Cosmographic analysis of redshift drift

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Abstract. Redshift drift is the phenomenon whereby the observed redshift between an emitter and observer comoving with the Hubble flow in an expanding FLRW universe will slowly evolve — on a timescale comparable to the Hubble time. There are nevertheless serious astrometric proposals for actually observing this effect. We shall however pursue a more abstract theoretical goal, and perform a general cosmographic analysis of this effect, eschewing (for now) dynamical considerations in favour of purely kinematic symmetry considerations and Taylor series expansions based on FLRW spacetimes. We shall develop various exact results and series expansions for the redshift drift (and its derivatives) in terms of the present day Hubble, deceleration, jerk, snap, crackle, and pop parameters, as well as the present day redshift of the source. In particular, potential observation of this redshift drift effect is intimately related to the universe exhibiting a nonzero deceleration parameter.

Keywords: redshift drift; cosmography; cosmokinetics; FLRW symmetries; jerk; snap; crackle; pop; series expansions.

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1. Introduction

The concept of “redshift drift” dates back (at least) some 58 years, to 1962, arising in coupled papers by Sandage [1] and McVittie [2]. Relatively little direct follow-up work took place in the 20th century, with Loeb’s 1998 article [3] as a stand-out exception. However, over the last 15 years the concept of redshift drift has become much more mainstream [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. The basic idea is this: If in a FLRW universe emitter and observer are comoving with the Hubble flow then the null curve connecting them is slowly evolving on a timescale set by the Hubble parameter — this implies that the redshift is slowly evolving. Despite the fact that the magnitude of the redshift drift is extremely small — the spectral shift is of order one part in $10^9$ to $10^{10}$ over the period of a decade — the realistic possibility of the detectability of this effect has been explored in the subsequent literature [3, 4, 5, 6, 7, 8, 9, 10].

The key equation (which we shall re-derive and subsequently extend below) is:

$$\dot{z} = (1 + z)H_0 - H(z). \tag{1.1}$$

See specifically McVittie, and in fact all of references [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. A second key result at low redshift is presented by Neben & Turner [8], citing McVittie [2], where they assert

$$\dot{z} = -z q_0 H_0 + O(z^2). \tag{1.2}$$

(See also Martins et al. [18].) Note that this is intimately related to the deceleration parameter, so that the presence of a redshift drift is a direct signature of acceleration or deceleration. We shall soon see that higher derivatives in the redshift drift are related to higher orders in the cosmographic expansion.

Specifically we shall extend this result to include the jerk, snap, crackle, and pop; in principle we could go to even higher order in the cosmographic expansion. It should be noted that the terminology is not entirely uniform or standardized. Instead of using “jerk” to denote the third derivative $\ddot{a}$ some authors use “jolt” or “super-acceleration” or more rarely (and with significant risk of confusion) “lurch”, “pulse”, “impulse”, “bounce”, “surge”, or “shock”. Instead of using “snap” to denote the fourth derivative $\dot{\ddot{a}}$ some authors use “lerk” or “jounce”. Instead of “crackle” for the fifth derivative $d^5a/dt^5$ some authors simply use “m”. There seems to be little to no competition for using “pop” to denote the sixth derivative $d^6a/dt^6$. Rather rarely, we have encountered specialized names for even higher derivatives: “lock” ($d^7a/dt^7$), “drop” ($d^8a/dt^8$), “shot” ($d^9a/dt^9$), and “put” ($d^{10}a/dt^{10}$). A discussion of some these terminological issues can be found in references [22, 23, 24]. Our choice of terminology, jerk, snap, crackle, and pop, seems to be reasonably well-established in the wider community.
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The only reason that we stop our expansions at the sixth derivative “pop” is a purely pragmatic one — the expansion formulae simply get too long and unwieldy. In counterpoint, one does want to go to high enough order that one has a good basis for looking for possible patterns in the expansion.

One way of writing one of our central results is this:

\[
\begin{align*}
\dot{z} &= -H_0 z \left\{ q_0 + \frac{1}{2!}(j_0 - q_0^2)z - \frac{1}{3!}(s_0 + 3j_0 + 4j_0q_0 - 3q_0^2 - 3q_0^3)z^2 \\
& \quad + \frac{1}{4!}\{(c_0 + 8s_0 + 12j_0 - 4j_0^2) + (7s_0 + 32j_0)q_0 + (25j_0^2 - 12)q_0^2 - 24q_0^3 - 15q_0^4\}z^3 \\
& \quad - \frac{1}{5!}\{(p_0 + 15c_0 + 60s_0 + 60j_0 - 15s_0j_0 - 60j_0^2) + (11c_0 + 105s_0 + 240j_0 - 70j_0^2)q_0 \\
& \quad + (60s_0 + 375j_0 - 60)q_0 + (210j_0 - 180)q_0^3 - 225q_0^4 - 105q_0^5\}z^4 \\
& \quad + O(z^5) \right\}.
\end{align*}
\]

(1.3)

This determines the (first-order) redshift drift in terms of the present day Hubble, deceleration, jerk, snap, crackle, and pop parameters, as well as the present day redshift of the source.

The article is outlined as follows: In Section 2 we systematically develop cosmographic expansions for the emission time as a function of redshift \(t(z)\), the Hubble parameter \(H(z)\), and for the deceleration, jerk, snap, crackle, and pop parameters. Section 3 presents various series expansions for the redshift drift in terms of the present day Hubble, deceleration, jerk, snap, crackle, and pop parameters, as well as the present day redshift of the source. We discuss convergence criteria in Section 4, and introduce a modified \(y\)-redshift in Section 5. Finally, we conclude in Section 6.

