We study some consequences of noncommutativity to homogeneous cosmologies by introducing a deformation of the commutation relation between the minisuperspace variables. The investigation is carried out for the Kantowski-Sachs model by means of a comparative study of the universe evolution in four different scenarios: the classical commutative, classical noncommutative, quantum commutative, and quantum noncommutative. The comparison is rendered transparent by the use of the Bohmian formalism of quantum trajectories. As a result of our analysis, we found that noncommutativity can modify significantly the universe evolution, but cannot alter its singular behavior in the classical context. Quantum effects, on the other hand, can originate non-singular periodic universes in both commutative and noncommutative cases. The quantum noncommutative model is shown to present interesting properties, as the capability to give rise to non-trivial dynamics in situations where its commutative counterpart is necessarily static.

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I. INTRODUCTION

Recently, there has been a great amount of work devoted to noncommutative theories (see, e.g., [1, 2]). The boom of interest in noncommutativity of the canonical type was triggered by works establishing its connection with string and M-theory [3], although previous investigations in the context of semiclassical gravity [4] have pointed out the relevance of noncommutative field theories. In addition to its relevance to string theory, the study of noncommutative theories is justified in its own by the opportunity it gives us to deal with interesting properties, such as the IR/UV mixing and nonlocality [3], Lorentz violation [6], and new physics at very short scale distances [1, 2, 7].

In the latest two years, several investigations have been carried out to clarify the possible role of noncommutativity in the cosmological scenario in a great variety of contexts. Among them, we quote the Newtonian cosmology [8], cosmological perturbation theory and noncommutative inflationary cosmology [9], noncommutative gravity [10], and quantum cosmology [11]. The latter, in particular, provides an interesting arena for speculation on the possible connection between noncommutativity and quantum gravity. The common claim that noncommutativity leads to fuzzyness renders obscure the application of noncommutative ideas to the description of a quantum universe, which, according to the Copenhagen interpretation of quantum theory, has no objective reality. Indeed, the inadequacy of the Copenhagen interpretation for quantum cosmology has been stressed long time ago by several physicists, as Everett [12], Feynman [13], and Bell [14]. Recently, 't Hooft [15] has argued that a reconsideration of hidden-variables theories is in turn necessary to account for the difficulties that appear in the unification of General Relativity with Quantum Theory.

Perhaps the great riddle of quantum gravity is the comprehension of the behavior of spacetime (if this concept has a meaning) at the Planck scale. At very early times, when the universe was small and hot, even when its characteristic length scale was larger than the Planckian one, noncommutativity may have played a relevant role in its evolution. The aim of this work is to exploit this possibility by carrying out a comparative study of the universe evolution in four different scenarios: classical commutative, classical noncommutative, quantum commutative, and quantum noncommutative. As our object of analysis, we chose the Kantowski-Sachs

*Electronic address: gbarbosa@cbpf.br
†Electronic address: nelsonpn@cbpf.br
universe (see, e.g., [16, 17, 18, 19, 20, 21]). A noncommutative version of the Kantowski-Sachs universe was previously considered in [11], where a Moyal deformation of the Wheeler-DeWitt equation in the minisuperspace approximation was introduced. In the present work, we explore another possibility that can be present in noncommutative quantum cosmology using the same framework pioneered in [11]. The noncommutative geometry considered here is a property of the minisuperspace observables which refer to the physical metric. This, as will be detailed later, entails an identification of the universe degrees of freedom as a set of variables that differs from the one adopted in [11]. As quoted in that reference, what renders the Kantowski-Sachs model attractive for investigation in the noncommutative context is the opportunity its noncommutative quantum version provides us to deal with nonperturbative effects of noncommutative geometry and quantum gravity comprised in the same model. This is assured by the analytic solutions admitted by the noncommutative Wheeler-DeWitt equation.

As an interpretation for quantum theory, we are adopting Bohm’s ontological one [22, 23, 24, 25]. Such an interpretation has also been employed in other works on quantum cosmology and quantum gravity (see, e.g., [26, 27, 28, 29]). In this work, with the aid of the Bohmian minisuperspace trajectories, we will show how it is possible to conceive a “noncommutative quantum universe” with minisuperspace operators satisfying a noncommutative algebra. We shall study the quantum universe as a well defined entity, without appeal to any external observer, which would find it “fuzzy” in a supposed measurement process. Independent of the orientation with respect to the foundations of quantum theory one may have, the Bohmian formalism of trajectories can always be adopted as a useful tool in providing an intuitive interpretation for the quantum phenomena. Indeed, the interest in the Bohmian approach is growing in a broad community (see, e.g., [30]).

The paper is organized as follows. In section 2 we summarize the essential aspects of canonical quantum gravity and Bohmian quantum physics necessary for the remaining sections. Sections 3, 4 and 5 are devoted to a comparative study of the commutative and noncommutative classical Kantowski-Sachs universes. In section 6 the commutative quantum version of the model is studied in the language of Bohmian trajectories. An extension of the Bohmian formalism is proposed and applied to the noncommutative quantum Kantowski-Sachs universe in section 7. In Section 8 we end up with a general discussion and summary of the main results.

II. QUANTUM GRAVITY AND THE BOHMIAN FORMALISM

Before introducing noncommutativity in the cosmological scenario, it is interesting to comment some aspects of the standard Hamiltonian and Bohmian quantum gravity formalisms.

The Hamiltonian of General Relativity is usually expressed in the ADM formulation [31]. In this formalism, the line element is written as

\[ ds^2 = \left( N_i N^i - N^2 \right) dt^2 + 2N_i dx^i dt + h_{ij} dx^i dx^j, \]

where \( N \) represents the lapse function, \( N^i \) is the shift vector, and \( h_{ij} \) is the three-metric of a three-surface embedded in spacetime. The dynamics of the spacetime is described in terms of the evolution of \( h_{ij} \) in superspace, the space of all three-geometries.

The Hamiltonian of General Relativity without matter is\(^1\)

\[ H = \int d^3x \left( N \mathcal{H} + N_j \mathcal{H}^j \right), \]

where

\[ \mathcal{H} = G_{ijkl} \Pi^i \Pi^j - h^{1/2} R^{(3)}, \quad \mathcal{H}^j = D_i \Pi^i. \]

Units are chosen such that \( \hbar = c = 16\pi G = 1 \). The quantity \( R^{(3)} \) is the intrinsic curvature of the spacelike hypersurfaces, \( D_i \) is the covariant derivative with respect to \( h_{ij} \), and \( h \) is the determinant of \( h_{ij} \). The momentum

\(^1\) We shall restrict all of our considerations to pure gravity, since this is the case of interest in this work.
\( \Pi_{ij} \) canonically conjugated to \( h^{ij} \), and the DeWitt metric \( G_{ijkl} \) are
\[
\Pi_{ij} = -h^{1/2} (K_{ij} - h_{ij} K),
\]
\[
G_{ijkl} = \frac{1}{2} h^{-1/2} (h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl}),
\]
where \( K_{ij} = -(\partial_i h_{ij} - D_i N_j - D_j N_i)/(2N) \) is the second fundamental form.

As long as one follows the Dirac quantization procedure, the constraints of the theory become conditions imposed on the possible states of the wavefunctional of the universe. The super-Hamiltonian constraint \( \mathcal{H} \approx 0 \) yields the Wheeler-DeWitt equation
\[
\left( G^{ijkl} \frac{\delta}{\delta h^{ij}} \frac{\delta}{\delta h^{kl}} + h^{1/2} R^{(3)} \right) \Psi[h^{ij}] = 0,
\]
which determines the evolution of the wavefunctional. Up to now, the implications of this equation to quantum cosmology are still under debate. Among the variety of technical and conceptual problems under discussion, there are the issue of time and the definition of probability (see [32, 33] and references therein). One way to circumvent them is by adopting the non-epistemological interpretation for quantum theory proposed by Bohm [22, 23, 24, 25].

