Ultrametric properties of the attractor spaces for random iterated linear function systems

To cite this article: A G Buchovets and P V Moskalev 2018 J. Phys.: Conf. Ser. 973 012028

View the article online for updates and enhancements.
Ultrametric properties of the attractor spaces for random iterated linear function systems

A G Buchovets and P V Moskalev
Voronezh State Agricultural University, 1 Michurin street, Voronezh, 394087, Russia
E-mail: abuchovets@mail.ru, moskaleff@mail.ru

Abstract. We investigate attractors of random iterated linear function systems as independent spaces embedded in the ordinary Euclidean space. The introduction on the set of attractor points of a metric that satisfies the strengthened triangle inequality makes this space ultrametric. Then inherent in ultrametric spaces the properties of disconnectedness and hierarchical self-similarity make it possible to define an attractor as a fractal. We note that a rigorous proof of these properties in the case of an ordinary Euclidean space is very difficult.

1. Introduction
Expanding the practical use of ultrametric spaces is an important problem in mathematical modeling. The classical solution of this problem consists in mapping the model into p-adic spaces, as it is done in mathematical physics and cognitive sciences [9, 7], in astrophysics and genome research [6, 2], in number theory and complex systems [1, 8] and so on. In this case, the problems of mapping a model into an ultrametric space are solved in various ways. For example, in [11] the author constructs an ultrametric space from $R^n$ by stochastic generating of new points and increasing the dimension of the original space. Another approach is to determine the parameter that characterizes the mapping of the Euclidean metric in the ultrametric. In this paper, we use cluster analysis to determine the ultrametric classification space associated with the original characteristic space by randomized procedures.

2. Materials and methods
Iterated function system (IFS) in the most general form represent a certain set of functions $\{f_j\}_{j=1}^k$ that are executed in a given sequence. As a result of the implementation of this IFS, there is some compact set, which, as a rule (but not necessarily), has fractal properties [3]. However, for computer implementation it is more convenient to execute on each iteration only one of the functions $f_i$ in accordance with a given probability distribution $\{f_j|p_j\}_{j=1}^k$, where $p_j > 0$, $\sum_{j=1}^k p_j = 1$ [10]. The use of a random order in the execution of functions is called randomization, and the system obtained in this way is called a random iterated function system (RIFS) [5, 4].
2.1. Random iterated linear function systems

The simplest form for functions \( f_j \) from the IFS is linear, and under additional constraints the function is a convex combination:

\[
    x_{i+1} = \xi x_i + (1 - \xi) z_{i,j},
\]

(1)

where \( \xi \) — parameter; \( Z = \{ z_{j} \}_{j=1}^{k} \) — parameters defining the form of linear functions, \( Z \subset R^p \); \( i = 1, 2, \ldots, n \) — iteration number.

The iterative process begins at an arbitrary point \( x_0 \in R^p \), not necessarily, but preferably \( x_0 \in \text{conv}(Z) \). After \( n \) iterations \( x_{i+1} = f_j(x_i) \), where \( i = 1, 2, \ldots, n, j \in [1, k] \), a set of points \( X = \{ x_i \}_{i=1}^{n} \), called protofractals, will be obtained. The limit set, called the RIFS attractor, was shown in [4] to be compact, open-closed, has zero Lebesgue measure and fractional fractal dimension [9]. In the general case, the attractor RIFS is a homomorphic mapping of the Cantor set. This way of constructing the RIFS attractor will be denoted by F1.

