A proof of the multiplicity one conjecture
for $GL_n$ in $GL_{n+1}$

Avraham Aizenbud and Dmitry Gourevitch

February 5, 2008

Abstract

Let $F$ be a non-archimedean local field of characteristic zero. We consider
distributions on $GL(n + 1, F)$ which are invariant under the adjoint action of
$GL(n, F)$. We prove that any such distribution is invariant with respect to
transposition. This implies that the restriction to $GL(n)$ of any irreducible
smooth representation of $GL(n + 1)$ is multiplicity free.

Our paper is based on the recent work [RS] of Steve Rallis and Gérard
Schiffmann where they made a remarkable progress on this problem.

In [RS], they also show that our result implies multiplicity one theorem
for restrictions from the orthogonal group $O(V \oplus F)$ to $O(V)$.

Acknowledgements

We would like to thank our teacher Joseph Bernstein for teaching us most of the
mathematics we know.

We cordially thank Joseph Bernstein and Eitan Sayag for guiding us
through this project. Their contribution was very important.

We would also like to thank Vladimir Berkovich, Yuval Flicker, Erez
Lapid, Omer Offen and Yiannis Sakellaridis for useful remarks.

Contents

1 Introduction 2
2 Preliminaries and notations 4
3 Reformulations of the problem 5
4 Proof of the main theorem 6
5 Proof of the key lemma 8
6 Weil representation and proof of theorem 2.8 10
7 Appendix: Proof of theorem 4.2 11
1 Introduction

Let \( \mathbb{F} \) be a non-archimedean local field of characteristic zero. Consider the standard imbedding \( \text{GL}(n, \mathbb{F}) \subset \text{GL}(n+1, \mathbb{F}) \). Let \( \text{GL}(n, \mathbb{F}) \) act on \( \text{GL}(n+1, \mathbb{F}) \) by conjugation. The goal of this paper is to prove the following theorem.

**Theorem 1.1** Every \( \text{GL}(n, \mathbb{F}) \)-invariant distribution on \( \text{GL}(n+1, \mathbb{F}) \) is invariant with respect to transposition.

This theorem is important in representation theory, since it implies the following multiplicity one theorem (see e.g. [RS], section 2).

**Theorem 1.2** Let \( \pi \) be an irreducible smooth representation of \( \text{GL}(n+1, \mathbb{F}) \) and \( \rho \) be an irreducible smooth representation of \( \text{GL}(n, \mathbb{F}) \). Then

\[
\dim \text{Hom}_{\text{GL}(n, \mathbb{F})}(\pi|_{\text{GL}(n, \mathbb{F})}, \rho) \leq 1.
\]

In their recent paper [RS], in part II, Steve Rallis and Gérard Schiffmann have shown that theorem 1.1 also implies similar theorems for distributions on unitary and orthogonal groups, which in turn imply multiplicity one results for those groups.

In [RS], Rallis and Schiffmann have also made a remarkable progress in proving the above theorem 1.1, and our paper is based on their results. For the benefit of the reader we present the proofs of all the statements from [RS] that we use.

Theorem 1.1 also gives another proof of Bernstein’s theorem about \( P \)-invariant distributions on \( \text{GL}(n) \) (see [Ber]) which proves Kirillov’s conjecture in the non-archimedean case.

1.1 Reformulation of the main theorem

Let \( G := G_n := \text{GL}(n, \mathbb{F}) \). Consider the action of the 2-element group \( S_2 \) on \( G \) given by the involution \( g \mapsto t^g \). It defines a semidirect product \( \tilde{G} := \tilde{G}_n := G_n \rtimes S_2 \).

Let \( V := V_n = \mathbb{F}^n \) and \( X := X_n := \text{sl}(V_n) \times V_n \times V_n^* \) where \( \text{sl}(V) \subset \text{End}(V) \) is the space of operators with zero trace.

The group \( \tilde{G} \) acts on \( X \) by

\[
g(A, v, \phi) := (gAg^{-1}, gv, g^{-1}^* \phi) \quad \text{and} \quad T(A, v, \phi) := (t^A, t^\phi, t^v)
\]

where \( g \in G \) and \( T \) is the generator of \( S_2 \). Here, \( t^A \) denotes the transposed matrix in \( \text{sl}_n \), \( t^\phi \in V_n \) denotes the column vector corresponding to the row vector \( \phi \in V_n^* \), and \( t^v \) denotes the row vector corresponding to the column vector \( v \in V_n \). Also for any operator \( g : V \to V \), we denote by \( g^* : V^* \to V^* \) the adjoint operator.

Note that \( \tilde{G} \) acts separately on \( \text{sl}(V) \) and on \( V \times V^* \). Define a character \( \chi \) of \( \tilde{G} \) by \( \chi(g, s) := \text{sign}(s) \).

Theorem 1.1 can be deduced from the following theorem.

**Theorem 1.3** Any \( (\tilde{G}, \chi) \)-equivariant distribution on \( X \) is zero.

The deduction was first proven in [RS] (section 5). We prove it in section 3 in a slightly different way. In section 3 we also give a coordinate-free definition of the group \( \tilde{G} \) and its action on \( X \).
1.2 Sketch of our proof

The theorem will be proved by induction on $n$. Let $S$ denote the closure of the union of the supports of all $(\tilde{G}, \chi)$-equivariant distributions on $X$. We would like to prove that $S = \emptyset$.

