THE AUTOMORPHISM GROUP OF A MULTI-GGS GROUP

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Abstract. A multi-GGS-group is a group of automorphisms of a regular rooted tree, generalising the Gupta–Sidki $p$-groups. We compute the automorphism groups of all non-constant multi-GGS-groups.

1. Introduction

The family of Grigorchuk–Gupta–Sidki-groups, hereafter abbreviated ‘GGS-groups’, is best known as a source of groups with exotic properties, e.g. just-infinite groups or infinite finitely generated periodic groups. It generalises earlier examples constructed by its three namesakes in the 80’s, see [6, 9]. These groups are defined as groups of automorphisms of a $p$-regular rooted tree $X^*$, for an odd prime $p$. They are two-generated, and one of the generators is defined according to a one-dimensional subspace $E \subseteq \mathbb{F}_p^{p-1}$. Allowing $E$ to be more than one-dimensional yields a natural generalisation, these ‘higher-dimensional’ GGS-groups are called multi-GGS-groups or multi-edge spinal groups. We prefer the first term.

In many regards, multi-GGS-groups do not differ overly much from their one-dimensional counterparts, e.g. they are periodic under similar conditions, see [1, Theorem 3.2], they possess the congruence subgroup property, see [5], and they allow similar branching structures. Their virtue, aside from extending the list of subgroups of $\text{Aut}(X^*)$ with remarkable properties, lies therein that many conditions on GGS-groups, when generalised to the higher-dimensional counterparts, reveal themselves as linear conditions. In this sense, multi-GGS-groups are the more natural class.

We are concerned with the computation of the automorphism group of a given multi-GGS-group. The automorphisms of groups acting on rooted trees have been investigated before, e.g. in [2, 11]. In general, such groups are quite rigid objects, and their automorphisms are induced by homeomorphisms of the tree. Indeed, in many cases all automorphisms are actually induced by automorphisms of the tree, cf. [8, 11], and for some specific classes the automorphism groups can be uniformly computed, see [2]. More generally, the (abstract) commensurator of groups acting on rooted trees has been investigated, cf. [13]; this is the group of ‘almost automorphisms’, i.e. automorphisms between two finite-index subgroups.

However, there are only few explicit computations of the automorphism group of GGS- and related groups. Sidki computed the automorphism group of the Gupta–Sidki 3-group in [14], and building on the approach for this group, the automorphism groups of the first Grigorchuk group [7], the Fabrikowski–Gupta and the constant
GGS-group on the ternary tree [14] have been computed. The first and the last two examples give a complete list of GGS-groups acting on the ternary tree.

We now state our main result.

**Theorem 1.1.** Let $G$ be a non-constant multi-GGS-group, and let $U$ be the maximal subgroup of $\mathbb{F}_p^\infty$ such that $E$ is invariant under the permutation action induced by $U$ by reordering the columns according to multiplication, and $W$ the maximal subgroup of $\mathbb{F}_p^\infty$ of elements $\lambda$ such that $E \subseteq \text{Eig}_\lambda(u)$ for some $u \in U$. Then the following holds.

(i) If $G$ is regular, then
\[
\text{Aut}(G) \cong (G \rtimes \prod_\omega C_p) \rtimes (U \times W).
\]

(ii) If $G$ is symmetric, then
\[
\text{Aut}(G) \cong (G \rtimes C_p) \rtimes (U \times W).
\]

The definitions of ‘regular’ and ‘symmetric’ can be found in Section 2.2. The slightly obscure definitions of $U$ and $W$ are made more transparent in Section 5. We can immediately derive the following corollary.

**Corollary 1.2.** Let $G$ be a non-constant multi-GGS-group. Then the following statements hold.

(i) The outer automorphism group of $G$ is finite if and only if $G$ is a symmetric GGS-group.

(ii) The outer automorphism group of $G$ is non-trivial.

(iii) The automorphism group of $G$ contains elements of order coprime to $p$ if and only if $E$ is invariant under a permutation induced by multiplication in $\mathbb{F}_p$.

(iv) The automorphism group of $G$ is a $p$-group if and only if $G$ is periodic and $E$ is not invariant under any permutation induced by multiplication in $\mathbb{F}_p$.

We also explicitly compute the automorphism group for a selection of examples, e.g. all Gupta–Sidki $p$-groups, see Section 6.

Our proof combines the methods developed by Sidki in [14] (cf. [2] for a sketch of the strategy used in Sidki’s paper) with techniques used by the author to determine the isomorphism classes of GGS-groups in [12]. This, together with a theorem on the rigidity of branch groups of Grigorchuk and Wilson [8], allows to reduce the complexity of the computations. On the other hand, the inclusion of the symmetric GGS-groups complicates some of the arguments.

2. Higher dimensional Grigorchuk–Gupta–Sidki-groups

2.1. Regular rooted trees and their automorphisms. Let $p$ be an odd prime, and denote by $X$ the set $\{0, \ldots, p-1\}$. The Cayley graph $X^*$ of the free monoid on $X$ is a $p$-regular rooted tree. We think of the vertices of $X^*$ as words in $X$. The root of the tree is the empty word $\varnothing$. We write $X^n$ for the set of all words of length $n$, called the $n$-th layer of $X^*$, and we identify $X$ and $X^1$.

Every (graph) automorphism $g \in \text{Aut}(X^*)$ necessarily fixes the root, since it has a smaller valency than every other vertex. Consequently, every automorphism $g$ leaves all layers $X^n$ invariant. We write $\text{Stab}(n)$ for the (setwise) stabiliser of $X^n$, and $\text{Stab}_G(n)$ for its intersection with some subgroup $G \leq \text{Aut}(X^*)$. We call a group $G \leq \text{Aut}(X^*)$ spherically transitive if it acts transitively on all layers $X^n$. 
The group $\text{Aut}(X^*)$ inherits the self-similar structure of $X^*$, and decomposes as a wreath product

$$\text{Aut}(X^*) \cong \text{Aut}(X^*) \wr \text{Sym}(X).$$

We deduce that $\text{Aut}(X^*) \cong \text{Aut}(X^*) \wr (\text{Sym}(X) \wr \text{Sym}(X))$, for every $n \in \mathbb{N}$, where the finite iterated wreath product acts on $X^n$ as on the leaves of the the finite rooted $p$-regular tree with $n$ layers. The base group of the $n$th such wreath product decomposition is equal to the $n$th layer stabiliser. We denote the induced isomorphism $\text{Stab}(n) \to \text{Aut}(X^*) \times p^n \times \text{Aut}(X^*)$ by $\psi_n$. For $v \in X^n$, we denote the projection to the $v$th component of the base group by $|v| : \text{Aut}(X^*) \to \text{Aut}(X^*)$, this so-called section map is a group homomorphism on the pointwise stabiliser $\text{stab}(v)$ of $v$. We call a subgroup $G \leq \text{Aut}(X^*)$ self-similar if $G|v \subseteq G$ for all $v \in X^*$, and we call it fractal if $\text{Stab}_G(1)|x \leq G$ for all $x \in X$.

The image of an element $g \in \text{Aut}(X^*)$ in $\text{Sym}(X)$ under the quotient by $\text{Stab}(1)$ is denoted $g|\emptyset$, and we write $g|v = g|v|\emptyset^G$, for any $v \in X^*$, for the label of $g$ at $v$. Any automorphism is uniquely determined by the collection of its labels.

We fix an embedding $rt : \text{Sym}(X) \to \text{Aut}(X^*)$ by $rt(\sigma)|\emptyset^G = \sigma$ and $rt(\sigma)|v = 1$ for all $v \in X^* \setminus \{\emptyset\}$. We call the elements $rt \text{Sym}(X)$ rooted automorphisms.

Let $\Gamma \leq \text{Sym}(X)$ be a permutation group. We define the $\Gamma$-labelled subgroup of $\text{Aut}(X^*)$ by

$$\text{lab}(\Gamma) = \{ g \in \text{Aut}(X^*) | g|v \in \Gamma \text{ for all } v \in X^* \}.$$ 

It is a well-known fact that if $\Gamma$ is of order $p$, the subgroup $\text{lab}(\Gamma)$ is a Sylow pro-$p$ subgroup of $\text{Aut}(X^*)$.

Let $(x_i)_{i \in \mathbb{N}}$ be a sequence of elements $x_i \in X$. The words $\{x_0 \cdots x_k | k \in \mathbb{N}\}$ form a half-infinite ray $R$ in $X^*$ (or, equivalently, a point of the boundary). Write $\overrightarrow{x}$ for the ray associated to the constant sequence $(x_i)_{i \in \mathbb{N}}$. An $R$-directed automorphism is an automorphism $g$ fixing $R$ such that for all $v \in X^*$ either $v$ connected by an edge to an element of $R$, or

$$g|v = \text{id}.$$ 

A spherically transitive group $G \leq \text{Aut}(X^*)$ is regular branch over $K$, for a finite-index subgroup $K \leq G$, if $K$ contains $\psi_1^{-1}(K \times \cdot^p \times K)$ as a subgroup of finite index.

