Symplectic forms on the space of embedded symplectic surfaces and their reductions

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Abstract

Let \((M, \omega)\) be a symplectic manifold, and \((\Sigma, \sigma)\) a closed connected symplectic 2-manifold. We construct a weakly symplectic form \(\omega^D(\Sigma, \sigma)\) on the space of immersions \(\Sigma \to M\) that is a special case of Donaldson’s form. We show that the restriction of \(\omega^D(\Sigma, \sigma)\) to any orbit of the group of Hamiltonian symplectomorphisms through a symplectic embedding \((\Sigma, \sigma) \hookrightarrow (M, \omega)\) descends to a weakly symplectic form \(\omega^D_{\text{red}}\) on the quotient by \(\text{Sympl}(\Sigma, \sigma)\), and that the obtained symplectic space is a symplectic quotient of the subspace of symplectic embeddings \(\mathcal{S}_e(\Sigma, \sigma)\) with respect to the \(\text{Sympl}(\Sigma, \sigma)\)-action. We also compare \(\omega^D(\Sigma, \sigma)\) and its reduction \(\omega^D_{\text{red}}\) to another 2-form on the space of immersed symplectic \(\Sigma\)-surfaces in \(M\). We conclude by a result on the restriction of \(\omega^D(\Sigma, \sigma)\) to moduli spaces of \(J\)-holomorphic curves.

1 Introduction

Let \((M, \omega)\) be a compact finite-dimensional symplectic manifold, and \(\Sigma\) a closed connected 2-manifold. Fix a symplectic form \(\sigma\) on \(\Sigma\). We identify the tangent space to \(C^\infty(\Sigma, M)\) at \(f: \Sigma \to M\) with the space \(\Omega^0(\Sigma, f^*(TM))\) of smooth vector fields \(\tau: \Sigma \to f^*(TM)\).

**Definition 1.1** Define a 2-form on \(C^\infty(\Sigma, M)\) by

\[
(\omega^D(\Sigma, \sigma))_f(\tau_1, \tau_2) := \int_\Sigma \omega_{f(x)}(\tau_1(x), \tau_2(x)) \sigma,
\]

where \(\tau_1, \tau_2 \in T_f(C^\infty(\Sigma, M))\).

The form \(\omega^D(\Sigma, \sigma)\) is a special case of the two-form on the space of smooth maps \(S \to M\) of a compact oriented manifold \(S\) equipped with a fixed volume form \(\eta\), introduced by Donaldson in \([3]\). Under some topological conditions, e.g., that \(H^1(S) = 0\) and for all \(i \in C^\infty(\Sigma, M)\) the class \(i^* [\omega]\) is the zero class in \(H^2(S)\), Donaldson described a moment map for the action of the Lie group of volume preserving diffeomorphisms \(\text{Diff}(S, \eta)\) on \(C^\infty(S, M)\). This action restricts to a Hamiltonian action on the subspace of embeddings \(\text{Emb}(S, M)\). In \([6]\), Brian Lee gives a rigorous formulation of Donaldson’s heuristic construction, in the “Convenient Setup” of Frölicher, Kriegl, and Michor \([5]\), and shows that the form reduces to the image of \(\text{Ham}(M, \omega)\)-orbits through isotropic embeddings in \(\text{Emb}(S, M)\) under the projection to the quotient \(\text{Emb}(S, M)/\text{Diff}(S, \eta)\). Lee’s result does not assume \(H^1(S) = 0\). In this paper, we will omit also the condition \(i^* [\omega] = 0\), and instead of looking at orbits through isotropic embeddings, we will look at orbits through symplectic embeddings. Denote by

\[S_e(\Sigma, \sigma)\]
the subspace of symplectic embeddings \((\Sigma, \sigma) \to (M, \omega)\). The Lie group Sympl(\(\Sigma, \sigma\)) of diffeomorphisms of \(\Sigma\) that pull back \(\sigma\) to itself acts freely on \(\mathcal{S}_c(\Sigma, \sigma)\) on the right. In this paper we study the reduction of \(\omega^D(\Sigma, \sigma)\) to \(\mathcal{S}_c(\Sigma, \sigma)\) modulo Sympl(\(\Sigma, \sigma\)) and to moduli spaces of un-parametrized \(J\)-holomorphic curves. The terms smooth manifold and map, tangent space, and differential form are interpreted in the “Convenient Setup”. In this framework, the local model is the convenient vector space: a locally convex vector space \(E\) with the property that for any smooth (infinitely differentiable) curve \(c_1: \mathbb{R} \to E\) there is a curve \(c_2: \mathbb{R} \to E\) such that \(c_2' = c_1\), with the \(c^\infty\)-topology: the finest topology for which all smooth curves \(\mathbb{R} \to E\) are continuous. (The \(c^\infty\)-topology is finer than the locally convex topology on \(E\). If \(E\) is a Frechet space, (i.e., a complete and metrizable locally convex space), then the two topologies coincide.) A map between convenient vector spaces is smooth if it sends smooth curves to smooth curves. Smooth manifolds are modeled on convenient vector spaces via charts, whose transition functions are smooth; a map between smooth manifolds is smooth if it maps smooth curves to smooth curves. (See [5] and [6, Sec. 2].)

In the Appendix we show that the 2-form \(\omega^D(\Sigma, \sigma)\) is closed and its restriction to the space of immersions \(\Sigma \to M\) is weakly non-degenerate. We also show that for an almost complex structure \(J: TM \to TM\) that is compatible with \(\omega\), the induced almost complex structure \(\tilde{J}: T^\infty(\Sigma, M) \to T^\infty(\Sigma, M)\) is compatible with \(\omega^D(\Sigma, \sigma)\). In Section 2 we prove that \(\mathcal{S}_c(\Sigma, \sigma)\) is a smooth manifold and describe its tangent bundle, see Proposition 2.2. We show that the restriction of the form \(\omega^D(\Sigma, \sigma)\) to \(\mathcal{S}_c(\Sigma, \sigma)\) is weakly symplectic, see Proposition 2.10. In Section 2 we prove the following theorem.

**Theorem 1.2.** Let \(\mathcal{N}\) be a Ham(\(M, \omega\))-orbit in \(\mathcal{S}_c(\Sigma, \sigma)\). The restriction of \(\omega^D(\Sigma, \sigma)\) to \(\mathcal{N}\) descends to a closed weakly non-degenerate 2-form \(\omega^D_{\text{rod}}\) on the image \(\mathcal{O}\) in the orbit space under the projection \(q: \mathcal{S}_c(\Sigma, \sigma) \to \mathcal{S}_c(\Sigma, \sigma)/\text{Sympl}(\Sigma, \sigma)\). The symplectic space \((\mathcal{O}, \omega^D_{\text{rod}})\) is a symplectic quotient of \(\mathcal{S}_c(\Sigma, \sigma)\) with respect to the \(\text{Sympl}(\Sigma, \sigma)\)-action.

The notion of a symplectic quotient here does not depend on having a moment map, see Definition 3.1. It is motivated by the optimal reduction method of Ortega and Ratiu [11].

We also compare \(\omega^D(\Sigma, \sigma)\) to the 2-form we defined in [2] on the space of immersed symplectic \(\Sigma\)-surfaces in \(M\). Denote by

\[ \text{ev} : C^\infty(\Sigma, M) \times \Sigma \to M \]

the *evaluation map*

\[ \text{ev}(f, x) := f(x). \]

**Definition 1.3** Define a 2-form on \(C^\infty(\Sigma, M)\) as the push-forward of the 4-form \(\text{ev}^*(\omega \wedge \omega)\) along the coordinate-projection \(\pi_{C^\infty(\Sigma, M)}: C^\infty(\Sigma, M) \times \Sigma \to C^\infty(\Sigma, M)\) by

\[ (\omega_{C^\infty(\Sigma, M)})_f(\tau_1, \tau_2) := \int_{f}^{\Sigma} \ell^1(\tau_1 \wedge \tau_2) \text{ev}^*(\omega \wedge \omega). \tag{1.1} \]

Here \(\ell_i \in T(C^\infty(\Sigma, M) \times \Sigma)\) is a lifting of \(\tau_i \in T_f(C^\infty(\Sigma, M))\), i.e.,

\[ d(\pi_{C^\infty(\Sigma, M)})_f(\ell_i(f, x)) = \tau_i \text{ at each point } (f, x) \in \pi_{C^\infty(\Sigma, M)}^{-1}(f). \]

Denote

\[ \mathcal{S}_i(\Sigma) := \{ f: \Sigma \to M \mid f \text{ is an immersion, } f^*\omega \text{ is a symplectic form on } \Sigma \}. \]
The space $\mathcal{S}_1(\Sigma)$ is an open subset of $C^\infty(\Sigma, M)$ in the $C^\infty$-topology. Let

$$\omega_{\mathcal{S}_1(\Sigma)}$$

be the 2-form on $\mathcal{S}_1(\Sigma)$ given by the restriction of $\omega_{C^\infty(\Sigma, M)}$. We showed in [2] that the 2-form $\omega_{C^\infty(\Sigma, M)}$ on $C^\infty(\Sigma, M)$ is well defined and closed, and $\omega_{C^\infty(\Sigma, M)}(\tau, \cdot)$ vanishes at $f$ if $\tau$ is everywhere tangent to $f(\Sigma)$. Furthermore,

$$\omega_{\mathcal{S}_1(\Sigma)}(\tau, \cdot) = 0 \iff \tau \text{ is tangent to } f(\Sigma) \text{ at every } x \in \Sigma.$$ 

We say that a vector field $\tau: \Sigma \to f^*(TM)$ is tangent to $f(\Sigma)$ at $x$ if $\tau(x) \in df_x(T_x\Sigma)$. See also [7].

Consider the space of $\omega$-compatible almost complex structures $\mathcal{J} = \mathcal{J}(M, \omega)$ on $(M, \omega)$. Fix $\Sigma = (\Sigma, j)$, where $j$ is a complex structure on $\Sigma$. The moduli space $\mathcal{M}_i(A, \Sigma, J)$ is the space of simple immersed $(j, J)$-holomorphic $\Sigma$-curves in a homology class $A \in H_2(M, \mathbb{Z})$. The moduli space $\mathcal{M}_e(A, \Sigma, J)$ is the space of embedded $(j, J)$-holomorphic $\Sigma$-curves in a homology class $A \in H_2(M, \mathbb{Z})$. We look at almost complex structures that are regular for the projection map

$$p_A: \mathcal{M}_i(A, \Sigma, J) \to J,$$

for such a $J$, the spaces $\mathcal{M}_i(A, \Sigma, J)$ and $\mathcal{M}_e(A, \Sigma, J)$ are finite-dimensional manifolds. (The set of $p_A$-regular $\omega$-compatible almost complex structures is of the second category in $\mathcal{J}$.) See [9] Thm 3.1.5. There is merit to the form $\omega_{\mathcal{S}_1(\Sigma)}$ in the fact that it is degenerate along directions tangent to $f(\Sigma)$, hence descends to a well defined form on the quotient space $\tilde{\mathcal{M}}_i(A, \Sigma, J)$ of $\mathcal{M}_i(A, \Sigma, J)$ by the proper action of the group $\text{Aut}(\Sigma, j)$ of bi-holomorphisms of $\Sigma$: this enables us to apply Gromov’s compactness theorem and get a well defined invariant of $(M, \omega)$. If $J_* \in J_{\text{reg}}(A)$ is integrable, then the restriction of the form $\omega_{\mathcal{S}_1(\Sigma)}$ to $\mathcal{M}_i(A, \Sigma, J_*)$ is non-degenerate, up to reparametrizations; see [2] Prop. 4.4. We obtained results on the existence of $J$-holomorphic curves in a homology class $A$ for some subset of $\mathcal{J}$, and in some cases for a generic $J$, see [2] Cor. 1.3.