In the Bohmian approach to quantum theory an ontology is given to the physical systems (particles, fields, etc.), which evolve continuously in time obeying a deterministic law of motion [22, 23, 24, 25]. When applying the Bohmian formalism in the study of three-space geometry evolution in quantum cosmology, we expect that the notion of space and time should have an objective meaning, in a similar way as the notion of trajectories has in Bohmian non-relativistic quantum mechanics [25]. Indeed, this is exactly the case in Bohmian quantum gravity, which has the evolution law for the three-space metric \( h_{ij} \) given by
\[
\Pi_{ij} = -h^{1/2} (K_{ij} - h_{ij} K) = \text{Re} \left\{ \frac{1}{\Psi^* \Psi} \left[ \Psi^* \left( -i \frac{\delta}{\delta h^{ij}} \right) \Psi \right] \right\} = \frac{\delta S}{\delta h^{ij}},
\]
where \( S \) is found by writing the wavefunctional in the polar form \( \Psi = A \exp(iS) \). An intuitive picture of the deviation of the classical behavior present in (7) may be constructed by substituting \( \Psi = A \exp(iS) \) in (6) and separating the real and imaginary parts. As a result, we obtain the equations
\[
G^{ijkl} \frac{\delta S}{\delta h^{ij}} \frac{\delta S}{\delta h^{kl}} - h^{1/2} R^{(3)} + Q = 0,
\]
\[
G^{ijkl} \frac{\delta S}{\delta h^{ij}} \left( A^2 \frac{\delta S}{\delta h^{kl}} \right) = 0,
\]
where
\[
Q = -\frac{1}{A} G^{ijkl} \frac{\delta^2 A}{\delta h^{ij} \delta h^{kl}}.
\]
Expression (8) can interpreted as a quantum Hamilton-Jacobi equation. The quantity \( Q \), absent in the classical Hamilton-Jacobi equation, is known as the quantum potential. It is responsible for the quantum effects present in the three-space geometry evolution. The classical limit of the theory is found in the regime where \( Q \) is negligible if compared with the other terms present in (8). When this is the case, the theory is clearly reduced to classical General Relativity in the Hamilton-Jacobi formulation. Note that, for the evolution law (9) to be consistent, the wavefunctional need not necessarily be normalizable. In Bohmian quantum gravity probability is a concept that is derived from the ontology of the theory.

\(^2\) A particular factor ordering was assumed in its derivation.
III. THE KANTOWSKI-SACHS UNIVERSE

The Kantowski-Sachs universe \[16\] is one of the most investigated anisotropic cosmological models. Part of the interest in this universe model is due to the wide set of analytical solutions it admits, even if particular types of matter are coupled to gravity. In other anisotropic models, such as the Bianchi IX, e.g., the computation of analytical solutions is a rather complicated task \[17\]. Several investigations of the Kantowski-Sachs model have been carried out recently in a great variety of contexts, such as braneworld cosmology \[18\], scalar field cosmology \[19\], and quantum cosmology \[17, 20\]. In addition to its cosmological relevance, the Kantowski-Sachs geometry might be useful in the description of the black holes. It has the same symmetries as the spatially homogeneous interior region of the extended vacuum Kruskal solution that represents the late stage of evolution of an isotropic black hole when the matter can be neglected. Indeed, a possible connection between the Kantowski-Sachs metric with quantum black holes and quantum wormholes has been proposed \[21\].

The Kantowski-Sachs line element is \[16\]

\[
ds^2 = -N^2 dt^2 + X^2(t) dv^2 + Y^2(t) \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) \, .
\]

In the Misner parametrization, \[11\] is written as

\[
ds^2 = -N^2 dt^2 + e^{2\sqrt{3}\beta} dv^2 + e^{-2\sqrt{3}\beta} e^{-2\sqrt{3}\Omega} \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) \, .
\]

From \[2\] and \[3\], the Hamiltonian of General Relativity for this metric is found to be

\[
H = N\mathcal{H} = N \exp \left( \sqrt{3}\beta + 2\sqrt{3}\Omega \right) \left[ -\frac{P_\beta^2}{24} + \frac{P_\varphi^2}{24} - 2 \exp \left( -2\sqrt{3}\Omega \right) \right] .
\]

A good characterization of the evolution of the spacetime metric \[12\] is provided by the study of its volume expansion \( \Theta \equiv V^\alpha_\alpha \) with respect to the comoving observer using proper time, \( V^\alpha = \delta^\alpha_0/N \), and the shear \( \sigma^2 = \sigma^{a\beta} \sigma_{a\beta}/2 \), where \( \sigma^{a\beta} = (h^a_\alpha h^\beta_\nu + h^a_\beta h^\nu_\alpha)V_{\mu\nu}/2 - \Theta b_{a\beta}/3 \). The semicolon stands for four-dimensional covariant derivative, and \( h^a_\alpha = \delta^a_\alpha + V^\mu V_\alpha \) is the projector orthogonal to the observer \( V^\mu \). A characteristic length scale \( l \) can also be defined in terms of the volume expansion through \( \Theta = 3/(lN) \). In the gauge \( N = 24 \exp \left( -\sqrt{3}\beta - 2\sqrt{3}\Omega \right) \), the volume expansion, shear, and characteristic volume for the Kantowski-Sachs metric read

\[
\Theta(t) = \frac{1}{N} \left( \frac{\dot{X}}{X} + 2\frac{\dot{Y}}{Y} \right) = -\frac{\sqrt{3}}{24} \left( \beta + 2\dot{\Omega} \right) \exp \left( \sqrt{3}\beta + 2\sqrt{3}\Omega \right) ,
\]

\[
\sigma(t) = \frac{1}{N\sqrt{3}} \left( \frac{\dot{X}}{X} - \frac{\dot{Y}}{Y} \right) = \frac{1}{24} \left( 2\beta + \dot{\Omega} \right) \exp \left( \sqrt{3}\beta + 2\sqrt{3}\Omega \right) ,
\]

\[
l^3(t) = X(t) Y^2(t) = \exp \left( -\sqrt{3}\beta(t) - 2\sqrt{3}\Omega(t) \right) .
\]

IV. COMMUTATIVE CLASSICAL MODEL

In order to distinguish individually the role of the quantum and noncommutative effects in our noncommutative quantum universe, it is interesting to start our study of the Kantowski-Sachs geometry from its commutative classical version, which will be our reference for comparison later. The Poisson brackets for the classical phase space variables are

\[
\{ \Omega, P_\Omega \} = 1, \quad \{ \beta, P_\beta \} = 1, \quad \{ P_\Omega, P_\beta \} = 0, \quad \{ \Omega, \beta \} = 0.
\]

For the metric \[12\], the super-Hamiltonian constraint \( \mathcal{H} \approx 0 \) is reduced to

\[
\mathcal{H} = \xi h \approx 0 ,
\]
where 

\[ \xi = \frac{1}{24} \exp \left( \sqrt{3} \beta + 2 \sqrt{3} \Omega \right), \quad h = -P_{\Omega}^2 + P_{\beta}^2 - 48 \exp \left( -2 \sqrt{3} \Omega \right) \approx 0. \]  

(17)

The classical equations of motion for the phase space variables \( \Omega, P_{\Omega}, \beta \) and \( P_{\beta} \) are

\[ \dot{\Omega} = N \{ \Omega, \mathcal{H} \} = -2 P_{\Omega}, \]

\[ \dot{P}_{\Omega} = N \{ P_{\Omega}, \mathcal{H} \} = -96 \sqrt{3} e^{-2 \sqrt{3} \Omega}, \]

\[ \dot{\beta} = N \{ \beta, \mathcal{H} \} = 2 P_{\beta}, \]

\[ \dot{P}_{\beta} = N \{ P_{\beta}, \mathcal{H} \} = 0, \]  

(18)

where we have used the constraint \( h \approx 0 \) and fixed the gauge \( N = \xi^{-1} = 24 \beta = 24 \exp \left( -\sqrt{3} \beta - 2 \sqrt{3} \Omega \right) \). From now on, including the sections on quantum cosmology, we shall restrict all of our considerations to this gauge. Note that this gauge choice does not correspond to the comoving cosmological time.

As solutions for \( \Omega \) and \( \beta \) we find

\[ \Omega(t) = \sqrt{3} \frac{6 \ln \left\{ \frac{48}{P_{\beta_0}^2} \cosh^2 \left[ 2 \sqrt{3} P_{\beta_0} (t - t_0) \right] \right\}}{6 \ln \left\{ \frac{48}{P_{\beta_0}^2} \cosh^2 \left[ 2 \sqrt{3} P_{\beta_0} (t - t_0) \right] \right\}}, \]

\[ \beta(t) = 2 P_{\beta_0} (t - t_0) + \beta_0. \]  

(19)

From (14) and (19) we can evaluate

\[ \Theta(t) = \frac{4 \sqrt{3}}{P_{\beta_0}} \left\{ \cosh^2 \left[ 2 \sqrt{3} P_{\beta_0} (t - t_0) \right] + \sinh \left[ 4 \sqrt{3} P_{\beta_0} (t - t_0) \right] \right\} \exp \left[ \sqrt{3} \left( 2 P_{\beta_0} (t - t_0) + \beta_0 \right) \right], \]

\[ \sigma(t) = \frac{2}{P_{\beta_0}} \left\{ 4 \cosh^2 \left[ 2 \sqrt{3} P_{\beta_0} (t - t_0) \right] + \sinh \left[ 4 \sqrt{3} P_{\beta_0} (t - t_0) \right] \right\} \exp \left[ \sqrt{3} \left( 2 P_{\beta_0} (t - t_0) + \beta_0 \right) \right]. \]  

