Finite field dependent BRST transformations and its applications to gauge field theories

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Dedicated to My Family
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ABSTRACT

The Becchi-Rouet-Stora and Tyutin (BRST) transformation plays a crucial role in the quantization of gauge theories. The BRST transformation is also very important tool in characterizing the various renormalizable field theoretic models. The generalization of the usual BRST transformation, by making the infinitesimal global parameter finite and field dependent, is commonly known as the finite field dependent BRST (FFBRST) transformation. In this thesis, we have extended the FFBRST transformation in an auxiliary field formulation and have developed both on-shell and off-shell FF-anti-BRST transformations. The different aspects of such transformation are studied in Batalin-Vilkovisky (BV) formulation. FFBRST transformation has further been used to study the celebrated Gribov problem and to analyze the constrained dynamics in gauge theories. A new finite field dependent symmetry (combination of FFBRST and FF-anti-BRST) transformation has been invented. The FFBRST transformation is shown useful in connection of first-class constrained theory to that of second-class also. Further, we have applied the Batalin-Fradkin-Vilkovisky (BFV) technique to quantize a field theoretic model in the Hamiltonian framework. The Hodge de Rham theorem for differential geometry has also been studied in such context.
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Due to the lack of hard empirical data, symmetry principles have been proved to be the most invaluable tools in describing physical phenomenon. Gauge field theories (based on the local gauge invariance of the Lagrangian density of the theories) have enormous importance in describing all the fundamental interactions of nature and play the central role in understanding the present state of the art of modern particle physics. The standard model of particle physics which describes strong, weak and electromagnetic interactions on the same footing is a non-Abelian gauge theory (Yang-Mills theory) [1].

However, one faces various problems to develop the quantum version of such theories with local gauge invariance consistently. In particular, the generating functional,

\[ Z = \int \mathcal{D}A \ e^{i \int d^4 x L}, \]

for such gauge theories becomes ill-defined due to the over counting of physically equivalent gauge configuration. This in turn leads to the ill-defined Green’s functions of these theories. Therefore, it is necessary to eliminate the redundant degrees of freedom from the functional integral representation of the generating functional \( Z \). This can be achieved by adding a gauge variant term, called as gauge-fixing term, to the Lagrangian density \( L \) of the theory. The generating functional is made well-defined in the cost of the gauge symmetry. The gauge-fixing was achieved by adding an extra term consisting of arbitrary function of the gauge field and arbitrary gauge parameter. This of course solves the problem of over counting but the physical theory now depends on arbitrary function of gauge field and/or arbitrary parameter which is not desirable for any physical theory. Faddeev-Popov (FP) resolved this problem by introducing unphysical ghost fields which are scalars but behave like Grassmannian [2]. These unphysical fields compensate the effect of arbitrary function and in term preserves the unitarity of the theory. Various difficulties in different situations occur due to the gauge non-invariance of the theory; For example, the choices of the counter terms in the renormalization program in such theories are no more restricted to the gauge invariant terms as the gauge invariance is broken.

C. Becchi, A. Rouet and R. Stora and independently I. V. Tyutin came to resolve the situation by discovering a new symmetry of the FP effective theory known as BRST symmetry [3, 4]. This BRST transformation is characterized by (i) infinitesimal, (ii) global (i.e. does not depend on the space-time) and (iii) anticommuting parameter. Such BRST transformation leaves the effective action, including gauge-fixing and ghost parts, invariant and is nilpotent in nature. Sometimes the nilpotency is proved using equation of motion of one or more fields then it is referred as on-shell nilpotent. However, BRST transformation can be made off-shell nilpotent by introducing Nakanishi-Lautrup type auxiliary fields to the theory. BRST symmetry is extremely useful in quantizing different gauge field theoretic models and the renormalization program is greatly facilitated by the use of such symmetry [3, 4, 5, 6].
To cover the wider class of gauge theories, including open or reducible gauge theories, a powerful technique was introduced by I. A. Batalin and G. A. Vilkovisky [5, 6, 7, 8, 9], known as field/antifield (or BV) formulation. The main idea of this formulation is to construct an extended action by introducing the antifields for each field in the theories. The antifields satisfy the opposite statistics corresponding to that of fields and have the ghost number equal to \(-gh(\phi)-1\), where \(gh(\phi)\) is the ghost number of the fields. However, the extended action satisfies the certain rich mathematical formula known as quantum master equation which reflects the gauge symmetry in the zeroth order of antifields and in the first order of antifields it reflects the nilpotency of BRST transformation. These extended theories work extremely well in the frame of gauge theories which are always endowed with first-class constraints in the language of Dirac’s constraints analysis [10, 11, 12, 13]. The systems with second-class constraints are quantized by converting these to a first-class theory in an extended phase space [3, 10, 11, 12, 14]. This procedure has been introduced by I. A. Batalin, E. S. Fradkin and I. V. Tyutin [15, 16] and has been applied to the various models [17, 18, 19, 20, 21]. Another way of approaching the problem, which is very different from the Dirac’s method, is the BFV (due to I. A. Batalin, E. S. Fradkin and G. A. Vilkovisky) quantization [5, 22, 23]. The main features of BFV approach are as follows: (I) it does not require closure off-shell of the gauge algebra and therefore does not need an auxiliary fields, (II) this formalism relies on BRST transformation which is independent of gauge-fixing condition and (III) it is also applicable to the first order Lagrangian. Hence it is more general than the strict Lagrangian approach.

In all these approaches of studying gauge theories the main ingredient is the underlying BRST symmetry of the FP effective theory. Therefore, any modification or reformulation or generalization of BRST transformation is extremely important in the study of fundamental interactions which are described by gauge theories. With various motivations and goals, BRST transformation has been generalized in many different ways. M. Lavelle and D. Mcmullan had found a generalized BRST symmetry adjoint to usual BRST symmetry in the case of QED which is nonlocal and noncovariant [24]. The motivation behind the emergence of this symmetry was to refine the characterization of physical states given by the BRST charge. Later, Z. Tang and D. Finkelstein had found another generalized BRST symmetry which is nonlocal but covariant [25]. Such a BRST symmetry is not nilpotent generally and additional conditions are required in auxiliary field formulation to make them nilpotent. H. S. Yang and B. H. Lee had presented a local and noncovariant BRST symmetry in the case of Abelian gauge theories [26]. Finite field dependent BRST (FFBRST) transformation, where the parameter is finite and field dependent but still anticommuting in nature, is the most important among the generalizations of BRST symmetry which was developed by S. D. Joglekar and B. P. Mandal for the first time in 1995 [27]. They had shown that the usual infinitesimal, global BRST transformation can be integrated out to construct the FFBRST transformation [27]. The parameter in such a transformation is anticommuting, finite in nature, depends on the fields, and does not depend on space-time explicitly. FFBRST transformation is also the symmetry of the effective theories and maintains the on-shell nilpotency property. Moreover, FFBRST transformation is capable of connecting the generating functionals of two different effective field theories with suitable choice of the finite field dependent parameters [27]. For example, this transformation was used to connect the FP effective action in Lorentz gauge with a gauge parameter \(\lambda\) to (i) the most general BRST/anti-BRST symmetric action in Lorentz gauge [27], (ii) the FP effective action in axial gauge [28, 29, 30, 31, 32], (iii) the FP effective action in Coulomb gauge [33], (iv) FP effective
action with another distinct gauge parameter $\lambda'$ \[27\] and (v) the FP effective action in quadratic gauge \[27\]. The FFBRST transformation was also used to connect the generating functionals corresponding to different solutions of the quantum master equation in field/antifield formulation \[34\]. The choice of the finite parameter is crucial in connecting different effective gauge theories by means of the FFBRST transformation. The path integral measure in the expression of generating functional is not invariant under FFBRST transformation. The nontrivial Jacobian of such FFBRST transformation is the source for new results. The FFBRST formulation has many applications \[28, 29, 30, 31, 32, 35, 36, 37, 38, 39, 40, 41\] on the gauge theories. A correct prescription for the poles in the gauge field propagators in noncovariant gauges has been derived by connecting effective theories in covariant gauges to the theories in noncovariant gauges by using FFBRST transformation \[37\]. The divergent energy integrals in the Coulomb gauge are regularized by modifying the time like propagator using FFBRST transformation \[33\]. The FFBRST transformation, which is discussed so far in literature, is only on-shell nilpotent \[27, 28, 29, 36\].

In this thesis, we would like to address different issues of BRST transformation, its generalizations and applications to different gauge field theoretic models. We further generalize the FFBRST transformation and find new applications. We develop the off-shell nilpotent FFBRST transformation by introducing a Nakinishi-Lautrup type auxiliary field and show that such transformation is more elegant to use in certain specific cases \[42\]. The anti-BRST transformation, where the role of ghost and antighost fields are interchanged with some coefficients, does not play as fundamental role as BRST symmetry itself but it is a useful tool in geometrical description \[43\] of BRST transformation, in the investigation of perturbative renormalization \[44\]. We develop both the on-shell and off-shell nilpotent finite field dependent anti-BRST (FF-anti-BRST) transformations for the first time which play similar role as FFBRST transformation \[42\].

We study these transformations in the context of higher form gauge theories \[45\]. The gauge theories of Abelian rank-2 antisymmetric tensor field play crucial role in studying the theory for classical strings \[46\], vortex motion in an irrotational, incompressible fluid \[10, 47\] and the dual formulation of the Abelian Higgs model \[48, 49\]. Abelian rank-2 antisymmetric tensor fields are also very useful in studying supergravity multiplets \[50\], excited states in superstring theories \[51, 52\] and anomaly cancellation in certain superstring theories. Geometrical aspects of Abelian rank-2 antisymmetric tensor fields are studied in a U(1) gauge theory in loop space. We extend the FFBRST formulation to study Abelian rank-2 tensor field theories. We establish the connection between different effective 2-form gauge theories using the FFBRST and FF-anti-BRST transformations. The FF-anti-BRST transformation plays similar role to connect different effective theories. We further extend these FFBRST and FF-anti-BRST transformations to the field/antifield formulation of 2-form gauge theory \[45\].

In non-Abelian gauge theories even after gauge-fixing the redundancy of gauge fields is not completely removed in certain gauges for large gauge fields (Gribov problem) \[53\]. The Yang-Mills (YM) theories in those gauges contain so-called Gribov copies. Gribov copies play a crucial role in the infrared (IR) regime while it can be neglected in the perturbative ultraviolet (UV) regime \[53, 54, 55\]. This topic has become very exciting currently due to the fact that color confinement is closely related to the asymptotic behavior of the ghost and gluon propagators in deep IR regime \[56\]. In order to resolve the Gribov problem, Gribov and Zwanziger (GZ) proposed a theory, which restricts the domain of integration in the functional integral within
the first Gribov horizon [54]. This restriction to first Gribov horizon is achieved by adding a nonlocal term, commonly known as horizon term, to the YM action [54, 55, 57]. But the YM action restricted in Gribov region (i.e. GZ action) does not exhibit the usual BRST invariance, due to the presence of the nonlocal horizon term [58]. The famous Kugo-Ojima (KO) criterion for color confinement [12] is based on the assumption of an exact BRST invariance of YM theory in the manifestly covariant gauge. Recently, a nilpotent BRST transformation which leaves the GZ action invariant has been obtained and can be applied to KO analysis of the GZ theory [59]. The BRST symmetry in presence of the Gribov horizon has great applicability in order to solve the nonperturbative features of confining YM theories [60, 61], where the soft breaking of the BRST symmetry exhibited by the GZ action is converted into an exact invariance [62].

We consider FFBRST transformation in Euclidean space to show the mapping between the generating functional of GZ theory to that of YM theory [63]. Such a mapping is also shown to exists in field/antifield formulation of GZ theory [64].

So far we have seen that FFBRST and FF-anti-BRST transformations are symmetry of the effective action but do not leave the generating functionals invariant. The Jacobians of the path integral measure in the expression of generating functional are not unity as these transformations are finite in nature. We address the important question whether it is possible to develop a finite nilpotent symmetry for both effective action as well as the generating functional? In search of answer to this question, we propose the finite version of mixed BRST (combination of BRST and anti-BRST) transformation. Such a finite mixed BRST (FFMBRST) transformation is shown to be nilpotent as well as the symmetry of both the effective action and the generating functional [65]. Our results are established with the help of several explicit examples. This formulation is further extended to field/antifield formulation [65].

In another problem, we study the systems with constraints in the framework of FFBRST transformation. The gauge variant model for the single self-dual chiral boson in (1+1) dimensions (2D) is well known example of the second-class theory [66, 67, 68, 69, 70, 71, 72]. This model was made gauge invariant by adding the Wess-Zumino (WZ) term and had been studied using BFV formulation [73, 74]. Such a model is very useful in the study of certain string theoretic models [75] and plays a crucial role in the study of quantum Hall effect [76]. The Proca model in (1+3) dimensions (4D) for massive spin 1 vector field also is another example of a system with the second-class constraint as the gauge symmetry is broken by the mass term of the theory. However, Stueckelberg converted this theory to a first-class theory by introducing a scalar field [77, 78, 79]. Such a gauge invariant description for massive spin 1 field has many applications in gauge field theories as well as in string theories [80, 81, 82]. We establish the connection between the generating functionals for the first-class theories and the generating functionals for the second-class theories using FFBRST transformation [83]. The generating functional of the Proca model is obtained from the generating functional of the Stueckelberg theory for massive spin 1 vector field using FFBRST transformation with appropriate choice of the finite parameter. In the other example we relate the generating functionals for the gauge invariant and the gauge variant theory for self-dual chiral boson by constructing suitable FFBRST transformation. Thus, the complicated nonlocal Dirac bracket analysis in the study of the second-class theories is avoided in our formulation.

In a different problem, we study the analogy between the conserved charges of different BRST and co-BRST symmetry transformations with exterior and co-exterior derivatives [73]. In the BRST formulation of gauge theories one requires that the physical subspace of total Hilbert
space of states contains only those states that are annihilated by the nilpotent and conserved BRST charge $Q_b$ i.e. $Q_b |\text{phys}\rangle = 0$ [13, 84]. The nilpotency of the BRST charge ($Q_b^2 = 0$) and the physicality criterion ($Q_b |\text{phys}\rangle = 0$) are the two essential ingredients of BRST quantization. In the language of differential geometry defined on compact, orientable Riemannian manifold, the cohomological aspects of BRST charge is realized in a simple, elegant manner. The nilpotent BRST charge is connected with exterior derivative (de Rham cohomological operator $d = dx^\mu \partial_\mu$, with $d^2 = 0$)[85, 86, 87, 88, 89, 90, 91, 92, 93, 94]. It has been found that the co-BRST transformation which is also the symmetry of the action and leaves the gauge-fixing part of the action invariant separately. The conserved charge corresponding to the co-BRST transformation is shown to be analog to the co-exterior derivative ($\delta = \pm \ast d \ast$, with $\delta^2 = 0$)[92].

This thesis is divided into following nine chapters. The detail contents of these chapters are given below.

General introductions to i) BRST transformation, its generalizations and applications, ii) basic techniques of field/antifield formulation and BFV formulation are presented in chapter one. We brief the contents of the different chapters of the thesis.

In Chapter two, we provide the different mathematical techniques which will be used to construct the thesis. In particular FFBRST transformation, BV formulation and BFV technique are discussed in brief.

In chapter three, we formulate the FFBRST transformation in auxiliary field formulation to make it off-shell nilpotent [42]. We consider several examples to demonstrate that the off-shell nilpotent FFBRST transformation also leads to similar results in connecting the different generating functionals. We further construct the finite field dependent anti-BRST (FF-anti-BRST) transformation analogous to the FFBRST transformation by integrating infinitesimal anti-BRST transformation. By considering several choices of finite field dependent parameter in FF-anti-BRST transformation we show that FF-anti-BRST transformation also plays the same role as FFBRST transformation in connecting different effective theories but with different parameters. Finally, we consider the formulation of FF-anti-BRST transformation in auxiliary field formulation also to make it off-shell nilpotent.

In chapter four, we study the quantization of the Abelian rank-2 antisymmetric tensor field by using the FFBRST transformation [45]. We show that it is possible to construct the Abelian rank-2 tensor field theory in noncovariant gauges by using FFBRST transformation. In particular, we show that the generating functional for Abelian rank-2 tensor field theory in covariant gauges transformed to the generating functional for the same theory in a noncovariant gauges for a particular choice of the finite parameter in FFBRST transformation. The new results arise from the nontrivial Jacobian of the path integral measure under such finite BRST transformation. The connections between the theories in two different noncovariant gauge, namely axial gauge and Coulomb gauge are also established explicitly. Further, we consider the field/antifield formulation of Abelian rank-2 tensor field theory by introducing the antifield $\phi^*$ corresponding to each field $\phi$ with opposite statistics to study the role of FFBRST transformation in such formulation. We show that the FFBRST transformation changes the generating functional corresponding to one gauge-fixed fermion to the generating functional to another gauge-fixed fermion. Thus, the FFBRST transformation connects the different solutions of the master equation in field/antifield formulation. We show this by considering explicit examples.

Chapter five is devoted for the construction of FFBRST transformation in Euclidean space
to study the GZ theory \cite{63, 64}. By constructing an appropriate finite field dependent parameter we show that such FFBRST transformation relates the generating functional for GZ theory to the generating functional in YM theory. Thus, we are able to connect the theories with and without Gribov copies.

In chapter six, we construct the finite version of MBRST transformation having finite and field dependent parameters \cite{65}. The usual FFBRST and FF-anti-BRST transformations are the symmetry transformations of the effective action only but do not leave the generating functional invariant as the path integral measure in the definition of generating functional transforms in a nontrivial manner \cite{27, 42}. Unlike the usual FFBRST and FF-anti-BRST transformations, this finite field dependent MBRST (FFMBRST) transformation is shown to be the symmetry of the both effective action as well as the generating functional of the theory. We construct the finite parameters in the FFMBRST transformation in such a way that the Jacobian contribution due to FFBRST part compensates the same due to FF-anti-BRST part. Thus, we are able to construct the finite nilpotent transformation which leaves the generating functional as well as the effective action of the theory invariant. We further show that the effect of FFMBRST transformation is equivalent to the effect of successive operations of FFBRST and FF-anti-BRST transformations. Our results are supported by several explicit examples. First of all we consider the gauge invariant model for single self-dual chiral boson in (1+1) dimensions \cite{66, 67, 73} to show our results. (1+3) dimensional Abelian as well as non-Abelian YM theory in the Curci-Ferrari-Delbourgo-Jarvis (CFDJ) gauge \cite{95, 96, 97} are also considered to demonstrate the above finite nilpotent symmetry. To study the role of FFMBRST transformation in field/antifield formulation we consider the same three simple models in BV formulation. We show that the FFMBRST transformation does not change the generating functional written in terms of extended quantum action in BV formulation. Hence the FFMBRST transformation leaves the different solutions of the quantum master equation in field/antifield formulation invariant.

In chapter seven, we study the theories with constraints in the context of FFBRST transformation \cite{83}. Theories with first-class constraints are shown to be related to the theories with second-class constraints through FFBRST transformation. The generating functional of Stueckelberg theory for massive spin 1 vector field is related to the generating functional of Proca model for the same theory through FFBRST and FF-anti-BRST transformations. Similar relationship is also established between the gauge invariant and gauge variant models for single self-dual chiral boson through FFBRST and FF-anti-BRST formalism.

In chapter eight, we present BFV formulation for the model of single self-dual chiral boson \cite{73}. Along with the usual nilpotent BRST symmetry, anti-BRST, co-BRST and anti-co-BRST symmetries are investigated in this framework. The generators of all these continuous symmetry transformations are shown to obey the algebra of de Rham cohomological operators of differential geometry. The Hodge decomposition theorem in the quantum Hilbert space of states is also discussed. We show that the classical theory for a self-dual chiral boson is a field theoretic model for Hodge theory.

The last chapter is devoted for summary and conclusion.
Chapter 2

The mathematical basis

The aim of this chapter is to provide the basic techniques and mathematical tools to prepare the necessary background relevant to this thesis. In particular, we briefly outline the basic ideas of the on-shell finite field dependent BRST transformation, BV formulation of gauge theories and BFV technique. We start with on-shell FFBRST transformation in the next section.

2.1 On-shell finite field dependent BRST (FFBRST) transformation

We begin with the on-shell FFBRST formulation of pure gauge theories [27]. The usual BRST transformation for the generic fields \( \phi \) of an effective theory is defined compactly as

\[
\delta_b \phi = s_b \phi \Lambda,
\]

(2.1)

where \( s_b \phi \) is the BRST variation of the fields with infinitesimal, anticommuting and global parameter \( \Lambda \). Such transformation is on-shell nilpotent, i.e. \( s_b^2 = 0 \), with the use of some equation of motion for fields and leaves the FP effective action invariant. It was observed by Joglekar and Mandal [27] that \( \Lambda \) needs neither to be infinitesimal, nor to be field-independent to maintain the symmetry of the FP effective action of the theory as long as it does not depend explicitly on space-time. They made it infinitesimally field dependent and then integrated the infinitesimal field dependent BRST transformation to construct FFBRST transformation which preserves the same form as

\[
\delta_b \phi = s_b \phi \Theta_b[\phi],
\]

(2.2)

where \( \Theta_b[\phi] \) is an \( x \)-independent functional of fields \( \phi \).

We briefly mention the important steps to construct FFBRST transformation. We start with the fields, \( \phi(x, \kappa) \), which are made to depend on some parameter, \( \kappa : 0 \leq \kappa \leq 1 \), in such a manner that \( \phi(x, \kappa = 0) = \phi(x) \) is the initial field and \( \phi(x, \kappa = 1) = \phi'(x) \) is the transformed field. The infinitesimal parameter \( \Lambda \) in the BRST transformation is made field dependent and hence the BRST transformation can be written as

\[
\frac{d}{d\kappa} \phi(x, \kappa) = s_b \phi(x, \kappa) \Theta'_b[\phi(x, \kappa)],
\]

(2.3)

where \( \Theta'_b \) is an infinitesimal field dependent parameter. By integrating these equations from \( \kappa = 0 \) to \( \kappa = 1 \), it has been shown [27] that the \( \phi'(x) \) are related to \( \phi(x) \) by the FFBRST transformation as

\[
\phi'(x) = \phi(x) + s_b \phi(x) \Theta_b[\phi(x)],
\]

(2.4)
2.1. On-shell finite field dependent BRST (FFBRST) transformation

where $\Theta_b[\phi(x)]$ is obtained from $\Theta'_b[\phi(x)]$ through the relation [27]

$$
\Theta_b[\phi(x, \kappa)] = \Theta'_b[\phi(x, 0)] \frac{\exp f[\phi(x, 0)] - 1}{f[\phi(x, 0)]},
$$

and $f$ is given by $f = \sum_i \delta \Theta_i(x) s_i \phi_i(x)$. This transformation is nilpotent and symmetry of the effective action. The generating functional, defined as

$$
Z = \int [\mathcal{D} \phi] e^{i S_{\text{eff}}},
$$

is not invariant under such FFBRST transformation as the Jacobian in the above expression is not invariant under it. Under FFBRST transformation Jacobian changes as

$$
\mathcal{D} \phi = J[\phi(\kappa)] \mathcal{D} \phi(\kappa).
$$

It has been shown [27] that under certain condition this nontrivial Jacobian can be replaced (within the functional integral) as

$$
J[\phi(\kappa)] \rightarrow e^{i S_1[\phi(\kappa)]},
$$

where $S_1[\phi(\kappa)]$ is some local functional of $\phi(x)$. The condition for existence of $S_1$ is

$$
\int [\mathcal{D} \phi] \left[ \frac{1}{J} \frac{dJ}{d\kappa} - i \frac{dS_1}{d\kappa} \right] \exp i[S_{\text{eff}} + S_1] = 0.
$$

Thus,

$$
Z \left( - \int [\mathcal{D} \phi] e^{i S_{\text{eff}}} \right) \xrightarrow{\text{FFBRST}} Z' \left( - \int [\mathcal{D} \phi] e^{i [S_{\text{eff}}(\phi) + S_1(\phi) + S_1]} \right).
$$

$S_1[\phi]$ depends on the finite field dependent parameter. Therefore, the generating functional corresponding to the two different effective theories can be related through FFBRST transformation with appropriate choices of finite parameters. The FFBRST transformation has also been used to solve many of the long outstanding problems in quantum field theory [28, 29, 30, 31, 32, 45, 33, 36, 37, 38]. For example, the gauge field propagators in noncovariant gauges contain singularities on the real momentum axis [30]. Proper prescriptions for these singularities in gauge field propagators have been found by using FFBRST transformation [33].

2.1.1 Evaluation of Jacobian

Due to the finiteness of FFBRST transformation the Jacobian is not unity and hence it is important to consider the Jacobian contribution. In this subsection we present the general method to evaluate the nontrivial Jacobian of path integral measure for FFBRST transformation. Here we utilize the fact that FFBRST transformation can be written as a succession of infinitesimal transformation given in Eq. (2.3). Now, one defines the path integral measure as

$$
\mathcal{D} \phi = J(\kappa) \mathcal{D} \phi(\kappa) = J(\kappa + d\kappa) \mathcal{D} \phi(\kappa + d\kappa).
$$
2.2. Batalin-Vilkovisky (BV) formalism

Since the transformation $\phi(\kappa)$ to $\phi(\kappa + d\kappa)$ is an infinitesimal one, then the equation reduces to

$$\frac{J(\kappa)}{J(\kappa + d\kappa)} = \int d^4x \sum_\phi \frac{\delta\phi(x, \kappa + d\kappa)}{\delta\phi(x, \kappa)}. \quad (2.12)$$

where $\Sigma_\phi$ sums over all fields in the measure and $\pm$ refers to whether $\phi$ is bosonic or fermionic field. Using the Taylor’s expansion in the above equation, the expression for infinitesimal change in Jacobian is obtained as follows:

$$\frac{1}{J(\kappa)} \frac{dJ(\kappa)}{d\kappa} = -\int d^4x \sum_\phi \left[ \pm s_b \frac{\delta\Theta(\phi(x, \kappa))}{\delta\phi(x, \kappa)} \right]. \quad (2.13)$$

The nontrivial Jacobian is the source of new results in the FFBRST formulation.

2.2 Batalin-Vilkovisky (BV) formalism

The BV formulation (also known as field/antifield formulation) is a powerful technique in the Lagrangian framework to deal with more general gauge theories. This method is applicable to gauge theories with both reducible/open as well as irreducible/close algebras \cite{6, 7, 8}. The basic idea in this approach is to introduce the so-called antifield ($\phi^*$) for each field ($\phi$) in the theory. The antifields satisfy opposite statistics with ghost number $-gh(\phi) - 1$, where $gh(\phi)$ is ghost number of field $\phi$.

The effective action $S_{eff}$ is then extended with the antifields as

$$S_{eff}[\phi, \phi^*] = I[\phi] + (s_b \phi)\phi^*, \quad (2.14)$$

where $I[\phi]$ is gauge invariant action. The antifields $\phi^*$ are obtained from gauge-fixing fermion $\Psi$ as

$$\phi^* = \frac{\delta\Psi}{\delta\phi}. \quad (2.15)$$

The extended effective action is then written in terms of $\Psi$ as

$$S_{eff}[\phi] = I[\phi] + s_b \Psi. \quad (2.16)$$

The effective action $S_{eff}$ satisfies certain rich mathematical relation which is known as ‘quantum master equation’ as follows:

$$\left( S_{eff}, S_{eff} \right) - 2i\Delta S_{eff} = 0, \text{ at } \phi^* = \frac{\delta\Psi}{\delta\phi}, \quad (2.17)$$

where the antibracket of effective action, $(S_{eff}, S_{eff})$, is defined by

$$\left( S_{eff}, S_{eff} \right) = \frac{\delta_r S_{eff}}{\delta\phi} \frac{\delta_l S_{eff}}{\delta\phi^*} - \frac{\delta_r S_{eff}}{\delta\phi^*} \frac{\delta_l S_{eff}}{\delta\phi}, \quad (2.18)$$

and $\Delta$ is defined with left and right differentials ($\delta_l$ and $\delta_r$ respectively) as

$$\Delta = \frac{\delta_r}{\delta\phi^*} \frac{\delta_l}{\delta\phi}. \quad (2.19)$$
2.3. Batalin-Fradkin-Vilkovisky (BFV) formulation

Usually it is easy to construct an action that satisfies the ‘classical master equation’

\[(S_{eff}, S_{eff}) = 0.\]  \hspace{1cm} (2.20)

The generating function given in Eq. (2.6) can also be written in compact form as

\[Z = \int [D\phi] \exp [iW_\Psi(\phi, \phi^*)], \]  \hspace{1cm} (2.21)

where \(W_\Psi(\phi, \phi^*)\) is an extended action satisfying following ‘quantum master equation’ \[6\]:

\[\Delta e^{iW_\Psi[\phi, \phi^*]} = 0.\] \hspace{1cm} (2.22)

The quantum master equation in the zeroth order of antifields gives the condition of gauge invariance. On the other hand it reflects the nilpotency of BRST transformation in the first order of antifields. We will be using this formulation in the framework of FFBRST formulation in different contexts.

2.3 Batalin-Fradkin-Vilkovisky (BFV) formulation

We briefly mention the BFV formalism \[5, 22\] which is applicable for the general theories with first-class constraints. This formalism is developed on an extended phase space with finite number of canonically conjugate variables. The basic features of this approach are as follows: i) it does not require closure off-shell of the gauge algebra and therefore does not need an auxiliary field, ii) heavily relies on BRST transformation which is independent of the gauge condition and iii) it is even applicable to Lagrangian which are not quadratic in velocities and hence is more general than the strict Lagrangian approach. The action, in terms of canonical Hamiltonian density \(H_0\) and first-class constraints, \(\Omega_a(a = 1, 2, ..., m)\), can be written in this formalism as

\[S = \int d^4x (p^\mu \dot{q}_\mu - H_0 - \lambda^a \Omega^a),\] \hspace{1cm} (2.23)

where \((q_\mu, p^\mu)\) are the canonical variables and \(\lambda^a\) are the Lagrange multiplier associated with first-class constraints. In this method, the Lagrange multipliers \(\lambda^a\) are treated as the canonical variables. Therefore, one introduces its canonical conjugate momenta \(p^a\), where \(p^a\) further imposes new constraints such that the dynamics of the theory must not be changed. In order to make the extended theory to be consistent with the initial theory a pair of canonically conjugate anticommuting ghost coordinate and momenta \((c^a, \pi^a)\) is introduced for each constraint. These canonically conjugate ghosts satisfy the following anticommutation relation:

\[\{c^a(x, t), \pi^b(y, t)\} = -i\delta^{ab}\delta^3(x - y),\] \hspace{1cm} (2.24)

where \(c_a\) and \(\pi_a\) have ghost number 1 and \(-1\), respectively. The generating functional for this extended theory is then defined as

\[Z_\Psi = \int [D\phi] e^{iS_{eff}[\phi]},\] \hspace{1cm} (2.25)
where \([\mathcal{D}\phi]\) is the path integral measure and the effective action \(S_{\text{eff}}\) is

\[
S_{\text{eff}} = \int d^4x (p^\mu \dot{q}_\mu + p^a \dot{\lambda}^a + \pi^a \dot{\epsilon}^a - \mathcal{H}_\Psi).
\]

(2.26)

\(\mathcal{H}_\Psi\) is the extended Hamiltonian and is written as

\[
\mathcal{H}_\Psi = \mathcal{H}_0 + \{Q_b, \Psi\},
\]

(2.27)

where \(\Psi\) is the gauge-fixing fermion and \(Q_b\) is the nilpotent BRST charge which has the following general form:

\[
Q_b = c^a \Omega^a + \frac{1}{2} \pi^a f^{abc} c^b c^c,
\]

(2.28)

where the \(f^{abc}\) is a structure constant.

The BRST symmetry transformation for fields \(\phi\) can be calculated with BRST charge \(Q_b\) using the relation

\[
s_b \phi = -[\phi, Q_b]_\pm,
\]

(2.29)

where + and − signs, respectively, denote anticommutator and commutator for the fermionic and bosonic nature of fields \(\phi\).

In this chapter we have provided the basic mathematical techniques which are relevant for the later part of the thesis. In the next chapter we would like to deal with the off-shell FFBRST and FF-anti-BRST transformations.
Chapter 3

Off-shell nilpotent FFBRST transformation

The FFBRST transformation which have been discussed in the previous chapter is on-shell nilpotent. In this chapter we extend this formulation using auxiliary fields to make FFBRST transformation off-shell nilpotent [42]. Several explicit examples of off-shell nilpotent FFBRST transformation are considered. We further construct both on-shell and off-shell nilpotent FF-anti-BRST transformations and show that such transformations also play the similar role.

3.1 FFBRST transformation for Faddeev-Popov (FP) effective theory: short survey

Let us now briefly review the FFBRST formulation, particularly, in case of FP effective theory as an example of pure gauge theories [27]. In this case the Jacobian of the path integral measure changes the generating functional corresponding to FP effective theory to the generating functional for a different effective theory. The meaning of these field transformations is as follows. We consider the vacuum expectation value of a gauge invariant functional $G[A]$ in some effective theory,

$$
<< G[A] >> \equiv \int [\mathcal{D} \phi] \, G[A] \exp(iS_{eff}[\phi]),
$$

(3.1)

where $\phi$ is generic notation for all fields and the FP effective action is defined as

$$
S_{eff} = S_0 + S_{gf} + S_g.
$$

(3.2)

Here, $S_0$ is the pure YM action$^1$

$$
S_0 = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a \right],
$$

(3.3)

$^1$Here we adopt the following notations throughout the thesis. The $d$ dimensional metric tensor in Minkowski space-time is defined as $g_{\mu\nu} = g^{\mu\nu} = \text{diag}(+1, -1, -1, -1, ...)$ and the quantity $X_{\mu}X^{\mu} = X \cdot X$ is a Lorentz scalar. We always use the small letters (a,b,c,...) in superscript to denote the group indices. This should not be confused with the small letters (b, ab, d, ad,...) used in subscript which denote the different forms of BRST transformations (e.g. BRST, anti-BRST, co-BRST, anti-co-BRST,...).
and the gauge-fixing and ghost part of the effective action in Lorentz gauge are given as

\begin{align}
S_{gf} &= -\frac{1}{2\lambda} \int d^4x (\partial \cdot A^a)^2, \\
S_g &= -\int d^4x \left[ \bar{c}^a \partial^\mu D^a_{\mu b} c^b \right].
\end{align}

(3.4)

The covariant derivative is defined as \( D^{ab}_\mu[A] \equiv \delta^{ab} \partial_\mu + g f^{abc} A^c_\mu \).

Now we perform the FFBRST transformation \( \phi \rightarrow \phi' \) given by Eq. (2.4). Then we have

\[ \langle\langle G[A] \rangle\rangle = \langle\langle G[A'] \rangle\rangle = \int [D\phi'] J[\phi'] G[A'] \exp(i S_{eff}^F[\phi']) \],

(3.5)

on account of BRST invariance of \( S_{eff} \) and the gauge invariance of \( G[A] \). Here \( J[\phi'] \) is the Jacobian associated with FFBRST transformation and is defined as

\[ D\phi = D\phi(\kappa) J[\phi(\kappa)]. \]

(3.6)

Note that unlike the usual infinitesimal BRST transformation, the Jacobian for FFBRST is not unity. In fact, this nontrivial Jacobian is the source of the new results in this formulation. As shown in Ref. [27] for the special case \( G[A] = 1 \), the Jacobian \( J[\phi(\kappa)] \) can always be replaced by \( e^{i S_1[\phi(\kappa)]} \), where \( S_1[\phi(\kappa)] \) is some local functional of the fields and can be added to the action at \( \kappa = 1 \),

\[ S_{eff}[\phi'] + S_1[\phi'] = S'_{eff}[\phi']. \]

(3.7)

Thus, the FFBRST transformation changes the FP effective action of the theory [37].

### 3.2 Off-shell nilpotent FFBRST transformation

In this section, we intend to generalize the FFBRST transformation in an auxiliary field formulation. We only mention the necessary steps of the FFBRST formulation in presence of auxiliary field. For simplicity, we consider the case of pure YM theory described by the effective action in Lorentz gauge

\[ S_{eff}^L = \int d^4x \left[ -\frac{1}{4} F^{a\mu\nu} F_{a\mu\nu} + \frac{\lambda}{2} (B^a)^2 - B^a \partial \cdot A^a - \bar{c}^a \partial^\mu D^a_{\mu b} c^b \right]. \]

(3.8)

This effective action is invariant under an off-shell nilpotent usual BRST transformation with infinitesimal parameter. Following the procedure outlined in the previous chapter, it is straightforward to construct FFBRST transformation under which the above \( S_{eff}^L \) remains invariant. The transformation is as follows:

\begin{align}
A^a_\mu &\rightarrow A^a_\mu + D^{ab}_\mu c^b \Theta_b(A,c,\bar{c},B), \\
c^a &\rightarrow c^a - \frac{g}{2} f^{cde} c^d \Theta_b(A,c,\bar{c},B), \\
\bar{c}^a &\rightarrow \bar{c}^a + B^a \Theta_b(A,c,\bar{c},B), \\
B^a &\rightarrow B^a.
\end{align}

(3.9)
The finite parameter, $\Theta_b(A, c, \bar{c}, B)$ depends also on the auxiliary field $B$ and hence the non-trivial modification arises in the calculation of Jacobian for this FFBRST in an auxiliary field formulation. The Jacobian is defined as

$$
\mathcal{D}A(x)\mathcal{D}c(x)\mathcal{D}\bar{c}(x)\mathcal{D}B(x) = J(x, k)\mathcal{D}A(x, k)\mathcal{D}c(x, k)\mathcal{D}\bar{c}(x, k)\mathcal{D}B(x, k)
$$

$$
= J(k + dk)\mathcal{D}A(k + dk)\mathcal{D}c(k + dk)\mathcal{D}\bar{c}(k + dk)\mathcal{D}B(k + dk).
$$

The transformation from $\phi(k)$ to $\phi(k + dk)$ is an infinitesimal one and one has, for its Jacobian

$$
\frac{J(k)}{J(k + dk)} = \int d^4x \sum_\phi \pm \frac{\delta \phi(x, k + dk)}{\delta \phi(x, k)}.
$$

dropping those terms which do not contribute on account of the antisymmetry of structure constant. We calculate the infinitesimal change in Jacobian, as mentioned in \cite{27}, as

$$
\frac{1}{J(k)} \frac{dJ(k)}{dk} = -\int d^4x \left[ (s_b A^a_\mu) \frac{\delta \Theta'_b}{\delta A^a_\mu} - (s_b c^a) \frac{\delta \Theta'_b}{\delta c^a} - (s_b \bar{c}) \frac{\delta \Theta'_b}{\delta \bar{c}} + (s_b B^a) \frac{\delta \Theta'_b}{\delta B^a} \right].
$$

Further, it can be shown that the Jacobian in Eq. (3.10) can be expressed as $e^{iS_1[\phi]}$ if it satisfies the condition given in Eq. (2.9).

