Supersymmetric KdV equation: Darboux transformation and discrete systems

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Abstract
For the supersymmetric KdV equation, a proper Darboux transformation is presented. This Darboux transformation leads to the Bäcklund transformation found earlier by Liu and Xie (2004 Phys. Lett. A 325 139–43). The Darboux transformation and the related Bäcklund transformation are used to construct integrable super differential–difference and difference–difference systems. The continuum limits of these discrete systems and of their Lax pairs are also considered.

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1. Introduction
In a seminal paper [2], Manin and Radul proposed a supersymmetric KP hierarchy. This integrable hierarchy was later studied intensively (see [3, 4] and references therein). As in the classical case, the reductions to a finite number of fields of the supersymmetric KP hierarchy have been studied. The simplest and most important reduction is the supersymmetric KdV equation, which reads

\[ \alpha_t = \alpha_{xxx} + 3(\alpha D\alpha)_x, \]  

where \( \alpha = \alpha(t,x,\theta) \) is a fermionic super field depending on a temporal variable \( t \) and the spatial super variables \( (x, \theta) \), and \( D = \partial_\theta + \theta \partial_x \) is the corresponding super space derivative. \( \theta \) is a fermionic anti-commuting variable, while \( x \) is a bosonic commuting variable. Thus, the fermionic (odd) variable \( \theta \) is nilpotent and we can carry out the Taylor expansion of the fermionic super field \( \alpha \):

\[ \alpha = \rho(x,t) + \theta v(x,t), \]

where the component fields \( \rho(x,t) \) and \( v(x,t) \) are the fermionic and bosonic quantities, respectively. In this way, the supersymmetric KdV equation (1) may be decomposed into the following system:
\[ \begin{align*}
\psi_t &= \psi_{xxx} + 6\psi\psi_x - 3\rho\rho_{xx}, \\
\rho_t &= \rho_{xxx} + 3(\psi\rho)_x. 
\end{align*} \]

This system may be compared with the super KdV equation proposed by Kupershmidt \[34\]:
\[ \begin{align*}
\psi_t &= -\psi_{xxx} + 6\psi\psi_x - 3\rho\rho_{xx}, \\
\rho_t &= -4\rho_{xxx} + 6\psi\rho + 3\psi\rho. 
\end{align*} \]

As observed by Mathieu \[8\], even though the above two systems (2) and (3) appear similar, they are very different. In fact, only (2) is invariant under
\[ \tilde{\psi} = \epsilon \rho_x, \quad \tilde{\rho} = \epsilon \psi, \]
where \( \epsilon \) is a fermionic parameter. Thus, (2) is known as the supersymmetric KdV equation while (3) is the fermionic KdV equation. More details on this may be found in \[8\] (see also \[34\]).

We can also write (1) in the potential form, i.e.
\[ \begin{align*}
\beta_t &= \beta_{xxx} + 3\beta_x D\beta_x, \\
\alpha &= \beta_x.
\end{align*} \]

Like the KdV equation, the supersymmetric KdV equation has many interesting properties: it is a bi-Hamiltonian system \[5, 6\], has a prolongation structure \[7\] and a Bäcklund transformation \[1\], possesses an infinite number of conserved quantities and satisfies the Painlevé property \[8, 9\], can be studied in the framework of Hirota’s bilinear method \[10–12\], etc (see also \[13–16\] for the most recent developments).

The existence of Darboux transformations (DTs) is important for integrable systems (see, for example, the monographs \[17–19\] and the recent review article \[20\]). In general, one can introduce two kinds of DTs, the elementary DT and the binary DT, and from them we can obtain the Bäcklund transformations. For the supersymmetric KdV equation (1), the construction of DTs was initiated in \[21\] and continued in \[22\] and both the elementary DT and binary DT have been constructed. Moreover, with the use of a super Gardner transformation, one has derived a Bäcklund transformation for the supersymmetric KdV equation \[1\], which reads
\[ \begin{align*}
(\beta_{[1]} + \beta)_x - 2p_1(\beta_{[1]} - \beta) + \frac{1}{2}(\beta_{[1]} - \beta)D(\beta_{[1]} - \beta) = 0,
\end{align*} \]
where \( p_1 \) is the Bäcklund parameter. Moreover, (5) yields a superposition formula which may be applied to construct the multi-soliton solutions.

