Classical solvability of the multidimensional free boundary problem for the thin film equation in the case of partial wetting.

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To the memory of my dear Big chief Academician I.V.Skrypnik

Abstract

We prove locally in time the existence of the unique smooth solution (including smooth interface) to the multidimensional free boundary problem for the thin film equation in the case of partial wetting. We also obtain the Schauder estimates and solvability for the Dirichlet and the Neumann problem for a linear degenerate parabolic equation of fourth order.

Key words: thin film equation, free boundary problem, degenerate parabolic equation, Schauder’s estimates, smooth solution.

MSC: 35R35, 35K55, 35K65

1 Introduction.

The present paper is devoted to the studying of a local in time smooth solution to a free boundary problem for the thin film equation in multidimensional setting. The literature on the thin film equation is so numerous that it is impossible to give the complete overview in this brief introduction. Among papers on the thin film equation in the one dimensional or multidimensional setting we mention only the papers [1] - [18] and we refer the reader to these papers on questions on physical origins of the model. At the same time the literature on smoothness of the solutions to the thin film models are far not so numerous even for the case of one spatial variable. Regularity and smoothness of the solution and it’s free boundary in the one dimensional setting was obtained in the papers [1] - [5]. As for the case of more than one spatial variable (the multidimensional setting), the author is aware only of the paper [15] (see also the paper [17] in this connection). It is well known that multidimensional setting is fundamentally differ from the one-dimensional one. In the one dimensional case the free boundary is just a point at each moment of time. So there is no the question about the smoothness of the free boundary with respect to the
spatial variables. Instead, in the multidimensional setting the problem requires the studying of the smoothness of the free boundary not only with respect to time but also with respect to the spatial variables.

In the present paper we consider the free boundary model for the thin film equation in the case of partial wetting. In fact, the present paper can be seen as a generalization to the multidimensional setting of the paper [1]. So all physical foundations for the mathematical model below can be found in [1].

Let us turn to the formal mathematical statement of the problem. Let \( N \geq 1 \) be an integer, \( T > 0 \). Let \( Q \) be a (non-cylindrical) bounded domain in \( \mathbb{R}^N \times [0, T] \) with the lateral boundary \( S_T \). Denote also for each \( t \in [0, T] \) the open section \( Q_t = \{(y, \tau) \in Q : \tau = t \in [0, T]\} \subset \mathbb{R}^N \). We denote independent variables by \( (y, \tau) \) in view of a subsequent transformation of the problem to new variables.

We denote for further \( Q_0 = \{(y, 0) \in Q \} = \Omega \), where \( \Omega \) is a given domain in \( \mathbb{R}^N \). In this notation \( S_T = \{(y, \tau) : \tau \in [0, T], y \in \partial Q_\tau \} \). The domain \( Q \) is unknown and has to be determined together with the unknown nonnegative function \( h(y, \tau), (y, \tau) \in Q \), by the conditions

\[
\frac{\partial h}{\partial \tau} + \nabla (h^2 \nabla \Delta h) = 0, \quad (y, \tau) \in Q, \quad (1.1)
\]

\[
h(y, \tau) = 0, \quad y \in \partial Q_\tau, \quad \text{that is} \quad (y, \tau) \in S_T, \quad (1.2)
\]

\[
\frac{\partial h}{\partial \tau}(y, \tau) = g(y, \tau), \quad y \in \partial Q_\tau, \quad \text{that is} \quad (y, \tau) \in S_T, \quad (1.3)
\]

\[
h(y, \tau) > 0, \quad y \in Q_\tau, \quad \text{in open} \quad Q_\tau, \quad (1.4)
\]

\[
h(y, 0) = h_0(y), \quad y \in \overline{Q_0} = \overline{\Omega}. \quad (1.5)
\]

Here \( \nabla = (\partial/\partial y_1, ..., \partial/\partial y_N) \), \( \Delta \) is the Laplace operator, \( \Omega \) is a given domain in \( \mathbb{R}^N \), \( g(y, \tau) \) is a given function on \( \mathbb{R}^N \times [0, T] \), \( h_0(y) \) is a given function on \( \Omega \), \( \overline{\partial Q_\tau} \) is the unite outward normal to \( \partial Q_\tau \). To formulate strict conditions on the data \( \Omega, g(y, \tau), h_0(y) \) we have to introduce some function spaces we use below.

Let \( M \) be a positive integer. In the space \( \mathbb{R}^M \) we use standard Hölder spaces \( C^{\text{\overline{\tau}}}(\mathbb{R}^M) \), where \( \overline{\tau} = (l_1, l_2, ..., l_M), l_i \) are arbitrary positive non-integers. The norm in such spaces is defined by

\[
\|u\|_{C^{\text{\overline{\tau}}}(\mathbb{R}^M)} = |u|_{R^M}^{\text{\overline{\tau}}} = |u|_{R^M}^{(0)} + \sum_{i=1}^{M} \langle u \rangle_{x_i, R^M}^{(l_i)}, \quad (1.6)
\]

\[
\langle u \rangle_{x_i, R^M}^{(l_i)} = \sup_{x \in R^M, h > 0} \left| \frac{D_{\overline{x}}^{[li]} u(x_1, ..., x_i + h, ..., x_M) - D_{\overline{x}}^{[li]} u(x)}{h^{[li]} \overline{|x|}^{[li]}} \right|, \quad (1.7)
\]
\[ \langle u \rangle^\circ \equiv \sum_{i=1}^{M} \langle u \rangle_{x_i, R^M}^{(l_i)}, \]

where \([l_i]\) is the integer part of the number \(l_i\), \(D_{x_i}^{[l_i]} u\) is the derivative of order \([l_i]\) with respect to the variable \(x_i\) of a function \(u\).

**Proposition 1** Seminorm (1.7) can be equivalently defined by (19), (20), (21)

\[ \langle u \rangle_{x_i, R^M}^{(l_i)} \simeq \sup_{x \in R^M, h > 0} \frac{\Delta_k u(x)}{h^{l_i}}, \quad k > l_i, \quad (1.8) \]

where \(\Delta_{h,x_i} u(x) = u(x_1, ..., x_i + h, ..., x_N) - u(x)\) is the difference from a function \(u(x)\) with respect to the variable \(x_i\) with a step \(h\), \(\Delta_{h,x_i}^k u(x) = \Delta_{h,x_i}^{k-1} u(x)\) = \((\Delta_{h,x_i})^k u(x)\) is the difference of power \(k\).

The same is also valid not only for the whole space \(R^M\) but also for its subsets of the form \(R^M \cap \{x_i_1, x_i_2, ..., x_i_K \geq 0\}\) with \(K \leq M\). It is known that functions from the space \(C^1(R^M)\) have also mixed derivatives up to definite orders and all derivatives are Hölder continuous with respect to all variables with some exponents in accordance with ratios between the exponents \(l_i\). Namely, if \(k = (k_1, ..., k_M)\) with nonnegative integers \(k_i, k_i \leq [l_i]\), and

\[ \omega = 1 - \sum_{i=1}^{N} \frac{k_i}{l_i} > 0, \quad (1.9) \]

then (see for example [20])

\[ D_{x_i}^k u(x) \in \mathcal{C}^{d_i}(R^M), \quad \|D_{x_i}^k u\|_{\mathcal{C}^{d_i}(R^M)} \leq C \|u\|_{\mathcal{C}^{d_i}(R^M)}, \quad (1.10) \]

where

\[ \mathcal{d} = (d_1, ..., d_M), \quad d_i = \omega l_i. \quad (1.11) \]

Moreover, relation (1.10) is valid not only for \(R^M\) but for any domain \(\Omega \subset R^M\) with sufficiently smooth boundary and we have

\[ \|D_{x_i}^k u\|_{C^{\mathcal{d}}(\Omega)} \leq C \|u\|_{C^{\mathcal{d}}(\Omega)}, \quad (1.12) \]

For special domains of the form \(\Omega_+ = R^M \cap \{x_i_1, x_i_2, ..., x_i_K \geq 0\}\) we have even more strong inequality just for seminorms

\[ \sum_{i=1}^{M} \sum_{\mathcal{T}} \langle D_{x_i}^k u \rangle_{x_i, \mathcal{T}_+}^{(d_i)} \leq C \sum_{i=1}^{M} \langle u \rangle_{x_i, \mathcal{T}_+}^{(l_i)}. \quad (1.13) \]

Here the sum is taken over all \(k\) with the property (1.9) and \(d_i\) are defined in (1.11).
The analog of this estimate for an arbitrary smooth domain $\Omega$ (including bounded domains) is
\[
\sum_{k=1}^{M} \sum_{i=1}^{\hat{d}_i} \langle D_k^{\gamma} u \rangle_{\hat{d}_i} \leq C \left( \sum_{i=1}^{M} (u)^{\delta_i} + |u(0)| \right)
\] (1.14)
with arbitrary $\hat{d}_i \leq d_i$.

Now we define weighted Hölder spaces for problem (1.1)-(1.5). These spaces are a particular case of spaces from \([22]\) (see the preprint version in \([23]\)).

Let $\gamma \in (0, 1)$. Let $\Omega$ has the boundary $\Gamma = \partial \Omega$ of the class $C^{4+\gamma}$. Let $d(x)$ be a function of the class $C^{1+\gamma}(\Omega)$ with the property
\[
\nu \cdot \text{dist}(x, \partial \Omega) \leq d(x) \leq \nu^{-1} \cdot \text{dist}(x, \partial \Omega), \quad \text{dist}(x, \partial \Omega) \leq 1, \quad \nu > 0.
\] (1.15)
As such a function can serve, for example, the bounded solution of the problem
\[
\Delta d(x) = -1, \quad x \in \Omega, \quad d(x)|_{\partial \Omega} = 0.
\]
For $x, \bar{x} \in \Omega$ we denote $|d(x, \bar{x})| = \max\{|d(x)|, |d(\bar{x})|\}$ and for a function $u(x)$ denote
\[
\langle u \rangle_{\gamma/2} = \sup_{x, \bar{x} \in \Omega} |u(\bar{x}) - u(x)|/|\bar{x} - x|^{\gamma/2}.
\] (1.16)
Note that weighted seminorm (1.16) is equivalent to the usual Hölder seminorm with respect to some Carnot-Caratheodory metric for equation (1.1) (see \([26]\), \([25]\), \([15]\), \([22]\) for the definitions and see \([24]\), \([22]\) for the equivalence).

Define the space $C^{1+\gamma/2}(\Omega)$ as the space of functions $u(x)$ with the finite norm
\[
|u|_{1+\gamma/2, \Omega} = \|u\|_{C^{1+\gamma/2}(\Omega)} = |u(0)|_{\Omega} + \langle u \rangle_{\gamma/2, \Omega},
\] (1.17)
where $|u(0)|_{\Omega} = \max_{\Omega}|u(x)|$. And define the space $C^{4+\gamma, 4+\gamma/2}(\Omega)$ as the space of continuous in $\Omega$ functions $u(x)$ with the finite norm
\[
\|u\|_{C^{4+\gamma, 4+\gamma/2}(\Omega)} = |u(0)|_{\Omega} + \sum_{|\alpha|=4} \langle d(x)^2 D_\alpha^2 u(x) \rangle_{\gamma/2, \Omega},
\] (1.18)
where $\alpha = (\alpha_1, ..., \alpha_N)$ is a multiindex, $|\alpha| = \alpha_1 + ... + \alpha_N$.

For $T > 0$ denote $\Omega_T = \{(x, t) : x \in \Omega, t \in (0, T)\}$ and define the space $C^{4+\gamma, 4+\gamma/2}(\Omega_T)$ as the space of continuous in $\Omega_T$ functions $u(x, t)$ with the finite norm
\[
\|u\|_{C^{4+\gamma, 4+\gamma/2}(\Omega_T)} = |u(0)|_{\Omega_T} + \langle u \rangle_{4+\gamma, 2, \Omega_T},
\] (1.19)
where
\[ \langle u \rangle_{2, \gamma/2, \tau}^{(4+\gamma)} \equiv \sum_{|\alpha|=4} \langle d(x)^2 D^\alpha_x u(x, t) \rangle_{\gamma/2, \tau}^{(\gamma)} + \langle D_t u \rangle_{\gamma/4, \tau}^{(\gamma/4)}. \]  

(1.20)

Note that functions from \( C^{4+\gamma, \gamma/2}_2(\Omega), C^{4+\gamma, \gamma/2}_2(\Omega_T) \) have unweighted first order and some second order derivatives with respect to \( x \) - see the next section for details. Note also that all norms defined for different functions \( d(x) \in C^{1+\gamma}(\Omega) \) with properties (1.15) are equivalent.

We can formulate now our assumptions on the data of problem (1.1)-(1.5). Let \( \gamma \in (0, 1) \) be fixed and fix \( \gamma' \in (\gamma, 1) \). We suppose that the initial domain \( \Omega \) is sufficiently smooth,

\[ \Gamma \equiv \partial \Omega \in C^{7+\gamma}. \]  

(1.21)

Here and everywhere below we denote \( \Gamma \equiv \partial \Omega, \Omega_T \equiv \Omega \times [0,T], \Gamma_T \equiv \Gamma \times [0,T] \) - the lateral boundary of \( \Omega_T \). For the initial data \( h_0(y) \) we assume that

\[ |h_0|^{(4+\gamma')}_{2, \gamma/2, \Omega} \leq \mu < \infty, \quad h_0(y) > 0, y \in \Omega, \quad \frac{\partial h_0}{\partial \mathbf{n}} \bigg|_{\partial \Omega} \geq \nu > 0, \]  

(1.22)

where \( \mathbf{n} \) is the unit inward normal to \( \Gamma \equiv \partial \Omega \). Here and below we denote by the same symbols \( \mu, \nu, C \) all absolute constants or constants depending only on fixed data of the problem. About the boundary condition \( g(y, \tau) \) in (1.3) we suppose that

\[ g(y, \tau) \in C^2(\mathbb{R}^N \times [0,T]), \quad g(y, \tau) \leq -\nu < 0. \]  

(1.23)

We suppose also that the following agreement condition for \( \tau = 0, y \in \partial \Omega \) is fulfilled

\[ h_0(y) = 0, y \in \Gamma \equiv \partial \Omega, \quad \frac{\partial h_0}{\partial \mathbf{n}} \bigg|_{\partial \Omega} = g(y, 0)\big|_{\partial \Omega}. \]  

(1.24)

Formulate now the main result of the paper.

**Theorem 2.** Under assumptions (1.21)-(1.24) problem (1.1)-(1.5) has the unique solution \( h(y, \tau) \in C^{4+\gamma, \gamma/4}_{2, \gamma/2}(\Omega_T) \) for some \( T \leq T_0(\Omega, g, h_0) \) and the free boundary \( S_T \) belongs to the class \( C^{2,1+\gamma/4}_{x,t}. \)

The method of the proving of Theorem 2 consists of reducing of the problem to some nonlinear operator equation and applying the Inverse Function theorem as it was done in [24]. So we formulate a variant of such theorem.

**Theorem 3 ([27], Theorem 1.2 and its proof.)**

Let \( U \) be open in a Banach space \( H \), and let \( F : U \to Y \) be continuously differentiable on \( U \), where \( Y \) is a Banach space. Let \( x_0 \in U \) and assume that \( F'(x_0) : H \to Y \) is a toplinear isomorphism (i.e. invertible as a continuous
linear map). Then $F$ is a local diffeomorphism at $x_0$ and there exists $d > 0$ such that the inverse mapping $F^{-1}$ is defined on the ball $K_d = \{ y \in Y : \| y - y_0 \| \leq d \}$, $y_0 = F(x_0)$.

Here $d = d(M_1, M_2, \omega_0)$ depends only on $M_1, M_2, \omega_0$, where

$$\begin{align*}
M_1 &= \| F'(x_0) \|_{H \rightarrow Y}, \\
M_2 &= \| (F'(x_0))^{-1} \|_{Y \rightarrow H}, \\
\omega_0 &= \sup_{x_1, x_2 \in U} \| F'(x_1) - F'(x_2) \|_{H \rightarrow Y}.
\end{align*}$$

Due to this classical theorem we have the following very simple but fundamentally important for us assertion, where $x_0$ serves as an approximate solution to the equation $F(x) = 0$.

**Corollary 4** Let the conditions of Theorem 3 are satisfied. Then there exists $\varepsilon_0 = \varepsilon_0(M_1, M_2, \omega_0) > 0$ such that if $\| F(x_0) \|_Y = \| y_0 \|_Y \leq \varepsilon_0$, then for some $x^* \in U$ we have $F(x^*) = 0$.

**Proof.** This corollary immediately follows from Theorem 3 if we choose $\varepsilon_0 = d/2$ so that $0 \in K_d$. ■

Note that since we are going to use this corollary, the key ingredient of our proof of Theorem 2 is the proof of the fact that $f'(x_0)$ is invertible.

The rest of the paper is organised as follows. In Section 2 we collect some auxiliary assertion we need below, including some properties of the weighted Hölder spaces. Section 3 is devoted to a reformulation of the original problem as a nonlinear problem in a fixed domain with some additional unknown function for a parametrization of the free boundary. In Section 4 we calculate the Frechet derivative of the nonlinear operator of the problem from section 3. Section 5 is devoted to obtaining the Schauder estimates in weighted Hölder classes for some model problems for the linearised thin film equation and in Section 6 we show the solvability of slightly different model problems. In Section 7 we consider the Neumann and the Dirichlet problem for the linearised thin film equation in an arbitrary smooth domain. Section 8 shows the invertibility of the Frechet derivative from Section 4 by proving the unique solvability of some linear problem. At last, Section 9 completes the proof of Theorem 2.

### 2 Auxiliary assertions.

Let $\Omega$ be a domain in $\mathbb{R}^N$ with the boundary $\Gamma = \partial \Omega$ of the class $C^{4+\gamma}, \Omega_T = \Omega \times [0, T], T > 0$. Denote also $H = R^N_+ = \{ x \in R^N : x_N \geq 0 \}$. We need for further use two technical lemmas.

**Lemma 5** Let a function $u(x) \in C^\gamma_0(\Omega)$. Then $u(x)$ belongs to the unweighted class $C^{\gamma/2}(\Omega)$ and

$$\| u \|_{C^\gamma_0(\Omega)} \leq C \| u \|_{C^{\gamma/2}(\Omega)}. \quad (2.1)$$
Lemma 6 \cite{22} Let \( K \subseteq \overline{\Omega} \) be a compact set. Let \( U \subset C_{\gamma/2}^J(K) \) be a bounded subset in \( C_{\gamma/2}^J(K) \) that is

\[
 u(x) \in U \Rightarrow \|u\|_{C_{\gamma/2}^J(K)} \leq M \tag{2.2}
\]

for some constant \( M > 0 \). Then there exists a sequence \( \{u_n(x)\} \subset U \) and a function \( u_0(x) \in C_{\gamma/2}^J(K) \) from the same space \( C_{\gamma/2}^J(K) \) such that for any \( \gamma' \in (0, \gamma) \)

\[
 \|u_n - u_0\|_{C_{\gamma'/2}^J(K)} + \|u_n - u_0\|_{C_{\gamma'/2}^J(K)} \to n \to \infty 0, \quad \|u_0\|_{C_{\gamma'/2}^J(K)} \leq M. \tag{2.3}
\]

Lemmas 5 and 6 were proved in \cite{22} for the case \( \Omega = \mathbb{R}_+^N \) but the general case is completely similar.

Below in this section we collect for the further use some assertions about spaces \( C_{2,\gamma/2}^{4+\gamma}(\Omega), C_{2,\gamma/2}^{4+\gamma}(\Omega_T) \). For the proofs we refer the reader to the paper \cite{22} (see also the preprint version \cite{23}), where the more general spaces \( C_{n,\omega}^{m+\gamma/2}(\Omega_T) \) are considered.

First of all, functions from the space \( C_{2,\gamma/2}^{4+\gamma}(\Omega_T) \) has finite some weighted and unweighted lower order derivatives.

Proposition 7 \cite{22}

Let \( u(x,t) \in C_{2,\gamma/2}^{4+\gamma}(\Omega_T) \). There is an absolute constant \( C = C(\Omega, \gamma) \) with

\[
 \sum_{|\alpha|=1} |D^\alpha_x u(x,t)|_{(\gamma)_{\gamma/2,\Omega_T}} + \sum_{|\alpha|=1} |D^\alpha_t u(x,t)|_{(\gamma)_{\gamma/2,\Omega_T}} + \sum_{|\alpha|=3} |d(x)D^\alpha_x u(x,t)|_{(\gamma)_{\gamma/2,\Omega_T}} \leq C|u|_{C_{2,\gamma/2,\Omega_T}^{4+\gamma}}, \tag{2.4}
\]

where for a function \( v(x,t) \) on \( \Omega_T \)

\[
 |v|_{(\gamma)_{\gamma/2,\Omega_T}} \equiv \sup_{\frac{t}{\theta+2t} \in \Omega_T} \frac{|\Delta_{\theta,t} v(x,t)|}{|\theta|^{\gamma/4}},
\]

\[
 \Delta_{\theta,t} v(x,t) = v(x,t+\theta) - v(x,t), \quad \Delta_{\theta,t} v(x,t) = v(x+\theta, t) - v(x,t), \quad \Delta^2_{\theta,t} v(x,t) = \Delta_{\theta,t} \left( \Delta_{\theta,t} v(x,t) \right).
\]

Thus first derivatives \( D^\alpha_x u(x,t) \) with \( |\alpha| = 1 \) belong to the Zigmund space \( Z^1 \) with the additional smoothness in \( t \). At the same time the second derivatives \( D^\alpha_x u(x,t) \) with \( |\alpha| = 2 \) may be unbounded and in general \( |D^2_x u(x,t)| \sim C|\ln d(x)| \) as \( x \to \partial \Omega \) - see \cite{1}, \cite{22}.

Lemma 8 \cite{22} Let \( u(x,t) \in C_{2,\gamma/2}^{4+\gamma}(\Omega_T) \). Then for \( |\alpha| = 2 \)

\[
 |D^\alpha_x u(x,t)| \leq C|\ln d(x)||d(x)|D^2_x u|_{C_{\gamma/2,\Omega_T}^{(\gamma)}}, \tag{2.6}
\]
where
\[ |d(x)D^3_u(0)|_{\Omega_T} = \sum_{|\alpha|=3} |d(x)D^3_u(0)|_{\Omega_T}. \]

We denote by \( C_{2,\gamma/2,0}^{4+\gamma/2} (\Omega_T) \) the closed subspace of \( C_{2,\gamma/2}^{4+\gamma/2} (\Omega_T) \) consisting of functions \( u(x,t) \) with the property \( u(x,0) \equiv u_t(x,0) \equiv 0 \) in \( \Omega \).

**Proposition 9** ([22]) Let \( u(x,t) \in C_{2,\gamma/2,0}^{4+\gamma/2} (\Omega_T), \ T \leq 1. \ Then for \( |\alpha|=2,3 \) and with some \( \delta > 0 \)
\[ |d(x)D^2_u(\gamma/4)|_{\Omega_T} \leq CT^\delta \| u \|_{C_{2,\gamma/2,0}^{4+\gamma/2} (\Omega_T)}. \] (2.7)

And for \( |\alpha| < 2 \)
\[ |D^2_u(\gamma/4)|_{\Omega_T} \leq CT^\delta \| u \|_{C_{2,\gamma/2,0}^{4+\gamma/2} (\Omega_T)}. \] (2.8)

Such inequalities for usual Hölder norms in \( C_{0}^{l_1,l_2} (\Omega_T) \) are well studied ([32], [33]) and we have \((l'_1 > l_1, l'_2 > l_2)\)
\[ \| u \|_{C_{0}^{l_1,l_2} (\Omega_T)} \leq CT^\delta \| u \|_{C_{0}^{l'_1,l'_2} (\Omega_T)}. \] (2.9)

Let \( \overline{\gamma} \) be outward normal to \( \Gamma \). We consider now the question of traces of \( u(x,t) \in C_{2,\gamma/2}^{4+\gamma/2} (\Omega_T) \) at \( \Gamma_T \).

**Proposition 10** ([22]) A function \( u(x,t) \in C_{2,\gamma/2}^{4+\gamma/2} (\Omega_T) \) and it’s derivative \( \partial u/\partial \overline{\gamma} \) on \( \Gamma_T \) have traces at \( \Gamma_T \) from the spaces \( u(x,t)|_{\Gamma_T} \in C_{x',t}^{2+\gamma/2,1+\frac{\gamma}{2}}(\Gamma_T), \partial u/\partial \overline{\gamma} \in C_{x',t}^{1+\gamma/2,1+\frac{\gamma}{2}}(\Gamma_T) \) \((x' \in \Gamma)\) and
\[ \| u(x',t) \|_{C_{x',t}^{2+\gamma/2,1+\frac{\gamma}{2}}(\Gamma_T)} \leq C \| u \|_{C_{x',t}^{4+\gamma}(\Gamma_T)}, \]
\[ \| \partial u/\partial \overline{\gamma} (x',t) \|_{C_{x',t}^{1+\gamma/2,1+\frac{\gamma}{2}}(\Gamma_T)} \leq C \| u \|_{C_{x',t}^{4+\gamma}(\Gamma_T)}. \] (2.10)

As for the extension of functions \( v(x',t) \) from the class \( C_{2,\gamma/2}^{2+\gamma/2,1+\frac{\gamma}{2}}(\Gamma_T) \) to the region \( \overline{\Omega_T} \), we have the following assertion.

Denote a neighbourhood of \( \Gamma_T \)
\[ N_\lambda \equiv \{(x,t) \in \overline{\Omega_T} : \ dist(x,\Gamma) \leq \lambda \}, \]
where \( \lambda > 0 \) is sufficiently small.

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Proposition 11 ([22]) For any sufficiently small \( \lambda > 0 \) there exists an operator \( E : C^{2+\gamma/2, 1+\gamma/2}_{x,x'}(\Gamma_T) \to C^{2+\gamma/2, 1+\gamma/2}_{x,x'}(\Gamma_T) \) and \( E : C^{\gamma/2, \gamma/4}_{x,x'}(\Gamma_T) \to C^{\gamma/2, \gamma/4}_{x,x'}(\Gamma_T) \) with the property:

for a given function \( v(x', t) \in C^{2+\gamma/2, 1+\gamma/2}_{x,x'}(\Gamma_T) \) the function \( w(x,t) = Ev \in C^{2+\gamma/2, 1+\gamma/2}_{x,x'}(\Gamma_T) \) has support in a neighborhood \( N_\lambda \) of \( \Gamma_T \) and satisfies

\[
w(x,t)|_{\Gamma_T} = v(x'), \quad \|w\|_{C^{2+\gamma/2, 1+\gamma/2}_{x,x'}(\Gamma_T)} \leq C_\lambda \|v\|_{C^{2+\gamma/2, 1+\gamma/2}_{x,x'}(\Gamma_T)},
\]

(2.11)

where the constant \( C \) does not depend on \( v \).

Besides, the operator \( E \) possesses the property

\[
\frac{\partial Ev}{\partial t} = Ev \quad \text{for} \quad (x, t) \in \Gamma_T.
\]

Propositions 10 and 11 were proved in [22] for the halfspace \( \Omega = \{ x : x_N \geq 0 \} \) but the general case is obtained in standard way by the localisation near \( \Gamma_T \). In the special case of \( \Omega = \{ x : x_N \geq 0 \} \equiv \mathbb{R}^N_+ \), \( \Omega_T = \mathbb{R}^N_+ \times [0, T] \equiv \mathbb{R}^N_+T \), and \( d(x) = x_N \) we have the following properties of the space \( C^{2+\gamma/2, \gamma/4}_{x,x'}(\mathbb{R}^N_+T) \).

Proposition 12 ([22])

Let \( u(x, t) \in C^{2+\gamma/2, \gamma/4}_{x,x'}(\mathbb{R}^N_+T) \). Then

\[
\langle u \rangle_{2, \gamma/2, \mathbb{Q}^N}^{(1+\gamma/2,4+\gamma)} \equiv \sum_{j=0}^{1} \sum_{|\alpha|=4-j} \langle x_N^{2-j} D_{x}^{\alpha} u \rangle_{\gamma/2, \mathbb{Q}^{N}}^{(\gamma/4)} + \sum_{j=0}^{2} \sum_{|\alpha|=4-j} \langle x_N^{2-j/2} D_{x}^{\alpha} u \rangle_{t, \mathbb{Q}^{N} \times T}^{(2+\gamma/2)} + \langle D_{x}^{\gamma/2} u \rangle_{\gamma/2, \mathbb{Q}^{N} \times T}^{(\gamma/4)}
\]

\[
+ \langle D_{x}^{\gamma/2} u \rangle_{\gamma/2, \mathbb{Q}^{N} \times T}^{(\gamma/4)}
\]

\[
+ \langle D_{x}^{\gamma/2} D_{x}^{\gamma/2} u \rangle_{\gamma/2, \mathbb{Q}^{N} \times T}^{(\gamma/4)}
\]

\[
+ \langle D_{x}^{\gamma/2} D_{x}^{\gamma/2} u \rangle_{\gamma/2, \mathbb{Q}^{N} \times T}^{(\gamma/4)}
\]

(2.12)

where \( \langle x_N^{2} D_{x}^{4} u \rangle_{\gamma/2, \mathbb{Q}^{N} \times T}^{(\gamma)} \), \( i = 1, N \), are the corresponding weighted Hölder constants with respect to only particular variables \( x_i \) of the corresponding fourth derivatives with respect to the same variable.

Moreover,

\[
x_N^{2-j} D_{x}^{\alpha} u(x, t) \to 0, x_N \to 0, \quad j = 0, 1, \alpha = (\alpha_1, ..., \alpha_N), |\alpha| = 4-j, \alpha_N < 4-j.
\]

(2.14)
It is important also that the following interpolation inequalities are valid.

**Theorem 13** \((2.13)\) Let a function \(u(x, t) \in C^{4+\gamma/2}_{2,\gamma/2}(\Omega_T)\) and \(\alpha = (\alpha_1, ..., \alpha_N), |\alpha| = 4\), be a multiindex, \(k \in \{1, 2, ..., N\}\). Then for any \(\varepsilon > 0\)

\[
\langle x_N D^2 u \rangle^{(\gamma)}_{\gamma/2, x_k, R_{+\infty}} \leq C\varepsilon^{-\alpha_k - \gamma} \sum_{i=1, i \neq k}^N \langle x_N D^4 u \rangle^{(\gamma)}_{\gamma/2, x_k, R_{+\infty}} + C\varepsilon^{4-\alpha_k} \langle x_N D^4 u \rangle^{(\gamma)}_{\gamma/2, x_N, R_{+\infty}}, \quad k < N,
\]

\[
\langle x_N D^2 u \rangle^{(\gamma/4)}_{1, T} \leq \varepsilon^{-\gamma/4} C \sum_{i=1}^N \langle x_N D^4 u \rangle^{(\gamma)}_{\gamma/2, x_i, R_{+\infty}} + C\varepsilon \langle D_t u \rangle^{(\gamma/4)}_{T, R_{+\infty}},
\]

\[
\langle D_t u \rangle^{(\gamma)}_{\gamma/2, x_k, R_{+\infty}} \leq C\varepsilon^{-\gamma} \sum_{i=1, i \neq k}^N \langle x_N D^4 u \rangle^{(\gamma)}_{\gamma/2, x_i, R_{+\infty}} + \varepsilon^{-\gamma/2} C \langle D_t u \rangle^{(\gamma/4)}_{T, R_{+\infty}} + \varepsilon^4 C \langle x_N D^4 u \rangle^{(\gamma)}_{\gamma/2, x_k, R_{+\infty}}, \quad k < N,
\]

\[
\langle D_t u \rangle^{(\gamma/4)}_{1, T} \leq C\varepsilon^{-2-\gamma/2} \sum_{i=1}^N \langle x_N D^4 u \rangle^{(\gamma)}_{\gamma/2, x_i, R_{+\infty}} + C\varepsilon^{-\gamma/2} \langle D_t u \rangle^{(\gamma/4)}_{T, R_{+\infty}} + C\varepsilon^2 \langle x_N D^4 u \rangle^{(\gamma)}_{\gamma/2, x_N, R_{+\infty}},
\]

where the constants \(C\) does not depend on \(\varepsilon, u\).

**Theorem 14** Let a function \(u(x, t) \in C^{4+\gamma/2}_{2,\gamma/2}(\Omega_T)\). Then for \(\varepsilon > 0\)

\[
\sum_{|\alpha|=4} |d^2(x)D^a_{x_k}u(x,t)|^{(0)}_{T} \leq \varepsilon \sum_{|\alpha|=4} \langle d^2(x)D^a_{x_k}u(x,t) \rangle^{(\gamma)}_{\gamma/2, x, \Omega_T} + \varepsilon C \sum_{|\alpha|=3} |d(x)D^a_{x_k}u(x,t)|^{(0)}_{T},
\]

10
\[
\sum_{|\alpha|=1} \langle d^2(x) D_x^\alpha u(x,t) \rangle_{x,\gamma/2, \Omega_T}^{(\gamma)} \leq \varepsilon \sum_{|\alpha|=4} |d^2(x) D_x^\alpha u(x,t)|_{\Omega_T}^{(0)} + \frac{C}{\varepsilon} \sum_{|\alpha|=3} |d(x) D_x^\alpha u(x,t)|_{\Omega_T}^{(0)},
\]

(2.21)

\[
\sum_{|\alpha|=3} |d(x) D_x^\alpha u(x,t)|_{\Omega_T}^{(0)} \leq \varepsilon \sum_{|\alpha|=3} \langle d(x) D_x^\alpha u(x,t) \rangle_{x,\gamma/2, \Omega_T}^{(\gamma)} + \frac{C}{\varepsilon} \sum_{|\alpha|=2} |d(x) D_x^\alpha u(x,t)|_{\Omega_T}^{(0)} + C \varepsilon |D_x^\alpha u(x,t)|_{\Omega_T}^{(0)},
\]

(2.22)

\[
\sum_{|\alpha|=2} \langle d(x) D_x^\alpha u(x,t) \rangle_{x,\gamma/2, \Omega_T}^{(\gamma)} \leq \varepsilon \sum_{|\alpha|=3} |d(x) D_x^\alpha u(x,t)|_{\Omega_T}^{(0)} + \frac{C}{\varepsilon} \sum_{|\alpha|=2} |d(x) D_x^\alpha u(x,t)|_{\Omega_T}^{(0)} + C \varepsilon |D_x^\alpha u(x,t)|_{\Omega_T}^{(0)},
\]

(2.23)

\[
\sum_{|\alpha|=2} |d(x) D_x^\alpha u(x,t)|_{\Omega_T}^{(0)} \leq \varepsilon \sum_{|\alpha|=3} \langle d(x) D_x^\alpha u(x,t) \rangle_{x,\gamma/2, \Omega_T}^{(\gamma)} + \frac{C}{\varepsilon} \sum_{|\alpha|=1} |D_x^\alpha u(x,t)|_{\Omega_T}^{(0)},
\]

(2.24)

\[
\sum_{|\alpha|=1} \langle D_x^\alpha u(x,t) \rangle_{x,\gamma/2, \Omega_T}^{(\gamma)} \leq \varepsilon \sum_{|\alpha|=1} \langle D_x^\alpha u(x,t) \rangle_{x,\gamma/2, \Omega_T}^{(\gamma)} + \frac{C}{\varepsilon} \sum_{|\alpha|=1} |D_x^\alpha u(x,t)|_{\Omega_T}^{(0)} \leq 
\]

(2.25)

\[
\varepsilon \sum_{|\alpha|=3} |d(x) D_x^\alpha u(x,t)|_{\Omega_T}^{(0)} + \frac{C}{\varepsilon} \sum_{|\alpha|=1} |D_x^\alpha u(x,t)|_{\Omega_T}^{(0)},
\]

(2.26)

\[
\sum_{|\alpha|=1} |D_x^\alpha u(x,t)|_{\Omega_T}^{(0)} \leq \varepsilon \sum_{|\alpha|=1} \langle D_x^\alpha u(x,t) \rangle_{x,\gamma/2, \Omega_T}^{(\gamma)} + \frac{C}{\varepsilon} |u(x,t)|_{\Omega_T}^{(0)},
\]

(2.27)

Proof. Due to the possibility of the localization it is enough to consider the case of \( \Omega = \{ x : x_N \geq 0 \} \equiv R^N_+ \), \( \Omega_T = R^N_+ \times [0,T] \equiv R^N_{+T}, d(x) = x_N \) and \( u(x,t) \) is a function with compact support in the set \( \{ |x| \leq R, t \in [0,T] \} \). Besides, below in the proof the argument \( t \) is fixed.

