Thermalization of isotopically disordered chain

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Abstract

Recently, the equipartition hypothesis in statistical physics has been verified in the case of one-dimensional lattices based on the wave turbulence theory. It has been shown that thermalization should occur for arbitrarily small nonlinearity. Particularly, in the lattices with the polynomial interaction potential $V(x) = x^2/2 + \lambda x^n/n$ for $n > 4$, the thermalization time $T_{eq}$ is proportional to $\lambda^{-2}$ in the thermodynamic limit. In this paper, we apply wave turbulence theory to study the dynamics of one-dimensional disordered chains with the polynomial potential. We show that the universal scaling law can be extended to $n = 3$ since the three-wave resonances are allowable due to the space-translation symmetry broken. Our results indicate that wave turbulence theory is still applicable to disordered systems.

I Introduction

The study of thermalization of nonlinear lattices has a long history. In 1955 the Fermi-Pasta-Ulam-Tsingou (FPUT) paradox was formulated [1] which suggested the non-equipartition of energy among normal modes of harmonic oscillators coupled with a quadratic or cubic nonlinearity. Since then extensive studies on thermalization for the FPUT model have been carried [2–19]. Nowadays, it has been clarified that the state observed in the original FPUT paper is a metastable state existing in a relatively short time. On a much longer time scale, the model may eventually evolve towards energy equipartition. However, researchers have not reached a consensus over whether there is a threshold above which thermalization would indeed occur. A method developed by Chirikov and Izrailev [2] predicts a threshold and numerical studies confirmed the presence of a threshold above which certain FPUT systems reach a fast thermalized state [4, 6, 7]. However, numerical simulations of the FPUT-α model [11, 13] have shown that even for small nonlinearity the system does reach a thermalized state in a sufficient long time. More recently, a detailed study of the FPUT-β model with a finite number of masses [16] shows that thermalization seems to occurs for any, even extremely small, levels of nonlinearity and that the time scale of equipartition matches the prediction based on exact wave-wave resonant interaction theory [20, 21]. Further studies [17, 19] have revealed that there is a universal scaling law of the equipartition time, $T_{eq}$, in the thermodynamic limit, i.e., $T_{eq} \propto \lambda^{-2} \varepsilon^{-(n-2)}$ if $n \geq 4$, for harmonic models perturbed by nonlinear polynomial interaction potential, i.e.,

$$V(x) = \frac{\lambda}{n} x^n. \quad (1)$$

Here $\lambda$ is a positive number and $\varepsilon$ denotes the energy density. Extensive numerical simulations have confirmed the validity for an even $n$, while slight correction is needed for an odd $n$. This scaling is also reported in Ref. [18], for FPUT-α, FPUT-β, and discrete nonlinear Klein-Gordon
The results presented above are limited to the crystalline models within which wave resonance theory works well. However, many practical materials are disordered in a certain extent. Hence effects of disorder are studied intensively from the viewpoint of fundamental physics, such as Anderson localization [22]. However, on one hand, as far as we know, few studies has been focused on the equipartition problem [15]. On the other hand, this topic is of fundamental importance. It is known that whether there is an amplitude threshold of perturbations below which the Anderson localization can keep stable is still an open problem [23–25]. It will imply the possibility of the absence of such a threshold if the recent results of equipartition can be extended to disordered systems. More generally, the effect of localization modes, as well as the consequence due to the space-translation symmetry breaking, should be essential issues need to be clarified.

In this paper, we show that due to the symmetry breaking, the constraint on the dispersion-relation between wave vectors and frequencies is relaxed. Thus the disorder can occur once the n-order polynomial potential in Eq. (1) is present. In these sense, we emphasize that the universal law of $T_{eq} \propto \lambda^{-2}$ should be extended to $n \geq 3$. Our extensive numerical simulations confirm that this scaling law is accurate for either even or odd $n$ in the weakly nonlinear region. The paper is organized as follows : the model and theoretical analysis are introduced in section II. The results of the numerical simulations as well as their comparison with theoretical predictions are given in section III. In section IV we draw some conclusions and point out some possible directions for further work.

\section{Models and Theoretical analysis}

We consider a 1D chain of $N$ atoms with nearest-neighbours interactions. The corresponding Hamiltonian of the mass-disordered chain is given by

$$ H = H_0 + H', $$

where

$$ H_0 = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \sum_{i=1}^{N} \frac{(q_{i+1} - q_i)^2}{2}. $$

Here $p_i$ and $q_i$ respectively represents the momentum and the displacement from the equilibrium position of the $i$-th atom of mass $m_i$. We set $q_0 = q_{N+1} = 0$ in our simulations. The random masses $m_i$ is chosen independently and identically from a uniform distribution between $m - \delta m$ to $m + \delta m$, the strength of disorder is then characterized by $\delta m$. The second term $H'$ describes the nonlinear polynomial interactions. For the sake of simplicity we introduce the shorthand notation $V_n$-model to denote the model (2) with nonlinear polynomial interaction (1). Note that $V_3$- and $V_4$- model is respectively known as the FPUT-\(\alpha\) [13] and FPUT-\(\beta\) model [16].

