Canonical extensions of manifolds with nef tangent bundle

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Abstract
To any compact Kähler manifold \((X, \omega)\) one may associate a bundle of affine spaces \(Z_X \to X\) called a canonical extension of \(X\). In this paper we prove that (assuming a well-known conjecture of Campana-Peternell to hold true) if the tangent bundle of \(X\) is nef, then the total space \(Z_X\) is a Stein manifold. This partially answers a question raised by Greb-Wong of whether these two properties are actually equivalent. We also complement some known results for surfaces in the converse direction.

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0 Introduction

Given a Kähler manifold \((X, \omega)\) one may define in a natural way a bundle \(p: Z_X \to X\) of affine spaces called a canonical extension of \(X\). One possible way to define \(Z_X\) is as the universal complex manifold on which the cohomology class \([p^*\omega] = 0\) vanishes.

Canonical extensions were introduced by [Don02] to prove regularity properties of solutions to the Monge-Ampère equation. They have subsequently also seen some uses related to K-stability and the existence of Kähler-Einstein metrics on Fano manifolds, see for example [Tia92] or [GKP22].

Recently, in [GW20] another point of view on canonical extensions was suggested. Namely, besides discussing some relations to the existence of complexifications, they suggested the following question:

**Question 0.1.** Let \(X\) be a compact Kähler manifold. Is it true, that the tangent bundle of \(X\) is nef if and only if some (resp. any) canonical extension of \(X\) is a Stein manifold?
The structure of compact Kähler manifolds possessing a nef tangent bundle is well-understood and by now classical (see Section 2). However, specifically in the Fano case some very interesting questions such as the conjecture of Campana-Peternell remain open. Thus, Question 0.1 is interesting as it suggests a possibly more geometric point of view on these problems.

In Section 3 we give the following partial answer to Question 0.1:

**Theorem 0.2.** Let $X$ be a compact Kähler manifold with nef tangent bundle. If the (weak) Campana-Peternell conjecture (Conjecture 2.5) holds true then any canonical extension $Z_X$ of $X$ is a Stein manifold.

The idea for the proof is quite simple: By a well-known result of Demailly-Peternell-Schneider, Cao (cf. Theorem 2.1) some finite étale cover of $X$ fibres over a complex torus $T$. Now, any canonical extension of a torus is Stein as remarked in [GW20, Proposition 2.13]. Moreover, assuming a weak form of the Campana-Peternell conjecture (see Conjecture 2.5) any canonical extension of any fibre $F$ is Stein as well. This was proved in [HP21, Theorem 1.2]. In conclusion, it remains to put both cases together. To this end, we understand canonical extensions of $X$ as fibre bundles in terms of the canonical extensions of $F$ and $T$ respectively.

In the converse direction of what can be said about manifolds admitting a canonical extension which is Stein, little is known (see [HP21] for some partial results). In fact, even for projective surfaces Question 0.1 is not completely settled yet, although it is known to hold in most cases by the work of [HP21] Theorem 1.13. In Section 4 we partially complement their results by ruling out the case of ruled surfaces over higher genus curves as well:

**Lemma 0.3.** Let $X = \mathbb{P}(\mathcal{E}) \to C$ be a ruled surface over a curve of genus $g(C) \geq 2$ defined by a semi-stable vector bundle $\mathcal{E}$. Then, no canonical extension of $X$ is Stein.

This only leaves the case of unstable ruled surfaces over elliptic curves.

Finally, in a short Appendix we provide some clarifications on our convention regarding integration along the fibres and some (maybe not so standard) formulae from multi-linear algebra used throughout the text.

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# 1 Canonical extensions of complex manifolds

## 1.1 A variety of constructions

In the following we provide a short overview of a general approach to constructing bundles of affine spaces over complex manifolds. This theory is necessary for the definition of canonical extensions in the next subsection.
Reminder 1.1. Let $\mathcal{F}$ be a coherent sheaf on a complex analytic variety $X$. Recall that the elements of the cohomology group $\text{Ext}^1(\mathcal{O}_X, \mathcal{F})$ are in one-to-one correspondence with isomorphism classes of extensions $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{O}_X \to 0$ of $\mathcal{O}_X$ by $\mathcal{F}$. Here, $\text{Ext}_\mathcal{O}(\mathcal{O}_X, -)$ coincides by definition with the right-derived functor

$$\text{Ext}_\mathcal{O}(\mathcal{O}_X, -) := \text{RHom}_\mathcal{O}(\mathcal{O}_X, -) = \text{R}\Gamma(X, -).$$

In other words, isomorphism classes of extensions $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{O}_X \to 0$ are in one-to-one correspondence with the elements of $H^1(X, \mathcal{F})$.

Now, fix a complex manifold $X$, a holomorphic vector bundle $\mathcal{E}$ on $X$ and any cohomology class $a \in H^1(X, \mathcal{E})$. In the following we describe three equivalent ways of constructing affine bundles over $X$ from the data $(\mathcal{E}, a)$:

Construction 1.2. (as torsors)
As we saw above, to $a \in H^1(X, \mathcal{E})$ we can associate an extension

$$0 \to \mathcal{E} \to \mathcal{V}_a \xrightarrow{p} \mathcal{O}_X \to 0$$

of holomorphic vector bundles on $X$. Consider the sub sheaf $Z_a := p^{-1}(1) \subseteq \mathcal{V}_a$ of sections of $\mathcal{V}_a$ mapping under $p$ to the constant function 1. Note that $Z_a$ is not a sheaf of $\mathcal{O}_X$-modules. However, it comes with a natural action of $\mathcal{E}$ by translations making $Z_a$ into an $\mathcal{E}$-torsor. In this sense, $Z_a$ is an affine bundle: Its underlying total space $Z_a := |Z_a| \to X$ is a fibre bundle over $X$ and the fibre $Z_a|_x$ over any point $x$ is in a natural way an affine vector space with group of translations $\mathcal{E}|_x$. In the following, we will call the total space $Z_a := |Z_a| \to X$ (an) extension of $X$. Sometimes we may also denote it by $Z_{\mathcal{E}, a}$ if we want to make explicit the dependence on the bundle $\mathcal{E}$.

A similar way to construct $Z_a$ is as follows: We may consider $p$ as a holomorphic map between the underlying total spaces $|p|: |\mathcal{E}| \to |\mathcal{O}_X| = X \times \mathbb{C}$. Then, $Z_a$ may be naturally identified with the pre-image

$$Z_a = |p|^{-1}(X \times \{1\}).$$

Since $p$ is a surjective morphism of vector bundles, $|p|$ is a submersion. In particular, we see from this that $Z_a$ is indeed a manifold and we also see that we may view the affine space structure on the fibres $Z_a|_x$ as arising from the embedding $Z_a|_x \subseteq \mathcal{V}_a|_x$. This is the definition of $Z_a$ used in [GW20].

Construction 1.3. (as complements of a hypersurface)
A second, possibly more geometric construction of $Z_a$ is as follows: Dualising the short exact sequence [Eq. (1)] we find the short exact sequence

$$0 \to \mathcal{O}_X \to (\mathcal{V}_a)^* \to \mathcal{E}^* \to 0$$

which defines an embedding

$$\mathbb{P}(\mathcal{E}^*) \hookrightarrow \mathbb{P}(\mathcal{V}_a^*).$$
Here, throughout this paper we will always use the convention that $\mathbb{P}(\mathcal{E})$ denotes the projective bundle of linear hyperplanes in $\mathcal{E}$.

We claim that there exists a natural identification of the affine bundle $Z_a$ with the complement $\mathbb{P}(\mathcal{V}_a^* \setminus \mathbb{P}(\mathcal{E}^*))$. Indeed, for any $x \in X$ the fibre $\mathbb{P}(\mathcal{V}_a^*)|_x$ is just the space of lines in $\mathcal{V}_a|_x$ passing through the origin. Now, of course any point in the affine space $Z_a|_x \subseteq \mathcal{V}_a|_x$ defines a unique line passing trough itself and the origin (here, we use that $0 \notin Z_a|_x = p_x^{-1}(1)$) and so $Z_a|_x \subset \mathbb{P}(\mathcal{V}_a^*)|_x$ in a natural way. Moreover, the set $\mathbb{P}(\mathcal{V}_a^*)|_x \setminus Z_a|_x$ consists precisely in those lines which are parallel to $Z_a|_x$, i.e. contained in $\mathbb{P}(\mathcal{E}^*)|_x$. This concludes the proof of the claim.

