Supercurrent Quantization in Narrow Channel SNS Junctions

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We determine the quasi-particle excitation spectrum in the normal region of a narrow ballistic superconductor–normal-metal–superconductor (SNS) Josephson contact. Increasing the effective chemical potential in the contact converts the electronic levels into Andreev-levels carrying supercurrent. The opening of these superchannels leads to a supercurrent quantization which exhibits a non-universal behavior in general and we discuss its dependence on the junction parameters.

The Josephson effect\textsuperscript{[1]}, a hallmark of superconductivity, is of fundamental interest and bears considerable potential for applications in superconducting electronics\textsuperscript{[2]}. Today, the miniaturization of electronic structures has reached the regime where the transport proceeds via few or even a single conducting channel\textsuperscript{[3]}. Using gated structures, junctions can be transformed from insulating SIS to superconducting SNS states with the quasi-particle spectrum evolving from phase-insensitive electronic- to phase-sensitive Andreev states carrying large supercurrents. The onset of superflow proceeds in steps associated with the opening of transverse channels, similar to the conductance quantization in a metallic contact\textsuperscript{[4]}. However, contrary to the universality of the quantization in a normal contact, the quantization of the critical supercurrent is perfect only in the limit of short junctions $L \ll \xi_0$ ($\xi_0 = \hbar v_F/\pi \Delta$ is the superconducting coherence length) but is non-universal in general\textsuperscript{[5]}. Indeed, while experiments on superconducting quantum point contacts do show steps in the critical current $I_c$, these are much less prominent than the corresponding steps in the conductance $G$\textsuperscript{[6]}. In this letter, we study the opening of superconducting channels in the metallic link of a narrow ballistic SNS Josephson contact and determine the evolution of the quasi-particle spectrum and the dependence of the supercurrent quantization on the junction parameters.

While the behavior of macroscopic SNS Josephson junctions is well understood\textsuperscript{[7]}, the present interest concentrates on gated structures of mesoscopic size. Such quantum point contacts are realized in heterostructures or via manipulations with a scanning tunneling microscope. Theoretically, the supercurrent-phase relation in mesoscopic SNS junctions with a $\delta$-scatterer has been analyzed\textsuperscript{[8]} and the phenomenon of supercurrent quantization has been studied in short junctions. Non-universal features of supercurrent quantization have first been observed by Furusaki et al.\textsuperscript{[9]} — unfortunately, these numerical results provide limited insight into the physical origin and the parametric dependence of these effects. Here, we present a detailed discussion of the opening of superchannels in mesoscopic SNS junctions using quasi-classical and scattering matrix techniques. We discuss the non-trivial evolution of the excitation spectrum as the chemical potential drops below the superconducting gap and analyze the transformation of the ballistic SNS structure into a SIS tunnel junction. The phase-dependence of the quasi-particle spectrum allows us to find the supercurrent quantization in short and long junctions, where the contribution from the continuous spectrum can be ignored.

Consider a narrow metallic lead (with few transverse channels) connecting two superconducting contacts as sketched in Fig. 1(a) [we assume piecewise constant gap parameters $\Delta(x < -L/2) = \Delta \exp(i \varphi_L)$, $\Delta(|x| < L/2) = 0$, and $\Delta(L/2 < x) = \Delta \exp(i \varphi_R)$]. For each transverse channel in the metallic wire the quasi-particle spectrum $\varepsilon_{\nu}$ is determined through the 1D Bogoliubov-de Gennes equation (we choose states with $\varepsilon_{\nu} \geq 0$)

$$
\left[ \begin{array}{cc}
\frac{\hbar^2 \partial^2}{2m} - \mu_e(x) & \Delta(x) \\
\Delta^*(x) & \frac{\hbar^2 \partial^2}{2m} + \mu_x(x)
\end{array} \right] \begin{bmatrix} u_{\nu}(x) \\ v_{\nu}(x) \end{bmatrix} = \varepsilon_{\nu} \begin{bmatrix} u_{\nu}(x) \\ v_{\nu}(x) \end{bmatrix},
$$

where $u_{\nu}$ and $v_{\nu}$ denote the electron- and hole-like components of the wave function $\Psi_{\nu}$. The spectrum splits into continuous and discrete parts and we concentrate

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig1.png}
\caption{Narrow channel SNS contact: (a) geometrical setup showing the adiabatic joining of the wire to the superconductor, (b) potential landscape with a flat potential in the wire center and smoothly dropping to the band bottom in the superconducting banks. While electrons and holes with small excitation energies $\varepsilon < \mu_e$ form current carrying Andreev states, the hole propagation is quenched at large energies $\varepsilon > \mu_x$ and the Andreev levels transform into localized electronic states.}
\end{figure}
on the latter part in the following, \( \varepsilon_r < \Delta \).

