Asymptotic density of Motzkin numbers modulo small primes

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Abstract

We establish the asymptotic density of the Motzkin numbers modulo 2, 4, 8, 3 and 5.

1 Introduction

The Motzkin numbers $M_n$ are defined by

$$M_n := \sum_{k \geq 0} \binom{n}{2k} C_k$$

where $C_k$ are the Catalan numbers.

There has been some work in recent years on analysing the Motzkin numbers $M_n$ modulo primes and prime powers. This work has often been done in concert with and using the same methods as work analysing the Catalan numbers. Deutsch and Sagan [1] provided a characterisation of Motzkin numbers divisible by 2, 4 and 5. They also provided a complete characterisation of the Motzkin numbers modulo 3 and showed that no Motzkin number is divisible by 8. Eu, Liu and Yeh [2] reproved some of these results and extended them to include criteria for when $M_n$ is congruent to $\{2, 4, 6\}$ mod 8. Krattenthaler and Müller [4] established identities for the Motzkin numbers modulo higher powers of 3 which include the modulo 3 result of [1] as a special case. Krattenthaler and Müller [3] have more recently extended this work to a full characterisation of $M_n$ mod 8 in terms of the binary expansion of $n$. Their characterisation is rather elaborate and less susceptible to analysis than that provided in [2]. The results in [4] and [3] are obtained by expressing the generating function of $M_n$ as a polynomial involving a special function. Rowland and Yassawi [5] investigated $M_n$ in the general setting of automatic sequences. The values of $M_n$ (as well as other sequences) modulo prime powers can be computed via automata.
Rowland and Yassawi provided algorithms for creating the relevant automata. They established results for \( M_n \) modulo small prime powers, including a full characterisation of \( M_n \) modulo 8 (modulo \( 5^2 \) and \( 13^2 \) are available from Rowlands website). They also established that 0 is a forbidden residue for \( M_n \) modulo 8, \( 5^2 \) and \( 13^2 \). In theory the automata can be constructed for any prime power but computing power and memory quickly becomes a barrier. For example, the automata for \( M_n \) modulo \( 13^2 \) has over 2000 states. Rowland and Yassawi also went on to describe a method for obtaining asymptotic densities of \( M_n \).

We will use the above results to establish asymptotic densities of \( M_n \) modulo 2, 4, 8, 3 and 5. Here, the asymptotic density of a subset \( S \) of \( \mathbb{N} \) is defined to be

\[
\lim_{N \to \infty} \frac{1}{N} \# \{ n \in S : n \leq N \}
\]

if the limit exists, where \( \#S \) is the number of elements in a set \( S \). In contrast to the results for the Catalan numbers \( C_n \), the set of Motzkin numbers congruent to 0 mod \( n \) is not expected to have asymptotic density 1 for a general \( n \in \mathbb{N} \). The results here show that this expectation holds for small values of \( n \).

## 2 Asymptotic density of certain forms of numbers

The main method in the literature of characterising \( M_n \mod q \) is to divide the natural numbers into classes of the form

\[
S(q, r, s, t) = \{(qi + r)q^{sj+t} + c : i, j \in \mathbb{N}\}
\]

for various choices of \( r, s, t \) and \( c \). It will therefore be useful to know how these types of sets behave asymptotically. We can disregard the \( c \) term as this does not change the asymptotic behaviour. So the set of interest is

\[
S(q, r, s, t) = \{(qi + r)q^{sj+t} : i, j \in \mathbb{N}\}
\]

for integers \( q, r, s, t \).

**Theorem 1.** Let \( q, r, s, t \in \mathbb{Z} \) with \( q, s > 0 \), \( t \geq 0 \) and \( 0 \leq r < q \). Then the asymptotic density of the set \( S \) is \( (q^{s+1} - (q^s - 1))^{-1} \).