(20)

\[ l^3(t) = \frac{P_{\beta_0}^2}{48} \frac{1}{\cosh^2 \left[ 2 \sqrt{3} P_{\beta_0} (t - t_0) \right]} \exp \left[ -\sqrt{3} \left( 2 P_{\beta_0} (t - t_0) - \beta_0 \right) \right]. \]

From (20), it can be seen that the universe volume expansion \( \Theta(t) \) decreases monotonically passing by zero at \( t = t_0 - \sqrt{3} \ln (3) / (12 P_{\beta_0}) \). The characteristic volume \( l^3(t) \) departs from zero at \( t = -\infty \), increases up to \( 3 \sqrt{3} P_{\beta_0}^2 \exp \left[ -\sqrt{3} \beta_0 \right] / 4 \) at \( t = t_0 - \sqrt{3} \ln (3) / 12 P_{\beta_0} \), where it achieves its maximum, and then decreases to zero\(^3\) at \( t = \infty \). The universe shape, on the other hand, departs from a highly asymmetric state, achieves a configuration of minimum anisotropy, and again return to a highly asymmetric condition. The typical behavior of \( \Theta(t), \sigma(t) \) and \( l^3(t) \) is depicted in the thick curves of Figs. 1a, b, and c for given values of \( P_{\beta_0} \).

\[ ^3 \text{It can be shown that the vanishing of } l^3(t) \text{ at } t = \pm \infty \text{ correspond to singularities achieved at finite values of the cosmic time} \]
FIG. 1. The typical behavior of the universe characteristic volume $l^3(t)$, volume expansion $\Theta(t)$, and shear $\sigma(t)$ in the commutative and noncommutative versions of the classical Kantowski-Sachs universe. The initial conditions chosen are $\beta_0 = -11, P_{\beta_0} = 2/5, t_0 = 0$. In the plots (a), (b) and (c): $\theta = 0$ (thick lines) and $\theta = 5$ (thin lines). In plot (d): $\theta = 0$ (thick line) and $\theta = -5$ (thin line).

V. NONCOMMUTATIVE CLASSICAL MODEL

The natural step to follow before the introduction of the quantum effects in the Kantowski-Sachs universe is the investigation of the implications of noncommutative geometry for the classical version of this model. Let us introduce a noncommutative classical geometry in the model by considering a Hamiltonian that has the same functional form as (13) but is valued on variables that satisfy the deformed Poisson brackets

$$\{\Omega, P_\Omega\} = 1, \quad \{\beta, P_\beta\} = 1, \quad \{P_\Omega, P_\beta\} = 0, \quad \{\Omega, \beta\} = \theta.$$  

The solutions for $\Omega(t)$ and $\beta(t)$ are

$$\Omega(t) = \frac{\sqrt{3}}{6} \ln \left\{ \frac{48}{P_{\beta_0}^2} \cosh^2 \left[ 2\sqrt{3} P_{\beta_0} (t - t_0) \right] \right\},$$

$$\beta(t) = 2P_{\beta_0} (t - t_0) + \beta_0 - \theta P_{\beta_0} \tanh \left[ 2\sqrt{3} P_{\beta_0} (t - t_0) \right].$$

(23)
This is the most direct way to obtain the metric variables. A parallel with the calculation procedure adopted in the quantum case, however, is rendered easier by the use of the auxiliary canonical variable formalism. We shall therefore present this alternative approach below.

Instead of working directly with the physical variables $\Omega$ and $\beta$, we may achieve the solutions above by making use of the auxiliary canonical variables $\Omega_c$ and $\beta_c$, defined as

$$\Omega_c = \Omega + \frac{\theta}{2} P_\beta, \quad \beta_c = \beta - \frac{\theta}{2} P_\Omega,$$  \hspace{1cm} (24)

The Poisson brackets for these variables are

$$\{\Omega_c, P_{\Omega_c}\} = 1, \quad \{\beta_c, P_{\beta_c}\} = 1, \quad \{P_{\Omega_c}, P_{\beta_c}\} = 0, \quad \{\Omega_c, \beta_c\} = 0.$$  \hspace{1cm} (25)

As the equations of motion in the canonical formalism in the gauge $N = 24 \exp (-\sqrt{3} \beta - 2 \sqrt{3} \Omega)$ we have

$$\ddot{\Omega}_c = -2P_{\Omega_c},$$

$$\dot{P}_{\Omega_c} = -96\sqrt{3}e^{-2\sqrt{3}\Omega},$$

$$\ddot{\beta}_c = 2P_{\beta_c} - 48\sqrt{3} \theta e^{-2\sqrt{3}\Omega},$$

$$\dot{P}_{\beta_c} = 0,$$  \hspace{1cm} (26)

whose solutions are

$$\Omega_c(t) = \frac{\sqrt{3}}{6} \ln \left\{ \frac{48}{P_{\beta_0}^2} \cosh^2 \left[ 2\sqrt{3} P_{\beta_0} (t - t_0) \right] + \frac{\theta}{2} P_{\beta_0} \right\},$$

$$\beta_c(t) = 2P_{\beta_0} (t - t_0) + \beta_0 - \frac{\theta}{2} P_{\beta_0} \tanh \left[ 2\sqrt{3} P_{\beta_0} (t - t_0) \right],$$

$$P_{\Omega_c}(t) = -P_{\beta_0} \tanh \left[ 2\sqrt{3} P_{\beta_0} (t - t_0) \right],$$

$$P_{\beta_c}(t) = P_{\beta_0}.$$  \hspace{1cm} (27)

Finally, from (24) and (26) we can recover the solution (23).

Having the noncommutative $\Omega(t)$ and $\beta(t)$ from (24), we can evaluate the expression for $\Theta(t), \sigma(t)$ and $l^3(t)$ as

$$\Theta(t) = \left\{ -\frac{4\sqrt{3}}{P_{\beta_0}} \left( \cosh^2 \left[ 2\sqrt{3} P_{\beta_0} (t - t_0) \right] + \sinh \left[ 4\sqrt{3} P_{\beta_0} (t - t_0) \right] \right) + 4\sqrt{3} \theta \right\} \times \exp \left[ \sqrt{3} \left( 2P_{\beta_0} (t + \beta_0) - \theta P_{\beta_0} \tanh \left( 2\sqrt{3} P_{\beta_0} (t - t_0) \right) \right) \right],$$

$$\sigma(t) = \left\{ \frac{2}{P_{\beta_0}} \left( 4 \cosh^2 \left[ 2\sqrt{3} P_{\beta_0} (t - t_0) \right] + \sinh \left[ 4\sqrt{3} P_{\beta_0} (t - t_0) \right] \right) - 4\sqrt{3} \theta \right\} \times \exp \left[ \sqrt{3} \left( 2P_{\beta_0} (t + \beta_0) - \theta P_{\beta_0} \tanh \left( 2\sqrt{3} P_{\beta_0} (t - t_0) \right) \right) \right],$$

$$l^3(t) = \frac{P_{\beta_0}^2}{48} \sech^2 \left[ 2\sqrt{3} P_{\beta_0} (t - t_0) \right] \exp \left[ \sqrt{3} \left( 2P_{\beta_0} (t + \beta_0) - \theta P_{\beta_0} \tanh \left( 2\sqrt{3} P_{\beta_0} (t - t_0) \right) \right) \right].$$  \hspace{1cm} (28)

We start our comparison with the previous case by studying the behavior of the characteristic volume $l^3(t)$. As it can be seen from the expression for $l^3(t)$ in (28), the singular behavior of the classical noncommutative Kantowski-Sachs universe at $t = \pm \infty$ is the same as that of the classical commutative analog. The $\theta$ contribution in the expression for $\beta(t)$ in (28) does not modify the behavior of $l^3(t)$ in the infinite past or future. Instead,
its contribution is relevant near $t = t_0$, where the hyperbolic tangent varies fast. Near this time, the behavior of $l^3(t)$, $\Theta(t)$ and $\sigma(t)$ can differ appreciably from the commutative case, as it is shown in the thin curves of Figs. 1a, b, and c. While the difference in the behavior of $l^3(t)$ and $\Theta(t)$ is only qualitative, in shear function it is quantitative. The noncommutative universe can change its expansion directions, and become two times isotropic before being similar in shape to its commutative counterpart at late times. By varying the initial conditions $\beta_0$ and $P_{\beta_0}$, the deviation from the commutative behavior can become very large for $l^3(t)$, $\Theta(t)$ and $\sigma(t)$. For $\beta_0 = 0$, $P_{\beta_0} = 3$ and $\theta = 5$, e.g., we have $l^3_{\text{max}} \simeq 1.55 \cdot 10^8$, while for the same initial conditions in the commutative analog we have $l^3_{\text{max}} \simeq 0.24$. Other interesting aspect of the noncommutative universe solution is its dependence on the sign of the $\theta$ parameter. Fig. 1d presents a plot of $l^3(t)$ with the same initial conditions previously adopted in Fig. 1a, but with the $\theta$ sign inverted. As it can be seen from the figure, the $l^3(t)$ width and maximum value are considerably enlarged with respect to the similar curve of Fig. 1a. Therefore, depending on the initial conditions and on the $\theta$ sign, it is possible to obtain a appreciable deviation from the commutative behavior even with smaller values of $\theta$.

