Spatial Search via Memoryless Walk with Selfloop

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Abstract

The defining feature of memoryless quantum walks is that they operate on the vertex
space of a graph, and therefore can be used to produce search algorithms with minimal
memory. We present a memoryless walk that can find a unique marked vertex on a two-
dimensional grid. Our walk is based on the construction proposed by Falk, which tessel-
lates the grid with squares of size $2 \times 2$. Our walk uses minimal memory, $O(\sqrt{N \log N})$
applications of the walk operator, and outputs the marked vertex with vanishing error
probability. To accomplish this, we apply a selfloop to the marked vertex—a technique we
adapt from interpolated walks. We prove that with our explicit choice of selfloop weight,
this forces the action of the walk asymptotically into a single rotational space. We char-
acterize this space and as a result, show that our memoryless walk produces the marked
vertex with a success probability asymptotically approaching one.

1 Introduction

Search problems are one of the foundational applications of quantum algorithms, and are one
of the situations in which quantum algorithms are proven to provide a speedup over their
classical counterparts. For example, Grover’s search algorithm [Gro96] can find a single ele-
ment in an $N$-element database with $\Theta(\sqrt{N})$ quantum queries, while any classical algorithm
requires $\Omega(N)$ queries for the same task.

In spatial search, the goal is to find a “marked” element on a graph, with the restriction that
in a single time step, amplitude can only be moved between adjacent vertices on the graph.
Such restrictions can emerge from some underlying physical structure [AA05], or from the
computational cost of moving from one vertex to another [Amb07]. Quantum spatial search
was first considered by Benioff [Ben02], who showed that direct application of Grover’s search
on the grid does not yield a speedup over classical algorithms. A near-quadratic speed-up
was then discovered using a divide-and-conquer quantum algorithm [AA05] and quantum
walks [AKR05, CG04, San08].

Quantum walks are the quantum counterparts of classical random walks. Memoryless, or
coinless, walks are a less commonly studied type of quantum walk, but have several advan-
tages over other models. As the name suggests, they operate directly on the vertex space of
the graph. This differentiates them from coined or Szegedy-style quantum walks, which use
extra registers to encode the previous location of the walker. Memoryless walks can therefore be used to produce search algorithms with optimal memory requirements. They also have a simple structure, alternating two or more non-commuting reflections that can be derived from tessellations of the underlying graph. A proposal for implementing memoryless walks using superconducting microwave resonators is given by [MdOP17], and an implementation of memoryless walks on IBM quantum computers is in [AAMP20].

Staggered walks are a class of memoryless walks and were defined by Portugal et al. in [PSFG16]. Staggered walks are based on graph tessellations into cliques, which enforces the spatial search constraint and gives a method for constructing memoryless walks on general graphs. Ref. [PSFG16] shows that any quantum walk on a graph $G$ in the standard Szegedy model [Sze04] can be converted to a staggered walk on the underlying line graph of $G$. The conversion preserves the asymptotic cost, the success probability, and the space requirement. Staggered walks have also been used to derive relationships between memoryless, coined, Szegedy, and continuous quantum walks [PBF15, Por16a, Por16b, PdOM17, CP18, KPSS18].

A memoryless walk on the line was given by Patel, Raghunathan, and Rungta in [PRR05], who give and analyse a walk with Hamiltonians, and they note that their walk operator resembles the staggered fermion formalism. In [PRR10], Patel, Raghunathan and Rahaman present numerical simulations on the extension of this walk to the $N$-vertex grid. Their results show that it finds a unique marked vertex in $\Theta(\sqrt{N \log N})$ applications of the walk operator with success probability $\Theta(\frac{1}{\log N})$, and that by using an ancilla qubit, the success probability can be improved to $\Theta(1)$.

Falk [Fal13] gives a memoryless walk on the two-dimensional grid, constructing a discrete walk operator by reflecting about two alternating tessellations of the grid. The walk by Falk was analysed by Ambainis, Portugal and Nahimovs [APN15], who proved that it finds a unique marked vertex on the grid using $\Theta(\sqrt{N \log N})$ applications of the walk operator with success probability $\Theta(\frac{1}{\log N})$. Portugal and Fernandes [PF17] give a memoryless walk with Hamiltonians that also finds a unique marked vertex on the two-dimensional grid with $\Theta(\sqrt{N \log N})$ applications of the walk operator and success probability $\Theta(\frac{1}{\log N})$.

The success probability of [PRR05, PRR10], [Fal13] and [PF17] is sub-constant. The success probability can be improved to $\Theta(1)$ by applying amplitude amplification [BHMT02], but that would increase the number of steps by a factor of $\Theta(\sqrt{N \log N})$ and introduce additional operators in an implementation. Two alternative methods [APN15, PF17] for increasing the success probability are the post-processing local neighborhood search in [ABN+12] and Tulsi’s proposal of adding an ancilla qubit to the grid [PRR10, Tul08].

In this paper, we give a memoryless walk that finds the marked vertex in $\Theta(\sqrt{N \log N})$ steps with vanishing error probability. Our walk uses minimal memory and preserves the simple structure given by alternating tessellations of the grid. To do this, we augment the basic memoryless construction by showing how the interpolated walks of Krovi et al. [KMOR16] can be adapted to the vertex space of a graph. We define our memoryless walk using the tessellations shown in Figure 1, which divide the grid into squares of size $2 \times 2$. This is the same tessellation structure analysed by [APN15], who proved that using Falk’s construction, the maximum success probability of the corresponding memoryless walk scales with $\Theta(\frac{1}{\log N})$. By using a selfloop to force the action of the walk into a single two-dimensional subspace, we
modify their walk so that the maximum success probability asymptotically approaches one. Classically, the interpolated version of a random walk is constructed by adding weighted self-loop edges to marked vertices. Applying Szegedy’s isometry [Sze04] to interpolated walks produces quantum walks that can find a unique marked vertex on any graph [KMOR16]. This approach has recently been used to solve the spatial search problem on any graph with a quadratic speedup over classical random walks, even for the case of multiple marked vertices. Solutions of this form have been obtained for both the discrete [AGJK20, AGJ21] and continuous [ACNR21] models using quantum walks that operate on the edge space of a graph. Quantum interpolated walks can be simulated by controlled quantum walks [DH17], preserving the asymptotic cost, the success probability, and the space requirement.

To extend the approach of [KMOR16] to the memoryless setting, we introduce a new state corresponding to a selfloop on the marked vertex. Rather than reflecting about the marked vertex, we reflect about an interpolation between the selfloop state and the marked vertex. The result is a memoryless walk parametrized by the weight of the selfloop, $s$. We prove in our analysis that with our explicit choice of weight, this forces the evolution of the initial state into a single rotational subspace of our walk operator. As a result, our walk achieves a success probability $1 - O\left(\frac{1}{\log N}\right)$ with $\Theta(\sqrt{N \log N})$ steps and minimal memory.

Most of our analysis considers the eigenvector of an operator with the smallest positive eigenphase. We use the term “slowest eigenvector” to refer to this eigenvector, and “slowest rotational subspace” to refer to the subspace spanned by this eigenvector and its conjugate.

To prove our main result, we present a set of techniques for analysing memoryless walks. We analyse an operator $W$, whose spectra are completely known, composed with a two-dimensional rotation $F$. As part of our proof, we determine the asymptotic behaviour of both the smallest positive eigenphase of $WF$ and its associated eigenvector. The result is a precise asymptotic description of the slowest rotational subspace of our walk operator.

The techniques we use to obtain this description are general enough that they could be applied in other contexts as well.

We begin by defining our walk in Section 2. An overview of the paper layout and proof structure is given in Section 2.1.

## 2 Walk construction and main result

We consider the task of finding a unique marked vertex on a two-dimensional grid with $n_r$ rows and $n_c$ columns, where both $n_r$ and $n_c$ are even. The grid boundaries are those of a torus, so there are edges between vertices $(i, n_c - 1)$ and $(i, 0)$ for $0 \leq i < n_r$, and between vertices $(n_r - 1, j)$ and $(0, j)$ for $0 \leq j < n_c$. The total number of vertices is given by $N = n_r \times n_c$. After Lemma 7, we restrict to the case where $n_r = n_c = \sqrt{N}$.

We construct a memoryless quantum walk with a selfloop. Our walk operates on a space of dimension $N + 1$, which is optimal for this task. The vertex at position $(i, j)$ is represented by the quantum state $|i, j\rangle$. We introduce a new state $|\bigcirc\rangle$ corresponding to a selfloop on the marked vertex. Our approach and terminology are inspired by the interpolated walks of [KMOR16], which add selfloop edges to marked vertices in Szegedy-style walks.
Our walk applies two alternating reflections about the faces of the graph. Here, we follow the construction used in \cite{APN15}. For \(0 \leq i < \frac{n_r}{2}, 0 \leq j < \frac{n_c}{2}\), define

\[
|a_{ij}\rangle = \frac{1}{2} \sum_{i',j'=0}^{1} |2i + i', 2j + j'\rangle,
\]

\[
|b_{ij}\rangle = \frac{1}{2} \sum_{i',j'=0}^{1} |2i + 1 + i', 2j + 1 + j'\rangle.
\]

The sets \(\{|a_{ij}\rangle\}_{i,j}\) and \(\{|b_{ij}\rangle\}_{i,j}\) each specify a partition of the grid into \(2 \times 2\) squares, positioned at even and odd indices, respectively. These sets form the tessellations depicted in Figure 1.

Define projections onto the even and odd partitions as

\[
\Pi_e = \sum_{i=0}^{n_r-1} \sum_{j=0}^{n_c-1} |a_{ij}\rangle \langle a_{ij}|, \quad \Pi_o = \sum_{i=0}^{n_r-1} \sum_{j=0}^{n_c-1} |b_{ij}\rangle \langle b_{ij}|,
\]

and let

\[
A = 2\Pi_e + 2|\circ\rangle \langle \circ| - 1, \quad B = 2\Pi_o + 2|\circ\rangle \langle \circ| - 1.
\]

In \cite{APN15}, these reflections are alternated with a reflection about the marked state. This is the standard way of adding finding behaviour to memoryless walks, where an operator
composed of two or more non-commuting reflections is alternated with a reflection of the marked vertices. In our walk, we replace the reflection of the marked vertex with a reflection of an interpolated state with parameter \(0 \leq s \leq 1\). Letting \(|g\rangle\) denote the marked vertex, we define the interpolated state to be \(|\tilde{g}\rangle = \sqrt{s}|g\rangle + \sqrt{1-s}|\bigcirc\rangle\). Our input-dependent reflection is then

\[
\tilde{G} = 1 - 2|\tilde{g}\rangle\langle\tilde{g}|.
\]

By selecting an appropriate value for \(s\), we are able to force the action of the walk asymptotically into a single two-dimensional subspace. We set \(s = 1 - \frac{1}{N+1}\), which is close to the value \(s = 1 - \frac{1}{N}\) used by [KMOR16] for interpolated walks.

Given a fixed value for \(s\), we define a single step of our walk to be the operator

\[
U = B\tilde{G}A\tilde{G}.
\]

We apply the walk to the initial state \(|\pi\rangle\), which is defined by \(\langle i, j|\pi\rangle = \frac{1}{\sqrt{N}}\) for all vertices \((i, j)\), and with \(\langle \bigcirc|\pi\rangle = 0\). This allows us to present our main result.

**Theorem 1 (Main result)** Fix \(s = 1 - \frac{1}{N+1}\) and suppose \(n_r = n_c\). Then there exists a constant \(c > 0\) such that after \(c\sqrt{N \log N}\) applications of \(U\) to \(|\pi\rangle\), measuring the state will produce \(|\bigcirc\rangle\) with probability \(1 - e(N)\), where \(e(N) \in O\left(\frac{1}{N \log N}\right)\).

We remark that given the state \(|\bigcirc\rangle\), one can obtain the marked state \(|g\rangle\) through amplitude amplification [BHMT02]. This can be done by alternating the reflection \(\tilde{G}\) with a reflection about either \(|g\rangle\) or \(|\bigcirc\rangle\). After \(\left\lceil \frac{\pi}{2} (\arcsin(\frac{1}{\sqrt{N+1}}))^{-1} \right\rceil \in \Theta(\sqrt{N})\) steps of amplification, measuring the resulting state will produce \(|g\rangle\) with probability \(1 - e(N)\), where \(e(N) \in O\left(\frac{1}{N\log N}\right)\). Note that both the error probability and the query complexity for obtaining \(|g\rangle\) from \(|\bigcirc\rangle\) are dominated by the cost of finding the state \(|\bigcirc\rangle\) as in Theorem 1.

### 2.1 Proof strategy

Our proof of Theorem 1 is based on the analysis of an intermediate walk operator, which we define below.

To simplify the analysis, we also introduce a change of basis. This will allow us to compute necessary properties of the walk spectra, as the eigenvectors of BA factor into product states under this basis change. Let \(cz\) be an operator acting on the pair of least significant bits in a tensor product space, with the action defined by \(|i, j\rangle \mapsto -|i, j\rangle\) if both \(i\) and \(j\) are even, and being the identity otherwise. Let \(cz\) act trivially on \(|\bigcirc\rangle\). For any operator \(X\), let \(X_z = cz \circ X \circ cz\), and let \(|\pi_z\rangle = cz|\pi\rangle\).