Now it seems natural to ask ourselves which DT provides the Bäcklund transformation (5)? A careful consideration shows that none of the DTs constructed in \[21, 22\] gives the Bäcklund transformation (5). Thus, a proper DT is missing for the supersymmetric KdV equation and one of the purposes of this paper is to fill this gap.

Apart from their importance in the construction of solutions, Darboux and Bäcklund transformations are known to play a key role for the constructing nonlinear integrable discrete equations. Indeed, as showed by Levi and Benguria \[23\], Darboux and Bäcklund transformations can be used to provide the integrable discretizations of the continuous nonlinear integrable systems (see also \[24–27\]). Most recently, Grahovski and Mikhailov found integrable discretizations for a class of nonlinear Schrödinger equations on Grassmann algebras \[28\]. A generalization of this approach to supersymmetric integrable systems is up to now lacking and another purpose of this paper is to fill this gap. We will apply this idea to the supersymmetric KdV equation and show that the proper Darboux transformation enables us to discretize it and obtain an integrable super differential–difference system and an integrable super difference–difference system.

The paper is organized as follows. In section 2, we present a proper Darboux transformation for the supersymmetric KdV equation and derive its well-known Bäcklund
transformation. Then in section 3, we use the obtained transformations to construct discrete integrable super systems. Both differential–difference equations and difference–difference equations are obtained. In section 4, by performing various continuum limits, we show that our discrete systems are the proper discretizations of the potential supersymmetric KdV equation. The final section summarizes the results and indicates the further directions of research. We leave to an appendix the proof that the obtained nonlinear partial difference equation satisfies the compatibility around the cube but does not have the tetrahedron property.

2. Darboux transformations for the supersymmetric KdV

Our result is summarized by the following proposition.

Proposition 1. Let us suppose that the super field \( \phi(x,\lambda) \) satisfies the linear supersymmetric equation

\[
\phi_{xx} + \alpha D\phi = \lambda^2 \phi, \tag{6}
\]

and \( \phi[0](x, p_1) \) is a solution of (6) for \( \lambda = p_1 \). Let us define

\[
\phi[1] \equiv (\partial_x - \Lambda D + p_1)\phi, \tag{7}
\alpha[1] \equiv \alpha + \frac{2\Lambda}{\Lambda_1}, \quad \Lambda \equiv \frac{\phi[0], x + p_1 \phi[0]}{D\phi[0]}. \tag{8}
\]

Then, \( \phi[1](x, \lambda) \) will satisfy the following linear supersymmetric equation:

\[
\phi[1], xx + \alpha[1](D\phi[1]) = \lambda^2 \phi[1]. \tag{9}
\]

Proof. It is just a trivial direct calculation in which (7), (8) are introduced in (9) and \( \phi \) is shown to satisfy (6).

The above DT (7), (8) may be regarded as a nontrivial extension of the DT given in [21]. Indeed, in proposition 2.1 of [21] one has considered a solution of (6) with zero energy (namely \( p_1 = 0 \)); here our seed solution \( \phi[0] \) is an (odd) solution of (6) for an arbitrary value of \( p_1 \).

