THE BOUNDARY VALUE PROBLEM FOR YANG–MILLS–HIGGS FIELDS

WANJUN AI, CHONG SONG, AND MIAOMIAO ZHU

Abstract. We show the existence of Yang–Mills–Higgs (YMH) fields over a Riemann surface with boundary where a free boundary condition is imposed on the section and a Neumann boundary condition on the connection. In technical terms, we study the convergence and blow-up behavior of a sequence of Sacks–Uhlenbeck type $\alpha$-YMH fields as $\alpha \to 1$. For $\alpha > 1$, some regularity results for $\alpha$-YMH field are shown. This is achieved by showing a regularity theorem for more general coupled systems, which extends the classical results of Ladyzhenskaya–Ural’ceva and Morrey.

1. Introduction

The Yang–Mills–Higgs (YMH) theory arises from the research of electromagnetic phenomena and plays a fundamental role in modern physics, especially in quantum field theories. Due to its remarkable applications in both geometry and topology, the YMH theory has been extensively studied by mathematicians in the last several decades.

The general YMH theory can be modeled in the following setting. Suppose $\Sigma$ is a Riemannian manifold, $G$ is a compact Lie group with Lie algebra $\mathfrak{g}$, which is endowed with a left-invariant metric, and $P$ is a $G$-principal bundle on $\Sigma$. Let $F$ be a Riemannian manifold admitting a $G$-action, and $\mathcal{F} = P \times_G F$ be the associated fiber bundle. Suppose there is a generalized Higgs potential $\mu$ which is a smooth gauge invariant vector-valued function on $\mathcal{F}$. Let $\mathcal{A}$ denote the space of smooth sections of $\mathcal{F}$, and $\mathcal{A}$ denote the affine space of smooth connections on $P$. Then the YMH functional is defined for a pair $(A, \phi) \in \mathcal{A} \times \mathcal{A}$ by

$$L(A, \phi) = \|\nabla_A \phi\|^2_{L^2} + \|F_A\|^2_{L^2} + \|\mu(\phi)\|^2_{L^2},$$

where $F_A$ is the curvature of $A$, $\nabla_A$ is the covariant differential corresponding to $A$. The exterior extension of $\nabla_A$ is denoted by $D_A$, i.e., the exterior covariant differential. Critical points of the above YMH functional $L$ are called YMH fields, which satisfy the following Euler–Lagrange equation on $\Sigma$:

$$\begin{cases}
\nabla^*_A \nabla_A \phi + \mu(\phi) \cdot \nabla \mu(\phi) = 0, \\
D_A^* F_A + \langle \nabla_A \phi, \phi \rangle = 0.
\end{cases} \quad (1.1)$$

As the Lie groups and the manifolds differ, the above YMH framework covers many variants. For example, if $F$ is a point, then the YMH theory reduces to the usual Yang–Mills theory. If the Lie group $G$ is trivial, then the YMH fields are just harmonic maps (with potential). When $\mathcal{F}$ is a complex line bundle and $G = S^1$ is abelian, then we recover the classical Ginzburg–Landau equations in the theory of superconductivity [1,13]. In general, $G$ can be any compact and possibly non-abelian Lie group and the YMH model can also be viewed as a version of gauged sigma model with certain supersymmetries determined by the type of $F$ [1,2,41].

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particular important case is when both $\Sigma$ and $F$ are Kähler manifolds and there is a holomorphic structure on $F$, then the minimal points of the YMH functional satisfies a first-order equation and are usually referred to as vortices. The existence of vortices has deep relations with the notion of stability conditions, which is now known as the Hitchin–Kobayashi correspondence (see for example [3, 5, 9, 25, 36]). Moreover, the moduli space of vortices can be used to construct symplectic invariants of $F$ with respect to the group action, which is a generalization of the celebrated Gromov–Witten invariants [7, 26]. On the other hand, non-minimal YMH fields do exist, see for example [34, 35].

From now on, we assume $\Sigma$ is a compact Riemann surface with non-empty boundary $\partial \Sigma$, $F$ is a compact Riemannian manifold and $G$ is a connected compact Lie group. We will investigate the existence of general YMH fields satisfying the second-order Euler–Lagrange equations (1.1) under appropriate boundary conditions, namely, a free boundary condition imposed on the section and a Neumann boundary condition on the connection.

The existence of general YMH fields on a closed Riemann surface has been studied by Song [31]. The corresponding gradient flow of the YMH functional is investigated by Yu [43] under the name of gauged harmonic maps following the work of Lin–Yang [19] and by Song–Wang [33]. Song [32] also studied the convergence of YMH fields where the conformal structure of the underlying surface $\Sigma$ is allowed to vary and degenerate. When $\Sigma$ has possibly non-empty boundary, the minimal YMH fields in the holomorphic setting are studied by Xu [42] and Venugopalan [39]. See also the very recent paper by Lin–Shen [18] on the heat flow of the YMH functional over a compact Kähler manifold.

A closely related problem is harmonic maps from surfaces, which has been extensively studied. For example, the convergence of harmonic maps from degenerating surfaces was firstly systematically explored in [44] and the existence of harmonic maps with free boundary was studied via various approaches [11, 12, 21]. On one hand, since the Dirichlet energy $\|\nabla_A \phi\|^2_{L^2}$ is critical in dimension two, we shall follow the general scheme developed for two-dimensional harmonic map type problems to deal with the section part $\phi$. On the other hand, although the Yang–Mills energy $\|F_A\|^2_{L^2}$ is subcritical in dimension two, however, as we will see in this paper, the coupled system brings new technical difficulties caused by the connection part $A$. One of the main achievements in the present paper is to overcome them (see the remark after Theorem B for more details).

Now we shall describe our boundary value problem for YMH fields $(A, \phi)$ in more precise terms. Let $K \subset F$ be a closed sub-manifold which is invariant under the $G$-action. Let $\mathcal{K} = \mathcal{P} \times_G K$ be the sub-bundle of $\mathcal{F}$ with fiber $K$, define the space of smooth sections of $\mathcal{F}$ with free boundary as
\[
\mathcal{F}_K := \{ \phi \in \mathcal{F} : \phi|_{\partial \Sigma} \in \mathcal{K} \}.
\]
Clearly, the tangent space of $\mathcal{F}_K$ at $\phi$ is given by
\[
T_{\phi} \mathcal{F}_K = \{ \psi \in \Gamma(\phi^* T \mathcal{F}^\nu) : \psi(x) \in T_{\phi(x)} \mathcal{K}^\nu, \ x \in \partial \Sigma \},
\]
where $T\mathcal{F}^\nu$ denotes the vertical distribution of tangent bundle $T\mathcal{F}$. On the other hand, the affine space of connections of principal bundle over $\Sigma$ with $\partial \Sigma \neq \emptyset$ is still denoted by $\mathcal{A}$. The tangent space of $\mathcal{A}$ at $A$ is $T_A \mathcal{A} = \Omega^1(\mathfrak{g}_\mathcal{P})$, where $\mathfrak{g}_\mathcal{P} := \mathcal{P} \rtimes_{\text{Ad}} \mathfrak{g}$ is the Lie algebra vector bundle. A simple computation yields the first variation of $\mathcal{L}$ on $\mathcal{A} \times \mathcal{F}_K$,
\[
\delta_{\xi, \psi}(\mathcal{L}(A, \phi)) = 2 \int_\Sigma \langle \nabla_A^\nu \nabla_A \phi, \psi \rangle + \langle \mu(\phi) \cdot \nabla \mu(\phi), \psi \rangle + \langle D_A^* F_A, \xi \rangle + \langle \nabla_A \phi, \xi_\phi \rangle + 2 \int_{\partial \Sigma} \langle \psi, \nu \rangle \nabla_A \phi + \langle \xi, \nu \rangle F_A,
\]
where $\xi \in T_A \mathcal{A}$ and $\psi \in T_{\phi} \mathcal{F}_K$, $\nu$ is the unit outer normal vector field on $\partial \Sigma$ and $\nu \int$ denotes the contraction of a form with $\nu$. Therefore, a critical point $(A, \phi) \in \mathcal{A} \times \mathcal{F}_K$ of $\mathcal{L}$ satisfies
the Euler–Lagrangian equation (1.1) in the interior of Σ and satisfies the following boundary condition on ∂Σ,
\[
\mathcal{N}: \begin{cases} 
\nu \int_A \phi \perp T_\partial K^v, \\
\nu \int F_A = 0. 
\end{cases}
\]

**Definition 1.1.** A smooth pair \((A, \phi) \in \mathcal{A} \times \mathcal{J}_K\) is called a YMH field with free boundary on the section and Neumann boundary on the connection if it satisfies the system (1.1) in the interior of Σ and satisfies the boundary condition \(\mathcal{N}\) on the boundary ∂Σ.

Our main goal in this paper is to show the existence of such YMH fields on Σ. Note that in dimension two, the above condition for the connection \(A\) simply means \(*F_A = 0\) on ∂Σ, which is exactly the Neumann boundary condition in the study of Yang–Mills theory (see e.g. [22]). On the other hand, if we take \(K = F\) to be the total bundle, then the boundary condition for the section \(\phi\) reduces to the Neumann boundary condition \(\langle \nu, \nabla A \phi \rangle = 0\) on ∂Σ. In the case of Ginzburg–Landau theory with \(K = F\), this boundary condition \(\mathcal{N}\) coincides with the natural homogeneous de Gennes–Neumann boundary condition in the study of superconductivity (see e.g. [8, 20] for non-homogeneous condition of sections, and [6, 10, 27] for the homogeneous one).

To investigate the existence of YMH fields subject to the boundary condition \(\mathcal{N}\), in contrast to the Ginzburg–Landau case (see [27, Lem. 3.1]), \(\mathcal{L}\) does not satisfy the Palais–Smale condition anymore and we follow the scheme of [28, 31] by considering the following perturbed \(\alpha\)-functional for \(\alpha > 1\):
\[
\mathcal{L}_\alpha(A, \phi) = \int_\Sigma (1 + |\nabla A \phi|)^2 + \|F_A\|^2_{L^2} + \|\mu(\phi)\|^2_{L^2}, \quad (A, \phi) \in \mathcal{A}^2 \times \mathcal{J}^{2\alpha}_{1,K},
\]

where \(\mathcal{A}^2\) and \(\mathcal{J}^{2\alpha}_{1,K}\) denote the corresponding Sobolev spaces which are defined as follows: for a fixed smooth connection \(A_0 \in \mathcal{A}\), the affine Sobolev space of \(L^p_1\) connections is defined as
\[
\mathcal{A}^p_1 : = \left\{ A \in A_0 + L^p_1 \left( \Omega^1(\mathfrak{g}_P) \right) \right\}.
\]
The spaces \(\mathcal{A}^p_1\) defined via different choices of \(A_0\) are isomorphic to each other. The Sobolev space of sections \(\mathcal{J}^{2\alpha}_{1,K}\) is defined by
\[
\mathcal{J}^{2\alpha}_{1,K} = \left\{ \phi \in L^2_1(E) : \phi(x) \in \mathcal{F} \text{ for a.e. } x \in \Sigma \text{ and } \phi(x) \in K \text{ for a.e. } x \in \partial \Sigma \right\},
\]
where we embed \(\mathcal{F}\) into a vector bundle \(E = \mathcal{P} \times_G \mathbb{R}^l\) for some large enough \(l\) such that \(F \hookrightarrow \mathbb{R}^l\) is an equivariant (with respect to the orthogonal representation \(\rho : G \to O(l)\)) isometrical embedding (see [23, Main Thm.]), and we view sections of \(\mathcal{F}\) as sections of \(E\), where the covariant differential induced by \(A\) is also defined. We refer to [40, Appx. B] for the definition of Sobolev norms on vector bundles and fiber bundles (e.g., the gauge group of \(\mathcal{P}\), \(\mathcal{J}^2_1 = L^2_1(\mathcal{P} \times_c G)\), where \(c\) is the conjugation).

It turns out that, the perturbed functional \(\mathcal{L}_\alpha\) with \(\alpha > 1\) satisfies the Palais–Smale condition on \(\mathcal{A}^2 \times \mathcal{J}^{2\alpha}_{1,K}\) (see Sect. 2.1), hence it admits critical points, which we call \(\alpha\)-YMH fields with free boundary on the section and Neumann boundary on the connection, by classical theory of calculus of variation. The Euler–Lagrange equation for a critical point \((A, \phi)\) of \(\mathcal{L}_\alpha\) is given by
\[
\begin{aligned}
\nabla_A^* \left( \alpha (1 + |\nabla A \phi|^2)^{\alpha - 1} \nabla_A \phi \right) &- \mu(\phi) \cdot \nabla \mu(\phi) = 0, \quad x \in \Sigma \\
D_A^* F_A + \alpha (1 + |\nabla A \phi|^2)^{\alpha - 1} \langle \nabla_A \phi, \phi \rangle & = 0, \quad x \in \Sigma \\
\nu \int F_A = 0, \quad x \in \Sigma \\
\nu \int \nabla_A \phi \perp T_\partial K^v, \quad x \in \partial \Sigma.
\end{aligned}
\]

Our first result is the following interior regularity theorem and boundary regularity theorem for \(\alpha\)-YMH fields under our boundary condition \(\mathcal{N}\).

**Theorem A.** Suppose \(\alpha > 1\), \(K \subset F\) is a \(G\)-invariant sub-manifold and \((A_\alpha, \phi_\alpha)\) is a critical point of \(\mathcal{L}_\alpha\) in \(\mathcal{A}^2 \times \mathcal{J}^{2\alpha}_{1,K}\). Then for any compact subset \(\Sigma'\) in the interior of \(\Sigma\), there exists...
a gauge transformation $\tilde{S} \in \mathscr{D}_2^2(\Sigma')$, such that $(\tilde{S}^* A_\alpha, \tilde{S}^* \phi_\alpha)$ is smooth on $\Sigma'$. If, in addition, $K \subset F$ is a totally geodesic sub-manifold, then there exists a gauge transformation $\tilde{S} \in \mathscr{D}_2^2(\Sigma)$, such that $(\tilde{S}^* A_\alpha, \tilde{S}^* \phi_\alpha)$ is smooth up to the boundary.

For $\alpha$-harmonic maps, which can be regarded as a special kind of $\alpha$-YMH fields, such regularity result was proved by Sacks-Uhlenbeck [28, Prop. 2.3] in the case of a closed domain and the free boundary case was considered by Fraser [11, Prop. 1.4]. The proof for the case of $\alpha$-harmonic maps simply follows from a classic regularity theorem by Morrey [24, Thm. 1.11.1'], extending the one by Ladyzhenskaya-Ural’ceva [16, Chap. 8, Thm. 2.1, p. 412]. However, Morrey’s theorem in [24, Thm. 1.11.1'] cannot be applied to the coupled system of $\alpha$-YMH field $(A, \phi) \in \mathscr{A}_1^2 \times \mathscr{A}_1^2 \mathscr{F}$, because the corresponding ellipticity condition in (1.10.8) of [24, Thm. 1.11.1'] cannot be verified, due to the feature of the non-trivial coupling between the two fields. Therefore, we need to develop a new regularity theorem to handle coupled systems of more general type, in particular, to include the system of $\alpha$-YMH fields.

In this paper, we succeed in deriving such a more general regularity result, which itself is interesting and might lead to applications to various other coupled systems emerging from geometry and physics.

Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain, we will consider a 2-coupled system,

\[
\begin{align*}
-\sum_{\alpha=1}^{n} \partial_\alpha q_{1i}(x, z, \nabla z) + w_{1i}(x, z, \nabla z) &= 0, \quad i = 1, 2, \ldots, m_1, \\
-\sum_{\alpha=1}^{n} \partial_\alpha q_{2i}(x, z, \nabla z) + w_{2i}(x, z, \nabla z) &= 0, \quad i = 1, 2, \ldots, m_2,
\end{align*}
\]

(1.3)

where $x = (x^1, \ldots, x^n) \in \Omega$, $\partial_\alpha = \partial_{x^\alpha}$; $z = z(x) = (z_1(x), z_2(x)) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$, and $\nabla z = (\nabla z_1, \nabla z_2) \in \mathbb{R}^{m_1n} \times \mathbb{R}^{m_2n}$ is the gradient of $z$; $q = (q_1, q_2)$, $q_\alpha = (q_\alpha^i)_{m_\alpha \times 1}$; $w = (w_1, w_2)$, $w_\alpha = (w_{\alpha i})_{m_\alpha \times 1}$. A vector valued function $z(x) = (z_1(x), z_2(x)) \in L^k_1(\Omega, \mathbb{R}^{m_1}) \times L^k_1(\Omega, \mathbb{R}^{m_2})$, $k_1 \geq 2$, $k_2 \geq 2$, is called a weak solution of (1.3) if

\[
\begin{align*}
\int_{\Omega} \sum_{i=1}^{m_1} \left( \sum_{\alpha=1}^{n} \partial_\alpha \xi_1^i(x) q_{1i}(x, z, \nabla z) + \xi_1^i(x) w_{1i}(x, z, \nabla z) \right) = 0, \quad \forall \xi_1 \in L^{k_1}_1(\Omega, \mathbb{R}^{m_1}), \\
\int_{\Omega} \sum_{i=1}^{m_2} \left( \sum_{\alpha=1}^{n} \partial_\alpha \xi_2^i(x) q_{2i}(x, z, \nabla z) + \xi_2^i(x) w_{2i}(x, z, \nabla z) \right) = 0, \quad \forall \xi_2 \in L^{k_2}_1(\Omega, \mathbb{R}^{m_2}).
\end{align*}
\]

(1.4)

We will assume that the coefficients $q = q(x, z, p) \in L^1_{1, \text{loc}} \cap C^0(\Omega \times \mathbb{R}^n \times \mathbb{R}^{mn})$ and $w = w(x, z, p) \in L^1_{1, \text{loc}} \cap C^0(\Omega \times \mathbb{R}^n \times \mathbb{R}^{mn})$, $m = m_1 + m_2$, satisfy the following natural structure conditions: for almost all $x \in \Omega$, we have

\[
\begin{align*}
(|w_1| + |w_{1x}|, |w_2| + |w_{2x}|) &\leq \Lambda(R) \left(V_{1}^{k_1} + V_{2}^{k_2-1}, V_{2}^{k_2} \right) ; \\
(|q_1| + |q_{1x}|, |q_2| + |q_{2x}|) &\leq \Lambda(R) \left(V_{1}^{k_1-1}, V_{2}^{k_2-1} \right) ; \\
\pi \cdot w_{z} \cdot \pi^T &\leq \Lambda(R) \left( \sum_{\alpha=1}^{n} V_{\alpha}^{k_\alpha} |\pi_\alpha|^2 + V_{2}^{k_2-1} |\pi_1|^2 \right) ; \\
|q_\alpha| + |w_{\alpha x}| &\leq \Lambda(R) \left( V_{1}^{k_1-2}, \left( V_{2}^{k_2-2} \right) \right) ; \\
|q_\alpha| + |w_{\alpha T}| &\leq \Lambda(R) \left( \sum_{\alpha=1}^{n} V_{\alpha}^{k_\alpha-2} |\pi_\alpha|^2 \right) ; \\
\pi \cdot q_\alpha \cdot \pi^T &\geq \Lambda(R) \left( \sum_{\alpha=1}^{n} V_{\alpha}^{k_\alpha-2} |\pi_\alpha|^2 \right) ;
\end{align*}
\]

(1.5)

This technical issue was overlooked by Song in [31] and here we take our opportunity to fix the gap by extending Morrey’s theorem to Theorem B.
where the mixed derivatives with respect to \( x, z \) and \( p \) are simply denoted by subscripts; 
\[ \Lambda = \lambda(R) > \lambda = \lambda(R) > 0 \] are constants depending on \( R \), and \( R > 0 \) is the upper bound of \( (x, z) \), i.e., \( |x|^2 + |z|^2 \leq R^2 \); 
\[ V_0 := (1 + |p_a|^2)^{1/2}, \quad a = 1, 2; \quad \pi = (\pi_1, \pi_2), \quad \pi_a = (\pi_a^i)_{m \times n}, \quad \alpha = (\alpha^i)_{m \times 1}; \]
\[ \tilde{\pi} = (\tilde{\pi}_1, \tilde{\pi}_2), \quad \pi_a = (\tilde{\pi}_a^i)_{m \times n} \] are any constant matrices; Here we basically follow the notations of \cite{24}, Thm. 1.11.1.
In addition, \( |\cdot| \) is the maximum norm, and for two \( 2 \times 2 \) block non-symmetric real matrices \( M_1, M_2 \), \( |M_1| \leq M_2 \) means \( |M_{1,ab}| \leq |M_{2,ab}| \) for all \( a, b = 1, 2 \); similar notations are adopted for \( 1 \times 2 \) block matrices. In particular, the first condition for \( q_p \) implies that 
\[ q_p = \begin{pmatrix} \partial_{p_1}q_1 & \partial_{p_2}q_1 \\ \partial_{p_1}q_2 & \partial_{p_2}q_2 \end{pmatrix} \] is a block diagonal matrix, the second condition for \( q_p \) is the ellipticity.