Note that by construction $\mathbb{P}(\mathcal{E}^*)$ is embedded as a smooth hypersurface in the linear series of $\mathcal{O}_{\mathbb{P}(\mathcal{V}_a^*)}(1)$. In particular, its normal bundle is given by

$$N_{\mathbb{P}(\mathcal{E}^*)/\mathbb{P}(\mathcal{V}_a^*)} = \mathcal{O}_{\mathbb{P}(\mathcal{E}^*)}(1).$$

This is the preferred point of view in [HP21].

**Construction 1.4. (via a universal property)**

Finally, $Z_a \xrightarrow{\sim} X$ enjoys the following universal property (which of course determines it uniquely): Let $Y$ be any complex manifold and let $h: Y \to X$ be any holomorphic map such that the cohomology class $h^*a = 0 \in H^1(Y, f^*\mathcal{E})$ vanishes. Then, $h$ factors uniquely (up to translation by an element of $H^0(Y, f^*\mathcal{E})$) through $Z_a \xrightarrow{\sim} X$. In this sense, $Z_a \to X$ is the universal manifold on which the cohomology class $a$ vanishes. A more precise version of this statement may be found in [GW20, Lemma 1.16.(c)].

**Corollary 1.5. (cf. [GW20, Remark 2.4.])**

Let $X$ be a complex manifold, let $\mathcal{E}$ be a holomorphic vector bundle on $X$ and fix any cohomology class $a \in H^1(X, \mathcal{E})$. Then, for any $\lambda \in \mathbb{C}^\times$ there exists an isomorphism of affine bundles

$$Z_a = Z_{\lambda a},$$

which is canonical up to translation.

**Proof.** Both bundles share the same universal property described in Construction 1.4 hence are canonically isomorphic. Compare also [GW20, Remark 2.4.].

The construction of extensions is clearly functorial:

**Proposition 1.6. (see also [GW20, Lemma 1.16(b)])**

Let $f: X \to T$ be a holomorphic map between complex manifolds. Let $\mathcal{E}$ be a holomorphic vector bundle on $T$ and fix any cohomology class $a \in H^1(T, \mathcal{E})$.

There exists a natural isomorphism of affine bundles

$$Z_{f^*\mathcal{E}, f^*a} \cong f^*Z_{\mathcal{E}, a} = Z_{\mathcal{E}, a} \times_T X.$$

We will denote the induced map $Z_{f^*\mathcal{E}, f^*a} \to Z_{\mathcal{E}, a}$ by $Z_f$. 
1.2 Canonical extensions and positivity of curvature

Definition 1.7. Let \((X, \omega)\) be a complex Kähler manifold. Then, \(\omega\) is a \(\bar{\partial} \partial\)-closed form and hence defines a cohomology class \([\omega] \in H^1(X, \Omega^1_X)\). The associated extension \(Z_{[\omega]}\) is called (a) canonical extension of \(X\). Alternatively, we also write \(Z_{X,[\omega]}\) if we want to stress the dependence on \(X\) or simply \(Z_X\) if the dependence on \([\omega] \in H^1(X, \Omega^1_X)\) is not important in that situation.

In the preceding subsection we have seen three equivalent constructions for \(Z_{[\omega]}\):

1. As a bundle of affine spaces over \(X\) (more precisely: as a \(\Omega^1_X\)-torsor).
2. As the complement \(Z_{[\omega]} = \mathbb{P}(\mathcal{V}^*_{[\omega]}) \setminus \mathbb{P}(T_X)\) of the smooth hypersurface \(\mathbb{P}(T_X)\) which is an element in the linear series of \(\mathcal{O}_{\mathbb{P}(\mathcal{V}^*)}(1)\). The normal bundle of \(\mathbb{P}(T_X)/\mathbb{P}(\mathcal{V}^*) = \mathcal{O}_{\mathbb{P}(T_X)}(1)\) (this was part of Construction 1.3).
3. As the universal manifold on which the cohomology class \([\omega]\) vanishes.

In the following we are going to use all of these constructions interchangeably.

The following conjecture arose out of the work of [GW20] and [HP21] on canonical extensions:

Conjecture 1.8. (Greb-Wong, Höring-Peternell)

Let \(X\) be a compact Kähler manifold. Then, the tangent bundle \(T_X\) is nef (respectively big and nef) if and only if some canonical extension \(Z_X\) of \(X\) is Stein (respectively affine).

In this context, recall that a vector bundle \(\mathcal{E}\) on a complex manifold is said to be nef (resp. big) if and only if the tautological bundle \(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)\) is nef in the sense of [DPS94, Definition 1.2.] (resp. big in the classical sense, see [Laz04, Definition 2.2.1.]). Conjecture 1.8 is interesting as it promises a possibly more geometric way to study manifolds of positive curvature. See Section 2 below for an overview of the (expected) structure theory of manifolds with a nef tangent bundle.

Let us quickly summarise what is known thus far about [Conjecture 1.8] in general:

- The conjecture is known to hold for curves and (most) projective surfaces by [HP21, Theorem 1.13.]. More details on the cases left open will be provided in Section 4.
- If \(X\) is projective and some canonical extension of \(X\) is affine, then the tangent bundle of \(X\) is big by [GW20, Corollary 4.4.]. Conversely, if \(T_X\) is nef and big, then all canonical extensions of \(X\) are affine. The latter result is due to [HP21, Theorem 1.2.]. Thus, (at least modulo the nef case) the big case is settled.
- Building on the work of [GW20] and [HP21], in Section 3 we are going to prove that if the tangent bundle of \(X\) is nef (and if the weak form of the conjecture of Campana and Peternell holds true, cf. [Conjecture 2.5]), then the canonical extensions of \(X\) are always Stein.
- The remaining case is to prove that if a canonical extension of \(X\) is Stein then the tangent bundle of \(X\) is nef. This problem is still almost completely open.
Proposition 1.9. Let $\pi : X' \to X$ be an étale cover between Kähler manifolds. Then, for any Kähler form $\omega_X$ on $X$ there exists a natural isomorphism of affine bundles:

$$Z_{X'}[\pi^*\omega_X] \cong \pi^*Z_X[\omega_X] := Z_X[\omega_X] \times_X X'.$$

Moreover, if $\pi$ is finite then $Z_X[\omega_X]$ is Stein if and only if $Z_{X'}[\pi^*\omega_X]$ is so.

Proof. First of all, it follows from Proposition 1.6 that

$$Z_X[\omega_X] \times_X X' = \pi^*Z_X[\omega_X] \cong Z_{\pi^*T_X}[\pi^*\omega_X].$$

Since $\pi$ is étale the natural morphism $d\pi : T_{X'} \to \pi^*T_X$ is an isomorphism. Thus, Eq. (2) is proved. Regarding the second assertion, the identification

$$Z_{X'}[\pi^*\omega_X] \cong Z_X[\omega_X] \times_X X'$$

which we just proved shows that together with $\pi : X' \to X$ also the holomorphic map $Z_{\pi} : Z_{X'} \to Z_X$ is a finite étale cover. But in general, if $Z' \to Z$ is any finite map between complex manifolds then $Z$ is Stein if and only if $Z'$ is Stein (see e.g. [Nar62, Lemma 2.]). $\square$

2 Structure theory of manifolds with nef tangent bundle

For the convenience of the reader, in this section we want to provide a short summary of the (conjectural) structure theory of manifolds with a nef tangent bundle. The following result summarises the successive work of [CP91], [DPS93], [DPS94] and [Cao13]:

Theorem 2.1. (Cao, Demailly-Peternell-Schneider)

Let $X$ be a compact Kähler manifold possessing a nef tangent bundle. There exists a finite étale cover $X' \to X$ such that the Albanese map $\alpha : X' \to \text{Alb}(X')$ is a locally constant analytic fibre bundle. The typical fibre is a Fano manifold with a nef tangent bundle.

Here, a fibre bundle is said to be locally constant if it satisfies one of the following equivalent characterisations:

Lemma 2.2. Let $\alpha : X \to T$ be a proper holomorphic fibre bundle with fibre $F$. Let $\tilde{T} \to T$ denote the universal cover of $T$. Then, the following assertions are equivalent:

1. The transition functions of $\alpha : X \to T$ may be chosen to be locally constant.
2. There exists a representation $\rho : \pi_1(T) \to \text{Aut}(F)$ and a biholomorphism of fibre bundles

$$(\tilde{T} \times F)/\pi_1(T) \cong X.$$

Here, $\pi_1(T)$ acts on $\tilde{T}$ in the natural way and on $F$ through $\rho.$
(3) The short exact sequence \(0 \to T_{X/T} \to T_X \to \alpha^* T_T \to 0\) admits a global holomorphic splitting establishing \(\alpha^* T_T\) as an integrable sub bundle of \(T_X\).