We show now that the DT (7), (8) leads to the Bäcklund transformation (5) derived by other techniques in [1]. In fact, let us introduce the potential \( \beta[1] \) such that \( \alpha[1] = \beta[1], \) then the first equation in (8) gives

\[
\Lambda = \frac{1}{2} (\beta[1] - \beta), \tag{10}
\]

where the constant of integration is set to zero. On the other hand, taking into account that

\[
\phi[0], x = \Lambda(D\phi[0]) - p_1 \phi[0]
\]

and

\[
\phi[0], xx = p_1^2 \phi[0] - \alpha(D\phi[0]),
\]

we obtain

\[
\Lambda_x + \beta_x - 2p_1\Lambda + \Lambda(D\Lambda) = 0,
\]

i.e. the Bäcklund transformation (5) for the supersymmetric KdV (1).

3. Discrete equations from the supersymmetric DT

Let us rewrite the Darboux transformation (7), (8) in the matrix form. Introducing the vector \( \Psi = (\phi, \phi_x, D\phi, D\phi_x)^T \), with the help of (5) one may rewrite (6) and (7) as the following systems:
\[ \Psi_t = L \Psi, \quad L = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \lambda^2 & 0 & -\beta_x & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \beta_x & \lambda^2 - D\beta_x & 0 \end{pmatrix}, \quad (11) \]

\[ \Psi_{[1]} = W \Psi, \quad W = \begin{pmatrix} p_1 & 1 & -\Lambda & 0 \\ \lambda^2 & p_1 & (D\Lambda - 2p_1)\Lambda & -\Lambda \\ 0 & \Lambda & p_1 - D\Lambda & 1 \\ \lambda^2 \Lambda & (2p_1 - D\Lambda)\Lambda & \lambda^2 - 2p_1(D\Lambda) + (D\Lambda)^2 & p_1 - D\Lambda \end{pmatrix}, \quad (12) \]

where $\Lambda$ is given by (10). The compatibility of the two linear systems (11), (12) is

\[ W_t + WL - L_{[1]}W = 0, \quad (13) \]

equivalent to the Backlund transformation (5). It is worthwhile to note here that the matrix $W$ is a function, through $\Lambda$, of both $\beta$ and $\beta_{[1]}$. We can interpret (13) as a differential–difference equation by the following identifications:

\[ \beta \equiv \beta_n(x), \quad \beta_{[1]} \equiv \beta_{n+1}(x). \]

Equation (13) will admit as a Lax pair (11) and (12), therefore, it will be integrable, at least in the Lax sense.

Let us introduce a new Darboux transformation

\[ \Psi_{[2]} = V \Psi, \quad (14) \]

where the matrix $V$ is equal to the matrix $W$ with $p_1$ and $\beta_{[1]}$ replaced by $p_2$ and $\beta_{[2]}$, respectively, i.e. the matrix $V$ is a function through $\Lambda$ of both $\beta$ and $\beta_{[2]}$. If we consider the compatibility of (12) with (14), i.e. the Bianchi permutability of a Backlund transformation (13) of parameter $p_1$ with one of parameter $p_2$, $(\Psi_{[1]}|_{[2]} = (\Psi_{[2]}|_{[1]}$, with $\beta_{[12]} = \beta_{[21]}$, we obtain that the following consistency condition

\[ W_{[2]}V = V_{[1]}W \quad (15) \]

must be true. Equation (15) leads to

\[ \beta_{[12]} = \beta + \frac{2(p_1 + p_2)(\beta_{[1]} - \beta_{[2]})}{2(p_2 - p_1) + D(\beta_{[1]} - \beta_{[2]})} \quad (16) \]

and

\[ (\beta_{[1]} - \beta_{[2]})(2(\beta_{[1]} - \beta_{[2]})(\beta_{[1]} - \beta_{[2]})(4(p_2 - p_1) + D(\beta_{[1]} - \beta_{[2]}))) = 0. \quad (17) \]

Equation (16) is the superposition formula considered in [1], while (17) is an extra constraint.

To obtain a partial difference system, we use the following identifications:

\[ \beta \equiv \beta_{n,m}(x), \quad \beta_{[1]} \equiv \beta_{n+1,m}(x), \quad \beta_{[2]} \equiv \beta_{n,m+1}(x), \quad \beta_{[12]} \equiv \beta_{n+1,m+1}(x), \]

thus, the system (16), (17) can be interpreted as an integrable differential-partial difference system as it contains the super derivative $D$.

