Numerical Bianchi I solutions in semi-classical gravitation

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It is believed that soon after the Planck era, spacetime should have a semi-classical nature. In this context we consider quantum fields propagating in a classical gravitational field and study the backreaction of these fields, using the expected value of the energy-momentum tensor as source of the gravitational field. According to this theory, the escape from General Relativity theory is unavoidable. Two geometric counter-term are needed to regularize the divergences which come from the expected value. There is a parameter associated to each counter-term and in this work we found numerical solutions of this theory to particular initial conditions, for general Bianchi Type I spaces. We show that even though there are spurious solutions some of them can be considered physical. These physical solutions include de Sitter and Minkowski that are obtained asymptotically.

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I. INTRODUCTION

The semi-classical theory considers the backreaction of quantum fields in a classical geometric background. It began about forty years ago with the introduction of the quadratic terms in the Riemann tensor made by the renormalization of the energy-momentum tensor, and continued with a more complete work made by De Witt, and since then, its consequences and applications are still under research, see for example. Some previous work, studied the effect of particle creation in a Bianchi Type I spaces, and the renormalized coefficients of some of the higher derivative terms obtained by renormalization were set to 0.

Different from the usual Einstein-Hilbert action, the predicted gravitational action allows differential equations with fourth order derivatives, which is called the full theory, see also [6].

There are several problems connected to the full theory which prove that it is not consistent with our expectation of the present day physical world. Instability of flat space, Planck scale tidal forces, tachyonic propagation of gravitational particles, and violation of the positive energy theorem, see for example [7]. Some of these results are obtained through linearization.

In the very interesting articles [8], Simon and Parker-Simon, suggest that the theory should be seen in a perturbative fashion. Their method results in differential equations for the metric to only second order. The order reduction allows for the elimination of the nonphysical effects. For instance in absence of a classical energy momentum source, Einstein’s General Relativity is identically recovered in [8]. The above mentioned problems with the full theory are removed. The order reduction formalism should be the correct physical description in which the higher order terms are seen as a very small perturbative correction.

The full higher order theory was previously studied by Starobinsky, and more recently, also by Shapiro, Pelinson and others. Starobinsky idea was that the higher order terms could mimic a cosmological constant. In only the homogeneous and isotropic space time is studied.

The full theory with four time derivatives is addressed, which apparently was first investigated in Tomita’s article for general Bianchi I spaces. They found that the presence of anisotropy contributes to the formation of the singularity. Berkin’s work shows that a
quadratic Weyl theory is less stable than a quadratic Riemann scalar $R^2$. Barrow and Hervik found exact and analytic solutions for anisotropic quadratic gravity with $\Lambda \neq 0$ that do not approach a de Sitter space time. In that article Barrow and Hervik discuss Bianchi types $II$ and $VI_h$ and also a very interesting stability criterion concerning small anisotropies. There is also a recent article by Clifton and Barrow in which Kasner type solutions are addressed. H. J. Schmidt does a recent and very interesting review of higher order gravity theories in connection to cosmology [12].

In this present work, only the vacuum energy momentum classical source is considered in the higher derivative theory. It should be valid soon after the Planck era neglecting any particle creation that took place at that time. The full theory predicts explosions and formation of physical singularities depending in the parameters and initial conditions. From this point of view the full theory is certainly not a complete theory. In fact, the exact numerical solutions for Bianchi I spaces seem to reproduce all the problems mentioned above, this question was not investigated in this present work.

But for some values of the parameters and very particular initial conditions, physically consistent solutions are obtained. The isotropization of spacetime also occurs with zero cosmological constant. The scalar Riemann four curvature oscillates near a constant value with decreasing amplitude.

The following conventions and unit choice are taken $R_{abcd} = \Gamma_{bd,c}^{a} - \ldots$, $R_{ab} = R_{acb}^c$, $R = R_{a}^{a}$, metric signature $-+++$, Latin symbols run from $0 - 3$, Greek symbols run from $1 - 3$ and $G = \hbar = c = 1$.

II. FULL THEORY AND NUMERICAL SOLUTIONS

The Lagrange function is,

$$\mathcal{L} = \sqrt{-g} \left[ \Lambda + R + \alpha \left( R_{ab}^a R_{ab} - \frac{1}{3} R^2 \right) + \beta R^2 \right] + \mathcal{L}_c. \quad (1)$$

Metric variations in the above action results in

$$E_{ab} = G_{ab} + \left( \beta \frac{1}{3} \alpha \right) H^{(1)}_{ab} + \alpha H^{(2)}_{ab} - T_{ab} - \frac{1}{2} \Lambda g_{ab}, \quad (2)$$
where

\[ H_{ab}^{(1)} = \frac{1}{2} g_{ab} R^2 - 2 R R_{ab} + 2 R^a_{\, a} - 2 \Box g_{ab}, \]
\[ H_{ab}^{(2)} = \frac{1}{2} g_{ab} R_{mn} R^{mn} + R_{ab} - 2 R^c_{\, c} R_{cbna} - \Box R_{ab} - \frac{1}{2} \Box g_{ab}, \]
\[ G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R, \]

and \( T_{ab} \) is the energy momentum source, which comes from the classical part of the Lagrangian \( \mathcal{L}_c \). Only classical vacuum solutions \( T_{ab} = \mathcal{L}_c = 0 \) will be considered in this section since it seems the most natural condition soon after the Planck era.

