Abstract

We consider a network of banks that optimally choose a strategy of asset liquidations and borrowing in order to cover short term obligations. The borrowing is done in the form of collateralized repurchase agreements, the haircut level of which depends on the total liquidations of all the banks. Similarly the fire-sale price of the asset obtained by each of the banks depends on the amount of assets liquidated by the bank itself and by other banks. By nature of this setup, banks’ behavior is considered as a Nash equilibrium. This paper provides two forms for market clearing to occur: through a common closing price and through an application of the limit order book. The main results of this work are providing sufficient conditions for existence and uniqueness of the clearing solutions (i.e., liquidations, borrowing, fire sale prices, and haircut levels).

Keywords Finance, Systemic Risk, Price-Mediated Contagion, Repurchase Agreements.

1 Introduction

Historically, financial risk was typically measured for individual firms separately. After the financial crisis of 2007-2009, a new understanding that risk can spread through the entire financial system has emerged. This is referred to as systemic risk – the risk that the distress of several banks can spread throughout the system to a degree that it may affect the viability of the entire system or a significant part of it. Such a propagation of risk is known as financial contagion. Two types of contagion are usually distinguished: those that happen due to local connections (e.g., obligations between banks in the network), and those that happen due to their influence on the entire network (e.g., impact to asset prices). This study focuses on a form of global contagion through asset

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prices, and investigates the existence and uniqueness of Nash equilibrium in a model of fire sales and collateralized borrowing.

A seminal paper in the systemic risk literature is Eisenberg and Noe (2001); that work studied an equilibrium payment model in a network of banks. That network model of interbank payments has been applied in numerous follow-up studies, e.g., Anand et al. (2013); Halai and Kok (2015); Boss et al. (2004); Elsinger et al. (2013); Upper (2011); Gai et al. (2011); Bardoscia et al. (2017). The Eisenberg and Noe (2001) model has also been extended to consider bankruptcy costs and asset fire sale dynamics (see Elsinger (2009); Rogers and Veraart (2013); Elliott et al. (2014); Glasserman and Young (2015); Weber and Weske (2017); Capponi et al. (2016); Elsinger (2009); Elliott et al. (2014); Weber and Weske (2017); Gouri´eroux et al. (2012); Cifuentes et al. (2005); Nier et al. (2007); Gai and Kapadia (2010); Amini et al. (2013); Chen et al. (2016); Weber and Weske (2017); Amini et al. (2016); Feinstein (2017); Feinstein and El-Masri (2017); Feinstein (2019)). We refer to, e.g., Weber and Weske (2017); Staum (2013); Hüser (2015) for detailed review of this literature.

In this paper we extend the model of Bichuch and Feinstein (2019); that work considered a network of banks facing shortfalls on their obligations which can be met through borrowing or by liquidating assets. The primary goal of this current paper is to consider the effects of repurchase agreement (repo) markets on financial stability. Such markets require banks to post collateral above the value of the loan in order to secure short term financing. In this construction, each bank seeks to optimize their strategy between asset liquidations and borrowing in the repo market. As in the traditional fire sale literature (see, e.g., Amini et al. (2016)), asset liquidations cause price impacts and, thus, the actions of one bank influence the decisions of all other institutions as well, i.e., we consider the Nash equilibrium of strategies. Based on the static setting traditionally followed in the literature, we assume that all the trading happens simultaneously and instantaneously.

Previously, prices were provided by an inverse demand function which was used to price liquidations as well as provide mark-to-market accounting. In undertaking this study, we consider two classical and realistic pricing functions in the fire sale process: Volume Weighted Average Pricing (VWAP) and a Limit Order Book (LOB) based pricing scheme. Both of these schemes can be viewed as pricing limits as order sizes decrease to zero, but with different rates of liquidation. This allows us to incorporate notions of time dynamics into the static model proposed. The VWAP scheme determines prices if firms place orders at a rate proportional to their total desired liquidations; this, ultimately, results in the same average price for every bank. Such a pricing scheme was introduced in Banerjee and Feinstein (2019). The LOB setting distinguishes prices by assuming all firms place orders at the same speed; banks with smaller order volumes will receive a higher price than those with a larger order volume (as the latter will continue to eat through the book even after the former are done liquidating).

As highlighted above, the innovation of this work is two-fold. First, we consider realistic pricing schemes that allow for banks to receive different prices based on the quantity of assets sold instead of the, more standard, assumption that there is a unique price at which all transactions occur. Second, we consider collateralized borrowing of illiquid assets in a repo market in which the haircut
of this collateral also depends on the mark-to-market value of the asset. As opposed to the realized liquidation prices, the haircut remains bank independent and only depends on the entire sale volume of the entire banking system since the deal depends on the value of the collateralized asset rather than the riskiness of the individual banks. Under these constructions, we are able to investigate the sensitivity of the resulting market prices to the prevailing repo interest rate. In particular, regulators use interest rates as the primary control for financial stability. This was seen in the emergency liquidity injection by the Federal Reserve in September 2019, in order to stabilize the repo market [Ihrig et al., 2020; Afonso et al., 2020]. In fact, Gorton and Metrick (2012); Brunnermeier (2009) consider the 2007-2009 financial crisis as a run on the repo market. Therefore systematic consideration of repo markets and the impact of interest rates is of paramount importance.

The organization of this paper is as follows. Section 2 introduces the general model with general inverse demand pricing functions. In that section we provide the existence of Nash equilibrium under a minimal set of assumptions. Section 3 introduces the VWAP and LOB based inverse demand pricing functions and discusses the conditions needed for maximal clearing solutions and uniqueness of Nash equilibrium. Numerical case studies and comparison of VWAP and LOB inverse demand pricing functions is in Section 4. The proofs for the main results are provided in the Appendix. Additionally in the Appendix, under the uniqueness conditions, we investigate the sensitivity of the clearing solutions to the prevailing repo rate.

2 Financial setting

We begin by assuming a system of \( n \) banks. In contrast to works that explicitly depend on the network of interbank obligations, e.g., in Eisenberg and Noe (2001); Cifuentes et al. (2005); Amini et al. (2016); Feinstein (2017), herein we will consider only fire sale effects and price mediated contagion as in, e.g., Greenwood et al. (2015); Braouezec and Wagalath (2018, 2019); Feinstein (2020); Banerjee and Feinstein (2013). We will, for simplicity, assume that all the banks \( i = 1, ..., n \) are facing a (cash) shortfall \( h_i > 0 \), all while holding \( a_i > 0 \) shares of illiquid assets; any banks without either a shortfall or illiquid asset holdings will not participate in any fire sale or borrowing and thus are extraneous to the considerations of this model. The banks are faced with the task of finding the optimal strategy to raise \( h_i \) cash in order to cover this shortfall. We assume that they can do so by either selling their illiquid asset, borrowing, or both. It will be assumed that the borrowing is going to be collateralized using the same illiquid asset. As is standard in the literature, due to the illiquidity, the price of the illiquid asset declines as assets are being sold; this is due to supply-demand dynamics so that the equilibrium is maintained. The same effect is assumed for the collateral value of the asset.

Herein we introduce two “pricing” functions. Let \( \hat{f}_i : \mathbb{R}^n_+ \to [0, 1] \) denote the average price obtained by bank \( i = 1, ..., n \) given the set of system liquidations \((s_1, ..., s_n) \in \mathcal{D} := \prod_{j=1}^{n}[0, a_j]\). Note that we implicitly impose a no short selling constraint throughout this work. Here, without

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loss of generality, it was assumed that the current, highest price of the asset, before any sales happened is 1, and it can only decrease thereafter. Notably, the construction of \( \bar{f}_i \) implies that different banks may obtain different prices in the market due to the market design or different order sizes. Let \( g : \mathbb{R}_+ \rightarrow [0, 1] \) denote the price of the collateralized asset in the repurchase agreements under study, i.e., the function \( g \left( \sum_{j=1}^{n} s_j \right) \) encodes the haircut on the asset as a mapping of the total liquidations by all the banks. Note that while the price obtained by bank \( i \) may be unique due to the different quantities different banks are selling, since the repo transaction is collateralized it is assumed that the repo market offers the same repo rate \( r \) to all banks and uses the same haircut \( g \left( \sum_{j=1}^{n} s_j \right) \). Though we call \( g \) the “haircut”, it is more appropriate to denote \( 1 - g \) to be the true haircut on the asset in the repo market. At various times in this work we will refer to \( g \) as the haircut and others \( 1 - g \) will be given that name.

Assuming banks sell \( s := (s_1, \ldots, s_n) \in \mathcal{D} \), the realized loss to bank \( i \) from the sale is \( s_i(1 - \bar{f}_i(s)) \). The bank obtained \( s_i \bar{f}_i(s) \) through this sale, therefore it needs to borrow an additional \( (h_i - s_i \bar{f}_i(s)) \) for the cost of \( r(h_i - s_i \bar{f}_i(s)) \). We will abuse notation and denote for convenience \( \bar{f}_i \) to be both \( \bar{f}_i(s, s_{-i}) \) and \( \bar{f}_i(s) \). Therefore, bank \( i \) seeks to optimize:

\[
s_i^* = s_i^*(s_{-i}) = \arg\min_{s_i \in [0, a_i]} s_i \left( 1 - \bar{f}_i(s_i, s_{-i}) \right) + r \left( h_i - s_i \bar{f}_i(s_i, s_{-i}) \right)
\]

\[\text{s.t. } s_i \leq \frac{h_i}{\bar{f}_i(s_i, s_{-i})}, \quad s_i \geq \frac{h_i - a_i g \left( \sum_{j=1}^{n} s_j \right)}{\bar{f}_i(s_i, s_{-i}) - g \left( \sum_{j=1}^{n} s_j \right)} . \quad (2.1)\]

Here, the first inequality ensures that bank \( i \) does not obtain more than \( h_i \) through the asset sale, and the second inequality constraint is used to ensure that \( h_i - s_i \bar{f}_i(s_i, s_{-i}) \leq (a_i - s_i) g \left( \sum_{j=1}^{n} s_j \right) \), i.e., after the sale, bank \( i \) has enough collateral \((a_i - s_i) g \left( \sum_{j=1}^{n} s_j \right)\) to cover its loan. The paper of \textbf{Bichuch and Feinstein (2019)} considers the case in which no haircut is taken, i.e., \( g \equiv 1 \).

In \( (2.1) \), it follows that bank \( i \) is solvent if and only if

\[
\frac{h_i - a_i g \left( \sum_{j=1}^{n} s_j \right)}{\bar{f}_i(s_i, s_{-i}) - g \left( \sum_{j=1}^{n} s_j \right)} \leq \frac{h_i}{\bar{f}_i(s_i, s_{-i})} \leq a_i.
\]

By construction of the haircut for repurchase agreements \( 0 \leq g \left( \sum_{j=1}^{n} s_j \right) < \bar{f}_i(s_i, s_{-i}) \). Under such a construction bank \( i \) is solvent if and only if \( h_i \leq a_i \bar{f}_i(s_i, s_{-i}) \), i.e., if at the current price realized by bank \( i \) it is possible for said bank to cover its shortfall by liquidations alone. If bank \( i \) is insolvent then we will assume that it is forced to liquidate all of its asset holdings, i.e., \( s_i^* = a_i \).

For convenience, for the remainder of this work, denote \( \bar{q}_i = \bar{f}_i(s_i, s_{-i}), \quad i = 1, \ldots, n \), and \( q = g \left( \sum_{j=1}^{n} s_j \right) \). With this notation, we modify \( (2.1) \) (similarly as in \textbf{Bichuch and Feinstein (2019)}).

4
Proof of Theorem 2.2. Theorem 2.2 with \( g \) every argument with \( \bar{g} \).

For \( i \) 1,

\[
\text{let } \sum_{j=1}^{n} s_j^* = 0 \]

We assume that \( \bar{g} \) functions \( f \) and \( 
\text{and, in such a situation, } s_i^* = a_i. \text{ Our goal is primarily to find conditions for existence and uniqueness of this Nash game in the financial system. In order to do that we need assumptions on the inverse demand functions } \tilde{f}_i \text{ and } g. \]

**Assumption 2.1.** Let \( M \geq \sum_{i=1}^{n} a_i \) be the total initial market capitalization of the illiquid asset. For \( i = 1, ..., n \) we assume that \( \tilde{f}_i: \mathbb{R}^n_+ \rightarrow [0, 1] \) are each continuous and strictly decreasing in every argument with \( \tilde{f}_i(0, ..., 0) = 1 \) and \( \tilde{f}_i(a_1, ..., a_n) > 0 \). Additionally, for \( i = 1, ..., n \) and \( s_{-i} \in \prod_{j=1, j \neq 1}^{n} [0, a_j] \) we assume that \( s_i \mapsto s_i \tilde{f}_i(s, s_{-i}) \) is concave.

The haircut function \( g: \mathbb{R}_+ \rightarrow [0, 1] \) is continuous and strictly decreasing, with \( \min_{1 \leq i \leq n} \tilde{f}_i(s) > g \left( \sum_{j=1}^{n} s_j^* \right) \) for every \( s \in \mathcal{D} \).

Existence of a Nash equilibrium easily follows as a consequence of Brouwer’s fixed-point theorem:

**Theorem 2.2 (Existence of Nash Equilibrium).** Assume the inverse demand functions \( \tilde{f}_i, i = 1, ..., n \) and haircut function \( g \) satisfy Assumption 2.1. Then there exists a Nash equilibrium liquidating strategy \( s^{**} \in \mathcal{D} \) with equilibrium prices \((q^{**}, q_1^{**}, ..., q_n^{**}) = (g \left( \sum_{i=1}^{n} s_i^{**} \right), \tilde{f}_1(s^{**}), ..., \tilde{f}_n(s^{**})) \).

