On the Schrödinger equation with singular potentials

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Abstract
We study the Cauchy problem for the non-linear Schrödinger equation with singular potentials. For point-mass potential and nonperiodic case, we prove existence and asymptotic stability of global solutions in weak-\(L^p\) spaces. Specific interest is given to the point-like \(\delta\) and \(\delta'\) impurity and for two \(\delta\)-interactions in one dimension. We also consider the periodic case which is analyzed in a functional space based on Fourier transform and local-in-time well-posedness is proved.

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1 Introduction

We are interested in this paper in the the Cauchy problem for the following Schrödinger model

\[
\begin{cases}
i \partial_t u + \Delta u + \mu(x)u = F(u), & x \in \mathbb{R}^n, \ t \in \mathbb{R} \\
u(x, 0) = u_0(x),
\end{cases}
\]  

(1.1)
in weak-\(L^p\) spaces (Marcinkiewicz spaces) and in a space based on Fourier transform. In the weak-\(L^p\) spaces we consider the case \(n = 1\) and \(\mu(x) = \sigma \delta, \mu(x) = \sigma (\delta(-a) + \delta(+a))\) (two Dirac’s \(\delta\) potentials place at the points \(\pm a \in \mathbb{R}\)) or \(\mu(x) = \sigma \delta'\) where \(\delta\) represents the delta function in the origin and \(\sigma \in \mathbb{R}, \ F(u) = \lambda |u|^\rho - 1 u, \) where \(\lambda = \pm 1\) and \(\rho > 1\). In the space based on Fourier transform we consider \(n\) arbitrary and \(\mu(x)\) being a bounded continuous function with a Fourier transform being a finite Radon measure and \(F(u) = \lambda u^\rho, \) where \(\lambda = \pm 1\) and \(\rho \in \mathbb{N}.\) The case \(F(u) = \lambda |u|^\rho - 1 u\) is also commented.

The non-linear Schrödinger model (1.1) in the case \(\mu(x) = \sigma \delta(x)\) (called the non-linear Schrödinger equation with a \(\delta\)-type impurity, the NLS-\(\delta\) equation henceforth) arise in
different areas of quantum field theory and are essential for understanding a number of phenomena in condensed matter physics. At the experimental side, the recent interest in point-like impurities (defects) is triggered by the great progress in building nanoscale devices. More exactly, the NLS-$\delta$ model with an impurity at the origin in the repulsive ($\sigma < 0$) case and in the attractive ($\sigma > 0$) is described by the following boundary problem (see Caudrelier&Mintchev&Ragoucy [9])

$$
\begin{align*}
&i\partial_t u(x,t) + u_{xx}(x,t) = \lambda |u(x,t)|^{\rho-1} u(x,t), \quad x \neq 0 \\
&\lim_{x \to 0^+} [u(x,t) - u(-x,t)] = 0, \\
&\lim_{x \to 0^+} [\partial_x u(x,t) - \partial_x u(-x,t)] = \sigma u(0,t) \\
&\lim_{x \to \pm\infty} u(x,t) = 0,
\end{align*}
$$

hence $u(x,t)$ must be solution of the non-linear Schrödinger equation on $\mathbb{R}^-$ and $\mathbb{R}^+$, continuous at $x = 0$ and satisfy a “jump condition” at the origin and it also vanishes at infinity.

The equations in (1.2) are a particular case of a more general model considering that the impurity is localized at $x = 0$; in fact the equation of motion

$$
i\partial_t u(x,t) + u_{xx}(x,t) = \lambda |u(x,t)|^{\rho-1} u(x,t), \quad x \neq 0,$$

with the impurity boundary conditions

$$
\begin{pmatrix}
u(0+,t) \\
\partial_x u(0+,t)
\end{pmatrix}
= \alpha
\begin{pmatrix}a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}u(0-,t) \\
\partial_x u(0-,t)
\end{pmatrix}
$$

with

$$
\{a, b, c, d \in \mathbb{R}, \alpha \in \mathbb{C} : ad - bc = 1, |\alpha| = 1\}.
$$

(1.3)

The equation (1.3) captures the interaction of the “field” $u$ with the impurity [8]. The parameters in (1.4) label the self-adjoint extensions of the (closable) symmetric operator $H_0 = -\frac{d^2}{dx^2}$ defined on the space $C_0^\infty(\mathbb{R} - \{0\})$ of smooth functions with compact support separated from the origin $x = 0$. In fact, by von Neumann-Krein’s theory of self-adjoint extensions for symmetric operators on Hilbert spaces, it is not difficult to show that there is a 4-parameter family of self-adjoint operators which describes all one point interactions in one-dimension of the second derivative operator $H_0$. Such a family can be equivalently described through the family of boundary conditions at the origin

$$
\begin{pmatrix}
\psi(0+) \\
\psi'(0+)
\end{pmatrix}
= \alpha
\begin{pmatrix}a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}\psi(0-) \\
\psi'(0-)
\end{pmatrix}
$$

with $a, b, c, d$ and $\alpha$ satisfying the conditions in (1.3) (see Theorem 3.2.3 in [4]).

Here we are interested in two specific choices of the parameters in (1.4), which are relevant in physics applications (see [8]-[9]). The first choice $\alpha = a = d = 1, b = 0, c = \sigma \neq 0$ corresponds to the case of a pure Dirac $\delta$ interaction of strength $\sigma$ (see Theorem
The second one $\alpha = a = d = 1$, $c = 0$, $b = \beta \neq 0$ corresponds to the case of the so-called $\delta'$ interaction of strength $\beta$ (see Theorem 4.1 below).

In section 2 below for convenience of the reader we present a precise formulation for the point interaction determined by the formal linear differential operator

$$-\Delta_{\varphi} = -\frac{d^2}{dx^2} + \sigma \delta,$$

which will be match with the singular boundary condition in (2.2) at every time $t$.

Existence and uniqueness of local and global-in-time solutions of problem (1.1) with $\mu(x) = 0$ and $F(u) = \lambda |u|^\rho - 1 u$ have been much studied in the framework of the Sobolev spaces $H^s(\mathbb{R}^n)$, $s \geq 0$, i.e., the solutions and their derivatives have finite energy (see Cazenave’s book [10] and the reference therein). In the case of $\delta$-interaction, namely, $\mu(x) = \sigma \delta$ the existence of global solution in $H^1(\mathbb{R})$ and $L^2(\mathbb{R})$ has been addressed in Adami & Noja [1] (we can also to apply Theorem 3.7.1 in [10] for obtaining a local-in-time well-posedness theory in $H^1(\mathbb{R})$).

The first study of infinite $L^2$-norm solutions for $\mu(x) = 0$ and $F(u) = \lambda |u|^\rho - 1 u$ was addressed by Cazenave & Weissler in [12] where they consider the space

$$X_\rho = \{ u : \mathbb{R} \to L^{\rho+1}(\mathbb{R}^n) \ \text{Bochner meas.} : \sup_{-\infty < t < \infty} |t|^\vartheta \| u(t) \|_{L^{\rho+1}} < \infty \},$$

where $\vartheta = \frac{1}{\rho - 1} - \frac{n}{2(\rho + 1)}$ and $\| \cdot \|_{L^{\rho+1}}$ denotes the usual $L^{\rho+1}$ norm. Under a suitable smallness condition on the initial data, they prove the existence of global solution in $X_\rho$, for $\rho_0(n) < \rho < \gamma(n)$ where $\rho_0(n) = \frac{n + 2 + \sqrt{n^2 + 4n + 4}}{2n}$ is the positive root of the equation $n\rho^2 - (n + 2)\rho - 2 = 0$ and $\gamma(n) = \infty$ if $n = 1, 2$ and $\gamma(n) = \frac{n + 2}{n - 2}$ in otherwise.

Later on, in Cazenave & Vega & Vilela [11] the Cauchy problem was studied in the framework of weak-$L^p$ spaces. Using a Strichartz-type inequality, the authors obtained existence of solutions in the class $L^{(p, \infty)} (\mathbb{R}^{n+1}) \equiv L^{(p, \infty)} (L^{(p, \infty)} (\mathbb{R}^n))$, where $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ and $p = \frac{n(\rho - 1) + n + 2}{2\rho}$, for $\rho$ in the range

$$\rho_0 < \frac{4(n + 1)}{n(n + 2)} < \rho - 1 < \frac{4(n + 1)}{n^2} < \frac{4}{n - 2}.$$

More recently, in Braz e Silva & Ferreira & Villamizar-Roa [7] the Cauchy problem was studied in the Marcinkiewicz space $L^{(p+1, \infty)}$. Using bounds for the Schrödinger linear group in the context of Lorentz spaces, the authors showed existence and uniqueness of local-in-time solutions in the class

$$\{ u : \mathbb{R} \to L^{(p+1, \infty)} \ \text{Bochner meas.} : \sup_{-T < t < T} |t|^\vartheta \| u(t) \|_{L^{(p+1, \infty)}} < \infty \},$$

where $1 < \rho < \rho_0(n)$ and $\frac{n(\rho - 1)}{2(\rho + 1)} = \zeta_0 < \zeta < \frac{1}{\rho}$. Since $\rho_0(n) < \frac{4}{n}$, the range for $\rho$ is different from the ones in Cazenave & Weissler [12] and Cazenave et al. [11]. The existence of global solutions is showed in norms of type $\sup_{|t| > 0} |t|^\vartheta \| u(t) \|_{L^{(p+1, \infty)}}$, where $\vartheta = \frac{1}{\rho - 1} - \frac{n}{2(\rho + 1)}$ and $\rho_0(n) < \rho < \gamma(n)$. 

3
Our approach is based in some ideas in [7], so via real interpolation techniques we establish bounds for the Schrödinger linear group \( G_\sigma(t) = e^{i(\partial_x^2 + \sigma^2)t} \) in the context of Lorentz spaces in the one-dimensional case. The cases \( n = 2, 3 \) remain open. The fundamental solution of the corresponding linear time-dependent Schrödinger equation, namely

\[
iu_t = -(\Delta + \sigma\partial_x)u,
\]

is now well known for \( n = 1, 2, 3 \); see Albeverio et al. [3] for instance. However, surprisingly, a “good formula” of the unitary group \( G_\sigma(t)\phi = e^{i(\Delta + \sigma^2)t}\phi \) depending of the free linear propagator \( e^{i\Delta t}\phi \) was found explicitly only for the one-dimensional case (see Holmer et al. [20]-[19]). In fact, by using scattering techniques, it was established in [20] the convenient formula (for the case \( \sigma \geq 0 \))

\[
G_\sigma(t)\phi(x) = e^{i\Delta t}(\phi * \tau_\sigma)(x)\chi_+^0 + \left[e^{it\Delta}\phi(x) + e^{it\Delta}(\phi * \rho_\sigma)(-x)\right]\chi_-^0
\]

(1.7)

where

\[
\rho_\sigma(x) = -\frac{\sigma}{2}e^{\frac{\sigma x}{2}}\chi_0^- \quad \tau_\sigma(x) = \delta(x) + \rho_\sigma(x),
\]

with \( \chi_+^0 \) the characteristic function of \([0, +\infty)\) and \( \chi_-^0 \) the characteristic function of \((-\infty, 0]\). For the case \( \sigma < 0 \) see [19] and Theorem 4.3 below. Here we show in the Appendix how to obtain the formula (1.7) based in the fundamental solution found in Albeverio et al. [3], which does not use scattering ideas. Nice formulas as that in (1.7) are not known for the cases \( n = 2, 3 \). Now, in the case \( \sigma \geq 0 \), one can show from (1.7) the dispersive estimate in Lorentz spaces (see Lemma 5.1 below)

\[
\|G_\sigma(t)f\|_{(\rho', d)} \leq C|t|^{-\frac{1}{2}(\frac{2}{p} - 1)}\|f\|_{(\rho, d)},
\]

(1.8)

for \( 1 \leq d \leq \infty, \rho' \in (2, \infty), \rho \in (1, 2) \) and \( \rho' \) satisfying \( \frac{1}{\rho} + \frac{1}{\rho'} = 1 \), where \( C > 0 \) is independent of \( f \) and \( t \neq 0 \). Then, under a suitable smallness condition on the initial data \( u_0 \), the existence of global solutions for (1.1) is proved in the space (see Theorem 5.3)

\[
\mathcal{L}^\infty_\sigma = \{u : \mathbb{R} \rightarrow L^{(\rho+1, \infty)} \text{ Bochner meas. : } \sup_{-\infty < t < \infty} |t|^{\vartheta} \|u\|_{(\rho+1, \infty)} < \infty \},
\]

for \( \sigma \geq 0 \), where \( \vartheta = \frac{1}{\rho-1} - \frac{1}{2(\rho+1)} \) and \( \rho_0 = \frac{3\rho + \sqrt{9\rho^2 - 12}}{2} > 1 \) is the positive root of the equation \( \rho^2 - 3\rho - 2 = 0 \). We also analyze the asymptotic stability of the global solutions (see Theorem 5.4). For \( \sigma < 0 \) our approach in general is not applicable because in this case the operator \(-\Delta_\sigma\) has a non-trivial negative point spectrum. But, in this case it is possible to show the existence of an invariant manifold of periodic orbits in Lorentz spaces (see Section 6).

