ON NON-WEIGHT REPRESENTATIONS OF THE UNTWISTED $N = 2$ SUPERCONFORMAL ALGEBRAS

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Abstract. In this paper, we construct a family of non-weight modules over the untwisted $N = 2$ superconformal algebras. Those modules when regarded as modules over the Cartan subalgebra are free of rank 2. We give a classification of isomorphism classes of such modules. Moreover, all submodules of such modules are precisely determined. In particular, those modules we construct are not simple.

Key words: untwisted $N = 2$ superconformal algebra, Cartan subalgebra, non-weight module

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1. Introduction

Superconformal algebras were first constructed by Ademollo et al. [1] and Kac [12] from the point of view of mathematics and physics respectively in 1970s. They are the supersymmetry extensions of the Virasoro algebra, and play important roles in conformal field theory and string theory. In 2002, Fattori and Kac [9] (see also [13]) gave a complete classification of superconformal algebras. Among those, the $N = 2$ superconformal algebras play a fundamental role in the mirror symmetry theory.

The $N = 2$ superconformal algebras fall into four types: the Ramond $N = 2$ algebra, the Neveu-Schwarz $N = 2$ algebra, the topological $N = 2$ algebra, and the twisted $N = 2$ algebra (i.e. with mixed boundary conditions for the fermionic fields). The first three algebras are isomorphic, and called the untwisted $N = 2$ superconformal algebras. The Ramond $N = 2$ algebra and the Neveu-Schwarz $N = 2$ algebra are isomorphic by the spectral flow map [20]. As the symmetry algebra of topological conformal field theory in two dimensions, the topological $N = 2$ algebra was presented by Dijkgraaf, Verlinde and Verlinde [6] in 1991. This algebra can be obtained from the Neveu-Schwarz $N = 2$ algebra through modifying the stress-energy tensor by adding the derivative of the $U(1)$ current procedure known as “topological twist” (see [7, 22]).

The representation theory of superconformal algebras, including weight representations and non-weight representations, is of interest to both mathematicians and physicists. It is known that representations of the superconformal algebras get more and more complicated

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with increasing the number of fermionic currents $N$. The classification of all simple Harish-Chandra modules over the $N = 1$ and untwisted $N = 2$ superconformal algebras was achieved respectively in [21, 16]. Moreover, some special weight modules over the $N = 2$ superconformal algebras were studied (see [8, 11, 14, 19, 25]).

Recently, some authors constructed and studied an important class of non-weight modules on which the Cartan subalgebra acts freely. These modules are called free $U(\mathfrak{h})$-modules, where $U(\mathfrak{h})$ is the universal enveloping algebra of the Cartan subalgebra $\mathfrak{h}$. Free $U(\mathfrak{h})$-modules were first constructed by Nilsson for the complex matrices algebra $\mathfrak{sl}_{n+1}$ in [17]. After that, such modules for finite-dimensional simple Lie algebras [18] and some infinite-dimensional Lie algebras, such as the Virasoro algebra [2, 23], the Heisenberg-Virasoro algebra [3], the algebra $\text{Vir}(a, b)$ [10], the Schrodinger-Virasoro algebra [26] and the Block algebra [4, 15], have been studied. The paper [5] studied free $U(\mathfrak{h})$-modules over the basic Lie superalgebras. It was shown that $\mathfrak{osp}(1|2n)$ is the only basic Lie superalgebra that admits such modules. The $N = 1$ superconformal algebras fall into two types: the Ramond $N = 1$ algebra and the Neveu-Schwarz $N = 1$ algebra. In [24], the authors gave a complete classification of free $U(\mathfrak{h})$-modules of rank 1 over the Ramond $N = 1$ algebra, and free $U(\mathfrak{h})$-modules of rank 2 over the Neveu-Schwarz $N = 1$ algebra. In this paper we aim to study free $U(\mathfrak{h})$-modules over the untwisted $N = 2$ superconformal algebras.

The present paper is organized as follows. In Section 2, we recall the definition of the $N = 2$ superconformal algebras and construct a family of non-weight modules over the Ramond $N = 2$ superconformal algebra. Section 3 is devoted to studying free $U(\mathfrak{h})$-modules of rank 2 over the Ramond $N = 2$ superconformal algebra. To be precise, we classify the free $U(\mathfrak{h})$-modules of rank 2 over the Ramond $N = 2$ supeconformal algebra, and determine the isomorphism classes of such modules. Moreover, all submodules of such modules are precisely determined. In particular, those modules we construct are not simple.

2. A family of non-weight modules over $\mathcal{R}$

Throughout the paper, we denote by $\mathbb{C}, \mathbb{C}^*, \mathbb{Z}$ and $\mathbb{N}$ the sets of all complex numbers, nonzero complex numbers, integers and positive integers, respectively. We always assume that the base field is the complex number field $\mathbb{C}$. All vector superspaces (resp. superalgebras, supermodules) $V = V_0 \oplus V_1$ are defined over $\mathbb{C}$, and sometimes simply called spaces (resp. algebras, modules). We call elements in $V_0$ and $V_1$ odd and even, respectively. Both odd and even elements are referred to homogeneous ones. Throughout this paper, a module $M$ of a superalgebra $A$ always means a supermodule, i.e., $A_i \cdot M_j \subseteq M_{i+j}$ for all $i, j \in \mathbb{Z}_2$. There is a parity change functor $\Pi$ from the category of $A$-modules to itself. That is, for any module $M = M_0 \oplus M_1$, we have a new module $\Pi(M)$ with the same underlining space with the parity exchanged, i.e., $(\Pi(M))_0 = M_1$ and $(\Pi(M))_1 = M_0$.

In this section, we construct a family of non-weight modules over the untwisted $N = 2$ superconformal algebra of Ramond type, which are actually free of rank 2 when restricted as modules over the Cartan subalgebra.
Let us first recall the definition of the untwisted $N = 2$ superconformal algebras, which include three types, that is, the Ramond, the Neveu-Schwarz and the topological $N = 2$ superconformal algebras.

