TODA SHOCK WAVES: JUSTIFICATION OF ASYMPTOTICS

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Abstract. This paper is a continuation of research started in [17], where formulas for the leading terms of the long-time asymptotics for the Toda shock wave were derived by the method of nonlinear steepest descent, in all principal domains of the space-time half plane. In the present paper we study the same Riemann–Hilbert problem in more detail, including investigations of the influence of resonances and eigenvalues on the asymptotics in the elliptic zone. Our main task was to solve the local parametrix RH problems and to implement a rigorous conclusive analysis (an analog of “small norm” arguments) which justifies the asymptotics obtained previously.

1. Introduction

A Toda shock wave is a solution of the initial value problem for the Toda lattice (32, 33)
\begin{align*}
\dot{b}(n,t) &= 2(a(n,t)^2 - a(n-1,t)^2), \\
\dot{a}(n,t) &= a(n,t)(b(n+1,t) - b(n,t)), \\
(n,t) &\in \mathbb{Z} \times \mathbb{R}_+,
\end{align*}
with a steplike initial profile
\begin{align*}
a(n,0) &\to a, \quad b(n,0) \to b, \quad \text{as } n \to -\infty, \\
2a &< -1,
\end{align*}
where $a > 0$, $b \in \mathbb{R}$ satisfy the condition
\begin{align*}
b + 2a &< -1.
\end{align*}

The Toda shock problem is understood as a study of the long time asymptotic behavior of the Cauchy problem solution for (1.1)–(1.3).

Initially, the Toda shock wave was associated with symmetric initial data (34)
\begin{align*}
a(n,0) &= a(-n,0) \to \frac{1}{2}, \\
b(-n,0) &= -b(n,0) \to \pm b, \\
n &\to \pm \infty, \\
b &> 1.
\end{align*}
Such a model is closely related to the motion of driving particles in a container filled with gas ahead of a piston compressing the content of the container. From the point of view of spectral theory, this model corresponds to two non-intersecting spectral intervals of equal length, which are associated with left and right background Jacobi operators with constant coefficients, moreover, the left background spectrum lies to the left. It is natural therefore to extend the notion of shock wave to background spectra of different lengths. By scaling and shifting of the spectral parameter, one can always assume that the right spectrum coincides with the interval $[-1, 1]$. 

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The Toda shock problem was studied first for the initial data \((1.4)\) on a physical level of rigor by Bloch and Kodama in \([3, 4]\) using the Whitham approach, and by Venakides, Deift and Oba in \([34]\) (see also \([24]\)) by use of the Lax–Levermore approach. In \([34]\) it was shown that in a middle region of the half plane \((n, t) \in \mathbb{Z} \times \mathbb{R}_+\) the solution to \((1.1), (1.4)\) is asymptotically close to a periodic Toda lattice solution of period two with spectrum \([-1 - b, 1 - b] \cup [-1 + b, 1 + b]\). There were several investigations based on the classical inverse scattering transform, which proves its efficiency in the soliton regions \([33]\), and in a transition region behind the leading wave front, where the train of asymptotic solitons was evaluated \([5, 6]\). Further essential progress in the asymptotic analysis of the Toda shock waves is due to the development of the nonlinear steepest descent (NSD) method, which yields the most interesting results in the regime \(n \to \infty, t \to \infty\) with the ratio \(n/t\) close to a constant. The application of NSD to the Toda shock problem has much in common with the analogous problem for the KdV shock wave. In particular, a matrix statement of the associated Riemann–Hilbert (RH) problem is also ill-posed for certain values of \(n\) and \(t\) (cf. \([18]\)), and it is natural to consider vector statements. However, the investigation of the Toda case is more complicated. First of all, one has to describe two “potential” functions \(a(n, t)\) and \(b(n, t)\), and it requires additional asymptotic expansions. Moreover, values of the vector-functions which participate in the statements of the RH problems (both initial and model RHP) are not well controllable at any point on the spectral plane, including infinity. Nevertheless, a description of the asymptotic behavior of the Toda shock wave is done, both qualitatively and quantitatively. In particular, in \([17]\) it was shown that for the solution of \((1.1) – (1.3)\) there are five principal sectors in the \((n, t)\) half-plane with different qualitative behavior (see \([28]\) for an overview). To describe them in more detail recall that the continuous spectrum of the underlying Jacobi operator

\[
H(t)y(n) = a(n - 1, t)y(n - 1) + b(n, t)y(n) + a(n, t)y(n + 1)
= \lambda y(n), \quad \lambda \in \mathbb{C},
\]

consists of two intervals \([b - 2a, b + 2a]\) and \([-1, 1]\) which are the spectra of the background left and right constant Jacobi operators,

\[
H_Ly(n) = ay(n - 1) + ay(n + 1) + by(n),
H_Ry(n) = \frac{1}{2}y(n - 1) + \frac{1}{2}y(n + 1), \quad n \in \mathbb{Z}.
\]

In addition to the spectral parameter \(\lambda\) we will use two parameters \(z\) and \(\zeta\) associated with the respective backgrounds. They are connected with \(\lambda\) by the Joukovski transformation

\[
\lambda = \frac{1}{2} \left( z + z^{-1} \right) = b + a \left( \zeta + \zeta^{-1} \right), \quad |z| \leq 1, \quad |\zeta| \leq 1.
\]

In the NSD approach it is well known that the behavior of the solution essentially depends on the location of the stationary phase points, that is, the nodal points of the level lines where the real part of the phase function vanishes. In our case both the right phase function

\[
\Phi(z, \xi) = \frac{z - z^{-1}}{2} + \xi \log z, \quad \xi := \frac{n}{t},
\]

and the left phase function

\[
\Phi_L(z, \xi) = a(\zeta^{-1} - \zeta) - \xi \log \zeta,
\]
take part in this characterization. First of all, two soliton regions corresponding to the domains of \( n \) and \( t \) for which \( \frac{n}{2} > \xi_{cr} \) or \( \frac{n}{2} < \xi_{cr,1} \) are naturally identified. In these domains the solution to (1.1)–(1.3) is asymptotically close to the respective constant background solution plus solitons generated by the discrete spectrum (if any), as \( t \to \infty \). Here the right leading wave front

\[
(1.9) \quad \xi_{cr} = \frac{\sqrt{(2a-b)^2 - 1}}{\log(2a - b + \sqrt{(2a-b)^2 - 1})}
\]

corresponds to the case when the level line \( \text{Re} \Phi(z, \xi) = 0 \) crosses the real line at the point \( z(b-2a) \) (the left edge of the left spectrum). In analogy, the left wave front

\[
(1.10) \quad \xi_{cr,1} = \frac{\sqrt{(1-b)^2 - 4a^2}}{\log(2a) - \log(1-b + \sqrt{(1-b)^2 - 4a^2})}
\]

is the value where the stationary phase point of the left phase \( \Phi_t \) coincides with the right edge of the right spectrum, i.e. \( z = 1 \). In turn, the region \( \xi_{cr,1} < \frac{n}{2} < \xi_{cr} \) consists of three sectors with different type of quasi-periodic behavior of the solution. These sectors are divided by rays corresponding to the critical values \( \xi'_{cr,1} \) and \( \xi''_{cr} \) of the parameter \( \xi \) such that \( \xi_{cr,1} < \xi'_{cr,1} < \xi''_{cr} < \xi_{cr} \). We have shown in [17] that in the regions \( \xi_{cr,1} < \frac{n}{2} < \xi'_{cr,1} \) and \( \xi'_{cr} < \frac{n}{2} < \xi_{cr} \), the main terms of the expansion of the solution with respect to large \( t \) are modulated elliptic waves. In the middle region \( \xi'_{cr,1} < \frac{n}{2} < \xi''_{cr} \) the solution is asymptotically close to a finite gap (two band) solution of the Toda equation provided that the discrete spectrum is absent in the gap \( (b+2a, -1) \).

Let us give a more detailed description of these modulated elliptic waves. As it is well known, a finite gap solution of the Toda equation is completely characterized by the geometry of its continuous spectrum and by the initial Dirichlet divisor on the hyperelliptic Riemann surface associated with the spectrum. In our shock wave case, we deal with spectra consisting of two bands and one initial Dirichlet eigenvalue in the gap between the bands. The sign necessary to lift this eigenvalue to lift this eigenvalue to the Riemann surface is the sign of the respective half-axis, where the corresponding eigenvector is supported.

First consider \( \xi \in [\xi'_{cr}, \xi_{cr}) \). For any such \( \xi \) consider a point \( \gamma(\xi) \in (b-2a, b+2a] \) which moves monotonically and continuously with respect to \( \xi \) covering the interval \( (b-2a, b+2a] \), with \( \gamma(\xi') = b+2a \). Consider the two-sheeted Riemann surface \( \mathcal{M}(\xi) \) associated with the set

\[
(1.11) \quad \sigma(\xi) := [b-2a, \gamma(\xi)] \cup [-1, 1].
\]

The upper sheet of \( \mathcal{M}(\xi) \) is treated as the complex plane of the spectral parameter \( \lambda \) with cuts along \( \sigma(\xi) \). Let \( \Omega(\lambda, \xi) \) and \( \omega(\lambda, \xi) \) be the normalized Abel integrals of the second and the third kind on the upper sheet of \( \mathcal{M}(\xi) \), with zero \( a \)-periods along the gap \( \gamma(\xi), -1 \). The linear combination \( g(\lambda, \xi) = \Omega(\lambda, \xi) + \xi \omega(\lambda, \xi) \) is also an Abel integral with zero \( a \)-period. Respectively, the nominator of the function \( \frac{\partial g(\lambda, \xi)}{\partial a} \) has two real zeros \( \nu(\xi) \) and \( \mu(\xi) \) with at least one zero in the gap, say \( \mu(\xi) \in (\gamma(\xi), -1) \). Moreover, for \( \lambda \to \infty \) the function \( g(\lambda, \xi) \) has the same asymptotic behavior as the phase function \( \Phi(z(\lambda), \xi) \) up to a constant term. These properties hold for any choice of point \( \gamma(\xi) \). The peculiarity of our choice for \( \gamma(\xi) \) when \( \xi \in [\xi'_{cr}, \xi_{cr}) \) is
that we require the second zero \( \nu(\xi) \) of \( g(\lambda, \xi) \) to match with \( \gamma(\xi) \), that is,

\[
\frac{\partial}{\partial \lambda}(\Omega(\lambda, \xi) + \xi \omega(\lambda, \xi)) = \frac{(\lambda - \mu(\xi))\sqrt{\lambda - \gamma(\xi)}}{\sqrt{(\lambda - b + 2a)(\lambda^2 - 1)}}
\]

Such a point \( \gamma(\xi) \) is unique for every \( \xi \), and \( \gamma(\xi) \) satisfies continuity and monotonicity properties as above in the region under consideration. In fact, it defines the moving edge of the Whitham zone (cf. [31]).

In contradistinction to the phase function the properties of \( g(\lambda, \xi) \) allow us to apply the lens mechanism of the RH problem approach for all contours where we need it. For this reason we replace the phase function by \( g(\lambda, \xi) \), which will play the role of the "\( g \)-function" (cf. [10]) in the NSD method.

Given the point \( \gamma(\xi) \), let \( \{a(n, t, \xi), b(n, t, \xi)\} \) be the finite gap solution for the Toda lattice associated with the spectrum \( \{1.11\} \) and with an initial Dirichlet eigenvalue defined via the initial scattering data for \( (1.2), (1.3) \) by the Jacobi inversion problem. This Dirichlet eigenvalue was computed in [14, Equ. (5.25)], and it is a smooth function of \( \xi \). Evidently, the functions

\[
\{a(n, t, \frac{\xi}{\tau}), b(n, t, \frac{\xi}{\tau})\}
\]

are well defined in the region

\[
\{(n, t) \in \mathbb{Z} \times \mathbb{R}_+: \frac{\nu}{\tau} \in [\xi^*_b, \xi^*_a - \varepsilon]\},
\]

where \( \varepsilon \) is arbitrary small. In analogy to KdV shock waves we call \( (1.3) \) modulated elliptic waves. From this construction it follows that

\[
\xi^*_b = -2a - \frac{\int_{b+2a}^{-1} \lambda Q(\lambda)d\lambda}{\int_{b+2a}^{1} \lambda Q(\lambda)d\lambda}, \quad Q(\lambda) = \sqrt{\frac{\lambda - b - 2a}{(\lambda - b + 2a)(\lambda^2 - 1)}}.
\]

Thus, the asymptotics in the region \( (1.14) \) can be derived using the \( g \)-function \( (1.12) \), and for the ending value of \( \xi \) we get

\[
\sigma(\xi^*_b) = [b - 2a, b + 2a] \cup [-1, 1].
\]

In turn, the region \( \frac{\nu}{\tau} \in (\xi^*_b, 1, \xi^*_a) \), where

\[
\xi^*_{cr,1} = b + 1 - \frac{\int_{b+2a}^{-1} \lambda Q_1(\lambda)d\lambda}{\int_{b+2a}^{1} \lambda Q_1(\lambda)d\lambda}, \quad Q_1(\lambda) = \sqrt{\frac{\lambda + 1}{((\lambda - 1)^2 - 4a^2)(\lambda - 1)}}
\]

is associated with the gap \((b + 2a, -1)\), although we cannot claim that the stationary phase point of \( \Phi(z(\lambda), \xi) \) for such \( \xi \) is located namely in the gap. However, a suitable \( g \)-function here is simply \( \Omega(\lambda) + \xi \omega(\lambda) \), where \( \Omega(\lambda) \) and \( \omega(\lambda) \) are the Abel integrals as defined above associated with the spectrum

\[
\sigma(\xi^*_b) = \sigma(\xi^*_b, 1) = [b - 2a, b + 2a] \cup [-1, 1].
\]

The level line \( \text{Re} \ g(\lambda, \xi) = 0 \) in this case intersects the real axis at a point \( \lambda_0(\xi) \) inside the gap, which moves continuously along the gap when \( \xi \) moves along the interval \( (\xi^*_b, \xi^*_a) \). The main asymptotic term for the solution is the classic two band solution of the Toda lattice \( \{a(n, t), b(n, t)\} \) with the initial Dirichlet eigenvalue depending on \( \xi \) if the discrete spectrum inside the gap is not empty. Indeed, the phase summand in the theta function representation for this two band solution (cf. [32, Equ. (9.48)]) contains information on the initial scattering data for
and it undergoes a shift when \( \lambda_0(\xi) \) hits a point of the discrete spectrum. This is in agreement with the effect of adding a single eigenvalue as can be done using the double commutation method (cf. \([20]\) and \([32\, \text{Lem. 11.26}]\)). Note that for \( \xi_t = \xi'_{cr} \), the modulated elliptic wave (1.13) coincides with the two band solution \( \{ \hat{a}(n, t), \hat{b}(n, t) \} \). The same is valid for the second boundary of the middle region, that is, for \( \xi = \xi'_{cr,1} \), because the construction of the \( g \)-function in the region \( [\xi'_{cr,1}, \xi_{cr,1}] \) is the same as for the modulated elliptic waves above. It is associated with the set \([b - 2a, b + 2a] \cup [\gamma(\xi), 1]\) where \( \gamma(\xi) \in [-1, 1) \).

A numerical computation of the Toda lattice solution generated by a "pure step" initial data \( a(n, 0) = \frac{1}{2}, \, b(n, 0) = 0 \) as \( n \geq 0 \) and \( a(n, 0) = a, \, b(n, 0) = b \) as \( n < 0 \) is given in Fig. 1 for \( a = 1, \, b = -4 \). The picture demonstrates the behavior of the solution for large (but fixed) time \( t \). Due to scaling of the space variable axis neighboring points almost merge, and we partly observe solid lines instead of single points. On this picture we can clearly distinguish the three regions described above. The left and right regions correspond to the modulated elliptic waves, the middle region corresponds to the quasi-periodic or periodic finite gap solution of (1.1) with the spectrum on two bands \([{-6}, {-2}] \cup [{-1}, 1]\). Recall that quasi-periodicity or pure periodicity of the finite gap solution with spectrum \([b - 2a, b + 2a] \cup [-1, 1]\) depends on the ratio \( r \) of the frequencies of its quasi-momentum \( \omega(\lambda) \),

\[
\frac{\omega(1) - \omega(-1)}{\omega(b + 2a) - \omega(b - 2a)} = r.
\]
If \( r = \frac{p}{q} \in \mathbb{Q} \), where \( \frac{p}{q} \) is an irreducible fraction, then the respective finite gap solution is in fact periodic with period \( p + q \). In particular, for \( a = \frac{1}{2} \) we get a solution of period 2. Moreover, if \( \omega(\lambda, \xi) \) is the quasi-momentum associated with the sets \([b - 2a, \gamma(\xi)] \cup [-1, 1]\) or \([b - 2a, b + 2a] \cup [\gamma(\xi), 1]\) and

\[
r(\xi) = \frac{\omega(1, \xi) - \omega(-1, \xi)}{\omega(\gamma(\xi), \xi) - \omega(b - 2a, \xi)}, \quad \text{or} \quad r(\xi) = \frac{\omega(1, \xi) - \omega(\gamma(\xi), \xi)}{\omega(b + 2a, \xi) - \omega(b - 2a, \xi)}
\]

is rational, then for such a point \( \xi \) the solution is periodic; the period may be quite large, see domains B in Fig. 1. In particular, domain A corresponds to a modulated wave near \( \xi_0 \) such that \( \gamma(\xi_0) = -4 \), where the solution is of period 2.

