On Wilson loops for two touching circles with opposite orientation

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Abstract
We study the Wilson loops for contours formed by a consecutive passage of two touching circles with a common tangent, but opposite orientation. The calculations are performed in lowest nontrivial order for $\mathcal{N} = 4$ SYM at weak and strong coupling and for QCD at weak coupling. After subtracting the standard linear divergence proportional to the length, as well the recently analysed spike divergence, we get for the renormalised Wilson loops

$$\log W_{\text{ren}} = 0.$$  

The result holds for circles with different radii and arbitrary angle between the discs spanned by them.

Keywords: Wilson loops, renormalisation, AdS/CFT

(Some figures may appear in colour only in the online journal)

1. Introduction

Ultraviolet divergences of Wilson loops for smooth contours, as well as for those with cusps and intersecting points, have been studied in much detail from the early eighties to present time. Especially the cusp anomalous dimension has drawn a lot of attention since it is also related to various other physical situations, see e.g. [1] and references therein. It diverges in the limit of vanishing opening angle. However, the removal of the regularisation does not commute with that limit, and only recently we have started the investigation of renormalisation in the presence of zero opening angle cusps i.e. spikes [2].

A spike turned out to be responsible for a divergence proportional to the inverse of the square root of the product of the dimensional cutoff times the jump in the curvature. The analysis has been performed in lowest order at weak coupling both for $\mathcal{N} = 4$ SYM and QCD and at strong coupling via holography in the supersymmetric case. In addition, the spike generates in the SUSY case, at least at weak coupling, an additional logarithmic divergence, which could be related to the breaking of zig–zag symmetry [3, 4].
Although the lowest order setting in [2] was very simple, the safe extraction of terms beyond the leading divergence required some technical effort. In the present paper we go one step further and want to evaluate also the finite terms, which after subtraction of the divergences define the renormalised Wilson loops. This we will do for a special contour. It is formed out of two touching circles with a common tangent in the following way. After starting at the common point one traverses the first circle and then continues along the second circle in just the opposite direction. The discs related to the circles are allowed to form an angle $\beta$.

The paper is organised as follows. The next section is devoted to lowest order at weak coupling for the locally supersymmetric Wilson loop. Then section 3 contains the holographic analysis at strong 't Hooft coupling. In section 4 we comment on the situation without supersymmetry by subtracting the scalar contributions from the result in section 2. After the concluding section follow two appendices containing the technical details of the asymptotic estimates of the necessary integrals.

2. Lowest order at weak coupling in $\mathcal{N} = 4$ SYM

In $\mathcal{N} = 4$ SYM the Euclidean local supersymmetric Wilson loop for a closed contour parameterised by $x^{\mu}(\tau)$ is given by [4–6]

$$W = \frac{1}{N} \langle \text{tr Pexp} \int (iA_\mu \dot{x}^\mu + |\dot{x}| \theta^I \theta^J) \, d\tau \rangle. \tag{1}$$

For simplicity we consider only the case of fixed $\theta^I \in S^5$.

Our contour of interest has been characterised in the introduction. Let the two circles with radii $R_1 > R_2$ be parameterised by

$$\vec{x}_1(\varphi_1) = R_1(\sin \varphi_1, 1 - \cos \varphi_1, 0),
\vec{x}_2(\varphi_2) = R_2(\sin \varphi_2, \cos \beta(1 - \cos \varphi_2), \sin \beta(1 - \cos \varphi_2)). \tag{3}$$

Then the contour to be used in (1) is given by

$$\vec{x}(\tau) = \vec{x}_1(\tau), \quad 0 \leq \tau \leq 2\pi,
\vec{x}(\tau) = \vec{x}_2(4\pi - \tau), \quad 2\pi \leq \tau \leq 4\pi. \tag{4}$$

The situation for a fixed larger circle and smaller partner circles at various values of the angle $\beta$ is illustrated in figure 1.

Then the perturbative expansion of this Wilson loop is given by

$$\log W = \frac{g^2 C_F}{4\pi^2} \left( I_1 + I_2 + I_{12} \right) + \mathcal{O}(g^4). \tag{5}$$

The integrals $I_1$ and $I_2$ correspond to the contributions, where both endpoints of the propagators are on the same circle. In $I_{12}$ the propagators connect the two circles. This means ($\epsilon$ denotes a dimensionful parameter for UV regularisation)

$$I_j = \int_0^{2\pi} \int_0^{\varphi_2} \frac{R_j^2(1 - \cos(\varphi_1 - \varphi_2))}{2R_j^2(1 - \cos(\varphi_1 - \varphi_2)) + \epsilon^2} \, d\varphi_1 d\varphi_2. \tag{6}$$
Performing one trivial integration one gets
\[ I_j = \frac{\pi}{2} \int_0^{2\pi} \frac{(1 - \cos x) \, dx}{1 - \cos x + \frac{x^2}{2R_j^2}} = \pi^2 + \mathcal{O}(\epsilon). \] (7)

Furthermore \( I_{12} \) is given by
\[ I_{12} = \int_0^{2\pi} \int_0^{2\pi} \frac{R_1 R_2 (1 + \cos \varphi_1 \cos \varphi_2 + \cos \beta \sin \varphi_1 \sin \varphi_2) \, d\varphi_1 \, d\varphi_2}{(x_1 - x_2)^2 + \epsilon^2}. \] (8)

