Finite element method for singularly perturbed problems with two parameters on a Bakhvalov-type mesh in 2D

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Abstract
On a Bakhvalov-type mesh widely used for boundary layers, we consider the finite element method for singularly perturbed elliptic problems with two parameters on the unit square. It is a very challenging task to analyze uniform convergence of finite element method on this mesh in 2D. The existing analysis tool, quasi-interpolation, is only applicable to one-dimensional case because of the complexity of Bakhvalov-type mesh in 2D. In this paper, a powerful tool, Lagrange-type interpolation, is proposed, which is simple and effective and can be used in both 1D and 2D. The application of this interpolation in 2D must be handled carefully. Some boundary correction terms must be introduced to maintain the homogeneous Dirichlet boundary condition. These correction terms are difficult to be handled because the traditional analysis do not work for them. To overcome this difficulty, we derive a delicate estimation of the width of some mesh. Moreover, we adopt different analysis strategies for different layers. Finally, we prove uniform convergence of optimal order. Numerical results verify the theoretical analysis.

Keywords Singular perturbation · Convection–diffusion equation · Finite element method · Bakhvalov-type mesh · Two parameters

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1 Introduction

In this paper, we consider the singularly perturbed elliptic problems of the form

\[ Lu := -\varepsilon_1 \Delta u + \varepsilon_2 b(x) u_x + c(x) u = f(x, y) \quad \text{in } \Omega := (0, 1) \times (0, 1), \]

\[ u |_{\partial \Omega} = 0, \]

with

\[ b(x) \geq \lambda > 0, \quad c(x) \geq \beta > 0 \quad \text{for } x \in [0, 1], \]

\[ c(x) - \frac{1}{2} \varepsilon_2 b'(x) \geq \gamma > 0, \]

\[ f(0, 0) = f(0, 1) = f(1, 1) = f(1, 0) = 0. \]

Above, \( \lambda, \beta, \gamma \) are positive constants, and we assume that \( b(x), c(x) \) and \( f(x, y) \) are sufficiently smooth functions on \( \hat{\Omega} \). Here we only discuss the case of \( 0 < \varepsilon_1, \varepsilon_2 \ll 1 \) (see [7]), and the cases of \( \varepsilon_2 = 0 \) and \( \varepsilon_2 = 1 \) are analyzed in [11] and its references. In this paper, we use \( b(x) \) and \( c(x) \) in our problem instead of \( b(x, y) \) and \( c(x, y) \) because the prior information of the solution in the latter case is still open at present; see [10].

Moreover, these assumptions and condition (4) ensure that there exists a unique solution \( u \in C^{3,\alpha_0}(\hat{\Omega}) \) with \( \alpha_0 \in (0, 1) \) (see [6]), which is characterized by exponential layers at \( x = 0 \) and \( x = 1 \), parabolic layers at \( y = 0 \) and \( y = 1 \), and corner layers at four corners of the domain. For the treatment of boundary layer, researchers usually use a class of special meshes which are very fine on the layer region of the solution. Compared to the quasi-uniform mesh, this kind of meshes can capture the change of layers better. Among those the most representative ones are Shishkin mesh [9] and Bakhvalov mesh [2], and numerical experiments show the superiority of Bakhvalov mesh.

In fact, up to now, there are few articles about finite element method on Bakhvalov mesh, because the application of Lagrange interpolation is not workable. In [8], Roos clearly stated the difficulty in convergence analysis on Bakhvalov-type mesh in 1D, and obtained the optimal convergence order by using quasi-interpolation. Brdar and Zarin analyzed a singularly perturbed problem with two-parameter in 1D using the same method in [3]. However, quasi-interpolation is not feasible in the cases of higher-order finite element methods or higher-dimensional problems. Recently, Zhang and Liu proposed a new interpolation which is much simpler to construct and analyze for the 1D one-parameter problem in [12], and can be directly extended to higher-order cases. But, it is not straightforward to generalize Zhang and Liu’s idea to 2D problems, because compared with 1D case, new difficulties arise in the case of 2D, such as analysis of boundary correction terms and how to choose different analysis strategies for different layers. However, due to the lack of powerful tools to solve these difficulties, convergence analysis of finite element method on Bakhvalov-type mesh in 2D is still in a blank state.

In this paper, we prove uniformly convergence of optimal order for problem (1), which is solved by the finite element method of any order on a Bakhvalov-type mesh.
Firstly, we propose a new Lagrange-type interpolation according to the characteristics of layers, which makes convergence analysis possible. Secondly, after applying this interpolation, the boundary correction terms must be introduced to maintain the homogeneous Dirichlet boundary condition. However, these correction terms are difficult to be handled because the traditional analysis do not work for them. To overcome this difficulty, we make use of Lemma 3.4, which gives a delicate estimation of the width of mesh scale of the strong exponential layer near the transition point. Thirdly, different strategies are used to analyze different layers according to the characteristics of layers, especially the strong exponential layer and its related corner layers. Finally, we prove uniform convergence of optimal order.

The rest of this article is organized as follows. In the Section 2, the prior estimation of the solution of the continuous problem is given. In the Section 3, we will construct the Bakhvalov-type mesh, and give some mesh properties, and finally establish the finite element method. The new Lagrange-type interpolation will appear in the Section 4, and we will also prove some results of Lagrange interpolation error. The convergence analysis is carried out in Section 5. Finally, numerical experiments are given in Section 6 to verify our conclusion.

Throughout the paper, we shall use $C$ to denote a generic positive constant independent of $\varepsilon_1, \varepsilon_2$ and $N$, which can take different values at different places. For any domain $D$ of $\Omega$, we use the standard notation for Banach spaces $L^p(D)$, Sobolev spaces $W^{k,p}(D)$, $H^k(D) = W^{k,2}(D)$. Define $\| \cdot \|_{L^\infty(D)}$, $\| \cdot \|_D$ to be $\| \cdot \|_{L^\infty(D)}$, $\| \cdot \|_D$ to be $\| \cdot \|_{L^2(D)}$, and $| \cdot |_D$ to be the seminorms of $\| \cdot \|_{W^1(D)}$; The scalar product in $L^2(D)$ is denoted with $(\cdot, \cdot)_D$. And we will drop the subscript $D$ from the notation for simplicity when $D = \Omega$.

2 A priori estimates of solution of the continuous problem

Compared with the parabolic layer, the exponential layer changes more dramatically. So in order to describe the exponential layers at $x = 0$ and $x = 1$, we introduce the characteristic equation as following

$$-\varepsilon_1 g^2(x) + \varepsilon_2 b(x) g(x) + c(x) = 0.$$ 

This equation defines two continuous functions $g_0, g_1 : [0, 1] \to \mathbb{R}$ with $g_0(x) < 0$, $g_1(x) > 0$. Let

$$\mu_0 = -\max_{0 \leq x \leq 1} g_0(x), \quad \mu_1 = \min_{0 \leq x \leq 1} g_1(x).$$

For the sake of simplicity, we take

$$\mu_0 = \frac{-\varepsilon_2 b_* + \sqrt{\varepsilon_2^3 b_*^2 + 4\varepsilon_1 b}}{2\varepsilon_1}, \quad \mu_1 = \frac{\varepsilon_2 \lambda + \sqrt{\varepsilon_2^3 \lambda^2 + 4\varepsilon_1 \beta}}{2\varepsilon_1},$$
with $b_* = \max_{0 \leq x \leq 1} b(x)$, which is the same as [11, (6)]. Then, we give some properties of $\mu_0$ and $\mu_1$ (see [10]):

\[ \mu_0 \leq \mu_1, \quad \max\{\mu_0^{-1}, \epsilon_1 \mu_1\} \leq C(\epsilon_2 + \epsilon_1^2), \quad (5) \]

\[ \epsilon_2 \mu_0 \leq \lambda^{-1} \|c\|_{\infty}, \quad \epsilon_2 (\epsilon_1 \mu_1)^{-\frac{1}{2}} \leq C \epsilon_2^\frac{1}{2}. \quad (6) \]

These properties will play an important role in the subsequent analysis.

In this paper, we assume that

\[ \mu_1^{-1} \leq \mu_0^{-1} \leq N^{-1}, \quad (7) \]

and it is worth noting that there is no such limitation in practice. By direct computations of (7), we can obtain

\[ \epsilon_1 \leq c_0 N^{-2}, \quad \epsilon_2 \leq c_1 N^{-1}, \quad (8) \]

with $c_0 = \beta$ and $c_1 = \beta \frac{b_*}{\beta}$. On the basis of the prior estimation of the solution given in [10], we make the following assumption about the decomposition of the solution and the prior estimation of each component. In the subsequent analysis, $k$ is a fixed positive integer and $k \geq 1$.

**Assumption 2.1** Let there be given elliptic problem (1) on the unit square $\tilde{\Omega}$ satisfying conditions (2)–(4), and let $p \in (0, 1)$ and $k_0 \in (0, \frac{1}{2})$ be arbitrary. Assume that

\[ 2 \|b'\|_{\infty} \epsilon_2 \leq k_0(1 - p)\beta. \]

Furthermore, let $\delta$ be a positive constant satisfying

\[ \delta^2 \leq \frac{(1 - p)\beta}{2}. \]

