Calculation methods and problems of optimizing the effective characteristics of rods made of composite materials

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Abstract. A pipe made of composite material is considered. Two approaches are proposed. The first is based on the averaging method. In this case, an effective model is a model of a hollow circular cylinder. In the second approach, a one-dimensional beam is an effective model. We also represent the problem of minimizing the weight of the pipe at fixed stiffness of the layers. The paper shows that this problem is equivalent to a well-studied nonlinear programming problem.

1. Introduction

In many areas of construction, problems associate with inhomogeneous materials. To improve the quality of products, you can use several layers of different materials. Layers differ in their elastic characteristics. After applying asymptotic averaging, the layered material turns into homogeneous, which can be described by averaged equations. After applying asymptotic averaging [1-3], the layered material becomes homogeneous. The averaged equations are applicable to it. Examples of the application of this method for problems of the theory of elasticity are given in [4,5]. This asymptotic method in the works [6-8] was applied to composite materials. Also, the averaging method was used to study inhomogeneous elastic creeping materials [9-12].

This paper deals with loaded cylindrical multi-layer composite tube. Our goal is to replace this pipe with a one-dimensional beam, which has corresponding effective characteristics for bending, tension, torsion in explicit analytical form (see subsection 2.2). Similar tasks were posed and solved in [13-17]. The asymptotic analysis of bar systems is also considered in [18-19].

In this article we consider a rod with a periodic structure. According to the formulation of the problem, our task is a special case of [13]. For the accepted conditions, we can analytically determine the averaged, that is, "effective" modules of a given rod. This follows from the possibility of exact solution of partial differential equations on a periodicity cell. In a cylindrical coordinate system, these additional problems on the periodicity cell are one-dimensional, with unknown functions depending on the radial variable. The divergent form of these systems of equations allows us to obtain an analytical solution. By condition, the solution does not depend on the axial variable; therefore, the periodicity conditions along the axis of the rod are satisfied. Since we solve the Neumann problem, there are constant matrices in the solutions of cell problems. Using these matrices, it is possible to determine such important characteristics of a loaded rod as bending stiffness, tensile and torsional
stiffness in an analytical form. Different cases of loading can be considered. For example, the load can act perpendicular or parallel to the axis of the rod. It also could be torsion around the axis of the rod. The solution of the problem gives the stress-strain state of the loaded rod.

Another approach, considered in this paper and proposed as an “alternative”, is to average the equations of elasticity theory for a cylindrical body with a rapidly oscillating periodic layered structure with the radial symmetry in a three-dimensional cylindrical region, without a transition to one-dimensional spatial variable structures. It is described in subsection 2.1 and is an implementation of the approach developed in [1] for analyzing layered structures. The result for further analysis requires the use of numerical methods using large finite elements or differential grids with a large step (much larger than the layer thickness). This drastically reduces the amount of computation compared to direct numerical calculation, but, of course, does not lead to an analytical representation for solutions. In this case, it is possible to obtain numerically a more complete picture for the stress-strain state of the cylindrical body under consideration, than this can be done in the transition to one-dimensional elastic models. The coefficients of the averaged system of elasticity theory will depend on the slow variable which is the distance to the axis of the cylinder.

2. Mathematical model

2.1. The first case

We have system of equilibrium equations [1] in Cartesian coordinates $x_1, x_2, x_3$:

$$\frac{\partial}{\partial x_j} \left( A^{ij}(x) \frac{\varphi(x)}{\varepsilon} \right) \frac{\partial u}{\partial x_i} = \tilde{f}$$

(1)

for the next pipe $P = \{ 0 < x^2 + x^2 < R^2, \ x_3 \in (0; z) \},$ $(i, j = 1, 2, 3)$. Matrices $A^{ij}$ periodically depend on the scalar variable $t$, $\varphi = \varphi(x_1, x_2)$ is the function with a continuous derivative, in the problem under consideration, the function $\varphi = \sqrt{x_1^2 + x_2^2}$, vector $\vec{u}$ is the displacement vector, $\tilde{f}$ is mass forces vector, $\varepsilon$ is the small parameter, $\varepsilon \in (0; 1)$. Let be $\varphi_k$ the derivative of the function $\varphi$ with respect to the appropriate variable $x_k$ ($k = 1, 2, 3$), and $y = (x_1, x_2, x_3)$.

