UNIQUENESS OF STEADY STATE, SMOOTH SHAPES IN A NONLOCAL GEOMETRIC PDE

ANDRÁS A. SIPOS

Abstract. We investigate steady state solutions of a nonlocal geometric PDE that serves as a simple model of simultaneous contraction and growth of grains called ooids in geosciences. As a main result of the paper we demonstrate that the parameters associated with the physical environment determine a unique, steady state solution of the equation among smooth, convex curves embedded in $\mathbb{R}^2$. It is also revealed that any steady state solution possesses D2 symmetry.

Key words. nonlocal PDE, shape evolution, uniqueness of steady state solutions

AMS subject classifications. 35Q86, 35B06, 34A26

1. Introduction. We consider a geometric, non-local PDE to model the shape evolution of mm-sized grains typically formed in shallow tropical seas called ooids in geosciences. Shape evolution of particles is widely investigated both in the mathematical and in the geoscientific literature (e.g. [2, 3] and the citations therein). Most of the treated models are local ones, i.e. the evolution is determined by some pointwise law, for instance the curve-shortening flow [5] is a good example for such a model in two spatial dimensions. Having a strictly inward flow a steady state solution can be considered via some rescaling (like fixing the area or the arch-length of the curve) [6]. Another way of investigating some particular shapes is to track the flow in the backward infinite time limit [4, 1]. In case the direction of the flow is not prescribed a-priori, i.e shrinkage or growth either take effect during the evolution, one may inquire about the existence (and some properties) of a bounded (finite) steady state shape. Regarding the proposed model we address such a question.

The model investigated in this paper grabs the three crucial physical effects of ooid-growth: a chemical process leading to radial accumulation of material, abrasion of the grain due to collisions to the seabed and finally sliding (friction) which takes effect at shallow shores, which landform is widely anticipated as the principal venue for ooid formation. While the velocity of growth is independent of the size the particle, abrasion and friction both governed by mass-dependent laws. Whence the material quantity in the particle cannot be omitted from a realistic model; it should be a non-local one. In this paper we focus on two spatial dimensions, non-locality manifests as an area-dependent speed of contraction.

This paper is devoted to the rigorous investigation of the existence and uniqueness of steady state solutions of the model in $\mathbb{R}^2$ establishing further work aiming to compare model predictions against observable shapes in nature.

2. Description of the model. Shape evolution might be interpreted as a process that moves any point of a closed, non-self-intersecting curve $\Gamma$ embedded in $\mathbb{R}^2$ to the normal direction with a velocity depending on some physical features of the environment. In our model the evolution of $\Gamma$ is defined via

$$\Gamma_t = c_3 (-1 + c_1 A\kappa + c_2 A\gamma \cos \gamma) \mathbf{n},$$

where $A$ is the - time dependent - area enclosed by $\Gamma$ and the $t$ subscript refers to differentiation respect to time. $\kappa$ and $\mathbf{n}$ stand for the curvature and the unit (inward
directed) normal of the curve at time \( t \), respectively. Without loss of generality we assume the curve possesses a unique maximal diameter (line \( e \) between points \( P \) and \( P' \) in Fig. 1.), which is designated to be the \( x \) axis of an orthogonal basis located at the middle point of the \( PP' \) segment. \( \gamma \) denotes the angle between the \( x \)-direction and the local tangent to the curve. \( c_1, c_2 \) and \( c_3 \) are positive real parameters of the problem associated with the physical environment and they are assumed to be time-independent during the evolution. The three key physical components of the proposed model can be easily identified: in the brackets the first, negative term stands for the \textit{growing} of the particle, in the second term \textit{abrasion} is assumed to be a curvature-driven process and finally the affine term is associated with \textit{friction}. As the right-hand side of eq. (2.1) consists both positive and negative terms, a natural question arises as is there any steady state solution \( \Gamma^* \) of the flow? In specific, we seek shapes that fulfill

\[
-1 + c_1 A \kappa + c_2 A y \cos \gamma = 0,
\]
whole along the curve \( \Gamma \). Note, that the steady state shape is independent of \( c_3 \) as it scales solely the time and cannot be reconstructed by pure observation of \( \Gamma^* \). In further work we intend to investigate the question, whether the parameter pair \( (c_1, c_2) \) can be reproduced just from the observation of a steady state shape (in case it exists)? In this paper we demonstrate, that among smooth, convex curves any steady state shape must possess \( D_2 \) symmetry and for a given parameter pair \( (c_1, c_2) \) this shape is unique, thus the answer for the question is positive.

