Creeping flow around a spherical particle covered by semipermeable shell in presence of magnetic field

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Abstract. This paper deals the MHD slow viscous motion of electrically conducting fluid over a rigid sphere surrounded by a concentric permeable sphere. Darcy’s equation is adopted to describe the motion in semipermeable region and Stokes equation is applied to describe the flow of viscous fluid region. For both flow fields, stream functions are calculated. The resistance force on composite sphere is obtained and variation of drag force with respect to different parameters has been plotted graphically. Some limiting cases are deduced and compared with the solution derived in other research papers. We observed that drag force increases with increasing magnetic field.

1. Introduction

There are many important applications of the problem of flow through porous particles, such as petroleum reservoir rocks, enhance oil reservoir recovery, filtration process, the extraction of energy from the geothermal zone, the underground spread of chemical waste and the investigation of flow sedimentation, etc.

Usually, the Stokes equation is used to formulate the creeping motion of viscous fluid. In the irrigation and oil industry, Darcy's equation [1] is generally applied to analyze the slow motion of viscous fluid in porous region for low permeability. Joseph and Tao [2] examined the impact of permeability on the creeping flow of viscous fluid past a permeable sphere. Padmavathi and Amaranath [3] have solved the Brinkman equations in infinite series form and applied this method to analyze the arbitrary Stokes flow past a composite sphere. Shapovalov [4] provided a solution to the problem of viscous fluid flow around a semipermeable particle. The creeping motion of micropolar fluid through an impermeable sphere surrounded by permeable sphere was studied by Mishra and Gupta [5].

Many authors have recently researched MHD flow over a spherical particle and discussed its application. MHD stratified flow past a sphere has been studied by Devi and Raghavachar [6]. Stream functions for the creeping motion of conducting fluid over a rigid sphere covered by a permeable spherical shell were calculated by Jayalakshmamma et al. [7]. Srivastava et al. [8] implemented cell model technique to explain the magneto hydrodynamics effect on swarm of porous sphere for an incompressible viscous fluid. Saad [9] studied the magneto hydrodynamics effect on the motion of permeable particles with cell
model technique. More recently, Prasad and Bucha [10, 11] investigated the effect of MHD flow over perfect and approximate semipermeable spherical particles using Darcy’s and Brinkman’s models respectively and concluded that the drag coefficient increases when the magnetic parameter increases. Inspired by all of the above analysis and gaps in the literature, we are decided to solve the problem of creeping MHD flow around a rigid sphere surrounded by a concentric semipermeable sphere.

In the present work, we have studied the impact of MHD flow on the semipermeable sphere that surrounded a solid sphere. The flow within the semipermeable sphere is governed by the Darcy’s law and for outside area, modified Stokes equation is applied. Stream function and pressure are calculated for both the regions. For the drag force, the explicit expression is determined numerically. The effect of MHD flow and other parameters on the drag coefficient has been studied and depicted graphically.

2. Problem formulation and solution

Fig. 1 illustrates the steady, axis-symmetric creeping flow of electrically conducting fluid with uniform velocity \( U \) past a concentric semipermeable sphere with solid core under magnetic impact \( (H_0) \) in transverse direction. Let \((r, \theta, \phi)\) denotes the system of spherical polar coordinates and taking \( \theta = 0 \) as an axis along the uniform flow direction. There are two parts of the flow area, region I on the outside of the semi permeable sphere and region II on the inside of the semipermeable sphere. Magnetic force \( F \) is defined by \( F = \mu_0^2 \sigma \times (q \times H) \times H \), where \( H \) is the magnetic field, \( \sigma \) and \( H \) are the electric conductivity and electromagnetic induction vector respectively. We assumed that the velocity vector \( q \) and the magnetic permeability \( \mu_0 \) are same for the both region.
Using equation (11) we eliminate the pressure term from equations (7) and (9). The azimuthal component of velocity parameter \( W \) we substitute in these variables in equation (11) to (4) and omitting bar above the dimensionless variables, we get

\[
\nabla . \mathbf{q}^{(1)} = 0, \quad (1)
\]