**Proof.** We have, for fixed \( j \geq 0 \),

\[
\# \{ i \geq 0 : (qi + r)q^{sj+t} \leq N \} = \frac{N}{q^{sj+t+1}} - \frac{r}{q} - E(j, N, q, r, s, t)
\]

where \( 0 \leq E(j, N, q, r, s, t) < 1 \) is an error term introduced by not rounding down to the nearest integer. So, letting

\[
U(N, s, t) := \left\lfloor \frac{\log_q(N) - t - 1}{s} \right\rfloor
\]
, we have
\[
\#\{ n < N : n = (q_i + r) q_j \text{ for some } i, j \in \mathbb{N} \}
\]
\[
= \sum_{j \geq 0} \left( \frac{N}{q^{s_j+1}} - \frac{r}{q} - E(j, N, q, r, s, t) \right)
\]
\[
= \sum_{j=0}^{U} \left( \frac{N}{q^{s_j+1}} - \frac{r}{q} - E(j, N, q, r, s, t) \right)
\]
\[
= \frac{N}{q^{t+1}} \sum_{j=0}^{U} \left( \frac{1}{q^j} \right)^{j} - E'(N, q, r, s, t)
\]
where the new error term $E'(N, q, r, s, t)$ satisfies
\[
0 < E'(N, q, r, s, t) < 2(U + 1).
\]
Then
\[
\#\{ n < N : n = (q_i + r) q_j \text{ for some } i, j \in \mathbb{N} \}
\]
\[
= \left( \frac{N}{q^{t+1}} \right) \left( 1 - \left( \frac{1}{q^s} \right)^{U+1} \right) \left( 1 - \frac{1}{q^s} \right)^{-1} - E'.
\]
Since $\lim_{N \to \infty} \frac{E'(N, q, r, s, t)}{N} = 0$ and $\lim_{N \to \infty} \frac{1}{N} \left( \frac{1}{q^s} \right)^{U+1} = 0$ we have
\[
\lim_{N \to \infty} \frac{1}{N} \#\{ n < N : n = (q_i + r) q_j \text{ for some } i, j \in \mathbb{N} \}
\]
\[
= \left( q^{t+1-s} (q^s - 1) \right)^{-1}.
\]
\[
\square
\]
**Remark.** Sometimes we will need to consider the set
\[
S'(q, r, s, t) = \{(q_i + r) q_j : i, j \in \mathbb{N}, j \geq 1 \}
\]
for integers $q, r, s, t$. The asymptotic density of the set $S'$ can be derived from theorem 1 as $\left( q^{t+1}(q^s - 1) \right)^{-1}$.

3 Motzkin numbers modulo 2, 4 and 8

The following result is established in [2]
Theorem 2. (Theorem 5.5 of [2]). The $n$th Motzkin number $M_n$ is even if and only if

$$n = (4i + \epsilon)4^j + 1 - \delta$$

for $i, j \in \mathbb{N}, \epsilon \in \{1, 3\}$ and $\delta \in \{1, 2\}$.

Moreover, we have

$$M_n \equiv 4 \mod 8 \text{ if } (\epsilon, \delta) = (1, 1) \text{ or } (3, 2)$$

$$M_n \equiv 4y + 2 \mod 8 \text{ if } (\epsilon, \delta) = (1, 2) \text{ or } (3, 1)$$

where $y$ is the number of digit 1s in the base 2 representation of $4i + \epsilon - 1$.

Remark. The 4 choices of $(\epsilon, \delta)$ in the above theorem give 4 disjoint sets of numbers $n = (4i + \epsilon)4^j + 1 - \delta$.

Theorem 3. Each of the 4 disjoint sets defined by the choice of $(\epsilon, \delta)$ in Theorem 2 has asymptotic density $\frac{1}{12}$ in the natural numbers.

Proof. Use the result of Theorem 1 for the set $S$ with $q = 4, r = \epsilon, s = 1, t = 1$. \qed

Corollary 4. The asymptotic density of

$$\{n < N : M_n \equiv 0 \mod 2\}$$

is $\frac{1}{3}$.

The asymptotic density of

$$\{n < N : M_n \equiv 4 \mod 8\}$$

is $\frac{1}{6}$.

The asymptotic density of each the sets

$$\{n < N : M_n \equiv 2 \mod 8\} \text{ and } \{n < N : M_n \equiv 6 \mod 8\}$$

is $\frac{1}{12}$.

Proof. The first 2 statements of the corollary follow immediately from theorem 2 and theorem 3. The third statement follows from the observation that the numbers of 1’s in the base 2 expansion of $i$ is equally likely to be odd or even and therefore the same applies to the the number of 1’s in the base 2 expansion of $4i + \epsilon - 1$. Since the asymptotic density of the 2 sets combined is $\frac{1}{6}$ (from theorem 3), each of the two sets has asymptotic density $\frac{1}{12}$. \qed

Remark. Rowland and Yassawi [5] proved the first two results of the corollary and also established that the asymptotic density of the sets of $M_n$ congruent to 2 modulo 4 is $\frac{1}{6}$. 