This procedure was implemented by the authors in the RIFS package [10] with the free programming language and software environment for statistical computing and graphics R. The following listing shows the source code that implements the procedure F1:

```r
> R2ngon <- function(n1=3, n2=1, r=1, o=c(0,0), cycle=FALSE) {
+   phi <- seq(0, 2*pi, length=n1+1)
+   x <- approx(o[1] + r*cos(phi), n=n1*n2+1)$y
+   y <- approx(o[2] + r*sin(phi), n=n1*n2+1)$y
+   if (cycle) return(cbind(x = x, y = y))
+   else return(cbind(x = x[-1], y = y[-1]))
+ }
> preRIFS <- function(n=10000, Z=R2ngon(),
+   p=rep(1/nrow(Z),nrow(Z)), M=rep(1,nrow(Z))) {
+   z <- sample.int(nrow(Z), size=n, prob=p, replace=TRUE)
+   X <- array(0, dim=c(n,ncol(Z)))
+   X[1,] <- colMeans(Z)
+   for (i in 2:n) X[i,] <- (X[i-1,] + M[z[i]]*Z[z[i],])/(1+M[z[i]])
+   return(list(pre=X, proto=Z, distr=cbind(p=P, mu=M), index=z))
+ }
```

Examples of using this code for fractal sets simulation are shown in Fig. 1.

![Figure 1](image-url)
Figure 1 (a) shows the implementation of a fractal set known as the Sierpinski triangle. The parameter $\xi$ is equal to one for it, and the set $Z$ corresponds to an equilateral triangle with equiprobable vertices $Z_{31} = \{(-\frac{1}{2}, -\frac{\sqrt{3}}{2}), (-\frac{1}{2}, \frac{\sqrt{3}}{2}), (1,0)\}$. Figure 1 (b) shows the implementation of a fractal set, which can be called the Sierpinski triangle of the second order. The parameter $\xi$ is equal to two for it, and the set $Z$ corresponds to the union of the vertices of an equilateral triangle with the bases of its medians $Z_{32} = Z_{31} \cup \{(-\frac{1}{2}, 0), (\frac{1}{4}, \frac{\sqrt{3}}{4}), (\frac{1}{4}, -\frac{\sqrt{3}}{4})\}$. All points of the set $Z_{32}$ are also equiprobable.

When implementing the iterative process, we can obtain the following representation for $x_n$ value [3]:

$$x_n = \beta_{n-1}\xi + \beta_{n-2}\xi^2 + \ldots + \beta_0\xi^n,$$

(2)

where $\beta_i = (1 - \xi)z_{i,j}$ for $j = 1, 2, \ldots, k$, and the limiting value of $x^*$ after grouping elements with respect to $z_j$ takes the form

$$x^* = \left(1 - \xi\right)\left(z_1 \sum_s \xi^s + z_2 \sum_t \xi^t + \ldots + z_k \sum_u \xi^u\right) = \sum_{j=1}^{k} a_j z_j,$$

where $a_j = (1 - \xi) \sum_j \xi^t$; $a_j > 0$; $\sum_{j=1}^{k} a_j = 1$.

Consequently, the value of $x^*$ is finite and represents a convex combination of protofractal points $Z = \{z_j\}_{j=1}^{k}$. We can say that the set of coefficients $a_j$ is the barycentric coordinates of the point $x^*$ in the affine coordinate system.

Relation (2) shows that the initial value of $x_0$ entering the term with the highest degree of $\xi$, when iteration is repeated, gets less and less weight in the limiting value $x^*$. In other words, the limiting value of $x^*$ is practically independent of the initial value $x_0$. This fact, already noted in publications (see, for example, [5, p. 99]), allows us to move from procedure F1 to another randomized procedure, which we will denote as F2.

In the case where the parameter $\xi$ is constant and the same for all functions, we can form the RIFS attractor using some analog of the urn scheme. To this end, we introduce the series

$$\mu \sum_{t=1}^{\infty} \xi^t = 1,$$

(3)

where $\mu$ is the valuation constant associated with the parameter $\xi$ by the relation $\mu = (1 - \xi)/\xi$. Elements of the series (3) are distributed over the $k$ classes in accordance with the probability distribution: $\{p_j\}_{j=1}^{k}$, where $p_j > 0$, $\sum_{j=1}^{k} p_j = 1$.