Inequality (2.21) was proved in \[22\] so we start with (2.21). Let \( |\alpha| = 3 \) and let \( x, \varpi \in R^N_+, x_N \leq \varpi_N \). Consider the ratio
A ≡ x_N^{\gamma/2} \frac{|x_N^2 D_x^\alpha u(x, t) - \bar{x}_N^2 D_x^\alpha u(\bar{x}, t)|}{|x - \bar{x}|^{\gamma}}.

Let we are given an $\varepsilon > 0$ and consider two cases. Let first $|x - \bar{x}| \leq \varepsilon$. Then with some $x_\theta \in [x, \bar{x}]$

$$A \leq x_N^{\gamma/2} |x - \bar{x}|^{1-\gamma} \frac{|x_N^2 D_x^\alpha u(x, t) - \bar{x}_N^2 D_x^\alpha u(\bar{x}, t)|}{|x - \bar{x}|} \leq$$

$$\leq C_R \varepsilon^{1-\gamma} |\nabla \left( x_\theta^2 N D_x^\alpha u(x \theta, t) \right) | \leq$$

$$\leq C_R \varepsilon^{1-\gamma} \left( \sum_{|\beta|=4} |x_N^2 D_x^\alpha u(x, t)|^{(0)}_{\Pi_G} + \sum_{|\beta|=3} |x_N D_x^\alpha u(x, t)|^{(0)}_{\Pi_G} \right).$$

If now $|x - \bar{x}| > \varepsilon$, then

$$A \leq C_R \varepsilon^{-\gamma} \left( |x_N^2 D_x^\alpha u(x, t)| + |\bar{x}_N^2 D_x^\alpha u(\bar{x}, t)| \right) \leq$$

$$\leq \frac{C_R}{\varepsilon^{\gamma}} \sum_{|\beta|=3} |x_N D_x^\alpha u(x, t)|^{(0)}_{\Pi_G}.$$

Substituting now $\varepsilon^{1-\gamma}$ instead of $\varepsilon$, we obtain (2.21) from the last two estimates in view of the definition of the expression $A$.

Consider now (2.22). Let $|\alpha| = 3$ and let $x, y \in R^N_+$. We have

$$x_N D_x^\alpha u(x, t) = \frac{(x_N D_x^\alpha u(x, t) - y_N D_x^\alpha u(y, t))}{|x - y|^{\gamma/2}} |x - y|^{\gamma/2} + y_N D_x^\alpha u(y, t).$$

Integrate this inequality in $y$ over the set $Q_\varepsilon = \{ y \in R^N_+ : |x_i - y_i| \leq \varepsilon, i = 1, N \}$.

According to Lemma [5] we have, dividing by $C \varepsilon^N$,

$$|x_N D_x^\alpha u(x, t)| \leq \varepsilon^{\gamma/2} \langle x_N D_x^\alpha u(x, t) \rangle_{x, \gamma/2, R^N_+} + \frac{C}{\varepsilon^N} \int_{Q_\varepsilon} y_N D_x^\alpha u(y, t) dy.$$

Integrating in the integral by parts and taking into account that the point $x$ is arbitrary, we obtain (2.22).

The proofs of the others inequalities are completely analogous with the taking into account (2.6).
Lemma 15 Let \( u(x, t) \in C^{4+\gamma, 4+\gamma}_{2,\gamma/2,0} (\Omega_T) \), \( T \leq 1 \). Then for \( |\alpha| = 2, 3 \) and with some \( \delta > 0 \)

\[
|d(x) D^{\alpha}_x u(x,t)|_{\gamma/2,\gamma/2} \leq C T^\delta \|u\|_{C^{4+\gamma, 4+\gamma}_{2,\gamma/2,0} (\Omega_T)}, \quad |\alpha| = 3, \quad (2.28)
\]

\[
|d(x) D^{\alpha}_x u(x,t)|_{\gamma/2,\gamma/2} \leq C T^\delta \|u\|_{C^{4+\gamma, 4+\gamma}_{2,\gamma/2,0} (\Omega_T)}, \quad |\alpha| = 2, \quad (2.29)
\]

\[
|\nabla_x u|_{\gamma/2,\gamma/2} + |u|_{\gamma/2,\gamma/2} \leq C T^\delta \|u\|_{C^{4+\gamma, 4+\gamma}_{2,\gamma/2,0} (\Omega_T)}. \quad (2.30)
\]

Proof. Due to the possibility of the localization it is enough to consider the case of \( \Omega = \{ x : x_N \geq 0 \} \equiv R^N_+ \), \( \Omega_T = R^N_+ \times [0,T] \equiv R^N_+ \), \( d(x) = x_N \) and \( u(x,t) \) is a function with compact support in the set \( \{ |x| \leq R, t \in [0,T] \} \).

Consider first the case \(|\alpha| = 3\). Let \( t, T \in [0,T] \). Then it follows from (2.13) that

\[
\frac{|x_N^{3/2} D^\alpha_x u(x,t) - x_N^{3/2} D^\alpha_x u(x,t)|}{|t - T|^{1/4}} \leq C T^\delta \|u\|_{C^{4+\gamma, 4+\gamma}_{2,\gamma/2,0} (R^N_+, T)} \quad (2.31)
\]

This means that

\[
\left( x_N^{3/2} D^\alpha_x u(x,t) \right)_{t, R^N_+} \leq C T^\delta \|u\|_{C^{4+\gamma, 4+\gamma}_{2,\gamma/2,0} (R^N_+, T)} \quad (2.32)
\]

Consider now the properties \( x_N^{3/2} D^\alpha_x u(x,t) \) with respect to \( x \). Let \( x, T \in R^N_+ \) and let \( x_N \leq T_N \). Consider two cases. Let first \( h = |x - T| \leq x_N \). Then we have

\[
A \equiv x_N^{\gamma/2} \frac{|x_N^{3/2} D^\alpha_x u(x,t) - T_N^{3/2} D^\alpha_x u(T,N)|}{|x|^{1/4}} \leq x_N^{\gamma/2} \frac{|D^\alpha_x u(x,t) - D^\alpha_x u(T,N)|}{|x - T|^{1/4}} + \quad + x_N^{\gamma/2} \frac{|x_N^{3/2} - T_N^{3/2}|}{|x - T|^{1/4}} |D^\alpha_x u(T,N)| \equiv A_1 + A_2 \quad (2.33)
\]

For \( A_1 \), since \( \gamma \in (0, 1/2) \) and since \( u(x, 0) \equiv 0 \), we have with some \( x_0 \in [x, T] \)

\[
A_1 = x_N^{-\gamma/2} |x_N^{-3/2} |x - T|^{1/4} \frac{|D^\alpha_x u(x,t) - D^\alpha_x u(T,N)|}{|x - T|^{1/4}} \leq x_N^{-\gamma/2} x_N^{3/2} x_N^{1-\gamma} |x_N^{3/2} |\nabla D^\alpha_x u(x_0, t)| \leq C R \frac{|x_N^{3/2} |\nabla D^\alpha_x u(x,t)|}{|R^N_+|} \leq C R T^\delta \left( x_N^{3/2} |\nabla D^\alpha_x u(x,t)| \right)_{t, R^N_+} \leq C R T^\delta \|u\|_{C^{4+\gamma, 4+\gamma}_{2,\gamma/2,0} (R^N_+, T)} \quad (2.34)
\]
And analogously for $A_2$ (since $h = |x - \overline{x}| \leq x_N$, $x_N \sim \overline{x}_N \sim x_N^0$)

$$A_2 \leq C x_N^{\gamma/2} x_N^0 \gamma |D_x^2 u(\overline{x}, t)| \leq C_R |x_N D_x^2 u(x, t)|_{R_N^+}^{(0)} \leq$$

$$\leq C_R T^{\frac{\gamma}{4}} \langle x_N D_x^2 u(x, t) \rangle_{1, R_N^+}^{(\gamma/4)} \leq C_R T^{\frac{\gamma}{4}} \| u \|_{C_{\gamma/2, 0}^{\frac{\gamma}{4}} (R_N^+)}.$$

Thus we obtain in the case $h = |x - \overline{x}| \leq x_N$

$$A \leq C_R T^{\frac{\gamma}{4}} \| u \|_{C_{\gamma/2, 0}^{\frac{\gamma}{4}} (R_N^+)} \cdot \ h = |x - \overline{x}| \leq x_N. \quad (2.36)$$

Let now $h = |x - \overline{x}| > x_N$ and we note that in this case $\overline{x}_N \leq x_N + h < 2h$. We have

$$A \leq \left( \frac{x_N}{h} \right)^{\gamma} x_N^{(1-\gamma)/2} |x_N^{3/2} D_x^2 u(x, t)| + \left( \frac{\overline{x}_N}{h} \right)^{\gamma} x_N^{(1-\gamma)/2} |x_N^{3/2} D_x^2 u(\overline{x}, t)| \leq$$

$$\leq C_R |x_N D_x^2 u(x, t)|_{R_N^+}^{(0)} \leq C_R T^{\frac{\gamma}{4}} \| u \|_{C_{\gamma/2, 0}^{\frac{\gamma}{4}} (R_N^+)} \cdot \quad (2.37)$$

From $(2.30), (2.37)$ it follows that

$$\left( x_N^{3/2} D_x^2 u(x, t) \right)^{(\gamma)}_{\gamma/2, x, R_N^+} \leq C_R T^{\frac{\gamma}{4}} \| u \|_{C_{\gamma/2, 0}^{\frac{\gamma}{4}} (R_N^+)} \cdot \quad (2.38)$$

At last,

$$|x_N^{3/2} D_x^2 u(x, t)|_{R_N^+}^{(0)} \leq C_R T^{\frac{\gamma}{4}} \langle x_N D_x^2 u(x, t) \rangle_{1, R_N^+}^{(\gamma/4)} \leq C_R T^{\frac{\gamma}{4}} \| u \|_{C_{\gamma/2, 0}^{\frac{\gamma}{4}} (R_N^+)} \cdot \quad (2.39)$$

Estimates $(2.32), (2.35)$, and $(2.39)$ prove $(2.28)$.

Consider inequality $(2.29)$. Represent $x_N^{1/2} D_x^2 u(x, t)$ as

$$x_N^{1/2} D_x^2 u(x, t) = -x_N^{1/2} \int_{x_N}^{R} \xi^{-3/2} a(x', \xi, t) d\xi,$$

where $a(x', \xi, t) \equiv \xi^{3/2} D_N^2 D_x^2 u(x', \xi, t)$. Analogously $(2.31)$ we have

$$|x_N^{1/2} D_x^2 u(x, t) - x_N^{1/2} D_x^2 u(x', \overline{t})| \leq x_N^{1/2} T^{1/4} \int_{x_N}^{R} \xi^{-3/2} \frac{|a(x', \xi, t) - a(x', \xi, \overline{t})|}{|t - \overline{t}|^{\gamma/4}} \leq$$

$$\leq C_R T^{1/4} \left( x_N^{3/2} D_N D_x^2 u(x, t) \right)^{\gamma/4+1/4}_{1, R_N^+} \leq C_R T^{1/4} \| u \|_{C_{\gamma/2, 0}^{\frac{\gamma}{4}} (R_N^+)} \cdot$$
This means that
\[
\left\langle x_N^{1/2} D_N^2 u(x,t) \right\rangle_{\gamma/2, x, R_N^+, T} \leq C R T^{1/4} \| u \|_{C^{4+\gamma, 4+\gamma/2} (R_N^+, T)} \cdot (2.40)
\]
The properties of \( x_N^{1/2} D_N^2 u(x,t) \) with respect to the \( x \) variables are considered analogously to (2.38) on the base of (2.6) and this gives
\[
\left\langle x_N^{1/2} D_N^2 u(x,t) \right\rangle_{\gamma/4, x, R_N^+, T} \leq C R T^{1/4} \| u \|_{C^{4+\gamma, 4+\gamma/2} (R_N^+, T)} \cdot (2.41)
\]
Now (2.29) follows from (2.40), (2.41).

The proof of (2.30) is completely analogous due to the Newton-Leibnitz formula and (2.6).

Below we will use also the following inequality for functions \( u(x,t), v(x,t) \in C^{\gamma, \gamma/4, \gamma/2} (\Omega_T) \)
\[
|uv|_{\gamma/2, \Omega_T} \leq CT^{\gamma/4} |u|_{\gamma/2, \Omega_T} |v|_{\gamma/2, \Omega_T}. (2.42)
\]
This inequality is completely analogous to the well known unweighted case.
We have the following assertion (22) Let \( Q^+ = R_N^+ \times [0, \infty) \)

**Lemma 16** Let a function \( u(x,t) \in C^{4+\gamma, 4+\gamma/2} (\Omega_T) \). Denote
\[
a_u = \lim_{(x,t) \to (0,0)} x_N^{2} D_N^4 u(x,t). (2.43)
\]
and denote
\[
\tilde{Q}_u(x_N) = -a \ln(2) x_N, (2.44)
\]
where
\[
\ln(2) x_N \equiv \int_0^{x_N} \frac{\xi}{0} d\xi \int_0^{\xi} \ln \xi d\xi. (2.45)
\]
Denote further
\[
Q_u(x,t) = -a_u \ln(2) x_N + \sum_{|\alpha| \leq 2} \frac{a^\alpha}{\alpha!} (x - \bar{x})^\alpha + a^{(1)} t, (2.46)
\]
where \( \alpha = (\alpha_1, ..., \alpha_N) \), \( \alpha! = \alpha_1! ... \alpha_N! \), \( \bar{x} = (0, ..., 1) \), \( x - \bar{x} = x_N^{\alpha_1} ... x_N^{\alpha_N - 1} (x_N - 1)^{\alpha_N} \)
\[ a_\alpha = D_x^\alpha (u - \tilde{Q}_u(x_N))|_{x=\pi,t=0}, \quad a^{(1)} = D_t(u - \tilde{Q}_u(x_N))|_{x=\pi,t=0}. \]

Then the function \( Q_u(x,t) \) has the following properties

\[ x_N^{2-j}D_x^\alpha [u(x,t) - Q_u(x,t)]|_{(x,t)=(0,0)} = 0, \quad j = 0, 1, \quad |\alpha| = 4 - j, \quad (2.47) \]

\[ D_x^\alpha [u(x,t) - Q_u(x,t)]|_{(x,t)=\pi,0} = 0, \quad |\alpha| \leq 2, \quad D_t[u(x,t) - Q_u(x,t)]|_{(x,t)=\pi,0} = 0, \quad (2.48) \]

\[ x_N^{\alpha N}Q_u(x,t) \equiv \text{const}, \quad |\alpha| = 4 - j, \quad 0 \leq j \leq 2, \quad \alpha_N < 2, \quad D_tQ_u(x,t) \equiv \text{const}. \quad (2.49) \]

At last for \( j \leq 2 \) and \( |\alpha| = 2 - j \)

\[ D_x^\alpha D_x^{\alpha N}Q_u(x,t) \quad \text{does not depend on} \quad x' \quad \text{and} \quad t. \]

In what follows we will use also the following Liouville theorem. Consider in the domain \( Q_+ = \{(x,t): x \in \mathbb{R}^N_+, -\infty < t < \infty\} \) the homogeneous boundary value problem for an unknown function \( u(x,t) \)

\[ \frac{\partial u}{\partial t} + \nabla(x_N^2 \nabla u) = 0, \quad (x,t) \in Q_+, \quad (2.50) \]

\[ u(x',0,t) = 0, \quad x_N = 0, \quad (2.51) \]

or, instead of boundary condition (2.51), the boundary condition

\[ \frac{\partial u(x',0,t)}{\partial x_N} = 0, \quad x_N = 0. \quad (2.52) \]

**Theorem 17** Let a solution to problem (2.50), (2.51) (or (2.50), (2.52)) belongs to the class \( C^{4+\gamma,4+\gamma/2}_{2,\gamma/2}(K_R) \) for any compact set \( K_R = \{|x| \leq R, |t| \leq R\} \cap Q_+, \quad R > 0 \), and has a power growth

\[ |u(x,t)|^{(0)}_{K_R} \leq CR^A, \]

where \( C \) and \( A \) are some positive constants. Then \( u(x,t) \) is a polynomial with respect to the variables \( x' \) and \( t \).

If in addition the function \( u(x,t) \) satisfies the initial condition \( u(x,0) \equiv 0 \), then \( u(x,t) \equiv 0 \).

This theorem was proved in [18] in the case of boundary condition (2.51) by the method of local integral estimates, but all the reasoning of the proof a word to a word is applicable also to boundary condition (2.52). Condition (2.51) is used in [18] only in the places of the proof, where some boundary integrals over \( \{x_N = 0\} \) vanish. But all those boundary integrals vanish also under condition (2.52). So we refer the reader to [18] for the proof.
Corollary 18 Let a function \( u(x) \) does not depend on \( t \) and satisfy the conditions of Theorem 17, that is \( u(x) \) is a solution of power growth to the corresponding elliptic problem. Then \( u(x) \) is a polynomial with respect to \( x' \)-variables.

3 Reduction of the problem to a fixed domain.

We will show below, that the free (unknown) boundary \( S_T \) can be parameterized in terms of its deviation from the given surface \( \Gamma_T = \Gamma \times [0,T] \). We follow to [28] to give the exact formulation (compare [28], [24]).

Let \( \omega = (\omega_1, ..., \omega_{N-1}) \) is some local curvilinear coordinates in a domain \( \Theta \) on \( \Gamma \). In some small neighbourhood \( N \) in \( R^N \) of the surface \( \Gamma \) we introduce the coordinates \((\omega, \lambda)\) in the way that for any \( x \in N \) we have the following unique representation

\[
x = x'(x) + n(x') \lambda \equiv x(\omega) + n(\omega) \lambda,
\]

where \( x'(x) = x'(\omega) \) is the point in the domain \( \Theta \) on the surface \( \Gamma \) with the coordinates \( \omega \), \( n(\omega) \) - the normal vector to \( \Gamma \) at the point \( x(\omega) \) with the direction into \( \Omega \). The coordinate \( \lambda \in R \) means, in fact, the deviation of a point \( x \) from \( \Gamma \), \( \pm \lambda > 0 \) for \( x \in \Omega \) or \( x \in R^N \setminus \Omega \). We assume that the neighbourhood \( N \) of the surface \( \Gamma \) is the set

\[
N = \{ x \in \Omega : |\lambda(x)| < \gamma_0 \},
\]

where \( \gamma_0 \) is sufficiently small and will be chosen below.

Let \( \rho(x', t) \equiv \rho(\omega, t) \) is a sufficiently small and regular function and \( \rho(x', t) \equiv \rho(\omega, t) \) is defined on the surface \( \Gamma_T \). Let us note that here and in what follows we use the notation \( \rho(\omega, t) \) with the argument \( \omega \) instead of \( \rho(x', t) \) for all functions on the surface \( \Gamma \) if it does not cause ambiguity. We do that just for simplification of the notation, bearing in mind that in each local domain \( \Theta \) on \( \Gamma \) we can introduce local coordinates \( \omega \). At the same time the coordinate \( \lambda \) in (3.1) does not depend on a choice of local coordinates \( \omega \).

We parameterize the unknown surface \( S_T \) with the help of the unknown function \( \rho(\omega, t) \) as follows

\[
S_T \equiv \Gamma_{\rho,T} = \{ (x, t) \in \Omega_T : x = x' + \rho(x', t) n(x') = x(\omega) + \rho(\omega, t) n(\omega) \},
\]

where \( x' = x(\omega) \in \Gamma \). Note that this definition of the surface \( S_T \equiv \Gamma_{\rho,T} \) does not depend on a choice of local coordinates \( \omega \) in a particular local domain on \( \Gamma \). Thus, the unknown function \( \rho(\omega, t) \) means, in fact, deviation of the surface \( \Gamma_{\rho,T} = S_T \) from the given surface \( \Gamma_T \).

Along with \( Q \) in (1.1) we use the notation \( \Omega_{\rho,T} = Q \). Let further \( \rho(x, t) \) is an extension of the function \( \rho(\omega, t) \) from the surface \( \Gamma_T \) to the whole domain \( \Omega_T \) to a function with support in the neighborhood \( N_T = N \times [0,T] \) of the surface \( \Gamma_T \), \( \rho(x, t) = E\rho(\omega, t) \), where \( E \) is some fixed extension operator from Proposition
Define a mapping \( e_\rho(x,t) \) from \( \mathbb{R}^N \times [0,T] \) on itself with the help of the formula
\[
e_\rho: (x,t) \rightarrow (y,\tau),
\]
where, according to the notations of (3.1),
\[
y = \begin{cases} x'(x) + \overrightarrow{n}(x')(\lambda(x) + \rho(x,t)), & x \in \mathcal{N}, \\
x, & x \in \overline{\mathcal{N}} \setminus \mathcal{N}, \\
\tau = t,
\end{cases}
\]
or, with the help of the local coordinates \( \omega \),
\[
y = \begin{cases} x'(\omega(x)) + \overrightarrow{n}(\omega(x))(\lambda(x) + \rho(x,t)), & x \in \mathcal{N}, \\
x, & x \in \overline{\mathcal{N}} \setminus \mathcal{N}, \\
\tau = t.
\end{cases}
\]
Here \( x'(x) \in \Gamma, \omega(x), \lambda(x) \) are \((\omega,\lambda)\)-coordinates of a point \( x \) in the neighbourhood \( \mathcal{N} \). Note that the definition of the mapping \( e_\rho \) does not depend on a choice of local coordinates \( \omega \) on the surface \( \Gamma \). We choose \( \gamma_0 \) sufficiently small so that under the condition
\[
|\rho|_{\Gamma}^{1+\gamma} \leq \frac{\gamma_0}{2}
\]
the mapping \( e_\rho \) is a diffeomorphism of \( \mathbb{R}^N \times [0,T] \) on itself and also the mapping \( e_\rho(x,t) \) is the identical mapping out of the neighbourhood \( \mathcal{N}_T \) of \( \Gamma_T \). Note that since \( \Gamma \) is the initial position of the unknown surface \( \Gamma_{\rho,T} \),
\[
\rho(\omega,0) \equiv 0, \omega \in \Gamma, \quad \text{and thus} \quad \rho(x,0) = E\rho(\omega,0) \equiv 0, x \in \overline{\mathcal{N}}
\]
and thus \( e_\rho(x,0) \) is the identical mapping of \( \overline{\mathcal{N}} \) onto itself.

We make in problem (1.1)-(1.5) the change of the independent variables \( (y,\tau) = e_\rho(x,t) \) and for simplicity denote the new function \( h(y,\tau) \circ e_\rho(x,t) = h(x,t) \) by the same symbol. In the new variables \((x,t)\) the problem become
\[
\frac{\partial h}{\partial t} - b_{h,\rho} \lambda - \nabla \rho \left( h^2 \nabla \rho \left( \nabla^2 h \right) \right) = 0, \quad (x,t) \in \Omega_T,
\]
\[
h(x,t) = 0, \quad (x,t) \in \Gamma_T,
\]
\[
\frac{\partial h}{\partial n}(x,t) \frac{1}{(1+\rho)} \left[ 1 + \sum_{i,j=1}^{N-1} m_{ij}(x,\rho) \rho_{\omega_i} \rho_{\omega_j} \right]^{1/2} - g_\rho(x,t) = 0, \quad (x,t) \in \Gamma_T,
\]
\[
h(x,t) > 0, \quad \text{in open} \quad \Omega_T,
\]
\[
h(x,0) = h_0(x), \quad \rho(x,0) = 0, \quad x \in \overline{\mathcal{N}}.
\]
\( \rho(x,t) = E \rho(\omega, t), \quad (x,t) \in \Omega_T, \omega(t) \in \Gamma_T. \) \hspace{1cm} (3.13)

Here \( \nabla \rho \equiv E \rho \nabla_x, \) \( \rho = \{e_{ij}(x,\rho, \nabla \rho) : i,j = 1, 2, \ldots, N\} \) is the transition matrix from the differentiating in \( y \) to the differentiating in \( x \) under the change of the variables \((3.3)\), that is for any function \( u(y, \tau) \) we have \( (\nabla_y u(y, \tau)) \circ e_{\rho}(x,t) = \nabla_x (u(y, \tau) \circ e_{\rho}(x,t)) \). Note that by the definition of \( e_{\rho}(x,t) \) the elements \( e_{ij}(x,\rho, \nabla \rho) \) of the matrix \( E_{\rho} \) depend only on the surface \( \Gamma \) and they are given smooth functions of their arguments of the class \( C^{\gamma+\gamma} \). Further, in \((3.8)\)

\[ b_{h,\rho} \equiv \frac{\partial h}{\partial \lambda} / (1 + \rho \lambda), \]

\[ g_{\rho}(x,t) \equiv g(\omega, \tau) \circ e_{\rho}(x,t) \text{ in } (3.10) \]

and \( m_{ij}(x,\rho) \) in \((3.10)\) are given functions of their coordinates of the class \( C^{\gamma+\gamma} \). The functions \( m_{ij}(x,\rho) \) depend not only on \( \Gamma \) but also on the choice of the local coordinates \( \omega \) on \( \Gamma \) in the way that the expression \( N^{i=1} m_{ij}(x,\rho) \rho \omega_i \rho \omega_j \) in brackets does not depend on such choice (because all other terms in \((3.10)\) do not depend on \( \omega \)). Note that the function \( h_0(x) \) in \((3.12)\) is the original initial function from \((1.5)\) because the mapping \( e_{\rho}(x,0) \) is identical at \( t = 0 \).

Let us explain for completeness the obtaining of conditions \((3.8), (3.10)\) from conditions \((1.1), (1.3)\) - compare \([24]\). Denote by \((\omega_x, \lambda_x)\) the \((\omega, \lambda)\)-coordinates of the point \( y \) and by \((\omega_y, \lambda_y)\) the \((\omega, \lambda)\)-coordinates of the point \( x \) in \( N \). The expression \( \frac{\partial h}{\partial \tau} - b_{h,\rho} \rho_t \) is the recalculated in the variables \( x, t \) derivative \( \frac{\partial h}{\partial \tau} \) after change of variables \((3.5)\):

\[
\frac{\partial h}{\partial \tau} \circ e_{\rho} = \frac{\partial h}{\partial t} - \frac{\partial h}{\partial \lambda} / (1 + \rho \lambda) \rho_t = \frac{\partial h}{\partial t} - b_{h,\rho} \rho_t.
\]

Here in fact \( \frac{\partial t}{\partial \tau} = 1, \frac{\partial \omega_{xi}}{\partial \tau} = 0 \).

And for the value of \( \frac{\partial \lambda_x}{\partial \tau} \) due to the relation

\[ \lambda_x = \lambda_y - \rho(x,t) \circ e_{\rho}^{-1}, \]

and taking into account \((3.14)\) we have

\[
\frac{\partial \lambda_x}{\partial \tau} = - \frac{\partial}{\partial \tau} [\rho(x,t) \circ e_{\rho}(x,t)^{-1}] =
\]

\[
- \frac{\partial \rho}{\partial t} \frac{\partial t}{\partial \tau} - \frac{\partial \rho}{\partial \lambda_x} \frac{\partial \lambda_x}{\partial \tau} - \frac{\partial \rho}{\partial \omega_{xi}} \frac{\partial \omega_{xi}}{\partial \tau} = - \rho_t - \rho \lambda_x \frac{\partial \lambda_x}{\partial \tau}.
\]

So in the variables \( x \) and \( t \)

\[
\frac{\partial \lambda_x}{\partial \tau} = - \rho_t / (1 + \rho \lambda_x). \hspace{1cm} (3.15)
\]

Thus, from \((3.14)\) and \((3.15)\) it follows that

\[
\frac{\partial h}{\partial \tau} \circ e_{\rho} = \frac{\partial h}{\partial t} - \frac{\partial h}{\partial \lambda} / (1 + \rho \lambda) \rho_t = \frac{\partial h}{\partial t} - b_{h,\rho} \rho_t.
\]
We explain further the transition from condition (1.3) to condition (3.10) under change of variables (3.5), as we shall need in the future the exact explicit form of this condition. Define in the neighborhood \( N_T \) of the surface \( \Gamma_T \) the function

\[
\Phi_{\rho}(y, \tau) = \lambda_x \circ e^{-1}_{\rho}(y, \tau) = \lambda_y - \rho(x, t) \circ e^{-1}_{\rho}(y, \tau) = \lambda(y) - \rho(y, \tau),
\]

where for simplicity we have retained for the function \( \rho(x, t) \circ e^{-1}_{\rho}(y, \tau) \) the same notation \( \rho(y, \tau) \). By the definition we have \( |\Phi_{\rho}(y, \tau)| > 0 \) for \( (y, \tau) \notin \Gamma_{\rho,T} \) and \( \Phi_{\rho}(y, \tau) = 0 \) for \( (y, \tau) \in \Gamma_{\rho,T} \). Hence, in (1.3)

\[
\cos(\vec{n}_z, y_i) = \frac{\Phi_{\rho y}}{|\nabla y \Phi_{\rho}|}.
\]

Therefore, relation (1.3) can be written as follows

\[
(\nabla_y h, \nabla_y \Phi_{\rho}) = g(y, \tau)|\nabla y \Phi_{\rho}|.
\]

(3.17)

Under change of variables (3.5) we have

\[
(\nabla_y h, \nabla_y \Phi_{\rho}) \circ e_{\rho}(x, t) = (\nabla_{\rho} h, \nabla_{\rho} \lambda_x), \quad |\nabla_{\rho} \Phi_{\rho}| \circ e_{\rho}(x, t) = |\nabla_{\rho} \lambda_x|.
\]

(3.18)

Denote by \( \Lambda(x) \) the transition matrix from the gradient with respect to the variables \( x \) to the gradient with respect to the variables \( (\omega_x, \lambda_x) \), that is,

\[
\nabla_x = \Lambda(x) \nabla_{(\lambda_x, \omega_x)}, \quad (\nabla_y = \Lambda(y) \nabla_{(\lambda_y, \omega_y)}),
\]

(3.19)

where

\[
\Lambda(x) = \begin{pmatrix}
\frac{\partial \lambda}{\partial x_1} & \frac{\partial \omega_1}{\partial x_1} & \cdots & \frac{\partial \omega_{N-1}}{\partial x_1} \\
\frac{\partial \lambda}{\partial x_2} & \frac{\partial \omega_1}{\partial x_2} & \cdots & \frac{\partial \omega_{N-1}}{\partial x_2} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{\partial \lambda}{\partial x_N} & \frac{\partial \omega_1}{\partial x_N} & \cdots & \frac{\partial \omega_{N-1}}{\partial x_N}
\end{pmatrix},
\]

(3.20)

and similarly for the variables \( y \). Then in the variables \( (x, t) \)

\[
(\nabla_{\rho} h, \nabla_{\rho} \lambda_x) = (\mathcal{E}_{\rho} \Lambda \nabla_{(\lambda_x, \omega_x)} h, \mathcal{E}_{\rho} \Lambda \nabla_{(\lambda_y, \omega_y)} \lambda_x).
\]

Note that \( \nabla_{(\lambda_x, \omega_x)} \lambda_x = \{1, 0, \ldots, 0\} \), and also \( h \equiv 0 \) on \( \Gamma_T \), hence \( \partial h / \partial \omega_i = 0 \), and therefore

\[
\nabla_{(\lambda_x, \omega_x)} h = \{ \partial h / \partial \lambda_x, 0, \ldots, 0 \} = \begin{pmatrix} \partial h / \partial \lambda_x \end{pmatrix} \nabla_{(\lambda_x, \omega_x)} \lambda_x.
\]

Thus we obtain

\[
(\nabla_y h, \nabla_y \Phi_{\rho}) \circ e_{\rho}(x, t) = (\nabla_{\rho} h, \nabla_{\rho} \lambda_x) = \frac{\partial h}{\partial \lambda_x} (\nabla_{\rho} \lambda_x, \nabla_{\rho} \lambda_x),
\]

(3.21)

that is in view of (3.18) in the new variables condition (1.3) become

\[
\frac{\partial h}{\partial \lambda_x} [(\nabla_{\rho} \lambda_x, \nabla_{\rho} \lambda_x)]^{\frac{1}{2}} = g_{\rho}(x, t).
\]

(3.22)
On the other hand, due to the definition of $\Phi_\rho(y, \tau)$

$$(\nabla_\rho \lambda_x, \nabla_\rho \lambda_x) = (\nabla_y (\lambda_x \circ e_\rho^{-1}), \nabla_y (\lambda_x \circ e_\rho^{-1})) \circ e_\rho = (\nabla_y \Phi_\rho, \nabla_y \Phi_\rho) \circ e_\rho. \quad (3.23)$$

Making use of introduced in (3.19) matrix $\Lambda(y)$, we have

$$(\nabla_y \Phi_\rho, \nabla_y \Phi_\rho) = (\Lambda(y) \nabla_{(\lambda_y, \omega_y)} \Phi_\rho, \Lambda(y) \nabla_{(\lambda_y, \omega_y)} \Phi_\rho) =$$

$$= (\nabla_{(\lambda_y, \omega_y)} \Phi_\rho, \Lambda(y)^* \Lambda(y) \nabla_{(\lambda_y, \omega_y)} \Phi_\rho). \quad (3.24)$$

First, by the definition of $\Phi_\rho$

$$\frac{\partial \Phi_\rho}{\partial \lambda_y} = \frac{\partial}{\partial \lambda_y} (\lambda_y - \rho(y, \tau)) = 1 - \rho_{\lambda_y},$$

$$\frac{\partial \Phi_\rho}{\partial \omega_{yi}} = \frac{\partial}{\partial \omega_{yi}} (\lambda_y - \rho(y, \tau)) = -\rho_{\omega_{yi}}. \quad (3.25)$$

In addition, since the coordinate $\lambda_y$ is counted along the normal to $\Gamma$, and $\omega_{yi}$ are coordinates on the surface $\Gamma$, we have

$$(\nabla_y \lambda, \nabla_y \lambda) = 1, \quad (\nabla_y \lambda, \nabla_y \omega_i) = 0, \quad i = 1, ..., N - 1.$$

Therefore the matrix $\Lambda^*(y) \Lambda(y)$ has the form

$$\Lambda^*(y) \Lambda(y) = \begin{pmatrix}
1 & 0 & 0 & ... & 0 \\
0 & m_{11} & m_{12} & ... & m_{1(N-1)} \\
... & ... & ... & ... & ... \\
0 & m_{(N-1)1} & m_{(N-1)2} & ... & m_{(N-1)(N-1)}
\end{pmatrix}, \quad (3.26)$$

where

$$m_{ij} = m_{ji} = (\nabla_y \omega_i(y), \nabla_y \omega_j(y)) - \quad (3.27)$$

are some smooth functions. Thus,

$$(\nabla_{(\lambda_y, \omega_y)} \Phi_\rho, \Lambda^*(y) \Lambda(y) \nabla_{(\lambda_y, \omega_y)} \Phi_\rho) =$$

$$= (1 - \rho_{\lambda_y})^2 + \sum_{i,j=1}^{N-1} m_{ij}(y)\rho_{\omega_{yi}} \rho_{\omega_{yj}}. \quad (3.28)$$

Make now in (3.28) change of variables (3.5), and recalculate the derivatives of $\rho$ with respect to $(\lambda_y, \omega_y)$ in terms of the derivatives with respect to $(\lambda_x, \omega_x)$. We have

$$\rho_{\lambda_y} \circ e_\rho = \rho_{t} \frac{\partial t}{\partial \lambda_y} + \rho_{\lambda_x} \frac{\partial \lambda_x}{\partial \lambda_y} + \sum_{i=1}^{N-1} \rho_{\omega_{xi}} \frac{\partial \omega_{xi}}{\partial \lambda_y}. \quad (3.29)$$

From the definition of the mapping $e_\rho$ it follows that

$$\frac{\partial t}{\partial \lambda_y} = 0, \quad \frac{\partial \omega_{xi}}{\partial \lambda_y} = 0. \quad (3.30)$$
At the same time by (3.29), (3.30)
\[
\frac{\partial \lambda_x}{\partial \lambda_y} = 1 - \rho_x = 1 - \rho_x \frac{\partial \lambda_x}{\partial \lambda_y},
\]
that is,
\[
\frac{\partial \lambda_x}{\partial \lambda_y} = \frac{1}{1 + \rho_x}. \tag{3.31}
\]
Therefore, by (3.29), (3.30) and (3.31)
\[
\rho_x \circ e_{\rho} = \frac{\rho_x}{1 + \rho_x}. \tag{3.32}
\]
Further,
\[
\rho_{\omega_{yi}} \circ e_{\rho} = \rho_t \frac{\partial}{\partial \omega_{yi}} + \rho_x \frac{\partial \lambda_x}{\partial \omega_{yi}} + \sum_{j=1}^{N-1} \rho_{\omega_{xi}} \frac{\partial \omega_{xj}}{\partial \omega_{yi}}, \tag{3.33}
\]
and
\[
\frac{\partial t}{\partial \omega_{yi}} = 0, \quad \frac{\partial \omega_{xj}}{\partial \omega_{yi}} = \delta_{ij}, \quad i, j = 1, ..., N - 1. \tag{3.34}
\]
At the same time
\[
\frac{\partial (\lambda_x \circ e_{\rho})}{\partial \omega_{yi}} = \left[ \frac{\partial}{\partial \omega_{yi}} (\lambda_y - \rho(y, \tau)) \right] \circ e_{\rho} = -\rho_{\omega_{yi}} \circ e_{\rho},
\]
That is, by virtue of (3.33) and (3.34),
\[
\rho_{\omega_{yi}} \circ e_{\rho} = \rho_x (-\rho_{\omega_{yi}} \circ e_{\rho}) + \rho_{\omega_{xi}}, \tag{3.35}
\]
hence by (3.35),
\[
\rho_{\omega_{yi}} \circ e_{\rho} = \frac{\rho_{\omega_{xi}}}{1 + \rho_x}. \tag{3.36}
\]
Thus, from (3.21), (3.28), (3.32) and (3.30) it follows that in (3.21)
\[
(\nabla_{\rho} \lambda_x, \nabla_{\rho} \lambda_x) = \frac{1}{(1 + \rho_x)^2} \left[ 1 + \sum_{i,j=1}^{N} m_{ij}(x, \rho) \rho_{\omega_{xi}} \rho_{\omega_{xj}} \right]. \tag{3.37}
\]
Taking into account (3.22) and condition (3.6) with \(\gamma_0\) sufficiently small, we arrive at (3.10).

