Stiff self-interacting string near QCD deconfinement point

A. Bakry,1,* X. Chen,1 M. Deliyergiyev,1 A. Galal,1 S. Xu,1 and P.M. Zhang1

1Institute of Modern Physics, Chinese Academy of Sciences, Gansu 730000, China

(Dated: July 11, 2017)

We investigate the implications of Nambu-Goto (NG), Lüscher-Weisz (LW) and Polyakov-Kleinert (PK) string actions for the Casimir energy and the width of the quantum delocalization of the string at two loop order. We focus our numerical study on the 4-dim pure SU(3) Yang-Mills lattice gauge theory at two temperature scales near the deconfinement point. The region under scrutiny is the long and the intermediate source separation distance before the string breaks in full QCD. The numerical data corresponding to the quark-antiquark (QQ) potential and the width profile of the color tube are found to match the leading order free string predictions at the temperature near the end of QCD plateau. However, at a temperature closer to the critical point, we found that the next to leading order (NLO) contributions of the self-interacting (NG) string, boundary terms in (LW) string action in addition to the term proportional to the extrinsic curvature in (PK) string action to yield a static potential and broadening profile that is in a good agreement with the lattice data at the intermediate and long distances scales. The string’s self-interactions account for the suppressed broadening of the width and the geometrical features along the string profile in the intermediate distance.

PACS numbers: 12.38.Gc, 12.38.Lg, 12.38.Aw

I. INTRODUCTION

The formulation of a string theory for hadrons has been an attractive proposal induced by the phenomenological success in explaining Veneziano formula [1] and even before the formulation of quantum chromodynamics (QCD). Despite the difficulties encountered in the quantization scheme in a fundamental string theory, the proposal to to describe the long-distance dynamics of strong interactions inside the hadrons by a low energy effective string [2, 3] has remained an alluring conjecture.

String formation is realized in many strongly correlated systems and is not an exclusive property of the QCD color tubes [4–8]. The normalization group equations imply that the system flows towards a roughening phase where the string is no longer rigid and oscillations transverse to the classical world sheet with predictable measurable effects that can be and verified in numerical simulations in the rough phase of lattice gauge theories (LGT).

In the leading Gaussian approximation of the NG action, the quantum fluctuations of the string bring forth a universal quantum correction to the linearly rising potential well known as the Lüscher term in the meson and geometry-dependent Lüscher-like terms [9, 10] in baryonic configurations. The width due to the quantum delocalizations of the string grows logarithmically [11] as two color sources are pulled apart. Logarithmic broadening is expected for the baryonic junction [12] as well.

Precise lattice measurements of the QQ potential in SU(3) gauge model are in consistency with the Lüscher subleading term for color source separation commencing from distance $R = 0.5$ fm [3]. This is typically the distance scale where the effects of the intrinsic thickness of the flux tube $1/T_c$ [13] diminishes and the effective description is expected to hold. Many gauge models have unambiguously identified the Lüscher correction to the potential with unprecedented accuracy [14–19]. The string model predicts, in addition, a logarithmic broadening for the width profile of the string delocalization. This also has been observed in several lattice simulations corresponding to the different gauge groups [14, 20–30]. An overlapping spectrum of the string’s excited states would manifest in the high temperature regime of lattice gauge models. The free string approximation implies a decrease in the slope of the QQ potential or in other-words a temperature-dependent effective string tension [31, 32]. The leading-order correction for the mesonic potential turns into a logarithmic term of Dedekind $\eta$ function which encompasses Lüscher term as the zero temperature $T = 0$ limiting term [3, 33, 34]. For large source separation distances the string’s broadening exhibits a linear pattern before the deconfinement is reached from below [35–38].

Nevertheless, the simplified picture of the bosonic string derived on the basis of the leading order formulation of NG action poorly describe the numerical data in the intermediate distances at high temperatures. For instance, substantial deviations [39–42] from the free string behavior have been found for the lattice data corresponding to temperatures very close to the deconfinement point.

A comparison with the lattice Monte-Carlo data showed the validity of the leading-order approximation at sources separations larger than $R = 0.9$ fm [39, 40, 42] for both the QQ potential and color-tube width profile. The region extends behind source separation distances at which leading-order string model predictions are valid [3] at zero temperature. In the baryon [43–45],

*abakry@impcas.ac.cn
taking into account the length of the Y-string between any two quarks, we found a similar behavior [46].

The fact that the lattice data substantially deviate from the free string description in the intermediate distance and high temperatures has induced many numerical experiments to verify the validity of higher-order model dependent corrections for the NG action [47, 48].

Even in the Nambu-Goto (NG) framework there is no reason to believe that all orders of the power expansion are relevant to the correct behavior of QCD strings [49]. For example, a first-order term deviating from the universal behavior has been determined unambiguously in 3D percolation model [49], no numerical evidence indicating universal features of the corrections beyond the Lüscher term have been encountered among $Z(2)$, $SU(2)$ and $SU(3)$ confining gauge models [47]. Numerical simulations of different gauge models in different dimensions may culminate in describing both the intermediate and long string behavior by different effective strings [47].

The pure $SU(3)$ Yang-Mills theory in four dimensions is the closest approximation to full QCD. Even though, we are lacking detailed understanding for the string behavior at high temperature and intermediate distance scale. In this region the deviations from the free string behavior occurs on scales that is relevant to full QCD before the string breaks [30]. The nature of the QCD strings at finite temperatures can be very relevant to many portrayals involving high energy phenomena [51, 52] such as mesonic spectroscopy [53–55], glueballs [56, 57] and string fireballs [58], for example. This calls for a discussion concerning the validity higher order string effects at temperature scales near QCD critical point which is the target of this report.

The paper is organized as follows: In section(II), we review the most relevant string model to QCD and discuss the lattice data corresponding to the Casimir energy versus different approximation schemes. In section(III), the profile of the density distribution is compare to the mean-square width of the string fluctuations of both the Nambu-Goto (NG) and Polyakov-Kleinert (PK) strings. Concluding remarks are provided in the last section.

II. STRING PHEMENOLOGY AND LATTICE DATA

A. String actions and Casimir energy

The linear rise property of the confining potential advocated the conjecture that Yang-Mills (YM) vacuum admits the existence of a very thin string object [2] transmitting the strongly interacting forces between the color sources. The intuition is in consistency [59] with the dual superconductive [16, 60–64] QCD vacuum and the squeeze of the color fields into a confining thin string dual to the Abrikosov line by the virtue of the dual Meissner effect.

The formation of the string condensate spontaneously breakdown the transnational and rotational symmetries of the YM-vacuum and entails the manifestation of $(D-2)$ massless transverse Goldstone modes in addition to their interactions.

To establish an effective string description, a string action can be constructed from the derivative expansion of collective string co-ordinates satisfy Poincare and parity invariance. One particular form of this action is the Lüscher and Weisz [2, 3] which encompasses built-in surface/boundary terms to account for the interaction of an open string with boundaries. The Lüscher and Weisz [28] $(LW)$ effective action up to four-derivative term reads

$$S^{LW}[X] = \sigma A + \frac{\sigma}{2} \int d\zeta_1 \int d\zeta_2 \left( \frac{\partial X}{\partial \zeta_\alpha} \cdot \frac{\partial X}{\partial \zeta_\alpha} \right) + \sigma \int d\zeta_1 \int d\zeta_2 [\kappa_2 \left( \frac{\partial X}{\partial \zeta_\alpha} \cdot \frac{\partial X}{\partial \zeta_\alpha} \right)^2 + \kappa_3 \left( \frac{\partial X}{\partial \zeta_\alpha} \cdot \frac{\partial X}{\partial \zeta_\beta} \right)^2] + S_b$$

Invariance under parity transform would keep only even number derivative terms. The vector $X^\mu(\zeta_1, \zeta_2)$ maps the region $\mathcal{C} \subset \mathbb{R}^2$ into $\mathbb{R}^4$ and couplings $\kappa_1, \kappa_2$ are effective low-energy parameters. The boundary term $S_b$ describes the interaction of the effective string with the Polyakov loops at the fixed ends of the string and is given by

$$S_b = \int d\zeta_0 [b_1 \partial^2 X_i \partial^2 X^i + b_2 \partial_2 \partial_1 X_i \partial_2 \partial_1 X^i + .......].$$

Consistency with the open-closed string duality [28] implies a vanishing value of the first boundary coupling $\delta_1 = 0$, the leading order corrections due to second boundary terms with the coupling $b_2$ appears at higher order than the four derivative term in the bulk.

For the next-to-leading order terms in $D$ dimension the open-closed duality [28] imposes further constraint on the kinematically-dependent couplings to

$$(D - 2)\kappa_2 + \kappa_3 = \left( \frac{D - 4}{2\sigma} \right).$$

Moreover, it has been shown [65] through a nonlinear Lorentz transform in terms of the string collective variables $X_i$ [66] that the action is invariant under $SO(1, D - 1)$. By this symmetry the couplings Eq. (3) of the four derivative term in the Lüscher-Weisz (LW)
action Eq. (1) are not arbitrary and are fixed in any dimension D by
\[ \kappa_2 + \kappa_3 = \frac{-1}{8\sigma}, \] (4)
Condition Eq. (3) implies that all the terms with only first derivatives in the effective string action Eq. (1) coincide with the corresponding one in Nambu-Goto action in the derivative expansion. The Nambu-Goto action is the most simple form of string actions proportional to area of the world-sheet
\[ S^{NG}[X] = \sigma \int d\zeta_1 \int d\zeta_2 \sqrt{g}, \] (5)
where g is the two-dimensional induced metric on the world sheet embedded in the background \( \mathbb{R}^4 \). On the quantum level the Weyl invariance of the NG action is broken in four dimensions; however, the anomaly is known to vanish at large distances \([59]\). The physical gauge \( X^1 = \zeta_1, X^4 = \zeta_2 \) restricts the string fluctuations to transverse directions \( \mathcal{C} \). The action after gauge fixing reads
\[ S^{NG}[X] = \sigma \int_0^L d\zeta_1 \int_0^R d\zeta_2 \sqrt{1 + (\partial_\zeta_1 X_\perp)^2 + (\partial_\zeta_2 X_\perp)^2}}. \] (6)
The expansion of the above NG action up to the leading and next to leading order terms is
\[ S^{NG}_{lo}[X] = S_{LW}[X] = \sigma A + \frac{\sigma}{2} \int d\zeta_1 \int d\zeta_2 \left( \frac{\partial X}{\partial \zeta_\alpha} \cdot \frac{\partial X}{\partial \zeta_\beta} \right), \] (7)
and
\[ S^{NG}_{nlo}[X] = \sigma \int d\zeta_1 \int d\zeta_2 \left( \frac{\partial X}{\partial \zeta_\alpha} \cdot \frac{\partial X}{\partial \zeta_\beta} \right)^2 + (\partial X/\partial \zeta_\alpha \partial X/\partial \zeta_\beta)^2, \] (8)
respectively. A simple generalization of the Nambu-Goto string \([59,67,68]\) has been proposed by Polyakov \([69]\) and Kleinert \([70]\) to stabilize the NG action and was first investigated in the context of fluid membranes. This is a bosonic string with a term proportional to the extrinsic curvature of the surface as a next operator after NG action \([69,70]\). The action of the bosonic (Polyakov) string with the extrinsic curvature term reads
\[ S^{PK} = \frac{\sigma}{2} \int d^2\zeta \sqrt{g} \rho^\alpha \rho^\beta \partial_\alpha X \partial_\beta + S^{Ext}[X]. \] (9)
\( S^{Ext} \) is defined as
\[ S^{Ext}[X] = \alpha \int d^2\zeta \sqrt{g} \mathcal{K}^2, \] (10)
with the extrinsic curvature \( \mathcal{K} \) defined as
\[ \mathcal{K} = \Delta(g) \partial_\alpha [\sqrt{g} \rho^\alpha \rho^\beta \partial_\beta], \] (11)
where \( \Delta \) is Laplace operator. The term could also be considered as an additional term consistent with Poincare and parity invariance contributing to (LW) action (1).