VI. COMMUTATIVE QUANTUM MODEL

Now let us consider the quantum Kantowski-Sachs model. Due to the technical difficulty in dealing with the quantum cosmology is usually based on the minisuperspace construction of homogeneous models\cite{32, 33}. With this approach, it is possible to access a nonperturbative sector of quantum gravity by paying the price of freezing out infinite degrees of freedom. For that, an ansatz of the type of \cite{12} is introduced in \cite{30}, and the spatial dependence of the metric is integrated out. The Wheeler-DeWitt equation is thereby reduced to a Klein-Gordon equation. For the Kantowski-Sachs universe, such an equation is\footnote{We are assuming a particular factor ordering.}

\begin{equation}
\left[-\hat{P}_\Omega^2 + \hat{P}_\beta^2 - 48 \exp\left(-2\sqrt{3}\Omega\right)\right] \Psi(\Omega, \beta) = 0, \tag{29}
\end{equation}

where $\hat{P}_\Omega = -i\partial/\partial\Omega$ and $\hat{P}_\beta = -i\partial/\partial\beta$. A solution to equation \footnote{Since the index in $\nu$ is continuous, in the most general case the sum can be replaced by an integral.} \footnote{This is not the case beyond minisuperspace. For details see \cite{32}.} is \footnote{This is not the case beyond minisuperspace. For details see \cite{32}.}

\begin{equation}
\Psi(\Omega, \beta) = e^{i\nu\sqrt{3}\beta} K_{i\nu} \left(4 e^{-\sqrt{3}\Omega}\right), \tag{30}
\end{equation}

where $K_{i\nu}$ is a modified Bessel function and $\nu$ is a real constant. Once a quantum state of the universe is given, as, e.g., a superposition of states\footnote{This is not the case beyond minisuperspace. For details see \cite{32}.}

\begin{equation}
\Psi(\Omega, \beta) = \sum_{\nu} C_{\nu} e^{i\nu\sqrt{3}\beta} K_{i\nu} \left(4 e^{-\sqrt{3}\Omega}\right) = R e^{iS}, \tag{31}
\end{equation}

the universe evolution can be determined by integrating the guiding equation \footnote{This is not the case beyond minisuperspace. For details see \cite{32}.}. In the minisuperspace approach, the analog of that equation is

\begin{align}
P_\Omega &= -\frac{1}{2}\hat{\Omega} = \text{Re} \left\{ \frac{[\Psi^* (-i\hbar \partial_\Omega)] \Psi}{\Psi^* \Psi} \right\} = \frac{\partial S}{\partial \Omega}, \\
P_\beta &= \frac{1}{2}\hat{\beta} = \text{Re} \left\{ \frac{[\Psi^* (-i\hbar \partial_\beta)] \Psi}{\Psi^* \Psi} \right\} = \frac{\partial S}{\partial \beta}. \tag{32}
\end{align}

As before, we have fixed the gauge $N = 24l^3 = 24 \exp\left(-\sqrt{3}\beta - 2\sqrt{3}\Omega\right)$. Usually, different choices of time yield different quantum theories\footnote{This is not the case beyond minisuperspace. For details see \cite{32}.}. However, when one uses the Bohmian interpretation in minisuperspace models the situation is identical to that of the classical case\footnote{This is not the case beyond minisuperspace. For details see \cite{32}.}. different choices yield the same theory\footnote{This is not the case beyond minisuperspace. For details see \cite{32}.}. Hence,
as long as $l^3(t)$ does not pass through zero (which would mean that the universe has reached a singularity) the
above choice for $N(t)$ is valid for the description of all the universe history.

The minisuperspace analog of the Hamilton Jacobi equation \( S \) is

\[
-\frac{1}{24} \left( \frac{\partial S}{\partial \Omega} \right)^2 + \frac{1}{24} \left( \frac{\partial S}{\partial \beta} \right)^2 - 2e^{-2\sqrt{3}\Omega} + \frac{1}{24R} \left( \frac{\partial^2 R}{\partial \Omega^2} - \frac{\partial^2 R}{\partial \beta^2} \right) = 0.
\]  

(33)

In what follows, we shall consider some wavefunctions and apply the Bohmian formalism to investigate the
properties of the universe they represent by means of quantum trajectories.

### A. Case 1

The wavefunction is of the type \( \Psi(\Omega, \beta) \). Since the Bessel function $K_{i\nu}(x)$ is real for \( \nu \) real and $x > 0$ \( \Re \), the
phase can be read directly from the exponential in \( \Psi(\Omega, \beta) \): \( S = \nu \sqrt{3} \beta \). The equations of motion are therefore

\[
\dot{\Omega} = 0, \quad \dot{\beta} = 2\sqrt{3} \nu,
\]

(34)

whose solutions are

\[
\Omega = \Omega_0, \quad \beta = 2\sqrt{3} \nu (t - t_0) + \beta_0.
\]

(35)

By substituting \( \beta \) into \( \Theta(t), \sigma(t), \) and \( l^3(t) \) as

\[
\Theta(t) = -\frac{\nu}{4} \exp \left[ 6\nu (t - t_0) + 2\sqrt{3} \Omega_0 + \sqrt{3} \beta_0 \right],
\]

\[
\sigma(t) = \frac{\sqrt{3} \nu}{6} \exp \left[ 6\nu (t - t_0) + 2\sqrt{3} \Omega_0 + \sqrt{3} \beta_0 \right],
\]

(36)

\[
l^3(t) = \exp \left[ -6\nu (t - t_0) - 2\sqrt{3} \Omega_0 - \sqrt{3} \beta_0 \right].
\]

From \( \Theta(t) \) it is easy to see that, according to the sign of \( \nu \), there are two possibilities for the universe evolution.
The first (\( \nu > 0 \)) corresponds to a universe that starts with infinitely large and isotropic volume in the remote
past and evolves contracting to a configuration of small and distorted volume. The second (\( \nu > 0 \)) is a universe
whose volume is infinitely small and distorted in the remote past, and evolves expanding to a large and isotropic
configuration in the infinite future. This qualitative different behavior from the classical counterpart can be intuitively understood by evaluating the quantum potential. From \( Q \) and \( \Theta(t) \) we can calculate

\[
Q = \frac{1}{24R} \left( \frac{\partial^2 R}{\partial \Omega^2} - \frac{\partial^2 R}{\partial \beta^2} \right) = 2e^{-2\sqrt{3}\Omega} - \frac{\nu^2}{8}.
\]

(37)

Since \( Q \) does not depend on \( \beta \), we expect this variable to have a classical behavior, while \( \Omega \) should encode all the quantum effects. This is the main reason for the solution for \( \beta \) in \( l^3(t) \) if we identify $P_{\beta_0} = \sqrt{3} \nu$, while $\Omega = \Omega_0$ is radically different.

### B. Case 2

The wavefunction is a superposition of two solutions of the type \( \Psi(\Omega, \beta) \)

\[
\Psi(\Omega, \beta) = A_1 K_{i\mu} \left( 4e^{-\sqrt{3}\Omega} \right) e^{i\sqrt{3}\mu \beta} + A_2 K_{i\nu} \left( 4e^{-\sqrt{3}\Omega} \right) e^{i\sqrt{3}\nu \beta}.
\]

(38)

The corresponding phase is

\[
S(\Omega, \beta) = \arctan \left[ \frac{A_1 K_{i\mu} \left( 4e^{-\sqrt{3}\Omega} \right) \sin (\sqrt{3} \mu \beta) + A_2 K_{i\nu} \left( 4e^{-\sqrt{3}\Omega} \right) \sin (\sqrt{3} \nu \beta)}{A_1 K_{i\mu} \left( 4e^{-\sqrt{3}\Omega} \right) \cos (\sqrt{3} \mu \beta) + A_2 K_{i\nu} \left( 4e^{-\sqrt{3}\Omega} \right) \cos (\sqrt{3} \nu \beta)} \right],
\]

(39)
where the $A_1$ and $A_2$ are chosen as real coefficients. The equations of motion for this state are

$$\frac{d\Omega}{dt} = 8\sqrt{3} \frac{A_1 A_2 \left[K''_{\mu}K_{\nu} - K'_{\mu}K_{\nu}''\right] \exp \left[-\sqrt{3}\Omega\right] \sin \left[\sqrt{3} (\mu - \nu) \beta\right]}{(A_1K_{\mu})^2 + (A_2K_{\nu})^2 + 2A_1A_2K_{\mu}K_{\nu}\cos \left[\sqrt{3} (\mu - \nu) \beta\right]},$$

$$\frac{d\beta}{dt} = 2\sqrt{3} \frac{\mu A_1^2 K_{\mu}^2 + \nu A_2^2 K_{\nu}^2 + (\mu + \nu) A_1 A_2 K_{\mu} K_{\nu}\cos \left[\sqrt{3} (\mu - \nu) \beta\right]}{(A_1K_{\mu})^2 + (A_2K_{\nu})^2 + 2A_1A_2K_{\mu}K_{\nu}\cos \left[\sqrt{3} (\mu - \nu) \beta\right]},$$

where prime means derivative with respect to the argument.