2. Cosmographic expansion

2.1. Emission time as a function of redshift \(t(z)\)

The idea of cosmography (cosmokinetics) dates back (at least) to Weinberg’s 1972 textbook [25]. The central idea is to maximize the use of the symmetries of FLRW spacetime,

\[
ds^2 = -dt^2 + a(t)^2 \left\{ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta \, d\phi^2) \right\},
\]

and delay the explicit use of the Einstein equations for as long as possible. Cosmographic ideas have become increasingly popular over the last two decades. See references [22, 23] and [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39].
Our first goal will be to invert the standard relationship $1 + z = \frac{a_0}{a(t)}$ to find $t(z)$ which we can formally define as $t(z) = a^{-1}(a_0/(1 + z))$. We shall aim for a power series expansion of $t(z)$. Using only the symmetries of FLRW spacetime, together with the definitions

$$H = \frac{\dot{a}}{a}; \quad q = -\frac{\ddot{a}}{a \dot{a}^2} = -\frac{\ddot{a}}{a H^2}; \quad j = \frac{a^2 \dddot{a}}{(\dot{a})^3} = \frac{\dddot{a}}{a H^3};$$

of the Hubble, deceleration, jerk, snap, crackle, and pop parameters, a (truncated) Taylor series expansion around the current epoch yields

$$a(t) = a_0 \left\{ 1 + H_0(t - t_0) - \frac{q_0}{2!}[H_0(t - t_0)]^2 + \frac{j_0}{3!}[H_0(t - t_0)]^3 + \frac{s_0}{4!}[H_0(t - t_0)]^4 + \frac{c_0}{5!}[H_0(t - t_0)]^5 + \frac{p_0}{6!}[H_0(t - t_0)]^6 + O([t - t_0]^7) \right\}. \tag{2.3}$$

Note the presence, on the left hand side of the equation, of a pole at $z = -1$, which corresponds to $a \to \infty$, that is, the instant that the universe has expanded to infinite size.

This series is now easily reverted \cite{44, 45} to yield emission time as a function of redshift.
To order $O(z^6)$, that is, including terms up to the pop $p_0$, one has:

$$t(z) = t_0 + \frac{z}{H_0} \left\{ -1 + \left( 1 + \frac{q_0}{2!} \right) - \left( 1 - \frac{j_0 - 6q_0 - 3q_0^2}{3!} \right) z^2 \right. \\
+ \left( 1 - \frac{s_0 + 12j_0 - 36q_0 + 10j_0q_0 - 36q_0^2 - 15q_0^3}{4!} \right) z^3 \\
\left. - \left( 1 - \frac{1}{5!} \{(c_0 + 20s_0 + 120j_0 - 10j_0^2) + (15s_0 + 200j_0 - 240)q_0 \\
+ (105j_0 - 360)q_0^2 - 300q_0^3 - 105q_0^4 \} \right) z^4 \right. \\
+ \left( 1 - \frac{1}{6!} \{(p_0 + 30c_0 + 300s_0 + 1200j_0 - 35j_0s_0 - 300j_0^2) \\
+ (21c_0 + 450s_0 + 3000j_0 - 1800 - 280j_0^2)q_0 \\
+ (3150j_0 + 210s_0 - 3600)q_0^2 \\
+ (1260j_0 - 4500)q_0^3 - 3150q_0^4 - 945q_0^5 \} \right) z^5 + O(z^6) \right\}.$$  \hfill (2.5)

While this reversion could in principle be done by hand, at least for the first few terms, use of a symbolic algebra package is certainly advantageous. In contrast to what happens for luminosity distance, this expansion for $t(z)$ does not depend on the spatial curvature parameter $k \in \{-1, 0, +1\}$. Note that the quantity $t_0 - t(z)$ is often called the “lookback time”.

We have gone to such a high order in the cosmographic expansion largely in the hope of finding useful patterns in the coefficients. One immediately useful pattern is the alternating $\pm 1$ leading terms at each order in redshift. However it must be admitted that the series expansion of $t(z)$ in terms of $z$ quickly becomes somewhat clumsy.

It is useful to note that the series for $t(z)$ can be partially summed as follows:

$$t(z) = t_0 - \frac{1}{H_0} \frac{z}{1 + z} + \frac{z^2}{H_0} \left\{ \frac{q_0}{2!} + \left( \frac{j_0 - 6q_0 - 3q_0^2}{3!} \right) \right. \\
- \left( \frac{s_0 + 12j_0 - 36q_0 + 10j_0q_0 - 36q_0^2 - 15q_0^3}{4!} \right) z^2 \\
+ \left( \frac{1}{5!} \{(c_0 + 20s_0 + 120j_0 - 10j_0^2) + (15s_0 + 200j_0 - 240)q_0 \\
+ (105j_0 - 360)q_0^2 - 300q_0^3 - 105q_0^4 \} \right) z^3 \\
- \left( \frac{1}{6!} \{(p_0 + 30c_0 + 300s_0 + 1200j_0 - 35j_0s_0 - 300j_0^2) \\
+ (21c_0 + 450s_0 + 3000j_0 - 1800 - 280j_0^2)q_0 \\
+ (3150j_0 + 210s_0 - 3600)q_0^2 \\
+ (1260j_0 - 4500)q_0^3 - 3150q_0^4 - 945q_0^5 \} \right) z^5 + O(z^6) \right\}.$$  \hfill (2.6)
This is perhaps the first indication that the variable \( y = \frac{z}{1+z} \) may prove useful.

A distinct framework that is plausibly cosmographic in nature is the use of Padé rational polynomial approximants, both for the Hubble function \( H(z) \) itself, and also for other related cosmological functions such as \( q(z) \). See references [40, 41, 42, 43]. A drawback of the Padé approximant approach is that the coefficients in the Padé rational polynomials typically do not have an immediate physical interpretation. So in the present article we shall work with the usual implementation of cosmography in terms of Taylor series.

2.2. Hubble parameter \( H(z) \)

Inserting the truncated Taylor series for \( t(z) \) into the definition of the Hubble parameter \( H(t) = \dot{a}(t)/a(t) \) and expanding one finds

\[
H(z) = H_0 \left\{ 1 + (1 + q_0)z + \frac{1}{2!} (j_0 - q_0^2)z^2 - \frac{1}{3!} (s_0 + 3j_0 + 4j_0q_0 - 3q_0^2 - 3q_0^3)z^3 \\
+ \frac{1}{4!} \{(c_0 + 8s_0 + 12j_0 - 4j_0^2) + (7s_0 + 32j_0)q_0 + (25j_0 - 12)q_0^2 \\
- 24q_0^3 - 15q_0^4\}z^4 \\
- \frac{1}{5!} \{(p_0 + 15c_0 + 60s_0 + 60j_0 - 60j_0^2 - 15j_0s_0) \\
+ (11c_0 + 105s_0 + 240j_0 - 70j_0^2)q_0 \\
+ (60s_0 + 375j_0 - 60)q_0^2 + (210j_0 - 180)q_0^3 \\
- 225q_0^4 - 105q_0^5\}z^5 + O(z^6) \right\}.
\]

(2.7)

Note that this result for \( H(z) \) again does not depend on the spatial curvature parameter \( k \in \{ -1, 0, +1 \} \). This expansion is purely cosmographic, no dynamics is required in deriving this result. The expansion for \( H(z) \) can easily be extended to higher order in the redshift, it just becomes increasingly more tedious and messy to write down.