We define

\[
W = (BA)_z,
\]

\[
F = (AG\tilde{A}G)_z,
\]
so that

\[ U = (BA) (A\tilde{G}A\tilde{G}) = (WF)z. \] 

(7)

This is analogous to the decomposition of \( U \) used by \[ \text{APN15} \] in their analysis. Note that \( W \) is input-independent, while \( F \) depends on both \( s \) and the marked vertex. We show in Section 3 that \( F \) is a two-dimensional rotation, and therefore can be decomposed as the product of two reflections, which we write as \( F = F_1F_2 \).

Our intermediate walk consists of \( W \) composed with only \( F_1 \). This choice of intermediate walk has the advantage that it consists of a real operator whose spectra are completely known, composed with a one-dimensional reflection. This allows us to apply existing results about operators of this type, including the eigenvector analysis of \[ \text{Amb07} \] and the flip-flop theorem from \[ \text{DH17} \]. An overview of these results is given in Appendix 8.1.

Our proof is based on a tight characterization of the slowest rotational subspace of the intermediate walk \( WF_1 \). This characterization is developed in Section 5, where we prove asymptotic properties of the rotational angle and the spanning vectors. To our knowledge, this is the first case of the flip-flop theorem being used to derive properties of a subspace in this way. We show that with our choice of selfloop, the action of \( WF_1 \) on \( |\pi_z\rangle \) can be reduced asymptotically to a Grover-like rotation in the slowest two-dimensional rotational subspace of \( WF_1 \). This key property is what allows our main algorithm to achieve a success probability asymptotically close to 1.

To prove our main result, we relate the slowest rotational subspaces of \( WF_1 \) and \( WF \) in Section 6. We show that composing \( WF_1 \) with the reflection \( F_2 \) does not significantly alter the slowest rotational subspace, and therefore that \( WF \) has the same asymptotic behaviour as \( WF_1 \) when applied to \( |\pi_z\rangle \). The proof of Theorem 1 follows from the basis-change relationship between \( U \) and \( WF \) given in equation (7).

With our approach, we are able to derive precise statements about the behaviour of an operator composed with a two-dimensional rotation. This addresses a more general challenge in analysing quantum algorithms, and may have applications outside of memoryless walks.

3 Decomposition of \( F \)

In this section, we derive the exact form of the rotation \( F \). We show that it can be decomposed into two one-dimensional reflections, \( F_1 \) and \( F_2 \), which we compose sequentially with \( W \) in Sections 5 and 6.

Without loss of generality, assume \( |g\rangle = |0,0\rangle \) is the marked vertex. Consider the three-dimensional subspace spanned by \( |g\rangle, |\varnothing\rangle \) and \( |a_0\rangle \), the even-indexed square containing \( |0,0\rangle \). The operator \( F \) only acts non-trivially in this subspace, which is spanned by \( |\varnothing\rangle \) and the two orthonormal states

\[ |+\rangle = \frac{1}{\sqrt{3}}(|g\rangle + |a_0\rangle), \]  

(8)

\[ |−\rangle = (|g\rangle - |a_0\rangle). \]  

(9)
Lemma 2 Let $0 \leq \eta \leq \frac{\pi}{3}$ be such that

$$\sin^2(\eta) = \frac{3}{4}s.$$ \tag{10}

Set $\lambda = e^{i4\eta}$. Then $F = I + (\lambda - 1)|f_+\rangle\langle f_+| + (\lambda^{-1} - 1)|f_-\rangle\langle f_-|$, where the two non-trivial eigenvectors are

$$|f_+\rangle = \frac{1}{\sqrt{2}} \left[|+\rangle - i \frac{1}{\sqrt{4 - 3s}} \left(\sqrt{s}|-\rangle - 2\sqrt{1-s}|\bigcirc\rangle\right)\right], \tag{11}$$

$$|f_-\rangle = \frac{1}{\sqrt{2}} \left[|+\rangle + i \frac{1}{\sqrt{4 - 3s}} \left(\sqrt{s}|-\rangle - 2\sqrt{1-s}|\bigcirc\rangle\right)\right]. \tag{12}$$

Proof Observe that $F$ is a real-valued operator that only acts non-trivially on the span of $|+\rangle$, $|−\rangle$ and $|\bigcirc\rangle$. Therefore, any complex eigenvalues of $F$ must come in conjugate pairs, and $F$ can have at most three non-trivial eigenvectors. Using the observation that $(A\tilde{G})_z$ is real-valued, any $(-1)$-eigenspace of $F$ would necessarily have even dimension. Thus, $F$ must have either two or zero non-trivial eigenvectors.

Define $|f_{+\text{un}}\rangle = \sqrt{2(4 - 3s)}|f_+\rangle$ and compute

$$(A\tilde{G})_z|f_{+\text{un}}\rangle = \left[- (2 - 3s)\frac{\sqrt{4 - 3s}}{2} - i\sqrt{3s}(4 - 3s)\right]|+\rangle$$

$$+ \left[- \frac{\sqrt{4 - 3s}}{2}\sqrt{3s} + i\sqrt{5}(2 - 3s)\right]|−\rangle$$

$$+ \left[\sqrt{3s(1-s)(4 - 3s)} - i\sqrt{1-s}(2 - 3s)\right]|\bigcirc\rangle$$

$$= -\frac{1}{2} \left( (2 - 3s) + i\sqrt{3s}(4 - 3s) \right)|f_{+\text{un}}\rangle$$

$$= -e^{i2\eta}|f_{+\text{un}}\rangle.$$ 

This shows that $|f_+\rangle$ is an eigenvector of $F = (A\tilde{G})_z(A\tilde{G})_z$ with eigenvalue $(-e^{i2\eta})^2 = \lambda$. It follows that the entrywise conjugate of $|f_+\rangle$, given by $|f_-\rangle$, must be an eigenvector of $F$ with eigenvalue $\lambda^{-1}$. \hfill \Box

Lemma 2 shows that $F$ is a rotation by $4\eta$ of a single two-dimensional space, spanned by $|f_+\rangle$ and $|f_-\rangle$. Therefore, $F$ can be decomposed as the product of two one-dimensional reflections. We choose these reflections as follows.

Fact 3 Define

$$|f_1\rangle = \frac{1}{\sqrt{4 - 3s}} \left(\sqrt{s}|−\rangle - 2\sqrt{1-s}|\bigcirc\rangle\right), \tag{13}$$

$$|f_2\rangle = \sin(2\eta)|+\rangle + \cos(2\eta)|f_1\rangle, \tag{14}$$
and let

\[ F_1 = I - 2|f_1\rangle\langle f_1|, \quad (15) \]
\[ F_2 = I - 2|f_2\rangle\langle f_2|. \quad (16) \]

Then \( F = F_1F_2 \).

4 Structure of \( W \)

We give exact formulas for the eigenvectors and eigenphases of \( W \), as well as a decomposition of the \((N+1)\)-dimensional domain into subspaces that are invariant under \( W \). These properties are required for the precise characterisation of \( WF_1 \) and \( WF \) in later sections.

4.1 Spectra of \( W \)

Recall that \(|\varnothing\rangle\) is trivially a \((+1)\)-eigenvector of \( W \). The remaining \( N \) eigenvectors of \( W \) can be indexed by \( k \) and \( l \) as follows.

For \( 0 \leq k < n_r/2 \), let \( \tilde{k} = \frac{2\pi k}{n_r} \), and for \( 0 \leq l < n_c/2 \), let \( \tilde{l} = \frac{2\pi l}{n_c} \). Let \( \epsilon_k = \text{sign}(\cos \tilde{k}) \) and \( \epsilon_l = \text{sign}(\cos \tilde{l}) \). Define the sign of zero to be \(+1\). This case occurs when \( n_r \) or \( n_c \) is divisible by \( 4 \), since \( \cos(\tilde{k}) = 0 \) when \( k = n_r/4 \) and \( \cos(\tilde{l}) = 0 \) when \( l = n_c/4 \).

Define

\[ p_{kl} = \sqrt{1 - \cos^2 \tilde{k} \cos^2 \tilde{l}}, \quad (17) \]
\[ \theta_{kl} = \epsilon_k \epsilon_l \cos(1 - 2p_{kl}^2), \quad (18) \]

and

\[ r_{kl}^\pm = \sqrt{2 \left( 1 \pm \frac{\sin \tilde{k} \cos \tilde{l}}{p_{kl}} \right)} = \sqrt{1 + \frac{\sin \tilde{l}}{p_{kl}}} \pm \epsilon_l \sqrt{1 - \frac{\sin \tilde{l}}{p_{kl}}}, \]
\[ c_{kl}^\pm = \sqrt{2 \left( 1 \pm \frac{\cos \tilde{k} \sin \tilde{l}}{p_{kl}} \right)} = \sqrt{1 + \frac{\sin \tilde{k}}{p_{kl}}} \pm \epsilon_k \sqrt{1 - \frac{\sin \tilde{k}}{p_{kl}}}. \]

If \( k = l = 0 \), then \( p_{00} = 0 \) and the division by zero is ill-defined. In this case, we define

\[ r_{00}^+ = r_{00}^- = c_{00}^+ = c_{00}^- = \sqrt{2}. \]

For each \( 0 \leq k < n_r/2 \) and \( 0 \leq l < n_c/2 \), there is an eigenvector \(|w_{kl}\rangle\) of \( W \) with eigenvalue \( e^{i\theta_{kl}} \). This \(|w_{kl}\rangle\) is the product state

\[ |w_{kl}\rangle = |u_{kl}\rangle \otimes |v_{kl}\rangle, \quad (19) \]

where the factors are given by the normalized states
\[ |u_{kl}⟩ = \sqrt{2}|ϕ^k_c⟩ \circ (|1_{n_r/2}⟩ \otimes |r_{kl}⟩), \]
\[ |v_{kl}⟩ = \sqrt{2}|ϕ^l_c⟩ \circ (|1_{n_c/2}⟩ \otimes |c_{kl}⟩), \]
\[ |r_{kl}⟩ = \frac{1}{2} \begin{bmatrix} r_{kl}^- \\ r_{kl}^+ \end{bmatrix}, \]
\[ |c_{kl}⟩ = \frac{1}{2} \begin{bmatrix} c_{kl}^- \\ c_{kl}^+ \end{bmatrix}. \]

Here, \( \circ \) denotes entry-wise multiplication and \(|1_n⟩\) is the all-ones vector of dimension \( n \). The Fourier states are \( |ϕ^k_c⟩ = \frac{1}{\sqrt{n_r}} \sum_{i=0}^{n_r-1} \omega^{ik}_{nr} |i⟩ \) and \( |ϕ^l_c⟩ = \frac{1}{\sqrt{n_c}} \sum_{i=0}^{n_c-1} \omega^{il}_{nc} |i⟩ \), where \( \omega_n = e^{2πi/n} \) denotes the \( n \)-th root of unity.

Let
\[ |r_{kl}^1⟩ = XZ|r_{kl}⟩, \]
\[ |c_{kl}^1⟩ = XZ|c_{kl}⟩. \]

Then replacing \(|r_{kl}⟩\) and \(|c_{kl}⟩\) with \(|r_{kl}^1⟩\) and \(|c_{kl}^1⟩\) in the above construction yields an eigenvector with eigenphase \(-θ_{kl}\), and replacing either one of the two yields an eigenvector with eigenvalue 1. Let \(|u^1_{kl}⟩\) and \(|v^1_{kl}⟩\) be defined with \(|r_{kl}^1⟩\) and \(|c_{kl}^1⟩\), respectively. We denote \(|u^0_{kl}⟩ = |u_{kl}⟩\), \(|v^0_{kl}⟩ = |v_{kl}⟩\). Let \(|w^B_{kl}⟩\) be defined accordingly for \( B \in \{00, 01, 10, 11\} \).

By this definition, the \( N \) eigenvectors \( \{|w^B_{kl}\rangle\} \) of \( W \) constitute an orthonormal basis for the grid, where the eigenvector \(|w^B_{kl}\rangle\) has eigenphase \( θ_{kl}, 0, 0, −θ_{kl} \) for \( B = 00, 01, 10, 11 \).

**Fact 4** Both \( |π⟩ \) and \( |π_z⟩ \) are \((+1)\)-eigenvectors of \( W \).

**Proof** Based on equation (18), we note \( θ_{00} = 0 \). The fact follows from the observation that
\[ |π⟩ = |w^0_{00}⟩, \]
\[ |π_z⟩ = \frac{1}{2} \left( |w^0_{00}⟩ + |w^0_{01}⟩ + |w^0_{10}⟩ − |w^0_{11}⟩ \right). \]

\[ \square \]

### 4.2 Invariant subspaces

We partition the domain of \( W \) into subspaces \( W_{kl} \). The number of subspaces depends on the parity of \( \frac{n_c}{2} \). If \( \frac{n_c}{2} \) is odd, there are \( \binom{n_r+2}{n_c-2} \) + 1 invariant subspaces, and if \( \frac{n_c}{2} \) is even, there are \( \frac{(n_r+2)n_c}{8} + \lfloor \frac{n_c}{4} \rfloor - 1 \) invariant subspaces. Each \( W_{kl} \) is spanned by a set of eigenvectors of \( W \) with eigenphase \( θ_{kl} \), and is therefore invariant under the action of \( W \). The projection onto subspace \( W_{kl} \) is denoted \( Π_{kl} \).

First, observe that \( θ_{00} = 0 \), so the eigenvectors \(|w^0_{00}⟩\) and \(|w^{11}_{00}⟩\) are both \((+1)\)-eigenvectors. The \((+1)\)-eigenspace of \( W \) therefore has dimension \( \frac{n_c}{2} + 2 + 1 \), where the last dimension comes from the selfloop state. We denote the \((+1)\)-eigenspace as \( W_{00} \) and its associated projection as \( Π_{00} \).

