Universal deformation rings of group representations, with an application of Brauer's generalized decomposition numbers

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Abstract. We give an introduction to the deformation theory of linear representations of profinite groups which Mazur initiated in the 1980's. We then consider the case of representations of finite groups. We show how Brauer's generalized decomposition numbers can be used in some cases to explicitly determine universal deformation rings.

1. Introduction

Let $p$ be a prime number and let $(K, \mathcal{O}, k)$ be a $p$-modular system, where $\mathcal{O}$ is a complete discrete valuation ring of characteristic 0 with maximal ideal $m_\mathcal{O}$, $K = \text{Frac}(\mathcal{O})$ is its fraction field and $k = \mathcal{O}/m_\mathcal{O}$ is its residue field of characteristic $p$. Let $G$ be a group and let $V$ be a $kG$-module of finite $k$-dimension. It is a classical question to ask whether $V$ can be lifted to an $\mathcal{O}$-free $\mathcal{O}G$-module. Green showed in [33] that this is possible if $\text{Ext}_{kG}^2(V, V) = 0$. However, this is only a sufficient criterion, as there are many cases when $\text{Ext}_{kG}^2(V, V) \neq 0$ and $V$ can still be lifted to an $\mathcal{O}$-free $\mathcal{O}G$-module. Even if we know that there is such a lift of $V$ to $\mathcal{O}$, we may still not be able to characterize all possible lifts to $\mathcal{O}$. On the other hand, if $V$ cannot be lifted to $\mathcal{O}$, it is natural to ask if $V$ can be lifted to other complete local commutative rings with residue field $k$. This can be formulated as the following two natural questions:

(i) How can all possible lifts of $V$ to $\mathcal{O}$ be described?
(ii) Over which complete local commutative rings with residue field $k$ can $V$ be lifted? Is there one particular such complete local ring from which all these lifts arise?

To answer both of these questions, it is necessary to develop a systematic way to study isomorphism classes of lifts, also called deformations, of $kG$-modules $V$. In the 1980’s, Mazur developed a deformation theory of representations of profinite Galois groups over finite fields which can be used to give answers to these questions in certain cases.

2010 Mathematics Subject Classification. Primary 20C20; Secondary 20C15, 16G10.

Key words and phrases. Universal deformation rings; Brauer’s generalized decomposition numbers; tame blocks; dihedral defect groups; semidihedral defect groups; generalized quaternion defect groups.

The author was supported in part by NSA Grant H98230-11-1-0131.
The goal of this paper is to give an introduction to this deformation theory, with an emphasis on the particular case of deformations of representations of finite groups.

In section 2 we will first focus on Mazur’s general deformation theory by studying the deformation functor, universal deformation rings, tangent spaces, obstructions, and deformation rings in number theory. In section 3 we will then concentrate on deformation rings and deformations of representations of finite groups. In section 4 we will describe how Brauer’s generalized decomposition numbers can be used to explicitly determine the universal deformation rings of certain representations belonging to blocks of tame representation type.

We close this introduction by discussing a few classical examples which show that lifts occur naturally both in representation theory and number theory.

Examples 1.1.

(i) Let $G$ be a finite group and consider permutation modules for $kG$. Recall that direct summands of permutation modules are also called $p$-permutation modules or trivial source modules. Scott proved in [47] that $p$-permutation modules can be lifted to $\mathcal{O}$. Rickard used this result, for example, in [43] to show that tilting complexes defining splendid equivalences can be lifted from $k$ to $\mathcal{O}$.

Another class of permutation modules is given by endo-permutation modules for $kG$, i.e. $kG$-modules $V$ for which $\text{End}_k(V)$ is a permutation module. A special subclass is provided by the endo-trivial modules, for which $\text{End}_k(V)$ is isomorphic as a $kG$-module to a direct sum of the trivial $kG$-module and a projective $kG$-module. Alperin proved in [2] that if $G$ is a $p$-group and $V$ is an endo-trivial $kG$-module, then $V$ can be lifted to an endo-trivial $\mathcal{O}G$-module.

(ii) Let $k = \mathbb{F}_p = \mathbb{Z}/p$ and let $\mathcal{O} = \mathbb{Z}_p$ be the ring of $p$-adic integers. Suppose $E$ is an elliptic curve over $\mathbb{Q}$ and define $G = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then $G$ acts on the torsion points $E[p^n] \cong \mathbb{Z}/p^n \times \mathbb{Z}/p^n$ for all $n \geq 1$. In particular, for $n = 1$, we obtain a $kG$-module $V$ of $k$-dimension 2. Since the action of $G$ on $E[p^n]$ commutes with the multiplication by $p$ on $E[p^n]$, $G$ acts naturally on the Tate module $T_p(E) = \lim_{\leftarrow} E[p^n] \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence $T_p(E)$ defines a lift of $V$ over $\mathcal{O} = \mathbb{Z}_p$.

2. Mazur’s deformation theory

In the 1980’s, Mazur developed a deformation theory of Galois representations to systematically study $p$-adic lifts of representations of profinite Galois groups [38]. His initial motivation seems to have come from Hida’s work on ordinary $p$-adic modular forms (see for example Gouvêa’s summary in [32, Lecture 2]). Mazur’s deformation theory became a fundamental tool in number theory when it was shown to play a crucial role in the proof of Fermat’s Last Theorem, and more generally in the proof of the modularity conjecture for elliptic curves over $\mathbb{Q}$ (see [49, 48, 21]). For more information and background, we also refer the reader to the survey articles [31, 32, 39] on Mazur’s deformation theory and the collection of articles in [40, 23] on the proof of Fermat’s Last Theorem by Wiles and Taylor.
2.1. The deformation functor. Let $k$ be a perfect field of positive characteristic $p$, and let $W = W(k)$ be the ring of infinite Witt vectors over $k$. Recall that since $k$ is perfect, $W$ is the unique (up to isomorphism) complete discrete valuation ring of characteristic zero which is absolutely unramified, meaning that the valuation of $p$ is 1 so that $p$ is a generator of the maximal ideal $m_W$. Let $\hat{C}$ denote the category whose objects are all complete local commutative Noetherian rings with residue field $k$ and whose morphisms are local homomorphisms of complete local Noetherian rings that induce the identity on $k$. Strictly speaking, each object of $\hat{C}$ is a pair $(R, \pi_R)$ consisting of a complete local commutative Noetherian ring $R$ and a fixed reduction map $\pi_R : R \to k$ inducing an isomorphism $R/m_R \cong k$. Note that all rings $R$ in $\hat{C}$ have a natural $W$-algebra structure, which means that the morphisms in $\hat{C}$ can also be viewed as continuous $W$-algebra homomorphisms inducing the identity on the residue field $k$.