2.3. Deceleration, jerk, snap, crackle, and pop parameters in terms of \( z \)

In a similar fashion one easily derives cosmographic expansions for the deceleration, jerk, snap, crackle, and pop parameters.
Deceleration:

\[ q(z) = q_0 + (j_0 - q_0 - 2q_0^2)z - \frac{1}{2!} (s_0 + 4j_0 - 2q_0 + 7q_0j_0 - 8q_0^2 - 8q_0^3)z^2 \]
\[ + \frac{1}{3!} [(c_0 + 9s_0 + 18j_0 - 7j_0^2) + (11s_0 + 63j_0 - 6)q_0 + (59j_0 - 36)q_0^2 \]
\[ - 72q_0^3 - 48q_0^4]z^3 \]
\[ - \frac{1}{4!} [(p_0 + 16c_0 + 72s_0 + 96j_0 - 25s_0j_0 - 112j_0^2) \]
\[ + (16c_0 + 176s_0 + 504j_0 - 160j_0^2 - 24)q_0 \]
\[ + (125s_0 + 944j_0 - 192)q_0^2 + (605j_0 - 576)q_0^3 - 768q_0^4 - 384q_0^5]z^4 \]
\[ + O(z^5). \]  

(2.8)

Jerk:

\[ j(z) = j_0 - (s_0 + 2j_0 + 3q_0j_0)z + \frac{1}{2!} [(c_0 + 6s_0 + 6j_0 - 3j_0^2) + (7s_0 + 18j_0)q_0 + 15j_0q_0^2]z^2 \]
\[ - \frac{1}{3!} [(p_0 + 12c_0 + 36s_0 + 24j_0 - 13s_0j_0 - 36j_0^2) + (12c_0 + 108j_0 + 84s_0 - 48j_0^2)q_0 \]
\[ + (57s_0 + 180j_0)q_0^2 + 105j_0q_0^3]z^3 + O(z^4). \]  

(2.9)

Snap:

\[ s(z) = s_0 - (c_0 + 3s_0 + 4s_0q_0)z + \frac{1}{2!} [(p_0 + 8c_0 + 12s_0 - 4s_0j_0) \]
\[ + (9c_0 + 32s_0)q_0 + 24s_0q_0^2]z^2 + O(z^3). \]  

(2.10)

Crackle:

\[ c(z) = c_0 - (p_0 + 4c_0 + 5c_0q_0)z + O(z^2). \]  

(2.11)

Pop:

\[ p(z) = p_0 + O(z). \]  

(2.12)

Despite the relatively messy form of some of these expansions, there are some definite patterns here.

For instance, working at lowest non-trivial order in redshift, from the above we see

\[ q(z) = q_0 - \frac{\dot{q}_0}{H_0} z + O(z^2); \]  

(2.13)

\[ j(z) = j_0 - \frac{\ddot{j}_0/dt}{H_0} z + O(z^2); \]  

(2.14)

\[ s(z) = s_0 - \frac{\dot{s}_0}{H_0} z + O(z^2); \]  

(2.15)

\[ c(z) = c_0 - \frac{\dot{c}_0}{H_0} z + O(z^2). \]  

(2.16)
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In fact with a bit more work

\[ p(z) = p_0 - \frac{\dot{p}_0}{H_0} z + O(z^2). \]  

(2.17)

This can easily be generalized to \( n \)th order.

3. Redshift drift in terms of \( z \)

To see where the redshift drift comes from, start with the utterly standard FLRW result

\[ 1 + z = \frac{a_0}{a_e} = \frac{dt_0}{dt_e}. \]  

(3.1)

Here the subscript 0 denotes the current epoch (reception of the photon) while the subscript \( e \) denotes the emission event.

3.1. First-order redshift drift

By the chain rule we have

\[ \dot{z} = \frac{\dot{a}_0}{a_e} - \frac{a_0}{a_e} \left( \frac{da_e}{dt_e} \right) = \frac{\dot{a}_0}{a_e} - \frac{a_0}{a_e} \left( \frac{da_e}{dt_e} \right) \left( \frac{dt_e}{dt_0} \right). \]  

(3.2)

Simplifying

\[ \dot{z} = \frac{\dot{a}_0}{a_e} \frac{a_0}{a_e} - \frac{da_e}{dt_e} \left( \frac{1}{a_e} \right) = \left( 1 + z \right) H_0 - H_e. \]  

(3.3)

That is

\[ \dot{z} = \left( 1 + z \right) H_0 - H(z). \]  

(3.4)

This is McVittie’s result [2].

Once we have this key exact result, combining this with our cosmographic expansion for \( H(z) \) easily yields

\[ \dot{z} = - H_0 z \left\{ q_0 + \frac{1}{2!} (j_0 - q_0^2) z - \frac{1}{3!} (s_0 + 3j_0 + 4j_0q_0 - 3q_0^2 - 3q_0^3) z^2 \right. \]

\[ + \frac{1}{4!} \left[ (c_0 + 8s_0 + 12j_0 - 4j_0^2) + (7s_0 + 32j_0)q_0 + (25j_0 - 12)q_0^2 - 24q_0^3 - 15q_0^4 \right] z^3 \]

\[ - \frac{1}{5!} \left[ (p_0 + 15c_0 + 60s_0 + 60j_0 - 15s_0j_0 - 60j_0^2) + (11c_0 + 105s_0 + 240j_0 - 70j_0^2)q_0 \right. \]

\[ + (60s_0 + 375j_0 - 60)q_0^2 + (210j_0 - 180)q_0^3 - 225q_0^4 - 105q_0^5 \right] z^4 \]

\[ + O(z^5) \}. \]  

(3.5)

The lowest-order term is the Neben & Turner [8] result

\[ \dot{z} = -z q_0 H_0 + O(z^2). \]  

(3.6)
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Note that the timescale for the redshift drift is of order the Hubble time. This makes potential observations challenging \[4, 6\], though other authors are considerably more optimistic \[3, 5, 8, 9\].