**Definition 2.1.** (cf. [1]) Let $\mathcal{L}$ be an infinite dimensional Lie superalgebra whose even part is spanned by $\{L_m, H_m, C \mid m \in \mathbb{Z}\}$ and odd part is spanned by $\{G^\epsilon_r \mid r \in \epsilon + \mathbb{Z}\}$ ($\epsilon = 0$ or $\frac{1}{2}$) subject to the following relations:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}C, \quad [L_m, H_n] = -nH_{m+n},$$

$$[H_m, H_n] = \frac{1}{3}m\delta_{m+n,0}C, \quad [L_m, G^\pm_r] = (\frac{1}{2}m - r)G^\pm_{m+r},$$

$$[H_m, G^\pm_r] = \pm G^\pm_{m+r}, \quad [G^-_r, G^+_s] = 2L_{r+s} - (r-s)H_{r+s} + \frac{1}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}C,$$

$$[G^+_r, G^-_s] = [G^+_r, G^+_s] = 0, \quad [\mathcal{L}, C] = 0$$

for $m, n \in \mathbb{Z}$, $r, s \in \epsilon + \mathbb{Z}$. If $\mathcal{L} = \text{span}_\mathbb{C} \{L_m, H_m, G^\pm_m, C \mid m \in \mathbb{Z}\}$, it is called the Ramond $N = 2$ superconformal algebra, and denoted by $\mathcal{R}$. If $\mathcal{L} = \text{span}_\mathbb{C} \{L_m, H_m, G^\pm_m, C \mid m \in \mathbb{Z}, r \in \frac{1}{2} + \mathbb{Z}\}$, it is called the Neveu-Schwarz $N = 2$ superconformal algebra, and denoted by $\mathcal{NS}$.

Let $\sigma: \mathcal{NS} \to \mathcal{R}$ be the spectral flow map (see [20]) defined by

$$L_m \mapsto L_m + \frac{1}{2}H_m + \frac{1}{24}\delta_{m,0}C, \quad H_m \mapsto H_m + \frac{1}{6}\delta_{m,0}C, \quad G^\pm_r \mapsto G^\pm_{r + \frac{1}{2}}C, \quad C \mapsto C,$$

for $m \in \mathbb{Z}, r \in \frac{1}{2} + \mathbb{Z}$. It is straightforward to show that $\sigma$ is an isomorphism between the Neveu-Schwarz $N = 2$ superconformal algebra and the Ramond $N = 2$ superconformal algebra.

**Definition 2.2.** (cf. [6]) The topological $N = 2$ superconformal algebra is a Lie superalgebra

$$\mathcal{T} = \bigoplus_{m \in \mathbb{Z}} \mathbb{C}L_m \bigoplus_{m \in \mathbb{Z}} \mathbb{C}H_m \bigoplus_{m \in \mathbb{Z}} \mathbb{C}G_m \bigoplus_{m \in \mathbb{Z}} \mathbb{C}Q_m \bigoplus \mathbb{C}C$$

with the following brackets:

$$[L_m, L_n] = (m-n)L_{m+n}, \quad [L_m, H_n] = -nH_{m+n} + \frac{1}{6}(m^2 + m)\delta_{m+n,0}C,$$

$$[H_m, H_n] = \frac{1}{3}m\delta_{m+n,0}C, \quad [L_m, G_n] = (m-n)G_{m+n},$$

$$[L_m, Q_n] = -nQ_{m+n}, \quad [H_m, G_n] = G_{m+n},$$

$$[H_m, Q_n] = -Q_{m+n}, \quad [G_m, Q_n] = 2L_{m+n} - 2nH_{m+n} + \frac{1}{3}(m^2 + m)\delta_{m+n,0}C,$$

$$[G_m, G_n] = [Q_m, Q_n] = 0, \quad [\mathcal{T}, C] = 0$$

for $m, n \in \mathbb{Z}$. 

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It is also known that the following map from $T$ to $R$ defined by
\begin{equation}
L_m \mapsto L_m + \frac{1}{2}(m + 1)H_m + \frac{1}{8}\delta_m C, \quad H_m \mapsto H_m + \frac{1}{6}\delta_m C, \quad G_m \mapsto G^+_{m+1}, \quad Q_m \mapsto G^-_{m-1}, \quad C \mapsto C,
\end{equation}
is an isomorphism, where $m \in \mathbb{Z}$. Indeed, it is the composition of the inverse of the topological twists $\tau$ from $\mathcal{NS}$ to $T$ (see [7, 22]) defined by
\begin{equation}
L_m \mapsto L_m - \frac{1}{2}(m + 1)H_m, \quad H_m \mapsto H_m, \quad G^+_{m+1} \mapsto G_m, \quad G^-_{m-1} \mapsto Q_m, \quad C \mapsto C,
\end{equation}
and the map $\sigma$ defined in (2.1). Thus the Ramond, the Neveu-Schwarz and the topological $N = 2$ superconformal algebras are isomorphic. They are called \textit{untwisted $N = 2$ superconformal algebras}. In this paper, we consider the Ramond $N = 2$ superconformal algebra as the representative of the untwisted case. More precisely, we study free $U(\mathfrak{h})$-modules over the Ramond $N = 2$ superconformal algebra $R$, where $\mathfrak{h} = \mathbb{C}L_0 \oplus \mathbb{C}H_0$ is the canonical Cartan subalgebra of $R$. Since $[L_0, H_0] = 0$, we have $U(\mathfrak{h}) = \mathbb{C}[L_0, H_0]$.

Let $\mathbb{C}[x, y]$ and $\mathbb{C}[s, t]$ be the polynomial algebras in the indeterminates $x, y$ and $s, t$, respectively. For $\lambda, \alpha \in \mathbb{C}^*$, denote by $\Omega(\lambda, \alpha) = \mathbb{C}[x, y] \oplus \mathbb{C}[s, t]$ the $\mathbb{Z}_2$-graded vector space with $\Omega(\lambda, \alpha)_0 = \mathbb{C}[x, y]$ and $\Omega(\lambda, \alpha)_1 = \mathbb{C}[s, t]$. For any $m \in \mathbb{Z}$, $f(x, y) \in \mathbb{C}[x, y]$ and $g(s, t) \in \mathbb{C}[s, t]$, we define the action of $R$ on $\Omega(\lambda, \alpha)$ as follows
\begin{align}
L_m f(x, y) &= \lambda^m (x + \frac{1}{2}my) f(x + m, y), \quad L_m g(s, t) = \lambda^m (s + \frac{1}{2}mt + m) g(s + m, t), \\
H_m f(x, y) &= \lambda^m y f(x + m, y), \quad H_m g(s, t) = \lambda^m t g(s + m, t), \\
G^+_m f(x, y) &= 0, \quad G^+_m g(s, t) = \lambda^m \frac{2}{\alpha} (x + my) g(x + m, y - 1), \\
G^-_m f(x, y) &= \lambda^m \alpha f(s + m, t + 1), \quad G^-_m g(s, t) = 0, \\
C f(x, y) &= C g(s, t) = 0.
\end{align}

\textbf{Proposition 2.3.} For $\lambda, \alpha \in \mathbb{C}^*$, $\Omega(\lambda, \alpha)$ is an $R$-module under the action defined by (2.3)-(2.7). Moreover, $\Omega(\lambda, \alpha)$ is free of rank 2 as a module over $\mathbb{C}[L_0, H_0]$.