To find a differential–difference system, we rewrite the super fields in terms of its components. Let us assume that

\[ \beta \equiv \xi + \theta u, \quad \phi = \varphi + \theta f, \]

and define

\[ \xi \equiv \xi_n(x), \quad \xi_{[1]} \equiv \xi_{n+1}(x), \quad u \equiv u_n(x), \quad u_{[1]} \equiv u_{n+1}(x). \]

Then, the Backlund transformation (13) is split into

\[ (\xi_{n+1} + \xi_n) = 2p_1(\xi_{n+1} - \xi_n) - \frac{1}{2}(u_{n+1} - u_n)(\xi_{n+1} - \xi_n), \quad (18a) \]
where the matrix \( \chi \) and \( \xi \) are defined by

\[
\chi = \begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 \\
\xi_1 & \xi_2 & \cdots & \cdots & \xi_n \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \ddots & \xi_n \\
0 & \cdots & \cdots & \cdots & 1
\end{pmatrix},
\]

and

\[
\xi = \begin{pmatrix}
\xi_1 \\
\xi_2 \\
\vdots \\
\xi_n \\
1
\end{pmatrix}.
\]

Then, the compatibility condition of (19) and (20) gives

\[
\xi_{n+1} = \mathcal{W}_n \xi_n,
\]

and

\[
\mathcal{W}_n = \begin{pmatrix}
p_1 & 0 & \cdots & \cdots & 0 \\
-l^{-1} & 1 & \cdots & \cdots & 0 \\
0 & \xi_1 & \cdots & \cdots & \xi_n \\
0 & \cdots & \cdots & \cdots & -1
\end{pmatrix},
\]

with

\[
\eta = \frac{1}{2}(\xi_{n+1} - \xi_n),
\]

\[
g = \frac{1}{2}(u_{n+1} - u_n).
\]

Then, the compatibility condition of (19) and (20) gives

\[
(\mathcal{W}_n)_{n+1} + \mathcal{W}_n \mathcal{L}_n - \mathcal{L}_{n+1} \mathcal{W}_n = 0
\]

which holds if and only if (18) is satisfied. In the following section, we will show that, after a continuum limit, the system (18) leads to the potential supersymmetric KdV equation, thus, it constitutes a discretization of the potential supersymmetric KdV equation.

To find a difference–difference system, we define

\[
\xi = \xi_n, \quad \xi_1 = \xi_{n+1}, \quad \xi_2 = \xi_{n+1}, \quad \xi_3 = \xi_{n+1}, \quad \xi_4 = \xi_{n+1},
\]

\[
u = u_n, \quad u_1 = u_{n+1}, \quad u_2 = u_{n+1}, \quad u_3 = u_{n+1}, \quad u_4 = u_{n+1}.
\]

As above, we consider another Darboux transformation, equivalent to (14), which introduces \( \chi_{n,m} = ((\varphi_{n,m}), (\varphi_{m,n}), (f_{n,m}), (f_{m,n}))^T \), now reads

\[
\chi_{n,m+1} = \mathcal{W}_{n,m} \chi_{n,m},
\]

where the matrix \( \mathcal{W}_{n,m} \) is the matrix \( \mathcal{W}_n \) of (20) with \( p_1 \), \( \xi_{n+1} \), \( u_{n+1} \), \( u_{n+1} \) replaced by \( p_2 \), \( \xi_{n+1} \), \( u_{n+1} \), \( u_{n+1} \), respectively.