\[
\nabla p^{(1)} + \mu \nabla \times \nabla \times \mathbf{q}^{(1)} - \mu^2 \sigma_L (\mathbf{q}^{(1)} \times \mathbf{H}^{(1)}) \times \mathbf{H}^{(1)} = 0, \quad (2)
\]

\[
\nabla . \mathbf{q}^{(2)} = 0, \quad (3)
\]

\[
\nabla p^{(2)} + \frac{\mu}{k} q^{(2)} - 2 \mu^2 \sigma_L (\mathbf{q}^{(2)} \times \mathbf{H}^{(2)}) \times \mathbf{H}^{(2)} = 0, \quad (4)
\]

where viscosity coefficient \( \mu \) is assumed to be the same for both regions, \( k \) designates the permeability and \( \varepsilon \) represents porosity of the semi permeable region.

The following dimensionless parameters are used to make the governing equation into dimensionless:

\[
\quad r = b \tilde{r}, \quad \mathbf{q}^{(i)} = \mathbf{U} \tilde{q}^{(i)}, \quad \mathbf{v} = \frac{\mathbf{v}}{b}, \quad p^{(i)} = \frac{\mu U}{b} \tilde{p}^{(i)}, \quad \mathbf{H}^{(i)} = H \tilde{H}^{(i)} \quad (5)
\]

where \( U \) is the velocity of the fluid away from the hindrance, and \( \tilde{r}, \tilde{q}, \tilde{v}, \tilde{p}, \tilde{H}, \ldots \) are dimensionless variables.

Substituting these variables in equations (1) to (4) and omitting bar above the dimensionless variables, we get

\[
\nabla . \mathbf{q}^{(1)} = 0, \quad (6)
\]

\[
\nabla p^{(1)} + \nabla \times \nabla \times \mathbf{q}^{(1)} - \alpha^2 (\mathbf{q}^{(1)} \times \mathbf{H}^{(1)}) \times \mathbf{H}^{(1)} = 0, \quad (7)
\]

\[
\nabla . \mathbf{q}^{(2)} = 0, \quad (8)
\]

\[
\nabla p^{(2)} + \xi_2 q^{(2)} - \xi_2 \xi_2 (\mathbf{q}^{(2)} \times \mathbf{H}^{(2)}) \times \mathbf{H}^{(2)} = 0, \quad (9)
\]

where the Hartmann numbers \( \alpha = \sqrt{\frac{\mu U H^2 b^2}{\mu}} \) and \( \xi_2 = \sqrt{\frac{\mu U H^2 b^2}{\varepsilon \mu}}, \) dimensionless permeability parameter \( \beta^2 = \xi_1^2 + \xi_2^2 \) with \( \xi_1^2 = \frac{b^2}{k} \).

The azimuthal component of velocity \( q^{(i)}_\phi \) is zero for axisymmetric creeping flow and the velocity components can be expressed as

\[
\mathbf{q}^{(i)} = q^{(i)}_r (r, \theta) \mathbf{e}_r + q^{(i)}_\theta (r, \theta) \mathbf{e}_\theta, \quad i = 1, 2 \quad (10)
\]

Relation between the stream function \( \psi^{(i)} \) and velocity vector \( q^{(i)}_r, q^{(i)}_\theta, 0 \) are given by

\[
q^{(i)}_r = -\frac{1}{r \sin \theta} \frac{\partial \psi^{(i)}}{\partial \theta}, \quad q^{(i)}_\theta = \frac{1}{r \sin \theta} \frac{\partial \psi^{(i)}}{\partial r}; \quad i = 1, 2 \quad (11)
\]

Using equation (11) we eliminate the pressure term from equations (7) and (9) we get

Figure 1. Flow diagram
\[ E^2(E^2 - \alpha^2)\psi^{(1)} = 0, \quad (12) \]

and
\[ E^2\psi^{(2)} = 0, \quad (13) \]

where \( E^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} - \frac{\cot \theta}{r^2} \frac{\partial}{\partial \theta} \) is the dimensionless operator.