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4 Motzkin numbers modulo 5

The following result is established in [1]

**Theorem 5.** (Theorem 5.4 of [1]). The Motzkin number $M_n$ is divisible by 5 if and only if $n$ is one of the following forms

$$(5i + 1)5^{2j} - 2, (5i + 2)5^{2j-1} - 1, (5i + 3)5^{2j-1} - 2, (5i + 4)5^{2j} - 1$$

where $i, j \in \mathbb{N}$ and $j \geq 1$.

**Theorem 6.** The asymptotic density of numbers of the first form in theorem 5 is $1/120$. Numbers of the fourth form also have asymptotic density $1/120$. The asymptotic density of numbers of the second and third forms in theorem 5 is $1/24$ each.

**Proof.** Firstly consider numbers of the form $(5i + r)5^{2j} - 2$. As we are interested in asymptotic density it is enough to look at numbers of the form $(5i + r)5^{2j}$. We can now use the remark 2 at the end of theorem 1 for the set $S'$ with $q = 5, s = 2$ and $t = 0$. From the remark the asymptotic density of the set

$$\{ n \in \mathbb{N} : n = (5i + r)5^{2j} \text{ with } i, j \in \mathbb{N} \text{ and } j \geq 1 \}$$

is $(5 \times (5^2 - 1))^{-1} = \frac{1}{120}$. For numbers of the second and third forms we shift the $j$ index so that it starts from 0 and use theorem 1 for the set $S$ with $q = 5, s = 2$ and $t = 1$. From theorem 1 the asymptotic density of the set

$$\{ n \in \mathbb{N} : n = (5i + r)5^{2j+1} \text{ with } i, j \in \mathbb{N} \text{ and } j \geq 0 \}$$

is $(5^0(5^2 - 1))^{-1} = \frac{1}{24}$.

**Corollary 7.** The asymptotic density of $\#\{ n < N : M_n \equiv 0 \mod 5 \}$ is $1/10$.

**Proof.** The corollary follows immediately from theorem 5 and theorem 6 and the disjointness of the 4 forms of integers listed in theorem 5.

**Remark.** Numerical tests also show that roughly 22.5% of Motzkin numbers are congruent to each of $1, 2, 3, 4 \mod 5$.

5 Motzkin numbers modulo 3

The structure of the Motzkin numbers modulo 3 is based on a set $T(01)$ which was defined by Deutsch and Sagan in [1]. The set $T(01)$ is the set of natural numbers which have a base 3 expansion containing only the digits 0 and 1. The following theorem from [1] will be used in this section.
Theorem 8. (Corollary 4.10 of [1]). The Motzkin numbers satisfy
\[ M_n \equiv -1 \mod 3 \text{ if } n \in 3T(01) - 1, \]
\[ M_n \equiv 1 \mod 3 \text{ if } n \in 3T(01) \text{ or } n \in 3T(01) - 2, \]
\[ M_n \equiv 0 \mod 3 \text{ otherwise.} \]

We will first examine the nature of the set \( T(01) \). We have,

**Theorem 9.** The asymptotic density of the set \( T(01) \) is zero.

**Proof.** Let \( N \in \mathbb{N} \) and choose \( k \in \mathbb{N} : 3^k \leq N < 3^{k+1} \). Then \( k = \lfloor \log_3(N) \rfloor \) and
\[
\frac{1}{N} \# \{ n \leq N : n \in T(01) \} \leq \frac{2^{k+1}}{N} \leq \frac{2^{k+1}}{3^k} \to 0 \text{ as } N \to \infty.
\]

**Theorem 10.** The asymptotic density of the set \( \{ n \leq N : M_n \equiv 0 \mod 3 \} \) is 1.

**Proof.** Since the asymptotic density of \( T(01) \) is zero, so is the asymptotic density of the sets \( 3T(01) - k \) for \( k \in \{0, 1, 2\} \). Therefore theorem 8 implies that
\[
\lim_{N \to \infty} \frac{1}{N} \# \{ n \leq N : M_n \equiv \pm 1 \mod 3 \} = 0
\]
and the result follows.

**References**

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