Then, performing the \( \varphi_2 \)-integration, we get\(^1\)
\[ I_{12} = 4\pi \int_0^\pi (f(R_1, R_2, \beta, \varphi) + g(R_1, R_2, \beta, \epsilon, \varphi)) \, d\varphi, \] (9)

with
\[ f(R_1, R_2, \beta, \varphi) = \frac{1}{2} \frac{R_1 (R_1 \cos \beta (\cos \varphi - 1) - R_2 \cos \varphi)}{R_1^2 \sin^2 \varphi + (R_2 + R_1 \cos \beta (\cos \varphi - 1))^2} \] (10)

and, using the abbreviation
\[ U(R_1, R_2, \beta, \varphi) = R_1 R_2 \cos \beta (1 - \cos \varphi) - R_2^2, \]
\[ g(R_1, R_2, \beta, \epsilon, \varphi) = \frac{R_1 R_2 - \left( \epsilon^2 + 2R_1^2 (1 - \cos \varphi) - 2U \right) \left( R_1 R_2 \cos \varphi - R_2^2 \cos \beta \sin^2 \varphi \right)}{\sqrt{\left( \epsilon^2 + 2R_1^2 (1 - \cos \varphi) - 2U \right)^2 - 4U^2 - 4R_1^2 R_2^2 \sin^2 \varphi}}. \] (11)

\(^1\) Since the remaining integrand depends on \( \cos \varphi \) only, the original integral over \((0, 2\pi)\) can be written as twice that over \((0, \pi)\).
The indefinite integral over \( f(R_1, R_2, \beta, \varphi) \) is
\[
- \frac{1}{2} \arctan \left( \frac{R_1 \sin \varphi}{R_2 - R_1 \cos \beta (1 - \cos \varphi)} \right).
\]
It is zero at both ends of the integration interval of the definite integral needed in (9). However one has to be careful, since for
\[
\cos \beta > \frac{R_2}{2R_1}
\]
the argument of \( \arctan \)-function passes infinity within the integration interval \( \varphi \in [0, \pi] \). This leads to (\( \Theta \) denoting the step function)
\[
\int_0^\pi f(R_1, R_2, \beta, \varphi) \, d\varphi = - \frac{\pi}{2} \Theta \left( \cos \beta - \frac{R_2}{2R_1} \right).
\]
(12)
For the integral over \( g(R_1, R_2, \beta, \epsilon, \varphi) \) we change the integration variable via
\[
\frac{1}{2} B_\epsilon^2 (1 - \cos \varphi) = x^2,
\]
where we introduced the abbreviations
\[
B_\epsilon = \sqrt{\frac{2R_1 R_{12}}{\epsilon R_2}}
\]
(14)
and
\[
R_{12}(R_1, R_2, \beta) = \sqrt{R_1^2 + R_2^2 - 2R_1 R_2 \cos \beta}.
\]
(15)
\( R_{12} \) is just the distance between the centers of the two circles. It is also via
\[
\frac{R_{12}}{R_1 R_2} = |\vec{k}_1 - \vec{k}_2|
\]
(16)
related to the difference of the curvature vectors at the touching point.

Then we arrive with (9), (11) and (12) at
\[
I_{12} = - 2\pi^2 \Theta \left( \cos \beta - \frac{R_2}{2R_1} \right) + \frac{4\pi}{\sqrt{\epsilon}} \sqrt{\frac{2R_1 R_2}{R_{12}}} \int_0^{\beta_e} h(R_1, R_2, \beta, \epsilon, x) \, dx.
\]
(17)
In the above equation use has been made of the following definitions
\[
h(R_1, R_2, \beta, \epsilon, x) = \sqrt{\frac{1 + x^4}{1 + x^4 + \frac{\epsilon^2}{4R_1^2} + h_3 \epsilon x^2}}.
\]
(18)
with
\[
h_1(R_1, R_2, \beta, \epsilon) = \frac{R_2 (4(R_1^2 - R_2^2) + 2R_{12}^2) + (R_2 \cos \beta - R_2) \epsilon^2}{4R_1 R_2 R_{12}},
\]
(19)
\[
h_2(R_1, R_2, \beta) = \frac{(R_1^2 + R_2^2) \cos \beta - 2R_1 R_2}{2R_1 R_2 R_{12}}.
\]
(20)
\[ h_3(R_1, R_2, \beta) = \frac{R_1 - R_2 \cos \beta}{R_1 R_{12}}. \]  

(21)

We are interested in the finite piece of \( I_{12} \) at \( \epsilon \to 0 \). Therefore, we have to keep control also over the \( O(\sqrt{\epsilon}) \) contribution to the integral in (17). Now for each fixed \( x \) the nominator in the integrand of (17) is \( h = 1 + O(\epsilon) \). But, unfortunately, this estimate does not hold uniformly in the whole integration range \((0, B_\epsilon)\). Hence the necessary analysis requires some detailed care and is put into appendix A. Inserting its result (A.19) for the integral into (17) we get

\[ I_{12} = \sqrt{\frac{2 \pi R_1 R_2}{R_{12}}} \left( \Gamma \left( \frac{1}{4} \right) \right)^2 \frac{1}{\sqrt{\epsilon}} - \frac{2 \pi^2}{2} + o(\epsilon^0). \]

(22)

As one should have expected, the discontinuities at \( \cos \beta = \frac{R_2}{R_1} \), i.e. \( R_{12} = R_1 \), showing up in both (12) and (A.19), cancel in the final result for \( I_{12} \).