Now, we consider different choices of the parameter $\Theta'_b$ (which is related to $\Theta_b$ through the relation in Eq. (2.5)) to show the connection between different pairs of effective theories.

### 3.2.1 Connecting YM theory in Lorentz gauge to the same theory in axial gauge

To show the connection between YM theories in Lorentz gauge and axial gauge we start with the Lorentz gauge YM theory in the auxiliary field formulation, described by the effective action given in Eq. (3.8) and choose the infinitesimal field dependent parameter as

$$
\Theta'_b = i \int d^4x \cdot \bar{c}^a \left[ \gamma_1 \lambda B^a + \gamma_2 (\partial \cdot A^a - \eta \cdot A^a) \right],
$$

where $\gamma_1$, $\gamma_2$ are arbitrary constants and $\lambda$ is a gauge parameter. Using Eq. (3.13), we calculate the change in the Jacobian of such transformation as

$$
\frac{1}{J(k)} \frac{dJ}{dk} = i \int d^4x \left[ \gamma_1 (B^a)^2 + \gamma_2 B^a (\partial \cdot A^a - \eta \cdot A^a) + \gamma_2 \bar{c}^a \left( M^{ab} \bar{c}^b - \bar{M}^{ab} c^b \right) \right].
$$

The small letter (b) used in the subscript of Eq. (3.13), which denotes BRST, should not be confused with the group index which is always written in the superscript.
where $M^{ab} = \partial \cdot D^{ab}$ and $\tilde{M}^{ab} = \eta \cdot D^{ab}$. We further make an ansatz for $S_1$ in this case as

$$S_1 [\phi(\kappa), \kappa] = \int d^4 x \left[ \xi_1(\kappa) (B^a)^2 + \xi_2(\kappa) B^a \partial \cdot A^a + \xi_3(\kappa) B^a \eta \cdot A^a + \xi_4(\kappa) \bar{c}^a M^{ab, b} + \xi_5(\kappa) \bar{c}^a \tilde{M}^{ab, b} \right]. \quad (3.16)$$

The constants $\xi_i(\kappa)$ depend on $\kappa$ explicitly and satisfies the following initial condition:

$$\xi_i(\kappa = 0) = 0. \quad (3.17)$$

Using Eq. (2.3) we calculate

$$\frac{dS_1}{d\kappa} = \int d^4 x \left[ \frac{d\xi_1}{d\kappa} (B^a)^2 + \frac{d\xi_2}{d\kappa} B^a \partial \cdot A^a + \frac{d\xi_3}{d\kappa} B^a \eta \cdot A^a + \frac{d\xi_4}{d\kappa} \bar{c}^a M^{ab, b} + \xi_2 B^a M^{ab, c} \Theta'_b + \xi_3 B^a \tilde{M}^{ab, c} \Theta'_b - \xi_4 B^a M^{ab, b} \Theta'_b \right. \quad \left. - \xi_5 B^a \tilde{M}^{ab, c} \Theta'_b \right]. \quad (3.18)$$

From the condition mentioned in Eq. (2.9), we obtain

$$\int [D \phi] e^{i(S_{eff}^L + S_1)} \left\{ M^{ab, b} \Theta'_b [B^a(\xi_2 - \xi_4)] + \tilde{M}^{ab, b} \Theta'_b [B^a(\xi_3 - \xi_5)] \right\} + (B^a)^2 \left( \frac{d\xi_2}{d\kappa} - \gamma_1 \lambda \right) + B^a \partial \cdot A^a \left( \frac{d\xi_2}{d\kappa} - \gamma_2 \right) + B^a \eta \cdot A^a \left( \frac{d\xi_3}{d\kappa} + \gamma_2 \right) + \bar{c}^a M^{ab, b} \left( \frac{d\xi_4}{d\kappa} - \gamma_2 \right) + \bar{c}^a \tilde{M}^{ab, b} \left( \frac{d\xi_5}{d\kappa} + \gamma_2 \right) \right\} = 0. \quad (3.19)$$

The last two terms in the integrand of Eq. (3.19) depend on $\bar{c}$ in a local fashion. The contribution of these terms vanish by antighost equation of motion [27, 37]

$$\int D\bar{c}^a \frac{\delta}{\delta \bar{c}^a} e^{i(S_{eff}^L + S_1)} = 0. \quad (3.20)$$

This can only happen if the ratio of coefficients of the two terms is identical to the ratio of coefficients of $\bar{c}^a M^{ab, b}$ and $\bar{c}^a \tilde{M}^{ab, b}$ in $S_{eff}^L + S_1$. This requires that

$$\frac{d\xi_4/d\kappa - \gamma_2}{\xi_4 - 1} = \frac{d\xi_5/d\kappa + \gamma_2}{\xi_5}. \quad (3.21)$$

The nonlocal $\Theta'_b$ dependent terms are canceled by converting them to local terms using antighost equation of motion [37]. This can only work if the two $\Theta'_b$ dependent terms in a certain manner, depending again on the ratio of coefficients of $\bar{c}^a M^{ab, b}$ and $\bar{c}^a \tilde{M}^{ab, b}$ in terms in $S_{eff}^L + S_1$. This requires that

$$\frac{\xi_2 - \xi_4}{\xi_4 - 1} = \frac{\xi_3 - \xi_5}{\xi_5}. \quad (3.22)$$

When the above two equations (3.21) and (3.22) are satisfied, the nonlocal $\Theta'_b$ dependent terms get converted to local terms. The coefficients of local terms $(B^a)^2$, $B^a \partial \cdot A^a$, $B^a \eta \cdot A^a$, $\bar{c}^a M^{ab, b}$,
and $\bar{c}^{a}\bar{M}^{ab}c^{b}$, independently, vanish and are giving rise to following differential equations, respectively:

\[
\begin{align*}
\frac{d\xi_1}{d\kappa} - \gamma_1\lambda + \gamma_1\lambda(\xi_2 - \xi_1) + \gamma_1\lambda(\xi_3 - \xi_5) &= 0, \\
\frac{d\xi_2}{d\kappa} - \gamma_2 + \gamma_2(\xi_2 - \xi_4) + \gamma_2(\xi_3 - \xi_5) &= 0, \\
\frac{d\xi_3}{d\kappa} + \gamma_2 - \gamma_2(\xi_2 - \xi_4) - \gamma_2(\xi_3 - \xi_5) &= 0, \\
\frac{d\xi_4}{d\kappa} - \gamma_2 &= 0, \\
\frac{d\xi_5}{d\kappa} + \gamma_2 &= 0.
\end{align*}
\] (3.23)

The above equations can be solved for various $\xi_i(\kappa)$ using the boundary conditions given by Eq. (3.17) and the solutions ($\gamma_2 = 1$) are given as

\[
\begin{align*}
\xi_1 &= \gamma_1\lambda\kappa, \\
\xi_2 &= \kappa, \\
\xi_3 &= -\kappa, \\
\xi_4 &= \kappa, \\
\xi_5 &= -\kappa.
\end{align*}
\] (3.24)

Putting the above values in Eq. (3.16), we get

\[
S_L = \gamma_1\lambda\kappa(B^a)^2 + \kappa B^a\partial \cdot A^a - \kappa B^a\eta \cdot A^a + \kappa \bar{c}^{a} M^{ab}c^b - \kappa \bar{c}^{a} \bar{M}^{ab}c^b.
\] (3.25)

FFBRST transformation in Eq. (3.9) with the parameter given in Eq. (3.14) connects the generating functional in Lorentz gauge,

\[
Z_L = \int[D\phi] e^{iS_L_{\text{eff}}},
\] (3.26)
to the generating functional corresponding to the effective action

\[
S'_{\text{eff}} = \int d^4x \left[ \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \frac{\zeta}{2} (B^a)^2 - B^a\eta \cdot A^a - \bar{c}^{a} \bar{M}^{ab}c^b \right] = S'_{\text{eff}}^{A},
\] (3.27)

where $S'_{\text{eff}}^{A}$ is nothing but FP effective action in axial gauge with the gauge parameter $\zeta = (2\gamma_1 + 1)\lambda$.

### 3.2.2 Relating theories in Coulomb gauge and Lorentz gauge

We again start with Lorentz gauge theory given in Eq. (3.8) and choose another parameter

\[
\Theta'_b = i \int d^4x \bar{c}^{a} \left[ \gamma_1\lambda B^a + \gamma_2 \left( \partial \cdot A^a - \partial^j A_j^a \right) \right]; \quad j = 1, 2, 3,
\] (3.28)

to show the connection with theory in the Coulomb gauge. The change in the Jacobian due to this FFBRST transformation is calculated using Eq. (3.13) as

\[
\frac{1}{J} \frac{dJ}{dk} = i \int d^4x \left[ \gamma_1\lambda(B^a)^2 + \gamma_2 B^a \left( \partial \cdot A^a - \partial^j A_j^a \right) + \gamma_2 \bar{c}^{a} \left( M^{ab}c^b - \bar{M}^{ab}c^b \right) \right],
\] (3.29)
where $\tilde{M}' = \partial^j D_j^b$. We try the following ansatz for $S_1$ for this case:

$$S_1 = \int d^4 x \left[ \xi_1(\kappa)(B^a)^2 + \xi_2(\kappa)B^a \partial \cdot A^a + \xi_3(\kappa)B^a \partial^j A_j^a \right] + \xi_4(\kappa)e^a(Mc)^a + \xi_5(\kappa)e^a(\tilde{M}'c)^a \right].$$

(3.30)

Now, using the condition Eq. (2.39) for replacing the Jacobian as $e^{iS_1}$ and following the similar procedure as discussed in the previous case, we obtain exactly same solutions as given in Eq. (3.24) for the coefficients $\xi_i$. Putting these solutions in Eq. (3.30) we obtain

$$S_1 = \int d^4 x \left[ \gamma_1 \lambda \kappa(B^a)^2 + \kappa B^a \partial \cdot A^a - \kappa B^a \partial^j A_j^a + \kappa \tilde{c}^a(Mc)^a - \kappa \tilde{c}^a(\tilde{M}'c)^a \right].$$

(3.31)

The transformed effective action

$$S'_{\text{eff}} = S_{\text{eff}}^L + S_1(\kappa = 1),$$

$$= \int d^4 x \left[ -\frac{1}{4} F_{\mu\nu}F_{\mu\nu} + \frac{\lambda}{2}(B^a)^2 - B^a \partial^j A_j^a - c^a(\tilde{M}'c)^a \right],$$

(3.32)

which is the FP effective action in Coulomb gauge with gauge parameter $\zeta$.

Thus, FFBRST transformation with parameter given in equation (3.28) connects the generating functional for YM theories in Lorentz gauge to the generating functional for the same theory in Coulomb gauge.

### 3.2.3 FFBRST transformation to link FP effective action in Lorentz gauge to quadratic gauge

Next we consider theories in quadratic gauges which are often useful in doing calculations. The effective action in quadratic gauge in terms of auxiliary field can be written as

$$S_{\text{eff}}^Q = \int d^4 x \left[ -\frac{1}{4} F_{\mu\nu}F_{\mu\nu} + \frac{\lambda}{2}(B^a)^2 - B^a \partial^j A_j^a + d^{abc}A^b_{\mu}A^c_{\mu} \right] - c^a\partial^a(D_\mu c)^a - 2d^{abc}c^aA^c(D_\mu c)^b \right],$$

(3.33)

where $d^{abc}$ is structure constant symmetric in $b$ and $c$. This effective action is invariant under the FFBRST transformation mentioned in Eq. (3.39).

For this case, we start with the following choice of the infinitesimal field dependent parameter:

$$\Theta_b = i \int d^4 x \ c^a \left[ \gamma_1 \lambda B^a + \gamma_2 d^{abc}A^b_{\mu}A^c_{\mu} \right].$$

(3.34)

We calculate the Jacobian change as

$$\frac{1}{J} \frac{dJ}{d\kappa} = i \int d^4 x \left[ \gamma_1 \lambda(B^a)^2 + \gamma_2 B^a d^{abc}A^b_{\mu}A^c_{\mu} + 2\gamma_2 d^{abc}c^a(D_\mu c)^b A^c_{\mu} \right].$$

(3.35)

We make an ansatz for $S_1$ for this case as

$$S_1 = \int d^4 x \left[ \xi_1(\kappa)(B^a)^2 + \xi_2(\kappa)B^a d^{abc}A^b_{\mu}A^c_{\mu} + \xi_3(\kappa)d^{abc}c^a(D_\mu c)^b A^c_{\mu} \right].$$

(3.36)
The unknown coefficients $\xi_i$ are determined by using the condition in equation (2.9) and the initial condition in equation (3.17), to get

$$S_1(\kappa = 1) = \gamma_1 \lambda (B^a)^2 - B^a d^{abc} A^b A^c \lambda - 2d^{abc} c^a (D_\mu c)^b A^c,$$

(3.37)

and $S'_\text{eff} = S'_\text{eff} \text{L} + S_1(\kappa = 1) = S'_\text{eff} \text{Q}$, which is effective action in quadratic gauge as given in Eq. (3.33) with gauge parameter $\zeta$.

Thus, the FFBRST transformation with parameter given in equation (3.34) connects Lorentz gauge theory to the theory for quadratic gauge.

### 3.2.4 FP action to the most general BRST/anti-BRST invariant action

The most general BRST/anti-BRST invariant action for YM theories in Lorentz gauge is given as [99]

$$S^{AB}_{\text{eff}} = \int d^4x \left[ -\frac{1}{4} F^{a\mu\nu} F_{a\mu\nu} - \frac{(\partial \cdot A^a)^2}{2\lambda} + \partial^a \partial^b \partial^c - \frac{1}{2} f^{abc} e^a (D_\mu e)^b c^c \right].$$

(3.38)

This effective action has the following global symmetries.

**BRST:**

$$\delta_b A^a_a = (D_\mu c)^a_a \Lambda, \quad \delta_b e^a = -\frac{1}{2} g f^{abc} e^b \Lambda,$$

(3.39)

**Anti-BRST:**

$$\delta_{ab} A^a_a = (D_\mu c)^a_a \Lambda, \quad \delta_{ab} e^a = -\frac{1}{2} g f^{abc} e^b \Lambda,$$

(3.40)

The above most general BRST/anti-BRST effective action can be re-expressed in the auxiliary field formulation as,

$$S^{AB}_{\text{eff}} = \int d^4x \left[ -\frac{1}{4} F^{a\mu\nu} F_{a\mu\nu} + \frac{\lambda}{2} (B^a)^2 - B^a (\partial \cdot A^a - \frac{\alpha g \lambda}{2} f^{abc} e^b \Lambda) \right] + \partial^a e^a D_\mu e^b \Lambda - \frac{1}{8} \alpha \lambda g^2 f^{abc} e^b \Lambda f^{alm} c^m c^l.$$  

(3.41)

The off-shell nilpotent, global BRST/anti-BRST symmetries for this effective action are given as

**BRST:**

$$\delta_b A^a = (D_\mu c)^a \Lambda, \quad \delta_b e^a = -\frac{1}{2} g f^{abc} e^b \Lambda,$$

(3.42)

$$\delta_b A^a = B^a \Lambda, \quad \delta_b B^a = 0.$$
Anti-BRST:
\[ \delta_{ab} A^a_\mu = (D_\mu \bar{c})^a \Lambda, \quad \delta_{ab} \bar{c} = -\frac{1}{2} g f^{abc} \bar{c}^b c^c \Lambda, \]
\[ \delta_{abc} e^a = (-B^a - g f^{abc} \bar{c}^b c^c) \Lambda, \quad \delta_{ab} B^a = -g f^{abc} B^b c^c \Lambda. \]  
(3.43)

To obtain the generating functional corresponding to this theory, we apply the FFBRST transform-

\[ \Theta'_b = i \int d^4 x \bar{c}^a \left[ \gamma_1 \lambda B^a + \gamma_2 f^{abc} \bar{c}^b c^c \right], \]  
(4.44)

on the generating functional given in Eq. (3.26). Using Eq. (3.13), change in Jacobian can be calculated as
\[ \frac{1}{J} \frac{dJ}{d\kappa} = i \int d^4 x \left[ \gamma_1 \lambda (B^a)^2 + 2\gamma_2 f^{abc} B^a \bar{c}^b c^c - \frac{g}{2} \gamma_2 f^{abc} \bar{c}^b c^c f^{alm} c^l c^m \right]. \]  
(3.45)

We further make an ansatz for \( S_1 \) as
\[ S_1 = \int d^4 x \left[ \xi_1(\kappa)(B^a)^2 + \xi_2(\kappa) B^a f^{abc} \bar{c}^b c^c + \xi_3(\kappa) f^{abc} \bar{c}^b c^c f^{alm} c^l c^m \right]. \]  
(3.46)

The condition given in equation (2.9), then be written for this case as
\[ \int [D \phi] \ e^{i(S_{eff} + S_1)} \left[ \left( \frac{d\xi_1}{d\kappa} - \gamma_1 \lambda \right) (B^a)^2 + \left( \frac{d\xi_2}{d\kappa} - 2\gamma_2 \right) B^a f^{abc} \bar{c}^b c^c + \left( \frac{d\xi_3}{d\kappa} + \frac{g}{2} \gamma_2 \right) f^{abc} \bar{c}^b c^c f^{alm} c^l c^m - \left( \frac{g}{2} \xi_2 + 2\xi_3 \right) f^{abc} B^b c^c f^{alm} c^l c^m \Theta'_b \right] = 0. \]  
(3.47)

We look for a special solution corresponding to the condition
\[ \frac{g}{2} \xi_2 + 2\xi_3 = 0. \]  
(3.48)

Comparing the different coefficients, we get the following differential equations for \( \xi_i(\kappa) \):
\[ \frac{d\xi_1}{d\kappa} - \gamma_1 \lambda = 0, \quad \frac{d\xi_2}{d\kappa} - 2\gamma_2 = 0, \quad \frac{d\xi_3}{d\kappa} + \frac{g}{2} \gamma_2 = 0. \]  
(3.49)

Solutions of the above equations subjected to the initial condition given in Eq. (3.17) are,
\[ \xi_1 = \gamma_1 \lambda \kappa, \quad \xi_2 = 2\gamma_2 \kappa, \quad \xi_3 = -\frac{g}{2} \gamma_2 \kappa. \]  
(3.50)

These solutions are consistent with condition in Eq. (3.38). Since \( \gamma_2 \) is arbitrary, we choose \( \gamma_2 = 4 \alpha g \zeta \) to get
\[ S_{eff}^L + S_1(\kappa = 1) = \int d^4 x \left[ -\frac{1}{4} F_{\mu \nu}^a F^{a \mu \nu} + \frac{\zeta}{2} (B^a)^2 - B^a (\partial \cdot A^a) - \frac{\alpha g \zeta}{2} f^{abc} \bar{c}^b c^c + \partial^\mu c^a D^\mu_{ab} \bar{c}^b - \frac{1}{8} \alpha g^2 f^{abc} \bar{c}^b c^c f^{alm} c^l c^m \right] \equiv S_{eff}^{AB}, \]  
(3.51)
which is the same effective action as mentioned in Eq. (3.41), where $\zeta$ is the gauge parameter.

Thus, even in the auxiliary field formulation the different generating functionals corresponding to the different effective theories are connected through off-shell nilpotent FFBRST transformation with different choices of the finite parameter which also depends on the auxiliary field.

### 3.3 Finite field dependent anti-BRST (FF-anti-BRST) formulation

In this section, we construct the FF-anti-BRST transformation analogous to FFBRST transformation. For simplicity, we consider the pure YM theory in Lorentz gauge described by the effective action in Eq. (3.2) which is invariant under the following on-shell anti-BRST transformation:

$$
\delta_{ab} A_{\mu}^a = (D_{\mu} \bar{c})^a \Lambda,
$$

$$
\delta_{ab} \bar{c}^a = -\frac{1}{2} g f^{abc} \bar{c}^b \bar{c}^c \Lambda,
$$

$$
\delta_{ab} c^a = \left( -\frac{\partial \cdot A^a}{\lambda} - g f^{abc} \bar{c}^b \bar{c}^c \right) \Lambda,
$$

(3.52)

where $\Lambda$ is infinitesimal, anti commuting and global parameter. This anti-BRST transformation is special case ($\alpha = 0$) of general anti-BRST transformation given in Eq. (3.40). Following the procedure similar to the construction of FFBRST transformation as outlined in the chapter 2, we can easily construct the FF-anti-BRST transformation for the pure YM theory as

$$
\delta_{ab} A_{\mu}^a = (D_{\mu} \bar{c})^a \Theta_{ab},
$$

$$
\delta_{ab} \bar{c}^a = -\frac{1}{2} g f^{abc} \bar{c}^b \bar{c}^c \Theta_{ab},
$$

$$
\delta_{ab} c^a = \left( -\frac{\partial \cdot A^a}{\lambda} - g f^{abc} \bar{c}^b \bar{c}^c \right) \Theta_{ab},
$$

(3.53)

where $\Theta_{ab}(A,c,\bar{c})$ is finite, field dependent and anticommuting parameter.

In FF-anti-BRST formulation, the infinitesimal change in fields $\phi$ is written as

$$
\frac{d}{d\kappa} \phi(x, \kappa) = s_{ab} \phi(x, \kappa) \Theta'_{ab}[\phi(x, \kappa)],
$$

(3.54)

where $s_{ab}$ is anti-BRST variation of fields $\phi$ and $\Theta'_{ab}$ is an infinitesimal field dependent parameter which is related with finite field dependent parameter $\Theta_{ab}(A,c,\bar{c})$ as

$$
\Theta_{ab}[\phi(x, \kappa)] = \Theta'_{ab}[\phi(x, 0)] \exp \left[ \frac{g[\phi(x, 0)] - 1}{g[\phi(x, 0)]} \right] - 1,
$$

(3.55)

and $g$ is given by $g = \sum_i \frac{\delta \Theta'_{ab}(x)}{\delta \phi_i(x)} s_{ab} \phi_i(x)$.

In FF-anti-BRST formulation, we use the following relation to calculate the infinitesimal change in Jacobian:

$$
\frac{1}{J(\kappa)} \frac{dJ(\kappa)}{d\kappa} = - \int d^4 x \sum_{\phi} \left[ \pm \delta_{ab} \phi \frac{\delta \Theta'_{ab}[\phi(x, \kappa)]}{\delta \phi(x, \kappa)} \right].
$$

(3.56)
3.3. Finite field dependent anti-BRST (FF-anti-BRST) formulation

Now, we would like to investigate the role of such transformation by considering different infinitesimal field dependent parameters $\Theta_{ab}(A,c,\bar{c})$

3.3.1 FF-anti-BRST transformation to change the gauge parameter $\lambda$

We consider a very simple example to show that a simple FF-anti-BRST transformation can transfer the generating functional corresponding to YM effective action in Lorentz gauge with a gauge parameter $\lambda$ to the generating functional corresponding to same effective action with a different gauge parameter $\lambda'$. We start with the Lorentz gauge effective action given in Eq. (3.8) with the gauge parameter $\lambda$ and consider

$$\Theta_{ab} = -i\gamma \int d^4x \, e^a(x,\kappa)\partial \cdot A^a(x,\kappa),$$

(3.57)

with $\gamma$ as arbitrary parameter.

Using Eq. (3.56), we calculate the infinitesimal change in Jacobian as

$$\frac{1}{J} \frac{dJ}{d\kappa} = i\gamma \int d^4x \left( \frac{(\partial \cdot A^a)^2}{\lambda} + \bar{c}^a M^{ab} c^b \right).$$

(3.58)

We choose

$$S_1 = \xi(\kappa) \int d^4x \frac{(\partial \cdot A^a)^2}{\lambda}.$$  

(3.59)

The condition, for replacing the Jacobian of the FF-anti-BRST transformation with parameter given in Eq. (3.57) as $e^{iS_1}$, is given in Eq. (2.9) and is calculated as

$$\int [D\phi] e^{i(S_{eff}^L + S_1)} \left[ \frac{(\partial \cdot A^a)^2}{\lambda}(\xi' - \gamma) + 2\xi \frac{\partial \cdot A^a}{\lambda} M^{ab} c^b \Theta_{ab} - \bar{c}^a M^a \right] = 0.$$  

(3.60)

The last term of above equation gives no contribution due to dimensional regularization and we can substitute

$$\int d^4x \frac{\partial \cdot A^a}{\lambda} M^{ab} c^b \Theta_{ab} \rightarrow \gamma \frac{(\partial \cdot A^a)^2}{\lambda}.$$  

(3.61)

Thus, the L.H.S. of Eq. (3.60) is vanish iff

$$\xi' - \gamma + 2\xi\gamma = 0.$$  

(3.62)

We solve this equation subjected to the initial condition given in Eq. (3.17) to obtain

$$\xi = \frac{1}{2}(1 - e^{-2\gamma\kappa}).$$

(3.63)

Thus, at $\kappa = 1$ the extra term in the net effective action from the Jacobian is

$$S_1 = \frac{1}{2}(1 - e^{-2\gamma}) \int d^4x \frac{(\partial \cdot A^a)^2}{\lambda}.$$  

(3.64)

The new effective action becomes $S_{eff}' = S_{eff}^L + S_1$. In this case,

$$S_{eff}^L + S_1 = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2\lambda} (\partial \cdot A^a)^2 - \bar{c}^a M^{ab} c^b \right],$$

(3.65)

which is effective action in Lorentz gauge with gauge parameter $\lambda' = \lambda/e^{-2\gamma}$. Thus, the FF-anti-BRST transformation in Eq. (3.53) with parameter given in Eq. (3.57) connects two effective theories which differ only by a gauge parameter.
3.3. Finite field dependent anti-BRST (FF-anti-BRST) formulation

3.3.2 Lorentz gauge to axial gauge theory

In this subsection we show that FF-anti-BRST transformation plays exactly same role of FF-BRST transformation in connecting different effective theories. For this purpose we consider same pair of theories which were connected by FFBRST transformation. The effective action in Lorentz gauge given in Eq. (3.2) can be written as

\[ S^L_{\text{eff}} = \int d^4x \left[ -\frac{1}{4} F_{\mu
u}^a F^{a\mu\nu} - \frac{1}{2\lambda} (\partial \cdot A)^a + c^a M^{ab} \bar{c}^b - g f^{abc} \bar{c}^c (\partial \cdot A)^a \right] , \] (3.66)

where we have interchanged the position of \( c, \bar{c} \) in the ghost term for the seek of convenience. This action is invariant under anti-BRST transformation given in Eq. (3.52). Similarly, the effective action in axial gauge can be written as

\[ S^A_{\text{eff}} = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2\lambda} (\eta \cdot A)^a + c^a \tilde{M} \bar{c}^a - g f^{abc} \bar{c}^c (\eta \cdot A)^a \right] , \] (3.67)

which is invariant under the following anti-BRST symmetry transformation:

\[ \delta_b A^a_{\mu} = (D_\mu \bar{c})^a , \delta_b c^a = -\frac{1}{2} g f^{abc} \bar{c}^b \Lambda, \delta_b \bar{c}^a = \left( -\frac{\eta \cdot A}{\lambda} - g f^{abc} \bar{c}^b \right) \Lambda. \] (3.68)

Now, we show that the generating functionals corresponding to these two effective action are related through FF-anti-BRST transformation.

To show the connection, we choose

\[ \Theta'_{ab} = -i\gamma \int d^4x \ c^a (\partial \cdot A^a - \eta \cdot A^a) . \] (3.69)

We calculate the change in Jacobian corresponding to this FF-anti-BRST transformation using Eq. (3.56) as

\[ \frac{1}{J} \frac{dJ}{d\kappa} = i\gamma \int d^4x \left[ \frac{1}{\lambda} (\partial \cdot A)^a - \frac{1}{\lambda} (\eta \cdot A^a)(\eta \cdot A^a) + g f^{abc} \bar{c}^c (\partial \cdot A^a) \
- g f^{abc} \bar{c}^c (\eta \cdot A^a) - c^a M^{ab} \bar{c}^b + c^a \tilde{M}^{ab} \bar{c}^b \right] . \] (3.70)

We make an ansatz for \( S_1 \) as the following:

\[ S_1 = \int d^4x \left[ \xi_1(\kappa)(\partial \cdot A)^a + \xi_2(\kappa)(\eta \cdot A^a)^2 + \xi_3(\kappa)(\partial \cdot A^a)(\eta \cdot A^a) + \xi_4(\kappa)\left( c^a M^{ab} \bar{c}^b - g f^{abc} \bar{c}^c (\partial \cdot A^a) \right) + \xi_5(\kappa)\left( c^a \tilde{M}^{ab} \bar{c}^b - g f^{abc} \bar{c}^c (\eta \cdot A^a) \right) \right] , \] (3.71)

where \( \xi_i(\kappa) \) are parameters to be determined. The condition mentioned in Eq. (2.9) to replace
3.3. Finite field dependent anti-BRST (FF-anti-BRST) formulation

the Jacobian as $e^{S_1}$ for this case is

$$\int [D\phi] e^{i(S^L_{eff} + S_1)} \left[ (M^{ab\bar{c}b} - gf^{abc\bar{b}} \epsilon \cdot \bar{A}^c) \Theta'_{\bar{a}b} \left( \frac{\partial \cdot A^a}{\lambda} \right) 2\xi_1 + \frac{\xi_4}{\lambda} \right] + \eta \cdot A^a \xi_3 \right) + \left( \frac{\partial}{\xi_3} - \frac{\gamma}{\xi_4} \right) (\partial \cdot A^a)^2 + \frac{\lambda}{\xi_5} (\eta \cdot A^a)^2 \right]
+ \left( (\partial \cdot A^a)(\eta \cdot A^a) \left( \frac{\xi_3}{\xi_4} + \frac{\gamma}{\xi_5} \right) + \left( \frac{\xi_4}{\xi_4} + \frac{\gamma}{\xi_5} \right) e^a M^{ab\bar{c}b} + \left( \frac{\xi_5}{\xi_5} - \gamma \right) e^a \bar{M}^{ab\bar{c}b} \right)
- \left( g^{abc\bar{b}c}(\partial \cdot A^a) \left( \frac{\xi_4}{\xi_4} + \gamma \right) - g^{abc\bar{b}c}(\eta \cdot A^a) \left( \frac{\xi_5}{\xi_5} - \gamma \right) \right] = 0. \quad (3.72)

The last four terms in the integrand of Eq. (3.72) are dependent on $c$ in a local fashion. The contribution of these terms can possibly vanish by ghost equation of motion \[26, 37\].

$$\int D\phi \frac{\delta}{\delta c^a} e^{i(S^L_{eff} + S_1)} = 0. \quad (3.73)

This can only happen if the ratio of coefficients of the four terms is identical to the ratio of coefficients of $\hat{c}^a M^{ab\bar{c}b}$ and $\hat{c}^a \bar{M}^{ab\bar{c}b}$ in $S^L_{eff} + S_1$. This requires that

$$\frac{d\xi_4/d\kappa + \gamma}{\xi_4 + 1} = \frac{d\xi_5/d\kappa - \gamma}{\xi_5}. \quad (3.74)

The nonlocal $\Theta'_{ab}$ dependent terms are cancelled by converting them to local terms using ghost equation of motion \[37\]. This occurs only if the two $\Theta'_{ab}$ dependent terms combine in a certain manner, depending again on the ratio of coefficients of $\hat{c}^a M^{ab\bar{c}b}$ and $\hat{c}^a \bar{M}^{ab\bar{c}b}$ in terms in $S^L_{eff} + S_1$, i.e.

$$\frac{2\xi_1 + \xi_4/\lambda}{\xi_4 + 1} = \frac{\xi_3 + \xi_5/\lambda}{\xi_5},
\frac{\xi_3}{\xi_4 + 1} = \frac{2\xi_2}{\xi_5},
\frac{d\xi_4/d\kappa + \gamma}{\xi_4 + \gamma} = \frac{d\xi_5/d\kappa - \gamma}{\xi_5}. \quad (3.75)

Comparing the coefficients of $(\partial \cdot A^a)^2$, $(\eta \cdot A^a)^2$, $(\partial \cdot A^a)(\eta \cdot A^a)$, $(c^a M^{ab\bar{c}b} - gf^{abc\bar{b}c}(\partial \cdot A^a)$ and $(c^a \bar{M}^{ab\bar{c}b} - gf^{abc\bar{b}c}(\eta \cdot A^a)$ respectively, we get

$$\frac{d\xi_1}{d\kappa} + \gamma \left( \frac{2\xi_1 + \xi_4/\lambda}{\lambda} \right) + \gamma \left( \frac{\xi_3 + \xi_5}{\xi_5} \right) = 0, \quad (3.76)
\frac{d\xi_2}{d\kappa} - \gamma \xi_3 - 2\gamma \xi_2 = 0, \quad (3.77)
\frac{d\xi_3}{d\kappa} + \gamma \xi_3 - \gamma \left( \frac{2\xi_1 + \xi_4/\lambda}{\lambda} \right) + 2\gamma \xi_2 - \gamma \left( \frac{\xi_3 + \xi_5}{\xi_5} \right) = 0, \quad (3.78)
\frac{d\xi_4}{d\kappa} + \gamma = 0,
\frac{d\xi_5}{d\kappa} - \gamma = 0. \quad (3.79)
The solutions of the above equations (3.76) to (3.79) (for $\gamma = 1$) are

\[
\xi_1 = \frac{1}{2\lambda} \left[ 1 - (\kappa - 1)^2 \right], \quad \xi_2 = -\frac{\kappa^2}{2\lambda}, \\
\xi_3 = \frac{1}{\lambda} \kappa (\kappa - 1), \quad \xi_4 = -\kappa, \quad \xi_5 = \kappa.
\] (3.80)

Putting these in the expression for $S_1$, we have

\[
S_1(\kappa = 1) = \int d^4x \left[ \frac{(\partial \cdot A^a)^2}{2\lambda} - \frac{(\eta \cdot A^a)^2}{2\lambda} - c^a M^{ab} \bar{c}^b + c^a \tilde{M}^{ab} \bar{c}^b \\
+ g f^{abc} \bar{c}^c \partial \cdot A^a - g f^{abc} \bar{c}^c \partial_j A^a \right].
\] (3.81)

The new effective action becomes

\[
S_{\text{eff}}' = S_{\text{eff}}^L + S_1.
\]

In this case,

\[
S_{\text{eff}}^L + S_1 = \int d^4x \left[ \frac{-1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\lambda} (\eta \cdot A^a)^2 + c^a \tilde{M}^{ab} \bar{c}^b \\
- g f^{abc} \bar{c}^c (\eta \cdot A)^a \right] = S_{\text{eff}}^A,
\] (3.82)

which is nothing but the FP effective action in axial gauge.

Thus, the generating functional corresponding to Lorentz gauge and axial gauge can also be related by FF-anti-BRST transformation. We observe FF-anti-BRST transformation plays exactly the same role as FFBFRST transformation in this example.

### 3.3.3 Lorentz gauge and Coulomb gauge in YM theory

To show the connection between generating functional corresponding to the effective action in Lorentz gauge to that of the effective action in Coulomb gauge through FF-anti-BRST transformation, we choose the parameter,

\[
\Theta'_{ab} = -i\gamma \int d^4x \ c^a (\partial \cdot A^a - \partial_j A^a).
\] (3.83)

Using equation (3.56), we calculate the change in Jacobian as

\[
\frac{1}{J} \frac{dJ}{d\kappa} = i\gamma \int d^4x \left[ \frac{(\partial \cdot A^a)^2}{\lambda} - \frac{(\partial \cdot A^a)(\partial_j A^a)}{\lambda} - c^a M^{ab} \bar{c}^b + c^a (\tilde{M}' \bar{c})^a \\
+ g f^{abc} \bar{c}^c \partial \cdot A^a - g f^{abc} \bar{c}^c \partial_j A^a \right],
\] (3.84)

where $\tilde{M}' = \partial_j D^j$.

We make an ansatz for $S_1$ looking at the different terms in the effective action in Lorentz gauge and Coulomb gauge as

\[
S_1 = \int d^4x \left[ \xi_1(\kappa)(\partial \cdot A^a)^2 + \xi_2(\kappa)(\partial_j A^a)^2 + \xi_3(\kappa)(\partial \cdot A^a)(\partial_j A^a) \\
+ \xi_4(\kappa) \left( c^a M^{ab} \bar{c}^b - g f^{abc} \bar{c}^c \partial \cdot A^a \right) + \xi_5(\kappa) \left( c^a (\tilde{M}' \bar{c})^a \\
- g f^{abc} \bar{c}^c \partial_j A^a \right) \right].
\] (3.85)
Following the similar procedure as in the previous subsection, we obtain $S_3$ as

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then, corresponding to the above $\Theta'$ will be the part of the new effective action if and only if the condition in Eq. (2.9) is satisfied.

Following the similar procedure as in the previous subsection, we obtain $S_1$ at $\kappa = 1$ as

$$S_1 = \int d^4x \left[ \frac{(\partial \cdot A)^2}{2\lambda} - \frac{(\partial_j A^j)^2}{2\lambda} - e^a M^{ab} \bar{c}^b + c^a (\bar{M} e)^a + g f^{abc} \bar{c}^c \partial \cdot A^a \right] - g f^{abc} \bar{c}^c \partial_j A^j. \quad (3.87)$$

Adding this part to $S_{eff}^L$ we obtain

$$S_{eff}^L + S_1(\kappa = 1) = \int d^4x \left[ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{(\partial_j A^j)^2}{2\lambda} + c^a (\bar{M} e)^a - g f^{abc} \bar{c}^c \partial_j A^j \right]$$

$$= \int d^4x \left[ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{(\partial_j A^j)^2}{2\lambda} - e^a (\bar{M} e)^a \right]$$

$$= S_{eff}^L, \quad (3.88)$$

which is effective action in Coulomb gauge. Thus, the generating functionals corresponding to Lorentz gauge and Coulomb gauge can also be related by FF-anti-BRST transformation.