Let $G$ be a profinite group, and let $V$ be a finite dimensional vector space over $k$ with a continuous $k$-linear action of $G$ on $V$, which is given by a continuous group homomorphism from $G$ to the discrete group $\text{Aut}_k(V)$. A lift of $V$ over a ring $R$ in $\hat{C}$ is defined to be a pair $(M, \phi)$ consisting of a finitely generated free $R$-module $M$ on which $G$ acts continuously together with a $G$-equivariant isomorphism $\phi : k \otimes_R M \to V$ of $k$-vector spaces. We say two lifts $(M, \phi)$ and $(M', \phi')$ of $V$ over $R$ are isomorphic if there exists an $R$-linear $G$-equivariant isomorphism $f : M \to M'$ satisfying $\phi' \circ (k \otimes f) = \phi$. We define the set $\text{Def}_G(V, R)$ of deformations of $V$ over $R$ to be the set of isomorphism classes of lifts of $V$ over $R$. The deformation functor $F_V : \hat{C} \to \text{Sets}$ is defined to be the covariant functor which sends a ring $R$ in $\hat{C}$ to $\text{Def}_G(V, R)$ and a morphism $\alpha : R \to R'$ in $\hat{C}$ to the set map

$$F_V(\alpha) : \text{Def}_G(V, R) \to \text{Def}_G(V, R')$$

$$[M, \phi] \mapsto [R' \otimes_{R, \alpha} M, \phi \circ \alpha]$$

where $\phi \circ \alpha$ is the composition $k \otimes_{R'} (R' \otimes_{R, \alpha} M) \cong k \otimes_R M \stackrel{\phi}{\to} V$.

Sometimes it is useful to describe the deformation functor $F_V$ in terms of matrix groups. Choosing a $k$-basis of $V$, we can identify $V$ with $k^n$ where $n = \dim_k V$. The $G$-action on $V$ is then given by a continuous homomorphism $\overline{\rho} : G \to \text{GL}_n(k)$, which is called a residual representation. Let $R$ be a ring in $\hat{C}$ and denote the reduction map $\text{GL}_n(R) \to \text{GL}_n(k)$ induced by $\pi_R : R \to k$ also by $\pi_R$. By a lift of $\overline{\rho}$ over $R$ we mean a continuous homomorphism $\rho : G \to \text{GL}_n(R)$ such that $\pi_R \circ \rho = \overline{\rho}$. Such a lift defines a $G$-action on $M = \mathbb{R}^n$, and with the obvious isomorphism $\phi : k \otimes_R M \to V$ such a lift defines a deformation $[M, \phi]$ of $V$ over $R$. Two lifts $\rho, \rho' : G \to \text{GL}_n(R)$ of $\overline{\rho}$ over $R$ give rise to the same deformation if and only if they are strictly equivalent, that is, if one can be brought into the other by conjugation by a matrix in the kernel of $\pi_R$. In this way, the choice of a basis of $V$ gives rise to an identification of $\text{Def}_G(V, R)$ with the set $\text{Def}_G(\overline{\rho}, R)$ of strict equivalence classes of lifts of $\overline{\rho}$ over $R$. In particular, the deformation functor $F_{\overline{\rho}} : \hat{C} \to \text{Sets}$ associated to strict equivalence classes of lifts of $\overline{\rho}$ is naturally isomorphic to the deformation functor $F_V$. Note that if $\alpha : R \to R'$ is a morphism in $\hat{C}$, then $F_{\overline{\rho}}(\alpha)$ is the set map $\text{Def}_G(\overline{\rho}, R) \to \text{Def}_G(\overline{\rho}, R')$ which sends $[\rho]$ to $[\alpha \circ \rho]$, where we denote the morphism $\text{GL}_n(R) \to \text{GL}_n(R')$ induced by $\alpha$ also by $\alpha$. In the following, we identify $F_V = F_{\overline{\rho}}$.

2.2. Universal deformation rings. Assume the notation from the previous subsection. The functor $F_V$ is representable if there is a ring $R(G, V)$ in $\hat{C}$ and a
lift \((U(G, V), \phi_U)\) of \(V\) over \(R(G, V)\) such that for all \(R\) in \(\hat{C}\) the map

\[
f_R : \text{Hom}_\mathcal{C}(R(G, V), R) \to \text{Def}_G(V, R)
\]

\[
\alpha \mapsto F_V(\alpha)([U(G, V), \phi_U])
\]

is bijective. Put differently, this is the case if and only if \(F_V\) is naturally isomorphic to the Hom functor \(\text{Hom}_{\hat{\mathcal{C}}}(R(G, V), -)\). If this is the case then \(R(G, V)\) is called the \textit{universal deformation ring} of \(V\) and \([U(G, V), \phi_U]\) is called the \textit{universal deformation} of \(V\). The defining property determines \(R(G, V)\) and \([U(G, V), \phi_U]\) uniquely up to a unique isomorphism.

A slightly weaker notion can be useful if the functor \(F_V\) is not representable. The ring \(k[\epsilon]\) of dual numbers with \(\epsilon^2 = 0\) has a \(W\)-algebra structure such that the maximal ideal \(m_W\) of \(W\) annihilates \(k[\epsilon]\). One says \(R(G, V)\) is a \textit{versal deformation ring} of \(V\) if the maps \(f_R\) are surjective for all \(R\), and bijective for \(R = k[\epsilon]\). These conditions determine \(R(G, V)\) uniquely up to isomorphism, but the isomorphism need not be unique.

\[
\text{Theorem 2.1. \([38 \S 1.2], [25 \text{ Prop. 7.1}]\)} \text{ Suppose } G \text{ satisfies the following } p\text{-finiteness condition:}
\]

\((\Phi_p)\) \text{ For every open subgroup } J \text{ of finite index in } G, \text{ there are only finitely many continuous homomorphisms from } J \text{ to } \mathbb{Z}/p.\]

Then every finite dimensional continuous representation \(V\) of \(G\) over \(k\) has a versal deformation ring. If \(\text{End}_{kG}(V) = k\), then \(V\) has a universal deformation ring.

To prove the existence of versal deformation rings, Mazur verified Schlessinger’s criteria [46] of pro-representability of Artin functors for the deformation functor \(F_V\). He also proved that \(F_V\) is continuous, meaning that for all objects \(R\) in \(\hat{C}\) we have \(F_V(R) = \lim_{\leftarrow} F_V(R/m_R^i)\). Note that Mazur assumed \(k\) to be a finite field in [38]. However, his proofs in [38 \S 1.2] go through in the more general case we are considering.

In the case when \(\text{End}_{kG}(V) = k\), de Smit and Lenstra took a different approach in [25] which proceeds in three main steps: First they let \(G\) be finite and considered the functor which assigns to each \(R\) a certain set of homomorphisms \(G \to \text{GL}_n(R)\). They showed that this functor is representable by defining the corresponding universal ring by generators and relations. Taking projective limits, they obtained a similar result for profinite \(G\). Finally, they concluded the construction by passing to a suitable closed subring, which is either generated by the traces of the elements of \(G\) if \(V\) is absolutely irreducible, or by a larger collection of elements as suggested by Faltings if \(\text{End}_{kG}(V) = k\).

In the case when \(V\) is absolutely irreducible, another approach was given by Rouquier in [44], using pseudo-characters and the results in [41]. The main point in this latter construction is that being a pseudo-character function has a universal solution, providing another description of the universal deformation ring in this case.

\[2.3. \text{Pseudocompact modules.}\] In this subsection, we briefly describe another viewpoint of lifts and deformations using pseudocompact modules. This is useful, for example, when generalizing deformations of group representations to deformations of objects in derived categories (see for example [7, 8]). Pseudocompact
rings, algebras and modules have been studied, for example, in \[29, 30, 22\], which also serve as references for the following statements.

Assume the notation from subsection \[2.1\]. For \( R \in \text{Ob}(\mathcal{C}) \), let \( R[[G]] \) be the completed group algebra of the usual abstract group algebra \( RG \) of \( G \) over \( R \), i.e. \( R[[G]] \) is the projective limit of the usual group algebras \( R[G/U] \) as \( U \) ranges over the open normal subgroups of \( G \) (where we put brackets around \( G/U \) for better readability). Giving a finitely generated free \( R \)-module \( M \) on which \( G \) acts continuously is the same as giving a topological \( R[[G]] \)-module \( M \) which is finitely generated and free as an \( R \)-module.