3.2. Second-order redshift drift

We can evaluate \( \ddot{z} \) as follows:

\[
\ddot{z} = \frac{d}{dt} [(1 + z)H_0 - H(z)] = \dot{z}H_0 + (1 + z)\dot{H}_0 - \ddot{H}(z). \tag{3.7}
\]

In the usual manner

\[
\dot{H}_0 = \frac{d}{dt} \left( \frac{\dot{a}}{a} \right) = -(1 + q_0)H_0^2. \tag{3.8}
\]

A trifle more subtle is the chain rule result

\[
\dot{H}(z) = \frac{dt_e}{dt} \frac{d}{dt_e} \left( \frac{\dot{a}}{a} \right) = -\frac{1 + q(z)}{1 + z}H(z)^2. \tag{3.9}
\]

Combining these results

\[
\ddot{z} = \dot{z}H_0 - (1 + z)H_0^2(1 + q_0) + \frac{1 + q(z)}{1 + z} H(z)^2
\]

\[
= ((1 + z)H_0 - H(z))H_0 - (1 + z)H_0^2(1 + q_0) + \frac{1 + q(z)}{1 + z} H(z)^2
\]

\[
= -H_0H(z) - (1 + z)q_0H_0^2 + \left[ \frac{1 + q(z)}{1 + z} \right] H(z)^2. \tag{3.10}
\]

That is

\[
\ddot{z} = -(1 + z)q_0H_0^2 - H_0H(z) + \left[ \frac{1 + q(z)}{1 + z} \right] H(z)^2. \tag{3.11}
\]

This formula is, within the framework of FLRW spacetimes, exact.

Inserting our previous expansions for \( H(z) \) and \( q(z) \) one now obtains the cosmographic expansion

\[
\ddot{z} = z H_0^2 \left\{ j_0 - \frac{1}{2!} (s_0 + j_0 + j_0q_0 - q_0^2)z \\
+ \frac{1}{3!} ((c_0 + 5s_0 + 3j_0 - j_0^2) + (3s_0 + 8j_0)q_0 + (3j_0 - 3)q_0^2 - 3q_0^3)z^2 \\
- \frac{1}{4!} ((\rho_0 + 11c_0 + 28s_0 + 12j_0 - 5s_0j_0 - 14j_0^2) + (6c_0 + 37s_0 + 52j_0 - 10j_0^2)q_0 \\
+ (15s_0 + 55j_0 - 12)q_0^2 + (15j_0 - 24)q_0^3 - 15q_0^4)z^3 + O(z^4) \right\}. \tag{3.12}
\]

At lowest order this agrees with Martins et al. \[18\], who in their equation (24) state

\[
\ddot{z} = z j_0 H_0^2 + O(z^2). \tag{3.13}
\]
3.3. Third-order redshift drift

Differentiating yet a third time we obtain

\[
\ddot{\dot{z}} = \frac{d}{dt} \left[ -(1 + z)q_0 H_0^2 - H_0 H(z) + \left( \frac{1 + q(z)}{1 + z} \right) H(z)^2 \right].
\]  

(3.14)

Now we have already seen

\[
\dot{H}_0 = -(1 + q_0) H_0^2; \quad \dot{H}(z) = -\frac{1 + q(z)}{1 + z} H(z)^2.
\]  

(3.15)

The new ingredient is

\[
\dot{q}_0 = -(j_0 - q_0 - 2q_0^2) H_0; \quad \dot{q}(z) = -\frac{1}{1 + z} \left[ j(z) - q(z) - 2q(z)^2 \right] H(z).
\]  

(3.16)

Combining these results we see

\[
\ddot{\dot{z}} = -\ddot{z} q_0 H_0^2 + (1 + z)(j_0 - q_0 - 2q_0^2) H_0^3 + 2(1 + z)q_0(1 + q_0) H_0^3

+ (1 + q_0) H_0^2 H(z) + \frac{1 + q(z)}{1 + z} H_0 H(z)^2 - \dot{z} \left( \frac{1 + q(z)}{(1 + z)^2} \right) H(z)^2

- \left( \frac{j(z) - q(z) - 2q(z)^2}{(1 + z)^2} \right) H(z)^3 - 2 \left[ \frac{1 + q(z)}{1 + z} \right]^2 H(z)^3.
\]  

(3.17)

That is

\[
\ddot{\dot{z}} = -(1 + z) j_0 H_0^3 + (1 + 2q_0) H_0^3 H(z) - \left( \frac{1 + j(z) + 2q(z)}{(1 + z)^2} \right) H(z)^3.
\]  

(3.18)

There are a significant number of cancellations, leading to the relatively pleasant result

\[
\ddot{\dot{z}} = (1 + z) j_0 H_0^3 + (1 + 2q_0) H_0^3 H(z) - \frac{1 + j(z) + 2q(z)}{(1 + z)^2} H(z)^3.
\]  

(3.19)

This formula is, within the framework of FLRW spacetimes, exact.

Inserting our previous expansions for \(H(z), q(z),\) and \(j(z)\) one obtains

\[
\ddot{z} = z H_0^3 \left\{ s_0 - \frac{1}{2!} \left( (c_0 + 2s_0) + (s_0 + 2j_0) q_0 \right) z

+ \frac{1}{3!} \left( (p_0 + 7c_0 + 8s_0 - 8s_0 j_0 - 4j_0^2) + (3c_0 + 9s_0 + 8j_0) q_0 + (3s_0 + 6j_0) q_0^2 \right) z^2

+ O(z^3) \right\}.
\]  

(3.20)

Note that at lowest order

\[
\ddot{z} = z s_0 H_0 + O(z^2).
\]  

(3.21)
3.4. Fourth-order redshift drift

We start by noting

\[
\ddot{z} = \frac{d}{dt} \left\{ (1+z)j_0H_0^3 + (1+2q_0)H_0^2H(z) - \frac{[1+j(z)+2q(z)]H(z)^3}{(1+z)^2} \right\}. \quad (3.22)
\]

We already have

\[
\dot{H}_0 = -(1 + q_0)H_0^2; \quad \dot{H}(z) = -\frac{1+q(z)}{1+z} H(z)^2; \quad (3.23)
\]

and

\[
\dot{q}_0 = -(j_0 - q_0 - 2q_0)H_0; \quad \dot{q}(z) = -\frac{1}{1+z} \left[ j(z) - q(z) - 2q(z)^2 \right] H(z). \quad (3.24)
\]

The new ingredient is

\[
\frac{d}{dt}j_0 = (s_0 + (2 + 3q_0)j_0)H_0; \quad \frac{d}{dt}j(z) = \frac{1}{1+z} \left[ s(z) + (2 + 3q(z))j(z) \right] H(z). \quad (3.25)
\]

Thence, a little tedious algebra leads to

\[
\ddot{z} = (1+z)s_0H_0^4 - (2 + 3j_0 + 4q_0)H_0^3H(z) - \frac{(1 + q(z))(1 + 2q_0)}{1+z} H_0^2H(z)^2
\]
\[
+ \frac{2(1 + j(z) + 2q(z))}{(1+z)^2} H_0H(z)^3
\]
\[
+ \frac{1 - s(z) + j(z) + 3q(z) + 2q(z)^2}{(1+z)^3} H(z)^4. \quad (3.26)
\]

This result is still (within the framework of a FLRW universe) exact.

