Density Elimination for Semilinear Substructural Logics

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Abstract

We present a uniform method of density elimination for several semilinear substructural logics. Especially, the density elimination for the involutive uninorm logic $IUL$ is proved. Then the standard completeness of $IUL$ follows as a lemma by virtue of previous work by Metcalfe and Montagna.

Keywords: Density elimination, Involutive uninorm logic, Standard completeness of IUL, Semilinear substructural logics, Fuzzy logic

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1. Introduction

The problem of the completeness of Łukasiewicz infinite-valued logic ($L$ for short) was posed by Łukasiewicz and Tarski in the 1930s. It was twenty-eight years later that it was syntactically solved by Rose and Rosser [18]. Chang [4] developed at almost the same time a theory of algebraic systems for $L$, which is called $MV$-algebras, with an attempt to make $MV$-algebras correspond to $L$ as Boolean algebras to the classical two-valued logic. Chang [5] subsequently finished another proof for the completeness of $L$ by virtue of his $MV$-algebras.

It was Chang who observed that the key role in the structure theory of $MV$-algebras is not locally finite $MV$-algebras but linearly ordered ones. The observation was formalized by Hájek [12] who showing the completeness for his basic fuzzy logic ($BL$ for short) with respect to linearly ordered $BL$-algebras. Starting with the structure of $BL$-algebras, Hájek [13] reduced the problem of the standard completeness of $BL$ to two formulas to be provable in $BL$. Here and thereafter, by the standard completeness we mean that logics are complete with respect to algebras with lattice reduct [0, 1]. Cignoli et al. [6] subsequently proved the standard completeness of $BL$, i.e., $BL$ is the logic of continuous t-norms and their residua.

Hajek’s approach toward fuzzy logic has been extended by Esteva and Godo in [9], where the authors introduced the logic $MTL$ which aims at capturing the tautologies of left-continuous t-norms and their residua. The standard completeness of $MTL$ was proved by Jenei and Montagna in [15], where the major step is to embed linearly ordered $MTL$-algebras into the dense ones under the situation that the structure of $MTL$-algebras has been unknown as yet.

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Esteva and Godo’s work was further promoted by Metcalfe and Montagna [16] who introduced the uninorm logic $\mathbf{UL}$ and involutive uninorm logic $I\mathbf{UL}$ which aim at capturing tautologies of left-continuous uninorms and their residua and those of involutive left-continuous ones, respectively. Recently, Cintula and Noguera [8] introduced semilinear substructural logics which are substructural logics complete with respect to linearly ordered models. Almost all well-known families of fuzzy logics such as $\mathbf{L}$, $\mathbf{BL}$, $\mathbf{MTL}$, $\mathbf{UL}$ and $I\mathbf{UL}$ belong to the class of semilinear substructural logics.

Metcalfe and Montagna’s method to prove standard completeness for $\mathbf{UL}$ and its extensions is of proof theory in nature and consists of two key steps: Firstly, they extended $\mathbf{UL}$ with the density rule of Takeuti and Titani [19]:

\[
\Gamma \vdash (A \rightarrow p) \lor (p \rightarrow B) \lor C \quad (D)
\]

where $p$ does not occur in $\Gamma, A, B$ or $C$, and then prove the logics with $(D)$ are complete with respect to algebras with lattice reduct $[0, 1]$; Secondly, they give a syntactic elimination of $(D)$ that was formulated as a rule of the corresponding hypersequent calculus.

Hypersequents are a natural generalization of sequents which were introduced independently by Avron [1] and Pottinger [17] and have proved to be particularly suitable for logics with pre-linearity [2, 16]. Following the spirit of Gentzen’s cut elimination, Metcalfe and Montagna succeeded to eliminate the density rule for $\mathbf{UL}$ and several extensions of $\mathbf{UL}$ by induction on the height of a derivation of the premise and shifting applications of the rule upwards, but failed for $I\mathbf{UL}$ and therefore left it as an open problem.

There are several relevant works about the standard completeness of $I\mathbf{UL}$ as follows. With an attempt to prove the standard completeness of $I\mathbf{UL}$, we generalized Jenei and Montagna’s method for $\mathbf{MTL}$ in [20], but our effort was only partially successful. It seems that the subtle reason why it does not work for $\mathbf{UL}$ and $I\mathbf{UL}$ is the failure of FMP of these systems [21]. Jenei [14] constructed several classes of involutive $\mathbf{FL}_e$-algebras, as he said, in order to gain a better insight into the algebric semantic of the substructural logic $\mathbf{UL}$, and also to the longstanding open problem about its standard completeness. Ciabattoni and Metcalfe [7] introduced the method of density elimination by substitutions which is applicable to a general classes of (first-order) hypersequent calculi but fails to the case of $I\mathbf{UL}$.

We reconsidered Metcalfe and Montagna’s proof-theoretic method to investigate the standard completeness of $I\mathbf{UL}$, because they have proved the standard completeness of $\mathbf{UL}$ by their method and we can’t prove such a result by the Jenei and Montagna’s model-theoretic method. In order to prove the density elimination for $\mathbf{UL}$, they prove that the following generalized density rule $(D_1)$:

\[
G_0 \equiv \{ \Gamma, i, p \Rightarrow \Delta_i \}_{i=1, \ldots, n} \{ \Sigma_k, (\mu_k+1) \Rightarrow p \}_{k=1, \ldots, m} \{ \Pi_j \Rightarrow p \}_{j=1, \ldots, o} (D_1)
\]

is admissible for $\mathbf{UL}$, where they set two constraints to the form of $G_0$: (i) $n, m \geq 1$ and $\lambda_i \geq 1$ for some $1 \leq i \leq n$ and (ii) $p$ does not occur in $\Gamma, \Delta_i, \Pi_j, \Sigma_k$ for $i = 1 \ldots, n$, $j = 1 \ldots, m$, $k = 1 \ldots, o$.

In this paper, $G_1 \equiv G_2$ means that symbol $G_1$ denote a complex hypersequent $G_2$ temporarily for convenience. We may regard $(D_1)$ as a procedure whose input and output are the premise and conclusion of $(D_1)$, respectively. We denote the conclusion of $(D_1)$ by $D_1(G_0)$ when its premise is $G_0$. Observe that Metcalfe and Montagna had succeeded to define the suitable conclusion for almost arbitrary premise in $(D_1)$, but it seems impossible for $I\mathbf{UL}$ (See
Section 3 for an example). We then define the following generalized density rule \((D_0)\) for GIUL, GMITL and prove its admissibility in Section 9.

**Lemma 1.1 (Main Lemma).** Let \(n, m \geq 1, p\) does not occur in \(G', \Gamma_i, \Delta_i, \Pi_j\) or \(\Sigma_j\) for all \(1 \leq i \leq n, 1 \leq j \leq m\). Then the strong density rule

\[
G_0 \in G' \{\Gamma_i, p \Rightarrow \Delta_i\}_{i=1}^n \{\Pi_j \Rightarrow p, \Sigma_j\}_{j=1}^m (D_0)
\]

is admissible in GL.

In proving admissibility of \((D_1)\), Metcalfe and Montagna made some restriction on the proof \(\tau\) of \(G_0\), i.e., converted \(\tau\) into an \(r\)-proof. The reason why they need an \(r\)-proof is that they set the constraint \((i)\) to \(G_0\). We may imagine the restriction on \(\tau\) and the constraints to \(G_0\) as two pallets of a balance, i.e., one is strong if another is weak and vice versa. Observe that we select the weakest form of \(G_0\) that guarantees the validity of \((D)\) in \((D_0)\). Then it is natural that we need make the strongest restriction on the proof \(\tau\) of \(G_0\). But it seems extremely difficult to follow such a way to prove the admissibility of \((D_0)\).

In order to overcome such difficulty, we first of all choose Avron-style hypersequent calculi as the underlying systems (See Appendix 1). In Section 4, a proof \(\tau^*\) in a restricted subsystem GIUL is built up from a given proof \(\tau\) in GL by a systematic novel manipulations such as sequent-inserting operation, derivation-pruning operation, eigenvariable-replacing operation and eigenvariable-labeling operation so on. In Section 5, we define the generalized density rule \((D)\) in GIUL and prove that it is admissible. More derivations are constructed from \(\tau^*\) by derivation-pruning operation in Section 6 and applied to build up much more complicated derivations by the separation algorithm of one branch in Section 7. We give the separation algorithm of multiple branches by derivation-grafting operation in Section 8 and prove that the density elimination theorem in GL holds in Section 9. Especially, the standard completeness of IUL follows from Theorem 62 of [16].

**Global notations**

- \(G_0\) – Upper hypersequent of strong density rule in Lemma 1.1.
- \(\tau\) – A cut-free proof of \(G_0\) in GL, in Lemma 1.1.
- \(\mathcal{P}(H)\) – The position of \(H \in \tau\), Def. 2.13.
- \(\tau^*\) – The proof of \(G^*\) in GIUL resulting from preprocessing of \(\tau\), Notation 4.13.
- \(G^*\) – The root of \(\tau^*\) corresponding to the root \(G_0\) of \(\tau\), Notation 4.13.
- \(H_i^e\) – The \(i\)-th \((pEC)\)-node in \(\tau^*\), the superscript ‘c’ means contraction, Notation 4.14.
- \(S_j^e\) – The focus sequent of \(H_j^e\)’s, Notation 4.14.
- \(S_i^e\) or \(S_{i^e}\) or \(S_{j^e}\) or one copy of \(S_j^e\)’s, Notation 4.14.
- \(\{H_1^e, \ldots, H_n^e\}\) – The set of all \((pEC)\)-nodes in \(\tau^*\), Notation 4.14.
- \(I = \{H_1^e, \ldots, H_n^e\}\) – A subset of \(\{H_1^e, \ldots, H_n^e\}\), Notation 6.10.
- \(\mathcal{I} = \{S_{i^e}, \ldots, S_{n^e}\}\) – A subset of \((pEC)\)-sequents to \(I\), Definition 6.14.
- \(I' = \{G_0 \mid S_{i^e}, \ldots, G_{n^e}\}\) – A set of closed hypersequents to \(I\), Def. 6.14.
- \(X := Y\) – Define \(X\) as \(Y\) for two hypersequents \((set\ or\ derivations)\) \(X\) and \(Y\).

**2. Preliminaries**

In this section, we recall the basic definitions and results involved, which are mainly from [16]. Substructural fuzzy logics are based on a countable propositional language with formulas
FOR built inductively as usual from a set of propositional variables \( \text{VAR} \), binary connectives \( \odot, \to, \land, \lor \), and constants \( \bot, \top, t, f \) with definable connectives:

\[
\neg A := A \to f.
\]

\[
A \odot B := \neg(\neg A \odot \neg B).
\]

\[
A \leftrightarrow B := (A \to B) \land (B \to A).
\]

**Definition 2.1.** ([1, 16]) A sequent is an ordered pair \((\Gamma, \Delta)\) of finite multisets (possibly empty) of formulas, which we denote by \( \Gamma \Rightarrow \Delta \). \( \Gamma \) and \( \Delta \) are called the antecedent and succedents, respectively, of the sequent and each formula in \( \Gamma \) and \( \Delta \) is called a sequent-formula. A hypersequent \( G \) is a finite multiset of the form \( \Gamma_1 \Rightarrow \Delta_1 | \cdots | \Gamma_n \Rightarrow \Delta_n \), where \( \Gamma_i \Rightarrow \Delta_i \) is a sequent and called component of \( G \) for each \( 1 \leq i \leq n \). If \( \Delta_i \) contains at most one formula for \( i = 1 \cdots n \), then the hypersequent is single-conclusion, otherwise it is multiple-conclusion.

**Definition 2.2.** Let \( S \) be a sequent and \( G = S \mid \cdots \mid S_m \) a hypersequent. We say that \( S \in G \) if \( S \) is one of \( S_1, \cdots, S_m \).

**Notation 2.3.** Let \( G_1 \) and \( G_2 \) be two hypersequents. We will assume from now on that all set terminology refers to multisets, adopting the conventions of writing \( \Gamma, \Delta \Rightarrow S \) for the multiset union of \( \Gamma \) and \( \Delta \), \( A \) for the singleton multiset \( \{A\} \), and \( \lambda \Gamma \) for the multiset union of \( \lambda \) copies of \( \Gamma \) for \( \lambda \in \mathbb{N} \). By \( G_1 \subseteq G_2 \) we mean that \( S \in G_2 \) for all \( S \in G_1 \) and the multiplicity of \( S \) in \( G_1 \) is not more than that of \( S \) in \( G_2 \). We will use \( G_1 \cap G_2 \), \( G_1 \cup G_2 \), \( G_1 \setminus G_2 \) by their standard meaning for multisets by default and we will declare when we use them for sets. We sometimes write \( S_1 | \cdots | S_m \) and \( G \mid S \mid \cdots \mid S \) as \( \{S_1, \cdots, S_m\} \), \( G \mid S^n \) (or \( G \mid \{S\}^n \)), respectively.

**Definition 2.4.** ([1]) A hypersequent rule is an ordered pair consisting of a sequence of hypersequents \( G_1, \cdots, G_n \) called the premises (upper hypersequents) of the rule, and a hypersequent \( G \) called the conclusion (lower hypersequent), written by \( \frac{G_1 \cdots G_n}{G} \). If \( n = 0 \), then the rule has no premise and is called an initial sequent. The single-conclusion version of a rule adds the restriction that both the premises and conclusion must be single-conclusion; otherwise the rule is multiple-conclusion.

**Definition 2.5.** ([16]) \( \text{GUL} \) and \( \text{GIUL} \) consist of the single-conclusion and multiple-conclusion versions of the following initial sequents and rules, respectively:

**Initial Sequents**

\[
\begin{align*}
A & \Rightarrow A \quad \text{(ID)} \\
\Gamma & \Rightarrow \top, \Delta \quad \text{(Tr)} \\
\Gamma, \bot & \Rightarrow \bot \quad \text{(Lt)} \\
f & \Rightarrow f \quad \text{(f)}
\end{align*}
\]

**Structural Rules**

\[
\begin{align*}
\frac{G \mid \Gamma \Rightarrow A \mid \Delta}{G \mid \Gamma \Rightarrow A} & \quad \text{(EC)} \\
\frac{G \mid \Gamma \Rightarrow A}{G \mid \Gamma \Rightarrow A} & \quad \text{(EW)} \\
\frac{G_1 \mid \Gamma_1, \Pi_1 \Rightarrow \Sigma_1, \Delta_1}{G_1 \mid G_2 \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} & \quad \text{(COM)}
\end{align*}
\]

4
Logical Rules

\[
\frac{G \vdash \Gamma \Rightarrow \Delta}{G \vdash \Gamma, \Gamma, \Gamma \Rightarrow \Delta} (t_i) \]

\[
\frac{G_1 \vdash \Gamma_1 \Rightarrow \Delta_1 \quad G_2 \vdash \Gamma_2 \Rightarrow \Delta_2}{G_1 \mid G_2 \vdash \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} (\rightarrow_i)
\]

\[
\frac{G \vdash \Delta_1, \Delta_2}{G \vdash \Delta, \Delta} (f_r)
\]

\[
\frac{G \vdash \Gamma \Rightarrow \Delta}{G \vdash \Gamma, A \Rightarrow \Delta} (\emptyset_i)
\]

\[
\frac{G \vdash \Gamma, A \Rightarrow \Delta}{G \vdash \Gamma, A \Rightarrow \Delta} (\wedge_i)
\]

\[
\frac{G \vdash \Delta_1, \Delta_2}{G \vdash \Delta, \Delta} (\lor_i)
\]

\[
\frac{G \vdash \Gamma, A \Rightarrow \Delta}{G \vdash \Gamma, A \Rightarrow \Delta} (\emptyset_r)
\]

\[
\frac{G \vdash \Gamma, A \Rightarrow \Delta}{G \vdash \Gamma, A \Rightarrow \Delta} (\wedge_r)
\]

\[
\frac{G \vdash \Gamma, \bar{A} \Rightarrow \Delta}{G \vdash \Gamma, \bar{A} \Rightarrow \Delta} (\vee_i)
\]

\[
\frac{G_1 \vdash \Gamma_1 \Rightarrow \Delta_1 \quad G_2 \vdash \Gamma_2 \Rightarrow \Delta_2}{G_1 \mid G_2 \vdash \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} (\text{CUT})
\]

Definition 2.6. ([16]) GMTL and GMUML are GUL and GIUL plus the single conclusion and multiple-conclusion versions, respectively, of:

\[
\frac{G \vdash \Delta_1, \Delta_2}{G \vdash \Delta} (\text{WL}), \quad \frac{G \vdash \Delta_1, \Delta_2}{G \vdash \Delta, \Delta} (\text{WR}).
\]

Definition 2.7. (i) \( I \in \{ (t_i), (f_r), (\rightarrow_i), (\emptyset_i), (\wedge_i), (\emptyset_r), (\vee_i), (\Lambda_r), (\Lambda_r), (\vee_i) \} \).

(ii) By \( G \vdash [G' \mid G''] \vdash \Lambda \) (or \( G' \vdash [G' \mid G''] \vdash \Lambda \)) we mean that \( \Lambda \) is a two-premise inference rule (one-premise inference rule for \( I \) accordingly) of GL, where \( S' \) and \( S'' \) are its focus sequents and \( H' \) is its principle sequent (for \( \rightarrow_i \), \( \emptyset_i \), \( \Lambda_r \) and \( \vee_i \)) or hypersequent (for \( \text{COM} \), \( \wedge_r \) and \( \vee_i \)) as defined in 4.2.

Definition 2.8. ([16]) GLD is GL extended with the following density rule:

\[
\frac{G \vdash \Gamma \Rightarrow \Delta_1, \Delta_2 \mid \vdash \Gamma_1 \Rightarrow p, \Delta_2}{G \vdash \Gamma \Rightarrow \Delta_1, \Delta_2} (D)
\]

where \( p \) does not occur in \( G, \Gamma_1, \Gamma_2, \Delta_1 \) or \( \Delta_2 \).

Definition 2.9. ([11]) A derivation \( \tau \) of a hypersequent \( G \) from hypersequents \( G_1, \ldots, G_n \) in a hypersequent calculus GL is a labeled tree with the root labeled by \( G \), leaves labeled initial sequents or some \( G_1, \ldots, G_m \), and for each node labeled \( G_0 \) with parent nodes labeled \( G_1', \ldots, G_m' \) (where possibly \( m = 0 \)), \( G_1' \ldots G_m' / G_0 \) is an instance of a rule of GL.
Notation 2.10. (i) $G_1\cdots G_n/_{G_0}$ denotes that $\tau$ is a derivation of $G_0$ from $G_1,\cdots,G_n$.

(ii) Let $H$ be a hypersequent. $H \in \tau$ denotes that $H$ is a node of $\tau$. We call $H$ a leaf hypersequent if $H$ is a leaf of $\tau$, the root hypersequent if it is the root of $\tau$. $G'_1\cdots G'_m/_{G'_0}$ $\in \tau$ denotes that $G'_0 \in \tau$ and its parent nodes are $G'_1,\cdots,G'_m$.

(iii) Let $H \in \tau$ then $\tau(H)$ denotes the subtree of $\tau$ rooted at $H$.

(iv) $\tau$ determines a partial order $\leq$ with the root as the least element. $H_1 \downarrow H_2$ denotes $H_1 \leq \tau$ $H_2$ and $H_2 \not\leq \tau$ $H_1$ for any $H_1, H_2 \in \tau$. By $H_1 = \tau H_2$ we mean that $H_1$ is the same node as $H_2$ in $\tau$. We sometimes write $\leq_s$ as $\leq$.

(v) $G'/S^n(\text{EC}^n) \in \tau$ denotes the full external contraction, where $n \geq 2$, $G'/S$ results from $G'/S^n$ by multiple applications of (EC) and, $G'/S^n$ is not a lower hypersequent of an application of (EC) whose contraction sequent is $S$ and $G'/S$ not an upper one in $\tau$.

Definition 2.11. Let $\tau$ be a derivation of $G$ and $H \in \tau$. The thread $Th_r(\tau)$ of $\tau$ at $H$ is a sequence $H_0,\cdots,H_n$ of node hypersequents of $\tau$ such that $H_0 = \tau H, H_n = \tau G, H_{k+1} \in \tau$ or there exists $G' \in \tau$ such that $H_k G' \in \tau_{H_{k+1}}$ or $G' H_k \in \tau_{H_{k+1}}$ in $\tau$ for all $0 \leq k \leq n - 1$.

Proposition 2.12. Let $H_1, H_2 \in \tau$. Then

(i) $H_1 \leq H_2$ if and only if $H_1 \in Th_r(H_2)$;

(ii) $H_1, H_2$ and $H_1 \leq H_2$ imply $H_2 \parallel H_2$;

(iii) $H_1 \leq H_3$ and $H_2 \leq H_3$ imply $H_1 \parallel H_2$.

Definition 2.13. Let $H \in \tau$ and $Th(H) = (H_0,\cdots,H_n)$. Let $b_n := 1$,

$$b_k := \begin{cases} 1 & \text{if } \frac{G'}{H_{k+1}} \in \tau, \\
0 & \text{if } \frac{H_k}{H_{k+1}} \in \tau \text{ or } \frac{G'}{H_{k+1}} \in \tau \end{cases}$$

for all $0 \leq k \leq n - 1$. Then $P(H) := \sum_{k=0}^{n} 2^k b_k$ and call it the position of $H$ in $\tau$.

Definition 2.14. A rule is admissible for a calculus GL if whenever its premises are derivable in GL, then so is its conclusion.

Lemma 2.15. (161) Cut-elimination holds for GL, i.e., proofs using (CUT) can be transformed syntactically into proofs not using (CUT).

3. Proof of the main lemma: A computational example

In this section, we present an example to illustrate the proof of Main lemma.

Let $G_0 \equiv \equiv p, B/B \Rightarrow p, A \otimes \sim A \Rightarrow p \Rightarrow C[C, p \Rightarrow A \otimes A$. $G_0$ is a theorem of IUL and a cut-free proof $\tau$ of $G_0$ is shown in Figure 1. Note that we denote three applications of (EC) in $\tau$.
respectively by \((EC)_1, (EC)_2, (EC)_3\) and three \((\odot_r)\) by \((\odot_r)_1, (\odot_r)_2\) and \((\odot_r)_3\).

### Figure 1: A proof \(\tau\) of \(G_0\)

By applying (D) to free combinations of all sequents in \(\Rightarrow p, B|B|B \Rightarrow p, \neg A \odot \neg A\) and in \(p \Rightarrow C|C, p \Rightarrow A \odot A\), we get that \(H_0 \equiv B, C|C \Rightarrow A \odot A, B|B \Rightarrow C, \neg A \odot \neg A|C, B \Rightarrow A \odot A, \neg A \odot \neg A\). \(H_0\) is a theorem of \(\text{IUL}\) and a cut-free proof \(\rho\) of \(H_0\) is shown in Figure 2. It supports the validity of the generalized density rule \((D_0)\) in Section 1, as an instance of \((D_0)\).
Our task is to construct \( \rho \), starting from \( \tau \). The tree structure of \( \rho \) is more complicated than that of \( \tau \). Compared with UL, MTL and IMTL, there is no one-to-one correspondence between nodes in \( \tau \) and \( \rho \).

Following the method given by G. Metcalfe and F. Montagna, we need to define a generalized density rule for IUL. We denote such an expected unknown rule by \((D_\tau)\) for convenience. Then \(D_\tau(H)\) must be definable for all \( H \in \tau \). Naturally, \(D_\tau(p \Rightarrow p) = t; D_\tau(A \Rightarrow p \Rightarrow A) = A \Rightarrow A; D_\tau(p, p \Rightarrow A) = p, A \Rightarrow A; D_\tau(p, A \Rightarrow p) = p, A \Rightarrow A; D_\tau(p, A \Rightarrow A) = p, A \Rightarrow A; D_\tau(A \Rightarrow A) = A \Rightarrow A\).

However, we couldn’t find a suitable way to define \(D_\tau(H_{xx})\) and \(D_\tau(H_x)\) for \(H_{xx}\) and \(H_x\) in \(\tau\), see Figure 2. This is the biggest difficulty we encounter in the case of IUL such that it is hard to prove density elimination for IUL. A possible way is to define \(D_\tau(\Rightarrow p, p \Rightarrow A) = p, A \Rightarrow A\) as \(\Rightarrow, A \Rightarrow A, A \Rightarrow A\). Unfortunately, it is not a theorem of IUL.

Notice that two upper hypersequents \( p, A \Rightarrow A, p \Rightarrow A \otimes A \) are permissible inputs of \((D_\tau)\). Why is \(H_{xx}\) an invalid input? One reason is that, two applications \((EC)_1\) and \((EC)_2\) cut off two sequents \( A \Rightarrow p \) such that two \( p \)'s disappear in all nodes lower than upper hypersequent of \((EC)_1\) or \((EC)_2\), including \(H_{xx}\). These make occurrences of \(p \)'s to be incomplete in \(H_{xx}\). We then perform the following operation in order to get complete occurrences of \(p \)'s in \(H_{xx}\).

**Step 1 (Preprocessing of \( \tau \).** Firstly, we replace \( H \) with \( H|S' \) for all \( G^i|S'|S'(EC)_k \in \tau \).

\( H \leq G^i|S' \) then replace \( G^i|S'|S'(EC)_k \) with \( G^i|S'|S' \) for all \( k = 1, 2, 3 \). Then we construct a proof without \((EC)\), which we denote by \( \tau_1 \), as shown in Figure 3. We call such manipulations
sequent-inserting operations, which eliminate applications of (EC) in \( \tau \).

\[
\begin{array}{c}
p \Rightarrow p A \Rightarrow A \quad p \Rightarrow p A \Rightarrow A \quad p \Rightarrow p A \Rightarrow A \\
A \Rightarrow p| A \Rightarrow p| p \Rightarrow A \\
A \Rightarrow p| A \Rightarrow p| p \Rightarrow A \quad A \Rightarrow p| A \Rightarrow p| p \Rightarrow A \\
A \Rightarrow p| A \Rightarrow p, f| p, p \Rightarrow A \quad A \Rightarrow p| A \Rightarrow p, f| p, p \Rightarrow A \\
A \Rightarrow p| A \Rightarrow p, \neg A \quad A \Rightarrow p, p \Rightarrow A \quad A \Rightarrow p, p \Rightarrow A \quad A \Rightarrow p, p \Rightarrow A \quad A \Rightarrow p, p \Rightarrow A
\end{array}
\]

\[
\begin{array}{c}
B \Rightarrow B \\
C \Rightarrow C
\end{array}
\]

\[
H'_x \equiv A \Rightarrow p| \quad p, p, \neg A \Rightarrow A \quad p \Rightarrow p \neg A \Rightarrow \quad A \Rightarrow p| A \Rightarrow p| p \Rightarrow A \quad A \Rightarrow p| A \Rightarrow p| p \Rightarrow A \quad A \Rightarrow p| A \Rightarrow p| p \Rightarrow A
\]

\[
A \Rightarrow p| A \Rightarrow p, B| B \Rightarrow p, A \quad A \Rightarrow p| A \Rightarrow p, A \quad A \Rightarrow p| A \Rightarrow p, A \Rightarrow A \quad A \Rightarrow p| A \Rightarrow p, A \Rightarrow A
\]

\[
FIGURE 3 \quad A \text{ proof } \tau_1
\]

However, we also can’t define \( D_\epsilon(H'_x) \) for \( H'_x \in \tau_1 \) in that \( \Rightarrow p, p, \neg A \Rightarrow A[p, p \Rightarrow A \subseteq H'_x \}. The reason is that the origins of \( p \)'s in \( H'_x \) are indistinguishable if we regard all leaves in the form \( p \Rightarrow p \) as the origins of \( p \)'s which occur in the inner node. For example, we don’t know which \( p \) comes from the left subtree of \( \tau_1(H'_x) \) and which from the right subtree in two occurrences of \( p \)'s in \( \Rightarrow p, p, \neg A \Rightarrow A \in H'_x \). We then perform the following operation in order to make all occurrences of \( p \)'s in \( H'_x \) distinguishable.