\textbf{Proposition 2.1.} Any smooth, convex, steady state solution \( \Gamma^* \) of eq. (2.1) with positive parameters \( (c_1 > 0 \text{ and } c_2 > 0) \) embedded in \( \mathbb{R}^2 \) possesses \( D_2 \) symmetry.

\textbf{Proposition 2.2.} Smooth, convex, steady state solutions of eq. (2.1) are uniquely determined by \( c_1 \) and \( c_2 \), and for any positive values of the two parameters there exists a \( \Gamma^* \) curve.

We prove the first Proposition in Section 3. where we assume that \( A \) is known \textit{a-priori}, this case is refereed as a \textit{local equation} to distinguish it from the general, \textit{non-local equation}. Section 4. is devoted to prove Proposition 2.2. via a bijective mapping between the parameter spaces of the local and non-local equations.

\textbf{3. Proof of Proposition 2.1.} For a moment let us assume that the area of the invariant curve is known \textit{a-priori}. As we investigate smooth, closed curves without self-intersections the derivation can be substantially simplified (without loss of generality) by considering solely the curve segment \( \overline{\Gamma} \) between the leftmost point \( P \) and the one that possesses a horizontal tangent and a positive \( y \) coordinate. This latest is point \( Q \). In order to simplify the derivations we use several parametrizations of the
curve segment in the sequel: the natural parametrization respect to the arc length, the parametrization respect to the $y$ coordinate and finally the parametrization respect to the $\gamma$ inclination of the tangent of the curve.

In this section we employ the parametrization of $\bar{\Gamma}$ respect to $y$, and $(\cdot)'$ refers to the first derivative respect to $y$. By this parametrization equation (2.2) can be written as

\begin{equation}
-1 + c_1 A \kappa(y) + c_2 A y \cos(\gamma(y)) = 0,
\end{equation}

where there is a triple of parameters $(c_1, c_2, A)$, all of them assumed to be fixed. As both $c_1$ and $c_2$ is multiplied by $A$, a convenient notation is defined via $\hat{c}_1 = c_1 A$ and $\hat{c}_2 = c_2 A$ which renders eq. (3.1) to

\begin{equation}
-1 + \hat{c}_1 \kappa(y) + \hat{c}_2 y \cos(\gamma(y)) = 0.
\end{equation}

We aim to rewrite this equation to make it solely depend on $\gamma(y)$ and its derivatives. This step is similar to the case of the curve shortening flow: there an equation solely depending on the curvature $\kappa$ reveals important features [4], here the form with $\gamma(y)$ provides the most convenient choice. (Nonetheless, an ODE with $\kappa(y)$ as the unknown function can be obtained as well.)

For a moment we reconsider the natural parametrization of the curve. As we investigate two, arbitrary close points along the curve, by the chain rule we derive the following expression between $\kappa$ and $\gamma$ (using the fact, that the derivative of the slope respect to the arch length equals the curvature):

\begin{equation}
\kappa(y) = -\frac{d\gamma}{ds} = -\frac{d\gamma}{dy} \frac{dy}{ds} = -\gamma'(y) \sin(\gamma(y)).
\end{equation}

Note, that the negative sign relates to the fact, that by definition $\gamma(y)$ is decreasing between points $P$ and $Q$ (Fig. 1. b)). In the virtue (3.3) eq. (3.2) takes the following form, which is a first order, nonlinear ODE:

\begin{equation}
-1 - \hat{c}_1 \gamma'(y) \sin(\gamma(y)) + \hat{c}_2 y \cos(\gamma(y)) = 0.
\end{equation}

