The Mathematical Proof Steps of Mathematics Study Program Students in the Subject of Real Analysis

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Abstract. The present paper describes the second year-research entitled the learning trajectory on mathematical proof of the students majoring in mathematics education. It applies the product of the first-year research namely the framework of the learning trajectory of mathematical proof and the design of its means in the topic of the real number system and the real number sequence. The present research aims at obtaining the mathematical proof steps in the learning trajectory which has been already set in the first-year research. Design research was determined as the research method aimed at, according to Gravemeijer (2013), developing a Local Instruction Theory (LIT). The LIT is expected to improve the quality of learning by developing the sequence of activities and understanding the empirical phenomena about how something works. In this research, the steps of mathematical proof were hypothetically designed i.e., (1) understanding the statement which will be proved, (2) deciding to choose either direct proof or indirect proof, (3) writing the proof specifically, and (4) verifying the validity of the proof. After the activities had been applied, retrospective analysis was undertaken to identify whether the stages were in line to the learning trajectory set in the first-year research. The learning trajectory itself includes: (a) proof with simple procedures of which the proof of a statement is done with single thinking procedure, (b) existential proof covering existential proof and non-existential fact or concept, (c) proof with complex thinking procedure, and (d) proof through construction. The findings suggest that the steps of the previously hypothesized proof occur with a support which is called scaffolding by visualizing a diagram of proof as the guide for writing the proof and there are still difficulties of writing the proof symbolically and describing mathematical argument which relates each step of the proof process.

Keywords: Steps of mathematical proof, learning trajectory of mathematical proof

1. INTRODUCTION

In the last two years, the research team has designed a mathematical proof learning trajectory in the subject of real analysis course. It has created a mathematical proof learning trajectory following the difficulty level of the types of proof. The learning trajectory begins with providing the experience of understanding proof through rigor definitions, writing down the negation of the given definition, and explaining the definition through visual illustrations. The order of presenting the proof problems really should take into account the order in which the previous concepts, definitions, and theorems are
presented. The next steps of the proof learning trajectory are applied successively, namely proof with simple procedures, existential proof covering existential proof and non-existential fact or concept, proof with complex thinking procedure, and proof through construction. The learning trajectory has been evaluated through retrospective analysis, and the obtained hypothetical learning trajectory through the planned pilot experiment has been undertaken according to what has been expected.

However, when the level of the difficulty of the proposition to be proven increased, the framework formulated by the students could not be completed properly. Lecturers began to review the proof steps taken by students by asking whether the student understood the proposition, to clarify, what was known from the proposition, and what would be proved. Furthermore, through scaffolding efforts, the lecturer directed students to plan the proof by asking students to make proof flowcharts and also asking students to explain intuitively what would be done later. The rest of the students were asked to proof by themselves and check the validity of the proof.

The experience of teaching mathematical proof by the team in the real analysis course was made as a basis for learning design research.

It then led the hypothesis of the mathematical proof stages, namely: (1) understanding the proposition, (2) planning / choosing the type of proof (3) writing the details of the proof, and (4) checking the validity of the proof.

The proof steps describe the sequence of proof that must be taken and what concepts related to the proposition that is proven by students so that students can prove it completely. The steps of proof must be formulated before testing their implementation in learning. The formulated steps of proof are hypothetical, so it is called the hypothetical steps of proof.

2. RESEARCH METHOD

The present research design is related to the objectives of this study. It is based on the theory of the design research proposed by Gravemeijer and Cobb (2013), which is characterized by a cyclic process of preparing for the experiment (preparation stage), conducting the experiment (experimental implementation stage), and the retrospective analysis. It is also classified as development studies (Plomp, 2013), because it is a systematic study of designing, developing, and evaluating educational interventions as a solution to solving complex problems in educational practice.