For any \( 0 \leq k < n_r/2 \) and \( 0 \leq l < n_c/2 \), define
\[ k' = \begin{cases} \frac{n_r}{2} - k & \text{if } 0 < k < \frac{n_r}{2} \\ 0 & \text{if } k = 0 \end{cases} \]
and
\[ l' = \begin{cases} \frac{n_c}{2} - l & \text{if } 0 < l < \frac{n_c}{2} \\ 0 & \text{if } l = 0. \end{cases} \]
Note that \( k = k' \) exactly when \( k = 0 \) or \( k = \frac{n_r}{4} \), and similarly for \( l = l' \).

For \( 0 < k < \frac{n_r}{4}, k \neq \frac{n_r}{4} \) and \( 0 < l < \frac{n_c}{4} \), define

\[ W_{kl} = \text{span}\{ |w_{kl}^{00}\rangle, |w_{k'l'}^{00}\rangle, |w_{kl}^{11}\rangle, |w_{k'l'}^{11}\rangle \}. \]

Each of these subspaces has an associated eigenphase \( \theta_{kl} \neq 0, \pi \). When \( n_r \) or \( n_c \) is a multiple of 4, \( W \) also has a \((-1)\)-eigenspace. In this case, there are subspaces with eigenphase \( \pi \) given by

\[ W_{kl} = \text{span}\{ |w_{kl}^{00}\rangle, |w_{k'l'}^{00}\rangle, |w_{kl}^{11}\rangle, |w_{k'l'}^{11}\rangle \}. \]

for \( k = \frac{n_r}{4}, 0 < l < \frac{n_c}{4} \) and \( l = \frac{n_c}{4}, 0 < k \leq \frac{n_r}{4} \). These four vectors will be distinct unless both \( k = \frac{n_r}{4} \) and \( l = \frac{n_c}{4} \), in which case \( \dim(W_{kl}) = 2 \).

For \( k = 0 \) and \( 0 < l < \frac{n_r}{2} \), the corresponding invariant subspace is

\[ W_{kl} = \text{span}\{ |w_{kl}^{00}\rangle, |w_{kl}^{11}\rangle \}, \]

and similarly for \( l = 0 \) and \( 0 < k < \frac{n_r}{2} \),

\[ W_{kl} = \text{span}\{ |w_{kl}^{00}\rangle, |w_{kl}^{11}\rangle \}. \]

Partitioning the domain in this way allows us to closely analyse the behaviour of \( |+\rangle \) and \( |--\rangle \) under the action of \( W \).

**Lemma 5** The following statements hold.

\[
\begin{align*}
\Pi_{kl}|+\rangle &\perp \Pi_{kl}|-\rangle \quad &\text{for all subspaces } W_{kl} \\
\|\Pi_{kl}|+\rangle\|^2 &= \frac{2}{3} \frac{\dim(W_{kl})}{N} \quad &\text{for all } k, l \text{ not both } 0 \\
\|\Pi_{kl}|-\rangle\|^2 &= \frac{2}{3} \frac{\dim(W_{kl})}{N} \quad &\text{for all } k, l \text{ not both } 0 \\
\|\Pi_{00}|+\rangle\|^2 &= \frac{2}{3} \frac{N+2}{N} \quad &\text{for all } k, l \text{ not both } 0 \\
\|\Pi_{00}|-\rangle\|^2 &= \frac{4}{N}.
\end{align*}
\]

**Proof** See Appendix 8.2. \( \square \)

### 5 Reduction to the slowest subspace

Our memoryless walk, given in Theorem 1, achieves a success probability asymptotically close to 1. This is possible because our choice of \( s \) reduces the walk asymptotically to a rotation in a single two-dimensional subspace. As we show, this subspace is exactly the slowest rotational subspace of the applied walk operator. We prove this by giving a tight description
of the smallest positive eigenvalue of the walk operator and its associated eigenvector. We also show that the two-dimensional rotation induced by the walk maps the initial state to the desired state, $\langle \bigcirc \rangle$.

We consider two walk operators. The first consists of the real operator $W$ composed with the one-dimensional reflection $F_1$. The second consists of $W$ composed with the two-dimensional rotation $F$. We discuss the first operator in this section, and then use the results to derive properties of the second operator in Section 6.

First, we prove three key lemmas about the slowest rotational subspace of $WF_1$. Lemma 8 gives a tight bound on the smallest positive eigenphase of $WF_1$, and Lemmas 10 and 11 characterize the asymptotic behaviour of its associated eigenvector. These three lemmas show that the action of $WF_1$ on $|\pi_z\rangle$ can be reduced to a rotation in the slowest rotational subspace. These results are the basis of the methods used in Section 6 to prove our main result.

The intermediate operator $WF_1$ is fundamentally a tool for analysis. However, it is interesting to note that $WF_1$ can be used directly to find the marked state. The proof of the corollary below follows from a similar argument to our proof of the main theorem.

**Corollary 6** Fix $s = 1 - \frac{1}{N+1}$ and suppose $n_r = n_c$. Then there exists a constant $c > 0$ such that after $c\sqrt{N\log N}$ applications of $WF_1$ to $|\pi_z\rangle$, measuring the state will produce $|\bigcirc\rangle$ with probability $1 - e(N)$, where $e(N) \in O\left(\frac{1}{\log N}\right)$.

Comparing Corollary 6 with Theorem 1 shows that most of the finding behaviour of the memoryless walk comes from the first reflection $F_1$. In Section 6, we show that composing $WF_1$ with the second reflection $F_2$ only changes the behaviour of the walk slightly, leading to our proof of Theorem 1.

Our proofs of both Corollary 6 and our main theorem rely on the following observation.

**Lemma 7** The (unnormalized) vector

\[
|U_0\rangle = |\pi_z\rangle - \sqrt{\frac{s}{(1-s)N}}|\bigcirc\rangle
\]  

is a $(+1)$-eigenvector for each of $W$, $F_1$, and $F_2$.

**Proof** Both $|\pi_z\rangle$ and $|\bigcirc\rangle$ are $(+1)$-eigenvectors of $W$, so $|U_0\rangle$ is a $(+1)$-eigenvector of $W$. To show $|U_0\rangle$ is a $(+1)$-eigenvector of $F_1$, we compute

\[
\langle U_0 | f_1 \rangle = \frac{2\sqrt{s}}{\sqrt{N(4-3s)}} + \frac{\sqrt{s}}{\sqrt{4-3s}} \langle \pi_z | - \rangle = 0.
\]

Finally, $|+\rangle$ is orthogonal to both $|\pi_z\rangle$ and $|\bigcirc\rangle$, so $|U_0\rangle$ is orthogonal to $|f_2\rangle$. Therefore, $|U_0\rangle$ is also a $(+1)$-eigenvector of $F_2$. $\Box$

For the rest of the paper, we assume a square grid, so $n_r = n_c = \sqrt{N}$. We also fix $s = 1 - \frac{1}{N+1}$. This choice of selfloop weight means $|U_0\rangle = |\pi_z\rangle - |\bigcirc\rangle$, so we can decompose the initial state $|\pi_z\rangle$ as

\[
|\pi_z\rangle = \frac{1}{2}|U_0\rangle + \frac{1}{2}(|\pi_z\rangle + |\bigcirc\rangle).
\]
We show in Section 5.2 that for our chosen $s$, $|\pi_z\rangle + |\bigcirc\rangle$ lies asymptotically in the slowest rotational subspace of $WF_1$. Therefore, $WF_1$ can be used to apply a negative phase to this portion of the initial state. This rotates the state $|\pi_z\rangle$ to the state $\frac{1}{2}|U_0\rangle - \frac{1}{2}(|\pi_z\rangle + |\bigcirc\rangle) = -|\bigcirc\rangle$, as we make precise in our proof of Corollary 6.

When $s = 1 - \frac{1}{N+1}$, note that the vector $|f_1\rangle$ has the form

$$|f_1\rangle = \sqrt{\frac{N}{N+4}}|\rangle - \frac{2}{\sqrt{N+4}}|\bigcirc\rangle.$$  \hspace{1cm} (27)

### 5.1 Smallest eigenphase of $WF_1$

We choose the operator $WF_1$ as our intermediate step in the analysis of $WF$ because it is the composition of a well-characterized real operator with a one-dimensional reflection. This allows us to apply results from the literature about operators of this type. Here, we show how these results can be used to obtain a tight bound on the smallest positive eigenphase of $WF_1$.

An overview of the applied results is given in Appendix 8.1.

**Lemma 8** The smallest positive eigenphase $\phi_1$ of $WF_1$ satisfies $\phi_1 \in O\left(\frac{1}{\sqrt{N \log N}}\right)$.

**Proof** First, we note that the smallest positive eigenvalue of $W$ is given by

$$\theta_{10} = \cos \left( 2 \cos^2 \left( \frac{2\pi}{\sqrt{N}} \right) - 1 \right) = \frac{4\pi}{\sqrt{N}}.$$  \hspace{1cm} (28)

We know by Lemma 5 and the decomposition in equation (27) that $|f_1\rangle$ satisfies

$$\|\Pi_{00}|f_1\rangle\|^2 = \frac{8}{N+4},$$

$$\|\Pi_{kl}|f_1\rangle\|^2 = \frac{2 \dim(W_{kl})}{N+4} \hspace{1cm} \text{for all } k, l \text{ not both } 0.$$  

In particular, $|f_1\rangle$ overlaps all eigenspaces of $W$, so by Theorem 17, $\phi_1 < \theta_{10} = \frac{4\pi}{\sqrt{N}}$.

By Lemma 16, $\phi_1$ must also satisfy the equation

$$\|\Pi_{00}|f_1\rangle\|^2 \cot \left( \frac{\phi_1}{2} \right) + \sum_{kl \neq 00} \|\Pi_{kl}|f_1\rangle\|^2 \cot \left( \frac{\phi_1 - \theta_{kl}}{2} \right) = 0,$$  \hspace{1cm} (29)

where the sum is taken over all invariant subspaces of $W$ except the $(+1)$-eigenspace, $W_{00}$. Note that the $(-1)$-eigenspace is included this sum. Because $\phi_1 > 0$ and $\phi_1 \in o(1)$, we have

$$\|\Pi_{00}|f_1\rangle\|^2 \cot \left( \frac{\phi_1}{2} \right) \in \Theta \left( \frac{1}{\phi_1 N} \right).$$

Therefore, for equation (29) to hold, it must be the case that

$$\sum_{kl \neq 00} \|\Pi_{kl}|f_1\rangle\|^2 \cot \left( \frac{\theta_{kl} - \phi_1}{2} \right) \in \Theta \left( \frac{1}{\phi_1 N} \right).$$  \hspace{1cm} (30)
We argue that there cannot be a solution \( \varphi_1 \in \Theta(\frac{1}{\sqrt{N}}) \). By Fact 19, we know that for such a \( \varphi_1 \), the sum in equation (30) has order \( \Omega(\frac{\log N}{\sqrt{N}}) \). Therefore, it must be the case that \( \varphi_1 \in o(\frac{1}{\sqrt{N}}) \).

By Fact 19, we also know that if \( \varphi_1 \in o(\frac{1}{\sqrt{N}}) \), then the sum in equation (30) has order \( \Theta(\varphi_1 \log N) \). Due to the requirement \( \varphi_1 \log N \in \Theta(\frac{1}{\varphi_1 N}) \), the only possible solution is \( \varphi_1 \in \Theta(\frac{1}{\sqrt{N} \log N}) \), as stated. \( \square \)

5.2 Slowest eigenvector of \( \WF_1 \)

In this section, we use a constraint-solving approach to analyse the eigenvector of \( \WF_1 \) associated with eigenphase \( \varphi_1 \). By determining its asymptotic behaviour, we show that the \( |\pi_z\rangle + |\circ\rangle \) component of equation (26) lies in the span of this eigenvector and its conjugate. This shows that \( \WF_1 \) can be applied to rotate the initial state \( |\pi_z\rangle \) to the target state \( |\circ\rangle \).

Define \( |\perp\rangle = |\rangle + \frac{2}{\sqrt{N}}|\pi_z\rangle \) to be the (unnormalized) component of \( |\rangle \) that is orthogonal to \( |\pi_z\rangle \). Note that by Lemma 5, this vector is orthogonal to \((+1)\)-eigenspace of \( W \).

Let \( |\zeta\rangle \) be an unnormalized eigenvector of \( \WF_1 \) with eigenphase \( \alpha \neq 0, \pi \) and \( \langle \zeta |\pi_z\rangle \neq 0 \), scaled such that \( \langle \zeta |\pi_z\rangle = \frac{1}{2} \). Because \( |\zeta\rangle \) is perpendicular to \( |\psi_0\rangle \), this implies \( \langle \zeta |\circ\rangle = \frac{1}{2} \). We decompose \( |\zeta\rangle \) as

\[
|\zeta\rangle = a|\rangle + \frac{1}{2}|\pi_z\rangle + \frac{1}{2}|\circ\rangle + |\psi\rangle,
\]

(31)

where \( |\psi\rangle \) is an unnormalized vector orthogonal to \( |\rangle \). By analysing the asymptotic behaviour of \( a \) and \( |\psi\rangle \), we show that the real part of \( |\zeta\rangle \) tends to \( \frac{1}{2}(|\pi_z\rangle + |\circ\rangle) \) when \( \alpha = \varphi_1 \).