The completed group algebra \( R[[G]] \) is a so-called pseudocompact ring, i.e. it is a complete Hausdorff topological ring which admits a basis of open neighborhoods of 0 consisting of two-sided ideals \( J \) for which \( R[[G]]/J \) is an Artinian ring. In particular, \( R[[G]] \) is the projective limit of Artinian quotient rings having the discrete topology. Since \( R \) is a commutative pseudocompact ring and \( R[[G]] \) is an \( R \)-algebra and since the open neighborhood basis of 0 can be chosen to consist of two-sided ideals \( J \) for which \( R[[G]]/J \) has finite length as \( R \)-module, \( R[[G]] \) is moreover a pseudocompact \( R \)-algebra. A complete Hausdorff topological \( R[[G]] \)-module \( M \) is said to be a pseudocompact \( R[[G]] \)-module if \( M \) has a basis of open neighborhoods of 0 consisting of submodules \( N \) for which \( M/N \) has finite length. It follows that an \( R[[G]] \)-module is pseudocompact if and only if it is the projective limit of \( R[[G]] \)-modules of finite length having the discrete topology. We denote the category of pseudocompact \( R[[G]] \)-modules by \( \text{PCMod}(R[[G]]) \). Note that \( \text{PCMod}(R[[G]]) \) is an abelian category with exact projective limits.

A pseudocompact \( R[[G]] \)-module \( M \) is said to be topologically free on a set \( X = \{x_i\}_{i \in I} \) if \( M \) is isomorphic to the product of a family \( (R[[G]]_i)_{i \in I} \) where \( R[[G]]_i = R[[G]] \) for all \( i \). In particular, a topologically free pseudocompact \( R[[G]] \)-module on a finite set is the same as a finitely generated abstractly free \( R[[G]] \)-module.

As before, assume \( V \) is a finite dimensional vector space over \( k \) with a continuous \( k \)-linear action of \( G \) on \( V \), which is given by a continuous group homomorphism from \( G \) to the discrete group \( \text{Aut}_k(V) \). Then \( V \) is a pseudocompact \( k[[G]] \)-module. Moreover, any lift of \( V \) over a ring \( R \) in \( \mathcal{C} \) is given by a pseudocompact \( R[[G]] \)-module \( M \) which is finitely generated and abstractly free as an \( R \)-module together with an isomorphism \( \phi : k \otimes_R M \to V \) in \( \text{PCMod}(k[[G]]) \). Note that in principle, we should use the completed tensor product \( \hat{\otimes}_R \) (see \[22\] \S2) rather than the usual tensor product \( \otimes_R \). However, since \( k \) is finitely generated as a pseudocompact \( R \)-module, it follows that the functors \( k \otimes_R - \) and \( k \hat{\otimes}_R - \) are naturally isomorphic.

Every topologically free pseudocompact \( R[[G]] \)-module is a projective object in \( \text{PCMod}(R[[G]]) \), and every pseudocompact \( R[[G]] \)-module is the quotient of a topologically free \( R[[G]] \)-module. Hence \( \text{PCMod}(R[[G]]) \) has enough projective objects. If \( M \) and \( N \) are pseudocompact \( R[[G]] \)-modules, then we define the right derived functors \( \text{Ext}^n_{R[[G]]}(M, N) \) by using a projective resolution of \( M \).

For all positive integers \( n \), we have an isomorphism of \( k \)-vector spaces

\[
\text{H}^n(G, \text{Hom}_k(V, V)) \cong \text{Ext}^n_{k[[G]]}(V, V)
\]

where \( \text{H}^n \) is “continuous” cohomology. Since \( V \) is finite dimensional and discrete, this isomorphism can be deduced by looking at direct limits over all open normal subgroups \( U \) of \( G \) which act trivially on \( V \).
2.4. Tangent space. We continue to assume the notation from subsection 2.1. Moreover, we assume that $G$ satisfies the $p$-finiteness condition ($\Phi_p$) from Theorem 2.1. For simplicity, we also assume that $V$ has a universal deformation ring $R(G, V)$. In other words, the deformation functor $F_V$ is represented by $R(G, V)$ and thus naturally isomorphic to the Hom functor $\text{Hom}_{\mathcal{C}}(R(G, V), -)$.

The tangent space of $F_V$ is defined as

$$t_{F_V} = F_V(k[e]) \cong \text{Hom}_{\mathcal{C}}(R(G, V), k[e])$$

where, as before, $k[e]$ is the ring of dual numbers with $e^2 = 0$. Since in $\mathcal{C}$ there is only one morphism $R(G, V) \to k$, we obtain an isomorphism

$$\text{Hom}_{\mathcal{C}}(R(G, V), k[e]) \cong \text{Hom}_k\left(\frac{m_R(G, V)}{m_R^2(G, V) + pR(G, V)}, k\right)$$

of $k$-vector spaces. For $R$ in $\mathcal{C}$, we call

$$t^*_R = \frac{m_R}{m_R^2 + pR}$$

the Zariski cotangent space of $R$. Hence we obtain an isomorphism of tangent spaces

$$t_{F_V} \cong \text{Hom}_k(t^*_R(G, V), k) = t_{R(G, V)}.$$  

The following result gives a connection of Theorem 2.1. For simplicity, we also assume that $R$ and thus naturally isomorphic to the Hom functor $\text{Hom}_{\mathcal{C}}(R(G, V), -)$.

Proposition 2.2. (38 §1.2], 39 §22]) There is a natural isomorphism of $k$-vector spaces

$$t_{F_V} \cong H^1(G, \text{Hom}_k(V, V)) \cong \text{Ext}^1_{k[[G]]}(V, V).$$

If $r = \text{dim}_k\text{Ext}^1_{k[[G]]}(V, V)$ then $R(G, V)$ is isomorphic to a quotient algebra of the power series algebra $W[[t_1, \ldots, t_r]]$ in $r$ commuting variables and $r$ is minimal with this property.

As in the previous subsection, $\text{Ext}^1_{k[[G]]}(V, V)$ means $\text{Ext}^1$ in the category of pseudocompact $k[[G]]$-modules; see also Equation 2.1. The main idea of the proof is to notice that if $(M, \phi)$ is a lift of $V$ over $k[e]$ then by restricting the scalars from $k[e]$ to $k$ we may view $M$ as a $k$-vector space of dimension $2 \cdot \text{dim}_k V$, with a $k$-linear continuous action of $G$. Identifying the $k[[G]]$-modules $\epsilon M$ and $M/\epsilon M$ with $V$ (using $\phi$), we then see $M$ as an extension of $V$ by $V$ in the category of pseudocompact $k[[G]]$-modules:

$$E : 0 \to V \rightarrowtail M \twoheadrightarrow V \to 0.$$  

Sending the element of $t_{F_V}$ corresponding to the isomorphism class $[M, \phi]$ to the element of $\text{Ext}^1_{k[[G]]}(V, V)$ corresponding to $E$, we obtain a well-defined map

$$s : t_{F_V} \to \text{Ext}^1_{k[[G]]}(V, V)$$

which is a $k$-vector space homomorphism. The inverse map of $s$ is obtained by going backward: Given an extension $E$ as above, we define a $k[e]$-structure on $M$ by letting $\epsilon$ act as the composition $\epsilon \circ \tau$, enabling us to view $M$ as a lift of $V$ over $k[e].$
2.5. Obstructions. We continue to assume the notation from subsection 2.1 that \( G \) satisfies \((\Phi_p)\) and that \( V \) has a universal deformation ring \( R(G, V) \). Let \( \overline{\rho} : G \rightarrow \text{GL}_n(k) \) be a residual representation corresponding to \( V \).