The extrinsic-curvature term filters out the sharply curved string configurations. The rigidity parameter provides the relative weight between the term proportional to the surface area and the smoothing term in the effective string \([71,72]\). In non-abelian gauge theories this ratio remains constant when taking the continuum limit \([73]\). Rigid string effects may manifest in the IR region of SU(N) non-abelian gauge theories and could perhaps account for some fine structure deviations from the simple NG string.

Prior to set a comparison with the numerical Yang-mills lattice data, we review in the following the corresponding expression for the Casimir energy due to each string action. Generally, the Casimir potential is extracted from the string partition function as
\[ V(R,T) = -\frac{1}{T} \log(Z(R,T)). \] (12)
The partition function of the NG model in the physical gauge is functional integrals over all the world sheet configurations swept by the string,
\[ Z(R,T) = \int \mathcal{D}X \exp(-S^{NG}(X)). \] (13)
For a periodic boundary condition along the time direction with extend equals to the inverse of the temperature \( L_T = \frac{1}{T} \)
\[ X(\zeta^0 = 0, \zeta^1) = X(\zeta^0 = L_T = \frac{1}{T}, \zeta^1), \] (14)
and Dirichlet boundary condition at the sources position
\[ X(\zeta^0, \zeta^1 = 0) = X(\zeta^0, \zeta^1 = R) = 0. \] (15)
the path integral of Eq. (13) and Eq. (12) yield the static potential for the leading order contribution Eq. (7) of the NG action using \( \zeta \) function regularization scheme \([74]\) as
\[ V_{lo}(R,T) = \sigma R + (D - 2)T \ln \eta(i\tau) + \mu(T), \] (16)
where \( \mu(T) \) is a renormalization parameter and \( \eta \) is the Dedekind eta function defined according to
\[ \eta(\tau) = \frac{1}{2\pi} \prod_{n=1}^{\infty} (1 - q^n); \quad q = e^{-\pi \sqrt{\frac{r}{2 \pi T}}}, \] (17)
where \( \tau = \frac{L_T}{2T} \) is the modular parameter of the cylinder. The second term on the right hand side encompasses the Lüscher term of the interquark potential. This term signifies a universal quantum effect which is a characteristic of the CFT in infrared free-string limit and is independent of the interaction terms of the effective theory.

Deitz and Filk \([74]\) extracted the second model-independent corrections \([75]\) to the Casimir effect from the explicit calculation of the two-loop approximation with the same regularization scheme as
\[ V_{n\ell_0}(R, T) = \sigma R + (D - 2)T \ln \eta(i\tau) - T \ln \left( 1 - \frac{(D - 2)\pi^2 T}{1152\sigma_o R^4} \left[ 2E_4(\tau) + (D - 4)E_4^2(\tau) \right] \right) + \mu(T). \quad (18) \]

\[ E_2k(\tau) = 1 + (-1)^k \frac{4k}{B_k} \sum_{n=1}^{\infty} \frac{n^{2k-1} \omega^n}{1 - \omega^n}. \quad (19) \]

The string tension as a function of the temperature which is the slope of the leading linear term of the potential. From Eq. (18) the string tension up to the next to leading order is given by

\[ \sigma(T) = \sigma_0 - \frac{\pi(D - 2)}{6} T^2 - \frac{\pi^2(D - 2)^2}{72\sigma_0} T^4 + O(T^6) \quad (20) \]

The coefficients of additional higher-order corrections \( T^6 \) can be inducted \([49, 67]\) and leads to the string tension

\[ \sigma(T) = \sigma_0 - \frac{\pi(D - 2)}{6} T^2 - \frac{\pi^2(D - 2)^2}{72\sigma_0} T^4 - \frac{(D - 2)^3 \pi^3 T^6}{432\sigma_0} + O(T^7) \quad (21) \]

In addition to the consecutive expansion terms in NG action, the boundary term \( S_b \) in Lüscher-Weisz action encodes the interactions of the string with the boundaries.

In Refs. [65, 76] the first nonvanishing Lorentz-Invariant boundary term contribution has been calculated. The modification to the potential received from this interaction for Dirichelet boundary condition is given by

\[ V_b = (D - 2)b_2 \frac{\pi^3 L T}{60 R^4} E_4(q), \quad (22) \]

where \( b_2 \) is a fit parameter.

Apart from the corrections of the area expansion of the worldsheet in NG action, the static potential can be given as a function of geometrical characteristics such as the extrinsic curvature Eq. (9). The Casmir and free energies contribution due to an extrinsic curvature term was evaluated in [77–80] for stiff strings. Employing zeta function regularization [81] the finite-temperature contribution is calculated [82, 83] to the first loop.

Recalling that the rigid string action is defined as additional extrinsic curvature term to the ordinary NG action, the quark-antiquark potential can be considered in conjunction with two subsequent orders in the NG perturbative expansion. In four dimensions the total static potential of rigid string in the leading order term

\[ V_{\text{ng}}^{\text{Stiff}}(R, T) = \sigma R + T \ln \eta(i\tau) + T \sum_{n=0}^{\infty} \ln \left( 1 - e^{-2R\sqrt{\Omega_n + \omega_q^2}} \right) - \frac{\pi RT^2}{6} + \frac{T}{4} \ln \left( \frac{1}{2TR} \right) + \mu(T), \quad (23) \]

with \( \Omega_n = 2\pi nT \). The above expression treats the oscillations of two dimensional scalar field but with mass equal to \( \omega_q = \sigma_0/\ell_o \) in the massless limit [82] Eq. (23) yields Eq. (16). The potential of the stiff string with the next leading order NG contribution is

\[ V_{\text{ng}}^{\text{Stiff}}(R, T) = V_{\text{Stiff}}^{\ell_0}(R, T) - T \ln \left( 1 - \frac{(D - 2)\pi^2 T}{1152\sigma_o R^4} \left[ 2E_4(\tau) + (D - 4)E_4^2(\tau) \right] \right). \quad (24) \]

In the limit of short distance and low temperature the coefficient of the Lüscher term is doubled. The string tension of rigid string is given in large \( d \) limit in Ref. [78].

One way to address the string tension dependency on the temperature is to consider a universal relation that utilizes an overlap formalism of the two point correlators discussed in Ref. [84]. The partition function of a closed string with Dirichelet boundaries can be related to a tower of energy states [28].

\[ V_{QQ} = -T \ln \left( \sum_n c_n K_0(RE_n) \right) + \mu(T) \quad (25) \]

which are functions in the energy levels of the closed NG
string [28, 84] given by

\[
K_0(R E_n) = \sqrt{\frac{\pi}{2 E_n^c(\sigma, R, T)}} \left(1 + \frac{4n^2 - 1}{8 E_n^c(\sigma, R, T)} + \frac{16n^4 - 40n^2 + 9}{128(E_n^c)^2(\sigma, R, T)} e^{-E_n^c(\sigma, R, T)},
\]

(26)

and

\[
E_n^c(\sigma, R, T) = \sigma RT \sqrt{1 + \frac{8\pi}{\sigma T^2} \left(n - \frac{1}{12}\right)}.
\]

(27)

The parameterization of the static potential through this expression is very relevant to the large distance behavior of the static potential. On the assumption that this behavior is dominated by the lowest state in the Bessel functions sum, which is justified for \(RT > 1\), the static potential is

\[
V_{Q\bar{Q}} = -T \ln (K_0(R E_0)) + \mu(T)
\]

(28)

In the following, we numerically measure the Polyakov-loop two point correlators and explore to what extend each of the above string actions can be a sufficiently good description for the potential between two static color sources.

### B. Fit Analysis of QQ potential

At fixed temperature \(T\), the Polyakov loop correlators address the free energy of a system of two static color charges coupled to a heatbath [85]. Within the transfer matrix formalism \[3\] the two point Polyakov-loop correlators are the partition function of the string. The Monte Carlo evaluation of the temperature dependent quark-antiquark potential at each \(R\) is calculated through the expectation value of the Polyakov loop correlators

\[
P_{2Q} = \int d[U] P(0) P^\dagger(R) \exp(-S_w),
\]

\[
= \exp(-V(R, T)/T).
\]

(29)

with the Polyakov loop defined as

\[
P(\vec{r}_i) = \frac{1}{3} \text{Tr} \left[ \prod_{n=1}^{N_t} U_{\mu=4}(\vec{r}_i, n_i) \right],
\]

(30)

Making use of the space-time symmetries of the torus, the above correlator is evaluated at each point of the lattice and then averaged. We perform simulations on large enough lattice sizes to gain high statistics in a gauge-independent manner [86] in addition to reduce correlations across the boundaries. The two lattices employed in this investigation are of a typical spatial size of \(3.6^3 \text{ fm}^3\) with a lattice spacing \(a = 0.1 \text{ fm}\). We chose to perform our analysis with lattices with temporal extents of \(N_t = 8\), and \(N_t = 10\) slices at a coupling of value \(\beta = 6.00\). The two lattices correspond to temperatures \(T/T_c = 0.9\) just before the deconfinement point, and \(T/T_c = 0.8\) near the end of QCD plateau [87].