In [17] three of us derived the precise formulas for the modulated elliptic wave (1.13) using the NSD approach for vector RH problems on the two-sheeted Riemann surface of the spectral parameter \( \lambda \). However, the initial data generating the shock wave in [17] are more restrictive than in the present paper: it was assumed that there are no resonances on the edges of the spectrum of \( H(t) \), and that the discrete spectrum consists of just one point located in the spectral gap. Moreover, the first terms of asymptotics obtained in [17] were not justified. It was conjectured however that the next term of the asymptotic expansion for the solution of (1.1–1.3) is of order \( O(t^{-1}) \). The aim of the present paper is to establish this fact by solving local parametrix problems and by performing the conclusive analysis rigorously. We will implement a rigorous asymptotic analysis in the region

\[
D := \{(n, t) \in \mathbb{Z} \times \mathbb{R}_+: \frac{b}{q} \in \mathcal{I}_\varepsilon := [\xi_{cr} + \varepsilon, \xi_{cr} - \varepsilon]\}.
\]

Since the main terms of the asymptotic expansion in the other two regions are already evaluated (cf. [17]), the remaining asymptotic analysis can be conducted analogously to that for (1.14). However, under condition (2.1) in the region \( \frac{b}{q} \in (\xi_{cr,1}', \xi_{cr,2}') \) the estimate on the error term can be essentially improved. This estimate, and the influence of resonances and the discrete spectrum in the gap on the asymptotic behavior of the solution will be discussed in [15].

In the same way as in [17], in this paper we deal with the vector statements of RH problems. It should be emphasized that for the Toda equation, the vector statement is more natural than the matrix one, because the matrix statements are ill-posed in the class of invertible matrices with \( L^2 \)-integrable singularities. This fact for the Toda lattice can be established similarly as for the KdV shock wave (cf. [18]). In turn, the vector statement of the RH problem requires additional symmetries to be posed on the contours, jump matrices and on the solution itself to guarantee uniqueness of the solution. In [17] the RH problem was stated in terms of \( \lambda \), that is, on the two-sheeted Riemann surface with sheets glued along the cuts \([b - 2a, b + 2a] \cup [-1, 1]\). In the present paper we use the standard approach via the Joukovski map \( z(\lambda) \) in (1.6): the upper sheet of the Riemann surface is identified with the inner part of the circle \(|z| < 1\) without the cut \([z(b - 2a), z(b + 2a)]\), and the lower sheet with \(|z| > 1\) without \([z^{-1}(b - 2a), z^{-1}(b + 2a)]\). We formulate the initial RH problem and reformulate all transformations (conjugations and deformations) which lead to the model RH problem in terms of \( z \) taking into account the presence of the discrete spectrum and resonances, which produce singularities in the jump matrix, and require a more sophisticated analysis and additional proofs of the uniqueness results. We also solve the vector model problem independently, and derive asymptotics by use of a new more convenient formula (2.14).
For the KdV shock case there are two ways to justify asymptotics within the frameworks of NSD. The first approach, proposed recently in \[31\], does not provide a matrix solution of the model RH problem, but instead evaluates the smallness of difference between the initial and the model vector solutions as solutions of the associated singular integral equations with small differing kernels. The second, ”matrix approach” is more traditional, and it requires first to solve the matrix analog of the model RH problem, then to find matrix solutions of the local parametrix RH problems, and finally to derive the singular integral equation for the error vector and to estimate its norm.

The first approach is suitable for the KdV case, but does not seem to work for the Toda equation, because we cannot control well the behavior of the components of the vector solutions \( m(\lambda) \) of the initial and model RH problems at infinity. Indeed, for KdV it is known that \( m(\lambda) \to (1, 1) \) as \( \lambda \to \infty \), but for Toda we only know that \( m_1(\lambda)m_2(\lambda) \to 1 \), which is not sufficient to apply the technique of \[31\]. On the other hand, implementing the traditional matrix approach also leads to complications in the case of KdV and Toda due to a ”partial absence”\(^\text{1}\) of the matrix solution for the model RH problem (cf. \[18\]).

Recently A. Minakov \((29, 30)\) proposed a new singular matrix model solution, which in case of KdV has a pole at \( \lambda = 0 \), however, the respective error vector does not have pole-like singularities. We use this idea to construct the matrix model solution for Toda shock. It has simple poles at the edges of the right background spectrum, but the error vector does not (see Theorem 5.4 below).

To justify that the expansion with respect to \( z \) of the product of components of the solution to the initial RH problem is asymptotically close to the expansion for the model RH problem, one should establish that the error vector \( \nu(z) \) (cf. (6.10), (6.14)) is asymptotically close to the vector \((1, 1)\) up to a term \( O(t^{-1}) \). However, initially we only know that \( \lim_{z \to 0} \nu_1(z)\nu_2(z) = 1 \). To derive and investigate a singular integral equation associated with \( \nu \), we use a curious property of this vector which appears due to the symmetry condition. Indeed, by preserving the symmetry for all vector and matrix solutions on each transformation step, we obtain \( \nu(z^{-1}) = \nu(z)\sigma_1 \), where \( \sigma_1 \) is the first Pauli matrix. Since the point \( z = 1 \) turns out to be regular for \( \nu(z) \), this implies \( \nu_1(1) = \nu_2(1) \). This property is already sufficient to analyze the respective singular integral equation if we take the kernel with zero at \( z = 1 \) as the Cauchy kernel (see (3.14) below). Our main result in this paper is Theorem 7.3.

2. Statement of the Riemann–Hilbert problem

In this section we cover some basic facts of the inverse scattering transform and fix notation. For a detailed account of scattering theory for Jacobi operators with steplike backgrounds see \[14, 15, 16\], with zero background see \[32\], Chapter 10.

Under the assumption that the coefficients of the initial data \((1.2)\) tend to the background constants sufficiently fast\(^\text{2}\) the spectrum of the Jacobi operator \( H(t) \) consists of an absolutely continuous part of multiplicity one,

\[
\sigma_{ac}(H) = [b - 2a, b + 2a] \cup [-1, 1],
\]

\(^\text{1}\)By partial absence we refer to the absence of the invertible matrix solution with \( L^2 \) singularities for some pairs of \((n, t)\) sufficiently large, such that \( \frac{n}{t} = \xi, \forall \xi \in I_\epsilon \).

\(^\text{2}\)For example, with finite first moments of perturbation.
plus a finite simple pure point part,
\[ \{ \lambda_j : j = 1, \ldots, N \} \subset \mathbb{R} \setminus \sigma_{ac}(H). \]
To simplify further considerations we assume that the initial data \( (1.2) \) decay to their backgrounds exponentially fast
\[
(2.1) \quad \sum_{n=1}^{\infty} e^{\rho n} \left( |a(-n,0) - a| + |b(-n,0) - b| + |a(n,0) - \frac{1}{2}| + |b(n,0)| \right) < \infty,
\]
where \( \rho > 0 \) is a positive number. The operator \( H(t) \) is self-adjoint and the diagonal elements of its Green function \( G(\lambda, n, m, t) \) (that is, the kernel of the resolvent operator \( (H(t) - \mu I)^{-1} \)) have the following expansion as \( \lambda \to \infty \) ([22 Sec. 6.1])
\[
(2.2) \quad G(\lambda, n, m, t) = -\frac{1}{\lambda} \left( 1 + b(n,t) + \frac{a(n,t)^2 + a(n-1,t)^2 + b(n,t)^2}{\lambda^2} \right).
\]
As already mentioned in the introduction, instead of the spectral parameter \( \lambda \) we use its Joukovski transformation,
\[
z(\lambda) = \lambda - \sqrt{\lambda^2 - 1},
\]
which maps the sides of the cut along the interval \([-1, 1] \) to the unit circle \( \mathbb{T} = \{ z : |z| = 1 \} \). The map \( z \mapsto \lambda \) is one-to-one between the closed domains \( \text{clos}(Q) \) and \( \text{clos}(\mathbb{C} \setminus \sigma_{ac}(H(t))) \), where\(^3\)
\[
(2.3) \quad Q := \{ z : |z| < 1 \} \setminus [q_1, q],
\]
The points \( q_1 = z(b + 2a) \) and \( q = z(b - 2a) \) correspond to the edges of \( \sigma(H_{\ell}) \) and \( z = -1 \) and \( z = 1 \) correspond to the edges of \( \sigma(H_{r}) \). The eigenvalues \( \lambda_j \) are mapped to \( z_j \in ((-1,0) \cup (0,1)) \setminus [q_1, q] \), for \( j = 1, \ldots, N \); we denote them by
\[
\sigma_d = \{ z_j, j = 1, \ldots, N \}.
\]
In addition to \( z \) we also use the Joukovski transformation \( \zeta = \zeta(\lambda) \) associated with the left background and given by \( (1.6) \).
Recall that the Jacobi equation \( (1.5) \) has two Jost solutions \( \psi(z, n, t), \psi_{r}(z, n, t) \) for each \( z \in Q \) with asymptotic behavior
\[
\lim_{n \to \infty} z^{-n} \psi(z, n, t) = 1, \quad |z| \leq 1; \quad \lim_{n \to -\infty} \zeta^{n} \psi_{r}(z, n, t) = 1, \quad |\zeta| \leq 1.
\]
As functions of \( z \) they have slightly different properties on \( Q \). Indeed, since \( \zeta^n \psi_{r}(z, n, t) \) is in fact an analytical function of \( \zeta \) as \( |\zeta| < 1 \), this function has complex conjugated values on the sides of the cut along \([q_1, q]\), which we denote as \([q_1, q] \pm i0\). It has equal real values at \( z, z^{-1} \in \mathbb{T} \). As for \( \psi(z, n, t) \), it has complex conjugated values at conjugated points of \( \mathbb{T} \), but
\[
\psi(z - i0, n, t) = \psi(z + i0, n, t) \in \mathbb{R}, \quad \text{for } z \in [q_1, q].
\]
Recall that \( \psi_{r}(z, n, 0) \) admits a representation via the transformation operator
\[
\psi_{r}(z, n, 0) = \sum_{-\infty}^{n} K(n, m) \zeta^{-m}, \quad |\zeta| \leq 1.
\]
\(^3\) We treat closure as adding to the boundary the points of the upper and lower sides along the cuts, while considering them as distinct points.
Under condition (2.1) in the domain $1 \leq |\zeta| < e^\sigma$ there exists an analytic function which is an extension of $\tilde{\psi}_r$,

$$\tilde{\psi}_r(z, n) := \sum_{\ell = -\infty}^{n} K(n, m) \zeta^m, \quad \tilde{\psi}_r(z \pm i0, n) = \overline{\psi}_r(z \pm i0, n, 0), \quad z \in [q_1, q].$$

The Jost solutions of (1.5) are connected by the scattering relation

$$T(z, t) \psi_r(z, n, t) = \overline{\psi}(z, n, t) + R(z, t) \psi(z, n, t), \quad |z| = 1,$$

where $R(z, t)$ and $T(z, t)$ are the right reflection and transmission coefficients. Their time evolution is given by

$$R(z, t) = R(z)e^{(z-z^{-1})t}, \quad z \in \mathbb{T}, \quad |T(z, t)|^2 = |T(z)|^2 e^{(z-z^{-1})t}, \quad z \in [q_1, q],$$

where $R(z) = R(z, 0), T(z) = T(z, 0)$. The right norming constants

$$\gamma_j(t) = \left( \sum_{n \in \mathbb{Z}} \psi^2(z_j, n, t) \right)^{-2}$$

corresponding to $z_j \in \sigma_d$ evolve as $\gamma_j(t) = \gamma_j e^{(z_j-z_j^{-1})t}, \quad \gamma_j = \gamma_j(0) > 0$. Let

$$W(z, t) = a(n-1, t)(\psi(z, n-1, t)\psi(z, n, t) - \psi(z, n, t)\psi(z, n-1, t))$$

be the Wronskian of the Jost solutions and $W(z) := W(z, 0)$.

**Resonant points.** The point $\tilde{q} \in \{-1, 1, q, q_1\}$ is called a resonant point if $W(\tilde{q}) = 0$. If $W(\tilde{q}) \neq 0$, then $z$ is non-resonant. Note that $W(\tilde{q}, t) = 0$ iff $W(\tilde{q}) = 0$, that is, the property of being resonant (or not) is preserved with $t$.

Under a much weaker decaying condition than (2.1),

$$\sum_{n=1}^{\infty} n |a(-n, 0) - a| + |b(-n, 0) - b| + |a(n, 0) - \frac{1}{2}| + |b(n, 0)| < \infty,$$

with a finite first moment of perturbation, the set of the associated right initial scattering data

$$\{R(z), z \in \mathbb{T}; \chi(z), z \in [q_1, q]; (z_j, \gamma_j), z_j \in \sigma_d\},$$

where

$$\chi(z) = -2a \frac{\zeta(z - i0) - \zeta^{-1}(z - i0)}{z - z^{-1}} |T(z)|^2, \quad z \in [q_1, q],$$

defines the solution of the Cauchy problem (1.1)–(1.3) uniquely. For each $t$ this solution also has finite first moments of perturbations (cf. (1.5)). The scattering data (2.7) satisfy the following properties (cf. (1.4)):

- The function $R(z)$ is continuous on $\mathbb{T}$ and $R(z^{-1}) = \overline{R(z)} = R^{-1}(z)$ for $z \in \mathbb{T}$.
- If $z = -1$ is non-resonant, then $R(-1) = -1$, and if $z = -1$ is resonant, then $R(-1) = 1$.
- The function $T(z)$ can be restored uniquely for $z \in Q$ from the data (2.7); it is meromorphic with simple poles at $z_j$.
- The function $\chi(z)$ is continuous for $z \in (q_1, q)$ and vanishes at $\tilde{q} \in \{q, q_1\}$ with

$$\chi(z) = C(z - \tilde{q})^{1/2}, \quad z \to \tilde{q} \in \{q, q_1\},$$

if $\tilde{q}$ is a non-resonant point. If $\tilde{q}$ is a resonant point, then

$$\chi(z) = C(z - \tilde{q})^{-1/2}(1 + o(1)), \quad z \to \tilde{q} \in \{q, q_1\}.$$
To apply the nonlinear steepest descent approach in the most general situation which assumes resonances, we will suppose that the number \( \rho > 0 \) in (2.1) is the minimal one to provide an inclusion \([q_1, q] \subset \{ z : e^{-\rho} < |z| < 1 \}\), that is,

\[
\rho > -\log |q|.
\]

Under condition (2.1) the scattering data have additional properties:

- The function \( R(z) \) admits an analytic continuation to \([ z : e^{-\rho} < |z| < 1 \} \setminus [q_1, q] \) with simple poles at the points of the discrete spectrum located in this domain.
- The function \( \chi(z) \) has an analytic continuation \( X(z) \) in a vicinity of \([q_1, q] \) with

\[
\chi(z) = i|\chi(z)| = X(z - i0), \quad z \in [q_1, q],
\]

where

\[
X(z) = \frac{a(\zeta - \zeta^{-1})(z - z^{-1})}{2W(z)W(\psi_\ell, \psi)(z)}.
\]

Here \( \psi_\ell(z, n) \) is defined by (2.4) and \( W(\psi_\ell, \psi)(z) \) is the Wronskian of \( \psi_\ell(z, n, 0) \) and \( \psi(z, n, 0) \).

Treating the values \( n \) and \( t \) as parameters, we define a vector-valued function \( m(z) = (m_1(z, n, t), m_2(z, n, t)) \) on \( Q \) by

\[
m(z, n, t) = (T(z, t)\psi_\ell(z, n, t)z^n, \psi(z, n, t)z^{-n}).
\]

The first component \( m_1(z) \) is a meromorphic function in \( Q \) with poles at \( z_j \). It has continuous limits as \( z \) approaches the boundary of \( Q \) except, possibly, at points \( q \) and \( q_1 \), where a square root singularities may appear in the case of resonance. The second component of this vector is a holomorphic function in \( Q \) with continuous limits to the boundary. Both functions have finite positive limits as \( z \to 0 \) (cf. [17]). For our purpose it will be sufficient to control the product of the components.

**Lemma 2.1.** For \( z \to 0 \) the following asymptotics hold:

\[
m_1(z, n, t)m_2(z, n, t) = 1 + 2zb(n, t)
\]

\[
+ 4z^2 \left( \frac{a(n - 1, t)^2 + a(n, t)^2 + b(n, t)^2 - \frac{1}{2}}{2} \right) + O(z^3).
\]

**Proof.** The Jost solutions \( \psi \) and \( \psi_\ell \) can be considered as the Weil solution of \( H(t) \), and therefore the Green function \( \eqref{eq:2.2} \) considered as a function of \( z \) can be represented as

\[
G(\lambda(z), n, n, t) = \frac{\psi(z, n, t)\psi_\ell(z, n, t)}{W(z, t)}.
\]

Recall that

\[
T(z, t) = \frac{z - z^{-1}}{2W(z, t)}.
\]

that is,

\[
m_1(z, n, t)m_2(z, n, t) = \frac{z - z^{-1}}{2} G(\lambda(z), n, n, t).
\]

Taking into account that \( \frac{1}{\lambda} = \frac{2z}{1+z} \) and \( \frac{z-z^{-1}}{2} = -\sqrt{\lambda^2 - 1} \), we obtain \( \eqref{eq:2.14} \).
Let $Q^* := \{ z : |z| > 1 \} \setminus [q_1^{-1}, q_1^{-1}]$ be the image of the domain $Q$ under the map $z \mapsto z^{-1}$. We extend $m$ to $Q^*$ by $m(z^{-1}) = m(z)\sigma_1$, where $\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the first Pauli matrix. With this extension, the second component $m_2(z)$ is a meromorphic function on $Q^*$ with poles at $z_j^{-1}$, $z_j \in \sigma_d$, and the first component $m_1$ is holomorphic. Being defined now on $\mathbb{C} \setminus \Sigma$, where

$$
(2.16) \quad \Sigma = T \cup [q, q_1] \cup [q_1^{-1}, q^{-1}],
$$

the function $m(z)$ may have jumps along $\Sigma$. For convenience, from here on we encode the orientation of the contours in $\mathbb{R}$ as follows: assume $-\infty \leq c < d \leq \infty$, then we write $[d, c]$ for the interval $[c, d]$ with orientation right-to-left. In particular, the contours $[q, q_1]$, and $[q_1^{-1}, q^{-1}]$ in $(2.16)$ are oriented right-to-left, and the unit circle $T$ is oriented counterclockwise.