Note that any eigenvector of \( W \) with eigenphase \( \theta_{kl} \) that is orthogonal to \( |f_1\rangle \) is also an eigenvector of \( \WF_1 \) with eigenphase \( \theta_{kl} \). Therefore, \( \Pi_{kl}|\rangle \) is some scalar multiple of \( \Pi_{kl}|\rangle \) for each \( k, l \). We determine this scalar factor in Lemma 9.

We further decompose both \( |\rangle \) and \( |\psi\rangle \) into the invariant subspaces of \( W \). Both \( |\rangle \) and \( |\psi\rangle \) are orthogonal to the \((+1)\)-eigenspace \( \WF_00 \), so we write the decomposition as

\[
|\rangle = \sum_{kl \neq 00} m_{kl} |\rangle_{kl},
\]

(32)

\[
|\psi\rangle = \sum_{kl \neq 00} |\psi_{kl}\rangle,
\]

(33)

where the vectors \( |\rangle_{kl} \) are normalized for all \( k, l \). We know by Lemma 5 that \( m_{kl} = \sqrt{\frac{2 \dim(W_{kl})}{N}} \).

The vectors \( |\psi_{kl}\rangle \) in the decomposition of \( |\psi\rangle \) are unnormalized.

**Lemma 9** The following equations must be satisfied.

\[
\frac{8a(N - 4)}{\sqrt{N}(N + 4)} - \frac{16}{N + 4} = e^{i\alpha} - 1
\]

(34)

\[
\langle -kl|\psi_{kl}\rangle = m_{kl} \left[ a - \frac{\sqrt{N}}{4}(e^{i\alpha} - 1) \left( \frac{1}{1 - e^{i(\alpha - \theta_{kl})}} \right) \right] \quad \text{for all } k, l \text{ not both 0.}
\]

(35)
Proof By definition, $|\zeta\rangle$ is an eigenvector of $WF_1$ with eigenphase $\alpha$. We obtain the lemma by expanding the equation $WF_1|\zeta\rangle = e^{i\alpha}|\zeta\rangle$ and solving for constraints.

Observe that

$$\langle f_1|\zeta\rangle = \frac{1}{\sqrt{N+4}} \left( \frac{a(N-4)}{\sqrt{N}} - 2 \right).$$

Using this property, we compute

$$WF_1|\zeta\rangle = W|\zeta\rangle - 2W\langle f_1|\zeta\rangle |f_1\rangle$$

$$= \gamma_- |\zeta\rangle^\perp + \gamma_* (|\pi_z\rangle + |\zeta\rangle) + W|\psi\rangle,$$

where

$$\gamma_- = a - \frac{2}{N+4} \left( a(N-4) - 2\sqrt{N} \right),$$

$$\gamma_* = \frac{1}{2} + \frac{4}{\sqrt{N(N+4)}} \left( a(N-4) - 2\sqrt{N} \right).$$

Setting $WF_1|\zeta\rangle = e^{i\alpha}|\zeta\rangle$ and comparing coefficients on $|\pi_z\rangle$, we get

$$\gamma_* = \frac{1}{2} e^{i\alpha},$$

which can be expanded to give equation (34).

To get equation (35), we first solve $WF_1|\zeta\rangle = e^{i\alpha}|\zeta\rangle$ on the subspace $W_{kl}$ to get

$$\langle -kl|\psi_{kl}\rangle = m_{kl} \left( \frac{ae^{i\alpha} - \gamma_- e^{i\theta_{kl}}}{e^{i\theta_{kl}} - e^{i\alpha}} \right).$$

(36)

Next, we use equation (34) to rewrite $\gamma_-$ as

$$\gamma_- = a - \frac{\sqrt{N}}{4} (e^{i\alpha} - 1).$$

Substituting this expression for $\gamma_-$ into equation (36) produces equation (35).

Now, fix $|\zeta\rangle$ to be the eigenvector with eigenphase $\varphi_1$. By Lemma 16, we know that $\langle \zeta |\pi_z\rangle \neq 0$, so the constraints given in Lemma 9 apply. These constraints, together with the bound on $\varphi_1$ from Lemma 8, define asymptotic bounds on $a$ and the real part of $|\psi\rangle$. We use this to show that as $N$ increases, the real part of $|\zeta\rangle$ converges to $\frac{1}{2} |\pi_z\rangle + |\zeta\rangle$.

Let $|\overline{\zeta}\rangle$ denote the entrywise conjugate of $|\zeta\rangle$. Then $|\zeta\rangle$ and $|\overline{\zeta}\rangle$ span the slowest rotational subspace of $WF_1$. In this way, the following lemmas provide a close description of the spanning eigenvectors for the slowest rotational subspace of $WF_1$.

Lemma 10 Let $\alpha = \varphi_1$. Then $|a| \in \Theta(\frac{1}{\sqrt{\log N}})$. 

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Proof By Lemma 8, we know that $|e^{i\varphi_1} - 1| \in \Theta\left(\frac{1}{\sqrt{N \log N}}\right)$. Applying this to equation (34) produces the stated bound. \hfill \Box

Lemma 11 Let $\alpha = \varphi_1$. Let $|\psi\rangle = \Re(|\psi\rangle) + i \Im(|\psi\rangle)$, where both $\Re(|\psi\rangle)$ and $\Im(|\psi\rangle)$ are vectors with real entries. Then $\|\Re(|\psi\rangle)\| \in O\left(\frac{1}{\sqrt{\log N}}\right)$.

Proof From equation (35), we know that $|\psi\rangle = \sum_{kl \neq 00} m_{kl} \left[ a - \frac{\sqrt{N}}{4} (e^{i\alpha} - 1) \left(\frac{1}{1 - e^{i(\alpha - \theta_{kl})}}\right) \right] |\neg kl\rangle$

$= a|\neg\rangle^\perp + \rho |v\rangle,$

where we define

$\rho = -\frac{\sqrt{N}}{4} (e^{i\alpha} - 1),$

$|v\rangle = \sum_{kl \neq 00} m_{kl} \left(\frac{1}{1 - e^{i(\alpha - \theta_{kl})}}\right) |\neg kl\rangle.$

We know from Lemma 10 that $\|a|\neg\rangle^\perp\| \in \Theta\left(\frac{1}{\sqrt{N \log N}}\right)$. Thus, it remains to consider $\rho |v\rangle$.

Examining $\rho$ shows that for $\alpha \in \Theta\left(\frac{1}{\sqrt{N \log N}}\right)$,

$|\Re(\rho)| \in \Theta\left(\frac{1}{\sqrt{N \log N}}\right),$

$|\Im(\rho)| \in \Theta\left(\frac{1}{\sqrt{\log N}}\right).$

Therefore, we can prove the lemma by showing that $\|\Re(|v\rangle)\| \in O\left(\sqrt{N \log N}\right)$ and $\|\Im(|v\rangle)\| \in O(1)$. Observe that for all $k, l$,

$\Re\left(\frac{1}{1 - e^{i(\alpha - \theta_{kl})}}\right) = \frac{1}{2},$

$\Im\left(\frac{1}{1 - e^{i(\alpha - \theta_{kl})}}\right) = -\frac{1}{2} \cot\left(\frac{\theta_{kl} - \alpha}{2}\right).$

Using this property, we split the coefficients of $|v\rangle$ into their real and imaginary parts, giving

$|v\rangle = \frac{1}{2} \sum_{kl \neq 00} m_{kl} |\neg kl\rangle - \frac{i}{2} \sum_{kl \neq 00} m_{kl} \cot\left(\frac{\theta_{kl} - \alpha}{2}\right) |\neg kl\rangle$

$= \frac{1}{2} |\neg\rangle^\perp - \frac{i}{2} |v'\rangle,$

where

$|v'\rangle = \sum_{kl \neq 00} m_{kl} \cot\left(\frac{\theta_{kl} - \alpha}{2}\right) |\neg kl\rangle.$

Note that $|\neg\rangle^\perp$ is real-valued and has norm $\Theta(1).$
We now bound the real and imaginary parts of $|\psi\rangle$. Recall that for each subspace $W_{kl}$ with eigenphase $0 < \theta_{kl} < \pi$, there is a corresponding subspace with eigenphase $-\theta_{kl}$, which we denote $\overline{W_{kl}}$. Because $|\psi\rangle^\perp$ is real-valued, it must be the case that the normalized projection of $|\psi\rangle^\perp$ onto $\overline{W_{kl}}$ is $|\psi_{kl}\rangle$, the entrywise conjugate of $|\psi_{kl}\rangle$. Using this property, we decompose $|\psi\rangle$ as

$$|\psi\rangle = \sum_{0 < \theta_{kl} < \pi} m_{kl} \left[ \cot \left( \frac{\theta_{kl} - \alpha}{2} \right) |\psi_{kl}\rangle - \cot \left( \frac{\theta_{kl} + \alpha}{2} \right) |\psi_{kl}\rangle \right]$$

$$+ \sum_{\theta_{kl} = \pi} m_{kl} \cot \left( \frac{\theta_{kl} - \alpha}{2} \right) |\psi_{kl}\rangle$$

$$= |v_1\rangle + |v_2\rangle + |v_3\rangle,$$

where

$$|v_1\rangle = \sum_{0 < \theta_{kl} < \pi} m_{kl} \left[ \cot \left( \frac{\theta_{kl} + \alpha}{2} \right) |\psi_{kl}\rangle - \cot \left( \frac{\theta_{kl} + \alpha}{2} \right) |\psi_{kl}\rangle \right],$$

$$|v_2\rangle = \sum_{0 < \theta_{kl} < \pi} m_{kl} \left[ \cot \left( \frac{\theta_{kl} - \alpha}{2} \right) - \cot \left( \frac{\theta_{kl} + \alpha}{2} \right) \right] |\psi_{kl}\rangle,$$

$$|v_3\rangle = \sum_{\theta_{kl} = \pi} m_{kl} \cot \left( \frac{\theta_{kl} - \alpha}{2} \right) |\psi_{kl}\rangle.$$

Note that the sums are taken over the invariant subspaces of $W$ whose eigenphases lie in the indicated range. We bound the norms of these three components individually. First, observe that

$$|v_3\rangle = \tan \left( \frac{\alpha}{2} \right) \sum_{\theta_{kl} = \pi} m_{kl} |\psi_{kl}\rangle,$$

where $\sum_{\theta_{kl}=\pi} m_{kl} |\psi_{kl}\rangle$ is the projection of $|\psi\rangle$ onto the $(-1)$-eigenspace of $W$. Therefore, $|v_3\rangle$ must be entirely real-valued, with norm $||v_3\rangle|| \in O\left(\frac{1}{\sqrt{N \log N}}\right)$ by Lemma 8.

The vector $|v_1\rangle$ is entirely imaginary-valued, with norm

$$||v_1||^2 = \sum_{0 < \theta_{kl} < \pi} m_{kl}^2 \left[ \cot \left( \frac{\theta_{kl} + \alpha}{2} \right)^2 + \cot \left( \frac{\theta_{kl} + \alpha}{2} \right)^2 \right]$$

$$\leq \frac{16}{N} \sum_{0 < \theta_{kl} < \pi} \cot \left( \frac{\theta_{kl} + \alpha}{2} \right)^2$$

$$= \frac{16}{N} \sum_{0 < \theta_{kl} < \pi} \left( \frac{\cot \left( \frac{\theta_{kl}}{2} \right) \cot \left( \frac{\alpha}{2} \right) - 1}{\cot \left( \frac{\alpha}{2} \right) + \cot \left( \frac{\theta_{kl}}{2} \right)} \right)^2$$

$$\leq \frac{16}{N} \cot^2 \left( \frac{\alpha}{2} \right) \sum_{0 < \theta_{kl} < \pi} \left( \frac{\cot \left( \frac{\theta_{kl}}{2} \right)}{\cot \left( \frac{\alpha}{2} \right) + \cot \left( \frac{\theta_{kl}}{2} \right)} \right)^2.$$
There are $O(N)$ terms in the final sum, each of which is at most 1, so $||v_1||^2 \in O(N \log N)$.

Finally, the vector $|v_2\rangle$ has both real and imaginary parts, and has norm

$$
||v_2||^2 = \sum_{0<\theta_{kl}<\pi} m_{kl}^2 \left[ \cot \left( \frac{\theta_{kl} + \alpha}{2} \right) - \cot \left( \frac{\theta_{kl} - \alpha}{2} \right) \right]^2
$$

$$
= \frac{2}{N} \sum_{0<\theta_{kl}<\pi} \dim(W_{kl}) \left[ \cot \left( \frac{\theta_{kl} + \alpha}{2} \right) - \cot \left( \frac{\theta_{kl} - \alpha}{2} \right) \right]^2
$$

$$
= \frac{1}{N} \sum_{kl \neq 0} \dim(W_{kl}) \left[ \cot \left( \frac{\theta_{kl} + \alpha}{2} \right) - \cot \left( \frac{\theta_{kl} - \alpha}{2} \right) \right]^2.
$$

By Fact 20, this implies $||v_2||^2 \in O\left(\frac{1}{\log N}\right)$.