### 3.3.4 FP theory to most general BRST/anti-BRST invariant theory

In all previous examples we consider the effective theory in different gauges. However, similar to FFBRST transformation, FF-anti-BRST transformation can also relate the different effective theories. In order to connect two different theories viz. YM effective action in Lorentz gauge and the most general BRST/anti-BRST invariant action in Lorentz gauge, we consider infinitesimal field dependent parameter as

$$\Theta'_{ab} = -i\gamma \int d^4x \ e^a f^{abc} \bar{c}^c. \quad (3.89)$$

Then, corresponding to the above $\Theta'_{ab}$ the change in Jacobian, using the Eq. (3.58), is calculated as

$$\frac{1}{J} \frac{dJ}{d\kappa} = i\gamma \int d^4x \ \left[ 2 \frac{\partial \cdot A^a}{\lambda} f^{abc} \bar{c}^c + g f^{abc} \bar{c}^c f^{alm} \bar{c}^m \right]. \quad (3.90)$$
Looking at the kind of terms present in the FP effective action in Lorentz gauge and in the most general BRST/anti-BRST invariant effective action in Lorentz gauge, we try an ansatz for $S_1$ as

$$S_1 = \int d^4x \left[ \xi_1(\kappa) f^{abc} \partial \cdot A^a \partial_c^e c^e + \xi_2(\kappa) f^{abc} \partial_{\bar{c}}^e f^{alm} \partial^m c^m \right].$$  

(3.91)

$S_1$ can be expressed as $e^{iS_1}$ iff it satisfies the condition mentioned in Eq. (2.9). The condition for this case is calculated as

$$\int [D\phi] e^{i(S_{L\text{eff}} + S_1)} \left[ f^{abc} \partial \cdot A^a \partial_c^e c^e \left( \frac{d\xi_1}{d\kappa} - \frac{2\gamma}{\lambda} \right) \right.$$  

$$+ f^{abc} \partial_{\bar{c}}^e f^{alm} \partial^m c^m \left( \frac{d\xi_2}{d\kappa} + \frac{\gamma g}{2} - \gamma \xi_1 \right)$$  

$$\left. + f^{abc} \partial_{\bar{c}}^e f^{alm} \partial^m \partial \cdot A^m \Theta_{ab}^e \left( \frac{\xi_1^2}{2} - \frac{\gamma g_1}{2} - \frac{2\xi_2}{\lambda} \right) \right] = 0.$$  

(3.92)

We look for a special solution corresponding to the condition

$$\frac{\xi_1}{2} - \frac{g\xi_1}{2} - \frac{2\xi_2}{\lambda} = 0.$$  

(3.93)

The coefficient of $f^{abc} \partial \cdot A^a \partial_c^e c^e$ and $f^{abc} \partial_{\bar{c}}^e f^{alm} \partial^m c^m$ gives respectively

$$\frac{d\xi_1}{d\kappa} - \frac{2\gamma}{\lambda} = 0,$$  

(3.94)

$$\frac{d\xi_2}{d\kappa} + \frac{g\gamma}{2} - \gamma \xi_1 = 0.$$  

(3.95)

For a particular $\gamma = \frac{\alpha \lambda g}{4}$, the solutions of above two equations are

$$\xi_1 = \frac{\alpha}{2} g\kappa,$$  

(3.96)

$$\xi_2 = -\frac{\alpha}{8} g^2 \lambda \kappa + \frac{\alpha^2}{16} \lambda g^2 \kappa^2.$$  

(3.97)

At $\kappa = 1$

$$S_1 = \int d^4x \left[ \frac{\alpha}{2} g f^{abc} \partial \cdot A^a \partial_c^e c^e - \frac{\alpha}{8} \left( 1 - \frac{\alpha}{2} \right) \lambda g^2 f^{abc} \partial_{\bar{c}}^e f^{alm} \partial^m c^m \right].$$  

(3.98)

Hence,

$$S_{\text{eff}}^L + S_1 = \int d^4x \left[ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{(\partial \cdot A^a)^2}{2\lambda} + \partial \mu \partial^\nu D_{ab}^{\mu} \right.$$  

$$\left. + \frac{\alpha}{2} g f^{abc} \partial \cdot A^a \partial_c^e c^e - \frac{\alpha}{8} \alpha (1 - \frac{1}{2} \alpha) \lambda g^2 f^{abc} \partial_{\bar{c}}^e f^{alm} \partial^m c^m \right]$$  

$$= S_{\text{eff}}^{AB}[A, c, \bar{c}].$$  

(3.99)

which is most general BRST/anti-BRST invariant effective action.

Thus, the generating functional corresponding to most general effective actions in Lorentz gauge can also be related through FF-anti-BRST transformation. We observe that FF-anti-BRST transformation plays the same role as of the FFBRST transformation but with different parameters.
3.4 Off-shell nilpotent FF-anti-BRST transformation

The FF-anti-BRST transformation, we have constructed in previous section, is on-shell nilpotent. In this section, we construct FF-anti-BRST transformation which is off-shell nilpotent. For this purpose, we consider the following effective action for YM theories in auxiliary field formulation in Lorentz gauge

\[
S_{\text{eff}}^L = \int d^4x \left[ -\frac{1}{4} F^{a\mu\nu} F_{a}^{\mu\nu} + \frac{\lambda}{2} (B^a)^2 - B^a \partial \cdot A^a + c^a M^{ab} \bar{c}^b - g f^{abc} \bar{c}^b c^c (\partial \cdot A)^a \right].
\]

(3.100)

This effective action is invariant under anti-BRST transformation mentioned in Eq. (3.43). Following the procedure outlined in the section 3.2, we obtain the FF-anti-BRST transformation in auxiliary field formulation as,

\[
\begin{align*}
\delta_b A^a_{\mu} &= (D_\mu \bar{c})^a \Theta_{ab}(A, c, \bar{c}, B), \\
\delta_b c^a &= -\frac{1}{2} g f^{abc} \bar{c}^b c^c \Theta_{ab}(A, c, \bar{c}, B), \\
\delta_b B^a &= -g f^{abc} B^b \bar{c}^c \Theta_{ab}(A, c, \bar{c}, B),
\end{align*}
\]

(3.101)

which also leaves the effective action in Eq. (3.100) invariant. Now, we consider the different choices of the parameter \( \Theta'_{ab}(A, c, \bar{c}, B) \) in auxiliary field formulation to connect different theories. We redo the same examples using off-shell FF-anti-BRST transformation.

3.4.1 YM theory in Lorentz gauge to Coulomb gauge

The effective action for YM theory in Coulomb gauge can be written after rearranging the ghost term as

\[
S_{\text{eff}}^C = \int d^4x \left[ -\frac{1}{4} F^{a\mu\nu} F_{a}^{\mu\nu} + \frac{\lambda}{2} (B^a)^2 - B^a \partial \cdot A^a + c^a (\tilde{M} \bar{c})^a \right].
\]

(3.102)

To show the connection of this theory with the theory in Lorentz gauge, we choose the following infinitesimal field dependent parameter:

\[
\Theta'_{ab} = -i \int d^4x \ c^a [\gamma_1 \lambda B^a + \gamma_2 (\partial \cdot A^a - \partial^j A^a_j)].
\]

(3.103)

Using the above \( \Theta'_{ab} \), we find the change in Jacobian as

\[
\frac{1}{J} \frac{dJ}{dk} = i \int d^4x \left[ \lambda_1 (B^a)^2 + \gamma_2 B^a \partial \cdot A^a - \gamma_2 B^a \partial^j A^a_j + \gamma_2 f^{abc} \bar{c}^b c^c \partial^j A^a_j - \gamma_2 g f^{abc} \bar{c}^b c^c (\tilde{M} \bar{c})^a \right].
\]

(3.104)
We make an ansatz for $S_1$ as

$$S_1 = \int d^4x \left[ \xi_1(\kappa) (B^a)^2 + \xi_2(\kappa) B^a \partial \cdot A^a + \xi_3(\kappa) \bar{B}^a \partial^j A^a_j \right. $$

$$\left. + \xi_4(\kappa) \left( c^a M^{ab} c^b - g f^{abc} c^b c^c \partial \cdot A^a \right) + \xi_5(\kappa) \left( c^a (\bar{M}^c)^a \right. 
\right. $$

$$\left. - g f^{abc} c^b c^c \partial_j A^a_j \right]. \tag{3.105}$$

The essential requirement for replacing the Jacobian as $e^{iS_1}$ mentioned in Eq. (3.101) is satisfied iff

$$\int [D\phi] e^{i(S_{eff}^L + S_1)} \left[ (B^a)^2 \left( \frac{d\xi_1}{d\kappa} - \lambda \gamma_1 \right) + B^a \partial \cdot A^a \left( \frac{d\xi_2}{d\kappa} - \gamma_2 \right) \right. $$

$$\left. + B^a \partial^j A^a_j \left( \frac{d\xi_3}{d\kappa} + \gamma_2 \right) - \bar{c}^a M^{ab} c^b \left( \frac{d\xi_4}{d\kappa} + \gamma_2 \right) - \bar{c}^a (\bar{M}^c)^a \left( \frac{d\xi_5}{d\kappa} - \gamma_2 \right) \right. $$

$$\left. + \left\{ M^{ab} c^b - g f^{abc} c^b \partial \cdot A^c \right\} \Theta_{ab} \left\{ B^a (\xi_2 + \xi_4) \right\} + \left\{ (\bar{M}^c)^a - g f^{abc} \bar{c}^b \partial^j A^a_j \right\} \Theta_{ab} \right] \tag{3.106}$$

The last two terms in the integrand of Eq. (3.106) are dependent on $c$ in a local fashion. The contribution of these terms can possibly vanish by ghost equation of motion given in Eq. (3.73). This can only happen if the ratio of coefficients of the two terms is identical to the ratio of coefficients of $c^a M^{ab} c^b$ and $\bar{c}^a (\bar{M}^c)^a$ in $S^L_{eff} + S_1$. The nonlocal terms become local, only if, it satisfy the following conditions:

$$\frac{d\xi_1/d\kappa + \gamma_2}{\xi_4 + 1} = \frac{d\xi_5/d\kappa - \gamma_2}{\xi_5}, \tag{3.107}$$

and

$$\frac{\xi_2 + \xi_4}{\xi_4 + 1} = \frac{\xi_3 + \xi_5}{\xi_5}. \tag{3.108}$$

We further obtain equations for the parameter $\xi_i$ by vanishing the coefficient of different independent terms in the L.H.S. of the Eq. (3.106) as

$$\frac{d\xi_1}{d\kappa} - \gamma_1 \lambda + \gamma_1 \lambda (\xi_2 + \xi_4) + \gamma_1 \lambda (\xi_3 + \xi_5) = 0,$$

$$\frac{d\xi_2}{d\kappa} - \gamma_2 + \gamma_2 (\xi_2 + \xi_4) + \gamma_2 (\xi_3 + \xi_5) = 0,$$

$$\frac{d\xi_3}{d\kappa} + \gamma_2 - \gamma_2 (\xi_2 + \xi_4) - \gamma_2 (\xi_3 + \xi_5) = 0,$$

$$\frac{d\xi_4}{d\kappa} + \gamma_2 = 0,$$

$$\frac{d\xi_5}{d\kappa} - \gamma_2 = 0. \tag{3.109}$$

We determine the parameter $\xi_i$ subjected to the initial condition in Eq. (3.17) as

$$\xi_1 = \gamma_1 \lambda, \quad \xi_2 = \kappa, \quad \xi_3 = -\kappa, \quad \xi_4 = -\kappa, \quad \xi_5 = \kappa. \tag{3.110}$$
Using the above solutions for $\xi_i$, we write $S_1$ at $\kappa = 1$ as

$$
S_1 = \int d^4x \left[ \gamma_1 B^a (B^a)^2 + B^a \partial \cdot A^a - B^a \partial_j A^a_{\bar{i}a} - c^a M^a_{\bar{i}b} c^b \right. \\
+ \left. c^a (\tilde{M}'^a)^a - gf^{abc} c^c \partial \cdot A^c \right].
$$

(3.111)

Now, when this $S_1$ is added to the effective action $S^L_{\text{eff}}$, it provides effective action in Coulomb gauge as

$$
S^L_{\text{eff}} + S_1 = \int d^4x \left[ \frac{-1}{4} F^{a\mu\nu} F^a_{\mu\nu} + \frac{\zeta}{2} (B^a)^2 - B^a \partial^i A^a_i + c^a (\tilde{M}'^a)^a \\
- \frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} + \frac{\zeta}{2} (B^a)^2 - B^a \partial^i A^a_i - c^a (\tilde{M}'^a)^a \right],
$$

(3.112)

which is the effective action in Coulomb gauge with gauge parameter $\zeta$.

Thus, FF-anti-BRST in auxiliary field formulation produces the same result as expected, even though the finite finite parameter is different.

### 3.4.2 Lorentz gauge to axial gauge theory

We repeat the same steps as in the previous subsection again for this case, with a different finite field dependent parameter obtainable from

$$
\Theta'_{ab} = -i \int d^4x c^a \left[ \gamma_1 \lambda B^a + \gamma_2 (\partial \cdot A^a - \eta \cdot A^a) \right],
$$

(3.113)

using Eq. (3.55) and consider the ansatz for $S_1$ as

$$
S_1 = \int d^4x \left[ \xi_1 (\kappa) (B^a)^2 + \xi_2 (\kappa) B^a \partial \cdot A^a + \xi_3 (\kappa) B^a \eta \cdot A^a \\
+ \xi_4 (\kappa) (c^a M^a_{\bar{i}b} c^b - gf^{abc} c^c \partial \cdot A^c) + \xi_5 (\kappa) (c^a M^a_{\bar{i}b} c^b - gf^{abc} c^c \partial \cdot A^c) \right].
$$

(3.114)

The condition for which the Jacobian of FF-anti-BRST transformation in auxiliary field formulation corresponding to the parameter given in Eq. (3.113) can be replaced as $e^{S_1}$ is,

$$
\int [D\phi] e^{i(S^L_{\text{eff}} + S_1)} \left\{ M^a_{\bar{i}b} c^b - gf^{abc} (\partial \cdot A^c) \right\} \Theta'_{ab} \{ B^a (\xi_2 + \xi_5) \}
$$

+ \left\{ M^a_{\bar{i}b} c^b - gf^{abc} (\partial \cdot A^c) \right\} \Theta'_{ab} \{ B^a (\xi_3 + \xi_5) \} + (B^a)^2 \left( \frac{d\xi_1}{d\kappa} - \lambda \gamma_1 \right)

+ B^a \partial \cdot A^a \left( \frac{d\xi_2}{d\kappa} - \gamma_2 \right) + B^a \eta \cdot A^a \left( \frac{d\xi_3}{d\kappa} + \gamma_2 \right) + c^a M^a_{\bar{i}b} \left( \frac{d\xi_4}{d\kappa} + \gamma_2 \right)

+ c^a M^a_{\bar{i}b} \left( \frac{d\xi_5}{d\kappa} - \gamma_2 \right) - gf^{abc} c^c \partial \cdot A^a \left( \frac{d\xi_4}{d\kappa} + \gamma_2 \right)

- gf^{abc} c^c \eta \cdot A^a \left( \frac{d\xi_5}{d\kappa} - \gamma_2 \right) = 0.
$$

(3.115)
3.4. Off-shell nilpotent FF-anti-BRST transformation

Following the previous subsection, we solve the differential equations, which are obtained from the above condition, to calculate the values of parameters $\xi_i$.

After calculating the exact values of $\xi_i$, the extra piece of the action $S_1$ becomes as

$$S_1 = \int d^4x \left[ \lambda \gamma_1 (B^a)^2 + B^a \partial \cdot A^a - B^a \eta \cdot A^a + \bar{\epsilon}^a M^{ab} c^b - \bar{\epsilon}^a \bar{M}^{ab} c^b \right]. \quad (3.116)$$

Now,

$$S_{eff}^L + S_1 = \int d^4x \left[ -\frac{1}{4} F^{a\mu\nu} F_{a\mu\nu} + \frac{\zeta}{2} (B^a)^2 - B^a \eta \cdot A^a - \bar{\epsilon}^a \bar{M}^{ab} c^b \right] = S_{eff}^{A'}, \quad (3.117)$$

where $S_{eff}^{A'}$ is the effective action in axial gauge with gauge parameter $\zeta$.

This implies off-shell nilpotent FF-anti-BRST also produces the same result even though calculations are different.

### 3.4.3 Lorentz gauge theory to most general BRST/anti-BRST invariant theory

We consider one more example in FF-anti-BRST formulation using auxiliary field. We show that the most general BRST/anti-BRST invariant theory can be obtained from FP theory in Lorentz gauge. The most general effective action which is invariant under BRST/anti-BRST transformation in auxiliary field is given in Eq. (3.41). We choose

$$\Theta'_{ab} = -i \int d^4x \epsilon^a \left[ \gamma_1 \lambda B^a + \gamma_2 f^{abc} c^b c^c \right]. \quad (3.118)$$

and make an ansatz for $S_1$ as

$$S_1 = \int d^4x \left[ \xi_1(\kappa) (B^a)^2 + \xi_2(\kappa) B^a f^{abc} c^b c^c + \xi_3(\kappa) f^{abc} c^b c^c f^{alm} c^l c^m \right]. \quad (3.119)$$

The condition in Eq. (2.9) leads to

$$\int [D\phi] e^{i(S_{eff}^L + S_1)} \left[ \left( \frac{d\xi_1}{d\kappa} - \gamma_1 \lambda \right) B^a + \left( \frac{d\xi_2}{d\kappa} - 2\gamma_2 \right) B^a f^{abc} c^b c^c + \left( \frac{d\xi_3}{d\kappa} + \frac{g}{2} \gamma_2 \right) f^{abc} c^b c^c f^{alm} c^l c^m \right] = 0. \quad (3.120)$$

Using the same procedure we obtain the solutions for the parameter $\xi_i$, exactly same as in Eq. (3.50). Even if the finite parameter is different in FFBRST and FF-anti-BRST, we obtain the same contribution from Jacobian in this case as

$$S_1 = \int d^4x \left[ \gamma_1 \lambda (B^a)^2 + \frac{ag}{2} f^{abc} c^b c^c - \frac{1}{8} ag^2 \gamma f^{abc} c^b c^c f^{alm} c^l c^m \right]. \quad (3.121)$$

Now, $S_{eff}^L + S_1 = S_{eff}^{AB'}$, which is nothing but most general BRST/anti-BRST invariant effective action with gauge parameter $\zeta$ mentioned in Eq. (3.51). In all three cases we show that off-shell nilpotent FF-anti-BRST transformation plays exactly same role as on-shell nilpotent FF-anti-BRST transformation but with different finite parameters.
3.5 Conclusions

In this chapter we have reformulated the FFBRST transformation in an auxiliary field formulation where the BRST transformation is off-shell nilpotent. We have considered several examples with different choices of finite parameter to connect the different effective theories. Most of the results of the FFBRST transformation are also obtained in auxiliary field formulation. We have further introduced and developed for first time the concept of the FF anti-BRST transformation analogous to the FFBRST transformation. FF-anti-BRST transformation is also used to connect the different generating functionals corresponding to different effective theories. Several examples have been worked out explicitly to show that the FF-anti-BRST transformation also plays the same role as of FFBRST transformation but with different finite parameters. Lastly, we consider the FF-anti-BRST transformation in auxiliary field formulation to make it off-shell nilpotent. The overall multiplicative antighost fields in the finite parameters of the FFBRST transformation are replaced by ghost fields in case of the FF-anti-BRST transformation. Even though the finite parameters and hence the calculations are different, the same results are also produced in an auxiliary field formulation of the FF-anti-BRST transformation. The BRST and the anti-BRST transformations are not independent transformations in the YM theories. We observe that the FF-anti-BRST transformation plays exactly the same role in connecting theories in 1-form gauge theory as expected. In 2-form gauge theories the BRST and anti-BRST transformations play some sort of independent roles. Therefore, it will be interesting to study the finite field dependent BRST and anti-BRST transformations in 2-form gauge theories [45], which we will discuss in the next chapter.
Chapter 4

FFBRST formulation in 2-form gauge theory

In this chapter we extend FFBRST formulation to Abelian 2-form gauge theories and show that it relates generating functional corresponding to different effective theories \[45\]. We further construct FF-anti-BRST transformation for such theories to show that it plays same role as in the case of 1-form gauge theories. Field/antifield formulation of 2-form gauge theories are also studied in the context of FFBRST transformation.

4.1 Preliminary: gauge theory of Abelian rank-2 antisymmetric tensor field

We consider the Abelian gauge theory for rank-2 antisymmetric tensor field $B_{\mu\nu}$ defined by the action

$$S_0 = \frac{1}{12} \int d^4 x F_{\mu\nu\rho} F^{\mu\nu\rho},$$

where $F_{\mu\nu\rho} \equiv \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$. This action is invariant under the gauge transformation

$$\delta B_{\mu\nu} = \partial_\mu \zeta_\nu - \partial_\nu \zeta_\mu$$

with a vector gauge parameter $\zeta_\mu(x)$.

To quantize this theory using BRST transformation, it is necessary to introduce the following ghost and auxiliary fields: anticommuting vector fields $\rho_\mu$ and $\tilde{\rho}_\mu$, a commuting vector field $\beta_\mu$, anticommuting scalar fields $\chi$ and $\tilde{\chi}$, and commuting scalar fields $\sigma, \varphi$, and $\tilde{\sigma}$. The BRST transformation is then defined for $B_{\mu\nu}$ by replacing $\zeta_\mu$ in the gauge transformation by the ghost field $\rho_\mu$.

The complete effective action for this theory in covariant gauge, using the BRST formulation, is given by

$$S_{eff} = S_0 + S_{gf} + S_{gh},$$

with the gauge-fixing and ghost term

$$S_{gf} + S_{gh} = \int d^4 x \left[ -i \partial_\mu \tilde{\rho}_\nu (\partial^\mu \rho^\nu - \partial^\nu \rho^\mu) + \partial_\mu \tilde{\sigma} \partial^\mu \sigma + \beta_\nu (\partial_\mu B^{\mu\nu} + \lambda_1 \beta^\nu - \partial^\nu \varphi) - i \tilde{\chi} \partial_\mu \rho^\mu - i \chi (\partial_\mu \tilde{\rho}^\mu - \lambda_2 \tilde{\chi}) \right],$$

where $\lambda_1$ and $\lambda_2$ are gauge parameters. This effective action is invariant under following BRST and anti-BRST symmetries.
BRST:

\[
\delta_b B_{\mu\nu} = - (\partial_{\mu} \rho_{\nu} - \partial_{\nu} \rho_{\mu}) \Lambda, \quad \delta_b \rho_{\mu} = -i \partial_{\mu} \sigma \Lambda, \\
\delta_b \sigma = 0, \quad \delta_b \beta_{\mu} = i \beta_{\mu} \Lambda, \quad \delta_b \bar{\sigma} = -\chi \Lambda, \\
\delta_b \bar{\beta}_{\mu} = 0, \quad \delta_b \bar{\chi} = 0, \quad \delta_b \bar{\varphi} = -\chi \Lambda, \quad \delta_b \beta_{\chi} = 0,
\]

(4.4)

anti-BRST:

\[
\delta_{ab} B_{\mu\nu} = - (\partial_{\mu} \tilde{\rho}_{\nu} - \partial_{\nu} \tilde{\rho}_{\mu}) \Lambda, \quad \delta_{ab} \tilde{\rho}_{\mu} = -i \partial_{\mu} \tilde{\sigma} \Lambda, \\
\delta_{ab} \tilde{\beta}_{\mu} = 0, \quad \delta_{ab} \tilde{\sigma} = 0, \quad \delta_{ab} \beta_{\mu} = -i \beta_{\mu} \Lambda, \quad \delta_{ab} \bar{\sigma} = -\chi \Lambda, \\
\delta_{ab} \bar{\beta}_{\mu} = 0, \quad \delta_{ab} \bar{\chi} = 0, \quad \delta_{ab} \bar{\varphi} = 0,
\]

(4.5)

where the BRST parameter \( \Lambda \) is global, infinitesimal and anticommuting in nature. The anti-BRST transformation is similar to the BRST transformation, where the role of ghost and antighost field is interchanged with some modification in coefficients. The generating functional for this Abelian rank 2 antisymmetric tensor field theory in a covariant gauge is defined as,

\[
Z^L = \int D\phi \exp[i S^L_{eff}[\phi]],
\]

(4.6)

where \( \phi \) is the generic notation for all the fields \((B_{\mu\nu}, \rho_{\mu}, \tilde{\rho}_{\mu}, \beta_{\mu}, \varphi, \sigma, \bar{\sigma}, \chi, \bar{\chi})\).

The BRST and anti-BRST transformations in Eqs. (4.4) and (4.5) respectively leave the above generating functional invariant as, the path integral measure \( D\phi \equiv DBD\bar{\rho}D\beta D\sigma D\bar{\sigma}D\chi D\bar{\chi} \) is invariant under such transformations.

4.2 FFBRST formulation of Abelian rank 2 anti-symmetric tensor field

To generalize the BRST transformation for this theory we follow the method outlined in chapter 2. We start by making the infinitesimal BRST parameter field dependent by introducing a parameter \( \kappa \) (0 \( \leq \kappa \leq 1 \)). All the fields \( \phi(x, \kappa) \) dependent on \( \kappa \) in such a way that \( \phi(x, \kappa = 0) = \phi(x) \) and \( \phi(x, \kappa = 1) = \phi'(x) \), the transformed field. It can easily be shown that such off-shell nilpotent BRST transformation with finite field dependent parameter is symmetry of the effective action in Eq. (4.2). However, the path integral measure in Eq. (4.6) is not invariant under such transformation as the BRST parameter is finite.

The Jacobian of the path integral measure for such transformations can be evaluated for some particular choices of the finite field dependent parameter, \( \Theta_b[\phi(x)] \), as

\[
DB'\bar{D}'\rho'\bar{D}'\beta'\bar{D}'\sigma'\bar{D}'\bar{\sigma}'\bar{D}'\chi'\bar{D}'\bar{\chi}' = J(\kappa)DB(\kappa)\bar{D}\rho(\kappa)\bar{D}\beta(\kappa)\bar{D}\sigma(\kappa) \\
\bar{D}\bar{\sigma}(\kappa)\bar{D}\chi(\kappa)\bar{D}\bar{\chi}(\kappa).
\]

(4.7)

The Jacobian, \( J(\kappa) \) can be replaced (within the functional integral) as

\[
J(\kappa) \rightarrow \exp[i S_1[\phi(x, \kappa)]],
\]

(4.8)

iff the condition given in Eq. (2.9) is satisfied. However, the infinitesimal change in the \( J(\kappa) \) can be calculated using Eq. (2.13).
By choosing appropriate $\Theta_b$, we can change $S_{\text{eff}}$ either to another effective action for same theory or to an effective action for another theory. The resulting effective action also be invariant under same BRST transformation.

4.3 FFBRST transformation in 2-form gauge theory: examples

In this section, we would like to show explicitly that the FFBRST transformation interrelates the different effective 2-form gauge theories. In particular, we are interested to obtain the effective theories for Abelian rank-2 tensor field in noncovariant gauges by applying FFBRST transformation to the effective theories in covariant gauge.

4.3.1 Effective theory in axial gauge

We start with the generating functional corresponding to the effective theory in Lorentz gauge given in Eq. (4.6), where the Lorentz gauge effective action $S_L^{\text{eff}}$ is invariant under following FFBRST:

\begin{align*}
\delta_b B_{\mu \nu} &= - (\partial_\mu \rho_\nu - \partial_\nu \rho_\mu) \Theta_b[\phi] \\
\delta_b \rho_\mu &= - i \partial_\mu \sigma \Theta_b[\phi], \quad \delta_b \sigma = 0 \\
\delta_b \tilde{\rho}_\mu &= i \beta_\mu \Theta_b[\phi], \quad \delta_b \beta_\mu = 0 \\
\delta_b \tilde{\sigma} &= - \chi \Theta_b[\phi], \quad \delta_b \chi = 0 \\
\delta_b \phi &= - \chi \Theta_b[\phi], \quad \delta_b \chi = 0,
\end{align*}

(4.9)

where $\Theta_b$ is a finite, anticommuting BRST parameter depends on the fields in global manner.

To obtain the effective theories in axial gauge we choose the finite parameter obtainable from

\begin{align*}
\Theta'_b &= \int d^4 x [\gamma_1 \tilde{\rho}_\nu (\partial_\mu B^{\mu \nu} - \eta_\mu B^{\nu \mu} - \partial^\nu \varphi - \eta^\nu \varphi) + \gamma_2 \lambda_1 \tilde{\rho}_\nu \beta^\nu \\
&+ \gamma_1 \tilde{\sigma} (\partial_\mu \rho^\mu - \eta_\mu \rho^\mu) + \gamma_2 \lambda_2 \tilde{\sigma} \chi],
\end{align*}

(4.10)

where $\gamma_1$ and $\gamma_2$ are arbitrary parameters (depend on $\kappa$) and $\eta_\mu$ is arbitrary constant four vector.

Now we apply this FFBRST transformation to the generating functional $Z_L^{\text{eff}}$ given in Eq. (4.6). The path integral measure is not invariant and give rise a nontrivial functional $e^{i S^A_1}$ (explicitly shown in Appendix A), where

\begin{align*}
S^A_1 &= \int d^4 x [ - \beta_\nu \partial_\mu B^{\mu \nu} + \beta_\mu \eta_\nu B^{\nu \mu} + \gamma_2 \lambda_1 \beta_\nu \beta^\nu - i \tilde{\rho}_\nu \partial_\mu (\partial^\mu \rho^\nu - \partial^\nu \rho^\mu) \\
&+ i \tilde{\rho}_\nu \eta_\mu (\partial^\mu \rho^\nu - \partial^\nu \rho^\mu) + i \gamma_2 \lambda_2 \chi \tilde{\chi} + i \tilde{\chi} \partial_\mu \rho^\mu - i \chi \eta_\mu \rho^\mu \\
&+ \tilde{\sigma} \partial_\mu \partial^\mu \sigma - \partial_\nu \eta_\mu \partial^\nu \sigma - \partial_\mu \beta^\mu \varphi + \eta_\mu \beta^\mu \varphi + i \chi \partial_\mu \rho^\mu - i \chi \eta_\mu \tilde{\rho}^\mu].
\end{align*}

(4.11)

Now, adding this $S^A_1$ to $S^L_{\text{eff}}$, we get

\begin{align*}
S^L_{\text{eff}} + S^A_1 &= S^A_1,
\end{align*}

(4.12)
where

\[ S_{eff}^A = \int d^4x \left[ \frac{1}{12} F_{\mu\nu\rho} F^{\mu\nu\rho} + i\bar{\theta}_\nu \eta_{\mu}(\partial^\mu \rho^\nu - \partial^\nu \rho^\mu) - \bar{\sigma} \eta_{\mu} \partial^\mu \sigma + \beta_\nu (\eta_{\mu} B^{\mu\nu} + \lambda_1' \beta^\nu + \eta^\nu \varphi) - i\bar{\chi} \eta_{\nu} \rho^\mu - i\chi (\eta_{\mu} \beta^\mu - \lambda_2' \chi) \right], \tag{4.13} \]

is the effective action in axial gauge with new gauge parameters \( \lambda_1' = (1 + \gamma_2) \lambda_1 \) and \( \lambda_2' = (1 + \gamma_2) \lambda_2 \). Thus, the FFBRST transformation with the finite parameter given in Eq. (4.9) takes

\[ Z^L \left( = \int D\phi^A \right) \rightarrow FFBRST \rightarrow Z^A \left( = \int D\phi^A \right). \tag{4.14} \]

The effective theory of Abelian rank-2 antisymmetric field in axial gauge is convenient in many different situations. The generating functional in axial gauge with a suitable axis is same as the generating functional obtained by using Zwanziger’s formulation for electric and magnetic charges [51, 100]. Using the FFBRST transformation with the parameter given in Eq. (4.10) we have linked generating functionals corresponding to the effective theories in covariant and noncovariant gauges.

### 4.3.2 Effective theory in Coulomb gauge

The generating functional for the effective theories in Coulomb gauge is also obtained by using the FFBRST transformation with a different parameter

\[ \Theta'_C = \int d^4x \left[ \gamma_1 \bar{\rho}_\nu (\partial^\mu B^{\mu\nu} - \partial_i B^{\mu\nu} - \partial^\nu \varphi) + \gamma_1 \bar{\rho}_i \partial^\nu \varphi + \gamma_2 \lambda_1 \bar{\rho}_\nu \beta^\nu + \gamma_1 \bar{\sigma} (\partial_\mu \rho^\mu - \partial_i \rho^i) + \gamma_2 \lambda_2 \bar{\sigma} \chi \right]. \tag{4.15} \]

The effective action in Lorentz gauge, \( S_{eff}^L \), as given in Eq. (4.12) is invariant under FFBRST transformation in Eq. (4.9) corresponding to above mentioned finite parameter. Now we consider the effect of this FFBRST transformation on the generating functional in Lorentz gauge.

In Appendix A, it has been shown that the Jacobian for the path integral measure corresponding to this FFBRST transformation is replaced by \( e^{iS_1^C} \), where \( S_1^C \) is the local functional of fields calculated as

\[ S_1^C = \int d^4x \left[ -\beta_\nu \partial_\mu B^{\mu\nu} + \beta_\nu \partial_i B^{\mu\nu} + \gamma_2 \lambda_1 \beta_\nu \beta^\nu - i\bar{\rho}_\nu \partial_\mu (\partial^\mu \rho^\nu - \partial^\nu \rho^\mu) + i\gamma_2 \lambda_2 \chi \bar{\chi} + i\bar{\chi} \partial_\mu \rho^\mu - i\chi \partial_\mu \rho^\mu \right] + \tilde{\partial}_\mu \bar{\rho}^\mu \sigma - \tilde{\partial}_i \bar{\rho}^i \sigma - \tilde{\partial}_\mu \beta^\mu \varphi + \partial_\nu \beta^\nu \varphi + i\chi \partial_\mu \beta^\mu - i\chi \partial_\nu \beta^\nu \right], \tag{4.16} \]

and this extra piece of the action can be added to the effective action in covariant gauge to lead a new effective action

\[ S_{eff}^L + S_1^C = \int d^4x \left[ \frac{1}{12} F_{\mu\nu\rho} F^{\mu\nu\rho} + i\bar{\theta}_\nu \partial_i (\partial^\mu \rho^\nu - \partial^\nu \rho^\mu) - \bar{\sigma} \partial_i \partial^\sigma \right. \]

\[ + \beta_\nu (\partial_i B^{\mu\nu} + \lambda_1' \beta^\nu) - \beta_\nu \chi \partial^\nu \varphi - i\bar{\chi} \partial_i \mu \beta^\nu - i\chi (\partial_i \rho^i - \lambda_2' \chi) \]

\[ \equiv S_{eff}^C, \tag{4.17} \]

which is an effective action in Coulomb gauge for Abelian rank 2 tensor field. Thus, we study the Abelian 2-form gauge theory in Coulomb gauge more rigorously through its connection with Lorentz gauge via finite BRST transformation.
4.4 FF-anti-BRST transformation in 2-form gauge theories

In this section, we consider the generalization of anti-BRST transformation following the similar method as discussed in section 2 and show it plays exactly similar role in connecting the generating functionals in different effective theories of Abelian rank-2 antisymmetric tensor field.

4.4.1 Lorentz to axial gauge theory using FF-anti-BRST formulation

For sake of convenience we recast the effective action in covariant gauge given in Eq. (4.2) as

\[ S_{\text{eff}}^L = \int d^4x \left[ \frac{1}{12} F_{\mu\nu} F^{\mu\nu\rho} + i \partial_\mu \rho_\nu (\partial^\mu \tilde{\rho}^\nu - \partial^\nu \tilde{\rho}^\mu) + \partial_\mu \sigma \partial^\mu \tilde{\sigma} + \beta_\nu (\partial_\mu B^{\mu\nu} + \lambda_1 \beta^\nu - \partial^\nu \varphi) - i \chi \partial_\mu \tilde{\rho}^\mu - i \tilde{\chi} (\partial_\mu \rho^\mu + \lambda_2 \chi) \right], \]

(4.18)

which is invariant under following FF-anti-BRST transformation:

\[
\begin{align*}
\delta_{ab} B_{\mu\nu} &= - (\partial_\mu \tilde{\rho}_\nu - \partial_\nu \tilde{\rho}_\mu) \Theta_{ab}[\phi] \\
\delta_{ab} \tilde{\rho}_\mu &= - i \tilde{\rho}_\mu \sigma \Theta_{ab}[\phi], \
\delta_{ab} \beta_\nu &= 0 \\
\delta_{ab} \sigma &= - \chi \Theta_{ab}[\phi], \
\delta_{ab} \varphi &= - \tilde{\chi} \Theta_{ab}[\phi],
\end{align*}
\]

(4.19)

where \( \Theta_{ab} \) is finite field dependent anti-BRST parameter.

To obtain the generating functional in axial gauge using FF-anti-BRST transformation we choose,

\[
\Theta'_{ab} = - \int d^4x \left[ \gamma_1 \rho_\nu (\partial_\mu B^{\mu\nu} - \eta_\mu B^{\mu\nu} - \partial^\mu \varphi - \eta^\mu \varphi) + \gamma_2 \lambda_1 \rho_\nu \beta^\nu \\
- \gamma_1 \sigma (\partial_\mu \tilde{\rho}^\mu - \eta_\mu \tilde{\rho}^\mu) + \gamma_2 \lambda_2 \sigma \tilde{\chi} \right].
\]

(4.20)

This parameter is similar to \( \Theta'_{ab} \) in Eq. (4.10) except the (anti)ghost and ghost of (anti)ghost fields are replaced by their antighost fields respectively.

Similar to FFBRST case, this FF-anti-BRST transformation changes the Jacobian in the path integral measure by a factor \( e^{iS^A_1} \), where \( S^A_1 \) is a local functional of fields and is given by

\[
\begin{align*}
S^A_1 &= \int d^4x \left[ - \beta_\nu \partial_\mu B^{\mu\nu} + \beta_\nu \eta_\mu B^{\mu\nu} + \gamma_2 \lambda_1 \beta_\nu \beta^\nu + i \rho_\nu \partial_\mu (\partial^\mu \tilde{\rho}^\nu - \partial^\nu \tilde{\rho}^\mu) \\
&- i \rho_\nu \eta_\mu (\partial^\mu \tilde{\rho}^\nu - \partial^\nu \tilde{\rho}^\mu) + i \gamma_2 \lambda_2 \chi \tilde{\chi} + i \chi \partial_\mu \tilde{\rho}^\mu - i \eta_\mu \tilde{\rho}^\mu + \sigma \partial_\mu \tilde{\rho}^\mu - \sigma \eta_\mu \partial^\mu \tilde{\sigma} - \tilde{\beta}_\nu \tilde{\rho}^\nu \varphi + \eta_\mu \beta^\mu \varphi + i \tilde{\chi} \partial_\mu \rho^\mu - i \tilde{\chi} \eta_\mu \rho^\mu \right].
\end{align*}
\]

(4.21)

It is easy to verify that

\[
S^L + S^A_1 = \int d^4x \left[ \frac{1}{12} F_{\mu\nu\rho} F^{\mu\nu\rho} - i \rho_\nu \eta_\mu (\partial^\mu \tilde{\rho}^\nu - \partial^\nu \tilde{\rho}^\mu) - \sigma \eta_\mu \partial^\mu \tilde{\sigma} + \beta_\nu (\eta_\mu B^{\mu\nu} + \lambda_1 \beta^\nu + \eta^\nu \varphi) - i \tilde{\chi} \eta_\mu \rho^\mu - i \chi (\eta_\mu \tilde{\rho}^\mu - \lambda_2 \chi) \right] \\
\equiv S^A,
\]

(4.22)
which is the action in axial gauge in 2-form gauge theory. Thus, the FF-anti-BRST transformation with the finite parameter given in Eq. (4.19) takes
\[
Z^L \left( = \int \mathcal{D}\phi e^{iS^L_{eff}} \right) \to - \to Z^A \left( = \int \mathcal{D}\phi e^{iS^A_{eff}} \right), \tag{4.23}
\]
which shows that the FF-anti-BRST transformation plays the similar role as FFBRST transformation in 2-form gauge theory but with different finite field dependent parameter.