Throughout this paper, plus (+) and minus (−) sides of a contour correspond to the left and right sides by orientation, that is, the + side of an oriented contour lies to the left as one traverses the contour in the direction of its orientation. And $m_{\pm}(z)$ denote the boundary values of $m(z)$ as $z$ tends to the contour from the ± side. Using this notation implicitly assumes that the limit exists (in the sense that $m(z)$ extends to a continuous function on the boundary except probably at a finite number of points). In this paper, all contours are symmetric with respect to the boundary values of $m(z)$, i.e. they contain with each point $z$ also $z^{-1}$. The orientation on these contours (or the form of the jump matrices on them) should be chosen in such a way that the following symmetry is preserved for the jump matrix of the vector RH problem and for its solution.

**Symmetry condition.** Let $\Sigma$ be a symmetric oriented contour. Then the jump matrix of the vector problem $m_+(z) = m_-(z)v(z)$ satisfies

$$
(2.17) \quad (v(z))^{-1} = \sigma_1 v(z^{-1}) \sigma_1, \quad z \in \Sigma.
$$

Moreover,

$$
(2.18) \quad m(z) = m(z^{-1})\sigma_1, \quad z \in \mathbb{C} \setminus \Sigma.
$$

To preserve the symmetry condition we will always choose symmetric deformations of the contours. Moreover, we will only use conjugations with diagonal matrices which respect the symmetry condition, as outlined in the next lemma.

**Lemma 2.2 (Conjugation, [25]).** Let $m$ be the solution on $\mathbb{C}$ of the RH problem $m_+(z) = m_-(z)v(z)$, $z \in \hat{\Sigma}$, which satisfies the symmetry condition. Let $d : \mathbb{C} \setminus \Sigma \to \mathbb{C}$ be a sectionally analytic function. Set

$$
(2.19) \quad \tilde{m}(z) = m(z) \begin{pmatrix} d(z)^{-1} & 0 \\ 0 & d(z) \end{pmatrix} = m(z)[d(z)]^{-\sigma_3}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

If $d$ satisfies $d(z^{-1}) = d(z)^{-1}$ for $z \in \mathbb{C} \setminus \Sigma$, then $(2.19)$ respects the symmetry condition. The jump matrix of the problem $\tilde{m}_+ = \tilde{m}_-v$ is given by

$$
\tilde{v} = \begin{cases} 
\begin{pmatrix} v_{11} & v_{12}d^2 \\ v_{21}d^{-2} & v_{22} \end{pmatrix}, & z \in \hat{\Sigma} \setminus \Sigma, \\
\begin{pmatrix} \frac{dz}{dz} v_{11} & v_{12}d_+d_- \\ v_{21}d_+d_-d_1^{-1} & \frac{dz}{dz} v_{22} \end{pmatrix}, & z \in \Sigma.
\end{cases}
$$

In particular, the contour $(2.16)$ is symmetric with respect to $z \mapsto z^{-1}$. To satisfy the symmetry condition for the extended function $(2.13)$ we continue $(2.8)$
to $[q_1^{-1}, q^{-1}]$ as an odd function, $\chi(z) = -\chi(z^{-1})$. This implies that we can extend $X(z)$ given by (2.12) as an even function in a vicinity $O_\rho$ of $[q, q_1] \cup [q_1^{-1}, q^{-1}]$ (which is possible by the condition on $\rho$ in (2.11)).

\begin{equation}
X(z^{-1}) = X(z), \quad z \in \mathbb{C} \setminus (\Sigma \cup \sigma_d \cup \sigma_2^+) \cap O_\rho.
\end{equation}

In $\mathbb{C} \setminus (-\infty, 0]$ introduce the phase function

\begin{equation}
\Phi(z) := \Phi(z, \xi) = \frac{1}{2} (z - z^{-1}) + \xi \log z, \quad \xi := \frac{n}{\ell},
\end{equation}

which is odd with respect to $z \to z^{-1}$, that is, $\Phi(z^{-1}) = -\Phi(z)$. Note that $e^{2\Phi(z)} = z^{2n} e^{t(z-z^{-1})}$ is well defined in $\mathbb{C} \setminus \{0\}$. The vector function (2.13) extended to $Q^*$ by symmetry (2.18) solves the following RH problem (cf. [7, 26, 18]):

**RH problem 1.** [Initial "meromorphic" RH problem statement] Find a vector-valued function $m : \mathbb{C} \setminus \Sigma \to \mathbb{C}^{1 \times 2}$ which is meromorphic in $Q \cup Q^*$, continuous up to $\Sigma$ except at possibly the points $q, q_1, q^{-1}, q_1^{-1}$. It has simple poles at $z_j^{\pm 1}$, $j = 1, \ldots, N$, and satisfies:

- **the jump condition** $m_+(z) = m_-(z) v(z)$, where

\begin{equation}
v(z) = \begin{cases}
0 & z \in \mathbb{T}, \\
\begin{pmatrix}
R(z) e^{2i\Phi(z)} & -\overline{R(z)} e^{-2i\Phi(z)} \\
\chi(z) e^{2i\Phi(z)} & 1
\end{pmatrix}, & z \in [q, q_1], \\
\begin{pmatrix}
1 & \chi(z) e^{-2i\Phi(z)} \\
\chi(z) e^{2i\Phi(z)} & 1
\end{pmatrix}, & z \in [q_1^{-1}, q^{-1}];
\end{cases}
\end{equation}

- **the residue conditions**:

\begin{align*}
\text{Res}_{z = z_j} m(z) &= \lim_{z \to z_j} m(z) \begin{pmatrix} 0 & 0 \\ -z_j \gamma_j e^{2i\Phi(z_j)} & 0 \end{pmatrix}, \quad j = 1, \ldots, N, \\
\text{Res}_{z = z_j^{-1}} m(z) &= \lim_{z \to z_j^{-1}} m(z) \begin{pmatrix} 0 & 0 \\ z_j^{-1} \gamma_j e^{2i\Phi(z_j)} & 0 \end{pmatrix}, \quad j = 1, \ldots, N;
\end{align*}

- **the symmetry condition**: $m(z^{-1}) = m(z) \sigma_1$.
- **the normalization condition**: $m_1(0) \cdot m_2(0) = 1$ and $m_1(0) > 0$.
- **the resonant/non-resonant condition**: If $\chi(z)$ satisfies (2.9) at $\tilde{q}$ then $m(z)$ has finite limits $m(z \to \tilde{q}^{\pm 1}) \in \mathbb{R}^{1 \times 2}$ as $z \to \tilde{q}^{\pm 1}$, $\tilde{q} \in \{q, q_1\}$. If (2.10) is fulfilled then

\begin{align}
m(z) &= \begin{pmatrix}
C_1 & C_2 \\
(z - \tilde{q})^{1/2} & C_2
\end{pmatrix} (1 + o(1)), \quad C_1 C_2 \neq 0, \quad \text{or} \\
m(z) &= \begin{pmatrix}
C_1 & C_2(z - \tilde{q}) \end{pmatrix} (1 + o(1)), \quad z \to \tilde{q}, \quad C_1 C_2 \neq 0.
\end{align}

Respectively, at point $\tilde{q}^{-1}$ the analog of (2.21) holds by symmetry (2.18).

**Lemma 2.3.** Suppose that the initial data of the Cauchy problem (1.1)–(1.3) satisfy (2.0) and let (2.7) be the associated initial right scattering data. Then the vector function $m(z) = m(z, n, t)$ defined by (2.13) and (2.18) is the unique solution of RH problem 1.

The proof of uniqueness is completely analogous to the KdV shock case ([18]).

The behavior of solutions of such RH problems is determined mostly by the behavior of the real part of the phase function $\Phi(z, \xi)$ which depends on the value of the parameter $\xi = \frac{n}{\ell}$. As already mentioned, we intend to study thoroughly the
asymptotics of the solution (1.1)–(1.3) in the domain (1.15). The signature table of Re Φ(z, ξ) for this region is depicted in Fig. 2. One part of the eigenvalues in Q lies in the set Re Φ(z) > 0 (namely z_j ∈ (0, q)), while the remaining eigenvalues belong to the domain Re Φ(z) < 0 (z_k ∈ (−1, q_1) ∪ (0, 1)). As outlined in [7, 25], one can redefine m(z) on Q (an associated transformation on Q^* follows immediately from the symmetry condition) conjugating it with an invertible bounded matrix-function in a way that the residue conditions at z_j ∈ σ_d will be replaced by jump conditions along non-intersecting small circles around points of σ_d. Moreover, the respective jump matrices will be exponentially close as t → ∞ to the unit matrices for all further transformations of RH problem 1. With this transformation, the main contour Σ is not changed and the structure of the jump matrices on it remains qualitatively the same as in RH problem 1 regarding decaying/oscillating properties with respect to t and symmetry properties.

Indeed, let ϵ > 0 be sufficiently small such that the circles T_j = \{z : |z - z_j| = ϵ\}, z_j ∈ σ_d, do not intersect, do not contain the origin, and lie away from T ∪ [q, q_1] (the precise value of ϵ will be chosen later). Denote their images under the map z → z^{-1} by T_j^* and orient both T_j and T_j^* counterclockwise. The curves T_j^* are not circles, but they surround z_j^{-1} with minimal distance from the curve to z_j^{-1} given by \(\frac{z_j^{-1}}{z_j - (z_j - ϵ)}\). Introduce the Blaschke product

\[
\Pi(z) = \prod_{z_j ∈ (q, 0)} \frac{|z_j|}{|z - z_j|} \frac{z - z_j^{-1}}{z_j - z},
\]

and note that \(\Pi(z^{-1}) = \Pi^{-1}(z)\), \(\Pi(0) > 0\). Set

\[
A(z) = \begin{cases} 
\frac{1}{z_j - 1}, & |z - z_j| < ϵ, \quad z_j ∈ (q, 0), \\
\frac{1}{z_j - 1}, & |z - z_j| < ϵ, \quad z_j ∈ (-1, q_1) ∪ (0, 1), \\
\sigma_1(A(z^{-1}))^{-1}σ_1, & |z^{-1} - z_j| < ϵ, \\
\text{I}, & \text{else}.
\end{cases}
\]
Then $m_{\text{ini}}(z) = m_{\text{ini}}(z, n, t)$ defined by
\begin{equation}
(2.23) \quad m_{\text{ini}}(z) = m(z) \Lambda(z)[\Pi(z)]^{-\sigma_1}, \quad z \in \mathbb{C},
\end{equation}
is the unique solution of the following RH problem:

**RH problem 2.** [Holomorphic statement of RH problem for $\xi \in [\xi_{cr}, \xi_{\text{cr}}]$] Find a holomorphic vector function away from $\Sigma \cup \bigcup_{j=1}^{N} (T_j \cup T'_j)$ and satisfying

- the jump condition $m_{+}^1(z) = m_{-}^1(z)v_{+}^1(z)$, where

$$v_{+}^1(z) = \begin{cases} 
\left( \begin{array}{cc}
0 & \frac{-H^2(z)R(z)}{e^{2\Phi(z)}} \\
1 & \Pi^{-2}(z) \chi(z) e^{2\Phi(z)}
\end{array} \right), & z \in \mathbb{T}, \\
1 & 0, & z \in [q, q_1], \\
\left( \begin{array}{cc}
z_{j} \gamma e^{2\Phi(z)} & 0 \\
(z - z_{j})H(z) & 1
\end{array} \right), & z \in T_j, z_j \in (q, 0), \\
\sigma_1 (v_{+}^1(z))^{-1} \sigma_1, & z \in \bigcup_{j=1}^{N} T'_j \cup [q_1^{-1}, q^{-1}];
\end{cases}$$

- $m_{\text{ini}}(z^{-1}) = m_{\text{ini}}(z) \sigma_1$;
- $m_{1}^\text{ini}(0) \cdot m_{2}^\text{ini}(0) = 1$, $m_{\text{ini}}(0) > 0$;
- The resonant/non-resonant condition of RH problem 1 holds for $m_{\text{ini}}^1(z)$ too.

To summarize, for all values of $\xi \in [\xi_{cr}, \xi_{\text{cr}}]$ we performed a one-to-one transformation, replacing the meromorphic RH statement by the holomorphic RH statement,

$$[m(z, n, t); \text{RH problem 1}] \longrightarrow [m_{\text{ini}}^1(z, n, t); \text{RH problem 2}].$$

In the next section we will list some results established in [17], represented in terms of the variable $z$ with modifications which take into account resonances and the additional discrete spectrum.

### 3. Reduction to the Model RH Problem

Let $\xi \in [\xi_{cr}, \xi_{\text{cr}}]$. Before we describe transformations which are applicable in (1.15) and lead to the statement of the model problem used in this region, let us recall that in the case of the shock wave the $g$-function mechanism proved its efficiency for numerous completely integrable equations (cf. [23, 12]). In our case the $z$-analog of the $g$-function constructed in [17] looks as follows. Set

\begin{equation}
(3.1) \quad Q(z) = \sqrt{\frac{z - y}{z - q} \frac{z - y^{-1}}{z - q^{-1}}}, \quad z \in \mathbb{C} \setminus ([q, y] \cup [y^{-1}, q^{-1}]),
\end{equation}

where $y$ is a point\footnote{The point $\gamma(\xi)$ mentioned in the introduction is connected with $y = y(\xi)$ by $\gamma = \frac{x + y^{-1}}{2}$. In the present paper we will use the notation $\lambda_y$ instead of $\gamma$ (see Remark 3.2).} which can be computed implicitly from the condition

\begin{equation}
(3.2) \quad \int_{-1}^{y} P(s) Q(s) \frac{ds}{s} = 0,
\end{equation}

\begin{equation}
(3.3) \quad \int_{-1}^{y} P(s) Q(s) \frac{ds}{s} = 0,
\end{equation}

\begin{equation}
(3.4) \quad \int_{-1}^{y} P(s) Q(s) \frac{ds}{s} = 0.
\end{equation}
with

\[ P(s) := s + s^{-1} + 2\xi + \frac{1}{2}(y + y^{-1} - q - q^{-1}). \]

As is shown in [17], equation (3.2) has the unique solution \( y = y(\xi) \in (q_1, q) \) for any \( \xi \in [\xi_{cr}, \xi_{cr}]. \) The function \( y(\xi) \) is continuous and monotonous, with \( y(\xi_{cr}) = q_1 \) and \( y(\xi_{cr}) = q. \) Moreover, \( y(\xi) \) is differentiable with respect to \( \xi \in (\xi_{cr}, \xi_{cr}) \) (cf. [17, App.]). In turn, for any \( y \) the function \( g(\xi) \) satisfies \( Q^2(z^{-1}) = Q^2(z). \) This implies the evenness \( Q(z^{-1}) = Q(z) \) because \( Q(1^{-1}) = Q(1). \) With the chosen orientation on \( [q, y] \cup [y^{-1}, q^{-1}] \) denote \( Q_+(z) = Q(z - i0). \) From the evenness of \( Q \) outside of \( [q, y] \cup [y^{-1}, q^{-1}] \) we obtain oddness of \( Q_+, Q_+(s) = -Q_+(s^{-1}) \) for \( s \in [q, y] \cup [y^{-1}, q^{-1}]. \) Note, that we choose the square root in (3.1) such that \( Q(z) > 0 \) for \( z \in (q, +\infty). \) Introduce the \( g \)-function by

\[ g(z) = g(z; \xi) = \frac{1}{2} \int_{\xi}^{z} P(s)Q(s) \frac{ds}{s}, \quad z \in \mathbb{C} \setminus (-\infty, 1). \]

**Lemma 3.1.** The function \( g(z) \) satisfies the following properties:

(a) \( g(z) \) is single valued on \( \mathbb{C} \setminus [q^{-1}, q], \) moreover,

\[ g(z^{-1}) = -g(z) \quad \text{for} \quad z \in \mathbb{C} \setminus [q^{-1}, q]; \]

(b) \( \text{Re } g(z) = 0 \) for \( z \in [q, y] \cup [y^{-1}, q^{-1}] \cup \{ z : |z| = 1 \}; \)

(c) \( g(q) = g(q^{-1}) = 0; \)

(d) \( g_{-}(z) = -g_{+}(z) \) for \( z \in [q, y] \cup [y^{-1}, q^{-1}]; \)

(e) \( \Phi(z) - g(z) = K(\xi) + O(z) \) as \( z \to 0, \) where \( K(\xi) \in \mathbb{R}; \)

(f) \( g_{+}(z) - g_{-}(z) = 2iB \) for \( z \in [q, y], \) where

\[ B := -i \int_{q}^{y} P(s)Q_{+}(s) \frac{ds}{s} \in \mathbb{R}_{+}. \]

In particular, \( g_{\pm}(y^{-1}) = \pm iB. \)

**Proof.** (a)--(c) Since \( Q(z^{-1}) = Q(z) \) and \( P(z^{-1}) = P(z) \) for \( z \in \mathbb{C} \setminus [q^{-1}, q], \) then choosing a contour from 1 to \( z \) which does not have common points with the interval \( [q^{-1}, q], \) we obtain (3.5) by the simple change of variables \( s \to s^{-1}. \) Condition (3.2) implies \( g_{\pm}(y) = g_{\pm}(y^{-1}). \) Since the integrand \( P(s)Q_{\pm}(s)s^{-1} \) is purely imaginary for \( s \in [q, y] \cup [y^{-1}, q^{-1}], \) we have

\[ \text{Re } g(q) = \text{Re } g_{\pm}(y) = \text{Re } g_{\pm}(y^{-1}) = \text{Re } g_{\pm}(q^{-1}). \]

Oddness of \( Q_{+}(s) \) also implies \( \text{Im } g(q) = \text{Im } g_{\pm}(q^{-1}) = 0. \) Together with (3.7) and the oddness of \( g \) this yields \( g_{+}(q^{-1}) = g_{-}(q^{-1}) = g(q) = 0. \) Moreover, for \( |s| = 1 \) we have \( Q^2(s) = Q^2(s) \in \mathbb{R}_{+}, \) and therefore \( \text{Im } Q(s)P(s) = 0. \) Since \( \frac{ds}{s} \in \mathbb{R}, \) this implies \( \text{Re } g(z) = 0 \) for \( |z| = 1. \) Items (b)--(c) are thus proved. They imply that \( g(z) \) does not have a jump along \( (-\infty, q^{-1}), \) and this justifies (a). Note that these properties improve [17, Lemma 3.2] (see, e.g. [17, Equ. (3.25)]). The above considerations imply an additional property,

\[ \text{Re } g_{\pm}(-1) = 0. \]

Items (e)--(f) are \( z \)-analog of [17, Equ. (3.21) and (3.24)], and can be obtained by a simple change of variable \( (\lambda, +) \mapsto z. \) Evidently, the constant \( B = B(\xi) \) in (f) is the same as [17, Equ. (3.21)].  \( \square \)
**Remark 3.2.** The point \( y(\xi) \) defines the edge \( \lambda_y = \frac{1}{2}(y+y^{-1}) \) of the Whitham zone for the Toda shock case. The point \( y(\xi) \) coincides with the stationary phase point \( z_0(\xi) \) for \( \Phi(z,\xi) \) at \( \xi = \xi_{cr} \), that is, \( z_0(\xi_{cr}) = q \). One can see that \( y(\xi'_{cr}) \neq z_0(\xi'_{cr}) \).