The effective chemical potential \( \mu_x(x) = \varepsilon_r - \varepsilon_\perp(x) \) accounts for the transverse energy \( \varepsilon_\perp(x) \) of the channel, see Fig. 1(b). Within a quasi-classical formulation we describe the quasi-particles in terms of their kinetic energies \( K_\pm = \hbar^2 k_\pm^2/2m = \mu_x(x) \pm \varepsilon \) and assume transmission and reflection to be ideal (the excitation energies \( \varepsilon = E - \varepsilon_r > 0 \) are measured with respect to the Fermi energy \( \varepsilon_r \)). An electron with energy \( \varepsilon < \Delta \) below the gap is reflected back from the superconductor as a hole with kinetic energy \( K_- = \mu_x - \varepsilon \), injecting a Cooper-pair into the superconducting contact, a process known as Andreev reflection [1]. A second reflection at the opposite NS boundary transforms the hole state back into the original electron state, thus producing a phase sensitive Andreev level carrying the supercurrent across the normal metal lead. The hole-part associated with the Andreev level can propagate only if its kinetic energy is positive, \( K_+ > 0 \), see Fig. 1(b). Otherwise, the hole is back-reflected from the normal potential in the junction and transformed into an electron at the NS boundary — the incident electron is effectively reflected back as an electron and a phase-insensitive electronic level is formed. Hence, the superchannel starts being modified when the chemical potential \( \mu_x \) drops below the gap \( \Delta \) and is quenched completely with all Andreev levels transformed into electronic ones when \( \mu_x \) becomes negative.

Going beyond quasi-classics, the above physics is conveniently described through the scattering matrix formalism [12][13]. We define scattering states in the normal region and characterize them through the energy dependent transmission and reflection coefficients \( t \exp(i\chi^t) \) and \( r \exp(i\chi^r) \) describing the propagation of quasi-particles incident from the left through the junction. Matching these states with the evanescent modes in the superconductors we obtain (within the Andreev approximation [1]): \( (K_+ - K_-)/(K_+ + K_-) \ll 1 \) at the NS interface) the quantization condition,

\[
\cos(S_+ - S_- - \alpha) = r_+ r_- \cos \beta + t_+ t_- \cos \varphi,
\]

where the \((+(-))\) signs refer to the positive and negative energies \( \pm \varepsilon \) of the electron(hole)-like quasi-particles and \( S_\pm(\varepsilon) = \chi^t_\pm + k_{0,\pm} L \), with \( k_{0,\pm} L = \sqrt{2m(\varepsilon_F \pm \varepsilon)} L/\hbar \) the phase for free propagation (while the phase \( S \) refers to the propagation from \(-L/2\) to \(L/2\), the scattering phases \( \chi^t \) and \( \chi^r \) refer to the origin). Andreev scattering at the NS boundaries introduces the phase \( \alpha = 2\arccos(\varepsilon/\Delta) \) ranging from \( \pi \) at \( \varepsilon = 0 \) to \( \Delta \) at \( \varepsilon = \Delta \), as well as the phase difference \( \varphi = \varphi_r - \varphi_l \) between the two superconducting banks. The phase \( \beta = (\chi^t_+ - \chi^t_-) - (\chi^r_+ - \chi^r_-) \) reduces to \( \beta = 0 \) for a symmetric barrier in the absence of perfect resonances [as follows from the unitarity of the scattering matrix; for a symmetric potential shifted by \( a \) from the center we have \( \beta = 2(k_{0,+} - k_{0,-})a \)]. The secular equation (1) involves two main energy dependencies originating from the propagation through the normal layer \((S\pm)\) and from scattering at the NS boundaries, e.g., due to potential steps or barriers. Here, we concentrate on the case where the transport through the junction is dominated by the normal metallic wire — we will comment on the effect of resonances introduced by an additional scattering at the NS boundaries below.