Here we show that the 2-forms $2\omega^D(\Sigma, \sigma)$ and $\omega_{\mathcal{S}_1(\Sigma)}$ coincide in exact direction, hence on the quotient of a $\text{Ham}(M, \omega)$-orbit with respect to the $\text{Sympl}(\Sigma, \sigma)$-action. The difference between $\omega^D$ and $\omega_{\mathcal{S}_1(\Sigma)}$ is that $\iota_v \omega^D$ is degenerate along vectors in $T\mathcal{S}_1(\Sigma, \sigma)$ everywhere tangent to $\Sigma$ iff $v = \iota^* V_H$ for a Hamiltonian vector field $V_H$ on $M$ whereas $\iota_v \omega_{\mathcal{S}_1(\Sigma)}$ is degenerate along vectors everywhere tangent to $\Sigma$ for every $v$ in $T\mathcal{S}_1(\Sigma)$, see Remark 3.14. Due to this difference, Theorem 1.2 does not hold for $\omega_{\mathcal{S}_1(\Sigma)}$. On the other hand, we do not get a well defined reduction of $\omega^D(\Sigma, \sigma)$ on the quotient of $\mathcal{M}_i(A, \Sigma, J)$ by the action of $\text{Aut}(\Sigma, j)$, as we did for $\omega_{\mathcal{S}_1(\Sigma)}$. However we do get a partial result. For $J \in \mathcal{J}(M, \omega)$, denote by $\text{Ham}^J(M, \omega)$ the subgroup of $\text{Ham}(M, \omega)$ of $J$-holomorphic Hamiltonian symplectomorphisms. Let $\mathcal{N}$ be an orbit of $\text{Ham}^J(M, \omega)$ through an embedded $(j, J)$-holomorphic curve $f: \Sigma \to M$ for which $f^*\omega = \sigma$. The orbit $\mathcal{N}$ is a subset of $\mathcal{M}_e(A, \Sigma, J)$, where

$$A \in H_2(M, \mathbb{Z})$$

is the class for which the area $f^*\omega(\Sigma) = \sigma(\Sigma)$ for (every) $f \in A$.

**Corollary 1.4.** Assume that the symplectic form $\sigma$ on $\Sigma$ is compatible with the complex structure $j$ on $\Sigma$. Let $J \in \mathcal{J}(M, \omega)$, assume that $J$ is integrable and regular for $A$. Let $\mathcal{N}$ be an orbit of $\text{Ham}^J(M, \omega)$ through a $(j, J)$-holomorphic embedding $f: \Sigma \to M$ for which $f^*\omega = \sigma$.

The forms $\omega^D(\Sigma, \sigma)$ and $\omega_{\mathcal{S}_1(\Sigma)}$ descend to well defined symplectic forms $\omega_{\text{red}}^D$ and $\omega_{\mathcal{S}_1(\Sigma)}^{\text{red}}$ on the quotient of $\mathcal{N}$ with respect to $\text{Aut}(\Sigma, j)$. The form $\omega_{\mathcal{S}_1(\Sigma)}^{\text{red}}$ coincides with the form $2\omega_{\text{red}}^D$ on the quotient.
2 The space $S_c(\Sigma, \sigma)$

**Proposition 2.1.** The 2-form $\omega^D(\Sigma, \sigma)$ on $C^\infty(\Sigma, M)$ is closed and its restriction to the space of immersions $\Sigma \to M$ is weakly non-degenerate.

The space $C^\infty(\Sigma, M)$ is a smooth manifold in the Convenient Setup, modeled on spaces $\Gamma(f^*TM)$ of sections of the pullback bundle along $f \in C^\infty(\Sigma, M)$ [5] 42.1. The space $\Gamma(f^*TM)$ has a natural convenient structure [5] 30.1.

A 2-form $\Omega$ on a manifold $X$ (possibly infinite-dimensional) is called weakly non-degenerate if for every $x \in X$ and $0 \neq v \in T_x X$ there exists a $w \in T_x X$ such that $\Omega_x(v, w) \neq 0$. This is equivalent to its associated vector bundle homomorphism $\Omega^0: TX \to T^*X$ being injective. If $\Omega^0: TX \to T^*X$ is an isomorphism, i.e., invertible with a smooth inverse, then $\Omega$ is called strongly non-degenerate. In this paper, by non-degenerate we mean weakly non-degenerate. If $\Omega$ is closed and weakly non-degenerate, it is called weakly symplectic.

For the proof of Proposition 2.1 and required facts on compatible almost complex structures, see the Appendix.

**Notation:**
For every embedding $i: \Sigma \to M$, for $v \in \Gamma(i^*TM)$, let $\alpha_v \in \Omega^1(\Sigma)$ denote the form
\[(\alpha_v)_x(\xi) := \omega_x(v(x), d_x\xi) \text{ for } \xi \in T_x \Sigma.\]

Also, set
\[\Gamma_{\text{closed}}(i^*TM) := \{v \in \Gamma(i^*TM) \mid \alpha_v \text{ is a closed 1-form on } \Sigma\},\]
and
\[\Gamma_{\text{exact}}(i^*TM) := \{v \in \Gamma(i^*TM) \mid \alpha_v \text{ is an exact 1-form on } \Sigma\}.

For a vector field $v \in T_i C^\infty(\Sigma, M)$ denote by
\[\xi_v + \tau_v \quad (2.2)\]
the decomposition of $v$ to a vector field $\xi_v$ everywhere $\omega$-orthogonal to $\Sigma$ and a vector field $\tau_v$ everywhere tangent to $\Sigma$. Such a decomposition exists and is unique, e.g., by Remark A.9 and Corollary A.8.

We say that a vector field $\tau: \Sigma \to TM$ is tangent to $\Sigma$ at $x$ if $\tau(x) \in T_{i(x)} i(\Sigma)$. We say that a vector field $\xi: \Sigma \to TM$ is $\omega$-orthogonal to $\Sigma$ at $x$ if $\xi(x) \in (T_{i(x)} i(\Sigma))^\omega$.

Denote by
\[S_c(\Sigma, \sigma)\]
the set of embeddings $(\Sigma, \sigma) \to (M, \omega)$ such that $i^*\omega = \sigma$.

**Proposition 2.2.** The set $S_c(\Sigma, \sigma)$ is a smooth manifold modeled on $\Gamma_{\text{exact}}(i^*TM) \oplus \mathcal{X}(\Sigma, \sigma)$, where
\[\mathcal{X}(\Sigma, \sigma) = \{\xi \text{ a vector field on } \Sigma \mid \mathcal{L}_\xi \sigma = 0\}.

To prove the proposition, we first recall the symplectic tubular neighbourhood Theorem of Weinstein.

2.3 Consider a symplectic embedding $i: (\Sigma, \sigma) \hookrightarrow (M, \omega)$. The symplectic normal bundle
\[N\Sigma = \{(x, v) \mid x \in \Sigma, v \in T_{i(x)} M/T_x i(\Sigma)\} \to \Sigma.

The minimal coupling form, due to Sternberg [12], is a closed 2-form $\omega_{\Sigma}$ with the following properties:
1. Its pullback to the fibers coincide with the fiberwise symplectic forms.

2. Its pullback to the zero section coincides with \( \sigma \).

3. At the points of the zero section, the fibers of \( N\Sigma \) are \( \omega_{N\Sigma} \)-orthogonal to the zero section.

Consequently, \( \omega_{N\Sigma} \) is non-degenerate near the zero section.

The symplectic normal bundle \( N\Sigma \) can be realized as a subbundle of \( TM \): the symplectic orthocomplement of \( T\Sigma = Ti(\Sigma) \) in \( TM|_{i(\Sigma)} \). In other words, the fiber \( N_x \Sigma \) at \( x \in \Sigma \) is identified with

\[
(T_i(x) i(\Sigma))^\omega = \{ v \in T_i(x) M \, | \, \omega(v, w) = 0 \text{ for every } w \in T_i(x) i(\Sigma) \}
\]

with the symplectic form \( \omega|_{(T_i(x) i(\Sigma))^\omega} \).

By the classical tubular neighbourhood theorem in differential topology combined with a theorem of Weinstein [13, Theorem 4.1], there exists a neighbourhood \( U \) of the zero section in \( N\Sigma \) and a symplectic open embedding

\[
\Phi_i: (U, \omega_{N\Sigma}) \to (M, \omega)
\]

whose restriction to the zero section is \( i \), and whose differential is \( di \) at every point of \( \Sigma \).

**Lemma 2.4.** Let \( \Sigma = (\Sigma, \sigma) \xrightarrow{i} (M, \omega) \) be an embedded closed connected symplectic submanifold of dimension 2. Let \( v \in T_i C^\infty(\Sigma, M) \). If \( v \) is everywhere \( \omega \)-orthogonal to \( \Sigma \), then \( v \) equals the restriction \( i^*V_H \) to \( i(\Sigma) \) of a Hamiltonian vector field \( V_H \) on \( M \).

**Proof.** By [2.3] we can consider \( \xi_v \), as a vector field \( \xi_0 \) on the zero section in \( N\Sigma \); it is enough to show that \( \xi_0 \) extends to a Hamiltonian vector field \( \xi \) on a neighbourhood of the zero section in \( N\Sigma \), since then the push forward of \( \xi \) via \( \Phi_i \) in [2.3] is a Hamiltonian vector field on a neighbourhood of \( \Sigma \) in \( M \). Then \( \xi_v \) can be extended to a Hamiltonian vector field on \( M \), using a cut-off function with a support that is close enough to \( \Sigma \).

By assumption, for every \( x \) in the zero section, \( \xi_0(x) \) is in \( N_x \Sigma = (T_i(x) i(\Sigma))^\omega \). Each of the fibers

\[
((T_i(x) i(\Sigma))^\omega, \omega_{N\Sigma}|(T_i(x) i(\Sigma))^\omega) = ((T_i(x) i(\Sigma))^\omega, \omega|_{(T_i(x) i(\Sigma))^\omega})
\]

is a symplectic vector space; to each vector \( \xi_0(x) \in (T_i(x) i(\Sigma))^\omega \) there corresponds a linear function \( \omega_{N\Sigma}(\xi_0(x), \cdot) = \omega(\xi_0(x), \cdot) \) from the fiber to \( \mathbb{R} \). Taking the union of these functions over the points of the zero section, we get a function

\[
H: N\Sigma \to \mathbb{R}
\]

which is smooth in a neighbourhood of the zero section in \( (N\Sigma, \omega_{N\Sigma}) \). Take \( \xi \) to be the vector field defined by

\[
dH = \omega_{N\Sigma}(\xi, \cdot)
\]

in a neighbourhood of the zero section on which \( \omega_{N\Sigma} \) is non-degenerate.