However, as it was shown in [17], there are proper \( g \)-functions in the whole diapason \( \xi \in (\xi_{cr},\xi_{cr,1}) \), and the respective Whitham point \( y_1(\xi) \) for \( \xi \to \xi_{cr,1} \) will end at \( z_0,\ell(\xi_{cr,1}) = 1 \), where \( z_0,\ell(\xi) \) is the stationary phase point for the left phase function \( \Phi_\ell(z,\xi) \) in (1.8) connected with the left initial scattering data.

The signature table for the real part of \( g \) is depicted in Fig. 3. The points \( y \) and \( y^{-1} \) are nodal points for the curves \( \text{Re } g(z) = 0 \). This signature table allows us to

\[
\begin{array}{c|c|c|c}
\text{Re } g < 0 & \text{Re } g > 0 \\
T_j & T_k & T_y \\
q^{-1} & y^{-1} & q^{-1} \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
\text{Re } g < 0 & \text{Re } g > 0 \\
T_{q'} & y & 0 \\
q_{1}' & q'_1 & 0 \\
\end{array}
\]

**Figure 3.** Signature table of \( \text{Re } g(z,\xi) \) for \( \xi \in (\xi'_{cr},\xi_{cr}) \).

choose the radius \( \epsilon \) of the circles \( T_j \) so small that for \( \Phi_j(z) := \Phi(z_j) - \Phi(z) + g(z) \) we will have

\[
(3.8) \quad \text{Re } \Phi_j(z) < 0 \quad \text{for all } j = 1, \ldots, N \quad \text{and } |z - z_j| = \epsilon.
\]

The radius \( \epsilon \) should also satisfy

\[
8\epsilon < \min \{ \min_{j \neq k} |z_j - z_k|; \min_j |z_j + 1|; \min_j |z_j - 1|; \min_j |z_j - q|; \min_j |z_j - q_1| \}.
\]

Moreover, since we intend to justify the asymptotics uniformly in the regions (1.14) or (1.15) for arbitrary small but fixed positive \( \epsilon \), we also assume that

\[
(3.9) \quad 4\epsilon < |y(\xi_{cr} - \epsilon) - q|,
\]

and additionally that

\[
(3.10) \quad 4\epsilon < |y(\xi'_{cr} + \epsilon) - q_1|.
\]

Choosing such a value of \( \epsilon \), we next perform three transformations which lead to a model problem. The transformations are analogous to those in [17], modified by additional deformations to wipe out the non-\( L^2 \) singularities of the jump matrix in case of resonances at \( q \) or \( q_1 \).

**Step 1:** On \( \mathbb{T} \) one can factorize \( v^{\text{ini}} \) using Schur complements

\[
v^{\text{ini}} = \begin{pmatrix} 1 & -\Pi^2 e^{-2t\Phi} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\Pi e^{2t\Phi} & 1 \end{pmatrix}.
\]

Let \( \xi \in \mathcal{I}_\epsilon \), where \( \mathcal{I}_\epsilon \) is defined by (1.15). Let \( y = y(\xi) \) and let \( \tau, q_1 < \tau < y \), be a point in a small vicinity of \( y \) as depicted in Fig. 4 with

\[
(3.11) \quad \frac{\epsilon}{2} \leq |\tau - y| \leq \epsilon.
\]

Introduce a closed contour \( \mathcal{C}_\tau \) oriented counterclockwise, which starts at \( \tau \) and encloses the interval \([q_1, \tau]\) passing through the point \( q_1 - \epsilon \). Denote the domain
inside this contour by Ωε (with [q1, r] excluded). Let Ωε be an open annulus between

![Figure 4. Contour deformation of Step 1.](image)

circles T and Cε = \{ z : |z| = 1 − ε \} oriented counterclockwise, with Ωε* and Ωε the images of these domains under the map z \mapsto z^{-1}. According to (2.11), the reflection coefficient can be continued as a meromorphic function in the domain \{ z : 1 > |z| > e^{-ρ} \}, which covers the interval [q1, y(ξr − ε)] by (2.11). Thus the reflection coefficient is a holomorphic function in Ωε ∪ Ωε, because these domains do not contain points of the discrete spectrum due to our choice of ε. We extend R(z) to Ωε* ∪ Ωε by R(z) = R(z^{-1}). Redefine m^{ini} by

\begin{equation}
(3.12) \quad m^{(1)}(z) = m^{ini}(z) \begin{cases}
    \begin{pmatrix}
        1 & \Pi^{-2}(z)R(z)e^{2i\Phi(z)} & 0 \\
        -\Pi^{-2}(z)R(z)e^{2i\Phi(z)} & 1 & 0 \\
        0 & 1 & 1
    \end{pmatrix}, & z \in \Omega_{\epsilon} \cup \Omega_{\epsilon}^*, \\
    \begin{pmatrix}
        1 & 0 & 0 \\
        0 & 1 & 0 \\
        0 & 0 & 1
    \end{pmatrix}, & z \in \Omega_{\epsilon}^* \cup \Omega_{\epsilon}^*, \\
    \begin{pmatrix}
        1 & 0 & 0 \\
        0 & 1 & 0 \\
        0 & 0 & 1
    \end{pmatrix}, & \text{else},
\end{cases}
\end{equation}

and orient Cε counterclockwise, Cε* clockwise. Then the jump along T disappears as well as the jump along [r, q1], since the Plücker identity implies that \( R_-(z) - R_+(z) + \chi(z) = 0 \) for \( z \in [r, q_1] \) (compare [12, Lemma 3.2]). Moreover, since the continuation of the reflection coefficient \( R(z) \) is in agreement with the scattering relation (2.5) and (2.13) is the unique solution of RH problem 3, it is straightforward to check that \( m^{(1)}(z) \) given by (2.23), (3.12) does not have singularities at \( q_1 \) and \( q_1^{-1} \) both in the resonant and non-resonant case. In summary, \( m^{(1)}(z) \) satisfies

**RH problem 3.**

- \( m^{(1)}_+ (z, n, t) = m^{(1)}_- (z, n, t)v^{(1)}(z, n, t) \), where

\begin{equation}
v^{(1)}(z) = \begin{cases}
    \begin{pmatrix}
        1 & \Pi^{-2}(z)\chi(z)e^{2i\Phi(z)} & 0 \\
        0 & 1 & 0 \\
        0 & 0 & 1
    \end{pmatrix}, & z \in [q, r], \\
    \begin{pmatrix}
        1 & \Pi^{-2}(z)R(z)e^{2i\Phi(z)} & 0 \\
        0 & 1 & 0 \\
        \sigma_1((v^{(1)}(z^{-1}))^{-1})\sigma_1 & 0 & 1
    \end{pmatrix}, & z \in C_{\epsilon} \cup C_{\epsilon}^*, \\
    v^{ini}(z), & z \in \bigcup_j (T_j \cup T^*_j);
\end{cases}
\end{equation}

- \( m^{(1)}(z^{-1}) = m^{(1)}(z)\sigma_1; \)
- \( m^{(1)}_1(0) \cdot m^{(1)}_2(0) = 1, m^{(1)}_1(0) > 0; \)

\footnote{From here on we (mostly) state RH problems only in terms of their conditions.}
• The resonant/non-resonant condition of RHP \ref{RHP1} holds for \( m^{(1)}(z) \) only at \( q, q^{-1} \).

Step 2: The jump matrix on \([q, r] \cap \{ z : \text{Re} \Phi(z) > 0 \}\) contains off-diagonal elements which are exponentially increasing in time. One can get rid of this exponential growth by replacing the phase function with the \( g \)-function, which is purely imaginary on \([q, y]\) and has negative real part on \([y, r]\). For \( z \in \mathbb{C}\) set

\[
m^{(2)}(z) = m^{(1)}(z) e^{-t(\Phi(z) - g(z))} = m^{(1)}(z) \begin{pmatrix} e^{-t(\Phi(z) - g(z))} & 0 \\ 0 & e^{t(\Phi(z) - g(z))} \end{pmatrix},
\]

Then Lemma \ref{lem:1} and (3.6) imply that \( m^{(2)}(z) \) is the unique holomorphic solution of the following problem:

**RH problem 4.**

- \( m^{(2)}_+(z, n, t) = m^{(2)}(z, n, t) v^{(2)}(z, n, t) \), where

\[
v^{(2)}(z) = \begin{cases} 
\begin{pmatrix} e^{t(g_+ - g_-)} & 0 \\
\Pi^{-2} \chi & e^{-t(g_+ - g_-)} 
\end{pmatrix}, & z \in [q, y], \\
\begin{pmatrix} e^{2itB} & 0 \\
\Pi^{-2} \chi e^{2it \text{Re} g} & e^{-2itB} 
\end{pmatrix}, & z \in [y, r], \\
\begin{pmatrix} e^{2itB} & 0 \\
0 & e^{-2itB} 
\end{pmatrix}, & z \in [r, r^{-1}], \\
\sigma_1(v^{(2)}(z^{-1}))^{-1} \sigma_1, & z \in [r^{-1}, q^{-1}], \\
e^{-t(g - g)g_+} v^{(1)}(z) e^{t(g - g)g_+}, & z \in \mathcal{K},
\end{cases}
\]

and

\[
\mathcal{K} := \mathcal{C}_r \cup \mathcal{C}_r^* \cup \mathcal{C}_e \cup \mathcal{C}_e^* \cup \bigcup_{j=1}^N (T_j \cup T_j^*) ;
\]

- \( m^{(2)}(z^{-1}) = m^{(2)}(z) \sigma_1; \)
- \( m^{(2)}_1(0) \cdot m^{(2)}_2(0) = 1, m^{(2)}(0) > 0. \)
- The vector \( m^{(2)}(z) \) does not have singularities at \( q_1 \) and \( q_1^{-1}. \) It has bounded values at \( r \) and \( r^{-1}. \) Its behavior at \( q \) and \( q^{-1} \) is the same as for \( m(z) \) in (2.21).

**Remark 3.3.** Our choice of \( y, r \) and \( \xi \) in (3.8)–(3.9), (3.11) guarantees that

\[
v^{(2)}(z) = I + O(e^{-c(z)t}), \quad \text{for} \quad z \in \mathcal{K}, \quad \text{as} \quad t \to \infty,
\]

uniformly for \( \xi \in \mathcal{I}_\xi. \)

Step 3: The last step involves the lense mechanism to remove the oscillating terms (with respect to \( t \)) in the jump matrix on \([q, y] \cup [y^{-1}, q^{-1}].\) To this end introduce the function

\[
\Omega(z, s) = \frac{1}{2} \left( \frac{s + z}{s - z} - \frac{s + 1}{s - 1} \right) \frac{1}{s},
\]

which can be considered as the Cauchy kernel for symmetric contours which do not contain point \( z = 1, \) because \( \Omega(z, s) = \frac{1}{z - s} (1 + o(1)) \) as \( z \to s \neq 1, \) and this kernel vanishes at \( z = 1. \) Moreover, it is easy to check that if \( f(s) = f(s^{-1}) \) is a piecewise-continuous function, then

\[
\int_{c-1}^{c} f(s) \Omega(z, s) ds = - \int_{c-1}^{c} f(s) \Omega(z^{-1}, s) ds.
\]
Set
\begin{equation}
Q_1(z) = \sqrt{\frac{(z-q)(z-y)(z-y^{-1})(z-q^{-1})}{z^2}}, \quad z \in \text{clo} \mathcal{E},
\end{equation}
where
\begin{equation}
\mathcal{E} = \mathbb{C} \setminus ([q, y] \cup [y^{-1}, q^{-1}]).
\end{equation}
This function satisfies the following symmetries: \(Q_1(z^{-1}) = Q_1(z)\) for \(z \in \mathcal{E}\) and \([Q_1]_-(z) = [Q_1]_+(z^{-1}) = -[Q_1]_+(z)\) for \(z \in [q, y] \cup [y^{-1}, q^{-1}]\). Define
\begin{equation}
p(z) = \frac{1}{2\pi i} \int_q^{q^{-1}} \Omega(z, s)f(s)ds, \quad f(s) := \begin{cases} 
\log(\Pi^{-2}(s)|\chi(s)|)_{[Q_1]_+(s)}, & s \in [q, y], \\
f(s^{-1}), & s \in [y^{-1}, q^{-1}], \\
\chi s^{-1}, & s \in [y, y^{-1}], 
\end{cases}
\end{equation}
where
\begin{equation}
\Delta = \Delta(\xi) = -2i \int_q^{y} \log(\Pi^{-2}(s)|\chi(s)|) ds s \left( \int_q^{y^{-1}} \frac{ds}{sQ_1(s)} \right)^{-1}.
\end{equation}
It is straightforward to verify that \(p(z)\) solves the following scalar RH problem:
\begin{align*}
p_+(z) &= p_-(z) + f(z), \quad z \in [q, q^{-1}], \\
p(z^{-1}) &= -p(z), \quad z \in \mathcal{E}, \\
p(z) &= O(1), \quad z \to 0.
\end{align*}
The above considerations imply for \(F(z)\) defined by
\begin{equation}
F(z) = e^{Q_1(z)p(z)}, \quad z \in \mathbb{C} \setminus [q, q^{-1}],
\end{equation}
the following properties.