## 2.2. Multi-GGS-groups.

Fix the permutation $\sigma = (0 \ldots p - 1)$. Write $a = rt(\sigma)$ and $A = \langle a \rangle$, as well as $\Sigma = \langle \sigma \rangle$. Let $E$ be an $r$-dimensional subspace of $\mathbb{F}_p^{p-1}$, for $r > 0$. Choose an ordered basis $(b_1, \ldots, b_r)$ of $E$, and denote the standard basis of $\mathbb{F}_p$ by $(s_1, \ldots, s_p)$. Let $E \in \text{Mat}(r, p - 1; \mathbb{F}_p)$ be the matrix with the basis elements as rows. The columns of $E$ are denoted $e_i$ for $i \in \{1, \ldots, p - 1\}$; thinking of $E$ as a subspace of $\{0\} \times \mathbb{F}_p^{p-1} \leq \mathbb{F}_p^p$, we will also write $e_0$ for the zero column vector of length $r$. Define, for all $j \in \{1, \ldots, r\}$, the $\overrightarrow{0}$-rooted automorphisms

$$b^{s_j} := \psi_1^{-1}(b^{s_j}, a^{s_j} e_1, \ldots, a^{s_j} e_{p-1}).$$

Since $a$ has order $p$, we may extend this definition to arbitrary vectors $\mathbf{n} \in \mathbb{F}_p^r$, such that $\psi_1(b^{\mathbf{n}}) = (b^{\mathbf{n}}, a^{\mathbf{n}E})$, where for any $\mathbf{m} = (m_1, \ldots, m_{p-1}) \in \mathbb{F}_p^{p-1}$ we set $a^{\mathbf{m}}$ to be the tuple $(a^{m_1}, \ldots, a^{m_{p-1}})$ (and tuples are combined appropriately). The associated map $b^* : \mathbb{F}_p^r \to \text{Aut}(X^*)$ is an injective group homomorphism. We write $B$ for the image $b^*$.
Definition 2.1. The multi-GGS-group associated to $E$ is the group $G_E$ of automorphisms generated by the set

$$A \cup B.$$ 

The subgroup $A$ (shared by all multi-GGS-groups) is called the root group, and the subgroup $B$ is called the directed group. The generating set in the definition is clearly not minimal; a minimal generating set is given by $\{a\} \cup \{b^j \mid j \in \{1, \ldots, r\}\}$.

If the dimension $r$ of $E$ is 1, one usually speaks of a GGS-group rather than a multi-GGS-group. In this case, abusing notation, we write $b$ for $b^1$.

Depending on the space $E$, we distinguish three classes of multi-GGS-groups:

(i) If $E = \{(\lambda, \ldots, \lambda) \in \mathbb{F}_p^{r-1} \mid \lambda \in \mathbb{F}_p\}$, we call $G_E$ the constant GGS-group. This special case behaves very differently to all other multi-GGS-groups; we will, for the most part, exclude it from our considerations.

(ii) If $r = 1$, the space $E$ is contained in $\{(\lambda_1, \ldots, \lambda_p-1) \in \mathbb{F}_p^{p-1} \mid \lambda_i = \lambda_{p-i} \text{ for all } i \in \{1, \ldots, p-1\}\}$, and $G_E$ is not the constant GGS-group, we call $G_E$ a symmetric GGS-group.

(iii) If $G_E$ is neither constant nor symmetric, we call it a regular multi-GGS-group.

We record some of the key properties of multi-GGS-groups that have been established in the literature, cf. [10, Proposition 3.3] & [5, Lemma 2] for statement (i), [10, Proposition 4.3 and Proposition 3.2] for statements (ii) and (iii), [3, Lemma 3.5] for (iv), [4, Theorem C] for (v), and [1, Proposition 3.1] for (vi).

Theorem 2.2. Let $G = G_E$ be a multi-GGS-group. The following statements hold.

(i) If $G$ is regular, it is regular branch over the derived subgroup $G'$, and the equality $\psi_1(\text{Stab}_G(1)) = G' \times \ldots \times G'$ holds.

(ii) The abelisation of $G$ is an elementary abelian $p$-group of rank $r + 1$.

(iii) If $G$ is not constant, it is regular branch over $\gamma_3(G)$, such that $\psi_1(\gamma_3(\text{Stab}_G(1))) = \gamma_3(G) \times \ldots \times \gamma_3(G)$.

(iv) If $G$ is a symmetric GGS-group, the intersection $\psi_1(G) \cap (G' \times \ldots \times G')$ fulfils

$$\psi_1(G) \cap (G' \times \ldots \times G') = \{(g_0, \ldots, g_{p-1}) \mid \prod_{i=0}^{p-1} g_i \in \gamma_3(G)\},$$

and thus is of index $p$ in $G' \times \ldots \times G'$.

(v) Every multi-GGS-group is self-similar and fractal.

It is fruitful to introduce the following overgroup to deal with the special case of symmetric GGS-groups.

Definition 2.3. Let $G$ be a multi-GGS-group. Set $\zeta = \psi_1^{-1}([b^u, a], \text{id}, \ldots, \text{id})$. The regularisation $G_{\text{reg}}$ of $G$ is the group

$$G_{\text{reg}} = \langle G \cup \{\zeta\} \rangle.$$ 

We record the following lemma on the regularisation of a multi-GGS-group.

Lemma 2.4. Let $G$ be a non-constant multi-GGS-group. Then the following statements hold.

(i) $G_{\text{reg}} = G$ if and only if $G$ is regular.

(ii) If $G$ is symmetric, then $|G_{\text{reg}} : G| = p$.

(iii) The derived subgroups of $G$ and $G_{\text{reg}}$ are equal.
The first two statements are immediate consequences of Theorem 2.2. Also the last statement follows, in view of
\[ \psi_1([b^a, c]) = ([b^a, [b, a]], [b, a], \ldots, [b, a]) \in \gamma_3(G) \times \ldots \times \gamma_3(G), \]
and of
\[ \psi_1([a, c]) = ([b, a], [b, a], \ldots, [b, a]) \in \{(g_0, \ldots, g_{p-1}) | \prod_{i=0}^{p-1} g_i \in \gamma_3(G)\}, \]
from Theorem 2.2 (iii) and (iv).

2.3. Constructions within Aut(X∗). We introduce some notation. Let \( g \in \text{Aut}(X^*) \) and \( n \in \mathbb{N} \). We define the \( n \)-th diagonal of \( g \) as the element
\[ \kappa_n(g) = \psi_1^{-1}(g, \ldots, g). \]
Analogously, for any subset \( G \subseteq \text{Aut}(X^*) \) we define \( \kappa_n(G) = \{ \kappa_n(g) | g \in G \} \). Note that if \( G \) is a group, the set \( \kappa_n(G) \) is a group isomorphic to \( G \).

Definition 2.5. Let \( S \subseteq \text{Aut}(X^*) \) be a set of tree automorphisms. The diagonal closure of \( S \) is the set
\[ \overline{S} = \left\{ \prod_{i=0}^{\infty} \kappa_i(s_i) \bigg| s_i \in S \text{ for } i \in \mathbb{N} \right\}. \]

Since the \( n \)-th factor of the infinite product is contained in \( \text{Stab}(n) \), the product is defined as \( \text{Aut}(X^*) \) is closed in the (profinite) topology induced by the layer stabilisers. Note that the diagonal closure is in general not a subgroup, even if \( S \subseteq \text{Aut}(X^*) \) is one.

Definition 2.6. Let \( S = \text{rt}(\Sigma) \) be a group of rooted automorphisms of \( \text{Aut}(X^*) \). The group
\[ \kappa_{\infty}(S) = \{ \kappa_n(s) | n \in \mathbb{N}, s \in S \} \]
is called the group of layerwise constant labels in \( \Sigma \).

It is easy to see that \( \kappa_{\infty}(S) \cong \prod_{\omega} S \), and \( \overline{\kappa_{\infty}(S)} = \overline{S} \).

2.4. Coordinates for multi-GGS-groups. We first establish the following lemma, that allows us to uniquely describe elements of the first layer stabiliser in terms of ‘coordinates’. To be precise, we construct an isomorphism
\[ \text{Stab}_{\text{reg}}(1) \cong (G' \times \ldots \times G') \rtimes B. \]
This uses the fact that, also modulo \( \psi_1^{-1}(G' \times \ldots \times G') \), the labels \( g|_{\mathbb{Z}} \) at first layer vertices of an element \( g \in \text{Stab}_{\text{reg}}(1) \) are completely determined by the image of \( g \) in \( G/G' \). Recall that \( e_i \) is the \( i \)-th column of \( E \), and that \( e_0 \) denotes the zero column vector of length \( r \).

Lemma 2.7. Let \( G \) be a non-constant multi-GGS-group. Let \( g_0, \ldots, g_{p-1} \in G \) be a collection of elements of \( G \). Then
\[ \psi_1^{-1}(g_0, \ldots, g_{p-1}) \in \text{Stab}_{\text{reg}}(1) \]
if and only if there exist \( n_k \in \mathbb{F}_p^* \) and \( y_k \in G' \) for \( k \in \{0, \ldots, p-1\} \) such that
\[ g_k = a^{s_k} b^{n_k} y_k, \quad \text{where} \quad s_k = \sum_{i=0}^{p-1} n_i \cdot e_k - i. \]
Theorem 2.2 (iv), it is enough to show that have only one non-trivial section, which for regular

\(8\).

described above. The normaliser of \(G\) in \(\text{lab}(\Sigma)\), its normaliser is contained in \(\text{lab}(\text{Nor}_G)\) for some \(\tilde{G}\) of a multi-GGS-group \(G\) with the multiplication action on \(\Sigma \sim G\) out to be closely related. Apart from \(G\) only comes into play when considering the normaliser of \(G\).