**Theorem B.** Suppose that \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), \( z = (z_1, z_2), \ z_2 \in L^k_{1/2} \cap C^0(\Omega, \mathbb{R}^m) \), for some \( 0 < \mu < 1 \) and \( k_a \geq 2, \ a = 1, 2 \), is a weak solution of the \( 2 \)-coupled system \( (1.3) \), with the coefficients \( q = q(x, z, p) \in L^1_{1, loc} \cap C^0(\Omega \times \mathbb{R}^m \times \mathbb{R}^m) \) and \( w = w(x, z, p) \in L^1_{1, loc} \cap C^0(\Omega \times \mathbb{R}^m \times \mathbb{R}^m) \), \( m = m_1 + m_2, \) and satisfying the natural structure conditions \( (1.5) \). If \( z_1 \in L^k_{1/2}(\Omega, \mathbb{R}^{m_1}) \), then \( z \in L^2_{1/2}(\Omega, \mathbb{R}^m) \).

**Remark.** On one hand, taking \( z_1 = 0 \) or taking \( z_2 = 0 \) and \( k_2 = k_1/2 \) in Theorem B gives Morrey’s theorem \cite[Thm. 1.11.1’]{24}. On the other hand, applying \cite[Thm. 1.11.1’]{24} to a coupled system for \( z = (z_1, z_2) \) does not simply imply the results in Theorem B, because the conditions \( (1.5) \) given in Theorem B are different from those in \cite[Thm. 1.11.1’]{24} for a coupled system of \( z = (z_1, z_2) \). In fact, there are two main differences. The first one is that the coupling relation in conditions \( (1.5) \) is expressed in terms of \( w_1, w_{1z}, w_z, q_z \) and \( w_p \), which will produce cross terms as expressed by the terms of the last parentheses in \( (2.9) \), see Sect. 2.2. To control these extra terms, we need to make additional regularity assumption for \( z_1 \), which is natural for coupled systems. The second one is that, the conditions \( (1.5) \) are only required to be held almost everywhere in \( \Omega \) and the assumption on the regularity of the coefficients \( q \) and \( w \) is also weakened. We can check the well-definiteness of the weak solution in \( (1.4) \) under these regularity assumptions. The latter is useful when dealing with some coupled systems with non-smooth coefficients.

**Remark.** The coupling relation expressed by \( w_1, w_{1z}, w_z, q_z \) and \( w_p \) in conditions \( (1.5) \) is sharp and delicate in some sense. From the coupled condition of \( w_1, \ w_{1z}, \ w_z \), it seems that one can add some lower order perturbed terms such as \( V_2^{k_2 - 1} \), however, the coupled condition of \( q_z \) and \( w_p \) shows that this principle is not true anymore. This is because if we change the upper corner \( 0 \) to \( V_2^{k_2 - 2} \) or any other nonzero lower order term of \( V_2 \), it will then produce some new coupled terms, which cannot be analytically controlled anymore. For the same reason, the transpose of \( w_p \) is also crucial here.

To get the regularity up to the boundary for \( \alpha \)-YMH fields satisfying the boundary condition \( (\mathcal{N}) \), we shall locally rectify both the section \( \phi \) and the connection \( A \) across the free boundary naturally and derive a new coupled system for the reflected fields, then we apply the regularity results in Theorem B to this new coupled system to get the interior regularity of the reflected fields, which gives the regularity up to the boundary of the original one.

Next we study the existence of YM fields under our boundary condition \( (\mathcal{N}) \) by exploring the limiting behavior of a sequence of \( \alpha \)-YM fields as \( \alpha \rightarrow 1 \). Since the Dirichlet energy \( \|\nabla A_\phi\|_{L^2}^2 \) is conformally invariant in dimension two, energy concentration and bubbling phenomena can possibly occur, which is similar to various harmonic map type problems. Actually, in \cite{31,32}, it was shown that when the surface \( \Sigma \) is closed, a sub-sequence of the \( \alpha \)-YM fields converges to a YM fields away from at most finitely many blow-up points where the energies concentrate. At each blow-up point, a harmonic sphere can split off. In the situation considered in this paper, where \( \Sigma \) has non-empty boundary, it is sufficient to focus on the blow-up behavior near the boundary \( \partial \Sigma \). For \( \alpha \)-harmonic maps with free boundary, we refer to \cite{11}.
Our main result, in analogy to the closed case (see [31]), is the following boundary bubbling convergence theorem for a sequence of \( \alpha \)-YMH fields under our boundary conditions.

**Theorem C.** There exists a constant \( \alpha_0 > 1 \), such that if \( \{(A_\alpha, \phi_\alpha)\} \subset \mathcal{S} \times \mathcal{K} \) is a sequence of smooth \( \alpha \)-YMH fields with \( \alpha \in (1, \alpha_0) \) and \( \mathcal{L}_\alpha(A_\alpha, \phi_\alpha) \leq \Lambda < +\infty \), then the blow-up set \( S \) of \( \{(A_\alpha, \phi_\alpha)\} \) defined by

\[
S := \left\{ x \in \Sigma : \lim_{\alpha \to 1} \inf_{r \to 0} \int_{U_r(x)} |\nabla A_\alpha \phi_\alpha|^2 \geq \varepsilon_0 \right\},
\]

is a set of at most finitely points; where \( \varepsilon_0 > 0 \) is a constant depending on the geometry of the bundle (see Lemma 3.2) and \( U_r(x) \) is a geodesic ball of radius \( r \) centered at \( x \) in \( \Sigma \). Moreover, as \( \alpha \to 1 \), after taking a sub-sequence of \( \{(A_\alpha, \phi_\alpha)\} \), we have

(a) \( A_\alpha \to A_\infty \) in \( C^\infty_{\text{loc}}(\Sigma \setminus S) \cap C^0(\Sigma) \) and \( \phi_\alpha \to \phi_\infty \) in \( C^\infty_{\text{loc}}(\Sigma \setminus S) \) module gauge. Moreover, \( (A_\infty, \phi_\infty) \) extends to a smooth YM-H fields on \( \Sigma \) satisfying the boundary condition \( (\mathcal{N}) \).

(b) For each \( x \in S \cap \partial \Sigma \), there exist either a non-trivial harmonic spheres \( \omega : S^2 \to F \) or a non-trivial harmonic discs \( w : B \to F \) with free boundary on \( K \).

**Remark.** In Theorem C, if \( F \) admits no non-trivial harmonic 2-spheres, then either \( (A_\alpha, \phi_\alpha) \) subconverges smoothly to a YM-H field \( (A_\infty, \phi_\infty) \) over \( \Sigma \), where \( \phi_\infty \) and \( \phi_\alpha \) are in the same homotopy class, or there exists at least one minimal 2-disc in \( F \) with free boundary on \( K \).

The rest of the paper is organized as follows. In Sect. 2, we study the perturbed YM-H functional and \( \alpha \)-YM-H fields. We start with the verification of Palais–Smale condition in Sect. 2.1, then prove the regularity Theorem B in Sect. 2.2, from which Theorem A follows in Sect. 2.3. In Sect. 3, we derive local estimates for both the connection and the section. The blow-up argument is demonstrated in Sect. 4, which is the content of Theorem C. Finally, we collect some classical boundary estimates and regularity theorems of free boundary problems in Appx. A.

2. The \( \alpha \)-YM-H Functional

We first show in Sect. 2.1 that \( \mathcal{L}_\alpha, \alpha > 1 \), satisfies the Palais–Smale condition so that there exist critical points of \( \mathcal{L}_\alpha \) which solve the Euler–Lagrange equation of \( \mathcal{L}_\alpha \) weakly. To improve the regularity of the weak solution, we generalize a classical regularity result of elliptic systems to coupled elliptic systems in Sect. 2.2 and then rewrite the weak solution into strong form, from which the smoothness of the solution when \( \alpha > 1 \) follows from classical elliptic estimate (up to the boundary) and bootstrap as sketched in Sect. 2.3.

2.1. The Palais–Smale condition. It is well-known that the Palais–Smale condition is crucial in deriving the existence of certain kinds of critical points in variational problems. For \( \alpha \)-harmonic maps, we refer to [38, Sect. 3.2] for the case of closed surfaces and [11, Prop. 1.1] for the free boundary case. The same idea is applied to \( \alpha \)-YM-H functional in [31, Lem. 3.2] for the case of a closed surface \( \Sigma \). In what follows, we verify the Palais–Smale condition for \( \mathcal{L}_\alpha \) when \( \partial \Sigma \neq \emptyset \) and the boundary condition \( (\mathcal{N}) \) is imposed.

Recall the following weak compactness theorem of connections on manifolds with boundary.

**Theorem 2.1 ([40, Thm. 7.1, p. 108]).** Suppose \( M \) is a Riemannian manifold with boundary. Let \( 2p > \text{dim } M \geq 2 \) and \( \{A_n\} \subset \mathcal{A}^p_\mathcal{L} \) be a sequence of connections with \( \|F_{A_n}\|_{L^p(M)} \leq \Lambda < +\infty \). Then, there exists a sub-sequence, still denoted by \( \{A_n\} \), and a sequence of gauge transformations \( S_n \in \mathcal{B}^p_\mathcal{L} \) such that \( \{S_n^*A_n\} \) converges weakly in \( \mathcal{A}^p_\mathcal{L} \). That is, a sub-sequence of \( \{A_n\} \) converges weakly in \( \mathcal{A}^p_\mathcal{L} \) module gauge.

With the help of above theorem, we will show that \( \mathcal{L}_\alpha, \alpha > 1 \), satisfies the Palais–Smale condition.

**Lemma 2.2.** For any \( \alpha > 1 \), \( \mathcal{L}_\alpha \) satisfies the Palais–Smale condition on the product space \( \mathcal{A}^p_\mathcal{L} \times \mathcal{A}^p_{\mathcal{L}, K} \). That is, for any sequence \( \{(A_n, \phi_n)\} \in \mathcal{A}^p_\mathcal{L} \times \mathcal{A}^p_{\mathcal{L}, K} \), if
\( \mathcal{L}_\alpha(A_n, \phi_n) \leq \Lambda < +\infty; \)
\( \|D\mathcal{L}_\alpha(A_n, \phi_n)\| \to 0, \) where the norm is taken in \( T^+_{(A_n, \phi_n)} \mathcal{A}^2_1 \times \mathcal{J}^{2\alpha}_{1,K}; \)
then there exists a sub-sequence which converges strongly in \( \mathcal{A}^2_1 \times \mathcal{J}^{2\alpha}_{1,K} \) module gauge.

\textbf{Proof.} In what follows, for simplicity, we don't distinguish a sequence and its sub-sequences.

\textbf{Step 1.} We first show that \( \{A_n\} \) converges strongly in \( \mathcal{A}^2_1 \) to some \( A_\infty. \)

By assumption (a) of \( \mathcal{L}_\alpha, \|F_{A_n}\|_{L^2(\Sigma)}^2 \leq \Lambda \) and we can apply Theorem 2.1 to show that \( \{S_n^* A_n\} \) converges weakly to \( A_\infty \) in \( \mathcal{A}^2_1 \) for some sequence \( \{S_n\} \subset \mathcal{B}^2_2. \) For simplicity, we still denote \( \{S_n^* A_n\} \) by \( \{A_n\}, \) then
\[ A_n \to A_\infty \text{ in } \mathcal{A}^2_1, \quad \|A_n - A_0\|_{L^q_1} \leq C, \] (2.1)
where \( A_0 \) is the reference connection. By assumption (b) of \( \mathcal{L}_\alpha, \)
\[ |\langle D\mathcal{L}_\alpha(A_n, \phi_n), (A_n - A_\infty, 0) \rangle| \leq \|D\mathcal{L}_\alpha(A_n, \phi_n)\| \cdot \|A_n - A_\infty\|_{L^2_1} \to 0. \] (2.2)

Similar to the computation of Euler–Lagrange equation of \( \mathcal{L}_\alpha, \) we have
\[ \langle D\mathcal{L}_\alpha(A_n, \phi_n), (A_n - A_\infty, 0) \rangle = \int_\Sigma \langle F_{A_n}, D_{A_n}(A_n - A_\infty) \rangle + \int_\Sigma \alpha(1 + |\nabla A_n \phi_n|^2)^{\alpha - 1} \nabla A_n \phi_n, (A_n - A_\infty) \phi_n \rangle =: I + II. \]

Since \( A_n - A_\infty \to 0 \) in \( L^q_1, \) it follows that \( \|A_n - A_\infty\|_{L^q_1} \to 0 \) for any \( 1 \leq q < +\infty \) by the Sobolev embedding theorems. Now, by Hölder’s inequality
\[ |II| \leq \alpha \|1 + |\nabla A_n \phi_n|^2\|_{L^q_1}^{\alpha - 1} \cdot \|\nabla A_n \phi_n\|_{L^2_\alpha} \cdot \|\phi_n\|_{L^\infty} \cdot \|A_n - A_\infty\|_{L^2_\alpha} \]
\[ \leq C(\Lambda) \|A_n - A_\infty\|_{L^2_\alpha} \to 0, \]
as \( n \to \infty. \) For I, we can compute, for \( a_n = A_n - A_0 \in \Omega^1(g_P) \) and \( a_\infty = A_\infty - A_0 \in \Omega^1(g_P), \)
\[ I = \int_\Sigma \langle F_{A_0} + D_{A_0} a_n + a_n \wedge a_n, D_{A_0} (a_n - a_\infty) + [a_n \wedge (a_n - a_\infty)] \rangle \]
\[ = \int_\Sigma |D_{A_0} (a_n - a_\infty)|^2 + \int_\Sigma \langle D_{A_0} a_\infty + a_n \wedge a_n, D_{A_0} (a_n - a_\infty) \rangle \]
\[ + \int_\Sigma \langle F_{A_0}, [a_n \wedge (a_n - a_\infty)] \rangle. \]

Note that \( D_{A_0} a_\infty \in L^2, \|F_{A_0}\|_{L^2} < \Lambda, \) and (2.1) implies that \( \|a_n \wedge a_n\|_{L^2} < C\|a_n\|_{L^4}^2 < C'\|A_n - A_0\|_{L^2_1}^2 < C'' \) by the Sobolev embedding. Thus, by the definition of weak convergence and the Hölder’s inequality, we know that the last two terms in I tend to 0 as \( n \to \infty \) and
\[ I \to \|D_{A_0} (A_n - A_\infty)\|_{L^2_1}. \]

Inserting the estimates of I and II into (2.2), we obtain that
\[ \|D_{A_0} (A_n - A_\infty)\|_{L^2_1} \to 0. \]

Since \( A_n - A_\infty \to 0 \) in \( L^2_1 \) and \( A_n - A_\infty \to 0 \) strongly in \( L^2, \) we conclude that \( A_n \to A_\infty \) strongly in \( \mathcal{A}^2_1. \)

\textbf{Step 2.} Next, we show that for fixed \( A_\infty \in \mathcal{A}^2_1, \mathcal{L}_\alpha(A_\infty, \cdot) \) satisfies the Palais–Smale condition in \( \mathcal{J}^{2\alpha}_{1,K}. \) Recall, for \( \alpha > 1, \mathcal{J}^{2\alpha}_{1,K} \) is defined as a subspace of \( L^\infty_1(\mathcal{E}), \)
\[ \mathcal{J}^{2\alpha}_{1,K} = \{ \phi \in L^\infty_1(\mathcal{E}) : \phi(x) \in \mathcal{F} \text{ for a.e. } x \in \Sigma \text{ and } \phi(x) \in \mathcal{K} \text{ for a.e. } x \in \partial \Sigma \}. \]

We note the following facts:
• As a closed sub-manifold of Banach manifold $L_1^{2\alpha}(\mathcal{E})$, $\mathcal{S}_{1,K}^{2\alpha}$ can be given a structure of smooth Banach manifold. In particular, $\mathcal{S}_{1,K}^{2\alpha}$ is complete under the pull-back Finsler metric $\|\cdot\|_{L_1^{2\alpha}(\mathcal{E})}$.

• The tangent space of $\mathcal{S}_{1,K}^{2\alpha}$ is given by

$$T_{\phi}\mathcal{S}_{1,K}^{2\alpha} = \left\{ \psi \in L_1^{2\alpha}(\phi^* TF^\nu) : \psi(x) \in T_{\phi(x)}K^\nu \text{ for a.e. } x \in \partial \Sigma \right\}.$$ 

Now define

$$\mathcal{L} : L_1^{2\alpha}(\mathcal{E}) \rightarrow \mathbb{R}, \quad \mathcal{L}(\phi) = \int_{\Sigma} (1 + |\nabla_{A_N} \phi|^2)^{\alpha} + |F_{A_N}|^2 + |\mu(\phi)|^2,$$

where $\mu$ to the sections of $\mathcal{E}$ by $\bar{\mu}(\phi) = \eta(x) \cdot \mu(\pi_N(\phi))$, here $\pi_N$ is the nearest projection from a neighborhood $N$ of $\mathcal{F}$ in $\mathcal{E}$ to $\mathcal{F}$, and $\eta$ is a cutoff function supported on $N$ and equals to 1 when restricted to $\mathcal{F}$. Clearly, $\mathcal{L}(\phi) = \mathcal{L}_\alpha(A_\infty, \phi)$ when we restrict $\phi$ to $\mathcal{S}_{1,K}^{2\alpha}$, which is denoted by $\mathcal{J}(\phi)$. We can imitate the argument of [38, Sect. 3.2, p. 105ff] to show

(a) $\mathcal{J}$ is a $C^2$ function on $\mathcal{S}_{1,K}^{2\alpha}$.

(b) There exists a positive constant $C$ (depending on $\mu$) such that for any $\phi_1, \phi_2 \in L_1^{2\alpha}(\mathcal{E})$,

$$(D\mathcal{L}_{\phi_1} - D\mathcal{L}_{\phi_2})(\phi_1 - \phi_2) \geq C \left( \|\phi_1 - \phi_2\|_{L_1^{2\alpha}} - \|\phi_1 - \phi_2\|_{L_2^{2\alpha}} - \|\phi_1 - \phi_2\|_{L_2^{2\alpha}} \right).$$

(2.3)

(c) Suppose $\{\phi_n\} \subset \mathcal{S}_{1,K}^{2\alpha}$ is a bounded sequence under the norm of $L_1^{2\alpha}(\mathcal{E})$, then there exists a sub-sequence such that

$$\|(\text{Id} - \Pi_{\phi_n})(\phi_n - \phi_m)\|_{L_1^{2\alpha}} \rightarrow 0, \quad \text{as } m, n \rightarrow \infty,$$

where Id is the identity map, and $\Pi_{\phi_n}$ is the fiber-wise orthogonal projection from $\mathcal{E}$ to $T\mathcal{S}_{1,K}^{2\alpha}$ at $\phi_n$. Here we should be careful about the projection at the boundary. As we require that the projected section lies in $T_{\phi_n}\mathcal{S}_{1,K}^{2\alpha}$, which requires, at the boundary, it is a vector of $T_{\phi_n(\partial \Sigma)}K^\nu$. This is accomplished by first defining the projections

$$\Pi_{\phi}^E : L_1^{2\alpha}(\mathcal{E}) \rightarrow L_1^{2\alpha}(\phi^* TF^\nu)$$

$$\psi \mapsto \Pi_{\phi}^E(\psi), \quad [\Pi_{\phi}^E(\psi)](x) = \Pi_{\phi}(\phi(x))(\psi(x)),$$

and

$$\Pi_{\phi}^{K^\perp} : L_1^{2\alpha}(\mathcal{E}) \rightarrow L_1^{2\alpha}(\phi^* TK^\nu)$$

$$\psi \mapsto \Pi_{\phi}^{K^\perp}(\psi), \quad [\Pi_{\phi}^{K^\perp}(\psi)](x) = \Pi_{\phi}(\phi(x))(\psi(x)).$$

Here $\Pi_{\phi}^E(y)$ denotes the orthogonal projection of the fiber $\mathcal{E}_y$ onto the tangent space $T_y\mathcal{F}^\nu$ for $y \in \mathcal{F}^\nu$, and $\Pi_{\phi}^{K^\perp}(y)$ denotes the orthogonal projection of $\mathcal{E}_y$ onto $T_yK^\nu$ for $y \in K^\nu$ with respect to the decomposition $\mathcal{E}_y = T_y\mathcal{F}^\nu \oplus T_yK^\nu$. Then our real projection $\Pi_{\phi}$ is defined by

$$\Pi_{\phi} : L_1^{2\alpha}(\mathcal{E}) \rightarrow T_{\phi}\mathcal{S}_{1,K}^{2\alpha}$$

$$\psi \mapsto \Pi_{\phi}(\psi) = \Pi_{\phi}^E \left( \psi - \theta \left( \Pi_{\phi}^{K^\perp} (\psi|_{\partial \Sigma}) \right) \right),$$

where $\theta : L_1^{2\alpha}(\mathcal{E}|_{\partial \Sigma}) \rightarrow L_1^{2\alpha}(\mathcal{E})$ is a continuous linear extension operator from the trace space of $L_1^{2\alpha}(\mathcal{E})$.

