Mathematical understanding and reasoning abilities related to cognitive stage of senior high school students

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Abstract. The goal of this study is to analyse the relationships among students’ mathematical understanding, reasoning abilities and their cognitive stage. The study is descriptive-analytic research involving 414 eleventh-grade students (17,43 years old) from seven senior high schools. The instruments of this study are tests on mathematical understanding and reasoning abilities, the Longeot Test and the Test of Logical Thinking (TOLT). The study revealed there were two kinds of attributes of the findings. First, based on the subject's age (17,43 years old) there were different findings on students' cognitive stages from the Piaget's theory, there were many students did not reach formal operational stage yet, namely 21 % students were still on the concrete stage, 34% of students were still in the transition stage, and only 45% were on a formal stage. Second, there were two kinds findings those were firm toward Piaget’s theory, those were: a) the higher subject's school classes, the number of formal subjects; b) on students’ mathematical understanding and reasoning abilities and mathematics formative and summative tests, formal students attained higher grades than transition students, and transition students attained higher grades than concrete students.

1. Introduction

The ability to understand and reason mathematically are two basic mathematical abilities that need to be mastered by high school students. The reason behind the statement is the ability included in the purpose of teaching mathematics (mathematics curriculum), namely students able to analyze arguments, draw conclusions, analyze information or data, and understand the mathematical concepts, principles, rules, regulations, and conclusions obtained. This implies that to study the complex mathematical contents an individual must first master higher cognitive abilities as well. Cognitive ability is related to the stage of child development. Classify children's cognitive development into four main stages sequentially [1], namely the sensory motoric stage, the pre-operational stage, the concrete operational stage, and the formal operational stage. Then they identify what types of cognitive abilities are capable and unable to be performed by each stage of cognitive development.

These cognitive abilities contain logical reasoning abilities such as controlling variables, operating proportions, probabilistic operations, combinatorial operations, correlational operations, and propositional operations. To determine the stage of cognitive development, an appropriate instrument, observation, and individual interview is needed. Based on that reason, and considering indicators of logical reasoning ability [1], some experts as Sheehan developed the Longeot Test [2] and developed a
Logical Thinking Test (TOLT) to determine the cognitive developmental stage of the subject [3]. Piaget realized that his conclusions about the stages of children's cognitive development based on observations of several students from some of the best schools in Geneva could not be generalized to students from different cultures and conditions [4].

Therefore it is necessary to conduct cross-cultural studies of instruments that have been developed, namely the Longeot Test [2] and the Logical Thinking Test [3] to determine the stage of cognitive development of a child or student. Other conditions and culture. Based on this reason, several others cross-culture the Longeot and TOLT to various age groups, school level, and culture [5-8]. The research findings concluded and confirmed that both tests were valid to determine the cognitive stage of students based on Piaget's theory. Considering that a higher cognitive stage is needed or logical reasoning ability to study higher and more complex mathematical products and processes, he raises the hypothesis that there is a correlation between the development of the cognitive stage and the achievement of mathematical understanding and reasoning ability.

2. Method

This study is a descriptive–analytic survey which has a goal to analyse deeply relation between mathematical understanding and reasoning and mathematics learning achievement of senior high and their stage of cognitive development. The design of this study is in figure 1.

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Figure 1. Design of study.
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Note:
- Pra-survey: Compiling instruments and their try-out (6 tests, for one semester)
- TOLT: Tobin dan Capie [1] (Indonesian version)
- Tes Longeot: Mcdonald dan Sheehan [2] (Indonesian version)
- TMU1 and TMU2: Test of Mathematical Understanding (1) and (2)
- TMR1 and TMR2: Test of mathematical reasoning (1) and (2)
- TFM and TSM: Formative and summative mathematics test

The subjects of this study are 414 eleventh grade of students of science and mathematics class, came from State Senior High Schools in seven cities. The subject is determined purposively which their teachers are participants of a teacher training program that conducted by Government Education Services. Compiling and try-out all of the instruments are conducted in some schools from junior up to senior high for about one semester. The true survey is conducted for one semester after pre-survey. The content of TMU1 and TMU2, TMR1 and TMR2 are related to the mathematics content of eleventh grade of senior high mathematics curriculum.

This study involved six tests compiled by the researcher and two mathematics tests of mathematics teacher from senior high schools in seven cities. In the beginning, the number of the subject was 503 students. Then, because of the mathematics content were too broad for to test in a term of the test (90 minutes), then each of mathematical understanding test and mathematical reasoning test was compiled into two part: for mid-semester 1 and the end of semester 1. So, the six instruments were: Test of Logical Thinking (TOLT) and the Longeot tests in Indonesian version, a test of mathematical understanding (TMU1 and TMU2), a test of mathematical reasoning (TMR1 dan TMR2). Further, as the six tests were conducted at the different time relevant to the design of this study, there were some students did not
participate in a certain test. For further analysis, it was determined the subjects of this study be 414 students those were who participated in all instruments.

