Global existence to a 3D chemotaxis-Navier-stokes system with nonlinear diffusion and rotation

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Abstract

This paper is concerned with the following quasilinear chemotaxis–Navier–Stokes system with nonlinear diffusion and rotation

\[
\begin{aligned}
  n_t + u \cdot \nabla n &= \Delta n^m - \nabla \cdot (n S(x, n, c) \cdot \nabla c), & x \in \Omega, \ t > 0, \\
  c_t + u \cdot \nabla c &= \Delta c - nc, & x \in \Omega, \ t > 0, \\
  u_t + \kappa (u \cdot \nabla) u + \nabla P &= \Delta u + n \nabla \phi, & x \in \Omega, \ t > 0, \\
  \nabla \cdot u &= 0, & x \in \Omega, \ t > 0
\end{aligned}
\]

\hspace{1cm} (CNF)

is considered under the no-flux boundary conditions for $n, c$ and the Dirichlet boundary condition for $u$ in a three-dimensional convex domain $\Omega \subseteq \mathbb{R}^3$ with smooth boundary, which describes the motion of oxygen-driven bacteria in a fluid. Here $\kappa \in \mathbb{R}$ and $S$ denotes the strength of nonlinear fluid convection and a given tensor-valued function, respectively. Assume $m > \frac{10}{9}$ and $S$ fulfills $|S(x, n, c)| \leq S_0(c)$ for all $(x, n, c) \in \bar{\Omega} \times [0, \infty) \times [0, \infty)$ with $S_0(c)$ nondecreasing on $[0, \infty)$, then for any reasonably regular

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initial data, the corresponding initial-boundary problem ($CNF$) admits at least one global weak solution.

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1 Introduction

In biological contexts, many simple life-forms exhibit a complex collective behavior. This effect, called chemotaxis, is presumed to have a deep impact on the time evolution of a bacteria population. The chemotaxis system is proposed by Keller and Segel [12] in 1970. During the past four decades, the chemotaxis system, the chemical substrate can be produced or consumed by the cells has been well studied in mathematical biology ([1, 10, 19, 27]). For the more related works in this direction, we mention that a corresponding quasilinear version has been deeply investigated by [18, 22, 38, 37, 42].

Furthermore, in [5] it can be observed experimentally that bacteria are suspended in the fluid, which is influenced by the gravitational forcing generated by the aggregation of cells. Then the movement of bacteria is effected by composite factors, namely, random diffusion, chemotactic migration towards gradients of oxygen and transport through the fluid. Taking into account all these processes, Tuval et al. ([21]) proposed the model

\[
\begin{align*}
    n_t + u \cdot \nabla n &= \Delta n - C_S \nabla \cdot (nS(c) \cdot \nabla c), \quad x \in \Omega, t > 0, \\
    c_t + u \cdot \nabla c &= \Delta c - nf(c), \quad x \in \Omega, t > 0, \\
    u_t + \kappa(u \cdot \nabla u) &= \nabla P + \Delta u + n\nabla \phi, \quad x \in \Omega, t > 0, \\
    \nabla \cdot u &= 0, \quad x \in \Omega, t > 0
\end{align*}
\] (1.1)

for the unknown bacterial density \(n\), the oxygen concentration \(c\), the fluid velocity field \(u\) and the associated pressure \(P\) in the physical domain \(\Omega \subset \mathbb{R}^N\). Here, \(\kappa \in \mathbb{R}\) is related to the strength of nonlinear fluid convection, the functions \(S(c)\), \(f(c)\) and \(\phi\) denotes the chemotactic sensitivity, the consumption rate of the oxygen by the bacteria and the gravitational potential, respectively. System (1.1) describes the movement of the cells towards the higher concentration of the oxygen that is consumed by the cells. For system (1.1), by making use of energy-type functionals, there have been many literatures studied the existence of global solutions in the bounded domain or the whole space under some assumption on \(f(c), S(c)\) and initial data (see [11, 13, 6, 14, 31, 28, 32, 36] and references therein). In fact, in the two-dimensional setting, if \(\kappa = 0\), Duan et al. ([6]) proved global existence of weak solutions for the Cauchy problem of (1.1), under smallness assumptions on either \(\nabla \phi\) or the
initial data for oxygen concentration. Winkler [28] proved that (1.1) has a unique global classical solution in a bounded convex domain $\Omega \subset \mathbb{R}^2$ with smooth boundary for large data with suitable regularity. While for three-dimensional chemotaxis-(Navier)-Stokes system (1.1), when $\kappa = 0$, Winkler [28] proved that the chemotaxis-Stokes system (1.1) possesses at least one global weak solution.

When $\Delta n$ is replaced by $\nabla \cdot (D(n)\nabla n)$ in the first equation in (1.1), some authors used some energy-type functionals to prove the global or local existence of the solutions to system (1.1) (see [7, 8, 35] and references therein). Indeed, the porous medium diffusion function $D$ satisfies

$$D \in C^\iota_{\text{loc}}([0, \infty)) \quad \text{for some } \iota > 0, \quad C_D n^{m-1} \geq D(n) \geq C_D n^{m-1} \quad \text{for all } n > 0 \quad (1.2)$$

with some $m > 0$ and $C_D \geq C_D$. Zhang and Li [35] used some energy-type functionals to prove the global existence of the weak solutions to system (1.1) when $m \geq \frac{2}{3}$. Recently, if $\kappa = 0$, Tao and Winkler [20] proved the locally bounded global of weak solution of (1.1) in $\mathbb{R}^3$ as $m > \frac{8}{7}$. These energy-type functionals play key roles in their proofs.

Generally speaking, more recent modeling approaches (see DiLuzio et al. [4], Winkler [29, 30], Xue et al. [33]) suggest that chemotactic migration is not directed to the gradient of the chemical substance but with a rotation, and that accordingly, the chemotactic sensitivity should be a tensor which may have nontrivial off-diagonal entries. Motivated by the above works, we will investigate the chemotaxis-Navier-stokes system with nonlinear diffusion and the rotational sensitivity in this paper. Precisely, we shall consider the following initial-boundary problem

$$\begin{cases}
    n_t + u \cdot \nabla n = \nabla \cdot (D(n)\nabla n) - \nabla \cdot (nS(x, n, c) \cdot \nabla c), & x \in \Omega, t > 0, \\
    c_t + u \cdot \nabla c = \Delta c - nc, & x \in \Omega, t > 0, \\
    u_t + \kappa (u \cdot \nabla) u + \nabla P = \Delta u + n \nabla \phi, & x \in \Omega, t > 0, \\
    \nabla \cdot u = 0, & x \in \Omega, t > 0, \\
    (D(n)\nabla n - nS(x, n, c) \cdot \nabla c) \cdot \nu = \nabla c \cdot \nu = 0, u = 0, & x \in \partial \Omega, t > 0, \\
    n(x, 0) = n_0(x), c(x, 0) = c_0(x), u(x, 0) = u_0(x), & x \in \Omega,
\end{cases} \quad (1.3)$$

where $\Omega \subset \mathbb{R}^3$ be a bounded convex domain with smooth boundary, $S(x, n, c)$ is a tensor-
valued function, satisfying

\[ S \in C^2(\Omega \times [0, \infty)^2; \mathbb{R}^{3 \times 3}) \]  \hspace{1cm} (1.4)

and

\[ |S(x, n, c)| \leq S_0(c) \quad \text{for all} \quad (x, n, c) \in \Omega \times [0, \infty)^2 \]  \hspace{1cm} (1.5)

with some nondecreasing \( S_0 : [0, \infty) \rightarrow \mathbb{R} \), which indicates the rotational effect. \( D, \nabla \phi, \kappa, n(x, t), u(x, t), c(x, t) \) and \( P(x, t) \) are denoted as before. Due to the significance of the biological background, many mathematicians have studied (1.3) and made more progress in the past years (Ishida [11], Zheng [39, 41], Wang et al. [23, 25], Winkler [30], Wang and Li [24], Cao and Lankeit [2]). In contrast to the chemotaxis-(Navier-)Stokes system (1.1), chemotaxis-(Navier-)Stokes system (1.3) with tensor-valued sensitivity loses some natural gradient-like structure, which gives rise to considerable mathematical difficulties and some new analysis is needed. Therefore, as far as I know that only very few results appear to be available on chemotaxis-(Navier-)Stokes with such tensor-valued sensitivities. To this end, if \( \kappa = 0 \) in (1.3) and \( D \) satisfies (1.2) with \( m > \frac{7}{6} \), Winkler ([30]) showed that the three space dimensions of the chemotaxis–Stokes system (\( \kappa = 0 \) in (1.3)) possessed at least one bounded weak solution which stabilizes to the spatially homogeneous equilibrium \( (\bar{n}_0, 0, 0) \) with \( \bar{n}_0 := \frac{1}{|\Omega|} \int_{\Omega} n_0 \) as \( t \to \infty \). While, if \( \kappa \neq 0 \), assuming that (1.4)–(1.5) hold and \( D(n) = mn^{m-1} \), Ishida ([11]) showed that (1.3) admits a bounded global weak solution in two space dimensions. More recently, if \( D \equiv 1 \), \( S \) satisfies that (1.4)–(1.5) and the initial data satisfied certain smallness conditions, Cao and Lankeit ([2]) proved that (1.3) possessed a global classical solution and gave the decay properties of these solutions on three space dimensions. However, for three space dimensions of full nonlinear chemotaxis-Navier-Stokes system (1.3) without the assuming of smallness conditions, there is still a open problem.

Before formulating our main results, we first explain the notations and conventions used throughout this paper. Throughout this paper, let \( A_r \) denote the Stokes operator with domain \( D(A_r) := W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega) \cap L_r^r(\Omega) \), and \( L_r^r(\Omega) := \{ \varphi \in L^r(\Omega) | \nabla \cdot \varphi = 0 \} \) for \( r \in (1, \infty) \) ([17]).
Theorem 1.1. Let

\[ \phi \in W^{1,\infty}(\Omega). \] (1.6)

Moreover, assume that the initial data \((n_0, c_0, u_0)\) satisfy

\[
\begin{cases}
  n_0 \in C^\kappa(\bar{\Omega}) & \text{for certain } \kappa > 0 \text{ with } n_0 \geq 0 \text{ in } \Omega, \\
  c_0 \in W^{1,\infty}(\bar{\Omega}) & \text{with } c_0 \geq 0 \text{ in } \bar{\Omega}, \\
  u_0 \in D(A_\gamma^r) & \text{for some } \gamma \in (\frac{3}{4}, 1) \text{ and any } r \in (1, \infty).
\end{cases}
\] (1.7)

and suppose that \(m\) and \(S\) satisfy (1.2) and (1.4)–(1.5), respectively. If

\[ m > \frac{10}{9}, \] (1.8)

then it holds that there exists at least one global weak solution (in the sense of Definition 5.1 above) of problem (1.3).

Remark 1.1. (i) If \(S(x, n, c) := C_S\) and \(\kappa = 0\), Theorem 1.1 extends the results of Theorem 1.1 of Tao and Winkler [20], who proved the possibility of locally bounded global solutions, in the case that \(m > \frac{8}{7}\).

(ii) If \(\kappa = 0\), Theorem 1.1 extends the results of Theorem 1.1 of Zheng [41], who proved the possibility of locally bounded global solutions, in the case that \(m > \frac{9}{8}\).

(iii) In view of Theorem 1.1, if the flow of fluid is ignored or the fluid is stationary in (1.3), \(S(x, n, c) := C_S\), and \(N = 3\), Theorem 1.1 is consistent with the result of Theorem 2.1 of Zheng and Wang [42], who proved the possibility of global existence, in the case that \(m > \frac{9}{8}\).

(iv) If \(m > \frac{10}{9}\), Theorem 1.1 is hold without requirement of the small initial data (see Cao and Lankeit [2]).

Before proving our main results about the model (1.3) in the next part, let us mention the following Keller-Segel-(Navier)-Stokes model (accounting for terms \(+n - c\) in place of \(-nc\) in the second equation of (1.3)), which is a closely related variant of (1.3)

\[
\begin{align*}
n_t + u \cdot \nabla n &= \nabla \cdot (D(n) \nabla n) - \nabla \cdot (nS(x, n, c)\nabla c), \quad x \in \Omega, t > 0, \\
c_t + u \cdot \nabla c &= \Delta c - c + n, \quad x \in \Omega, t > 0, \\
u_t + \kappa(u \cdot \nabla)u + \nabla P &= \Delta u + n\nabla \phi, \quad x \in \Omega, t > 0, \\
\nabla \cdot u &= 0, \quad x \in \Omega, t > 0.
\end{align*}
\] (1.9)
In contrast to (1.3), in the classical Keller-Segel system the chemoattractant is produced by the bacteria themselves and not consumed, and models of Keller-Segel-(Navier)-Stokes type have also been considered (see Wang and Xiang [25, 26], Liu and Wang [15], Zheng [40]).

The crucial step of our approaches is to establish the natural gradient-like energy functional

\[
\int_{\Omega} n_{\varepsilon}(\cdot, t) \ln n_{\varepsilon}(\cdot, t) + \int_{\Omega} |\nabla c_{\varepsilon}(\cdot, t)|^2 + \int_{\Omega} |u_{\varepsilon}(\cdot, t)|^2, \quad \text{if} \quad \frac{10}{9} < m \leq 2,
\]

\[
\int_{\Omega} [n_{\varepsilon}(\cdot, t)^{m-1}(\cdot, t) + c_{\varepsilon}^2(\cdot, t) + |u_{\varepsilon}(\cdot, t)|^2], \quad \text{if} \quad m > 2,
\]

which is a new estimate of chemotaxis–Navier–Stokes system with rotation (see Lemmata 2.6–2.10), although, the part of (1.10) (\(m < 2\)) has been used to solve the chemotaxis-(Navier)-Stokes system without rotation (see [1, 13, 28]). Here \((n_{\varepsilon}, c_{\varepsilon}, u_{\varepsilon})\) is the solution of the suitable approximate problem of (1.3). We guess that (1.10) can also be dealt with other types of systems, e.g., Keller-Segel-Navier-Stokes system with nonlinear diffusion (see our recent paper [40]). Then, in view of the estimates (1.10), the suitable interpolation arguments (see Lemma 2.3) and the basic a priori information (see Lemma 2.4), one can obtain boundedness of

\[
\|n_{\varepsilon}(\cdot, t)\|_{L^p(\Omega)} \leq C \quad \text{for all} \quad t \in (0, T_{\max, \varepsilon}) \quad \text{with} \quad p_0 > 3 \tag{1.11}
\]

and \(C := C(\varepsilon)\) depends on \(\varepsilon\) (see Lemmata 3.1–3.3). With estimate of (1.11) at hand, by using variation-of-constants, smoothing properties of the Stokes semigroup and Moser-type iteration, we can show that our approximate solutions \((n_{\varepsilon}, c_{\varepsilon}, u_{\varepsilon})\) are actually global in time. Finally, by the interpolation inequality, we derive a priori estimates for the approximate solutions \((n_{\varepsilon}, c_{\varepsilon}, u_{\varepsilon})\) to the approximate problems of problem (1.3) and complete the proof of main results by an approximation procedure.

The rest of this paper is organized as follows. In the following section, we state our main results, introduce the regularized system of (1.3) and collect some basic estimates which will be useful for proofs later on. In Section 3, we derive a series of useful estimates which depend on \(\varepsilon\) and then obtain the global existence of the regularized problems. In Section 4, in light of the Gagliardo-Nirenberg inequality and the other some basic analysis, we derive some \(\varepsilon\)-independent boundedness of the time derivatives of certain powers of \(n_{\varepsilon}, c_{\varepsilon}\) and \(u_{\varepsilon}\). In the
final step, it is proved that (1.3) possesses at least one weak solution by the Aubin–Lions lemma, the standard parabolic regularity theory and the Egorov theorem.

