ON T-DIVISORS AND INTERSECTIONS IN $\overline{M}_{1,3}$

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Abstract. The moduli space of stable surfaces with $K_X^2 = 1$ and $\chi(X) = 3$ has at least two irreducible components that contain surfaces with T-singularities. We show that the two known components intersect transversally in a divisor. Moreover, we exhibit other new boundary divisors and study how they intersect one another.

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1. INTRODUCTION

Surfaces of general type and their moduli spaces is a classical subject of study that goes back to the beginning of the 20th century. The moduli spaces $\overline{M}_{K^2,X}$ classifying canonical models of surfaces of general type were constructed by Gieseker [Gie77] and we now have a modular compactification $\overline{M}_{K^2,X}$, the moduli space of stable surfaces, due mainly to work of Kollár, Shepherd-Barron, and Alexeev [KSB88, Ale94].

In this paper we continue the investigation of I-surfaces, that is, stable surfaces with $K_X^2 = 1$ and $\chi(X) = 3$, refining a conjectural picture developed by the last four authors in [FPRR21], where I-surfaces with one T-singularity were studied. Note that T-singularities are precisely the quotient singularities that can occur in the closure of the Gieseker components of $\overline{M}_{K^2,X}$.

Classical I-surfaces, i.e. with canonical singularities, or more generally I-surfaces with $K_X$ Cartier, are very well understood. They are double covers of the quadric cone in $\mathbb{P}^3$ branched over a quintic section and the vertex [FPR17]. Nonetheless, progress on understanding their degenerations, or better all I-surfaces has been slow.
A philosophically sound method consists of imposing the existence of one T-singularity (i.e. locally of the form $\frac{1}{dn}(1, dna - 1)$). If the singularity imposes independent conditions on the stable surface then we get a stratum whose codimension coincides with the dimension of the local deformation space of the singularity.

For example, there is a T-divisor in $\overline{\mathcal{M}}_{1,3}$ consisting of the stratum of I-surfaces with a singularity of type $\frac{1}{4}(1,1)$, obtained by allowing the branch locus of the double cover to pass through the vertex. In [FPRR21] it was shown that the other I-surfaces with one T-singularity do not conform to this simple pattern.

Our starting point is the observation [FPRR21, Cor. 1.2] that $\overline{\mathcal{M}}_{1,3}$ has at least two irreducible components, both of dimension 28: the Gieseker component $\mathcal{M}_{1,3}$ and the component $\mathcal{M}_{RU}$ whose general element is an I-surface with a unique $\frac{1}{25}(1,14)$ singularity constructed by Rana and Urzúa in [RU19].

Confirming a suspicion from [FPRR21] we show that the two components intersect in a divisor parametrising RU-surfaces of cuspidal type. We go on to investigate the intersections of the various T-divisors obtained so far.

The schematic picture in Figure 1 illustrates our results, where we label each stratum by the singularities of its general element.

More precisely, we prove the following:

1. I-surfaces with a $\frac{1}{25}(1,14)$ singularity and of cuspidal type are smoothable; they form a divisor which is the intersection of the two components. (Section 4)
2. While the $\frac{1}{9}(1,5)$ singularity does not occur in the closure of the Gieseker component, there is a divisor in the RU-component of surfaces with T-singularities $\frac{1}{9}(1,5) + \frac{1}{25}(1,14)$. (Section 5.3)
(3) The divisor of surfaces with an $\frac{1}{18}(1,5)$ singularity, known to be in the closure of the Gieseker component, intersects the aforementioned divisors in a subset of codimension two parametrising surfaces with singularities $\frac{1}{18}(1,5) + \frac{1}{25}(1,14)$. (Prop. 5.12 and Ex. 5.13)

(4) The divisors parametrising surfaces with a $\frac{1}{4}(1,1)$ singularity respectively a $\frac{1}{18}(1,5)$ singularity intersect, as expected, in a subset of codimension two, whose general element is a surface with singularities $\frac{1}{4}(1,1) + \frac{1}{18}(1,5)$. (Section 5.4)

(5) We suspect the remaining intersection to be empty, but can only show that the combination $\frac{1}{4}(1,1) + \frac{1}{25}(1,14)$ cannot occur on an I-surface (Section 5). If the two divisors intersect at all, then they do so in considerably more singular surfaces, and presumably in high codimension.

Two complementary approaches are used to establish these claims: geometric study of the minimal resolution and algebraic study of the canonical ring. Both points of view give us descriptions of the general surface in each stratum. Canonical rings are needed to establish deformations and (partial) smoothings via explicit equations, and the study of configurations of rational curves on the minimal resolution gives the non-existence result.

Having established a complicated format for the canonical ring of the RU-surface in Corollary 3.11, we include a very surprising alternative description of the same surface as a hypersurface in weighted projective space in Section 3.7.

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2. Preliminaries

We work over the complex numbers. Linear equivalence is denoted by $\sim$.

For a Hirzebruch surface $\mathbb{F}_n$, we denote by $\sigma_\infty$ the negative section and by $\Gamma$ the class of a ruling, so that a section $\sigma_0$ disjoint from $\sigma_\infty$ is linearly equivalent to $n\Gamma + \sigma_\infty$.

An I-surface $X$ is a stable surface with $K_X^2 = 1$, $p_g = 2$, $q = 0$ (see [FPRR21]).

\footnote{To exclude some more pathological (non-smoothable) examples (compare [Rol21]) we could be more specific and fix the Hilbert series of the canonical divisor to be $h(t) = \frac{1-t^6}{(1-t^2)(1-t^5)(1-t^7)}$.}
A curve is an effective divisor, not necessarily irreducible or reduced. For a positive integer, an irreducible \((-n)\)-curve \(D\) on a smooth surface is a smooth rational curve with \(D^2 = -n\).

A \(T\)-singularity \(Q\) is either a rational double point or a 2-dimensional quotient singularity of type \(\frac{1}{d a^2}(1, d a - 1)\), where \(n > 1\) and \(d, a > 0\) are integers with \(a\) and \(n\) coprime. These are precisely the quotient singularities that admit a \(\mathbb{Q}\)-Gorenstein smoothing, that is, that can occur on smoothable stable surfaces (cf. [KSB88, §3]).

The exceptional divisor of the minimal resolution of a \(T\)-singularity \(\frac{1}{d a^2}(1, d a - 1)\) is a so-called \(T\)-string, a string of rational curves with self-intersections \(-b_1, -b_2, \ldots, -b_r\) given by the Hirzebruch–Jung continued fraction expansion \([b_1, b_2, \ldots, b_r]\) of \(\frac{d a^2}{d a - 1}\) (see, e.g. [CLS11, Chapter 10]). The index of \(X\) at \(Q\) is \(n\).

3. Computing the canonical ring of the RU-surface

For the reader’s convenience we recall the construction of a general RU-surface.

**Example 3.1.** Let \(Y\) be an elliptic surface with \(p_g(Y) = 2\), \(q(Y) = 0\) such that:

- \(Y\) has a \((-3)\)-section \(A\)
- all the elliptic fibers are irreducible.

By [FPRR21, Lemma 3.8], the surface \(Y\) is a double cover \(\pi: Y \to \mathbb{F}_6\) branched on a smooth divisor \(D \in |\sigma_\infty + 3\sigma_0|\).

Let \(F_1\) be a singular fiber and let \(q\) be its singular point: blow up \(F_1\) at \(q\) and then at a point \(q_1\) infinitely near to \(q\) and lying on the strict transform of \(F_1\) to get a surface \(\tilde{Y}\). The strict transform of \(F_1\) is a \((-5)\)-curve \(B\), the strict transform of \(A\) (that we still denote by \(A\)) is a \((-3)\)-curve, the strict transform of the curve of the first blow up is a \((-2)\)-curve, which we call \(C\), so that \(A, B, C\) is a string of type \([3, 5, 2]\).

Note that this is true both for \(F_1\) nodal and for \(F_1\) cuspidal. Then the string \(A, B, C\) can be blown down to obtain an I-surface with unique singularity of type \(\frac{1}{125}(1, 14)\), compare Figure 2.

Since we assumed the fiber \(F_1\) to be irreducible it is either a nodal curve (type I) or a cuspidal curve (type II) and we call \(X\) a nodal or cuspidal RU-surface respectively.

We know from [FPRR21, Prop. 3.13] that nodal RU-surfaces form an open subset of an irreducible component \(\mathcal{M}_{RU}\) of the moduli space. In order to understand the interaction of this component with the Gieseker component, we study RU-surfaces from the point of view of canonical rings.
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Figure 2. Construction of an RU-surface, nodal case

3.1. Strategy. Let us explain the strategy underlying the algebraic computations along the following diagram.

\[
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{\text{contract } A + B + C} & \mathcal{X} \\
\xrightarrow{\eta} & \xrightarrow{\text{min. elliptic}} & \mathbb{P}^1
\end{array}
\]

\[
\begin{array}{ccc}
\hat{\mathcal{X}} & \xleftarrow{\text{two toric blow ups}} & \hat{\mathcal{F}} \\
\xleftarrow{f} & \xleftrightarrow{\epsilon} & \xrightarrow{2:1} \\
\mathcal{X} & \xrightarrow{\text{contract } A + B + C} & \mathcal{Y} \\
\xrightarrow{\text{two toric blow ups}} & \xrightarrow{\text{two toric blow ups}} & \xrightarrow{\text{two toric blow ups}} \\
\hat{\mathcal{F}} & \xrightarrow{\text{two toric blow ups}} & \mathbb{P}^1
\end{array}
\]

(1) Construct $Y$ in toric bundle $\mathcal{F}$ over $\mathbb{P}^1$.
(2) Blow up to $\hat{\mathcal{Y}}$ in a new toric variety $\hat{\mathcal{F}}$.
(3) Construct the canonical model of $X$ by writing $R(X, K_X)$ as a subring of the Cox ring of $\hat{\mathcal{Y}}$.
(4) Study $\mathbb{Q}$-Gorenstein deformations of $X$ by deforming the canonical ring.