In general, normal modes of disordered systems can be obtained by numerically diagonalizing the harmonic matrix. The elements of the harmonic matrix physically represent the harmonic force constants of the bonds connecting various atoms. Mathematically, the harmonic matrix is defined as

$$ \Phi = \Phi_{ij} = \frac{1}{\sqrt{m_im_j}} \frac{\partial^2 V}{\partial q_i \partial q_j}. $$

There exists a unitary transformation matrix $U$, whose columns are the normal modes $w^k$, such that

$$ U^\dagger \Phi U = \Omega^2, $$

where $\Omega$ is a diagonal matrix whose elements are the normal mode frequencies, namely $\Omega_{kk} = \omega_k$. Spectral index $k$ is ordered according to the value of frequency, so that $\omega_k \leq \omega_{k+1}$.

Motivated by previous works [13, 16–18], we prefer to work in a new space spanned by an orthonormal basis set $\{u^k\}$. We define the direct and inverse discrete transform of the $q_j$ variables as

$$ \begin{cases} Q_k = \sum_j \sqrt{m_j} q_j u^k_j, \\ q_j = \sum_k Q_k u^k_j / \sqrt{m_j}. \end{cases} $$

We then introduce the complex amplitude of a normal mode $\alpha(k, t)$ as

$$ \alpha_k = \frac{1}{\sqrt{2\omega_k}} (P_k - i\omega_k Q_k), $$

with

$$ H_0 = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \sum_{i=1}^{N} \frac{(q_{i+1} - q_i)^2}{2}. $$

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where $P_k = \dot{Q}_k$. Substituting the above definitions into Eq. (2) and introducing the nondimensional variable

$$a_k' = \frac{a_k}{\sqrt{\sum_k \omega_k a_k^2(k,0)/N}} = \frac{a_k}{\sqrt{\varepsilon}}, \quad H' = \frac{H}{\varepsilon},$$

we get the following Hamiltonian for $V_n$-model,

$$H = \sum_k \omega_k a_k a_k^* + \frac{\lambda \varepsilon^{(n-2)/2}}{n} \sum_{k_1, \cdots, k_n} A_{1, \cdots, n} \prod_{s=1}^n (a_{-k_s}^* + a_{k_s}),$$

where primes have been omitted for brevity and the matrix $A_{1, \cdots, n}$ weights the transfer of energy between modes indexed by $k_1, k_2, \cdots, k_n$ and is given by

$$A_{1, \cdots, n} = (-i)^n \tilde{A}_{1, \cdots, n} \prod_{s=1}^n \frac{\sqrt{2\omega_{k_s}}}{2\omega_{k_s}},$$

where

$$\tilde{A}_{1, \cdots, n} = \sum_j \prod_{s=1}^n \left( \frac{u_{s+1}^{k_j}}{\sqrt{m_{s+1}}} - \frac{u_s^{k_j}}{\sqrt{m_s}} \right).$$

Then, the equation of motion for the $k_1$th complex normal mode reduces to

$$i\dot{a}_{k_1} = \omega_{k_1} a_{k_1} + \lambda \varepsilon^{(n-2)/2} \sum_{s=2}^n A_{1, \cdots, n} \prod_{s=2}^n (a_{-k_s}^* + a_{k_s}).$$

The evolution equation (12) has a Hamiltonian structure with canonical variables \{ $i\dot{a}_k, a_k^*$ \}, describing the time evolution of the amplitudes of the normal modes of the system. To evaluate the equipartition time, it is convenient to introduce the wave action spectral density $D_i \delta_i^3 = \langle a_k a_k^* \rangle$ following the wave resonance approach. We then derive the $n$-wave kinetic equation in the thermodynamic limit

$$\dot{D}_1 = \lambda^2 \varepsilon^{n-2} \int_{-1}^1 |A_{1, \cdots, n}|^2 \mathfrak{F}(D_1, n) \delta(\omega_{1,n}) dk_2 \cdots dk_n,$$