Then, the solution $u$ of problem (1) can be decomposed as

\[ u = S + E_{10} + E_{11} + E_{20} + E_{21} + E_{31} + E_{32} + E_{33} + E_{34}, \]

where for all $(x, y) \in \tilde{\Omega}$ and $0 \leq i + j \leq k + 1$, the regular part $S$ satisfies

\[ \left| \frac{\partial^{i+j} S}{\partial x^i \partial y^j} \right| \leq C, \]

the exponential and parabolic layer components satisfy

\[ \left| \frac{\partial^{i+j} E_{10}}{\partial x^i \partial y^j} \right| \leq C \mu_0^i e^{-p \mu_0 x}, \quad (9) \]

\[ \left| \frac{\partial^{i+j} E_{11}}{\partial x^i \partial y^j} \right| \leq C \mu_1^i e^{-p \mu_1 (1-x)}, \quad (10) \]

\[ \left| \frac{\partial^{i+j} E_{20}}{\partial x^i \partial y^j} \right| \leq C \epsilon_1^{-\frac{j}{2}} e^{-\frac{\delta y}{\sqrt{\epsilon_1}}}, \]

\[ \left| \frac{\partial^{i+j} E_{21}}{\partial x^i \partial y^j} \right| \leq C \epsilon_1^{-\frac{j}{2}} e^{-\frac{\delta(1-y)}{\sqrt{\epsilon_1}}}, \]
while the corner layer components satisfy the following estimates

\[
\left| \frac{\partial^{i+j} E_{31}}{\partial x^i \partial y^j} \right| \leq C \epsilon_1^{-\frac{i}{2}} \mu_0^{1/2} e^{-p \mu_0 x} e^{-\frac{\delta y}{\sqrt{\epsilon_1}}},
\]

\[
(11)
\]

\[
\left| \frac{\partial^{i+j} E_{32}}{\partial x^i \partial y^j} \right| \leq C \epsilon_1^{-\frac{i}{2}} \mu_1^{1/2} e^{-p \mu_1 (1-x)} e^{-\frac{\delta y}{\sqrt{\epsilon_1}}},
\]

\[
(12)
\]

\[
\left| \frac{\partial^{i+j} E_{33}}{\partial x^i \partial y^j} \right| \leq C \epsilon_1^{-\frac{i}{2}} \mu_1^{1/2} e^{-p \mu_1 (1-x)} e^{-\frac{\delta (1-y)}{\sqrt{\epsilon_1}}},
\]

\[
\left| \frac{\partial^{i+j} E_{34}}{\partial x^i \partial y^j} \right| \leq C \epsilon_1^{-\frac{i}{2}} \mu_0^{1/2} e^{-p \mu_0 x} e^{-\frac{\delta (1-y)}{\sqrt{\epsilon_1}}}.
\]

Remark 1 If Assumption 2.1 is still valid after \(b(x)\) and \(c(x)\) is replaced by \(b(x, y)\) and \(c(x, y)\), then the analysis and conclusion in this paper will be also applicable.

3 Bakhvalov-type mesh and finite element method

3.1 Bakhvalov-type mesh

Let \(N \in \mathbb{N}, N \geq 8\), can be divisible by 4. Define

\[
\sigma_{x,i} := \frac{\tau}{p \mu_i} \ln \mu_i \quad i = 0, 1 \quad \text{and} \quad \sigma_y := \frac{\tau}{\delta \sqrt{\epsilon_1}} \ln \frac{1}{\epsilon_1},
\]

where \(\tau \geq k + 1\) is a user-chosen parameter and \(p \in (0, 1)\) is the parameter from Assumption 2.1. On \(x\)-axis, we set \(\sigma_{x,0}\) and \(1 - \sigma_{x,1}\) as transition points, where the mesh changes from fine to coarse and vice versa. On \(y\)-axis, we set \(\sigma_y\) and \(1 - \sigma_y\) as transition points. For technical reasons, we also assume

\[
\sigma_{x,i} \leq \frac{1}{4} \quad i = 0, 1 \quad \text{and} \quad \sigma_y \leq \frac{1}{4}.
\]

(13)

Now, we define a Bakhvalov-type mesh for problem (1), which is introduced in [8]. The mesh points \(x_i, i = 0, 1, \ldots, N\), are defined by

\[
x_i = \begin{cases} 
\frac{\tau}{p \mu_0} \varphi_0(t_i) & i = 0, 1, \ldots, \frac{N}{4}, \\
\sigma_{x,0} + 2(t_i - \frac{1}{4})(1 - \sigma_{x,0} - \sigma_{x,1}) & i = \frac{N}{4}, \frac{3N}{4} + 1, \ldots, \frac{3N}{4}, \\
1 - \frac{\tau}{p \mu_1} \varphi_1(t_i) & i = \frac{3N}{4}, \frac{3N}{4} + 1, \ldots, N,
\end{cases}
\]

(14)

where \(t_i = \frac{i}{N}, i = 0, 1, \ldots, N\) and

\[
\varphi_0(t) = -\ln(1 - 4(1 - \mu_0^{-1})t), \quad \varphi_1(t) = -\ln(1 - 4(1 - \mu_1^{-1})(1 - t)).
\]

It can be seen from (14) that the characteristic of mesh is graded on \([x_0, \sigma_{x,0}]\) and \([1 - \sigma_{x,1}, x_N]\), and is uniform on \([\sigma_{x,0}, 1 - \sigma_{x,1}]\).
Similarly, the mesh points $y_j, j = 0, 1, \cdots, N$, are defined by

$$y_j = \begin{cases} \frac{\tau}{\delta} \sqrt{\varepsilon_1} \phi_0(t_j) & j = 0, 1, \cdots, \frac{N}{4}, \\ \sigma_y + 2(t_j - \frac{1}{4})(1 - 2\sigma_y) & j = \frac{N}{4}, \frac{N}{4} + 1, \cdots, \frac{3N}{4}, \\ 1 - \frac{\tau}{\delta} \sqrt{\varepsilon_1} \phi_1(t_j) & j = \frac{3N}{4}, \frac{3N}{4} + 1, \cdots, N, \end{cases}$$

(15)

where $t_j = \frac{j}{N}, j = 0, 1, \cdots, N$ and 

$$\phi_0(t) = -2 \ln(1 - 4(1 - \sqrt{\varepsilon_1})t), \quad \phi_1(t) = -2 \ln(1 - 4(1 - \sqrt{\varepsilon_1})(1 - t)).$$

With mesh points $\{(x_i, y_j)\}$, we obtain a tensor-product rectangular mesh $\mathbb{T}$.

Set $h_{x,i} := x_{i+1} - x_i$ and $h_{y,j} := y_{j+1} - y_j$ are the mesh sizes in the $x$ and $y$ directions, respectively. Also set $\mathcal{T}_{i,j} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ be any mesh rectangle in $\mathbb{T}$.

According to [12, Lemma 2], we have the following three lemmas.

**Lemma 3.1** From (14), we can get the mesh size in the $x$ direction as follows

$$C \mu_0^{-1} N^{-1} \leq h_{x,0} \leq h_{x,1} \leq \cdots \leq h_{x,\frac{N}{4} - 2},$$

$$\frac{\tau}{4p} \mu_0^{-1} \leq h_{x,\frac{N}{4} - 2} \leq \frac{\tau}{p} \mu_0^{-1},$$

$$\frac{\tau}{2p} \mu_0^{-1} \leq h_{x,\frac{N}{4} - 1} \leq \frac{4\tau}{p} N^{-1},$$

$$N^{-1} \leq h_{x,i} \leq 2N^{-1} \quad \frac{N}{4} \leq i \leq \frac{3N}{4} - 1,$$

$$\frac{\tau}{2p} \mu_1^{-1} \leq h_{x,\frac{N}{4}} \leq \frac{4\tau}{p} N^{-1},$$

$$\frac{\tau}{4p} \mu_1^{-1} \leq h_{x,\frac{3N}{4} + 1} \leq \frac{\tau}{p} \mu_1^{-1},$$

$$h_{x,\frac{3N}{4} + 1} \geq h_{x,\frac{3N}{4} + 2} \geq \cdots \geq h_{x,N - 1} \geq C \mu_1^{-1} N^{-1},$$

$$1 - x_{\frac{3N}{4} + 2} \leq C \mu_1^{-1} \ln N.$$

**Lemma 3.2** From (15), we can get the mesh size in the $y$ direction as follows

$$C \sqrt{\varepsilon_1} N^{-1} \leq h_{y,0} \leq h_{y,1} \leq \cdots \leq h_{y,\frac{N}{4} - 2},$$

$$C_1 \sqrt{\varepsilon_1} \leq h_{y,\frac{N}{4} - 2} \leq \frac{2\tau}{\delta} \sqrt{\varepsilon_1},$$

$$C_2 \sqrt{\varepsilon_1} \leq h_{y,\frac{N}{4} - 1} \leq \frac{8\tau}{\delta} N^{-1},$$

$$N^{-1} \leq h_{y,j} \leq 2N^{-1} \quad \frac{N}{4} \leq j \leq \frac{3N}{4} - 1,$$

$$C_2 \sqrt{\varepsilon_1} \leq h_{y,\frac{3N}{4}} \leq \frac{8\tau}{\delta} N^{-1},$$

$$C_1 \sqrt{\varepsilon_1} \leq h_{y,\frac{3N}{4} + 1} \leq \frac{2\tau}{\delta} \sqrt{\varepsilon_1},$$

$$h_{y,\frac{3N}{4} + 1} \geq h_{y,\frac{3N}{4} + 2} \geq \cdots \geq h_{y,N - 1} \geq C \sqrt{\varepsilon_1} N^{-1},$$
where \( C_1 = \frac{\tau}{\delta(\sqrt{c_0}+8)} \), \( C_2 = \frac{\tau}{\delta(\sqrt{c_0}+4)} \).

**Lemma 3.3** For \( 0 \leq i \leq \frac{N}{4} - 2 \) and \( 0 \leq m \leq \tau \), one has
\[
h_{x,i}^m e^{-p\mu_0 x_i} \leq C_{\mu_0}^{-m} N^{-m}. \tag{16}
\]

For \( \frac{3N}{4} + 1 \leq i \leq N - 1 \) and \( 0 \leq m \leq \tau \), one has
\[
h_{x,i}^m e^{-p\mu_1(1-x_i+1)} \leq C_{\mu_1}^{-m} N^{-m}. \tag{17}
\]

For \( 0 \leq j \leq \frac{N}{4} - 2 \) and \( 0 \leq m \leq \tau \), one has
\[
h_{y,j}^m e^{-\delta y_j \sqrt{\varepsilon}} \leq C_{\varepsilon}^{-m} N^{-m}. \tag{18}
\]

For \( \frac{3N}{4} + 1 \leq j \leq N - 1 \) and \( 0 \leq m \leq \tau \), one has
\[
h_{y,j}^m e^{-\delta(1-y_j+1) \sqrt{\varepsilon}} \leq C_{\varepsilon}^{-m} N^{-m}.
\]

Also we need to re-estimate \( h_{x,\frac{3N}{4}} \) for our convergence analysis.

