Scaling analysis of a model Hamiltonian for Ce\(^{3+}\) impurity in a cubic metal

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Abstract

We introduce various exchange interactions in a model Hamiltonian for Ce\(^{3+}\) ions in cubic symmetry with three configurations \((f^0,f^1,f^2)\). With the impurity pseudo spin \(S_I = 1/2\), our Hamiltonian includes: (i) One-channel \(S_c = 1/2\) Anderson model; (ii) Two-channel \(S_c = 1/2\) Anderson model; (iii) An unforseen one-channel \(S_c = 3/2\) Anderson model with a non-trivial fixed point; (iv) Mixing exchange interaction between the \(\Gamma_{6,7}\) and the \(\Gamma_8\) conduction electron partial wave states; (v) Multiple conduction electron partial wave states. Using the third-order scaling (perturbative renormalization group) analysis, we study stability of various fixed points relevant to various exchange interactions for Ce\(^{3+}\) ions in cubic symmetry.

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I. INTRODUCTION.

Since the introduction of the orbitally nondegenerate Anderson model [1], realistic extensions of this model have been considered [2,3]. Hirst [2] described the Anderson model including the spin-orbit interaction and the crystal electric field effect based on the group theory. A detailed study of possible low temperature physics along this line was given by Nozières and Blandin [3] for the magnetic ions in normal metal hosts (primarily transition metal ions). Nozières and Blandin discovered the multichannel Kondo models which have non-trivial fixed points in the overscreened cases, but concluded that the realization of this model has a difficulty in real materials due to the channel asymmetry. Cox introduced the two-channel quadrupolar Anderson model [4] to explain the physics of UBe$_{13}$, and listed possible crystal symmetries [5] for the existence of the two-channel exchange interaction in rare earth and actinide compounds or alloys. However, a detailed study is required for the stability of the two-channel fixed point against possible symmetry-breaking fields. Though in a different context, the metallic two-level system (electron-assisted atom tunneling in a double well) was shown to map into the two-channel Kondo model [6].

In this paper, we study in detail a one-impurity Anderson model for Ce$^{3+}$ with three configurations in cubic symmetry in the same spirit as Nozières and Blandin did for the transition metal ions. Our study can be generalized to any $f$ atomic configurations in cubic symmetry. As we shall show below, the inclusion of the detailed energy structure in the atomic orbital leads to rich physics. In summary, we find various exchange interactions with the impurity pseudo spin $S_I = 1/2$: (a) One-channel $S_c = 1/2$ exchange interactions for $\Gamma_{6,7}$ conduction electrons (magnetic “Kramers”’ doublet orbitals); (b) Two-channel $S_c = 1/2$ exchange interaction for $\Gamma_8$ conduction electrons (quartet of orbitals); (c) One-channel $S_c = 3/2$ exchange interaction for $\Gamma_8$ conduction electrons; (d) Mixing exchange interactions between $\Gamma_{6,7}$ and $\Gamma_8$ conduction electrons. In addition, (e) the multiple conduction electron partial wave states enhance the exchange couplings. In connection to the normal states of the cuprate superconductors [7] and the recent discoveries of non-Fermi liquid systems [8] in
heavy fermion materials, the relevant question is whether the ground states of the models (b) or (c) can be realized in the presence of other exchange interactions.

We will answer this question using perturbative renormalization group (third-order scaling) arguments. We find three kinds of stable fixed points with the impurity spin \( S_I = 1/2 \): one-channel Fermi liquid fixed points; two-channel non-Fermi liquid fixed point; and three-channel non-Fermi liquid fixed points. In addition, a “zoo” of unstable fixed points are discovered. At these unstable fixed points, all the possible exchange interactions can exist in the octuplet manifolds of the conduction electrons. Furthermore the multiple conduction electron partial waves generate the multi-channel unstable fixed points. Some of our fixed points possess dimensionless couplings which are rigorously in the perturbative regime, while for the above mentioned one-, two-, and three-channel cases this is not so. Nonetheless, since we are only after the existence of the fixed points, the scaling theory has proven qualitatively accurate for non-trivial two- and three-channel fixed points, and the one-channel fixed points are well known to flow to strong coupling. Thus we have confidence in the correctness of our qualitative conclusions regarding stability of the various fixed points, provided we understand the modification for the one-channel case.

Similar results and conclusions have been reached by Zaránd [9] in the two-level system Kondo model (electron assisted tunneling of an atom in a double well) where he employed a large number of channels (large real spin) and thus rigorously controlled the perturbative renormalization group theory. We have verified the essential correctness of our scaling results for the \( S_c = 3/2 \) conduction electrons with a numerical renormalization group study of the model (also combined with some conformal field theory analysis of the model) [10,11]. A recent conformal field theory study of single channel, large conduction spin Kondo models, with special emphasis on the \( S_c = 3/2 \) case; the results are compatible with ours [12]. Since this fixed point is ultimately unstable against partial wave mixing exchange terms, it may be of little physical relevance. However, an interesting theoretical question is raised by our study of the fixed point, bolstered by an independent numerical renormalization group and conformal field theory study [11]. Namely, the universality class of the fixed point is not
that of the three-channel $S_f = 1/2$ Kondo model as conjectured by Tsvelik and Wiegman [13] and utilized in their Bethe-Ansatz treatment of the multichannel model, a point also noted by Zarând in his study of the novel unstable fixed points of the TLS Kondo model [9]. These ideas will be discussed more extensively in Ref. [11].

Our paper is organized as follows: Section II contains an overview of the generalized Anderson model Hamiltonian and the underlying group theoretical motivations for understanding the various couplings which arise. Section III contains a precise and systematic discussion of the hybridization matrix elements which feed into the model together with derivation of the effective Schrieffer-Wolff couplings between conduction and $4f$ electrons upon integration out of virtual charge fluctuations. Section IV contains the scaling analysis of various models which may be obtained in limiting cases of the full Hamiltonian including all possible conduction electron symmetry states without adding in multiple partial waves. Section V discusses the effect of including multiple conduction electron partial waves in each symmetry channel. Finally, we summarize and offer conclusions in Section VI. Three appendices detail finer points about the derivation of mixing matrix elements (App. A), the multiplicative renormalization group applied here (App. B), and the general form of the third order scaling equations for our models (App. C).

II. MODEL HAMILTONIAN I: OVERVIEW OF ENERGETICS AND SYMMETRY.

The model Hamiltonian we consider throughout this paper is the generalized Anderson model [12]. Schematically, this Hamiltonian has three parts, defined by

$$H = H_{\text{cond}} + H_f + H_{\text{hyb}}$$

where $H_{\text{cond}}$ defines a broad flat conduction band of width $D$, $H_f$ defines the quasi-atomic $4f$ states on the Ce site as modified by metallic screening of the direct Coulomb interaction and the crystalline electric field (CEF), and $H_{\text{hyb}} = H_{01} + H_{12}$ describes the hybridization between
the 4f and conduction electrons. $H_{01}$ describes mixing of $f^0 - f^1$ excitations with conduction electrons, and $H_{12}$ describes mixing of $f^1 - f^2$ excitations with conduction electrons. We shall describe $H_{hyb}$ in detail in the next section; in this section we shall focus on the symmetry and energetic aspects of the first two terms. In particular, we shall emphasize the symmetries of tensor operators from the conduction and f sectors which may be coupled together in the effective exchange interactions mediated by virtual interconfiguration excitations.

Before proceeding further, we provide a brief physical description of the different irrep labels of the cubic group:

(1) $\Gamma_6$ and $\Gamma_7$ are magnetic Kramers’ doublet ($\text{Kramers’}$). Two degenerate states can be transformed into each other with time reversal symmetry operator on them and will be labeled with pseudo-spin $\uparrow$ or $\downarrow$. That is, the $\Gamma_6$ and $\Gamma_7$ CEF states are similar to the $J = 1/2$ angular momentum manifold.

(2) $\Gamma_8$ is a magnetic quartet ($\Gamma_8 = \Gamma_3 \otimes \Gamma_7$). With the time reversal symmetry operator, the four-fold degeneracy can be decomposed into two pairs. These two pairs are disjoint in terms of time reversal and will be labeled by orbital indices, ±, which correspond to the quadrupolar or orbital $\Gamma_3$ degree of freedom – stretched ($3z^2 - r^2$ like) or squashed ($x^2 - y^2$ like) local orbitals guaranteed degenerate by the cubic symmetry. The $\Gamma_7$ degree of freedom will be labeled as $\uparrow$ and $\downarrow$ as above. Overall, the $\Gamma_8$ quartet is labeled by $\pm \uparrow / \downarrow$. Under certain circumstances it is more favorable to view the $\Gamma_8$ manifold as an effective $S = 3/2$ manifold, as we shall discuss below.

(3) The orbital singlets $\Gamma_1$ and $\Gamma_2$ do not need any further label. The $\Gamma_2$ is a pseudoscalar, the lowest even parity version being $(x^2 - y^2)(y^2 - z^2)(z^2 - x^2)$.

(4) The orbital (non-magnetic, or non-Kramers’) doublet $\Gamma_3$ will be labeled by ± as in the above. In this case, time reversal symmetry operator does not transform one state to the other as for a Kramers’ doublet.

(5) The magnetic triplets $\Gamma_4$ and $\Gamma_5$ are labeled by 0, ±1. Under the time reversal symmetry operation, the ±1 states transform into the ±1 states, respectively. On the other hand, the state 0 transforms into itself. That is, the $\Gamma_4$ and $\Gamma_5$ CEF states are similar to the $J = 1$
multiplets under time reversal symmetry operation.

In the absence of energetic considerations, there are 14 (91) different degenerate eigenstates in the $f^1$ ($f^2$) configuration. These states can be progressively split by the strong on-site Coulomb “Hund’s rule” interactions ($f^2$), spin-orbit coupling, and the crystalline electric field (CEF). The energy hierarchy in Cerium is, roughly, that the interconfiguration splitting are several eV, the exchange splittings are about 0.2 – 0.5 eV for the $f^2$, the spin orbit splittings are order 0.3 eV, and the typical CEF splittings are from a few to tens of meV. Specifically, the $f^0 - f^1$ interconfiguration excitation energy is $E(f^1) - E(f^0) \approx -2\text{eV}$, and the $f^1 - f^2$ energy difference is $E(f^2) - E(f^1) \approx 4\text{eV}$ [14]. Traditionally, $E(f^1) - E(f^0)$ is referred to as $\epsilon_f$ while $E(f^2) - E(f^1)$ is referred to as $\epsilon_f + U$, meaning the effective Coulomb interaction is about 6 eV.

For the $f^1$ configuration, the energy level structure is simple due to the absence of the on-site Coulomb interaction. Spin-orbit coupling first results in $J = 5/2, 7/2$ multiplets. In most Ce$^{3+}$ compounds and alloys, this energy splitting is always about 0.3 eV [15]. Furthermore the crystalline electric field will split each $J = 5/2, 7/2$ multiplet. The CEF can mix two different $J$’s but with the second order perturbation correction of order $|< J = 5/2 | H_{CEF} | J = 7/2 > | ^2 / \Delta_{LS}$ which is of the order of $0.1 \times \Delta_{CEF}$. Neglecting the mixing of two different $J$’s in the presence of the CEF, the irreducible representations (irreps) of the full rotation group, $D_{5/2}, D_{7/2}$ decompose in the cubic field as [16]

$$D_{5/2} = \Gamma_7 \oplus \Gamma_8,$$

(2)

$$D_{7/2} = \Gamma_6 \oplus \Gamma_7 \oplus \Gamma_8.$$

(3)

Here $\Gamma_6$ and $\Gamma_7$ are the magnetic Kramers doublet’s irreps and $\Gamma_8$ is the magnetic quartet irrep of the cubic double group. The eigenstates of $\Gamma_{7,8}$ for $J = 5/2$ multiplets are given in Table I [17]. We will remove the superscript $J = 5/2$ below when the context is clear. The eigenstates of $\Gamma_{6,7,8}$ for $J = 7/2$ multiplets are listed in Table [1]. The choice of the overall phase is arbitrary in defining the CEF eigenstates. The above convention will make the effective exchange interaction a simple form. Throughout this work we shall assume that
the $J = 5/2, \Gamma_7$ doublet lies lower than the $J = 5/2, \Gamma_8$ quartet.

For the $f^2$ configuration, the existence of the strong on-site Coulomb interaction complicates the energy level structure. The LS Russel-Saunders scheme applies to the $f$ electron in Ce \cite{15}. First in the absence of both spin-orbit coupling and CEF, total orbital angular momentum $\vec{L}$ and total spin angular momentum $\vec{S}$ are good quantum numbers, and thus energy eigenstates can be sorted out by these quantum numbers. In the presence of spin-orbit coupling, the total angular momentum $\vec{J} = \vec{L} + \vec{S}$ is a good quantum number. There are seven possible values of $J$ starting from 0 through 6. When Ce$^{3+}$ ions are sitting in a cubic environment, each $J$ multiplet further splits into CEF energy levels (see Table II). For example, the $f^2$ Hund’s rule ground multiplet $J = 4(3H_4)$ decomposes as

\[ D_4 = \Gamma_1 \oplus \Gamma_3 \oplus \Gamma_4 \oplus \Gamma_5. \] (4)

All in all, there are 7 $\Gamma_1$’s, 3 $\Gamma_2$’s, 9 $\Gamma_3$’s, 9 $\Gamma_4$’s, and 12 $\Gamma_5$’s in the $f^2$ configuration. Here $\Gamma_1$ and $\Gamma_2$ are singlet irreps, $\Gamma_3$ the non-Kramers doublet irrep (nonmagnetic), and $\Gamma_4,5$ the magnetic triplet irreps of the cubic point group. The eigenstates of $\Gamma_{1,3,4,5}$ for $J = 4$ multiplets are in Table II.

We now consider the conduction electrons. According to the Anderson model picture, the conduction electrons can hop on and off the atomic orbitals at the impurity site. The Bloch electrons may be projected at the impurity site into three irreps in the cubic symmetry: $\Gamma_6, \Gamma_7, \Gamma_8$. For the $f$ shell in Ce, we expect the $l = 3$ conduction electron partial waves are most strongly coupled to the $f$ shell. For the isotropic hybridization, only the $l = 3$ components can hybridize with the $f$ orbitals. In the presence of the spin-orbit coupling, the total angular momentum is a good quantum number. Thus we have $J = 5/2, 7/2$ conduction electron partial waves. these $J$ multiplets further split into CEF irreps in crystal environments as mentioned above. When the real charge fluctuations are removed from the model system in the Kondo limit, we have to construct the tensor operators representing each CEF state for the $f^1$ configuration and the projected conduction electron CEF states. The relevant tensor operators are for $\Gamma_{6,7,8}$ CEF states. We list the decomposition of conduction
electron operators into irreducible tensor operators in Table IV using the character table [16].

In the direct product, the first CEF states are written as ket, and the second as bra. The \( \Gamma_{6,7} \) tensor operator (\( 2 \times 2 \) tensor) is a direct sum of a charge operator (\( \Gamma_1 \)) and a pseudo spin operator (\( \Gamma_4 \)). Indeed, the Schrieffer-Wolff transformation leads to two interaction terms: the spin exchange interaction and the pure potential scattering term. The relevant \( \Gamma_8 \) tensor operators are the \( 2\Gamma_4 \) triplets for our model. In the conduction electron \( \Gamma_8 \) tensor space, one of the two \( \Gamma_4 \)'s gives rise to the pseudo spin \( S_c = 1/2 \) operators with two degenerate orbital channels and the other to the pseudo spin \( S_c = 3/2 \) operator with one channel. The interpretation of the orbital channels is that these are shape degrees of freedom of the local \( \Gamma_8 \) states—one is “stretched” or \( 3z^2-r^2 \) like, and the other “squashed” or \( x^2-y^2 \) like. Thus, the \( \Gamma_{6,7} \) conduction electrons provide two different channels with the pseudo spin \( S_c = 1/2 \), while the \( \Gamma_8 \) conduction electrons provide two degenerate channels with \( S_c = 1/2 \) or one channel with \( S_c = 3/2 \).

As a further complication, mixing is possible between \( \Gamma_{6,7} \) and \( \Gamma_8 \) conduction electrons. The mixing tensor operators can be read off from Table IV. That is, each mixing can provide one pseudo spin representation. Hence from the group theory, we conclude that there are in total 6 possible exchange interactions between the \( f^1 J = 5/2 \) \( \Gamma_7 \) pseudo-spin and the three-symmetry conduction electrons. We are going to derive these in the next section.

III. MODEL HAMILTONIAN II: ANDERSON HYBRIDIZATION.

In this section we derive explicitly the forms of the hybridization matrix elements between excited \( f^0, f^2 \) states and \( f^1 \) states which are mediated by conduction electrons hopping on and off the impurity. We also derive the contributions to the effective exchange interactions obtained by integrating out virtual charge fluctuations to the \( f^0, f^2 \) states.

The conventional theory of \( Ce^{3+} \) impurities assumes an infinite on-site Coulomb interaction, which removes the \( f^2 \) configuration from consideration. When we relax our assumption about the strong on-site Coulomb interaction and we include the detailed atomic energy
structure in cubic crystals, we can construct a variety of model Hamiltonians. In this work we assume that the magnetic doublet $f^1 J = 5/2 \Gamma_7$ states lies lower than the $f^1 J = 5/2 \Gamma_8$ quartet states. The one-channel $S_c = S_I = 1/2$ Anderson hybridization is present between $f^0$ and $f^1 J = 5/2 \Gamma_7$ states. This gives rise to the conventional single channel Kondo coupling. From the hybridization between $f^1$ and $f^2$ configurations, various exchange interactions arise from virtual charge fluctuations, which include:

(i) the one-channel exchange interaction for magnetic doublet $\Gamma_{6,7}$ conduction electron orbitals;
(ii) the two-channel $S_c = 1/2$ exchange interaction for $\Gamma_8$ conduction quartets;
(iii) the one-channel $S_c = 3/2$ exchange interactions for $\Gamma_8$ conduction electrons;
(iv) mixing exchange interactions between the $\Gamma_8$ and the $\Gamma_{6,7}$ conduction electrons;
(v) multiple conduction electron partial wave states.

Below we list all the possible terms arising from the Anderson hybridization in the cubic symmetry. All the possible selection rules for the hybridization are listed in Table V. Since the Anderson hybridization is irreducible tensor of $\Gamma_1$, the selection rule is solely determined by the three components: the conduction electron partial wave states and the other two neighboring atomic configuration states. In the $f^1$ configuration, we only consider the ground CEF states $f^1 J = 5/2 \Gamma_7$. In the $f^2$ configuration, there are all $7\Gamma_1$, $3\Gamma_2$, $9\Gamma_3$, $9\Gamma_4$, and $12\Gamma_5$ CEF states.

For definiteness, we will consider isotropic hybridization. By this we mean that the original hybridization potential is taken as isotropic and this will be projected down to the appropriate irreps of the cubic point group symmetry at the Ce ion. Then $J_c = 5/2, 7/2$ conduction electron partial waves are mediating the tunneling process between neighboring configurations of the atomic orbitals.