The sums of the series (3) obtained at each step are written as rows $A_r = (a_{r1}, a_{r2}, \ldots, a_{rk})$ for the matrix $A$ of size $n \times k$. In this case, the result of the procedure will be represented as the product of two matrices

$$X = AZ.$$  

(4)

Such a method of constructing the RIFS attractor, as mentioned above, will be denoted by F2. The proposed F2 procedure allows the calculation of new matrices $X$, changing the matrix $Z$, and preserving the matrix $A$, which gives an effective method of forming sets $X$ different from each other.

This procedure was also implemented by the authors in the RIFS package [10] with the free programming language and software environment for statistical computing and graphics R. The following listing shows the source code that implements the procedure F2:

```r
> preRSum0 <- function(n=10000, mu=1, eps=1e-09, 
+     Z=R2ngon(), P=rep(1/nrow(Z),nrow(Z))) {
+     
+     }
Fig. 2 shows the commutative diagram for obtaining the RIFS attractor. By commutativity in this case we mean that there is a nonzero probability that equality \( X_{i+1} = A_{i+1} Z \) will be satisfied.

\[
\mu \sum_{t=1}^{\infty} \xi^t = \mu(\xi + \xi^2 + \ldots + \xi^t + \ldots) = 1
\]

(5)

Note that the space formed by row vectors of the input data matrix is usually called a characteristic space, and the space formed by \( F^2 \) will be called the classification space [4]. It is obvious that the set of rows from the matrix \( A \) does not depend on the values of \( X \), but characterizes the structural features of the matrix \( X \), and in this sense is an invariant of the classification problem.

2.2. Spaces of random partitions

We consider the collection of different subsets of the series formed by the procedure \( F^2 \)

\[
\alpha = \mu \sum_{t'=1}^{\infty} \xi^{t'},
\]

(6)

where \( t' \) are the indices of randomly chosen elements of the series (5). Note that if in the fraction \( \xi = 1/(1 + \mu) \), the denominator \( 1 + \mu \equiv p \) has properties analogous to the properties of prime numbers, then we can assert that these numbers are, in some sense, \( p \)-adic norms of numbers \( 1 + \mu \).

Now consider the set of subsets of the elements of the series (5), which will correspond to the set of numbers formed from the elements of the sequence (5). Obviously, the collection of elements of the sequence (5) is a countable set, while the set of all its subsets has the cardinality of the continuum.

The numbers (6) introduced above can be bijectively mapped into a binary set of digits \( \{0, 1\} \). Each such number can be associated with a sequence of digits 0 and 1, depending on whether the element of the series (5) was excluded or included when forming the number (6).
The previously introduced set of all subsets of the elements of the series (5) is combined with the limiting values 0, 1 and denoted by $A_{\xi} = \{\{0\}\cup0\cup1\}$. This set, as noted above, has the cardinality of the continuum.

For two arbitrary elements $\alpha, \beta \in A_{\xi}$ we define operation $\alpha \otimes \beta = \alpha \land \beta$, where $\alpha$ and $\beta$ are represented by the corresponding binary code. Then $A_{\xi}$ is regarded as an commutative multiplicative group with generator $\xi$ and neutral element 1.