2 Preliminaries and main results

Due to hypothesis (1.2), \(\kappa \neq 0\) and the presence of tensor-valued \(S\) in system (1.3), we need to consider an appropriately regularized problem of (1.3) at first. Indeed, following the idea of [32] (see also [20, 35]), the corresponding regularized problem is introduced as follows:

\[
\begin{aligned}
\begin{cases}
 n_{\varepsilon t} + u_{\varepsilon} \cdot \nabla n_{\varepsilon} = \nabla \cdot (D_{\varepsilon}(n_{\varepsilon})\nabla n_{\varepsilon}) - \nabla \cdot (n_{\varepsilon}F_{\varepsilon}(n_{\varepsilon})S_{\varepsilon}(x, n_{\varepsilon}, c_{\varepsilon}) \cdot \nabla c_{\varepsilon}), & x \in \Omega, t > 0, \\
 c_{\varepsilon t} + u_{\varepsilon} \cdot \nabla c_{\varepsilon} = \Delta c_{\varepsilon} - n_{\varepsilon}c_{\varepsilon}, & x \in \Omega, t > 0, \\
 u_{\varepsilon t} + \nabla P_{\varepsilon} = \Delta u_{\varepsilon} - \kappa(Y_{\varepsilon}u_{\varepsilon} \cdot \nabla)u_{\varepsilon} + n_{\varepsilon}\nabla \phi, & x \in \Omega, t > 0, \\
 \nabla \cdot u_{\varepsilon} = 0, & x \in \Omega, t > 0, \\
 \nabla n_{\varepsilon} \cdot \nu = \nabla c_{\varepsilon} \cdot \nu = 0, u_{\varepsilon} = 0, & x \in \partial \Omega, t > 0, \\
 n_{\varepsilon}(x, 0) = n_{0}(x), c_{\varepsilon}(x, 0) = c_{0}(x), u_{\varepsilon}(x, 0) = u_{0}(x), & x \in \Omega,
\end{cases}
\end{aligned}
\]  

(2.1)

where

\[
Y_{\varepsilon}w := (1 + \varepsilon A)^{-1}w \quad \text{for all } w \in L^{2}_{0}(\Omega)
\]  

(2.2)

is the standard Yosida approximation, \(D_{\varepsilon}(s) := D_{\varepsilon}(s + \varepsilon)\),

\[
S_{\varepsilon}(x, n, c) := \rho_{\varepsilon}(x)S(x, n, c), \quad x \in \bar{\Omega}, \quad n \geq 0, \quad c \geq 0 \quad \text{and} \quad \varepsilon \in (0, 1)
\]

(2.3)

and

\[
F_{\varepsilon}(s) = \frac{1}{1 + \varepsilon s} \quad \text{for } s \geq 0.
\]

(2.4)

Here \((\rho_{\varepsilon})_{\varepsilon \in (0, 1)} \in C^{\infty}_{0}(\Omega)\) be a family of standard cut-off functions satisfying \(0 \leq \rho_{\varepsilon} \leq 1\) in \(\Omega\) and \(\rho_{\varepsilon} \to 1\) in \(\Omega\) as \(\varepsilon \to 0\).

With the help of the Schauder fixed point theorem, the standard regularity theory of parabolic equations and the Stokes system, in light of a straightforward adaptation of a corresponding procedure in Lemma 2.1 of [28] to the present setting (see also Lemma 2.1 of [30]), we can easily obtain following local existence result of (2.1):

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Lemma 2.1. Let $\Omega \subseteq \mathbb{R}^3$ be a bounded convex domain with smooth boundary. Assume that the initial data $(n_0, c_0, u_0)$ fulfills (1.7). Then there exist $T_{\max, \varepsilon} \in (0, \infty]$ and a classical solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ of (2.1) in $\Omega \times (0, T_{\max, \varepsilon})$ such that

\[
\begin{aligned}
&n_\varepsilon \in C^0(\Omega \times [0, T_{\max, \varepsilon})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max, \varepsilon})), \\
c_\varepsilon \in C^0(\Omega \times [0, T_{\max, \varepsilon})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max, \varepsilon})), \\
u_\varepsilon \in C^0(\Omega \times [0, T_{\max, \varepsilon})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max, \varepsilon})), \\
&P_\varepsilon \in C^{1,0}(\bar{\Omega} \times (0, T_{\max, \varepsilon})),
\end{aligned}
\]  

classically solving (2.1) in $\Omega \times (0, T_{\max, \varepsilon})$. Moreover, it holds that $n_\varepsilon$ and $c_\varepsilon$ are nonnegative in $\Omega \times (0, T_{\max, \varepsilon})$, and

\[
\limsup_{t \nearrow T_{\max, \varepsilon}} \left( ||n_\varepsilon(\cdot, t)||_{L^\infty(\Omega)} + ||c_\varepsilon(\cdot, t)||_{W^{1,\infty}(\Omega)} + ||A^\gamma u_\varepsilon(\cdot, t)||_{L^2(\Omega)} \right) = \infty, \tag{2.6}
\]

where $\gamma$ is given by (1.7).

Lemma 2.2. (13) Let $w \in C^2(\bar{\Omega})$ satisfy $\nabla w \cdot \nu = 0$ on $\partial \Omega$.

(i) Then

\[
\frac{\partial |\nabla w|^2}{\partial \nu} \leq C_{\partial \Omega} |\nabla w|^2,
\]

where $C_{\partial \Omega}$ is an upper bound on the curvature of $\partial \Omega$.

(ii) Furthermore, for any $\delta > 0$ there is $C(\delta) > 0$ such that every $w \in C^2(\bar{\Omega})$ with $\nabla w \cdot \nu = 0$ on $\partial \Omega$ fulfills

\[
||w||_{L^2(\partial \Omega)} \leq \delta ||\Delta w||_{L^2(\Omega)} + C(\delta)||w||_{L^2(\Omega)}.
\]

(iii) For any positive $w \in C^2(\bar{\Omega})$

\[
||\Delta w^\frac{1}{2}||_{L^2(\Omega)} \leq \frac{1}{2}||w^\frac{1}{2}\Delta \ln w||_{L^2(\Omega)} + \frac{1}{4}||w^{-\frac{3}{2}}|\nabla w|^2||_{L^2(\Omega)}. \tag{2.7}
\]

(iv) There are $C > 0$ and $\delta > 0$ such that every positive $w \in C^2(\bar{\Omega})$ fulfilling $\nabla w \cdot \nu = 0$ on $\partial \Omega$ satisfies

\[
-2 \int_\Omega \frac{|\Delta w|^2}{w} + \int_\Omega \frac{|\nabla w|^2 \Delta w}{w^2} \leq -\delta \int_\Omega \frac{w}{w^2} D^2 \ln w - \delta \int_\Omega \frac{|\nabla w|^4}{w^3} + C \int_\Omega w. \tag{2.8}
\]
Lemma 2.3. (Lemma 3.8 of [30]) Let \( q \geq 1 \),
\[
\lambda \in [2q + 2, 4q + 1]
\]  
(2.9)
and \( \Omega \subset \mathbb{R}^3 \) be a bounded convex domain with smooth boundary. Then there exists \( C > 0 \) such that for all \( \varphi \in C^2(\bar{\Omega}) \) fulfilling \( \varphi \cdot \frac{\partial \varphi}{\partial n} = 0 \) on \( \partial \Omega \) we have
\[
\| \nabla \varphi \|_{L^\lambda(\Omega)} \leq C \| \nabla \varphi \|^q \| D^2 \varphi \|_{L^2(\Omega)} \frac{4^\lambda}{(2q + 1)^\lambda} \| \varphi \|_{L^\infty(\Omega)} + C \| \varphi \|_{L^\infty(\Omega)}. 
\]  
(2.10)

Let us state two well-known results of solution of (2.1).

Lemma 2.4. The solution of (2.1) satisfies
\[
\| n_\varepsilon(\cdot, t) \|_{L^1(\Omega)} = \| n_0 \|_{L^1(\Omega)} \quad \text{for all} \quad t \in (0, T_{\max, \varepsilon})
\]  
(2.11)
and
\[
\| c_\varepsilon(\cdot, t) \|_{L^\infty(\Omega)} \leq \| c_0 \|_{L^\infty(\Omega)} \quad \text{for all} \quad t \in (0, T_{\max, \varepsilon}).
\]  
(2.12)

Lemma 2.5. Let \( m > \frac{4}{3} \). There exists \( C > 0 \) such that the solution of (2.1) satisfies
\[
\frac{d}{dt} \int_{\Omega} | u_\varepsilon |^2 + \int_{\Omega} | \nabla u_\varepsilon |^2 \leq \frac{1}{8} \| \nabla n_\varepsilon^{m-1} \|_{L^2(\Omega)}^2 + C \quad \text{for all} \quad t \in (0, T_{\max, \varepsilon}),
\]  
(2.13)

Proof. Multiplying the third equation of (2.1) by \( u_\varepsilon \), and then integrating by parts over \( \Omega \) and using \( \nabla \cdot u_\varepsilon = 0 \), it follows that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} | u_\varepsilon |^2 + \int_{\Omega} | \nabla u_\varepsilon |^2 = \int_{\Omega} n_\varepsilon u_\varepsilon \cdot \nabla \varphi \quad \text{for all} \quad t \in (0, T_{\max, \varepsilon}).
\]  
(2.14)

Here we use the H"older inequality, (1.6) and the continuity of the embedding \( W^{1,2}(\Omega) \hookrightarrow L^6(\Omega) \) and to find \( C_1 > 0 \) such that
\[
\int_{\Omega} n_\varepsilon u_\varepsilon \cdot \nabla \varphi \leq \| \nabla \varphi \|_{L^{\infty}(\Omega)} \| n_\varepsilon \|_{L^{\frac{m}{2}}(\Omega)} \| \nabla u_\varepsilon \|_{L^2(\Omega)}
\]
\[
\leq C_1 \| n_\varepsilon \|_{L^{\frac{m}{2}}(\Omega)} \| \nabla u_\varepsilon \|_{L^2(\Omega)} \quad \text{for all} \quad t \in (0, T_{\max, \varepsilon}),
\]  
(2.15)

According to the Gagliardo–Nirenberg inequality and (2.11), it is readily to see that
\[
\| n_\varepsilon \|_{L^{\frac{m}{2}}(\Omega)} = \| n_\varepsilon^{m-1} \|_{L^{\frac{m-1}{m-\frac{m}{2}}}(\Omega)}^{\frac{1}{m-1}} \| n_\varepsilon^{m-1} \|_{L^{\frac{m}{2}}(\Omega)}^{\frac{1}{m-1}}
\]
\[
\leq C_2 \| \nabla n_\varepsilon^{m-1} \|_{L^2(\Omega)}^{\frac{1}{m-1}} \| n_\varepsilon^{m-1} \|_{L^{\frac{m}{2}}(\Omega)}^{\frac{1}{m-1}} \| n_\varepsilon^{m-1} \|_{L^{\frac{m}{2}}(\Omega)}^{\frac{1}{m-1}}
\]
\[
\leq C_3 \| \nabla n_\varepsilon^{m-1} \|_{L^2(\Omega)}^{\frac{1}{m-1}} \| n_\varepsilon^{m-1} \|_{L^{\frac{m}{2}}(\Omega)}^{\frac{1}{m-1}} + 1 \quad \text{for all} \quad t \in (0, T_{\max, \varepsilon})
\]  
(2.16)
with some positive constants $C_2$ and $C_3$ independent of $\varepsilon$. Next, by (2.15)–(2.16) and using the Young inequality and $m > \frac{4}{3}$ yields
\[
\int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla \phi \leq \frac{1}{2} \| \nabla u_\varepsilon \|_{L^2(\Omega)}^2 + C_4(\| \nabla n_\varepsilon^{m-1} \|_{L^{\frac{2}{m-2}}(\Omega)}^2 + 1) \\
\leq \frac{1}{2} \| \nabla u_\varepsilon \|_{L^2(\Omega)}^2 + \frac{1}{8} \| \nabla n_\varepsilon^{m-1} \|_{L^2(\Omega)}^2 + C_5 \text{ for all } t \in (0, T_{\text{max},\varepsilon}).
\] (2.17)

Here $C_4$ and $C_5$ are positive constants independent of $\varepsilon$. Finally, putting (2.17) into (2.15), one obtains (2.13).

\[\square\]

**Lemma 2.6.** Let $m > \frac{2}{3}$. There exists $C > 0$ such that for every $\delta_1 > 0$, the solution of (2.7) satisfies
\[
\frac{d}{dt} \int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 \leq \delta_1 \int_\Omega \frac{D_\varepsilon(n_\varepsilon)|\nabla n_\varepsilon|^2}{n_\varepsilon} + C \text{ for all } t \in (0, T_{\text{max},\varepsilon}),
\] (2.18)

**Proof.** We begin with (2.15), the Gagliardo–Nirenberg inequality and (2.11) ensure
\[
\| n_\varepsilon \|_{L^\frac{m}{m-2}(\Omega)} = \| n_\varepsilon^\frac{m}{m-2} \|_{L^{\frac{m}{m-2}}(\Omega)} \\
\leq C_1 \| \nabla n_\varepsilon^\frac{m}{m-2} \|_{L^2(\Omega)} \| n_\varepsilon^\frac{m}{m-2} \|_{L^\frac{m}{m-2}(\Omega)} \\
\leq C_2(\| \nabla n_\varepsilon^\frac{m}{m-2} \|_{L^2(\Omega)} + 1) \text{ for all } t \in (0, T_{\text{max},\varepsilon}),
\] (2.19)

where $C_1$ and $C_2$ are positive constants independent of $\varepsilon$. Next, substituting (2.19) into (2.15) and using the Young inequality, (1.2) and $m > \frac{2}{3}$ yields
\[
\| n_\varepsilon \|_{L^\frac{m}{m-2}(\Omega)} \leq \frac{1}{2} \| \nabla u_\varepsilon \|_{L^2(\Omega)}^2 + C_3(\| \nabla n_\varepsilon^\frac{m}{m-2} \|_{L^2(\Omega)}^2 + 1) \\
\leq \frac{1}{2} \| \nabla u_\varepsilon \|_{L^2(\Omega)}^2 + \frac{\delta_1}{2C_4} \| \nabla n_\varepsilon^\frac{m}{m-2} \|_{L^2(\Omega)}^2 + C_4 \\
\leq \frac{1}{2} \| \nabla u_\varepsilon \|_{L^2(\Omega)}^2 + \frac{\delta_1}{2} \int_\Omega \frac{D_\varepsilon(n_\varepsilon)|\nabla n_\varepsilon|^2}{n_\varepsilon} + C_4 \text{ for all } t \in (0, T_{\text{max},\varepsilon}),
\] (2.20)

where $C_3$ and $C_4$ are positive constants independent of $\varepsilon$. Finally, collecting (2.15) and (2.20), we can conclude (2.18).

\[\square\]