The mixing between the $f^0$ and the $f^1$ configurations is simple and straightforward to evaluate, since the $f^0$ is a singlet with $\Gamma_1$ character. This is given by the Hamiltonian terms

$$H^{(0)}_{01} = V \sum_{\epsilon} \sum_{j, m_c} c_{\epsilon j, m_c}^\dagger |f^0 > < f^1; j, m_c| + h.c. \quad (5)$$
in the absence of the CEF, or

\[
H_{01} = V \sum_\epsilon \sum_{j_c \Gamma_c d_e} c_{\epsilon j_c \Gamma_c d_e}^\dagger f^0 < f^1; j_c \Gamma_c d_e | + h.c. \tag{6}
\]

in the presence of the CEF. Here the conduction electron Bloch states are symmetry adapted into appropriate partial wave states labeled by cubic irrep indices \( \Gamma_c \) at the impurity site. These projected quantum labels play an important role in determining the ground state properties of the model system.

However, the mixing between the \( f^1 \) and the \( f^2 \) configurations becomes complicated due to the fractional parentage and the Clebsch-Gordan coefficients. First we consider the case in the absence of the CEF.

\[
H_{12}^{(0)} = V \sum_\epsilon \sum_{j_c m_c} \sum_{j_1 m_1} \sum_{LSJM} \Lambda(j_c m_c; f^1 j_1 m_1 | f^2 LSJM) \\
\times c_{\epsilon j_c m_c}^\dagger | f^1 j_1 m_1 > < f^2 LSJM | + h.c. \tag{7}
\]

Here we use the reduced matrix elements are defined

\[
\Lambda(f^1 j_1 m_1; j_c m_c | f^2 LSJM) \equiv < j_1 m_1 | f_{j_c m_c} | f^2 LSJM > \\
= K(j_c; f^1 j_1 | f^2 LSJ) < j_c m_c; j_1 m_1 | J M > . \tag{8}
\]

In the last line we used the Wigner-Eckart theorem. The prefactor \( K(j_c; f^1 j_1 | f^2 LSJ) \) is the fractional parentage coefficient (see the Appendix A for more details). Though we projected the atomic Fock space using the symmetry quantum labels, these states are not energy eigenstates of the atomic Hamiltonian. The Coulomb (exchange) interaction will mix multiplets with the same \( J \). This will introduce another unitary transformation from the symmetry states \( |f^2 LSJM > \) to the energy eigenstates. These energy eigenstates still respect the total angular momentum and are labeled by \( JM \) with extra energy quantum label. We eschew this complication in favor of working in the simpler LS coupling limit, which is accurate for Ce ions in any case.

As will be discussed below, the cubic Clebsch-Gordan coefficients for different \( J \)'s are equal to within a prefactor for \( f^1 \Gamma_7 \) CEF states. Hence the mixing between the different
multiplets with the same value of $J$ will not change the matrix form (hybridization or mixing matrix).

In passing we note that the same approach was undertaken for the $f^2$ and $f^3$ configurations including only the Hund’s rule ground multiplets \[19\]. In principle, the discussion we give here for the $f^1, J = 5/2, \Gamma_7$ doublet should equally well apply to the $f^3, J = 9/2, \Gamma_6$ doublet considered in Ref. \[19\], which however contends that only a fermi liquid state is possible. We shall defer a discussion about this different conclusion between the two works to the last section of the paper.

We now turn to a systematic discussion of the hybridization matrix elements between $f^1$ and $f^2$ and the effective exchange interactions mediated by virtual $f^2$ fluctuations.

**A. The hybridization between $f^1 J = 5/2 \Gamma_7$ and $f^2 \Gamma_1, \Gamma_2$.**

As shown in Table \[V\], only the $\Gamma_7 (\Gamma_6)$ conduction electrons can mix with the $f^1 \Gamma_7$ to reach the $f^2 \Gamma_1 (\Gamma_2)$ states. The corresponding cubic Clebsch-Gordan coefficients can be written in a very simple form

$$< j_c \Gamma_7 \alpha; \frac{5}{2} \Gamma_7 \beta | J_2 \Gamma_1 > = (-1)^{1/2-\alpha} (\beta, \bar{\alpha}) < j_c \Gamma_7; \frac{5}{2} \Gamma_7 | J_2 \Gamma_1 >,$$

$$< j_c \Gamma_6 \alpha; \frac{5}{2} \Gamma_7 \beta | J_2 \Gamma_2 > = (-1)^{1/2-\alpha} (\beta, \bar{\alpha}) < j_c \Gamma_7; \frac{5}{2} \Gamma_7 | J_2 \Gamma_2 > .$$

The phase dependence derives from time reversal symmetry. The common factors, which are independent of the CEF degeneracy labels, $< j_c \Gamma_7; \frac{5}{2} \Gamma_7 | J_2 \Gamma_1 >$ and $< j_c \Gamma_7; \frac{5}{2} \Gamma_7 | J_2 \Gamma_2 >$ are listed in Table \[VI\] and \[VII\].

Then the Anderson hybridization can be written as

$$H_{12}(\Gamma_1) = V \sum_{\alpha} \sum_{j_c LSJ} (-1)^{\alpha-1/2} K(j_c; f^1 \frac{5}{2} \Gamma_7 | f^2 LS : J) < j_c \Gamma_7; \frac{5}{2} \Gamma_7 | J \Gamma_1 >$$

$$\times c^\dagger_{j_c \Gamma_7 \alpha} | f^1 \frac{5}{2} \Gamma_7 \bar{\alpha} > < f^2 LS : J \Gamma_1 | \text{h.c.},$$

$$H_{12}(\Gamma_2) = V \sum_{\alpha} \sum_{j_c LSJ} (-1)^{\alpha-1/2} K(j_c; f^1 \frac{5}{2} \Gamma_7 | f^2 LS : J) < j_c \Gamma_6; \frac{5}{2} \Gamma_7 | J \Gamma_2 >$$

$$\times c^\dagger_{j_c \Gamma_6 \alpha} | f^1 \frac{5}{2} \Gamma_7 \bar{\alpha} > < f^2 LS : J \Gamma_2 | \text{h.c.}.$$
In the mixing, the degeneracy index for a time reversal pair is active, i.e., can change. In the local moment limit, the Schrieffer-Wolff transformation [20] leads to the one-channel antiferromagnetic exchange interaction

\[ H_{1}(\Gamma_{1}) = \sum_{ij=5/2,7/2} J_{ij} \vec{S}_{c\gamma ij}(0) \cdot \vec{S}_{\Gamma_{7}}, \quad J_{ij} = \sqrt{J_{i}J_{j}}, \]  

and

\[ H_{1}(\Gamma_{2}) = J \vec{S}_{c\Gamma_{6}}(0) \cdot \vec{S}_{\Gamma_{7}}, \]  

\[ J = \sum_{LSJ} \frac{2|V|^{2}}{E_{2}(LSJ) - E_{1}} |K(j_{c}; f^{1}5/2 | f^{2}LS : J)|^{2} |< j_{c}\Gamma_{7}; 5/2\Gamma_{7}|J\Gamma_{2} > |^{2}. \]  

Here the conduction pseudo-spin is \( S_{c} = 1/2 \). We also introduced the energy levels for atomic states in the \( f^{1} \) and the \( f^{2} \) configurations. \( E_{1} \) is the energy level for the \( f^{1} J = 5/2 \Gamma_{7} \) state and \( E_{2}(LSJ) \) for the \( f^{2} \) atomic state with quantum labels of orbital \( (L) \), spin \( (S) \), and total angular momenta \( (J) \).

**B. The hybridization between \( f^{1}J = 5/2 \Gamma_{7} \) and \( f^{2}\Gamma_{3} \).**

Only the \( \Gamma_{8} \) conduction electrons can mix between the \( f^{1}J = 5/2\Gamma_{7} \) and the \( f^{2}\Gamma_{3} \)'s. The relevant cubic Clebsch-Gordan coefficients are

\[ < j_{c}\Gamma_{8}n\alpha'; 5/2\Gamma_{7}\alpha|J_{2}\Gamma_{3}n > = (-1)^{\alpha+1/2} (\alpha', \bar{\alpha}) (n', n) < j_{c}\Gamma_{8}; 5/2\Gamma_{7}|J_{2}\Gamma_{3} > . \]  

The overall phase factor derives from time reversal symmetry. The common factor \( < j_{c}\Gamma_{8}; 5/2\Gamma_{7}|J_{2}\Gamma_{3} > \) is listed in Table [VIII]. Hence the hybridization can be written as

\[ H_{12}(\Gamma_{3}) = V \sum_{j_{c}LSJ} \sum_{n=\pm} \sum_{\alpha=\uparrow, \downarrow} (-1)^{\alpha-1/2} K(j_{c}; f^{1}5/2 | f^{2}LS : J) < j_{c}\Gamma_{8}; 5/2\Gamma_{7}|J_{2}\Gamma_{3} > \]

\[ \times c_{j_{c}\Gamma_{8}n\alpha}^{\dagger} |f^{1}5/2\Gamma_{7}\bar{\alpha} > < f^{2}LS : J\Gamma_{3}n| + h.c. \]  

(18)
In the hybridization, the time reversal pair index $\alpha = \uparrow, \downarrow$ is active while the orbital index $n = \pm$ does not change and is not active. Thus the time reversal pair (\uparrow / \downarrow) gives rise to the pseudo-spin and the orbital doublet ($\pm$) provides two channels. The Schrieffer-Wolff transformation leads to the two-channel exchange interaction

$$H_1 = \sum_{ij=5/2, 7/2} \sum_{n=\pm} J_{ij} \bar{S}_{G_{\text{spin}}} \cdot \bar{S}_{\Gamma_7}, \quad J_{ij} = \sqrt{J_i J_j},$$

$$J_{\Gamma} = \sum_{\text{LSJ}} \frac{2|V|^2}{E_2(\text{LSJ}) - E_1} |K(j; f^5/2 | f^2LS : J)|^2 | < j; \Gamma_8; \frac{5}{2} \Gamma_7 | J \Gamma_3 > |^2.$$  \hspace{1cm} (19)

Here the conduction pseudo-spin is $S_c = 1/2$.

**C. The hybridization between $f^1J = 5/2 \Gamma_7$ and $f^2\Gamma_4$.**

The $\Gamma_7$ or $\Gamma_8$ conduction electrons can mix between the $f^1J = 5/2 \Gamma_7$ and the $f^2\Gamma_4$'s. The hybridization has the following form

$$H_{12}(\Gamma_4) = V \sum_{\epsilon} \sum_{j; \text{LSJ}} K(j; f^5/2 | f^2LS : J) < j; \Gamma_8; \frac{5}{2} \Gamma_7 | J \Gamma_4 >$$

$$\times \left\{ -\sum_{\alpha} \sqrt{2} c_{\epsilon; \Gamma_8}^\dagger | f^1\alpha > < f^2LSJ \Gamma_40 | \right\}$$

$$\times \left\{ + c_{\epsilon; \Gamma_8}^\dagger | f^1 \uparrow > -\sqrt{3} c_{\epsilon; \Gamma_8}^\dagger | f^1 \downarrow > < f^2LSJ \Gamma_41 | \right\}$$

$$+ V \sum_{\epsilon} \sum_{j; \text{LSJ}} K(j; f^5/2 | f^2LS : J) < j; \Gamma_7; \frac{5}{2} \Gamma_7 | J \Gamma_4 >$$

$$\times \left\{ \sum_{\alpha} c_{\epsilon; \Gamma_7}^\dagger | f^1\alpha > < f^2LSJ \Gamma_40 | \right\}$$

$$\times \left\{ + \sqrt{2} c_{\epsilon; \Gamma_7}^\dagger | f^1 \uparrow > < f^2LSJ \Gamma_41 | \right\}$$

$$+ \sqrt{2} c_{\epsilon; \Gamma_7}^\dagger | f^1 \downarrow > < f^2LSJ \Gamma_41 | \right\}.$$  \hspace{1cm} (21)

The hybridization looks more complicated compared to those for $\Gamma_1, \Gamma_2$ and $\Gamma_3$ CEF states. However, this complication comes from our special labeling of the CEF degeneracy as we shall show below. We list the numerical common factors $< j; \Gamma_8; \frac{5}{2} \Gamma_7 | J \Gamma_4 >$ and $< j; \Gamma_8; \frac{5}{2} \Gamma_7 | J \Gamma_4 >$ in Table IX. Note that the explicit dependence on the degeneracy labels is left out in extracting the prefactors.
The Schrieffer-Wolff transformation leads to a new exchange interaction model with the
$S_c = 3/2$ conduction electrons interacting with the impurity pseudo spin $S_I = 1/2$ in the $\Gamma_8$
conduction electron space

$$H_1 = \sum_{ij} J_{ij} \mathbf{S}_{c\Gamma_8(0)} \cdot \mathbf{S}_{\Gamma_7}, \quad J_{ij} = \sqrt{J_i J_j}.$$  \hspace{1cm} (22)

$$J_{i=jc} = \sum_{LSJ} \frac{2|V|^2}{E_2(LSJ) - E_1} |K(jc; f^2LS: J)|^2 < j_c \Gamma_8; \frac{5}{2} \Gamma_7|J \Gamma_4 > |^2.$$  \hspace{1cm} (23)

Here $S_c = 3/2$. The exchange coupling can be rewritten in the matrix form. The corresponding representations of the angular momentum operators are ($|+\uparrow>, |+\downarrow>, |-\uparrow>, |-\downarrow>$)

$$S_c^z = \frac{1}{2} \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \left( \begin{array}{cc} -3 & 0 \\ 0 & 1 \end{array} \right) \otimes \frac{1}{2} \sigma^z,$$

$$S_c^+ = \begin{pmatrix} 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & -2 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} \end{pmatrix}, \quad S_c^- = \begin{pmatrix} 0 & \sqrt{3} \\ \sqrt{3} & -2 \end{pmatrix} \otimes \frac{1}{2} \sigma^x,$$

$$S_c^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix}, \quad S_c^y = \begin{pmatrix} 0 & -\sqrt{3} \\ -\sqrt{3} & -2 \end{pmatrix} \otimes \frac{1}{2} \sigma^y.$$  \hspace{1cm} (24)

We can show that these matrices transform into the canonical $SU(2)$ form after the rearrangement of the states:

$$(|+\uparrow>, |+\downarrow>, |-\uparrow>, |-\downarrow>) \rightarrow (|\frac{3}{2}\frac{3}{2}>, |\frac{3}{2}\frac{3}{2}>, |\frac{3}{2}\frac{1}{2}>, |\frac{3}{2}\frac{1}{2}>).$$

This extra complication comes from our biased labeling of the CEF degeneracy toward the
two-channel exchange interaction. From a viewpoint of the two-channel exchange interaction, the induced exchange interaction can be written as
\[ H_1 = J \otimes (-\tau^0 - 2\tau^i) \otimes S^i_{\Gamma_8} S^i_{\Gamma_7}, \]  
(25)

\[ \tau^x = -\frac{1}{2} \tau^3 - \frac{\sqrt{3}}{2} \tau^1, \quad \tau^y = -\frac{1}{2} \tau^3 + \frac{\sqrt{3}}{2} \tau^1, \quad \tau^z = \tau^3. \]  
(26)

Thus we can see that the exchange interaction generated by the \( f^2 \Gamma_4 \) CEF states are harmful to the two-channel Kondo effect.

On the other hand, the \( \Gamma_7 \) conduction electrons are coupled to the impurity spin with the ferromagnetic exchange coupling

\[ H_1 = -\sum_{ij} J_{ij} \vec{S}_{\Gamma_{ij}}(0) \cdot \vec{S}_{\Gamma_7}, \quad J_{ij} = \sqrt{J_i J_j}, \]  
(27)

\[ J_{i=j_c} = \sum_{l,s,l'} \frac{2|V|^2}{E_2(LSJ) - E_1} |K(j_c; f^{5/2} | f^2LS : J)|^2 |< j_c \Gamma_7; f^5/2 \Gamma_4 >|^2. \]  
(28)

Here \( S_c = 1/2 \). In this case, the exchange interaction, mediated by the \( f^2 \Gamma_4 \) states, degrade the one-channel exchange coupling which is ever present between the \( f^0 \) and the \( f^1 \) configurations.

Another complication arises through the \( f^2 \Gamma_4 \) CEF states. Two conduction electrons of \( \Gamma_7 \) and \( \Gamma_8 \) can mix with each other. Since it can be easily generalized to the multiple conduction partial waves, we present the model for the one partial conduction wave case. In the \( \Gamma_7 \oplus \Gamma_8 \) conduction electron space, all the possible exchange interactions are

\[ H_1 = J^i \otimes S^i_c S^i_{\Gamma_7}, \]  
(29)

\[ J^i = J_7 \kappa^0 + J_8 \tau^0 + J_{78} \kappa^i + J_{88} \tau^i, \]

\[ \kappa^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \kappa^x = -\frac{1}{2} \begin{pmatrix} 0 & \sqrt{3} & 1 \\ \sqrt{3} & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \]

\[ \kappa^y = -\frac{1}{2} \begin{pmatrix} 0 & -\sqrt{3} & 1 \\ -\sqrt{3} & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \kappa^z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]  
(30)

\[ \tau^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau^x = -\frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \sqrt{3} \\ \sqrt{3} & 0 & -1 \end{pmatrix}, \]

\[ \tau^y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau^z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \sqrt{3} \\ \sqrt{3} & 0 & -1 \end{pmatrix}. \]
\[
\tau^y = -\frac{1}{2} \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & -\sqrt{3} \\
0 & -\sqrt{3} & -1
\end{pmatrix}, \quad \tau^z = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}.
\]

(31)

D. The hybridization between \(f^1J = 5/2 \Gamma_7\) and \(f^2\Gamma_5\).

The \(\Gamma_6\) or \(\Gamma_8\) conduction electrons can mix between the \(f^1J = 5/2\Gamma_7\) and the \(f^2\Gamma_5\)'s.