With (7) and (22) into (5) one gets, after subtraction of the \( \frac{1}{\sqrt{\epsilon}} \) spike divergence [2],

\[ \log W_{\text{ren}} = 0 + O(g^4). \]  

(23)

3. Holographic evaluation at strong coupling

To generate the two circles as the image of two straight lines after an inversion on the unit sphere, we have to choose for these lines

\[
\begin{align*}
\vec{y}_1(\tau) &= \left( \tau, \frac{1}{2R_1}, 0 \right), \\
\vec{y}_2(\tau) &= \left( \tau, \frac{\cos \beta}{2R_2}, \frac{\sin \beta}{2R_2} \right),
\end{align*}
\]

(24)

with

\[ -\infty < \tau < \infty. \]

(25)

The distance between them is

\[ L = \frac{R_{12}}{2R_1 R_2}, \]

(26)

with \( R_{12} \) from (15).

As a result one gets the circles in the form

\[
\begin{align*}
\vec{x}_1(\tau) &= \frac{4R_1^2}{1 + 4R_1^2 \tau^2} \left( \tau, \frac{1}{2R_1}, 0 \right), \\
\vec{x}_2(\tau) &= \frac{4R_2^2}{1 + 4R_2^2 \tau^2} \left( \tau, \frac{\cos \beta}{2R_2}, \frac{\sin \beta}{2R_2} \right).
\end{align*}
\]

(27)

An example of this mapping is shown in figure 2.

The minimal surface in AdS, approaching the two straight lines (24) on the boundary, is given by (in Poincaré coordinates \( x_1, x_2, x_3, z \), with \( z = 0 \) as boundary,

\[ ds^2 = (dz^2 + dx_1^2 + dx_2^2 + dx_3^2)/z^2 \]) [6]

\[ o(\epsilon^0) \] denotes terms vanishing for \( \epsilon \to 0 \).
The function \( r(\sigma) \) is defined via

\[
\begin{align*}
    r(-\sigma) &= r(\sigma) \\
    \sigma &= r_0 \int_0^1 \frac{y^2 dy}{\sqrt{1-y^4}} \quad \text{for } 0 < \sigma < \frac{L}{2},
\end{align*}
\]  

with \( r_0 \) fixed by

\[
L = 2r_0 \int_0^1 \frac{y^2 dy}{\sqrt{1-y^4}} = \frac{(2\pi)^{3/2} r_0}{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}.
\]  

The AdS isometry

\[
\begin{align*}
    x_\mu &\mapsto \frac{x_\mu}{x^2 + z^2}, \quad z &\mapsto \frac{z}{x^2 + z^2}
\end{align*}
\]  

acts on the boundary \( (z = 0) \) as inversion on the unit sphere, mapping the straight lines (24) and circles (27) to each another. Therefore, the minimal surface in AdS, approaching the two circles (27) is given by the image of (28) under the map (31), i.e. by

**Figure 2.** In red: larger circle with radius \( R_1 = 1.2 \) and parts of its preimage. In blue: the same for smaller circle with \( R_2 = 1.1, \ \beta = \pi/4 \).
\[
\begin{pmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    z
\end{pmatrix}
= \left( \tau^2 + r^2(\sigma) + \frac{\sin^2 \beta}{4R_{12}^2} + \left( \sigma + \frac{R_1^2 - R_2^2}{4R_1R_2R_{12}} \right)^2 \right)^{-1}
\begin{pmatrix}
    \tau \\
    \frac{1}{2R_1} \left( \frac{1 - \sigma}{\tau} \right) + \frac{1}{2R_2} \left( \frac{1}{\tau} + \sigma \right) \cos \beta \\
    \frac{1}{2R_1} \left( \frac{1}{\tau} + \sigma \right) \sin \beta \\
    r(\sigma)
\end{pmatrix}.
\tag{32}
\]

The regularised area \( A_\epsilon \), needed for the holographic evaluation of our Wilson loop, is then just the area of that part of (32), for which \( z > \epsilon \). Its boundary, as parameterised by \( \sigma \) and \( \tau \), is given by
\[
\epsilon = \frac{r(\sigma)}{\tau^2 + r^2(\sigma) + \frac{\sin^2 \beta}{4R_{12}^2} + \left( \sigma + \frac{R_1^2 - R_2^2}{4R_1R_2R_{12}} \right)^2}.
\tag{33}
\]

Based on the isometric character of the map (31), we prefer as in [2] to calculate \( A_\epsilon \) on the preimage (28). There the induced metric is independent of \( \tau \) and areas are given by
\[
A_\epsilon = A_\epsilon^+ + A_\epsilon^-.
\tag{34}
\]

The integration regions \( B_\epsilon^\pm \) are defined by
\[
\frac{r}{\tau^2 + r^2 + (M(\pm \sigma(r))^2 + \frac{\sin^2 \beta}{4R_{12}^2}) > \epsilon},
\tag{36}
\]

with
\[
M(R_1, R_2, \beta) = \frac{R_1^2 - R_2^2}{4R_1R_2R_{12}}.
\tag{37}
\]