We assign the unique identification number to each leaf in the form \( p \Rightarrow p \in \tau_1 \) and transfer these identification numbers from leaves to the root, as shown in Figure 4. We denote the proof of \( G|G^* \) resulting from this step by \( \tau^* \), where \( G \equiv p, B| B \Rightarrow p, \neg A \Rightarrow A p \Rightarrow A \Rightarrow A \) in which each sequent is a copy of some sequent in \( G_0 \) and \( G^* \equiv A \Rightarrow p| A \Rightarrow p, p \Rightarrow A \Rightarrow A \) in which each sequent is a copy of some external contraction sequent in \( (EC) \)-node of \( \tau \). We call such manipulations eigenvariable-labeling operations, which make us to trace eigenvariables in \( \tau \).

\[
\begin{array}{c}
p_1 \Rightarrow p_1 A \Rightarrow A \quad p_2 \Rightarrow p_2 A \Rightarrow A \quad p_3 \Rightarrow p_3 A \Rightarrow A \quad p_4 \Rightarrow p_4 A \Rightarrow A \\
A \Rightarrow p_1| A \Rightarrow p_2| p_2 \Rightarrow A \quad A \Rightarrow p_3| p_3 \Rightarrow A \quad p_4| p_4 \Rightarrow A
\end{array}
\]

\[
\begin{array}{c}
B \Rightarrow B \\
C \Rightarrow C
\end{array}
\]

\[
H'_x \equiv A \Rightarrow p| A \Rightarrow p, B| B \Rightarrow p, A \quad A \Rightarrow p| A \Rightarrow p, A \quad A \Rightarrow p| A \Rightarrow p, A
\]

\[
A \Rightarrow p| A \Rightarrow p, B| B \Rightarrow p, A \quad A \Rightarrow p| A \Rightarrow p, A \quad A \Rightarrow p| A \Rightarrow p, A
\]

\[
FIGURE 4 \quad A \text{ proof } \tau^* \text{ of } G|G^*
\]

Then all occurrences of \( p \) in \( \tau^* \) are distinguishable and we regard them as distinct eigenvariables (See Definition 4.16 (i)). Firstly, by selecting \( p_1 \) as the eigenvariable and applying \( (D) \) to \( G|G^* \), we get

\[
G^\prime \equiv A \Rightarrow C| \Rightarrow p_2, B| B \Rightarrow p_4, \neg A \Rightarrow A C, p_2 \Rightarrow A \Rightarrow A A \Rightarrow p_1| A \Rightarrow p_1| A \Rightarrow p_3| p_3 \Rightarrow A \Rightarrow A
\]

Secondly, by selecting \( p_2 \) and applying \( (D) \) to \( G^\prime \), we get

\[
G^\prime \equiv A \Rightarrow C| \Rightarrow p_4, \neg A \Rightarrow A C \Rightarrow B, A \Rightarrow A | A \Rightarrow p_3| p_3 \Rightarrow A \Rightarrow A
\]

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Repeatedly, we get

\[ G'''' \equiv A \Rightarrow C|A, B \Rightarrow A \OD A, \neg A \OD \neg A|C \Rightarrow A \OD A, B. \]

We define such iterative applications of \((D)\) as \(D\)-rule (See Definition 5.4). Lemma 5.8 shows that \(\text{GUL}D(G|G^*)\) if \(\text{GUL}G|G^*\). Then we obtain \(\text{GUL}D(G|G^*)\), i.e., \(\text{GUL}G''''\).

A miracle happens here! The difficulty that we encountered in GIUL is overcome by converting \(H_{x}' = A \Rightarrow p \Rightarrow p, p, \neg A \OD \neg A|p, p \Rightarrow A \OD A|A \Rightarrow p|p, p \Rightarrow A \OD A \) into \(A \Rightarrow p_1 \Rightarrow p_2, p_4, \neg A \OD \neg A|p_1, p_2, p_3 \Rightarrow A \OD A|A \Rightarrow p_3, p_4 \Rightarrow A \OD A \) and using \((D)\) to replace \((D.)\).

Why do we assign the unique identification number to each \(p \Rightarrow p \in \tau_1\)? We would return back to the same situation as that of \(\tau_1\) if we assign the same indices to all \(p \Rightarrow p \in \tau_1\) or, replace \(p_1 \Rightarrow p_3\) and \(p_4 \Rightarrow p_4\) by \(p_2 \Rightarrow p_2\) in \(\tau^*\).

Note that \(D(G|G^*) = H_1\). So we have built up a one-one correspondence between the proof \(\tau^*\) of \(G|G^*\) and that of \(H_1\). Observe that each sequent in \(G^*\) is not a copy of any sequent in \(G_0\). In the following steps, we work on eliminating these sequents in \(G^*\).

**Step 2 (Extraction of Elimination Rules).** We select \(A \Rightarrow p_2\) as the focus sequent in \(H_1\) in \(\tau^*\) and leave \(A \Rightarrow p_1\) alone (See Figure 4). Then \(A \Rightarrow p_1\), keep unchanged from \(H_1\) downward to \(G|G^*\). So we extract a derivation from \(A \Rightarrow p_2\) by pruning some sequents (or hypersequents) in \(\tau^*\), which we denote by \(\tau_{H_1;A \Rightarrow p_2}^\ast\), as shown in Figure 5.

\[ \begin{array}{c|c}
\hline
p_1 \Rightarrow p_1 & A \Rightarrow A \\
p_4 \Rightarrow p_4 & A \Rightarrow A \\
A \Rightarrow p_2 & A \Rightarrow p_3 A \Rightarrow p_4 p_3, p_4 \Rightarrow A \OD A \\
A \Rightarrow p_2, f & A \Rightarrow p_3 A \Rightarrow p_4, f[p_3, p_4 \Rightarrow A \OD A \\
B \Rightarrow B & \Rightarrow p_2, A \Rightarrow p_1, A \Rightarrow p_2, A \Rightarrow p_4 A \OD A \\
B \Rightarrow B & \Rightarrow p_2, B|B \Rightarrow p_4, A \OD A | A \Rightarrow p_1, p_3, p_4 \Rightarrow A \OD A \\
\hline
\end{array} \]

**FIGURE 5** A derivation \(\tau_{H_1;A \Rightarrow p_2}^\ast\) from \(A \Rightarrow p_2\)

A derivation \(\tau_{H_1;A \Rightarrow p_1}^\ast\) from \(A \Rightarrow p_1\) is constructed by replacing \(p_2\) with \(p_1, p_3\) with \(p_5, p_4\) with \(p_6\) in \(\tau_{H_1;A \Rightarrow p_2}^\ast\), as shown in Figure 6. Notice that we assign new identification numbers to new occurrences of \(p\) in \(\tau_{H_1;A \Rightarrow p_1}^\ast\).

\[ \begin{array}{c|c}
\hline
p_5 \Rightarrow p_5 & A \Rightarrow A \\
p_6 \Rightarrow p_6 & A \Rightarrow A \\
A \Rightarrow p_5 & A \Rightarrow p_5 p_5, p_6 \Rightarrow A \OD A \\
A \Rightarrow p_1, f & A \Rightarrow p_3 A \Rightarrow p_6, f[p_5, p_6 \Rightarrow A \OD A \\
B \Rightarrow B & \Rightarrow p_1, A \Rightarrow p_5, A \Rightarrow p_6, A | A \Rightarrow p_5, p_6 \Rightarrow A \OD A \\
B \Rightarrow B & \Rightarrow p_1, p_6, A \OD A | A \Rightarrow p_5, p_6 \Rightarrow A \OD A \\
\hline
\end{array} \]

**FIGURE 6** A derivation \(\tau_{H_1;A \Rightarrow p_1}^\ast\) from \(A \Rightarrow p_1\)
Next, we apply $\tau^*_{H_C^*;A \to p_3}$ to $A \Rightarrow p_1$ in $G[G^*]$. Then we construct a proof $\tau^*_G$ as shown in Figure 7, where $G' \equiv G[G^* \{ A \Rightarrow p_1 \}].$

\[ G' \Rightarrow B \]
\[ G' \Rightarrow p_1, f \]
\[ G' \Rightarrow p_1, A \]
\[ G' \Rightarrow p_1, p_6, A \]
\[ G' \Rightarrow p_5, p_6, A \]
\[ G' \Rightarrow p_6, A \]
\[ G' \Rightarrow p_5, p_6 \]

\[ G^*_G \equiv G'[p_1, B[B \Rightarrow p_6, \neg A \Rightarrow A \Rightarrow p_3, p_5 \Rightarrow A \Rightarrow A] \Rightarrow p_1, B[B \Rightarrow p_5, p_6 \Rightarrow A \Rightarrow A] \Rightarrow p_3, p_5 \Rightarrow A \Rightarrow A] \Rightarrow p_3, p_5 \Rightarrow A \Rightarrow A]

**FIGURE 7** A proof $\tau^*_G$ of $G^*_G$.

However, $G^*_G \Rightarrow p_2, B[B \Rightarrow p_6, \neg A \Rightarrow A \Rightarrow p_3, p_5 \Rightarrow A \Rightarrow A] \Rightarrow p_3, p_5 \Rightarrow A \Rightarrow A$ contains more copies of sequent from $G^*$ and seems more complex than $G[G^*].$ We will present a unified method to tackle with it in the following steps. Other derivations are shown in Figures 8, 9, 10, 11.

\[ A \Rightarrow p_4 \]
\[ A \Rightarrow p_4, f \]
\[ A \Rightarrow p_4, \neg A \]
\[ A \Rightarrow p_4, A \]
\[ A \Rightarrow p_4, \neg A \]

**FIGURE 8** A derivation $\tau^*_G$ from $A \Rightarrow p_4$

\[ A \Rightarrow p_5 \]
\[ A \Rightarrow p_5, \neg A \]
\[ A \Rightarrow p_5, A \]
\[ A \Rightarrow p_5, \neg A \]

**FIGURE 9** A derivation $\tau^*_G$ from $A \Rightarrow p_3$

\[ A \Rightarrow p_2 \]
\[ A \Rightarrow p_2, f \]
\[ A \Rightarrow p_2, \neg A \]
\[ A \Rightarrow p_2, A \]

\[ A \Rightarrow p_3 \]
\[ A \Rightarrow p_3, f \]
\[ A \Rightarrow p_3, \neg A \]
\[ A \Rightarrow p_3, A \]

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Step 3 (Separation of one branch). A proof $\tau^{(2)}_{H_i;G^*}$ is constructed by applying sequentially

$$\tau^*_{H_i;G^*}(p_1, p_2, A \odot A) \quad \tau^*_{H_i;G^*}(p_3, p_4, A \odot A)$$

to $p_1, p_4 \Rightarrow A \odot A$ and $p_3, p_5 \Rightarrow A \odot A$ in $G^{(1)}$, as shown in Figure 12, where $G'' \equiv G^{(1)}_{H_i;G^*} \setminus \{p_3, p_4 \Rightarrow A \odot A, p_5, p_6 \Rightarrow A \odot A\}$

\[
\begin{array}{c}
C \Rightarrow C \quad C \Rightarrow C \quad G'' \mid p_3, p_4 \Rightarrow A \odot A \mid p_5, p_6 \Rightarrow A \odot A \\
p_3 \Rightarrow C \mid C, p_1 \Rightarrow A \odot A \mid p_5, p_6 \Rightarrow A \odot A \\
G^{(2)}_{H_i;G^*} \equiv G'' \mid p_3 \Rightarrow C \mid C, p_4 \Rightarrow A \odot A \mid p_5 \Rightarrow C \mid C, p_6 \Rightarrow A \odot A \\
\end{array}
\]

FIGURE 12 A proof $\tau^{(2)}_{H_i;G^*}$ of $G^{(2)}_{H_i;G^*}$.

$G^{(2)}_{H_i;G^*} \Rightarrow p_2, B \Rightarrow p_4, \neg A \odot \neg A \mid p_1 \Rightarrow C \mid C, p_2 \Rightarrow A \odot A \mid A \Rightarrow p_3 \Rightarrow p_1, B$

$B \Rightarrow p_6, \neg A \odot \neg A \mid A \Rightarrow p_2 \Rightarrow p_3 \Rightarrow C \mid C, p_4 \Rightarrow A \odot A \mid p_5 \Rightarrow C \mid C, p_6 \Rightarrow A \odot A.$

Notice that

$$D(B \Rightarrow p_2, \neg A \odot \neg A \mid A \Rightarrow p_3 \Rightarrow C \mid C, p_4 \Rightarrow A \odot A)$$

$$= D(B \Rightarrow p_6, \neg A \odot \neg A \mid A \Rightarrow p_5 \Rightarrow p_5 \Rightarrow C \mid C, p_6 \Rightarrow A \odot A)$$

$$= A \Rightarrow C \mid C, B \Rightarrow A \odot A, \neg A \odot \neg A.$$

Then it is permissible to cut off the part

$$B \Rightarrow p_6, \neg A \odot \neg A \mid A \Rightarrow p_5 \Rightarrow C \mid C, p_6 \Rightarrow A \odot A$$

of $G^{(2)}_{H_i;G^*}$, which corresponds to applying $(EC)$ to $D(G^{(2)}_{H_i;G^*})$. We regard such a manipulation as a constrained contraction rule applied to $G^{(2)}_{H_i;G^*}$ and denote it by $(EC_1)$. Define $G^{(2)}_{H_i;G^*}$ to be

$\Rightarrow p_2, B \Rightarrow p_4, \neg A \odot \neg A \mid p_1 \Rightarrow C \mid C, p_2 \Rightarrow A \odot A |$

$A \Rightarrow p_3 \Rightarrow p_1, B \Rightarrow p_3 \Rightarrow C \mid C, p_4 \Rightarrow A \odot A.$
Then we construct a proof of $G^\Sigma_{H^*_1;G^G}$ by
\[
\frac{G^\Sigma_{H^*_1;G^G}(EC_\Omega)}{G^\Sigma_{H^*_1;G^G}}(13)
\]
which guarantees the validity of
\[
\vdash_{\text{GUL}} D(G^\Sigma_{H^*_1;G^G})
\]
under the condition
\[
\vdash_{\text{GUL}} D(G^\Sigma_{H^*_1;G^G}).
\]

A change happens here! There is only one sequent which is a copy of sequent in $G^*$ in $G^\Sigma_{H^*_1;G^G}$. It is simpler than $G|G^*$. So we are moving forward. The above procedure is called the separation of $G|G^*$ as a branch of $H^*_1$ and reformulated as follows (See Section 7 for details).

The separation of $G|G^*$ as a branch of $H^*_2$ is constructed by a similar procedure as follows.

Note that $D(G^\Sigma_{H^*_1;G^G}) = H^*_2$ and $D(G^\Sigma_{H^*_1;G^G}) = H^*_1$. So we have built up one-one correspondences between proofs of $G^\Sigma_{H^*_1;G^G}$, $G^\Sigma_{H^*_1;G^G}$, and those of $H^*_2$, $H^*_3$.

**Step 3 (Separation algorithm of multiple branches).** We will prove $\vdash_{\text{GUL}} D_0(G_0)$ in a direct way, i.e., only the major step of Lemma 8.2 is presented in the following. Recall that
\[
G^\Sigma_{H^*_1;G^G} \Rightarrow p_2, B \Rightarrow p_4, -A \odot -A|p_1 \Rightarrow C|C, p_2 \Rightarrow A \odot A|
\]
\[
A \Rightarrow p_3 \Rightarrow p_1, B|p_3 \Rightarrow C|C, p_4 \Rightarrow A \odot A,
\]
\[
G^\Sigma_{H^*_2;G^G} \Rightarrow p_1 \Rightarrow p_2, B|p_2 \Rightarrow p_4, -A \odot -A|p_1 \Rightarrow C|C, p_2 \Rightarrow A \odot A|
\]
\[
B \Rightarrow p_3, -A \odot -A|p_3 \Rightarrow C|C, p_4 \Rightarrow A \odot A.
\]

By reassigning identification numbers to occurrences of $p’s$ in $G^\Sigma_{H^*_1;G^G}$,
\[
G^\Sigma_{H^*_2;G^G} \Rightarrow A \Rightarrow p_5 \Rightarrow p_6, B|B \Rightarrow p_8, -A \odot -A|p_5 \Rightarrow C|C, p_8 \Rightarrow A \odot A|
\]
\[
B \Rightarrow p_7, -A \odot -A|p_7 \Rightarrow C|C, p_8 \Rightarrow A \odot A.
\]
By applying $\tau^*_{\{H_i|A \Rightarrow p_3\},H_i'\Rightarrow p_1} \rightarrow A \Rightarrow p_3$ in $G^\Box_{H_i|G|G^*}$ and $A \Rightarrow p_5$ in $G^\Box_{H_i'|G|G^*}$, we get $\nu_{G} G'$, where

$$
\begin{align*}
G' \equiv & \quad p_2, B \Rightarrow p_4, \neg A \land \neg A | p_1 \Rightarrow C, p_2 \Rightarrow A \land A | p_1, B | \\
p_3 \Rightarrow & \quad C, p_4 \Rightarrow A \land A | p_2, B \Rightarrow p_6, B | B \Rightarrow p_6, \neg A \land \neg A | p_5 \Rightarrow C, p_5 \Rightarrow A \land A | \\
B \Rightarrow & \quad p_7, \neg A \land \neg A | p_7 \Rightarrow C, p_8 \Rightarrow A \land A | p_5, B \Rightarrow p_3, \neg A \land \neg A.
\end{align*}
$$

Why do you reassign identification numbers to occurrences of $p'$s in $G^\Box_{H_i'|G|G^*}$? It makes different occurrences of $p'$s to be assigned different identification numbers in two nodes $G^\Box_{H_i|G|G^*}$ and $G^\Box_{H_i'|G|G^*}$ of the proof of $G'$.

By applying $(\nu_{G} E^*_{\bar{G}})$ to $G'$, we get $\nu_{G} G^\Box_{1}$, where

$$
\begin{align*}
G^\Box_{1} \equiv & \quad p_2, B \Rightarrow p_4, \neg A \land \neg A | p_1 \Rightarrow C, p_2 \Rightarrow A \land A | p_1, B | \\
p_3 \Rightarrow & \quad C, p_4 \Rightarrow A \land A | B \Rightarrow p_3, \neg A \land \neg A.
\end{align*}
$$

A great change happens here! We have eliminated all sequents which are copies of some sequents in $G^*$ and convert $G|G^*$ into $G^\Box_{1}$ in which each sequent is some copy of a sequent in $G^*_0$.

Then $\nu_{G} D(G^\Box_{1})$ by Lemma 5.6, where $D(G^\Box_{1}) = H_0 = \\
\Rightarrow C, B | C \Rightarrow B, A \land A | B \Rightarrow C, \neg A \land \neg A | C, B \Rightarrow A \land A, \neg A \land \neg A.$

So we have built up one-one correspondences between the proof of $G^\Box_{1}$ and that of $H_0$, i.e., the proof of $H_0$ can be constructed by applying $(D)$ to the proof of $G^\Box_{1}$. The major steps of constructing $G^\Box_{1}$ are shown in the following figure, where $D(G|G^*) = H_1, D(G^\Box_{H_i|G|G^*}) = H_2, D(G^\Box_{H_i'|G|G^*}) = H_3$ and $D(G^\Box_{0}) = H_0$.

In the above example, $D(G^\Box_{1}) = D_0(G_0)$. But it is not always the case. In general, we can prove that $\nu_{G} D_0(G_0)$ if $\nu_{G} D(G^\Box_{1})$, which is shown in the proof of Main lemma in Page 46. This example shows that the proof of Main lemma essentially presents an algorithm to construct a proof of $D_0(G_0)$ from $\tau$. 

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4. Preprocessing of Proof Tree

Let $\tau$ be a cut-free proof of $G_0$ in Main Lemma in $\text{GL}$ by Lemma 2.15. Starting with $\tau$, we will construct a proof $\tau^*$ which contains no application of $(EC)$ and have some other properties in this section.

**Lemma 4.1.** (i) If $\vdash_{\text{GL}} \Gamma_1 \Rightarrow A, \Delta_1$ and $\vdash_{\text{GL}} \Gamma_2 \Rightarrow B, \Delta_2$ then $\vdash_{\text{GL}} \Gamma_1 \Rightarrow A \land B, \Delta_1 | \Gamma_2 \Rightarrow A \land B, \Delta_2$;

(ii) If $\vdash_{\text{GL}} \Gamma_1, A \Rightarrow \Delta_1$ and $\vdash_{\text{GL}} \Gamma_2, B \Rightarrow \Delta_2$ then $\vdash_{\text{GL}} \Gamma_1, A \lor B \Rightarrow \Delta_1 | \Gamma_2, A \lor B \Rightarrow \Delta_2$.

**Proof.** (i)

\[ A \Rightarrow A \Rightarrow A \Rightarrow B \text{ (COM)} \]
\[ B \Rightarrow B \Rightarrow A \Rightarrow B \Rightarrow A \text{ (\(\land_r\))} \]
\[ \Gamma_1 \Rightarrow A, \Delta_1 \Rightarrow A \Rightarrow B, \Delta_2 \Rightarrow A \Rightarrow B, \Delta_2 \text{ (CUT)} \]
\[ \Gamma_2 \Rightarrow B, \Delta_2 \Rightarrow A \Rightarrow B, \Delta_2 \text{ (CUT)} \]

(ii) is proved by a procedure similar to that of (i) and omitted. \[ \square \]

We introduce two new inference rules by Lemma 4.1.

**Definition 4.2.** $G_1[|\Gamma_1 \Rightarrow A, \Delta_1 \Rightarrow G_2 | \Gamma_2 \Rightarrow B, \Delta_2 \Rightarrow (\land_w)]$

and $G_1[\Gamma_1, A \Rightarrow \Delta_1 \Rightarrow G_2[\Gamma_2, B \Rightarrow \Delta_2 \Rightarrow (\lor_w)]$ are called the generalized $(\land_r)$ and $(\lor_r)$ rules, respectively.

Now, we begin to process $\tau$ as follows.

**Step 1** A proof $\tau'$ is constructed by replacing inductively all applications of

\[ G_1[\Gamma \Rightarrow A, \Delta, G_2[\Gamma \Rightarrow B, \Delta \Rightarrow \land_r)] \text{ (or) } G_1[\Gamma, A \Rightarrow \Delta, G_2[\Gamma, B \Rightarrow \Delta \Rightarrow \lor_r)] \]

in $\tau$ with

\[ G_1[\Gamma \Rightarrow A, \Delta, G_2[\Gamma \Rightarrow B, \Delta \Rightarrow \land_w)] \]

\[ G_1[\Gamma, A \Rightarrow \Delta, G_2[\Gamma, B \Rightarrow \Delta \Rightarrow \lor_w)] \]

(accordingly $G_1[\Gamma, A \Rightarrow \Delta, G_2[\Gamma, B \Rightarrow \Delta \Rightarrow \lor_r]$)

The replacements in Step 1 are local and the root of $\tau^*$ is also labeled by $G_0$.

**Definition 4.3.** We sometimes may regard $G'_0$ as a structural rule of $\text{GL}$ and denote it by $(ID_{\Omega})$ for convenience. The focus sequent for $(ID_{\Omega})$ is undefined.
Lemma 4.4. Let $\frac{G[S^m]}{G'|S}$ be $\tau'$, $Th\tau'(G'|S) = (H_0, H_1, \cdots, H_n)$, where $H_0 = G'|S$ and $H_n = G_0$. A tree $\tau_1$ is constructed by replacing each $H_k$ in $\tau'$ with $H_k|S^m$ for all $0 \leq k \leq n$. Then $\tau_1$ is a proof of $G_0|S^m$.

Proof. The proof is by induction on $n$. Since $\tau'(G'|S^m)$ is a proof and $\frac{G[S^m]}{G'|S}$ is a proof, $\frac{G[S^m]}{G_0|S^m}$ is valid in $GL$, then $\tau_1(H_0|S^m)$ is a proof. Suppose that $\tau_1(H_{n-1}|S^m)$ is a proof. Since $\frac{H_{n-1}}{H_n}G''(I)$

(Step 2) Let $\frac{G''}{G'}|S^m, \cdots, G''|S^m} = \{EC\}$ be all applications of $\{EC\}$ in $\tau'$ and $G''_n := \{S^m)^{m-1}, \cdots, \{S^m\}_{n-1}$. By repeatedly applying sequent-inserting operation, we construct a proof of $H|G''_n$ in $GL$ and denote it by $\tau''$.

Remark 4.6. (i) $\tau''$ is constructed by converting $\{EC\}$ into $(ID\omega)$; (ii) Each node of $\tau''$ has the form $H_0|H''_0$, where $H_0 \in \tau$ and $H''_0$ is a (possibly empty) subset of $G''_n$.

Construction 4.7. Let $H \in \tau''$, $H' \subseteq H$ and $Th\tau''(H) = (H_0, \cdots, H_n)$, where $H_0 = H$, $H_n = G_0|G''_n$. Hyper sequents $(H_k)|H''_k$ and trees $\tau''_H((H_k)|H''_k)$ for all $0 \leq k \leq n$ are constructed inductively as follows.

(i) $(H_0)|H''_0 := H'$ and $\tau''_H((H_0)|H''_0)$ be built up with $H'$.

(ii) Let $\frac{G'|S'}{G''|S''} = \{EC\}$ be in $\tau''$, $H_k = G'|S'$ and $H_{k+1} = G''|S''$ for some $0 \leq k \leq n - 1$.

If $S' \in (H_k)|H''_k$, $(H_{k+1})|H''_k := (H_k)|H''_k \setminus \{S'\}|G''|H''$ and $\tau''_H((H_{k+1})|H''_k)$ is constructed by combining trees $\tau''_H((H_k)|H''_k), \tau''(G''|S'')$ with $\frac{H_{k+1}|H''_k}{H_k}$ and $\tau''_H((H_{k+1})|H''_k)$ for (I)

otherwise $(H_{k+1})|H''_k := (H_k)|H''_k$ and $\tau''_H((H_{k+1})|H''_k)$ is constructed by combining $\tau''_H((H_k)|H''_k), \tau''(G''|S'')$ with $\frac{H_{k+1}|H''_k}{H_k}$.
\[(i)\] Let \(\frac{G'}{G'}(EW) \in \tau''\), \(H_k = G'\) and \(H_{k+1} = G'|S'\) then \((H_{k+1})_{H|H'} := (H_k)_{H|H'}\) and \(\tau''_{H|H'}((H_{k+1})_{H|H'})\) is constructed by combining \(\tau''_{H|H'}((H_k)_{H|H'})\) with \(\frac{(H_k)_{H|H'}}{(H_{k+1})_{H|H'}}(ID_\Omega)\).

**Lemma 4.8.** (i) \(\langle H_k \rangle_{H|H'} \subseteq H_k\) for all \(0 \leq k \leq n\);
(ii) \(\tau''_{H|H'}((H_k)_{H|H'})\) is a derivation of \((H_k)_{H|H'}\) from \(H'\) without \((EC)\).

**Proof.** The proof is by induction on \(k\). For the base step, \((H_0)_{H|H'} = H'\) and \(\tau''_{H|H'}((H_0)_{H|H'})\) is built up with \(H'\). Then \((H_0)_{H|H'} \subseteq H_0 = H\), \(\tau''_{H|H'}((H_0)_{H|H'})\) is a derivation of \((H_0)_{H|H'}\) from \(H'\) without \((EC)\).

For the induction step, suppose that \((H_k)_{H|H'}\) and \(\tau''_{H|H'}((H_k)_{H|H'})\) be constructed such that (i) and (ii) hold for some \(0 < k < n - 1\). There are two cases to be considered.

**Case 1** Let \(\frac{G'|S''}{G'|S''}(I) \in \tau''\), \(H_k = G'|S'\) and \(H_{k+1} = G'|S''\). If \(S' \in (H_k)_{H|H'}\) then \((H_{k+1})_{H|H'} \subseteq (\{S'\} \subseteq G') \subseteq H_k = G'|S'\). Thus \((H_{k+1})_{H|H'} = ((H_k)_{H|H'} \setminus \{S'\})S'' \subseteq G'|S'' = H_{k+1}\). Otherwise \(S' \notin (H_k)_{H|H'}\) then \((H_{k+1})_{H|H'} \subseteq G' \subseteq (H_k)_{H|H'} \subseteq H_k = G'|S'\). Hence \((H_{k+1})_{H|H'} \subseteq (H_k)_{H|H'} \subseteq G' \subseteq H_k\). \(\tau''_{H|H'}((H_{k+1})_{H|H'})\) is a derivation of \((H_{k+1})_{H|H'}\) from \(H'\) without \((EC)\) since \(\tau''_{H|H'}((H_k)_{H|H'})\) is such one and \(\frac{(H_k)_{H|H'}}{(H_{k+1})_{H|H'}}(I)\) is a valid inference instance of \(GL\). The case of applications of two-premise rule is proved by a similar procedure and omitted.