(d) There exists a sub-sequence of $\{\phi_n\}$, such that

$$D\mathcal{L}_{\phi_n}(\phi_n - \phi_m) \rightarrow 0, \quad \text{as } m, n \rightarrow \infty.$$ 

(2.4)

Now, we continue the verification of Palais–Smale condition of $\mathcal{L}_\alpha(A_\infty, \cdot)$. Since $\mathcal{L}_\alpha(A_n, \phi_n) \leq \Lambda$ and $A_n \rightarrow A_\infty$ strongly in $\mathcal{S}_{1,K}^2$, we know that $\mathcal{L}_\alpha(A_n, \phi_n) \rightarrow \mathcal{L}_\alpha(A_\infty, \phi_n)$ as $n \rightarrow \infty$. In
particular, \{\phi_n\} is bounded in \(L^4_{\Omega}(E)\). By (2.4), we can choose a sub-sequence such that
\[ D\mathcal{L}_{\phi_n}(\phi_n - \phi_m) \to 0 \] as \(m, n \to \infty\). It is clear that
\[ (D\mathcal{L}_{\phi_n} - D\mathcal{L}_{\phi_m})(\phi_n - \phi_m) \to 0, \quad \text{as } m, n \to \infty. \]

Note also that \{\phi_n\} is bounded in \(L^4_{\Omega}(E)\), the weak compactness of this Sobolev space implies that there is a convergent sub-sequence and so, for such a sequence we have
\[ \|\phi_n - \phi_m\|_{L^2} \to 0 \quad \text{and} \quad \|\phi_n - \phi_m\|_{L^1} \to 0, \quad \text{as } m, n \to \infty. \]

Thus, (2.3) implies that
\[ \|\phi_n - \phi_m\|_{L^2} \to 0, \quad \text{as } m, n \to \infty, \]
i.e., \{\phi_n\} is a Cauchy sequence in \(\mathcal{S}^{2\alpha}_{1,\bar{K}}\). It is convergent because \(\mathcal{S}^{2\alpha}_{1,\bar{K}}\) is complete. As we have shown that \(A_n\) converges to \(A_{\infty}\) in \(\mathcal{S}^{2\alpha}_{1,\bar{K}}\) up to sub-sequence, and for such sub-sequence, \{\phi_n\} converges to some \(\phi_{\infty}\) in \(\mathcal{S}^{2\alpha}_{1,\bar{K}}\), it follows that \(L_\alpha\) satisfies the Palais–Smale condition.

2.2. A regularity theorem for coupled equations. We prove in this section a regularity theorem for Hölder continuous weak solution of some coupled equations, which is an extension of the classical regularity results by Ladyzhenskaya-Ural’tseva and Morrey. The idea is that, when the coupling relations of the coupled system satisfies the conditions in (1.5), then the bad terms appeared due to the coupling relations are controllable.

**Proof of Theorem B.** For any fixed \(x_0 \in \Omega\), by the relation between weak derivatives and difference quotients, we only need to show the uniform boundedness of \(\{\|\nabla z_h\|_{L^2(B_r)}\}_{0 < h < r}\), where \(B_r = B_r(x_0)\), and \(r > 0\) is a small real number to be determined latter. Currently, we only assume \(0 < 8r < \text{dist}(x_0, \partial \Omega) = r_0\). \(z_h\) is the difference quotient defined as follows: for any fixed coordinate direction \(e_\gamma\), and real number \(h, 0 < |h| < r\),
\[ z_h = (z_{1h}, z_{2h}), \quad z_{ih}^j = \Delta_{h} z_{ih}^j = \frac{z_{ih}^j(x + he_\gamma) - z_{ih}^j(x)}{h}, \quad a = 1, 2, i = 1, 2, \ldots, m_a. \]

Now, let us fix some \(D' \subseteq \subseteq \Omega_r := \{x \in \Omega | \text{dist}(x, \partial \Omega) \geq r\}, x_0 \in D'\) and let \(\xi = (\xi_1, \xi_2)\) be a test function with \(\text{supp} \xi \subseteq \subseteq D'\). We denote by \(\xi_h\) the difference quotient of \(\xi\) and substitute \(\xi\) in (1.4) by \(\xi_{-h}\), hereafter the repeated indices \(\alpha, \beta, a, b, i, j\) and sum are,
\[ 0 = \int_{D'} \partial_{\alpha} \xi_h a_{\alpha i}^h \Delta_{\beta} g_{ai}^h + \xi_{\beta}^h \Delta_{\gamma} w_{ai}. \tag{5.5} \]

If we set \(\Delta x := he_\gamma, \Delta z = z(x + \Delta x) - z(x)\) and \(\Delta p = p(z(x + \Delta x)) - p(z(x))\), then since \(q_a \in L^4_{\text{loc}} \cap C^0(\Omega \times \mathbb{R}^m \times \mathbb{R}^{mn})\) and \(z_a \in L^4_{\Omega}(\Omega)\), by the fundamental theorem of calculus for distributions (see [17, Thm. 6.9]) and the chain rule (see [17, Thm. 6.16]), for a.e. \(x \in \Omega\), we have
\[ \Delta_{\gamma} g_{ai}^h = \frac{1}{h} \int_0^1 \frac{d}{dt} g_{ai}^h(x + t \Delta x, z(x) + t \Delta z, p(z(x)) + t \Delta p) \, dt \]
\[ = \int_0^1 \left( \partial_{\gamma} g_{ai}^h(t) \partial_{\gamma} z_{bh}^j + \partial_{\beta}^t g_{ai}^h(t) \partial_{\beta} z_{bh}^j \right) \, dt, \quad a = 1, 2, \]
where \(g_{ai}^h(t) := g_{ai}^h(x + t \Delta x, z(x) + t \Delta z, p(z(x)) + t \Delta p)\). Define \(w_{ai}(t)\) similarly, we have
\[ \Delta_{\gamma} w_{ai} = \int_0^1 \left( \partial_{\gamma} w_{ai}(t) + \partial_{\gamma} z_{bh}^j \right) \, dt, \quad a = 1, 2. \]

Therefore, we can rewrite (2.5) in matrix form as
\[ 0 = \int_{D'} \int_0^1 \nabla \xi : (q_p(t) \cdot \nabla z_h + q_z(t) \cdot z_h + q_{\bar{w}}(t)) + \int_{D'} \int_0^1 \xi : (w_p(t) \cdot \nabla z_h + w_z(t) \cdot z_h + w_{\bar{w}}(t)). \tag{6.6} \]

Next, let \(\eta \in C_0^\infty(\Omega)\) be a cutoff function satisfying
\[ 0 \leq \eta \leq 1, \quad \eta|_D \equiv 1, \quad \text{supp} \eta \subseteq \subseteq D', \quad |\nabla \eta| \leq 8/r, \]
where $D \subset \subset D' \subset \subset D'' \subset \subset \Omega_r$, and $D'_r := \{ x \in D'| \text{dist}(x, \partial D') \geq r \}$. If we set $Z'_a = \eta z'_a$ and $\xi'_a = \eta Z'_a$, then
\[
\eta \nabla z_h = \nabla Z - \nabla \eta z_h, \quad \nabla \xi = \eta (\nabla Z + \nabla \eta z_h).
\]
By (2.6), and note that we did not assume $q_p$ is symmetric (i.e., $\partial_{\rho_j} q^{\alpha \beta}_{ia} \neq \partial_{\rho_i a} q^{\alpha \beta}_{ja}$ in general),
\[
\int_{D'} \left[ \nabla Z \cdot \int_0^1 q_p(t) dt \cdot \nabla Z \right] = \int_{D'} \nabla Z \cdot \int_0^1 q_p(t) dt \cdot \nabla \eta z_h - \int_{D'} \nabla \eta z_h \cdot \int_0^1 q_p(t) \cdot \nabla Z \\
+ \int_{D'} \left[ \nabla \eta z_h \cdot \int_0^1 q_p(t) dt \cdot \nabla z_h \right] - \int_{D'} \left[ (\nabla Z + \nabla \eta z_h) \cdot \int_0^1 q_z(t) dt \cdot Z \right] \\
- \int_{D'} \left[ \eta (\nabla Z + \nabla \eta z_h) \cdot \int_0^1 q_x(t) dt \right] - \int_{D'} \left[ Z \cdot \int_0^1 w_p(t) dt \cdot (\nabla Z - \nabla \eta z_h) \right] \\
- \int_{D'} \left[ Z \cdot \int_0^1 w_z(t) dt \cdot Z \right] - \int_{D'} \left[ \eta Z \cdot \int_0^1 w_x(t) dt \right].
\]
To simplify the notations, let us set
\[
A_{ah} := \int_0^1 V_{a}^{k_a-2}(t) dt, \quad A_{ah} P_{ah} := \int_0^1 V_{a}^{k_a-1}(t) dt, \quad A_{ah} Q_{ah} := \int_0^1 V_{a}^{k_a}(t) dt,
\]
where $V_a(t) := (1 + |p_a(z(x)) + t \Delta p_a|^2)^{1/2}$. Clearly,
\[
A_{ah} \geq 1, \quad P_{ah} \geq 1, \quad Q_{ah} \geq 1, \quad P_{ah}^2 \leq Q_{ah}.
\]
Although our condition (1.5) is only satisfied almost everywhere on $\Omega$, we essentially use these conditions in integral form and the value on a subset of measure zero will not affect the result. The ellipticity condition in (1.5) implies
\[
\int_{D'} \left[ \nabla Z \cdot \int_0^1 q_p(t) dt \cdot \nabla Z \right] \geq \lambda(R) \int_{D'} A_{ah} |\nabla Z_a|^2,
\]
where $R$ is the upper bound for $x \in D'$ and $z = z(x)$, i.e., $|x|^2 + |z|^2 \leq R^2$. The right-hand side terms of (2.7) can be controlled by condition (1.5) and Cauchy-Schwarz inequality. Here, we demonstrate the estimates of the terms $q_z$ and $w_p$, which explains that the coupling structure of $q_z$ and $w_p$ in (1.5) is crucial.
\[
- \int_{D'} \nabla Z \cdot \int_0^1 q_z(t) dt \cdot Z \leq \Lambda(R) \int_{D'} \left[ A_{ah} P_{ah} |\nabla Z_a||Z_a| + A_{2h} |\nabla Z_2||Z_1| \right] \\
\leq \Lambda(R) \int_{D'} \left[ \epsilon A_{ah} |\nabla Z_a|^2 + \frac{1}{4 \epsilon} A_{ah} P_{ah}^2 |Z_a|^2 + \frac{1}{4 \epsilon} A_{2h} |Z_1|^2 \right] \\
- \int_{D'} \left[ Z \cdot \int_0^1 w_p(t) dt \cdot \nabla Z \right] \leq \Lambda(R) \int_{D'} \left[ \epsilon A_{ah} |\nabla Z_a|^2 + \frac{1}{4 \epsilon} A_{ah} P_{ah}^2 |Z_a|^2 + \frac{1}{4 \epsilon} A_{2h} |Z_1|^2 \right].
\]
Therefore (recall that $\Lambda = \Lambda(R)$ and $\lambda = \lambda(R)$),
\[
\int_{D'} A_{ah} |\nabla Z_a|^2 \leq C(\Lambda, \lambda) \int_{D'} \left[ A_{ah} |\nabla \eta|^2 |z_{ah}|^2 + A_{ah} Q_{ah} (1 + |Z|^2) + A_{2h} P_{2h} |Z_1|^2 \right]. \tag{2.8}
\]
Now, we need the following claim to handle $\int_{D'} A_{ah} Q_{ah} |Z_a|^2$.

**Claim** ([15, Lem. 2; 24, Lem. 5.9.1]). With the assumption of Theorem B, we have for any $\delta > 0$ and any $x_0 \in \Omega$, there exists $\rho$, $0 < \rho \leq 4r$, where $0 < 8r = r_0$, depending on $\delta$, $|x_0|$, $|z(x_0)|$, $\lambda$, $\Lambda$ and $h_{2i} (\Omega_r)$ — the modulus of (Hölder) continuity of the solution over $\Omega_r$, such that for $B_{\rho} = B_{\rho}(x_0)$,
\[
\int_{B_{\rho}} V_{a}^{k_a} c_{\eta a}^2 dx \leq \delta \int_{B_{\rho}} V_{a}^{k_a-2} |\nabla \xi_a|^2 + \int_{B_{\rho}} V_{a}^{k_a-1} c_{\xi a}^2, \quad \forall \xi_a \in L_{1,0}^k \cap C^0(B_{\rho}).
\]
In fact, by the assumption of Theorem B and note that $z \in C^4(\Omega)$, if $\rho \leq \max \{4r,1\}$, then we have for almost all $x \in B_\rho$, $|x|^2 + |z(x)|^2 \leq R_1^2$, where $R_1 > 0$ is a constant depending on $|x_0|$, $|z(x_0)|$ and $h_\alpha (\Omega_r)$, and the natural structure condition (1.5) holds for $R = R_1$ over $B_\rho$.

Let $\zeta_a^\prime (x): = \xi_a^\prime (x) \left( z_a^\prime (x) - z_a^\prime (x_0) \right) \in L^2_{1,0} \cap C^0 (B_\rho)$ be the test function in (1.4), we have

$$\int_{B_\rho} 2\xi_a \partial_\alpha \xi_a \Theta_a^\alpha_i (z_i^\prime - z_i^\prime (x_0)) + \xi_a^2 \left( p_{\alpha i} q_{\alpha i} + \left( z_i^\prime - z_i^\prime (x_0) \right) w_{\alpha i} \right) = 0.$$  

The condition given in (1.5) implies that

$$\xi_a^2 \xi_a q_{\alpha i} (x, z, p) = \xi_a^2 \xi_a q_{\alpha i} (x, z, 0) + \xi_a^2 \xi_a \frac{1}{\partial p_{i \beta}} p_{i \beta} dt$$

$$\geq \xi_a^2 \xi_a q_{\alpha i} (x, z, 0) + \lambda \int_0^1 \xi_a^2 |p_a|^2 (1 + |p_a|^2)^{k_a/2 - 1} dt$$

$$\geq \lambda \int_0^1 \xi_a^2 |p_a|^2 |q_a|^2 - \xi_a^2 |p_a| |q_a| (x, z, 0)|$$

$$\geq \lambda \xi_a^2 \frac{1}{k_a} |p_a|^2 - \Lambda \xi_a^2 |p_a|.$$  

Since

$$V_a^k \geq (1 + |p_a|^2)^{k_a/2} \geq 2^{k_a/2} \left( 1 + \frac{|p_a|}{k_a} \right)$$

and

$$\lambda |p_a| = \epsilon^{-1/k_a} \Lambda \cdot \xi^{1/k_a} |p_a| \leq \epsilon \frac{|p_a|}{k_a} + \epsilon^{-k_a/k_a} \Lambda \xi_a^2,$$

we conclude that, for some constants $\lambda', \Lambda'$ depending on $\lambda = \lambda (R_1), \Lambda = \Lambda (R_1)$,

$$\xi_a^2 \xi_a q_{\alpha i} (x, z, p) \geq \lambda \xi_a^2 V_a^k - \Lambda \xi_a^2.$$

Also, from (1.5)

$$|q_a| \leq \Lambda V_a^{k_a - 1}, \forall a \in \{1, 2\}, \quad |w_1| \leq \Lambda \left( V_1^{k_1} + V_2^{k_2 - 1} \right), \quad |w_2| \leq \Lambda V_2^{k_2}.$$  

It follows that

$$\int_{B_\rho} \xi_a^2 V_a^k \leq C (\Lambda, \lambda) \int_{B_\rho} \xi_a^2 - \int_{B_\rho} \xi_a (2 \partial_\alpha \xi_a q_{\alpha i} + \xi_a w_{\alpha i}) \left( z_i^\prime - z_i^\prime (x_0) \right)$$

$$\leq C (\Lambda, \lambda) \int_{B_\rho} \left[ \xi_a^2 + \sup_{B_\rho} |z - z(x_0)| \left( \left( V_a^{k_a - 2} |
\nabla \xi_a|^2 + V_a^{k_a} \xi_a^2 \right) + V_a^{k_a - 2} \xi_1^2 \right) \right].$$

Now, the Poincaré inequality implies that (note that $\xi_a \in L^1_{k_a} (B_\rho))$,

$$\int_{B_\rho} \xi_a^2 \leq C \rho^2 \int_{B_\rho} |\nabla \xi_a|^2 \leq C \rho^2 \int_{B_\rho} |\nabla \xi_a|^2 V_a^{k_a - 2}.$$  

The claim follows from the fact that $z_a \in C^4 (B_\rho)$ and $\sup_{B_\rho} |z - z(x_0)|$ can be chosen as small as we need, provided that $\rho$ is small enough.

To apply the above claim, we take $r = \rho/4$ further small, where $\rho$ is the constant in the above claim, $D' = B_{3r} = B_{3r} (x_0), D = B_r = B_r (x_0)$, clearly $B_{4r} = B_{4r} (x_0) \subset \Omega$, and $\zeta_a = z (\cdot + h_{\alpha}) \in C^4 (B_3r)$ for any $0 < |h| < r$. Moreover, $\tilde{z} = (\tilde{z}_1, \tilde{z}_2)$ solves (1.4) with $\tilde{q}_1 = q (x + h_{\alpha}, \cdot)$ and $\tilde{w} = w (x + h_{\alpha}, \cdot)$; and as coefficients they satisfy the condition (1.5) on $B_{3r}$ with $R = R_1$. Thus, we can apply the above claim in $B_{3r}$ for $\tilde{z}$ to obtain (note that $4r \leq \rho$)

$$\int_{B_{3r}} \tilde{V}_{a+2} \xi_a^2 \leq \delta \int_{B_{3r}} \tilde{V}_{a+2} \left( V_a^{k_a - 2} |
\nabla \xi_a|^2 + \int_{B_{3r}} \tilde{V}_2^{k_a - 2} \xi_1^2 \right), \quad \forall \xi_a \in L^1_{k_a} \cap C^0 (B_{3r}),$$
where
\[ \tilde{V}_a^2 = 1 + |p_a(x + he_\gamma)|^2 = 1 + |p_a + \Delta p_a|^2. \]

Since \( \text{supp} \eta \subset D'_r = B_{2r} \) and \( z_{ah} \in L^k_{2d} \cap C^0(B_{3r}) \), we can take \( \xi_a = z_a = \eta z_{ah} \) to obtain
\[
\int_{B_{3r}} \tilde{V}_a^{k_a} |z_a|^2 \leq \delta \int_{B_{3r}} \tilde{V}_a^{k_a-2} \left| \nabla z_a \right|^2 + \int_{B_{3r}} \tilde{V}_2^{k_2-1} |Z_1|^2.
\]
Clearly,
\[
\int_{B_{3r}} V_a^{k_a} |z_a|^2 \leq \delta \int_{B_{3r}} V_a^{k_a-2} \left| \nabla z_a \right|^2 + \int_{B_{3r}} V_2^{k_2-1} |Z_1|^2.
\]

Now, we can estimate
\[
\int_{B_{3r}} A_{ah} Q_{ah} |z_a|^2 = \int_{B_{3r}} \int_0^1 \left( 1 + |p_a + t\Delta p_a|^2 \right)^{k_a/2} |z_a|^2 dt \leq C \int_{B_{3r}} \left( \tilde{V}_a^{k_a} + V_a^{k_a} \right) |z_a|^2
\]
\[
\leq C\delta \int_{B_{3r}} \left( \tilde{V}_a^{k_a-2} + V_a^{k_a-2} \right) \left| \nabla z_a \right|^2 + C \int_{B_{3r}} \left( V_2^{k_2-1} + \tilde{V}_2^{k_2-1} \right) |Z_1|^2
\]
\[
\leq \frac{C\delta}{c} \int_{B_{3r}} \int_0^1 \left( 1 + |p_a + t\Delta p_a|^2 \right)^{k_a/2-1} \left| \nabla z_a \right|^2 + C \int_{B_{3r}} \left( V_2^{k_2-1} + \tilde{V}_2^{k_2-1} \right) |Z_1|^2
\]
\[
= \frac{C\delta}{c} \int_{B_{3r}} A_{ah} |\nabla z_a|^2 + \left( V_2^{k_2-1} + C \int_{B_{3r}} \tilde{V}_2^{k_2-1} \right) |Z_1|^2,
\]
where in the second and fourth lines, we used the following elementary inequalities (see [24, p. 189, (5.9.4)]). For \( q = k_a/2 \) or \( q = k_a/2 - 1 \), there exist some constants \( c, C \) such that
\[
c \left( (1 + |p_a|^2)^q + (1 + |p_a + \Delta p_a|^2)^q \right) \leq \int_0^1 (1 + |p_a + t\Delta p_a|^2)^q dt \leq C \left( (1 + |p_a|^2)^q + (1 + |p_a + \Delta p_a|^2)^q \right).
\]

Thus, by (2.8), if we take \( \delta \) small enough
\[
\int_{B_r} A_{ah} \left| \nabla z_{ah} \right|^2 \leq C(\Lambda, \lambda) \int_{B_{3r}} \left[ A_{ah} \left( \frac{|z_{ah}|^2}{r^2} + Q_{ah} \right) + (A_{2h} P_{2h} + V_2^{k_2-1} + \tilde{V}_2^{k_2-1}) \left| z_{ah} \right|^2 \right].
\]