From now on this equation is called local. There exist a closed-form solution for the local equation:

\begin{equation}
\gamma(y) = \arccos \left( \frac{\sqrt{\pi} \text{erf} \left( \sqrt{\frac{2}{\hat{c}_1}} \hat{y} \right) - C i \sqrt{\frac{2}{\hat{c}_1}} \hat{c}_1 \exp \left( \frac{i \hat{c}_2 y^2}{2 \hat{c}_1} \right)}{\hat{c}_1 \sqrt{\frac{2 \hat{c}_1}{\hat{c}_1}}} \right),
\end{equation}

where $i = \sqrt{-1}$ and the error function is given by its usual definition, $\text{erf}(x) := 2\sqrt{\pi}^{-1} \int_0^x \exp(-t^2)dt$. Formal substitution verifies that this expression up to the arbitrary constant $C$ solves the local equation. In this text we focus on smooth curves with $\gamma(0) = \pi/2$ at point $P$, which restricts $C = 0$. (It means $C \neq 0$ opens the gate for non-smooth, steady state shapes with two or more vertices in case the equation is assumed to apply on smooth segments of a piecewise smooth curve.) Substitution of the solution in eq. (3.5) into the right-hand-side of (3.3) yields

\begin{equation}
\kappa(y) = \frac{1}{\hat{c}_1} + \frac{\sqrt{\pi}}{\hat{c}_1} \sqrt{\frac{\hat{c}_2}{2 \hat{c}_1}} \exp \left( -\frac{\hat{c}_2 y^2}{2 \hat{c}_1} \right) \text{erf} \left( \sqrt{\frac{\hat{c}_2}{2 \hat{c}_1}} \hat{y} \right) \hat{y}.
\end{equation}
This is the unique solution of the local equation, it can be demonstrated by a routine technique (i.e. demonstrating contradiction from the assumption about another solution which is not given by (3.6)). Detailed investigation of the properties of \( \kappa(y) \) is needed for further development as these govern properties of the steady state solutions. First we carry out a convenient change of parameters via

\[
q := \sqrt{\frac{c_2}{2c_1}},
\]

whence the solution (keeping \( C = 0 \)) in (3.6) can be reformulated as

\[
\kappa(y) = \frac{1}{c_1} \left( 1 + \sqrt{\pi} \frac{\text{erf}(qyi)}{\exp(q^2 y^2)} qyi \right)
\]

The following properties of \( \kappa(y) \) can be settled (proof is provided in the Appendix):

1. \( \kappa(y) \) is real (\( \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R} \)).
2. \( \kappa(y) \) is continuous.
3. \( \kappa(0) \) is positive and equals \( \hat{c}_1^{-1} \).
4. \( \kappa(y) \) has a maximum at \( y = 0 \).
5. \( \kappa(y) \rightarrow 0 \) as \( y \rightarrow \infty \).
6. There is exactly one point, denoted to \( y_0 \), where \( \kappa(y) \) vanishes and \( y_0 \) solely depends on \( q \).
7. There is no local extrema for \( \kappa(y) \) between \( 0 < y < y_0 \), thus it is monotonic in this range.

To realize a steady state shape \( \Gamma^\ast \) we need \( \gamma(y) \) itself. By the virtue of (3.3) \( \gamma(y) \) fulfills

\[
\gamma(y) = \arccos \left( \int_0^y \kappa(\eta) \, d\eta \right).
\]

Since \( \arccos(x) \) is monotonic decreasing in \([0, 1]\), the area below the solution function \( \kappa(y) \) determines \( \gamma \). In other words \( \hat{c}_1 \) and \( q \) (i.e. \( \hat{c}_1 \) and \( \hat{c}_2 \)) determine a unique steady state solution of eq. (3.2), we aim to determine the parameter range, where the curve associated with the solution in smooth. Apparently, if the area under \( \kappa(y) \) between \( 0 \leq y \leq y_0 \) exceeds 1, then we can construct a smooth shape: at the unique \( \tilde{y} < y_0 \) the area below \( \kappa(y) \) equals 1, i.e. this corresponds to point Q with a tangent parallel to the axis \( x \). For \( 0 \leq y \leq \tilde{y} \) the connection between \( \gamma \) and \( y \) is one to one, thus we can draw the physical realization. For cases, at which the area below \( \kappa(y) \) is smaller than 1, the physical shapes are non-smooth (in fact, they become concave as the curvature flips sign above \( y_0 \) and there is no other zero for \( \kappa(y) \)). As we have seen, \( \kappa(0) \) depends solely on \( \hat{c}_1 \) and for fixed \( q \) the value of \( y_0 \) is fixed, too. This leads to the conclusion that for any fixed \( q \) there exist a \( \hat{c}_{1, \text{crit}} \), critical value at which