Let $\text{pr}_1 : X \to \text{sl}(V)$ and $\text{pr}_2 : X \to V \oplus V^*$ be the natural projections. Rallis and Schiffman have shown that the induction hypothesis implies:

(i) $\text{pr}_2(S)$ is contained in $Y := \{(v, \phi) \in V \oplus V^* | \langle \phi, v \rangle = 0\}$

(ii) $\text{pr}_1(S)$ is contained in the nilpotent cone $N$.

Part (i) follows from the localization principle and Frobenius reciprocity.

Part (ii) is proven using Harish-Chandra descent method.

We will present a proof of this statement in the appendix (section 7).

For any vector $v$ and covector $\phi$ let $v \otimes \phi$ denote the operator of rank one defined by them. Let $\lambda \in \mathbb{F}$ be a scalar. We use a family of automorphisms $\nu_\lambda$ of $X$ defined by

$$\nu_\lambda(A, v, \phi) := (A + \lambda v \otimes \phi - \lambda \frac{\langle \phi, v \rangle}{n} \text{Id}, v, \phi).$$

Note that the automorphisms $\nu_\lambda$ commute with the action of $\tilde{G}$ and hence preserve $S$.

Let $N_i \subset \text{sl}(V)$ be the union of all nilpotent orbits of dimensions $\leq i$. We prove by downward induction that $S \subset N_i \times Y$ for all $i$. Suppose $S \subset N_i \times Y$. We have to show $S \subset N_{i-1} \times Y$. Since $\nu_\lambda(S) = S$, $S$ is contained in the intersection $\bigcap \nu_\lambda(N_i \times Y)$.

We have to show that for any nilpotent orbit $O$ of dimension $i$, the restriction of any $(\tilde{G}, \chi)$-equivariant distribution $\xi$ to $O \times Y$ is zero. As we have seen, the support of $\xi|_{O \times Y}$ is contained in $(O \times Y) \cap (\bigcap \nu_\lambda(N_i \times Y))$, which we denote by $\tilde{O}$. Using the fact that the Fourier transform of a $(\tilde{G}, \chi)$-equivariant distribution is also $(\tilde{G}, \chi)$-equivariant, the theorem boils down to the following key lemma.

**Lemma 1.4 (Key lemma)** Let $O$ be a nilpotent orbit. Let $\zeta \in S^*(O \times V \times V^*)^{\tilde{G}, \chi}$. Suppose that both $\zeta$ and $\mathcal{F}(\zeta)$ are supported in $\tilde{O}$. Then $\zeta = 0$.

Using Frobenious reciprocity, the key lemma reduces to a statement about distributions on $V \oplus V^*$.

Namely, fix $A \in O$. Let $R_A$ denote the fiber over $A$ of the projection $\tilde{O} \to O$. Then $\zeta$ corresponds to a distribution $\eta$ on $V \oplus V^*$ with the following properties:

(i) $\eta$ is supported in $R_A$

(ii) $\mathcal{F}(\eta)$ is supported in $R_A$

(iii) $\eta$ is $\chi$-equivariant with respect to the stabilizer of $A$ in $\tilde{G}$.

We have to show $\eta = 0$. We will prove that $R_A$ is contained in

$$Q_A := \{(v, \phi) \in V \oplus V^* | v \otimes \phi \in [A, \text{gl}(V)]\}.$$ 

It is convenient to work with $Q_A$ since its description is linear.

For example, we will show that $Q_{A_1 \oplus A_2} \subset Q_{A_1} \times Q_{A_2}$. This will allow us to decompose the problem into Jordan blocks (see section 5.1).

Then we will solve the case of one Jordan block (in section 5.2) using an important result by Rallis and Schiffmann which is proven using the Weil representation.
2 Preliminaries and notations

We will use the standard terminology of l-spaces introduced in [BZ], section 1. We denote by $S(Z)$ the space of Schwartz functions on an l-space $Z$, and by $S^*(Z)$ the space of distributions on $Z$ equipped with the weak topology.

We fix a nontrivial additive character $\psi$ of $\mathbb{F}$.

Notation 2.1 (Fourier transform) Let $W$ be a finite dimensional vector space over $\mathbb{F}$. Let $B$ be a nondegenerate symmetric bilinear form on $W$. We denote by $\mathcal{F}_B : S^*(W) \rightarrow S^*(W)$ the Fourier transform defined using $B$ and the self-dual measure on $W$.

By abuse of notation, we also denote by $\mathcal{F}_B$ the partial Fourier transform $\mathcal{F}_B : S^*(Z \times W) \rightarrow S^*(Z \times W)$ for any l-space $Z$.

If $W = U \oplus U^*$ then it has a canonical symmetric bilinear form given by the quadratic form $Q((v, \phi)) := \langle \phi, v \rangle := \phi(v)$. We will denote the Fourier transform defined by it simply by $\mathcal{F}_W$. If there is no ambiguity, we will denote it simply by $\mathcal{F}$.

Proposition 2.2 Let $W_1 \oplus W_2$ be finite dimensional vector spaces. Let $B_1$ and $B_2$ be nondegenerate symmetric bilinear forms on $W_1$ and $W_2$ respectively. Let $Z \subset W_1$ be a closed subset. Let $\xi \in S^*(W_1 \oplus W_2)$ be a distribution. Suppose that $\mathcal{F}_{B_1 \oplus B_2}(\xi)$ is supported in $Z \times W_2$. Then $\mathcal{F}_{B_1}(\xi)$ is also supported in $Z \times W_2$.