### 4.5 Field/Antifield formulation of Abelian rank-2 antisymmetric tensor field theory

In this section we construct field/antifield formulation for Abelian rank-2 antisymmetric tensor field theory to show that techniques of FFBRST formulation can also be applied in this modern approach of quantum field theory. For this purpose we express the generating functional in Eq. (4.6) in field/antifield formulation by introducing antifield \( \phi^* \) corresponding to each field \( \phi \) with opposite statistics as,
\[
Z^L = \int [\mathcal{D}\phi] \exp \left[ i \int d^4x \left\{ \frac{1}{12} F_{\mu\nu\lambda} F^{\mu\nu\lambda} - B_{\mu\nu}^{\ast} (\partial_\mu \rho_\nu - i\partial_\nu \rho_\mu) - \rho^{\mu\ast} \partial_\mu \sigma + i\tilde{\rho}^{\mu\ast} \beta_\nu - \tilde{\sigma}^{\ast} \tilde{\chi} - \varphi^\ast \varphi \right\} \right]. \tag{4.24}
\]
This can be written in compact form as
\[
Z^L = \int [\mathcal{D}\phi] \exp [i W_{\Psi^L}(\phi, \phi^*)], \tag{4.25}
\]
where \( W_{\Psi^L}(\phi, \phi^*) \) is an extended action for 2-form gauge theory in Lorentz gauge corresponding the gauge-fixed fermion \( \Psi^L \) having Grassmann parity 1 and ghost number -1. The expression for \( \Psi^L \) is
\[
\Psi^L = -i \int d^4x [\tilde{\rho}_\nu (\partial_\mu B^{\mu\nu} + \lambda_1 \beta^\nu) + \tilde{\sigma} \partial_\mu \rho^\mu + \varphi (\partial_\mu \tilde{\rho}^\mu - \lambda_2 \tilde{\chi})]. \tag{4.26}
\]
The generating functional \( Z^L \) does not depend on the choice of gauge-fixed fermion. This extended quantum action, \( W_{\Psi^L}(\phi, \phi^*) \), satisfies the quantum master equation given in Eq. (2.22). The antifields \( \phi^* \) corresponding to each field \( \phi \) for this particular theory can be obtained from the gauge-fixed fermion as
\[
\begin{align*}
B^{\mu\ast} &= \frac{\delta \psi^L}{\delta B^{\mu\nu}} = i\partial^\mu \tilde{\rho}^\nu, & \tilde{\rho}^{\mu\ast} &= \frac{\delta \psi^L}{\delta \tilde{\rho}_\nu} = -i(\partial_\mu B^{\mu\nu} + \lambda_1 \beta^\nu - \partial^\nu \varphi), \\
\rho^{\mu\ast} &= \frac{\delta \psi^L}{\delta \rho_\mu} = i\partial^\mu \sigma, & \tilde{\sigma}^\ast &= \frac{\delta \psi^L}{\delta \tilde{\sigma}} = -i\partial_\nu \rho^\mu, \\
\sigma^\ast &= \frac{\delta \psi^L}{\delta \sigma} = 0, & \beta^{\nu\ast} &= \frac{\delta \psi^L}{\delta \beta_\nu} = -i\lambda_1 \tilde{\rho}^\nu, \\
\varphi^\ast &= \frac{\delta \psi^L}{\delta \varphi} = -i(\partial_\mu \tilde{\rho}^\mu - \lambda_2 \tilde{\chi}), & \tilde{\chi}^\ast &= \frac{\delta \psi^L}{\delta \tilde{\chi}} = i\lambda_2 \varphi, \\
\chi^\ast &= \frac{\delta \psi^L}{\delta \chi} = 0.
\end{align*} \tag{4.27}
\]
4.6 Conclusions

Now we apply the FFBRST transformation with the finite parameter given in Eq. (4.10) to this generating functional in Lorentz gauge. But the path integral measure is not invariant under such a finite transformation and give rise to a factor which can be written as $e^{iS_1}$, where the functional $S_1$ is calculated in Appendix A and also given in Eq. (4.11). The transformed generating functional

$$Z' = \int \mathcal{D}\phi \exp[i\{W_{\Psi L} + S_1\}],$$

$$= \int \mathcal{D}\phi \exp[iW_{\Psi A}] \equiv Z^A. \quad (4.28)$$

The generating functional in axial gauge

$$Z^A = \int [DBd\rho D\sigma D\varphi D\chi D\beta] \exp \left[i \int d^4x \left\{ \frac{1}{12} F_{\mu\nu\lambda} F^{\mu\nu\lambda} \right. \right.$$

$$\left. \left. - i \tilde{B}^{\mu*} (\partial_\mu \rho_\nu - \partial_\nu \rho_\mu) - \bar{\rho}^{\mu*} \partial_\nu \sigma + i \bar{\beta}^{\mu*} \beta_\nu - \bar{\sigma}^* \chi - \bar{\varphi}^* \chi \right\} \right]. \quad (4.29)$$

The extended action $W_{\Psi A}$ for 2-form gauge theory in axial gauge also satisfies the quantum master equation given in (2.22). The gauge-fixed fermion for axial gauge

$$\Psi^A = -i \int d^4x [\tilde{\rho}_\nu (\eta_\mu B^{\mu\nu} + \lambda_1 \beta^\nu) + \tilde{\sigma} \eta_\mu \rho^\mu + \varphi (\eta_\mu \beta^\nu - \lambda_2 \chi)]. \quad (4.30)$$

and corresponding antifields are

$$\tilde{B}^{\mu*} = \frac{\delta \psi^A}{\delta B_{\mu\nu}} = -i \eta^{\mu} \tilde{\rho}^{\nu}, \quad \tilde{\rho}^{\mu*} = \frac{\delta \psi^A}{\delta \rho_\nu} = -i (\eta_\mu B^{\mu\nu} + \lambda_1 \beta^\nu + \eta^\nu \varphi)$$

$$\tilde{\sigma}^* = \frac{\delta \psi^A}{\delta \sigma} = 0, \quad \tilde{\beta}^{\mu*} = \frac{\delta \psi^A}{\delta \beta_\nu} = -i \lambda_1 \tilde{\rho}^\nu$$

$$\tilde{\varphi}^* = \frac{\delta \psi^A}{\delta \varphi} = -i (\eta_\mu \beta^\mu - \lambda_2 \chi), \quad \tilde{\chi}^* = \frac{\delta \psi^A}{\delta \chi} = i \lambda_2 \varphi \quad (4.31)$$

Thus, the FFBRST transformation with finite parameter given in Eq. (4.10) relates the different solutions of quantum master equation in field/antifield formulation.

4.6 Conclusions

The usual BRST transformation has been generalized for the Abelian rank-2 antisymmetric tensor field theory by making the parameter finite and field dependent. Such FFBRST transformation is nilpotent and leaves the effective action invariant. However, being finite in nature such transformations do not leave the path integral measure invariant. We have shown that for certain choices of finite field dependent parameter, the Jacobian for the path integral of such
FFBRST transformation always can be written as $e^{iS_1}$, where $S_1$ is some local functional of fields and depends on the choice of the finite BRST parameter. $S_1$ can be added with $S_{\text{eff}}^\text{GL}$ to produce the new effective action. Thus, the generating functional corresponding to one effective theory is then linked to the generating functional corresponding to another effective theory through the FFBRST transformation. In this present work we have shown that the generating functional corresponding to covariant gauge viz. Lorentz gauge is connected to the generating functional in noncovariant gauges viz. axial gauge and Coulomb gauge. Thus, the generalized BRST transformation is helpful in the study of Abelian rank-2 antisymmetric tensor field theory in noncovariant gauges, which is very useful in certain situation \cite{100}. We further have considered the generalization of anti-BRST transformation and show that even FF-anti-BRST transformation can connect generating functionals for different effective theories. The FFBRST transformation is also very useful in modern approach of quantum field theory, namely field/antifield formulation. With the help of an explicit example we have shown that the different solutions of the master equation are related through FFBRST transformation in the field/antifield formulation of Abelian 2-form antisymmetric tensor field theory.
Chapter 5

FFBRST formulation for Gribov-Zwanziger (GZ) theory

In this chapter, we develop the FFBRST formulation for the multiplicatively renormalizable GZ theory which is free from Gribov ambiguity. The mapping between GZ theory and Yang-Mills (YM) theory, which contains the Gribov copies, is established in Euclidean space [63]. We further extend this formulation using BV techniques [64].

5.1 GZ theory: brief introduction

In YM theories even after gauge-fixing the redundancy of gauge fields is not completely removed in certain gauges for large gauge fields (Gribov ambiguity) [53]. In order to resolve such problem, Gribov and Zwanziger proposed a theory, which restricts the domain of integration in the functional integral within the first Gribov horizon [54]. It has been shown in Ref. [55] that the restriction to the Gribov region \( \Omega \) (defined in such a way that the Faddeev-Popov (FP) operator is strictly positive. i.e.

\[
\Omega \equiv \{ A_{\mu}^{a}, \partial_{\mu} A^{\mu a} = 0, \mathcal{M}^{ab} > 0 \} ,
\]

(5.1)
can be imposed by adding a nonlocal term \( S_h \) to the standard YM action

\[
S_{YM} = S_0 + S_{GF+FP},
\]

(5.2)
where \( S_0 \) is the kinetic part and \( S_{GF+FP} \) is the ghost and gauge(Landau gauge)-fixing part of the YM action respectively,

\[
S_0 = \int d^4 x \left[ \frac{1}{4} F_{\mu \nu}^{a} F^{\mu \nu a} \right],
\]

\[
S_{GF+FP} = \int d^4 x \left[ B^{a} \partial_{\mu} A^{\mu a} + e^{a} \partial^{a} D_{\mu}^{ab} e^{b} \right],
\]

(5.3)
and the nonlocal horizon term in 4-dimensional Euclidean space is written as

\[
S_h = \int d^4 x h(x),
\]

(5.4)
where the integrand \( h(x) \) is called the horizon function. There exist many different choices for the horizon function in literature [59]. One such horizon term is

\[
h_1(x) = \gamma^4 \int d^4 y \, g^2 f^{abc} A_{\mu}^{b}(x) (\mathcal{M}^{-1})^{ce}(x, y) f^{ade} A^{\mu e}(y).
\]

(5.5)
(M^{-1})^{ce}$ is the inverse of the Faddeev-Popov operator $M^{ab} = -\partial^\mu D^{ab}_\mu = -\partial^\mu (\partial_\mu \delta^{ab} + gf^{abc} A^c_\mu)$. The Gribov parameter $\gamma$ can be obtained in a consistent way by solving a gap equation (also known as horizon condition) \[^{54}^{55}\]

\[ \langle h(x) \rangle = 4(N^2 - 1), \tag{5.6} \]

where $N$ is the number of colors. Another horizon term which gives the correct multiplicative renormalizability of the GZ theory is given as \[^{59}\]

\[ h_2(x) = \lim_{\gamma(x) \to \gamma} \int d^4y \left[ (D^{ac}_\mu (x) \gamma^2(x)) \left( (M^{-1})^{ce}(x,y) \left( D^{\mu ac}(y) \gamma^2(x) \right) \right) \right]. \tag{5.7} \]

The nonlocal term \[^{53}\] corresponding to the horizon function \[^{57}\] can be localized as \[^{51}^{55}\]

\[ e^{-S_{h2}} = \int D\phi D\bar{\phi} D\omega D\bar{\omega} e^{S_{loc}}, \tag{5.8} \]

with

\[ S_{loc} = \int d^4x \left[ \bar{\varphi}_i^{\alpha} \partial^\mu D^{ab}_\mu \varphi_i^{\alpha b} - \bar{\omega}_i^{\alpha} \partial^\mu D^{ab}_\mu \omega_i^{\alpha b} - \gamma^2 D^{\mu ac}(\varphi_i^{ac}(x) + \bar{\varphi}_i^{ac}(x)) \right], \tag{5.9} \]

where a pair of complex conjugate bosonic field $(\varphi_i^{\alpha}, \bar{\varphi}_i^{\alpha}) = (\varphi_i^{ac}, \bar{\varphi}_i^{ac})$ and anticommuting auxiliary fields $(\omega_i^{\alpha}, \bar{\omega}_i^{\alpha}) = (\omega_i^{ac}, \bar{\omega}_i^{ac})$, with composite index $i = (\nu, c)$, have been introduced. As at the level of the action, total derivatives are always neglected, $S_{loc}$ becomes

\[ S_{loc} = \int d^4x \left[ \bar{\varphi}_i^{\alpha} \partial^\mu D^{ab}_\mu \varphi_i^{\alpha b} - \bar{\omega}_i^{\alpha} \partial^\mu D^{ab}_\mu \omega_i^{\alpha b} - \gamma^2 gf^{abc} A^{\mu a}(\varphi_i^{bc}(x) + \bar{\varphi}_i^{bc}(x)) \right]. \tag{5.10} \]

Here it is concluded that at the local level horizon functions \[^{(5.5)}\] and \[^{(5.7)}\] are same. So that the localized GZ action becomes

\[ S_{GZ} = S_{YM} + S_{loc} = S_{YM} + \int d^4x \left[ \bar{\varphi}_i^{\alpha} \partial^\mu D^{ab}_\mu \varphi_i^{\alpha b} - \bar{\omega}_i^{\alpha} \partial^\mu D^{ab}_\mu \omega_i^{\alpha b} - \gamma^2 gf^{abc} A^{\mu a}(\varphi_i^{bc}(x) + \bar{\varphi}_i^{bc}(x)) \right]. \tag{5.11} \]

Thus, the local action $S_{GZ}$ and the nonlocal action $S_{YM} + S_h$ are related as the following:

\[ \int [D\phi_1] e^{-\{S_{YM} + S_{h2}\}} = \int [D\phi] e^{-S_{GZ}}, \tag{5.12} \]

with $\int [D\phi_1] \equiv \int [DADBDCd\bar{c}]$ and $\int [D\phi] \equiv \int [DADBDCd\bar{c}D\varphi D\bar{\varphi} D\omega D\bar{\omega}]$. By differentiating Eq. \[^{(5.12)}\] with respect to $\gamma^2$ and noting $\langle \partial^\mu \varphi_\mu^{\alpha a} \rangle = \langle \partial^\mu \bar{\varphi}_\mu^{\alpha a} \rangle = 0$, the horizon condition in Eq. \[^{(5.6)}\] is recast as

\[ \left( gf^{abc} A^{\mu a}(\varphi_i^{bc}(x) + \bar{\varphi}_i^{bc}(x)) \right) + 8\gamma^2(N^2 - 1) = 0. \tag{5.13} \]

The horizon condition can further be written as \[^{51}^{52}\]

\[ \frac{\partial \Gamma}{\partial \gamma^2} = 0, \tag{5.14} \]
with \( \Gamma \), the quantum action defined as

\[
e^{-\Gamma} = \int [D\phi] e^{-S_{GZ}}.
\]  

(5.15)

We see that the horizon condition (5.14) is equivalent to

\[
\left\langle 0 \mid g f^{abc} A^{\mu a} \varphi^b_{\mu} \mid 0 \right\rangle + \left\langle 0 \mid g f^{abc} A^{\mu a} \bar{\varphi}^b_{\mu} \mid 0 \right\rangle = -8\gamma^2 (N^2 - 1),
\]

(5.16)

which, owing to the discrete symmetry of the action \( S \) becomes

\[
\varphi^a_{\mu} \rightarrow \varphi^a_{\mu}, \quad \varphi^a_{\mu} \rightarrow \bar{\varphi}^a_{\mu}, \quad B^a \rightarrow (B^a - g f^{amn} \varphi^{mc} \varphi^{mnc}),
\]

(5.17)

becomes

\[
\left\langle 0 \mid g f^{abc} A^{\mu a} \varphi^b_{\mu} \mid 0 \right\rangle = \left\langle 0 \mid g f^{abc} A^{\mu a} \bar{\varphi}^b_{\mu} \mid 0 \right\rangle = -4\gamma^2 (N^2 - 1).
\]

(5.18)

Further, the constant term \( 4\gamma^4 (N^2 - 1) \) is introduced in \( S_{GZ} \), to incorporate the effect of horizon condition in the action as

\[
S_{GZ} = S_{YM} + \int d^4x \left[ \bar{\varphi}^a_i \partial^\mu D^b_{\mu} \varphi_i^b - \bar{\omega}^{ai} \partial^\mu D^b_{\mu} \omega_i^b \right.
\]

\[
- \gamma^2 g f^{abc} A^{\mu a} (\varphi^b_{\mu} + \bar{\varphi}^b_{\mu}) - 4(N^2 - 1)\gamma^4 \right].
\]

(5.19)

For the GZ action to be renormalizable, it is crucial to shift the field \( \omega_i^a \),

\[
\omega_i^a(x) \rightarrow \omega_i^a + \int d^4y (M^{-1})^{ab}(x,y) g f^{bkl} \partial^\mu [D^b_{\mu} \bar{\epsilon}^c(y) \varphi^{tal}(y)],
\]

(5.20)

so that the complete GZ action becomes

\[
S_{GZ} = S_{YM} + \int d^4x \left[ \bar{\varphi}^a_i \partial^\mu D^b_{\mu} \varphi_i^b - \bar{\omega}^{ai} \partial^\mu D^b_{\mu} \omega_i^b - g f^{abc} \partial^\mu \bar{\varphi}^{ai} D^b_{\mu} \bar{\epsilon}^c \right.
\]

\[
- \gamma^2 \left( f^{abc} A^{\mu a} \varphi^b_{\mu} + f^{abc} A^{\mu a} \bar{\varphi}^b_{\mu} + \frac{4}{g} (N^2 - 1)\gamma^2 \right),
\]

(5.21)

which has been shown to be multiplicative renormalizable [59].

### 5.2 The nilpotent BRST transformation of GZ action

The complete GZ action after localizing the nonlocal horizon term in D dimensional Euclidean space can be recast as

\[
S_{GZ} = S_{\text{exact}} + S_{\gamma}
\]

(5.22)

with \( S_{\text{exact}} \), the BRST exact action and \( S_{\gamma} \), the action for horizon term, defined as [59]

\[
S_{\text{exact}} = S_{YM} + \int d^4x \left[ \bar{\varphi}^a_i \partial^\mu D^b_{\mu} \varphi_i^b - \bar{\omega}^{ai} \partial^\mu D^b_{\mu} \omega_i^b - g f^{abc} \partial^\mu \bar{\varphi}^{ai} D^b_{\mu} \bar{\epsilon}^c \right],
\]

\[
S_{\gamma} = -\gamma^2 g \int d^4x \left[ f^{abc} A^{\mu a} \varphi^b_{\mu} + f^{abc} A^{\mu a} \bar{\varphi}^b_{\mu} + \frac{4}{g} (N^2 - 1)\gamma^2 \right].
\]

(5.23)
5.2. The nilpotent BRST transformation of GZ action

The conventional BRST transformation for all the fields is given by

\[ \delta_b A^a_\mu = D_\mu^{ab} e_b^a \Lambda, \quad \delta_b c^a = \frac{1}{2} g f^{abc} e_b^a c^c \Lambda, \quad \delta_b \varphi^a = B^a \Lambda, \quad \delta_b B^a = 0, \]

\[ \delta_b \varphi^a_i = -\omega^a_i \Lambda, \quad \delta_b \omega^a_i = 0, \quad \delta_b \bar{\omega}^a_i = \bar{\varphi}^a_i \Lambda, \quad \delta_b \bar{\varphi}^a_i = 0, \]

(5.24)

where \( \Lambda \) is usual infinitesimal BRST parameter. But one can check that the BRST symmetry is broken softly for the GZ action [54],

\[ \delta_b S_{GZ} = \delta_b (S_{exact} + S_\gamma) = \delta_b S_\gamma \]

\[ = \gamma^2 g \int d^4 x f^{abc} \left( A^{\mu a} \omega^{bc}_\mu - (D^{\mu a m} e^m)(\bar{\varphi}^{bc} + \varphi^{bc}_\mu) \right), \]

(5.25)

the breaking is due to the presence of \( \gamma \) dependent term, \( S_\gamma \).

To discuss the renormalizability of \( S_{GZ} \), \( S_\gamma \) is embedded into a larger action with 3 doublets of sources \( (U_\mu^a, M_\mu^a), (V^a_i, N_i^a) \) and \( (T_\mu^a, R_\mu^a) \) as [59]

\[ \Sigma_\gamma = \delta_b \int d^4 x \left( -U_\mu^a D^{\mu ab} \varphi_i^b - V^a_i D^{\mu ab} \omega_i^b - U_\mu^a V_i^a + g f^{abc} T^a_\mu D^{\mu bd} \bar{e}^{d} \bar{\omega}^c_i \right) \]

\[ = \int d^4 x \left( -M_\mu^a D^{\mu ab} \varphi_i^b - g f^{abc} U_\mu^a D^{\mu bd} \bar{e}^{d} \bar{\omega}^c_i + U_\mu^a D^{\mu ab} \omega_i^b \right) \]

\[ - N_i^a D^{\mu ab} \omega_i^b - V^a_i D^{\mu ab} \varphi_i^b + g f^{abc} V^a_i D^{\mu bd} \bar{e}^{d} \bar{\omega}^c_i \]

\[ - M_\mu^a V_i^a + U_\mu^a N_i^a - g f^{abc} R^a_\mu D^{\mu bd} \bar{e}^{d} \bar{\omega}^c_i + g f^{abc} T^a_\mu D^{\mu bd} \bar{e}^{d} \bar{\omega}^c_i \), \]

(5.26)

whereas the sources involved \( M_\mu^a, V^a_i, R^a_\mu \) are commuting and \( U_\mu^a, N_i^a, T^a_\mu \) are fermionic in nature. The above action is invariant under following BRST transformation:

\[ \delta_b U_\mu^a = M_\mu^a \Lambda, \quad \delta_b M_\mu^a = 0, \quad \delta_b V_i^a = -N_i^a \Lambda, \]

\[ \delta_b N_i^a = 0, \quad \delta_b T^a_\mu = -p^a_\mu \Lambda, \quad \delta_b R^a_\mu = 0. \]

(5.27)

Therefore, the BRST symmetry has been restored at the cost of introducing new sources. The different quantum numbers (to study the system properly) of fields and sources, involved in this theory, are discussed in Ref. [59]. However, we do not want to change our original theory [5.23] and therefore we choose the sources to have the following values at the end:

\[ U_\mu^a|_{phys} = N_i^a|_{phys} = T^a_\mu|_{phys} = 0 \]

\[ M_\mu^a|_{phys} = V^a_i|_{phys} = R^a_\mu|_{phys} = \gamma^2 g f^{abc} \delta^{ab} \delta_{\mu \nu}. \]

(5.28)

It follows that \( \Sigma_\gamma|_{phys} = S_\gamma \).

The generating functional for the effective GZ action in Euclidean space is defined as

\[ Z_{GZ} = \int [D \phi] e^{-S_{GZ}}, \]

(5.29)

where \( \phi \) is generic notation for all fields used in GZ action.
5.3 FFBRST transformation in Euclidean space

The Jacobian, $J(\kappa)$, of the path integral measure (as given in Eq. (2.7)) in the Euclidean space can be replaced (within the functional integral) as

$$J(\kappa) \rightarrow \exp[-S_1[\phi(x,\kappa)]]$$

iff the following condition is satisfied [35]:

$$\int [D\phi(x)] \left[ \frac{1}{J} \frac{dJ}{d\kappa} + \frac{dS_1[\phi(x,\kappa)]}{d\kappa} \right] \exp[-(S_{GZ} + S_1)] = 0$$

(5.31)

where $S_1[\phi]$ is local functional of fields. The infinitesimal change in the $J(\kappa)$ can be calculated using Eq. (2.13).

Now, we generalize the BRST transformation given in Eqs. (5.24) and (5.27) by making usual BRST parameter finite and field dependent as

$$\delta_b A^a_\mu = D^{ab}_\mu c^b \Theta_b, \quad \delta_b c^a = \frac{1}{2} \epsilon^{abc} c^b c^c \Theta_b, \quad \delta_b \bar{c}^a = B^a_c \Theta_b,$$

$$\delta_b \varphi_1^a = -\omega_1^a \Theta_b, \quad \delta_b \bar{\varphi}_1^a = \bar{\varphi}_1^a \Theta_b, \quad \delta_b U_{\mu}^{ai} = M_{\mu}^{ai} \Theta_b, \quad \delta_b V_{\mu}^{ai} = -N_{\mu}^{ai} \Theta_b,$$

$$\delta_b T_{\mu}^{ai} = -R_{\mu}^{ai} \Theta_b,$$

(5.32)

where $\Theta_b$ is finite, field dependent, anticommuting and space-time independent parameter. One can easily check that the above FFBRST transformation is also symmetry of the effective GZ action ($S_{GZ}$).

5.4 A mapping between GZ theory and YM theory

In this section we establish the connection between the theories with GZ action and YM action by using finite field dependent BRST transformation. In particular, we show that the generating functional for GZ theory in path integral formulation is directly related to that of YM theory with proper choice of finite field dependent BRST transformation. The nontrivial Jacobian of the path integral measure is responsible for such a connection. For this purpose we choose a finite field dependent parameter $\Theta_b$ obtainable from

$$\Theta_b' = \int d^4x \left[ \varphi_1^a \partial^\mu D^{ab}_\mu \varphi_1^b - U_{\mu}^{ai} D^{ab}_\mu \varphi_1^b - V_{\mu}^{ai} D^{ab}_\mu \varphi_1^b - U_{\mu}^{ai} V_{\mu}^{ai} + T^{ai} g f^{abc} D^{bd}_\mu c^d \omega_1^c \right],$$

(5.33)

using Eq. (2.25). The infinitesimal change in Jacobian for above $\Theta_b'$ using Eq. (2.13) is calculated as

$$\frac{1}{J} \frac{dJ}{d\kappa} = - \int d^4x \left[ -\varphi_1^a \partial^\mu D^{ab}_\mu \varphi_1^b + \bar{\varphi}_1^a \partial^\mu D^{ab}_\mu \omega_1^b + g f^{abc} \partial^\mu \omega_1^a D^{bd}_\mu c^d \varphi_1^c + M_{\mu}^{ai} D^{ab}_\mu \varphi_1^b - U_{\mu}^{ai} D^{ab}_\mu \omega_1^b + g f^{abc} U_{\mu}^{ai} D^{bd}_\mu c^d \varphi_1^c + M_{\mu}^{ai} V_{\mu}^{ai} - U_{\mu}^{ai} M_{\mu}^{ai} + g f^{abc} R_{\mu}^{ai} D^{bd}_\mu c^d \varphi_1^c - g f^{abc} T_{\mu}^{ai} D^{bd}_\mu c^d \varphi_1^c \right].$$

(5.34)
Now, the Jacobian for path integral measure in the generating functional \([5.29]\) can be replaced by \(e^{-S_1}\) if condition \([5.31]\) is satisfied. We consider an ansatz for \(S_1\) as

\[
S_1 = \int d^4 x \left[ \frac{\beta}{\alpha} \partial^\mu \partial^\nu \partial^\rho \partial^\sigma \partial^\lambda \partial^\mu \partial^\nu \partial^\rho \partial^\sigma \partial^\lambda \phi^a \partial^a \phi^b \partial^b \phi^c \partial^c \phi^d \partial^d \phi^e \partial^e \phi^f \partial^f \phi^g \partial^g \phi^h \partial^h \phi^i \partial^i \phi^j \partial^j \phi^k \partial^k \phi^l \partial^l \phi^m \partial^m \phi^n \partial^n \phi^o \partial^o \phi^p \partial^p \phi^q \partial^q \phi^r \partial^r \phi^s \partial^s \phi^t \partial^t \phi^u \partial^u \phi^v \partial^v \phi^w \partial^w \phi^x \partial^x \phi^y \partial^y \phi^z \partial^z \phi \right] + \frac{1}{2} \sum_{\nu=1}^{N} \phi^a \partial^a \phi^b \partial^b \phi^c \partial^c \phi^d \partial^d \phi^e \partial^e \phi^f \partial^f \phi^g \partial^g \phi^h \partial^h \phi^i \partial^i \phi^j \partial^j \phi^k \partial^k \phi^l \partial^l \phi^m \partial^m \phi^n \partial^n \phi^o \partial^o \phi^p \partial^p \phi^q \partial^q \phi^r \partial^r \phi^s \partial^s \phi^t \partial^t \phi^u \partial^u \phi^v \partial^v \phi^w \partial^w \phi^x \partial^x \phi^y \partial^y \phi^z \partial^z \phi \]

where \(\chi_j(\kappa) (j = 1, 2, \ldots, 13)\) are arbitrary constants which depend on the parameter \(\kappa\) and satisfy following initial conditions:

\[
\chi_j(\kappa = 0) = 0. \tag{5.36}
\]

The condition \([5.31]\) with the above \(S_1\) leads to

\[
\int [D\phi] e^{-(S_{eff} + S_1)} \left[ \frac{\beta}{\alpha} \partial^\mu \partial^\nu \partial^\rho \partial^\sigma \partial^\lambda \partial^\mu \partial^\nu \partial^\rho \partial^\sigma \partial^\lambda \phi^a \partial^a \phi^b \partial^b \phi^c \partial^c \phi^d \partial^d \phi^e \partial^e \phi^f \partial^f \phi^g \partial^g \phi^h \partial^h \phi^i \partial^i \phi^j \partial^j \phi^k \partial^k \phi^l \partial^l \phi^m \partial^m \phi^n \partial^n \phi^o \partial^o \phi^p \partial^p \phi^q \partial^q \phi^r \partial^r \phi^s \partial^s \phi^t \partial^t \phi^u \partial^u \phi^v \partial^v \phi^w \partial^w \phi^x \partial^x \phi^y \partial^y \phi^z \partial^z \phi \right] + \frac{1}{2} \sum_{\nu=1}^{N} \phi^a \partial^a \phi^b \partial^b \phi^c \partial^c \phi^d \partial^d \phi^e \partial^e \phi^f \partial^f \phi^g \partial^g \phi^h \partial^h \phi^i \partial^i \phi^j \partial^j \phi^k \partial^k \phi^l \partial^l \phi^m \partial^m \phi^n \partial^n \phi^o \partial^o \phi^p \partial^p \phi^q \partial^q \phi^r \partial^r \phi^s \partial^s \phi^t \partial^t \phi^u \partial^u \phi^v \partial^v \phi^w \partial^w \phi^x \partial^x \phi^y \partial^y \phi^z \partial^z \phi \]

where prime denotes the differentiation with respect to the parameter \(\kappa\). Equating the coefficient of terms \(\frac{\beta}{\alpha} \partial^\mu \partial^\nu \partial^\rho \partial^\sigma \partial^\lambda \partial^\mu \partial^\nu \partial^\rho \partial^\sigma \partial^\lambda \phi^a \partial^a \phi^b \partial^b \phi^c \partial^c \phi^d \partial^d \phi^e \partial^e \phi^f \partial^f \phi^g \partial^g \phi^h \partial^h \phi^i \partial^i \phi^j \partial^j \phi^k \partial^k \phi^l \partial^l \phi^m \partial^m \phi^n \partial^n \phi^o \partial^o \phi^p \partial^p \phi^q \partial^q \phi^r \partial^r \phi^s \partial^s \phi^t \partial^t \phi^u \partial^u \phi^v \partial^v \phi^w \partial^w \phi^x \partial^x \phi^y \partial^y \phi^z \partial^z \phi \)

from both sides of above condition, we get following differential equations:

\[
\begin{align*}
\chi_1' + 1 &= 0, \quad \chi_2' - 1 = 0, \quad \chi_3' - 1 = 0, \quad \chi_4' - 1 = 0, \\
\chi_5' - 1 &= 0, \quad \chi_6' + 1 = 0, \quad \chi_7' - 1 = 0, \quad \chi_8' - 1 = 0, \\
\chi_9' + 1 &= 0, \quad \chi_{10}' - 1 = 0, \quad \chi_{11}' + 1 = 0, \quad \chi_{12}' - 1 = 0, \\
\chi_{13}' + 1 &= 0.
\end{align*} \tag{5.38}
\]
The Θ′_b dependent terms will be cancelled separately and comparing the coefficients of Θ′_b dependent terms, we obtain

\[ \chi_1 + \chi_2 = \chi_1 + \chi_3 = \chi_2 - \chi_3 = \chi_4 - \chi_5 = 0 \]
\[ \chi_4 + \chi_6 = \chi_5 + \chi_6 = \chi_7 - \chi_8 = \chi_7 + \chi_9 = 0 \]
\[ \chi_8 + \chi_9 = \chi_{10} + \chi_{11} = \chi_{12} + \chi_{13} = 0. \]  
(5.39)

The particular solution of Eq. (5.38) subjected to the condition (5.36) and Eq. (5.39) is

\[ \chi_1 = -\kappa, \quad \chi_2 = \kappa, \quad \chi_3 = \kappa, \quad \chi_4 = \kappa \]
\[ \chi_5 = \kappa, \quad \chi_6 = -\kappa, \quad \chi_7 = \kappa, \quad \chi_8 = \kappa \]
\[ \chi_9 = -\kappa, \quad \chi_{10} = \kappa, \quad \chi_{11} = -\kappa, \quad \chi_{12} = \kappa \]
\[ \chi_{13} = -\kappa. \]  
(5.40)

Therefore, the expression for S_1 in term of κ is

\[ S_1 = \int d^4x \left[ -\kappa \varphi^a_i \partial^\mu D_{\mu}^{ab} \varphi_i^b + \kappa \bar{\omega}^a_i \partial^\mu D_{\mu}^{ab} \omega_i^b + \kappa g f^{abc} \partial^\mu \omega^a_i D_{\mu}^{bd} c^d \varphi_i^c \right. \]
\[ + \kappa M_{\mu}^{ab} \omega_i^b + \kappa U_{\mu}^{ai} D_{\mu}^{ab} \omega_i^b - \kappa g f^{abc} U_{\mu}^{ai} D_{\mu}^{bd} c^d \varphi_i^c \]
\[ + \kappa N_{\mu}^{ai} D_{\mu}^{ab} \omega_i^b + \kappa V_{\mu}^{ai} D_{\mu}^{ab} \omega_i^b - \kappa g f^{abc} V_{\mu}^{ai} D_{\mu}^{bd} c^d \omega_i^c + \kappa M_{\mu}^{ai} V_{\mu}^{ia} \]
\[ - \kappa U_{\mu}^{ai} N_{\mu}^{ia} + \kappa g f^{abc} R_{\mu}^{ai} D_{\mu}^{bd} c^d \omega_i^c - \kappa g f^{abc} T_{\mu}^{ai} D_{\mu}^{bd} c^d \omega_i^c \left]. \right] \]  
(5.41)

The transformed action is obtained by adding S_1(κ = 1) to S_GZ as,

\[ S_GZ + S_1 = \int d^4x \left[ \frac{1}{4} F_{\mu \nu}^a F^{\mu \nu a} + B^a \partial^\mu A_\mu^a + \bar{c}^a \partial^\mu D_{\mu}^{ab} c^b \right]. \]  
(5.42)

We left with the YM effective action in Landau gauge.

\[ S_GZ + S_1 = S_{YM}. \]  
(5.43)

Note the new action is independent of horizon parameter γ, and hence horizon condition \( \left( \frac{\partial \Gamma}{\partial \gamma} = 0 \right) \) leads trivial relation for S_{YM}. Thus, using FFBRST transformation we have mapped the generating functionals in Euclidean space as

\[ Z_{YM} = \left( \int [D\phi] e^{-S_{YM}} \right) \xrightarrow{FFBRST} Z_{YM} = \left( \int [D\phi] e^{-S_{YM}} \right), \]  
(5.44)

where Z_{YM} is the generating functional for Yang-Mils action S_{YM}.

### 5.5 Connecting GZ theory and YM theory in BV formalism

The generating functional of YM theory in the BV formulation can be written by introducing antifields φ* corresponding to the all fields φ with opposite statistics as,

\[ Z_{YM} = \int [D\phi]^c e^{-\int d^4x \left( \frac{1}{4} F_{\mu \nu}^a F^{\mu \nu a} + A_\mu^a D_{\mu}^{ab} c^b + \bar{c}^a B^b \right)}, \]  
(5.45)
Thus, using FFBRST transformation we connect the GZ theory to YM theory in BV formulation.