**Lemma 2.7.** Let $\frac{10}{9} < m \leq 2$. There exist $\mu_0, C > 0$ such that for every $\varepsilon > 0$ and $\delta_i(i = 2, 3, 4, 5) > 0$
\[
\frac{d}{dt} \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + \mu_0 \int_\Omega c_\varepsilon |D^2 \ln c_\varepsilon|^2 + (\mu_0 - \frac{\delta_2}{4} - \frac{\delta_3}{4}) \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} \\
\leq \left( \frac{\delta_4}{4} + \frac{\delta_5}{4} \right) \int_\Omega \frac{D_\varepsilon(n_\varepsilon)|\nabla n_\varepsilon|^2}{n_\varepsilon} + \frac{4}{\delta_2} \| c_0 \|_{L^\infty(\Omega)} \int_\Omega |\nabla u_\varepsilon|^2 + C \text{ for all } t \in (0, T_{\text{max},\varepsilon})
\] (2.21)
Proof. We begin by computing \(\frac{d}{dt} \int_{\Omega} \frac{\lvert \nabla c_{\varepsilon} \rvert^2}{c_{\varepsilon}}\). For any \(t \in (0, T_{\text{max, } \varepsilon})\), we have

\[
\frac{d}{dt} \int_{\Omega} \frac{\lvert \nabla c_{\varepsilon} \rvert^2}{c_{\varepsilon}} = 2 \int_{\Omega} \frac{\nabla c_{\varepsilon} \cdot \nabla c_{\varepsilon t}}{c_{\varepsilon}} - \int_{\Omega} \frac{\lvert \nabla c_{\varepsilon} \rvert^2 c_{\varepsilon t}}{c_{\varepsilon}^2} = -2 \int_{\Omega} \nabla c_{\varepsilon} c_{\varepsilon t} + \int_{\Omega} \frac{\lvert \nabla c_{\varepsilon} \rvert^2 c_{\varepsilon t}}{c_{\varepsilon}^2} = -2 \int_{\Omega} \lvert \Delta c_{\varepsilon} \rvert^2 + 2 \int_{\Omega} \frac{\Delta c_{\varepsilon} n_{\varepsilon} c_{\varepsilon}}{c_{\varepsilon}^2} + 2 \int_{\Omega} \frac{\Delta c_{\varepsilon} u_{\varepsilon} \cdot \nabla c_{\varepsilon}}{c_{\varepsilon}^2} + \int_{\Omega} \nabla c_{\varepsilon} |\nabla c_{\varepsilon}|^2 + \int_{\Omega} \frac{\nabla c_{\varepsilon} u_{\varepsilon} \cdot \nabla c_{\varepsilon}}{c_{\varepsilon}^2}.
\]  

(2.22)

From (vi) of Lemma 2.2, by Young inequality, there exist \(\mu_0 > 0\) and \(C(\mu_0) > 0\) such that

\[
-2 \int_{\Omega} \frac{\lvert \Delta c_{\varepsilon} \rvert^2}{c_{\varepsilon}} + \int_{\Omega} \frac{\lvert \nabla c_{\varepsilon} \rvert^2 \Delta c_{\varepsilon}}{c_{\varepsilon}^2} \leq -\mu_0 \int_{\Omega} c_{\varepsilon} |D \ln c_{\varepsilon}|^2 - \mu_0 \int_{\Omega} \frac{\lvert \nabla c_{\varepsilon} \rvert^4}{c_{\varepsilon}^3} + C(\mu_0) \int_{\Omega} c_{\varepsilon}
\]

for all \(t \in (0, T_{\text{max, } \varepsilon})\). As to the terms containing \(u_{\varepsilon}\), we note that for all \(\varepsilon > 0\)

\[
2 \int_{\Omega} \frac{\Delta c_{\varepsilon}}{c_{\varepsilon}} (u_{\varepsilon} \cdot \nabla c_{\varepsilon}) = 2 \int_{\Omega} \frac{\lvert \nabla c_{\varepsilon} \rvert^2}{c_{\varepsilon}^2} u_{\varepsilon} \cdot \nabla c_{\varepsilon} - 2 \int_{\Omega} \frac{1}{c_{\varepsilon}} \nabla c_{\varepsilon} \cdot (\nabla u_{\varepsilon} \nabla c_{\varepsilon}) - 2 \int_{\Omega} \frac{1}{c_{\varepsilon}} u_{\varepsilon} \cdot D^2 c_{\varepsilon} \nabla c_{\varepsilon}
\]

for all \(t \in (0, T_{\text{max, } \varepsilon})\) and

\[
\int_{\Omega} \frac{\lvert \nabla c_{\varepsilon} \rvert^2}{c_{\varepsilon}^2} u_{\varepsilon} \cdot \nabla c_{\varepsilon} = 2 \int_{\Omega} \frac{1}{c_{\varepsilon}} u_{\varepsilon} \cdot D^2 c_{\varepsilon} \nabla c_{\varepsilon}
\]

for all \(t \in (0, T_{\text{max, } \varepsilon})\), so that due to the Young inequality and Lemma 2.4 for any \(\delta_2 > 0\)

\[
2 \int_{\Omega} \frac{\Delta c_{\varepsilon}}{c_{\varepsilon}} (u_{\varepsilon} \cdot \nabla c_{\varepsilon}) - \int_{\Omega} \frac{\lvert \nabla c_{\varepsilon} \rvert^2}{c_{\varepsilon}^2} u_{\varepsilon} \cdot \nabla c_{\varepsilon} = -2 \int_{\Omega} \frac{1}{c_{\varepsilon}} \nabla c_{\varepsilon} \cdot (\nabla u_{\varepsilon} \nabla c_{\varepsilon}) = \leq \frac{\delta_2}{4} \int_{\Omega} \frac{\lvert \nabla c_{\varepsilon} \rvert^4}{c_{\varepsilon}^3} + \frac{4}{\delta_2} \int_{\Omega} \lvert \nabla u_{\varepsilon} \rvert^2 = \leq \frac{\delta_2}{4} \int_{\Omega} \frac{\lvert \nabla c_{\varepsilon} \rvert^4}{c_{\varepsilon}^3} + C_1 \int_{\Omega} \lvert \nabla u_{\varepsilon} \rvert^2
\]

for all \(t \in (0, T_{\text{max, } \varepsilon})\)

(2.24)

with some \(C_1 := \frac{4}{\delta_2} \lVert c_0 \rVert_{L^\infty(\Omega)}\). And an integration by parts, the Young inequality, (1.2) and (2.12) shows

\[
2 \int_{\Omega} \frac{\Delta c_{\varepsilon} n_{\varepsilon}}{c_{\varepsilon}} c_{\varepsilon} = -2 \int_{\Omega} \frac{\lvert \nabla n_{\varepsilon} \rvert^2}{c_{\varepsilon}^2} c_{\varepsilon} \leq \frac{\delta_3}{4} \int_{\Omega} \frac{\lvert \nabla c_{\varepsilon} \rvert^4}{c_{\varepsilon}^3} + 2^\frac{4}{3} \delta_3 \int_{\Omega} \frac{\lvert \nabla n_{\varepsilon} \rvert^4}{c_{\varepsilon}^3} \leq \frac{\delta_3}{4} \int_{\Omega} \frac{\lvert \nabla c_{\varepsilon} \rvert^4}{c_{\varepsilon}^3} + \frac{\delta_4}{4C_D} \int_{\Omega} \frac{n_{\varepsilon}^{m-2} \lvert \nabla n_{\varepsilon} \rvert^2 + C_2 \int_{\Omega} n_{\varepsilon}^{4-2m} c_{\varepsilon}^3 \leq \frac{\delta_3}{4} \int_{\Omega} \frac{\lvert \nabla c_{\varepsilon} \rvert^4}{c_{\varepsilon}^3} + \frac{\delta_5}{4} \int_{\Omega} \frac{D_{\varepsilon}(n_{\varepsilon}) \lvert \nabla n_{\varepsilon} \rvert^2}{c_{\varepsilon}^3} + C_3 \int_{\Omega} n_{\varepsilon}^{4-2m} \text{ for all } t \in (0, T_{\text{max, } \varepsilon}),
\]

(2.25)
where $\delta_3, \delta_4, C_2 := C_2(\delta_3, \delta_4), C_3 := C_3(\delta_3, \delta_4, \|c_0\|_{L^\infty(\Omega)})$ are positive constants.

Case $\frac{10}{9} < m < \frac{3}{2}$: It is easy to deduce from the Gagliardo–Nirenberg inequality and (2.11) that

$$C_3 \int_{\Omega} n_\varepsilon^{4-2m} = C_3 \|n_\varepsilon^m\|_L^{\frac{2(4-2m)}{(4-2m)m}}(\Omega)$$

$$\leq C_4 \|D n_\varepsilon^m\|_{L^2(\Omega)}^\frac{m}{2} \|n_\varepsilon^m\|_{L^{\frac{2(4-2m)(1-\mu)}{m}}(\Omega)}^\frac{2(4-2m)}{m} + \|n_\varepsilon^m\|_{L^\infty(\Omega)}^\frac{2(4-2m)}{m}$$

$$\leq C_5 (\|D n_\varepsilon^m\|_{L^2(\Omega)}^\frac{m}{2} + 1)$$

$$= C_5 (\|D n_\varepsilon^m\|_{L^2(\Omega)}^\frac{m}{2} + 1) \text{ for all } t \in (0, T_{max,\varepsilon}),$$

where $C_4$ and $C_5$ are positive constants,

$$\mu_1 = \frac{3m}{2} - \frac{3m}{3m-1} \in (0, 1).$$

Now, in view of $m > \frac{10}{9}$, with the help of the Young inequality and (2.26), for any $\delta_5 > 0$, we have

$$C_3 \int_{\Omega} n_\varepsilon^{4-2m} \leq \frac{\delta_5}{4C_D} \|D n_\varepsilon^m\|_{L^2(\Omega)}^2 + C_6 \text{ for all } t \in (0, T_{max,\varepsilon})$$

(2.27)

with some $C_6 > 0$.

Case $\frac{3}{2} \leq m \leq 2$: With the help of the Young inequality and (2.11), we derive that

$$C_3 \int_{\Omega} n_\varepsilon^{4-2m} \leq \int_{\Omega} n_\varepsilon + C_7$$

$$\leq C_8 \text{ for all } t \in (0, T_{max,\varepsilon}),$$

(2.28)

where $C_7$ and $C_8$ are positive constants independent of $\varepsilon$. Finally, we utilize (2.23)–(2.28) and (2.22) to deduce the results.

**Lemma 2.8.** Let $\frac{10}{9} < m \leq 2$ and $\delta > 0$. There is $C > 0$ such that for any $\delta_6$ and $\delta_7$

$$\frac{d}{dt} \int_{\Omega} n_\varepsilon \ln n_\varepsilon + (1 - \frac{\delta_7}{4}) \int_{\Omega} \frac{D_\varepsilon(n_\varepsilon)|\nabla n_\varepsilon|^2}{n_\varepsilon} \leq \frac{\delta_6}{4} \int_{\Omega} \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + C \text{ for all } t \in (0, T_{max,\varepsilon}).$$

(2.29)

**Proof.** Firstly, using the first equation of (2.1) and (2.4), from integration by parts we obtain...
from (1.5)

\[
\frac{d}{dt} \int_{\Omega} n_{\varepsilon} \ln n_{\varepsilon} = \int_{\Omega} n_{\varepsilon t} \ln n_{\varepsilon} + \int_{\Omega} n_{\varepsilon t} \\
= \int_{\Omega} \nabla \cdot (D_{\varepsilon}(n_{\varepsilon})\nabla n_{\varepsilon}) \ln n_{\varepsilon} - \int_{\Omega} \ln n_{\varepsilon} \nabla \cdot (n_{\varepsilon}F_{\varepsilon}(n_{\varepsilon})S_{\varepsilon}(x, n_{\varepsilon}, c_{\varepsilon}) \cdot \nabla c_{\varepsilon}) - \int_{\Omega} \ln n_{\varepsilon} u_{\varepsilon} \cdot \nabla n_{\varepsilon} \\
\leq -\int_{\Omega} \frac{D_{\varepsilon}(n_{\varepsilon})|\nabla n_{\varepsilon}|^2}{n_{\varepsilon}} + \int_{\Omega} S_{0}(c_{\varepsilon})|\nabla n_{\varepsilon}| |\nabla c_{\varepsilon}| \\
\leq \frac{\delta_{0}}{4} \int_{\Omega} |\nabla c_{\varepsilon}|^4 c_{\varepsilon}^3 + \frac{\delta_{7}}{4} \int_{\Omega} \frac{D_{\varepsilon}(n_{\varepsilon})|\nabla n_{\varepsilon}|^2}{n_{\varepsilon}} + C_{1} \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}) \tag{2.30}
\]

for all \( t \in (0, T_{\text{max}, \varepsilon}) \). Now, in view of (2.12), employing the same argument of (2.25)–(2.28), for any \( \delta_{6} > 0 \) and \( \delta_{7} > 0 \), we conclude that

\[
\int_{\Omega} S_{0}(c_{\varepsilon})|\nabla n_{\varepsilon}| |\nabla c_{\varepsilon}| \leq \frac{\delta_{0}}{4} \int_{\Omega} |\nabla c_{\varepsilon}|^4 c_{\varepsilon}^3 + \frac{\delta_{7}}{4} \int_{\Omega} \frac{D_{\varepsilon}(n_{\varepsilon})|\nabla n_{\varepsilon}|^2}{n_{\varepsilon}} + C_{1} \quad \text{for all } t \in (0, T_{\text{max}, \varepsilon}) \tag{2.31}
\]

with \( C_{1} > 0 \) independent of \( \varepsilon \). Now, in conjunction with (2.30) and (2.31), we get the results. This completes the proof of Lemma 2.8.

Properly combining Lemmata 2.6–2.8, we arrive at the following Lemma, which plays a key role in obtaining the existence of solutions to (2.1).

**Lemma 2.9.** Let \( \frac{10}{9} < m \leq 2 \) and \( S \) satisfy (1.4)–(1.5). Suppose that (1.2) and (1.6)–(1.7) holds. Then there exists \( C > 0 \) independent of \( \varepsilon \) such that the solution of (2.1) satisfies

\[
\int_{\Omega} n_{\varepsilon} \ln n_{\varepsilon} + \int_{\Omega} |\nabla \sqrt{c_{\varepsilon}}|^2 + \int_{\Omega} |u_{\varepsilon}|^2 \leq C 
\tag{2.32}
\]

for all \( t \in (0, T_{\text{max}, \varepsilon}) \). Moreover, for each \( T \in (0, T_{\text{max}, \varepsilon}) \), one can find a constant \( C > 0 \) independent of \( \varepsilon \) such that

\[
\int_{0}^{T} \int_{\Omega} n_{\varepsilon}^{m-2} |\nabla n_{\varepsilon}|^2 \leq C, \tag{2.33}
\]

\[
\int_{0}^{T} \int_{\Omega} |\nabla u_{\varepsilon}|^2 \leq C \tag{2.34}
\]

and

\[
\int_{0}^{T} \int_{\Omega} |\nabla c_{\varepsilon}|^4 \leq C \tag{2.35}
\]
as well as
\[
\int_0^T \int_{\Omega} c_\varepsilon |D^2 \ln c_\varepsilon|^2 \leq C. 
\]  
(2.36)

**Proof.** Take an evident linear combination of the inequalities provided by Lemmata 2.6–2.8, we conclude
\[
\frac{d}{dt} \left( \int_{\Omega} \frac{\nabla c_\varepsilon}{c_\varepsilon} \cdot \nabla c_\varepsilon + L \int_{\Omega} n_\varepsilon \ln n_\varepsilon + K \int_{\Omega} |u_\varepsilon|^2 \right) + (K - \frac{4}{\delta_2} \|c_0\|_{L^\infty(\Omega)}) \int_{\Omega} |\nabla u_\varepsilon|^2 + \mu_0 \int_{\Omega} c_\varepsilon |D^2 \ln c_\varepsilon|^2 \\
+ \left[ (\mu_0 - \frac{\delta_2}{4} - \frac{\delta_3}{4}) - L \frac{\delta_6}{4} \right] \int_{\Omega} |\nabla c_\varepsilon|^4 + [L(1 - \frac{\delta_7}{4}) - \frac{\delta_1}{4} - \frac{\delta_5}{4} - K \delta_1] \int_{\Omega} \frac{D_\varepsilon(n_\varepsilon)|\nabla n_\varepsilon|^2}{n_\varepsilon} \\
\leq C \quad \text{for all} \quad t \in (0, T_{\text{max}, \varepsilon}),
\]  
(2.37)

where \(K, L\) are positive constants. Now, choosing \(\delta_7 = 1, \delta_6 = \frac{\mu_0}{L}, \delta_3 = \mu_0, \delta_4 = \delta_5 = L, \delta_1 = \frac{L}{8K}\) and \(\delta_2 = \frac{8}{K} \|c_0\|_{L^\infty(\Omega)}\) and \(K\) large enough such that \(\frac{8}{K} \|c_0\|_{L^\infty(\Omega)} < \mu_0\) in (2.37), one may arrive at
\[
\frac{d}{dt} \left( \int_{\Omega} \frac{\nabla c_\varepsilon}{c_\varepsilon} \cdot \nabla c_\varepsilon + L \int_{\Omega} n_\varepsilon \ln n_\varepsilon + K \int_{\Omega} |u_\varepsilon|^2 \right) + \frac{K}{2} \int_{\Omega} |\nabla u_\varepsilon|^2 + \mu_0 \int_{\Omega} c_\varepsilon |D^2 \ln c_\varepsilon|^2 \\
+ \frac{\mu_0}{4} \int_{\Omega} |\nabla c_\varepsilon|^4 + \frac{L}{8} \int_{\Omega} \frac{D_\varepsilon(n_\varepsilon)|\nabla n_\varepsilon|^2}{n_\varepsilon} \\
\leq C \quad \text{for all} \quad t \in (0, T_{\text{max}, \varepsilon}).
\]  
(2.38)

As a result, we immediately obtain (2.32)–(2.36) after integrating (2.38) over \((0, T)\). \(\square\)

In what follows, we are in a position to discuss the case \(m > 2\), we first give the following Lemma which plays a key role in obtaining the existence of solution to main results.