The gauge configurations were generated using the standard Wilson gauge action employing a pseudo-heatbath algorithm [88, 89] updating the corresponding to three \(SU(2)\) subgroup elements [90]. Each update step/sweep consists of one heatbath and 5 microcanonical reflections. The gauge configurations are thermalized following 2000 sweeps. The measurements are taken on 500 bins. Each bin consists of 4 measurements separated by 100 sweeps of updates.

The correlator Eq. (32) is evaluated after averaging the time links \[91\] in Eq. (32)

\[
\bar{U}_t = \frac{\int d[U] e^{-T r(Q U^\dagger + U Q)} \bar{U} t}{\int d[U] e^{-T r(Q U^\dagger + U Q)}},
\]

(31)

\[
V_{QQ}(R) = \frac{1}{T} \log < P(x) P(x + R) >
\]

(32)

We look first at the fit and parameterization behavior according to the overlap of the Bessel function Eq. (25) for the QQ potential. Fig. 1 shows the lattice data of the QQ potential, normalized to the value retrieved at \(R = 1.2\), at both temperatures \(T/T_c = 0.8\) and \(T/T_c = 0.9\). The corresponding fits to the overlap potential with the lowest state Eq. (28). The values of \(\chi^2_{\text{ dof}}\) in Table 1 are indicating a good fit behavior for all considered fit region at \(T/T_c = 0.8\) with the string tension set as a free fit parameter. Nevertheless, the plots depict that the best fits is attainable when considering the whole fit region.

At the temperature \(T/T_c = 0.9\) only for large distances commencing from \(R \geq 1.0 \text{ fm}\) a good fit is returned (Table 1). However, the fits are returning the same value of

\[
\begin{array}{cccc}
\text{Fit Range} & n = R/a & \sigma a^2 & \chi^2/N_{\text{DOF}} \\
\hline
\text{at } T/T_c = 0.8 & \text{7-12} & 0.051855 \pm 7.3921 \times 10^{-4} & 2.7581/6 \\
\text{at } T/T_c = 0.9 & \text{7-12} & 0.049810 \pm 5.9853 \times 10^{-4} & 2.7091/6 \\
\end{array}
\]

TABLE I: The returned values of the string tension and the corresponding \(\chi^2\) from the fit to Eq. (28) at \(T/T_c = 0.8\) and \(T/T_c = 0.9\).
the string tension as that received at the other temperature $T/T_c = 0.8$ on the same fit interval. The returned value of the zero temperature string tension $\sigma_0 = 0.049$ is within the standard value of the string tension $\sqrt{\sigma} = 440$ MeV [93]. As a consistency check for our lattice data, we reproduce the same value of the string tension taken as a fit parameter as in Ref. [31] and [32] with the corresponding function of the static potential [3, 33] and fit domain.

To set comparison with the string model predictions, we fit the $Q\bar{Q}$ potential to that derived from the NG string Eqs. (16) and (18) for the exact expression of leading and next-to-leading order, separately. Similarly, we set the string tension $\sigma a^2$ and the renormalization constant $\mu(T)$ as a free fitting parameters. Table II enlists the returned value of the string tension $\sigma a^2$ and $\chi^2_{dof}$ for various source separations commencing from $R = 0.4, 0.5, 0.6$ and $0.7$ fm and extending to $R = 1.2$ fm.

A large value of $\chi^2$ is returned for fits of color sources separations commencing from $R = 0.4$. For separations distance $R \leq 0.4$ fm the NG string description is showing increasingly significant deviations from the LGT data due to the short distance physics and the one dimensional idealization of NG string. In Ref. [13] the intrinsic thickness of the flux-tube has been discussed.

Excluding the point $R = 0.4$ fm dramatically decreases the returned value of $\chi^2$ for both the leading order LO and the next to leading approximation NLO Eqs. (16) and Eqs. (18), respectively. The returned values of the string tension parameter quickly reaches stability even by the exclusion of further points at short distances $R = 0.5$ fm and $R = 0.6$ fm from the fit range. At this temperature, the string tension settles at a stable value of $\sigma a^2 = 0.0445$ measured in lattice units.

| $T/T_c = 0.8$ | Fit Range | $\sigma a^2$ | $\chi^2/N_DOF$ |
|---------------|-----------|-------------|---------------|
| $\bar{V}_{c0}$ | 4-12      | 0.043555±3.9772 × 10^{-4} | 18.6977/9 |
|               | 5-12      | 0.044589±3.8418 × 10^{-4} | 1.5966/8  |
|               | 6-12      | 0.044572±4.6472 × 10^{-4} | 1.5950/7  |
|               | 7-12      | 0.044102±5.3562 × 10^{-4} | 1.0899/6  |
|               | 8-12      | 0.043221±5.8316 × 10^{-4} | 0.4754/5  |
| $\bar{V}_{c0}$ | 4-12      | 0.042235±3.3247 × 10^{-4} | 120.703/9 |
|               | 5-12      | 0.044899±3.3302 × 10^{-4} | 2.9297/8  |
|               | 6-12      | 0.045387±4.3017 × 10^{-4} | 1.2004/7  |
|               | 7-12      | 0.045104±5.1431 × 10^{-4} | 1.0079/6  |
|               | 8-12      | 0.044295±5.6790 × 10^{-4} | 0.4679/5  |

TABLE II: The returned values of the string tension and the corresponding $\chi^2$, the table compares both values for fits to the leading order (LO) Eq. (16) and next to leading order (NLO) Eq. (18).

Considering the fit of the same data of $Q\bar{Q}$ potential to the two-loop expression of the NG string Eq. (18), the
value of $\chi^2_{\text{col}}$ does not apprise significant differences for source separation distances commencing from $R = 0.5$ fm. As shown in Table 2, for different fit ranges with a fixed end point at $R = 12$ fm, the fit return acceptable values of $\chi^2$ with subtle changes in the free fit parameter $\sigma a^2$.

The numerical data for the $Q\bar{Q}$ potential match both the free leading-order NG string Eq. (16) and the NLO self-interacting form of Eq. (18). Approximately the same difference in the value of the string tension is retrieved for fit domains involving short to large $Q\bar{Q}$ separation distances. The enclosure of the fourth derivative term of the NG/LW action appears, thereof, neither to alter the parameterisation behavior nor to indicate significant changes on the value of the string tension shown in Table 2. The absence of the mismatch between Eq. (18) and the numerical data at this temperature scale does not rule out the validity at this temperature scale.

This points out to the minor role of the higher order modes at the end of the QCD plateau $T/T_c = 0.8$. The pale out of the thermal effects together with a flat plateau region at this temperature is present as well in the string tension measurements [31] and the more recent Monte-Carlo measurements [94] which reproduces the same value of 0.044 of the zero temperature string tension.

Thermal effects are more noticeable in the present SU(3) Yang-Mills model [31, 87] if the temperature is scaled down to $T/T_c = 0.9$ close to the critical point. The lattice data corresponding to the measured $Q\bar{Q}$ potential are depicted in Fig. 3.

We follow a different connive to disclose the fit behavior of the lattice data at this temperature scale with respect to the LO and NLO approximation. We schematically inspect the returned values of $\chi^2$ for an interval of selected values of the string tension $\sigma a^2 \in [0.035, 0.046]$. The residuals and normalization constant $\mu(T)$ for the corresponding $\sigma a^2$ are inlisted in Table. III. The fits to the $Q\bar{Q}$ potential are for two fit ranges.

For the next to leading approximation, gradual descend of the string tension parameter from 0.045 to 0.041 reduces dramatically the values of $\chi^2$-inlisted in Table III- till a minimum is reached at $\sigma a^2 = 0.041$ for a fit interval from $R = [0.5, 1.2]$ fm. Even so, excluding points at short distance, i.e., considering a fit interval $R \in [0.9, 1.2]$, results in a smaller value of $\chi^2$ with a shifted minimal at $\sigma a^2 = 0.039$ as depicted in Fig. 4.

The fit of the numerical data to the leading order approximation Eq. (16) produce similar reduction in the residuals by excluding short distance points. The fits return a minimal of $\chi^2$ at $\sigma a^2 = 0.039$ on $R \in [0.5, 1.2]$ fm and $\sigma a^2 = 0.037$ on $R \in [0.9, 1.2]$ fm. However, the values of $\chi^2$ are outstandingly higher than the corresponding returned values considering the next-to-leading approximation Eq. (18).

In spite of some improvements by stretching out the

---

**FIG. 2:** The quark-antiquark potential measured at temperature $T/T_c = 0.8$, the solid and dashed lines correspond to fits to the LO and NLO string potential of Eqs. (16) and Eq. (18), respectively.

**TABLE III:** The $\chi^2$ values returned from fits for each corresponding value of the string tension, the table compares both values for fits to the leading order (LO) Eq. (16) and next to leading order (NLO) Eq. (18).
fits to the string’s self interactions beyond the Gaussian approximation, the inclusion of the NLO terms does not provide an acceptable optimization for the potential data. Nevertheless, higher order terms in the free energy provide fine corrections for the free-parameter $\sigma_0 a^2$ interpreted as the zero temperature string tension, i.e., the value returned from fits at $T/T_c = 0.8$, or measured at zero temperature [94].

A probable role that could be played by even higher order terms of the power expansion of NG string action may be discussed in the context of the string tension dependency on the temperature. Equation (20) sets out the perturbative temperature dependence of the string tension at both the fourth and six powers of temperature.

In Fig 5 each theoretical curve is a plot corresponding to the respective order in the NG power expansion. A single data point fixes the curve of the string tension corresponding to each temperature. A correct string tension behavior versus the temperature would entail that the other data point fall into the same line.

The measured value of $\sigma a^2$ at $T/T_c = 0.8$ making use of the fits of LO and NLO approximations are the same within the standard deviation of the measurements. We take this value of the string tension as a reference value for the zero-temperature string tension $\sigma_0 = 0.044$ measured also in [94].

The lattice point at $T/T_c = 0.8$ fixes the string tension curve as depicted in Fig. 5. The measurement taken for the string tension using formula Eq. (20) for the lattice data at $T/T_c = 0.9$ produces a minimal of the residual at $\sigma_0 = 0.041$ for the zero temperature string tension. The corresponding curve is shown in Fig. 5 as solid line. The value of the string tension $\sigma(T)a^2 = 0.0192051$ for $\sigma_0 = 0.039$ and $\sigma(T)a^2 = 0.0245952$ for $\sigma_0 = 0.044$ at the fourth power of the temperature. However, very small correction is received from the term proportional to the six power in the temperature. At this order the string tension is $\sigma(T)a^2 = 0.0234638$ at $\sigma_0 = 0.044$ and is $\sigma(T)a^2 = 0.017765$ for $\sigma_0 = 0.039$.