Inserting our cosmographic series

\[
\ddot{z} = z H_0^4 \left\{ c_0 - \frac{1}{2!} \left[ (p_0 + 3c_0 - 2j_0^2) + (c_0 + s_0)q_0 \right] z + O(z^2) \right\}. \quad (3.27)
\]

Note that at lowest order, as we have by now begun to expect,

\[
\ddot{z} = z c_0 H_0^4 + O(z^2). \quad (3.28)
\]

3.5. Fifth-order redshift drift

At fifth-order it is useful to simplify the argument by considering

\[
\ddot{z} = \frac{d}{dt} \left\{ \ddot{z} \right\} = \frac{d}{dt} \left\{ z c_0 H_0^4 + O(z^2) \right\} = \frac{d}{dt} \left\{ z c_0 H_0^4 \right\} + O(z^2). \quad (3.29)
\]

Here we have used the fact that \( \dot{z} = O(z) \). Then

\[
z^{(5)} = \left\{ \dot{z} c_0 H_0^4 + z c_0 H_0^4 + 4z c_0 H_0^3 \dot{H}_0 \right\} + O(z^2). \quad (3.30)
\]
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But \( \dot{z} = -zq_0 H_0 + O(z^2) \) and \( \dot{H}_0 = -(1 + q_0)H_0^2 \) while

\[
\dot{c}_0 = [p_0 + (4 + 5q_0)c_0] H_0. \tag{3.31}
\]

Combining the above

\[
z^{(5)} = z p_0 H_0^5 + O(z^2). \tag{3.32}
\]

We shall now extend this to a general \( n \)-th-order result.

3.6. \( n \)-th-order redshift drift

Let us now define an \( n \)-th-order dimensionless generalized acceleration parameter, which we evaluate for convenience at the current epoch, as

\[
k_n = \left( \frac{a^{(n)}(t) a(t)^{n-1}}{[\dot{a}(t)]^n} \right)_{t=t_0}. \tag{3.33}
\]

Here \( a^{(n)}(t) \) denotes as usual the \( n \)-th derivative. Then \( k_1 = 1, k_2 = -q_0, k_3 = j_0, \) and \( k_4 = s_0, k_5 = c_0, \) and \( k_6 = p_0. \)

Based on what we have already seen above, it seems plausible that the redshift drift satisfies

\[
z^{(n)} = z k_{n+1} H_0^n + O(z^2); \quad \forall n \geq 1. \tag{3.34}
\]

Certainly, as explicitly verified above, this is true for \( n \in \{1, 2, 3, 4, 5\} \), and we shall now extend this to arbitrary \( n \) by induction. First note that

\[
\dot{k}_n = \left( \frac{a^{(n+1)} a^{n-1}}{[\dot{a}]^n} \right) + (n-1) \left( \frac{a^{(n)} a^{n-2}}{[\dot{a}]^{n-1}} \right) - n \left( \frac{a^{(n)} a^{n-1} \ddot{a}}{[\dot{a}]^{n+1}} \right), \tag{3.35}
\]

which we can recast as

\[
\dot{k}_n = \{k_{n+1} + (n-1)k_n + nq_0k_n\} H_0. \tag{3.36}
\]

But from the discussion above we also know

\[
\dot{H}_0 = -(1 + q_0)H_0^2; \quad \dot{z} = -zq_0 H_0 + O(z^2) \tag{3.37}
\]

So if we assume the induction hypotheses then

\[
z^{(n+1)}(t) = \frac{dz^{(n)}(t)}{dt} = \frac{d}{dt} \left( z k_{n+1} H_0^n \right) + O(z^2) \tag{3.38}
\]
But then
\[
\frac{d}{dt} \left( z k_{n+1} H_0^n \right) = - z q_0 k_{n+1} H_0^{n+1} + z \left\{ k_{n+2} + n k_{n+1} + (n+1) q_0 k_{n+1} \right\} H_0^{n+1} \\
- n z (1+q_0) k_{n+1} H_0^{n+1} + O(z^2) \\
= - z k_{n+2} H_0^{n+1} + O(z^2).
\]
(3.39)

This completes the proof of the inductive step.

In view of the previous explicit verification for \( n \in \{1, 2, 3, 4, 5\} \) this now completes the full proof that
\[
z^{(n)} = z k_{n+1} H_0^n + O(z^2); \quad \forall n \geq 1.
\]
(3.40)

While direct measurement of these higher-order redshift drifts \( z^{(n)} \) is likely to be technologically infeasible, they do have a nice theoretical interpretation in terms of the cosmographic parameters.

4. Convergence issues

One of the problematic issues with cosmographic methods is that in the usual (naive) formulation one is dealing with truncated Taylor series in \( z \) but often wishes to apply the formulae at large redshift \( z > 1 \). Does the Taylor series converge? In fact, there are good mathematical and physical reasons for believing that these Taylor series in terms of \( z \) cannot possibly converge for \( z > 1 \). See particularly references [30, 31, 32]. This follows from a variant of the Dyson argument [46] that is normally used in quantum field theory (QFT) to argue that the Feynman diagram expansion cannot possibly be convergent. Even after renormalization to eliminate the infinities, the Feynman diagram expansion is at best asymptotic.

In the present cosmographic context we argue as follows: If any of these Taylor series, (either for \( t(z), H(z), q(z), j(z), s(z), c(z), p(z), \dot{z}(z) \), or any of the \( z^{(n)} \), were to converge for some region \( z < z_* \) with \( z_* > 1 \) then it is a standard result of real (or complex) analysis that the Taylor series must also converge for the reflected region \( z > -z_* \) with \( -z_* < -1 \). But \( z = -1 \) corresponds to infinite expansion, so \( z < -1 \) corresponds to making predictions after the universe has reached infinite size, which is physically unreasonable.