Suppose we have a surjective morphism \( R_1 \rightarrow R_0 \) in \( \hat{\mathcal{C}} \) and assume that the kernel \( I \) satisfies \( I \cdot m_{R_1} = 0 \), so \( I \) has the structure of a \( k \)-vector space. Suppose we have a lift \( \rho_0 : G \rightarrow \text{GL}_n(R_0) \) of \( \overline{\rho} \) over \( R_0 \). We want to describe the obstruction to lifting \( \rho_0 \) to \( R_1 \).

Let \( \gamma_1 : G \rightarrow \text{GL}_n(R_1) \) be a set-theoretic lift of \( \rho_0 \). Since \( \gamma_1 \) is a homomorphism modulo \( I \), we obtain a 2-cocycle

\[
c : \quad G \times G \rightarrow 1 + \text{Mat}_n(I)
\]

\[
(g_1, g_2) \mapsto \gamma_1(g_1g_2)\gamma_1(g_2)^{-1}\gamma_1(g_1)^{-1}
\]

Identifying the multiplicative group \( 1 + \text{Mat}_n(I) \subset \text{GL}_n(R_1) \) with the additive group \( \text{Mat}_n(I)^+ \), we obtain the following isomorphisms:

\[
1 + \text{Mat}_n(I) \cong \text{Mat}_n(I)^+ \cong \text{Mat}_n(k) \otimes_k I \cong \text{Hom}_k(V, V) \otimes_k I.
\]

Here the action of \( G \) on \( \text{Mat}_n(k) \), respectively \( \text{Hom}_k(V, V) \), is given by the usual conjugation action; note that this is also called the adjoint representation of \( \overline{\rho} \) and denoted by \( \text{ad}(\overline{\rho}) \). If we replace \( \gamma_1 \) by a different set-theoretic lift, this changes \( c \) by a 2-coboundary. Therefore we obtain an element

\[
[c] \in H^2(G, \text{Hom}_k(V, V) \otimes_k I) \cong H^2(G, \text{Hom}_k(V, V)) \otimes_k I
\]

which gives the obstruction to lifting \( \rho_0 \) to \( R_1 \). The class \([c]\) is sometimes called the obstruction class of \( \rho_0 \) relative to the morphism \( R_1 \rightarrow R_0 \).

**Proposition 2.3.** \([38, \S 1.6], [9, \text{Thm. 2.4}]\) If \( r = \dim_k \text{Ext}^1_{k[[G]]}(V, V) \)

and \( s = \dim_k \text{Ext}^2_{k[[G]]}(V, V) \) then \( R(G, V) \) is isomorphic to a quotient algebra \( W[[t_1, \ldots, t_r]]/J \) where \( s \) is an upper bound on the minimal number of generators of \( J \).

The main idea of the proof is as follows: Let \( \rho_u : G \rightarrow \text{GL}_n(R(G, V)) \) be a universal lift of \( \overline{\rho} \). Let \( \hat{R} = W[[t_1, \ldots, t_r]] \) and consider the sequence

\[
0 \rightarrow J' = J/(Jm_{\hat{R}}) \rightarrow R' = \hat{R}/(Jm_{\hat{R}}) \rightarrow R(G, V) \rightarrow 0.
\]

Since \( m_{\hat{R}} \) annihilates \( J' \), we have an obstruction \([c] \in H^2(G, \text{Hom}_k(V, V)) \otimes_k J' \) to lifting \( \rho_u \) to \( R' \). This obstruction depends only on the strict equivalence class of \( \rho_u \) and not on the chosen representation \( \rho_u \). One then shows that the \( k \)-linear map

\[
\text{Hom}_k(J', k) \rightarrow H^2(G, \text{Hom}_k(V, V)) \cong \text{Ext}^2_{k[[G]]}(V, V)
\]

\[
f \mapsto (1 \otimes f)([c])
\]

is injective, which implies the proposition.

2.6. Deformation rings in number theory. One of the main uses of deformation rings in number theory has been to establish a relationship between certain Galois representations and automorphic forms. More precisely, let \( \overline{\rho} : G_{\mathbb{Q}} \rightarrow \text{GL}_2(k) \) be an absolutely irreducible representation, where \( k \) is a finite field of characteristic \( p > 2 \) and \( G_{\mathbb{Q}} \) is the absolute Galois group of \( \mathbb{Q} \). Suppose that \( \overline{\rho} \) is modular in the sense that it corresponds to a modular form (modulo \( p \)) which is an eigenfunction of Hecke operators. The idea is to prove that all reasonable lifts of \( \overline{\rho} \) to \( p \)-adic representations are modular by establishing an isomorphism between a
universal deformation ring, which parameterizes lifts of $\overline{\rho}$ with bounded ramification and satisfying appropriate deformation conditions, and a Hecke algebra, which parameterizes certain lifts of $\overline{\rho}$ which are modular of some fixed level.

Taylor and Wiles established such an isomorphism in [48], which then led to the proof of Fermat’s Last Theorem in [49]. The Taylor-Wiles method has been further refined by many people, such as Diamond [26] and Fujiwara [28]. In [35], Khare gave an alternative approach for semistable $\overline{\rho}$ by establishing an isomorphism $R_Q \cong T_Q$, where $Q$ is a so-called auxiliary set of primes, $R_Q$ is the universal deformation ring for lifts of $\overline{\rho}$ minimally ramified away from $Q$ and satisfying appropriate deformation conditions and $T_Q$ is the analogous Hecke algebra.

Apart from the proof of Fermat’s Last Theorem in [48, 49] and the proof of the general Shimura-Taniyama-Weil conjecture in [21], deformation rings also played an important role in the proof of Serre’s modularity conjecture [45] by Khare-Wintenberger [36] and Kisin [37], which asserts that every absolutely irreducible representation $\rho: G \rightarrow \text{GL}_2(k)$ with odd determinant is modular.

Suppose $k$ is a finite field of characteristic $p > 2$, $K$ is a number field, $S$ is a finite set of primes of $K$ containing the primes over $p$ and the infinite ones, and $\overline{\rho}: G_K \rightarrow \text{GL}_2(k)$ is an absolutely irreducible representation unramified outside $S$. It is often desirable to have an explicit presentation of a universal deformation ring $R$, which parameterizes lifts of $\overline{\rho}$ satisfying certain deformation conditions, in terms of a power series algebra over $W = W(k)$ modulo an ideal given by a (minimal) number of generators. In the following, we describe several results of Böckle in this respect, some of which also played an important role in [36].

Since the relations occurring in the universal deformation ring $R$ often come from the obstructions of the associated local deformation problems $\overline{\rho}_p : G_{K_p} \rightarrow \text{GL}_2(k)$ for $p \in S$, Böckle considered in [10] the problem of finding the universal deformation, or a smooth cover of it, in the local case where the relevant pro-$p$ group is an arbitrary Demuškin group. He showed that the corresponding universal deformation ring is a complete intersection, flat over $W$, and with the (minimal) number of generators given by the $k$-dimension of $H^2(G_{K_p}, \text{ad}\overline{\rho})$. Moreover, he applied his local results to the global situation. For example, he gave conditions under which the universal deformation ring of an odd, absolutely irreducible representation $G_Q \rightarrow \text{GL}_2(k)$, unramified outside $S$, can be described explicitly, thus generalizing a result of Boston [13].