![Graph](image.png)

**TABLE IV:** The $\chi^2$ values returned from fits for each corresponding value of the string tension, the table compares both values for fits to the next to leading order (NLO) static potential with boundary terms Eq. (33).

| $T/T_c = 0.9$ | String tension | Fit Range[5-12] | Fit Range[7-12] |
|---------------|----------------|-----------------|-----------------|
| $\sigma$      | $\chi^2$       | $b_2$           | $\chi^2$       | $b_2$           |
| 0.043         | 717.74         | -1.89           | 332.49          | -40.92          |
| 0.042         | 261.90         | -0.57           | 169.42          | -21.15          |
| 0.041         | 70.49          | 0.75            | 63.98           | -1.34           |
| 0.040         | 145.07         | 2.07            | 16.45           | 18.51           |
| 0.0397        | 249.74         | 2.60            | 13.53           | 24.8            |
| 0.039         | 487.30         | 3.40            | 27.19           | 38.40           |
| 0.038         | 1099.03        | 4.73            | 96.54           | 58.34           |

FIG. 3: The quark-antiquark $QQ$ potential measured at temperature $T/T_c = 0.9$, the left and right plots correspond to the fits to LO and NLO perturbative expansion of Nambu-Goto string Eq. (16) and Eq. (18), respectively.
FIG. 4: (a) The returned $\chi^2_{\text{dof}}$ versus the string tension $\sigma a^2$, scaled by factor of $10^2$, from the fits of $Q\bar{Q}$ potential to leading order approximation of Nambu-Goto string Eq. (16) at $T/T_c = 0.9$. (b) Similar to (a), however, the fits are for the next to leading order approximation Eq. (18).

FIG. 5: The temperature-dependence of the string tension for Nambu-Goto string action at LO, NLO and NNLO perturbative expansion. The dashed lines correspond to $\sigma_0 a^2 = 0.044$ and solid line corresponds to $\sigma_0 a^2 = 0.039$.

FIG. 6: (a) The returned $\chi^2_{\text{dof}}$ versus the string tension $\sigma a^2$, scaled by factor of $10^2$, from the fits of $Q\bar{Q}$ potential to Nambu-goto string with boundary terms Eq. (33) at $T/T_c = 0.9$. (b) The corresponding fits to strings with extrinsic curvature added to leading order and next leading order terms of NG action Eq. (23) and Eq. (24).

The deviations of the theoretical lines from the standard measured value at $\sigma_0 a^2 = 0.044$ indicate that model-dependent corrections received even from the six derivative terms in NG do not provide a precise match with the present four-dimensional Yang-Mills model for the correct behavior of the temperature-dependent string tension.

Effects such as the interaction of the string with the boundaries may relevant to the discrepancy in the string description for intermediate distance and the correct string tension dependency on the temperature. The contribution to the Casimir energy due to the leading nonva-

| $T/T_c$ = 0.9 String tension | Fit Range[5-12] | Fit Range[7-12] |
|-----------------------------|----------------|----------------|
| 0.046                       | $60.89$       | $60.7$         |
| 0.045                       | $42.15$       | $34.87$        |
| 0.044                       | $43.01$       | $20.25$        |
| 0.043                       | $58.45$       | $13.92$        |
| 0.042                       | $84.60$       | $13.74$        |
| 0.041                       | $118.41$      | $18.07$        |
| 0.040                       | $157.51$      | $25.67$        |
| 0.039                       | $200.03$      | $35.55$        |
| 0.038                       | $244.46$      | $46.96$        |
| 0.037                       | $289.64$      | $59.28$        |

TABLE V: The $\chi^2$ values returned from fits for each corresponding value of the string tension, the table compares both values for fits to the leading order (LO) approximation of the stiff string Eq. (34).
The quark-antiquark $Q\bar{Q}$ potential data at temperature $T/T_c = 0.8$ and 0.9 are fit to the static potential

$$V_{Q\bar{Q}} = V_{nLO} + V_b,$$

with $V_{nLO}$ and $V_b$ are given by Eq. (22) and Eq. (18), respectively. The returned values of $\chi^2$ are listed in Table IV considering two fit intervals.

The boundary fit parameter $b_2$ is negative valued and is sensitive to the considered fit interval. The values of $\chi^2$ are still high when considering the fit interval $R \in [0.5, 1.2]$ fm. Even though, none trivial improvements in the values of $\chi^2$ are retrieved as shown in Table IV and Fig. 6 compared to that obtained by merely considering NG action Eqs. (16) and (18) (Table. III). Moreover, the fits to the static potential with boundary term produces acceptable $\chi^2_{dof}$ value for the fit intervals commencing from $R \in [0.7, 1.2]$ fm.

Despite of the reductions in the values of $\chi^2_{dof}$, the consideration of the first boundary term does not significantly alter the value of string tension at the minimal of $\chi^2$. As shown in Fig. 6 and Table IV acceptable value of $\chi^2$ returns zero temperature string tension $\sigma_{00} = 0.0397$ on the fit interval $[0.7, 1.2]$ fm.

The absence of correct description of the $Q\bar{Q}$ potential data and the thermodynamical behavior of string tension on the basis of string models near the deconfinement point has been a long withstanding issue. Effects such as the rigidity of QCD flux tube may become noticeable in confined phase at high temperatures.

![Graph](image-url)
In order to clearly appreciate the changes on the fits when stiffness effects are taken into account, we discuss the modified static potential by virtue of the rigidity in conjunction with both the leading and next to leading approximations to NG action, separately. That is, the fit to \( \bar{Q}Q \) potential data, we consider Eq. (23), (24) and (22) such that

\[
V_{\bar{Q}Q} = V_{\text{stiff}},
\]

and

\[
V_{\bar{Q}Q} = V_{\text{stiff}} + V_b.
\]

We proceed fitting the data in a similar way by varying the string tension leaving the rigidity factor, \( \alpha_{\text{rig}} \) which weighs the extrinsic curvature term in Polyakov-Kleinert action, and the ultraviolet cutoff \( \mu \) as free fit parameters.

Table. VI summarizes values of \( \chi^2 \) obtained from fits to Eq. (34) and (35) at the temperature \( T/T_c = 0.9 \) close to the deconfinement point. We remark the following points:

- Significant improvement in the fit behavior when considering both the leading Eq. (34) and the next to leading order approximation (NLO) Eq. (35) together with stiffness effects.

- The consideration of the stiffness correction only with the leading order NG potential Eq. (34) does provide a smaller \( \chi^2 \) compared to the fits to LW static potential (NLO of NG potential in addition to leading nonvanishing boundary term) Eqs. (33) and (22). In Fig. 8 the \( \chi^2 \) show more sharper peaks of \( \chi^2_{dof} \) minimal around the corresponding values of the string tension.

- The returned values of the rigidity parameter approaches a stable value of the rigidity factor \( \alpha_{\text{rig}} = 1.12 \) with good \( \chi^2 \) for fits to Eq. (35) and flows towards this value considering shorter intervals \( R \in [0.8, 1.2] \) fm for fits to Eq. (34).

- The residuals are minimized at values of the string tension that is shifted from the one obtained by considering merely the static potential of NG action for both LO and NLO approximations. The fit to Eq. (35) results in a value of the string tension which is produced at \( T/T_c = 0.8 \) using all the approximation schemes Eqs. (16), (18), (34) and (35).

- The above mentioned points and the comparison of the fit behavior of Eq. (33) and Eqs. (34) with Eq. (35) points to the remark that the improvement received in the fit by considering a possible stiffness physics of the flux tube is not as merely due to acquiring additional adjustable parameters. At the temperature \( T/T_c = 0.8 \) and string tension value of \( \sigma_0 a^2 = 0.044 \), fits to Eq. (35) are returning acceptable values of \( \chi^2 \) for the same value of the rigidity parameter \( \alpha_{\text{rig}} = 1.12 \). This is consistent with the fact that at relatively lower temperatures the string’s smooth fluctuations are dominant.

The reduction in residual from the fits to Eq. (35) and the subsequent retrieve of the correct behavior of the string tension would suggest a string with a suppressed sharp fluctuations with higher-order effects such as self-interactions and interactions with boundaries to significantly manifest by QCD string at high temperature.
III. THE STRING WIDTH PROFILE

A. Width of Nambu-Goto string

The mean-square width of the string is defined as the second moment of the transverse fluctuations
\[ W^2(\zeta;\zeta^0) = \langle X^2(\zeta;\zeta^0) \rangle \]
\[ = \frac{\int e^{[D X] X^2 \exp(-S^{NG}[X])}}{\int e^{[D X] \exp(-S^{NG}[X])}}, \quad (36) \]
where \( \zeta = (\zeta^1, i\zeta^0) \) is a complex parameterization of the world sheet, such that \( \zeta^1 \in [-R/2, R/2], \zeta^0 \in [-L_T/2, L_T/2] \).

Lüscher, Münster and Weisz \[11\] have shown long-ago the famed property of the logarithmic divergence of the mean square width of the free NG string at the middle plane in the zero temperature limit
\[ W^2 \sim \frac{1}{\pi \sigma} \log \left( \frac{R}{R_0} \right), \quad (37) \]
where \( R_0 \) is an ultra-violet scale. Considering the NG action in the leading order limit, Casselle et al. \[20\] employed the point-split technique to regularize the divergence of the quadratic operator \( (36) \).

The expectation value \[35, 36\] of the mean square width corresponds to the Green function correlator of the free bosonic string theory in two dimensions.

In \( D \) dimension and for cylindrical boundary conditions Eq. \((14)\) and Eq. \((15)\) the mean-square width would read
\[ W^2_{lo}(\zeta, \tau) = \frac{D - 2}{2\pi \sigma} \log \left( \frac{R}{R_0(\zeta)} \right) + \frac{D - 2}{2\pi \sigma} \log \left| \frac{\theta_2(\pi \zeta/R; \tau)}{\theta_1(0; \tau)} \right| \quad (38) \]

where \( \theta \) are Jacobi elliptic functions
\[ \theta_1(\zeta; \tau) = 2 \sum_{n=0}^{\infty} (-1)^n q_1^{(n+1)+\frac{1}{2}} \sin((2n + 1) \zeta), \]
\[ \theta_2(\zeta; \tau) = 2 \sum_{n=0}^{\infty} q_1^{(n+1)+\frac{1}{2}} \cos((2n + 1) \zeta), \quad (39) \]
with \( q_1 = e^{-\pi \tau}, \) with \( \tau = L_T / R \) being the modular parameter of the cylinder, and \( L_T = 1/T \) is the temporal extent governing the inverse temperature and \( R_0(\zeta) \) is the UV cutoff which has been generalized to be dependent on distances from the sources.