With regard to the more two singular cases: two Dirac’s \( \delta \) potentials placed at the points \( \pm a \in \mathbb{R}, \mu(x) = \alpha(\delta(\cdot - a) + \delta(\cdot + a)), \) and \( \mu(x) = \beta\delta' \), i.e., the derivative of a \( \delta \), a similar analysis to that above for the case of a \( \delta \)-potential can be established. In these cases, it is not known an explicit expression for the associated time propagator as that in (1.7) for the case of \( \mu(x) = \sigma\delta \). However, by using a formula for the integral kernel of the time propagator associated (see [24] and [2]), we obtain an estimate similar to (1.8).
For the case $n \geq 4$ we do not have Schrödinger operators with point interactions. In fact, the Schrödinger operators with point interactions, namely, perturbations of the Laplace operators by “measures” supported on a discrete set (supported at zero for simplicity, namely, by the Dirac delta measure $\delta$ centered at zero) are usually defined by means of von Neumann–Krein theory of self-adjoint extensions of symmetric operators, and so as one of a whole family of self-adjoint (in $L^2(\mathbb{R}^n)$) extensions of an operator $A$, $D(A) = C_0^\infty(\mathbb{R}^n - \{0\})$, $Au = -\Delta u$, $u \in D(A)$. In the case $n \geq 4$ it is well known that the theory trivializes where there is only one self-adjoint extension of $A$ (see Albeverio et al. [2]).

Next, let $\mathcal{M}$ be the set of finite Radon measure endowed with the norm of total variation, that is, $\|\omega\|_{\mathcal{M}} = |\omega|(\mathbb{R}^n)$ for $\omega \in \mathcal{M}$, $n \geq 1$. Then, by considering $\mu(x)$ in (1.1) being a bounded continuous function with a Fourier transform such that $\hat{\mu} \in \mathcal{M}(\mathbb{R}^n)$, we show a local-in-time well-posedness result in the Banach space

$$I = [\mathcal{M}(\mathbb{R}^n)]^\vee = \{f \in S'(\mathbb{R}^n) : \hat{f} \in \mathcal{M}(\mathbb{R}^n)\}$$

whose norm is given by $\|f\|_I = \|\hat{f}\|_{\mathcal{M}}$. We also obtain a similar result in the periodic case (see Section 7).

2 The one-center $\delta$-interaction in one dimension

In this subsection for convenience of the reader we establish initially a precise formulation for the point interaction determined by the formal linear differential operator

$$-\Delta_\sigma = -\frac{d^2}{dx^2} + \sigma \delta,$$

defined on functions on the real line. The parameter $\sigma$ represents the coupling constant or strength attached to the point source located at $x = 0$. We note that there are many approaches for studying the operator in (2.1), for instance, by the use of quadratic forms or by the self-adjoint extensions of symmetric operators. We also note that the quantum mechanics model in (2.1) has been studied into a more general framework when it is associated with the Kronig-Penney model in solid state physics (see Chapter III.2 in Albeverio et al. [2]) or when it is associated to singular rank one perturbations (Albeverio et al. [4]).

By following [4], we consider the operator $A = -\frac{d^2}{dx^2}$ with domain $D(A) = H^2(\mathbb{R})$ and the (closeable) symmetric restriction $A^0 \equiv A|_{D(A^0)}$ with dense domain $D(A^0) = \{\psi \in D(A) : (\delta, \psi) \equiv \psi(0) = 0\}$. Then we obtain that the deficiency subspaces of $A^0$,

$$D_+ = \text{Ker}(A^0* - i), \quad \text{and} \quad D_- = \text{Ker}(A^0* + i),$$

have dimension (deficiency indexes) equal to 1. It is no difficult to see that these subspaces are generated, respectively, by $g_+ i \equiv (A - i)^{-1} \delta$ and $g_- i \equiv (A + i)^{-1} \delta$, called deficiency elements and given explicitly by (see [4]),

$$g_{\pm i}(x) = \frac{i}{2\sqrt{\pm i}} e^{i\sqrt{\pm i}|x|}, \quad \text{Im} \sqrt{\pm i} > 0.$$

5
We note that the Fourier transform of $g_{\pm i}$ are given by $
abla g_{\pm i}(\xi) = \frac{1}{\xi^2 + 1}$.

Next we present explicitly all the self-adjoint extensions of the symmetric operator $A^0$, which will be parameterized by the strength $\sigma$. By normalizing the deficiency elements $\nabla g_{\pm i} = \frac{g_{\pm i}}{\|g_{\pm i}\|}$ and for convenience of notation we will continue to use $g_{\pm i}$, we have from the von Neumann's theory of self-adjoint extensions for symmetric operators (see [26]) that all the closed symmetric extensions of $A^0$ are self-adjoint and coincides with the restriction of the operator $A^0*$. Moreover, for $\theta \in [0, 2\pi)$ the self-adjoint extension $A^0(\theta)$ of $A^0$ is defined as follows:

\[
\begin{cases}
D(A^0(\theta)) = \{\psi + \lambda g + \lambda e^{i\theta} g_{-i} : \psi \in D(A^0), \lambda \in \mathbb{C}\}, \\
A^0(\theta)(\psi + \lambda g + \lambda e^{i\theta} g_{-i}) = A^0* (\psi + \lambda g + \lambda e^{i\theta} g_{-i}) = A^0\psi + i\lambda g - i\lambda e^{i\theta} g_{-i}. 
\end{cases}
\]

(2.4)

Thus from (2.4) and (2.3) we obtain that for $\zeta \in D(A^0(\theta))$, in the form $\zeta = \psi + \lambda g + \lambda e^{i\theta} g_{-i}$, we have the basic expression

\[
\zeta'(0+) - \zeta'(0-) = -\lambda(1 + e^{i\theta}).
\]

(2.5)

Next we find $\sigma$ such that $\sigma\zeta(0) = -\lambda(1 + e^{i\theta})$. Indeed, $\sigma$ is given by the formula

\[
\sigma(\theta) = \frac{-2\cos(\theta/2)}{\cos(\frac{\theta}{2} - \frac{\pi}{4})}.
\]

(2.6)

So, from now on we parameterize all self-adjoint extensions of $A^0$ with the help of $\sigma$. Thus we get:

**Theorem 2.1.** All self-adjoint extensions of $A^0$ are given for $-\infty < \sigma \leq +\infty$ by

\[
\begin{cases}
-\Delta_\sigma = -\frac{d^2}{dx^2} \\
D(-\Delta_\sigma) = \{\zeta \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} - \{0\}) : \zeta'(0+) - \zeta'(0-) = \sigma\zeta(0)\}.
\end{cases}
\]

(2.7)

The special case $\sigma = 0$ just leads to the operator $-\Delta$ in $L^2(\mathbb{R})$,

\[
-\Delta = -\frac{d^2}{dx^2}, \quad D(-\Delta) = H^2(\mathbb{R}),
\]

(2.8)

whereas the case $\sigma = +\infty$ yields a Dirichlet boundary condition at zero,

\[
D(-\Delta_{+\infty}) = \{\zeta \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} - \{0\}) : \zeta(0) = 0\}.
\]

(2.9)

**Proof.** By the arguments sketched above we obtain easily that $A^0(\theta) \subset -\Delta_\sigma$, with $\sigma = \sigma(\theta)$ given in (2.6). But $-\Delta_\sigma$ is symmetric in the corresponding domain $D(-\Delta_\sigma)$ for all $-\infty < \sigma \leq +\infty$, which implies the relation $A^0(\theta) \subset -\Delta_\sigma \subset (-\Delta_\sigma)^* \subset A^0(\theta)$. It completes the proof of the Theorem.

Next, we recall the basic spectral properties of $-\Delta_\sigma$ which will be relevant for our results (see [2]).
Theorem 2.2. Let $-\infty < \sigma \leq +\infty$. Then the essential spectrum of $-\Delta_\sigma$ is the nonnegative real axis, $\sigma_{ess}(-\Delta_\sigma) = [0, +\infty)$.

If $-\infty < \sigma < 0$, $-\Delta_\sigma$ has exactly one negative, simple eigenvalue, i.e., its discrete spectrum $\sigma_{dis}(-\Delta_\sigma)$ is $\sigma_{dis}(-\Delta_\sigma) = \{-\sigma^2/4\}$, with a strictly (normalized) eigenfunction

$$\Psi_\sigma(x) = \sqrt{-\sigma^2/2} e^\sigma|x|.$$ 

If $\sigma \geq 0$ or $\sigma = +\infty$, $-\Delta_\sigma$ has not discrete spectrum, $\sigma_{dis}(-\Delta_\sigma) = \emptyset$.

3 Two symmetric $\delta$-interaction in one dimension

The one-dimensional Schrödinger operator with two symmetric delta interactions of strength $\alpha$ and placed at the point $\pm a$ is given formally by the linear differential operator

$$-\Delta_\alpha = -\frac{d^2}{dx^2} + \alpha(\delta(\cdot - a) + \delta(\cdot + a)), \quad (3.1)$$

defined on functions on the real line. By using the same notations as in last section, the symmetric operator $A^1 = A|_{D(A^1)}$ with dense domain

$$D(A^1) = \{\psi \in H^2(\mathbb{R}) : \psi(\pm a) = 0\},$$
has deficiency indices $(2, 2)$, and so from the Von Neumann-Krein theory we have that all self-adjoint extensions of $A^1$ are given by a four-parameter family of self-adjoint operators. Here we restrict to the case of so-called separated boundary conditions at each point $\pm a$. More specifically, we have the following theorem (see [3]).

Theorem 3.1. There is a family of self-adjoint extensions of $A^1$ given for $-\infty < \alpha \leq +\infty$ by

$$\begin{cases} 
-\Delta_\alpha = -\frac{d^2}{dx^2} + \alpha(\delta(\cdot - a) + \delta(\cdot + a)), \\
D(-\Delta_\alpha) = \{\zeta \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} - \{\pm a\}) : \zeta'(\pm a+) - \zeta'(\pm a-) = \alpha \zeta(\pm a)\},
\end{cases} \quad (3.2)$$

The special case $\alpha = 0$ just leads to the operator $-\Delta$ in $L^2(\mathbb{R})$,

$$-\Delta = -\frac{d^2}{dx^2}, \quad D(-\Delta) = H^2(\mathbb{R}), \quad (3.3)$$

whereas the case $\alpha = +\infty$ yields a Dirichlet boundary condition at the point $\pm a$,

$$D(-\Delta_{+\infty}) = \{\zeta \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} - \{\pm a\}) : \zeta'(\pm a+) = \zeta'(\pm a-) = 0\}. \quad (3.4)$$

Next, we establish the basic spectral properties of $-\Delta_\alpha$ which will be relevant for our results (see [2]).
Theorem 3.2. Let $-\infty < \alpha \leq +\infty$. Then the essential spectrum of $-\Delta_\alpha$ is the nonnegative real axis, $\Sigma_{\text{ess}}(-\Delta_\alpha) = [0, +\infty)$.

1) If $-\infty < \alpha < 0$, then the discrete spectrum of $-\Delta_\alpha$, $\Sigma_{\text{dis}}(-\Delta_\alpha)$, consists of negative eigenvalues $\gamma$ given by the implicit equation
\[
(-2i\eta + \alpha)^2 = \alpha^2 e^{4i\alpha}, \quad \eta = \sqrt{\gamma}, \quad \text{Im} \eta > 0.
\]
Moreover, we have that:

1) if $a \leq \frac{1}{\alpha}$, then $\Sigma_{\text{dis}}(-\Delta_\alpha) = \{\gamma_1(a, \alpha)\}$, where $\gamma_1(a, \alpha)$ is defined by
\[
\gamma_1(a, \alpha) = -\frac{1}{4a^2} [W(-a\alpha e^{a\alpha}) - a\alpha]^2,
\]
where $W(\cdot)$ is the Lambert special function (or product logarithm) defined by the equation $W(x)e^{W(x)} = x$.

2) if $a > -\frac{1}{\alpha}$, then $\Sigma_{\text{dis}}(-\Delta_\alpha) = \{\gamma_1(a, \alpha), \gamma_2(a, \alpha)\}$, where $\gamma_2(a, \alpha)$ is defined by
\[
\gamma_2(a, \alpha) = -\frac{1}{4a^2} [W(a\alpha e^{a\alpha}) - a\alpha]^2.
\]

II) If $\alpha \geq 0$ or $\alpha = +\infty$, $-\Delta_\alpha$ has not discrete spectrum, $\Sigma_{\text{dis}}(-\Delta_\alpha) = \emptyset$.