In [9, 11], Böckle studied in more detail the connection between local and global deformation functors. In [9], he presented a rather general class of (global) deformation functors of $\overline{\rho}$ that satisfy local deformation conditions and investigated for those, under what conditions the global deformation functor is determined by the local deformation functors corresponding to primes $p \in S$. Böckle gave precise conditions under which the local functors are sufficient to describe the global functor. These conditions involve the vanishing of a second Shafarevich-Tate group and auxiliary primes as introduced by Taylor and Wiles in [48]. In [11], Böckle provided generalizations and simplified proofs for some of the results in [9].

3. Universal deformation rings of modules for finite groups

Assume the notation from subsection [2.1] If $\text{End}_{kG}(V) = k$, the construction of the universal deformation ring $R(G, V)$ by de Smit and Lenstra in [25] shows that $R(G, V)$ is the inverse limit of the universal deformation rings $R(H, V)$ when
$H$ ranges over all finite discrete quotients of $G$ through which the $G$-action on $V$ factors. Thus to answer questions about the ring structure of $R(G, V)$, it is natural to first consider the case when $G = H$ is finite.

For the remainder of this paper, we assume that $G$ is finite. The representation theory of $kG$ when $p$ divides $\#G$ is very beautiful but difficult. To avoid rationality questions and to simplify notation, we assume that $k$ is algebraically closed. More precisely, we make the following assumptions.

**Hypothesis 3.1.** Let $k$ be an algebraically closed field of positive characteristic $p > 0$, let $W = W(k)$ be the ring of infinite Witt vectors over $k$, let $G$ be a finite group, and let $V$ be a finitely generated $kG$-module.

It follows as before that $V$ has a universal deformation ring if its endomorphism ring $\text{End}_{kG}(V)$ is isomorphic to $k$. Note that $\text{End}_{kG}(V) \cong H^0(G, \text{Hom}_k(V, V))$. When $G$ is finite, Tate cohomology groups often play an important role. Therefore, the question arises if we can use the 0-th Tate cohomology group,

$$\hat{H}^0(G, \text{Hom}_k(V, V)) = \text{End}_{kG}(V)/(s_G \cdot \text{End}_k(V))$$

where $s_G = \sum_{g \in G} g$, to obtain a criterion for the existence of a universal deformation ring of $V$. Let $\text{PEnd}_{kG}(V)$ denote the ideal of $\text{End}_{kG}(V)$ consisting of all $kG$-module endomorphisms of $V$ factoring through a projective $kG$-module. Then $\text{PEnd}_{kG}(V)$ is equal to $s_G \cdot \text{End}_k(V)$, which implies that $\hat{H}^0(G, \text{Hom}_k(V, V)) = \text{End}_{kG}(V)/\text{PEnd}_{kG}(V)$. The quotient ring $\text{End}_{kG}(V)/\text{PEnd}_{kG}(V)$ is called the **stable endomorphism ring** of $V$ and is denoted by $\hat{\text{End}}_{kG}(V)$. Note that in general $\text{PEnd}_{kG}(V) \neq 0$, i.e. $\text{End}_{kG}(V)$ properly surjects onto $\hat{\text{End}}_{kG}(V)$. We have the following result:

**Proposition 3.2.** ($\square$ Prop. 2.1, $\square$ Rem. 2.1) Assume Hypothesis 3.1 and that the stable endomorphism ring $\hat{\text{End}}_{kG}(V)$ is isomorphic to $k$. Then $V$ has a universal deformation ring. Moreover, if $R$ is in $\hat{\mathcal{C}}$ and $(M, \phi)$ and $(M', \phi')$ are lifts of $V$ over $R$ such that $M$ and $M'$ are isomorphic as $RG$-modules then $[M, \phi] = [M', \phi']$.

The main point of the proof of this proposition is to show that if $(M, \phi)$ is a lift of $V$ over an Artinian object $R$ in $\hat{\mathcal{C}}$, then the ring homomorphism $R \to \hat{\text{End}}_{kG}(M)$ coming from the action of $R$ on $M$ via scalar multiplication is surjective.

**Remark 3.3.** The last statement of Proposition 3.2 means that the particular $kG$-module isomorphism $k \otimes_R M \xrightarrow{\phi} V$ from a lift $(M, \phi)$ is not important when $\hat{\text{End}}_{kG}(V) \cong k$, which significantly simplifies computations.

Note that this is not true in general, as can be seen in the following example. Let $G = \langle \sigma \rangle$ be a cyclic group of order $p$ and let $V = k \oplus k$ with trivial $G$-action. Let $M$ be the $k[\epsilon]G$-module with $M = k[\epsilon] \oplus k[\epsilon]$ and $\sigma$ acting as $\begin{bmatrix} 1 & \epsilon \\ 0 & 1 \end{bmatrix}$. Consider $\phi, \phi' : k \otimes_{k[\epsilon]} M \to V$ with $\phi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\phi' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $(M, \phi)$ and $(M, \phi')$ are two non-isomorphic lifts of $V$ over $k[\epsilon]$.

The following result analyzes $R(G, V)$ further in the case when $\hat{\text{End}}_{kG}(V) \cong k$. Here $\Omega$ denotes the syzygy functor or Heller operator, i.e. if $\pi : P_V \to V$ is a projective $kG$-module cover of $V$ then $\Omega(V)$ denotes the kernel of $\pi$ (see, for example, $\square$ §20).
Proposition 3.4. ([6] Cors. 2.5 and 2.8]). Assume Hypothesis 3.1 and that the stable endomorphism ring \( \text{End}_{kG}(V) \) is isomorphic to \( k \).

(i) Then \( \text{End}_{kG}(\Omega(V)) \cong k \), and \( R(G, V) \) and \( R(G, \Omega(V)) \) are isomorphic.

(ii) There is a non-projective indecomposable \( kG \)-module \( V_0 \) (unique up to isomorphism) such that \( \text{End}_{kG}(V_0) \cong k \), \( V \) is isomorphic to \( V_0 \oplus Q \) for some projective \( kG \)-module \( Q \), and \( R(G, V) \) and \( R(G, V_0) \) are isomorphic.

The main ideas of the proof are as follows: Since the deformation functor \( F_V \) is continuous, most of the arguments can be carried out for the restriction of \( F_V \) to the full subcategory \( C \) of \( \hat{C} \) of Artinian objects. For part (i), one shows that the syzygy functor \( \Omega \) induces an isomorphism between the restrictions of the functors \( F_V \) and \( F_{\Omega(V)} \) to \( C \). For part (ii), one uses that the projective \( kG \)-module \( Q \) can be lifted to a projective \( RG \)-module \( Q_R \) for every \( R \) in \( C \) to show that there is an isomorphism between the restrictions of the functors \( F_V \) and \( F_{V_0} \) to \( C \).

Recall that \( kG \) can be written as a finite direct product of blocks

\[
kG = B_1 \times \cdots \times B_r
\]

where the blocks \( B_1, \ldots, B_r \) are in one-to-one correspondence with the primitive central idempotents of \( kG \). (For a good introduction to block theory, we refer the reader to [1] Chap. IV.) If \( B \) is a block of \( kG \), there is associated to it a conjugacy class of \( p \)-subgroups of \( G \), called the defect groups of \( B \). The defect groups measure how far \( B \) is away from being a full matrix ring; they also determine the representation type of \( B \). More precisely, \( B \) has finite representation type if and only if its defect groups are cyclic; \( B \) has infinite tame representation type if and only if \( p = 2 \) and the defect groups of \( B \) are dihedral, semi-dihedral or generalized quaternion; and \( B \) has wild representation type in all other cases. (This result follows from [34, 20, 12]. A description of this result together with an introduction to the representation type can also be found in [27] Intro. and Sect. I.4.)