**Lemma 3.4** For any fixed \( \eta \in (0, 1] \), one has
\[
h_{x,\frac{3N}{4}} \leq C_{\mu_1}^{\eta-1} N^{-\eta}. \tag{19}
\]

**Proof** For any fixed \( \eta \in (0, 1] \), standard arguments show
\[
\ln x \leq \frac{x^\eta}{\eta} \quad x \in [1, +\infty).
\]

Combine (14) to get
\[
h_{x,\frac{3N}{4}} = \frac{\tau}{p\mu_1} \ln \frac{\mu_1^{-1} + 4(1 - \mu_1^{-1})N^{-1}}{\mu_1^{-1}}
\leq \frac{\tau}{p\mu_1} \ln(N^{-1} \mu_1) \leq \frac{1}{\eta} \frac{\tau}{p\mu_1} N^{-\eta} \mu_1^\eta
\leq C_{\mu_1}^{\eta-1} N^{-\eta}.
\]

\[ \square \]

### 3.2 Finite element method

The weak form of problem (1) is to find \( u \in H_0^1(\Omega) \) such that
\[
a(u, v) = (f, v) \quad \forall v \in H_0^1(\Omega), \tag{19}
\]
where
\[
a(u, v) := \varepsilon_1(\nabla u, \nabla v) + (\varepsilon_2 b u_x + c u, v) \tag{20}
\]
and \((\cdot, \cdot)\) denotes the standard scalar product in \( L^2(\Omega) \).
Define the finite element space on the Bakhvalov-type mesh
\[ V_N = \{ w \in C(\bar{\Omega}) : w|_{\partial\Omega} = 0, w|_{\mathcal{T}} \in \mathcal{Q}_k(\mathcal{T}) \ \forall \mathcal{T} \in \mathcal{T} \}, \]
where \( \mathcal{Q}_k(\mathcal{T}) = \sum_{0 \leq i,j \leq k} \alpha_{ij}x^i y^j \) with constants \( \alpha_{ij} \in \mathbb{R} \).

The finite element method for (19) is of \( u^N \in V_N \) such that
\[ a(u^N, v^N) = (f, v^N) \ \forall v^N \in V_N. \] (21)

The energy norm associated with \( a(\cdot, \cdot) \) is defined by
\[ \| v \|_E^2 := \varepsilon_1 |v_1|^2_1 + \| v \|_2^2 \ \forall v \in H^1(\Omega). \]

Using (3), it’s easy to prove coercivity
\[ a(v^N, v^N) \geq C \| v^N \|_E^2 \text{ for all } v^N \in V_N. \] (22)

It follows that \( u^N \) is well defined by (21) (see [4] and references therein).

### 4 Interpolation errors

In this section, we will introduce a new Lagrange-type interpolation. The structure of this interpolation is similar to one in [12]. Set \( x^s_i := x_i + \frac{s}{k} h_{x,i} \) and \( y^t_j := y_j + \frac{t}{k} h_{y,j} \) for \( i, j = 0, 1, \ldots, N - 1 \) and \( s, t = 0, 1, 2, \ldots, k \).

For any \( v \in C^0(\bar{\Omega}) \) its Lagrange interpolation \( v^I \in V_N \) on the Bakhvalov-type mesh is defined by
\[ v^I(x, y) = \sum_{i=0}^{N-1} \sum_{s=0}^{k-1} \sum_{j=0}^{N-1} \sum_{t=0}^{k-1} v(x^s_i, y^t_j) \theta^s_i \theta^t_j(x, y) + v(x^0_i, y^0_N) \theta^0_i \theta^0_j(x, y), \]
where \( \theta^s_i \theta^t_j(x, y) \in V_N \) is the piecewise \( k \)-th order Lagrange basis function satisfying the well-known delta properties associated with the nodes \( (x^s_i, y^t_j) \). We define the Lagrange-type interpolation \( \Pi u \) to the solution \( u \) by
\[ \Pi u := S^I + E^I_{10} + \pi_1 E_{11} + E^I_{20} + E^I_{21} + E^I_{31} + \pi_2 E_{32} + \pi_3 E_{33} + E^I_{34}, \] (23)
where
\[ \pi_i E_i(x, y) = E^I_i - P E_i + \Theta E_i \ \text{ for } i = 11, 32, 33 \] (24)
with
\[ (P E_a)(x, y) = \sum_{i=\frac{3N}{4}}^k \sum_{s=1}^{N-1} \sum_{t=0}^{k-1} \left( \sum_{j=0}^{N-1} E_a(x^s_i, y^t_j) \theta^s_i \theta^t_j(x, y) + E_a(x^s_i, y^0_N) \theta^s_i \theta^0_j(x, y) \right) \]
\[ (\Theta E_a)(x, y) = \sum_{s=1}^k E_a(x^s_N, y^0_N) \theta^s_N \theta^0_j(x, y) + \sum_{s=1}^k E_a(x^s_N, y^0_N) \theta^s_N \theta^0_N \quad a = 11, 32, 33. \]
From (23) and (24), we can easily get $\Pi u \in V^N$ and

$$\Pi u = u^l - \sum_{i=11,32,33} (PE_i - \Theta E_i). \tag{25}$$

Next, we will prove the Lagrange interpolation estimation. From [1, Theorem 2.7], we have the following anisotropic interpolation results.

**Lemma 4.1** Let $\mathscr{T} \in \mathbb{T}$ and $v \in H^{k+1}(\mathscr{T})$. Then, there exists a constant $C$ such that Lagrange interpolation $v^l$ satisfies

$$\| v - v^l \|_{\mathscr{T}} \leq C \sum_{i+j=k+1} h^j_{x,\mathscr{T}} h^j_{y,\mathscr{T}} \| \frac{\partial^{k+1} v}{\partial x^i \partial y^j} \|_{\mathscr{T}},$$

$$\| (v - v^l)_x \|_{\mathscr{T}} \leq C \sum_{i+j=k} h^j_{x,\mathscr{T}} h^j_{y,\mathscr{T}} \| \frac{\partial^{k+1} v}{\partial x^{i+1} \partial y^j} \|_{\mathscr{T}},$$

$$\| (v - v^l)_y \|_{\mathscr{T}} \leq C \sum_{i+j=k} h^j_{x,\mathscr{T}} h^j_{y,\mathscr{T}} \| \frac{\partial^{k+1} v}{\partial x^i \partial y^{j+1}} \|_{\mathscr{T}},$$

where $h_{x,\mathscr{T}}$ and $h_{y,\mathscr{T}}$ are respectively the mesh size in $x$ direction and $y$ direction on the rectangular interval $\mathscr{T}$.

**Lemma 4.2** Assume $\tau \geq k + 1$. On Bakhvalov-type mesh $\mathbb{T}$, one has

$$\| E_i - E^l_i \| \leq C N^{-(k+1)} \quad i = 10, 11, 20, 21, 31, 32, 33, 34.$$

**Proof** To consider $\| E_{10} - E^l_{10} \|$, we decompose it as follows

$$\| E_{10} - E^l_{10} \|^2 = \| E_{10} - E^l_{10} \|^2_{[x_{N-1},-1] \times [0,1]} + \| E_{10} - E^l_{10} \|^2_{[x_{N-1},x_N] \times [0,1]} =: A_1 + A_2.$$

Using (9), Lemmas 3.1, 3.2, 4.1 and (16) with $m = l$ we obtain

$$A_1 = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \| E_{10} - E^l_{10} \|^2_{\mathscr{T},ij} \leq C \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \sum_{l+r=k+1} h_{x,i}^{2l} h_{y,j}^{2r} \left( \mu_0 2^l e^{-2p\mu_0 x_i h_{x,i} h_{y,j}} \right) \| \frac{\partial^{k+1} E_{10}}{\partial x^l \partial y^r} \|^2_{\mathscr{T},ij},$$

$$\leq C \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \sum_{l+r=k+1} \left( \mu_0 2^l N^{-2l} \right) (\mu_0 2^l N^{-2l}) h_{x,i}^{2l} h_{y,j}^{2r+1} \| \frac{\partial^{k+1} E_{10}}{\partial x^l \partial y^r} \|^2_{\mathscr{T},ij},$$

$$\leq C \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \mu_0^{-1} N^{-(k+1)-1} \leq C 2 \mu_0^{-1} N^{-(k+1)}, \tag{26}$$
and after a simple calculation, we get $|E_{10}(x, y)\|_{[\frac{x}{4}-1, xN] \times [0,1]} \leq CN^{-\tau}$. Then, the triangle inequality yields

$$A_2 \leq C(\|E_{10}\|_{[\frac{x}{4}-1, xN] \times [0,1]} + \|E_{10}'\|_{[\frac{x}{4}-1, xN] \times [0,1]})$$

$$\leq C(\|E_{10}\|_{\infty, [\frac{x}{4}-1, xN] \times [0,1]} + \|E_{10}'\|_{\infty, [\frac{x}{4}-1, xN] \times [0,1]})$$

$$\leq C\|E_{10}\|_{\infty, [\frac{x}{4}-1, xN] \times [0,1]}$$

$$\leq CN^{-2\tau}. \quad (27)$$

From (7), (26), and (27), we could prove our conclusion. Using the same method, we could get the estimates of $\|E_i - E_i'\|$ with $i = 11, 20, 21, 31, 32, 33, 34$. For the cases of $i = 31, 32, 33, 34$, we divide the whole interval into three pieces not two pieces in the case of $E_{10} - E_{10}'$. For example, for $\|E_{31} - E_{31}'\|$, we can break it down into

$$\|E_{31} - E_{31}'\|^2 = \|E_{31} - E_{31}'\|_{[x_0, xN-1] \times [y_0, yN-1]}^2$$

$$+ \|E_{31} - E_{31}'\|_{[x_0, xN-1] \times [yN-1, yN]}^2 + \|E_{31} - E_{31}'\|_{[xN-1, xN] \times [0,1]}^2$$

$$=: B_1 + B_2 + B_3.$$

Similar to (26), we get

$$B_1 \leq C\varepsilon_1^2 \mu_0^{-1} N^{-2k}.$$

And similar to (27), one has

$$B_2 + B_3 \leq CN^{-2\tau}.$$

Lemma 4.3 Assume $\tau \geq k + 1$. On Bakhvalov-type mesh $T$, one has

$$\|E_i - E_i'\|_E \leq C(\varepsilon_1^{\frac{1}{2}} + \varepsilon_2)^{\frac{1}{2}} N^{-k} + CN^{-(k+1)} \quad i = 10, 11, 20, 21,$$