Combining the bounds on $|v_1\rangle$, $|v_2\rangle$ and $|v_3\rangle$, we bound the norm of the real and imaginary parts of $|v'\rangle$. Thus,

$$
||\mathbb{R}(|v'\rangle)|| \leq ||v_3|| + ||v_2|| \in O\left(\frac{1}{\sqrt{\log N}}\right),
$$

$$
||\mathbb{I}(|v'\rangle)|| \leq ||v_1|| + ||v_2|| \in O\left(\sqrt{N \log N}\right).
$$

Because $|v\rangle = \frac{1}{2}|-\rangle - \frac{i}{2}|v'\rangle$, this shows in particular that $||\mathbb{R}(|v\rangle)|| \in O\left(\sqrt{N \log N}\right)$ and $||\mathbb{I}(|v\rangle)|| \in O(1)$. We combine this with the bounds on $\rho$ to obtain $||\mathbb{R}(|\psi\rangle)|| \in O\left(\frac{1}{\sqrt{\log N}}\right)$ as stated.

Lemmas 10 and 11 show that the real part of $|\zeta\rangle$ tends to $\frac{1}{2}(|\pi_z\rangle + |\phi\rangle)$ as $N$ increases. This implies that $|\pi_z\rangle + |\phi\rangle$ lies asymptotically in the slowest rotational subspace of $WF_1$. We make this precise in the following lemma.

**Lemma 12** Let $\Pi_{\varphi_1}$ denote the projection onto the slowest rotational subspace of $WF_1$, which is spanned by the eigenvectors with eigenphases $\pm \varphi_1$. Then

$$
\left\| \Pi_{\varphi_1}\left(|\pi_z\rangle + |\phi\rangle\right) \right\| = \sqrt{2} - O\left(\frac{1}{\log N}\right).
$$

**Proof** Let $\mathbb{R}(a)$ denote the real part of $a$. Observe that

$$
|\zeta\rangle + |\overline{\zeta}\rangle = |\pi_z\rangle + |\phi\rangle + 2\mathbb{R}(a)|-\rangle + 2\mathbb{R}(|\psi\rangle).
$$

By Lemma 10, we have $|\mathbb{R}(a)|| \in O\left(\frac{1}{\sqrt{\log N}}\right)$, and by Lemma 11 we have $||\mathbb{R}(|\psi\rangle)|| \in O\left(\frac{1}{\sqrt{\log N}}\right)$. Therefore,

$$
|||\zeta\rangle + |\overline{\zeta}\rangle|| = \sqrt{2} + O\left(\frac{1}{\log N}\right).
$$

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We can also see that
\[
\left( \langle \zeta | + \langle \overline{\zeta} | \right) \left( |\pi z\rangle + |\varnothing\rangle \right) = 2.
\]
Noting that \(\Pi_{\varphi_1}\) denotes the projection onto span\(\{|\zeta\rangle, |\overline{\zeta}\rangle\}\), this implies that
\[
\left\| \Pi_{\varphi_1} \left( |\pi z\rangle + |\varnothing\rangle \right) \right\| \geq \sqrt{2} - O\left(\frac{1}{\log N}\right).
\]

5.3 Proof of corollary

Applying the operator \(WF_1\) to the initial state \(|\pi z\rangle\) yields an optimal algorithm for finding \(|\varnothing\rangle\). The proof of this corollary follows from a similar argument to the proof of Theorem 1. We apply our characterization of the slowest rotational subspace of \(WF_1\), given by Lemmas 8 and 12, to show the action of \(WF_1\) on \(|\pi z\rangle\) is asymptotically restricted to this single subspace. The result is a Grover-like algorithm that rotates \(|\pi z\rangle\) to our desired state \(|\varnothing\rangle\).

Corollary 6 Fix \(s = 1 - \frac{1}{N+1}\) and suppose \(n_r = n_c\). Then there exists a constant \(c > 0\) such that after \(c\sqrt{N\log N}\) applications of \(WF_1\) to \(|\pi z\rangle\), measuring the state will produce \(|\varnothing\rangle\) with probability \(1 - e(N)\), where \(e(N) \in O\left(\frac{1}{\log N}\right)\).

Proof Recall the decomposition of \(|\pi z\rangle\) in equation (26). By Lemma 7, we know that \(|U_0\rangle\) is a (+1)-eigenvector of \(WF_1\). Letting \(\Pi_{\varphi_1}\) denote the projection onto the slowest rotational subspace of \(WF_1\), we decompose \(|\pi z\rangle + |\varnothing\rangle\) as
\[
|\pi z\rangle + |\varnothing\rangle = \Pi_{\varphi_1} \left( |\pi z\rangle + |\varnothing\rangle \right) + |\perp\rangle.
\]
for some vector \(|\perp\rangle\). By Lemma 12, we know that \(\|\|\perp\rangle\| \in O\left(\frac{1}{\log N}\right)\). We also know from Lemma 8 that the slowest rotational subspace has eigenphase \(\varphi_1 \in \Theta\left(\frac{1}{\sqrt{N\log N}}\right)\). Therefore, there exists a constant \(c\) such that \(c\sqrt{N\log N} = \|\frac{\pi}{\varphi_1}\| = k\). After \(k\) applications of \(WF_1\) to \(|\pi z\rangle\), we get the state
\[
(WF_1)^k |\pi z\rangle = \frac{1}{2} \left( |\pi z\rangle - |\varnothing\rangle \right) - \frac{1}{2} \Pi_{\varphi_1} \left( |\pi z\rangle + |\varnothing\rangle \right) + |\rho\rangle
= -|\varnothing\rangle + \frac{1}{2} |\perp\rangle + |\rho\rangle.
\]

Here, \(|\rho\rangle\) is some state that captures both the result of applying \((WF_1)^k\) to \(|\perp\rangle\) and the small error incurred by the rounding of \(\frac{\pi}{\varphi_1}\), and has norm \(\|\|\rho\rangle\| \in O\left(\frac{1}{\log N}\right)\). Thus, measuring the state will produce \(|\varnothing\rangle\) with probability \(1 - e(N)\), where \(e(N) \in O\left(\frac{1}{\log N}\right)\) as stated. \(\square\)
6 Finding with a memoryless walk

We now present the proof of our main theorem, which is stated as follows.

**Theorem 1 (Main result)** Fix $s = 1 - \frac{1}{N+1}$ and suppose $n_r = n_c$. Then there exists a constant $c > 0$ such that after $c\sqrt{N\log N}$ applications of $U$ to $|\pi\rangle$, measuring the state will produce $|\bigcirc\rangle$ with probability $1 - e(N)$, where $e(N) = O\left(\frac{1}{\log N}\right)$.

Our proof uses the decomposition of $|\pi_z\rangle$ given in equation (26). We show that the state $\frac{1}{2}(|\pi_z\rangle + |\bigcirc\rangle)$ lies asymptotically in the slowest rotational subspace of $W$. Therefore, $W$ can be used to rotate $|\pi_z\rangle$ to a state close to $-|\bigcirc\rangle$. Applying the change of basis $cz$ yields the result as stated.

The proof is based on relating the slowest rotational subspaces of $WF_1$ and $WF = (WF_1)F_2$. We continue to apply the results from Appendix 8.1, this time to analyse the real operator $WF_1$ composed with the one-dimensional reflection $F_2$. We show the slowest rotational subspaces of $WF_1$ and $WF$ have the same asymptotic bound on the rotational angle, and that both asymptotically contain $\frac{1}{2}(|\pi_z\rangle + |\bigcirc\rangle)$. By using $WF_1$ as an intermediate operator, we are thus able to tightly characterize the slowest rotational subspace of a real operator $W$, composed with a two-dimensional rotation $F$.

### 6.1 Relationship with $WF_1$

We begin by relating the slowest rotational subspaces of $WF$ and $WF_1$. Note that when $s = 1 - \frac{1}{N+1}$, the vector $|f_2\rangle$ has the form

$$|f_2\rangle = \frac{\sqrt{3N}\sqrt{N+4}}{2(N+1)}|+\rangle + \frac{2-N}{2(N+1)}|f_1\rangle. \quad (38)$$

Let the eigenphases of $WF_1$ different from 0, $\pi$ be denoted by $\pm \varphi_k$ for $1 \leq k \leq m$, where $0 < |\varphi_1| \leq |\varphi_2| \leq \cdots \leq |\varphi_m| < \pi$. Let the associated eigenvectors be $|A_k^\pm\rangle$. Both $|f_2\rangle$ and $WF_1$ are real-valued, so we can decompose $|f_2\rangle$ into the eigenbasis of $WF_1$ as

$$|f_2\rangle = g_0|A_0\rangle + \sum_{k=1}^m g_k (|A_k^+\rangle + |A_k^-\rangle) + g_{-1}|A_{-1}\rangle, \quad (39)$$

where $|A_0\rangle$ is a $(+1)$-eigenvector, $|A_{-1}\rangle$ is a $(-1)$-eigenvector, and all $g_i$ are non-negative real numbers.

By Lemma 5, $\Pi_{kl}|+\rangle \perp \Pi_{kl}|f_1\rangle$ for each invariant subspace $W_{kl}$ of $W$ (including $W_{00}$). Therefore, the eigenvectors $\Pi_{kl}|+\rangle$ of $W$ are also eigenvectors of $WF_1$ with the same eigenphases $\theta_{kl}$.

**Lemma 13** The decomposition of $|f_2\rangle$ in equation (39) satisfies:

$$g_0^2 = \frac{1}{2} + O\left(\frac{1}{\sqrt{N}}\right) \quad (40)$$

$$g_1^2 \in O\left(\frac{1}{\log N}\right). \quad (41)$$
Proof Recall that $0 < \varphi_1 < |\theta_{kl}|$ for all nonzero eigenphases $\theta_{kl}$ of $W$. Using the property that $\Pi_{kl} |+\rangle$ is an eigenvector of $WF_1$ with eigenvalue $\varphi_1$, this implies that $\langle A_1^+ | \Pi_{kl} | + \rangle = 0$ for all $k,l$, so $\langle A_1^+ | + \rangle = 0$. Let $\Pi_\{1\}$ denote the projection onto the $(+1)$-eigenspace of $WF_1$. Then we have $\|\Pi_\{1\} | + \rangle\|^2 = \|\Pi_{00} | + \rangle\|^2 = \frac{2(N+1)}{3N}$ by Lemma 5.

We bound $\|\Pi_\{1\} | f_1 \rangle\|^2$ by observing that

$$\|\Pi_\{1\} | f_1 \rangle\|^2 = \sum_{\theta_{kl} = \pi} \|\Pi_{kl} | f_1 \rangle\|^2$$
$$= \frac{N}{N+4} \sum_{\theta_{kl} = \pi} \|\Pi_{kl} | - \rangle\|^2$$
$$= \frac{N}{N+4} \sum_{\theta_{kl} = \pi} \frac{2 \dim(W_{kl})}{N} \in O\left(\frac{1}{\sqrt{N}}\right),$$

where the final bound follows from the property that there are $O(\sqrt{N})$ terms in the sum.

Thus, using equation (38) and the property that $\Pi_\{1\} | + \rangle$ and $\Pi_\{1\} | f_1 \rangle$ are orthogonal,

$$g_0^2 = \frac{3N(N+4)}{4(N+1)^2} \|\Pi_\{1\} | + \rangle\|^2 + \frac{(2-N)^2}{4(N+1)^2} \|\Pi_\{1\} | f_1 \rangle\|^2$$
$$= \frac{3N(N+4)}{4(N+1)^2} \left(\frac{2(N+2)}{3N}\right) + O\left(\frac{1}{\sqrt{N}}\right)$$
$$= \frac{1}{2} + O\left(\frac{1}{\sqrt{N}}\right).$$

To bound $g_1^2$, consider the eigenvector decomposition given in equation (31) in the case where $\alpha = \varphi_1$. Then $|A_1^+\rangle$ is the normalized version of $|\zeta\rangle$. By definition, we have $\||\zeta\rangle\|^2 \geq \frac{1}{2}$, so

$$g_1^2 = |\langle A_1^+ | f_2 \rangle|^2$$
$$= \cos^2(2\eta) |\langle A_1^+ | f_1 \rangle|^2 + \sin^2(2\eta) |\langle A_1^+ | + \rangle|^2$$
$$\leq |\langle \zeta | f_1 \rangle|^2 / \|\zeta\|^2$$
$$\leq 2|\langle \zeta | f_1 \rangle|^2$$
$$\leq 2 \left| a \sqrt{\frac{N}{N+4}} \left(1 - \frac{4}{N}\right) - \frac{2}{\sqrt{N+4}} \right|^2 \in O\left(\frac{1}{\log N}\right),$$

where the final bound follows from Lemma 10. \qed

Lemma 13 shows that $| f_2 \rangle$ has a large constant overlap with the $(+1)$-eigenspace of $WF_1$, and a vanishing overlap with the slowest rotational space. Intuitively, this suggests the reflection $F_2$ will have little effect on the slowest rotational subspace of the intermediate walk $WF_1$. As discussed in Section 5, this subspace is where most of the action of the walk takes place. This observation is the basis for the proof of Lemma 15.
In the next lemma, we use the constraints from Lemma 16 to show that the smallest positive eigenphase of $WF$ is asymptotically close to $\varphi_1$.

**Lemma 14** Let $\beta$ denote the smallest eigenphase of $WF$. Then

$$\beta = \varphi_1 - O\left(\frac{1}{\sqrt{N \log N}}\right)^{3/2}. \quad (42)$$

Note that in particular, this implies $\beta \in \Theta\left(\frac{1}{\sqrt{N \log N}}\right)$.

