ON THE AKSZ FORMULATION OF THE POISSON SIGMA MODEL

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Abstract. We review and extend the Alexandrov–Kontsevich–Schwarz–Zaboronsky construction of solutions of the Batalin–Vilkovisky classical master equation. In particular, we study the case of sigma models on manifolds with boundary. We show that a special case of this construction yields the Batalin–Vilkovisky action functional of the Poisson sigma model on a disk. As we have shown in a previous paper, the perturbative quantization of this model is related to Kontsevich’s deformation quantization of Poisson manifolds and to his formality theorem. We also discuss the action of diffeomorphisms of the target manifolds.

1. Introduction

In this paper we continue our study [4] of the Poisson sigma model on surfaces with boundary. In particular, we clarify the relation of our construction with the AKSZ method [1] (see also [15], [11] for more detailed descriptions), and comment on the action of diffeomorphisms.

In [4] we discussed the quantization of Poisson sigma models—topological sigma models whose target space is a Poisson manifold—on a disk in the framework of perturbative path integrals, and derived in this context the Kontsevich formula [9] for deformation quantization [3].

The starting point is the classical action functional $S$, defined on the space of bundle maps from the tangent bundle $T\Sigma$ of a two-dimensional oriented manifold $\Sigma$, possibly with boundary, to the cotangent bundle $T^*M$ of a Poisson manifold $M$. If we denote such a bundle map by a pair $(X, \eta)$, where $X: \Sigma \to M$ is the base map, and $\eta$, the map between fibers, is a section in $\Gamma(\Sigma, \text{Hom}(T\Sigma, X^*(T^*M))) = \Omega^1(\Sigma, X^*(T^*M))$,
then the expression for the classical action functional is

$$S(X, \eta) = \int_\Sigma \langle \eta, dX \rangle + \frac{1}{2}(\alpha \circ X)(\eta, \eta).$$

Here the pairing is between the tangent and cotangent bundle of $M$ and the Poisson tensor $\alpha$ is viewed as a bilinear form on the cotangent bundle. The boundary conditions used in [4] are $j_0^* \partial \eta = 0$ where $j_0: \partial \Sigma \to \Sigma$ denotes the inclusion of the boundary. Among the critical points of $S$ a special role is played by the trivial classical solutions for which $X$ is constant and $\eta = 0$. They are in one-to-one correspondence with points of $M$. The Feynman perturbation expansion around these trivial solutions was developed in the case of a disk $\Sigma$ in [4]. The gauge symmetry of the action functional was taken into account by applying the Batalin–Vilkovisky (BV) method [2] (see [14] for a more precise mathematical description). The Feynman rules for the resulting BV action functional yield, after gauge fixing, the Kontsevich formula for the star product [9].

The AKSZ method [1] is a method to construct solutions of the BV master equation directly, without starting from a classical action with a set of symmetries, as is done in the BV method. The classical action is then recovered \textit{a posteriori} by setting the fields of non-zero degree to zero. This method is applied in [1] to some examples and it is shown that the BV action of the Chern–Simons theory, the Witten A- and B-models are special cases of their construction.

Here we adapt the AKSZ method to the case of manifolds with boundary, and show that the BV action of the Poisson sigma model derived in [4], with the same boundary conditions, can be obtained in this framework. Closely related results have been obtained recently in [11].

In the case of the Poisson sigma model, the construction goes as follows: to an oriented two-dimensional manifold $\Sigma$, possibly with boundary, one associates the supermanifold $\Pi T \Sigma$, the tangent bundle with reversed parity of the fiber. The algebra of functions on $\Pi T \Sigma$ is isomorphic to the (graded commutative) algebra of differential forms on $\Sigma$ with values in a graded commutative ground ring $\Lambda$, which we take, following [16], to be the infinite-dimensional exterior algebra $\lim \bigwedge R^k$. The integration of differential forms and the de Rham differential are, in the language of supermanifolds, a measure $\mu$ and a self-commuting vector field $D$, respectively, on $\Pi T \Sigma$. Let $M$ be a Poisson manifold and $\Pi T^* M$ be the cotangent bundle of $M$ with reversed parity of the fiber. This supermanifold has a canonical symplectic structure (with exact symplectic form) just as any cotangent bundle. It is odd, because of
the parity reversion, meaning that the corresponding Poisson bracket is graded skew-symmetric only after shifting the degree by one. The algebra of functions on $\Pi T^*M$ is isomorphic to the algebra of multivector fields on $M$ with values in $\Lambda$. In particular, the Poisson bivector field on $M$ can be identified with a function on $\Pi T^*M$. Its Hamiltonian vector field is an odd vector field $Q$. It commutes with itself owing to the Jacobi identity. The space of maps $\Pi T\Sigma \to \Pi T^*M$ inherits then an odd symplectic form and an odd Hamiltonian self-commuting vector field: the symplectic form is obtained from the symplectic form on $\Pi T^*M$ upon integration over $\Pi T\Sigma$ with respect to the measure $\mu$. The odd vector field is the sum of the commuting vector fields $\hat{D}$ and $\hat{Q}$ obtained from $D$ and $Q$ by acting on maps $\Pi T\Sigma \to \Pi T^*M$ by the corresponding infinitesimal diffeomorphisms of $\Pi T\Sigma$ on the left and of $\Pi T^*M$ on the right. The Hamiltonian function of this vector field is then the BV action functional, if $\Sigma$ has no boundary. It is a function on the space of maps from $\Pi T\Sigma$ to $\Pi T^*M$, whose Poisson bracket with itself vanishes, i.e., it solves the BV classical master equation. If $\Sigma$ has a boundary, suitable boundary conditions must be imposed. They are first-class constraints, and the correct action functional is obtained after Hamiltonian reduction.

We describe this construction in detail in Sect. 3 after reviewing and extending the AKSZ construction in Sect. 2. In the first part of Sect. 2, we present the original construction of [1], see Theorem 2.7. In the typical case, the data are a closed manifold $\Sigma$ and an odd or even (depending on the dimension of $\Sigma$) symplectic supermanifold $Y$ with a Hamiltonian odd self-commuting vector field $Q$. The construction produces a solution of the classical BV master equation on the space of maps from $\Pi T\Sigma$ to $Y$. In the case where $\Sigma$ has a boundary, a similar result holds if the symplectic form on $Y$ is exact, and it depends on the choice of a symplectic potential—a 1-form whose differential is the symplectic form. Moreover the Hamiltonian function of $Q$ must vanish on the zero set of the symplectic potential. Under these hypotheses, we construct a BV action functional on a Hamiltonian reduction of the space of maps from $\Pi T\Sigma$ to $Y$ obeying suitable boundary conditions, see Theorem 2.10. In the case when $Y$ has global Darboux coordinates, similar results—but with a different treatment of boundary conditions—were obtained in [11].