We implement the method of separation of variables to solve equations (12) and (13), and we find the stream functions as
\[
\psi^{(1)} = \frac{1}{2} \left[ r^2 + \frac{A}{r} + B \sqrt{r}K_\frac{3}{2}(ar) \right]\sin^2 \theta, \quad (14)
\]
\[
\psi^{(2)} = \frac{1}{2} \left[ Cr^2 + \frac{D}{r} \right]\sin^2 \theta. \quad (15)
\]

Using equations (14) and (15) into equations (7) and (9) and integrating the resulting equations, we get the pressure for both regions are
\[
p^{(1)} = \alpha^2 \left( r - \frac{A}{2r} \right) \cos \theta, \quad (16)
\]
\[
p^{(2)} = \beta^2 \left( Cr - \frac{D}{2r} \right) \cos \theta. \quad (17)
\]

The coefficients \( A, B, C, \) and \( D \) are undetermined constants and can be calculated with the help of various boundary conditions.

3. Boundary conditions and arbitrary constant

On the surface of the semi-permeable and impermeable region, these boundary conditions to be satisfied and described in terms of stream functions \( \psi^{(i)}, i = 1, 2 \):

- Continuity of normal velocities:
  \[
  \frac{\partial \psi^{(1)}}{\partial \theta} = \frac{\partial \psi^{(2)}}{\partial \theta}, \quad \text{on} \quad r = b \quad (18)
  \]
- Vanishing of tangential velocities:
  \[
  \frac{\partial \psi^{(1)}}{\partial r} = 0, \quad \text{on} \quad r = b \quad (19)
  \]
- Continuity of pressure:
  \[
  p^{(1)} = p^{(2)}, \quad \text{on} \quad r = b \quad (20)
  \]
- Vanishing of normal velocities:
  \[
  \frac{\partial \psi^{(2)}}{\partial \theta} = 0, \quad \text{on} \quad r = a \quad (21)
  \]

Putting the values of \( \psi^{(1)}, \psi^{(2)}, p^{(1)} \) and \( p^{(2)} \) in boundary conditions (18) to (21), we get
\[
A + BK_{\frac{3}{2}}(\alpha) - C - D = -1, \quad (22)
\]
\[
A + \left( \alpha K_{\frac{3}{2}}(\alpha) + K_{\frac{3}{2}}(\alpha) \right) B = 2, \quad (23)
\]
\[
A\frac{\alpha^2}{2} - C\beta^2 + D\frac{\beta^2}{2} = \alpha^2, \quad (24)
\]
\[ C l^2 + \frac{D}{l} = 0, \]  

where \( l = \frac{a}{b} \) is the separation parameter. Solving equations (22) to (25), we get

\[ A = \frac{[2 \alpha^2(1-l^3)+\alpha \beta^2(2+l^3)] K_1(\alpha)+[2 \alpha^2(1-l^3)+3 \beta^2(2+l^3)] K_3(\alpha)}{2 \alpha \beta}, \]  

\[ B = -\frac{3 \beta^2(2+l^3)}{\alpha \beta}, \]  

\[ C = -\frac{3 \alpha K_1(\alpha)+\alpha K_3(\alpha)}{2 \beta}, \]  

\[ D = \frac{3 \beta^2(2+l^3)}{\alpha \beta}, \]  

where, \( \nabla = (\alpha^2(1-l^3) - \beta^2(2+l^3)) K_1(\alpha) + \alpha(1-l^3) K_3(\alpha). \)

4. Computation of drag force

The formula

\[ F_D = \pi \mu U b \int_0^\pi \left[ r \bar{w}^2 \frac{\partial}{\partial r} \left( \frac{E^2 \psi^{(1)}}{\bar{w}^2} \right) \right] dr \]  

provides the value of drag on spherical body due to motion of fluid and magnetic field. Inserting \( \bar{w} = r \sin \theta \) and \( E^2 \psi^{(1)} = \frac{1}{3} B \alpha^2 \sqrt{r} K_1(\alpha) \sin^2 \theta \) in equation (30) and simplifying it, we get

\[ F_D = \frac{2}{3} \pi \mu b U a^2 \left( A - 2 B K_3(\alpha) \right), \]  

substituting the value of A and B in equation (31), we get

\[ F_D = -2\pi \mu U b \beta^2 \left[ \frac{(a^2+3a+3)(2+l^3)}{(2 \beta^2+a^2+a+1)+\beta^2-a^2-a-1)} \right]. \]  