4 The $\delta'$-interaction in one dimension

In this subsection for convenience of the reader we establish a precise formulation for the point interaction determined by the formal linear differential operator
\[
-\Delta_\beta = -\frac{d^2}{dx^2} + \beta \delta',
\]
defined on functions on the real line. The parameter $\beta$ represents the coupling constant or strength attached to the point source located at $x = 0$ and $\delta'$ is the derivative of the $\delta$. By following Albeverio et al. [2]-[4], the elements in the domain of the operator $-\Delta_\beta$ are characterized by suitable bilateral singular boundary conditions at the singularity (see (1.5)), while the real true action coincides with the laplacian out the singularity. At variance with the $\delta$ interaction, whose domain is contained in $H^1(\mathbb{R}) \cap H^2(\mathbb{R} - \{0\})$ (in particular in a continuous function set, see (3.2)), the latter has a domain contained only in $H^2(\mathbb{R} - \{0\})$ and so by allowing discontinuities of the elements at the position of the defect. More precisely, for $A^2 = A|_{D(A^2)}$ being considered with dense domain
\[
D(A^2) = \{\psi \in H^2(\mathbb{R}) : \psi(0) = \psi'(0) = 0\}.
\]
$A^2$ has deficiency indices $(2, 2)$ and hence it has a four-parameter family of self-adjoint. We are interested in the following one-parameter family of self-adjoint extensions (see [2]-[4]).
Theorem 4.1. There is a family of self-adjoint extensions of $A^2$ given for $-\infty < \beta \leq +\infty$ by

$$
\begin{cases}
-\Delta_\beta = -\frac{d^2}{dx^2} \\
D(-\Delta_\beta) = \{ \zeta \in H^2(\mathbb{R} - \{0\}) : \zeta'(0+) = \zeta'(0-) = \zeta(0-) = \beta \zeta'(0-) \},
\end{cases}
$$

(4.2)

The special case $\beta = 0$ just leads to the operator $-\Delta$ in $L^2(\mathbb{R})$,

$$
-\Delta = -\frac{d^2}{dx^2}, \quad D(-\Delta) = H^2(\mathbb{R}),
$$

(4.3)

whereas the case $\beta = +\infty$ yields a Neumann boundary condition at zero and decouples $(-\infty, 0)$ and $(0, +\infty)$, i.e.,

$$
D(-\Delta_{+\infty}) = \{ \zeta \in H^2(\mathbb{R} - \{0\}) : \zeta'(0+) = \zeta'(0-) = 0 \}.
$$

(4.4)

Note that the functions in the domain of $\delta'$ have a jump at the origin, and the left and right derivatives coincide. Next, we establish the basic spectral properties of $-\Delta_\beta$ which will be relevant for our results (see [2]).

Theorem 4.2. Let $-\infty < \beta \leq +\infty$. Then the essential spectrum of $-\Delta_\beta$ is the non-negative real axis, $\Sigma_{ess}(-\Delta_\beta) = [0, +\infty)$.

If $-\infty < \beta < 0$, $-\Delta_\beta$ has exactly one negative simple eigenvalue, i.e., its discrete spectrum $\Sigma_{dis}(-\Delta_\beta)$ is $\Sigma_{dis}(-\Delta_\beta) = \{-\frac{4}{\beta^2}\}$, with a (normalized) eigenfunction

$$
\Phi_\beta(x) = \left( -\frac{2}{\beta} \right)^\frac{1}{2} \text{sign}(x) e^{\frac{2}{\beta}|x|}.
$$

If $\beta \geq 0$ or $\beta = +\infty$, $-\Delta_\beta$ has not discrete spectrum, $\Sigma_{dis}(-\Delta_\beta) = \emptyset$.

4.1 The linear propagator

4.1.1 The case $\mu(x) = \sigma \delta$.

Next we determine the linear propagator $G_\sigma = e^{i(\Delta + \sigma \delta)t}$ (unitary group) determined by the linear system associated with (1.1),

$$
\begin{cases}
i\partial_t u = -(\Delta u + \sigma \delta) u \equiv H_\sigma u, \\
u(0) = u_0,
\end{cases}
$$

(4.5)

where we are using the notation $H_\sigma = -\Delta_{-\sigma}$.

We will use the representation of the propagator in terms of the eigenfunctions (associated to discrete eigenvalues) and generalized eigenfunctions (see Iorio [21], Holmer et al. [20] and Duchêne et al. [13]). Indeed, the family of generalized eigenfunctions $\{\psi_\lambda\}_{\lambda \in \mathbb{R}}$ will be such that satisfy

$$
\begin{cases}
H_\sigma \psi_\lambda = \lambda^2 \psi_\lambda, \quad \psi_\lambda \text{ continuous and} \\
\psi'_\lambda(0+) - \psi'_\lambda(0-) = \sigma \psi_\lambda(0).
\end{cases}
$$

(4.6)
Hence we obtain the following family of special solutions, $e_{\pm}(x, \lambda)$ to (4.6), as follows

$$e_{\pm}(x, \lambda) = t_\sigma(\lambda)e^{\pm i\lambda x} + (e^{\pm i\lambda x} + r_\sigma(\lambda)e^{\mp i\lambda x})\chi^0_\pm,$$

where $\chi^0_+$ is the characteristic function of $[0, +\infty)$ and $\chi^0_-$ is the characteristic function of $(-\infty, 0]$. $t_\sigma$ and $r_\sigma$ are the transmission and reflection coefficients:

$$t_\sigma(\lambda) = \frac{2i\lambda}{2i\lambda - \sigma}, \quad r_\sigma(\lambda) = \frac{\sigma}{2i\lambda - \sigma}. \quad (4.8)$$

They satisfy the following two equations:

$$|t_\sigma(\lambda)|^2 + |r_\sigma(\lambda)|^2 = 1, \quad r_\sigma(\lambda) + 1 = t_\sigma(\lambda). \quad (4.9)$$

Next, by defining the family $\{\psi_\lambda\}_{\lambda \in \mathbb{R}}$ as

$$\psi_\lambda(x) = \begin{cases} e_+(x, \lambda) & \text{for } \lambda \geq 0 \\ e_-(x, -\lambda) & \text{for } \lambda < 0 \end{cases}$$

we obtain from Theorem [2.2] the following relations (see [13], [21]):

1) $\int_{\mathbb{R}} \overline{\Psi_\sigma(x)\psi_\lambda(x)}dx = 0$, for all $\lambda \in \mathbb{R}$ and $\sigma < 0$,

2) $\int_{\mathbb{R}} \overline{\psi_\mu(x)\psi_\lambda(x)}dx = \delta(\lambda - \mu)$, for all $\mu, \lambda \in \mathbb{R}$,

3) $\Psi_\sigma(x)\Psi_\sigma(y) + \int_{\mathbb{R}} \overline{\psi_\lambda(x)\psi_\lambda(y)}d\lambda = \delta(x - y), \quad \lambda \in \mathbb{R}, \sigma < 0$.

We recall that the relation 3) above, called the completeness relations, in the case $\sigma > 0$ is reads as $\int_{\mathbb{R}} \overline{\psi_\lambda(x)\psi_\lambda(y)}d\lambda = \delta(x - y)$ (the proof of 3) for the family $\{\psi_\lambda\}_{\lambda \in \mathbb{R}}$ can be showed by following the ideas in the proof of Theorem [4.3] below). Moreover, the family $\{\psi_\lambda\}_{\lambda \in \mathbb{R}}$ allows us to define the generalized Fourier transform

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}} f(x)\overline{\psi_\lambda(x)}dx, \quad (4.10)$$

and its formal adjoint $\mathcal{G}(g)(x) = \int_{\mathbb{R}} \overline{\psi_\lambda(x)}g(\lambda)d\lambda$. Hence, from 2) we obtain immediately that $\mathcal{G}$ is the inverse Fourier transform, namely,

$$f(\lambda) = f * \delta(\lambda) = \int_{\mathbb{R}} \overline{\psi_\lambda(x)}f(\mu)\psi_\mu(x)d\mu dx = \mathcal{F}(\mathcal{G}g)(\lambda).$$

Moreover, from the completeness relations 3) we obtain for every $f \in L^2(\mathbb{R})$ the following (orthogonal) expansion in eigenfunctions of $H_\sigma$,

$$f = \langle f, \Phi_\sigma \rangle \Phi_\sigma + \int_{\mathbb{R}} \mathcal{F}(f)(\lambda)\psi_\lambda(x)d\lambda. \quad (4.11)$$
Thus for $u \in C(\mathbb{R}; L^2(\mathbb{R}))$ being a solution of (4.5), the method of separation of variables implies that

$$u(x, t) = e^{-itH_a} u_0(x) = e^{i\xi^2 t} \langle u_0, \Phi_\sigma \rangle \Phi_\sigma(x) + \int_{\mathbb{R}} e^{-i\lambda^2 t} \mathcal{F}(u_0)(\lambda) \psi_\lambda(x) d\lambda. \quad (4.12)$$

In the next Theorem we describe explicitly the propagator $e^{-itH_a}$ in terms of the free propagator of the Schrödinger equation $e^{it\Delta}$ (see Holmer et al. [20], Datchev & Holmer [19]). In the Appendix we present a different proof based in the fundamental solution associated to (4.5).

**Theorem 4.3.** Suppose that $\phi \in L^1(\mathbb{R})$ with supp $\phi \subset (-\infty, 0]$. Then,

1) Para $\sigma \geq 0$, we have

$$e^{-itH_a} \phi(x) = e^{it\Delta} (\phi * \tau_\sigma)(x)\chi_+^0 + [e^{it\Delta} \phi(x) + e^{it\Delta} (\phi * \rho_\sigma)(-x)]\chi_-^0 \quad (4.13)$$

where

$$\rho_\sigma(x) = -\frac{\sigma}{2} e^{\frac{\sigma^2}{4}} \chi_-^0, \quad \tau_\sigma(x) = \delta(x) + \rho_\sigma(x).$$

2) Para $\sigma < 0$, we have

$$e^{-itH_a} \phi(x) = e^{i\sigma^2 t} P\phi(x) + e^{it\Delta} (\phi * \tau_\sigma)(x)\chi_+^0 + [e^{it\Delta} \phi(x) + e^{it\Delta} (\phi * \rho_\sigma)(-x)]\chi_-^0 \quad (4.14)$$

where

$$\rho_\sigma(x) = \frac{\sigma}{2} e^{\frac{\sigma^2}{4}} \chi_-^0, \quad \tau_\sigma(x) = \delta(x) + \rho_\sigma(x),$$

and $P$ is the $L^2(\mathbb{R})$-orthogonal projection onto the eigenfunction $\Phi_\sigma$, $P\phi = \langle \phi, \Phi_\sigma \rangle \Phi_\sigma$.

**Remark 4.1.** We observe the following:

1) The Fourier transform de $\rho_\sigma$ for every sign of $\sigma$ is given by $\widehat{\rho_\sigma}(\lambda) = r_\sigma(\lambda)$ and so $\widehat{\tau_\sigma}(\lambda) = 1 + r_\sigma(\lambda) = t_\sigma(\lambda)$.

2) Formula (4.13) and (4.14) will allow us to estimate the operator norm of $e^{-itH_a}$ using $e^{it\Delta}$.

**Proof.** We only consider the case $\sigma \geq 0$. From (4.12), without the term of projection, we have from the definition of the family $\{\psi_\lambda\}$ and a change of variable that

$$e^{-itH_a} \phi(x) = \int_{\mathbb{R}} \phi(y) \int_{0}^{\infty} e^{-it\lambda^2} \left(e_+(x, \lambda)e_+(y, \lambda) + e_-(x, \lambda)e_-(y, \lambda)\right) d\lambda dy. \quad (4.15)$$

Next, we compute first

$$\int_{\mathbb{R}} \phi(y)e_+(y, \lambda) dy = \int_{-\infty}^{0} \phi(y)e^{-i\lambda y} dy + r_\sigma(\lambda) \int_{0}^{\infty} \phi(y)e^{i\lambda y} dy = \widehat{\phi}(\lambda) + r_\sigma(-\lambda)\widehat{\phi}(-\lambda),$$

$$\int_{\mathbb{R}} \phi(y)e_-(y, \lambda) dy = t_\sigma(-\lambda)\widehat{\phi}(-\lambda), \quad (4.16)$$
so for \( x > 0 \) we have from (4.15) and the fact \( r_\sigma(-\lambda)t_\sigma(\lambda) + r_\sigma(\lambda)t_\sigma(-\lambda) = 0 \) that

\[
e^{-itH_\sigma}\phi(x) = \int_{\mathbb{R}} e^{-it\lambda^2} t_\sigma(\lambda)\widehat{\phi}(\lambda)e^{i\lambda x} d\lambda = e^{it\Delta}(\tau_\sigma * \phi)(x),
\]

(4.17)

where \( \tau_\sigma(\lambda) = t_\sigma(\lambda) \). Similarly, since \( r_\sigma(-\lambda)r_\sigma(\lambda) + t_\sigma(-\lambda)t_\sigma(\lambda) = 1 \), we have for \( x < 0 \)

\[
e^{-itH_\sigma}\phi(x) = \int_{\mathbb{R}} e^{-it\lambda^2}(\widehat{\phi}(\lambda)e^{i\lambda x} + r_\sigma(\lambda)\widehat{\phi}(\lambda)e^{-i\lambda x})d\lambda = e^{it\Delta}\phi(x) + e^{it\Delta}(\rho_\sigma * \phi)(-x),
\]

(4.18)

where \( \rho_\sigma(\lambda) = r_\sigma(\lambda) \).

\[ \blacksquare \]

4.1.2 The case \( \mu(x) = \alpha(\delta(\cdot - a) + \delta(\cdot + a)) \).

Next we determine the linear propagator \( M_\alpha(t) = e^{-itU_\alpha} \) (unitary group) determined by the linear system associated with (4.14),

\[
\begin{aligned}
  i\partial_t u &= -\Delta u + \alpha(\delta(\cdot - a) + \delta(\cdot + a))u \\
  u(0) &= u_0,
\end{aligned}
\]

(4.19)

where we are using the notation \( U_\alpha = -\Delta_{-\alpha} \).