**Remark 2.1.** Due to the strong nonlinear term \((u \cdot \nabla)u\), the methods used in [41] to derive the higher-order estimates on the solutions \((n_\varepsilon, c_\varepsilon, u_\varepsilon)\), which guarantee the solutions obtained are indeed a locally bounded one, cannot be applied any more. To overcome this difficulty, we need some new careful analysis.

**Lemma 2.10.** Let \(m > 2\). Then there exists \(C > 0\) independent of \(\varepsilon\) such that the solution of (2.1) satisfies
\[
\int_{\Omega} (n_\varepsilon + \varepsilon)^{m-1} + \int_{\Omega} c_\varepsilon^2 + \int_{\Omega} |u_\varepsilon|^2 \leq C \quad \text{for all} \quad t \in (0, T_{\text{max}, \varepsilon}).
\]  
(2.39)
In addition, for each $T \in (0, T_{\text{max}, \varepsilon})$, one can find a constant $C > 0$ independent of $\varepsilon$ such that
\[
\int_0^T \int_\Omega \left[ (D_\varepsilon(n_\varepsilon))^\frac{2m-4}{m-1} |\nabla n_\varepsilon|^2 + (n_\varepsilon + \varepsilon)^{2m-4} |\nabla n_\varepsilon|^2 + |\nabla c_\varepsilon|^2 + |\nabla u_\varepsilon|^2 \right] \leq C. \tag{2.40}
\]

**Proof.** Multiplying $c_\varepsilon$ on both sides of the second equation of (2.1) and using $\nabla \cdot u_\varepsilon = 0$, one has after integration by parts that
\[
\frac{1}{2} \frac{d}{dt} \|c_\varepsilon\|^2_{L^2(\Omega)} + \int_\Omega |\nabla c_\varepsilon|^2 = - \int_\Omega n_\varepsilon c_\varepsilon^2, \tag{2.41}
\]
which together with $n_\varepsilon \geq 0$, $c_\varepsilon \geq 0$ and the Gronwall inequality implies that
\[
\int \Omega c_\varepsilon^2 \leq C_1 \text{ for all } t \in (0, T_{\text{max}, \varepsilon}) \tag{2.42}
\]
and
\[
\int_0^T \int_\Omega |\nabla c_\varepsilon|^2 \leq C_1 \text{ for all } T \in (0, T_{\text{max}, \varepsilon}) \tag{2.43}
\]
with some positive constant $C_1$. Next, multiply the first equation in (2.1) by $(n_\varepsilon + \varepsilon)^{p-1}$ and combining with the second equation and using $\nabla \cdot u_\varepsilon = 0$ and (2.4) implies that
\[
\frac{1}{p} \frac{d}{dt} \|n_\varepsilon + \varepsilon\|^p_{L^p(\Omega)} + \frac{C_D(p-1)}{2} \int_\Omega (n_\varepsilon + \varepsilon)^{m+p-3} |\nabla n_\varepsilon|^2 \leq \frac{(p-1)S_0(\|c_0\|_{L^\infty(\Omega)})^2}{2C_D} \int_\Omega (n_\varepsilon + \varepsilon)^{p+1-m} |\nabla c_\varepsilon|^2. \tag{2.44}
\]
Now, choosing $p = m - 1$ in (2.44) and using (2.43) yields to
\[
\int \Omega (n_\varepsilon + \varepsilon)^{m-1} \leq C_2 \text{ for all } t \in (0, T_{\text{max}, \varepsilon}) \tag{2.45}
\]
and
\[
\int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{2m-4} |\nabla n_\varepsilon|^2 \leq C_2 \text{ for all } T \in (0, T_{\text{max}, \varepsilon}) \tag{2.46}
\]
and some positive constant $C_2$. Next, collecting (2.13) and (2.46) and with some basic calculation, we conclude that there exists a positive constant $C_3$ such that
\[
\int \Omega |u_\varepsilon|^2 \leq C_3 \text{ for all } t \in (0, T_{\text{max}, \varepsilon}) \tag{2.47}
\]
and
\[
\int_0^T \int_\Omega |\nabla u_\varepsilon|^2 \leq C_3. \tag{2.48}
\]
Finally, combining (2.42)–(2.43) and (2.45)–(2.48) and using (1.2), we can get (2.39) and (2.40). \qed
3 Global existence of the regularized problems

To prove global existence of the regularized problems (2.1), whose proof will be postponed to the end of this subsection, we need to give a series of useful estimates. For notational convenience, throughout this section we denote by $C$ or $C_i$ ($i = 1, 2, \ldots$) the generic positive constants which may depend on $\varepsilon$. To this end, we intend to supplement Lemmata 2.9–2.10 with bounds on $n_\varepsilon$. This will be the purpose of the following lemmata.

**Lemma 3.1.** Assuming that $\frac{10}{9} < m \leq 2$ and $T_{\text{max,}\varepsilon} < +\infty$. Then there exists a positive constant $C$ depends on $\varepsilon$ such that

$$\|n_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \leq C \text{ for all } t \in (0, T_{\text{max,}\varepsilon})$$

(3.1)

with $p < \frac{38}{9}$.

**Proof.** Multiplying the first equation in (2.44) by $n_\varepsilon^{p-1}$ with $p \in [m + 1, 2(m + 1)]$ and using integration by parts and the Young inequality, we obtain

$$\frac{1}{p} \frac{d}{dt} \|n_\varepsilon\|_{L^p(\Omega)}^p + C_D(p - 1) \int_{\Omega} n_\varepsilon^{m+p-3} |\nabla n_\varepsilon|^2$$

$$= \int_{\Omega} -\nabla \cdot (n_\varepsilon F_\varepsilon(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon) n_\varepsilon^{p-1}$$

$$= (p - 1) \int_{\Omega} n_\varepsilon^{p-2} F_\varepsilon'(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \nabla n_\varepsilon \cdot \nabla c_\varepsilon$$

$$\leq (p - 1) \int_{\Omega} n_\varepsilon^{p-2} |\nabla n_\varepsilon||\nabla c_\varepsilon|$$

(3.2)

and some positive constants $C_1$ and $C_2$. Finally, we obtain (3.1) after by using (2.35) and the Gronwall inequality. The proof of Lemma 3.1 is completed. \qed

**Lemma 3.2.** Suppose that $m > 2$ and $T_{\text{max,}\varepsilon} < +\infty$. Then there exists a positive constant $C$ depends on $\varepsilon$ such that

$$\|n_\varepsilon(\cdot, t)\|_{L^{m+1}(\Omega)} \leq C \text{ for all } t \in (0, T_{\text{max,}\varepsilon}).$$

(3.3)
Proof. Multiplying the first equation in (2.44) by $n_\varepsilon^m$, and integrating them by parts over $\Omega$, one easily deduces from the Young inequality that there exists a positive constant $C_1$ such that

$$\frac{1}{m+1} \frac{d}{dt} \|n_\varepsilon\|_{L^{m+1}(\Omega)}^2 + C_D m \int_\Omega n_\varepsilon^{2m-2} |\nabla n_\varepsilon|^2$$

$$= \int_\Omega -\nabla \cdot (n_\varepsilon F_\varepsilon(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon) n_\varepsilon^m$$

$$= m \int_\Omega n_\varepsilon^m F_\varepsilon(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \nabla n_\varepsilon \cdot \nabla c_\varepsilon$$

$$\leq \frac{m S_0 (\|c_0\|_{L^\infty(\Omega)})}{\int_\Omega n_\varepsilon^{m-1} |\nabla n_\varepsilon| |\nabla c_\varepsilon|} \int_\Omega n_\varepsilon^{m-1} |\nabla n_\varepsilon|^2$$

$$\leq \frac{C_D m}{2} \int_\Omega n_\varepsilon^{2m-2} |\nabla n_\varepsilon|^2 + C_1 \int_\Omega |\nabla c_\varepsilon|^2$$

for all $t \in (0, T_{\text{max}, \varepsilon})$.

Thus, in view of (2.35), an application of the Gronwall inequality immediately leads to (3.3).

Properly combining Lemmata 3.1–3.2, we arrive at the following.

**Lemma 3.3.** Assuming that $m > \frac{10}{9}$ and $T_{\text{max}, \varepsilon} < +\infty$. Then there exists a positive constant $C$ such that

$$\|n_\varepsilon(\cdot, t)\|_{L^{p_0}(\Omega)} \leq C \text{ for all } t \in (0, T_{\text{max}, \varepsilon}) \text{ with } p_0 > 3.$$  \hspace{1cm} (3.5)

**Proof.** If $\frac{10}{9} < m \leq 2$, by Lemma 3.1, we obtain that there exists a positive constant $C_1$ such that

$$\|n_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \leq C_1 \text{ for all } t \in (0, T_{\text{max}, \varepsilon})$$  \hspace{1cm} (3.6)

with $p < \frac{38}{9}$. While, if $m > 2$, then by Lemma 3.2 we derive that we can find a positive $C_2$ such that

$$\|n_\varepsilon(\cdot, t)\|_{L^{m+1}(\Omega)} \leq C_2 \text{ for all } t \in (0, T_{\text{max}, \varepsilon}).$$  \hspace{1cm} (3.7)

This combined with (3.6) gives (3.1) and finishes the proof of Lemma 3.3.

With Lemma 3.3 at hand, we can proceed to show that our approximate solutions are actually global in time.

**Lemma 3.4.** Let $m > \frac{10}{9}$. Then for all $\varepsilon \in (0, 1)$, the solution of (2.1) is global in time.
Proof. Assuming that $T_{\max,\varepsilon}$ be finite for some $\varepsilon \in (0, 1)$. Firstly, testing the projected Stokes equation $u_{\varepsilon t} + Au_{\varepsilon} = \mathcal{P}(-\kappa(Y_{\varepsilon} u_{\varepsilon} \cdot \nabla) u_{\varepsilon} + n_{\varepsilon} \nabla \phi)$ by $Au_{\varepsilon}$ shows that

$$
\frac{1}{2} \frac{d}{dt} \| A^{\frac{1}{2}} u_{\varepsilon} \|_{L^{2}(\Omega)}^{2} + \int_{\Omega} |Au_{\varepsilon}|^{2} \\
= \int_{\Omega} Au_{\varepsilon} \kappa(Y_{\varepsilon} u_{\varepsilon} \cdot \nabla) u_{\varepsilon} + \int_{\Omega} n_{\varepsilon} \nabla \phi Au_{\varepsilon} \\
\leq \frac{3}{4} \int_{\Omega} |Au_{\varepsilon}|^{2} + \kappa^{2} \int_{\Omega} |(Y_{\varepsilon} u_{\varepsilon} \cdot \nabla) u_{\varepsilon}|^{2} + \|\nabla \phi\|_{L^{\infty}(\Omega)}^{2} \int_{\Omega} n_{\varepsilon}^{2} \text{ for all } t \in (0, T_{\max,\varepsilon}),
$$

(3.8)

where $\mathcal{P}$ denotes the Helmholtz project from $L^{2}(\Omega)$ into $L^{2}_{\sigma}(\Omega)$. Next, we observe that $D(1 + \varepsilon A) := W^{2,2}(\Omega) \cap W^{1,2}_{0,\sigma}(\Omega) \hookrightarrow L^{\infty}(\Omega)$, we can find $C_{3} > 0$ and $C_{4} > 0$ such that

$$
\|Y_{\varepsilon} u_{\varepsilon}(\cdot, t)\|_{L^{\infty}(\Omega)} = \|(I + \varepsilon A)^{-1} u_{\varepsilon}(\cdot, t)\|_{L^{\infty}(\Omega)} \leq C_{3}\|u_{\varepsilon}(\cdot, t)\|_{L^{2}(\Omega)} \leq C_{4} \text{ for all } t \in (0, T_{\max,\varepsilon}).
$$

(3.9)

Now, we derive from the Hölder inequality, (2.32) (or (2.39)) and (3.9) that

$$
\|Y_{\varepsilon}(u_{\varepsilon}(\cdot, t) \cdot \nabla) u_{\varepsilon}(\cdot, t)\|_{L^{p}(\Omega)} \\
\leq \|Y_{\varepsilon} u_{\varepsilon}(\cdot, t)\|_{L^{\infty}(\Omega)} \|u_{\varepsilon}(\cdot, t)\|_{L^{2}(\Omega)} |\Omega|^{\frac{2}{p-2}} \\
\leq C_{5} \text{ for all } t \in (0, T_{\max,\varepsilon}).
$$

(3.10)

On the other hand, by (3.9), we derive that

$$
\kappa^{2} \int_{\Omega} |(Y_{\varepsilon} u_{\varepsilon} \cdot \nabla) u_{\varepsilon}|^{2} \leq \kappa^{2} \|Y_{\varepsilon} u_{\varepsilon}\|_{L^{\infty}(\Omega)}^{2} \int_{\Omega} |\nabla u_{\varepsilon}|^{2} \\
\leq \kappa^{2} \|Y_{\varepsilon} u_{\varepsilon}\|_{L^{\infty}(\Omega)}^{2} \int_{\Omega} |\nabla u_{\varepsilon}|^{2} \\
\leq \kappa^{2} C_{4}^{2} \int_{\Omega} |\nabla u_{\varepsilon}|^{2} \text{ for all } t \in (0, T_{\max,\varepsilon}),
$$

(3.11)

where $C_{4}$ is the same as (3.9).

Plugging substitution of (3.11) into (3.8), we derive from (3.5) and the Gronwall inequality that there exists a positive constant $C_{6}$ such that

$$
\int_{\Omega} |\nabla u_{\varepsilon}(\cdot, t)|^{2} \leq C_{6} \text{ for all } t \in (0, T_{\max,\varepsilon}).
$$

(3.12)

Let $h_{\varepsilon}(x, t) = \mathcal{P}[-\kappa(Y_{\varepsilon} u_{\varepsilon} \cdot \nabla) u_{\varepsilon} + n_{\varepsilon} \nabla \phi]$. Then along with (3.5) and (3.11)–(3.12), this in turn provides $C_{7} > 0$ such that $\|h_{\varepsilon}(\cdot, t)\|_{L^{2}(\Omega)} \leq C_{7}$ for all $t \in (0, T_{\max,\varepsilon})$. Thus if we pick an arbitrary $\gamma \in (\frac{3}{4}, 1)$, then by smoothing properties of the Stokes semigroup ([9]) entail that
Since $\gamma > \frac{3}{4}$, $D(A^\gamma)$ is continuously embedded into $L^\infty(\Omega)$, hence, (3.13) yields to
\[
\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C_9 \|A^\gamma u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq C_{10} \quad \text{for all } t \in (0, T_{max, \varepsilon})
\]
for some positive constants $C_9$ and $C_{10}$.