**Lemma 3.4.** The function \(F(z)\) satisfies
\begin{enumerate}[(a)]
\item \(F_+(z)F_-(z) = \Pi^{-2}(z)|\chi(z)|\) for \(z \in [q, y]\); \\
\item \(F_+(z)F_-(z) = \Pi^{-2}(z)|\chi(z)|^{-1}\) for \(z \in [y^{-1}, q^{-1}]\); \\
\item \(F_+(z) = F_-(z)e^{\Delta}\) for \(z \in [y, y^{-1}]\); \\
\item \(F(z^{-1}) = F^{-1}(z)\) for \(z \in \mathbb{C} \setminus [q, q^{-1}]\); \\
\item \(F(0) > 0, \ F(1) = 1\); \\
\item If \(\chi(z)\) satisfies \(\ref{2.10}\) at \(q^{\pm 1}\) then \(F(z) = C(z-q^{\pm 1})^{1/4}(1+o(1))\) as \(z \to q^{\pm 1}\). \\
\item If \(\chi(z)\) satisfies \(\ref{2.10}\) at \(q^{\pm 1}\) then \(F(z) = C(z-q^{\pm 1})^{1/4}(1+o(1))\) as \(z \to q^{\pm 1}\).
\end{enumerate}

Set
\begin{equation}
G(z) = \begin{pmatrix}
F^{-1}(z) & -\Pi^2(z)F(z) e^{-2\gamma_2(z)} \\
0 & X(z) F(z)
\end{pmatrix},
\end{equation}
where the function \(X(z)\) is well defined by \(\ref{2.12}\), \(\ref{2.20}\) in vicinities of \([q, y]\) and \([y^{-1}, q^{-1}]\) and satisfies the property \(X_+(z) = \pm i|\chi(z)|\) for \(z \in [q, y]\) and \(X_+(z) = \mp i|\chi(z)|\) for \(z \in [y^{-1}, q^{-1}]\). Recalling that \(g_+(z) = -g_-(z)\) for \(z \in [q, y] \cup [y^{-1}, q^{-1}]\), we observe that \(\psi^{(2)}(z)\) can be factorized by
\begin{equation}
\psi^{(2)}(z) = \begin{pmatrix}
G_-(z) & 0 & i \\
0 & 0 & 0 \\
0 & -i & 0
\end{pmatrix} \begin{pmatrix}
G_+(z)^{-1} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
\sigma_1G(z^{-1})^{-1}\sigma_1 & 0 & 0 \\
0 & \sigma_1G(z^{-1}) & 0 \\
0 & 0 & \sigma_1G(z^{-1})
\end{pmatrix}, \quad z \in [q, y],
\end{equation}
\begin{equation}
\psi^{(2)}(z) = \begin{pmatrix}
G_+(z) & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
\sigma_1G(z^{-1}) & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
\sigma_1G(z^{-1})^{-1}\sigma_1 & 0 & 0 \\
0 & \sigma_1G(z^{-1}) & 0 \\
0 & 0 & \sigma_1G(z^{-1})
\end{pmatrix}, \quad z \in [y^{-1}, q^{-1}].
\end{equation}
In the domain of existence of $X(z)$ introduce subdomains $\Omega$ and $\Omega^*$ as depicted in Fig. 5. These domains should not contain or intersect $T_j$ and $T_j^*$ and should be situated inside the regions $\text{Re } g > 0$ and $\text{Re } g < 0$, respectively, as well as their boundaries $\mathcal{C}$ and $\mathcal{C}^*$ (as oriented in Fig. 5). Define $m^{(3)}(z)$ by

\begin{equation}
(3.20) \quad m^{(3)}(z) = m^{(2)}(z) \begin{cases}
G(z), & z \in \Omega, \\
\sigma_1 G(z^{-1}) \sigma_1, & z \in \Omega^*, \\
(F(z))^{-\sigma_3}, & z \in \mathbb{C} \setminus (\Omega \cup \Omega^*).
\end{cases}
\end{equation}

Theorem 3.5. For $\xi \in \mathcal{T}_\varepsilon$, RH problem \[3\text{ is equivalent to the following RH problem: to find a vector-function holomorphic in } \mathbb{C} \setminus (\mathcal{K} \cup \{q, q^{-1}\}) \text{ (cf. (3.13)) which has continuous limits on the sides of the contour } \mathcal{K} \cup \{q, q^{-1}\} \text{ except of points } q, q^{-1}, y, y^{-1} \text{ and satisfies}

- the jump condition $m^{(3)}_+(z, n, t) = m^{(3)}_-(z, n, t) v^{(3)}(z, n, t)$, where

\[ v^{(3)}(z) = \begin{cases}
\begin{pmatrix} i\sigma_1 & -i\sigma_1 \end{pmatrix}, & z \in [q, y], \\
\begin{pmatrix} e^{2itB-i\Delta} & 0 \\
0 & e^{-2itB+i\Delta} \end{pmatrix}, & z \in [r, y^{-1}], \\
\begin{pmatrix} e^{2itB-i\Delta} & 0 \\
\frac{\Pi^2(z)F_+(z)F_-(z)}{X(z)} e^{-2itB+i\Delta} & 0 \\
0 & \frac{1}{X(z)} \end{pmatrix}, & z \in [y, t], \\
\sigma_1 (v^{(3)}(z^{-1}))^{-1} \sigma_1, & z \in \mathcal{C} \cup [r^{-1}, y^{-1}], \\
\left[ F(z) \right]^{-\sigma_3} v^{(2)}(z) \left[ F(z) \right]^{\sigma_3}, & z \in \mathcal{K},
\end{cases}\]

- the symmetry condition $m^{(3)}(z^{-1}) = m^{(3)}(z) \sigma_1$;

- and the normalization condition: $m^{(3)}_1(0) \cdot m^{(3)}_2(0) = 1$, $m^{(3)}_1(0) > 0$.

At the points $\{q, q^{-1}, y, y^{-1}, r, r^{-1}\}$ of discontinuity of the jump matrix, $m^{(3)}(z)$ has the following behavior: it has at most a fourth root singularity

\begin{equation}
(3.21) \quad m^{(3)}(z) = O(z - \kappa)^{-1/4}, \quad \text{as } z \to \kappa \in \{q, q^{-1}\}, \text{ and}
\end{equation}

\begin{equation}
(3.22) \quad \lim_{z \to 1} m^{(3)}(z) = (\eta, \eta), \quad \eta \in \mathbb{R}.
\end{equation}
Here $B$ and $\Delta$ are defined by (3.6) and (3.18), $g(z)$ by (3.4), and $F(z)$ by (3.15), (3.17), (3.19). For large $z$, $m^{(3)}(z)$ and the solution $m(z)$ of the initial RH problem are connected via

$$m^{(3)}(z) = m(z) \left[ \Pi(z) F(z) e^{t(\Phi(z) - g(z))} \right]^{-\sigma_3}. \quad (3.23)$$

Proof. The jump condition is immediate from Lemmas 3.4, 3.1, which imply

$$\frac{F_-(z)}{F_+(z)} e^{i(g_+(z) - g_-(z))} = e^{2itB - i\Delta}, \quad z \in [y, y^{-1}]. \quad (3.24)$$

The claims to be discussed in more detail are (3.21) and (3.22). Transformation (3.20) implies that in a vicinity of $q$

$$m^{(3)}(z) = \left( F^{-1}(z)m^{(2)}_1(z), -\frac{\Pi^2(z)F(z)}{X(z)} m^{(2)}_1(z)e^{-2tg(z)} + F(z)m^{(2)}_2(z) \right).$$

In the non-resonant case (2.9) we have three possibilities for $m$, and therefore for $m^{(2)}$, which include possible zeros of the Jost solutions,

(i) $m^{(2)}(q) = (C_1, C_2)$
(ii) $m^{(2)}(z) = (C_1(z - q)^{1/2}, C_2)(1 + o(1))$
(iii) $m^{(2)}(z) = (C_1, C_2(z - q))(1 + o(1))$, where $C_1C_2 \neq 0$.

The symmetry condition implies the respective behavior at $q^{-1}$. In the non-resonant case (2.10) we have (2.21). By use of (f) and (g) of Lemma 3.4 we obtain (3.21).

Since $m^{(3)}(z)$ does not have a jump at $z = 1$, then the symmetry condition implies $m_1(1) = m_2(1^{-1}) = m_2(1)$, that is, $\lim_{z \to 1} m^{(3)}(z) = (\eta, \eta)$. \hfill $\square$

In summary, we have transformed the initial RH problem $[m^{ini}(z, n, t); \text{RHP 2}]$ by Steps 1-3 to an equivalent RH problem $[m^{(3)}(z, n, t); \text{Theorem 3.5}]$ with jump matrix $v^{(3)}$ of the form $v^{(3)} = v^{\text{mod}} + v^{\text{err}}$, where

$$v^{\text{mod}} = \begin{cases} 
  i\sigma_1, & z \in [g, y], \\
  -i\sigma_1, & z \in [y^{-1}, q^{-1}], \\
  e^{2itB - i\Delta}, & z \in [y, y^{-1}]. 
\end{cases} \quad (3.25)$$

The matrix $v^{\text{err}}$ is defined by $v^{\text{err}}(z) = v^{(3)}(z) - I$ for $z \in \mathcal{K} \cup \mathcal{C} \cup \mathcal{C}^*$. The matrix $v^{\text{mod}}$ on $[g, q^{-1}]$ is the jump matrix for an explicitly solvable RH problem and its solution will yield the principal term of the long-time asymptotic expansion of the solution for the initial value problem (1.1)–(1.3), (2.1). We will solve this model RH problem in the next section. The jump matrix $v^{\text{err}}$ on the contour $\mathcal{K} \cup \mathcal{C} \cup \mathcal{C}^* \cup [\xi, \xi^{-1}]$ is exponentially close to the identity matrix as $t \to \infty$ except for small neighborhoods of the critical (parametrix) points $y, y^{-1}$. To estimate the error term one has to rescale the equivalent RH problem in neighborhoods of the parametrix points and solve the respective local problems, which can be analyzed and controlled individually. This will be done in Section 6.

4. Solution of the vector model RH problem

We have to solve the following jump problem

**Model RH problem.** Find a holomorphic vector-function in $\mathbb{C} \setminus [q^{-1}, q]$ satisfying

- the jump condition $m^{\text{mod}}_r(z) = m^{\text{mod}}_l(z)v^{\text{mod}}(z)$ with $v^{\text{mod}}(z)$ given by (3.25);
- $m^{\text{mod}}(z^{-1}) = m^{\text{mod}}(z)\sigma_1$. 


• $m_1^\text{mod}(0) \cdot m_2^\text{mod}(0) = 1$, $m_1^\text{mod}(0) > 0$;

• The vector $m^\text{mod}(z)$ has continuous limits as $z$ approaches the jump contour except of the end points $q$ and $q^{-1}$, and the points of discontinuity of the jump matrix $y$, $y^{-1}$ where the forth-root singularities are admissible.

The uniqueness of the solution to this problem is proved in [17].

Consider the two-sheeted Riemann surface $\mathbb{X}$ associated with the function

$$R(z) = \sqrt{(z - q)(z - y)(z - y^{-1})(z - q^{-1})}$$

such that $R(1) \in \mathbb{R}_+$ and $R(-1) \in \mathbb{R}_-$. The sheets of $\mathbb{X}$ are glued along the cuts $[q^{-1}, y^{-1}]$ and $[y, q]$. Points on $\mathbb{X}$ are denoted by $(z, \pm)$. We first choose a canonical homology basis of cycles $\{a, b\}$ on $\mathbb{X}$, see Fig. 6. The $b$ cycle surrounds the interval $[y, q]$ counterclockwise on the upper sheet and the $a$ cycle passes from $y$ to $y^{-1}$ on the upper sheet and back from $y^{-1}$ to $y$ on the lower sheet.

![Figure 6. Homology basis on $\mathbb{X}$. Solid curves lie on upper sheet, dotted one lies on lower sheet.](image)

Consider the normalized holomorphic Abel differential

$$\zeta = \frac{dz}{\Gamma R(z)}, \quad \Gamma = \int_a \frac{dz}{R(z)} = 2 \int_y^{y^{-1}} \frac{dz}{R(z)} > 0,$$

then $\int_a \zeta = 1$ and $\tau = \tau(\xi) = \int_a \zeta \in \mathbb{i}\mathbb{R}_+$. From here on we work on the upper sheet of $\mathbb{X}$ and identify it with the domain $\mathcal{E}$ given by (3.16). On $\mathcal{E}$ introduce the Abel map $A(z) = \int^z \zeta$ for $z$. Its properties are determined by those of $R(z)$, that is, we will take into account that

$$\frac{dz}{R(z)} = - \frac{d(z^{-1})}{R(z^{-1})}, \quad z \in \mathcal{E};$$

(4.1) \[
\frac{ds}{R_-(s)} = \frac{d(s^{-1})}{R_-(s^{-1})}, \quad s \in [q^{-1}, y^{-1}] \cup [y, q], \quad R_-(z) = R(z + i0).
\]

**Lemma 4.1.** The Abel map $A(z)$ satisfies

(4.2) \[
A(z^{-1}) = -A(z) + \frac{1}{2}, \quad A(q^{-1}) = \frac{1}{2}, \quad A(y) = \frac{\tau}{2} \left( \text{mod } \tau \right),
\]

(4.3) \[
A_{\pm}(-1) = \frac{1}{4} \mp \frac{\tau}{2}, \quad A(1) = \frac{1}{4}, \quad A(0) = \frac{1}{2} - A(\infty),
\]

(4.4) \[
A_+(z) = -A_-(z), \quad z \in [q, y],
\]

(4.5) \[
A_+(z) = A_-(z) - \tau, \quad z \in [y, y^{-1}],
\]

(4.6) \[
A_+(z) = -A_-(z) + 1, \quad z \in [y^{-1}, q^{-1}].
\]
Associated with $X$ is the Riemann theta function

$$\theta(z) = \theta(z \mid \tau) = \sum_{k \in \mathbb{Z}} \exp \left( \pi i k^2 \tau + 2\pi i k z \right).$$

It satisfies $\theta(-z) = \theta(z)$ and $\theta(z + l + k \tau) = \exp(-2\pi i k z) \theta(z)$ for $l, k \in \mathbb{Z}$.

**Lemma 4.2.** [Vector solution of the model RHP] On $\mathbb{C} \setminus [q, q^{-1}]$, define

$$\delta(z) = \frac{\theta(A(z) - \frac{1}{2} + \frac{1}{2} B - \frac{\Delta}{2}) \theta(A(z) + \frac{1}{2} B - \frac{\Delta}{2})}{\theta(A(z) - \frac{1}{2}) \theta(A(z))},$$

and

$$H(z) = \sqrt{\frac{(y - z)(y^{-1} - z)}{(q - z)(q^{-1} - z)}}.$$

Then the vector function

$$m^{\text{mod}}(z) = (\delta(z), \delta(z^{-1})) \frac{H(z)}{\sqrt{\delta(0)\delta(\infty)}}$$

is the unique solution of the model RH problem. Moreover,

$$\delta(z) = \frac{\theta(2A(z) - \frac{1}{2} + \frac{1}{2} B - \frac{\Delta}{2} | 2\tau)}{\theta(2A(z) - \frac{1}{2} | 2\tau)}.$$

**Proof.** Using the formula $\theta(v \mid \tau)\theta(v - \frac{1}{2} \mid \tau) = \theta(2v - \frac{1}{2} \mid 2\tau)\theta(\frac{1}{2} \mid 2\tau)$ (cf. [11]) we can rewrite $\delta(z)$ as a quotient of two theta functions with double period $2\tau$ as it is written in (4.10). In turn, applying Lemma 4.1 to (4.7) and using that

$$H(z^{-1}) = H(z), \quad z \in \mathbb{C} \setminus [q, y] \cup [y^{-1}, q^{-1}] ; \quad H(0) = 1 ;$$

$$H_+(z) = iH_-(z), \quad z \in [q, y]; \quad H_+(z) = -iH_-(z), \quad z \in [y^{-1}, q^{-1}],$$

it is straightforward to check that (4.9) satisfies the jump (3.25) as far as the symmetry and normalization conditions. In fact, (4.8)–(4.9) are the $z$-analog of [17 Equ. (5.22)], where the vector model problem solution (4.9) was computed on the Riemann surface $\mathbb{M}(\xi)$ of the function

$$R^{1/2}(\lambda) = -\sqrt{(\lambda^2 - 1)(\lambda - \lambda_q)(\lambda - \lambda_y)}$$

with $\lambda_q = b - 2a = \frac{1}{2}(q + q^{-1})$ and $\lambda_y = \frac{1}{2}(y + y^{-1})$.

Recall that $B = B(\xi)$ depends on $n$ and $t$. By [17 Lem. 5.3],

$$2itB = -n\Lambda - tU,$$

where $\Lambda$ and $U$ are the $b$-periods of the normalized Abel differentials $\Omega_0$ and $\omega_{\infty_+ \infty_-}$ of the second and third kind on $\mathbb{M}(\xi)$ (cf. [32 Ch. 9]). They do not correspond to the respective Abel differentials on $X$, but due to (4.1) the constants $\Lambda$ and $U$ can be easily expressed in terms of the variable $z$. In particular,

$$\Lambda = 2 \int_y^q \frac{(s - h)(s - h^{-1}) ds}{W_-(s)} s,$$

\[6\text{In fact, } \Omega_0 = \frac{\partial}{\partial \Lambda} \Omega(\lambda, \xi) d\lambda \text{ and } \omega_{\infty_+ \infty_-} = \frac{\partial}{\partial \Lambda} \omega(\lambda, \xi) d\lambda \text{ from [1,12].} \]
where \( \lambda_h = \frac{h + h^{-1}}{2} \) is the zero of \( \omega_{-\infty, -\infty} \). Note that \( \Lambda \) is connected with the Abel map \( A(z) \) by (cf. [32])

\[
(4.12) \quad A(\infty) = A(0) - \frac{\Lambda}{4\pi i}.
\]

**Remark 4.3.** Observe that by (4.3),

\[
\delta_\pm(-1) = \frac\theta\left(-\frac14 \mp \frac\pi2 + \frac{\theta\left(\frac14 \mp \frac\pi2 - \frac{\Delta}{4\pi}\right)}{\theta\left(-\frac14 \mp \frac\pi2\right)}\right).
\]

This implies that \( m_n^{\text{mod}}(-1) = (0, 0) \) if \( 2tB - \Delta = \pi + 2\pi k, k \in \mathbb{Z} \). As it is shown in [13], for those pairs of \( n \) and \( t \) which satisfy

\[
n\Lambda + tU = i(2k + 1)\pi, \quad k \in \mathbb{Z},
\]

a bounded and invertible (in \( E \)) matrix solution of the model jump problem (3.25) with integrable isolated singularities on the jump contour \([y, y^{-1}]\) does not exist.

In the next section we propose a matrix model solution \( M^{\text{mod}}(z) \) which is invertible for all \( n \) and \( t \), but has poles at the edges of the right background (at \( z = 1 \) and \( z = -1 \)). We will establish that the determinant of this matrix does not have singularities at these points, and therefore is a nonzero constant. Moreover, \( m_{n}^{(3)}(z)[M^{\text{mod}}(z)]^{-1} \) does not have singularities at these points too, and hence is a suitable vector for the conclusive asymptotic analysis.

### 5. The matrix model RH problem

Let \( \omega(p) = \int_{R_{-2\pi}}^{p} \omega_{-\infty, -\infty} \) be the Abel integral of the third kind on the Riemann surface \( M(\xi) \) of (4.11), as introduced in [17]. Let \( I(\xi) \) be the closed contour on \( M(\xi) \) with projection on the interval \([\lambda_y, 1] \), which starts at \( \lambda_y \), passes to \(-1 \) on the upper sheet and returns on the lower sheet. Then

\[
(5.1) \quad \omega_+(p) - \omega_-(p) = -\Lambda, \quad p \in I(\xi).
\]

We associate \( z \in Q(\xi) := \{ z : |z| < 1 \} \setminus [q, y] \) with \( p = (\lambda, +) \) on the upper sheet of \( M(\xi) \), and \( z^{-1} \), \( z \in Q(\xi) \), with \( p^* = (\lambda, -) \) on the lower sheet. Calculating the \( z \)-analog of (5.1) and taking into account the symmetry property \( \omega(p) = -\omega(p^*) \), we obtain that \( e^{-\omega(p)} = G(z) \), defined on \( z \in E \) if and only if \( p \in M(\xi) \setminus I(\xi) \), admits the representation

\[
G(z) = \exp\left(\int_q^z \frac{(s - h)(s - h^{-1})}{\mathcal{R}(s)} ds\right), \quad z \in E,
\]

and has the following properties.