$$\|PE_{11}\|_E \leq CN^{-\tau - \frac{1}{2}},$$

$$\|\Theta E_{11}\|_E \leq C\varepsilon_1^{\frac{1}{4}} N^{-\tau}.$$

Proof We only consider $\|E_{10} - E_{10}'\|_E$, because the remaining terms could be analyzed in a similar way. Clearly, one has

$$\|E_{10} - E_{10}'\|^2 = \|(E_{10} - E_{10}')_x\|^2 + \|(E_{10} - E_{10}')_y\|^2.$$
From (9), Lemmas 3.2, 4.1 and (16) with \( m = l + \frac{1}{2} \), we could obtain

\[
\| (E_{10} - E_{10}^l) x \|^{2}_{[x_0, x_{N-1}] \times [0,1]} = \sum_{i=0}^{N-1} \sum_{j=0}^{N-2} \| (E_{10} - E_{10}^l) x \|^{2}_{\mathcal{H}_{i,j}} 
\]

\[
\leq C \sum_{i=0}^{N-1} \sum_{j=0}^{N-2} \sum_{l+r=k} h_{x,i}^2 h_{y,j}^2 \| \frac{\partial^{k+1} E_{10}}{\partial x^l \partial y^r} \|^2_{\mathcal{H}_{i,j}} 
\]

\[
\leq C \sum_{i=0}^{N-1} \sum_{j=0}^{N-2} \sum_{l+r=k} h_{x,i}^2 h_{y,j}^2 (\mu_0^{2(l+1)} e^{-2p\mu_0 x_i h_{x,i} h_{y,j}}) \leq C \mu_0 N^{-2k} . 
\]

Note \( \| (E_{10}) x \|^{2}_{[x_{N-1}^{-1}, x_N] \times [0,1]} \leq C \mu_0 \frac{1}{N^\tau} \). Then, one has

\[
\| (E_{10} - E_{10}^l) x \|^{2}_{[x_0, x_{N-1}] \times [0,1]} \leq C \| (E_{10}) x \|^{2}_{[x_{N-1}^{-1}, x_N] \times [0,1]} + C \| (E_{10}^l) x \|^{2}_{[x_{N-1}^{-1}, x_N] \times [0,1]} 
\]

\[
\leq C \| (E_{10}) x \|^{2}_{[x_0, x_{N-1}] \times [0,1]} + C \sum_{i=0}^{N-1} \sum_{j=0}^{N-2} \| (E_{10}^l) x \|^{2}_{\mathcal{H}_{i,j}} 
\]

\[
\leq C \mu_0 N^{-2\tau} + C \mu_1 N^{2-2\tau} 
\]

\[
\leq C \mu_1 N^{2-2\tau} , 
\]

where inverse inequality \[ 5, \text{Theorem 3.2.6}, \] (9), Lemmas 3.1 and 3.2 yield

\[
\sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \| (E_{10}) x \|^{2}_{\mathcal{H}_{i,j}} \leq C \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} h_{x,i}^{-2} \| E_{10}^l \|^{2}_{\mathcal{H}_{i,j}} 
\]

\[
\leq C \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} h_{x,i}^{-2} \| E_{10}^l \|^{2}_{\mathcal{H}_{i,j}} \leq C \mu_1 N^{2-2\tau} . 
\]

Similar to (28), we can get

\[
\| (E_{10} - E_{10}^l) y \|^{2}_{[x_0, x_{N-1}] \times [0,1]} \leq C \mu_0^{-1} N^{-2k} . 
\]
Similar to (29), one has
\[
\| (E_{10} - E_{10}^f) \|_E^2 \leq C \varepsilon_1^{-\frac{1}{2}} N^{2-2\tau}. \tag{31}
\]

From (28)–(31), (5), and Lemma 4.2, we can easily obtain
\[
\| E_{10} - E_{10}^f \|_E^2 \leq C (\varepsilon_1 \mu_1 N^{-2(k+1)} + \varepsilon_1^{\frac{1}{2}} N^{-2(k+1)} N^2 + C N^{-2(k+1)})
\]
\[
\leq C ((\varepsilon_1^{\frac{1}{2}} + \varepsilon_2) N^{-2(k+1)} + \varepsilon_1^{\frac{1}{2}} N^{-2(k+1)} + C N^{-2(k+1)}),
\]
i.e.,
\[
\| E_{10} - E_{10}^f \|_E \leq C (\varepsilon_1^{\frac{1}{2}} + \varepsilon_2) N^{-k} + C N^{-(k+1)}.
\]

For \( \| PE_{11} \|_E \) and \( \| \Theta E_{11} \|_E \), Lemmas 3.1, 3.2, (5), and (8) yield
\[
\| PE_{11} \|_E^2 \leq C N^{-2\tau} \sum_{s=1}^{k} \left( \sum_{j=0}^{N-1} \sum_{t=0}^{N-1} \| \theta_{3N^s,j}^x \|_E^2 + \| \theta_{3N^s,N}^x \|_E^2 \right)
\]
\[
\leq C N^{-2\tau} \sum_{j=0}^{N-1} (\varepsilon_1 h^{-1}_x 3^{N} h_y, j + \varepsilon_1 h^{-1}_x 3^{N} h_y, j + h_x, 3^{N} h_y, j)
\]
\[
\leq C N^{-2\tau} (\varepsilon_1 \mu_1 + \varepsilon_1^{\frac{1}{2}} N + N^{-1}),
\]
\[
\leq C N^{-2\tau} (\varepsilon_2 + \varepsilon_1^{\frac{1}{2}} N + N^{-1})
\]
\[
\leq C N^{-2\tau},
\]
and
\[
\| \Theta E_{11} \|_E^2 \leq C N^{-2\tau} \left( \sum_{s=1}^{k} \| \theta_{3N^s,0}^x \|_E^2 + \sum_{s=1}^{k} \| \theta_{3N^s,N}^x \|_E^2 \right)
\]
\[
\leq C N^{-2\tau} (\varepsilon_1 h^{-1}_x 3^{N} h_y, 0 + \varepsilon_1 h^{-1}_x 3^{N} h_y, 0 + h_x, 3^{N} h_y, 0
\]
\[
+ \varepsilon_1 h^{-1}_x 3^{N} h_y, N-1 + \varepsilon_1 h^{-1}_x 3^{N} h_y, N-1 + h_x, 3^{N} h_y, N-1)
\]
\[
\leq C N^{-2\tau} (\varepsilon_1 \mu_1 \varepsilon_1^{\frac{1}{2}} + \varepsilon_1 N^{-1} \varepsilon_1^{-\frac{1}{2}} N + N^{-1} \varepsilon_1^{\frac{1}{2}})
\]
\[
\leq C N^{-2\tau} (\varepsilon_1^{\frac{1}{2}} (\varepsilon_1^{\frac{1}{2}} + \varepsilon_2) + \varepsilon_1^{\frac{1}{2}})
\]
\[
\leq C \varepsilon_1^{\frac{1}{2}} N^{-2\tau}.
\]
Lemma 4.4 For interpolation error estimates of corner layers, we have

\[ \| E_i - E_i^I \|_E \leq C N^{-(k+1)} \quad i = 31, 32, 33, 34, \]
\[ \| P E_j \|_E \leq C N^{-\tau - \frac{1}{2}} \quad j = 32, 33, \]
\[ \| \Theta E_j \|_E \leq C \varepsilon_1^\tau N^{-\tau} \quad j = 32, 33. \]

Proof We have omitted the proofs of \( \| P E_j \|_E \) and \( \| \Theta E_j \|_E \) with \( j = 32, 33 \) here, because they are similar to ones of \( \| P E_{11} \|_E \) and \( \| \Theta E_{11} \|_E \), respectively.

In order to analyze \( \| (E_{31} - E_{31}^I) \|_E \), we set \( D_{0,0} := [x_0, x_{N^4 - 1}] \times [y_0, y_{N^4 - 1}] \).

Then,
\[
\| (E_{31} - E_{31}^I) \|_E^2 \Omega \setminus D_{0,0} \leq C \| (E_{31})_x \|_E^2 \Omega \setminus D_{0,0} + C \| (E_{31}^I)_x \|_E^2 \Omega \setminus D_{0,0} \\
\leq C \| (E_{31})_x \|_E^2 \Omega \setminus D_{0,0} + C \sum_{i = \frac{N}{4} - 1}^{N-1} \sum_{j = 0}^{N-1} \| (E_{31}^I)_x \|_E^2 T_{i,j} + C \sum_{i = 0}^{N-1} \sum_{j = \frac{N}{4} - 1}^{N-1} \| (E_{31}^I)_x \|_E^2 T_{i,j} \\
=: D_1 + D_2 + D_3. \]

Inverse inequality, (11), Lemmas 3.1, and 3.2 yield
\[
D_1 = \| (E_{31})_x \|_E^2 \Omega \setminus D_{0,0} \leq \int_{x_0}^{x_{N^4 - 1}} \int_{y_0}^{y_{N^4 - 1}} \mu_0^2 e^{-2p\mu_0 x_i} e^{-\frac{2y_j}{\sqrt{\varepsilon_1}}} dx dy \\
+ \int_{x_{N^4 - 1}}^{x_{N^4 - 1}} \int_{y_0}^{y_{N^4 - 1}} \mu_0^2 e^{-2p\mu_0 x_i} e^{-\frac{2y_j}{\sqrt{\varepsilon_1}}} dx dy \leq C \mu_1 \varepsilon_1 \mu_0 N^{-2\tau}. \]

and
\[
D_2 = \sum_{i = \frac{N}{4} - 1}^{N-1} \sum_{j = 0}^{N-1} \| (E_{31}^I)_x \|_E^2 T_{i,j} \leq C \sum_{i = \frac{N}{4} - 1}^{N-1} \sum_{j = 0}^{N-1} h_{x,i}^{-2} \| E_{31}^I \|_E^2 T_{i,j} \\
\leq C \sum_{i = \frac{N}{4} - 1}^{N-1} \sum_{j = 0}^{N-1} h_{x,i}^{-2} \| E_{31} \|_E^2 T_{i,j} h_{x,i} h_{y,j} \leq C \mu_1 N^{2-2\tau}. \]