Proof. The equivalence of (1) and (2) is clear. Moreover, if (2) is satisfied, then the holomorphic vector bundle \(pr_T^* \tilde{T}_F\) on \(\tilde{T} \times F\) is clearly \(\pi_1(T)\)-invariant and, hence, descends to a holomorphic vector bundle on \(X \cong (\tilde{T} \times F)/\pi_1(T)\) providing an integrable splitting of the short exact sequence \(0 \to T_{X/T} \to T_X \to \alpha^* T_T \to 0\).

Finally, assume (3) to hold true and fix an integrable, holomorphic sub bundle \(T \subseteq T_X\) splitting the sequence \(0 \to T_{X/T} \to T_X \to \alpha^* T_T \to 0\). Then, any local holomorphic frame \(V_1, \ldots, V_m\) of \(T (\text{locally})\) admits a unique lift to a (local, holomorphic) frame \(\tilde{V}_1, \ldots, \tilde{V}_m\) for \(T\). Since \(\alpha\) is proper the flows \(\phi^1, \ldots, \phi^m: X \times D \to X\) to \(\tilde{V}_1, \ldots, \tilde{V}_m\) are well-defined local automorphisms of \(X\). Here, \(D \subseteq \mathbb{C}\) is a sufficiently small open disc. Then, the map \(\psi: F \times D^m \to X, (y, z_1, \ldots, z_m) \mapsto \phi^1_{z_1}(\ldots(\phi^m_{z_m}(y)))\) gives a local trivialisation of \(X\). Since \(T\) was chosen to be integrable, this trivialisation respects \(T\) in the sense that \(d\psi(T_{D^m}) = T\). In particular, if \(\psi_1, \psi_2\) are any two such trivialisations, then \(d(\psi_2^{-1} \circ \psi_1)(T_{D^m}) = T_{D^m}\), i.e. \(\psi_1, \psi_2\) differ by a locally constant transition function.

It is conjectured that in the situation of Theorem 2.1 much more can be said about the fibre of \(\alpha:\)

**Conjecture 2.3.** (Campana-Peternell, [CP91])
Every Fano manifold with a nef tangent bundle is homogeneous.

As is well-known, the group of holomorphic automorphisms of a compact complex manifold is always a complex Lie group and its Lie algebra may be identified with \(H^0(F, T_F)\). In particular, \(F\) is homogeneous if and only if its tangent bundle is globally generated.

Conjecture 2.3 has seen attention by quite a number of authors and is by now verified for manifolds of dimension at most five by [Kan17] (see also the introduction thereof for a short summary of contributions to this problem or alternatively [Muñ+15] for a survey on the topic). In full generality however it has not even been proved yet that the tangent bundle must be semi ample. For now, we only have the following characterisation:

**Lemma 2.4.** Let \(F\) be a Fano manifold with nef tangent bundle.

(1) If the tangent bundle \(T_F\) is generated by global sections, then \(T_F\) is also big.

(2) If the tangent bundle \(T_F\) is big, then it is also semi ample (in the sense that \(O_{\mathbb{P}(T_X)}(1)\) is semi ample).

Proof. A proof of (1) using the theory of canonical extensions may be found in [GW20, Corollary 4.4.]. Alternatively, a more general argument is provided in [Hsi15, Corollary 1.3.].

The second statement seems to be a well-known consequence of the basepoint-free theorem, cf. [Muñ+15 Proposition 5.5.].

In this sense, we will record the following weak version of Conjecture 2.3

**Conjecture 2.5.** (weak Campana-Peternell conjecture)
If the tangent bundle of a Fano manifold is nef then it is also big.
3 Canonical extensions of manifolds with nef tangent bundle

In this section we want to give a proof of Theorem 0.2. To this end, let $X$ be a compact Kähler manifold with nef tangent bundle. According to Theorem 2.1 there exists a finite étale cover $X'$ of $X$ whose Albanese map $\alpha: X' \to \text{Alb}(X') =: T$ is a flat fibre bundle. Moreover, the typical fibre $F$ of $\alpha$ is (assuming the weak Campana-Peternell conjecture) a Fano manifold with big and nef tangent bundle. Now, in the extremal cases $X = T$ and $X = F$ the result is already known:

3.1 Summary of known results

**Theorem 3.1.** (Greb-Wong, [GW20, Proposition 2.13.])

Let $T = \mathbb{C}^q/\Gamma$ be a complex torus. Fix any Kähler form $\omega_T$ on $T$. Then the canonical extension of $T$ with respect to $\omega_T$ is a Stein manifold. In fact, there exists a biholomorphism $Z_{T,\omega_T} \cong (\mathbb{C}^\times)^{2q}$.

The proof of Theorem 3.1 uses that on a torus any Kähler class contains a unique constant Kähler metric. In the latter case, the extension may be computed explicitly. Moreover,

**Theorem 3.2.** (Höring-Peternell, [HP21, Theorem 1.2.])

Let $F$ be a Fano manifold with big and nef tangent bundle. Then, any canonical extension of $F$ is affine and, hence, Stein.

The proof of Theorem 3.2 uses some basic birational geometry.

**Remark 3.3.** Assuming the Campana-Peternell conjecture another proof of Theorem 3.2 is given in [GW20, Proposition 2.23.]: Therein, the authors provide an explicit description of the canonical extension of a homogeneous Fano manifold $F$ from which it follows that $Z_F$ is affine. For concreteness, let us only make this explicit in case $F = \mathbb{P}^n$. To this end, let us abbreviate $G := \text{PGL}_n = \text{Aut}(\mathbb{P}^n)$. Then, we may identify $\mathbb{P}^n = G/P$, where $P := \{A \in G | Ae_1 = \lambda e_1\}$. Let $L = (\mathbb{C}^\times \times \text{GL}_n)/\mathbb{C}^\times \subset P$ be the subgroup of block diagonal matrices (note that $L$ is a Levi subgroup of $P$). Then, the bundle $Z_{\mathbb{P}^n} \to \mathbb{P}^n$ may naturally be identified with $G/L \to G/P$.

3.2 The general case

In this subsection we are going to prove that (in the notation at the beginning of this section) any canonical extension $Z_X$ of $X'$ may be viewed in a natural way as a fibre bundle over a canonical extension $Z_T$ of $T$ and with fibre $Z_F$ a canonical extension of $F$. This will immediately imply that all canonical extensions of $X$ are Stein, thus partially confirming Conjecture 1.8.

To explain the existence of the fibre bundle structure on $Z_X$ we need the following technical result which may be found in [HP21]:

...
Proposition 3.4. (Höring-Peternell, [HP21, Lemma 5.5])
Let \((X, \omega_X)\) be a Kähler manifold. Assume that one may decompose \(T_X = E \oplus F\) into holomorphic sub bundles. Let \([\omega_X] = [\omega_E] + [\omega_F]\) be the induced decomposition in
\[
\text{Ext}^1_{\mathcal{O}} \left( \mathcal{O}_X, \Omega^1_X \right) = \text{Ext}^1_{\mathcal{O}} \left( \mathcal{O}_X, E^* \right) \oplus \text{Ext}^1_{\mathcal{O}} \left( \mathcal{O}_X, F^* \right).
\]
Then, there exists a natural isomorphism of affine bundles over \(X\)
\[
Z_{[\omega_X]} \cong Z_{[\omega_E]} \times_X Z_{[\omega_F]}.
\]

Corollary 3.5. Let \((X, \omega_X)\) be a compact Kähler manifold with nef tangent bundle. Assume the Albanese morphism \(\alpha : X \to \text{Alb}(X) =: T\) is a locally constant holomorphic fibre bundle. Then, there exists a natural isomorphism of affine bundles
\[
Z_{T_X,[\omega_X]} \cong Z_{T_X/T,[\omega_{X/T}]} \times_X Z_{\alpha^* T_T}.
\]
Here, by \([\omega_{X/T}]\) we denote the image of \([\omega_X]\) under the natural homomorphism
\[
H^1 \left( X, \Omega^1_X \right) \to H^1 \left( X, \Omega^1_{X/T} \right).
\]

Remark 3.6. Within the statement of Corollary 3.5 above, we leave the extension class that \(Z_{\alpha^* T_T}\) is build from ambiguous on purpose. Indeed, the proof below will implicitly determined this class but the given description is not all that useful for us. Our next order of business will thus be to have a closer look at this class and also give a more explicit description of it.