3. Results and discussion

Based on Longeot test and TOLT the study found that percentage of the cognitive stage of the subjects (17.43 years old) of 11th-grade students of science and mathematics class (n = 414) was illustrated in table 1 below. The percentage of concrete students based on both tests were same that was 30%, however the number of formal students based on Longeot test (70%) greater than the percentage of formal students based on TOLT (48%). It was understandable because in TOLT there were three classifications (concrete, transition, and formal stage) while in Longeot test there were only two classifications (concrete and formal stage). In further analysis, a student was classified as a concrete stage or formal stage when he or she was on the same classification on both Longeot test and TOLT. When a student was classified on different stage based on Longeot test and TOLT, then he or she would be classified on transition stage.

Table 1. Percentage of concrete subjects and formal subjects based on Longeot Test and TOLT of 11th grade students of science and mathematics class TOLT dan Tes Longeot (mean age: 17.43 years old).

| Test         | n  | Concrete Stage | Transition Stage | Formal Stage |
|--------------|----|----------------|------------------|--------------|
| Longeot Test | 414| 124            | 30               | 290          |
| TOLT         | 414| 127            | 30               | 197          |
| Union*       | 414| 86             | 21               | 185          |

Note: *In union tests, subjects were a concrete or formal stage if they were in same stage (concrete or formal); if the subject was indifferent stage then he or she was in a transition stage.

The study found that many students of 17.43 years old (55%) did not attain formal stage yet, namely 21% were still on the concrete stage, and 34% were still in the transition stage. Based on the student’s age, that findings were different from Piaget’s theory that an individual will reach the formal operational stage at 12-13 years old up to 14-15 years old. Likewise, the findings of this study were also different from other findings as well, namely there were many (72%) talented elementary students (11 years old) were on formal stage Longeot in Sumarmo [3] and all (100%) 7th and 8th grade of talented students on science and mathematics (12 – 15 years old) were already on formal stage Farmer in Sumarmo [3], see table 1.

However, the findings of this study were not against to latter Piaget’s hypothesis that normal students in other condition will reach the formal stage at 18-20 years old. The implication of that statement and findings of this study supported a statement that decision of cognitive stage of a subject was not based on her or his biology age but based on her or his logical thinking ability. Other findings of this study were related to mathematical understanding and mathematical reasoning, and mathematics achievement in formative and summative tests of students. Those findings were illustrated in table 2.

Table 2. Findings on Longeot Test, TOLT, TMU, TMR, TFM and TSM of This Study (491 ≤ n ≤ 503).

| Tests     | n  | Number of item Test | Mean score | Standard deviation | Ideal Score (IS) | % of mean out of IS | Test Validity |
|-----------|----|---------------------|------------|--------------------|------------------|---------------------|---------------|
| Tes Longeot| 502| 26                  | 17.77      | 3.83               | 26               | 69                  | Concurrent Validity |
| TOLT      | 502| 10                  | 5.11       | 2.63               | 10               | 51                  | r = 0.60 |
| TMU₁      | 505| 28                  | 13.64      | 4.06               | 28               | 49                  | Content validity |
| TMR₁      | 584| 28                  | 10.59      | 3.33               | 28               | 38                  | estimated by relevancy the grid of test and item tests |
| TMU₂      | 503| 21                  | 18.49      | 8.80               | 45               | 41                  |              |
| TMR₂      | 491| 29                  | 17.00      | 6.33               | 44               | 39                  |              |
Table 2. Cont.

| Teacher’s Tests | TF M | X 6.22 | 1.51 | 10 | 62 |
|-----------------|------|--------|------|----|----|
| Assumed valid   | TSM  | 491    | X    | 6.12| 1.33| 10 | 61 |

The number of subjects in table 2 was varied (491 ≤ n ≤ 503). Then, because there were eight tests which conducted at different time, so there some students did not participate in the certain test. For that reason, and in order to make a significant conclusion, in further analysis researcher decided the subjects of this study be students who participated in all tests, that they were 414 students. The students’ scores on the eight tests (in T score) were illustrated in figure 2.