\textbf{Proof.} For any $m, n \in \mathbb{Z}$, $f(x, y) \in \mathbb{C}[x, y], g(s, t) \in \mathbb{C}[s, t]$, by (2.3), we have
\begin{align*}
L_m L_n f(x, y) &= \lambda^n L_m (x + \frac{1}{2}ny) f(x + n, y) \\
&= \lambda^{m+n} (x + \frac{1}{2}my)(x + \frac{1}{2}ny + m) f(x + m + n, y),
\end{align*}
and
\begin{align*}
L_m L_n g(s, t) &= \lambda^n L_m (s + \frac{1}{2}nt + n) g(s + n, t) \\
&= \lambda^{m+n} (s + \frac{1}{2}mt + m)(s + \frac{1}{2}nt + m + n) g(s + m + n, t),
\end{align*}

which implies that

\[ L_m L_n f(x, y) - L_n L_m f(x, y) \]

\[ = \lambda^{m+n}((x + \frac{1}{2} my)(x + \frac{1}{2} ny + m) - (x + \frac{1}{2} my)(x + \frac{1}{2} my + n)) f(x + m + n, y), \]

\[ = \lambda^{m+n}(m - n)((x + \frac{1}{2}(m + n)y)) f(x + m + n, y) \]

\[ = ((m - n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}C) f(x, y), \]

and

\[ L_m L_n g(s, t) - L_n L_m g(s, t) \]

\[ = \lambda^{m+n}((s + \frac{1}{2} mt + m)(s + \frac{1}{2} nt + m + n) \]

\[ - (s + \frac{1}{2} nt + n)(s + \frac{1}{2} mt + m + n)) g(s + m + n, t) \]

\[ = \lambda^{m+n}(m - n)((s + \frac{1}{2}(m + n)t + m + n)) g(s + m + n, t) \]

\[ = ((m - n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}C) g(s, t). \]

If we take into account (2.3) and (2.4), it follows that

\[ L_m H_n f(x, y) - H_n L_m f(x, y) \]

\[ = \lambda^n L_m y f(x + n, y) - \lambda^m H_n(x + \frac{1}{2} my) f(x + m, y) \]

\[ = \lambda^{m+n}((x + \frac{1}{2} my) - (x + \frac{1}{2} my + n)) y f(x + m + n, y) \]

\[ = -n\lambda^{m+n} y f(x + m + n, y) \]

\[ = -nH_{m+n} f(x, y). \]

Similarly, it is easy to prove that

\[ L_m H_n g(s, t) - H_n L_m g(s, t) = -nH_{m+n} g(s, t). \]

It follows from (2.3) and (2.5) that

\[ L_m G_n^+ g(s, t) = \frac{2}{\alpha} \lambda^n L_m(x + ny) g(x + n, y - 1) \]

\[ = \frac{2}{\alpha} \lambda^{m+n}(x + \frac{1}{2} my)(x + ny + m) g(x + m + n, y - 1), \]

and

\[ G_n^+ L_m g(s, t) = \lambda^m G_n^+(s + \frac{1}{2} mt + m) g(s + m, t) \]

\[ = \frac{2}{\alpha} \lambda^{m+n}(x + ny)(x + \frac{1}{2} my + \frac{1}{2} m + n) g(x + m + n, y - 1). \]
Hence,
\[
L_m G^+_n g(s, t) - G^+_n L_m g(s, t)
= \frac{2}{\alpha} \lambda^{m+n} \left( (x + \frac{1}{2} my)(x + ny + m) \right.
\left. - (x + ny)(x + \frac{1}{2} my + \frac{1}{2} m + n) \right) g(x + m + n, y - 1)
= \left( \frac{1}{2} m - n \right) \frac{2}{\alpha} \lambda^{m+n} (x + (m+n)y) g(x + m + n, y - 1)
= \left( \frac{1}{2} m - n \right) G^+_{m+n} g(s, t).
\]

According to (2.3), (2.4) and (2.5), we obtain
\[
L_m G^-_n f(x, y) - G^-_n L_m f(x, y)
= \alpha \lambda^n L_m f(s + n, t + 1) - \lambda^m G^-_n (x + \frac{1}{2} my) f(x + m, y)
= \alpha \lambda^{m+n} \left( s + \frac{1}{2} mt + m - (s + n + \frac{1}{2} m(t + 1)) \right) f(s + m + n, t + 1)
= \left( \frac{1}{2} m - n \right) \alpha \lambda^{m+n} f(s + m + n, t + 1)
= \left( \frac{1}{2} m - n \right) G^-_{m+n} f(x, y),
\]
and
\[
H_m G^+_n g(s, t) - G^+_n H_m g(s, t)
= \frac{2}{\alpha} \lambda^n H_m (x + ny) g(x + n, y - 1) - \lambda^m G^+_n t g(s + m, t)
= \frac{2}{\alpha} \lambda^{m+n} \left( (x + ny + m) - (x + ny)(y - 1) \right) g(x + m + n, y - 1)
= \frac{2}{\alpha} \lambda^{m+n} (x + (m+n)y) g(x + m + n, y - 1)
= G^+_{m+n} g(s, t).
\]

In a similar way one sees that
\[
[H_m, G^-_n] f(x, y) = -G^-_{m+n} f(x, y).
\]

By (2.3)-(2.7), we have
\[
G^-_m G^+_n f(x, y) + G^+_n G^-_m f(x, y)
= \alpha \lambda^m G^+_n f(s + m, t + 1)
= 2 \lambda^{m+n} (x + ny) f(x + m + n, y),
\]
and
\[
2L_{m+n}f(x, y) - (m - n)H_{m+n}f(x, y) + \frac{1}{3}(m^2 - \frac{1}{4})\delta_{m+n,0}\delta_{m+n,0} f(x, y)
\]
\[= \lambda^{m+n}(2(x + \frac{1}{2}(m+n)y) - (m-n)y)f(x + m + n, y)
\]
\[= 2\lambda^{m+n}(x + ny)f(x + m + n, y).
\]
That is
\[
[G_m^-, G_n^+]f(x, y) = (2L_{m+n} - (m - n)H_{m+n} + \frac{1}{3}(m^2 - \frac{1}{4})\delta_{m+n,0}\delta_{m+n,0}) f(x, y).
\]

A similar computation yields that
\[
[G_m^-, G_n^+]g(s, t) = (2L_{m+n} - (m - n)H_{m+n} + \frac{1}{3}(m^2 - \frac{1}{4})\delta_{m+n,0}\delta_{m+n,0}) g(s, t).
\]

Moreover, it is easy to show that
\[
[L_m, G_n^+]f(x, y) = (\frac{1}{2}m - n)G_n^+ f(x, y) = 0, \quad [H_m, G_n^+]f(x, y) = G_n^+ f(x, y) = 0;
\]
\[
[H_m, G_n^-]g(s, t) = -G_n^- g(s, t) = 0, [H_m, H_n]f(x, y) = [H_m, H_n]g(s, t) = 0;
\]
\[
[L_m, G_n^-]g(s, t) = (\frac{1}{2}m - n)G_n^- g(s, t) = 0,
\]
\[
[G_m^+, G_n^+]f(x, y) = [G_m^-, G_n^-]f(x, y) = [G_m^+, G_n^+]g(s, t) = [G_m^-, G_n^-]g(s, t) = 0,
\]
and
\[
[R, C]f(x, y) = [R, C]g(s, t) = 0.
\]

In conclusion, we have shown that \(\Omega(\lambda, \alpha)\) is an \(R\)-module. By taking \(m = 0\) in (2.3) and (2.4), we see that \(\Omega(\lambda, \alpha)\) is free of rank 2 as a module over \(\mathbb{C}[L_0, H_0]\). We complete the proof.

\[\square\]

3. Free \(U(\mathfrak{h})\)-modules of rank 2 over \(R\)

We first present the following easy observation.

