Extensivity of Irreversible Current and Stability in Causal Dissipative Hydrodynamics

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We extended our formulation of causal dissipative hydrodynamics [T. Koide et al., Phys. Rev. C75, 034909 (2007)] to be applicable to the ultra-relativistic regime by considering the extensiveness of irreversible currents. The new equation has a non-linear term which suppresses the effect of viscosity. We found that such a term is necessary to guarantee the positive definiteness of the inertia term and stabilize numerical calculations in ultra-relativistic initial conditions. Because of the suppression of the viscosity, the behavior of the fluid is more close to that of the ideal fluid. Our result is essentially same as that from the extended irreversible thermodynamics, but is different from the Israel-Stewart theory. A possible origin of the difference is discussed.

I. INTRODUCTION

It is widely believed that the basic features of collective motion in relativistic heavy-ion collisions can be well described by using the (almost) ideal hydrodynamic model. Several studies on the effects of viscosity are available in various works and seem to support such a vision. However, when the relaxation time, which characterizes the memory effect, is small compared to the variation scale of the fluid. Navier-Stokes theory, because thermodynamic forces instantaneously produce irreversible currents. That is, the time exists an

finite

assumption in the derivation of hydrodynamics is the local equilibrium ansatz: at any space point, there should exist an

finite

extension of the fluid which is described by the thermodynamic laws in equilibrium. We refer such an element of the fluid to a fluid cell. The irreversible currents are phenomenologically defined so that the second law of thermodynamics is satisfied for each fluid cell. These fluid cells permit superpositions and need not to be exclusive, each other, but they have to be of finite size to apply thermodynamics. However, in the usual hydrodynamic formulation, the finiteness of the fluid cell is considered to be irrelevant and the irreversible currents are defined by applying the second law for densities of extensive quantities. This does not give rise to problems in the (relativistic) Navier-Stokes theory, because thermodynamic forces instantaneously produce irreversible currents. That is, the time scale of deformation of a fluid cell is considered infinitely large in comparison with this scale. This is also the case when the relaxation time, which characterizes the memory effect, is small compared to the variation scale of the fluid. However, when the relaxation time is of the same order as the variation scale, we have to distinguish the quantities affected by the change in internal degrees of freedom from the global kinematic degrees of freedom in introducing memory effects.

In this paper, we rederive the causal dissipative hydrodynamics by taking the finite size of the fluid cell into account. For simplicity, we consider the 1+1 dimensional system. We found that the equation of the bulk viscosity has a nonlinear term. Because of the nonlinear term, the effect of bulk viscosity is suppressed and the behavior of the fluid is closer to that of the ideal fluid. More importantly, we found that this effect is indispensable to implement stable numerical calculations in ultra-relativistic initial conditions.

This paper is organized as follows. In Sec.II we discuss the new formulation of relativistic dissipative hydrodynamics by taking the finite size of the fluid cell into account. The restriction for the parameters which are consistent with causality is discussed in Sec.III. In Sec.IV, we apply our theory to the 1+1 dimensional scaling solution of the Bjorken model. In Sec.V, we implements numerical simulations. We apply the smoothed particle formulation to solve the hydrodynamics numerically. Here, we discussed three examples, the shock formation, the expansion to vacuum with Landau initial condition and the appearance of nonperiodic oscillations similar to turbulence. We show that the result of our formulation can be justified from the viewpoint of the extended irreversible thermodynamics in Sec.VI. The relation between our theory and the Israel-Stewart theory is discussed in Sec.VII. Sec.VIII is devoted to concluding
II. EXTENSIVE MEASURE FOR THE IRREVERSIBLE CURRENT

For simplicity, we consider the case of vanishing baryon chemical potential for the simple 1+1 dimensional system. In this case, the hydrodynamic equations of motion can be written as only the conservation of the energy-momentum tensor,

$$\partial_{\mu}T^{\mu\nu} = 0,$$

(1)

together with the thermodynamic relations. We adopt the local equilibrium ansatz in the energy local rest frame, as proposed by Landau-Lifshitz [20]. The energy-momentum tensor is given by

$$T^{\mu\nu} = (\varepsilon + P + \Pi)u^\mu u^\nu - (P + \Pi)g^{\mu\nu},$$

(2)

where $\varepsilon$, $P$, $u^\mu$ and $\Pi$ are, respectively, the energy density, pressure, four-velocity and bulk viscosity. From Eq. (1), we obtain the entropy production rate in terms of densities,

$$\partial_{\mu}(su^\mu) = \frac{1}{T}\Pi\partial_{\mu}u^\mu.$$

(3)

We should remember that the local thermal equilibrium ansatz must be applied to a fluid cell which, in principle, has a finite volume defined by the coarse-grained size of microscopic degrees of freedom. The thermodynamic laws should be applied to the integrated quantities of the fluid inside each cell. To see this more clearly, let us introduce the volume $V^*$ of such a fluid cell. That is, $V^*$ is the volume of the fluid at the point $\vec{r}$, inside of which the fluid is considered to be homogeneous and satisfies the thermodynamic laws in equilibrium. The flow of the fluid deforms such a cell so that its volume is a function of time. If we follow the fluid flow given by the velocity field $\vec{v}$, the time variation of $V^*$ is given by

$$\frac{1}{V^*} \frac{dV^*}{dt} = \nabla \cdot \vec{v}$$

(4)

or in a covariant form,

$$\partial_{\mu}(\sigma u^\mu) = 0,$$

(5)

where we have introduced the proper reference density $\sigma$ by

$$\sigma = \frac{1}{V} = \frac{\gamma}{V^*},$$

(6)

where $\gamma$ is the Lorentz factor.

Now, let us denote the extensive measure of the entropy inside this volume as $\tilde{s} = sV = s/\sigma$. Then we rewrite Eq. (3) as

$$T \frac{d\tilde{s}}{d\tau} = -\tilde{J}F = -\tilde{\Pi}\partial_{\mu}u^\mu,$$

(7)

with

$$\tilde{\Pi} = \Pi V = \frac{\Pi}{\sigma},$$

(8)

is the extensive measure inside the fluid cell of the irreversible current $\Pi$.