Similar to \( D_2 \), we have
\[
D_3 \leq C \mu_0 N^{2-4\tau}. \]

Combination of (32), (33), and (34), we have
\[
\| (E_{31} - E_{31}^I) \|_E^2 \Omega \setminus D_{0,0} \leq C \mu_1 N^{2-2\tau}. \]
From (11), Lemma 4.1, (16) with \(m = l + \frac{1}{2}\) and (18) with \(m = r + \frac{1}{2}\) yield

\[
\| \left( E_{31} - E^l_{31} \right)_x \|^2_{D_{0,0}} = \sum_{i=0}^{N_l-2} \sum_{j=0}^{N_r-2} \| \left( E_{31} - E^l_{31} \right)_x \|^2_{\mathcal{F}_{i,j}} \leq C \sum_{i=0}^{N_l-2} \sum_{j=0}^{N_r-2} h_{x,i}^2 h_{y,j}^2 \left| \frac{\partial^{k+1} E_{31}}{\partial x^{l+1} \partial y^r} \right|_{\mathcal{F}_{i,j}} \leq C \sum_{i=0}^{N_l-2} \sum_{j=0}^{N_r-2} \sum_{l+r=k} \mu_0 \varepsilon_1 \left| \frac{\partial^{k+1} E_{31}}{\partial x^{l+1} \partial y^r} \right|_{\mathcal{F}_{i,j}} \leq C \mu_0 \varepsilon_1^{\frac{1}{2}} N^{-2k}. \]

For \(\| \left( E_{31} - E^l_{31} \right)_y \|\), we use the same processing technique as \(\| \left( E_{31} - E^l_{31} \right)_x \|\) to obtain

\[
\| \left( E_{31} - E^l_{31} \right)_y \|^2_{D_{0,0}} \leq C \mu_0 \varepsilon_1 N^{-2k}, \quad \| \left( E_{31} - E^l_{31} \right)_y \|^2_{D_{0,0}} \leq C \varepsilon_1 N^{-2} \tau. \]

From (35)–(38), we can easily obtain

\[
\| E_{31} - E^l_{31} \|^2 \leq C \mu_1 N^{-2k} + \varepsilon_1^{\frac{1}{2}} N^{-2k}. \]

By combining Lemma 4.2 and (5), we get

\[
\| E_{31} - E^l_{31} \|^2_E \leq C \varepsilon_1 (\mu_1 N^{-2k} + \varepsilon_1^{\frac{1}{2}} N^{-2k}) + C N^{-2(k+1)} \leq C(\varepsilon_1 \mu_1 N^{-2k} + \varepsilon_1^{\frac{1}{2}} N^{-2k}) + C N^{-2(k+1)} \leq C(\varepsilon_1^{\frac{1}{2}} + \varepsilon_2) N^{-2k} + C N^{-2(k+1)} \leq C(\varepsilon_1^{\frac{1}{2}} + \varepsilon_2) N^{-2k} + C N^{-2(k+1)}.
\]

Similarly, we have

\[
\| E_i - E^l_i \|^2_E \leq C(\varepsilon_1^{\frac{1}{2}} + \varepsilon_2) N^{-2k} + C N^{-2(k+1)} \quad i = 32, 33, 34.
\]

When calculating the interpolation error of \(\| (E_i - E^l_i)_x \|(i = 10, 20, 21, 31, 34)\), we use a different technique.
Lemma 4.5 Assume $\tau \geq k + 1$. On Bakhvalov-type mesh $\mathbb{T}$, one has

$$\| (E_i - E_i^l) x \| \leq C \mu_0 N^{-(k+1)} \quad i = 10, 20, 21, 31, 34.$$  

Proof Here, we only prove the conclusion of the boundary layer at $x = 0$, because the proof for other boundary layers is similar. The analysis of the two corner layers is also similar, so we only present the proof of one of them.

For $\| (E_{10} - E_{10}^l) x \|$, on the interval $[x_0, x_{N-1}] \times [0, 1]$, using (28) to get

$$\| (E_{10} - E_{10}^l) x \|_{[x_0, x_{N-1}] \times [0, 1]}^2 \leq C \mu_0 N^{-2k}. \quad (39)$$

But, on the interval $[x_{N-1}, x_N] \times [0, 1]$, instead of using the inverse inequality in (29), we use the triangle inequality and (9) yield

$$\| (E_{10} - E_{10}^l) x \|_{[x_{N-1}, x_N] \times [0, 1]} \leq \| (E_{10} x) \|_{\infty, [x_{N-1}, x_N] \times [0, 1]} + \| (E_{10}^l x) \|_{\infty, [x_{N-1}, x_N] \times [0, 1]} \leq C \mu_0 N^{-\tau}. \quad (40)$$

From (39) and (40), we get $\| (E_{10} - E_{10}^l) x \| \leq C \mu_0 N^{-(k+1)}$.

For $\| (E_{31} - E_{31}^l) x \|$, we decompose it as follows

$$\| (E_{31} - E_{31}^l) x \| \leq \| (E_{31} - E_{31}^l) x \|_{[x_0, x_{N-1}] \times [y_0, y_{N-1}]} + \| (E_{31} - E_{31}^l) x \|_{[x_{N-1}, x_N] \times [y_{N-1}, y_N]} + \| (E_{31} - E_{31}^l) x \|_{[x_{N-1}, x_N] \times [0, 1]} \leq C \mu_0 N^{-(k+1)},$$

where similar to (36), we have

$$\| (E_{31} - E_{31}^l) x \|_{[x_0, x_{N-1}] \times [y_0, y_{N-1}]}^2 \leq \mu_0 \epsilon_1^2 N^{-2k},$$

and similar to (40) we obtain

$$\| (E_{31} - E_{31}^l) x \|_{[x_{N-1}, x_N] \times [y_{N-1}, y_N]} + \| (E_{31} - E_{31}^l) x \|_{[x_{N-1}, x_N] \times [0, 1]} \leq \mu_0 N^{-\tau}.$$
Theorem 4.6 Assume $\tau \geq k + 1$. On the Bakhvalov-type mesh $\mathbb{T}$, one has

\[
\sum_i \| \pi_i E_i - E_i \| \leq CN^{-(k+1)}, \quad i = 11, 32, 33,
\]

\[
\| u - u^I \|_E + \| u - \Pi u \|_E \leq C (\varepsilon_1^{1/2} + \varepsilon_2^{1/2}) N^{-k} + N^{-(k+1)}.
\]

Proof From Lemma 4.2 and the proof of Lemma 4.3, we could obtain

\[
\| \pi E_{11} - E_{11} \| \leq \| E_{11} - EI_{11} \| + \| PE_{11} \| + \| \Theta E_{11} \| \leq CN^{-(k+1)}.
\]

Similarly, we could get estimates for $\| \pi E_i - E_i \|$ with $i = 32, 33$.

By a simple calculation, we get

\[
\| S - SI \| \leq CN^{-(k+1)} \quad \text{and} \quad | S - SI | \leq CN^{-k}.
\]

Then by combining Lemmas 4.2, 4.3, and 4.4, we prove

\[
\| u - u^I \|_E \leq C (\varepsilon_1^{1/2} + \varepsilon_2^{1/2}) N^{-k} + N^{-(k+1)}.
\]

Finally using (25), we have

\[
\| u - \Pi u \|_E \leq C (\varepsilon_1^{1/2} + \varepsilon_2^{1/2}) N^{-k} + N^{-(k+1)}.
\]

5 Uniform convergence

Set $\chi := \Pi u - u^N$. Using (20), (22), (23), integration by parts, and Galerkin orthogonality, we have

\[
a \| \chi \|^2_E \leq a(\chi, \chi) = a(\Pi u - u, \chi)
\]

\[
= \varepsilon_1 \int_\Omega \nabla (\Pi u - u) \nabla \chi dxdy + \varepsilon_2 \int_\Omega (S^l - S)_x \chi dxdy
\]

\[
+ \sum_{l=10,20,21,30,34} \varepsilon_2 \int_\Omega (E^l_{11} - E_{11})_x \chi dxdy - \varepsilon_2 \int_\Omega (\pi_{11} E_{11} - E_{11}) b_x \chi dxdy
\]

\[
- \sum_{i=32,33} \varepsilon_2 \int_\Omega (\pi_i E_i - E_i) b_{x,i} \chi dxdy - \sum_{j=11,32,33} \varepsilon_2 \int_\Omega (\pi_j E_j - E_j) b_x \chi dxdy
\]

\[
+ \int_\Omega c(\Pi u - u) \chi dxdy =: I + II + III + IV + V + VI + VII.
\]

Theorem 4.6 yields

\[
| (I + VII) + VI | \leq C \| \Pi u - u \|_E \| \chi \|_E + \sum_{j=11,32,33} \| \pi_j E_j - E_j \| \| \chi \|
\]

\[
\leq C ((\varepsilon_1^{1/2} + \varepsilon_2^{1/2}) N^{-k} + N^{-(k+1)}) \| \chi \|_E.
\]
Using (6), Lemma 4.5, and Hölder inequality, we can get

\[
|II + III| \leq C(\varepsilon_2 \|(S^I - S)_x\| + \varepsilon_2 \sum_{I=10,20,21,31,34} \|(E^{I}_i - E_i)_x\| \|\chi\|)
\]

\[
\leq C\varepsilon_2(N^{-k} + \mu_0 N^{-(k+1)}) \|\chi\|
\]

\[
\leq C(\varepsilon_2 N^{-k} + N^{-(k+1)}) \|\chi\|_E.
\]

(42)

For IV and V, we have the following two lemmas.