We first prove that the full normaliser splits as a semidirect product of the normaliser of \(G\) within said closure, and normaliser of \(G\) within an appropriate subgroup of \(\text{lab}(\Delta)\). At last, we compute the normaliser of \(G\) within \(\text{lab}(\Delta)\), and combine our results.

Proof. We first prove that every element with its sections determined by a collection of vectors and elements of the commutator subgroup defines an element of the regularisation. Fix some \(n_k \in \mathbb{F}_p^*\) and \(y_k \in G'\) for all \(k \in X\). Then the element

\[ g = b^{n_0} (b^{n_1})^{a^{p-1}} \cdots (b^{n_{p-1}})^a \in \text{Stab}_G(1) \]

fulfils

\[ g|_k = a^{n_0 e_k} a^{n_1 e_k-1} \cdots a^{n_{k-1} e_k-1} b^{n_k} a^{n_{k+1} e_k-1} \cdots a^{n_{p-1} e_k-1} \]

for some \(\tilde{y}_k \in G'\). We have

\(\text{Stab}_{G\text{reg}}(1) \geq \langle \text{Stab}_G(1)' \cup \{\xi\} \rangle^G = \psi_1^{-1}(G' \times \cdots \times G')\),

which follows directly from Theorem 2.2 (i) for regular \(G\). For symmetric \(G\), by Theorem 2.2 (iv), it is enough to show that \(\langle \xi \rangle^G = \psi_1^{-1}(G' \times \cdots \times G')\). Clearly \([b, a]\) normally generates \(G'\). The conjugates of \(\xi\) have only one non-trivial section, which is equal to \([b, a]\). The statement follows, since \(G\) is fractal.

Thus the element \(y = \psi_1^{-1}(\tilde{y}_0^{-1} y_0, \ldots, \tilde{y}_{p-1}^{-1} y_{p-1})\) is contained in \(G\text{reg}\), so the element

\[ gy = \psi_1^{-1}(a^{s_0 b^{n_0} y_0}, \ldots, a^{s_{p-1} b^{n_{p-1}} y_{p-1}}) \in G\text{reg} \]

has the prescribed sections.

Now let \(g \in \text{Stab}_{G\text{reg}}(1)\). Up to \(\psi_1^{-1}(G' \times \cdots \times G')\), i.e. up to the choice of \(y_k \in G'\) for \(k \in \{0, \ldots, p-1\}\), we may calculate modulo the subgroup \(L := (\text{Stab}_G(1)' \cup \{\xi\})^G \leq G\text{reg}\). Thus there are \(n_k \in \mathbb{F}_p^*\) for \(k \in X\) such that

\[ g \equiv_L b^{n_0} (b^{n_1})^{a^{p-1}} \cdots (b^{n_{p-1}})^a. \]

Taking sections as we did above shows that \(g|_k \equiv_{G'} a^{s_k} b^{n_k}. \)

Given \(g \in \text{Stab}_{G\text{reg}}(1)\), we call the vectors \(n_k\) introduced in Lemma 2.7 the \(B\)-coordinates of \(g\), and the collection of elements \(y_k \in G'\) the \(L\)-coordinates of \(g\). The elements \(s_k\) (since they are fixed by the \(B\)-coordinates) are called the forced \(A\)-coordinates of \(g\).

2.5. Strategy for the proof of Theorem 1.1. By [8, Theorem 1] and [10, Proposition 3.7], the automorphism group of \(G\) coincides with the normaliser of \(G\) in \(\text{Aut}(X^*)\). Hence we compute this normaliser \(\text{Nor}(G)\). In general, for any \(H \leq \text{Aut}(X^*)\), we denote by \(\text{Nor}(H)\) (without subscript) the normaliser of \(H\) in \(\text{Aut}(X^*)\).

The normaliser of \(\Sigma\) in \(\text{Sym}(X)\) has the form \(\text{Nor}_{\text{Sym}(X)}(\Sigma) \cong \Sigma \times \Delta\), where \(\Delta \cong \mathbb{F}_p^\times\) with the multiplication action on \(\Sigma \cong \mathbb{F}_p^*\). Heuristically, the automorphism group of a multi-GGS-group \(G\) allows for a similar decomposition. Since \(G\) is contained in \(\text{lab}(\Sigma)\), its normaliser is contained in \(\text{lab}(\text{Nor}_{\text{Sym}(X)}(\Sigma))\), which decomposes as described above. The normaliser of \(G\) within \(\text{lab}(\Sigma)\) is not identical to \(G\), but turns out to be closely related. Apart from \(G\) being symmetric or not, the structure of \(E\) only comes into play when considering the normaliser of \(G\) in \(\text{lab}(\Delta)\).

We first consider the normaliser of \(G\) in an appropriate closure within \(\text{lab}(\Sigma)\). Then we prove that the full normaliser splits as a semidirect product of the normaliser of \(G\) within said closure, and normaliser of \(G\) within an appropriate subgroup of \(\text{lab}(\Delta)\). At last, we compute the normaliser of \(G\) within \(\text{lab}(\Delta)\), and combine our results.
3. The normaliser in $G_{\text{reg}}$

We begin our study of elements normalising $G$. Adapting the strategy of Sidki in [14], we start not with the normaliser in the full automorphism group, but rather in the group $G_{\text{reg}} \geq G$. This a natural candidate, since it contains the normaliser of the rooted group $A$ (cf. Lemma 3.2) and the group $G$ itself.

**Lemma 3.1.** Let $G$ be a non-constant multi-GGS-group. Then
\[ \kappa_1(G_{\text{reg}}) \leq \kappa_\infty(A) \cdot G_{\text{reg}}. \]

*Proof.* We check that the generators of $\kappa_1(G_{\text{reg}})$ are contained in the group on the right hand side. Clearly $\kappa_1(a) \in \kappa_\infty(A)$.

To see that $\kappa_1(b^s_j)$ is contained in $\kappa_\infty(A) \cdot G_{\text{reg}}$ for a given $j \in \{1, \ldots, r\}$, we use Lemma 2.7. We have no choice for the set of $B$-coordinates; since $\kappa_1(b^s_j)|_x = b^s_j$ for all $x \in X$ they are all equal to $s_j$. Thus we compute the forced $A$-coordinates
\[ s_k = \sum_{i=0}^{p-1} n_i \cdot e_{k-i} = \sum_{i=0}^{p-1} s_j \cdot e_{k-i} = \sum_{i=0}^{p-1} e_{j,k-i}, \]
where $e_{j,k-i}$ is the respective entry of $E$. Consequently, all forced $A$-coordinates are equal to some fixed $s \in \mathbb{F}_p$ and independent of $k$. Hence
\[ \kappa_1(a^s b^s_j) \in G_{\text{reg}}. \]
Since we have already established that $\kappa_1(a) \in \kappa_\infty(A)$, this implies that $\kappa_1(b^s_j)$ is contained in $\kappa_\infty(A) \cdot G_{\text{reg}}$ for all $j \in \{1, \ldots, r\}$.

Finally, in the case that $G$ is symmetric, we have $[\kappa_1(a), b] = \Sigma$ and hence
\[ \psi_1([\kappa_2(a), \kappa_1(b)]) = \kappa_1([\kappa_1(a), b]) = \kappa_1(\Sigma). \]

The problem to determine the normaliser is easily solved for the rooted group $A$. To determine the normaliser of $B$ is significantly harder.

**Lemma 3.2.** The centraliser and the normaliser of the rooted group $A$ are given by
\[ \text{Cen}(A) = \kappa_1(\text{Aut}(X^*)) \rtimes A, \quad \text{and} \quad \text{Nor}(A) = \kappa_1(\text{Aut}(X^*)) \rtimes \text{rt}(\text{Nor}_{\text{Sym}(X)}(\Sigma)). \]

*Proof.* Given $x \in X$, we have $a^0|_x = \text{id}$ if and only if $g|_x = g|_{x+1}$, hence we have $a^g|_x = a^j|_x$ for some $j \in \{1, \ldots, p-1\}$ if and only if $g|_0 = g|_x$ for all $x \in X$. The image of $a$ under conjugation with $g$ now only depends on $g|_\Sigma$, hence we only need to observe $\text{Cen}_{\text{Sym}(X)}(\Sigma) = \Sigma$. \Box

**Lemma 3.3.** Let $G$ be a non-constant multi-GGS-group. Then $\text{Nor}(B) \leq \text{stab}(0)$, the point stabiliser of the vertex $0 \in X$, and
\[ \text{Nor}(B)|_0 \leq \text{Nor}(B) \quad \text{and} \quad \text{Cen}(B)|_0 \leq \text{Cen}(B). \]

*Proof.* Let $g \in \text{Nor}(B)$. Then there is some $n \in \mathbb{F}_p^r \setminus \{0\}$ such that $(b^s_1)^g = b^n$. If $0^{g^{-1}} = x \neq 0$, we see that
\[ b^n = b^n|_0 = (b^s_1)^g|_0 = (a^{b_1|x})^{g|_0}. \]
But a rooted automorphism cannot be conjugate to a directed automorphism. Thus $g \in \text{stab}(0)$. Similarly, we find
\[ b^n = (b^s_1)^g|_0 = (b^s_1|_0)^{g|_0} = (b^s_1)^{g|_0}. \]
This shows (allowing $n = s_1$) both other statements. \Box
For the next lemma we introduce the (word) length function \( \ell : G \to \mathbb{N} \), with respect to the generating set \( A \cup B \), i.e. the mapping

\[
\ell(g) = \min\{ n \in \mathbb{N} \mid g \text{ can be written as a product of length } n \text{ in } A \cup B \}.
\]

It is well-known that this length function is contracting, i.e. that \( \ell(g|x) \leq g \) for \( x \in X \). We need some finer analysis to establish a strict inequality in certain cases. Note that the strictness of the inequality above, for a more general class of self-similar groups \( G \), is related to \( G \) being a periodic group.