- The function \( G(z) \) is holomorphic on \( E \) and satisfies \( G(z^{-1}) = G^{-1}(z), z \in E \).
- Its jumps are given by
  \[
  G_+(z) = G_-(z)e^{-\Lambda}, \quad z \in [y, y^{-1}],
  \]
  \[
  G_\pm(z) = [G_\pm(z^{-1})]^{-1}, \quad z \in [q, y] \cup [y^{-1}, q^{-1}].
  \]
- The following asymptotic expansion is valid,
  \[
  G(z) = -\frac{\tilde{a}}{2z}\left(1 + 2bz + O(z^2)\right), \quad G(z^{-1}) = -\frac{2z}{\tilde{a}}\left(1 - 2bz + O(z^2)\right).
  \]
Here $\tilde{a}$ and $\tilde{b}$ are the coefficients of the asymptotic expansion for $\omega(p)$ as $p \to \infty_{\pm}$ (cf. [32 Equ. (9.44)]),

$$e^{\omega(p)} = -\left(\frac{\tilde{a}}{\lambda}\right)^{\pm 1} \left(1 + \frac{\tilde{b}}{\lambda} + O(\lambda^{-2})\right).$$

Note that in all our considerations the values $y$, $\tau$, $h$ etc. depend on $n$ via $\xi$. To emphasize the dependence of (4.10) on $n$, we abbreviate (5.3)

$$\alpha_n(z) := \delta(z) \frac{H(z)}{\sqrt{\delta(0)\delta(\infty)}} = \alpha_n H(z) \frac{\theta(2A(z) - \frac{1}{2} - \frac{nA}{2\pi} - \frac{it}{2\pi} - \frac{A}{2\pi} | 2\tau)}{\theta(2A(z) - \frac{1}{2} | 2\tau)},$$

Then

$$\alpha_n := \frac{\theta(2A(\infty) - \frac{1}{2} | 2\tau)}{\sqrt{\theta(2A(\infty) - \frac{1}{2} - \frac{nA}{2\pi} - \frac{it}{2\pi} - \frac{A}{2\pi} | 2\tau)\theta(2A(0) - \frac{1}{2} - \frac{nA}{2\pi} - \frac{it}{2\pi} - \frac{A}{2\pi} | 2\tau)}}.$$

We fix $y$, $\tau$ and $h$ in (5.3) and consider this expression shifted to $n+1$,

$$\alpha_{n+1}(z) = \alpha_{n+1} H(z) \frac{\theta(2A(z) - \frac{1}{2} - \frac{(n+1)A}{2\pi} - \frac{it}{2\pi} - \frac{A}{2\pi} | 2\tau)}{\theta(2A(z) - \frac{1}{2} | 2\tau)}.$$

Lemma 5.1. Introduce the vector function (holomorphic in $E$)

$$m^\#(z) = (\beta_n(z), \beta_n(z^{-1})), \quad \text{where} \quad \beta_n(z) := \alpha_{n+1}(z) G(z^{-1}).$$

Then $m^\#(z)$ solves the jump problem of the model RH problem, that is,

$$m^\#_+(z,n,t) = m^\#_-(z,n,t) v^{\text{mod}}(z), \quad \text{where}$$

$$v^{\text{mod}}(z) = \begin{cases} i\sigma_1, & z \in [q,y], \\ -i\sigma_1, & z \in [y^{-1}, q^{-1}], \\ e^{nA+itU+i\Delta}, & z \in [y, y^{-1}], \\ 0 & \text{otherwise}. \end{cases}$$

It also satisfies the symmetry condition $m^\#(z^{-1}) = m^\#(z) \sigma_1$. The normalization condition is not fulfilled, instead $\lim_{z \to 0} m^\#_1(z) m^\#_2(z) = 1$. More precisely,

$$m^\#_1(z) = -\frac{2z}{a} \alpha_{n+1}(0)(1 + O(z)), \quad m^\#_2(z) = \frac{\tilde{a}}{2z} \alpha_{n+1}(\infty)(1 + O(z)), \quad \text{as } z \to 0.$$

Introduce two functions defined on $E \setminus \{1, -1\}$,

$$\Psi_1(z) = m^\text{mod}_1(z) + \rho(z)m^\text{mod}_1(z) = \alpha_n(z) + \rho(z) \beta_n(z),$$

$$\Psi_2(z) = m^\text{mod}_2(z) + \rho(z)m^\text{mod}_2(z) = \alpha_n(z^{-1}) + \rho(z) \beta_n(z^{-1}),$$

where

$$\rho(z) = -\rho(z^{-1}) = \frac{K_n}{\tilde{a} (z^{-1} - z^{-1})}, \quad K_n = \sqrt{\frac{\theta(2A(0) - \frac{1}{2} - \frac{(n+1)A}{2\pi} - \frac{it}{2\pi} - \frac{A}{2\pi} | 2\tau)}{\theta(2A(0) - \frac{1}{2} - \frac{nA}{2\pi} - \frac{it}{2\pi} - \frac{A}{2\pi} | 2\tau)}}.$$

Lemma 5.2. The following formula is valid,

$$K_n^{-1} = \alpha_n(0) \alpha_{n+1}(\infty).$$

Proof. The proof is straightforward using (4.12) and Lemma 4.1. \qed
Lemma 5.3.  

(i) The matrix

\[ M^{\text{mod}}(z) = \begin{pmatrix} \Psi_1(z) & \Psi_2(z) \\ \Psi_2(z^{-1}) & \Psi_1(z^{-1}) \end{pmatrix}, \quad z \in \mathbb{C} \setminus ([q^{-1}, q] \cup \{1, -1\}) , \]

is a sectionally holomorphic matrix solution for the model jump problem

\[ M^{\text{mod}}_+(z) = M^{\text{mod}}_-(z) v^{\text{mod}}(z), \quad z \in [q, q^{-1}], \]

with \( v^{\text{mod}}(z) \) given by (3.25) (or by the equivalent matrix (5.5)). It has simple poles at \( z = \pm 1 \).

(ii) \( M^{\text{mod}}(z) \) satisfies the symmetry

\[ M^{\text{mod}}(z^{-1}) = \sigma_1 M^{\text{mod}}(z) \sigma_1. \]

(iii) The vector function \((1, 1) M^{\text{mod}}(z)\) has removable singularities at \(1, -1\) and integrable singularities at \([q, q^{-1}, y, y^{-1}]\) of order \( O((z - \zeta)^{-\frac{1}{2}}) \) as \( z \to \zeta \in \{q, q^{-1}, y, y^{-1}\} \).

(iv) The determinant of \( M^{\text{mod}}(z) \) is constant,

\[ \det M^{\text{mod}}(z) = 1, \quad z \in \mathbb{C}. \]

Proof. Items (i) and (ii) follow from Lemmas 4.2 and 5.1. To prove (iii) observe that

\[ \Psi_2(z^{-1}) = m_1^{\text{mod}}(z) - \rho(z)m_1^\#(z), \quad \Psi_1(z^{-1}) = m_2^{\text{mod}}(z) - \rho(z)m_2^\#(z). \]

Thus,

\[ m^{\text{mod}}(z) = (m_1^{\text{mod}}(z), m_2^{\text{mod}}(z)) = (1, 1) M^{\text{mod}}(z). \]

The vector-function \( m^{\text{mod}}(z) \) given by (4.7), (4.8) and (4.9), does not have singularities at \( z = \pm 1 \), and it has fourth root singularities at \([q, q^{-1}, y, y^{-1}]\) which proves (iii). We emphasize that (5.13) provides a connection between the unique solution of the vector model RH problem and the matrix model problem solution.

(iv) Evaluating \( \det M^{\text{mod}}(z) \) as \( z \to 0 \) by use of (5.2) and (5.7), we get

\[ \det M^{\text{mod}}(z) = \rho(z) \left( m_1^\#(z)m_2^{\text{mod}}(z) - m_2^\#(z)m_1^{\text{mod}}(z) \right), \]

that is,

\[ \det M^{\text{mod}}(z) = -2\rho(z)\alpha_{n+1}(z^{-1})G(z)\alpha_n(z) + O(z^2), \]

\[ = \frac{K_n}{\lambda} \left( -\frac{\lambda}{2z} \right) \alpha_n(0)\alpha_{n+1}(\infty)(1 + O(z)), \]

\[ = 1 + O(z), \quad z \to 0. \]

By (5.9), \( \det M^{\text{mod}}(z) \) does not have jumps in \( \mathbb{C} \) and by (5.10), it is an even function,

\[ \det M^{\text{mod}}(z^{-1}) = \det M^{\text{mod}}(z). \]

Therefore, \( \det M^{\text{mod}}(\infty) = 1 \). This function is even and bounded outside of small vicinities of 1 and \(-1\) and may have simple poles at \( \pm 1 \). The singularities at the points \([q, q^{-1}, y, y^{-1}]\) are at most of square root order, and therefore removable. Since the Abel map \( A(z) \) and \( G(z) \) are single valued functions in a vicinity of 1 with \( A(1) = 1/4 \) and \( G(1) = 1 \), we have

\[ m_1^{\text{mod}}(1) = m_2^{\text{mod}}(1), \quad m_1^\#(1) = m_2^\#(1). \]

This implies together with (5.14) the absence of a singularity at 1. Hence we have a function holomorphic in \( \mathbb{C} \setminus \{-1\} \) and bounded at infinity, which has at most a
simple pole at the only point $z = -1$ and the additional symmetry \((5.15)\). The only function which satisfies these properties is a constant. \(\square\)

**Theorem 5.4.** The vector function $\nu(z) = m^{(3)}(z)[M^{\text{mod}}(z)]^{-1}$ does not have singularities at $q, q^{-1}, 1, -1$ and satisfies

\[(5.16) \lim_{z \to 1} \nu(z) = (\tilde{\nu}, \nu),\]

with $\tilde{\nu}$ defined by \((5.17)\).

**Proof.** By \((3.25)\) and \((3.20)\) the vector $\nu(z)$ does not have jumps in small vicinities of $q, q^{-1}$ and $1$. By \((3.21), (4.8)\), \((5.3)\), \((5.4)\), \((5.6)\), \((5.8)\) and \((5.11)\) we conclude that $\nu(z) = O(z - q^{1/2})^{-1/2}$, and therefore it has no singularities at $q$ and $q^{-1}$.

At $z = 1$ both $m^{(3)}(z)$ and $M^{\text{mod}}(z)$ do not have jumps. The same is true for $m^{\text{mod}}(z), m^{\#}(z)$ and $G(z)$, which means that the equalities

\[G(z^{-1}) = G^{-1}(z), \quad m_1^{(3)}(z) = m_2^{(3)}(z^{-1}),\]
\[m_1^{\text{mod}}(z) = m_2^{\text{mod}}(z^{-1}), \quad m_1^{\#}(z) = m_2^{\#}(z^{-1}),\]

can be applied in a vicinity of $z = 1$. Moreover, the differences

\[G(z^{-1}) - G(z), \quad m_1^{(3)}(z) - m_1^{(3)}(z^{-1}), \quad m_1^{\text{mod}}(z) - m_1^{\text{mod}}(z^{-1}), \quad m_1^{\#}(z) - m_1^{\#}(z^{-1}),\]

are all of order $O(z - 1)$ as $z \to 1$. Thus, from \((5.6)\) and \((5.12)\) it follows that

\[
\Psi_2(z) - \Psi_1(z) = O(z - 1) + \rho(z)(m_1^{\#}(z) - m_1^{\#}(z^{-1})) \to 0, \quad z \to 1,
\]
\[
\Psi_1(z^{-1}) - \Psi_2(z^{-1}) = O(z - 1) - \rho(z)(m_1^{\#}(z) - m_1^{\#}(z^{-1})) \to 0, \quad z \to 1,
\]

where

\[\Psi_0 = \lim_{z \to 1} \frac{K_n}{\overline{\alpha}(z^{-1} - z)}(\beta_n(z) - \beta_n(z^{-1})).\]

Hence

\[
\nu(z) = m^{(3)}(z) \begin{pmatrix} \Psi_1(z^{-1}) & -\Psi_2(z) \\ -\Psi_2(z^{-1}) & \Psi_1(z) \end{pmatrix}
= \begin{pmatrix} m_1^{(3)}(z)\Psi_1(z^{-1}) - m_2^{(3)}(z)\Psi_2(z^{-1}) \\ m_2^{(3)}(z)\Psi_1(z) - m_1^{(3)}(z)\Psi_2(z) \end{pmatrix}.
\]

Evidently, (cf. \((3.22)\))

\[(5.17) \quad \nu(z) \to \eta \Psi_0(1, 1) := \tilde{\nu}(1, 1), \quad \text{where} \quad \eta = \lim_{z \to 1} m_1^{(3)}(z).\]

Now we investigate the behavior of $\nu(z)$ in a vicinity of $z = -1$. Since $m^{(3)}(z)$ and $M^{\text{mod}}(z)$ have the same constant jump $v^3(z) = v^{\text{mod}}(z) = e^{(2itB - i\Delta)s_3}$ in a vicinity of this point, we conclude that $\nu(z)$ does not have jumps here, and therefore $z = -1$ is an isolated singularity, which is at most a simple pole. From the symmetry condition it follows that both components $\nu_1(z)$ and $\nu_2(z)$ of $\nu(z)$ have the same behavior, either simple poles or removable singularities. To prove that in fact it is a removable singularity, it is sufficient to check that $f(z) = \nu_1(z)\nu_2(z)$ increases not faster than $o((z + 1)^{-2})$ from some direction. The behavior of

\[f(z) = \left( m_1^{(3)}(z)\Psi_1(z^{-1}) - m_2^{(3)}(z)\Psi_2(z^{-1}) \right) \left( m_2^{(3)}(z)\Psi_1(z) - m_1^{(3)}(z)\Psi_2(z) \right)\]
is determined by the summand which contains ρ²(z) (cf. (5.6) and (5.12)). Computing this term we get
\[ f(z) \sim \rho²(z) \left( |m₂¹(z)|²|m₁³(z)|² + |m₁²(z)|²|m₂³(z)|² \right. \
\[ \left. - 2m₁²(z)m₂²(z)m₁³(z)m₂³(z) \right) = \rho²(z)\tilde{f}(z). \]

The function \( \tilde{f}(z) \) has finite limiting values on the sides of the contour \([r, r^{-1}]\), and in particular at \( z = -1 \). Using the symmetry condition we get \( m₁⁺(-1) = m₂⁺(-1), \)
\( m₁⁻(-1) = m₂⁻(-1), \) therefore
\[ m₁⁺(1) = m₂⁺(1) = e^{2itB-\frac{\eta}{2}}, \quad m₁⁻(1) = m₂⁻(1) = e^{2itB+\frac{\eta}{2}}, \]
that is, \( m₂⁺(1)m₁⁻(1) = m₂⁻(1)m₁⁺(1) = -1 \). Thus
\[ \tilde{f}(±1) = m₂⁺(±1)m₂⁻(±1)m₁⁺(±1)m₁⁻(±1) = 0. \]

6. Solution of the parametrix RH problems

In this section we solve local RH problems in vicinities of the points \( y, y^{-1} \), where the error jump matrix (introduced at the end of Section 3) is not small as \( t \to \infty \).

Indeed, recall that on the contours with \( y^{-1} \) as a nodal point we have
\[ v^{(3)}(z) = \begin{cases} 
-\imath \sigma_1, & z \in [y^{-1}, q^{-1}], \\
0, & z \in [r, r^{-1}], \\
\begin{pmatrix}
F_-(z) e^{\imath (g_-(z) - g_+(z))} & U(z) e^{-\imath \Re g(z)} \\
0 & F_+(z) e^{\imath (g_+(z) - g_-(z))}
\end{pmatrix}, & z \in [r^{-1}, y^{-1}], \\
\begin{pmatrix}
\frac{1}{\imath \eta(z)} & 0 \\
0 & 1
\end{pmatrix}, & z \in \mathbb{C}^*,
\end{cases} \]
where we denoted
\[ U(z) = -\imath |\chi(z)|\Pi²(z)F_+(z)F_-(z), \quad U_1(z) = -\Pi²(z)\Pi²(z)X(z), \]
and used (3.24). Respectively,
\[ v^{err}(z) = v^{(3)}(z) - v^{mod}(z) = \begin{cases} 
0, & z \in [r^{-1}, y^{-1}], \\
0, & z \in [r, r^{-1}], \\
\begin{pmatrix}
0 & U(z) e^{-\imath \Re g(z)} \\
0 & 0 \\
\frac{2\imath \eta(z)}{U_1(z)} & 0
\end{pmatrix}, & z \in \mathbb{C}^*,
\end{cases} \]
does not vanish as \( t \to \infty \) since \( \Re g(y^{-1}) = 0, U(y^{-1})U_1(y^{-1}) \neq 0 \). The local (parametrix) RH problems are similar to those of the KdV shock wave analysis (see e.g. [15]). So let us summarize next the results obtained in [18 Sec. 6], adapted for the Toda equation.