**Proof of Theorem 2.2** Fix bank \( i \) and consider (2.2) as a function of \((s_{-i}, q, \bar{q}_i)\) such that \( 0 \leq q < \bar{q}_i \), with \( f_i(a_1, ..., a_n) \leq \bar{q}_i \), and \( s_{-i} \in \prod_{j=1, j \neq i}^{n} [0, a_j] \). Since the objective function of (2.2) is convex in \( s_i \) and the constraint set is a convex interval, the set of minimizers for a fixed set of parameters \((s_{-i}, q, \bar{q}_i)\) is convex. An application of Berge maximum theorem (on \( \bar{q}_i \geq \frac{h_i}{a_i} \) because of the continuity of the objective and constraint functions) yields upper continuity and convex-valuedness of the set of maximizers. This is extended for the region of insolvency by the assumption that \( s_i^* = a_i \) on \( h_i > a_i \bar{q}_i \). Thus a joint equilibrium \((s^{**}, q^{**}, q_1^{**}, ..., q_n^{**})\) can be found via Kakutani’s fixed point theorem.

It turns out that the conditions for existence of equilibrium are very mild, compared to the uniqueness conditions. This is not surprising considering the following example.

**Example 2.3.** Consider an \( n = 1 \) bank setting with \( r = 0 \) repo rate. This bank has assets and shortfall so that \( a_1 \tilde{f}_1(a_1) < h_1 < a_1 g(0) \). Therefore, two possible solutions exist:
1. If the bank liquidates no assets then \((q^{**}, \bar{q}_1^{**}, s_1^{**}) = (g(0), 1, 0)\) is an equilibrium solution;
2. If the bank defaults and liquidates all its assets then \((q^{**}, \bar{q}_1^{**}, s_1^{**}) = (g(a_1), \bar{f}_1(a_1), a_1)\) is an equilibrium solution.

We now concentrate our efforts into understanding properties of these equilibria and find conditions to guarantee their uniqueness. In what follows we will concentrate on two examples for the inverse demand functions \(\bar{f}_i\), \(i = 1, \ldots, n\).

3 Main results

We now concentrate our efforts into understanding when the above equilibrium is unique. In what follows we will concentrate on two sample functions. However, instead of specifying the inverse demand functions \(\bar{f}_i\) directly, we derive them from a density function of limit order book together with some trading rules. Let this density be given by \(f : \mathbb{R}_+ \rightarrow [0, 1]\). Alternatively, this \(f\) can be viewed as a the price of the next infinitely small trade. We concentrate on two realistic examples of price constructions given the liquidations, i.e., market rules, to construct the price of the trade with functional forms \(\bar{f}_i : \mathbb{R}_+^n \rightarrow [0, 1]\) which provides the average price obtained by firm \(i\) given the set of system liquidations.

1. **Volume Weighted Average Price (VWAP):** For \(i = 1, \ldots, n\) set
\[
\bar{f}_i(s) := \frac{\sum_{j=1}^{n} s_j f(\sigma) d\sigma}{\sum_{j=1}^{n} s_j}.
\]
Note that in this case \(\bar{f}_i(s) = \bar{f}_j(s)\) for \(i, j \in \{1, 2, \ldots, n\}\).

2. **Limit Order Book Based Price (LOB):** For \(i = 1, \ldots, n\) set
\[
\bar{f}_i(s) := \frac{1}{s_i} \sum_{j=1}^{k} \frac{1}{n - (j - 1)} \int_{\sum_{l=1}^{j-1} (n - (l-1)) (s_l - s_{l-1})}^{\sum_{l=1}^{j} (n - (l-1)) (s_l - s_{l-1})} f(\sigma) d\sigma,
\]
where \(0 =: s_{[0]} \leq s_{[1]} \leq s_{[2]} \leq \ldots \leq s_{[n]}\) are the order statistics and \(s_i = s_{[k]}\).

Note that the VWAP example corresponds to how some exchanges calculate the closing price (e.g., in Mexico, India and Saudi Arabia\(^1\)). Therefore, given our assumption that this is an illiquid asset, this is a good representation of price paid by banks given the amounts of trades they (collectively) want to make. Whereas the LOB example is an example of how to price market trades all coming at the same time using an existing limit order trades already in the book. This is a very interesting and novel example, as in this case, different banks pay different prices. As far as the authors are aware, this LOB construction has never previously been formulated.

Alternatively, these specific pricing functionals can be viewed as a limit as order sizes decrease to zero at different rates. VWAP can be viewed as the limit when all banks submit their orders at a rate proportional to the total desired liquidation; as such, every bank finishes trading at the same “time” and thus all banks obtain the same average price. In contrast, the LOB is the limit when

\(^1\)research.ftserussell.com/products/downloads/Closing_Prices_Used_For_Index_Calculation.pdf
all banks submit their orders at the same rate; as such, banks finish their transactions at different times based on the desired quantity of assets to be liquidated which generates heterogeneous prices for different trading strategies. Therefore, though this model is static, these constructions allow us to approximate simple time dynamics.

The following assumptions are placed on the order book density function \( f \):

**Assumption 3.1.** Let \( M \geq \sum_{i=1}^{n} a_i \) be the total initial market capitalization of the illiquid asset. The order book density function \( f : \mathbb{R}_+ \rightarrow [0, 1] \) is strictly decreasing and twice continuously differentiable, with \( f(0) = 1 \) and \( f(s) > 0 \) for any \( s \in [0, M] \). Additionally it will be assumed that the first derivative \( f' : \mathbb{R}_+ \rightarrow -\mathbb{R}_+ \) is nondecreasing.

Throughout the remainder of this work we often wish to consider a comparison of vectors of \((q, \bar{q})\); this is accomplished in the usual way, i.e., \((q^1, \bar{q}^1) \geq (q^2, \bar{q}^2)\) if and only if \(q^1 \geq q^2\) and \(\bar{q}^1_i \geq \bar{q}^2_i\) for every \(i = 1, \ldots, n\).

Our next goal is to ultimately establish uniqueness-type properties of the Nash equilibrium. In order to do so, similarly to Bichuch and Feinstein (2019), we consider the problem with fixed liquidation price(s) and the haircut value as described in (2.2). As opposed to Theorem 2.2 above, we first show that there exist unique Nash equilibrium liquidations for these fixed prices as shown in Proposition 3.2 below, the proof of which is delayed until Appendix A.1.

**Proposition 3.2.** Let \( \hat{Q} := \{(q, \bar{q}) \in (0, 1] \times (0, 1]^n \mid q < \bar{q}_i \forall i = 1, 2, \ldots, n\} \). Under VWAP or LOB structure and Assumption 3.1, given \((q, \bar{q}_1, \ldots, \bar{q}_n) \in \hat{Q}\) there exists a unique set of equilibrium liquidations \(\bar{s}(q, \bar{q}_1, \ldots, \bar{q}_n) = s^*(\bar{s}(q, \bar{q}_1, \ldots, \bar{q}_n), q, \bar{q}_1, \ldots, \bar{q}_n)\) to (2.2).

From Example 2.3 it is clear that the uniqueness of the equilibrium does not hold without further assumptions, but we can show, as done in Theorem 3.3, that the set of all fixed points prices \((q^*, \bar{q}^*)\) in the Nash equilibrium of (2.2) is a lattice under a VWAP pricing scheme; notably, as demonstrated below in Example 3.4, the LOB pricing scheme does not satisfy the typical conditions for this result. The proof of the theorem is presented in the Appendix A.2.

**Theorem 3.3.** Under the VWAP structure and Assumption 3.1, the set of clearing haircuts and prices is a lattice; in particular, there exists a greatest and least clearing haircut and set of clearing prices: \((q^*, \bar{q}^*_1, \ldots, \bar{q}^*_n) \geq (q^*, \bar{q}^*_1, \ldots, \bar{q}^*_n)\).

**Sketch of proof.** Taking advantage of Proposition 3.2 we find that the sum \(\sum_{i=1}^{n} \bar{s}_i\) is monotonic in \((q, \bar{q}_1, \ldots, \bar{q}_n)\). Therefore we apply Tarski’s fixed point theorem. The details are provided in Appendix A.2.

Importantly, Theorem 3.3 is not applied to the LOB structure. To demonstrate the lack of such a result for the LOB, we present here a counterexample to the monotonicity property utilized in the proof of Theorem 3.3.

**Example 3.4.** Consider an \( n = 2 \) bank system with \( r = 5\% \) repo rate. Let bank 1 have shortfall \( h_1 = 1.98 \) and \( a_1 = 10 \) assets. Let bank 2 have shortfall \( h_2 = 5 \) and \( a_2 = 10 \) assets. Consider
a linear order density function $f(s) = 1 - 0.0095s$ and haircut function $g(s) = 0.9 - 0.0095s$. We then consider two possible inputs to the clearing system $(q^1, \bar{q}^1) := (0.9, (1,1))$ and $(q^2, \bar{q}^2) := (0.9, (0.99, 0.98))$.

1. Under initial prices of $(q^1, \bar{q}^1)$, the banks sell $\bar{s}(q^1, \bar{q}^1) = (1.98,3.02)$ assets each. This results in LOB based prices of $\bar{q}^1_{2} := 0.9811$ and $\bar{q}^1_{1} := 0.9729$ for banks 1 and 2 respectively. The resulting haircut is $q^{1,\dagger} := 0.8524$.

2. Under initial prices of $(q^2, \bar{q}^2)$, the banks sell $\bar{s}(q^2, \bar{q}^2) = (2,3)$ assets each. This results in LOB based prices of $\bar{q}^2_{2} := 0.9810$ and $\bar{q}^2_{1} := 0.9730$ for banks 1 and 2 respectively. The resulting haircut is $q^{2,\dagger} := 0.8524$.

Notably, though $(q^1, \bar{q}^1) \geq (q^2, \bar{q}^2)$, this monotonicity does not hold for the resulting prices as $(q^{1,\dagger}, \bar{q}^{1,\dagger}) \nless (q^{2,\dagger}, \bar{q}^{2,\dagger})$.

Finally, we introduce additional assumptions and establish uniqueness of the equilibrium in Theorem 3.6 below, the proof of which is delayed until the Appendix A.3. For such a result we introduce a simplified notation, let $\partial_x := \frac{\partial}{\partial x}$ denote the partial derivative operator with respect to some variable $x$.

**Definition 3.5.** We will say that bank $i \in \{1, ..., n\}$ is fundamentally solvent if it is able to cover its shortfall in any case, that is if $h_i \leq a_i \bar{f}(\mathbf{a})$, where $\mathbf{a} = (a_1, ..., a_n)^\top$.

**Remark 1.** If bank $i$ is fundamentally solvent then there is a feasible solution to the maximization problem (2.2), provided $(q, \bar{q}) = (g(\sum_{i=1}^n s_i), \bar{f}(\mathbf{s}))$ for some $\mathbf{s} \in D$, since the feasible region is non-empty. Indeed,

1. $\frac{h_i}{\bar{q}_i} \leq a_i$ if and only if $h_i \leq a_i \bar{q}_i$.

2. $\frac{h_i - a_i q}{\bar{q}_i - q} \leq a_i$ if and only if $h_i \leq a_i \bar{q}_i$.

3. $\frac{h_i - a_i q}{\bar{q}_i - q} \leq \frac{h_i}{\bar{q}_i}$ if and only if $h_i \leq a_i \bar{q}_i$.

**Theorem 3.6.** Assume all banks are fundamentally solvent. Under VWAP or LOB structure and Assumption 3.7 if additionally, $-c M (\min_{j,k} \partial_{s_i} \bar{f}_j(0_n) \land g'_i(0)) < \min_{s \in D} (\bar{f}_j(s) - g(\sum_{i=1}^n s_i))$ with $c = 3$ and $c = n$ in case of VWAP and LOB, respectively, then there exists a unique clearing haircut and set of actualized prices $(q^*, \bar{q}^*)$.

**Sketch of proof.** The proof follows from the Banach fixed point theorem and is presented in Appendix A.3.\qed

**Remark 2.** At this point we wish to recall Example 2.3 which highlights a case of non-uniqueness of the clearing solution. In that single bank setting, the bank is not fundamentally solvent since, by construction, $a_1 \bar{f}_1(a_1) < h_1$. This highlights the importance of the assumption that all banks are fundamentally solvent in Theorem 3.6 for the uniqueness of the clearing prices.

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Remark 3. With the consideration of existence and uniqueness of the clearing solution, the sensitivity of the equilibrium liquidations and prices to the repo rate $r$ is of great interest. This is studied mathematically in Appendix B. Intuitively, we expect that as the repo rate rises, and borrowing becomes more expensive the liquidation of the illiquid asset increase. It also follows from here that, the higher the interest rate, the lower the terminal asset price. Alternatively, from a regulator’s perspective, if the goal is to limit the extent of the fire sales, it can be achieved by controlling the interest rates, as was done recently in September 2019, and was also used extensively during the crisis (see Quinn et al. (2020) and Cecchetti (2009) respectively). We refer to the case studies in Section 4 for visualizations of this notion.