**Lemma 3.4.** Let \( G \) be a non-constant multi-GGS-group, and let \( g \in G \) be an element with \( \ell(g) > 1 \). Then there is some \( i \in X \setminus \{0\} \) such that \( \ell(g|_0 g^{-1}_i) < \ell(g) \).

**Proof.** Write \( g = a^{i_0}b^{n_0} \cdots a^{i_{m-1}}b^{n_{m-1}}a^{i_m} \), where \( m \in \mathbb{N}, n_k \in \mathbb{F}_p^*, i_k \in \mathbb{Z} \setminus \{0\} \) for \( k \in \{0, \ldots, m-1\} \), and \( i_n \in \mathbb{Z} \). Passing to a conjugate if necessary, every \( g \in G \) can be written in this way. Taking sections, we see that

\[
g|_k = b^{n_0}|_{k-i_0}(b^{n_1})|_{k-i_0-i_1} \cdots (b^{n_{m-1}})|_{k-\sum_{t=0}^{m-1}i_t}
\]

for any \( k \in X \), hence \( \ell(g|_k) \leq m \). Since every \( B \)-letter contributes at most one \( B \)-letter to one of the sections, we have \( \sum_{k=0}^{p-1} \ell(g|_k) \leq \ell(g) + p - 1 \). Assume that \( \ell(g|_0 g^{-1}_i) \geq \ell(g) \). Then \( \ell(g|_0) + \ell(g|_1) \geq \ell(g) \), hence \( \ell(g|_0) = \ell(g|_1) = m \). This can only be the case if every \( B \)-letter contributes its only section that is contained in \( B \) either to \( g|_0 \) or \( g|_1 \), i.e. \( g|_k \in A \) for all other \( k \in X \). Thus, if \( m > 2 \), we have

\[
\ell(g|_0 g^{-1}_i) \leq \ell(g|_0) + \ell(g|_k) \leq m + 1 \leq 2m - 1 < \ell(g),
\]

If \( m = 2 \), we see that \( g = a^{i_0}b^{n_0} \), implying \( g|_0 \in A \). Since there is at least a second section contained in \( A \), the result follows. \( \square \)

**Lemma 3.5.** Let \( G \) be a non-constant multi-GGS-group. Then

\[
\text{Nor}_{Cen(A)}(G) \cap (Cen(B) \cdot G) \subseteq \kappa_\infty(A) \cdot G_{\text{reg}}.
\]

**Proof.** Let \( g \in \text{Nor}_{Cen(A)}(G) \cap (Cen(B) \cdot G) \) and \( h \in G \) be an element of minimal length such that we may write \( g = g'h \) for some \( g' \in Cen(B) \). The proof uses induction on the length of \( h \).

First assume that \( h \) has length one, i.e. \( h \in A \cup B \). If \( h \) is in \( B \), we find that \( g \in Cen(B) \). Thus \( h \) centralises \( G \), but it is well-known that the centraliser of a branch group in \( \text{Aut}(X^*) \) is trivial; hence \( g = \text{id} \). If \( h \) is a power of \( a \), the same holds for \( gh^{-1} \), hence \( g \in A \leq G_{\text{reg}} \).

Now we assume that \( \ell(h) > 1 \). By Lemma 3.2 we may write \( g = \kappa_1(g|_0)a^k \) for some \( k \in \mathbb{Z} \), yielding for any \( n \in \mathbb{F}_p^* \)

\[
((b^n)^{g|_0}, (a^{n-e_1})^{g|_0}, \ldots, (a^{n-e_{p-1}})^{g|_0})a^k = \psi_1((b^n)^g) = \psi_1((b^n)^h) = ((b^n)^{h|_0}, (a^{n-e_1})^{h|_1}, \ldots, (a^{n-e_{p-1}})^{h|_{p-1}})^{h|_0}.
\]

Since \( a \) and \( b^n \) are not conjugate in \( \text{Aut}(X^*) \), this shows that \( g|_0 |_0 = h|_0 \) and \( a^{g|_0} = a^{h|_i} \) for all \( i \in X \setminus \{0\} \). Thus \( g|_0 h|_{i-1} \) centralises \( A \), and by Lemma 3.3 we find \( (b^n)^{g|_0 h|_{i-1}} = (b^n)^{h|_0 h|_{i-1}} \in B^G \), hence \( g|_0 h|_{i-1} \) normalises \( G \). By Lemma 3.4, there is some \( i \in X \setminus \{0\} \) such that \( \ell(h|_0 h|_{i-1}) < \ell(h) \), so by induction we see that \( g|_0 h|_{i-1} \in \kappa_\infty(A) \cdot G_{\text{reg}} \). Since \( h|_i \in G \), we have \( g|_0 \in \kappa_\infty(A) \cdot G_{\text{reg}} \), and

\[
g = \kappa_1(g|_0)a^k = a^k \kappa_1(g|_0) \in A \cdot \kappa_1(\kappa_\infty(A) \cdot G_{\text{reg}}) = \kappa_\infty(A) \cdot \kappa_1(G_{\text{reg}}).
\]

Now Lemma 3.1 yields \( g \in \kappa_\infty(A) \cdot G_{\text{reg}} \). \( \square \)
With a little care, we can use the same idea to extend the result to $G_{\text{reg}}$.

**Lemma 3.6.** Let $G$ be a non-constant multi-GGS-group. Then

$$\text{Nor}_{\text{Cen}(\alpha)}(G) \cap (\text{Cen}(b) \cdot G_{\text{reg}}) \leq \kappa_{\infty}(A) \cdot G_{\text{reg}}.$$  

*Proof.* In view of the previous lemma, we may restrict to symmetric $G$. Let $g \in \text{Nor}_{\text{Cen}(\alpha)}(G) \cap (\text{Cen}(B) \cdot G_{\text{reg}})$ and choose $g' \in \text{Cen}(B)$, $h \in G$ and $j \in \mathbb{Z}$, such that $g = g' \zeta^j h$. Write $g = \kappa_1(g_0|_0) a^k$ for some $k \in \mathbb{Z}$, and calculate,

$$\begin{align*}
(b^{|0|}, (a^{e_1})^{g_0|_0}, \ldots, (a^{e_{p-1}})^{g_0|_0}) a^k &= \psi_1(b^g) = \psi_1(\zeta^j h) \\
&= ((\zeta^j |_0 h_0|^1, (a^{e_1})^1, \ldots, (a^{e_{p-1}})^{|b|_{p-1}})^{h|_0}).
\end{align*}$$

As we did in the proof of Lemma 3.5, we may conclude that $h|_0 = a^k$. Consequently, for all $i \in X \setminus \{0\}$, the element $g_0|_0 h_{i}^{-1}$ centralises $a$. By Lemma 3.3, the element $g_0|_0$ is in $\text{Cen}(B) \cdot \zeta^j |_0 h_0$. Since $\zeta^j |_0 = [b, a]^j \in G$, this implies that $g_0|_0 h_{i}^{-1}$, for all $i \in X \setminus \{0\}$, is an element in $\text{Nor}_{\text{Cen}(\alpha)}(G) \cap (\text{Cen}(B) \cdot G)$.

By Lemma 3.5, the element $g_0|_0 h_{i}^{-1}$, and consequently also $g_0|_0$ is contained in $\kappa_{\infty}(A) \cdot G_{\text{reg}}$. Finally, by Lemma 3.1, we find $g = \kappa_1(g_0|_0) g|_0^2 \in \kappa_{\infty}(A) \cdot G_{\text{reg}}$. \hfill $\square$

**Lemma 3.7.** Let $G$ be a non-constant multi-GGS-group. Then

$$\text{Nor}_{\text{lab}(\Sigma)}(A) \leq \kappa(A) \quad \text{and} \quad \text{Nor}_{\text{lab}(\Sigma)}(B) \leq \kappa(B).$$