Next, let $T \in (0, T_{max, \varepsilon})$ and $M(T) := \sup_{t \in (0, T)} \|\nabla c_\varepsilon(\cdot, t)\|_{L^4(\Omega)}$. Now, employing $\Delta$ to both sides of the variation-of-constants formula for $c_\varepsilon$, we derive that
\[
c_\varepsilon(\cdot, t) = e^{t\Delta}c_0 - \int_0^t e^{-(t-s)\Delta}(n_\varepsilon c_\varepsilon + u_\varepsilon \cdot \nabla c_\varepsilon)(\cdot, s)ds \quad \text{for all } t \in (0, T_{max, \varepsilon}),
\]
hence,
\[
\|\nabla c_\varepsilon(\cdot, t)\|_{L^4(\Omega)} \\
\leq \|\nabla e^{t\Delta}c_0\|_{L^4(\Omega)} + \int_0^t \|\nabla e^{-(t-s)\Delta}(n_\varepsilon c_\varepsilon)(\cdot, s)\|_{L^4(\Omega)}ds + \int_0^t \|\nabla e^{-(t-s)\Delta}(u_\varepsilon \cdot \nabla c_\varepsilon)(\cdot, s)\|_{L^4(\Omega)}ds
\]
for all $t \in (0, T_{max, \varepsilon})$. In the following, we will estimate the right-hand side of (3.15).

Indeed, due to the hypothesis of $c_0$ and the $L^p$-$L^q$ estimates we conclude that there exists $C_{11} > 0$ such that
\[
\|\nabla e^{t\Delta}c_0\|_{L^4(\Omega)} \leq C_{11}t^{-\frac{1}{2}} \|c_0\|_{L^4(\Omega)} \quad \text{for all } t > 0.
\]

Since, $-\frac{1}{2} - \frac{3}{2}(\frac{1}{2} - \frac{1}{4}) > -1$, by $L^p$-$L^q$ estimate for Neumann semigroup and Lemma 2.4 and Lemma 3.3, we can find $C_{12} > 0, C_{13} > 0$ and $\lambda_1 > 0$ such that
\[
\int_0^t \|e^{-(t-s)\Delta}(n_\varepsilon c_\varepsilon)(\cdot, s)\|_{L^4(\Omega)}ds \\
\leq \int_0^t C_{12}(1 + (t-s)^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{2} - \frac{1}{4})})e^{-\lambda_1(t-s)}\|n_\varepsilon(\cdot, s)c_\varepsilon(\cdot, s)\|_{L^2(\Omega)}ds \\
\leq \int_0^t C_{12}(1 + (t-s)^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{2} - \frac{1}{4})})e^{-\lambda_1(t-s)}\|n_\varepsilon(\cdot, s)\|_{L^2(\Omega)}\|c_\varepsilon(\cdot, s)\|_{L^\infty(\Omega)}ds \\
\leq C_{13} \quad \text{for all } t \in (0, T_{max, \varepsilon}).
\]
Now, with the help of the Hölder inequality, we conclude that there exists a positive constant $C_{14}$ such that

$$\int_0^t \| \nabla e^{(t-s)\Delta}(u_\varepsilon \cdot \nabla c_\varepsilon)(\cdot, s) \|_{L^2(\Omega)} ds \leq \int_0^t C_{14}(1 + (t - s)^{-\frac{1}{2}})(1 + (t - s)^{-\frac{1}{4}}) e^{-\lambda_1(t-s)} \| u_\varepsilon(\cdot, s) \nabla c_\varepsilon(\cdot, s) \|_{L^\infty(\Omega)} ds$$

(3.18)

$$\leq \int_0^t C_{14}(1 + (t - s)^{-\frac{1}{2}})(1 + (t - s)^{-\frac{1}{4}}) e^{-\lambda_1(t-s)} \| u_\varepsilon(\cdot, s) \|_{L^\infty(\Omega)} \| \nabla c_\varepsilon(\cdot, s) \|_{L^{\frac{18}{5}}(\Omega)} ds$$

for all $t \in (0, T_{\text{max, } \varepsilon})$. On the other hand, due to the interpolation inequality, we get that

$$\| \nabla c_\varepsilon(\cdot, s) \|_{L^{\frac{18}{5}}(\Omega)} \leq C_{14}[\| \nabla c_\varepsilon(\cdot, s) \|_{L^{\frac{2}{3}}(\Omega)}^\frac{\bar{q}}{L^{\frac{18}{5}}(\Omega)} + \| c_\varepsilon(\cdot, s) \|_{L^\infty(\Omega)}] \quad \text{for all } t \in (0, T_{\text{max, } \varepsilon}).$$

Plugging the above inequality into (3.18) and applying (3.14), we have

$$\int_0^t \| \nabla e^{(t-s)\Delta}(u_\varepsilon \cdot \nabla c_\varepsilon)(\cdot, s) \|_{L^2(\Omega)} ds \leq C_{15} M^\frac{\bar{q}}{3}(T) + C_{15} \quad \text{for all } t \in (0, T_{\text{max, } \varepsilon})$$

(3.19)

and some positive constant $C_{15}$. Now, collecting (3.15)–(3.18) and (3.19), we can derive

$$\| \nabla c_\varepsilon(\cdot, t) \|_{L^2(\Omega)} \leq C_{16} \quad \text{for all } t \in (\tau, T_{\text{max, } \varepsilon})$$

(3.20)

with $\tau \in (0, T_{\text{max, } \varepsilon})$ and some positive constant $C_{16}$. In order to get the boundedness of $\| \nabla c_\varepsilon(\cdot, t) \|_{L^\infty(\Omega)}$, we rewrite the variation-of-constants formula for $c_\varepsilon$ in the form

$$c_\varepsilon(\cdot, t) = e^{(\Delta - 1)\varepsilon} u_0 + \int_0^t e^{(t-s)(\Delta - 1)} (c_\varepsilon - n_\varepsilon c_\varepsilon - u_\varepsilon \cdot \nabla c_\varepsilon)(\cdot, s) ds \quad \text{for all } t \in (0, T_{\text{max, } \varepsilon}).$$

Now, picking $\theta \in (\frac{1}{2} + \frac{3}{2q_0}, 1)$, then the domain of the fractional power $D((-\Delta + 1)^\theta) \hookrightarrow W^{1, \infty}(\Omega)$ (34), where $q_0 := \min\{p_0, 4\} > 3$ and $p_0$ is the same as (3.3).

Hence, in view of $L^p$-$L^q$ estimates associated heat semigroup, (1.7), (3.5), (3.14) and (3.20), we conclude that

$$\| \nabla c_\varepsilon(\cdot, t) \|_{W^{1, \infty}(\Omega)} \leq C_{17} t^{-\theta} e^{-\lambda t} |c_0|_{L^{\infty}(\Omega)}$$

$$+ \int_0^t (t-s)^{-\theta} e^{-\lambda s} \| (c_\varepsilon - n_\varepsilon c_\varepsilon - u_\varepsilon \cdot \nabla c_\varepsilon)(s) \|_{L^\infty(\Omega)} ds$$

$$\leq C_{18} t^{-\theta} + C_{18} \int_0^t (t-s)^{-\theta} e^{-\lambda s} ds + C_{18} \int_0^t (t-s)^{-\theta} e^{-\lambda s} \| n_\varepsilon(s) \|_{L^\infty(\Omega)} + \| \nabla c_\varepsilon(s) \|_{L^\infty(\Omega)} ds$$

$$\leq C_{19} \quad \text{for all } t \in (\tau, T_{\text{max, } \varepsilon})$$

(3.21)
for some positive constant $C_{17}, C_{18}$ and $C_{19}$.

Finally, for all $p > 1$, multiplying the first equation in (2.44) by $n^p_\varepsilon - 1$, after integrating by parts and using the Young inequality, we easily deduce from (3.21) that

$$\frac{1}{p} \frac{d}{dt} \| n_\varepsilon \|^p_{L^p(\Omega)} + \frac{C_D(p - 1)}{2} \int_\Omega n_\varepsilon^{m+p-3} |\nabla n_\varepsilon|^2 \leq C_{20} \int_\Omega n_\varepsilon^{p+1-m}$$

$$\leq C_{21} \int_\Omega n_\varepsilon^p + C_{14} \text{ for all } t \in (\tau, T_{max, \varepsilon})$$

(3.22)

and some positive constants $C_{20}$ and $C_{21}$.

Therefore, integrating the above inequality with respect to $t$, we derive that there exists a positive constant $C_{22}$ such that

$$\| n_\varepsilon(\cdot, t) \|_{L^p(\Omega)} \leq C_{22} \text{ for all } p \geq 1 \text{ and } t \in (\tau, T_{max, \varepsilon}).$$

(3.23)

Next, using the outcome of (3.23) with suitably large $p$ as a starting point, we may invoke Lemma A.1 in [18] which by means of a Moser-type iteration applied to the first equation in (2.1) establishes

$$\| n_\varepsilon(\cdot, t) \|_{L^\infty(\Omega)} \leq C_{23} \text{ for all } t \in (\tau, T_{max, \varepsilon})$$

(3.24)

and a positive constant $C_{23}$. In view of (3.14), (3.21) and (3.24), we apply Lemma 2.1 to reach a contradiction.

4 Time regularity

In order to pass to the limit in (2.1), we shall need an appropriate boundedness property of the time derivatives of certain powers of $n_\varepsilon, c_\varepsilon$ and $u_\varepsilon$. We first give the following lemma, which gives some estimates for $n_\varepsilon, c_\varepsilon$ and $n_\varepsilon$.

**Lemma 4.1.** Let (1.6) and (1.7) hold, and suppose that $m$ and $S$ satisfy (1.2) and (1.4)–(1.5), respectively. Then any small $\varepsilon > 0$ ($\varepsilon < 1$), one can find $C > 0$ independent of $\varepsilon$ such that for all $T \in (0, \infty)$

$$\int_0^T \int_\Omega \left( |\nabla u_\varepsilon|^2 + |u_\varepsilon|^{14} \right) \leq C(T + 1).$$

(4.1)
Moreover, if $\frac{10}{9} < m \leq 2$, then we have
\[
\int_0^T \int_\Omega \left[ (n_\varepsilon + \varepsilon)^{\frac{3m+2}{2}} + |\nabla n_\varepsilon|^{\frac{3m+2}{2}} + |\nabla c_\varepsilon|^4 + |D_\varepsilon(n_\varepsilon) \nabla n_\varepsilon|^{\frac{3m+2}{3m+4}} \right] \leq C(T + 1).
\] (4.2)

While if $m > 2$, then there exists $C > 0$ independent of $\varepsilon$ such that
\[
\int_\Omega \left[ (n_\varepsilon + \varepsilon)^{m-1} + c_\varepsilon^2 + |u_\varepsilon|^2 \right] \leq C \quad \text{for all } t > 0
\] (4.3)

and
\[
\int_0^T \int_\Omega \left[ (n_\varepsilon + \varepsilon) + (n_\varepsilon + \varepsilon)^{2m-4} |\nabla n_\varepsilon|^2 + |\nabla c_\varepsilon|^2 + |\nabla u_\varepsilon|^2 + |D_\varepsilon(n_\varepsilon) \nabla n_\varepsilon|^{\frac{8(m-1)}{4m-4}} \right] 
\leq C(T + 1) \quad \text{for all } T > 0.
\] (4.4)

**Proof.** Case $\frac{10}{9} < m \leq 2$: Due to Lemma 2.9, there exists $C_1 > 0$ such that the solution of (2.1) satisfies
\[
\int_\Omega n_\varepsilon \ln n_\varepsilon + \int_\Omega |\nabla \sqrt{c_\varepsilon}|^2 + \int_\Omega |u_\varepsilon|^2 \leq C_1 \quad \text{for all } t > 0
\] (4.5)

and
\[
\int_0^T \int_\Omega \left( |\nabla u_\varepsilon|^2 + \frac{D_\varepsilon(n_\varepsilon) |\nabla n_\varepsilon|^2}{n_\varepsilon} + n_\varepsilon^{m-2} |\nabla n_\varepsilon|^2 + |\nabla c_\varepsilon|^4 + c_\varepsilon^2 |D_\varepsilon \ln c_\varepsilon|^2 \right) \leq C_1(T + 1)
\] (4.6)

for all $T > 0$. Now, applying the Gagliardo-Nirenberg inequality, (2.1) and $\varepsilon < 1$, we derive that there exist $C_i (i = 2 \ldots 6)$ such that
\[
\int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{\frac{3m+2}{2}} = \int_0^T \|(n_\varepsilon + \varepsilon)^{\frac{m}{2}}\|_{L^{\frac{2(3m+2)}{3m+4}}(\Omega)}^{\frac{2(3m+2)}{3m}} \leq C_2 \int_0^T \left( \|\nabla (n_\varepsilon + \varepsilon)^{\frac{m}{2}}\|_{L^2(\Omega)} \| (n_\varepsilon + \varepsilon)^{\frac{m}{2}} \|_{L^{\frac{3m}{2}}(\Omega)}^{\frac{m}{3}} + \|(n_\varepsilon + \varepsilon)^{\frac{m}{2}}\|_{L^{\frac{3m}{2}}(\Omega)}^{\frac{3m}{2}} \right) 
\leq C_3 \int_0^T \left( \|\nabla (n_\varepsilon + \varepsilon)^{\frac{m}{2}}\|_{L^2(\Omega)} \left[ \int_\Omega n_\varepsilon + |\Omega| \right]^{\frac{m}{2}} \right) 
\leq C_4(T + 1) \quad \text{for all } T > 0
\] (4.7)

and
\[
\int_0^T \int_\Omega |u_\varepsilon|^{\frac{10}{3}} = \int_0^T \|u_\varepsilon\|_{L^{\frac{10}{3}}(\Omega)}^{\frac{10}{3}} \leq C_5 \int_0^T \left( \|\nabla u_\varepsilon\|_{L^2(\Omega)}^{2} \|u_\varepsilon\|_{L^2(\Omega)}^{\frac{4}{3}} + \|u_\varepsilon\|_{L^2(\Omega)}^{\frac{10}{3}} \right) 
\leq C_6(T + 1) \quad \text{for all } T > 0.
\] (4.8)
Now, the estimates (4.6)–(4.7) together with the Young inequality ensures
\[
\int_0^T \int_\Omega |\nabla n_\varepsilon|^{\frac{3m+2}{m-1}} \leq C_7 \left( \int_0^T \int_\Omega n_\varepsilon^{m-2} |\nabla n_\varepsilon|^2 + \int_0^T \int_\Omega n_\varepsilon^{\frac{3m+2}{3m+1}} \right)
\leq C_8(T + 1) \quad \text{for all } T > 0
\]
and some positive constants $C_7$ and $C_8$.

Utilizing (1.2), (4.6) and the Hölder inequality, it yields from (4.7) that we can find $C_9 > 0$ and $C_{10} > 0$ such that
\[
\int_0^T \int_\Omega |D_\varepsilon(n_\varepsilon) \nabla n_\varepsilon|^{\frac{3m+2}{m-1}} \leq \left[ \int_0^T \int_\Omega |D_\varepsilon(n_\varepsilon)|^{2} n_\varepsilon \right]^{\frac{3m+2}{m+2}} \left[ \int_0^T \int_\Omega |D_\varepsilon(n_\varepsilon)|^{\frac{3m+2}{3m+1}} n_\varepsilon \right]^{\frac{3m}{m+2}}
\leq C_9 \left[ \int_0^T \int_\Omega |D_\varepsilon(n_\varepsilon)|^{2} n_\varepsilon \right]^{\frac{3m+2}{m+2}} \left[ \int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{\frac{3m+2}{4}} \right]
\leq C_{10}(T + 1) \quad \text{for all } T > 0.
\]

(4.10)

Case $m > 2$: By virtue of (1.2) and Lemma 2.10, it follows that
\[
\int_\Omega (n_\varepsilon + \varepsilon)^{m-1} + \int_\Omega c_\varepsilon^2 + \int_\Omega |u_\varepsilon|^2 \leq C_{11} \quad \text{for all } t > 0
\]
and
\[
\int_0^T \int_\Omega \left[ (D_\varepsilon(n_\varepsilon))^{\frac{2m-4}{m-1}} |\nabla n_\varepsilon|^2 + (n_\varepsilon + \varepsilon)^{2m-4} |\nabla n_\varepsilon|^2 + |\nabla c_\varepsilon|^2 + |\nabla u_\varepsilon|^2 \right] \leq C_{11}(T + 1)
\]
for all $T > 0$ and a positive constant $C_{11} > 0$ independent of $\varepsilon$.