We will use the fundamental solution \( F_\alpha(x, y; t) \) to the Schrödinger equation (4.26) for obtaining the propagator (unitary group). Then we have the representation

\[
e^{-itU_\alpha}f(x) = \int_{\mathbb{R}} F_\alpha(x, y; t)f(y)dy.
\]

(4.20)

Indeed, from [3] and [24] we have for \( S(x; t) \) denoting the free propagator in \( \mathbb{R} \), i.e.

\[
S(x, t) = \frac{e^{-x^2/4it}}{(4\pi t)^{1/2}}, \quad t > 0
\]

(4.21)

and so \( e^{it\Delta}f(x) = S(x; t) *_x f(x) \), the following expression for \( a\alpha \neq -1 \):

1) For \( \alpha > 0 \)

\[
F_\alpha(x, y; t) = S(x - y; t) - \frac{1}{2\pi i} \int_{\mathbb{R}} e^{-i\xi^2t} \frac{f_\alpha(x, y; \xi)}{(2\xi + i\alpha)^2 + \alpha^2e^{4i\xi\alpha}} d\xi
\]

(4.22)

with \( f_\alpha(x, y; \xi) = \sum_{j=1}^{4} L^j_\alpha(x, y; \xi) \) and

\[
\begin{align*}
L^1_\alpha(x, y; \xi) &= -\alpha(2\xi + i\alpha)e^{i\xi[x+a]}e^{i\xi[y+a]}, \\
L^2_\alpha(x, y; \xi) &= i\alpha^2e^{2i\xi\alpha}e^{i\xi[x+a]}e^{i\xi[y-a]}, \\
L^3_\alpha(x, y; \xi) &= L^1_\alpha(-x, -y; \xi), \\
L^4_\alpha(x, y; \xi) &= L^1_\alpha(-x, -y; \xi).
\end{align*}
\]

(4.23)

(4.24)
2) For $\alpha < 0$,
\[
F_\alpha(x, y; t) = e^{-it\gamma_1} \Gamma_1(x) \Gamma_1(y) + e^{-it\gamma_2} \Gamma_2(x) \Gamma_2(y) + F_{-\alpha}(x, y; t) \tag{4.25}
\]
where $\Gamma_1$ and $\Gamma_2$ are the normalized eigenfunction associated with the eigenvalues $\gamma_1$ and $\gamma_2$.

**Remark 4.2.** We observe the following:

1) The case $a\alpha = -1$ is assumed for technical reasons (see [24]).

2) For $a\alpha = -1$ we obtain that only $\gamma_1$ remains as an eigenvalue in the discrete spectrum.

### 4.1.3 The case $\mu(x) = \beta\delta'$.

Next we determine the linear propagator $J_\beta(t) = e^{i(\Delta + \beta\delta')t}$ (unitary group) determined by the linear system associated with (1.1),
\[
\begin{align*}
  i\partial_t u &= -(\Delta u + \beta\delta')u = K_\beta u, \\
  u(0) &= u_0,
\end{align*}
\tag{4.26}
\]
where we are using the notation $K_\beta = -\Delta_{-\beta}$.

We will use the fundamental solution $S_\beta(x, y; t)$ to the Schrödinger equation (4.26) for obtaining the propagator (unitary group). Then we have the representation
\[
e^{-itK_\beta} f(x) = \int_{\mathbb{R}} S_\beta(x, y; t) f(y) dy. \tag{4.27}
\]

Indeed, from [2] we have for $S(x; t)$ denoting the free propagator in (4.21) the following:

1) For $\beta > 0$
\[
S_\beta(x, y; t) = S(x - y; t) + \text{sgn}(xy)S(|x| + |y|; t) + \frac{2}{\beta} \int_0^\infty \text{sgn}(xy) e^{-\frac{2}{\beta} s} S(s + |x| + |y|; t) ds \tag{4.28}
\]

2) For $\beta < 0$
\[
S_\beta(x, y; t) = S(x - y; t) + \text{sgn}(xy)S(|x| + |y|; t) + e^{\frac{i}{\beta^2} t} \Phi_\beta(x) \Phi_\beta(y) \tag{4.29}
\]
\[
- \frac{2}{\beta} \int_0^\infty \text{sgn}(xy) e^{\frac{2}{\beta} s} S(s - |x| - |y|; t) ds
\]
where $\Phi_\beta$ is defined in Theorem 4.2.
4.1.4 Dispersive Estimates

The following proposition extends the well known estimates for the free propagator $e^{it\Delta}$,  
\[ \|e^{it\Delta} f(t)\|_{p'} \leq C_0 t^{-\frac{1}{2}(\frac{2}{p}-1)} \|f\|_p, \]  
(4.30)

to the the groups $G_\sigma(t) = e^{-itH_\sigma}$, $M_\alpha(t) = e^{-itU_\alpha}$ and $J_\beta(t) = e^{i(\Delta+\beta\sigma)t}$ in the one-dimensional case. We denote by $W_1$ the group $G_\sigma$, $W_2$ the group $M_\alpha$ and by $W_3$ the group $J_\beta$.

**Proposition 4.4.** Let $p \in [1,2]$ and $p'$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then we have:

Suppose that $u(x,t) = W_i(t)f(x)$, $i = 1, 2, 3$, is the solution of the linear equation (4.19), (4.26), respectively. Then:

1) for $\sigma \geq 0$, $\alpha \geq 0$ and $\beta \geq 0$,  
\[ \|W_i(t)f\|_{p'} \leq C |t|^{-\frac{1}{2}(\frac{2}{p}-1)} \|f\|_p, \]  
(4.31)

where $C > 0$ is independent of $f$ and $t \neq 0$.

2) for $\sigma < 0$, $a \leq -\frac{1}{\alpha}$ ($\alpha < 0$) and $\beta < 0$,  
\[ \|W_i(t)f - e^{i\alpha t}P_i f\|_{p'} \leq C |t|^{-\frac{1}{2}(\frac{2}{p}-1)} \|f\|_p, \]  
(4.32)

where $\alpha_1 = \frac{a^2}{4}$, $P_1 f = \langle f, \Psi_\sigma \rangle \Psi_\sigma$, $\alpha_2 = -\gamma_1(a,\alpha)$, $P_2 f = \langle f, \Gamma_1 \rangle \Gamma_1$ and $\alpha_3 = \frac{4}{\beta^2}$, $P_3 f = \langle f, \Phi_\beta \rangle \Phi_\beta$, and $C > 0$ is independent of $f$ and $t \neq 0$.

3) for $\alpha < 0$, $a > -\frac{1}{\alpha}$,  
\[ \left\| W_2(t)f - \sum_{j=1}^{2} e^{-i\gamma_j t} Q_j f \right\|_{p'} \leq C |t|^{-\frac{1}{2}(\frac{2}{p}-1)} \|f\|_p, \]  
(4.33)

where $\gamma_j = \gamma_j(a,\alpha)$, $j = 1, 2$, $Q_1 f = \langle f, \Gamma_1 \rangle \Gamma_1$, $Q_2 f = \langle f, \Gamma_2 \rangle \Gamma_2$, and $C > 0$ is independent of $f$ and $t \neq 0$.

**Proof.** i) We consider $\sigma > 0$. Initially $G_\sigma(t)$ is a unitary group on $L^2(\mathbb{R})$, $\|G_\sigma f\|_2 = \|f\|_2$ for all $t \in \mathbb{R}$. Let $\phi \in L^1(\mathbb{R})$ and $R\phi(x) = \phi(-x)$. Then for $\phi^\pm = \phi R^\mp_\sigma$ we have the decomposition $\phi = \phi^- + R\phi^+$. Hence since sup $\phi^+ \subset (-\infty, 0]$, $RG_\sigma = G_\sigma R$ and $R(f * Rg) = (Rf) * g$ we obtain from the following equality,  
\[ G_\sigma \phi(t) = [e^{it\Delta} \phi^- + e^{it\Delta} (\phi^- * \rho_\sigma)] \chi^-_0 + e^{it\Delta} (\phi^- * \tau_\sigma) \chi^+_0. \]  
(4.34)

\[ + [e^{it\Delta} R\phi^+ + e^{it\Delta} (\phi^+ * \rho_\sigma)] \chi^+_0 + e^{it\Delta} (R\phi^+ * \tau_\sigma) \chi^-_0. \]  
(4.35)

Therefore from (4.30) and applying Young’s inequality we obtain for $t \neq 0$
\[ \|G_\sigma \phi(t)\|_{\infty} \leq \frac{C_0}{\sqrt{|t|}} (\|\phi^-\|_1 + \|\phi^- * \rho_\sigma\|_1 + \|\phi^- * \tau_\sigma\|_1 + \|R\phi^+\|_1 + \|\phi^+ * \rho_\sigma\|_1 + \|R\phi^+ * \tau_\sigma\|_1) \]  
(4.36)

\[ \leq \frac{C_0}{\sqrt{|t|}} (3 + 4\|\rho_\sigma\|_1) \|\phi\|_1 = \frac{C}{\sqrt{|t|}} \|\phi\|_1 \]  
(4.37)
where \( C = C(\sigma) \). By the Riesz-Thorin interpolation theorem we obtain (4.31). The case \( \sigma < 0 \) follows similarly from the expression (4.14).

ii) Let \( \beta > 0 \). From (4.28) we obtain immediate for \( x, y \in \mathbb{R} \) that
\[
|S_\beta(x, y, t)| \leq C|t|^{-\frac{1}{2}}.
\]
So, from (4.27) we have
\[
\|J_\beta(t)\phi\|_\infty \leq \frac{C}{\sqrt{|t|}}\|\phi\|_1.
\]
By following a similar analysis as in the later case we obtain (4.31).

iii) From the representations in (4.14) and (4.29) we get immediately (4.32).

iv) The case of the group \( W_2 \) for \( a\alpha \neq -1 \), it follows of the estimate
\[
\|e^{-itU_\alpha}P_c f\|_\infty \leq Ct^{-1/2}\|f\|_1,
\]
for \( t > 0 \) and \( P_c \) being the spectral projector of \( U_\alpha \) on its continuous spectrum.

5 Weak-\( L^p \) Solutions

In this section we focus our study of global solutions for the Cauchy problem
\[
\begin{aligned}
\left\{
\begin{array}{ll}
    i\partial_t u + \Delta u + \mu(x)u = \lambda |u|^{\rho-1}u, & x \in \mathbb{R}, \ t \in \mathbb{R}, \\
    u(x, 0) = u_0,
\end{array}
\right.
\end{aligned}
\]  
(5.1)

for \( \mu(x) = \sigma \delta \) in the spaces \( L^{(p, \infty)}(\mathbb{R}) \), which are called weak-\( L^p \) or Marcinkiewicz spaces. The cases \( \mu(x) = \alpha(\delta(\cdot - a) + \delta(\cdot + a)) \) and \( \mu(x) = \beta \delta' \) are treatment similarly.

We start by recalling some facts about the weak spaces \( L^{(p, \infty)}(\mathbb{R}) \). For \( 1 < r \leq \infty \), we recall that a measurable function \( f \) defined on \( \mathbb{R} \) belongs to \( L^{(r, \infty)}(\mathbb{R}) \) if the norm
\[
\|f\|_{(r, \infty)} = \sup_{t > 0} t^{\frac{1}{r}} f^{**}(t)
\]
is finite, where
\[
f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \, ds,
\]
and \( f^* \) is the decreasing rearrangement of \( f \) with regard to the Lebesgue measure \( \nu \), namely,
\[
f^*(t) = \inf \{ s > 0 : \nu(\{ x \in \mathbb{R} : |f(x)| > s \}) \leq t \}, \ t > 0,
\]
The space \( L^{(r, \infty)} \) with the norm \( \|f\|_{(r, \infty)} \) is a Banach space. We have the continuous inclusion \( L^r(\mathbb{R}) \subset L^{(r, \infty)}(\mathbb{R}) \). Moreover, the Hölder’s inequality holds true in this framework, namely
\[
\|fg\|_{(r, \infty)} \leq C\|f\|_{(q_1, \infty)}\|g\|_{(q_2, \infty)},
\]
(5.2)
for $1 < q_1, q_2 < \infty$, $\frac{1}{q_1} + \frac{1}{q_2} < 1$ and $\frac{1}{r} = \frac{1}{q_1} + \frac{1}{q_2}$, where $C > 0$ depends only on $r$. Lastly, we have the Lorentz spaces $L^{(p,q)}(\mathbb{R})$ that can be constructed via real interpolation; indeed, $L^{(p,q)}(\mathbb{R}) = (L^1(\mathbb{R}), L^\infty(\mathbb{R}))_{1 / 2, q}$. For $1 < p < \infty$. They have the interpolation property

$$
(L^{(p_0,q_0)}(\mathbb{R}), L^{(p_1,q_1)}(\mathbb{R}))_{\theta,q} = L^{(p,q)}(\mathbb{R}),
$$

provided $0 < p_0 < p_1 < \infty$, $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $1 \leq q_0, q_1, q \leq \infty$, where $(\cdot, \cdot)_{\theta,q}$ stands for the real interpolation spaces constructed via the $K$-method. For further details about weak-$L^p$ and Lorentz spaces see \cite{6} and Grafakos \cite{16}.

From (5.3) we obtain our main estimate for the group $G_\sigma$ in Lorentz spaces. A similar result is obtained for the groups $M_\alpha$ and $J_\beta$.