Along the way, we also describe some by-products, such as a geometric explanation of the supersymmetry discovered in [4] and generalized in [11]. We also comment on the action of diffeomorphisms of the target manifold $M$ of the Poisson sigma model. We show that diffeomorphisms of $M$ induce canonical transformations of the space of maps
from $\Pi T\Sigma$ to $\Pi T^*M$. This means that one should expect the partition function to be invariant under such diffeomorphisms. However, when we introduce boundary observables in the path integrals, the variation of the correlation functions under a canonical transformation is singular, and the diffeomorphism invariance is spoiled after regularization, albeit in a controlled way. These ideas, closely related to ideas in [10], are a motivation for the globalization results in the form described in [5].

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2. The AKSZ construction

The Batalin–Vilkovisky (BV) method in quantum field theory [2] relies on extending the space of fields of a theory to a larger superspace with a $QP$-structure (in the terminology introduced in [14] and [1]). The AKSZ method [1] is a way of inducing a $QP$-structure on a space of supermaps (given certain structures on the domain and on the target supermanifolds). We begin by giving a short description of $Q$-, $P$- and $QP$-structures.

2.1. Some structures on differentiable supermanifolds. Let $N$ be a differentiable supermanifold. We will denote by $\text{Fun}(M)$ its $\mathbb{Z}_2$-graded algebra of smooth functions. Following [16], we view $\text{Fun}(M)$ as a $\Lambda$-module, where

$$\Lambda := \operatorname{lim} \bigwedge^\cdot \mathbb{R}^k$$

is the exterior algebra of $\mathbb{R}^\infty$.

2.1.1. $Q$-structures. A $Q$-structure on a supermanifold is the choice of an odd self-commuting vector field. In homogeneous local coordinates $\{y_1, \ldots, y_n\}$,

$$Q = Q^i \partial_i,$$

with $\partial_i = \partial/\partial y^i$. “Odd” means that $\deg Q^i \equiv \epsilon_i + 1 \mod 2$, where by $\epsilon_i$ we denote the degree of the coordinate $y^i$. “Self-commuting” means

$$[Q, Q] = 2Q^2 = 2(Q^i \partial_j Q^j)\partial_i = 0.$$

In other words, a $Q$-structure is the choice of a differential on the algebra of functions of $N$. 
Example 2.1. Let $N = \Pi T\Sigma$, with $\Sigma$ an ordinary manifold. By this we denote the tangent bundle of $\Sigma$ with reversed parity on the fiber. By definition the algebra of functions on $\Pi T\Sigma$ is isomorphic to the algebra of differential forms on $\Sigma$ tensor $\Lambda$. Let us denote by $\phi$ this isomorphism. Then the exterior derivative $d$ defines a $Q$-structure: $D_f := \phi^{-1}(d\phi(f)), f \in Fun(\Pi T\Sigma)$. Choosing local coordinates $\{u^1, \ldots, u^s\}$ on $\Sigma$ together with their odd counterparts $\{\theta^1, \ldots, \theta^s\}$, we can write

$$D = \theta^\mu \frac{\partial}{\partial u^\mu}.$$ 

Another example of differential is the contraction $\iota_v$ by a vector field $v$. We define then the $Q$-vector field $K_v := \phi^{-1}\iota_v\phi$ and can write

$$K_v = v^\mu(u)\frac{\partial}{\partial \theta^\mu}$$

in local coordinates. Finally, observe that the Lie bracket $[D, K_v] =: L_v$ is an even vector field corresponding to the Lie derivative $L_v$ via the formula $L_v = \phi^{-1} L_v \phi$.

2.1.2. $P$-structures. A symplectic structure is a closed nondegenerate 2-form on $N$. With respect to the $\mathbb{Z}_2$-grading of $Fun(M)$, it can be even or odd. In local coordinates we write

$$\omega = \frac{1}{2} dy^i \omega_{ij} dy^j,$$

with $\wedge$-symbols suppressed and the Koszul sign rule

$$dy^i dy^j = (-1)^{\epsilon_i \epsilon_j} dy^i dy^j.$$ 

(2.1)

An even symplectic structure satisfies $\text{deg} \omega_{ij} \equiv \epsilon_i + \epsilon_j$, while an odd symplectic structure (shortly a $P$-structure) satisfies $\text{deg} \omega_{ij} \equiv \epsilon_i + \epsilon_j + 1$. Moreover,

$$\omega_{ji} = \begin{cases} (-1)^{(\epsilon_i + 1)(\epsilon_j + 1)} \omega_{ij} & \text{even case}, \\ (-1)^{1+\epsilon_i \epsilon_j} \omega_{ij} & \text{odd case}. \end{cases}$$

A symplectic structure associates to each function $f$ a vector field $X_f$ by the identity $\iota_{X_f} \omega = df$. Observe that $X_f$ has the same (opposite) parity of $f$ if $\omega$ is even (odd). One defines the bracket of two functions $f$ and $g$ by $(f, g) := X_f(g) = \iota_{X_f} \iota_{X_g} \omega$. In the even case, it has the properties of a Poisson bracket, while in the odd case it is a Gerstenhaber bracket (but is referred to in the quantum field theory literature as a BV bracket).

Example 2.2. Let $N = \Pi T^* M$, with $M$ an ordinary manifold. The algebra of functions on $N$ can be identified with the algebra of multivector fields on $M$ tensor the infinite-dimensional exterior algebra $\Lambda$.
The algebra of multivector fields admits a nondegenerate Gerstenhaber bracket: viz., the Schouten–Nijenhuis bracket \([\cdot,\cdot]\). This bracket is determined by the canonical odd symplectic form \(\omega\) that, using local coordinates \(x_1, \ldots, x_m\) on \(M\) and their odd counterparts \(p_1, \ldots, p_m\), can be written as \(\omega = dp_i dx^i\).

2.1.3. \(QP\)-structures. A vector field \(Q\) and a symplectic structure are said to be compatible if \(L_Q \omega = d\iota_Q \omega = 0\). Here and in the following, \(L_X = d\iota_X + \iota_X d\) denotes the Lie derivative w.r.t. the (odd or even) vector field \(X\). If in particular \(\iota_Q \omega = dS\) for some function \(S\), then \(Q\) is said to be Hamiltonian. A \(QP\)-manifold is a supermanifold with compatible \(Q\)- and \(P\)-structures; in case \(Q\) is Hamiltonian, its Hamiltonian function \(S\) is even and satisfies the so-called master equation \((S, S) = 0\).