**Case 2** Let \(\frac{G'|S'}{G'/S'}(EW) \in \tau''\), \(H_k = G'|S'\) and \(H_{k+1} = G'|S'\). Then \((H_{k+1})_{H|H'} \subseteq H_k = G'|S'\). \((H_{k+1})_{H|H'} = (H_k)_{H|H'} \subseteq H_{k+1}\). \(\tau''_{H|H'}((H_{k+1})_{H|H'})\) is a derivation of \((H_{k+1})_{H|H'}\) from \(H'\) without \((EC)\) since \(\tau''_{H|H'}((H_k)_{H|H'})\) is such one and \(\frac{(H_k)_{H|H'}}{(H_{k+1})_{H|H'}}(ID_\Omega)\) is valid by Definition 4.3.

**Definition 4.9.** The manipulation described in Construction 4.7 is called derivation-pruning operation.

**Notation 4.10.** We denote \((H_n)_{H|H'}\) by \(G'|S'\), \(\tau''_{H|H'}((H_n)_{H|H'})\) by \(\tau''_{H|H'}\) and say that \(H'\) is transformed into \(G'_n\) in \(\tau''\).

Then Lemma 4.8 shows that \(\frac{H'}{G'_n}(\tau''_{H|H'}), G''_{H|H'} \subseteq G_0|G'_0\). Now, we continue to process \(\tau\) as follows.

**Step 3** Let \(\frac{G'}{G'/S'}(EW) \in \tau''\) then \(\tau''_{G'/S'}((H_n)_{G'/S'})\) is a derivation of \((H_n)_{G'/S'}\) from \(G'\) thus a proof of \((H_n)_{G'/S'}\) is constructed by combining \(\tau''(G')\) and \(\tau''_{G'/S'}((H_n)_{G'/S'})\) with \(G'(ID_\Omega)\). By repeatedly applying the procedure above, we construct a proof \(\tau''\) without \((EW)\) of \(G_1|G'_1\) in \(GL\), where \(G_1 \subseteq G_0, G'_1 \subseteq G'_0\) by Lemma 4.8 (i).

**Step 4** Let \(\Gamma, p, \perp \Rightarrow \Delta \in \tau''\) (or \(\Gamma, p \Rightarrow \tau, \Delta\), \(\frac{G[\Gamma \Rightarrow \Delta]}{G[\Gamma, p \Rightarrow \Delta]}(WL)\)) then there exists \(\Gamma' \Rightarrow \Delta' \in H\) such that \(p \in \Gamma'\) for all \(H \in Th_{\tau''}(\Gamma, p, \perp \Rightarrow \Delta)\) (accordingly \(H \in Th_{\tau''}(\Gamma, p \Rightarrow \tau, \Delta)\), \(H \in Th_{\tau''}(\Gamma, p \Rightarrow \Delta)\)) thus a proof is constructed by replacing top-down \(p\) in each \(\Gamma'\) with \(\tau\).
Let $\Gamma, \bot \Rightarrow p, \Delta$ (or $\Gamma \Rightarrow \tau, p, \Delta$). $G[\Gamma] \Rightarrow \Delta (WR)$ is a leaf of $\tau''$ then there exists $\Gamma' \Rightarrow \Delta' \in H$ such that $p \in \Delta'$ for all $H \in Th_{\tau''}(\Gamma \Rightarrow \bot, p, \Delta)$ (accordingly $H \in Th_{\tau''}(\Gamma \Rightarrow \tau, p, \Delta)$ or $H \in Th_{\tau}(\Gamma \Rightarrow p, \Delta)$) thus a proof is constructed by replacing top-down $p$ in each $\Gamma'$ with $\bot$.

Repeatedly applying the procedure above, we construct a proof $\tau'''$ of $G[\Gamma'] \Rightarrow \Delta$ in $GL$ such that there doesn’t exist occurrence of $p$ in $\Gamma$ or $\Delta$ at each leaf labeled by $\Gamma, \bot \Rightarrow \Delta$ or $\Gamma \Rightarrow \tau, \Delta$, or $p$ is not the weakening formula $A$ in $G[\Gamma] \Rightarrow \Delta (WR)$ or $G[\Gamma, A] \Rightarrow \Delta$ (WL) when (WR) or (WL) is available. Define two operations $\sigma_1$ and $\sigma_2$ on sequents by $\sigma_1(\Gamma, p \Rightarrow \Delta) := \Gamma, \tau \Rightarrow \Delta$ and $\sigma_2(\Gamma \Rightarrow p, \Delta) := \Gamma \Rightarrow \bot, \Delta$. Then $G[\Gamma] \Rightarrow \Delta$ is obtained by applying $\sigma_1$ and $\sigma_2$ to some designated sequents in $G[\Gamma]$.

Definition 4.11. The manipulation described in Step 4 is called eigenvariable-replacing operation.

Step 5 A proof $\tau'$ is constructed from $\tau'''$ by assigning inductively one unique identification number to each occurrence of $p$ in $\tau'''$ as follows.

One unique identification number, which is a positive integer, is assigned to each leaf of the form $p \Rightarrow p$ in $\tau'''$ which corresponds to $p_k \Rightarrow p_k$ in $\tau'$. Other nodes of $\tau'''$ are processed as follows.

- Let $G[\Gamma, \lambda \Rightarrow p, \Delta] \Rightarrow \mu p, \Delta$ and $G[\Gamma', \lambda \Rightarrow p, \Delta] \Rightarrow \mu p, \lambda \Rightarrow p, \Delta$. Suppose that all occurrences of $p$ in $G[\Gamma, \lambda \Rightarrow p, \Delta]$ are assigned identification numbers and have the form $G[\Gamma, \{p_{i_k}\}_{k=1}^j \Rightarrow \{p_{j_k}\}_{k=1}^j, \Delta]$ in $\tau''$, which we often write as $G[\Gamma, \{p_{i_k}\}_{k=1}^j \Rightarrow \{p_{j_k}\}_{k=1}^j, \Delta]$. Then $G[\Gamma', \lambda \Rightarrow p, \Delta]$ have the form $G[\Gamma', \{p_{i_k}\}_{k=1}^j \Rightarrow \{p_{j_k}\}_{k=1}^j, \lambda \Rightarrow p, \Delta]$.

- Let $G[\Gamma, (\land \mu)] \Rightarrow \tau'''$, where $G[\Gamma', \lambda \Rightarrow p, \Delta] \Rightarrow \mu p, A, \Delta$ and $\mu, A \Rightarrow B, \Delta$. Suppose that $G'$ and $G''$ have the forms $G[\Gamma, \{p_{i_k}\}_{k=1}^j \Rightarrow \{p_{j_k}\}_{k=1}^j, A, \Delta]$ and $G[\Gamma', \{p_{i_k}\}_{k=1}^j \Rightarrow \{p_{j_k}\}_{k=1}^j, \lambda \Rightarrow p, B, \Delta]$. Then $G''$ have the form $G''[\Gamma, \{p_{i_k}\}_{k=1}^j \Rightarrow \{p_{j_k}\}_{k=1}^j, A \Rightarrow B, \Delta]$. All applications of $\land \mu$ are processed by the procedure similar to that of $(\land \nu)$.

- Let $G[\Gamma, (\lor \mu)] \Rightarrow \tau'''$, where $G' \equiv G[\Gamma, \lambda \Rightarrow p, \Delta] \Rightarrow \mu p, A, \Delta$, $G'' \equiv G[\Gamma, \lambda \Rightarrow p, \Delta] \Rightarrow \mu p, A \Rightarrow B, \Delta$. Suppose that $G'$ and $G''$ have the forms $G[\Gamma, \{p_{i_k}\}_{k=1}^j \Rightarrow \{p_{j_k}\}_{k=1}^j, A, \Delta]$ and $G[\Gamma', \{p_{i_k}\}_{k=1}^j \Rightarrow \{p_{j_k}\}_{k=1}^j, \Delta \Rightarrow p, B, \Delta]$. Then $G''$ have the form $G''[\Gamma, \{p_{i_k}\}_{k=1}^j \Rightarrow \{p_{j_k}\}_{k=1}^j, A \Rightarrow B, \Delta]$. All applications of $\lor \mu$ are processed by the procedure similar to that of $(\lor \nu)$.

- Let $G[\Gamma, (\lor \mu)] \Rightarrow \tau'''$, where $G' \equiv G[\Gamma, \lambda \Rightarrow p, \Delta] \Rightarrow \mu p, A, \Delta$, $G'' \equiv G[\Gamma, \lambda \Rightarrow p, \Delta] \Rightarrow \mu p, A \Rightarrow B, \Delta$. Suppose that $G'$ and $G''$ have the forms $G[\Gamma, \{p_{i_k}\}_{k=1}^j \Rightarrow \{p_{j_k}\}_{k=1}^j, A, \Delta]$ and $G[\Gamma', \{p_{i_k}\}_{k=1}^j \Rightarrow \{p_{j_k}\}_{k=1}^j, \Delta \Rightarrow p, B, \Delta]$. Then $G''$ have the form $G''[\Gamma, \{p_{i_k}\}_{k=1}^j \Rightarrow \{p_{j_k}\}_{k=1}^j, A \Rightarrow B, \Delta]$. All applications of $\lor \mu$ are processed by the procedure similar to that of $(\lor \nu)$.
Definition 4.12. The manipulation described in Step 5 is called eigenvariable-labeling operation.

Notation 4.13. Let $G$ and $G_2$, where $G_2 = (C, \tau, \Sigma_1, \Sigma_2)$, be members of $G$. So we assume that all $H_i$ is converted to $G_i$, respectively. Then $\tau^*$ is a proof of $G|G^*$.

In preprocessing of $\tau$, each $G_i$ is converted into $G_i^{\tau}$ in Step 2, where $G_i^{\tau}$ is a pseudo-EC $(G_i^{\tau} | \Sigma_1, \Sigma_2)$, as described in Step 4. Each occurrence of $p \in \tau^*$ is assigned the unique identification number in Step 5. The whole preprocessing above is depicted by Figure 13.

![Figure 13 Preprocessing of $\tau$](image)

Notation 4.14 (See Appendix 2). Let $G_i^{\tau} | \Sigma_1, \Sigma_2$ be a pseudo-EC $(G_i^{\tau} | \Sigma_1, \Sigma_2)$, $1 \leq i \leq N$ be all $(G_i^{\tau} | \Sigma_1, \Sigma_2)$-nodes of $\tau^*$ and $G_i^{\tau} | \Sigma_1, \Sigma_2$ is a proof of $G_i^{\tau} | \Sigma_1, \Sigma_2$ in $\tau^*$. Note that there are no identification numbers for occurrences of variable $p$ in $S_i^{\tau} \in G_i^{\tau} | \Sigma_1, \Sigma_2$ while the are assigned to $p$ in $S_i^{\tau} \in G_i^{\tau} | \Sigma_1, \Sigma_2$. But we use the same notations for $S_i^{\tau}$ and $S_i^{\tau}$ for simplicity.

In the whole paper, let $G_i^{\tau} | \Sigma_1, \Sigma_2$ denote the unique node of $\tau^*$ such that $G_i^{\tau} | \Sigma_1, \Sigma_2$ is the focus sequent of $G_i^{\tau}$, $1 \leq i \leq N$. We sometimes denote $G_i^{\tau}$ also by $G_i^{\tau}$ in Step 4, Step 5 and others $S_i^{\tau}$ among $S_i^{\tau}$ for simplicity. We then write $G_i^{\tau}$ as $\{S_i^{\tau} \}_{i=1}^{i=N}$.

We call $H_i^{\tau}$ the $i$-th pseudo-(EC) node of $\tau^*$ and pseudo-(EC) sequent, respectively. We abbreviate pseudo-EC as pEC. Let $H \in \tau^*$, by $S_i^{\tau} \in H$ we mean that $S_i^{\tau} \in H$ for some $1 \leq u \leq m$. It is possible that there doesn’t exist $H_i^{\tau} \in G_i^{\tau} | \Sigma_1, \Sigma_2$, such that $S_i^{\tau}$ is the focus sequent of $H_i^{\tau}$ in $\tau^*$, in which case $\{S_i^{\tau} \}_{i=1}^{i=N} \subseteq G|G^*$, then it hasn’t any effect on our argument to treat all such $S_i^{\tau}$ as members of $G$. So we assume that all $H_i^{\tau}$ are always defined for all $G_i^{\tau} | \Sigma_1, \Sigma_2$ in $\tau^*$, i.e., $H_i^{\tau} \in H$.

Proposition 4.15. (i) $\{S_i^{\tau} \}_{i=1}^{i=N} \subseteq H$ for all $H \leq H_i^{\tau}$; (ii) $G^{\tau} = \{S_i^{\tau} \}_{i=1}^{i=N}$.

Now, we replace locally each $G_i^{\tau} | \Sigma_1, \Sigma_2$ in $\tau^*$, with $G_i^{\tau}$ and denote the resulting proof also by $\tau^*$, which has no essential difference with the original one but could simplify subsequent arguments. We introduce the system $GL_{\Omega}$ as follows.
Definition 4.16. \(GL_\Omega\) is a restricted subsystem of \(GL\) such that

(i) \(p\) is designated as the unique eigenvariable by which we mean that it is not used to built up any formula containing logical connectives and only used as a sequent-formula.

(ii) Each occurrence of \(p\) in a hypersequent of \(GL\) is assigned one unique identification number \(i\) and written as \(p_i\) in \(GL_\Omega\). Initial sequent \(p \Rightarrow p\) of \(GL\) has the form \(p_i \Rightarrow p_j\) in \(GL_\Omega\).

(iii) Each sequent \(S\) of \(GL\) in the form \(\Gamma, \lambda p \Rightarrow \mu p, \Delta\) has the form

\[\Gamma, \{p_i\}_{k=1}^i \Rightarrow \{p_h\}_{k=1}^j, \Delta\]

in \(GL_\Omega\), where \(p\) does not occur in \(\Gamma\) or \(\Delta\), \(i_k \neq j_l\) for all \(1 \leq k < l \leq \lambda, j_k \neq j_l\) for all \(1 \leq k < l \leq \mu\). Define \(v_i(S) = \{i_1, \ldots, i_k\}\) and \(v_j(S) = \{j_1, \ldots, j_l\}\). Let \(G\) be a hypersequent of \(GL_\Omega\) in the form \(S \vdash \cdots \vdash S\), then \(v_i(S_k) \cap v_i(S_l) = \emptyset\) and \(v_j(S_k) \cap v_j(S_l) = \emptyset\) for all \(1 \leq k < l \leq n\). Define \(v_i(G) = \bigcup_{k=1}^i v_i(S_k)\), \(v_j(G) = \bigcup_{j=1}^j v_j(S_j)\).

(iv) A hypersequent \(G\) of \(GL_\Omega\) is called closed if \(v_i(G) = v_i(G)\). Two hypersequents \(G'\) and \(G''\) of \(GL_\Omega\) are called disjoint if \(v_i(G') \cap v_i(G'') = \emptyset\), \(v_j(G') \cap v_j(G'') = \emptyset\), \(v_j(G') \cap v_j(G'') = \emptyset\) and \(v_i(G') \cap v_i(G'') = \emptyset\). \(G''\) is a copy of \(G'\) if they are disjoint and there exist two bijections \(\sigma_r : v_i(G') \rightarrow v_i(G'')\) and \(\sigma_r : v_j(G') \rightarrow v_j(G'')\) such that \(G''\) can be obtained by applying \(\sigma_r\) to antecedents of sequents in \(G'\) and \(\sigma_r\) to succedents of sequents in \(G'\), i.e., \(G'' = \sigma_r \circ \sigma_r(G')\).

(v) A closed hypersequent \(G''|G''|G''\) can be contracted as \(G''|G''\) in \(GL_\Omega\) under the condition that \(G''\) and \(G''\) are closed and \(G''\) is a copy of \(G''\). We call it the constraint external contraction rule and denote by

\[\frac{G''|G''|G''}{G''|G''}(EC_\Omega)\]

Furthermore, if there doesn’t exist two closed hypersequents \(H', H'' \subseteq G''|G''\) such that \(H''\) is a copy of \(H'\) then we call it the fully constraint contraction rule and denote by

\[\frac{G''|G''|G''}{G''|G''}(EC_\Omega)\]

(vi) \((EW)\) and \((CUT)\) of \(GL\) are forbidden. \((EC)\), \((\wedge_p)\) and \((\vee_l)\) of \(GL\) are replaced with \((EC_\Omega)\), \((\wedge)\) and \((\vee)\) in \(GL_\Omega\), respectively.

(vii) \(G_1|S_1\) and \(G_2|S_2\) are closed and disjoint for each two-premise inference rule \(\frac{G_1|S_1 \quad G_2|S_2}{G_1|G_2|\wedge'} (H)\) of \(GL_\Omega\) and \(G'\) is closed for each one-premise inference rule \(\frac{G'|S'}{G'|S'} (I)\).\n
(viii) \(p\) doesn’t occur in \(\Gamma\) or \(\Delta\) for each initial sequent \(\Gamma, \lambda \Rightarrow \mu, \Delta\) and, \(p\) doesn’t act as the weakening formula \(A\) in \(\frac{G|\Gamma \Rightarrow \Delta}{G|\Gamma \Rightarrow A, \Delta} (WR)\) or \(\frac{G|\Gamma \Rightarrow \Delta}{G|\Gamma, A \Rightarrow \Delta} (WL)\) when \((WR)\) or \((WL)\) is available.

Lemma 4.17. Let \(\tau\) be a cut-free proof of \(G_\Omega\) in \(L\) and \(\tau^*\) be the tree resulting from preprocessing of \(\tau\).

(i) If \(\frac{G''|S'' |G''|S'' }{G''|S'' |S'' } (I) \in \tau^*\) then \(v_i(G''|S'' ) = v_j(G''|S'' ) = v_i(S'' ) = v_j(S'' )\):

(ii) If \(\frac{G''|S'' |G''|S'' }{G''|S'' |S'' } (I) \in \tau^*\) then \(v_i(G''|S'' |H') = v_j(G''|S'' |H') = v_j(S'' ) = v_j(S'' )\):

(iii) If \(H \in \tau^*\) and \(k \in v_i(H)\) then \(k \in v_i(H)\):

(iv) If \(H \in \tau^*\) and \(k \in v_i(H)\) then \(H \leq p_k \Rightarrow p_i\):

(v) \(\tau^*\) is a proof of \(G''|G''\) in \(GL_\Omega\) without \((EC_\Omega)\):

(vi) If \(H', H'' \in \tau^*\) and \(H' \in H''\) then \(v_i(H') \cap v_i(H'') = \emptyset, v_j(H') \cap v_j(H'') = \emptyset\).
Proof. Claims from (i) to (iv) are immediately from Step 5 in preprocessing of \(\tau\) and Definition 4.16. (v) is from Notation 4.13 and Definition 4.16. Only (vi) is proved as follows.

Suppose that \( k \in v_l(H') \cap v_l(H'') \). Then \( H' \leq p_k \Rightarrow p_k, H'' \leq p_k \Rightarrow p_k \) by Claim (iv). Thus \( H' \leq H'' \) or \( H'' \leq H' \), a contradiction with \( H'' \mid H' \) hence \( v_l(H') \cap v_l(H'') = \emptyset \).

5. The generalized density rule (D) for GL\(_{\Omega}\)

In this section, we define the generalized density rule (D) for GL\(_{\Omega}\) and prove that it is admissible in GL\(_{\Omega}\).

Definition 5.1. Let \( G \) be a closed hypersequent of GL\(_{\Omega}\) and \( S \in G \). Define \([S]_G = \bigcap \{ H : S \in H \leq G, v_r(H) = v_r(H) \} \), i.e., \([S]_G\) is the minimal closed sub-hypersequent of \( G \) such that \( S \in [S]_G \subseteq G \). In general, for \( G' \subseteq G \), define \([G']_G = \bigcap \{ H : G' \subseteq H \leq G, v_r(H) = v_r(H) \} \).

Clearly, \([S]_G = S\) if \( v_r(S) = v_r(S) \) or \( p \) does not occur in \( S \).

Construction 5.2. Let \( G \) and \( S \) be as above. A sequence \( G_1, G_2, \ldots, G_n \) of hypersequents is constructed recursively as follows. (i) \( G_1 = \{ S \} \); (ii) Suppose that \( G_k \) is constructed for \( k \geq 1 \). If \( v_l(G_k) \neq v_r(G_k) \) then there exists \( i_k \in v_l(G_k) \setminus v_r(G_k) \) (or \( i_k \in v_r(G_k) \setminus v_l(G_k) \)) then there exists the unique \( S_{k+1} \in G G_3 \) such that \( i_k + 1 \in v_l(S_{k+1}) \setminus v_r(G_k) \) (or \( i_k + 1 \in v_r(S_{k+1}) \setminus v_l(G_k) \)) by \( v_l(G) = v_r(G) \) and Definition 4.16 then let \( G_{k+1} = G_k[S_{k+1}] \) otherwise the procedure terminates and \( n = k \).

Lemma 5.3. (i) \( G_n = [S]_G \); (ii) Let \( S' \in [S]_G \) then \([S']_G = [S]_G\); (iii) Let \( G' \equiv G'/H', G'' \equiv G'H'' \), \( v_r(G') = v_r(G'), v_l(G''') = v_r(G''), v_r(H'') = v_r(H'') \) then \([H']_G \setminus [H'']_G = [H''']_G \setminus [H''']_G\), where \( A \setminus B \) is the symmetric difference of two multisets \( A, B \); (iv) Let \( v_l(G_k) = v_l(G_k) \cap v_r(G_k) \) then \( |v_l(G_k)| + 1 \geq |G_k| \) for all \( 1 \leq k < n \); (v) \( v_l([S]_G) + 1 \geq [S]_G \).

Proof. (i) Since \( G_k \subseteq G_{k+1} \) for \( 1 \leq k \leq n - 1 \) and \( S \in G_1 \) then \( S \subseteq G_n \subseteq G \) thus \([S]_G \subseteq S[G] \) by \( v_l(G_k) = v_r(G_k) \). We prove \( G_k \subseteq [S]_G \) for \( 1 \leq k \leq n \) by induction on \( k \) in the following. Clearly, \( G_1 \subseteq [S]_G \). Suppose that \( G_k \subseteq [S]_G \) for some \( 1 \leq k \leq n - 1 \). Since \( i_k + 1 \in v_l(G_k) \setminus v_r(G_k) \) (or \( i_k + 1 \in v_r(G_k) \setminus v_l(G_k) \)) and \( i_k + 1 \in v_l(S_{k+1}) \) (or \( i_k + 1 \in v_r(S_{k+1}) \)) then \( S_{k+1} \subseteq [S]_G \) by \( G_k \subseteq [S]_G \) and \( v_l(S_{k+1}) \subseteq [S]_G \), \( v_l(S_{k+1}) \subseteq [S]_G \) thus \( G_{k+1} \subseteq [S]_G \). Then \( G_n \subseteq [S]_G \) thus \( G_n = [S]_G \).

(ii) By (i), \([S]_G = S[s_2]\cdots[s_n] \), where \( S_1 = S \). Then \( S' = S_k \) for some \( 1 \leq k \leq n \) thus \( i_k \in v_l(S_k) \setminus v_r(S_k) \) (or \( i_k \in v_l(S_k) \setminus v_r(S_k) \)) hence there exists the unique \( k' \) such that \( i_k \in v_l(S_k) \setminus v_r(S_k) \) (or \( i_k \in v_l(S_k) \setminus v_r(S_k) \)) if \( k \geq k' \) hence \( S_{k'} \in [S_k]_G \). Repeatedly, \( S_k \subseteq [S_k]_G \), \( S_k' \subseteq [S_k']_G \), \( S_k' \subseteq [S_k']_G \) by \( S' \subseteq [S]_G \) then \([S']_G = [S]_G \).

(iii) It holds immediately from Construction 5.2 and (i).

(iv) The proof is by induction on \( k \). For the base step, let \( k = 1 \) then \( |G_k| = 1 \) thus \( |v_l(G_k)| + 1 \geq |G_k| \) by \( v_l(G_k) > 0 \). For the induction step, suppose that \( |v_l(G_k)| + 1 \geq |G_k| \) for some \( 1 \leq k < n \). Then \( |v_l(G_{k+1})| \geq |v_l(G_k)| + 1 \) by \( i_k + 1 \in v_l(G_{k+1}) \setminus v_l(G_k) \) and \( v_l(G_k) \subseteq v_l(G_{k+1}) \). Then \( |v_l(G_{k+1})| + 1 \geq |G_{k+1}| \) by \( |G_{k+1}| = |G_k| + 1 = k + 1 \).

(v) It holds by (iv) and \( v_l([S]_G) = v_l([S]_G) \).
Definition 5.4. Let $G = S_1 | \cdots | S_r$ and $S_1$ be in the form $\Gamma_i: \{p_i^k\}_{k=1}^i \Rightarrow \{p_i^k\}_{k=1}^i, \Delta_i$ for $1 \leq i \leq r$.

(i) If $S \in G$ and $[S]_G$ be $S_k | \cdots | S_k$ then $D_G(S)$ is defined as
$$
\Gamma_{k_1} \cdots \Gamma_{k_s} \Rightarrow ([S_1]_G) - \{[S_k]_G\} + 1, r, \Delta_k, \cdots, \Delta_k ;
$$
(ii) Let $\bigcup_{k=1}^r [S_k]_G = G$ and $[S_k]_G \cap [S_k]_G = \varnothing$ for all $1 \leq k < l \leq r$ then $D_G(G)$ is defined as
$$
D_G(S) = \bigcup_{k=1}^r D_G(S_k) ;
$$
(iii) We call $(D)$ the generalized density rule of $GL_1$, whose conclusion $D(G)$ is defined by
(ii) if its premise is $G$.

Clearly, $D(p_k \Rightarrow p_k)$ is $\Rightarrow t$ and $D(S) = S$ if $p$ does not occur in $S$.

Lemma 5.5. Let $G' \equiv G[S$ and $G'' \equiv G[S_1]_G \cap [S_2]_G = \varnothing$, where $S_1 = \Gamma_1, \{p_i^k\}_{k=1}^1 \Rightarrow \{p_i^k\}_{k=1}^1, \Delta_1$; $S_2 = \Gamma_2, \{p_i^k\}_{k=1}^1 \Rightarrow \{p_i^k\}_{k=1}^1, \Delta_2$;

$S = \Gamma_1, \Gamma_2, \{p_i^k\}_{k=1}^1, \{p_i^k\}_{k=1}^1 \Rightarrow \{p_i^k\}_{k=1}^1, \{p_i^k\}_{k=1}^1, \Delta_1, \Delta_2 ;$ $D_{G'}(S_1) = \Gamma_1, \Sigma_1 \Rightarrow \Pi_1, \Delta_1$ and $D_{G''}(S_2) = \Gamma_2, \Sigma_2 \Rightarrow \Pi_2, \Delta_2$. Then $D_G(S) = \Gamma_1, \Gamma_2, \Sigma_1 \Rightarrow \Pi_1, \Delta_1, \Pi_2, \Delta_2$.