**Lemma 2.5.** Let \( \Sigma = (\Sigma, \sigma) \xrightarrow{i} (M, \omega) \) be an embedded closed connected symplectic submanifold of dimension 2. Let \( v \in \Gamma_{\text{closed}}(i^*TM) \). The following are equivalent.

1. The vector field \( v \) equals the restriction \( i^*V_H \) to \( i(\Sigma) \) of a Hamiltonian vector field \( V_H \) on \( M \).

2. The form \( \alpha_v = \omega(v, di(\cdot)) \) on \( \Sigma \) is exact.
3. \((\omega^D_{(\Sigma,\sigma)})_\xi(v,w) = 0\) for every \(w\) that is everywhere tangent to \(\Sigma\) and satisfies \(\mathcal{L}_{(\xi)}^{-1}w\sigma = 0\).

4. \(v\) is everywhere \(\omega\)-orthogonal to \(\Sigma\).

Proof.

1 \(\Rightarrow\) 2 If \(V_H\) is a Hamiltonian vector field on \(M\), then on \(\Sigma\) the form \(\alpha_i\circ V_H\) equals \(dh\) with \(h = H \circ i\).

2 \(\Rightarrow\) 3 If \(\alpha_v = dh\) for a function \(h: \Sigma \rightarrow \mathbb{R}\) then for every \(w\) everywhere tangent to \(\Sigma\) such that \((di)^{-1}w \in \mathcal{X}(\Sigma,\sigma)\),

\[
(\omega^D_{(\Sigma,\sigma)})_\xi(v,w) = \int_{\Sigma} \omega(v,w)\sigma = \int_{\Sigma} dh((di)^{-1}w)\sigma \\
= \int_{\Sigma} (\mathcal{L}_{(di)}^{-1}w h)\sigma = \int_{\Sigma} \mathcal{L}_{(di)}^{-1}w(h\sigma) = \mathcal{L}_{(di)}^{-1}w \int_{\Sigma} h\sigma \\
= \lim_{t \to 0} \frac{\phi_t^* \int_{\Sigma} h\sigma - \int_{\Sigma} h\sigma}{t} = 0. \quad (2.5)
\]

The fourth equality is since \(\mathcal{L}_{(di)}^{-1}w\sigma = 0\) and the fact that \(\mathcal{L}_{(di)}^{-1}w(h\sigma) = (\mathcal{L}_{(di)}^{-1}w h)\sigma + h(\mathcal{L}_{(di)}^{-1}w\sigma)\); the fifth equality is since \(\Sigma\) is compact, and the last equality is since for an orientation preserving integral curve \(t \rightarrow \phi_t\) and a 2-form \(\gamma\) on \(\Sigma\), \(\int_{\Sigma} \gamma\) is invariant under pulling back by \(\phi_t\).

3 \(\Rightarrow\) 4 Assume that \(\int_{\Sigma} \alpha_v(w)\sigma = 0\) for every \(w\) that is everywhere tangent to \(\Sigma\) and satisfies \(\mathcal{L}_{(di)}^{-1}w\sigma = 0\). Decompose \(v = \xi_v + \tau_v\), to a vector field \(\xi_v\) everywhere \(\omega\)-orthogonal to \(\Sigma\) and a vector field \(\tau_v\) everywhere tangent to \(\Sigma\), as in (2.2), so \(\alpha_v(\cdot) = \omega(\xi_v,di(\cdot)) + \omega(\tau_v,di(\cdot))\). By assumption \(\alpha_v\) is closed; by Lemma [2.4] and the step 1 \(\Rightarrow\) 2 above, \(\alpha_{\xi_v}\) is closed, hence \(\alpha_{\tau_v}\) is closed. When we consider \(\tau_v: \Sigma \rightarrow di(T\Sigma) \cdot (di)^{-1} \rightarrow T\Sigma\) as a vector field on \(\Sigma\), we conclude that \(\sigma((di)^{-1}\tau_v,\cdot)\) is closed on \(\Sigma\). By Cartan’s formula and since \(\sigma\) is closed, we get that \((di)^{-1}\tau_v \in \mathcal{X}(\Sigma,\sigma)\). By the assumption on \(v\) and the choice of \(\xi_v\), for every \(w\) that is everywhere tangent to \(\Sigma\) and satisfies \(\mathcal{L}_{(di)}^{-1}w\sigma = 0\),

\[
0 = \int_{\Sigma} \alpha_v(w)\sigma = \int_{\Sigma} \sigma((di)^{-1}\tau_v, w)\sigma.
\]

In particular,

\[
\int_{\Sigma} \sigma((di)^{-1}\tau_v, (di)^{-1}\tau_v)\sigma = 0.
\]

Thus (since \(\Sigma\) is connected and \(\sigma\) is a volume form) \(\tau_v = 0\), so \(\alpha_v = \xi_v\) is everywhere \(\omega\)-orthogonal to \(\Sigma\).

4 \(\Rightarrow\) 1 By Lemma [2.4].

Remark 2.6 Notice that the steps 1 \(\Rightarrow\) 2, 2 \(\Rightarrow\) 3 and 4 \(\Rightarrow\) 1 hold for every \(v \in T_iC^\infty(\Sigma, M)\). Only the step 3 \(\Rightarrow\) 4 requires \(v \in \Gamma_{\text{closed}}(i^*TM)\).
Lemma 2.7. Let \( \Sigma = (\Sigma, \sigma) \hookrightarrow (M, \omega) \) be an embedded closed connected symplectic submanifold of dimension 2. The map \( v \mapsto di^{-1}\tau_v \) from \( \Gamma_{\closed} (i^* TM) \) is onto \( \mathcal{X}(\Sigma, \sigma) \) and restricts to a one-to-one and onto map from the subspace \( \{ \tau_v \mid v \in \Gamma_{\closed} (i^* TM) \} \).

Proof. First, we show that the image is a subset of \( \mathcal{X}(\Sigma, \sigma) \): by assumption, \( v \in \Gamma_{\closed} (i^* TM) \); by Lemma 2.4 and 1 \( \Rightarrow \) 2 in Lemma 2.5, \( \xi_v \in \Gamma_{\exact} (i^* TM) \subset \Gamma_{\closed} (i^* TM) \), hence, \( \tau_v = v - \xi_v \) is in the space \( \Gamma_{\closed} (i^* TM) \). In other words,

\[
d_{\tau_v} \omega |_{(\mathcal{X}(\Sigma))} = 0. \tag{2.6}
\]

Therefore, since \( i \) is a symplectic embedding, \( d_{\tau_v} \omega |_{(\mathcal{X}(\Sigma))} = 0 \). Thus, by Cartan’s formula, since \( \sigma \) is a closed form, \( L_{d_{\tau_v} \omega} \sigma = 0 \). Reversing the argument, we get that for every \( \tau \in \mathcal{X}(\Sigma, \sigma) \), the vector \( d\tau \) is a vector in \( \Gamma_{\closed}(i^* TM) \) that is everywhere tangent to \( \Sigma \), hence the map is onto.

By (2.6), the space \( \{ \tau_v \mid v \in \Gamma_{\closed}(i^* TM) \} \) is a subspace of \( \Gamma_{\closed}(i^* TM) \). By the above argument, the map \( \tau_v \mapsto di^{-1}\tau_v \) on it is onto \( \mathcal{X}(\Sigma, \sigma) \).

Corollary 2.8. Let \( \Sigma = (\Sigma, \sigma) \hookrightarrow (M, \omega) \) be an embedded closed connected symplectic submanifold of dimension 2. Then

\[
\Gamma_{\closed}(i^* TM) = \Gamma_{\exact}(i^* TM) \oplus \mathcal{X}(\Sigma, \sigma).
\]

The splitting gives a convenient space structure on \( \Gamma_{\exact}(i^* TM) \oplus \mathcal{X}(\Sigma, \sigma) \).

Proof. A vector \( v \in \Gamma_{\closed}(i^* TM) \) decomposes as \( \xi_v + \tau_v \), where \( \xi_v \) is everywhere \( \omega \)-orthogonal to \( \Sigma \) and \( \tau_v \) is everywhere tangent to \( \Sigma \). Such a decomposition exists and is unique, e.g., by Remark A.9 and Corollary A.8.

By Lemma 2.5, the space \( \{ \xi_v \mid v \in \Gamma_{\closed}(i^* TM) \} \) equals \( \Gamma_{\exact}(i^* TM) \). By Lemma 2.7, the space \( \{ \tau_v \mid v \in \Gamma_{\closed}(i^* TM) \} \) is identified with \( \mathcal{X}(\Sigma, \sigma) \). Notice that the maps \( \xi \mapsto (\xi, v, d\xi \wedge \omega) \) and \( (\xi, v) \mapsto (\xi + d\xi \wedge \omega) \) send smooth curves to smooth curves. Moreover, for \( c_1 : \mathbb{R} \to \Gamma_{\exact}(i^* TM) \oplus \mathcal{X}(\Sigma, \sigma) \), if \( c_2 : \mathbb{R} \to \Gamma_{\closed}(i^* TM) \) satisfies \( c_2' = h_2(c_1) \) then \( (h_1(c_2))' = c_1 \).

The space \( \Gamma_{\closed}(i^* TM) \) is convenient since it is the kernel of the continuous map \( v \mapsto \alpha_v \) composed on \( \alpha \to d\alpha \), from the convenient space \( \Gamma(i^* TM) \) to the space \( \Omega^1(\Sigma) \) of 1-forms on \( \Sigma \) and then to \( \Omega^2(\Sigma) \).

Proof of Proposition 2.2. Given \( i \in \mathcal{S}_e(\Sigma, \sigma) \), by Weinstein’s symplectic tubular neighbourhood theorem (see [3.3]), the symplectic embedding \( i \) can be extended on a neighbourhood \( U \) of the zero section in \( N\Sigma \) to a symplectic embedding \( \Phi_i : U \to M \). By the identification of each fiber \( N_x \Sigma \) with \( (\Gamma(i^* TM))_{ij} \), and Lemma 2.5, the elements of \( U \) are of the form \( (y, \xi(y)) \) where \( \xi \in \Gamma_{\exact}(i^* TM) \). The space \( \mathcal{X}(\Sigma, \sigma) \) is the Lie algebra of \( \text{Sympl}(\Sigma, \sigma) \), see [5, 43.12]. Let \( V_e \) be a chart neighbourhood of the identity map \( e \in \text{Sympl}(\Sigma, \sigma) \) and denote by

\[
\psi_e : V_e \to \mathcal{X}(\Sigma, \sigma)
\]

the corresponding chart in an atlas on \( \text{Sympl}(\Sigma, \sigma) \). Define

\[
W_i := \{ \ell \in \mathcal{S}_e(\Sigma, \sigma) \mid \ell(x) = \Phi_i(b(x), \xi(b(x))) \text{ for } \xi \in \Gamma_{\exact}(i^* TM) \text{, } b \in V_e \text{ s.t. } (b(x), \xi(b(x))) \in U \forall x \in \Sigma \},
\]

\[
\phi_i : W_i \to \Gamma_{\exact}(i^* TM) \oplus \mathcal{X}(\Sigma, \sigma), \quad \phi_i(\ell) := (\xi, \psi_e(b)).
\]

By part (2) of Corollary 2.8, \( \Gamma_{\exact}(i^* TM) \oplus \mathcal{X}(\Sigma, \sigma) \) is a convenient space. The set \( \{ (b(x), \xi(b(x))) \mid U \forall x \in \Sigma \} \) is \( C^\infty \)-open in \( \Gamma_{\exact}(i^* TM) \oplus \mathcal{X}(\Sigma, \sigma) \). Thus \( \phi_i \) is a bijection of \( W_i \) onto a \( C^\infty \)-open subset of \( \Gamma_{\exact}(i^* TM) \oplus \mathcal{X}(\Sigma, \sigma) \). The collection \( (W_i, \phi_i)_{i \in \mathcal{S}_e(\Sigma, \sigma)} \) defines a smooth atlas on \( \mathcal{S}_e(\Sigma, \sigma) \): the chart changings \( \phi_{ik} \) are smooth by smoothness of the exponential map and of each symplectic embedding \( \Phi_i \).
Lemma 2.9. Let $\Sigma = (\Sigma, \sigma) \hookrightarrow (M, \omega)$ be an embedded closed connected symplectic submanifold of dimension 2.