The hybridization has the following form

\[
H_{12}(\Gamma_5) = V \sum_{\epsilon} \sum_{j_{cLS}} K(j_{c}; f^1\frac{5}{2} | f^2LS : J) < j_{c}\Gamma_8; \frac{5}{2}\Gamma_7 | \Gamma_5 J > \\
\times \left\{ \frac{\sqrt{2}}{\sqrt{3}} c_{j_{c}\Gamma_8-\uparrow}^\dagger f^1 \uparrow | f^1 \uparrow < \frac{f^2LSJ\Gamma_50}{f^2LSJ\Gamma_51} \right\} \\
+ V \sum_{\epsilon} \sum_{j_{cLS}} K(j_{c}; f^1\frac{5}{2} | f^2LS : J) < j_{c}\Gamma_6; \frac{5}{2}\Gamma_7 | \Gamma_5 J > \\
\times \left\{ -\sum_{\alpha} c_{j_{c}\Gamma_6\alpha}^\dagger f^1 \uparrow | f^1 \uparrow < \frac{f^2LSJ\Gamma_50}{f^2LSJ\Gamma_51} \right\}.
\]

(32)

The Schrieffer-Wolff transformation leads to a new exchange interaction with the \(S_c = 3/2\) conduction electrons interacting with the impurity pseudo spin \(S_I = 1/2\) in the \(\Gamma_8\) conduction electron space

\[
H_1 = \sum_{ij} J_{ij} \vec{S}_{c\Gamma_8ij}(0) \cdot \vec{S}_{\Gamma_7}, \quad J_{ij} = \sqrt{J_iJ_j}, \quad (33)
\]

\[
J_{i=j_{c}} = \sum_{LSJ} \frac{2|V|^2}{E_2(LLS) - E_1} |K(j_{c}; f^1\frac{5}{2} | f^2LS : J)|^2 | < j_{c}\Gamma_8; \frac{5}{2}\Gamma_7 | \Gamma_4 J > |^2.
\]

(34)

Here \(S_c = 3/2\). This conduction electron pseudo spin is different from that defined in the previous section. The corresponding representations of the angular momentum operators are \(|+ \uparrow>, |+ \downarrow>, |− \uparrow>, |− \downarrow>\).
\[
S_c^z = \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -3 & 0 \\
0 & 0 & 0 & 3
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & -3
\end{pmatrix} \otimes \frac{1}{2} \sigma^z,
\]
\[
S_c^+ = \begin{pmatrix}
0 & 0 & \sqrt{3} & 0 \\
0 & 0 & 0 & 0 \\
-\sqrt{3} & 0 & 0 & 0 \\
0 & 0 & 0 & -\sqrt{3}
\end{pmatrix}, \\
S_c^- = \begin{pmatrix}
-2 & 0 & 0 & 0 \\
0 & -\sqrt{3} & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
S_c^x = \begin{pmatrix}
-2 & \sqrt{3} \\
-\sqrt{3} & 0
\end{pmatrix} \otimes \frac{1}{2} \sigma^x,
\]
\[
S_c^y = \begin{pmatrix}
-2 & -\sqrt{3} \\
\sqrt{3} & 0
\end{pmatrix} \otimes \frac{1}{2} \sigma^y.
\]

We can show that these matrices transform into the canonical form after the rearrangement of the states:

\[
(|+\uparrow>, |+\downarrow>, |-\uparrow>, |-\downarrow>) \to (-|\frac{1}{2} \uparrow>, |\frac{3}{2} \uparrow>, -|\frac{1}{2} \downarrow>, |\frac{3}{2} \downarrow>).
\]

From a viewpoint of the two-channel exchange, the induced exchange interaction can be written as

\[
H_1 = J \otimes (-\tau^0 + 2\tau^i) \otimes S_{\Gamma_8}^i S_{\Gamma_7}^i,
\]
\[
\tau^x = -\frac{1}{2} \tau^3 - \frac{\sqrt{3}}{2} \tau^1, \quad \tau^y = -\frac{1}{2} \tau^3 + \frac{\sqrt{3}}{2} \tau^1, \quad \tau^z = \tau^3.
\]

Thus in the \(\Gamma_8\) conduction electron space, there are three competitors: two-channel Kondo effect and the other two \(S_c = 3/2\) Kondo effect (overscreened).

On the other hand, the \(\Gamma_6\) conduction electrons are coupled to the impurity pseudo spin with the ferromagnetic exchange coupling

\[
H_1 = -\sum_{ij} J_{ij} \vec{S}_i \cdot \vec{S}_f, \quad J_{ij} = \sqrt{J_i J_j}, 
\]

\[
J_{i=j_c} = \sum_{LSJ} \frac{2|V|^2}{E_2(LSJ) - E_1} |K(j_c; f)^{1/2} | f^2 LS : J)|^2 < j_c \Gamma_6; \frac{5}{2} \Gamma_7 | J \Gamma_4 > |^2.
\]

\[
(35)
\]

\[
(36)
\]

\[
(37)
\]

\[
(38)
\]

\[
(39)
\]
Here $S_c = 1/2$. In this case, the exchange interaction, mediated by the $f^2\Gamma_4$ states, degrade the one-channel exchange coupling.

Another complication arises through the $f^2\Gamma_5$ CEF states. Two conduction electrons of $\Gamma_6$ and $\Gamma_8$ can mix with each other. In the $\Gamma_6 \otimes \Gamma_8$ conduction electron space, all the possible exchange interactions are

\[ H_1 = J^i \otimes S^i_c \otimes S^{i\tau}_\gamma, \]  
\[ J^i = J_6 \gamma^0 + J_8 \tau^0 + J_{68} \gamma^i + J_{88} \tau^i, \]  
\[ \gamma^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \gamma^x = \frac{1}{2} \begin{pmatrix} 0 & 1 & -\sqrt{3} \\ 1 & 0 & 0 \\ -\sqrt{3} & 0 & 0 \end{pmatrix}, \]  
\[ \gamma^y = -\frac{1}{2} \begin{pmatrix} 0 & 1 & \sqrt{3} \\ 1 & 0 & 0 \\ \sqrt{3} & 0 & 0 \end{pmatrix}, \quad \gamma^z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \]  
\[ \tau^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau^x = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \sqrt{3} \\ 0 & \sqrt{3} & -1 \end{pmatrix}, \]  
\[ \tau^y = -\frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -\sqrt{3} \\ 0 & -\sqrt{3} & -1 \end{pmatrix}, \quad \tau^z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \]  

E. The $\Gamma_6 \oplus \Gamma_7 \oplus \Gamma_8$ conduction electron space.

In cubic symmetry, when we keep the lowest lying $f^1J = 5/2 \Gamma_7$ CEF states, the most general exchange interaction form to each conduction electron partial wave species is

\[ H_1 = J^i \otimes S^i_c \otimes S^{i\tau}_\gamma, \]  
\[ J^i = J_6 \gamma^0 + J_7 \kappa^0 + J_8 \tau^0 + J_{68} \gamma^i + J_{78} \kappa^i + J_{88} \tau^i, \]
The matrix algebra for $\gamma, \kappa, \tau$ is listed below. We note that in the subspaces, $\Gamma_7 \oplus \Gamma_8$ or $\Gamma_6 \oplus \Gamma_8$, the exchange interactions are self-contained. No new interactions are generated with renormalization. Also in the whole $\Gamma_6 \oplus \Gamma_7 \oplus \Gamma_8$ conduction electron space, the exchange interactions are self-contained.

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^x = -\frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & -\sqrt{3} \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -\sqrt{3} & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^y = -\frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

(46)

$$\kappa^0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \kappa^x = -\frac{1}{2} \begin{pmatrix} 0 & 0 & \sqrt{3} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\kappa^y = -\frac{1}{2} \begin{pmatrix} 0 & 0 & -\sqrt{3} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \kappa^z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

(47)

$$\tau^0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tau^x = -\frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & -1 \end{pmatrix},$$

$$\tau^y = -\frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\sqrt{3} \\ 0 & 0 & -\sqrt{3} & -1 \end{pmatrix}, \quad \tau^z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

(48)

The matrix algebra for $\gamma, \kappa, \tau$ is listed below. We note that in the subspaces, $\Gamma_7 \oplus \Gamma_8$ or $\Gamma_6 \oplus \Gamma_8$, the exchange interactions are self-contained. No new interactions are generated with renormalization. Also in the whole $\Gamma_6 \oplus \Gamma_7 \oplus \Gamma_8$ conduction electron space, the exchange interactions are self-contained.

$$\gamma^i \gamma^j = \gamma^0 + \frac{1}{2} \tau^0 + \frac{1}{2} \tau^i,$$
\[ \gamma^i + \gamma^j = -\gamma^k \text{ (cyclic),} \] (50)
\[ \gamma^0 \gamma^i + \gamma^i \gamma^0 = \gamma^i, \] (51)
\[ \gamma^i \gamma^j + \gamma^j \gamma^i = -\gamma^0 - \frac{1}{2} \tau^0 + \tau^k \text{ (cyclic).} \] (52)
\[ \gamma^0 \tau^i + \tau^i \gamma^0 = 0, \text{ for } i = 0, x, y, z, \] (53)
\[ \tau^0 \gamma^i + \gamma^i \tau^0 = \gamma^i, \] (54)
\[ \tau^i \gamma^j + \gamma^j \tau^i = \gamma^k \text{ (cyclic),} \] (55)
\[ \gamma^i \tau^j + \tau^j \gamma^i = \gamma^k \text{ (cyclic),} \] (56)
\[ \gamma^i \tau^i + \tau^i \gamma^i = \gamma^i. \] (57)
\[ \tau^i \tau^i = \tau^0, \] (58)
\[ \tau^0 \tau^i + \tau^i \tau^0 = 2\tau^i, \] (59)
\[ \tau^i + \tau^j = -\tau^k \text{ (cyclic),} \] (60)
\[ \tau^x \tau^y = \tau^y \tau^x = \tau^z \tau^x = -\frac{1}{2} \tau^0 - \frac{\sqrt{3}}{2} \tau^2, \] (61)
\[ \tau^y \tau^x = \tau^z \tau^y = \tau^x \tau^z = -\frac{1}{2} \tau^0 + \frac{\sqrt{3}}{2} \tau^2. \] (62)

\[ \kappa^i \kappa^i = \kappa^0 + \frac{1}{2} \tau^0 - \frac{1}{2} \tau^i, \] (63)
\[ \kappa^i + \kappa^j = -\kappa^k \text{ (cyclic),} \] (64)
\[ \kappa^0 \kappa^i + \kappa^i \kappa^0 = \kappa^i, \] (65)
\[ \kappa^i \kappa^j + \kappa^j \kappa^i = -\kappa^0 - \frac{1}{2} \tau^0 - \tau^k \text{ (cyclic).} \] (66)
\[ \kappa^0 \tau^i + \tau^i \kappa^0 = 0, \text{ for } i = 0, x, y, z, \] (67)
\[ \tau^0 \kappa^i + \kappa^i \tau^0 = \kappa^i, \] (68)
\[ \tau^i \kappa^j + \kappa^j \tau^i = -\kappa^k \text{ (cyclic),} \] (69)
\[ \kappa^i \tau^j + \tau^j \kappa^i = -\kappa^k \text{ (cyclic),} \] (70)
\[ \kappa^i \tau^i + \tau^i \kappa^i = -\kappa^i. \] (71)
\[ \kappa^0 \gamma^i + \gamma^i \kappa^0 = 0, \] (72)
\[ \gamma^0 \kappa^i + \kappa^i \gamma^0 = 0, \] (73)
\[ \kappa^i \gamma^j + \gamma^j \kappa^i = \frac{\sqrt{3}}{2} \chi, \] (74)
\[ \gamma^i \kappa^j + \kappa^j \gamma^i = - \frac{\sqrt{3}}{2} \chi, \] (75)
\[ \kappa^i \gamma^j + \gamma^j \kappa^i = 0, \] (76)
\[ \kappa^0 \chi + \chi \kappa^0 = \chi, \] (77)
\[ \gamma^0 \chi + \chi \gamma^0 = \chi. \] (78)

Here \( \chi \) is given by
\[
\chi = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}. \tag{79}
\]

Though this new term \( \chi \) arises in the intermediate calculation step, this term cancels out in the final result.

In deriving the scaling equations, the following relations are useful
\[
J^i J^j + J^j J^i = (2J_6^2 - J_{68}^2) \gamma^0 + (2J_7^2 - J_{78}^2) \kappa^0 + \left[ 2J_8^2 - J_{88}^2 - \frac{1}{2} (J_{68}^2 + J_{78}^2) \right] \tau^0 \\
+ J_{68} \left( 2J_{88} - J_6 - J_7 \right) \gamma^k + J_{78} \left( -2J_{88} - J_7 - J_8 \right) \kappa^k \\
+ \left( J_{68}^2 - J_{78}^2 - 2J_8 J_{88} \right) \tau^k. \tag{80}
\]

Here \((i, j, k)\) is cyclic, and
\[
J^i J^i = \left( J_6^2 + J_{68}^2 \right) \gamma^0 + \left( J_7^2 + J_{78}^2 \right) \kappa^0 + \left[ J_8^2 + J_{88}^2 + \frac{1}{2} (J_{68}^2 + J_{78}^2) \right] \tau^0 \\
+ J_{68} \left( J_{88} + J_6 + J_8 \right) \gamma^i + J_{78} \left( -J_{88} + J_7 + J_8 \right) \kappa^i \\
+ \left[ 2J_8 J_{88} + \frac{1}{2} (J_{68}^2 - J_{78}^2) \right] \tau^i. \tag{81}
\]
F. Summary

In summary, we discovered various exchange interactions. In particular, a new type of exchange interaction is discovered: the $S_c = 3/2$ conduction electrons interact with the impurity pseudo spin $S_I = 1/2$. All the possible exchange interactions in our model study are summarized in Fig. [I].

The $\Gamma_6$ conduction electrons (pseudo spin $S_c = 1/2$) are coupled to the impurity pseudo spin $S_I = 1/2$ with the antiferromagnetic exchange coupling through $f^2\Gamma_2$ states, and with the ferromagnetic exchange coupling through $f^2\Gamma_5$ states. The mixing between $\Gamma_6$ and $\Gamma_8$ conduction electrons is also present. That is, the $\Gamma_6$ conduction electrons can change their symmetry states ($\Gamma_8$) after scattered off the impurity pseudo spin.

The $\Gamma_7$ conduction electrons (pseudo spin $S_c = 1/2$) are coupled to the impurity pseudo spin $S_I = 1/2$ with the antiferromagnetic exchange coupling through $f^2\Gamma_1$ states, and with the ferromagnetic exchange coupling through $f^2\Gamma_4$ states. The mixing between $\Gamma_7$ and $\Gamma_8$ conduction electrons is also present. That is, the $\Gamma_7$ conduction electrons can change their symmetry states ($\Gamma_8$) after scattered off the impurity spin.

The $\Gamma_8$ conduction electrons can generate two different conduction electron pseudo-spins: $S_c = 1/2$ with two degenerate orbital channels and and $S_c = 3/2$ with one channel. The $\Gamma_8$ conduction electrons are coupled to the impurity pseudo spin in a two-channel exchange interaction through $f^2\Gamma_3$, and in a one-channel exchange interaction through $f^2\Gamma_{4,5}$ states. Both models lead to the overcompensation of the impurity pseudo spin $S_I = 1/2$. The two-channel case has been studied extensively using several techniques and leads to a non-Fermi liquid fixed point. Using the numerical renormalization group and the conformal field theory, the one-channel $S_c = 3/2$ case is also shown to lead to the non-Fermi liquid fixed point [11].

We can give some plausible argument why some of exchange couplings are ferromagnetic while the others are antiferromagnetic. This argument follows from the addition of two angular momenta. As noted before, the CEF manifolds in the $f^2$ configuration can be
roughly evisaged as angular momentum multiplets.

To hop into the orbital singlet $\Gamma_1$ ($\Gamma_2$) in the $f^2$ configuration, the $\Gamma_7 (\Gamma_6)$ conduction electrons have to be in the opposite pseudo-spin state compared to the $f^1 \Gamma_7$ (impurity) pseudo spin state. Thus the coupling should be antiferromagnetic.

Now we consider the orbital doublet $\Gamma_3$ in the $f^2$ configuration. This orbital doublet is also represented as $J = 0$ singlet in terms of time reversal. In this case, the relevant $\Gamma_8$ conduction electrons are represented by pseudo-spin $S_c = 1/2$ and should be in the opposite pseudo spin state compared to the $f^1 \Gamma_7$ (impurity) pseudo spin state. Thus the coupling should be antiferromagnetic. From this argument, we can clearly see why the $\Gamma_8$ conduction electron cannot be represented by pseudo-spin $S_c = 3/2$ for the virtual charge fluctuation to the $\Gamma_3$ CEF states.

When we consider the magnetic triplets $\Gamma_4$ and $\Gamma_5$ in the $f^2$ configuration, the exchange coupling can be ferromagnetic or antiferromagnetic depending on which CEF symmetry electrons hop into the atomic orbitals. These magnetic triplets can be represented by the $J = 1$ manifolds. For the $\Gamma_6$ or the $\Gamma_7$ conduction electrons which are represented by pseudo-spin $S_c = 1/2$, the coupling should be ferromagnetic to form the $J = 1$ triplet state from the addition of $S_c = 1/2$ and $S_I = 1/2$. As we noted before, the $\Gamma_8$ conduction electrons can be represented in both ways: $S_c = 3/2$ with one orbital channel or $S_c = 1/2$ with two orbitally degenerate channels. Hence depending on the spin representation, the coupling has different sign. With $S_c = 1/2$ representation, the coupling should be ferromagnetic as explained above. This picture in fact agrees with our results that the virtual charge fluctuations to the triplets $\Gamma_4$ and $\Gamma_5$ in the $f^2$ are harmful to the two-channel Kondo effect for the $\Gamma_8$ conduction electrons. With $S_c = 3/2$ representation, the coupling should be antiferromagnetic which can be deduced from the angular momentum addition of $S_c = 3/2$ and $S_I = 1/2$ which has to be $J = 1$. 

IV. SCALING ANALYSIS I: VARIOUS FIXED POINTS

In this section we analyze our various model exchange interaction terms using the third order scaling argument - perturbative renormalization group (RG) \cite{22,23}. At temperature $T$, only the conduction electrons (thermally excited) inside the band of order $T$ with respect to the Fermi level play an important role in determining physical properties. Thus we can integrate out the band edge states (virtually excited states) to find the effective Hamiltonian. Though the following analysis is restricted to the perturbative regime (weak coupling limit), we can derive qualitatively correct results out of this. For quantitative results, a full numerical renormalization group (NRG) study is required.

Our strategy is as follows: we start with each exchange interaction term and identify the relevant fixed points. After that, we include other interaction terms step by step to see if the relevant fixed points remain stable. Finally, in the next section, we consider the effect of the multiple conduction electron partial wave states on the various Kondo effects.

In summary, we find that three types of fixed points are stable in the presence of the full exchange interactions:

(i) one-channel Fermi liquid fixed points ($\Gamma_6$ and $\Gamma_7$ conduction electrons);
(ii) two-channel non-Fermi liquid fixed point ($\Gamma_8$ conduction electrons);
(iii) three-channel non-Fermi liquid fixed points (some linear combination of $\Gamma_8$ and $\Gamma_6$ or $\Gamma_7$ conduction electrons). In addition to these stable fixed points, we find a “zoo” of unstable fixed points at which various exchange interactions we can imagine are realized. When we consider the multiple conduction electron partial waves, the relevant stable fixed points remain stable with enhanced bare exchange couplings and new unstable multichannel fixed points are generated.
A. Structure of the fixed points

Before discussing the scaling equations for various exchange interactions, we wish to establish some mathematical structure of the fixed points generated by the third order scaling equations. We consider the following generic exchange interactions

\[ H_1 = J^i \otimes S^i_c(0) S^i_I. \]  

(82)

Here \( S_c = S_I = 1/2 \) are spin operators and \( J^i \) is the exchange coupling matrices whose dimension is \( 4 \times 4 \) in our model case. We first consider some arbitrary \( SU(2) \) angular momentum operators

\[ L^i \equiv M^i \otimes \frac{1}{2} \sigma^i. \]  

(83)

Here \( \sigma^i \) are Pauli matrices and \( M^i \) are undetermined at this point. Now we demand that \( L^i \) satisfy the \( SU(2) \) spin algebra, that is,

\[ [ L^i, L^j ] = i \epsilon_{ijk} L^k. \]  

(84)

Here \( \epsilon_{ijk} \) is Levi-Civita antisymmetry tensor. Then we find the condition for \( M^i \)'s to satisfy

\[ \{ M^i, M^j \} = 2 M^k. \]  

(85)

Here \( (i, j, k) \) is cyclic. Up to this point, all is a pure mathematical fun. Now we are going to infuse some physics into the above algebra.