The covariant divergence of the above tensors are identically zero due their variational definition. The following Bianchi Type I line element is considered

\[ ds^2 = -dt^2 + e^{2a_1(t)} dx^2 + e^{2a_2(t)} dy^2 + e^{2a_3(t)} dz^2, \]

which is a general spatially flat and anisotropic space, with proper time \( t \). With this line element all the tensors which enter the expressions are diagonal. The substitution of (6) in (2) with \( T_{ab} = 0 \), results for the spatial part of (2), in differential equations of the type

\[ \frac{d^4}{dt^4} a_1 = f_1 \left( \frac{d^3}{dt^3} a_i, \ddot{a}_i, \dot{a}_i \right), \]
\[ \frac{d^4}{dt^4} a_2 = f_2 \left( \frac{d^3}{dt^3} a_i, \ddot{a}_i, \dot{a}_i \right), \]
\[ \frac{d^4}{dt^4} a_3 = f_3 \left( \frac{d^3}{dt^3} a_i, \ddot{a}_i, \dot{a}_i \right), \]

where the functions \( f_i \) involve the derivatives of \( a_1, a_2, a_3 \), in a polynomial fashion. The very interesting article [13] shows that the theory which follows from (11) has a well posed initial value problem. In [13], the differential equations for the metric are written in a form suitable for the application of the theorem of Leray [14].

Instead of going through the general construction given in [13], in this particular case, the existence and uniqueness of the solutions of (7)-(9) are more simply understood in [15].

Besides the equations (7)-(9), we have the temporal part of (2). To understand the role of this equation we have first to study the covariant divergence of the equation (2),

\[ \nabla_a E^{ab} = \partial_a E^{ab} + \Gamma^a_{ac} E^{cb} + \Gamma^b_{ac} E^{ac} = 0, \]

but since we are using (7)-(9) to integrate the system numerically \( E^{ii} \equiv 0 \), then

\[ \partial_0 E^{00} + \Gamma^0_{a0} E^{00} + \Gamma^0_{00} E^{00} = 0. \]
Therefore if $E_{00} = 0$ initially, it will remain zero at any instant. Therefore the equation $E_{00}$ acts as a constraint on the initial conditions and we use it to test the accuracy of our results.

For a space-like vector $v^\alpha$ and a time-like vector $t^\alpha = (1, 0, 0, 0)$ tidal forces are given by the geodesic deviation equations

\[
t^\alpha \nabla_a v^\alpha = R^\alpha_{\mu\nu\beta} t^\mu t^n v^\beta
\]

\[
t^\alpha \nabla_a v^\alpha = R^\alpha_{00\beta} v^\beta
\]

\[
R^\alpha_{00\beta} = \delta_{\alpha\beta} (\ddot{a}_\alpha + \dot{a}_\alpha).
\]

The theory predicted by (11) is believed to be correct if the tidal forces are less than 1 in Plank units,

\[
|R^\alpha_{00\beta}| \leq 1 \quad \text{(no summation.)} \quad (11)
\]

When this condition is not satisfied, quantum effects could introduce further modifications into (1).

In order to integrate these equations we used an open source and well tested C library, the GSL GNU Scientific Library [16], we used several algorithms provided by this library like Embedded Runge-Kutta-Fehlberg (4, 5) method and Implicit Bulirsch-Stoer method of Bader and Deuflhard, we also used Maple to calculate the equations and a Perl script developed by us to generate the C source code from the Maple output, and finally we used a AMD Athlon(TM) XP 2600+ to integrate the equations.

A. Numerical Solutions

For particular initial conditions and values for the parameters $\alpha$, $\beta$, $\Lambda$ consistent numerical solutions for (2) are shown in FIG. 1 and 2. In FIG. 1 $\alpha = 0.1$, $\beta = -0.1$, $\Lambda = 0$ and in FIG. 2 $\alpha = 1000$, $\beta = -1000$, $\Lambda = 0$

The only non null initial conditions are chosen

| $\dot{a}_1(0)$ | $\dot{a}_2(0)$ | $\dot{a}_3(0)$ | $d^3a_1/dt^3$ |
|----------------|----------------|----------------|----------------|
| $1 \times 10^{-1}$ | $2 \times 10^{-1}$ | $7 \times 10^{-1}$ | $-4.3248 \times 10^{-1}$ |

for FIG. 1 and

| $\dot{a}_1(0)$ | $\dot{a}_2(0)$ | $\dot{a}_3(0)$ | $d^3a_1/dt^3$ |
|----------------|----------------|----------------|----------------|
| $1 \times 10^{-1}$ | $2 \times 10^{-1}$ | $7 \times 10^{-1}$ | $1.1076 \times 10^{-1}$ |
FIG. 1: Using $\alpha = 0.1$, $\beta = -0.1$, $\Lambda = 0$