4 Comparative statics

Before considering specific examples, we will first introduce a consideration for the computation of the clearing prices $(q, \bar{q}) = (g(\sum_{i=1}^{n} \bar{s}_i(q, \bar{q})), \bar{f}(\bar{s}(q, \bar{q})))$. This approach will always converge for the VWAP setting due to Theorem 3.3 though the LOB setting does not satisfy monotonicity, this algorithm converged for every choice of parameters attempted by the authors indicating a stronger result than found thus far. Specifically, these are computed via Picard iterations beginning from $(q^0, \bar{q}^0) := 1_{n+1}$. However, $\bar{s}(q, \bar{q})$ will require consideration for computation itself due to its game theoretic construction. As provided in Proposition 3.2 these liquidations exist and are unique. In fact, due to the construction of the problem as discussed in the proof of that proposition, we are able to apply the algorithm provided in Rosen (1965). This is summarized in Algorithm 2 of Bichuch and Feinstein (2019) for the VWAP setting. We wish to note that in the LOB setting, the computation can be improved significantly via an iterative approach of determining the banks liquidating the fewest number of assets.

In this section we will consider two primary case studies. The first is a consideration of the VWAP and LOB structures to determine their relative ordering, i.e., is one better than the other. This is important from a mechanism design perspective as different markets consider the closing price using different rule sets. The second case study we will consider is an implementation of European banking data to determine the impacts of interest rates and haircut functions on the clearing prices.

4.1 Mechanism design

In this first case study, we will investigate two networks in detail in order to show that some system constructions find that VWAP has more total liquidations with less system-wide use of the repo markets than LOB, while other constructions have the reverse ordering. In particular, we will first consider a system of $n$ identical banks and second a specific system of $n = 2$ banks only.
4.1.1 Symmetric case study

Consider a system of \( n \) identical banks. Each of these banks has shortfall \( h > 0 \) and assets \( a > 0 \). The prevailing repo rate is provided by \( r \in (0, \frac{1}{4}) \). For the purposes of this example, consider the order book density \( f(s) = 1 - \alpha s \) and haircut function \( g(s) = \frac{1}{2} - \alpha s \) for \( \alpha \in \left( \frac{2r}{(1+r)(n+1)a}, \frac{1}{2n} \right) \); notably these constructions satisfy Assumption 3.1 and taken so as to construct an example in which firms have a choice of behavior. Consider now our two market mechanisms: VWAP and LOB.

1. VWAP: By construction \( f_i(s) := 1 - \frac{2}{n} \sum_{j=1}^{n} s_j \) for every bank \( i \) in the VWAP construction.

Additionally, we take advantage of the symmetric setup to conclude that all banks should follow the same strategy, i.e., \( s_i^* = \frac{1}{n} \sum_{j=1}^{n} s_j \). Consider game (2.2) for fixed values \( (q, \bar{q}) \) with \( q < \bar{q} \):

\[
\begin{align*}
\left( s_i^*, \bar{q} \right)_n = & \arg \min_{s_i \in [0, a]} \left\{ \alpha (1 + r) \left( \sum_{j \neq i} s_j^*(q, q \mathbf{1}_n) + s_i \right) s_i + r(h - s_i) : s_i \in \left[ \frac{h - aq, h}{\bar{q} - q}, \frac{h}{\bar{q}} \right] \right\} \\
= & \arg \min_{s_i \in [0, a]} \left\{ \frac{\alpha}{2} (1 + r) s_i^2 + \frac{\alpha}{2} (1 + r)(n - 1) s^*_{VWAP}(q, \bar{q}) - r \right\} s_i + rh : s_i \in \left[ \frac{h - aq, h}{\bar{q} - q}, \frac{h}{\bar{q}} \right] \\
= & \frac{h - aq}{\bar{q} - q} \sqrt{\frac{r}{(1 + r)\alpha} - \frac{n - 1}{2} s^*_{VWAP}(q, \bar{q})} \wedge \frac{h}{\bar{q}},
\end{align*}
\]

if \( h < a\bar{q} \) (and \( s^*_{VWAP}(q, \bar{q}) = a \) if \( h \geq a\bar{q} \)). In particular, this provides a single fixed point problem to find \( s^*_{VWAP}(q, \bar{q}) \), i.e.,

\[
\begin{align*}
\frac{h - aq}{\bar{q} - q} \wedge \frac{h}{\bar{q}} = & \frac{r}{(1 + r)\alpha} - \frac{n - 1}{2} s^*_{VWAP}(q, \bar{q}) \\
\Rightarrow s^*_{VWAP}(q, \bar{q}) = & \frac{h - aq}{\bar{q} - q} \wedge \frac{2r}{\alpha(1 + r)(n + 1)} \wedge \frac{h}{\bar{q}}
\end{align*}
\]

if \( h < a\bar{q} \). We wish to note that the existence of \( s^*_{VWAP}(q, \bar{q}) \) justifies our choice of \( s^*_{VWAP} = s^*_{VWAP} \mathbf{1}_n \) as, due to Proposition 3.2, \( s^*_{VWAP} \) is unique and thus must equal \( s^*_{VWAP} \mathbf{1}_n \). Finally, it remains to find the equilibrium prices \( (q^*_{VWAP}, \bar{q}^*_{VWAP}) \):

\[
q^*_{VWAP} = \begin{cases} 
-\frac{1}{2} + \sqrt{1 - 2a\alpha h} & \text{if } h \in \mathcal{H}^2_{VWAP}, \\
\frac{1}{2} - \frac{2rn}{(1+r)(n+1)} & \text{if } h \in \mathcal{H}^1_{VWAP}, \\
1 - \alpha a - \frac{1}{2} \sqrt{1 + 8a\alpha h - 4(\alpha a)^2} & \text{if } h \in \mathcal{H}^3_{VWAP}, \\
\frac{1}{2} - \alpha a & \text{if } h \in \mathcal{H}^4_{VWAP}.
\end{cases}
\]
with borrowing/liquidation regions

$$\mathcal{H}^{VWAP}_1 = \left[ 0, \frac{2r}{\alpha(1 + r)(n + 1)} \left(1 - \frac{rn}{(1 + r)(1 + n)}\right) \right],$$

$$\mathcal{H}^{VWAP}_2 = \left[ \frac{2r}{\alpha(1 + r)(n + 1)} \left(1 - \frac{rn}{(1 + r)(1 + n)}\right) , \frac{2r}{\alpha(1 + r)(n + 1)} \left(\frac{1}{2} + \frac{rn}{(1 + r)(1 + n)}\right) \right],$$

$$\mathcal{H}^{VWAP}_3 = \left[ \frac{2r}{\alpha(1 + r)(n + 1)} \left(\frac{1}{2} + \frac{rn}{(1 + r)(1 + n)}\right) + a \left(\frac{1}{2} - \frac{2rn}{(1 + r)(n + 1)}\right), a \left(1 - \frac{\alpha}{2}rn\right) \right],$$

$$\mathcal{H}^{VWAP}_4 = \left[a \left(1 - \frac{\alpha}{2}rn\right), \infty \right].$$

We wish to note that all square roots are well defined on the intervals on which they are considered. Additionally, $q^{VWAP}$ and $\tilde{q}^{VWAP}$ are continuous in $h$; as such the closures of the bounding intervals can be chosen arbitrarily. Though this setting does not satisfy the uniqueness conditions of Theorem 3.6, the simplicity of the symmetric system still admits a unique clearing solution.

2. **LOB:** By construction $\tilde{f}_i(s) := 1 - \frac{\alpha}{2s[\bar{i}]} \left[\sum_{k=1}^{i-1} s[k](2s[i] - s[k]) + (n - (i - 1))s[\bar{i}]^2\right]$ for every bank $[i]$ (i.e., the bank liquidating the $i^{th}$ most assets) in the LOB construction. Additionally, we take advantage of the symmetric setup to conclude that all banks should follow the same strategy, i.e., $s^{LOB} = \tilde{s}^{LOB} \mathbf{1}_n$ for some singleton $s^{LOB} \in [0, a]$ and $\tilde{q}^{LOB} = \tilde{q}^{LOB} \mathbf{1}_n$ for some singleton $\tilde{q}^{LOB} \in [0, 1]$. Consider game (2.2) for fixed values $(q, \tilde{q})$ with $q < \tilde{q}$:

$$s^*_i(q, \tilde{q} \mathbf{1}_n) = \arg \min_{s_i \in [0, a]} \left\{ \frac{2}{a(1 + r)} \left[ n_1(s_i \leq s^{LOB}(q, \tilde{q})) + (n - 1)s^{LOB}(q, \tilde{q})^2 + 2(n - 1)s^{LOB}(q, \tilde{q})s_i + s^*_i(s_i > s^{LOB}(q, \tilde{q})) \right] + r(h - s_i) \right\}$$

$$= \begin{cases} \frac{h - aq}{\tilde{q} - q} \wedge \frac{h}{\tilde{q}} & \text{if } \frac{r}{\alpha(1 + r)n} \leq s^{LOB}(q, \tilde{q}), \\ \frac{h - aq}{\tilde{q} - q} \vee \frac{r}{\alpha(1 + r)n} - (n - 1)s^{LOB}(q, \tilde{q}) \wedge \frac{h}{\tilde{q}} & \text{if } \frac{r}{\alpha(1 + r)n} > s^{LOB}(q, \tilde{q}), \end{cases}$$

if $h < a\tilde{q}$ (and $s^{LOB}(q, \tilde{q}) = a$ if $h \geq a\tilde{q}$). In particular, this provides a single fixed point.
problem to find $s_{LOB}(q, \bar{q})$, i.e., if $h < aq$

$$s_{LOB}(q, \bar{q}) = \begin{cases} \frac{h-aq}{\bar{q}-q} \lor \frac{r}{\alpha(1+r)n} \wedge \frac{h}{\bar{q}} \quad & \text{if } \frac{r}{\alpha(1+r)n} \leq s_{LOB}(q, \bar{q}), \\ \frac{h-aq}{\bar{q}-q} \lor \frac{r}{\alpha(1+r)} - (n-1)s_{LOB}(q, \bar{q}) \wedge \frac{h}{\bar{q}} \quad & \text{if } \frac{r}{\alpha(1+r)n} > s_{LOB}(q, \bar{q}) \end{cases}$$

$$\Rightarrow s_{LOB}(q, \bar{q}) = \frac{h-aq}{\bar{q}-q} \lor \frac{r}{\alpha(1+r)n} \wedge \frac{h}{\bar{q}}$$

as both provided cases result in the same fixed point. We wish to note that the existence of $s_{LOB}(q, \bar{q})$ justifies our choice of $s_{LOB} = s_{LOB}1_n$ as, due to Proposition 3.2, $s_{LOB}$ is unique and thus must equal $s_{LOB}1_n$. Finally, it remains to find the equilibrium prices $(q^{LOB}, \bar{q}^{LOB})$:

$$q^{LOB} = \begin{cases} -\frac{1}{2} + \sqrt{1 - 2\alpha nh} \quad & \text{if } h \in H_1^{LOB} \\ \frac{1}{2} - \frac{r}{1+r} \quad & \text{if } h \in H_2^{LOB} \\ 1 - \alpha a - \frac{1}{2} \sqrt{1 + 8\alpha n(h-a) + 4(\alpha a)^2} \quad & \text{if } h \in H_3^{LOB} \\ \frac{1}{2} - \alpha a \quad & \text{if } h \in H_4^{LOB} \end{cases}$$

$$\bar{q}^{LOB} = \begin{cases} \frac{1+\sqrt{1 - 2\alpha nh}}{2} \quad & \text{if } h \in H_1^{LOB} \\ 1 - \frac{2r}{2(1+r)} \quad & \text{if } h \in H_2^{LOB} \\ \frac{5}{4} - \frac{\alpha a}{2} - \frac{1}{4} \sqrt{1 + 8\alpha n(h-a) + 4(\alpha a)^2} \quad & \text{if } h \in H_3^{LOB} \\ 1 - \frac{\alpha a}{2} \quad & \text{if } h \in H_4^{LOB} \end{cases}$$

with borrowing/liquidation regions

$$H_1^{LOB} = \left[ 0, \frac{r}{\alpha(1+r)n} \left( 1 - \frac{r}{2(1+r)} \right) \right)$$

$$H_2^{LOB} = \left[ \frac{r}{\alpha(1+r)n} \left( 1 - \frac{r}{2(1+r)} \right), \frac{r}{2\alpha(1+r)n} \left( 1 + \frac{r}{1+r} \right) \right]$$

$$H_3^{LOB} = \left[ \frac{r}{2\alpha(1+r)n} \left( 1 + \frac{r}{1+r} \right) + a \left( \frac{1}{2} - \frac{r}{1+r} \right), a \left( 1 - \frac{\alpha a}{2} \right) \right]$$

$$H_4^{LOB} = \left[ a \left( 1 - \frac{\alpha a}{2} \right), \infty \right).$$

We wish to note that all square roots are well defined on the intervals on which they are considered. Additionally, $q^{VWAP}$ and $\bar{q}^{VWAP}$ are continuous in $h$; as such the closures of the bounding intervals can be chosen arbitrarily. Though this setting does not satisfy the uniqueness conditions of Theorem 3.6, the simplicity of the symmetric system still admits a unique clearing solution.