The deformations in Step (4) were originally found by considering all deformations over the base $\text{Spec } \mathbb{C}[\epsilon]/(\epsilon^k)$ for $k = 2$ and extending them to $k = 3, 4$ etc. according to ideas of Reid [Rei90]. Rather than reproducing these unwieldy computations, we express the final results using formats for Gorenstein rings.

3.2. The elliptic surface. Instead of using the double cover $Y \to \mathbb{F}_6$, which leads to an elliptic surface with fibers polarised in degree 2, we construct $Y$ via the halfpolarisation $A$, so that the fibers have degree 1. Consider the toric variety $\mathcal{F}$ with Cox ring
\[
\begin{pmatrix}
t_0 & t_1 & s_1 & s_0 & \zeta \\
1 & 1 & -3 & 0 & 0 \\
0 & 0 & 1 & 2 & 3
\end{pmatrix}
\]
and irrelevant ideal $I = (t_0, t_1) \cap (s_0, s_1, \zeta)$. Geometrically, this is a $\mathbb{P}(1, 2, 3)$-bundle over $\mathbb{P}_{t_0, t_1}^3$. Alternatively, $\mathcal{F} \cong (\mathbb{C}^3 \setminus V(I))/(\mathbb{C}^*)^2$.
where the action is determined by the columns of the above matrix. The subvariety $\zeta = 0$ is isomorphic to $\mathbb{F}_6$. Write $\Gamma$ for the fiber of $\mathbb{F}_6 \rightarrow \mathbb{P}^1$ with base coordinates $t_0, t_1$. The fiber coordinates are $s_0, s_1^2$; $(s_0 = 0)$ being the positive section $\sigma_0$ and $(s_1^2 = 0)$ being the negative section $\sigma_\infty$. The elliptic surface $Y$ is a relative sextic in $\mathcal{F}$ and the halfpolarisation $A$ is a section cutting out one Weierstrass point on each fiber.

**Lemma 3.2.** Let $Y$ be a double cover of $\mathbb{F}_6$ branched over a smooth element $D$ of $|-6\Gamma + 4\sigma_0|$. Then $Y$ is a hypersurface in $\mathcal{F}$ of bidegree $(0, 6)$. One can choose coordinates so that $Y$ has a nodal or cuspidal fiber $B$ over $(0, 1)$ with singularity at the point with coordinates $(0, 1; 0, 1, 0)$. The equation of $Y$ can be written in the form:

\[
(\zeta - \theta t_2^3 s_0 s_1)\zeta = s_0^6 + t_0 k_{11} s_0 s_1^4 + t_0 (t_0 l_{16} + \tau t_{17}) s_1^6,
\]

where $k'_{11}(t_0, t_1), l_{16}(t_0, t_1)$ are general polynomials in $t_0, t_1$ of respective degrees 11, 16 and $\theta, \tau$ are parameters. In particular, the special fiber is nodal if $(\theta, \tau \neq 0)$, cuspidal if $(\theta = 0)$, type $I_2$ if $(\tau = 0)$ and type $III$ if $(\theta = \tau = 0)$.\(^2\)

**Proof.** The linear system $|-6\Gamma + 4\sigma_0|$ on $\mathbb{F}_6$ decomposes into $A + |3\sigma_0|$. By the above discussion, the branching over $A$ is inherited from the structure of the toric variety, and thus $Y$ has equation:

\[
\zeta^2 = s_0^3 + j_6(t_0, t_1) s_0^2 s_1^2 + k_{12}(t_0, t_1) s_0 s_1^4 + l_{18}(t_0, t_1) s_1^6
\]

where $\zeta$ is the double cover variable and $j, k, l$ are general polynomials in $t_0, t_1$ of respective degrees 6, 12, 18 so that the right hand side of the equation cuts out a general element of $|3\sigma_0|$. The coefficient of $s_0^6$ is non-zero because otherwise two components of $D$ would intersect giving a singularity. We normalise this coefficient to be 1 and then we use the Tschirnhausen transformation $s_0 \mapsto s_0 + \frac{1}{3} j_6 s_1^2$ to remove $j_6$.

The elliptic fibration $Y \rightarrow \mathbb{P}^1$ has singular fibers when the discriminant $\Delta := 4k^3 + 27\beta^2$ vanishes. Assume that the fiber $B$: $(t_0 = 0)$ is singular, so that $\Delta|_B$ vanishes. Note that $\Delta|_B = 4\alpha^3 + 27\beta^2$, where $\alpha$ (resp. $\beta$) is the coefficient of $t_{12}^3$ in $k$ (resp. $t_{18}^3$ in $l$). Hence there exists $\varepsilon$ with $\alpha = -3\varepsilon^2$ and $\beta = 2\varepsilon^3$.

Next we use the coordinate change $s_0 \mapsto s_0 + \varepsilon t_{11}^6 s_1^2$ to move the singularity of $B$ onto the section $\sigma_0$: $(s_0 = 0)$, and the equation of $\tilde{Y}$ becomes

\[
\zeta^2 = s_0^3 + 3\varepsilon t_{11}^6 s_0^2 s_1^2 + t_0 k_{11}(t_0, t_1) s_0 s_1^4 + t_0 (t_0 l_{16}(t_0, t_1) + \tau t_{17}) s_1^6.
\]

If $\varepsilon$ is zero then the fiber $B$ is cuspidal. We write now $\theta^2 := 3\varepsilon$ and use the coordinates change $\zeta \mapsto \zeta + \theta t_3^4 s_0 s_1$ to move the $3\varepsilon t_{11}^6$-term to the left side. This gives the claimed equation.

\(^2\)More precisely, in the latter two cases, the surface $Y$ is singular and its minimal resolution has a fiber of the given type.
Close to \((0, 1; 0, 1, 0)\) we can set \(t_1 = s_1 = 1\) so that the equation becomes
\[
\zeta^2 = s_0^2 + 3\varepsilon s_0^2 + t_0 k_{11}'(t_0, 1)s_0 + t_0(t_0 k_{16}'(t_0, 1) + \tau) \\
= \tau t_0 + (3\varepsilon s_0^2 + t_0 s_0 k_{11}'(0, 1) + t_0^2 k_{16}'(0, 1)) + \text{h.o.t.,}
\]
which defines a smooth surface if \(\tau \neq 0\). Otherwise, if \(\tau = 0\) (resp. \(\varepsilon = \tau = 0\)), we get a fiber of type \(I_2\), respectively \(III\) after resolving the \(A_1\) surface singularity.

Note that the choice of square-root \(\theta\) corresponds to a choice of branch for the second blowup of the nodal fiber (cf. Figure 2). The curious change of coordinates on \(\zeta\) ensures that the blowups are at torus fixed points and thus in the next Section they can be expressed in terms of toric geometry.

### 3.3. The resolution \(\tilde{Y}\). Consider now the toric variety \(\tilde{F}\) with Cox ring
\[
\begin{pmatrix}
  t_0 & t_1 & s_1 & s_0 & \zeta & c & e \\
  1 & 1 & -3 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 2 & 3 & 0 & 0 \\
  2 & 0 & 0 & 1 & 1 & -1 & 0 \\
  1 & 0 & 0 & 0 & 1 & 1 & -1 \\
\end{pmatrix}
\]

and irrelevant ideal
\[(t_0, t_1) \cap (s_0, s_1, \zeta) \cap (c, t_1) \cap (c, s_1) \cap (t_0, s_0, \zeta) \cap (e, t_1) \cap (e, s_0) \cap (e, s_1) \cap (t_0, \zeta, c).
\]
This is the toric blow up of \(F\) at the subscheme \(t_0^2 = s_0 = \zeta = 0\) with exceptional divisor \((c = 0) \cong \mathbb{P}(1, 1, 2)\), followed by the toric blowup at the subscheme \(t_0 = \zeta = c = 0\) with exceptional divisor \((e = 0) \cong \mathbb{P}^2\).

Indeed, consider the subvariety \((e = 0)\). The structure of the irrelevant ideal implies that \(t_1, s_0, s_1\) are non-zero. Thus we apply part of the \((\mathbb{C}^*)^4\)-action (first three rows of the matrix \((1)\)) to rescale these three coordinates \(t_1 = s_0 = s_1 = 1\). We are left with the \(\mathbb{C}^*\)-quotient of \(C_{t_0, \zeta, c}^3\) induced by the last row with irrelevant ideal \((t_0, \zeta, c)\). In the same way, \((c = 0)\) is the toric subvariety with Cox ring
\[
\begin{pmatrix}
  t_0 & s_0 & \zeta & e \\
  2 & 1 & 1 & 0 \\
  1 & 0 & 1 & -1 \\
\end{pmatrix}
\]

and irrelevant ideal \((s_0, e) \cap (t_0, \zeta)\). That is, the blowup of \(\mathbb{P}(2_{t_0}, 1_{s_0}, 1_{\zeta})\) at the point \((0, 1, 0)\).

### Remark 3.3. A reference for this approach to toric varieties is Chapter 14 of the book [CLS11]. The generators of the Cox ring (columns of \((1)\)) correspond to primitive generators of the rays in the fan \(\Sigma\) of
\(v_{so}\)

\(v_{s1}\)

\(v_{t0}\)

\(v_\zeta\)

\(v_c\)

\(v_s\)

\(v_t\)

\(v_1\)

\(v_0\)

\(v_{\zeta}\)

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\(v_s\)

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\(v_t\)
Recall that the parameter \( \tau \) is the coefficient of \( t_1^{17} \) in \( l_1^{17} \).

Thus \( C \) is usually an irreducible rational curve. If \( \theta = 0 \), then nothing especially interesting happens to \( C \). On the other hand, if \( \tau = 0 \), then \( C \) breaks into two rational curves meeting in a singularity on \( \tilde{Y} \). In this situation, \( Y \) has an \( A_1 \)-singularity at the node of the fiber \( B \). This case is treated in more detail later. If \( \theta = \tau = 0 \), then \( C \) is a nonreduced double curve and \( Y \) has an \( A_1 \) singularity at the node of \( B \).