**Lemma 5.1.** Let $1 \leq d \leq \infty$, $p' \in (2, \infty)$, and $p \in (1, 2)$. If $p'$ satisfies $\frac{1}{p'} + \frac{1}{p'} = 1$, then there exists a constant $C > 0$ such that:

1) for $\sigma \geq 0$,

$$
\|G_\sigma(t)f\|_{(p',d)} \leq C|t|^{-\frac{1}{2}(\frac{2}{p'} - 1)}\|f\|_{(p,d)},
$$

for all $f \in L^{(p,d)}(\mathbb{R})$ and all $t \neq 0$.

2) for $\sigma < 0$,

$$
\left\|G_\sigma(t)f - e^{i\frac{\sigma^2}{4}t}P_1f\right\|_{(p',d)} \leq C|t|^{-\frac{1}{2}(\frac{2}{p'} - 1)}\|f\|_{(p,d)},
$$

for all $f \in L^{(p,d)}(\mathbb{R})$ and all $t \neq 0$.

**Proof.** We only consider the case $\sigma \geq 0$ because for $\sigma < 0$ the analysis is similar. Let fixed $t \neq 0$ and let $1 < p_0 < p < p_1 < 2$. From the $L^p = L^{(p,p)}$ estimate of the Schrödinger group in Proposition 4.4 we have that $G_\sigma(t): L^{p_0} \to L^{p'_0}$ and $G_\sigma(t): L^{p_1} \to L^{p'_1}$ satisfy

$$
\|G_\sigma(t)\|_{p_0 \to p'_0} \leq C|t|^{-\frac{1}{2}(\frac{2}{p_0} - 1)}, \quad \|G_\sigma(t)\|_{p_1 \to p'_1} \leq C|t|^{-\frac{1}{2}(\frac{2}{p_1} - 1)}
$$

with $\frac{1}{p_0} + \frac{1}{p'_0} = 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Hence, for $\lambda \in (0, 1)$, $\frac{1}{p} = \frac{1-\lambda}{p_0} + \frac{\lambda}{p_1}$, and $\frac{1}{p'} = \frac{1-\lambda}{p'_0} + \frac{\lambda}{p'_1}$, we obtain from (5.3) that

$$
\|G_\sigma(t)\|_{(p,d)} \leq \|G_\sigma(t)\|_{p_0 \to p'_0} \|G_\sigma(t)\|_{p_1 \to p'_1} \|G_\sigma(t)\|_{(1-\lambda,p'_0)} \|G_\sigma(t)\|_{(\lambda,p'_1)} \leq C|t|^{-\frac{1}{2}(\frac{2}{p} - 1)},
$$

which gives (5.4).

Next we establish the main results of this section. From now on we focus in the case of $\mu(x) = \sigma\delta$ in (5.1), since for the case of the potential being the two symmetric deltas and the derivative of the Dirac- delta we have a similar analysis. We start by defining $L^\infty_\theta$ as the Banach space of all Bochner measurable functions $u: \mathbb{R} \to L^{(p+1,\infty)}$ endowed with the norm

$$
$$
\[ \|u\|_{L^\infty_\vartheta} = \sup_{-\infty < t < \infty} |t|^{\vartheta} \|u(t)\|_{(\rho+1, \infty)}, \]  

(5.6)

where

\[ \vartheta = \frac{1}{\rho - 1} - \frac{1}{2(\rho + 1)}. \]  

(5.7)

Let us also define the initial data space \( E_0 \) as the set of all \( \varrho \in S'(\mathbb{R}) \) such that the norm

\[ \|\varrho_0\|_{E_0} = \sup_{-\infty < t < \infty} |t|^\vartheta \|G_\sigma(t)\varrho_0\|_{(\rho+1, \infty)} \]

is finite. Throughout this paper we stand for \( \rho_0 = \frac{3+\sqrt{17}}{2} > 1 \) the positive root of the equation \( \rho^2 - 3\rho - 2 = 0 \).

From Duhamel’s principle, (5.1) is formally equivalent to the integral equation

\[ u(t) = G_\sigma(t)\varrho_0 - i\lambda \int_0^t G_\sigma(t-s)\|u(s)|^{\rho-1}u(s)|ds, \]  

(5.8)

where \( G_\sigma(t) = e^{i(\Delta+\sigma\delta)t} \) is the group determined by the linear system associated with (5.1).

**Definition 5.2.** A mild solution of the initial value problem (5.1) is a complex-valued function \( u \in L^\infty_\vartheta \) satisfying (5.8).

Our main results of this section read as follows.

**Theorem 5.3.** Let \( \sigma \geq 0, \rho_0 < \rho < \infty \) and \( \varrho_0 \in E_0 \). There is \( \varepsilon > 0 \) such that if \( \|\varrho_0\|_{E_0} \leq \varepsilon \) then the IVP (5.7) has a unique global-in-time mild solution \( u \in L^\infty_\vartheta \) satisfying \( \|u\|_{L^\infty_\vartheta} \leq 2\varepsilon \). Moreover, the data-solution map \( \varrho_0 \mapsto u \) from \( E_0 \) into \( L^\infty_\vartheta \) is locally Lipschitz.

**Remark 5.1.** The proof of Theorem 5.3 is based in an argument of fixed point, so by using the implicit function theorem is not difficult to show that the data-solution map \( \varrho_0 \mapsto u \) from \( E_0 \) into \( L^\infty_\vartheta \) is smooth.

**Remark 5.2.** (Local-in-time solutions) Let \( 1 < \rho < \rho_0, \ d_0 = \frac{1}{2}(\rho - 1), \) and \( \rho_0 < \zeta < \frac{1}{\rho} \). For \( 0 < T < \infty \), consider the Banach space \( L^T_\zeta \) of all Bochner measurable functions \( u : (-T, T) \to L^{(\rho+1, \infty)} \) endowed with the norm

\[ \|u\|_{L^T_\zeta} = \sup_{-T < t < T} |t|^\zeta \|u(\cdot, t)\|_{(\rho+1, \infty)}. \]

A local-in-time existence result in \( L^T_\zeta \) could be proved for (5.7) by considering \( \varrho_0 \in L^{(\rho+1, \infty)}(\mathbb{R}) \) and small \( T > 0 \) (see [7]).

In the sequel we give an asymptotic stability result for the obtained solutions.
The condition (5.10) holds, in particular, for \( u, v \) for all if only if Lemma 5.5.

The nonlinear part of the integral equation (5.8). We have the following estimate in order respect. We have that \( d \) which is finite for all \( \nu > 0 \) and \( \eta > 0 \). Let \( u_0 \) and \( v_0 \) be two solutions of (5.8) obtained through Theorem 5.3 with initial data \( v \) and \( \nu > 0 \). Theorem 5.4.

\( \text{(Asymptotic Stability)} \) Under the hypotheses of Theorem 5.3 let \( u \) and \( v \) be two solutions of (5.8) obtained through Theorem 5.3 with initial data \( u_0 \) and \( v_0 \), respectively. We have that

\[
\lim_{|t| \to \infty} |t|^\theta \|u(\cdot, t) - v(\cdot, t)\|_{(\rho+1, \infty)} = 0
\]  

(5.9)

if only if

\[
\lim_{|t| \to \infty} |t|^\theta \|G_\sigma(t)(u_0 - v_0)\|_{(\rho+1, \infty)} = 0
\]  

(5.10)

The condition (5.10) holds, in particular, for \( u_0 - v_0 \in L^{\frac{\mu+1}{\rho}}(\infty) \).

5.1 Nonlinear Estimate

In this subsection we give the nonlinear estimate essential in the proof of Theorem 5.3. We start by recalling the Beta function

\[
B(\nu, \eta) = \int_0^1 (1 - s)^{\nu-1} s^{\eta-1} ds,
\]

which is finite for all \( \nu > 0 \) and \( \eta > 0 \). So, for \( k_1, k_2 < 1 \) and \( t > 0 \), the change of variable \( s \to st \) yields

\[
\int_0^t (t - s)^{-k_1} s^{-k_2} ds = t^{1-k_1-k_2} \int_0^1 (1 - s)^{-k_1} s^{-k_2} ds = t^{1-k_1-k_2} B(1 - k_1, 1 - k_2) < \infty.
\]

Next we denote by

\[
\mathcal{N}(u) = -i\lambda \int_0^t G_\sigma(t - s)[|u(s)|^{\rho-1} u(s)] ds
\]  

(5.12)

the nonlinear part of the integral equation (5.8). We have the following estimate in order to apply a point fixed argument.

**Lemma 5.5.** Let \( \sigma \geq 0 \) and \( \rho_0 < \rho < \infty \). There is a constant \( K > 0 \) such that

\[
\|\mathcal{N}(u) - \mathcal{N}(v)\|_{L^\sigma} \leq K\|u - v\|_{L^\sigma} (\|u\|_{L^\sigma}^{\rho-1} + \|v\|_{L^\sigma}^{\rho-1})
\]  

(5.13)

for all \( u, v \in L^\sigma \).

**Proof.** Without loss of generality, we assume \( t > 0 \). It follows from (5.4) with \( p = \frac{\rho+1}{\rho} \), \( d = \infty \) and Hölder inequality (5.2) and \( \|f\|_{(p, \infty)} = \|f\|_{(p, \infty)} \) that

\[
\|\mathcal{N}(u) - \mathcal{N}(v)\|_{(\rho+1, \infty)} \leq \int_0^t \|G_\sigma(t - s)(|u|^{\rho-1} u - |v|^{\rho-1} v)\|_{(\rho+1, \infty)} ds
\]

\[
\leq C \int_0^t (t - s)^{-\frac{1}{2} \frac{2p}{\rho+1} - 1} (\|u - v\|_{(\rho+1, \infty)} (\|u\|_{(\rho+1, \infty)}^{\rho-1} + \|v\|_{(\rho+1, \infty)}^{\rho-1}) ds
\]

\[
\leq C \int_0^t (t - s)^{-\zeta} \|u - v\|_{(\rho+1, \infty)} (\|u\|_{(\rho+1, \infty)}^{\rho-1} + \|v\|_{(\rho+1, \infty)}^{\rho-1}) ds.
\]

(5.14)
Next, notice that $\zeta = \frac{1(\rho-1)}{2(\rho+1)} < 1$ and $\vartheta \rho < 1$ when $\rho_0 < \rho < \infty$. Thus, by using (5.11), the r.h.s of (5.14) can be bounded by

$$\leq C \left( \sup_{0 < t < \infty} t^\vartheta \|u - v\|_{(\rho+1, \infty)} \sup_{0 < t < \infty} \left( t^{\vartheta(\rho-1)} \|u\|_{(\rho+1, \infty)}^{\rho-1} + t^{\vartheta(\rho-1)} \|v\|_{(\rho+1, \infty)}^{\rho-1} \right) \right) \times \int_0^t \tau \, ds$$

$$= CB(1 - \zeta, 1 - \vartheta \rho) t^{1 - \vartheta \rho - \zeta} \left( \|u - v\|_{\mathcal{L}^\infty_\rho} (\|u\|_{\mathcal{L}^\infty_\rho}^{\rho-1} + \|v\|_{\mathcal{L}^\infty_\rho}^{\rho-1}) \right),$$

which implies (5.13), because $\zeta + \vartheta \rho = -\vartheta - 1$.

5.2 Proof of Theorem 5.3

Consider the map $\Phi$ defined on $\mathcal{L}^\infty_\rho$ by

$$\Phi(u) = G_\sigma(t)u_0 + \mathcal{N}(u) \quad (5.15)$$

where $\mathcal{N}(u)$ is given in (5.12). Let $\mathcal{B}_\varepsilon = \{u \in \mathcal{L}^\infty_\rho; \|u\|_{\mathcal{L}^\infty_\rho} \leq 2\varepsilon\}$ where $\varepsilon > 0$ will be chosen later. Lemma 5.5 implies that

$$\|\Phi(u) - \Phi(v)\|_{\mathcal{L}^\infty_\rho} = \|\mathcal{N}(u) - \mathcal{N}(v)\|_{\mathcal{L}^\infty_\rho} \leq K \|u - v\|_{\mathcal{L}^\infty_\rho} (\|u\|_{\mathcal{L}^\infty_\rho}^{\rho-1} + \|v\|_{\mathcal{L}^\infty_\rho}^{\rho-1})$$

$$\leq 2^\rho \varepsilon^{\rho-1} K \|u - v\|_{\mathcal{L}^\infty_\rho}, \quad (5.16)$$

for all $u, v \in \mathcal{B}_\varepsilon$. Since

$$\|G_\sigma(t)u_0\|_{\mathcal{L}^\infty_\rho} = \|u_0\|_{\mathcal{L}^\infty_\rho} \leq \varepsilon,$$

and by using inequality (5.13) with $v = 0$ we obtain

$$\|\Phi(u)\|_{\mathcal{L}^\infty_\rho} \leq \|G_\sigma(t)u_0\|_{\mathcal{L}^\infty_\rho} + \|\mathcal{N}(u)\|_{\mathcal{L}^\infty_\rho} \leq \|G_\sigma(t)u_0\|_{\mathcal{L}^\infty_\rho} + K \|u\|_{\mathcal{L}^\infty_\rho}^{\rho-1}$$

$$\leq \varepsilon + 2^\rho \varepsilon^{\rho-1} K \leq 2\varepsilon, \quad (5.17)$$

provided that $2^\rho \varepsilon^{\rho-1} K < 1$ and $u \in \mathcal{B}_\varepsilon$. It follows that $\Phi : \mathcal{B}_\varepsilon \rightarrow \mathcal{B}_\varepsilon$ is a contraction, and then it has a fixed point $u \in \mathcal{B}_\varepsilon$. $\Phi(u) = u$, which is the unique solution for the integral equation (5.8) satisfying $\|u\|_{\mathcal{L}^\infty_\rho} \leq 2\varepsilon$.