Now, the compatibility condition of (20), written as \( \mathcal{W}_{n,m} = \mathcal{W}_{n,m} \chi_{n,m} \), and (21), namely

\[
\mathcal{W}_{n,m+1} \mathcal{W}_{n,m} = \mathcal{W}_{n,m+1} \mathcal{W}_{n,m}
\]

yields an integrable difference–difference system

\[
\xi_{n+1} = \xi_n + \frac{2(p_1 + p_2)(\xi_{n+1} - \xi_{n+1})}{2(p_2 - p_1) + u_{n+1} - u_{n+1}}
\]

\[
u_{n+1} = u_n + \frac{2(p_1 + p_2)(u_{n+1} - u_{n+1})}{2(p_2 - p_1) + u_{n+1} - u_{n+1}}
\]

\[
u_{n+1} = u_n + \frac{(p_1 + p_2)(4(p_2 - p_1) + u_{n+1} - u_{n+1})}{2(p_2 - p_1) + u_{n+1} - u_{n+1}} \left( \xi_{n+1} - \xi_{n+1} \right)
\]

\[
(22a)
\]

In the following section, we will show that, after a double continuum limit, the system (22) leads to the potential supersymmetric KdV equation, thus, it constitutes a discretization of the potential supersymmetric KdV equation.
Remark.

(i) The difference–difference system (22) may also be derived from (16) and (17) directly. Indeed, one can write (16) in components and obtain (22a) and

\[ u_{n+1,m+1} = u_{n,m} + \frac{2(p_1 + p_2)(u_{n+1,m} - u_{n,m+1})}{2(p_2 - p_1) + u_{n+1,m} - u_{n,m+1}} + \frac{2(p_1 + p_2)(\xi_{n+1,m} - \xi_{n,m+1})(\xi_{n+1,m} - \xi_{n,m+1})}{2(p_2 - p_1) + u_{n+1,m} - u_{n,m+1}}. \]  

(23)

Equation (17) gives

\[ 2(\xi_{n+1,m} - \xi_{n,m+1})(\xi_{n+1,m} - \xi_{n,m+1}), \]

from which we can eliminate the differential part of (23) and obtain (22b).

(ii) If \( \xi = 0 \), (22) reduces to the well-known lattice potential KdV equation

\[ u_{n+1,m+1} = u_{n,m} + \frac{2(p_1 + p_2)(u_{n+1,m} - u_{n,m+1})}{2(p_2 - p_1) + u_{n+1,m} - u_{n,m+1}}, \]

from which we can eliminate the differential part of (23) and obtain (22b).

(iii) The system (22) is integrable in the sense that it possesses Lax representation. In the appendix, it will be shown that this system is consistent around a cube, i.e. it has the consistency-around-the-cube property [29].

4. Continuum limits of the discrete equations obtained in the previous section

We can characterize the discrete systems obtained in the previous section by analyzing their continuum limits [30, 31]. We obtained the differential–difference system (18) and the difference–difference system (22) as the discrete versions of the potential supersymmetric KdV equation. We now justify this claim by considering their continuum limits.

4.1. The continuous limit of (18)

To carry out the continuous limit of (18), we define the new continuous variable \( \tau \) as

\[ \xi_n(x) \equiv \xi(x, \tau), \quad u_n(x) \equiv u(x, \tau), \quad \tau = \frac{n}{p_1}. \]

Then,

\[ \xi_{n+1}(x) \equiv \xi(x, \tau + \frac{1}{p_1}), \quad u_{n+1}(x) \equiv u(x, \tau + \frac{1}{p_1}), \]

can be expanded in \( \frac{1}{p_1} \), and defining a new independent variable \( t \) in term of \( \tau \) and \( x \) such that

\[ \partial_\tau = \partial_x + \frac{1}{12 p_1^2} \partial_t, \]

we obtain in the continuous limit up to terms of order \( \frac{1}{p_1} \)

\[ \xi_t = \xi_{xxx} + 3\xi_t u_x, \quad u_t = u_{xxx} + 3u_x^2 + 3\xi_x \xi_x, \]

which is the potential form of the supersymmetric KdV equation (4).