where $\mathfrak{F}(D_1, n)$ is a function of $D_1, D_2, \ldots, D_n$, and $\delta(\omega_{1,n})$ is the shorthand notation of delta function $\delta(\omega_{k_1} \pm \omega_{k_2} \pm \cdots \pm \omega_{k_n})$. Following the weak turbulence theory [20, 21], the $T_{eq}$ is inversely proportional to the amplitude of Eq. (13) if the integral doesn’t vanish, and thus one obtains

$$T_{eq} \propto \lambda^{-2} \varepsilon^{-(n-2)}.$$  

To guarantee a non-vanishing integral for $\mathfrak{F}(D_1, n)$, it should has

$$\omega_{k_1} \pm \omega_{k_2} \pm \cdots \pm \omega_{k_n} = 0,$$

i.e., the $n$-wave resonant condition for the frequencies should be satisfied. This condition is relatively easy to be satisfied for a sufficiently large system. Firstly, the frequencies will become dense when the system tends to infinitely large. Secondly, each frequency will have a certain degree of broadening due to the nonlinearity. Therefore, one can expect that this condition is satisfied for a finite but sufficiently large system.

For the term $A_{1, \cdots, n}$ one can easily show that it reduces to $\delta(k_{1,n})$ multiplying a constant factor if the lattice turns to be uniform, which agrees with the result obtained in Ref. [17]. In this case, the integral does not vanish only when the spectral indexes (the so-called wave numbers in ordered lattices) satisfy the resonant condition, i.e.,

$$k_1 \pm k_2 \pm \cdots \pm k_n \mod N = 0.$$  

Otherwise, by applying the spatial translation invariance condition and the normal mode solutions of a uniform lattice to Eq. (11) one can prove that the integral must vanish. Therefore, to get a non-vanishing integral in the uniform lattice, one needs to seek for the satisfaction of both resonance conditions (15) and (16). For $n = 3$, it is known that these two conditions can not be satisfied simultaneously due to the restriction of the dispersion relation between the spectral index and the frequency.

However, in a disordered system the $A_{1, \cdots, n}$ given by Eq. (11) should not vanish generally due to the absence of the spatial translation invariance. To confirm this argument, we employ the $V_3$-model as an example to calculate the probability $P$ that a specific term of $A_{1,2,3}$ will be non-vanishing, i.e., for a realization of disorder of $\delta_{mn}$ we numerically calculate $\tilde{A}_{1,2,3}$ by Eq. (11) for all of the combinations of $k_1, k_2, \text{and } k_3$ to find the proportion of
non-vanishing $A_{1,2,3}$. Figure 1 shows $P$ versus $\delta m$ for several system sizes. We see that there is a clear transition from $\delta m = 0$ to $\delta m \neq 0$, with probability changing suddenly from $P \approx 0$ to $P \approx 1$. In other words, once the spatial translation invariance is removed, the weight $A_{1,2,3}$ becomes non-vanishing. Here we define $A_{1,2,3}$ to be vanishing if $A_{1,2,3} < 10^{-6}$. Therefore, removing the restriction for spectral indexes, we can expect that the universal scaling law of $T_{eq} \propto \lambda^{-2}$ should be observed for all of the $V_n$-model with $n \geq 3$.

![Figure 1: The probabilities of non-vanishing $A_{1,2,3}$ for different disorder strengths and system sizes.](image)

### III Numerical Verification

We adopt the method presented in Ref. [12] to calculate equilibration time. The energy of the $k$th mode is $E_k = \frac{1}{2}(P_k^2 + \omega_k^2 Q_k^2)$. The indicator of thermalization, $\xi(t) = 2\xi(t)e^{\eta(t)}/N$, is adopted, where $\eta(t) = -\sum_{k=N/2}^N w_k(t) \log w_k(t)$ is the spectral entropy, in which $w_k = E_k(t)/\sum_{i=1}^N E_i(t)$, $\xi(t) = 2\sum_{N/2}^N \hat{E}_k(t)/\sum_1^N E_k(t)$, and $\hat{E}_k(T) = \frac{1}{(1-\mu)T} \int_0^T E_k (P(t), Q(t)) dt$ is the average energy of the $k$th normal mode. Here, parameter $\mu$ controls the size of the time window for averaging and is fixed at $\mu = 2/3$ in our simulation. The equilibration time $T_{eq}$ is measured as that satisfying $\xi(T_{eq})=1/2$.

In this paper, the initial energies are randomly distributed among the 10% lowest frequency normal modes. We use the eighth-order Yoshida method [26] to integrate the equations of motion derived by the Hamiltonian (2). The typical integration time step is set to $\Delta t = 0.1$. In order to reduce the fluctuations, a further average on 120 different random choices of the phases uniformly distributed in $[0, 2\pi]$ is introduced for each realization of disorder. In the following, when there is no risk of confusion we use $\tilde{E}_k(t)$ or $\xi(t)$ to denote both variable itself and its average value over 120 phases. Average over 10 realizations of disorder is also taken into consideration and we show that this makes no significant difference on our conclusion.