One can see that Eq. (7) has the structure that the net entropy production in the cell is given by the product of the irreversible displacement $\tilde{\Pi}$ occurred in the cell and the corresponding thermodynamic force field $F = \partial_{\mu}u^\mu$ in the cell. It should be noted that, from Eq. (5), the thermodynamic force is reexpressed as

$$F = \partial_{\mu}u^\mu = \sigma \frac{d}{d\tau} \left( \frac{1}{\sigma} \right).$$

(9)

This result means that the thermodynamic force for the fluid cell is given by the change of the cell volume, indicating clearly the physical meaning of the bulk viscosity: the resistance to the change of the volume of the system.
In the relativistic Navier-Stokes theory (the Landau-Lifshitz theory), it is assumed that the bulk viscosity per volume element is produced by the thermodynamic force without any retardation,

\[ \tilde{\Pi} = -\tilde{\eta}F = -\frac{\zeta}{\sigma}\partial_\mu u^\mu, \]  

(10)

where \( \tilde{\eta} \) represents an extensive measure of coupling (total charge) for the whole matter inside the cell. We thus identify \( \zeta \) as the bulk viscosity coefficient. By multiplying \( \sigma \) for both sides, we reproduce the usual result of the Landau-Lifshitz theory [20]. That is, the finite size effect does not affect the definition of irreversible currents in the Navier-Stokes theory. However, it is by now well known that the derived equation has the problem of acausality and instability [8, 9, 10]. To solve these difficulties, we introduce a memory effect to the irreversible current by using a memory function.

When microscopic and macroscopic scales are clearly separated, the time scale of the variation of the fluid cell itself is infinitely large compared to the microscopic scales, and only the change in the internal degrees of freedom is relevant. If, however, this is not true, they are entangled and we have to distinguish their roles. One is the change of the internal degrees of freedom with the transient effects (memory effect) and the other the motion of the global kinematic degrees of freedom (fluid cell deformation). As the simplest memory function which can be reduced to the differential equation, we apply

\[ G(\tau, \tau') = \frac{1}{\tau_R(\tau')} \exp \left( -\int_{\tau'}^{\tau} \frac{d\tau''}{\tau_R(\tau'')} \right). \]

(11)

Here, \( \tau_R \) is a relaxation time which characterizes the time scale of the retardation. In the previous work [8], we have applied this memory function to the thermodynamic force \( \eta F \) (not \( \tilde{\eta}F \)) to introduce the retardation. As a consequence, the final form of the derived equation for \( \Pi \) is same as that of the so-called truncated form of the IS equation, where nonlinear terms are ignored. However as we mentioned before, when we deal with memory effects, we should not use densities. This is because memory effects relate different thermodynamic states of the matter for different times and this depends on the size of the system not necessarily in an extensive manner. Then the memory effect should be applied to the integrated quantity of a fluid cell.

In this paper, we introduce the memory effect of the thermodynamic force field on an extensive measure for the irreversible current. Then the bulk viscosity consistent with causality is given by

\[ \tilde{\Pi}(\tau) = -\int_{\tau_0}^{\tau} d\tau' G(\tau, \tau') \frac{\zeta}{\sigma}\partial_\mu u^\mu + \tilde{\Pi}_0 \exp \left( -\int_{\tau_0}^{\tau} \frac{d\tau''}{\tau_R(\tau'')} \right), \]

(12)

where \( \Pi_0 \) is the initial value given at \( \tau_0 \). This integral form is equivalent to the solution of the following differential equation,

\[ \tau_R \frac{d}{d\tau} \tilde{\Pi} + \tilde{\Pi} = -\frac{\zeta}{\sigma}\partial_\mu u^\mu. \]

(13)

This equation can be reexpressed in terms of the density as follows,

\[ \tau_R \frac{d}{d\tau} \Pi + \Pi = - (\zeta + \tau_R \Pi) \partial_\mu u^\mu. \]

(14)

On the other hand, the corresponding equation in our previous paper [8] where the memory effect is directly applied to the densities has the form,

\[ \tau_R \frac{d}{d\tau} \Pi + \Pi = -\zeta \partial_\mu u^\mu. \]

(15)

The difference of Eq. (13) from Eq. (15) is the presence of the term \( -\tau_R \Pi \partial_\mu u^\mu \).

It should be noted that Eq. (13) also does not explicitly depend on the cell volume \( 1/\sigma \). This is because the size of the cell volume is irrelevant as far as the length is much larger than the mean-free path and much smaller than the typical hydrodynamic scale. For later convenience, we call the causal dissipative hydrodynamics without the finite size effect, Eq. (13) as "linear causal dissipative hydrodynamics (LCDH)", whereas the one with finite size effect, Eq. (15) as "nonlinear causal dissipative hydrodynamics (NLCDH)".

An important effect due to the nonlinear term is the lower bound of the bulk viscosity. In LCDH, the bulk viscosity can, in principle, take any negative value. In NLCDH, when the bulk viscosity becomes negatively large, the effective
bulk viscosity coefficient $\zeta_{\text{eff}} = \zeta + \tau_R \Pi$ eventually changes sign and the bulk viscosity start to increase. Thus, the bulk viscosity in NLCDH cannot be smaller than

$$\Pi_{\text{min}} = -\frac{\zeta}{\tau_R},$$

(16)

when the initial value of $\Pi$ is larger than $\Pi_{\text{min}}$. This aspect plays an important role for the stability of numerical simulations of ultra-relativistic cases, as we will see Sec. V.

III. PROPAGATION SPEED OF SOUND

We parametrize the bulk viscosity coefficient and the relaxation time as follows,

$$\zeta = as,$$

(17)

$$\tau_R = \frac{\zeta}{\varepsilon + P^b},$$

(18)

where $a$ and $b$ are arbitrary constants.

As was pointed out in [8, 9, 10], LCDH can be acausal depending on the choice of the parameters. To see the limitation of the parameters, we have to calculate the propagation speed of NLCDH. Following [8, 9, 10], we discuss the linear perturbation around the hydrostatic state. Then the nonlinear term of the equation of the bulk viscosity (14) disappears and the dispersion relation is same as that of LCDH [9, 10]. Thus, by assuming the group velocity gives the propagating speed of the dissipative fluid, we found

$$v_c = \sqrt{1/b + \alpha},$$

(19)

where $\alpha = \partial P / \partial \varepsilon$. To satisfy causality $v_c \leq 1$, the parameter $b$ should be less than $1/(1 - \alpha)$. This is completely the same restriction as the case of LCDH [9, 10].
IV. SCALING SOLUTION

We apply NLCDH to the one dimensional scaling solution of the Bjorken model. Then the hydrodynamic equations are given by

\[
\frac{\partial \tau}{\partial \tau} \varepsilon + \left( \frac{\varepsilon + P + \Pi}{\tau} \right) = 0,
\]

\[
\tau_R \frac{\partial \Pi}{\partial \tau} + \Pi = -\frac{\zeta + \tau R \Pi}{\tau},
\]

where \( t = \tau \cosh y \) and \( x = \tau \sinh y \).

We adopt the massless ideal gas equation of state where \( \alpha = 1/3 \). To satisfy causality, the parameter \( b \) should be larger than \( 3/2 \). In this calculation, we choose \( b = 6 \). For the initial condition, we set \( \varepsilon (\tau_0) = 1 \text{ GeV/fm}^3 \) and \( \Pi (\tau_0) = 0 \) at the initial proper time \( \tau_0 = 0.1 \text{ fm} \).