Notably, $s_{VWAP}(q, \bar{q}) \geq s_{LOB}(q, \bar{q})$ for any choice of $(q, \bar{q})$ by construction. In fact, if there exists $n \geq 2$ banks, then this inequality is strict at equilibrium on $H_2^{VWAP} \cap H_2^{LOB}$, i.e.,

$$h \in \left( \frac{r}{\alpha(1+r)n} \left( 1 - \frac{r}{2(1+r)} \right), \frac{2r}{\alpha(1+r)(n+1)} \left( \frac{1}{2} + \frac{rn}{(1+r)(n+1)} \right) + a \left( \frac{1}{2} - \frac{2rn}{(1+r)(n+1)} \right) \right).$$
In contrast, by construction of the order book density \( f \), the borrowing by each firm at equilibrium (and therefore total system wide borrowing) is greater under the VWAP framework than the LOB framework, i.e., \( h - s^{VWAP}(q^{VWAP}, \bar{q}^{VWAP})\bar{q}^{VWAP} \leq h - s^{LOB}(q^{LOB}, \bar{q}^{LOB})\bar{q}^{LOB} \), with strict ordering on the same interval as given above.

### 4.1.2 A counterexample to the symmetric ordering

In contrast to the symmetric system above, we now wish to consider a system in which the VWAP setting results in fewer liquidations and more borrowing than the LOB framework. To do this, let’s consider a simple heterogeneous \( n = 2 \) bank setting with \( r = 0.01 \), \( a = (1, 2) \), and \( h = (0.3, 1.2) \). For this example consider the same order book density function \( f(s) = 1 - \alpha s \) and haircut function \( g(s) = \frac{1}{2} - \alpha s \), but with the specific price impact parameter \( \alpha = 0.05 \). With this construction, the clearing liquidations and prices can be determined numerically to be

- \( s^{VWAP} = (0, 0.4853) \) with \( q^{VWAP} = 0.4757 \) and \( \bar{q}^{VWAP} = (0.9879, 0.9879) \).
- \( s^{LOB} = (0.0990, 0.5080) \) with \( q^{LOB} = 0.4696 \) and \( \bar{q}^{LOB} = (0.9950, 0.9828) \).

As desired at the beginning of this example, total liquidations are less for both banks (i.e., \( s^{VWAP} < s^{LOB} \)), but borrowing by both banks has the opposite order (i.e., \( h_i - s_i^{VWAP} q_i^{VWAP} > h_i - s_i^{LOB} q_i^{LOB} \), \( i = 1, 2 \)). This is the opposite order from the symmetric case study considered above; as such, there is no consistent order between the VWAP and LOB settings that can be determined.

### 4.1.3 Discussion

As shown in the prior two examples, there is no consistent ordering between the VWAP and LOB settings. For symmetric systems and, more generally, systems close to symmetric, if the market regulators wish to promote borrowing over liquidations, then the LOB framework is preferable; however, for certain heterogeneous systems, the VWAP framework may be preferable to that same regulator. As such, the use of stress testing of different market mechanisms is of the paramount importance in order to determine the optimal market mechanism.

We wish to make one final consideration on the comparison of the VWAP and LOB frameworks. We conjecture that the distinction between the two setting occurs only if some bank is both liquidating and borrowing. Most prior works, e.g., [Amini et al. (2016)](https://ssrn.com/abstract=3701847), consider only the situation in which firms can only liquidate in order to cover their liabilities. Without borrowing allowed, the VWAP and LOB frameworks will always coincide at the aggregate level. As such the mechanism choice of \( \bar{f} \) is irrelevant when considered in the standard literature (which is written in a VWAP style manner).

### 4.2 EBA case study

We conclude this work with a consideration of financial system calibrated to 2011 European banking data. This stress test data has been utilized in numerous prior studies for studying interbank...
liability networks (e.g., Gandy and Veraart (2016); Chen et al. (2016); Feinstein (2019)). We will calibrate and utilize this EBA dataset in much the same way as in Bichuch and Feinstein (2019), i.e., to have a more realistic system but one that still requires heuristics and, as such, is for demonstration purposes only.

As a stylized bank balance sheet, we will consider two categories of assets: cash assets $c_i$ and illiquid assets $a_i$. We will additionally consider two categories of liabilities: external liabilities $\bar{p}_i$ and capital $C_i$. In order to determine these values, we calibrate the system as in Bichuch and Feinstein (2019) but ignoring all interbank obligations considered as cash so as to discount default contagion and focus solely on price-mediate contagion as discussed in the remainder of this work. The total assets $T_i$ and capital $C_i$ are provided by this dataset directly for each bank $i$. The external liabilities $\bar{p}_i = T_i - C_i$ are computed by balance sheet construction. It remains to split the total assets into cash and illiquid assets; we make this split according to the tier 1 capital ratio $R_i$, i.e., $c_i = R_i T_i$ and $a_i = (1 - R_i) T_i$.

In order to complete our model, we need to consider the remaining parameters of the system. We set the market capitalization $M = \sum_{i=1}^{n} a_i$ to be the total number of shares of the illiquid assets held by the banks. For this example we consider the linear order book density function $f(s) = 1 - \alpha s$ and haircut function $g(s) = \frac{7}{10} - \alpha s$ for $\alpha = \frac{1}{30 M}$ (i.e., a 0.30 euro haircut is charged on top of the “market price” $f(s)$). By construction, this setting satisfies all conditions of Theorem 3.6. We will focus on the impacts of altering the interest rate environment in order to compare the VWAP and LOB settings. This is undertaken in the prevailing low interest rate environment during the period from which this data is collected. For this study, no external shock is applied to the financial system.

For our consideration, we compare the VWAP and LOB settings while varying the interest rate environment. The results of varying the interest rate is displayed in Figure 1. As expected,
total liquidations (Figure 1a) increase as the interest rate increases, whereas the total borrowing (Figure 1b) is exactly the reverse of the total liquidations and, as such, is decreasing as the interest rate increases. Notably, as discussed in the case studies of Section 4.1, under some interest rate environments VWAP encourages more borrowing than LOB and vice versa under other interest rate environments. We find that, system-wide, there is less selling and more borrowing in the LOB setting for higher interest rates. Notably, the LOB setting results in a non-smooth response as a function of the interest rate $r$. This results from the heterogeneous prices actualized by all banks; due to these varying prices, each bank switches strategies at varying interest rates. This is in contrast to the VWAP setting in which, though the banks are heterogeneous, the strategies of the banks mostly overlap. With this notion, it becomes clear that LOB provides greater flexibility for an intervention to control fire sales through the manipulation of interest rates.

5 Conclusion

In this work, we have considered a model of a system of banks that need to raise funds to cover their liquidity shortfalls. These firms decide on an optimal combination to raise the money through borrowing in a repo market and selling an illiquid asset in a fire-sale, with both the haircut and the fire-sale prices dependent on actions of other banks. We focused on two frameworks to determine the fire-sale prices: the volume weighted average price and a notion of the limit order book in order to capture notions of pricing dynamics in this, otherwise, static model. We found sufficient conditions for existence and uniqueness of the Nash equilibrium in this game. Finally, we have compared the VWAP and the LOB settings analytically when the banks are identical and perform a numerical study using the 2011 EBA data.

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A Proofs of Section 3

A.1 Proof of Proposition 3.2

In both the VWAP and LOB settings, for a fixed \((q, \bar{q}_1, \ldots, \bar{q}_n) \in \hat{\mathcal{Q}}\) the existence of an equilibrium \(\bar{s}(q, \bar{q}_1, \ldots, \bar{q}_n)\) follows along the same steps as the proof of Theorem 2.2. We next show the uniqueness of \(\bar{s}(q, \bar{q}_1, \ldots, \bar{q}_n)\) by utilizing the results of [Rosen 1965] on convex games.

A.1.1 Volume weighted averaged price

**Proof.** In this case, the uniqueness of \(\bar{s}(q, \bar{q}_1, \ldots, \bar{q}_n)\) follows from [Bichuch and Feinstein 2019] [Theorem 3.2], as soon as we verify that the assumptions of that theorem hold.

Recall that \(\hat{f}_i\) is independent of the index \(i\). Also, note that we can write \(\hat{f}_i\) as a function of the total liquidation \(s_{-i}\), where

\[
\hat{f}_i(s_i, s_{-i}) = \hat{f}(s_i + s_{-i}) = \frac{1}{s_i + s_{-i}} \int_{0}^{s_i + s_{-i}} f(s) ds.
\]

We assume that \(f\) satisfies Assumption 3.1 and proceed to verify that [Bichuch and Feinstein 2019] [Assumption 2.1] is also satisfied by \(\hat{f}\). Indeed, \(\hat{f}'(s) = -\frac{1}{s^2} \int_0^s f(u) du + \frac{f(s)}{s} \leq -\frac{1}{s^2} f(s) + \frac{f(s)}{s} = 0\). It is also easily seen that \(\frac{\partial^2}{\partial s^2} (\hat{f}(s)) = f'(s) < 0\).

Lastly we need to show that \(\hat{f}'' \geq 0\). Calculate that \(s^2 \hat{f}''(s) = \frac{2}{s} \int_0^s f(u) du - 2f(s) + sf'(s)\). Since \(f\) is convex, we have that \(f(s) - f(0) \leq sf'(s)\), and thus it is sufficient to show that \(\frac{2}{s} \int_0^s f(u) du - f(0) - f(s) \geq 0\).

Using the fact that \(f\) is convex, we have that \(\lambda f(s_1) + (1-\lambda)f(s_2) \leq f(\lambda s_1 + (1-\lambda)s_2)\), \(\lambda \in [0, 1]\). Integrating over \(\lambda \in [0, 1]\) gives \(\frac{(s_1) + f(s_2)}{2} \leq \frac{1}{s_2 - s_1} \int_{s_1}^{s_2} f(u) du\), which gives the desired result.

A.1.2 Limit order book

**Proof.** Recall from [Rosen 1965] that for \(s \in \mathbb{R}^n\), the function \(s \mapsto H(s; \rho)\) is diagonally strictly convex, if for some (fixed) \(\rho \in \mathbb{R}_+^n\) and for every \(s^0, s^1 \in \mathbb{R}^n\), \(s^0 \neq s^1\), we have \((s^1 - s^0)^\top \gamma(s^0; \rho) - (s^1 - s^0)^\top \gamma(s^1; \rho) < 0\), where

\[
\gamma(s; \rho) = \begin{pmatrix}
\partial_{s_1} H_1(s; \rho_1) \\
\vdots \\
\partial_{s_n} H_n(s; \rho_n)
\end{pmatrix} =
\begin{pmatrix}
\rho_1 \left(1 - (1+r)f \left(\sum_{\ell=1}^{k_1} (n-(\ell-1))(s[\ell]-s[\ell-1])\right)\right) \\
\vdots \\
\rho_n \left(1 - (1+r)f \left(\sum_{\ell=1}^{k_n} (n-(\ell-1))(s[\ell]-s[\ell-1])\right)\right)
\end{pmatrix},
\]

where \(k_i\) is such that \(s[k_i] = s_i\). Additionally, [Rosen 1965, Theorem 6] shows that a sufficient condition for \(H\) to be diagonally strictly convex is if \(\Gamma(s; \rho) + \Gamma(s; \rho)^\top\) is a symmetric positive definite matrix for every \(s \in \mathbb{R}^n\) and some \(\rho \in \mathbb{R}_+^n\), where \(\Gamma\) is the Jacobian matrix of \(\gamma\) with respect to \(s\). Without loss of generality, for fixed value \(s\), assume \(k_i = i\) for every bank.

Set \(\rho_i = \frac{1}{1+r}\), then

\[
[\Gamma(s; \rho) + (\Gamma(s; \rho))^\top]_{ij} = -(1 + \mathbb{I}_{i=j}) [2(n-i) + 1])f(\sum_{\ell=1}^{k_i} s_\ell + (n - i \vee j - 1)s_{i \vee j}).
\]
Thus, in full matrix notation, we find

$$\Gamma(s; \rho) + \Gamma(s; \rho)\top = A(s) + \sum_{j=1}^{n} B_j(s)$$

$$A(s) = -\operatorname{diag}\left(\left[2(n-i) + 1\right]f^\prime\left(\sum_{\ell=1}^{i-1} s_{\ell} + (n - (i - 1))s_i\right)\right)$$

$$B_j(s) = \begin{cases} 
    f^\prime\left(\sum_{\ell=1}^{j-1} s_{\ell} + (n - (j - 1))s_j\right) - f^\prime\left(\sum_{\ell=1}^{j} s_{\ell} + (n - j)s_{j+1}\right) & \text{if } j < n \\
    -f^\prime(\sum_{\ell=1}^{n} s_{\ell}) 1_{n \times n} & \text{if } j = n.
\end{cases}$$

For any liquidations \(s\), by construction, the matrix \(A(s)\) is positive definite and \(B_j(s)\) is positive semidefinite (by nondecreasing property of \(f^\prime\)). The uniqueness of \(\bar{s}(q, \bar{q}_1, ..., \bar{q}_n)\) follows from [Rosen (1965) [Theorem 2]].

**A.2 Proof of Theorem 3.3**

Our goal is to apply Tarski’s fixed point theorem, to do which, we need to prove that

$$\sum_{i=1}^{n} \bar{s}_i(q, \bar{q} 1_n) \geq \sum_{i=1}^{n} \bar{s}_i(q^*, \bar{q}^*)$$

for \((q^*, \bar{q}^*) \leq (q^\top, \bar{q}^\top)\).