4.2. The semi-continuous limits of (22)

We present here two different results obtained by implementing different continuous limits of (22), at first when we send to infinity just one of the discrete variables and secondly when we send to infinity a combination of both discrete variables.
4.2.1. Straight continuum limit. The system (22) may be regarded as a discrete analogue of the differential–difference system (18). To see it, let us define

\[ \xi_{n,m} \equiv \xi_n(x), \quad u_{n,m} \equiv u_n(x), \quad x = \frac{m}{p_2} \]

For \( \frac{1}{p_2} \) small, we have the following Taylor series expansions:

\[ \xi_{k,m+1} = \xi_k(x + \frac{1}{p_2}) = \xi_k + \frac{1}{p_2} \xi_{k,x} + O\left(\frac{1}{p_2^2}\right), \]
\[ u_{k,m+1} = u_k(x + \frac{1}{p_2}) = u_k + \frac{1}{p_2} u_{k,x} + O\left(\frac{1}{p_2^2}\right) \]

where \( k \) can take the value \( n \) or \( n + 1 \). Substituting the above expansions into (22), the leading terms yield

\[ (\xi_{n+1} + \xi_n) - 2p_1 (\xi_{n+1} - \xi_n) - \frac{1}{2} (\xi_{n+1} - \xi_n)(u_{n+1} - u_n), \]
\[ (u_{n+1} + u_n) - 2p_1 (u_{n+1} - u_n) - \frac{1}{2} (u_{n+1} - u_n)^2 - (\xi_{n+1} - \xi_n)\xi_{n,x}, \]

i.e. (18).

4.2.2. Skew continuum limit. In this case, we introduce the new discrete variable

\[ N = n + m, \quad p_2 = p_1 + \epsilon, \quad \tau = \epsilon m, \]

and

\[ \xi_{n,m} \equiv \xi_N(\tau), \quad u_{n,m} \equiv u_N(\tau), \]
\[ \xi_{n+1,m} \equiv \xi_{N+1}(\tau), \quad u_{n+1,m} \equiv u_{N+1}(\tau), \]
\[ \xi_{n+1,m+1} \equiv \xi_{N+2}(\tau + \epsilon), \quad u_{n+1,m+1} \equiv u_{N+2}(\tau + \epsilon), \]

and develop all dependent variables in the Taylor series in \( \epsilon \). After inserting them into (22), the leading order terms give

\[ \xi_{N+1,\tau} = -\frac{1}{4p_1} (\xi_{N+2} - \xi_N)(2 - u_{N+1,\tau}), \]
\[ (u_{N+2} - u_N)(2 - u_{N+1,\tau})^2 + 4p_1 u_{N+1,\tau} (2 - u_{N+1,\tau}) = 2p_1 \xi_{N+1,\tau} (4 - u_{N+1,\tau})(\xi_{N+1} - \xi_N). \]

Solving this system with respect to \( \xi_{N+1,\tau} \) and \( u_{N+1,\tau} \), and shifting \( N + 1 \) to \( N \), we obtain, either

\[ \xi_{N,\tau} = \frac{2(\xi_{N+1} - \xi_{N-1})}{u_{N+1} - u_{N-1} - 4p_1}, \]
\[ u_{N,\tau} = \frac{2(u_{N+1} - u_{N-1})}{u_{N+1} - u_{N-1} - 4p_1} + \frac{(u_{N+1} - u_{N-1} - 8p_1)}{(u_{N+1} - u_{N-1} - 4p_1)^2}(\xi_{N+1} - \xi_{N-1})(\xi_N - \xi_{N-1}) \]

or the trivial solution

\[ \xi_{N,\tau} = 0, \quad u_{N,\tau} = 2. \]

If the fermionic variable \( \xi_N \) is null, the system (28) is reduced to

\[ u_{N,\tau} = \frac{2(u_{N+1} - u_{N-1})}{u_{N+1} - u_{N-1} - 4p_1}, \]

a differential–difference equation related to the Kac–van Moerbeke equation [32]. Thus, with this limit, the system (28) is a super extension of the Kac–van Moerbeke equation.
4.3. Full continuum limit

Two different semi-continuum limits have been considered for (22) and two differential–
difference systems, namely (25) and (28), are obtained. As (25) is equal to (18) its continuous
limit is contained in section (4.1) and it leads to the potential supersymmetric KdV equation
(4). In the following, we will consider the continuum limit of (28) and it will turn out that it
also leads to the potential supersymmetric KdV equation (4).