In the scaling analysis (the perturbative renormalization group), we remove the high energy states progressively (renormalize) to find the effective Hamiltonian which describes the same low temperature physics of the original Hamiltonian. When the effective Hamiltonian does not change further with renormalization (removal of the high energy states), it is called the fixed point Hamiltonian. For the above general exchange interactions, the scaling equations up to the third order diagrams of Fig. 2 are (see the Appendices B & C)

\[ \frac{\partial g^k}{\partial x} = \frac{1}{2} \left[ g^i g^j + g^j g^i \right] - \frac{1}{4} g^k \text{Tr} \left[ g^i g^j + g^j g^i \right]. \]  

(86)
Here \((i, j, k)\) is cyclic. All our exchange interactions we are going to study satisfy the cyclic property such that \(\text{Tr}[\ ]\) is independent of the \(x, y, z\) indices. Then we can see that the equations for the fixed points have the same structure as in the \(SU(2)\) spin algebra. That is, the exchange coupling matrices have the \(SU(2)\) spin structure at the fixed points

\[
g^i = g^* M^i, \tag{87}
\]
\[
g^* = \frac{6}{\text{Tr} M^2}, \tag{88}
\]
\[
\Lambda = \frac{1}{4} M^2. \tag{89}
\]

The fixed point can be further specified by \(\Lambda\) which measures the conduction electron spin size. The number of channels is determined by the ratio of the degeneracy of the relevant manifold and the spin size. Below we are going to illustrate this point explicitly.

In passing we note that the same approach was independently applied to the metallic two-level system [9].

**B. The one-channel \(S_c = 3/2, S_I = 1/2\) exchange interaction**

As shown in chapter II, the \(\Gamma_8\) conduction electrons are coupled to the impurity pseudo spin \(S_I = 1/2\) with their pseudospin \(S_c = 3/2\) through the virtual fluctuations to the \(f^2\Gamma_{4,5}\) triplet states.

For comparison, it is well understood that the one-channel \(S_c = S_I = 1/2\) exchange interactions (\(\Gamma_6\) and \(\Gamma_7\) conduction electrons) lead to the Fermi liquid strong coupling fixed point, while the two-channel \(S_c = S_I = 1/2\) exchange interactions (\(\Gamma_8\) conduction electrons) give rise to the non-Fermi liquid non-trivial fixed point.

The third order scaling equation is

\[
\frac{\partial g}{\partial x} = g^2 - \frac{1}{3} s_c(s_c + 1)(2s_c + 1) g^3, \quad g = N(0) J. \tag{90}
\]

Here \(s_c = 3/2\). We can find the stable fixed point \(g^* = 1/5\).

We now argue that this new exchange interaction can lead to a non-Fermi liquid fixed point. In Fig. 3, we schematically show how the screening of the impurity spin can happen...
by the conduction electron spin cloud. With the antiferromagnetic coupling, the energy is minimized when two conduction electrons of $S^z_c = -3/2, -1/2$ are coupled to the impurity spin $S^z_I = 1/2$. This picture does not violate the Pauli exclusion principle. Thus the effective impurity spin is $S^z_I = 3/2$. In the next shell screening, only the $S^z_c = 3/2, 1/2$ can come closer to the effective impurity spin due to the Pauli exclusion principle. Hence the effective exchange interaction is antiferromagnetic. From this argument, we can see that the strong coupling fixed point cannot remain stable. Since the weak coupling fixed point is unstable, as we shall show, an intermediate fixed point must exist [3]. This is a non-Fermi liquid fixed point. We are separately studying this simple new exchange interaction using the numerical renormalization group(NRG) and conformal field theory [11].

We are next going to show that this new exchange interaction competes with the two-channel Kondo effect.

C. The $\Gamma_8$ conduction electron space

In the $\Gamma_8$ conduction electron space, two different spins can be realized as shown in section III. In summary, three different exchange interactions are possible via the $f^2\Gamma_{3,4,5}$:

\begin{align*}
H_1(\Gamma_3) &= J_3 \sum_{n=\pm} \vec{S}_{cn}(0) \cdot \vec{S}_{\Gamma_7} = J_3 \tau^0 \otimes \vec{S}_c(0) \cdot \vec{S}_{\Gamma_7}, \\
H_1(\Gamma_4) &= J_4 \vec{L}_c(0) \cdot \vec{S}_{\Gamma_7} = J_4 (-\tau^0 - 2\tau^i) \otimes S^i_c(0) S^i_{\Gamma_7}, \\
H_1(\Gamma_5) &= J_5 \vec{L}'_c(0) \cdot \vec{S}_{\Gamma_7} = J_5 (-\tau^0 + 2\tau^i) \otimes S^i_c(0) S^i_{\Gamma_7}.
\end{align*}

Here $\vec{S}_{\Gamma_7}$ is the impurity pseudo spin for the $f^1J = 5/2 \Gamma_7$ state. $\vec{L}_c(0)$ and $\vec{L}'_c(0)$ are two different conduction electron pseudo spin representations for the same $S_c = 3/2$ state manifold. Here we have rewritten the $S_c = 3/2$ operators from a view point of the two-channel exchange interaction.

All combined in the $\Gamma_8$ conduction electron space, we find

\begin{align*}
H_1 &= J^i \otimes S^i_c S^i, \\
J^i &= J_8 \tau^0 + J_{88} \tau^i.
\end{align*}
\[ J_8 = J_3 - J_4 - J_5, \quad J_{88} = J_5 - J_4. \]  (96)

Physically, the \( \tau^0 \) term leads to the ordinary two-channel exchange interaction while the \( \tau^i \) terms are channel-symmetry breaking and channel-mixing. Then we can derive the scaling equation for the matrix \( g^i = N(0)J^i \) in a compact way. In the \( \Gamma_8 \) conduction electron space, the third order scaling equations are

\[
\begin{align*}
\frac{\partial g_8}{\partial x} &= g_8^2 - \frac{1}{2} g_{88}^2 - g_8 \left[ g_8^2 + g_{88}^2 \right], \quad g_8 = N(0)J_8 > 0, \quad (97) \\
\frac{\partial g_{88}}{\partial x} &= -g_8 g_{88} - g_{88} \left[ g_8^2 + g_{88}^2 \right], \quad g_{88} = N(0)J_{88}. \quad (98)
\end{align*}
\]

We can identify the two-channel fixed point \((g^*_8, g^*_{88}) = (1, 0)\) and two other interesting fixed points \((g^*_8, g^*_{88}) = (-1/5, \pm 2/5)\)

\[
\begin{align*}
g^* &= \frac{1}{5} \bar{M}, \\
M^i &= -\tau^0 \pm 2 \tau^i, \\
\Lambda &= \frac{3}{2} \cdot \frac{5}{2} \tau^0.
\end{align*}
\]  (99)  (100)  (101)

corresponding to the exchange interaction models generated by \( f^2\Gamma_{4,5} \) in this restricted space, respectively. The linear analysis shows that all these fixed points are stable. Solving the above scaling equations, we can construct the flow diagram (see Fig. 4). There are three linear separatrices along which all the exchange interactions derived purely from the \( f^2\Gamma_{3,4}, \Gamma_4, \) or \( \Gamma_5 \) CEF states lie. Along the \( g_8 \) axis, the two-channel \( S_c = S_I = 1/2 \) exchange interaction due to virtual \( f^2\Gamma_3 \) charge fluctuations is located. Along the line \( g_{88} = 2g_8 \), the one-channel \( S_c = 3/2, S_I = 1/2 \) exchange interaction due to virtual \( f^2\Gamma_4 \) charge fluctuations is located. Along the line \( g_{88} = -2g_8 \), the one-channel \( S_c = 3/2, S_I = 1/2 \) exchange interaction due to virtual \( f^2\Gamma_3 \) charge fluctuations is located. From this flow diagram, we can get the following qualitative results: (1) When \( 2g_8 > |g_{88}| \), two-channel fixed point ground state is realized; (2) When \( g_{88} > 0 \) and \( 2g_8 < |g_{88}| \), the new fixed point ground state (generated by \( f^2\Gamma_5 \)) is realized. (3) When \( g_{88} < 0 \) and \( 2g_8 < |g_{88}| \), the new fixed point ground state (generated by \( f^2\Gamma_4 \)) is realized.
From a viewpoint of the two-channel exchange interaction, the above model Hamiltonian has channel asymmetry and exchange coupling anisotropy without the $\tau^1$ terms. From a well-known scaling argument \[3,25,26\], the model will flow to the one-channel fixed point (the channel with larger $z$ component of exchange coupling) in the absence of the channel-mixing term. Though the $\tau^3$ term breaks channel symmetry, the additional term from $\tau^1$ mixes the two different channels. That is, the channel asymmetry due to $\tau^3$ term is apparently washed out by the channel mixing term restoring to the two-channel fixed point in the two-channel Kondo effect parameter regime.

We note that the $S_c = 3/2$ fixed points have a net coupling strength (along the separatrices) of $g^* = 1/5$, safely in the perturbative regime so that the scaling analysis is trustworthy.

**D. One-channel & two-channel exchange interaction**

When we keep only the three atomic states $f^0, f^1J = 5/2 \Gamma_7$, and $f^2\Gamma_3$, we find the one-channel ($\Gamma_7$) & two-channel ($\Gamma_8\pm$) Anderson hybridization interactions with the impurity pseudo spin $S_I = 1/2$. This simple model shows competition between the Fermi liquid fixed point of the one-channel Kondo effect and the non-Fermi liquid fixed point of the two-channel Kondo effect \[24\]. For completeness, we will include the scaling analysis of this model here. We studied this simplified model using the non-crossing approximation \[24\].

It can be deduced from the scaling theory that the low temperature and the low energy physics is dominated by the one-channel or the two-channel Kondo effects depending on their relative magnitude of the antiferromagnetic couplings. With the introduction of the exchange coupling matrix, it is more convenient for the derivation of the scaling equations to rewrite the one-channel and two-channel exchange interactions in the following form

$$\tilde{H}_1 = \mathbf{J} \otimes \tilde{S}_c(0) \cdot \tilde{S}_I,$$

$$\mathbf{J} = \begin{pmatrix} J_7 & 0 & 0 \\ 0 & J_8 & 0 \\ 0 & 0 & J_8 \end{pmatrix}.$$
Here $\vec{S}_c$ and $\vec{S}_I$ are $S = 1/2$ operators. The scaling equations of our simple model Hamiltonian up to the third order diagrams are

$$\frac{\partial g}{\partial x} = g^2 - \frac{1}{2} g \text{ Tr}[g^2] \cdots, \quad (104)$$

$$g = N(0)J. \quad (105)$$

The scaling equations in components are

$$\frac{\partial g_7}{\partial x} = g^2_7 - \frac{1}{2} g_7 \left[ g^2_7 + 2g^2_8 \right], \quad g_7 = N(0)J_7 > 0, \quad (106)$$

$$\frac{\partial g_8}{\partial x} = g^2_8 - \frac{1}{2} g_8 \left[ g^2_7 + 2g^2_8 \right], \quad g_8 = N(0)J_8 > 0. \quad (107)$$

Here $x = \log(D/T)$. We can identify three fixed points related to one-, two-, and three-channel Kondo effects. The one-channel, strong coupling fixed point $(g^*_7, g^*_8) = (\infty, 0)$ is beyond the perturbative regime, but is stable leading to the singlet ground state and Fermi liquid spectrum [27]. The three-channel fixed point $(2/3, 2/3)$ is stable along the line $g_7 = g_8$ in the $g_7 - g_8$ plane, but unstable for any small perturbation from $g_7 = g_8$. Finally, the two-channel fixed point $(0, 1)$ is stable leading to the logarithmically divergent thermodynamics at zero temperature. From the scaling analysis, we can infer the ground state physics: one-channel for $J_7 > J_8$; two-channel for $J_7 < J_8$; and three-channel for $J_7 = J_8$.

**E. Complete $\Gamma_7 \oplus \Gamma_8$ exchange coupling space**

In the $J_c = 5/2 \Gamma_7 \oplus \Gamma_8$ conduction electron subspace allowing for other than virtual $f^2 \Gamma_3$ charge fluctuations, all the possible exchange interactions can be written in a simple, compact form

$$H_1 = J^i \otimes S^i_c(0) S^i_{\Gamma_7}, \quad (108)$$

$$J^i = J_7 \kappa^0 + J_8 \tau^0 + J_{78} \kappa^i + J_{88} \tau^i. \quad (109)$$

The matrices are defined in the section III. We already studied this model in some specific cases in the previous sections. We can find the scaling equations up to the third order (see the Appendices [3 & 4] as
\[
\begin{align*}
\frac{\partial g_7}{\partial x} &= g_7^2 - \frac{1}{2} g_{78}^2 - \frac{1}{2} \, g_7 \left[ g_7^2 + 2(g_{78}^2 + g_8^2 + g_{88}^2) \right], \\
\frac{\partial g_{78}}{\partial x} &= -g_{78} \left[ g_{88} + \frac{1}{2}(g_7 + g_8) \right] - \frac{1}{2} \, g_{78} \left[ g_7^2 + 2(g_{78}^2 + g_8^2 + g_{88}^2) \right], \\
\frac{\partial g_8}{\partial x} &= g_8^2 - \frac{1}{2} g_{88}^2 - \frac{1}{4} g_{78}^2 - \frac{1}{2} \, g_8 \left[ g_7^2 + 2(g_{78}^2 + g_8^2 + g_{88}^2) \right], \\
\frac{\partial g_{88}}{\partial x} &= -g_8 g_{88} - \frac{1}{2} g_{78}^2 - \frac{1}{2} \, g_{88} \left[ g_7^2 + 2(g_{78}^2 + g_8^2 + g_{88}^2) \right],
\end{align*}
\]
\[
g_7 = N(0) J_7, \quad g_{78} = N(0) J_{78}, \quad g_8 = N(0) J_8, \quad g_{88} = N(0) J_{88}.
\]

From these scaling equations, a linearized analysis reveals that some of the fixed points we found in the previous sections remain stable. The relevant fixed points \((g^*_7, g^*_{78}, g^*_8, g^*_{88})\) are:

(i) \((\infty, 0, 0, 0)\): one-channel fermi liquid fixed point;

(ii) \((0, 0, 1, 0)\): two-channel non-fermi liquid fixed point remains stable against any small perturbation away from this fixed point

\[
g^* = M, \quad M^i = \tau^0,
\]
\[
\Lambda = \frac{1}{2} \cdot \frac{3}{2} \tau^0.
\]

As discussed previously, \(\Lambda\) measures the conduction pseudo-spin size and the number of channels. Here the conduction electrons are described with the pseudo-spin \(S_c = 1/2\) and with two orbitally degenerate channels (\(\text{Tr} \tau_0 = 2\));

(iii) \((2/3, 0, 2/3, 0)\): three-channel non-fermi liquid fixed point

\[
g^* = \frac{2}{3} M, \quad M^i = \kappa^0 + \tau^0,
\]
\[
\Lambda = \frac{1}{2} \cdot \frac{3}{2} (\kappa^0 + \tau^0).
\]

is stable against the \(\Gamma_7 - \Gamma_8\) mixing interaction or the \(\Gamma_8 \pm\) channel mixing interaction, but unstable for any perturbation away from \(g_7 = g_8\);

(iv) \((0, 0, -1/5, \pm 2/5)\): \(S_c = 3/2\) fixed points are

\[
g^i_\pm = \frac{1}{5} M^i_\pm, \quad M^i_\pm = -\tau^0 \pm 2 \tau^i.
\]
The $g_7^*$ fixed point is stable, but the other $S_c = 3/2$ fixed point $g_8^*$ becomes unstable against the $\Gamma_7$ and $\Gamma_8$ mixing exchange interaction ($J_{78}$). As we shall show below, the mixing interaction between $\Gamma_6$ and $\Gamma_8$ conduction electrons is relevant to the $g_7^*$ fixed point.

There are another two unstable fixed points which we did not consider before. The fixed points are $g_7^* = 2/11$, $g_8^* = -2/11$, $g_{78}^* = 0$, $g_{88}^* = \pm 4/11$

$$g^i = \frac{2}{11} M^i, \quad (123)$$

$$M^i = \kappa^0 - \tau^0 \pm 2 \tau^i, \quad (124)$$

$$\Lambda = \frac{1}{2} \cdot \frac{3}{2} \kappa^0 + \frac{1}{2} \cdot \frac{5}{2} \tau^0. \quad (125)$$

At the fixed points the reduced exchange interaction is

$$H_1 = \frac{2}{11} \left( \vec{S}_c(0) + \vec{S}_c'(0) \right) \cdot \vec{S}_{\Gamma_7}. \quad (126)$$

Here $S_c = 1/2$ and $S_c' = 3/2$. That is, two different conduction “bands” with different spins are coupled to the impurity spin.

All the fixed points we considered so far have $g_{78}^* = 0$. Since the $\Gamma_7 - \Gamma_8$ mixing interaction is relevant perturbation for some fixed point, we might imagine that some fixed points with a finite value of $g_{78}$ are possible. There are in total 6 fixed points of which only two are stable

$$A: g_7^* = -\frac{2}{9}, \quad g_8^* = -\frac{2}{9}, \quad g_{78}^* = \pm \frac{4\sqrt{2}}{9}, \quad g_{88}^* = -\frac{4}{9}, \quad (127)$$

$$B: g_7^* = \frac{1}{3}, \quad g_8^* = -\frac{1}{6}, \quad g_{78}^* = \pm \frac{\sqrt{2}}{6}, \quad g_{88}^* = -\frac{1}{3}, \quad (128)$$

$$C: g_7^* = -\frac{2}{21}, \quad g_8^* = \frac{2}{15}, \quad g_{78}^* = \pm \frac{8\sqrt{5}}{105}, \quad g_{88}^* = -\frac{8}{105}. \quad (129)$$

Out of these, only the type $A$ fixed points are stable. The type $B$ & $C$ fixed points are unstable against a perturbation along one principal direction.

The $A$ fixed points are
\[ g_A^* = \frac{2}{3} M, \quad (130) \]
\[ M^i = -\frac{1}{3} \left( \kappa^0 + \tau^0 \right) \pm \frac{2\sqrt{2}}{3} \kappa^i - \frac{2}{3} \tau^i, \quad (131) \]
\[ \Lambda = \frac{1}{2} \cdot \frac{3}{2} \left( \kappa^0 + \tau^0 \right). \quad (132) \]

The fixed points can be represented with the direct sum of three \( S_c = 1/2 \) conduction electron pseudo spins. We further note that the complex coupling matrices can be simplified using the orthogonal transformation in the conduction electron channel space

\[
M^x = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad M^y = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M^z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (133)
\]

After rearrangement and redefinition of the states with some phase and another orthogonal transformation to diagonalize the spin-flipping terms in the channel space, we can rewrite the exchange interaction at the fixed points in a concise form

\[ H_1 = J \sum_{n=1}^{3} \vec{S}_{cn}(0) \cdot \vec{S}_I. \quad (134) \]

This is the three-channel \( S_c = S_I = 1/2 \) exchange interaction. We further note that the following exchange interactions are all equivalent and result in the same physics, because the \( SU(2) \) algebra does not change (we call this the chiral – “handedness” – symmetry).

\[ H_1 = J \vec{S}_c(0) \cdot \vec{S}_I, \quad (135) \]
\[ H_1 = J \left( -S_c^x S_I^x - S_c^y S_I^y + S_c^z S_I^z \right), \quad (136) \]
\[ H_1 = J \left( S_c^x S_I^x - S_c^y S_I^y - S_c^z S_I^z \right), \quad (137) \]
\[ H_1 = J \left( S_c^x S_I^x - S_c^y S_I^y - S_c^z S_I^z \right). \quad (138) \]

We can show the equivalence between them by redefining the states with the introduction of additional phase.