**Proof.** Recall the definitions of \(\hat{f}\) and \(s_{-i}\) given in (A.2) and (A.1) respectively. Therefore, we can assume that \(\bar{q} := \bar{q}_1 = ... = \bar{q}_n\). First we note that for \(i = 1, ..., n\)

$$\bar{s}_i(q, \bar{q} 1_n) = \begin{cases} 
    a_i & \text{if } h_i \geq a_i \bar{q}, \\
    \left(\frac{h_i - a_i \bar{q}}{q - \bar{q}}\right)^+ & \text{if } h_i < a_i \bar{q}, \\
    \left[s_i^0(\bar{s}_{-i}(q, \bar{q} 1_n)) \wedge \frac{h_i}{q}\right] & \text{if } h_i < a_i \bar{q},
\end{cases}$$

(A.4)

where \(s_i^0\) is the solution to

$$1 - (1 + r)(\hat{f}(s_i^0 + \bar{s}_{-i}(q, \bar{q} 1_n)) + s_i^0 \hat{f}(s_i^0 + \bar{s}_{-i}(q, \bar{q} 1_n))) = 0.$$  

(A.5)

With this construction we wish to note that bank \(i\) is defaulting and has no other option but to liquidate all its assets if and only if \(h_i \geq a_i \bar{q}\). Indeed, as noted previously in the body of this work, in the opposite case \(h_i < a_i \bar{q}\), we have that:

1. \(\frac{h_i}{q} < a_i\) if and only if \(h_i < a_i \bar{q}\).
2. \(\frac{h_i - a_i \bar{q}}{q - \bar{q}} < a_i\) if and only if \(h_i < a_i \bar{q}\).
3. \(\frac{h_i - a_i \bar{q}}{q - \bar{q}} < \frac{h_i}{q}\) if and only if \(h_i < a_i \bar{q}\).

Therefore, \(\bar{s}_i(q, \bar{q} 1_n), \ i = 1, ..., n\) is well defined in (A.4), and \(\bar{s}_i(q, \bar{q}) < a_i\) in all those cases.

First consider the case when all banks keep at the same liquidation strategy, in other words the definition of \(\bar{s}_i\) in (A.4) is equal to the same term (i.e., among \(a_i, \frac{h_i}{q} + \left(\frac{h_i - a_i \bar{q}}{q - \bar{q}}\right)^+, \) and \(s_i^0\)). Then for \(i = 1, ..., n:\)

- If \(\bar{s}_i = a_i\) then \(\partial_q \bar{s}_i, \partial_q \bar{s}_i = 0\).
• If $\bar{s}_i = \frac{h_i}{q}$, then $\partial_y \bar{s}_i = 0, \partial_q \bar{s}_i < 0$.

• If $\bar{s}_i = \left(\frac{h_i - a_i q}{q - q} \right)^+$, first assume that $h_i \geq a_i q$, in addition to $h_i < a_i \bar{q}$. The former results in $\partial_q \bar{s}_i \leq 0$, while it follows from the latter that $\partial_q \bar{s}_i < 0$. If, instead, $h_i < a_i q$ then $\partial_q \bar{s}_i = \partial_q \bar{s}_i = 0$. Note that we have also used our assumption that $\bar{q} > q$.

• The last case to consider is when $\bar{s}_i = s_i^0$. This is the most interesting case because $(s_i^0)' \in (-1, 0]$ as shown in [Bichuch and Feinstein (2019)] [Theorem 3.2], therefore the derivative has the opposite sign $\partial_q s_i^0 = (s_i^0)' \partial_q s_i < 0$. This case, requires a more careful analysis as follows.

Let $I_0$ be the set of banks $j = 1, \ldots, n$, such that $\bar{s}_j = s_j^0$, then differentiating (A.5) w.r.t. $\bar{q}$, and using the fact that $(s_i^0)' \in (-1, 0]$, we see that

$$\partial_q \bar{s}_I_0 = -\left(\text{diag}(1_{|I_0|} - c) + c 1_{|I_0|}^\top \right)^{-1} c \sum_{j \notin I_0} \partial_q \bar{s}_j,$$

for some $c \in [0, 1]^{I_0}$. First, we wish to show that $\text{diag}(1_{|I_0|} - c) + c 1_{|I_0|}^\top$ is invertible:

$$\det \left(\text{diag}(1_{|I_0|} - c) + c 1_{|I_0|}^\top\right) = \det \begin{pmatrix}
1 & c_{i_1} & \cdots & c_{i_1} \\
c_{i_2} & 1 & \cdots & c_{i_2} \\
\vdots & \vdots & \ddots & \vdots \\
c_{i_{|I_0|}} & c_{i_{|I_0|}} & \cdots & 1
\end{pmatrix}
$$

$$= \det \begin{pmatrix}
1 & -(1 - c_{i_1}) & \cdots & -(1 - c_{i_1}) \\
c_{i_2} & 1 - c_{i_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
c_{i_{|I_0|}} & 0 & \cdots & 1 - c_{i_{|I_0|}}
\end{pmatrix}
$$

$$= \left(1 + (1 - c_{i_1}) \sum_{i \in I_0 \setminus \{i_1\}} \frac{c_i}{1 - c_i}\right) \prod_{i \in I_0 \setminus \{i_1\}} (1 - c_i)
$$

$$= \left(1 + (1 - c_{i_1}) \sum_{i \in I_0} \frac{c_i}{1 - c_i} - (1 - c_{i_1}) \frac{c_{i_1}}{1 - c_{i_1}}\right) \prod_{i \in I_0 \setminus \{i_1\}} (1 - c_i)
$$

$$= \left(1 + \sum_{i \in I_0} \frac{c_i}{1 - c_i}\right) \prod_{i \in I_0} (1 - c_i),
$$

where the 2nd line follows from subtracting the first column from every subsequent column and the 3rd line by using the Schur complement to determine the determinant. Thus we find that

$$\det \left(\text{diag}(1_{|I_0|} - c) + c 1_{|I_0|}^\top\right) = \left(1 + \sum_{i \in I_0} \frac{c_i}{1 - c_i}\right) \prod_{i \in I_0} (1 - c_i) > 0.$$

Taking all this together:

$$\sum_{i=1}^n \partial_q \bar{s}_i = \sum_{j \notin I_0} \partial_q \bar{s}_j - 1_{|I_0|}^\top \left(\text{diag}(1_{|I_0|} - c) + c 1_{|I_0|}^\top\right)^{-1} c \sum_{j \notin I_0} \partial_q \bar{s}_j
$$

$$= \left(1 - 1_{|I_0|}^\top \left(\text{diag}(1_{|I_0|} - c) + c 1_{|I_0|}^\top\right)^{-1} c \right) \sum_{j \notin I_0} \partial_q \bar{s}_j. \quad (A.6)$$
Moreover $1 - \mathbf{1}_{[t_0]}^\top (\text{diag}(1_{[t_0]} - \mathbf{c}) + \mathbf{c} \mathbf{1}_{[t_0]})^{-1} \mathbf{c} \geq 0$, since:

$$1_{[t_0]}^\top (\text{diag}(1_{[t_0]} - \mathbf{c}) + \mathbf{c} \mathbf{1}_{[t_0]})^{-1} \mathbf{c} = 1_{[t_0]}^\top \left( \text{diag}(1_{[t_0]} - \mathbf{c})^{-1} - \frac{\text{diag}(1_{[t_0]} - \mathbf{c})^{-1} \mathbf{c} \mathbf{1}_{[t_0]} \text{diag}(1_{[t_0]} - \mathbf{c})^{-1}}{1 + \mathbf{1}_{[t_0]}^\top \text{diag}(1_{[t_0]} - \mathbf{c})^{-1} \mathbf{c}} \right) \mathbf{c}$$

$$= \sum_{i \in I_0} \frac{c_i}{1 - c_i} - \frac{1}{1 + \sum_{i \in I_0} \frac{c_i}{1 - c_i} \sum_{j \in I_0} \frac{c_j}{1 - c_j}} \sum_{i \in I_0} \frac{c_i c_j}{1 - c_i (1 - c_j)}$$

$$= 1 + \sum_{i \in I_0} \frac{c_i}{1 - c_i} \sum_{i \in I_0} \frac{c_i}{1 - c_i} \leq 1,$$

where the first equality follows from the Sherman-Morrison matrix identity.

It now follows from (A.6) that $\sum_{i=1}^n \partial_q \bar{s}_i \leq 0$, as desired. The same calculation also shows that $\sum_{i=1}^n \partial_q \bar{s}_i \leq 0$, and therefore (A.3) holds.

Finally, in the case, that some banks may switch liquidation strategies, we use the fact that the mappings $\bar{s}_i(\cdot, \cdot), \bar{s}_i^*(\cdot) i = 1, \ldots, n$ are continuous. If there is a switch in strategies for bank $i$ at some fixed point $q_0, \bar{q}_0$, by continuity, of all the mappings in (A.4) it follows that both one sided derivatives $\sum_{i=1}^n \partial_{q_0} \bar{s}_i, \sum_{i=1}^n \partial_{\bar{q}_0} \bar{s}_i \leq 0$. Therefore $\sum_{i=1}^n \bar{s}_i$ is decreasing in $\bar{q}$. Similar result also holds for $q$. We conclude that (A.3) holds.

**A.3 Proof of Theorem 3.6**

To show uniqueness, we consider the equilibrium prices, as a mapping of $(q^*, \bar{q}^*)$ to liquidating positions of banks $\bar{s}(q^*, \bar{q}^*)$, and then to the resulting prices, and show the uniqueness of a fixed point to this mapping. To simplify notation throughout this proof, let $Q_i, i \in \{1, n\}$, denote the set of attainable prices. The case of VWAP corresponds to $i = 1$ and $Q_1 := \{(g(s), \hat{f}(s)) \mid s \in [0, M]\}$ and the case of LOB corresponds to $i = 1$, and $Q_n := \{(g(\sum_{i=1}^n s_i), \hat{f}_1(s), \ldots, f_n(s)) \mid s \in D\}$. Moreover, for convenience define

$$I_0 := \{i \in \{1, \ldots, n\} \mid \bar{s}_i = s_i^0\}, \quad I_U := \left\{i \in \{1, \ldots, n\} \mid \bar{s}_i = \frac{h_i}{q_i}\right\}, \quad I_L := \left\{i \in \{1, \ldots, n\} \mid \bar{s}_i = \left(\frac{h_i - a_i q}{q_i - q}\right)^+\right\}.$$  

(A.7)

(A.8)

As before, we divide the proof into the VWAP and LOB cases:

**A.3.1 Volume weighted average price**

Proof. We first fix $\bar{q} = \hat{f}(s), q = g(s)$ for some $s \in [0, M]$ (recall the definition of $\hat{f}$ from (A.2)) and look for an equilibrium $\bar{s}_i(q, \bar{q}, \mathbf{1}_n) = s_i^0(\sum_{j \neq i} \bar{s}_j(q, \bar{q}, \mathbf{1}_n), q, \bar{q}, \mathbf{1}_n)$ for all $i = 1, \ldots, n$. That is for the modified Nash equilibrium given by (2.2) and formulated explicitly in (A.4).

The next goal is to show $(q, \bar{q}) \mapsto (\Phi(q, \bar{q}), \bar{\Phi}(q, \bar{q})) = (g(\sum_{j=1}^n \bar{s}_j(q, \bar{q}, \mathbf{1}_n)), \hat{f}(\sum_{j=1}^n \bar{s}_j(q, \bar{q}, \mathbf{1}_n)))$, is a contraction mapping. That is, our goal is to show that $\|\Phi(q^1, \bar{q}^1) - \Phi(q^2, \bar{q}^2)\| \leq L \|\Phi(q^1, \bar{q}^1) - (q^2, \bar{q}^2)\|_\infty$, and $|\Phi(q^1, \bar{q}^1) - \Phi(q^2, \bar{q}^2)| \leq L \|\Phi(q^1, \bar{q}^1) - (q^2, \bar{q}^2)\|_\infty$, with $L, \bar{L} < 1$ for any attainable set of prices $(q^1, \bar{q}^1)$,
(q^2, \tilde{q}^2) \in Q_1$. Without loss of generality, for this proof we will assume $q^1 \leq q^2$; therefore $(q^1, \tilde{q}^2) \in \tilde{Q}$ as well.