Consider first the point \( y^{-1} \). Let \( B^* = B^*(\varepsilon) \) be a neighborhood of \( y^{-1} \) such that its boundary contains \( r^{-1} \) given by (3.11). To describe the boundary of \( B^* \) in
more detail, introduce a local change of variables

\[ w^{3/2}(z) = \frac{3t}{2} (g(z) - g_{\pm}(y^{-1})) , \quad z \in B^*, \]

with the cut along the interval \( J = [q^{-1}, y^{-1}] \cap \overline{B^*} \). From (3.4) and item (b) of Lemma 3.1 we have

\[
\frac{3}{2} (g(z) - g_{\pm}(y^{-1})) = \frac{3}{2} \int_{y^{-1} \pm \iota} P(s) \tilde{Q}(s) \sqrt{y^{-1} - s} \frac{ds}{2s} \\
= \frac{2}{y} P(y) \tilde{Q}(y)(z - y^{-1})^{3/2} (1 + o(1)), \quad z \to y^{-1},
\]

with \( P(s) \) given by (3.3) and

\[
\tilde{Q}(s) := \sqrt{\frac{s - y}{(s - q)(s - q^{-1})}}.
\]

Evidently, \( \tilde{Q}(y) > 0 \). For \( \xi \in (\xi_{cr}, \xi_{cr}') \) we have (see [17]) \( P(s) = s^2 - \zeta - \zeta^{-1} \), where \( \zeta = \zeta(\xi) \in (-1, y) \). Thus, \( P(y^{-1}) < 0 \) and \( T := 2y^{-1} P(y) \tilde{Q}(y) > 0 \). Then

\[ w(z) = T^{2/3} y^{2/3} (z - y^{-1})(1 + o(1)), \quad z \to y^{-1}, \quad (Tt)^{2/3} > 0. \]

Hence \( w(z) \) is a holomorphic function in \( B^* \).

![Figure 7](https://example.com/figure7.png)

**Figure 7.** The local change of variables \( w(z) \).

Uttill now we did not specify the particular shape of the boundary of \( B^* \) and the shape of the contour \( \mathcal{C} \) inside \( B^* \). Treating \( w(z) \) as a conformal map, let us think of \( B^* \) as a pre-image of a disc \( \mathcal{O} \) of radius \( T^{2/3} y^{-1} - \tau^{-1} t^{2/3} \) centered at the origin. Then \( w(z) \) maps the interval \( J = [q^{-1}, y^{-1}] \cap \overline{B^*} \) to the negative half axis and \( J' = [y^{-1}, \tau^{-1}] \to \) the positive half axis. Let us emphasize that we reverse the orientation on \( J, J' \) (cf. (6.1)) to simplify our further considerations for the solution of the parametrix problem. We also choose the contour \( \mathcal{C}^* \cap \overline{B^*} \) to be contained in the pre-image of the rays \( \arg w = \pm \frac{2\pi i}{3} \) and divide it in two parts \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \), with orientation as depicted in Fig. 7. With these new orientations,

\[
v^{\text{mod}}(z) = \begin{cases} 
\iota \sigma_1 & z \in J \\
\iota e^{(-2\pi i B + i\Delta)} \sigma_3 & z \in J' 
\end{cases}
\]

and \( v^{\text{err}}(z) \) changes to \([v^{\text{err}}(z)]^{-1} \). In \( \mathcal{B}^* \) we introduce the function

\[
r(z) := \frac{e^{\pm i\pi} e^{-itB}}{\sqrt{X(z)\Pi(z)F(z)}}, \quad z \in \mathcal{B}^* \cap \{z : \pm \Im z > 0\},
\]
Since
\begin{equation}(6.6)\end{equation}
\[
X_\pm(z) = \mp i|\chi(z)|, \quad z \in J \cup J',
\end{equation}
we see that
\[
q_\pm(z) = \frac{e^{itB}}{\sqrt{|\chi(z)|\Pi(z)F_\pm(z)}}, \quad z \in J \cup J'.
\]
By items (b) and (c) of Lemma 3.4, taking into account the change of direction for the contour in (c), we get
\begin{equation}(6.7)\end{equation}
\[
q_+(z)q_-(z) = 1, \quad z \in J; \quad q_+(z) = q_-(z)e^{i\Delta - 2itB}, \quad z \in J'.
\]
Due to (6.6), \(\sqrt{X_+(z)X_-(z)} = |\chi(z)|\) for \(z \in J'.\) Thus, by use of (6.3), for the off-diagonal elements of \([\nu^{err}]^{-1}\) (cf. (6.2)) we have
\begin{equation}(6.8)\end{equation}
\[
-U(z)e^{-2t\Re g(z)} = i|\chi(z)|\Pi^2(z)F_+(z)F_-(z)e^{-t(g_-(z) + g_+(z))}
\]
\[= i e^{-t(g_+(z) - g_-(y^{-1}))}e^{-t(g_-(z) - g_-(y^{-1}))} - i e^{-\frac{4}{3}w^{3/2}(z)}
\]
\[= e^{2t\imath(z)}U_1(z) = \pm \imath r^2(z)e^{\pm 2itB + 2t\imath g(z)} = \pm \imath r^2(z)e^{2t\imath g(z) - g_-(y^{-1})}
\]
\[= \pm i r^2(z) e^{\pm \imath \frac{4}{3}w^{3/2}(z)}, \quad \pm \imath \text{Im} z > 0.
\]
Redefine \(m^{(3)}(z)\) and \(m^{\text{mod}}(z)\) as well as the matrix \(M^{\text{mod}}(z)\) inside \(B^*\) by
\begin{equation}(6.9)\end{equation}
\[
\hat{m}^{(3)}(z) = m^{(3)}(z)q(z)^{-\sigma_3}, \quad \hat{m}^{\text{mod}}(z) = m^{\text{mod}}(z)r(z)^{-\sigma_3},
\]
\[
\hat{M}^{\text{mod}}(z) = M^{\text{mod}}(z)r(z)^{-\sigma_3}, \quad z \in B^*.
\]
By use of (6.5), (6.7), (6.8), and (6.9) we obtain that inside \(B^*\)
\[
\hat{m}^{(3)}(z) = \hat{m}^{(3)}(z)\hat{v}^{(3)}(z), \quad \hat{M}^{\text{mod}}(z) = \hat{M}^{\text{mod}}(z)\hat{v}^{\text{mod}}(z),
\]
where
\[
\hat{v}^{\text{mod}}(z) = i\sigma_1, \quad z \in J; \quad \hat{v}^{\text{mod}}(z) = \mathbb{I}, \quad z \in J'.
\]
\[
\hat{v}^{(3)}(z) = \begin{cases} i\sigma_1, & z \in J, \\ 1 & z \in J', \\ 0 & 1 \\ 1 & 0 \\ -i e^{4/3w^{3/2}(z)} & 1, \quad z \in L_1 \\ 1 & 0 \\ i e^{4/3w^{3/2}(z)} & 1, \quad z \in L_2.
\end{cases}
\]
By (6.4) we conclude that \(w^{1/4}(z)\) has the following jump along \(J,
\[
w^{1/4}_+(k) = w^{1/4}_-(k)i, \quad k \in J.
\]
Recall that \(O = w(B^*).\) It is now straightforward to check that the matrix
\[
N(w) = \frac{1}{\sqrt{2}} \begin{pmatrix} w^{1/4} & w^{1/4} \\ -w^{-1/4} & w^{-1/4} \end{pmatrix}, \quad w \in O,
\]
solves the jump problem
\[
N_+(w(z)) = iN_-(w(z))\sigma_1, \quad z \in J.
\]
Therefore, in $\mathcal{B}^*$ we have $\hat{M}^{\text{mod}}(z) = \mathcal{H}(z)N(w(z))$, where $\mathcal{H}(z)$ is a holomorphic matrix function in $\mathcal{B}^*$. Since $\det N(w) = \det[r(z)^{\sigma_3}] = 1$, we have

$$
\det \mathcal{H}(z) = \det M^{\text{mod}}(z) = \det \hat{M}^{\text{mod}}(z) = 1.
$$

According to (6.11) we then get

$$
M^{\text{mod}}(z) = \mathcal{H}(z)N(w(z))r(z)^{\sigma_3}, \quad z \in \partial \mathcal{B}^*.
$$

By property (c) of Lemma 3.1 $w_+(z)^{3/2} = -w_-(z)^{3/2}$ for $z \in J$, that is,

$$
\tilde{\gamma}(3)(z) = d_-(z)^{-\sigma_3} S d_+(z)^{\sigma_3}, \quad z \in \mathcal{B}^*,
$$

where

$$
d(z) := \tilde{d}(w(z)), \quad \tilde{d}(w) = e^{2/3w^{3/2}},
$$

and

$$
S = \begin{cases}
S_1, & z \in L_1, \\
S_2, & z \in J, \\
S_3, & z \in L_2, \\
S_4, & z \in J'.
\end{cases}
$$

Here

$$
S_1 = \begin{pmatrix}
1 & 0 \\
i & 1
\end{pmatrix}; \quad S_2 = \begin{pmatrix}
0 & i \\
0 & 0
\end{pmatrix}; \quad S_3 = \begin{pmatrix}
1 & 0 \\
-1 & 1
\end{pmatrix}; \quad S_4 = \begin{pmatrix}
1 & i \\
0 & 1
\end{pmatrix}.
$$

Consider $S$ as the jump matrix on the contour $w(J \cup J' \cup L_1 \cup L_2)$ in $\mathcal{O}$. Let $\mathcal{A}(w)$ be the matrix solution of the jump problem

$$
\mathcal{A}_+(w) = \mathcal{A}_-(w)S, \quad w \in w(J \cup J' \cup L_1 \cup L_2),
$$

satisfying the boundary condition

$$
\mathcal{A}(w) = N(w)\Psi(w)\tilde{d}(w)^{-\sigma_3}, \quad w \in \partial \mathcal{O}, \quad t \to \infty,
$$

where

$$
\Psi(w) = \mathbb{I} + \frac{C}{w^{3/2}} \left( 1 + O(w^{-3/2}) \right), \quad w \to \infty,
$$

is an invertible matrix, and $C$ is a constant matrix with respect to $w$, $t$ and $\xi$. The solution $\mathcal{A}(w)$ can be expressed via Airy functions and their derivatives in a standard manner (see, for example, [8], [2, Ch. 3], [19] or [11]). In particular, in the sector between the contours $w(J')$ and $w(L_1)$ in $\mathcal{O}$ we have

$$
\mathcal{A}(w) = \mathcal{A}_1(w) = \sqrt{2\pi} \begin{pmatrix}
-y_1'(w) & iy_2'(w) \\
y_1(w) & iy_2(w)
\end{pmatrix},
$$

where $y_1(w) = Ai(w)$ and $y_2(w) = e^{-2\pi i}Ai(e^{-2\pi i}w)$. In the sector between the lines $w(L_1)$ and $w(J)$ we get

$$
\mathcal{A}(w) = \mathcal{A}_2(w) = \mathcal{A}_1(w)S_1 = \sqrt{2\pi} \begin{pmatrix}
y_3'(w) & iy_2'(w) \\
y_3(w) & iy_2(w)
\end{pmatrix},
$$

where $y_3(w) = e^{2\pi i}Ai(e^{2\pi i}w)$. Here we used the standard equality $y_1(w) + y_2(w) + y_3(w) = 0$. Changing orientation on $J$ and $L_2$ we obtain between $w(J)$ and $w(L_2)$

$$
\mathcal{A}(w) = \mathcal{A}_3(w) = -i\mathcal{A}_2(w)\sigma_1,
$$

and between the lines $w(L_2)$ and $w(J')$, correspondingly,

$$
\mathcal{A}(w) = \mathcal{A}_4(w) = \mathcal{A}_3(w)S_4^{-1}.
$$
The last conjugation with the matrix $S_4$ will lead to matrix $A_1(w)$ again, because $S_1S_2^{-1}S_3^{-1}S_4 = I$. Note that the constant matrix $C$ is the same for all regions,

$$C = \frac{1}{48} \begin{pmatrix} -1 & 6 \\ -6 & 1 \end{pmatrix}.$$ 

The precise formulas for $A_j(w)$ are in fact not important for us. The matrix

$$M_{par}(z) := \mathcal{H}(z)A(w(z))d(z)^{\sigma_3}, \quad z \in B^*,$$

solves in $B^*$ the jump problem

$$M_{+}^{par}(z) = M_{-}^{par}(z)e^{\mathbf{i}(3)}(z), \quad z \in J \cup J' \cup \mathcal{L}_1 \cup \mathcal{L}_2,$$

and satisfies for sufficiently large $t$ the boundary condition

$$M^{par}(z) = \mathcal{H}(z)N(w(z))\Psi(w(z)) = \widetilde{M}^{mod}(z)\Psi(w(z)) = M^{mod}(z)r(z)^{-\sigma_3}\Psi(w(z)), \quad z \in \partial B^*.$$ 

Note that (6.13) and (6.3) yield

$$\det \Psi(w(z)) = 1 + O(t^{-1}), \quad z \in B^*, \quad t \to \infty,$$

uniformly with respect to $\xi \in \mathcal{L}$. This implies with (6.12) invertibility of $M^{par}(z)$ in $\overline{B}$. In summary, we constructed a matrix with the following properties:

**Lemma 6.1.** The vector function

(6.14) $\nu(z) = \hat{m}^{(3)}(z)M^{par}(z)^{-1} = m^{(3)}(z)r(z)^{-\sigma_3}M^{par}(z)^{-1}, \quad z \in B^*$, does not have jumps and isolated singularities in $B^*$, it is holomorphic there. The function $\nu(z)$ has piecewise continuous limiting values as $z$ approaches $\partial B^*$ from inside, given by

(6.15) $\nu(z) = m^{(3)}(z)r(z)^{-\sigma_3}\Psi(w(z))^{-1}r(z)^{\sigma_3}M^{mod}(z)^{-1}, \quad z \in \partial B^*$.

Introduce a vicinity $B$ of $y$ which is symmetric to $B^*$ with respect to $z \to z^{-1}$, 

$$B = \{ z : z^{-1} \in B^* \}.$$ 

Respectively, set

$$r(z) = r^{-1}(z^{-1}), \quad m^{(4)}(z) = m^{(4)}(z^{-1})\sigma_1, \quad \hat{M}(z) = \sigma_1\hat{M}(z^{-1})\sigma_1,$$

$$M^{par}(z) = \sigma_1M^{par}(z^{-1})\sigma_1, \quad \nu(z) = \nu(z^{-1})\sigma_1, \quad z \in B.$$ 

With this extension, the definitions (6.10) and (6.14) remain valid in $B$ and $\nu(z)$ is holomorphic in $B$ too. Let us extend the definition of $\nu(z)$ to $\mathbb{C} \setminus (\overline{B^*} \cup \overline{B})$ by

(6.16) $\nu(z) = m^{(3)}(z)M^{mod}(z)^{-1}, \quad z \in \mathbb{C} \setminus (\overline{B^*} \cup \overline{B}).$

Theorem 5.4 implies that this function does have jumps on the contour $[q, q^{-1}]$ outside of $\overline{B^*} \cup \overline{B}$. We label the parts of $C$ and $C^*$ outside $B$ and $B^*$ by $\mathcal{L}_3$ and $\mathcal{L}_3^*$, see Fig. 5. The jumps of $\nu(z)$ on $K \cup \mathcal{L}_3 \cup \mathcal{L}_3^*$ (cf. (6.13)) are exponentially small with respect to $t \to \infty$. Let us compute the jump of this vector on the boundaries $\partial B$ and $\partial B^*$, which we treat as clockwise oriented contours. Since neither $m^{(3)}(z)$, $r(z)$ nor $M^{mod}(z)$ have jumps on these contours, we obtain from (6.15) and (6.16)

$$m^{(3)}(z) = \nu_{\text{err}}(z)M^{mod}(z)^{-1}r(z)^{-\sigma_3}\Psi(w(z))r(z)^{\sigma_3} = \nu_{\text{err}}(z)M^{mod}(z), \quad z \in \partial B^*.$$ 

Taking into account (6.13) we find the jump

$$\nu_{\text{err}}(z) = \nu_{\text{err}}(z)(I + V^{err}(z)), \quad z \in \partial B^* \cup \partial B,$$
where
\begin{align}
V_{err}(z) &= M_{mod}(z)r(z)^{-\sigma_3}(\Psi(w(z)) - \bar{w})r(z)^{\sigma_3}M_{mod}(z)^{-1}, \quad z \in \partial B^*; \\
V_{err}(z) &= \sigma_1V_{err}(z)^{-1}\sigma_1, \quad z \in \partial B.
\end{align}

The results of this section together with Theorem 5.4 and (5.13) can be formulated as follows.

