S_{i,j}\) are the analogous angular momentum operators along the \(x\), \(y\), and \(z\) axes for the \(i\)th qubit. For a detailed derivation of the covariances time derivatives see Appendix B. In order to tackle Eq. (3) we consider the projector

\[
P = \sum_{a,\beta=0}^{1} \otimes_{i=1}^{3} P_i (2n + \alpha) \otimes \ell_{2,\alpha=2} (\beta),
\]

(4)

where \(P_i (n) = |n\rangle \langle n| \) is the projector onto the bosonic mode state with \(n\) photons or excitations and \(\ell_{2,\alpha=2} (\beta)\) is \(|\beta\rangle \langle \beta|\) if \(q = 0\), the projector onto the qubit ground state, or \(|\epsilon\rangle \langle \epsilon|\) if \(q = 1\), the projector onto the qubit excited state. We find that this projector is a conserved quantity of the system. Consider the following motivation behind its definition: The Hamiltonian in Eq. (1) allows for some transitions between the stationary Hamiltonian eigenstates. In particular, it allows for transitions that change all three modes in one photon (via the 3SPDC process) as well as transitions changing a qubit-mode pair in one excitation (that is, any combination of creating or destroying a photon while exciting or relaxing the qubit). However, there are many other transitions that are not allowed: creating or destroying a pair of photons but not a third one, spontaneously exciting or relaxing a qubit without changing photon number, and so on. Then \(P\) is built to project onto all of the eigenstates the vacuum can transition to, while excluding those the vacuum cannot leak into. For further information about the derivation of \(P\), as well as proof of how it commutes with the Hamiltonian, see Appendix C. The expectation value of \(P\) for the initial state \(|0\rangle|0\rangle|0\rangle\) is 1. Therefore, the time evolution of \(|0\rangle|0\rangle|0\rangle\) will never leave the subspace \(P\) projects onto, which we denote the dynamical subspace,

\[
\psi(t) = \sum_{a,\beta=0}^{\infty} \otimes_{i=1}^{3} \alpha_{a,\beta,i,n} (t) \langle 2n + \alpha | \otimes | \beta \rangle.
\]

With this we can evaluate many of the expectation values in the covariances time derivatives in Eq. (3). In particular, all time derivatives become zero, except for the \(\Delta^2 S_i S_{i,j}\)

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FIG. 2. Value of the witness $G_{CV}$ defined in Eq. (6) as a function of time $t$ and 3SPDC coupling $g_0$ in units of the lowest-frequency mode $\omega_1$ when the initial state $|0\rangle |0\rangle |0\rangle$ evolves under the Hamiltonian in Eq. (1). The other mode frequencies are $\omega_2 = 2\omega_1$ and $\omega_3 = \omega_1$. The qubits are resonant with their modes so that $\Omega_2 = \omega_i$ and their couplings are all equal $g_i = 0.01\omega_1$. The witness reports non-Gaussian entanglement in the modes, that is, it is greater than zero, for short times. Note that entanglement is 0 at $t = 0$.

Therefore, the reduced state of the three modes cannot contain Gaussian entanglement: It has the same covariances as a clearly separable state, the vacuum $|000\rangle$, but the state gets entangled with time, as we proved in [13] for the qubitless system. In that work we built a genuine tripartite entanglement witness defined as

$$G_{CV} = |\langle a_1 a_2 a_3 | - \sum_{i,j,k=1,2,3} \sqrt{|a_i^\dagger a_j^\dagger a_k^\dagger a_k|},$$

so that when $G_{CV} > 0$, genuine tripartite entanglement is detected. In fact, since the publication of [13] we have found an improved witness

$$G_{CV} = |\langle a_1 a_2 a_3 | - \max_{i,j,k=1,2,3} \sum_{i,j,k=1,2,3} \sqrt{|a_i^\dagger a_j^\dagger a_k^\dagger a_k|},$$

by following the derivation in [13] and making use of the fact that the expectation values of a mixed state cannot be larger than the largest of its pure components. Figure 2 shows the value of the genuine tripartite entanglement witness $G_{CV}$ for different times and 3SPDC coupling strength. We conclude that the field contains non-Gaussian entanglement at times not much longer than $g_0 t = 1$. For longer times, all we know is that Gaussian witnesses will fail, but if there is any entanglement in the modes non-Gaussian witnesses might succeed.

FIG. 3. Value of the witness $G_{DV}$ defined in Eq. (7) as a function of time $t$ and 3SPDC coupling $g_0$ in units of the lowest-frequency mode $\omega_1$ in the same conditions as in Fig. 2. The witness reports non-Gaussian entanglement in the qubits, that is, it is greater than zero, for a broad parameter regime. Note that entanglement is 0 at $t = 0$.