### 4 A linearisation of the problem (3.8)- (3.12).

We will consider the set of left hand sides of (3.8)- (3.12) as some nonlinear operator on the pair \((h, \rho)\). In this section we describe an approximate solution \((w, \sigma) \approx (h, \rho)\) to (3.8)- (3.12) for small \(T > 0\) and extract the linear parts of (3.8)- (3.12) around this approximate solution in terms of \((h - w, \rho - \sigma)\).
From relations (3.8), (3.9), and (3.12) as ∂h/∂t = 0 on Γ_T, it follows that we can determine the value of ∂ρ(ω,0)/∂t on Γ at t = 0. Namely, from (3.8) it follows that
\[ \rho^{(1)}(\omega) \equiv \frac{\partial \rho(\omega,0)}{\partial t} = \left( \nabla(h_0^2(x)\nabla^2 h_0(x))/\partial \mathbf{n} \right) \bigg|_{\Gamma}. \] (4.1)

From condition (1.22) it follows first that \( h_0(x) \sim \nu d(x) \) near Γ and then it can be checked directly by the definitions and from Lemma 5 that
\[ |\rho^{(1)}(\omega)|^{(\gamma/2)} \leq C(|h_0|^{(4+\gamma')}_{\gamma/2,\mathbf{1}}). \] (4.2)

From (4.2) it follows that
\[ \rho^{(1)}(x) = E\rho^{(1)}(\omega) \in C^{\gamma'/2}(\Omega), \quad |\rho^{(1)}(x)|^{(\gamma')}_{\gamma'/2} \leq C(|h_0|^{(4+\gamma')}_{\gamma/2,\mathbf{1}}). \] (4.3)

Now we can determine the initial value of ∂h/∂t from equation (3.8). We have
\[ h^{(1)}(x) \equiv \frac{\partial h}{\partial t}(x,0) = \frac{\partial h_0(x)}{\partial t} \rho^{(1)}(x) - \nabla(h_0^2(x)\nabla^2 h_0(x)). \] (4.4)

From (4.3) and (4.4) it follows that
\[ |h^{(1)}(x)|^{(\gamma')}_{\gamma'/2,\mathbf{1}} \leq C(|h_0|^{(4+\gamma')}_{\gamma/2,\mathbf{1}}). \] (4.5)

Consider functions \( w(x,t) \in C^{4+\gamma',4+\gamma'}_{2,\gamma'/2}(\Omega_T), \ \sigma(\omega,t) \in C^{2+\gamma'/2,1+\gamma'/4}(\Gamma_T), \ \sigma(x,t) = E\sigma(\omega,t) \in C^{4+\gamma',4+\gamma'}_{2,\gamma'/2}(\Omega_T) \) with the properties
\[ |w|^{(4+\gamma')}_{2,\gamma'/2,\mathbf{1}_T} + |\sigma|^{(4+\gamma')}_{2,\gamma'/2,\mathbf{1}_T} \leq C(|h_0|^{(4+\gamma')}_{\gamma/2,\mathbf{1}} + |\rho^{(1)}(x)|^{(\gamma')}_{\gamma'/2,\mathbf{1}} + |h^{(1)}(x)|^{(\gamma')}_{\gamma'/2,\mathbf{1}}) \leq C(|h_0|^{(4+\gamma')}_{\gamma/2,\mathbf{1}}). \] (4.6)

\[ w(x,0) = h(x,0) = h_0(x), \quad \frac{\partial w}{\partial t}(x,0) = \frac{\partial h}{\partial t}(x,0) = h^{(1)}(x), \quad x \in \Omega, \] (4.7)

\[ w(x,t) = 0, \quad (x,t) \in \Gamma_T, \] (4.8)

\[ \sigma(x,0) = \rho(x,0) = 0, \quad \frac{\partial \sigma}{\partial t}(x,0) = \frac{\partial \rho}{\partial t}(x,0) = \rho^{(1)}(x) \quad x \in \Omega. \] (4.9)

The way of constructing such functions will be given below in Section 7, Proposition 4.
Lemma 19 By choosing the length $T$ of the time interval sufficiently small we can assume that

$$
\left. \frac{\partial w(x,t)}{\partial n} \right|_{\Gamma_T} \geq \nu > 0, \quad w(x,t) > 0, \quad x \in \Omega, \ t \in [0,T].
$$

\textbf{Proof.} Really, in view of properties (1.22) of $h_0(x)$ and (2.10) we have for $(x,t) \in \Gamma_T$

$$
\frac{\partial w(x,t)}{\partial n} = \frac{\partial h_0(x)}{\partial n} + \left( \frac{\partial w(x,t)}{\partial n} - \frac{\partial h_0(x)}{\partial n} \right) \geq
$$

$$
\geq \nu - |w|^{(4+\gamma')}_{2,\gamma'/2,\mu} T^{2+\gamma'} \geq \nu - C(\|h_0\|^{(4+\gamma')}_{\gamma'/2,\mu}) T^{2+\gamma'} \geq \nu/2 > 0,
$$

that is the first relation in (4.10) if $T$ is sufficiently small. Now from this and from (4.8) it follows that for some sufficiently small $\mu > 0$

$$
w(x,t) > 0, \ 0 < \text{dist}(x, \partial \Omega) \leq \mu, \ t \in [0,T].
$$

Denote the rest of $\Omega$ by $\Omega_\mu = \{ x \in \Omega : \mu \leq \text{dist}(x, \partial \Omega) \}$. In view of (1.22) on this compact set $h_0(x) = w(x,0) \geq \nu > 0$ and therefore for $x \in \Omega_\mu$

$$
w(x,t) = h_0(x) + (w(x,t) - h_0(x)) \geq
$$

$$
\geq \nu - |w|^{(0)}_{\mu} T \geq C(\|h_0\|^{(4+\gamma')}_{\gamma'/2,\mu}) T \geq \nu/2 > 0
$$

if $T$ is sufficiently small. Thus for such $T = T(h_0)$ we have (4.10).

Denote by $C^{4+\gamma, 4+\gamma}_{2,\gamma'/2,0}(\overline{\Omega_T})$ the closed subspace of $C^{4+\gamma, 4+\gamma}_{2,\gamma'/2,0}(\overline{\Omega_T})$ consisting of function $u(x,t)$ with the property

$$
u(x,t) = 0, \quad (x,t) \in \Gamma_T.
$$

Define the space $H = C^{4+\gamma, 4+\gamma}_{2,\gamma'/2,0}(\overline{\Omega_T}) \times C^{2+\gamma/2,1+\gamma/4}(\Gamma_T)$ and define for $r > 0$ a ball $B_r = B_r(0)$ in $H$ as

$$
B_r \equiv \{ \psi = (u, \delta) \in H : \|\psi\| = |u|^{(4+\gamma)}_{2,\gamma'/2,\mu} + \|\delta\|_{C^{2+\gamma/2,1+\gamma/4}(\Gamma_T)} \leq r \}.
$$

We suppose that $r \leq \gamma_0$, where $\gamma_0$ is from condition (3.0) and we will choose sufficiently small $r$ below. We represent unknown functions $h(x,t)$ and $\rho(\omega, t)$ in (3.8)- (3.13) as $h(x,t) = w(x,t) + u(x,t)$, $\rho(\omega, t) = \sigma(\omega, t) + \delta(\omega, t)$ with new unknown functions $u(x,t) \in C^{4+\gamma, 4+\gamma}_{2,\gamma'/2,0}(\overline{\Omega_T})$ and $\delta(\omega, t) \in C^{2+\gamma/2,1+\gamma/4}(\Gamma_T)$. Such defined functions $h$ and $\rho$ satisfy initial conditions (3.12) and condition (3.16) automatically. Analogously to the proof of (4.10) we can choose the radius $r \leq r(h_0)$ of $B_r$ so small that for any $\psi = (u, \delta) \in B_r$ we have
\[
\frac{\partial (w + u)}{\partial n} \bigg|_{\Gamma_T} \geq \nu > 0, \quad w(x, t) + u(x, t) > 0, \quad x \in \Omega, \quad t \in [0, T]. \quad (4.13)
\]

The proof is similar to the proof of (4.10). For example,
\[
\frac{\partial (w + u)}{\partial n} \bigg|_{\Gamma_T} \geq \nu - \frac{\partial u}{\partial n} \bigg|_{\Gamma_T} \geq \nu - r > \nu/2 > 0,
\]
that is the first relation in (4.13). The second relation is also proved similar to the second relation in (4.10). Thus for
\[
T \leq T(h_0), \quad r \leq r(h_0)
\]
relation (3.11) is also satisfied automatically for \((u, \delta) \in B_r\).

Write conditions (3.8), (3.10) as \((\psi = (u, \delta) \in B_r, h \equiv w + u, \rho \equiv \sigma + \delta)\)

\[
F_1(\psi) \equiv \frac{\partial h}{\partial t} - \frac{\partial h}{\partial \lambda}/(1 + \rho_\lambda)\rho_\lambda + \nabla_\rho (h^2 \nabla_\rho (\nabla_\rho^2 h)) = 0, \quad (x, t) \in \Omega_T, \quad (4.15)
\]

\[
F_2(\psi) \equiv \frac{\partial h}{\partial n}(x, t) \frac{1}{(1 + \rho_\lambda)} \left[ 1 + \sum_{i,j=1}^{N-1} m_{ij}(x, \rho) \rho_\omega_i \rho_\omega_j \right]^{1/2} - g_\rho(x, t) = 0, \quad (x, t) \in \Gamma_T,
\]

\[
(4.16)
\]

\[
\delta(x, t) = E\delta(\omega, t), \quad (x, t) \in \Omega_T, (\omega, t) \in \Gamma_T.
\]

**Lemma 20** The values \(T(h_0), r(h_0)\) in (4.14) can be chosen in a way that the mapping \(F(\psi) \equiv (F_1(\psi), F_2(\psi))\) is well defined as a mapping from \(B_r\) to \(C^{\gamma/2}(\Omega_T) \times C^{1+\gamma/2, (2+\gamma)/4}(\Gamma_T)\) and this mapping is continuous Frechet differentiable on \(B_r\).

**Proof.** Consider first the operator \(F_2(\psi)\). Since \(\sigma(x, 0) \equiv 0, \delta(x, 0) \equiv 0\) and the functions \(m_{ij}(x, \rho)\) are smooth functions of their arguments, exactly as at the reasonings for the proof of (4.13) we can choose \(T(h_0), r(h_0)\) so small that in (4.10)
\[
|\rho_\lambda| \leq 1/2, \quad \sum_{i,j=1}^{N-1} m_{ij}(x, \rho) \rho_\omega_i \rho_\omega_j \leq 1/2, \quad (u, \delta) \in B_r, \quad (4.18)
\]

where \(\rho \equiv \sigma + \delta\). Further, from Proposition 10 it follows that for \((u, \delta) \in B_r\)
\[
\frac{\partial h}{\partial n} = \frac{\partial (w + u)}{\partial n}, \rho_\lambda = \sigma_\lambda + \delta_\lambda, \rho_\omega_i = \sigma_\omega_i + \delta_\omega_i \in C^{1+\gamma/2, (2+\gamma)/4}(\Gamma_T). \quad (4.19)
\]

25
In addition, since \( g(y, \tau) \in C^2(R^N \times [0, T]) \) and the functions \( m_{ij}(x, \rho) \) are smooth functions of their arguments, it follows from the same proposition that the compositions

\[
g_\rho (x, t) = g(y, \tau) \circ e_\rho (x, t)|_{\Gamma_T}, m_{ij}(x, \rho) \in C^{1+\gamma/2, (2+\gamma)/4} (\Gamma_T).
\]  

(4.20)

From (4.18) - (4.20) it follows that the operator \( F_2(\psi) \) is well defined as an operator from \( B_r \) to \( C^{1+\gamma/2, (2+\gamma)/4} (\Gamma_T) \). Moreover, since the functions \( g(y, \tau) \) and \( m_{ij}(x, \rho) \) are smooth, under the condition (4.13) the right hand side of (4.16) is a \( C^2 \)-continuous function of its arguments \( \partial u/\partial n, \delta, \delta_x, \delta_{\omega} \) for \( (u, \delta) \in B_r \). Thus, \( F_2(\psi) \) defines a Frechet continuously differentiable mapping from \( B_r \) to \( C^{1+\gamma/2, (2+\gamma)/4} (\Gamma_T) \).

Consider now \( F_1(\psi) \). Directly from the definition of \( \tilde{C}^{1+\gamma, \frac{\gamma}{2}+\gamma}_{2, 0, 0} (\bar{\Omega}_T) \) it follows that the terms \( \partial h/\partial t \) and \( [\partial h/(1 + \rho \lambda)] \rho_{\lambda} \) in the definition of \( F_1(\psi) \) are continuously differentiable mappings from \( B_r \) to \( C^{\gamma/2, (\gamma/2)}/(\Gamma_T) \) (one should take into account also the condition \( |\rho \lambda| \leq 1/2 \) in (4.18)). Write the third term in (4.15) as \( (h = u + \rho, \rho = \sigma + \delta) \)

\[
\nabla_{\rho} (h^2 \nabla_{\rho} (\nabla_{\rho}^2 h)) = h^2 \nabla_{\rho}^2 (\nabla_{\rho}^2 h) + 2h \left( \nabla_{\rho} h, \nabla_{\rho} (\nabla_{\rho}^2 h) \right) \equiv f_1(\psi) + f_2(\psi).
\]  

(4.21)

Consider the term with the highest order \( f_1(\psi) \) since the situation with \( f_2(\psi) \) is completely similar. Let \( d(x) \) is the function in (1.15) from the definition of the space \( \tilde{C}^{1+\gamma, 2\gamma}_{2, 0, 0} (\bar{\Omega}_T) \). From (4.18) it follows that we can choose sufficiently small \( \mu > 0 \) with

\[
|\nabla (w + u)| \geq \nu > 0, \quad x \in \Omega_\mu = \{ x \in \Omega : \mu \leq \text{dist}(x, \partial \Omega) \}, t \in [0, T].
\]

Choose a small \( \mu \in (0, \gamma_0/4) \), where \( \gamma_0 \) is from (3.32), denote \( \Omega_{\mu, T} = \Omega_\mu \times [0, T], \Omega_\mu = \{ x \in \Omega : \mu \leq \text{dist}(x, \partial \Omega) \} \), and represent the expression \( h(x, t) = w(x, t) + u(x, t) \) in \( \Omega_{\mu, T} \) as (we use also \( (\omega, \lambda) \) coordinates)

\[
h(x, t) = \lambda(x) \int_0^1 \frac{\partial h}{\partial \lambda} (\omega(x), \theta \lambda(x), t) d\theta =
\]

\[
d(x) \left( \frac{\lambda(x)}{d(x)} \int_0^1 \frac{\partial h}{\partial \lambda} (\omega(x), \theta \lambda(x), t) d\theta \right) \equiv d(x) \overline{A}(u).
\]  

(4.22)

Then we have in \( \overline{\Omega}_T \)

\[
h(x, t) = d(x) A(u), \quad (x, t) \in \overline{\Omega}_T,
\]

26
where
\[
A(u) = \begin{cases} 
\tilde{A}(u), & (x, t) \in \Omega_{\mu, T}, \\
h(x, t)/d(x), & (x, t) \in \Omega_{T} \setminus \Omega_{\mu/2, T}
\end{cases}
\quad (4.23)
\]
and directly from the definition of \(A(u)\) it follows that \(A(u)\) is a bounded linear map from \(B_r\) to \(C_{\gamma/4, \gamma/2}^\gamma(\Omega_T)\). Thus
\[
h^2(x, t) = d^2(x) (A(u))^2 \quad (x, t) \in \Omega_T
\quad (4.24)
\]
and evidently that the mapping \(u \to (A(u))^2\) is continuously differentiable from \(B_r\) to \(C_{\gamma/4, \gamma/2}^\gamma(\Omega_T)\).

Consider now \(f_1(\psi)\) from (4.21). Since
\[
(\nabla \rho)_i = \sum_{j=1}^d e_{ij}(x, \rho, \nabla \rho) \frac{\partial}{\partial x_j},
\]
we have
\[
\nabla^2_\rho (\nabla^2_\rho h) =
\]
\[
= \left(\sum a_{ij}^{}(x, \rho, \nabla \rho) \frac{\partial^2}{\partial x_i \partial x_j} + \left(\sum b_{ijk}(x, \rho, \nabla \rho) \frac{\partial^2 \rho}{\partial x_i \partial x_k} \right) \frac{\partial}{\partial x_j}\right)^2 h =
\]
\[
= \sum_{|\alpha|=4} a_{\alpha}^{(1)} D_{x}^\alpha h + \sum_{|\alpha|=3,|\beta|=2} a_{\alpha,\beta}^{(2)} D_{x}^\alpha h D_{x}^\beta \rho + \sum_{|\alpha|=2,|\beta|=2,|\omega|=2} a_{\alpha,\beta,\omega}^{(3)} D_{x}^\alpha h D_{x}^\beta \rho D_{x}^\omega \rho + \sum_{|\alpha|=2,|\beta|=3} a_{\alpha,\beta}^{(4)} D_{x}^\alpha h D_{x}^\beta \rho + \sum_{|\alpha|=1,|\beta|=4} a_{\alpha,\beta}^{(5)} D_{x}^\alpha h D_{x}^\beta \rho +
\]
\[
+ \sum_{|\alpha|=1,|\beta|=2,|\omega|=3} a_{\alpha,\beta,\omega}^{(6)} D_{x}^\alpha h D_{x}^\beta \rho D_{x}^\omega \rho + \sum_{|\alpha|=1,|\beta|=2,|\omega|=2,|\chi|=2} a_{\alpha,\beta,\omega,\chi}^{(7)} D_{x}^\alpha h D_{x}^\beta \rho D_{x}^\omega \rho D_{x}^\chi \rho \equiv
\]
\[
\equiv \sum_{i=1}^7 \tilde{A}^{(i)}(\psi) = \sum_{i=1}^7 \tilde{A}^{(i)}(u, \delta), \quad (4.25)
\]
where the coefficients \(a^{(k)} = a^{(k)}(x, \rho, \nabla \rho)\) are some smooth functions of their arguments. Consequently,
\[
f_1(\psi) = h^2 \nabla^2_\rho (\nabla^2_\rho h) = (A(u))^2 \sum_{i=1}^7 d^2(x) \tilde{A}^{(i)}(\psi) \equiv (A(u))^2 \sum_{i=1}^7 A^{(i)}(\psi). \quad (4.26)
\]
Directly from the representation (4.26) and from the fact that \( D_x^2 \rho(x, t) \in C^{\gamma/4}_x(\Omega_T) \) (due to the properties of the extension operator \( E\rho(\omega, t) \)) it follows that \( A^{(1)}(\psi) \) and thus \( f_1(\psi) \) are continuously differentiable mappings from \( B_r \) to \( C^{\gamma/2}_x(\Omega_T) \). Really, for example for \( A^{(1)}(\psi) \) we have

\[
A^{(1)}(\psi) = \sum_{|\alpha|=4} a^{(1)}_\alpha(x, \sigma + \delta, \nabla \sigma + \nabla \delta) \left( d^2(x) D_x^\sigma w + d^2(x) D_x^\sigma u \right).
\]

This expression is affine with respect to \( d^2(x) D_x^\sigma u \) and is smooth with respect to \( \delta, \nabla \delta \) and thus it is smooth with respect to \( \psi = (u, \delta) \) from \( B_r \) to \( C^{\gamma/4}_x(\Omega_T) \).

Analogously for \( A^{(5)}(\psi) \)

\[
A^{(5)}(\psi) = \sum_{|\alpha|=1, |\beta|=4} a^{(5)}_{\alpha, \beta}(x, \sigma + \delta, \nabla \sigma + \nabla \delta) \left( D_x^\alpha \sigma + d^2(x) D_x^\beta \delta \right)
\]

and this mapping is also smooth with respect to \( \psi = (u, \delta) \) from \( B_r \) to \( C^{\gamma/4}_x(\Omega_T) \).

Other operators in (4.26) are considered in the same way.

Now we are going to find explicit representations for the Fréchet derivative \( F'(0) \) of the operator \( F(\psi) = (F_1(\psi), F_2(\psi)) \) at \( \psi = 0 \). For this we note first that for a \( C^1 \)-smooth function \( f(x, t) \) the Fréchet derivative of the composition \( f \circ e_\rho(x, t) = f \circ e_{\sigma + \delta}(x, t) \) with respect to \( \delta \) is the linear operator (30)

\[
[f \circ e_\rho(x, t)]'_{\delta} = \frac{d}{d\varepsilon} f \circ e_{\sigma + \varepsilon \delta}(x, t) \big|_{\varepsilon=0} = \frac{\partial f \circ e_{\sigma}(x, t)}{\partial \lambda} \delta(x, t). \tag{4.27}
\]

In fact, (4.27) follows directly from the definitions with the help of \((\omega, \lambda)\)-coordinates. We have

\[
\frac{d}{d\varepsilon} f \circ e_{\sigma + \varepsilon \delta}(x, t) \big|_{\varepsilon=0} = \frac{d}{d\varepsilon} f(\omega(x), \lambda(x) + \sigma(x, t) + \varepsilon \delta(x, t), t) \big|_{\varepsilon=0}
\]

that is (4.27).

Consider first the operator \( F_2(\psi) \) in (4.16). The Fréchet derivative \( F'_2(0) \) can be found directly from the definition of \( F_2(\psi) \) and we have \((\partial/\partial \lambda = \partial/\partial \mathbf{n}) \) on \( \Gamma_T \)

\[
F'_2(0) [(u, \delta)] = a^{(1)} \frac{\partial u}{\partial \mathbf{n}}(x, t) - a^{(2)} \frac{\partial \delta}{\partial \mathbf{n}}(x, t) + \sum_{i=1}^{N-1} a^{(3)}_i \delta_{\omega_i} + a^{(4)} \delta, \tag{4.28}
\]

where
\[ a^{(1)} = \frac{1}{1 + \sigma \lambda} \left[ 1 + \sum_{i,j=1}^{N-1} m_{ij}(x, \sigma) \sigma_{\omega_i} \sigma_{\omega_j} \right]^{\frac{1}{2}}, \quad (4.29) \]

\[ a^{(2)} = \frac{\partial w}{\partial n} \frac{1}{(1 + \sigma \lambda)^2} \left[ 1 + \sum_{i,j=1}^{N-1} m_{ij}(x, \sigma) \sigma_{\omega_i} \sigma_{\omega_j} \right]^{\frac{1}{2}}, \quad (4.30) \]

\[ a^{(3)} = \frac{1}{2} \frac{\partial w}{\partial n} \frac{1}{(1 + \sigma \lambda)} \left[ 1 + \sum_{i,j=1}^{N-1} m_{ij}(x, \sigma) \sigma_{\omega_i} \sigma_{\omega_j} \right]^{-\frac{1}{2}} \times \]

\[ \times \left( \sum_{j=1}^{N-1} (m_{ji}(x, \sigma) + m_{ij}(x, \sigma)) \sigma_{\omega_j} \right), \quad (4.31) \]

\[ a^{(4)} = \frac{1}{2} \frac{\partial w}{\partial n} \frac{1}{(1 + \sigma \lambda)} \left[ 1 + \sum_{i,j=1}^{N-1} m_{ij}(x, \sigma) \sigma_{\omega_i} \sigma_{\omega_j} \right]^{-\frac{1}{2}} \times \]

\[ \times \left( \sum_{i,j=1}^{N-1} \frac{\partial m_{ij}(x, \sigma)}{\partial \sigma} \sigma_{\omega_i} \sigma_{\omega_j} \right) + \frac{\partial g \circ e_{\sigma}(x, t)}{\partial n}. \quad (4.32) \]

Consider now the operator \( F_1(\psi) \). This is a usual quasilinear differential operator inside \( \Omega_T \) with the subsequent closer of the result (in the usual way) up to the closed domain \( \overline{\Omega}_T \). Thus it’s Frechet derivative is a linear differential operator inside \( \Omega_T \) with the subsequent closer of the result (in the usual way) up to the closed domain \( \overline{\Omega}_T \). Since all operations in the definition of \( F_1(\psi) \) are local, the coefficients of this linear differential operator at any point \((x_0, t_0)\) are completely defined by the behavior of \( w, u, \sigma, \) and \( \delta \) at any small neighbourhood around \((x_0, t_0)\). This permits us with the aim of calculating the explicit form of the derivative \( [F_1'(0)] [(u, \delta)] \) to fix arbitrary point \((x_0, t_0)\) in \( \Omega_T \) and suppose that \( w(x, t) \) and \( u(x, t) \) have compact supports in a small neighbourhood of \((x_0, t_0)\). The goal of this is to have the compositions of the form \( w \circ e_{\sigma}(x, t) \) well defined in \( \Omega_T \).

\[ \frac{\partial h}{\partial t} - \left( \frac{\partial h}{\partial \lambda} / (1 + \rho \lambda) \right) \rho_t + \nabla_{\rho} (h^2 \nabla_{\rho} \left( \nabla_{\rho}^2 h \right)) \]

Consider the expression

\[ f_1(\psi) \equiv \nabla_{\rho} (h^2 \nabla_{\rho} \left( \nabla_{\rho}^2 h \right)) \]

in the definition of \( F_1(\psi) \). It can be checked directly that

\[ f_1(0)[u, \delta] = \frac{d}{d\varepsilon} \nabla_{\sigma + \varepsilon \delta} ((w + \varepsilon u)^2 \nabla_{\sigma + \varepsilon \delta} \left( \nabla_{\sigma + \varepsilon \delta}^2 (w + \varepsilon u) \right))|_{\varepsilon=0} = \]
By simple algebraic transformations we obtain for the first and the second terms

\[ = \nabla_\sigma (2wu \nabla_\sigma (\nabla_\sigma^2 w)) + \nabla_\sigma (w^2 \nabla_\sigma (\nabla_\sigma^2 w)) + \]

\[ + \frac{d}{d\varepsilon} \nabla_{\sigma+\varepsilon\delta} (w^2 \nabla_{\sigma+\varepsilon\delta} (\nabla_\sigma^2 w))|_{\varepsilon=0}. \tag{4.34} \]

To calculate the last derivative, note that according to the definition of \( \nabla_\rho \), we have for any differential operator \( L(\nabla) \) with constant coefficients and for any function \( v \)

\[ [L(\nabla)v] \circ e_\rho(x,t) = L(\nabla_\rho)(v \circ e_\rho(x,t)). \tag{4.35} \]

Note also that according to the definition of the transformation \( e_\rho(x,t) \) (which is in fact the \( \rho \)-shift along the \( \lambda \)-coordinate)

\[ e_{\rho_1+\rho_2}(x,t) = e_{\rho_1}(x,t) \circ e_{\rho_2}(x,t), \quad e_{\rho}^{-1} = e_{-\rho}(x,t). \tag{4.36} \]

Thus the last term in \( 4.34 \) can be represented as

\[ \nabla_{\sigma+\varepsilon\delta} (w^2 \nabla_{\sigma+\varepsilon\delta} (\nabla_\sigma^2 w)) = \left( \nabla ((w \circ e_{-\sigma-\varepsilon\delta})^2 \nabla \left( \nabla^2 w \circ e_{-\sigma-\varepsilon\delta} \right)) \right) \circ e_{\sigma+\varepsilon\delta} = \]

\[ = \left( \nabla_\sigma ((w \circ e_{-\varepsilon\delta})^2 \nabla_\sigma (\nabla_\sigma^2 w \circ e_{-\varepsilon\delta})) \right) \circ e_{\varepsilon\delta}. \]

Therefore

\[ \frac{d}{d\varepsilon} \nabla_{\sigma+\varepsilon\delta} (w^2 \nabla_{\sigma+\varepsilon\delta} (\nabla_\sigma^2 w))|_{\varepsilon=0} = \]

\[ = -\nabla_\sigma (2w \frac{\partial w}{\partial \lambda} \delta \nabla_\sigma \nabla_\sigma^2 w) - \nabla_\sigma (w^2 \nabla_\sigma \nabla_\sigma^2 \frac{\partial w}{\partial \lambda} \delta) + \left( \frac{\partial}{\partial \lambda} \nabla_\sigma (w^2 \nabla_\sigma \nabla_\sigma^2 w) \right) \delta. \tag{4.37} \]

By simple algebraic transformations we obtain for the first and the second terms

\[ - \nabla_\sigma (2w \frac{\partial w}{\partial \lambda} \delta \nabla_\sigma \nabla_\sigma^2 w) = -\nabla_\sigma (2w \frac{\partial w}{\partial \lambda} \nabla_\sigma \nabla_\sigma^2 w) \cdot \delta + \sum |\alpha|=1 a_\alpha^{(0)} D_\alpha \delta, \tag{4.38} \]

\[ - \nabla_\sigma (w^2 \nabla_\sigma \nabla_\sigma^2 \frac{\partial w}{\partial \lambda} \delta) = \nabla_\sigma (2w \frac{\partial w}{\partial \lambda} \nabla_\sigma \nabla_\sigma^2 w) \cdot \delta - \left( \frac{\partial}{\partial \lambda} \nabla_\sigma (w^2 \nabla_\sigma \nabla_\sigma^2 w) \right) \delta - \]

\[ - \frac{\partial w}{\partial \lambda} \nabla_\sigma (w^2 \nabla_\sigma \nabla_\sigma^2 \delta) + R_1[\delta], \tag{4.39} \]

where
\[
R_1[\delta] \equiv \sum_{|\alpha|+|\beta|=4, |\alpha|<4, |\beta|<4} a^{(1)}_{\alpha,\beta} w^2 D_x^{\alpha} \left( \frac{\partial w}{\partial \lambda} \right) D^\beta \delta + \sum_{|\alpha|+|\beta|=3, |\alpha|<3, |\beta|<3} a^{(2)}_{\alpha,\beta} w D_x^{\alpha} \left( \frac{\partial w}{\partial \lambda} \right) D^\beta \delta + \]
\[
+ \sum_{|\alpha|+|\beta|=2, |\alpha|<2, |\beta|<2} a^{(3)}_{\alpha,\beta} w D_x^{\alpha} \left( \frac{\partial w}{\partial \lambda} \right) D^\beta \delta + a^{(4)} \delta
\]  

(4.40)

and

\[
\sum_{\alpha}|a^{(0)}_{\gamma/2,\Pi_r}| + \sum_{i=1}^3 \sum_{\alpha,\beta}|a^{(i)}_{\gamma/2,\Pi_r}| + |a^{(4)}_{\gamma/2,\Pi_r}| \leq C(h_0).
\]  

(4.41)

From (4.34), (4.37)-(4.39) it follows that

\[
f_1'(0)[u, \delta] = \nabla_\sigma (w^2 \nabla_\sigma (\nabla_\sigma^2 u)) - \frac{\partial w}{\partial \lambda} \nabla_\sigma (w^2 \nabla_\sigma \nabla_\sigma^2 \delta) + \nabla_\sigma (2wu \nabla_\sigma (\nabla_\sigma^2 u)) + R_1[\delta].
\]  

(4.42)

The Frechet derivative of the rest of the operator \( F_1(\psi) \)

\[
f_2(\psi) \equiv \frac{\partial h}{\partial t} - \left[ \frac{\partial h}{\partial \lambda}/(1 + \rho_\lambda) \right] \rho t
\]  

(4.43)

can be calculated directly and we have

\[
f_2'(0)[u, \delta] = \frac{\partial u}{\partial t} - \left( (1 + \sigma_\lambda)^{-1} \frac{\partial w}{\partial \lambda} \right) \frac{\partial \delta}{\partial t} + R_2[u, \delta].
\]  

(4.44)

From (4.42) and (4.44) it follows that

\[
F_1'(0)[u, \delta] = \left[ \frac{\partial u}{\partial t} + \nabla_\sigma (w^2 \nabla_\sigma (\nabla_\sigma^2 u)) \right] -
\]
\[
- \left[ \left( (1 + \sigma_\lambda)^{-1} \frac{\partial w}{\partial \lambda} \right) \frac{\partial \delta}{\partial t} + \frac{\partial w}{\partial \lambda} \nabla_\sigma (w^2 \nabla_\sigma \nabla_\sigma^2 \delta) \right] + R[u, \delta],
\]  

(4.45)

where

\[
R[u, \delta] \equiv R_2[u, \delta] + \sum_{|\alpha|=1} a^{(0)}_{\alpha} D_x^{\alpha} \delta + \nabla_\sigma (2wu \nabla_\sigma (\nabla_\sigma^2 u)) + R_1[\delta].
\]  

(4.46)
Note that the linear operator \( R[u, \delta] \) contains only lower order terms and we will show below that its norm can be made arbitrary small by choosing sufficiently small \( T > 0 \). Thus we obtain the following proposition.

**Proposition 21** The operator \( F(\psi) = (F_1(\psi), F_2(\psi)) \) is Fréchet-smooth on \( B_r \rightarrow C^{\gamma/4}(\overline{\Omega}_T) \times C^{1+\gamma/2}(\Gamma_T) \) and its Fréchet-derivative \( F'(0)[u, \delta] = (F_1'(0)[u, \delta], F_2'(0)[u, \delta]) \) is defined by (4.28), (4.45).