Proof. (of Corollary 3.5)
Since \(\alpha\) is a locally constant bundle the short exact sequence \(0 \to T_{X/T} \to T_X \to \alpha^* T_T \to 0\) admits a global holomorphic splitting (we may even assume that \(\alpha^* T_T \subseteq T_X\) is integrable; see Lemma 2.2). Hence,
\[
Z_{T_X,[\omega_X]} \cong Z_{T_{X/T},[\omega_{X/T}]} \times_X Z_{\alpha^* T_T}.
\]
according to Proposition 3.4 above. Here, the class defining the affine bundle \(Z_{T_{X/T}}\) is the image of \([\omega_X]\) under the induced map
\[
\text{Ext}^1_{\mathcal{O}} \left( \mathcal{O}_X, \Omega^1_X \right) \to \text{Ext}^1_{\mathcal{O}} \left( \mathcal{O}_X, \Omega^1_{X/T} \right).
\]
Modulo the identification \(\text{Ext}^1_{\mathcal{O}}(\mathcal{O}_X,-) = H^1(X,-)\) this is the proclaimed class. \(\square\)

As explained in Remark 3.6 our next goal is to give an explicit description of the cohomology class defining the extension \(Z_{\alpha^* T_T}\) in Corollary 3.5 above. To this end, we will require some auxiliary results.
Proposition 3.7. Let \( f: X \to T \) be a holomorphic submersion of relative dimension \( m \) between compact Kähler manifolds. Let us denote by \( F_t \) the fibres of \( f \) and fix a Kähler form \( \omega_X \) on \( X \). Then, the function

\[
\text{vol}(F_t, \omega_X|_{F_t}) := \frac{1}{m!} \int_{F_t} (\omega_X|_{F_t})^m
\]

is constant (i.e. does not depend on \( t \)).

Proof. Note that by definition

\[
\text{vol}(F_t, \omega_X|_{F_t}) = \frac{1}{m!} f_*\left(\omega_X^m\right)
\]

where \( f_* \) denotes the integration along the fibres (cf. Definition 5.1). In particular, since \( f_* \) commutes with the exterior derivative (Proposition 5.2) and since \( \omega_X \) is \( d \)-closed (\( X \) being Kähler) also the function \( t \mapsto \text{vol}(F_t) \) is \( d \)-closed, i.e. constant. \( \square \)

Corollary 3.8. Let \( f: X \to T \) be a holomorphic submersion between compact Kähler manifolds. Suppose that every fibre \( F_t \) of \( f \) is Fano and denote \( m := \text{dim}(F_t) \). Fix a Kähler form \( \omega_X \) on \( X \) and recall that by Proposition 3.7 the volume \( \text{vol}(F_t) \) of any fibre is the same. Then, the composition

\[
P: H^q(X, f^*\Omega_T^p) \xrightarrow{i_*} H^q(X, \Omega_X^p) \xrightarrow{\int_{F_t} \wedge^m} H^{q+m}(X, \Omega_X^{p+m}) \xrightarrow{f_*} H^q(T, \Omega_T^p)
\]

is an isomorphism for all \( p, q \). In fact, the inverse is given (up to a factor of \( \frac{1}{\text{vol}(F_t)} \)) by the natural map

\[
f^*: H^q(T, \Omega_T^p) \to H^q(X, f^*\Omega_T^p).
\]

Proof. First, let us prove that \( P \circ f^* = \text{vol}(F) \cdot \text{id} \) using Dolbeaut representatives: Fix any integers \( p, q \) and any closed differentiable \((p,q)\)-form \( \eta \) on \( T \). Using the properties of the push forward we compute

\[
P(f^*([\eta])) = \frac{1}{m!} \left[ f_* \left( f^* \eta \wedge \omega_X^m \right) \right] = \frac{1}{m!} \left[ \eta \wedge f_* \omega_X^m \right] = \left[ \eta \right] \cdot \text{vol}(F)
\]

so that indeed \( P \circ f^* = \text{vol}(F) \cdot \text{id} \). Since both \( H^q(T, \Omega_T^p), H^q(X, f^*\Omega_T^p) \) are finite dimensional vector spaces to complete the proof of our result it thus suffices to prove that \( f^* \) is an isomorphism. Then, (modulo a scalar factor) \( P \) will automatically be its inverse and, hence, an isomorphism itself.
But indeed, since every fibre $F_t$ is Fano the relative Kodaira vanishing theorem yields
\[
R^j f_* O_X = R^j f_* O_X(-K_X + K_X) = 0, \quad \forall \, j > 0.
\]
It follows that also \( R^j f_∗ f^∗ \Omega^p T = \Omega^p T \otimes R^j f_* O_X = 0 \) vanishes for all \( j > 0 \) and so \( f^∗ : H^q(T, \Omega^p_T) \to H^q(X, f^∗ \Omega^p_T) \) is an isomorphism as follows from the Leray spectral sequence. Combining this with Eq. (3) we are done. □

**Proposition 3.9.** Let \( f : X \to T \) be a holomorphic submersion of relative dimension \( m \) between compact Kähler manifolds. Assume that the natural short exact sequence
\[
0 \to f^∗ \Omega^1_T \to \Omega^1_X \to \Omega^1_{X/T} \to 0
\]
admits a global holomorphic splitting \( s : \Omega^1_X \to f^∗ \Omega^1_T \) (recall that this is always true provided that \( f \) is a flat fibre bundle).

Fix a Kähler form \( \omega_X \) on \( X \), consider the decomposition
\[
[\omega_X] = [\omega_{X/T}] + a_T \in H^1 \left( X, \Omega^1_X \right) = H^1 \left( X, \Omega^1_{X/T} \right) \oplus H^1 \left( X, f^∗ \Omega^1_T \right)
\]
according to the splitting \( s \) (i.e. \( a_T = H^1(s)([\omega_X]) \)) and let \( \omega_T := f_*(\omega^{m+1}_X) \) denote the Kähler form on \( T \) obtained from \( \omega_X \) by integration along the fibres. Then,
\[
a_T = \frac{1}{(m+1)!} \cdot \text{vol}(F) \cdot [f^∗ \omega_T] \in H^1 \left( X, \Omega^1_X \right).
\]

**Corollary 3.10.** Let \((X, \omega_X)\) be a compact Kähler manifold with nef tangent bundle and assume that its Albanese \( \alpha : X \to \text{Alb}(X) =: T \) is a locally constant holomorphic fibre bundle.

Then, there exists a natural isomorphism of affine bundles
\[
Z_{T, \omega_X} \cong Z_{T_{X/T}, \omega_{X/T}} \times_X Z_{\alpha^∗ \omega_T, [\alpha^∗ \omega_T]} \cong Z_{T_{X/T}, \omega_{X/T}} \times_T Z_{T, \omega_T}.
\]
Here, by \([\omega_{X/T}]\) we denote the image of \([\omega_X]\) under the natural homomorphism
\[
H^1 \left( X, \Omega^1_X \right) \to H^1 \left( X, \Omega^1_{X/T} \right)
\]
and we denote \( \omega_T := \alpha_∗(\omega^{m+1}_X) \), where \( m := \text{dim } F \).

**Proof.** Since \( \alpha \) is locally constant the short exact sequence \( 0 \to \alpha^∗ \Omega^1_T \to \Omega^1_X \to \Omega^1_{X/T} \to 0 \) splits. According to Proposition 3.9 above, the decomposition of the cohomology class \([\omega_X]\) according to this splitting is given by
\[
[\omega_X] = [\omega_{X/T}] + \lambda \cdot [\alpha^∗ \omega_T] \in \text{Ext}^1_0 \left( O_X, \Omega^1_X \right) = \text{Ext}^1_0 \left( O_X, \Omega^1_{X/T} \right) \oplus \text{Ext}^1_0 \left( O_X, \alpha^∗ \Omega^1_T \right),
\]
where $\lambda := \frac{1}{\text{vol}(F)} > 0$ is some positive real number. In effect, an application of Proposition 3.4 yields

$$Z_{T_X,[\omega_X]} \cong Z_{T_{X/T},[\omega_{X/T}]} \times_X Z_{\alpha^* T_{\alpha},[\alpha^* \omega_T]}.$$  

Since extensions only depend on their defining cohomology class up to scaling by Corollary 1.5 it follows that

$$Z_{T_X,[\omega_X]} \cong Z_{T_{X/T},[\omega_{X/T}]} \times_X Z_{\alpha^* T_{\alpha},[\alpha^* \omega_T]} \cong Z_{T_{X/T},[\omega_T]} \times T T_{X}.$$  