1. For a section $v: \Sigma \to i^* TM$ that is in $T_i S_\Sigma(\Sigma, \sigma)$, the form

   $$\alpha_v = \omega(v, di(\cdot))$$

   is a closed form on $\Sigma$.

2. Every vector $v \in \Gamma_{\text{closed}}(i^* TM)$ can be extended to a vector field $\tilde{v}$ on a neighbourhood of $i(\Sigma)$ in $M$ such that $L_{\tilde{v}} \omega = 0$.

Proof. 1. For a section $v$, let $t \to \phi_t$ be the integral curve of $v$ starting at $i$. Since $v$ is tangent to $S_\Sigma(\Sigma, \sigma)$, for $w_1, w_2$ in $T \Sigma$, we get $\phi_t^* \omega(diw_1, diw_2) = \sigma(w_1, w_2) = \omega(diw_1, diw_2)$, where $di: T \Sigma \to T(i(\Sigma))$. Hence on $i(\Sigma)$,

   $$L_v \omega = \lim_{t \to 0} \frac{\phi_t^* \omega - \omega}{t} = 0.$$ 

   By Cartan’s formula, this implies $dt \omega = 0$ as a form on $i(\Sigma)$, i.e., $\alpha_v$ is a closed form on $\Sigma$.

2. By Cartan’s formula and the fact that $\omega$ is a closed form, we need to extend $v$ to $\tilde{v}$ on a neighbourhood of $i(\Sigma)$ in $M$ such that the 1-form $\omega(\tilde{v}, \cdot)$ is a closed form. By the decomposition (2.2) and Lemma 2.4 it is enough to extend $\tau_v \in \Gamma_{\text{closed}}(i^* TM)$ that is everywhere tangent to $\Sigma$ to such a vector. The closed form $\iota_{(di)^{-1} \tau_v} \omega$ on the zero section of $(N \Sigma, \omega_{N \Sigma})$ pulls back (through the projection of $N \Sigma$ to the zero section) to a closed one-form on $N \Sigma$ that is consistent with $\iota_{(di)^{-1} \tau_v} \omega$ on the zero section and zero on directions $\omega_{N \Sigma}$-orthogonal to the zero section. By Weinstein’s symplectic tubular neighbourhood theorem, the push forward of this form via the symplectic embedding $\Phi_i: (U, \omega_{N \Sigma}) \to (M, \omega)$ of (2.3) is a closed one-form $\tilde{\alpha}_{\tau_v}$ on a neighbourhood of $i(\Sigma)$ in $M$ that is consistent with $\iota_{\tau_v} \omega$ on vectors tangent to $i(\Sigma)$. Define $\tilde{\tau}_v$ to be the vector field such that $\tilde{\alpha}_{\tau_v}(\cdot) = \omega(\tilde{\tau}_v, \cdot)$. The vector $\tilde{\tau}_v$ is well defined since $\omega$ is non-degenerate.

Denote by $\omega_{S_\Sigma(\Sigma, \sigma)}$ the pullback (through inclusion) of $\omega_{S(\Sigma)}$ to $S_\Sigma(\Sigma, \sigma)$, and by $\omega^D_{S_\Sigma(\Sigma, \sigma)}$ the pullback of $\omega^D(\Sigma, \sigma)$ to $S_\Sigma(\Sigma, \sigma)$.

Proposition 2.10. The 2-form $\omega^D_{S_\Sigma(\Sigma, \sigma)}$ on $S_\Sigma(\Sigma, \sigma)$ is closed and weakly non-degenerate.

Proof. The form is closed as the restriction of the closed form $\omega^D(\Sigma, \sigma)$ (see proposition 2.1) to the manifold $S_\Sigma(\Sigma, \sigma)$ (see proposition 2.2). We need to show that it is weakly non-degenerate.

1. For $0 \neq \tau_v \in T_i S_\Sigma(\Sigma, \sigma)$ that is everywhere tangent to $\Sigma$,

   $$\omega^D(\tau_v, \tau_v) = \int_\Sigma \omega(\tau_v, \tau_v)\sigma = \int_\Sigma \sigma(d\iota^{-1} \tau_v, d\iota^{-1} \tau_v)\sigma \neq 0.$$ 

   The last inequality is since $\Sigma$ is connected, $\sigma$ is a volume form, $\tau_v \neq 0$ and $di: T \Sigma \to T(i(\Sigma))$ is an isomorphism.
2. Suppose that \(w \in T_x(S_\varepsilon(\Sigma, \sigma))\) is not tangent to \(\Sigma\) at \(x \in \Sigma\). By Lemma \[2.4\] \(w = \xi_w + \tau_w\), where \(\xi_w = i^*\xi\) with \(\xi\) a Hamiltonian vector field on \(M\) and \(\tau_w\) everywhere tangent to \(\Sigma\). In particular, \(\xi_w(x) \not= 0\). Let \(w_1\) be a vector in \((i^*(TM))_x\) such that

\[\omega(\xi_w(x), w_1) > 0,\]

and \(w_1\) is symplectically orthogonal to \(di_x(T_x \Sigma)\). (For example, \(w_1 = J\xi_w(x)\) for an almost complex structure \(J\) that is \(\omega\)-compatible. See part (2) of Claim \[A.7\] and Remark \[A.9\].

Now extend \(w_1\) to a section \(w_1: \Sigma \to i^*(TM)\) in \(T_1 S_\varepsilon(\Sigma, \sigma)\) such that \(\omega(\xi_w(y), w_1(y)) > 0\) and \(w_1(y)\) is symplectically orthogonal to \(di_y(T_y \Sigma)\) for \(y\) in a small neighborhood of \(x\), and vanishing outside it. By Lemma \[2.4\] \(w_1 = i^*W_H\) with \(W_H\) a Hamiltonian vector field on \(M\). By Lemma \[2.9\] \(w \in \Gamma_{\text{closed}}(i^*TM)\), hence (see Lemma \[2.7\]), \(\tau_w\) is an everywhere tangent to \(\Sigma\) vector that satisfies \(L_{\tau^{-1} \tau_w} = 0\). Then

\[(\omega^D S_\varepsilon(\Sigma, \sigma))_\xi(w, w_1) = (\omega^D S_\varepsilon(\Sigma, \sigma))_\xi(\xi_w, w_1) = \int_\Sigma \omega(\xi_w, w_1) \sigma \neq 0,\]

where the first equality follows from \(1 \Rightarrow 3\) in Lemma \[2.5\] and the last inequality follows from the choice of \(w_1\).

\[\square\]

3. The quotient of \(S_\varepsilon(\Sigma, \sigma)\) by symplectic reparametrizations

The forms \(\omega_{S_\varepsilon(\Sigma, \sigma)}\) and \(\omega^D_{S_\varepsilon(\Sigma, \sigma)}\) coincide in exact directions

Claim 3.1. For tangent vectors \(v_1, v_2 \in T_f S_\varepsilon(\Sigma, \sigma)\), the integrand \(i_{((v_1,0) \wedge (v_2,0))}\ ev^*(\omega \wedge \omega)\) equals

\[2\omega(v_1, v_2) \omega(df, df) + 2\omega(v_1, df) \wedge \omega(v_2, df) = 2 \omega(v_1, v_2) f^*\omega(\cdot, \cdot) + 2 \omega(v_1, df) \wedge \omega(v_2, df),\]

which, since \(f \in S_\varepsilon(\Sigma, \sigma)\), equals

\[2 \omega(v_1, v_2) \sigma(\cdot, \cdot) + 2 \omega(v_1, df) \wedge \omega(v_2, df).\]  

(3.7)

Lemma 3.2. For \(u \in T_f S_\varepsilon(\Sigma, \sigma)\) such that \(\omega(u, df)\) on \(\Sigma\) is exact, we get that

\[i_u(\omega_{S_\varepsilon(\Sigma, \sigma)} f) = 2 \int_\Sigma \omega(u, \cdot) \sigma = 2 i_u(\omega^D_{S_\varepsilon(\Sigma, \sigma)}) f,\]

i.e., the two forms coincide in exact directions, up to multiplication by a constant.

Proof. Indeed, since \(\omega(u, df) = dh\) for a function \(h: \Sigma \to \mathbb{R}\), and for every \(v \in T_f S_\varepsilon(\Sigma, \sigma)\) the form \(\alpha_v\) on \(\Sigma\) is closed (see Lemma \[2.9\]), the integration of the second term in (3.7) along \(f(\Sigma)\) vanishes:

\[\int_\Sigma \omega(v, df) \wedge \omega(u, df) = \int_\Sigma \alpha_v \wedge dh = \int_\Sigma \alpha_v \wedge dh - d\alpha_v \wedge h = \int_\Sigma d(\alpha_v \wedge h) = \int_{\partial(\Sigma)} \alpha_v \wedge h = 0.\]

\[\square\]

Note that (in the second equality) we used the fact that \(\alpha_v\) on \(\Sigma\) is closed (by Lemma \[2.9\]), so the argument works in the space of symplectic embeddings and not for arbitrary embeddings.
Directions of degeneracy for the form $\omega_{\mathcal{S}_0(\Sigma, \sigma)}$

**Lemma 3.3.** Let $i \in \mathcal{S}_0(\Sigma, \sigma)$.

1. If $w \in T_i \mathcal{S}_0(\Sigma, \sigma)$ is not everywhere tangent to $\Sigma$, then there exists $w_1 \in T_i \mathcal{S}_0(\Sigma, \sigma)$ satisfying $w_1 = i^*W_H$ with $W_H$ a Hamiltonian vector field on $M$ such that $(\omega^D_{\mathcal{S}_0(\Sigma, \sigma)})_i(w, w_1) \neq 0$.

2. If $\tau \in T_i \mathcal{S}_0(\Sigma, \sigma)$ is everywhere tangent to $\Sigma$, then $\iota_\tau (\omega_{\mathcal{S}_0(\Sigma, \sigma)})_i = 0$.