Performing the trivial \( \tau \)-integration (see [2]) we arrive at
\[
A_\epsilon^\pm = \frac{2r_0^2}{\sqrt{\epsilon}} \int_{r_\pm^L}^{r_\pm^U} \sqrt{\frac{r - \epsilon (M \pm \sigma(r))^2 + \frac{\sin^2 \beta}{4R_{12}^2}}{r^2 \sqrt{r_0^2 - r^4}}} \, dr.
\tag{38}
\]

The lower boundaries \( r_\pm^L \) are defined as solutions of
\[
r_\pm^L - \epsilon \left( \left( M \pm \sigma(r_\pm^L) \right)^2 + \frac{\sin^2 \beta}{4R_{12}^2} \right) = 0.
\tag{39}
\]

\(^3 M \) depends on \( \beta \) via \( R_{12}(R_1, R_2, \beta) \). For \( \beta = 0 \) this agrees with the formulas in [2] of course.
The evaluation of these integrals for $\epsilon \to 0$ up to divergent and $O(\epsilon^0)$ terms is performed in appendix B. After applying some $\Gamma$-function arithmetic to the result (B.14) we get

$$A_\epsilon = \frac{2\pi (R_1 + R_2)}{\epsilon} - \frac{32\pi^2 \sqrt{2} + 3}{(\Gamma(\frac{1}{2}))^2} \frac{1}{\sqrt{\epsilon|k_1 - k_2|}} + O(\sqrt{\epsilon}).$$

(40)

The leading divergent term is due to the standard $1/\epsilon$ divergence proportional to the length of the boundary contour. The next-leading $1/\sqrt{\epsilon}$ divergence is just twice the spike divergence analysed in [2]. After subtracting these divergences the remainder tends to zero for $\epsilon \to 0$, hence

$$A_{\text{ren}} = 0.$$  

(41)

Then via the holographic Wilson loop formula [6] we get at large $N$ and strong 't Hooft coupling $\lambda = g^2 N$

$$\log W_{\text{ren}} = 0.$$  

(42)

Before closing this section we have to mention a certain subtlety. There is still another potentially competing surface, the disconnected one, built out of the surfaces for the two single circles. First of all it is discriminated by the fact, that the regularised contour generated by cutting at $z = \epsilon$ is not connected. Furthermore, its regularised area is [4, 7]

$$A^{\text{disconn}}_\epsilon = \frac{2\pi (R_1 + R_2)}{\epsilon} - 4\pi + O(\epsilon).$$

(43)

For applications to the holographic evaluation of Wilson loops the common leading $1/\epsilon$-divergence is cancelled by a boundary term induced by a necessary Legendre transformation [4]. For small $\epsilon$ the disconnected surface is once more discriminated, since (43) both as it stands as well as after subtraction of the leading term is larger than (40). However it would win, if the values of the finite pieces would have to decide. To my knowledge so far this alternative did not play any role in papers studying the Gross–Ooguri phase transition [8], since there the competing areas had the same divergent parts. Only in a recent paper [9] on the cross anomalous dimensions a comparison of areas with differing divergent terms was relevant and the decisions were based also on the full regularised areas.

4. Comment on the ordinary Wilson loop

The ordinary (not supersymmetric) Wilson loop is given by (1) without the coupling of the contour to the scalars. According to the recipe for its holographic evaluation, as formulated in [10, 11], at leading order strong coupling it coincides with the supersymmetric Wilson loop as studied in the previous section.

To handle the leading order at weak coupling, we have to subtract the scalar contributions from those in section 2. The result is then valid both for the ordinary Wilson loop in $N = 4$ SYM and QCD.

There are the two trivial terms with both points of the propagator on the same circle

$$I_j^{\text{scalar}} = \frac{\pi}{2} \int_0^{2\pi} \frac{dx}{1 - \cos x + \frac{x^2}{2R_j^2}} = \frac{\pi^2 R_j}{\epsilon} + O(\epsilon).$$

(44)

$^4$Up to the touching point on the boundary of AdS.
For
\[ I_{12}^{\text{scalar}} = \int_0^{2\pi} \int_0^{2\pi} \frac{R_1 R_2 \, d\varphi_1 \, d\varphi_2}{(\vec{x}_1 - \vec{x}_2)^2 + \epsilon^2} \]  
we get after performing the \( \varphi_2 \)-integration\(^5\)
\[ I_{12}^{\text{scalar}} = 4\pi \int_0^{\pi} \frac{R_1 R_2 \, d\varphi}{\sqrt{\epsilon^4 + 4\epsilon^2 R_2^2 + 4\epsilon^2 R_1 R_1 R_2 (1 - \cos \varphi) + 4R_1^2 R_2^2 (1 - \cos \varphi)^2}}. \]  

After the change of integration variable as indicated in (13) this becomes
\[ I_{12}^{\text{scalar}} = 2\pi \sqrt{\epsilon} \sqrt{\frac{2R_1 R_2}{R_{12}}} \int_0^{\beta} \frac{dx}{\sqrt{(1 - \frac{x^2}{\beta^2})(1 + x^4 + \frac{x^2}{2\beta^2} + \frac{x^4}{4\beta^4})}}. \]  

Now an analysis analogously to appendix A yields
\[ I_{12}^{\text{scalar}} = \sqrt{\frac{\pi R_1 R_2}{2R_{12}}} \left( \Gamma\left(\frac{1}{4}\right) \right)^2 \frac{1}{\sqrt{\epsilon}} + o(\epsilon^0). \]  

Note that both (44) and (48) beyond the divergent terms contain no finite term remaining in the limit \( \epsilon \to 0 \).