Proof. Since $[S_1]_G \cap [S_2]_G = \varnothing$ then $[S]_G = [S_1]_G \cup [S_2]_G \cup \{[S_2]_G \}$ by $\forall v(S) = v(S_1) + v(S_2)$ and Lemma 5.3 (iii). Thus
$$
|v((S_1)_G)| = |v((S_1)_G)| + |v((S_2)_G)|, |[S_1]_G| = |[S_1]_G| + |[S_2]_G| - 1.
$$
Hence
$$
|v((S_1)_G)| - |[S_1]_G| + 1 = |v((S_1)_G)| - |[S_1]_G| + 1 + |v((S_2)_G)| - |[S_2]_G| + 1.
$$
Therefore $D_G(S) = \Gamma_1, \Gamma_2, \Sigma_1 \Rightarrow \Pi_1, \Delta_1, \Pi_2, \Delta_2$ by
$$
\Pi_1 = \{v((S_1)_G) - |[S_1]_G| + 1\}, \Pi_2 = \{v((S_2)_G) - |[S_2]_G| + 1\}, \Delta_1, \Delta_2 ;
$$
$$
\Pi_1 = \{v((S_1)_G) - |[S_1]_G| + 1\}, \Pi_2 = \{v((S_2)_G) - |[S_2]_G| + 1\}, \Delta_1, \Delta_2 ;
$$
$$
\Pi_1 = \{v((S_1)_G) - |[S_1]_G| + 1\} \Rightarrow \Pi_2 = \{v((S_2)_G) - |[S_2]_G| + 1\} \Rightarrow \Delta_1, \Delta_2 ;
$$
$$\Delta_1, \Delta_2 ;$$
where $\lambda = \{1, \ldots, t\}$.

Lemma 5.6. If there exists a proof $\tau$ of $G$ in $GL_1$ then there exists a proof of $D(G)$ in $GL$, i.e., $(D)$ is admissible in $GL_1$.

Proof. We proceed by induction on the height of $\tau$. For the base step, if $G$ is $p_k \Rightarrow p_k$ then $D(G)$ is $\Rightarrow t$ otherwise $D(G)$ is $G$ then $v_G. D(G)$ holds. For the induction step, the following cases are considered.

* Let
$$
\frac{G', S'}{G''} \in \tau
$$
where
$$
S' \equiv A, \Gamma, \{p_i^k\}_{k=1}^1 \Rightarrow \{p_i^k\}_{k=1}^1, \Delta, B,
$$
$$
S'' \equiv \Gamma, \{p_i^k\}_{k=1}^1 \Rightarrow \{p_i^k\}_{k=1}^1, \Delta, A \Rightarrow B.
$$
Then $[S'']_G \cap [S']_G \equiv [S']_G \cap [S''_G$ by $v(S') = v(S''_G) = v(S''_G)$ and Lemma 5.3 (iii). Let $D_{G'}(S') = \Gamma, \Gamma' \Rightarrow \Delta''_G, A \Rightarrow B$ then $D_{G''}(S'') = \Gamma, \Gamma'' \Rightarrow \Delta''_G, A \Rightarrow B$ thus a proof of
\( D(G'|S'') \) is constructed by combining the proof of \( D(G'|S') \) and \( \text{if } D(G'|S') \rightarrow r \). Other rules of type (I) are processed by a procedure similar to above.

• Let

\[
\frac{G_1|S_1 \ G_2|S_2}{G_1 G_2|S_3} \ (\lor_r) \in \tau
\]

where

\[
S_1 \equiv \Gamma_1, \{ p_{i_1}^1 \}_{k=1} \Rightarrow \{ p_{i_1}^2 \}_{k=1} A, \Delta_1
\]

\[
S_2 \equiv \Delta_2, \{ p_{i_2}^1 \}_{k=1} \Rightarrow \{ p_{i_2}^2 \}_{k=1} A \odot B, \Delta_1, \Delta_2
\]

Let

\[
D_{G_1|S_1}(S_1) = \Gamma_1, \Gamma_1 \Rightarrow \Delta_1, \{ \nu_1([S_1]_{G_1|S_1}) - [S_1]_{G_1|S_1} + 1 \} t, A, \Delta_1,
\]

\[
D_{G_2|S_2}(S_2) = \Gamma_2, \Gamma_2 \Rightarrow \Gamma_2, \{ \nu_2([S_2]_{G_2|S_2}) - [S_2]_{G_2|S_2} + 1 \} t, B, \Delta_2.
\]

Then \( D_{G_1|G_2|S_3}(S_3) \) is

\[
\Gamma_1, \Gamma_2, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \Delta_1 \odot B, \Delta_1, \Delta_2,
\]

\[
\{ \nu_1([S_1]_{G_1|S_1}) - [S_1]_{G_1|S_1} \} + \{ \nu_2([S_2]_{G_2|S_2}) - [S_2]_{G_2|S_2} \} + 2 t
\]

by \([S_3]_{G_1|G_2|S_3} = ([S_1]_{G_1|S_1} \setminus \{ S_1 \}) \cup ([S_2]_{G_2|S_2} \setminus \{ S_2 \}) \cup \{ S_3 \} \). Then the proof of \( D(G_1|G_2|S_3) \) is constructed by combining \( r_{GL} D(G_1|S_1) \) and

\[
\text{if } D(G_2|S_2) \rightarrow \text{if } D(G_1|G_2|S_3) \ (\lor_r) \in \tau.
\]

• Let

\[
\frac{G'}{G''} \ (\wedge_{rw}) \in \tau
\]

where

\[
G' \equiv G_1|S_1, \quad G'' \equiv G_2|S_2, \quad G''' \equiv G_1|G_2|S_3|S_2',
\]

\[
S_w \equiv \Gamma_w, \{ p_{i_1}^1 \}_{k=1} \Rightarrow \{ p_{i_1}^2 \}_{k=1} A_w, \Delta_w,
\]

\[
S_w' \equiv \Gamma_w, \{ p_{i_2}^1 \}_{k=1} \Rightarrow \{ p_{i_2}^2 \}_{k=1} A_1 \wedge A_2, \Delta_w
\]

for \( w = 1, 2 \). Then \([S_1]_{G'''} = [S_1]_{G'} \setminus \{ S_1 \} \setminus \{ S_2 \}, [S_2]_{G'''} = [S_2]_{G'} \setminus \{ S_2 \} \setminus \{ S_2 \}\) by Lemma 5.3 (iii).

Let

\[
D_{G_1|S_w}(S_w) = \Gamma_w, \Gamma_w \Rightarrow \Delta_w, \{ \nu_1([S_w]_{G_1|S_w}) - [S_w]_{G_1|S_w} + 1 \} t, A_w, \Delta_w
\]

for \( w = 1, 2 \). Then

\[
D_{G'''}(S_w') = \Gamma_w, \Gamma_w \Rightarrow \Delta_w, \{ \nu_1([S_w]_{G_1|S_w}) - [S_w]_{G_1|S_w} + 1 \} t, A_1 \wedge A_2, \Delta_w
\]

for \( w = 1, 2 \). Then the proof of \( D(G''') \) is constructed by combining \( r_{GL} D(G') \) and \( r_{GL} D(G'') \) with

\[
\text{if } D(G_1) \rightarrow \text{if } D(G_2') \rightarrow D(G''') \ (\wedge_{rw}) \in \tau.
\]

All applications of \((\lor_{rw})\) are processed by a procedure similar to that of \((\wedge_{rw})\) and omitted.
• Let
\[
\frac{G' G''}{G'''} (\text{COM}) \in \tau
\]

where
\[
G' \equiv G_1[S_1], \quad G'' \equiv G_2[S_2], \quad G''' \equiv G_1[S_2]S_4
\]

\[
S_1 \equiv G_1, \Pi_1, \{p_{_1}^{_{_1}k_{_1}}\}_{k_{_1}=1}^{_{_4}}, \Delta_1, \Sigma_1;
S_2 \equiv G_2, \Pi_2, \{p_{_2}^{_{_1}k_{_1}}\}_{k_{_1}=1}^{_{4}}, \Delta_2, \Sigma_2;
S_3 \equiv G_1, \Pi_1, \{p_{_3}^{_{_1}k_{_1}}\}_{k_{_1}=1}^{_{4}}, \Delta_1, \Sigma_1, \Delta_2;
S_4 \equiv G_1, \Pi_2, \{p_{_4}^{_{_1}k_{_1}}\}_{k_{_1}=1}^{_{4}}, \Delta_2, \Sigma_1, \Sigma_2;
\]

where \(\{p_{_l}^{_{_1}k_{_1}}\}_{k_{_1}=1}^{_{4}}\).\(\cup\{p_{_l'}^{_{_1}k_{_1}}\}_{k_{_1}=1}^{_{4}}\) = \(\{p_{_l}^{_{_1}k_{_1}}\}_{k_{_1}=1}^{_{4}}\) for w = 1, 2.

**Case 1** \(S_3 \in [S_4]_{G'''}.\) Then \([S_3]_{G'''} = [S_4]_{G'''}\) by Lemma 5.3 (ii) and \([S_3]_{G'''} = [S_1]_{G'''} \cup [S_2]_{G'''}\) by Lemma 5.3 (iii). Then
\[
|v_1([S_1]_{G'''})| - |[S_3]_{G'''}| + 1 = |v_1([S_1]_{G'''})| + |v_1([S_2]_{G'''})| - |[S_1]_{G'''}| - |[S_2]_{G'''}| + 1 \geq 0.
\]

Thus \(|v_1([S_1]_{G'''})| - |[S_1]_{G'''}| + 1 \geq 1\) or \(|v_1([S_2]_{G'''})| - |[S_2]_{G'''}| + 1 \geq 1\). Hence we assume that, without loss of generality,
\[
\mathcal{D}_{G'}(S_1) = G_1, \Pi_1, \Gamma' \Rightarrow \Delta', \Sigma_1, \Delta_1;
\]
\[
\mathcal{D}_{G'''}(S_2) = G_2, \Pi_2, \Gamma'' \Rightarrow \Delta'', \Sigma_2, \Delta_2.
\]

Then
\[
\mathcal{D}_{G'''}(S_3|S_4) = G_1, \Pi_1, \Gamma', \Pi_2, \Gamma'' \Rightarrow \Delta', \Sigma_1, \Delta_2.
\]

Thus the proof of \(\mathcal{D}_{G''}(S_3|S_4)\) is constructed by
\[
\Gamma_1, \Pi_1, \Gamma', \Delta', \Sigma_1, \Delta_1 \quad \Gamma_2, \Pi_2, \Gamma'' \Rightarrow \Delta', \Sigma_2, \Delta_2
\]
\[
\Gamma_1, \Pi_1, \Gamma', \Pi_2, \Gamma'' \Rightarrow \Delta'', \Sigma_1, \Delta_2 (\text{CUT}).
\]

**Case 2** \(S_3 \notin [S_4]_{G'''}\). Then \([S_3]_{G'''} \cap [S_4]_{G'''} = \emptyset\) by Lemma 5.3 (ii). Let
\[
S_{3w} \equiv G_w, \{p_{_{_2}w_{_1}}^{_{_1}k_{_1}}\}_{k_{_1}=1}^{_{4}}, \Delta_w, \Sigma_w;
S_{4w} \equiv G_w, \{p_{_{_2}w_{_1}}^{_{_1}k_{_1}}\}_{k_{_1}=1}^{_{4}}, \Delta_w, \Sigma_w;
\]

for w = 1, 2. Then
\[
[S_3]_{G'''} = [S_{31}]_{G_1[S_3;S_4]} \cup [S_{32}]_{G_2[S_3;S_4]} \cup [S_{33}] \cup [S_{34}];
[S_4]_{G'''} = [S_{41}]_{G_1[S_3;S_4]} \cup [S_{42}]_{G_2[S_3;S_4]} \cup [S_{43}] \cup [S_{44}];
\]

by \(v_1(S_3) = v_1(S_{32})\), \(v_1(S_3) = v_1(S_{31})S_{41}\), \(v_1(S_2) = v_1(S_{32})S_{42}\) and \(v_1(S_4) = v_1(S_{41})S_{42}\). Let
\[
\mathcal{D}_{G_w[S_w;S_{3w}]}(S_{3w}) = G_w, X_{3w} \Rightarrow \Psi_{3w}, \Delta_w,
\]
\[ \Delta G_\epsilon | S_n | S_w (S_{4w}) = \Pi_w, X_{4w} = \Psi_{4w}, \Sigma_w \]

for \( w = 1, 2 \). Then

\[ \Delta G_\epsilon (S_1) = \Gamma_1, \Pi_1, X_{31}, X_{41} = \Psi_{31}, \Psi_{41}, \Sigma_1, \Delta_1, \]
\[ \Delta G_\epsilon (S_2) = \Gamma_2, \Pi_2, X_{32}, X_{42} = \Psi_{32}, \Psi_{42}, \Sigma_2, \Delta_2, \]
\[ \Delta G_\epsilon (S_3) = \Gamma_1, X_{31}, \Gamma_2, X_{32} = \Psi_{31}, \Delta_1, \Psi_{32}, \Delta_2, \]
\[ \Delta G_\epsilon (S_4) = \Pi_1, X_{41}, \Pi_2, X_{42} = \Psi_{41}, \Sigma_1, \Psi_{42}, \Sigma_2 \]

by Lemma 5.5, \([S_3]_{G_\epsilon} \cap [S_4]_{G_\epsilon} = \emptyset, [S_3]_{G_\epsilon}, [S_4]_{G_\epsilon} \cap [S_4]_{G_\epsilon}, [S_3]_{G_\epsilon} = \emptyset, [S_2]_{G_\epsilon}, [S_4]_{G_\epsilon} \cap [S_4]_{G_\epsilon}, [S_2]_{G_\epsilon} = \emptyset. \] Then the proof of \( \Delta G_\epsilon (S_3) \) is constructed by combining the proofs of \( \Delta G_\epsilon (S_1) \) and \( \Delta G_\epsilon (S_2) \) with \( \frac{\Delta G_\epsilon (S_1)}{\Delta G_\epsilon (S_2)} \) \((COM)\).

- \[ \frac{G'|G'' |G''' (EC_\Omega) \in \tau. \] Then \( G', G'' \) and \( G''' \) are closed and \( G''' \) is a copy of \( G''' \) thus \( \Delta G_\epsilon (G') = \Delta G_\epsilon (G'') \) hence a proof of \( \Delta (G'|G'') \) is constructed by combining the proof of \( \Delta (G'|G') \) and \( \Delta (G''|G''') \) \((EC^*)\).

The following two lemmas are corollaries of Lemma 5.6.

**Lemma 5.7.** If there exists a derivation of \( G_\epsilon \) from \( G_1, \ldots, G_r \) in \( GL_\Omega \) then there exists a derivation of \( \Delta (G_\Omega) \) from \( \Delta (G_1), \ldots, \Delta (G_r) \) in \( GL \).

**Lemma 5.8.** Let \( \tau \) be a cut-free proof of \( G_\epsilon \) in \( GL \) and \( \tau^* \) be the proof of \( G|G^* \) in \( GL_\Omega \) resulting from preprocessing of \( \tau \). Then \( \tau_\Omega \in \Delta (G(G^*)) \).

### 6. Extraction of Elimination Rules

In this section, we will investigate Construction 4.7 further to extract more derivations from \( \tau^* \).

Any two sequents in a hypersequent seem independent of one another in the sense that they can only be contracted into one by \((EC)\) when it is applicable. Note that one-premise logical rules just modify one sequent of a hypersequent and two-premise rules associate a sequent in a hypersequent with one in a different hypersequent. \( \tau^* \) (or any proof without \((EC_\Omega)\) in \( GL_\Omega \)) has an essential property, which we call the distinguishability of \( \tau^* \), i.e., any variables, formulas, sequents or hypersequents which occur at the node \( H \) of \( \tau^* \) occur inevitably at \( H' < H \) in some forms.

Let \( H \equiv G'|S'|S'' \in \tau^* \). If \( S' \) is equal to \( S'' \) as two sequents then the case that \( \tau_{H,S'} \) is equal to \( \tau_{H,S''} \) as two derivations would happen. This means that both \( S' \) and \( S'' \) are the focus sequent of one node in \( Th_{H'} (H) \) when \( G^*_{H,S'} \neq S' \) and \( G^*_{H,S''} \neq S'' \). which contradicts that each node has the unique focus sequent in any derivation. Thus we need distinguish \( S' \) from \( S'' \) for all \( G^*|S'|S'' \in \tau^* \).

Define \( S^\perp \in \tau^* \) such that \( G'|S'|S'' \in S^\perp \) and \( S' \in S^\perp \) is the principal sequent of \( S^\perp \) and define \( N_{S'} = 0 \) if \( S^\perp \) has the unique principal sequent otherwise \( N_{S'} = 1 \) (or \( N_{S''} = 2 \)) to indicate that \( S' \) is the left principal sequent (or accordingly \( N_{S''} = 2 \) for the right one) of such an application as \((COM)\), \((\wedge_n)\) or \((\vee_n)\). Then we may regard \( S' \) as \( (S', \mathcal{P}(S^\perp), N_{S'}) \). Thus \( S' \) is always different from \( S'' \) by \( \mathcal{P}(S^\perp) \neq \mathcal{P}(S^\perp) \) or \( \mathcal{P}(S') = \mathcal{P}(S^\perp) \) and \( N_{S'} \neq N_{S''} \). We formulate it by the following construction.
Construction 6.1. A labeled tree $\tau^*$, which has the same tree structure as $\tau^*$, is constructed as follows.

(i) If $S$ is a leaf $\tau^*$, define $\overline{S} = S$, $N_\overline{S} = 0$ and the node $P(S)$ of $\tau^{**}$ is labeled by $(S, P(\overline{S}), N_\overline{S})$;

(ii) If $H \equiv G[S'] \in \tau^*$ and $P(G'[S'])$ be labeled by $G'[\{S', P(\overline{S}), N_\overline{S}\}]$ in $\tau^{**}$. Then define $\overline{S'} = H$, $N_{\overline{S'}} = 0$ and the node $P(H)$ of $\tau^{**}$ is labeled by $G'[\{S'', P(\overline{S''}), N_{\overline{S''}}\}]$.

(iii) If $H \equiv G'[G'[H']] \in \tau^*$, $P(G'[S'])$ and $P(G''[S''])$ be labeled by $G'[\{S', P(\overline{S'}), N_{\overline{S'}}\}]$ and $G''[\{S'', P(\overline{S''}), N_{\overline{S''}}\}]$ in $\tau^{**}$, respectively. If $H' = S_1 \equiv G' \in \tau^*$ then define $\overline{S_1} = S_2 = H, N_{\overline{S_1}} = 1, N_{S_2} = 2$ and the node $P(H)$ in $\tau^{**}$ is labeled by $G'[\{S', P(\overline{S'}), N_{\overline{S'}}\}]$ for some $1 \leq u \leq m$, by Notation 4.14. Thus $S'_i \in H'_i$ also by Notation 4.14. Hence $H \leq S'_i$ and $H'_i \leq S'_i$ by Construction 6.1. Therefore $H \leq H'_i$ or $H'_i \leq H$.

Proposition 6.2. (i) $G[S'] \in \tau^*$ implies $\{S'\} \in \{S''\} = 0$; (ii) $H \in \tau^*$ and $H' \in \tau^*$ imply $H' \in \tau^*$ implies $H \in \tau^*$ and $S'_j \in H$ then $H \leq H'_i$ or $H'_i \leq H$.

Proof. (iii) Let $S'_i \in H$ then $S''_i = S''_i$ for some $1 \leq u \leq m$, by Notation 4.14. Thus $S'_i \in H'_i$ also by Notation 4.14. Hence $H \leq S''_i$ and $H'_i \leq S''_i$ by Construction 6.1. Therefore $H \leq H'_i$ or $H'_i \leq H$.

Lemma 6.3. Let $H \in \tau^*$ and $Th(H) = (H_0, \ldots, H_n)$, where $H_0 = H$, $H_n = G[G^*$, $G_k \in H$ for $1 \leq k \leq 3$.

(i) If $G_3 = G_1 \cup G_2$ then $\langle H_i \rangle_{H_{G_1}} = \langle H_i \rangle_{H_{G_1}} \cap \langle H_i \rangle_{H_{G_2}}$ for all $0 \leq i \leq n$.

(ii) If $G_3 = G_1 \cup G_2$ then $\langle H_i \rangle_{H_{G_1}} = \langle H_i \rangle_{H_{G_1}} \cap \langle H_i \rangle_{H_{G_2}}$ for all $0 \leq i \leq n$.

Proof. The proof is by induction on $i$ for $0 \leq i < n$. Only (i) is proved as follows and (ii) by a similar procedure and omitted.

For the base step, $(H_0)_{H_{G_1}} = (H_0)_{H_{G_1}} \cap (H_0)_{H_{G_2}}$ holds by $(H_0)_{H_{G_1}} = G_1$, $(H_0)_{H_{G_2}} = G_2$, $(H_0)_{H_{G_1}} = G_1$ and $G_3 = G_1 \cup G_2$.

For the induction step, suppose that $(H_i)_{H_{G_1}} = (H_i)_{H_{G_1}} \cap (H_i)_{H_{G_2}}$ for some $0 \leq i < n$. Only is the case of one-premise rule given in the following and other cases are omitted.

Let $G'[\{S', \overline{S'}, \overline{S''}\}] \in \tau^*$, $H_i = G'[\{S', \overline{S'}, \overline{S''}\}]$, and $H_{i+1} = G'[\overline{S''}]$.

Let $S' \in (H_i)_{H_{G_1}}$. Then $(H_{i+1})_{H_{G_1}} = (H_i)_{H_{G_1}} \cap (H_i)_{H_{G_2}}$ for all $0 \leq i \leq n$.

$(H_{i+1})_{H_{G_1}} = (H_i)_{H_{G_1}} \cap (H_i)_{H_{G_2}} = \langle H_i \rangle_{H_{G_1}} \cap \langle H_i \rangle_{H_{G_2}}$.

$(H_{i+1})_{H_{G_1}} = (H_i)_{H_{G_1}} \cap (H_i)_{H_{G_2}}$.

$(H_{i+1})_{H_{G_1}} = (H_i)_{H_{G_1}} \cap (H_i)_{H_{G_2}}$.

$(H_{i+1})_{H_{G_1}} = (H_i)_{H_{G_1}} \cap (H_i)_{H_{G_2}}$.

$(H_{i+1})_{H_{G_1}} = (H_i)_{H_{G_1}} \cap (H_i)_{H_{G_2}}$.

$(H_{i+1})_{H_{G_1}} = (H_i)_{H_{G_1}} \cap (H_i)_{H_{G_2}}$.

The case of $S' \in (H_i)_{H_{G_1}}$, $S'' \in (H_i)_{H_{G_2}}$ is proved by a similar procedure and omitted. \[26\]
Lemma 6.4. (i) Let $G''|S'' \in \tau^*$ then $G''|_{S''|S'} \cap G''|_{S''|G''} = \emptyset, G''|_{S''|G''} G''|_{S''|S'}, = G|G''$;

(ii) $H \in \tau^*$; $H'|H'' \subseteq H$ then $G''|_{H'|H''} = G''|_{H'|G''}$.

Proof. (i) and (ii) are immediately from Lemma 6.3.

Notation 6.5. We write $\tau^*_S, \tau^*_{S'}, G^*_{S''}, G^*_{S''}$ as $\tau^*_S, G^*_{S''},$ respectively, for the sake of simplicity.

Lemma 6.6. (i) $G^*_{S''} \subseteq G|G^*$;

(ii) $\tau^*_S$ is a derivation of $G^*_{S''}$ from $S''_i$, which we denote by $\frac{S''_i}{G^*_{S''}} \{\tau^*_S\}$;

(iii) $G^*_{S''} = S^c_u$ and $\tau^*_S$ is built up with $S^c_u$ for all $2 \leq u \leq m$;

(iv) $v_i(G^*_{S''}) \cup v_1(G^*_{S''}) \cup v_2(G^*_{S''})$;

(v) $(H)S'' \subseteq \tau^*_S$ implies $H \subseteq H''$. Note that $(H)S''$ is undefined for any $H \in H''$ or $H|H''$.

(vi) $S'' \in G^*_{S''}$ implies $H'' \notin H''$.

Proof. Claims from (i) to (v) are immediately from Construction 4.7 and Lemma 4.8.

(vi) Since $S''_u \in G^*_{S''} \subseteq G|G^*$ then $S''_u$ has the form $S^c_u$ for some $u \geq 2$ by Notation 4.14. Then $G^*_{S''} = S^c_u$ by (iii). Suppose that $H'' \subseteq H''$. Then $S''_u$ is transferred from $H''$ downward to $H''$ and in side-hyprusequent of $H''$ by Notation 4.14 and $G|G^* < H'' \subseteq H''$. Thus $S'' \subseteq \emptyset$ at $H''$ since $S''_u$ is the unique focus sequent of $H''$. Hence $S'' \notin G^*_{S''}$ by Lemma 6.3 and (iii), a contradiction therefore $H'' \notin H''$.

Lemma 6.7. Let $\frac{G''|S'' \cap G''|S'}{H \in G''|S'' \cap G''|S'} (II) \in \tau^*$. (i) If $S''_u \in G''|H''$ then $H'' \subseteq H$ or $H''|G''$; (ii) If $S''_u \in G''|G''$ then $H'' \subseteq H$ or $H''|G''|S''$.

Proof. (i) We impose a restriction on (II) such that each sequent in $H''$ is different from $S''_u$ or $S''_u$ otherwise we treat it as an (EW)-application. Since $S''_u \in G''|H'' \subseteq G|G''$ then $S''_u$ has the form $S''_u$ for some $u \geq 2$ by Notation 4.14. Thus $G''|G'' = G''|S''_u$. Suppose that $H'' \subseteq H''$. Then $S''_u$ is transferred from $H''$ downward to $H''$. Thus $S''_u \subseteq H''$ by $G''|S''_u = G''|G''$ and Lemma 6.3. Hence $S''_u \subseteq S''_u \subseteq H''$, a contradiction with the restriction above. Therefore $H'' \subseteq H$ or $H''|H''$.

(ii) Let $S''_u \in G''|G''$. If $H'' \subseteq H$ then $S''_u \subseteq H$ by Proposition 4.15(i) and thus $S''_u \in G''|G''$ by Lemma 6.3 and, hence $H'' \parallel G''|S''_u \parallel G''|S''_u \parallel G''|S''_u \parallel G''|S''_u$. If $H''|H''$ then $H'' \parallel G''|S''_u$ by $H'' \subseteq G''|S''_u$. Thus $H'' \subseteq H''$ or $H''|G''|S''_u$.

Definition 6.8. (i) By $H'' \sim H''$ we mean that $S''_u \in G''|S''_u$ for some $2 \leq u \leq m$; (ii) By $H'' \sim H''$ we mean that $H'' \sim H''$ and $H'' \sim H''$; (iii) $H'' \sim H''$ means that $S''_u \notin G''|S''_u$ for all $2 \leq u \leq m$.

Then Lemma 6.6 (vi) shows that $H'' \sim H''$ implies $H'' \notin H''$.

Lemma 6.9. Let $H''|H'', H'' \to H'' \parallel G''|S''_u \parallel G''|S''_u (II) \in \tau^*$ such that $G''|S'' \subseteq H''$, $G''|S'' \subseteq H''$.

Then $S'' \subseteq (G''|S''|S''_u)_{S''_u}$.

Proof. Suppose that $S'' \notin (G''|S''|S''_u)_{S''_u}$. Then $(G''|S''|S''_u)_{S''_u} \subseteq G'' \parallel (G''|S''|S''_u)_{S''_u} \subseteq G''|S''_u \subseteq G''|S'' \parallel (G''|S''|H''|S''_u) \subseteq (G''|S''_u)_S''_u$ by Construction 4.7. Thus $(G''|S''_u|S''_u)_{S''_u} \subseteq G''$. Hence $G''|S'' \subseteq (G''|S''|H''|S''_u) = (G''|S''|S''_u)_{S''_u}$ by Construction 4.7. Thus $S'' \notin (G''|S''|H''|S''_u)_{S''_u}$ by Proposition 6.2 (ii). Therefore $S'' \notin (G''|S''|S''_u)_{S''_u}$ for all $1 \leq u \leq m$ by Lemma 6.3, i.e., $H'' \sim H''$, a contradiction and hence $S'' \subseteq (G''|S''|S''_u)_{S''_u}$.

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Lemma 6.6 (ii) shows that $\tau^{\Lambda}_{\Lambda}$ is a derivation of $G^{*}_{\Lambda}$ from one premise $S^{\star}_{\Lambda}$. We generalize it by introducing derivations from multiple premises in the following. In the remainder of this section, let $I = \{H_{i1}, \ldots, H_{im}\} \subseteq \{H_{i}, \ldots, H_{n}\}$, $H_{i} \leftrightarrow H_{i}^{*}$ for all $1 \leq k < q \leq m$. Then $H_{i}^{*} \lneq H_{i}^{*}$ and $H_{i}^{*} \lneq H_{i}$ by Lemma 6.6 (vi) thus $H_{i}^{*} \lneq H_{i}^{*}$ for all $1 \leq k < q \leq m$.