IV. NON-GAUSAON THREE-QUBIT ENTANGLEMENT

The nature of the three-qubit entanglement is however more difficult to determine: Since the $z$ covariances do change in time we need to answer the question of whether or not a Gaussian witness exists that uses only the $z$ spin covariances. We find that the answer is no, and therefore the qubit entanglement, if there is any, is non-Gaussian too. See Appendix D for a proof.

In order to detect whether there is actually entanglement, we need a suitable non-Gaussian entanglement witness. The same proof [13] that leads to the construction of $G_{CV}$ in CV systems can be extended to a DV witness by replacing the canonical variables with spin variables

$$G_{DV} = |\langle \sigma_1^+ \sigma_2^- \sigma_3^- \rangle | - \max_{i,j,k=1,2,3} \sum_{i,j,k=1,2,3} \sqrt{|\langle \sigma_i^+ \sigma_j^- \sigma_k^- \rangle|^2},$$

which works as $G_{CV}$ but in DV systems; it reports genuine tripartite entanglement whenever $G_{DV} > 0$. Figure 3 shows the value of $G_{DV}$ for different times and 3SPDC coupling strengths. We conclude that the qubits are indeed entangled in a non-Gaussian way for a broad parameter regime. Indeed, it seems that the qubits are entangled in a wider regime of parameters, suggesting that swapping the entanglement from the photons to the qubits could be a way to exploit the multipartite entanglement generated in 3SPDC radiation. However, notice that there could be other witnesses detecting entanglement where ours fails. Note also that, as usual, an entanglement witness only tells us about the existence of entanglement, not necessarily its degree, which would require the use of an entanglement measure.

V. CONCLUSION AND OUTLOOK

In summary, we have presented a setup in which three qubits are coupled to a 3SPDC source. We have shown that there is genuine tripartite entanglement generated both among
the three modes of the electromagnetic field and among the qubits. Moreover, we have proved the non-Gaussian nature of this entanglement, as well as that of the GHZ state, suggesting that Gaussianity might be an extension to CV and mixed states of the W and GHZ classes. We have introduced witnesses of genuine tripartite entanglement for both the field and the qubits. Interestingly, in the case of the qubits, entanglement is detected for a wider regime of parameters, which suggests that our setup could provide an efficient way of exploiting the genuine non-Gaussian multiparticle entanglement generated in 3SPDC interactions. In particular, qubits with non-Gaussian entanglement display useful properties for quantum metrology and quantum computing applications.

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APPENDIX A: DYNAMICS OF STATISTICAL MOMENTS

In this Appendix we will derive the expression for the time derivatives of the canonical and spin variable covariances. We will be particularly interested in the cases when the moments are constant. If that is the case, Gaussian entanglement cannot be generated. We start with the Heisenberg equation of motion

\[ i\hbar \delta O(t) = [O(t), H(t)], \]

which immediately yields expressions for the time derivatives of the first-order statistical moments, the means

\[ i\hbar \delta \langle O \rangle = \langle [O(t), H(t)] \rangle. \]  

(A2)

In order to derive a similar expression for second-order statistical moments, that is, variances and covariances, we follow a similar approach. We recall the definition of the covariances of two observables \( O_i \) and \( O_j \),

\[ \Delta^2 O_i O_j = \langle O_i O_j \rangle - \langle O_i \rangle \langle O_j \rangle, \]

and by taking its time derivative we arrive at

\[ i\hbar \delta \Delta^2 O_i O_j = ([O_i O_j, H]) - \langle [O_i, [O_j, H]] \rangle - \langle [O_j, [O_i, H]] \rangle - \langle O_i \rangle \langle [O_j, H] \rangle - \langle [O_j, H] \rangle \langle O_i \rangle. \]

This equation gives us conditions systems must follow in order not to generate or destroy Gaussian entanglement

\[ i\hbar \delta \Delta^2 O_i O_j = 0 \]

\[ \Leftrightarrow \]

\[ \langle [O_i O_j, H] \rangle = \langle [O_i, [O_j, H]] \rangle + \langle [O_j, [O_i, H]] \rangle + \langle O_i \rangle \langle [O_j, H] \rangle + \langle [O_j, H] \rangle \langle O_i \rangle. \]  

(A3)

Note that if the averages of \( O_i \) and \( O_j \) are zero, then the condition states that in order not to change the covariances, the operator \( O_i O_j \) must be a conserved quantity in the subspace spanned by the state during all that time.

Summarizing, we have obtained expressions for the time derivatives of the means and covariances of general observables. Those equations led to Hamiltonian conditions in Eq. (2) that will tell when the covariances (and Gaussian entanglement) are constant in a particular system. We will consider particular Hamiltonians in the calculations to come.