The QCD Wilson loop becomes
\[ \log W_{\text{QCD}} = \frac{g^2 C_F}{4\pi^2} \left( \frac{\pi^2 (R_1 + R_2)}{\epsilon} + \sqrt{\frac{\pi R_1 R_2}{2R_{12}}} \left( \Gamma\left(\frac{1}{4}\right) \right)^2 \frac{1}{\sqrt{\epsilon}} + o(\epsilon^0) \right) + O(g^4). \]  

Then after subtraction of the standard \( 1/\epsilon \) divergence proportional to the length and the QCD spike divergence [2] our final result for the renormalised Wilson loop is\(^6\)
\[ \log W_{\text{ren}}^{\text{QCD}} = 0 + O(g^4). \]  

5. Conclusions

In \( \mathcal{N} = 4 \) SYM we obtained for the locally supersymmetric as well as for the ordinary Wilson loop in lowest nontrivial order
\[ W_{\text{ren}} = 1 \]  
both at weak and strong coupling.

This result holds also at weak coupling for QCD. Furthermore, it is independent of the angle between the discs spanned by the circles. Because no logarithmic divergences showed up\(^3\), it is free of any renormalisation group ambiguity.

Of course the main open question is, whether this result is an accident of the lowest orders or whether it extends to all orders. In further work in higher orders one has to take into account also the mixing with the correlation function for the two Wilson loops for the single circles.

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\(^5\)To keep formulas short, we write down only the case \( \beta = 0 \). Then \( R_{12} = R_1 - R_2 \).

\(^6\)Since for lowest order weak coupling the scalars contribute no finite nonvanishing term, \( \log W_{\text{ren}} = 0 \) holds in this approximation also for the family of interpolating Wilson loops considered in [11, 12].

\(^3\)Although present in the supersymmetric case for single spikes at weak coupling, there appears no logarithmic term for the case of two touching spikes. This has been noticed already in [2].
Using modifications of AdS, proposed for holographic QCD, see e.g. [13] and references therein, it should be straightforwardly to get the strong coupling result for QCD.

In speculating about physical properties, which could be related to our issue, ones mind is crossed by zig–zag symmetry [3] and conformal invariance. Zig–zag symmetry means that a part of a contour which is backtracked contributes only a factor 1. Classically it is realised for the ordinary Wilson loops, i.e. gauge parallel transporters, but is violated for the local supersymmetric loop due to the coupling to the scalars, which is not sensitive to the orientation. It is expected to hold in all orders of perturbation theory for ordinary Wilson loops, and there are arguments, that for the local supersymmetric loops it should be restored in the strong coupling limit [4].

With this assumption (51) holds as an all order result in QCD for $R_1 = R_2$ and $\beta = 0$, i.e. the exact backtracking case. For $R_1 > R_2$ there is only local backtracking and the Wilson loop for the single circles become different due the scale dependence of the renormalised coupling constant.

On the other side, in $\mathcal{N} = 4$ SYM conformal symmetry is unbroken. The Wilson loops for single circles are independent of their radius and known as an all order result [14, 15].

A last comment concerns the relation of our result to the symmetry breaking under conformal transformations, which map one point of the contour to infinity. The seminal discussion of this issue in [15] applies to cases where the respective point is on a smooth piece of the contour. In our case this point is just the singular point at the tip of the spikes, i.e. it is not of the type considered in [15] and one should not imperatively expect that their universal anomaly factor also governs the relation between the touching circles and anti-parallel straight lines. Some details for the comparison with the case of two anti-parallel lines are collected in appendix C.

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Appendix A

This appendix is devoted to the evaluation of the integral

$$J(R_1, R_2, \beta, \epsilon) = \int_0^{B_\epsilon} \frac{h(R_1, R_2, \beta, \epsilon, x) \, dx}{\sqrt{(1 - x^2 B_\epsilon^2)(1 + x^4)}}$$  \hspace{1cm} (A.1)

for $\epsilon \to 0$. $h$ and $B_\epsilon$ are defined in (18)–(21) and (14), respectively.

We start with the integral $J_0$ where, compared to $J$, $h$ is replaced by 1. It can be expressed in terms of the complete elliptic integral $K$ of the first kind via

$$J_0(R_1, R_2, \beta, \epsilon) = \int_0^{B_\epsilon} \frac{dx}{\sqrt{(1 - x^2 B_\epsilon^2)(1 + x^4)}} = \frac{B_\epsilon}{\sqrt{1 + iB_\epsilon^2}} K\left(\frac{2B_\epsilon^2}{B_\epsilon^2 - i}\right)$$

$$= \Re\left(K\left(\frac{1}{2} - \frac{i}{2B_\epsilon^2}\right)\right) + \Im\left(K\left(\frac{1}{2} - \frac{i}{2B_\epsilon^2}\right)\right).$$  \hspace{1cm} (A.2)

$^8$IIt has been derived for Euclidean contours. Variations have been observed also for lightlike polygons [16].
$K(y)$ is near $y = 1/2$ an analytic function. The deviation from $1/2$ in the second line of (A.2) is proportional to $\epsilon$, see (14). Then expressing $K(1/2)$ in terms of the Gamma function, we get

$$J_0(R_1, R_2, \beta, \epsilon) = \frac{(\Gamma(\frac{1}{4}))^2}{4\sqrt{\pi}} + \mathcal{O}(\epsilon).$$

