What is the Reference Frame of an Accelerated Observer?

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Abstract
The general construction of extended reference frames for non-inertial observers in flat space is studied. It is shown that, if the observer moves inertially before and after an arbitrary acceleration and rotation, the region where reference frames can coincide with an inertial system is bounded for final velocities exceeding $0.6 \, c$.

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1 Introduction

One of the most basic concepts in modern physics is that of an inertial system, i.e., the extended reference frame of an observer with constant velocity in flat space. This is not only a keystone in special relativity. Also Einstein’s derivation of the general theory of relativity was inspired by the idea to extend the equivalence of the inertial frames in Special Relativity to an equivalence of all frames, including those of accelerated observers. It is therefore of interest to ask how such a frame may be realized in general relativity.

A second and more important reason for the study of extended reference frames in general relativity is Mashhoon’s discovery of the problem with the hypothesis of locality which is needed for the principle of general covariance [1]. This problem is based on the fact that an acceleration $a$ introduces a length scale $c^2/a$. Mashhoon argued that the hypothesis of locality, i.e., the local equivalence between an accelerated observer and an inertial observer having the same velocity at a given space-time point, holds only for wavelengths much smaller than this scale. Later [2] he demonstrated this problem with an example based on Fermi coordinates. It is the working hypothesis of the present paper that it may be possible to overcome the restriction to small wavelengths by introducing an extended reference frame for accelerated observers in which the coordinates of an event in space-time are identified with the proper time and certain proper lengths between the worldline and the event even if it is far away from the worldline. An extended frame of reference may circumvent the problem because it was (possibly) constructed by use of a non-local procedure. The price one has to pay for this solution is that the equivalence of all coordinate systems is lost for the observer who obviously distinguishes the extended frame of reference from other systems. The principle that all extended coordinate systems are equally valid is not affected, however, as long as we do not consider a particular observer.

A very good approximation to the notion of a frame of reference in the context of general relativity are Fermi coordinates [3, 4]. In this coordinate system the metric of space-time is Minkowskian on the complete world line of the observer. In the limit of weak gravitational fields the Newtonian potential can be recovered in this coordinate system [5]. An important point is that the values of the spatial coordinates $x^i$ are directly related to the proper length $s$ of geodesics connecting the observer with other space-time points by $[(x^1)^2 + (x^2)^2 + (x^3)^2]^{1/2} = s$. They are therefore measurable quantities. The success of Fermi coordinates in the neighborhood of the observer’s worldline may justify the following criterion for a frame of reference: A necessary condition for a coordinate system to be the reference frame of a particular observer is that the metric in this coordinate system is Minkowskian on the whole worldline.

Despite their advantages Fermi coordinates have a severe shortcoming: they cannot cover the complete space-time or even the past of the observer if an acceleration is present. An instructive example for this to occur is Rindler space which is identi-
cal to the Fermi coordinate system of an observer with constant acceleration in two dimensional Minkowski space. It is well known that Rindler space encloses only the right wedge of Minkowski space. The rest of Minkowski space cannot be covered by this coordinate system.

To circumvent these problems a modification of the original construction scheme for Fermi coordinates was proposed in Ref. [6]. In an ordinary Fermi coordinate system the spatial coordinate lines are identified with geodesics emanating from the world line and being orthogonal to the four-velocity of the observer. In the modified approach these curves are not geodesics but are found by a non-local construction scheme which guarantees that the coordinates are directly related to the proper length of these curves. In this point of view the problem to find a reference frame for an arbitrary observer is essentially identical to the problem of choosing a sequence of spacelike hypersurfaces belonging to this observer and defining his slices of constant proper time. Fermi coordinates are a means to find at least a local approximation to these planes of simultaneity; a modification of Fermi coordinates should have the goal to construct the complete hypersurfaces of constant proper time. It should be remarked that the construction of an adapted coordinate system is not the only way to deal with non-inertial observers. Mashhoon [7] has made a non-local ansatz to overcome the problems with the hypothesis of locality, for instance.

The central result of this paper will be a statement about the general behaviour of any possible extended reference frame, regardless how it may be constructed. This result holds for observers in flat space with arbitrary worldlines and with any rotation provided both acceleration and rotation will be non-zero only during a finite period of proper time $T$. The study of this behaviour was inspired by the examination of the particular proposal for an extended frame of reference made in Ref. [6]. For illustration this special case is added in the last section of this paper.