We consider the following matrices

$$B^i_{s}(t, y) = B^i \left( \frac{t}{\varepsilon}, y \right) = \left[ \varphi_s(y) \varphi_s(y) A^{ij}(y) \left( \frac{t}{\varepsilon} \right) \right]^{-1},$$

(2)

$$B^i_{s}(t, y) = B^i \left( \frac{t}{\varepsilon}, y \right) = \left[ \varphi_s(y) \varphi_s(y) A^{ij}(y) \left( \frac{t}{\varepsilon} \right) \right]^{-1} \varphi_s(y) A^{ij}(y) \left( \frac{t}{\varepsilon} \right)$$

(3)

$$B^i_{s}(t, y) = B^i \left( \frac{t}{\varepsilon}, y \right) = \varphi_s(y) A^{ij}(y) \left( \frac{t}{\varepsilon} \right) \left[ \varphi_s(y) \varphi_s(y) A^{ij}(y) \left( \frac{t}{\varepsilon} \right) \right]^{-1} \varphi_s(y) A^{ij}(y) \left( \frac{t}{\varepsilon} \right) - A^{ij}(y) \left( \frac{t}{\varepsilon} \right)$$

(4)

Matrices $\hat{A}^{ij}$ describing the “averaged” or “effective” properties of an elastic cylindrical body $P$ is given by the formulas $\hat{A}^{ij} = \left< B^i \right> \left< B^j \right>^{-1} \left< B^i \right> - \left< B^{i,s} \right>$. 

$$\hat{A}^{ij} = \left< B^i \right> \left< B^j \right>^{-1} \left< B^i \right> - \left< B^{i,s} \right>.$$ 

(5)
Here \( \langle B' \rangle^* \) is the matrix conjugate to \( \langle B' \rangle \), and \( \langle f \rangle \) is the period average: \( \langle f \rangle = \frac{1}{\tau} \int_0^\tau f(t) \, dt \), \( \tau \) is the period value.

After applying the averaging method, we solve the following problem:

\[
\frac{\partial}{\partial x_i} \left( \hat{A}^{i*} \frac{\partial u}{\partial x_i} \right) = \hat{f} (\bar{x}) \tag{6}
\]

and boundary conditions at the ends of the pipe \( P \) and at his lateral surface.

If the pipe has a circular cross section, \( \hat{A}^{i*} \) depend on the “slow” radial variable \( r = \sqrt{x_1^2 + x_2^2} \) as well as \( x_1, x_2 \). This is a consequence of the formulas (2 - 4).

2.2. The second case

Suppose that the thickness of an elastic cylindrical body (pipe) tends to zero, the layers do not have a periodic structure, but the “winding” has radial symmetry, that is, the coefficients of the matrices \( A^i_j \) have the form \( F \left( \frac{r}{\varepsilon} \right) \), where \( r = \sqrt{x_1^2 + x_2^2}, \varepsilon \to 0 \).

Using the results \([13]\), we can approximately replace this “thin” elastic body with a one-dimensional rod, the stress-strain state of which is described by functions of only one spatial variable \( x_3 \), and the “effective” equation is a rod (beam) equation that can be solved explicitly. To determine the characteristics of Young’s modulus and flexural rigidity, it is necessary, according to \([13]\), to solve "auxiliary" problems for next functions \( N_1, N_2, N_3 \):

\[
\frac{\partial}{\partial \xi_1} \left( A^{i1}(\xi_1) \frac{\partial N_1}{\partial \xi_1} \right) = -\frac{\partial}{\partial \xi_1} A^{i1}(\xi_1), \quad \xi_1 \in Q \tag{7}
\]

\[
n_1 A^{i1}(\xi_1) \frac{\partial N_1}{\partial \xi_1} \Bigg|_{\xi_1} = -n_1 A^{i1}, \quad \xi_1 \in Q
\]

\[
\frac{\partial}{\partial \xi_1} \left( A^{i2}(\xi_2) N_1(\xi_2) \right) + A^{i1}(\xi_2) \frac{\partial N_1}{\partial \xi_2}(\xi_2) = -A^{i2}(\xi_2) - h_2 - \frac{\partial}{\partial \xi_1} \left( A^{i1}(\xi_1) \frac{\partial N_2}{\partial \xi_1} \right), \quad \xi_2 \in Q \tag{8}
\]

\[
n_2 A^{i2}(\xi_2) \frac{\partial N_2}{\partial \xi_2} \Bigg|_{\xi_2} = -n_2 A^{i2} N_2, \quad \xi_2 \in Q
\]

\[
\frac{\partial}{\partial \xi_2} \left( A^{i3}(\xi_3) N_2(\xi_3) \right) + A^{i2}(\xi_3) \frac{\partial N_2}{\partial \xi_3}(\xi_3) = -A^{i3}(\xi_3) - h_3 - \frac{\partial}{\partial \xi_2} \left( A^{i1}(\xi_1) \frac{\partial N_1}{\partial \xi_1} \right), \quad \xi_3 \in Q \tag{9}
\]

\[
n_3 A^{i3}(\xi_3) \frac{\partial N_3}{\partial \xi_3} \Bigg|_{\xi_3} = -n_3 A^{i3} N_3, \quad \xi_3 \in Q
\]
Here \( Q \) is the next domain: \( Q=\{\xi \in B: 0<\xi_1<1\} \), \( B \) is a 1-periodic in the direction \( \xi_1 \) set in three-dimensional space, \( \xi=\{\xi_1, \xi_2, \xi_3\} \), the symbol \( \Sigma \) stands for the lateral boundary of \( Q \): \( \Sigma=\{\xi \in \partial B: 0<\xi_1<1\} \).