Indeed, with the convention that 0/0 = 0:

$$
\frac{\|\Phi(q^1, q^1) - \Phi(q^2, \tilde{q}^2)\|}{\|q^1 - q^2\|} \leq \frac{\|\Phi(q^1, q^1) - \Phi(q^2, \tilde{q}^2)\|}{\|q^1 - q^2\|} + \frac{\|\Phi(q^2, \tilde{q}^2) - \Phi(q^2, \tilde{q}^2)\|}{\|q^1 - q^2\|}
$$

$$
= \frac{1}{|q^1 - q^2|} \left( \sum_{j=1}^{n} \bar{s}_j(q^1, \tilde{q}^2 1_n) - \tilde{f} \left( \sum_{j=1}^{n} \bar{s}_j(q^1, \tilde{q}^2 1_n) \right) \right)
$$

$$
+ \frac{1}{|q^1 - q^2|} \left( \sum_{j=1}^{n} \bar{s}_j(q^1, \tilde{q}^2 1_n) - \tilde{f} \left( \sum_{j=1}^{n} \bar{s}_j(q^2, \tilde{q}^2 1_n) \right) \right)
$$

$$
\leq -\tilde{f}'(0) \left( \max_{(q, \tilde{q}) \in Q_1} \sum_{j=1}^{n} \partial_q \bar{s}_j(q, \tilde{q} 1_n) \right) + \max_{(q, \tilde{q}) \in Q_1} \sum_{j=1}^{n} \partial_q \bar{s}_j(q, \tilde{q} 1_n) \right) \right) < 1,
$$

$$
\left( \max_{(q, \tilde{q}) \in Q_1} \sum_{j=1}^{n} \partial_q \bar{s}_j(q, \tilde{q} 1_n) \right) \right) < 1.
$$

In order to show this, consider the sensitivity of $\bar{s}(q, \tilde{q} 1_n)$ with respect to $q, \tilde{q}$. Recall the construction of $\bar{s}$ given by (A.4). Recall the definitions of $I_U, I_L, I_0$ from (A.7) and (A.8). Assume that $a_i, \frac{h_i}{q-q}$, $s_i^0(\sum_{j \neq i} \bar{s}_j(q, \tilde{q} 1_n))$, $\frac{h_i-a_i q}{q-q}$ are all different for all $i = 1, \ldots, n$, so that together with the continuity of $s^0$ it follows that $\bar{s}$ is differentiable with respect to $q, \tilde{q}$ and its derivatives for a given bank $i$ are given by

$$
\partial_q \bar{s}_i(q, \tilde{q} 1_n) = \left( -\mathbb{I}_{(q \in I_U)} \frac{h_i}{q^2} - \mathbb{I}_{(q \in I_L)} \frac{h_i-a_i q}{(q-q)^2} + (s_i^0)'(\sum_{j \neq i} \bar{s}_j(q, \tilde{q} 1_n))(\sum_{j \neq i} \partial_q \bar{s}_j(q, \tilde{q} 1_n)) \mathbb{I}_{(q \in I_0)} \right),
$$

$$
\partial_q \bar{s}_i(q, \tilde{q} 1_n) = \left( \mathbb{I}_{(q \in I_U)} \frac{h_i-a_i q}{(q-q)^2} + (s_i^0)'(\sum_{j \neq i} \bar{s}_j(q, \tilde{q} 1_n))(\sum_{j \neq i} \partial_q \bar{s}_j(q, \tilde{q} 1_n)) \mathbb{I}_{(q \in I_0)} \right). \quad (A.10)
$$

Here, the derivative of the optimal liquidations $(s_i^0(s_{-i}))$ can be found via implicit differentiation: $(s_i^0)'(s_{-i}) = -\frac{f'(s_{-i}+s_i^0(s_{-i}))+s_i^0(s_{-i})f''(s_{-i}+s_i^0(s_{-i}))}{2f'(s_{-i}+s_i^0(s_{-i}))+s_i^0(s_{-i})f''(s_{-i}+s_i^0(s_{-i}))}$. Therefore $(s_i^0)'(s_{-i}) \in (-1, 0]$ for all banks $i$ such that $\bar{s}_i = s_i^0$ if $\tilde{f}'(s) + s \tilde{f}''(s) \leq 0$ for every $s \in [0, M]$. 

Electronic copy available at: https://ssrn.com/abstract=3701847
Solving the system (A.10), it follows that

\[
\partial_q \tilde{s}(q, \bar{q}) = - \left( I - \text{diag} \left( \left( s^0_j \right) \left( \sum_{j \neq i} \tilde{s}_j(q, \bar{q}1_n) \right) \left( \sum_{j \neq i} \partial_q \tilde{s}_j(q, \bar{q}1_n) \right) \mathbb{I}_{\{i \in I_0\}} \right) \right)^{-1} \times \left( \text{diag} \left( \mathbb{I}_{\{i \in I_U\}} \right) \frac{h \mathbb{I}_{\{d_i = 0, i \in I_U\}}}{q^2} + \text{diag} \left( \mathbb{I}_{\{d_i = 0, i \in I_L\}} \right) \frac{h - qa}{(q - q)^2} \right).
\]

Using the fact that \((s^0_j)'(s - i) \in (-1, 0)\) for \(i = 1, \ldots, n\) as follows from the sufficient assumption of the theorem, it thus follows that

\[
\left| 1_n^T \partial_q \tilde{s}(q, \bar{q}) \right| 
\leq \max_{d \in \{0, 1\}^n} \left| 1_n^T (I + \text{diag}(d)(1_{n \times n} - I))^{-1} \left( \text{diag} \left( \mathbb{I}_{\{d_i = 0, i \in I_U\}} \right) \frac{h \mathbb{I}_{\{d_i = 0, i \in I_U\}}}{q^2} + \text{diag} \left( \mathbb{I}_{\{d_i = 0, i \in I_L\}} \right) \frac{h - qa}{(q - q)^2} \right) \right|.
\]

To compute this maximum, let \(B(d) := I + \text{diag}(d)(1_{n \times n} - I) = \text{diag}(1_n - d) + d1_n^T\). By the Sherman-Morrison formula \(B(d)^{-1} = \text{diag}(1_n - d)^{-1} - \frac{1}{1 + \sum_{k=1}^n \text{diag}(1_n - d)^{-1} d_k} \text{diag}(1_n - d)^{-1} d1_n^T \text{diag}(1_n - d)^{-1}\). It now follows that for any \(j = 1, \ldots, n\)

\[
\sum_{i=1}^n \left( B(d)^{-1} \right)_{ij} \mathbb{I}_{\{d_j = 0\}} = \frac{1}{1 + \sum_{k=1}^n \frac{d_k}{1 - d_k}} \left( 1 + \sum_{k=1}^n \frac{d_k}{1 - d_k} - \sum_{k \neq j} \frac{d_k}{1 - d_k} \right) \mathbb{I}_{\{d_j = 0\}} = \frac{\mathbb{I}_{\{d_j = 0\}}}{1 + \sum_{k=1}^n \frac{d_k}{1 - d_k}}.
\]

Together with Remark 3 we conclude that

\[
\max_{(q, \bar{q}) \in \mathbb{Q}_1} \left| 1_n^T \partial_q \tilde{s}(q, \bar{q}) \right| \leq \max_{(q, \bar{q}) \in \mathbb{Q}_1, d \in \{0, 1\}^n} \left| 1_n^T \tilde{B}(d) \right| \times \left( \text{diag} \left( \mathbb{I}_{\{d_i = 0, i \in I_U\}} \right) \frac{h \mathbb{I}_{\{d_i = 0, i \in I_U\}}}{q^2} + \text{diag} \left( \mathbb{I}_{\{d_i = 0, i \in I_L\}} \right) \frac{h - qa}{(q - q)^2} \right) \leq \max_{\bar{q} \in [\bar{f}(M), 1]} \left| 1_n^T \frac{h \wedge \bar{a}}{\bar{q}} \right| + \max_{(q, \bar{q}) \in \mathbb{Q}_1} \left| 1_n^T \frac{h - qa}{q - q} \right| \leq \max_{\bar{q} \in [\bar{f}(M), 1]} \left( \sum_{i=1}^n a_i \frac{\bar{q}}{\bar{q} - q} \right) + \max_{(q, \bar{q}) \in \mathbb{Q}_1} \sum_{i=1}^n \frac{a_i}{\bar{q} - q} \leq \frac{M}{\min_{s \in [0, M]} \left( \bar{f}(s) - g(s) \right)}.
\]
Similarly,

$$\max_{(q,\bar{q}) \in \Omega_1} \left| \frac{\partial}{\partial q} \bar{s}(q, \bar{q}) \right| \leq \max_{(q,\bar{q}) \in \Omega_1} \left| \frac{h_i - a_i}{\bar{q} - q} \right| \leq \max_{(q,\bar{q}) \in \Omega_1} \left| \frac{h_i - a_i}{\bar{q} - q} \right| \leq \frac{M}{\min_{s \in [0,M]} (\hat{f}(s) - g(s))},$$

where in the last inequality we have used that fact that $a_i \geq \frac{h_i - a_i}{\bar{q} - q} \geq \frac{h_i - a_i}{\bar{q} - q} = -a_i + \frac{h_i - a_i}{\bar{q} - q} \geq -a_i$. Recalling \(A.9\), we conclude that \((\Phi, \bar{\Phi})\) is a contraction mapping if \(-3M(\hat{f}'(0) \wedge g'(0)) < \min_{s \in [0,M]} (\hat{f}(s) - g(s))\).

Finally, it can be seen that \(\hat{f}'(s) = \frac{\hat{f}(s) - f(s)}{s}.\) Therefore, \(\hat{f}'(0) = \frac{1}{2} f'(0)\).

Recall that it was assumed that $a_i, b_i, h_i - a_i, q_i$, $s_i^0(\sum_{j \neq i} \bar{s}_j(q, \bar{q} 1_n))$ are all different. If this assumption is violated, say $s_i^0(\sum_{j \neq i} \bar{s}_j(q, \bar{q} 1_n)) < \frac{h_i - a_i}{\bar{q} - q} = \frac{h_i}{\bar{q}}$, then we need to consider one-sided derivatives. In that case, the derivative from the right \(\partial_{\bar{q}} \bar{s}_i(q, \bar{q} 1_n) = -\frac{h_i}{\bar{q}}\), while the derivative from the left \(\partial_{\bar{q}} \bar{s}_i(q, \bar{q} 1_n) = -\frac{h_i - a_i}{\bar{q} - q}\).

In this case, both one-sided derivatives would satisfy \(A.11\). The other cases, can be treated similarly.

\[\square\]

A.3.2 Limit order book

Proof. We first fix $\bar{q} = \bar{s}(s), q = g(\sum_{i=1}^n s_i)$ for some $s \in D$ and look for an equilibrium $\bar{s}_i(q, \bar{q}) = s_i^0(\bar{s}_i(q, \bar{q}), q, \bar{q})$ which is explicitly provided by

$$\bar{s}_i(q, \bar{q}) = \begin{cases} a_i & \text{if } h_i \geq a_i \bar{q}_i \\ \left(\frac{h_i - a_i}{\bar{q} - q}\right) + \sqrt{\left(s_i^0(\bar{s}_i(q, \bar{q})) \wedge \frac{h_i}{\bar{q}}\right)} & \text{if } h_i < a_i \bar{q}_i \end{cases} \quad (A.12)$$

where $s_i^0(\bar{s}_i(q, \bar{q}))$ solves the first order condition

$$1 - (1 + r)(\sum_{j=1}^n \sum_{s_j < s_i} \bar{s}_j + (n - (k_i - 1)) s_i) = 0, \quad (A.13)$$

where we recall that $k_i$ is such that $s_{[k_i]} = s_i$. For simplicity, we will continue to assume that $q_1 \geq q_2 \geq ... \geq q_n$.

The next goal is to show $(q, \bar{q}) \mapsto (\Phi(q, \bar{q}), \bar{\Phi}(q, \bar{q})) = (g(\sum_{j=1}^n \bar{s}_j(q, \bar{q})), \bar{s}(s(q, \bar{q})))$ is a contraction mapping, i.e., to show that $\|\Phi(q^1, q^1) - \Phi(q^2, q^2)\|_\infty \leq L \|\phi^1 - \phi^2\|_\infty$ and $\Phi(q^1, q^1) - \Phi(q^2, q^2)\|_\infty \leq L \|\phi^1 - \phi^2\|_\infty$ with $L, L < 1$ for any $(q^1, q^1), (q^2, q^2) \in \Omega_n$. Without loss of generality, for this proof we will assume $q^1 \leq q^2$; therefore $(q^1, q^2) \in \Omega.$
Indeed, with the convention that $0/0 = 0$, for any $1 \leq j \leq n$:
\[
\left| \Phi_j(q^1, \bar{q}^1) - \Phi_j(q^2, \bar{q}^2) \right| \leq \frac{1}{\|q^1 - q^2\| \|l\|_\infty} \left| \Phi_j(q^1, \bar{q}^1) - \Phi_j(q^2, \bar{q}^2) \right| + \frac{1}{\|q^1 - q^2\| \|l\|_\infty} \left| \Phi_j(q^1, \bar{q}^1) - \Phi_j(q^2, \bar{q}^2) \right|
\]
\[
\leq \sum_{k=1}^n \left| \Phi_j(q^1, \bar{q}^1_{(k \in \mathbb{Z})}) - \Phi_j(q^2, \bar{q}^2_{(k \in \mathbb{Z})}) \right| + \frac{1}{\|q^1 - q^2\| \|l\|_\infty} \left| \Phi_j(q^1, \bar{q}^1) - \Phi_j(q^2, \bar{q}^2) \right|
\]
\[
= \sum_{k=1}^n \frac{1}{\|q^1 - q^2\|} \left| \bar{f}_j \left( (s(q^1, \bar{q}^1_{(k \in \mathbb{Z})}), s(q^2, \bar{q}^2_{(k \in \mathbb{Z})})) \right) - \bar{f}_j \left( (s(q^1, \bar{q}^1_{(k \in \mathbb{Z})}), s(q^2, \bar{q}^2_{(k \in \mathbb{Z})})) \right) \right|
\]
\[
+ \frac{1}{\|q^1 - q^2\|} \left| \bar{f}_j \left( (s(q^1, \bar{q}^1)), s(q^2, \bar{q}^2) \right) - \bar{f}_j \left( (s(q^1, \bar{q}^1)), s(q^2, \bar{q}^2) \right) \right|
\]
\[
\leq \sum_{k=1}^n \max_{(q,k) \in \mathbb{Q}} \sum_{i=1}^n \partial_{q_k} \bar{f}_j(q_0) \partial_{q_i} \bar{s}_i(q, \bar{q}) + \max_{(q,k) \in \mathbb{Q}} \sum_{i=1}^n \partial_{q_k} \bar{f}_j(q_0) \partial_{q_i} \bar{s}_i(q, \bar{q}) < 1,
\]
(A.14)
\[
-g'(0) \left( \sum_{k=1}^n \max_{(q,k) \in \mathbb{Q}} \sum_{i=1}^n \partial_{q_k} \bar{s}_i(q, \bar{q}) - \max_{(q,k) \in \mathbb{Q}} \sum_{i=1}^n \partial_{q_k} \bar{s}_i(q, \bar{q}) \right) < 1.
\]