The results of $\tilde{E}_k(T)$ versus $k/N$ and the dependence of $\xi(t)$ on time $t$ are shown in Fig. III. Our focus here is on the $V_3$-model, while results of the models with other interactions are not presented, since the scenarios in these models are quite similar. It is seen from Fig. 2(a) that the energy of the initial excited modes decreases continuously with time. Meanwhile, the energy of other frequency modes grows continuously with time. Eventually, the energy of all modes approaches to the global equipartition. The metastable state, in which the $\tilde{E}_k(T)$ keeps its profile nearly unchanged in a very long range of initial time scale with weak nonlinearity and was found in the FPUT models [10] and $\varphi^4$ model [3], is not found here. This phenomenon of the the disordered model is similar to that found in the FK model [14].

![Figure 2: (a) The function $\tilde{E}_k(T)$ versus $k/N$ at different times $T$ and (b) the dependence of $\xi(t)$ on time $t$ for $V_3$-model.](image)

Note that $H' = \varepsilon H$ under the scaling transformation $q_i' = q_i \varepsilon^{1/2}$ for the $v_n$ model, hence, the nonlinear parameter $\lambda$ and the energy density $\varepsilon$ has a rigid scaling relation $\lambda' = \lambda \varepsilon^{(n-2)/2}$. Therefore, it is equivalent to studying the scaling of $\lambda$ by fixing $\varepsilon$ or that of $\varepsilon$ by fixing $\lambda$. Here, we perform the latter with fixed $\lambda = 1$. However, numerical simulation can only be performed with finite-size system. The strategy is to investigate the tendency with increasing
system size. This is shown in Fig. 3(a) by taking the case of \( n = 4 \) as an example. We see that with the increase of the system size the result converges to the theoretical prediction \( T_{eq} \propto \varepsilon^{-2} \) perfectly. We thus infer that in the thermodynamical limit this scaling law can be extended to arbitrarily small energy density. Note that in our calculation only one realization of disorder for a fixed-size system is used, since we have checked that computation efforts on average over realization, say 10 realizations in our simulation, make no difference within the accuracy of calculation.

Figure 3(b) shows the results for the \( V_3^- \), \( V_4^- \), and \( V_5^- \) model. It can be saw that they all agree with the theoretical prediction. Particularly, for the \( V_3^- \) model it shows a more fast convergence to \( T_{eq} \propto \varepsilon^{-1} \), which is equivalent to \( T_{eq} \propto \lambda^{-2} \). Therefore, the universal scaling law can be extended to \( n = 3 \) for the disordered system. Furthermore, comparing to the ordered counterpart, we find that the theoretical prediction is satisfied perfectly either for even and odd \( n \) of \( V_n^- \) model. We conjecture that it is should be also due to the spatial translation symmetry breaking.

Figure 3: The equipartition time \( T_{eq} \) as a function of \( \varepsilon \) in log-log scale (a) for the \( V_4^- \)-model with different size \( N = 511, 1023, 2047 \) from top to bottom. (b) for \( V_5^- \), \( V_4^- \) and \( V_3^- \)-model from bottom to top, fixed \( N = 1023 \). The purpose of solid lines with slopes \(-1, -2 \) and \(-3 \) in the figure is to facilitate comparison with the theoretical results. All simulation results are obtain with \( m = 1 \) and \( \delta m = 0.2 \).

IV Conclusions

We show that major theoretical and conceptual frameworks in the filed of WT theory still works well for the disordered systems. These results provide a general prediction for discussion and further research to reach a generally conclusion about long time dynamics of one-dimensional real materials, since disorder is usually encountered when one begins to deal with real materials in laboratory.

References

[1] David K Campbell, Phillip Rosenau, and George M Zaslavsky. Introduction: The fermi-pasta-ulam problem-the first fifty years. Chaos: An Interdisciplinary Journal of Nonlinear Science, 15(1):015101, 2005.

[2] FM Izrailev and BV Chirikov. Stochasticity of the simplest dynamical model with divided phase space. Sov. Phys. Dokl, 11(1):30, 1966.

[3] F Fucito, F Marchesoni, E Marinari, G Parisi, L Peliti, S Ruffo, and A Vulpiani. Approach to equilibrium in a chain of nonlinear oscillators. Journal de Physique, 43(5):707–713, 1982.