The antifields can be evaluated from $\Psi^*$ as

$$A^a_{\mu} = \frac{\delta \Psi_1}{\delta A^a_{\mu}} = -\partial_\mu \bar{c}^a, \quad \bar{c}^{a*} = \frac{\delta \Psi_1}{\delta \bar{c}^a} = \partial_\mu A^a_{\mu}, \quad B^{a*} = \frac{\delta \Psi_1}{\delta B^a} = 0.$$  

Similarly, the generating functional of GZ theory in BV formulation can be written as,

$$Z_{GZ} = \int [D\phi] e^{-[W_{\Psi_2}(\phi, \phi^*)]}, \quad Z_{YM} = \int [D\phi] e^{-[W_{\Psi_1}(\phi, \phi^*)]}.$$  

The antifields are obtained from $\Psi_1$ as

$$A^a_{\mu} = \partial_\mu e^a - g f^{abc} \partial_\mu \bar{c}^b \varphi^c_{\mu} - g f^{abc} U^a_{\mu} \varphi^c_{\mu} - g f^{abc} V^a_{\mu} \bar{w}^b_{\mu} \bar{c}^c_{\mu}, \quad \bar{c}^{a*} = \partial_\mu A^a_{\mu},$$

$$U^{ai*} = -D^a_{\mu} \varphi^i_{\mu} - V^a_{\mu} \bar{w}^i_{\mu}, \quad \bar{w}^{ai*} = \partial_\mu D^a_{\mu} \varphi^i_{\mu} - V^a_{\mu} D^b_{\mu} \varphi^i_{\mu} + g f^{abc} T^a_{\mu} D^b_{\mu} \bar{c}^c_{\mu},$$

and

$$V^{ai*} = -D^a_{\mu} \bar{w}^i_{\mu} - U^a_{\mu} T^{ai*} = g f^{abc} T^a_{\mu} D^b_{\mu} \varphi^i_{\mu} + U^a_{\mu} D^b_{\mu} \bar{c}^c_{\mu} \varphi^i_{\mu}.$$

To connect these two theories we construct the following finite field dependent parameter $\Theta_b[\phi]$:

$$\Theta_b[\phi, \phi^*] = \int_0^1 d\kappa \int d^4x \left[ \varphi^b_{\mu} \varphi^i_{\mu} + \bar{w}^b_{\mu} \bar{c}^c_{\mu} \bar{w}^i_{\mu} + V^{ai*} \varphi^i_{\mu} \right].$$  

The Jacobian of path integral measure in the generating functional \textcolor{red}{5.49} for the FFBRST with this parameter can be replaced by $e^{-S_1}$ iff condition \textcolor{red}{5.51} is satisfied. To find $S_1$ we start with an ansatz for $S_1$ as

$$S_1 = \int d^4x \left[ \chi_1 \varphi^b_{\mu} \varphi^i_{\mu} + \chi_2 \varphi^a \bar{w}^i_{\mu} + \chi_3 U^{ai*} M^{i\mu} + \chi_4 V^{ai*} N^{i\mu} + \chi_5 T^{ai*} R^{i\mu} \right].$$

where $\chi_j(\kappa)$ ($j = 1, 2, \ldots, 5$) are arbitrary but $\kappa$-dependent constants and satisfy the following initial conditions: $\chi_j(\kappa = 0) = 0$. These constants are calculated using Eq. \textcolor{red}{5.31} subjected to the initial condition to find the $S_1$ as

$$S_1 = \int d^4x \left[ \kappa \varphi^b_{\mu} \varphi^i_{\mu} - \kappa \varphi^a \bar{w}^i_{\mu} - \kappa U^{ai*} M^{i\mu} + \kappa V^{ai*} N^{i\mu} + \kappa T^{ai*} R^{i\mu} \right].$$

By adding $S_1(\kappa = 1)$ to $S_{GZ}$, we get $S_{GZ} + S_1(\kappa = 1) = S_{YM}$. Hence,

$$Z_{GZ} = \int [D\phi] e^{-W_{\Psi_2}} \xrightarrow{\text{FFBRST}} Z_{YM} = \int [D\phi] e^{-W_{\Psi_1}(\phi, \phi^*)}. $$

Thus, using FFBRST transformation we connect the GZ theory to YM theory in BV formulation.
5.6 Conclusions

The GZ theory which is free from Gribov copies as the domain of integration is restricted to the first Gribov horizon, is not invariant under usual BRST transformation due to the presence of the nonlocal horizon term. Hence the KO criterion for color confinement in a manifestly covariant gauge fails for GZ theory. A nilpotent BRST transformation which leaves GZ action invariant was developed recently and can be applied to KO analysis for color confinement. This nilpotent BRST symmetry is generalized by allowing the transformation parameter finite and field dependent. This generalized BRST transformation is nilpotent and symmetry of the GZ effective action. We have shown that this nilpotent BRST with an appropriate choice of finite field dependent parameter relates the GZ theory with a correct horizon term to the YM theory in Euclidean space where horizon condition becomes a trivial one. We have shown the same connection in BV formulation also by considering the appropriate finite parameter. Thus, we have shown that the theory, free from Gribov copies (i.e. GZ theory with proper horizon term), is related through a nilpotent BRST transformation with a finite parameter to a theory with Gribov copies (i.e. YM theory in Euclidean space). The nontrivial Jacobian of such finite transformation is responsible for this important connection. This implies our formulation is very useful for the better understanding of Gribov ambiguity.
Chapter 6

Finite nilpotent symmetry for gauge theories

Earlier it was shown that FFBRST and FF-anti-BRST transformations are symmetry of the effective action but not of the generating functional. In this chapter we construct a nilpotent finite BRST transformation which leaves both the effective action and generating functional invariant \[65\]. To construct such a finite transformation we combine usual BRST and anti-BRST transformations with finite parameters. The field/antifield (or BV) formulation in the context such transformation is also studied in this chapter.

6.1 The infinitesimal mixed BRST (MBRST) transformation

The generating functional for the Green’s function in an effective theory described by the effective action \[S_{\text{eff}}[\phi]\] is defined as

\[
Z = \int [\mathcal{D}\phi] \, e^{iS_{\text{eff}}[\phi]},
\]

(6.1)

\[
S_{\text{eff}}[\phi] = S_0[\phi] + S_{gf}[\phi] + S_{gh}[\phi],
\]

(6.2)

where \(\phi\) is the generic notation for all fields involved in the effective theory. The infinitesimal BRST (\(\delta_b\)) and anti-BRST (\(\delta_{ab}\)) transformations are defined as

\[
\delta_b\phi = s_b\phi \delta\Lambda_1, \quad s_b^2 = 0
\]

(6.3)

\[
\delta_{ab}\phi = s_{ab}\phi \delta\Lambda_2, \quad s_{ab}^2 = 0
\]

(6.4)

where \(\delta\Lambda_1\) and \(\delta\Lambda_2\) are infinitesimal, anticommuting but global parameters. Such transformations leave the generating functional as well as effective action invariant, separately, as

\[
\delta_b Z = 0 = \delta_b S_{\text{eff}},
\]

(6.5)

\[
\delta_{ab} Z = 0 = \delta_{ab} S_{\text{eff}}.
\]

(6.6)

This implies that the effective action \(S_{\text{eff}}[\phi]\) and the generating functional are also invariant under the MBRST \((\delta_m = \delta_b + \delta_{ab})\) transformation

\[
\delta_m Z = 0 = \delta_m S_{\text{eff}}.
\]

(6.7)

Further such MBRST transformation is nilpotent because,

\[
\{s_b, s_{ab}\} = 0.
\]

(6.8)
Now, in the next section we construct the finite version of the following infinitesimal MBRST symmetry transformation:

\[ \delta_m \phi = s_b \phi \delta \Lambda_1 + s_{ab} \phi \delta \Lambda_2. \] (6.9)

### 6.2 Construction of finite field dependent MBRST (FFMBRST) transformation

To construct the FFMBRST transformation, we follow the similar method of constructing FF-BRST transformation [27]. However, in this case unlike FFBRST transformation we have to deal with two parameters, one for the BRST transformation and the other for anti-BRST transformation. We introduce a numerical parameter \( \kappa \) (0 \( \leq \) \( \kappa \) \( \leq \) 1) and make all the fields \( \phi(x, \kappa) \) \( \kappa \)-dependent in such a way that \( \phi(x, \kappa = 0) \equiv \phi(x) \) and \( \phi(x, \kappa = 1) \equiv \phi'(x) \), the transformed field. Further, we make the infinitesimal parameters \( \delta \Lambda_1 \) and \( \delta \Lambda_2 \) field dependent as

\[ \delta \Lambda_1 = \Theta'_b[\phi(x, \kappa)]d \kappa \] (6.10)
\[ \delta \Lambda_2 = \Theta'_{ab}[\phi(x, \kappa)]d \kappa, \] (6.11)

where the prime denotes the derivative with respect to \( \kappa \) and \( \Theta'_i[\phi(x, \kappa)] (i = b, ab) \) are infinitesimal field dependent parameters. The infinitesimal but field dependent MBRST transformation, thus can be written generically as

\[ \frac{d\phi(x, \kappa)}{d \kappa} = s_b \phi(x, \kappa) \Theta'_b[\phi(x, \kappa)] + s_{ab} \phi(x, \kappa) \Theta'_{ab}[\phi(x, \kappa)]. \] (6.12)

Following the work in Ref.[27] it can be shown that the parameters \( \Theta'_i[\phi(x, \kappa)] (i = b, ab) \), contain the factors \( \Theta'_i[\phi(x, 0)] (i = b, ab) \), which are considered to be nilpotent. Thus \( \kappa \) dependency from \( \delta_b \phi(x, \kappa) \) and \( \delta_{ab} \phi(x, \kappa) \) can be dropped. Then Eq. (6.12) can be written as

\[ \frac{d\phi(x, \kappa)}{d \kappa} = s_b \phi(x, 0) \Theta'_b[\phi(x, \kappa)] + s_{ab} \phi(x, 0) \Theta'_{ab}[\phi(x, \kappa)]. \] (6.13)

The FFMBRST transformation with the finite field dependent parameters then can be constructed by integrating such infinitesimal transformations from \( \kappa = 0 \) to \( \kappa = 1 \), such that

\[ \phi' \equiv \phi(x, \kappa = 1) = \phi(x, \kappa = 0) + s_b \phi(x) \Theta_b[\phi(x)] + s_{ab} \phi(x) \Theta_{ab}[\phi(x)], \] (6.14)

where

\[ \Theta_i[\phi(x)] = \int^1_0 d \kappa \Theta'_i[\phi(x, \kappa')], \] (6.15)

are the finite field dependent parameters with \( i = b, ab \).

Therefore, the FFMBRST transformation corresponding to MBRST transformation mentioned in Eq. (6.9) is given by

\[ \delta_m \phi = s_b \phi \Theta_b + s_{ab} \phi \Theta_{ab} \] (6.16)

It can be shown that above FFMBRST transformation with some specific choices of the finite parameters \( \Theta_b \) and \( \Theta_{ab} \) is the symmetry transformation of the both effective action and the generating functional as the path integral measure is invariant under such transformation.
6.3 Method for evaluating the Jacobian

To show the invariance of the generating functional we need to calculate the Jacobian of the path integral measure in the expression of generating functional. The Jacobian of the path integral measure for FFMBRST transformation \( J \) can be evaluated for some particular choices of the finite field dependent parameters \( \Theta_b[\phi(x)] \) and \( \Theta_{ab}[\phi(x)] \). We start with the definition,

\[
D\phi = J(\kappa) \frac{d\phi(\kappa)}{d\kappa} = J(\kappa + d\kappa) \frac{d\phi(\kappa + d\kappa)}{d\kappa},
\]

(6.17)

Now the transformation from \( \phi(\kappa) \) to \( \phi(\kappa + d\kappa) \) is infinitesimal in nature, thus the infinitesimal change in Jacobian can be calculated as

\[
\frac{J(\kappa)}{J(\kappa + d\kappa)} = \int d^4x \sum_{\phi} \frac{\pm \delta\phi(x,\kappa)}{\delta\phi(x,\kappa + d\kappa)}
\]

(6.18)

where \( \Sigma_\phi \) sums over all fields involved in the path integral measure and \( \pm \) sign refers to whether \( \phi \) is a bosonic or a fermionic field. Using the Taylor expansion we calculate the above expression as

\[
1 - \frac{1}{J(\kappa)} \frac{dJ(\kappa)}{d\kappa} d\kappa = 1 + \int d^4x \sum_{\phi} \left[ \pm s_b \phi(x,\kappa) \frac{\delta\Theta'_b[\phi(x,\kappa)]}{\delta\phi(x)} \right. \\
\left. \pm s_{ab} \phi(x,\kappa) \frac{\delta\Theta'_{ab}[\phi(x,\kappa)]}{\delta\phi(x)} \right] d\kappa.
\]

(6.19)

The Jacobian, \( J(\kappa) \), can be replaced (within the functional integral) as

\[
J(\kappa) \rightarrow e^{i(S_1[\phi] + S_2[\phi])}
\]

(6.20)

iff the following condition is satisfied:

\[
\int [D\phi(x)] \left[ 1 + \frac{dS_1[\phi(x,\kappa)]}{d\kappa} - i \frac{dS_2[\phi(x,\kappa)]}{d\kappa} \right] e^{i(S_{eff} + S_1 + S_2)} = 0,
\]

(6.21)

where \( S_1[\phi] \) and \( S_2[\phi] \) are some local functionals of fields and satisfy the initial condition

\[
S_i[\phi(\kappa = 0)] = 0, \quad i = 1, 2.
\]

(6.22)

The finite parameters \( \Theta_b \) and \( \Theta_{ab} \) are arbitrary and we can construct them in such a way that the infinitesimal change in Jacobian \( J \) (Eq. (6.19)) with respect to \( \kappa \) vanishes

\[
\frac{1}{J} \frac{dJ}{d\kappa} = 0.
\]

(6.23)

Therefore, with the help of Eqs. (6.21) and (6.23), we see that

\[
\frac{dS_1[\phi(x,\kappa)]}{d\kappa} + S_2[\phi(x,\kappa)] = 0.
\]

(6.24)

It means the \( S_1 + S_2 \) is independent of \( \kappa \) (fields) and must vanish to satisfy the initial condition given in Eq. (6.22) satisfied. Hence the generating functional is not affected by the Jacobian \( J \) as \( J = e^{i(S_1 + S_2)} = 1 \). The nontrivial Jacobian arising from finite BRST parameter \( \Theta_b \) compensates the same arising due to finite anti-BRST parameter \( \Theta_{ab} \). It is straightforward to see that the effective action \( S_{eff} \) is invariant under such FFMBRSRT transformation.
6.4 Examples

To demonstrate the results obtained in the previous section we would like to consider several explicit examples in 1D as well as in 4D. In particular we consider the bosonized self-dual chiral model in 2D, Maxwell’s theory in 4D, and non-Abelian YM theory in 4D. In all these cases we construct explicit finite parameters $\Theta_b$ and $\Theta_{ab}$ of FFMBRST transformation such that the generating functional remains invariant.

### 6.4.1 Bosonized chiral model

We start with the generating functional for the bosonized self-dual chiral model as [66, 67]

$$Z_{CB} = \int [D\phi] \, e^{iS_{CB}}, \quad (6.25)$$

where $[D\phi]$ is the path integral measure in generic notation. The effective action $S_{CB}$ in 2D is given as

$$S_{CB} = \int d^2x \left[ \pi \varphi \dot{\varphi} + \pi \vartheta \dot{\vartheta} + p_u \dot{u} - \frac{1}{2} \pi^2 + \frac{1}{2} \pi^2 \pi(\varphi' - \vartheta' + \lambda) + \pi \varphi \lambda + \frac{1}{2} B^2 + B(\dot{\lambda} - \varphi - \vartheta) + \dot{\bar{c}} - 2\bar{c}c \right], \quad (6.26)$$

where the fields $\varphi, \vartheta, u, B, c$ and $\bar{c}$ are the self-dual field, Wess-Zumino field, multiplier field, auxiliary field, ghost field and antighost field, respectively. The nilpotent BRST and anti-BRST transformations for this theory are

**BRST:**

$$\begin{align*}
\delta_b \varphi &= c \delta \Lambda_1, \quad \delta_b \lambda = -\dot{c} \delta \Lambda_1, \quad \delta_b \vartheta = c \delta \Lambda_1, \\
\delta_b \pi \varphi &= 0, \quad \delta_b u = 0, \quad \delta_b \pi \vartheta &= 0, \quad \delta_b \bar{c} = B \delta \Lambda_1, \\
\delta_b B &= 0, \quad \delta_b c = 0, \quad \delta_b p_u = 0,
\end{align*} \quad (6.27)$$

**anti-BRST:**

$$\begin{align*}
\delta_{ab} \varphi &= -\bar{c} \delta \Lambda_2, \quad \delta_{ab} \lambda = \dot{\bar{c}} \delta \Lambda_2, \quad \delta_{ab} \vartheta = -\bar{c} \delta \Lambda_2, \\
\delta_{ab} \pi \varphi &= 0, \quad \delta_{ab} u = 0, \quad \delta_{ab} \pi \vartheta &= 0, \quad \delta_{ab} c = B \delta \Lambda_2, \\
\delta_{ab} B &= 0, \quad \delta_{ab} \bar{c} = 0, \quad \delta_{ab} p_u = 0,
\end{align*} \quad (6.28)$$

where $\delta \Lambda_1$ and $\delta \Lambda_2$ are infinitesimal, anticommuting and global parameters. Note that $s_b$ and $s_{ab}$ are absolutely anticommuting i.e. $(s_b s_{ab} + s_{ab} s_b) \phi = 0$. In this case the MBRST symmetry transformation ($\delta_m \equiv \delta_b + \delta_{ab}$), as constructed in section II, reads

$$\begin{align*}
\delta_m \varphi &= c \delta \Lambda_1 - \bar{c} \delta \Lambda_2, \quad \delta_m \lambda = -\dot{c} \delta \Lambda_1 + \dot{\bar{c}} \delta \Lambda_2, \\
\delta_m \vartheta &= c \delta \Lambda_1 - \bar{c} \delta \Lambda_2, \quad \delta_m \pi \varphi = 0, \quad \delta_m u = 0, \\
\delta_m \pi \vartheta &= 0, \quad \delta_m \bar{c} = B \delta \Lambda_1, \quad \delta_m B = 0, \quad \delta_m c = B \delta \Lambda_2, \\
\delta_m p_u &= 0.
\end{align*} \quad (6.29)$$
The FFMBRST transformation corresponding to the above MBRST transformation is constructed as
\[
\begin{align*}
\delta_m \varphi &= c \Theta_b - \dot{c} \Theta_{ab}, \quad \delta_m \lambda = -\dot{c} \Theta_b + \dot{c} \Theta_{ab}, \quad \delta_m \vartheta = c \Theta_b - \ddot{c} \Theta_{ab} \\
\delta_m \pi &= 0, \quad \delta_m u = 0, \quad \delta_m \pi_\theta = 0, \quad \delta_m c = B \Theta_{ab}, \quad \delta_m B = 0,
\end{align*}
\] (6.30)
where \( \Theta_b \) and \( \Theta_{ab} \) are finite field dependent parameters and are still anticommuting in nature.

We construct the finite parameters \( \Theta_b \) and \( \Theta_{ab} \) as
\[
\begin{align*}
\Theta_b &= \int \Theta_b' d\kappa = \gamma \int d\kappa \int d^2 x [\dot{c}(\lambda - \varphi - \vartheta)], \\
\Theta_{ab} &= \int \Theta_{ab}' d\kappa = -\gamma \int d\kappa \int d^2 x [c(\lambda - \varphi - \vartheta)],
\end{align*}
\] (6.31) (6.32)
where \( \gamma \) is an arbitrary parameter.

Using Eq. (6.19), the infinitesimal change in Jacobian for the FFMBRST transformation given in Eq. (6.30) is calculated as
\[
\frac{1}{J} \frac{dJ}{d\kappa} = 0.
\] (6.33)
The contributions from second and third terms in the R.H.S. of Eq. (6.19) cancel each other. This implies that the Jacobian for path integral measure is unit under FFMBRST transformation. Hence, the generating functional as well as the effective action are invariant under FFMBRST transformation
\[
Z_{CB} = \int [D\phi] e^{iS_{CB}} \xrightarrow{FFMBRST} Z_{CB}.
\] (6.34)

Now, we would like to consider of the effect of FFBRST transformation with finite parameter \( \Theta_b \) and FF-anti-BRST transformation with finite parameter \( \Theta_{ab} \) independently. The infinitesimal change in Jacobian \( J_1 \) for the FFBRST transformation with the parameter \( \Theta_b \) is calculated as
\[
\frac{1}{J_1} \frac{dJ_1}{d\kappa} = \gamma \int d^4 x \left[ B(\dot{\lambda} - \varphi - \vartheta) + \dot{c}\dot{c} - 2\ddot{c}c \right].
\] (6.35)
To write the Jacobian \( J_1 \) as \( e^{iS_1} \) in case of BRST transformation, we make the following ansatz for \( S_1 \):
\[
S_1 = i \int d^4 x \left[ \xi_1(\kappa) B(\dot{\lambda} - \varphi - \vartheta) + \xi_2(\kappa) \dot{c}\dot{c} + \xi_3(\kappa) \ddot{c}c \right],
\] (6.36)
where \( \xi_i (i = 1, 2, 3) \) are arbitrary \( \kappa \)-dependent constants and satisfy the initial conditions \( \xi_i (\kappa = 0) = 0 \).

The essential condition in Eq. (6.21) satisfies with Eqs. (6.35) and (6.30) iff
\[
\int d^4 x \left[ -B(\dot{\lambda} - \varphi - \vartheta)(\xi_1' + \gamma) - \dot{c}\dot{c}(\xi_2' + \gamma) - \ddot{c}c(2\gamma - \xi_3') \right] + B\dot{c}\Theta_\lambda' (\xi_1 - \xi_2) + B\dot{c}\Theta_\lambda' (2\xi_1 + \xi_3) = 0,
\] (6.37)
where prime denotes the derivative with respect to $\kappa$. Equating the both sides of the above equation, we get the following equations:

\[
\xi'_1 + \gamma = 0, \quad \xi'_2 + \gamma = 0, \quad \xi'_3 - 2\gamma = 0, \quad \xi_1 - \xi_2 = 0 = 2\xi_1 + \xi_3. \tag{6.38}
\]

The solution of above equations satisfying the initial conditions is

\[
\xi_1 = -\gamma\kappa, \quad \xi_2 = -\gamma\kappa, \quad \xi_3 = 2\gamma\kappa. \tag{6.39}
\]

Then, the expression for $S_1$ in terms of $\kappa$ becomes

\[
S_1 = i \int d^4x \left[ -\gamma\kappa B(\dot{\lambda} - \varphi - \vartheta) - \gamma\kappa \dot{c}\dot{c} + 2\gamma\kappa \bar{c}\bar{c} \right]. \tag{6.40}
\]

On the other hand the infinitesimal change in Jacobian $J_2$ for the FF-anti-BRST parameter $\Theta_{ab}$ is calculated as

\[
\frac{1}{J_2} \frac{dJ_2}{d\kappa} = -\gamma \int d^4x \left[ B(\dot{\lambda} - \varphi - \vartheta) + \dot{c}\dot{c} - \bar{c}\bar{c} \right]. \tag{6.41}
\]

Similarly, to write the Jacobian $J_2$ as $e^{iS_2}$ in the anti-BRST case, we make an ansatz for $S_2$ as

\[
S_2 = i \int d^4x \left[ \xi_4(\kappa) B(\dot{\lambda} - \varphi - \vartheta) + \xi_5(\kappa) \dot{c}\dot{c} + \xi_6(\kappa) \bar{c}\bar{c} \right], \tag{6.42}
\]

where arbitrary $\kappa$-dependent constants $\xi_i (i = 4, 5, 6)$ have to be calculated.

The essential condition in Eq. \((6.21)\) for the above Jacobian $J_2$ and functional $S_2$ provides

\[
\int d^4x \left[ B(\dot{\lambda} - \varphi - \vartheta)(\xi'_4 - \gamma) + \dot{c}(\xi'_5 - \gamma) - \bar{c}(2\gamma + \xi'_6) + B\bar{c}\Theta_{ab}(\xi_4 - \xi_5) + B\bar{c}\Theta_{ab}(2\xi_4 + \xi_6) \right] = 0. \tag{6.43}
\]

Comparing the L.H.S. and R.H.S. of the above equation, we get the following equations:

\[
\xi'_4 - \gamma = 0, \quad \xi'_5 - \gamma = 0, \quad \xi'_6 + 2\gamma = 0, \quad \xi_4 - \xi_5 = 0 = 2\xi_4 + \xi_6. \tag{6.44}
\]

Solving the above equations, we get the following values for the $\xi_i$'s:

\[
\xi_4 = \gamma\kappa, \quad \xi_5 = \gamma\kappa, \quad \xi_6 = -2\gamma\kappa. \tag{6.45}
\]

Putting these values in expression of $S_2$, we get

\[
S_2 = i \int d^4x \left[ \gamma\kappa B(\dot{\lambda} - \varphi - \vartheta) + \gamma\kappa \dot{c}\dot{c} + 2\gamma\kappa \bar{c}\bar{c} \right]. \tag{6.46}
\]

Thus, under successive FFBRST and FF-anti-BRST transformations the generating functional transformed as

\[
Z_{CB} \left( = \int [D\phi] \ e^{iS_{CB}} \right) \ (FFBRST)(FF-anti-BRST) \rightarrow Z_{CB} \left( = \int [D\phi] \ e^{iS_{CB} + S_1 + S_2} \right). \tag{6.47}
\]

Note, for the particular choices of $\Theta_b$ and $\Theta_{ab}$, the $S_1$ and $S_2$ cancel each other. Hence $Z_{CB}$ remains invariant under successive FFBRST and FF-anti-BRST transformations. It is interesting to note that the effect of FFMBRST transformation is equivalent to successive operation of FFBRST and FF-anti-BRST transformations.
Maxwell’s theory

We consider FFMBRST transformation here for a more basic model. The generating functional for Maxwell theory, using Nakanishi-Lautrup type auxiliary field \( B \), can be given as

\[
Z_M = \int [D\phi] e^{iS^M_{\text{eff}}},
\]

(6.48)

where the effective action in covariant (Lorentz) gauge with the ghost term is

\[
S^M_{\text{eff}} = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \lambda B^2 - B \partial_\mu A^\mu - \bar{c} \partial_\mu \partial^\mu c \right].
\]

(6.49)

The infinitesimal off-shell nilpotent BRST and anti-BRST transformations under which the effective action \( S^M_{\text{eff}} \) as well as generating functional \( Z_M \) remain invariant, are given as

- **BRST:**

\[
\begin{align*}
\delta_b A_\mu &= \partial_\mu c \delta \Lambda_1, \quad \delta_b c = 0 \\
\delta_b \bar{c} &= B \delta \Lambda_1, \quad \delta_b B = 0.
\end{align*}
\]

(6.50)

- **Anti-BRST:**

\[
\begin{align*}
\delta_{ab} A_\mu &= \partial_\mu \bar{c} \delta \Lambda_2, \quad \delta_{ab} \bar{c} = 0, \\
\delta_{ab} c &= -B \delta \Lambda_2, \quad \delta_{ab} B = 0.
\end{align*}
\]

(6.51)

The nilpotent BRST transformation \( (s_b) \) and anti-BRST transformation \( (s_{ab}) \) mentioned above are absolutely anticommuting in nature i.e. \( \{s_b, s_{ab}\} = s_b s_{ab} + s_{ab} s_b = 0 \). Therefore, the sum of these two transformations \( (s_b, s_{ab}) \) is also a nilpotent symmetry transformation. Let us define MBRST transformation \( (\delta_m \equiv \delta_b + \delta_{ab}) \) in this case, which is characterized by two infinitesimal parameters \( \delta \Lambda_1 \) and \( \delta \Lambda_2 \), as

\[
\begin{align*}
\delta_m A_\mu &= \partial_\mu c \delta \Lambda_1 + \partial_\mu \bar{c} \delta \Lambda_2, \\
\delta_m c &= -B \delta \Lambda_2, \\
\delta_m \bar{c} &= B \delta \Lambda_1, \\
\delta_m B &= 0.
\end{align*}
\]

(6.52)

The FFMBRST symmetry transformation for this theory is then constructed as

\[
\begin{align*}
\delta_m A_\mu &= \partial_\mu c \Theta_b + \partial_\mu \bar{c} \Theta_{ab}, \\
\delta_m c &= -B \Theta_{ab}, \\
\delta_m \bar{c} &= B \Theta_b, \\
\delta_m B &= 0.
\end{align*}
\]

(6.53)

where \( \Theta_b \) and \( \Theta_{ab} \) are finite, field dependent and anticommuting parameters. We choose particular \( \Theta_b \) and \( \Theta_{ab} \) in this case as

\[
\Theta_b = \int \Theta_b' d\kappa = \gamma \int d\kappa \int d^4x [\bar{c} \partial_\mu A^\mu],
\]

(6.54)
and

$$\Theta_{ab} = \int \Theta'_{ab} d\kappa = \gamma \int d\kappa \int d^4x [c \partial_\mu A^\mu],$$  \tag{6.55}

where $\gamma$ is an arbitrary parameter. The infinitesimal change in Jacobian using Eq. \ref{eq:6.19} for the FFMBRST transformation with the above finite parameters vanishes. It means that the path integral measure and hence the generating functional is invariant under FFMBRST transformation.

Now, the infinitesimal change in Jacobian for the FFBRST transformation with the parameter $\Theta^b$ is calculated as

$$\frac{1}{J_1} \frac{dJ_1}{d\kappa} = \gamma \int d^4x [B \partial_\mu A^\mu + \bar{c} \partial_\mu \partial^\mu c].$$  \tag{6.56}

To write the Jacobian $J_1$ as $e^{iS_1}$ in the BRST case, we make the following ansatz for $S_1$:

$$S_1 = i \int d^4x [\xi_1 B \partial_\mu A^\mu + \xi_2 \bar{c} \partial_\mu \partial^\mu c].$$  \tag{6.57}

The essential condition in Eq. \ref{eq:6.21} is satisfied subjected to

$$\int d^4x \left[ B \partial_\mu A^\mu (\xi_1' + \gamma) + \bar{c} \partial_\mu \partial^\mu c (\xi_2' + \gamma) - B \partial_\mu \partial^\mu \bar{c} \Theta'_{ab} (\xi_1 - \xi_2) \right] = 0.$$  \tag{6.58}

where the prime denotes the derivative with respect to $\kappa$. Equating the both sides of the above equation, we get the following:

$$\xi_1' + \gamma = 0, \quad \xi_2' + \gamma = 0, \quad \xi_1 - \xi_2 = 0.$$  \tag{6.59}

The solution of above equations satisfying the initial conditions $\xi_i = 0, (i = 1, 2)$ is

$$\xi_1 = -\gamma \kappa, \quad \xi_2 = -\gamma \kappa.$$  \tag{6.60}

Putting these value in Eq. \ref{eq:6.57}, the expression of $S_1$ becomes

$$S_1 = -i \gamma \kappa \int d^4x [B \partial_\mu A^\mu + \bar{c} \partial_\mu \partial^\mu c].$$  \tag{6.61}

However, the infinitesimal change in Jacobian $J_2$ for the FF-anti-BRST transformation with the parameter $\Theta_{ab}$ is calculated as

$$\frac{1}{J_2} \frac{dJ_2}{d\kappa} = -\gamma \int d^4x [B \partial_\mu A^\mu + \bar{c} \partial_\mu \partial^\mu c].$$  \tag{6.62}

Similarly, to write the Jacobian $J_2$ as $e^{iS_2}$ in the anti-BRST case, we make the following ansatz for $S_2$:

$$S_2 = i \int d^4x [\xi_3 B \partial_\mu A^\mu + \xi_4 \bar{c} \partial_\mu \partial^\mu c].$$  \tag{6.63}

The essential condition in Eq. \ref{eq:6.21} for Eqs. \ref{eq:6.62} and \ref{eq:6.63} provides

$$\int d^4x \left[ B \partial_\mu A^\mu (\xi_3' - \gamma) + \bar{c} \partial_\mu \partial^\mu c (\xi_4' - \gamma) - B \partial_\mu \partial^\mu \bar{c} \Theta'_{ab} (\xi_3 - \xi_4) \right] = 0.$$  \tag{6.64}
Comparing the L.H.S. and R.H.S. of the above equation we get the following equations:

\[ \xi_3' - \gamma = 0, \quad \xi_4' - \gamma = 0, \quad \xi_3 - \xi_4 = 0. \]  

(6.65)

Solving the above equations, we get the following values for the \( \xi \)'s:

\[ \xi_3 = \gamma \kappa, \quad \xi_4 = \gamma \kappa. \]  

(6.66)

Plugging back these value of \( \xi_i \) (\( i = 3, 4 \)) in Eq. (6.63), we obtain

\[ S_2 = i \gamma \kappa \int d^4x \left[ B \partial_\mu A^\mu + \bar{c} \partial_\mu \partial^\mu c \right]. \]  

(6.67)

From Eqs. (6.61) and (6.67), one can easily see that \( S_1 + S_2 = 0 \). Therefore, under successive FFBRST and FF-anti-BRST transformations with these particular finite parameters \( \Theta_b \) and \( \Theta_{ab} \), respectively, the generating functional transformed as

\[ Z_M \left( = \int [D\phi] e^{iS^M_{FF BRST} + iS^M_{FF - antiBRST}} \right) \rightarrow Z_M \left( = \int [D\phi] e^{iS^M_M + S_1 + S_2} \right). \]  

(6.68)

Hence, the successive operations of FFBRST and FF-anti-BRST transformations also leave the generating functional \( Z_M \). This reconfirms that FFMBRST transformation has same effect on both the effective action and the generating functional as the successive FFBRST and FF-anti-BRST transformations.

**6.4.3 Non-Abelian YM theory in Curci-Ferrari-Delbourgo-Jarvis (CFDJ) gauge**

The examples studied so far were Abelian gauge theories. We now consider an example with non-Abelian gauge theory. CFDJ gauge is a standard gauge-fixing which has been studied extensively in non-Abelian gauge theories \[101\]. The generating functional for non-Abelian YM theory in CFDJ gauge is written as

\[ Z'^{CF}_{YM} = \int [D\phi] e^{iS^F_{YM}[\phi]}, \]  

(6.69)

where \( \phi \) is generic notation for all the fields in the effective action \( S^F_{YM} \)

\[ S^F_{YM} = \int d^4x \left[ -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{\xi}{2} (h^a)^2 + i h_a^a \partial_\mu A^\mu A^a + \frac{1}{2} \partial_\mu \bar{c}^a (D^\mu c)^a \right. \]

\[ + \left. \frac{1}{2} (D_\mu \bar{c})^a \partial^\mu c^a - \xi \frac{g^2}{8} (f^{abc} \bar{c}^b c^c)^2 \right], \]  

(6.70)

with the field strength tensor \( F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \) and \( h^a \) is the Nakanishi-Lautrup type auxiliary field. The effective action as well as the generating functional are invariant under the following infinitesimal BRST and anti-BRST transformations.
Corresponding FFMBRST symmetry transformation is constructed as

\[
\begin{align*}
\text{BRST: } & \delta_b A^a_\mu = -(D_\mu c)^a \delta \Lambda_1, \quad \delta_b c^a = -\frac{g}{2} f^{abc} c^b c^c \delta \Lambda_1 \\
& \delta_b c^a = \left(i h^a - \frac{g}{2} f^{abc} c^b c^c \right) \delta \Lambda_1 \\
& \delta_b (i h^a) = -\frac{g}{2} f^{abc} \left(i h^b c^c + \frac{g}{4} f^{cde} c^d c^e \right) \delta \Lambda_1,
\end{align*}
\]

anti-BRST: \( \delta_{ab} A^a_\mu = -(D_\mu \bar{c})^a \delta \Lambda_2, \quad \delta_{ab} c^a = -\frac{g}{2} f^{abc} c^b c^c \delta \Lambda_2 \\
\delta_{ab} c^a = \left(-i h^a - \frac{g}{2} f^{abc} c^b c^c \right) \delta \Lambda_2 \\
\delta_{ab} (i h^a) = -\frac{g}{2} f^{abc} \left(i h^b c^c + \frac{g}{4} f^{cde} c^d c^e \right) \delta \Lambda_2,
\]  

where \( \delta \Lambda_1 \) and \( \delta \Lambda_2 \) are infinitesimal, anticommuting and global parameters. The infinitesimal MBRST symmetry transformation (\( \delta_m = \delta_b + \delta_{ab} \)) in this case is written as:

\[
\begin{align*}
\delta_m A^a_\mu &= -D_\mu c^a \delta \Lambda_1 - D_\mu \bar{c}^a \delta \Lambda_2, \\
\delta_m c^a &= -\frac{g}{2} f^{abc} c^b c^c \delta \Lambda_1 - \left(i h^a + \frac{g}{2} f^{abc} c^b c^c \right) \delta \Lambda_2 \\
\delta_m \bar{c}^a &= \left(i h^a - \frac{g}{2} f^{abc} c^b c^c \right) \delta \Lambda_1 - \frac{g}{2} f^{abc} \left(i h^b c^c + \frac{g}{4} f^{cde} c^d c^e \right) \delta \Lambda_2 \\
\delta_m (i h^a) &= -\frac{g}{2} f^{abc} \left(i h^b c^c + \frac{g}{4} f^{cde} c^d c^e \right) \delta \Lambda_1 - \frac{g}{2} f^{abc} \left(i h^b c^c + \frac{g}{4} f^{cde} c^d c^e \right) \delta \Lambda_2.
\end{align*}
\]

Corresponding FFMBRST symmetry transformation is constructed as

\[
\begin{align*}
\delta_m A^a_\mu &= -D_\mu c^a \Theta_b - D_\mu \bar{c}^a \Theta_{ab}, \\
\delta_m c^a &= -\frac{g}{2} f^{abc} c^b c^c \Theta_b - \left(i h^a + \frac{g}{2} f^{abc} c^b c^c \right) \Theta_{ab} \\
\delta_m \bar{c}^a &= \left(i h^a - \frac{g}{2} f^{abc} c^b c^c \right) \Theta_b - \frac{g}{2} f^{abc} \left(i h^b c^c + \frac{g}{4} f^{cde} c^d c^e \right) \Theta_{ab} \\
\delta_m (i h^a) &= -\frac{g}{2} f^{abc} \left(i h^b c^c + \frac{g}{4} f^{cde} c^d c^e \right) \Theta_b - \frac{g}{2} f^{abc} \left(i h^b c^c + \frac{g}{4} f^{cde} c^d c^e \right) \Theta_{ab}.
\end{align*}
\]

with two arbitrary finite field dependent parameters \( \Theta_b \) and \( \Theta_{ab} \). The generating functional \( Z^{CF}_{YM} \) is made invariant under the above FFMBRST transformation by constructing appropriate finite parameters \( \Theta_b \) and \( \Theta_{ab} \). We construct the finite nilpotent parameters \( \Theta_b \) and \( \Theta_{ab} \) as

\[
\Theta_b = \int \Theta_b d\kappa = \gamma \int d\kappa \int d^4x [e^a \partial_\mu A^{a\mu}], \quad (6.74)
\]

\[
\Theta_{ab} = \int \Theta_{ab} d\kappa = \gamma \int d\kappa \int d^4x [e^a \partial_\mu A^{a\mu}], \quad (6.75)
\]

where \( \gamma \) is an arbitrary parameter. Following the same method elaborated in the previous two examples, we show that the Jacobian for path integral measure due to FFMBRST transformation given in Eq. (6.73) with finite parameters \( \Theta_b \) and \( \Theta_{ab} \) becomes unit. It means that under
such FFMBRST transformation the generating functional as well as effective action remain invariant. The Jacobian contribution for path integral measure due to FFBRST transformation with parameter $\Theta_b$ compensates the same due to FF-anti-BRST transformation with parameter $\Theta_{ab}$. Therefore, under the successive FFBRST and FF-anti-BRST transformations the generating functional remains invariant as

$$Z_{YM}^{CF} = \int [D\phi] e^{iS_{YM}^{CF}} \quad \text{(FFBRST)(FF-anti-BRST)} \rightarrow Z_{YM}^{CF}. \quad (6.76)$$

Again we see the equivalence between FFMBRST transformation and successive operations of FFBRST and FF-anti-BRST transformations.