Proposition 3.4(ii) says that if the stable endomorphism ring of \( V \) is isomorphic to \( k \) and we want to determine the universal deformation ring \( R(G, V) \) then we may assume that \( V \) is non-projective indecomposable. But then \( V \) belongs to a unique block of \( kG \), and we can use the theory of blocks, as introduced by Brauer and developed by many other authors, to determine the universal deformation ring \( R(G, V) \).

4. Brauer’s generalized decomposition numbers and universal deformation rings

We continue to assume Hypothesis 3.1. Our goal in this section is to show how Brauer’s generalized decomposition numbers can be used in certain cases to determine the isomorphism type of the universal deformation ring \( R(G, V) \). We first give a brief introduction to these generalized decomposition numbers.

4.1. Brauer’s generalized decomposition numbers. The usual decomposition numbers were introduced by Brauer and Nesbitt in [18] (see also [19]). They allow us to express the values of the ordinary irreducible characters of \( G \) on \( p \)-regular elements of \( G \), i.e. elements of order prime to \( p \), by means of the absolutely irreducible \( p \)-modular characters of \( G \). More precisely, if \( \zeta_1, \zeta_2, \ldots \) are the ordinary
irreducible characters of $G$ and $\varphi_1, \varphi_2, \ldots$ are the absolutely irreducible $p$-modular characters of $G$, then we have a formula
\[
\zeta_\mu(g) = \sum_\nu d_{\mu\nu} \varphi_\nu(g)
\]
provided $g$ is a $p$-regular element of $G$. The $d_{\mu\nu}$ are non-negative integers, called the decomposition numbers of $G$ for $p$. As Brauer wrote in [14, p. 192]:

“We may say that the group characters $\zeta_\mu$ of $G$ are built up by the modular characters $\varphi_\nu$, and it is possible to obtain a deeper insight into the nature of the ordinary group characters by the use of the modular characters and their properties. However, it is disturbing that we have to restrict ourselves to $p$-regular elements.”

For this reason, Brauer introduced generalized decomposition numbers in [14]. The value $\zeta_\mu(g)$ on an element $g \in G$ whose order is divisible by $p$ is then expressed by means of the absolutely irreducible $p$-modular characters of certain subgroups $C_i$ of $G$. The corresponding generalized decomposition numbers $d_{\mu\nu}$ are not necessarily rational integers, but they are algebraic integers in a cyclotomic field of $p$-power order roots of unity. More precisely, Brauer defined $d_{\mu\nu}$ as follows.

Suppose $\#G = p^a m'$ where $m'$ is relatively prime to $p$, and let $P$ be a fixed Sylow $p$-subgroup of $G$. Let $u_0 = 1, u_1, u_2, \ldots, u_h$ be a complete system of representatives of $G$-conjugacy classes of $p$-power order elements in $G$ with $u_i \in P$ for all $1 \leq i \leq h$. Every conjugacy class of $G$ contains an element of the form $u_v$ where $v \in \{0, 1, \ldots, h\}$ is uniquely determined by the class and $v$ is a $p$-regular element in the centralizer $C_G(u_i)$. For each $0 \leq i \leq h$, let $v_{i,1}, \ldots, v_{i,\ell_i}$ be a complete system of representatives of $C_G(u_i)$-conjugacy classes of $p$-regular elements in $C_G(u_i)$ with $v_{i,1} = 1$. Then $\{u_i v_{i,j} \mid 0 \leq i \leq h, 1 \leq j \leq \ell_i\}$ is a complete set of representatives of the conjugacy classes of $G$. Moreover, for each $0 \leq i \leq h$, there are precisely $\ell_i$ absolutely irreducible $p$-modular characters of $C_G(u_i)$, which we denote by $\varphi_{i,1}, \ldots, \varphi_{i,\ell_i}$. As before, let $\zeta_1, \zeta_2, \ldots$ be the ordinary irreducible characters of $G$. Then
\[
\zeta_\mu(u_i v_{i,j}) = \sum_{\nu=1}^{\ell_i} d_{\mu\nu}^i \varphi_\nu^i(v_{i,j})
\]
for all $0 \leq i \leq h$, $1 \leq j \leq \ell_i$. The $d_{\mu\nu}^i$ are called the generalized decomposition numbers of $G$. For $i = 0$, we have $u_0 = 1$ and $C_G(u_0) = G$, and the $d_{\mu\nu}$ coincide with the usual decomposition numbers $d_{\mu\nu}$ of $G$ in Equation (4.1). In general, $d_{\mu\nu}^i$ is an algebraic integer in the field of the $p^{\alpha_{\mu\nu}}$-th roots of unity where $p^{\alpha_{\mu\nu}}$ is the order of $u_\mu$. In particular, $d_{\mu\nu}^0$ can be viewed to belong to $W[\omega_i]$ if $\omega_i$ is a primitive $p^{\alpha_{\mu\nu}}$-th root of unity. In [15, Sect. 6], Brauer moreover showed that if $\zeta_\mu$ belongs to the block $B$ of $kG$, then the generalized decomposition number $d_{\mu\nu}^i$ vanishes if $\varphi_\nu^i$ belongs to a block of $kC_G(u_i)$ whose Brauer correspondent in $G$ is not equal to $B$.

4.2. Universal deformation rings of certain modules belonging to infinite tame blocks. We now focus on a certain class of modules belonging to blocks of infinite tame representation type for which Brauer’s generalized decomposition numbers can be used to determine their universal deformation rings. This subsection is based on the paper [5], and details can be found there. Recall from section 3 that a block $B$ of $kG$ has infinite tame representation type if and only if $p = 2$ and the defect groups of $B$ are either dihedral, or semi-dihedral, or generalized quaternion.
In [16, 17, 42], Brauer and Olsson determined the generalized decomposition numbers for all the ordinary irreducible characters belonging to infinite tame blocks. Moreover, they proved that an infinite tame block has at most three isomorphism classes of simple modules. In [27], Erdmann classified all infinite tame blocks up to Morita equivalence by providing a list of quivers and relations for their basic algebras.

We make the following assumptions.

**Hypothesis 4.1.** Assume Hypothesis 3.1 Additionally, assume that \( p = 2 \), \( V \) is indecomposable with \( \text{End}_{kG}(V) \cong k \), and that \( V \) belongs to a non-local block \( B \) of \( kG \) of infinite tame representation type with a defect group \( D \) of order \( 2^n \). Let \( F \) be the fraction field of \( k \G \), and let \( \overline{F} \) be a fixed algebraic closure of \( F \).

We want to concentrate on those \( V \) for which Brauer’s generalized decomposition numbers carry the most information. More precisely, we call a module \( V \) as in Hypothesis 4.1 *maximally ordinary* if the 2-modular character of \( V \) is the restriction to the 2-regular conjugacy classes of an ordinary irreducible character \( \chi \) such that for every \( \sigma \in D \) of maximal 2-power order, Brauer’s generalized decomposition numbers corresponding to \( \sigma \) and \( \chi \) do not all lie in \( \{0, \pm 1\} \). In other words, using the notation of Equation (4.2), if \( \chi = \zeta_\mu \) and \( \sigma \) is conjugate in \( G \) to \( u \), then there exists an absolutely irreducible 2-modular character \( \phi'_\nu \) of \( C_G(u_i) \) such that \( d^\mu_{\nu} \notin \{0, \pm 1\} \).