Now we use the special way of construction of the functions \( w(x, t) \) and \( \sigma(\omega, t) \) and show that the value \( \|F(0)\| \) is sufficiently small for small values of the time interval \( T \). This means that \( \psi = 0 \) is an approximate solution of the equation \( F(\psi) = 0 \) for small \( T > 0 \). Consider \( F_2(0) \),

\[
F_2(0)(x, t) = \frac{\partial w}{\partial \mu}(x, t) \frac{1}{(1 + \sigma \lambda)} \left[ 1 + \sum_{i,j=1}^{N-1} m_{ij}(x, \sigma) \sigma_{ij} \right]^{1/2} - g_{\sigma}(x, t).
\]

Since \( w \in C^{2+\gamma'/(2+\gamma'),1+\gamma'/4}(\Gamma_T) \) and functions \( m_{ij}(x, \sigma) \) are smooth, it follows from Proposition 10 about traces that \( F_2(0)(x, t) \in C^{1+\gamma'/2}(\Gamma_T) \). Moreover, since \( \sigma(\omega, 0) \equiv 0 \) and \( w(x, 0) = h_0(x) \),

\[
F_2(0)(x, 0) = \frac{\partial h_0}{\partial \mu}(x) - g(x, 0) \equiv 0, \quad x \in \Gamma
\]

as it follows from compatibility condition (4.24). Thus \( F_2(0)(x, t) \in C^{1+\gamma'/2}(\Gamma_T) \) and since \( \gamma' > \gamma \), it follows by (2.9) that

\[
\|F_2(0)\|_{C^{1+\gamma'/2}(\Gamma_T)} \leq CT^\mu \|F_2(0)\|_{C^{1+\gamma'/2}(\Gamma_T)} \leq C(h_0, g)T^\mu.
\]  

(4.47)

The considerations for \( F_1(0) \) are completely analogous. We have

\[
F_1(0)(x, t) = \frac{\partial w}{\partial t} - \frac{\partial w}{\partial \lambda}/(1 + \sigma \lambda) \sigma_t + \nabla_\sigma (w^2 \nabla_\sigma (\nabla_\sigma^2 w)) \in C^{\gamma'/2}(\overline{\Omega}_T)
\]

and from the way of construction of \( w \) and \( \sigma \) it follows that \( F_1(0)(x, 0) \equiv 0 \) that is \( F_1(0)(x, t) \in C^{\gamma'/2}(\overline{\Omega}_T) \). Thus we have on the base of (2.9)

\[
\|F_1(0)\|_{C^{\gamma'/2}(\overline{\Omega}_T)} \leq CT^\mu \|F_1(0)\|_{C^{\gamma'/2}(\overline{\Omega}_T)} \leq C(h_0)T^\mu.
\]  

(4.48)

From (4.47) and (4.48) it follows that

\[
\|F(0)\|_{C^{\gamma'/2}(\overline{\Omega}_T) \times C^{1+\gamma'/2}(\Gamma_T)} \leq C(h_0, g)T^\mu.
\]  

(4.49)
5 The Schauder estimates for model problems in the half-space.

Denote $H = R^N_+ = \{x = (x', x_N) \in R^N : x_N \geq 0\}$, $Q^+ = \{(x, t) : x \in H, t \geq 0\}$, $Q = \{(x, t) : x \in H, -\infty < t < \infty\}$, $G^+ = \{(x', t) : x' \in R^{N-1}, t \geq 0\}$. For a function $u(x, t) \in C^{4+\gamma/2}_{2,\gamma/2}(Q^+)$ we define

$$
\langle \langle x_N^2 D^4_{x_i} u \rangle \rangle^{(\gamma)(10)}(\epsilon+ \Rightarrow (x, t) \in Q^+, h \geq \epsilon \|x\|_N) \equiv \sup_{(x, t) \in Q^+, h} x_N^{2\gamma/2} \frac{\Delta_{h,x_i}^{10} (x_N^2 D^4_{x_i} u(x, t))}{h^{\gamma}}, i = 1, N, \tag{5.1}
$$

$$
\langle \langle x_N^2 D^4_{x_i} u \rangle \rangle^{(\gamma)(10)}(\epsilon+ \Rightarrow (x, t) \in Q^+, h > 0) \equiv \sup_{(x, t) \in Q^+, h} x_N^{2\gamma/2} \frac{\Delta_{h,x_i}^{10} (x_N^2 D^4_{x_i} u(x, t))}{h^{\gamma}}, i = 1, N, \tag{5.2}
$$

$$
\langle \langle u \rangle \rangle^{(4+\gamma)(10)}(\epsilon+ \Rightarrow (x, t) \in Q^+, h \geq \epsilon \|x\|_N) \equiv \sum_{i=1}^{N-1} \langle \langle x_N^2 D^4_{x_i} u \rangle \rangle^{(\gamma)(10)}(\epsilon+ \Rightarrow (x, t) \in Q^+, h) + \langle \langle D_t u \rangle \rangle^{(\gamma/4)(10)}(\epsilon+ \Rightarrow (x, t) \in Q^+, h), \tag{5.3}
$$

$$
\langle \langle u \rangle \rangle^{(4+\gamma)(10)}(\epsilon+ \Rightarrow (x, t) \in Q^+, h \geq \epsilon \|x\|_N) \equiv \langle \langle u \rangle \rangle^{(4+\gamma)(10)}(\epsilon+ \Rightarrow (x, t) \in Q^+, h), \tag{5.4}
$$

and analogously with respect to all variables

$$
\langle \langle u \rangle \rangle^{(4+\gamma)(10)}(\epsilon+ \Rightarrow (x, t) \in Q^+, h \geq \epsilon \|x\|_N) \equiv \sum_{i=1}^{N} \langle \langle x_N^2 D^4_{x_i} u \rangle \rangle^{(\gamma)(10)}(\epsilon+ \Rightarrow (x, t) \in Q^+, h) + \langle \langle D_t u \rangle \rangle^{(\gamma/4)(10)}(\epsilon+ \Rightarrow (x, t) \in Q^+, h),
$$

where for a function $v(x, t)$ we denote $\Delta_{h,x} v(x, t) = v(x_1, ... x_i + h, ... x_N, t) - v(x, t)$, $\Delta_{h,t} v(x, t) = v(x, t + h) - v(x, t)$, $\Delta_{h,x}^n v(x, t) = \Delta_{h,x} \left( \Delta_{h,x}^{n-1} v(x, t) \right)$.

It is important that it was proved in [22] that seminorms $\langle \langle u \rangle \rangle^{(4+\gamma)(10)}$ and $\langle \langle u \rangle \rangle^{(4+\gamma)}$ are equivalent

$$
\langle \langle u \rangle \rangle^{(4+\gamma)(10)}_{2,\gamma/2, Q^+} \simeq \langle \langle u \rangle \rangle^{(4+\gamma)}_{2,\gamma/2, Q^+}. \tag{5.5}
$$

Lemma 22 Let functions $f(x, t)$, $g(x', t)$, $\varphi(x', t)$, and $\psi(x)$ have compact supports and

$$
f(x, t) \in C^{7+\gamma/4}_{2,\gamma/2}(Q^+), g(x', t) \in C^{1+\gamma/2,1/2+\gamma/4}_{2,\gamma/2}(G^+),
$$

$$
\varphi(x', t) \in C^{2+\gamma/2,1+\gamma/4}_{2,\gamma/2}(G^+), \psi(x) \in C^{4+\gamma/2}_{2,\gamma/2}(R^N). \tag{5.6}
$$
Let a function \( u(x,t) \in C^{4+,\gamma}_{2,\gamma/2}(Q^+) \) with a compact support satisfy the following initial boundary value problem in \( Q^+ \):

\[
L_{x,t}u \equiv \frac{\partial u}{\partial t} + \nabla(x_N \nabla \Delta u) = f(x,t), \quad (x,t) \in Q^+,
\]

\[
\frac{\partial u}{\partial x_N}(x',0,t) = g(x',t), \quad (x',t) \in G^+,
\]

\[
u(x,0) = \psi(x), \quad x \in \mathbb{R}^N
\]

and

\[
\langle \langle u \rangle \rangle_{2,\gamma/2,x',t,Q^+}^{(4+\gamma)(10)} \geq \frac{1}{2} \langle \langle u \rangle \rangle_{2,\gamma/2,x',t,Q^+}^{(4+\gamma)(10)} + \mu \langle x_N^2 D_{x_N}^4 u \rangle_{2,\gamma/2,x',t,Q^+}^{(\gamma)}.
\]

Then for any \( \varepsilon, \mu > 0 \) there exists a constant \( C_{\varepsilon,\mu} > 0 \) with the property

\[
\langle \langle u \rangle \rangle_{2,\gamma/2,x',t,Q^+}^{(4+\gamma)(10)} \leq C_{\varepsilon,\mu} \left( \langle f \rangle_{1/2,2,Q^+}^{(\gamma)} + \langle g \rangle_{2,\gamma/2,1/2,1/2+\gamma/4}^{(2+\gamma/2,1/2+\gamma/4)} + \langle \psi \rangle_{2,\gamma/2,\mathbb{R}^N}^{(4+\gamma)} + \mu \langle x_N^2 D_{x_N}^4 u \rangle_{2,\gamma/2,x',t,Q^+}^{(\gamma)} \right)
\]

If instead of (5.8) the function \( u(x,t) \) satisfies

\[
u(x',0,t) = \varphi(x',t), \quad (x',t) \in G^+,
\]

then

\[
\langle \langle u \rangle \rangle_{2,\gamma/2,x',t,Q^+}^{(4+\gamma)(10)} \leq C_{\varepsilon,\mu} \left( \langle f \rangle_{1/2,2,Q^+}^{(\gamma)} + \langle \varphi \rangle_{2,\gamma/2,1/2,1/2+\gamma/4}^{(2+\gamma/2,1/2+\gamma/4)} + \langle \psi \rangle_{2,\gamma/2,\mathbb{R}^N}^{(4+\gamma)} + \mu \langle x_N^2 D_{x_N}^4 u \rangle_{2,\gamma/2,x',t,Q^+}^{(\gamma)} \right)
\]

Proof.

We prove only (5.11) since the proof of (5.13) is absolutely identical to that of (5.11). The idea of the proof is taken from [31] and is adopted to the weighted spaces for the degenerate equation with variable coefficients as it was done in [22].

The proof is by contradiction. Suppose that (5.11) is not valid. Then there exist \( \mu > 0 \) and a sequence \( \{ u_p(x,t) \} \subset C^{4+,\gamma}_{2,\gamma/2}(Q^+), \quad p = 1, 2, ..., \), with the property (5.10) and with

\[
\langle \langle u_p \rangle \rangle_{2,\gamma/2,2,Q^+}^{(4+\gamma)(10)} \geq p \left( \langle f_p \rangle_{1/2,2,Q^+}^{(\gamma)} + \langle g_p \rangle_{2,\gamma/2,1/2,1/2+\gamma/4}^{(2+\gamma/2,1/2+\gamma/4)} + \langle \psi_p \rangle_{2,\gamma/2,\mathbb{R}^N}^{(4+\gamma)} + \mu \langle x_N^2 D_{x_N}^4 u_p \rangle_{2,\gamma/2,x',t,Q^+}^{(\gamma)} \right)
\]

\[
+ \mu \langle x_N^2 D_{x_N}^4 u_p \rangle_{2,\gamma/2,x',t,Q^+}^{(\gamma)}.
\]

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where \( f_p, g_p, \) and \( \psi_p \) correspond to \( u_p \) in relations \([5,7]-[5,9]\). From \([5,10]\) it follows also that

\[
\langle\langle u_p \rangle\rangle_{2,\gamma/2,x',t,\mathcal{O}}^{(4+\gamma)(10)(\varepsilon+)} \leq \langle\langle u_p \rangle\rangle_{2,\gamma/2,x',t,\mathcal{O}}^{(4+\gamma)(10)} \leq 2 \langle\langle u_p \rangle\rangle_{2,\gamma/2,x',t,\mathcal{O}}^{(4+\gamma)(10)(\varepsilon+)}.
\]

Denote \( v_p(x,t) \equiv u_p(x,t)/\langle\langle u_p \rangle\rangle_{2,\gamma/2,x',t,\mathcal{O}}^{(4+\gamma)(10)(\varepsilon+)} \). Functions \( v_p(x,t) \) satisfy \([5,7]-[5,9]\) with the right hand sides

\[
f_p^{(1)} = f_p/\langle\langle u_p \rangle\rangle_{2,\gamma/2,x',t,\mathcal{O}}^{(4+\gamma)(10)(\varepsilon+)}, \quad g_p^{(1)} = g_p/\langle\langle u_p \rangle\rangle_{2,\gamma/2,x',t,\mathcal{O}}^{(4+\gamma)(10)(\varepsilon+)}, \quad \psi_p^{(1)} = \psi_p/\langle\langle u_p \rangle\rangle_{2,\gamma/2,x',t,\mathcal{O}}^{(4+\gamma)(10)(\varepsilon+)}.\]

For the functions \( \{v_p\} \) we have from \([5,14]\)

\[
1 = \langle\langle v_p \rangle\rangle_{2,\gamma/2,x',t,\mathcal{O}}^{(4+\gamma)(10)(\varepsilon+)} \geq p \left( \langle f_p^{(1)} \rangle_{\gamma/2,Q^+}^{(\gamma)} + \langle g_p^{(1)} \rangle_{G^+}^{(1+\gamma/2,1/2+\gamma/4)} + \langle \psi_p^{(1)} \rangle_{2,\gamma/2,R^N}^{(4+\gamma)} \right) + \mu \langle x_N^2 D_{x,N}^4 v_p \rangle_{\gamma/2,x,N,Q^+}^{(\gamma)}.
\]

And from the last inequality and from \([5,15]\) we infer that

\[
\langle f_p^{(1)} \rangle_{\gamma/2,Q^+}^{(\gamma)} + \langle g_p^{(1)} \rangle_{G^+}^{(1+\gamma/2,1/2+\gamma/4)} + \langle \psi_p^{(1)} \rangle_{2,\gamma/2,R^N}^{(4+\gamma)} \leq \frac{1}{\mu}.
\]

\[
1 \leq \langle\langle v_p \rangle\rangle_{2,\gamma/2,x',t,\mathcal{O}}^{(4+\gamma)(10)} \leq 2 \langle\langle v_p \rangle\rangle_{2,\gamma/2,x',t,\mathcal{O}}^{(4+\gamma)(10)(\varepsilon+)} \leq 2, \quad \langle x_N^2 D_{x,N}^4 v_p \rangle_{\gamma/2,x,N,Q^+}^{(\gamma)} \leq \frac{1}{\mu}.
\]

The last two inequalities together with \([5,5]\) imply that

\[
1 \leq \langle\langle v_p \rangle\rangle_{2,\gamma/2,Q^+}^{(4+\gamma)} \leq C(\mu).
\]

Since \( 1 = \langle\langle v_p \rangle\rangle_{2,\gamma/2,x',t,\mathcal{O}}^{(4+\gamma)(10)(\varepsilon+)} \), there is a term in the definition of \( \langle\langle v_p \rangle\rangle_{2,\gamma/2,x',t,\mathcal{O}}^{(4+\gamma)(10)(\varepsilon+)} \) which is not less than some absolute constant \( \nu = \nu(N) > 0 \). This is valid at least for a subsequence of indexes \( \{p\} \). We suppose, for example, that for some multindex \( \hat{\alpha}, |\hat{\alpha}| = 4, \alpha_N = 0, \)

\[
\langle x_N^2 D_{x}^5 v_p \rangle_{\gamma/2,x',t,\mathcal{O}}^{(\gamma)(10)(\varepsilon+)} \geq \nu > 0, \quad p = 1, 2, \ldots.
\]

The all reasonings below are completely the same for all other terms in the definition of \( \langle\langle v_p \rangle\rangle_{(m+\gamma)(2s)(\varepsilon+)}^{(m+\gamma)(2s)(\varepsilon+)} \). From \([5,16]\) and from the definition of \( \langle x_N^2 D_{x}^5 v_p \rangle_{\gamma/2,x',t,\mathcal{O}}^{(\gamma)(10)(\varepsilon+)} \) it follows that there exist sequences of points \( \{\{x^{(p)},t^{(p)}\} \in \mathcal{O}\} \) and vectors \( \{\hat{h}^{(p)} \in \mathcal{H}\} \) with
\[ h_p \equiv |\tilde{h}^{(p)}| \geq \varepsilon x_N^{(p)}, \quad p = 1, 2, \ldots \] (5.19)

and with

\[
\left( x_N^{(p)} \right)^{\gamma/2} \frac{\Delta^{10}_{\tilde{R}^N}}{\tilde{h}_p^\gamma} \left[ \left( x_N^{(p)} \right)^2 \tilde{D}_x v_p(x^{(p)}, t^{(p)}) \right] \geq \frac{\nu}{2} > 0. \] (5.20)

We make in the functions \( \{v_p\} \) the change of the independent variables \((x, t) \rightarrow (y, \tau)\)

\[ x_i = x_i^{(p)} + y_i h_p, \quad i = 1, N - 1, \quad x_N = y_N h_p; \quad t = t^{(p)} + h_p^2 \tau \] (5.21)

and denote

\[ w_p(y, \tau) = h_p^{-(2+\gamma/2)} v_p(x^{(p)} + y'h_p, y_N h_p, t^{(p)} + \tau h_p^2), \] (5.22)

\[ f_p^{(2)}(y, \tau) = h_p^{-\gamma/2} f_p^{(1)}(x^{(p)} + y'h_p, y_N h_p, t^{(p)} + \tau h_p^2), \] (5.23)

\[ g_p^{(2)}(y, \tau) = h_p^{-(1+\gamma/2)} g_p^{(1)}(x^{(p)} + y'h_p, y_N h_p, t^{(p)} + \tau h_p^2), \] (5.24)

\[ \psi_p^{(2)}(y) = h_p^{-(2+\gamma/2)} \psi_p^{(1)}(x^{(p)} + y'h_p, y_N h_p). \] (5.25)

It can be checked directly that the rescaled functions \( w^{(p)}(y, \tau) \) satisfy relations (5.7) - (5.9) with the right hand sides \( f_p^{(2)}(y, \tau), g_p^{(2)}(y, \tau), \psi_p^{(2)}(y) \) in the domain

\[ Q^{(p)} = \{ (y, \tau) : y_N \geq 0, \tau \geq \tau^{(p)} \equiv -t^{(p)}/h_p^2 \}, \] (5.26)

\[ L_{y,\tau} w_p(y, \tau) = \frac{\partial w_p}{\partial \tau} + \nabla_y (y_N^2 \nabla_y \Delta_y w_p) = f_p^{(2)}(y, \tau), \quad (y, \tau) \in Q^{(p)}, \] (5.27)

\[ \frac{\partial w_p}{\partial y_N}(y', 0, \tau) = g_p^{(2)}(y', \tau), \quad (y', \tau) \in G^{(p)} = \overline{Q^{(p)}} \cap \{ y_N = 0 \}, \] (5.28)

\[ w_p(y, \tau^{(p)}) = \psi_p^{(2)}(y), \quad y \in R^N. \] (5.29)

And also it can be checked directly from the definitions that

\[
\langle w_p \rangle_{2, \gamma/2, \overline{Q}^{(p)}}^{(4+\gamma)} = \langle v_p \rangle_{2, \gamma/2, \overline{Q}^{(p)}}^{(4+\gamma)}, \quad \langle f_p^{(2)} \rangle_{\gamma/2, \overline{Q}^{(p)}}^{(\gamma)} = \langle f_p^{(1)} \rangle_{\gamma/2, \overline{Q}^{(p)}}^{(\gamma)}, \] (5.30)

\[
\langle g_p^{(2)} \rangle_{\overline{Q}^{(p)}}^{(1+\gamma/2, 1/2+\gamma/4)} = \langle g_p^{(1)} \rangle_{\overline{Q}^{(p)}}^{(1+\gamma/2, 1/2+\gamma/4)}, \quad \langle \psi_p^{(2)} \rangle_{2, \gamma/2, R^N}^{(4+\gamma)} = \langle \psi_p^{(2)} \rangle_{2, \gamma/2, R^N}^{(4+\gamma)}.
\] (5.31)
Thus from (5.17) it follows that
\[ 1 \leq (w_p^{(4+\gamma)})_{2,\gamma/2,\mathcal{Q}}^p \leq C(\mu). \] (5.28)
and from (5.16) we have
\[ \left\langle f_p^{(2)} \right\rangle_{\gamma/2,Q(p)}^{(2)} + \left\langle g_p^{(2)} \right\rangle_{G(p)}^{(1+\gamma/2,1/2+\gamma/4)} + \left\langle \psi_p^{(2)} \right\rangle_{2,\gamma/2,\mathcal{R}_N}^{(4+\gamma)} \leq \frac{1}{\rho}. \] (5.29)
Besides, from (5.20) we obtain
\[ \left( y_N^{(p)} \right)^{\omega_0} \left| \Delta_{x^{(p)}} D_{y}^{\alpha} y_N^{(p)} n D_{y}^{\alpha} w_p^p(P^{(p)}, 0) \right| \geq \nu / 2, \] (5.30)
where
\[ y_N^{(p)} \equiv x_N^{(p)} / h_p, \quad e^{(p)} \equiv h^{(p)} / h_p, \quad |e^{(p)}| = 1, \quad P^{(p)} \equiv (y_N^{(p)}). \] (5.31)
Denote by \( Q_p(y, \tau) \equiv Q_{w_p}(y, \tau) \) the "Taylor" function \( Q_{w_p}(y, \tau) \) for the function \( w_p(y, \tau) \), which was constructed in Lemma 16.
\[ Q_{w_p}(y, \tau) = Q_p(y, \tau) = -a w_p \ln^{(2)} y_N + \sum_{|\alpha| \leq 2} \frac{a_\alpha}{\alpha!} (y - \tau)^\alpha + a^{(1)} \tau \]
Denote \( r_p(y, \tau) \equiv w_p(y, \tau) - Q_p(y, \tau) \). From Lemma 16 it follows that
\[ y_N^{2-j} D_{y}^{\alpha} r_p(y, \tau)|_{(y,\tau)=(0,0)} = 0, \quad j < 2, \quad |\alpha| = 4 - j, \] (5.32)
\[ D_{y}^{\alpha} r_p(y, \tau)|_{(y,\tau)=(\tau,0)} = 0, \quad |\alpha| \leq 2, \quad D_{\tau} r_p(y, \tau)|_{(y,\tau)=(\tau,0)} = 0. \] (5.33)
Recall that
\[ y_N^{2-j} D_{y}^{\alpha} Q_p(y, \tau) \equiv const, \quad |\alpha| = 4 - j, \quad j = 0, 1, \quad D_{\tau} Q_p(y, \tau) \equiv const. \] (5.34)
Consequently, from (5.34) and from the definition of Hölder classes in view of (5.28) it follows that
\[ \langle r_p \rangle_{2,\gamma/2,\mathcal{Q}}^{(4+\gamma, \frac{4+\gamma}{2})} = (w_p - Q_p(y, \tau))_{2,\gamma/2,\mathcal{Q}}^{(4+\gamma, \frac{4+\gamma}{2})} \leq C. \] (5.35)
Besides, from (5.24)- (5.26) it follows that the functions \( r_p(y, \tau) \) satisfy relations (5.24)- (5.26) with the functions
\[ f_p^{(3)}(y, \tau) = f_p^{(2)}(y, \tau) - C_0, \quad g_p^{(3)}(y, \tau) = g_p^{(2)}(y, \tau) - \sum_{i=1}^{N-1} C_i y_i - C_N, \] (5.36)
\[ \psi^{(3)}(y) = \psi^{(2)}(y) - Q_p(y, 0). \]

Thus analogously to (5.35) we have

\[
\left\langle f_p^{(3)}(\gamma) \right\rangle_{\gamma/2, \Omega(p)} = \left\langle f_p^{(2)}(\gamma) \right\rangle_{\gamma/2, \Omega(p)} + \left\langle g_p^{(3)}(\gamma/2,1/2+\gamma/4) \right\rangle_{\sigma(p)} = \left\langle g_p^{(2)}(\gamma/2,1/2+\gamma/4) \right\rangle_{\sigma(p)},
\]

and therefore

\[
\left\langle \psi_p^{(3)}(\gamma) \right\rangle_{2, \gamma/2, R_N} = \left\langle \psi_p^{(2)}(\gamma) \right\rangle_{2, \gamma/2, R_N}
\]

where

\[
\psi_p^{(3)}(\gamma) = \left\langle f_p^{(3)}(\gamma) \right\rangle_{\gamma/2, \Omega(p)} + \left\langle g_p^{(3)}(\gamma/2,1/2+\gamma/4) \right\rangle_{\sigma(p)} + \left\langle \psi_p^{(3)}(\gamma) \right\rangle_{2, \gamma/2, R_N} \leq \frac{1}{p}. \tag{5.37}
\]

From (5.30) we have also

\[
\left( y^{(p)} \right)^{\gamma/2} \left| \Delta_{\gamma/2}^{2p} \right| \left( y^{(p)} \right)^2 D_y^2 r_p(P^{(p)}, 0) \right| \geq \nu. \tag{5.38}
\]

Further, from (5.35) and from the properties of \( r_p(y, \tau) \) in (5.32), (5.38) it follows that for any compact set \( K \subset \Omega(p) \)

\[
\| r_p(y, \tau) \|_{C^{4+\gamma, 1/2}_{\gamma/2}(K)} \leq C(K), \quad \left\| g_p^{(3)}(y', \tau) = \frac{r_p(y', 0, \tau)}{\partial x_N} \right\|_{C^{1+\gamma, 1/2+\gamma/4}_{\gamma/2}(K')} \leq C(K'), \tag{5.39}
\]

where \( K' = K \cap \{ x_N = 0 \} \). Moreover, from the properties of \( r_p(y, \tau) \) (5.32), (5.33), (5.35) it follows that for any compact set \( K_R \) of the form \( K_R = \{(y, \tau) \in \Omega(p) : |y| \leq R, \tau \leq R \} \)

\[
\| r_p(y, \tau) \|^{(0)}_{K_R} \leq CR^{4+\gamma}. \tag{5.40}
\]

Besides, in view of (5.36), (5.37), (5.39), and of the properties of \( \psi^{(3)}(y) \),

\[
\psi^{(3)}(y) = w_p(y, 0) - Q_p(y, 0),
\]

for any \( K \subset Q(p), K' \subset G(p), K'' \subset R_N \cap Q(p) \)

\[
\left\| f_p^{(3)}(y, \tau) \right\|_{C^{4+\gamma, 1/2}_{\gamma/2}(K)} + \left\| g_p^{(3)}(y', \tau) \right\|_{C^{1+\gamma, 1/2+\gamma/4}_{\gamma/2}(K')} + \left\| \psi_p^{(3)}(y) \right\|_{C^{4+\gamma, 1/2}_{\gamma/2}(K'')} \leq C(K, K', K''), \tag{5.41}
\]

where \( K'' \subset R = K \cap \{ t = 0 \} \).

We consider two cases of the behaviour of \( \tau^{(p)} = -t^{(p)}/\delta^2 \). It can go to a finite limit or to infinity as \( p \to \infty \). Let first \( \tau^{(p)} \to -\infty \) as \( p \to \infty \). Thus
...and also for any compact set \( K \subset \overline{Q}^+ \)

\[
r_p \to r \text{ in } C^{2+\gamma_1/2,1+\gamma_1/4}(K), \quad p \to \infty, \quad \forall K \subset Q, \quad \gamma_1 < \gamma.
\]

Besides, for any compact sets \( K \subset \overline{Q}, \ K' \subset G, \ K'' \subset \overline{R}^N \) (at least for a subsequence) for \( p \to \infty \)

\[
f_p^{(3)} \to C^{-\gamma_1/2}_N (y) \quad 0, \quad g_p^{(3)} \to C^{1+\gamma_1/2,1+\gamma_1/4}(K') \quad 0, \quad \psi_p^{(3)}(y) \to C_N^{1+\gamma_1/2}(K'') \quad 0.
\]

At the same time, since the sequences \( \{y_N^{(p)}\}, \ \{\tau^{(p)}\}, \ \text{and } \{P^{(p)}\} \) are bounded (recall that \( y_N^{(p)} = x_N^{(p)}/h_p \leq \varepsilon^{-1} \))

\[
y_N^{(p)} \to y_N^{(0)}, \quad \tau^{(p)} \to \tau^{(0)}, \quad P^{(p)} \to P^{(0)}, \quad p \to \infty,
\]

where \( y_N^{(0)} \) is a nonnegative number, \( \tau^{(0)} \in \overline{H} \) is a unit vector, \( P^{(0)} = (0', \ y_N^{(0)}) \in \overline{H} \). From (5.32) and (5.33) (together with (55) and the Arzela theorem) it follows that the functions \( y_N^2 D_{y}^{\delta} r_p(y, \tau) \) are uniformly convergent (for a subsequence) on any compact set \( K_R \subset \overline{Q} \cap \{0 \leq y_N \leq R\}, \ R > 0 \)

\[
y_N^2 D_{y}^{\delta} r_p(y, \tau) \to y_N^2 D_{y}^{\delta} r(y, \tau), \quad p \to \infty.
\]

Thus we can choose a compact set \( K_R \) and take the limit of relation (5.38) on this set. This gives

\[
|\Delta y_N^{(0)}(y_N^{(0)})^2 D_{y}^{\delta} r(P^{(0)}, 0)| \geq \nu > 0.
\]

Moreover, from Lemma 5 and (5.35) it follows that uniformly in \( p \)

\[
\left\langle y_N^2 D_{y}^{\delta} r_p \right\rangle_{y, \tau}^{\gamma/2} + \left\langle y_N^2 D_{y}^{\delta} r_p \right\rangle_{y, \tau}^{\gamma/4} \leq C.
\]

Together with (5.32), this means that the sequence \( \{y_N^2 D_{y}^{\delta} r_p\} \) is bounded in the space \( C^{\gamma/2, \gamma}(K_R) \) for any compact set \( K_R \). Therefore for any \( \gamma_1 < \gamma \) the sequence \( \{y_N^2 D_{y}^{\delta} r_p\} \) converges to \( y_N^2 D_{y}^{\delta} r \) in the space \( C^{\gamma_1, \gamma}(K_R) \) and for the limit \( y_N^2 D_{y}^{\delta} r \) we have with the same exponent \( \gamma \)

\[
\left\langle y_N^2 D_{y}^{\delta} r \right\rangle_{y, \tau}^{\gamma/2} + \left\langle y_N^2 D_{y}^{\delta} r \right\rangle_{y, \tau}^{\gamma/4} \leq C.
\]
Further, from (5.41) and (5.44) it follows that the function \( r(y, \tau) \) satisfies in \( Q \) the homogeneous problem without initial conditions

\[
L_y r(y, \tau) \equiv \frac{\partial r}{\partial \tau} + \nabla_y (y_N \Delta_y r) = 0, \quad (y, \tau) \in Q,
\] (5.49)

\[
\frac{\partial r}{\partial y_N}(y', 0, \tau) = 0, \quad (y', \tau) \in G = \overline{Q} \cap \{ y_N = 0 \}.
\] (5.50)

From this together with (5.40) and from Theorem 17 it follows that \( r(y, \tau) \) is a polynomial with respect to \( y' \) an \( \tau \). And from (5.46) it follows that \( r(y, \tau) \) is a nonconstant polynomial. But nonconstant polynomial in unbounded domain \( Q \) can not have a finite seminorms as those in (5.48). This contradiction proves the lemma in the case \( \tau(p) = \frac{-t(p)}{h^2} \to -\infty \).

In the other case, If \( \tau(p) = \frac{-t(p)}{h^2} \to -\tau_0 > -\infty \) the reasonings are completely the same. The difference is only that instead of the relations (5.49), (5.50) in view of (5.44) we obtain for the function \( r(y, \tau) \) in \( Q(0) = Q \cap \{ \tau \geq -\tau_0 \} \)

\[
L_y r(y, \tau) \equiv \frac{\partial r}{\partial \tau} + \nabla_y (y_N \Delta_y r) = 0, \quad (y, \tau) \in Q(0),
\] (5.51)

\[
\frac{\partial r}{\partial y_N}(y', 0, \tau) = 0, \quad (y', \tau) \in G(0) = Q(0) \cap \{ y_N = 0 \},
\] (5.52)

\[
r(y, -\tau_0) \equiv 0, \quad y \in \mathbb{R}^N.
\] (5.53)

In this case in view of (5.40) again from the Theorem 17 it follows that \( r(y, \tau) \equiv 0 \) and this contradicts to (5.46).

Note again that all the above reasonings for the term \( \langle \langle x_N^2 D^5_x v_p \rangle \rangle^{(10)(\varepsilon+)}_{\gamma/2, x', Q} \) are completely the same for \( \langle \langle D_t u \rangle \rangle^{(\gamma/4)(10)(\varepsilon+)}_{t, Q^+} \). For this term we obtain an analog of relations (5.40) and (5.48) with the same contradiction.

Thus the lemma is proved for the condition (5.8). The proof for the condition (5.12) is absolutely similar with some another but evident rescaling for the function \( \varphi(x', t) \).

This finishes the proof of the lemma.

\[ \square \]

Denote for a function \( u(x, t) \in C^{4+\gamma, 4+\gamma}_{2, \gamma/2}(Q^+) \)

\[
\langle \langle x_N^2 D^4_x u \rangle \rangle^{(10)(\varepsilon-)}_{\gamma/2, x, Q^+} \equiv \sup_{(x, t) \in Q^+, h \leq \varepsilon x_N} x_N^{\gamma/2} \left| \frac{\Delta_{h,x,i} \left( x_N^2 D^4_x u(x, t) \right)}{h^\gamma} \right|, \quad i = 1, N,
\] (5.54)

\[
\langle \langle D_t u \rangle \rangle^{(\gamma/4)(10)(\varepsilon-)}_{t, Q^+} \equiv \sup_{(x, t) \in Q^+, h \leq \varepsilon x_N} \left| \frac{\Delta_{h,t} \left( D_t u(x, t) \right)}{h^{\gamma/4}} \right|,
\] (5.55)
\[
\langle \langle u \rangle \rangle^{(4+\gamma)\leq 0,2,\gamma/2,4,x,t,Q^+} = \sum_{i=1}^{N-1} \langle \langle x_N^2 D^4_{x_i} u \rangle \rangle_{\gamma/2,2,\gamma/2} \leq 0 \begin{array}{c} + \langle \langle D_t u \rangle \rangle_{\gamma/2,4,4,\epsilon} + \langle \langle D_i u \rangle \rangle_{\gamma/2,4,2,10,\epsilon} \end{array}
\]

Lemma 23 Let a function \( u(x,t) \) be \( C^{4+\gamma}u^{1+\gamma/2,1} \) with a compact support satisfy in \( Q^+ \) relations \((5.7), (5.9)\) or relations \((5.7), (5.9), (5.12)\). Then for \( 0 < \epsilon < \epsilon_0 \), where \( \epsilon_0 \in (0,1) \) is an absolute constant,

\[
\langle \langle u \rangle \rangle^{(4+\gamma)\leq 0,2,\gamma/2,2,4,x,t,Q^+} = \sum_{i=1}^{N-1} \langle \langle x_N^2 D^4_{x_i} u \rangle \rangle_{\gamma/2,2,\gamma/2} \leq 0 \begin{array}{c} + \langle \langle D_t u \rangle \rangle_{\gamma/2,2,\gamma/2} + \langle \langle D_i u \rangle \rangle_{\gamma/2,2,\gamma/2} \end{array}
\]

Proof.

Consider some particular index \( i = 1, N-1 \) and consider the derivative with respect to tangent variable \( x_i, D^4_{x_i} u \),

\[
\langle \langle x_N^2 D^4_{x_i} u \rangle \rangle_{\gamma/2,2,\gamma/2} \leq 0 \begin{array}{c} = \sup_{(x,t) \in Q^+, h \leq \epsilon x_N} x_N^{\gamma/2} \frac{\partial^{10} \epsilon \\Delta_{h,x_i}^1 (x_N^2 D^4_{x_i} u(x,t))}{h^\gamma} \end{array}
\]

We represent the expression in \( (5.58) \) as

\[
\langle \langle x_N^2 D^4_{x_i} u \rangle \rangle_{\gamma/2,2,\gamma/2} = \sup_{(x,t) \in Q^+, h \leq \epsilon x_N} x_N^{\gamma/2} \frac{\partial^{10} \epsilon \\Delta_{h,x_i}^1 (x_N^2 D^4_{x_i} u(x,t))}{h^\gamma}
\]

where \( v(x,t) = \Delta_{h,x_i}^5 u(x,t) \).