In Fig. 1 we show the energy density \( \varepsilon \) as a function of the proper time \( \tau \). The dotted, dashed and solid lines correspond to the ideal hydrodynamics, LCDH and NLCDH, respectively. Because of the memory effect, the behaviors of LCDH and NLCDH are similar to that of the ideal fluid at the early stage of the time evolution. After the time larger than the relaxation time, the behaviors of LCDH deviates from that of the ideal hydrodynamics. On the other hand, the behavior of NLCDH stays close to that of the ideal hydrodynamics, that is, the effect of the bulk viscosity is suppressed in NLCDH compared to LCDH. This is directly observed from the behavior of the bulk viscosity as is shown in Fig. 2.

It should be noted that the scaling solution of the bulk viscosity including the similar nonlinear term was already discussed in [4], where the equation derived by Israel and Stewart is discussed. The relation of our equation and the Israel-Stewart equation will be discussed in Sec. VII.

V. NUMERICAL SIMULATIONS

A. Smoothed particle formulation

To solve numerically the hydrodynamic equations, we use the Smoothed Particle Hydrodynamic (SPH) method. The original idea of the SPH method is to obtain an approximate solution of hydrodynamics by parameterizing the fluid into a set of effective particles [22]. Because of its flexibility to adapt to complex geometries, the SPH method has also been extensively applied to the relativistic heavy ion reactions to perform an event-by-event analysis of the data [21]. For the sake of convenience, we reproduce below the ideas and basic equations of the SPH approach shown in [9].
Let us consider a distribution $a(\mathbf{r}, t)$ of any extensive physical quantity, $A$. In a system like the hot and dense matter created in heavy ion collisions, the behavior of $a(\mathbf{r}, t)$ contains the effects of whole microscopic degrees of freedom. We are not interested in the extremely short wavelength behavior of $a(\mathbf{r}, t)$ but rather in global behaviors which are related directly to the experimental observables. Therefore, we would like to introduce a coarse-graining procedure for $a$. To do this, we introduce the kernel function $W(\mathbf{r} - \mathbf{r}, h)$ which maps the original distribution $a$ to a coarse-grained version $a_{CG}$ as,

$$a_{CG}(\mathbf{r}, t) = \int a(\mathbf{r}, t) W(\mathbf{r} - \mathbf{r}, h) \, d\mathbf{r}$$  \hspace{1cm} (22)

where $W$ is normalized,

$$\int W(\mathbf{r}, h) \, d\mathbf{r} = 1,$$  \hspace{1cm} (23)

and has a bounded support of the scale of $h$,

$$W(\mathbf{r}, h) \to 0, \quad |\mathbf{r}| \gtrsim h,$$  \hspace{1cm} (24)

satisfying

$$\lim_{h \to 0} W(\mathbf{r}, h) = \delta(\mathbf{r}).$$  \hspace{1cm} (25)

Here, $h$ is a typical length scale for the coarse-graining in the sense that the kernel function $W$ introduces a cut-off in short wavelength of the order of $h$. Thus we will take this value as the scale of coarse graining in the QCD dynamics (i.e., the mean-free path of partons) to obtain the hydrodynamics of QGP ($h \simeq 0.1\, \text{fm}$).

The second step is to approximate this coarse-grained distribution $a_{CG}(\mathbf{r}, t)$ by replacing the integral in Eq.\,(22) by a summation over a finite and discrete set of points, $\{\mathbf{r}_\alpha(t), \alpha = 1, \ldots, N_{\text{SPH}}\}$,

$$a_{\text{SPH}}(\mathbf{r}, t) = \sum_{\alpha=1}^{N_{\text{SPH}}} A_\alpha(t) W(|\mathbf{r} - \mathbf{r}_\alpha(t)|).$$  \hspace{1cm} (26)

If the choice of $\{A_\alpha(t), \alpha = 1, \ldots, N_{\text{SPH}}\}$ and $\{\mathbf{r}_\alpha(t), \alpha = 1, \ldots, N_{\text{SPH}}\}$ are appropriate, the above expression should converge to the coarse-grained distribution $a_{CG}$ for large $N_{\text{SPH}}$. Parameters $\{A_\alpha(t), \alpha = 1, \ldots, N_{\text{SPH}}\}$ and $\{\mathbf{r}_\alpha(t), \alpha = 1, \ldots, N_{\text{SPH}}\}$ should be determined from the dynamics of the system. In practice, we first choose the reference density $\sigma^*$ which is conserved,

$$\frac{\partial \sigma^*}{\partial t} + \nabla \cdot \mathbf{j} = 0,$$  \hspace{1cm} (27)

where $\mathbf{j}$ is the current associated with the density $\sigma^*$. Then, we note that the following ansatzes,

$$\sigma_{\text{SPH}}^*(\mathbf{r}, t) = \sum_{\alpha=1}^{N_{\text{SPH}}} \nu_\alpha W(|\mathbf{r} - \mathbf{r}_\alpha(t)|),$$  \hspace{1cm} (28)

$$\mathbf{j}_{\text{SPH}}(\mathbf{r}, t) = \sum_{\alpha=1}^{N_{\text{SPH}}} \nu_\alpha \frac{d\mathbf{r}_\alpha(t)}{dt} W(|\mathbf{r} - \mathbf{r}_\alpha(t)|),$$  \hspace{1cm} (29)

satisfies the equation,

$$\frac{\partial \sigma_{\text{SPH}}^*}{\partial t} + \nabla \cdot \mathbf{j}_{\text{SPH}} = 0,$$  \hspace{1cm} (30)

where $\nu_\alpha$’s are constant. By using the normalization of $W$, we have

$$\int_{\text{SPH}} \sigma^*(\mathbf{r}, t) \, d^3\mathbf{r} = \sum_{\alpha=1}^{N_{\text{SPH}}} \nu_\alpha.$$  \hspace{1cm} (31)

Then we can interpret the quantity $\nu_\alpha$ as the conserved quantity attached at the point $\mathbf{r} = \mathbf{r}_\alpha(t)$. Therefore, the distribution $\sigma_{\text{SPH}}^*(\mathbf{r}, t)$ is a sum of small piece-wise distribution, carrying the density,

$$\nu_\alpha W(|\mathbf{r} - \mathbf{r}_\alpha(t)|).$$  \hspace{1cm} (32)
These pieces are referred to as ”SPH-particles”.