The second logarithmic term in Eq. \((38)\) encrypts the dependency of the width on the modular parameter of the cylinder and implies an increase in the width with the temperature and the color source separations. In the limit of large separation distances \( R > L_T \), a modular transform of Eq. \((38)\) yields a linear broadening pattern in the string’s width \[35, 36\]. Moreover, this term signifies a geometrical fine structure of the free string due to the non-constant width along the transverse planes. The curvature is more pronounced with the temperature and the string’s length increase.

F. Gliozzi. M. Pepe and Wiese \[29, 36\] computed analytically the width of the string at next-to-leading order
\[ W^2(\zeta) = W^2_{lo}(\zeta) + W^2_{nlo}(\zeta) \quad (40) \]
with the leading order term \( W^2_{lo} \) in accord to Eq. \((38)\) and the next to leading term is given by
\[ W^2_{nlo}(\zeta) = \frac{\pi}{12 \sigma R^2} [E_2(i\tau) - 4E_2(2i\tau)] \left( W^2_{lo}(\zeta) - \frac{D - 2}{4\pi \sigma} \right) + \frac{(D - 2)\pi}{12 \sigma^2 R^2} \left( \frac{d}{d\tau} - \frac{D - 2}{12} E_2(i\tau) \right) \left( [E_2(2i\tau) - E_2(i\tau)] - \frac{D - 2}{8\pi} E_2(i\tau) \right) \quad (41) \]

B. The mean-square width of smooth string

The smooth configurations of quantum fluctuations swept in the Euclidean space-time by the Nambu-Goto string are favored by adding a new term resulting from the geometrical second fundamental form or the so-called extrinsic curvature/rigidity/stiffness term of the world-sheet.
The second fundamental form (or the shape tensor) in the differential geometer notation defines a quadratic form on the tangent plane of a smooth surface in the three-dimensional Euclidean space. With a smooth choice of the unit normal vector at each point, this quadratic form is generalized for a smooth hypersurface in a Riemannian manifold.

The mean-squared width of the Polyakov-Kleinert string is given by

\[ W^2(\zeta) = \frac{\int DX (X(\zeta, \zeta^0)X_0)^2 \exp(S^P[X])}{\int DX \exp(S^P[X])}, \quad (42) \]

with the action defined as

\[ S^P[X] = S_{fo}^{NG}[X] + S_{nto}^{NG}[X] + S^{Ext}[X], \quad (43) \]

and \( S^{Ext}[X], S_{fo}^{NG}[X] \) and \( S_{nto}^{NG}[X] \) are in accord with Eq. (10), Eq. (6), (7) and Eq. (8), respectively.

The field is replaced by \( X(x, t) \to X(x, t) + \partial_\mu \partial_\mu X(x, t) \) at the next-to-leading order, where \( \gamma \) is a low-energy parameter \([36, 95]\). Expanding around the free-string action Eq. (7) the squared width of the string is then

\[ W^2(\zeta) = W_{fo}^2(\zeta) + (X(\zeta, \zeta^0)^2 (S_{nto}^{NG} + S^{Ext}))[0] \]

\[ + 2 \gamma (\partial_\mu X(\zeta, \zeta^0))_0^2 + \gamma^2 (\partial_\mu \partial_\mu X(\zeta, \zeta^0))_0^2 \]

\[ - \beta \gamma^2 \int d\zeta d\zeta' d\zeta'' (\partial_\mu \partial_\mu X(\zeta, \zeta^0)\partial_\mu \partial_\mu X(\zeta', \zeta''))_0. \quad (44) \]

where \( <>_0 \) represents the vacuum expectation value with respect to the free-string action.

In the following, we calculate the width of the rigid string up to one loop order. The extrinsic curvature term and the corresponding mean-square width are accordingly

\[ S^{Ext} = \alpha_{rig} \int_0^{LT} \int_0^R d\zeta^0 \int_0^{R} d\zeta [\{\partial_\xi \partial_\zeta X + (\partial_\zeta \partial_\xi X)^2\}] \]

The two loop expansion Eq. (44) is then

\[ W^2(\zeta) = W_{fo}^2(\zeta) + W_{nto}^2(\zeta) + W_{Ext}^2, \quad (45) \]

\[ \langle X(\zeta, \zeta^0)^2 S^{Ext} \rangle = (D - 2) \lim_{\epsilon, \epsilon' \to 0} \int_0^R d\zeta' \int_0^{LT} d\zeta'' (G(\zeta, \zeta^0; \zeta', \zeta'') \partial_\mu \partial_\mu \partial_\mu \partial_\mu G(\zeta, \zeta^0; \zeta', \zeta'') + \partial_\mu \partial_\mu G(\zeta, \zeta^0; \zeta', \zeta'') \partial_\mu \partial_\mu \partial_\mu \partial_\mu G(\zeta, \zeta^0; \zeta', \zeta'')). \quad (47) \]

The limit \( \epsilon, \epsilon' \to 0 \) is such that the integral is ultimately regularized using the point-split method.

The Gaussian correlator \( G(\zeta, \zeta^0; \zeta', \zeta'') \) on a cylindrical sheet of surface area RL with Dirichlet and periodic boundary condition in \( \zeta^0 \) with period \( L_T \) according to Eqs. (14) and (15) is

\[ G(\zeta, \zeta^0; \zeta', \zeta'') = \frac{1}{\pi \sigma} \sum_{n=1}^{\infty} \frac{1}{n} (1 - q^n) \sin \left( \frac{\pi n \zeta}{R} \right) \sin \left( \frac{\pi n \zeta'}{R} \right) \left( q^n e^{\frac{\pi n (\zeta - \zeta')}{R}} + e^{-\frac{\pi n (\zeta - \zeta')}{R}} \right), \quad (48) \]

The nonlocal contribution to the mean-square width from the extrinsic curvature component of the generalized smooth string can be calculated in detail \([96]\) and
is given by
\[ W_{\text{Ext}}^2 = (D - 2) \frac{\alpha_{\text{fix}} \pi^2 T}{24 R^3 \sigma^2} E_2^2(q) \] (49)

where \( E_{2n}(q) \) is Eisenstein Series.

IV. ACTION DENSITY ON THE LATTICE

A. Width of the Action Density

In the following, we measure the mean square width of the action density in SU(3) gluonic configurations. The action density is related to the chromo-electromagnetic fields via \( \frac{1}{2}(E^2 - B^2) \) and is evaluated via a three-loop improved lattice field-strength tensor \([97]\).

Constructing a color-averaged infinitely-heavy static quark-antiquark \( \bar{Q}Q \) state by means of two Polyakov lines

\[ \mathcal{P}_{2Q}(\vec{r}_1, \vec{r}_2) = P(\vec{r}_1) P^\dagger(\vec{r}_2), \]

A scalar field characterizing the action density distribution in the Polyakov vacuum or in the presence of color sources \([98]\) can be defined as

\[ C(\vec{\rho}; \vec{r}_1, \vec{r}_2) = \frac{\langle \mathcal{P}_{2Q}(\vec{r}_1, \vec{r}_2) S(\vec{\rho}) \rangle}{\langle \mathcal{P}_{2Q}(\vec{r}_1, \vec{r}_2) \rangle \langle S(\vec{\rho}) \rangle}, \] (50)

with the vector \( \vec{\rho} \) referring to the spatial position of the energy probe with respect to some origin, and the bracket \( < ...... > \) stands for averaging over gauge configurations and lattice symmetries.

We make use of the symmetry of the four dimensional torus, that is, the measurements taken at a fixed color source’s separations \( R \) are repeated at each point of the three-dimensional torus and time slice then averaged. The lattice size is sufficiently large to avoid mirror effects or correlations from the other side of the finite size periodic lattice. The characterization Eq. (50) yields \( C \rightarrow 1 \) away from the quarks by virtue of the cluster decomposition of the operators.

To eliminate statistical fluctuations, uncompromising the physical observables are left intact, only 20 sweeps of UV filtering using an over-improved algorithm \([99, 100]\) have been applied on all gauge configurations.

Different UV filtering schemes can be calibrated \([39, 101]\) in terms of the corresponding radius of the Brownian motion. The prescribed number of stout-link sweeps would be the equivalent of 10 sweeps of APE \([102]\) algorithm \([39, 101]\) with an averaging parameter \( \alpha = 0.7 \).

A careful analysis performed earlier \([39]\) shows that for the prescribed number of sweeps no effects are detectable on either the quark-antiquark \( \bar{Q}Q \) potential or the energy density profile for color source separation distances \( R \geq 0.5 \text{ fm} \) which is the distance region under scrutiny here.

To estimate the mean-square width of the gluonic action density a long each transverse plane to the quark-antiquark axis. Taking into consideration the axial cylindrical symmetry of the tube, we choose a double Gaussian function of the same amplitude, \( A \), and mean value \( \mu = 0 \)

\[ G(r, \theta; z) = A (e^{-r^2/\sigma_1^2} + e^{-r^2/\sigma_2^2}) + \kappa, \] (51)

In the above form the constraint \( \sigma_1 = \sigma_2 \) corresponds to the standard Gaussian distribution. Table. VII compares the returned value of the \( \chi^2 \) for both optimization ansatz, namely, the constrained form \( \sigma_1 = \sigma_2 \) and \( \sigma_1 \neq \sigma_2 \) unconstrained form. The fits of the double Gaussian form return acceptable values of \( \chi^2 \) at the intermediate distances.

Good \( \chi^2 \) values are returned as well when fitting the action density profile to a convolution of the Gaussian with an exponential \([43, 103]\), however, considering statistical uncertainties at large distances (see Fig. (9)) we opt to the form Eq. 51 with \( \sigma_1 \neq \sigma_2 \) for stable fits.

A measurement of the width of the string’s action density may be taken by fitting the density distribution \( C(\vec{\rho}; z) \) to Eq. (51) through each transverse to the cylinder’s axis \( z \) to Eq. (51)

\[ C(r, \theta; z) = 1 - G(r, \theta; z) \] (52)

with \( r^2 = x^2 + y^2 \) in each selected transverse plane \( \vec{\rho}(r, \theta; z) \). The second moment of the action density dis-
FIG. 10: The mean-square width of the density distribution in the middle of the tube $z = R/2$ at the temperature $T/T_c \approx 0.9$. The solid and dashed lines correspond to the free and self-interacting NG string Eq. (38) and Eq. (41), respectively.