Now, since (4.11) and (4.12), employing the Gagliardo-Nirenberg inequality and the Hölder inequality, we conclude that there exist positive constants $C_i (i = 12 \ldots 15)$ such that
\[
\int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{\frac{n(m-1)}{4}} = \int_0^T \|(n_\varepsilon + \varepsilon)^{m-1}\|_{L^\frac{8}{n}(\Omega)}^\frac{8}{n}
\leq C_{12} \int_0^T \left( \|\nabla(n_\varepsilon + \varepsilon)^{m-1}\|_{L^2(\Omega)}^2 \|n_\varepsilon + \varepsilon\|_{L^1(\Omega)}^{\frac{2}{n}} + \|(n_\varepsilon + \varepsilon)^{m-1}\|_{L^1(\Omega)}^{\frac{8}{n}} \right)
\leq C_{13}(T + 1) \quad \text{for all } T > 0.
\]

(4.13)
and
\[
\int_0^T \int_\Omega |D_\varepsilon(n_\varepsilon) \nabla n_\varepsilon|^{\frac{2}{3m-1}} \leq \left[ \int_0^T \int_\Omega (D_\varepsilon(n_\varepsilon))^\frac{2m-4}{m-1} |\nabla n_\varepsilon|^2 \right]^\frac{2}{4m-1} \left[ \int_0^T \int_\Omega (D_\varepsilon(n_\varepsilon))^\frac{8}{3m-1} \right]^\frac{3}{4m-1} \leq C_{14} \left[ \int_0^T \int_\Omega (D_\varepsilon(n_\varepsilon))^\frac{8}{3m-1} \right]^\frac{3}{4m-1} \leq C_{15}(T+1) \text{ for all } T > 0.
\]

(4.14)

Collecting (4.5)–(4.14), we can obtain (4.1)–(4.4).

As a last preparation for main results, we intend to supplement Lemmata 2.9–2.10 with bounds on time-derivatives.

**Lemma 4.2.** Let (1.6) and (1.7) hold, and suppose that \(m\) and \(S\) satisfy (1.2) and (1.4)–(1.5), respectively. Then for any \(T > 0\), one can find \(C > 0\) independent if \(\varepsilon\) such that

\[
\begin{aligned}
&\int_0^T \left\| \partial_t n_\varepsilon(\cdot, t) \right\|_{(W^{2,q}(\Omega))} dt \leq C(T + 1), \quad \text{if } \frac{10}{9} < m < \frac{26}{21}, \\
&\int_0^T \left\| \partial_t n_\varepsilon(\cdot, t) \right\|_{(W^{1,3m+2}(\Omega))} dt \leq C(T + 1), \quad \text{if } \frac{26}{21} \leq m \leq 2,
\end{aligned}
\]

(4.15)

as well as

\[
\begin{aligned}
&\int_0^T \left\| \partial_t n_\varepsilon^{m-1}(\cdot, t) \right\|_{(W^{2,q}(\Omega))} dt \leq C(T + 1), \quad \text{if } m > 2
\end{aligned}
\]

(4.16)

and

\[
\begin{aligned}
&\int_0^T \left\| \partial_t c_\varepsilon(\cdot, t) \right\|_{(W^{1,3m+2}(\Omega))} dt \leq C(T + 1), \quad \text{if } \frac{10}{9} < m < \frac{4}{3}, \quad \text{if } m > 2
\end{aligned}
\]

(4.17)

where \(q > 3\).

**Proof.** Case \(\frac{26}{21} \leq m \leq 2\): Firstly, testing the first equation of (2.1) by certain \(\varphi \in C^\infty(\bar{\Omega})\),
we have

\[
\left| \int_{\Omega} \partial_t n_\varepsilon(\cdot, t) \varphi \right| = \int_{\Omega} \left| \nabla \cdot (D_\varepsilon(n_\varepsilon)) \nabla n_\varepsilon - \nabla \cdot (n_\varepsilon F_\varepsilon(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon) - u_\varepsilon \cdot \nabla n_\varepsilon \cdot \varphi \right|
\]

\[
= - \int_{\Omega} D_\varepsilon(n_\varepsilon) \nabla n_\varepsilon \cdot \nabla \varphi + \int_{\Omega} n_\varepsilon F_\varepsilon(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon \cdot \nabla \varphi + \int_{\Omega} n_\varepsilon u_\varepsilon \cdot \nabla \varphi
\]

\[
\leq \left\{ \| D_\varepsilon(n_\varepsilon) \nabla n_\varepsilon \| \frac{3m+2}{3m+1} L^{3m+1+\varepsilon}(\Omega) + \| n_\varepsilon F_\varepsilon(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon \| \frac{3m+2}{3m+1} L^{3m+1+\varepsilon}(\Omega) + \| n_\varepsilon u_\varepsilon \| \frac{3m+2}{3m+1} L^{3m+1+\varepsilon}(\Omega) \right\} \| \varphi \| W^{1,3m+2}(\Omega)
\]

(4.18)

for all \( t > 0 \). Hence, with the help of (4.2) and (1.5), we derive that

\[
\int_{0}^{T} \| \partial_t n_\varepsilon(\cdot, t) \| \frac{3m+2}{3m+1} (W^{1,3m+2}(\Omega)), dt
\]

\[
\leq \int_{0}^{T} \left\{ \| D_\varepsilon(n_\varepsilon) \nabla n_\varepsilon \| \frac{3m+2}{3m+1} L^{3m+1+\varepsilon}(\Omega) + \| n_\varepsilon F_\varepsilon(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon \| \frac{3m+2}{3m+1} L^{3m+1+\varepsilon}(\Omega) + \| n_\varepsilon u_\varepsilon \| \frac{3m+2}{3m+1} L^{3m+1+\varepsilon}(\Omega) \right\} \| \varphi \| W^{1,3m+2}(\Omega)
\]

(4.19)

for all \( T > 0 \) and some positive constants \( C_1 \) and \( C_2 \). In what follows, we shall estimate each term on the right-hand side of (4.19) term by term. Next, applying (4.1), (4.2), (4.6), the Young inequality and employing \( m \geq \frac{26}{27} \), we conclude that

\[
\int_{0}^{T} \int_{\Omega} |n_\varepsilon \nabla c_\varepsilon| \frac{3m+2}{3m+1} \leq \int_{0}^{T} \int_{\Omega} n_\varepsilon \frac{3m+2}{3m+1} + \int_{0}^{T} \int_{\Omega} |\nabla c_\varepsilon| \frac{3m+2}{3m+1} \leq C_3(T+1) \text{ for all } T > 0
\]

(4.20)

and

\[
\int_{0}^{T} \int_{\Omega} |n_\varepsilon u_\varepsilon| \frac{3m+2}{3m+1} \leq \int_{0}^{T} \int_{\Omega} n_\varepsilon \frac{3m+2}{3m+1} + \int_{0}^{T} \int_{\Omega} |u_\varepsilon| \frac{3m+2}{3m+1} \leq \int_{0}^{T} \int_{\Omega} n_\varepsilon \frac{3m+2}{3m+1} + \int_{0}^{T} \int_{\Omega} |u_\varepsilon| \frac{3m+2}{3m+1} \leq C_4(T+1) \text{ for all } T > 0.
\]

(4.21)

and some positive constants \( C_3 \) and \( C_4 \). Now, collecting (4.19) - (4.21) yields to

\[
\int_{0}^{T} \| \partial_t n_\varepsilon(\cdot, t) \| \frac{3m+2}{3m+1} (W^{1,3m+2}(\Omega)), dt \leq C_5(T+1) \text{ for all } T > 0.
\]

(4.22)

and a constant \( C_5 > 0 \).
Case $\frac{10}{9} < m < \frac{26}{21}$: Similarly, we also derive that
\[
\left| \int_{\Omega} \partial_{\nu} n_\epsilon(x, t) \varphi \right|
\leq \left\{ \| D_\epsilon(n_\epsilon) \nabla n_\epsilon \|_{L^1(\Omega)} + \| n_\epsilon F_\epsilon(n_\epsilon) S_\epsilon(x, n_\epsilon, c_\epsilon) \cdot \nabla c_\epsilon \|_{L^1(\Omega)} + \| n_\epsilon u_\epsilon \|_{L^1(\Omega)} \right\} \| \varphi \|_{W^{1,\infty}(\Omega)}
\] (4.23)
for all $t > 0$. Hence, due to the embedding $W^{2,q}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ $(q > 3)$, we deduce from $C_6$ and $C_7$ such that
\[
\int_0^T \| \partial_{\nu} n_\epsilon(\cdot, t) \|_{(W^{2,q}(\Omega))^*} dt
\leq C_6 \left\{ \int_0^T \int_{\Omega} |D_\epsilon(n_\epsilon) \nabla n_\epsilon|^{\frac{3m+2}{3m+1}} + \int_0^T \int_{\Omega} |\nabla c_\epsilon|^4 + \int_0^T \int_{\Omega} n_\epsilon^{\frac{3m+2}{3m+1}} + \int_0^T \int_{\Omega} |u_\epsilon|^{\frac{10}{3}} + T + 1 \right\}
\leq C_7 (T + 1)
\] (4.24)
for all $T > 0$. Next, in view of (4.1), (4.2), (4.6), the Young inequality and $m \geq \frac{26}{21}$, we may find some positive constants $C_8$ and $C_9$ such that
\[
\int_0^T \int_{\Omega} |n_\epsilon \nabla c_\epsilon|^{\frac{3m+2}{3m+1}} \leq \int_0^T \int_{\Omega} n_\epsilon^{\frac{3m+2}{3m+1}} + \int_0^T \int_{\Omega} |\nabla c_\epsilon|^{\frac{3m+2}{3m+1}}
\leq \int_0^T \int_{\Omega} n_\epsilon^{\frac{3m+2}{3m+1}} + \int_0^T \int_{\Omega} |\nabla c_\epsilon|^4 + |\Omega| T
\leq C_8 (T + 1) \text{ for all } T > 0
\] (4.25)
and
\[
\int_0^T \int_{\Omega} |n_\epsilon u_\epsilon|^{\frac{3m+2}{3m+1}} \leq \int_0^T \int_{\Omega} n_\epsilon^{\frac{3m+2}{3m+1}} + \int_0^T \int_{\Omega} |u_\epsilon|^{\frac{3m+2}{3m+1}}
\leq \int_0^T \int_{\Omega} n_\epsilon^{\frac{3m+2}{3m+1}} + \int_0^T \int_{\Omega} |u_\epsilon|^{\frac{10}{3}} + |\Omega| T
\leq C_9 (T + 1) \text{ for all } T > 0.
\] (4.26)

Case $m > 2$: Next, testing the first equation of (2.1) by certain $(m - 1)n_\epsilon^{m-2} \varphi \in C^\infty(\Omega)$
and using (1.2), we have

\[
\left| \int_\Omega (n_\varepsilon^{m-1})_t \varphi \right| \\
= \left| \int_\Omega \left[ \nabla \cdot (D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon) - \nabla \cdot (n_\varepsilon F_\varepsilon(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon) - u_\varepsilon \cdot \nabla n_\varepsilon \right] \cdot (m-1)n_\varepsilon^{m-2} \varphi \right| \\
\leq (m-1) \int_\Omega \left[ D_\varepsilon(n_\varepsilon) n_\varepsilon^{m-2} \nabla n_\varepsilon \cdot \nabla \varphi + (m-2)D_\varepsilon(n_\varepsilon)n_\varepsilon^{m-3} |\nabla n_\varepsilon|^2 \varphi \right] \\
+ (m-1) \int_\Omega \left[ (m-2)n_\varepsilon^{m-2} F_\varepsilon(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \nabla n_\varepsilon \cdot \nabla c_\varepsilon \varphi + n_\varepsilon^{m-1} F_\varepsilon(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \nabla c_\varepsilon \cdot \nabla \varphi \right] \\
+ \left| \int_\Omega n_\varepsilon^{m-1} u_\varepsilon \cdot \nabla \varphi \right| \\
\leq (m-1) \int_\Omega \left[ C_D(n_\varepsilon + \varepsilon)^{m-1} n_\varepsilon^{m-2} |\nabla n_\varepsilon||\nabla \varphi| + (m-2)C_D(n_\varepsilon + \varepsilon)^{m-1} n_\varepsilon^{m-3} |\nabla n_\varepsilon|^2 |\varphi| \right] \\
+ (m-1) \int_\Omega \left[ (m-2)n_\varepsilon^{m-2} F_\varepsilon(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \nabla n_\varepsilon \cdot \nabla c_\varepsilon \varphi + n_\varepsilon^{m-1} F_\varepsilon(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \nabla c_\varepsilon \cdot \nabla \varphi \right] \\
+ \left| \int_\Omega n_\varepsilon^{m-1} u_\varepsilon \cdot \nabla \varphi \right| \\
\leq m(m-1)C_D \left\{ \int_\Omega \left[ \left( n_\varepsilon + \varepsilon \right)^{m-1} n_\varepsilon^{m-2} |\nabla n_\varepsilon| + \left( n_\varepsilon + \varepsilon \right)^{m-1} \n_\varepsilon^{m-3} |\nabla n_\varepsilon|^2 \right] \right\} \|\varphi\|_{W^{1,\infty}(\Omega)} \\
+ (m-1)^2 [\nu_0^{\infty}(\Omega) + 1] \left\{ \int_\Omega \left[ n_\varepsilon^{m-2} |\nabla n_\varepsilon||\nabla c_\varepsilon| + \n_\varepsilon^{m-1} |\nabla c_\varepsilon| + \n_\varepsilon^{m-1} |u_\varepsilon| \right] \right\} \|\varphi\|_{W^{1,\infty}(\Omega)} \\
(4.27)
\]

for all \( t > 0 \). Hence, observe that the embedding \( W^{2,q}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)(q > 3) \), due to (1.21), (2.40) and (4.8), applying \( m > 2 \) and the Young inequality, we derive \( C_1, C_2 \) and \( C_3 \) such that

\[
\int_0^T \|\partial_t n_\varepsilon^{m-1}(\cdot, t)\|_{(W^{2,q}(\Omega))} \text{d}t \\
\leq C_1 \left\{ \int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{2m-4} |\nabla n_\varepsilon|^2 + \int_0^T \int_\Omega n_\varepsilon^{2m-2} + \int_0^T \int_\Omega |\nabla c_\varepsilon|^2 + \int_0^T \int_\Omega |u_\varepsilon|^2 \right\} \\
\leq C_2 \left\{ \int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{2m-4} |\nabla n_\varepsilon|^2 + \int_0^T \int_\Omega |\nabla c_\varepsilon|^2 + \int_0^T \int_\Omega n_\varepsilon^{\frac{8(m-1)}{3}} + \int_0^T \int_\Omega |u_\varepsilon|^\frac{10}{3} + T \right\} \\
\leq C_3(T+1) \text{ for all } T > 0; \\
(4.28)
\]

which leads directly to

\[
\int_0^T \|\partial_t n_\varepsilon^{m-1}(\cdot, t)\|_{(W^{2,q}(\Omega))} \text{d}t \leq C_4(T+1) \\
(4.29)
\]

for a positive constant \( C_4 \). 

Now, in view of (1.22), (1.24) and (1.29), we can derive (1.15).