Using the above reference density and the extensive nature of \( A \), we can write \( A_\alpha \) in Eq.(26) as

\[
A_\alpha (t) = \nu_\alpha \frac{a(r_\alpha, t)}{\sigma^* (r_\alpha, t)}
\]

which represents the quantity \( A \) carried by the SPH particle at the position \( r = r_\alpha(t) \). In fact, the total amount of \( A \) of the system at the instant \( t \) is given by

\[
A(t) = \sum_{\alpha=1}^{N_{\text{SPH}}} A_\alpha(t).
\]

In the ideal fluid, the entropy density is chosen as the reference density and the dynamics of the parameters \( \{ r_\alpha(t), \alpha = 1, \ldots, N_{\text{SPH}} \} \) are determined from the variational principle from the action of ideal hydrodynamics. The entropy density is, however, not conserved for the dissipative fluid. Thus we introduce a new conserved quantity, the specific proper density \( \sigma \), which is defined by the flow of the fluid,

\[
\partial_{\mu} (\sigma u^\mu) = 0,
\]

and we will use it as the reference density for viscous fluids. Here, the four-velocity \( u^\mu \) is defined in terms of the local rest frame of the energy flow (Landau frame). The specific density is expressed in the SPH form as

\[
\sigma^* (r, t) = \sum_{\alpha=1}^{N_{\text{SPH}}} \nu_\alpha W (|r - r_\alpha(t)|),
\]

where \( \sigma^* = \sigma u^0 \) is the specific density in the laboratory frame and \( \nu_\alpha \) is the inverse of the specific volume of the SPH particle \( \alpha \). In this work, the specific volume should be interpreted as the volume of the fluid cell, and hence \( \nu_\alpha \) is the inverse of the cell volume. However, as we showed, the final results do not depend on this choice and we set \( \nu_\alpha = 1 \) for simplicity. As for the kernel \( W (r) \), we use the spline function.

It should be mentioned that this procedure is only possible provided that the lines of flow in space defined by the velocity field \( u^\mu \) do not cross each other during the evolution in time. That is, if there appear turbulence or singularities in the flow lines, the above definition of Lagrange coordinates can fail.

Now we apply this method to NLCDH in 1 + 1 dimension. We have to solve the evolution equation of the bulk viscosity in the SPH scheme. For this, we express the viscosity as

\[
\Pi = \sum_{\alpha=1}^{N_{\text{SPH}}} \nu_\alpha \left( \frac{\Pi}{\gamma_\alpha} \right)_\alpha W (|r - r_\alpha(t)|),
\]

The time evolution of the term \( \Pi_\alpha \) can be calculated as

\[
\frac{d\Pi_\alpha}{dt} = -\frac{\zeta}{\sigma_\alpha \tau_R} (\partial_{\mu} u^\mu)_\alpha - \frac{1}{\gamma_\alpha \tau_R} \Pi_\alpha
\]

where \( \gamma_\alpha \) is the Lorentz factor of the \( \alpha \)-th particle. In the following, we denote the quantity in the observable frame with the asterisk. It should be noted that we solve Eq. (13) instead of (14) in the numerical calculations.

At the same time, using the SPH expression for the entropy density \( s^* \) in the observable frame,

\[
s^* = \sum_{\alpha=1}^{N_{\text{SPH}}} \nu_\alpha \left( \frac{s}{\gamma_\alpha} \right)_\alpha W (|r - r_\alpha(t)|),
\]

and the evolution of the entropy per SPH particle is given by

\[
\frac{d}{dt} \left( \frac{s}{\gamma_\alpha} \right)_\alpha = -\frac{1}{T} \frac{\Pi_\alpha}{\sigma_\alpha} (\partial_{\mu} u^\mu)_\alpha.
\]

where \( s = s^*/u^0 \) is the proper entropy density. In the above expressions, the relaxation time \( \tau_R \), viscosity coefficient \( \zeta \) and temperature \( T \) are functions of space and time, so that they should be evaluated at the position of each particle \( \alpha \).
Finally, we need to express the momentum conservation equation by the SPH variables. We write the space component of energy-momentum equation of continuity in terms of the reference density,

\[ \sigma \frac{d}{dr} \left( \frac{\epsilon + P + \Pi}{\sigma} u^i \right) + \partial_i (P + \Pi) = 0. \]  

(41)

It should be noted that there exist ambiguities within the resolution of the coarse-graining size \( h \) to express the equation of motion in the SPH form. However, in the ideal fluid, the SPH equation of motion can be derived by the variational method uniquely. Thus, we obtain the equation of motion by using the same SPH parametrization to Eq.(41),

\[ \sigma \alpha \frac{d}{d\tau} \left( \frac{\epsilon_{\alpha} + P_{\alpha} + \Pi_{\alpha}}{\sigma_\alpha} u^i_{\alpha} \right) = \sum_{\beta=1}^{N_{\text{SPH}}} \nu_{\beta} \sigma^*_{\beta} \left( \frac{P_{\beta} + \Pi_{\beta} \sigma_{\beta}}{(\sigma^*_\beta)^2} + \frac{P_{\alpha} + \Pi_{\alpha} \sigma_{\alpha}}{(\sigma^*_\alpha)^2} \right) \partial_i W \left( |r_{\alpha} - r_{\beta}(t)| \right), \]  

(42)

where the right hand side of Eq.(42) corresponds to the term \( \partial_i (P + \Pi) \) written in terms of the SPH parametrization. We remark that in the case of vanishing viscosity our result is reduced to the expression derived with variational principle for ideal fluids.

By separating the acceleration and force terms in Eq.(42), we obtain our final expression of the equation of motion for each SPH particle,

\[ M \frac{du}{dt} = F, \]  

(43)

where the mass matrix \( M \) and the force term \( F \) are defined as

\[ M_{ij} = \gamma (\epsilon + P + \Pi) \delta_{ij} + A u_i u_j, \]  

(44)

\[ F_j = -\partial_j (P + \Pi) + B u_j, \]  

(45)

with

\[ A = -\frac{1}{\gamma} \left[ (\epsilon + P + \Pi) + \frac{\zeta}{\tau_R} + \Pi \right], \]  

(46)

\[ B = A \frac{\gamma^2 \sigma^*}{\sigma^*_\alpha} \frac{d\sigma^*_\alpha}{d\tau} + \frac{\Pi}{\tau_R}, \]  

(47)

where \( \alpha = \partial P / \partial \epsilon \). It should be noted that the expression of \( A \) in NLCDH is different from that of in LCDH [9],

\[ A_{\text{LCDH}} = -\frac{1}{\gamma} \left[ (\epsilon + P + \Pi) + \frac{\zeta}{\tau_R} \right]. \]  

(48)

To carry out the calculation in this scheme, the mass matrix \( M \) must be nonsingular. However, this is not guaranteed in LCDH. For example, consider the 1+1 dimensional system with ultra-relativistic fluid velocity, \( u \approx \gamma \). Then the mass in LCDH becomes

\[ M_{\text{LCDH}} \approx \gamma \left[ (1 - \alpha) (\epsilon + P + \Pi) - \frac{\zeta}{\tau_R} \right]. \]  

(49)