Design research initiated by Gravemeijer and Cobb is focused on developing the order of presentation of material in mathematics learning. It starts with a thought experiment (though experiment), thinking of the learning route/trajectory that students will go through. The results of the next thought experiment are experimented in class. By reflecting on the experimental results in class, the next thought experiment is carried out. In a long term process, these two activities can be seen as a cumulative cyclic process as shown in the figure 1.

Source : Gravemeijer & Cobb, 2006

Figure 1. Cyclic Thought Process & Classroom Experiment

The present part describes the activities carried out in each phase of the design research:

2.1. Preparation Stage (Preparing for The Experiment)
Based on the results of the observations and the experiences of the mathematical proof learning in the real analysis course, the steps of the proof are formulated for each mathematical proof activity which
include (1) understanding the proposition, (2) deciding the type of proof, (3) writing the details of the proof, and (4) checking the validity of the proof. In the preparation stage, the problem of proof is prepared for the topic of function limit and questions about the reasons for the steps of the proof.

2.2. Experiment Stage (Conducting the Experiment)
At this stage, the hypothesized proof step will guide the researcher in observing the learning process and as a guide for providing proof problems. In this study, the stage was carried out in small groups consisting of four or five students. The research subjects used in the study were students of the Mathematics Education Study Program at UNM Makassar. In addition, a revision was made to the topic of limit function as the topic of real analysis subject.

2.3. Retrospective Analysis
The main objective at this stage is to contribute to the development of a hypothetical proof steps in supporting students' understanding of the Real Analysis subject. Specifically, this stage is applied to evaluate whether the hypothetical proof steps that have been planned occur in the learning. The plan of the proof steps used in the retrospective analysis is set as the main guide and the reference in answering the research questions. The role of the hypothetical proof steps in this stage is to serve as a guide in determining the focus of the analysis. The analysis process is not only on the factors that support the success of the proof but also on some hypothetical learning that does not get a response from the subjects. The explanations obtained are used to draw conclusions and answer the research questions.

3. DISCUSSION
The findings of the study are based on the steps of proving activities that are designed to be the proof steps in the real analysis course. The data were collected through the observation and the discussion when students presented the given tasks.

3.1. Preparation Stage
In this stage, the research team reviewed the proposed proof steps. At the step of understanding the propositions, the subject should be able to analyze the information given and what thing which should be proved in the proposition, to switch a proposition into a simpler statement which is easier to prove, for example by changing it to its contraposition or taking into account its negation.

In the stage of planning/deciding the type of proof, it is considered that the subject should be able to (1) convert the proposition into a mathematical statement, (2) explain the support of the given statement for the proposition, (3) if possible, the subject is asked to make a flow chart of the proof. The proof flowchart is discussed by the subjects. In the stage of writing the details of the proof, the subjects works independently, but metacognitive scaffolding can be given, if they have difficulty. In the stage of checking the validity of the proof, the subject must be able to provide reasons why the process of the proof is valid. (1) In a direct proof, the proposition is in the form of implication of which the subject take the advantage of the given statements to obtain statements that will be proven, (2) In proving contradictions, the subject assumes the statement to be proven as false and uses the given statements to obtain contradictions.

3.2. The Experiment Stage and the Retrospective Analysis Stage
The formulation of the mathematical proof steps in the preparation stage is applied in the experimental stage and evaluated at the retrospective analysis stage, whether the formulated proof steps occur according to what is expected. The findings of these stages are arranged in the form of learning activities as follows:

Step 1. Understanding Proposition
The step of understanding propositions is the initial step of mathematical proof. A number of propositions that will be proved are generally well understood by students, especially single propositions, propositions in the form of implications, or that of in the form of biimplications. Students can identify the key information and what need to prove from the given propositions. To prove the proposition,
students need to change those statements into a form that is in accordance with definition or theorem to obtain mathematical statements that are ready to be proven. The results of the observations and the discussion during the experiment stage suggests the need for students to understand proof through strict definitions, to write down the negations of the given definitions, and explain the definitions through visual illustrations. The order of presenting the problems of proof should take into account the order in which the concepts, definitions and theorems.