In order to show this, consider the sensitivity of $\bar{s}(q, \bar{q})$ with respect to $q, \bar{q}$. Recall again the definitions of $I_U, I_L, I_0$ from (A.7) and (A.8). Assume that $a_i, b_i, q_i = 0$ for all $i = 1, \ldots, n$. Note that for different $i$, some of these quantities may be equal, namely, we must have $s_i^0 = s_j^0$, since if there is a solution $s^0$, it is unique. Similar to the proof in Section A.3.1 otherwise, one sided derivatives can be considered. Together with the continuity of $s^0$ it follows that $\bar{s}_i$ is differentiable with respect to $q, \bar{q}$ and its derivatives for a given bank $i$ are given by
\[
\partial_{q_k} \bar{s}_i(q, \bar{q}) = \left( - \bar{I}_{(i=k, i \in I_U)} \frac{h_i}{q^2} - \bar{I}_{(i,k, i \in I_L)} \frac{h_i}{q^2} - \bar{I}_{(i=k, i \in I_L)} \frac{h_i - a_i q}{(q - q^2)^2} + \nabla s_i^0(\bar{s}_{-i}(q, \bar{q})) \cdot \left[ \partial_{q_k} \bar{s}_j(q, \bar{q}) \bar{l}_{(s_j \in s_{-i}^0)} \right]_{j=1,...,n,j \neq i} \right) \bar{l}_{(i \in I_0)},
\]
\[
\partial_{q_k} \bar{s}_i(q, \bar{q}) = \left( \bar{I}_{(i \in I_U)} \frac{h_i}{q^2} - \bar{I}_{(i \in I_L)} \frac{h_i - a_i q}{(q - q^2)^2} + \nabla s_i^0(\bar{s}_{-i}(q, \bar{q})) \cdot \partial_{q_i} \bar{s}_{-i}(q, \bar{q}) \bar{l}_{(i \in I_0)} \right).
\]
Here, the derivative of the optimal liquidations $(s_{-i}^0(\bar{s}_{-i}))$ can be found via implicit differentiation of $1 - (1 + r)f \left( \sum_{j=1}^n \bar{l}_{(i < j < s_i^0)} \bar{s}_j + (n - k_i + 1) s_i \right) = 0$ to be $\partial_{s_i} s_{-i}^0(\bar{s}_{-i}) = - \frac{\bar{l}_{(i < j < s_i^0)}}{n - (k_i - 1)}$. Set $q' = \min_{j,k} \partial_{s_i} \bar{f}_j(q_0) / \min_{j,k} \partial_{s_i} \bar{f}_j(q_0) < 0$. Similarly to before, we have that $\partial_{s_i} \bar{f}_j(q_0) = \frac{1}{2} f'(0) \bar{I}_{(1 \leq k \leq j)}$. Recall that
\[
\frac{h_i}{q^2} = \frac{h_i - a_i q}{(q - q^2)^2} \leq \min_{j,k} \frac{h_i}{q^2} \frac{h_i - a_i q}{(q - q^2)^2} \leq \min_{s_{-i}^0} \left( f_j(s_j) - g(\sum_{i=1}^n s_i) \right) \frac{a_j}{\min_{s_{-i}^0} \left( f_j(s_j) - g(\sum_{i=1}^n s_i) \right)}.
\]
Thus, for any $i_0 \in I_0$, we have that $\sum_{i \in I_0} \partial_{q_i} \bar{s}_i(q, \bar{q}) > -\bar{l}_{(i < j < s_i^0)} \sum_{k \in \{ k : s_k < s_{i_0}^0 \}} \frac{a_k}{\min_{s_{-i}^0} \left( f_j(s_j) - g(\sum_{i=1}^n s_i) \right)} \leq \frac{M}{\min_{s_{-i}^0} \left( f_j(s_j) - g(\sum_{i=1}^n s_i) \right)}$
Similarly, since $|\sum_{i=1}^{n} \partial_q \tilde{s}_i(q, \tilde{q})| \leq \frac{M}{\min_{s \in \mathcal{D}} (f_j(s) - g(\sum_{i=1}^{n} s_i))}$, we get that
\[
\max_{(q, \tilde{q}) \in \mathcal{Q}} \left| \sum_{i=1}^{n} \partial_q f_j(0_n) \partial_q \tilde{s}_i(q, \tilde{q}) \right| \leq |\tilde{q}'| \frac{nM}{\min_{s \in \mathcal{D}} (f_j(s) - g(\sum_{i=1}^{n} s_i))}.
\]
Recalling (A.14) we conclude that $\tilde{a}$ is a contraction mapping if
\[
-nM \min_{j,k} \partial_{s_k} f_j(0_n) < \min_{j \in \mathcal{D}} \left( f_j(s) - g(\sum_{i=1}^{n} s_i) \right).
\]

\[
\text{Similarly, } -nMg'(0) < \min_{j \in \mathcal{D}} \left( f_j(s) - g(\sum_{i=1}^{n} s_i) \right) \text{ ensures that } \Phi \text{ is a contraction mapping.}
\]

\section*{B Sensitivity of the clearing solutions to interest rates $r$}

In this section we consider the assumptions of Theorem 3.6 with the goal of investigating the sensitivity of the (unique) equilibrium to the interest rate $r$. To simplify notation, for this section we write $\bar{s}_i := \bar{s}_i(q, \tilde{q})$ or $\bar{s}_i := \bar{s}_i(q, \bar{q})$ where the values of $(q, \tilde{q})$ and $(q, \bar{q})$ is clear from context for the VWAP and LOB settings respectively. In the following, we derive $\partial_r \bar{s}_i$, the derivatives of the equilibrium liquidations w.r.t. $r$. We then provide conditions under which the intuition of Remark 3 holds, i.e., the system-wide total liquidations increase with increase of $r$.

\subsection*{B.1 Volume weighted average price}

Initially, as in the prior proofs, assume that for each $i = 1, ..., n$, the possible solutions to the optimization for $\bar{s}_i$ from (A.4), namely $a_i, \frac{h_i}{q}, \frac{h_i - a_i}{q - q}, s_i^0(\sum_{j \neq i} \bar{s}_j)$, are all different. We want to study $\partial_r \bar{s}_i$ for $i = 1, ..., n$. From the previous assumption it follows that
\[
\partial_r \bar{s}_i = \begin{cases} 
0 & \text{if } i \in I_a \\
-\frac{q}{h_i} \partial_r \tilde{q} & \text{if } i \in I_U \\
\frac{h_i - a_i}{(q - q)^2} \partial_r q - \frac{h_i - a_i}{(q - q)^2} \partial_r \tilde{q} & \text{if } i \in I_L \\
\partial_r s_i^0 & \text{if } i \in I_0 \end{cases}
\]

where $I_a, I_U, I_L, I_0$ were defined in (A.7) and (A.8).

Before continuing, we will consider $\partial_r \bar{s}_i$ for $i \in I_0$. By construction, we have
\[
- (f(\bar{s}_i^0 + \sum_{j \neq i} \bar{s}_j) + s_i^0 f'(s_i^0 + \sum_{j \neq i} \bar{s}_j)) - (1 + r) (2f'(s_i^0 + \sum_{j \neq i} \bar{s}_j) + s_i^0 f''(s_i^0 + \sum_{j \neq i} \bar{s}_j)) \partial_r \bar{s}_i
\]
\[
- (1 + r) \sum_{j \neq i} (f'(s_i^0 + \sum_{j \neq i} \bar{s}_j) + s_i^0 f''(s_i^0 + \sum_{j \neq i} \bar{s}_j)) \partial_r \bar{s}_j = 0.
\]
Recall that every bank \( i \in I_0 \) will satisfy the same condition, i.e., \( \partial_r \bar{s}_i = \partial_q \bar{s}_j \) for every \( i, j \in I_0 \). For notational simplicity let \( s^0 = s^0_i, \partial_r s^0 = \partial_r s^0_i \) for arbitrary \( i \in I_0 \). Let \( c = \frac{f'(I_0 | s^0 + \sum_{j \in I_0} \bar{s}_j |) + s^0 f''(I_0 | s^0 + \sum_{j \in I_0} \bar{s}_j |)}{2f'(I_0 | s^0 + \sum_{j \in I_0} \bar{s}_j |) + s^0 f''(I_0 | s^0 + \sum_{j \in I_0} \bar{s}_j |)} \) and \( d = -\frac{1}{f'(I_0 | s^0 + \sum_{j \in I_0} \bar{s}_j |) + s^0 f''(I_0 | s^0 + \sum_{j \in I_0} \bar{s}_j |)} \). Recall that by our Assumption 2.1 \( 0 \leq c < 1 \) and \( d > 0 \). Therefore, it can be shown that

\[
\partial_r s^0 = \frac{d}{1 + c(|I_0| - 1)} - \frac{c}{1 + c(|I_0| - 1)} \sum_{j \notin I_0} \partial_r \bar{s}_j.
\]

We can now consider the joint sensitivity of the haircut \( q \) and price \( \bar{q} \) to interest rates:

\[
\begin{align*}
\partial_r q &= \left[ \sum_{i \in I_0} \partial_r s^0 + \sum_{i \notin I_0} \partial_r \bar{s}_i \right] g'(\sum_{i=1}^{n} \bar{s}_i) \\
&= \left[ \frac{|I_0|d}{1 + c(|I_0| - 1)} + \frac{1 - c}{1 + c(|I_0| - 1)} \sum_{j \notin I_0} \partial_r \bar{s}_i \right] g'(\sum_{i=1}^{n} \bar{s}_i) \\
\partial_r \bar{q} &= \left[ \sum_{i \in I_0} \partial_r s^0 + \sum_{i \notin I_0} \partial_r \bar{s}_i \right] \bar{f}'(\sum_{i=1}^{n} \bar{s}_i) \\
&= \left[ \frac{|I_0|d}{1 + c(|I_0| - 1)} + \frac{1 - c}{1 + c(|I_0| - 1)} \sum_{j \notin I_0} \partial_r \bar{s}_i \right] \bar{f}'(\sum_{i=1}^{n} \bar{s}_i).
\end{align*}
\]

To simplify notation, let \( \tilde{c} = \frac{1 - c}{1 + c(|I_0| - 1)} \) and \( \tilde{d} = \frac{|I_0|d}{1 + c(|I_0| - 1)} \). Therefore

\[
\partial_r q = \left[ \tilde{d} + \tilde{c} \left( \frac{h - a \bar{q}}{(q - q)^2} \|I_{i \in I_L}\|_i - \frac{h}{q^2} \right) \right] \partial_r q - \tilde{c} \left( \frac{h - a \bar{q}}{(q - q)^2} \|I_{i \in I_L}\|_i - \frac{h}{q^2} \right) \partial_r \bar{q} \right] g'(\sum_{i=1}^{n} \bar{s}_i),
\]

\[
\partial_r \bar{q} = \left[ \tilde{d} + \tilde{c} \left( \frac{h - a \bar{q}}{(q - q)^2} \|I_{i \in I_L}\|_i - \frac{h}{q^2} \right) \right] \partial_r q - \tilde{c} \left( \frac{h - a \bar{q}}{(q - q)^2} \|I_{i \in I_L}\|_i - \frac{h}{q^2} \right) \partial_r \bar{q} \right] \bar{f}'(\sum_{i=1}^{n} \bar{s}_i).
\]

That is, the sensitivity of the haircut and prices \((q, \bar{q})\) w.r.t. the interest rate \( r \) is the solution of a linear system

\[
\begin{pmatrix} \partial_r q \\ \partial_r \bar{q} \end{pmatrix} = (I - W)^{-1} \begin{pmatrix} \tilde{d} & \tilde{c} g'(\sum_{i=1}^{n} \bar{s}_i) \frac{h - a \bar{q}}{(q - q)^2} \|I_{i \in I_L}\|_i - \frac{h}{q^2} \|I_{i \in I_L}\|_i + \frac{h}{q^2} \|I_{i \in I_U}\|_i \\ \tilde{d} & \tilde{c} \bar{f}'(\sum_{i=1}^{n} \bar{s}_i) \frac{h - a \bar{q}}{(q - q)^2} \|I_{i \in I_L}\|_i - \frac{h}{q^2} \|I_{i \in I_L}\|_i + \frac{h}{q^2} \|I_{i \in I_U}\|_i \end{pmatrix} \tilde{c}
\]

\[
\times \begin{pmatrix} \tilde{d} & \tilde{c} g'(\sum_{i=1}^{n} \bar{s}_i) \\ \tilde{d} & \tilde{c} \bar{f}'(\sum_{i=1}^{n} \bar{s}_i) \end{pmatrix},
\]

\[
W = \begin{pmatrix} \tilde{d} & \tilde{c} g'(\sum_{i=1}^{n} \bar{s}_i) \\ \tilde{d} & \tilde{c} \bar{f}'(\sum_{i=1}^{n} \bar{s}_i) \end{pmatrix} \begin{pmatrix} \|I_{i \in I_L}\|_i - \frac{h}{q^2} \|I_{i \in I_L}\|_i + \frac{h}{q^2} \|I_{i \in I_U}\|_i \\ \tilde{c} \end{pmatrix}.
\]

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Moreover, it also follows that

\[
\partial_r \sum_{i=1}^{n} \bar{s}_i = \frac{\partial_r q}{g'(\sum_{i=1}^{n} \bar{s}_i)}
\]

\[
= 1 + \frac{1}{1 - \tilde{c}} \frac{\partial}{\partial r} \left( \frac{\tilde{c}}{(q - q)^2} \right) \left[ \sum_{i=1}^{n} \bar{s}_i \right] - \frac{h-aq}{(q - q)^2} \left[ \sum_{i=1}^{n} \bar{s}_i \right] \left\{ \sum_{i=1}^{n} \bar{s}_i \right\}
\]

\[
= \frac{1}{1 - \tilde{c}} \left( \frac{\tilde{c}}{(q - q)^2} \right) \left[ \sum_{i=1}^{n} \bar{s}_i \right] - \frac{h-aq}{(q - q)^2} \left[ \sum_{i=1}^{n} \bar{s}_i \right] \left\{ \sum_{i=1}^{n} \bar{s}_i \right\}
\]

It follows that \( \partial_r \sum_{i=1}^{n} \bar{s}_i > 0 \) if \( \left( \frac{h-aq}{(q - q)^2} \right) \left[ \sum_{i=1}^{n} \bar{s}_i \right] \left\{ \sum_{i=1}^{n} \bar{s}_i \right\} < \frac{1}{\tilde{c}} \), which happens if, for example, \( \tilde{c} \) is small enough.