We end up the section with conclusion that in all the three cases the FFMBRST transformation with appropriate finite parameters is the finite nilpotent symmetry of the effective action as well as the generating functional of the effective theories. Here, we also note that the successive operations of FFBRST and FF-anti-BRST also leave the generating functional as well as effective action invariant and hence equivalent to FFMBRST transformation.

### 6.5 FFMBRST symmetry in BV formulation

In this section, we consider the BV formulation using MBRST transformation. Unlike BV formulation using either BRST or anti-BRST transformation, we need two sets of antifields in BV formulation for MBRST transformation. We construct FFMBRST transformation in this context. The change in Jacobian under FFBRST transformation in the path integral measure in the definition of generating functional is used to adjust with the change in the gauge-fixing fermion $\Psi_1$ \[64\]. Hence, the FFBRST transformation is used to connect the generating functionals of different solutions of quantum master equation \[45\]. However, in case of BV formulation for FFMBRST transformation we need to introduce two gauge-fixing fermions $\Psi_1$ and $\Psi_2$. We construct the finite parameters in FFMBRST transformation in such a way that contributions from $\Psi_1$ and $\Psi_2$ adjust each other to leave the extended action invariant. This implies that we can construct appropriate parameters in FFMBRST transformation such that generating functionals corresponding to different solutions of quantum master equations remain invariant under such transformation. These results can be demonstrated with the help of explicit examples. We would like to consider the same examples of previous section for this purpose.

#### 6.5.1 Bosonized chiral model in BV formulation

We recast the generating functional for (1+1) dimensional bosonized chiral model given in Eq. \[6.25\] using both BRST and anti-BRST exact terms as

$$Z_{CB} = \int D\phi \ e^{iS_{CB}} = \int D\phi \ \exp \left[ i \int d^2x \left\{ \pi_\phi \dot{\phi} + \pi_{\bar{\phi}} \dot{\bar{\phi}} + p_u \dot{u} - \frac{1}{2} \pi_\phi^2 - \frac{1}{2} \pi_{\bar{\phi}}^2 + \pi_{\bar{\phi}} (\phi' - \bar{\phi}' + \lambda) + \frac{1}{2} \pi_\phi \lambda + \frac{1}{2} s_b \Psi_1 + \frac{1}{2} s_{ab} \Psi_2 \right\} \right]. \quad (6.77)$$

Here, the Lagrange multiplier field $u$ is considered as dynamical variable and expression for gauge-fixing fermions for BRST symmetry ($\Psi_1$) and anti-BRST symmetry ($\Psi_2$), respectively,
are

\[ \Psi_1 = \int d^2x \bar{c}(\ddot{\lambda} - \varphi - \vartheta + \frac{1}{2}B). \]  
(6.78)

\[ \Psi_2 = \int d^2x c(\ddot{\lambda} - \varphi - \vartheta + \frac{1}{2}B). \]  
(6.79)

The effective action \( S_{CB} \) is invariant under combined BRST and anti-BRST transformations given in Eq. (6.29). The generating functional \( Z_{CB} \) can be written in terms of antifields \( \phi^*_1 \) and \( \phi^*_2 \) corresponding to all fields \( \phi \) as

\[
Z_{CB} = \int D\phi \exp \left[ i \int d^2x \left\{ \pi_\varphi \dot{\varphi} + \pi_\vartheta \dot{\vartheta} + p_u \dot{u} - \frac{1}{2} \pi_\varphi^2 \right. \right. \\
+ \left. \left. \frac{1}{2} \pi_\vartheta^2 + \pi_\varphi (\varphi' - \vartheta' + \lambda) + \pi_\vartheta \lambda + \frac{1}{2} \varphi^*_1 c - \frac{1}{2} \varphi^*_2 \bar{c} \right. \right. \\
+ \left. \left. \frac{1}{2} \vartheta^*_1 c - \frac{1}{2} \vartheta^*_2 \bar{c} + \frac{1}{2} \varphi^*_1 B - \frac{1}{2} \varphi^*_2 B - \frac{1}{2} \lambda^*_1 \dot{c} + \frac{1}{2} \lambda^*_2 \dot{\bar{c}} \right. \right. \\
+ \left. \left. \frac{1}{2} \lambda^*_1 \dot{c} - \frac{1}{2} \lambda^*_2 \dot{\bar{c}} \right] \right],
\]  
(6.80)

where \( \phi^*_i (i = 1, 2) \) is a generic notation for antifields arising from gauge-fixing fermions \( \Psi_i \). The above relation can further be written in compact form as

\[
Z_{CB} = \int [D\phi] e^{iW_{\Psi_1+\Psi_2}[\phi,\phi^*_i]},
\]  
(6.81)

where \( W_{\Psi_1+\Psi_2}[\phi,\phi^*_i] \) is an extended action for the theory of self-dual chiral boson corresponding to all fields \( \phi \).

This extended quantum action, \( W_{\Psi_1+\Psi_2}[\phi,\phi^*_i] \) satisfies certain rich mathematical relations commonly known as quantum master equation \([6]\), given by

\[
\Delta e^{iW_{\Psi_1+\Psi_2}[\phi,\phi^*_i]} = 0 \quad \text{with} \quad \Delta \equiv \frac{\partial}{\partial\phi} \frac{\partial}{\partial\phi^*_i} (-1)^{i+1}.
\]  
(6.82)

The generating functional does not depend on the choice of gauge-fixing fermions \([5]\) and therefore extended quantum action \( W_{\Psi_i} \) with all possible \( \Psi_i \) are the different solutions of quantum master equation. The antifields \( \phi^*_i \) corresponding to each field \( \phi \) for this particular theory can be obtained from the gauge-fixed fermion \( \Psi_i \) as

\[
\varphi^*_1 = \frac{\delta \Psi_1}{\delta \varphi} = -\bar{c}, \quad \vartheta^*_1 = \frac{\delta \Psi_1}{\delta \vartheta} = -\bar{c}, \quad c^*_1 = \frac{\delta \Psi_1}{\delta c} = 0, \quad \bar{c}^*_1 = \frac{\delta \Psi_1}{\delta \bar{c}} = 0, \\
\varphi^*_2 = \frac{\delta \Psi_2}{\delta \varphi} = -c, \quad \vartheta^*_2 = \frac{\delta \Psi_2}{\delta \vartheta} = -c, \\
c^*_2 = \frac{\delta \Psi_2}{\delta c} = (\dot{\lambda} - \varphi - \vartheta + \frac{1}{2}B), \quad \bar{c}^*_2 = \frac{\delta \Psi_2}{\delta \bar{c}} = 0, \\
B^*_1 = \frac{\delta \Psi_1}{\delta B} = \frac{1}{2} \bar{c}, \quad \lambda^*_1 = -\frac{\delta \Psi_1}{\delta \lambda} = -\dot{\bar{c}}.
\]  
(6.83)

Similarly, the antifields \( \phi^*_2 \) can be calculated from the gauge-fixing fermion \( \Psi_2 \) as

\[
\varphi^*_2 = \frac{\delta \Psi_2}{\delta \varphi} = -c, \quad \vartheta^*_2 = \frac{\delta \Psi_2}{\delta \vartheta} = -c, \\
c^*_2 = \frac{\delta \Psi_2}{\delta c} = (\dot{\lambda} - \varphi - \vartheta + \frac{1}{2}B), \quad \bar{c}^*_2 = \frac{\delta \Psi_2}{\delta \bar{c}} = 0, \\
B^*_2 = \frac{\delta \Psi_2}{\delta B} = \frac{1}{2} c, \quad \lambda^*_2 = -\frac{\delta \Psi_2}{\delta \lambda} = -\dot{\bar{c}}.
\]  
(6.84)
Now, we apply the FFMBRST transformation given in Eq. (6.30) with the finite parameters written in Eqs. (6.31) and (6.32) to this generating functional. We see that the path integral measure in Eq. (6.81) remains invariant under this FFMBRST transformation as the Jacobian for path integral measure is 1. Therefore,

$$Z_{CB} = \int [D\phi] \, e^{iW_{\Psi_1+\Psi_2}} (\text{FFBRST})(\text{FFantiBRST}) \rightarrow Z_{CB}. \quad (6.85)$$

Thus, the solutions of quantum master equation in this model remain invariant under FFMBRST transformation as well as under consecutive operation of FFBRST and FF-anti-BRST transformations. However, the FFBRST (FF-anti-BRST) transformation connects the generating functionals corresponding to the different solutions of the quantum master equation [34].

### 6.5.2 Maxwell’s theory in BV formulation

The generating functional for Maxwell’s theory given in Eq. (6.48) can be recast using BRST and anti-BRST exact terms as

$$Z_M = \int [D\phi] \, e^{i\int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} A_{\mu} \partial^\mu c + \frac{1}{2} A_{\mu} \partial^\mu \bar{c} + \frac{1}{2} \bar{c}^2 - \frac{1}{2} c^2 B \right]} \quad (6.86)$$

where the expressions for gauge-fixing fermions $\Psi_1$ and $\Psi_2$ are

$$\Psi_1 = \int d^4x \, \bar{c} (\lambda B - \partial \cdot A),$$

$$\Psi_2 = -\int d^4x \, c (\lambda B - \partial \cdot A). \quad (6.87)$$

The generating functional for such theory can further be expressed in fields/antifields formulation as

$$Z_M = \int [D\phi] \, e^{i\int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} A_{\mu} \partial^\mu \phi + \frac{1}{2} A_{\mu} \partial^\mu \phi^* + \frac{1}{2} \phi^* B - \frac{1}{2} \phi B \right]} \quad (6.88)$$

In the compact form above generating functional is written as

$$Z_M = \int [D\phi] \, e^{iW_{\Psi_1+\Psi_2}[\phi,\phi^*]} \quad (6.89)$$

where $W_{\Psi_1+\Psi_2}[\phi,\phi^*]$ is an extended action for Maxwell’s theory corresponding to the gauge-fixing fermions $\Psi_1$ and $\Psi_2$.

The antifields for gauge-fixed fermion $\Psi_1$ are calculated as

$$A^*_{\mu 1} = \frac{\delta \Psi_1}{\delta A^\mu} = \partial_\mu \bar{c}, \quad c^*_1 = \frac{\delta \Psi_1}{\delta \bar{c}} = (\lambda B - \partial \cdot A),$$

$$c^*_1 = \frac{\delta \Psi_1}{\delta c} = 0, \quad B^*_1 = \frac{\delta \Psi_1}{\delta B} = \lambda \bar{c}. \quad (6.90)$$

The antifields $\phi^*_2$ can be calculated from the gauge-fixed fermion $\Psi_2$ as

$$A^*_{\mu 2} = \frac{\delta \Psi_2}{\delta A^\mu} = -\partial_\mu c, \quad c^*_2 = \frac{\delta \Psi_2}{\delta \bar{c}} = 0,$$

$$c^*_2 = \frac{\delta \Psi_2}{\delta c} = - (\lambda B - \partial \cdot A), \quad B^*_2 = \frac{\delta \Psi_2}{\delta B} = -\lambda c. \quad (6.91)$$
Now, implementing the FFMBRST transformation mentioned in Eq. (6.53) with parameters given in Eqs. (6.54) and (6.55) to this generating functional we see that the Jacobian for the path integral measure for such transformation becomes unit. Hence, the FFMBRST transformation given in Eq. (6.53) is a finite symmetry of the solutions of quantum master equation for Maxwell’s theory.

Now, we focus on the contributions arising from the Jacobian due to independent applications of FFBRST and FF-anti-BRST transformations. We construct the finite parameters of FFBRST and FF-anti-BRST transformations in such a way that Jacobian remains invariant. Therefore, the generating functional remains invariant under consecutive operations of FFBRST and FF-anti-BRST transformations with appropriate parameters as

\[ Z_M = \int [D\phi] \, e^{iW_{\Psi_1 + \Psi_2}} \rightarrow Z_M. \]  

(6.92)

It also implies that the effect of consecutive FFBRST and FF-anti-BRST transformations is same as the effect of FFMBRST transformation on \( Z_M \).

6.5.3 Non-Abelian YM theory in BV formulation

The generating functional for this theory can be written in both BRST and anti-BRST exact terms as

\[ Z_{YM}^{CF} = \int [D\phi] \, e^{i\int d^4x \left\{ -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2} s_\phi \Psi_1 + \frac{1}{2} s_{\phi^*} \Psi_2 \right\}}, \]  

(6.93)

with the expressions of gauge-fixing fermions \( \Psi_1 \) and \( \Psi_2 \) as

\[ \Psi_1 = -\int d^4x \, \bar{c}^a (i \frac{\xi}{2} h^a - \partial \cdot A^a), \]
\[ \Psi_2 = \int d^4x \, c^a (i \frac{\xi}{2} h^a - \partial \cdot A^a). \]  

(6.94)

We re-write the generating functional given in Eq. (6.69) using field/antifield formulation as,

\[ Z_{YM}^{CF} = \int [D\phi] \exp \left[ i \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2} A_{\mu2}^a D^\mu c^a - \frac{1}{2} A_{\mu2}^a D^\mu \bar{c}^a \right. \right. \]
\[ + \frac{1}{2} \bar{c}^a \bar{c}^a \left( i h^a - \frac{g}{2} f^{abc} \bar{c}^b c^c \right) - \frac{1}{2} \bar{c}^a \left( i h^a + \frac{g}{2} f^{abc} c^b \bar{c}^c \right) - \frac{1}{2} h^a \left( \frac{g}{2} f^{abc} h^b \bar{c}^c \right) \]
\[ \left. - \frac{g^2}{8} f^{abc} f^{cde} \bar{c}^b c^d \bar{c}^e \right] \bigg\}. \]  

(6.95)

This generating functional \( Z_{YM}^{CF} \) can be written compactly as

\[ Z_{YM}^{CF} = \int [D\phi] \, e^{iW_{\Psi_1 + \Psi_2}[\phi, \phi^*]}, \]  

(6.96)

where \( W_{\Psi_1 + \Psi_2}[\phi, \phi^*] \) is an extended quantum action for the non-Abelian YM theory in CFDJ gauge.
The antifields are calculated with the help of gauge-fixed fermion \( \Psi_1 \) as

\[
A_{\mu 1}^{\ast} = \frac{\delta \Psi_1}{\delta A^\mu a} = -\partial_\mu \bar{c}^a, \quad c_1^{\ast} = \frac{\delta \Psi_1}{\delta \bar{c}^a} = 0, \\
\bar{c}_1^{\ast} = \frac{\delta \Psi_1}{\delta c^a} = -(i\frac{\xi}{2} h^a - \partial \cdot A^a), \\
h_1^{\ast} = \frac{\delta \Psi_1}{\delta h^a} = \frac{i}{2} \xi \bar{c}^a.
\] (6.97)

The explicit value of antifields can be calculated with \( \Psi_2 \) as

\[
A_{\mu 2}^{\ast} = \frac{\delta \Psi_2}{\delta A^\mu a} = \partial_\mu c^a, \quad \bar{c}_2^{\ast} = \frac{\delta \Psi_2}{\delta \bar{c}^a} = 0, \\
c_2^{\ast} = \frac{\delta \Psi_2}{\delta c^a} = (i\frac{\xi}{2} h^a - \partial \cdot A^a), \\
h_2^{\ast} = \frac{\delta \Psi_2}{\delta h^a} = \frac{i}{2} \xi c^a.
\] (6.98)

We observe here again that the Jacobian for path integral measure in the expression of generating functional \( Z_{YM}^{CF} \) arising due to FFMBRST transformation and due to successive operations of FFBRST and FF-anti-BRST transformations remains unit for appropriate choice of finite parameters. Thus, the consequence of FFMBRST transformation given in Eq. (6.73) with the finite parameters given in Eqs. (6.74) and (6.75) is equivalent to the subsequent operations of FFBRST and FF-anti-BRST transformations with same finite parameters.

### 6.6 Conclusions

FFBRST and FF-anti-BRST transformations are nilpotent symmetries of the effective action. However, these transformations do not leave the generating functional invariant as the path integral measure changes in a nontrivial way under these transformations. We have constructed infinitesimal MBRST transformation which is the combination of infinitesimal BRST and anti-BRST transformations. Even though infinitesimal MBRST transformation does not play much significant role, its finite version has very important consequences. We have shown that it is possible to construct the finite field dependent MBRST (FFMBRST) transformation which leaves the effective action as well as the generating functional invariant. The finite parameters in the FFMBRST transformation have been chosen in such a way that the Jacobian contribution from the FFBRST part compensates the same arising from FF-anti-BRST part. We have considered several explicit examples with diverse character in both gauge theories as well as in field/antifield formulation to show these results. It is interesting to point out that the effect of FFMBRST transformation is equivalent to successive operations of FFBRST and FF-anti-BRST transformations. We have further shown that the generating functionals corresponding to different solutions of quantum master equation remain invariant under such FFMBRST transformation whereas the independent FFBRST and FF-anti-BRST transformations connect the generating functionals corresponding to the different solutions of the quantum master equation. It will be interesting to see whether this FFMBRST transformation puts further restrictions on the relation of different Green’s functions of the theory to simplify the renormalization program. In particular, such FFMBRST transformation may be helpful for the theories where BRST and anti-BRST transformations play independent role.
Chapter 7

FFBRST transformation and constrained systems

This chapter is devoted to study the different class of constraints theories in the framework of FFBRST transformation. Here we develop the FFBRST and FF-anti-BRST transformations for first-class theories \[83\]. Then we show that such generalization helps to connect first-class theories to second-class theories. The results are established with the help of two explicit examples.

7.1 The theories with constraints: examples

In the various field theories, all the dynamical phase space variables are not independent rather some of the variables satisfy the constraints emerging from the structure of the theories. In other words, the relations between various dynamical variables are known as constraints of the theories. The usual Poisson brackets may not represent the true brackets as they need not to satisfy the constraints of the theories in such constrained systems. A system is said to be first-class constrained system if all the Poisson brackets among the constraints vanish weakly. On the other hand, if there exists at least one non-zero Poisson bracket among the constraints then the theory is called as second-class. In this section, we briefly outline the essential features of second-class and first-class theories. In particular, we discuss the Proca theory for massive spin 1 vector field theory and gauge variant theory for the self-dual chiral boson, which are second-class theories. Corresponding first-class theories i.e. the Stueckelberg theory for massive spin 1 vector fields and gauge invariant theory for the self-dual chiral boson are also outlined in this section.

7.1.1 Theory for massive spin 1 vector field

Proca model

We start with the action for a massive charge neutral spin 1 vector field \(A_\mu\) in 4D

\[ S_P = \int d^4x \, \mathcal{L}_P, \tag{7.1} \]

where the Lagrangian density is given as

\[ \mathcal{L}_P = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{M^2}{2} A_\mu A^\mu. \tag{7.2} \]
7.1. The theories with constraints: examples

The field strength tensor is defined as\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \] We recall the convention for \( g^{\mu\nu} \), i.e., \( g^{\mu\nu} = \text{diagonal} (1, -1, -1, -1) \), where \( \mu, \nu = 0, 1, 2, 3 \). The canonically conjugate momenta for \( A_\mu \) field is\[ \Pi^\mu = \frac{\partial L}{\partial A_\mu^0} = F^{\mu0}. \] (7.3)

This implies that the primary constraint of the theory is\[ \Omega_1 \equiv \Pi^0 \approx 0. \] (7.4)

The Hamiltonian density of the theory is given by\[ H = \Pi^\mu \dot{A}_\mu^0 - L = \Pi_i \partial_i A^0 - \frac{1}{2} \Pi_i^2 + \frac{1}{2} F^{ij} F_{ij} - \frac{1}{2} M^2 A_\mu A^\mu. \] (7.5)

The time evolution for the dynamical variable \( \Pi^0 \) can be written as\[ \dot{\Pi}^0 = [\Pi^0, H], \] (7.6)

where the Hamiltonian \( H = \int d^3 x \mathcal{H} \). The constraints of the theory should be invariant under time evolution and using (7.6) we obtain the secondary constraint\[ \Omega_2 \equiv [\Pi^0, H] = \partial_i \Pi^i + M^2 A^0 \approx 0. \] (7.7)

The constraint \( \Omega_2 \) contains \( A^0 \) which implies that \( [\Omega_1, \Omega_2] \neq 0 \). Hence, the Proca theory for massive spin 1 vector field is endowed with the second-class constraint.

The propagator for this theory can be written in a simple manner as\[ iG_{\mu\nu}(p) = -\frac{i}{p^2 - M^2} \left( \eta_{\mu\nu} - \frac{p_\mu p_\nu}{M^2} \right). \] (7.8)

Note that the propagator in this theory does not fall rapidly for large values of the momenta. This leads to difficulties in establishing renormalizability of the (interacting) Proca theory for massive photons. Hence the limit \( M \to 0 \) of the Proca theory is clearly difficult to perceive.

The generating functional for the Proca theory is defined as\[ Z_P \equiv \int [DA] e^{iS_P}. \] (7.9)

### Stueckelberg theory

To remove the difficulties in the Proca model, Stueckelberg considered the following generalized action\[ S_{ST} = \int d^4 x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{M^2}{2} \left( A_\mu - \frac{1}{M} \partial_\mu B \right)^2 \right], \] (7.10)

by introducing a real scalar field \( B \).

This action is invariant under the following gauge transformation:

\[
\begin{align*}
A_\mu(x) &\to A'_\mu(x) = A_\mu(x) + \partial_\mu \lambda(x), \\
B(x) &\to B'(x) = B(x) + M \lambda(x),
\end{align*}
\] (7.11) (7.12)
7.1. The theories with constraints: examples

where $\lambda$ is the gauge parameter. For the quantization of such theory one has to choose a gauge condition. By choosing the 't Hooft gauge condition, $L_{gf} = -\frac{1}{2}(\partial^\mu A_\mu + \chi MB)^2$ where $\chi$ is any arbitrary gauge parameter, it is easy to see that the propagators are well behaved at high momentum. As a result, there is no difficulty in establishing renormalizability for such theory.

Now we turn to the BRST symmetry for the Stueckelberg theory. Introducing a ghost ($c$) and an antighost field ($\bar{c}$) the effective Stueckelberg action can be written as

$$S_{ST} = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{M^2}{2} \left( A_\mu - \frac{1}{M} \partial_\mu B \right)^2 - \frac{1}{2\chi} (\partial^\mu A_\mu + \chi MB)^2 - \bar{c}(\partial^2 + \chi M^2)c \right].$$

(7.13)

This action is invariant under the following on-shell BRST transformation:

$$\delta_b A_\mu = \partial_\mu c \Lambda, \quad \delta_b B = M c \Lambda,$$

$$\delta_b c = 0, \quad \delta_b \bar{c} = -\frac{1}{\chi} (\partial_\mu A_\mu + \chi MB) \Lambda,$$

(7.14)

where $\Lambda$ is an infinitesimal, anticommuting and global parameter. The generating functional for the Stueckelberg theory is defined as

$$Z_{ST} = \int [D\phi] e^{iS_{ST}[\phi]},$$

(7.15)

where $\phi$ is the generic notation for all fields involved in the theory. All the Green functions in this theory can be obtain from $Z_{ST}$.

7.1.2 Theory for self-dual chiral boson

A self-dual chiral boson can be described by the gauge variant as well as the gauge invariant model. The purpose of this section is to introduce such models for a self-dual chiral boson.

Gauge variant theory for self-dual chiral boson

We start with the gauge variant model [66] in 2D for a single self-dual chiral boson. The effective action for such a theory is given as

$$S_{CB} = \int d^2x L_{CB} = \int d^2x \left[ \frac{1}{2} \dot{\varphi}^2 - \frac{1}{2} \varphi'^2 + \lambda(\varphi - \varphi') \right],$$

(7.16)

where over dot and prime denote time and space derivatives respectively and $\lambda$ is the Lagrange multiplier. The field $\varphi$ satisfies the self-duality condition $\dot{\varphi} = \varphi'$ in this case. We choose the Lorentz metric $g^{\mu\nu} = (1, -1)$ with $\mu, \nu = 0, 1$. The associated momenta for the field $\varphi$ and the Lagrange multiplier are calculated as

$$\pi_\varphi = \frac{\partial L_{CB}}{\partial \dot{\varphi}} = \dot{\varphi} + \lambda, \quad \pi_\lambda = \frac{\partial L_{CB}}{\partial \lambda} = 0,$$

(7.17)
which show that the model has the following primary constraint: \( \Omega_1 = \pi_\lambda \approx 0 \). The Hamiltonian density corresponding to the above Lagrangian density \( \mathcal{L}_{CB} \) in Eq. (7.16) is

\[
\mathcal{H}_{CB} = \pi_\phi \dot{\phi} + \pi_\lambda \dot{\lambda} - \mathcal{L}_{CB} = \frac{1}{2}(\pi_\phi - \lambda)^2 + \frac{1}{2} \phi'^2 + \lambda \phi'.
\] (7.18)

Further, we can write the total Hamiltonian density corresponding to \( \mathcal{L}_{CB} \) by introducing the Lagrange multiplier field \( \eta \) for the primary constraint \( \Omega_1 \) as

\[
\mathcal{H}^T_{CB} = \frac{1}{2}(\pi_\phi - \lambda)^2 + \frac{1}{2} \phi'^2 + \lambda \phi' + \eta \pi_\lambda.
\] (7.19)

Following the Dirac prescription [13], we obtain the secondary constraint in this case as

\[
\Omega_2 \equiv \dot{\pi}_\lambda = [\pi_\lambda, \mathcal{H}_{CB}] = \pi_\phi - \lambda - \phi' \approx 0.
\] (7.20)

The constraints \( \Omega_1 \) and \( \Omega_2 \) are of second-class as \([\Omega_1, \Omega_2] \neq 0\). This is an essential feature of a gauge variant theory.

This model is quantized by establishing the following commutation relations [66]

\[
[\phi(x), \pi_\phi(y)] = [\phi(x), \lambda(y)] = +i\delta(x-y),
\] (7.21)

\[
2[\lambda(x), \pi_\phi(y)] = [\lambda(x), \lambda(y)] = -2i\delta'(x-y),
\] (7.22)

where prime denotes the space derivative. The rest of the commutator vanishes.

The generating functional for the gauge variant theory for a self-dual chiral boson is defined as

\[
Z_{CB} = \int [\mathcal{D} \phi] \ e^{iS_{CB}},
\] (7.23)

where \( D\phi \) is the path integral measure and \( S_{CB} \) is the effective action for a self-dual chiral boson.

**Gauge invariant theory for self-dual chiral boson**

To construct a gauge invariant theory corresponding to the gauge non-invariant model for chiral bosons, one generally introduces the WZ term in the Lagrangian density \( \mathcal{L}_{CB} \). For this purpose we need to enlarge the Hilbert space of the theory by introducing a new quantum field \( \vartheta \), called the WZ field, through the redefinition of fields \( \phi \) and \( \lambda \) as follows [21]: \( \phi \rightarrow \phi - \vartheta, \ \lambda \rightarrow \lambda + \dot{\vartheta} \).

With this redefinition of fields the modified Lagrangian density becomes

\[
\mathcal{L}_{CB}^I = \mathcal{L}_{CB} + \mathcal{L}_{CB}^{WZ},
\] (7.24)

where the WZ term

\[
\mathcal{L}_{CB}^{WZ} = -\frac{1}{2} \vartheta'^2 - \frac{1}{2} \vartheta'^2 + \varphi' \vartheta' + \dot{\vartheta} \vartheta' - \dot{\vartheta} \varphi' - \lambda (\dot{\vartheta} - \vartheta').
\] (7.25)
7.2 Relating the first-class and second-class theories through FFBRST formulation: examples

The above Lagrangian density in Eq. (7.24) is invariant under time-dependent chiral gauge transformation:

\[
\delta \varphi = \mu(x,t), \quad \delta \vartheta = \mu(x,t), \quad \delta \lambda = -\dot{\mu}(x,t), \\
\delta \pi_{\varphi} = 0, \quad \delta \pi_{\vartheta} = 0, \quad \delta p_\lambda = 0,
\]

(7.26)

where \(\mu(x,t)\) is an arbitrary function of the space and time.

The BRST invariant effective theory for the self-dual chiral boson [73] can be written as

\[
S_{CB}^{II} = \int d^2x \mathcal{L}_{CB}^{II},
\]

(7.27)

where \(\mathcal{L}_{CB}^{II} = \frac{1}{2} \dot{\varphi}^2 - \frac{1}{2} \varphi'^2 + \lambda (\dot{\varphi} - \varphi') - \frac{1}{2} \dot{\vartheta}^2 - \frac{1}{2} \vartheta'^2 + \varphi' \vartheta' + \dot{\vartheta} \varphi' - \lambda (\dot{\vartheta} - \varphi') - \frac{1}{2} (\dot{\varrho} - \vartheta)^2 + \dot{c} \bar{c} - 2\bar{c}c.
\]

(7.28)

c and \(\bar{c}\) are ghost and antighost fields respectively. The corresponding generating functional for the gauge invariant theory for the self-dual chiral boson is given as

\[
Z_{CB}^{II} = \int [D\phi] e^{iS_{CB}^{II}},
\]

(7.29)

where \(\phi\) is generic notation for all fields involved in the effective action. The effective action \(S_{CB}^{II}\) and the generating functional \(Z_{CB}^{II}\) are invariant under the following nilpotent BRST transformation:

\[
\delta_b \varphi = c \Lambda, \quad \delta_b \lambda = -\dot{c} \Lambda, \quad \delta_b \vartheta = c \Lambda, \\
\delta_b \bar{c} = -(\dot{\lambda} - \varphi - \vartheta) \Lambda, \quad \delta_b c = 0,
\]

(7.30)

where \(\Lambda\) is the infinitesimal and anticommuting BRST parameter.

7.2 Relating the first-class and second-class theories through FFBRST formulation: examples

In this section, we consider two examples to show the connection between the generating functionals for theories with first-class and second-class constraints. First we show the connection between the Stueckelberg theory and the Proca theory for massive vector fields. In the second example we link the gauge invariant and the gauge variant theory for the self-dual chiral boson.

7.2.1 Connecting Stueckelberg and Proca theories

We start with the linearized form of the Stueckelberg effective action (7.13) by introducing a Nakanishi-Lautrup type auxiliary field \(B\) as

\[
S_{ST} = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} M^2 \left( A_{\mu} - \frac{1}{M} \partial_{\mu} B \right)^2 + \frac{\chi}{2} B^2 - B (\partial_\mu A^\mu + \chi M B) - \dot{c} (\dot{\varrho}^2 + \chi M^2) c \right],
\]

(7.31)
which is invariant under the following off-shell nilpotent BRST transformation:
\[
\delta_b A_\mu = \partial_\mu c \ \Lambda, \quad \delta_b B = M c \ \Lambda, \quad \delta_b c = 0, \quad \delta_b \bar{c} = B \ \Lambda, \quad \delta_b B = 0.
\]
(7.32)
The FFBRST transformation corresponding to the above BRST transformation is constructed as,
\[
\delta_b A_\mu = \partial_\mu c \ \Theta_b[\phi], \quad \delta_b B = M c \ \Theta_b[\phi], \quad \delta_b c = 0, \quad \delta_b \bar{c} = B \ \Theta_b[\phi], \quad \delta_b B = 0,
\]
(7.33)
where \(\Theta_b[\phi]\) is an arbitrary finite field dependent parameter but still anticommuting in nature.

To establish the connection we construct a finite field dependent parameter \(\Theta_b\) obtainable from
\[
\Theta'_b = i\gamma \int d^4x \left[ \bar{c} \left( \chi MB - \frac{\chi}{2} B + \partial_\mu A^\mu \right) \right],
\]
(7.34)
via Eq. (2.5), where \(\gamma\) is an arbitrary parameter.

Using Eq. (2.7) the infinitesimal change in nontrivial Jacobian is calculated for this finite field dependent parameter as
\[
\frac{1}{J} \frac{dJ}{d\kappa} = i\gamma \int d^4x \left[ B \left( \chi MB - \frac{\chi}{2} B + \partial_\mu A^\mu \right) \right],
\]
(7.35)
where the equation of motion for the antighost field, \((\partial^2 + \chi M^2)c = 0\), has been used.

We now make the following ansatz for \(S_1\):
\[
S_1 = \int d^4x \left[ \xi_1(\kappa) B^2 + \xi_2(\kappa) B \partial_\mu A^\mu + \xi_3(\kappa) \chi MBB \right],
\]
(7.36)
where \(\xi_i, \ (i = 1, 2, 3)\) are arbitrary \(\kappa\) dependent parameter and satisfy the following initial conditions: \(\xi_i(\kappa = 0) = 0\). Now, using the relation in Eq. (2.3) we calculate \(\frac{dS_1}{d\kappa}\) as
\[
\frac{dS_1}{d\kappa} = \int d^4x \left[ B^2 \xi'_1 + B \partial_\mu A^\mu \xi'_2 + \chi MBB \xi'_3 \right],
\]
(7.37)
where prime denotes the differentiation with respect to \(\kappa\). The Jacobian contribution can be written as \(e^{S_1}\) if the essential condition in Eq. (2.9) is satisfied. This leads to
\[
\int [D\phi] \ e^{i(S_{ST}+S_1)} \left[ iB^2(\xi'_1 + \gamma \frac{\chi}{2}) + iB\partial_\mu A^\mu(\xi'_2 - \gamma) + i\chi MBB(\xi'_3 - \gamma) \right] = 0.
\]
(7.38)
Equating the coefficients of terms \(iB^2, iB\partial_\mu A^\mu,\) and \(i\chi MBB\) from both sides of the above condition, we get the following differential equations:
\[
\xi'_1 + \gamma \frac{\chi}{2} = 0, \quad \xi'_2 - \gamma = 0, \quad \xi'_3 - \gamma = 0.
\]
(7.39)
To obtain the solution of the above equations we put \(\gamma = 1\) without any loss of generality. The solutions satisfying initial conditions are given as
\[
\xi_1 = -\frac{\chi}{2} \bar{c}, \quad \xi_2 = \kappa, \quad \xi_3 = \kappa.
\]
(7.40)
The transformed action can be obtained by adding $S_1(\kappa = 1)$ to $S_{ST}$ as

$$S_{ST} + S_1 = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} M^2 \left( A_\mu - \frac{1}{M} \partial_\mu B \right)^2 - \bar{c}(\partial^2 + \chi M^2)c \right]. \quad (7.41)$$

Now the generating functional under FFBRST transforms as

$$Z' = \int [DA_\mu] e^{i(S_{ST} + S_1)} = Z_{P}. \quad (7.42)$$

Integrating over the $B, c, \bar{c}$ fields, the above expression reduces to the generating functional for the Proca model up to some renormalization constants as follows:

$$Z' = \int [DA_\mu] e^{iS_P} = Z_{P}. \quad (7.43)$$

Therefore,

$$Z_{ST} = \int [D\phi] e^{iS_{ST}} \rightarrow Z_{P} = \int [DA_\mu] e^{iS_P}. \quad (7.44)$$

Thus, by constructing appropriate finite field dependent parameter (given in Eq. (7.33)) we have shown that the generating functional for Stueckelberg theory is connected to the generating functional for the Proca theory through FFBRST transformation. This indicates that the Green functions in these two theories are related through FFBRST formulation.

### 7.2.2 Relating gauge invariant and the gauge variant theory for chiral boson

To see the connection between the gauge invariant and variant theories for chiral boson, we start with the effective action for the gauge invariant self-dual chiral boson theory as

$$S^{II}_{CB} = \int d^2x \left[ \frac{1}{2} \dot{\varphi}^2 - \frac{1}{2} \varphi'^2 + \lambda(\dot{\varphi} - \varphi') - \frac{1}{2} \dot{\vartheta}^2 - \frac{1}{2} \vartheta'^2 + \varphi' \vartheta' + \dot{\vartheta} \vartheta' - \dot{\varphi} \varphi' - \lambda(\dot{\varphi} - \varphi') + \frac{1}{2} B^2 + B(\dot{\lambda} - \varphi - \vartheta) + \dot{\vartheta} c - 2\dot{\varphi} \bar{c} \right], \quad (7.45)$$

where we have linearized the gauge-fixing part of the effective action by introducing the extra auxiliary field $B$. This effective action is invariant under the following infinitesimal BRST transformation:

$$\delta_b \varphi = c \Lambda, \quad \delta_b \lambda = -\dot{c} \Lambda, \quad \delta_b \vartheta = c \Lambda,$$

$$\delta_b \bar{c} = B \Lambda, \quad \delta_b B = 0, \quad \delta_b c = 0. \quad (7.46)$$

Corresponding FFBRST transformation is written as

$$\delta_b \varphi = c \Theta_b[\phi], \quad \delta_b \lambda = -\dot{c} \Theta_b[\phi], \quad \delta_b \vartheta = c \Theta_b[\phi],$$

$$\delta_b \bar{c} = B \Theta_b[\phi], \quad \delta_b B = 0, \quad \delta_b c = 0, \quad (7.47)$$

where $\Theta_b[\phi]$ is arbitrary finite field dependent parameter, which we have to constructed. In this case we construct the finite field dependent BRST parameter $\Theta_b[\phi]$ obtainable from

$$\Theta_b = i\gamma \int d^2x \left[ \bar{c}(\dot{\lambda} - \varphi - \vartheta + \frac{1}{2} B) \right], \quad (7.48)$$
7.2. Relating the first-class and second-class theories through FFBRST formulation: examples

using Eq. (2.5) and demand that the corresponding BRST transformation will lead to the gauge variant theory for self-dual chiral boson.