Type \( B \) & \( C \) fixed points can be written in a standard form
\[ \mathbf{g}_B = \frac{1}{4} M, \quad (139) \]

\[ M^i = \frac{4}{3} \kappa^0 - \frac{2}{3} \tau^0 \pm \frac{2\sqrt{2}}{3} \kappa^i - \frac{4}{3} \tau^i, \quad (140) \]

\[ \Lambda = 1 \cdot 2 \left( \kappa^0 + \tau^0 \right), \quad (141) \]

The spin representation is \( S_c = 1 \).

\[ \mathbf{g}_C = \frac{2}{35} M, \quad (142) \]

\[ M^i = -\frac{5}{3} \kappa^0 + \frac{7}{3} \tau^0 \pm \frac{4\sqrt{5}}{3} \kappa^i - \frac{4}{3} \tau^i, \quad (143) \]

\[ \Lambda = \frac{5}{2} \cdot \frac{7}{2} \left( \kappa^0 + \tau^0 \right), \quad (144) \]

The spin representation is \( S_c = 5/2 \).

In the \( \Gamma_6 \oplus \Gamma_8 \) conduction electron space, we get the same physics as in the above analysis. The difference is that the \( S_c = 3/2 \) fixed point stable against the \( \Gamma_7 - \Gamma_8 \) mixing becomes unstable against the \( \Gamma_6 - \Gamma_7 \) conduction electron mixing. Hence both of the \( S_c = 3/2 \) fixed points cannot remain stable in the presence of the mixing interactions between the \( \Gamma_8 \) and the \( \Gamma_{6,7} \) conduction electrons.

**F. The \( \Gamma_6 \oplus \Gamma_7 \oplus \Gamma_8 \) conduction electron space**

In this section, we are going to study the full exchange interactions possible in our model approach. The extension of the scaling analysis to the full interactions is straightforward.

\[ H_1 = J^i \otimes S_c^i(0) S_{\Gamma_7}^i, \quad (145) \]

\[ J^i = J_6 \gamma^0 + J_7 \kappa^0 + J_8 \tau^0 + J_{68} \gamma^i + J_{78} \kappa^i + J_{88} \tau^i. \quad (146) \]

Third order scaling equations are

\[ \frac{\partial g_6}{\partial x} = g_6^2 - \frac{1}{2} g_{68}^2 - \frac{1}{2} g_6 \left[ g_6^2 + g_7^2 + 2(g_{68}^2 + g_{78}^2 + g_8^2 + g_{88}^2) \right], \quad (147) \]

\[ \frac{\partial g_7}{\partial x} = g_7^2 - \frac{1}{2} g_{78}^2 - \frac{1}{2} g_7 \left[ g_6^2 + g_7^2 + 2(g_{68}^2 + g_{78}^2 + g_8^2 + g_{88}^2) \right], \quad (148) \]

\[ \frac{\partial g_8}{\partial x} = g_8^2 - \frac{1}{2} g_{88}^2 - \frac{1}{4} \left[ g_{68}^2 + g_{78}^2 \right] - \frac{1}{2} g_8 \left[ g_6^2 + g_7^2 + 2(g_{68}^2 + g_{78}^2 + g_8^2 + g_{88}^2) \right], \]
\begin{align}
\frac{\partial g_{68}}{\partial x} &= g_{68} \left[ g_{88} - \frac{1}{2} (g_6 + g_8) \right] - \frac{1}{2} g_{68} \left[ g_6^2 + g_7^2 + 2(g_{68}^2 + g_{78}^2 + g_8^2 + g_{88}^2) \right], \\
\frac{\partial g_{78}}{\partial x} &= -g_{78} \left[ g_{88} + \frac{1}{2} (g_7 + g_8) \right] - \frac{1}{2} g_{78} \left[ g_6^2 + g_7^2 + 2(g_{68}^2 + g_{78}^2 + g_8^2 + g_{88}^2) \right], \\
\frac{\partial g_{88}}{\partial x} &= -g_{88} g_{88} + \frac{1}{2} \left[ g_{68}^2 - g_{78}^2 \right] - \frac{1}{2} g_{88} \left[ g_6^2 + g_7^2 + 2(g_{68}^2 + g_{78}^2 + g_8^2 + g_{88}^2) \right],
\end{align}

\begin{align}
g_6 &= N(0)J_6, \quad g_7 = N(0)J_7, \quad g_{78} = N(0)J_{78}, \quad g_8 = N(0)J_8, \quad g_{88} = N(0)J_{88}.
\end{align}

We can find various fixed points of the above scaling equations. We already considered the cases (1) $g_{78}^* = g_{68}^* = 0$; (2) $g_{78}^* \neq 0, g_{68}^* = 0$; (3) $g_{78}^* = 0, g_{68}^* \neq 0$. One relevant question is to ask if the two-channel fixed point of the $\Gamma_8$ conduction electron manifolds remains stable in the presence of the full interactions. The two-channel fixed point turns out to be stable. There arises the four-channel fixed point due to the expansion of the conduction electron space

\begin{align}
g^* &= \frac{1}{2} M, \\
M^i &= \gamma^0 + \kappa^0 + \tau^0, \\
\Lambda &= \frac{1}{2} \cdot \frac{3}{2} \left( \gamma^0 + \kappa^0 + \tau^0 \right).
\end{align}

This fixed point as well as the three-channel fixed points remain stable against a small perturbation of the mixing interactions ($J_{68}, J_{78}$), but they are unstable away from $g_6 = g_7 = g_8$. There is also an unstable two-channel fixed point when we fine tune $g_6 = g_7$

\begin{align}
g^* &= M, \\
M^i &= \gamma^0 + \kappa^0, \\
\Lambda &= \frac{1}{2} \cdot \frac{3}{2} \left( \gamma^0 + \kappa^0 \right).
\end{align}
This fixed point remains stable against the conduction electron mixing interactions but becomes unstable away from $g_6 = g_7$.

The two $S_c = 3/2$ fixed points are unstable in the presence of the full mixing interactions between the $\Gamma_6, \Gamma_7$ and the $\Gamma_8$ conduction electrons. Since the $g_{68}$ and $g_{78}$ terms are relevant perturbations to the $S_c = 3/2$ fixed points, we may imagine that these fixed points flow to the other fixed points with a finite value of mixing exchange couplings. We shall address this question shortly.

Another relevant question is to ask if the new type $A$ three-channel fixed points discussed in the previous section remain stable. These two fixed points which have

$$g^*_A = \frac{2}{3} M,$$

$$M^i = -\frac{1}{3} (\kappa^0 + \tau^0) \pm \frac{2\sqrt{2}}{3} \kappa^i - \frac{2}{3} \tau^i,$$

$$\Lambda = \frac{1}{2} \cdot \frac{3}{2} (\kappa^0 + \tau^0),$$

are stable, modulo the separate equality of exchange constants in the $\Gamma_6$ or $\Gamma_7$ space with those of the $\Gamma_8$ space.

Due to the expansion of the conduction electron space ($\Gamma_6$), other fixed points related to the types $A, B, C$ (in the $\Gamma_7 \oplus \Gamma_8$ conduction electron space) considered in the previous section arise

$$g^*_A = \frac{1}{2} M,$$

$$M^i = \gamma^0 - \frac{1}{3} (\kappa^0 + \tau^0) \pm \frac{2\sqrt{2}}{3} \kappa^i - \frac{2}{3} \tau^i,$$

$$\Lambda = \frac{1}{2} \cdot \frac{3}{2} (\gamma^0 + \kappa^0 + \tau^0),$$

$$g^*_B = \frac{2}{9} M,$$

$$M^i = \gamma^0 + \frac{4}{3} \kappa^0 - \frac{2}{3} \tau^0 \pm \frac{2\sqrt{2}}{3} \kappa^i - \frac{4}{3} \tau^i,$$

$$\Lambda = \frac{1}{2} \cdot \frac{3}{2} \gamma^0 + 1 \cdot 2 (\kappa^0 + \tau^0),$$

$$g^*_C = \frac{1}{18} M.$$
\[ M^i = \gamma^0 - \frac{5}{3} \kappa^0 + \frac{7}{3} \tau^0 \pm \frac{4 \sqrt{5}}{3} \kappa^i - \frac{4}{3} \tau^i, \]  
\[ \Lambda = \frac{1}{2} \cdot \frac{3}{2} \gamma^0 + \frac{5}{2} \cdot \frac{7}{2} \left( \kappa^0 + \tau^0 \right). \]  

All these fixed points are unstable.

We now consider the case: \( g^*_7 \neq 0, g^*_6 \neq 0 \). There are 8 new fixed points which are not discussed in the previous sections

\[ g^* = \frac{1}{42} M, \]  
\[ M^i = 3 \gamma^0 - \frac{7}{3} \kappa^0 - \frac{7}{3} \tau^0 \pm 2 \sqrt{3} \gamma^i \pm \frac{2 \sqrt{35}}{3} \kappa^i + \frac{4}{3} \tau^i, \]  
\[ M^i = -\frac{7}{3} \gamma^0 + 3 \kappa^0 - \frac{7}{3} \tau^0 \pm \frac{2 \sqrt{35}}{3} \gamma^i \pm 2 \sqrt{3} \kappa^i - \frac{4}{3} \tau^i, \]  
\[ \Lambda = \frac{7}{2} \cdot \frac{9}{2} \left( \gamma^0 + \kappa^0 + \tau^0 \right). \]  

These 8 fixed points are all unstable. The spin representation is \( S_c = 7/2 \).

In summary, various fixed points have been identified. See Table XI for a list of all the possible fixed points found from the third order scaling equations. The stable fixed points are: (i) one-channel fixed points; (ii) two-channel fixed point; (iii) three-channel fixed point. We also identified a “zoo” of unstable fixed points. At these unstable fixed points, various exchange interactions are possible: (iv) the higher conduction electron pseudo spins \((S_c = 3/2, 5/2, 7/2)\) are interacting with the impurity pseudo spin \( S_I = 1/2 \); (v) the multi-channel conduction electrons with different size of pseudo spins interact with the impurity pseudo spin. The higher conduction electron pseudo spin models overscreen the impurity pseudo spin and thus can lead to non-trivial non-Fermi liquid fixed points.

V. SCALING ANALYSIS II: MULTIPLE CONDUCTION PARTIAL WAVE STATES.

Up to now, we have not included the multiple conduction electron partial wave states. By this we mean, e.g., additional \( \Gamma_7 \) states of odd spatial parity about the Ce\(^{3+} \) site which may mix with the principle \( J = 5/2 \) \( \Gamma_7 \) states. In this section we are going to study the
effect of them on various Kondo effects. The point is that the multiple partial waves lead to the enhancement in the exchange coupling of the original exchange interaction model.

In the cases of simple one-channel or two-channel exchange interactions, the algebra is very simple. When various exchange interactions are studied at the same time, the third order scaling equations become complex, large matrix differential equations. As an example, we analyze the matrix scaling equations in the $\Gamma_8$ conduction electron space.

**A. Two-channel exchange interaction model.**

We consider the two-channel exchange interaction with some arbitrary number ($N$) of $\Gamma_8$ conduction electron states. Then the Anderson hybridization interaction reads in the two-channel sector

$$H_1 = \sum \sum_{\epsilon} \sum_{n\alpha} (-1)^{\alpha-1/2} V_i c_{\epsilon \Gamma_8 n\alpha} |f^1 \Gamma_7 \bar{\alpha} > < f^2 \Gamma_3 n| + h.c. \quad (176)$$

We focus on one $f^2 \Gamma_3$ CEF state for the moment. Here the index $i$ labels the different $\Gamma_8$’s of the conduction electron partial waves projected at the atomic site. Then the Schrieffer-Wolff transformation leads to the effective exchange interaction

$$\tilde{H}_1 = \sum_{ij} \sum_{n=\pm} J_{ij} \tilde{S}_\epsilon |\epsilon \Gamma_8 i | n \alpha \rangle \cdot \tilde{S}_\epsilon |\epsilon \Gamma_7 \bar{\alpha} \rangle, \quad (177)$$

$$\tilde{S}_\epsilon |\epsilon \Gamma_8 i | n \alpha \rangle = \sum_{\epsilon' \epsilon''} c^{\dagger}_{\epsilon' \Gamma_8 n \alpha} \frac{1}{2} \sigma_{\alpha\beta} c_{\epsilon'' \Gamma_8 n \beta}, \quad (178)$$

$$J_{ij} = \sqrt{J_i J_j}, \quad J_i = \frac{2V_i^2}{\epsilon_2 - \epsilon_1}. \quad (179)$$

Here $\epsilon_1(\epsilon_2)$ is the energy of the atomic state $f^1 \Gamma_7(f^2 \Gamma_3)$, respectively. Note that the orbital degeneracy index $n$ in the $\Gamma_8$ manifold is not active in the exchange interaction and thus provides the two degenerate channels. Except for the unphysical case of the same form of the projected conduction electron DOS for each $\Gamma_8$ symmetry, we can not diagonalize both the projected conduction electron Hamiltonian and the above interaction at the same time.

We demonstrate below how the physics can be simplified in this unrealistic, simple case: the same form of the projected conduction band DOS for each $\Gamma_8$ symmetry. $J_{ij} \tilde{S}_\epsilon |\epsilon \Gamma_8 i | n \alpha \rangle$
can be diagonalized in the $\Gamma_8$ label space with the orthogonal transformation. Since the conduction electron part is a unit matrix in the $\Gamma_8$ label space, the diagonalization in the $\Gamma_8$ label space can be done at the same time for the conduction electron part and the exchange interaction part

$$\tilde{H}_1 = J \sum_{n=\pm} \vec{S}_{cn}(0) \cdot \vec{S}_{\Gamma_7}, \quad (180)$$

$$\vec{S}_{cn}(0) = \sum_{\epsilon \alpha} \sum_{\epsilon' \beta} c^\dagger_{\epsilon \alpha} \frac{1}{2} \vec{\sigma}_{\alpha \beta} c_{\epsilon' \beta}, \quad (181)$$

$$c_{\epsilon \alpha} = \sum_i \sqrt{\frac{J_i}{J}} c_{\epsilon \Gamma_8 \alpha}, \quad (182)$$

$$J = \sum_i J_i. \quad (183)$$

We note that only one eigenvalue is non-zero and is given by $J$. All the other eigenvalues are identically zero. We have not included the other transformations for the conduction electrons which are irrelevant. The other components of the transformed conduction electrons (nonbonding combinations) are decoupled from the impurity spin. We conclude that the multiple $\Gamma_8$ conduction electrons give rise to the enhanced two-channel exchange coupling with $J = \sum_i J_i$ and the nonbonding combinations decouple.

We now consider more general case with all the excited $f^2\Gamma_3$ CEF states considered. In this case, the exchange coupling is symmetric $J_{ij} = J_{ji}$ but without the relation $J_{ij} = \sqrt{J_iJ_j}$. In general, the simplification as above does not occur when we diagonalize $J_{ij} \vec{S}_{\epsilon \Gamma_8 i j n}$. After diagonalization, we may enumerate the eigenvalues of $J_{ij}$ in the descending order from the maximum positive value ($J_1$) to the smallest value ($J_N$). That is, $J_1 \geq J_2 \geq \cdots \geq J_N$. Then we can write the interaction in the diagonal form

$$\tilde{H}_1 = \sum_{i=1}^N \sum_{n=\pm} J_i \vec{S}_{cin}(0) \cdot \vec{S}_{\Gamma_7}. \quad (184)$$

If there are any negative $J$'s, their magnitude will be renormalized to zero at zero temperature. If $J_i$'s are nondegenerate, as generically expected, the model system will flow to the two-channel fixed point for the most strongly coupled set of conduction electron states. If $J_1 = J_2 \neq J_3$, the system will flow to the four-channel fixed point, and so forth.
Though we have already diagonalized the exchange couplings in the $\Gamma_8$ conduction electron label space for the same projected DOS, we can give the scaling argument for more general forms of both the symmetric exchange couplings and the projected DOS

$$\tilde{H}_1 = \sum_{ij} \sum_{n=\pm} J_{ij} \vec{S}_{cijn}(0) \cdot \vec{S},$$

$$J_{ij} = J_{ji}, \quad J_{ii} > 0. \quad (185)$$

The index $n$ labels the orbital degeneracy in the $\Gamma_8$ manifold and the indices $i, j$ label the conduction electron partial waves. In the scaling analysis, it is more convenient to introduce the exchange coupling matrix $g_{ij} = \sqrt{N_i(0)N_j(0)}J_{ij}$. Here $N_i(\epsilon)$ is the projected DOS for the $i$-th $\Gamma_8$ conduction electron. Then the third order scaling equations can be written in a very compact form ($g$ is an $N \times N$ matrix)

$$\frac{\partial g}{\partial x} = g^2 - g \text{ Tr} g^2. \quad (187)$$

Now we can diagonalize the exchange coupling matrix and we assume it is already done. Then we can write the scaling equations for each diagonal component $G_i$

$$\frac{\partial G_i}{\partial x} = G_i^2 - G_i \sum_j G_j^2. \quad (188)$$

Whenever the diagonal elements are positive, they will grow initially, e.g., with decreasing temperatures. However, the largest $G$ will determine the low temperature physics. There are several fixed points of which only one fixed point is stable according to third order scaling analysis. Third order scaling analysis says that the fixed point ($G_1^* = 1, G_2^* = G_3^* = \cdots = G_N^* = 0$) is stable. This fixed point is none other than the two-channel fixed point with the initial enhanced exchange couplings.

In general, the $2M$-channel fixed point is $G_1^* = G_2^* = \cdots = G_M^* = 1/M, G_i^* = 0$ for $i \geq M + 1$ or

$$g^* = \frac{1}{M} \begin{pmatrix} I_M & 0 \\ 0 & 0 \end{pmatrix} \quad (189)$$

In the fixed point coupling matrix, $I_M$ is an $M \times M$ unit matrix. This $2M$-channel fixed point has one eigenvalue of $-1/M$ (irrelevant), $M - 1$ eigenvalues of $1/M$ (relevant), and the
remaining eigenvalues of $-1/M$ (irrelevant, corresponding to $G_i^* = 0$ for $i \geq M + 1$). This 2M-channel fixed point is unstable along the $M - 1$ principal directions and stable along the other remaining principal directions. Hence this 2M-channel fixed point is stable only when the bare couplings satisfy the relation: $G_1 = \cdots = G_M > G_i$ for $i \geq M + 1$. In the $M = 1$ case, this fixed point becomes stable as mentioned above, since the unstable principal directions are absent.