**Lemma 3.1.** Let \(\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1\) be a Lie superalgebra with a Cartan subalgebra \(\mathfrak{h} \subseteq \mathfrak{g}_0\), and \([\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_0\). Then there do not exist \(\mathfrak{g}\)-modules which are free of rank 1 as \(U(\mathfrak{h})\)-modules.

The Ramond \(N = 2\) superconformal algebra \(R\) has a canonical Cartan subalgebra \(\mathfrak{h} = \mathbb{C}L_0 \oplus \mathbb{C}H_0 \subseteq \mathcal{R}_0\). Moreover, \(R\) is generated by odd elements \(G_m^\pm, m \in \mathbb{Z}\). Hence, by Lemma 3.1, there do not exist \(R\)-modules which are free over \(U(\mathfrak{h})\) of rank 1.

In this section, we classify the free \(U(\mathfrak{h})\)-modules of rank 2 over the Ramond \(N = 2\) superconformal algebra \(\mathcal{R}\). Moreover, we determine the isomorphism classes of such modules.
and we precisely give all submodules of such modules. In particular, the free $U(h)$-modules of rank 2 over the Ramond $N = 2$ superconformal algebra $\mathcal{R}$ are not simple.

Let $M = M_0 \oplus M_1$ be an $\mathcal{R}$-module such that it is free of rank 2 as a $U(h)$-module with two homogeneous basis elements $v$ and $w$. Obviously, $v$ and $w$ have different parities. Set $v = 1_0 \in M_0$ and $w = 1_1 \in M_1$. We may assume

$$M = U(h)1_0 \oplus U(h)1_1 = \mathbb{C}[L_0, H_0]1_0 \oplus \mathbb{C}[L_0, H_0]1_1$$

with $M_0 = \mathbb{C}[L_0, H_0]1_0$ and $M_1 = \mathbb{C}[L_0, H_0]1_1$.

We need the following preliminary result for later use.

**Lemma 3.2.** For any $m \in \mathbb{Z}$, $n \in \mathbb{N}$, we have

1. $L_m L_0^m = (L_0 + m)^n L_m$, $L_m H_0^n = H_0^n L_m$.
2. $H_m L_0^m = (L_0 + m)^n H_m$, $H_m H_0^n = H_0^n H_m$.
3. $G_m^\pm L_0^m = (L_0 + m)^n G_m^\pm$, $G_m^\pm H_0^n = (H_0 \mp 1)^n G_m^\pm$.

**Proof.** According to Definition 2.1, it is easy to check that

$$L_m L_0 = (L_0 + m) L_m, \quad L_m H_0 = H_0 L_m, \quad H_m L_0 = (L_0 + m) H_m,$$

$$H_m H_0 = H_0 H_m, \quad G_m^\pm L_0 = (L_0 + m) G_m^\pm, \quad G_m^\pm H_0 = (H_0 \mp 1) G_m^\pm.$$

Then the lemma can be proven by induction on $n$. \qed

The following assertion on the action of $H_m$ ($m \in \mathbb{Z}$) is crucial for our further discussion.

**Lemma 3.3.** For any $m \in \mathbb{Z}$, we have $H_m 1_0 = c_m(H_0) 1_0$, $H_m 1_1 = c_m'(H_0) 1_1$ with $c_m(H_0), c_m'(H_0) \in \mathbb{C}[H_0]$.

**Proof.** Suppose

$$H_m 1_0 = \sum_{i=0}^{k_m} c_{m,i} L_0^i 1_0,$$

where $c_{m,i} \in \mathbb{C}[H_0], k_m \in \mathbb{N}$ and $c_{m,k_m} \neq 0$. Having in mind $[H_m, H_n] = 0$ for $m + n \neq 0$ and by applying Lemma 3.2 (2), we have

$$0 = H_m H_n 1_0 - H_n H_m 1_0$$

$$= \sum_{i=0}^{k_m} c_{m,i} (L_0 + m)^i H_m 1_0 - \sum_{i=0}^{k_m} c_{m,i} (L_0 + n)^i H_n 1_0$$

$$= c_{m,k_m} c_{n,k_n} (L_0 + m)^{k_n} L_0^{k_m} 1_0 - c_{m,k_m} c_{n,k_n} (L_0 + n)^{k_m} L_0^{k_n} 1_0$$

$$+ \sum_{i=0}^{k_n-1} c_{n,i} (L_0 + m)^i H_m 1_0 - \sum_{i=0}^{k_n-1} c_{n,i} (L_0 + n)^i H_n 1_0$$

$$= c_{m,k_m} c_{n,k_n} (nk_m - mk_n) L_0^{k_n + k_m - 1} 1_0 + \text{lower degree terms with respect to } L_0^i 1_0.$$

8
Since $c_{m,k_m}c_{n,k_n} \neq 0$, we obtain $nk_m - mk_n = 0$ for all $m, n \in \mathbb{Z}$ with $m+n \neq 0$. This implies $k_m = 0$ for all $m \in \mathbb{Z}$. Thus $H_m1_0 \in \mathbb{C}[H_0]1_0$. Similar arguments yield $H_m1_1 \in \mathbb{C}[H_0]1_1$ for all $m \in \mathbb{Z}$.

The following assertion on the trivial action of the central element follows directly from Lemma 3.3.