The subjects are given daily problem experiences in the form of propositions and how to prove these propositions. In addition, the subjects are reminded of some of the basic meanings of conjunction words and quantifier statements.

Furthermore, the findings of the retrospective analysis suggest that the step is well applied by the subjects. They can explain the given information, the statements which should be proven. Moreover, they can write notations, the terms that are expressed in the propositions. When they are not able to explain it, they are asked to review the definition, notation, or unfamiliar terms in the used textbook.

**Step 2. Planning/Deciding the Type of Proof**

In the step of planning the proof, the subject is asked to change the propositions into mathematical statements. In order to change a proposition into a mathematical statement, the subject must first change the statement which is proved into a mathematical statement, then change the given information of the proposition into a mathematical form that corresponds to the mathematical form of the statement to be proved. In this case, the subject needs to rigorously know the definition or theorem associated with the proposition to be proved. The next step is to ask the subjects to describe the planned flowchart of the proof. This section is carried out in a discussion among them. Specifically, in this case, it is undertaken through a discussion group by uploading the chart and then it is commented by other subjects.

**Step 3. Writing Down the Detail of the Proof**

At this step, the subjects are asked to write down the detail of the proof, guided by the proof flowchart. After completing the proof, the research team conducted an interview to explore whether this step occurs properly.

One of the works is shown in the figure 2:

**Figure 2. Subject Answer**

Based on the figure 2, the subject converts the proposition into a mathematical form using the definition of the limit function. The last form is a statement in the form of implications and can be solved through algebraic operations.

**Figure 3. Subject Answer**
Based on the figure 3, it can be suggested that the subject changes the statement to be proven into a mathematical form using the definition of the limit function. This form is broken down into a form corresponding to the known mathematical form of the proposition.

The detail of the proof of one of the subjects is shown in the figure 4:

\[
\begin{align*}
| f(x) \cdot g(x) - LM | &= | f(x) \cdot g(x) - M(f(x)) + M(f(x)) - LM | \\
&= | f(x) \cdot (g(x) - M(f(x))) + M(f(x)) - LM | \\
&= | f(x) | | g(x) - M(f(x)) | + | M(f(x)) - LM | \\
\end{align*}
\]

**Figure 4. Subject Answer**

This form will be adjusted to the form of the given statement in the proposition which will be proven

\[
\begin{align*}
\lim_{x \to C} f(x) = L & \text{ ada, maka jika dibuktikan } \delta > 0 \text{ maka kita } \\
\text{muncul pada } | f(x) - L | < \epsilon / 2 | M | & \text{ atau } \lim_{x \to C} g(x) = M \text{ ada, } \\
\text{muncul jika dibuktikan } \delta > 0 \text{ maka kita muncul pada } | g(x) - M | < \epsilon / 2 | L | \\
\end{align*}
\]

**Figure 5. Subject Answer**

Based on the figure 5, the subject is interviewed to identify the reason why the subject applied the steps and confirmed there were errors in writing the notation and why the mathematical notations were used.

In general, the justification for the steps follows the correct flowchart of proof. The errors in writing notations and the existence of notations stem from the inability of the subject to write the proof in detail. Some of the ideas are left in the mind of the subject of which they have difficulty to represent in writing.