Example 2.3. Again let \(N = \Pi T^* M\). Let \(\alpha\) be a multivector field on \(M\) of even degree satisfying \([\alpha, \alpha] = 0\). Then \(\alpha\) can be seen as the Hamiltonian function of an odd self-commuting vector field \(Q\) that acts on multivector fields by \([\alpha, \cdot]\). For example, if \(\alpha\) is a Poisson bivector field, we may write in local coordinates

\[
S_\alpha(x, p) = \frac{1}{2} \alpha^{ij}(x)p_ip_j, \tag{2.2a}
\]

\[
Q_\alpha(x, p) = \alpha^{ij}(x)p_j \frac{\partial}{\partial x^i} + \frac{1}{2} \partial_i \alpha^{jk} p_j p_k \frac{\partial}{\partial p_i}. \tag{2.2b}
\]

2.1.4. Measures. We recall that a measure on an \((m, n)\)-supermanifold \(N\) is a linear functional on the algebra of function \(\text{Fun}(M)\) that kills all components of homogeneous degree in the odd coordinates less than \(n\). More precisely, a measure is a section of the Berezinian bundle. Our notation for a measure applied to a function \(f\) is \(\int_N f \mu\), and we will often call \(\mu\) the measure. We say that a measure is nondegenerate if its composition with the product yields a nondegenerate bilinear form on \(\text{Fun}(M)\). Finally, given a vector field \(D\) on \(N\), we say that the measure is \(D\)-invariant if \(\int_N Df \mu = 0, \forall f \in \text{Fun}(N)\).

Example 2.4. Let \(N = \Pi T\Sigma\) as in Example 2.1. Its canonical measure \(\mu\) is defined by \(\int_{\Pi T\Sigma} f \mu = \int_{\Sigma} \phi(f), \; f \in \text{Fun}(\Pi T\Sigma)\), and is clearly nondegenerate. In local coordinates, assuming the rules of Berezinian integration (\(\int \theta^\mu d\theta^\nu = \delta^\mu{}^\nu\)), we may write

\[
\mu = d\theta^s \cdots d\theta^1 \; dx^1 \cdots dx^s.
\]

This measure is compatible with the vector field \(D\) defined in Example 2.1 in the following sense. If \(\Sigma\) has a boundary \(\partial\Sigma\), we denote by \(\mu^\partial\) the canonical measure on \(\Pi T\partial\Sigma\) and by \(i_\partial\) the inclusion \(\Pi T\partial\Sigma \hookrightarrow \Pi T\Sigma\).
induced from $\partial \Sigma \hookrightarrow \Sigma$. Then Stokes’ theorem can be reformulated as follows:

$$\int_{\Pi T\Sigma} D f \mu = \int_{\Pi T\partial \Sigma} i^*_0 f \mu^\partial. \quad (2.3)$$

In particular, if $\partial \Sigma = \emptyset$, the canonical measure $\mu$ is $D$-invariant. Finally, for every vector field $v$ on $\Sigma$, the canonical measure $\mu$ is $K_v$-invariant, with $K_v$ defined in Example 2.1, since the top form component of $\iota_v \phi(f)$ vanishes for every $f \in \text{Fun}(\Pi T\Sigma)$.

More generally, let $N$ and $L$ be two supermanifolds and let $\mu$ be a measure on $N$. We may define a chain map $\mu_* : \Omega^*(N \times L) \rightarrow \Omega^*(L)$, $\forall k$, by the rule

$$(\mu_* \omega)(z)(\lambda_1, \ldots, \lambda_k) = \int_{y \in N} \omega(y, z)(\lambda_1, \ldots, \lambda_k) \mu(y),$$

with $z \in L$ and $\lambda_1, \ldots, \lambda_k \in T_z L$. Observe that, if $N$ is of type $(m, n)$ with $n$ odd (even), then $\mu_*$ changes (preserves) the parity. Moreover, if $\mu$ is $D$-invariant, one obtains $\mu_* L_{D_1} = 0$, where $D_1$ is the lift to $N \times L$ of the vector field $D$ defined on $N$.

In the case $N = \Pi T\Sigma$ described in Example 2.4, the generalization of (2.3) is instead

$$\mu_* L_{D_1} = \mu^\partial (\iota_0 \times \text{id})^*, \quad (2.4)$$

where $\mu$ is the canonical measure, and $D$ is the vector field defined in Example 2.1.

2.1.5. $\mathbb{Z}$-grading. In the application of the BV formalism to quantum field theories, one usually considers a $\mathbb{Z}$-grading (ghost number) instead of just a $\mathbb{Z}_2$-grading. This is obtained as follows. Let $A$ be a $\mathbb{Z}_2$-graded supercommutative algebra (e.g., the algebra of functions on some supermanifold). Define $A := A[\mathbb{Z}]$, where the action of the nontrivial element $\epsilon \in \mathbb{Z}_2$ on $n \in \mathbb{Z}$ is given by $\epsilon n = (-1)^n n$, while its action on $A_+$ is trivial and $\epsilon a = -a$ for $a \in A_-$. We can decompose $A$ as $\bigoplus_{j \in \mathbb{Z}} A_j$ with

- $A_{2j} = A_+[2j]$,
- $A_{2j+1} = A_-[2j+1]$.

We say then that an element of $A_j$ has ghost number $j$. It follows that $A$ is a $\mathbb{Z}$-graded supercommutative algebra.

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1 More precisely, let $j_\partial$ be the inclusion of $\partial \Sigma$ into $\Sigma$. Then $i_\partial^* : \text{Fun}(\Pi T\Sigma) \rightarrow \text{Fun}(\Pi T\partial \Sigma)$ is defined as $\phi^{-1}(j_\partial \otimes \text{id}) \phi$, where $\phi : \text{Fun}(\Pi T\Sigma) \rightarrow \Omega^*(\Sigma) \otimes \Lambda$ and $\phi_\partial : \text{Fun}(\Pi T\partial \Sigma) \rightarrow \Omega^*(\partial \Sigma) \otimes \Lambda$ are the isomorphisms that define the functions on these supermanifolds.
We finally have to define an inclusion of the original algebra $A$ into $A$. Consider the case when $A$ is the algebra of functions on a supermanifold of the form $\Pi E$, with $E$ a vector bundle over an ordinary manifold: $A = \text{Fun}(E) \simeq \Gamma(\wedge^\bullet E^*) \otimes \Lambda$. We define then the inclusion by sending the even coordinates to $A_0$ and the linear odd coordinates (i.e., those corresponding to linear functions on the fiber of $E$) to $A_1$. (That is, we assign ghost number zero to the former and one to the latter.)