3. If $w \in T_i \mathcal{S}_0(\Sigma, \sigma)$ is not everywhere tangent to $\Sigma$, then there exists $w_1 \in T_i \mathcal{S}_0(\Sigma, \sigma)$ satisfying $w_1 = i^*W_H$ with $W_H$ a Hamiltonian vector field on $M$ such that $(\omega_{\mathcal{S}_0(\Sigma, \sigma)})_i(w, w_1) \neq 0$.

**Proof.**

1. This is shown in the proof of Proposition [2.10](#).

2. Suppose that $\tau$ is everywhere tangent to $\Sigma$. Lift $\tau$ to a vector field $\ell = (\tau, 0)$; let $\tau_2 \in T_i \mathcal{S}_0(\Sigma, \sigma)$ and $\ell_2$ a lifting of $\tau_2$. We show that the integrand $\iota_{\ell \wedge \ell_2} \ev^* (\omega \wedge \omega)$ vanishes when restricted to $T(\{i\} \times \Sigma)$. Indeed, for $z_1, z_2 \in T_x(\{i\} \times \Sigma)$, by definition and Lemma [3.5](#) below,

$$
\iota_{\ell \wedge \ell_2} \ev^* (\omega \wedge \omega)(z_1, z_2) = \iota_{\tau \wedge d\ev(\ell)}(\omega \wedge \omega)(\ev(z_1), \ev(z_2)).
$$

So it is enough to show that

$$
\iota_{\tau \wedge d\ev(\ell)}(\omega \wedge \omega)_{\ev} = 0
$$

vanishes. This follows from Lemma [3.6](#) since, by assumption, $\tau(x) \in \ev(T_x \Sigma)$ and $\ev(T_x \Sigma) \subset T_\ell M$ is a two-dimensional subspace.

3. By item (2) and Lemma [2.4](#) it is enough to prove item (3) with the assumption that $w = i^*V_H$ with $V_H$ a Hamiltonian vector field on $M$. By Lemma [3.2](#) this case follows from item (1).

**Remark 3.4** By the same argument we get that also on $\mathcal{S}_1(\Sigma)$, we have $\omega_{\mathcal{S}_1(\Sigma)}(\tau, \cdot) = 0$ at $f$ iff $\tau$ is tangent to $f(\Sigma)$ at every $x \in \Sigma$. See [2](#) Thm 1).

**Lemma 3.5.** For $(\nu, v_{\Sigma}) \in T(C^\infty(\Sigma, M) \times \Sigma)$,

$$
d(\ev)(f, x)(\nu, v_{\Sigma}) = \nu_f(x) + d_f(x)(v_{\Sigma})
$$

In particular,

$$
d(\ev)|_{T(f) \times \Sigma} = df,
$$

and

$$
d(\ev)(f, x)(\nu, 0) = \nu_f(x).
$$

**Lemma 3.6.** Let $W$ be a vector space, and let $\alpha$ be a 4-form: $\alpha: \wedge^4 W \rightarrow \mathbb{R}$. Let $V \subset W$ be a subspace of dimension $\leq 2$. Then $\left(\iota_\omega \wedge \omega \right) \alpha|_V = 0$ for all $v \in V$, $w \in W$.

**Proof.** This is since any three vectors in $V$ are linearly dependent.

As a result of Lemma [3.3](#) and Lemma [2.4](#) we get an extension of Lemma [3.2](#).

**Corollary 3.7.** For every $u \in T_f \mathcal{S}_0(\Sigma, \sigma)$, there is $w \in T_f \mathcal{S}_0(\Sigma, \sigma)$ that is everywhere tangent to $f$, such that $\omega(u + w, df(\cdot))$ on $\Sigma$ is exact and

$$
\iota_u(\omega_{\mathcal{S}_0(\Sigma, \sigma)})_f = \iota_{u+w}(\omega_{\mathcal{S}_0(\Sigma, \sigma)})_f = 2 \int \omega(u + w, \cdot) \sigma = 2 \iota_{u+w}(\omega^D_{\mathcal{S}_0(\Sigma, \sigma)})_f,
$$

$w$ equals zero if $\omega(u, df(\cdot))$ on $\Sigma$ is exact.
A symplectic form on the quotient by symplectic reparametrizations

The group $\text{Symp}(\Sigma, \sigma)$ of symplectomorphisms of $(\Sigma, \sigma)$ is a Lie group in the Convenient Setup, its Lie algebra is $\mathcal{X}(\Sigma, \sigma)$ [5 43.12]. The Lie group $\text{Symp}(\Sigma, \sigma)$ acts freely on $\mathcal{S}_e(\Sigma, \sigma)$ on the right by

$$\psi.i = i \circ \psi^{-1}. \quad (3.10)$$

Denote the quotient map

$$q: \mathcal{S}_e(\Sigma, \sigma) \to \mathcal{S}_e(\Sigma, \sigma)/\text{Symp}(\Sigma, \sigma). \quad (3.11)$$

Denote by $\text{Ham}(M, \omega)$ the Lie group of Hamiltonian symplectomorphisms of $(M, \omega)$; its Lie algebra $\text{ham}$ is the space of Hamiltonian vector fields [5 43.12, 43.13]. (A vector field $X$ on $M$ is Hamiltonian if the form $\iota_X \omega$ is exact; a symplectomorphism of $(M, \omega)$ is Hamiltonian if it is the time one flow of a time dependent Hamiltonian vector field.) The group $\text{Ham}(M, \omega) \subset \text{Symp}(M, \omega)$ acts freely on $\mathcal{S}_e(\Sigma, \sigma)$ on the left by

$$\phi.i = \phi \circ i.$$ 

The action descends to the quotient $\mathcal{S}_e(\Sigma, \sigma)/\text{Symp}(\Sigma, \sigma)$ as

$$\phi.\lbrack i \rbrack = \lbrack \phi \circ i \rbrack.$$ 

Denote by $\text{Emb}(\Sigma, M)$ the space of embeddings of $\Sigma$ into $M$. (An open set in the manifold $C^\infty(\Sigma, M)$.) The Lie group $\text{Diff}(\Sigma)$ [5 43.1] of reparametrizations of $\Sigma$ acts freely on $\text{Emb}(\Sigma, M)$ on the right by

$$\psi.i = i \circ \psi^{-1}.$$ 

The group $\text{Ham}(M, \omega)$ acts on $\text{Emb}(\Sigma, M)$, (and its quotient by $\text{Diff}(\Sigma)$), by $\phi.i = \phi \circ i$.

**Lemma 3.8.** A path-connected component of the space $\mathcal{S}_e(\Sigma, \sigma)$ modulo the action of $\text{Symp}(\Sigma, \sigma)$ is identified with a path-connected component of the space $\text{Emb}(\Sigma, M)$ modulo the action of $\text{Diff}(\Sigma)$ as follows. Let $f: \Sigma \to M$ be an embedding that is connected to an element $i \in \mathcal{S}_e(\Sigma, \sigma)$ through a path in $\text{Emb}(\Sigma, M)$. There exists a $\Sigma$-reparametrization $\varphi_f$ such that $f \circ \varphi_f \in \mathcal{S}_e(\Sigma, \sigma)$; the $\Sigma$-reparametrization $\varphi_f$ is unique up to symplectic $\Sigma$-reparametrizations. The map $\lbrack f \rbrack \mapsto \lbrack f \circ \varphi_f \rbrack$ from the quotient of the path-connected component of $i \in \mathcal{S}_e(\Sigma, \sigma)$ in $\text{Emb}(\Sigma, M)$ by $\text{Diff}(\Sigma)$ to $\mathcal{S}_e(\Sigma, \sigma)/\text{Symp}(\Sigma, \sigma)$ is well defined, one-to-one and its image is the quotient of the path-connected component of $i$ in $\mathcal{S}_e(\Sigma, \sigma)$ by the action of $\text{Symp}(\Sigma, \sigma)$.

The map $f \mapsto f \circ \varphi_f$ sends a $\text{Ham}(M, \omega)$-orbit in $\text{Emb}(\Sigma, M)$ to a $\text{Ham}(M, \omega)$-orbit in $\mathcal{S}_e(\Sigma, \sigma)$ modulo symplectic reparametrizations.

**Proof.** Let $\{f_t\}_{0 \leq t \leq 1}$ be a path in $\text{Emb}(\Sigma, M)$ with $f_0 = i \in \mathcal{S}_e(\Sigma, \sigma)$ and $f_1 = f$. Set

$$\omega_t = f_t^* \omega.$$ 

By definition $\omega_t$ is closed. By the Homotopy invariance of de Rham cohomology, $[\omega_t] = [\omega_0]$ for all $t$. Therefore

$$\int_{\Sigma} f_t^* \omega = \int_{\Sigma} f_0^* \omega = \int_{\Sigma} d\eta = \int_{\partial \Sigma} \eta = 0, \quad (3.12)$$

where the last equality is since $\Sigma$ is closed and the equality before it is by Stoke’s theorem. Since $f_0^* \omega = i^* \omega = \sigma$, equation (3.12) implies that for all $t$, the integral $\int_{\Sigma} \omega_t = \int_{\Sigma} f_t^* \omega \neq 0$ therefore, the 2-form $\omega_t$ is
a non-vanishing volume form on the 2-dimensional manifold $\Sigma$ hence non-degenerate. We can then apply Moser’s theorem \[10\] to get an isotopy $\varphi: \Sigma \times \mathbb{R} \to \Sigma$ such that

$$(f_t \circ \varphi_t)^* \omega = \varphi_t^* \omega_t = \omega_0 = \sigma$$

for $0 \leq t \leq 1$. (3.13)

Set $\varphi_f := \varphi_1$.

Notice that if $\psi_1, \psi_2 \in \text{Diff}(\Sigma)$ are such that $(f \circ \psi_1)^* \omega = \sigma$, then

$$(\psi_2^{-1} \circ \psi_1)^* \sigma = (\psi_2^{-1} \circ \psi_1)^*(f \circ \psi_2)^* \omega = (f \circ \psi_2 \circ \psi_2^{-1} \circ \psi_1)^* \omega = (f \circ \psi_1)^* \omega = \sigma,$$

so $[f \circ \psi_1] = [f \circ \psi_2]$ in $\mathcal{S}_e(\Sigma, \sigma)$ modulo $\text{Sympl}(\Sigma, \sigma)$. Therefore the class in $\mathcal{S}_e(\Sigma, \sigma)$ modulo $\text{Sympl}(\Sigma, \sigma)$ we associated to $f$ is independent of the choice of path and isotopy. Similarly, it is independent of the choice of the representative of $[f]$; if $g: \Sigma \to M$ and $\alpha, \beta \in \text{Diff}(\Sigma)$ are such that $f = g \circ \alpha$ and $(g \circ \beta)^* \omega = \sigma$, then

$$(\beta^{-1} \circ \alpha \circ \varphi_f)^* \sigma = (\beta^{-1} \circ \alpha \circ \varphi_f)^*(g \circ \beta)^* \omega = (g \circ \beta \circ \beta^{-1} \circ \alpha \circ \varphi_f)^* \omega = (f \circ \varphi_f)^* \omega = \sigma,$$

so $[f \circ \varphi_f] = [g \circ \beta]$. Therefore the assignment $[f] \mapsto [f \circ \varphi_f]$ is well defined. It is also one-to-one: if $g$ and $f$ are not in the same class in $\text{Emb}(\Sigma, M)$ modulo $\text{Diff}(\Sigma)$, the associated $g \circ \varphi_g$ and $f \circ \varphi_f$ cannot be equal up to symplectic reparametrization of $\Sigma$. By the construction of $\varphi_f$ (see (3.13)), the map $f \mapsto f \circ \varphi_f$ sends paths starting from $i$ to paths starting from $i$ composed on a symplectic reparametrization of $(\Sigma, \sigma)$, hence the image of $[f] \mapsto [f \circ \varphi_f]$ is contained in the quotient of the path-connected component of $i$ in $\mathcal{S}_e(\Sigma, \sigma)$ modulo $\text{Sympl}(\Sigma, \sigma)$; it is onto this quotient since for every symplectic embedding $h$ that is path-connected to $i$ in $\mathcal{S}_e(\Sigma, \sigma)$, the class of $h$ in $C$ modulo $\text{Diff}(\Sigma)$ is sent to the class of $h$ in $\mathcal{S}_e(\Sigma, \sigma)$ modulo $\text{Sympl}(\Sigma, \sigma)$.