Since \( p_{\alpha \gamma} = \partial_\gamma z_a \in L^{k_a}(B_{3r}) \) by assumption, the relation of weak derivatives and differential quotients implies \( z_{ah} = \Delta h z_a \in L^{k_a}(B_{3r}) \) too and \( z_{ah} \rightarrow \partial_\gamma z_a \) in \( L^{k_a}(B_{3r}) \). Since \( V_a(t) = (1 + |p_a + t\Delta p_a|^2)^{1/2} \in L^{k_a}(B_{3r}) \), we know that \( V_a^{k_2-1}(t) \in L^{k_a/(k_2 - 2)}(B_{3r}) \) and \( A_{ah} = \int_0^1 V_a^{k_2-1}(t) dt \rightarrow A_a = V_a^{k_2-2} \) in \( L^{k_a/(k_2 - 2)}(B_{3r}) \) by [24, Thm. 3.6.8]. A similar argument shows that \( A_{ah} P_{2h} = \int_0^1 V_a^{k_2-1}(t) dt \rightarrow V_a^{k_2-1} \) in \( L^{k_a/(k_2 - 1)}(B_{3r}) \) and \( A_{ah} Q_{ah} = \int_0^1 V_a^{k_2-1}(t) dt \rightarrow V_a^{k_2} \) in \( L^1(B_{3r}) \). Applying Hölder’s inequality, we know that the right-hand side of (2.9) is uniformly bounded (independent of \( h \)). Here, we need the additional assumption \( z_{ah} \in L^{2k_2}_{2d}(B_{4r}) \) to conclude that the terms in the second parentheses of (2.9) are uniformly bounded. Since \( A_{ah} \geq 1 \), we conclude from (2.9) that \( \| \nabla z_{ah} \| L^2(B_{3r}) \) is uniformly bounded. But \( z_{ah} \in L^{k_a}(B_r) \) with \( k_a \geq 2 \) thus \( \tilde{z}_{ah} \) is uniformly bounded in \( L^2_1(B_r) \). The weak compactness implies \( \tilde{z}_{ah} \rightarrow v_a \) in \( L^2_1(B_r) \) for some sub-sequence \( h \rightarrow 0 \). The compact embedding \( L^2_d(B_r) \rightarrow L^{k_a}(B_r) \) implies that, after taking a further sub-sequence, \( \tilde{z}_{ah} \rightarrow v_a \) in \( L^{k_a}(B_r) \), but we already shown \( \tilde{z}_{ah} \rightarrow \partial_\gamma z_a \in L^{k_a}(B_r) \) \( (k_a \geq 2) \), thus \( \partial_\gamma z_a = v_a \in L^2_d(B_r) \). Since \( \gamma \) is arbitrary, it shows that \( z_a \in L^2_d(B_r) \) and the proof is completed by the arbitrariness of \( x_0 \).
2.3. The smoothness of perturbed solution. We first write down the Euler–Lagrange equation of $L_\alpha$ locally in terms of Fermi coordinates, then the $L^2$-interior regularity follows directly from Theorem B. To prove the boundary regularity, we extend the solution from half disc to the whole disc via a reflection argument. It turns out that such reflected solution satisfies an equation that is similar to the original one (with coefficients extended properly), c.f. Lemma 2.4. The verification of this fact is given by decomposing the test function through parity and check the parity of each coefficient. It is notable that in general the coefficient involving the Christoffel symbols of extended solution is only $L^\infty$ near the free boundary, and we cannot apply Theorem B directly to the extended solution to show the boundary regularity. This explains the additional requirement that $K \subset F$ is totally geodesic. Finally, the smoothness up to the boundary of critical points of $L_\alpha$ for $\alpha - 1$ small follows from a bootstrap of the $L^2_\alpha$-strong solution.

Locally, we take coordinate systems near the boundary as $\{U; x = (x^1, x^2)\}$ with $\partial \Sigma \cap U = \{x^2 = 0\}$ and for any $(x^1, 0) \in \partial \Sigma \cap U$, let $x^2 \mapsto (x^1, x^2)$ be a regular geodesic orthogonal to $\partial \Sigma$. Let $B = \{x \in \Sigma : |x| < 1\}$ be the unit disc in $\Sigma$. $D = \{x \in \Sigma : |x| < 1, x^2 \geq 0\}$ be the unit upper half disc in $\Sigma$, $\partial^0 D = \{x \in \partial D : x^2 = 0\}$ and $\partial^+ D = \{x \in \partial D : |x| = 1\}$. For simplicity, we use $U$ to denote either $B$ or $D$. The following theorem implies that locally we can always choose a representative that is in Coulomb gauge.

Theorem 2.3 ([22, Thm. 3.2’–3.3’; 37, Thm. 2.1]). Suppose $p \geq 1$, $G$ is a compact Lie group and $U = U \times G$ is the trivial bundle on a disc/half disc $U \subset \mathbb{R}^2$ with flat metric on $U$. Then, there exists a uniform constant $\delta_0 > 0$, such that any connection $\hat{A} \in L^1_0(\Omega^1(U \times_M g))$ with $\|F_{\hat{A}}\|_{L^1(\Omega^2(U \times_M g))} \leq \delta_0$ is gauge equivalent to a connection $d + A \in L^1_0(\Omega^1(U \times_M g))$, i.e., for some $S \in L^0_2(U \times_c G)$, $S^* \hat{A} = d + A$, where $A$ satisfies

(a) $d^* A = 0$, where $*$ is the Hodge star operator with respect to the flat metric;
(b) $\nu \cdot A = 0$ for any $x \in \partial U$;
(c) $\|A\|_{L^p} \leq C\|F_A\|_{L^p}$.

Suppose $\sigma : U \times F \rightarrow \pi^{-1}(U)$ is a local trivialization of $\mathcal{F}$. Under this trivialization, we write the section $\phi(x) = (x, u(x)) \in U \times F$ and identify $\phi$ with $u$, $\nabla_A \phi$ with $\nabla_A u$ and $\mu(u)$ with $\mu(u)$, since their values are determined by $u$. With these notations, when the metric on $U$ is Euclidean and $u$ is regular enough, we can rewrite (1.2) as (under Coulomb gauge)

\[
\begin{align*}
\nabla_x \nabla A u - \frac{1}{4} \langle dY, \nabla A u \rangle - \frac{d^*A}{4}u \cdot \nabla \mu(u) &= 0, \quad x \in U \\
\nabla A = \langle dA, A \rangle - \langle A, [A, A] \rangle + \mathcal{U}(\nabla A u, u) &= 0, \quad x \in U \\
\nu \cdot \nabla A \phi &\perp T_0 K^\nu, \quad x \in \partial^0 U \\
\nu \cdot F_A &= 0, \quad x \in \partial^+ U \\
\nu \cdot A &= 0, \quad x \in \partial U,
\end{align*}
\]

(2.10)

where $\partial^0 U: = \partial \Sigma \cap U$, $\mathcal{U} = \alpha(1 + |\nabla_A u|^2)^{\alpha - 1}$, $\Delta A = d^* dA + dd^* A$ is the Laplace operator on 1-forms, and we use

\[
\nabla_A(f \nabla A \phi) = -(d \langle f, \nabla A \phi \rangle + f \nabla_A f \nabla_A \phi).
\]

Note that, by definition $\nabla_A u = du + A \cdot u = (\partial_t u + A_t \cdot u)dx^i := u_t dx^i$, where $A_t \in g$, which acts on $u$ as follows

\[
A_t \cdot u := \frac{d}{dt}\bigg|_{t=0} \exp(t A_t) \cdot u,
\]

(2.11)

here, exp is the exponential map of $G$. It is clear that $\nabla_A u$ is a tangent vector of $F$ at $u$, we will write $A_t \cdot u = A^f_t(u)$ be the fundamental vector field corresponding to $A_t$ at $u$. Similarly, for a tangent vector fields $v \in \Gamma(u^* TF)$, we have $\nabla_A v = \nabla v + A \cdot v = (\nabla_{\partial_t} v + A_t \cdot v)dx^i$, where $\nabla_{\partial_t}$ is the pullback connection, and

\[
A_t \cdot v := \frac{\nabla}{dt}\bigg|_{t=0} (d(\exp(t A_t) \cdot u))(v) = \nabla_A^f v,
\]

(2.12)
where $\nabla$ is the Levi-Civita connection of $F$. Suppose $g = \text{span} \{v_1, \ldots, v_m\}$, and denote $V_\alpha$ the fundamental vector field generated by $v_\alpha$, then for $A_i(x) = a_i^\alpha(x)v_\alpha$, we have

$$A_i \cdot u = A_i^\alpha(u) = a_i^\alpha(x)V_\alpha(u).$$

Now, a direct computation shows that the local equation is given by

$$
\begin{cases}
\Delta_{\Sigma} u - 2(\alpha - 1)\frac{\langle \nabla^2_{\alpha} u, \nabla A u \rangle \nabla A u}{1 + \|\nabla A u\|^2} - \Phi_\alpha(A, u) = 0, & x \in U \\
\Delta A - \Psi_\alpha(A, u) = 0, & x \in U \\
\frac{\partial u}{\partial \nu} \perp T_u K, & x \in \partial^0 U \\
A_2 = 0, & x \in \partial U, \\
\frac{\partial A_1}{\partial \nu} = 0, & x \in \partial^0 U,
\end{cases} \tag{2.13}
$$

where

$$
\Phi_\alpha(A, u) = \Gamma(u)(du, du) + 2A \cdot du + A \cdot u + \frac{1}{\Gamma} \mu(u) \cdot \nabla \mu(u), \tag{2.14}
$$

$\Delta_{\Sigma}$ is the Laplace-Beltrami operator on functions over $\Sigma$, $\Delta = dd^* + d^*d$ is the Laplace operator of 1-forms, $\Gamma(u)$ is the second fundamental form of $F \rightarrow \mathbb{R}^l$, and

$$
\Psi_\alpha(A, u) = (dA, A) + [A, [A, A]] - \Upsilon(\nabla A u, u). \tag{2.15}
$$

The boundary condition is localized as follows: let $\{e_1, e_2\}$, $e_2|_{\Sigma} = \nu$, be a moving frame near the boundary and $\{\omega^1, \omega^2\}$ be the dual frame. If we write $A = A_i\omega^i$, then

$$
\begin{cases}
\nu \int A = 0, & x \in \partial U \\
\nu \int F_A = 0, & x \in \partial^0 U
\end{cases}
\quad \implies \begin{cases}
A_2 = 0, & x \in \partial U \\
\frac{\partial A_1}{\partial \nu} = 0, & x \in \partial^0 U.
\end{cases}
$$

The boundary condition for the section $\phi$ is given by

$$
\nu \int \nabla A \phi \perp T_\phi K^\nu, \quad x \in \partial^0 U,
$$

which is equivalent to

$$
\frac{\partial u}{\partial \nu} \perp T_u K, \quad x \in \partial^0 U,
$$

since $\nu \int A = 0$ on $\partial^0 U$.

Remark. The boundary condition imposed on $\partial^0 U$ in (2.13) is empty if $U$ is an interior neighborhood. For the boundary neighborhood, the free-boundary is only prescribed at the flat part $\partial^0 U$. We should remark also that $A_2 = 0$ is exactly the local Coulomb gauge boundary condition given by $\nu \int A = 0$ as in Theorem 2.3.

Before we get involved into the proof of Theorem A, we state the reflecting technique as follows, which will be needed in the proof of boundary regularity. For simplicity, we will assume that the underlying metric on a local chart $U$ is flat in the following context. Recall that the metric of a two dimensional surface $\Sigma$ is locally conformal to the standard Euclidean metric, i.e., $g = e^{2\nu} g_0$, where $g_0$ is the Euclidean metric. Then, the YMH energy has the form

$$
\mathcal{L}_\alpha(A, u) = \int \left( (1 + \|\nabla_A u\|^2\alpha + |F_A|^2 + \|\mu(u)\|^2) \right) dg
$$

$$
= \int \left( (1 + e^{-2\nu}\|\nabla_A u\|^2\alpha + e^{-4\nu}|F_A|^2 + \|\mu(u)\|^2) \right) e^{2\nu} d\nu g_0
$$

$$
= \int \left( e^{-2(\alpha-1)\nu}(e^{2\nu} + \|\nabla_A u\|^2\alpha + e^{-2\nu}|F_A|^2 + e^{2\nu}|\mu(u)\|^2) \right) d\nu g_0,
$$

thus the Euler-Lagragian equation under the conformal metric $g_0$ is given by

$$
\begin{cases}
\nabla_A^* \left( \alpha e^{-2(\alpha-1)\nu}(e^{2\nu} + \|\nabla_A u\|^2\alpha - 1\nabla_A u) + e^{2\nu}\mu(u)\nabla \mu(u) = 0, \\
D_A^* e^{-2\nu} F_A + \alpha e^{-2(\alpha-1)\nu}(e^{2\nu} + \|\nabla_A u\|^2\alpha - 1\nabla_A u, u) = 0.
\end{cases}
$$
Although the YMH field equation is not conformally invariant, it is clear from equation (1.2) and (2.13) that after a conformal change of the metric, the structure of the equation will not change, except that some additional lower order terms (which can be analytically well controlled) emerge. Therefore our arguments blow still works for non-flat metrics. Especially, in the proofs of Theorem A and Theorem C, we only consider the case that the metric is locally flat for simplicity and omit the general case with the conformal factor $v$.

Now for $x_0 \in \partial \Sigma$, without loss of generality, we assume the local trivialization chart $U$ of $x_0$ is an upper half disc $D_\rho$ centered at $x_0 = 0$ and the flat boundary is settled on $\partial \Sigma$. Moreover, since $u \in L^2_1(\Sigma; \mathbb{R}^l) \hookrightarrow C^0(\bar{\Sigma}; \mathbb{R}^l)$, we can take $\rho$ small enough such that the following reflection is well-defined. A more geometric way can be found in [29, Sect. 3]. For $p = u(x_0) \in K$, we choose Fermi coordinates $(f^1, \ldots, f^n)$ on an open neighborhood $V$ of $p$ in $F$, such that

- $V \cap K = \{ f^{k+1} = 0, \ldots, f^n = 0 \}$;
- For any fixed $q \in K$ and $a \in \{ k + 1, \ldots, n \}$, the $f^a$-coordinate curve start from $q$ is a geodesic in $V \subset F$, which is perpendicular to $K$.

In order to keep the extension as smooth as possible, it turns out that the extension depends on the “type” of boundary condition. More precisely, for homogeneous Neumann boundary we use the even extension and for Dirichlet boundary, we use the odd extension. These two types of boundary conditions root in the free boundary condition, the $n-k$ Dirichlet conditions come from the fact that $u(\partial^0 U) \subset K$. The remaining $k$ boundary conditions come from the constraint in calculus of variation. To write down these boundary conditions in Fermi coordinates, we note first that

$$
\frac{\partial u}{\partial \nu} = -\frac{\partial u}{\partial x^2} = \frac{\partial u^a}{\partial x^2} \frac{\partial}{\partial f^a},
$$

where $u^a := f^a \circ u$. Then, the local boundary condition in (2.13) of $u$ is given by

$$
\frac{\partial u^a}{\partial x^2}(x) = 0, \quad x \in \partial^0 U, \quad a \in I_1 := \{1, \ldots, k\}.
$$

The constraint $u(\partial^0 U) \subset K$ transforms to

$$
u^a(x) = 0, \quad x \in \partial^0 U, \quad a \in I_2 := \{k + 1, \ldots, n\}.
$$

Next, we extend various quantities from $D_\rho$ to $B_\rho$. Let us illustrate the basic idea by the extension of $u$. Suppose $x^* = (x^1, -x^2) \in D_\rho$ is the reflection of $x = (x^1, x^2)$ with respect to $\partial^0 D_\rho$ and $r(x) = x^*$ is the reflection map. The reflection of $F$ with respect to $K$ is defined as follows,

$$
\gamma : F|_V \rightarrow F
$$

where $q = \varphi^{-1}(f^1, \ldots, f^n) \mapsto q^* = \varphi^{-1}(f^1, \ldots, f^k, -f^{k+1}, \ldots, -f^n)$, and $\varphi : F|_V \rightarrow \mathbb{R}^n$ is the coordinate map. The extension of $u$ is given by, for $x \in D_\rho^- := B_\rho \setminus D_\rho^*$,

$$
\tilde{u} = \gamma \circ u \circ r.
$$

In order to check the parity, we note that, for $x \in D_\rho^-$,

$$
\tilde{u}^a(x) = \begin{cases} u^a(x^*), & a \in I_1 \\ -u^a(x^*), & a \in I_2 \end{cases}.
$$

By our boundary conditions (2.16) and (2.17) of $u$, it is easy to see, $\tilde{u} \in L^2_1(B_\rho, F)$ for any $u \in L^2_1(D_\rho, F)$.

For a vector field $v(x) = \nu^a(x) \partial_{f^a}(u(x))$ along $u$, we define the extended vector filed along $\tilde{u}$ as, for $x \in D_\rho^-$,

$$
\tilde{v} := \gamma_* \circ v \circ r.
$$

In what follows, we always omit the trivial relation that the extended quantity restricting to $D_\rho$ equals to the original one for simplicity.
In particular, for \( x \in D_\rho^- \),
\[
\tilde{\partial}_{f^a}(u) = \gamma_s(\partial_{f^a}(u(x^*))) = (-1)^{j-1}\partial_{f^a}(\tilde{u}(x)), \quad a \in I_j,
\]
and if we write \( \tilde{v}(x) = \tilde{v}^b(x)\partial_{f^a}(\tilde{u}(x)) \), then
\[
\tilde{v}^a(x) = (-1)^{j-1}v^a(x^*), \quad a \in I_j.
\]

The metric \( h \) is extended by \( \hat{h} = \gamma^*h \). It is easy to show, in the coordinates \( (f^1, \ldots, f^n) \), for \( x \in D_\rho^- \),
\[
\hat{h}_{ab}(\tilde{u}(x)) = \begin{cases} -h_{ab}(u(x^*)), & \text{if } (a, b) \in -IA := I_1 \times I_2 \cup I_2 \times I_1 \\ h_{ab}(u(x^*)), & \text{otherwise} . \end{cases}
\]
The extended Christoffel symbol \( \hat{\Gamma}^\circ(\tilde{u})^* \) is defined by the extended metric \( \hat{h}(\tilde{u}) \).

We extend the connection one form \( A \) from \( D_\rho \) to the whole disc \( B_\rho \) evenly, i.e., we define \( \hat{A} \) by the following relation,
\[
\gamma^*A = \hat{A}.
\]
If we write \( A \) as a \( g \)-valued 1-form \( A_i(x)dx^i \) locally, then for \( x \in D_\rho^- \),
\[
\hat{A}_i(x) = A_i(x^*), \quad \hat{A}_2(x) = -A_2(x^*).
\]
Locally, let \( g = \text{span}\{v_1, \ldots, v_m\} \), the fundamental vector field generated by \( v_\alpha \) is denoted by \( V_\alpha = \lambda^\alpha_i\partial_{f^a} \). If we extend \( V_\alpha \) by, for \( x \in D_\rho^- \),
\[
\hat{V}_\alpha(\tilde{u}) = \lambda^\alpha_i(\tilde{u})\partial_{f^a}(\tilde{u}) = \gamma_s \circ V_\alpha(u) \circ r,
\]
then
\[
\hat{\gamma}_\alpha(u(x^*)) = (-1)^{j-1}\gamma^\alpha_i(u(x^*)), \quad a \in I_j.
\]
(2.18)

If we write \( A_i(x) = a^\alpha_i(x)v_\alpha \) and \( \hat{A}_i(x) = \tilde{a}^\alpha_i(x)v_\alpha \), then for \( x \in D_\rho^- \),
\[
\tilde{a}^\alpha_i(x) = (-1)^{j-1}a^\alpha_i(x^*).
\]
(2.19)

Note that the local boundary condition of \( A \) is given by, for any \( \alpha = 1, 2, \ldots, m \) and any \( x \in \partial^0U \),
\[
\frac{\partial a^\alpha_i}{\partial x^j}(x) = 0, \quad a^\alpha_2(x) = 0,
\]
which clearly implies that \( \hat{A} \in L^2_\gamma(B, \Omega^1(g)) \) for any \( A \in L^2_\gamma(B, \Omega^1(g)) \).