By [17, 42], there are precisely \( 2^{n-2} - 1 \) ordinary irreducible characters of height 1 belonging to \( B \) if \( n \geq 4 \). Moreover, they all define the same 2-modular character when they are restricted to the 2-regular conjugacy classes. If \( n = 3 \), then there are either 1 or 3 ordinary irreducible characters of height 1 belonging to \( B \), depending on whether \( D \) is dihedral or quaternion. If \( n = 2 \), then there are no ordinary irreducible characters of height 1 belonging to \( B \). Recall that the height of an ordinary irreducible character \( \chi \) belonging to \( B \) is \( b - a + n \), where \( 2^a \) (resp. \( 2^b \)) is the maximal 2-power dividing \( \#G \) (resp. \( \text{deg}(\chi) \)). Since \( n \) is the defect of the block \( B \), it follows that \( b - a + n \) is a non-negative integer (see, for example, 24, Sect. 56.E and Cor. (57.19)).

Suppose \( n \geq 4 \). By [17, 42], exactly one of the \( 2^{n-2} - 1 \) ordinary irreducible characters of height 1 belonging to \( B \) is realizable over \( F \), i.e., it corresponds to an absolutely irreducible \( FG \)-module. Moreover, the remaining \( 2^{n-2} - 2 \) characters of height 1 are precisely the ordinary irreducible characters belonging to \( B \) for which the generalized decomposition numbers corresponding to maximal 2-power order elements in \( D \) do not all lie in \( \{0, \pm 1\} \).

We have the following result:

**Theorem 4.2.** ([5 Thm. 1.1 and Cor. 6.2]). Assume Hypothesis 4.1 Then \( V \) is maximally ordinary if and only if \( n \geq 4 \) and the 2-modular character of \( V \) is equal to the restriction to the 2-regular conjugacy classes of an ordinary irreducible character of \( G \) of height 1. Suppose \( V \) is maximally ordinary. There exists a monic polynomial \( q_n(t) \in W[t] \) of degree \( 2^{n-2} - 1 \) which depends on \( D \) but not on \( V \) and which can be given explicitly such that either

\[
\begin{align*}
&\text{(i) } R(G, V)/2R(G, V) \cong k[[t]]/(t^{2^{n-2}} - 1), \text{ in which case } R(G, V) \text{ is isomorphic to } W[[t]]/(q_n(t)), \\
&\text{(ii) } R(G, V)/2R(G, V) \cong k[[t]]/(t^{2^{n-2}}), \text{ in which case } R(G, V) \text{ is isomorphic to } W[[t]]/(t q_n(t), 2 q_n(t)).
\end{align*}
\]
In all cases, the ring $R(G, V)$ is isomorphic to a subquotient ring of $WD$, and it is a complete intersection if and only if we are in case (i).

A precise description of the maximally ordinary modules $V$ belonging to $B$ is given in [5] Lemma 6.1 and Cor. 6.2. A formula for the polynomials $q_n(t)$ can be found in [5] Def. 5.3 and Rem. 5.4.

We now discuss the main ideas of the proof of Theorem 4.2. For details we refer the reader to [5]. The first statement of the theorem follows from the results in [16, 17, 42]. Suppose now that $n \geq 4$. As noted above, there are then precisely $2^{n-2}-1$ ordinary irreducible characters of height 1 belonging to $B$. Moreover, these characters fall into $n-2$ Galois orbits under the action of $\text{Gal}(\overline{F}/F)$:

$$\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_{n-3}$$

where $\# \mathcal{F}_j = 2^j$ for $0 \leq j \leq n-3$. If $\xi$ is the ordinary character which is the sum of all the characters of height 1, then $\xi$ can be realized by an $FG$-module

$$X = X_0 \oplus X_1 \oplus \cdots \oplus X_{n-3}$$

where each $X_j$ is a simple $FG$-module with Schur index 1 corresponding to the orbit $O_j$.

The main steps to prove Theorem 4.2 are as follows: Suppose $V$ is maximally ordinary. First, we use Erdmann’s description of the basic algebra of $B$ to show that $R(G, V)$ is a quotient algebra of $W[[t]]$ and that $V$ can be lifted to $k[[t]]/((t^{n-2} - 1)$. Moreover, we show that this lift is given by an indecomposable $B$-module $U'$ of $V$ such that $\text{End}_{kG}(U') \cong k[[t]]/((t^{n-2} - 1)$. Next, we use the usual decomposition numbers, together with the description of the projective indecomposable $B$-modules and [24 Prop. (23.7)] and [8, Lemma 2.3.2] to show that $U'$ can be lifted to $W$. Moreover, we show that this lift is given by an indecomposable $WG$-module $U'$ which is free over $W$ with $F \otimes_W U' \cong X$. Then we use Brauer’s generalized decomposition numbers to show that $\text{End}_{WG}(U') \cong W[[t]]/(q_n(t))$ and that $U'$ is free as a module for $\text{End}_{WG}(U')$. This then implies that $W[[t]]/(q_n(t))$ is a quotient ring of the universal deformation ring $R(G, V)$.

To complete the proof of Theorem 4.2 we use again Erdmann’s description of the basic algebra of $B$ to determine the universal mod 2 deformation ring $R(G, V)/2R(G, V)$. It follows that the isomorphism type of $R(G, V)/2R(G, V)$ depends on whether or not the stable Auslander-Reiten quiver $\Gamma_s(B)$ of $B$ contains 3-tubes. Note that if $D$ is dihedral then $\Gamma_s(B)$ always contains 3-tubes, whereas if $D$ is generalized quaternion then $\Gamma_s(B)$ never contains 3-tubes, and if $D$ is semi-dihedral then $\Gamma_s(B)$ may or may not contain 3-tubes. We show that $R(G, V)/2R(G, V) \cong k[[t]]/((t^{n-2} - 1)$ if $\Gamma_s(B)$ contains no 3-tubes, and that $R(G, V)/2R(G, V) \cong k[[t]]/((t^{n-2} - 2)$ if $\Gamma_s(B)$ does contain 3-tubes. In the first case, the universal deformation ring of $V$ is $R(G, V) \cong W[[t]]/(q_n(t))$, whereas in the second case we use [8, Lemma 2.3.3] to show that $R(G, V) \cong W[[t]]/(t q_n(t), 2 q_n(t))$.