To proceed, we note that the square root factor in the definition of $h$ in (18) allows an uniform estimate $1 + \mathcal{O}(\epsilon)$. The first factor does not, but is at least bounded in the whole integration interval. Let us define

$$\hat{h}_1(R_1, R_2, \beta) = \frac{2(R_1^2 - R_2^2) + R_{12}^2}{2R_1R_2R_{12}},$$

and

$$\hat{h}_i(R_1, R_2, \beta) = h_i(R_1, R_2, \beta), \; i = 2, 3$$

as well as

$$\hat{J}(R_1, R_2, \beta, \epsilon) = \int_0^{B_2} \frac{\hat{h}(R_1, R_2, \beta, \epsilon, x) \, dx}{\sqrt{(1 - x^2)}(1 + x^4)}.$$

Then we get

$$J(R_1, R_2, \beta, \epsilon) = \hat{J}(R_1, R_2, \beta, \epsilon) + \mathcal{O}(\epsilon).$$

Now we split the integration over $x$ in two pieces via

$$\hat{J}(R_1, R_2, \beta, \epsilon) = \hat{J}_{\text{lower}}(R_1, R_2, \beta, \epsilon) + \hat{J}_{\text{upper}}(R_1, R_2, \beta, \epsilon)$$

$$= \int_0^{b} \frac{\hat{h}(R_1, R_2, \beta, \epsilon, x) \, dx}{\sqrt{(1 - x^2)}(1 + x^4)} + \int_{\frac{b}{\sqrt{2}}}^{B_2} \frac{\hat{h}(R_1, R_2, \beta, \epsilon, x) \, dx}{\sqrt{(1 - x^2)}(1 + x^4)},$$

with $b > 0$ a fixed number and

$$\frac{1}{4} < \alpha < \frac{3}{8}.$$

Then the deviation of $\hat{h}$ from 1 in $\hat{J}_{\text{lower}}$ is uniformly $\mathcal{O}(\epsilon^{2\alpha})$, hence

$$\hat{J}_{\text{lower}} = \int_0^{b} \frac{dx}{\sqrt{(1 - x^2)}(1 + x^4)} + \mathcal{O}(\epsilon^{2\alpha}).$$

For the estimate of $\hat{J}_{\text{upper}}$ we use

$^9$Of course terms containing $\epsilon$ without a factor $x^2$, or $\epsilon^3 x^2$ are also irrelevant for our analysis.

$^{10}$Concerning only $\hat{J}_{\text{lower}}$, we could allow $\alpha$ even up to $\frac{1}{2}$. 

\[ \text{J. Phys. A: Math. Theor. 52 (2019) 095401} \]
\[
\frac{1}{\sqrt{1+x^2}} = \frac{1}{x^2} (1 + O(e^{2-4\alpha}))
\]  
(A.12)

to get with (A.6) and (A.9)

\[
J_{\text{upper}} = \int_{\frac{1}{2}}^{1/\sqrt{1-x^4}} \hat{h}(R_1, R_2, \beta, \epsilon, x) \, dx 
= \int_{\frac{1}{2}}^{1/\sqrt{1-x^4}} \frac{dx}{x^2} + V_1 + V_2 + O(e^{2-4\alpha}),
\]  
(A.13)

where

\[
V_1(R_1, R_2, \beta, \epsilon) = (\hat{h}_1 - 2\hat{h}_3) \cdot \epsilon \int_{\frac{1}{2}}^{1/\sqrt{1-x^4}} \frac{dx}{\sqrt{1 - \frac{x^2}{R_1^2}}} (1 + 2h_3\epsilon x^2 - \frac{\sin^2 \beta}{R_1^2} \epsilon^2 x^4),
\]

\[
V_2(R_1, R_2, \beta, \epsilon) = (\hat{h}_2 + \frac{\sin^2 \beta}{R_{12}^2}) \cdot \epsilon^2 \int_{\frac{1}{2}}^{1/\sqrt{1-x^4}} \frac{x^2 \, dx}{\sqrt{1 - \frac{x^2}{R_1^2}}} (1 + 2h_3\epsilon x^2 - \frac{\sin^2 \beta}{R_{12}^2} \epsilon^2 x^4).
\]  
(A.14)

Adding (A.11) and (A.13) we can reinstall the factor $1/\sqrt{1+x^4}$ instead of $1/x^2$ in the first term on the rhs of (A.13) and arrive with (A.9), (A.2) and (A.8) at

\[
J(R_1, R_2, \beta, \epsilon) = J_0 + \sum_{n=1}^{2} V_n(R_1, R_2, \beta, \epsilon) + O(\epsilon) + O(e^{2-4\alpha}) + O(\epsilon^{2\alpha}).
\]  
(A.15)