**Proof** First, we derive an upper bound on $\beta$ using the flip-flop theorem. By Lemma 5, $|+\rangle$ intersects every eigenspace of $W$. Each of the eigenvectors $\Pi_{kl}|+\rangle$ is also an eigenvector of $WF$, so in particular, this implies that $g_0 > 0$, $g_{-1} > 0$, and $g_k > 0$ for the $k$ corresponding to the eigenphases $\theta_{kl}$. Therefore, by Theorem 17, $0 < \beta < \varphi_1$. Applying Lemma 8, we obtain $\beta \in O\left(\frac{1}{\sqrt{N \log N}}\right)$.

To obtain the upper bound on $\varphi_1 - \beta$, we apply Lemma 16, which states that $\beta$ must satisfy

$$g_0^2 \cot\left(\frac{\beta}{2}\right) + \sum_{k=1}^{m} g_k^2 \left[\cot\left(\frac{\varphi_k + \beta}{2}\right) - \cot\left(\frac{\varphi_k - \beta}{2}\right)\right] - g_{-1}^2 \tan\left(\frac{\beta}{2}\right) = 0. \quad (43)$$

We apply trigonometric identities to rewrite this as

$$g_0^2 \cot\left(\frac{\beta}{2}\right) - 2 \cot\left(\frac{\beta}{2}\right) \sum_{k=1}^{m} g_k^2 \left(\frac{\cot^2\left(\frac{\varphi_k}{2}\right) + 1}{\cot^2\left(\frac{\beta}{2}\right) - \cot^2\left(\frac{\varphi_k}{2}\right)}\right) - g_{-1}^2 \tan\left(\frac{\beta}{2}\right) = 0. \quad (43)$$

Applying our upper bound on $\beta$, we know that $g_{-1}^2 \tan^2(\frac{\beta}{2}) \in O\left(\frac{1}{N \log N}\right)$, so we have

$$2 \sum_{k=1}^{m} g_k^2 \frac{\cot^2\left(\frac{\varphi_k}{2}\right) + 1}{\cot^2\left(\frac{\beta}{2}\right) - \cot^2\left(\frac{\varphi_k}{2}\right)} = g_0^2 - O\left(\frac{1}{N \log N}\right) = \frac{1}{2} + O\left(\frac{1}{\sqrt{N}}\right), \quad (44)$$

where the last equality follows by Lemma 13. Because the smallest positive eigenphase of $W$ has order $\Theta\left(\frac{1}{\sqrt{N}}\right)$, Theorem 17 implies that $\varphi_k \in \Omega\left(\frac{1}{\sqrt{N}}\right)$ for $k \geq 2$. Therefore, $\cot^2\left(\frac{\varphi_k}{2}\right) \in O(N)$ for $k \geq 2$, while $\cot^2\left(\frac{\beta}{2}\right) \in \Omega(N \log N)$. Also note that $\sum_k g_k^2 \leq 1$. This means that

$$\sum_{k=2}^{m} g_k^2 \frac{\cot^2\left(\frac{\varphi_k}{2}\right) + 1}{\cot^2\left(\frac{\beta}{2}\right) - \cot^2\left(\frac{\varphi_k}{2}\right)} \in O\left(\frac{1}{\log N}\right). \quad (45)$$

Combining equation (44) and equation (45), we get

$$2 g_1^2 \frac{\cot^2\left(\frac{\varphi_1}{2}\right) + 1}{\cot^2\left(\frac{\beta}{2}\right) - \cot^2\left(\frac{\varphi_1}{2}\right)} = \frac{1}{2} - O\left(\frac{1}{\log N}\right).$$
We know that \( \cot^2(\frac{\varphi_1}{2}) \in O(N \log N) \) by Lemma 8. We also have \( g_1^2 \in O(\frac{1}{\log N}) \) by Lemma 13. This implies that

\[
\cot^2\left(\frac{\varphi_1}{2}\right) - \cot^2\left(\frac{\varphi_1}{2}\right) \in O(N).
\]

Because \( \cot(\frac{\varphi_1}{2}) + \cot(\frac{\varphi_1}{2}) \in \Omega(\sqrt{N \log N}) \), it must be the case that

\[
\cot\left(\frac{\varphi_1}{2}\right) - \cot\left(\frac{\varphi_1}{2}\right) \in O\left(\frac{\sqrt{N}}{\sqrt{\log N}}\right).
\]

Applying the Taylor expansion for cotangent, we get

\[
\frac{1}{\beta} - \frac{1}{\varphi_1} = \frac{\varphi_1 - \beta}{\varphi_1 \beta} \in O\left(\frac{\sqrt{N}}{\sqrt{\log N}}\right).
\]

We know that \( \frac{1}{\varphi_1} \in \Theta(\sqrt{N \log N}) \), so this implies that \( \frac{1}{\beta} \in \Theta(\sqrt{N \log N}) \). Thus,

\[
\varphi_1 - \beta \in O\left(\frac{1}{\sqrt{N(\log N)^{3/2}}}\right).
\]

Finally, we show that \( |\pi_z\rangle + |\varnothing\rangle \) lies asymptotically in the slowest rotational subspace of \( WF \). To do so, we apply our bounds from Lemmas 13 and 14 to show that the slowest eigenvectors of \( WF \) are asymptotically close to the slowest eigenvectors of \( WF_1 \).

**Lemma 15** Let \( \Pi_\beta \) denote the projection onto the slowest rotational subspace of \( WF \), which is spanned by the eigenvectors with eigenphases \( \pm \beta \). Then

\[
\left\| \Pi_\beta \left( |\pi_z\rangle + |\varnothing\rangle \right) \right\| = \sqrt{2} - O\left(\frac{1}{\log N}\right).
\]

**Proof** We use the decomposition of \( |f_2\rangle \) in equation (39). By Lemma 16, the (unnormalized) eigenvector of \( WF \) associated with eigenphase \( \beta \) is \( |e_\beta\rangle = |f_2\rangle + i|e_\perp\rangle \), where

\[
|e_\perp\rangle = g_0 \cot\left(\frac{\beta}{2}\right) |A_0\rangle + \sum_{k=1}^m g_k \left[ \cot\left(\frac{\beta - \varphi_k}{2}\right) |A_k^+\rangle + \cot\left(\frac{\beta + \varphi_k}{2}\right) |A_k^-\rangle \right] - g_{-1} \tan\left(\frac{\beta}{2}\right) |A_{-1}\rangle.
\]

Let \( |B_1^+\rangle \) denote the normalization of \( |e_\beta\rangle \), and let \( |B_1^-\rangle \) denote the conjugate of \( |B_1^+\rangle \). Then \( \Pi_\beta \) is a projection onto the span of \( |B_1^+\rangle \) and \( |B_1^-\rangle \). We know from the proof of Lemma 12 that

\[
\left| \left( \langle A_1^+ | + \langle A_1^- | \right) \left( |\pi_z\rangle + |\varnothing\rangle \right) \right| = 2 - O\left(\frac{1}{\log N}\right),
\]
where $|A_1^+\rangle, |A_1^-\rangle$ are the normalizations of $|\zeta\rangle$ and $|\bar{\zeta}\rangle$, respectively. We prove Lemma 15 by showing that $|A_1^+\rangle$ and $|A_1^-\rangle$ have large overlap with $|B_1^+\rangle$ and $|B_1^-\rangle$, respectively.

We know from equation (47) that
\[
\||e_\beta\rangle\|^2 = \||e_\beta^\perp\rangle\|^2 + \|f_2\|^2 \\
= g_0^2 \cot^2\left(\frac{\beta}{2}\right) + \sum_{k=1}^{m} g_k^2 \left[ \cot^2\left(\frac{\beta - \varphi_k}{2}\right) + \cot^2\left(\frac{\beta + \varphi_k}{2}\right) \right] + g_{-1}^2 \tan^2\left(\frac{\beta}{2}\right) + 1.
\]

By Lemmas 13 and 14, we know that $g_0^2 \cot^2\left(\frac{\beta}{2}\right) \in \mathcal{O}(N \log N)$ and that $g_{-1}^2 \tan^2\left(\frac{\beta}{2}\right) \in \mathcal{O}\left(\frac{1}{N \log N}\right)$.

Recall from equation (45) that
\[
\sum_{k=2}^{m} g_k^2 \left[ \cot^2\left(\frac{\beta - \varphi_k}{2}\right) + \cot^2\left(\frac{\beta + \varphi_k}{2}\right) \right] \in \mathcal{O}\left(\frac{1}{\log N}\right).
\]

Finally, we know from Lemma 14 that $\cot^2\left(\frac{\beta - \varphi_1}{2}\right) \in \mathcal{O}(N \log N)$. Thus,
\[
\||e_\beta\rangle\|^2 = g_1^2 \cot^2\left(\frac{\beta - \varphi_1}{2}\right) + \mathcal{O}(N \log N).
\]

By Lemma 13, $g_1^2 \in \mathcal{O}\left(\frac{1}{\log N}\right)$, and by Lemma 14, $\cot^2\left(\frac{\beta - \varphi_1}{2}\right) \in \mathcal{O}(N(\log N)^3)$. Therefore,
\[
|\langle A_1^+ | B_1^+ \rangle|^2 = \frac{|\langle A_1^+ | e_\beta \rangle|^2}{\||e_\beta\rangle\|^2} \\
= \frac{|\langle A_1^+ | f_2 \rangle + i \langle A_1^+ | e_\beta^\perp \rangle|^2}{\||e_\beta\rangle\|^2} \\
= \frac{|g_1 + ig_1 \cot\left(\frac{\beta - \varphi_1}{2}\right)|^2}{\||e_\beta\rangle\|^2} \\
= 1 - \mathcal{O}\left(\frac{1}{\log N}\right),
\]
so $|\langle A_1^+ | B_1^+ \rangle| = 1 - \mathcal{O}\left(\frac{1}{\log N}\right)$. Similarly, one can show that
\[
|\langle A_1^- | B_1^+ \rangle| \in \mathcal{O}\left(\frac{1}{\log^2 N}\right),
\]
\[
|\langle A_1^+ | B_1^- \rangle| \in \mathcal{O}\left(\frac{1}{\log^2 N}\right),
\]
\[
|\langle A_1^- | B_1^- \rangle| = 1 - \mathcal{O}\left(\frac{1}{\log N}\right).
\]

Combining these results, we get
\[
\left|\left(\langle B_1^+ | + \langle B_1^- \rangle\right)\left(\langle A_1^+ \rangle + \langle A_1^- \rangle\right)\right| = 2 - \mathcal{O}\left(\frac{1}{\log N}\right).
\]
Therefore,
\[
\| \Pi_\beta (|\pi_z\rangle + |\circ\rangle) \| \geq \frac{1}{\sqrt{2}} \left| \left( \langle B_1^+ | + \langle B_1^- | \right) (|\pi_z\rangle + |\circ\rangle) \right| \\
\geq \frac{1}{2\sqrt{2}} \left| \left( \langle B_1^+ | + \langle B_1^- | \right) (|A_1^+\rangle + |A_1^-\rangle) \right| \left| \left( \langle A_1^+ | + \langle A_1^- | \right) (|\pi_z\rangle + |\circ\rangle) \right| \\
= \sqrt{2} - O\left( \frac{1}{\log N} \right).
\]

\[\square\]

### 6.2 Proof of main result

Through Lemmas 14 and 15, we have an asymptotic description of the slowest rotational subspace of WF. The description shows that as \( N \) increases, the action of WF on \( |\pi_z\rangle \) approaches a rotation in this slowest rotational subspace. This property is what allows us to map our initial state to the target state \( |\circ\rangle \) with probability approaching 1. Applying the relationship \( WF = (U)_z \), we thus obtain the proof of our main result.

**Theorem 1 (Main result)** Fix \( s = 1 - \frac{1}{\sqrt{N}} \) and suppose \( n_r = n_c \). Then there exists a constant \( c > 0 \) such that after \( c\sqrt{N \log N} \) applications of \( U \) to \( |\pi\rangle \), measuring the state will produce \( |\circ\rangle \) with probability \( 1 - e(N) \), where \( e(N) \in O\left( \frac{1}{\log N} \right) \).

**Proof** First, observe that for any \( k \),
\[
U^k |\pi\rangle = cz(WF)^k cz |\pi\rangle = cz(WF)^k |\pi_z\rangle.
\]
Thus, we prove that after \( c\sqrt{N \log N} \) applications of WF to \( |\pi_z\rangle \), measuring the state will produce \( cz|\circ\rangle = |\circ\rangle \) with the stated probability.

Recall from equation (26) that \( |\pi_z\rangle \) can be decomposed as
\[
|\pi_z\rangle = \frac{1}{2} |U_0\rangle + \frac{1}{2} \left( |\pi_z\rangle + |\circ\rangle \right).
\]

By Lemma 7, we know that \( |U_0\rangle \) is a \((+1)\)-eigenvector of WF. Letting \( \Pi_\beta \) denote the projection onto the slowest rotational subspace of WF, we decompose \( |\pi_z\rangle + |\circ\rangle \) as
\[
|\pi_z\rangle + |\circ\rangle = \Pi_\beta \left( |\pi_z\rangle + |\circ\rangle \right) + |\perp\rangle.
\]
for some vector \( |\perp\rangle \). By Lemma 15, we know that \( \| |\perp\rangle \| \in O\left( \frac{1}{\log N} \right) \). We also know from Lemma 14 that applying WF to a vector in the slowest rotational subspace will result in a rotation of the vector by the angle \( \beta \in \Theta\left( \frac{1}{\sqrt{N \log N}} \right) \). Therefore, there exists a constant \( c \) such that \( c\sqrt{N \log N} = \left\lfloor \frac{n}{\beta} \right\rfloor = k \). After \( k \) applications of WF to \( |\pi_z\rangle \), we get the state
\[
(WF)^k |\pi_z\rangle = \frac{1}{2} \left( |\pi_z\rangle - |\circ\rangle \right) - \frac{1}{2} \Pi_\beta \left( |\pi_z\rangle + |\circ\rangle \right) + |\rho\rangle
\]
\[
= -|\circ\rangle + \frac{1}{2} |\perp\rangle + |\rho\rangle.
\]

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Here, $|\rho\rangle$ is some state that captures both the result of applying $(WF)^k$ to $|\perp\rangle$ and the small error incurred by the rounding of $\frac{2}{\rho}$, and has norm $\|\rho\| \in O\left(\frac{1}{\log N}\right)$. Thus, measuring the state will produce $|\bigcirc\rangle$ with probability $1 - e(N)$, where $e(N) \in O\left(\frac{1}{\log N}\right)$.