**Notation 6.10.** $H^{*}_{i}$ denotes the intersection node of $H_{i1}, \ldots, H_{im}$. We sometimes write the intersection node of $H_{i}^{*}$ and $H_{j}^{*}$ as $H_{i}^{*}$.

Let $G^{*}_{I} = G^{*}_{I} \odot G^{*}_{I}$ from $I_1 \{H_{i1}, \ldots, H_{im}\}$ and $I = \{H_{i1}, \ldots, H_{im}^{*}\}$, which occur in the left subtree $\tau^{*}(G^{*}_{I})$ and right subtree $\tau^{*}(G^{*}_{I})$ of $\tau^{*}(G^{*}_{I})$ respectively.

Let $I = \{S^{*}_{i1}, \ldots, S^{*}_{im}\}$, $I = \{S^{*}_{i1}, \ldots, S^{*}_{im}\}$, $I = \{S^{*}_{i1}, \ldots, S^{*}_{im}\}$ such that $I = I_1 \cup I_2$. A derivation $\tau^{*}_{I}$ of $(G^{*})_{I}$ from $S^{*}_{i1}, \ldots, S^{*}_{im}$ is constructed by induction on $|I|$. The base case of $|I| = 1$ has been done by Construction 4.7. For the induction case, suppose that derivations $\tau^{*}_{I_1}$ of $(G^{*})_{I_1}$ from $S^{*}_{i1}, \ldots, S^{*}_{im}$ and $\tau^{*}_{I_2}$ of $(G^{*})_{I_2}$ from $S^{*}_{i1}, \ldots, S^{*}_{im}$ be constructed. Then $\tau^{*}_{I}$ of $(G^{*})_{I}$ from $S^{*}_{i1}, \ldots, S^{*}_{im}$ is constructed as follows.

**Construction 6.11 (See Appendix 5.1).** (i) 

\[ \langle H \rangle_{I} := \langle H \rangle_{I_1} \text{ for all } G^{*}_{I} \leq G^{*}_{I_2} \text{ for some } H_{i}^{*} \in I, \]

\[ \langle H \rangle_{I} := \langle H \rangle_{I_2} \text{ for all } G^{*}_{I} \leq G^{*}_{I_2} \text{ for some } H_{i}^{*} \in I, \]

\[ \tau^{*}_{I}((G^{*})_{I}) := \tau^{*}_{I_1}((G^{*})_{I_1}) \quad \tau^{*}_{I}((G^{*})_{I_2}) := \tau^{*}_{I_2}((G^{*})_{I_2}) \]

(ii) 

\[ \langle G^{*}_{I} \rangle_{I} := \langle G^{*}_{I} \rangle_{I_1} \langle G^{*}_{I} \rangle_{I_2} \langle H^{*} \rangle_{I} \]

and

\[ \langle G^{*}_{I} \rangle_{I_1} \langle G^{*}_{I} \rangle_{I_2} \langle H^{*} \rangle_{I} \langle H^{*} \rangle_{I} \in \tau^{*}_{I} \]

(iii) Other nodes of $\tau^{*}_{I}$ are built up by Construction 4.7 (ii).

The following lemma is a generalization of Lemma 6.6.

**Lemma 6.12.** Let $Th(H_{i}^{e}) = (H_{i1}^{e}, \ldots, H_{im}^{e})$, where $1 \leq k \leq m$, $H_{i0}^{e} = H_{i}^{e}$ and $H_{i}^{e} H_{j}^{e} = G^{*}$. Then, for all $0 \leq u \leq n$,

(i) 

\[ \langle H_{i}^{e} \rangle_{u} = \bigcap \{ \{ H_{i}^{e} : H_{i}^{e} \in I, H_{i}^{e} \leq H_{j}^{e} \} \} \]

(ii) 

\[ \{ H_{i}^{e} : H_{i}^{e} \in I, H_{i}^{e} \leq H_{j}^{e} \} \]

(iii) 

\[ \{ H_{i}^{e} : H_{i}^{e} \in I, H_{i}^{e} \leq H_{j}^{e} \} \]

(iv) $\langle H \rangle_{I} \in \tau^{*}_{I}$ if and only if $H \leq H_{i}^{e}$ for some $H_{i}^{e} \in I$. Note that $\langle H \rangle_{I}$ is undefined if $H > H_{i}^{e}$ or $H \nleq H_{i}^{e}$ for all $H_{i}^{e} \in I$. 

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Proof: (i) is proved by induction on \(|I|\). For the base step, let \(|I| = 1\) then the claim holds clearly. For the induction step, let \(|I| \geq 2\) then \(|I| \geq 1\) and \(|I| \geq 1\). Then \(S' \in \langle G' \mid S' \rangle_{S_j}^\dagger\) for all \(H_j^i \in I\) by Lemma 6.9 and \(H_j^i \Rightarrow H_j^i\) for all \(H_j^i \in I\). \(\langle G' \mid S' \rangle_{S_j}^\dagger = \bigcap_{H_j^i \in I} \langle G' \mid S' \rangle_{S_j}^\dagger\) by the induction hypothesis then \(S' \in \langle G' \mid S' \rangle_{S_j}^\dagger\), thus \(\langle G' \mid G'' \mid H' \rangle_{S_j}^\dagger = \langle G' \mid G'' \mid H' \rangle_{S_j}^\dagger\) by \(G' \mid S' \leq H_j^i\).

\(\langle G' \mid G'' \mid H' \rangle_{S_j}^\dagger = \langle G' \mid G'' \mid H' \rangle_{S_j}^\dagger \cap \langle G'' \mid H' \rangle_{S_j}^\dagger\)

by \(\langle G' \mid S' \rangle_{S_j}^\dagger \subseteq G'\) and \(\langle G'' \mid S' \rangle_{S_j}^\dagger \subseteq G''\). Other claims hold immediately from Construction 6.11.

Lemma 6.13. (i) Let \(G'_z\) denote \(\langle G' \mid S' \rangle_{S_j}^\dagger\) then \(G'_z = \bigcap_{H_j^i \in I} G'_z\):  

\(S'_{1;1} \cdots S'_{n;1}(\tau'_z)\);

(ii) \(v_1(G'_z) \cup_{H_j^i \in I} v_1(S'_{j;1}) = v_1(G'_z) \cup_{H_j^i \in I} v_1(S'_{j;1})\);

(iii) \(\forall_i(G'_z) \leq \bigcap_{H_j^i \in I} \forall_i(S'_{j;1})\);

(iv) \(S'_j \in G'_z\) implies \(H_j^i \leq H_j^i\) for all \(H_j^i \in I\).

Proof. (i), (ii) and (iii) are immediately from Lemma 6.12. (iv) holds by (i) and Lemma 6.6 (vi).

Lemma 6.13 (iv) shows that there exists no copy of \(S'_{j;1}\) in \(G'_z\) for all \(1 \leq k \leq m\). Then we may regard them to be eliminated in \(\tau'_z\). We then call \(\tau'_z\) an elimination derivation.

Let \(\mathcal{I}' = \{S'_{i;1}, \ldots, S'_{i;u} \}_{i, u}^c\) be another set of sequents to \(I\) such that \(G'' \equiv S'_{i;u} \ldots | S'_{i;u}^c\) is a copy of \(G'' \equiv S'_{i;1} \cdots | S'_{i;1}^c\). Then \(G'\) and \(G''\) are disjoint and there exist two bijections \(\sigma_1 : v_1(G') \rightarrow v_1(G'')\) and \(\sigma_2 : v_1(G') \rightarrow v_1(G''\rangle\) such that \(\sigma_1 \circ \sigma_2 (G') = G''\). By applying \(\sigma_2 \circ \sigma_1\) to \(\tau'_z\), we construct a derivation from \(S'_{i;1} \cdots, S'_{i;u} \in S'_{i;1}^c\) and denote it by \(\tau''_z\) and its root by \(G''_z\).

Let \(\mathcal{P}' = \{b_{1|1}, \ldots, b_{m|1}\}^{S'_{i;1}^c}\) be a set of hypersequents to \(I\), where \(b_{1|1} <_{S'_{i;1}} \ldots <_{S'_{i;1}} b_{m|1}\) be closed for all \(1 \leq k \leq m\). By applying \(\tau''_z\) to \(S'_{i;1} \cdots, S'_{i;u} \in S'_{i;1}^c\) in \(G''_z\), \(b_{1|1} \cdots, b_{m|1}\), we construct a derivation from \(b_{1|1} \cdots, b_{m|1}\) and denote it by \(\tau''_z\) and its root by \(G''_z\).

Definition 6.14. We will use all \(\tau''_z\) as inference rules of \(\text{GL}_{\text{B}}\) and call them elimination rules. Further, we call \(S'_{i;1} \cdots, S'_{i;u} \in S'_{i;1}^c\) focus sequents and, all sequents in \(G'_z\) principal sequents and, \(b_{1|1} \cdots, b_{m|1}\) side-hypersequents of \(\tau''_z\).

Remark 6.15. We regard Construction 4.7 as a procedure \(\mathcal{F}\), whose inputs are \(\tau'', H, H'\) and output \(\tau''_{H,H'}\). With such a viewpoint, we write \(\tau''_{H,H'}\) as \(\mathcal{F}_{H,H'}(\tau'')\). Then \(\tau''_z\) can be constructed by iteratively applying \(\mathcal{F}\) to \(\tau''_z\), i.e., \(\tau'_z = \mathcal{F}_{H_1,H_2}(\cdots \mathcal{F}_{H_1,H_2}(\cdots \mathcal{F}_{H_1,H_2}(\tau''_z\).

We replace locally each \(\mathcal{F}_{H_1,H_2}(\tau''_z\) in \(\tau''_z\) with \(G'\) and denote the resulting derivation also by \(\tau''_z\). Then each non-root node in \(\tau''_z\) has the focus sequent.

Let \(H \in \tau''_z\). Then there exists a unique node in \(\tau''_z\), which we denote by \(\mathcal{O}(H)\) such that \(H\) comes from \(\mathcal{O}(H)\) by Construction 4.7 and 6.11. Then the focus sequent of \(\mathcal{O}(H)\) in \(\tau''_z\) is the focus of \(H\) in \(\tau''_z\). If \(H\) is a non-root node then, \(\mathcal{O}(H) = H\) or \(H \subseteq \mathcal{O}(H)\) as two hypersequents.
Since the relative position of any two nodes in \( \tau^* \) keep unchanged in constructing \( \tau^*_{\tau}, H_1 \leq \tau_{\tau} H_2 \) if and only if \( \mathcal{O}(H_1) \leq \tau_{\tau} \mathcal{O}(H_2) \) for any \( H_1, H_2 \in \tau^*_{\tau} \). Especially, \( \mathcal{O}(S^c_{\tau_{\tau}}) = H^c_{\tau} \) for \( S^c_{\tau_{\tau}} \in \tau^*_{\tau} \).

Let \( H \in \tau^*_{\tau} \). Then \( H^c = \sigma_H \sigma_{\tau}(H) \in \tau^*_{\tau} \) and \( H^c = \{ G_{b_\nu} : H \leq \tau_{\tau} S^c_{\tau_{\tau}} \} \) and \( 1 \leq k \leq m \) \( |H^c| \subseteq \tau^*_{\tau}\). Define \( \mathcal{O}(H^c) = \mathcal{O}(H^c) = \mathcal{O}(H) \). Then \( \mathcal{O}(G^c) = G^c \) and \( \mathcal{O}(G^c_{b_\nu}) = H^c_{b_\nu} \) for all \( G_{b_\nu} S^c_{\tau_{\tau}} \in \tau^*_{\tau}\).

7. Separation of one branch

In this section of the paper, we assume that \( p \) occur at most one time for each sequent in \( G_0 \) as the one in Main lemma, \( \tau \) be a cut-free proof of \( G_0 \) in \( \text{GL} \) and \( \tau^* \) the proof of \( G^c \) in \( \text{GL}_\Omega \) resulting from preprocessing of \( \tau \). Then \( |\nu(S) + |\nu(S) | \leq 1 \) for all \( S \in G \), which plays a key role in discussing the separation of branches.

**Definition 7.1.** By \( S' \in G' \) we mean that there exists some copy of \( S' \) in \( G' \). \( G' \subseteq G' \) if \( S' \in G' \) for all \( S' \in G' \). \( G' \leq G' \) if \( G' \subseteq G' \) and \( G' \subseteq G' \). Let \( G_1, \ldots, G_m \) be \( m \) copies of \( G_1 \) then we denote \( G'[G_1] \cdots [G_m] by (G'_1 \cdots G'_m)\} \).

**Definition 7.2.** Let \( I = \{ H'_0, \ldots, H'_m \} \subseteq \{ H'_0, \ldots, H'_m \} \). \( H'_c \parallel H'_c \) for all \( 1 \leq k < l \leq m \). \( [S^c_{\tau_{\tau}}] \), is called a branch of \( H'_c \) to \( I \) if it is a closed hypersequent such that \( [S^c_{\tau_{\tau}}] \subseteq G^c \). \( S^c_{\tau_{\tau}} \in [S^c_{\tau_{\tau}}] \) and \( S^c_{\tau_{\tau}} \in [S^c_{\tau_{\tau}}] \) implies \( H^c_{\tau} \leq H^c_{\tau} \) or \( H^c_{\tau} \neq H^c_{\tau} \) for all \( H^c_{\tau} \in I \).

Then (i) \( S^c_{\tau_{\tau}} \in [S^c_{\tau_{\tau}}] \) for all \( 1 \leq k, l \leq m, k \neq l \); (ii) \( S^c_{\tau_{\tau}} \in [S^c_{\tau_{\tau}}] \) and \( H^c_{\tau} \neq H^c_{\tau} \) imply \( H^c_{\tau} \in I \). In this section, let \( I = \{ H'_c \} \), \( I = \{ [S^c_{\tau_{\tau}}] \} \), we will give an algorithm to eliminate all \( S^c_{\tau_{\tau}} \in [S^c_{\tau_{\tau}}] \) that \( H^c_{\tau} \leq H^c_{\tau} \).

**Construction 7.3.** A sequence of hypersequents \( G^c_1 \) and their derivations \( \tau^c_1 \) from \( [S^c_{\tau_{\tau}}] \) for all \( q > 0 \) are constructed inductively as follows.

For the base case, define \( G^c_1 = [S^c_{\tau_{\tau}}] \). For the induction case, suppose that \( G^c_1 \) and \( G^c_1 \) be constructed for some \( 0 \leq q \). If there exists no \( S^c_1 \in G^c_1 \) such that \( H^c_{\tau} \leq H^c_{\tau} \) the procedure terminates and define \( A_1 \) to be \( q \) otherwise define \( A_1 \) such that \( S^c_1 \in G^c_1 \), \( H^c_{\tau} \leq H^c_{\tau} \) and \( H^c_{\tau} \leq H^c_{\tau} \) for all \( S^c_1 \in G^c_1 \), \( H^c_{\tau} \leq H^c_{\tau} \). Let \( S^c_1, \ldots, S^c_{m_1} \) be all copies of \( S^c_1 \) in \( G^c_1 \) then define \( G^c_{1+1} = G^c_1 \) \( \langle \{ S^c_1 \} \cup \cdots \cup \{ S^c_{m_1} \} \} \) and its derivation \( \tau^c_1 \) is constructed by sequentially applying \( \tau^c_1, \ldots, \tau^c_1 \) to \( \{ S^c_1 \} \cup \cdots \cup \{ S^c_{m_1} \} \), respectively. Notice that we assign new identification numbers to new occurrences of \( p \) in \( \tau^c_1 \). for all \( 0 \leq q \leq J_1, 1 \leq u \leq m_1 \).

**Lemma 7.4.** (i) \( H^c_{\tau} = H^c_{\tau} \) and \( H^c_{\tau} < H^c_{\tau} \) for all \( 0 \leq q \leq J_1 - 2 \); (ii) \( G^c_1 \subseteq G^c_1 \) is closed for all \( 0 \leq q \leq J_1 \); (iii) \( \{ S^c_{\tau_{\tau}} \} \leq G^c_1 \) for all \( 0 \leq q \leq J_1 \), especially \( \{ S^c_{\tau_{\tau}} \} \leq G^c_1 \); (iv) \( S^c_1 \in G^c_1 \) implies \( H^c_{\tau} \parallel H^c_{\tau} \) and \( S^c_1 \in G^c_1 \) for some \( \tau^c_{\tau} \) \( \tau^c_{\tau} \) \( \tau^c_{\tau} \) for \( 0 \leq q \leq J_1 - 1, 1 \leq u \leq m_1 \).
Proof. (i) Since $S'_{i} \in G_{i}^{(q_{1})}$ by $S'_{i} \in [S'_{i}]_{i}$, $G_{j}^{(q_{1})}$ and, $H_{j} \leq H_{i}$ for all $S'_{j} \in G_{i}^{(q_{1})}$, $H_{j} \leq H_{i}$, then $H_{j} = H_{i}$. If $S'_{j+1} \in G_{j+1}^{(q_{2})}$ then $H_{j+1} \leq H_{i}$. If $S'_{j+1} \in G_{j+1}^{(q_{1})}$ then $H_{j+1} \leq H_{i}$ thus $H_{j+1} \leq H_{i}$ by all copies of $S'_{i} \in G_{i}^{(q)}$ being collected in $\{S'_{i}\}_{i=1}^{m_{2}}$. If $S'_{j} \in G_{j}^{(q_{2})}$ then $H_{j} \leq H_{i}$, $H_{j+1} \leq H_{i}$, and $H_{j+1} < H_{i}$ by $\tau_{j}^{(q_{1})} < H_{i}$ by Lemma 6.6 (vi) thus $H_{j+1} < H_{i}$, $H_{j+1} < H_{i}$, and $H_{j+1} < H_{i}$ by $G_{1}^{(q+1)}(i) = G_{1}^{(q)}(i) \{G_{j}^{(q_{2})}\}_{j=1}^{m_{2}}$. Note that $H_{j}^{(q_{1})}$ is undefined in Construction 7.3.

(ii) $\nu_{j}(G_{j}^{(q_{1})}) = \nu_{j}(G_{j}^{(q_{1})}) \leq G|G^{*}$ by $G_{j}^{(q_{1})} = [S'_{i}]_{i}$. Suppose that $\nu_{j}(G_{j}^{(q)}) = \nu_{j}(G_{j}^{(q)}) \leq G|G^{*}$ then $\nu_{j}(G_{j}^{(q+1)}) = \nu_{j}(G_{j}^{(q+1)}) \leq G|G^{*}$ by $G_{j}^{(q+1)} = G_{j}^{(q)} \{G_{j}^{(q_{2})}\}_{j=1}^{m_{2}}$. $\nu_{j}(G_{j}^{(q+1)}) = \nu_{j}(G_{j}^{(q+1)}) \leq G|G^{*}$ for all $1 \leq m_{2}$. Given $\frac{[S'_{i}]_{i}}{G_{i}^{(q_{1})}} \{\tau_{i}^{(q_{1})}\}$ then $\frac{[S'_{i}]_{i}}{G_{i}^{(q_{1})}} \{\tau_{i}^{(q_{1})}\}$ is constructed by linking up the conclusion of the previous derivation to the premise of its successor in the sequence of derivations

\[
\frac{[S'_{i}]_{i}}{G_{i}^{(q)}} \{\tau_{i}^{(q)}\} = G_{i}^{(q)} \{S'_{i}\}_{i=1}^{m_{2}} \{G_{j}^{(q_{2})}\}_{j=1}^{m_{2}} \{\tau_{i}^{(q)}\}
\]