Since the $\text{Ham}(M, \omega)$-actions are from the left while the actions of $\text{Diff}(\Sigma)$ and $\text{Sympl}(\Sigma, \sigma)$ are from the right, the map $f \mapsto f \circ \varphi_f$ sends an $\text{Ham}(M, \omega)$-orbit in $\text{Emb}(\Sigma, M)$ to an $\text{Ham}(M, \omega)$-orbit in $\mathcal{S}_e(\Sigma, \sigma)$ modulo symplectic reparametrizations. For $\phi \in \text{Ham}(M, \omega)$, the embedding $h = \phi \circ f$ is connected to $\phi \circ i \in \mathcal{S}_e(\Sigma, \sigma)$ through a path in $\text{Emb}(\Sigma, M)$, hence there exists $\varphi_h \in \text{Diff}(\Sigma)$ such that $h \circ \varphi_h \in \mathcal{S}_e(\Sigma, \sigma)$. Since

$$(\varphi_f^{-1} \circ \varphi_h)^* \sigma = (\varphi_f^{-1} \circ \varphi_h)^*(f \circ \varphi_f)^* \omega = (f \circ \varphi_f)^* \omega = (\phi \circ \varphi_h)^* \omega = (h \circ \varphi_h)^* \omega = (h \circ \varphi_h)^* \omega = \sigma,$$

the symplectic embeddings $h \circ \varphi_h = (\phi \circ f) \circ \varphi_h$ and $\phi \circ (f \circ \varphi_f)$ are in the same class of $\mathcal{S}_e(\Sigma, \sigma)$ modulo $\text{Sympl}(\Sigma, \sigma)$.

\[\square\]

3.9 A neighbourhood of an element $[i]$ in $\text{Emb}(\Sigma, M)$ modulo $\text{Diff}(\Sigma)$ is identified with a neighbourhood of the zero section in the space of sections of the Normal bundle $N\Sigma$. The identification is as follows: choose a Riemannian metric on $M$ such that $N\Sigma$ is the orthogonal complement of $T\Sigma = T(i\Sigma)$ in $TM|_{i\Sigma}$; the exponential map with respect to that metric sends a neighbourhood of the zero section to a neighbourhood of the submanifold $i(\Sigma)$. This defines an atlas on $\text{Emb}(\Sigma, M)/\text{Diff}(\Sigma)$, all translation functions are smooth by the smoothness of the exponential map. Hence, by Lemma 3.8, we get a smooth structure on $\mathcal{S}_e(\Sigma, \sigma)$ modulo $\text{Sympl}(\Sigma, \sigma)$.

The map $\exp$ sends an orbit of the linearized action of $\text{Ham}(M, \omega)$ on $N\Sigma$ onto an orbit of $\text{Ham}(M, \omega)$ in $\text{Emb}(\Sigma, M)$ modulo reparametrizations.
Corollary 3.10. A path of symplectic embeddings $(\Sigma, \sigma) \to (M, \omega)$ through $f$ can be written, up to symplectic reparametrizations of $(\Sigma, \sigma)$, as $\Psi_t \circ f$, where $\Psi_t$ is a path in $\text{Ham}(M, \omega)$.

Proof. Note that for a time dependent vector field $\xi_t \in T_t S_e(\Sigma, \sigma) \times \mathbb{R}$, the decomposition (2.2) is smooth with respect to the time parameter. Thus the corollary follows from Lemma 2.4 and Lemma 2.7.

Definition 3.11 [6, Definition 10]. Let $(X, \Omega)$ be a weakly symplectic manifold. Let $G \curvearrowright X$ be a free action of a Lie group $G$ on $X$, such that $g^*\Omega = \Omega$ for all $g \in G$ ($g: X \to X$ denotes the action $x \to g.x$). The collection of subspaces

$$D_x := \{v \in T_x X \mid \Omega_x(v, \xi_X(x)) = 0 \forall \xi \in \mathfrak{g}\}$$

for $x \in X$ defines a distribution $D$ on $X$. Let $i_N: N \hookrightarrow X$ be a maximal integral manifold of $D$ and let $q: X \to X/G$ denote the projection to the orbit space. Suppose that the quotient induces a smooth structure on $q(N)$, in which there exists a unique weak symplectic structure $\Omega_{\text{red}}$ on $q(N)$ such that $(q|_N)^*\Omega_{\text{red}} = i_N^*\Omega$. Then the weakly symplectic manifold $(q(N), \Omega_{\text{red}})$ will be called a reduction or symplectic quotient of $(X, \Omega)$ with respect to the $G$-action.

A distribution on a smooth manifold $M$ assigns to each point $x \in M$ a $c^\infty$-closed subspace $D_x$ of $T_x M$. (The $c^\infty$-topology on a locally convex space $E$ is the finest topology for which all smooth curves $c: \mathbb{R} \to E$ are continuous.) If $D = \{D_x\}$ is a distribution on a manifold $M$ and $i: N \hookrightarrow M$ is the inclusion map of a path-connected submanifold $N$ of $M$, then $N$ is called an integral manifold of $D$ if $d_i(T_x N) = D_{i(x)}$ for all $x \in N$. An integral manifold of $D$ is called maximal if it is not properly contained in any other integral manifold. (Note that since the local model is a locally convex vector space, a manifold is path-connected if and only if it is connected.)

3.12 The motivation for Definition 3.11 comes from the standard reduction of a finite dimensional symplectic manifold $(X, \Omega)$ with respect to a Hamiltonian $G$-action with a moment map $\phi$. For a regular value $r$ of $\phi$, the tangent space at $p$ to the level surface $\phi^{-1}(r)$ is equal to the set $D_p$ of all vectors $v \in T_p X$ satisfying $\Omega(v, \xi_X(p)) = 0$ for all $\xi \in \mathfrak{g}$. The subspaces $D_p$ give a distribution $D$ on $X$, defined even in the absence of a moment map. If $G \curvearrowright X$ is a free symplectic action, then this distribution can be taken as the starting point of the optimal reduction method of Ortega and Ratiu [11].

Theorem 3.13. Consider $(S_e(\Sigma, \sigma), \omega^{D_{S_e(\Sigma, \sigma)}})$ with the action (3.10) of $\text{Symp}(\Sigma, \sigma)$. The $\text{Ham}(M, \omega)$-orbit $N$ through $i \in S_e(\Sigma, \sigma)$ is a maximal integral manifold of the distribution $D$. The restriction of $\omega^{D_{S_e(\Sigma, \sigma)}}$ to $N$ descends to a weak symplectic structure $\omega^{D_{\text{red}}}$ on the image $\mathcal{O} := q(N)$ in the orbit space under the projection $q: S_e(\Sigma, \sigma) \to S_e(\Sigma, \sigma)/\text{Symp}(\Sigma, \sigma)$. Thus the symplectic space $(\mathcal{O}, \omega^{D_{\text{red}}})$ is a reduction of $S_e(\Sigma, \sigma)$ with respect to the $\text{Symp}(\Sigma, \sigma)$-action.

Proof. By Proposition 2.2 and Proposition 2.10 $(S_e(\Sigma, \sigma), \omega^{D_{S_e(\Sigma, \sigma)}})$ is a weakly symplectic manifold. Let $i \in S_e(\Sigma, \sigma)$. By definition,

$$D_i = \{v \in T_i S_e(\Sigma, \sigma) \mid \omega^D_i(v, \xi_{S_e(\Sigma, \sigma)}(i)) = 0 \forall \xi \in \mathcal{X}(\Sigma, \sigma)\}.$$

By $1 \iff 3$ in Lemma 2.5 and the fact that $T_i S_e(\Sigma, \sigma) \subseteq \Gamma_{\text{closed}}(i^*TM)$ (by Lemma 2.9),

$$T_i(\text{Ham}(M, \omega).i) = \{v \in T_i S_e(\Sigma, \sigma) \mid \omega^D_i(v, \xi) = 0 \forall \xi \in \Gamma_{\text{closed}}(i^*TM) \text{ that is everywhere tangent to } \Sigma\}.$$
By Lemma 2.7, the space of vector fields in $\Gamma_{\text{closed}}(i^*TM)$ that are everywhere tangent to $\Sigma$ is identified with the space $\mathcal{X}(\Sigma, \sigma)$. So $\text{Ham}(M, \omega)$-orbits are integral manifolds of $\mathcal{D}$.

To see that maximal, assume that the $\text{Ham}(M, \omega)$-orbit $\mathcal{N}$ is properly contained in a (path-connected) integral manifold $\tilde{N}$. By Lemma 3.10 and Corollary 2.8, a path in $\tilde{N}$ from an element $f \in \mathcal{N}$ to an element in $\tilde{N} \setminus \mathcal{N}$ can be written as $\Psi_t \circ f \circ \alpha_t$, where $\Psi_t$ is a path in $\text{Ham}(M, \omega)$ and $\alpha_t : \Sigma \rightarrow \Sigma$ are in $\text{Sympl}(\Sigma, \sigma)$; moreover, the generating vector field of the path $v_t$ decomposes uniquely as $\xi_{v_t} + \tau_{v_t}$, where $\xi_{v_t}$ is symplectically orthogonal to $\Sigma$ and $\tau_{v_t}$ is everywhere tangent to $\Sigma$, and $\xi_{v_t}$ is the vector field generating $\Psi_t$ and $\tau_{v_t}$ is generating $\alpha_t$. Since $d\iota_{\xi_{v_t}}(T_x \tilde{N}) = D\iota_{\xi_{v_t}}(x)$ for all $x \in \tilde{N}$ (where $\iota_{\tilde{N}}$ is the inclusion map of $\tilde{N}$ into $M$), and by 3 $\Leftrightarrow$ 4 in Lemma 2.5 for every $v_t \in d\iota_{\xi_{v_t}}(T_x \tilde{N})$, the vector $v_t = \xi_{v_t}$ and $\tau_{v_t} = 0$. Hence $\alpha_t = \text{Id}$ for every $t$, and we get a path in $\text{Ham}(M, \omega)$ connecting an element in a $\text{Ham}(M, \omega)$-orbit with an element outside the orbit: a contradiction.