In the case of the one-channel exchange interaction models ($\Gamma_7$ and $\Gamma_6$ conduction electrons), we get the same results as in the two-channel exchange interaction model case.

B. Multiple $\Gamma_8$ conduction electron partial waves.

Now we study the effect of multiple $\Gamma_8$ conduction partial waves on the Kondo effect in detail allowing for the $S_c = 3/2$ fixed point

$$H_1 = [J_8 \otimes \tau^0 + J_{ss} \otimes \tau^i] \otimes S_c^i(0)S^i. \quad (190)$$

Then it is straightforward to find the scaling equations up to third order (see the Appendices B & C)

$$\frac{\partial g_8}{\partial x} = g_8^2 - \frac{1}{2} g_{ss}^2 - g_8 \text{Tr}(g_8^2 + g_{ss}^2), \quad (191)$$

$$\frac{\partial g_{ss}}{\partial x} = -\frac{1}{2} [g_8 g_{ss} + g_{ss} g_8] - g_{ss} \text{Tr}(g_8^2 + g_{ss}^2). \quad (192)$$

Here $g_8$ and $g_{ss}$ are reduced exchange coupling matrices. In general, when we consider the multiple conduction electron partial waves, the exchange couplings become matrices.

For definiteness, we first consider two $\Gamma_8$ conduction partial waves, say, coming from $J_c = 5/2, 7/2$ multiplets. In this case, the exchange coupling matrices are $2 \times 2$ matrices

$$J_8 = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}; \quad J_{ss} = \begin{pmatrix} \tilde{J}_{11} & \tilde{J}_{12} \\ \tilde{J}_{21} & \tilde{J}_{22} \end{pmatrix}. \quad (193)$$

To solve the above scaling equations we may introduce the following decomposition of the exchange coupling matrices
\[ g_8 = g_0 \rho^0 + g_1 \rho^1 + g_3 \rho^3, \]  
(194)  
\[ g_{88} = \tilde{g}_0 \rho^0 + \tilde{g}_1 \rho^1 + \tilde{g}_3 \rho^3. \]  
(195)  

Here \( \rho^0 \) is a \( 2 \times 2 \) unit matrix, and \( \rho^i \)'s are Pauli matrices. These matrices are defined in the space of the conduction electron partial waves where the \( \Gamma_8 \) manifolds are derived. Since the exchange coupling matrices are real and symmetric, there is no \( \rho^2 \) term. Then the scaling equations in components read

\[
\frac{\partial g_0}{\partial x} = |g|^2 - \frac{1}{2} |\tilde{g}|^2 - 2g_0 ( |g|^2 + |\tilde{g}|^2 ),
\]  
(196)  
\[
\frac{\partial g_1}{\partial x} = 2g_0 g_1 - \tilde{g}_0 \tilde{g}_1 - 2g_1 ( |g|^2 + |\tilde{g}|^2 ),
\]  
(197)  
\[
\frac{\partial g_3}{\partial x} = 2g_0 g_3 - \tilde{g}_0 \tilde{g}_3 - 2g_3 ( |g|^2 + |\tilde{g}|^2 ),
\]  
(198)  
\[
\frac{\partial \tilde{g}_0}{\partial x} = -g \cdot \tilde{g} - 2\tilde{g}_0 ( |g|^2 + |\tilde{g}|^2 ),
\]  
(199)  
\[
\frac{\partial \tilde{g}_1}{\partial x} = -\frac{1}{2} ( g_0 \tilde{g}_1 + \tilde{g}_0 g_1 ) - 2\tilde{g}_1 ( |g|^2 + |\tilde{g}|^2 ),
\]  
(200)  
\[
\frac{\partial \tilde{g}_3}{\partial x} = -\frac{1}{2} ( g_0 \tilde{g}_3 + \tilde{g}_0 g_3 ) - 2\tilde{g}_3 ( |g|^2 + |\tilde{g}|^2 ),
\]  
(201)  
\[
|g|^2 \equiv g_0^2 + g_1^2 + g_3^2,
\]  
(202)  
\[
|\tilde{g}|^2 \equiv \tilde{g}_0^2 + \tilde{g}_1^2 + \tilde{g}_3^2,
\]  
(203)  
\[
g \cdot \tilde{g} \equiv g_0 \tilde{g}_0 + g_1 \tilde{g}_1 + g_3 \tilde{g}_3.
\]  
(204)  

There are several fixed points. First of all, the weak coupling fixed point \( g_8^* = g_{88}^* = 0 \) is unstable. The four-channel fixed point

\[
g_8^* = \frac{1}{2} \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \quad g_{88}^* = \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right)
\]  
(205)  

is stable along \( g_0 \)-axis, while unstable away from \( g_1 = g_3 = 0 \). The two-channel fixed point is

\[
g_8^* = \frac{1}{2} \left( \rho^0 + \sin \theta \rho^1 + \cos \theta \rho^3 \right) \to \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right).
\]  
(206)  

Here \( \theta \) is an arbitrary real number. We get the diagonal matrix along the principal directions after an orthogonal transformation. This fixed point is stable with enhanced two-channel exchange coupling.
There is another class of the fixed points when $g_{ss}^* = \pm 2g_8^*$. The fixed point $g_0^* = -1/10, g_1^* = g_3^* = 0$ or

$$g_8^* = -\frac{1}{10} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is unstable, which corresponds to the two-channel $S_c = 3/2$ exchange interaction. The relevant stable fixed points are $g_0^* = -1/5, |g_1^*|^2 + |g_3^*|^2 = 1/25$ or

$$g_8^* = -\frac{1}{5} \begin{pmatrix} 1 + \cos \theta & \sin \theta \\ \sin \theta & 1 - \cos \theta \end{pmatrix} \rightarrow -\frac{1}{5} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

which are none other than new $S_c = 3/2$ non-Fermi liquid fixed points with enhanced exchange coupling after the orthogonal transformation at the fixed points.

We can generalize the above analysis with arbitrary number of conduction electron partial waves ($N$). We can determine the stability of the fixed points relatively easily. The case $g_{ss}^* = 0$ is equivalent to the case of the two-channel exchange interaction with multiple partial waves. There are several multi-channel fixed points of which only the two-channel fixed point is stable. In this case, the fixed points $g_8^*$ are solutions of

$$g_8^2 - \lambda g_8 = 0, \quad \lambda = \text{Tr} g_8^2.$$

The trivial weak coupling fixed point $g_8^* = 0$ is unstable. To find non-trivial fixed points, we introduce the orthogonal transformation to diagonalize the above matrix equation. Then we find the general $2M$-channel fixed points (after the orthogonal transformation)

$$g_8^* = \frac{1}{M} \begin{pmatrix} I_M & 0 \\ 0 & 0 \end{pmatrix}, \quad g_{ss}^* = 0$$

with $\lambda = 1/M$. Here $I_M$ is an $M \times M$ unit matrix. $M$ can have any value of $M = 1, 2, \cdots, N$. Now we consider the stability of the fixed points. We will show that only the fixed point for $M = 1$ is stable. Close to the fixed points, we can linearize the scaling equations to find

$$\frac{\partial \delta g_8}{\partial x} = g_8^* \delta g_8 + \delta g_8 g_8^* - \lambda \delta g_8 - g_8^* \text{Tr} \{ g_8^* \delta g_8 + \delta g_8 g_8^* \},$$

$$\frac{\partial \delta g_{ss}}{\partial x} = -\frac{1}{2} \begin{pmatrix} g_8^* \delta g_{ss} + \delta g_{ss} g_8^* \end{pmatrix} - \lambda \delta g_{ss},$$

$$\lambda = \text{Tr} \{ g_8^* g_8^* \} = \frac{1}{M}.$$
Diagonalization of the first equation can be done by the same orthogonal transformation introduced for finding the fixed points. Here it is enough to consider a small perturbation along the principal directions, since any other forms of perturbation can be written as a linear combination of those along the principal directions. Then along the principal directions, we find

$$\frac{\partial \delta g_i}{\partial x} = \frac{1}{M} \delta g_i - \frac{2}{M^2} \sum_{j=1}^{M} \delta g_j, \quad i = 1, 2, \cdots, M, \quad (214)$$

$$\frac{\partial \delta g_i^j}{\partial x} = -\frac{1}{M} \delta g_i^j, \quad i = M + 1, M + 2, \cdots, N. \quad (215)$$

Thus this 2M-channel fixed point is stable along the N − M principal directions (corresponding to i = M + 1, M + 2, · · ·, N). And it is stable along one direction and unstable along the other M − 1 directions after the diagonalization for i = 1, 2, · · ·, M. For a small perturbation along the principal directions of $g_{ss}^* = 0$ (in fact, those of $g_s^*$), we find

$$\frac{\partial \delta g_{ss}^i}{\partial x} = -\frac{2}{M} \delta g_{ss}^i, \quad i = 1, 2, \cdots, M, \quad (216)$$

$$\frac{\partial \delta g_{ss}^i}{\partial x} = -\frac{1}{M} \delta g_{ss}^i, \quad i = M + 1, M + 2, \cdots, N. \quad (217)$$

Thus the 2M-channel fixed point is stable for any small perturbation of $g_{ss}^*$. We conclude that only the two-channel fixed point for $M = 1$ is stable when we consider the multiple conduction electron partial waves.

Another class of fixed points $g_{ss}^* = \pm 2g_s^*$ are analyzed in the same way. The identification of the fixed points are simple: there are M-channel $S_c = 3/2$ fixed points of which only the case $M = 1$ is stable. The M-channel $S_c = 3/2$ fixed point matrix is (after diagonalization)

$$g_{ss}^* = \pm 2g_s^*, \quad (218)$$

$$g_s^* = \frac{1}{5M} \begin{pmatrix} I_M & 0 \\ 0 & 0 \end{pmatrix}. \quad (219)$$

Here $I_M$ is an $M \times M$ unit matrix. Now we study the stability of this fixed point using the linearized scaling equations close to the fixed point

$$\frac{\partial \delta g_s}{\partial x} = g_s^* \delta g_s + \delta g_s - \frac{1}{2} [g_{ss}^* \delta g_{ss} + \delta g_{ss} \ g_{ss}^*] - \lambda \delta g_s$$
\[
- \mathbf{g}_s^* \text{Tr}\{ \mathbf{g}_s^* \delta \mathbf{g}_s + \delta \mathbf{g}_s \mathbf{g}_s^* + \mathbf{g}_{ss}^* \delta \mathbf{g}_{ss} + \delta \mathbf{g}_{ss} \mathbf{g}_{ss}^* \},
\]

\[
\frac{\partial \delta \mathbf{g}_{ss}}{\partial x} = - \frac{1}{2} [ \mathbf{g}_{ss}^* \delta \mathbf{g}_s + \delta \mathbf{g}_s \mathbf{g}_{ss}^* + \mathbf{g}_s^* \delta \mathbf{g}_{ss} + \delta \mathbf{g}_{ss} \mathbf{g}_s^* ] - \lambda \delta \mathbf{g}_{ss}
\]

\[
\frac{\partial \delta \mathbf{g}_{ss}}{\partial x} = - \frac{1}{2} \left[ \mathbf{g}_{ss}^* \delta \mathbf{g}_s + \delta \mathbf{g}_s \mathbf{g}_{ss}^* + \mathbf{g}_s^* \delta \mathbf{g}_{ss} + \delta \mathbf{g}_{ss} \mathbf{g}_s^* \right] - \lambda \delta \mathbf{g}_{ss}
\]

\[
\begin{align*}
\lambda &= \text{Tr}\{ \mathbf{g}_s^* \mathbf{g}_s^* + \mathbf{g}_{ss}^* \mathbf{g}_{ss}^* \} = \frac{1}{5M} .
\end{align*}
\]

Again it is enough to consider the linear stability along the principal directions. Then we find for \( \mathbf{g}_{ss}^* = 2 \mathbf{g}_s^* \)

\[
\frac{\partial \delta \mathbf{g}_s^i}{\partial x} = - \frac{3}{5M} \delta \mathbf{g}_s^i + \frac{2}{5M} \delta \mathbf{g}_{ss}^i - \frac{2}{25M^2} \sum_{j=1}^M \left( \delta \mathbf{g}_s^j + 2 \delta \mathbf{g}_{ss}^j \right), \quad i = 1, 2, \ldots, M,
\]

\[
\frac{\partial \delta \mathbf{g}_{ss}^i}{\partial x} = \frac{2}{5M} \delta \mathbf{g}_s^i - \frac{4}{25M^2} \sum_{j=1}^M \left( \delta \mathbf{g}_s^j + 2 \delta \mathbf{g}_{ss}^j \right), \quad i = 1, 2, \ldots, M,
\]

\[
\begin{align*}
\frac{\partial \delta \mathbf{g}_s^i}{\partial x} &= - \frac{1}{5M} \delta \mathbf{g}_s^i, \quad i = M + 1, M + 2, \ldots, N, \\
\frac{\partial \delta \mathbf{g}_{ss}^i}{\partial x} &= - \frac{1}{5M} \delta \mathbf{g}_{ss}^i, \quad i = M + 1, M + 2, \ldots, N.
\end{align*}
\]

As we can see, the fixed point is stable for any small perturbation along the principal directions with \( i = M + 1, M + 2, \ldots, N \). For a small perturbation along the principal directions of \( i = 1, 2, \ldots, M \), we can rewrite the linearized scaling equations as

\[
\frac{\partial \delta \mathbf{g}_s^i}{\partial x} = \left( - \frac{3}{5M} \delta_{ij} - \frac{2}{25M^2} A_{ij} \right) \delta \mathbf{g}_s^j + \left( \frac{2}{5M} \delta_{ij} - \frac{4}{25M^2} A_{ij} \right) \delta \mathbf{g}_{ss}^j,
\]

\[
\frac{\partial \delta \mathbf{g}_{ss}^i}{\partial x} = \left( \frac{2}{5M} \delta_{ij} - \frac{4}{25M^2} A_{ij} \right) \delta \mathbf{g}_s^j - \frac{8}{25M^2} A_{ij} \delta \mathbf{g}_{ss}^j.
\]

Here \( \delta_{ij} \) is the Kronecker delta and \( A_{ij} = 1 \) for all \( i, j = 1, 2, \ldots, M \). We further note that the combinations \( \delta g \equiv \delta g_s^i + 2 \delta g_{ss}^i \) and \( \tilde{\delta} g \equiv 2 \delta g_s^i - \delta g_{ss}^i \) have simple scaling equations

\[
\frac{\partial \delta \mathbf{g}_s^i}{\partial x} = \left( \frac{1}{5M} \delta_{ij} - \frac{2}{25M^2} A_{ij} \right) \delta \mathbf{g}_s^j,
\]

\[
\frac{\partial \tilde{\delta} \mathbf{g}_s^i}{\partial x} = - \frac{2}{25M^2} \delta \mathbf{g}_s^j.
\]

Note that the diagonalization of the combination \( g \) leads to one eigenvalue of \(-1/M\) and \( M - 1 \) eigenvalues of \( 1/M \). The combination \( \tilde{g} \) is an irrelevant perturbation, while the
combination $g$ is an relevant perturbation for $M > 1$. Hence we conclude that only the one-channel $S_c = 3/2$ fixed point is stable, while the $M$-channel $S_c = 3/2$ fixed point is unstable for $M > 1$.

Now we are going to close this section after discussing how to extend our analysis to the full interaction case. In this case, the exchange interaction is a little bit complicated

$$H_1 = J^i \otimes S^i_c S^i_{1\gamma},$$

(231)

$$J^i = J^i_6 \otimes \gamma^0 + J^i_7 \otimes \epsilon^0 + J^i_8 \otimes \tau^0 + J^i_{68} \otimes \gamma^i + J^i_{78} \otimes \epsilon^i + J^i_{88} \otimes \tau^i.$$  

(232)

Here $J^i$’s are matrices whose dimension is determined by the number of the conduction electron partial waves. Applying the third order scaling equations, we find more fixed points. However, only the stable fixed points we considered in the previous section – one-, two-, and three-channel fixed points – remain stable in the presence of the multiple conduction electron partial waves. As we found in the analysis of Sec. IV, a big zoo of new unstable multi-channel fixed points are generated.

Our overall conclusion is that the one-, two-, and three-channel fixed points remain stable with the enhanced bare exchange couplings, even in the presence of the multiple conduction electron partial waves.

**VI. DISCUSSION AND CONCLUSION.**

We have introduced and studied a realistic model Hamiltonian for Ce$^{3+}$ impurities with three configurations ($f^0$, $f^1$, $f^2$), which are embedded in cubic normal metals. Using the third order scaling theory, we analyzed a various terms in our model Hamiltonian.

In our study, a new exchange interaction has been discovered, which may lead to the non-Fermi liquid $S_c = 3/2$ fixed point. This model is under study using the numerical renormalization group (NRG) and conformal field theory [11]. Unfortunately, this new fixed point becomes unstable against the mixing interactions between the $\Gamma_8$ and the $\Gamma_{6,7}$ conduction electrons.
Within our model study, we found that the one-channel (the $\Gamma_6$ or the $\Gamma_7$ conduction electron manifold), the two-channel (the $\Gamma_8$ conduction electron manifold), and the three-channel (the $\Gamma_6 \oplus \Gamma_8$ or the $\Gamma_7 \oplus \Gamma_8$ manifold) $S_c = S_I = 1/2$ non-Fermi liquid fixed points are stable. From the scaling equations, we also identified several unstable fixed points. Of special interest are the fixed points with the higher conduction electron spins $S_c = 3/2, 5/2, 7/2$, which may generate non-Fermi liquid physics. The multi-channel exchange interactions with the different conduction electron spin size are also identified at the unstable fixed points. We may ask what the physical meaning of these unstable fixed points are. When some specific exchange interaction coupling is strong enough, the system will show the physics at high temperatures relevant to this exchange interaction model before flowing finally to the stable fixed point at zero temperature.

The multiple conduction electron partial waves generate many unstable multi-channel fixed points. Again we find stability of the one-channel, two-channel and three-channel fixed points.

We close by noting a slight discrepancy between our work and that of Koga and Shiba [19], who studied the possible non-Fermi liquid ground states of an $f^3, J = 9/2, \Gamma_6$ doublet interacting with conduction electrons. The exchange coupling with conduction electrons is mediated by virtual $f^3 - f^2$ charge fluctuations. In an NRG study of this model with a particular choice of coupling strengths, it was found that a fermi liquid ground state always arose. However, while the initially derived hamiltonian is identical in form to that considered in our paper for the $\Gamma_6 \oplus \Gamma_7 \oplus \Gamma_8$ sector of conduction states, we find that the NRG Hamiltonian used in this paper has explicit broken channel symmetry for the effective two-channel coupling. We believe that this may be the source of the divergent conclusions between our work and theirs.
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APPENDIX A: MIXING MATRIX ELEMENTS

In this Appendix, we shall discuss briefly how to project the atomic electron operators to the atomic electron Fock space. We will use the Wigner-Eckart theorem in the projection and calculate the fractional parentage coefficients.