Let a layered pipe has along its radius pairs of alternating homogeneous layers of two elastic isotropic materials. In this case the problem can be reduced to systems of ordinary differential equations and solved explicitly. For such a problem Lame equations are in the cylindrical coordinate system \((r; \theta; z)\) [20]:

\[
\begin{align*}
(\lambda+2\mu) \frac{\partial \Delta}{\partial r} - \frac{2\mu}{r} \frac{\partial \omega_j}{\partial \theta} + 2\mu \frac{\partial \omega_0}{\partial z} &= f_r, \\
(\lambda+2\mu) \frac{1}{r} \frac{\partial \Delta}{\partial \theta} - \frac{2\mu}{r} \frac{\partial \omega_j}{\partial z} + 2\mu \frac{\partial \omega_0}{\partial r} &= f_\theta, \\
(\lambda+2\mu) \frac{\partial \omega_0}{\partial z} - \frac{2\mu}{r} \frac{\partial \omega_j}{\partial r} + 2\mu \frac{\partial \omega_0}{\partial \theta} &= f_z.
\end{align*}
\]

Here \( \Delta = \frac{1}{r} \frac{\partial (ru_j)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_j}{\partial z} \); \( 2\omega_j = \frac{1}{r} \frac{\partial u_j}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \); \( 2\omega_0 = \frac{\partial u_\theta}{\partial r} - \frac{\partial u_j}{\partial z} \); \( 2\omega_z = \frac{1}{r} \left( \frac{\partial (ru_j)}{\partial \theta} - \frac{\partial u_j}{\partial \theta} \right) \).

Then using the next formulas [13]

\[
\begin{align*}
\hat{c}_i &= \frac{1}{|Q|} \int_Q \left[ A^i(\xi) \left( \frac{\partial N_j}{\partial \zeta}(\xi) + \delta_{ij} E \right) \right] d\xi, \\
\hat{H} &= \frac{1}{|Q|} \int_Q \left[ A^i(\xi) \left( \frac{\partial N_j(\xi)}{\partial \zeta} + \delta_{ij} N_j(\xi) \right) \right] d\xi, \\
\hat{c}_z &= \frac{1}{|Q|} \int_Q \left[ A^i(\xi) \left( \frac{\partial N_3}{\partial \zeta}(\xi) + \delta_{ij} N_j(\xi) \right) \right] d\xi, \\
\hat{c}_3 &= \frac{1}{|Q|} \int_Q \left[ A^i(\xi) \left( \frac{\partial N_3}{\partial \zeta}(\xi) + \delta_{ij} N_j(\xi) \right) \right] d\xi.
\end{align*}
\]

we can explicitly obtain the elastic characteristics of an “effective” rod. In (11) - (14) the symbol \( E \) stands for the unit matrix, and \( |Q| \) stands for the volume of \( Q \). These explicit expressions are obtained by converting matrices \( N_i \) \((i=1+3)\) from Cartesian coordinates to cylindrical coordinates.

Calculations show that these systems decompose into scalar boundary value problems with boundary conditions of the third kind. The calculations show that these systems decompose into scalar boundary value problems with boundary conditions of the third kind, which greatly simplifies the construction of the analytical solution. We formulate separately the statement that we use.

In the set \( B \) we consider the boundary value problem for the elasticity system [13]:

\[
\begin{align*}
(\lambda+2\mu) \frac{\partial \Delta}{\partial r} - \frac{2\mu}{r} \frac{\partial \omega_j}{\partial \theta} + 2\mu \frac{\partial \omega_0}{\partial z} &= f_r, \\
(\lambda+2\mu) \frac{1}{r} \frac{\partial \Delta}{\partial \theta} - \frac{2\mu}{r} \frac{\partial \omega_j}{\partial z} + 2\mu \frac{\partial \omega_0}{\partial r} &= f_\theta, \\
(\lambda+2\mu) \frac{\partial \omega_0}{\partial z} - \frac{2\mu}{r} \frac{\partial \omega_j}{\partial r} + 2\mu \frac{\partial \omega_0}{\partial \theta} &= f_z.
\end{align*}
\]
\[
\frac{d}{dr} \left( A(r) \frac{dw}{dr} \right) = \frac{d}{dr} F_1(r) + F_2(r), \quad \left[ \frac{dw}{dr} + u(r)w - g(r) \right]_{r=R_0} = 0. \tag{15}
\]