**Lemma 6.2.** The vector \(\nu(z)\) is a holomorphic function in the domain \(\mathbb{C} \setminus \Gamma\), where
\begin{equation}
\Gamma := K \cup \mathcal{L}_3 \cup \mathcal{L}_3^* \cup \partial B \cup \partial B^*,
\end{equation}
and bounded on the closure of this domain. On the contour \(\Gamma\), \(\nu(z)\) has the jump
\begin{equation}
\nu_+(z) = \nu_-(z)(1 + V_{err}(z)),
\end{equation}
where \(V_{err}(z)\) is given by (6.17) on \(\partial B \cup \partial B^*\) and
\begin{equation}
V_{err}(z) = M_{mod}^-(z)(\nu(z) - 1)M_{mod}^+(z)^{-1}, \quad z \in K \cup \mathcal{L}_3 \cup \mathcal{L}_3^*.
\end{equation}
The vector \(\nu(z)\) satisfies
\begin{equation}
\nu(z)\big|_{z = 1} = \nu_1, \quad \text{and has equal coordinates as } z \to 1.
\end{equation}

![Figure 8. \(\Gamma = \bigcup_{j=1}^N (T_j \cup T_j^*) \cup C \cup C^*_r \cup C \cup C^*_l \cup L_3 \cup \mathcal{L}_3^* \cup \partial B \cup \partial B^*\).](image)

**7. Completion of the asymptotic analysis**

The aim of this section is to establish that the solution \(m^{(3)}(z)\) is well approximated by \(m_{mod}(z) = (1 - \bar{1})M_{mod}(z)\) as \(z \to 0\). We follow the well-known approach via singular integral equations (see e.g., [9], [21], [22, Ch. 4], [27]). A peculiarity of this approach applied to the Toda equation is generated by the type of normalization condition of the vector RH problem. Indeed, we only know the product of the coordinates. The only point with additional information is point 1. As soon as it ceases to be a point of conjugation due to the shift of the jump contour, for any symmetric vector solution we have \(m_1(1) = m_2(1)\). Therefore we will attach the integral equations to \(z = 1\), and use the Cauchy kernel (5.1). Recall that the contour \(\Gamma = \Gamma(\xi)\) in (6.18), Fig. 8, is bounded and separated from points \(1, -1, q, q^{-1}, y, y^{-1}\) where the model matrix solution has singularities of different kind. The matrix \(V_{err}(z)\) is bounded on \(\Gamma\) and admits an estimate \(V_{err}(z) = O(t^{-1})\) uniformly with respect to \(z \in \Gamma(\xi)\) and \(\xi \in \mathcal{I}_\varepsilon = [\xi_\varepsilon, + \varepsilon, \xi_\varepsilon - \varepsilon]\). This implies

**Lemma 7.1.** Uniformly with respect to \(\xi \in \mathcal{I}_\varepsilon\),
\begin{equation}
\|z^jV_{err}(z)\|_{L^p(\Gamma)} \leq C(\varepsilon)t^{-1}, \quad p \in [1, \infty], \quad j = 0, 1.
\end{equation}
Now we are ready to apply the technique of singular integral equations. Let $\mathcal{C}$ denote the Cauchy operator associated with $\Gamma$,

$$(\mathcal{C}h)(z) = \frac{1}{2\pi i} \int_{\Gamma} h(s) \Omega(z, s) ds, \quad s \in \mathbb{C} \setminus \Gamma,$$

where $h = (h_1, h_2) \in L^2(\Gamma)$ and $\Omega(z, s)$ is defined by (3.14). By this definition, $(\mathcal{C}h)(1) = 0$ for any $h \in L^2(\Gamma)$. Let $(\mathcal{C}+h)(z)$ and $(\mathcal{C}-h)(z)$ be the non-tangential limiting values of $(\mathcal{C}h)(z)$ from the left and right sides of $\Gamma$, respectively. As usual, we introduce the operator $\mathcal{C}_V : L^2(\Gamma) \cap L^\infty(\Gamma) \to L^2(\Gamma)$ by $\mathcal{C}_V h = \mathcal{C}_-(hV^{err})$, where $V^{err}$ is the error matrix (6.17), (6.19). Then

$$\|\mathcal{C}_V\| = \|\mathcal{C}_V\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq C\|V^{err}\|_{L^\infty(\Gamma)} \leq O(t^{-1}),$$

as well as

$$(7.2) \quad \|I - \mathcal{C}_V\|_{L^2(\Gamma) \to L^2(\Gamma)} \leq \frac{1}{1 - O(t^{-1})}$$

for sufficiently large $t$. Consequently, for $t \gg 1$, on $\Gamma$ we define a vector function

$$\mu(s) = (\tilde{\nu}, \tilde{v}) + (I - \mathcal{C}_V)^{-1}\mathcal{C}_V((\tilde{\nu}, \tilde{v}))(s),$$

with $\tilde{\nu}$ defined by (5.17). Then by (7.1) and (7.2)

$$\|\mu(s) - (\tilde{\nu}, \tilde{v})\|_{L^2(\Gamma)} \leq \|(I - \mathcal{C}_V)^{-1}\|\|\mathcal{C}_-\||V^{err}\|_{L^\infty(\Gamma)} = O(t^{-1}).$$

(7.3)

With the help of $\mu$, the vector function $\nu(z)$ can be represented as

$$\nu(z) = (\tilde{\nu}, \tilde{v}) + \frac{1}{2\pi i} \int_{\Gamma} \mu(s)V^{err}(s)\Omega(z, s) ds,$$

and by virtue of (7.3) and Lemma 7.1 we obtain as $z \to 0$

$$\nu(z) = (\tilde{\nu}, \tilde{v}) + \frac{1}{2\pi i} \int_{\Gamma} (\tilde{\nu}, \tilde{v})V^{err}(s)\Omega(z, s) ds + E(z),$$

where

$$|E(z)| \leq \frac{1}{\epsilon} \|V^{err}\|_{L^2(\Gamma)}\|\mu(s) - (\tilde{\nu}, \tilde{v})\|_{L^2(\Gamma)} \leq C(z) \left( \frac{1}{t^2} + \frac{e^{-C(t)\epsilon}}{t^2} \right).$$

Here $\epsilon$ is the minimal distance between the point 1 and the contour $\mathcal{C}_e$. Since

$$\Omega(z, s) = \frac{1}{s(1 - s)} + \frac{1}{2s}(z + O(z^2)), \quad z \to 0, \quad s \in \Gamma,$$

we have

$$(\tilde{\nu}, \tilde{v}) \int_{\Gamma} V^{err}(s)\Omega(z, s) ds = \frac{(h_1^0, h_2^0)}{t} z + O(t^{-1})O(z^2) + O(t^{-2}),$$

where the components of the vectors $h_j^0 = h_j^0(\xi, t)$ are uniformly bounded for $t \to \infty$ and $\xi \in I_e$. Furthermore, the terms $O(z^2)$ are vector-functions depending only on $z$ and the estimates $O(t^{-s})$ are uniform with respect to $\xi \in I_e$. Hence, for $z \to 0$

$$m^{(3)}(z) = \nu(z)M^{mod}(z) = \tilde{v}m^{mod}(z) + \frac{(h_1^0, h_2^0(\xi, t))}{t} M^{mod}(z) + O(t^{-1})O(z),$$

where we took into account (5.13). Recall that $m^{(3)}(z)$ and $m^{mod}(z)$ satisfy the normalization condition, which means that $\tilde{\nu} > 0$ and $\tilde{v}^2 = 1 + O(t^{-1})$. We proved

$$m^{(3)}(z) = m^{mod}(z) + O(t^{-1})O(z) + O(t^{-1})O(z^2), \quad z \to 0, \quad t \to \infty.$$
In particular, by use of \(1.8-1.9\) and \(3.23\) for the solution of RH problem \(1\) we obtain the following result.

**Lemma 7.2.** The following representation holds for \(t \to \infty\)

\[
m_1(z)m_2(z) = H^2(z) \frac{\delta(z)\delta(z^{-1})}{\delta(0)\delta(\infty)} + \beta_1(\xi,t) + \beta_2(\xi,t)z + \beta_2(\xi,t)O(z^2), \quad z \to 0,
\]

where \(|\beta_j(\xi,t)| \leq C\) uniformly with respect to \(\xi \in \mathcal{I}_x\).

Our next aim is to clarify the properties of the function

\[
f(z) := \frac{\delta(z)\delta(z^{-1})}{\delta(0)\delta(\infty)}.
\]

To simplify notations denote \(\frac{tU}{4\pi} - \frac{\Delta}{4\pi} = s \in \mathbb{R}\). Then by \(4.7\)

\[
f(z) = \frac{\theta(A(z) - \frac{1}{2} + s)\theta(A(z) + s)\theta(A(z^{-1}) - \frac{1}{2} + s)\theta(A(z^{-1}) + s)}{\theta(A(z) - \frac{1}{2})\theta(A(z)\theta(A(z^{-1}) - \frac{1}{2})\theta(A(z^{-1}) + s)}.
\]

Since \(s \in \mathbb{R}\), the properties of the Abel integral \(A(z)\) listed in Lemma \(4.1\) imply that \(\theta(A(z) - \frac{1}{2} + s)\theta(A(z) + s)\) has a simple zero at \((\mu, \pm)\) on one of the sheets of the Riemann surface \(\mathcal{X}\) with projection \(\mu \in [y^{-1}, y]\), and a zero \((\mu^{-1}, \mp)\) on the other sheet. The function \(\theta(A(z^{-1}) - \frac{1}{2} + s)\theta(A(z^{-1}) + s)\) has zeros at \((\mu, \mp), (\mu^{-1}, \pm)\). The denominator of \(f(z)\) has double zeros (as points on the Riemann surface) at \(y\) and \(y^{-1}\). Since \(y\) and \(y^{-1}\) are the branching points on \(\mathcal{X}\), then these zeros are simple in the variable \(z\) on the complex plane. We observe that \(f(z)\), being considered as a function in the domain \(\mathbb{C}\setminus [q^{-1}, q]\) identified with the upper sheet of \(\mathcal{X}\), does not have jumps on \([q^{-1}, q]\) and tends to 1 as \(z \to \infty\). Hence \(f(z)\) is a rational function with simple poles at \(y\) and \(y^{-1}\) and simple zeros at \(\mu\) and \(\mu^{-1}\). Therefore,

\[
f(z) = \frac{(z - \mu)(z - \mu^{-1})}{(z - y)(z - y^{-1})},
\]

and

\[
m_1^{mod}(z)m_2^{mod}(z) = H^2(z) f(z) = \frac{(z - \mu)(z - \mu^{-1})}{\sqrt{(z - q)(z - q^{-1})(z - y)(z - y^{-1})}} = \frac{z + z^{-1} - \mu - \mu^{-1}}{\sqrt{(z + z^{-1} - q - q^{-1})(z + z^{-1} - y - y^{-1})}} = \frac{\lambda - \lambda(n,t)}{\sqrt{(\lambda - (b - 2a))(\lambda - \lambda_y)}},
\]

where \(\lambda(n,t) = \frac{\mu + \mu^{-1}}{2} \in [\lambda_y, -1]\). We emphasize that \(\mu = \mu(s)\) depends on \(n\) and \(t\) via

\[
s = -\frac{n\lambda}{4\pi} - \frac{tU}{4\pi} - \frac{\Delta}{4\pi}.
\]

Let \(\Psi(n,t,p,\xi)\), \(p \in \mathbb{M}(\xi)\), be the Baker–Akhiezer function of a finite gap Toda lattice solution \(\{\hat{a}(n,t,\xi), \hat{b}(n,t,\xi)\}\) associated with the spectrum on the set \([b - 2a, \lambda_y] \cup [-1, 1]\) and with initial divisor point \((\lambda(0,0), \pm)\), where we choose sign + if \(\mu(0,0) \in [-1, y]\) and sign − if \(\mu(0,0) \in [y^{-1}, -1]\). Here we took into account that the set \(\{z : |z| < 1\}\) \(\setminus [y, q]\) is in one-to-one correspondence with the upper sheet of \(\mathbb{M}(\xi)\) (cf. Section 5). Then (cf. 33)

\[
\Psi(n,t,p,\xi)\Psi(n,t,p^*,\xi) = \frac{\lambda - \lambda(n,t)}{\lambda - \lambda(0,0)}.
\]
where \(\lambda(n,t)\) is the projection of the zero divisor for \(\Psi\). Equation (1.10) and our considerations above justify this claim. In particular, according to the trace formulas we have
\[
\hat{b}(n,t,\xi) = \frac{1}{2} (b - 2a + \lambda_y(\xi) - 2\lambda(n,t)),
\]
\[
\hat{a}(n,t,\xi)^2 + \hat{a}(n-1,t,\xi)^2 = \frac{1}{4} \left( 2 + (b - 2a)^2 + \lambda_y(\xi)^2 - 2\lambda(n,t)^2 - \frac{1}{2} (b - 2a + \lambda_y(\xi) - 2\lambda(n,t))^2 \right).
\]
The operator \(\hat{H(t)} = \hat{H(t,\xi)}\) associated with these coefficients is reflectionless (since it is finite gap) and has the following Green’s function (cf. [32])
\[
\begin{equation}
\hat{G}(\lambda, n,n,t) = -\frac{\lambda - \lambda(n,t)}{\sqrt{(\lambda^2 - 1)(\lambda - (b - 2a))}} = -\frac{m_1^{mod}(z)m_2^{mod}(z)}{\sqrt{\lambda^2 - 1}}.
\end{equation}
\]
Combining (2.2, 2.14), (2.15), (7.4), (7.5) and (7.6) we arrive at the following

**Theorem 7.3.** For \((n,t) \in \mathcal{D}, n,t \to \infty\) uniformly with respect to \(\xi \in \mathcal{I}_x\), the following asymptotic is valid for the Toda shock wave \(\{a(n,t), b(n,t)\}\) given by (1.1) - (1.3), (2.1),
\[
a(n,t)^2 + a(n-1,t)^2 = \hat{a}(n,t, \hat{n})^2 + \hat{a}(n-1,t, \hat{n})^2 + O(t^{-1}),
\]
\[
b(n,t) = \hat{b}(n,t, \hat{n}) + O(t^{-1}),
\]
where \(\{\hat{a}(n,t,\xi), \hat{b}(n,t,\xi)\}\) is the finite gap solution of the Toda lattice associated with the spectrum \([b - 2a, \lambda_y] \cup [-1, 1]\) and the initial divisor \((\lambda(0,0), \pm)\) which is the only zero of the function \(\theta \left( 2A(z) - \frac{1}{2} - \frac{A}{2\pi} | 2\tau \right)\) on the Riemann surface \(\mathcal{M}(\xi)\) with projection on the gap \([\lambda_y, -1]\).

8. Discussions

In this section we briefly discuss how to derive and justify the asymptotics in the region \(\mathcal{I}_{1,\eps} := [\xi_{cr,1} + \eps, \xi'_{cr,1} - \eps]\). A justification of asymptotics in the middle region \((\xi'_{cr,1}, \xi'_{cr})\) which takes into account the presence of resonances and the discrete spectrum in the gap \((b + 2a, -1)\) will be done in [13].

First of all, if left and right background spectra are of equal length, that is, in case \(a = 1\), there is no need for an independent extensive study. Indeed, for arbitrary \(a > 0\) let us consider the Toda lattice associated with the functions
\[
\hat{a}(n,t) = \frac{1}{2a} a(-n-1, \frac{n}{2a}), \quad \hat{b}(n,t) = \frac{1}{2a} \left( b - b(-n, \frac{n}{2a}) \right),
\]
where \(\{a(n,t), b(n,t)\}\) is the solution of (1.1) - (1.3), (2.1). It is straightforward to check that \(\{\hat{a}(n,t), \hat{b}(n,t)\}\) satisfy (1.1) associated with the initial profile
\[
\hat{a}(n,0) \to \frac{1}{2a}, \quad \hat{b}(n,0) \to \frac{b}{2a}, \quad \text{as } n \to -\infty,
\]
\[
\hat{a}(n,0) \to \frac{1}{2}, \quad \hat{b}(n,0) \to 0, \quad \text{as } n \to +\infty.
\]
If \(a = 1\), the region \((\xi'_{cr}, \xi_{cr})\) for \(\{\hat{a}(n,t), \hat{b}(n,t)\}\) coincides with \((\xi'_{cr,1}, \xi_{cr,1})\) for \(\{a(n,t), b(n,t)\}\), and therefore we can simply apply the results of Theorem 7.3. This approach is applicable for arbitrary \(a\) when \(b - 2a > 1\), that is, for rarefaction
waves. Unfortunately, for the shock waves and $a \neq 1$ we are not able to match these regions. Moreover, even if they would match, this approach would still require cumbersome computations. Indeed, applying the asymptotics from Theorem 7.3 to

\[ a(m, \tilde{t}) = 2a\tilde{a}(n, t), \quad b(m, \tilde{t}) = b - 2a\tilde{b}(n + 1, t), \quad \tilde{t} = \frac{t}{2a}, \quad m = -n - 1, \]

one has to take into account the new time and space variables and recompute $b$ and $\tau$ periods in theta functions, as far as the initial Dirichlet eigenvalues.

As we see from the above considerations, the form of the $g$-function in $I_{1, \varepsilon}$ is dictated by the spectrum $[b - 2a, b + 2a] \cup [\gamma(\xi), 1]$, where $\gamma(\xi) \in [-1, 1]$. Therefore on the $z$-plane the images of this point will belong to $\mathbb{T} \setminus \{-1, 1\}$. Denote them by $z_0$ and $\overline{z_0}$. It is clear that the piece-wise constant jump matrix for the respective model problem will appear on the union of the real interval and the arc,

\[ \{q, q^{-1}\} \cup \{z \in \mathbb{T} : \text{Re} \, z < \text{Re} \, z_0 \}. \]

A construction of the vector and matrix model solutions in theta functions for such a contour in terms of $z$ is quite bulky and not transparent for further analysis. For this reason it is much more convenient to study the asymptotics for $\xi \in I_{1, \varepsilon}$ using the other vector RH problem stated with respect to the left scattering data $R_{\ell}(\zeta, t), T_{\ell}(\zeta, t)$ on the $\zeta$-plane (cf. (1.6)). It will be done in terms of the left phase function (1.8) which will be replaced with a suitable $g$-function. The structure of the jump matrices and further analysis is completely analogous to the one given in this paper. It allows us to conclude that the error term in this region is described in terms of Airy function and is of order $O(t^{-1})$. The order of the error term in the middle region is an open question.

Our last remark concerns condition (2.1). Evidently, the value of $\rho$ given by (2.11) can be significantly reduced up to any $\rho > 0$ if the point $q_1$ is non-resonant, because in the non-resonant case we do not need to apply the lens mechanism around domains $\Omega_\varepsilon$ and $\Omega_{\varepsilon}^*$. It was performed to remove a possible singularity of $m$ at $q_1$. Moreover, the condition $\rho > -\log |q_1|$ is sufficient to remove the singularity. Condition (2.11) was chosen to avoid additional unwieldiness in the formulas for the jump matrices. Note that condition (3.10) is only essential in the resonant case, and one can expect that the asymptotics in Theorem 7.3 hold in the region (1.14) if $q_1$ is non-resonant.

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