*Proof.* We use the description of $\kappa(A)$ given in Lemma 3.2. Let $g \in \text{Nor}_{\text{lab}(\Sigma)}(A)$. For all $i \in \text{lab}(\Sigma)$, we have $h|_0 \in \langle \sigma \rangle$. Thus we see that $\alpha_i g|_0 \sigma \alpha_i^{-1} g|_0 \sigma^\prime = a \sigma a^{-1} g|_0 \sigma = a$. Now let $g \in \text{Nor}_{\text{lab}(\Sigma)}(B)$, let $n \in \mathbb{N}^r$ be arbitrary and let $m \in \mathbb{N}^r$ be such that $(b^n)^g = b^m$. Then

$$((b^n)^g|_0, (a^n)^{|e_1|}, \ldots, (a^n)^{|e_{p-1}|}) = b^m = (b^n)^g.$$  

The label $g|_0^2$ is a power of $a|_0^2$. Since $b^n$ and $a$ are not conjugate in $\text{Aut}(X^*)$, the element $g|_0^2$ must stabilise the vertex 0, thus it is trivial. Varying $n$, we see that $g|_i$ normalises $A$ for all $i \in X \setminus \{0\}$ for which $e_i \neq 0$. Now since $\text{lab}(\Sigma)$ is self-similar, this implies $g|_i \in \text{Cen}(A)$ for the first part of this lemma, hence $a^{n e_i} = a^m e_i$ for all $i \in X \setminus \{0\}$. Thus $b^m = b^n$. Since $g_0|_0 \in \text{Nor}_{\text{lab}(\Sigma)}(B)$ by Lemma 3.3, we can argue in the same way for $g|_0$, hence $g \in \kappa(B)$. \hfill $\square$

**Lemma 3.8.** Let $G$ be a non-constant multi-GGS-group. Then

$$\overline{\text{Nor}_{\text{reg}}(G)} \subseteq \kappa_{\infty}(A) \cdot G_{\text{reg}}.$$  

*Proof.* Let $g \in \overline{\text{Nor}_{\text{reg}}(G)}$. There is a sequence $(g_i)_{i \in \mathbb{N}}$ with $g_i \in G_{\text{reg}}$ such that

$$g = \prod_{i=0}^{\infty} \kappa_i(g_i).$$

Write $h_n$ for the partial product $\prod_{i=0}^{n} \kappa_i(g_i)$. By Lemma 3.1, we find $h_n \in \kappa_{\infty}(A) \cdot G_{\text{reg}}$ for all $n \in \mathbb{N}$. We may write

$$g|_0^n = h_n|_0^n \left( \prod_{i=n+1}^{\infty} \kappa_i(g_i) \right) |_0^n \cdot h_n = h_n|_0^n \prod_{i=1}^{\infty} \kappa_i(g_{i+n}).$$

In view of Lemma 3.2, we conclude that $h_n|_0^{-1} g|_0^n \in \text{Cen}(A)$. There is nothing special about $0^n$; indeed, we see that $h_n|_v^{-1} g|_v = h_n|_0^{-1} g|_0^n$ for all $v \in X^n$. By [12, Lemma 3.4],
there exists an integer \( n \in \mathbb{N} \) such that \( g|_{v_0} \in \text{Cen}(B) \). By Lemma 3.7 \( g|_{v_0} \in \text{Cen}(B) \). Consequently

\[
B_{g|_{v_0}}^{-1} h_n|_{v_0} = B_{h_n|_{v_0}}.
\]

Since \( h_n|_v \in \text{reg} \) for all \( v \in X^n \), we may use Lemma 3.6 and obtain \( g|_{v_0}^{-1} h_n|_{v_0} \in \kappa_{\infty}(A) \cdot \text{G}_{\text{reg}} \). Using Lemma 3.1 again, we find \( \kappa_n(h_n|_{v_0}^{-1} g|_{v_0}) \in \kappa_{\infty}(A) \cdot \text{G}_{\text{reg}} \), and moreover

\[
g = h_n \psi^{-1}_n(h_n|_{v_0}^{-1} g|_{v_0}, \ldots, h_n|_{(p-1)v_0}^{-1} g|_{(p-1)v_0}) = h_n \kappa_n(h_n|_{v_0}^{-1} g|_{v_0}) \in \kappa_{\infty}(A) \cdot \text{G}_{\text{reg}}.
\]

**Lemma 3.9.** Let \( G \) be a non-constant multi-GGS-group. Write \( G_{\text{lay}} \) for the product set \( \kappa_{\infty}(A) \cdot \text{G}_{\text{reg}} \).

(i) If \( G \) is regular, then we have \( \kappa_{\infty}(A) \leq \text{Nor}_{\text{Aut}(X^*)}(G) \), hence \( G_{\text{lay}} \) acquires the structure of a semidirect product.

(ii) If \( G \) is symmetric, we find \( \text{Nor}_{\kappa_{\infty}(A)}(G) = A \).

(iii) Write \( G_{\text{lay}} \) for product set in (i). Then

\[
\text{Nor}_{G_{\text{lay}}}(G) = \begin{cases} 
G_{\text{lay}} & \text{if } G \text{ is regular,} \\
\text{G}_{\text{reg}} & \text{if } G \text{ is symmetric.}
\end{cases}
\]

**Proof.** Let \( n \in \mathbb{N} \). Clearly \( a_{\kappa_n(a)}(a) = a \), and for all \( j \in \{1, \ldots, r\} \)

\[
[b^a_j, \kappa_n(a)] = \psi^{-1}_1([b^a_j, \kappa_{n-1}(a)], [a^{b^a_{j-1}}, \kappa_{n-1}(a)], \ldots, [a^{b^a_{j-1}}, \kappa_{n-1}(a)])
\]

\[
= \psi^{-1}_1([b^a_j, a], a, \ldots, a) \in \psi^{-1}_1(G' \times \cdots \times G') \leq G.
\]

This shows (i), and it also shows that \( \kappa_n(a) \) does not normalise a symmetric GGS-group \( G \) for \( n > 0 \), since \( \psi^{-1}_1([b, a], a, \ldots, a) \notin G \). Thus (ii) is proven.

Statement (iii) is a consequence of (i) in case \( G \) is regular, and an immediate consequence of Lemma 2.4 (iii) in case \( G \) is symmetric.

**Proposition 3.10.** Let \( G \) be a non-constant multi-GGS-group. Then

\[
\text{Nor}_{G_{\text{reg}}}(G) = \begin{cases} 
G \rtimes \kappa_{\infty}(A), & \text{if } G \text{ is regular,} \\
\text{G}_{\text{reg}}, & \text{if } G \text{ is symmetric.}
\end{cases}
\]

**Proof.** Assume that \( G \) is regular. By Lemma 3.9, the set \( \kappa_{\infty}(A) \cdot \text{G}_{\text{reg}} \) is a group. In view of Lemma 3.8 and \( G = \text{G}_{\text{reg}} \), this proves the first case. If \( G \) is symmetric, the result follows from Lemma 3.8 and Lemma 3.9. \( \square \)

### 4. The Normaliser as a Product

We now prove that the normaliser of \( G \) in \( \text{Aut}(X^*) \) decomposes as a semi-direct product. To begin with, we prove the following generalisation of [14, 2.2.5(i)], which is an interesting proposition in its own right.

**Proposition 4.1.** Let \( G \) be a non-constant multi-GGS-group. Every element of \( G \) that has order \( p \) is either contained in \( \text{Stab}_G(1) \) or is conjugate to a power of \( a \) in \( \text{G}_{\text{reg}} \).

**Proof.** We have to prove that, given \( g \in \text{Stab}_G(1) \) and \( i \in \mathbb{Z} \), every element \( a^i g \) of order \( p \) may be written \( (a^h)^i \) for some \( h \in \text{G}_{\text{reg}} \). Passing to an appropriate power of \( a \), we may assume that \( i = 1 \). From \( (ag)^p = 1 \) we derive the equations

\[
id = (ag)^p|_0 = g|_0 \ldots g|_{p-1}, \text{ resp.}
\]

\[
g|_{p-1} = g|_{p-2}^{-1} \ldots g|_0^{-1}.
\]
Since \( g \in \text{Stab}_G(1) \), by Lemma 2.7 there exists a set of \( B \)-coordinates \( n_k \in \mathbb{F}_p^* \) and a set of \( L \)-coordinates \( y_k \in G' \) uniquely describing \( g \). Reformulated in these \( B \)-coordinates, the condition above reads
\[
\sum_{i=0}^{p-2} n_i = -n_{p-1}.
\]
Given some integer \( s \in \mathbb{Z} \), we define an element
\[
h_s = \psi_1^{-1}(a^s, a^s g|_0, a^s g|_1, \ldots, a^s g|_{p-2}) \in \psi_1^{-1}(G \times F \times G).
\]
Since
\[
a^{h_s}|_k = h_s|_{k-1} h_s|_{k+1} = (a^s \prod_{i=0}^{k-1} g|_i)^{-1} a^s \prod_{i=0}^{k} g|_i
\]
the conjugate \( a^{h_s} \) is equal to \( ag \). It remains to prove that \( h_s \in G_{\text{reg}} \) for some \( s \in \mathbb{Z} \). If it is contained in \( G_{\text{reg}} \), the element \( h_s \) has the \( B \)-coordinates \( h_k = \sum_{i=0}^{k-1} n_i \) (and some commutators \( z_k \) that we shall not need to specify). We have to prove that the corresponding forced \( A \)-coordinates \( s_k = \sum_{i=0}^{p-1} h_i \cdot e_{k-i} \) are equal to the actual \( a \)-exponents of the corresponding sections of \( h \). Since it is enough to show that \( h_s \in G_{\text{reg}} \) for one \( s \), we fix \( s = s_0 \), so that the proposed equality holds in the first component by definition.