In view of (5.16), if $u, v$ are two integral solutions with respective data $u_0, v_0$, then

$$\|u - v\|_{\mathcal{L}^\infty_\rho} = \|G_\sigma(t)(u_0 - v_0)\|_{\mathcal{L}^\infty_\rho} + \|\mathcal{N}(u) - \mathcal{N}(v)\|_{\mathcal{L}^\infty_\rho}$$

$$\leq \|u_0 - v_0\|_{\mathcal{L}^\infty_\rho} + 2^\rho \varepsilon^{\rho-1} K \|u - v\|_{\mathcal{L}^\infty_\rho},$$

which, due to $2^\rho \varepsilon^{\rho-1} K < 1$, yields the Lipschitz continuity of the data-solution map

\[ \square \]
5.3 Proof of Theorem 5.4

We will only prove that (5.10) implies (5.9). The converse follows similarly (in fact it is easier) and it is left to the reader. For that matter, we subtract the integral equations verified by $u$ and $v$ in order to obtain

$$t^\vartheta \|u(\cdot, t) - v(\cdot, t)\|_{(\rho+1, \infty)} \leq t^\vartheta \|G_\sigma(t)(u_0 - v_0)\|_{(\rho+1, \infty)} +$$

$$+ t^\vartheta \left\| \int_0^t G_\sigma(t - s)(|u|^\rho - |v|^\rho) \, ds \right\|_{(\rho+1, \infty)}. \quad (5.18)$$

Since $\|u\|_{L_\vartheta^\infty}, \|v\|_{L_\vartheta^\infty} \leq 2\epsilon$, we can estimate the second term in R.H.S. of (5.18) as follows.

$$I(t) = t^\vartheta \left\| \int_0^t G_\sigma(t - s)(|u|^\rho - |v|^\rho) \, ds \right\|_{(\rho+1, \infty)} \leq C t^\vartheta \int_0^t (t - s)^{-\frac{1}{2}\left(\frac{2\rho}{\rho+1} - 1\right)} s^{-\vartheta \rho} \|u(\cdot, s) - v(\cdot, s)\|_{(\rho+1, \infty)} \left( \|u\|_{L_\vartheta^\infty}^{\rho-1} + \|v\|_{L_\vartheta^\infty}^{\rho-1} \right)$$

$$\leq C 2^\rho \epsilon^{\rho-1} t^\vartheta \int_0^t (t - s)^{-\zeta} s^{-\vartheta \rho} \|u(\cdot, s) - v(\cdot, s)\|_{(\rho+1, \infty)} \, ds, \quad (5.19)$$

where $\zeta = \frac{1}{2}\left(\frac{2\rho}{\rho+1} - 1\right)$. Now, recalling that $\zeta + \vartheta \rho - \vartheta - 1 = -\vartheta$, the change of variable $s \mapsto ts$ in (5.19) leads us to

$$I(t) \leq C 2^\rho \epsilon^{\rho-1} \int_0^1 (1 - s)^{-\zeta} s^{-\vartheta \rho} \|u(\cdot, ts) - v(\cdot, ts)\|_{(\rho+1, \infty)} \, ds. \quad (5.20)$$

Set

$$L = \limsup_{t \to \infty} t^\vartheta \|u(\cdot, t) - v(\cdot, t)\|_{(\rho+1, \infty)} < \infty \quad (5.21)$$

and recall from proofs of Lemma 5.5 and Theorem 5.3 that

$$K = C \int_0^1 (1 - s)^{-\zeta} s^{-\vartheta \rho} \, ds \quad \text{and} \quad 2^\rho \epsilon^{\rho-1} K < 1.$$ 

Then, computing $\limsup_{t \to \infty}$ in (5.18) and using (5.20), we get

$$L \leq \left( C 2^\rho \epsilon^{\rho-1} \int_0^1 (1 - s)^{-\zeta} s^{-\vartheta \rho} \, ds \right) L$$

$$= 2^\rho \epsilon^{\rho-1} KL$$

and therefore $L = 0$, as required.

6 Existence of an invariant manifold of periodic orbits

It is not clear for us whether the approach applied in the proof of Theorem 5.3 for the case $\mu(x) = \sigma \delta$ in (5.1) with $\sigma \geq 0$ can be applied for the case $\sigma < 0$. Similar situation
is happening for the cases \( \mu(x) = \alpha(\delta(\cdot - a) + \delta(\cdot + a)) \) and \( \mu(x) = \beta \delta' \) with \( \alpha < 0 \) and \( \beta < 0 \), respectively.

But, for instance, in the case \( \sigma < 0 \) we can establish a nice qualitative behavior associated to the linear flow generated by equation (5.1). In fact, it follows from Theorem 2.2 that the linear part of the NLS-\( \delta \) equation (5.1) has a two-dimensional manifold of periodic orbits, namely,

\[ E^p = \{ \gamma e^{i\theta} \Phi_\sigma(x) : \gamma \geq 0 \text{ and } \theta \in [0, 2\pi) \}. \]

So, the estimate (5.5) will imply immediately that all solutions \( u(t) \) of (5.1) with \( \lambda = 0 \) and with initial conditions \( u_0 \in L^{(p,d)}(\mathbb{R}) \) will approach to one of the periodic orbits \( \gamma e^{i(\frac{2}{4}t+\theta) \Phi_\sigma} \in E^p \). More exactly, we have the following theorem.

**Theorem 6.1.** Let \( \sigma < 0 \). For \( d \in [1, \infty], p' \in [1, \infty] \) and \( p \in [1, 2] \), we have that for \( p' \) satisfying \( \frac{1}{p} + \frac{1}{p'} = 1 \), the solution \( u(t) \) of the linear equation associated to (5.1) with initial data \( u(0) = u_0 \in L^{(p,d)}(\mathbb{R}) \) satisfies

\[ \lim_{t \to \pm \infty} \| u(t) - \gamma_0 e^{i(\frac{2}{4}t+\theta) \Phi_\sigma} \|_{(p',d)} = 0, \]

for \( \gamma_0 = |\langle u_0, \Phi_\sigma \rangle| \) and some \( \theta \in [0, 2\pi] \).

**Remark 6.1.** 1) A similar result to that in Theorem 6.1 can be obtained for the linear system of equation (5.1) in the case of \( \mu(x) = \beta \delta' \) with \( \beta < 0 \) and for the linear system (4.19) in the case of \( \mu(x) = \alpha(\delta(\cdot - a) + \delta(\cdot + a)) \) with \( \alpha \leq -\frac{1}{\delta} \) and \( \alpha < 0 \).

2) Note that \( \gamma_0 < \infty \). In fact, it is not difficult to see that for \( \Psi_\sigma(x) = \sqrt{-\frac{-\sigma}{2}} e^{\frac{\pi}{4}|x|^2} \) we have for \( s \geq 0 \) that \( \Psi_\sigma^*(s) = \sqrt{-\frac{-\sigma}{2}} e^{\frac{\pi}{4} s^2} \). So for all \( p, q \in (0, \infty) \) we obtain that

\[ \| \Psi_\sigma \|_{L^{(p,q)}}^q = \int_0^\infty \left( t^{\frac{q}{2}} \Psi_\sigma^*(t) \right)^q \frac{dt}{t} = \left( \frac{-\sigma}{2} \right)^{\frac{q}{2}} \left( \frac{-4}{q\sigma} \right)^{\frac{q}{2}} \Gamma \left( \frac{q}{p} \right), \]

where \( \Gamma \) represents the Gamma function. The case \( q = \infty \) is immediate. Next, by the Hardy-Littlewood inequality for decreasing rearrangements and the Hölder inequality in the classical \( L^p(dv) \) spaces, we obtain for \( p \in [1, 2], p' \) such that \( \frac{1}{p} + \frac{1}{p'} = 1 \) and for \( r \) such that \( \frac{1}{r} + \frac{1}{r'} = 1 \), with \( d \geq 1 \), the estimate

\[ \beta_0 \leq \int_\Omega |u_0(x)||\Psi_\sigma(x)| dx \leq \int_0^\infty u_0^*(t)\Psi_\sigma^*(t) dt = \int_0^\infty t^{\frac{1}{p'}} u_0^*(t) t^{\frac{1}{r'}} \Psi_\sigma^*(t) \frac{dt}{t} \leq \| u_0 \|_{(p,d)} \| \Psi_\sigma \|_{(p',r)} < \infty. \]

### 7 Spaces based on Fourier transform

In this section we consider the nonlinear Schrödinger equation

\[
\begin{cases}
  i\partial_t u + \Delta u + \mu(x)u = \lambda u^p, & x \in \Omega, \ t \in \mathbb{R}, \\
  u(x, 0) = u_0, & x \in \Omega
\end{cases}
\]  
(7.1)
where $\mu \in BC(\mathbb{R}^n)$ (the space of all bounded continuous functions on $\mathbb{R}^n$), $\lambda = \pm 1$ and $\rho \in \mathbb{N}$. Here we will consider $\Omega = \mathbb{T}^n$ and $\Omega = \mathbb{R}^n$, i.e. the periodic and nonperiodic cases, respectively. The nonlinearity $\lambda |u|^{p-1}u$ could be considered in (7.1), however we prefer $u^\rho$ for the sake of simplicity of the exposition. For more details, see Remark 7.2 below.

We start by defining the spaces for the nonperiodic case. We recall that if $\mathcal{M}(\mathbb{R}^n)$ denotes the space of complex Radon measures on $\mathbb{R}^n$, then it is a vector space and for $\nu \in \mathcal{M}(\mathbb{R}^n)$, $\|\nu\|_\mathcal{M} = |\nu|(\mathbb{R}^n)$ is a norm on it, where $|\nu|$ is the total variation of $\nu$ (we note that every measure in $\mathcal{M}(\mathbb{R}^n)$ is automatically a finite Radon measure. Moreover, we can embed $L^1(\mathbb{R}^n, dm)$ into $\mathcal{M}(\mathbb{R}^n)$ by identifying $f \in L^1(\mathbb{R}^n, dm)$ with the complex measure $d\nu = f dm$, and $\|\nu\|_\mathcal{M} = \int |f| dm$. Next, every $\nu \in \mathcal{M}(\mathbb{R}^n)$ defines a tempered distribution by $T_\nu(\varphi) = \int_{\mathbb{R}^n} \varphi(x) d\nu$, thereby identifying $\mathcal{M}(\mathbb{R}^n)$ with a subspace of $S'$.

The Fourier transform on $L^1(\mathbb{R}^n)$ can be extended of a natural form to $\mathcal{M}(\mathbb{R}^n)$; if $\nu \in \mathcal{M}(\mathbb{R}^n)$, the Fourier transform of $\nu$ is the function $\hat{\nu}$ defined by

$$\hat{\nu}(\xi) = \int e^{-2\pi i \xi \cdot x} d\nu(x), \quad \xi \in \mathbb{R}^n. \quad (7.2)$$

Using that $e^{-2\pi i \xi \cdot x}$ is uniformly continuous in $x$, it is not difficult to check that $\hat{\nu} \in BC(\mathbb{R}^n)$ and that $\|\hat{\nu}\|_{\infty} \leq \|\nu\|_\mathcal{M}$ (see Folland [14]). Similarly, we define the inverse Fourier transform $\check{\nu}$ of $\nu$ by $\check{\nu}(\xi) = \hat{\nu}(-\xi)$ for $\xi \in \mathbb{R}^n$. Moreover, if $\mathcal{F}$ represents the Fourier transform on $S'$, then for every $\nu \in \mathcal{M}(\mathbb{R}^n)$ we have $\mathcal{F}(\nu) = \hat{\nu}$. Similarly, $\mathcal{F}^{-1}(\nu) = \check{\nu}$.

Recall that the space $\mathcal{M}(\mathbb{R}^n)$ can be identified with a subspace of $S'$. Hence, if we assume that $\mu \in S'$ and $\mathcal{F}(\mu)$ is a finite Radon measure, then for $\nu = \mathcal{F}(\mu)$ we have

$$\mu = \mathcal{F}^{-1} \mathcal{F}(\mu) = \mathcal{F}^{-1}(\nu) = \hat{\nu} \in BC(\mathbb{R}^n).$$

Next, by using the above identification between $\mathcal{F}$ and $\check{\nu}$, we define the Banach space

$$\mathcal{I} = [\mathcal{M}(\mathbb{R}^n)]' = \{ f \in S'((\mathbb{R}^n) : \hat{\mu} \in \mathcal{M}(\mathbb{R}^n) \} \subset BC(\mathbb{R}^n), \quad (7.3)$$

with norm

$$\|f\|_{\mathcal{I}} = \|\hat{\mu}\|_\mathcal{M}. \quad (7.4)$$

We note that in general $\mu \in \mathcal{I}$ may not to belong to $L^p(\mathbb{R}^n)$, nor to $L^{p,\infty}(\mathbb{R}^n)$, with $p \neq \infty$. In particular, $\mu \in \mathcal{I}$ may have infinite $L^2$-mass; for instance, if $\mu \equiv 1$ then $\mathcal{F}(\mu) = \delta_0 \in \mathcal{M}(\mathbb{R}^n)$.