Proof. Let $p_1$ denote the projection $W_1 \oplus W_2 \rightarrow W_1$. Since $\mathcal{F}_{B_2}$ does not change the projection of the support of a distribution to $W_1$,

$$p_1(\text{Supp}(\mathcal{F}_{B_1}(\xi))) = p_1(\text{Supp}(\mathcal{F}_{B_2} \circ \mathcal{F}_{B_1}(\xi))) = p_1(\mathcal{F}_{B_1 \oplus B_2}(\xi)) \subset Z$$

We will use the localization principle, formulated in [Ber], section 1.4.

Theorem 2.3 (Localization principle) Let $q : Z \rightarrow T$ be a continuous map of l-spaces. Denote $Z_t := q^{-1}(t)$. Consider $S^*(Z)$ as $S(T)$-module. Let $M$ be a closed subspace of $S^*(Z)$ which is an $S(T)$-submodule. Then $M = \bigoplus_{t \in T}(M \cap S^*(Z_t)).$

Informally, it means that in order to prove a certain property of distributions on $Z$ it is enough to prove that distributions on every fiber $Z_t$ have this property.

Corollary 2.4 Let $q : Z \rightarrow T$ be a continuous map of l-spaces. Let an l-group $H$ act on an l-space $Z$ preserving the fibers of $q$. Let $\mu$ be a character of $H$. Suppose that for any $t \in T$, $S^*(q^{-1}(t))H,\mu = 0$. Then $S^*(Z)^H,\mu = 0$

Corollary 2.5 Let $H_i \subset \widetilde{H}_i$ be l-groups acting on l-spaces $Z_i$ for $i = 1, \ldots, k$. Suppose that $S^*(Z_i)^{H_i} = S^*(Z_i)^{\widetilde{H}_i}$ for all $i$. Then $S^*(\prod Z_i)^{\prod H_i} = S^*(\prod Z_i)^{\prod \widetilde{H}_i}$.

We will use the following version of the Frobenious reciprocity. It can be easily deduced from the Frobenious reciprocity described in [Ber], section 1.5.

Theorem 2.6 (Frobenious reciprocity) Let a unimodular l-group $H$ act transitively on an l-space $Z$. Let $\varphi : E \rightarrow Z$ be an $H$-equivariant map of l-spaces. Let
Suppose that its stabilizer $\text{Stab}_H(x)$ is unimodular. Let $W$ be the fiber of $x$. Let $\mu$ be a character of $H$. Then

(i) There exists a canonical isomorphism $\text{Fr} : S^*(E)^{H,\mu} \to S^*(W)^{\text{Stab}_H(x),\mu}$.

(ii) For any distribution $\xi \in S^*(E)^{H,\mu}$, $\text{Supp}(\text{Fr}(\xi)) = \text{Supp}(\xi) \cap W$.

(iii) Frobenius reciprocity commutes with Fourier transform.

Namely, let $W$ be a finite dimensional linear space over $\mathbb{F}$ with a nondegenerate bilinear form $B$. Let $H$ act on $W$ linearly preserving $B$.

Then for any $\xi \in S^*(Z \times W)^{H,\mu}$, we have $\mathcal{F}_B(\text{Fr}(\xi)) = \text{Fr}(\mathcal{F}_B(\xi))$ where $\text{Fr}$ is taken with respect to the projection $Z \times W \to Z$.

**Definition 2.7** Let $W$ be a finite dimensional vector space over $\mathbb{F}$. We call a distribution $\xi \in S^*(W)$ **abs-homogeneous of degree** $d$ if for any function $f \in S(W)$,

$$|\xi(h_{t^{-1}}(f))| = |t|^{-d}|\xi(f)|$$

where $(h_{t^{-1}}(f))(v) = f(tv)$.

For example, a Haar measure on $W$ is abs-homogeneous of degree $\text{dim } W$ and the $\delta$-distribution is abs-homogeneous of degree 0.

A crucial step in the proof of the main theorem is the following special case of a result by Rallis and Schiffmann ([RS], lemma 8.1).

**Theorem 2.8** (Rallis-Schiffmann) Let $W$ be a finite dimensional vector space over $\mathbb{F}$ and $B$ be a nondegenerate symmetric bilinear form on $W$. Denote $Z(B) := \{v \in W|B(v,v) = 0\}$. Let $\xi$ be a distribution on $W$. Suppose that both $\xi$ and $\mathcal{F}_B(\xi)$ are supported in $Z_B$.

Then $\xi$ is abs-homogeneous of degree $\frac{1}{2}\text{dim } W$.

For the benefit of the reader we reproduce the proof of this theorem in section 6.

**Remark 2.9** Let $Z$ be an $l$-space and $Q \subset Z$ be a closed subset. We will identify $S^*(Q)$ with the space of all distributions on $Z$ supported on $Q$. In particular, we can restrict a distribution $\xi$ to any open subset of the support of $\xi$.

### 3 Reformulations of the problem

In this section we will prove the following proposition.

**Proposition 3.1** Theorem 1.3 implies theorem [LL].

We will divide this reduction to several propositions.

Consider the action of $\tilde{G}_n$ on $G_{n+1}$ and on $\text{gl}_{n+1}$ where $G_n$ acts by conjugation and the generator of $S_2$ acts by transposition.

**Proposition 3.2** If $S^*(G_{n+1})^{\tilde{G}_n,\chi} = 0$ then theorem [LL] holds.

The proof is straightforward.