We can formulate this more precisely in terms of the radius of convergence \( R_* \), which is determined by the distance from the origin \( z = 0 \) to the nearest mathematical singularity. Looking into the future, suppose the universe has a future singularity, or turnaround event, or asymptotically approaches some finite size, at some \( a_{\text{max}} > a_0 \), where we set \( a_{\text{max}} \rightarrow \infty \) if the universe expands to infinite size. Then the Taylor series in \( z \) converges for \( |z| < R_* \) where we bound \( R_* \) by
\[
R_* = |z_{\text{nearest singularity}}| \leq \left| \frac{a_0}{a_{\text{max}}} - 1 \right| = 1 - \frac{a_0}{a_{\text{max}}} \leq 1.
\]
(4.1)
Since certainly $R_s \leq 1$, it makes no sense to push the Taylor series expansion into the region $z > 1$. (In principle we should also look for past singularities, but since looking to the future already gives the convergence bound $|z| < 1$, and since we have strong confidence in the absence of physical singularities in the cosmologically recent past, such considerations are unnecessary for present purposes.) Fortunately there are workarounds to side-step this convergence issue \cite{30, 31, 32}. Basically, one should rearrange the Taylor series to improve convergence. Indeed, mathematicians have developed an impressively large body of techniques for dealing with naively divergent series. (See for instance \cite{47}.)

5. Modified $y$-redshift

A physically well-motivated improved redshift parameterization that was mooted in references \cite{30, 31, 32} is to set

$$1 - y = \frac{a}{a_0} = \frac{1}{1 + z}, \quad (5.1)$$

so that

$$y = \frac{z}{1 + z}; \quad z = \frac{y}{1 - y}. \quad (5.2)$$

Physically, in terms of the change in wavelength this corresponds to

$$z = \frac{\Delta \lambda}{\lambda_e}; \quad y = \frac{\Delta \lambda}{\lambda_0}. \quad (5.3)$$

So when working with $y$ instead of $z$ all one is doing is redefining the redshift by normalizing it in terms of the arguably more physically relevant present-day value of the wavelength \cite{30, 31, 32}. Though physically equivalent to $z$, the $y$-redshift has much better mathematical convergence properties.

Now suppose the universe has a past singularity, or turnaround event, or asymptotically approaches some finite size, at some $a_{\text{min}} < a_0$, where we set $a_{\text{min}} \to 0$ if the universe emerges from a big bang singularity. Then the Taylor series in $y = z/(1 + z)$ converges for $|y| < R_s$ where now

$$R_s = |y_{\text{nearest singularity}}| \leq 1 - \frac{a_{\text{min}}}{a_0} \leq 1. \quad (5.4)$$

But this corresponds to convergence, of the power series in $y$, for $z$ in the asymmetric region

$$z \in \left( -\frac{R_s}{1 + R_s}, +\frac{R_s}{1 - R_s} \right) \subseteq \left( -\frac{1}{2}, \infty \right). \quad (5.5)$$

Other functional relationships $z(y)$ and $y(z)$ have been mooted by other authors, see for instance references \cite{21} and \cite{28}. In this article we concentrate on the particularly simple choice $z = y/(1 - y), \ y = z/(1 + z)$ because it does minimal physical violence to the
usual notion of redshift, being essentially just a change in normalization. Furthermore with this definition of the $y$-redshift the mathematical analysis of convergence properties is particularly easy. Alternative choices mooted in the literature include

$$y(z) = \tan^{-1}\left(\frac{z}{1+z}\right); \quad y(z) = \tan^{-1} z; \quad y(z) = \frac{z}{1+z^2}. \quad (5.6)$$

These alternative choices seem to us to be not particularly well motivated in terms of the underlying physics, and from a mathematical perspective to also require significantly more subtle convergence analyses.

5.1. Hubble parameter $H(y)$

To see this in action, first note that in terms of this $y$-redshift one has

$$1 - y = 1 + H_0(t - t_0) - \frac{q_0}{2!}[H_0(t - t_0)]^2 + \frac{j_0}{3!}[H_0(t - t_0)]^3 + \frac{s_0}{4!}[H_0(t - t_0)]^4$$

$$+ \frac{c_0}{5!}[H_0(t - t_0)]^5 + \frac{p_0}{6!}[H_0(t - t_0)]^6 + O([t - t_0]^7). \quad (5.7)$$

Reversion of this power series \cite{44, 45} yields:

$$t(y) = t_0 + \frac{y}{H_0} \left\{ -1 + \frac{1}{2!}q_0 y - \frac{1}{3!} \left(j_0 - 3q_0^2\right)y^2 - \frac{1}{4!} \left(s_0 + 10q_0 j_0 - 15q_0^3\right)y^3$$

$$+ \frac{1}{5!} \left(c_0 + 15s_0 q_0 - 10j_0^2 + 105j_0 q_0^2 - 105q_0^4\right)y^4$$

$$- \frac{1}{6!} \left(p_0 + 21c_0 q_0 - 35s_0 j_0 + 210s_0 q_0^2 - 280j_0^2 q_0 + 1260j_0 q_0^3 - 945q_0^4\right)y^5$$

$$+ O(y^6) \right\}. \quad (5.8)$$
Consequently for the Hubble parameter we now see

\[ H(y) = H_0 \left\{ 1 + (1 + q_0)y + \left[ 1 + q_0 + \frac{1}{2}(j_0 - q_0^2) \right] y^2 \right. \]

\[ + \left[ 1 + q_0 - \frac{1}{3!}(s_0 - 3j_0 + 4j_0q_0 + 3q_0^2 - 3q_0^3) \right] y^3 \]

\[ + \left[ 1 + q_0 + \frac{1}{4!}(c_0 - 4s_0 + 12j_0 - 4j_0^2 + (7s_0 - 16j_0)q_0 \right. \]

\[ + (25j_0 - 12)q_0^2 + 12q_0^3 - 15q_0^4) \right] y^4 \]

\[ + \left[ 1 + q_0 - \frac{1}{5!}((p_0 - 5c_0 + 20s_0 - 60j_0 + 20j_0^2 - 15j_0s_0) \]

\[ + (11c_0 - 35s_0 + 80j_0 - 70j_0^2)q_0 + (60s_0 - 125j_0 + 60)q_0^2 \]

\[ + (210j_0 - 60)q_0^3 + 75q_0^4 - 105q_0^5) \right] y^5 \]

\[ + O(y^6) \} . \] (5.9)