![Figure 2](image)

Note:
- TMU₁ and TMU₂ : Test of mathematical understanding
- TMR₁ and TMR₂ : Test of mathematical reasoning
- TMF and TMS : Test of mathematics formative and Test of mathematics summative

**Figure 2.** Standard Score T of formal subject, transition subject, and k concrete subject on TMU₁ and TMU₂, TMR₁ and TMR₂, TMF and TMS.

Based on figure 2 in all tests formal subjects’ abilities were fairly good, (more than mean score of T standard score), whereas abilities of transition subjects were close to medium (few below mean score of T standard score), and abilities of concrete subjects were low (less of mean score of T standard score). Those findings pointed out that there were significantly differences on mathematics abilities between formal subjects and concrete subjects. The concrete subjects realised many difficulties in solving mathematical understanding and reasoning tasks, and in mathematics formative and summative as well. Likewise, study found there was significant correlation between logical reasoning (union scores on Longeot test and TOLT) with TMU and TMR (0.50 ≤ r ≤ 0.70), and with TMF and TMS (0.43 ≤ r ≤ 0.55). Those findings on correlation pointed out that TMU and TMR possessed closer cognitive demand to logical reasoning than TMF and TMS to cognitive demand for logical reasoning.

Some of at the time of this study, some studies reported superiority of formal subjects than concrete subjects on obtaining mathematics abilities as well. For examples, McDonald reported that formal 10th-grade students possessed geometry structure came close to teacher’s geometry structure and they attained better grade on geometry than concrete students [3]. Baharuddin found that formal students obtained better grade on compiling physics model [3]. Lawson reported that formal college students were more able to compile arguments and to test the hypothesis that the concrete students [4]. Likewise, using Indonesian version of TOLT, adopting and modifying instruments on mathematical understanding and reasoning reported the superiority of formal senior high students than concrete subjects on solving...
mathematical understanding and reasoning abilities [9]. Those findings supported to or in line with the statement that formal subjects were more able to think deductively.

Likewise, some recent studies such as Rohaeta, Herman, Hendriana and Dewanto adopted indicators and modified item tests of mathematical understanding of this study [10-13]. Those studies reported that on mathematical understanding students taught by using various innovative teaching attained higher grades than students taught by conventional teaching. On those studies, students’ mathematical understanding grades were at medium level (about 60% out of ideal score). Those findings were better than the findings of this study that 11th-grade students attained low grade (about 41% out of ideal score) on mathematical understanding. Besides that, some other recent studies among others were reported that students taught by various innovative teaching approaches attained higher grades on mathematical reasoning than students taught by conventional teaching. However, on those studies, the mathematics reasoning grades were at low-medium level (about 50%-60% out of ideal score), and those were better than the students’ grade (45%-50% out of ideal score) [14-18].

All studies above found that on mathematical understanding students’ grades were in medium level (about 60% out of ideal score) and on mathematical reasoning students’ grades were in low-medium level (about 50%-60% out of ideal score). These findings illustrated that mathematical understanding and reasoning and logical reasoning (illustrated in the Indonesian version of Longeot test and TOLT) were still difficult tasks for high and university students. Moreover, those studies found that various teaching approaches took better role than conventional teaching on improving those mathematics abilities. Actually, in those various innovative teaching approaches students were assisted more individually relevant to their need such as their cognitive stage. That statement connoted that various innovative teaching approaches of those studies which based on constructivism philosophy fitted to the wish of this study. Moreover, findings of those reported studies pointed out that indicators and their instruments on logical reasoning (Indonesian version of TOLT and Longeot Test), mathematical understanding and mathematical reasoning of this study were still relevant for being developed in further studies.

4. Conclusion

Many (55%) of students of 17,43 years old did not reach the formal operational stage yet, and only (45%) students were already informal operational stage were in line with Piaget’s hypothesis that “normal (not talented) students will reach formal operational stage at 12-13 years old up to 14-15 years old and in other condition between 18-20 years old”. The difference of percentage of the formal operational stage between 11th-grade students of social class and science and mathematics class supported Piaget’s opinion that achieving of formal operational stage depends on maturity, equilibrium, socialisation, and transmission of education and culture of the students. Moreover, the findings that on logical reasoning, mathematical understanding, mathematical reasoning, and mathematics achievement the grades of formal students which greater than the grades of concrete students were in line or firm with Piaget’s hypothesis that in solving HOT in mathematics (which need formal thinking) formal subjects were more capable than concrete subjects. Formal students attained fairly good grade on logical reasoning, mathematical understanding, mathematical reasoning, and on mathematics achievement, while concrete and transitional students obtained low grades and they realised many difficulties on solving high order thinking (HOT) of mathematics tasks. Logical reasoning ability was a good predictor for achieving HOT mathematics tasks.

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