**Lemma 5.1** Assuming that \(\tau \geq k + 1\), on the Bakhvalov-type mesh \(\mathcal{T}\), one has

\[
|IV| \leq C(\varepsilon_2 N^{-k} + \varepsilon_2^2 N^{-(k+1)}) \|\chi\|_E.
\]

**Proof** After analysis, we do the following decomposition

\[
\int_{\Omega} (\pi_{11} E_{11} - E_{11}) b \chi_x dx dy = \int_{x_0}^{x_{3N}} \int_0^1 b(E^{I}_i - E_i) \chi_x dx dy
\]

\[+ \int_{x_{3N}}^{x_{3N}+1} \int_0^1 b(\pi_{11} E_{11} - E_{11}) \chi_x dx dy
\]

\[+ \int_{x_{3N}+2}^{x_N} \int_0^1 b(E^{I}_i - E_i) \chi_x dx dy =: F_1 + F_2 + F_3 + F_4.
\]

(43)

First, using (10), the inverse inequality, Lemmas 3.1, 3.2, and 4.1, we can obtain

\[
|F_1| \leq \sum_{i=0}^{3N-1} \sum_{j=0}^{N-1} \|E^{I}_i - E_{11}\|_{\mathcal{G}_{i,j}} \|\chi_x\| \leq C \sum_{i=0}^{3N-1} \sum_{j=0}^{N-1} \sum_{l=0}^{l+r=k+1} h_{x,i}h_{y,j} \left\|\frac{\partial E^{I}_i}{\partial x}\right\|_{\mathcal{G}_{i,j}} \|\chi\|_{\mathcal{G}_{i,j}}
\]

\[
\leq C \sum_{i=0}^{3N-1} \sum_{j=0}^{N-1} \sum_{l=0}^{l+r=k+1} h_{x,i}h_{y,j} \mu_1 e^{-\mu_1(1-x_i+1)} h_{x,i}^2 h_{y,j}^2 h_{x,i}^{-1} \|\chi\|_{\mathcal{G}_{i,j}}
\]

\[
\leq C \sum_{i=0}^{3N-1} \sum_{j=0}^{N-1} \sum_{l=0}^{l+r=k+1} \mu_1 \mu_1^{-\tau} h_{x,i} h_{y,j} \left\|\frac{\partial E^{I}_i}{\partial x}\right\|_{\mathcal{G}_{i,j}} \leq C \mu_1^{1-\tau} \sum_{i=0}^{3N-1} \sum_{j=0}^{N-1} N^{-(k+1)} \|\chi\|_{\mathcal{G}_{i,j}}
\]

\[
\leq C \mu_1^{1-\tau} \left(\sum_{i=0}^{3N-1} \sum_{j=0}^{N-1} N^{-2(k+1)}\right)^{\frac{1}{2}} \left(\sum_{i=0}^{3N-1} \sum_{j=0}^{N-1} \|\chi\|^2_{\mathcal{G}_{i,j}}\right)^{\frac{1}{2}}
\]

\[
\leq CN^{-k} \|\chi\|_{[x_0, x_{3N}] \times [0,1]} \leq CN^{-k} \|\chi\|_{E,[x_0, x_{3N}] \times [0,1]}.
\]

(44)
Next, using (10), (17) with \( m = l \), Lemmas 3.1, 3.2, and 4.1, we can obtain

\[
|F_4| \leq \sum_{i=\frac{N}{3}+2}^{N-1} \sum_{j=0}^{N-1} \left| E_1 \| T_{i,j} \| x \| T_{i,j} \right| \leq C \sum_{i=\frac{N}{3}+2}^{N-1} \sum_{j=0}^{N-1} h_{i,j}^k h_{i,j}^r \left\| \frac{\partial^k E_{11}}{\partial x^k} \right\| T_{i,j} \| x \| T_{i,j}
\]

\[
\leq C \sum_{i=\frac{N}{3}+2}^{N-1} \sum_{j=0}^{N-1} \sum_{l+r=k+1} \left( \mu_i^{-l} N^{-i} \right) \mu_j^{-r} h_{i,j}^k h_{i,j}^r \| x \| T_{i,j} \leq C \mu^{-\frac{1}{2}} \sum_{i=\frac{N}{3}+2}^{N-1} \sum_{j=0}^{N-1} N^{-(k+1)-\frac{1}{2}} \| x \| T_{i,j}
\]

\[
\leq C \mu^{-\frac{1}{2}} N^{-k} \| x \| \| T \| \times [0,1]
\]

\[
\leq C \mu^{-\frac{1}{2}} N^{-k+1} \| x \| \| T \| \times [0,1].
\]

(45)

Then, on the interval \([x_{\frac{3N}{4}+1}, x_{\frac{3N}{4}+2}] \times [0, 1] \), we notice that

\[
\pi_{11} E_{11} = E_{11} - \sum_{j=0}^{N-1} \sum_{t=0}^{k-1} E_{11}(x_{\frac{3N}{4}+1}, y_j) \theta_{\frac{3N}{4}+1}^t - E_{11}(x_{\frac{3N}{4}+1}, y_N) \theta_{\frac{3N}{4}+1}^0, N.
\]

Thus, we have

\[
|F_3| \leq C \sum_{j=0}^{N-1} \left| E_1 \| T_{\frac{3N}{4}+1,j} \| x \| T_{\frac{3N}{4}+1,j} \right|
\]

\[
+ C \sum_{j=0}^{N-1} \sum_{t=0}^{k-1} \left| E_{11}(x_{\frac{3N}{4}+1}, y_j) \right| \left\| \theta_{\frac{3N}{4}+1,j}^t \right\| T_{\frac{3N}{4}+1,j} \| x \| T_{\frac{3N}{4}+1,j}
\]

\[
+ |E_{11}(x_{\frac{3N}{4}+1}, y_N) \right| \left\| \theta_{\frac{3N}{4}+1,N}^0 \right\| T_{\frac{3N}{4}+1,N} \| x \| T_{\frac{3N}{4}+1,N}
\]

\[
=: R_1 + R_2 + R_3
\]

\[
\leq C \mu^{-\frac{1}{2}} N^{-k} \| x \| \| T \| \times [0,1].
\]

where same as (45), we get

\[
R_1 \leq C \mu^{-\frac{1}{2}} N^{-k} \| x \| \| T \| \times [0,1],
\]

\[
\text{Springer}
\]
and (10), Lemmas 3.1 and 3.2 yield
\[ \mathcal{B}_2 \leq C \sum_{j=0}^{N-1} N^{-\tau} h_{x, \frac{x_N}{4}}^j h_{y, j} \| \chi \|_{\mathcal{T}_{\frac{3N}{4}+1,j}} \leq C \mu_1^{-\frac{1}{2}} \sum_{j=0}^{N-1} N^{-\tau} N^{-\frac{1}{2}} \| \chi \|_{\mathcal{T}_{\frac{3N}{4}+1,j}}. \]

\[ \leq C \mu_1^{-\frac{1}{2}} \left( \sum_{j=0}^{N-1} N^{-2\tau-1} \right) \left( \sum_{j=0}^{N-1} \| \chi \|_{\mathcal{T}_{\frac{3N}{4}+1,j}}^2 \right)^{\frac{1}{2}} \]

\[ \leq C \mu_1^{-\frac{1}{2}} N^{-\tau} \| \chi \|_{[x_{\frac{3N}{4}+1}, x_{\frac{3N}{4}+2}] \times [0, 1]} \]

\[ \leq C \varepsilon_1^{-\frac{1}{2}} \mu_1^{-\frac{1}{2}} N^{-\tau} \| \chi \|_{E, [x_{\frac{3N}{4}+1}, x_{\frac{3N}{4}+2}] \times [0, 1]} . \] (47)

In the same way, one has
\[ \mathcal{B}_3 \leq C \varepsilon_1^{-\frac{1}{2}} \mu_1^{-\frac{1}{2}} N^{-(\tau+\frac{1}{2})} \| \chi \|_{E, \mathcal{T}_{\frac{3N}{4}+1,N-1}}. \]

Last, for \( F_2 \), on the interval \([x_{\frac{3N}{4}}, x_{\frac{3N}{4}+1}] \times [0, 1]\),
\[ \pi_{11} E_{11} = \sum_{j=0}^{N-1} \sum_{t=0}^{k-1} E_{11}(x_{\frac{3N}{4}}, y_j) \theta_{N,j}^{0,t} + \sum_{s=1}^{k} E_{11}(x_{\frac{3N}{4}}, y_0) \theta_{N,0}^{0,s} + \sum_{s=0}^{k} E_{11}(x_{\frac{3N}{4}}, y_N) \theta_{N,0}^{N,s} . \]

Thus, we have
\[ |F_2| \leq C \sum_{j=0}^{N-1} \sum_{t=0}^{k-1} |E_{11}(x_{\frac{3N}{4}}, y_j)| \| \theta_{N,j}^{0,t} \|_{\mathcal{T}_{\frac{3N}{4}+1,j}} \| \chi \|_{\mathcal{T}_{\frac{3N}{4}+1,j}} + C \| E_{11} \|_{[x_{\frac{3N}{4}+1}, x_{\frac{3N}{4}+2}] \times [0, 1]} \| \chi \|_{[x_{\frac{3N}{4}+1}, x_{\frac{3N}{4}+2}] \times [0, 1]} \]

\[ + C \sum_{s=1}^{k} |E_{11}(x_{\frac{3N}{4}}, y_0)| \| \theta_{N,0}^{0,s} \|_{\mathcal{T}_{\frac{3N}{4}+1,0}} \| \chi \|_{\mathcal{T}_{\frac{3N}{4}+1,0}} \]

\[ + C \sum_{s=0}^{k} |E_{11}(x_{\frac{3N}{4}}, y_N)| \| \theta_{N,0}^{N,s} \|_{\mathcal{T}_{\frac{3N}{4}+1,N-1}} \| \chi \|_{\mathcal{T}_{\frac{3N}{4}+1,N-1}} \] (48)

\[ = \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4 \]

\[ \leq C \varepsilon_1^{-\frac{1}{2}} \mu_1^{-\frac{1}{2}} N^{-(k+1)} + C \varepsilon_1^{-\frac{1}{2}} \mu_1^{-\frac{1}{2}} N^{-(\tau+\frac{1}{2})} \| \chi \|_{E, [x_{\frac{3N}{4}+1}, x_{\frac{3N}{4}+2}] \times [0, 1]} . \]

The proof is as follows, same as (47), we get
\[ \mathcal{J}_1 + \mathcal{J}_2 \leq C \varepsilon_1^{-\frac{1}{2}} \mu_1^{-\tau} N^{-\frac{1}{2}} \| \chi \|_{E, [x_{\frac{3N}{4}+1}, x_{\frac{3N}{4}+2}] \times [0, 1]} . \]