In addition, on this set for the binary representations of the numbers $\alpha, \beta \in A_{\xi}$ we define operation $\alpha \oplus \beta = \alpha + \beta - (\alpha \land \beta)$, where $\alpha + \beta = \alpha \lor \beta$, $\operatorname{sign} \; "-"$ corresponds to the opposite element, such that $\alpha + (-\alpha) = 0$. Then $A_{\xi}$ is regarded as an associative group with neutral element 0.

With the operations introduced above, the set $A_{\xi}$ will represent a ring without zero divisors. We define a norm on the set $A_{\xi}$ as follows

$$|\alpha|_{\xi} = \left|\mu \left(\xi^{s} + \xi^{s'} + \xi^{s''} + \ldots\right)\right|_{\xi} = \xi^{s}. \quad (7)$$

The properties of symmetry and homogeneity of this norm are easily verified by direct transformations. For example, if $s < t$

$$|\alpha \oplus \beta|_{\xi} = \left|\mu \left(\xi^{s} + \xi^{s'} + \xi^{s''} + \ldots\right) + \mu \left(\xi^{t} + \xi^{t'} + \xi^{t''} + \ldots\right)\right|_{\xi} =$$

$$= \left|\mu \left(1 + \xi^{s'-s} + \xi^{s''-s} + \ldots\right) + \mu \left(\xi^{t-s} + \xi^{t'-s} + \xi^{t''-s} + \ldots\right)\right|_{\xi} =$$

$$= \xi^{s} \leq \max\{\xi^{s}, \xi^{t}\} = \max\{|\alpha|_{\xi}, |\beta|_{\xi}\} \leq |\alpha|_{\xi} + |\beta|_{\xi}.$$ 

Then for any $\alpha, \beta, \delta \in A_{\xi}$, we can define an ultrametric

$$d_{\xi}(\alpha, \beta) = |\alpha - \beta|_{\xi} \leq \max\{d_{\xi}(\alpha, \delta), d_{\xi}(\delta, \beta)\}.$$ 

Indeed,

$$d_{\xi}(\alpha, \beta) = |\alpha - \beta|_{\xi} = |(\alpha - \delta) - (\beta - \delta)|_{\xi} \leq$$

$$\leq \max\{|\alpha - \delta|_{\xi}, |\beta - \delta|_{\xi}\} = \max\{d_{\xi}(\alpha, \delta), d_{\xi}(\delta, \beta)\} < d_{\xi}(\alpha, \delta) + d_{\xi}(\delta, \beta).$$

We define the set $A_{\xi}^{\times k}$ as the Cartesian product $A_{\xi}^{\times k} = A_{\xi} \times A_{\xi} \times \cdots \times A_{\xi}$, and $B$ as the subset of $A_{\xi}^{\times k}$ consisting of the objects $A_{b}$

$$B = \left\{ A_{b} = \{a_{b1}, a_{b2}, \ldots, a_{bk}\} \mid a_{bj} \geq 0, \sum_{j=1}^{k} a_{bj} = 1 \right\}.$$

We define a norm on the set $B$ as follows

$$||B||_{\xi} = \max_{1 \leq j \leq k} |a_{ij}|_{\xi},$$

where $a_{j} \in A_{\xi}$, and $B_{i} \in A_{\xi}^{\times k}$. For this norm the triangle inequality is satisfied

$$||B_{s} + B_{t}||_{\xi} = \max_{1 \leq j \leq k} (a_{sj} + a_{tj}) \leq \max_{1 \leq j \leq k} \left(\max\{|a_{sj}|_{\xi}, |a_{tj}|_{\xi}\}\right) =$$

$$= \max\left(\max_{1 \leq j \leq k} \left(|a_{sj}|_{\xi}, |a_{tj}|_{\xi}\right)\right) = \max\left(\max_{1 \leq j \leq k} \left(|a_{sj}|_{\xi}\right), \max_{1 \leq j \leq k} \left(|a_{tj}|_{\xi}\right)\right) =$$

$$= \max\{||B_{s}||_{\xi}, ||B_{t}||_{\xi}\} \leq ||B_{s}||_{\xi} + ||B_{t}||_{\xi}.$$
Then the natural metric in the space $B$ satisfies the properties of homogeneity and symmetry. The third property of the metric is related to the triangle inequality

$$d_\xi(B_s, B_t) = \|(B_s - B_t) - (B_s - B_q)\|_\xi \leq \max\{\|B_s - B_q\|_\xi, \|B_t - B_q\|_\xi\} = \max\{d_\xi(B_s, B_q), d_\xi(B_t, B_q)\} < d_\xi(B_s, B_q) + d_\xi(B_t, B_q).$$

As you can see, in this case not only the triangle inequality is satisfied, but also the stronger inequality — the ultrametric.

$$d_\xi(B_s, B_t) \leq \max\{d_\xi(B_s, B_q), d_\xi(B_t, B_q)\},$$

where $B_s$, $B_t$, and $B_q \in A_k^\xi$.

3. Results and discussion