As we discussed the bulk viscosity in LCDH does not have a lower bound, we see that \( M_{\text{LCDH}} \) can be zero, while in NLCDH, we get

\[ M_{\text{NLCDH}} \approx \gamma \left[ (1 - \alpha) (\epsilon + P + \Pi) - \frac{\zeta}{\tau_R} \right] \geq \gamma (1 - \alpha) \left( \epsilon + P - \frac{\zeta}{\tau_R} \right) \]

\[ = \gamma (1 - \alpha) (\epsilon + P) \left( 1 - \frac{1}{b} \right) > 0. \]  

(50)

Here, we used the expression of the bulk viscosity coefficient and the relaxation time, Eqs. (17) and (18). The mass does not vanish in NLCDH and the simulation is stable even for the ultra-relativistic situations.
As was discussed in [9], in the LCDH scheme, the numerical calculation can be carried out with the help of the additional viscosity for $\gamma = 2$. Here, we show that this scheme becomes unstable for ultra-relativistic initial conditions. We use the same additional viscosity as the one proposed in [9]. In Figs. 3 and 4 we show, respectively, the temperature and velocity profiles calculated in LCDH with $a = 1$ at $t = 0.75$ fm for the initial trigger velocity $\gamma = 5$. The dotted lines are for the initial condition. We can see that the calculation becomes unstable and rapid oscillations appear around $x = 0$ and the calculation eventually collapses. This is due to the vanishing of the mass term. The behavior of the mass term is shown in Fig. 5. We can see that the mass becomes zero at the point where the calculation shows the rapid oscillation.

On the other hand, as was shown in the previous section, the mass does not vanish in NLCDH. Thus we can implement the numerical calculation with the ultra-relativistic initial condition without numerical singularities. In

**B. Shock formation**

FIG. 3: The temperature in the shock formation calculated in LCDH with $a = 1$ at $t = 0.75$ fm, starting from the homogeneous initial condition (dotted line).

FIG. 4: The velocity in the shock formation calculated in LCDH with $a = 1$ at $t = 0.75$ fm. The initial velocity (dotted line) at the maximum is $\gamma = 5$.

FIG. 5: The mass matrix in the shock formation calculated in LCDH with $a = 1$ at $t = 0.75$ fm. The mass matrix crosses zero around the minimum.
Figs. 6 and 7 we show, respectively, the temperature and velocity distributions calculated in NLCDH with $a = 1$ and $b = 6$ at $t = 0.75$ fm, starting from the same initial condition. We can see that the calculation remains stable, because the mass does not vanish as is shown in Fig. 8.

In all simulations of the present work, we consistently use the additional viscosity in the NLCDH scheme. That is, we introduce the nonlinear term for the coarse-grain viscosity used in [9].

C. Landau initial condition

Here, we discuss the expansion of the fluid to vacuum and compare the cooling process of LCDH and NLCDH. We use the Landau initial condition where the initial temperature is 590 MeV and the initial size is 0.7 fm. In Fig. 9 we...
show the evolution of the temperature with $a = 0.1$ and $b = 6$ for $t = 1$, 2 and 4 fm, from the top. The solid and dotted lines represents the results of NLCDH and LCDH, respectively. One can see that the cooling and expansion of the fluid of the NLCDH are faster than that of LCDH, similarly to the case of the scaling solution. This is because the bulk viscosity in NLCDH is suppressed by the nonlinear term in comparison with LCDH. This is explicitly shown in Fig. 10, where the evolutions of the bulk viscosity are plotted.

As was discussed in [9], the propagation to vacuum in LCDH, a stationary wave is formed and the pressure and the bulk viscosity should satisfy the relation $P = -\Pi$ at the boundary. In Fig. 11 the pressure (dotted line) and the bulk viscosity (solid line) of NLCDH are shown. One can see that the relation $P = -\Pi$ at the boundary is still satisfied even in NLCDH.
D. Nonperiodic oscillations

In [10], we pointed out that the numerical calculation of LCDH becomes unstable near the central rapidity region and the nonperiodic oscillations appear. It is interesting to note that this scenario persists even in the NLCDH scheme. In Fig. 12, we show the evolution of the temperature calculated in NLCDH with $a = 1$, $b = 6$ for $t = 1.44$, 2.04 and 2.64 fm from the top. We use the Landau initial condition with the initial temperature 590 MeV and the initial size 0.7 fm. One can see that the nonperiodic oscillations evolve with time in the center of the fluid.

Interestingly, the appearance of nonperiodic oscillations has a regularity. We investigate the parameter dependence of how these oscillations emerge. In Fig. 13, we plot the trajectories of the fluid element at the central rapidity region as a function of $\hat{\tau}$ and $R_0$ for $a = 0.5, 0.8, 1, 2, 3, 4$ and 5 from the top fixing $b = 6$. We follow the trajectory of the fluid element in this plane up to the point where the oscillation emerges for each value of $a$. Thus, the line formed by the endpoints of these trajectories defines the critical line for the appearance of the oscillations, indicated by the solid line. One note that the trajectories without oscillation, $a = 0.5$ and 0.8, do not cross this line. We confirm numerically that, for various sets of parameters, the oscillations appear only when a trajectory crosses the critical line. Instabilities have also been analyzed for the scaling solution in the framework of the first order theory [11].

VI. EXTENDED IRREVERSIBLE THERMODYNAMICS

In this paper, we derived our equation by introducing the memory effect and the finite size effect. As was pointed out, there are several different approaches to derive the relativistic hydrodynamics consistent with causality. In this section, we briefly review the derivation based on the extended irreversible thermodynamics (EIT) [16] and show that our formulation and the extended irreversible thermodynamics give the same conclusion.

The usual thermodynamics describes the thermal equilibrium state which can be described by the so-called thermodynamic variables; energy, volume and number of particles. The extended irreversible thermodynamics is the extension of the usual thermodynamics so as to describe the non-equilibrium state, which is characterized by not only the thermodynamic variables but also irreversible currents. Then the first law of thermodynamics in the local rest frame is

$$dS = \frac{\partial S}{\partial E} dE + \frac{\partial S}{\partial V} dV + \frac{\partial S}{\partial N} dN + \frac{\partial S}{\partial \Pi} d\Pi + \frac{\partial S}{\partial \pi^i} d\pi^i + \frac{\partial S}{\partial \nu^i} d\nu^i,$$