In a similar way, we find that the second exceptional curve is \( E \): \( (e = 0) \), which implies \( t_1 = s_0 = s_1 = 1 \) with equation
\[
E: (\theta \zeta = c + \tau t_0) \subset \mathbb{P}^2_{t_0, \zeta, c}.
\]

### 3.4. Canonical rings from Cox rings.

We denote the Cox ring of \( \tilde{F} \) by \( S \). This is a \( \mathbb{Z}^4 \)-graded polynomial ring with generators \( t_0, t_1, s_0, s_1, \zeta, c, e \). There is a \( \mathbb{Z} \)-linear coordinate change on \( A_1(\tilde{F}) \), i.e., applied to the degrees, which shifts the \( \mathbb{Z}^4 \)-grading on \( S \) to:

\[
\begin{pmatrix}
s_1 & t_0 & c & e & t_1 & s_0 & \zeta \\
1 & 0 & 2 & 3 & 1 & 6 & 9 \\
1 & 2 & 11 & 17 & 1 & 3 & 17 & 25
\end{pmatrix}
\]

By Lemma 3.4, for \( d \in \mathbb{Z}^4 \) we have maps
\[
S_d \rightarrow H^0(\tilde{Y}, d_1 A + d_2 B + d_3 C + d_4 E)
\]
where the divisor classes are

\( A: (s_1 = 0), \quad B: (t_0 = 0), \quad C: (c = 0), \quad E: (e = 0), \quad \Gamma: (t_1 = 0) \)

on \( \tilde{Y} \). Note that \( (s_1 = 0) \) cuts out \( A \) because \( A \) is pulled back from the irreducible component \( (s_1^2 = 0) \) of the branch locus of the double cover \( Y \rightarrow \mathbb{F}_6 \). Thus the grading records the various linear equivalences on \( \tilde{Y} \):

\[
B \sim \Gamma - 2C - 3E,
\]
\[
(s_0 = 0) \sim 2A + 6B + 11C + 17E,
\]
\[
(\zeta = 0) \sim 3A + 9B + 17C + 25E.
\]

The pullback of \( K_X \) to \( \tilde{Y} \) is
\[
\Gamma + C + 2E + \frac{1}{5}(3A + 4B + 2C) \sim B + 3C + 5E + \frac{1}{5}(3A + 4B + 2C) \\
\sim 5E + \frac{1}{5}(3A + 9B + 17C).
\]

### 3.5. Generators.

We want to construct the canonical ring \( R(X, K_X) \) as the image of a map from the Cox ring \( S \). To avoid having to keep track of corrections to multiplication maps in \( R(X, K_X) \) due to rounding, we introduce formal 5-th roots of \( s_1, t_0, e \): \( \alpha^5 = s_1, \beta^5 = t_0, \gamma^5 = e \).
The extension $S[\alpha, \beta, \gamma]$ is then a $(\frac{1}{5} \mathbb{Z})^3 \oplus \mathbb{Z}$-graded ring containing $S$.

In what follows, we consider the ring homomorphism

$$
\iota: \bigoplus_{n \geq 0} S[\alpha, \beta, \gamma] \cdot (\frac{1}{5} n, \frac{9}{5} n, \frac{17}{5} n, 5n) \to \bigoplus_{n \geq 0} H^0(\tilde{Y}, n f^* K_X) \cong R(X, K_X)
$$

The proof that this ring homomorphism is in fact surjective, is Corollary 3.12.

Let $R'$ be the image of $\iota$ and $X' = \text{Proj}(R')$.

**Lemma 3.6.** The graded ring

$$R' \subseteq \bigoplus_{n \geq 0} H^0 \left( \tilde{Y}, n(5E + \frac{1}{5}(3A + 9B + 17C)) \right)$$

is generated by

\[
\begin{align*}
x_0 &= \alpha^3 \beta^9 \gamma^{17} e^3 & \text{deg} & 1 \\
x_1 &= \alpha^3 \beta^4 \gamma^7 e^2 t_1 & \text{deg} & 1 \\
y &= \alpha^6 \beta^3 \gamma^4 e^4 t_1^3 & \text{deg} & 2 \\
w &= \alpha^9 \beta^2 \gamma^4 t_1^2 & \text{deg} & 3 \\
u_0 &= \alpha^2 \beta^6 \gamma^3 e^3 s_0 & \text{deg} & 4 \\
u_1 &= \alpha^2 \beta \gamma^3 t_1 s_0 & \text{deg} & 4 \\
z &= \zeta & \text{deg} & 5 \\
t &= \alpha \beta^3 \gamma^9 e^2 s_0^2 & \text{deg} & 7 \\
g &= \alpha \beta^3 \gamma^{14} s_0^5 & \text{deg} & 17
\end{align*}
\]

**Proof.** The generators can be determined algorithmically as follows. The grading on $S[\alpha, \beta, \gamma]$ is induced by the map $\delta: \mathbb{Z}^7 \to (\frac{1}{5} \mathbb{Z})^3 \oplus \mathbb{Z}$,

$$\delta(n_1, \ldots, n_7) = \left( \frac{n_1}{5}, \frac{n_2}{5}, \frac{n_3}{5}, n_4, n_5, n_6, n_7 \right) \left( \begin{array}{cccccc} 1 & 0 & 2 & 3 \\ 1 & 1 & 6 & 9 \\ 1 & 2 & 11 & 17 \\ 1 & 3 & 17 & 25 \end{array} \right)^t$$

via $\text{deg}(n_1 \beta^{n_2} \gamma^{n_3} e^{n_4} t_1^{n_5} s_0^{n_6} \zeta^{n_7}) = \delta(n_1, \ldots, n_7)$. Let $\Sigma = \mathbb{Z}^+ \cdot (\frac{3}{5}, \frac{9}{5}, \frac{17}{5}, 5)$ be the cone in $(\frac{1}{5} \mathbb{Z})^3 \oplus \mathbb{Z}$ generated by $\frac{1}{5}(3A + 9B + 17C) + 5E$, the pullback of $K_X$. Then the intersection of the preimage $\delta^{-1} \Sigma$ with the positive octant $(\frac{1}{5} \mathbb{Z}^+)^3 \oplus \mathbb{Z}^+$, is the cone of monomials in $R'$. The primitive generators of this cone can be found using standard Hilbert basis algorithms for lattice cones (see e.g. Chapter 7 of [MS05]) and these are the generators of $R'$.

**Remark 3.7.** It follows from Lemma 3.9 below, that the generator $g$ in degree 17 is eliminated by relations.
Remark 3.8. At this stage, we already note that the fixed part of the canonical linear system is the image of the curve $E$, because both $x_0 = \alpha^3 \beta^9 \gamma^17 e^5$ and $x_1 = \alpha^3 \beta^4 \gamma^7 e^2 t_1$ are divisible by powers of $e$. The other common factors correspond to curves $A, B, C$ which are contracted to the $\frac{1}{25}(1, 14)$-point. A more precise description of the fixed part can be found in Remark 3.14.

3.6. Relations. There are 10 binomial relations between the generators found in Lemma 3.6 (excluding those involving $g$). These define a cone over the degree 5 generator $z$ in $\mathbb{P}(1, 1, 2, 3, 4, 4, 5, 7)$:

$$
R_1: x_0 y - x_1^3 = 0 \quad R_2: x_0 w - x_1^2 y = 0 \\
R_3: x_1 w - y^2 = 0 \quad R_4: x_0 u_1 - x_1 u_0 = 0 \\
R_5: x_1^2 u_1 - y u_0 = 0 \quad R_6: x_1 y u_1 - w u_0 = 0 \\
R_7: x_0 t - u_0^2 = 0 \quad R_8: x_1 t - u_0 u_1 = 0 \\
R_9: x_1 u_1^2 - y t = 0 \quad R_{10}: y u_1^2 - w t = 0
$$

The equation (2) defining $\tilde{Y}$ induces several relations which cut out $X'$ inside this cone:

Lemma 3.9. There are four relations induced by (2) and these can be written as:

$$
R_{11}: x_0 P + x_1^2 (\theta u_1 z + \tau w^3) + u_0 t = 0 \\
R_{12}: x_1 P + y (\theta u_1 z + \tau w^3) + u_1 t = 0 \\
R_{13}: y P + w (\theta u_1 z + \tau w^3) + u_1^2 = 0 \\
R_{14}: u_0 P + x_1 u_1 (\theta u_1 z + \tau w^3) + t^2 = 0
$$

where $P_{10} = z^2 + \ldots$ is a general homogeneous form of degree 10 and $\theta, \tau$ are parameters. The generator $g$ of degree 17 is eliminated by a fifth relation $t P = \cdots + g$.

Proof. The leading term of the equation (2) cutting out $\tilde{Y}$ in $\tilde{F}$ is $e \zeta^2$. By Lemma 3.6 the unique generator involving $\zeta$ is $z$, and $e$ appears in $x_0, x_1, y, u_0, t$, thus we expect five relations induced by (2).

We first study the relation $R_{11}$ more precisely. Using Lemma 3.6 we can write in $S[\alpha, \beta, \gamma]$

$$
x_0 z^2 = \left( \alpha^3 \beta^9 \gamma^17 e^5 \right) \zeta^2 = \left( \alpha^3 \beta^9 \gamma^17 e^4 \right) \left( e \zeta^2 \right),
$$

and multiplying (2) with the excess monomial $\alpha^3 \beta^9 \gamma^17 e^4$, we get for the left hand side

$$
\alpha^3 \beta^9 \gamma^17 e^4 \left( e \zeta - \theta t_1^3 s_0 s_1 \right) \zeta = x_0 z^2 - \theta \left( \alpha^3 \beta^4 \gamma^7 e^2 t_1 \right)^2 \left( \alpha^2 \beta \gamma^3 t_1 s_0 \right) \zeta \\
= x_0 z^2 - \theta x_1^2 u_1 z
$$
and for the right hand side
\[
\begin{align*}
\alpha^3\beta^9\gamma^{17}e^4 \left( cs_0^3 + cet_0k'_{11}s_0s_1^4 + t_0(c^2e^3t_0t'_{16} + \tau t_{17}^4)s_1^6 \right) \\
= \left( \alpha^2\beta^6\gamma^{13}e^3s_0 \right) \left( \alpha^3\beta^4\gamma^7e^2t_1 \right)^2 \left( \alpha^9\beta^2\gamma^5 \right)^3 \\
+ \left( \alpha^3\beta^9\gamma^{17}e^5 \right) \left( s_1^2 \left( ct_0k'_{11}s_0s_1^2 + t_0c^2e^3t'_{16}s_1^4 \right) \right)
\end{align*}
\]
\[
= u_0t + \tau x_1^2w^3 + x_0\tilde{P}
\]

Setting the two expressions equal and rearranging the terms we get a relation of the form
\[
x_0P + x_1^2(\theta u_1z + \tau w^3) + u_0t = 0.
\]

Note that the monomial \( u_0t \) corresponds to the term \( cs_0^3 \) from (2), and most of the terms (including \( x_0z^2 \) itself) are wrapped up in the general element \( P \) of degree 10.