Note that the function \( v(x,t) \) satisfies in \( Q^+ \) the equation

\[
L_{x,t} v(x,t) = \frac{\partial v}{\partial t} + \nabla_x (x_N^2 \nabla_x v) = f_1(x,t) \equiv \Delta_{h,x}^5 f(x,t), \quad (x,t) \in Q^+,
\]

and the initial condition

\[
v(x,0) = \psi_1(x) \equiv \Delta_{h,x}^5 \psi(x), \quad x \in \overline{\Omega}.
\]

Let a point \((x,t) = (x_0,t_0) = (x_0',x_N^0,t_0)\) be fixed and fix also a vector \( h > 0 \), \( h \leq \epsilon x^0_N \). Suppose that \( \epsilon \in (0,1/128) \). Consider the expression
\[ A \equiv (x_N^0)^{\gamma/2} \frac{|(x_N^0)^2 \Delta_{h,x_0}^5 D_{x_0}^4, v(x_0, t_0)|}{h^\gamma}, \ |\alpha| = 4. \] (5.60)

Make in the functions \( u(x, t) \) and \( v(x, t) \) the change of variables \((x, t) \rightarrow (y, \tau)\), \( v(x, t) \rightarrow v(y, \tau) \)

\[ x' = x_0' + (x_N^0) y', \quad x_N = (x_N^0) y_N, \quad t = t_0 + (x_N^0)^2 \tau \] (5.61)

and denote \( d = h/x_N^0 \leq \varepsilon < 1/128, P_1 \equiv (y_0, \tau_0) \equiv (0', 1, 0) \), that is \((x_0, t_0) \rightarrow (y_0, \tau_0)\). In the new variables the expression \( A \) takes the form

\[ A = (x_N^0)^{\gamma/2+2-4-\gamma} \frac{|\Delta_{d,y}^5 D_{y}^4 v(0', 1, 0)|}{d^\gamma}. \] (5.62)

Denote for \( \rho < 1 \)

\[ Q_\rho \equiv \{(y, \tau) \in Q : |y'| \leq \rho, |y_N - 1| \leq \rho, |\tau| \leq \rho^2\}, Q_\rho^+ = Q_\rho \cap \{t \geq 0\}. \] (5.63)

and consider the function \( v(y, \tau) \) on the cylinder \( Q_{3/4}^+ \). On this cylinder \( v(y, \tau) \) satisfies the equation

\[ \frac{\partial v}{\partial \tau} + \nabla_y(y_N^2 \nabla_y v) = (x_N^0)^2 f_1(y, \tau) \] (5.64)

and the initial condition

\[ v(y, \tau_0) = \psi_1(y) \] (5.65)

if \( \tau_0 \equiv -t_0/ (x_N^0)^2 \geq -3/4\). Note that since \( y_N \in (1/4, 5/4) \) on \( Q_{3/4} \), the function \( v(y, \tau) \) belongs to the usual smooth class \( C^{4+\gamma, 3+\gamma/4}(\overline{Q_{3/4}}) \). Consider this function on \( \overline{Q_{1/4}} \subset \overline{Q_{3/4}} \). Applying known local estimates for parabolic equations (see, for example, [32]) we obtain

\[ \frac{|\Delta_{d,y}^5 D_{y}^4 v(0', 1, 0)|}{d^\gamma} \leq C \left( \langle D_{y}^4 v(y, \tau) \rangle_{y, \overline{Q_{3/4}}}^{(\gamma)} \right) \leq (5.66) \]

\[ \leq C \left( (x_N^0)^2 |f_1(y, \tau)|_{\overline{Q_{1/2}}}^{(\gamma)} + |\psi_1(y)|_{\overline{Q_{1/2}}}^{(4+\gamma)} + |v(y, \tau)|_{\overline{Q_{1/2}}}^{(0)} \right)^4_{\overline{Q_{1/2}}}^{(\gamma)}. \n\]

Note that the relation of the norm \( |f_1(y, \tau)|_{\overline{Q_{1/2}}}^{(\gamma)} \) in variables \((y, \tau)\) an \((x, t)\) is

\[ |f_1(y, \tau)|_{\overline{Q_{1/2}}}^{(\gamma)} = \langle f_1(y, \tau) \rangle_{y, \overline{Q_{1/2}}}^{(\gamma)} + \langle f_1(y, \tau) \rangle_{\tau, \overline{Q_{1/2}}}^{(\gamma/4)} + |f_1(y, \tau)|_{\overline{Q_{1/2}}}^{(0)}. \]

\[ = (x_N^0)^{\gamma} \langle f_1(x, t) \rangle_{x, \overline{Q_{(1/2), x}}}^{(\gamma)} + (x_N^0)^{\gamma/2} \langle f_1(x, t) \rangle_{x, \overline{Q_{(1/2), x}}}^{(\gamma/4)} + |f_1(x, t)|_{\overline{Q_{(1/2), x}}}^{(0)}, \] (5.67)
where

\[ Q_{x_N^0} = \{ (x, t) \in Q : |x| \leq \rho x_N^0, |x_N - x_N^0| \leq \rho x_N^0, |t - t_0| \leq (\rho x_N^0)^2 \} \cap \{ t \geq 0 \}. \]  

(5.68)

Analogously,

\[
\psi_1(y) \left|\frac{Q_{x_N^0}^{(4+\gamma)}}{Q_{x_N^0}^{(2)}}\right. = \sum_{|a|=4} \left< D_{y}^{a} \psi_1(y) \right|_{Q_{x_N^0}^{(2)}}^{(4\gamma)} + \psi_1(y) \left|\frac{Q_{x_N^0}^{(4)}}{Q_{x_N^0}^{(2)}}\right. =
\]

\[
= (x_N^0)^{4+\gamma} \sum_{|a|=4} \left< D_{x}^{a} \psi_1(x) \right|_{Q_{x_N^0}^{(2)}}^{(4\gamma)} + \psi_1(x) \left|\frac{Q_{x_N^0}^{(4)}}{Q_{x_N^0}^{(2)}}\right. . \]  

(5.69)

Now we go back to the variables \((x, t)\) in estimate (5.66) and obtain \(|\alpha| = 4\)

\[
(x_N^0)^{4+\gamma} \left|\frac{\Delta h_{x_i} D_{x_i}^4 \psi(x_0, t_0)}{h^\gamma}\right. =
\]

\[
\leq C \left( (x_N^0)^{2+\gamma} \left< f_1(x, t) \right|_{Q_{x_N^0}^{(2)}}^{(4\gamma)} + (x_N^0)^{2+\gamma/2} \left< f_1(x, t) \right|_{Q_{x_N^0}^{(2)}}^{(4\gamma/2)} +
\right.
\]

\[
+ (x_N^0)^{2} \left< f_1(x, t) \right|_{Q_{x_N^0}^{(2)}}^{(4\gamma/2)} +
\]

\[
+ (x_N^0)^{4+\gamma} \sum_{|a|=4} \left< D_{x}^{a} \psi_1(x) \right|_{Q_{x_N^0}^{(2)}}^{(4\gamma)} + \psi_1(x) \left|\frac{Q_{x_N^0}^{(4)}}{Q_{x_N^0}^{(2)}}\right. + \psi_1(x) \left|\frac{Q_{x_N^0}^{(4)}}{Q_{x_N^0}^{(2)}}\right. . \]  

(5.70)

Before proceeding further with the estimate of the expression \(A\) in (5.66), note that since \(x_N \sim x_N^0\) on the set \(Q_{x_N^0}^{(2)}\), and we have just from the definition of the Hölder constants

\[
(x_N^0)^{2+\gamma/2} \left< D_{x}^{4} \psi_1 \right|_{Q_{x_N^0}^{(2)}}^{(4\gamma)} \leq
\]

\[
\leq C \left( x_N^0 \right)^{\gamma/2} \left< x_N^{2} D_{x}^{4} \psi_1 \right|_{Q_{x_N^0}^{(2)}}^{(4\gamma)} + (x_N^0)^{2-\gamma} |D_{x}^{4} \psi_1| \left|\frac{Q_{x_N^0}^{(4)}}{Q_{x_N^0}^{(2)}}\right. . \]  

(5.71)

And at the same time

\[
|D_{x}^{4} \psi_1| \left|\frac{Q_{x_N^0}^{(4)}}{Q_{x_N^0}^{(2)}}\right. = |\Delta h_{x_i} D_{x_i}^{4} \psi(x)| \left|\frac{Q_{x_N^0}^{(4)}}{Q_{x_N^0}^{(2)}}\right. \leq C \gamma \left< x_N^{2} D_{x}^{4} \psi \right|_{Q_{x_N^0}^{(2)}}^{(4\gamma)} \leq
\]

\[
\leq C \varepsilon \gamma \left( x_N^0 \right)^{\gamma} \left< x_N^{2} D_{x}^{4} \psi \right|_{Q_{x_N^0}^{(2)}}^{(4\gamma)} ,
\]

(5.72)

and since

\[
(x_N^0)^{\gamma/2} \left< x_N^{2} D_{x}^{4} \psi \right|_{Q_{x_N^0}^{(2)}}^{(4\gamma)} \leq C \left|\psi\right|_{Q_{x_N^0}^{(2)}}^{(4\gamma/2)} \left|\frac{Q_{x_N^0}^{(2)}}{Q_{x_N^0}^{(2)}}\right. ,
\]

we have
\[(x_N^0)^{2+\gamma/2} \langle D_x^4 \psi_1 \rangle_{x, Q(1/2), x_N^0} \leq C \langle \psi \rangle_{2, \gamma/2, Q(3/4), x_N^0}^{(4+\gamma)} \]  

(5.73)

Analogously, since the difference \( \Delta^5_N \) is taken with respect to the tangent variables \( x' \) only

\[|\psi(0)|_{Q(1/2), x_N^0} \leq C h^{4+\gamma} \langle D_x^4 \psi \rangle_{x, Q(3/4), x_N^0} \leq C \varepsilon^{4+\gamma} \langle \psi \rangle_{2, \gamma/2, Q(3/4), x_N^0}^{(4+\gamma)} . \]

(5.74)

By the exactly same reasonings we obtain

\[(x_N^0)^2 |f_1(x, t)|_{Q(1/2), x_N^0} \leq (x_N^0)^2 |\Delta^5_{h, x} f(x, t)|_{Q(1/2), x_N^0} \leq (x_N^0)^2 h^{\gamma} \langle f(x, t) \rangle_{x, Q(3/4), x_N^0} \leq \varepsilon^{4+\gamma} (x_N^0)^{2+\gamma/2} \langle f(x, t) \rangle_{x, Q(3/4), x_N^0} \.\]

(5.75)

\[(x_N^0)^{2+\gamma} \langle f_1(x, t) \rangle_{x, Q(1/2), x_N^0} \leq C (x_N^0)^{2+\gamma/2} \langle f(x, t) \rangle_{x, Q(3/4), x_N^0} \.\]

(5.76)

and, at last, as in (5.74), since \( h \leq \varepsilon x_N^0 \)

\[|v(0)|_{Q(1/2), x_N^0} \leq |\Delta^5_{h, x} u(0)|_{Q(1/2), x_N^0} \leq C h^{4+\gamma} \langle D_x^4 u \rangle_{x, Q(3/4), x_N^0} \leq C \varepsilon^{4+\gamma} (x_N^0)^{2+\gamma/2} \langle D_x^4 u \rangle_{x, Q(3/4), x_N^0} \.\]

(5.77)

Substituting estimates (5.73) - (5.77) in (5.70) and dividing both parts by \( (x_N^0)^{2+\gamma/2} \), we obtain

\[(x_N^0)^{\gamma/2} \frac{|\Delta_{h, x} (x_N^0)^2 D_x^4 u(x_0, t_0)|}{\hbar} \leq C \left( \langle f(x, t) \rangle_{x, Q(3/4), x_N^0}^{(4+\gamma)} + \langle u \rangle_{2, \gamma/2, Q(3/4), x_N^0}^{(4+\gamma)} \right) + \]

\[+ C \varepsilon^{4+\gamma} \sum_{|\alpha|=4, \alpha N=0} \langle x_N^0 D_x^4 u \rangle_{x, Q(3/4), x_N^0}^{(4+\gamma)} \leq \]

\[\leq C \left( \langle f(x, t) \rangle_{x, Q(3/4), x_N^0}^{(4+\gamma)} + \langle u \rangle_{2, \gamma/2, Q(3/4), x_N^0}^{(4+\gamma)} \right) + C \varepsilon^{4+\gamma} \langle u \rangle_{2, \gamma/2, Q(3/4), x_N^0}^{(4+\gamma)} .\]

Since the point \((x_0, t_0)\), the step \(h\) and the index \(i\) are arbitrary, the last estimate proves the lemma.
Lemma 24 Let a function $u(x,t) \in C_{2,\gamma/2}^{a+\gamma,4\alpha/2}(Q^+)$ has compact support in the set $Q^+_R = Q^+ \cap \{ x_N \leq R \}$, $R > 0$, and satisfy relations (5.7), (5.9). Then for $0 < \varepsilon < \varepsilon_0$, where $\varepsilon_0 \in (0, 1)$ is an absolute constant,

$$
\langle \langle D_t u \rangle \rangle (x,t)_{\varepsilon} \leq C \left( \langle \psi \rangle_{2,\gamma/2,R_N} + |f|_{\gamma/2,Q^+} \right) + \sum_{i=1}^{N-1} \langle \langle x_N^i D^i x,u \rangle \rangle (x,t)_{\varepsilon} + C \varepsilon \langle \langle x_N^i D^i x,u \rangle \rangle (x,t)_{\varepsilon}.
$$

Proof. Let a point $(x_0, t_0) \in Q^+$ be fixed and fix $h > 0$,

$$
x_0 = (x_0^0, x_N^0), x_N^0 > 0, \quad 0 < h < \varepsilon^2 \left( x_N^0 \right)^2.
$$

We consider separately two cases of the value of $t_0$. Let first

$$
t_0 \geq 20 \varepsilon^2 \left( x_N^0 \right)^2.
$$

Then we can proceed exactly as in the Lemma 23 and consider the expression

$$
A \equiv \| \Delta_{h,t}^{10} D_t u(x_0, t_0) \|_{h^{\gamma/4}} = \| \Delta_{h,t}^{10} D_t v(x_0, t_0) \|_{h^{\gamma/4}},
$$

where $v(x,t) = \Delta_{h,t}^{10} u(x,t)$. As in Lemma 23 consider $v(x,t)$ on the cylinder $Q_{x_N^0} = Q_{\Delta_{h,t}^{10} u(x_0, t_0)}$ from (5.68). Since $t_0 \geq 20 \varepsilon^2 \left( x_N^0 \right)^2$ this cylinder is included in the set $Q^+ \cap \{ t \geq 19 \varepsilon^2 \left( x_N^0 \right)^2 \}$. Moreover since $h < \varepsilon^2 \left( x_N^0 \right)^2$, for $(x,t) \in Q_{x_N^0}(x_0, t_0)$ we have

$$
t_i = t \pm ih \geq 9 \varepsilon^2 \left( x_N^0 \right)^2 > 0, \quad i = 0, 10.
$$

Thus for $(x,t) \in Q_{x_N^0}(x_0, t_0)$ the arguments of the functions $u$, $D_t u$, and $D^2 u$ in the expressions $\Delta_{h,t}^{10} u(x,t)$, $\Delta_{h,t}^{10} D_t u(x,t)$, and $\Delta_{h,t}^{10} D^2 u$ belong to the set $Q^+ \cap \{ t \geq 9 \varepsilon^2 \left( x_N^0 \right)^2 > 0 \}$. Therefore we can consider the equation (5.74) for the function $v(x,t) = \Delta_{h,t}^{10} u(x,t)$ on $Q_{x_N^0}(x_0, t_0)$ without initial and boundary data. This permits to estimate the expression $A$ from (5.82) exactly as it was done in Lemma 23 and we obtain

$$
A \equiv \| \Delta_{h,t}^{10} D_t u(x_0, t_0) \|_{h^{\gamma/4}} \leq C \left\langle f(x,t) \right\rangle_{x_N^0} + C \varepsilon \left\langle u \right\rangle_{2,\gamma/2,Q^+} \leq \langle f(x,t) \rangle_{x_N^0} + C \varepsilon \langle u \rangle_{2,\gamma/2,Q^+}.
$$

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Consider now the case
\[ t_0 < 20\varepsilon^2 \left( x_N^{(0)} \right)^2. \] (5.84)

Note that \( \langle \langle D_t u \rangle \rangle_{t, Q^+}^{(\gamma/4)(10)(\varepsilon)} = \langle \langle D_t (u - \psi) \rangle \rangle_{t, Q^+}^{(\gamma/4)(10)(\varepsilon)} \). Besides the function \( v = u - \psi \) satisfies relation \( 5.7 \) with the righthand side \( f_1 = f - \nabla(x_N^2 \nabla \psi) \) and relation \( 5.9 \) with \( \psi_1 \equiv 0 \). Thus considering the function \( u - \psi \) instead of \( u \) we can assume that \( \psi \equiv 0 \) in \( 5.9 \).

Further, since the support of \( u(x, t) \subset \{ x_N \leq R \} \), we can consider only such \( x_N \) when estimate \( \langle \langle D_t u \rangle \rangle_{t, Q^+}^{(\gamma/4)(10)(\varepsilon)} \). Consider the function \( u(x, t) \) in a neibourhood of \( (x_0, t_0) \). We can assume that \( \partial u / \partial t(x_0, 0) = 0 \) and \( f(x_0, 0) = 0 \). If it is not the case, we can consider the function \( v = u(x, t) - tf(x_0, 0) \) instead of \( u(x, t) \). For such function the right hand side of \( 5.7 \) become \( f(x, t) - f(x_0, 0) \) with the desired property. So we assume that the function \( u(x, t) \) satisfies the relations

\[
\frac{\partial u}{\partial t} + \nabla_x(x_N^2 \nabla x \Delta u) = f(x, t), \quad f(x_0, 0) = 0, \quad (x, t) \in \bar{Q}_{\rho x_N^{(0)}}(x_0, t_0), \quad (5.85)
\]

\[
u(x_0, 0) = 0, \quad (5.86)
\]

where \( \bar{Q}_{\rho x_N^{(0)}}(x_0, t_0) \) and \( \bar{Q}_{(x_0, t_0)}^{(\rho)} \) are defined as \( (\rho \in (0, 1)) \)

\[
\bar{Q}_{\rho x_N^{(0)}}(x_0, t_0) = \{ (x, t) \in Q^+: |x - x_0^0| \leq \rho x_N^{(0)}, |x_N - x_N^{(0)}| \leq \rho x_N^{(0)}, |t - t_0| \leq 60\varepsilon^2 x_N^{(0)^2} \}.
\] (5.87)

\[
\bar{Q}_{(x_0, t_0)}^{(\rho)} = \{ (y, \tau) \in Q^+: |y| \leq \rho, |y_N - 1| \leq \rho, |\tau| \leq 60\varepsilon^2 \rho^2, \tau \geq t_0 - t_0 / \left( x_N^{(0)^2} \right) \}.
\] (5.88)

Make in relations \( 5.85, 5.86 \) the change of the variables \( 5.61 \). These relations take the form

\[
\frac{\partial u}{\partial \tau} + \nabla_y(y_N^2 \nabla y \Delta_y u) = \left(x_N^{(0)^2} \right)^2 f(y, \tau), \quad f(P_0, -\tau_0) = 0, \quad (y, \tau) \in \bar{Q}_{(x_0, t_0)}^{(\rho)}, \quad (5.89)
\]

\[
u(y, -\tau_0) = 0, \quad -\tau_0 = -t_0 / \left( x_N^{(0)^2} \right)^2 \geq -20\varepsilon^2, \quad (5.90)
\]

where the point \( P_0 = (0', 1) \). Denote

\[
Q_{1/2}^{(x_0, t_0)} = \bar{Q}_{(x_0, t_0)}^{(\rho)} \cap \{ |y| \leq 1/2, |y_N - 1| \leq 1/2 \}.
\]

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Thus, substituting (5.92), (5.93) in (5.91), we obtain
\[ f \text{ in view of the property } f(P_0, -\tau_0) = 0. \]

Since \( y_N \in [1/4, 7/4] \) on \( \overline{Q}^{(x_0, \tau_0)} \), from classical local interior estimates for parabolic initial value problems with respect to spatial variables it follows that with some absolute constant \( C > 0 \) (see, for example, [32])

\[
\langle D_\tau u \rangle_{\tau, \overline{Q}^{(x_0, \tau_0), \tau_0}}^{(\gamma/4)} \leq C \left( \left( x_N^0 \right)^2 |f|_{\gamma/2, \overline{Q}^{(x_0, \tau_0), \tau_0}}^{(\gamma/4)} + |u|_{\overline{Q}^{(x_0, \tau_0), \tau_0}}^{(0)} \right) \leq \tag{5.91}
\]

\[
\leq C \left( \left( x_N^0 \right)^2 \langle f \rangle_{y, \overline{Q}^{(x_0, \tau_0), \tau_0}}^{(\gamma/4)} + \left( x_N^0 \right)^2 \langle f \rangle_{\tau, \overline{Q}^{(x_0, \tau_0), \tau_0}}^{(\gamma/4)} \right) + C |u|_{\overline{Q}^{(x_0, \tau_0), \tau_0}}^{(0)}.
\]

Since
\[
|f|_{\overline{Q}^{(x_0, \tau_0), \tau_0}}^{(0)} \leq C \left( \langle f \rangle_{y, \overline{Q}^{(x_0, \tau_0), \tau_0}}^{(\gamma/4)} + \langle f \rangle_{\tau, \overline{Q}^{(x_0, \tau_0), \tau_0}}^{(\gamma/4)} \right)
\]
in view of the property \( f(P_0, -\tau_0) = 0 \). The height \( H_Q \) of the cylinder \( \overline{Q}^{(x_0, \tau_0)} \)
is equal \( H_Q = \tau_0 + 45\varepsilon^2 \leq 20\varepsilon^2 + 45\varepsilon^2 = C\varepsilon^2 \). Since \( u(y, -\tau_0) \equiv 0 \), we have

\[
|u|_{\overline{Q}^{(x_0, \tau_0), \tau_0}}^{(0)} \leq \int_{-\tau_0}^{\tau} D_\tau u(y, \theta) d\theta |_{\overline{Q}^{(x_0, \tau_0), \tau_0}}^{(0)} \leq H_Q |D_\tau u|_{\overline{Q}^{(x_0, \tau_0), \tau_0}}^{(0)} \leq C\varepsilon^2 |D_\tau u|_{\overline{Q}^{(x_0, \tau_0), \tau_0}}^{(0)}. \tag{5.92}
\]

On the other hand, from relations (5.89), (5.90) it follows that
\[
D_\tau u(P_0, -\tau_0) = \left( x_N^0 \right)^2 f(P_0, -\tau_0) = 0.
\]

Thus
\[
|D_\tau u|_{\overline{Q}^{(x_0, \tau_0), \tau_0}}^{(0)} \leq C \left( \langle D_\tau u \rangle_{y, \overline{Q}^{(x_0, \tau_0), \tau_0}}^{(\gamma/4)} + \langle D_\tau u \rangle_{\tau, \overline{Q}^{(x_0, \tau_0), \tau_0}}^{(\gamma/4)} \right). \tag{5.93}
\]

Substituting (5.92), (5.93) in (5.91), we obtain

\[
\langle D_\tau u \rangle_{\tau, \overline{Q}^{(x_0, \tau_0), \tau_0}}^{(\gamma/4)} \leq C \left( \left( x_N^0 \right)^2 \langle f \rangle_{y, \overline{Q}^{(x_0, \tau_0), \tau_0}}^{(\gamma/4)} + \left( x_N^0 \right)^2 \langle f \rangle_{\tau, \overline{Q}^{(x_0, \tau_0), \tau_0}}^{(\gamma/4)} \right) + 
\]

\[ + C\varepsilon^2 \left( \langle D_\tau u \rangle_{y, \overline{Q}^{(x_0, \tau_0), \tau_0}}^{(\gamma/4)} + \langle D_\tau u \rangle_{\tau, \overline{Q}^{(x_0, \tau_0), \tau_0}}^{(\gamma/4)} \right). \tag{5.94}\]

Going back to the variables \((x, t)\) in (5.94), dividing both parts by \(\left( x_N^0 \right)^{2+\gamma/2}\), and repeating the reasoning of Lemma 23, we arrive at
Let a function \( u \in Q \) and for any \( x, t \), this proves the lemma. 

Taking into account interpolation inequality (2.18) and the fact that \( h \leq \varepsilon^2 \left( x_N^0 \right) \) in (5.82), we obtain

\[
A \equiv \frac{\| \Delta_{h/4} D_t u(x_0, t_0) \|_{h^{-3/4}}}{C} \leq C \langle f \rangle_{\gamma/2, Q^+} + C \varepsilon \langle D_t u \rangle_{\gamma/2, Q^+},
\]

where

\[
Q_{x_N/2}(x_0, t_0) = \{(x, t) \in Q^+: |x' - x_0'| \leq x_N^0/2, |x_N - x_0| \leq x_N^0/2, |t - t_0| \leq 15\varepsilon^2 \left( x_N^0 \right)^2 \}.
\]

Since \( (x_0, t_0) \) and \( h \) are arbitrary, we infer (5.79) from the last inequality and this proves the lemma. ■

**Proposition 25** Let functions \( f(x, t) \), \( g(x', t) \), \( \varphi(x', t) \), and \( \psi(x) \) have compact supports and

\[
f(x, t) \in C^2_{\gamma/2}(Q^+), g(x', t) \in C^{1+\gamma/2,1/2+\gamma/4}(G^+),
\]

\[
\varphi(x', t) \in C^{2+\gamma/2,1+\gamma/4}(G^+), \psi(x) \in C^4_{2,\gamma/2}(R^N).
\]

Let a function \( u(x, t) \in C^{4+\gamma/2,4+\gamma}(Q^+) \) with a compact support satisfy the following initial boundary value problem in \( Q^+ \)

\[
L_{x,t} u \equiv \frac{\partial u}{\partial t} + \nabla (x_N^2 \nabla u) = f(x, t), \quad (x, t) \in Q^+,
\]

\[
\frac{\partial u}{\partial x_N}(x_0', 0, t) = g(x', t), \quad (x', t) \in G^+,
\]

\[
u(x_0, 0) = \psi(x), \quad x \in R^N
\]

Then for any \( \varepsilon, \mu > 0 \) there exists a constant \( C_{\varepsilon, \mu} > 0 \) with the property

\[
\langle u \rangle_{2,\gamma/2,x',Q^+} \leq C \langle \langle u \rangle \rangle_{2,\gamma/2,x',Q^+} \leq C_{\varepsilon, \mu} \left( \langle f \rangle_{\gamma/2,Q^+} + \langle g \rangle_{G^+}^{1+\gamma/2,1/2+\gamma/4} + \langle \psi \rangle_{2,\gamma/2,R^N}^{(1+\gamma)} \right) + \mu \langle x_N^2 D_{x_N}^4 u \rangle_{\gamma/2,x_N,Q^+}.
\]

(5.100)
If instead of (5.98) the function \( u(x, t) \) satisfies

\[
u(x', 0, t) = \varphi(x', t), \quad (x', t) \in G^+,
\]

then

\[
\langle u \rangle_{(4+\gamma)}^{(4+\gamma)} \leq C \langle \{ u \} \rangle_{(4+\gamma)}^{(4+\gamma)(10)} 
\]

\[
\leq C_{\mu} \left( \langle f \rangle_{\gamma/2, Q^+} + \langle \varphi \rangle_{\gamma/2, 2, x', t, Q^+} + \langle \psi \rangle_{\gamma/2, 2, R^+} \right) + \mu \langle x_N^2 D_{x_N}^4 u \rangle_{\gamma/2, 2, x, Q^+},
\]

where

\[
\langle u \rangle_{(4+\gamma)}^{(4+\gamma)} = \sum_{i=1}^{N-1} \langle x_N^2 D_{x_N}^4 u \rangle_{1/2, x, Q^+} + \langle D_i u \rangle_{4/2, Q^+}.
\]

**Proof.** For any \( \varepsilon > 0 \) we have

\[
\langle \{ u \} \rangle_{(4+\gamma)(10)}^{(4+\gamma)(10)} \leq \langle \{ u \} \rangle_{(4+\gamma)(10)}^{(4+\gamma)(10)} + \langle \{ u \} \rangle_{(4+\gamma)(10)}^{(4+\gamma)(10)}.
\]

It is evident that

\[
\langle \{ u \} \rangle_{(4+\gamma)(10)}^{(4+\gamma)(10)} \geq \frac{1}{2} \langle \{ u \} \rangle_{(4+\gamma)(10)}^{(4+\gamma)(10)} + \langle \{ u \} \rangle_{(4+\gamma)(10)}^{(4+\gamma)(10)}.
\]

Then from lemmas (22)-(24) it follows that (in the case of condition (5.98))

\[
\langle \{ u \} \rangle_{(4+\gamma)(10)}^{(4+\gamma)(10)} \leq \]

\[
\leq C_{\varepsilon, \mu} \left( \langle f \rangle_{\gamma/2, Q^+} + \langle g \rangle_{\gamma/2, 1/2, 2, x, t, Q^+} \right) + 
\]

\[
+ C_{\varepsilon} \langle \{ u \} \rangle_{(4+\gamma)(10)}^{(4+\gamma)(10)} + \mu \langle x_N^2 D_{x_N}^4 u \rangle_{\gamma/2, 2, x, Q^+}.
\]

Absorbing now the term with \( \varepsilon \) from the righthand side in the lefthand side for sufficiently small \( \varepsilon \), we arrive at (5.100) (in view of (5.5)). Estimate (5.102) is analogous.

**Theorem 26** Let functions \( f(x, t) \), \( g(x', t) \), \( \varphi(x', t) \), and \( \psi(x) \) have compact supports and satisfy (5.96).

Let a function \( u(x, t) \in C_{2, \gamma/2}^{4+\gamma}(Q^+) \) with a compact support satisfy initial boundary value problem (5.97), (5.99) or problem (5.97), (5.99), (5.101). Then

\[
\langle u \rangle_{(4+\gamma)}^{(4+\gamma)} \leq C \left( \langle f \rangle_{\gamma/2, Q^+} + \langle g \rangle_{\gamma/2, 1/2, 2, 2, x, t, Q^+} \right) (5.104)
\]
\[
(u)_\gamma^{(4+\gamma)} \leq C \left( (f)_\gamma^{(\gamma)} + (\varphi)_{G+}^{(2+\gamma/2,1+\gamma/4)} + (\psi)_{2,\gamma/2,R^N}^{(4+\gamma)} \right), \tag{5.105}
\]

where the constants \( C \) do not depend on \( f, \psi, g, \varphi \).

**Proof.**

We show only estimate \( \Box \) since \( \Box \) is completely similar. Consider equation \( \Box \). We leave only the pure derivatives with respect to the variable \( x_N \) in the left hand side and write this equation in the form

\[
D_{x_N} \left( x_N^2 D_x^3 u \right) = f_1(x,t), \tag{5.106}
\]

where

\[
f_1(x,t) = f(x,t) - \frac{\partial u}{\partial t} - \sum_{|\alpha|=4, \alpha_N<4} \delta_{\alpha} x_N^2 D_\alpha^2 u - \sum_{|\alpha|=3, \alpha_N<3} \delta_{\alpha} x_N D_\alpha^2 u
\]

and \( \delta_{\alpha} \) are some absolute constants. Let us show that

\[
\left( x_N^2 D_x^4 u \right)_{\gamma/2,x_N,Q^+} \leq C \left( f_{1} \right)_{\gamma/2,x_N,Q^+} . \tag{5.107}
\]

Since \( (x_N^2 D_x^3 u) |_{x_N=0} = 0 \) we have from (5.106)

\[
D_x^3 u(x,t) = \frac{1}{x_N} \int_0^{x_N} f_1(x',\xi,t)d\xi.
\]

Thus,

\[
x_N^2 D_x^4 u(x,t) = -\frac{2}{x_N} \int_0^{x_N} f_1(x',\xi,t)d\xi + f_1(x',x_N,t) =
\]

\[
= -2 \int_0^1 f_1(x',x_N\omega,t)d\xi + f_1(x',x_N,t),
\]

where we made the change of the variable \( \xi = \omega x_N \) in the integral. From this representation the obtaining of estimate \( \Box \) is straightforward. Therefore, we have the estimate

\[
\left( x_N^2 D_x^4 u \right)_{\gamma/2,x_N,Q^+} \leq C \left( f \right)_{\gamma/2,x_N,Q^+} + \tag{5.108}
\]

\[
+ C \left( (D_t u)_{\gamma/2,x_N,Q^+} + \sum_{|\alpha|=4, \alpha_N<4} \left( x_N^2 D_x^\alpha u \right)_{\gamma/2,x_N,Q^+} + \sum_{|\alpha|=3, \alpha_N<3} \left( x_N D_x^\alpha u \right)_{\gamma/2,x_N,Q^+} \right).
\]

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Let use now the interpolation inequalities of Theorem 13. This gives \((\varepsilon \in (0, 1))\)

\[
\langle D_t u \rangle_{\gamma/2, x N, Q^+}^{(\gamma)} + \sum_{|\alpha|=4, \alpha_N<4} \langle x_N^2 D_x^2 u \rangle_{\gamma/2, x N, Q^+}^{(\gamma)} + \sum_{|\alpha|=3, \alpha_N<3} \langle x_N D_x u \rangle_{\gamma/2, x N, Q^+}^{(\gamma)} \leq 
\]

\[
\leq C \varepsilon \langle u \rangle_{\gamma/2, x N, t, Q^+}^{(4+\gamma)} + \varepsilon \langle x_N^2 D_x^4 u \rangle_{\gamma/2, x N, Q^+}^{(\gamma)}. \quad (5.109)
\]

Substituting estimate (5.109) in (5.108) and absorbing the term with \(\varepsilon\) in the left hand side, we obtain

\[
\langle x_N^2 D_x^4 u \rangle_{\gamma/2, x N, Q^+}^{(\gamma)} \leq C \langle f \rangle_{\gamma/2, x N, Q^+}^{(\gamma)} + C \langle u \rangle_{\gamma/2, x N, t, Q^+}^{(4+\gamma)}. \quad (5.110)
\]

Thus, making use of estimate (5.109) of Proposition 25 for the full highest seminorm of the function \(u\) we have

\[
\langle u \rangle_{\gamma/2, x N, t, Q^+}^{(4+\gamma)} = \langle x_N^2 D_x^4 u \rangle_{\gamma/2, x N, Q^+}^{(\gamma)} + \langle u \rangle_{\gamma/2, x N, t, Q^+}^{(4+\gamma)} \leq 
\]

\[
\leq C \mu \left( \langle f \rangle_{\gamma/2, x N, Q^+}^{(\gamma)} + \langle g \rangle_{G^+}^{(1+\gamma/2, 1/2, 1/4)} + \langle \psi \rangle_{2, x N, R^N}^{(4+\gamma)} \right) + \mu \langle x_N^2 D_x^4 u \rangle_{\gamma/2, x N, Q^+}^{(\gamma)}. \quad (5.104)
\]

Absorbing now the term with \(\mu\) in the left hand side, we arrive at estimate (5.105).

Estimate (5.105) is completely similar and this finishes the proof of the theorem.

Consider now the elliptic variant of the problems of Theorem 26.

**Theorem 27** Let functions \(f(x)\), \(g(x')\), and \(\varphi(x')\) have compact supports and

\[
f(x) \in C^\gamma_{\gamma/2}(R^N_+), g(x') \in C^{1+\gamma/2}(R^{N-1}_+), \varphi(x', t) \in C^{2+\gamma/2}(R^{N-1}_+). \quad (5.111)
\]

Let a function \(u(x) \in C^{2+\gamma/2}_{\gamma/2}(R^N_+)\) with a compact support satisfy the following boundary value problem in \(R^N_+\)

\[
L_x u \equiv \nabla (x_N^2 \nabla \Delta u) = f(x), \quad x \in R^N_+, \quad (5.112)
\]

\[
\frac{\partial u}{\partial x_N}(x', 0) = g(x'), \quad x_N = 0. \quad (5.113)
\]

Then

\[
\langle u \rangle_{\gamma/2, x N, Q^+}^{(4+\gamma)} \leq C \langle f \rangle_{\gamma/2, x N, Q^+}^{(\gamma)} + \langle g \rangle_{R^{N-1}_+}^{(1+\gamma/2)}. \quad (5.114)
\]

If instead of (5.113) the function \(u(x)\) satisfies
\[ u(x', 0) = \varphi(x'), \quad x_N = 0, \quad (5.115) \]

then

\[ \langle u \rangle_{\gamma/2, R_N}^{(4+\gamma)} \leq C \langle f \rangle_{\gamma/2, R_N}^{(\gamma)} + \langle \varphi \rangle_{R_N-1}^{(2+\gamma/2)}. \quad (5.116) \]

where

\[ \langle u \rangle_{\gamma/2, R_N}^{(4+\gamma)} \equiv \sum_{i=1}^{N} \langle x_N D_{x_i}^4 u \rangle_{\gamma/2, R_N}^{(\gamma)}. \quad (5.117) \]

The proof of this theorem is an evident simplification of the proof of Theorem \(26\) on the base of Corollary \(18\).

6 Solvability of model problems.

In this section we consider two model problems in simple special domains for the model linearized thin film equation with two different boundary conditions at \(\{x_N = 0\}\). We will use these problems to prove the solvability of boundary value problems for the linearized thin film equation in arbitrary smooth domain by the standard way of the regularizator (near inverse operator) constructing. Throughout this section we denote \(I = [0, 1]\).