The integrals in both $V_1$ and $V_2$ can be expressed in terms of inverse trigonometric functions, and after some algebra we get

\[
V_1(R_1, R_2, \epsilon) = \frac{\pi \sqrt{\epsilon}}{4} \sqrt{\frac{2R_1 R_2 R_{12}}{|R_{12} - R_1|}} \cdot \left( \frac{R_{12} \cos \beta}{R_1} \Theta \left( \cos \beta = \frac{R_2}{2R_1} \right) + \left( \frac{R_2}{R_1} - \cos \beta \right) \Theta \left( \frac{R_2}{2R_1} - \cos \beta \right) \right) + O(\epsilon^{1\alpha}).
\]  
(A.16)

\[
V_2(R_1, R_2, \epsilon) = \frac{\pi \sqrt{\epsilon}}{4} \sqrt{\frac{2R_1 R_2 R_{12}}{|R_{12} - R_1|}} \cdot \left( \frac{R_{12} \cos \beta}{R_1} \Theta \left( \cos \beta = \frac{R_2}{2R_1} \right) + \cos \beta \Theta \left( \frac{R_2}{2R_1} - \cos \beta \right) \right) + O(\epsilon^{1\alpha}).
\]  
(A.17)

This implies

\[
\sum_{n=1}^{2} V_n = -\frac{\pi \sqrt{\epsilon}}{4} \sqrt{\frac{2R_{12}}{R_1 R_2}} \Theta \left( \frac{R_2}{2R_1} - \cos \beta \right) + O(\epsilon^{1\alpha}).
\]  
(A.18)

Inserting this in (A.15) and using (A.3) as well as (A.10) we arrive at \(^{11}\)

\[
J(R_1, R_2, \epsilon) = \left( \frac{\Gamma \left( \frac{1}{2} \right) \epsilon}{4\sqrt{\pi}} \right)^2 \frac{\pi \sqrt{\epsilon}}{4} \sqrt{\frac{2R_{12}}{R_1 R_2}} \Theta \left( \frac{R_2}{2R_1} - \cos \beta \right) + o(\sqrt{\epsilon}).
\]  
(A.19)

\(^{11}\) By $o(\sqrt{\epsilon})$ we denote terms vanishing faster than $\sqrt{\epsilon}$.
Appendix B

We need the $\epsilon \to 0$ behaviour of the integrals (38). The corresponding analysis follows closely the lines of [2], see also footnote\textsuperscript{12}. Nevertheless there are two reasons to present it here in detail. At first with the angle $\beta$ an additional new parameter is present, and secondly we now have to be more careful, since we need also the finite term, which was not of interest in [2].

Adding under the square root in the nominator of the integrand a zero in the form of the lhs of (39) we get

$$A_{\pm} = \frac{2r_0^2}{\sqrt{\epsilon}} \int_{r_0^2}^{r_0^2 + \epsilon} \sqrt{r - r_0^2} \sqrt{1 - \epsilon f_\pm (r, r_0^2)} \frac{dr}{r^2 \sqrt{r_0^2 - r^2}}, \quad (B.1)$$

with

$$f_\pm (r, r_0^2) = \frac{(M \pm \sigma(r))^2 - (M \pm \sigma(r_0^2))^2}{r - r_0^2} + (r + r_0^2). \quad (B.2)$$

This has the same form as the corresponding equation for $\beta = 0$ in [2]. The $\beta$-dependence enters here only via that of $M$, see (37). There is more $\beta$-dependence in the equation for $r_\pm$, (39). But for the expansion at small $\epsilon$ we can use

$$(M \pm L/2)^2 + \frac{\sin^2 \beta}{4R_1^2} = \frac{1}{4R_2^2}, \quad \text{or} \quad \frac{1}{4R_1^2}, \quad (B.3)$$

which follows from (15), (26) and (37). Then with (29) and (30) we get

$$r_+ = \frac{\epsilon}{4R_2^2} + \frac{\epsilon^3}{16R_2^4} + \mathcal{O}(\epsilon^4), \quad (B.4)$$

$$r_- = \frac{\epsilon}{4R_1^2} + \frac{\epsilon^3}{16R_1^4} + \mathcal{O}(\epsilon^4).$$

Note that these expansions up to terms $\propto \epsilon^3$ do not depend on $\beta$.

Now we split $A_\pm$ in two pieces

$$A_{\pm} = A_{\pm, \text{lead}} + A_{\pm, \text{rem}}, \quad (B.5)$$

with

$$A_{\pm, \text{lead}} = \frac{2}{\sqrt{\epsilon r_0^2}} \int_{r_0^2}^{1} \sqrt{x - r_0^2} \ dx \quad (B.6)$$

and

$$A_{\pm, \text{rem}} = \frac{2}{\sqrt{\epsilon r_0^2}} \int_{r_0^2}^{1} \sqrt{x - r_0^2} \left( \sqrt{1 - \epsilon f_\pm (xr_0^2, r_0^2)} - 1 \right) \ dx. \quad (B.7)$$

For the estimate of $A_\pm$ we use

\textsuperscript{12} An error in the journal version has been indicated in the erratum enclosed in the citation. The related correction of the relevant appendix can be found in the updated arXiv version.
\[
\int_0^1 \frac{\sqrt{x - \delta}}{x^2 \sqrt{1 - x^2}} \, dx = \frac{\pi}{2 \sqrt{3}} - \frac{2 \sqrt{3} \Gamma(\frac{3}{2})}{\Gamma(\frac{5}{2})} \, _2F_1\left(-\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{1}{8} \right) = \delta + \delta^2 b_2(\delta^2) + \delta^3 b_3(\delta^2),
\]

where \(b_1, b_2, b_3\) are given by different hypergeometric \(_2F_1\)'s of argument \(\delta^4\) times some numerical factors. With (B.4) this implies \((j(+) = 2, j(-) = 1)\)

\[
A_{\epsilon, \text{lead}}^\pm = \frac{2 \pi R_0^{(\pm)}}{\epsilon} - \frac{4 \sqrt{3} \Gamma(\frac{3}{2})}{\Gamma(\frac{5}{2})} \frac{1}{\sqrt{\epsilon}} + O(\sqrt{\epsilon}).
\]