**Proposition 3.3** If $S^*(\text{gl}_{n+1})^{\tilde{G}_n,\chi} = 0$ then $S^*(G_{n+1})^{\tilde{G}_n,\chi} = 0$.
Proof. Let $ξ \in S^*(G_{n+1})\tilde{G}_{n,χ}$. We have to prove $ξ = 0$. Assume the contrary. Take $p \in \text{Supp}(ξ)$. Let $t = \det(p)$. Let $f \in S(ℙ)$ be such that $f(0) = 0$ and $f(t) \neq 0$. Consider the determinant map $\det : G_{n+1} \to ℙ$. Consider $ξ' := (f \circ \det) \cdot ξ$. It is easy to check that $ξ' \in S^*(G_{n+1})\tilde{G}_{n,χ}$ and $p \in \text{Supp}(ξ')$. However, we can extend $ξ'$ by zero to $ξ'' \in S^*(g_{n+1})\tilde{G}_{n,χ}$, which is zero by the assumption. Hence $ξ'$ is also zero. Contradiction. □

**Proposition 3.4** If $S^*(s_{n+1})\tilde{G}_{n,χ} = 0$ then $S^*(g_{n+1})\tilde{G}_{n,χ} = 0$

Proof. Consider the trace map $\text{tr} : g_{n+1} \to ℙ$. By the localization principle (Corollary 2.4), it is enough to prove that for any $t \in ℙ$, $S^*(\text{tr}^{-1}(t))\tilde{G}_{n,χ} = 0$. However, all $\text{tr}^{-1}(t)$ are isomorphic as $\tilde{G}_{n}$-equivariant $t$-spaces to $s_{n+1}$ by $A \mapsto A - \frac{\text{tr}(A)}{n}\text{Id}$. □

**Proposition 3.5** If $S^*(X_n)\tilde{G}_{n,χ} = 0$ then $S^*(s_{n+1})\tilde{G}_{n,χ} = 0$.

Proof. Consider the map $q : s_{n+1} \to ℙ$ given by $q(B) := B_{n+1,n+1}$. By the localization principle (corollary 2.4), it is enough to prove that for any $t \in ℙ$, $S^*(q^{-1}(t))\tilde{G}_{n,χ} = 0$. However, all $q^{-1}(t)$ are isomorphic as $\tilde{G}_{n}$-equivariant $t$-spaces to $X_n$ by

$$(A_{n \times n} \quad v_{n \times 1} \\
φ_{1 \times n} \quad λ) \mapsto (A + \frac{λ}{n}\text{Id}, v, φ)$$

□

This finishes the proof of proposition 3.1.

**Remark 3.6** One can give a coordinate free description of $\tilde{G}$ and of its action on $X$. Namely, $\tilde{G}$ is isomorphic to the disjoint union of the group $G = \text{Aut}(V)$ of automorphisms of $V$ and the set $\text{Iso}(V, V^*)$ of isomorphisms between $V$ and $V^*$ with the following multiplication. Let $g, g' \in \text{Aut}(V)$ and $h, h' \in \text{Iso}(V, V^*)$.

$$g \times g' := g \circ g' \quad h \times g := h \circ g$$

$$g \times h := g^{-1} \circ h \quad h \times h' := h^{-1} \circ h'$$

The action of $\tilde{G}$ on $X$ is given by

$$g(A, v, φ) = (gA, gv, (g^*)^{-1}φ) \quad \text{and} \quad h(A, v, φ) = ((hAh^{-1})^*, (h^*)^{-1}φ, hv)$$

4 **Proof of the main theorem**

Rallis and Schiffmann have proven various properties of the support of $(\tilde{G}, χ)$-equivariant distributions on $X$. We will now summarize those that we need in our proof.

**Notation 4.1** Denote the cone of nilpotent operators in $s_{\text{L}}(V)$ by $N$. Denote also

$$Y := \{(v, φ) \in V \oplus V^* | \langle φ, v \rangle = 0\}.$$  

\(^1\)This proof is analogous to the proof of an analogous statement in [Ber], section 2.2.


**Theorem 4.2 (Rallis-Schiffmann)** Suppose that the main theorem holds for all dimensions smaller than $n$ for all finite extensions $E$ of $F$. Let $\xi$ be a $(\tilde{G}, \chi)$-equivariant distribution on $X$. Then $\xi$ is supported in $N \times Y$.

We reproduce the proof of this theorem in the appendix (section 7).

Now we will stratify the nilpotent cone and reduce the support of the distribution stratum by stratum.

**Notation 4.3** For any $i$ we denote by $N_i$ the union of all nilpotent orbits of dimensions $\leq i$. Note that $N_i$ are Zariski closed, $N_i = N$ for $i$ large enough and $N_{-1} = \emptyset$.

In order to excise the support of the distribution we will use a family of automorphisms of the problem, which play a crucial role in our proof.

**Notation 4.4** For any $\lambda \in F$ we denote by $\nu_\lambda : X \to X$ the homeomorphism defined by
\[
\nu_\lambda(A, v, \phi) := (A + \lambda v \otimes \phi - \frac{\lambda \langle \phi, v \rangle}{n} \text{Id}, v, \phi).
\]

A simple but important observation is that $\nu_\lambda$ commutes with the action of $\tilde{G}$.