\[
\int_0^{y_0} \kappa(\eta) \, d\eta = 1.
\]

For further convenience for a fixed \( q \) we define the set

\[
\chi_q := \{ \hat{c}_1 \mid 0 < \hat{c}_1 \leq \hat{c}_{1, \text{crit}} \}
\]

It follows, that for \( \hat{c}_1 > \hat{c}_{1, \text{crit}} \) the integral on the left-hand-side of (3.10) is smaller than one which means the associated curve cannot have a horizontal tangent at any
point. Having assumed convex, smooth curves this parameter-range is not in our interest. In case \( \hat{c}_1 \in \chi_q \), the integral on the left-hand side is bigger than 1 thus the shape can be realized. As \( y \) provides a possible parametrization of the curve segment \( \hat{\Gamma} \), the unique closed form solution in (3.5) can be realized as a unique curve in \( \mathbb{R}^2 \). Finally we prove uniqueness for \( \Gamma^* \) itself. So far we know that for a proper \( \hat{c}_1 \) and \( \hat{c}_2 \) the curve segment \( \hat{\Gamma}^* \) is uniquely determined. Note, that \( \hat{\Gamma}^* \) has vertical tangent at \( P \) and horizontal tangent at \( Q \). As we consider smooth shapes the only way to glue the \( \hat{\Gamma}^* \) curve-segments to form a closed, non-intersecting curve is reflection along the \( x \) and \( y \) axes. It clearly hints to that a smooth steady state shape must possess \( D_2 \) symmetry.

It is worthy to remark that for \( \hat{c}_2 = 0 \) we have \( \kappa(y) = 1/(\hat{c}_1) = \kappa(0) \) implying that in this case the steady state shape is a circle. As the term of friction (the one with parameter \( \hat{c}_2 \)) represents an affine flow, a first intuition says that the steady state curve should be an ellipse. In the appendix we show that this is not the case, ellipses are not steady state as long as \( \hat{c}_2 \) is strictly positive.

Solution of the local equation establish a solution for the non-local case (eq. 2.2) as well. To see this, let us fix the two parameters, \( \hat{c}_1 \) and \( \hat{c}_2 \), follow the lines in this section to obtain a steady state solution \( \Gamma^* \). In case there exists such a solution, measure the \( A \) area enclosed by the curve. It simply leads to the parameters of the non-local equation via \( c_1 = \hat{c}_1/A \) and \( c_2 = \hat{c}_2/A \). In the other way round, if one knows a steady state solution of the non-local equation, calculation of the parameters in the local is straightforward. These observations imply that a smooth solution of the non-local case must possess \( D_2 \) symmetry as well and it completes the proof. In the next section we investigate the connection between the local and non-local models via the relations between their parameters.

4. Proof of Proposition 2.2. We turn to investigate steady state solutions of the non-local equation (2.2). As we found that they possess \( D_2 \) symmetry, we keep investigating a curve segment \( \hat{\Gamma} \) (c.f. Figure 1.). To investigate uniqueness of solutions in (2.2) let us assign \((\hat{c}_1, \hat{c}_2)\) and \((c_1, c_2)\) if they result in an identical curve as a steady state solution of the proper model (i.e. it fulfills 3.4) in case \((\hat{c}_1, \hat{c}_2)\) and (2.2) in case \((c_1, c_2)\)). In this sense we can talk about a mapping between the parameter spaces. Observe, that

\[
\sqrt{\frac{\hat{c}_2}{2} \frac{A}{\hat{c}_1}} = q = \sqrt{\frac{c_2}{2c_1}}
\]

holds, implying that \( q \) is invariant under the above mentioned map. In order to facilitate this observation, instead of \( \hat{c}_2 \) and \( c_2 \) we use \( q \) as one of the parameters in the problem. Based on the previous section, in the local model only \( \hat{c}_1 \in \chi_q \) can result in a smooth curve. Let us formally define the map \( F \) at a fixed value of \( q \) as:

\[
F : \chi_q \rightarrow \mathbb{R}^+ \\
\hat{c}_1 \rightarrow c_1.
\]