Consider the inclusion-quotient diagram

$$
\begin{array}{ccc}
\mathcal{N} & \xrightarrow{i_N} & S_\text{e}(\Sigma, \sigma) \\
\downarrow & & \downarrow \\
\mathcal{O} & \xrightarrow{\text{quot}} & O
\end{array}
$$

By Lemma 3.8 and 3.9, the quotient induces a smooth structure on $\mathcal{O} = q(\mathcal{N})$. The pullback under the inclusion $i_N$ of the form $\omega^D_{S_\text{e}(\Sigma, \sigma)}$ to $\mathcal{N}$ is closed, horizontal (by Lemma 2.5 and Lemma 2.7) and invariant to the action of $\text{Sympl}(\Sigma, \sigma)$, hence basic. Therefore it is the pullback under the quotient $q$ of a closed 2-form $\omega^D_{\text{red}}$ on $\mathcal{O}$. The reduced form is given by

$$
\omega^D_{\text{red}}([v_1], [v_2]) = \int_\Sigma \omega(v_1, v_2)\sigma.
$$

By part (1) of Lemma 3.3, the form $\omega^D_{\text{red}}$ is weakly non-degenerate. \hfill \Box

**Remark 3.14** Similarly we get a closed 2-form $\omega^\text{red}_{S_\text{e}(\Sigma, \sigma)}$ on $\mathcal{O}$:

$$
\omega^\text{red}_{S_\text{e}(\Sigma, \sigma)}([v_1], [v_2]) = \omega_{S_\text{e}(\Sigma, \sigma)}(v_1, v_2),
$$

well defined and weakly non-degenerate by Lemma 3.3. By Lemma 3.2, we get $\omega^\text{red}_{S_\text{e}(\Sigma, \sigma)} = 2\omega^D_{\text{red}}$ on $\mathcal{O}$. However, in that case the $\text{Ham}(M, \omega)$-orbits are not integral manifolds of the distribution $\mathcal{D}$ given by

$$
D_i = \{v \in T_i S_\text{e}(\Sigma, \sigma) \mid \omega_{S_\text{e}(\Sigma, \sigma)}(v, \xi_{S_\text{e}(\Sigma, \sigma)}(i)) = 0 \forall \xi \in \mathcal{X}(\Sigma, \sigma)\}.
$$

The reason for the difference is that for $v \in T_i S_\text{e}(\Sigma, \sigma)$ the form $\iota_v \omega^D_{S_\text{e}(\Sigma, \sigma)} |_{\mathcal{X}(\Sigma, \sigma)} \equiv 0$ iff $v = i^* V_H$ for a Hamiltonian vector field $V_H$ on $M$ (by Lemma 2.5 and Lemma 2.7), whereas $\iota_v \omega^D_{S_\text{e}(\Sigma, \sigma)} |_{\mathcal{X}(\Sigma, \sigma)} \equiv 0$ for every $v$ in $T S_\text{e}(\Sigma, \sigma)$, by part (2) of Lemma 3.3. From the same reason, the form $\omega_{S_\text{e}(\Sigma, \sigma)}$ is degenerate. \hfill \Box

## 4 Corollary to moduli spaces of $J$-holomorphic $\Sigma$-curves

For $J \in \mathcal{J}(M, \omega)$, denote by

$$
\text{Ham}^J(M, \omega)
$$

the subset of $\text{Ham}(M, \omega)$ of $J$-holomorphic Hamiltonian symplectomorphisms.
Lemma 4.1. If $f$ is an embedded $(j, J)$-holomorphic curve $f : \Sigma \to M$, and $g = \phi \circ f$ for $\phi \in \text{Ham}^J(M, \omega)$ then $g \in \mathcal{M}_c(A, \Sigma, J)$. If, in addition, $f^*\omega = \sigma$, then $g^*\omega = \sigma$.

Proof. If $g = \phi \circ f$ for $\phi \in \text{Ham}^J(M, \omega)$ then $g$ is an embedding $\Sigma \to M$, and
\[ dg \circ j = d\phi \circ df \circ j = d\phi \circ J \circ df = J \circ d\phi \circ df = J \circ dg, \]
i.e., $g$ is a $(j, J)$-holomorphic $\Sigma$-curve in $M$. If $f^*\omega = \sigma$, then
\[ g^*\omega = (\phi \circ f)^*\omega = f^*\phi^*\omega = f^*\omega = \sigma. \]

\[ \square \]

Lemma 4.2. Consider $f_1 : \Sigma \to M$ and $f_2 : \Sigma \to M$. Assume that $f_i^*\omega = \sigma$ for $i = 1, 2$. If $\phi$ is a diffeomorphism $\Sigma \to \Sigma$ such that $f_1 \circ \phi = f_2$, then $\phi \in \text{Sympl}(\Sigma, \sigma)$.

Proof.
\[ \sigma(d\phi(\cdot), d\phi(\cdot)) = f_1^*\omega(d\phi(\cdot), d\phi(\cdot)) = \omega(df_1 d\phi(\cdot), df_1 d\phi(\cdot)) \]
\[ = \omega(df_2(\cdot), df_2(\cdot)) = f_2^* \omega(\cdot, \cdot) \]
\[ = \sigma(\cdot, \cdot) \]

\[ \square \]

Lemma 4.3. Consider two $(j, J)$-holomorphic immersions $f_1 : \Sigma \to M$ and $f_2 : \Sigma \to M$. If $\phi$ is a diffeomorphism $\Sigma \to \Sigma$ such that $f_1 \circ \phi = f_2$, then $\phi \in \text{Aut}(\Sigma, j)$.

Proof.
\[ df_1 \circ d\phi \circ j = df_2 \circ j = J \circ df_2 = J \circ df_1 \circ d\phi = df_1 \circ j \circ d\phi. \]

Since $f_1$ is an immersion, we conclude $d\phi \circ j = j \circ d\phi$.

\[ \square \]

Lemma 4.4. Let $J$ be an almost complex structure on $M$, and $f : \Sigma \to M$ an immersed $(j, J)$-holomorphic curve. Assume that $J$ is $\omega$-compatible. If $v$ is not everywhere tangent to $f(\Sigma)$, then $\omega^D_{\text{red}}([v], [\tilde{J}v]) \neq 0$.

Proof. Decompose $v = \xi_v + \tau_v$, with $\xi_v$, everywhere $\omega$-orthogonal to $\Sigma$ and $\tau_v$, everywhere tangent to $\Sigma$. By assumption $\xi_v \neq 0$. Then $\tilde{J}v = \tilde{J}\xi_v + \tilde{J}\tau_v$ with $\tilde{J}\xi_v \neq 0$. By Lemma 4.7, $[\tilde{J}v] = [\tilde{J}\xi_v] \neq 0$. Thus, since $J$ is $\omega$-tamed and $\sigma$ is an area form, $\omega^D_{\text{red}}([v], [\tilde{J}v]) = \omega^D_{\text{red}}([\xi_v], [\tilde{J}\xi_v]) = \int_{\Sigma} \omega(\xi_v, J(\xi_v)) \sigma \neq 0$.

\[ \square \]

Proof of Corollary 4.4. $\text{Ham}^J(M, \omega)$ is a closed subgroup of $\text{Ham}(M, \omega)$. (Notice that a diffeomorphism $\phi : M \to M$ is $J$-holomorphic iff $\phi^{-1}$ is.) Since $J$ is regular for the projection map $p_A : \mathcal{M}_c(A, \Sigma, J) \to \mathcal{J}$, the space $\mathcal{M}_c(A, \Sigma, J)$ of embedded $(j, J)$-holomorphic $\Sigma$-curves in a homology class $A \in H_2(M, \mathbb{Z})$ is a finite-dimensional manifold [9, Thm 3.1.5]. By Lemma 4.1, the group $\text{Ham}^J(M, \omega)$ acts on $\mathcal{M}_c(A, \Sigma, J)$ by composition on the left. Since the action of $\text{Ham}^J(M, \omega)$ on $C^\infty(\Sigma, M)$ preserves both $\omega^D_{\Sigma, \sigma}$ and the almost complex structure $\tilde{J} : TC^\infty(\Sigma, M) \to TC^\infty(\Sigma, M)$ induced from $J : TM \to TM$, it also preserves the map defined by $\tilde{g}(\tau_1, \tau_2) = \omega^D_{\Sigma, \sigma}(\tau_1, \tilde{J}\tau_2)$.
By Lemma [A,5], the manifold $\mathcal{M}_c(A, \Sigma, J)$ is closed under $\tilde{J}$, hence by Lemma [A,4] the restriction of $\omega_{(\Sigma, \sigma)}^D$ to $\mathcal{M}_c(A, \Sigma, J)$ is non-degenerate, so the map $\tilde{g}$ restricts to a metric on $\mathcal{M}_c(A, \Sigma, J)$, and $\Ham^J(M, \omega)$ acts on the moduli space as a subgroup of the isometry group. Since the action of the isometry group on a finite-dimensional manifold is proper and $\Ham^J(M, \omega)$ is a closed subgroup, every orbit of $\Ham^J(M, \omega)$ is an embedded submanifold of $\mathcal{M}_c(A, \Sigma, J)$.

Let $N$ be an orbit of $\Ham^J(M, \omega)$ through an embedded $(j, J)$-holomorphic curve $f: \Sigma \to M$ for which $f^*\omega = \sigma$. By Lemma [4,1] the orbit $N \subset \mathcal{M}_c(A, \Sigma, J) \cap \mathcal{S}_c(\Sigma, \sigma)$. Thus, by Lemma [2,5] and Lemma [3,3] the reductions $\omega_{\text{red}}^D$ and $\omega_{\text{red}}^S(\Sigma)$ are well defined on the quotient $N$ modulo $\text{Sympl}(\Sigma, \sigma)$. By Lemma 4.2 and 4.3, $N$ modulo $\text{Aut}(\Sigma, j)$ is the same as $N$ modulo $\text{Sympl}(\Sigma, \sigma)$. By Lemma 4.4, the form $\omega_{\text{red}}^D$ restricted to $N$ modulo $\text{Aut}(\Sigma, j)$ is non-degenerate. By Lemma 3.2 the form $\omega_{\text{red}}^S(\Sigma)$ coincides with the form $2\omega_{\text{red}}^D$ on the quotient.

**Remark 4.5** For examples of regular integrable compatible almost complex structures, we look at Kähler manifolds whose automorphism groups act transitively. By [9, Proposition 7.4.3], if $(M, \omega_0, J_0)$ is a compact Kähler manifold and $G$ is a Lie group that acts transitively on $M$ by holomorphic diffeomorphisms, then $J_0$ is regular for every $A \in H_2(M, \mathbb{Z})$. This applies, e.g., when $M = \mathbb{C}P^n, \omega_0$ the Fubini-Study form, $J_0$ the standard complex structure on $\mathbb{C}P^n$, and $G$ is the automorphism group $\text{PSL}(n+1)$.

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### A Appendix : a symplectic form and compatible almost complex structures on the space of immersed surfaces

To prove the closedness part of Proposition [2,1] we first show the following claim.