The system constitutes an autonomous set of nonlinear coupled differential equations. Although it is hard to solve analytically this system, the global properties of the solutions can be easily grasped by considering the associated field of velocities. A first inspection on the RHS of (40) reveals that the velocity field has its direction inverted by the replacement $\mu \rightarrow -\mu$, $\nu \rightarrow -\nu$. Therefore, to have a qualitative picture of the velocity field, it is sufficient to consider $\mu > 0$ and study the cases where $\nu > 0$ and $\nu < 0$. For simplicity, let us fix $A_1 = A_2 = 1/\sqrt{2}$. The most interesting of the two cases, $\nu < 0$, gives rise to the velocity field that is depicted normalized in Fig. 2.

FIG. 2. The normalized field of velocities corresponding to the Bohmian differential equations for the commutative Kantowski-Sachs universe with $\mu = 1/10$ and $\nu = -1/5$.

A simple inspection on (40) suggests that there should exist stability points appearing periodically with period $2\pi/\sqrt{3} |\mu - \nu|$ along the $\beta$ direction. Indeed, for the values given to $\mu$ and $\nu$ in the plots of Fig. 2 we have $2\pi/\sqrt{3} |\mu - \nu| = 12.09$, which matches exactly with the observed period of appearance of the stability points along $\beta$ direction.

By varying the initial conditions and the values of $\mu$ and $\nu$, we can find a great variety of solutions of (40). In what follows we shall exhibit some of them, giving preference for the ones which correspond to non-singular universes.

From the field plot of Fig. 2 we can obtain information about the qualitative behavior of the solutions for $l^i(t)$, $\Theta(t)$ and $\sigma(t)$. If $\Omega(t)$ and $\beta(t)$ are periodic, equations (4) show that $l^i(t)$, $\Theta(t)$ and $\sigma(t)$ have the same
behavior. This is the solution type we expect to find when integrating the system \( \text{(10)} \) around the stability point near \( \Omega = 6 \) and \( \beta = -24 \), for example. In Fig. 2 it can also be seen, from the flow emerging around \( \Omega = 11 \) or \( \Omega = 27 \), that there exist solutions monotonically increasing in \( \beta \) and oscillatory in \( \Omega \). In addition to the periodic universes, we therefore expect to find universe solutions that contract (expand) with \( l^3(t) \) passing by a sequence of bounces and becoming singular in the infinite future (past).

We start our numerical study with the periodic solutions. In order to find them, we shall focus our attention on the vertical straight line that crosses the \( \Omega \) axes near \( \Omega = 6 \) in Fig. 2. Without loss of generality, let us consider the stability point located near \( \Omega = 6 \) and \( \beta = -24 \). By solving \( \text{(10)} \) numerically with initial conditions \( \Omega(0) = 2 \) and \( \beta(0) = -24 \) and computing the respective \( \ln[l^3(t)] \), \( \Theta(t) \) and \( \sigma(t) \), we verify that these three quantities indeed present a periodic behavior (Figs. 3a, c, and e). An interesting information that can be read directly from the plot of \( \ln[l^3(t)] \) is the number of \( e \)-folds between the maximum and minimum universe volumes (\( \simeq 29 \) \( e \)-folds). It is easy to see that the number of \( e \)-folds and the value of the minimum volume can be adjusted by varying the initial conditions or changing the stability point. For a larger number of \( e \)-folds, it is enough to enlarge the orbit radius by choosing \( \Omega(0) \) and \( \beta(0) \) appropriately.

By studying the \( l^3(t) \) function, we can know how many times the universe volume is larger than the Planckian volume \( l_p^3 \). In the unit system adopted, \( l_p^3 \sim 10^{-33} \). From \( \text{(10)} \) we can see that \( l^3_{\text{min}} \) can be rendered larger (smaller) if we decrease (increase), e.g., \( \beta_{\text{max}} \). This is accomplished by changing to a similar orbit around a stability point immediately below (above) moving vertically along the \( \beta \) direction. If one keeps \( \Omega(0) \) unaltered and decreases \( \beta(0) \) by the spacing between the orbit centers, the difference in \( \beta_{\text{max}} \) will be decreased or increased about 12. The corresponding decrease or increase in \( l^3_{\text{min}} \) will therefore be about \( \exp[12\sqrt{3}] \simeq 21 \) \( e \)-folds.

In addition to \( l^3(t) \), the variables \( \Theta(t) \) and \( \sigma(t) \) provide relevant information about the universe behavior during each of its periodic cycles. A general inspection on Fig. 3 reveals that the universe in question alternates between configurations of large volume, almost uniform shape and small volume expansion, with configurations of small volume, distorted shape and large volume expansion. In each of its cycles the universe is isotropic at two times (Figs 3g and h).

Let us now turn our attention to the flow emerging around \( \Omega = 11 \) or \( \Omega = 27 \) in Fig. 2. The solutions in these regions correspond to universes that start at \( t = -\infty \) with infinite volume and contract passing by a sequence of bounces up to a singularity, as is shown in the logarithmic plot of \( l^3(t) \) in Fig. 3b. The logarithms of the corresponding volume expansion and shear appear plotted in Figs. 3d and 3f, respectively. In the same way as in the periodic solutions, the regions where the universe is small correspond exactly to the ones of maximum anisotropy. This is verified by moving from the local minimums at Fig. 3b and moving down vertically to arrive near the local maximums at Fig. 3d. The small creases in the top of the picks in Fig. 3d account for the abrupt change in direction of the expansion that occur in each of the bouncing regions. As stated before, the velocity field in Fig. 2, has its direction inverted whenever the replacement \( \mu \to -\mu \), \( \nu \to -\nu \) is made. We can use this property to construct an expanding solution where \( l^3(t) \) starts from a singularity and increases up to infinity passing by a sequence of bounces from the solution depicted in Figs. 3b, d and f.

Another different solution type is present in the case where \( \nu = -\mu \). The phase of such a state is \( S = 0 \). We have therefore an static universe of arbitrary size, a genuinely quantum behavior. Indeed the quantum potential for this state,

\[
Q = 2e^{-2\sqrt{3}\Omega},
\]
cancels exactly the classical potential. This justifies the highly nonclassical behavior observed.
FIG. 3. The evolution of the universe characteristic volume, volume expansion, and shear in the commutative quantum Kantowski-Sachs universe with $\mu = 1/10$ and $\nu = -1/5$. (a), (c), and (e) : $\Omega(0) = 2, \beta(0) = -24$. (b), (d), and (f) : $\Omega(0) = 7.7, \beta(0) = -21.7$. (g) and (h) : enlarged plots of parts of (e).

All the solutions discussed in this section are interesting in their own by the mathematically allowed universes they represent. From the phenomenological point of view, however, it is interesting to know that there exist non-
singular (periodic) dynamic solutions that can account for our expectation that the minimum length achieved by the universe be larger than the Planckian one. Near this length scale, the effective quantum gravity theory based on the Wheeler-DeWitt equation is no more expected to be valid. For the solution plotted in Fig. 3a, e.g., $l_{\text{min}} \simeq 20 \sim 200 l_p$.

VII. THE NONCOMMUTATIVE QUANTUM MODEL

Having studied the quantum and noncommutativity effects in isolated examples, we now have joined elements to analyze the combination of them, which is realized in the noncommutative quantum model. In the construction of this model, we shall follow a prescription similar to that proposed in [11], where a canonical deformation in the algebra of the minisuperspace operators is introduced. As in the ordinary quantum mechanics, such a kind of deformation is usually defined with respect to a “preferred” set of Cartesian coordinates, where the noncommutative parameter is taken to be constant. As this preferred set, we shall take the one constituted by the configuration variables $\Omega$ and $\beta$,

$$[\hat{\Omega}, \hat{\beta}] = i\theta. \quad (41)$$

According to the Weyl quantization procedure [2, 3], the realization of the commutation relation (41) between the observables $\hat{\Omega}$ and $\hat{\beta}$ in terms of commutative functions is made by the Moyal star product, defined as below

$$f(\Omega_c, \beta_c) \star g(\Omega_c, \beta_c) = f(\Omega_c, \beta_c) e^{i\frac{\theta}{2}(\partial_{\Omega_c} \beta_c - \partial_{\beta_c} \Omega_c)} g(\Omega_c, \beta_c). \quad (42)$$

The commutative coordinates $\Omega_c$ and $\beta_c$ are called Weyl symbols of the operators $\hat{\Omega}$ and $\hat{\beta}$, respectively. The notation here is suggestive because these symbols match exactly with the canonical variables as defined by (24), which may be seen directly from the properties of the Moyal product.