We can repeat this procedure to get the remaining relations, but it is easier to derive them from \( R_{11} \). Using \( R_1 \) and \( R_4 \) we have:
\[
\begin{align*}
x_0 : x_1 = u_0 : u_1. \\
\text{Thus writing } R_{12} = \frac{2x}{x_0} R_{11} \text{ and applying these ratios gives the claimed relation.}
\end{align*}
\]

The relation \( tP = \ldots \) always eliminates \( g \), because \( g \) corresponds to the term \( cs_0^3 \) in (2), which always appears with nonzero coefficient.

\begin{remark}
Recall the following characterisation of \( T \)-singularities via their index 1 canonical covers:
\[
\frac{1}{dn^2}(1, dna - 1) \cong (xy - z^{dn}) \subset \frac{1}{n}(1_x, -1_y, a_z),
\]
see e.g. [KSB88, Prop. 3.10] or [Hac16, Ex. 2.1.8].
\end{remark}

\begin{corollary}
The 14 relations defining \( X' \subset \mathbb{P}(1, 1, 2, 3, 4, 4, 5, 7) \) fit into a Pfaffian presentation as follows:
\[
Pf_4 M = MV = 0
\]
where
\[
M = \begin{pmatrix}
0 & 0 & x_0 & x_1^2 & u_0 & u_1 \\
0 & 0 & x_1 & y & u_1 & t \\
u_0 & x_1u_1 & u_1 & t & -\left( \tau w^3 + \theta u_1z \right) \\
-sym & t & -\left( \tau w^3 + \theta u_1z \right) & P
\end{pmatrix}, \quad V = \begin{pmatrix}
0 \\
-u_1^2 \\
0 \\
w \\
0 \\
y \\
0
\end{pmatrix},
\]
and we omit the diagonal zero entries of the 6 \( \times \) 6 skew matrix \( M \). For general choices of \( P_{10} \) and \( \tau \) the singular locus of \( X' \) is one point of type \( \frac{1}{25}(1, 14) \).

\begin{proof}
Write out the matrix product and the Pfaffians and compare this with the list of relations.

For the singular locus, \( X' \) has a covering by orbifold affine charts \( U_m: (m \neq 0) \) centred at each coordinate point \( P_m \) of the ambient space.
\end{proof}
Despite the number of equations, it is straightforward to show the nonsingularity (except \( U_2 \)) of each chart. Here are a couple of sample computations. The chart \( U_y \) is contained in \( U_x \cap U_w \) because \( R_{3\mid y=1} \) gives \( x_1w = 1 \), thus we ignore \( U_y \). Indeed, similar considerations show that \( U_x = U_y \subset U_{x_0} \cap U_w \) and \( U_{u_0} \subset U_{x_0} \cap U_t \). Hence we only need to check nonsingularity of four charts \( U_{x_0}, U_w, U_{u_1} \) and \( U_t \). For \( U_{x_0} \), we use \( R_1, R_2, R_4, R_7 \) to eliminate \( y, w, u_1, t \) respectively, giving a hypersurface in \( \mathbb{C}^{1, u_{0, z}} \) induced by any one of \( R_{11}, \ldots, R_{14} \). It is then easy to show that this chart is nonsingular. The other three charts work in a similar way.

Since we know that the other charts of \( X' \) are nonsingular, we only need to consider the orbifold chart \( U \) in an analytic neighbourhood of \( P \in X \). We use \( R_{11}, R_{12}, R_{13}, R_{14} \) to eliminate \( x_0, x_1, y, u_0 \) respectively. Thus the local coordinates near \( P \) are \( w, u_1, t \) and \( R_{10} \) defines \( X' \) locally as the hypersurface \( u_0^5 - wt + \text{h.o.t} = 0 \) in \( \frac{1}{5}(3w, 4u_1, 2t) \cong \frac{1}{2}(1, 3, 4) \) after substituting \( y = u_0^3 + \text{h.o.t.} \) using \( R_{13} \). By Remark 3.10, this is a \( \frac{1}{2}(1, 14) \) singularity.

**Corollary 3.12.** The coordinate ring \( R' \) of \( X' \subset \mathbb{P}(1, 1, 2, 3, 4, 4, 5, 7) \) described in Corollary 3.11 is the canonical ring \( R(X, K_X) \) and \( X' \cong X \) is an RU-surface of nodal type if \( \theta \neq 0 \) and of cuspidal type if \( \theta = 0 \).

**Proof.** A standard computer calculation shows that the minimal free resolution of \( \mathcal{O}_{X'} \) as an \( \mathcal{O}_\mathbb{P} \)-module is

\[
0 \to \mathcal{O}(-28) \to \mathcal{O}(-28) \otimes \mathcal{L}_1' \to \mathcal{O}(-28) \otimes \mathcal{L}_2' \to \mathcal{L}_2 \to \mathcal{L}_1 \to \mathcal{O} \to \mathcal{O}_{X'},
\]

where \( \mathcal{L}_1 = \bigoplus_{d \in L_1} \mathcal{O}(-d) \) with

\[
L_1 = \langle 3, 4^2, 5, 6, 7, 8^2, 9, 10, 11^2, 12, 14 \rangle
\]

and \( \mathcal{L}_2 = \bigoplus_{d \in L_2} \mathcal{O}(-d) \) with

\[
L_2 = \langle 5, 6, 7^2, 8^3, 9^3, 10^3, 11^4, 12^3, 13^4, 14^3, 15^3, 16^2, 17, 18^2, 19 \rangle.
\]

Thus the coordinate ring \( R' \) is Cohen–Macaulay by the Auslander–Buchsbaum formula [BH93, §1.3]. Combining this with regularity in codimension 1, which follows from Corollary 3.11, we see that \( X' \) is projectively normal by Serre’s criterion for normality. The dualising sheaf of \( X' \) is \( \omega_{X'} = \mathcal{E}xt^2_\mathcal{O}(\mathcal{O}_{X'}, \omega_\mathbb{P}) = \mathcal{O}_{X'}(28 - 1 - 2 - 3 - 4 - 4 - 5 - 7) = \mathcal{O}_{X'}(1) \) where \( \mathcal{E}xt^2_\mathcal{O}(\mathcal{O}_{X'}, \mathcal{O}) = \mathcal{O}(28) \) can be read off from the resolution above [BH93, §3.6]. By projective normality, we obtain \( H^0(X, nK_X) \cong H^0(X', \mathcal{O}_{X'}(n)) \). Hence we have \( X' \cong X \), \( p_8(X) = h^0(\mathcal{O}_{X'}(1)) = 2 \), and \( K_X^2 = 1 \) follows from \( P_2(X) = h^0(\mathcal{O}_{X'}(2)) = 4 \) and the Riemann–Roch formula of Blache [Bla95].

**Remark 3.13.** The vanishing of \( \theta \) corresponds to a cuspidal singular fiber as explained above. The vanishing of \( \tau \) imposes an extra \( \frac{1}{2}(1, 5) \) point on \( X' \) (these are two independent conditions in moduli).
Remark 3.14. As promised in Remark 3.8, we describe the fixed part $F$ of $|K_X|$. Indeed, $e$ divides all generators except $w, u_1, z$. Thus the image of $E$ is obtained by restricting the relation $R_{13}$ to the locus $x_0 = x_1 = y = u_0 = t = 0$:

$$F: (w(\theta u_1 z + \tau w^3) + u_1^3 = 0) \subset \mathbb{P}(3w, 4u_1, 5z).$$

For general $\theta, \tau$, the curve $F$ passes through the index 5 point and has a node there. If $\theta = 0$ then $F$ is a cone with vertex $P_z$. If $\tau = 0$ then $F$ has two components each passing through $P_w$ and $P_z$. If $\theta = \tau = 0$ then $F$ is the triple line joining $P_w$ and $P_z$.

3.7. A hypersurface in weighted projective space. We give an alternative description of the general RU-surface, which is algebraically much simpler. The following result is inspired by the observation that $\tilde{F}$ is birational to $\mathbb{P}(1, 3, 17, 25)$, as can be read off from the last row of the weight matrix (3). The decomposition of this map into extremal contractions is briefly described in Figure 4, where the red loci denote exceptional locus of each factor (for more details on how to compute this, see [CLS11, Chapter 15]). Indeed, modulo flips, the birational map is a composition of three divisorial contractions $D_{s_1}, D_c$ and $D_{t_0}$ whose restrictions to $\tilde{Y}$ contract the T-chain $A + B + C$.

Proposition 3.15. A general quasismooth hypersurface $S$ of weighted degree 51 in $\mathbb{P}(1, 3, 17, 25)$ is an RU-surface (here quasismooth means that the affine cone is smooth outside of the vertex).

Proof. By adjunction we have $K_S = \mathcal{O}_S(51 - 25 - 17 - 3 - 1) = \mathcal{O}_S(5)$ which has two global sections and $K^2_S = (51 \cdot 5^2)/(1 \cdot 3 \cdot 17 \cdot 25) = 1$.

Write $e, t_1, s_0, \zeta$ for the coordinates on the weighted projective space then the equation of degree 51 can be expressed as

$$eP_{50} + \tau t_1^{17} + \theta t_1^3 s_0 \zeta + s_0^3$$

where $P_{50}(e, t_1, s_0, \zeta)$ is a polynomial of weighted degree 50 and $\theta, \tau$ are parameters. Note that there are only three forms of degree 51 that are not divisible by $e$. It is easy to check by hand that $S$ is quasismooth.