2 Where can reference frames be Minkowskian?

In the study of the modified Fermi coordinates introduced in Ref. [6] it turned out that for an observer constantly accelerated during a finite period of time $T$ the coordinate lines of proper time $\tau$ can become spacelike far away from the worldline $z^\mu(\tau)$. This will be briefly described in the next section. Here it will be argued that this phenomenon occurs for a very broad class of possible extended reference frames.

To demonstrate this we consider an observer which is in inertial motion and without rotation before the proper time $\tau = 0$ and after $\tau = T$. For simplicity we will first consider the two dimensional case, in which a rotation is excluded, and switch to an inertial system in which the observer is at rest before $\tau = 0$ and where $z^\mu(\tau = 0)$ lies in the origin. Attached to the worldline is a tetrad $e_a^\mu$ with $e_0^\mu$ being the tangent to the worldline and the spatial tetrads $e_i^\mu$ being orthogonal to it. Throughout the paper greek indices run from 0 to 3, latin indices run from 1 to 3, and underlined indices are tetrad indices. The spatial coordinate lines start on the worldline with their directions being given by the spatial tetrads (as for Fermi coordinates). This guarantees that the metric is Minkowskian on the whole
worldline.

Without referring in any way to the actual construction of the reference frame one can make two assumptions on how it may look like.

1) As long as the observer is at rest its frame of reference should certainly be an inertial system at every point with causal connection to the world line. This is the case for any point inside the lower light cone in Fig. 1.

2) After the observer has reached his final velocity and is not accelerated anymore the frame of reference should again be an inertial system in a certain region around the worldline. The central question of this paper is how large this region may be.

Since the coordinate lines of an inertial system in flat space are straight lines one can infer that (in two dimensions) the coordinate transformation to the reference frame in the inertial regions takes the form

\[
x^\mu(\tau, r) = z^\mu(T) + (\tau - T)e_0^\mu(T) + re_1^\mu(T) \quad \text{for} \quad \tau > T
\]

\[
x^\mu(\tau, r) = z^\mu(0) + \tau e_0^\mu(0) + re_1^\mu(0) \quad \text{for} \quad \tau < 0
\]

(we use the convention \(\text{sgn}(g_{\mu\nu}) = +2\)). In these equations \(r\) denotes the proper distance between the point \(z^\mu(\tau)\) on the worldline and the point \(x^\mu(\tau, r)\) (inside an inertial region) on the spatial coordinate line that is parametrized by \(r\). Hence two points \(x^\mu(\tau, r)\) and \(x^\mu(\tau', r)\) having the same parameter \(r\) belong to the same line of constant proper distance \(r\) even if they lie in different inertial regions (compare Fig. 1). These lines are just the coordinate lines of proper time.

It is now reasonable to demand that the coordinate lines of the proper time are timelike everywhere. A necessary condition for this to hold is that the distance between two points having the same parameter \(r\) is always timelike. To check if this condition is always fulfilled we choose two points. One point \(x^\mu(\tau = r, r)\) is placed on the past light cone of the origin (units with \(c = 1\) are used and \(r < 0\) is assumed) and therefore lies on the borderline of the lower inertial region. The other point \(x^\mu(\tau', r)\) belongs to the outgoing inertial region (compare Fig. 1). In the inertial system that was chosen the various tetrad vectors in Eq. (1) can be represented as

\[
e_0^\mu(0) = (1, 0), \quad e_1^\mu(0) = (0, 1), \quad e_0^\mu(T) = (\cosh \alpha, \sinh \alpha), \quad \text{and} \quad e_1^\mu(T) = (\sinh \alpha, \cosh \alpha)
\]

so that \(\Delta x^\mu := x^\mu(\tau', r) - x^\mu(\tau = r, r)\) can be calculated explicitly. It is clear that \(\Delta x^\mu\) is a timelike vector if \(r\) is small. If it becomes spacelike for growing \(|r|\) there must be a value \(\hat{r}\) of \(r\) for which it is lightlike. To determine \(\hat{r}\) one has to solve the equation \(\Delta x^\mu \Delta x_\mu = 0\) or, in two dimensions and since both \(x^\mu(\tau', r)\) and \(x^\mu(\tau = r, r)\) are on the left of the (timelike) worldline, \(\Delta x^0 + \Delta x^1 = 0\). Inserting Eq. (1) into this expression leads to

\[
de^\beta + (\tau' - T)e_0^\alpha + \hat{r}(e^\alpha - 2) = 0
\]

where the timelike vector \(z^\mu(T) - z^\mu(0)\) was written as \((d \cosh \beta, d \sinh \beta)\) with \(d\) being positive.