A quick calculation shows that, for all \( k \in X \setminus \{0\} \),
\[
s_k - s_{k-1} = \sum_{i=0}^{p-1} h_i \cdot e_{k-i} - \sum_{i=0}^{p-1} h_i \cdot e_{k-1-i} = \sum_{i=0}^{p-1} (h_i - h_{i-1}) \cdot e_{k-i} = \sum_{i=0}^{p-1} n_{i-1} \cdot e_{k-i} = s_{k-1},
\]
and consequently the \( a \)-exponent of \( h|_k \) is equal to
\[
s + \sum_{i=0}^{k-1} s_i = s_0 + \sum_{i=0}^{k-1} s_i = \tilde{s}_k,
\]
for all \( k \in X \). But the values \( s_i \) are the forced \( A \)-coordinates of \( g \), hence, comparing with the definition of \( h_s \), we see that the forced \( A \)-coordinates of \( h_k \), for \( k \in X \), and the actual \( a \)-exponents of \( h|_k \) coincide. Hence \( h \in G_{\text{reg}} \).

Notice that for a symmetric GGS-group, we do not have to pass to \( G_{\text{reg}} \) to make this statement true: take the element \( d = ([b, a], [b, a]^{-1}, \text{id}, \ldots, \text{id}) \in G \). Clearly \( a^{[b, a]} = d \), but assume for contradiction that there is another element \( h \in \text{Stab}_G(1) \) such that \( a^{h^{-1}} = d \). Then \( gh \) centralises \( a \), hence
\[
[b, a]h|_0 = (gh)|_0 = (gh)|_i = h_i,
\]
for all \( i \in \mathbb{F}_p^* \). Counting the powers of \([b, a]\) in \( h|_k \) as in Theorem 2.2 (iv), we see that if \( h|_1 \equiv_{\gamma_3(G)} a^s(b^n([b, a])^v \) the sum of the \([b, a]\)-exponents over all sections mod \( \gamma_3(G) \) equals \( v - 1 + (p - 1)v \equiv p - 1 \), contradicting Theorem 2.2 (iv). Thus there is no such \( h \in \text{Stab}_G(1) \).
Recall that the group $\Delta$ is $\text{Nor}_{\text{Sym}(\Sigma)}(\Sigma) \cap \text{stab}_{\text{Sym}(\Sigma)}(0)$. Set $D = \text{rt}(\Delta)$, i.e. the group of rooted automorphisms normalising but not centralising $a$.

**Lemma 4.2.** Let $G$ be a non-constant multi-GGS-group. Then

$$\text{Nor}(G) \subseteq \overline{G_{\text{reg}} \cdot D}.$$  

**Proof.** Let $g_0 \in \text{Nor}(G)$. Let $k \in \mathbb{Z}$ be such that $(a^{g_0})^k|_\Sigma = a$. By Proposition 4.1 there exists an element $h_0 \in G_{\text{reg}}$ such that $(a^{g_0})^k = a^{h_0}$. Consequently $h_0^{-1}g_0 \in \text{Nor}(A)$. Using Lemma 3.2 and the fact that $\text{Nor}(G)$ is self-similar, cf. [12, Lemma 3.3], we may write

$$h_0^{-1}g_0 = \kappa_1((h_0^{-1}g_0)|_0) \text{rt}((h_0^{-1}g_0)|_\Sigma)$$

for $h_0^{-1}g_0|_0 = g_1 \in \text{Nor}(G)$. Since $g_0h_0^{-1}|_\Sigma$ normalises $\sigma$, we may write $g_0h_0^{-1}|_\Sigma = a^{k_0}d_0$ for some $d_0 \in D$ and $k_0 \in \mathbb{Z}$, so that we obtain the equation

$$g_0 = h_0\kappa_1(g_1)a^{k_0}d_0 = h_0a^{k_0}\kappa_1(g_1)d_0,$$

using the fact that $\kappa_1(\text{Aut}(X^*))$ normalises $a$ for the second equality. Repeating the procedure for $g_1$, we obtain $g_2 \in \text{Nor}(G), h_1 \in G_{\text{reg}}, d_1 \in D$ and $k_1 \in \mathbb{Z}$ such that

$$g_0 = h_0a^{k_0}\kappa_1(h_1a^{k_1}\kappa_1(g_2)d_1)d_0$$

$$= h_0a^{k_0}\kappa_1(h_1a^{k_1})\kappa_2(g_2)\kappa_1(d_1)d_0$$

$$= h_0a^{k_0}\kappa_1(h_1a^{k_1})\kappa_2(g_2)d_0\kappa_1(d_1).$$

In the last step we have used the fact that $\overline{D}$ is abelian. Going on, we obtain a sequence of products

$$t_n = \prod_{i=0}^{n-1} \kappa_1(h_ia^{k_i}) \prod_{i=0}^{n-1} \kappa_1(d_i) = \prod_{i=0}^{n-1} \kappa_1(h_ia^{k_i}) \prod_{i=0}^{n-1} \kappa_1(d_i),$$

such that $t_n \equiv_{\text{stab}(n+1)} g_0$, i.e. that are converging to $g_0$ in the topology induced by the layer stabilisers. Since both $\overline{D}$ and $\overline{G_{\text{reg}}}$ are closed sets, the corresponding limits are well-defined. We obtain

$$g_0 = \prod_{i=0}^{\infty} \kappa_1(h_ia^{k_i}) \prod_{i=0}^{\infty} \kappa_1(d_i).$$

This shows $g_0 \in \overline{G_{\text{reg}} \cdot D}$. \hfill $\square$

**Lemma 4.3.** Let $G$ be a non-constant multi-GGS-group and let $g \in G$ be an element directed along $\overline{0}$. Then $g \in B$.

**Proof.** Consider that, since $G \leq \text{lab}(\Sigma)$, there are $(x_1, \ldots, x_{p-1}) \in \mathbb{F}_\mathbb{P}^{p-1}$ such that

$$\psi_1(g) = (g|_0, a^{x_1}, \ldots, a^{x_{p-1}}).$$

Since directed elements stabilise the first layer, there exist $B$-coordinates $n_0, \ldots, n_{p-1}$ and $y_0, \ldots, y_{p-1} \in G'$ for $g$. The equation above shows that $n_1 = \cdots = n_{p-1} = 0$ and $y_1 = \cdots = y_{p-1} = \text{id}$. Thus the forced $A$-coordinate at 0 fulfils

$$s_0 = \sum_{i=0}^{p-1} n_i \cdot c_{p-i} = 0,$$

hence $g|_0 = a^{s_0}b^{n_0} = b^{n_0} \in B$, and in consequence $g = b^{n_0}\psi_1^{-1}(y_0, \text{id}, \ldots, \text{id})$. Since the set of elements directed along $\overline{0}$ forms a subgroup, the element $\psi_1^{-1}(y_0, \text{id}, \ldots, \text{id})$,
and consequently also \( y_0 \) is directed along \( 0 \). We can argue as above for \( y_0 \), but since \( y_0 \in G' \), by Theorem 2.2 (ii), the sum \( \sum_{i=0}^{p-1} n_i = 0 \). Thus \( n_0 = 0 \), and, chasing down the spine, we find \( y_0 = id \). Thus \( g \in B \). \( \square \\

**Lemma 4.4.** Let \( G \) be a non-constant multi-GGS-group, and let \( h \in G_{\text{reg}} \). Then 
\[
a^h \equiv_{G'} a. 
\]

**Proof.** Let \( (g_i)_{i \in \mathbb{N}} \) be a sequence of elements \( g_i \in G_{\text{reg}} \) for \( i \in \mathbb{N} \) such that 
\[
h = \prod_{i=0}^{\infty} \kappa_i(g_i) = g_0 \prod_{i=1}^{\infty} \kappa_i(g_i) = g_0 \kappa_1 \left( \prod_{i=0}^{\infty} \kappa_i(g_{i+1}) \right).
\]
By Lemma 3.2, the element \( \kappa_1 \left( \prod_{i=0}^{\infty} \kappa_i(g_{i+1}) \right) \) centralises \( a \). Thus it is sufficient to consider \( h = g_0 \). The statement now follows from Lemma 2.4 (iii). \( \square \\

**Lemma 4.5.** Let \( G \) be a non-constant multi-GGS-group. Then 
\[
\text{Nor}(G) = \text{Nor}_{G_{\text{reg}}}(G) \times \text{Nor}_{\overline{G}}(G)
\]

**Proof.** Assume that \( \text{Nor}(G) \) is equal to the product set \( \text{Nor}_{G_{\text{reg}}}(G) \cdot \text{Nor}_{\overline{G}}(G) \). By Proposition 3.10, \( \text{Nor}_{G_{\text{reg}}}(G) \) is equal to \( G \times \kappa_\infty(A) \) or to \( G_{\text{reg}} \). Both groups are normalised by \( \text{Nor}_{\overline{G}}(G) \), the first one since \( \overline{G} \) normalises \( \kappa_\infty(A) \), and the second one since for every \( d_0 \kappa_1(d_1) \) with \( d_0 \in D \) and \( d_1 \in \overline{D} \),
\[
\mathcal{L}^{d_0 \kappa_1(d_1)} = \psi_1^{-1}((b, a)^{d_1}, id, \ldots, id)^{d_0} \in \psi_1^{-1}(G' \times \mathbb{P} \times G') \leq G_{\text{reg}}.
\]
Thus the product set is in fact a semidirect product. It remains to show the equality \( \text{Nor}(G) = \text{Nor}_{G_{\text{reg}}}(G) \cdot \text{Nor}_{\overline{G}}(G) \).