APPENDIX B: DERIVATION OF THE COVARIANCES’ TIME DERIVATIVES

In this Appendix we will take Hamiltonian in Eq. (1) and compute the covariances’ time derivatives as instructed by Eq. (2). Note that the Hamiltonian can be written in terms of the canonical and spin variables alone

\[ H = \frac{p_i^2}{2m_i} + \frac{1}{2}m_i \omega_i^2 x_i^2 + \Omega_i^2 \]

\[ + \hat{g}_0 \cos \left( \sum_i \omega_i t \right) x_1 x_2 x_3 + \sum_i g_i \sigma_i^x x_i. \]

Then the field’s position covariances have the time derivatives

\[ [x_i, H] = \frac{1}{2m_i} \left[ x_i, p_i^2 \right] \]

\[ = \frac{1}{2m_i} \left( [x_i, p_i] p_i + p_i [x_i, p_i] \right) \]

\[ = \frac{i\hbar}{m_i} p_i, \]

\[ [x_i, x_j, H] = x_i [x_j, H] + [x_i, H] x_j \]

\[ = i\hbar \left( \frac{x_i p_j}{m_j} + \frac{x_j p_i}{m_i} \right), \]

\[ \partial_t \Delta^2 x_i x_j = \left( \frac{x_i p_j}{m_j} + \frac{x_j p_i}{m_i} \right) - \langle x_i \rangle \left( \frac{p_j}{m_j} \right) - \langle x_j \rangle \left( \frac{p_i}{m_i} \right). \]  

(B1)

For the momentum’s covariances

\[ [p_i, H] = \frac{m_i \omega_i^2}{2} \left[ p_i, p_i^2 \right] + g_i \sigma_i [p_i, x_i] + \hat{g}(t) [p_i, x_1 x_2 x_3] \]

\[ = -i\hbar m_i \omega_i^2 x_i - i\hbar \hat{g} \sigma_i x_i \]

\[ - i\hbar \hat{g}(t) x_j x_k \]

\[ = -i\hbar \left( m_i \omega_i^2 p_i x_j + m_i \omega_i^2 x_i p_j \right) \]

\[ - i\hbar (g_i \sigma_i x_i p_j + g_i \sigma_i p_j x_i) \]

\[ - i\hbar \hat{g}(t) (p_i x_j x_k + x_j p_i x_k), \]

which results in a time derivative of the momenta’s covariances

\[ \partial_t \Delta^2 p_i p_j = -\left( m_i \omega_i^2 p_i x_j + m_i \omega_i^2 x_i p_j \right) \]

\[ - i\hbar (g_j \sigma_i p_j + g_i \sigma_i p_j) \]

\[ - \hat{g}(t) (p_i x_j x_k + x_j p_i x_k) \]

\[ + \left( m_i \omega_i^2 x_i + g_i \sigma_i + \hat{g}(t) x_i x_k \right) (p_j) \]

\[ + \left( m_i \omega_i^2 x_j + g_j \sigma_j + \hat{g}(t) x_j x_k \right) (p_i). \]  

(B2)

The conditions derived in Eq. (2) not only apply to continuous-variable systems, but discrete ones as well. By plugging the spin variables \( S_i^x, S_i^y \), and \( S_i^z \) and the Hamiltonian
in Eq. (1) we derive
\[ \partial_t \Delta^2 S_i^j = \Omega_i (\sigma_i^+ \sigma_i^-) + \Omega_i (\sigma_i^- \sigma_i^+), \]
\[ \partial_t \Delta^2 S_i^j = \Omega_i (\sigma_i^+ \sigma_i^-) + \Omega_i (\sigma_i^- \sigma_i^+), \]
\[ - 2 g_i (\sigma_i^+ \sigma_i^-) - 2 g_i (\sigma_i^- \sigma_i^+), \]
\[ \partial_t \Delta^2 S_i^j = \frac{g_i}{2} (\sigma_i^+ \sigma_i^-) + \frac{g_i}{2} (\sigma_i^- \sigma_i^+). \]

**APPENDIX C: CONSERVED QUANTITIES**

In this Appendix we provide proof of the conserved quantity \( P \) in Eq. (4). Note that \( P \) projects onto the subspace that contains every eigenstate with the same parity of qubit plus photon excitation on each pair of qubits and modes. That is, for every eigenstate in that subspace, the addition of the number of photons on the first mode plus the number of excitations on the first qubit (that is, zero for \(| g \rangle \) or one for \(| e \rangle \)) will always be the same as the addition of the number of photons and qubit excitations in the second qubit-mode pair. That is, each mode has to change its number of photons in one unit. Therefore, the parity of each mode-qubit pair. In order to gain some insight into why that particular projector is a conserved quantity, we first argue for its construction with perturbation theory. Then an actual proof calculating the commutator with the Hamiltonian is provided. Finally, we will compute some elementary expectation values within the image of \( P \) that happen to appear in the covariances’ time derivatives.