More generally, if $A = \Omega^\bullet(N)$ is the algebra of differential forms on a supermanifold $N = \Pi E$, the inclusion is defined by sending even coordinates and their differentials to $A_0$ and linear odd coordinates and their differentials to $A_1$. Then $\Omega^\bullet(N)$ is a bigraded algebra and we follow the Koszul sign rule (2.1) for the products. As a consequence of this, the chain map $\mu_*$ defined above can be extended to a chain map $\Omega^\bullet(N \times L) \to \Omega^\bullet(L)$ that lowers the ghost number by $n$ if $N = \Pi E$ with $E$ a vector bundle of rank $n$.

Observe that the vector fields $D$ and $K_v$ defined in Example 2.1 have ghost number 1 and $-1$ respectively, while the vector field $Q$ in (2.2b) has ghost number 1 being generated by the ghost-number-zero function $S$ in (2.2a) using the canonical symplectic form $\omega$ of Example 2.2 that has ghost number 1.

In this framework, one usually speaks of a $Q$-structure only if the self-commuting vector field $Q$ has ghost number 1 and of $P$-structure only if the symplectic form $\omega$ has ghost number $-1$ (i.e., $\text{gh} Q^i = \epsilon_i + 1$ and $\text{gh} \omega_{ij} = -\epsilon_i - \epsilon_j - 1$, with $\epsilon_i$ the ghost number of the $i$th coordinate). So the Hamiltonian function $S$ of $Q$, when existing, has ghost number 0 and is usually called the BV-action.

2.2. Spaces of maps. The AKSZ construction is a way of defining a $QP$-structure on the space $Y^X$ of smooth maps from the supermanifold $X$ to the supermanifold $Y$.

2.2.1. $Q$-structure. Diffeomorphisms of $X$ and of $Y$ act on $Y^X$. As the former act from the right and the latter from the left, they (super)commute. At the infinitesimal level, this means that we can associate to vector fields $D$ on $X$ and $Q$ on $Y$ the (super)commuting vector fields $\hat{D}$ and $\hat{Q}$ on $Y^X$. Recall that $T_fY^X \simeq \Gamma(X,f^*TY)$, $f \in Y^X$. So a vector field on $Y^X$ assigns to each $x \in X$ and each $f \in Y^X$ an element of $T_{f(x)}Y$. So we can write

$$\hat{D}(x,f) = df(x)D(x), \quad \hat{Q}(x,f) = Q(f(x)).$$

In particular, if both $D$ and $Q$ are odd and self-commuting, any linear combination of $\hat{D}$ and $\hat{Q}$ defines a $Q$-structure on $Y^X$. 


2.2.2. \textit{Y}-\textit{induced} QP-structure. Let $\mu$ be a nondegenerate measure on $X$ and $\omega$ a symplectic structure on $Y$. We assume $\omega$ to be even (odd) if $X$ is of type $(m, n)$ with $n$ odd (even). Consider the evaluation map

$$\text{ev}: X \times Y^X \to Y, \quad (x, f) \mapsto f(x).$$

This induces a chain map $\mu^* \text{ev}^*: \Omega^*(Y) \to \Omega^*(Y^X)$ that lowers the ghost number by $n$ if $X$ is of type $(m, n)$. Then $\omega := \mu^* \text{ev}^* \omega$ defines a P-structure on $Y^X$.

It is easy to verify that $\tilde{Q}$ is compatible with $\omega$. In fact, $\iota_{\tilde{Q}} \omega = \mu^* \text{ev}^* \iota_Q \omega$. In particular, if $Q$ is Hamiltonian with Hamiltonian function $S$, then so is $\tilde{Q}$ with Hamiltonian function

$$\tilde{S} = \mu^* \text{ev}^* S. \quad (2.5)$$

In fact,

$$d \tilde{S} = \mu^* \text{ev}^* dS = \mu^* \text{ev}^* \iota_Q \omega = \iota_{\tilde{Q}} \omega.$$

In the following, we will denote by $(\ , \ )$ the Gerstenhaber bracket induced by $\omega$ on $\text{Fun}(Y^X)$.

\textbf{Proposition 2.5.} $\mu^* \text{ev}^*: (\text{Fun}(Y), (\ , \ )) \to (\text{Fun}(Y^X), (\ , \ ))$ is a Lie algebra homomorphism.

\textit{Proof.} Let $F$ and $G$ be two functions on $Y$, and denote by $Q$ the Hamiltonian vector field of $F$. So $(F, G) = \iota_Q dG$. As already observed, $\tilde{Q}$ is the Hamiltonian vector field of $\mu^* \text{ev}^* F$. As a consequence,

$$(\mu^* \text{ev}^* F, \mu^* \text{ev}^* G) = \iota_{\tilde{Q}} d\mu^* \text{ev}^* G = \mu^* \text{ev}^* \iota_Q dG = \mu^* \text{ev}^* (F, G).$$

\hfill \Box

2.2.3. \textit{The AKSZ QP-structure.} We have constructed above a P-structure $\omega$ on $Y^X$ as well as a Q-structure $Q := \hat{D} + \tilde{Q}$. We have also checked that $(\tilde{Q}, \omega)$ is a QP-structure. What is left to check is whether $\hat{D}$ is also compatible with $\omega$. We need the following

\textbf{Lemma 2.6.} \textit{If} $\mu$ \textit{is} $\hat{D}$-invariant, \textit{then} $L_{\hat{D}} \mu^* \text{ev}^* = 0$.

\textit{Proof.} Since the evaluation map is invariant under the simultaneous action of the inverse of a diffeomorphism on $X$ and of the induced diffeomorphism on $Y^X$—i.e., $\text{ev}(\phi^{-1}(x), f \circ \phi) = \text{ev}(x, f)$—, we obtain, at the infinitesimal level,

$$L_{\hat{D}_1} \text{ev}^* = L_{\hat{D}_2} \text{ev}^*,$$
where $D_1$ and $\hat{D}_2$ are the lifts of $D$ and $\hat{D}$ to $X \times Y^X$. On the other hand, by definition of $\mu_*$, we have $\iota_{\hat{D}} \mu_* = \mu_* \iota_{\hat{D}_2}$. Combining these identities, we get

$$L_{\hat{D}} \mu_* \text{ev}^* = \mu_* L_{\hat{D}_2} \text{ev}^* = \mu_* L_{D_1} \text{ev}^* = 0.$$ 

From this we obtain the

**Theorem 2.7.** If $D$ and $Q$ are $Q$-structures on $X$ and $Y$ respectively, $\mu$ is a nondegenerate $D$-invariant measure on $X$ and $\omega$ is a symplectic form on $Y$ (of parity opposite to the odd dimension of $X$), then $(\hat{D} + \hat{Q}, \omega)$ is a $QP$-structure on $Y^X$. If moreover $\omega = d\vartheta$, then $\hat{D}$ is Hamiltonian with Hamiltonian function

$$\hat{S} = -\iota_{\hat{D}} \vartheta,$$

with $\vartheta = \mu_* \text{ev}^* \vartheta$. If in addition $Q$ admits a Hamiltonian function $S$, then $(\hat{S}, \hat{S}) = 0$, with $\hat{S}$ defined in (2.5). In particular $S = \hat{S} + \hat{S}$ satisfies the master equation $(S, S) = 0$.