Our program is to show that \( F \) is injective and surjective, thus it is bijective implying smooth solutions of the non-local equation are unique as we had uniqueness of solutions for eq. (3.4).

4.1. \( F \) is injective. As we have seen, \( \hat{c}_1 \in \chi_q \) results in a smooth curve enclosing some positive area \( A \) in \( \mathbb{R}^2 \). Based on our construction, \( c_1(\hat{c}_1) := \hat{c}_1A^{-1} \) can be
readily computed. It means, injectivity of \( F \) hangs on strict monotonicity of the \( c_1(\tilde{c}_1) \) function over \( \chi_q \). To prove this, let us consider two smooth solutions (at a fixed value of \( q \)) of the local equation in (3.4) identified by the letters \( i \) and \( j \). Let us relate their parameters via

\[
\hat{c}_1^j = (1 + \varepsilon)\hat{c}_1^i,
\]

where without loss of generality \( \varepsilon > 0 \). By the virtue of eq. (3.8) is is clear, that not just the parameters, but the \( \kappa(y) \) functions behind the steady state curve segments \( \tilde{\Gamma}_i^* \) and \( \tilde{\Gamma}_j^* \) are related as

\[
\kappa^j(y) = \frac{1}{1 + \varepsilon} \kappa^i(y).
\]

**Fig. 2.** Comparison of two, steady state curve-segments, \( \tilde{\Gamma}_i^* \) and \( \tilde{\Gamma}_j^* \) at a fixed \( q \). a) depicts the two segments and denote the arbitrary point-pair with a fix \( \gamma_0 \), which is used to determine the relation between the areas under the curve-segments. b) the graphs of \( \kappa(y) \) curvature functions for \( \tilde{\Gamma}_i^* \) and \( \tilde{\Gamma}_j^* \), respectively.

We choose two points along \( \tilde{\Gamma}_i^* \) and \( \tilde{\Gamma}_j^* \) (Fig. 2.), one for each, such way that their tangent direction, \( \gamma_0 \) is identical, the \( (\cdot) \) will refer to any quantity evaluated at these points (e.g. \( \tilde{y}^i \) is the parameter of the curve at the chosen point along \( \tilde{\Gamma}_i^* \)). As \( \gamma(y) \) is monotonic along \( \tilde{\Gamma} \), the position of the two points is well-defined. As we have seen in the previous Section \( \kappa(y) \) and \( \gamma(y) \) is related via (3.9), thus for our two curves we see, that

\[
\int_0^{\tilde{y}^i} \kappa^i(\eta)d\eta = \cos(\gamma_0) = \int_0^{\tilde{y}^j} \kappa^j(\eta)d\eta
\]

must hold, which by eq. (4.4) implies \( \tilde{y}^i < \tilde{y}^j \). By the properties of \( \kappa(y) \) and (4.4) it is easy to see, that the curvatures at the chosen point-pair must fulfill

\[
\kappa^i(\tilde{y}^i) > (1 + \varepsilon)\kappa^j(\tilde{y}^j),
\]

because their parameter fulfill \( \tilde{y}^i < \tilde{y}^j \). From this observation and the positivity of all the involved quantities we conclude, that

\[
\frac{\tilde{y}^i}{(1 + \varepsilon)\kappa^j(\tilde{y}^j)} > \frac{\tilde{y}^i}{(\kappa^j(\tilde{y}^j))}.
\]
We switch to the parametrization of \( \bar{\Gamma} \) respect to the tangent direction \( \gamma \). Based on eq. (3.3) we see, that the \( \bar{A} \) area under \( \bar{\Gamma} \) can be computed as

\[
\bar{A} = \int_0^{\pi/2} \frac{d}{d\gamma} \cos(\gamma) g(\gamma) d\gamma = \int_0^{\pi/2} \frac{y(\gamma)}{\kappa(\gamma)} \cos(\gamma) d\gamma.
\]