6.1 A model problem with the Newman condition at \(\{x_N = 0\}\).

We first consider an axillary model problem for an elliptic equation.

Let \(P = \{x = (x', x_N) : 0 \leq x_N \leq 1, |x_i| < \pi, i = 1, N-1\}, \quad P' = P \cap \{x_N = 0\}\). Let a function \(f(x) \in C^\gamma_{\gamma/2}(\overline{P})\) and let also \(f(x)\) be \(2\pi\)-periodic in each variable \(x_i, \quad i = 1, N-1\). Consider the following problem for the unknown \(2\pi\)-periodic with respect to the variables \(x_i, \quad i = 1, N-1\), function \(u(x)\):

\[ \nabla (x_N^2 \nabla^2 u) = f(x), \quad x \in P, \quad (6.1) \]

\[ \frac{\partial u}{\partial x_N}(x', 0) = 0, \quad x' \in P', \quad (6.2) \]

\[ \frac{\partial}{\partial x_N} \Delta u(x', 1) + \Delta u(x', 1) = 0, \quad (6.3) \]

\[ u(x', 1) = 0, \quad (6.4) \]

and the periodicity conditions
\[
\frac{\partial^n u}{\partial x_i^n}(x) = \frac{\partial^n u}{\partial x_i^n}(x) \quad \text{at} \quad i = 1, N - 1.
\]

(6.5)

Thus we consider in fact the periodic functions \(f(x)\) and \(u(x)\). Note that the boundary conditions at \(\{x_N = 1\}\) are chosen just from technical reasons. They do not play any special role when we construct the regularisation of the problem in an arbitrary smooth domain. The all we need that such conditions at \(\{x_N = 1\}\) make the problem well posed.

Lemma 28

Problem (6.1) - (6.5) has the unique solution \(u(x)\) with

\[
|u|^{(4+\gamma)}_{2, \gamma/2, \mathcal{P}} \leq C|f|^{(\gamma)}_{\gamma/2, \mathcal{P}}.
\]

(6.6)

Proof.

Let first \(f(x) = f(x', x_N)\) be of the class \(C^\infty\) with respect to the variables \(x'\). We are going to find the smooth periodic solution of the problem in the form

\[
u(x', x_N) = \sum_{\omega \in \mathbb{Z}^{N-1}} v(\omega, x_N)e^{-i\omega x'},
\]

(6.7)

where \(\omega = (\omega_1, ..., \omega_{N-1})\), \(\omega_i = 0, \pm 1, \pm 2, ..., \omega x' = \omega_1 x_1 + ... + \omega_{N-1} x_{N-1}\), and \(v(\omega, x_N)\) are unknown functions. Correspondingly, we represent the function \(f(x)\) as

\[
f(x', x_N) = \sum_{\omega \in \mathbb{Z}^{N-1}} h(\omega, x_N)e^{-i\omega x'}.
\]

(6.8)

Here in fact \(v(\omega, x_N)\) and \(h(\omega, x_N)\) are discrete Fourier transforms of \(u(x)\) and \(f(x)\) correspondingly. Since \(f(x) \in C_0^\infty(P)\), it is well known that for any \(K > 0\)

\[
|h(\omega, x_N)|_{\omega, \gamma/2, \mathcal{I}} \leq C_K (1 + \omega^2)^{-K},
\]

(6.9)

where \(\omega^2 = \omega_1^2 + ... + \omega_{N-1}^2\). Substituting representations (6.7), (6.8) in relations (6.1), (6.5), we in standard way arrive at the following problem for an ordinary differential equation on \(x_N \in I = [0, 1]\) with the parameter \(\omega\) for the unknown function \(v(\omega, x_N)\)

\[
(x_N v'''(\omega, x_N))' - 2\omega^2 x_N v'' - 2\omega^2 x_N v' + (\omega^2)^2 x_N v(\omega, x_N) = h(\omega, x_N), x_N \in I,
\]

(6.10)

\[
v'(\omega, 0) = 0,
\]

(6.11)

\[
v(\omega, 1) = 0,
\]

(6.12)
v'''(\omega, 1) + v''(\omega, 1) - \omega^2 v'(\omega, 1) = 0.  \quad (6.13)

Note that, in the author’s opinion, it is not so easy to solve ODE (6.10) explicitly. Therefore we are going to use the method of the extension with respect to a parameter (see, for example, [34]). For this we consider the following problem with the parameter \( \lambda \in [0, 1] \)

\[
(x_N^2 v'''(\omega, x_N))' - 2\lambda \omega^2 x_N^2 v'' - 2\lambda \omega^2 x_N v' + \lambda (\omega^2)^2 x_N^2 v(\omega, x_N) = h(\omega, x_N), \quad x_N \in I,  
\]

\[
v'(\omega, 0) = 0,  \quad (6.14)
\]

\[
v(\omega, 1) = 0,  \quad (6.15)
\]

\[
v'''(\omega, 1) + v''(\omega, 1) - \lambda \omega^2 v'(\omega, 1) = 0.  \quad (6.16)
\]

Consider first this problem for the initial value of the parameter \( \lambda = 0 \). Then equation (6.14) and boundary condition (6.17) became

\[
(x_N^2 v'''(\omega, x_N))' = h(\omega, x_N), \quad x_N \in I,  
\]

\[
v''(\omega, 1) + v'(\omega, 1) = a,  \quad (6.18)
\]

where \( a \) is a prescribed complex constant. We can find the solution of this simplified problem explicitly. Taking in mind that due to (6.6) we must have

\[
|x_N v'''(\omega, x_N)| \leq C, \quad x_N \in I,  \quad (6.19)
\]

we obtain from (6.18) with arbitrary constant \( C_1 \)

\[
v'''(\omega, x_N) = \frac{1}{x_N^2} \int_0^{x_N} h(\omega, \xi) d\xi + C_1 = \frac{1}{x_N^2} \int_0^{x_N} h(\omega, \xi) d\xi,  \quad (6.20)
\]

where from (6.20) it follows that we must have \( C_1 = 0 \). This is exactly the place, where the class of the solution serves instead of additional boundary condition at \( \{x_N = 0\} \). Then we find from (6.21)

\[
v''(\omega, x_N) = - \int_0^{x_N} v'''(\omega, \eta) d\eta + C_2 =
\]

\[
= - \left( \frac{1}{x_N} - 1 \right) \int_0^{x_N} h(\omega, \xi) d\xi - \int_{x_N}^{1} \left( \frac{1}{\xi} - 1 \right) h(\omega, \xi) d\xi + C_2.  \quad (6.21)
\]
From (6.21), (6.22) and from boundary condition (6.19) it follows that

\[ C_2 = a - \int_0^1 h(\omega, \xi) d\xi \]

and hence

\[ v''(\omega, x_N) = -\left(\frac{1}{x_N} - 1\right) \int_{x_N}^0 h(\omega, \xi) d\xi - \int_{x_N}^1 \left(\frac{1}{\xi} - 1\right) h(\omega, \xi) d\xi - \int_0^1 h(\omega, \xi) d\xi + a. \]

(6.23)

Now from representation (6.21) analogously to (5.107) it follows that

\[ \left| x_N v'''(\omega, x_N) \right| \leq C \left( |h(\omega, x_N)|^{(\gamma)}_{\gamma/2, I} \right) \frac{1}{x_N} \ln \frac{1}{x_N} \]

and then from equation (6.18) we have

\[ \left| x_N^2 v''''(\omega, x_N) \right| \leq C \left( |h(\omega, x_N)|^{(\gamma)}_{\gamma/2, I} \right) \frac{1}{x_N} \ln \frac{1}{x_N}. \]

(6.25)

From representation (6.23) we directly infer that

\[ \left| v''(\omega, x_N) \right| \leq C \left( |h(\omega, x_N)|^{(\gamma)}_{\gamma/2, I} \right) \left( 1 + \ln \frac{1}{x_N} \right) \]

and then from boundary conditions \( v'(\omega, 0) = 0 \) and \( v(\omega, 1) = 0 \) we can find \( v'(\omega, x_N), v(\omega, x_N) \) and obtain

\[ \left| v'(\omega, x_N) \right|^{(\gamma)}_{\gamma/2, I} + \left| v(\omega, x_N) \right|^{(\gamma)}_{\gamma/2, I} \leq C \left( |h(\omega, x_N)|^{(\gamma)}_{\gamma/2, I} + a \right). \]

(6.26)

Estimates (6.24) to (6.26) mean that we find the solution \( v(\omega, x_N) \in C^{4+\gamma}_{2,\gamma/2}(I) \) and

\[ \left| v(\omega, \cdot) \right|^{(4+\gamma)}_{1/2, I} \leq C \left( |h(\omega, \cdot)|^{(\gamma)}_{\gamma/2, I} + a \right), \]

(6.27)

where the constant \( C \) does not depend on \( \omega \) and \( h(\omega, x_N), a. \)

Denote by \( \tilde{C}^{4+\gamma}_{2,\gamma/2}(I) \) the subspace of \( C^{4+\gamma}_{2,\gamma/2}(I) \) with boundary conditions (6.13), (6.16). Then (6.27) means that the operator \( L_0 : \tilde{C}^{4+\gamma}_{2,\gamma/2}(I) \to C^{\gamma}_{\gamma/2}(I) \times C \) of problem (6.18), (6.19) is an invertible operator. Now we consider the equation \( (\lambda \in [0, 1]) \)

\[ (L_0 + \lambda T) v = (h(\omega, x_N), a), \quad v \in \tilde{C}^{4+\gamma}_{2,\gamma/2}(I). \]

(6.28)

Here operator \( T \) is defined by the terms with \( \lambda \) in expressions in (6.14) and (6.17), that is, in particular, instead of (6.17) we have nonhomogeneous condition

55
\[ v'''(\omega, 1) + v''(\omega, 1) - \lambda \omega^2 v'(\omega, 1) = a. \]  
\[ (6.29) \]

We first obtain uniformly in \( \lambda \in [0, 1] \) an a-priory estimate of \( L_2(I) \) norm of the possible solution \( v(\omega, x_N) \) to equation (6.28). So let \( v \in \tilde{C}^{4+\gamma}_{2,1/2}(I) \) and satisfy (6.14), (6.29). Since \( v'(0, \omega) = 0 \), the function \( v'(x_N, \omega)/x_N \) is bounded. Let \( \overline{v}(x_N, \omega) \) is the complex conjugate of \( v(x_N, \omega) \). Multiply (6.14) by \( v'(x_N, \omega) \) and integrate by parts over \( I \). We have for each term in (6.14) the following expressions.

\[ J_1 = \int_0^1 (x_N^2 v'''(\omega, x_N))' \frac{\overline{v}'(x_N, \omega)}{x_N} dx_N = v'''(\omega, 1) \overline{v}'(1, \omega) - \]

\[ \int_0^1 x_N v'''(\omega, x_N) \overline{v}'(x_N, \omega) dx_N + \int_0^1 v'''(\omega, x_N) \overline{v}'(x_N, \omega) dx_N = \]

\[ = v'''(\omega, 1) \overline{v}'(1, \omega) - \int_0^1 x_N v'''(\omega, x_N) \overline{v}'(x_N, \omega) dx_N + \]

\[ + v''(\omega, 1) \overline{v}'(1, \omega) - \int_0^1 |v''(\omega, x_N)|^2 dx_N. \]

Adding up \( J_1 \) and it’s complex conjugate \( J_1 \), we obtain, integrating by parts again,

\[ J_1 + \overline{J_1} = -2 \int_0^1 |v''(\omega, x_N)|^2 dx_N - \]

\[ - \int_0^1 x_N (|v''(\omega, x_N)|^2)' dx_N + 2 \text{Re} [(v'''(\omega, 1) + v''(\omega, 1)) \overline{v}'(\omega, 1)] = \]

\[ - \int_0^1 |v''(\omega, x_N)|^2 dx_N - |v''(\omega, 1)|^2 + 2 \text{Re} [(v'''(\omega, 1) + v''(\omega, 1)) \overline{v}'(\omega, 1)]. \]

Making use of boundary condition (6.29), we arrive at

\[ J_1 + \overline{J_1} = - \int_0^1 |v''(\omega, x_N)|^2 dx_N - |v''(\omega, 1)|^2 + 2 \lambda \omega^2 |v'(\omega, 1)|^2 + 2 \text{Re}[a \overline{v}'(\omega, 1)]. \]

\[ (6.30) \]
Further,
\[ J_2 \equiv -2\lambda \omega^2 \int_0^1 x_N^2 v'' \frac{\nabla (x_N, \omega)}{x_N} dx_N. \]

Therefore,
\[ J_2 + \overline{J}_2 = -2\lambda \omega^2 \int_0^1 x_N (\nabla' v' + v'' \nabla') = -2\lambda \omega^2 \int_0^1 x_N (|v'|^2)' dx_N = \]
\[ = -2\lambda \omega^2 |v'(\omega, 1)|^2 + 2\lambda \omega^2 \int_0^1 |v'|^2 dx_N. \quad (6.31) \]

For the next term in (6.14) we have
\[ J_3 \equiv -2\lambda \omega^2 \int_0^1 x_N v' \frac{\nabla (x_N, \omega)}{x_N} dx_N, \]
and thus
\[ J_3 + \overline{J}_3 = -4\lambda \omega^2 \int_0^1 |v'|^2 dx_N. \quad (6.32) \]

Now,
\[ J_4 \equiv \lambda (\omega^2)^2 \int_0^1 x_N^2 v(\omega, x_N) \frac{\nabla (x_N, \omega)}{x_N} dx_N = \lambda (\omega^2)^2 \int_0^1 x_N v(\omega, x_N) \frac{\nabla (x_N, \omega)}{x_N} dx_N. \]

Therefore,
\[ J_4 + \overline{J}_4 = \lambda (\omega^2)^2 \int_0^1 x_N (|v|^2)' dx_N = -\lambda (\omega^2)^2 \int_0^1 |v|^2 dx_N. \quad (6.33) \]

At last,
\[ J_5 \equiv \int_0^1 h(\omega, x_N) \frac{\nabla (x_N, \omega)}{x_N} dx_N, \]
and thus
\[ J_5 + \mathcal{J}_5 = 2 \text{Re} \int_0^1 h(\omega, x_N) \frac{\nu'(x_N, \omega)}{x_N} dx_N. \]  

(6.34)

Taking into account that \( J_1 + J_2 + J_3 + J_4 = J_5 \) and adding up relations (6.30)—(6.1), we obtain

\[- \int_0^1 |v''(\omega, x_N)|^2 dx_N - |v''(\omega, 1)|^2 + 2 \text{Re}[a \nu'(\omega, 1)] -
\]

\[-2\lambda \omega^2 \int_0^1 |v'|^2 dx_N - \lambda (\omega^2)^2 \int_0^1 |v|^2 dx_N =
\]

\[= 2 \text{Re} \int_0^1 h(\omega, x_N) \frac{\nu'(x_N, \omega)}{x_N} dx_N. \]

Thus we infer from the last relation

\[\int_0^1 |v''(\omega, x_N)|^2 dx_N + 2\lambda \omega^2 \int_0^1 |v'|^2 dx_N + \lambda (\omega^2)^2 \int_0^1 |v|^2 dx_N \leq \]

\[\leq 2 \int_0^1 |h(\omega, x_N)| \left| \frac{\nu'(x_N, \omega)}{x_N} \right| dx_N + 2 |a||v'(\omega, 1)| \equiv I_1 + I_2. \]  

(6.35)

Due to the Hardy inequality we have the following estimates with an arbitrary small \( \varepsilon > 0 \) for the terms in the right hand side of the last inequality.

\[ I_1 \leq \varepsilon \int_0^1 \left| \frac{\nu'(x_N, \omega)}{x_N} \right|^2 dx_N + \frac{4}{\varepsilon} \int_0^1 |h(\omega, x_N)|^2 dx_N \leq \]

\[\leq \varepsilon C \int_0^1 |v''(\omega, x_N)|^2 dx_N + C_\varepsilon \left( |h(\omega, \cdot)|_{\gamma/2, 1} \right)^2, \]

(6.36)

and since

\[|v' (\omega, 1)| = \left| \int_0^1 v''(\omega, x_N) dx_N \right| \leq \left( \int_0^1 |v''(\omega, x_N)|^2 dx_N \right)^{1/2}, \]

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\[ I_2 \leq \varepsilon \int_0^1 |v''(\omega, x_N)|^2 dx_N + C_\varepsilon |a|^2. \]  

(6.37)

Substituting these estimates in (6.35), choosing sufficiently small \( \varepsilon \), and absorbing the terms with \( \varepsilon \) by the left hand side of (6.35), we obtain

\[ \int_0^1 |v''(\omega, x_N)|^2 dx_N + 2\lambda \omega^2 \int_0^1 |v'|^2 dx_N + \lambda (\omega^2)^2 \int_0^1 |v|^2 dx_N \leq C \left( \left( |h(\omega, \cdot)|_{\gamma/2,1}^{(\gamma)} \right)^2 + |a|^2 \right). \]  

Taking into account that \( v'(\omega, 0) = 0 \), \( v(\omega, 1) = 0 \), and making use of the Poincare inequality, we arrive at the estimate

\[ \int_0^1 |v|^2 dx_N \leq C \int_0^1 |v'|^2 dx_N \leq C \int_0^1 |v''|^2 \leq C \left( \left( |h(\omega, \cdot)|_{\gamma/2,1}^{(\gamma)} \right)^2 + |a|^2 \right). \]  

(6.39)

This is uniform in \( \lambda \) estimate for the \( L_2(I) \) - norm of the possible solution of (6.28). Now we obtain uniform in \( \lambda \) estimate for the \( C^{4+\gamma}_{2,\gamma/2}(I) \) - norm of the possible solution of (6.28). For this we just move all terms with \( \lambda \) to the right hand sides of relations (6.14), (6.17). Then making use of estimate (6.27) for the simplest problem and applying interpolation inequalities (2.20)-(2.27), we obtain

\[ |v(\omega, \cdot)|_{2,\gamma/2,1}^{(4+\gamma)} \leq C \left( |h(\omega, \cdot)|_{\gamma/2,1}^{(\gamma)} + a \right) \]

\[ \epsilon (1 + \omega^2)^{A_1} |v(\omega, \cdot)|_{2,\gamma/2,1}^{(4+\gamma)} + C \varepsilon^{-A_2} (1 + \omega^2)^{A_1} \left( \int_0^1 |v|^2 dx_N \right)^{\frac{1}{2}}, \]

where \( A_1 \) and \( A_2 \) some positive exponents. Substituting here instead \( \varepsilon \) the expression \( \varepsilon/(1 + \omega^2)^{A_1} \), making use of (6.39), and absorbing the terms with \( \varepsilon \) by the left hand side, we obtain finally

\[ |v(\omega, \cdot)|_{2,\gamma/2,1}^{(4+\gamma)} \leq C (1 + \omega^2)^A \left( |h(\omega, \cdot)|_{\gamma/2,1}^{(\gamma)} + a \right), \]

(6.40)

where \( A \) is some fixed positive exponent and constants \( C, A \) do not depend on \( \omega, \lambda \).

Thus, problem (6.28) has the unique solution for \( \lambda = 0 \) with estimate (6.27) and for \( \lambda \in (0, 1] \) the possible solution of this problem has uniform in \( \lambda \) a priori
estimate (6.40). According to the method of the extension along a parameter, this means that problem (6.28) has the unique solution for any \(\lambda \in [0,1]\), including \(\lambda = 1\), with the estimate (6.40). Therefore we infer that problem (6.10)-(6.13) has the unique solution \(v(x)\) for any \(\omega \in \mathbb{Z}^{N-1}\) and
\[
|v(\omega, \cdot)|_{2, \gamma/2, 1}^{(4+\gamma)} \leq C(1 + \omega^2)^{1/2} h(\omega, \cdot)|^{(\gamma)}_{\gamma/2, 1}.
\] (6.41)
From this estimate and from (6.9) we have also for any \(K > 0\)
\[
|v(\omega, \cdot)|_{2, \gamma/2, 1}^{(4+\gamma)} \leq C_K (1 + \omega^2)^{-K}.
\] (6.42)
From (6.41), (6.42) and from the way of constructing function \(v(x)\) it directly follows that the function \(u(x')\) from (6.7) gives a solution to problem (6.1)-(6.5). This solution is infinitely differentiable with respect to \(x\) and it is of the class \(C^{4+\gamma}_0(\mathcal{P})\) with respect to \(x\). Thus \(u(x', x_N) \in C^{4+\gamma}_0(\mathcal{P})\).

Let us turn now to the estimate (6.6). This estimate is obtained in completely standard way of the Schauder technique on the base of estimate (6.41). We multiply equation (6.1) by cut-off functions \(\eta_{\nu,0}(x)\) with the small supports in a neighbourhood of a point \(x^0 \in \mathcal{P}\) and obtain a simple model problems in whole space \(\mathbb{R}^N\) (for the inner points of \(\mathcal{P}\) or for the points with \(x_i = \pm \pi, i = 1, N - 1\) or in the half space (for points with \(x_N = 0\) or \(x_N = 1\)) for the function \(u(x)\eta_{\nu,0}(x)\). For points \(x^0\) with \(x_N^0 = 0\) we use estimate (5.114). Other points correspond to non-degenerate case \((x_N > \nu > 0)\) and for them we use well known results for elliptic problems - see, for example, [35]. To estimate emerging lower order terms we use interpolation inequalities (2.20)-(2.27), and the standard interpolation inequalities. This process is completely standard to nowdays and we omit it.

As a result for any \(f \in C^0_\gamma(\mathcal{P})\) we obtain the estimate with the lower order term
\[
|u|_{2, \gamma/2, 1}^{(4+\gamma)} \leq C|f|_{\gamma/2, \mathcal{P}}^{(\gamma)} + C|u|_{\mathcal{P}}^{(0)}.
\] (6.43)
is known also that if we have the uniqueness for problem (6.1)-(6.5) then the lower order term \(C|u|_{\mathcal{P}}^{(0)}\) can be omitted. The proof of this fact is by contradiction on the base of (6.43) and the fact of uniqueness and is standard. The proof can be found, for example, in [36] or in [37]. Thus it is enough to show the uniqueness of the solution to problem (6.1)-(6.5). So let \(u(x) \in C^{4+\gamma}_0(\mathcal{P})\) satisfy problem (6.1)-(6.5) with \(f \equiv 0\).

Multiply equation (6.1) by \(\Delta u(x)\) and integrate by parts over \(\mathcal{P}\). With the taking into account of the boundary conditions and \(u(x) \in C^{4+\gamma}_0(\mathcal{P})\), we obtain
\[
\int_{\mathcal{P}} u_N^2 (\nabla u)^2 dx - \int_{\mathcal{P} \cap \{x_N = 1\}} (\Delta u)^2 dx' = 0.
\]
Since both integrals in this equality have the same sign, we conclude that, \(\Delta u(x) \equiv \text{const}\) and \(\Delta u \equiv 0\) at \(x_N = 0\), that is \(\Delta u \equiv 0\) in \(\mathcal{P}\). Taking into account...
boundary conditions (6.2) and (6.4) we infer in standard way that \( u(x) \equiv 0 \) in \( \overline{P} \). This proves the uniqueness for the problem and thus we have the estimate

\[
|u|_{2, \gamma/2, \overline{P}}^{(4+\gamma)} \leq C|f|_{\gamma/2, \overline{P}}^{(\gamma)},
\]

where the constant \( C \) depends only on \( N \) and \( \gamma \).

Free ourselves now from the assumption that \( f(x', x_N) \) is of class \( C^\infty \) with respect to \( x' \). Let \( \omega_\varepsilon(x') \in C^\infty(\mathbb{R}^{N-1}) \) be a nonnegative mollifier kernel with the parameter \( \varepsilon \) and with the support in a set \( \{|x'| \leq C\varepsilon\} \). Denote

\[
f_\varepsilon(x) = \omega_\varepsilon(x') *_{x'} f(x', x_N) = \int_{\mathbb{R}^{N-1}} \omega_\varepsilon(x' - y') f(y', x_N) dy'.
\]

(6.44)

Evidently, \( f_\varepsilon(x) \) is \( 2\pi \)-periodic, \( f_\varepsilon(x) \in C^\infty \) with respect to \( x' \), and it is straightforward to check that

\[
|f_\varepsilon|_{\gamma/2, \overline{P}}^{(\gamma)} \leq |f|_{\gamma/2, \overline{P}}^{(\gamma)}, \quad \varepsilon > 0.
\]

(6.45)

From this estimate and from Lemma 5 and Lemma 6 it follows that at least for a subsequence

\[
|f - f_\varepsilon|_{\gamma/2, \overline{P}}^{(\gamma/2)} \to 0, \quad \varepsilon \to 0, \quad \gamma_1 \in (0, \gamma).
\]

(6.46)

By above, problem (6.1)-(6.5) with \( f_\varepsilon(x) \) instead of \( f(x) \) has the unique solution \( u_\varepsilon(x) \) with the estimate

\[
|u_\varepsilon|_{2, \gamma/2, \overline{P}}^{(4+\gamma)} \leq C|f_\varepsilon|_{\gamma/2, \overline{P}}^{(\gamma)} \leq C|f|_{\gamma/2, \overline{P}}^{(\gamma)}
\]

(6.47)

By (6.47) and by Lemma 6 there exists a function \( u(x) \in C^{4+\gamma}_{2, \gamma/2} (\overline{P}) \) with

\[
|u - u_\varepsilon|_{2, \gamma_1/2, \overline{P}}^{(4+\gamma_1)} \to 0, \quad \varepsilon \to 0, \quad \gamma_1 \in (0, \gamma)
\]

(6.48)

and

\[
|u|_{2, \gamma/2, \overline{P}}^{(4+\gamma)} \leq C|f|_{\gamma/2, \overline{P}}^{(\gamma)}.
\]

Now (6.46) and (6.48) permit us to go to the limit in problem (6.1)-(6.5) and infer that \( u(x) \) is the solution of this problem with estimate (6.6).

This completes the proof of the lemma. ■

Before to proceed to a parabolic problem we present some simple variant of well known Hardy’s inequality in \( P \). The difference is that the function in the inequality does not vanish at \( x_N = 1 \).
Lemma 29 Let complex valued \( u(x) \) be defined on \( P \) and let \( \Delta u \) and \( x_N \nabla \Delta u \) be square integrable over \( P \).

Then

\[
\int_P |\Delta u|^2 \, dx \leq C \left( \int_P x_N^2 |\nabla \Delta u|^2 \, dx + \int_{\mathbb{P}^{N-1}} |\Delta u(x',1)|^2 \, dx' \right). \tag{6.49}
\]

Proof. Let complex valued function \( v(x_N) \) be defined on \([0,1]\) and let \( v(x_N) \) and \( x_N v'(x_N) \) be square integrable on \([0,1]\). Consider the equality

\[
(x_N|v|^2)' = |v|^2 + x_N v' \nabla + x_N v \nabla \bar{v} = |v|^2 + 2x_N \text{Re} v' \nabla.
\]

Integrating this equality over \([0,1]\), we obtain

\[
\int_0^1 |v|^2 \, dx_N = |v|^2(1) - 2x_N \text{Re} v' \nabla dx_N.
\]

Estimating the last integral by the Cauchy inequality with \( \varepsilon \), we get

\[
\int_0^1 |v|^2 \, dx_N \leq |v|^2(1) + \varepsilon \int_0^1 |v|^2 \, dx_N + C \int_0^1 x_N^2 |v'|^2 \, dx_N
\]

and we conclude that with some absolute \( C \)

\[
\int_0^1 |v|^2 \, dx_N \leq C \left( |v|^2(1) + \frac{1}{x^2_N} \left| \frac{dv}{dx_N} \right|^2 \, dx_N \right). \tag{6.50}
\]

Now we substitute \( \Delta u(x',x_N) \) in eqref{6.47.02} instead of \( v(x_N) \) and integrate the result with respect to \( x' \) to obtain (6.49).

We consider now a parabolic problem of the kind (6.1)–(6.5). Let \( P_T = P \times [0,T] \), \( T > 0 \), and let \( P_\infty = P \times [0,\infty) \). Let a function \( f(x,t) \) be \( 2\pi \)-periodic with respect to the variables \( x_i, i = 1,N-1 \),

\[
f(x,t) \in C^2_{\gamma/2}(P_T) \text{ with } f(x,0) \equiv 0, \tag{6.51}
\]

and with the support in \( P_{T_1} \), \( 0 < T_1 < T \). Consider the following problem for a unknown \( 2\pi \)-periodic with respect to the variables \( x_i, i = 1,N-1 \), function \( u(x,t) \):

\[
\frac{\partial u}{\partial t} + \nabla(x_N^2 \nabla u) = f(x,t), \quad x \in P_T, \tag{6.52}
\]

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\[
\frac{\partial u}{\partial x_N}(x', 0, t) = 0, \quad x' \in P' \times [0, T], \tag{6.53}
\]

\[
\frac{\partial}{\partial x_N} \Delta u(x', 1, t) + \Delta u(x', 1, t) = 0, \quad x' \in P' \times [0, T], \tag{6.54}
\]

\[
u(x', 1, t) = 0, \quad x' \in P' \times [0, T], \tag{6.55}
\]

the initial condition

\[
u(x, 0) \equiv 0, \quad x \in P \tag{6.56}
\]

and the periodicity conditions

\[
\left. \frac{\partial^n u}{\partial x_i^n}(x, t) \right|_{x_i = -\pi} = \left. \frac{\partial^n u}{\partial x_i^n}(x, t) \right|_{x_i = \pi}, \quad n = 0, 1, 2, 3, \quad i = 1, N - 1. \tag{6.57}
\]

**Lemma 30** Problem (6.52) - (6.57) has the unique solution \( u(x, t) \in C^{\frac{\gamma}{2}, \gamma} \) \((P_T)\) and

\[
|u(x, t)|^{(\frac{1}{4} + \frac{1}{2})} \leq C_T (f(x, t)). \tag{6.58}
\]

**Proof.** Since the support of \( f(x, t) \) is included in \( P_T \), we can extend this function by the identical zero over \( T \) and have \( f(x, t) \in C^{\frac{\gamma}{2}, \gamma} \). We suppose first that \( f(x, t) \) is of the class \( C^\infty \) with respect to the variable \( t \) and has the property

\[
\left. \frac{\partial^n f(x, t)}{\partial t^n} \right|_{t=0} = 0, \quad n = 0, 1, 2, \ldots \tag{6.59}
\]

Denote for a complex number \( p \) with \( \text{Re} p > 0 \)

\[
v(x, p) \equiv \int_0^\infty u(x, t)e^{-pt}dt, \quad h(x, p) \equiv \int_0^\infty f(x, t)e^{-pt}dt- \tag{6.60}
\]

the Laplace transforms of the functions \( u(x, t) \) and \( f(x, t) \) respectively. Because of the properties of the function \( f(x, t) \) including (6.59) we have for the function \( h(x, p) \) for each \( p \) and for an arbitrary \( K > 0 \)

\[
|h(x, p)|^{(\frac{\gamma}{2}, \gamma)} \leq C_K (1 + |p|)^{-K}. \tag{6.61}
\]

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Making in problem (6.52)-(6.57) the Laplace transform (6.60) we arrive at the following elliptic problem with the parameter $p$

\[ Av + pKv \equiv \nabla(x_N^2 \nabla v) + pv = h(x, p), \quad x \in P, \quad (6.62) \]

\[ \frac{\partial v}{\partial x_i}(x', 0, p) = 0, \quad x' \in P', \quad (6.63) \]

\[ \frac{\partial}{\partial x_N} \Delta v(x', 1, p) + \Delta u(x', 1, p) = 0, \quad x' \in P', \quad (6.64) \]

\[ v(x', 1, p) = 0, \quad x' \in P', \quad (6.65) \]

and the periodicity conditions

\[ \frac{\partial^n v}{\partial x_i^n}(x, p) \bigg|_{x_i = -\pi} = \frac{\partial^n v}{\partial x_i^n}(x, p) \bigg|_{x_i = \pi}, \quad n = 0, 1, 2, 3, \quad i = 1, N - 1. \quad (6.66) \]

Denote by $\tilde{C}^{4+\gamma}_{2,\gamma/2}(\overline{P})$ the closed subspace of periodic functions from $C^{4+\gamma}_{2,\gamma/2}(\overline{P})$, defined by homogeneous boundary conditions (6.63)-(6.65). It was proved in Lemma 28 that the operator $Av \equiv \nabla(x_N^2 \nabla v)$ from (6.62) is a bounded linear operator $A : \tilde{C}^{4+\gamma}_{2,\gamma/2}(\overline{P}) \to \tilde{C}^{\gamma}_{\gamma/2}(\overline{P})$, where $\tilde{C}^{\gamma}_{\gamma/2}(\overline{P})$ is the space of periodic functions from $C^{\gamma}_{\gamma/2}(\overline{P})$. Moreover, the operator $pKv \equiv pv$ is evidently a compact operator from $\tilde{C}^{4+\gamma}_{2,\gamma/2}(\overline{P})$ to $\tilde{C}^{\gamma}_{\gamma/2}(\overline{P})$. Thus equation (6.62) is a Fredholm equation and it is uniquely solvable for the all right hand sides from $\tilde{C}^{\gamma}_{\gamma/2}(\overline{P})$ if and only if the kernel of the operator $A + pK$ consists from zero only. We will prove that for $\text{Re} \ p > 0$ the kernel of the operator $A + pK$ consists from zero only by obtaining an a-priory estimate for a possible solution $v(x, p)$ for (6.62) from $\tilde{C}^{4+\gamma}_{2,\gamma/2}(\overline{P})$. Similar to Lemma 28 we start with the estimate of $L_2(P)$-norm of $v(x, p)$.

Multiply equation (6.62) by $\Delta v(x, p)$ and integrate by parts over $P$ with the taking into account the boundary conditions. Considering only real part of the obtaining expression and changing it’s sign, we get

\[ \int_P x_N^2 |\nabla v|^2 \, dx + \int_{P \cap \{x_N = 1\}} |\Delta v(x', 1, p)|^2 \, dx' + \text{Re} \, p \int_P |\nabla v|^2 \, dx = -\text{Re} \, \int_P h \Delta v \, dx. \quad (6.67) \]

We estimate the right hand side by the Cauchy inequality with $\varepsilon$

\[ \left| \int_P h \Delta v \, dx \right| \leq \varepsilon \int_P |\Delta v|^2 \, dx + \frac{C}{\varepsilon} \int_P |h|^2 \, dx. \]
On the base of (6.49) we can absorb the term with \( \varepsilon \) by the left hand side of (6.67) and we get from this inequality

\[
\int_P |\nabla v|^2 \, dx \leq \frac{C}{\text{Re} \, p} \int_P |h|^2 \, dx \leq \frac{C}{\text{Re} \, p} \left( |h|_{\gamma/2, \mathcal{P}}^{(\gamma)} \right)^2 .
\]

Finally, since \( v(x', 1, p) \equiv 0 \), we obtain by the Poincare inequality

\[
\int_P |v|^2 \, dx \leq \frac{C}{\text{Re} \, p} \left( |h|_{\gamma/2, \mathcal{P}}^{(\gamma)} \right)^2 .
\] (6.68)

This inequality means that the operator \( A + pK \) has zero kernel for \( \text{Re} \, p > 0 \) and thus equation (6.62) has the unique solution \( v(x, p) \in \tilde{C}^{4+\gamma}_{\gamma/2}(\mathcal{P}) \) for any \( h(x, p) \in C^{\gamma}_{\gamma/2}(\mathcal{P}) \).

To obtain estimate for the \( \tilde{C}^{4+\gamma}_{\gamma/2}(\mathcal{P}) \)- norm of \( v(x, p) \) we proceed exactly as in Lemma 28. That is we move the term \( pv \) to the right hand side of (6.62), use estimate (6.6), interpolation inequalities (2.20)-(2.27), and estimate (6.68). As a result we obtain for \( \text{Re} \, p > 1 \)

\[
|v(x, p)|^{(4+\gamma)}_{\gamma/2, \mathcal{P}} \leq C(1 + |p|)^{4} |h(x, p)|^{(\gamma)}_{\gamma/2, \mathcal{P}},
\] (6.69)

where \( A \) is some positive exponent. Thus in view of (6.61) we have also for any \( K > 0 \)

\[
|v(x, p)|^{(4+\gamma)}_{\gamma/2, \mathcal{P}} \leq CK(1 + |p|)^{-K}.
\] (6.70)

The last estimate permits us to take the inverse Laplace transform \( u(x, t) \) from \( v(x, p) \) and thus to obtain the solution to problem (6.52)-(6.56). Note that in view of (6.71) and the properties of the inverse Laplace transform, initial condition (6.56) is also satisfied. The solution \( u(x, t) \in C^{4+\gamma}_{\gamma/2}(\mathcal{P}_\infty) \) and moreover this solution is infinitely differentiable in \( t \).