(51)
where $\tilde{\Pi}, \tilde{\pi}^{ij}$ and $\tilde{\nu}^i$ are, respectively, the bulk viscosity, the shear viscosity and the heat conduction times $V = 1/\sigma$, following the definition of this paper. On the other hand, the entropy is expanded around the equilibrium state of $(E, V, N)$ as

$$S(E, V, N, \tilde{\Pi}, \tilde{\pi}, \tilde{\nu}) - S(E, V, N) = \frac{1}{2} \frac{\partial^2 S}{\partial \tilde{\Pi}^2} |_{0} \tilde{\Pi}^2 + \frac{1}{2} \frac{\partial^2 S}{\partial \tilde{\pi}^{ij} \partial \tilde{\pi}^{lm}} |_{0} \tilde{\pi}^{ij} \tilde{\pi}^{lm} + \frac{1}{2} \frac{\partial^2 S}{\partial \tilde{\nu}^i \partial \tilde{\nu}^j} |_{0} \tilde{\nu}^i \tilde{\nu}^j,$$

where the suffix 0 denotes $(\tilde{\Pi}, \tilde{\pi}^{ij}, \tilde{\nu}^i) = 0$. It should be noted that we omitted the mixed derivatives with different tensority like $\partial^2 S/\partial \tilde{\Pi} \partial \tilde{\nu}^i$, for simplicity. As a matter of fact, such a term can be small according to the Curie principle [23]. Here, we used that the first derivatives vanishes because the entropy should be maximum in the equilibrium, and ignored higher order derivatives. By comparing the two expressions, we have

$$\frac{\partial S}{\partial \tilde{\Pi}} = -\beta_1 \frac{1}{T} \tilde{\Pi},$$

$$\frac{\partial S}{\partial \tilde{\pi}^{ij}} = -\beta_2 \frac{1}{T} \tilde{\pi}^{ij},$$

$$\frac{\partial S}{\partial \tilde{\nu}^i} = -\beta_3 \frac{1}{T} \tilde{\nu}^i,$$

where $\beta_i/T$ is a function only of the usual thermodynamic variables, $(E, V, N)$. Inserting these results into the first law (51), we obtain

$$TdS = dE + PdV - \mu dN - \beta_1 \Pi \tilde{\Pi} - \beta_2 \pi^{\mu\nu} \tilde{\pi}^{\mu\nu} - \beta_3 \nu^{\mu} \tilde{\nu}^\mu,$$

expressed in a covariant form. From the equation of continuity of the energy-momentum tensor and particle flux, we have

$$\sigma \frac{d}{dT} \tilde{\varepsilon} + (P + \Pi) \partial_\mu u^\mu + u^\mu \partial_\mu \pi^{\mu\nu} = 0,$$

$$\sigma \frac{d}{dT} \tilde{n} + \partial_\mu \nu^\mu = 0.$$

By combining Eqs. (56), (57) and (58) together, we have

$$\partial_\mu S^\mu = \sigma \frac{d}{dT} \tilde{\varepsilon} - \partial_\mu \left( \frac{\mu}{T} \nu^\mu \right) = Q,$$

where

$$Q = -\frac{\Pi}{T} \left( \beta_1 \sigma \frac{d\Pi}{dT} + \partial_\mu u^\mu \right) - \nu_\mu \left( \frac{\beta_3}{T} \sigma \frac{d\tilde{\pi}^{\mu\nu}}{dT} + \partial_\mu \frac{\mu}{T} \right) - \frac{\pi^{\mu\nu}}{T} \left( \beta_2 \sigma \frac{d\tilde{\pi}^{\mu\nu}}{dT} + \partial_\mu \nu^\nu \right),$$

and the entropy four flux is defined by

$$S^\mu = su^\mu - \frac{\mu}{T} \nu^\mu.$$  

To satisfy the algebraic positivity of this entropy production, we obtain the equations of the irreversible current,

$$\partial_\mu u^\mu + \beta_1 \sigma \frac{d\Pi}{dT} = -\alpha_1 \Pi,$$

$$P^{\mu\alpha\beta} \left( \partial_\alpha u_\beta - \beta_2 \sigma \frac{d\tilde{\pi}^{\alpha\beta}}{dT} \right) = \alpha_2 \pi^{\mu\nu},$$

$$P^{\mu\nu} \left( \partial_\nu \frac{\mu}{T} + \frac{\beta_3}{T} \sigma \frac{d\tilde{\nu}_\nu}{dT} \right) = -\alpha_3 \nu^\mu,$$

where the projection operators are defined by

$$P^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu,$$

$$P^{\mu\alpha\beta} = \frac{1}{2} \left( P^{\mu\alpha} P^{\nu\beta} + P^{\mu\beta} P^{\nu\alpha} \right) - \frac{1}{D} P^{\mu\nu} P^{\alpha\beta},$$

(66)
VII. RELATION BETWEEN NLCDH AND THE ISRAEL-STEWART THEORY

As was shown in the previous section, the equation obtained in our formulation can be derived also from the extended irreversible thermodynamics.

Exactly speaking, the similar non-linear term appears even in the IS theory, although in the so-called truncated version of IS theory the non-linear terms are ignored. Then the equation of the bulk viscosity in the original IS theory is given by

$$\tau_R \frac{d\Pi}{d\tau} + \Pi = -\zeta \partial_\mu u^\mu - \frac{\tau_R}{2} \Pi \partial_\mu u^\mu - \frac{\zeta T}{2} \Pi \frac{d}{d\tau} \left( \frac{\tau_R}{\zeta T} \right).$$

(67)

The last two terms on the r. h. s. are ignored in the truncated IS theory. For detailed derivation, see Appendix A. Here we used the relation $\zeta = 1/\alpha_1$ and $\tau_R = \beta_0/\alpha_1$.

One can easily see that there are two differences between NLCDH and IS theory. One is the coefficient of the non-linear term $\tau_R \Pi \partial_\mu u^\mu$. In NLCDH, the coefficient is given by just 1, but it is 1/2 in the IS theory. The other is the last term of Eq. (67), which does not appear in NLCDH.

As was shown in Sec. V A, the hydrodynamic equation is stabilized by the non-linear term in NLCDH. In the IS theory, we could not show the stability of the IS theory analytically because of the last term of Eq. (67). However, the numerical simulation shows that the IS theory is more stable than the truncated IS theory (or LCDH). We calculate the shock formation and the fluid expansion with the same parameters and initial conditions, as was discussed in Sec. V B and V C in the IS theory. We found that the numerical calculations are stable as in the case of NLCDH and the behaviors of the IS theory is similar to that of the NLCDH. In Fig 14, we plot the entropy production as a function of time in the calculation of the expansion to vacuum with the Landau initial condition. One can see that the entropy production of NLCDH and the IS theory is smaller than that of LCDH (or the truncated IS theory) and the IS theory is most close to the Ideal fluid.

The similar reduction due to non-linear terms also appears even for the shear viscosity as was numerically studied in A. In the bulk case, we can show explicitly that the suppression happens because of the minimum value of $\Pi$ guaranteed by the nonlinear term. In the case of the full IS theory, it is not obvious why this happens, but surprisingly the net effect is very close to ours.