1. Construction of a conserved quantity

We will begin with the first-order perturbative corrections to the time evolution of \( H_{\text{3SPDC+3qubits}} \)
\[ \psi^{(1)}(t) = \frac{1}{\hbar} \int_0^t dt' \psi^{(1)}(t') \int_0^t dt' \psi^{(1)}(t') \int_0^t dt' \psi^{(1)}(t') = \alpha |11gge \rangle + \beta |100ggg \rangle + \gamma |010egg \rangle + \delta |001gge \rangle, \]
where \( H_{\text{int}} \) is the Hamiltonian in the interaction picture. The important fact to note here is that all kets share some sort of parity. If we add together the number of photons in the first mode and the number of excitations in the first qubit, we obtain 2 or 0, even numbers. The same happens with every mode-qubit pair and for every ket.

The second-order correction takes the form
\[ \psi^{(2)}(t) = \frac{1}{\hbar^2} \int_0^t \int_0^t dt' \psi^{(2)}(t') \int_0^t \int_0^t dt' \psi^{(2)}(t') \int_0^t \int_0^t dt' \psi^{(2)}(t') \in \text{span}(000ggg, 002ggg, 020ggg, 022ggg, 020ggg, 022ggg), \]
\[ \text{span}(022ggg, 010ggg, 011ggg, 012ggg, 010ggg, 000ggg, 002ggg, 010ggg, 012ggg, 010ggg, 011ggg, 000ggg, 002ggg, 001ggg, 011ggg, 010ggg, 011ggg, 010ggg, 011ggg, 000ggg, 002ggg), \]
\[ \text{span}(000ggg, 002ggg, 020ggg, 022ggg, 010ggg, 011ggg, 012ggg, 010ggg, 011ggg, 000ggg, 002ggg, 010ggg, 012ggg, 010ggg, 011ggg, 000ggg, 002ggg, 001ggg, 011ggg, 010ggg, 011ggg, 010ggg, 011ggg, 000ggg, 002ggg), \]
Again, all the kets involved in the second-order correction share a notion of parity, but it appears to be a different, or more general, parity than the first-order corrections. Some kets have an even number of photons plus qubit excitations (e.g., \(|222ggg\rangle\)) and other kets have an odd number of photons plus qubit excitations (e.g., \(|110ggge\rangle\)), but there are no kets that mix odd and even numbers of photons plus qubit excitations (e.g., there is no \(|211ggge\rangle\)).

The reader might have noticed that we are now in position to finish a proof by induction. We have proven that the first-order corrections are composed of kets with an even number of fields plus qubit excitations. We have proven that the second-order corrections are a superposition of kets with an odd or even number (but no mixtures) of fields plus qubit excitations. Now we will prove that if the \( n \)th-order correction is such a superposition, the \((n+1)\)th correction has that same parity. In order to do so, we will study the effects each of the pieces of the Hamiltonian has on the parity of a ket.

First, the 3SPDC piece has the form \( g(t)(a_i^+ + a_i)(a_j^+ + a_j)(a_k^+ + a_k) \). Note that the result of the application of this piece of the Hamiltonian on a vector with well-defined parity is to completely change the parity of each mode-qubit pair. That is, each mode has to change its number of photons in one unit, up or down, but their interacting qubit will remain the same. Therefore, the result is a superposition of vectors with the same parity on each qubit-mode pair.

Second, the Rabi piece has the form \( g(t)a_i^+ \). The result of applying this piece of the Hamiltonian on a vector with well-defined parity is a superposition of vectors of the same parity. This is due to the fact that the \( i \)th qubit must change its quantum number and the same \( i \)th mode must change its number of photons in one unit. Therefore, the parity of that part will be the same.

Because of these two facts, the parities of the kets forming the superposition that is the evolution of the vacuum will never mix. Therefore, the state must remain in the subspace of vectors with well-defined qubit plus mode excitation parity. The operator that projects onto the subspace of vectors with that well-defined excitation parity is
\[ P = \frac{3}{\beta} \sum_{\beta=0}^{\infty} P(2\beta) \otimes P_{2,2}(0) \]
\[ + \frac{3}{\beta} \sum_{\beta=0}^{\infty} P(2\beta + 1) \otimes P_{2,2}(0) \]
\[ + \frac{3}{\beta} \sum_{\beta=0}^{\infty} P(2\beta) \otimes P_{2,2}(1) \]
\[ + \frac{3}{\beta} \sum_{\beta=0}^{\infty} P(2\beta + 1) \otimes P_{2,2}(1) \]
\[ = \sum_{\alpha, \beta=0}^{\infty} \sum_{\alpha=0}^{\infty} P(2\alpha + \beta) \otimes P_{2,2}(\beta), \]
the projector onto the \( S_z \) lower eigenstate if \( q = 0 \) or onto the higher eigenstate if \( q = 1 \).