In order to compare evolutions with the same time parameter as in the previous cases, we again fix the gauge $N = 24 \exp(-\sqrt{3} \beta - 2\sqrt{3} \Omega)$ in (13). The Wheeler-DeWitt equation for the noncommutative Kantowsky-Sachs model is [11]

$$\left[-\hat{P}^2_{\Omega_c} + \hat{P}^2_{\beta_c} - 48 \exp(-2\sqrt{3} \Omega_c)\right] \star \Psi(\Omega_c, \beta_c) = 0, \quad (43)$$

which is the Moyal deformed version of (29). By using the properties of the Moyal product, it is possible to write the potential term (which we denote by $V$ to include the general case) as

$$V(\Omega_c, \beta_c) \star \Psi(\Omega_c, \beta_c) = V\left(\Omega_c + i\frac{\theta}{2} \partial_{\beta_c}, \beta_c - i\frac{\theta}{2} \partial_{\Omega_c}\right) \Psi(\Omega_c, \beta_c) = V\left(\hat{\Omega}, \hat{\beta}\right) \Psi(\Omega_c, \beta_c) \quad (44)$$

where

$$\hat{\Omega} = \hat{\Omega}_c - \frac{\theta}{2} \hat{P}_{\beta_c}, \quad \hat{\beta} = \hat{\beta}_c + \frac{\theta}{2} \hat{P}_{\Omega_c}. \quad (45)$$

Equation (45) is nothing but the operatorial version of equation (24). The Wheeler-DeWitt equation then reads

$$\left[-\hat{P}^2_{\Omega_c} + \hat{P}^2_{\beta_c} - 48 \exp(-2\sqrt{3} \Omega_c + \sqrt{3} \theta \hat{P}_{\beta_c})\right] \Psi(\Omega_c, \beta_c) = 0. \quad (46)$$

Two consistent interpretations for the cosmology which emerges from equations (11) - (24) are possible. The first consists in considering the Weyl symbols $\Omega_c$ and $\beta_c$ as the constituents of the physical metric. In this
case the theory is essentially commutative with a modified interaction. In the second interpretation, which is adopted, e.g., in \[37, 38\], the Weyl symbols are considered as auxiliary coordinates, in the same way as in the classical case discussed in the previous section. Such an interpretation is closer to the spirit of this work, which is to study the evolution of a noncommutative quantum universe. Since it is the algebra of $\hat{\Omega}$ and $\hat{\beta}$, rather than the algebra of $\hat{\Omega}_c$ and $\hat{\beta}_c$, that satisfies (41), we shall interpret the $\hat{\Omega}$ and $\hat{\beta}$ as the operators associated with the physical metric. Moreover, the adoption of $\hat{\Omega}$ and $\hat{\beta}$ as the operators associated with the physical metric is also in accordance with the Dirac quantization procedure
\[
\{ , \} \rightarrow \frac{1}{i} [ , ],
\]
if one departs from the noncommutative classical analog discussed before. In the context of ordinary quantum mechanics, the two points of view provide the same energy spectrum, which is the physical quantity calculated in many works (see, e.g., \[39\]). However, as long as one wants to give an ontology to the theory (to circumvent fundamental problems of quantum cosmology), a precise specification of the objects the theory refers to must be made.

### A. Bohmian Formalism for Noncommutative Minisuperspaces

In order to go on in our comparative study of the Kantowski-Sachs universe, it is important to develop a Bohmian formulation for noncommutative quantum cosmology. This will be carried out here in the context of the minisuperspace formalism, which is our systematic tool in the study of quantum cosmological models.

In Bohmian noncommutative quantum cosmology we want to deal with a formalism that allow us to trace a clear picture of the universe evolution in a similar way as in the commutative quantum cosmology. One could ask how is this possible, since we are dealing with noncommutative coordinates that satisfy (41). The answer is that the operatorial formalism of quantum mechanics with operators acting in a Hilbert space of states is not a primary concept in Bohmian quantum mechanics. This is exactly one of the features of Bohmian formalism that renders it interesting for application in quantum cosmology, where there is no external observer. In commutative Bohmian quantum mechanics it is possible to describe particles with well defined position and momentum at each instant of time, although their position and momentum operators satisfy (for details see \[25\])
\[
[\hat{x}^i, \hat{p}^j] = i\hbar \delta^{ij}.
\]
Thus, it is reasonable to expect that in Bohmian noncommutative quantum cosmology it should be possible to describe the metric variables as well defined entities, although the operators $\hat{\Omega}$ and $\hat{\beta}$ satisfy (41). Indeed, this is exactly the case in the formulation proposed here.\footnote{For a related work on the Bohmian interpretation in the context of ordinary non-relativistic quantum mechanics see \[38\].}

The key ingredients in our Bohmian formalism are the wavefunction, which contain information about the universe evolution, and the metric variables $\Omega$ and $\beta$. To these quantities we want to give an objective meaning. The wavefunction can be obtained by solving (43). What is missing therefore is the evolution law for $\Omega$ and $\beta$. A simple and direct way to find this evolution law is by extending the formalism of section 6 employing the mapping described below.

To the Hermitian operator $\hat{A}(\hat{\Omega}_c, \hat{\beta}_c, \hat{P}_{\hat{\Omega}_c}, \hat{P}_{\hat{\beta}_c})$ it is possible to associate a function $A(\Omega_c, \beta_c)$ according to the rule
\[
\hat{A} \rightarrow \text{Re} \left[ \frac{\Psi^*(\Omega_c, \beta_c) \hat{A} (\Omega_c, \beta_c, -i\hbar \partial_{\Omega_c}, -i\hbar \partial_{\beta_c}) \Psi (\Omega_c, \beta_c)}{\Psi^*(\Omega_c, \beta_c) \Psi (\Omega_c, \beta_c)} \right] = A(\Omega_c, \beta_c),
\]
where the real value was taken to account for the hermiticity of $\hat{A}$. The operation \[48\] could be called “beable
with the beables associated with their time evolution. In our time gauge Hamiltonian $A_{\Omega}$ a solution to (43) is \[11\]

Once a quantum state of the universe is given as, e.g., a superposition of states $\hat{\Psi}(\Omega_{c}, \beta_{c})$, the beables corresponding to the operators $\Omega$ and $\beta$ we find

\[\Omega(\Omega_{c}, \beta_{c}) = B[\Omega] = \frac{\text{Re} \left[ \Psi^{\ast}(\Omega_{c}, \beta_{c}) \hat{\Omega}(\Omega_{c}, -i\hbar \partial_{\beta_{c}}) \Psi(\Omega_{c}, \beta_{c}) \right]}{\Psi^{\ast}(\Omega_{c}, \beta_{c}) \Psi(\Omega_{c}, \beta_{c})} = \Omega_{c} - \frac{\theta}{2} \partial_{\beta_{c}} S \tag{49}\]

\[\beta(\Omega_{c}, \beta_{c}) = B[\beta] = \frac{\text{Re} \left[ \Psi^{\ast}(\Omega_{c}, \beta_{c}) \hat{\beta}(\beta_{c}, -i\hbar \partial_{\Omega_{c}}) \Psi(\Omega_{c}, \beta_{c}) \right]}{\Psi^{\ast}(\Omega_{c}, \beta_{c}) \Psi(\Omega_{c}, \beta_{c})} = \beta_{c} + \frac{\theta}{2} \partial_{\Omega_{c}} S. \tag{50}\]

The strategy to find $\Omega(t)$ and $\beta(t)$ now becomes clear. The relevant information for universe evolution can be extracted from the guiding wave $\Psi(\Omega_{c}, \beta_{c})$ by first computing the associated canonical position tracks $\Omega_{c}(t)$ and $\beta_{c}(t)$. After that, we can obtain $\Omega(t)$ and $\beta(t)$ by evaluating (49) and (50) at $\Omega_{c} = \Omega_{c}(t)$ and $\beta_{c} = \beta_{c}(t)$, in a close similarity with the procedure of the second route to calculate $\Omega(t)$ and $\beta(t)$ in section 5.