For (28), defining
\[ \xi_N(t) \equiv \xi(s, t), \quad u_N(t) \equiv u(s, t), \quad s = \frac{N}{p_1}, \]
expanding
\[ \xi_{N \pm 1}(t) \equiv \xi(s \pm \frac{1}{p_1}, t), \quad u_{N \pm 1}(t) \equiv u(s \pm \frac{1}{p_1}, t) \]
in \( \frac{1}{p_1} \), and redefining the independent variables from \( s, t \) to \( x, t \) such that
\[ \partial_s = \partial_x, \quad \partial_t = -\frac{1}{p_1^2} \partial_n - \frac{1}{6p_1^4} \partial_t, \]
we obtain once again (24).

4.4. Semi-continuous limits of the Lax pair

The continuum limits considered above were carried out on the level of the nonlinear systems.
We actually can carry out also the continuum limits of the corresponding Lax pair (20) and
(21):
\[ (p_1 - \lambda) \chi_{n+1,m} = \mathcal{W}_{n,m} \chi_{n,m}, \quad (p_2 - \lambda) \chi_{n,m+1} = \mathcal{V}_{n,m} \chi_{n,m}, \]
in which the matrices \( \mathcal{W} \) and \( \mathcal{V} \) are defined in section 3. We now carry out at first the skew
continuum limit (26) on the Lax pair in order to obtain a Lax pair for (28). Using (26), (27)
the first element of the Lax pair (29) reads
\[ (p_1 - \lambda) \chi_{N+1} = \mathcal{W}_N \chi_N, \quad \mathcal{W}_N = \begin{pmatrix}
  p_1 & \lambda^2 & 0 \\
  1 & p_1 & -(h-2p_1) \xi \\
  \zeta & p_1 - h & 1
\end{pmatrix}, \]
with
\[ t_{23} \equiv -p_1 \xi_{N+1, \tau} - 2\zeta + \frac{1}{2} (h \xi_{N+1, \tau} + \xi u_{N+1, \tau}), \]
\[ t_4 \equiv -p_1 u_{N+1, \tau} - 2h + hu_{N+1, \tau}. \]
Next with the help of (30), we obtain as the coefficient of the leading term of order \( O(\epsilon) \) the
following equation:
\[ (p_1 - \lambda) \chi_{N+1, \tau} = T_N \chi_N - \chi_{N+1}. \]

The consistency condition of (30) and (31)
\[ W_{N+1} W_{N-1} + W_N T_{N-1} - T_N W_{N-1} = 0 \]
leads to (28), thus (30), (31) constitute a Lax pair for the supersymmetric Kac–van Moerbeke equation (28).

5. Conclusion

In this paper, we give a new Darboux transformation for the supersymmetric KdV equation. By means of this Darboux transformation, the supersymmetric KdV equation is discretized. Both differential–difference system and difference–difference system are obtained, their integrability is shown and various continuum limits are considered.

This work opens the way to the construction of many discrete supersymmetric equations. We are planning to apply the idea to other supersymmetric integrable equations. In particular, we are considering the supersymmetric Schrödinger equation [33] and the Kupershmidt super KdV equation [34], which is a fermionic rather than the supersymmetric extension of the KdV equation.

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Appendix. CAC property of the system (22)

An integrable partial difference equation defined on a square lattice is characterized by the consistency-around-the-cube (CAC) property [29, 35, 36], equivalent to the Bianchi permutability theorem for the Bäcklund transformations of integrable nonlinear partial differential equations. In the following, we will show that also the system (22) possesses such a property.