2. Proof that \( P \) is a conserved quantity

In this section we let the \( i \) indices drop as they are redundant notation. The projector \( P \) clearly commutes with the Hamiltonian’s stationary part. In order to prove that it commutes with the interacting pieces as well, we need to introduce some notation

\[
x_{1,2,3} \rightarrow \otimes_{i=1}^{3} x \otimes \mathbb{I}_{2 \times 2},
\]

\[
\sigma_{x} \rightarrow \otimes_{j=1}^{3} [\delta_{ij} x \otimes \sigma_{x} + (1 - \delta_{ij}) I \otimes \mathbb{I}_{2 \times 2}].
\]

First, we will show that \( x_{1,2,3} \) commutes with \( P \),

\[
x_{1,2,3}P = \sum_{\alpha, \beta = 0}^{1} \sum_{n=0}^{\infty} \left( \sum_{i=1}^{3} P(2n + \alpha) \otimes P_{2 \times 2}(\beta) \right) = \sum_{\alpha, \beta = 0}^{1} \sum_{n=0}^{\infty} xP(2n + \alpha) \otimes P_{2 \times 2}(\beta)
\]

\[
= \sum_{\alpha, \beta = 0}^{1} \sum_{i=1}^{3} \sum_{n=0}^{\infty} (\sqrt{2n + \alpha} 2n + \alpha - 1)(2n + \alpha) \otimes P_{2 \times 2}(\beta)
\]

\[
= \sum_{\alpha, \beta = 0}^{1} \sum_{i=1}^{3} \sum_{n=0}^{\infty} \sqrt{2n + \alpha} 2n + \alpha - 1)(2n + \alpha) \otimes P_{2 \times 2}(\beta)
\]

where we understand that if \( 2n + \alpha - 1 < 0 \) then \( 2n + \alpha - 1 = 0 \). We have split the summation on \( n \) into two different summations. We will perform a change of variables in the first one so that \( n \rightarrow n + 1 \). Note that in that case the summation index starts at \(-1\),

\[
x_{1,2,3}P = \sum_{\alpha, \beta = 0}^{1} \sum_{i=1}^{3} \left( \sum_{n=1}^{\infty} \sqrt{2n + \alpha} 2n + \alpha + 1)(2n + \alpha) \otimes P_{2 \times 2}(\beta) \right)
\]

Now compare both summations over \( n \). They contain the same ket \( |2n + \alpha + 1\rangle \) and have different coefficients and bras. Those coefficients and bras match the result of applying the \( x \) operator to the projector \( P(2n + \alpha + 1) \) from the right. Therefore,

\[
x_{1,2,3}P = \sum_{\alpha, \beta = 0}^{1} \sum_{i=1}^{3} \left( \sqrt{2n + \alpha} 2n + \alpha + 1)(2n + \alpha) \otimes P_{2 \times 2}(\beta) \right)
\]

The term \( \sqrt{2n + \alpha} \) is due to one of the summations over \( n \) starting at \( n = -1 \). That term, however, is different from zero only when \( \alpha = 1 \). To regroup the term with the rest of the summations it is easier to study the cases \( \alpha = 0 \) and \( \alpha = 1 \) separately,

\[
x_{1,2,3}P = \sum_{\alpha, \beta = 0}^{1} \sum_{i=1}^{3} \left( \sum_{n=0}^{\infty} P(2n + 1 + \alpha)x \otimes P_{2 \times 2}(\beta) \right) + \sum_{\alpha, \beta = 0}^{1} \sum_{i=1}^{3} \left( \sum_{n=0}^{\infty} P(2n + 2 + \alpha)x \otimes P_{2 \times 2}(\beta) \right)
\]

The second line of the one representing the case \( \alpha = 1 \). Note that \( \langle 0 | 1 \rangle \) is the result of applying \( x \) to the projector \( |0 \rangle \) from the right. Additionally, we can change the variable in the summation on \( n \) so that \( n \rightarrow n + 1 \) and put \( P(0)x \) together with the rest of the summation

\[
x_{1,2,3}P = \sum_{\alpha, \beta = 0}^{1} \sum_{i=1}^{3} \left( \sum_{n=0}^{\infty} P(2n + 1 + \alpha)x \otimes P_{2 \times 2}(\beta) \right) + \sum_{\alpha, \beta = 0}^{1} \sum_{i=1}^{3} \left( \sum_{n=0}^{\infty} P(2n + 2 + \alpha)x \otimes P_{2 \times 2}(\beta) \right)
\]

Finally, this expression can be formulated in terms of a new summation over \( \alpha \),

\[
x_{1,2,3}P = \sum_{\alpha, \beta = 0}^{1} \sum_{i=1}^{3} \left( \sum_{n=0}^{\infty} P(2n + \alpha)x \otimes P_{2 \times 2}(\beta) \right) = P_{x_1} x_{2,3}.
\]

Therefore, \( [P, x_{1,2,3}] = 0 \).