Differential equations for the canonical positions $\Omega_{c}(t)$ and $\beta_{c}(t)$ may be found by identifying $\Omega_{c}(t)$ and $\beta_{c}(t)$ with the beables associated with their time evolution. In our time gauge $N = 24 \exp(-\sqrt{3} \beta - 2\sqrt{3} \Omega)$, the Hamiltonian $H = N \xi \hbar$, with $\xi$ and $\hbar$ defined in Eq. (17), reduces simply to $\hbar$. We can therefore use $\hbar$ to generate time displacements and obtain the equations of motion for $\Omega_{c}(t)$ and $\beta_{c}(t)$ as

\[\dot{\Omega}_{c}(t) = B \left( \frac{1}{i} \hat{\Omega}_{c}, \hbar \right) \bigg|_{\Omega_{c} = \Omega_{c}(t), \beta_{c} = \beta_{c}(t)} = -2 \frac{\partial S}{\partial \Omega_{c}} \bigg|_{\Omega_{c} = \Omega_{c}(t), \beta_{c} = \beta_{c}(t)}, \tag{51}\]

\[\dot{\beta}_{c}(t) = B \left( \frac{1}{i} \hat{\beta}_{c}, \hbar \right) \bigg|_{\Omega_{c} = \Omega_{c}(t), \beta_{c} = \beta_{c}(t)} = \left[ 2 \frac{\partial S}{\partial \beta_{c}} - 48 \sqrt{3} \theta \text{Re} \left\{ \exp \left( -2\sqrt{3} \Omega_{c} - i\theta \sqrt{3} \beta_{c} \right) \left( R e^{iS} \right) \right\} \right] \bigg|_{\Omega_{c} = \Omega_{c}(t), \beta_{c} = \beta_{c}(t)}. \tag{52}\]

As long as $\Omega_{c}(t)$ and $\beta_{c}(t)$ are known, the minisuperspace trajectories are given by

\[\Omega(t) = \Omega_{c}(t) - \frac{\theta}{2} \partial_{\beta_{c}} S [\Omega_{c}(t), \beta_{c}(t)], \tag{53}\]

\[\beta(t) = \beta_{c}(t) + \frac{\theta}{2} \partial_{\Omega_{c}} S [\Omega_{c}(t), \beta_{c}(t)]. \tag{54}\]

A solution to (43) is \[11\]

\[\Psi_{\nu}(\Omega_{c}, \beta_{c}) = e^{i\nu \sqrt{3} \beta_{c}} K_{iv} \left\{ 4 \exp \left[ -\sqrt{3} \left( \Omega_{c} - \frac{\sqrt{3}}{2} \nu \theta \right) \right] \right\}. \tag{55}\]

Once a quantum state of the universe is given as, e.g., a superposition of states

\[\Psi(\Omega_{c}, \beta_{c}) = \sum_{\nu} C_{\nu} e^{i\nu \sqrt{3} \beta_{c}} K_{iv} \left\{ 4 \exp \left[ -\sqrt{3} \left( \Omega_{c} - \frac{\sqrt{3}}{2} \nu \theta \right) \right] \right\} = R e^{iS}, \tag{56}\]

8 In the context of ordinary non-relativistic quantum mechanics, where a probability interpretation can be given to $\rho = \Psi^{\ast} \Psi$, the same procedure is to assign the local expectation value $[22]$. Such a nomenclature is clearly senseless here, where $\rho = \Psi^{\ast} \Psi$ does not have a probabilistic interpretation.

9 Although the operators $\hat{\Omega}$ and $\hat{\beta}$ do not commute, we shall refer to $\Omega$ and $\beta$ as their respective beables $[11]$ by the ontology of the spacetime metric encoded in these variables.
the universe evolution can be determined by solving the system of equations constituted by (51) and (52) and substituting the solution in (53) and (54).

Before applying the formalism to practical calculations, it is interesting to understand the meaning of all the terms in equation (52). This is accomplished by considering the associated Hamilton-Jacobi equation. The generalized Hamilton-Jacobi equation for noncommutative quantum cosmology is obtained by plugging \( \Psi(\Omega, \beta) = R e^{iS} \) into (43) and taking the real piece. As a result, we find

\[
-\frac{1}{24} \left( \frac{\partial S}{\partial \Omega} \right)^2 + \frac{1}{24} \left( \frac{\partial S}{\partial \beta} \right)^2 + V + V_{nc} + Q_K + Q_I = 0,
\]

(57)

where

\[
V = -2e^{-2\sqrt{3}\Omega_c},
\]

\[
V_{nc} = 2e^{-2\sqrt{3}\Omega_c} - 2e^{-2\sqrt{3}\Omega_c+\sqrt{3}\theta \partial \beta_c} S,
\]

\[
Q_K = \frac{1}{24R} \left( \frac{\partial^2 R}{\partial \Omega_c^2} - \frac{\partial^2 R}{\partial \beta_c^2} \right),
\]

(58)

\[
Q_I = -2 \text{Re} \left\{ \exp \left( -2\sqrt{3}\Omega_c - i\theta \sqrt{3} \partial \beta_c \right) \left( R e^{iS} \right) \right\} + 2e^{-2\sqrt{3}\Omega_c+\theta \sqrt{3} \partial \beta_c} S.
\]

The term \( V_{nc} \) is the noncommutative part of the classical potential, while \( Q_K \) and \( Q_I \) are denominated as the kinetic and interaction quantum potentials. Due to the noncommutative corrections to the wavefunction, it is clear that, although functionally similar, \( Q_K \) differs from \( Q \) previously defined in section 4. Equation (52) can now be written as

\[
\dot{\beta}_c(t) = \left[ 2 \frac{\partial S}{\partial \beta_c} - 48\sqrt{3}\theta e^{-2\sqrt{3}\Omega_c} + 24\sqrt{3} \theta (V_{nc} + Q_I) \right] \bigg|_{\Omega_c=\Omega_c(t), \beta_c=\beta_c(t)}.
\]

(59)

Noncommutative effects are therefore manifest not only via \( S \), which is functionally different from its commutative quantum analog, but also directly in the equation of motion for the canonical variables. This entails a series of consequences for the model. The first of them is the condition for the classical limit, which is now that the terms containing \( Q_K \) and \( Q_I \) be negligible in (57) and (58). The presence of the \( V_{nc} \) and \( Q_I \) terms in (59) tells us also that noncommutativity can induce dynamics in situations where it is impossible in the commutative case. Real wavefunctions (\( S = 0 \)), which represent universes that are necessarily static in the commutative formulation, can yield dynamic universes in noncommutative quantum cosmology.

In what follows we present examples of application of the formalism proposed.

B. Case 1

The wavefunction is of the type (55). In this case we have \( S = \nu \sqrt{3} \beta \). The equations of motion are therefore

\[
\dot{\Omega}_c = 0, \quad \dot{\beta}_c = 2\sqrt{3}\nu - 48\sqrt{3}\theta \exp \left( -2\sqrt{3}\Omega_c + 3\theta \nu \right)
\]

(60)

whose solution is

\[
\Omega_c = \Omega_{c_0}, \quad \beta_c(t) = 2\sqrt{3}\alpha_C (t - t_0) + \beta_0,
\]

(61)

where

\[
\alpha_C = \left[ \nu - 24\theta \exp \left( -2\sqrt{3}\Omega_{c_0} + 3\theta \nu \right) \right].
\]

(62)

10 Real wavefunctions are priviledged, e.g., by the no-boundary proposal for the initial conditions of the universe.
From (53) and (54) we have

\[ \Omega(t) = \Omega_{c_0} - \frac{\theta \sqrt{3}}{2} t \]
\[ \beta(t) = \beta_c(t) = 2\sqrt{3} \alpha_c (t - t_0) + \beta_0 \] (63)

Except by the \( \theta \) contributions that appear shifting the values of the constants, the time dependence of the solutions is exactly the same as the commutative counterpart discussed in section 6. The qualitative behavior assumed by the universe in this case are therefore identical to the one discussed there.

C. Case 2

Let us now consider a wavefunction that is the combination of two solutions of the type (55),

\[ \Psi(\Omega_c, \beta_c) = A_1 K_{i\mu} \left( 4e^{-\sqrt{3} \Omega_c + \frac{3\theta \mu}{2}} \right) e^{i\sqrt{3} \mu \beta_c} + A_2 K_{i\nu} \left( 4e^{-\sqrt{3} \Omega_c + \frac{3\theta \nu}{2}} \right) e^{i\sqrt{3} \nu \beta_c} \] (64)