Notice that this can be partially summed

\[ H(y) = H_0 \left\{ 1 + (1 + q_0)\frac{y}{1 - y} + \left[ \frac{1}{2}(j_0 - q_0^2) \right] y^2 \right. \]

\[ - \left[ \frac{1}{3!}(s_0 - 3j_0 + 4j_0q_0 + 3q_0^2 - 3q_0^3) \right] y^3 \]

\[ + \left[ \frac{1}{4!}(c_0 - 4s_0 + 12j_0 - 4j_0^2 + (7s_0 - 16j_0)q_0 \right. \]

\[ + (25j_0 - 12)q_0^2 + 12q_0^3 - 15q_0^4) \right] y^4 \]

\[ - \left[ \frac{1}{5!}((p_0 - 5c_0 + 20s_0 - 60j_0 + 20j_0^2 - 15j_0s_0) \]

\[ + (11c_0 - 35s_0 + 80j_0 - 70j_0^2)q_0 + (60s_0 - 125j_0 + 60)q_0^2 \]

\[ + (210j_0 - 60)q_0^3 + 75q_0^4 - 105q_0^5) \right] y^5 \]

\[ + O(y^6) \} . \] (5.10)

The pole in this expression is at \( y = 1 \) which corresponds to the big-bang singularity. Note, for instance, that at \( z \approx 4 \) we have \( y \approx \frac{4}{3} < 1 \), so these \( y \)-expansions will be somewhat better behaved than the original \( z \)-expansions.

5.2. Deceleration, jerk, snap, crackle, and pop parameters in terms of \( y \)

Similarly the deceleration, jerk, snap, crackle, and pop parameters can now easily be expanded in terms of the \( y \)-redshift.
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Deceleration:

\[ q(y) = q_0 + (j_0 - q_0 - 2q_0^2)y + \frac{1}{2!} (s_0 + 2j_0 + 7j_0q_0 - 4q_0^2 - 8q_0^3) y^2 \]
\[ + \frac{1}{3!} (c_0 + 3s_0 - 7j_0^2 + (11s_0 + 21j_0)q_0 + 59j_0q_0^2 - 24q_0^3 - 48q_0^4)y^3 \]
\[- \frac{1}{4!} (p_0 + 4c_0 - 25s_0j_0 - 28j_0^2 + (16c_0 + 44s_0 - 160j_0^2)q_0 + (125s_0 + 236j_0)q_0^2 \]
\[ + 605j_0q_0^3 - 192q_0^4 - 384q_0^5) y^4 + O(y^5). \] (5.11)

Jerk:

\[ j(y) = j_0 - (s_0 + 2j_0 + 3q_0j_0)y + \frac{1}{2!} (c_0 + 4s_0 + 2j_0 - 3j_0^2 + (7s_0 + 12j_0)q_0 + 15j_0q_0^2) y^2 \]
\[ - \frac{1}{3!} (p_0 + 6c_0 + 6s_0 - 13j_0s_0 - 18j_0^2 + (12c_0 + 42s_0 + 18j_0 - 48j_0^2)q_0 \]
\[ + (57s_0 + 90j_0)q_0^2 + 105j_0q_0^3) y^3 + O(y^4). \] (5.12)

Snap:

\[ s(y) = s_0 - (c_0 + 3s_0 + 4s_0q_0)y + \frac{1}{2!} \{ p_0 + 6c_0 + 6s_0 - 4s_0j_0 + (9c_0 + 24s_0)q_0 + 24s_0q_0^2 \} y^2 \]
\[ + O(y^3). \] (5.13)

Crackle:

\[ c(y) = c_0 - (p_0 + 4c_0 + 5c_0q_0) y + O(y^2). \] (5.14)

Pop:

\[ p(y) = p_0 + O(y). \] (5.15)

5.3. Redshift drift in terms of $y$

First-order in $y$: Note

\[ \dot{y} = \frac{\dot{z}}{(1+z)^2} = \frac{(1+z)H_0 - H(z)}{(1+z)^2} = (1 - y)H_0 - (1 - y)^2H(y). \] (5.16)

This result is so far exact (within the context of FLRW cosmology).
Thence, inserting our cosmographic expansions, we explicitly have
\[ \dot{y} = -H_0 y \left\{ q_0 + \frac{1}{2!} (j_0 - 2q_0 - q_0^2)y - \frac{1}{3!} (s_0 + 3j_0 + 4j_0q_0 - 3q_0^2 - 3q_0^3)y^2 \\ + \frac{1}{4!} (c_0 + 4s_0 - 4j_0^2 + (7s_0 + 16j_0)q_0 + 25j_0q_0^2 - 12q_0^3 - 15q_0^5)y^3 \\ - \frac{1}{5!} (p_0 + 5c_0 - 15s_0j_0 - 20j_0^2 + (11c_0 + 35s_0 - 70j_0^2)q_0 + (60s_0 + 125j_0)q_0^2 \\ + 210j_0q_0^3 - 75q_0^4 - 105q_0^5)y^4 + O(y^5) \right\}. \] (5.17)

This is our key cosmographic result for redshift drift in terms of the \( y \)-redshift.

Note that at small redshift (where \( y \approx z \)) we have
\[ \dot{y} = -y_0 H_0 + O(y^2). \] (5.18)

**Second-order in \( y \):** Similarly
\[ \ddot{y} = \frac{d}{dt} \left( \frac{\dot{z}}{(1 + z)^2} \right) = \frac{\ddot{z}}{(1 + z)^2} - \frac{2(\dot{z})^2}{(1 + z)^3} = (1 - y)^2 \ddot{z} - 2(1 - y)^3 \dot{z}^2. \] (5.19)

Thence, substituting \( z \to \frac{y}{1-y} \) in our previous expressions for \( \ddot{z} \) and \( \dot{z} \) one has
\[ \ddot{y} = y H_0^2 \left\{ j_0 - \frac{1}{2!} (s_0 + 3j_0 + j_0q_0 - 3q_0^2)y \\ + \frac{1}{3!} (c_0 + 5s_0 + 3j_0 - j_0^2 + (3s_0 + 14j_0)q_0 + (3j_0 - 9)q_0^2 - 9q_0^3)y^2 \\ - \frac{1}{4!} (p_0 + 7c_0 + 8s_0 - 5s_0j_0 - 16j_0^2 + (6c_0 + 33s_0 + 44j_0 - 10j_0^2)q_0 \\ + (15s_0 + 87j_0)q_0^2 + (15j_0 - 36)q_0^3 - 45q_0^4)y^3 + O(y^4) \right\}. \] (5.20)