We are missing a second step to prove that \( P \) is a conserved quantity: It has to commute with the interaction Hamiltonians of the qubits and modes. In order to do so, we will prove that \( x_{i} \sigma_{x,i} P = P_{x} \sigma_{x,i} \),
\[ x_0 \sigma_x P = \left( \bigotimes_{j=1}^{3} \delta_{ij} \sigma_x + (1 - \delta_{ij}) \mathbb{1} \otimes 1_{2 \times 2} \right) \times \left( \sum_{\alpha, \beta = 0}^{1} \sum_{n=0}^{\infty} P(2n + \alpha) \otimes P_{2 \times 2}(\beta) \right) \]

\[ = \sum_{\alpha, \beta = 0}^{1} \sum_{j=1}^{3} \sum_{n=0}^{\infty} \left[ \delta_{ij} \sqrt{2n + \alpha + 1} \right] \left( 2n + \alpha + 1 \right) \} P(2n + \alpha) \otimes \sigma_x P_{2 \times 2}(\beta) \right] (1 - \delta_{ij}) P(2n + \alpha) \otimes P_{2 \times 2}(\beta) \right] .
\]

Now we will study the action of \( x \) on \( P(2n + \alpha) \),

\[ x_0 \sigma_x P = \sum_{\alpha, \beta = 0}^{1} \sum_{j=1}^{3} \sum_{n=0}^{\infty} \left[ \delta_{ij} \sqrt{2n + \alpha + 1} \right] \left( 2n + \alpha + 1 \right) \} P(2n + \alpha) \otimes \sigma_x P_{2 \times 2}(\beta) \right] (1 - \delta_{ij}) P(2n + \alpha) \otimes P_{2 \times 2}(\beta) \right] .
\]

As it happened with \( x_1 x_2 x_3 P \), we will make a change in the variable \( n \) so that \( n \rightarrow n + 1 \) only in the first line,

\[ x_0 \sigma_x P = \sum_{\alpha, \beta = 0}^{1} \sum_{j=1}^{3} \sum_{n=0}^{\infty} \left[ \delta_{ij} \sqrt{2n + \alpha + 1} \right] \left( 2n + \alpha + 1 \right) \} P(2n + \alpha) \otimes \sigma_x P_{2 \times 2}(\beta) \right] (1 - \delta_{ij}) P(2n + \alpha) \otimes P_{2 \times 2}(\beta) \right] .
\]

The same way as before, the summation can be rewritten in terms of \( P(2n + \alpha + 1) \) acting on \( x \),

\[ x_0 \sigma_x P = \sum_{\alpha, \beta = 0}^{1} \sum_{j=1}^{3} \sum_{n=0}^{\infty} \left[ \delta_{ij} \sqrt{2n + \alpha + 1} \right] \left( 2n + \alpha + 1 \right) \} P(2n + \alpha) \otimes \sigma_x P_{2 \times 2}(\beta) \right] (1 - \delta_{ij}) P(2n + \alpha) \otimes P_{2 \times 2}(\beta) \right] .
\]

Now we will study the action of \( \sigma_x \) on \( P_{2 \times 2}(\beta) \),

\[ \sigma_x P_{2 \times 2}(\beta) = |\beta - 1\rangle \langle \beta | + |\beta + 1\rangle \langle \beta | ,
\]

where we understand that if \( \beta - 1 < 0 \) then \( |\beta - 1\rangle = 0 \) and if \( \beta + 1 > 0 \) then \( |\beta + 1\rangle = 0 \). Plugging this equation into the last expression for \( x_0 \sigma_x P \) results in

\[ x_0 \sigma_x P = \sum_{\alpha, \beta = 0}^{1} \sum_{j=1}^{3} \sum_{n=0}^{\infty} \left[ \delta_{ij} \sqrt{2n + \alpha + 1} \right] \left( 2n + \alpha + 1 \right) \} P(2n + \alpha) \otimes \sigma_x P_{2 \times 2}(\beta) \right] (1 - \delta_{ij}) P(2n + \alpha) \otimes P_{2 \times 2}(\beta) \right] .
\]