Since we expect \( \hat{\nabla}_A\hat{u} \) and \( \hat{\nabla}_A\hat{v} \) are vector fields along \( \hat{u} \), we extend them, respectively, as follows, for \( x \in D_\rho^- \),
\[
\hat{\nabla}_\hat{A}\hat{u}(x) = \hat{\nabla}_A u(x) = \gamma_s(\nabla_A u(x^*)), \quad \hat{\nabla}_\hat{A}\hat{v}(x) = \hat{\nabla}_A v(x) = \gamma_s(\nabla_A v(x^*)).
\]
Locally, if we write
\[
\hat{\nabla}_\hat{A}\hat{u}(x) = \left(\hat{\nabla}_{\partial_i}\hat{u}(x) + \hat{A}_i\hat{u}(x)\right) \otimes dx^i
\]
\[
\hat{\nabla}_\hat{A}\hat{v}(x) = \left(\hat{\nabla}_{\partial_i}\hat{v}(x) + \hat{A}_i\hat{v}(x)\right) \otimes dx^i,
\]
where \( \hat{\nabla}_{\partial_i} \) is the pullback connection along \( \hat{u} \), then the above extension requires that \( \hat{A}_i\hat{u} \) and \( \hat{A}_i\hat{v} \) satisfy the following relations, respectively, for \( x \in D_\rho^- \):
\[
\hat{A}_i\hat{u}(x) = (-1)^{j-1}A_i \cdot u(x) = (-1)^{j-1}\gamma_s(A_i \cdot u(x^*)), \quad (2.20)
\]
\[
\hat{A}_i\hat{v}(x) = (-1)^{j-1}A_i \cdot v(x) = (-1)^{j-1}\gamma_s(A_i \cdot v(x^*)). \quad (2.21)
\]
Recall that
\[
A_i \cdot u = A^a_i(u) = a^\alpha_i(x)V_\alpha(u) = a^\alpha_i(x)\lambda^\alpha_i(u)\partial_{f^a}(u) =: A^\alpha_i(u)\partial_{f^a}(u),
\]
\( \Gamma^\alpha_{ab}(\hat{u}(x)) = \Gamma^\alpha_{ab}(u(x^*)) \) and \( \Gamma^\alpha_{ab}(\hat{u}(x)) = -\Gamma^\alpha_{ab}(u(x^*)) \) otherwise.
If we write
\[ \tilde{A}_i \tilde{\nu} := \tilde{\alpha}^a_i(x) \tilde{\lambda}_a(u(x)) \partial f^a(\tilde{u}(x)) =: \tilde{A}_i^a(\tilde{u}) \partial f^a(\tilde{u}), \]
where \( \tilde{\lambda}_a(\tilde{u}) = (-1)^{i-1} \lambda^a(u \circ r) \) for \( x \in D^-_{\rho} \) and \( a \in I_j \) (by (2.18)), then by (2.20), for \( x \in D^-_{\rho} \),
\[ \tilde{A}_i^a(\tilde{u}(x)) = (-1)^{i+j} A_i^a(u(x^*)), \quad a \in I_j. \]
To simplify the notation further, let \( \tilde{u}_i^a(x) := \partial_i \tilde{\nu}^a(x) + \tilde{A}_i^a(\tilde{u}(x)) \), then for \( x \in D^-_{\rho} \),
\[ \tilde{u}_i^a(x) = (-1)^{i+j} u_i^a(x^*), \quad a \in I_j, \]
where
\[ u_i^a(x^*) := \partial_i u^a(x^*) + A_i^a(u(x^*)) \text{ and } \nabla_{\lambda}(u^a) = u_i^a(x^*) \partial f^a(u(x^*)) \otimes dx^i. \quad (2.22) \]

In order to show the local expression of \( \tilde{\nabla}_{\lambda} \tilde{\nu} \), recall that
\[ \tilde{\nabla}_{\lambda} \tilde{v}(x) = \left( \tilde{\nabla}_{\lambda_i} \tilde{v}(x) + \tilde{A}_i \tilde{v}(x) \right) \otimes dx^i, \]
where \( \tilde{\nabla}_{\lambda_i} \) is the pullback connection along \( \tilde{u} \). More precisely,
\[ \tilde{\nabla}_{\lambda_i} \tilde{v}(x) = \tilde{\nabla}_{\lambda_i} (\tilde{v}^a(x) \partial f^a(\tilde{u}(x))) \]
\[ = \partial_i \tilde{v}^a(x) \partial f^a(\tilde{u}(x)) + \tilde{v}^a(x) \tilde{\nabla}_{\lambda_i} \partial f^a(\tilde{u}(x)) \]
\[ = \left( \partial_i \tilde{v}^c(x) + \tilde{v}^a(x) \partial_i \tilde{\nu}^b(x) \tilde{\Gamma}_ab(\tilde{u}(x)) \right) \partial f^c(\tilde{u}(x)) \]
\[ = (-1)^{i-1} \tilde{\nabla}_{\lambda_i} \nu(x), \]
where
\[ \nabla_{\lambda_i} \nu(x) = \left( \partial_i \tilde{v}^c(x) + \tilde{v}^a(x) \partial_i u^b(x) \Gamma_{abc}(u(x)) \right) \partial f^c(u(x)). \quad (2.23) \]

In order to write down \( \tilde{A}_i \tilde{\nu}(x) = \tilde{\nabla}_{\lambda} \left( \tilde{A}_i \tilde{\nu}(x) \right) \) locally, we adopt the following notation
\[ \tilde{A}_i \partial f^a(\tilde{u}(x)) := \tilde{A}_{ai}^b(\tilde{u}(x)) \partial f^a(\tilde{u}(x)), \]
then \( \tilde{A}_i \tilde{\nu}(x) = \tilde{\nu}^a(x) \tilde{A}_{ai}^b(\tilde{u}(x)) \partial f^a(\tilde{u}(x)) \). By (2.21), we know that
\[ \tilde{A}_i \partial f^a(\tilde{u}(x)) = (-1)^{i+j+k} \tilde{A}_{ai}^b(\tilde{u}(x^*)), \quad a \in I_j, \]
which implies, for \( a \in I_j, b \in I_k \) and \( x \in D^-_{\rho} \),
\[ \tilde{A}_{ai}^b(\tilde{u}(x)) = (-1)^{i+j+k+1} \tilde{A}_{ai}^b(\tilde{u}(x^*)), \]
where
\[ \tilde{A}_{ai}^b(u(x)) = \partial f^a A_i^b(u(x)) + A_i^b(u(x)) \Gamma_{abc}^b(u(x)) \]
\[ = a_i^a(x) \left( \partial f^a \Lambda_{ac}^b(u(x)) + \Lambda_{ac}^c(u(x)) \Gamma_{abc}(u(x)) \right). \quad (2.24) \]

The extension of \( \mu \) is given by, for \( x \in D^-_{\rho} \),
\[ \tilde{\mu}(\tilde{u}(x)) = \mu(u(x^*)). \]
It is easy to show, if we define
\[ \tilde{\nabla}_{\partial f^a} \tilde{\mu}(\tilde{u}(x)) := \left( \nabla_{\partial f^a} \tilde{\mu}(\tilde{u}(x)) \right) = \gamma_+ \left( \nabla_{\partial f^a} \mu \right)(u(x^*)), \]
then, for \( b \in I_j \) and \( x \in D^-_{\rho} \),
\[ \tilde{\nabla}_{\partial f^a} \tilde{\mu}(\tilde{u}(x)) = (-1)^{i-1} \tilde{\nabla}_{\partial f^a} \mu(u(x^*)). \]
We should remark that, if we view \( \tilde{h}, \tilde{\Gamma} \) and \( \tilde{\mu} \) as functions of \( \tilde{u} \), then they maybe multi-valued, but they are still single-valued when restrict to \( \tilde{u}(B_\rho) \) for \( \rho \) is small enough, and we can apply Theorem B to improve the regularity.
The following lemma asserts that under the above extension, \((\bar{A}, \bar{v})\) solves weakly an equation that is similar to (2.10).

**Lemma 2.4.** Suppose \((\bar{A}, \bar{v})\) is the extension of \((A, u)\) as above, where \((A, u) \in L^2_t(D_\rho, \Omega^1(g)) \times L^{2\alpha}_t(D_\rho, F)\) solves (2.10) weakly in \(D_\rho\). Then \((\bar{A}, \bar{v}) \in L^2_t(B_\rho, \Omega^1(g)) \times L^{2\alpha}_t(B_\rho, F)\), and for all \((B, v) \in C^0_0(B_\rho, \Omega^1(g)) \times C^0_0(B_\rho, \bar{u}^*(TF))\), there holds

\[
\left\{ \begin{array}{l}
\int_{D_\rho} \alpha (1 + |\nabla A|^2)^{\alpha - 1} \left( \nabla_A \bar{u} \bar{v} \right) + \left( \bar{\mu} \bar{v} \right) = 0, \\
\int_{B_\rho} \alpha (1 + |\nabla A|^2)^{\alpha - 1} \left( \nabla_A \bar{u} \nabla A v + B \cdot u \right) + \left( F_A, D_A B \right) + \left( \mu(u), \nabla \mu(v) \right).
\end{array} \right.
\tag{2.25}
\]

**Proof.** The weak form of (2.10) is given by, for any \((B, v) \in L^2_t \cap C^0_0(D_\rho, \Omega^1(g)) \times L^{2\alpha}_t(D_\rho, \bar{u}^*(TF))\) and \(v|_{\partial D_\rho} \in T_u K\),

\[
0 = \int_{D_\rho} \alpha (1 + |\nabla A|^2)^{\alpha - 1} \left( \nabla_A u, \nabla A v + B \cdot u \right) + \left( F_A, D_A B \right) + \left( \mu(u), \nabla \mu(v) \right).
\]

That is

\[
\left\{ \begin{array}{l}
\int_{D_\rho} \alpha (1 + |\nabla A|^2)^{\alpha - 1} \left( \nabla_A u, \nabla_A v \right) + \left( \mu(u), \nabla \mu(v) \right) = 0, \\
\int_{B_\rho} \alpha (1 + |\nabla A|^2)^{\alpha - 1} \left( \nabla_A u, B \cdot u \right) + \left( F_A, D_A B \right) = 0.
\end{array} \right.
\tag{2.26}
\]

Let us write down (2.26) exactly in local Fermi coordinates. Note that the test functions \(B \) and \(v\) are vector valued, and we will test each component. For \(v(x) = v^b(x) \partial_j u^d(x) \in \Gamma(u^*TF)\), by (2.22), (2.23) and (2.24),

\[
\langle \nabla_A u, \nabla_A v \rangle = h^{ab} u^a_i \left( \partial_i v^b(x) + v^b(x) \partial_i u^d(x) \Gamma_i^d u(x) + v^b(x) A_b^d(u(x)) \right),
\]

where \(u^a_i := \partial_i u^a + A^a_i(u) = \partial_i u^a + a^a_i \lambda^a_i(u)\). For any fix \(b\), if we take \(v^b(x) = \varphi(x)\), where \(\varphi \in C^\infty(D_\rho)\) for \(b \in I_1\) and \(\varphi \in C^0_0(D_\rho)\) for \(b \in I_2\), then by the first equation of (2.26),

\[
0 = \int_{D_\rho} \alpha \left(1 + |\nabla A|^2\right)^{\alpha - 1} \left(h^{ab}(u) \partial_i \varphi v^a_i + h^{ad}(u) \varphi u_i^a \partial_i \varphi \Gamma_i^d(u) + h_{ad}(u) \varphi u^a_i A^d_{ib} + \varphi(u) \cdot \langle \nabla \varphi, \mu \rangle(u) \right).
\tag{2.27}
\]

By the definition of induced connection,

\[
D_A B = dB + [A \wedge B] = (- \partial_j B_i + [A_i, B_j]) dx^i \wedge dx^j.
\]

Recall also that

\[
F_A = dA + [A \wedge A] = (\partial_i A_j + A_i A_j) dx^i \wedge dx^j = A_{ij} dx^i \wedge dx^j, \quad F_{ij} = \frac{1}{2} \left( A_{ij} - A_{ji} \right),
\]

In order to write down the above equations locally, suppose \(B_i(x) = b^i(u(x)) v_\alpha, \langle v_\alpha, v_\beta \rangle = \delta_{\alpha \beta}, \partial_i v_\alpha = 0\) and \(\langle v_\alpha, v_\beta \rangle = g_{\alpha \beta} v_\gamma\), then

\[
2F_A = 2F_{ij} dx^i \wedge dx^j = (\partial_i A_j - \partial_j A_i + [A_i, A_j]) dx^i \wedge dx^j = \left( \partial_i a^j_\gamma - \partial_j a^i_\gamma \right) v_\gamma dx^i \wedge dx^j = 2F_{ij} v_\gamma dx^i \wedge dx^j,
\]

\[
D_A B = (\partial_i B_j + [A_i, B_j]) dx^i \wedge dx^j = \left( \partial_i b^j_\gamma + a^a_i b^j_\delta g_{\alpha \beta} \right) v_\gamma dx^i \wedge dx^j,
\]

\[
\langle F_A, D_A B \rangle = \left( \partial_i b^j_\gamma + a^a_i b^j_\delta g_{\alpha \beta} \right) F_{ij}.
\]

Note also that \(A_i^j(u(x)) = a_i^j(u(x)) V_{\alpha}(u(x))\) and \(B_i \cdot u = B_i^j(u) = b_i^j(u(x)) \lambda_i^a(u) \partial_a(u)\), thus we can write \(\langle \nabla_A u, B \cdot u \rangle\) locally as follows:

\[
\langle \nabla_A u, B \cdot u \rangle = h_{ab}(u) u^a_i b^j_i(x) \lambda_i^b(u),
\]
Therefore, for any fixed \( j, \beta \), if we take \( b_j^\beta = \vartheta \), such that

\[
\vartheta \in \begin{cases} 
C^\infty(D_\rho), & j = 1, \\
C_0^\infty(D_\rho), & j = 2.
\end{cases}
\]

then by the second equation of (2.26),

\[
0 = \int_{D_\rho} \alpha^j \left( 1 + |\nabla_A u|^2 \right)^{a-1} \partial\mu \varphi \vartheta + F_{ij}^\beta \partial_i \varphi \vartheta + F_{ij}^\alpha \tilde{\alpha}^\gamma \vartheta.
\] (2.28)

The next step is to show that, if we extend \( h_{ab}, \Gamma^c_{\tilde{a} b}, A, u, \mu \) as before, and use prime to distinguish the equations obtained by replacing all quantities in (2.27) and (2.28) with their extensions, then \( \tilde{u} \) and \( \tilde{A} \) satisfy (2.27)' and (2.28)' respectively. Clearly, (2.27)' holds for \( (\tilde{A}, \tilde{u}) \) and any \( \varphi \in C_0^\infty(B_\rho) \) when \( b \in I_1 \) and any \( \varphi \in C_0^\infty(D_\rho) \) with \( \varphi \equiv 0 \) on \( \partial^0 D_\rho \) when \( b \in I_2 \). We only need to check that when \( b \in I_2, \) (2.27)' holds for any \( \varphi \in C_0^\infty(B_\rho) \). Write

\[
\varphi = \varphi_e + \varphi_o, \quad \varphi_e(x) = \frac{1}{2} (\varphi(x) + \varphi(x^*)) , \quad \varphi_o(x) = \frac{1}{2} (\varphi(x) - \varphi(x^*) ),
\]

then clearly, for \( x \in D_\rho^- \),

\[
\partial_i \varphi_o(x) = (-1)^i \partial_i \varphi_e(x^*), \quad \partial_i \varphi_e(x) = (-1)^{i-1} \partial_i \varphi_e(x^*).
\]

Note that for \( b \in I_2 \), it is easy to check the parity of the components in (2.27)' for any \( x \in D_\rho^- \),

\[
\tilde{h}_{ab}(\tilde{u}) \tilde{u}^a_{i|x} = (-1)^i \tilde{h}_{ab}(u) u^a_{i|x}, \quad \tilde{h}_{ad}(\tilde{u}) \tilde{u}^a_{i|\tilde{a} b c} = \tilde{h}_{ad}(u) u^a_{i|\tilde{a} bc},
\]

Now, clearly, \( |\tilde{\nabla}_{\tilde{A}} \tilde{u}^a_{b}(x)| = |\nabla_A u^a_{b}(x^*)| \), for any \( x^* \in D_\rho \). We can compute the extended weak equation (2.27)' as

\[
\int_{B_\rho} \alpha(1 + |\tilde{\nabla}_{\tilde{A}} \tilde{u}^a_{b}|)^{a-1} (\tilde{h}_{ab}(\tilde{u}) \partial_i \varphi \tilde{u}^a_{i|\tilde{a} b c} + \tilde{h}_{ad}(\tilde{u}) \varphi \tilde{u}^a_{i|\tilde{a} c b})
\]

\[
= \int_{B_\rho} \alpha(1 + |\tilde{\nabla}_{\tilde{A}} \tilde{u}^a_{b}|)^{a-1} (\tilde{h}_{ab}(\tilde{u}) \partial_i \varphi \tilde{u}^a_{i|\tilde{a} b c} + \tilde{h}_{ad}(\tilde{u}) \varphi \tilde{u}^a_{i|\tilde{a} c b} \tilde{\Gamma}_{\tilde{a} b c}^{\tilde{d}}(\tilde{u}))
\]

\[
= \int_{B_\rho} \alpha(1 + |\tilde{\nabla}_{\tilde{A}} \tilde{u}^a_{b}|)^{a-1} (\tilde{h}_{ab}(\tilde{u}) \partial_i \varphi \tilde{u}^a_{i|\tilde{a} b c} + \tilde{h}_{ad}(\tilde{u}) \varphi \tilde{u}^a_{i|\tilde{a} c b} \tilde{\Gamma}_{\tilde{a} b c}^{\tilde{d}}(\tilde{u}))
\]

\[
= 2 \int_{D_\rho} \alpha(1 + |\nabla_A u|^2)^{a-1} (h_{ab}(u) \partial_i \varphi u^a_{i|x} + h_{ad}(u) \varphi u^a_{i|x} u^i u^\mu \Gamma^\mu_{\tilde{a} b c}(u))
\]

\[
= 0,
\]

the last equality follows from (2.27) and the fact that \( \varphi_o = 0 \) on \( \partial^0 D_\rho \). This shows that the extended solution \( (\tilde{A}, \tilde{u}) \) solves (2.27)' weakly.

Lastly, we verify (2.28)' for extended \( (\tilde{A}, \tilde{u}) \). Actually, we only need the following symmetries for \( j = 2 \). By the symmetry of \( a^\alpha_i \) (see (2.19))

\[
F_{ij}^\gamma(x) = \frac{1}{2} \left( \partial_x \tilde{\alpha}^\gamma_j(x) - \partial_{x_\beta} \tilde{\alpha}^\gamma_j(x) + 2 \tilde{a}^\alpha_i(x) \tilde{a}^\beta_i(x) g_{\alpha \beta}^\gamma \right) = (-1)^{i+j} F_{ij}^\gamma(x^*).
\]
It is easy to check, the following symmetries

\[
\tilde{F}^\gamma_{ij}(x)a^\delta_i(x) = (-1)^{j-1}F^\gamma_{ij}(x^*)a^\delta_i(x^*),
\]

\[
\tilde{h}_{ab}(\tilde{u})\tilde{\omega}^a_{ij}\lambda^b_{\gamma}(\tilde{u}) = (-1)^{j-1}h_{ab}(u(x^*))u^a_i(x^*)\lambda^b_{\gamma}(u(x^*)).
\]

With these parities in hand, we can decompose \( \vartheta \) into even part and odd part as \( \varphi \) and the verification of (2.28)' is the same.

Now, we are ready to prove Theorem A. We first rewrite the extended weak equation to standard form, and then check the condition (1.5) is satisfied, Theorem B shows that the weak solution is strong, and we can bootstrap the regularity of the strong solution to show the smoothness up to the boundary.

**Proof of Theorem A.** By Lemma 2.2, for \( \alpha > 1 \), there exists a weak solution \( (A_\alpha, \phi_\alpha) \in \mathcal{A}^2 \times \mathcal{A}^{2^2} \) of (1.2). We will improve the regularity of this weak solution module gauge and prove Theorem A.