When dealing with \( \mathcal{J}_3 \) and \( \mathcal{J}_4 \), the mesh scale in Lemma 3.1 is not enough, so we still use the analysis method of (47), but we use the mesh scale of Lemma 3.4 with
\( \eta = \frac{1}{2} \), thus we have

\[
\mathcal{J}_3 + \mathcal{J}_4 \leq C \varepsilon_1^{-\frac{1}{2}} \mu_1^{-\frac{1}{2}} N^{-(\tau + \frac{1}{2})} \| \chi \|_E, [x_{\frac{3N}{4}}, x_{\frac{3N}{4} + 1}] \times [0, 1].
\]

So, by combining (6), (7), (43), (44), (45), (46), and (48) to obtain

\[
|IV| \leq C (\varepsilon_2 N^{-k} + \varepsilon_4 \varepsilon_1^{-\frac{1}{2}} \mu_1^{-\frac{1}{2}} N^{-(k + \frac{1}{2})} + C \varepsilon_2 \varepsilon_1^{-\frac{1}{2}} \mu_1^{-\frac{1}{2}} N^{-(k + \frac{3}{2})}) \| \chi \|_E
\]

\[
\leq C (\varepsilon_2 N^{-k} + \varepsilon_4 \varepsilon_1^{-\frac{1}{2}} N^{-(k + \frac{1}{2})} + \varepsilon_2 \varepsilon_1^{-\frac{1}{2}} N^{-(k + \frac{3}{2})}) \| \chi \|_E
\]

\[
\leq C (\varepsilon_2 N^{-k} + \varepsilon_4 \varepsilon_1^{-\frac{1}{2}} N^{-(k + \frac{1}{2})}) \| \chi \|_E.
\]

\( \square \)

**Lemma 5.2** Assuming that \( \tau \geq k + 1 \), on the Bakhvalov-type mesh \( \mathbb{T} \), one has

\[
|V| \leq C \varepsilon_2^{\frac{1}{2}} N^{-(k + \frac{1}{2})} \| \chi \|_E.
\]

**Proof** To simplify the analysis, we decompose \( V \) as follows

\[
V = \varepsilon_2 \int_{x_0}^{x_{3N/4}} \int_0^1 (E_{32} - E_{32}) b \chi_x dx dy + \varepsilon_2 \int_{x_{3N/4}}^{x_{3N/2} + 1} \int_0^1 (\pi_{32} E_{32} - E_{32}) b \chi_x dx dy
\]

\[
+ \varepsilon_2 \int_{x_{3N/4} + 2}^{x_{3N/2} + 1} \int_0^1 (\pi_{32} E_{32} - E_{32}) b \chi_x dx dy + \varepsilon_2 \int_{x_{3N/4}}^{x_{3N/2}} \int_0^1 (E_{32} - E_{32}) b \chi_x dx dy
\]

\[
=: M_1 + M_2 + M_3 + M_4.
\]

For \( M_1 \), using triangle inequality, (6) and (12), we can get

\[
|M_1| \leq C \varepsilon_2 (\| E_{32} \|_\infty, [x_0, x_{3N/4}] \times [0, 1]) + \| E_{32} \|_\infty, [x_0, x_{3N/4}] \times [0, 1]) \| \chi_x \|_E, [x_0, x_{3N/4}] \times [0, 1]
\]

\[
\leq C \varepsilon_2 \| E_{32} \|_\infty, [x_0, x_{3N/4}] \times [0, 1]) \| \chi_x \|_E, [x_0, x_{3N/4}] \times [0, 1]
\]

\[
\leq C \varepsilon_2 \varepsilon_1^{-\frac{1}{2}} \mu_1^{-\tau} \| \chi \|_E, [x_0, x_{3N/4}] \times [0, 1]
\]

\[
\leq C \varepsilon_2 \varepsilon_1^{-\frac{1}{2}} \mu_1^{-\tau} \| \chi \|_E, [x_0, x_{3N/4}] \times [0, 1].
\]

(49)

On the interval \([x_{3N/4}, x_{3N/4} + 1] \times [0, 1]\), we notice

\[
\pi_{32} E_{32} = \sum_{j=0}^{N-1-k} \sum_{r=0}^{k-1} E_{32}(x_j^0, y_r^0) \theta_{\frac{N}{4}}^r, \quad + \sum_{s=1}^{k} E_{32}(x_{\frac{3N}{4}}, y_s^0) \theta_{\frac{3N}{4}, 0}^s, \quad + \sum_{s=0}^{k} E_{32}(x_{\frac{3N}{4}}, y_s^0) \theta_{\frac{3N}{4}}^s, N.
\]
Thus,
\[ |M_2| \leq C \varepsilon \sum_{j=0}^{N-1} \sum_{t=0}^{k-1} |E_{32}(x_{3N}^0, y_j^t)| \| \theta_{3N}^{0,t} \| \mathcal{T}_{3N} \| X_x \| \mathcal{T}_{3N} \]
\[ + C \varepsilon \sum_{s=1}^{k} |E_{32}(x_{3N}^s, y_0^s)| \| \theta_{3N}^{t,0} \| \mathcal{T}_{3N} \| X_x \| \mathcal{T}_{3N} \]
\[ + C \varepsilon \sum_{s=0}^{N} |E_{32}(x_{3N}^s, y_0^s)| \| \theta_{3N}^{s,0} \| \mathcal{T}_{3N} \| X_x \| \mathcal{T}_{3N} \]
\[ =: \mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3 + \mathcal{V}_4 \]
\[ \leq C \left( \frac{1}{2} N^{t+\frac{1}{2}} + \frac{1}{2} \varepsilon \right) \| X \| \mathcal{E}, \]

where similar to (47), one has \( \mathcal{V}_1 \leq C \varepsilon \| \mu^{-\frac{1}{2}} \| M^{-\frac{1}{2}} N^{-1} \| X \| \mathcal{E} \), then use (6) we obtain
\[ \mathcal{V}_1 \leq C \varepsilon \| \mu^{-\frac{1}{2}} \| M^{-\frac{1}{2}} N^{-1} \| X \| \mathcal{E}. \] (51)

In the same way, one has
\[ \mathcal{V}_2 + \mathcal{V}_3 \leq C \varepsilon \| N^{-t-\frac{1}{2}} \| X \| \mathcal{E}. \]
\[ \mathcal{V}_4 \leq C \varepsilon \| N^{-t} \| X \| \mathcal{E}. \]

On the interval \([x_{3N}^{N}, x_{3N}^{N+1}] \times [0, 1],\)
\[ \pi_{32} E_{32} = E_{32}^I - \sum_{j=0}^{N-1} \sum_{t=0}^{k-1} E_{32}(x_{3N}^0, y_j^t) \theta_{3N}^{0,t} + E_{32}(x_{3N}^0, y_0^t) \theta_{3N}^{0,0}. \]

Thus,
\[ |M_3| \leq C \varepsilon \sum_{j=0}^{N-1} \| E_{32}^I - \sum_{t=0}^{k-1} E_{32}(x_{3N}^0, y_j^t) \theta_{3N}^{0,t} + E_{32}(x_{3N}^0, y_0^t) \theta_{3N}^{0,0} \| \mathcal{T}_{3N+1} \| X_x \| \mathcal{T}_{3N+1} \]
\[ + C \varepsilon \sum_{j=0}^{N-1} \sum_{t=0}^{k-1} |E_{32}(x_{3N}^0, y_j^t)| \| \theta_{3N}^{0,t} \| \mathcal{T}_{3N+1} \| X_x \| \mathcal{T}_{3N+1} \]
\[ + C \varepsilon \| E_{32}(x_{3N}^0, y_0^t) \| \| \theta_{3N}^{0,0} \| \mathcal{T}_{3N+1, N} \| X_x \| \mathcal{T}_{3N+1, N} \]
\[ =: \mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3 \leq C \varepsilon \| N^{-(k+\frac{1}{2})} \| X \| \mathcal{E}, \]

where same as (49), one has
\[ \mathcal{V}_1 \leq C \varepsilon \| N^{\frac{1}{2}-t} \| X \| \mathcal{E}. \]
and in the same way as (51), it can obtain

$$
\mathcal{W}_2 \leq C \varepsilon_2^\frac{1}{2} N^{-\tau} \| \chi \|_E,
$$

$$
\mathcal{W}_3 \leq C \varepsilon_1^{\frac{1+\tau}{2}} \varepsilon_1^\frac{1}{2} N^{-\tau} \| \chi \|_E.
$$

For $M_4$, Hölder inequality yields

$$
|M_4| \leq C \varepsilon_2 \| E_{32}^I - E_{32} \|_{\infty, [x_{3N}^2, x_N] \times [y_N^2 - \tau, y_N]} \| \chi \|_{[x_{3N}^2, x_N] \times [y_N^2 - \tau, y_N]} 
+ C \varepsilon_2 \| E_{32}^I - E_{32} \|_{[x_{3N}^2, x_N] \times [y_N^2 - \tau, y_N]} \| \chi \|_{[x_{3N}^2, x_N] \times [y_N^2 - \tau, y_N]}
\leq: \mathcal{Z}_1 + \mathcal{Z}_2 \leq C (\varepsilon_2 \mu_1^{-\frac{1}{2}} N^{-k} + \varepsilon_2^\frac{1}{2} N^{\frac{1}{2} - \tau}) \| \chi \|_E,
$$

(53)

where similar to (45), one has

$$
\mathcal{Z}_1 \leq C \varepsilon_2 \mu_1^{-\frac{1}{2}} N^{-k} \| \chi \|_E,
$$

and Hölder inequality, (12), Lemma 3.1 and (6) yield

$$
\mathcal{Z}_2 \leq C \varepsilon_2 \| E_{32}^I - E_{32} \|_{\infty, [x_{3N}^2, x_N] \times [y_N^2 - \tau, y_N]} (1 - \frac{1}{2} \ln N)^{\frac{1}{2}} \| \chi \|_E 
\leq C \varepsilon_2 N^\tau \mu_1^{-\frac{1}{2}} N^{\frac{1}{2}} \| \chi \|_E
\leq C \varepsilon_2 \varepsilon_1^{-\frac{1}{2}} \mu_1^{-\frac{1}{2}} N^\tau N^{\frac{1}{2}} \| \chi \|_E
\leq C \varepsilon_2^\frac{1}{2} N^{\frac{1}{2} - \tau} \| \chi \|_E.
$$

Thus, from (7), (8), (49), (50), (52), and (53), we can obtain our conclusion.