as shown in the following figure.

```
\[
\begin{array}{c}
\frac{[S'_{i}]_{i}}{G_{i}^{(q)}} \{\tau_{i}^{(q)}\} = G_{i}^{(q)} \{S'_{i}\}_{i=1}^{m_{2}} \{G_{j}^{(q_{2})}\}_{j=1}^{m_{2}} \{\tau_{i}^{(q)}\}
\end{array}
\]
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A derivation of $G_{i}^{(q+1)}$ from $G_{i}^{(q)}$

(iv) Let $S'_{j} \in G_{j}^{(q_{1})}$ such that $S'_{j} \in G_{i}^{(q_{1})}$ by the definition of $J_{i}$. If $S'_{j} \in [S'_{i}]_{i}$ then $H_{j} \leq H_{i}$ by $H_{i} \leq H_{j}$ and the definition of $[S'_{i}]_{i}$ otherwise, by Construction 7.3, there exists some $\tau_{j}^{(q_{1})}$ in $G_{i}^{(q_{1})}$ such that $S'_{j} \in G_{j}^{(q_{1})}$. Then $H_{j} \leq H_{j}$ by Lemma 6.6 (vi). Thus $H_{j} \leq H_{j}$ by $H_{j} \leq H_{j}$. Hence $H_{j} \leq H_{j}$. \(\square\)
Lemma 7.4 shows that Construction 7.3 presents a derivation \( \tau_{1}^{\circ} \) of \( G_{\circ}^{1} \) from \( [S_{u}] \) such that there doesn’t exist \( S_{j} \in G_{1}^{1} \) such that \( H_{j} \not\in H_{u} \), i.e., all \( S_{j} \in [S_{u}] \) such that \( H_{j} \not\in H_{u} \) are eliminated by Construction 7.3. We generalize this procedure as follows.

Construction 7.5. Let \( H \in \tau^{+} \), \( H_{1} \subseteq H \) and \( H_{2} \subseteq G[G^{*}] \). Then \( G_{H:H_{1}}^{\circ} \) and its derivation \( \tau_{H:H_{1}}^{\circ} \) for \( l = 1, 2 \) are constructed by procedures similar to that of Construction 7.3 such that \( H_{j} \not\in H \) for all \( S_{j} \in G_{H:H_{1}}^{\circ} \), where \( G_{H:H_{1}}^{\circ} := G_{H:H_{1}}, \tau_{H:H_{1}}^{\circ} := \tau_{H:H_{1}} \), which are defined by Construction 4.7.

We sometimes write \( J_{1}, J_{H:H} \) as \( J \) for simplicity. Then the following lemma holds clearly.

Lemma 7.6. (i) \( \frac{H_{1}}{G_{H:H}} \left( \tau_{H:H_{1}}^{\circ} \right) \). \( H_{j} \not\in H \) for all \( S_{j} \in G_{H:H_{1}}^{\circ} \).

(ii) If \( S_{j} \not\in H \) and \( H_{j} > H \) then \( G_{H:S_{j}}^{\circ} = S_{j} \).

(iii) If \( S \in G \) or \( S \in G^{*} \) is a copy of \( S_{j} \) and \( H_{j} \not\in H \) then \( G_{H:S}^{\circ} = S \).

(iv) Let \( H \subseteq H^{*} \subseteq H \). Then \( G_{H:H^{*}}^{\circ} = G_{H:H}^{\circ} \) by suitable assignments of identification numbers to new occurrences of \( p \) in constructing \( \tau_{H:H^{*}}^{\circ}, \tau_{H:H}^{\circ} \) and \( \tau_{H:S}^{\circ} \).

(v) \( G_{H}^{\circ} = \bigcup \{ G_{H:S_{j}}^{\circ} : S_{j} \in [S_{u}] \} \bigcup \{ S_{j} : S_{j} \in [S_{u}] \}, H_{j} \not\in H_{u} \} \bigcup \{ S : S \in [S_{u}] \} \).

Proof. (i) is proved by a procedure similar to that of Lemma 7.4 (iii), (iv) and omitted.

(ii) Since \( S_{j}^{c} \) is the focus sequent of \( H_{j} \) then it is revised by some inference rule at the node lower than \( H_{j} \). Thus \( S_{j}^{c} \in H \) is some copy of \( S_{j} \) by \( H_{j} > H \). Hence \( S_{j}^{c} \) has the form \( S_{j}^{c} \) for some \( u \geq 2 \). Therefore it is transferred downward to \( G[G^{*}] \), i.e., \( S_{j}^{c} \in G[G^{*}] \). Then \( G_{H:S_{j}}^{\circ} = G_{H:S_{j}}^{\circ} \).

Since there exists no \( S_{j}^{c} \in G_{H:S_{j}}^{\circ} \), \( H_{j} \not\in H \) then \( J = 0 \). Thus \( G_{H:S_{j}}^{\circ} = S_{j}^{c} \).

(iii) is proved by a procedure similar to that of (ii) and omitted.

(iv) Since \( H \subseteq H^{*} \subseteq H \) then \( H \cap H^{*} = \emptyset \) by Proposition 6.2. Thus \( G_{H:H^{*}}^{\circ} = G_{H:H}^{\circ} \) due to \( G_{H:H}^{\circ} \).

Then \( G_{H:H}^{\circ} = G_{H:H}^{\circ} \) due to \( G_{H:H}^{\circ} \).

Therefore we can construct \( G_{H:H}^{\circ} \), \( G_{H:H}^{\circ} \) and \( G_{H:H}^{\circ} \) simultaneously and assign the same identification numbers to new occurrences of \( p \) in \( G_{H:H}^{\circ} \) and \( G_{H:H}^{\circ} \) as the corresponding one in \( G_{H:H}^{\circ} \).

Hence \( G_{H:H}^{\circ} = G_{H:H}^{\circ} \) due to \( G_{H:H}^{\circ} \). Then \( G_{H:H}^{\circ} = G_{H:H}^{\circ} \) due to \( G_{H:H}^{\circ} \).

Note that the requirement is imposed only on one derivation that distinct occurrence of \( p \) has different identification number. We permit \( G_{H:H}^{\circ} = G_{H:H}^{\circ} \) or \( G_{H:H}^{\circ} = G_{H:H}^{\circ} \) in the proof above, which has no essential effect on the proof of the claim.

(v) is immediately from (iv).

Lemma 7.6(v) shows that \( G_{H}^{\circ} \) could be constructed by applying \( \tau_{H:H}^{\circ} \) sequentially to each \( S \in [S_{u}] \).

Thus it is not necessary that \( H_{j}^{0} \not\in H_{j}^{0} \) in Construction 7.3, but which make the termination of the procedure obvious.

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Construction 7.7. Apply \((EC^n_\Omega)\) to \(G^\Omega_1\) and denote the resulting hypersequent by \(G^\Omega_1\) and its derivation by \(\tau^\Omega_1\). It is possible that \((EC^n_\Omega)\) is not applicable to \(G^\Omega_1\) in which case we apply \((ID_\Omega)\) to it for the regularity of the derivation.

Lemma 7.8.  
(i) \(\frac{[S^c_\Omega]_U}{G^\Omega_1} (\tau^\Omega_1)\), \(G^\Omega_1\) is closed and \(H'^*_j \| H'^*_u\) for all \(S^c_j \in G^\Omega_1\);

(ii) \(\tau^\Omega_1\) is constructed by applying elimination rules, say, \(G^\Omega_b|G^\Omega^\ast_j|\langle \tau^\Omega_{G_b}[S^c_\Omega]_{\xi^\prime}\rangle\), and the fully constraint contraction rules, say, \(G^\Omega_1/(EC^n_\Omega)\), where \(H'^*_q \leq H'^*_u\), \(G^\Omega_b|S^c_\Omega|\) is closed for \(0 \leq q \leq J - 1\), \(1 \leq u \leq m^c_\Omega\).

Proof. Immediately from Lemma 7.4.

Definition 7.9. Let \(G' \in \tau^\Omega_1\), \(H' \subseteq G'\) and \(|v_r(H')| + |v_r(H')| \geq 1\). \(H'\) is called separable in \(\tau^\Omega_1\) if there exists \(H \subseteq G^\Omega_1\) such that \(|v_r(S)| + |v_r(S)| = 1\) for all \(S \in H\), \(v_r(H) = v_r(H')\) and \(v_r(H) = v_r(H')\), and \(H'\) is called to be separated into \(H\) and \(\overline{H'} = H\).

Note that \(\tau^\Omega_1\) is a derivation without \((EC_\Omega)\) in \(GL_\Omega\). Then we can extract elimination derivations from it by Construction 4.7.

Notation 7.10. Let \(H' \subseteq G' \in \tau^\Omega_1\). \(\tau^\Omega_1(G'\subseteq H'\subseteq G_1)\) denotes the derivation from \(H'\), which extracts from \(\tau^\Omega_1\) by Construction 4.7, and denote its root by \(G^\Omega_1(G'\subseteq H'\subseteq G_1)\).

The following two lemmas show that Construction 7.3 and 7.5 force some sequents in \([S^c_\Omega]_U\) or \(H'\) to be separable.

Lemma 7.11. Let \(G^\Omega_1|S' \subseteq T_{H_2} \subseteq \{\tau^\Omega_1\}, S' \in \langle G^\Omega_1|S' \rangle_{\xi^\prime}\) by \(G^\Omega_1|H' \subseteq H_1 \in \tau^\Omega_1\), \(H_1 \equiv G^\Omega_b|G^\Omega^\ast_j|G^\Omega^\ast_j|H^\prime|(I)\) \(\subseteq \tau^\Omega_1\).

Then \(H'\) is separable in \(\tau^\Omega_1\) and there is a unique copy of \(\widetilde{S^c_\Omega}(G^\Omega_1)\) in \(G^\Omega_1\).

Proof. We write \(\xi^\prime\) as \(\xi^\prime\) for simplicity. Clearly, \(G^\Omega_1(G'\subseteq H'\subseteq G_1)\) is a copy of \(G^\Omega_1(G'\subseteq H'\subseteq G_1)\) and, \(\tau^\Omega_1(G'\subseteq H'\subseteq G_1)\) has no difference with \(\tau^\Omega_1(G'\subseteq H'\subseteq G_1)\) except some applications of \((ID_\Omega)\) and identification numbers of some \(p'\)s.

By Construction 4.7, \(G^\Omega_1|S' \subseteq T_{H_2} \subseteq \{\tau^\Omega_1\}, S' \in \langle G^\Omega_1|S' \rangle_{\xi^\prime}\) by \(G^\Omega_1|H' \subseteq H_1 \in \tau^\Omega_1\), \(H_1 \equiv G^\Omega_b|G^\Omega^\ast_j|G^\Omega^\ast_j|H^\prime|(I)\) \(\subseteq \tau^\Omega_1\).

Then \(G^\Omega_1|S' \subseteq G^\Omega^\ast_j|G^\Omega^\ast_j|S' \subseteq T_{H_2} \subseteq \{\tau^\Omega_1\}, S' \in \langle G^\Omega_1|S' \rangle_{\xi^\prime}\) by \(G^\Omega_1|H' \subseteq H_1 \in \tau^\Omega_1\). Then \(G^\Omega_1|S' \subseteq G^\Omega^\ast_j|G^\Omega_1|S' \subseteq T_{H_2} \subseteq \{\tau^\Omega_1\}, S' \in \langle G^\Omega_1|S' \rangle_{\xi^\prime}\) by Lemma 6.6(v) and 6.4(i), \(G^\Omega_1(G'\subseteq H'\subseteq G_1) \subseteq G^\Omega_1\) by Lemma 4.8(i).

Let \(S^c_j \in G^\Omega_1(G'\subseteq H'\subseteq G_1)\). Then \(H^c_j \subseteq \tau^\Omega_1\) by \(G^\Omega_1(G'\subseteq H'\subseteq G_1) \subseteq G^\Omega_1\). Suppose that \(S^c_j \in [S^c_\Omega]_U\). Then \(H^c_j \subseteq \tau^\Omega_1\) by \(G^\Omega_1(G'\subseteq H'\subseteq G_1) \subseteq G^\Omega_1\). Then \(H^c_j \subseteq \tau^\Omega_1\) by \(G^\Omega_1(G'\subseteq H'\subseteq G_1) \subseteq G^\Omega_1\). Then, by Lemma 7.4(iv), there exists some \(\tau^\Omega_1|S^c_j|\). Then, by Lemma 7.4(iv), there exists some \(\tau^\Omega_1|S^c_j|\).
Lemma 7.14. It holds by all Definition 7.15. Definition 7.13. \( \tau(i) \) and (ii) are proved by a procedure similar to that of Lemma 7.11 and omitted.

8. Separation algorithm of multiple branches

We will generalize the separation algorithm of one branch to that of multiple branches. Roughly, we give an algorithm to eliminate all \( S_j^c \in G^* \) that \( H'_j \leq H'^{c} \) for some \( H'^{c} \in I \).
Notation 8.1 (See Appendix 5.2). Given $H_i^c \in \{H_1^c, \ldots, H_N^c\}$, let $\{H_{i_0}^c, H_{i_1}^c, \ldots, H_{i_n}^c\} := \{H_j^c : H_j^c \leq H_i^c, 1 \leq j \leq N\}$ such that $H_{i_0}^c = H_i^c, H_{i_n}^c > H_{i_{n-1}}^c$ for all $0 \leq t \leq n - 1$.

Lemma 8.2 (See Appendix 4 and 5.2). Let $I = \{[S_0^c], \ldots, [S_m^c]\}$. Then there exist one closed hypersequent $G_i^c \leq c G^*$ and its derivation $\tau_i^0$ from $[S_0^c], \ldots, [S_m^c]$ in $GL_\Omega$ such that

(i) $\tau_i^0$ is constructed by applying elimination rules, say,

$$
\frac{G_{b_1}|S_i^c, \ldots, G_{b_n}|S_i^c}{G_i|_{\Omega}^c} = \{G_{b_1}|S_i^c, \ldots, G_{b_n}|S_i^c\}^{\tau_i^0},
$$

and the fully constraint contraction rules, say $G_i|_{\Omega}^{F}(EC_{\Omega}^*), \text{ where } 1 \leq w \leq m, \{j_1, \ldots, j_w\} \subseteq \{i_1, \ldots, i_m\}$ and $H_{j_k}^c \leq H_i^c$ for all $1 \leq k \leq w, H_{j_k}^c \leq H_{j_{k-1}}^c$ for all $1 \leq k < l \leq w, \delta_i(\Omega) = \{H_{j_k}^c, \ldots, H_{j_{k-1}}^c\}, \tau_i^{F} = \{S_i^c, \ldots, S_i^c\}, I_i^{(t)} = \{G_{b_1}|S_i^c, \ldots, G_{b_n}|S_i^c\}$ and $G_{b_k}|S_i^c$ is closed for all $1 \leq k \leq w$. Then $H_i^c \leq H_j^c$ for all $S_i^c \in G_i^c$ and $H_i \in I$.

(ii) For all $H \in \tau_i^{F}$, let

$$
\partial_{\Omega}^{F}(H) := \begin{cases}
G|G^* & \text{is the root of } \tau_i^{F} \text{ or } G_2 \text{ in } G_1(\Omega) = \tau_i^{F}, \\
H_{j_k}^c & \text{is } G_{b_k}|S_i^c \text{ in } \tau_i^{F} \text{ for some } 1 \leq k \leq w, \\
\end{cases}
$$

where $\tau_i^{F}$ is the skeleton of $\tau_i^{F}$ which is defined as Definition 7.13. Then $\partial_{\Omega}^{F}(G_i|_{\Omega}^{F}) = \partial_{\Omega}^{F}(G_{b_k}|S_i^c)$ for some $1 \leq k \leq w$ in $\tau_i^{F}$.

(iii) Let $H \in \tau_i^{F}, G|G^* < \partial_{\Omega}^{F}(H) \leq H_{j_k}^c$ then $G_{b_k}|S_i^c \in \tau_i^{F}$ and it is built up by applying the separation algorithm along $H_{j_k}^c$ to $H$, and is an upper hypersequent of either $\{EC_{\Omega}^*\}$ if it is applicable, or $\{ID_{\Omega}\}$ otherwise.

(iv) $S_i^c \in G_i^c$ implies $H_i^c \leq H_j^c$ for all $H_i \in I$ and, $S_i^c \in G_i^c$ for some $\tau_i^{F} \in \tau_i^{F}$ or $S_i^c \in [S_m^c]$ for some $H_i \in I, H_j \leq H_{j_k}^c$.

Note that in Claim (i), bold $t$ and $j$ in $I_i^{(t)}$ or $I_i^{(t)}$ indicate $w$-tuples $(t_1, \ldots, t_w)$ and $(j_1, \ldots, j_w)$ in $S_i^{(j_1)}, \ldots, S_i^{(j_w)}$, respectively. Claim (iv) shows the final aim of Lemma 8.2, i.e., there exists no $S_j^c \in G_i^c$ such that $H_j^c \leq H_i^c$ for some $H_i \in I$. It is almost impossible to construct $\tau_i^{F}$ in a non-recursive way. Thus we use Claims (i), (ii) and (iii) in Lemma 8.2 to characterize the structure of $\tau_i^{F}$ in order to construct it recursively.

Let

$$
\frac{G_i^c|S_i^c}{G_i^c|H_i^c|S_i^c} \in \tau_i^*, \text{ where } G_i^c|G_i^c=H_i^c. \text{ Then } \{H_i^c, \ldots, H_m^c\} \text{ is divided into two subsets}
$$

$$
I_l = \{H_i^c, \ldots, H_m^c\}, I_r = \{H_m^c, \ldots, H_{m-1}^c\},
$$

which occur in the left subtree $\tau^*(G_i^c|S_i^c)$ and right subtree $\tau^*(G_i^c|S_i^c)$ of $\tau^*(H_i^c)$, respectively. Then $m_i + m_r = m$. Let

$$
H_l = \{[S_{i_1}^c], \ldots, [S_{i_{m_l}}^c]\}, H_r = \{[S_{i_{m_l+1}}^c], \ldots, [S_{i_m}^c]\}.
$$
\( \tau^0 \) is constructed by induction on \(|l| \). For the base case, let \(|l| = 1 \). Then \( \tau^0 \) is built up by Construction 7.3 and 7.7. Here, Claim (i) holds by Lemma 7.8(ii), Lemma 7.4(i) and Lemma 6.6 (vi), Claim (ii) by Lemma 7.4(i), (iii) is clear and (iv) by Lemma 7.4(iv).

For the induction case, suppose that derivations \( \tau^0 \) of \( G^0_n \) and \( \tau^0 \) of \( G^0_k \) are constructed such that Claims from (i) to (iv) hold. We present some properties of \( \tau^0 \) derived from Claims from (i) to (iv) and they are applied to \( \tau^0 \) or \( \tau^0 \) under the induction hypothesis.

**Property (A.1)** \( \tau^0 \) is an m-ary tree and, \( \tau^0 \) is a binary tree;

(A.2) Let \( H \in \tau^0 \) then \( \partial_{>0} (H) \leq H^0_k \) for some \( 1 \leq k \leq m \);

(A.3) Let \( H \in \tau^0 \) then \( H^0_k \parallel \partial_{>0} (H) \);

(A.4) Let \( w > 1 \) in \( \tau^0 \) then \( H^0_k < H^0_k \) for all \( 1 \leq k \leq w \).

(A.5) Let \( \tau^0 \in \tau^0 \), \( \partial_{>0} (G_B (S^c, \gamma)) \leq H^0_k \) for some \( 1 \leq k' \leq w \). Then \( w = 1 \).

**Proof.** (A.1) is immediately from Claim (i), (A.2) by \( G^0 \leq H^0 \leq H^0 \), (A.3) from Proposition 2.12(ii), (A.2) and \( H^0_k \leq H^0_k \). For (A.4), let \( w > 1 \). Then for each \( 2 \leq k \leq w \), \( H^0_k \parallel H^0_k \) by Claim (i). Thus \( H^0_k \parallel H^0_k \) by Proposition 2.12(ii). Hence \( H^0_k \parallel H^0_k \) by \( H^0_k \parallel H^0_k \) by Claim (i). Thus \( H^0_k \parallel H^0_k \) and \( H^0_k \parallel H^0_k \) by (A.3), \( H^0_k \leq H^0_k \). Hence \( H^0_k \parallel H^0_k \) for all \( 1 \leq k \leq w \). (A.5) is from (A.4). \( \square \)

**Property (B)** Let \( \frac{H_1 \cdots H_m}{H_{1,1}} \frac{\lambda^0_{(i)}}{H_{1,1}} \frac{\partial^0_{(i)}(H_{1,i})}{H_{1,1}} \in \tau^0 \) for all \( 1 \leq i \leq n \) such that \( \partial^0_{(i)}(H_{0,1}) = G^0 \) and \( \partial^0_{(i)}(H_{0,1}) \leq H^0_k \) for all \( 1 \leq i \leq n \).

**Proof.** The proof is by induction on \( n \). Let \( n = 1 \) then \( w_1 = 1 \) by Property (A.5) and \( \partial^0_{(1)}(H_{1,1}) \leq H^0_k \). For the induction step, let \( \partial^0_{(i)}(H_{1,i}) \leq H^0_k \) for some \( 1 \leq i \leq n \) then \( w_i = 1 \) by Property (A.5).

Since \( \frac{H_1 \cdots H_m}{H_{1,1}} \frac{\lambda^0_{(i)}}{H_{1,1}} \frac{\partial^0_{(i)}(H_{1,i})}{H_{1,1}} \in \tau^0 \) then \( \partial^0_{(i)}(H_{1,i}) \leq \partial^0_{(i)}(H_{1,i}) \) for some \( 1 \leq k \leq w_i \) by Claim (ii).

Then \( \partial^0_{(i)}(H_{1,i}) \leq \partial^0_{(i)}(H_{1,i}) \leq H^0_k \) by \( w_i = 1 \). Thus \( w_{i-1} = 1 \) by Property (A.5). \( \square \)

**Definition 8.3.** Let \( \frac{G_2}{G_1} (E^0_c) \in \tau^0 \). The module of \( \tau^0 \) at \( G_2 \), which we denote by \( \tau^0_{G_2} \), is defined as follows: (1) \( G_2 \in \tau^0_{G_2} \); (2) \( \frac{H_1 \cdots H_0}{H_0} \frac{\lambda^0_{(i)}}{H_{1,1}} \frac{\partial^0_{(i)}(H_{1,i})}{H_{1,1}} \in \tau^0_{G_2} \) if \( H_0 \in \tau^0_{G_2} \); (3) \( H_1 \notin \tau^0_{G_2} \) if \( \frac{H_1}{H_0} (E^0_c) \in \tau^0 \).

Each node of \( \tau^0_{G_2} \) is determined bottom-up, starting with \( G_2 \), whose root is \( G_2 \) and leaves may be branches, leaves of \( \tau^0 \) or lower hypersequents of \( (E^0_c) \)-applications. While each node of \( \tau^0_{G,H} \) is determined top-down, starting with \( H^0 \), whose root is a subset of \( G^0 \) and leaves contain \( H^0 \) and some leaves of \( \tau^0 \).

**Property (C.1)** \( \tau^0_{G_2} \) is a derivation without \( (E^0_c) \) in \( GL_0 \).

(C.2) Let \( H^0 \in \tau^0_{G_2} \) and \( \partial^0_{(H)}(H') > H^0 \). Then \( \partial^0_{(H)}(H') > H^0 \) for all \( H \in \tau^0_{G_2} \) and \( H \geq H^0 \).

**Proof.** (C.1) is clear and (C.2) immediately from Property (B). \( \square \)
Given

$$G_{b_k} | S^e_{j(a)} \rightarrow G_{b_2} | S^e_{j(2)} \rightarrow \cdots \rightarrow G_{b_n} | S^e_{j(a)} \rightarrow \tau^e_{j(a)} \in \tau^e_D$$

such that $S'' \in \langle G''|S'' \rangle_{T_{(a)}}$ and $H_{j(a)} > H_j$ for all $1 \leq k \leq v$, where, $1 \leq v \leq m$, $\{j_1, \cdots, j_n\} \subseteq \{i_{r_1}, \cdots, i_{m_c}\}$, $H_{j(a)} \leq H_{j(a)}$ and $G_{b_h} | S^e_{j(a)}$ is closed for all $1 \leq k \leq v$,

$$J_{k}^{(l)} = \{H_{j_1}^{(l)}, H_{j_2}^{(l)}, \cdots, H_{j_{l}}^{(l)}\},$$

$$T_{k}^{(l)} = \{S_{j_1}^{(l)}, S_{j_2}^{(l)}, \cdots, S_{j_{l}}^{(l)}\},$$

$$I_{k}^{(l)} = \{G_{b_1} | S_{j_{l}}^{(l)}, \cdots, G_{b_h} | S_{j_{l}}^{(l)}\}.$$

Then $H_{j_1}^{(l)} \geq G''|S''$ by $H_{j_2}^{(l)} > H_{j_1}^{(l)}$ for all $1 \leq k \leq v$. Thus $H_{j_1} \rightarrow H_{j_2}$ for all $H_{j_1} \in I_{k}^{(l)}$ and $H_{j_2} \in H$ by $S'' \in \langle G''|S'' \rangle_{T_{(a)}}$ and Construction 6.11. If there exists no such $\tau^e_{j(a)}$ in $\tau^e_D$ then the procedure terminates and $\tau^e_D := \tau^e_D$.

**Notation 8.4.** Let

$$G_{\hat{z}} := \hat{S}^e(G_{\hat{z}}^{(l)} | G_{\hat{z}}^{(l)} \setminus \hat{S}^e)$$

and

$$G_{\hat{z}} := \{G_{b_1} | \hat{S}^e(G_{\hat{z}}^{(l)} | G_{\hat{z}}^{(l)} \setminus \hat{S}^e)\}$$

be two close hypersequents, $G_{\hat{z}} \subseteq H$ for some $H \in \tau^e_D$ and $G_{\hat{z}} \setminus \{G_{b_1}\}_{k=1} \subseteq H$ for some $H \in \tau^e_D$.

Generally, $\hat{S}^e \subseteq G_{\hat{z}}$ is a copy of $\hat{S}^e \subseteq G_{\hat{z}}$, i.e., eigenvariables in $\hat{S}^e \subseteq G_{\hat{z}}$ have different identification numbers with those in $\hat{S}^e \subseteq G_{\hat{z}}$, so are $H', \hat{G}''$, $\hat{S}'$.

**Property** (D) $S_{j_1} \in G_{\hat{z}}$ implies $H_{j_1} \parallel G''|S''$.

**Proof.** Let $S_{j_2} \in G_{\hat{z}} \subseteq G_{\hat{z}}^{(l)} \hat{G}_{\hat{z}}^{(l)} | G_{\hat{z}}^{(l)} | H''$. Then $H_{j_2} \notin H''$ by Lemma 7.6(i). Thus $H_{j_1} > H_{j_2}^l$ or $H_{j_1}^l \parallel H''$. If $H_{j_1}^l \parallel H''$ then $H_{j_2}^l \parallel G''|S''$ by $H_{j_2}^l \parallel G''|S''$ and Proposition 2.12(ii). If $H_{j_1}^l > H_{j_2}^l$ then $S_{j_2} \in H_{j_2}^l$ by Proposition 4.15(i). Thus $S_{j_2} \in G''$ by Lemma 6.3, Lemma 6.7(i). Hence $H_{j_2}^l \parallel G''|S''$. If $H_{j_2}^l \parallel G''|S''$ then $S_{j_2} \in G''$. If $H_{j_2}^l \parallel G''|S''$ then $H_{j_2}^l \parallel G''|S''$. $\square$

**Stage 1 Construction of Subroutine $\tau^e_{K_{(a)}}(\tau^e_{V_{(a)}}).** We construct a derivation $\tau^e_{K_{(a)}}(\tau^e_{V_{(a)}})$ in which you may regard $\tau^e_{K_{(a)}}$ as a subroutine and $\tau^e_{V_{(a)}}$ as its input. Roughly speaking, $\tau^e_{K_{(a)}}(\tau^e_{V_{(a)}})$ is constructed by replacing some nodes $\tau^e_{V_{(a)}} \in \tau^e_D$ with $\tau^e_{K_{(a)}}(\tau^e_{V_{(a)}})$ in post-order. However, the ordinal postorder-algorithm cannot be used to construct $\tau^e_{K_{(a)}}(\tau^e_{V_{(a)}})$ because the tree structure of $\tau^e_{K_{(a)}}(\tau^e_{V_{(a)}})$ is generally different from that of $\tau^e_{K_{(a)}}$ at some nodes $H \in \tau^e_D$ that $\partial_{G_{K_{(a)}}}(H) < H''$. Thus we construct a sequence $\tau^e_{K_{(a)}}(\tau^e_{V_{(a)}})$ of trees for all $q \geq 0$ inductively as follows.

For the base case, we mark all $\langle EC_{\hat{z}} \rangle$-applications in $\tau^e_D$ as unprocessed and define such marked derivation to be $\tau^e_{K_{(a)}}(\tau^e_{V_{(a)}})$. For the induction case, let $\tau^e_{K_{(a)}}(\tau^e_{V_{(a)}})$ be constructed. If all applications
of $\langle EC_\Omega \rangle$ in $\tau_{\Omega}^{\Omega(q)}$ are marked as processed, we firstly delete the root of the tree resulting from the procedure and then, apply $\langle EC_\Omega \rangle$ to the root of the resulting derivation if it is applicable otherwise add an $(ID_\Omega)$-application to it and finally, terminate the procedure. Otherwise we select one of the outermost unprocessed $\langle EC_\Omega \rangle$-applications in $\tau_{\Omega}^{\Omega(q)}$, say, $\frac{G_{q+1}^\circ}{G_{q+1}^\circ} \langle EC_\Omega \rangle_{q+1}$, and perform the following steps to construct $\tau_{\Omega}^{\Omega(q+1)}$ in which $\frac{G_{q+1}^\circ}{G_{q+1}^\circ} \langle EC_\Omega \rangle_{q+1}$ be revised as $\frac{G_{q+1}^\circ}{G_{q+1}^\circ} \langle EC_\Omega \rangle_{q+1}$ such that

(a) $\tau_{\Omega}^{\Omega(q+1)}$ is constructed by locally revising $\tau_{\Omega}^{\Omega(q)}$ and leaving other nodes of $\tau_{\Omega}^{\Omega(q)}$ unchanged, particularly including $G_{q+1}^\circ$;

(b) $\tau_{\Omega}^{\Omega(q+1)}(G_{q+1}^\circ)$ is a derivation in $G_{1}:G_{q+1}^\circ$;