**Corollary 3.4.** The central element $C$ acts on $M$ trivially.

**Proof.** It suffices to show that $C1_0 = C1_1 = 0$. For that, we note that $[H_1, H_{-1}] = \frac{1}{3}C$. Then it follows from Lemma 3.3 that

\[
C1_0 = 3H_1H_{-1}1_0 - 3H_{-1}H_11_0 \\
= 3H_{c-1}(H_0)1_0 - 3H_{-1}c_1(H_0)1_0 \\
= 3c_{-1}(H_0)H_11_0 - 3c_1(H_0)H_{-1}1_0 \\
= 3c_{-1}(H_0)c_1(H_0)1_0 - 3c_1(H_0)c_{-1}(H_0)1_0 \\
= 0.
\]

Similar argument yields that $C1_1 = 0$. We complete the proof. \qed

**Lemma 3.5.** For any $m \in \mathbb{Z}$, we have $L_m1_0 \neq 0$, $L_m1_1 \neq 0$, $H_m1_0 \neq 0$, $H_m1_1 \neq 0$.

**Proof.** Suppose on the contrary that $L_m01_0 = 0$ and $H_n01_0 = 0$ for some nonzero integer $m_0$ and $n_0$. By Lemma 3.2 (1) and Lemma 3.2 (2), for any $f(L_0, H_0)1_0 \in M_0$, we know that

\[
L_{m_0}f(L_0, H_0)1_0 = f(L_0 + m_0, H_0)L_{m_0}1_0 = 0, \\
H_{n_0}f(L_0, H_0)1_0 = f(L_0 + n_0, H_0)H_{n_0}1_0 = 0.
\]

Since $[L_{m_0}, L_{-m_0}]1_0 = 2m_0L_01_0$ and $[L_{-n_0}, H_{n_0}]1_0 = -n_0H_01_0$, it follows that

\[
2m_0L_01_0 = L_{m_0}L_{-m_0}1_0 - L_{-m_0}L_{m_0}1_0 = 0, \\
-n_0H_01_0 = L_{-n_0}H_{n_0}1_0 - H_{n_0}L_{-n_0}1_0 = 0,
\]

which is a contradiction. Similarly, we can prove $L_m1_1 \neq 0$ and $H_m1_1 \neq 0$ for all $m \in \mathbb{Z}$. \qed

**Lemma 3.6.** For any $m \in \mathbb{Z}$, the following one and only one case happens.

1. $G_m^+1_0 = G_m^-1_1 = 0, G_m^+1_1 \neq 0, G_m^-1_0 \neq 0$.
2. $G_m^+1_1 = G_m^-1_0 = 0, G_m^+1_0 \neq 0, G_m^-1_1 \neq 0$.

**Proof.** Assume that

\[
G_m^+1_0 = f_m^+(L_0, H_0)1_1, \quad G_m^-1_0 = f_m^-(L_0, H_0)1_1, \\
G_m^+1_1 = g_m^+(L_0, H_0)1_0, \quad G_m^-1_1 = g_m^-(L_0, H_0)1_0.
\]
where \( f_m^\pm(L_0, H_0), g_m^\pm(L_0, H_0) \in \mathbb{C}[L_0, H_0] \). It follows from Lemma 3.2 (3) and \([G^+, G^+] = 0\) that

\[
0 = (G^+)^2 1_0
= G^+ f^+_0(L_0, H_0) 1_1
= f^+_0(L_0, H_0 - 1) G^+_0 1_1
= f^+_0(L_0, H_0 - 1) g^+_0(L_0, H_0) 1_0.
\]

Then we have either \( f^+_0(L_0, H_0 - 1) = 0 \) or \( g^+_0(L_0, H_0) = 0 \). Hence, \( f^+_0(L_0, H_0) = 0 \) or \( g^+_0(L_0, H_0) = 0 \), i.e.,

either \( G^+_0 1_0 = 0 \) or \( G^+_0 1_1 = 0 \).

By similar discussion, we have

(3.3) either \( G^-_0 1_0 = 0 \) or \( G^-_0 1_1 = 0 \).

**Case 1: \( G^+_0 1_0 = 0 \).**

From \([G^-_0, G^+_0] = 2L_0 - \frac{1}{12} C\), one sees that

\[
G^+_0 G^-_0 1_0 = G^-_0 G^+_0 1_0 + G^+_0 G^-_0 1_0 = 2L_0 1_0 \neq 0.
\]

Thus \( G^-_0 1_0 \neq 0 \). This together with (3.3) forces \( G^-_0 1_1 = 0 \). Using this result and the relation

\[
G^-_0 G^-_0 1_1 = G^-_0 G^+_0 1_1 + G^+_0 G^-_0 1_1 = 2L_0 1_1 \neq 0,
\]

we further obtain that \( G^+_0 1_1 \neq 0 \). For any \( m \in \mathbb{Z} \), since \([G^+_0, G^+_m] = 0\), we have

\[
0 = G^+_0 G^+_m 1_0 + G^+_m G^+_0 1_0 = G^+_0 f^+_m(L_0, H_0) 1_1 = f^+_m(L_0, H_0 - 1) G^+_0 1_1,
\]

and

\[
0 = G^-_0 G^+_m 1_1 + G^+_m G^-_0 1_1 = G^-_0 g^+_m(L_0, H_0) 1_0 = g^+_m(L_0, H_0 + 1) G^-_0 1_0.
\]

These force \( f^+_m(L_0, H_0 - 1) = 0 \) and \( g^+_m(L_0, H_0 + 1) = 0 \). Thus we have \( f^+_m(L_0, H_0) = 0 \) and \( g^+_m(L_0, H_0) = 0 \), i.e.,

\[
G^+_m 1_0 = 0, \quad G^-_m 1_1 = 0, \quad \forall m \in \mathbb{Z}.
\]

This together with \([G^-_m, G^+_m] = 2L_2m + \frac{1}{3}(m^2 - \frac{1}{4}) \delta_{m,0} C\), Corollary 3.4 and Lemma 3.5 yield that

\[
G^+_m G^-_m 1_0 = G^-_m G^+_m 1_0 + G^+_m G^-_m 1_0 = 2L_2m 1_0 \neq 0,
\]

\[
G^-_m G^+_m 1_1 = G^-_m G^+_m 1_1 + G^+_m G^-_m 1_1 = 2L_2m 1_1 \neq 0, \quad \forall m \in \mathbb{Z}.
\]

Thus

\[
G^-_m 1_0 \neq 0, \quad G^+_m 1_1 \neq 0, \quad \forall m \in \mathbb{Z}.
\]

Now part (1) holds in this case.

**Case 2: \( G^+_0 1_1 = 0 \).**

In this case, by using similar arguments, we can prove part (2). \(\square\)
By exchanging the parity, we know that a module $M$ satisfying Lemma 3.6 (1) is isomorphic to a module $M$ satisfying Lemma 3.6 (2). Thus in the following, we always assume

\begin{equation}
G_m^+1_0 = G_m^-1_1 = 0, G_m^+1_1 \neq 0, G_m^-1_0 \neq 0 \text{ for all } m \in \mathbb{Z}.
\end{equation}

**Lemma 3.7.** Up to a parity, for any $m \in \mathbb{Z}$, we have $G_m^-1_0 = a_m1_1, G_m^+1_1 = \frac{2}{a_m}(L_0 + mH_0)1_0$, where $0 \neq a_m \in \mathbb{C}$.