By Lemma 4.2, we may write \( g \in \text{Nor}(G) \) as a product \( g = h' \cdot d \) with \( h' \in G_{\text{reg}} \) and \( d \in \overline{D} \). Clearly \( \overline{D} \) normalises \( A \). Thus it is enough to prove: For all \( n \in \mathbb{F}_p^r \) such that \( (b^n)^d \notin G \), then \( h' \cdot d \notin \text{Nor}(G) \) for all \( h' \in G_{\text{reg}} \). We to prove that \( (h'^{-1} \cdot d)^{-1} \notin \text{Nor}(G) \).

Since \( \overline{D} \) is a group, we may replace \( d \) by its inverse. Write \( h \) for \( h'^{-1} \). Notice that 
\[
h = \kappa_1(h_1)h_0
\]
for some \( h_1 \in G_{\text{reg}} \) and \( h_0 \in G_{\text{reg}} \). Since \( G_{\text{reg}} \) normalises \( G \), we may assume that \( h_0 = id \), and thus \( h \cdot h_0^2 = id \). Let \( d = \prod_{i=0}^{\infty} \kappa_i(d_i) \) for a sequence \( (d_i)_{i \in \mathbb{N}} \) of elements \( d_i \in D \) such that, for all \( i \in \mathbb{N} \), we have \( a^d_i = a^{\kappa_j}j_i \) for some \( j_i \in \mathbb{Z} \). Then, for all \( n \in \mathbb{F}_p^r \),
\[
\psi_1((b^n)^d) = ((b^n)^{d_0}, (a^{n \kappa_1(d_1)})^{d_0}, \ldots, (a^{n \kappa_{p-1}(d_{p-1})})^{d_0}) = ((b^n)^{d_0}, a^{j_1 \cdot n \cdot e_1 \cdot d_0}, \ldots, a^{j_1 \cdot n \cdot e_{(p-1)} \cdot d_0}).
\]
Write \( x_{k,i} = j_1 \cdot n \cdot e_i \cdot d_0 \) for all \( i \in \{1, \ldots, p-1\} \). Since \( \overline{D} \) is self-similar, we see that \( (b^n)^d \) is directed along \( 0 \), and we write \( x_k = (x_{k,1}, \ldots, x_{k,p-1}) \) for the A-exponents of the sections at \( i \in \{1, \ldots, p-1\} \) of \( (b^n)^d |_{q_k-1} \).

Now, using Lemma 4.3, we see that if \( (b^n)^d \in G \), then actually \( (b^n)^d \in B \). Assume that the \( \overline{0} \)-directed element \( (b^n)^d \) is not a member of \( B \). Then there are two possibilities:

(i) there exists some \( k \in \mathbb{N} \) such that \( x_k \) is not contained in the row space of \( E \), or,

(ii) if for all \( k \in \mathbb{N} \) the vector \( x \) is contained in the row space of \( E \), but there exists some \( k \in \mathbb{N} \) such that \( x_k \neq x_{k+1} \).
In both cases, we may assume that \( k = 0 \), since \( G_{\text{reg}} \) and \( \overline{D} \) are self-similar.

Given \( (b^n)^d \), we compute the conjugate by \( dh \),
\[
\psi_1((b^n)^{dh}) = ((b^n)^d|0, a^{x_1}, \ldots, a^{x_p-1})^h
= (((b^n)^d|0)^h|0, (a^{x_1})^h|1, \ldots, (a^{x_p-1})^h|p-1).
\]
Since \( G_{\text{reg}}^{-1} \) is self-similar, we may apply Lemma 4.4, and we find
\[
(*) \quad \psi_1(((b^n)^{dh} \equiv_{G' \times \cdots \times G'} ((b^n)^d|0)^h|0 \text{ mod } G', a^{x_1}, \ldots, a^{x_p-1}).
\]
Assume that we are in case (i), i.e. that \( x_0 \) is not contained in the row space of \( E \). Then, by (*) and Lemma 2.7, also \((b^n)^{dh} \notin G\).

Assume that we are in case (ii), i.e. that \( x_0 \neq x_1 \), but both represent the forced \( a \)-exponents of an element \( b^m_0 \) and \( b^{m_1} \), respectively. Thus by (*) and Lemma 2.7,
\[
(b^n)^{dh} \equiv_{\psi_1^{-1}(G' \times \cdots \times G')} (b^{m_0}) \quad \text{and} \quad (b^n)^{dh}|0 \equiv_{\psi_1^{-1}(G' \times \cdots \times G')} (b^{m_1})^h|0.
\]
Thus
\[
(b^{m_1})^h|0 \equiv ((b^n)^{dh}|0 \equiv b^{m_0}|0 \equiv b^{m_0} \text{ mod } \psi_1^{-1}(G' \times \cdots \times G').
\]
Since \( \psi_1^{-1}(G' \times \cdots \times G') \cap G = \text{Stab}_G(1)' \), this implies \((b^{m_1})^a^k \equiv_{\text{Stab}_G(1)} b^{m_0} \) for some \( k \in \mathbb{Z} \), hence \( b^{m_1} = b^{m_0} \) and \( m_0 = m_1 \). But then \( x_0 = x_1 \), a contradiction. \( \square \)

5. Elements normalising \( G \) with labels in \( \Delta \)

Recall that the permutation group \( \Delta = \langle \delta \rangle \) is isomorphic to \( \mathbb{F}_p^\times \). The rooted automorphism \( d = \text{rt } \delta \) acts in two different ways on \( G = \text{Stab}_G(1) \times A \). It raises \( a \) to a power, i.e. it acts my multiplication on the exponent of \( a \); and it acts on an element of \( g \in \text{Stab}_G(1) \) by permuting the tuple \( \psi_1(g) \), i.e. by multiplication of the indices of said tuple. Note that the vertex 0 is fixed by \( \delta \).

Recall that \( B \) is isomorphic to \( E \leq \mathbb{F}_p^{\times} \). We now show that \( B \) is normalised by every normaliser of \( G \) in \( \overline{D} \).

**Lemma 5.1.** Let \( G \) be a non-constant multi-GGS-group. Then \( \text{Nor}_{\overline{D}}(G) = \text{Nor}_{\overline{B}}(B) \).

**Proof.** Since \( \overline{D} \leq \text{Nor}(A) \), the inclusion \( \text{Nor}_{\overline{D}}(G) \geq \text{Nor}_{\overline{B}}(B) \) is obvious. We now prove the other inclusion. Let \( g \in \text{Nor}_{\overline{D}}(G) \). By [12, Lemma 3.4], there exists an integer \( k \in \mathbb{N} \) such that \( g|0^k \) normalises \( B \). Thus it is enough to prove that if \( g|0 \) normalises \( B \), also \( g \) normalises \( B \).

Assume \( g|0 \in \text{Nor}(B) \), and let \( m \) and \( n \in \mathbb{F}_p^\times \) be such that \((b^n)^{g|0} = b^m \). We may write \( g = \kappa_1(g|0)g|^2 \), where \( g|^2 \) normalises \( a|^2 \). Hence there exists \( j \in \mathbb{Z} \) such that \( a^{g|0} = a^j \). We calculate
\[
(b^n)^{-1}(b^m)^g = (b^n)^{-1}((b^m)^{g|0}, a^{j\cdot m\cdot e_1}, \ldots, a^{j\cdot m\cdot e_{p-1}})^g|0
= (id, a^{m\cdot e_1}, a^{m\cdot e_{p-1}}, \ldots, a^{m\cdot e_{p-1}|0}).
\]
We see that the commutator coordinates of \((b^n)^{-1}(b^m)^g \) are trivial, the \( B \)-coordinates are all zero, and hence the forced \( A \)-coordinates are also \( s_k = \sum_{i=0}^{p-1} n_i \cdot c_{k-i} = 0 \). Thus \( b^n = b^m \). \( \square \)
Thus, we may restrict our attention to the group $B$. It is fruitful to consider $B$ as a subgroup of the directed subgroup $b^{\mathbb{F}_p^{-1}}$ of the multi-GGS-group associated to the full space $\mathbb{F}_p^{-1}$ (with standard basis), which is, by the previous lemma, also invariant under $\text{Nor}_{\overline{\mathcal{G}}}(G)$. Write $\mu: D \to \mathbb{F}_p^{\times}$ for the isomorphism induced by $\alpha^\delta = \alpha^{\mu}$, where the second operation is taking the power, and define a map $P_* : D \to \text{GL}_p(p-1)$ such that $P_d$ is the permutation matrix corresponding to the permutation $d|_{\overline{\mathcal{G}}} = \delta$.