![Figure 2](image)

**FIG. 2.** Discrete energy spectrum [(a) qualitative sketch for a flat potential, (b) smooth parabolic potential]: For \( \mu_{x0} > \varepsilon \) (region I) both electrons and holes propagate, forming phase \((\varphi)\) sensitive Andreev levels carrying supercurrent. The double degeneracy of the Andreev states is lifted by a finite phase drop \( \varphi \) across the junction as well as a finite reflection in the wire (see (c)), the latter becoming relevant upon decreasing \( \mu_{x0} \). As \( \mu_{x0} \) drops below \( \varepsilon \) the Andreev levels first transform into electronic states (regions II and III) and finally turn into boundary states trapped at the NS interfaces when \( \mu_{x0} \lesssim -\varepsilon \) (region V). Within the shaded regions around \( \varepsilon = \pm \mu_{x0} \) the transmission drops from unity to zero for holes and electrons. (c): graphical solution of (1) along the cut A-A in (b).

A rough understanding of the transformation from a metallic to an insulating junction is obtained in the quasi-classical approximation using a flat potential, see Fig. 1(b): For a large chemical potential \( \mu_{x} \gg \varepsilon \) we have \( r_\pm = 0, t_\pm = 1 \) in (1) and we obtain the Bohr-Sommerfeld quantization condition for the (phase sensitive) Andreev levels \( S_+ - S_- - \alpha \pm \varphi = 2n\pi \). Evaluating this condition for a flat potential (ignoring contributions to \( S_\pm = k_{x,\pm} L \sqrt{1 \pm \varepsilon/\mu_{x}} \) originating from the adiabatic joints) we obtain the level scheme shown in Fig. 2(a). On the other hand, for \(-\varepsilon < \mu_{x} < \varepsilon \) the right hand side of (1) vanishes and using \( S_- = 3\pi/2 \) (assuming a hard wall potential) we find the quantization condition \( 2S_+ - 2\alpha = 2n\pi \), for the electronic levels, see Fig. 2(a) for a qualitative result in a flat potential. Note that we have twice as many electronic than Andreev levels as the latter are doubly degenerate at \( \varphi = 0 \) — the exact transformation of the Andreev levels in electronic ones at \( \varepsilon \approx \mu_{x} \) requires a more careful analysis accounting for the non-ideal transmission and reflection through the normal channel, see below. Finally, as \( \mu_{x} \) drops be-
low $-\varepsilon$ both the electron- and hole-like trajectories are quenched. Note the evolution of the r.h.s. of (7), going from $\cos \varphi$ at large positive $\mu_x > \varepsilon$, to a small value in the intermediate region $-\varepsilon < \mu < \varepsilon$, and back to unity at large negative $\mu_x < -\varepsilon$. This provides us with a first rough understanding of the SNS to SIS transformation.

In a more accurate study of the evolution of the bound state spectrum from a SNS to a SIS junction we assume a smooth potential $\mu_x(x)$ with a small curvature $m\Omega^2 = \partial_x^2 \mu_x$, $\hbar \Omega < \Delta$, producing a sharp switching between transmission and reflection within the energy interval $\hbar \Omega$ (a $\delta$-function scatterer [11] does not describe a pronounced transformation from a SNS to a SIS junction). Adiabatic joining of the wire to the superconducting banks requires that $m\Omega^2(L/2)^2/2 \sim \varepsilon_\nu$ and allows us to make use of the Andreev approximation while avoiding the appearance of resonances (this condition can be relaxed as the Andreev approximation requires $m\Omega^2(L/2)^2/2 \gg \Delta$, while a step in the potential $\Delta V < 0.9\varepsilon_\nu$ produces only weak resonances). In summary, a smooth contact without resonances requires the parameter setting $\sqrt{\varepsilon_\nu \varepsilon_\kappa} < \hbar \Omega < \Delta$ where $\varepsilon_\kappa \equiv h^2 \pi^2/2mL^2$; this condition implies a long junction $L > \xi_0$ and hence a relatively large number $n \sim \sqrt{\varepsilon_\nu/\Delta}$ of trapped levels. For such a smooth potential the Kemble formula is valid and the transmission probabilities take the form $t_\pm = t_0^2 = 1/\{1 + \exp[-2\pi(\mu_x(0) \pm \varepsilon)/\hbar \Omega]\}$ [11].