Estimate (6.58) is obtained in standard way by the Schauder technique on the base of estimate (6.103) for the problem in the half space, estimates for parabolic problems for non-degenerate equations (see, for example, [32]), and interpolations inequalities (2.20)-(2.27). In this way, due to inequalities (2.20)-(2.30), we first prove estimate (6.58) on a sufficiently small time interval \([0, T]\), which does not depend on \( f \). Then (and this is also standard way of reasonings - [32], [38]) we consider on the interval \([T/2, 3T/2]\) the function \( u(x, t) - u(x, T/2) \) with zero initial value at \( t = T/2 \) and repeat the estimates. In this way we obtain (6.58) on an arbitrary interval \([0, T]\) but with time dependent constant \( C_T \).

Thus we have proved that if \( f(x, t) \) is infinitely differentiable in \( t \) and if it satisfies condition (6.59), then problem (6.52)-(6.56) has the unique solution and estimate (6.58) is valid. Let now \( f(x, t) \) satisfy (6.51). Since \( f(x, 0) \equiv 0 \), we can extend \( f(x, t) \) by the identical zero in the domain \( P \times (-\infty, 0] \) with
the preserving of $C_{2/4}^\gamma(\mathcal{P}_T)$ - norm. Let $\omega_\varepsilon(t) \in C^\infty(R^1)$ be a nonnegative mollifier kernel with the parameter $\varepsilon$ and with the support in $[-\varepsilon, \varepsilon]$. We define

$$f_\varepsilon(x, t) \equiv \omega_\varepsilon(t) \ast_1 f(x, t - 2\varepsilon) =$$

$$= \int_{-\infty}^{\infty} \omega_\varepsilon(t - \tau)f(x, \tau - 2\varepsilon)d\tau = \int_{-\infty}^{\infty} \omega_\varepsilon(\tau)f(x, t - \tau - 2\varepsilon)d\tau. \quad (6.71)$$

It can be checked directly that the functions $f_\varepsilon(x, t)$ posses the properties:

$$|f_\varepsilon|^{(\gamma, \gamma/4)}_{\gamma/2, \mathcal{P}_T} \leq |f|^{(\gamma, \gamma/4)}_{\gamma/2, \mathcal{P}_T}, \quad |f - f_\varepsilon|^{(\gamma_1, \gamma_{1/4})}_{\gamma/2, \mathcal{P}_T} \rightarrow 0, \varepsilon \rightarrow 0, \gamma_1 \in (0, \gamma), \quad (6.72)$$

$$\left. \frac{\partial^n f_\varepsilon(x, t)}{\partial t^n} \right|_{t=0} \equiv 0, n = 0, 1, 2, \ldots. \quad (6.73)$$

By the proved above, problem (6.52)-(6.56) with $f_\varepsilon(x, t)$ instead of $f(x, t)$ has the unique solution $u_\varepsilon(x, t)$ and by (6.58), (6.72)

$$|u_\varepsilon(x, t)|^{(\gamma, \gamma/4)}_{\gamma/2, \mathcal{P}_T} \leq C_T|f_\varepsilon|^{(\gamma, \gamma/4)}_{\gamma/2, \mathcal{P}_T} \leq C_T|f|^{(\gamma, \gamma/4)}_{\gamma/2, \mathcal{P}_T}, \quad (6.73)$$

$$|u_{\varepsilon_1} - u_{\varepsilon_2}|^{(\gamma, \gamma_{1/4})}_{\gamma/2, \mathcal{P}_T} \rightarrow 0, \quad \varepsilon_1, \varepsilon_2 \rightarrow 0. \quad (6.74)$$

From (6.72)-(6.74) on the base of Lemma 6 it follows that $u_\varepsilon(x, t)$ converges (at least for a subsequence) as $\varepsilon \rightarrow 0$ to the solution $u(x, t)$ of problem (6.52)-(6.56) and for this solution estimate (6.58) is valid.

This completes the proof of the lemma.

Let a function $f(x, t) \in C_{\gamma/2}^\gamma(\mathcal{P}_T)$ and periodic in $x'$ and let at $(x_N = 1, t = 0)$

$$f(x', 1, 0) \equiv 0, \quad x \in \mathcal{P}. \quad (6.75)$$

We extend $f(x, t)$ over $\{x_N = 1\}$ by the constant with respect to $x_N$, and over $\{x_N = 0\}$ and $\{t = 0\}$ in the even way by $(x_N \in [0, 1], t \in [0, T])$ up to the function $\bar{f}$

$$\bar{f}(x', -x_N, t) = f(x', x_N, t), \quad \bar{f}(x', 1 + x_N, t) = f(x', 1, t), \quad \bar{f}(x, -t) = f(x, t). \quad (6.76)$$

Denote $\bar{\mathcal{P}}_T = \{ (x, t) : |x_i| \leq \pi, i = 1, \mathcal{N} - 1, 0 \leq x_N \leq 2, 0 \leq t \leq T \}$. From the definition of $C_{\gamma/2}^\gamma(\mathcal{P}_T)$ it directly follows that
\[
|\tilde{f}^{(\gamma, \gamma/4)}_{\gamma/2, \mathbb{P}_T}| \leq C|f^{(\gamma, \gamma/4)}_{\gamma/2, \mathbb{P}_T}|. 
\] (6.77)

Let \( \omega(x) \in C^\infty(R^N) \) with the support in \( \{|x| \leq 1\} \) and the unit integral, \( \omega(x) \geq 0 \). Let also \( \chi(t) \in C^\infty(R^1) \) with the support in \( \{|t| \leq 1\} \) and the unit integral, \( \chi(t) \geq 0 \). Denote the mollifier kernels \( \omega_\varepsilon(x) \equiv \varepsilon^{-N}\omega(x/\varepsilon) \), \( \chi_\varepsilon(t) = \varepsilon^{-1}\chi(t/\varepsilon) \), \( \varepsilon \in (0, 1/2) \). We consider in \( \mathbb{P}_T \) the mollified function

\[
\tilde{f}_\varepsilon(x, t) = \chi_\varepsilon(t) * \big( \omega_\varepsilon(x) * \tilde{f}(x, t) \big) =
\]

\[
= \int_{-\infty}^{\infty} d\tau \chi_\varepsilon(t - \tau) \int_{\mathbb{R}^N} \omega_\varepsilon(x - \xi) \tilde{f}(\xi, \tau) d\xi = \int_{-\infty}^{\infty} d\tau \chi_\varepsilon(t - \tau) \int_{\mathbb{R}^N} \omega_\varepsilon(\xi) \tilde{f}(x - \xi, \tau) d\xi.
\] (6.78)

And we define

\[
f_\varepsilon(x, t) \equiv \tilde{f}_\varepsilon(x', x_N, t) - \tilde{f}_\varepsilon(x', 1, 0). \] (6.79)

**Lemma 31** For the function \( f_\varepsilon(x, t) \in C^\infty(\mathbb{P}_T) \) in (6.78) we have uniformly in \( f \) and \( \varepsilon \)

\[
|f_\varepsilon|^{(\gamma, \gamma/4)}_{\gamma/2, \mathbb{P}_T} \leq C|f^{(\gamma, \gamma/4)}_{\gamma/2, \mathbb{P}_T}| 
\] (6.80)

and, at least for a subsequence,

\[
f_\varepsilon(x', 1, 0) \equiv 0, x' \in \mathbb{P}, |f_\varepsilon|^{(\gamma_1, \gamma_1/4)}_{\gamma_1/2, \mathbb{P}_T} \rightarrow 0, \varepsilon \rightarrow 0, \gamma_1 \in (0, \gamma). \] (6.81)

**Proof.**

From the properties of the function \( f(x, t) \) and from the definition of mollifiers it follows that for the proof of (6.80) it is enough to prove the estimate

\[
\langle k_\varepsilon(x, t) \rangle^{(\gamma)}_{\gamma/2, x, \mathbb{P}_T} \leq C|f^{(\gamma, \gamma/4)}_{\gamma/2, \mathbb{P}_T}|, 
\] (6.82)

where

\[
k_\varepsilon(x, t) \equiv \int_{\mathbb{R}^N} \omega_\varepsilon(\xi) \tilde{f}(x - \xi, t) d\xi.
\] (6.83)

Let \( x, \overline{x} \in P, x_N \leq \overline{x}_N \). Consider two cases. Let first \( x_N > 2\varepsilon \). Then, since in (6.83) in fact \( x_N/2 < x_N - \overline{x}_N < 3x_N/2 \),

\[
x_N^{\gamma/2} \frac{|k_\varepsilon(x, t) - k_\varepsilon(\overline{x}, t)|}{|x - \overline{x}|^\gamma} \leq
\]

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\[
\leq \int_{\mathbb{R}^N} \omega_\varepsilon(\xi) \left[ \frac{x_N^{\gamma/2}}{(x_N - \xi_N)^{\gamma/2}} \right] (x_N - \xi_N)^{\gamma/2} \left( \frac{|f(x - \xi, t) - \tilde{f}(\mathbf{x} - \xi, t)|}{|x - \mathbf{x}|^\gamma} \right) d\xi \leq (6.84)
\]

\[
\leq C |f|^{(\gamma, \gamma/4)}_{\gamma/2, \mathcal{P}_T} \leq C |f|^{(\gamma, \gamma/4)}_{\gamma/2, \mathcal{P}_T}.
\]

Let now \( x_N < 2\varepsilon \). Then

\[
x_N^{\gamma/2} \left| k_\varepsilon(x, t) - k_\varepsilon(\mathbf{x}, t) \right| \leq C \varepsilon^{\gamma/2} |f|^{(\gamma, \gamma/4)}_{\gamma/2, \mathcal{P}_T}.
\]

\[
\leq \sum_{j=1}^{3} \int_{B_j} \omega_\varepsilon(\xi) x_N^{\gamma/2} \left| \frac{f(x - \xi, t) - \tilde{f}(\mathbf{x} - \xi, t)}{|x - \mathbf{x}|^\gamma} \right| d\xi = I_1 + I_2 + I_2,
\]

where

\[
B_1 = \{ |\xi| \leq \varepsilon : x_N - \xi_N \geq x_N - \xi \geq 0 \},
\]

\[
B_2 = \{ |\xi| \leq \varepsilon : x_N - \xi_N < 0, x_N - \xi \geq 0 \},
\]

\[
B_3 = \{ |\xi| \leq \varepsilon : x_N - \xi_N < x_N - \xi < 0 \}.
\]

For the set \( B_1 \) we have \( (x_N < 2\varepsilon) \)

\[
I_1 \leq C \varepsilon^{\gamma/2} \int_{B_1} \omega_\varepsilon(\xi) (x_N - \xi_N)^{\gamma/2} \left[ (x_N - \xi_N)^{\gamma/2} \left( \frac{|f(x - \xi, t) - f(\mathbf{x} - \xi, t)|}{|x - \mathbf{x}|^\gamma} \right) \right] d\xi \leq
\]

\[
\leq C \varepsilon^{\gamma/2} |f|^{(\gamma, \gamma/4)}_{\gamma/2, \mathcal{P}_T} \left[ C \varepsilon^{-N} \int_{|\xi| \leq \varepsilon} (x_N - \xi_N)^{-\gamma/2} d\xi \right] \leq
\]

\[
\leq C \varepsilon^{\gamma/2} |f|^{(\gamma, \gamma/4)}_{\gamma/2, \mathcal{P}_T}, C \varepsilon^{-N + (N - \gamma/2)} = C |f|^{(\gamma, \gamma/4)}_{\gamma/2, \mathcal{P}_T}.
\] (6.85)

On the set \( B_3 \) we have

\[
\tilde{f}(x - \xi, t) = f((x - \xi)^*, t), \tilde{f}(\mathbf{x} - \xi, t) = f((\mathbf{x} - \xi)^*, t),
\]

where \((x - \xi)^* = (x' - \xi, (x_N - \xi_N)), (\mathbf{x} - \xi)^* = (\mathbf{x}' - \xi, - (\mathbf{x}_N - \xi_N))\). Since a shift and a reflection are isometries, \(|(x - \xi)^* - (\mathbf{x} - \xi)^*| = |x - \mathbf{x}|\). Therefore the integral \( I_3 \) is estimated exactly as it was done for \( I_1 \) and we have

\[
I_3 \leq C |f|^{(\gamma, \gamma/4)}_{\gamma/2, \mathcal{P}_T}.
\] (6.86)

On the set \( B_3 \)
\[ \bar{f}(x - \xi, t) = f((x - \xi)^*, t), \bar{f}(\bar{x} - \xi, t) = f(\bar{x} - \xi, t). \]

According to the triangle inequality we have after the shift and reflection
\[ |(x - \xi)^* - (\bar{x} - \xi)| \leq |x - \bar{x}|. \]

Therefore, as above, denoting \( |x_N - \xi_N| = \min\{ |x_N - \xi_N|, |\bar{x}_N - \xi_N| \} \),
\[ I_2 \leq C \varepsilon^{\gamma/2} \int_{B_2} \omega_{\xi}(\xi) |x_N - \xi_N|^{-\gamma/2} \left[ |x_N - \xi_N|^{\gamma/2} \frac{|f((x - \xi)^*, t) - f(\bar{x} - \xi, t)|}{|(x - \xi)^* - (\bar{x} - \xi)|^{\gamma/2}} \right] d\xi \leq \]
\[ \leq C \varepsilon^{\gamma/2 - N} \int_{|\xi| \leq \varepsilon} |x_N - \xi_N|^{-\gamma/2} d\xi \leq \]
\[ \leq C \varepsilon^{\gamma/2 - N} \int_{|\eta| \leq 4\varepsilon} |\eta|^{-\gamma/2} d\eta = C |f|^{(\gamma, \gamma/4)}_{\gamma/2, P_T}. \] (6.87)

Thus we have in the case \( x_N < 2\varepsilon \)
\[ \tilde{\gamma}^{\gamma/2} \frac{|k_\gamma(x, t) - k_\gamma(\bar{x}, t)|}{|x - \bar{x}|^{\gamma}} \leq C |f|^{(\gamma, \gamma/4)}_{\gamma/2, P_T}. \] (6.88)

Estimate (6.82) follows now from (6.84) and (6.88), thus (6.80) is proved. Relations (6.81) follow now from (6.80) and from (6.75) by construction of \( f_\varepsilon(x, t) \).

This finishes the proof of the lemma.

Consider now the following nonhomogeneous problem for an unknown \( 2\pi \)-periodic with respect to the variables \( x_i, i = 1, N - 1 \), function \( u(x, t) \):
\[ \frac{\partial u}{\partial t} + \nabla (x_N^3 \nabla u) = f(x, t), \quad x \in P_T, \] (6.89)
\[ \frac{\partial u}{\partial x_N}(x', 0, t) = g(x', t), \quad x' \in P_T' \equiv P' \times [0, T], \] (6.90)

\[ \frac{\partial}{\partial x_N} \Delta u(x', 1, t) + \Delta u(x', 1, t) = 0, \quad x' \in P_T' \equiv P' \times [0, T], \] (6.91)
\[ u(x', 1, t) = 0, \quad x' \in P_T' \equiv P' \times [0, T], \] (6.92)
the initial condition
\[ u(x, 0) \equiv \psi(x), \quad x \in \bar{P} \] (6.93)
and the periodicity conditions
\[ \frac{\partial^n u}{\partial x_i^n}(x,t) \bigg|_{x_i=-\pi} = \frac{\partial^n u}{\partial x_i^n}(x,t) \bigg|_{x_i=\pi}, n = 0, 1, 2, 3, \quad i = 1, N-1. \] (6.94)

Here \( f, g, \psi \) are given \( 2\pi \)-periodic with respect to the variables \( x_i, i = 1, N-1 \),

functions with
\[ f(x,t) \in C^{{4+\gamma/4}}_{2,\gamma/2}(\mathbb{P}_T), \quad g(x',t) \in C^{1+\gamma/2,1/2+\gamma/4}_{x',t}(\mathbb{P}'), \quad \psi(x) \in C^{{4+\gamma}}_{2,\gamma/2}(\mathbb{P}). \] (6.95)

Without loss of generality we can suppose that the functions \( f(x,t) \) and \( g(x',t) \) vanish for \( t \geq T - \delta \) with some small \( \delta \). In the general case we can extend \( f(x,t) \) and \( g(x',t) \) over \( t = T \) with the preserving of their classes and then we can cut them off to obtain finite in \( t \) functions on a new interval \([0, T'], T' \in (T, 2T)\).

The way of extending over \( t = T \) is described in, for example, [38]. We assume also the compatibility condition at \( t = 0, x_N = 0 \)
\[ \frac{\partial \psi(x',x_N)}{\partial x_N} \bigg|_{x_N=0} = g(x',0), \quad x' \in \mathbb{P}' \] (6.96)
and at \( t = 0, x_N = 1 \)
\[ \left( \frac{\partial \Delta \psi(x',x_N)}{\partial x_N} + \Delta \psi(x',x_N) \right) \bigg|_{x_N=1} = 0, \] (6.97)
\[ \psi(x',1) = 0, \quad f(x',1,0) - \nabla(x_N^2 \nabla \psi)(x',1,0) = 0, \quad x' \in \mathbb{P}'. \]

**Proposition 32** Under conditions \( (6.95)-(6.97) \) problem \( (6.89)-(6.94) \) has the unique periodic solution \( u(x,t) \in C^{{4+\gamma}}_{2,\gamma/2}(\mathbb{P}_T) \) and
\[ |u(x,t)|_{2,\gamma/2,\mathbb{P}_T} \leq C_T \left( |f|_{2,\gamma/2,\mathbb{P}_T}^{4+\gamma} + |g|_{C^{1+\gamma/2,1/2+\gamma/4}_{x',t}(\mathbb{P}')}^{4+\gamma} + |\psi|_{2,\gamma/2,\mathbb{P}}^{4+\gamma} \right). \] (6.98)

**Proof.** The proof is just by reduction to the conditions of Lemma [40]. First, the change of the unknown \( u = u_1 + \psi(x) \) reduces the problem to the case \( \psi \equiv 0 \). For the function \( u_1 \) we have problem \( (6.89)-(6.94) \) with the right hand side in \( (6.89) \) \( f \to f_1 \)
\[ f_1(x,t) = f(x,t) - \nabla(x_N^2 \nabla \psi(x)) \in C^{4+\gamma/4}_{\gamma/2}(\mathbb{P}_T), \]
\[ f_1(x',1,0) = 0, \] (6.99)
with the boundary condition in \( (6.90) \) \( g \to g_1 \).
\( g_1(x', t) = g(x', t) - \frac{\partial \psi(x', 0)}{\partial x_N}, \quad g_1(x', 0) \equiv 0, \) \hspace{1cm} (6.100)

and with the same boundary conditions \( (6.91), (6.92) \) (because of conditions \( (6.97) \)). Note that

\[
|f_1|^{(\gamma, \gamma/4)}_{\gamma/2, P_T} \leq C \left( |f_1|^{(\gamma, \gamma/4)}_{\gamma/2, P_T} + |\psi|^{(4+\gamma)}_{2, \gamma/2, P_T} \right),
\]

\[
|g_1|^{1+\gamma/2, 1/2+\gamma/4}_{x', t}(P_T') \leq \left( |g|^{1+\gamma/2, 1/2+\gamma/4}_{x', t}(P_T') + |\psi|^{(4+\gamma)}_{2, \gamma/2, P_T} \right). \hspace{1cm} (6.101)
\]

After this we can obtain estimate \( (6.98) \) for a possible solution by the standard Schauder’s technique as it was described in lemmas \( 28, 30 \).

Now we apply several steps of smoothings and changes of unknown to reduce the problem to the conditions of Lemma \( 30 \) and to prove the existence of the solution. Let \( f_{1\varepsilon}(x, t) \in C^\infty(P_T) \) be constructed on the base of \( f_1(x, t) \) as in Lemma \( 31 \). At least for a subsequence we have

\[
|f_{1\varepsilon}(x, t) - f_1(x, t)|^{(\gamma_1, \gamma_1/4)}_{\gamma_1/2, P_T} \to 0, \varepsilon \to 0, \quad \gamma_1 \in (0, \gamma).
\]

and also at \( (x_N = 1, t = 0) \)

\[
f_{1\varepsilon}(x', 1, 0) \equiv 0 \hspace{1cm} (6.102)
\]

so that the compatibility conditions of the kind \( (6.97) \) at \( (x_N = 1, t = 0) \) are satisfied. On the base of estimate \( (6.98) \) we will prove the existence of the solution for \( f_1(x, t) \) if we have the solution for \( f_{1\varepsilon}(x, t) \), as it was done in lemmas \( 28, 30 \). So we can assume that \( f_1(x, t) \in C^\infty(P_T) \). Make now the change of the unknown \( u_1 = u_2 + tf_1(x, 0) \). Then we obtain the problem for \( u_2 \), where \( f_1(x, t) \) in \( (6.89) \) is replaced by

\[
f_2(x, t) \equiv f_1(x, t) - f_1(x, 0) - t\nabla(x_N^2 \nabla \Delta f_1(x, 0)).
\]

The boundary conditions become

\[
\frac{\partial u_2}{\partial x_N}(x', 0, t) = g_2(x', t), \quad x' \in P'_T \equiv P' \times [0, T], \hspace{1cm} (6.103)
\]

\[
\frac{\partial}{\partial x_N} \Delta u_2(x', 1, t) + \Delta u_2(x', 1, t) = h_2(x', t), \quad x' \in P'_T \equiv P' \times [0, T], \hspace{1cm} (6.104)
\]

\[
u_2(x', 1, t) = k_2(x', t), \quad x' \in P'_T \equiv P' \times [0, T], \hspace{1cm} (6.105)
\]

where
\[g_2(x', t) \equiv g_1(x', t) - t \frac{\partial f_1(x, 0)}{\partial x_N} \bigg|_{x_N=0}, \quad (6.106)\]

\[h_2(x', t) \equiv -t \left( \frac{\partial}{\partial x_N} \Delta f_1(x, 0) + \Delta f_1(x, 0) \right) \bigg|_{x_N=1}, \quad (6.107)\]

\[k_2(x', t) \equiv -tf_1(x, 0) \bigg|_{x_N=1} \equiv 0. \quad (6.108)\]

We also have zero initial condition

\[u_2(x, 0) \equiv 0, \quad x \in \overline{P}. \quad (6.109)\]

From (6.99), (6.100), and (6.102) it follows that the compatibility conditions up to the first order at \((x_N = 0, t = 0)\) and at \((x_N = 1, t = 0)\) are satisfied. In particular

\[f_2(x, 0) \equiv 0, x \in \overline{P}, \quad g_2(x', 0) \equiv 0, \quad h_2(x', 0) \equiv 0, x' \in \overline{P'} \quad (6.110)\]

Besides, we have

\[f_2(x, t) \in C^\infty(\overline{P_T}), \quad g_2(x', t) \in C^{1+\gamma/2, 1/2+\gamma/4} (P'_T), \quad h_2(x', t) \in C^{1+\gamma, 1+\gamma/4} (P'_T).\]

Since \(g_1(x', 0) \equiv 0, h_2(x', 0)\) we can extend these functions to the domain \(P' \times \{t \leq 0\}\) by the identical zero with the preservation of the classes \(C^{1+\gamma/2, 1/2+\gamma/4} (P'_T)\) and \(C^{1+\gamma, 1+\gamma/4} (P'_T)\) correspondingly.

Then we define the shifted and smoothed functions

\[g_{2\varepsilon}(x', t) \equiv \theta_{\varepsilon}(x', t) * g_2(x', t - 2\varepsilon) = \]

\[= \int_{\mathbb{R}^{n-1} \times \mathbb{R}} \theta_{\varepsilon}(x' - \xi', t - \tau) g_2(\xi', \tau - 2\varepsilon) d\xi' d\tau,\]

\[h_{2\varepsilon}(x', t) \equiv \theta_{\varepsilon}(x', t) * h_2(x', t - 2\varepsilon) = \]

\[= \int_{\mathbb{R}^{n-1} \times \mathbb{R}} \theta_{\varepsilon}(x' - \xi', t - \tau) h_2(\xi', \tau - 2\varepsilon) d\xi' d\tau,\]

where \(\theta_{\varepsilon}(x', t)\) is a mollifier kernel with the support in \(\{|x'| \leq \varepsilon, |t| \leq \varepsilon\}\). The functions \(g_{2\varepsilon}(x', t)\) and \(h_{2\varepsilon}(x', t)\) have properties

\[|g_{2\varepsilon}(x', t) - g_2(x', t)| \rightarrow 0, \varepsilon \rightarrow 0, \quad \gamma_1 \in (0, \gamma),\]
\[ |h_{2\varepsilon}(x',t) - h_2(x',t)|_{C^{1,\gamma}(\mathcal{P}_T')} \to 0, \varepsilon \to 0, \quad \gamma_1 \in (0, \gamma) \]

And besides,
\[ g_{2\varepsilon}(x',t) \equiv 0, \quad h_{2\varepsilon}(x',t) \equiv 0, \quad 0 \leq t \leq \varepsilon. \quad (6.111) \]

On the base of estimate \[6.98\] we will prove the existence of the solution for \( g_2(x',t) \) and \( h_2(x',t) \) if we have the solution for \( g_2(x',t) \) and \( h_2(x',t) \), as it was done in lemmas \[28, 30\]. So we can assume that \( g_2(x',t) \in C^\infty(\mathcal{P}_T' \mathcal{T}), h_2(x',t) \in C^\infty(\mathcal{P}_T' \mathcal{T}) \), and condition \[6.111\] is satisfied for these functions.

Denote
\[
G(x,t) \equiv x_N g_2(x',t) \eta_0(x_N),
\]
\[
H(x,t) \equiv \frac{(x_N - 1)^2}{2} h_2(x',t) \eta_0(x_N),
\]
where \( \eta_0(x_N), \eta_1(x_N) \in C^\infty([0,1]), \eta_0(x_N) \equiv 1 \) on \([0,1/4], \eta_0(x_N) \equiv 0 \) on \([3/4,1], \eta_1(x_N) \equiv 1 \) on \([3/4,1], \eta_1(x_N) \equiv 0 \) on \([0,1/4] \). Now it can be checked directly that the change of the unknown \( u_2(x,t) = u_3(x,t) + G(x,t) + H(x,t) \) reduces the problem to the problem for the function \( u_3(x,t) \) with exactly the conditions of Lemma \[30\].

This finishes the proof of the proposition.

6.2 A model problem with the Dirichlet condition at \([x_N = 0] \).

Let \( P, P', P_T, P_T' \) be defined in the previous subsection. Consider the following problem for the unknown \( 2\pi \)-periodic with respect to the variables \( x_i, i = 1,N-1 \), function \( u(x,t) \):
\[
L_{x,t}u \equiv \frac{\partial u}{\partial t} + \nabla(x_N^2 \nabla u) = f(x,t), \quad (x,t) \in P_T,
\]
\[
u(x',0,t) = \varphi(x',t), \quad (x',t) \in P_{T}', \quad (6.113)
\]
\[
u(x',1,t) = 0, \quad \frac{\partial^2 u(x',1,t)}{\partial x_N^2} = 0, \quad (x',t) \in P_{T}', \quad (6.114)
\]
\[
u(x,0) = \psi(x), \quad x \in \mathcal{P}, \quad (6.115)
\]
and the periodicity conditions
\[
\frac{\partial^n u}{\partial x_i^n}(x, t) \bigg|_{x_i = -\pi} = \frac{\partial^n u}{\partial x_i^n}(x, t) \bigg|_{x_i = \pi}, \ n = 0, 1, 2, 3, \ i = 1, N - 1. \quad (6.116)
\]

We suppose that
\[
f(x, t) \in C^{\gamma/4}_{\gamma/2}(P_T), \varphi(x', t) \in C^{2+\gamma/2,1+\gamma/4}_{x',t}(P'_T), \psi(x) \in C^{4+\gamma}_{2,\gamma/2}(P) \quad (6.117)
\]
and all these functions are \(2\pi\)-periodic in each variable \(x_i, i = 1, N - 1\). We suppose also that the given functions satisfy the following compatibility conditions at \((t = 0, x_N = 0)\) and \((t = 0, x_N = 1)\)
\[
\varphi(x', 0) = \psi(x', 0), \frac{\partial \varphi}{\partial t}(x', 0) = -\left[\nabla(x_N^2 \nabla \Delta \psi)\right]_{x_N = 0} + f(x', 0, 0), x' \in P', \quad (6.118)
\]
\[
\psi(x', 1) \equiv 0, -\left[\nabla(x_N^2 \nabla \Delta \psi)\right]_{x_N = 1} + f(x', 1, 0) \equiv 0, \frac{\partial^2 \psi(x', 1)}{\partial x_N^2} \equiv 0, x' \in P'. \quad (6.119)
\]

**Proposition 33** Under conditions \((6.117)-(6.119)\) problem \((6.112)-(6.116)\) has the unique periodic solution \(u(x, t) \in C^{4+\gamma}_{2,\gamma/2}(P_T)\)

\[
|u(x, t)|_{2,\gamma/2(P_T)}^{(4+\gamma,\frac{\gamma}{4})} \leq C_T \left( |f|_{\gamma/2(P_T)}^{(\gamma,\gamma/4)} + |\varphi|_{x',t}^{(2+\gamma/2,1+\gamma/4)(P'_T)} + |\psi|_{2,\gamma/2(P)}^{(4+\gamma)} \right). \quad (6.120)
\]

**Proof.**

We give only outline of the proof because it is very similar (and even simpler) to the proof of Proposition 32. We emphasize only the principal moment of obtaining the analog of estimates \((6.39)\), \((6.68)\). Note that in the case of the Dirichlet condition \((6.113)\) we need not to consider auxiliary elliptic problem.

First of all, on the base of Proposition 32 we can consider the auxiliary problem with the Neumann condition at \(x_N = 0\) and with the given initial datum \(\psi(x)\) and with the right hand side \(f(x, 0)\) in equation \((6.112)\). This reduces the problem to the case \(\psi(x) \equiv 0, f(x, 0) \equiv 0\). At the same time the boundary condition at \(x_N = 0\) is reduced to the case \(\varphi(x', 0) \equiv \varphi_2(x', 0) \equiv 0\).

And the same is applied to the boundary conditions at \(x_N = 1\). After this by the standard Schauder technique we obtain estimate \((6.120)\) as it was done in Proposition 32. We can also reduce the problem to the zero boundary conditions on the base on well known results on extensions of functions from standard Hölder classes and on the base of Proposition 11.

The further aim is to prove the existence of smooth solution for the smoothed right hand side \(f(x, t)\) in \((6.112)\). For this we make in problem \((6.112)-(6.116)\)
the Laplace transform and then represent as the Fourier series as it was done in
the previous subsection. Thus we denote
\[
\tilde{u}(x, p) = \int_0^\infty e^{-pt} u(x, t) dt, \quad \tilde{f}(x, p) = \int_0^\infty e^{-pt} f(x, t) dt,
\]
\[
\tilde{u}(x, p) = \tilde{u}(x', x_N, p) = \sum_{\omega \in \mathbb{Z}} v(\omega, p, x_N) e^{-i\omega x'},
\]
\[
\tilde{f}(x, p) = \tilde{f}(x', x_N, p) = \sum_{\omega \in \mathbb{Z}} h(\omega, p, x_N) e^{-i\omega x'},
\]
where
\[
|h(\omega, p, x_N)| \leq C_K (1 + \omega^2 + |p|^2)^{-K}, \quad K > 0.
\]
For the unknown function \( v(\omega, p, x_N) \) the original problem become the following
boundary value problem for an ordinary differential equation with the parameters \( p \) and \( \omega \)
\[
(x_N^2 v''')' + pv - 2\omega^2 x_N^2 v'' - 2\omega^2 x_N v' + (\omega^2)^2 x_N^2 v = h(\omega, p, x_N), x_N \in I = [0, 1],
\]
(6.121)
\[
v(\omega, p, 0) = 0,
\]
(6.122)
\[
v(\omega, p, 1) = 0,
\]
(6.123)
\[
v''(\omega, p, 1) = 0.
\]
(6.124)
As before, we define \( \tilde{C}^{4+\gamma}_{2,\gamma/2}(I) \) as the closed subspace of \( C^{4+\gamma}_{2,\gamma/2}(I) \) with boundary conditions (6.122) - (6.124). We also consider instead of (6.121) the equation with the parameter \( \lambda \) \in [0, 1]
\[
(L_0 + \lambda T)v \equiv (x_N^2 v''')' + \]
\[
+ \lambda pv - \lambda 2\omega^2 x_N^2 v'' - \lambda 2\omega^2 x_N v' + (\omega^2)^2 x_N^2 v = h(\omega, p, x_N), x_N \in I. \quad (6.125)
\]
It can be checked directly, as it was done in the previous subsection, that for \( \lambda = 0 \) the operator \( L_0 v \equiv (x_N^2 v''')' \) has a bounded inverse operator \( L_0^{-1} : C_{\gamma/2}(I) \to \tilde{C}^{4+\gamma}_{2,\gamma/2}(I) \). Thus, it is enough to obtain a uniform in \( \lambda \) estimate for possible solution of (6.125) in the space \( \tilde{C}^{4+\gamma}_{2,\gamma/2}(I) \). This is also done completely analogous to the previous subsection. The key ingredient of such estimate is the uniform estimate of \( L_2(I)-\)norm of the solution \( v(\omega, p, x_N) \), an analog of estimates (6.39), (6.68). We now demonstrate this estimate.
Since \( v(\omega, p, 0) = 0 \), we can multiply (6.125) by \( \overline{\varphi}/x_N \) and integrate by parts over \( I \). We consider each term in (6.125) separately. We have

\[
J_1 \equiv \int_0^1 \left( x_N^2 v''' \right) \frac{\overline{\varphi}}{x_N} dx_N = - \int_0^1 x_N v''' \overline{\varphi} dx_N + \int_0^1 v''' \overline{\varphi} dx_N =
\]

\[
= \int_0^1 x_N v'' \varphi' dx_N + \int_0^1 v' \varphi' dx_N - \int_0^1 v'' \overline{\varphi} dx_N = \int_0^1 x_N |v''|^2 dx_N,
\]

\[
J_2 \equiv \int_0^1 \lambda p v \varphi dx_N = \lambda p \int_0^1 \left| \frac{v'}{x_N} \right|^2 dx_N,
\]

\[
J_3 \equiv - \lambda (\omega^2)^2 \int_0^1 x_N v'' \frac{\overline{\varphi}}{x_N} dx_N = \lambda (\omega^2)^2 \int_0^1 x_N |v'|^2 dx_N + \lambda 2\omega^2 \int_0^1 v' \varphi dx_N,
\]

\[
J_4 \equiv - \lambda (\omega^2)^2 \int_0^1 x_N v' \frac{\overline{\varphi}}{x_N} dx_N = - \lambda (\omega^2)^2 \int_0^1 v' \varphi dx_N,
\]

\[
J_5 \equiv \lambda (\omega^2)^2 \int_0^1 x_N v \frac{\overline{\varphi}}{x_N} dx_N = \lambda (\omega^2)^2 \int_0^1 x_N |v|^2 dx_N,
\]

\[
J_6 \equiv \int_0^1 h(\omega, p, x_N) \frac{\overline{\varphi}}{x_N} dx_N.
\]

Since \( \sum_{j=1}^5 J_j = J_6 \), adding up the above integrals and taking the real part, we obtain the relation

\[
\int_0^1 x_N |v''|^2 dx_N + \lambda 2\omega^2 \int_0^1 x_N |v'|^2 dx_N + \lambda \int_0^1 \left[ (\omega^2)^2 + \text{Re} p \right] \left| \frac{v}{x_N} \right|^2 dx_N =
\]

\[
= \text{Re} \int_0^1 h(\omega, p, x_N) \frac{\overline{\varphi}}{x_N} dx_N.
\]

Applying the Cauchy inequality with \( \varepsilon \) to the right hand side and making use of the Hardy inequality, we get
\begin{equation}
\int_{0}^{1} x_N |v''|^2 \, dx_N \leq \varepsilon \int_{0}^{1} \frac{|v'|^2}{x_N} \, dx_N + C \varepsilon \int_{0}^{1} |h|^2 \, dx_N \leq \varepsilon C \int_{0}^{1} |v'|^2 \, dx_N + C \varepsilon \int_{0}^{1} |h|^2 \, dx_N.
\end{equation}

Note now that from boundary conditions (6.122)-(6.124) it follows that

\begin{equation}
\int_{0}^{1} |v'|^2 \, dx_N \leq C \int_{0}^{1} x_N |v''|^2 \, dx_N.
\end{equation}

Really, integrating over I the identity \((x_N v')' = v' + x_N v''\) and taking into account that the integral from \(v'\) over \(I\) is equal to \(v(1) - v(0) = 0\), we obtain

\[
|v'(1)| = \left| \int_{0}^{1} x_N v'' \, dx_N \right| \leq \left( \int_{0}^{1} x_N |v''|^2 \, dx_N \right)^{\frac{1}{2}}.
\]

Now we make use of (6.50) and arrive at (6.127). Applying (6.127) to (6.126) we get for a sufficiently small \(\varepsilon\)

\[
\int_{0}^{1} |v'|^2 \, dx_N \leq C \int_{0}^{1} |h|^2 \, dx_N,
\]

and thus

\[
\int_{0}^{1} |v|^2 \, dx_N \leq C \int_{0}^{1} |h|^2 \, dx_N \leq C \left( |h|^{(\gamma)}_{\gamma/2, I} \right)^2.
\]

The further reasoning of the proof of the present proposition are completely analogous to the previous subsection. By this we finish the proof.