Here, in the last step we used that we know from Proposition 1.6 that there exists a natural identification $Z_{\alpha^* T_{\alpha},[\alpha^* \omega_T]} \cong Z_{T_{\alpha},[\omega_T]} \times T X$. This concludes the proof. \hfill $\square$

*Proof (of Proposition 3.9).* We will verify Eq. (4) by an explicit calculation using Dolbeaut representatives. To this end, recall that $s: \Omega^1_X \to f^* \Omega^1_T$ induces maps of sections $s(0,1): A^{0,1}(\Omega^1_X) \to A^{0,1}(f^* \Omega^1_T)$ and the class

$$i_*(a_T) = i_* \left( H^1(s)([\omega_X]) \right) \in H^1 \left( X, f^* \Omega^1_T \right) \overset{i_*}{\to} H^1 \left( X, \Omega^1_X \right)$$  

is represented by the form $\zeta := i_*(s(0,1)(\omega_X))$. Below, we will show that

$$f_*(\zeta \wedge \omega^m_X) = \frac{f_*(\omega^{m+1}_X)}{m+1}$$  

(6)

This will immediately yield the result because assuming Eq. (6) we compute

$$i_*(a_T) =: [\zeta] \overset{\text{Corollary 3.8}}{=} \frac{1}{\text{vol}(F)} \cdot i_* \left[ f^* f_* \left( \zeta \wedge \omega^m_X \right) \right] \overset{\text{Eq. (6)}}{=} \frac{1}{\text{vol}(F)} \cdot \frac{1}{(m+1)!} \cdot i_* \left[ f^* f_* (\omega^{m+1}_X) \right] \overset{\text{Corollary 3.8}}{=} \frac{1}{\text{vol}(F) \cdot (m+1)!} \cdot i_* \left[ f^* \omega_T \right].$$  

(7)

which, using that by Corollary 3.8 $i_*$ is injective, is the equation to prove. In conclusion, it remains to verify Eq. (6). To this end, fix a point $t \in T$ and vectors $v \in T_t^{(1,0)}$, $w \in T_t^{(0,1)}$. Let $\tilde{V} := s^*(v)$, $\tilde{W} := s^*(w)$ be the differentiable vector fields along $F_t$ induced by the dual splitting $s^*: f^* T_T \hookrightarrow T_X$. Then, $\tilde{V}$, $\tilde{W}$ are of type $(1,0)$ (respectively $(0,1)$) and lift $v$, $w$, i.e.

$$df(\tilde{V}|_x) = v, \quad df(\tilde{W}|_x) = w, \quad \forall x \in F_t.$$  

By definition it holds that

$$f_*(\zeta \wedge \omega^m_X)(v,w) = \int_{F_t} i_{\tilde{V} \tilde{W}} \omega_X^m, \quad f_*(\omega^{m+1}_X)(v,w) = \int_{F_t} i_{\tilde{V} \tilde{W}} \omega^{m+1}_X$$  

(8)

and we need to prove the equality of both expressions (modulo a scalar factor). Clearly it suffices to prove equality of the integrands (as differential forms) and this is what we will do: Fix a point $x \in F_t$ and denote $\tilde{v} := \tilde{V}|_x$, $\tilde{w} := \tilde{W}|_x$. 


Step 1: For all tangent vectors \( v', w' \in T^{1,0}_x X, w' \in T^{0,1}_x X \) it holds that
\[
\zeta(v', w') := \left(i_*(s^{0,1}(\omega_X))(v', w') = \Omega_X (s^*(df(v'))(w')\right.
\]
Indeed, if more generally \( \phi: \mathcal{E} \to \mathcal{F} \) is any morphism between holomorphic vector bundles, then the induced map \( \phi^{(0,1)}: \mathcal{A}^{0,1}(\mathcal{E}) \to \mathcal{A}^{0,1}(\mathcal{F}) \) is determined by the rule \( \phi^{(0,1)}(\sigma \otimes d\bar{z}) = \phi(\sigma) \otimes d\bar{z} \). Accordingly, if \((z^j)\) are some local coordinates centred at \( x \in F_i \) and if with respect to these coordinates \( \omega_X = \sum h_{k,\ell} dz^k \wedge d\bar{z}^{\ell} \), then \( s^{(0,1)}(\omega_X) \) is locally given by the expression
\[
s^{(0,1)}(\omega_X) = s^{(0,1)} \left( \sum h_{k,\ell} dz^k \wedge d\bar{z}^{\ell} \right) = \sum h_{k,\ell} s(dz^k) \otimes d\bar{z}^{\ell}.
\]
Similarly, \( i_*: \mathcal{A}^{0,1}(f^*\Omega_X^1) \to \mathcal{A}^{0,1}(\Omega_X^1) \) is by construction the map induced by the bundle morphism \( (df)^*: f^*\Omega_X^1 \to \Omega_X^1 \). In other words,
\[
\left. i_*(s^{0,1}(\omega_X))(v', w') = \left( \sum h_{k,\ell} X \right) ((df^* \circ s)(dz^k))(v') \otimes d\bar{z}^{\ell}(w') \right.
\]
\[
= \sum h_{k,\ell} dz^{\ell} (s^*(df(v'))(v')) \otimes d\bar{z}^{\ell}(w')
\]
\[
= \left( \sum h_{k,\ell} dz^{\ell} \otimes d\bar{z}^{\ell} \right) (s^*(df(v'))(w') = \omega_X (s^*(df(v'))(w')\right.
\]
Step 2: The following identity holds true:
\[
\left. i_{\bar{v}}(\zeta \wedge \omega^m_X) |_{F_i} = \omega_X (\bar{v}, \bar{w}) \cdot \omega^m_X - \iota_{\bar{v}}(\omega_X \wedge \bar{\omega}^m_X) \right|_{F_i}.
\]
Using the formula in Proposition 5.3 regarding contractions by vectors of wedge products we compute
\[
\left. i_{\bar{v}}(\zeta \wedge \omega^m_X) = \iota_{\bar{v}}(\zeta \wedge \omega^m_X + (-1)^2 \zeta \wedge \iota_{\bar{v}}(\omega^m_X)) \right.
\]
\[
= \zeta(\bar{v}, \bar{w}) \cdot \omega^m_X + (-1) \zeta(\bar{v}) \wedge \iota_{\bar{v}}(\omega^m_X)
\]
\[
+ (-1)^2 \zeta(\omega^m_X) + (-1)^4 \zeta \wedge \iota_{\bar{v}}(\omega^m_X). \quad (10)
\]
Now, according to Step 1 it holds that
\[
\zeta(v', -) = \omega_X (s^*(df(v')), -), \quad \forall v' \in T^{0,1}_x X. \quad (11)
\]
In particular, if \( v' \) is tangent along the fibres, then \( df(v') = 0 \) and so \( \iota_{\bar{v}} \zeta = 0 \). This immediately implies that
\[
\iota_{\bar{v}}(\zeta)|_{F_i} = \zeta|_{F_i} = 0. \quad (12)
\]
On the other hand, consider the case \( v' = \bar{v} \) in Eq. (11) above. Then,
\[
s^*(df(\bar{v})) = \frac{df(\bar{v})}{=v} s^*(v) =: \tilde{v}
\]
by definition of \( \tilde{v} \). In view of [Eq. (11)] this implies that
\[
\zeta(\tilde{v}, \tilde{w}) = \omega_X(\tilde{v}, \tilde{w}), \quad \iota_{\tilde{v}}(\zeta) = \iota_{\tilde{v}}(\omega_X).
\] (13)

Substituting the terms in [Eq. (10)] above using [Eq. (12)] and [Eq. (13)] we find
\[
\iota_{\tilde{v}, \tilde{w}}(\zeta \wedge \omega_m^X) \big|_{F_t} = \left. (\omega_X(\tilde{v}, \tilde{w}) \cdot \omega_m^X - \iota_{\tilde{v}}(\omega_X) \wedge \iota_{\tilde{w}}(\omega_m^X) + 0) \right|_{F_t}.
\]
which is the identity in question.

**Step 3:** It holds that
\[
\iota_{\tilde{v}, \tilde{w}}(\omega_m^{m+1} X) = (m + 1) (\omega_X(\tilde{v}, \tilde{w}) \cdot \omega_m^X - \iota_{\tilde{v}}(\omega_X) \wedge \iota_{\tilde{w}}(\omega_m^X)).
\]

Using again [Proposition 5.3] we compute
\[
\iota_{\tilde{v}, \tilde{w}}(\omega_m^{m+1} X) = (m + 1) \cdot (\omega_X(\tilde{v}, \tilde{w}) \cdot \omega_m^X - \iota_{\tilde{v}}(\omega_X) \wedge \iota_{\tilde{w}}(\omega_m^X)).
\]

This finishes the proof of **Step 3**.