From (22), by adding a third direction, we have the following equations:
\[ \xi_{ij} = \xi + f_{ij}(\xi_i - \xi_j), \quad u_{ij} = u + f_{ij}(u_i - u_j) + g_{ij}(\xi_i - \xi_j)(\xi_j - \xi), \]
where
\[ f_{ij} = \frac{2(p_i + p_j)}{2(p_j - p_i) + u_{ij} - u_{ij}}, \quad g_{ij} = \frac{(p_i + p_j)(4(p_j - p_i) + u_{ij} - u_{ij})}{(2(p_j - p_i) + u_{ij} - u_{ij})^2}, \]
with \(1 \leq i, j \leq 3\). Then, we obtain
\[ \xi_{ij} = \xi_i + f_{ij}(\xi_i - \xi_j), \]
\[ u_{ij} = u_i + f_{ij}(u_i - u_j) + g_{ij}(\xi_i - \xi_j)(\xi_j - \xi). \]
where
\[ f_{ij} = \frac{2(p_i + p_j)}{2(p_j - p_i) + u_{ij} - u_{ij}}, \quad g_{ij} = \frac{(p_i + p_j)(4(p_j - p_i) + u_{ij} - u_{ij})}{(2(p_j - p_i) + u_{ij} - u_{ij})^2}. \]
Consistency means that \( \xi_{[123]} = \xi_{[231]} = \xi_{[312]} \), \( u_{[123]} = u_{[231]} = u_{[312]} \). Indeed, by direct calculation we find

\[
\xi_{[123]} = \frac{1}{H} \sigma_{ij} \xi_{[ij]} (p_i + p_j) (p_i + p_j) (2(p_j - p_i) - u_{ij}) + \frac{1}{2H} (\xi_{[1]} - \xi_{[2]} - \xi_{[3]} - \xi) (p_1 + p_2) (p_1 + p_3) (p_2 + p_3) \sigma_{ij} (p_i - p_j) u_{ij},
\]

\[
u_{[123]} = \frac{1}{H} \sigma_{ij} [u_{ij} (u_{ij} - u_{ij}) + 2p_j (u_{ij} - u_{ij}) - 2p_j (u_{ij} - u_{ij})]
= \frac{1}{2H} (p_1 + p_2) (p_1 + p_3) (p_2 + p_3) [\sigma_{ij} \xi_{[ij]} (2(p_i - p_j) + u_{ij}) - u_{ij})]
\times [p_i (u_{ij} - u_{ij}) + p_j (u_{ij} - u_{ij}) + p_j (u_{ij} - u_{ij})]
+ \sigma_{ij} \xi_{[ij]} [8(p_i - p_j) (p_i - p_j) (p_j - p_j) + (p_j - p_i) (u_{ij} - u_{ij}) (u_{ij} - u_{ij})]
- 4u_{ij} (p_i - p_j) (p_i - p_j) - 2u_{ij} (p_i - p_j) (p_j - p_j) + 2p_j
+ 2u_{ij} (p_i - p_j) (p_i - p_j) + 2p_j]
\]

where

\[
H \equiv 2(p_1 - p_2) (p_1 - p_3) (p_2 - p_3) + \sigma_{ij} [p_i^2 - p_j^2] u_{ij},
\]

and \( \sigma_{ij} \) denotes the cyclic sum over the subscripts \((i, j, l) = (1, 2, 3), (2, 3, 1), (3, 1, 2)\). Then, \( \xi_{[123]} \) and \( u_{[123]} \) are symmetric under any permutation of \((1, 2, 3)\), thus (22) obeys the CAC property. We note that since both \( \xi_{[123]} \) and \( u_{[123]} \) are independent of \( u \) but dependent on \( \xi \), the system (22) does not have the tetrahedron property.

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