References

1. J. L. Alperin, Local representation theory. Modular representations as an introduction to the local representation theory of finite groups. Cambridge Studies in Advanced Mathematics, vol. 11, Cambridge University Press, Cambridge, 1986.
2. J. L. Alperin, Lifting endo-trivial modules. J. Group Theory 4 (2001), 1–2.
3. F. M. Bleher, Universal deformation rings and dihedral defect groups. Trans. Amer. Math. Soc. 361 (2009), 3661–3705.
4. F. M. Bleher, Universal deformation rings and generalized quaternion defect groups. Adv. Math. 225 (2010), 1499–1522.
5. F. M. Bleher, Brauer’s generalized decomposition numbers and universal deformation rings. In press, Trans. Amer. Math. Soc., 2013, [arXiv:1204.0071]
6. F. M. Bleher and T. Chinburg, Universal deformation rings and cyclic blocks. Math. Ann. 318 (2000), 805–836.
7. F. M. Bleher and T. Chinburg, Deformations and derived categories. Ann. Institut Fourier (Grenoble) 55 (2005), 2285–2359.
8. F. M. Bleher and T. Chinburg, Obstructions for deformations of complexes. In press, Ann. Inst. Fourier (Grenoble), 2013, [arXiv:0901.0101]
9. G. Böckle, A local-to-global principle for deformations of Galois representations. J. reine angew. Math. 599 (1999), 199–236.
10. G. Böckle, Demuškin groups with group actions and applications to deformations of Galois representations. Compositio Math. 121 (2000), 109–154.
11. G. Böckle, Presentations of universal deformation rings. In: L-functions and Galois representations, London Math. Soc. Lecture Note Ser., 320, Cambridge Univ. Press, Cambridge, 2007, pp. 24–58.
12. V. M. Bondarenko and J. A. Drozd, The representation type of finite groups. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 71 (1977), 24–41. English translation: J. Soviet Math. 20 (1982), 2515–2528.
13. N. Boston, Explicit deformation of Galois representations. Invent. Math. 103 (1991), 181–196.
14. R. Brauer, On the Connection Between the Ordinary and The Modular Characters of Groups of Finite Order. Ann. Math. (2) 42 (1941), 926–935.
15. R. Brauer, Zur Darstellungstheorie der Gruppen endlicher Ordnung. II. Math. Zeitschr. 72 (1939), 25–46.
16. R. Brauer, Some applications of the theory of blocks of characters of finite groups. IV. J. Algebra 17 (1971), 489–521.
17. R. Brauer, On 2-blocks with dihedral defect groups. Symposia Mathematica, vol. XIII (Convegno di Gruppi e loro Rappresentazioni, INDAM, Rome, 1972), pp. 367–393, Academic Press, London, 1974.
18. R. Brauer and C. Nesbitt, On the modular representations of groups of finite order. University of Toronto Studies, Math. Series No. 4, 1937.
19. R. Brauer and C. Nesbitt, On the modular characters of groups. Ann. of Math. (2) 42 (1941), 556–590.
20. S. Brenner, Modular representations of $p$ groups. J. Algebra 15 (1970) 89–102.
21. C. Breuil, B. Conrad, F. Diamond and R. Taylor, On the modularity of elliptic curves over $\mathbb{Q}$: Wild 3-adic exercises. J. Amer. Math. Soc. 14 (2001), 843–939.
22. A. Brumer, Pseudomodular algebras, profinite groups and class formations. J. Algebra 4 (1966), 442–470.
23. G. Cornell, J. H. Silverman and G. Stevens (eds.), Modular Forms and Fermat’s Last Theorem (Boston, 1995). Springer-Verlag, Berlin-Heidelberg-New York, 1997.
24. C. W. Curtis and I. Reiner, Methods of representation theory. Vols. I and II. With applications to finite groups and orders. John Wiley & Sons, Inc., New York, 1981 and 1987.
25. B. de Smit and H. W. Lenstra, Explicit construction of universal deformation rings. In: Modular Forms and Fermat’s Last Theorem (Boston, 1995), Springer-Verlag, Berlin-Heidelberg-New York, 1997, pp. 313–326.
26. F. Diamond. The Taylor-Wiles construction and multiplicity one. Invent. Math., 128, (1997), 379–391.
27. K. Erdmann, Blocks of Tame Representation Type and Related Algebras. Lecture Notes in Mathematics, vol. 1428, Springer-Verlag, Berlin-Heidelberg-New York, 1990.
28. K. Fujiwara, Deformation rings and Hecke algebras in the totally real case. Preprint, 2006, [arXiv:math/0602606v2]
29. P. Gabriel, Des catégories abéliennes. Bull. Soc. math. France 90 (1962), 323–448.
30. P. Gabriel, Étude infinitesimale des schémas en groupes. In: A. Grothendieck, SGA 3 (with M. Demazure), Schémas en groupes I, II, III, Lecture Notes in Math. 151, Springer Verlag, Heidelberg, 1970, pp. 476–562.
31. F. Q. Gouvêa, Deforming Galois representations: a survey. In: Seminar on Fermat’s Last Theorem (Toronto, ON, 1993–1994), CMS Conf. Proc., 17, Amer. Math. Soc., Providence, RI, 1995, pp. 179–207.
32. F. Q. Gouvêa, Deformations of Galois representations. Appendix 1 by Mark Dickinson, Appendix 2 by Tom Weston and Appendix 3 by Matthew Emerton. In: IAS/Park City Math. Ser., 9, Arithmetic algebraic geometry (Park City, UT, 1999), Amer. Math. Soc., Providence, RI, 2001, pp. 233–406.
33. J. A. Green, A lifting theorem for modular representations. Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, Vol. 252, No. 1268 (Aug. 25, 1959), pp. 135-142.
34. D. Higman, Indecomposable representations at characteristic $p$. Duke Math. J. 21 (1954), 377–381.
35. C. Khare, On isomorphisms between deformation rings and Hecke rings. With an appendix by Gebhard Böckle. Invent. Math. 154 (2003), 199–222.
36. C. Khare and J.-P. Wintenberger, Serre’s modularity conjecture. I and II. Invent. Math. 178 (2009), 485–504, 505–586.
37. M. Kisin, Modularity of 2-adic Barsotti-Tate representations. Invent. Math. 178 (2009), 587–634.
38. B. Mazur, Deforming Galois representations. In: Galois groups over $\mathbb{Q}$ (Berkeley, 1987), Springer-Verlag, Berlin-Heidelberg-New York, 1989, pp. 385–437.
39. B. Mazur, Deformation theory of Galois representations. In: Modular Forms and Fermat’s Last Theorem (Boston, MA, 1995), Springer Verlag, Berlin-Heidelberg-New York, 1997, pp. 243–311.
40. V. Kumar Murty (ed.), Seminar on Fermat’s Last Theorem. Papers from the seminar held at the Fields Institute for Research in Mathematical Sciences, Toronto, Ontario, 1993–1994. CMS Conference Proceedings, 17. Published by the American Mathematical Society, Providence, RI; for the Canadian Mathematical Society, Ottawa, ON, 1995.
41. L. Nyssen, Pseudo-représentations. Math. Ann. 306 (1996), 257–283.
42. J. B. Olsson, On 2-blocks with quaternion and quasidihedral defect groups. J. Algebra 36 (1975), 212–241.
43. J. Rickard, Splendid equivalences: Derived categories and permutation modules. Proc. London Math. Soc. (3) 72 (1996) 331–358.
44. R. Rouquier, Caractérisation des caractères et pseudo-caractères. J. Algebra 180 (1996), 571–586.
45. J.-P. Serre, Sur les représentations modulaires de degré 2 de $\text{Gal} (\overline{\mathbb{Q}}/\mathbb{Q})$. Duke Math. J. 54 (1987), 179–230.
46. M. Schlessinger, Functors of Artin Rings. Trans. Amer. Math. Soc. 130 (1968) 208–222.
47. L. L. Scott, Modular permutation representations. Trans. Amer. Math. Soc. 175 (1973), 101–121.
48. R. Taylor and A. Wiles, Ring-theoretic properties of certain Hecke algebras. Ann. of Math. 141 (1995), 553–572.
49. A. Wiles, Modular elliptic curves and Fermat’s last theorem. Ann. of Math. 141 (1995), 443–551.

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