2.2.4. **The case $X = \Pi T\Sigma$.** As observed in Example 2.4, the canonical measure $\mu$ on $\Pi T\Sigma$ is $D$-invariant when $D$ is the vector field defined in Example 2.1 and $\partial \Sigma = \emptyset$. This is the typical case in which the AKSZ formalism is applied. We want however to consider also the case when $\Sigma$ has a boundary. In this case we cannot apply Lemma 2.6. We have however the following generalization.

**Lemma 2.8.** Let $\mu^\partial$ be the canonical measure on $\Pi T\partial \Sigma$, and $i_{\partial}$ the inclusion $\partial \Sigma \hookrightarrow \Sigma$. Define $\text{ev}^*_\partial = (i_{\partial} \times \text{id})^* \text{ev}^* : \Omega^*(Y) \to \Omega^*(\Pi T\partial \Sigma \times Y^\Pi T\Sigma)$. Then

$$L_{\hat{D}} \mu_* \text{ev}^* = \mu^\partial_* \text{ev}^*.$$ 

**Proof.** Proceed as in the proof of Lemma 2.6 till the last equation which is now replaced by

$$L_{\hat{D}} \mu_* \text{ev}^* = \mu_* L_{\hat{D}_2} \text{ev}^* = \mu_* L_{D_1} \text{ev}^* = 0 = \mu^\partial_* (i_{\partial} \times \text{id})^* \text{ev}^*;$$

upon using (2.4). 

From now on we assume $\omega = d\vartheta$. As in Theorem 2.7, we define the odd 1-form $\vartheta = \mu_* \text{ev}^* \vartheta$. We also introduce the even 1-form $\tau = \mu^\partial_* \text{ev}^* \vartheta$. Observe that $\tau$ is the obstruction to having $\hat{D}$ Hamiltonian; in fact, by Lemma 2.8 we have

$$\iota_{\hat{D}} \omega = \iota_{\hat{D}} d\vartheta = -d \iota_{\hat{D}} \vartheta + \tau.$$

In order to proceed we consider then the zero locus $\mathcal{C}$ of $\tau$. This can also be described as the common zero locus of a set of functions: For
every vector field $\xi$ on $Y^{\Pi T\Sigma}$, we define $H_\xi := \iota_\xi \tau$; then $C = \{ f \in Y^{\Pi T\Sigma} : H_\xi(f) = 0 \forall \xi \}$. Observe that $H_\xi(f)$ actually depends only on the values that $\xi(f)$, seen as an element of $\Gamma(\Pi T\Sigma, f^*TY)$, assumes on $\partial\Sigma$. If we denote by $Z(\vartheta)$ the zero locus of $\vartheta$, we also obtain $C = \{ f \in Y^{\Pi T\Sigma} : \iota^*_\vartheta f(\Pi T\partial\Sigma) \subset Z(\vartheta) \}$.

The above construction turns out however to be too singular, for no Hamiltonian vector fields correspond to the functions $H_\xi$. We regularize then as follows. Let $U$ be an open collar of $\partial\Sigma$ in $\Sigma$. We choose on $U$ a normal coordinate $x_n$ and note by $\theta_n$ the corresponding odd counterpart to $x_n$ on $\Pi T U$. Then we define $\tau_U = \mu^U_*(\text{ev}_U^* \vartheta \theta_n)$, with obvious meaning of notations. Thus, for every vector field $\xi$ on $Y^{\Pi T U}$, we can write

$H^U_\xi(f) := \iota_\xi \tau_U = \int_{\Pi T U} \xi^i(f(u)) \partial_i(f(u)) \theta_n \mu(u)$.

Let $X^U$ be the space of vector fields $\xi$ on $Y^{\Pi T\Sigma}$ with the property that $\xi(f)$, seen as an element of $\Gamma(\Pi T\Sigma, f^*TY)$, has support in $U$ for any $f \in Y^{\Pi T\Sigma}$. Any element of $X^U$ determines a vector field on $Y^{\Pi T U}$ which, by abuse of notation, we will denote by the same letter. So we can define

$C^U = \{ f \in Y^{\Pi T\Sigma} : H^U_\xi(f) = 0 \forall \xi \in X^U \}
= \{ f \in Y^{\Pi T\Sigma} : \theta_n f(\Pi T U) \subset Z(\vartheta) \}$.

We have then the following obvious

**Lemma 2.9.** For every two collars $U$ and $V$ with $U \subset V$, one has $C^V \subset C^U$. Moreover, $C^U \subset C$ for every collar $U$.

Now observe that the functions $H^U_\xi$, $\xi \in X^U$, define first-class constraints on $Y^{\Pi T\Sigma}$. In fact, their Hamiltonian vector fields are proportional to $\theta_n$. So their application to any $H^U_\xi$ automatically vanishes thanks to $(\theta_n)^2 = 0$. Thus, we can define a new $P$-space by considering the quotient of $C^U$ by the foliation generated by the Hamiltonian vector fields of all $H^U_\xi$. We denote this reduced space by $Y^{\Pi T\Sigma}//\tau^U$. Then we have the following

**Theorem 2.10.** $\hat{D}$ is Hamiltonian on $Y^{\Pi T\Sigma}//\tau^U$ with Hamiltonian function $\hat{S} = -\iota_\beta \vartheta$ satisfying $(\hat{S}, \hat{S}) = 0$.

If moreover $\hat{Q}$ is Hamiltonian and its Hamiltonian function $S$ is locally constant on the zero locus of $\vartheta$, then the restriction of $\hat{S}$ to $C^U$ is an invariant function. So $\hat{S}$ descends to a function on $Y^{\Pi T\Sigma}//\tau^U$.
that we will denote by the same symbol. Of course, \((\tilde{S}, \tilde{S}) = 0\) still holds if \((S, S) = 0\).

Finally, if we further assume that \(S\) vanishes on the zero locus of \(\vartheta\), then \((\hat{S}, \hat{S}) = 0\). In particular, \(S = \hat{S} + \check{S}\) satisfies the master equation \((S, S) = 0\).