**Notation 4.5** Let $O$ be a nilpotent orbit of dimension $i$. We set
\[
\tilde{O} := (O \times Y) \cap \bigcap_{\lambda \in F} \nu_\lambda^{-1}(N_i \times Y).
\]

To proceed stratum by stratum we will need the following key lemma.

**Lemma 4.6 (Key lemma)** Let $O$ be a nilpotent orbit. Note that $O \times V \times V^*$ is $\tilde{G}$-invariant. Let $\zeta \in S^*(O \times V \times V^*)^{\tilde{G}, \chi}$. Suppose that $\text{Supp}(\zeta) \subset \tilde{O}$ and $\text{Supp}(F_{V \oplus V^*}(\zeta)) \subset \tilde{O}$. Then $\zeta = 0$.

The proof will be given in section 5 below.

Now we are ready to prove the main theorem.

**Theorem 4.7** Any $(\tilde{G}, \chi)$-equivariant distribution on $X$ is zero.

**Proof.** We prove by downward induction the following statement: for any $i$, any $\xi \in S^*(X)^{(\tilde{G}, \chi)}$ is supported in $N_i \times Y$. For $i$ large enough it is theorem 4.2. Suppose that the statement is true for $i$ and let us prove it for $i - 1$. Let $\xi \in S^*(X)^{(\tilde{G}, \chi)}$. We need to show that $\xi|_{(N_i \setminus N_{i-1}) \times Y} = 0$. For this it is enough to show that for any nilpotent orbit $O$ of dimension $i$, we have $\xi|_{O \times Y} = 0$.

Denote $\zeta = \xi|_{O \times V \times V^*}$. We know that $\text{Supp}(\zeta) \subset N_i \times Y$. On the other hand, $\nu_\lambda(\xi)$ is also $(\tilde{G}, \chi)$-equivariant for any $\lambda$. Therefore $\text{Supp}(\xi) \subset \bigcap_{\lambda \in F} \nu_\lambda^{-1}(N_i \times Y)$.

Thus
\[
\text{Supp}(\zeta) \subset (O \times Y) \cap \left(\bigcap_{\lambda \in F} \nu_\lambda^{-1}(N_i \times Y)\right) = \tilde{O}.
\]

Since the action of $\tilde{G}$ preserves the standard bilinear form on $V \oplus V^*$, $F_{V \oplus V^*}(\xi)$ is also $(\tilde{G}, \chi)$-equivariant. Note that $F_{V \oplus V^*}(\xi) = F_{V \oplus V^*}(\xi)|_{O \times V \times V^*}$ and hence $\text{Supp}(F_{V \oplus V^*}(\zeta))$ is also contained in $\tilde{O}$. Thus by the key lemma $\zeta = 0$.  

\[\Box\]
5 Proof of the key lemma

**Notation 5.1** Let $A \in \mathfrak{sl}(V)$ be a nilpotent element. Let $O$ be the orbit of $A$ and $i$ be the dimension of $O$. We denote by $R_A$ the fiber at $A$ of the projection $\tilde{O} \to O$. We consider $R_A$ as a subset of $V \oplus V^*$.

Note that $R_A \subset Y$.

**Notation 5.2** Let $A \in \mathfrak{sl}(V)$ be a nilpotent element. We denote $Q_A := \{(v, \phi) \in V \oplus V^* \mid v \otimes \phi \in [A, \mathfrak{gl}(V)]\}$.

**Lemma 5.3** $R_A \subset Q_A$

**Proof.** Let $(v, \phi) \in R_A$. Let $O$ be the orbit of $A$ and $i$ be the dimension of $O$. Consider the Zariski tangent space $T_A N_i$ to $N_i$ at $A$. It coincides with $T_A O = [A, \mathfrak{gl}(V)]$. Since $\langle \phi, v \rangle = 0$, we see that $N_i$ contains the line $\{A + \lambda v \otimes \phi\}$. Thus $v \otimes \phi \in T_A N_i = [A, \mathfrak{gl}(V)]$ and hence $(v, \phi) \in Q_A$.

**Notation 5.4** Let $A \in \mathfrak{sl}(V)$ be a nilpotent element. We denote by $C_A$ the stabilizer of $A$ in $G$ and by $\tilde{C}_A$ the stabilizer of $A$ in $\tilde{G}$.

It is known that $C_A$ is unimodular and hence $\tilde{C}_A$ is also unimodular.

The key lemma follows now from Frobenious reciprocity and the following proposition.

**Proposition 5.5** Let $A \in \mathfrak{sl}(V)$ be a nilpotent element. Let $\eta \in S^*(V \oplus V^*)^{C_A}$. Suppose that both $\eta$ and $F(\eta)$ are supported in $Q_A$. Then $\eta \in S^*(V \oplus V^*)^{\tilde{C}_A}$.

We will call a nilpotent element $A \in \mathfrak{sl}(V_k)$ 'nice' if the previous proposition holds for $A$. Namely, $A$ is 'nice' if any distribution $\eta \in S^*(V_k \oplus V_k^*)^{C_A}$ such that both $\eta$ and $F(\eta)$ are supported in $Q_A$ is also $\tilde{C}_A$-invariant.

Proposition 5.5 clearly follows from the following two lemmas and Jordan decomposition.

**Lemma 5.6** Let $A_1 \in \mathfrak{sl}(V_k)$ and $A_2 \in \mathfrak{sl}(V_l)$ be nice nilpotent elements. Then $A_1 \oplus A_2 \in \mathfrak{sl}(V_{k+l})$ is nice.