Now $S$ contains the index 5 coordinate point $(0, 0, 0, 1)$ and we get a $\frac{1}{25}(3, 17) \cong \frac{1}{25}(1, 14)$ singularity there because since $P_{50}$ contains the monomial $\zeta^2$. If $\tau$ is nonzero then $S$ does not contain the singular point.
of index 3. We assume that the coefficient of $s_0^3$ is nonzero (= 1) to avoid the index 17 point.

By [RU19, Theorem 3.2 (A1)] the surface constructed is an RU-surface.

**Remark 3.16.** In coordinates, the map $\tilde{\mathcal{F}} \rightarrow \mathbb{P}(1,3,17,25)$ is given by:

$$(s_1, t_0, c, e, t_1, s_0, \zeta) \mapsto (1, 1, 1, e, t_1, s_0, \zeta),$$

and this maps the equation (2) defining $\tilde{Y}$, to the equation (4) defining $S_{51}$. The degree 51 can also be read off from the multidegree $(6, 18, 34, 51)$ of $Y$ with respect to the weight matrix (3). Thus we have a generically 1-1 map between the moduli spaces parametrising hypersurfaces $\tilde{Y} \subset \tilde{\mathcal{F}}$ and hypersurfaces $S_{51} \subset \mathbb{P}(1,3,17,25)$. In particular, the general element in $\mathcal{M}_{RU}$ can be realised as such a hypersurface $S_{51}$.

This agrees with the naive parameter count. In $\mathbb{P}(1,3,17,25)$, the linear system $\mathbb{P}(H^0(\mathcal{O}(51)))$ has dimension 50 and the automorphism group has dimension 22, which suggests that quasismooth hypersurfaces of weighted degree 51 have 28 moduli.

We now show that the parameters occurring in (4) play the same role as in Section 3.

**Proposition 3.17.** The RU-hypersurface with $\tau = 0$ (respectively $\tau = \theta = 0$) has an additional $\frac{1}{9}(1,5)$ (resp. $\frac{1}{18}(1,5)$) singularity.

**Proof.** Near the index 3 point $(0, 1, 0, 0)$ the local analytic form of $S$ is

$$(\theta s_0 \zeta + e^3 + \text{h.o.t.}) = 0 \subset \frac{1}{3}(1,e,2s_0,1\zeta),$$

if we assume that the monomial $e^2t_1^{18}$ appears in $P_{50}$. By Remark 3.10, this is a $\frac{1}{9}(1,5)$ singularity.

Moreover, since $t_1^{11}s_0$, $\zeta^2$, and $et_1^8\zeta$ appear in $P_{50}$, if $\theta = 0$ we get $(es_0 + e\zeta^2 + e^2\zeta + \text{h.o.t.} = 0) \subset \frac{1}{3}(1,2,1)$ which is a $\frac{1}{18}(1,5)$ singularity.

**Proposition 3.18.** The canonical model of an RU-hypersurface is the same as the one described in Corollary 3.11.

**Proof.** As shown above, $K_S = \mathcal{O}_S(5)$ so we can write out generators and relations for the canonical ring directly:

$$x_0 = e^5, \; x_1 = e^2t_1, \; y = et_1^3, \; w = t_1^5, \; u_0 = e^3s_0, \; u_1 = t_1s_0,$$

$$z = \zeta, \; t = es_0^2, \; g = s_0^5$$

in degrees $1, 1, 2, 3, 4, 5, 7, 17$ respectively.

The easy monomial relations between these generators are the same as in Section 3.6 and the equation of degree 51 can be expressed in terms of these new generators in five different ways by multiplying it with each of $e^4, et_1, e^3t_1^2, t_1^3, s_0^2$ (cf. Lemma 3.9). The last of these
five equations involves $s_0^5$ and therefore we can use it to eliminate the spurious generator $g$ of degree 17 in the same way as Lemma 3.9.

Finally, we can fit these relations into the skew-matrix format of Corollary 3.11.

\textbf{Remark 3.19.} After an appropriate coordinate change, $P_{50}$ involves only even powers of $\zeta$. Thus the general RU-hypersurface has an involution $(e, t_1, s_0, \zeta) \mapsto (e, t_1, s_0, -\zeta)$ if and only if $\theta$ vanishes. This is another interpretation of the obstruction to $\mathbb{Q}$-Gorenstein smoothing of the general RU-surface cf. [FPRR21, Prop. 3.18].

\section{Cuspidal RU-surfaces are $\mathbb{Q}$-Gorenstein smoothable}

In this section we assume that $\theta = 0$, that is, we consider the cuspidal RU-surfaces. We exhibit a $\mathbb{Q}$-Gorenstein smoothing of the general cuspidal RU-surface. Since the relations $R_{11}, \ldots, R_{14}$ are only determined modulo $R_1, \ldots, R_{10}$, we use $R_3$ to rewrite $R_{14}$ as

\[ R_{14}: u_0P + \tau y^2 w^2 u_1 + t^2 = 0. \]

The new relation no longer fits into the previous Pfaffian format. This choice of $R_{14}$ is crucial in finding the $\mathbb{Q}$-Gorenstein smoothing:

\textbf{Proposition 4.1.} Consider the family $\mathcal{X}/\Lambda$ defined by relations

\begin{align*}
\tilde{R}_1 &: x_0y - x_3^4 + \lambda^3 \tau w = 0 \quad \tilde{R}_2 : x_0w - x_3^2 y + \lambda u_0 = 0 \\
\tilde{R}_3 &: x_1w - y^2 + \lambda u_1 = 0 \quad \tilde{R}_4 : x_0u_1 - x_1u_0 + \lambda^2 \tau y w = 0 \\
\tilde{R}_5 &: x_1^2 u_1 - y u_0 + \lambda^2 \tau w^2 = 0 \\
\tilde{R}_6 &: x_1y u_1 - w u_0 - \lambda t = 0 \\
\tilde{R}_7 &: x_0t - u_0^2 + \lambda \tau x_1 y^2 w = 0 \\
\tilde{R}_8 &: x_1 t - u_0 u_1 + \lambda \tau y w^2 = 0 \\
\tilde{R}_9 &: x_1 u_1^2 - y t - \lambda \tau w^3 = 0 \\
\tilde{R}_{10} &: y u_1^2 - w t + \lambda \tilde{P} = 0 \\
\tilde{R}_{11} &: x_0 \tilde{P} + \tau x_1^2 w^3 + u_0 t - \lambda^2 \tau w u_1^2 = 0 \\
\tilde{R}_{12} &: x_1 \tilde{P} + \tau y w^3 + u_1 t = 0 \\
\tilde{R}_{13} &: y \tilde{P} + \tau w^4 + u_1^3 = 0 \\
\tilde{R}_{14} &: u_0 \tilde{P} + \tau y^2 w^2 u_1 + t^2 = 0
\end{align*}

where $\lambda$ is the coordinate on $0 \in \Lambda \subset \mathbb{C}$ and $\tilde{P}$ is a general polynomial of degree 10 satisfying $\tilde{P}|_{\lambda=0} = P$. If $\tau \neq 0$ then the central fiber $\mathcal{X}_0$ is a cuspidal Rana–Urzúa surface with a single $\frac{1}{125}(1,14)$ point, and $\mathcal{X}/\Lambda$ is a $\mathbb{Q}$-Gorenstein smoothing of $\mathcal{X}_0$.

\textbf{Proof.} By construction, the fiber over $\lambda = 0$ is a Rana–Urzúa surface because the relations match those of §3.6. By Lemma 4.2 below, the general fiber with $\lambda \neq 0$ is a smooth surface. Hence $\mathcal{X}/\Lambda$ is flat.

The $\mathbb{Q}$-Gorenstein condition is only relevant near the singular point of $\mathcal{X}$ at $P_{\tilde{z}}$ over $\lambda = 0$. Substituting $z = 1$ into the equations $\tilde{R}_{10}$, $\tilde{R}_{11}$, $\tilde{R}_{12}$, $\tilde{R}_{13}$, $\tilde{R}_{14}$ allows us to eliminate $x_0, x_1, y, u_0$ respectively in
The same way as the proof of Corollary 3.11, so that we are left with 
\((u_1^5 - wt + \lambda + \text{h.o.t.} = 0)\) in \(\frac{1}{5}(1, 3, 4) \times \Lambda\). Thus the family \(\mathcal{X}/\Lambda\) induces a \(\mathbb{Q}\)-Gorenstein smoothing of the \(\frac{1}{25}(1, 14)\) point on the fiber over \(\lambda = 0\) (see e.g. [Hac16, Ex. 2.1.8]).

**Lemma 4.2.** The family \(\mathcal{X}/\Lambda\) fits into the matrix format

\[
Pf_4 M_1 = Pf_4 M_2 = M_1 V_1 = M_2 V_2 = 0,
\]

where

\[
M_1 = \begin{pmatrix}
\lambda & x_1 & w & u_1 \\
u_0 & y u_1 & t & \tau w^2 \\
\quad \text{sym} & \quad & \\
\end{pmatrix}, \quad M_2 = \begin{pmatrix}
\lambda & x_1 & y & t \\
y & w & u_1^2 & \tau w^3 \\
\quad \text{sym} & \quad & \quad \\
\end{pmatrix},
\]

\[
V_1 = t(-\lambda t y^2 w, u_0, -x_1 y, x_0, 0), \\
V_2 = t(u_0, -\lambda^2 t w, x_1^2, -x_0, 0).
\]

If \(\lambda \neq 0\) then the fiber \(\mathcal{X}_\lambda\) is isomorphic to a nonsingular hypersurface of weighted degree 10 in \(\mathbb{P}(1, 1, 2, 5)\).