(c) $G_{q+1}^\circ = G_{q+1}^\circ$ if $q' \notin \langle G(S') \rangle_{h_{q}}$ for all $\tau_{\Omega}^{\Omega(q)} \in \tau_{\Omega}^{\Omega(q+1)}$ otherwise $G_{q+1}^\circ = G_{q+1}^\circ | G_{q+1}^\circ$ for some $m_{q+1} \geq 1$.

Remark 8.5. By two superscripts $\circ$ and $\cdot$ in $\langle EC_\Omega \rangle_{q+1}$ or $\langle EC_\Omega \rangle_{q+1}$, we indicate the unprocessed state and processed state, respectively. This procedure determines an ordering for all $\langle EC_\Omega \rangle$-applications in $\tau_{\Omega}^{\Omega(q)}$ and the subscript $q + 1$ indicates that it is the $q + 1$-th application of $\langle EC_\Omega \rangle$ in a post-order transversal of $\tau_{\Omega}^{\Omega(q)}$. If $\langle EC_\Omega \rangle_{q+1}$ and $G_{q+1}^\circ$ are the premise and conclusion of $\langle EC_\Omega \rangle_{q+1}$, respectively.

Step 1 (Delete). Take the module $\tau_{\Omega}^{\Omega(q)}$ out of $\tau_{\Omega}^{\Omega(q)}$. Since $\langle EC_\Omega \rangle_{q+1}$ is the unique unprocessed $\langle EC_\Omega \rangle$-applications in $\tau_{\Omega}^{\Omega(q)}$, by its choice criteria, $\tau_{\Omega}^{\Omega(q)}$ is the same as $\tau_{\Omega}^{\Omega(q)}$ by Claim (a). Thus it is a derivation. If $\partial_{\Omega}^{\Omega}(H) \leq H_{t}$ for all $H \in \tau_{\Omega}^{\Omega(q)}$, delete all internal nodes of $\tau_{\Omega}^{\Omega(q)}$. Otherwise there exists

$$
G_{b_{r_{1}}} | S_{r_{1}}^{\circ} \quad G_{b_{r_{2}}} | S_{r_{2}}^{\circ} \quad \ldots \quad G_{b_{r_{u}}} | S_{r_{u}}^{\circ} \quad G_{r_{u+1}} | S_{r_{u+1}}^{\circ} 
$$

such that $\partial_{\Omega}^{\Omega}(G_{b_{r_{k}}}) | S_{r_{k}}^{\circ} | H_{t}$ for all $1 \leq k \leq u'$ and $\partial_{\Omega}^{\Omega}(G_{r_{u'}}) \leq H_{t}$ by Property (C.2) and $\partial_{\Omega}^{\Omega}(G_{r_{u+1}}) \leq H_{t}$, then delete all $H \in \tau_{\Omega}^{\Omega(q)}, G_{q+1}^\circ \leq H < G_{r}$. We denote the structure resulting from the deletion operation above by $\tau_{\Omega}^{\Omega(q)}(G_{q+1}^\circ)$. Since $\partial_{\Omega}^{\Omega}(G_{r}) \leq H_{t}$ then $\tau_{\Omega}^{\Omega(q)}(G_{q+1}^\circ)$ is a tree by Property (B). Thus it is also a derivation.

Step 2 (Update). For each $G_{q'}^\circ \in \tau_{\Omega}^{\Omega(q)}(G_{q+1}^\circ)$, which satisfies $\frac{G_{q'}^\circ}{G_{q'}^\circ} \langle EC_\Omega \rangle_{q'}^\circ \in \tau_{\Omega}^{\Omega(q)}$ and $S' \in \langle EC_\Omega \rangle_{q'}^\circ$, for some $\tau_{\Omega}^{\Omega(q)} \in \tau_{\Omega}^{\Omega(q)}(G_{q+1}^\circ)$, we replace $H$ with $H| G_{r}$ for each $H \in \tau_{\Omega}^{\Omega(q)}(G_{q+1}^\circ), G_{r} \leq H \leq G_{q'}^\circ$.

Since $\frac{G_{q'}^\circ}{G_{q'}^\circ} \langle EC_\Omega \rangle_{q'}^\circ \in \tau_{\Omega}^{\Omega(q)}(G_{q+1}^\circ)$ and $\langle EC_\Omega \rangle_{q+1}$ is the outermost unprocessed
\((EC^*_{\Omega})\)-application in \(\tau^\Omega_{l_0}\) then \(q' \leq q\) and \((EC^*_{\Omega})_{q'}\) has been processed. Thus Claims (b) and (c) hold for \(\tau^\Omega_{l_0}(G_{q'})\) by the induction hypothesis. Then \(G_{q'}^0 / G_{q'}^1\) and \(G_{q'}^0 / G_{q'}^1\) are valid, where \(G_{q'}^0 = G_{q'}^0 / G_{q'}^0 | G_{q'}^1 / G_{q'}^1\), \(G_{q'}^1 / G_{q'}^1\) is a valid \((EC^*_{\Omega})\)-application since

\[
G_{q'}^0 / G_{q'}^1 \quad \text{and} \quad G_{q'}^0 / G_{q'}^1
\]

\(G_{q'}^0 / G_{q'}^1\) and \(G_{q'}^0 / G_{q'}^1\) are valid, where \(G_{q'}^0 = G_{q'}^0 / G_{q'}^0 | G_{q'}^1 / G_{q'}^1\), \(G_{q'}^1 / G_{q'}^1\) is a valid \((EC^*_{\Omega})\)-application since

**Property E.** Let \(G_{l'} < H \leq G_{q'}\). Then \(\partial_{q'}(H) \geq G^*|S'\).

**Proof.** Since \(G_{l'} < H\) then \(G_{l'}|S'_{\gamma_{l'}} \leq H\) for some \(1 \leq k \leq u'). If \(\partial_{q'}(H) \geq H_{l'}^k\) then \(\partial_{q'}(H) \geq G|S'\). Otherwise all applications between \(G_{l'}\) and \(H\) are one-premise inferences by Property (B). Then \(H_{l'}^k \leq \partial_{q'}(H)\) by Claim (ii). Thus \(\partial_{q'}(H) \geq G|S'\) by \(H_{l'}^k < H_{l'}^k\), \(\partial_{q'}(H) \leq H_{l'}^k\) for some \(1 \leq k' \leq m\).

Since \(\partial_{q'}(H) \geq G|S'\) by Property (E) and \(H_{l'}^k \| G|S'\) for each \(S_{l'}^k \in G_{l'}\) by Property (D), then \(G_{l'} \leq H\) as side-hypersequent of \(H\). Thus this step updates the revision of \(G_{q'}\) downward to \(G_{l'}\).

Let \(m'\) be the number of \(G_{q'}^0\) satisfying the above conditions, \(\tau^\Omega_{l_0}(G_{q'}^0 / G_{q'}^1, 1), G_{q'}^0\) and \(G_{l'}|S'_{l'}\) for all \(1 \leq k \leq u'\) be updated as \(\tau^\Omega_{l_0}(G_{q'}^0 / G_{q'}^1, 1), G_{q'}^0, G_{l'}\) respectively. Then \(\tau^\Omega_{l_0}(G_{q'}^0 / G_{q'}^1, 1)\) is a derivation and \(G_{l'} = G_{l'}|S'_{l'}\).

**Step 3 (Replace).** All \(\tau_{l'}(\bar{\mathcal{S}}) \in \tau^\Omega_{l_0}(G_{q'}^0 / G_{q'}^1, 1)\) are processed in order. If \(H_{l'}^k \sim H_{l'}^k\) for all \(H_{l'}^k \in I_{l'}(k)\) and \(H_{l'}^k \in I_{l'}(k)\) it proceeds by the following procedure otherwise it remains unchanged.

Let \(\tau_{l'}(\bar{\mathcal{S}})\) be in the form

\[
G_{l'}|S'_{l'} = \left\{ G_{l'}(1) \mid G_{l'}(1) \right\}
\]

Then \(H_{l'}^k \geq G|S'\) for all \(1 \leq k \leq u\) by Property (E), \(G_{l'}|S'_{l'} \geq G_{l'}\).

Firstly, replace \(\tau_{l'}(\bar{\mathcal{S}})\) with \(\tau_{l'}(\bar{\mathcal{S}})\). We may rewrite the roots of \(\tau_{l'}(\bar{\mathcal{S}})\) and \(\tau_{l'}(\bar{\mathcal{S}})\) as

\[
G_{l'} = \left\{ G_{l'}(1) \mid G_{l'}(1) \right\}
\]

respectively.

Let \(G_{l'} < H \leq G_{l'}\). By Property (E), \(\partial_{q'}(H) \geq G^*|S'\). By Lemma 6.7, \(H_{l'}^k \leq H_{l'}^k < G^*|S'\) or \(H_{l'}^k \| G^*|S'\) for all \(S_{l'}^k \in G_{l'}^0 / G_{l'}^1\). Thus \(G_{l'}^0 / G_{l'}^1|H_{l'}^k < H_{l'}^k\), \(\partial_{q'}(H) \leq H_{l'}^k\) for all \(G_{l'} < H \leq G_{l'}\). Let \(m''\) be the number of \(\tau_{l'}(\bar{\mathcal{S}}) \in \tau^\Omega_{l_0}(G_{q'}^0 / G_{q'}^1, 1)\) satisfying the replacement conditions above, \(\tau^\Omega_{l_0}(G_{q'}^0 / G_{q'}^1, 1), G_{l'}\) and \(G_{l'}|S'_{l'}\)
for all $1 \leq k \leq u'$ be updated as $\tau^{\Omega(q)}_{L_{G_{p'}}^{\tau_1}(3)}$, $G_{pm}$, $\tau^{\Omega(q)}_{L_{H_{p'}}^{\tau_1}(3)}$, respectively. Then $\tau^{\Omega(q)}_{L_{G_{p'}}^{\tau_1}(3)}$ is a derivation of $G_{pm}$ and $G_{pm}$ in $G_{p''}$ is $G_{pm} \setminus \{G_{pm}^{\tau_1}_{H_{p'}^{\tau_1}(3)} | H_{p'}^{\tau_1}(3) \} \cup \{G_{pm}^{\tau_1}_{H_{p'}^{\tau_1}(3)} | H_{p'}^{\tau_1}(3) \} = G_{pm}$.

**Step 4 (Separation along $H_{1}^{\tau_1}$).** Apply the separation algorithm along $H_{1}^{\tau_1}$ to $G_{pm}$ and denote the resulting derivation by $\tau^{\Omega(q)}_{L_{G_{p'}}^{\tau_1}(4)}$ whose root is labeled by $G_{p''+1}$. Then all $G_{pm}^{\tau_1}_{H_{p'}^{\tau_1}(3)} | H_{p'}^{\tau_1}(3)$ in $G_{pm}$ are transformed into $G_{p''}^{\tau_1}_{H_{p'}^{\tau_1}(3)} | H_{p'}^{\tau_1}(3)$ in $\tau^{\Omega(q)}_{L_{G_{p'}}^{\tau_1}(4)}$. Since $G_{pm}^{\tau_1}_{H_{p'}^{\tau_1}(3)} | H_{p'}^{\tau_1}(3) = G_{pm}^{\tau_1}_{H_{p'}^{\tau_1}(3)} | H_{p'}^{\tau_1}(3) \in \tau^*$

$$H_{1}, S_{1}^{\tau_1}$$ and $S''_{1}$ are separable in $\tau^{\Omega(q)}_{L_{G_{p'}}^{\tau_1}(4)}$ by a procedure similar to that of Lemma 7.11. Let $S_{1}^{\tau_1}$ and $S''_{1}$ be separated into $S_{1}^{\tau_1}$ and $S''_{1}$, respectively. By Claim (iii), $G_{p''}^{\tau_1}_{H_{p'}^{\tau_1}(3)} = G_{q''+1}^{\tau_1}$.

$$G_{p''}^{\tau_1}_{H_{p'}^{\tau_1}(3)} = G_{q''+1}^{\tau_1} \setminus G_{m''}^{\tau_1} | G_{m''}^{\tau_1}$$ by Lemma 7.6(iv),

$$G_{q''+1}^{\tau_1} = G_{p''}^{\tau_1}_{H_{p'}^{\tau_1}(3)} \setminus G_{m''}^{\tau_1}$$

where $m_{q''+1} := m' + m''$.

**Step 5 (Put back).** Replace $\tau^{\Omega(q)}_{L_{G_{p''}^{\tau_1}(4)}^{\tau_1}}$ in $\tau^{\Omega(q)}_{L_{G_{p''}^{\tau_1}(4)}^{\tau_1}}$ with $\tau^{\Omega(q)}_{L_{G_{p''}^{\tau_1}(4)}^{\tau_1}}$ and mark $G_{q''+1}^{\tau_1}$ as processed, i.e., revise $\langle EC_{\Omega}^{\tau_1} \rangle_{q''+1}^{\tau_1}$ as $\langle EC_{\Omega}^{\tau_1} \rangle_{q''+1}^{\tau_1}$. Among leaves of $\tau^{\Omega(q)}_{L_{G_{p''}^{\tau_1}(4)}^{\tau_1}}$, all $G_{p''}$ are updated as $G_{p''}$ and others keep unchanged in $\tau^{\Omega(q)}_{L_{G_{p''}^{\tau_1}(4)}^{\tau_1}}$. Then this replacement is feasible, especially, $G_{q''+1}^{\tau_1}$ be replaced with $G_{p''+1}^{\tau_1}$. Define the tree resulting from Step 5 to be $\tau_{\Omega(q')}$ Then Claims (a), (b) and (c) hold for $q + 1$ by the above construction.

Finally, we construct a derivation of $G_{p}^{\tau_1} \setminus G_{\tau_1}^{\tau_1}$ from $\{S_{\tau_1}^{\tau_1} | \tau_{\Omega(q)}^{\tau_1}, \ldots, \{S_{\tau_1}^{\tau_1} | \tau_{\Omega(q)}^{\tau_1}, \ldots, G_{\tau_1}^{\tau_1} | \tau_{\Omega(q)}^{\tau_1}, \ldots, G_{\tau_1}^{\tau_1} \} \in GL_{\Omega}$, which we denote by $\tau_{\Omega(q')}$.
Remark 8.6. All elimination rules used in constructing \( \tau^\Omega_{\text{Ik}} \) are extracted from \( \tau^* \). Since \( \tau^*_{\text{Ik}(0)} \) is a derivation in \( \text{GL}_{\Omega} \) without \( (EC_{\Omega}) \), we may extract elimination rules from \( \tau^*_{\text{Ik}(0)} \) which we may use to construct \( \tau^\Omega_{\text{Ik}(0)} \) by a procedure similar to that of constructing \( \tau^\Omega_{\text{Ik}} \) with minor revision at every node \( H \) that \( \partial_{\text{Ik}v}(H) \leq H' \). Note that updates and replacements in Steps 2 and 3 are essentially inductive operations but we neglect it for simplicity.

We may also think of constructing \( \tau^\Omega_{\text{Ik}(0)} \) as grafting \( \tau^*_{\text{Ik}(0)} \) in \( \tau^\Omega_{\text{Ik}} \) by adding \( \tau^*_{\text{Ik}(0)} \) to some \( \tau^*_{\text{Ik}(0)} \in \tau^\Omega_{\text{Ik}} \). Since the rootstock \( \tau^\Omega_{\text{Ik}} \) of the grafting process is invariant in Stage 2, we encapsulate \( \tau^\Omega_{\text{Ik}(0)} \) as an inference rule in \( \text{GL}_{\Omega} \) whose premises are \( G_{b_1}|S_{\lambda_1}; G_{b_2}|S_{\lambda_2}; \ldots; G_{b_n}|S_{\lambda_n} \) and conclusion is \( \overline{S'}\{(G_{b_1})_{\lambda_1}^{E_1}|(G_{b_2})_{\lambda_2}^{E_2}\} \{\overline{S}|\overline{S}''\}\{G_{b_n}^{E_n} \} \) i.e.,

\[
\begin{align*}
G_{b_1}|S_{\lambda_1} &\quad \quad G_{b_2}|S_{\lambda_2} &\quad \quad \ldots &\quad \quad G_{b_n}|S_{\lambda_n} \\
\overline{S'}\{(G_{b_1})_{\lambda_1}^{E_1}|(G_{b_2})_{\lambda_2}^{E_2}\} &\quad \quad \{\overline{S}|\overline{S}''\}\{G_{b_n}^{E_n} \}
\end{align*}
\]

where, \( G_{b_1}^E = G_{b_1}^E \setminus G_{\Gamma} \) is closed.

**Stage 2 Construction of Routine** \( \tau^\Omega_{\text{Ik}}(\sigma^{(0)}_{\text{Ik}(0)}, \sigma^{(0)}_{\text{Ik}(0)}) \). A sequence \( \tau^\Omega_{\text{Ik}(q)} \) of trees for all \( q \geq 0 \) is constructed inductively as follows. \( \tau^\Omega_{\text{Ik}(0)} \), \( \tau^\Omega_{\text{Ik}(q+1)} \) are defined as those of Stage 1. Then we perform the following steps to construct \( \tau^\Omega_{\text{Ik}(q+1)} \) in which \( \overline{G_{q+1}^{E_3}(EC_{\Omega})_{q+1}} \)

be revised as \( \overline{G_{q+1}^{E_3}(EC_{\Omega})_{q+1}} \) such that Claims (a) and (b) are same as those of Stage 1 and (c) \( G_{q+1}^{E_3} \) if \( S'' \not\in \{G''|S''\}^{E_3}_{\lambda_1} \) for all \( \tau^*_{\text{Ik}(0)} \in \tau^\Omega_{\text{Ik}(q+1)} \) otherwise \( G_{q+1}^{E_3} = G_{q+1}^{E_3}\\{\overline{S}|\overline{S}''\}\{G_{b_n}^{E_n}\}^{m_{q+1}} \) for some \( m_{q+1} \geq 1 \).

**Step 1 (Delete).** \( \tau^\Omega_{\text{Ik}(q)} \) and \( \tau^\Omega_{\text{lk}(q+1)} \) are defined as before.

\[
\begin{align*}
G_{b_1}|S_{\lambda_1} &\quad \quad G_{b_2}|S_{\lambda_2} &\quad \quad \ldots &\quad \quad G_{b_n}|S_{\lambda_n} \\
\tau^*_{\text{Ik}(0)} &\quad \quad \tau^*_{\text{Ik}(0)} \\
\overline{G_{\lambda_1}} &\quad \quad \overline{G_{\lambda_2}} &\quad \quad \ldots &\quad \quad \overline{G_{\lambda_n}}
\end{align*}
\]

satisfies \( \partial_{\text{Ik}v}(G_{b_1}|S_{\lambda_1}) > H' \) for all \( 1 \leq k \leq v' \) and \( \partial_{\text{Ik}v}(G_{\lambda_i}) \leq H' \).

**Step 2 (Update).** For all \( G_{q+1}^{E_3} \in \tau^\Omega_{\text{Ik}(q+1)} \) which satisfy \( \overline{G_{q+1}^{E_3}(EC_{\Omega})_{q+1}} \), \( \overline{G_{q+1}^{E_3}} \) and \( S'' \not\in \{G''|S''\}^{E_3}_{\lambda_1} \) for some \( \tau^*_{\text{Ik}(0)} \in \tau^\Omega_{\text{Ik}(q+1)} \), we replace \( H \) with \( H'\{\overline{S}|G_{b_n}^{E_n}\} \) for all \( H \in \tau^\Omega_{\text{Ik}(q+1)}, \tau^*_{\text{Ik}(0)} \leq H \leq G_{q+1}^{E_3} \). Then Claims (a) and (b) are proved by a procedure as before. Let
Let \( m' \) be the number of \( G_{q'} \) satisfying the above conditions. \( \tau_{\Omega(q)}^{0,1}(G_{q'}) \) for all \( 1 \leq k \leq k' \) be updated as \( \tau_{\Omega(q)}^{0,1}(G_{q'}, G_{h_{k'}}|S|_{j_k}) \), respectively. Then \( \tau_{\Omega(q)}^{0,1}(G_{q'}) \) is a derivation and \( G_{q'} = G_r \{ S \}_{H_r}^{G_r(J)} \) is \( \tau_{\Omega(q)}^{0,1}(G_{q'}) \).

**Step 3 (Replace).** All \( \tau_{h_{k'}}^{1}(G_{q'}) \) are processed in post-order. If \( H_{r_j} \) for all \( H_{r_j} \in I_k \) it proceeds by the following procedure otherwise it remains unchanged. Let \( \tau_{h_{k'}}^{1}(G_{q'}) \) be in the form

\[
G_{b_{i_1}}|S|_{j_{i_1}} \ G_{b_{i_2}}|S|_{j_{i_2}} \ldots \ G_{b_{i_l}}|S|_{j_{i_l}}
\]

\( G_r = \{ G_{b_{i_1}} \}_{j_{i_1}}^{G_r(J)} \) and \( \tau_{h_{k'}}^{1}(G_{q'}) \) as

\( G_{r} = \{ G_{b_{i_1}} \}_{j_{i_1}}^{G_r(J)} | G_r^{G_r(J)} \}_{j_{i_1}}^{G_r(J)} \). We may rewrite the roots of \( \tau_{h_{k'}}^{1}(G_{q'}) \) as \( G_{r} = \{ G_{b_{i_1}} \}_{j_{i_1}}^{G_r(J)} | G_r^{G_r(J)} \}_{j_{i_1}}^{G_r(J)} \) respectively.

Let \( G_{r'} < H \leq G_{r} \). Then \( \partial_{\Omega(q)}(H) \equiv G_r^{\Omega(q)}[S] \) by Property (E). Thus \( G_r^{\Omega(q)}[S] \) \( H \geq G_r^{\Omega(q)}[S] \) for all \( \{ S_j \} : S_j \in G_r^{\Omega(q)}[S] \}. Define \( G_{r'}^{\Omega(q)} = \{ S_j \} : S_j \in G_r^{\Omega(q)}[S] \}. Then we replace \( H \) with

\[
H \{ G_r^{\Omega(q)}(S) \} \{ G_r^{\Omega(q)}(S) \} | G_r^{\Omega(q)}(S) | G_r^{\Omega(q)}(S) \}
\]

for all \( G_{b_{i_1}}|S|_{j_{i_1}}^{G_r(J)} \) \( H \leq G_r \).

Let \( m'' \) be the number of \( \tau_{h_{k'}}^{1}(G_{q'}) \) satisfying the replacement conditions as above, \( \tau_{\Omega(q)}^{0,1}(G_{q'}) \) for all \( 1 \leq k \leq k' \) be updated as \( \tau_{\Omega(q)}^{0,1}(G_{q'}, G_{h_{k'}}|S|_{j_k}) \), respectively. Then \( \tau_{\Omega(q)}^{0,1}(G_{q'}) \) is a derivation and \( G_{q''} = G_r \{ H_{r_j} \}^{\tau_{\Omega(q)}^{0,1}(G_{q'})} \), where

\[
H_1 = G_r^{\Omega(q)}(S) | H_{r_j} \}^{G_r^{\Omega(q)}(S)} \]

\[
H_2 = S \}^{G_r^{\Omega(q)}(S)} \}^{G_r^{\Omega(q)}(S)} \}
\]

**Step 4 (Separation along \( H_1^{1} \)).** Apply the separation algorithm along \( H_1^{1} \) to \( G_{q''} \) and denote the resulting derivation by \( \tau_{\Omega(q)}^{0,1}(G_{q''}) \) whose root is labeled by \( G_{q''} \).
By Claim (iii), $G_{H^c_i,G'}^{\omega} = G_{q+1}^\theta$.

$$G_{H^c_i,G'}^{\omega} = G_{q+1}^\theta \setminus \{ (G_{H^c_i,G'}^{\omega})_T \}^{m'} \setminus \{ (G_{H^c_i,G'}^{\omega})_{n'} \},$$

$$G_{H^c_i,H^d_i}^{\omega} = G_{H^c_i,(G)^{T}_{T_H}}^{\omega} \setminus \{ (G_{H^c_i,G'}^{\omega})_{H^d_i} \},$$

$$C_{H^c_i,H^d_i}^{\omega} = S_\theta \{ (G_{H^c_i,G'}^{\omega})_{H^d_i} \} \setminus \{ (G_{H^c_i,G'}^{\omega})_{H^d_i} \}, \{ (G_{H^c_i,G'}^{\omega})_{H^d_i} \} \{ (G_{H^c_i,G'}^{\omega})_{H^d_i} \} \{ (G_{H^c_i,G'}^{\omega})_{H^d_i} \},$$

Then

$$G_{H^c_i,G'}^{\omega} = G_{q+1}^\theta \setminus \{ (G_{H^c_i,G'}^{\omega})_T \}^{m''} \setminus \{ (G_{H^c_i,G'}^{\omega})_T \}^{m''} \setminus \{ (G_{H^c_i,G'}^{\omega})_T \}^{m''}.$$
for all $H_j^i \in I_i$, $H_j^i \notin H_j^i$ by $G''|S'' \leq H_j^i$. Then $H_j^i \notin H_j^i$ for all $H_j^i \in I_i$. Claims (ii) and (iii) follow directly from the induction hypothesis.

• For Claim (iv), let $S_j^i \in G''_h$. It follows from the induction hypothesis that $H_j^i | H_j^i$ for all $H_j^i \in I_i$ and, $S_j^i \in G''_{Z_{j,k}^{(2)}}$ for some $\tau_{k,h}^{\ast} \in \tau_k^{\Omega}$ or $S_j^i \in [S_{h,k}^i]$ for some $H_{h,k} \in I_i$, $H_j^i \notin H_{h,k}^{i}$. Then $H_j^i \notin H_j^i$ by $H_j^i | H_j^i$ for all $H_j^i \in I_i$.

If $S_j^i \in [S_{h,k}^i]$ for some $H_{h,k} \in I_i$, $H_j^i \notin H_{h,k}^{i}$ then $H_j^i | H_j^i$ for all $H_j^i \in I_i$ by the definition of branches to $I_i$. Thus we assume that $S_j^i \in G''_{Z_{j,k}^{(2)}}$ for some $\tau_{k,h}^{\ast} \in \tau_k^{\Omega}$ in the following. If $G''|S' \leq H_j^i$

then $H_j^i | H_j^i$ for all $H_j^i \in I_i$, thus $H_j^i | H_j^i$ for all $H_j^i \in I_i$. Thus let $G''|S' \leq H_j^i$ in the following. By the proof of Claim (i) above, $G''|S'' \leq H_j^i$. Then $H_j^i \notin H_j^i$ by $G''|S' \leq H_j^i$ and $G''|S'' \leq H_j^i$. Thus $H_j^i | H_j^i$. Hence $H_j^i | H_j^i$ for all $H_j^i \in I_i$.

Case 2 $S'' \notin (G''|S'')_{Z_{j,k}^{(2)}}$ for all $\tau_{k,h}^{\ast} \in \tau_k^{\Omega}$. This case is proved by a procedure similar to that of Case 1 and omitted.

Case 3 $S' \notin (G'|S')_{Z_{j,k}^{(2)}}$ for some $\tau_{k,h}^{\ast} \in \tau_k^{\Omega}$ and $S'' \notin (G''|S'')_{Z_{j,k}^{(2)}}$ for some $\tau_{k,h}^{\ast} \in \tau_k^{\Omega}$. Then $\tau_k^{\Omega} = \tau_k^{\Omega} \cup (\tau_k^{\Omega} \setminus \tau_k^{\Omega} \cap \tau_k^{\Omega})$ and $G''_h = G''_h \setminus (G''_h \setminus G''_h)\cap (G''_h \setminus G''_h)$.

• For Claim (i), (ii): Let $H_{h,j}^i \cdots H_{h,j}^i \tau_{j,h}^{\ast} \in \tau_k^{\Omega}$ and $S_j^i \in G''_{Z_{j,k}^{(2)}}$. Then $\partial_{\tau_k^{\Omega}}(H_{h,j}^i) \notin H_{h,j}^i$ for all $1 \leq k \leq w$ by Lemma 6.13(iv).

If $\partial_{\tau_k^{\Omega}}(H_{h,j}^i) \notin H_{h,j}^i$ for some $1 \leq k' \leq w$, then $H_{h,j}^i \notin H_{h,j}^i$ for all $H_{h,j}^i \in I_i$ by $\partial_{\tau_k^{\Omega}}(H_{h,j}^i) \notin H_{h,j}^i$. Thus Claim (i) holds and Claim (ii) holds by Property (A.5) and Lemma 7.6(i). Note that Property (A.5) is independent of Claims (ii) to (iv).

Otherwise $\tau_{k,h}^{\ast}$ is built up from $\tau_{k,h}^{\ast} \in \tau_k^{\Omega}$ or $\tau_{k,h}^{\ast} \in \tau_k^{\Omega}$ by keeping their focus and principal sequents unchanged and making their side-hypersequents possibly to be modified, but which have no effect on discussing Claim (ii) and then Claim (ii) holds for $\tau_k^{\Omega}$ by the induction hypothesis on Claim (ii) of $\tau_k^{\Omega}$ or $\tau_k^{\Omega}$.

If $\tau_{k,h}^{\ast}$ is from $\tau_{k,h}^{\ast} \in \tau_k^{\Omega}$, then $S' \notin (G''|S')_{Z_{j,k}^{(2)}}$ and $S'' \notin (G''|S'')_{Z_{j,k}^{(2)}}$ by the choice of $\tau_{k,h}^{\ast}$ and $\tau_{k,h}^{\ast}$ at Stage 1. By the induction hypothesis, $H_j^i \notin H_j^i$ for all $S_j^i \in G''_{Z_{j,k}^{(2)}}$, $H_{h,j}^i \in I_i$, and $H_j^i \notin H_j^i$ for all $S_j^i \in G''_{Z_{j,k}^{(2)}}$. Then $H_j^i \notin H_j^i$ for all $I_i \in I_i$. Then $G_j^i \notin H_j^i$ for all $S_j^i \in G''_{Z_{j,k}^{(2)}}$ and $H_j^i \in I_i$ by $G''_{Z_{j,k}^{(2)}} = G''_{Z_{j,k}^{(2)}} \cap G''_{Z_{j,k}^{(2)}}$, $I_i \in I_i$.

If $\tau_{k,h}^{\ast}$ is from $\tau_{k,h}^{\ast} \in \tau_k^{\Omega}$, then $S' \notin (G''|S')_{Z_{j,k}^{(2)}}$ by Step 3 at Stage 1. Then $(G''|H_j^i \setminus G''|H_j^i) \cap (G''|H_j^i) = \emptyset$. Thus $S_j^i \notin G''_{Z_{j,k}^{(2)}}$. Hence $G''|S'' \notin H_j^i$. Therefore $H_j^i \notin H_j^i$ for all $H_j^i \in I_i$. Then $G''|S'' \leq H_j^i$. Thus $H_j^i \notin H_j^i$ for all $H_j^i \in I_i$ by $G''|S'' \leq H_j^i$ and the induction hypothesis from $\tau_{k,h}^{\ast} \in \tau_k^{\Omega}$. The case of $\tau_{k,h}^{\ast}$ built up from $\tau_{k,h}^{\ast}$ is proved by a procedure similar to above and omitted.

• Claim (iii) holds by Step 4 at Stage 1 and 2. Note that in the whole of Stage 1, we treat $\{G_{h,k}\}_{k=1}^{\ast}$ as a side-hypersequent. But it is possible that there exists $S_j^i \in \{G_{h,k}\}_{k=1}^{\ast}$ such that $H_j^i \leq H_j^i$. Since we haven’t applied the separation algorithm to $\{G_{h,k}\}_{k=1}^{\ast}$ in Step 4 at Stage 1, then it could make Claim (iii) invalid. But it is not difficult to find that just move the
separation of such \( S_j \) to Step 4 at Stage 2. Of course, we can move it to Step 4 at Stage 1, but which make the discussion complicated.

- For Claim (iv), we prove (1) \( H'_j \models H'_j \) for all \( S_j \in \mathcal{G}_I \), \( H'_j \notin \mathcal{I} \) and (2) \( H'_j \models H'_j \) for all \( S_j \in \mathcal{G}_I \setminus \mathcal{S}^\mathcal{G}_I \), \( H'_j \notin \mathcal{I} \), which are derived from the induction hypothesis about \( \tau^0_{k_i} \) and \( \tau^0_{k_i} \). Only (1) is proved as follows and (2) by a similar procedure and omitted.

Let \( S_j \in \mathcal{G}_I \). Then \( S_j \in \mathcal{G}_I \) and \( S_j \notin \mathcal{S}^\mathcal{G}_I \setminus \mathcal{S}^\mathcal{G}_I \) by the definition of \( \mathcal{G}_I \). By a procedure similar to that of Claim (iv) in Case 1, we get \( H'_j \notin H'_j \) and assume that \( S_j \in \mathcal{G}_I \) for some \( \tau^*_{k_i} \in \mathcal{I} \) and let \( G' | S_j \notin H'_j \) in the following.