**Step 1.** We first show the \( L^2 \)-interior regularity, which is a direct application of Theorem B. Locally, if we write the solution as \( (A, u) \), where \( A \) is a \( g \)-valued 1-form on \( U = B_\rho \) and \( u \) is a map from \( U \) to \( F \), then the equation of \( (A, u) \) is given by (2.27) and (2.28). That is, for \((\vartheta, \varphi) \in C_0^\infty(B_\rho) \) (note that \( C_0^\infty(B_\rho) \) is dense in \( L^2 \cap C^0(B_\rho) \times L^2(B_\rho) \)),

\[
\begin{align*}
0 &= \int_{B_\rho} \left\{ \partial_i \vartheta^a_j F^\beta_{ij} + \varphi^a_i \left( \Upsilon h_{ab}(u) u^a_i \lambda^b_\beta(u) + F^\gamma_{ij} a^\delta_i g^\gamma_{\delta\beta} \right) \right\} \\
0 &= \int_{B_\rho} \partial_i \varphi^a_i \cdot \Upsilon h_{ab}(u) u^a_i \\
&\quad + \int_{B_\rho} \varphi^a_i \left( \Upsilon h_{ab}(u) u^a_i \left( \partial_i u^\gamma \Gamma^\beta_{ab}(u) + A^a_\gamma(u) \right) + \mu(u) \cdot [\nabla_{\partial, \mu}] u \right),
\end{align*}
\]

where \( \Upsilon = \alpha(1 + |\nabla A|^2)^{\alpha-1} \). It is well-known in Yang–Mills theory that \( D^A_A F_A = 0 \) is not a strict elliptic equation of \( A \), we need to module the gauge action. Applying Theorem 2.3 to \( A \) on \( B_\rho \), we can assume further that \( A \) is in Coulomb gauge. Since under local Coulomb gauge, there holds, for all \( \beta = 1, 2, \ldots, m, \sum_{i=1}^2 \partial_i a^\beta_i(x) = 0 \). Integration by parts shows

\[
2 \int_{B_\rho} \partial_i \vartheta^a_j F^\beta_{ij} = \int_{B_\rho} \partial_i \vartheta^a_j \left( \partial_i a^\beta_j - \partial_j a^\beta_i + 2a^\gamma_i a^\delta_j g^\gamma_{\delta\beta} \right)
\]

\[
= \int_{B_\rho} \partial_i \vartheta^a_j \partial_j a^\beta_i - 2\partial_j a^\gamma_i \partial_i a^\delta_j g^\gamma_{\delta\beta}.
\]

Thus, under Coulomb gauge, (2.29) transforms to

\[
\begin{align*}
0 &= \int_{B_\rho} \left\{ \partial_i \vartheta^a_j \partial_j a^\beta_i + 2\partial_j \left( -a^\gamma_i \partial_i a^\delta_j g^\gamma_{\delta\beta} + \Upsilon h_{ab}(u) u^a_i \lambda^b_\beta(u) + F^\gamma_{ij} a^\delta_i g^\gamma_{\delta\beta} \right) \right\} \\
0 &= \int_{B_\rho} \partial_i \varphi^a_i \cdot \Upsilon h_{ab}(u) u^a_i \\
&\quad + \int_{B_\rho} \varphi^a_i \left( \Upsilon h_{ab}(u) u^a_i \left( \partial_i u^\gamma \Gamma^\beta_{ab}(u) + A^a_\gamma(u) \right) + \mu(u) \cdot [\nabla_{\partial, \mu}] u \right),
\end{align*}
\]

where \( \Upsilon = \alpha \left( 1 + h_{ab}(u)(\partial_i u^a + a^\beta_i \lambda^b_\beta(u))(\partial_i u^b + a^\gamma_i \lambda^b_\gamma(u)) \right)^{\alpha-1} \), \( u^a_i = \partial_i u^a + a^\beta_i \lambda^b_\beta(u), \) \( F^\gamma_{ij} = (\partial_i a^\gamma_j - \partial_j a^\gamma_i) + 2a^\beta_i a^\delta_j g^\gamma_{\delta\beta}, \) \( A^a_\gamma(u) = a^\beta_i \left( \partial_i \lambda^b_\beta(u) + \lambda^b_\gamma'(u) \Gamma^b_{ab}(u) \right), \) and \( h, \Gamma, \lambda, \mu \) are smooth functions of \( u \). By definition \( [v_\beta, v_\gamma] = g^b_{\beta\gamma} v^b, \) \( \left\{ g^b_{\beta\gamma} \right\} \) are called the structure constants of the Lie algebra.
To apply Theorem B, let $\Omega = B_\rho$ and

$\kappa = 2, \quad k_2 = 2\alpha$,

$z = (z_1, z_2) = (A, u), \quad z_{2,1} = a_i^\beta, \quad z_2 = u^b$

$\xi = (\xi_1, \xi_2) = (\vartheta, \varphi), \quad \xi_{1,j} = \vartheta_j^\beta, \quad \xi_2 = \varphi^b$

$p = (p_1, p_2) = (\nabla A, \nabla u), \quad p_{1,i,j} = \partial_j a_i^\beta, \quad p_{2,j} = \partial_j u^b,$

then the equation \( (2.30) \) is of form \( (1.4) \) with coefficients

$\begin{align*}
q(x, z, p) &= (q_1(x, z, p), q_2(x, z, p)) \\
q_1(x, z, p) &= p_1 \\
q_2(x, z, p) &= \alpha(1 + h(z_2)(p_2 + z_1\lambda(z_2))^2)^{\alpha-1}h(z_2)(p_2 + z_1\lambda(z_2)) \\
w(x, z, p) &= (w_1(x, z, p), w_2(x, z, p)) \\
w_1(x, z, p) &= \alpha(1 + h(z_2)(p_2 + z_1\lambda(z_2))^2)^{\alpha-1}h(z_2)(p_2 + z_1\lambda(z_2)) + (p_1 + z_1^2)z_1 \\
w_2(x, z, p) &= \alpha(1 + h(z_2)(p_2 + z_1\lambda(z_2))^2)^{\alpha-1}h(z_2)(p_2 + z_1\lambda(z_2)) \cdot (p_2\Gamma(z_2) + z_1(\nabla\lambda(z_2) + \lambda(z_2)\Gamma(z_2)) + \mu(z_2)\cdot\nabla\mu(z_2),
\end{align*}$

where $h, \Gamma, \lambda$ and $\mu$ are smooth functions of $z_2$ and $h$ is positive definite, which is bounded from above and below. The verification of condition \( (1.5) \) is tedious but straightforward. We illustrate by the computation of $w_2$. A direct computation shows that for some $\Lambda(R)$ depending on $\alpha$, the geometry of $F$, $G$ and $\mu$,

$\begin{align*}
|\partial_{z_1} w_1| &\leq \Lambda(R) \left( V_1^{k_1} + V_2^{k_2} \right), \quad |\partial_{z_2} w_1| \leq \Lambda(R) V_2^{k_2-1}, \\
|\partial_{z_1} w_2| &\leq \Lambda(R) V_2^{k_2-1}, \quad |\partial_{z_2} w_2| \leq \Lambda(R) V_2^{k_2},
\end{align*}$

where $R$ is the upper bound for $x$ and $z$, i.e., $|x|^2 + |z|^2 \leq R^2$. It is clear that, for any vector $\pi = (\pi_1, \pi_2)$, we have

$\begin{align*}
\pi \cdot \nabla \left( \partial_{z_1} w_1, \partial_{z_2} w_2 \right) \cdot \pi^T &= \sum_{a,b} \pi_a \cdot \partial_{z_a} w_b \cdot \pi_b^T \\
&\leq \Lambda(R) \left( \left( V_1^{k_1} + V_2^{k_2} \right) |\pi_1|^2 + 2V_2^{k_2-1} |\pi_1||\pi_2| + V_2^{k_2} |\pi_2|^2 \right) \\
&\leq 2\Lambda(R) \left( \left( V_1^{k_1} + V_2^{k_2} \right) |\pi_1|^2 + V_2^{k_2} |\pi_2|^2 \right) \\
&\leq 2\Lambda(R) \left( \left( V_1^{k_1} + V_2^{k_2-1} \right) |\pi_1|^2 + V_2^{k_2} |\pi_2|^2 \right).
\end{align*}$

The verification of other conditions is more or less the same. Moreover, the additional regularity assumption in Theorem B can be shown as follows: for $A \in L_1^2(B_\rho, \Omega^1(\mathfrak{g}))$ and $u \in L_2^\infty(B_\rho, u^*[T])$ solve \( (2.25) \) weakly, then $A$ solves

$\Delta A - (dA, A) - \langle A, [A, A] \rangle + \alpha \left( 1 + |\nabla A u|^2 \right)^{\alpha-1} \langle \nabla A u, u \rangle = 0,$

weakly, and note that

$\begin{align*}
\langle dA, A \rangle &\in L^p, \quad 1 < p < 2, \quad \langle A, [A, A] \rangle \in L^q, \quad 1 < q < +\infty, \quad \nabla A u = \nabla u + A \cdot u \in L^{2\alpha}, \quad \left( 1 + |\nabla A u|^2 \right)^{\alpha-1} \in L^{\frac{2\alpha}{\alpha-1}}, \quad \alpha \left( 1 + |\nabla A u|^2 \right)^{\alpha-1} \langle \nabla A u, u \rangle \in L^{\frac{2\alpha}{\alpha-1}}.
\end{align*}$

Thus $\Delta A \in L^{\frac{2\alpha}{\alpha-1}}$ and $A \in \overset{\not\equiv}{L_2^{2\alpha-1}} \hookrightarrow L_1^{2\alpha-1} \hookrightarrow L_1^2$ when $\alpha - 1 > 0$ is small enough. Note that, by Sobolev embedding $(A, u) \in C^{\mu}(B_\rho)$ for some $\mu \in (0, 1)$. Finally, we apply Theorem B
to conclude that \((A, u) \in L^2(\mathcal{B}_p)\), this shows the \(L^2\)-interior regularity of the weak solution of \(\alpha\)-Yang–Mills–Higgs fields.

**Step 2.** Next, we show the \(L^2\)-boundary regularity. At a boundary point \(x_0 \in \partial \Sigma\), since \(\partial \Sigma\) is smooth, we can assume that the coordinate chart at \(x_0\) is the upper half disc \(D_\rho\) centered at origin (since we can “flatten out” a piece of the boundary by a bi-Lipschitz map). By Lemma 2.4, the equation of extended solution \((\hat{A}, \hat{u})\) is given by \((2.25)\). Comparing to \((2.26)\), we know that \((\hat{A}, \hat{u}) \in L^2(\mathcal{B}_\rho, \Omega^1(\mathfrak{g})) \times L^{2\alpha}(\mathcal{B}_\rho, F)\) satisfies a system of equations similar to \((2.30)\), with the coefficients \(h, \Gamma, \lambda\) and \(\mu\) extends properly as in Sect. 2.3. Although these extended coefficients are \(C^\infty\)-smooth on \(V_3 \setminus K\), where \(V_3 \subset F\) is a tubular neighborhood of \(K\), they are not \(C^\infty\)-smooth on \(V_3\) in general. However, under the assumption \(K \subset F\) is a totally geodesic sub-manifold, we have \(h, \lambda\) and \(\mu\) are in \(C^{1,\alpha'}(V_3)\) and \(\Gamma \in C^{\alpha'}(V_3)\) for some \(\alpha' \in (0, 1)\). In particular, the regularity requirement of Theorem B is satisfied. Moreover, \(h, \lambda, \mu, \Gamma\) as functions of \(\hat{u}\) may be multi-valued, but since \(u\) is continuous, if we take \(\rho\) small enough, then they are still single-valued as functions of \(\hat{u}\) when restricted to \(\hat{u}(\mathcal{B}_\rho)\). With these properties of the extended equations in mind, we can apply the Theorem B to show that \((\hat{A}, \hat{u}) \in L^2(\mathcal{B}_\rho)\) and \((A, u) \in L^2(D_\rho)\).

**Step 3.** As long as we show the \(L^2\)-regularity, \((2.13)\) holds strongly. If \(\alpha - 1\) is small, then the linear operator

\[
\Delta_{(A,u)}: L^p_{k+2}(U, F) \to L^p_{k}(U, F)
\]

\[
v \mapsto \Delta_{\Sigma} v - 2(\alpha - 1) \frac{\nabla_A v \cdot \nabla_A u}{1 + |\nabla_A u|^2}
\]

is invertible. Now the smoothness of weak solution can be proved by standard bootstrap argument with up to the boundary estimates. In fact, \(du \in L^2(\mathcal{B}_\rho, \Omega^k(\mathfrak{g}))\), \(A \in L^2(\mathcal{B}_\rho, \Omega^1(\mathfrak{g}))\), \(u \in C^0(U, F)\) and since \(\mu\) is smooth, \(\mu(u) \in L^2(\mathcal{B}_\rho, \Omega^1(\mathfrak{g}))\). By the Sobolev multiplications \(L^2 \otimes L^1 \to L^p\) for some \(p\) slightly smaller than 2, \(\Phi_{\alpha}(A, u) \in L^p(\mathcal{B}_\rho, F)\). The inevitability of \(\Delta_{(A,u)}\) shows that \(u \in L^2(\mathcal{B}_\rho, F)\). Also, since \(A \in L^2(U, \Omega^1(\mathfrak{g}))\), it is easy to show \(\Psi_{\alpha}(A, u) \in L^2(\mathcal{B}_\rho, F)\).

Thus, \((2.13)\) implies that \(A \in L^2(\mathcal{B}_\rho, \mathfrak{g})\). By Lemma A.1 if \(U \cap \partial \Sigma \neq \emptyset\), which in turn gives \(\Psi_{\alpha}(A, u) \in L^2(\mathcal{B}_\rho, \mathfrak{g})\) and \(A \in L^2(\mathcal{B}_\rho, \mathfrak{g})\).

**Step 4.** We should note that the above smoothness requires that \(A\) is under some \(L^2\)-Coulomb gauge. Let \(\{U_\beta\}\) be an open cover of \(\Sigma'\), where each \(U_\beta\) is an open ball \(\mathcal{B}_\rho\) such that the above interior smooth regularity holds under some \(L^2\)-Coulomb gauge. Since the \(\alpha\)-YMH functional is invariant under gauge transformation, we can patch these local gauges together to obtain a global gauge \(\hat{S} \in \mathcal{G}_2(\Sigma')\) in the same way as [31, Sect. 3], such that \((\hat{S}^*A_\alpha, \hat{S}^*\phi_\alpha)\) is smooth on \(\Sigma'\). In the same manner, we can patch the local gauges over an open cover of \(\Sigma\) to obtain a global gauge \(\hat{S} \in \mathcal{G}_2(\Sigma)\), such that \((\hat{S}^*A_\alpha, \hat{S}^*\phi_\alpha)\) is smooth on \(\Sigma\) up to the boundary. This finishes the proof of Theorem A.

\[\square\]

### 3. The main estimates

In this section we give some local uniform (independent of \(\alpha\)) estimates for critical points of \(\mathcal{L}_\alpha\), which serve as a preparation of blow-up analysis. We focus on the local boundary estimates, because the corresponding interior one follows as in [31, Sect. 4]. Suppose \(U\) is a domain in \(\Sigma\) and under a fixed trivialization we write \(\phi(x) = (x, u(x))\) and \(\nabla_A = d + A\) as before. Since \(u \in L^{2\alpha}(U, F) \subset C^0(U, F)\), for \(x_0 = 0 \in \partial \Sigma\), we can take Fermi coordinates \((f_1, \ldots, f_n)\) on an open neighborhood \(V\) of \(p = u(x_0) \in K\) such that \(V \cap K = \{f^{k+1} = \cdots = f^n = 0\}\) as in Sect. 2.3.

Take polar coordinates \((r, \theta)\) on \(U\), we always assume \(A\) is in Coulomb gauge with estimate \((c)\) in Theorem 2.3 holds. Under these assumptions, the Euler–Lagrange equation of \(\mathcal{L}_\alpha\) is given
by (see (2.13)),
\[
\begin{aligned}
\Delta_S u - 2(\alpha - 1) \frac{\langle \nabla^2_u u, \nabla A u \rangle}{1 + |\nabla A|^2} - \Phi_\alpha(A, u) &= 0, \quad x \in U \\
\Delta A - \Psi_\alpha(A, u) &= 0, \quad x \in U \\
A_2 &= 0, \quad x \in \partial U \\
\frac{\partial A_1}{\partial \nu} &= 0, \quad x \in \partial^0 U \\
\frac{\partial u^a}{\partial \nu} &= 0, \quad a = 1, 2, \ldots, k, \quad x \in \partial U \\
u^a &= 0, \quad a = k + 1, \ldots, n, \quad x \in \partial^0 U
\end{aligned}
\]
(3.1)

where \(\Phi_\alpha\) and \(\Psi_\alpha\) are defined by (2.14) and (2.15) respectively.

Similarly, the local equation for critical points of \(\mathcal{L}\) is given in the following lemma.

**Lemma 3.1.** Suppose \((A, \phi)\) is a critical point of \(\mathcal{L}\) on \(\mathcal{A} \times \mathcal{K}\), then locally, when we choose Coulomb gauge in \(U\), that is
\[
\begin{aligned}
d^* A &= 0, \quad x \in U \\
\nu \cdot A &= 0, \quad x \in \partial U,
\end{aligned}
\]
the Euler–Lagrange equation can be written as:
\[
\begin{aligned}
\Delta_S u - \Phi_1(A, u) &= 0, \quad x \in U \\
\Delta A - \Psi_1(A, u) &= 0, \quad x \in U \\
A_2 &= 0, \quad x \in \partial U \\
\frac{\partial A_1}{\partial \nu} &= 0, \quad x \in \partial^0 U \\
\frac{\partial u^a}{\partial \nu} \perp T_u \mathcal{K}, \quad a = 1, \ldots, k, \quad x \in \partial U \\
u^a &= 0, \quad a = k + 1, \ldots, n, \quad x \in \partial^0 U,
\end{aligned}
\]
(3.2)

where
\[
\Phi_1(A, u) = \Gamma(u)(du, du) + 2A \cdot du + A \cdot A \cdot u + \mu(u) \cdot \nabla u(\mu)
\]
and
\[
\Psi_1(A, u) = \langle dA, A \rangle + \langle A, [A, A] \rangle - \langle \nabla A u, u \rangle.
\]

3.1. \(\epsilon\)-regularity estimates. The main estimates in Sacks-Uhlenbeck’s method is the so-called \(\epsilon\)-regularity theorem. Here we prove an analogy for \(\phi \) with small energy \(\|\nabla A \phi\|_{L^2(U)}\).

**Lemma 3.2 (\(\epsilon\)-regularity).** There exist \(\epsilon_0 > 0\) and \(\alpha_0 > 1\) such that if \((A, u) \in \mathcal{A}(U) \times \mathcal{K}(U) := \mathcal{A}_1(U) \times \mathcal{K}(U)\) is a smooth pair satisfying (3.1) for \(1 \leq \alpha < \alpha_0\) with \(\|\nabla A u\|_{L^2(U, \Omega^1(F))} < \epsilon_0\) and \(\mathcal{L}_\alpha(A, u; U) \leq \Lambda < +\infty\), then for any \(U' \subset U\) and \(p > 1\), the following estimate holds uniformly in \(1 \leq \alpha < \alpha_0\),
\[
\|u - \bar{u}\|_{L^p(U', F)} \leq C \left(\|\nabla A u\|_{L^p(U, \Omega^1(F))} + \|F_A\|_{L^p(U, \Omega^2(\mathcal{A}))} + 1\right),
\]
where \(\bar{u}\) is the integral mean over \(U\) and \(C > 0\) is a constant depending on \(U, U', F, \Lambda, \|\mu\|_{L^p(F)}, p, \alpha_0, \epsilon_0\).

**Remark.** Note that (3.1) and (3.2) require that \(A\) is in Coulomb gauge. We remark that when the radius of \(U\) is small enough, this is always satisfied.

In fact, \(\mathcal{L}_\alpha(A, u; U) \leq \Lambda < +\infty\), in particular, \(\|F_A\|_{L^2(U)} \leq \Lambda\). Thus, there exists a small constant \(r_0\) (depending only on \(\Lambda\), the geometry of \(\Sigma\) and \(\delta_0\)), such that \(\|F_A\|_{L^1(U)} \leq \delta_0\), provided that the radius of \(U\) is smaller than \(r_0\), so we may assume \(A\) is in Coulomb gauge by Theorem 2.3.