As we have demonstrated in (4.7), the argument of the integral in the RHS of (4.8) is smaller for \( \bar{\Gamma}^*_i \) than for \( \bar{\Gamma}^*_j \), and this holds for any \( \gamma \in (0, \pi/2) \), whence we conclude

\[
\frac{1}{1 + \varepsilon} \bar{A}^j = \int_0^{\pi/2} \frac{y^j(\gamma)}{(1 + \varepsilon)\kappa^j(\gamma)} \cos(\gamma) d\gamma > \int_0^{\pi/2} \frac{y^i(\gamma)}{\kappa^i(\gamma)} \cos(\gamma) d\gamma = \bar{A}^i.
\]

Finally we apply (4.3) to obtain

\[
\frac{\bar{A}^j}{c_1^j} > \frac{\bar{A}^i}{c_1^i}.
\]

As a steady state \( \Gamma \) curve possesses D2 symmetry \( A = 4\bar{A} \) holds so we are left with the conclusion that

\[
\frac{\hat{c}_1^i}{\bar{A}^i} > \frac{\hat{c}_1^j}{\bar{A}^j},
\]

which is exactly the monotonicity of the \( c_1(\hat{c}_1) \) function. This proves that \( F \) is injective, as different elements in \( \chi_q \) cannot be mapped to an identical value in \( \mathbb{R}^+ \). It is also worthy to note, that for all \( \hat{c}_1 \in \chi_q \) the area is obviously positive thus \( c_1(\hat{c}_1) \) is a positive, monotonic, continuous function.

**4.2. \( F \) is surjective.** To prove surjectivity we has to investigate the limits of \( c_1(\hat{c}_1) \) as \( \hat{c}_1 \) is varied. First we turn to investigate the limit as \( \hat{c}_1 \to 0 \) (\( q \) is still fixed). From the previous section we know, that the curvature at point \( P (\kappa(0)) \) is maximal along \( \bar{\Gamma} \) and \( \kappa(0) = \hat{c}_1^{-1} \). Curvature of any planar curve is the reciprocal of the \( r \) radius of its osculating circle. It provides an estimate on the area under the curve via \( A > 0.25\pi r^2 \) as \( 0.25\pi \hat{c}_1^2 \). In a similar way we use the fact, that the curvature is minimal at point \( Q \) (and there \( \gamma = 0 \) as well) to obtain the following inequality:

\[
\frac{\hat{c}_1}{\bar{A}^i} \leq \frac{\hat{c}_1}{\bar{A}^j} \leq \frac{\hat{c}_1}{\pi \bar{A}^i}.
\]

As both the lower and the upper expression in the above inequality approach +\( \infty \) as \( \hat{c}_1 \to 0 \) we conclude

\[
\lim_{\hat{c}_1 \to 0} \frac{\hat{c}_1}{\bar{A}(\hat{c}_1)} = +\infty.
\]

Finally we investigate the \( \hat{c}_1 \to \hat{c}_{\text{crit}} \) limit. As \( \hat{c}_{\text{crit}} \) is finite it is enough to investigate the \( \bar{A}(\hat{c}_1) \) area in the limit. We consider the already used identity between the curvature and and arch length. Taking again the parametrization respect to \( \gamma \) we write

\[
\kappa(\gamma) = -\left(\frac{dS(\gamma)}{d\gamma}\right)^{-1},
\]
where $S(\gamma)$ is the arch length between point P and the point with tangent inclination $\gamma$. As at $\hat{c}_1 = \hat{c}_{\text{crit}}$ the curvature at point Q vanish we conclude, that

$$
\lim_{\gamma \to 0} \frac{dS(\gamma)}{d\gamma} = \lim_{\gamma \to 0} \frac{1}{\kappa(\gamma)} = \infty.
$$