Proof. The first statement follows from (2.6) and Lemma 2.9. We have first to check that \(\hat{S}\) restricted to \(\mathcal{U}\) is invariant. This is a consequence of the identities

\[\left(\hat{S}, H^U_{\xi}\right) = \left[\hat{D}, \xi\right] \tau^U = \xi \left[\hat{D}, \xi\right] \tau^U = 0\]

The last equality holds on \(\mathcal{C}^U\) since \(\left[\hat{D}, \xi\right] \in \mathfrak{X}^U\) for \(\xi \in \mathfrak{X}^U\) and because the restriction of \(\text{ev}_U^* \vartheta \theta\) to \(\Pi T \partial U\) vanishes (as one has to set \(\theta = 0\)). Finally, \((\hat{S}, \hat{S}) = \left[\hat{D}, \hat{S}\right] = \left[\hat{D}, \xi\right] \tau^U = 0\).

As for the second statement, let us denote by \(\theta_n^U X\) the Hamiltonian vector field of \(H_{\xi}^U\). Observe that \(X(f, \bullet)\) has support in \(U\), so

\[\left(H^U_{\xi}, \check{S}\right)(f) = \int_{\Pi T U} \theta_n X^i (f, u) \partial_i S(f(u)) \mu\]

However, if we restrict to \(\mathcal{C}^U\), then \(\theta_n f(u)\) belongs to the zero locus of \(\vartheta\) for all \(u \in U\), and \(S\) is constant on it by assumption. So \((H^U_{\xi}, \check{S}) = 0\).

Since the vector fields \(\hat{D}\) and \(Q\) commute, it follows immediately that the bracket \((\hat{S}, \hat{S})\) is a constant function, but we want this function to vanish identically. Actually, by Lemma 2.8 we have

\[\left(\hat{S}, \hat{S}\right) = \left[\hat{D}, \check{S}\right] = \left[\hat{D}, S\right] = 0\]

the last equality holding on \(\mathcal{C}\) (and so on \(\mathcal{C}^U\)) since \(f(u)\) must belong to the zero locus of \(\vartheta\) for \(u \in \partial \Sigma\), and \(S\) vanishes on it by assumption. \(\square\)

In the application of the above construction to the perturbative evaluation of a path integral, the idea is to define it first on \(Y^{\Pi T \Sigma} \slash \tau^U\) for a given collar \(U\). Next one should compute the propagators and the vertices and finally consider the limit for \(U\) shrinking to \(\partial \Sigma\).

Another possibility, see e.g. next section, is to fix representatives in \(Y^{\Pi T \Sigma} \slash \tau^U\) as elements of \(\mathcal{C}^U\) satisfying some extra conditions. Let us denote by \(\tilde{\mathcal{C}}^U\) this further constrained space isomorphic to \(Y^{\Pi T \Sigma} \slash \tau^U\). If \(U \subset V\) implies \(\tilde{\mathcal{C}}^V \subset \tilde{\mathcal{C}}^U\), we define \(\tilde{\mathcal{C}}\) as \(\bigcup \tilde{\mathcal{C}}^U\). Then the path integral may be defined on a Lagrangian submanifold (see Remark 2.17 in the next subsection) of \(\tilde{\mathcal{C}}\).
2.2.5. Remarks.

Remark 2.11 (dependency on \( \vartheta \)). In the general case, described in Theorem 2.7, the choice of a potential \( \vartheta \) for the symplectic form \( \omega \) is irrelevant. In fact, \( \vartheta \) enters only in the definition of \( \hat{S} \). But if we choose \( \vartheta' = \vartheta + d\sigma \), we obtain \( \hat{S}' := -\iota_D \vartheta' = \hat{S} - L_D \mu_* \text{ev}^* \sigma \), and this equals \( \hat{S} \) by Lemma 2.6.

In the case of \( X = \Pi T\Sigma \), \( \partial \Sigma \neq \emptyset \), the choice of \( \vartheta \) is instead essential as it determines the boundary conditions through \( \tau \). As different but cohomologous \( \vartheta \)'s may have different zero sets, this also affects which functions \( S \) on \( Y \) are allowed by Theorem 2.10. (Think, e.g., of \( \Pi T^* \mathbb{R}^m \) with \( \omega = dp_i dq^i \) and \( \vartheta = ap_i dq^i + (a-1)q^i dp_i \) for different choices of \( a \).)

Remark 2.12 (ghost number). As observed in 2.1.5, in quantum field theory one is interested in the full \( \mathbb{Z} \)-grading. So, in order to have \( \omega \) of ghost number \(-1\), we must choose \( \omega \) to have ghost number \( n-1 \), if \( X \) is of type \((m,n)\). Moreover, we must assume that \( Q \) has ghost number one, which implies that its Hamiltonian function \( S \) must have ghost number \( n \). In the case \( X = \Pi T\Sigma \), we require then \( \text{gh}\omega = s-1 \) and \( \text{gh} S = s \) with \( s = \text{dim} \Sigma \).

Remark 2.13 (twisted supersymmetry). Since the canonical measure \( \mu \) on \( \Pi T\Sigma \) is \( \dot{K}_v \)-invariant for every vector field \( v \) on \( \Sigma \), we conclude by Lemma 2.3 that \( \omega, \vartheta \) and \( \hat{S} \) are \( \dot{K}_v \)-invariant. (Actually, \( \dot{K}_v \) is even Hamiltonian with Hamiltonian function \(-\iota_{\dot{K}_v} \vartheta \).) We obtain then

\[
\left[ \dot{K}_v, Q \right] = \left[ \dot{K}_v, \dot{D} \right] = \dot{L}_v.
\]

In particular, we may locally choose constant vector fields \( v_\mu := \frac{\partial}{\partial u_\mu} \). We then have \( \dot{L}_{v_\mu} = \frac{\partial}{\partial u_\mu} \). This means that \( Q \) and \( \dot{K}_{v_\mu}, \mu = 1, \ldots, \text{dim} \Sigma \), are generators of a twisted supersymmetry.

Remark 2.14 (classical theory). The AKSZ method induces a posteriori a classical theory of which the above discussion is the BV version. Distinguish the components of a map \( f : X \to Y \) according to the ghost number. The components of nonnegative ghost number are called the fields, the others the antifields. Among the fields one further distinguish between the classical fields (zero ghost number) and the ghosts (positive ghost numbers). The classical action \( S^{\text{cl}} \) is obtained by setting all the antifields in \( S \) to zero; it depends only on the classical fields. The action of the \( Q \)-vector field on the fields at zero antifields is called the BRST operator and generates the infinitesimal symmetries of the theory.
Remark 2.15 (BV quantization). Given a solution of the master equation the rules of the game for defining (in perturbation theory) the functional integral of \(\exp iS^{cl}/\hbar\) over the classical fields are as follows: Choose a Lagrangian submanifold (gauge fixing) of the space of fields and antifields, and integrate over it the new weight \(\exp iS/\hbar\). This procedure works whenever the new action is nondegenerate on the chosen Lagrangian submanifold. Actually, the AKSZ construction has not enough room for this step, but one just has to enlarge the space \(Y^X\) to include enough Lagrange multipliers (\(\lambda\)) and antighosts (\(\bar{c}\))—together with their canonically conjugated variables \(\lambda^\dagger\) and \(\bar{c}^\dagger\)—in order to gauge-fix all symmetries in the extended action given by \(S\) plus terms of the form \(\bar{c}^\dagger\lambda\).