**Lemma 5.7** Let $A \in \mathfrak{sl}(V_r)$ be a nilpotent Jordan block. Then $A$ is nice.

5.1 Proof of lemma 5.6

We will need the following simple lemma.

**Lemma 5.8** $Q_{A_1 \oplus A_2} \subset Q_{A_1} \times Q_{A_2}$.

**Proof.** Let $(v, \phi) \in Q_{A_1 \oplus A_2}$. This means that $v \otimes \phi = [A_1 \oplus A_2, B]$. Let $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$, $v = v_1 + v_2$ and $\phi = \phi_1 + \phi_2$ be the decompositions corresponding
to the blocks of $A_1 \oplus A_2$. Then $v_1 \otimes \phi_1 = [A_1, B_{11}]$ and $v_2 \otimes \phi_2 = [A_2, B_{22}]$. Hence $(v_1, \phi_1) \in Q_{A_1}$ and $(v_2, \phi_2) \in Q_{A_2}$. 

Now let us prove lemma 5.6 Let $A_1 \in \text{sl}(V_{k_1})$ and $A_2 \in \text{sl}(V_{k_2})$ be nice operators. Let $\eta \in S^*(V_{k_1} \oplus V_{k_2} \oplus V_{k_1}^* \oplus V_{k_2}^*)^{C_{A_1} \oplus A_2}$. Suppose that both $\eta$ and $\mathcal{F}_{V_{k_1} \oplus V_{k_2}^*}(\eta)$ are supported in $Q_{A_1} \times Q_{A_2}$. We need to show that $\eta$ is $\tilde{C}_{A_1 \oplus A_2}$-invariant. Note that $\tilde{C}_{A_1}$ acts on $V_{k_1} \oplus V_{k_2} \oplus V_{k_1}^* \oplus V_{k_2}^*$. Denote

$$M := \{ \alpha \in S^*(V_{k_1} \oplus V_{k_1}^* \oplus V_{k_2} \oplus V_{k_2}^*)^{C_{A_1}} | \text{both } \alpha \text{ and } \mathcal{F}_{V_{k_1} \oplus V_{k_1}^*}(\alpha) \text{ are supported in } Q_{A_1} \times V_{k_2} \times V_{k_2}^* \}. $$

By proposition 2.2, $\eta \in M$. The following lemma follows from the fact that $A_1$ is nice using the localization principle (theorem 2.3).

**Lemma 5.9** $M = M\tilde{C}_{A_1}$. 

Therefore, $\eta$ is $\tilde{C}_{A_1}$-invariant. By similar reasons, $\eta$ is $\tilde{C}_{A_2}$-invariant. Since $\eta$ is $C_{A_1 \oplus A_2}$-invariant, we get that $\eta$ is invariant with respect to $\tilde{C}_{A_1 \oplus A_2}$. This completes the proof of lemma 5.6 ∎

### 5.2 Proof of lemma 5.7

Let $A \in \text{sl}_r$ be the standard nilpotent Jordan block.

**Notation 5.10** Denote

$$F^i := \text{Ker} A^i = \text{Im} A^{r-i}, \quad L^i := (F^{r-i})^\perp = \text{Im} (A^*)^{r-i} = \text{Ker} (A^*)^i \subset V_r^*$$

and $$Z := \bigcup_{i=0}^{r} F^i \oplus L^{r-i}.$$ 

We will first prove the following lemma from linear algebra.

**Lemma 5.11** $Q(A) \subset Z$. 

**Proof.** Let $(v, \phi) \in Q(A)$. Note that for any $i \geq 0$ and any element $B \in \text{gl}_r$, $\text{tr}(A^i[A, B]) = \text{tr}([A, A^i B]) = 0$. Hence $\langle \phi, A^i v \rangle = \text{tr}(A^i v \otimes \phi) = 0$. Therefore the spaces $W := \text{Span}\{A^i v\}$ and $\Psi := \text{Span}\{(A^*)^i \phi\}$ are orthogonal and thus $\dim W + \dim \Psi \leq r$. Denote $k := \dim W$ and $l := \dim \Psi$.

Consider the set of all non-zero vectors of the form $A^i v$. Since $A$ is nilpotent, it is easy to see that this set is linearly independent. Hence $v \in \text{Ker} A^k$ and by the same reasoning $\phi \in \text{Ker} (A^*)^l$. But since $l \leq r - k$, $\text{Ker} (A^*)^l \subset \text{Ker} (A^*)^{r-k}$. Hence $(v, \phi) \in F^k \oplus L^{r-k} \subset Z$. ∎

Now let $T : V_r \rightarrow V_r^*$ be the symmetric nondegenerate bilinear form defined by $T(e_i) = e^*_{r+1-i}$. By remark 3.6, $T$ can be viewed as an element of $\tilde{G}_r$. Since $(T A T^{-1})^* = A, T \in \tilde{G}_A$.

In order to finish the proof of lemma 5.7 it remains to prove the following lemma.
Lemma 5.12 Consider the action of $\mathbb{F}^\times$ on $V_r \oplus V_r^*$ defined by $\rho(\lambda)(v, \phi) := (\lambda v, \lambda^{-1}\phi)$. Let $\eta \in S^r(V_r \oplus V_r^*)^{\mathbb{F}^\times}$. Suppose that $T(\eta) = -\eta$ and that both $\eta$ and $\mathcal{F}(\eta)$ are supported in $Z$. Then $\eta = 0$.