7 Conclusion

We give a quantum walk that uses minimal memory and $\Theta(\sqrt{N \log N})$ steps to find a unique marked vertex on a two-dimensional grid. In doing so, we show how interpolated walks can be adapted to the memoryless setting. By adding a selfloop to the marked vertex, our walk boosts the probability of measuring the marked state from $O\left(\frac{1}{\log N}\right)$ to $1 - O\left(\frac{1}{\log N}\right)$, while preserving the simplicity of the tessellation-based structure.

We give a precise analysis of how the selfloop affects the walk dynamics by showing that our walk asymptotically reduces to a rotation in a single two-dimensional subspace. Applying this rotation evolves the initial state to the selfloop state, from which the marked state can be obtained by straightforward amplitude amplification.

As part of our proof, we give a precise description of the slowest rotational subspace of our memoryless walk operator. This is done using its decomposition into a real operator composed with a two-dimensional rotation. The techniques we use to analyse such an operator are general enough they have the potential to be used in the analysis of other walks as well. This includes developing and analysing memory-optimal spatial search algorithms for other types of graph, as well as for handling graphs with multiple marked vertices.

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8 Appendix

8.1 Composition with a reflection

We discuss techniques to characterize the spectra for the composition of a real operator with a one-dimensional reflection. Operators with this structure appear at multiple points in our work. In this section, we present two lemmas from the quantum walk literature that we apply in our analysis of both \(WF_1\) and \(WF\).

Consider an arbitrary real unitary operator \(T\) acting on a space \(\mathcal{H}\), and let \(|s\rangle \in \mathcal{H}\) be a state with real amplitudes. Define \(S = I - |s\rangle \langle s|\) to be the reflection of state \(|s\rangle\). The goal of this section is to describe the spectra of the operator \(TS\).

Because \(T\) is real-valued, its eigenvalues different from \(\pm 1\) come in complex conjugate pairs. We denote the eigenvalues as \(e^{\pm \phi_k}\) for \(k = 1, 2, \ldots, m\), corresponding to the eigenvectors \(|T_k^\pm\rangle\).

We then decompose \(|s\rangle\) into the eigenbasis of \(T\) as

\[
|s\rangle = s_0 |T_0\rangle + \sum_k s_k \left( |T_k^+\rangle + |T_k^-\rangle \right) + s_{-1} |T_{-1}\rangle.
\]

(48)

Here, \(|T_0\rangle\) and \(|T_{-1}\rangle\) are eigenvectors of \(T\) with eigenvalues \(+1\) and \(-1\), respectively. The coefficients \(s_0, s_{-1}\), and all \(s_k\) are chosen to be non-negative real numbers by multiplying the eigenvectors with appropriate phases. This decomposition allows us to state the following lemma, originally given by [Amb07].

**Lemma 16** Consider the (unnormalized) state \(|e_\alpha\rangle = |s\rangle + i|e_\alpha^-\rangle\), where

\[
|e_\alpha^-\rangle = s_0 \cot \left( \frac{\alpha}{2} \right) |T_0\rangle + \sum_k s_k \left[ \cot \left( \frac{\alpha - \phi_k}{2} \right) |T_k^+\rangle + \cot \left( \frac{\alpha + \phi_k}{2} \right) |T_k^-\rangle \right] - s_{-1} \tan \left( \frac{\alpha}{2} \right) |T_{-1}\rangle,
\]

(49)

and \(|e_\alpha^-\rangle\) is orthogonal to \(|s\rangle\). If \(\alpha\) is a solution of the equation

\[
s_0^2 \cot \left( \frac{\alpha}{2} \right) + \sum_k s_k^2 \left[ \cot \left( \frac{\alpha - \phi_k}{2} \right) + \cot \left( \frac{\alpha + \phi_k}{2} \right) \right] - s_{-1}^2 \tan \left( \frac{\alpha}{2} \right) = 0,
\]

(50)

then \(|e_\alpha\rangle\) is an eigenvector of \(TS\) with eigenvalue \(e^{i\alpha}\).

This lemma allows us to determine the eigenvectors and eigenvalues of \(TS\) by specifying a set of constraints they must satisfy. The lemma is applied in [Amb07], and with slight variations
in [AKR05], [Tul08] and [DH17], to obtain bounds on the smallest eigenphase of a walk operator. We use the lemma for the same purpose, applying it to obtain a lower bound for the smallest eigenphase of WF1 and WF in Lemmas 8 and 14. We also use a similar technique in our analysis of the eigenvector $|\zeta\rangle$ in Lemma 9, where we derive a set of constraints and use them to find properties of $a$ and $|\psi\rangle$.

The next theorem we state describes the behaviour of the eigenphases of $TS$ in relation to those of $T$. The flip-flop theorem of [DH17] describes how the eigenphases of the operators interlace, with the exact pattern of interlacing depending on the eigen spaces of $T$ that $|s\rangle$ intersects. We limit the theorem statement to the case we apply in this paper, where $|s\rangle$ intersects the $(1)$-eigenspace, the $(2)$-eigenspace, and at least one other eigenspace of $T$.

**Theorem 17 (Flip-flop theorem)** Consider any real unitary $T$ and let $|s\rangle$ be a state with real amplitudes in the same space. Denote the positive eigenphases of $T$ different from 0, $\pi$ by $0 < \phi_1 \leq \phi_2 \leq \cdots \leq \phi_m < \pi$. If $s_0 \neq 0$, $s_{-1} \neq 0$ and $s_k \neq 0$ for some $k$, then $TS$ has $m+1$ two-dimensional eigenspaces, and no $(1)$- or $(2)$-eigenspaces which overlap $|s\rangle$. The positive eigenphases $\alpha_j$ of $TS$ satisfy the inequality $0 < \alpha_0 < \phi_1 \leq \alpha_1 \leq \cdots \leq \phi_m \leq \alpha_m < \pi$.

We apply this theorem in Lemmas 8 and 14 to obtain an upper bound on the smallest positive eigenphases of $WF_1$ and $WF$, respectively. One of the contributions of our work is to show how Theorem 17 can be used in combination with Lemma 16 to tightly bound these eigenphases. We show that this approach can be used in the case of an operator composed with a reflection, and then by applying a second reflection, to an operator composed with a two-dimensional rotation.

### 8.2 Decomposition of $|+\rangle$ and $|-\rangle$

To analyse the behaviour of $WF_1$ and $WF$, we specify how $W$ acts on vectors in the non-trivial eigenspaces of $F$. Recall from the definitions in equation (8) and equation (9) that $|+\rangle$ and $|-\rangle$ are orthonormal vectors that have the same span as $|g\rangle$ and $|a_{00}\rangle$. Together with $|\zeta\rangle$, they span a space that includes the two-dimensional subspace on which $F$ acts non-trivially. In this appendix, we prove Lemma 5, which describes how $|+\rangle$ and $|-\rangle$ decompose into the invariant subspaces of $W$.

To simplify notation, define

$$s_{kl}^+ = \frac{1}{2} (r_{kl}^+ + r_{kl}^-) = \sqrt{1 + \frac{\sin \bar{I}}{p_{kl}}}$$

$$s_{kl}^- = \frac{1}{2} (r_{kl}^+ - r_{kl}^-) = \epsilon_1 \sqrt{1 - \frac{\sin \bar{I}}{p_{kl}}}$$
and
\[
d_{kl}^+ = \frac{1}{2}(c_{kl}^+ + c_{kl}^-) = \sqrt{1 + \frac{\sin k}{p_{kl}}} \\
d_{kl}^- = \frac{1}{2}(c_{kl}^+ - c_{kl}^-) = \epsilon_k \sqrt{1 - \frac{\sin k}{p_{kl}}}
\]

In the case where \( k = l = 0 \), we define \( s_{00}^+ = d_{00}^+ = \sqrt{2} \) and \( s_{00}^- = d_{00}^- = 0 \). Note that \( r_{kl}^+ = s_{kl}^+ \pm s_{kl}^- \) and \( c_{kl}^+ = d_{kl}^+ \pm d_{kl}^- \).

Recall that both \( |a_{00}\rangle \) and \( |g\rangle \) lie in the span of the basis states \( |00\rangle, |01\rangle, |10\rangle \) and \( |11\rangle \). We compute the projections of these basis states onto the components of the eigenvectors of \( W \).

\[
\begin{align*}
\langle 0|u_{kl}\rangle &= \sqrt{2}\langle 0|r_{kl}\rangle \langle 0|\phi^k_r\rangle = \frac{1}{\sqrt{2n_r}}r_{kl}^- \\
\langle 0|v_{kl}\rangle &= \sqrt{2}\langle 0|c_{kl}\rangle \langle 0|\phi^l_c\rangle = \frac{1}{\sqrt{2n_c}}c_{kl}^- \\
\langle 1|u_{kl}\rangle &= \sqrt{2}\langle 1|r_{kl}\rangle \langle 1|\phi^k_r\rangle = \frac{1}{\sqrt{2n_r}}r_{kl}^+ \omega_n^k \\
\langle 1|v_{kl}\rangle &= \sqrt{2}\langle 1|c_{kl}\rangle \langle 1|\phi^l_c\rangle = \frac{1}{\sqrt{2n_c}}c_{kl}^+ \omega_n^l
\end{align*}
\]

Now, using the property that
\[
\begin{align*}
\frac{1}{2}\left(r_{kl}^+\omega_n^k + r_{kl}^-ight) &= \frac{1}{2}\omega_n^{k/2}\left(r_{kl}^+\omega_n^{k/2} + r_{kl}^-\omega_n^{-k/2}\right) = \omega_n^{k/2}\left(\cos \left(\frac{k}{2}\right)s_{kl}^+ + \sin \left(\frac{k}{2}\right)s_{kl}^\prime\right) \\
\frac{1}{2}\left(r_{kl}^+\omega_n^k - r_{kl}^\prime\right) &= \frac{1}{2}\omega_n^{k/2}\left(r_{kl}^-\omega_n^{k/2} - r_{kl}^+\omega_n^{-k/2}\right) = \omega_n^{k/2}\left(-\cos \left(\frac{k}{2}\right)s_{kl}^- + \sin \left(\frac{k}{2}\right)s_{kl}^\prime\right)
\end{align*}
\]

we compute
\[
\begin{align*}
\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)|u_{kl}\rangle &= \frac{1}{\sqrt{n_r}}\omega_n^{k/2}\left(\cos \left(\frac{k}{2}\right)s_{kl}^+ + \sin \left(\frac{k}{2}\right)s_{kl}^-\right) \\
\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)|u_{kl}^1\rangle &= \frac{1}{\sqrt{n_r}}\omega_n^{k/2}\left(-\cos \left(\frac{k}{2}\right)s_{kl}^\prime + \sin \left(\frac{k}{2}\right)s_{kl}^\prime\right) \\
\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)|v_{kl}\rangle &= \frac{1}{\sqrt{n_c}}\omega_n^{l/2}\left(\cos \left(\frac{l}{2}\right)d_{kl}^+ + \sin \left(\frac{l}{2}\right)d_{kl}^-\right) \\
\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)|v_{kl}^1\rangle &= \frac{1}{\sqrt{n_c}}\omega_n^{l/2}\left(-\cos \left(\frac{l}{2}\right)d_{kl}^- + \sin \left(\frac{l}{2}\right)d_{kl}^\prime\right)
\end{align*}
\]

Excluding the case \( k = l = 0 \), the squared amplitudes of the projections are then
Using the property that

By definition of the invariant subspaces $W$, it can be derived that

For any subspace $W_{kl}$, Fact 18

\[
\langle g \Pi_{kl} | a_{00} \rangle = \begin{cases} 
\frac{1}{2} & \text{if } k = l = 0 \\
0 & \text{otherwise}
\end{cases} \tag{51}
\]