Let $g = \prod_{i=0}^{\infty} \kappa_i(d_i) \in \overline{\mathcal{G}}$ for a sequence $(d_i)_{i \in \mathbb{N}}$ of elements in $D$. Then, for all $j \in \{1, \ldots, p - 1\}$, we see that the action induced on $G$ of the multi-GGS-group associated to the isomorphism $b^\bullet$, the linear map

$$(d_1)^\mu P_{d_0}$$

on $\mathbb{F}_p^{-1}$. Returning to the directed group $B$, we see that every $g \in \text{Nor}_{\overline{\mathcal{G}}}(G)$ must be such that $P_{d_0}$ leaves $E$ invariant. Hence we define

$$U := \{ u \in D \mid EP_u = E \} = \text{stab}_D(E).$$

Furthermore, we define the subgroup

$$V := \{ v \in U \mid \text{for all } e \in E \text{ there exists } \lambda \in \mathbb{F}_p^{\times} \text{ such that } EP_v = \lambda e \}$$

$$= \{ v \in U \mid E \subseteq \text{Eig}_\lambda(P_v) \text{ for some } \lambda \in \mathbb{F}_p^{\times} \}.$$ 

Since $V \leq D$ is cyclic, the element $\lambda \in \mathbb{F}_p^{\times}$ generating a maximal subgroup is uniquely determined. Finally, define the subgroup

$$W := \langle \lambda \rangle.$$ 

**Proposition 5.2.** Let $G$ be a non-constant multi-GGS-group. Let $U$ and $W$ be defined as above. Then

$$\text{Nor}_{\overline{\mathcal{G}}}(G) \cong U \times W.$$ 

**Proof.** By $(\dagger)$, the action of a given element $g = \prod_{i=0}^{\infty} \kappa_i(d_i)$ is determined by $d_0$ and $d_1$. Furthermore, since $E$ must be invariant under $P_{d_0}$, we see that necessarily $d_0 \in U$. Since $\text{Nor}(G)$, by [12, Lemma 3.3], and $\overline{\mathcal{G}}$ are self-similar, we find, for all $k \in \mathbb{N}$,

$$g|_{0^k} = \prod_{i=0}^{\infty} \kappa_i(d_{i+k}) \in \text{Nor}_{\overline{\mathcal{G}}}(G).$$

Since, for all $n \in \mathbb{F}_p^{\times}$ and $g \in \text{Nor}_{\overline{\mathcal{G}}}(G)$,

$$(b^n)^g = (b^n)|_0^g = (b^n|_0)^g = (b^n)^g|_0,$$

we see that the action induced on $\mathbb{F}_p^{-1}$ by all elements $g|_{0^k}$, for $k \in \mathbb{N}$, are equal, i.e. that the following equalities of matrices hold,

$$(d_{k+1})^\mu P_{d_k} = (d_1)^\mu P_{d_0}, \quad \text{hence} \quad I_{p-1} = (d_{k+1}d_{k+2})^\mu P_{d_kd_{k+1}}.$$
where $I_{p-1}$ is the identity matrix. Recall that, for all $k \in \mathbb{N}$, the matrix $P_{d_k d_{k+1}^{-1}}$ acts either does not act as a scalar on $E$, hence there is no $d_{k+1}$ fulfilling the equation above, or it acts as some scalar $\lambda^i$, for some $i \in \mathbb{Z}$. Thus, every difference $d_k d_{k+1}^{-1}$ must be an element of $W$, otherwise, $g$ cannot be normalising $G$.

On the other hand, for $d_0 \in U$ and $d_1 \in d_0^{-1}W$ there is a unique sequence $(d_i)_{i \in \mathbb{N}}$ that defines an element of $\text{Nor}_{\overline{D}}(G)$, since

$$d_{k+2} = d_{k+1}(d_k d_{k+1}^{-1})^r$$
$$= d_{k+1}(d_k((d_k d_{k+1}^{-1})^r)^{-1}d_{k+1}^{-1})^r$$
$$= d_{k+1}((d_k d_{k+1}^{-1})^{-1})^r$$
$$= d_{k+1}((d_0 d_1^{-1}(-1)^k)^{r^{k+1}}.$$

Thus $\text{Nor}_{\overline{D}}(G) \cong U \times W$. 

In particular, if $r = 1$, every linear map leaving $E$ invariant is a scalar multiplication, i.e. the subgroups $U$ and $V$ coincide. Clearly, if $W$ is the trivial group, the only elements of $\text{Nor}_M(G)$ are defined by the constant sequences. More generally, the sequence $(d_i)_{i \in \mathbb{N}}$ defining the normalising element with given $d_0$ and $d_1$ is periodic with periodicity prescribed by the order of $\lambda$.

Now all ingredients are ready for the proof of our main theorem.

**Proof of Theorem 1.1.** By [8, Theorem 1] and [10, Proposition 3.7], the automorphism group of $G$ coincides with the normaliser of $G$ in $\text{Aut}(X^*)$. By Lemma 4.5, this normaliser is the semidirect product

$$\text{Nor}_{\text{reg}}(G) \rtimes \text{Nor}_{\overline{D}}(G).$$

These two groups were computed in Proposition 3.10 and Proposition 5.2.

6. Examples

To illustrate the definitions of $U, V$ and $W$ we compute some explicit examples.

**Example 6.1.** Let $G$ be the GGS-group acting on the 5-adic tree with $E$ generated by $(1, 2, 2, 1)$. Clearly $G$ is symmetric. For every symmetric GGS-group, the space $E$ is by definition invariant under the permutation induced by $-1 \in \mathbb{F}_p^\times$. In fact, it always acts trivially, hence $-1 \in W$. In our case, this is the only non-trivial permutation leaving $E$ invariant, since

$$(1, 2, 2, 1)P_{x \rightarrow 2x} = (2, 1, 1, 2) = (1, 2, 2, 1)P_{x \rightarrow 3x},$$

while $(2, 1, 1, 2) \notin E$. Thus

$$\text{Aut}(G) = (G \rtimes \langle \omega \rangle) \rtimes \langle \prod_{i=0}^{\infty} \kappa_i(x \mapsto -x) \rangle \cong (G \rtimes \mathbb{C}_5) \rtimes \mathbb{C}_2.$$  

The group $G$ and the group defined by $(1, 4, 4, 1)$ are the multi-GGS-groups with the smallest possible outer automorphism group.

**Example 6.2.** Let $G$ be the (regular) GGS-group acting on the $p$-adic tree with $E$ generated by $b = (1, 2, \ldots, p-1)$. Let $(\lambda_1, \ldots, \lambda_{p-1})$ be the image of $b$ under $P_d$, for $d \in D$. Since

$$\lambda_i = b_{d^{-1}i} = (d^{-1})^\mu b_i,$$
we see that $bP_d = (d^{-1})\mu b$. Thus $W = V = U = D$, and the automorphism group is 'maximal',
\[ \text{Aut}(G) = (G \rtimes \kappa_\infty(A)) \rtimes (\mathbb{F}_p^\times)^2. \]

**Example 6.3.** The distinction between the subgroups $U, V$ and $W$ is not superficial. Consider the vector $b_1 = (1, 2, 11, 3, 12, 10, 12, 3, 11, 2, 1) \in \mathbb{F}_3^{12}$. An easy calculation shows that
\[ b_1P_{x \rightarrow 5x} = (12, 11, 2, 10, 1, 3, 1, 10, 2, 11, 12) = -b_1, \]
while $b_1P_{x \rightarrow 3x}$ is not a multiple of $b_1$. Set $b_2 = b_1P_{x \rightarrow 3x}$ and $b_3 = b_1P_{x \rightarrow 9x}$ and let $E$ be the space spanned by $b_1, b_2$ and $b_3$. Since $\mathbb{F}_3^{12}$ is generated by 3 and 5, the space $E$ is invariant under all permutations induced by index-multiplication, i.e. $U = D$. But only the multiplication by the multiples of 5 act by scalar multiplication of $E$, hence $V = \langle x \mapsto 5x \rangle$. The corresponding scalars are 1 and 12, hence $W$ is of order 2.

**Example 6.4.** Let $G_p$ be a Gupta–Sidki $p$-group, i.e. the GGS-group with $E$ spanned by $b = (1, -1, 0, \ldots, 0) \in \mathbb{F}_p^{-1}$. All Gupta–Sidki $p$-groups are regular. Let $n \in N$, and consider $bP_n$. Since the projection to the last $p - 3$ coordinates of $E$ is trivial, the index 1 must be mapped to 1 or 2 under $n$, and the same holds for 2. This is only possible if $n = 1$ or $n = 2$ and $2 \cdot 2 \equiv_p 1$, hence in case $p = 3$. Otherwise $U$ is trivial. If $p = 3$, the group $W$ is equal to $U$, since the non-trivial permutation induced by the index multiplication by 2 is equal to pointwise multiplication by 2. This recovers the result of [14], where the automorphism group of $G_3$ was first computed. Interestingly, this example is the ‘odd one out’, having automorphisms of order 2.

Concludingly, we found
\[ \text{Aut}(G_p) = \begin{cases} (G_p \rtimes \kappa_\infty(A)) \rtimes C_2^2 & \text{if } p = 3, \\ (G_p \rtimes \kappa_\infty(A)) & \text{otherwise.} \end{cases} \]

**Example 6.5.** Let $G_{p^{-1}}$ be the multi-GGS-group defined by the full space $\mathbb{F}_p^{-1}$. This group is regular, and every permutation $P_n$ leaves $\mathbb{F}_p^{-1}$ invariant. On the other hand, no non-trivial permutation acts on the full space as a multiplication. Thus
\[ \text{Aut}(G_{p^{-1}}) = (G_{p^{-1}} \rtimes \kappa_\infty(A)) \rtimes \prod_{i=0}^\infty \kappa_i(d') \mid d' \in D \]
\[ \cong (G_{p^{-1}} \rtimes \prod_{\omega} C_p) \rtimes C_{p^{-1}}. \]

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