Proof We will prove this lemma by induction on $r$. The case $r = 0$ is trivial. Suppose now that $r \geq 1$ and the lemma is true for all smaller $r$. By theorem 2.8, $\eta$ is abs-homogeneous of degree $r$. Consider $\eta|_{(V_r \oplus V_r^*) \setminus (F^{r-1} \oplus V_r^*)}$. Since $Z \setminus (F^{r-1} \oplus V_r^*) = (V_r \setminus F^{r-1}) \oplus \{0\}$, on $Z \setminus (F^{r-1} \oplus V_r^*)$, the action of $\mathbb{F}^\times$ coincides with homothety. Therefore $\eta|_{(V_r \oplus V_r^*) \setminus (F^{r-1} \oplus V_r^*)}$ is homothety invariant. On the other hand, it is abs-homogeneous of degree $r$. Hence it is zero. So $\eta$ is supported in $F^{r-1} \oplus V_r^*$. By the same reasons $\eta$ is supported in $V_r \oplus L^{r-1}$. Hence it is supported in $F^{r-1} \oplus L^{r-1}$.

By the same reasoning $\mathcal{F}(\eta)$ is supported in $F^{r-1} \oplus L^{r-1}$. Hence $\eta$ is invariant with respect to translations in $(F^{r-1} \oplus L^{r-1})^\perp$ which is equal to $F^1 \oplus L^1$. If $r = 1$ it implies $\eta = 0$. Otherwise it implies that $\eta$ is the pull back of a distribution $\alpha$ on the space $(F^{r-1} \oplus L^{r-1})/(F^1 \oplus L^1)$ which can be identified with $V_{r-2} \oplus V_{r-2}^*$.

It is easy to see that $\alpha$ satisfies the conditions of the lemma for dimension $r - 2$. Hence by the induction hypothesis $\alpha = 0$. \qed

6 Weil representation and proof of theorem 2.8

The goal of this section is to prove the following theorem.

Theorem 6.1 (Rallis-Schiffmann) Let $W$ be a finite dimensional vector space over $\mathbb{F}$ and $B$ be a nondegenerate symmetric bilinear form on $W$. Denote $Z(B) := \{v \in W | B(v, v) = 0\}$. Let $\xi$ be a distribution on $W$. Suppose that both $\xi$ and $\mathcal{F}_B(\xi)$ are supported in $Z_B$.

Then $\xi$ is abs-homogeneous of degree $\frac{1}{2} \dim W$.

For the proof we will need the Weil representation.

Notation 6.2 Let $t \in \mathbb{F}^\times$ be a scalar. We denote

$$a_t := \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad n_t := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \pi_t := \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$ 

The following standard lemma follows from Gauss elimination.

Lemma 6.3 The families $n_t$ and $\pi_t$ generate $\text{SL}(2, \mathbb{F})$.

The following theorem is well known.

Theorem 6.4 (Weil representation) Let $W$ be a vector space over $\mathbb{F}$ of dimension $d$. Let $B$ be a symmetric nondegenerate bilinear form on $W$. Then there exists a projective representation $\pi_B : \text{SL}(2, \mathbb{F}) \to \text{Aut}(S(W))$ such that

(i) for any $g, h \in \text{SL}(2, \mathbb{F})$, $\pi_B(gh) = u \pi_B(g) \pi_B(h)$ for some $u \in \mathbb{C}$ such that $|u| = 1$.
We denote the dual representation on $S(\pi)$.

6.1 Proof of theorem 2.8.

Since $\text{Supp}(\xi) \subset Z(B)$, we have $\pi_B(n_t)\xi = \xi$ for all $t$. 
Since $\text{Supp}(F_B(\xi)) \subset Z(B)$, we have

\[ \pi_B^*(J^{-1}n_tJ)\xi = u_1\xi \text{ where } |u_1| = 1. \]

Thus $\pi_B^*(n^{-1})\xi = u_1\xi$.
Since the families $n_t$ and $n_t$ generate $SL(2, \mathbb{F})$, we have

\[ \pi_B^*(a_t)\xi = u_2\xi \text{ where } |u_2| = 1. \]

Thus $\xi$ is abs-homogeneous of degree $\frac{\text{dim} W}{2}$. \qed

7 Appendix: Proof of theorem 4.2

The goal of this section is to prove the following theorem.

Theorem 7.1 (Rallis-Schiffmann) Suppose that the main theorem holds for all dimensions smaller than $n$ for all finite extensions $E$ of $\mathbb{F}$. Let $\xi$ be a $(\tilde{G}, \chi)$-equivariant distribution on $X$. Then $\xi$ is supported in $\mathcal{N} \times Y$.

Lemma 7.2 $\text{Supp}(\xi) \subset \text{sl}(V) \times Y$.

Proof. Let $U := X \setminus (\text{sl}(V) \times Y)$. We have to show $\mathcal{S}^*(U)\tilde{G} = 0$. Consider the map $p : U \rightarrow F^\times$ given by $p(A, v, \phi) = \langle \phi, v \rangle$. By the localization principle (corollary 2.4), it is enough to show that $\mathcal{S}^*(p^{-1}(\lambda))\tilde{G} = 0$ for any $\lambda \in F^\times$.