The non-periodic oscillation appears even in the IS theory. It should, however, be noted that in these simulations, we introduced the additional viscosity B. If we do not use the additional viscosity, we cannot implement stable numerical simulations even in the IS theory.

It is important to note that the IS theory also can be derived in our formulation. So far, we employed the memory effect between $\Pi$ and $F$ for simplicity reasons. Suppose we apply the same procedure to $\sqrt{\tau_R/\zeta T} \Pi$ and $\zeta \sqrt{\tau_R/\zeta T} \partial_\mu u^\mu$ by using the same memory function,

$$\sqrt{\frac{\tau_R}{\zeta T}} \Pi = -\int_{\tau_0}^{\tau} d\tau' G(\tau, \tau') \zeta \sqrt{\frac{\tau_R}{\zeta T}} \partial_\mu u^\mu.$$  

(68)

This leads to the following equation for $\Pi$,

$$\tau_R \frac{d\Pi}{d\tau} + \frac{1}{2} \tau_R \Pi \partial_\mu u^\mu + \frac{1}{2} \zeta T \frac{d}{d\tau} \tau_R = -\Pi - \zeta \partial_\mu u^\mu.$$  

(69)

This equation is nothing but the equation of the bulk viscosity in the IS theory (See Appendix A). It can also be derived in the framework of the internal-variable theory, although the concept of thermodynamics should be extended. See Appendix A and B respectively.
FIG. 14: The evolution of the entropy production with the Landau initial condition using $a = 0.1$. The dotted, solid and dashed lines represent the calculations of LCDH, NLCDH and the IS theory, respectively.

VIII. CONCLUDING REMARKS

| macroscopic | $1/\partial_{\mu}u^\mu \gg \tau_R$ | $1/\partial_{\mu}u^\mu > \tau_R$ | $1/\partial_{\mu}u^\mu \sim \tau_R$ |
| scale       | (non-relativistic)                  | (relativistic)                    | (ultra-relativistic)                |
| irreversible currents | $J = \eta F$ | $J = \int G \eta F$ | $\tilde{J} = \int G \tilde{\eta} F$ |

In this paper, we extended the previous derivation of the causal dissipative hydrodynamics to take into account the finiteness of fluid cells and the memory effects on the extensive measure of irreversible current inside the fluid cell. The new equation has a non-linear term which suppresses the effect of viscosity. Thus the behavior of this fluid is more close to that of the fluid described by the so-called truncated Israel-Stewart theory, where there is no finite size effect. More importantly, we found that the non-linear term is necessary to implement stable numerical simulations for the ultra-relativistic situations, like high initial velocity, high initial energy density and so on.

In this study, we found that there are three stages in the structure of hydrodynamics as is summarized in the above table. When the time scale of microscopic degrees of freedom are clearly separated from those of the hydrodynamic variables, we can assume that the irreversible currents $J$ are immediately produced by the thermodynamic forces $F$, that is, $J = \eta F$. This is realized in the non-relativistic cases, $1/\partial_{\mu}u^\mu \gg \tau_R$, because the time scale of constituent particle of the fluid is much faster than the velocity of the fluid. However, in the relativistic fluids where $1/\partial_{\mu}u^\mu > \tau_R$, the clear separation of the time scales is not necessarily borne out and we have to take into account the retardation effect in the formation of the irreversible currents by introducing memory functions $G$, that is, $J = \int G \eta F$. In the ultra-relativistic limit where the relaxation time is same order as the scale of inhomogeneity, $\tau_R \sim 1/\partial_{\mu}u^\mu$, it is important to consider the effect of the finite fluid cell volume $1/\sigma$, because the volume of the fluid cell changes during the hysteresis, $\tilde{J} = \int G \tilde{\eta} F$.

We further showed that our formulation and the extended irreversible thermodynamics lead to the same hydrodynamic equation. The only difference is that the thermodynamic variables $\varepsilon$ and $P$ in NLCDH satisfy usual thermodynamic relation, while the variables follows the extended thermodynamic relation in the hydrodynamic equation in the extended irreversible thermodynamics. This may indicate us the robustness of our equation.
On the other hand, our result is different from that of the IS theory. We showed that the IS theory also seems to be applicable to the ultra-relativistic cases, although we could not show the positivity of the mass explicitly. The quantitative difference between our theory and the IS theory is very small but the IS theory is more close to the behavior to the ideal fluid. It is also interesting to mention that the equation of the IS theory can be derived even in our formulation if we introduce a very peculiar form of thermodynamical variable to introduce the memory effect. Although our choice seems more natural and simple from the point of view of the memory function on thermodynamical forces, we need experimental and other theoretical supports to decide the appropriate forms of thermodynamical variables. The way of extending the thermodynamics to irreversible domain is not unique (See also the discussion in Appendix B and Ref. [16]). The schematic mapping of different theories are summarized in Fig. VIII. It should be noted that to derive the IS equation in the memory function method, we have to break the extensivity of currents because we employ the memory effect to the quantity proportional to $1/\sqrt{\sigma}$ as is shown in Eq. (68).

We are interested in the dynamics of fluid, which cannot be described by the simple Boltzmann equation. It is not obvious but if the theory is still applicable to the dynamics of a dilute gas as in the non-relativistic case, the equation should be justified from the kinetic argument such as the moment method. As a matter of fact, the problem of the simplest moment method is known and there are several proposals for the improvement [25]. The derivation of our equation from the kinetic point of view is still an open problem.

T. Koide acknowledges useful discussions with D. Jou. This work is supported by FAPERJ and CNPq.

APPENDIX A: ISRAEL-STEWART THEORY

Similar to the extended irreversible thermodynamics, the equations of irreversible currents are derived by applying the algebraic positivity of the entropy production in the IS theory. In the derivation of Israel and Stewart, they used two equations; one is the definition of the entropy four flux,

$$ S^\mu_{IS} = s(T, \mu) u^\mu - \frac{\mu}{T} u^\mu - Q^\mu, \quad (A1) $$

and the other is the expression of $Q^\mu$, which is assumed by the general quadratic form,

$$ T Q^\mu = \frac{1}{2} u^\mu \left( \beta_0 \Pi^2 + \beta_2 \pi_\mu \pi^{\mu\nu} + \beta_1 \nu_\mu \nu^\mu \right). \quad (A2) $$

Here $s(T, \mu)$ denotes the entropy density in equilibrium for given temperature $T$ and chemical potential $\mu$.