### TABLE VII

The mean-square width of the action density $W^2(z)$ and the corresponding $\chi^2$ at the temperature $T/T_c = 0.9$ in the middle transverse plane intersecting the $QQ$ line $z = R/2$. The width estimates and the relative differences are obtained in accord to Eq. (51), with $\sigma_1 = \sigma_2$ corresponding to the standard Gaussian.

| Fit Range | $QQ$ distance, $Ra^{-1}$ | Width of the action density $W^2(x)$ | $\chi^2_{\text{dof}}$ | Relative difference |
|-----------|----------------|---------------------------------|-----------------|---------------------|
|           | $\sigma_1 = \sigma_2$ | $\sigma_1 \neq \sigma_2$ | $\sigma_1 = \sigma_2$ | $\sigma_1 \neq \sigma_2$ |
| 5-9       | 10.1524±0.0563  | 13.1490±0.0826 | 63.2042        | 0.7365 | 22.79% |
| 6-9       | 11.5883±0.0832  | 14.9462±0.1336 | 39.2565        | 0.2515 | 22.47% |
| 7-9       | 13.4276±0.1277  | 17.2756±0.2028 | 21.7959        | 0.0382 | 22.26% |
| 8-9       | 15.4602±0.1985  | 19.8105±0.3578 | 10.2421        | 0.0052 | 21.96% |
| 9-9       | 17.4607±0.2986  | 22.0775±0.6263 | 4.6967         | 0.0043 | 20.91% |
| 10-9      | 19.6764±0.4636  | 24.0357±1.1423 | 1.9736         | 0.0007 | 18.14% |
| 11-9      | 21.9126±0.7022  | 25.7155±1.9140 | 1.1621         | 0.0094 | 14.79% |
| 12-9      | 24.3741±1.1399  | 27.2671±3.2948 | 0.8326         | 0.0249 | 10.61% |
| 5-9       | 199.681  | 374.824  | 79.6823  | 79.5311 |
| 6-9       | 115.849  | 149.802  | 50.6209  | 50.6209 |
| 7-9       | 24.3641  | 30.7772  | 82.9888  | 82.9888 |
| 4-12      | 1220.73  | 161.094  | 15.7182  | 15.7182 |
| 5-12      | 481.859  | 467.078  | 4.7030   | 4.7030 |
| 6-12      | 137.744  | 11.6294  | 2.8005   | 2.8005 |
| 7-12      | 36.8014  | 14.6037  | 0.0145   | 0.0145 |
| 8-12      | 10.9884  | 6.3758   | 0.2685   | 0.2685 |
| 10-12     | 0.3363   | 1.7424   | 10.61%   | 10.61% |

### TABLE VIII

The returned values of $\chi^2$ for fit to the free string (LO) Eq. (38) and self-interacting NG string (NLO) Eq. (41) at each selected transverse planes $z_i$ at $T/T_c = 0.9$ with the last column is the retrieved $\chi^2$ at the middle of the string $z = R/2$. 

| Fit Range | $T/\beta$ | $n = R/a$ | $z = 1$ | $z = 2$ | $\chi^2$ | $z = 3$ | $z = 4$ | $z = R/2$ |
|-----------|-----------|-----------|--------|--------|----------|--------|--------|-----------|
| 5-9       | 5-9       | 445.232   | 425.601| –      | –        | –      | –      | 79.5311  |
| 6-9       | 6-9       | 115.849   | 149.802| 91.0467| –        | –      | –      | 50.6209  |
| 7-9       | 7-9       | 24.3641   | 30.7772| 26.5945| 11.6436  | –      | –      | 14.2874  |
| 4-12      | 4-12      | 1220.73   | 620.346| 1018.86| –        | –      | –      | 84.183   |
| 5-12      | 5-12      | 481.859   | 467.078| –      | –        | –      | –      | 82.9888  |
| 6-12      | 6-12      | 137.744   | 178.265| 161.094| 39.245   | –      | –      | 53.3045  |
| 7-12      | 7-12      | 36.8014   | 48.0687| 78.9119| 38.7756  | –      | –      | 15.7182  |
| 8-12      | 8-12      | 10.9884   | 14.6037| 39.1034| 22.4106  | –      | –      | 1.9209   |
| 10-12     | 10-12     | 0.3363    | 1.7424 | 4.7030 | 0.0071   | –      | –      | 0.0145   |
| 5-9       | 5-9       | 199.681   | 374.824| –      | –        | –      | –      | 79.6823  |
| 6-9       | 6-9       | 37.0773   | 86.6316| 56.1896| –        | –      | –      | 28.6494  |
| 7-9       | 7-9       | 5.4318    | 11.6294| 12.0872| –        | –      | –      | 5.4264   |
| 4-12      | 4-12      | 692.334   | 1025.6 | 615.994| –        | –      | –      | 326.3    |
| 5-12      | 5-12      | 211.424   | 397.556| –      | –        | –      | –      | 80.8514  |
| 6-12      | 6-12      | 42.9046   | 99.4396| 92.2507| –        | –      | –      | 29.2459  |
| 7-12      | 7-12      | 8.3665    | 18.4657| 37.3376| 15.8853  | –      | –      | 5.5947   |
| 8-12      | 8-12      | 2.4583    | 5.1658 | 18.1117| 10.064  | –      | –      | 0.2685   |
| 10-12     | 10-12     | 0.1072    | 0.8282 | 3.5422 | 2.8005   | –      | –      | 0.0145   |
FIG. 11: The mean-square width $W^2(z)$ of the string versus $Q\bar{Q}$ separations $R$ at temperature $T/T_c \approx 0.9$ in lattice unit. Measurements are taken at consecutive planes $z = 1$, $z = 2$, $z = 3$ and $z = 4$ from the top to the bottom. The solid and dashed lines correspond to the one parameter fit to the string model, Eq. $(38)$ and $(41)$, respectively.

The mean-square width of the string on the lattice is

$$W^2(z) = \frac{\int dr r^2 G(r; \theta; z)}{\int dr G(r; \theta; z)},$$

which defines the mean square width of the tube on the lattice. The locus of the color sources corresponds to $z = 0$ or $z = R$, respectively.

In Table VII the numerical values of the mean-square width of the string at the middle plane between the two color sources are in-listed. The percentage differences in the measured width measured with the use of both ansatz

FIG. 12: The changes in the width from the middle plane $z = R/2$ at temperature $T/T_c \approx 0.9$. The co-ordinates $z$ are lattice coordinates (lattice units) and are measured from the quark position $z = 0$. The solid and dash lines represent the free-string model (LO) and self-interacting (NLO) string Eqs. $(41)$ and $(38)$, respectively.
in Table VII indicate an almost constant shift amounting approximately to 22% of that measured using unconstrained optimization \( \sigma_1 \neq \sigma_2 \).

Further measurements of the mean-square width at consecutive transverse planes \( z = 1 \) to \( z = 4 \) are enlisted in Table XI of Appendix A. The width is estimated in accord to Eqs. (51) and (53) at each selected plane \( z_i \) fixed with respect to one color source. We found the unconstrained optimization Eq. (51) is returning \( \sigma_1 \neq \sigma_2 \) at all color separation distances.

The numerical values in Table XI are indicating a broadening in mean-square width of the string at all transverse planes \( z_i \) as the color sources are pulled apart. The plot of the width at consecutive planes in Fig. 11 more clearly depicts an increasing slope in the pattern of growth as one considers farther planes from the quark sources up to a maximum slope in the middle plane.

### B. Width Profile and Nambu-Goto String

The broadening of the width at each selected transverse plane can be compared to that of the corresponding width of the quantum string Eqs. (38) and (41). The discussion in the previous section for the fit analysis of the two Polyakov loop correlator Eqs. (32) enlightens that both the LO and NLO approximations would return substantial differences at the closer to the deconfinement point at the temperature \( T/T_c \approx 0.9 \).

In Table VIII summarized are the resultant values of the fit considering various range of sources separations. For this fit the string tension is fixed to its value returned at \( T/T_c \approx 0.8 \) form fits of \( Q\bar{Q} \) data.

The leading order approximation would show a strong dependency on the fit range if the data points at small source separations are considered. The first three entries in Table VIII compares the value of \( \chi^2 \) for both approximations at source separations \( R = 0.5 \text{ fm up to } R = 0.9 \text{ fm} \), that is, excluding the last three points. The free string picture poorly describes the lattice data at short distances.

With the data points at short distances excluded from the fit the values of \( \chi^2 \) decrease gradually. For example, first four points excluded from the fit, the returned \( \chi^2 \) is smaller, indicating that only the data points at large source separation are parameterized by the string model formula. With the consideration of the next leading order solution of the NG action the values of \( \chi^2 \) are reduced. Nevertheless, the values of \( \chi^2 \) are still significantly too large to precisely match the numerical data in intermediate distances.

The highest values of \( \chi^2 \) are retrieved if the whole source separations \( R = 4a \) to \( R = 12a \) are included for both LO and NLO approximations. The fits in Table VIII divulge a strong dependency on the fit range if the points at small source separations are excluded. Indeed, a gradual decrease in the values of \( \chi^2 \) with the consideration of larger source separations manifests. This is relatively in favor of the string description in the two loop approximation.

The free string (LO) Eqs. (38) and self-interacting Eq. (41) (NLO) solutions are one parameter fit functions in the ultraviolet cutoff \( R(\xi) \). While in the LO formula the ultraviolet cutoff has the effect of a constant shift, this is not the case when considering the NLO formula.

In Figs. 10 and 11 are plots for the mean square width at the middle plane of the tube, \( z = R/2 \) together with the corresponding fits to the free string Eqs. (38) and the self-interacting NLO form Eq. (41). The fit range for the free string Eqs. (38) is chosen for color source separations extending from \( R = 1.0 \text{ fm} \) to \( R = 1.2 \text{ fm} \) however, for the NLO self interacting Eq. (41) string, the fit range includes two additional points \( R = 0.8 - 1.2 \text{ fm} \). The fit regions in Figs. 10 and 11 are chosen so that both approximations give almost the same behavior in the asymptotic region at large color source separations \( R \geq 10a \). Thus, in order to approach the NLO approximation in the asymptotic region fits to the leading order approximation Eqs. (38) should be considered on large distances.