The above norm defines an ultrametric on the set $B$, which turns the space $A_k^\xi$ into an ultrametric one. In practice, this leads to the fact that the geometric properties of many objects from this space will be counterintuitive to the properties of objects from a finite-dimensional Euclidean space. Indeed, many “strange” properties of fractal sets can be explained precisely by the properties of ultrametric space.

As an example, we compare the distances between pairs of points of a fractal set in the characteristic Euclidean space $R^2$ and in the classification space $A_{1/2}^3$.

![Figure 3. Fractal set simulation using the procedure F2.](image)

The distance between the points shown in Fig.3, in the Euclidean metric is equal to $\rho(CE) = 0.58$ and $\rho(DE) = 0.21$. Obviously, $\rho(CE) > \rho(DE)$, although the points $C$ and $E$ belong to the same cluster, and the point $D$ belongs to another cluster.
In the classification ultrametric space, the above distances will be equal to \( d_{1/2}(CE) = 0.22 \) and \( d_{1/2}(DE) = 0.25 \). In this case, already \( d_{1/2}(CE) < d_{1/2}(DE) \), which better reflects the location of the points.

The following properties of ultrametric spaces are known from the literature (see, for example, [7, 9]): 1) the ultrametric spaces are non-Archimedean and completely disconnected; 2) in any ultrametric space all triangles are isosceles; 3) any ball in an ultrametric space is an open-closed set, and any point inside the ball is its center; 4) any two balls in the ultrametric space either do not intersect, or one is inside the other; 5) any points \( C \) and \( D \) in an ultrametric space will not be \( \epsilon \)-connected if \( d_\epsilon(C, D) > \epsilon \) is true.

4. Conclusion

Interpretation of the properties of the RIFS attractor makes it possible to approximate data from the multidimensional classification problem by using a set of tuples from the space \( A^k_\xi \) that has ultrametric properties. Note that the classification ultrametric space \( A^k_\xi \) has all properties (1-5), whereas it is very difficult to find out their presence in a characteristic Euclidean space.

References

[1] Avetisov V A, Bikulov A H and Osipov V A 2003 J. Phys. A: Math. Gen. 36 4239 (DOI: 10.1088/0305-4470/36/15/301)
[2] Barnaby N, Biswas T and Cline J M 2007 J. High Energy Phys. JHEP04(2007)056 (DOI: 10.1088/1126-6708/2007/04/056)
[3] Bukhovets A G and Bukhovets E A 2012 Autom. Remote Control. 73 381–385 (DOI: 10.1134/S0005117912020154)
[4] Bukhovets A G and Borucinska T Y 2016 Proceedings of VSU. Series: Systems analysis and information technologies. 2 5–10 (URL: http://www.vestnik.vsu.ru/pdf/analiz/2016/02/2016-02-01.pdf)
[5] Crownover R M 1995 Introduction to fractals and chaos (Jones & Bartlett Pub.)
[6] Dragovich B. and Dragovich A 2007 Comput. J. 53 432–442 (DOI: 10.1093/comjnl/bxm083)
[7] Dragovich B, Khrennikov A Y, Kozyrev S V, Volovich I V and Zelenov E I 2017 p-Adic Num. Ultrametr. Anal. Appl. 9 87–121 (DOI: 10.1134/S2070046617020017)
[8] Greinecker F 2017 J. Fractal Geom. 4 105–126 (DOI: 10.4171/JFG/46)
[9] Khrennikov A Y 2004 Russ. J. Math. Phys. 11 45–70 (arXiv:nlin/0402042)
[10] Moskalev P V, Bukhovets A G and Biruchinskay T Ya 2012 RIFS: Random iterated function system (URL: https://cran.r-project.org/package=RIFS)
[11] Zubarev A P 2013 Vestnik SamGUPS. 2 55–64 (URL: https://elibrary.ru/item.asp?id=20245733)