The first two terms on the l.h.s. of Eq. (2) are always positive so that the last must be negative in order to fulfill the equality. Since \(r < 0\) this can only happen for \(\exp(\alpha) > 2\). This implies that the final velocity \(v\) of the observer must be larger than

\[
v = \tanh(\alpha) > 0.6 \quad \text{for} \quad e^\alpha > 2.
\]
For velocities larger than 0.6 one can determine $\tau'$ as function of $\hat{r}$ (or vice versa) from Eq. (2). This results in a borderline $x^\mu(\tau'(\hat{r}), \hat{r})$ above which a reference frame is allowed to behave as an inertial system without necessarily leading to a spacelike time coordinate. It is given by

$$x^\mu(\tau'(\hat{r}), \hat{r}) = z^\mu(T) - de^{\beta-\alpha}e^{\mu}_{0}(T) + \hat{r}\{(2e^{-\alpha} - 1)e^{\mu}_{2}(T) - e^{\mu}_{1}(T)\}.$$  

(4)

This is a spacelike straight line approaching the light cone if the velocity $v$ is close to 1. The line starts at the point $z^\mu(T) - de^{\beta-\alpha}e^{\mu}_{0}(T)$. It is not difficult to show that this point is given by the intersection point of the future light cone of the origin and a hypothetical inertial observer’s worldline which agrees with $z^\mu(\tau)$ for $\tau > T$.

Putting everything together we found that if in any construction of a reference frame the coordinate lines of proper time are always timelike then for a final velocity $v > 0.6$ the inertial system of the outgoing observer must lie above the spacelike straight line given by Eq. (4). This line approaches the light cone if $v$ approaches the velocity of light.

This result may be generalized in several ways. The first question is if the result also holds in four dimensions and in the presence of a rotation. This can be answered by working in an inertial system where the observer is initially at rest in the origin and where the final velocity points towards the $x^1$ direction. If only coordinate lines are studied which for $\tau < 0$ are in the $x^0 - x^1$ plane then for vanishing rotation the situation is the same as in the two dimensional example shown in Fig. 1 and Eq. (4). The new feature in four dimensional space is that in case of a rotated observer the outgoing coordinate line could be rotated, too. Hence this line need not to lie in the $x^0 - x^1$ plane for $\tau > T$.

More formally, if one introduces in the outgoing region a second set of four orthogonal vectors,

$$\hat{e}_{\underline{0}}^\mu = (\cosh \alpha, \sinh \alpha, 0, 0) \quad , \quad \hat{e}_{\underline{1}}^\mu = (\sinh \alpha, \cosh \alpha, 0, 0)$$

$$\hat{e}_{\underline{2}}^\mu = (0, 0, 1, 0) \quad , \quad \hat{e}_{\underline{3}}^\mu = (0, 0, 0, 1),$$  

(5)

the relevant tetrad vectors of the observer can be expressed as $\hat{e}_{\underline{0}}^\mu(T) = \hat{e}_{\underline{0}}^\mu$ and $\hat{e}_{\underline{1}}^\mu(T) = c_1\hat{e}_{\underline{1}}^\mu + c_2\hat{e}_{\underline{2}}^\mu + c_3\hat{e}_{\underline{3}}^\mu$ with $c_1^2 + c_2^2 + c_3^2 = 1$. In this notation $\Delta x^\mu$ is given by

$$\Delta x^\mu = z^\mu(T) + (\tau' - T)e_{\underline{0}}^\mu(T) + r\{c_1\hat{e}_{\underline{1}}^\mu + c_2\hat{e}_{\underline{2}}^\mu + c_3\hat{e}_{\underline{3}}^\mu - e_{\underline{0}}^\mu(0) - e_{\underline{1}}^\mu(0)\}.$$  

(6)