**Proof.** Since the interior case can be proved by a minor modification of the following boundary case, we assume \(U\) is a upper half disc centered at \(x_0 = 0 \in \partial \Sigma\). As \(F\) is compact and embedded
into Euclidean space, we can assume \( \bar{u} = 0 \) without loss of generality. In particular, we have the following Poincaré inequality,

\[
\|u\|_{L^p(U)} \leq C(U, p)\|du\|_{L^p(U)}.
\]

Since \( \nabla_A u = du + Au \),

\[
\|u\|_{L^p(U)} \leq C(U, F, p) \left(\|\nabla_A u\|_{L^p(U)} + \|A\|_{L^p(U)}\right).
\]

(3.3)

Now, note that the boundary condition of \( \{ u, A \} \) is either of homogeneous Dirichlet or Neumann type

\[
\left\{ \begin{array}{l}
\partial_{\nu}(\eta u) = 0, & a = 1, 2, \ldots, k, \\
\eta u^b = 0, & b = k + 1, \ldots, n.
\end{array} \right..
\]

By the standard \( L^p \) estimate (see Lemma A.1), the Sobolev embedding \( L^p \hookrightarrow L^{2p} \) and (3.3),

\[
\|\eta u\|_{L^p(U)} \leq C(U, U', F, \|\mu\|_{L^\infty_1}, p) \left( (\alpha - 1)\|d^2(\eta u)\|_{L^p(U)} + \|d(\eta u)\|_{L^{2p}(U)} \right)
\]

\[
+ \|\nabla_A u\|_{L^p(U)} + \|A\|_{L^p(U)} \right) + \|dA\| + \|A\|_{L^\infty_1} + \|du\| + |u|.
\]

(3.4)

First, for \( 1 < p < 2 \), by the Sobolev embedding \( L^p \hookrightarrow L^p \), \( p^* = 2p/(2 - p) \) and Hölder’s inequality,

\[
\|d(\eta u)\|_{L^p(U)} \leq \|d(\eta u)\|_{L^{2p}(U)} \|\nabla_A u\|_{L^2(U)}
\]

\[
\leq C(U, p)\|d(\eta u)\|_{L^p(U)} \|\nabla_A u\|_{L^2(U)}
\]

\[
||A||\nabla_A u\|_{L^p(U)} \leq C(U, p)\|A\|_{L^p(U)} \|\nabla_A u\|_{L^2(U)}
\]

and

\[
\|A\|_{L^p(U)} \leq ||A||\|\nabla_A u\|_{L^p(U)} \leq C(F)\|A\|_{L^p(U)} \|\nabla_A u\|_{L^p(U)}.
\]

Since \( A \) is in Coulomb gauge in \( U \), by (c) of Theorem 2.3,

\[
\|A\|_{L^p(U)} \leq C\|F_A\|_{L^p(U)}, \quad |A|_{L^p(U)} \leq C\|F_A\|_{L^p(U)}.
\]

Plugging these estimates into (3.4), when \( \alpha_0 - 1 \) is small enough,

\[
\|\eta u\|_{L^p(U)} \leq C \left( \|\nabla_A u\|_{L^2(U)} \left(\|d(\eta u)\|_{L^p(U)} + \|A\|_{L^p(U)}\right) \right)
\]

\[
+ \|F_A\|_{L^p(U)} + \|\nabla_A u\|_{L^p(U)} + 1\right).
\]

where \( C > 0 \) is a constant depending on \( U, U', F, \Lambda, \|\mu\|_{L^\infty_1}, p \) and \( \alpha_0 \). Therefore, if we take \( \|\nabla_A u\|_{L^2(U)} \leq \epsilon_0 \) small enough (in particular, it depends on \( \alpha_0 \)), then we can employ the estimate of Coulomb gauge again to conclude

\[
\|u\|_{L^p(U)} \leq C(U, U', F, \Lambda, \|\mu\|_{L^\infty_1}, p, \alpha_0, \epsilon_0)\left(\|\nabla_A u\|_{L^p(U)} + \|F_A\|_{L^p(U)} + 1\right).
\]
The general case of $p$ follows from a bootstrap argument. We only illustrate the case for $p = 2$ in what follows. Firstly, apply the above estimate for $p = 4/3$, then the Sobolev embedding $L^{4/3} \hookrightarrow L^4$ implies $du \in L^4$. Therefore,

$$
\|du\|_{L^2(U)}^2 \leq \|du\|_{L^4(U)}^2 \leq C \left( \|\nabla_A u\|_{L^2(U)} + \|F_A\|_{L^2(U)} + 1 \right)^2.
$$

Since $L^2_1 \hookrightarrow L^4$ and $\|A\|_{L^2_1(U)} \leq C\|F_A\|_{L^2(U)}$ by (c) of Theorem 2.3,

$$
\|Adu\|_{L^2(U)} \leq \|A\|_{L^2_1(U)}\|du\|_{L^2_1(U)} \leq C \left( \|\nabla_A u\|_{L^2(U)} + \|F_A\|_{L^2(U)} + 1 \right),
$$

where $C$ depends on $U, U', F, A, \|\mu\|_{L^\infty}, \alpha_0$ and $\epsilon_0$. Now, the standard $L^2$ estimate gives similar to (3.4),

$$
\|\eta u\|_{L^2_3(U)} \leq C(U, U', F, A, \|\mu\|_{L^\infty}) \left( (\alpha - 1)\|d^2(\eta u)\|_{L^2(U)} + \|\eta u\|_{L^2(U)} + \|Adu\|_{L^2(U)} + \|A\|_{L^2_3(U)} + \|A^2 u\|_{L^2(U)} + \|\nabla_A u\|_{L^2(U)} + 1 \right),
$$

and we can proceed as before to show the required estimate holds for $p = 2$. \qed

Since the equation of the connection $A$ is subcritical in dimension 2, we can prove the following lemma.

**Lemma 3.3.** For any $1 < p < 2$, there exists $\alpha_0 = \alpha_0(p) > 1$ such that for any $1 < \alpha < \alpha_0$, if $(A, u) \in \mathcal{U} \times U' \times F$ is a smooth pair which satisfies (3.1) for $1 < \alpha < \alpha_0$ with $\mathcal{L}_\alpha(A, u; U) \leq \Lambda < +\infty$, then

$$
\|A\|_{L^p_2(U', \Omega^1(\mathfrak{g}))} \leq C \left( \|\nabla_A u\|_{L^2(U', \Omega^1(\mathfrak{f}))} + \|F_A\|_{L^2(U', \Omega^2(\mathfrak{g}))} \right),
$$

where $U' \subset U$ and $C > 0$ is a constant depending on $U, U', F, A, p$.

**Proof.** Note that the equation for $A$ in (3.1) is given by,

$$
\begin{align*}
\Delta A - \Psi_\alpha(A, u) & = 0, & x & \in U, \\
A_2 & = 0, & x & \in \partial U, \\
\frac{\partial A}{\partial n} & = 0, & x & \in \partial^U,
\end{align*}
$$

where

$$
\Psi_\alpha(A, u) = \langle dA, A \rangle + \langle [A, [A, A]] \rangle - \alpha(1 + |\nabla_A u|^2)^{\alpha-1} \langle \nabla_A u, u \rangle.
$$

By Hölder’s inequality and the Sobolev embedding, for any $1 < p < 2$, let $p^* = 2p/(2 - p)$, we have

$$
\|\langle dA, A \rangle\|_{L^p(U)} \leq C\|dA\|_{L^2(U)}\|A\|_{L^{p^*}(U)} \leq C\|A\|_{L^2_1(U)}^{2},
$$

and since $L^2_1 \hookrightarrow L^q$, for any $1 < q < +\infty$,

$$
\|\langle [A, [A, A]] \rangle\|_{L^p(U)} \leq C\|A\|_{L^{2q}(U)} \leq C\|A\|_{L^2_1(U)}^{3}.
$$

As we already assumed that $A$ is in Coulomb gauge, by Theorem 2.3,

$$
\|A\|_{L^p_2(U)} \leq C\|F_A\|_{L^2(U)}.
$$

It is easy to show, for $\alpha^*$ with $\frac{1}{\alpha^*} = \frac{1}{2} + \frac{\alpha - 1}{\alpha}$,

$$
\left( 1 + |\nabla_A u|^2 \right)^{\alpha-1} \langle \nabla_A u, u \rangle_{L^{\alpha^*}(U)} \leq \left( 1 + |\nabla_A u|^2 \right)^{\alpha-1/\alpha} \|\nabla_A u\|_{L^1(U)} \|\langle \nabla_A u, u \rangle\|_{L^2(U)} \leq C(F)\Lambda^{\alpha-1/\alpha} \|\nabla_A u\|_{L^2(U)}.
$$
Thus, for any $1 < p < 2$, we can take $\alpha(p) = 2p/(3p-2) \in (1, 2)$, such that for any $1 < \alpha \leq \alpha(p)$, we have $p \leq \alpha^*$ and

$$\left\| \left( 1 + |\nabla A u|^2 \right)^{\alpha-1} (\nabla A u, u) \right\|_{L^p(U)} \leq C(F, \Lambda, p) \|\nabla A u\|_{L^2(U)}.$$

The $L^p$-estimate (see Lemma A.1) implies that, for any $U' \subset U$,

$$\|A\|_{L^p(U')} \leq C(U, U', F, \Lambda, p) \left( \|\nabla A u\|_{L^2(U)} + \|F_A\|_{L^2(U)} \right).$$

$\square$

In application, we also need the scaled version of small energy estimate. For any $r$, $0 < r < r_0 < 1$ (such that $\tilde{A}$ is in Coulomb gauge over $U_r$), and any fixed point $x_0 \in U_r$, define the scaling map $\lambda_r: U \to U_r$, $x \mapsto x_0 + rx$. If $(\tilde{A}, \tilde{u}) \in \mathcal{A}(U_r) \times \mathcal{F}_K(U_r)$ is a smooth pair which satisfies (3.1) with $\mathcal{L}_\alpha(A, u; U_r) \leq \Lambda < +\infty$, then it is easy to show, the pullback connection $\hat{A} := \lambda_r^* A$ (which is in Coulomb gauge over $U$) and the pullback section $\hat{u} := \lambda_r^* u = u \circ \lambda_r$ are locally given by

$$\hat{A}(x) := \lambda_r^* A(x) = rA(x_0 + rx), \quad \hat{u}(x) := \lambda_r^* u = u(x_0 + rx)$$

respectively. Therefore, $(\hat{A}, \hat{u})$ satisfies

$$\begin{align*}
\Delta \hat{u} - 2(\alpha - 1) \frac{(\nabla^2 \hat{u}, \nabla \hat{A}) \nabla \hat{u}}{r^2 + |\nabla \hat{A}|^2} - \hat{\Phi}_\alpha(\hat{A}, \hat{u}) &= 0, \quad x \in U \\
\Delta \hat{A} - \hat{\Psi}_\alpha(\hat{A}, \hat{u}) &= 0, \quad x \in U \\
\hat{u}^a &= 0, \quad a = k + 1, \ldots, n, \quad x \in \partial^0 U \\
\frac{\partial \hat{u}^a}{\partial \nu} &= 0, \quad a = 1, 2, \ldots, k, \quad x \in \partial^0 U \\
\hat{A}_1 &= 0, \quad x \in \partial U \\
\frac{\partial \hat{A}_2}{\partial \nu} &= 0, \quad x \in \partial^0 U,
\end{align*}$$

where $\partial^0 U = \{(x-x_0)/r : x \in \partial \Sigma \cap U_r\}$, $0 < r < r_0 < 1$,

$$\hat{\Phi}_\alpha(\hat{A}, \hat{u}) = \Gamma(\hat{u}) (d\hat{u}, d\hat{u}) + 2 \hat{A} \cdot d\hat{u} + \hat{A} \cdot \hat{u} + r^2 \frac{\nabla \mu(\hat{u})}{\alpha(1 + r^{-2}|\nabla \hat{A}|^2)^{\alpha-1}}$$

and

$$\hat{\Psi}_\alpha(\hat{A}, \hat{u}) = \langle \hat{A}, d\hat{A} \rangle + \langle \hat{A}, [\hat{A}, \hat{A}] \rangle - \alpha r^2 \left( 1 + r^{-2}|\nabla \hat{A}|^2 \right)^{\alpha-1} \langle \nabla \hat{A}, \hat{u} \rangle.$$

**Corollary 3.4.** There exist $\epsilon_0 > 0$ and $\alpha_0 > 0$, such that for any smooth $(\hat{A}, \hat{u}) \in \mathcal{A}(U) \times \mathcal{F}_K(U)$ which solves (3.6) for $1 < \alpha < \alpha_0$, and for any $p > 1$, if $\hat{A}, \hat{u}$ satisfies

$$\|\nabla \hat{u}\|_{L^2(U)} \leq \epsilon_0,$$

then for any $k = 2, 3, \ldots$,

$$\|\hat{u} - \tilde{u}\|_{L^p_k(U_{1/2}, F)} \leq C(\text{diam}(U), F, \Lambda, \|\mu\|_{L^\infty(U)}, p, k, \alpha_0, \epsilon_0) \left( \|\nabla \hat{u}\|_{L^2(U, \Omega^1(F))} + \|F_A\|_{L^2(U, \Omega^2(\mathbb{S}^3))} + 1 \right),$$

and

$$\|\hat{A}\|_{L^p_k(U_{1/2}, F)} \leq C(\text{diam}(U), F, \Lambda, p, k) \left( \|\nabla \hat{u}\|_{L^2(U, \Omega^1(F))} + \|F_A\|_{L^2(U, \Omega^2(\mathbb{S}^3))} \right),$$

where $\Lambda$ is the bound of $\mathcal{L}_\alpha(A, u; U_r)$.

**Proof.** Recall that harmonic maps are scaling invariant in dimension 2, although our coupled equation is not scaling invariant anymore, it behaves well under scaling. We only prove the case $1 < p < 2$ and $k = 2$, the general case follows from bootstrap argument as illustrated at the end of Sect. 2.3.
For the estimate of \( \hat{u} \), the proof is almost the same as Lemma 3.2. Note that

\[
\begin{align*}
\left| \frac{\langle \nabla^2 \hat{u}, \nabla \hat{u} \rangle}{r^2 + |\nabla \hat{u}|^2} \right| & \leq C|\nabla^2 \hat{u}|, \\
\left| \frac{r^2 \nabla \mu(\hat{u}) (\mu(\hat{u}))}{(1 + r^{-2}|\nabla \hat{u}|^2)^{\alpha - 1}} \right| & \leq C(\|\mu\|_{L_T^\infty}).
\end{align*}
\]

Multiplying the equation of \( \hat{u} \) by the cutoff function defined in Lemma 3.2 (note that \( |\nabla \eta| \leq C(\text{diam}(U)) \)), it is easy to show

\[
\Delta \Sigma (\eta \hat{u}) \leq C (\eta|\Delta \Sigma \hat{u}| + |d \hat{u}| + |\hat{u}|)
\]

\[
\leq C \left( (\alpha - 1)|\nabla \hat{u}| + \eta|\hat{\Phi}_a(\hat{A}, \hat{u})||d\hat{u}| + |u| \right)
\]

\[
\leq C \left( (\alpha - 1)|d^2(\eta \hat{u})| + |d(\eta \hat{u})||d\hat{u}| + |d \hat{A} | + |\hat{A} \hat{d} \hat{u}| + |d \hat{u}| + |\hat{u}| + 1 \right)
\]

\[
\leq C \left( (\alpha - 1)|d^2(\eta \hat{u})| + |d(\eta \hat{u})||\nabla \hat{A} \hat{u} | + |\nabla \hat{A} \hat{u} |\hat{A} + |\hat{A}^2| + |d \hat{A} | + |d \hat{u}| + |\hat{u}| + 1 \right),
\]

where \( C \) is a constant depending on \( \text{diam}(U) \), \( \Lambda, F, \|\mu\|_{L^I_\infty} \). Thus, we can control \( \Delta \Sigma (\eta \hat{u}) \) as in Lemma 3.2 and show the required estimate.

Next, we prove the required estimate for \( \hat{A} \). The proof is almost the same as Lemma 3.3, by noting that

\[
r^2 \left( 1 + r^{-2}|\nabla \hat{A} \hat{u}|^2 \right)^{\alpha - 1} \langle \nabla \hat{A} \hat{u}, \hat{u} \rangle = r^3 \left( 1 + |\nabla \hat{A} \hat{u}|^2 \right)^{\alpha - 1} \langle \nabla \hat{A} \hat{u}, \hat{u} \rangle,
\]

\[
\|r^2 \left( 1 + r^{-2}|\nabla \hat{A} \hat{u}|^2 \right)^{\alpha - 1} \langle \nabla \hat{A} \hat{u}, \hat{u} \rangle \|_{L^p(U)} \leq C r \left( 1 + |\nabla \hat{A} \hat{u}|^2 \right) \| \nabla \hat{A} \hat{u} \|_{L^2(U)} \\
\leq C \cdot \Lambda^{\frac{\alpha - 1}{2}} \cdot \| \nabla \hat{A} \hat{u} \|_{L^2(U)}.
\]

3.2. Removal of singularity for approximated harmonic maps. The following lemma is an extension of the classical removable singularity theorem for harmonic maps (see [11, Thm. 1.10; 28, Thm. 3.6]), which will be applied to the weak limit in the blow-up process to show that the isolated singularities are all removable. The proof given here is based on the regularity theorem of weak solution instead, comparing to the classical method involving energy decay estimates [11, 28]. Here we only state the boundary version, the interior case can be found in [31, Thm. 4.3].

Lemma 3.5 ([14, Thm. 3.6]). Suppose \( u \) is a \( L^2_{2, \text{loc}} \)-map from a neighborhood \( U^0 := U \setminus \{0\} \) of \( 0 \in \partial U \) to \( F \) with finite Dirichlet energy and satisfies the following equation weakly under Fermi coordinates (see also (3.2))

\[
\begin{align*}
\Delta \Sigma u - \Gamma(u)(du, du) = f & \in L^p(U^0), \quad x \in U^0 \\
\frac{\partial u^a}{\partial \nu} & = 0, \quad a = 1, 2, \ldots, k, \quad x \in \partial \Sigma \cap U^0 \\
u^a & = 0, \quad a = k + 1, \ldots, n, \quad x \in \partial \Sigma \cap U^0,
\end{align*}
\]

for some \( p \geq 2 \). Then \( u \) can be extended to a \( L^2_{2, \text{loc}} \)-map over \( U \) and it preserves the free boundary condition.

4. Convergence and blow-up

The following bubbling convergence argument is almost standard, the main difference is the possible phenomenon of boundary blow-ups.
Proof of Theorem C. Let $\alpha_0$ be the same constant in Lemma 3.2. Suppose $\{x_1,\ldots,x_L\} \subset S$. By the definition of $S$, for $r > 0$ small enough such that $\{U_r(x_j)\}_{j=1}^L$ are mutually disjoint and for all but finite many $\{\alpha\}$, we have

$$\int_{U_r(x_j)} |\nabla A_{\alpha} \phi_{\alpha}|^2 \geq \epsilon_0/2.$$ 

Summing over $j = 1,\ldots,L$, we see that

$$+\infty > \Lambda \geq \mathcal{L}_\alpha(A_{\alpha}, \phi_{\alpha}) \geq \sum_{j=1}^L \int_{U_r(x_j)} |\nabla A_{\alpha} \phi_{\alpha}|^2 \geq L\epsilon_0/2,$$

which clearly implies the finiteness of $S$.

To show the strong convergence over regular points in $\Sigma \setminus S$, we note first that, by the remark after Lemma 3.2, there exists $r_0 > 0$ independent of $\alpha$, such that $A_{\alpha}$ is in Coulomb gauge over $U_r$, provided that $r \leq r_0$. Then Lemma 3.3 implies, for $1 < p < 2$, there exists $\alpha(p) > 1$, such that for any $1 < \alpha < \alpha(p)$, $\|A_{\alpha}\|_{L^p_r(U_r, \Omega^1(\mathfrak{g}))}$ are uniformly bounded. Next, we show the $C^0$ convergence of $A_{\alpha} \rightarrow A_\infty$. For that purpose, covering $\Sigma$ with discs or half-discs with radius less than $r_0/2$, denote them by $\{U_i\}$. The above discussion shows that under some local trivialization $\{\sigma_{\alpha,i}: \mathcal{P}(\alpha) \rightarrow U_i \times G\}$, if we write $A_{\alpha}$ locally as $d + A_{\alpha,i}$, then for any $U_i$, any $1 < p < 2$ and any $1 < \alpha < \alpha(p)$,

$$\|A_{\alpha,i}\|_{L^p_\alpha(U_i)} \leq C(F, \Lambda, p).$$

(4.1)

Thus, we can assume that $A_{\alpha,i} \rightarrow A_i$ weakly in $L^p_\alpha(U_i)$ and strongly in $C^0(U_i)$ as $\alpha \rightarrow 1$.

Claim. $\{A_i\}$ represents a $L^p_\alpha$ connection $A_\infty$ on $\mathcal{P}$, i.e., $A_i \in L^p_\alpha(U_i, \Omega^1(\mathfrak{g}))$ and there exist transition functions $\{\tau_{ij} \in L^p(U_{ij}, \Omega^1(\mathfrak{g}))\}$, $U_{ij} = U_i \cap U_j$, such that

$$A_j = \tau_{ij}^* A_i = \tau_{ij}^{-1} d\tau_{ij} + \tau_{ij}^{-1} A_i \tau_{ij}.$$ 

In fact, on any $U_{ij} \neq \emptyset$, we have transition functions $\{\tau_{ij}: U_{ij} \rightarrow G\}$ such that $\pi_2 \circ \sigma_{\alpha,i} = \tau_{\alpha,i,j} \circ \pi_2 \circ \sigma_{\alpha,j}$, where $\pi_2$ is the projection to the second component. $\{A_{\alpha,i}\}$ transform as

$$A_{\alpha,j} = \tau_{\alpha,j}^{-1} d\tau_{\alpha,i,j} + \tau_{\alpha,i,j}^{-1} A_{\alpha,i} \tau_{\alpha,i,j} \iff d\tau_{\alpha,i,j} = \tau_{\alpha,i,j} A_{\alpha,j} - A_{\alpha,i} \tau_{\alpha,i,j}.$$ 

Since $G$ is compact, (4.1) and the above relation imply that, for $p^* = 2p/(2-p)$, 

$$\|d\tau_{\alpha,i,j}\|_{L^{p^*}(U_{ij})} \leq C(G) \left( \|A_{\alpha,i}\|_{L^{p^*}(U_i)} + \|A_{\alpha,j}\|_{L^{p^*}(U_j)} \right) \leq C(G) \left( \|A_{\alpha,i}\|_{L^p_\alpha(U_i)} + \|A_{\alpha,j}\|_{L^p_\alpha(U_j)} \right) \leq C(G, F, \Lambda, p).$$

Since $\tau_{\alpha,i,j} \in L^\infty(U_{ij})$ (because $G$ is compact), by employing the Sobolev multiplication theorems $L^p_\alpha \times L^{p^*}_\alpha \rightarrow L^p_\alpha$ and $L^p_\alpha \times L^p_\alpha \rightarrow L^q_\alpha$ (see [40, Lem. B.3]), we obtain the $L^p_\alpha(U_{ij})$-uniform boundedness of $\{\tau_{\alpha,i,j}\}$. By weak compactness, we may assume that $\tau_{\alpha,i,j}$ converges to some $\tau_{ij}$ weakly in $L^p_\alpha(U_{ij})$ and strongly in $C^0(U_{ij})$ as $\alpha \rightarrow 1$. It is clear that the co-cycle condition $\tau_{ik} = \tau_{ij} \circ \tau_{jk}$ is preserved and hence $\{\tau_{ij}\}$ defines a bundle isomorphic to $\mathcal{P}$. Moreover, the relation is preserved under weak limits,

$$d\tau_{ij} = \tau_{ij} A_j - A_i \tau_{ij} \iff A_j = \tau_{ij}^* A_i.$$ 

Thus, $\{A_i\}$ represents a connection $A_\infty \in \mathfrak{g}^*_\alpha$ on $\mathcal{P}$. This finishes the proof of the claim.