In the following we will consider $\mu, u_0 \in \mathcal{I}$. The Cauchy problem (7.1) is formally equivalent to the functional equation

$$u(t) = S(t) u_0 + B(u) + L_\mu(u), \quad (7.5)$$

where $S(t) = e^{i t \Delta}$ is the Schrödinger group in $\mathbb{R}^n$, and the operators $L_\mu(u), B(u)$ are defined via Fourier transform by

$$\check{L_\mu}(\mu) = \int_0^t e^{-i |s|^{2}(t-s)}(\mu * \hat{\mu})(\xi, s) ds, \quad (7.6)$$
and
\[ \hat{B}(u) = \lambda \int_0^t e^{-|\xi|^2(t-s)} \left( \hat{\mu} * \hat{\mu} * \cdots * \hat{\mu} \right)(\xi,s) ds, \quad (7.7) \]

for \( \mu \in \mathcal{I} \) and \( u \in L^\infty((-T,T);I) \). We recall that for arbitrary \( \mu, \nu \in \mathcal{M}(\mathbb{R}^n) \) their convolution \( \hat{\mu} * \hat{\nu} \in \mathcal{M}(\mathbb{R}^n) \) is defined by
\[ \hat{\mu} * \hat{\nu}(E) = \int \int \chi_E(x+y) d\mu(x) d\nu(y), \]

for every Borel set \( E \).

Let \( \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n \) stand for the \( n \)-torus. We say that functions on \( \mathbb{T}^n \) are functions \( f : \mathbb{R}^n \to \mathbb{C} \) that satisfy \( f(x+m) = f(x) \) for all \( x \in \mathbb{R}^n \) and \( m \in \mathbb{Z}^n \), which are called 1-periodic in every coordinate. Let \( \mathcal{P} = C^\infty_{per} = \{ f : \mathbb{R}^n \to \mathbb{C} : f \) is \( C^\infty \) and periodic with period 1 \}. So \( \mathcal{D}'(\mathbb{T}^n) \) is the set of all periodic distributions on \( \mathcal{P} \). We say that \( \mathcal{T} : \mathcal{P} \to \mathbb{C} \) is a periodic distribution if there exists a sequence \( (\Psi_j)_{j \geq 1} \subset \mathcal{P} \) such that
\[ \mathcal{T}(f) = \langle \mathcal{T}, f \rangle = \lim_{j \to \infty} \int_{[-1/2,1/2]^n} \Psi_j(x) f(x) dx, \quad f \in \mathcal{P}. \]

Above we have identify \( \mathbb{T}^n \) with \([ -1/2, 1/2 ]^n \). For a complex-valued function \( f \in L^1(\mathbb{T}^n) \) and \( m \in \mathbb{Z}^n \), we define
\[ \hat{f}(m) = \int_{[-1/2,1/2]^n} f(x) e^{-2\pi i x \cdot m} dx. \]

We call \( \hat{f}(m) \) the \( m \)-th Fourier coefficient of \( f \). The Fourier series of \( f \) at \( x \in \mathbb{T}^n \) is the series
\[ \sum_{m \in \mathbb{Z}^n} \hat{f}(m) e^{2\pi i x \cdot m}. \]

The Fourier transform of \( \mathcal{T} \in \mathcal{D}'(\mathbb{T}^n) \) is the function \( \hat{T} : \mathbb{Z}^n \to \mathbb{C} \) defined by the formula
\[ \hat{T}(m) = \langle \mathcal{T}, e^{-2\pi i x \cdot m} \rangle, \quad m \in \mathbb{Z}^n. \]

In the periodic case, we are going to study (7.1) in the space \( \mathcal{I}_{per} \) which is defined by
\[ \mathcal{I}_{per} = \{ f \in \mathcal{D}'(\mathbb{T}^n) : \hat{f} \in l^1(\mathbb{Z}^n) \} \quad (7.8) \]
endowed with the norm
\[ \| f \|_{\mathcal{I}_{per}} = \| \hat{f} \|_{l^1(\mathbb{Z}^n)}. \quad (7.9) \]

Here the IVP (7.1) is formally converted to
\[ u(t) = S_{per}(t) u_0 + B_{per}(u) + L_{\mu,per}(u), \quad (7.10) \]
where, similarly to above, we define the operators in (7.10) via Fourier transform in \( \mathcal{D}'(\mathbb{T}^n) \). Precisely, \( S_{per}(t) \) is the Schrodinger group in \( \mathbb{T}^n \)
\[ S_{per}(t) u_0 = \sum_{m \in \mathbb{Z}^n} \hat{u}_0(m) e^{-4\pi^2 t |m|^2} t e^{2\pi i x \cdot m}, \quad (7.11) \]
\[
L_{\mu, \text{per}}(u)(m, t) = \int_0^t e^{-4\pi^2 |m|^2 (t-s)} (\hat{\mu} * \hat{u})(m, s)\, ds \tag{7.12}
\]
and
\[
B_{\text{per}}(u)(m, t) = \int_0^t e^{-4\pi^2 |m|^2 (t-s)} (\hat{u} * \underbrace{\hat{u} * \cdots * \hat{u}}_{\rho \text{-times}})(m, s)\, ds, \tag{7.13}
\]
for \(u_0, \mu \in \mathcal{I}_{\text{per}}\) and \(u \in L^\infty((-T,T); \mathcal{I}_{\text{per}})\), where now the symbol * denotes the discrete convolution
\[
\hat{f} \ast \hat{g}(m) = \sum_{\xi \in \mathbb{Z}^n} \hat{f}(m-\xi)\hat{g}(\xi).
\]
Throughout this section, solutions of (7.5) or (7.10) will be called mild solutions for the IVP (7.1), according the respective case.

Theorem 7.1. Let \(1 \leq \rho < \infty\).

(1) (Periodic case) Let \(u_0 \in \mathcal{I}_{\text{per}}\) and \(\mu \in \mathcal{I}_{\text{per}}\). There is \(T > 0\) such that the IVP (7.1) has a unique mild solution \(u \in L^\infty((-T,T); \mathcal{I}_{\text{per}})\) satisfying
\[
\sup_{t \in (-T,T)} \|u(\cdot, t)\|_{\mathcal{I}_{\text{per}}} \leq 2 \|u_0\|_{\mathcal{I}_{\text{per}}}.
\]
Moreover, the data-map solution \(u_0 \to u\) is locally Lipschitz continuous from \(\mathcal{I}_{\text{per}}\) to \(L^\infty((-T,T); \mathcal{I}_{\text{per}})\).

(2) (Nonperiodic case) Let \(u_0 \in \mathcal{I}\) and \(\mu \in \mathcal{I}\). The same conclusion of item (1) holds true by replacing \(\mathcal{I}_{\text{per}}\) by \(\mathcal{I}\).

Remark 7.1. In item (2) of the above theorem, one can show that the solution \(u(x,t)\) verifies \(\hat{u}(\xi, t) \in L^1(\mathbb{R}^n)\), for all \(t \in (-T,T)\), provided that \(\hat{u}_0 \in L^1(\mathbb{R}^n)\) and \(\hat{\mu} \in L^1(\mathbb{R}^n)\). Then, due to Riemann-Lebesgue lemma, it follows that
\[
u(x, t) \to 0 \text{ as } |x| \to \infty, \text{ for each } t \in (-T,T).
\]

7.1 Nonlinear Estimate

Before proceeding with the proof of Theorem 7.1, let us recall the Young inequality for measures and discrete convolutions (see Folland [14] and Iorio&Iorio [22]). For \(\mu, \nu \in \mathcal{M}(\mathbb{R}^n)\) and \(f, g \in l^1 = l^1(\mathbb{Z}^n)\), we have the respective estimates
\[
\|\mu \ast \nu\|_{\mathcal{M}} \leq \|\mu\|_{\mathcal{M}} \|\nu\|_{\mathcal{M}} \tag{7.14}
\]
and
\[
\|f \ast g\|_{l^1} \leq \|f\|_{l^1} \|g\|_{l^1}. \tag{7.15}
\]

Lemma 7.2. Let \(1 \leq \rho < \infty\) and \(0 < T < \infty\).
(i) There exists a positive constant $K > 0$ such that

\[
\sup_{t \in (-T,T)} \|S_{\text{per}}(t)u_0\|_{\mathcal{I}_{\text{per}}} \leq \|u_0\|_{\mathcal{I}_{\text{per}}} \tag{7.16}
\]

\[
\sup_{t \in (-T,T)} \|L_{\mu,\text{per}}(u - v)\|_{\mathcal{I}_{\text{per}}} \leq T\|\mu\|_{\mathcal{I}_{\text{per}}} \sup_{t \in (-T,T)} \|u(\cdot, t) - v(\cdot, t)\|_{\mathcal{I}_{\text{per}}} \tag{7.17}
\]

\[
\sup_{t \in (-T,T)} \|B_{\text{per}}(u) - B_{\text{per}}(v)\|_{\mathcal{I}_{\text{per}}} \leq KT \sup_{t \in (-T,T)} \|u(\cdot, t) - v(\cdot, t)\|_{\mathcal{I}_{\text{per}}} \times \left( \sup_{t \in (-T,T)} \|u(\cdot, t)\|_{\mathcal{I}_{\text{per}}}^{\rho - 1} + \sup_{t \in (-T,T)} \|v(\cdot, t)\|_{\mathcal{I}_{\text{per}}}^{\rho - 1} \right), \tag{7.18}
\]

for all $u_0, \mu \in \mathcal{I}_{\text{per}}$ and $u, v \in L^\infty((-T,T);\mathcal{I}_{\text{per}})$.

(ii) The above estimates still hold true with $S(t), L_\mu(u), B(u)$ and $\mathcal{I}$ in place of $S_{\text{per}}(t), L_{\mu,\text{per}}(t), B_{\text{per}}(u)$ and $\mathcal{I}_{\text{per}}$, respectively.

**Proof.** We will only prove the item (i) because (ii) follows similarly by using (7.14) instead of (7.15). From definition of $S_{\text{per}}(t)$, we have that

\[
\sup_{t \in (-T,T)} \|S_{\text{per}}(t)u_0\|_{\mathcal{I}_{\text{per}}} = \left\| \left( \hat{u}_0(m) e^{-4\pi^2 j|m|^2 t} \right)_{m \in \mathbb{Z}^n} \right\|_{\ell^1(\mathbb{Z}^n)} \leq \|\hat{u}_0\|_{\ell^1(\mathbb{Z}^n)}.
\]

The operator $L_{\mu,\text{per}}$ can be estimated as

\[
\|L_{\mu,\text{per}}(u)\|_{\mathcal{I}_{\text{per}}} = \left\| \widehat{L_{\mu,\text{per}}(u)} \right\|_{\ell^1(\mathbb{Z}^n)} \leq \left\| \left( \int_0^t e^{-4\pi^2 j|m|^2(t-s)}(\hat{\mu} \ast \hat{u})(m, s)ds \right)_{m \in \mathbb{Z}^n} \right\|_{\ell^1(\mathbb{Z}^n)} \leq \sum_{m \in \mathbb{Z}^n} \left| \int_0^t e^{-4\pi^2 j|m|^2(t-s)}(\hat{\mu} \ast \hat{u})(m, s)ds \right| \leq \int_0^t \sum_{m \in \mathbb{Z}^n} |(\hat{\mu} \ast \hat{u})(m, s)| ds \leq \int_0^t \|\hat{\mu}\|_{\ell^1(\mathbb{Z}^n)} \|\hat{u}(\cdot, s)\|_{\ell^1(\mathbb{Z}^n)} ds = \|\mu\|_{\mathcal{I}_{\text{per}}} \sup_{t \in (-T,T)} \|u\|_{L^1(0,T;\mathcal{I}_{\text{per}})} \leq T \|\mu\|_{\mathcal{I}_{\text{per}}} \|u\|_{L^\infty(0,T;\mathcal{I}_{\text{per}})}.
\]

By elementary convolution properties and Young inequality (7.15), it follows that

\[
\left\| \left( \hat{u} \ast \hat{u} \ast \ldots \ast \hat{u} \right) - \left( \hat{v} \ast \hat{v} \ast \ldots \ast \hat{v} \right) \right\|_{\ell^1(\mathbb{Z}^n)} \leq \left\| \left( \hat{u} \ast \hat{v} \ast \ldots \ast \hat{v} \right) - \left( \hat{u} \ast \hat{v} \ast \ldots \ast \hat{v} \right) \right\|_{\ell^1(\mathbb{Z}^n)} \leq K \|\hat{u} \ast \hat{v}\|_{\ell^1(\mathbb{Z}^n)} \left( \|\hat{u}\|_{\ell^1}^{\rho - 1} + \|\hat{v}\|_{\ell^1}^{\rho - 1} \right).
\]
Therefore
\[ \|B_{\text{per}}(u)(t) - B_{\text{per}}(v)(t)\|_{I_{\text{per}}} = \|\hat{B}_{\text{per}}(u) - \hat{B}_{\text{per}}(v)\|_{11} \]
\[
\leq \int_0^t e^{-4\pi^2|\xi|^2(t-s)} \left[ \frac{(\hat{u} \ast \hat{u} \ast \ldots \ast \hat{u}) - (\hat{v} \ast \hat{v} \ast \ldots \ast \hat{v})}{\rho \text{-times}} \right] \|B\|_{\text{per}} \| u \|_{I_{\text{per}}} \| v \|_{I_{\text{per}}} \| \| \|_{I_{\text{per}}} ds \\
\leq K \int_0^t \|\hat{u} - \hat{v}\|_1 \left( \|\hat{u}\|_{I_{\text{per}}}^{\rho-1} + \|\hat{v}\|_{I_{\text{per}}}^{\rho-1} \right) ds \\
\leq KT \| u - v \|_{L^{\infty}(0,T;I_{\text{per}})} \left( \| u \|_{L^{\infty}(0,T;I_{\text{per}})}^{\rho-1} + \| v \|_{L^{\infty}(0,T;I_{\text{per}})}^{\rho-1} \right),
\]
as required.