Remark 2.16 (Applications). The AKSZ method was foreshadowed by Witten [19] in the sigma-model interpretation of Chern–Simons theory [18]. In [1] it was explained that Chern-Simons theory enters this scheme by choosing \(\Sigma\) a 3-manifold and \(Y = \mathfrak{g}[1] = \Pi g\), with \(g\) a metric Lie algebra; in the same paper the method was also applied to the A and B models [17]. In [8] the application to the Rozansky–Witten [12] theory was considered. In [6] the method has been applied to BF theories by choosing \(\Sigma\) an \(s\)-manifold and \(Y = \mathfrak{g}[1] \times \mathfrak{g}^*[s-2]\). In [11], topological open branes are defined by choosing \(\Sigma\) an \(s\)-manifold with boundary and \(Y = (\Pi T)^{s-2}\Pi T^* M\), with \(M\) an ordinary manifold.

3. The Poisson sigma model

Let \(\Sigma\) and \(M\) be ordinary manifolds. We want to apply the AKSZ construction described in the previous Section by choosing \(X = \Pi T \Sigma\) and \(Y = \Pi T^* M\), with the \(Q\)-vector field \(D\), the canonical measure \(\mu\) and the odd symplectic structure \(\omega\) introduced in Examples 2.1, 2.4 and 2.2 respectively. Since \(\omega\) has in this case ghost number 1 we will consider \(\Sigma\) to be 2-dimensional, according to the discussion in Remark 2.12.

3.1. The \(P\)-structure. An element of \(Y^X\) in the present case is a pair \((X, \eta)\) where \(X\) is a map \(\Pi T \Sigma \to M\) and \(\eta\) is a section of \(X^\ast \Pi T^\ast M\). In

\[\left( S, S \right) - 2i\hbar \Delta S = 0,\]

where \(\Delta\) is an operator that depends on the measure in the functional integral and so is well-defined only after a regularization has been chosen.
local coordinates on $\Pi T\Sigma$ (see Example 2.1), we may write
\[
X = X + \theta^\mu \eta^\mu - \frac{1}{2} \theta^\mu \theta^\nu \beta^+_{\mu\nu},
\]
\[
\eta = \beta + \theta^\mu \eta^\mu + \frac{1}{2} \theta^\mu \theta^\nu X^+_{\mu
u},
\]
with $X$ an ordinary map $\Sigma \to M$ and
\[
\eta^+ \in \Gamma(\Sigma, T^*\Sigma \otimes X^*TM) \otimes \Lambda_-[-1],
\]
\[
\beta^+ \in \Gamma(\Sigma, \wedge^2 T^*\Sigma \otimes X^*TM) \otimes \Lambda_+[-2],
\]
\[
\beta \in \Gamma(\Sigma, X^*TM) \otimes \Lambda_-[1],
\]
\[
\eta \in \Gamma(\Sigma, T^*\Sigma \otimes X^*TM) \otimes \Lambda_+[0],
\]
\[
X^+ \in \Gamma(\Sigma, \wedge^2 T^*\Sigma \otimes X^*TM) \otimes \Lambda_-[-1],
\]
where $\Lambda_\pm$ are the odd and even parts of $\Lambda$, and we follow the notations of 2.1.5. The classical fields (see Remark 2.14) are then $X$ and $\eta$, the only ghost is $\beta$, and the other components are antifields. By the symplectic form $\omega$, the superfields $X$ and $\eta$ are canonically conjugated. In terms of their components, each field is canonically conjugated to the corresponding antifield with an upper $+$. The canonical symplectic form is exact: $\omega = d\vartheta$. We choose $\vartheta = p_i dx^i = \langle p, dx \rangle$, where $\langle , \rangle$ is the canonical pairing of vectors and covectors on $M$, and define $\vartheta = \mu \ast ev^*\vartheta$ as in the previous section. To give a more explicit description of $\vartheta$, we observe that the tangent bundle of $\Pi T^*M^{\Pi T\Sigma}$ has a splitting $T(\Pi T^*M^{\Pi T\Sigma}) = \mathfrak{X} \oplus \mathfrak{E}$ given by vectors “in the directions of $X$ and in the directions of $\eta$” respectively; viz.,
\[
\mathfrak{X}_{(X,\eta)} = T_{(X,\eta)}M^{\Pi T\Sigma} \simeq \Gamma(\Pi T\Sigma, X^*TM),
\]
\[
\mathfrak{E}_{(X,\eta)} = T_{(X,\eta)}\Gamma(\Pi T\Sigma, X^*\Pi T^*M) \simeq \Gamma(\Pi T\Sigma, X^*\Pi T^*M).
\]
Then we can write
\[
\vartheta(X,\eta)(\xi \oplus e) = \int_{\Pi T\Sigma} \langle \xi, \eta \rangle \mu,
\]
for any $\xi \in \mathfrak{X}_{(X,\eta)}$ and $e \in \mathfrak{E}_{(X,\eta)}$.

3.2. The $QP$-structure and the classical action. As in the previous section we define a $Q$-structure on $Y^X$ starting from $Q$-structures on $X$ and on $Y$. On $Y = \Pi T^*M$ we look for a Hamiltonian vector field of ghost number 1. Since $\omega$ has ghost number 1, this means that we have to look for a Hamiltonian function $S$ of ghost number 2, i.e., for a
function corresponding to a bivector field as in Example 2.3; viz., we define $S_\alpha$ and $Q_\alpha$ as in (2.2). We recall that a manifold endowed with a bivector field $\alpha$ satisfying $[\alpha, \alpha] = 0$ is called a Poisson manifold, for $\alpha$ defines a Poisson bracket on the algebra of functions. By (2.3), it follows that the Hamiltonian function of the vector field $\dot{Q}_\alpha$ is

$$\dot{S}_\alpha = \frac{1}{2} \int_{\Pi T \Sigma} (\alpha \circ X)(\eta, \eta) \, \mu.$$ 

On $X = \Pi T \Sigma$ we take the $Q$-vector field $D$ of Example 2.1, which has ghost number 1. As in Theorem 2.10, we may define $\dot{S} = -\iota_D \vartheta$ getting

$$\dot{S} = - \int_{\Pi T \Sigma} \langle DX, \eta \rangle \, \mu,$$

where $DX$ is a short-hand notation for the projection of $\dot{D}$ to $X$.