[4] Roberto Livi, Marco Pettini, Stefano Ruffo, Massimo Sparpaglione, and Angelo Vulpiani. Equipartition threshold in nonlinear large hamiltonian systems: The fermi-pasta-ulam model. Physical Review A, 31(2):1039, 1985.

[5] Joseph Ford. The fermi-pasta-ulam problem: paradox turns discovery. Physics Reports, 213(5):271–310, 1992.

[6] Lapo Casetti, Monica Cerruti-Sola, Marco Pettini, and EGD Cohen. The fermi-pasta-ulam problem revisited: stochasticity thresholds in nonlinear hamiltonian systems. Physical Review E, 55(6):6566, 1997.
[7] J De Luca, AJ Lichtenberg, and S Ruffo. Finite times to equipartition in the thermodynamic limit. Physical Review E, 60(4):3781, 1999.

[8] GP Berman and FM Izrailev. The fermi–pasta–ulam problem: fifty years of progress. Chaos: An Interdisciplinary Journal of Nonlinear Science, 15(1):015104, 2005.

[9] Giovanni Gallavotti. The Fermi-Pasta-Ulam problem: a status report, volume 728. Springer, 2007.

[10] Giancarlo Benettin, Roberto Livi, and Antonio Ponno. The fermi-pasta-ulam problem: scaling laws vs. initial conditions. Journal of Statistical Physics, 135(5-6):873–893, 2009.

[11] Antonio Ponno, Helen Christodoulidi, Ch Skokos, and Sergei Flach. The two-stage dynamics in the fermi-pasta-ulam problem: from regular to diffusive behavior. Chaos: An Interdisciplinary Journal of Nonlinear Science, 21(4):043127, 2011.

[12] G Benettin and A Ponno. Time-scales to equipartition in the fermi–pasta–ulam problem: finite-size effects and thermodynamic limit. Journal of Statistical Physics, 144(4):793, 2011.

[13] Miguel Onorato, Lara Vozella, Davide Proment, and Yuri V Lvov. Route to thermalization in the α-fermi–pasta–ulam system. Proceedings of the National Academy of Sciences, 112(14):4208–4213, 2015.

[14] Zhenjun Zhang, Chunmei Tang, and Peiqing Tong. Dynamical thermalization of frenkel-kontorova model in the thermodynamic limit. Physical Review E, 93(2):022216, 2016.

[15] Zhenjun Zhang, Chunmei Tang, Jing Kang, and Peiqing Tong. Dynamical energy equipartition of the toda model with additional on-site potentials. Chinese Physics B, 26(10):100505, 2017.

[16] Yuri V Lvov and Miguel Onorato. Double scaling in the relaxation time in the β-fermi-pasta-ulam-tsingou model. Physical review letters, 120(14):144301, 2018.

[17] Weicheng Fu, Yong Zhang, and Hong Zhao. Universality of Energy Equipartition in One-dimensional Lattices. arXiv e-prints, page arXiv:1811.05697, November 2018.

[18] Lorenzo Pistone, Sergio Chibbaro, Miguel Bustamante, Yuri L’vov, and Miguel Onorato. Universal route to thermalization in weakly-nonlinear one-dimensional chains. arXiv e-prints, page arXiv:1812.08279, December 2018.

[19] Weicheng Fu, Yong Zhang, and Hong Zhao. Universal law of thermalization for one-dimensional perturbed Toda lattices. arXiv e-prints, page arXiv:1901.04245, Jan 2019.

[20] Vladimir E Zakharov, Victor S L’vov, and Gregory Falkovich. Kolmogorov spectra of turbulence I: Wave turbulence. Springer Science & Business Media, 2012.

[21] AJ Majda, DW McLaughlin, and EG Tabak. A one-dimensional model for dispersive wave turbulence. Journal of Nonlinear Science, 7(1):9–44, 1997.

[22] Elihu Abrahams. 50 years of Anderson Localization. world scientific, 2010.

[23] Jianjin Wang, Dahai He, Yong Zhang, Jiao Wang, and Hong Zhao. Effects of interaction symmetry on delocalization and energy transport in one-dimensional disordered lattices. Physical Review E, 92(3):032138, 2015.

[24] L Fleishman and PW Anderson. Interactions and the anderson transition. Physical Review B, 21(6):2366, 1980.

[25] Aart Lagendijk, Bart Van Tiggelen, and Diederik S Wiersma. Fifty years of anderson localization. Phys. Today, 62(8):24–29, 2009.

[26] Hiroshi Kinoshita, Haruo Yoshida, and Hiroshi Nakai. Symplectic integrators and their application to dynamical astronomy. Celestial Mechanics and Dynamical Astronomy, 50(1):59–71, 1990.