**Proof.** Note that $[G_m^-, G_m^+]_11 = 2L_01_0 - 2mH_01_0$ for $m \in \mathbb{Z}$. By (3.1), (3.2), (3.4) and Lemma 3.2 (3), we have

\[
G_m^-G_m^+1_0 + G_m^-G_m^-1_0 = G_m^-f_m^-(L_0, H_0)1_1 = f_m^-(L_0 - m, H_0 - 1)G_m^+1_1 = f_m^-(L_0 - m, H_0 - 1)g_m^+(L_0, H_0)1_0.
\]

Therefore,

\[
f_m^-(L_0 - m, H_0 - 1)g_m^+(L_0, H_0) = 2L_0 - 2mH_0.
\]

It follows that

\[
f_m^-(L_0 - m, H_0 - 1) = a_m, \quad g_m^+(L_0, H_0) = \frac{2}{a_m}L_0 - \frac{2m}{a_m}H_0,
\]

or

\[
f_m^-(L_0 - m, H_0 - 1) = \frac{2}{b_m}L_0 - \frac{2m}{b_m}H_0, \quad g_m^+(L_0, H_0) = b_m,
\]

where $a_m, b_m \in \mathbb{C}^*$. Next we prove the following claim.

**Claim:** only one of the following two cases can happen.

(i) $G_m^-1_0 = a_m1_1, G_m^+1_1 = \frac{2}{a_m}(L_0 - mH_0)1_0$ for all $m \in \mathbb{Z}, a_m \in \mathbb{C}^*$.

(ii) $G_m^-1_0 = \frac{2}{b_m}(L_0 - mH_0)1_1, G_m^-1_1 = b_m1_0$ for all $m \in \mathbb{Z}, b_m \in \mathbb{C}^*$.

Indeed, by the above discussion, without loss of generality, we can assume that $G_n^-1_0 = a_n1_1, G_n^+1_1 = \frac{2}{a_n}(L_0 - nH_0)1_0$ for some $n \in \mathbb{Z}$. If there exists some $m \in \mathbb{Z}$ such that $G_m^-1_0 = \frac{2}{b_m}(L_0 - (m + n)H_0)1_1, G_m^-1_1 = b_m1_0$. Then, on one hand, it follows from Lemma 3.2 (3) and Lemma 3.3 that

\[
[H_m, G_n^-]1_0 = H_mG_n^-1_0 - G_n^-H_m1_0 = a_nH_m1_1 - c_m(H_0)G_n^-1_0 = a_n(c'_m(H_0) - c_m(H_0))1_1 \in \mathbb{C}[H_0]1_1.
\]
On the other hand,

\[[H_m, G_n^-]1_0 = -G_{m+n}^- 1_0 = -\frac{2}{b_{m+n}}(L_0 - (m + n)H_0)1_1 \notin \mathbb{C}[H_0]1_1,\]

which is a contradiction. Hence, the claim follows, and we complete the proof. \(\square\)

**Lemma 3.8.** Up to a parity, there exist \(\lambda, \alpha \in \mathbb{C}^*\) such that for any \(m \in \mathbb{Z}\), we have

\[(3.5) \quad G_m^- 1_0 = \lambda^m \alpha 1_1, \quad G_m^+ 1_1 = \frac{2}{\alpha} \lambda^m (L_0 + mH_0)1_0,\]

\[(3.6) \quad H_m 1_0 = \lambda^m H_0 1_0, \quad H_m 1_1 = \lambda^m H_0 1_1.\]

**Proof.** For \(m, n \in \mathbb{Z}\), it follows from Lemma 3.2 (2), Lemma 3.2 (3), Lemma 3.3 and Lemma 3.7 that

\[
H_m G_n^+ 1_1 - G_n^+ H_m 1_1
\]

\[
= \frac{2}{a_n} H_m (L_0 + nH_0) 1_0 - G_n^+ c_m'(H_0) 1_1
\]

\[
= \frac{2}{a_n} (L_0 + m + nH_0) H_m 1_0 - c_m'(H_0 - 1) G_n^+ 1_1
\]

\[
= \frac{2}{a_n} ((L_0 + m + nH_0)c_m(H_0) - c_m'(H_0 - 1)(L_0 + nH_0))1_0
\]

\[
= \frac{2}{a_n} (mc_m(H_0) + (c_m(H_0) - c_m'(H_0 - 1)L_0 + (c_m(H_0) - c_m'(H_0 - 1))nH_0)1_0,
\]

where \(a_n \in \mathbb{C}^*\), \(c_m(H_0), c_m'(H_0) \in \mathbb{C}[H_0]\). Note that

\[(3.7) \quad [H_m, G_n^+] 1_1 = G_{m+n}^+ 1_1 = \frac{2}{a_{m+n}}(L_0 + (m + n)H_0)1_0.\]

Comparing the coefficients of \(L_0\) in (3.7), we obtain

\[(3.8) \quad c_m(H_0) - c_m'(H_0 - 1) = \frac{a_n}{a_{m+n}}, \quad \forall m, n \in \mathbb{Z}.\]

Using this result and comparing the coefficients of \(H_0\) in (3.7), we further obtain

\[
\frac{2}{a_n} (mc_m(H_0) + \frac{a_n}{a_{m+n}}nH_0) = \frac{2}{a_{m+n}}(m + n)H_0, \quad \forall m, n \in \mathbb{Z}.
\]

Then we get

\[
c_m(H_0) = \frac{a_n}{a_{m+n}} H_0, \quad \forall m, n \in \mathbb{Z}.
\]

Thus there exits nonzero \(\lambda, \alpha \in \mathbb{C}\) such that

\[
a_m = \lambda^{-m} \alpha, \quad c_m(H_0) = \lambda^m H_0, \quad \forall m \in \mathbb{Z}.
\]

Moreover, it follows from (3.8) that

\[
c_m'(H_0) = \lambda^m H_0, \quad \forall m \in \mathbb{Z}.
\]

This implies (3.5) and (3.6). The proof is done. \(\square\)
Lemma 3.9. Up to a parity, for any $m \in \mathbb{Z}$, we have

\begin{align}
L_m \bar{1}_0 &= \lambda^m (L_0 + \frac{1}{2} m H_0) \bar{1}_0, \quad L_m \bar{1}_1 = \lambda^m (L_0 + \frac{1}{2} m H_0 + m) \bar{1}_1.
\end{align}