Fig. 2(b) shows the refined results for the SNS to SIS transformation using the quadratic potential $\mu_x(x) = \mu_{x,0} + m\Omega^2 x^2/2$ with the parameter $\hbar \Omega(\mu_{x,0}) = (4/\pi)\sqrt{\varepsilon_\nu(\varepsilon_\nu - \mu_{x,0})}$ and $\mu_{x,0} = \mu_x(0)$, joining the band bottom of the superconductors at the two NS boundaries. For this case, the quasiclassical dimensionless action takes the form $S(E)/\hbar = (2E/\hbar \Omega)|\kappa^2/\sqrt{1 + \kappa^2} + \ln|\kappa|/(1 + \sqrt{1 + \kappa^2})|$ with $\kappa^2 = Q \hbar \Omega/E = \pi^2 \hbar^2 2^2/16 \varepsilon_\nu 2^{1/2} > 1$ a large parameter $|S_\pm = S(E = \mu_{x,0} \pm \varepsilon)|$: an additional phase $\pi$, which cannot be obtained within the quasiclassical scheme, is picked up over the energy interval $\hbar \Omega$ as $E$ goes through zero. As for the flat potential, the Andreev levels at large chemical potential $\mu_{x,0} > \varepsilon + \hbar \Omega$ (region I) are converted in steps (regions II $- \nu$) to the electronic states at negative potential $\mu_{x,0} < -\varepsilon - \hbar \Omega$: Upon entering region II the product $t_+ t_- \leq (1)\cdot(1)$ drops below unity and the Andreev levels split even for $\varphi = 0$. In region III the hole propagation is quenched and electronic levels with an exponentially weak phase sensitivity are formed. Entering region V the electronic propagation through the wire is suppressed and pairs of boundary states are formed near the NS interfaces [11] (for the flat potential these boundary states are lifted to $\varepsilon \approx \Delta$ as the Andreev phase $\alpha$ has to vanish). In region IV both the electron- and hole-components undergo finite reflection and the distinction between Andreev- and electronic levels is gone; this resembles the situation of a SNS junction with a $\delta$-scattering potential [11]. More details of this conversion between Andreev- and electronic levels will be given elsewhere [13]. Below, we concentrate on the quasi-classical region I and study the evolution of the critical supercurrent as the channel is switched on and off.

The supercurrent $I$ flowing through the junction splits into the two contributions from the discrete ($I_{\text{dis}}$) and the continuous ($I_{\text{con}}$) parts of the spectrum. Here, we concentrate on $I_{\text{dis}}$, which dominates the expression for the critical supercurrent in the quasi-classical regime I (requiring $\hbar \Omega < \mu_{x,0}$ is sufficient). The current of the $\nu$-th level (including a factor 2 for spin) can be obtained from the derivative $L_\nu = (2e/\hbar)\partial_\varphi \varepsilon_\nu = (2e/\tau T) \partial_\varphi \sin \varphi$, with the generalized travelling time $\tau = \sin(\delta S - \alpha)h\partial_\varphi[\delta S - \alpha] + h\partial_\varphi[(t_+ t_-) \cos \varphi + (t_+ t_-) \cos \beta]$. Within the quasi-classical region I, each Andreev level carries a finite supercurrent of amplitude $2e/[\tau_+ + \tau_- + 2\hbar/\sqrt{\Delta^2 - \varepsilon^2}]$, where $\tau_\pm = h\partial_\varphi S_\pm$ denote the propagation times for the electron- and hole-components; for the smooth quadratic potential we find $\tau(E) = \Omega^{-1}[2\ln|\kappa|/(1 + \sqrt{1 + \kappa^2})]$. For small energies the travel time increases logarithmically $\tau(E) \approx \Omega^{-1}\ln(4Q\hbar \Omega/E)$ within the interval $\hbar \Omega < E < Q\hbar \Omega$ and saturates at $\tau_\Omega \approx \Omega^{-1}\ln(4Q)$ as $E$ drops below $\hbar \Omega$, a result going beyond the quasi-classical approximation. At $\varphi = 0$, the pairwise degenerate levels produce equal currents of opposite sign and the sum over the discrete spectrum gives no current. Increasing $\varphi$, the degeneracy is lifted and the resulting miscancellation leads to a finite supercurrent. Each pair produces a monotonously growing current of the same sign, hence the largest current is reached at $\varphi = \pi^-$. However, at $\varphi = \pi^-$ the levels become degenerate again and their currents cancel pairwise, except for the lowest level which remains unpaired and thus carries all the supercurrent from the discrete part of the spectrum. The continuous part of the spectrum vanishes at $\varphi = \pi$, however, this is a priori not sufficient to guarantee that the critical current $I_c$ is the current $I_0$ carried by the lowest level — we have to show in addition that the maximum of $I = I_{\text{dis}} + I_{\text{con}}$ is reached at $\varphi = \pi^-$ [indeed, we could prove that this condition is fulfilled within a regime of the $L/\mu_{x,0}$ plane away from $(L \sim (\xi_0/k_F)^{1/2}, \mu_{x,0} \sim \Delta)$]. In the end, we arrive at a particularly simple expression for the critical current density in the quasi-classical region I,