To avoid tedious calculations it is convenient to calculate $\Delta x^\mu\Delta x_\mu$ only for the case $r$ and $\tau'$ being large. Then the term $z^\mu(T)$ in Eq. (3) is negligible and one finds

$$\Delta x^\mu\Delta x_\mu = -(\tau' - T)^2 + r^2[1 - 2c_1e^{-\alpha}] + 2r(\tau' - T)e^{-\alpha}.$$  

(7)

This expression agrees with the two dimensional result for $c_1 = 1$ if $z^\mu(T)$ is also neglected in the two dimensional case. For $c_1 < 1$ the expression for $\Delta x^\mu\Delta x_\mu$ becomes larger, that is ”more spacelike”. Thus for long times $\tau'$ a rotation will lead to even more stringent conditions on the inertial regions than those found in two dimensions. The result presented for the two dimensional case is therefore also a lower bound for the four dimensional case which includes a possible rotation of the observer.
3 A particular model

To demonstrate the result presented above for a particular construction of a possible reference frame the proposal of Ref. [6] will be used. In this model the hypersurfaces of constant proper time are constructed according to the following principle: (the tangents of) the spatial coordinate lines should, after parallel transport along a light ray, always be orthogonal to the four-velocity $d\mathbf{z}/d\tau$ of the observer at the point in the future of the coordinate line where the light ray intersects the world line. All details can be found in Ref. [6]. The motivation to study this proposal was to get rid of the shortcomings of the original construction for Fermi coordinates. In the present context it is essential that the spatial coordinate lines are constructed by using only the past light cone of the points on the worldline, not their future light cones. Therefore any point on the spatial coordinate lines contains only information about that part of the worldline which lies in its future. This implies that after the observer reached the final velocity all coordinate lines are constructed only with the aid of an inertial worldline and hence are straight lines.

Without going into any details we state here the results for an observer who is accelerated with constant acceleration $a$ between $\tau = 0$ and $\tau = T$. In this case the differentiable worldline is given by

$$
\begin{align*}
\left( z^0(\tau) \right. & = \begin{cases} 
\tau & \tau < 0 \\
1/a & 0 < \tau < T \\
\sinh(a\tau)/a & \tau > T 
\end{cases} \\
\left. z^1(\tau) \right) & = \begin{cases} 
\sinh(a\tau)/a & \tau < 0 \\
\cosh(a\tau)/a & 0 < \tau < T \\
\sinh(aT)/a + (\tau - T) \cosh(aT) & \tau > T 
\end{cases} 
\end{align*}
$$

It should be noted that for this worldline the spatial coordinate lines of the original Fermi coordinates do intersect far away from the worldline so that the time coordinate becomes undefined. This also shows that ordinary Fermi coordinates can only be considered as a local approximation to an extended reference frame. Following the steps of Ref. [6] the metric in the new "reference frame" is given by

$$
g_{\mu\nu} = \begin{pmatrix} 1 - 2 \exp(\phi) & \text{sgn}(r) [1 - \exp(\phi)] \\
\text{sgn}(r) [1 - \exp(\phi)] & 1 \end{pmatrix}
$$

where the function $\phi$ is defined by

$$
\phi(\tau, r) := \Theta(T - \tau) S(r, T - \tau) - \Theta(-\tau) S(r, -\tau)
$$

with $\Theta$ being the step function and

$$
S(x, q) := \begin{cases} 
q & x > q \\
x & -q < x < q \\
-q & x < -q
\end{cases}
$$

The coordinate lines of this coordinate system for $v = \tanh(aT) > 0.6$ are shown in Fig. 2. One can see that the crossing of spatial coordinate lines, which occurs in...
the original Fermi coordinates, is avoided. The price we have to pay for this is that
the time coordinate becomes spacelike far away from the worldline.

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Figure 1: The incoming and outgoing inertial regions of a possible reference frame. Lines of constant $r$ are drawn solid, and lines of constant $\tau$ are drawn dashed. The thick solid line represents the world line. The dashed line crossing the line of constant proper distance $r$ is the boundary below which the reference frame cannot be inertial.
Figure 2: The coordinate lines of the modified Fermi coordinates for a final velocity larger than $0.6 \, c$. In this case the lines of constant spatial coordinate $r$ can become spacelike. The hypersurface of constant proper time $\tau = T$ is drawn as a thick dashed line.
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