Some special results

I. When \( a = 0 \) i.e. \( l = \frac{a}{b} = 0 \), then the semipermeable sphere that contains a rigid sphere may be considered as fully permeable sphere and the drag force becomes

\[ F_D = -4\pi \mu U b \frac{\beta^2}{ \left[ \frac{(a^2+3a+3)(2+l^3)}{(2 \beta^2+a^2+a+1)+\beta^2-a^2-a-1)} \right]}. \]  

This result matches with the earlier work done by Prasad and Bucha [10].

II. Semipermeable sphere geometrically becomes a rigid sphere if \( a \to b \) or \( l \to 1 \) and the outcome is

\[ F_D = -2\pi \mu U b (3 + 3a + a^2) \]  

(33)
III. If permeability \( k = 0 \) i.e., \( \xi_1 \to \infty \), then semipermeable sphere geometrically becomes a rigid sphere in MHD flow and the outcome is

\[
F_D = -2\pi \mu lb (3 + 3\alpha + \alpha^2)
\]  

(35)

The equations (34) and (35) earlier reported by Prasad and Bucha [10].

IV. If magnetic fields \( H_0 \to 0 \) then \( \alpha = 0 \) and \( \xi_2 = 0 \), then the outcome is

\[
F_D = -6\pi \mu lb \left[ \frac{2\xi_1^2}{2\xi_1^2 + 1} \right]
\]

(36)

This result agrees with the previously established formula of Shapovalov [4].

V. If \( \xi_1 \to \infty \) in equation (36), then the drag is

\[
F_D = -6\pi \mu lb
\]

(37)

This is identical for Stokes drag past a solid sphere [12].

5. Results and Discussion

Non-dimensional drag by the semipermeable sphere is given by

\[
D_N = \frac{F_D}{-6\pi \mu lb}
\]

(38)

The parameter \( l \) measures the separation amid rigid and semipermeable spheres. Figure 2 depicts the variation of \( D_N \) against \( l \) for various values of \( H_0 \) and \( k = 0.5; \sigma_1 = 0.2; \sigma_2 = 0.2; b = 1; \mu = 100; \epsilon = 0.2; \mu_h = 0.2 \). From figure 2 it is noticed that as \( l \) increases the non-dimensional drag enhances.
Figure 2  Variation in drag coefficient $D_N$ versus separation parameter $l$

Figure 3 depicts the graph among non-dimensional drag $D_N$, permeability $k$ and $\mu$. This figure demonstrates that the non-dimensional drag $D_N$ decreases with permeability parameter $k$ increases for the fixed values of $l = 0.5; \sigma_1 = 0.02; \sigma_2 = 0.02; b = 1; e = 0.2; H_0 = 30; \mu_b = 0.25$. Physically it manifests that impervious sphere experiences high drag in comparison to permeable sphere.

Figure 3  Graph between drag coefficient $D_N$ and permeability parameter $k$. 
The variation of non-dimensional drag $D_N$ with respect to magnetic field $H_0$ is demonstrated in fig-4 for various values of permeability parameter $k$. Herein, we found that the drag increases with magnetic field $H_0$ for the fixed values of $I = 0.5; \sigma_1 = 0.02; \sigma_2 = 0.02; b = 1; \mu = 100; \epsilon = 0.2; \mu_h = 0.2$. From this graph, it is noted that the drag coefficient and magnetic field $H_0$ increases or decreases together. The growth in drag force occurs due to enhance in Lorentz force.

6. Conclusion

In this article, we have explored the stream function solution for the creeping motion of electrically conducting viscous fluid around a rigid sphere surrounded by a concentric permeable sphere. Drag force has been calculated and its variation has been plotted for different fluid parameters. We found that magnetic field enhances the drag on the composite sphere. Also, it is observed that impervious sphere experiences high drag in comparison to permeable sphere. Some special cases are recovered and compared with some previous work.

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