**Step 4:** Conclusion.

Combining the results of **Step 2** and **Step 3** we find that
\[
\iota_{\tilde{v}, \tilde{w}}(s(\omega_X) \wedge \omega_m^X) \big|_{F_t} = \frac{1}{m + 1} \cdot \iota_{\tilde{v}, \tilde{w}}(\omega_m^{m+1} X) \big|_{F_t}.
\]

Thus, the integrands in [Eq. (8)] and [Eq. (9)] above agree (up to scaling) and, hence,
\[
f_*(s(\omega_X) \wedge \omega_m^X)(v, w) = \frac{(f_* \omega_m^{m+1})(v, w)}{m + 1}, \quad \forall v \in T^{(1,0)} T, \forall w \in T^{(0,1)} T.
\]

This proves [Eq. (6)] and, as discussed above in [Eq. (7)], the result immediately follows. □

**Corollary 3.10** yields a splitting \( Z_X \cong Z_{X/T} \times_T Z_T \). Our next goal is to prove that the induced map \( Z_X \to Z_T \) makes \( Z_X \) into a holomorphic fibre bundle with typical fibre \( Z_F \).

To this end, we first need to take a closer look at \( Z_F \):

**Proposition 3.11.** Let \((F, \omega)\) be a compact Kähler manifold and denote \( G := \text{Aut}_0(F) \).
It is well-known that \( G \) is a complex Lie group (see for example [Akh95, Section 2.3.]). Moreover,

1. the natural action of \( G \) on \( H^*(F, \mathbb{R}) \) is trivial.
2. If \( H^1(F, \mathbb{R}) = 0 \), then the action of \( G \) on \( F \) extends naturally to an action by automorphisms of affine bundles on \( Z_{[\omega_F]} \).


Proof. Regarding the first statement, since $G$ is a Lie group, $G = \text{Aut}^0(F)$ is not only the connected component of the identity in $\text{Aut}(F)$ but also the path-connected component. Thus, for any $g \in G$ there exists a (smooth) path from $\text{id}_F$ to $g$ in $G$. But such a path is nothing but a (smooth) homotopy between $\text{id}_F$ and $g$, i.e. all maps in $G$ are null homotopic. In particular, they induce the identity maps on de Rahm cohomology.

For the second statement, note that any element $g \in G$ naturally induces an isomorphism of affine bundles

$$g : Z_{[\omega_F]} \to g^*Z_{[\omega_F]} = Z_{[g^*\omega_F]}.$$ 

Since the action of $G$ on $H^*(F, \mathbb{R})$ is trivial by item (1), in particular $[g^*\omega_F] = [\omega_F]$ for all $g \in G$. Hence, there exists an isomorphism of affine bundles $Z_{[g^*\omega_F]} \cong Z_{[\omega_F]}$. We claim, that in fact there exists only one such isomorphism. In particular, we may identify $Z_{[g^*\omega_F]}$ and $Z_{[\omega_F]}$ in a natural way and so the action of $G$ on $F$ lifts to $Z_F$ as required.

Regarding the claim, by construction any isomorphism as above is induced by an isomorphism of extensions or, in other words, by a commutative diagram as below:

$$\begin{array}{cccc}
0 & \to & \Omega^1_F & \to & V & \to & \mathcal{O}_F & \to & 0 \\
\| & & \| & & \| & & \| & & \\
0 & \to & \Omega^1_F & \to & V & \to & \mathcal{O}_F & \to & 0
\end{array}$$

It is now easily verified by a diagram chase that any morphism $\phi$ making the above diagram commute is of the form $\phi = \text{id} + p \cdot \eta$, where $\eta \in \text{Hom}(\mathcal{O}_F, \Omega^1_F) = H^0(F, \Omega^1_F)$.

and, as before, $V \xrightarrow{\phi} \mathcal{O}_X$. But $\dim \mathcal{O}_F = \dim \mathcal{O}_X = 0$ by the Hodge decomposition. Thus, there is only one isomorphism of affine bundles $Z_{[g^*\omega_F]} \cong Z_{[\omega_F]}$ and we are done.

Lemma 3.12. Let $f : X \to T$ be a holomorphic fibre bundle with structure group $G$ and with typical fibre $F$. Suppose that $X$ and $T$ are compact Kähler and fix a Kähler metric $\omega_X$ on $X$. Suppose moreover that $G \subseteq \text{Aut}^0(F)$ and that $H^1(F, \mathbb{C}) = 0$. Then, also

$$f \circ p : Z_{X/T} := Z_{T_X/T, [\omega_X/T]} \to X \to T$$

is a holomorphic fibre bundle. Its typical fibre is $Z_{T_F, [\omega_X|F]}$ and the structure group may be chosen to be $G$.

Note that $G$ indeed acts on $Z_{T_F, [\omega_X|F]}$ by Proposition 3.11 so that the assertion about the structure group of the bundle makes sense.

Proof. Since both $f : X \to T$ and $p : Z_{X/T} \to X$ are holomorphic fibre bundles, $f \circ p$ is at least a surjective holomorphic submersion. Moreover, it follows from the functoriality of the construction of $Z_-$ (see Proposition 1.6) that the fibre of $f \circ p$ over $t \in T$ is given by

$$(f \circ p)^{-1}(t) = p^{-1}(F_t) = Z_{X/T} \times_X F_t \xrightarrow{\text{Proposition 1.6}} Z_{T_{X/T}, [\omega_X|F_t]} = Z_{T_{F_t}, [\omega_X|F_t]}.$$
Now, fix \( t \in T \), denote \( F := f^{-1}(t) \) and choose a sufficiently small open polydisc \( t \in U \subset T \) so that \( f^{-1}(U) \cong U \times F \) is trivial. We want to show that there exists an isomorphism of fibre bundles

\[
Z_{X/T}|U \cong U \times Z_{TF}.\omega_X|F
\]

respecting the affine bundle structure on both sides. Indeed, since \( U \) is a polydisc it holds that \( H^j(U, \mathbb{C}) = 0 \) for all \( j > 0 \). Thus, according to the classical Künneth formula the map

\[
pr^*_F: H^*(F, \mathbb{C}) \to H^*(U \times F, \mathbb{C})
\]

is an isomorphism. Note that an inverse is clearly provided by the restriction map

\[
| t \rangle \times F: H^*(U \times F, \mathbb{C}) \to H^*(F, \mathbb{C}).
\]

In particular, we find that

\[
[\omega_X|U \times F] = pr^*_F[\omega_X|F].
\] (15)

Using again the functionality of extensions and the fact that \( \mathcal{T}_{U \times F/U} = pr^*_F \mathcal{T}_F \) we compute

\[
Z_{TX/T.|\omega_{X/T}}|U = Z_{TU \times F/U.|\omega_{X/T}} \overset{\text{Eq. (15)}}{=} Z_{pr^*_F \mathcal{T}_F.pr^*_F.|\omega_F}
\]

\[
\overset{\text{Proposition 1.6}}{=} pr^*_F Z_{\mathcal{T}_F.|\omega_F} := U \times Z_{TF.|\omega_F}.
\]

This proves Eq. (14) and, hence, that \( f \circ p \) is a holomorphic fibre bundle with fibre \( Z_F \).

The assertion about the structure group being \( G \) is clear, because we already saw as part of the proof of Proposition 3.11 that given any \( g \in G \), there is one and only one identification of \( Z_F \) and \( g^*Z_F \) as affine bundles. Hence, both \( f: X \to T \) and \( f \circ p: Z_{X/T} \to T \) are constructed using the same transition functions.

\( \square \)

**Remark 3.13.** Record for later reference that both the bundles \( f: X \to T \) and \( f \circ p: Z_{X/T} \to T \) are constructed using the same transition functions. In particular, the first is locally constant if and only if the latter is so.

**Corollary 3.14.** Let \( f: X \to T \) be a holomorphic fibre bundle. Assume that \( X \) and \( T \) are compact Kähler, fix a Kähler form \( \omega_X \) on \( X \) and suppose that the typical fibre \( F \) of \( f \) is a Fano manifold. Suppose moreover that the structure group \( G \) of \( f \) is contained in \( \text{Aut}^0(F) \) and that the short exact sequence

\[
0 \to T_{X/T} \to T_X \to f^*T_T \to 0
\]

admits a global holomorphic splitting (which is satisfied if for example \( f \) is locally constant).