**Proof.** Up to sign, the Pfaffians of \(M_1\) are \(\tilde{R}_{10}, \tilde{R}_{14}, \tilde{R}_{12}, \tilde{R}_6, \tilde{R}_8\) and the Pfaffians of \(M_2\) are \(\tilde{R}_{10}, \tilde{R}_{13}, \tilde{R}_{12}, \tilde{R}_3, \tilde{R}_0\). The product \(M_1 V_1\) gives \(\tilde{R}_2, y \tilde{R}_4, \tilde{R}_7, y \tilde{R}_8, \tilde{R}_{11} + \tau w(x_1 w - \lambda u_1) \tilde{R}_3\) and \(M_2 V_2\) gives \(\tilde{R}_1, \tilde{R}_2, \tilde{R}_4, \tilde{R}_5, \tilde{R}_{11}\). Thus taken all together these generate the ideal defining \(\mathcal{X}/\Lambda\).

Assume now that \(\lambda \neq 0\). We perform row and column operations on \(M_1\) preserving antisymmetry, and apply the complementary row operations to \(V_1\) so that the products \(M_i V_i\) are preserved. This gives new matrices \(M'_1\) and \(V'_1\):

\[
M'_1 = \begin{pmatrix}
\lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\quad \text{sym} & \quad & \quad \\
\end{pmatrix}, \quad M'_2 = \begin{pmatrix}
\lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\quad \text{sym} & \quad & \quad \\
\end{pmatrix},
\]

\[
V'_1 = t\left(\frac{u_0}{\lambda}, \frac{R_2}{\lambda}, -x_1 y, x_0, 0\right), \quad V'_2 = t\left(\frac{R_2}{\lambda}, \frac{R_3}{\lambda}, x_1^2, -x_0, 0\right).
\]

Thus the format reduces to \(\tilde{R}_1, \tilde{R}_2, \tilde{R}_3, \tilde{R}_6, \tilde{R}_8, \tilde{R}_9, \tilde{R}_{10}, y \tilde{R}_4\).

Moreover, the assumption \(\lambda \neq 0\) enables us to rewrite \(w, u_0, u_1, t\) using relations \(\tilde{R}_1, \tilde{R}_2, \tilde{R}_3, \tilde{R}_6\) respectively:

\[
x_0 y - x_1^3 = -\lambda^3 t w, \quad x_0 w - x_1^2 y = -\lambda u_0, \\
x_1 w - y^2 = -\lambda u_1, \quad x_1 y u_1 - w u_0 = \lambda t.
\]
Doing this transforms $\tilde{R}_8, \tilde{R}_9, y\tilde{R}_4$ into identities rather than relations. For example $\tilde{R}_4$ reduces to the identity:

$$x_0 u_1 - x_1 u_0 + \lambda^2 \tau y w$$

$$\equiv x_0 \left( \frac{x_1 w - y^2}{-\lambda} \right) - x_1 \left( \frac{x_0 w - x_1^2 y}{-\lambda} \right) + \lambda^2 \tau y \left( \frac{x_0 y - x_1^3}{-\lambda^3 \tau} \right)$$

$$\equiv \frac{1}{\lambda} \left( -x_0 x_1 w + x_0 y^2 + x_1 x_0 w - x_1^3 y - x_0 y^2 + x_1^3 y \right)$$

$$\equiv 0.$$

The remaining relation is $\tilde{R}_{10}$. By repeatedly using $\tilde{R}_1, \tilde{R}_2, \tilde{R}_3, \tilde{R}_6$ to eliminate $w, u_0, u_1, t$ as above, we are left with a relation between generators $x_0, x_1, y, z$, which we display here, multiplied by $\lambda^{11}$ for readability:

$$x_0 (x_0 y - x_1^3)^3 - 3\lambda^3 x_1^2 y (x_0 y - x_1^3)^2 + 3\lambda^6 \tau^{-1} x_1 y^3 (x_0 y - x_1^3) + \lambda^9 y^5 + \lambda^{12} \tilde{P} = 0.$$

Since $\tilde{P}$ is general, this is a nonsingular surface. □

The last equation in the proof can be interpreted as follows:

**Corollary 4.3.** Let $X \to \Lambda$ be the 1-parameter smoothing of a cuspidal RU-surface $X_0$ constructed above. Then there is a diagram

$$\xymatrix{X \ar[r]^\phi \ar[d] & \tilde{X} \ar[d] \ar[r]^\text{deg}^{10} & \mathbb{P}(1, 1, 2, 5) \times \Lambda \ar[d] \\
\Lambda & \Lambda}$

where $\phi$ is birational and an isomorphism outside the central fibers $X_0$ and $\tilde{X}_0 = (x_0(x_1^3 - x_0 y)^3 = 0)$.

This phenomenon could be taken as a starting point for a comparison of the closure of the Gieseker component and a GIT moduli space of hypersurfaces of degree 10 in $\mathbb{P}(1, 1, 2, 5)$.

5. I-surfaces with two T-singularities

In this Section, we apply and extend the results of Section 3 to understand how T-divisors in $\overline{M}_{1,3}$ intersect each other. We recall that for an I-surface $X$ with a unique non-canonical T-singularity $Q$, the point $Q$ is as in Table 1 by [FPRR21, Thm. 1.1].

Suppose $X$ is an I-surface with two distinct T-singularities $Q_1$ and $Q_2$ of index $i_1$, resp. $i_2$ (with $i_1 \neq i_2$) belonging to the above list, where for simplicity of exposition we restrict to the case $d = 1$ in the index 2 case. In particular, these surfaces should correspond to general points in the intersection of two T-divisors in $\overline{M}_{1,3}$. 
Table 1. T-singularities $\frac{1}{dn^2}(1, dna - 1)$ occurring individually

| Cartier index $n$ | $d$ | T-singularity | T-string |
|------------------|-----|---------------|----------|
| 2                | $d \leq 32$ | $\frac{1}{4d}(1, 2d - 1)$ | [4] or [3, 3] or [3, 2, ..., 3] |
| 3                | 2   | $\frac{1}{12}(1, 5)$ | [4, 3, 2] |
| 5                | 1   | $\frac{1}{25}(1, 14)$ | [2, 5, 3] |

Table 2. Codiscrepancy divisors

| T-singularity $\Delta_j$ | $\frac{1}{4}(1, 1)$ | $\frac{1}{2}A_j$ with $A_j^2 = -4$ |
|---------------------------|----------------------|-----------------------------------|
| $\frac{1}{12}(1, 5)$     | $\frac{1}{3}A_j + \frac{2}{3}B_j + \frac{1}{3}C_j$ with $A_j^2 = -4, B_j^2 = -3, C_j^2 = -2$ |
| $\frac{1}{25}(1, 14)$   | $\frac{3}{5}A_j + \frac{2}{5}B_j + \frac{1}{5}C_j$ with $A_j^2 = -3, B_j^2 = -5, C_j^2 = -2$ |

We let $f: \tilde{Y} \to X$ be the minimal desingularization and $\epsilon: \tilde{Y} \to Y$ be the morphism to a minimal model:

$$
\begin{array}{ccc}
X & \xrightarrow{f} & \tilde{Y} \\
\xleftarrow{\epsilon} & & \xrightarrow{\text{resolution}} \\
& & \xrightarrow{\text{sequence of blow-ups}} \\
& & Y
\end{array}
$$

Here $\epsilon$ is the composition of $k$ blow-ups at $P_1, \ldots, P_k$ (possibly infinitely near). We will denote the exceptional divisor of $\epsilon$ by $E = \sum_{i=1}^k E_i$, where the $E_i$’s are the $(-1)$-curves of each blow-up (possibly not irreducible, nor reduced). In particular, $K_{\tilde{Y}} = \epsilon^* K_Y + E$.

We write

$$
f^* K_X = K_{\tilde{Y}} + \Delta
$$

where $\Delta = \Delta_1 + \Delta_2$ is the codiscrepancy divisor of $f$ with $\Delta_j$ supported on $f^{-1}(Q_j)$. These are chains of smooth rational curves with self-intersections and coefficients as in Table 2.

Since $X$ has only rational singularities then we have $q(Y) = q(\tilde{Y}) = q(X) = 0$ and $p_g(Y) = p_g(\tilde{Y}) = p_g(X) = 2$, so the Kodaira dimension of $\tilde{Y}$ is positive and the minimal model $Y$ is unique. Arguing as in [FPRR21, §4] we see that $Y$ is a properly elliptic surface and moreover, by [Lee99, Prop. 20], we have

$$
K_{\tilde{Y}}^2 = 1 + (\Delta_1)^2 + (\Delta_2)^2 = d_1 - r_1 + d_2 - r_2 - 1
$$

where $r_j$ is the length of the T-string and $d_j$ is as in Table 1.
5.1. **Useful results.** We summarize some results that will be used throughout this section. First we recall some well known facts about the structure of the \(\epsilon\)-exceptional divisors.

**Remark 5.1.** Let \(\Gamma, \Gamma'\) be two distinct irreducible \((-1)\)-curves on \(\widetilde{Y}\). Then \(\Gamma \cap \Gamma' = \emptyset\) since \(\widetilde{Y}\) has Kodaira dimension 1.

**Remark 5.2.** Let \(E = \sum_{i=1}^{k} E_i\) be the exceptional divisor of \(\epsilon\), where the \(E_i\)'s are the \((-1)\)-curves of each blow-up. Then every \(E_i\) contains at least one irreducible \((-1)\)-curve \(\Gamma_i\). Moreover if \(D \subset E\) is an \(\epsilon\)-exceptional irreducible \((-n)\)-curve with \(n \geq 2\), then \(E - D \geq \sum m_h \Gamma_h\), with \(\Gamma_h\)'s irreducible \((-1)\)-curves and \(\sum m_h \geq k\).

Now we show some properties of the components of the T-strings (i.e. \(f\)-exceptional divisors) and the \(\epsilon\)-exceptional divisors.

From now on, we abuse notation and denote the pull back of \(K_Y\) to \(\widetilde{Y}\) by \(K_{\widetilde{Y}}\), and the pull back of \(K_X\) to \(\widetilde{Y}\) by \(K_{\widetilde{Y}}\). With this convention, we can write

\[
K_X = K_{\widetilde{Y}} + \Delta_1 + \Delta_2 = K_Y + E + \Delta_1 + \Delta_2.
\]

**Remark 5.3.** Let \(\Gamma \subset \widetilde{Y}\) be an \(\epsilon\)-exceptional irreducible \((-1)\)-curve. Then \(K_X \Gamma > 0\).