Thus the curve is unbounded. As the area $\bar{A}$ under $\bar{\Gamma}$ can be computed from the arc length ($y$ is finite!) we conclude

$$
\lim_{\hat{c}_1 \to \hat{c}_{\text{crit}}} S = \lim_{\hat{c}_1 \to \hat{c}_{\text{crit}}} A = \infty,
$$

which provides the required limit as

$$
\lim_{\hat{c}_1 \to \hat{c}_{\text{crit}}} \frac{\hat{c}_1}{A(\hat{c}_1)} = 0.
$$

It means, the range of $F$ is indeed $\mathbb{R}^+$ and based on the injectivity part of the proof the preimage is precisely $\chi_q$. As $F$ is injective and surjective we conclude that it must be one-to-one and onto. This means, the global equation has a unique solution among smooth curves for any positive $c_1$ and $c_2$.

Practical application of the results presented here and comparison of predicted steady state curves against observable shapes in nature will be carried out in a separate paper.

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Appendix A. Ellipses are not invariant shapes. Due to the $y$ dependence of the friction term (the one multiplied by $c_2$) for $c_2 > 0$ our natural expectation is to have ellipses as invariant shapes. We show that this is not the case. First let us investigate the $c_2 = 0$ case when

$$
-1 + c_1 A \kappa = 0
$$

holds, thus $\kappa \equiv \text{const}$ for all point of $\Gamma$. It implies circles are the only steady state shapes for $c_2 = 0$. For the general case ($c_2 \neq 0$) we use proof by contradiction. We assume an ellipse with $a > b$ semi axes is steady state. We parametrize the (in this case elliptic) arch between points A and P in the well-known way

$$
x(\phi) = a \cos \phi, \quad y(\phi) = b \sin \phi,
$$

where $\phi$ is the parameter, $0 \leq \phi \leq \pi/2$, $a = AO$ and $b = OP$. The curvature of the parametrically defined curve is given by

$$
\kappa(\phi) = \frac{|x' y'' - x'' y'|}{(x'^2 + y'^2)^{3/2}} = \frac{ab}{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{3/2}},
$$

with $(\cdot)'$ and $(\cdot)''$ denoting the first and second derivatives respect to $\phi$. Considering that the area of an ellipse is $A = ab\pi$ and $\cos \gamma = \cos(\arctan(y'/x'))$ we obtain, that the expression in eq. (2.1) can be written as

$$
-1 + c_1 \frac{a^2 b^2 \pi}{(b^2 \cos^2 \phi + a^2 \sin^2 \phi)^{3/2}} + c_2 \frac{ab^2 \pi \sin \phi}{\sqrt{1 + \frac{b^2 \cos^2 \phi}{a^2 \sin^2 \phi}}} = 0.
$$
At the endpoint of the major axis $\phi = 0$ and $c_2 = 0$. With this in hand, after simplification we obtain

\[(A.5)\]

\[c_1 = \frac{b}{a^2 \pi}.\]

In a very similar manner we substitute $\phi = \pi/2$, and using the value for $c_1$ from eq. (A.5) we obtain:

\[(A.6)\]

\[c_2 = \frac{a^3 - b^3}{a^4 b^2 \pi}.\]

Finally, we take a third value of $\phi$ to demonstrate, that with the derived constants $c_1$ and $c_2$ the equation is not satisfied. For example, after substitution of $c_1$, $c_2$ and $\phi = \pi/4$ into eq. (A.4) we obtain

\[(A.7)\]

\[-1 + \frac{b^3}{(0.5a^2 + 0.5b^2)^{3/2}} + \frac{\sqrt{2}}{2} \frac{a^3 - b^3}{a^3 \sqrt{1 + b^2/a^2}} \neq 0.\]

The left side of this equation is not identically zero, with truncating around $a = b$ one can show, that only $a = b$ makes it vanish. We found that among ellipses only the circle is a possible steady state candidate, which happens at $c_2 = 0$, as we have already demonstrated.

Appendix B. Proof of the properties of $\kappa(y)$. In Section 3 we listed several properties of $\kappa(y)$. The proofs are provided here, a graph of a typical $\kappa(y)$ function (given explicitly in (3.8)) is provided in Figure 3. below.