To treat the strong on-site Coulomb interaction accurately, we introduce the projected states belonging to the atomic electron Fock space. In our model for Ce$^{3+}$, we assume the strong on-site Coulomb interaction, strong spin-orbit coupling, and strong crystalline electric field (CEF) in the atomic electron space. We project the atomic electron Fock space to the three configurations $f^0, f^1, f^2$ which are relevant to the trivalent Ce impurities embedded in normal metals. The total orbital angular momentum ($\vec{L}$) and the total spin angular momentum ($\vec{S}$) operators are good quantum numbers only in the presence of the Coulomb interaction. When we include the LS coupling as well as the Coulomb interaction, now the total angular momentum operator, $\vec{J} = \vec{L} + \vec{S}$, is a good quantum number. Below we show the projection of the atomic electrons when the total angular momentum operator $\vec{J}$ commutes with the atomic Hamiltonian. The extension to the crystal electric field case is straightforward.

The projection of the atomic electron operators to the atomic electron Fock space is realized by

$$1 = |f^0><f^0| + \sum_i |f^1;i><f^1;i| + \sum_j |f^2;j><f^2;j|,$$

(A1)

$$f_\mu \to 1 f_\mu 1.$$  

(A2)
Here the right hand side of 1 is the restricted completeness relation to the three configurations $f^0, f^1, f^2$. The labels $i, j$ count all the possible states in the $f^1$ and $f^2$ configurations, respectively.

In the $f^1$ configuration, the atomic electron states are labeled by $j_1 = 5/2, 7/2$ and their azimuthal quantum numbers. That is,

$|f^1\frac{5}{2}\mu_1\rangle$ and $|f^1\frac{7}{2}\mu_1\rangle$.

In the $f^2$ configuration, the total angular momentum ranges from 0 through 6. The atomic electron states can be represented by $|f^2LSJ\mu_2\rangle$. For the spin singlet states, $J = L = 0, 2, 4, 6$. For the spin triplet states, $L = 1, 3, 5$ and $J = L, L \pm 1$. We are now in a position to find the projected atomic electron operator of $f_{j,c}\mu_c$.

$$f_{j,c} \rightarrow |f^0 > < f^0 | f_{j,c} \mu_c \sum_{j_1 \mu_1} |f^1 j_1 \mu_1 > < f^1 j_1 \mu_1|$$

$$+ \sum_{j_1 \mu_1} |f^1 j_1 \mu_1 > < f^1 j_1 \mu_1| f_{j,c} \mu_c \sum_{LSJ\mu_2} |f^2LSJ\mu_2 > < f^2LSJ\mu_2|$$

$$= |f^0 > < f^1 j_c \mu_c | + \sum_{j_1 \mu_1} \sum_{LSJ\mu_2} \Lambda(j_c \mu_c; f^1 j_1 \mu_1 | f^2LSJ\mu_2) |f^1 j_1 \mu_1 > < f^2LSJ\mu_2|,$$

(A3)

$$\Lambda(j_c \mu_c; f^1 j_1 \mu_1 | f^2LSJ\mu_2) \equiv < f^1 j_1 \mu_1 | f_{j,c} \mu_c | f^2LSJ\mu_2 > .$$

(A4)

Here $\Lambda$ measures the mixing strength between $f^1$ and $f^2$ atomic states. Applying the Wigner-Eckart theorem, we can rewrite the mixing strength $\Lambda$ as the Clebsch-Gordan coefficients multiplied by a prefactor

$$\Lambda(j_c \mu_c; f^1 j_1 \mu_1 | f^2LSJ\mu_2) = K(j_c; f^1 j_1 | f^2LSJ) < j_c \mu_c; j_1 \mu_1 | J \mu_2 > .$$

(A5)

Here the prefactor $K(j_c; f^1 j_1 | f^2LSJ)$ is the fractional parentage coefficient and is listed in Tables XII, XIII.

In the strong LS coupling limit, we now illustrate how we can calculate the matrix elements $< f^1 j_1 \mu_1 | f_{j,c} \mu_c | f^2LS : J \mu_2 >$ to find the fractional parentage coefficients. We construct symmetry eigenstates of the total angular momentum in the $f^2$ configuration.
\[ |f^2LS = 1 : J\mu_2 > \]
\[ = \sum_M \sum_{m_1m_2} < 3m_1; 3m_2|LM > \]
\[ \times \frac{1}{\sqrt{2}} \left( < 11; LM|J\mu_2 > f_{m_1\uparrow}^{\dagger}f_{m_2\uparrow}^{\dagger}|0 > + < 10; LM|J\mu_2 > \frac{1}{\sqrt{2}} ( f_{m_1\uparrow}^{\dagger}f_{m_2\downarrow}^{\dagger} + f_{m_1\downarrow}^{\dagger}f_{m_2\uparrow}^{\dagger} ) |0 > + < 1\bar{1}; LM|J\mu_2 > f_{m_1\downarrow}^{\dagger}f_{m_2\uparrow}^{\dagger}|0 > \right), \quad (A6) \]

\[ |f^2LS = 0 : J\mu_2 > \]
\[ = \sum_M \sum_{m_1m_2} < 3m_1; 3m_2|LM > \]
\[ \times < 00; LM|J\mu_2 > \frac{1}{2} ( f_{m_1\uparrow}^{\dagger}f_{m_2\downarrow}^{\dagger} - f_{m_1\downarrow}^{\dagger}f_{m_2\uparrow}^{\dagger} ) |0 > . \quad (A7) \]

These symmetry states are not energy eigenstates of the atomic Hamiltonian (Coulomb interaction and LS coupling) in the \( f^2 \) configuration. Mixing can occur among the same \( J \) multiplets, but this is neglected in the LS limit. Below we evaluate the desired matrix elements for the above symmetry eigenstates.

**Spin singlet case:** In this case, \( J = L \) and only even \( L = J \) terms are nonvanishing.

\[ < f^1j_1\mu_1 | f_{j_c\mu_c} | f^2LS = 0 : J\mu_2 > \]
\[ = \sum_{\alpha_c m_c m_1} (-1)^{1/2-\alpha_c} \frac{1}{2}\alpha_c; 3m_c > < j_1\mu_1 \frac{1}{2}\alpha_1; 3m_1 > < J\mu_2 | 3m_c; 3m_1 > \]
\[ = (-1)^{j_1+j_1+3/2} \sqrt{(2j_c+1)(2j_1+1)} \begin{pmatrix} j_c & j_1 & J \\ 3 & 3 & 1/2 \end{pmatrix} < J\mu_2 | j_c \mu_c; j_1\mu_1 > . \quad (A8) \]

The final result is proportional to the correct Clebsch-Gordan coefficients. Here we introduced the \( 6-j \) symbols \([28]\). The explicit calculation gives the fractional parentage coefficients in terms of \( 6-j \) symbols. The fractional parentage coefficients are listed in Table [XII].

**Spin triplet case:** In this case, only odd \( L \) terms are nonvanishing.

\[ < f^1j_1\mu_1 | f_{j_c\mu_c} | f^2LS = 1 : J\mu_2 > \]
\[ = \sqrt{2} \sum_{\alpha_1\alpha_c M_m} \sum_{m_1m_c} \sum < LM|3m_c; 3m_1 > < 1m|\frac{1}{2}\alpha_c; \frac{1}{2}\alpha_1 > \]

50
\[
\times \langle j_c \mu_e | \frac{1}{2} \alpha_c; 3m_c > \times \langle j_1 \mu_1 | \frac{1}{2} \alpha_1; 3m_1 > \times \langle 1m; LM | J \mu_2 > \quad (A9)
\]

\[
= K(j_c; f^1j_1 | f^2LS = 1 : J) \times \langle J \mu_2 | j_c \mu_e; j_1 \mu_1 >, \quad (A10)
\]

\[
K(j_c; f^1j_1 | f^2LS = 1 : J)
= \sqrt{6(2L + 1)(2j_1 + 1)(2j_c + 1)}
\]

\[
\times \sum_Q (2Q + 1) (-1)^P \begin{pmatrix} 3 & 3 & L \\ J & 1 & Q \end{pmatrix} \begin{pmatrix} 3 & j_c & 1/2 \\ 1/2 & 1 & Q \end{pmatrix} \begin{pmatrix} J & j_c & j_1 \\ 1/2 & 3 & Q \end{pmatrix} . \quad (A11)
\]

Here \( P = 2j_c + 2L + 2j_2 + 1 \). For a given \( L \), there are three values of \( J = L - 1, L, L + 1 \). The fractional parentage coefficients are listed in Table XIII.

**APPENDIX B: MULTIPLICATIVE RENORMALIZATION**

When we apply the perturbation theory to the Kondo problems, logarithmic divergences appear at all orders in the coupling. This infrared divergence can be handled by renormalizing the exchange coupling constant for the interaction vertex, and renormalizing the “mass enhancement” parameter for the pseudofermion Green’s function. Then we do the perturbation theory in the renormalized parameters in the hope that it leads to converging solutions.

Though the renormalization can be defined in several different ways, we may consider the progressive removal of the conduction band edge states for definiteness. At temperature \( T \), only the conduction electrons (thermally excited) inside the band of order \( T \) with respect to the Fermi level play an important role in determining physical properties. Thus we can integrate out the band edge states (virtually excited states) to find the effective Hamiltonian.

When the renormalized Hamiltonian is of the same form as the original one, we can find the scaling equations for the model parameters. Though the following analysis is restricted to the perturbative regime, we can derive qualitative results out of this. For quantitative results, the full numerical renormalization group method is required.
Here we just sketch how to generalize the multiplicative renormalization group \([22,23]\) to the following form of the exchange interaction.

\[
H_1 = \sum_i J^i \otimes S_c^i(0) S_I^i. \tag{B1}
\]

Here \(S_I = 1/2\) is the impurity spin, and \(S_c = 1/2\) is the conduction electron spin. The exchange coupling is arbitrary square matrix. For this exchange interaction, the interaction vertex is

\[
\Gamma_{pq\alpha\beta;\mu\nu}^a = J_{pr}^a \tilde{\Gamma}_{rq}^a S_{ca\beta}^a \frac{1}{2} \sigma_{\mu\nu}^a. \tag{B2}
\]

This exchange interaction describes the most general interaction introduced in chapter II. Though the Schrieffer-Wolff transformation generates the potential scattering terms, we are not going to include them here.

In the Kondo problem, the multiplicative renormalization is defined as \([22,23]\)

\[
G \rightarrow z_1 G, \tag{B3}
\]

\[
F \rightarrow z_2 F, \tag{B4}
\]

\[
\tilde{\Gamma}^a \rightarrow [Z^a]^{-1} \tilde{\Gamma}^a, \tag{B5}
\]

\[
J^a \rightarrow \frac{1}{z_1 z_2} J^a Z^a. \tag{B6}
\]

Here \(G\) and \(F\) are the Green’s functions for the conduction electrons and the pseudo fermions representing the local spins. \(\tilde{\Gamma}\) is the scattering interaction vertex (matrix), and \(J\) is the exchange coupling (matrix). Since the renormalization of the conduction electron is negligible, \(z_1 \rightarrow 1\). Here we note that \(Z^a\) ia a matrix and \(a = x, y, z\).

After following the same algebraic steps as in Ref \([22,23]\), we find the following scaling equations.

\[
\frac{\partial J_{\text{inv}}^a}{\partial x} = f_d(J_{\text{inv}}) J_{\text{inv}}^a + J_{\text{inv}}^a F_{\tilde{\Gamma}}^a(J_{\text{inv}}). \tag{B7}
\]

The functions \(f_d\) and \(F_{\tilde{\Gamma}}^a\) (matrix) are found from the irreducible or skeleton digrams of \(d(\omega)\) and \(\Gamma(\omega)\) using the perturbation theory.
\[
d(\omega) = 1 + f_d(J)\ K(\omega), \quad (B8)
\]
\[
\tilde{\Gamma}^a(\omega) = 1 + F^a_0(J)\ K(\omega), \quad (B9)
\]
\[
K(\omega) = \log \left| \frac{D}{\omega} \right| + i\pi \theta(\omega). \quad (B10)
\]

APPENDIX C: SCALING EQUATIONS UP TO THIRD ORDER

We consider the following form of the interaction.
\[
\tilde{H}_1 = \sum_i J^i \otimes S^i_c(0)\ S^i_I. \quad (C1)
\]
Here \(S^i\) is the \(i\)-th component of the impurity spin operator \(S_I = 1/2\) and \(S^i_c(0)\) is the \(i\)-th component of the conduction electron spin operator \(S_c = 1/2\) projected at the impurity site. \(J^i\) is the exchange coupling matrix.

It is straightforward to derive the scaling equations. The second order diagrams give
\[
-N(0)K(\omega)\ S^j S^i\ J^j \otimes S^j_c J^i \otimes S^i_c \quad (C2)
\]
\[
N(0)K(\omega)\ S^i S^j\ J^i \otimes S^j_c J^i \otimes S^i_c \quad (C3)
\]
The first comes from the particle excitation, while the second from the hole excitation. Their contribution is
\[
N(0)K(\omega)\ i\epsilon_{ijk} S^k\ J^j \otimes S^j_c S^i_c
\]
\[
= \frac{1}{2} N(0)K(\omega)\ \epsilon_{ijk}\epsilon_{ijl} J^i J^j \otimes S^l_c S^k. \quad (C4)
\]
The third order diagram gives
\[
[N(0)]^2 K(\omega)\ S^k S^j S^i\ \text{Tr}[J^k \otimes S^k_c J^j \otimes S^j_c J^i \otimes S^i_c]
\]
\[
= \frac{1}{2} [N(0)]^2 K(\omega)\ \text{Tr}[J^j J^i]\ J^j \otimes S^j_c S^i S^j S^i
\]
\[
= \frac{1}{2} [N(0)]^2 K(\omega)\ \text{Tr}[J^j J^i]\ J^j \otimes S^j_c [\ i\epsilon_{ijk} S^k S^i + S^j S^i S^i ] \quad (C5)
\]
Then after the self energy correction, third order scaling equations are
\[
\frac{\partial g^i}{\partial x} = \frac{1}{2} \left[ g^j g^k + g^k g^j \right] - \frac{1}{4} g^i \text{Tr} \left[ g^j g^j + g^k g^k \right] \quad (C6)
\]
Here \((i, j, k)\) are cyclic.
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### TABLES

**Table I.** Crystal electric field energy eigenstates for $J = 5/2$ multiplet in the cubic symmetry.

| $J = 5/2$ Multiplet | States |
|----------------------|--------|
| $|\Gamma^{(5/2)}_7; \uparrow / \downarrow>$ | $-\sqrt{\frac{1}{6}} |\pm 5/2 > + \sqrt{\frac{5}{6}} |\mp 3/2 >$ |
| $|\Gamma^{(5/2)}_8; +, \uparrow / \downarrow>$ | $|\pm 1/2 >$ |
| $|\Gamma^{(5/2)}_8; -, \uparrow / \downarrow>$ | $\sqrt{\frac{5}{6}} |\pm 5/2 > + \sqrt{\frac{1}{6}} |\mp 3/2 >$ |

**Table I.** Crystal electric field energy eigenstates for $J = 7/2$ multiplet in the cubic symmetry.

| $J = 7/2$ Multiplet | States |
|----------------------|--------|
| $|\Gamma^{(7/2)}_6; \uparrow / \downarrow>$ | $\pm \sqrt{\frac{5}{12}} |\mp 7/2 > \pm \sqrt{\frac{7}{12}} |\pm 1/2 >$ |
| $|\Gamma^{(7/2)}_7; \uparrow / \downarrow>$ | $\pm \sqrt{\frac{3}{2}} |\pm 5/2 > \mp \frac{1}{2} |\mp 3/2 >$ |
| $|\Gamma^{(7/2)}_8; +, \uparrow / \downarrow>$ | $\pm \sqrt{\frac{7}{12}} |\mp 7/2 > \mp \sqrt{\frac{5}{12}} |\pm 1/2 >$ |
| $|\Gamma^{(7/2)}_8; -, \uparrow / \downarrow>$ | $\pm \frac{1}{2} |\pm 5/2 > \pm \sqrt{\frac{3}{2}} |\mp 3/2 >$ |
TABLE II. Decomposition of $J$-multiplets in cubic symmetry. All possible $J$-multiplets are listed for the $f^2$ configuration.

| $J$-multiplets $D_J$ | Decomposition                        |
|---------------------|-------------------------------------|
| $D_0$               | $\Gamma_1$                         |
| $D_1$               | $\Gamma_4$                         |
| $D_2$               | $\Gamma_3 \oplus \Gamma_5$        |
| $D_3$               | $\Gamma_2 \oplus \Gamma_4 \oplus \Gamma_5$ |
| $D_4$               | $\Gamma_1 \oplus \Gamma_3 \oplus \Gamma_4 \oplus \Gamma_5$ |
| $D_5$               | $\Gamma_3 \oplus 2\Gamma_4 \oplus \Gamma_5$ |
| $D_6$               | $\Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3 \oplus \Gamma_4 \oplus 2\Gamma_5$ |

TABLE III. Crystal electric field energy eigenstates for $J = 4$ multiplet in the cubic symmetry.

| $J = 4$ Multiplet | States                                                                 |
|-------------------|------------------------------------------------------------------------|
| $|\Gamma_1^{(4)}>$ | $\sqrt{\frac{5}{24}} |4> + \sqrt{\frac{7}{12}} |0> + \sqrt{\frac{5}{24}} |-4>$ |
| $|\Gamma_3^{(4)};+>$ | $\frac{1}{\sqrt{2}} ( |2> + |2>$ ) |
| $|\Gamma_3^{(4)};->$ | $-\sqrt{\frac{7}{24}} |4> + \sqrt{\frac{7}{12}} |0> - \sqrt{\frac{7}{24}} |-4>$ |
| $|\Gamma_4^{(4)};0>$ | $\frac{1}{\sqrt{2}} ( |4> - |-4>$ ) |
| $|\Gamma_4^{(4)};\pm 1>$ | $\frac{1}{\sqrt{8}} |\mp 3> + \sqrt{\frac{7}{8}} |\pm 1>$ |
| $|\Gamma_5^{(4)};0>$ | $\frac{1}{\sqrt{2}} ( |2> - |-2>$ ) |
| $|\Gamma_5^{(4)};\pm 1>$ | $\sqrt{\frac{7}{8}} |\mp 3> - \frac{1}{\sqrt{8}} |\pm 1>$ |
TABLE IV. Irreducible tensor operators for the conduction electrons. As noted in the text, the $\Gamma_4$ irrep is the spin operator. Hence there are all in all 6 possible conduction spin operators. The mixing between $\Gamma_6$ and $\Gamma_7$ ($\Gamma_6 \otimes \Gamma_7 = \Gamma_2 \oplus \Gamma_5$) does not have a spin degree of freedom.

| $\Gamma^{\text{cb}} \otimes \Gamma^{\text{cb}}$ | Irreducible tensor operators |
|-----------------------------------------------|-----------------------------|
| $\Gamma_6 \otimes \Gamma_6$                 | $\Gamma_1 \oplus \Gamma_4$ |
| $\Gamma_7 \otimes \Gamma_7$                 | $\Gamma_1 \oplus \Gamma_4$ |
| $\Gamma_8 \otimes \Gamma_8$                 | $\Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3 \oplus 2\Gamma_4 \oplus 2\Gamma_5$ |
| $\Gamma_6 \otimes \Gamma_8$                 | $\Gamma_3 \oplus \Gamma_4 \oplus \Gamma_5$ |
| $\Gamma_7 \otimes \Gamma_8$                 | $\Gamma_3 \oplus \Gamma_4 \oplus \Gamma_5$ |