Now, we present the main conclusions of this paper.

**Theorem 5.3** Assuming $\tau \geq k + 1$. On the Bakhvalov-type mesh $\mathbb{T}$, and based on Assumption 2.1, we have

$$
\| u^I - u_N \|_E + \| \Pi u - u_N \|_E \leq C (\varepsilon_1^{\frac{1}{2}} + \varepsilon_2) \frac{1}{2} N^{-k} + CN^{-(k+1)},
$$

$$
\| u - u_N \|_E \leq C (\varepsilon_1^{\frac{1}{2}} + \varepsilon_2) \frac{1}{2} N^{-k} + CN^{-(k+1)}.
$$

**Proof** From (41), (42), Lemmas 5.1 and 5.2 we can prove

$$
\| \Pi u - u_N \|_E \leq C (\varepsilon_1^{\frac{1}{2}} + \varepsilon_2) \frac{1}{2} N^{-k} + CN^{-(k+1)}.
$$
Table 1  $\|u - u^N\|_E$ in the case of $\varepsilon_2 = 10^{-3}$ and $k = 1$

| $\varepsilon_1$ | $N$  | 8  | 16  | 32  | 64  | 128  | 256  |
|----------------|------|----|-----|-----|-----|------|------|
| $10^{-6}$      |      | 0.19E-1 | 0.99E-2 | 0.50E-2 | 0.25E-2 | 0.13E-3 | 0.63E-3 |
|                |      | 0.97   | 0.98 | 0.99 | 1.00 | 1.00 | –    |
| $10^{-7}$      |      | 0.12E-1 | 0.57E-2 | 0.29E-2 | 0.14E-2 | 0.73E-3 | 0.36E-3 |
|                |      | 1.02   | 0.99 | 1.00 | 1.00 | 1.00 | –    |
| $10^{-8}$      |      | 0.84E-2 | 0.36E-2 | 0.18E-2 | 0.89E-3 | 0.44E-3 | 0.22E-3 |
|                |      | 0.99   | 1.00 | 1.00 | 1.00 | 1.00 | –    |
| $10^{-9}$      |      | 0.74E-2 | 0.26E-2 | 0.13E-2 | 0.63E-3 | 0.31E-3 | 0.16E-3 |
|                |      | 1.48   | 1.05 | 1.01 | 1.00 | 1.00 | –    |
| $10^{-10}$     |      | 0.70E-2 | 0.23E-2 | 0.11E-2 | 0.53E-3 | 0.26E-3 | 0.13E-3 |
|                |      | 1.62   | 1.08 | 1.02 | 1.01 | 1.00 | –    |

Combination of (25), Lemmas 4.3, and 4.4 yields

$$\|u^I - u^N\|_E \leq C(\varepsilon_1^{\frac{1}{2}} + \varepsilon_2)^{\frac{1}{2}} N^{-k} + CN^{-(k+1)}.$$ 

Finally, using Theorem 4.6, we prove that

$$\|u - u^N\|_E \leq C(\varepsilon_1^{\frac{1}{2}} + \varepsilon_2)^{\frac{1}{2}} N^{-k} + CN^{-(k+1)}.$$ 

□

Table 2  $\|u - u^N\|_E$ in the case of $\varepsilon_2 = 10^{-5}$ and $k = 1$

| $\varepsilon_1$ | $N$  | 8  | 16  | 32  | 64  | 128  | 256  |
|----------------|------|----|-----|-----|-----|------|------|
| $10^{-6}$      |      | 0.19E-1 | 0.99E-2 | 0.50E-2 | 0.25E-2 | 0.13E-3 | 0.63E-3 |
|                |      | 0.97   | 0.97 | 0.99 | 1.00 | 1.00 | –    |
| $10^{-7}$      |      | 0.11E-1 | 0.55E-2 | 0.28E-2 | 0.14E-2 | 0.71E-3 | 0.36E-3 |
|                |      | 0.97   | 0.97 | 0.99 | 1.00 | 1.00 | –    |
| $10^{-8}$      |      | 0.61E-2 | 0.31E-2 | 0.16E-2 | 0.80E-3 | 0.40E-3 | 0.20E-3 |
|                |      | 0.97   | 0.97 | 0.99 | 1.00 | 1.00 | –    |
| $10^{-9}$      |      | 0.35E-2 | 0.18E-2 | 0.09E-3 | 0.45E-3 | 0.23E-3 | 0.11E-3 |
|                |      | 0.98   | 0.97 | 0.99 | 1.00 | 1.00 | –    |
| $10^{-10}$     |      | 0.20E-2 | 0.99E-3 | 0.50E-3 | 0.25E-3 | 0.13E-3 | 0.63E-4 |
|                |      | 0.98   | 0.97 | 0.99 | 1.00 | 1.00 | –    |
Table 3  \( \| u - u^N \|_E \) in the case of \( \epsilon_2 = 10^{-7} \) and \( k = 1 \)

| \( \epsilon_1 \) | \( N \) | 8   | 16  | 32  | 64  | 128 | 256 |
|------------------|-------------|------|-----|-----|-----|-----|-----|
| \( 10^{-6} \)    |             | 0.19E-1 | 0.99E-2 | 0.50E-2 | 0.25E-2 | 0.13E-3 | 0.63E-3 |
|                  |             | 0.97   | 0.97 | 0.99 | 1.00 | 1.00 | –   |
| \( 10^{-7} \)    |             | 0.11E-1 | 0.55E-2 | 0.28E-2 | 0.14E-2 | 0.71E-3 | 0.36E-3 |
|                  |             | 0.97   | 0.97 | 0.99 | 1.00 | 1.00 | –   |
| \( 10^{-8} \)    |             | 0.61E-2 | 0.31E-2 | 0.16E-2 | 0.80E-3 | 0.40E-3 | 0.20E-3 |
|                  |             | 0.97   | 0.97 | 0.99 | 1.00 | 1.00 | –   |
| \( 10^{-9} \)    |             | 0.35E-2 | 0.18E-2 | 0.89E-3 | 0.45E-3 | 0.23E-3 | 0.11E-3 |
|                  |             | 0.97   | 0.97 | 0.99 | 1.00 | 1.00 | –   |
| \( 10^{-10} \)   |             | 0.19E-2 | 0.99E-3 | 0.50E-3 | 0.25E-3 | 0.13E-3 | 0.63E-4 |
|                  |             | 0.98   | 0.97 | 0.99 | 1.00 | 1.00 | –   |

Fig. 1  \( k = 2, \epsilon_2 = 10^{-3} \)
The purpose of this section is to verify that our main conclusions are correct. In order to do so, we study the performance of the method when applied to the test problem

$$-\varepsilon_1 \Delta u + \varepsilon_2 (2 - x)u_x + u = f(x, y) \quad \text{in } \Omega,$$

$$u|_{\partial \Omega} = 0,$$  \hspace{1cm} (54)

where the right-hand side is chosen such that

$$u(x, y) = \frac{1}{4} \left(1 - e^{-\mu_0 x}\right) \left(1 - e^{-\mu_1 (1-x)}\right) \left(1 - e^{\frac{y}{\varepsilon_1}}\right) \left(1 - e^{\frac{(1-y)}{\sqrt{\varepsilon_1}}}\right),$$

with

$$\mu_0 = \frac{-\varepsilon_2 + \sqrt{\varepsilon_2^2 + \varepsilon_1}}{\varepsilon_1}, \quad \mu_1 = \frac{\varepsilon_2 + \sqrt{\varepsilon_2^2 + 4\varepsilon_1}}{2\varepsilon_1},$$

is the exact solution.

In our example, we take $k = 1, 2, 3, p = 0.5, \delta = 0.25, N = 2^3, \cdots, 2^9$. Besides, we should choose a perturbation parameters range $R(\varepsilon_1, \varepsilon_2)$ that meets the conditions (7), (13) and the mesh is completely in the Bakhvalov-type. Thus, for problem (54), the value range $R(\varepsilon_1, \varepsilon_2)$ of perturbation parameters should be

$$R(\varepsilon_1, \varepsilon_2) = \{(\varepsilon_1, \varepsilon_2) | 0 < \varepsilon_1 \leq 10^{-6}, 0 < \varepsilon_2 \leq 10^{-3}\}.$$
To be more general, we take \( \varepsilon_1 = 10^{-6}, 10^{-7}, 10^{-8}, 10^{-9}, 10^{-10} \), \( \varepsilon_2 = 10^{-3}, 10^{-5}, 10^{-7} \).

For any fixed value of \( k \) and \( \varepsilon_2 \), energy norm error estimation will be calculated by

\[
e^N = \| u - u^N \|_E,
\]

where \( u \) is the exact solution given by (54) and \( u^N \) represents its numerical approximation. And its corresponding convergence rate is

\[
p^N = \frac{\ln e^N - \ln e^{2N}}{\ln 2}.
\]

In Tables 1, 2, and 3, we give the energy error estimations and convergence orders of \( k = 1 \) and \( \varepsilon_2 = 10^{-3}, 10^{-5}, 10^{-7} \). At the same time, we present the energy estimations in the cases of \( k = 2, \varepsilon_2 = 10^{-3}, 10^{-5}, 10^{-7} \) and \( k = 3, \varepsilon_2 = 10^{-3}, 10^{-5}, 10^{-7} \) in the figure below. As can be seen from Tables 1–3, our Theorem 5.1 is well verified when \( k = 1 \). With the increase of \( N \), from Figs. 1, 2, 3, 4, 5, and 6, we can observe that the order of convergence for the case of \( k = 2 \) and \( k = 3 \) gradually approaches 2 and 3, respectively. This implies that Theorem 5.1 is sharp.
Fig. 4 $k = 3, \epsilon_2 = 10^{-5}$

Fig. 5 $k = 2, \epsilon_2 = 10^{-7}$
Fig. 6  $k = 3, \varepsilon_2 = 10^{-7}$

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