By Theorem 2.7, the function $S_\alpha = \dot{S} + \dot{S}_\alpha$ satisfies the strongest assumption of Theorem 2.10 and we get a solution of the master equation for every collar $U$. We next describe $C_U$. First observe that the even 1-form $\tau_U$ vanishes when contracted with vectors in $E$. If instead $\xi$ is a vector in $X$—i.e., $\xi(X, \eta) \in \Gamma(\Pi T \Sigma, X^* T M)$—, then we have

$$H^U_\xi = \iota_\xi \tau_U = \int_{\Pi T U} \langle \xi, \eta \rangle \theta_n \mu^U.$$

Thus, $C_U$ is the space of maps $(X, \eta)$ satisfying $\theta_n \iota^*_U \eta = 0$. The Hamiltonian vector field $H_\xi$ corresponding to a constant vector field $\xi$ is also easily computed to be $\xi \theta_n$. This means that, on $\Pi T^* M^{\Pi T \Sigma} / \tau^U$, $X$ is defined up to translations in the normal direction supported in $U$. We can fix a representative by choosing any direction $n'$ transversal to $\partial \Sigma$ (e.g., $n' = n$) and by requiring that the $n'$-components of $X$ vanish. This

---

3 More generally, we may consider any multivector field such that the component of the term of order $k$ has ghost number $k - 2$.

4 If we work in the more general setting of footnote 3, we should however exclude from $\alpha$ any term of order zero, i.e., functions.
means $\beta^+ = 0$ and $\eta^- = 0$. We can finally remove the regularization by considering
\begin{equation}
\tilde{C} = \{(X, \eta) : \beta(u) = 0, \eta(u) = 0, \eta^+(u) = 0, \beta^+(u) = 0 \ \forall u \in \partial \Sigma\},
\end{equation}
where $t$ denotes the direction tangent to $\partial \Sigma$.

The above discussion leads exactly to the BV action of [4] with the same boundary conditions. We refer to it for the introduction of antighosts and Lagrange multipliers, for the discussion of the quantum master equation and for the relation between the Poisson sigma model on the disk and Kontsevich's formula [9] for the deformation quantization of $M$.

3.3. Target diffeomorphisms. We conclude with a brief discussion on the effect of a diffeomorphism $\phi$ of $M$ on the whole theory. We denote by $\Phi$ the canonical extension of $\phi$ to $\Pi T^* M$:
\[ \Phi(x, p) = (\phi(x), (d\phi(x))^{-1}p). \]
It leaves invariant the canonical symplectic form $\omega$ as well as the canonical 1-form $\vartheta$. Moreover, one has $\Phi^* S_\phi = S_\phi$. Let $\tilde{\Phi}$ be the corresponding right action on $\Pi T^* M^{\Pi T \Sigma}$:
\[ \tilde{\Phi}(f) = \Phi \circ f, \quad f \in \Pi T^* M^{\Pi T \Sigma}. \]
It follows that $\tilde{\Phi}$ is a symplectomorphism of $(\Pi T^* M^{\Pi T \Sigma}, \omega)$ with the property $\tilde{\Phi}^* \tilde{\vartheta} = \vartheta$. Moreover, since $\hat{D}$ corresponds to an (infinitesimal) left action and $\tilde{\Phi}^* \vartheta = \vartheta$, we also obtain that $\hat{S}$ is $\tilde{\Phi}$-invariant. So we conclude that
\begin{equation}
\tilde{\Phi}^* S_\phi = S_\phi.
\end{equation}
This means that the Poisson sigma model is invariant under target diffeomorphisms at the classical level, and it is so at the quantum level if the measure on the space of fields is invariant.

To show that this is so at the quantum level, we restrict for simplicity to infinitesimal diffeomorphisms. Let $\xi$ be a vector field on $M$ and $\Xi$ the corresponding vector field on $\Pi T^* M$. It is not difficult to see that $\Xi$ has Hamiltonian $S_{\xi}(x, p) = \xi^i(x)p_i$. Let $\hat{\Xi}$ be the corresponding vector field on $\Pi T^* M^{\Pi T \Sigma}$. It is a Hamiltonian vector field with Hamiltonian $\hat{\xi} = \mu_\ast ev^* S_{\xi}$. We recall that a canonical transformation in the BV formalism extends to the quantum level (i.e., it preserves the path integral measure), if it is in the kernel of the BV Laplacian (which is constructed using the measure). We proved in [4] that this is the case for a function like $\hat{\xi}$ (actually, with $\xi$ any multivector field) in reasonable regularizations.
It is however well-known that Kontsevich’s formula for the \( \star \)-product does not transform well under diffeomorphisms. We would like to end the paper with some comment on this apparent contradiction. First observe that the infinitesimal form of (3.2),

\[
L \xi S\alpha = S[\xi, \alpha],
\]

may be obtained by the identities

\[
L \xi \dot{S}_\alpha = (\dot{S}_\xi, \dot{S}_\alpha) = \dot{S}[\xi, \alpha],
\]

\[
L \xi \dot{S} = -(\dot{S}_\xi, \dot{S}_\xi) = -\dot{D}\dot{S}_\xi = -\mu^\alpha_{\partial C} \partial^{*}ev^{*}_{\partial} S_\xi = 0.
\]

Now the last equality follows from the fact that \( ev_{\partial}^{*} S_\xi \) vanishes upon using the boundary conditions in (3.1). However, in the definition of the \( \star \)-product from the path integral of the Poisson sigma model, one has to insert boundary observables (i.e., evaluate some functions on \( M \) at points \( X(u) \) with \( u \in \partial \Sigma \), see [4]. These introduce singularities that have to be removed by choosing a small neighborhood \( U \) of the boundary points of insertions and integrating on \( \Pi T(\Sigma \setminus U) \) instead of \( \Pi T \Sigma \). As a consequence, in the last equation the term \( S_{C}^{\xi} = \int_{\Pi T C}ev^{*}S_{\xi} \mu^{C} \) survives, where \( C \) is the boundary of \( U \) modulo the boundary of \( \Sigma \). This spoils the original invariance of the Poisson sigma model under target diffeomorphisms. It also suggests that the deformed action of a diffeomorphism on a function should be given by the expectation value of \( S_{C}^{\xi} \) (i.e., \( \sum_{n}(i\hbar)^{n}U_{n+1}(\xi, \alpha, \ldots, \alpha)/n! \), where \( U \) denotes Kontsevich’s \( L^{\infty} \)-morphism [9]). This idea, also related to observations in [10], has been used in [5] to clarify the globalization of Kontsevich’s formula.

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