Remark 7.2. The approach employed here could be used to treat (7.1) with the nonlinearity \(|u|^{\rho-1} u\) instead of \(u^\rho\). For \(\rho\) odd, it would be enough to write \(|u|^{\rho-1} u\) (in the above proof) as
\[
\left[\left( |u|^2 \right)^{\frac{\rho-1}{2}} u \right] = \left[ (u \cdot \nu) u \right] = (\hat{u} \ast \hat{u} \ast \ldots \ast \hat{u}) \ast (\hat{u} \ast \hat{u} \ast \ldots \ast \hat{u}) \ast u \\
\text{\(\frac{\rho-1}{2}\)-times} \quad \text{\(\frac{\rho-1}{2}\)-times}
\]
and to note that \(\hat{u}(\xi) = \hat{u}(-\xi)\) and \(\|\hat{u}(\xi)\|_{I_{\text{per}}} = \|u(\xi)\|_{I_{\text{per}}}\).

7.2 Proof of Theorem 7.1

Proof of (1). Consider the ball \(B_{\varepsilon} = \{u \in L^{\infty}(-T, T; I_{\text{per}}); \| u \|_{L^{\infty}(-T, T; I_{\text{per}})} \leq 2\varepsilon\}\) endowed with the complete metric \(Z(\cdot, \cdot)\) defined by
\[
Z(u, v) = \| u - v \|_{L^{\infty}(-T, T; I_{\text{per}})}.
\]
Let \(\varepsilon = \|u_0\|_{I_{\text{per}}}\) and \(T > 0\) such that
\[
T(2 \|u\|_{I_{\text{per}}} + 2^{\rho} \varepsilon^{\rho-1} K) < 1. \quad (7.19)
\]
Notice that \(\varepsilon\) can be large. We shall show that the map
\[
\Phi(u) = S_{\text{per}}(t)u_0 + L_{\mu, \text{per}}(u) + B_{\text{per}}(u)
\]
is a contraction on \((B_{\varepsilon}, Z)\). Lemma 7.2 with \(v = 0\) yields

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\[ \| \Phi(u) \|_{L^\infty(-T,T;\mathcal{I}_{per})} \leq \| S_{\text{per}}(t) u_0 \|_{L^\infty(-T,T;\mathcal{I}_{per})} + \| L_{\mu,\text{per}}(u) \|_{L^\infty(-T,T;\mathcal{I}_{per})} + \| B_{\text{per}}(u) \|_{L^\infty(-T,T;\mathcal{I}_{per})} \]

for all \( u \in \mathcal{B}_\varepsilon \) and thus \( \Phi(\mathcal{B}_\varepsilon) \subset \mathcal{B}_\varepsilon \). From Lemma 7.2 we also have

\[ \| \Phi(u) - \Phi(v) \|_{L^\infty(-T,T;\mathcal{I}_{per})} = \| L_{\mu,\text{per}}(u) - L_{\mu,\text{per}}(v) \|_{L^\infty(-T,T;\mathcal{I}_{per})} + \| B_{\text{per}}(u) - B_{\text{per}}(v) \|_{L^\infty(-T,T;\mathcal{I}_{per})} \]

for all \( u,v \in \mathcal{B}_\varepsilon \). In view of (7.19), (7.20) and (7.21), the map \( \Phi \) is a contraction in \( \mathcal{B}_\varepsilon \) and then, the Banach fixed point theorem assures the existence of a unique solution \( u \in L^\infty(-T,T;\mathcal{I}_{per}) \) such that \( \| u \|_{L^\infty(-T,T;\mathcal{I}_{per})} \leq 2 \| u_0 \|_{\mathcal{I}_{per}} \).

On the other hand if \( u,v \) are two solutions with respective initial data \( u_0,v_0 \) then, similarly in deriving (7.21), one obtains

\[ \| u - v \|_{L^\infty(-T,T;\mathcal{I}_{per})} \leq \| u_0 - v_0 \|_{L^\infty(-T,T;\mathcal{I}_{per})} + \| L_{\mu,\text{per}}(u) - L_{\mu,\text{per}}(v) \|_{L^\infty(-T,T;\mathcal{I}_{per})} + \| B_{\text{per}}(u) - B_{\text{per}}(v) \|_{L^\infty(-T,T;\mathcal{I}_{per})} \]

which, in view of (7.19), implies the desired local Lipschitz continuity.

**Proof of (2).** It follows by proceeding entirely parallel to the proof of item (1) by replacing \( \mathcal{I}_{per} \) by \( \mathcal{I} \).

\[ \]

8 Appendix

Next we present a different proof of Theorem 4.3. For \( \sigma > 0 \), from Theorem 3.1 in Albeverio et al. [2], the fundamental solution \( S_\sigma(x,y;t) \) to the Schrödinger equation (4.5) is given by

\[ S_\sigma(x,y,t) = S(x-y;t) - \frac{\sigma}{2} \int_0^\infty e^{-\frac{\sigma}{2}u} S(u+|x| + |y|;t)du \]

(8.1)
where \( S(x; t) \) is the free propagator in \( \mathbb{R} \), i.e.

\[
S(x, t) = \frac{e^{-x^2/4t}}{(4\pi t)^{1/2}}, \quad t > 0
\]

and \( e^{it\Delta} f(x) = S(x; t) \ast_x f(x) \). Then we have the representation

\[
e^{-itH_\sigma} f(x) = \int_{\mathbb{R}} S_\sigma(x, y, t) f(y) dy. \tag{8.2}
\]

Next, we consider \( x > 0 \) and \( f \in L^1(\mathbb{R}) \) with \( \text{supp} f \subset (-\infty, 0] \). Then, since

\[
S(u + x - y; t) = \left( e^{-it\xi^2} \right)^\vee (u + x - y) = \left( e^{-it\xi^2} e^{i\xi(x+u)} \right)^\vee (y)
\]

we obtain via Parseval identity that (8.2) can be re-write for \( x > 0 \) in the form

\[
e^{-itH_\sigma} f(x) = e^{it\Delta} f(x) \chi_0^+(x) + e^{it\Delta} (\rho_\sigma * f)(x) \chi_0^+(x) + \int_{\mathbb{R}} e^{-ity^2} \left( -\frac{\sigma}{2} \chi_0^-(s) e^{-isy} ds \right) \hat{f}(y) e^{iyx} dy \tag{8.3}
\]

where \( \rho_\sigma(x) = -\frac{\sigma}{2} e^{\frac{\sigma}{2} s^2} \chi_0^-(s) \). Similarly, for \( x < 0 \) we obtain the representation

\[
e^{itH_\sigma} f(x) = e^{it\Delta} f(x) \chi_0^-(x) + e^{it\Delta} (\rho_\sigma * f)(-x) \chi_0^-(x). \tag{8.4}
\]

From (8.3)-(8.4) we obtain the formula (4.13).

For \( \sigma < 0 \), Theorem 3.1 in [2] establishes that the fundamental solution \( S_\sigma(x, y; t) \) to the Schrödinger equation (4.5) is given by

\[
S_\sigma(x, y, t) = S(x - y; t) + e^{i\frac{\sigma}{2}t} \Psi_\sigma(x) \Psi_\sigma(y) + \frac{\sigma}{2} \int_0^\infty e^{\frac{\sigma}{2} u} S(u - |x| - |y|; t) du.
\]

where \( \Psi_\sigma \) is the (normalized) eigenfunction defined in Theorem 2.2. Then, a similar analysis as above produces the formula (4.14).

We note that since \(|S(x; t)| \leq C_0 t^{-1/2}\) for every \( x \in \mathbb{R} \) and \( t > 0 \) we obtain from (8.1) that \(|S_\sigma(x, y; t)| \leq 2C_0 t^{-1/2}\) for all \( x, y \in \mathbb{R} \) and \( t > 0 \). Therefore from (8.2) we obtain the dispersive estimate

\[
\|e^{-itH_\sigma} f\|_\infty \leq 2C_0 t^{-1/2} \|f\|_1,
\]

which implies the estimate (4.31) for \( \sigma > 0 \).

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References

[1] R. Adami and D. Noja, *Existence of dynamics for a 1D NLS equation perturbed with a generalized point defect*, J. Phys. A 42 (49) (2009), 495302, 19 pp.

[2] S. Albeverio, Z. Brzezniak and L. Dabrowski, *Fundamental solution of the Heat and Schrödinger equations with point interaction*, J. Funct. Anal. 130 (1995), 220-254.

[3] S. Albeverio, F. Gestezy, R. Koegh-Krohn and H. Holden, *Solvable models in quantum mechanics*, Spreing-Verlag, New York, 1988, Russian Transl., MIR, Moscow, 1991

[4] S. Albeverio and P. Kurasov, *Singular perturbations of differential operators*, London Mathematical Society Lecture Note Series 271, Cambridge University Press, Cambridge, 2000.

[5] C. Bennett and R. Sharpley, *Interpolation of operators*, Academic Press, Pure and Applied Mathematics 129, 1988.

[6] J. Bergh and J. Lofstrom, *Interpolation Spaces*, Springer-Verlag, Berlin-New York, 1976.

[7] P. Braise Silva, L.C.F Ferreira and E.J. Villamizar-Roa, *On the existence of infinite energy solutions for nonlinear Schrödinger equations*, Proc. Amer. Math. Soc. 137 (2009), 1977-1987.

[8] V. Caudrelier, M. Mintchev and E. Ragoucy, *The quantum non-linear Schrödinger equation with point-like defect*, J. Physics A: Mathematical and General. 37 (30) (2004).

[9] V. Caudrelier, M. Mintchev and E. Ragoucy, *Solving the quantum non-linear Schrödinger equation with δ-type impurity*, J. Math. Phys. 46 (4) (2005).

[10] T. Cazenave, *Semilinear Schrödinger Equations*, Courant Lecture Notes in Mathematics vol 10, Providence, RI: American Mathematical Society, 2003.

[11] T. Cazenave, L. Vega and M. C. Vilela, *A note on the nonlinear Schrödinger equation in weak $L^p$ spaces*, Commun. Contemp. Math. 3 (1) (2001), 153–162.

[12] T. Cazenave and F.B. Weissler, *Asymptotically self-similar global solutions of the nonlinear Schrödinger and heat equations*, Math. Z. 228 (1) (1998), 83–120.

[13] V. Duchêne, J. Marzuola and M. Weinstein, *Wave operator bounds for onedimensional Schrödinger operators with singular potentials and applications*, J. Math. Phys. 52 (2011).

[14] G. Folland, *Real Analysis*, Modern Techniques and Their Applications, John Wiley&Sons, New York, 1999.

[15] R. Fukuizumi, M. Ohta and T. Ozawa, *Nonlinear Schrödinger equation with a point defect*, Ann. Inst. H. Poincaré Anal. Non Linéaire 25 (5) (2008), 837–845.
[16] L. Grafakos, *Classical Fourier Analysis*, Graduate texts in Mathematics 249, Springer, New York, 2000.

[17] J. Holmer, J. Marzuola and M. Zworski, *Soliton splitting by external delta potentials*, J. Nonlinear Sci. 17 (4) (2007), 349–367.

[18] J. Holmer and M. Zworski, *Slow soliton interaction with delta impurities*, J. Mod. Dyn. 1 (4) (2007), 689–718.

[19] K. Datchev and J. Holmer, *Fast soliton scattering by attractive delta impurities*, Comm. Partial Differential Equations 34 (7-9) (2009), 1074–1113.

[20] J. Holmer, J. Marzuola and M. Zworski, *Fast soliton scattering by delta impurities*, Comm. Math. Phys. 274 (1) (2007), 187–216.

[21] R. J. Iorio, *Tópicos na teoria da equação de Schrödinger*, 16 Colóquio Brasileiro de Matemática, IMPA, 1987.

[22] R. J. Iorio and V. M. Iorio, *Fourier Analysis and Partial Differential Equations*, Cambridge Studies in Advanced Mathematics 70, Cambridge University Press, 2001.

[23] T. Kato, *An $L^p$-theory for nonlinear Schrödinger equations. Spectral and scattering theory and applications*, Adv. Stud. Pure Math. 23, 223–238, Math. Soc. Japan, Tokyo, 1994.

[24] H. Kovarik and A. Sachetti, *A nonlinear Schrödinger equation with two symmetric point interactions in one dimension*, J. Phys. A: Math. Theor. 43 (2010).

[25] K. Nakanishi and T. Ozawa, *Remarks on scattering for nonlinear Schrödinger equations*, NoDEA Nonlinear Differential Equations Appl. 9 (1) (2002), 45–68.

[26] M. Reed and B. Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, First Edition, Academic Press, New York-London, 1975.