By using the algebraic positivity of the entropy production $\partial_\mu S^\mu_{IS} \geq 0$ as usual, we obtain

$$ \alpha_1 \Pi = -\partial_\mu u^\mu - \frac{\beta_0}{2} \Pi \partial_\mu u^\alpha - \frac{T}{2} \Pi \frac{d}{dt} \left( \frac{\beta_0}{T} \right), \quad (A3) $$

$$ \alpha_2 \pi^{\mu\nu} = P^{\mu\nu\alpha\beta} \left( \partial_\alpha u_\beta - \frac{\beta_2}{2} \pi_{\alpha\beta} \partial_\lambda u^\lambda - \frac{T}{2} \pi_{\alpha\beta} \frac{d}{dt} \left( \frac{\beta_2}{T} \right) \right), \quad (A4) $$

$$ \alpha_3 \nu^\mu = P^{\mu \nu} \left( -\partial_\nu \frac{\mu}{T} - \frac{\beta_1}{2T} \nu_\nu \partial_\nu u^\alpha - \frac{T}{2} \nu_\nu \frac{d}{dt} \left( \frac{\beta_1}{T} \right) \right). \quad (A5) $$

The derivation of Israel and Stewart depends on the validity of the assumption (A1). To derive this equation, it is useful to derive the following relation in the equilibrium,

$$ dS^\mu_{(0)} = d(su^\mu) = \beta_\nu dT^{\mu\nu}_{(0)} - \frac{\mu}{T} dN^\mu_{(0)}, \quad (A6) $$

$$ S^\mu_{(0)} = P^\beta^\mu + \beta_\nu T^{\mu\nu}_{(0)} - \frac{\mu}{T} N^\mu_{(0)}, \quad (A7) $$
here, the subscript (0) indicates quantities in equilibrium and $\beta_\nu = u_\nu/T$.

The fundamental assumption used by Israel and Stewart is the so called “release of variations”, which assumes that\footnote{\ref{A6}} stays valid for a virtual displacement from an equilibrium state to an arbitrary neighbouring state,

$$dS^\mu = \beta_\nu dT^\mu\nu - \frac{\mu}{T} dN^\mu + O_2.$$ \hfill (A8)

Here, the last term $O_2$ denotes the contribution from the second order deviation from equilibrium. This postulate enables us to determine the form of the entropy flux in a near equilibrium system perturbatively. By addition of \eqref{A7} and \eqref{A8} the the entropy four flux in the IS theory is obtained,

$$S^\mu_{\text{IS}} = P\beta^\mu + T^\mu\nu\beta_\nu - \frac{\mu}{T} N^\mu - Q^\mu,$$ \hfill (A9)

where $Q^\mu$ is an undetermined second order term in the deviations $T^\mu\nu - T^\mu\nu_0$, $N^\mu - N^\mu_0$.

We introduce, again, the energy-momentum tensor and particle flux as follows;

$$T^\mu\nu = (\varepsilon + P + \Pi) u^\mu u^\nu - g^\mu\nu (P + \Pi) + \pi^\mu\nu,$$ \hfill (A10)

$$N^\mu = nu^\mu + \nu^\mu.$$ \hfill (A11)

It should be noted that, different from the extended irreversible thermodynamics, the thermodynamic variables $\varepsilon$, $P$ and $n$ satisfy the usual thermodynamic relations by construction. By using Eqs. \eqref{A10} and \eqref{A11}, we finally obtain Eq. \eqref{A1}.

\section*{APPENDIX B: INTERNAL-VARIABLE THEORY}

The internal-variable theories (IVT) is another approach to derive the generalized hydrodynamics\footnote{\ref{16,24}}. Similarly to the extended irreversible thermodynamics, the IVT includes additional variables except for the usual thermodynamic variables. In this appendix, we use the idea of the IVT and rederive the IS theory.

First, we assume the following modified first law,

$$TdS = dE + pdV - \mu dN - TdQ,$$ \hfill (B1)

where $Q$ is an additional variable. As we will see later, this definition of the modified first law is different from that in the extended irreversible thermodynamics. Then, the entropy production is given by

$$\sigma \frac{d\tilde{s}}{d\tau} - \partial_\mu (\alpha u^\mu) = -\frac{\Pi}{T} \beta_\mu u^\mu + \frac{\pi^\mu\nu}{T} \partial_\nu u_\nu - \nu^\mu \beta_\mu \alpha - \sigma \frac{dQ}{d\tau}.$$ \hfill (B2)

To obtain the same result as the IS theory, we assume $Q$ as follows,

$$Q = \frac{\sigma}{2T} \left( \beta_1 \tilde{\Pi}^2 + \beta_2 \tilde{\pi}_{\mu\nu} \tilde{\pi}^{\mu\nu} + \beta_3 \tilde{\nu}_\mu \tilde{\nu}^\mu \right).$$ \hfill (B3)

Then, for the positivity of the r. h. s. of Eq. \eqref{B2}, we obtain the IS theory,

$$\left( \partial_\alpha u_\beta - \beta_2 \frac{d\pi_{\alpha\beta}}{d\tau} - \frac{\pi_{\alpha\beta}}{T} \frac{d}{d\tau} (\beta_1 T) + \beta_1 \frac{\pi_{\alpha\beta}}{T} \partial_\mu u^\mu \right) = -\alpha_1 \Pi,$$ \hfill (B4)

$$P^{\mu\nu\alpha\beta} \left( \partial_\alpha u_\beta - \beta_2 \frac{d\pi_{\alpha\beta}}{d\tau} - \frac{\pi_{\alpha\beta}}{T} \frac{d}{d\tau} (\beta_1 T) - \beta_2 \frac{\pi_{\alpha\beta}}{T} \partial_\mu u^\mu \right) = \alpha_2 \pi^{\mu\nu},$$ \hfill (B5)

$$P^{\mu \nu} \left( \partial_\mu \alpha + \beta_3 \frac{d\nu_\mu}{T} \frac{d}{d\tau} (\beta_3 T) + \frac{\nu_\mu}{T} \frac{d}{d\tau} (\beta_3 T) + \beta_3 \nu_\mu \partial_\mu u^\mu \right) = -\alpha_3 \nu^\nu.$$ \hfill (B6)

The main difference between the extended irreversible thermodynamics and the IVT is the definition of the first law. By substituting Eq. \eqref{B3} into Eq. \eqref{B1}, we obtain

$$TdS = dE + PdV - \mu dN - \beta_1 \Pi d\tilde{\Pi} - \beta_2 \pi_{\mu\nu} d\tilde{\pi}^{\mu\nu} - \beta_3 \nu_\mu d\tilde{\nu}^\mu - T\tilde{\Pi}^2 d \left( \frac{\sigma \beta_1}{2T} T \tilde{\pi}_{\mu\nu} \tilde{\pi}^{\mu\nu} \right) - T\tilde{\pi}_{\mu\nu} \tilde{\pi}^{\mu\nu} d \left( \frac{\sigma \beta_2}{2T} \right) - T\tilde{\nu}_\mu \tilde{\nu}^\mu d \left( \frac{\sigma \beta_3}{2T} \right).$$ \hfill (B7)
One can easily see that the last three terms on the r. h. s. do not exist in the extended irreversible thermodynamics.

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