### B.2 Limit order book

Initially, again assume that for each \( i = 1, \ldots, n \), the possible solutions \( (a_i, \frac{h_i}{q_i}, \frac{h_i-aq_i}{q_i}, s_i^0(\sum_{j \neq i} \bar{s}_j)) \) to the optimization (A.12) are all different. As in the VWAP case, we want to study \( \partial_r \bar{s}_i \) for \( i \in \{1, \ldots, n\} \). From the previous assumption it follows that

\[
\partial_r \bar{s}_i = \begin{cases} 
0 & \text{if } i \in I_a, \\
-\frac{h_i}{q_i} \partial_r \bar{q}_i & \text{if } i \in I_U, \\
\frac{h_i-aq_i}{(q_i-q)^2} \partial_r q - \frac{h_i-aq_i}{(q_i-q)^2} \partial_r \bar{q}_i & \text{if } i \in I_L, \\
\partial_r s_i^0 & \text{if } i \in I_0,
\end{cases}
\]

where \( I_a, I_U, I_L, I_0 \) were defined in (A.7) and (A.8).

Recall \( s_i^0(\bar{s}_{-i}) \) solves the first order condition

\[
1 - (1+r) f \left( \sum_{j=1}^{n} \mathbb{1}_{\{s_j < s_i\}} \bar{s}_j + (n - (k_i - 1))s_i \right) = 0,
\]

where \( k_i \) is such that \( s_{[k_i]} = s_i \). As noted in the proof of Theorem 3.6 in the LOB case, we have that \( \bar{s}_i = s_i^0 \) is unique, and independent of \( i \). In fact, \( s_i^0 = s_j^0 \) for every \( i, j \in I_0 \). We will denote this common value as \( s^0 \).

If \( \bar{s}_i = s^0 \) then from implicit differentiation of (A.13), we get that

\[
\partial_r \bar{s}_i = -\frac{f \left( \sum_{j=1}^{n} \mathbb{1}_{\{s_j < s^0\}} \bar{s}_j + (n - (|I_0 \cup I_L| - 1))s^0 \right)}{(1+r)(f' \left( \sum_{j=1}^{n} \mathbb{1}_{\{s_j < s^0\}} \bar{s}_j + (n - (|I_0 \cup I_L| - 1))s^0 \right))} - \frac{\sum_{i \in I_0} \partial_r \bar{s}_i}{n - (|I_0 \cup I_L| - 1)}
\]

Now we want to consider the case of \( \partial_r \bar{s}_i \) for \( i \in I_U \). Notably, \( \bar{s}_i < s^0 \) for \( i \in I_U \) by construction (see (A.12)). Therefore for such banks, there is no change to the attained prices \( \bar{q}_i \) by a change in the interest rate, i.e., \( \partial_r \bar{q}_i = 0 \) for \( i \in I_U \). This allows us to simplify \( \partial_r s^0 \).
We can now consider the joint sensitivity of the haircut \( q \) and prices \( \mathbf{q} \) to interest rates:

\[
\partial_r \mathbf{q} = \left[ \sum_{i \in I_0 \cup I_L} \partial_{\bar{s}_i} \right] \left[ \sum_{i=1}^{n} \bar{s}_i \right] g' \left( \sum_{i=1}^{n} \bar{s}_i \right) \,
\]

\[
= -\frac{|I_0| f \left( \sum_{j=n+1}^{\infty} \delta_{j < n+1} \bar{s}_j + \left( n - (|I_0 \cup I_L| - 1) \right) s^0 \right)}{(1 + r) f' \left( \sum_{j=1}^{n} \delta_{j < n} \bar{s}_j + \left( n - (|I_0 \cup I_L| - 1) \right) s^0 \right) \left( n - (|I_0 \cup I_L| - 1) \right)} + \sum_{i \in I_L} \partial_{\bar{s}_i} \g'(\sum_{i=1}^{n} \bar{s}_i),
\]

\[
\partial_{\bar{r}} \bar{q}_i = \begin{cases} 
0 & \text{if } i \in I_a \cup I_U, \\
\frac{f \left( \sum_{j=n+1}^{\infty} \delta_{j < n+1} \bar{s}_j + \left( n - (|I_0 \cup I_L| - 1) \right) s^0 \right)}{(1 + r) f' \left( \sum_{j=1}^{n} \delta_{j < n} \bar{s}_j + \left( n - (|I_0 \cup I_L| - 1) \right) s^0 \right) \left( n - (|I_0 \cup I_L| - 1) \right)} \partial_{\bar{s}_i} \tilde{f}_i(\bar{s}) & \text{if } i \in I_0, \\
\frac{|I_0| f \left( \sum_{j=n+1}^{\infty} \delta_{j < n+1} \bar{s}_j + \left( n - (|I_0 \cup I_L| - 1) \right) s^0 \right) \partial_{\bar{s}_i} \tilde{f}_i(\bar{s}) + \sum_{j \in I_L \bar{s}_j \leq \bar{s}_i} (\partial_{\bar{s}_j} \bar{s}_j) \partial_{\bar{s}_i} \tilde{f}_i(\bar{s}) & \text{if } i \in I_L,
\end{cases}
\]

for arbitrary \( i_0 \in I_0 \).

To simplify notation, let \( \mathbf{c} = -\frac{f \left( \sum_{j=n+1}^{\infty} \delta_{j < n+1} \bar{s}_j + \left( n - (|I_0 \cup I_L| - 1) \right) s^0 \right)}{(1 + r) f' \left( \sum_{j=1}^{n} \delta_{j < n} \bar{s}_j + \left( n - (|I_0 \cup I_L| - 1) \right) s^0 \right) \left( n - (|I_0 \cup I_L| - 1) \right)} \). It then follows that

\[
\begin{pmatrix}
\partial_r \mathbf{q} \\
\partial_{\bar{r}} \mathbf{q}
\end{pmatrix} = \mathbf{W}^{-1} \mathbf{b},
\]

where

\[
\mathbf{W} = (w_{i,j})_{1 \leq i,j \leq n+1},
\]

\[
\mathbf{b} = (b_j)_{j=1,\ldots,n+1},
\]

\[
w_{i,j} = \mathbb{I}_{\{i=j<n+1\}} + \mathbb{I}_{\{i,j \in I_L, s_i \geq s_j\}} \frac{h_j - a_j \bar{q}_j}{(\bar{q}_j - q)^2} \partial_{\bar{s}_i} \tilde{f}_i(\bar{s}) - \mathbb{I}_{\{i \in I_L, j=n+1\}} \sum_{k \in I_L, s_k \leq s_i} \frac{h_k - a_k \bar{q}_k}{(\bar{q}_k - q)^2} \partial_{\bar{s}_k} \tilde{f}_k(\bar{s}),
\]

\[
w_{n+1,j} = \mathbb{I}_{\{j \in I_L\}} \frac{h_j - a_j \bar{q}_j}{(\bar{q}_j - q)^2} g' + \mathbb{I}_{\{j=n+1\}} \left( 1 - g' \sum_{i \in I_L} \frac{h_i - a_i \bar{q}_i}{(\bar{q}_i - q)^2} \right),
\]

\[
b_j = \partial_{\bar{s}_i} \tilde{f}_i(\bar{s}) \mathbb{I}_{\{i \in I_0\}} + |I_0| \partial_{\bar{s}_i} \tilde{f}_i(\bar{s}) \mathbb{I}_{\{i \in I_L\}} + |I_0| \partial_{\bar{q}} g' \mathbb{I}_{\{i=n+1\}}.
\]

We note, without loss of generality, that if we assume that for any \( i \in I_a, j \in I_U, k \in I_0, l \in I_L \), we have that \( i < j < k < l \), and that for any \( i, j \in I_L \), such that \( i < j \) then \( \bar{s}_i \leq \bar{s}_j \), we then have that \( \mathbf{W} \) is lower triangular, but has an addition of one full \( n + 1 \) column. \( \mathbf{W} \) is invertible, and we can find its inverse as follows: Note that \( \mathbf{W} \) can be written as

\[
\mathbf{W} = \left( \mathbf{W}_0 + (0, 0, \ldots, 0, 1)^T \right)^T \left[ \mathbb{I}_{\{j \in I_L\}} \frac{h_j - a_j \bar{q}_j}{(\bar{q}_j - q)^2} g' - \mathbb{I}_{\{j=n+1\}} g' \sum_{i \in I_L} \frac{h_i - a_i \bar{q}_i}{(\bar{q}_i - q)^2} \right]_{j=1,\ldots,n+1}^T,
\]

where \( \mathbf{W}_0 = \mathbf{D} (I + \mathbf{N}) \) with

\[
\mathbf{D} = \text{diag} \left[ \begin{array}{c}
1 + \mathbb{I}_{\{j \in I_L\}} \frac{h_j - a_j \bar{q}_j}{(\bar{q}_j - q)^2} \partial_{\bar{s}_j} \tilde{f}_j(\bar{s})
\end{array} \right]_{j=1,\ldots,n}
\]

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is a diagonal matrix and a nilpotent matrix

\[ \mathbf{N} = \begin{bmatrix} \mathbb{I}_{\{i,j \in I_L, s_i > s_j\}} \left( \frac{h_j - a_j q_j}{\bar{q}_j - q} \right) \partial_r \bar{f}_j(\bar{s}) \end{bmatrix}_{j,i} 1 \leq i,j \leq n+1. \]

Note that \( \mathbf{N} \) is such that \( \mathbf{N}^{n+1} = 0 \). Therefore, we have that

\[ \mathbf{W}_0^{-1} = (I + \mathbf{N})^{-1} \mathbf{D}^{-1} = (I - \mathbf{N} + \mathbf{N}^2 + \cdots + (-1)^n \mathbf{N}^n) \mathbf{D}^{-1}. \]

Finally,

\[ \mathbf{W}^{-1} = \left( \mathbf{W}_0^{-1} - \frac{\mathbf{W}_0^{-1}(0,0,\ldots,0,1)\mathbb{T} \left[ \mathbb{I}_{\{j \in I_L\}} \left( \frac{h_j - a_j q_j}{\bar{q}_j - q} \right) \partial_r \bar{f}_j(\bar{s}) - \mathbb{I}_{\{j=n+1\}} g' \sum_{i \in I_L} \frac{h_i - a_i q_i}{\bar{q}_i - q} \right]_{j=1,\ldots,n+1} \mathbf{W}_0^{-1}}{1 + \mathbb{I}_{\{j \in I_L\}} \left( \frac{h_j - a_j q_j}{\bar{q}_j - q} \right) g' - \mathbb{I}_{\{j=n+1\}} g' \sum_{i \in I_L} \frac{h_i - a_i q_i}{\bar{q}_i - q}} \right) \mathbf{W}_0^{-1}(0,0,\ldots,0,1)\mathbb{I}. \]

To calculate \( \partial_r \sum_{i=1}^n \bar{s}_i \), recall that \( \partial_r \bar{q}_i = 0 \) for \( i \in I_U \), and from (B.2) it follows that

\[ \partial_r \sum_{i=1}^n \bar{s}_i = \sum_{i \in I_U \cup I_L} \partial_r \bar{s}_i = \frac{\partial_r q}{g' \left( \sum_{i=1}^n \bar{s}_i \right)} \frac{(0,0,\ldots,0,1)\mathbb{I} \mathbf{W}^{-1} \mathbf{b}}{g' \left( \sum_{i=1}^n \bar{s}_i \right)}. \]

It follows that \( \partial_r \sum_{i=1}^n \bar{s}_i \geq 0 \) if \( (0,0,\ldots,0,1)^\mathbb{T} \mathbf{W}^{-1} \mathbf{b} \leq 0. \)