Case \( m > 2 \) : Likewise, given any \( \varphi \in C^\infty(\bar{\Omega}) \), we may test the second equation in (2.1)
against \( \varphi \) to conclude that
\[
\left| \int_\Omega \partial_t c_\varepsilon(\cdot, t) \varphi \right| = \left| \int_\Omega \left[ \Delta c_\varepsilon - n_\varepsilon c_\varepsilon - u_\varepsilon \cdot \nabla c_\varepsilon \right] \cdot \varphi \right|
\]
\[
= \left| - \int_\Omega \nabla c_\varepsilon \cdot \nabla \varphi - \int_\Omega n_\varepsilon c_\varepsilon \varphi + \int_\Omega c_\varepsilon u_\varepsilon \cdot \nabla \varphi \right|
\]
\[
\leq \left\{ \| \nabla c_\varepsilon \|_{L^2(\Omega)} + \| n_\varepsilon c_\varepsilon \|_{L^2(\Omega)} + \| c_\varepsilon u_\varepsilon \|_{L^2(\Omega)} \right\} \| \varphi \|_{W^{1,2}(\Omega)} \text{ for all } t > 0.
\]  
(4.30)

from which, after using (2.12), (4.1) and (4.4), we conclude that there exist positive constants \( C_{15}, C_{16} \) and \( C_{17} \) such that
\[
\int_0^T \| \partial_t c_\varepsilon(\cdot, t) \|_{W^{1,2}(\Omega)}^2 dt 
\leq C_{15} \int_0^T \int_\Omega |\nabla c_\varepsilon|^2 + C_{15} \int_0^T \int_\Omega n_\varepsilon^2 + C_{15} \int_0^T \int_\Omega |u_\varepsilon|^2 
\leq C_{15} \int_0^T \int_\Omega |\nabla c_\varepsilon|^2 + C_{15} \| c_\varepsilon \|_{L^2(\Omega)}^2 + C_{15} \int_0^T \int_\Omega |u_\varepsilon|^2 + C_{15} T 
\leq C_{17}(T + 1) \text{ for all } T > 0.
\]  
(4.31)

Case \( \frac{10}{9} < m \leq 2 \): For any given \( t > 0 \) and \( \varphi \in C^\infty(\Omega) \), multiplying the second equation of (2.1) by \( \varphi \), we derive that
\[
\left| \int_\Omega \partial_t c_\varepsilon(\cdot, t) \varphi \right| \leq \left\{ \| \nabla c_\varepsilon \|_{L^{\frac{2m+4}{3m+2}}(\Omega)} + \| n_\varepsilon c_\varepsilon \|_{L^{\frac{2m+4}{3m+2}}(\Omega)} + \| u_\varepsilon c_\varepsilon \|_{L^{\frac{2m+4}{3m+2}}(\Omega)} \right\} \| \varphi \|_{W^{1, \frac{2m+4}{3m+2}}(\Omega)}.
\]

Hence, (2.12), (4.1) and (4.2) imply that
\[
\int_0^T \| \partial_t c_\varepsilon(\cdot, t) \|_{W^{1, \frac{2m+4}{3m+2}}(\Omega)}^2 dt 
\leq C_{18} \int_0^T \int_\Omega |\nabla c_\varepsilon|^2 + C_{18} \int_0^T \int_\Omega |n_\varepsilon|^2 + C_{18} \int_0^T \int_\Omega |u_\varepsilon|^2 + C_{19} T 
\leq C_{19}(T + 1) \text{ for all } T > 0.
\]  
(4.32)

and some positive constants \( C_{18}, C_{19} \) and \( C_{20} \).

Now, combining (4.31) and (4.32), we can get (4.39).

Finally, for any given \( \varphi \in C^\infty_0(\Omega; \mathbb{R}^3) \), we infer from the third equation in (2.1) that
\[
\left| \int_\Omega \partial_t u_\varepsilon(\cdot, t) \varphi \right| = \left| - \int_\Omega \nabla u_\varepsilon \cdot \nabla \varphi - \int_\Omega (Y_\varepsilon u_\varepsilon \otimes u_\varepsilon) \cdot \nabla \varphi + \int_\Omega n_\varepsilon \nabla \varphi \cdot \varphi \right| \text{ for all } t > 0.
\]  
(4.33)
Case $\frac{10}{9} < m < \frac{4}{3}$: Due to (4.1), (4.2) and (3.9), there exist some positive constants $C_{21}, C_{22}, C_{23}$ and $C_{24}$ such that

\[
\int_0^T \| \partial_t u_\epsilon(\cdot, t) \|^3 \omega^{\frac{3m+2}{3}}_{(W^{1,2}(\Omega))} \, dt \\
\leq C_{21} \int_0^T \int_\Omega |\nabla u_\epsilon|^3 + C_{21} \int_0^T \int_\Omega |Y_\epsilon u_\epsilon \otimes u_\epsilon|^3 + C_{21} \int_0^T \int_\Omega n_\epsilon^{\frac{3m+2}{3}} \\
\leq C_{21} \int_0^T \int_\Omega |\nabla u_\epsilon|^2 + C_{21} \int_0^T \int_\Omega |Y_\epsilon u_\epsilon|^2 + C_{21} \int_0^T \int_\Omega n_\epsilon^{\frac{3m+2}{3}} + C_{22}T \\
\leq C_{21} \int_0^T \int_\Omega |\nabla u_\epsilon|^2 + C_{21} \int_0^T \int_\Omega |u_\epsilon|^{10} + C_{21} \int_0^T \int_\Omega n_\epsilon^{\frac{3m+2}{3}} + C_{23}T \\
\leq C_{24}(T+1) \text{ for all } T > 0.
\]

Case $\frac{4}{3} \leq m \leq 2$: Similarly, by (4.1), (4.2) and (3.9), we may find some positive constants $C_{25}, C_{26}, C_{27}$ and $C_{28}$ such that

\[
\int_0^T \| \partial_t u_\epsilon(\cdot, t) \|^2 \omega^{\frac{3m+2}{3}}_{(W^{1,2}(\Omega))} \, dt \\
\leq C_{25} \int_0^T \int_\Omega |\nabla u_\epsilon|^2 + C_{25} \int_0^T \int_\Omega |Y_\epsilon u_\epsilon|^2 + C_{25} \int_0^T \int_\Omega n_\epsilon^2 \\
\leq C_{25} \int_0^T \int_\Omega |\nabla u_\epsilon|^2 + C_{25} \int_0^T \int_\Omega |Y_\epsilon u_\epsilon|^2 + C_{25} \int_0^T \int_\Omega n_\epsilon^{\frac{3m+2}{3}} + C_{26}T \\
\leq C_{25} \int_0^T \int_\Omega |\nabla u_\epsilon|^2 + C_{25} \int_0^T \int_\Omega |u_\epsilon|^{10} + C_{25} \int_0^T \int_\Omega n_\epsilon^{\frac{3m+2}{3}} + C_{27}T \\
\leq C_{28}(T+1) \text{ for all } T > 0.
\]

Case $m > 2$: Now, in view of (4.1), (4.4) and (3.9), we also derive that

\[
\int_0^T \| \partial_t u_\epsilon(\cdot, t) \|^2 \omega^{\frac{3m+2}{3}}_{(W^{1,2}(\Omega))} \, dt \\
\leq C_{29} \int_0^T \int_\Omega |\nabla u_\epsilon|^2 + C_{29} \int_0^T \int_\Omega |u_\epsilon|^{10} + C_{29} \int_0^T \int_\Omega n_\epsilon^{\frac{8(m-1)}{3}} + C_{29}T \\
\leq C_{30}(T+1) \text{ for all } T > 0.
\]

and some positive constants $C_{29}$ and $C_{30}$. Finally, in conjunction with (4.34)–(4.36), we can get the results.

In the following Lemma, we shall give some spatial estimates on $n_\epsilon F_\epsilon(n_\epsilon)S_\epsilon(x, n_\epsilon, c_\epsilon) \cdot \nabla c_\epsilon$ and $u_\epsilon \cdot \nabla c_\epsilon$, which is crucial to derive the existence of weak solution to problem (1.3).

**Lemma 4.3.** Assume that

\[
\gamma_1 := \begin{cases} 
\frac{4(3m+2)}{3m+14}, & \text{if } \frac{10}{9} < m \leq 2, \\
\frac{8(m-1)}{4m-1}, & \text{if } m > 2
\end{cases}
\]
and
\[
\gamma_2 := \begin{cases}
\frac{20}{11}, & \text{if } \frac{10}{9} < m \leq 2, \\
\frac{5}{4}, & \text{if } m > 2.
\end{cases}
\] (4.38)

Let \( m > \frac{10}{9} \), (1.6) and (1.7) hold. Then for any \( T > 0 \), one can find \( C > 0 \) independent of \( \varepsilon \) such that
\[
\int_0^T \int_\Omega |n_\varepsilon F_\varepsilon(n_\varepsilon)S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon|^{\gamma_1} \leq C(T + 1)
\] (4.39)
and
\[
\int_0^T \int_\Omega |u_\varepsilon \cdot \nabla c_\varepsilon|^{\gamma_2} \leq C(T + 1).
\] (4.40)

**Proof.** Case \( \frac{10}{9} < m \leq 2 \): Due to (1.5), (4.1), (4.2), (4.6) and the Hölder inequality, we derive that there exist positive constants \( C_1 \) and \( C_2 \) such that
\[
\int_0^T \int_\Omega |n_\varepsilon F_\varepsilon(n_\varepsilon)S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon|^{\frac{4(3m+2)}{3m+14}} \\
\leq S_0(\|c_0\|_{L^\infty(\Omega)})^{\frac{4(3m+2)}{3m+14}} \left( \int_0^T \int_\Omega |\nabla c_\varepsilon|^4 \right)^{\frac{3m+2}{3m+14}} \left( \int_0^T \int_\Omega n_\varepsilon^{\frac{3m+2}{3m+14}} \right)^{\frac{12}{3m+14}} \] (4.41)
and
\[
\int_0^T \int_\Omega |u_\varepsilon \cdot \nabla c_\varepsilon|^{\frac{20}{11}} \leq \left( \int_0^T \int_\Omega |\nabla c_\varepsilon|^4 \right)^{\frac{5}{11}} \left( \int_0^T \int_\Omega |u_\varepsilon|^{\frac{10}{3}} \right)^{\frac{6}{11}} \] \leq C_2(T + 1) \text{ for all } T > 0.
\] (4.42)

Case \( m > 2 \): In view of (1.5), (1.1), (4.4) and the Hölder inequality, it follows that there exist positive constants \( C_3 \) and \( C_4 \) such that
\[
\int_0^T \int_\Omega |n_\varepsilon F_\varepsilon(n_\varepsilon)S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \cdot \nabla c_\varepsilon|^{\frac{8(m-1)}{4m-1}} \\
\leq S_0(\|c_0\|_{L^\infty(\Omega)})^{\frac{8(m-1)}{4m-1}} \left( \int_0^T \int_\Omega |\nabla c_\varepsilon|^2 \right)^{\frac{4m-1}{4m-3}} \left( \int_0^T \int_\Omega n_\varepsilon^{\frac{8(m-1)}{4m-3}} \right)^{\frac{3}{4m-3}} \] (4.43)
and
\[
\int_0^T \int_\Omega |u_\varepsilon \cdot \nabla c_\varepsilon|^{\frac{5}{2}} \leq \left( \int_0^T \int_\Omega |\nabla c_\varepsilon|^2 \right)^{\frac{5}{4}} \left( \int_0^T \int_\Omega |u_\varepsilon|^{\frac{10}{3}} \right)^{\frac{3}{4}} \] \leq C_4(T + 1) \text{ for all } T > 0.
\] (4.44)

Finally, combining (4.41)–(4.44), we can derive (4.39) and (4.40). This completes the proof of Lemma 4.3.
5 Passing to the limit. Proof of Theorem 1.1

With the a-priori estimates obtained in Section 2 and Section 4, we shall give the proof of Theorem 1.1. Before going to do it, let us first give the definition of weak solution. In what follows, for vectors \( v \in \mathbb{R}^3 \) and \( w \in \mathbb{R}^3 \), we use \( v \otimes w \) denote the matrix \((a_{ij})_{i,j \in \{1,2,3\}} \in \mathbb{R}^{3 \times 3}\) with \( a_{ij} := v_i w_j \) for \( i,j \in \{1,2,3\} \).

**Definition 5.1.** We call \((n,c,u)\) a global weak solution of (1.3) if

\[
\begin{cases}
    n \in L^1_{\text{loc}}(\bar{\Omega} \times [0,\infty)), \\
    c \in L^1_{\text{loc}}([0,\infty); W^{1,1}(\Omega)), \\
    u \in L^1_{\text{loc}}([0,\infty); W^{1,1}_0(\Omega; \mathbb{R}^3)),
\end{cases}
\]

such that \( n \geq 0 \) and \( c \geq 0 \) a.e. in \( \Omega \times (0,\infty) \),

\[
n c \in L^1_{\text{loc}}(\bar{\Omega} \times [0,\infty)), \quad u \otimes u \in L^1_{\text{loc}}(\bar{\Omega} \times [0,\infty); \mathbb{R}^{3 \times 3}), \quad \text{and} \quad D(n)\nabla n, \quad nS(x,n,c)\nabla c, \quad cu \quad \text{and} \quad nu \quad \text{belong to} \quad L^1_{\text{loc}}(\bar{\Omega} \times [0,\infty); \mathbb{R}^3),
\]

that \( \nabla \cdot u = 0 \) a.e. in \( \Omega \times (0,\infty) \), and that

\[
-\int_0^T \int_\Omega n \varphi_t - \int_\Omega n_0 \varphi(\cdot,0) = -\int_0^T \int_\Omega D(n)\nabla n \cdot \nabla \varphi + \int_0^T \int_\Omega n(S(x,n,c) \cdot \nabla c) \cdot \nabla \varphi + \int_0^T \int_\Omega n u \cdot \nabla \varphi
\]

for any \( \varphi \in C^\infty_0(\bar{\Omega} \times [0,\infty)) \) as well as

\[
-\int_0^T \int_\Omega c \varphi_t - \int_\Omega c_0 \varphi(\cdot,0) = -\int_0^T \int_\Omega \nabla c \cdot \nabla \varphi - \int_0^T \int_\Omega n c \cdot \varphi + \int_0^T \int_\Omega cu \cdot \nabla \varphi
\]

for any \( \varphi \in C^\infty_0(\bar{\Omega} \times [0,\infty)) \) and

\[
-\int_0^T \int_\Omega w \varphi_t - \int_\Omega w_0 \varphi(\cdot,0) = -\int_0^T \int_\Omega \nabla u \cdot \nabla \varphi - \int_0^T \int_\Omega n \nabla \phi \cdot \varphi
\]

for any \( \varphi \in C^\infty_0(\Omega \times [0,\infty); \mathbb{R}^3) \) fulfilling \( \nabla \varphi \equiv 0 \).