The string fluctuations have an almost constant cross-section at the intermediate distances \( 0.8 < R < 1.1 \text{ fm} \) which is not what is expected from the free string approximation Eqs. (38) [35, 39, 40] at this distance scale. The analysis of the lattice data has revealed curvatures along the planes transverse to the quark-antiquark line at large distances [39, 40]. In the intermediate distances the profile along the transverse planes is geometrically more flat than the free-string picture would imply.

Re-render of the mean-square width of lattice data together with fits to Eqs. (38) and (41) discloses the geometrical effects of the inclusion of NLO order terms. The width \( W^2(\xi) a^{-2} \) at the middle plane \( z = R/2 \) is subtracted from that at the plane \( W^2(\xi_i) \), shown in Fig 12 for two typical \( Q\bar{Q} \) configurations at \( R = 0.8 \text{ fm} \) and \( R = 0.9 \text{ fm} \). The string profile when considering the NLO terms in the effective action Eq. (41) show improvements in the match with lattice data. The suppression of the tube curvature and the constant width property at the intermediate region can be conceived as geometrical features due to the higher loops in the string interactions.

Although the statistical fluctuations increase with the decrease of the temperature (see Fig. 19), the width estimates obtained through fitting action density to Eq. (51) can be stabilized with the use of the standard Gaussian form \( \sigma_1 = \sigma_2 \) in Eq. (51) at the temperature \( T/T_c = 0.8 \) instead.

Our expectations from the fit behavior of \( Q\bar{Q} \) potential at \( T/T_c = 0.8 \) to both the LO and NLO formulas that higher order effects are negligible at this temperature scale.

Most of the considerations concerning the validity of both approximations to the \( Q\bar{Q} \) potential at temperature \( T/T_c = 0.8 \) seem to hold for the string profile. Considering the same fit range, the solid and dashed lines corresponding to approximations Eqs. (38) and (41) in
\( \frac{T}{T_c} = 0.8 \) 

| Fit Range | \( n = R/a \) | \( z = 1 \) | \( z = 2 \) | \( \chi^2 \) | \( z = 4 \) | mid.plane |
|-----------|----------------|--------|--------|--------|--------|---------|
| Free String (LO) | 4-12 | 578.806 | 453430 | – | – | 880374 |
| | 5-12 | 60.5188 | 36402.6 | 266299 | – | 168150 |
| | 6-12 | 44.1409 | 31.0112 | 126.596 | – | 189.559 |
| | 7-12 | 19.8978 | 10.4049 | 4.0910 | 3.0459 | 2.7246 |
| | 8-12 | 10.7254 | 5.3255 | 2.2323 | 2.6146 | 2.7243 |
| | 10-12 | 2.5551 | 2.0697 | 0.7103 | 0.6643 | 0.6747 |
| 2 Loops (NLO) | 4-12 | 241.135 | 157140 | – | – | 69091.1 |
| | 5-12 | 93.9843 | 7415.33 | – | – | 9118.51 |
| | 6-12 | 37.3147 | 18.86 | 26.4692 | – | 53.3679 |
| | 7-12 | 12.6149 | 14.91 | 9.0537 | 2.8830 | 4.1864 |
| | 8-12 | 7.0765 | 4.6559 | 2.1293 | 2.5598 | 2.6349 |
| | 10-12 | 2.0170 | 1.8054 | 0.5687 | 0.5212 | 0.5480 |

TABLE IX: Inlisted are the returned value of \( \chi^2 \) corresponding to the fit to Nambu-Goto string in the leading-order (LO) Eq. (38) and the next to leading order (NLO) Eq. (41) formulation at each selected transverse planes \( z_i \), the last column corresponds to resultant fit at the middle plane of the string \( z = R/2 \).

FIG. 13: The broadening of the mean-square width of the density-distribution at the center of the tube \( z = R/2 \) versus the \( QQ \) separation distance \( R \) at temperature \( \frac{T}{T_c} = 0.8 \). The solid and dashed lines correspond to the free and self-interacting Nambu-Goto (NG) string (LO) Eq. (38) and (NLO) Eq. (41), respectively.

Fig. 13 coincide, with exception of subtlety at the end point \( R = 0.5 \) fm, in both the asymptotic region and intermediate distances regardless of the adopted fit range. This mismatch at 0.5 fm is less obvious when considering fits at other transverse planes than the middle as can be seen in Fig 15. This can be attributed to the high value of \( \chi^2_{dof} \) (only at \( R = 0.5 \) and \( R = 0.6 \) fm) when measuring the width through the standard Gaussian distribution, i.e., \( \sigma_1 = \sigma_2 \) in Eq. (51).

The increase of the uncertainties in the string’s width for color source separation \( R > 1.0 \) fm does not loose our argument that the string model in both approximation scheme provide a good description for the string profile at this temperature. Indeed, if one considers 40 sweeps of UV filtering [40, 104] on the gauge links before the evaluation of the correlator Eq. (50). Inspection of the plot Fig. 14 for fits to non-normalized width [40] at the middle plane would yield the same reasoning.

FIG. 14: Similar to Fig. 13 except that the width measurements has been taken on gluonic configurations after \( n_{sw} = 40 \) sweeps of UV filtering.
FIG. 15: The mean-square width of the string $W^2(z)$ at $T/T_c \approx 0.8$ versus $QQ$ separations measured in planes $z = 1, z = 2, z = 3$, and $z = 4$ from the top to bottom. The dashed and solid line denote the leading and next to leading order Nambu-Goto string model Eq. (16) and Eq. (18), respectively.

In Table IX the fit to the LO approximation unveil good values of $\chi^2$ for color source separation up to $R = 0.6$ fm, the next to leading order fits, however, improves with respect to fit range when including these source separations $R = 0.5$ fm and $R = 0.6$ fm, this manifests at the middle and other the consecutive planes $z$ as well.

At the temperature $T/T_c = 0.8$, a re-render of the mean-square width differences $\delta W^2 = |W^2(z_i) - W^2(R/2)|$ of lattice data and the corresponding fits to string model is displayed in Fig. 16. The subtracted width from the middle plan $z = R/2$ unveils an almost constant width along the transverse planes to the

FIG. 16: The density distribution is exhibiting a nonuniform pattern along the transverse planes, even though the tube’s width is constant at all source separations.
FIG. 17: The mean-square width of the string $W^2(z)$ versus $Q\overline{Q}$ separations measured in the middle plane $R/2$ at $T/T_c \approx 0.9$. The solid and dashed line denote are fits to Nambu-Goto and Stiff string Eqs. (41) and (49) on the interval $R \in [0.5, 1.2]$ fm, respectively.

| Fit Range \( n = R/a \) | \( z = 1 \) | \( z = 2 \) | \( z = 3 \) | \( z = 4 \) | \( z = R/2 \) |
|--------------------------|---------|---------|---------|---------|---------|
| \( 5-9 \)               | 199.681 | 374.824 | -       | -       | 79.6823 |
| \( 6-9 \)               | 4.57    | 7.79266 | -       | -       | -       |
| \( 7-12 \)              | 9.3854  | 1.112   | 1.92674 | -       | 6.10655 |
| \( 8-12 \)              | 3.61765 | 6.2292  | 0.89879 | 1.057   | 1.55448 |
| \( 10-12 \)             | 0.123321| 0.529496| 0.293811| 0.286096| 0.300564|

$T/T_c = 0.8$

| Fit Range \( n = R/a \) | \( z = 1 \) | \( z = 2 \) | \( z = 3 \) | \( z = 4 \) | \( z = R/2 \) |
|--------------------------|---------|---------|---------|---------|---------|
| \( 5-9 \)               | 71.53   | 10.8   | -       | -       | 0.92    |
| \( 5-12 \)              | 87.56   | 15.60  | 87.60   | -       | 3.26    |
| \( 6-12 \)              | 6.27    | 3.66   | 3.90    | -       | 3.17    |
| \( 7-12 \)              | 1.05    | 0.67   | 3.17    | 3.17    | 1.54    |

$T/T_c = 0.9$

TABLE X: The corresponding $\chi^2$ retrieved from the fit to Polyakov-Kleinert string Eq. (45) for temperatures $T/T_c = 0.8$ and $T/T_c = 0.9$ at each selected transverse planes $z_i$, the last column summarizes the fit results at the middle of the string $z = R/2$.

The expectations of an almost flat width geometry along the transverse planes is consistent with the analysis shown in Figs. 12, which indicates that the thermal effects strongly diminishes near the QCD plateau region [31]. In Figs. 16 the render of the action densities corresponding to the both temperatures $T/T_c = 0.8$ and $T/T_c = 0.9$ unveils an independent prolate-shaped action density for the color map.

These are two typical instances where the string’s width profile is exhibiting a constant width along the tube. The first is due to the diminish of thermal effects near the end of QCD plateau, the second manifests at the intermediate color source separations and the temperature close to the deconfinement point as a result of the role played by the string-self interactions. This is culminated in the squeeze/suppression along the transverse planes. Figs. 12 and 16 disclose the fact that the geometry of the density isolines are quite independent from both the width profile.

The values of $\chi^2$ in Tables IX and VIII from the fits to the LO and the NLO approximations to NG string compare favorably to the temperature near the end of QCD plateau indicating that the NG string would successfully model QCD strings in the thermal regime up to temperatures as high as $T/T_c = 0.8$.

In spite of the improvements in the parameterization behavior at high temperatures $T/T_c = 0.9$ in the intermediate distance region, the yet large values of $\chi^2$ raises the question whether it is sufficient to consider the NG action
V. SUMMARY AND CONCLUSION

In this work we discussed the effective bosonic string model of confinement in the vicinity of critical phase transition point [105]. The corrections received from the Nambu-Goto (NG) action expanded up next to leading order terms have been set into comparison with the corresponding $SU(3)$ Yang-Mills lattice data in four dimensions. The effects of boundary terms in Liu-Scher-Weisz (LW) action and extrinsic curvature in Polyakov-Kleinert (PK) action have been also considered. The region under

C. Polyakov-Kleinert String and the Width Profile

Similar to the Nambu-Goto string, the broadening of the width at each selected transverse plane can be set into comparison with that of the corresponding width of the quantum stiff strings Eqs. (45) and (49).

Our expectations based on the fit analysis of the $Q\bar{Q}$ potential data of the ordinary Nambu-Goto string Eq. (5) and Polyakov-Kleinert action Eq. (9) would unveil substantial improvements in the fit behavior at the temperature very close to the deconfinement point $T/T_c \simeq 0.9$. This improvements manifests also in the case of compact $U(1)$ flux-tube [16].