Note that at small redshift (where \( y \approx z \)) we have
\[ \ddot{y} = y_0 j_0 H_0^2 + O(y^2). \] (5.21)

**Third-order in \( y \):** The pattern should now be clear. For the next derivative
\[ \dddot{y} = \frac{d}{dt} \left( \frac{\ddot{z}}{(1 + z)^2} - \frac{2(\dot{z})^2}{(1 + z)^3} \right) = \frac{\dddot{z}}{(1 + z)^2} - \frac{6\ddot{z} \dot{z}}{(1 + z)^3} + \frac{6(\dot{z})^3}{(1 + z)^4}. \] (5.22)

That is
\[ \dddot{y} = (1 - y)^2 \dddot{z} - 6(1 - y)^3 \dot{z}^2 + 6(1 - y)^4 \dot{z}^3. \] (5.23)

But, given that we already know that \( \dddot{z}, \ddot{z}, \) and \( \dot{z} \), we simply substitute \( z \to \frac{y}{1-y} \) which
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implies

\[ \dot{y} = y H_0^3 \left\{ s_0 - \frac{1}{2!} (c_0 + 4s_0 + s_0q_0) y \\ + \frac{1}{3!} (p_0 + 7c_0 + 8s_0 - s_0j_0 - j_0^2 + (3c_0 + 6s_0 - j_0)q_0 + 3s_0q_0^2 + 7779q_0^3) y^2 \\ + O(y^3) \right\}. \]  
(5.24)

At small redshift

\[ \ddot{y} = y s_0 H_0^3 + O(y^2). \]  
(5.25)

Fourth-order in \( y \):  
At fourth order

\[ \dddot{y} = \frac{d}{dt} \left( \frac{\ddot{z}}{(1+z)^2} - \frac{6\ddot{z}\dot{z}}{(1+z)^3} + \frac{6(\dot{z})^3}{(1+z)^4} \right), \]  
(5.26)

Thence

\[ \dddot{y} = \frac{\ddot{z}}{(1+z)^2} - \frac{8\ddot{z}\dot{z} + 6(\dot{z})^2}{(1+z)^3} + \frac{36\ddot{z}(\dot{z})^2}{(1+z)^4} - \frac{24(\dot{z})^4}{(1+z)^5}. \]  
(5.27)

That is

\[ \dddot{y} = (1 - y)^2 \dddot{z} - 8(1 - y)^3 [\ddot{z}\dot{z} + 6(\dot{z})^2] + 36(1 - y)^4 \ddot{z}(\dot{z})^2 - 24(1 - y)^5 (\dot{z})^4. \]  
(5.28)

But we have already evaluated each of these ingredients. So we can simply substitute

\[ z \rightarrow \frac{\dot{y}}{1-y}, \]  
which now implies

\[ \dddot{y} = y H_0^4 \left\{ c_0 - \frac{1}{2!} (p_0 + 5c_0 + c_0q_0 + 3s_0q_0 - 14j_0^2) y + O(y^2) \right\}. \]  
(5.29)

At small redshift

\[ \dddot{y} = y c_0 H_0^4 + O(y^2). \]  
(5.30)

Fifth-order in \( y \):  
At fifth-order in \( y \) we use a minor variant of the result for fifth-order in \( z \). We simplify the argument by considering

\[ y^{(5)} = \frac{d}{dt} \left\{ \cdots \right\} = \frac{d}{dt} \left\{ y c_0 H_0^4 + O(y^2) \right\} = \frac{d}{dt} \left\{ y c_0 H_0^4 \right\} + O(y^2). \]  
(5.31)

Here we have used the fact that \( \dot{y} = O(y) \). Then

\[ y^{(5)} = \left\{ y c_0 H_0^4 + y\dot{c}_0 H_0^4 + 4y c_0 H_0^3 \dot{H}_0 \right\} + O(y^2). \]  
(5.32)

But \( \dot{y} = -yg_0 H_0 + O(y^2) \) and \( \dot{H}_0 = -(1 + q_0) H_0^2 \) while

\[ \dot{c}_0 = (p_0 + (4 + 5q_0)c_0) H_0. \]  
(5.33)
Combining the above
\[ y^{(5)} = y p_0 H_0^5 + O(z^2). \]
We shall now extend this to a general \( n^{th} \)-order result.

\( n^{th} \)-order in \( y \): Finally we point out that to lowest order in \( y \), a minor variant of the argument used for \( z \) yields
\[ y^{(n)} = y k_{n+1} H_0^n + O(y^2); \quad \forall n \geq 1. \]
This completes our cosmographic analysis for redshift drift in terms of the \( y \)-redshift.

6. Discussion and Conclusions

What we have seen above is that the main central features of the redshift drift can be dealt with cosmographically, using only the symmetries of the FLRW spacetime. We have explicitly included present-epoch values of the jerk, snap, crackle, and pop parameters in the redshift-dependent cosmographic expansion for all of the Hubble, deceleration, jerk, snap, crackle, and pop parameters, and for the redshift drift and its derivatives \( \dot{z}, \ddot{z}, \ldots, z^{(5)} \). One could in principle go to even higher order, the relevant formulae just become messy and tedious. We have also derived a quite general result for \( z^{(n)} \) at lowest order in \( z \).

Since in applications one often wants to work at large redshift (\( z > 1 \)) we have shown how to ameliorate problematic convergence issues by rephrasing the discussion in terms of a modified notion of redshift, the \( y \)-redshift \( y = \frac{z}{1+z} \). All of our cosmographic expansions have also been expressed in terms of the \( y \)-redshift.

Of course very much more could be said by introducing *cosmo-dynamics*, (that is, invoking the Einstein equations, or more specifically the Friedmann equations). This would first allow us to build specific explicit analytic models for \( a(t) \) in ideal FLRW spacetimes, thereby largely side-stepping the cosmographic expansion. Subsequently we also plan to consider deviations from ideal FLRW universes, (peculiar motions, density fluctuations, etcetera), but we shall leave all such considerations for future work.

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