We should remark that the local convergence $A_{\alpha,i} \rightarrow A_i$ in $C^0(U_i)$ depends on the choice of trivialization $\sigma_{\alpha,i}$ and we cannot assert $A_{\alpha} \rightarrow A_\infty$ in $C^0$ directly. But we can apply the patching argument similar to the weak compactness of Yang–Mills connections (see [37, Thm. 3.6]) to show that there exist gauge transformations $\{S_{\alpha}\} \subset \mathcal{P}_\alpha$, such that $S_{\alpha}^* A_{\alpha} \rightarrow A_\infty$ strongly in $C^0$ sense. That is, $A_{\alpha} \rightarrow A_\infty$ in $C^0(\Sigma)$ modulo gauge. Since $\mathcal{L}(A_{\alpha}, \phi_{\alpha})$ is gauge invariant, we will identify $S_{\alpha}^* A_{\alpha}$ and $S_{\alpha}^* \phi_{\alpha}$ with $A_{\alpha}$ and $\phi_{\alpha}$ hereafter.
To show the strong convergence of sections \( \{ \phi_\alpha \} \) over \( \Sigma \setminus S \), we note first that, the above argument can be started with any cover with radii are less than \( r_0/2 \). Now, by the definition of regular set, for any \( x \in \Sigma \setminus S \), there exist \( r^0 \in (0, r_0) \) and \( \alpha^0 \in (0, \alpha_0) \), such that for any \( U_r(x) \subset S \) centered at \( x \) with radius \( r \leq r^0 \), we have

\[
\int_{U_r(x)} |\nabla_A \phi_\alpha|^2 < \epsilon_0, \quad \forall 1 < \alpha \leq \alpha^0.
\]

Since by the choice of \( U_r(x) \), \( r < r^0 < r_0 \), we can assume that \( A_\alpha \) is in Coulomb gauge with estimate (4.1) over \( U \subset U_r(x) \). If we denote the corresponding local trivialization by \( \sigma_\alpha \) and write

\[
\phi_\alpha(x) = \sigma_\alpha \circ \phi_\alpha(x) = (x, u_\alpha(x)),
\]

then Lemma 3.2 implies that, for any \( 1 < p \leq 2 \),

\[
\|u_\alpha\|_{L^p_0(U')} \leq C(U, U', F, A, \|\mu\|_{L^\infty_0(F)}, p, \alpha_0, \epsilon_0), \quad U' \subset U.
\]

Since \( (A_\alpha, u_\alpha) \) satisfies (3.1), we can bootstrap the regularity as in the proof of smoothness of critical points of \( \alpha \)-YMH functional (see the end of Sect. 2.3) and conclude that \( \{ A_\alpha \} \) converges to \( A_\infty \) in \( C^\infty(U') \) and \( \{ u_\alpha \} \) converges to some \( u \in C^\infty(U') \) as \( \alpha \to 1 \). By the arbitrariness of \( x \in \Sigma \setminus S \), we can construct a cover \( \{ U_i' \} \) of \( \Sigma \setminus S \) and local trivializations \( \sigma_{\alpha,i} \), such that for any \( i \), \( \alpha \to A_{\alpha,i} \) in \( C^\infty(U_i') \) and \( u_{\alpha,i} \to u_i \) in \( C^\infty(U_i') \) as \( \alpha \to 1 \). Since the consistence condition \( u_{ij} = \tau_{ij} u_i \) is preserved on each \( U_i' \cap U_j' \), \( \{ u_i \} \) represents a section \( \phi_\infty \in \mathcal{F}_K \) over \( \Sigma \setminus S \). A patching argument as before shows that, there exists some \( S_\alpha \in C^{\infty}_c(\Sigma \setminus S) \), such that \( S_\alpha^* \phi_\alpha \to \phi_\infty \) and \( S_\alpha^* A_\alpha \to A_\infty \) in \( C^{\infty}_c(\Sigma \setminus S) \). Clearly, by taking \( \alpha \to 1 \) in (3.1), \( \phi_\infty \) satisfies the first equation of (3.2) locally and the corresponding boundary condition over some neighborhood \( U \setminus \{ x \} \), \( x \in S \). The removal of regularity theorem (see Lemma 3.5) asserts that \( \phi_\infty \) extends to a smooth section over \( \Sigma \) and we finish the first part of the theorem.

To show the second part, without loss of generality, suppose that \( x_0 = 0 \in S \cap \partial \Sigma \) and \( U = U_1 \) (unit half disc) is a neighborhood of origin such that it is the unique isolated singularity in \( U \). Let \( \sigma_\alpha \) be a local trivialization over \( U \) and \( \{ u_\alpha \} \) be the local representation of \( \{ \phi_\alpha \} \) as before. Without loss of generality, we may assume that the radius of \( U \) is less than \( r_0 \), such that \( A_\alpha \) is in Coulomb gauge. Set

\[
1/r_\alpha = \max_U |\nabla A_\alpha u_\alpha| = |\nabla A_\alpha u_\alpha|(x_\alpha),
\]

and let \( \lambda_{r_\alpha}: x \mapsto x_\alpha + r_\alpha x \) be the scaling mapping. We already showed that the pullback connection and pullback section are locally given by (see (3.5))

\[
\hat{A}_\alpha(x) = \lambda_{r_\alpha}^* A_\alpha(x) = r_\alpha A_\alpha(x_\alpha + r_\alpha x),
\]

\[
\hat{u}_\alpha(x) = \lambda_{r_\alpha}^* u_\alpha(x) = u_\alpha \circ \lambda_{r_\alpha}(x) = u_\alpha(x_\alpha + r_\alpha x).
\]

The following blow-up argument is standard, and we summarize it in the following claim as a complement.

Claim. With the above notations and assumptions, we have

(a) \( r_\alpha \to 0 \) as \( \alpha \to 1 \);

(b) \( x_\alpha \to 0 \) as \( \alpha \to 1 \);

(c) Define \( (\hat{A}_\alpha, \hat{u}_\alpha) \) as above, then there are two cases, where harmonic spheres and harmonic disc split off respectively.

- Harmonic spheres: \( \text{dist}(x_\alpha, U \cap \partial \Sigma)/r_\alpha \to \infty \);

- Harmonic discs: \( \text{dist}(x_\alpha, U \cap \partial \Sigma)/r_\alpha \to \rho < +\infty \).

If (a) is not true, then \( \|\nabla A_\alpha u_\alpha\|_{L^\infty(U)} \) are uniformly bounded. This contradicts the fact that \( x_0 = 0 \) is a singularity of \( \{ u_\alpha \} \) in \( U \).
For (b), suppose that \( x_\alpha \to x^0 \neq 0 \) as \( \alpha \to 1 \), then since \( x^0 \) is a regular point, there exist \( \delta > 0 \) and \( \alpha^0 \in (1, \alpha_0) \), such that we can apply Lemma 3.2 and Lemma 3.3 to show that,

\[
\frac{1}{r_\alpha} = |\nabla A_\alpha u_\alpha(x_\alpha) - \nabla A_\alpha u_\alpha(x)| \leq \|\nabla A_\alpha u_\alpha\|_{L^\infty(U_\delta(x^0))} \leq C < +\infty,
\]

take \( \alpha \to 1 \) we see that it contradicts (a).

For (c), we only show the case of splitting-off of harmonic discs with free boundary, because the harmonic sphere case can be derived in a very similar way. Firstly, we can take a proper coordinate system with origin at \( x_0 = 0 \) and \( x_1 \)-axis pointing to the interior of \( \Omega \), \( x_2 \)-axis tangent to \( \partial \Omega \) at 0. The scaled maps \((\hat{A}_\alpha, \hat{u}_\alpha)\) satisfy (3.6) with \( r \) replaced by \( r_\alpha \). Since \( \nabla \hat{A}_\alpha \hat{u}_\alpha(x) = r_\alpha \nabla A_\alpha u_\alpha(x + r_\alpha x) \),

\[
\|\nabla \hat{A}_\alpha \hat{u}_\alpha\|_{L^\infty(U_1/r_\alpha)} = r_\alpha \|\nabla A_\alpha u_\alpha\|_{L^\infty(U)} = 1,
\]

by the choice of \( r_\alpha \), we can apply Corollary 3.4 on each \( \hat{U} \subset U_1/(2r_\alpha) \) to \((\hat{A}, \hat{u}_\alpha)\) and show that, for \( k = 1, 2, \ldots \),

\[
\|\hat{u}_\alpha\|_{C^k(U_1/(2r_\alpha))} \leq C(\text{diam}(U), \hat{F}, \Lambda, \|\mu\|_{L^\infty(F)}, \alpha_0, \epsilon_0, k),
\]

\[
\|\hat{A}_\alpha\|_{C^k(U_1/(2r_\alpha))} \leq C(\text{diam}(U), \hat{F}, \Lambda, k).
\]

Moreover, since \( \hat{A}_\alpha \) is in Coulomb gauge, by Theorem 2.3,

\[
\|\hat{A}_\alpha\|_{L^2(U_1/(2r_\alpha))} \leq C\|F_{\hat{A}_\alpha}\|_{L^2(U_1/(2r_\alpha))} = Cr_\alpha \|F_{A_\alpha}\|_{L^2(U_1/2)} \to 0.
\]

Therefore, we obtain the following strong convergence in \( C^\infty_{\text{loc}}(\mathbb{R}^2_\rho,+) \), where the right half plane \( \mathbb{R}^2_\rho,+ = \{ x = (x_1, x_2) : x_1 > -\rho \} \),

\[
\hat{A}_\alpha \to 0, \quad \hat{u}_\alpha \to w.
\]

Clearly, the equation of \( w \) is given by

\[
\begin{cases}
\Delta \Sigma w - \Gamma(w)(dw, dw) = 0, & x \in \mathbb{R}^2_\rho,+
\
\frac{\partial u_\alpha}{\partial x_1} = 0, & a = 1, 2, \ldots, k, \quad x_1 = -\rho
\
w^a = 0, & a = k + 1, \ldots, n, \quad x_1 = -\rho.
\end{cases}
\]

By the removal of singularity theorem for harmonic maps (see Lemma 3.5) and the conformal invariance of \( w, w \) extends to a harmonic map on the disc \( B \) with free boundary \( w(\partial B) \subset K \). Therefore, at each singularity \( x_0 \in S \cap \partial \Sigma \), we obtain a harmonic disc or a harmonic sphere, which is called a bubble. This finishes the proof of Theorem C. \( \square \)

APPENDIX A. SOME REGULARITY RESULTS AND ESTIMATES

It is well-known that for a weakly harmonic map \( u \), the equation of \( u \) has anti-symmetric structure \( \Omega \) with \( \|\Omega\|_{L^2} \leq C \|\nabla u\|_{L^2} \) and the following regularity and estimate hold.

**Lemma A.1** (see [30, Thm. 1.2]). Suppose \( u \in L^p_1(D_1, \mathbb{R}^n) \) is a weak solution of

\[
\begin{cases}
\Delta u + \Omega \cdot \nabla u = f \in L^p(D_1, \mathbb{R}^n), & x \in D_1
\
\frac{\partial u_\alpha}{\partial \nu} = g^a \in L^1_{2,\partial}(\partial^0 D_1, \mathbb{R}^n), & x \in \partial^0 D_1, \quad 1 \leq a \leq k
\
u^a = h^a \in L^1_{2,\partial}(\partial^0 D_1, \mathbb{R}^n), & x \in \partial^0 D_1, \quad k + 1 \leq a \leq n,
\end{cases}
\]

where \( \Omega \in L^2(D_1, \mathfrak{so}(n) \times \Lambda^1 \mathbb{R}^2) \), \( 1 < p < 2 \) and boundary Sobolev space is defined as

\[
L^p_{k,\partial}(\partial^0 D_1) = \left\{ f \in L^1(\partial^0 D_1) : f = \hat{f}|_{\partial^0 D_1}, \hat{f} \in L^p_k(D_1) \right\}
\]

with norm

\[
\|f\|_{L^p_{k,\partial}(\partial^0 D_1)} = \inf_{f \in L^p_k(D_1); f|_{\partial^0 D_1} = \hat{f}} \|\hat{f}\|_{L^p_k(D_1)}.
\]
Then, \( u \in L^p_\beta(D_{1/2}, \mathbb{R}^n) \) and
\[
\|u\|_{L^p_{\beta}(D_{1/2}, \mathbb{R}^n)} \leq C \left( \|f\|_{L^p(D_1, \mathbb{R}^n)} + \|g\|_{L^p_{\beta}(\partial D_1, \mathbb{R}^n)} + \|h\|_{L^p_{\beta}(\partial D_1, \mathbb{R}^n)} + \|u\|_{L^1(D_1, \mathbb{R}^n)} \right),
\]
provided that \( \|\Omega\|_{L^2(D_1)} \leq \eta_0 = \eta_0(p, n) \).

**References**

[1] L. Alvarez-Gaumé and D. Z. Freedman, Geometrical structure and ultraviolet finiteness in the supersymmetric \( \sigma \)-model, Comm. Math. Phys. 80 (1981), no. 3, 433–451. MR062710

[2] J. Bagger and E. Witten, The gauge invariant supersymmetric nonlinear sigma model, Phys. Lett. B 118 (1982), no. 1-3, 103–106. MR084194

[3] D. Benfield, Stable pairs and principal bundles, Q. J. Math. 51 (2000), no. 4, 417–436. MR1806450

[4] F. Bethuel, H. Brezis, and F. Hélein, Ginzburg-Landau vortices, Progress in Nonlinear Differential Equations and their Applications, vol. 13, Birkhäuser Boston, Inc., Boston, MA, 1994. MR1269538

[5] S. B. Bradlow, Special metrics and stability for holomorphic bundles with global sections, Internat. Math. Res. Notices 16 (2000), 831–882. MR1777852

[6] S. J. Chapman, S. D. Howison, and J. R. Ockendon, Macroscopic models for superconductivity, SIAM Rev. 34 (1992), no. 4, 529–560. MR1193011

[7] K. Cieliebak, A. R. Gaio, and D. A. Salamon, J-holomorphic curves, moment maps, and invariants of Hamiltonian group actions, Internat. Math. Res. Notices 16 (2000), 831–882. MR1777853

[8] P.-G. de Gennes, Superconductivity of metals and alloys, Advanced book classics, Advanced Book Program, Perseus Books, 1999.

[9] S. K. Donaldson, Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles, Proc. London Math. Soc. (3) 50 (1985), no. 1, 1–26. MR765366

[10] Q. Du, M. D. Gunzburger, and J. S. Peterson, Analysis and approximation of the Ginzburg-Landau model of superconductivity, SIAM Rev. 34 (1992), no. 1, 54–81. MR1156289

[11] A. M. Fraser, On the free boundary variational problem for minimal disks, Comm. Pure Appl. Math. 53 (2000), no. 8, 931–971. MR1755947

[12] R. Gulliver and J. Jost, Harmonic maps which solve a free-boundary problem, J. Reine Angew. Math. 381 (1987), 61–89. MR918841

[13] A. Jaffe and C. Taubes, Vortices and monopoles, Progress in Nonlinear Differential Equations and their Applications, vol. 13, Birkhäuser Boston, Inc., Boston, MA, 1994. MR1269538

[14] J. Jost, L. Liu, and M. Zhu, The qualitative behavior at the free boundary for approximate harmonic maps, Comm. Pure Appl. Math. 53 (2000), no. 11, 1631–1665. MR1805130

[15] O. A. Ladyzhenskaya and N. N. Ural’ceva, On the smoothness of weak solutions of quasilinear elliptic equations in several variables and of variational problems, Comm. Pure Appl. Math. 14 (1961), 481–495. MR0149076

[16] O. A. Ladyzhenskaya and N. N. Ural’ceva, Linear and quasilinear elliptic equations, Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis, Academic Press, New York-London, 1968. MR0244627

[17] E. H. Lieb and M. Loss, Analysis, Second, Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 2001. MR1817225

[18] A. Lin and L. Shen, Gradient flow of the norm squared of a moment map over Kahler manifolds, ArXiv e-prints (Feb. 2018), available at arXiv:1802.09314 [math.DG].

[19] F. Lin and Y. Yang, Gauged harmonic maps, Born-Infeld electromagnetism, and magnetic vortices, Comm. Pure Appl. Math. 56 (2003), no. 11, 1631–1665. MR1995872

[20] K. Lu and X.-B. Pan, Ginzburg-Landau equation with DeGennes boundary condition, J. Differential Equations 129 (1996), no. 1, 136–165. MR1400799

[21] L. Ma, Harmonic map heat flow with free boundary, Comment. Math. Helv. 66 (1991), no. 2, 279–301. MR1107842

[22] A. Marini, Dirichlet and Neumann boundary value problems for Yang-Mills connections, Comm. Pure Appl. Math. 45 (1992), no. 8, 1015–1050. MR1168118

[23] J. D. Moore and R. Schafly, On equivariant isometric embeddings, Math. Z. 173 (1980), no. 2, 119–133. MR583381

[24] C. B. Morrey Jr., Multiple integrals in the calculus of variations, Classics in Mathematics, Springer-Verlag, Berlin, 2008. Reprint of the 1966 edition [ MR0202511]. MR2492985

[25] I. Mundet i Riera, A Hitchin-Kobayashi correspondence for Kähler fibrations, J. Reine Angew. Math. 528 (2000), 41–80. MR1801657

[26] I. Mundet i Riera, Hamiltonian Gromov-Witten invariants, Topology 42 (2003), no. 3, 525–553. MR1953239
[27] Á. Nagy, *Irreducible Ginzburg-Landau fields in dimension 2*, J. Geom. Anal. 28 (2018), no. 2, 1853–1868. MR3790522

[28] J. Sacks and K. Uhlenbeck, *The existence of minimal immersions of 2-spheres*, Ann. of Math. (2) 113 (1981), no. 1, 1–24. MR604040

[29] C. Scheven, *Partial regularity for stationary harmonic maps at a free boundary*, Math. Z. 253 (2006), no. 1, 135–157. MR2206640

[30] B. Sharp and M. Zhu, *Regularity at the free boundary for Dirac-harmonic maps from surfaces*, Calc. Var. Partial Differential Equations 55 (2016), no. 2, Paper No. 27, 30. MR3465443

[31] C. Song, *Critical points of Yang-Mills-Higgs functional*, Commun. Contemp. Math. 13 (2011), no. 3, 463–486. MR2813498

[32] C. Song and C. Wang, *Heat flow of Yang-Mills-Higgs functionals in dimension two*, J. Funct. Anal. 272 (2017), no. 11, 4709–4751. MR3630638

[33] C. H. Taubes, *The existence of a nonminimal solution to the SU(2) Yang-Mills-Higgs equations on $\mathbb{R}^3$. I*, Comm. Math. Phys. 86 (1982), no. 2, 257–298. MR676188

[34] C. H. Taubes, *The existence of a nonminimal solution to the SU(2) Yang-Mills-Higgs equations on $\mathbb{R}^3$. II*, Comm. Math. Phys. 86 (1982), no. 3, 299–320. MR677000

[35] K. Uhlenbeck and S.-T. Yau, *On the existence of Hermitian-Yang-Mills connections in stable vector bundles*, Comm. Pure Appl. Math. 39 (1986), no. S, suppl., S257–S293. Frontiers of the mathematical sciences: 1985 (New York, 1985). MR861491

[36] K. K. Uhlenbeck, *Connections with $L^p$ bounds on curvature*, Comm. Math. Phys. 83 (1982), no. 1, 31–42. MR648356

[37] H. Urakawa, *Calculus of variations and harmonic maps*, Translations of Mathematical Monographs, vol. 132, American Mathematical Society, Providence, RI, 1993. Translated from the 1990 Japanese original by the author. MR1252178

[38] S. Venugopalan, *Yang-Mills heat flow on gauged holomorphic maps*, J. Symplectic Geom. 14 (2016), no. 3, 903–981. MR3548487

[39] K. Wehrheim, *Uhlenbeck compactness*, EMS Series of Lectures in Mathematics, European Mathematical Society (EMS), Zürich, 2004. MR2030823

[40] E. Witten, *Phases of $N = 2$ theories in two dimensions*, Nuclear Phys. B 403 (1993), no. 1-2, 159–222. MR1232617

[41] G. Xu, *The moduli space of twisted holomorphic maps with Lagrangian boundary condition: compactness*, Adv. Math. 242 (2013), 1–49. MR3055986

[42] Y. Yu, *The gradient flow for gauged harmonic map in dimension two II*, Calc. Var. Partial Differential Equations 50 (2014), no. 3-4, 883–924. MR3216838

[43] M. Zhu, *Harmonic maps from degenerating Riemann surfaces*, Math. Z. 264 (2010), no. 1, 63–85. MR2564932

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