Then, there exists an isomorphism of affine bundles

\[
Z_{TX.|\omega_X} \cong Z_{TX/T.|\omega_{X/T}} \times_T Z_{TF.|\omega_T}.
\] (16)
Here, $\omega_T := f_*(\omega_X^{n+1})$ is the Kähler form on $T$ obtained from $\omega_X$ by integration along the fibres. Moreover, the projection map

$$\tilde{f} : Z_{TX, [\omega_X]} \to Z_{TT, [\omega_T]}$$

makes $Z_X$ into a (flat if $f$ is flat) holomorphic fibre bundle over $Z_T$ with fibre $Z_{F, [\omega_X|_F]}$ and structure group $G$.

**Proof.** First of all, Eq. (16) has already been verified in Corollary 3.10. Regarding the second assertion, note that $H^1(F, \mathbb{C}) = 0$ as $F$ is Fano. Thus, Lemma 3.12 above applies and yields that

$$Z_{TX/T, [\omega_{X/T}]} \to T$$

is a (flat; see Remark 3.13) holomorphic fibre bundle with structure group $G$ and fibre $Z_F$. But Eq. (16) just says that

$$\tilde{f} : Z_{X, [\omega_X]} \to Z_{T, [\omega_T]}$$

is the pull back along $Z_T \to T$ of the bundle $Z_{X/T} \to T$. Hence, along with $Z_{X/T} \to T$ also $\tilde{f}$ is a (flat) holomorphic fibre bundle with structure group $G$ and fibre $Z_F$. □

The following trick may be used to show that the condition $G \subseteq \text{Aut}^0(F)$ in Corollary 3.14 above is essentially superfluous.

**Proposition 3.15.** Let $f : X \to T$ be a holomorphic fibre bundle with typical fibre $F$, where both $X$ and $T$ are compact complex manifolds. Suppose that the group $\text{Aut}(F)/\text{Aut}^0(F)$ is finite (by [Bri18, Corollary 2.17] this is satisfied for example if $F$ is Fano). Then, there exists a finite étale cover $T' \to T$ such that the structure group of the holomorphic fibre bundle $X \times_T T' \to T'$ may be chosen to be contained in $\text{Aut}^0(F)$.

**Proof.** Let us abbreviate $G := \text{Aut}(F)$ and $G^0 := \text{Aut}^0(F)$. Since $G = \text{Aut}(F)$ acts effectively on $F$, there exists a unique holomorphic principal $G$-bundle $\mathcal{G} \to T$ such that $X \to T$ is the associated bundle with typical fibre $F$. Then,

$$T' := \mathcal{G}/G^0 \to T$$

is a finite étale cover of $T$ (since $G/G^0$ is finite by assumption) and by construction the structure group of the principal $G$-bundle $\mathcal{G} \times_T T' \to T'$ may be reduced to $G^0$. In effect, the same is true of the associated bundle $X \times_T T' \to T'$ and so we are done. □

We are now finally ready to prove the main result of this chapter:

**Theorem 3.16.** Let $(X, \omega_X)$ be a compact Kähler manifold with nef tangent bundle. If the weak Campana-Peternell conjecture [Conjecture 2.5] holds true then the canonical extension

$$Z_{X, [\omega_X]}$$

is a Stein manifold.
Proof. According to Theorem 2.1 there exists a finite étale cover \( \pi: X' \to X \) such that the Albanese \( \alpha: X' \to \text{Alb}(X') = T \) is a locally constant holomorphic fibre bundle. Its fibres are Fano manifolds with nef (and, hence, assuming Conjecture 2.5 also big) tangent bundle. Possibly replacing \( X' \) by another finite étale cover we may moreover assume by Proposition 3.15 above that the structure group \( G \) of \( \alpha \) is contained in \( \text{Aut}^0(F) \). But in this situation Corollary 3.14 applies to the compact Kähler manifold \( (X', \pi^*\omega_X) \) and shows that there exists a natural map

\[
\bar{\alpha} : Z_{X, [\pi^*\omega_X]} \to Z_{T, [\omega_T]}
\]

making \( Z_{X'} \) into a flat holomorphic fibre bundle with structure group \( G \subseteq \text{Aut}^0(F) \) and fibre

\[
Z_{F, [\pi^*\omega_X | F]}
\]

Here, \( \omega_T \) in Eq. (17) above is some (explicitly determined) Kähler form on \( T \). Note that by Proposition 3.11 \( \text{Aut}^0(F) \) acts on \( Z_F \) so that we may well assume the structure group of \( \bar{\alpha} \) to be \( \text{Aut}^0(F) \). Note moreover, that we already proved in Theorem 3.1 that \( Z_T \) must be Stein as a canonical extension of a complex torus and we showed in Theorem 3.2 that \( Z_F \) must be Stein as a canonical extension of a Fano manifold with big and nef tangent bundle.

In summary, \( Z_{X'} \) is naturally a holomorphic fibre bundle over the Stein manifold \( Z_T \). The typical fibre of this bundle is \( Z_F \), a Stein manifold, and the structure group of the bundle may be chosen to be the connected group \( \text{Aut}^0(F) \). But it is a classical theorem by [MM60, Théorème 6.] that in this situation also the total space

\[
Z_{X', [\pi^*\omega_X]}
\]

of the bundle is Stein. Finally, since \( \pi: X' \to X \) is finite étale \( \text{Proposition 1.9} \) yields that also \( Z_{X, [\omega_X]} \) is Stein and so we are done. \( \square \)

4 The special case of surfaces

As was already mentioned above, regarding the converse question of whether the tangent bundle of a manifold that admits canonical extensions which are Stein is nef little is known. In [HP21, Corollary 1.7.] it is proved that the tangent bundle must at least be pseudo-effective (in the weak sense, i.e. \( \mathcal{O}_{P(T_X)}(1) \) must be pseudo-effective) but this is far less than the expected nefness. As this question seems very difficult it is natural to concentrate on the low-dimensional cases first. Indeed, in [HP21] it is proved that:

**Theorem 4.1.** (Höring-Peternell, [HP21, Theorem 1.13.])

Let \( X \) be a smooth projective surface. Assume that there exists some Kähler class \( \omega_X \) on \( X \) whose canonical extension is Stein. Then, one of the following holds true:

(1) \( X \) is an étale quotient of a complex torus.
(2) \( X \) is a homogeneous Fano surface, i.e. either \( X = \mathbb{P}^2 \) or \( X = \mathbb{P}^1 \times \mathbb{P}^1 \).

(3) \( X = \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} C \) is a ruled surface over a curve of genus \( g(C) \geq 1 \). Moreover, if \( g(C) \geq 2 \) then \( \mathcal{E} \) must be semi-stable.

Note that item (3) is not quite what we expect: First of all, if \( g(C) \geq 2 \), then the tangent bundle of \( X \) can not be nef and so we would not expect any canonical extension to be Stein. Here, the reason for the first assertion is the relative tangent bundle sequence \( 0 \to T_{X/C} \to T_X \to \pi^*T_C \to 0 \): If \( T_X \) were nef then so were its quotient \( \pi^*T_C \) and, hence, \( T_C \) itself.

Moreover, it is well-known that the tangent bundle of a ruled surface \( X = \mathbb{P}(\mathcal{E}) \) over an elliptic curve is nef if and only if the defining bundle \( \mathcal{E} \) is semi-stable (cf. [DPS94, Theorem 6.1]).

This raises the question of what is true in the remaining cases. Indeed, we are able to rule out the higher genus case as well; to this end, we need the following auxiliary result:

**Proposition 4.2.** Let \( X = \mathbb{P}(\mathcal{E}) \to C \) be a ruled surface. If \( \mathcal{E} \) is semi-stable, then \( \pi \) is a locally constant fibre bundle.

**Proof.** This fact is rather well-known, see for example [JR13, Theorem 1.5, Proposition 1.7]. \( \square \)

**Lemma 4.3.** Let \( X = \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} C \) be a ruled surface over a curve of genus \( g(C) \geq 2 \) defined by a semi-stable vector bundle \( \mathcal{E} \). Then, no canonical extension of \( X \) is Stein.

**Proof.** Assume to the contrary that there exists a Kähler metric \( \omega_X \) on \( X \) whose canonical extension \( Z_X \) is Stein.