The resultant fits to the smooth string consisting Eq. (45) of the stiffness terms Eqs. (49) in addition to the next to leading order solution of the NG action Eq. (41) are enlisted in Table X.

The values of $\chi^2$ in the second panel of Table X are indicating significant reduction in the values of $\chi^2$ at the temperature $T/T_c = 0.9$ compared to returned residuals (Table VIII) considering only the NG string Eq. (41). Moreover, the fit to stiff string Eq. (45) are returning good values of $\chi^2$ on the whole the intermediate source separation distances at all transverse planes along the tube.

The solid and dashed lines in the plot of Figs. 17 and 18 corresponding to the NG string in the interaction approximations Eqs. (41) and stiff strings Eqs. (45) and (49) shows the dramatic improvement in the fits with respect to the stiff strings when considering fit range covering the whole fit range $R \in [0.5 - 1.2]$ fm.

At temperature $T/T_c = 0.8$, the fit results summarized in Tables IX and of the LO and NLO versions of NG string return very close parameterization behavior in both the asymptotic and intermediate distances regions regardless of the selected fit range. Indeed, higher order effects are almost suppressed at this temperature scale.

The fit to the NG approximation Eq. (41) returns good values of $\chi^2$ for the mean square width of the string in the middle plane. However, the fit to the stiff string Eq. (49) exhibits improvements with respect the planes near to the color sources, this could indicate the relevance of stiff effects near the quark sources. The resultant values of fits considering various range of sources separations $T/T_c = 0.8$ are summarized in Table X.

We conclude that the free NG string can be only a good approximation to QCD strings up to temperatures near the end of QCD plateau $T/T_c = 0.8$. However, at higher temperatures other effects, implied in LW and PK actions, such as self-interactions, interactions with boundaries and the rigidity come into play and must be taken into account to extrapolate to the correct description of QCD string.
scrutiny is the source separation $R = 0.5$ to $R = 1.2$ fm for two temperatures scales near the end of QCD plateau and just before the critical point.

The theoretical predictions laid down by both the LO and the NLO approximations of Nambu-Goto string show a good fit behavior for the data corresponding to the $QQ$ potential near the end of the QCD plateau region at $T/T_c = 0.8$. The fit returns almost the same parameterization behavior with negligible differences for the measured zero temperature string tension $\sigma_0 a^2$. The returned value of this fit parameter is in agreement with the measurements at zero temperature [94].

Considering a higher temperature near the deconfinement point $T/T_c = 0.9$. The fits of the $QQ$ potential data to the Nambu-Goto string model considering either of its approximation schemes return large values of $\chi^2$ if the fit region span the whole source separation distances $R = 0.5$ fm to $R = 1.2$ fm. Even though the fits still compare favorably to the next to leading order approximation of the NG string on each corresponding fit interval. The values of the residuals decrease by the exclusion of the data points at short distances for both approximations.

The effective description based only on Nambu-Goto model does not accurately describe the $QQ$ potential data which occur as a deviation from the standard value of the string tension and the static potential data. The fit to the Casimir energy of the self-interacting string returns a value of the zero temperature string tension $\sigma_0 a^2 = 0.041$ which deviates by 11% of that measured at $T/T_c = 0.8$ and zero temperature. This motivated discussing other effects such as the interaction with the boundaries and stiffness of the flux-tube.

The inclusion of leading boundary term of L"uscher-Weisz action in the approximation scheme improves the fits at all the considered source separations, however, deviations from the value of the zero temperature string tension $\sigma_0 a^2$ do not diminish.

Near the deconfinement point, the fit of the static potential considering boundary terms of LW action and contributions from the extrinsic curvature of PK action show a significant improvement compared to that considering merely the ordinary Nambu-Goto string for the intermediate and asymptotic color source separation distances $R \in [0.5, 1.2]$. The fits reproduce an acceptable value of $\chi^2$ and a zero temperature string tension $\sigma_0 a^2$ measured at $T/T_c = 0.8$ or at $T = 0$ [94], thus, indicating a correct temperature dependence of the string tension.

Similarly, we consider the implication of the NG string for the mean-square width profile near the end of the QCD plateau region. We find that the mean-square width of the NG string in both LO and NLO approximations to fit well to the lattice data. Negligible differences are observed in the intermediate distances and the asymptotic long string limit. We conclude that the potential and mean square width extracted from the string partition function up to next leading order are consistent with the lattice data in the intermediate distances for temperature scales up to $T/T_c = 0.8$.

At higher temperature $T/T_c = 0.9$, the color tube exhibits a suppressed growth profile in the intermediate region. The fits considering both intermediate and asymptotic color source separation distances show noticeable improvement with respect to the string self-interacting picture (NLO) compared to that obtained on the basis of the free string approximation. Nevertheless, the next to leading approximation does not provide an accurate match to the numerical data. This manifests as significantly large values of the returned $\chi^2$ when considering distances less than $R < 0.8$ fm.

However, we found that the rigid string width profile accurately matches the width measured from the numerical lattice data near the deconfinement point. This suggests that the rigidity effects can be very relevant to the correct description of Monte-Carlo data of the field density and motivates scrutinizing the stiffness physics of QCD-flux-tube in other frameworks [106].

The oscillations of a free NG string fixed at the ends by Dirichlet boundaries traces out a nonuniform width profile with a geometrical curved fine structure. This is detectable [39] at source separations $R > 1.0$ fm and near to the critical temperature. However, in the intermediate region the lattice data are not in consistency with the curved width of the free fluctuating string. The fits to mean-square width extracted from the NLO expansion of NG string, however, indicate that self interactions flatten the width profile in the intermediate region. The string’s self-interactions accounts for the constant width along consecutive transverse plane to the tube in addition to the decrease in slop of the suppressed width broadening.

At the end of the QCD plateau region at temperature $T/T_c = 0.8$ the constant width property is manifesting at all source separation distances and is in consistency with the pure NG action. These results indicate not only the fade out of the thermal effects at this temperature but also indicate a form of the action density map independent from the geometrical changes induced by the temperature. That is, the main features of the density map would persist at lower and zero temperature.

In this investigation, we found that the consideration of higher-order loops of the string interactions together with the geometrically smooth string configurations to successfully eliminate the deviations of the string model from the Yang-Mills lattice data at high temperature.

**Acknowledgments**

We thank Thomas Filk for useful comments. This work has been funded by the Chinese Academy of Sciences President’s International Fellowship Initiative grants No.2015PM062 and No.2016PM043, the Recruitment Program of Foreign Experts, NSFC grants (Nos. 11035006, 11175215, 11175220) and the Hundred Talent Program of the Chinese Academy of Sciences (Y101020BR0).
FIG. 19: The action density profile of quark-antiquark $Q\bar{Q}$ separation distances $R = 5, 7, 9, 11$ at the center of the tube, $z = R/2$. The pad below show uncertainty distributions of the corresponding action densities. Profile are shown for the depicted temperatures $T/T_c = 0.8$ and $T/T_c = 0.9$.

| Fit Range $Ra^{-1}$ | $Q\bar{Q}$ distance, $z = 1$ | Width of the action density $W^2(z)$ | $T/T_c = 0.9$ |
|---------------------|-------------------------------|-------------------------------------|----------------|
| 5                   | 14.119 ± 0.098                | 13.149 ± 0.083                     | 13.149 ± 0.083 |
| 6                   | 16.179 ± 0.153                | 15.223 ± 0.128                     | 14.942 ± 0.124 |
| 7                   | 17.96 ± 0.239                 | 17.360 ± 0.203                     | 17.272 ± 0.203 |
| 8                   | 19.383 ± 0.363                | 19.121 ± 0.316                     | 19.542 ± 0.336 |
| 9                   | 20.776 ± 0.541                | 21.368 ± 0.532                     | 22.077 ± 0.626 |
| 7−28                | 22.524 ± 0.805                | 23.236 ± 0.817                     | 23.920 ± 1.019 |
| 10                  | 24.111 ± 1.166                | 26.253 ± 1.303                     | 26.604 ± 1.618 |
| 11                  | 23.786 ± 1.495                | 29.249 ± 0.628                     | 30.914 ± 0.804 |
| 12                  | 21.450 ± 1.711                | 31.802 ± 0.029                     | 34.775 ± 0.073 |
| 13                  | 19.735 ± 2.013                | 32.512 ± 1.067                     | 38.333 ± 1.394 |
| 14                  | 17.298 ± 0.042                | 18.234 ± 0.817                     | 22.077 ± 0.626 |

TABLE XI: The mean square width of the action density $W^2(z)$ measured at the temperature $T/T_c = 0.9$. The width is estimated with $\sigma_1 \neq \sigma_2$ in Eq. (51) at five consecutive transverse planes $z_i$ to the $Q\bar{Q}$ line.

| Fit Range $Ra^{-1}$ | $Q\bar{Q}$ distance, $z = 1$ | Width of the action density $W^2(z)$ | $T/T_c = 0.8$ |
|---------------------|-------------------------------|-------------------------------------|----------------|
| 5                   | 7.3822 ± 0.042                | 7.2984 ± 0.038                     | 7.3822 ± 0.042 |
| 6                   | 8.2016 ± 0.068                | 8.1932 ± 0.062                     | 8.2032 ± 0.063 |
| 7                   | 8.9061 ± 0.113                | 9.0557 ± 0.105                     | 9.2114 ± 0.109 |
| 8                   | 9.5545 ± 0.191                | 9.8594 ± 0.178                     | 10.2916 ± 0.192 |
| 9                   | 10.398 ± 0.321                | 10.7443 ± 0.298                    | 11.4887 ± 0.332 |
| 7−28                | 11.8279 ± 0.553               | 12.4012 ± 0.681                    | 13.7030 ± 0.853 |
| 10                  | 14.084 ± 0.992                | 14.0718 ± 1.201                    | 15.0550 ± 1.447 |
| 11                  | 17.4755 ± 1.863               | 18.5431 ± 1.702                    | 17.0604 ± 2.460 |
| 12                  | 21.2485 ± 3.256               | 24.1686 ± 3.066                    | 26.6827 ± 3.341 |
| 13                  | 22.462 ± 4.675                | 26.9322 ± 4.510                    | 31.9809 ± 5.093 |
| 14                  | 50.1043 ± 16.622              | 50.1043 ± 16.622                   | 39.9434 ± 7.260 |

TABLE XII: Similar to Table XI, the width of the action density $W^2(x)$ measured at the temperature $T/T_c = 0.8$ with the use of the fit formula Eq. (51) setting $\sigma_1 \neq \sigma_2$. 