The last step written by the subject demonstrates the success of equating the previously claimed form through algebraic operations as shown in the figure 6:

\[
\begin{align*}
\text{Definisi} \ \delta = \min \left\{ \delta_1, \delta_2 \right\} \\
0 < |x - C| < \delta \Rightarrow | f(x) \cdot g(x) - LM | \\
&\leq | f(x) | | g(x) - M(f(x)) + M(f(x)) - LM | \\
&\leq | f(x) | | g(x) - M(f(x)) | + | M(f(x)) - LM | \\
&\leq | f(x) | \frac{\epsilon}{2 | L |} + | M | \frac{\epsilon}{2 | M |} \\
&= \epsilon \frac{1}{2 | L |} + \frac{\epsilon}{2 | M |} \\
&= \epsilon
\end{align*}
\]

**Figure 6. Subject Answer**

The works of the other subjects were also analyzed. It shows that there are various difficulties of the subjects in writing the proof in detail, namely representing ideas in writing. The flowchart of proof at the step of planning the proof certainly helps the subject guide the direction of the proof. However, difficulties generally arise when writing the proof based on the flowchart. This difficulty is discussed again with the lecturer and in general, writing the proof for the second time is better than writing it for the first time. It shows the need for this step to be equipped with scaffolding to help the subject write a proof in detail.

The results of the retrospective analysis show that this step occur with the help of scaffolding by the lecturer.
Step 4. Verifying the Proof

In this step, the subjects verify whether the proof follows the proof rules and the sentences in the proof are well defined and follows the flowchart. One of the subjects’ works is shown in the figure 7 of which a subject attempts to prove that the function has no limit at \( c \neq 0 \).

![Figure 7. Subject Answer](image)

The proof is shown in the figure 8:

![Figure 8. Subject Answer](image)

In the figure 8, the subject applies the proof by contradiction by supposing that the function has a limit \( L \) at point \( c \neq 0 \). The sentence that shows the function does not have a limit at point \( c \) is shown in the figure 9.

![Figure 9. Subject Answer](image)

However, there are flaws of the previous paragraph as shown in the figure 10:

![Figure 10. Subject Answer](image)

The sentence in the figure 10 is a kind of a contradictive statement because it has unclear definition. In this case, the subject is given a scaffolding to revise the proof.

The phenomenon of writing the proof among the subjects suggests that the activity of proving contradictions always requires scaffolding, so that the step of "checking the validity of proof" can be passed by the subjects.

Observing the proving by the subjects in this research shows that the steps of proof, especially in the real analysis subject, require scaffolding from the lecturer. Some mistakes and difficulties of the subjects in proving propositions that appear at each step of proof and the ability of the subjects in proving become better suggest the need for these steps of proof should be taken into account by the subjects in learning how to prove mathematics. Even though the flowchart of the proof has been made by the subjects, the difficulty of checking the validity of the proof still take place due to the difficulty of the subjects in writing down the ideas, including the difficulty of writing symbols and the justification that link the series of statements in the proof. Therefore, the proposed proof steps require scaffolding by the lecturer.
4. CONCLUSION AND SUGGESTION

4.1. Conclusion

1) The steps of the mathematical proof obtained in this study are a sequence of proving activities including: (1) understanding the proposition, (2) planning/deciding the type of proof (3) writing the detail of the proof, and (4) checking the validity of the proof.

2) In the step of understanding the proposition, most subjects can well explain the given statements, statements to be proven, and the terms in the proposition. Moreover, they are able to write notations. If there are subjects who cannot explain it, they are asked to review the definition, notation, or unfamiliar terms in the textbook.

3) In the phase of planning/deciding the type of proof, the students can plan the flowchart of proof through discussion among them, in this case, it is done through the WhatsApp group by uploading the diagram and then it can be commented on by other subjects.

4) The students can pass the steps of writing the proof in detail and checking the validity of the proof through the scaffolding by the lecturer.

4.2. Suggestion

Based on the findings of this study, it can be suggested that:

1) The design of mathematical proof steps still requires refinement. Specifically, they can be broken into sub-steps of proof. Because it is necessary to conduct a more focused research to reveal the sub-steps of the learning trajectory in mathematical proof.

2) The steps of mathematical proof require scaffolding. It needs to be explored, especially scaffolding, about how it can be applied to help students in mathematical proof. Therefore, further research is needed on how to do scaffolding at each step of mathematical proof.

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