To justify our claim we calculate the change in Jacobian, using equation of motion for antighost field, as

$$\frac{1}{J} \frac{dJ}{d\kappa} = i\gamma \int d^2 x \left[ B(\lambda - \varphi - \vartheta + \frac{1}{2} B) \right]. \quad (7.49)$$

We make an ansatz for local functional $S_1$ as,

$$S_1 = \int d^2 x \left[ \xi_1(\kappa) B^2 + \xi_2(\kappa) B(\dot{\lambda} - \varphi - \vartheta) \right]. \quad (7.50)$$

The change in $S_1$ with respect to $\kappa$ is calculated as

$$\frac{dS_1}{d\kappa} = \int d^2 x \left[ \xi_1' B^2 + \xi_2' B(\dot{\lambda} - \varphi - \vartheta) \right]. \quad (7.51)$$

Now, the necessary condition in Eq. (2.9) leads to the following equation:

$$\int [D\phi] e^{i(S_{II}^{CB} + S_1)} \left[ iB^2(\xi_1' - \frac{\gamma}{2}) + iB(\dot{\lambda} - \varphi - \vartheta)(\xi_2' - \gamma) \right] = 0. \quad (7.52)$$

Equating the coefficients of terms $iB^2$ and $iB(\dot{\lambda} - \varphi - \vartheta)$ from both sides of above condition, we get following differential equations:

$$\xi_1' - \frac{\gamma}{2} = 0, \quad \xi_2' - \gamma = 0. \quad (7.53)$$

The solutions of above equations are $\xi_1 = -\frac{1}{2} \kappa$, $\xi_2 = -\kappa$, where we have taken the parameter $\gamma = -1$. The transformed action is obtained by adding $S_1(\kappa = 1)$ to $S_{II}^{CB}$ as

$$S_{II}^{CB} + S_1 = \int d^2 x \left[ \frac{1}{2} \dot{\varphi}^2 - \frac{1}{2} \varphi'^2 + \lambda(\dot{\varphi} - \varphi') - \frac{1}{2} \vartheta^2 - \frac{1}{2} \varphi' \vartheta' + \dot{\vartheta} \varphi' - \dot{\varphi} \vartheta' - \lambda(\dot{\vartheta} - \vartheta') + \dot{\bar{c}} \dot{c} - 2\bar{c} \bar{c} \right]. \quad (7.54)$$

Now the transformed generating functional becomes

$$Z' = \int [D\varphi D\lambda D\vartheta DC DC] e^{i(S_{II}^{CB} + S_1)}. \quad (7.55)$$

Performing integration over fields $\vartheta, c$ and $\bar{c}$, the above generating functional reduces to the generating functional for the self dual chiral boson up to some constant as

$$Z' = \int [D\varphi D\lambda] e^{iS_{CB}} = Z_{CB}. \quad (7.56)$$

Therefore,

$$Z_{II}^{CB} \left( = \int [D\phi] e^{iS_{II}^{CB}} \right) \Rightarrow Z_{CB} \left( = \int [D\varphi D\lambda] e^{iS_{CB}} \right). \quad (7.57)$$

Thus, the generating functionals corresponding to the gauge invariant and gauge non-invariant theory for self-dual chiral boson are connected through the FFBRST transformation given in Eq. (7.47).

We end up this section by making conclusion that using FFBRST formulation the generating functional for the theory with second-class constraint can be achieved by generating functional for theory with first-class constraint.
7.3 First-class and second-class theories: FF-anti-BRST formulation

In this section, we consider FF-anti-BRST formulation to show the same connection between the generating functionals for theories with first-class and second-class constraints with same examples. The FF-anti-BRST transformation is also developed in same fashion as FFBRST transformation, the only key difference is the role of ghost fields are interchanged with antighost fields and vice-versa.

7.3.1 Relating Stueckelberg and Proca theories

We start with anti-BRST symmetry transformation for effective action given in Eq. (7.31), as

\[
\begin{align*}
\delta_{ab} A_\mu &= \partial_\mu \bar{c} \Lambda, \quad \delta_{ab} B = M \bar{c} \Lambda, \quad \delta_{ab} c = -\mathcal{B} \Lambda, \\
\delta_{ab} \bar{c} &= 0, \quad \delta_{ab} B = 0,
\end{align*}
\]

(7.58)

where \( \Lambda \) is infinitesimal, anticommuting and global parameter. The FF-anti-BRST transformation corresponding to the above anti-BRST transformation is constructed as,

\[
\begin{align*}
\delta_{ab} A_\mu &= \partial_\mu \Theta_{ab}, \quad \delta_{ab} B = M \bar{c} \Theta_{ab}, \quad \delta_{ab} c = -\mathcal{B} \Theta_{ab}, \\
\delta_{ab} \bar{c} &= 0, \quad \delta_{ab} B = 0,
\end{align*}
\]

(7.59)

where \( \Theta_{ab} \) is an arbitrary finite field dependent parameter but still anticommuting in nature. To establish the connection we choose a finite field dependent parameter \( \Theta_{ab} \) obtainable from

\[
\Theta'_{ab} = -i\gamma \int d^4x \left[ c \left( \chi MB - \frac{\chi}{2} \mathcal{B} + \partial_\mu A^\mu \right) \right],
\]

(7.60)

using Eq. (3.55), where \( \gamma \) is an arbitrary parameter.

Using Eq. (3.56) the infinitesimal change in nontrivial Jacobian can be calculated for this finite field dependent parameter as

\[
\frac{1}{J} \frac{dJ}{d\kappa} = -i\gamma \int d^4x \left[ -B \left( \chi MB - \frac{\chi}{2} \mathcal{B} + \partial_\mu A^\mu \right) \right].
\]

(7.61)

To Jacobian contribution can be expressed as \( e^{iS_2} \). To calculate \( S_2 \) we make following ansatz:

\[
S_2 = \int d^4x \left[ \xi_5(\kappa) B^2 + \xi_6(\kappa) B \partial_\mu A^\mu + \xi_7(\kappa) \chi MBB \right],
\]

(6.62)

where \( \xi_i \), \( i = 5, \ldots, 7 \) are arbitrary \( \kappa \) dependent parameter and satisfy following initial conditions: \( \xi_i(\kappa = 0) = 0 \). Now, infinitesimal change in \( S_2 \) is calculated as

\[
\frac{dS_2}{d\kappa} = \int d^4x \left[ B^2 \xi'_5 + B \partial_\mu A^\mu \xi'_6 + \chi MBB \xi'_7 \right],
\]

(7.63)

where prime denotes the differentiation with respect to \( \kappa \).
Putting the expressions (7.61) and (7.63) in the essential condition given in Eq. (2.9), we obtain
\[
\int \left[ D\phi \right] e^{i(S_{ST} + S_2)} \left[ B^2 \left( \xi'_5 + \gamma \frac{X}{2} \right) + B \partial_\mu A^\mu (\xi'_6 - \gamma) + \chi MBB (\xi'_7 - \gamma) \right] = 0. \tag{7.64}
\]
Equating the coefficients of terms \(iB^2, iB \partial_\mu A^\mu,\) and \(i\chi MBB\) from both sides of above condition, we get following differential equations:
\[
\xi'_5 + \gamma \frac{X}{2} = 0, \quad \xi'_6 - \gamma = 0, \quad \xi'_7 - \gamma = 0. \tag{7.65}
\]
The solutions of the above differential equation for \(\gamma = 1\) are \(\xi_5 = -\frac{X}{2} \kappa, \xi_6 = \kappa, \xi_7 = \kappa.\) The transformed action can be obtained by adding \(S_2 (\kappa = 1)\) to \(S_{ST}\) as
\[
S_{ST} + S_2 = \int d^4 x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} M^2 \left( A_\mu - \frac{1}{M} \partial_\mu B \right)^2 - \bar{c} (\partial^2 + \chi M^2) c \right]. \tag{7.66}
\]
We perform integration over \(B, c\) and \(\bar{c}\) fields to remove the divergence of the transformed generating functional \(Z' = \int [DA_\mu D\bar{c} Dc] e^{i(S_{ST} + S_2)}\) and hence we get the generating functional for the Proca model as
\[
Z' = \int [DA_\mu] e^{iS_P} = Z_P. \tag{7.67}
\]
Therefore, \(Z_{ST,FF-anti-BRST} \rightarrow Z_P.\) Thus, by constructing the appropriate finite field dependent parameter (given in Eq. (7.59)), we have shown that the generating functional for the Stueckelberg theory is related to the generating functional for the Proca theory through the FF-anti-BRST transformation also. However, the finite field dependent parameter is different from that one involved in the FFBRST transformation. This indicates that the Green functions in these two theories are related through the FFBRST and the FF-anti-BRST transformations.

### 7.3.2 Mapping between the gauge invariant and the gauge variant theory for chiral boson

To connect the gauge invariant and gauge variant theories for the chiral boson through the FF-anti-BRST transformation, first of all we write the anti-BRST transformation for effective action (7.45) as
\[
\delta_{ab} \varphi = \bar{c} \Lambda, \quad \delta_{ab}\lambda = -\dot{\bar{c}} \Lambda, \quad \delta_{ab}\vartheta = \bar{c} \Lambda, \\
\delta_{ab} c = -B \Lambda, \quad \delta_{ab} B = 0, \quad \delta_{ab}\bar{c} = 0. \tag{7.68}
\]
The corresponding FF-anti-BRST transformation is written as
\[
\delta_{ab} \varphi = \bar{c} \Theta_{ab}[\phi], \quad \delta_{ab}\lambda = -\dot{\bar{c}} \Theta_{ab}[\phi], \quad \delta_{ab}\vartheta = \bar{c} \Theta_{ab}[\phi], \\
\delta_{ab} c = -B \Theta_{ab}[\phi], \quad \delta_{ab} B = 0, \quad \delta_{ab}\bar{c} = 0, \tag{7.69}
\]
where \(\Theta_{ab}[\phi]\) is the arbitrary finite field dependent parameter. In this case, we construct the finite field dependent anti-BRST parameter \(\Theta_{ab}[\phi]\) obtainable from
\[
\Theta'_{ab} = -i\gamma \int d^2 x \left[ c (\dot{\lambda} - \varphi - \vartheta + \frac{1}{2} B) \right], \tag{7.70}
\]
using Eq. (3.55) and demand that the corresponding anti-BRST transformation will lead to the
gauge variant theory for the self-dual chiral boson.

We make an ansatz for local functional $S_2$ in this case as

$$S_2 = \int d^2 x \left[ \xi_5(\kappa) B_2^2 + \xi_6(\kappa) B(\dot{\lambda} - \varphi - \vartheta) \right]. \tag{7.71}$$

Now, the necessary condition in Eq. (2.9) leads to the following equation:

$$\int [D\phi] e^{i(S_{II}^{\mu} + S_2)} \left[ iB_2^2 (\xi_5' - \frac{\gamma}{2}) + iB(\dot{\lambda} - \varphi - \vartheta)(\xi_6' - \gamma) \right] = 0. \tag{7.72}$$

Equating the coefficients of different terms on both sides of the above equation, we get the following differential equations:

$$\xi_5' - \frac{\gamma}{2} = 0, \quad \xi_6' - \gamma = 0. \tag{7.73}$$

The solutions of above equations are $\xi_5 = -\frac{1}{2}\kappa$, $\xi_6 = -\kappa$, where the parameter $\gamma = -1$. The transformed action can be obtained by adding $S_2(\kappa = 1)$ to $S_{II}^{\mu}$ as

$$S_{II}^{\mu} + S_2 = \int d^2 x \left[ \frac{1}{2} \dot{\varphi}^2 - \frac{1}{2} \varphi'' + \lambda(\dot{\varphi} - \varphi') - \frac{1}{2} \dot{\vartheta}^2 - \frac{1}{2} \vartheta'^2 + \varphi' \vartheta' - \dot{\varphi} \vartheta' + \lambda(\dot{\vartheta} - \vartheta') + \dot{\bar{c}} \bar{c} - 2 \bar{c} \bar{c} \right]. \tag{7.74}$$

After functional integration over fields $\vartheta, c$ and $\bar{c}$ in the expression of transformed generating functional, we get the generating functional as

$$Z' = \int [D\varphi D\lambda] e^{iS_{CB}} = Z_{CB}. \tag{7.75}$$

Therefore, $Z_{II}^{\mu} \longrightarrow \text{anti-BRST} \longrightarrow Z_{CB}$. Thus, the generating functionals corresponding to the gauge invariant and the gauge non-invariant theory for the self-dual chiral boson are also connected through the FF-anti-BRST transformation given in Eq. (7.69).

We end up this section by making comment that the generating functional for the theory with the second-class constraint can be obtained from the generating functional for theory with the first-class constraint using both FFBRST and FF-anti-BRST transformations. The finite parameters involved in the FFBRST transformation, which are responsible for these connection, are different from the parameters in FF-anti-BRST transformation.

## 7.4 Conclusions

The Stueckelberg theory for massive spin 1 field and the gauge invariant theory for the self-dual chiral boson are the first-class theories. On the other hand, the Proca theory for massive spin 1 field and the gauge variant theory for the self-dual chiral boson are theories with the second-class constraint. We have shown that the FFBRST transformation relates the generating functionals of second-class theory and first-class theory. The path integral measure in the definition of
generating functional is not invariant under such FFBRST transformation and is responsible for such connections. The Jacobian for path integral measure under such a transformation with an appropriate finite parameter cancels the extra parts of the first-class theory. Our result is supported by two explicit examples. In the first case we have related the generating functional of the Stueckelberg theory to the generating functional of the Proca model and in the second case the generating functionals corresponding to the gauge invariant theory and the gauge variant theory for the self-dual chiral boson have linked through FFBRST transformation with appropriate choices of the finite field dependent parameter. The same goal has been achieved by using the FF-anti-BRST transformation. These formulations can be applied to connect the generating functionals for any first-class and second-class theories provided appropriate finite parameters are constructed. The complicacy arises due to the nonlocal and field dependent Dirac brackets in the quantization of second-class theories can thus be avoided by using FFBRST/FF-anti-BRST formulations which relate the Green functions of second-class theories to the first-class theories. These formulations can be applied to connect the generating functionals for any first-class (e.g. non-Abelian gauge theories) and second-class theories provided appropriate finite parameters are constructed.
Chapter 8

Hodge-de Rham theorem in the BRST context

In this chapter we study the different forms of BRST symmetry \[73\]. In particular we investigate co-BRST and anti-co-BRST along with usual BRST and anti-BRS T symmetries. The nilpotent conserved charges for all these symmetries are calculated and have shown to satisfy the algebra analogous to the algebra satisfied by de Rham cohomological operators. These results are shown in a particular model namely (1+1) dimensional theory for a self-dual chiral boson.

8.1 Self-dual chiral boson: preliminary idea

We start with the gauge non-invariant model \[66\] in (1+1) dimensions for a single self-dual chiral boson. The Lagrangian density for such a theory is given by

\[ \mathcal{L} = \frac{1}{2} \dot{\varphi}^2 - \frac{1}{2} \varphi'{}^2 + \lambda (\dot{\varphi} - \varphi'), \quad (8.1) \]

where overdot and prime denote time and space derivatives, respectively, and \( \lambda \) is a Lagrange multiplier. The field \( \varphi \) satisfies the self-duality condition \( \dot{\varphi} = \varphi' \) in this case. We choose the Lorentz metric \( g_{\mu\nu} = (1, -1) \) where \( \mu, \nu = 0, 1 \). The associated momenta for the field \( \varphi \) and Lagrange multiplier are calculated as

\[ \pi_{\varphi} = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\varphi} + \lambda, \quad \pi_{\lambda} = \frac{\partial \mathcal{L}}{\partial \lambda} = 0, \quad (8.2) \]

which shows that the model has following primary constraint

\[ \Omega_1 \equiv \pi_{\lambda} \approx 0. \quad (8.3) \]

The expression for Hamiltonian density corresponding to above Lagrangian density \( \mathcal{L} \) is

\[ \mathcal{H} = \pi_{\varphi} \dot{\varphi} + \pi_{\lambda} \dot{\lambda} - \mathcal{L} = \frac{1}{2} (\pi_{\varphi} - \lambda)^2 + \frac{1}{2} \dot{\varphi}'{}^2 + \lambda \varphi'. \quad (8.4) \]

Further we write the total Hamiltonian density corresponding to this \( \mathcal{L} \) by introducing Lagrange multiplier field \( \omega \) for the primary constraint \( \Omega_1 \) as

\[ \mathcal{H}_T = \frac{1}{2} (\pi_{\varphi} - \lambda)^2 + \frac{1}{2} \dot{\varphi}'{}^2 + \lambda \varphi' + \omega \pi_{\lambda}. \quad (8.5) \]

Following the Dirac prescription \[13\], we obtain the secondary constraint in this case as

\[ \Omega_2 \equiv \dot{\pi}_{\lambda} = \{ \pi_{\lambda}, \mathcal{H} \} = \pi_{\varphi} - \lambda - \varphi' \approx 0. \quad (8.6) \]
8.1. Self-dual chiral boson: preliminary idea

The Poison bracket for primary and secondary constraint is nonvanishing, \( \{\Omega_1, \Omega_2\} \neq 0 \). This implies the constraints \( \Omega_1 \) and \( \Omega_2 \) are of second-class, which is an essential feature of a gauge variant theory (model).

This model is quantized by establishing the commutation relation [66]

\[
[\varphi(x), \pi_{\varphi}(y)] = [\varphi(x), \lambda(y)] = -i\delta(x-y) \\
2[\lambda(x), \pi_{\varphi}(y)] = [\lambda(x), \lambda(y)] = 2i\delta'(x-y),
\]

and the rest of the commutators vanish.

8.1.1 Wess-Zumino term and Hamiltonian formulation

To construct a gauge invariant theory corresponding to this gauge non-invariant model for chiral bosons, one generally introduces the Wess-Zumino term in the Lagrangian density \( L \). For this purpose one has to enlarge the Hilbert space of the theory by introducing a new quantum field \( \vartheta \), called as Wess-Zumino field, through the redefinition of fields \( \varphi \) and \( \lambda \) in the original Lagrangian density \( \text{Eq. (8.1)} \) as follows [21]:

\[
\varphi \rightarrow \varphi - \vartheta, \quad \lambda \rightarrow \lambda + \dot{\vartheta}.
\]

With these redefinition of fields the modified Lagrangian density becomes

\[
\mathcal{L}^I = L - \frac{1}{2} \dot{\vartheta}^2 - \frac{1}{2} \vartheta'^2 + \varphi' \vartheta' + \dot{\vartheta} \vartheta' - \dot{\vartheta} \varphi' - \lambda(\dot{\vartheta} - \vartheta')
= L + \mathcal{L}^{WZ},
\]

where

\[
\mathcal{L}^{WZ} = -\frac{1}{2} \dot{\vartheta}^2 - \frac{1}{2} \vartheta'^2 + \varphi' \vartheta' + \dot{\vartheta} \vartheta' - \dot{\vartheta} \varphi' - \lambda(\dot{\vartheta} - \vartheta'),
\]

is the Wess-Zumino part of the \( \mathcal{L}^I \). It is easy to check that the above Lagrangian density is invariant under time-dependent chiral gauge transformation:

\[
\delta \varphi = \mu(x,t), \quad \delta \vartheta = \mu(x,t), \quad \delta \lambda = -\dot{\mu}(x,t) \\
\delta \pi_{\varphi} = 0, \quad \delta \pi_{\vartheta} = 0, \quad \delta p_\lambda = 0,
\]

where \( \mu(x,t) \) is an arbitrary function of the space and time. The canonical momenta for this gauge-invariant theory are calculated as

\[
\pi_\lambda = \frac{\partial \mathcal{L}^I}{\partial \dot{\lambda}} = 0, \quad \pi_{\varphi} = \frac{\partial \mathcal{L}^I}{\partial \dot{\varphi}} = \varphi + \lambda \\
\pi_{\vartheta} = \frac{\partial \mathcal{L}^I}{\partial \dot{\vartheta}} = -\dot{\vartheta} - \varphi' + \vartheta' - \lambda.
\]

This implies the theory \( \mathcal{L}^I \) possesses a primary constraint

\[
\psi_1 \equiv \pi_\lambda \approx 0.
\]

The Hamiltonian density corresponding to \( \mathcal{L}^I \) is then given by

\[
\mathcal{H}^I = \pi_{\varphi} \dot{\varphi} + \pi_{\vartheta} \dot{\vartheta} + \pi_\lambda \dot{\lambda} - \mathcal{L}^I.
\]
The total Hamiltonian density after the introduction of a Lagrange multiplier field $u$ corresponding to the primary constraint $\Psi_1$ becomes

$$H_I^T = \frac{1}{2} \pi_\varphi^2 - \frac{1}{2} \pi_\vartheta^2 + \pi_\vartheta \dot{\vartheta}' - \pi_\varphi \dot{\varphi}' - \lambda \pi_\varphi - \lambda \pi_\vartheta + u \pi_\lambda. \quad (8.16)$$

Following the Dirac method of constraint analysis we obtain the secondary constraint

$$\psi_2 \equiv (\pi_\varphi + \pi_\vartheta) \approx 0. \quad (8.17)$$

In Dirac’s quantization procedure \cite{13}, we have to change the first-class constraints of the theory into second-class constraints. To achieve this we impose some additional constraints on the system in the form of gauge-fixing conditions

$$\partial_\mu \vartheta = 0 \quad (\partial_0 \vartheta = \dot{\vartheta} = 0 \text{ and } -\partial_1 \vartheta = -\vartheta' = 0) \quad (8.18).$$

With the above choice of gauge-fixing conditions the extra constraints of the theory are

$$\xi_1 \equiv -\vartheta' \approx 0, \quad \xi_2 \equiv (\pi_\varphi - \vartheta' + \varphi' + \lambda) \approx 0. \quad (8.19)$$

Now, the total set of constraints after gauge fixing are

$$\chi_1 = \psi_1 \equiv \pi_\lambda \approx 0, \quad \chi_2 = \psi_2 \equiv (\pi_\varphi + \pi_\vartheta) \approx 0$$

$$\chi_3 = \xi_1 \equiv -\vartheta' \approx 0, \quad \chi_4 = \xi_2 \equiv (\pi_\varphi - \vartheta' + \varphi' + \lambda) \approx 0. \quad (8.19)$$

The nonvanishing commutators of gauge invariant theory are obtained as

$$[\varphi(x), \pi_\varphi(y)] = [\varphi(x), \lambda(y)] = -i \delta(x - y) \quad (8.20)$$

$$2[\lambda(x), \pi_\varphi(y)] = [\lambda(x), \lambda(y)] = +2i \delta'(x - y) \quad (8.21)$$

$$[\vartheta(x), \pi_\vartheta(y)] = 2[\varphi(x), \pi_\vartheta(y)] = -2i \delta(x - y) \quad (8.22)$$

$$[\lambda(x), \pi_\vartheta(y)] = -i \delta'(x - y). \quad (8.23)$$

We end up the section with the conclusion that the above relations (8.20)–(8.23), together with $H_I^T$ [Eq. (8.10)], reproduce the same quantum system described by $L$ under the gauge condition (8.18). This is similar to the quantization of a gauge invariant chiral Schwinger model (with an appropriate Wess-Zumino term) \cite{102}.

### 8.2 BFV formulation for model of self-dual chiral boson

In the BFV formulation of self-dual chiral boson, we need to introduce a pair of canonically conjugate ghosts $(c, p)$ with ghost numbers 1 and $-1$ respectively, for the first-class constraint $\pi_\lambda = 0$, and another pair of ghosts $(\bar{c}, \bar{p})$ with ghost number $-1$ and 1, respectively, for the secondary constraint, $(\pi_\varphi + \pi_\vartheta) = 0$. The effective action for this $(1 + 1)$ dimensional theory with a single self-dual chiral boson in this extended phase space then becomes

$$S_{\text{eff}} = \int d^2x \left[ \pi_\varphi \dot{\varphi} + 2 \lambda \dot{\varphi} - \pi_\vartheta \dot{\vartheta} + p_u \dot{u} - \pi_\lambda \dot{\lambda} - \frac{1}{2} \pi_\varphi^2 + \frac{1}{2} \pi_\vartheta^2 \right] + \pi_\varphi (\vartheta' - \varphi') + \bar{c} \dot{p} + c \dot{\bar{p}} - \{Q_b, \Psi\}, \quad (8.24)$$

where $Q_b$ is the BRST charge and $\Psi$ is the gauge-fixed fermion.
The generating functional, for any gauge invariant effective theory having $\Psi$ as a gauge-fixed fermion, is defined as
\[
Z_{\Psi} = \int D\phi \exp \left[ i \int d^2 x \, S_{\text{eff}} \right],
\]
where $\phi$ is generic notation for all the dynamical field involved in the effective theory. The BRST symmetry generator for this theory is written as
\[
Q_b = i c (\pi_{\varphi} + \pi_{\vartheta}) - i \bar{p} \pi_{\lambda}. \tag{8.26}
\]
The canonical brackets are defined for all dynamical variables as
\[
\begin{align*}
[\vartheta, \pi_{\vartheta}] &= -i, \quad [\varphi, \pi_{\varphi}] = -i, \quad [\lambda, \pi_{\lambda}] = -i, \\
[u, p_u] &= -i, \quad \{c, \dot{c}\} = i, \quad \{\bar{c}, \dot{c}\} = -i,
\end{align*}
\]
and the rest of the brackets are zero. The nilpotent BRST transformation, using Eqs. (2.29) and (8.26), is then explicitly calculated as
\[
\begin{align*}
s_b \varphi &= -c, \quad s_b \lambda = \bar{p}, \quad s_b \vartheta = -c \quad s_b \pi_{\varphi} = 0, \quad s_b u = 0, \\
s_b \pi_{\vartheta} &= 0, \quad s_b p = (\pi_{\varphi} + \pi_{\vartheta}), \quad s_b \bar{c} = \pi_{\lambda}, \quad s_b \pi_{\lambda} = 0, \quad s_b c = 0, \quad s_b \bar{p} u = 0.
\end{align*}
\]
In BFV formulation the generating functional is independent of the gauge-fixed fermion \cite{5, 6}, hence we have the freedom to choose it in the convenient way as
\[
\Psi = p\lambda + \bar{c} \left( \vartheta + \varphi + \xi \pi_{\lambda} \right), \tag{8.29}
\]
where $\xi$ is arbitrary gauge parameter.

Putting the value of $\Psi$ in Eq. (8.24) and using Eq. (8.26), we get
\[
S_{\text{eff}} = \int d^2 x \left[ \pi_{\varphi} \dot{\varphi} + \pi_{\vartheta} \dot{\vartheta} + p_u \dot{u} - \pi_{\lambda} \dot{\lambda} - \frac{1}{2} \pi_{\varphi}^2 + \frac{1}{2} \pi_{\vartheta}^2 - \pi_{\vartheta} \varphi' - \varphi' \right] + \dot{c} p + \dot{\bar{c}} \bar{p} + \lambda (\pi_{\varphi} + \pi_{\vartheta}) + 2 \dot{c} \bar{c} - \bar{p} p + \pi_{\lambda} \left( \vartheta + \varphi + \xi \pi_{\lambda} \right).
\]

The generating functional for this effective theory is then expressed as
\[
Z_{\Psi} = \int D\phi' \exp \left[ i \int d^2 x \left\{ \pi_{\varphi} \dot{\varphi} + \pi_{\vartheta} \dot{\vartheta} + p_u \dot{u} - \pi_{\lambda} \dot{\lambda} \right. \right.
\]
\[
\left. \left. - \frac{1}{2} \pi_{\varphi}^2 + \frac{1}{2} \pi_{\vartheta}^2 - \pi_{\vartheta} (\varphi' - \varphi') + \dot{c} p + \dot{\bar{c}} \bar{p} + \lambda (\pi_{\varphi} + \pi_{\vartheta}) \\
+ 2 \dot{c} \bar{c} - \bar{p} p + \pi_{\lambda} \left( \vartheta + \varphi + \xi \pi_{\lambda} \right) \right\} \right]. \tag{8.31}
\]
Performing the integration over $p$ and $\bar{p}$ in the above functional integration we further obtain
\[
Z_{\Psi} = \int D\phi' \exp \left[ i \int d^2 x \left\{ \pi_{\varphi} \dot{\varphi} + \pi_{\vartheta} \dot{\vartheta} + p_u \dot{u} - \pi_{\lambda} \dot{\lambda} \right. \right.
\]
\[
\left. \left. - \frac{1}{2} \pi_{\varphi}^2 + \frac{1}{2} \pi_{\vartheta}^2 - \pi_{\vartheta} (\varphi' - \varphi') + \dot{c} \bar{c} + \lambda (\pi_{\varphi} + \pi_{\vartheta}) \\
+ 2 \dot{c} \bar{c} + \pi_{\lambda} \left( \vartheta + \varphi + \xi \pi_{\lambda} \right) \right\} \right], \tag{8.32}
\]
8.3 Nilpotent symmetries: many guises

where $D\phi'$ is the path integral measure for effective theory when integrations over fields $p$ and $\bar{p}$ are carried out. Taking the arbitrary gauge parameter $\xi = 1$ and performing the integration over $\pi\lambda$, we obtain an effective generating functional as

$$Z_\Psi = \int D\phi'' \exp \left[ i \int d^2x \left\{ \pi_\varphi \dot{\varphi} + \pi_\vartheta \dot{\vartheta} + p_u \dot{u} - \frac{1}{2} \pi_\varphi^2 + \frac{1}{2} \pi_\vartheta^2 + \pi_\varphi (\varphi' - \vartheta' + \lambda) + \pi_\vartheta (\varphi' - \vartheta' + \lambda) + \dot{\bar{c}}\dot{c} + \pi_\varphi \lambda - 2\dot{\bar{c}}\dot{c} - \frac{(\dot{\lambda} - \vartheta - \varphi)^2}{2} \right\} \right],$$

(8.33)

where $D\phi''$ denotes the measure corresponding to all the dynamical variable involved in this effective action. The expression for effective action in the above equation is exactly same as the BRST invariant effective action in Ref. [103]. The BRST symmetry transformation for this effective theory is

$$s_b \varphi = -c, \quad s_b \lambda = \dot{c}, \quad s_b \vartheta = -c, \quad s_b \pi_\varphi = 0, \quad s_b u = 0,$$

$$s_b \pi_\vartheta = 0, \quad s_b \bar{c} = -(\dot{\lambda} - \vartheta - \varphi), \quad s_b c = 0, \quad s_b p_u = 0,$$

(8.34)

which is on-shell nilpotent. Antighost equation of motion (i.e. $\ddot{\bar{c}} + 2\dot{\bar{c}} = 0$) is required to show the nilpotency.

8.3 Nilpotent symmetries: many guises

In this section we study the different forms of the nilpotent BRST symmetry of the system of single self-dual chiral boson. In particular, we discuss co-BRST anti-co-BRST, bosonic and ghost symmetries in the context of self-dual chiral boson.

8.3.1 Off-shell BRST and anti-BRST Symmetry

To study the off-shell BRST and anti-BRST transformations we incorporate Nakanishi-Lautrup type auxiliary field $B$ to linearize the gauge-fixing part of the effective action written as

$$S_{eff} = \int d^2x \mathcal{L}_{eff},$$

(8.35)

where,

$$\mathcal{L}_{eff} = \pi_\varphi \dot{\varphi} + \pi_\vartheta \dot{\vartheta} + p_u \dot{u} - \frac{1}{2} \pi_\varphi^2 + \frac{1}{2} \pi_\vartheta^2 + \pi_\varphi (\varphi' - \vartheta' + \lambda) + \pi_\vartheta (\varphi' - \vartheta' + \lambda) + \frac{1}{2} B^2 + B (\lambda - \varphi + \vartheta) + \dot{\bar{c}}\dot{c} - 2\dot{\bar{c}}\dot{c}. $$

(8.36)

This effective theory is invariant under the following off-shell nilpotent BRST transformation:

$$s_b \varphi = -c, \quad s_b \lambda = \dot{c}, \quad s_b \vartheta = -c,$$

$$s_b \pi_\varphi = 0, \quad s_b u = 0, \quad s_b \pi_\vartheta = 0,$$

$$s_b \bar{c} = B, \quad s_b B = 0, \quad s_b c = 0, \quad s_b p_u = 0.$$

(8.37)
The corresponding anti-BRST symmetry transformation for this theory is written as

\[ s_{ab} \varphi = -\bar{c}, \quad s_{ab} \lambda = \dot{\bar{c}}, \quad s_{ab} \vartheta = -\bar{c} \]
\[ s_{ab} \pi = 0, \quad s_{ab} u = 0, \quad s_{ab} \pi \vartheta = 0 \]
\[ s_{ab} c = -B, \quad s_{ab} B = 0, \quad s_{ab} \bar{c} = 0, \quad s_{ab} p = 0. \]  

(8.38)

The conserved BRST and anti-BRST charges \( Q_b \) and \( Q_{ab} \), respectively, which are the generator of the above BRST and anti-BRST transformations, are

\[ Q_b = i(\pi \varphi + \pi \vartheta) c - i \pi \lambda \bar{c}, \]  

(8.39)

\[ Q_{ab} = i(\pi \varphi + \pi \vartheta) \bar{c} - i \pi \lambda \bar{c}. \]  

(8.40)

Further by using the equations of motion

\[ B + \dot{\pi} \varphi = 0, \quad \dot{\varphi} + B + \pi \vartheta = 0, \quad \dot{\vartheta} + \pi \varphi + \varphi' + \dot{\varphi}' + \lambda = 0, \quad \dot{B} = \pi \varphi + \pi \vartheta \]
\[ \dot{u} = 0, \quad \dot{p} = 0, \quad \ddot{\bar{c}} + 2\bar{c} = 0, \quad \ddot{c} + 2c = 0 \]
\[ B + \lambda - \varphi - \vartheta = 0, \]  

(8.41)

it can be shown that these charges are constant of motion i.e. \( \dot{Q}_b = 0, \dot{Q}_{ab} = 0 \) and satisfy the relations

\[ Q_b^2 = 0, \quad Q_{ab}^2 = 0, \quad Q_b Q_{ab} + Q_{ab} Q_b = 0. \]  

(8.42)

To arrive at these relations, we have used the canonical brackets [Eq. (8.27)] of the fields and the definition of canonical momenta,

\[ \pi_\lambda = B, \quad \pi_\bar{c} = \dot{c}, \quad \pi_c = -\dot{\bar{c}}, \quad \pi_u = p_u. \]  

(8.43)

We come to the end of this section with the remark that the condition for the physical states \( Q_b |phys\rangle = 0 \) and \( Q_{ab} |phys\rangle = 0 \) leads to the requirement that

\[ (\pi \varphi + \pi \vartheta) |phys\rangle = 0 \]  

(8.44)

and

\[ \pi_\lambda |phys\rangle = 0. \]  

(8.45)

This implies that the operator form of the first-class constraints \( \pi_\lambda \approx 0 \) and \( (\pi \varphi + \pi \vartheta) \approx 0 \) annihilate the physical state of the theory. Thus, the physicality criterion is consistent with the Dirac method [84] of quantization.

### 8.3.2 Co-BRST and anti-co-BRST symmetries

In this subsection we investigate the nilpotent co-BRST and anti-co-BRST (alternatively known as dual and anti-dual-BRST respectively) transformations which are also the symmetry of the effective action. Further these transformations leave the gauge-fixing term of the action invariant independently and the kinetic energy term (which remains invariant under BRST and anti-BRST transformations) transforms under it to compensate the terms arises due to the transformation of the ghost terms. The gauge-fixing term has its origin in the co-exterior derivative \( \delta = \pm * d * \),
where $\ast$ represents the Hodge duality operator. The $\pm$ signs is dictated by the dimensionality of the manifold [85]. Therefore, it is appropriate to call these transformations a dual and an anti-dual-BRST transformation.

The nilpotent co-BRST transformation ($s^2_d = 0$) and anti-co-BRST transformation ($s^2_{ad} = 0$) which are absolutely anticommuting ($s_d s_{ad} + s_{ad} s_d = 0$) are

$$s_d \varphi = -\frac{1}{2} \dot{\bar{c}}, \quad s_d \lambda = -\bar{c}, \quad s_d \theta = -\frac{1}{2} \dot{\bar{c}}$$
$$s_d \pi \varphi = 0, \quad s_d u = 0, \quad s_d \pi \theta = 0, \quad s_d p_u = 0$$
$$s_d c = \frac{1}{2} (\pi \varphi + \pi \theta), \quad s_d B = 0, \quad s_d \bar{c} = 0. \quad (8.46)$$

$$s_{ad} \varphi = -\frac{1}{2} \dot{c}, \quad s_{ad} \lambda = -c, \quad s_{ad} \theta = -\frac{1}{2} \dot{c}$$
$$s_{ad} \pi \varphi = 0, \quad s_{ad} u = 0, \quad s_{ad} \pi \theta = 0, \quad s_{ad} p_u = 0$$
$$s_{ad} \bar{c} = -\frac{1}{2} (\pi \varphi + \pi \theta), \quad s_{ad} B = 0, \quad s_{ad} c = 0. \quad (8.47)$$

The conserved charges for the above symmetries are obtained using Noether’s theorem as

$$Q_d = i \frac{1}{2} (\pi \varphi + \pi \theta) \dot{\bar{c}} + i \pi \lambda \bar{c}, \quad (8.48)$$
$$Q_{ad} = i \frac{1}{2} (\pi \varphi + \pi \theta) \dot{c} + i \pi \lambda c. \quad (8.49)$$

$Q_d$ and $Q_{ad}$ generate the symmetry transformations in Eqs. (8.46) and (8.47), respectively. It is easy to verify the following relations satisfied by these conserved charges:

$$s_d Q_d = -\{Q_d, Q_d\} = 0, \quad (8.50)$$
$$s_{ad} Q_{ad} = -\{Q_{ad}, Q_{ad}\} = 0, \quad s_d Q_{ad} = -\{Q_{ad}, Q_d\} = 0, \quad s_{ad} Q_d = -\{Q_d, Q_{ad}\} = 0.$$

These relations reflect the nilpotency and anticommutativity property of $s_d$ and $s_{ad}$ (i.e. $s_d^2 = 0, s_{ad}^2 = 0$ and $s_d s_{ad} + s_{ad} s_d = 0$).