**Proof** Recall that $|g\rangle = |00\rangle$ and $|a_{00}\rangle = \frac{1}{2}(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)$. Consider any $k, l$ not both 0. Then

\[
\langle g | w_{k,l}^{11} \rangle \langle w_{k,l}^{11} | a_{00} \rangle \\
= \frac{1}{2N} (r_{k,l}^{+} c_{k,l}^{+}) \left( \omega_{n_{r}}^{k} / 2 \omega_{n_{c}}^{l} / 2 \right) \left( - \cos \left( \frac{\tilde{k}}{2} \right) s_{k,l}^{+} + \sin \left( \frac{\tilde{k}}{2} \right) s_{k,l}^{-} \right) \left( - \cos \left( \frac{\tilde{l}}{2} \right) d_{k,l}^{+} + \sin \left( \frac{\tilde{l}}{2} \right) d_{k,l}^{-} \right) \\
= \frac{1}{2N} (r_{k,l}^{-} c_{k,l}^{-}) \left( \omega_{n_{r}}^{k} / 2 \omega_{n_{c}}^{l} / 2 \right) \left( \cos \left( \frac{\tilde{k}}{2} \right) s_{k,l}^{+} + \sin \left( \frac{\tilde{k}}{2} \right) s_{k,l}^{-} \right) \left( \cos \left( \frac{\tilde{l}}{2} \right) d_{k,l}^{+} + \sin \left( \frac{\tilde{l}}{2} \right) d_{k,l}^{-} \right) \\
= \frac{1}{2N} (r_{k,l}^{-} c_{k,l}^{-}) \left( \omega_{n_{r}}^{k} / 2 \omega_{n_{c}}^{l} / 2 \right) \left( \cos \left( \frac{\tilde{k}}{2} \right) s_{k,l}^{+} + \sin \left( \frac{\tilde{k}}{2} \right) s_{k,l}^{-} \right) \left( \cos \left( \frac{\tilde{l}}{2} \right) d_{k,l}^{+} + \sin \left( \frac{\tilde{l}}{2} \right) d_{k,l}^{-} \right) \\
= -\langle g | w_{k,l}^{00} \rangle \langle w_{k,l}^{00} | a_{00} \rangle.
\]

Similarly, it can be derived that

\[
\langle g | w_{k,l}^{11} \rangle \langle w_{k,l}^{11} | a_{00} \rangle = -\langle g | w_{k,l}^{00} \rangle \langle w_{k,l}^{00} | a_{00} \rangle.
\]

Using the property that $(k')' = k$, this further implies that $\langle g | w_{k,l}^{00} \rangle \langle w_{k,l}^{00} | a_{00} \rangle = \langle g | w_{k,l}^{00} \rangle \langle w_{k,l}^{00} | a_{00} \rangle$. By definition of the invariant subspaces $W_{kl}$ in Section 4.2, this shows that for any $k, l$ not both 0, $\langle g | \Pi_{kl} | a_{00} \rangle = 0$. It follows that $\langle g | \Pi_{00} | a_{00} \rangle = \langle g | a_{00} \rangle = \frac{1}{2}$. \qed

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Lemma 5 The following statements hold.

\[ \Pi_{kl}|+\rangle \perp \Pi_{kl}|-\rangle \quad \text{for all subspaces } W_{kl} \]  
\[ \|\Pi_{kl}|+\rangle\|^2 = \frac{2}{3} \frac{\dim(W_{kl})}{N} \quad \text{for all } k, l \text{ not both } 0 \]  
\[ \|\Pi_{kl}|-\rangle\|^2 = \frac{2}{3} \frac{\dim(W_{kl})}{N} \quad \text{for all } k, l \text{ not both } 0 \]  
\[ \|\Pi_{00}|+\rangle\|^2 = \frac{2N + 2}{3} \]  
\[ \|\Pi_{00}|-\rangle\|^2 = \frac{4}{N}. \]  

Proof Observe that for any \( k, l \) not both zero,

\[
\|\langle g|w_{kl}^{00}\rangle\|^2 + \|\langle g|w_{kl}^{11}\rangle\|^2 + \|\langle g|w_{kl}^{11'}\rangle\|^2 + \|\langle g|w_{kl}^{00'}\rangle\|^2
\]

\[
= \frac{1}{N} \left(1 - \frac{\sin k \cos \tilde{l}}{p_{kl}}\right) \left(1 - \frac{\cos k \sin \tilde{l}}{p_{kl}}\right) + \frac{1}{N} \left(1 - \frac{\sin k \cos \tilde{l}}{p_{kl}}\right) \left(1 + \frac{\cos k \sin \tilde{l}}{p_{kl}}\right)
\]

\[
+ \frac{1}{N} \left(1 + \frac{\sin k \cos \tilde{l}}{p_{kl}}\right) \left(1 - \frac{\cos k \sin \tilde{l}}{p_{kl}}\right) + \frac{1}{N} \left(1 + \frac{\sin k \cos \tilde{l}}{p_{kl}}\right) \left(1 + \frac{\cos k \sin \tilde{l}}{p_{kl}}\right)
\]

\[
= \frac{4}{N} = \|\langle a_{00}|w_{kl}^{00}\rangle\|^2 + \|\langle a_{00}|w_{kl}^{11}\rangle\|^2 + \|\langle a_{00}|w_{kl}^{11'}\rangle\|^2 + \|\langle a_{00}|w_{kl}^{00'}\rangle\|^2.
\]

Therefore, for any subspace \( W_{kl} \) with \( kl \neq 0 \), we have \( \langle g|\Pi_{kl}|g\rangle = \langle a_{00}|\Pi_{kl}|a_{00}\rangle = \frac{\dim(W_{kl})}{N} \). Next, recall that \( W_{00} \) refers to the \((+1)\)-eigenspace of \( W \). We know that both \(|g\rangle\) and \(|a_{00}\rangle\) are normalized, so

\[ \langle g|\Pi_{00}|g\rangle = \langle a_{00}|\Pi_{00}|a_{00}\rangle = 1 - \sum_{kl \neq 0} \frac{\dim(W_{kl})}{N} = \frac{N + 4}{2N}. \]

Applying Fact 18, this implies that

\[ \sqrt{3}\langle -|\Pi_{kl}|+\rangle = \langle g|\Pi_{kl}|g\rangle + \langle g|\Pi_{kl}|a_{00}\rangle - \langle a_{00}|\Pi_{kl}|g\rangle - \langle a_{00}|\Pi_{kl}|a_{00}\rangle = 0, \]

for any subspace \( W_{kl} \). This proves equation (20).

To prove equation (21), we compute

\[ \langle +|\Pi_{kl}|+\rangle = \frac{1}{3} \left[ \langle g|\Pi_{kl}|g\rangle - \langle g|\Pi_{kl}|a_{00}\rangle - \langle a_{00}|\Pi_{kl}|g\rangle + \langle a_{00}|\Pi_{kl}|a_{00}\rangle \right] = \frac{2\dim(W_{kl})}{3N}, \]

and similarly for equation (22).

Using the property that \(|+\rangle\) and \(|-\rangle\) are normalized, this implies that

\[ \|\Pi_{00}|+\rangle\|^2 = 1 - \sum_{kl \neq 0} \frac{2\dim(W_{kl})}{3N} = \frac{2(N + 2)}{3N}, \]

which proves equation (23). We can similarly compute that \( \|\Pi_{00}|-\rangle\|^2 = \frac{4}{N} \), proving equation (24). \( \square \)
8.3 Sums

In this section, we prove asymptotic bounds on a set of sums over the spectra of $W$. We assume a square grid, with $n_r = n_c = \sqrt{N}$.

**Fact 19** Suppose $0 < \alpha < \theta_{kl}$ for all $k, l$, and consider the sum

$$\sum_{kl \neq 0} \dim(W_{kl}) \cot\left(\frac{\theta_{kl} - \alpha}{2}\right).$$

1. If $\alpha \in \Theta\left(\frac{1}{\sqrt{N}}\right)$, then the sum has order $\Omega(\sqrt{N} \log N)$.
2. If $\alpha \in o\left(\frac{1}{\sqrt{N}}\right)$, then the sum has order $\Theta(\alpha N \log N)$.

**Proof** Instead of taking the sum over the subspaces $W_{kl}$, which partition the domain of $W$, we convert to a sum over $k$ and $l$. Recall that each pair $0 \leq k, l \leq \sqrt{N}/2 - 1$ corresponds to two eigenvectors of $W$: $|w_{kl}^{00}\rangle$ with eigenphase $\theta_{kl}$ and $|w_{kl}^{11}\rangle$ with eigenphase $-\theta_{kl}$. Using this property, we rewrite the sum as

$$\sum_{kl \neq 0} \dim(W_{kl}) \cot\left(\frac{\theta_{kl} - \alpha}{2}\right)$$

$$= \sum_{k=0}^{\sqrt{N}-1} \sum_{l=0}^{\sqrt{N}-1} \cot\left(\frac{\theta_{kl} - \alpha}{2}\right) - \cot\left(\frac{\theta_{kl} + \alpha}{2}\right)$$

$$= 2 \sum_{k=0}^{\sqrt{N}-1} \sum_{l=0}^{\sqrt{N}-1} \frac{\cot\left(\frac{\theta_{kl}}{2}\right)\left(\cot^2\left(\frac{\theta_{kl}}{2}\right) + 1\right)}{\cot^2\left(\frac{\alpha}{2}\right) - \cot^2\left(\frac{\theta_{kl}}{2}\right)},$$

where the final equality follows from angle sum identities.

Next, observe that by the definition of $\theta_{kl}$,

$$\cot^2\left(\frac{\theta_{kl}}{2}\right) + 1 = \frac{2}{1 - \cos \theta_{kl}} = \frac{1}{1 - \cos^2 k \cos^2 l}.$$

We therefore consider the sum

$$\sum_{k=0}^{\sqrt{N}-1} \sum_{l=0}^{\sqrt{N}-1} \frac{1}{1 - \cos^2 k \cos^2 l},$$

For the terms where $l = 0$, we get

$$\sum_{k=1}^{\sqrt{N}-1} \frac{1}{1 - \cos^2 k} \in \Theta(N),$$

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and similarly for \( k = 0 \). The remaining terms satisfy

\[
\sum_{k=1}^{\sqrt{N}-1} \sum_{l=1}^{\sqrt{N}-1} \frac{1}{1 - \cos k \cos l} \in \Theta(N \log N).
\]

Now, consider equation (53) in the case where \( \alpha \in \Theta(\frac{1}{\sqrt{N}}) \). We have

\[
2 \sum_{k=0}^{\sqrt{N}-1} \sum_{l=0}^{\sqrt{N}-1} \cot(\frac{\alpha}{2}) \left( \cot^2\left(\frac{\theta_{kl}}{2}\right) + 1 \right) \frac{\cot^2\left(\frac{\alpha}{2}\right) - \cot^2\left(\frac{\theta_{kl}}{2}\right)}{\cot^2\left(\frac{\alpha}{2}\right)} \\
\geq 2 \sum_{k=0}^{\sqrt{N}-1} \sum_{l=0}^{\sqrt{N}-1} \cot(\frac{\alpha}{2}) \left( \cot^2\left(\frac{\theta_{kl}}{2}\right) + 1 \right) \frac{1}{1 - \cos^2 k \cos^2 l}.
\]

Therefore,

\[
\sum_{kl \neq 0} \dim(W_{kl}) \cot\left(\frac{\theta_{kl} - \alpha}{2}\right) \in \Omega(\sqrt{N} \log N).
\]

In the case where \( \alpha \in o\left(\frac{1}{\sqrt{N}}\right) \), the denominator in equation (53) is dominated by the term \( \cot^2\left(\frac{\alpha}{2}\right) \). Therefore, the expression has the same asymptotic order as

\[
\frac{1}{\cot^2\left(\frac{\alpha}{2}\right)} \sum_{k=0}^{\sqrt{N}-1} \sum_{l=0}^{\sqrt{N}-1} \frac{1}{1 - \cos^2 k \cos^2 l} \in \Theta(\alpha^2 N \log N),
\]

proving the second clause. \( \Box \)

**Fact 20** Suppose \( 0 < \alpha < \theta_{kl} \) for all \( k, l \), and that \( \alpha \in o\left(\frac{1}{\sqrt{N}}\right) \). Then

\[
\sum_{kl \neq 0} \dim(W_{kl}) \left[ \cot\left(\frac{\theta_{kl} + \alpha}{2}\right) - \cot\left(\frac{\theta_{kl} - \alpha}{2}\right) \right]^2 \in \Theta(\alpha^2 N^2). \tag{54}
\]

**Proof**

\[
\sum_{kl \neq 0} \dim(W_{kl}) \left[ \cot\left(\frac{\theta_{kl} + \alpha}{2}\right) - \cot\left(\frac{\theta_{kl} - \alpha}{2}\right) \right]^2 \\
= \sum_{k=0}^{\sqrt{N}-1} \sum_{l=0}^{\sqrt{N}-1} \left[ \cot\left(\frac{\theta_{kl} + \alpha}{2}\right) - \cot\left(\frac{\theta_{kl} - \alpha}{2}\right) \right]^2.
\]
By a similar derivation as in Fact (19), this sum has the same order as

\[
\frac{1}{\cot^2(\frac{\alpha}{2})} \sum_{k=0}^{\sqrt{N}-1} \sum_{l=0}^{\sqrt{N}-1} \left( \frac{1}{1 - \cos^2 k \cos^2 l} \right)^2.
\]

Using the Taylor expansion of cosine, for \( l = 0 \) we get

\[
\sum_{k=0}^{\sqrt{N}-1} \left( \frac{1}{1 - \cos^2 k} \right)^2 \in \Theta(N^2),
\]

and similarly for \( k = 0 \). Finally,

\[
\sum_{k=1}^{\sqrt{N}-1} \sum_{l=1}^{\sqrt{N}-1} \left( \frac{1}{1 - \cos^2 k \cos^2 l} \right)^2 \in \Theta(N^2).
\]

Therefore,

\[
\sum_{kl \neq 0} \dim(W_{kl}) \left[ \cot\left( \frac{\theta_{kl} + \alpha}{2} \right) - \cot\left( \frac{\theta_{kl} - \alpha}{2} \right) \right]^2 \in \Theta(\alpha^2 N^2).
\]

\( \square \)