**Lemma 5.4.** Let \(D \subset \widetilde{Y}\) be an irreducible \((-2)\)-curve.

Then \(K_Y D = ED = 0\). In particular:

(i) if \(D \subset \widetilde{Y}\) is not contracted by \(\epsilon\), then \(\epsilon(D) \subset Y\) is again a \((-2)\)-curve and \(D \cap E = \emptyset\);

(ii) if \(D \subset \widetilde{Y}\) is contracted by \(\epsilon\), then \(D\) intersects only one \((-1)\)-curve \(\Gamma \subset E\) and \(D \Gamma = 1\).

**Proof.** Since \(D^2 = -2\), by adjunction we have

\[
0 = K_{\widetilde{Y}} D = K_Y D + ED
\]

and \(K_Y D \geq 0\) because \(K_Y\) is nef.

If \(D\) is not \(\epsilon\)-exceptional then the second summand is non-negative, giving \(ED = 0\) and the first item.

If \(D\) is \(\epsilon\)-exceptional then the first summand is zero, so also the second. To conclude the proof of (ii) assume that \(D\) intersects two distinct \((-1)\)-curves \(\Gamma, \Gamma'\). Contracting \(\Gamma\) and \(\Gamma'\), \(D\) becomes a curve with non-negative self intersection and negative intersection with the canonical divisor. This is absurd since \(Y\) is an elliptic surface. If \(D \Gamma \geq 2\) then contracting \(\Gamma\) we obtain a curve which is not \(\epsilon\)-exceptional and has negative intersection with the canonical divisor, impossible on the minimal surface \(Y\). \(\square\)

**Lemma 5.5.** Let \(D \subset \widetilde{Y}\) be an irreducible \((-3)\)-curve that is not \(\epsilon\)-exceptional. Then \(\epsilon(D) \subset Y\) is either a \((-3)\)-curve or a \((-2)\)-curve.
Proof. Since $D^2 = -3$, by adjunction we have

$$1 = K_YD = KYD + ED,$$

$K_YD \geq 0$ because $K_Y$ is nef and $ED \geq 0$ since $D$ is not exceptional. If $ED = 0$, then $\epsilon(D)$ is again a $-3$-curve. So assume $K_YD = 0$ and $ED = 1$ and write $D = \epsilon^*(\epsilon(D)) - \sum m_iE_i$, with $m_i \geq 0$. We have $1 = DE = \sum m_i$, so $D$ is obtained by blowing up $\epsilon(D)$ once at a smooth point and $\epsilon(D)$ is a $-2$-curve.

Lemma 5.6. We have

$$K_XK_Y = (\Delta_1 + \Delta_2)K_Y \geq \frac{1}{2}, \quad K_XE \leq \frac{1}{2}$$

Proof. We have

$$K_XK_Y = (K_Y + E + \Delta_1 + \Delta_2)K_Y = (\Delta_1 + \Delta_2)K_Y$$

since $K^2_Y = K_YE = 0$. Moreover, we have $K_XK_Y > 0$ because $K_Y$ moves and $K_X$ is positive outside the support of the $f$-exceptional locus. Looking at the description of the codiscrepancy divisors we note that the divisors with coefficient less than $\frac{1}{2}$ are $(-2)$-curves. Then by the above Lemma 5.4 we obtain $(\Delta_1 + \Delta_2)K_Y \geq \frac{1}{2}$. The second inequality follows since $1 = K^2_Y = K_X(K_Y + E)$.

Proposition 5.7. Let $D \subset E \subset \tilde{Y}$ be an $\epsilon$-exceptional irreducible $(-n)$-curve, with $n \geq 2$. Then $D$ is also $f$-exceptional.

Proof. Suppose not, then we have $0 \leq K_XD = K_YD + (\Delta_1 + \Delta_2)D = n - 2 + (\Delta_1 + \Delta_2)D$. If $n \geq 3$ then we get $K_XD \geq 1$, which is absurd since $K_XD \leq K_XE \leq \frac{1}{2}$ by Lemma 5.6.

Therefore we may assume $n = 2$. We first consider the case where $Q_1, Q_2$ are T-singularities of index $i_1 = 5$, resp. $i_2 = 2$, so that by equation (6), $\epsilon: \tilde{Y} \to Y$ is the composition of three blow-ups. In this case $K_XD = (\Delta_1 + \Delta_2)D \geq \frac{2}{5}$ (see the codiscrepancies shown above). By Remark 5.2 and Remark 5.3, since we have three blow-ups and for every irreducible $(-1)$-curve $\Gamma \subset E$ it is $K_X\Gamma \geq \frac{1}{10}$, we obtain $K_X(E - D) \geq \frac{3}{10}$. Whence $K_XE = K_XD + K_X(E - D) \geq \frac{2}{5} + \frac{3}{10} > \frac{1}{2}$, which contradicts Lemma 5.6.

The cases $(i_1, i_2) = (5, 3)$ and $(i_1, i_2) = (3, 2)$ are similar. □

Corollary 5.8. Let $D \subset E \subset \tilde{Y}$ be an $\epsilon$-exceptional irreducible $(-n)$-curve. If $n \geq 2$ then $K_XD = 0$; if $n = 1$ then $K_XD \geq \frac{1}{112}$.

5.2. I-surfaces with a singularity of type $\frac{1}{25}(1,14)$ and a singularity of index 2 do not exist. In this section we are going to prove the following

Proposition 5.9. There are no T-singular I-surfaces with a singularity of type $\frac{1}{25}(1,14)$ and a singularity of type $\frac{1}{4}(1,1)$. 
Remark 5.10. A generalisation of the below proof shows that there are no $T$-singular $I$-surfaces with a singularity $\frac{1}{35}(1, 14)$ and a singularity of type $\frac{1}{7d}(1, 2d - 1)$. We do not include the details here but it involves keeping track of the possible intersections with the index 2 $T$-chain. Moreover, we do not know if there is an $I$-surface with more general singularities of index 5 and index 2.

Proof. Assume by a contradiction the existence of such a surface and consider the diagram 5 where the resolution of the two singular points yields a string of type $[3, 5, 2]$ and a string of type $[4]$. The strategy of the proof consists in studying the possible configuration of $\varepsilon$-exceptional irreducible $(-1)$-curves.

First note that by equation (6), $\varepsilon: \tilde{Y} \rightarrow Y$ is a composition of three blow-ups. Let $Q_1$ be the point of index 5 and $Q_2$ be the point of index 2. The codiscrepancy divisor corresponding to $Q_1$ (respectively $Q_2$) is $\Delta_1 = \frac{3}{5}A_1 + \frac{4}{5}B_1 + \frac{2}{5}C_1$ (resp. $\Delta_2 = \frac{1}{2}A_2$). Thus we can write

$$K_X = K_Y + \sum_{i=1}^{3} E_i + \Delta_1 + \Delta_2.$$

By Remark 5.2, Lemma 5.6, Corollary 5.8 we have $\frac{3}{10} \leq K_X E \leq \frac{1}{2}$ and $\frac{1}{2} \leq K_X K_Y \leq \frac{7}{10}$. Now

$$K_X K_Y = (\Delta_1 + \Delta_2) K_Y = (\frac{3}{5}A_1 + \frac{4}{5}B_1 + \frac{2}{5}C_1 + \frac{1}{2}A_2) K_Y,$$

hence the only possibilities for $K_X K_Y$ are $\frac{1}{2}$, $\frac{3}{5}$ and for $K_X E$ we get:

$$(7) \quad K_X E = \frac{1}{2} \quad \text{or} \quad K_X E = \frac{2}{5}.$$

Now let $\Gamma$ be an $\varepsilon$-exceptional irreducible $(-1)$-curve. We have $K_X \Gamma \geq \frac{1}{10}$ since $X$ has index 10. Now, since $K_X (E - \Gamma) \geq \frac{2}{10}$ we obtain $K_X \Gamma \leq \frac{7}{10}$. Hence since $K_X = K_Y + \Delta_1 + \Delta_2$ and $K_Y \Gamma = -1$, we get $\frac{11}{10} \leq (\Delta_1 + \Delta_2) \Gamma \leq \frac{13}{10}$.

We exclude the cases where $\Gamma C_1 = 2, 3$ and $\Gamma A_1 = 2$ since they contradict Lemma 5.4(i) and Lemma 5.5. Therefore we are left with the following possibilities:

$$\begin{align*}
(\Delta_1 + \Delta_2) \Gamma &= \frac{3}{5} + \frac{1}{2} = \frac{3}{5}A_1 \Gamma + \frac{1}{2}A_2 \Gamma \quad ; \quad K_X \Gamma = \frac{1}{10} \\
(\Delta_1 + \Delta_2) \Gamma &= \frac{4}{5} + \frac{2}{5} = \frac{4}{5}B_1 \Gamma + \frac{2}{5}C_1 \Gamma \quad ; \quad K_X \Gamma = \frac{2}{10} \\
(\Delta_1 + \Delta_2) \Gamma &= \frac{4}{5} + \frac{1}{2} = \frac{4}{5}B_1 \Gamma + \frac{1}{2}A_2 \Gamma \quad ; \quad K_X \Gamma = \frac{3}{10}
\end{align*}$$

We will show that any of the above configurations gives a contradiction. Note that, since in view of Prop. 5.7 none of the $\Gamma$’s gives a three step contraction on its own, we need to combine them.
Figure 5. Configurations of type (I), (III) and (II)

Configuration (II). Assume that there is a curve $\Gamma$ of type (II). Since $K_X(\sum_{i=1}^{3} E_i) = \frac{1}{2}$ or $\frac{2}{5}$, there is a second exceptional curve $\Gamma'$ of type (I).

First blow down $\Gamma$. Then the image of $C_1$ is a $(-1)$-curve which intersects the image of $B_1$ in two points. Blowing this down we obtain a nodal curve which intersects the image of $A_1$ in 1 point.