With the above compactness properties at hand, by means of a standard extraction procedure we can conclude that (1.3) is indeed globally solvable.
Lemma 5.1. Assume that (1.6) and (1.7) hold, and suppose that $m$ and $S$ satisfy (1.2) and (1.4)–(1.5), respectively. If $m > \frac{10}{9}$, then there exists $(\varepsilon_j)_{j \in \mathbb{N}} \subset (0,1)$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$, and such that as $\varepsilon := \varepsilon_j \searrow 0$ we have



\begin{equation}
 n_\varepsilon \to n \text{ a.e. in } \Omega \times (0, \infty), \tag{5.6}
\end{equation}

\begin{equation}
 c_\varepsilon \to c \text{ in } L^2(\Omega \times [0, \infty)) \text{ and a.e. in } \Omega \times (0, \infty), \tag{5.7}
\end{equation}

\begin{equation}
 u_\varepsilon \to u \text{ in } L^2_{\text{loc}}(\Omega \times [0, \infty)) \text{ and a.e. in } \Omega \times (0, \infty), \tag{5.8}
\end{equation}

\begin{equation}
 \nabla u_\varepsilon \rightharpoonup \nabla u \text{ in } L^2(\Omega \times [0, \infty)), \tag{5.9}
\end{equation}

\begin{equation}
 \nabla c_\varepsilon \rightharpoonup \nabla c \left\{ \begin{array}{ll}
 in \ L^4_{\text{loc}}(\Omega \times [0, \infty)), & \text{if } \frac{10}{9} < m \leq 2, \\
 in \ L^2_{\text{loc}}(\Omega \times [0, \infty)), & \text{if } m > 2,
\end{array} \right. \tag{5.10}
\end{equation}

\begin{equation}
 n_\varepsilon \to n \left\{ \begin{array}{ll}
 in \ L^{\frac{3m+2}{m+2}}_{\text{loc}}(\Omega \times [0, \infty)), & \text{if } \frac{10}{9} < m \leq 2, \\
 in \ L^{\frac{8(m-1)}{8m-1}}_{\text{loc}}(\Omega \times [0, \infty)), & \text{if } m > 2,
\end{array} \right. \tag{5.11}
\end{equation}

\begin{equation}
 D_\varepsilon(n_\varepsilon) \nabla n_\varepsilon \rightharpoonup D(n) \nabla n \left\{ \begin{array}{ll}
 in \ L^{\frac{3m+2}{m+2}}_{\text{loc}}(\Omega \times [0, \infty)), & \text{if } \frac{10}{9} < m \leq 2, \\
 in \ L^{\frac{8(m-1)}{8m-1}}_{\text{loc}}(\Omega \times [0, \infty)), & \text{if } m > 2,
\end{array} \right. \tag{5.12}
\end{equation}

\begin{equation}
 c_\varepsilon \overset{\ast}{\rightharpoonup} c \text{ in } L^\infty(\Omega \times (0, \infty)) \tag{5.13}
\end{equation}

as well as

\begin{equation}
 u_\varepsilon \rightharpoonup u \text{ in } L^{\frac{10}{3}}(\Omega \times [0, \infty)) \tag{5.14}
\end{equation}

and

\begin{equation}
 Y_\varepsilon u_\varepsilon \rightharpoonup u \text{ in } L^2_{\text{loc}}([0, \infty); L^2(\Omega)) \tag{5.15}
\end{equation}

with some triple $(n, c, u)$ which is a global weak solution of (1.3) in the sense of Definition 5.1.

Proof. Firstly, letting

\begin{equation}
 \beta_1 := \left\{ \begin{array}{ll}
 \frac{3m+2}{4}, & \text{if } \frac{10}{9} < m \leq 2, \\
 2, & \text{if } m > 2,
\end{array} \right. \tag{5.16}
\end{equation}

\begin{equation}
 \gamma := \left\{ \begin{array}{ll}
 1, & \text{if } \frac{10}{9} < m \leq 2, \\
 m - 1, & \text{if } m > 2,
\end{array} \right. \tag{5.17}
\end{equation}

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$$W_1 := \begin{cases} W^{2,q}(\Omega), & \text{if } \frac{10}{9} < m < \frac{26}{21} \text{ or } m > 2, \\ W^{1,3m+2}(\Omega) & \text{if } \frac{26}{21} \leq m \leq 2, \end{cases}$$

$$\beta_2 := \begin{cases} 4, & \text{if } \frac{10}{9} < m \leq 2, \\ 2, & \text{if } m > 2 \end{cases}$$

as well as

$$W_2 := \begin{cases} W^{1,\frac{3m+2}{3m-1}}(\Omega) & \text{if } \frac{10}{9} < m \leq 2, \\ W^{1,2}(\Omega) & \text{if } m > 2, \end{cases}$$

and

$$W_3 := \begin{cases} W^{1,\frac{3m+2}{3m-1}}(\Omega) & \text{if } \frac{10}{9} < m < \frac{4}{3}, \\ W^{1,2}(\Omega) & \text{if } m \geq \frac{4}{3}, \end{cases}$$

where \( q \) is given by Lemma 4.2. Now, in light of Lemma 2.9, Lemma 2.10 and Lemma 4.2, for some \( C_1 > 0 \) which is independent of \( \varepsilon \), we have

$$\|n_\varepsilon^2\|_{L_{\text{loc}}^{\beta_1}(0,\infty);W^{1,\beta_1}(\Omega)} \leq C_1(T + 1) \quad \text{and} \quad \|\partial_t n_\varepsilon^2\|_{L_{\text{loc}}^1(0,\infty);W^{1,\beta_1}_*} \leq C_1(T + 1) \quad (5.16)$$

as well as

$$\|c_\varepsilon\|_{L_{\text{loc}}^2(0,\infty);W^{1,2}(\Omega)} \leq C_1(T + 1) \quad \text{and} \quad \|\partial_t c_\varepsilon\|_{L_{\text{loc}}^1(0,\infty);W^{1,2}_*} \leq C_1(T + 1) \quad (5.17)$$

and

$$\|u_\varepsilon\|_{L_{\text{loc}}^2(0,\infty);W^{1,2}(\Omega)} \leq C_1(T + 1) \quad \text{and} \quad \|\partial_t u_\varepsilon\|_{L_{\text{loc}}^1(0,\infty);W^{1,2}_*} \leq C_1(T + 1). \quad (5.18)$$

Now, applying the Aubin-Lions lemma (16) to (5.16)–(5.18), we can derive that

$$(n_\varepsilon^2)_{\varepsilon \in (0,1)} \text{ is strongly precompact in } L_{\text{loc}}^{\beta_1}(\bar{\Omega} \times [0, \infty)) \quad (5.19)$$

as well as

$$(c_\varepsilon)_{\varepsilon \in (0,1)} \text{ is strongly precompact in } L_{\text{loc}}^2(\bar{\Omega} \times [0, \infty)) \quad (5.20)$$

and

$$(u_\varepsilon)_{\varepsilon \in (0,1)} \text{ is strongly precompact in } L_{\text{loc}}^2(\bar{\Omega} \times [0, \infty)). \quad (5.21)$$

Therefore, there exist a subsequence \( \varepsilon = \varepsilon_j \subset (0,1)_{j \in \mathbb{N}} \) and the limit functions \( n \) and \( c \) such that (5.7)–(5.10) holds. Moreover, for each fixed \( T \in (0, \infty) \), (5.8) implies that there exists
a null set $N_T \in (0, T)$ such that we can pick a subsequence which we still denote by $(\varepsilon_j)_{j \in \mathbb{N}}$ fulfilling
\[
    u_\varepsilon(\cdot, t) \to u(\cdot, t) \quad \text{in} \quad L^2(\Omega) \quad \text{for all} \quad t \in (0, T) \setminus N_T \quad \text{as} \quad \varepsilon = \varepsilon_j \downarrow 0. \quad (5.22)
\]

Next, in view of (5.16), an Aubin–Lions lemma (see e.g. [16]) applies to yield strong precompactness of $(n_\varepsilon^\gamma)_{\varepsilon \in (0, 1)}$ in $L^{\beta_3}(\Omega \times (0, T))$, whence along a suitable subsequence we may derive that $n_\varepsilon^\gamma \to z_1^\gamma$ and hence
\[
    n_\varepsilon \to z_1 \quad \text{a.e. in} \quad \Omega \times (0, \infty) \quad \text{for some nonnegative measurable} \quad z_1 : \Omega \times (0, \infty) \to \mathbb{R}. \quad (5.23)
\]

The above estimate (5.23) combined with energy inequality (5.19), (5.23), and the Egorov theorem ensures $z_1 = n$, therefore, we deduce that (5.6). Now, let
\[
    \beta_3 := \begin{cases} 
        \frac{3m+2}{3}, & \text{if} \quad \frac{10}{9} < m \leq \frac{38}{33}, \\
        \gamma_2, & \text{if} \quad m > \frac{38}{33},
    \end{cases}
\]

where $\gamma_2$ is given by (4.38). On the other hand, observing that
\[
    \beta_3 \leq \begin{cases} 
        \frac{3m+2}{3}, & \text{if} \quad \frac{10}{9} < m \leq 2, \\
        \frac{8(m-1)}{3}, & \text{if} \quad m > \frac{38}{33},
    \end{cases}
\]
in light of (2.12), (4.2), (4.4), applying the Young inequality, we derive that there exists a positive constant $C_1$ such that
\[
    \int_0^T \int_\Omega |n_\varepsilon c_\varepsilon|^{\beta_3} \leq C_1(T + 1). \quad (5.24)
\]

Next, let $g_\varepsilon(x, t) := -n_\varepsilon c_\varepsilon - u_\varepsilon \cdot \nabla c_\varepsilon$. Therefore, recalling $1 < \beta_3 \leq \gamma_2$, by some basic calculation, we can get
\[
    \int_0^T \int_\Omega |g_\varepsilon|^{\beta_3} \leq C_2(T + 1). \quad (5.25)
\]

for a positive constant $C_2$. From this, we may invoke the standard parabolic regularity theory to infer that $(c_\varepsilon)_{\varepsilon \in (0, 1)}$ is bounded in $L^{\beta_3}((0, T); W^{2, \beta_3}(\Omega))$. Thus, by (4.39) and the Aubin–Lions lemma we derive that the relative compactness of $(c_\varepsilon)_{\varepsilon \in (0, 1)}$ in $L^{\beta_3}((0, T); W^{1, \beta_3}(\Omega))$. We can pick an appropriate subsequence which is still written as $(\varepsilon_j)_{j \in \mathbb{N}}$ such that $\nabla c_{\varepsilon_j} \to z_2$ in $L^{\beta_3}(\Omega \times (0, T))$ for all $T \in (0, \infty)$ and some $z_2 \in L^{\beta_3}(\Omega \times (0, T))$ as $j \to \infty$, hence $\nabla c_{\varepsilon_j} \to z_2$...
a.e. in $\Omega \times (0, \infty)$ as $j \to \infty$. In view of (5.10) and the Egorov theorem we conclude that $z_2 = \nabla c$, and whence

$$\nabla c_\varepsilon \to \nabla c \quad \text{a.e. in } \Omega \times (0, \infty) \quad \text{as } \varepsilon = \varepsilon_j \searrow 0. \quad (5.26)$$

Now, we will conclude that the triplet $(n, c, u)$ is the desired solution in the sense of Definition 5.1. Indeed, we first notice that from the nonnegativity of $n_\varepsilon$ and $c_\varepsilon$, the estimate of (5.9) and $\nabla \cdot u_\varepsilon = 0$, it is easy to see that $n \geq 0$ and $c \geq 0$ and $\nabla \cdot u = 0$ a.e. in $\Omega \times (0, \infty)$. On the other hand, in view of (4.39), we can infer from (4.2) and (4.4) that

$$n_\varepsilon F_\varepsilon(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \nabla c_\varepsilon \rightharpoonup z_3 \quad \text{in } L^n(\Omega \times (0, T)) \quad \text{for each } T \in (0, \infty) \quad (5.27)$$

with $\gamma_1$ is given by (4.37). On the other hand, in view of (1.4), (2.4), (5.6), (5.7) and (5.26) imply that

$$n_\varepsilon F_\varepsilon(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \nabla c_\varepsilon \rightharpoonup nS(x, n, c) \nabla c \quad \text{a.e. in } \Omega \times (0, \infty) \quad \text{as } \varepsilon = \varepsilon_j \searrow 0. \quad (5.28)$$

Again by the Egorov theorem, we gain $z_3 = nS(x, n, c) \nabla c$, and hence (5.27) can be rewritten as

$$n_\varepsilon F_\varepsilon(n_\varepsilon) S_\varepsilon(x, n_\varepsilon, c_\varepsilon) \nabla c_\varepsilon \rightharpoonup nS(x, n, c) \nabla c \quad \text{in } L^n(\Omega \times (0, T)) \quad \text{for each } T \in (0, \infty) \quad (5.29)$$

as $\varepsilon = \varepsilon_j \searrow 0$. Next, employing almost exactly the same arguments as in the proof of (5.27)–(5.29) (the minor necessary changes are left as an easy exercise to the reader), and taking advantage of (2.12), (4.1)–(4.4) and (5.6)–(5.8), we conclude that (5.10)–(5.14) is true as well as

$$c_\varepsilon u_\varepsilon \to cu \quad \text{in } L^1_{loc}(\bar{\Omega} \times (0, \infty)) \quad \text{as } \varepsilon = \varepsilon_j \searrow 0, \quad (5.30)$$

$$n_\varepsilon c_\varepsilon \to nc \quad \text{in } L^{\gamma_4}_{loc}(\bar{\Omega} \times (0, \infty)) \quad \text{as } \varepsilon = \varepsilon_j \searrow 0 \quad (5.31)$$

and

$$n_\varepsilon u_\varepsilon \to nu \quad \text{in } L^{\gamma_5}(\Omega \times (0, T)) \quad \text{as } \varepsilon = \varepsilon_j \searrow 0 \quad (5.32)$$

for each $T \in (0, \infty)$, where

$$\gamma_4 : = \begin{cases} \frac{1}{\frac{4}{3} + \frac{m+2}{m}} & \text{if } \frac{10}{9} < m \leq 2, \\ \frac{1}{\frac{1}{2} + \frac{1}{8(m-1)}} & \text{if } m > 2 \end{cases} \quad (5.33)$$
Under

\[
\gamma_5 := \begin{cases}
\frac{1}{10^m + 3n + 2}, & \text{if } \frac{10}{9} \leq m \leq 2,
\frac{1}{10^m + 8(n-1)}, & \text{if } m > 2.
\end{cases}
\]  

(5.34)

Now, by (5.7)–(5.9), (5.11), we conclude that (5.1). Now, employing (5.8) and using the fact that \(\|Y_\varepsilon \varphi\|_{L^2(\Omega)} \leq \|\varphi\|_{L^2(\Omega)}(\varphi \in L^2_\sigma(\Omega))\) and \(Y_\varepsilon \varphi \to \varphi\) in \(L^2(\Omega)\) as \(\varepsilon \searrow 0\), we can obtain

\[
\|Y_\varepsilon u_\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)} \leq \|Y_\varepsilon [u_\varepsilon(\cdot, t) - u(\cdot, t)]\|_{L^2(\Omega)} + \|Y_\varepsilon u_\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)}
\leq \|u_\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)} + \|Y_\varepsilon u(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)}
\to 0 \text{ as } \varepsilon := \varepsilon_j \searrow 0.
\]  

(5.35)

On the other hand, observing that

\[
\|Y_\varepsilon u_\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)}^2 \leq \left(\|Y_\varepsilon u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} + \|u(\cdot, t)\|_{L^2(\Omega)}\right)^2
\leq \left(\|u_\varepsilon(\cdot, t)\|_{L^2(\Omega)} + \|u(\cdot, t)\|_{L^2(\Omega)}\right)^2
\leq C_2 \text{ for all } t \in (0, \infty) \text{ and } \varepsilon \in (0, 1)
\]  

(5.36)

with some \(C_2 > 0\). Now, thanks to (5.8), (5.35) and (5.36) and the dominated convergence theorem, we conclude that

\[
\int_0^T \|Y_\varepsilon u_\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)}^2 dt \to 0 \text{ as } \varepsilon := \varepsilon_j \searrow 0 \text{ for all } T > 0.
\]  

(5.37)

Thus, (5.15) holds. Now, in conjunction with (5.15) and (5.8), we can obtain

\[
Y_\varepsilon u_\varepsilon \otimes u_\varepsilon \to u \otimes u \text{ in } L^1_{\text{loc}}(\bar{\Omega} \times [0, \infty)) \text{ as } \varepsilon := \varepsilon_j \searrow 0.
\]  

(5.38)

Therefore, (5.29)–(5.32) and (5.38) imply the integrability of \(nS(x, n, c)\nabla c, nc, nu\) and \(cu, u \otimes u\) in (5.2). Based on (5.9)–(5.15), (5.29)–(5.32) and (5.38), the integral identities (5.3)–(5.5) can be achieved by standard arguments from the corresponding weak formulations in the regularized system (2.1) upon taking \(\varepsilon = \varepsilon_j \searrow 0\). The proof of Lemma 5.1 is completed. \(\square\)

**The proof of Theorem 1.1**  The statement is evidently implied by Lemma 5.1.

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