By doing two different changes of variable in \( \beta \) for each of the terms \( |\beta - 1\rangle \langle \beta | \) and \( |\beta + 1\rangle \langle \beta | \) and realizing that only one of those is nonzero for a particular value of \( \beta \), we conclude that

\[ x_0 \sigma_x P = \sum_{\alpha, \beta = 0}^{1} \sum_{j=1}^{3} \sum_{n=0}^{\infty} \left[ \delta_{ij} \sqrt{2n + \alpha + 1} \right] \left( 2n + \alpha + 1 \right) \} P(2n + \alpha) \otimes \sigma_x P_{2 \times 2}(\beta) \right] (1 - \delta_{ij}) P(2n + \alpha) \otimes P_{2 \times 2}(\beta) \right] .
\]

Finally, we study the cases \( \alpha = 0 \) and \( \alpha = 1 \) separately,
Again, in the $\alpha = 1$ case we can regroup the matrix element $|0\rangle\langle 1|$ as $P(0|x)$ and combine it with the summation on $n$, 
\[
  x_i \sigma_x |P \rangle = \sum_{\beta=0}^{1} \sum_{j=1}^{3} \left( \delta_{ij} \sum_{n=0}^{\infty} P(2n+1) \sigma_j (0) \otimes P_{2 \times 2}(\beta) \sigma_x + (1-\delta_{ij}) \sum_{n=0}^{\infty} P(2n) \otimes P_{2 \times 2}(\beta) \right)
\]
\[
+ \sum_{\beta=0}^{1} \sum_{j=1}^{3} \left( \delta_{ij} \sum_{n=0}^{\infty} P(2n+1) \otimes P_{2 \times 2}(\beta) \sigma_x + (1-\delta_{ij}) \sum_{n=0}^{\infty} P(2n) \otimes P_{2 \times 2}(\beta) \right).
\]

This expression can be condensed again in a summation over $\alpha$ so that 
\[
x_i \sigma_x |P \rangle = \sum_{\beta=0}^{1} \sum_{j=1}^{3} \left( \delta_{ij} \sum_{n=0}^{\infty} P(2n+1) \sigma_j (0) \otimes P_{2 \times 2}(\beta) \sigma_x + (1-\delta_{ij}) \sum_{n=0}^{\infty} P(2n) \otimes P_{2 \times 2}(\beta) \right).
\]

Therefore, we have proven that $[x_i \sigma_x, |P \rangle] = 0$.

Summarizing, the projector $P$ as defined in Eq. (4) commutes with each of the ingredients that compose the full 3SPDC + 3qubits Hamiltonian of Eq. (1). We conclude that $P$ is a conserved quantity, and since the initial value of $(P)$ for the initial state of vacuum $|0\rangle|0\rangle|0\rangle|0\rangle$ is 1, it must remain one at all times. In other words, the state remains in the subspace that the projector $P$ projects on at all times, regardless of whether or not the RWA is taken on any interaction:

\[
\psi(t) = \sum_{\alpha,\beta=0}^{1} \sum_{i=1}^{3} \sum_{n=0}^{\infty} c_{\alpha,\beta,i,n}(t) (2n+\alpha) \otimes |\beta\rangle.
\]  
(C2)

We define the dynamical subspace as the subspace that contains $\psi$ at all times, that is, the image of $P$.

3. Some expectation values in the dynamical subspace

With a closed expression of the dynamical subspace, that is, the subspace that contains the time evolution of vacuum under the Hamiltonian (prior to any RWA), it is possible to compute some expectation values, in particular single, pairs, and triplets of ladder operators, involving the fields or the qubits.

The expectation values of single creation operators on the modes are zero. In Eq. (C2) all eigenbras of the superposition $\psi(t)$ will be orthogonal to all eigenkets of that same superposition if a photon is added to each of them. That is, the $a_i^\dagger$ operator will produce kets with mixed parities and there are no bras at the other side of the expectation value with mixed parities. A similar argument holds for the annihilation operators on each mode, 
\[
  \langle a_i \rangle = \langle a_i^\dagger \rangle = 0.
\]

The expectation values of single creation operators on the qubits are zero too, because of the same argument 
\[
  \langle \sigma_i^+ \rangle = \langle \sigma_i^- \rangle = 0.
\]

The expectation values of pairs of creation or annihilation operators on modes are zero only if they act on different modes. If that is the case, the result is zero because of the same argument as before. If the operators act on the same mode, we are talking about the expectation value of the number operator, which must not be zero, as there is photon generation and that operator does not mix parities of the kets.
\[
  \langle a_i^\dagger a_j^\dagger \rangle = \langle a_i a_j \rangle = \langle a_i a_j^\dagger \rangle = \langle a_i a_j^\dagger \rangle = 0 \quad \text{provided } i \neq j.
\]