\section{The Neumann and the Dirichlet problems for a linearized thin film equation in an arbitrary smooth domain.}

In this section we formulate theorems on solvability and estimates of the solution for a linearized thin film equation in an arbitrary smooth domain. But first we need an important proposition on constructing a function \(u(x, t)\) from an appropriate class with given initial values of \(u(x, 0)\) and \(u_t(x, 0)\).

Let the domains \(\Omega \in C^{4+\gamma}, \Omega_T \in C^{4+\gamma}\), the function \(d(x) \in C^{1+\gamma}(\Omega)\), and the spaces \(C^{2+\gamma/2}(\Omega), C^{3+\gamma/2}(\Omega_T), C^{\gamma/2}(\Omega), C^{\gamma/4}(\Omega_T)\) be defined in Section refs1. Let we are given functions...
\( u_0(x) \in C_{2,\gamma/2}^{4+\gamma}(\Omega), \ u_1(x) \in C_{\gamma/2}^{4}(\Omega). \) \tag{7.1}

**Proposition 34** For any functions \( u_0(x) \) and \( u_1(x) \) in (7.1) there exists a function \( w(x,t) \in C_{2,\gamma/2}^{4+\gamma}(\Omega_T) \) with

\[
w(x,0) \equiv u_0(x), \quad \frac{\partial w}{\partial t}(x,0) \equiv u_1(x), \quad x \in \Omega, \tag{7.2}
\]

\[
|w|_{2,\gamma/2}^{4+\gamma}(\Omega_T) \leq C_T \left( |u_0|_{2,\gamma/2}^{4+\gamma}(\Omega_T) + |u_1|_{\gamma/2}^{(\gamma)}(\Omega_T) \right), \tag{7.3}
\]

where the constant \( C_T \) does not depend on \( u_0(x) \) and \( u_1(x) \). Moreover, if

\[
u_0(\partial \Omega) \equiv u_1(\partial \Omega) \equiv 0, \tag{7.4}
\]

then

\[
w(x,t) \equiv 0, \quad x \in \partial \Omega. \tag{7.5}
\]

**Proof.**

The way of constructing \( w(x,t) \) is similar to the corresponding reasoning from [38]. From Lemma 5 it follows that \( u_1(x) \) belongs to the usual unweighted space \( C^{\gamma/2}(\Omega) \) and

\[
|u_1|_{\gamma/2}^{(\gamma/2)} \leq C |u_1|_{\gamma/2}^{(\gamma)}(\Omega) \tag{7.6}
\]

It was proved in [38], Ch.IV that there exists a function \( \varphi(x,t) \in C^{2+\gamma/2,1+\gamma/4}(\Omega_T) \) with

\[
\varphi(x,0) \equiv 0, \quad \frac{\partial \varphi}{\partial t}(x,0) \equiv u_1(x), \quad x \in \Omega, \tag{7.6}
\]

\[
|\varphi|_{\Omega_T}^{(2+\gamma/2,1+\gamma/4)} \leq C |u_1|_{\Omega_T}^{(\gamma/2)} \leq C |u_1|_{\gamma/2}^{(\gamma)}(\Omega). \tag{7.7}
\]

Moreover, if (7.4) is satisfied, we can take \( \varphi(x,t) \) as the solution of the initial boundary value problem

\[
\frac{\partial \varphi(x,t)}{\partial t} - \Delta \varphi(x,t) = u_1(x), \quad (x,t) \in \Omega_T,
\]

\[
\varphi(x,t)|_{x \in \partial \Omega} \equiv 0. \tag{7.8}
\]

\[
\varphi(x,0) \equiv 0, \quad x \in \Omega.
\]

And thus we have (7.8) for \( \varphi(x,t) \). Analogous to [38], Ch.IV, let a collection of functions \( \{ \eta_k(x) \in C^\infty(\Omega), k = 1,M \} \) be a partition of unity on \( \Omega \) with sufficiently small supports and in the sense

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\[ \sum_{k=1}^{M} \eta_k^2(x) = 1, \quad x \in \Omega. \tag{7.9} \]

We suppose (analogous to [38], Ch.IV) that the diameters \( d_k \) of the supports of \( \eta_k(x) \) satisfy \( \nu \lambda \leq d_k \leq \nu^{-1} \lambda, \lambda > 0 \), and if \( \text{supp}(\eta_k) \cap \partial \Omega = \emptyset \) (the set of the corresponding numbers \( k \) we denote by \( M_2 \), the rest we denote by \( M_1 \)), then \( \text{dist}(\text{supp}(\eta_k), \partial \Omega) \geq \nu \lambda \). Denote

\[ u_0^{(k)}(x) = u_0 \eta_k, \quad u_1^{(k)}(x) = u_1 \eta_k, \quad \varphi^{(k)}(x, t) = \varphi \eta_k. \]

Note that

\[ \sum_{k=1}^{M} \eta_k(x) u_0^{(k)}(x) = u_0(x), \quad \sum_{k=1}^{M} \eta_k(x) u_1^{(k)}(x) = u_1(x). \]

For \( k \in M_1 \) (that is when \( \text{supp}(\eta_k) \cap \partial \Omega \neq \emptyset \)) we denote by \( y = E_k(x) \in C^{4+\gamma}(R^N) \) a mapping from a neighborhood of \( \text{supp}(\eta_k) \) to the half space \( R^N_+ = \{ y \in R^N : y_N \geq 0 \} \) with the straightening of the boundary \( \partial \Omega \), that is \( \partial \Omega \cap \text{supp}(\eta_k) \) is mapped into \( \{ y_N = 0 \} \). For \( k \in M_2 \) we denote by \( u^{(k)}(x, t) \) the solution of the Cauchy problem

\[ \frac{\partial u^{(k)}(x, t)}{\partial t} + \Delta^2 u^{(k)}(x, t) = u_1^{(k)}(x) + \Delta^2 u_0^{(k)}(x), \quad x \in R^N, t \geq 0, \tag{7.10} \]

\[ u^{(k)}(x, 0) = u_0^{(k)}(x), \quad x \in R^N. \tag{7.11} \]

It is well known (see, for example, [32]) that in usual unweighted spaces

\[ |u^{(k)}|^{4+\gamma}_{R^N \times [0, T]} \leq C_T (|u_0|^{4+\gamma}_{2\gamma/2, \Omega} + |u_1|^{\gamma}_{2\gamma/2, \Omega} \leq C_{T, \lambda} \left( |u_0|^{4+\gamma}_{2\gamma/2, \Omega} + |u_1|^{\gamma}_{2\gamma/2, \Omega} \right). \tag{7.12} \]

For \( k \in M_1 \) we denote by \( u^{(k)}(x, t) \) the functions \( u^{(k)}(x, t) = u^{(k)}(y, t) \circ E_k(x) \), where \( u^{(k)}(y, t) \) is the solution of the model initial boundary problem corresponding to \( (0.112) - (6.116) \)

\[ \frac{\partial u^{(k)}(y, t)}{\partial t} + \nabla(y_N^2 \nabla^2 u^{(k)}(y, t)) = f^{(k)}(y, t), \quad (y, t) \in P_T, \tag{7.13} \]

\[ u^{(k)}(y', 0, t) = \varphi^{(k)}(y', t), \quad (y', t) \in P_T', \tag{7.14} \]

\[ u^{(k)}(y', 1, t) = 0, \quad \frac{\partial^2 u^{(k)}(y', 1, t)}{\partial y_N^2} = 0, \quad (y', t) \in P_T'. \tag{7.15} \]
\[ u^{(k)}(y,0) = \psi^{(k)}(y), \quad y \in \mathcal{P}, \quad (7.16) \]

\[ \frac{\partial^n u^{(k)}}{\partial y_i^n}(y,t) \bigg|_{x_i=-\pi} = \frac{\partial^n u^{(k)}}{\partial y_i^n}(y,t) \bigg|_{x_i=\pi}, \quad n = 0, 1, 2, 3, \quad i = 1, N - 1. \quad (7.17) \]

Here we denote

\[ \psi^{(k)}(y) \equiv u_0^{(k)}(x) \circ E_k^{-1}(y), \quad \varphi^{(k)}(y',t) \equiv \left[ \varphi^{(k)}(x,t) \circ E_k^{-1}(y) + \psi^{(k)}(y) \right] \bigg|_{y_N=0}, \]

\[ f^{(k)}(y,t) \equiv u_1^{(k)}(x) \circ E_k^{-1}(y) + \nabla(y_N^2 \nabla \Delta \psi^{(k)}(y)). \quad (7.18) \]

Note that we choose \( \lambda \) in the definition of \( \{ \eta_k \} \) so small that supports of all functions \( \psi^{(k)}(y), \varphi^{(k)}(y',t), f^{(k)}(y) \) are included in \( P_T \) or \( \overline{P_T} \). From the way of the construction of the function \( \varphi(x,t) \) it follows that for problems \((7.13)-(7.17)\) compatibility conditions \((6.118), (6.119)\) are satisfied. Then from Proposition \((\ref{prop1})\) it follows that

\[
|u^{(k)}(y,t)|^{(4+\gamma,\frac{4+\gamma}{\gamma})}_{2,\gamma/2,\overline{P_T}} \leq C_T \left( |f^{(k)}|^{(\gamma,\gamma/4)}_{\gamma/2,\overline{P_T}} + |\varphi^{(k)}|^{(2+\gamma/2,1+\gamma)}_{\overline{P_T}} + |\psi^{(k)}|^{(4+\gamma)}_{2,\gamma/2,\overline{P_T}} \right) \leq C_{T,\lambda} \left( |u_0|^{(4+\gamma)}_{2,\gamma/2,\overline{P_T}} + |u_1|^{(\gamma)}_{\gamma/2,\overline{P_T}} \right). \quad (7.19)\]

Finally, we define

\[ w(x,t) = \sum_{k=1}^{M} \eta_k(x)u^{(k)}(x,t). \]

It can be checked directly by the definition that such defined \( w(x,t) \) satisfies \((7.12), (7.13)\). Moreover, if \((7.3)\) is satisfied, then we have \((7.20)\) for \( w(x,t) \). This completes the proof of the proposition.

Let \( \sigma(x,t), \nabla \sigma, \) and \( w(x,t) \) be as in Section \( \[4\] \) That is, in particular

\[ \sigma(x,t), w(x,t) \in C_{2,\gamma/2}^{4+\gamma,\frac{4+\gamma}{\gamma}}(\overline{P_T}), \quad \sigma(x,0) \equiv 0, x \in \overline{\Omega}, \]

\[ w(x,t)|_{\partial \Omega} \equiv 0, \quad \frac{\partial w(x,t)}{\partial \eta} \bigg|_{\partial \Omega} \leq -\nu < 0, \quad w(x,t) > 0, x \in \Omega, \]

where \( \eta \) is the outward normal to \( \partial \Omega \). Consider the following initial boundary value problem for an unknown function \( u(x,t) \)

\[ \frac{\partial u(x,t)}{\partial t} + \nabla \sigma(w^2 \nabla \sigma \nabla^2 u(x,t)) = f(x,t), \quad \sigma(x,t) \quad (x,t) \in \Omega_T, \quad (7.20) \]

\[ 80 \]
\[ u(x, 0) = \psi(x), \quad x \in \Omega, \]  
\[ \frac{\partial u(x, t)}{\partial \vec{n}} = g(x, t), \quad x \in \Gamma_T \equiv \partial \Omega \times [0, T], \]  
(7.21)

where \( \vec{n} \) is the outward normal to \( \partial \Omega \), \( f \), \( g \), and \( \psi \) are given functions,

\[ f(x, t) \in C_{\gamma/2}^{1+\gamma/4}(\Gamma_T), \quad g(x, t) \in C_{2+\gamma/4}^{1+\gamma/2}(\Gamma_T), \quad \psi(x) \in C_{2+\gamma/2}^{4+\gamma}(\Omega). \]  
(7.22)

We suppose that the functions \( g(x, t) \) and \( \psi(x) \) satisfy the compatibility condition

\[ \frac{\partial \psi(x)}{\partial \vec{n}} = g(x, 0), \quad x \in \Gamma \equiv \partial \Omega. \]  
(7.23)

**Theorem 35** Under conditions (7.23), (7.24), problem (7.20) - (7.21) has the unique solution \( u(x, t) \in C_{4+\gamma/2}^{4+\gamma/2}(\Omega_T) \) for some \( T \leq T_0(\sigma) \) and

\[ |u|_{2, \gamma/2, 2+\gamma/4}^{4+\gamma/2} \leq C_T \left( |f|_{\gamma/2, 2+\gamma/4}^{4+\gamma/2} + |g|_{2+\gamma/4, 1+\gamma/2}^{4+\gamma/2} + |\psi|_{2+\gamma/2, 2+\gamma/4}^{4+\gamma/2} \right). \]  
(7.25)

Instead of boundary condition (7.22) we also consider the Dirichlet condition

\[ u(x, t) = \varphi(x, t), \quad x \in \Gamma_T \equiv \partial \Omega \times [0, T], \]  
(7.26)

where \( \varphi(x, t) \) is a given function and

\[ \varphi(x, t) \in C_{2+\gamma/2, 1+\gamma/4}^{2+\gamma/2}(\Gamma_T). \]  
(7.27)

We suppose the following compatibility conditions at \( t = 0, x \in \partial \Omega \)

\[ \varphi(x, 0) = \psi(x), \quad x \in \partial \Omega, \]  
(7.28)

\[ \frac{\partial \varphi}{\partial t}(x, 0) = -\nabla(w^2(x, 0)\nabla \psi(x)) + f(x, 0), \quad x \in \partial \Omega. \]  
(7.29)

**Theorem 36** Under conditions (7.27), (7.29), problem (7.20), (7.21), (7.26) has the unique solution \( u(x, t) \in C_{2+\gamma/2, 2+\gamma/4}^{4+\gamma/2}(\Omega_T) \) and

\[ |u|_{2, \gamma/2, 2+\gamma/4}^{4+\gamma/2} \leq C_T \left( |f|_{\gamma/2, 2+\gamma/4}^{4+\gamma/2} + |\varphi|_{2+\gamma/2, 1+\gamma/4}^{4+\gamma/2} + |\psi|_{2+\gamma/2, 2+\gamma/4}^{4+\gamma/2} \right). \]  
(7.30)

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The proof of theorems 35 and 36 is standard - see, for example, [32], [38]. It is based on propositions 32, 33 about corresponding model problems and on Proposition 34. Therefore we give only the schema of the proof. First we construct a function $w(x, t)$ with the properties

$$w(x, 0) = u(x, 0) = \psi(x),$$

$$\frac{\partial w}{\partial t}(x, 0) = \frac{\partial u}{\partial t}(x, 0) = -\nabla(d(x)^2\nabla\psi(x)) + f(x, 0).$$

Then the change of the unknown $u(x, t) = v(x, t) + w(x, t)$ reduces the problem to a problem for the unknown $v(x, t) \in C^{4+\gamma, 0}_2(\Omega_T)$, that is the operator of the problem is considered in the spaces with zeros, where all functions and all their possible derivatives with respect to $t$ vanish at $t = 0$. In these spaces we construct the regularizator (near inverse operator) of the problem on the base of propositions 32, 33 and on the base of inequalities (2.28)- (2.30), (2.20)- (2.27). Note that the model problems for strictly inner points of the domain $\Omega$, where the equation is not a degenerate one, are well studied (see, for example, [32]). This process is completely standard and can be found in details in, for example, Ch.IV or in [32]. Note that we still need to have the sufficiently small time interval $[0, T]$ because for small $T$ condition (7.22) is close to the more natural condition

$$(\nabla_\sigma u(x, t), \vec{n}) = g(x, t), \quad x \in \Gamma_T \equiv \partial \Omega \times [0, T].$$

By this we finish the outline of the proof.

8 The linear problem, corresponding to the Frechet derivative of the operator of the original problem $F(\psi)$ from section 4.

In this section we show the invertibility of the Frechet derivative $F'(0)[u, \delta] = (F'_1(0)[u, \delta], F'_2(0)[u, \delta])$, where $F'_1(0)[u, \delta]$ and $F'_2(0)[u, \delta]$ are defined by relations (4.28), (4.35). We start with the corresponding model problem. Consider in $R^{N,N}_{+},T$ the following model problem for the unknown functions $u(x, t) \in C^{4+\gamma, 0, 1}_2(R^{N,N}_{+},T)$ and $\delta(x', t) \in C^{0, 2+\gamma, 1}_0(R^{N,N}_{+},T)$

$$\frac{\partial u(x, t)}{\partial t} + \nabla(x_N^2 \nabla u) - A \left( \frac{\partial \delta(x, t)}{\partial t} + \nabla(x_N^2 \nabla \delta(x, t)) \right) = f(x, t), \quad (x, t) \in R^{N,N}_{+},T,$$

$$\frac{\partial u(x, t)}{\partial x_N} - A \frac{\partial \delta(x, t)}{\partial x_N} \bigg|_{x_N=0} = g(x', t), \quad x' \in R^{N-1}_{+},$$

(8.1)
\begin{equation}
  u(x', 0, t) = \varphi(x', t), \quad x' \in R^{N-1}, \tag{8.3}
\end{equation}

\begin{equation}
  u(x, 0) \equiv 0, \quad x \in R^N, \quad \delta(x', 0) \equiv 0, \quad x' \in R^{N-1}, \tag{8.4}
\end{equation}

\begin{equation}
  \delta(x, t) = \delta(x', x_N, t) \equiv E\delta(x', t). \tag{8.5}
\end{equation}

Here \( A \) is a constant, \( A + A^{-1} \leq C \), \( E \) is some extension operator from \( C^{2+\gamma/2,1+\frac{\gamma}{4}} (R^N_T) \) to \( C^{4+\gamma, \frac{4+\gamma}{2}} (R^N_T) \), \( f, g, \) and \( \varphi \) are given functions with compact supports and

\begin{equation}
  f \in C^{4+\gamma, \frac{4+\gamma}{2}} (R^N_T), \quad g \in C^{2+\gamma/2,1+\frac{\gamma}{4}} (R^N_T), \quad \varphi \in C^{2+\gamma/2,1+\frac{\gamma}{4}} (R^N_T). \tag{8.6}
\end{equation}

Recall that zero at the bottom of the designation of a space means that all functions with all their possible derivatives vanish at \( t = 0 \).

**Lemma 37** Let functions \( u(x, t) \in C^{4+\gamma, \frac{4+\gamma}{2}} (R^N_T) \) and \( \delta(x', t) \in C^{2+\gamma/2,1+\frac{\gamma}{4}} (R^N_T) \) with compact supports satisfy problem \([8.1], [8.5]\). Then

\begin{equation}
  |u|^{(4+\gamma, \frac{4+\gamma}{2})}_{2, \gamma/2, R^N_T} + |\delta|^{(2+\gamma/2,1+\frac{\gamma}{4})}_{2, \gamma/2, R^N_T} \leq C_T \left( |f|^{(\gamma, \frac{\gamma}{4})}_{\gamma/2, R^N_T} + |g|^{(2+\gamma/2,1+\frac{\gamma}{4})}_{\gamma/2, R^N_T} \right). \tag{8.7}
\end{equation}

**Proof.**

Denote \( v(x, t) \equiv u(x, t) - A\delta(x, t) \). Then the function \( v(x, t) \) satisfies the problem

\begin{equation}
  \frac{\partial v(x, t)}{\partial t} + \nabla(x_N^2 \Delta v) = f(x, t), \quad (x, t) \in R^N_T,
\end{equation}

\begin{equation}
  \frac{\partial v(x, t)}{\partial x_N} \bigg|_{x_N=0} = g(x', t), \quad x' \in R^{N-1},
\end{equation}

\begin{equation}
  v(x, 0) \equiv 0, \quad x \in R^N. \tag{8.8}
\end{equation}

From Theorem 35 with \( \sigma \equiv 0 \) and \( w \equiv x_N \) it follows that

\begin{equation}
  |u|^{(4+\gamma, \frac{4+\gamma}{2})}_{2, \gamma/2, R^N_T} \leq C_T \left( |f|^{(\gamma, \frac{\gamma}{4})}_{\gamma/2, R^N_T} + |g|^{(2+\gamma/2,1+\frac{\gamma}{4})}_{\gamma/2, R^N_T} \right). \tag{8.9}
\end{equation}

But then from \([8.3]\) it follows that \( \delta(x', t) = (-v(x', 0, t) + \varphi(x', t))/A \) and therefore

\begin{equation}
  |\delta|^{(2+\gamma/2,1+\frac{\gamma}{4})}_{2, \gamma/2, R^N_T} \leq C \left( |u|^{(4+\gamma, \frac{4+\gamma}{2})}_{2, \gamma/2, R^N_T} + |\varphi|^{(2+\gamma/2,1+\frac{\gamma}{4})}_{2, \gamma/2, R^N_T} \right) \leq
\end{equation}

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\[
\leq C_T \left( |f|_{C^{3+\gamma/2,1+\gamma/4}(\Omega_T)} + |\varphi|_{C^{3+\gamma/2,1+\gamma/4}(\Omega_T)} + |g|_{C^{3+\gamma/2,1+\gamma/4}(\Omega_T)} \right).
\]

(8.9)

Finally, taking into account that \(u(x,t) = v(x,t) + A \cdot E \delta(x',t)\), we obtain (8.7) from (8.8) and (8.9).

Let \(\sigma(x,t), w(x,t)\) be defined in (4.6)-(4.9). Consider now the following linear problem for the unknown functions \(u(x,t) \in C^{2+\gamma/2,\gamma/4} \big( \Omega_T \big)\) and \(\delta(x,t) \in C^{2+\gamma/2,\gamma/4} \big( \Gamma_T \big)\)

\[
\frac{\partial u(x,t)}{\partial t} + \nabla_{\sigma}(w^2 \nabla_{\sigma}^2 u) - A(x,t) \left( \frac{\partial \delta(x,t)}{\partial t} + \nabla_{\sigma}(w^2 \nabla_{\sigma}^2 \delta(x,t)) \right) + Q_1[u, \delta] = f(x,t), (x,t) \in \Omega_T,
\]

(8.10)

\[
\frac{\partial u}{\partial n} - A(x,t) \frac{\partial \delta}{\partial n} + Q_2[u, \delta] = g(x,t), (x,t) \in \Gamma_T,
\]

(8.11)

\[
u(x,t) = \varphi(x,t), (x,t) \in \Gamma_T,
\]

(8.12)

\[
u(x,0) = 0, x \in \Omega, \quad \delta(\omega,0) = 0, \omega \in \Gamma,
\]

(8.13)

\[
\delta(x,t) = E \delta(\omega,t).
\]

(8.14)

Here \(n\) is the outward normal to \(\Gamma\), \(E : C^{2+\gamma/2,1+\gamma/4} \big( \Gamma_T \big) \to C^{3+\gamma/2,1+\gamma/4} \big( \Omega_T \big)\) is some fixed extension operator, \(Q_1[u, \delta]\) and \(Q_2[u, \delta]\) are linear expressions of the form

\[
Q_1[u, \delta] = q(x,t) \frac{\partial \delta(x,t)}{\partial t} + \sum_{|\beta| = 3} q_{\beta}^{(1)}(x,t) d_\beta^2(x) D_\beta^2 \delta + \sum_{|\beta| = 2} q_{\beta}^{(2)}(x,t) d_\beta(x) D_\beta^2 \delta + \sum_{|\beta| = 1} q_{\beta}^{(3)}(x,t) D_\beta^2 \delta + b^{(1)} \delta.
\]

(8.15)

\[
Q_2[u, \delta] = \sum_{|\beta| = 1} b^{(1)}_{\beta}(x,t) D_\beta^2 u + \sum_{|\beta| = 1} b^{(2)}_{\beta}(x,t) D_\beta^2 \delta + b^{(3)} \delta.
\]

(8.16)

The coefficients in expressions (8.15), (8.16) have the properties

\[
q, q_{\beta}^{(1)} \in C^{\gamma/2} \big( \Gamma_T \big), \quad q(x,0) \equiv 0, x \in \Omega.
\]

(8.17)
\( b^{(i)}_\beta, b^{(3)} \in C^{1+\gamma/2,1/2+\gamma/4}(\Gamma_T), \quad b^{(i)}_\beta(x,0) \equiv 0, i = 1, 2, x \in \Gamma, \)  \hspace{1cm} (8.18)

and the coefficient \( A(x) \) satisfies

\[
A(x,t) \in C^{\gamma/2, \gamma/4}(\Omega_T), \quad 0 < \nu \leq A(x,t).
\]  \hspace{1cm} (8.19)

About the given functions \( f, g, \varphi \) we suppose that

\[
f \in C^{\gamma, \gamma/4}(\Omega_T), \quad g \in C^{1+\gamma/2,1/2+\gamma/4}(\Gamma_T), \quad \varphi \in C^{2+\gamma/2,1+\gamma/4}(\Gamma_T).
\]  \hspace{1cm} (8.20)

It is important that the Frechet derivatives \( F'_1(0)[u, \delta] \) and \( F'_2(0)[u, \delta] \) from relations (4.28), (4.45) have exactly the form of the left hand sides of (8.10), (8.11).

As it is applied to these derivatives, we have, for example,

\[
A(x,t) = \frac{\partial w(x,t)}{\partial \lambda}, \quad q(x,t) = \frac{\partial w(x,t)}{\partial \lambda} \left[(1+\sigma\lambda)^{-1} - 1\right]
\]

and analogously for other coefficients with the taking into account that \( \sigma(x,0) \equiv 0 \).

We explain also the factors \( d^2\vec{x}(x) \) and \( d^2\vec{q}(x) \) in (8.15). Consider, for example, a term from the definition of \( R_1[\delta] \) in (4.40) for \(|\beta| = 3, |\alpha| = 1\)

\[
\sum_{|\alpha|=1} a^{(1)}_{\alpha,\beta} w^2 D^\alpha_x \left( \frac{\partial w}{\partial \lambda} \right) D^\beta \delta =
\]

\[
= \sum_{|\alpha|=1} a^{(1)}_{\alpha,\beta} \left( \frac{w}{d\vec{x}(x)} \right)^2 \left[ d^2\vec{x}(x) D^\alpha_x \left( \frac{\partial w}{\partial \lambda} \right) \right] d^2\vec{x}(x) D^\beta \delta \equiv q^{(1)}_{\beta}(x,t) d^2\vec{x}(x) D^\beta \delta.
\]

Here the expression \( w/d\vec{x}(x) \) is considered as in (4.22), (4.23) and the terms \( d^2\vec{x}(x) D^\alpha_x \left( \frac{\partial w}{\partial \lambda} \right) \) are considered on the base of Lemma 15 and this gives (8.17).

**Lemma 38** Expressions \( Q_1[u, \delta] \) and \( Q_2[u, \delta] \) satisfy with some \( \delta > 0 \)

\[
|Q_1[u, \delta]|^{\gamma/4}_{\gamma/2, \gamma/4} \leq CT^\delta \left( |u|_{\gamma, 2, T}^{(4+\gamma, \gamma/2, \gamma/4)} + |\delta|_{C^{2+\gamma/2,1+\gamma/4}(\Gamma_T)} \right), \hspace{1cm} (8.21)
\]

\[
|Q_2[u, \delta]|_{C^{1+\gamma/2,1+\gamma/4}(\Gamma_T)} \leq CT^\delta \left( |u|_{\gamma, 2, T}^{(4+\gamma, \gamma/2, \gamma/4)} + |\delta|_{C^{2+\gamma/2,1+\gamma/4}(\Gamma_T)} \right). \hspace{1cm} (8.22)
\]

**Proof.**

The proof is obtained by the direct estimates of each term in the definitions of \( Q_1[u, \delta] \) and \( Q_2[u, \delta] \) on the base of inequalities of Lemma 15 and (2.42) with the taking into account (8.17), (8.18).
Proposition 39 Let in relations (8.10) and (8.10) \( Q_1[u, \delta] \equiv 0 \) and \( Q_2[u, \delta] \equiv 0 \). Then problem (8.10)–(8.13) has the unique solution \((u, \delta)\) for some \( T \leq T_0 \) and

\[
\begin{align*}
|u|_{2, \gamma/2, \Omega_T}^{(4+\gamma, \gamma/4)} &+ |\delta|_{C^{2+\gamma/2, 1+\gamma/4}(\Gamma_T)} \leq \\
&\leq CT \left( |f|_{\gamma/2, \Omega_T}^{(\gamma, \gamma/4)} + |\varphi|_{C^{2+\gamma/2, 1+\gamma/4}(\Gamma_T)} + |g|_{C^{2+\gamma/2, 1+\gamma/4}(\Gamma_T)} \right).
\end{align*}
\]

(8.23)

Proof.

We start with estimate (8.23) for a possible solution. This estimate is obtained by the standard Schauder technique on the base of Lemma 38 about the model problem for neighborhood of the boundary \( \partial \Omega \). The model problems for inner points of \( \Omega \) outside of some neighborhood of the boundary \( \partial \Omega \) correspond to non-degenerate case and the estimates of solutions to such problems can be found in, for example, [32]. Therefore we have to prove just the existence of the solution.

Denote by \( A_{\varepsilon}(x, t) \in C^\infty(\overline{\Omega}_T) \) the mollified function \( A(x, t) \) with

\[
|A(x, t) - A_{\varepsilon}(x, t)|_{\gamma/2, \Omega_T}^{(\gamma, \gamma/4)} \to 0, \quad \varepsilon \to 0.
\]

(8.24)

The way to obtain such a function \( A_{\varepsilon}(x, t) \) is described in Lemma 31. Consider problem (8.10)–(8.13) with \( A_{\varepsilon}(x, t) \) instead of \( A(x, t) \). As in Lemma 38 introduce the new unknown function

\[
v(x, t) = u(x, t) - A_{\varepsilon}(x, t) \delta(x, t).
\]

(8.25)

Then for the function \( v(x, t) \) we have the Neumann problem

\[
\frac{\partial v(x, t)}{\partial \nu} + \nabla \sigma (w^2 \nabla \varphi)^2 \nabla v + S_1[\delta] = f(x, t), \quad (x, t) \in \Omega_T,
\]

(8.26)

\[
\frac{\partial v}{\partial \nu} + S_2[\delta] = g(x, t), \quad (x, t) \in \Gamma_T,
\]

(8.27)

\[
v(x, 0) \equiv 0, \quad x \in \overline{\Omega},
\]

(8.28)

where \( S_1[\delta] \) and \( S_2[\delta] \) are some expressions with lower order terms and they are completely analogous to \( Q_1[u, \delta] \) and \( Q_2[u, \delta] \). Similar to (8.21), (8.22) we have

\[
|S_1[\delta]|_{\gamma/2, \Omega_T}^{(\gamma, \gamma/4)} + |S_2[\delta]|_{C^{1+\gamma/2, 1+\gamma/4}(\Gamma_T)} \leq CT^4|\delta|_{C^{2+\gamma/2, 1+\gamma/4}(\Gamma_T)}.
\]

(8.29)

Define a linear operator \( M : C^{1+\gamma/2, 1+\gamma/4}(\Gamma_T) \to C^{1+\gamma/2, 1+\gamma/4}(\Gamma_T) \) in the following way. We substitute a given \( \delta \in C^{1+\gamma/2, 1+\gamma/4}(\Gamma_T) \) in \( S_1[\delta] \) and \( S_2[\delta] \) and on the base of Theorem 33 we find the solution \( v(x, t) \) of problem (8.20)–(8.28). Then from (8.24) and (8.28) we define

\[
M \delta \equiv \frac{(\varphi(x, t) - v(x, t)\gamma_T)}{A_{\varepsilon}(x, t)}.
\]

(8.30)
From estimate (7.25) and (8.29) it follows that the operator \( M \) is a linear contraction for a sufficiently small \( T > 0 \) and thus it has the unique fixed point \( \delta_\varepsilon(x,t) \). This gives us the unknown function \( \delta_\varepsilon(x,t) \). The unknown function \( u_\varepsilon(x,t) \) is then given by (by virtue of (8.25))

\[
    u_\varepsilon(x,t) = v(x,t) + A_\varepsilon(x,t)\delta_\varepsilon(x,t).
\]

This gives us the solution \( (u_\varepsilon(x,t), \delta_\varepsilon(x,t)) \) for a smoothed function \( A_\varepsilon(x,t) \).

Theorem 40

Under conditions (8.17) - (8.20) problem (8.10) - (8.14) has the unique solution \( u(x,t) \in C^{4+\gamma,\frac{4+\gamma}{4}}_0(\Omega_T) \), \( \delta(x,t) \in C^{2+\gamma/2,1+\gamma/4}(\Gamma_T) \) for a sufficiently small \( T \leq T_0 \) and

\[
    |u|_{C^{4+\gamma,\frac{4+\gamma}{4}}_0(\Omega_T)} + |\delta|_{C^{2+\gamma/2,1+\gamma/4}(\Gamma_T)} \leq C_T \left( |f|_{C^{\gamma/4}}(\gamma/4) + |\varphi|_{C^{2+\gamma/2,1+\gamma/4}(\Gamma_T)} + |g|_{C^{1+\gamma/2,1/2+\gamma/4}(\Gamma_T)} \right), \quad (8.31)
\]

Proof.

The proof resembles the proof of the previous proposition. Define a linear operator \( M \) from \( C^{4+\gamma,\frac{4+\gamma}{4}}_0(\Omega_T) \times C^{2+\gamma/2,1+\gamma/4}(\Gamma_T) \) to itself in the following way. We substitute a given element \( (u, \delta) \) in \( Q_1[u, \delta] \) and \( Q_2[u, \delta] \) and solve the obtained problem (8.10) - (8.14) on the base of Proposition 39. We put the obtained solution \( (u, \delta) \) as the value of the operator \( M \) at \( (u, \delta) \), \( (u, \delta) \equiv M(u, \delta) \). From (8.23) and (8.22) it follows that the operator \( M \) is a linear contraction for a sufficiently small \( T > 0 \) and this completes the proof of the theorem.

9 The Proof of Theorem 2

We now conclude the proof of Theorem 2

Let \( B_r \) be the ball from (4.12) and consider on this ball the operator \( F(\psi) \) from (4.15), (4.16). From Proposition 21 it follows that \( F(\psi) \) is Frechet - continuously differentiable on \( B_r \) and from Theorem 40 it follows that it’s Frechet derivative \( F'(0)[\psi] \) has the bounded inverse operator for a sufficiently small \( T \leq T_0 \). Besides, relation (4.49) means that the value \( ||F(0)|| \) can be made arbitrary small for a sufficiently small \( T \leq T_0 \). Thus, due to the Corollary 4 we conclude that for \( T \leq T_0 \) the equation \( F(\psi) = 0 \) has a solution \( \psi_0 \in B_r \). The uniqueness of such element \( \psi_0 \in B_r \) for a sufficiently small \( T \leq T_0 \) is proved exactly as in [24]. According to the way of the construction of the operator \( F(\psi) \) this gives the unique smooth solution to problem (1.1) - (1.5) and proves Theorem 2.
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