By writing it in the polar form we can find its phase as

\[ S(\Omega_c, \beta_c) = \arctan \left[ \frac{A_1 K_{i\mu} \left( 4e^{-\sqrt{3} \Omega_c + \frac{3\theta \mu}{2}} \right) \sin \left( \sqrt{3} \mu \beta_c \right) + A_2 K_{i\nu} \left( 4e^{-\sqrt{3} \Omega_c + \frac{3\theta \nu}{2}} \right) \sin \left( \sqrt{3} \nu \beta_c \right)}{A_1 K_{i\mu} \left( 4e^{-\sqrt{3} \Omega_c + \frac{3\theta \mu}{2}} \right) \cos \left( \sqrt{3} \mu \beta_c \right) + A_2 K_{i\nu} \left( 4e^{-\sqrt{3} \Omega_c + \frac{3\theta \nu}{2}} \right) \cos \left( \sqrt{3} \nu \beta_c \right)} \right] \] (65)

where the \( A_1 \) and \( A_2 \) are chosen as real coefficients. The equations of motion (51) and (52) for this state are

\[ \frac{d\Omega_c}{dt} = 8\sqrt{3} \frac{A_1 A_2 \left[ K_{i\mu} K_{i\nu} - K_{i\mu} K_{i\nu} \right] e^{-\sqrt{3} \Omega_c} \sin \left( \sqrt{3} (\mu - \nu) \beta_c \right)}{(A_1 K_{i\mu})^2 + (A_2 K_{i\nu})^2 + 2A_1 A_2 K_{i\mu} K_{i\nu} \cos \left( \sqrt{3} (\mu - \nu) \beta_c \right)} \]
\[ \frac{d\beta_c}{dt} = 2\sqrt{3} \frac{\mu A_1^2 K_{i\mu}^2 + \nu A_2^2 K_{i\nu}^2 + (\mu + \nu) A_1 A_2 K_{i\mu} K_{i\nu} \cos \left( \sqrt{3} (\mu - \nu) \beta_c \right)}{(A_1 K_{i\mu})^2 + (A_2 K_{i\nu})^2 + 2A_1 A_2 K_{i\mu} K_{i\nu} \cos \left( \sqrt{3} (\mu - \nu) \beta_c \right)} \] (66)

\[ - 48\sqrt{3} \theta e^{-2\sqrt{3} \Omega} \frac{e^{3\mu \theta} A_1^2 K_{i\mu}^2 + e^{3\nu \theta} A_2^2 K_{i\nu}^2 + (e^{3\mu \theta} + e^{3\nu \theta}) A_1 A_2 K_{i\mu} K_{i\nu} \cos \left( \sqrt{3} (\mu - \nu) \beta_c \right)}{(A_1 K_{i\mu})^2 + (A_2 K_{i\nu})^2 + 2A_1 A_2 K_{i\mu} K_{i\nu} \cos \left( \sqrt{3} (\mu - \nu) \beta_c \right)} \]

where prime means derivative with respect to the argument. As a reference for comparison with the commutative analog, let us fix \( A_1 = A_2 = 1/\sqrt{2} \) and consider first the case where \( \mu = 1/10 \) and \( \nu = -1/5 \). Figure 4 presents a plot of the velocity field associated with the differential equations (66) for \( \theta = -4 \). The field of velocities suggests that the ensemble of solutions in this case is similar as the one of the commutative counterpart. When comparing each individual commutative solution with its noncommutative analog we expect therefore to find it quantitatively corrected. If qualitatively different, it should assume a behavior similar to one of the previously described in section 6. We shall verify this by studying the evolution of \( l^2(t) \).

Fig. 5a presents a plot of \( \ln [l^2(t)] \) for a cyclic universe where \( \theta = 4 \) and the initial conditions identical\(^{11}\) to that of Fig. 3a. As it can be seen, the principal effect of noncommutativity in this case is to shorten the period of the cycles. In a similar way as in the classical noncommutative case, the noncommutative effects are sensible to the \( \theta \) sign. Fig. 5b presents the plot of the solution obtained by preserving the initial conditions of Fig. 5a.

\(^{11}\) In all cases of Fig. 5 initial conditions for \( \Omega_c \) and \( \beta_c \) were chosen judiciously in order that the associated \( \Omega(0) \) and \( \beta(0) \) calculated using (55) and (56) correspond to representative examples of the noncommutative quantum dynamics.
and inverting the sign of $\theta$. The orbit which was originally closed (corresponding to a non-singular universe) now becomes open, originating a singular contracting universe. Note that the RHS of the system (66) has its sign inverted by the change $\mu \rightarrow -\mu$, $\nu \rightarrow -\nu$ and $\theta \rightarrow -\theta$. By differentiating equations (53) and (54) and using equations (65) and (66), we can see that $\dot{\Omega}(t)$ and $\dot{\beta}(t)$ have their signs inverted by the same change. It is therefore possible to generate an expanding universe solution from the solution depicted in Fig. 5b by using this property. Noncommutativity can also close orbits which were originally open. An example of this property is depicted in Fig. 5c, which present a cyclic universe whose correspondent commutative analog is the universe solution of Fig. 3b. For large values of $\theta$ the effect of closing the orbit can be reverted, as is shown in Fig. 5d.

We close this section by considering the case where $\mu = -\nu$. As discussed before, although this case has a real wavefunction it can have a nontrivial dynamics. In fact, it can be shown that, depending on the initial conditions, it can give rise to non-singular periodic solutions similar to that of Fig. 5a or to singular universes similar to that of Fig. 5b.
FIG. 5. The evolution of the characteristic volume of the noncommutative quantum Kantowski-Sachs universe
(a) : $\mu = 1/10, \nu = -1/5, \theta = 4$ and $\Omega(0) = 2, \beta(0) = -24$. (b) : $\mu = 1/10, \nu = -1/5, \theta = -4$ and $\Omega(0) = 2, \beta(0) = -24$. (c) : $\mu = 1/10, \nu = -1/5, \theta = 1$ and $\Omega(0) = 7.7, \beta(0) = -21.7$. (d) : $\mu = 1/10, \nu = -1/5, \theta = 12$ and $\Omega(0) = 7.7, \beta(0) = -21.7$.

VIII. DISCUSSION AND OUTLOOK

In this work we investigated some possible consequences of noncommutativity for cosmology in the early stages of the universe via deformation of the commutation relation between the minisuperspace variables. As an object of analysis, we chose the Kantowski-Sachs universe, which is one of the most investigated homogeneous universe models. In order to have a clear picture of the impact of noncommutativity, a comparative study of the Kantowski-Sachs universe was carried out in four different scenarios: the classical commutative, classical noncommutative, quantum commutative and quantum noncommutative. Our comparative analysis of the four versions of the model was traced out in the common language of minisuperspace trajectories. In the quantum context, this is provided by the Bohmian interpretation. We were lead, therefore, to combine noncommutative geometry and Bohmian quantum physics. This fusion of two apparently opposite ways of thinking, one commonly associated with fuzzyness, and the other to ontological point particles, proved to be interesting from the conceptual as well as from the operational point of view.

In the theoretical framework, we have extended Bohmian formulation to comprise the noncommutativity of the minisuperspace operators of the Kantowski-Sachs model via the beable correspondence. Such a mapping between Hermitian operators and ordinary functions, commonly employed in Bohmian quantum mechanics, allows the association of each Hermitian operator with an element of ontology. In that context, it can be shown that by averaging the beable $B[\hat{A}]$ over an ensemble of particles with probability density $\rho = |\Psi|^2$ gives the same result obtained by computing the expectation value of the observable $\hat{A}$ applying the standard operatorial formalism, reason for the denomination “local expectation value” for $B[\hat{A}]$ [25]. In the Kantowski-Sachs universe, although the Wheeler-DeWitt equation is of Klein-Gordon type (it gives us no natural notion of probability), the beable mapping is well defined, even in the noncommutative case. In the commutative context, our formulation
is reduced to the Bohmian quantum gravity proposed by Holland in the minisuperspace approximation.

From the practical point of view, the formulation proposed proved to be easy to handle in the calculations. The worked examples showed that noncommutativity in the classical context (section 5) can modify quantitatively the universe volume evolution and qualitatively its shape at intermediary times, but cannot alter its singular behavior in the infinite future and in the past. Quantum effects, on the other hand, can radically modify the universe evolution and remove singularities. Some quantum universes start with infinite volume and decrease up to a singular configuration or start from a singularity in past and increase in volume up to infinity. More interesting are the quantum periodic solutions, where the singularities are totally removed. As showed in section 6, these universes can present a great number of e-folds. The minimum length, $l_{\text{min}}$, for these eternal universes can assume a wide range of values. It is not difficult to find solutions in which $l_{\text{min}}$ is sufficiently small to be in a scale where quantum gravity effects are expected to be relevant, but larger than the Planck length, where a fundamental theory of gravitation is expected to be valid.

In section 7 it was shown that noncommutativity can modify appreciably the universe evolution in the quantum context. A comparison between the two quantum versions of the Kantowski-Sachs universe revealed that noncommutative effects can not only introduce quantitative corrections in the universe evolution but also modify its qualitative behavior. Periodic solutions can change to exponentially contracting or expanding universes, and vice-versa. The presence of noncommutative terms in the Bohmian equation of motion for $\beta_{\text{c}}(t)$ tells us that noncommutativity can give rise to a nontrivial dynamics when the wavefunction is real.

Although the analysis carried out in this work was restricted to the Kantowski-Sachs model, part of the results may be valid to other homogeneous cosmologies. Since the Friedman-Robertson-Walker universe presents a Wheeler-DeWitt equation similar to the one discussed here (see, e.g., [33]) the formalism proposed in this work may also be applied in its description.

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