FIG. 2: Using $\alpha = 1000$, $\beta = -1000$, $\Lambda = 0$
FIG. 3: Using $\alpha = 0.1$, $\beta = 0.1$, $\Lambda = 0$

FIG. 4: Using $\alpha = -0.1$, $\beta = 0.1$, $\Lambda = 0$
FIG. 5: Using $\alpha = 0.1$, $\beta = 0.1$, $\Lambda = 0$

for FIG. 2 such that the 00 component of (2) vanishes initially. According to (10), it is numerically checked that $|E_{00}| < 10^{-16}$ for the time interval in FIG. 1 and $|E_{00}| < 10^{-12}$ for FIG. 2.

The condition given in (11) is satisfied initially, and during the time evolution in the solutions plotted in FIG. 1-2. Certainly these are physically accepted solutions since FIG. 1 gives Minkowski space and FIG. 2 gives de Sitter space asymptotically.

B. Explosions

In FIG. 4 the values of the parameters are different

$$\alpha = -0.1, \beta = 0.1, \Lambda = 0.$$ 

And again the only non null initial conditions are

| $\dot{a}_1(0)$ | $\dot{a}_2(0)$ | $\dot{a}_3(0)$ | $d^3a_1/dt^3$ |
|---------------|---------------|---------------|----------------|
| $1 \times 10^{-1}$ | $2 \times 10^{-1}$ | $7 \times 10^{-1}$ | $6.5412 \times 10^{-1}$ |

are such that the 00 component of (2) vanishes. According to (10), it is numerically checked that $|E_{00}| < 10^{-14}$ for the time interval in FIG. 4.
It can be seen that although the initial conditions satisfy (11), \(|R^a_{0a0}|\) (no summation), assumes arbitrary large increasing positive values, which indicates the existence of explosions. In FIG. 4 it is shown that the curvature scalar increases to arbitrary large positive values.

C. Formation of singularities

The following values for \(\alpha = 0.1, \beta = 0.1, \Lambda = 0,\)

are the same for FIG. 3 and FIG. 5. In FIG. 3 the non null initial conditions are

\[
\begin{array}{cccc}
\dot{a}_1(0) & \dot{a}_2(0) & \dot{a}_3(0) & d^3a_1/dt^3 \\
1 \times 10^{-1} & 2 \times 10^{-1} & 7 \times 10^{-1} & 7.3955 \times 10^{-1}
\end{array}
\]

and for FIG. 5

\[
\begin{array}{cccc}
\dot{a}_1(0) & \dot{a}_2(0) & \dot{a}_3(0) & d^3a_1/dt^3 \\
1 \times 10^{-1} & 2 \times 10^{-1} & -2 \times 10^{-1} & -8.1350 \times 10^{-1}
\end{array}
\]

such that the 00 component of (2) vanishes. According to (10), it is numerically checked that \(|E_{00}| < 10^{-10}\) for the time interval in FIG. 3 and FIG. 5. Since both the scalar curvature tensor \(R\) and the squared Ricci tensor \(R_{ab}R^{ab}\), increase abruptly, the solutions shown in FIG. 3 and FIG. 5 are understood as singular type. The numerical error increases very fast when the solution comes closer to the singularity, which is expected.

III. CONCLUSIONS

In the present work it is considered general anisotropic Bianchi I homogeneous spacetimes. The full theory with four time derivatives problem are addressed.

Only the vacuum energy momentum classical source is considered in the full theory. It should be valid soon after the Planck era and vacuum classical source seems the most natural condition. The full theory predicts explosions and formation of physical singularities depending in the parameters and initial conditions. From this point of view the full theory is certainly not a complete theory. In fact, the exact numerical solutions for Bianchi I spaces seem to reproduce all the problems mentioned above, this question was not investigated in this present work.
However, for some values of the parameters and very particular initial conditions, physically consistent solutions are obtained. The isotropization of space time also occurs with zero cosmological constant. The scalar Riemann four curvature oscillates near constant values with decreasing amplitude.

The above is understood as follows. The formation of singularities and Plank type explosions indicate that the theory certainly is not complete. The existence of physical consistent solutions show that the theory could have a space-time region of validity.

Although we did not attempt a detailed verification, the numerical solutions obtained in this present work show no contradiction to the interesting previous calculations concerning anisotropies[11]. The analytical solutions found by Barrow-Hervik and Clifton-Barrow can be understood as limit sets in the space of solutions of the quadratic theory[11].

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