Now blow down $\Gamma'$. We obtain a minimal elliptic surface $Y$ with $\epsilon(A_1)K_Y = \epsilon(B_1)K_Y = 0$, which implies that $\epsilon(A_1) + \epsilon(B_1)$ is contained in a fiber, contradicting Kodaira’s list.

So a type (II) configuration does not exist.

Configuration (III). Consider a type (III) curve $\Gamma$. Since a type (II) configuration does not exist, then by equation (7) there is a second exceptional curve $\Gamma'$ of type (I).

Then blowing down $\Gamma$ and $\Gamma'$ we see that there are no $(-1)$-curves arising from $\Delta_1 + \Delta_2$. Hence there exists a third curve $\Gamma''$ of type (I). Blowing it down, we obtain another $(-1)$-curve, which is absurd.

So a type (III) configuration does not exist.

Configuration (I). We are left with the case where all the exceptional curves are of type (I). If there exist two curves $\Gamma, \Gamma'$ of type (I), then blowing them down and arguing as in the previous case we obtain a curve having negative intersection with the canonical divisor, which is absurd. Since, as we noticed earlier, there are at least two irreducible $-1$-curves on $\tilde{Y}$, the proof is complete. \hfill $\square$

5.3. The divisor of surfaces with one singularity of index 3.

In this subsection we study the divisor of surfaces with an additional singularity of index 3, and show that it fits into the original Pfaffian format of §3.6.

Lemma 5.11. In the notation of §3.6, if $\tau = 0$ and $P_{10}, \theta$ are general then $X$ has a $\frac{1}{6}(1, 5)$ singularity in addition to the $\frac{1}{25}(1, 14)$ singularity.
Proof. The index 3 coordinate point \( P_w \) is contained in \( X \) because \( w^4 \) no longer appears in \( R_{13} \) when \( \tau = 0 \). In a neighbourhood of \( P_w \), the relations \( R_2, R_3, R_6, R_{10} \) eliminate \( x_0, x_1, u_0, t \) respectively. Thus the local coordinates at \( P_w \) are \( y, u_1, z \). Since \( \theta \neq 0 \) in general, \( R_{13} \) locally defines \( X \) as \( w^2u_1z = y^3 + \cdots \) in \( \frac{1}{3}(1, u_1, 2, 2) \) (because in general the monomial \( y^2w^2 \) appears in \( P_{10} \)). By Remark 3.10, this is a \( \frac{1}{9}(1, 5) \)-singularity. □

Proposition 5.12. Suppose that \( \tau = 0 \) and consider the surfaces \( X_{\lambda, \theta} \) in \( \mathbb{P}(1, 1, 2, 3, 4, 5, 7) \) defined by

\[
Pf_4 \tilde{M} = \tilde{M}\tilde{V} = 0
\]

where

\[
\tilde{M} = \begin{pmatrix}
0 & 0 & x_0 & x_1^2 & u_0 \\
0 & 1 & y & u_1 \\
u_0 & x_1u_1 & t & \theta u_1z \\
\text{sym} & & & &
\end{pmatrix}, \quad \tilde{V} = \begin{pmatrix}
0 \\
-w \theta \\
0 \\
-w \\
-y \\
\lambda
\end{pmatrix}
\]

and \( \lambda, \theta \) are parameters satisfying \( \lambda \theta = 0 \).

If \( \lambda = \theta = 0 \) then \( X_{0,0} \) is a surface with one \( \frac{1}{25}(1, 14) \)-singularity and one \( \frac{1}{18}(1, 5) \)-singularity.

If \( \lambda \neq 0, \theta = 0 \) then \( X_{\lambda,0} \) is a surface \( X_{3,10} \subset \mathbb{P}(1, 1, 2, 3, 5) \) with one \( \frac{1}{18}(1, 5) \) singularity.

If \( \lambda = 0, \theta \neq 0 \) then \( X_{0,\theta} \) is an RU-surface with an extra \( \frac{1}{5}(1, 5) \) singularity as in Lemma 5.11.

Proof. Clearly, if \( \theta \neq 0 \) then we are in the situation described by Lemma 5.11.

If \( \lambda \neq 0 \), we can adapt the proof of Proposition 4.1. This time, since \( \tau = 0 \), the generator \( w \) can not be eliminated. It turns out that the general fiber \( X_{\lambda,0} \) has equations

\[
X_{\lambda,0}: (x_0y - x_1^3 = \lambda^3P + y^5 - 3x_1y^3w + 3x_1^2yw^2 - x_0w^3 = 0) \subset \mathbb{P}(1, 1, 2, 3, 5).
\]

That is, \( X_{\lambda,0} \) has a \( \frac{1}{18}(1, 5) \) singularity (see [FPRR21]).

If \( \theta = \lambda = 0 \) then we get a RU-surface with an extra singularity of index 3. Near the coordinate point \( P_w \) we use \( R_2, R_3, R_6, R_{10} \) to eliminate \( x_0, x_1, u_0, t \) respectively. Then \( R_{13} \) cuts out a hypersurface

\[
(yu_1 + yz^2 + y^2z + \cdots = 0) \subset \frac{1}{3}(1, u_1, 2y, 2z),
\]

because the monomial \( w^2u_1 \) appears in \( P_{10} \). This is local analytically a \( \frac{1}{18}(1, 5) \)-singularity. □

We have thus shown that both the Gieseker component and the RU-component contain in their closures a divisor parametrising surfaces with an additional T-singularity of index 3 and these divisors meet
the intersection divisor, i.e., the divisor of cuspidal RU-surfaces in an irreducible subset of codimension two parametrising surfaces with one \( \frac{1}{18}(1, 14) \)-singularity and one \( \frac{1}{25}(1, 5) \)-singularity.

These correspond to the central fiber of the family \((\lambda = \tau = \theta = 0)\) and their minimal resolution is an elliptic surface with a \((-3)\)-section and a singular fiber of type III (see Lemma 3.2). Thus geometrically, these surfaces can be obtained as follows.

**Example 5.13.** We consider an elliptic surface with \( p_g = 2 \), a fiber of type III, and a \((-3)\)-section.

We blow-up the singular point \( p_1 \) and its infinitesimal point \( p_2 \) given by the intersection of the two branches of the singular fiber. We blow-up two more points as shown in Figure 6.

With this procedure we obtain a string \([2, 5, 3]\) and a string \([4, 3, 2]\) connected by a \((-1)\)-curve. Blowing down the two strings we obtain an I-surface with a singularity of type \( \frac{1}{18}(1, 5) \) and a singularity of type \( \frac{1}{25}(1, 14) \).

![Figure 6](image.png)

**Figure 6.** Construction of an I-surface with a singularity of type \( \frac{1}{18}(1, 5) \) and a singularity of type \( \frac{1}{25}(1, 14) \).

There is a second way to construct an I-surface with a singularity of type \( \frac{1}{18}(1, 5) \) and a singularity of type \( \frac{1}{25}(1, 14) \). Algebraically, we assume that \( \lambda = \tau = 0 \) and the coefficient of \( t_1^{16} \) in \( l_0(t_0, t_1) \) vanishes (again, see Lemma 3.2). Then the elliptic surface \( Y \) has an \( I_3 \) fiber over \((0, 1; 0, 1, 0)\). Since \( \theta \) is generic here, this example is not smoothable as can also be deduced from the non-existence of an involution.

**Example 5.14.** We consider an elliptic surface with \( p_g = 2 \), a fiber of type \( I_3 \), and a \((-3)\)-section.
We blow-up the singular points $p_1, p_2$ of the singular fiber, and two infinitesimal point $p_3, p_4$ as shown in Figure 7.

With this procedure we obtain a string $[2, 5, 3]$ and a string $[4, 3, 2]$ connected by two $(-1)$-curves. Blowing down the two strings we obtain an I-surface with a singularity of type $\frac{1}{18}(1, 5)$ and a singularity of type $\frac{1}{25}(1, 14)$.

**Remark 5.15.** Let $X$ be an I-surface with a singularity of type $\frac{1}{18}(1, 5)$ and a singularity of type $\frac{1}{25}(1, 14)$. Arguing as in Proposition 5.9 one can see that $X$ is as in the above Examples 5.13, 5.14.

### 5.4. I-surfaces with a singularity of index 2 and a singularity of index 3.

Let us write the results of [FPRR21] in a slightly different form to exhibit clearly the intersection of the index 2 and index 3 divisors: consider the surfaces

$$X = X_{\mu, \nu, f} : \left( \frac{x_0 y - x_1^3 - \mu u}{z^2 - \nu y^5 - \tilde{f}_{10}(x_0, x_1, y, u)} = 0 \right) \subset \mathbb{P}(1, 1, 2, 3, 5),$$

where $\mu, \nu$ are parameters and $\tilde{f}_{10}$ is sufficiently general but not containing the monomial $y^5$. This is an admissible family of stable surfaces and

- for $\mu \nu \neq 0$ we can eliminate the variable $u$ via the first equation and get a classical I-surface;
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A small dimension count thus confirms that these two T-divisors in the closure of the Gieseker components intersect as expected in an irreducible subset of codimension two whose general element is a surface as in the last item.

We complement this algebraic discussion by an explicit geometric construction of surfaces in the intersection.

Example 5.16. We consider an elliptic surface with $p_g = 2$, an $I_2$ fiber, an $I_r$ fiber ($r \geq 2$) and a $(-3)$-section.

We blow-up the singular points $p_1, p_2$ of the fiber of type $I_2$.

With this procedure we obtain a string $[4]$ and a string $[4, 3, 2]$ connected by two $(-1)$-curves (see Figure 8). Blowing down the two strings we obtain an $I$-surface with a singularity of type $\frac{1}{18}(1, 5)$ and a singularity of type $\frac{1}{4}(1, 1)$.

Remark 5.17. Let $X$ be an I-surface with a singularity of type $\frac{1}{18}(1, 5)$ and a singularity of type $\frac{1}{4}(1, 1)$. Arguing as in Proposition 5.9 one can see that $X$ is as in the above Example 5.16.

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