$$I_c = e/\tau_0 + \hbar \Omega. \tag{2}$$

The travel time $\tau_0$ is constant $(\Omega^{-1}\ln4Q)$ at the opening of the channel, decreases as $\Omega^{-1}\ln4Q\hbar \Omega/\mu_{x,0}$ for $\mu_{x,0} > \hbar \Omega$ and transforms to the free travel time $L/\nu\varepsilon$ for $\mu_{x,0} > Q\hbar \Omega$. As the channel becomes wide open at high energies, the critical current saturates to the expected value $I_c = ev_F/\Omega(L/\pi\xi_0)$. The above discussion dealt with the parameter settings $\sqrt{\varepsilon_\nu \varepsilon_\kappa} < \hbar \Omega < \Delta$ requiring a junction with $L > \xi_0$. Releasing the condition of small curvature and assuming $\Delta < \hbar \Omega$, the SNS to SIS transformation is smeared and
the region IV occupies all of the interesting crossover regime. For a flat potential $h\Omega < \sqrt{\varepsilon_F} \Delta$, the situation is complicated by the appearance of resonances due to reflection from the potential step at the NS boundary [15]. The situation simplifies for a very short junction with $L \ll \xi_0$, where we can again make use of [14] to produce a simple and universal result [17]; with $\delta S = S_+ - S_- \approx 0$ and $t_+ \approx \sqrt{T}$, $r_- \approx r_+ \approx \sqrt{R}$, we find that only one level remains trapped in the junction at $\varepsilon_0 = \Delta(1 - T\sin^2(\varphi/2))^{1/2}$. Here, we require a width $h\Omega > \Delta$ in order to avoid a strong energy dependence in the transmission probability $T$. Determining the current $I_0(\varphi)$ from $\varepsilon_0$ and maximizing, we obtain

$$I_c = (e\Delta/h)(1 - \sqrt{R}),$$

in marked difference from the result for the conductance quantization $G = (2e^2/h)(1 - R)$: a finite reflection $0 < R \ll 1$ will affect the supercurrent quantization much more strongly than the conductance quantization.

Finally, we discuss the supercurrent quantization ‘steps’ appearing as the gate potential is decreased to $\varepsilon_F$ (i.e., $L/\xi \sim 1, 10$; with the curvature $h\Omega/\Delta < 5$, 0.5 no smearing is visible at the supercurrent onset).

$$d_1 = \pi/k_F,$$

the critical current increases sharply $I_c \approx e\Omega/[4(\Delta^2 / \pi^2 - 1)]$ (the logarithmic singularity is cutoff at $h\Omega$) and saturates at $I_c \approx e\nu_L/(L + \pi\xi_0)$, see Fig. 3. Here, we have ignored the smearing near the on-set within the range $h\Omega$ due to a finite reflection — while the interesting evolution of the quasi-particle spectrum is washed out as $h\Omega$ increases beyond the gap $\Delta$, see Fig. 2, the (smooth) steps in the onset of the supercurrent are much more robust. On the other hand, the absence of sharp steps in the critical current onset is an intrinsic feature of the superconducting junction. For the short junction with $L \ll \xi$ the sharpness of the steps in $I_c$ is dictated by the reflection probability $R$ of the junction and thus is more similar to the steps in the conductance $G$. However, with $I_c \propto 1 - \sqrt{R}$, the steps in $I_c$ are always smoother than those in the conductance $G \propto (1 - R)$.

In the end, universal supercurrent quantization first seems to require short junctions, but the gate needed to switch the channels will produce backscattering and spoil the quantization. While going over to longer contacts helps to produce sharp conductance steps, the onset of supercurrent remains smooth due to the long travelling time for the Andreev states.

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