Concerning the estimate of \(A_{\epsilon, \text{rem}}^\pm\) we noted in [2], that \(f_\pm(r, r_0^\pm)\) is bounded for \(\epsilon \to 0\) uniformly with respect to \(r\). This allowed to conclude

\[A_{\epsilon, \text{rem}}^\pm = O(\epsilon) \cdot A_{\epsilon, \text{lead}}^\pm = O(\epsilon^0).\]

But we can be more efficiently. Expanding the square root in (B.7) and using again the uniform boundedness of \(f_\pm(r, r_0^\pm)\) we get

\[A_{\epsilon, \text{rem}}^\pm = -\frac{\epsilon}{\sqrt{\epsilon}} \int_0^1 \frac{\sqrt{x - r_0^\pm / r_0}}{x^2 \sqrt{1 - x^2}} f_\pm(x, r_0^\pm) \, dx + O(\epsilon^2) A_{\epsilon, \text{lead}}^\pm.\]

The integral without the factor \(f_\pm(r, r_0^\pm)\) would diverge for \(\epsilon \to 0\) according to (B.8). But due to the behaviour

\[f_\pm(r, r_0^\pm) = O(\epsilon) + (1 + O(\epsilon))(r - r_0^\pm) + O((r - r_0^\pm)^2)\]

for small \(\epsilon\), and near the lower boundary of the integral, it remains finite. This means

\[A_{\epsilon, \text{rem}}^\pm = O(\sqrt{\epsilon})\]

and together with (B.9), (B.5) and (34)

\[A_\epsilon = \frac{2 \pi (R_1 + R_2)}{\epsilon} - \frac{8 (2 \pi)^{\frac{3}{2}} \Gamma(\frac{7}{8})}{\Gamma(\frac{13}{8})} \frac{1}{\sqrt{\epsilon} \frac{|\tilde{k} - \tilde{k}_2|}{\tilde{k}_1 - \tilde{k}_2}} + O(\sqrt{\epsilon}).\]

Here use has been made also of the relations between \(r_0, L\) and the curvature difference \(|\tilde{k}_1 - \tilde{k}_2|\), i.e. (30), (26) and (16).

**Appendix C**

Here we collect some details for the comparison of the two touching circles with two antiparallel straight lines. Performing the trivial integrations for lines at distance \(L\) one gets

\[
\log W_{\text{parallel}} = \frac{g^2 C_F}{2 \pi^2} \left( \frac{2l}{L} \arctan \frac{l}{L} - \log(1 + \frac{l^2}{L^2}) \right) + O(g^4),
\]

\[
= \frac{g^2 C_F}{2 \pi^2} \left( \frac{\pi l}{L} - 2 \log \frac{l}{L} - 2 + O(\frac{L^2}{l^2}) \right) + O(g^4).
\]

To control the infrared problem, the integration has been restricted to straight lines of length \(l\), with the goal \(l \to \infty\). Contrary to the treatment of ultraviolet divergences, there is no recipe to give for infinitely extended contours the Wilson loop a finite meaning per se. Nevertheless it is the source for a meaningful physical quantity, the static quark-antiquark potential, via
\[ V(L) = \lim_{l \to \infty} W_{\text{parallel}}/l. \] Thus this potential is just given by the factor of the linear infrared divergence.

The Wilson loop for the touching circles is from (5), (7), (16) and (22)

\[ \log W = \frac{g^2 C_F}{4\pi^2} \left( \sqrt{2\pi \left( \Gamma\left(\frac{1}{4}\right)^2 \frac{1}{\sqrt{\epsilon |k_1 - k_2|}} \right)} + o(\epsilon^0) \right) + O(g^4). \] (C.2)

Our ultraviolet regularisation parameter \( \epsilon \), as used in chapter 2, mimics a universal cutoff in the distance between the two endpoints of the propagator.

The special situation near the touching point of the two circles could be regularised also by restricting the integrations to the image under inversion of the two straight lines of finite length \( l \). Then the minimum of the allowed propagator distances would be

\[ \min |\vec{x}_1 - \vec{x}_2| = \frac{2R_{12}}{\sqrt{(1 + PR_{1}^2)(1 + PR_{2}^2)}} = \frac{2 |\vec{k}_1 - \vec{k}_2|}{l^2} + O\left(\frac{1}{l^4}\right). \] (C.3)

Identifying this minimum with \( \epsilon \) one finds, starting from (C.1), the spiky ultraviolet \( 1/\sqrt{\epsilon} \) divergence as an image of the linear infrared divergence. But invariance under inversion is broken, resulting in different numerical coefficients. Furthermore, there are different finite terms and no logarithmic divergence for the circles.

Of course, this interplay between the IR for straight lines and the UV for the circles holds also for strong coupling. It is illustrated in an eye-catching manner in figure 3 of [2]. But due to symmetry breaking, also here the coefficients require independent calculations.

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