Suppose that \( G' | S'' \notin H'_j \). Then \( S'' \notin \mathcal{G}_I \) and \( S'' \notin \mathcal{S}^\mathcal{G}_I \) by the definition of \( \mathcal{G}_I \) and \( \mathcal{S}^\mathcal{G}_I \). Hence \( S_j \in \mathcal{G}_I \setminus \mathcal{S}^\mathcal{G}_I \) by \( H'_j \notin \mathcal{G}_I \). Then \( H'_j \notin H'_j \) by \( G' | S'' \notin H'_j \) and \( G'' | S'' \notin H'_j \). Thus \( H'_j | H'_j \). Hence \( H'_j | H'_j \) for all \( H'_j \notin \mathcal{I} \).

**Definition 8.7.** The manipulation described in Lemma 8.2 is called derivation-grafting operation.

**9. The proof of Main lemma**

Recall that in Main lemma \( G_0 \equiv \mathcal{G}' \{ \Gamma_i, p \Rightarrow \Delta_i \}_{i=0}^{m} \{ \Pi_j \Rightarrow p, \Sigma_j \}_{j=1}^{\ldots} \).

**Lemma 9.1.** (i) If \( G_2 = G_0 \{ \Gamma_i, p \Rightarrow \Delta_i \} \) and \( \vdash_{\mathcal{B}} \mathcal{D}_0 (G_2) \) then \( \vdash_{\mathcal{B}} \mathcal{D}_0 (G_0) \); (ii) If \( G_2 = G_0 \{ \Pi_j \Rightarrow p, \Sigma_j \} \) and \( \vdash_{\mathcal{B}} \mathcal{D}_0 (G_2) \) then \( \vdash_{\mathcal{B}} \mathcal{D}_0 (G_0) \); (iii) If \( G_2 = G_0 \{ \Gamma_i, p \Rightarrow \Delta_i \}, \{ \Pi_j \Rightarrow p, \Sigma_j \} \) and \( \vdash_{\mathcal{B}} \mathcal{D}_0 (G_2) \) then \( \vdash_{\mathcal{B}} \mathcal{D}_0 (G_0) \); (iv) If \( G_2 = G_0 \{ \Pi_j \Rightarrow p, \Sigma_j \}, \{ \Gamma_i, p \Rightarrow \Delta_i \} \) and \( \vdash_{\mathcal{B}} \mathcal{D}_0 (G_2) \) then \( \vdash_{\mathcal{B}} \mathcal{D}_0 (G_0) \); (v) If \( G_2 = G_0 \{ \Pi_j \Rightarrow p, \Sigma_j \}, \{ \Gamma_i, p \Rightarrow \Delta_i \}, \{ \Pi'_j \Rightarrow p, \Sigma'_j \} \) and \( \vdash_{\mathcal{B}} \mathcal{D}_0 (G_2) \) then \( \vdash_{\mathcal{B}} \mathcal{D}_0 (G_0) \).

**Proof.** (i) Since \( \mathcal{D}_0 (G_2) = \mathcal{G}' \{ \Gamma_i, \Pi_j \Rightarrow \Delta_i, \Sigma_j \}_{i=1}^{m} \mathcal{S}^{\mathcal{G}} \{ \Gamma_i, \Pi_j \Rightarrow \Delta_i, \Sigma_j \}_{i=1}^{m} \{ \Pi_j \Rightarrow p, \Sigma_j \} \) then \( \vdash_{\mathcal{B}} \mathcal{D}_0 (G_0) \) holds. If \( n = 1 \), we replace all \( p \) in \( \Pi_j \Rightarrow p, \Sigma_j \) with \( \bot \). Then \( \vdash_{\mathcal{B}} \mathcal{D}_0 (G_0) \) holds by applying \( (\text{CUT}) \) to \( \Gamma_i, \Pi_j \Rightarrow \Delta_i, \Sigma_j \) and \( G' \{ \Pi_j \Rightarrow \bot, \Sigma_j \} \). (ii), (iii) and (iv) are proved by a procedure respectively similar to those of (i), (ii) and (iii) and omitted.

Let \( I = \{ H'_j, \ldots, H'_N \} \subseteq \{ H'_i, \ldots, H'_N \} \). \( G_I \) denote a closed hyperssequent such that \( G_I \subseteq \mathcal{G}^* \) and \( H'_j | H'_j \) for all \( S_j \in \mathcal{G}_I \).

**Lemma 9.2.** There exists \( G_I \) such that \( \vdash_{\mathcal{B}} G_I \) for all \( I \subseteq \{ H'_1, \ldots, H'_N \} \).

**Proof.** The proof is by induction on \( |I| \). For the base step, let \( I = \emptyset \) then \( G_I := G | G^* \) and \( \vdash_{\mathcal{B}} G_I \) by Lemma 4.17(v).
For the induction step, suppose that \( m \geq 1 \) and there exists \( G_I \) such that \( \vdash_{\Gamma \fa} D_I \) for all \( |I| \leq m - 1 \). Let \( I = \{H_{i_1}, \ldots, H_{i_n}\} \subseteq \{H_{j_1}, \ldots, H_{k_m}\} \). Then there exist \( G_{I \cup \{H_{j_1}\}} \) for all \( 1 \leq k \leq m \) such that \( \vdash_{\Gamma \fa} G_{I \cup \{H_{j_1}\}} \) and \( H_{j_1} \vdash_{\Gamma \fa} H_{k_m} \) for all \( 1 \leq i \leq n \) and \( i \notin I \).

If \( H_{j_1} \vdash_{\Gamma \fa} H_{k_m} \) for all \( S_{j_1} \in G_{I \cup \{H_{j_1}\}} \) then \( G_I := G_{I \cup \{H_{j_1}\}} \) and the claim holds clearly. Otherwise there exists \( S_{j_1} \in G_{I \cup \{H_{j_1}\}} \) such that \( H_{j_1} \not\vdash_{\Gamma \fa} H_{k_m} \) or \( H_{j_1} > H_{k_m} \). Then rewrite \( G_{I \cup \{H_{j_1}\}} \) as \( [S_{j_1}']_{I \cup \{H_{j_1}\}} \), where we define \( H_{j_1}' \) such that \( S_{j_1}' \in G_{I \cup \{H_{j_1}\}} \) and, \( S_{j_1}' \in G_{I \cup \{H_{j_1}\}} \) implies \( H_{j_1}' \not\vdash_{\Gamma \fa} H_{k_m} \) or \( H_{j_1}' \not\vdash_{\Gamma \fa} H_{k_m} \) for all \( H_{j_1}' \in \{H_{j_1}'\} \cup I \{H_{j_1}'\} \). If we can’t define \( G_I \) to be \( G_{I \cup \{H_{j_1}\}} \) for each \( 1 \leq k \leq m \), let \( I' := \{H_{j_1}', \ldots, H_{k_m}\} \). Then \( G_{I'} \) is constructed by applying the separation algorithm of multiple branches (or one branch if \( m = 1 \)) to \( [S_{j_1}' \prime]_{I'}, \ldots, [S_{j_1}' \prime]_{I'} \). Then \( \vdash_{\Gamma \fa} G_{I'} \) by \( \vdash_{\Gamma \fa} [S_{j_1}' \prime]_{I'}, \ldots, \vdash_{\Gamma \fa} [S_{j_1}' \prime]_{I'} \).

**The proof of Main lemma:** Let \( I = \{H_{j_1}', \ldots, H_{k_m}\} \) in Lemma 9.2. Then there exists \( G_I \) such that \( \vdash_{\Gamma \fa} G_I \), \( G_I \subseteq G \) and \( H_{j_1}' \vdash_{\Gamma \fa} H_{k_m} \) for all \( S_{j_1} \in G_I \) and \( H_{j_1}' \in I \). Then \( \vdash_{\Gamma \fa} G_{I'} \) by \( \vdash_{\Gamma \fa} G_{I'} \subseteq G \). Then \( \vdash_{\Gamma \fa} G_{I'} \) by \( \vdash_{\Gamma \fa} G \subseteq G \).

By removing the identification number of each occurrence of \( p \) in \( G \), we obtain the sub-hypersequent \( G_2 \) of \( G \) which is the root of \( \tau'''' \), resulting from Step 4 in Section 4. Then \( \vdash_{\Gamma \fa} D_0(G_2) \) by \( \vdash_{\Gamma \fa} D_0(G_I) \) and \( G_I \subseteq G \). Since \( G_2 \) is constructed by adding or removing some \( \Gamma, p \Rightarrow \Delta \) or \( \Pi, p \Rightarrow \Sigma \) from \( G_0 \), or replacing \( \Gamma, p \Rightarrow \Delta \) with \( \Gamma, \top \Rightarrow \Delta \), or \( \Pi, p \Rightarrow \Sigma \) with \( \Pi, \top \Rightarrow \Sigma \), then \( \vdash_{\Gamma \fa} D_0(G_0) \) by Lemma 9.1. This completes the proof of Main lemma.

**Theorem 9.3.** Density elimination holds for all GL in \( \{\text{GUL, GIUL, GMTL, GIMTL}\} \).

**Proof.** It follows immediately from Main lemma.

10. Final remarks and open problems

It has often been the case in the past that metamathematical proofs of the standard completeness have the corresponding algebraic ones, and vise versa. In particular, Baldi and Terui [3] had given an algebraic proof of the standard completeness of UL and their method had also been extended by Galatos and Horcik [11]. A natural problem is whether there is an algebraic proof corresponding to our proof-theoretic approach. It seems difficult to obtain it by using the insights gained from the approach described in this paper because ideas and syntactic manipulations introduced here are complicated and specialized. In addition, Baldi and Terui [3] also mentioned some open problems. Whether our method could be applied to their problems is another research direction.

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Appendices

Appendix 1. Why do we adopt Avron-style hypersequent calculi?

A hypersequent calculus is called Pottinger-style if its two-premise inference rules are in the form of $\frac{G[S]}{S'}\frac{G[S'']}{S''}$ and, Avron-style if in the form of $\frac{G[S]}{S'}\frac{G[S'']}{G[H]}\frac{S''}{H}$ in (II). In the viewpoint of Avron-style systems, each application of two-premise inference rules contains implicitly applications of (EC) in Pottinger-style systems, as shown in the following.

| $\frac{G[S]}{S'}\frac{G[S'']}{S''}$ | corresponds to |
|---------------------------------|-----------------|
| $\frac{G[S]}{S'}\frac{G[S'']}{G[H]}\frac{S''}{H}$ | in Avron-style system |
| $\frac{G[S]}{S'}\frac{G[S'']}{G[H]}\frac{S''}{H}$ | in Pottinger-style system |

The choice of the underlying system of hypersequent calculus is vital to our purpose and it gives the background or arena. In Pottinger-style system, $G_0$ in Section 3 is proved without
application of \((EC)\) as follows. But it seems helpless to prove that \(H_0\) is a theorem of \(IUL\).

\[
\begin{array}{c|c|c}
C \Rightarrow \ C & B \Rightarrow B \\
\hline
C \Rightarrow C & p \Rightarrow p \\
& A \Rightarrow A \\
& A \Rightarrow p \Rightarrow p \\
& A \Rightarrow A \\
& p \Rightarrow p \\
& A \Rightarrow A \\
& p \Rightarrow p \\
& A \Rightarrow A \\
& p \Rightarrow p \\
& A \Rightarrow A \\
& p \Rightarrow p \\
& A \Rightarrow A \\
& \vdots
\end{array}
\]

4.14 as follows. In Figure 4, let
\[
S = \emptyset
\]
where
\[
\begin{align*}
\text{We use the example in Section 3 to answer this question. Firstly, we illustrate Notation} \\
\text{4.14 as follows. In Figure 4, let} \\
S = \emptyset
\end{align*}
\]

The peculiarity of our method is not only to focus on controlling the role of the external contraction rule in the hypersequent calculus but also introduce other syntactic manipulations. For example, we label occurrences of the eigenvariable \(p\) introduced by an application of the density rule in order to be able to trace these occurrences from the leaves (axioms) of the derivation to the root (the derived hypersequent).

Appendix 2. Why do we need the constrained external contraction rule?

4.14 as follows. In Figure 4, let
\[
S = \emptyset
\]
where
\[
\begin{align*}
\text{We use the example in Section 3 to answer this question. Firstly, we illustrate Notation} \\
\text{4.14 as follows. In Figure 4, let} \\
S = \emptyset
\end{align*}
\]

We denote the derivation \(\tau_{H'_i:A\Rightarrow p_1}^*\) of \(G_{H'_i:A\Rightarrow p_2}^*\) from \(A \Rightarrow p_2\) by
\[
\frac{A \Rightarrow p_2}{G_{H'_i:A\Rightarrow p_2}^*} \left( \tau_{H'_i:A\Rightarrow p_2}^* \right)
\]

in the separation algorithm, we abbreviate
\[
\frac{A \Rightarrow p_2}{G_{H'_i:A\Rightarrow p_2}^*} \left( \tau_{H'_i:A\Rightarrow p_2}^* \right)
\]

Further to
\[
\frac{1}{2} \left( \tau_{1}^* \right)
\]

Then the separation algorithm \(\tau_{H'_i;A\Rightarrow p_2}^*\) is abbreviated as
\[
\begin{align*}
\frac{1}{2} & \left( \tau_{1}^* \right) \\
\frac{2}{3} & \left( \tau_{2}^* \right) \\
\frac{1}{1} & \left( \tau_{3}^* \right) \\
\end{align*}
\]

where \(2'\) and \(3'\) are abbreviations of \(A \Rightarrow p_3\) and \(p_5, p_6 \Rightarrow A \Rightarrow A\), respectively. We also write \(2'\) and \(3'\) respectively as \(2\) and \(3\) for simplicity. Then the whole separation derivation is given as follows.

\[
\frac{1}{2} \left( \tau_{1}^* \right) \\
\frac{2}{3} \left( \tau_{2}^* \right) \\
\frac{1}{1} \left( \tau_{3}^* \right) \\
\end{align*}
\]

where \(\emptyset\) is an abbreviation of \(G''\) in Page 14 and means that all sequents in it are copies of sequents in \(G_0\). Note that the simplified notations become intractable when we decide whether
\( \langle EC_{\Omega} \rangle \) is applicable to resulting hypersequents. If no application of \( \langle EC_{\Omega} \rangle \) is used in it, all resulting hypersequents fall into the set \( \{ 1 | 2 | 3 | \cdots | 3_j, 2 | 2 | 3 | \cdots | 3, 1 | 1 | 3 | \cdots | 3 : l \geq 0, m \geq 0, n \geq 0 \} \)

and \( \emptyset \) is never obtained.

**Appendix 3. Why do we need the separation of branches?**

In Figure 11, \( p_1 \) and \( p_2 \) in the premise of \( p_1 \Rightarrow A \land A \) could be viewed as being tangled in one sequent \( p_1 \), \( p_2 \) \( \Rightarrow A \) \( \land A \) but in the conclusion of \( \{ \tau_{S_{11}}, \} \) they are separated into two sequents \( p_1 \Rightarrow C \) and \( C, p_2 \Rightarrow A \land A \), which are copies of sequents in \( G_{0} \). In Figure 5, \( p_2 \) in \( A \Rightarrow p_2 \) falls into \( \Rightarrow p_2, B \) in the root of \( \tau_{G_{H_{A}}} \) and \( \Rightarrow p_2, B \) is a copy of a sequent in \( G_{0} \). The same is true for \( p_4 \) in \( A \Rightarrow p_4 \) in Figure 8. But it’s not the case.

Lemma 6.6 (vi) shows that in the elimination rule \( \frac{\tau_{S_{11}}}{G_{S_{11}^2}} \), \( S'_{j} \in G_{S_{11}^2} \) implies \( H_{j}' \) \( \not<\) \( H_{i}' \) or \( H_{j}' \parallel H_{i}' \). If there exists no \( S'_{j} \in G_{S_{11}^2} \) such that \( H_{j}' \not<\) \( H_{i}' \) then \( S'_{j} \in G_{S_{11}^2} \) implies \( H_{j}' \parallel H_{i}' \) and thus each occurrence of \( p's \) in \( S_{11}^c \) is fallen into a unique sequent which is a copy of a sequent in \( G_{0} \). Otherwise there exists \( S'_{j} \in G_{S_{11}^2} \) such that \( H_{j}' \not<\) \( H_{i}' \), then we apply \( \{ \tau_{S_{11}} \} \) to \( S'_{j} \) in \( G_{S_{11}^2} \) and the whole operations can be written as

\[
\frac{S_{11}^c}{G_{S_{11}^2}^{(0)}} \equiv \frac{\{ \tau_{S_{11}} \}}{G_{S_{11}^2} \backslash \{ S'_{j} \} G_{S_{11}^2}^j} \frac{\{ \tau_{S_{11}} \}}{G_{S_{11}^2} \backslash \{ S'_{j} \} G_{S_{11}^2}^j}.
\]

Repeatedly we can get \( G_{S_{11}^2}^{(J)} \) such that \( S'_{j} \in G_{S_{11}^2}^{(J)} \) implies \( H_{j}' \parallel H_{i}' \). Then each occurrence of \( p's \) in \( S_{11}^c \) is fell in \( G_{S_{11}^2}^{(J)} \) into a unique sequent which is a copy of a sequent in \( G_{0} \). In such case, we call occurrences of \( p's \) in \( S_{11}^c \) are separated in \( G_{S_{11}^2}^{(J)} \) and call such a procedure the separation algorithm. It is the starting point of the separation algorithm. We introduce branches in order to tackle the case of multiple-premise separation derivations for which it is necessary to apply \( \langle EC_{\Omega} \rangle \) to the resulting hypersequents.

**Appendix 4. Some questions about Lemma 8.2**

In Lemma 8.2, \( \tau_{L}^{D} \) is constructed by induction on the number \( |I| \) of branches. As usual, we take the algorithm of \( |I| - 1 \) branches as the induction hypothesis. Why do we take \( \tau_{L}^{D} \) and \( \tau_{L}^{D} \) as the induction hypotheses?

Roughly speaking, it degenerates the case of \( |I| \) branches into the case of two branches in the following sense. The subtree \( \tau^*(G''|S'') \) of \( \tau^* \) is as a whole contained in \( \tau^*_{S_{b}} \) or not in it. Similarly, \( \tau^*(G'|S') \) of \( \tau^* \) is as a whole contained in \( \tau^*_{S_{b}} \) or not in it. It is such a division of \( I \) into \( I_l \) and \( I_r \) that makes the whole algorithm possible.
Appendix 5. Illustration of notations and algorithms

We use the example in Section 3 to illustrate some notations and algorithms in this paper.

5.1 Illustration of notations 6.10 and Construction 6.11

Let \( I = \{ H_I, H'_I \}, I_s = \{ H_s, H'_s \}, I_T = \{ S^{\prime}_{11}, S^{\prime}_{21}, T^{\prime}_{1}, T^{\prime}_{2} \}, I_T = \{ S^c_{21} \}, I_T = \{ S^c_{21} \}, \)

\[
\frac{G'|S'| G''|S'''}{G'|G''|H'} (\sim l) \in \tau^*,
\]

where \( G'|G''|H' = H_I ; G' \equiv A \Rightarrow p_1 | p_1, p_2 \Rightarrow A \odot A; S' \equiv \Rightarrow p_2, \neg A; \)

\( G'' \equiv A \Rightarrow p_3 | p_1, p_4 \Rightarrow A \odot A; S'' \equiv \Rightarrow p_4, \neg A; H' \equiv \Rightarrow p_2, p_4, \neg A \odot \neg A \) (See Figure 4).

\[
\langle G'|S'| \rangle_{I_T} \Rightarrow p_2, \neg A; \langle G''| \rangle_{I_T} = \varnothing; \langle G'|G''|H' \rangle_{I_T} = A \Rightarrow p_3 \Rightarrow p_2, p_4, \neg A \odot \neg A | p_3, p_4 \Rightarrow A \odot A; \langle G|G'' \rangle_{I_T} = G^*_I = G^*_T = \varnothing \Rightarrow p_2, | B,B \Rightarrow p_4, \neg A \odot \neg A | p_3, p_4 \Rightarrow A \odot A \) (See Figure 5).

\[
\langle G''|S''| \rangle_{I_T} \Rightarrow p_4, \neg A; \langle G'|G''|H' \rangle_{I_T} = A \Rightarrow p_1 \Rightarrow p_2, p_4, \neg A \odot \neg A | p_1, p_2 \Rightarrow A \odot A; \langle G|G'' \rangle_{I_T} = G^*_I = G^*_T = \varnothing \Rightarrow p_1 \Rightarrow p_2, | B,B \Rightarrow p_4, \neg A \odot \neg A | p_1 \Rightarrow C,C, p_2 \Rightarrow A \odot A \) (See Figure 8).

\[
\langle G'|G''|H' \rangle_{I_T} \Rightarrow p_2, p_4, \neg A \odot \neg A; \langle G|G'' \rangle_{I_T} = G^*_I = G^*_T = \varnothing \Rightarrow p_2, | B,B \Rightarrow p_4, \neg A \odot \neg A \) (See Figure 10).
5.2 Illustration of Notation 8.1 and Lemma 8.2

For an example of Notation 8.1, \( H_1^{(0)} = H_1, H_1^{(1)} = H_1, H_2^{(0)} = H_2^c, H_2^{(1)} = H_2^c, n_1 = n_2 = 1 \).

Note that sequents in \( \mathcal{S} \) are principal sequents of elimination rules in the following.

Let \( \mathcal{I} = \{ [S_1^T], [S_2^T] \}, \mathcal{I}_l = \{ [S_1^T], \mathcal{I}_l = \{ [S_2^T] \} \),

\[
\begin{align*}
[S_1^T] & = G^*_{H_1^{(0)}, G^{(1)}} = A \Rightarrow p_5, B \Rightarrow p_{10}, \neg A \odot \neg A \Rightarrow p_9, p_9 \Rightarrow A \odot A | \Rightarrow p_6, B | \\
B & \Rightarrow p_9, \neg A \odot \neg A \Rightarrow p_5, C \Rightarrow p_6, A \odot A | B \Rightarrow p_7, \neg A \odot \neg A | \\
p_7 & \Rightarrow C[C, p_8] \Rightarrow A \odot A, \\
G^*_{H_1^{(2)}, G^{(2)}} & \Rightarrow p_5, B \Rightarrow p_{10}, \neg A \odot \neg A | A \Rightarrow p_9, C \Rightarrow p_{10} \Rightarrow A \odot A | \\
& \Rightarrow p_6, B \Rightarrow p_8, \neg A \odot \neg A | p_5 \Rightarrow C[C, p_6 \Rightarrow A \odot A | \\
B & \Rightarrow p_7, \neg A \odot \neg A | p_7 \Rightarrow C[C, p_8 \Rightarrow A \odot A, \\
G^*_{H_2^{(1)}, G^{(1)}} & \Rightarrow p_5, B \Rightarrow p_{10}, \neg A \odot \neg A | p_9 \Rightarrow C[C, p_{10} \Rightarrow A \odot A | \\
G^*_{H_2^{(2)}, G^{(2)}} & \Rightarrow p_5, B \Rightarrow p_{10}, \neg A \odot \neg A | p_9 \Rightarrow C[C, p_{10} \Rightarrow A \odot A | \\
G^*_{H_2^{(1)}, G^{(1)}} & \Rightarrow p_5, B \Rightarrow p_{10}, \neg A \odot \neg A | p_9 \Rightarrow C[C, p_{10} \Rightarrow A \odot A | \\
G^*_{H_2^{(2)}, G^{(2)}} & \Rightarrow p_5, B \Rightarrow p_{10}, \neg A \odot \neg A | p_9 \Rightarrow C[C, p_{10} \Rightarrow A \odot A | \\
G^*_{H_2^{(1)}, G^{(1)}} & \Rightarrow p_5, B \Rightarrow p_{10}, \neg A \odot \neg A | p_9 \Rightarrow C[C, p_{10} \Rightarrow A \odot A | \\
G^*_{H_2^{(2)}, G^{(2)}} & \Rightarrow p_5, B \Rightarrow p_{10}, \neg A \odot \neg A | p_9 \Rightarrow C[C, p_{10} \Rightarrow A \odot A | \\
G^*_{H_2^{(1)}, G^{(1)}} & \Rightarrow p_5, B \Rightarrow p_{10}, \neg A \odot \neg A | p_9 \Rightarrow C[C, p_{10} \Rightarrow A \odot A |
\end{align*}
\]

Since there is only one elimination rule in \( \tau^*_l \), the case we need to process is \( \tau^*_l[H_1^{(0)} \Rightarrow p_9] \), i.e.,

\[
\tau^*_l[H_1^{(0)} \Rightarrow p_9] = \frac{[S_2^T]}{G^*_{H_2^{(1)}, G^{(1)}} \{ \tau^*_l[H_2^{(0)} \Rightarrow p_9] \}}.
\]

Then \( v = 1, S_{H_1^{(0)}} = A \Rightarrow p_3; H_{b_1} = p_2, B \Rightarrow p_4, \neg A \odot \neg A | p_1 \Rightarrow C | \\
C, p_2 \Rightarrow A \odot A | p_1 \Rightarrow p_1, B | p_3 \Rightarrow C[C, p_4 \Rightarrow A \odot A | p_1 \Rightarrow p_1, \neg A \odot \neg A | p_1 \Rightarrow C].
\]
Replacing $\tau$ where

Then since there is only one elimination rule in $\lceil S \rceil_1$, the case we need to process is $\tau^G_{H;A\Rightarrow p_1}$, i.e.,

$$\tau^*_{h_1} = \frac{\left[ S \right]_1}{\tau_{H;A\Rightarrow p_1}^G} \cdot \left( \tau_{H;A\Rightarrow p_1}^G \right).$$

Then $u = 1, S^G_{h_1} = A \Rightarrow p_s; G_{b_1} \Rightarrow p_6, B \Rightarrow p_8; \neg\neg A \Rightarrow \neg A$.

$p_5 \Rightarrow C[C, p_6 \Rightarrow A \Rightarrow p_7; \neg\neg A \Rightarrow p_7 \Rightarrow C; p_8 \Rightarrow A \Rightarrow A$ in $\tau^*_{h_1}$.

$\tau^*_{h_1}$ is replaced with $\tau^G_{h_1}$ in Step 3 of Stage 1, i.e.,

$$\tau^G_{h_1} \left( \tau^G_{h_1} \right) = \frac{\left[ S \right]_1}{\tau_{H;A\Rightarrow p_1}^G} \cdot \left( \tau_{H;A\Rightarrow p_1}^G \right) = \tau^G_{h_1} \left( \tau_{H;A\Rightarrow p_1}^G \right),$$

where $G_{b_1} \Rightarrow p_5, B \Rightarrow p_7; \neg\neg A \Rightarrow \neg A; G_{b_1} \Rightarrow p_8, B \Rightarrow B; \neg\neg A \Rightarrow \neg A$.

Replacing $\tau^G_{h_1}$ in $\tau^G_{h_1} \left( \tau_{H;A\Rightarrow p_1}^G \right)$, then deleting $G_{h_1}$ and after that applying $(EC^G_2)$ to $G_{h_1}$ and keeping $G_{b_1}$ unchanged, we get

$$\tau^G_{h_1} \left( \tau^G_{h_1} \right) = \frac{\left[ S \right]_1}{\tau_{H;A\Rightarrow p_1}^G} \cdot \left( \tau_{H;A\Rightarrow p_1}^G \right).$$

$G_{h_1} \Rightarrow p_5, B \Rightarrow B; \neg\neg A \Rightarrow \neg A; G_{b_1} \Rightarrow S; G_{h_1} \Rightarrow S; \neg\neg A \Rightarrow p_5, B; \neg\neg A \Rightarrow p_8, B \Rightarrow B; \neg\neg A \Rightarrow \neg A$. 

$\tau^G_{h_1}$ and $\tau^G_{H;A\Rightarrow p_1}$.
\[G^\Omega_{L_G^{r_i}} \Rightarrow p_5, B \Rightarrow p_6, B \Rightarrow p_7, \neg A \odot \neg A \Rightarrow C, p_6 \Rightarrow A \odot A\]
\[p_7 \Rightarrow C, p_8 \Rightarrow A \odot A \Rightarrow p_8, \neg A \odot \neg A.\]

Stage 2

\[\tau^\Omega_{L_G^{r_i}} = \tau^\Omega_{L_G^{r_i}(1)} = \tau^\Omega_{L_G^{r_i}(2)} = \frac{[S^+_i]}{G^\Omega_{L_G^{r_i}(1)}} \left\{ \tau^\Omega_{G^{r_i}:A=p_5} \right\}, \]
\[\tau^\Omega_{L_G^{r_i}(3)} = \tau^\Omega_{L_G^{r_i}(4)} = \frac{[S^-_i]}{G^{\Omega_{L_G^{r_i}(4)}}} \left\{ \tau^\Omega_{G^{r_i}:A=p_5} \right\}. \]

Replacing \(\tau^\Omega_{L_G^{r_i}(1)}\) in \(\tau^\Omega_{L_G^{r_i}(4)}\) with \(\tau^\Omega_{L_G^{r_i}(4)}\), then deleting \(G^\Omega_{L_G^{r_i}}\) and after that applying \((EC^\Omega_H)\) to \(G_{b_j}^\Omega(G_{H^+_i:H^+_i})_k^\Omega \land \{S^\Omega[S^\Omega]\}_j^{\Omega} G^\Omega_{L_G^{r_i}},\) we get \(\tau^\Omega_{L_G^{r_i}}\).