8.3.3 Bosonic symmetry

In this subsection we construct the bosonic symmetry out of different nilpotent BRST symmetries of the theory. The BRST ($s_b$), anti-BRST ($s_{ab}$), co-BRST ($s_d$) and anti-co-BRST ($s_{ad}$) symmetry operators satisfy the following algebra:

$$\{s_d, s_{ad}\} = 0, \quad \{s_b, s_{ab}\} = 0 \quad (8.51)$$
$$\{s_b, s_{ad}\} = 0, \quad \{s_d, s_{ab}\} = 0, \quad (8.52)$$
$$\{s_b, s_d\} \equiv s_w, \quad \{s_{ab}, s_{ad}\} \equiv s_{w}. \quad (8.53)$$
The anticommutators in Eq. (8.53) define the bosonic symmetry of the system. Under this bosonic symmetry transformation the field variables transform as

\[ s_w \varphi = -\frac{1}{2} (\dot{B} + \pi_\varphi + \pi_\vartheta), \quad s_w \lambda = -\frac{1}{2} (2B - \dot{\pi}_\varphi - \dot{\pi}_\vartheta), \]
\[ s_w \vartheta = -\frac{1}{2} (\dot{B} + \pi_\varphi + \pi_\vartheta), \quad s_w \pi_\varphi = 0, \quad s_w \pi_\vartheta = 0, \]
\[ s_w u = 0, \quad s_w p_u = 0, \quad s_w c = 0, \quad s_w B = 0, \quad s_w \bar{c} = 0. \quad (8.54) \]

However, the symmetry operator \( s_\bar{w} \) is not an independent bosonic symmetry transformation as shown by

\[ s_\bar{w} \varphi = \frac{1}{2} (\dot{B} + \pi_\varphi + \pi_\vartheta), \quad s_\bar{w} \lambda = \frac{1}{2} (2B - \dot{\pi}_\varphi - \dot{\pi}_\vartheta), \]
\[ s_\bar{w} \vartheta = \frac{1}{2} (\dot{B} + \pi_\varphi + \pi_\vartheta), \quad s_\bar{w} \pi_\varphi = 0, \quad s_\bar{w} u = 0, \]
\[ s_\bar{w} \pi_\vartheta = 0, \quad s_\bar{w} p_u = 0, \quad s_\bar{w} c = 0, \quad s_\bar{w} B = 0, \quad s_\bar{w} \bar{c} = 0. \quad (8.55) \]

Now, it is easy to see that the operators \( s_w \) and \( s_\bar{w} \) satisfy the relation \( s_w + s_\bar{w} = 0 \). This implies, from Eq. (8.53), that

\[ \{s_b, s_d\} = s_w = -\{s_{ab}, s_{ad}\}, \quad (8.56) \]

It is clear from the above algebra that the operator \( s_w \) is the analog of the Laplacian operator in the language of differential geometry and the conserved charge for the above symmetry transformation is calculated as

\[ Q_w = -i \left[ B^2 + \frac{1}{2} (\pi_\varphi + \pi_\vartheta)^2 \right]. \quad (8.57) \]

Using the equation of motion, it can readily be checked that

\[ \frac{dQ_w}{dt} = -i \int dx [2B \dot{B} + (\pi_\varphi + \pi_\vartheta)(\dot{\pi}_\varphi + \dot{\pi}_\vartheta)] = 0. \quad (8.58) \]

Hence \( Q_w \) is the constant of motion for this theory.

### 8.3.4 Ghost and discrete symmetries

Now we would like to mention yet another symmetry of the system namely the ghost symmetry. The ghost number of the ghost and antighost fields are 1 and \(-1\), respectively, the rest of the variables in the action of the this theory have ghost number zero. Keeping this fact in mind we can introduce a scale transformation of the ghost field, under which the effective action is invariant, as

\[ \varphi \rightarrow \varphi, \quad \vartheta \rightarrow \vartheta, \quad \pi_\varphi \rightarrow \pi_\varphi, \quad \pi_\vartheta \rightarrow \pi_\vartheta, \]
\[ u \rightarrow u, \quad p_u \rightarrow p_u, \quad \lambda \rightarrow \lambda, \quad B \rightarrow B, \]
\[ c \rightarrow e^\tau c, \quad \bar{c} \rightarrow e^{-\tau} \bar{c}. \quad (8.59) \]
where $\tau$ is a global scale parameter. The infinitesimal version of the ghost scale transformation can be written as

$$
\begin{align*}
sg\varphi &= 0, \quad sg\theta = 0, \quad sg\lambda = 0, \\
gs\pi_\varphi &= 0, \quad gs\pi_u = 0, \quad gs\pi_\phi = 0, \\
gs\rho_u &= 0, \quad gs\rho_c = c, \quad gsB = 0, \\
gs\bar{c} &= -\bar{c}.
\end{align*}
$$

The Noether conserved charge for the above symmetry transformations is

$$
Q_g = i[\dot{\bar{c}}c + \dot{c}\bar{c}].
$$

In addition to the above continuous symmetry transformation, the ghost sector respects the discrete symmetry transformations

$$
c \rightarrow \pm ic, \quad \bar{c} \rightarrow \pm ic.
$$

The above discrete symmetry transformation is useful to obtain the anti-BRST symmetry transformation from the BRST symmetry transformation and vice versa.

### 8.4 Geometrical cohomology

In this section we study the de Rham cohomological operators and their realization in terms of conserved charges which generate the symmetries for the theory of self-dual chiral boson. In particular we point out the similarity between the algebra obeyed by de Rham cohomological operators and that by different BRST conserved charges.

#### 8.4.1 Hodge-de Rham decomposition theorem and differential operators

The de Rham cohomological operators in differential geometry obey the following algebra:

$$
\begin{align*}
\delta^2 &= 0, \quad \Delta = (d + \delta)^2 = d\delta + \delta d \equiv \{d, \delta\} \\
[\Delta, \delta] &= 0, \quad [\Delta, d] = 0
\end{align*}
$$

where $d, \delta$ and $\Delta$ are the exterior, co-exterior and Laplace-Beltrami operator, respectively. The operators $d$ and $\delta$ are adjoint or dual to each other and $\Delta$ is self-adjoint operator [104]. It is well known that the exterior derivative raises the degree of a form by one when it operates on it (i.e. $df_n \sim f_{n+1}$). On the other hand, the dual-exterior derivative lowers the degree of a form by one when it operates on forms (i.e. $\delta f_n \sim f_{n-1}$). However, $\Delta$ does not change the degree of form (i.e. $df_n \sim f_n$). Here $f_n$ denotes an arbitrary n-form object.

Let $M$ be a compact, orientable Riemannian manifold; then an inner product on the vector space $E^n(M)$ of n-forms on $M$ can be defined as [105]

$$
(\alpha, \beta) = \int_M \alpha \wedge \ast \beta,
$$

(8.64)
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for $\alpha, \beta \in E^n(M)$ and $\ast$ represents the Hodge duality operator \[106\]. Suppose that $\alpha$ and $\beta$ are forms of degree $n$ and $n + 1$, respectively, then the following relation for inner product will be satisfied

$$\langle d\alpha, \beta \rangle = \langle \alpha, \delta \beta \rangle.$$  (8.65)

Similarly, if $\beta$ is a form of degree $n - 1$, then we have the relation $\langle \alpha, d\beta \rangle = (\delta \alpha, \beta)$. Thus, the necessary and sufficient condition for $\alpha$ to be closed is that it should be orthogonal to all co-exact forms of degree $n$. The form $\omega \in E^n(M)$ is called harmonic if $\Delta \omega = 0$. Now let $\beta$ be a $n$-form on $M$ and if there exists another $n$-form $\alpha$ such that $\Delta \alpha = \beta$, then for a harmonic form $\gamma \in H^n$,

$$\langle \beta, \gamma \rangle = \langle \Delta \alpha, \gamma \rangle = \langle \alpha, \Delta \gamma \rangle = 0,$$  (8.66)

where $H^n(M)$ denote the subspace of $E^n(M)$ of harmonic forms on $M$. Therefore, if a form $\alpha$ exists with the property that $\Delta \alpha = \beta$, then Eq. (8.66) is a necessary and sufficient condition for $\beta$ to be orthogonal to the subspace $H^n$. This reasoning leads to the idea that $E^n(M)$ can be partitioned into three distinct subspaces $\Lambda^0_\delta$, $\Lambda^1_\delta$ and $H^n$ which are consistent with exact, co-exact and harmonic forms, respectively. Therefore, the Hodge-de Rham decomposition theorem can be stated \[107\].

A regular differential form of degree $n$ ($\alpha$) may be uniquely decomposed into a sum of the harmonic form ($\alpha_H$), exact form ($\alpha_d$) and co-exact form ($\alpha_\delta$) i.e.

$$\alpha = \alpha_H + \alpha_d + \alpha_\delta,$$  (8.67)

where $\alpha_H \in H^n$, $\alpha_\delta \in \Lambda^0_\delta$ and $\alpha_d \in \Lambda^1_\delta$.

8.4.2 Hodge-de Rham decomposition theorem and conserved charges

The conserved charges for all the symmetry transformations satisfy the following algebra:

$$Q^2_b = 0, \quad Q^2_{ab} = 0, \quad Q^2_d = 0, \quad Q^2_{ad} = 0,$$

$$\{Q_b, Q_{ab}\} = 0, \quad \{Q_d, Q_{ad}\} = 0, \quad \{Q_b, Q_{ad}\} = 0$$

$$\{Q_d, Q_{ab}\} = 0, \quad [Q_g, Q_b] = Q_b, \quad [Q_g, Q_{ad}] = Q_{ad},$$

$$[Q_g, Q_d] = -Q_d, \quad [Q_g, Q_{ab}] = -Q_{ab},$$

$$\{Q_b, Q_d\} = Q_w = -\{Q_{ad}, Q_{ab}\}, \quad [Q_w, Q_r] = 0.$$  (8.68)

This can be checked easily by using the canonical brackets in Eq. \[27\], where $Q_r$ generically represents the charges for BRST symmetry ($Q_b$), anti-BRST symmetry ($Q_{ab}$), dual-BRST symmetry ($Q_d$), anti-dual-BRST symmetry ($Q_{ad}$) and ghost symmetry ($Q_g$). We note that the relations between the conserved charges $Q_b$ and $Q_d$ as well as $Q_{ab}$ and $Q_{ad}$ mentioned in the last line of (8.68) can be established by using the equation of motions only.

This algebra is reminiscent of the algebra satisfied by the de Rham cohomological operators of differential geometry given in Eq. \[63\]. Comparing \[63\] and (8.68) we obtain the following analogies:

$$(Q_b, Q_{ad}) \rightarrow d, \quad (Q_d, Q_{ab}) \rightarrow \delta, \quad Q_w \rightarrow \Delta.$$  (8.69)
Let $n$ be the ghost number associated with a particular state $|\psi\rangle_n$ defined in the total Hilbert space of states, i.e.,

$$iQ_g |\psi\rangle_n = n |\psi\rangle_n$$  \hspace{1cm} (8.70)

Then it is easy to verify the relations

$$Q_gQ_b |\psi\rangle_n = (n + 1)Q_b |\psi\rangle_n, \quad Q_gQ_{ad} |\psi\rangle_n = (n + 1)Q_{ad} |\psi\rangle_n,$$

$$Q_gQ_d |\psi\rangle_n = (n - 1)Q_b |\psi\rangle_n, \quad Q_gQ_{ab} |\psi\rangle_n = (n - 1)Q_b |\psi\rangle_n,$$

$$Q_gQ_w |\psi\rangle_n = nQ_w |\psi\rangle_n,$$  \hspace{1cm} (8.71)

which imply that the ghost numbers of the states $Q_b |\psi\rangle_n, Q_d |\psi\rangle_n$ and $Q_w |\psi\rangle_n$ are $(n+1), (n-1)$ and $n$, respectively. The states $Q_{ab} |\psi\rangle_n$ and $Q_{ad} |\psi\rangle_n$ have ghost numbers $(n-1)$ and $(n+1)$, respectively. The properties of $d$ and $\delta$ are mimicked by the sets $(Q_b, Q_{ad})$ and $(Q_d, Q_{ab})$, respectively. It is evident from Eq. (8.71) that the set $(Q_b, Q_{ad})$ raises the ghost number of a state by one and on the other hand the set $(Q_d, Q_{ab})$ lowers the ghost number of the same state by one. These observations, keeping the analogy with the Hodge-de Rham decomposition theorem, enable us to express any arbitrary state $|\psi\rangle_n$ in terms of the sets $(Q_b, Q_d, Q_w)$ and $(Q_{ad}, Q_{ab}, Q_w)$ as

$$|\psi\rangle_n = |w\rangle_n + Q_b |\chi\rangle_{n-1} + Q_d |\phi\rangle_{n+1},$$  \hspace{1cm} (8.72)

$$|\psi\rangle_n = |w\rangle_n + Q_{ad} |\chi\rangle_{n-1} + Q_{ab} |\phi\rangle_{n+1},$$  \hspace{1cm} (8.73)

where the most symmetric state is the harmonic state $|w\rangle_n$ that satisfies

$$Q_w |w\rangle_n = 0, \quad Q_b |w\rangle_n = 0, \quad Q_d |w\rangle_n = 0,$$

$$Q_{ab} |w\rangle_n = 0, \quad Q_{ad} |w\rangle_n = 0,$$  \hspace{1cm} (8.74)

analogous to Eq. (8.66). It is quite interesting to point out that the physicality criterion of all the fermionic charges $Q_b, Q_{ab}, Q_d$ and $Q_{ad}$, i.e.,

$$Q_b |phys\rangle = 0, \quad Q_{ab} |phys\rangle = 0,$$

$$Q_d |phys\rangle = 0, \quad Q_{ad} |phys\rangle = 0,$$  \hspace{1cm} (8.75)

leads to the following conditions:

$$\pi_\lambda |phys\rangle = 0, \quad (\pi_w + \pi_\delta) |phys\rangle = 0.$$  \hspace{1cm} (8.76)

This is the operator form of the first-class constraint which annihilates the physical state as a consequence of the above physicality criterion, which is consistent with the Dirac method of quantization of a system with first-class constraints.

### 8.5 Conclusions

The BFV technique plays an important role in the gauge theory to analyze the constraint and the symmetry of the system. We have considered such a powerful technique to study the theory of a self-dual chiral boson. In particular we have derived the nilpotent BRST symmetry transformation for this theory using the BFV technique. Further we have studied the dual-BRST
transformation which is also the symmetry of the effective action and which leaves the gauge-fixing part of effective action invariant separately. Interchanging the role of ghost and antighost fields the anti-BRST and anti-dual-BRST symmetry transformations are also constructed. We have shown that the nilpotent BRST and anti-dual-BRST symmetry transformations are analogous to the exterior derivative as the ghost number of the state $|\psi\rangle_n$ on the total Hilbert space is increased by one when these operate on $|\psi\rangle_n$ and the algebra followed by these are the same as the algebra obeyed by the de Rham cohomological operators. In a similar fashion the dual-BRST and anti-BRST symmetry transformations are linked to the co-exterior derivative. The anticommutator of either the BRST and the dual-BRST transformations or anti-BRST and anti-dual-BRST transformations leads to a bosonic symmetry which turns out to be the analog of the Laplacian operator. Further, the effective theory has a non-nilpotent ghost symmetry transformation which leaves the ghost terms of the effective action invariant independently. Further we have noted that the Hodge duality operator ($\ast$) does not exist for the theory of a self-dual chiral boson in (1+1) dimensions because effectively this theory reduces to a theory in (0+1) dimensions due to the self-duality condition of fields ($\dot{\varphi} = \varphi'$ as well as $\dot{\vartheta} = \vartheta'$). The algebra satisfied by the conserved charges is exactly the same in appearance as the algebra of the de Rham cohomological operators of differential geometry. These lead to the conclusion that the theory for self-dual chiral boson is a Hodge theory.
In this thesis we have considered the extension of the finite field dependent BRST transformation (FFBRST) and its applications on gauge field theories.

We have started with the brief introduction of the BRST formalism and its importance in the study of gauge field theories. We have not only discussed the BRST formalism for simple gauge theories where the gauge algebra is closed or irreducible but have also discussed the BRST quantization for the more general class of the gauge theories when the gauge group is allowed to be open and/or irreducible. In particular, BV (field/antifield) formulation and BFV technique have been discussed using the BRST transformation. There are various generalizations of the BRST transformation in literature. The finite field dependent BRST transformation is one of the most important generalizations of the BRST transformation. The FFBRST transformation is also the symmetry of the effective action and nilpotent. However, such a finite transformation does not leave the generating functional invariant as the Jacobian in the definition of generating functional is not invariant due to finiteness of the parameter. But this nontrivial Jacobian under certain condition can be replaced by $e^{iS_1}$, where $S_1$ is local functional of the fields. Thus, the FFBRST transformation is extremely useful in connecting two different effective theories.

In this thesis we mainly focused on further generalization of such transformation with new applications. We have started with some basic techniques and mathematical tools relevant for this thesis in chapter two.

The generalization of BRST transformation considered earlier was on-shell nilpotent as equation of motion for some of the fields were used to show the nilpotency. In chapter three, we have constructed the off-shell nilpotent FFBRST transformation in 1-form gauge theories by introducing a Nakanishi-Lautrup type auxiliary field. Further, we have developed both the on-shell and the off-shell FF-anti-BRST transformations, by making the infinitesimal anti-BRST parameter finite and field dependent, for 1-form gauge theories. The FF-anti-BRST transformation also plays the same role as the FFBRST transformation and connects the two different effective theories. Thus, same pair of the theories are shown to be connected through both FFBRST and FF-anti-BRST transformations but with the different finite parameters. We have shown these by connecting the YM theories in the different covariant and noncovariant gauges (like the Lorentz gauge, the axial gauge, the Coulomb gauge and the quadratic gauge). The connection between generating functional for the YM theory to that of the most general Faddeev-popov effective theory have also been established. We have noted that the nontrivial Jacobians of the path integral measures which arise due to the FFBRST and the FF-anti-BRST transformations are responsible for such connection. We have further observed that the off-shell FFBRST and FF-anti-BRST formulations are more simpler and elegant to that of the on-shell transformations.
In chapter four, we have constructed the FFBRST transformation for an Abelian rank-2 tensor field theory by making the infinitesimal BRST parameter finite and field dependent. It has been shown that such finite transformation plays a crucial role in studying the Abelian 2-form gauge theory in noncovariant gauges. We have considered the axial gauge and the Coulomb gauge as the possible candidates for the noncovariant gauges. The generating functional of the 2-form gauge theories in different gauges have been connected through such FFBRST transformation with the different choices of finite field dependent parameters. We have further derived the FF-anti-BRST transformation and have shown that the FF-anti-BRST transformation with different finite parameters also relates the generating functional for the different effective theories. The BV formulation for this Abelian 2-form gauge theory has also been studied in the context of the FFBRST transformation. The effective theories of Abelian rank-2 tensor field in different gauges are considered as the different solutions of the quantum master equation. We have shown that the FFBRST transformation with appropriate finite parameters connects the generating functional corresponding to the different solutions of the quantum master equation in BV formulation.

In the YM theories, even after fixing the gauge, the redundancy of the gauge fields is not completely removed in certain covariant gauges for the large gauge fields. This Gribov problem has been resolved in the GZ theory by adding an extra term, known as horizon term, to the YM effective action. The Kugo-Ojima (KO) criterion for color confinement in manifestly covariant gauge is not satisfied in the GZ theory due to the presence of the horizon term which breaks the usual BRST symmetry. However, this theory has been extended to restore the BRST symmetry and hence to satisfy the KO criterion for color confinement. In chapter five, we have developed the FFBRST transformation for the GZ theory with appropriate horizon term exhibiting the exact BRST invariance. By constructing appropriate finite field dependent parameter we have mapped the generating functional of the GZ theory, which is free from the Gribov copies to that of the Yang-Mills theory through FFBRST transformation. Thus, the theory with Gribov copies are related through a field transformation to the theory without Gribov copies. We have shown that same results also holds in the BV formulation of the GZ theory.

In chapter six, we have introduced a new finite nilpotent symmetry, namely finite field dependent mixed BRST (FFMBRST) symmetry. Mixed BRST transformation is the combination of usual BRST and anti-BRST symmetry. Infinitesimal mixed BRST transformation has been integrated to construct the FFMBRST transformation. These finite transformations are an exact nilpotent symmetry of both the effective action as well as the generating functional for the certain choices of the finite parameters. Further, it has been shown that the Jacobian contributions for the path integral measure arising from BRST and anti-BRST part compensate each other. Such symmetry transformation has also been considered in the field/antifield formulation to show that the solutions of the quantum master equation remain invariant under this.

We have shown yet another application of the FFBRST transformation by establishing the connection between the generating functional for the first-class theories and the second-class theories. The Stueckelberg theory for the massive spin 1 field and the gauge invariant theory for a self-dual chiral boson have considered as the first-class theories. However, the Proca theory for massive spin 1 field and gauge variant theory for self-dual chiral boson are considered as the second-class constraint theories. The connection have been made with the help of explicit calculations in the two above mentioned models. In the first example, the generating functional of the Proca model has been obtained from the generating functional of the Stueckelberg theory for massive spin 1 vector field. In the other example, we have related the generating functionals
for the gauge invariant and the gauge variant theory for self-dual chiral boson. Thus, we can obtain the Green functions of the second-class theories from the Green functions of the first-class theories in our formulation. Complicated nonlocal Dirac bracket analysis for the study of the second-class theory is thus avoided in our FFBRST formulation.

In chapter eight, the (1+1) dimensional theory for a single self-dual chiral boson has been considered as a classical model for gauge theory. Using the BFV technique, the nilpotent BRST and anti-BRST symmetry transformations for this theory have been explored. In this model other forms of the nilpotent symmetry transformations like co-BRST and anti-co-BRST, which leave the gauge-fixing part of the action invariant, have also been investigated. We have shown that the nilpotent charges for these symmetry transformations satisfy the algebra of the de Rham cohomological operators in the differential geometry. The Hodge decomposition theorem on compact manifold has also been studied in the context of the conserved charges. Further the theory for single self-dual chiral boson has been realized as a field theoretic model for the Hodge theory.

In this thesis we have made an attempt to extend the FFBRST formulation by incorporating it in different field theoretic models. We do believe that our formulation will find many more new applications in future. In particular, it will be helpful in removing the discrepancy of the anomalous dimension calculation for a gauge invariant operators [108]. Exploring our formulation in the context of the different field theoretic models having spontaneous symmetry breaking will also be very exciting.
Appendix A

Mathematical details of 2-form gauge theories

A.1 FFBRST in Axial gauge

Under the FFBRST transformation with finite parameter given in Eq. (4.10), the path integral measure for the generating functional in Eq. (4.6) transforms as

$$\int D\phi' = \int D\phi J(\kappa), \quad (A.1)$$

$J(\kappa)$ can be replaced by $e^{iS_1^A}$ if the condition in Eq. (2.9) satisfies. We start with an ansatz for $S_1^A$ to connect the theory in Lorentz to axial gauge as

$$S_1^A = \int d^4x \left[ \xi_1 \beta_\nu \partial_\mu B^{\mu\nu} + \xi_2 \beta_\nu \eta_\mu B^{\mu\nu} + \xi_3 \beta_\nu \beta^\nu + i \xi_4  \tilde{\rho}_\nu \partial_\mu (\partial^\mu \rho^\nu - \partial^\nu \rho^\mu) ight. \right.$$  

$$+ \ i \xi_5 \tilde{\rho}_\nu \eta_\mu (\partial^\mu \rho^\nu - \partial^\nu \rho^\mu) + i \xi_6 \chi \tilde{\chi} + i \xi_7 \tilde{\chi} \partial_\mu \rho^\mu + i \xi_8 \tilde{\chi} \eta_\mu \rho^\mu$$  

$$+ \ \xi_9 \tilde{\sigma} \partial_\mu \partial^\mu \sigma + \xi_{10} \tilde{\sigma} \eta_\mu \partial^\mu \sigma + \xi_{11} \partial_\mu \beta^\mu \varphi + \xi_{12} \eta_\mu \beta^\mu \varphi$$  

$$\left. + \ i \xi_{13} \chi \partial_\mu \tilde{\rho}^\mu + i \xi_{14} \chi \eta_\mu \tilde{\rho}^\mu \right]. \quad (A.2)$$

where $\xi_i (i = 1, 2, ..., 14)$ are explicit $\kappa$ dependent parameters to be determined by using Eq. (2.9). The infinitesimal change in Jacobian, using Eq. (2.13) with finite parameter in Eq. (4.10), is calculated as

$$\frac{1}{J} \frac{dJ}{d\kappa} = -\int d^4x \left[ -i\gamma_1 \beta_\nu (\partial_\mu B^{\mu\nu} - \eta_\mu B^{\mu\nu}) + \gamma_1 \tilde{\rho}_\nu \partial_\mu (\partial^\mu \rho^\nu - \partial^\nu \rho^\mu) ight. \right.$$  

$$- \ \gamma_1 \tilde{\rho}_\nu \eta_\mu (\partial^\mu \rho^\nu - \partial^\nu \rho^\mu) - i \gamma_2 \lambda_1 \beta_\nu \beta^\nu - \gamma_1 \tilde{\chi} \partial_\mu \rho^\mu$$  

$$+ \ \gamma_1 \tilde{\chi} \eta_\mu \rho^\mu + \gamma_2 \lambda_3 \chi \tilde{\chi} + i \gamma_1 \tilde{\sigma} \partial_\mu \partial^\mu \sigma + i \gamma_1 \tilde{\sigma} \eta_\mu \partial^\mu \sigma$$  

$$\left. + \ i \gamma_1 \beta_\mu (\partial^\mu \varphi + \eta^\mu \varphi) - \gamma_1 \chi \partial_\mu \tilde{\rho}^\mu + \gamma_1 \chi \eta_\mu \tilde{\rho}^\mu \right] \ . \quad (A.3)$$
The condition \( (2.9) \) will be satisfied iff

\[
\int [\mathcal{D}\phi] \, e^{i(S_{eff} + S_1^L)} \left[ i\beta_\nu \partial_\mu B^{\mu\nu}(\xi'_1 - \gamma_1) + i\beta_\nu \eta_\mu B^{\mu\nu}(\xi'_2 + \gamma_1) + i\beta_\nu \beta^\nu(\xi'_3 - \gamma_2 \lambda_1) \right] \\
- \tilde{\rho}_\nu \partial_\mu (\partial^\mu \rho^\nu - \partial^\nu \rho^\mu)(\xi'_1 - \gamma_1) - \tilde{\rho}_\nu \eta_\mu (\partial^\mu \rho^\nu - \partial^\nu \rho^\mu)(\xi'_5 + \gamma_1) \\
- \chi \tilde{\chi}(\xi'_6 - \gamma_2 \lambda_2) - \tilde{\chi} \partial_\mu \rho^\mu(\xi'_7 + \gamma_1) - \chi \eta_\mu \rho^\mu(\xi'_8 - \gamma_1) + i\sigma \partial_\mu \partial^\mu \sigma(\xi'_9 + \gamma_1) \\
+ \tilde{\sigma}_\eta \rho^\mu \sigma(\xi'_10 - \gamma_1) + i\sigma \partial_\mu \beta^\mu \varphi(\xi'_11 - \gamma_1) + i\eta_\mu \beta^\mu \varphi(\xi'_12 + \gamma_1) \\
- \chi \tilde{\partial}_\mu \beta^\mu(\xi'_13 + \gamma_1) - \chi \eta_\mu \beta^\mu(\xi'_14 - \gamma_1) + i\partial_\mu (\partial^\mu \rho^\nu - \partial^\nu \rho^\mu) \Theta^\nu_0(\beta_\mu(\xi'_4 - \xi_1)) \\
+ i\eta_\mu (\partial^\mu \rho^\nu - \partial^\nu \rho^\mu) \Theta^\nu_0(\beta_\mu(\xi_5 - \xi_2)) - i\partial_\mu \partial^\mu \sigma \Theta^\nu_0(\chi(\xi_7 - \xi_9)) \\
- i\eta_\mu \partial^\mu \sigma \Theta^\nu_0(\chi(\xi_8 - \xi_10)) - i\partial^\mu \chi \Theta^\nu_0(\beta_\mu(\xi_11 + \xi_13)) \\
- i\eta_\mu \chi \Theta^\nu_0(\beta_\mu(\xi_12 + \xi_14)) = 0
\]

(A.4)

The contribution of antighost and ghost of antighost can possibly vanish by using the equations of motion of the \( \tilde{\rho}_\mu \) and \( \tilde{\sigma} \). It will happen if the ratio of the coefficient of terms in the above equation and the similar terms in \( S_{eff}^L + S_1^L \) is identical [31]. This requires that

\[
\begin{align*}
\frac{\xi'_1 - \gamma_1}{\xi_1 + 1} &= \frac{\xi'_5 + \gamma_1}{\xi_5} \\
\frac{\xi'_6 + \gamma_1}{\xi_5 - 1} &= \frac{\xi'_7 - \gamma_1}{\xi_10 - 1} \\
\frac{\xi'_4 - \xi_1}{\xi_4 + 1} &= \frac{\xi'_5 - \xi_2}{\xi_5} \\
\frac{\xi_7 - \xi_9}{\xi_9 - 1} &= \frac{\xi_8 - \xi_10}{\xi_10} \\
\frac{\xi_{11} + \xi_{13}}{\xi_{13} - 1} &= \frac{\xi_{12} + \xi_{14}}{\xi_{14}}
\end{align*}
\]

(A.5)

The \( \Theta^\nu_0 \) dependent terms can be converted into local terms by the equation of motion of different fields. This can only work if the following conditions are satisfied

\[
\begin{align*}
\frac{\xi_4 - \xi_1}{\xi_4 + 1} &= \frac{\xi_5 - \xi_2}{\xi_5} \\
\frac{\xi_7 - \xi_9}{\xi_9 - 1} &= \frac{\xi_8 - \xi_10}{\xi_10} \\
\frac{\xi_{11} + \xi_{13}}{\xi_{13} - 1} &= \frac{\xi_{12} + \xi_{14}}{\xi_{14}}
\end{align*}
\]

Further by comparing the coefficients of different terms \( i\beta_\nu \partial_\mu B^{\mu\nu}, i\beta_\nu \eta_\mu B^{\mu\nu}, i\beta_\nu \beta^\nu, \tilde{\rho}_\nu \partial_\mu (\partial^\mu \rho^\nu - \partial^\nu \rho^\mu), \tilde{\rho}_\nu \eta_\mu (\partial^\mu \rho^\nu - \partial^\nu \rho^\mu), \chi \tilde{\chi}, \tilde{\chi} \partial_\mu \rho^\mu, \chi \eta_\mu \rho^\mu, i\tilde{\sigma} \partial_\mu \partial^\mu \sigma, i\sigma \eta_\mu \partial^\mu \sigma, i\partial_\mu \beta^\mu \varphi, i\tilde{\partial}_\mu \beta^\mu \varphi, \chi \partial_\mu \rho^\mu \text{ and}
\]
\(\chi \eta_\mu \tilde{\rho}^\mu\) in both sides of Eq. (A.4), we obtain the following conditions

\[
\begin{align*}
\xi'_1 - \gamma_1 + \gamma_1 (\xi_4 - \xi_1) + \gamma_1 (\xi_5 - \xi_2) &= 0 \\
\xi'_2 + \gamma_1 - \gamma_1 (\xi_4 - \xi_1) - \gamma_1 (\xi_5 - \xi_2) &= 0 \\
\xi'_3 - \gamma_2 \lambda_1 + \gamma_2 \lambda_1 (\xi_4 - \xi_1) + \gamma_2 \lambda_1 (\xi_5 - \xi_2) &= 0 \\
\xi'_4 + \gamma_1 &= 0 \\
\xi'_5 + \gamma_1 &= 0 \\
\xi'_6 - \gamma_2 \lambda_2 - \gamma_2 \lambda_2 (\xi_7 - \xi_9) - \gamma_2 \lambda_2 (\xi_8 - \xi_{10}) &= 0 \\
\xi'_7 + \gamma_1 - \gamma_1 (\xi_7 - \xi_9) - \gamma_1 (\xi_8 - \xi_{10}) &= 0 \\
\xi'_8 - \gamma_1 + \gamma_1 (\xi_7 - \xi_9) + \gamma_1 (\xi_8 - \xi_{10}) &= 0 \\
\xi'_9 + \gamma_1 &= 0 \\
\xi'_{10} - \gamma_1 &= 0 \\
\xi'_1 - \gamma_1 - \gamma_1 (\xi_{11} + \xi_{13}) - \gamma_1 (\xi_{12} + \xi_{14}) &= 0 \\
\xi'_2 + \gamma_1 + \gamma_1 (\xi_{11} + \xi_{13}) + \gamma_1 (\xi_{12} + \xi_{14}) &= 0 \\
\xi'_3 + \gamma_1 &= 0 \\
\xi'_4 - \gamma_1 &= 0 \\
\end{align*}
\]

where we have chosen the arbitrary parameter \(\gamma_1 = -1\). Putting the values of \(\xi_i\) in Eq. (A.2) we obtain \(S^A_1\) at \(\kappa = 1\) as

\[
S^A_1 = \int d^4x \left[ -\beta_\mu \partial_\mu B^{\mu
u} + \beta_\mu \eta_\mu B^{\mu
u} + \gamma_2 \lambda_1 \beta_\nu \beta^\nu - i \tilde{\rho}_\nu \partial_\mu (\partial_\nu \rho^\nu - \partial^\nu \rho^\mu) \\
+ i \tilde{\rho}_\nu \eta_\mu (\partial_\nu \rho^\mu - \partial^\nu \rho^\nu) + i \gamma_2 \lambda_2 \chi \tilde{\chi} + i \chi \partial_\mu \rho^\mu - i \tilde{\chi} \eta_\mu \rho^\nu \\
+ \tilde{\sigma} \partial_\mu \partial_\mu \sigma - \tilde{\sigma} \eta_\mu \partial_\mu \sigma - \partial_\mu \beta_\nu \phi + \eta_\mu \beta_\nu \phi + i \chi \partial_\mu \tilde{\rho}^\mu - i \chi \eta_\mu \tilde{\rho}_\mu \right].
\]

### A.2 FFBRST in Coulomb gauge

For the finite parameter given in Eq. (A.15) we make following ansatz for \(S^C_1\)

\[
S^C_1 = \int d^4x \left[ \xi_1 \beta_\mu \partial_\mu B^{\mu\nu} + \xi_2 \beta_\mu \partial_\mu B^{\mu\nu} + \xi_3 \beta_\nu \beta^\nu + i \xi_4 \tilde{\rho}_\nu \partial_\mu (\partial_\nu \rho^\nu - \partial^\nu \rho^\mu) \\
+ i \xi_5 \tilde{\rho}_\nu \partial_\mu (\partial_\nu \rho^\mu - \partial^\nu \rho^\mu) + i \xi_6 \chi \tilde{\chi} + i \xi_7 \tilde{\chi} \partial_\mu \rho^\mu + i \xi_8 \chi \partial_\mu \rho^\mu \\
+ \xi_9 \tilde{\sigma} \partial_\mu \partial_\mu \sigma + \xi_10 \tilde{\sigma} \partial_\mu \partial_\mu \sigma + \xi_11 \tilde{\sigma} \partial_\mu \beta_\nu \phi + \xi_12 \tilde{\sigma} \beta_\mu \phi + i \xi_13 \chi \partial_\mu \tilde{\rho}^\mu + i \xi_14 \chi \partial_\mu \tilde{\rho}^\mu \right].
\]
The infinitesimal change in Jacobian for Coulomb gauge is calculated as

$$\frac{1}{J} \frac{dJ}{dr} = - \int d^4x \left[ - i\gamma_1 \beta_\nu (\partial_\mu B^{\mu 1} - \partial_1 B^{\mu 1}) + \gamma_1 \tilde{\rho}_\nu \partial_\mu (\partial^\nu \rho^\nu - \partial^\nu \rho^\mu) 
- \gamma_1 \tilde{\rho}_\nu \partial_1 (\partial^\nu \rho^\nu - \partial^\nu \rho^\mu) - i\gamma_2 \lambda_1 \beta_\nu \beta^{\nu} - \gamma_1 \tilde{\chi} \partial_\mu \rho^\mu 
+ \gamma_1 \tilde{\chi} \partial_1 \rho^1 + \gamma_2 \lambda_2 \tilde{\chi} + i\gamma_1 \tilde{\sigma} \partial_\mu \rho^\sigma - i\gamma_1 \tilde{\sigma} \partial_1 \rho^1 
+ i\gamma_1 \beta_\nu \partial^\mu \varphi - i\gamma_1 \beta_1 \partial^1 \varphi - \gamma_1 \chi \partial_\mu \tilde{\rho}^\mu + \gamma_1 \chi \partial_1 \tilde{\rho}^1 \right]. \quad (A.11)$$

The condition will be satisfied iff

$$\int [D\phi] e^{i(S_{\mathcal{E}L} + S_{\mathcal{F}})} \left[ i\beta_\nu \partial_\mu B^{\mu 1} (\xi_1' - \gamma_1) + i\beta_\nu \partial_1 B^{\mu 1} (\xi_2' + \gamma_1) + i\beta_\nu \beta^{\nu} (\xi_3' - \gamma_2 \lambda_1) 
- \tilde{\rho}_\nu \partial_\mu (\partial^\nu \rho^\mu - \partial^\nu \rho^\mu) (\xi_4' - \gamma_1) - \tilde{\rho}_\nu \partial_1 (\partial^\nu \rho^\nu - \partial^\nu \rho^1) (\xi_5' + \gamma_1) 
- \chi \tilde{\xi}_6' - \gamma_2 \lambda_2) - \tilde{\chi} \partial_\mu \rho^\mu (\xi_1' + \gamma_1) - \tilde{\chi} \partial_1 \rho^1 (\xi_6' - \gamma_1) + i\tilde{\sigma} \partial_\mu \rho^\sigma (\xi_6' + \gamma_1) 
+ \tilde{\sigma} \partial_1 \rho^1 (\xi_6' - \gamma_1) + i\tilde{\sigma} \rho^\mu \chi (\xi_1' + \gamma_1) + i\tilde{\rho}_\nu \partial^\nu (\partial^\mu \rho^\nu - \partial^\nu \rho^\mu) \Theta_{\xi}^1 [\beta_\nu (\xi_4' - \gamma_1)] 
+ i\partial_\mu (\partial^\nu \rho^\nu - \partial^\nu \rho^1) \Theta_{\xi}^1 [\beta_\nu (\xi_5' - \gamma_2 \lambda_1)] - i\partial_1 \rho^1 \partial^1 \rho^1 \Theta_{\xi}^1 [\chi (\xi_6' - \gamma_2 \lambda_1)] 
- i\tilde{\sigma} \partial_1 \rho^1 \partial^1 \rho^1 \Theta_{\xi}^1 [\beta_\nu (\xi_1' + \gamma_1)] \right] = 0. \quad (A.12)$$

Following the method exactly similar to Appendix A, we obtain the solution for the parameters $\xi_i$ which is exactly same as in Eq. (A.8).

Thus we obtain $S_{1C}^\xi$ at $\kappa = 1$ as

$$S_{1C}^\xi = \int d^4x \left[ -\beta_\nu \partial_\mu B^{\mu 1} + \beta_\nu \partial_1 B^{\mu 1} + \gamma_2 \lambda_1 \beta_\nu \beta^{\nu} - i\tilde{\rho}_\nu \partial_\mu (\partial^\nu \rho^\nu - \partial^\nu \rho^\mu) 
+ i\tilde{\rho}_\nu \partial_1 (\partial^\nu \rho^\nu - \partial^\nu \rho^1) + i\gamma_2 \lambda_2 \tilde{\chi} + i\tilde{\chi} \partial_\mu \rho^\mu - i\tilde{\chi} \partial_1 \rho^1 
+ \tilde{\sigma} \partial_\mu \rho^\sigma - \tilde{\sigma} \partial_1 \rho^1 - \partial_\mu \beta^{\mu} - \partial_1 \beta^{1} - i\chi \partial_\mu \tilde{\rho}^\mu 
- i\chi \partial_1 \tilde{\rho}^1 \right]. \quad (A.13)$$
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