Proof. Since

\begin{align}
[G_m, G_0^+] \bar{1}_0 &= 2L_m \bar{1}_0 - m H_m \bar{1}_0, \quad [G_m, G_0^+] \bar{1}_1 = 2L_m \bar{1}_1 - m H_m \bar{1}_1,
\end{align}

it follows from (3.4), (3.5) and (3.6) that

\begin{align}
L_m \bar{1}_0 &= \frac{1}{2} (G_m^- G_0^+ \bar{1}_0 + G_0^+ G_m^- \bar{1}_0 + m H_m \bar{1}_0) \\
&= \frac{1}{2} (\lambda^m \alpha G_0^+ \bar{1}_1 + m \lambda^m H_0 \bar{1}_0) \\
&= \lambda^m (L_0 + \frac{1}{2} m H_0) \bar{1}_0,
\end{align}

and

\begin{align}
L_m \bar{1}_1 &= \frac{1}{2} (G_m^- G_0^+ \bar{1}_1 + G_0^+ G_m^- \bar{1}_1 + m H_m \bar{1}_1) \\
&= \frac{1}{2} (\frac{2}{\alpha} G_m^- L_0 \bar{1}_0 + m \lambda^m H_0 \bar{1}_1) \\
&= \frac{1}{2} (\frac{2}{\alpha} \lambda^m G_m^- L_0 + m \lambda^m H_0 \bar{1}_1) \\
&= \lambda^m (L_0 + \frac{1}{2} m H_0 + m) \bar{1}_1.
\end{align}

We complete the proof. \hfill \Box

We are now in the position to present the main result of this section, which gives a complete classification of free $U(\mathfrak{h})$-modules of rank 2 over the Ramond $N = 2$ superconformal algebra.

**Theorem 3.10.** Let $M$ be an $\mathcal{R}$-module such that the restriction of $M$ as a $\mathbb{C}[L_0, H_0]$-module is free of rank 2. Then up to a parity, $M \cong \Omega(\lambda, \alpha)$ for some $\lambda, \alpha \in \mathbb{C}^*$ with the $\mathcal{R}$-module structure defined as in (2.3)-(2.7).

Proof. The assertion follows directly from Lemma 3.2, Corollary 3.4, Lemma 3.6, Lemma 3.8, and Lemma 3.9. \hfill \Box

**Remark 3.11.** It follows from (2.1) and (2.2) that Theorem 3.10 also gives a complete classification of free $U(\mathfrak{h})$-modules of rank 2 over the Neveu-Schwarz and the topological $N = 2$ superconformal algebras.

The following result determines the isomorphism classes of the free $U(\mathfrak{h})$-modules of rank 2 over the Ramond $N = 2$ superconformal algebra $\mathcal{R}$.

**Theorem 3.12.** Let $\lambda, \mu, \alpha, \beta \in \mathbb{C}^*$. Then $\Omega(\lambda, \alpha) \cong \Omega(\mu, \beta)$ as $\mathcal{R}$-modules if and only if $\lambda = \mu$ and $\alpha = \beta$. 

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Proof. The sufficiency is clear. Now suppose that \( \varphi : \Omega(\lambda, \alpha) \to \Omega(\mu, \beta) \) is an \( \mathcal{R} \)-module isomorphism. Let \( 1_0 \in \mathbb{C}[x, y] \) and \( 1_1 \in \mathbb{Z}[s, t] \) be the basis elements of the even part and the odd part as \( \mathfrak{U}(\mathfrak{h}) \)-modules. Assume \( \varphi(1_0) = h(x, y) \in \mathbb{C}[x, y] \), \( \varphi(1_1) = h'(s, t) \in \mathbb{C}[s, t] \). Then we have

\[
\alpha h'(s, t) = \alpha \varphi(1_1) = \varphi(\alpha 1_1) = \varphi(G_0^- 1_0) = G_0^- \varphi(1_0) = G_0^- h(x, y) = \beta h(s, t + 1).
\]

This yields that

\[
h(s, t + 1) = \frac{\alpha}{\beta} h'(s, t).
\]

Hence,

\[
\frac{2}{\alpha} x h(x, y) = G_0^+ h(s, t + 1) = \frac{\alpha}{\beta} G_0^+ h'(s, t) = \frac{\alpha}{\beta} G_0^+ \varphi(1_1) = \frac{\alpha}{\beta} \varphi(G_0^+ 1_1) = \frac{\alpha}{\beta} \varphi(\frac{2}{\alpha} x) = \frac{2}{\beta} x h(x, y),
\]

which implies \( \alpha = \beta \).

Moreover, since

\[
\mu y h(x + 1, y) = H_1 \varphi(1_0) = \varphi(h_1 1_0) = \varphi(\lambda H_0 1_0) = \lambda H_0 \varphi(1_0) = \lambda y h(x, y),
\]

it follows that \( \lambda = \mu \), as desired. \( \square \)

We end this section with the following theorem in which all submodules of \( \Omega(\lambda, \alpha) \) are precisely determined for any \( \lambda, \alpha \in \mathbb{C}^* \). In particular, \( \Omega(\lambda, \alpha) \) is not simple.

**Theorem 3.13.** For any \( h(y) \in \mathbb{C}[y] \), let \( M_h = h(y) \mathbb{C}[x, y] \oplus h(t + 1) \mathbb{C}[s, t] \subseteq \Omega(\lambda, \alpha) \) and \( N_h = h(y)(x \mathbb{C}[x, y] + y \mathbb{C}[x, y]) \oplus h(t + 1) \mathbb{C}[s, t] \subseteq \Omega(\lambda, \alpha) \). Then the following statements hold.

1. The set \( \{ M_h, N_h \mid h(y) \in \mathbb{C}[y] \} \) exhausts all \( \mathcal{R} \)-submodules of \( \Omega(\lambda, \alpha) \).
2. For any nonzero \( h(y) \in \mathbb{C}[y] \), the quotient \( \Omega_R(\lambda, \alpha)/M_h \) is a free \( \mathbb{C}[L_0] \)-module of rank 2 deg \( h(y) \).
3. Any maximal submodule of \( \Omega_R(\lambda, \alpha) \) is of the form \( M_h \) for some \( h(y) \in \mathbb{C}[y] \) with \( \deg h(y) = 1 \).

**Proof.** (1) Suppose \( \mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1 \) is a nonzero submodule of \( \Omega(\lambda, \alpha) = \mathbb{C}[x, y] \oplus \mathbb{C}[s, t] \). Take a nonzero \( f(x, y) = \sum_{i=0}^{k} x^i f_i(y) \in \mathbb{C}[x, y] \). We first prove the following claim.