**Claim A.1.** For $\tau_1, \tau_2: \Sigma \to f^*(TM)$, we have

$$\omega(\tau_1, \tau_2) \sigma = \frac{1}{2} \iota(\tau_1, 0, \tau_2, 0) (e^{\ast}(\omega)) \land (\pi_{\Sigma}^k(\sigma)),$$

where $\pi_{\Sigma}: C^\infty(\Sigma, M) \times \Sigma \to \Sigma$ is the projection onto the second coordinate.

**Proof.** Indeed,

$$\iota(\tau_1, 0, \tau_2, 0) (e^{\ast}(\omega)) \land (\pi_{\Sigma}^k(\sigma))(\cdot, \cdot) = 2\omega(\tau_1, \tau_2) \sigma(\cdot, \cdot) + 2\omega(\tau_1, df(\cdot)) \land \sigma(d\pi_{\Sigma}(\tau_2, 0), \cdot).$$

Notice that the second summand in the right-hand term always vanishes.

**Proof of the closedness part of Proposition [2,1]** We will show that for any two surfaces $R_1$ and $R_2$ in $\mathcal{S}_c(\Sigma)$, that are homologous relative to a common boundary $\partial R$,

$$\int_{R_1} \omega^D(\Sigma, \sigma) = \int_{R_2} \omega^D(\Sigma, \sigma). \quad (A.14)$$

See also [6] Prop. 12] for a proof that $d\omega^D(\Sigma, \sigma) = 0$ in the Convenient Setup.
Let $R_1$ and $R_2$ be two surfaces that are homologous relative to a common boundary $\partial R$. By the above claim,

\[
\int_{R_i} \omega^D(\Sigma, \sigma) = \frac{1}{2} \int_{R_i \times \{x\}} \left( \int_{\{f\} \times \Sigma} (ev^*(\omega)) \wedge (\pi^{\Sigma}_*(\sigma)) \right) = \frac{1}{2} \int_{R_i \times \Sigma} (ev^*(\omega)) \wedge (\pi^{\Sigma}_*(\sigma)).
\]

Since $R_1 \times \Sigma$ is homologous to $R_2 \times \Sigma$ relative to the boundary $\partial R \times \Sigma$, and $(ev^*(\omega)) \wedge (\pi^{\Sigma}_*(\sigma))$ is closed, we have that:

\[
\int_{R_1 \times \Sigma} (ev^*(\omega)) \wedge (\pi^{\Sigma}_*(\sigma)) = \int_{R_2 \times \Sigma} (ev^*(\omega)) \wedge (\pi^{\Sigma}_*(\sigma)).
\]

Therefore, we get (A.14).

\[\square\]

**Compatible almost complex structures**

An *almost complex structure* on a manifold $M$ is an automorphism of the tangent bundle,

\[J: TM \to TM,\]

such that $J^2 = -\text{Id}$. The pair $(M, J)$ is called an *almost complex manifold*.

An almost complex structure is *integrable* if it is induced from a complex manifold structure. In dimension two any almost complex manifold is integrable (see, e.g., [8, Th. 4.16]). In higher dimensions this is not true [1].

**Definition A.2** Let $J$ be an almost complex structure on $M$. We define a map

\[\tilde{J}: \mathcal{T}C(\Sigma, M) \to \mathcal{T}C(\Sigma, M)\]

as follows: for $\tau: \Sigma \to f^*(TM)$, the vector $\tilde{J}(\tau)$ is the section $\tilde{J} \circ \tau$, where $\tilde{J}$ is the map defined by the commutative diagram

\[
\begin{array}{ccc}
 f^*(TM) & \xrightarrow{J} & f^*(TM) \\
 \downarrow & & \downarrow \\
 TM & \xrightarrow{J} & TM
\end{array}
\]

Due to the properties of the almost complex structure $J$, the map $\tilde{J}$ is an automorphism and $\tilde{J}^2 = -\text{Id}$.

**Claim A.3.** Let $J$ be an almost complex structure on $M$. Then $\tilde{J}$ is an almost complex structure on $\mathcal{C}(\Sigma, M)$.

An almost complex structure $J$ on $M$ is *tamed* by a symplectic form $\omega$ if $\omega_x(v, Jv) > 0$ for all non-zero $v \in T_x M$. If, in addition, $\omega_x(Jv, Jw) = \omega_x(v, w)$ for all $v, w \in T_x M$, we say that $J$ is *$\omega$-compatible*. The space $\mathcal{J} = \mathcal{J}(M, \omega)$ of $\omega$-compatible almost complex structures is non-empty and contractible, in particular path-connected [8, Prop. 4.1].

**Lemma A.4.** Let $J$ be an almost complex structure on $M$. 

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1. If $J$ is $\omega$-tamed, then the induced almost complex structure $\tilde{J}$ on the space of immersions $\Sigma \to M$ is $\omega_D(\Sigma, \sigma)$-tamed.

2. If $J$ is $\omega$-compatible, then $\tilde{J}$ is $\omega_D(\Sigma, \sigma)$-compatible.

Proof. 1. For a non-zero vector field $\tau: \Sigma \to f^*TM$,

$$\omega_D(\Sigma, \sigma)_f(\tau, \tilde{J}(\tau)) := \int_{\Sigma} \omega(\tau, J(\tau)) \sigma > 0,$$

the last inequality is since $J$ is $\omega$-tamed, $f$ is an immersion, and $\sigma$ is an area-form.

2. For $\tau_1, \tau_2: \Sigma \to f^*TM$,

$$\omega_D(\Sigma, \sigma)_f(\tilde{J}(\tau_1), \tilde{J}(\tau_2)) := \int_{\Sigma} \omega(J(\tau_1), J(\tau_2)) \sigma = \int_{\Sigma} \omega(\tau_1, \tau_2) \sigma = \omega_D(\Sigma, \sigma)_f(\tau_1, \tau_2),$$

since $J$ is $\omega$-compatible.

Proof of the non-degeneracy part of Proposition 2.1. It follows from part (1) of Lemma A.4, and the fact that the space of $\omega$-tamed structures is not empty.

Fix $\Sigma = (\Sigma, j)$. A smooth ($C^\infty$) curve $f: \Sigma \to M$ is called $J$-holomorphic if the differential $df$ is a complex linear map between the fibers $T_p(\Sigma) \to T_{f(p)}(M)$ for all $p \in \Sigma$, i.e.

$$df_p \circ j_p = J_{f(p)} \circ df_p.$$

A $J$-holomorphic curve is simple if it cannot be factored through a branched covering of $\Sigma$. The moduli space $\mathcal{M}_i(A, \Sigma, J)$ of simple immersed $J$-holomorphic $\Sigma$-curves in a homology class $A \in H_2(M, \mathbb{Z})$. We look at almost complex structures that are regular for the projection map

$$p_A: \mathcal{M}_i(A, \Sigma, J) \to J;$$

for such a $J$, the space $\mathcal{M}_i(A, \Sigma, J)$ is a finite-dimensional manifold [2, Thm. 3.1.5].

Lemma A.5. If $J$ is an integrable almost complex structure on $M$ that is regular for $A$, and $f: \Sigma \to M$ is a $J$-holomorphic map in $A$, then for $\tau \in T_{f(J)}M(A, \Sigma, J)$, the vector $J \circ \tau$ is also in $T_f\mathcal{M}(A, \Sigma, J)$.

For proof, see, e.g., [2, Lem. 3.3].

As a result of Lemma A.5 and Lemma A.4, we get the following proposition.

Proposition A.6. If $J$ is an integrable almost complex structure on $M$ that is compatible with $\omega$ and regular for $A$, then $\omega_D(\Sigma, \sigma)$ restricted to $\mathcal{M}_i(A, \Sigma, J)$ is symplectic.

Lemma A.7. Let $J$ be an almost complex structure on $M$. Assume that $f: \Sigma \to M$ is $J$-holomorphic. Then, at $x \in \Sigma$,

1. if $\tau_x \in df_x(T_x \Sigma)$ then $J_{f(x)}(\tau_x) \in df_x(T_x \Sigma)$;

2. if $J$ is $\omega$-compatible and $\phi_x$ is $\omega$-orthogonal to $df_x(T_x \Sigma)$, then so is $J_{f(x)}(\phi_x)$. 

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Proof. 1. By assumption $\tau_x = df_x(\alpha)$ for $\alpha \in T_x \Sigma$. Hence, since $f$ is $J$-holomorphic,

$$J_{f(x)}(\tau_x) = J_{f(x)}(df_x(\alpha)) = df_x(j_x \alpha).$$

2. By the previous item, $J_{f(x)}(df_x(T_x \Sigma)) \subseteq df_x(T_x \Sigma)$, hence, since $J^2 = -\text{Id}$,

$$J_{f(x)}(df_x(T_x \Sigma)) = df_x(T_x \Sigma).$$

Let $\tau_x \in df_x(T_x \Sigma)$, then there exists $\tau'_x \in df_x(T_x \Sigma)$ such that $\tau_x = J_{f(x)}(\tau'_x)$. By assumption, $\omega(\phi_x, \tau'_x) = 0$. Thus

$$\omega(J_{f(x)}(\phi_x), \tau_x) = \omega(J_{f(x)}(\phi_x), J_{f(x)}(\tau'_x)) = \omega(\phi_x, \tau'_x) = 0.$$

□

Corollary A.8. Let $J$ be an $\omega$-tamed almost complex structure on $M$. Assume that $f : \Sigma \to M$ is $J$-holomorphic. Then every $\mu \in T_f(C^\infty(\Sigma, M))$ can be uniquely decomposed as

$$\mu = \mu' + \mu'',$$

where $\mu'(x) \in df_x(T_x \Sigma)$ at every $x \in \Sigma$, and $\mu''(x)$ is $\omega$-orthogonal to $df_x(T_x \Sigma)$ at every $x \in \Sigma$.

Proof. At $x \in \Sigma$, if $v \in W_x = df_x(T_x \Sigma)$, then $J(v) \in W_x$ (by part (1) of Lemma A.7). Since $J$ is $\omega$-tamed, if $v \neq 0$, $\omega(v, J(v)) > 0$, hence $0 \neq v \in W_x$ is not in $W_x^\omega$. Thus $W_x \cap W_x^\omega = \{0\}$. Since $\dim W_x^\omega = \dim M - \dim W_x$, we deduce that $T_f(x) M = W_x \oplus W_x^\omega$.

To conclude the corollary, recall that if a bundle $E \to B$ equals the direct sum of sub-bundles $E_1 \to B$ and $E_2 \to B$, then the space of sections of $E$ equals the direct sum of the space of sections of $E_1$ and the space of sections of $E_2$.

□

Remark A.9 Fix a symplectic form $\sigma$ on and a complex structure $j$ that is $\sigma$-compatible on $\Sigma$. Given a symplectic embedding $f : (\sigma, \Sigma) \to (M, \omega)$ there is a $J \in \mathcal{J}(M, \omega)$ such that $f$ is $(j, J)$-holomorphic. Define $J|_{T_f(\Sigma)}$ such that $f$ is holomorphic. Extend it to a compatible fiberwise complex structure on the symplectic vector bundle $TM|_{f(\Sigma)}$. Then extend it to a compatible almost complex structure on $M$. See [8 Section 2.6].

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