By [Proposition 4.2] \( \pi \) is a locally constant fibre bundle. In other words, if we denote by \( C \xrightarrow{\tilde{\pi}} \tilde{C} \) the universal cover of \( C \), then there exists a group homomorphism \( \rho: \pi_1(C) \to \text{Aut}(\mathbb{P}^1) =: G \) such that

\[
X \cong \pi_1(C) \backslash (\tilde{C} \times \mathbb{P}^1).
\]

Here, the reason for exceptionally denoting the quotient as one from the left is that shortly we will introduce a second action of a group. It will be crucial below that both of these groups will act from different sides so that the actions commute.

In any case, as \( \pi: X \to C \) is a locally constant fibre bundle with fibre \( \mathbb{P}^1 \) - a Fano manifold with connected automorphism group - [Corollary 3.14] applies and shows that we may also consider \( Z_X \) as a flat fibre bundle over \( Z_C \) with typical fibre \( \mathbb{P}^1 \) and with the same transition functions as \( X \to C \). Here, for the latter assertion we use [Remark 3.13] and the fact, that by [Proposition 3.11] the action of \( \text{Aut}(\mathbb{P}^1) \) on \( \mathbb{P}^1 \) lifts uniquely to \( Z_{\mathbb{P}^1} \). In summary, we may identify

\[
Z_{X,|\omega_X|} \cong \pi_1(C) \backslash \left( (Z_{\tilde{C},|\rho^*\omega_C|} \times Z_{\mathbb{P}^1,|\omega_{\mathbb{P}^1}|}) = \pi_1(C) \backslash \left( (Z_{\tilde{C},|\rho^*\omega_C|} \times \text{G/L}) \right) \right). \quad (18)
\]
Here, $\omega_C := f_*(\omega_X \wedge \omega_X)$ is the induced Kähler form on $C$. Moreover, we used that according to Remark 3.3 there exists a canonical $G$-equivariant identification of canonical extensions

$$\left( Z_{\mathbb{P}^1} \to \mathbb{P}^1 \right) = (G/L \to G/P).$$

The precise definition of the group $L \subseteq P \subseteq G$ is contained in Remark 3.3; we will only use the fact that $L \cong \mathbb{G}_m$ is connected and Stein.

Now, let us consider the manifold

$$G := \pi_1(C) \setminus \left( Z_{\mathbb{C},[p^*\omega_C]} \times G \right).$$

The natural projection $G \to Z_C$ makes it into a (right) principal $G = \text{Aut}(\mathbb{P}^1)$-bundle. Then, clearly combining Eq. (19) with Eq. (18) we deduce that $Z_{X,[\omega_X]} \cong \pi_1(C) \setminus \left( Z_{\mathbb{C},[p^*\omega_C]} \times G/L \right) \cong \pi_1(C) \setminus \left( Z_{\mathbb{C},[p^*\omega_C]} \times G \right) / L = G/L.

In other words, $G \to Z_X$ is naturally a (right) principal $L$-bundle. Note that $Z_X$ is Stein by assumption and that $L$ is connected and Stein (cf. Remark 3.3). Therefore, [MM60, Théorème 6.] again applies and proves that also $G$ is Stein. On the other hand, $G \to Z_C$ is naturally a (right) $G = \text{Aut}(\mathbb{P}^1)$-bundle. Since quotients of Stein spaces by reductive groups are again Stein by [Sno82], we infer that also $Z_{C,[\omega_C]} = G/G$ is Stein. But this contradicts [GW20, Example 3.6.] as $g(C) \geq 2$. Thus, $Z_X$ can not be Stein after all and we are done.

**Remark 4.4.** Note that essentially ad verbatim the same argument also yields the following: Let $f : X \to Y$ be a locally constant fibration with fibre $F$ and assume that $F = G/P$ is a homogeneous Fano. If the exists a Kähler form $\omega_X$ on $X$ such that the canonical extension $Z_{X,[\omega_X]}$ is Stein, then there exists a Kähler form $\omega_Y$ on $Y$ (in fact, $\omega_Y = f_*(\omega_X^{p+1})$ does the job) so that also $Z_{Y,[\omega_Y]}$ is Stein.

The case of unstable ruled surfaces over elliptic curves however is still completely open:

**Question 4.5.** Let $X = \mathbb{P}(\mathcal{E}) \to E$ be a ruled surface over an elliptic curve defined by an unstable bundle $\mathcal{E}$ (so that $T_X$ is not nef). Is it true, that no canonical extension of $X$ Stein?

This question is interesting because such surfaces lie on the boundary of what is known: One can show that they belong to the very restricted class of surfaces whose tangent bundle is (strongly) pseudo-effective but not nef (compare the discussion in [HIM22]). Thus, an affirmative answer to Question 4.5 would provide a serious indication towards the correctness of Conjecture 1.8. On the other hand, it seems very much possible that the answer to Question 4.5 may turn out to be negative. In this case, it would of course be interesting to see how much positivity exactly one can infer from the Steinness of canonical extensions.
5 Appendix

5.1 Integration along fibres

Since there is no universally agreed upon convention regarding the definition of integration along fibres, let us quickly state below the one we use:

Definition 5.1. Let $f : X \to T$ be a proper holomorphic submersion with fibres $F_t$. Denote $m := \dim F$. Given any differentiable $k$-form $\eta \in \mathcal{A}_X^k$ on $X$, we define the $(k - 2m)$-form $f_* \eta$ on $T$ by the rule

$$(f_* \eta)(V_1, \ldots, V_{k-2m})|_t := \int_{F_t} \eta(\tilde{V}_1, \ldots, \tilde{V}_{k-2m}, -), \quad \forall V_1, \ldots, V_{k-2m} \in T^C T,$$

where $\tilde{V}_1, \ldots, \tilde{V}_{k-2m}$ are any locally defined lifts of $V_1, \ldots, V_{k-2m}$ to $X$. We call $f_* \eta$ the form obtained by integrating $\eta$ along the fibres or the push forward of $\eta$ by $f$.

With this convention, the following properties of the push-forward are straightforward to verify:

Proposition 5.2. Integration along the fibres induces well-defined $C$-linear maps

$$f_*: \mathcal{A}_X^k \to \mathcal{A}_T^{k-2m}.$$

Moreover, it satisfies the following formulae:

1. Push forward preserves type: If $\eta \in \mathcal{A}_X^{p,q}$, then $f_* \eta \in \mathcal{A}_T^{p-m,q-m}$.

2. Push forward commutes with the exterior derivative: $d \circ f_* = f_* \circ d$. In particular, $f_*$ induces morphisms

$$f_*: H^k(X, \mathbb{C}) \to H^{k-m}(T, \mathbb{C}).$$

Similarly, $f_*$ commutes also with $\partial, \bar{\partial}$.

3. Push forward satisfies the projection formula: For all differential forms $\zeta$ on $T$ and $\eta$ on $X$ it holds that

$$f_*(f^* \zeta \wedge \eta) = \zeta \wedge f_* \eta.$$

4. The push forward of a (strictly) positive form on $X$ is a (strictly) positive form on $T$.

In particular, if $\omega_X$ is a Kähler form on $X$, then $f_*(\omega_X^{m+1})$ is a strictly positive closed $(1,1)$-form on $T$, i.e. a Kähler form.
5.2 Some formulae from multi-linear algebra

While we are at it, let us state the following formulae used in the main text:

**Proposition 5.3.** Let $V$ be a complex vector space and let $\varphi \in \wedge^k V^*$, $\psi \in \wedge^\ell V^*$ and $\omega \in \wedge^{2k} V^*$ be skew-symmetric forms on $V$ of the indicated degree. Then, for all vectors $v, w \in V$ the following identities are satisfied:

\[
\iota_v(\varphi \wedge \psi) = \iota_v(\varphi) \wedge \psi + (-1)^k \varphi \wedge \iota_v(\psi),
\]
\[
\iota_v(\omega^m) = m \cdot \iota_v(\omega) \wedge \omega^{m-1},
\]
\[
\iota_w \iota_v(\omega^m) = m \cdot \iota_w \iota_v(\omega) \wedge \omega^{m-1} - m(m-1) \iota_v(\omega) \wedge \iota_w(\omega) \wedge \omega^{m-1}.
\]

Here, as per usual $\iota_v$ is the contraction by $v$: $\iota_v \varphi = \varphi(v, -)$.

**Proof.** The first identity is proved in [Lee13, Lemma 14.13.]. The second formula clearly follows from the first one by an induction argument (note that we assumed $\omega$ to be of even degree to avoid worries about the correct signs). Finally, the third one is obtained by applying the first identity to the second one. □

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