TABLE V. Selection rules for the Anderson hybridization between $f^1\Gamma_7$ and $f^2$ configurations. For example, we can read off from this table that only the $\Gamma_8$ conduction electrons can mix with $f^1\Gamma_7$ states to reach $f^2\Gamma_3$ states.

| $\Gamma^{\text{cb}} \otimes \Gamma_7(f^1)$ | $f^2$ CEF states |
|----------------------------------------------|-----------------|
| $\Gamma_6 \otimes \Gamma_7$                 | $\Gamma_2 \oplus \Gamma_5$ |
| $\Gamma_7 \otimes \Gamma_7$                 | $\Gamma_1 \oplus \Gamma_4$ |
| $\Gamma_8 \otimes \Gamma_7$                 | $\Gamma_3 \oplus \Gamma_4 \oplus \Gamma_5$ |
TABLE VI. We tabulate $\langle j_c \Gamma_7; \frac{5}{2} \Gamma_7 | J_2 \Gamma_1 \rangle$. An asterisk denotes the violation of the triangular inequality for the addition of two angular momenta. The $\Gamma_1$ irrep exists only in the $J = 0, 4, 6$ multiplets in the $f^2$ configuration.

| $j_c$ | $J_1$ | $J_2$ | $\langle j_c \Gamma_7; \frac{5}{2} \Gamma_7 | J_2 \Gamma_1 \rangle$ |
|-------|-------|-------|--------------------------------------------------|
| 5/2   | 5/2   | 0     | $\sqrt{\frac{1}{6}}$ |
|       |       | 4     | $\sqrt{\frac{1}{3}}$ |
|       |       | 6     | * |
| 7/2   | 0     | *     | |
|       | 4     | $\sqrt{\frac{5}{22}}$ | |
|       | 6     | $\sqrt{\frac{3}{11}}$ | |

TABLE VII. We tabulate $\langle j_c \Gamma_6; \frac{5}{2} \Gamma_7 | J_2 \Gamma_2 \rangle$. The $\Gamma_6$ irrep is absent in the $j_c = 5/2$ partial waves. The $\Gamma_2$ irrep exists only in the $J = 3, 6$ multiplets in the $f^2$ configuration.

| $j_c$ | $J_1$ | $J_2$ | $\langle j_c \Gamma_6; \frac{5}{2} \Gamma_7 | J_2 \Gamma_2 \rangle$ |
|-------|-------|-------|--------------------------------------------------|
| 7/2   | 5/2   | 3     | $\sqrt{7/18}$ |
|       |       | 6     | $\frac{1}{3}$ |
TABLE VIII. Table for $\langle j_c \Gamma_8; \frac{5}{2} \Gamma_7 | J_2 \Gamma_3 \rangle$. Note that $\sum_{J_2} | \langle j_c \Gamma_8; \frac{5}{2} \Gamma_7 | J_2 \Gamma_3 \rangle |^2 = 0.5$, which is none other than a correct normalization. The $\Gamma_3$ irrep exists only in the $J = 2, 4, 5, 6$ multiplets in the $f^2$ configuration.

| $j_c$ | $J_1$ | $J_2$ | $\langle j_c \Gamma_8; \frac{5}{2} \Gamma_7 | J_2 \Gamma_3 \rangle$ |
|-------|-------|-------|-------------------------------------------------|
| 5/2   | 5/2   | 2     | $\sqrt{\frac{5}{34}}$                           |
|       |       | 4     | $-\sqrt{\frac{1}{21}}$                         |
|       |       | 5     | $\frac{1}{7}$                                   |
| 7/2   | 2     | 2     | $\sqrt{\frac{25}{126}}$                        |
|       |       | 4     | $\sqrt{\frac{27}{154}}$                        |
|       |       | 5     | $\sqrt{\frac{1}{18}}$                          |
|       |       | 6     | $\sqrt{\frac{2}{99}}$                          |
TABLE IX. Table for $< j_c \Gamma_7; \frac{5}{2} \Gamma_7 | J_2 \Gamma_4 >$ and $< j_c \Gamma_8; \frac{5}{2} \Gamma_7 | J_2 \Gamma_4 >$ in Table IX. Since the two $\Gamma_4$ irreps are present in the $J = 5$ multiplets, the $J = 5 \Gamma_4$ CEF states are model parameter dependent. However, we note that the generic hybridization form is preserved [10].

| $j_c$ | $J_1$ | $J_2$ | $< j_c \Gamma_7; \frac{5}{2} \Gamma_7 | J_2 \Gamma_4 >$ | $< j_c \Gamma_8; \frac{5}{2} \Gamma_7 | J_2 \Gamma_4 >$ |
|-------|-------|-------|-------------------------------------------------|-------------------------------------------------|
| 5/2   | 5/2   | 1     | $-\sqrt{\frac{5}{126}}$                         | $\frac{2}{3\sqrt{7}}$                          |
|       |       | 3     | $-\sqrt{\frac{20}{81}}$                         | $\frac{1}{18\sqrt{2}}$                         |
|       |       | 4     | 0                                               | $\frac{1}{\sqrt{8}}$                          |
|       |       | 5     |                                                 |                                                 |
| 7/2   | 1     | 1     | $-\sqrt{\frac{2}{21}}$                         | $-\sqrt{\frac{1}{28}}$                         |
|       | 3     | 3     | $\frac{1}{\sqrt{24}}$                          | $-\frac{1}{12}$                                 |
|       | 4     | 4     | $\frac{7}{\sqrt{288}}$                         | $-\frac{1}{4\sqrt{11}}$                         |
|       | 5     | 5     |                                                 |                                                 |
|       | 6     | 6     | $\sqrt{\frac{7}{66}}$                          | $-\sqrt{\frac{7}{99}}$                         |

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TABLE X. Table for \(< j_c \Gamma_{6,8}; \frac{5}{2} \Gamma_7 | J_2 \Gamma_5 >\). Since the two \( \Gamma_5 \) irreps are present in the \( J = 6 \) multiplets, the CEF states are model parameter-dependent. However, we note that the generic hybridization form is preserved [10].

| \( j_c \) | \( J_1 \) | \( J_2 \) | \( < j_c \Gamma_{6}; \frac{5}{2} \Gamma_7 | J_2 \Gamma_5 > \) | \( < j_c \Gamma_{8}; \frac{5}{2} \Gamma_7 | J_2 \Gamma_5 > \) |
|----------|-------|-------|-----------------|-----------------|
| 5/2      | 5/2   | 2     | \( \frac{5}{42} \sqrt{5} \)   | \( -\frac{5}{18\sqrt{7}} \)   |
|          |       | 3     | \( \frac{5}{12} \)           | \( \frac{5\sqrt{3}}{36} \)     |
|          |       | 4     | \( -\frac{13}{4\sqrt{377}} \) | \( \frac{1}{9} \)             |
|          |       | 5     | \( \frac{\sqrt{70}}{18} \)   | \( \frac{1}{9} \)             |
|          |       | 6     | \( \frac{\sqrt{70}}{18} \)   | \( \frac{1}{9} \)             |

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TABLE XI. We list all the possible fixed points generated from the third order scaling equations of our model exchange interactions. Most of them are unstable. Some of them might not exist in the full calculations.

| $g^*$ | $M^i$ | $\Lambda$ | Stability |
|-------|-------|------------|-----------|
| $\infty$ | $\gamma^0$ | $\frac{1}{2} \cdot \frac{3}{2} \gamma^0$ | Stable |
| $\infty$ | $\kappa^0$ | $\frac{1}{2} \cdot \frac{3}{2} \kappa^0$ | Stable |
| 1 | $\tau^0$ | $\frac{1}{2} \cdot \frac{3}{2} \tau^0$ | Stable |
| $2/3$ | $-\frac{1}{3} (\gamma^0 + \tau^0) + \frac{2\sqrt{2}}{3} \gamma^i - \frac{2}{3} \tau^i$ | $\frac{1}{2} \cdot \frac{3}{2} (\gamma^0 + \tau^0)$ | Stable |
| $2/3$ | $-\frac{1}{3} (\kappa^0 + \tau^0) + \frac{2\sqrt{2}}{3} \kappa^i + \frac{2}{3} \tau^i$ | $\frac{1}{2} \cdot \frac{3}{2} (\kappa^0 + \tau^0)$ | Stable |
| 0 | 0 | 0 | Unstable |
| $1/5$ | $-\tau^0 + 2 \tau^i$ | $\frac{3}{2} \cdot \frac{5}{2} \tau^0$ | Unstable |
| $2/3$ | $\gamma^0 + \tau^0$ | $\frac{1}{2} \cdot \frac{3}{2} (\gamma^0 + \tau^0)$ | Unstable |
| $2/3$ | $\kappa^0 + \tau^0$ | $\frac{1}{2} \cdot \frac{3}{2} (\kappa^0 + \tau^0)$ | Unstable |
| $2/11$ | $\kappa^0 - 5 \tau^0 + 2 \tau^i$ | $\frac{1}{2} \cdot \frac{3}{2} \kappa^0 + \frac{3}{2} \cdot \frac{5}{2} \tau^0$ | Unstable |
| $2/11$ | $\gamma^0 - \tau^0 + 2 \tau^i$ | $\frac{1}{2} \cdot \frac{3}{2} \gamma^0 + \frac{3}{2} \cdot \frac{5}{2} \tau^0$ | Unstable |
| $1/4$ | $\frac{4}{3} \gamma^0 - \frac{3}{3} \tau^0 + \frac{2\sqrt{2}}{3} \gamma^i - \frac{1}{3} \tau^i$ | $1 \cdot 2 (\gamma^0 + \tau^0)$ | Unstable |
| $1/4$ | $\frac{4}{3} \kappa^0 - \frac{3}{3} \tau^0 + \frac{2\sqrt{2}}{3} \kappa^i + \frac{1}{3} \tau^i$ | $1 \cdot 2 (\kappa^0 + \tau^0)$ | Unstable |
| $2/35$ | $\frac{5}{3} \gamma^0 + \frac{3}{3} \tau^0 + \frac{2\sqrt{2}}{3} \gamma^i + \frac{4}{3} \tau^i$ | $\frac{5}{2} \cdot \frac{7}{2} (\gamma^0 + \tau^0)$ | Unstable |
| $2/35$ | $\frac{5}{3} \kappa^0 + \frac{3}{3} \tau^0 + \frac{2\sqrt{2}}{3} \kappa^i - \frac{4}{3} \tau^i$ | $\frac{5}{2} \cdot \frac{7}{2} (\kappa^0 + \tau^0)$ | Unstable |
| 1 | $\gamma^0 + \kappa^0$ | $\frac{1}{2} \cdot \frac{3}{2} (\gamma^0 + \kappa^0)$ | Unstable |
| $1/2$ | $\gamma^0 + \kappa^0 + \tau^0$ | $\frac{1}{2} \cdot \frac{3}{2} (\gamma^0 + \kappa^0 + \tau^0)$ | Unstable |
| $1/6$ | $\gamma^0 + \kappa^0 - \tau^0 + 2 \tau^i$ | $\frac{1}{2} \cdot \frac{3}{2} (\gamma^0 + \kappa^0) + \frac{5}{2} \cdot \frac{5}{2} \tau^0$ | Unstable |
| $1/2$ | $\kappa^0 - \frac{3}{3} (\gamma^0 + \tau^0) + \frac{2\sqrt{2}}{3} \gamma^i + \frac{2}{3} \tau^i$ | $\frac{1}{2} \cdot \frac{3}{2} (\gamma^0 + \kappa^0 + \tau^0)$ | Unstable |
| $1/2$ | $\gamma^0 - \frac{3}{3} (\kappa^0 + \tau^0) + \frac{2\sqrt{2}}{3} \gamma^i - \frac{2}{3} \tau^i$ | $\frac{1}{2} \cdot \frac{3}{2} (\gamma^0 + \kappa^0 + \tau^0)$ | Unstable |
| $2/9$ | $\kappa^0 + \frac{3}{3} \gamma^0 - \frac{3}{3} \tau^0 + \frac{2\sqrt{2}}{3} \gamma^i + \frac{4}{3} \tau^i$ | $\frac{1}{2} \cdot \frac{3}{2} \kappa^0 + 1 \cdot 2 (\gamma^0 + \tau^0)$ | Unstable |
| $2/9$ | $\gamma^0 + \frac{3}{3} \kappa^0 - \frac{3}{3} \tau^0 + \frac{2\sqrt{2}}{3} \kappa^i - \frac{4}{3} \tau^i$ | $\frac{1}{2} \cdot \frac{3}{2} \gamma^0 + 1 \cdot 2 (\kappa^0 + \tau^0)$ | Unstable |
| $1/18$ | $\kappa^0 - \frac{5}{3} \gamma^0 + \frac{7}{3} \tau^0 + \frac{2\sqrt{2}}{3} \gamma^i + \frac{4}{3} \tau^i$ | $\frac{1}{2} \cdot \frac{3}{2} \kappa^0 + \frac{5}{2} \cdot \frac{7}{2} (\gamma^0 + \tau^0)$ | Unstable |
\[
\begin{array}{|c|c|c|}
\hline
1/18 & \gamma^0 - \frac{5}{3} \kappa^0 + \frac{7}{3} \tau^0 \pm \frac{4\sqrt{35}}{3} \kappa^i - \frac{4}{3} \tau^i & \frac{1}{2} \cdot \frac{3}{2} \gamma^0 + \frac{5}{2} \cdot \frac{7}{2} ( \kappa^0 + \tau^0 ) \text{ Unstable} \\
1/42 & 3 \gamma^0 - \frac{7}{3} \kappa^0 - \frac{7}{3} \tau^0 \pm 2\sqrt{3} \gamma^i & \frac{7}{2} \cdot \frac{9}{2} ( \gamma^0 + \kappa^0 + \tau^0 ) \text{ Unstable} \\
1/42 & -\frac{7}{3} \gamma^0 + 3 \kappa^0 - \frac{7}{3} \tau^0 \pm \frac{2\sqrt{35}}{3} \gamma^i & \frac{7}{2} \cdot \frac{9}{2} ( \gamma^0 + \kappa^0 + \tau^0 ) \text{ Unstable} \\
\hline
\end{array}
\]

TABLE XII. Fractional parentage coefficients for \( K(j_c; f^1 \frac{5}{2} | f^2 LS = 0 : J) \). * means the triangular sum is violated. Note that for a given \((S, L, J)\), sum of the squared fractional parentage coefficients over all possible \(j_c = 5/2, 7/2\) is always 6/7.

\[
\begin{array}{|c|c|c|}
\hline
(S, L, J) & j_c = 5/2 & j_c = 7/2 \\
\hline
(0, 0, 0) & \sqrt{\frac{6}{7}} & \ast \\
(0, 2, 2) & \frac{6}{7} & \sqrt{\frac{6}{35}} \\
(0, 4, 4) & \sqrt{\frac{22}{39}} & \sqrt{\frac{20}{39}} \\
(0, 6, 6) & \ast & \sqrt{\frac{6}{7}} \\
\hline
\end{array}
\]
TABLE XIII. Fractional parentage coefficients for $K(j_c; f_{1\frac{1}{2}}^1 | f^2 LS = 1 : J)$. * means the triangular sum is violated. Hund’s rule ground multiplets correspond to $(S = 1, L = 5, J = 4)$. Note that for a given $(S, L)$, sum of the squared fractional parentage coefficients over all possible $j_c = 5/2, 7/2$ and over all $J = L, L \pm 1$ is always $20/7$.

| $(S, L, J)$ | $j_c = 5/2$ | $j_c = 7/2$ |
|-------------|-------------|-------------|
| (1, 1, 0)   | $\sqrt{\frac{3}{7}}$ | $\ast$ |
| (1, 1, 1)   | 0           | 1           |
| (1, 1, 2)   | $-\sqrt{\frac{5}{147}}$ | $-\sqrt{\frac{27}{147}}$ |
| (1, 3, 2)   | $\sqrt{\frac{54}{19}}$ | $-\sqrt{\frac{16}{19}}$ |
| (1, 3, 3)   | 0           | 1           |
| (1, 3, 4)   | $-\sqrt{\frac{8}{117}}$ | $-\sqrt{\frac{55}{117}}$ |
| (1, 5, 4)   | $\sqrt{\frac{220}{147}}$ | $-\sqrt{\frac{32}{147}}$ |
| (1, 5, 5)   | 0           | 1           |
| (1, 5, 6)   | $\ast$      | $-\sqrt{\frac{1}{7}}$ |
FIGURES

FIG. 1. Schematic display of all the possible exchange interactions. With the impurity pseudo-spin $S_I = 1/2$ for $f_1^1J = 5/2\Gamma_7$ doublet, there are in total 6 exchange interactions for $\Gamma_6, \Gamma_7,$ and $\Gamma_8$ conduction electrons. In addition to the well-known one-, and two-channel $S_c = 1/2$ exchange interactions with antiferromagnetic (AFM) coupling, there arise one-channel $S_c = 3/2$ exchange interactions with antiferromagnetic coupling (not known in the literature), and one-channel $S_c = 1/2$ exchange interactions with ferromagnetic (FM) coupling. There are also the mixing exchange interactions through which the conduction electrons can change their CEF symmetry labels after interacting with the impurity pseudo-spin.

FIG. 2. The scaling diagrams up to the third order. The solid lines denote the conduction electron propagator and the dotted lines the impurity spin. The external solid lines are confined to the conduction electrons close to the Fermi level, while the intermediate solid lines denote the conduction electrons virtually excited to the high energy states. At each vertex, the exchange coupling matrices are determined by the specific exchange interaction type. When the multiple conduction electron partial waves are considered, the exchange coupling matrices should be written as the direct product of the channel matrix and the exchange interaction matrix.

FIG. 3. Schematic screening of $S_I = 1/2$ by $S_c = 3/2$. In the first shell screening, the energy is minimized by the compensation of the impurity spin by the conduction electrons of $S_c = -3/2, -1/2$. This picture does not violate the Pauli exclusion principle. After the first shell screening, the effective impurity spin is $S_I = 3/2$. In the second shell, $S_c = -3/2, -1/2$ conduction electrons cannot come closer to the impurity site due to the Pauli exclusion principle. Hence the effective exchange coupling becomes antiferromagnetic. In this case, the effective impurity spin is $S_I = 1/2$. Thus the effective impurity spins alternate between $S_I = 1/2$ and $S_I = 3/2$ with onion shell screening process. This alternating picture will be verified in Chapter V using the numerical renormalization group technique.
FIG. 4. **Flow diagram in the $\Gamma_8$ conduction electron space.** There are three stable fixed points in the $\Gamma_8$ conduction electron space. Only the two-channel fixed point remains stable when the full exchange interactions are included. The other two fixed points derived from $S_c = 3/2$ conduction electrons become unstable against the mixing exchange interactions between the $\Gamma_{6,7}$ and the $\Gamma_8$ conduction electrons. See the text for more details.