The expectation values of pairs of ladder operators on the qubits are zero if and only if they act on different qubits, because of the same argument as for the modes, 
\[
  \langle \sigma_i^+ \sigma_i^- \rangle = \langle \sigma_i^- \sigma_i^+ \rangle = \langle \sigma_i^- \sigma_i^- \rangle = \langle \sigma_i^+ \sigma_i^+ \rangle = 0 \quad \text{provided } i \neq j.
\]

The expectation values of pairs of ladder operators on one mode and on one qubit are zero only if the former acts on a mode that does not interact with the qubit the latter acts on, that is, 
\[
  \langle a_i^\dagger \sigma_j^\dagger \rangle = \langle a_i^\dagger \sigma_j^- \rangle = \langle a_i \sigma_j^\dagger \rangle = \langle a_i \sigma_j^- \rangle = \langle a_i \sigma_j^- \rangle = 0 \quad \text{provided } i \neq j.
\]

The reason is the same as before: Each operator will change the parity of two different pairs of modes and qubits, but will leave one pair with the previous parity.

The expectation values of triplets of ladder operators on the modes are zero as long as they act on two modes. If that is the case, one of the ladder operators acts on one mode, and by the same argument as before, that expectation value must be zero, 
\[
  \langle a_i^\dagger a_i a_j \rangle = 0 \quad \text{provided } i \neq j.
\]

With these expressions we have enough information to prove that the covariances in the fields’ canonical variables and qubits’ $x$ and $y$ spin variables are constant in time.

APPENDIX D: THE $z$ SPIN COVARIANCES ALONE ARE NOT GAUSSIAN ENTANGLEMENT

In this Appendix we will prove that any three-qubit mixed state that has the same $x$ and $y$ covariances as a separable state and only different $z$ spin covariances has no Gaussian entanglement. The argument is very similar to those presented before: Separable states have access to a particular range of values of the $z$ spin covariance. If general three-qubit states have access to a bigger range of the $z$ spin covariances, then a Gaussian witness paying attention to only the $z$ covariances could report entanglement; however, if the separable and general ranges are the same, then no witness can tell the difference between those states with only one covariance. Then a state that differs only in those $z$ covariances from a separable state,
as is the case of the qubits state in the main text, cannot contain Gaussian entanglement.

For separable states the bound on the $z$ spin covariances is given by classical probability theory, in particular, the Cauchy-Schwarz and Popoviciu inequalities

$$|\Delta^2 S_{O} S_{O'}| \leq \sqrt{\Delta^2 O_1 \Delta^2 O'},$$

$$\leq \frac{1}{4} (\text{sup } O_i - \inf O_i)(\text{sup } O_j - \inf O_j),$$

where sup $O$ and inf $O$ are bounds to the values a measurement of the observable $O$ may take. In particular, for spin variables we have

$$|\Delta^2 S_{0} S_{1}| \leq \frac{1}{4}.$$  

The question remains whether this classical bound can be violated by some entangled state. The reader might suspect that the answer is negative, as in the many years of research on entanglement there are no Bell-like inequalities or witnesses built from covariances on only one axis. To prove that intuition, consider a pure two-qubit state $\psi = \sum_{i=0}^{1} \sum_{j=0}^{1} c_{ij} |q_i, q_j\rangle$ and the fact that the covariances of the spin variables can be expressed in terms of the covariance of the excitation projector’s covariance

$$\Delta^2 S_{O} S_{O'} = \Delta^2 P_{q_1} P_{q_2},$$

$$= (1 - |c_{00}|^2 - |c_{10}|^2) |c_{11}|^2 - |c_{11}|^4 - |c_{01}|^2 |c_{10}|^2,$$

where $P_{q_i}$ is the projector onto the excited state of the $i$th qubit and $c_{ij}$ are the coefficients of a two-qubit state in the computational basis. It is a simple exercise to find the pure two-qubit state that maximizes the covariance, which is a Bell state $\psi = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$, which yields a covariance $\Delta^2 S_{0} S_{1} \geq \frac{1}{4}$. Two-qubit mixed states cannot violate this bound and the expectation value of a mixture is never larger than the largest of its pure components. General systems that contain two qubits cannot beat this bound either, as their expectation values will be the same as those of the reduced density matrix on the two qubits.

Therefore, we have proven that no witness will be able to report entanglement by inspecting the $z$ covariances alone and a state that differs from a separable state only in those covariances will not contain Gaussian entanglement.

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4.2 Conclusions

This chapter 4, that revolves around the results presented in [3], strengthens our results about the non-Gaussian nature of the entanglement in the 3SPDC system presented in chapter 2. Additionally it generalizes our results to DV systems, displaying interesting analogies with the W and GHZ states, as well as proposing a device capable of producing non-Gaussian entanglement between qubits. That entanglement is reported in a wider parameter regime. Both facts could enhance the technological significance of non-Gaussian entanglement.

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