Fix $\lambda \in F^\times$. Denote $Z_\lambda := \{(v, \phi) \in V \oplus V^*|\langle \phi, v \rangle = \lambda\}$. Note that $Z_\lambda$ is a transitive $\tilde{G}$-equivariant $l$-space. Define $pr_2 : p^{-1}(\lambda) \rightarrow Z_\lambda$ by $pr_2(A, v, \phi) := (v, \phi)$. Note that $pr_2$ is $\tilde{G}$-equivariant. Let $\tilde{z}_0 = (e_n, \lambda e_n^*) \in Z_\lambda$ where $e_n$ is the last element of the standard basis of $V_n$ and $e_n^*$ is the last element of the dual basis of $V_n^*$.

Note that the stabilizer of $\tilde{z}_0$ is isomorphic to $\tilde{G}_{n-1}$ and the fiber $pr_2^{-1}(\tilde{z}_0)$ is isomorphic to $\text{sl}(V_n)$ as a $\tilde{G}_{n-1}$-equivariant $l$-space. Hence by Frobenious reciprocity (theorem 2.6), $\mathcal{S}^*(p^{-1}(\lambda))\tilde{G}_{n-1, \chi} = \mathcal{S}^*(\text{sl}(V_n))\tilde{G}_{n-1, \chi}$. By proposition 3.3, the main theorem for dimension $n - 1$ implies that $\mathcal{S}^*(\text{sl}(V_n))\tilde{G}_{n-1, \chi} = 0$. \qed

Lemma 7.3 Let $A \in \text{sl}(V)$ be a non-zero semisimple element. Let $C_A$ be the stabilizer of $A$ in $G$, and $\tilde{C}_A$ be the stabilizer of $A$ in $\tilde{G}$. Let $\mathcal{N}_A \subset \mathcal{N}$ be the subset of all nilpotent operators that commute with $A$. Then $\mathcal{S}^*(\mathcal{N}_A \times V \times V^*)^{C_A} = \mathcal{S}^*(\mathcal{N}_A \times V \times V^*)^{\tilde{C}_A}$.
Proof. It is known that a centralizer of a semisimple element is a Levi subgroup. Hence $C_A$ is isomorphic to $\prod_i G_{k_i}(\mathbb{E}_i)$ where $k_i < n$ are certain natural numbers and $\mathbb{E}_i$ are certain finite extensions of $\mathbb{F}$. It is easy to see (e.g. using remark 3.6) that $\tilde{C}_A$ can be identified with a subgroup of $\prod_i \tilde{G}_{k_i}(\mathbb{E}_i)$.

Therefore, the main theorem for $k_i$ and $\mathbb{E}_i$ and corollary 2.5 of the localization principle imply $S^*(\prod_i X_{k_i}(\mathbb{E}_i))^{C_A} = S^*(\prod_i X_{k_i}(\mathbb{E}_i))^{C_A}$.

The lemma follows now from the fact that $N_A \times V \times V^*$ can be identified with a closed subset of $\prod_i X_{k_i}(\mathbb{E}_i)$.

Proof of theorem 4.2. By lemma 7.2, $\xi$ is supported in $sl(V) \times Y$. Hence it is left to show that $\xi$ is supported in $N \times V \times V^*$.

Let $X := X_n$ be the set of all monic polynomials of degree $n$ in variable $\lambda$. Consider the map $\Delta := \Delta_n : X_n \rightarrow X_n$ that maps $(A, v, \phi)$ to the characteristic polynomial of $A$. By the localization principle (corollary 2.4) it is enough to show $S^*(\Delta^{-1}(P))^{\tilde{G} \times \chi} = 0$ for any $P \neq \lambda^n \in X_n$.

Let $\mathcal{R} \subset sl(V)$ be the set of all semisimple operators with characteristic polynomial $P$. Note that $\tilde{G}$ acts transitively on $\mathcal{R}$. We recall that by the Jordan decomposition theorem any operator $A$ can be decomposed in a unique way into a sum of commuting operators $A_s$ and $A_n$ such that $A_s$ is semi-simple and $A_n$ is nilpotent. This defines a map $J : \Delta^{-1}(P) \rightarrow \mathcal{R}$ by $J(A) := A_s$. It is easy to see that $J$ is continuous and $\tilde{G}$-equivariant.

The theorem follows now from lemma 7.3 by Frobenious reciprocity. \hfill \Box

References

[Ber] J. Bernstein: *P-invariant Distributions on GL(N) and the classification of unitary representations of GL(N) (non-archimedean case)* Lie group representations, II (College Park, Md., 1982/1983), 50–102, Lecture Notes in Math., \textbf{1041}, Springer, Berlin (1984).

[BZ] J. Bernstein, A.V. Zelevinsky: *Representations of the group GL(n, F), where F is a local non-Archimedean field.* Uspekhi Mat. Nauk \textbf{10}, No.3, 5-70 (1976).

[Gel] S. Gelbart: *Weil’s Representation and the Spectrum of the Metaplectic Group,* Lecture Notes in Math., \textbf{530}, Springer, Berlin-New York (1976).

[RS] S. Rallis, G. Schiffmann: *Multiplicity one Conjectures*, arXiv:0705.2168v1 [math.RT]

A. Aizenbud
Faculty of Mathematics and Computer Science, Weizmann Institute of Science POB 26, Rehovot 76100, ISRAEL and Hausdorff Institute for Mathematics, Bonn.
E-mail: aizenr@yahoo.com

D. Gourevitch
Faculty of Mathematics and Computer Science, Weizmann Institute of Science POB 26, Rehovot 76100, ISRAEL and Hausdorff Institute for Mathematics, Bonn.
E-mail: guredim@yahoo.com.