Let \( \Phi(r) = \frac{1}{A(r)} \left[ F_1(r) + \Pi \left( \frac{F_2(r)}{A(r)} \right) \right] \), \( X(r) = \frac{1}{A(r)} + u(r) \Pi \left( \frac{1}{A(r)} \right) \), \( Z(r) = g(r) - \Phi(r) - a(r) \Pi \left( \frac{\Phi(r)}{A(r)} \right) \) and let \( \tilde{c}_1, \tilde{c}_2 \) be a solution of the following second order linear system

\[
\begin{aligned}
\tilde{c}_1 X(r_0) + \tilde{c}_2 a(r_0) &= Z(r_0) \\
\tilde{c}_1 X(R_0) + \tilde{c}_2 a(R_0) &= Z(R_0)
\end{aligned}
\]

Then

\[
\begin{aligned}
w(r) &= \Pi[\Phi(r)] + \tilde{c}_1 \left( \frac{1}{A(r)} \right) + \tilde{c}_2 \\
\frac{dw}{dr} &= \Phi(r) + \frac{\tilde{c}_1}{A(r)} 
\end{aligned}
\tag{16}
\]

It is easy to show that the solution of systems (16) with the corresponding boundary conditions can be reduced to the solution of the boundary value problem (15).

Consider the problem of minimizing weight at a fixed pipe stiffness \( \tilde{c}_z \). Suppose a composite tube is composed of \( 2N \) layers of alternating materials, heavy, hard (for example, metal) and relatively light, less solid (for example, polymer). Layer thicknesses are denoted as \( \Delta x_i, \Delta y_i \), \( \Delta x + \Delta y = \delta \), \( \delta = \frac{R_0 - r_0}{N} \), where \( R_0, r_0 \) are outer and inner radii of the pipe. We denote the density of each of the materials as \( \rho_1, \rho_2 \). The problem of minimizing the weight of the pipe will take the form

\[
2\pi \sum_{i=1}^{N} (\rho_1 \Delta x_i + \rho_2 \Delta y_i) (r_0 + \delta(i + 0.5)) \rightarrow \text{min}. \tag{17}
\]

There is the linear form from the next variables: \( \Delta x_1, \Delta y_1, \ldots, \Delta x_N, \Delta y_N \). We denote this linear form in the following way \( L(\Delta x_1, \Delta y_1, \ldots, \Delta x_N, \Delta y_N) \). Thus, our optimization problem takes the form:

\[
\begin{aligned}
L(\Delta x_1, \Delta y_1, \ldots, \Delta x_N, \Delta y_N) &\rightarrow \text{min} \\
0 &\leq \Delta x \leq \delta, 0 \leq \Delta y \leq \delta, \\
\Delta x_i + \Delta y_i &= \delta, \delta = \frac{R_0 - r_0}{N}, \\
\tilde{c}_2 &\geq \tilde{c}_z^0, \\
\tilde{c}_z^0 &\text{ is a given number.} 
\end{aligned} \tag{18}
\]

In this case, it is possible to prove that the magnitude of the flexural rigidity \( \tilde{c}_z \) is a polynomial expression of the values \( \Delta x_i, \Delta y_i \) that determine the thickness of the layers. In fact, due to the symmetry of the construction, the auxiliary matrix functions \( N_1, N_2, N_3 \) depend only on the radius of
the pipe. Therefore, they are solutions of systems of ordinary differential equations with boundary conditions such as the Neumann conditions. These problems will be solvable by choosing matrices with constant elements \( h_2, h_3, h_4 \). In this case, the elements of the matrices \( A^0(\xi) \) are piecewise constant functions that take values \( \lambda_2, \mu_2, \lambda_3, \mu_3 \). Considering the equations for \( N_1, N_2, N_3 \) and analyzing the formula for \( \hat{c}_2 \), it is easy to get that the quantity \( \hat{c}_2 \) is a nonlinear expression of \( \Delta x_1, \Delta y_1, \ldots, \Delta x_N, \Delta y_N \). Thus, our problem takes the form of a well-studied nonlinear programming problem for which software packages are developed.

3. Conclusion
The results of this study make it possible to obtain an analytical solution for determining the stress-strain state of a layered cylinder. The condition is accepted that the properties of the materials forming the cylinder depend only on its radial coordinate. It is possible to obtain the elements of \( h_2 \) and \( h_3 \) matrices explicitly and they characterize the effective modulus of the pipe under tension, torsion, and bending. A similar approach can be applied in the future to study the creeping properties of a pipe made of composite materials. Using the Laplace transform, the problem of an elastic-creeping cylinder can be reduced to the one considered in this article. The coefficients of the equations will depend on the complex parameter. The inverse Laplace transform in the case of creep (relaxation) kernels having the form of decreasing exponentials can be calculated analytically. As a result our optimization problem takes the form of a well-studied nonlinear programming problem for which software packages are developed.

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