**Claim:** \( f(x, y) = \sum_{i=0}^{k} x^i f_i(y) \in \mathcal{M}_0 \iff f_0(y), x f_i(y) \in \mathcal{M}_0 \) for \( i = 1, 2, \ldots, k \).
The sufficient direction is obvious. For the necessary direction, take any \( m \in \mathbb{Z} \), by (2.3), we have
\[
\frac{1}{\lambda^m} L_m f(x, y) = (x + \frac{1}{2} my) \sum_{i=0}^{k} (x + m)^i f_i(y) \\
= (x + \frac{1}{2} my) \sum_{i=0}^{k} \sum_{j=0}^{k} \left( \begin{array}{c} i \\ j \end{array} \right) m^j x^{i-j} f_i(y) \\
= \sum_{i=0}^{k} \sum_{j=0}^{k} \left( \begin{array}{c} i \\ j \end{array} \right) m^j x^{i-j+1} f_i(y) + \frac{1}{2} \sum_{i=0}^{k} \sum_{j=0}^{k} \left( \begin{array}{c} i \\ j \end{array} \right) m^{j+1} x^{i-j} y f_i(y) \\
= \sum_{j=0}^{k+1} \sum_{i=j-1}^{k} \left( \begin{array}{c} i \\ j \end{array} \right) m^j x^{i-j+1} f_i(y) + \frac{1}{2} \sum_{j=0}^{k+1} \sum_{i=j-1}^{k} \left( \begin{array}{c} i \\ j-1 \end{array} \right) m^j x^{i-j+1} y f_i(y) \\
= \sum_{j=0}^{k+1} a_j m^j \in M_0,
\]

where
\[
a_j = \sum_{i=j-1}^{k} \left( \left( \begin{array}{c} i \\ j \end{array} \right) + \frac{1}{2} \left( \begin{array}{c} i \\ j-1 \end{array} \right) y \right) x^{i-j+1} f_i(y) \in \mathbb{C} [x, y], \ j = 0, 1, \cdots, k + 1.
\]

Taking \( m = 1, 2, \cdots, k + 2 \), we then obtain that \( a_j \in M_0 \) for \( j = 0, 1, \cdots, k + 1 \). Let \( j = k, k+1 \), we get
\[
(3.10) \quad a_k = x f_k(y) + \frac{1}{2} y f_{k-1}(y) + \frac{1}{2} k x y f_k(y) \in M_0, \quad a_{k+1} = \frac{1}{2} y f_k(y) \in M_0.
\]

Using (2.4), we have
\[
\frac{1}{\lambda^m} H_m f(x, y) = y \sum_{i=0}^{k} (x + m)^i f_i(y) = y \sum_{i=0}^{k} \sum_{j=0}^{k} \left( \begin{array}{c} i \\ j \end{array} \right) m^j x^{i-j} f_i(y) = \sum_{j=0}^{k} b_j m^j \in M_0.
\]

Taking \( m = 1, 2, \cdots, k + 1 \), we obtain that
\[
b_j = \sum_{i=j}^{k} \left( \begin{array}{c} i \\ j \end{array} \right) x^{i-j} y f_i(y) \in M_0 \quad \text{for} \quad j = 0, 1, \cdots, k.
\]

In particular, we have
\[
(3.11) \quad b_{k-1} = y f_{k-1}(y) + k x y f_k(y) \in M_0.
\]

Using (3.10) and (3.11), one concludes that \( x f_k(y) \in M_0 \). This implies \( x^k f_k(y) \in M_0 \). Thus
\[
f(x, y) - x^k f_k(y) = \sum_{i=0}^{k-1} x^i f_i(y) \in M_0.
\]
In a similar way, we have
\[ f_0(y), xf_i(y) \in M_0, \quad i = 1, 2, \cdots, k. \]

Thus the Claim holds.

Let \( g(y), xh(y) \in \mathcal{M}_0 \) be nonzero polynomials such that \( \deg_y g(y), \deg_y xh(y) \) are minimal. Then for any \( f(x, y) = \sum_{i=0}^k x^i f_i(y) \in \mathcal{M}_0 \), it follows from the Claim that \( g(y)|f_0(y), h(y)|f_i(y) \) for \( i = 1, 2, \cdots, k \).

Note that \( xg(y) = L_0g(y) \in \mathcal{M}_0 \), we have \( h(y)|g(y) \). Since \( H_0xh(y) = xyh(y) \in \mathcal{M}_0 \) and \( H_1xh(y) = \lambda(x+1)yh(y) \in \mathcal{M}_0 \), we have \( yh(y) \in \mathcal{M}_0 \). Thus \( g(y)|yh(y) \). Therefore, \( g(y) = c_1h(y) \) or \( g(y) = c_2yh(y) \) for some nonzero \( c_1, c_2 \in \mathbb{C} \). We divide the following discussion into two cases.

Case (i): \( g(y) = c_1h(y) \) for some \( c_1 \in \mathbb{C}^* \).

In this case, \( \mathcal{M}_0 \) is generated by \( h(y) \), i.e., \( \mathcal{M}_0 = h(y)\mathbb{C}[x, y] \). For any \( u(s, t) \in \mathbb{C}[s, t] \), we have \( h(t+1)u(s, t) = G_0^-(\frac{1}{\lambda}h(y)u(x, y-1)) \in \mathcal{M}_1 \). Hence, \( h(t+1)\mathbb{C}[s, t] \subseteq \mathcal{M}_1 \). On the other hand, take any \( g(s, t) \in \mathcal{M}_1 \), then \( G_0^+g(s, t) = \frac{2}{\lambda}xg(x, y-1) \in \mathcal{M}_0 = h(y)\mathbb{C}[x, y] \). This implies that \( g(s, t) \in h(t+1)\mathbb{C}[s, t] \), i.e., \( \mathcal{M}_1 \subseteq h(t+1)\mathbb{C}[s, t] \). Consequently, \( \mathcal{M}_1 = h(t+1)\mathbb{C}[s, t] \), and \( \mathcal{M} = M_h \).

Case (ii): \( g(y) = c_2yh(y) \) for some \( c_2 \in \mathbb{C}^* \).

In this case, \( \mathcal{M}_0 \) is generated by \( xh(y) \) and \( yh(y) \), i.e., \( \mathcal{M}_0 = (xh(y))(x\mathbb{C}[x, y] + yh(y)) \). For any \( u(s, t) \in \mathbb{C}[s, t] \),
\[ h(t+1)u(s, t) = G_1^-\left(\frac{1}{\lambda}h(y)xu(x-1, y-1)\right) - G_0^-(\frac{1}{\lambda}h(y)xu(x, y-1)) \in \mathcal{M}_1. \]

Hence, \( h(t+1)\mathbb{C}[s, t] \subseteq \mathcal{M}_1 \). On the other hand, take any \( g(s, t) \in \mathcal{M}_1 \), then
\[ G_0^+g(s, t) = \frac{2}{\lambda}xg(x, y-1) \in \mathcal{M}_0 = h(y)(x\mathbb{C}[x, y] + yh(y)). \]

This implies that \( g(s, t) \in h(t+1)\mathbb{C}[s, t] \), i.e., \( \mathcal{M}_1 \subseteq h(t+1)\mathbb{C}[s, t] \). Consequently, \( \mathcal{M}_1 = h(t+1)\mathbb{C}[s, t] \), and \( \mathcal{M} = N_h \).

(2) and (3) are obvious.

\[\square\]

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