**Fig. 3.** An example for the $\kappa(y)$ solution function at $\hat{c}_1 = 1, q = \sqrt{0.25}, \hat{c}_2 = 0.5$. With these parameters $y_0$ (the point with $\kappa(y) = 0$) is close to 2.

1. $\kappa(y)$ is real ($\mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}$).
   Proof: since for any $a \in \mathbb{R}$ erf(ai) = bi with $b \in \mathbb{R}$, $\kappa(y)$ must be real.
2. $\kappa(y)$ is continuous.
   Proof: $\exp(x) > 0$ for all $x \in \mathbb{R}$ and continuity of erf(ai)i implies the statement.
3. $\kappa(0)$ is positive and equals $\hat{c}_1^{-1}$.
   Proof: since erf(0) = 0 and $\exp(x) > 0$, $\kappa(0) = \hat{c}_1^{-1}$. 
4. \( \kappa(y) \) has a maximum at \( y = 0 \).
   Proof: substituting \( y = 0 \) into the first and second derivatives of \( \kappa(y) \) reveals, that
   \( \kappa'(y)|_{y=0} = 0 \) and \( \kappa''(y)|_{y=0} = -4q^2 \hat{c}_1^{-1} < 0 \) which indicates a maximum.

5. \( \kappa(y) \to 0 \) as \( y \to \infty \).
   Proof: We use the polynomial expansion of the \( \text{erf}(x) \) function. Since a limit
   of a fraction of two polynomials is determined by the highest powers of the
   polynomials, we write
   \[
   \lim_{y \to \infty} \kappa(y) = \frac{1}{\hat{c}_1} - \frac{2q^2}{\hat{c}_1} \lim_{n \to \infty} \frac{2i(qy)^{2n+1}}{\sqrt{\pi} n!(2n+1)^{y}} - \frac{2q^2}{\hat{c}_1} = \frac{1}{\hat{c}_1} - \frac{2q^2}{\hat{c}_1} \lim_{n \to \infty} \frac{n+1}{2n+1} = 0
   \]

6. There is exactly one point, denoted to \( y_0 \), where \( \kappa(y) \) vanishes and \( y_0 \) solely
   depends on \( q \).
   Proof: by the derivative of the \( \text{erf}(x) \) function in eq. (3.8)) can be also given
   by
   \[
   \kappa(y) = \frac{1}{\hat{c}_1} - 2 \left( \text{erf}(qyi) \right) \frac{q^2}{\hat{c}_1} y.
   \]

At any point, where \( \kappa(y) \) vanishes this form can be arranged as
\[
\frac{\text{erf}'(qyi)}{i} - 2 \frac{\text{erf}(qyi)q^2y}{\hat{c}_1} = \iota(y) - \zeta(y),
\]
where \( \iota(.) \) and \( \zeta(.) \) are real valued, monotonic increasing functions with \( \iota(0) = 2q/\sqrt{\pi} \) and \( \zeta(0) = 0 \). Algebraic manipulations reveal that for all positive \( y \)
values \( 0 < \iota'(y) \leq \zeta'(y) \) holds, which implies there is one, and only one point,
where \( \iota(y) = \zeta(y) \). This is exactly the point, \( y_0 \), where \( \kappa(y) = 0 \). Observe,
that \( y_0 \) is uniquely determined by \( q \).

7. There is no local extrema for \( \kappa(y) \) between \( 0 < y < y_0 \), thus it is monotonic
   in this range.
   Proof: After simple algebraic manipulations the derivative of (B.2) can be written as
   \[
   \kappa'(y) = (\zeta(y) - \iota(y)) y + \text{erf}(qyi)i.
   \]
   Based on points 1. and 6. above both terms in eq. (B.4) are negative as long
   as \( 0 < y \leq y_0 \), which implies a lack of local extrema.

Note, that the properties investigated in detail above ensure that the graph of
\( \kappa(\delta) \) is like the one plotted in Figure 3. for any \( \hat{c}_1 \in \chi_q \).

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