SEMI-PARALLEL SYMMETRIC OPERATORS FOR HOPF
HYPERSURFACES IN COMPLEX TWO-PLANE
GRASSMANNIANS

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Abstract. In this paper, we introduce new notions of semi-parallel shape
operators and structure Jacobi operators in complex two-plane Grassmanni-
ans \( G_2(\mathbb{C}^{m+2}) \). By using such a semi-parallel condition, we give a complete
classification of Hopf hypersurfaces in \( G_2(\mathbb{C}^{m+2}) \).

Introduction

The classification of real hypersurfaces in Hermitian symmetric space is one of
interesting parts in the field of differential geometry. Among them, we introduce
a complex two-plane Grassmannian \( G_2(\mathbb{C}^{m+2}) \) defined by the set of all complex
two-dimensional linear subspaces in \( \mathbb{C}^{m+2} \). It is a kind of Hermitian symmetric
space of compact irreducible type with rank 2. Remarkably, the man ifolds are
equipped with both a Kähler structure \( J \) and a quaternionic Kähler structure \( \mathfrak{J} \)
satisfying \( JJ_\nu = J_\nu J \) (\( \nu = 1, 2, 3 \)) where \( J_\nu \) is an orthonormal basis of \( \mathfrak{J} \). When
\( m = 1 \), \( G_2(\mathbb{C}^3) \) is isometric to the two-dimensional complex projective space \( \mathbb{C}P^2 \)
with constant holomorphic sectional curvature eight. When \( m = 2 \), we note that
the isomorphism \( \text{Spin}(6) \cong SU(4) \) yields an isometry between \( G_2(\mathbb{C}^4) \) and the real
Grassmann manifold \( G_2^+(\mathbb{R}^6) \) of oriented two-dimensional linear subspaces in \( \mathbb{R}^6 \).
In this paper, we assume \( m \geq 3 \). (see Berndt and Suh \[2\] and \[3\]).

Let \( M \) be a real hypersurface in \( G_2(\mathbb{C}^{m+2}) \) and \( N \) a local unit normal vector field
of \( M \). Since \( G_2(\mathbb{C}^{m+2}) \) has the Kähler structure \( J \), we may define a Reeb vector
field \( \xi \) defined by \( \xi = -JN \) and a 1-dimensional distribution \( \{\xi\} = \text{Span}\{\xi\} \). The
Reeb vector field \( \xi \) is said to be a Hopf if it is invariant under the shape operator
\( A \) of \( M \). The 1-dimensional foliation of \( M \) by the integral curves of \( \xi \) is said to be a
Hopf foliation of \( M \). We say that \( M \) is a Hopf hypersurface if and if the Hopf
foliation of \( M \) is totally geodesic. By the formulas in \[10\] Section 2], it can be easily
checked that \( \xi \) is Hopf if and only if \( M \) is Hopf.

From the quaternionic Kähler structure \( \mathfrak{J} \) of \( G_2(\mathbb{C}^{m+2}) \), there naturally exists
almost contact 3-structure vector field \( \xi_1, \xi_2, \xi_3 \) defined by \( \xi_\nu = -J_\nu N, \ \nu = 1, 2, 3 \).
Put $Q^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$, which is a 3-dimensional distribution in a tangent vector space $T_xM$ of $M$ at $x \in M$. In addition, $Q$ stands for the orthogonal complement of $Q^\perp$ in $T_xM$. It becomes the quaternionic maximal subbundle of $T_xM$. Thus the tangent space of $M$ consists of the direct sum of $Q$ and $Q^\perp$ as follows: $T_xM = Q \oplus Q^\perp$.

For two distributions $[\xi]$ and $Q^\perp$ defined above, we may consider two natural invariant geometric properties under the shape operator $A$ of $M$, that is, $A[\xi] \subset [\xi]$ and $AQ^\perp \subset Q^\perp$. By using the result of Alekseevskii [11], Berndt and Suh [2] have classified all real hypersurfaces with two natural invariant properties in $G_2(C^{m+2})$ as follows:

**Theorem A.** Let $M$ be a connected real hypersurface in $G_2(C^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $Q^\perp$ are invariant under the shape operator of $M$ if and only if

(A) $M$ is an open part of a tube around a totally geodesic $G_2(C^{m+1})$ in $G_2(C^{m+2})$, or

(B) $m$ is even, say $m = 2n$, and $M$ is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(C^{m+2})$.

In the case (A), we call $M$ is a real hypersurface of Type (A) in $G_2(C^{m+2})$. Similarly in the case (B) we call $M$ one of Type (B). Using Theorem A, many geometerians have given some characterizations for Hopf hypersurfaces in $G_2(C^{m+2})$ with geometric quantities, for example, shape operator, normal (or structure) Jacobi operator, Ricci tensor, and so on. In particular, Lee and Suh [10] gave a characterization for real hypersurfaces of Type (B) as follows:

**Theorem B.** Let $M$ be a connected orientable Hopf hypersurface in $G_2(C^{m+2})$, $m \geq 3$. Then the Reeb vector field $\xi$ belongs to the distribution $Q$ if and only if $M$ is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(C^{m+2})$, $m = 2n$, where the distribution $Q$ denotes the orthogonal complement of $Q^\perp$ in $T_xM$, $x \in M$. In other words, $M$ is locally congruent to real hypersurfaces of Type (B).

On the other hand, regarding the parallelism of tensor field $T$ of type $(1,1)$, that is, $\nabla T = 0$, on $M$ in $G_2(C^{m+2})$, $m \geq 3$, there are many well-known results. Among them, when $T = A$ where $A$ denotes the shape operator of $M$, some geometerians have verified non-existence properties and some characterizations for the shape operator $A$ with many kinds of parallelisms, such as Levi-civita parallel, $\mathfrak{g}$-parallel, $Q^\perp$-parallel, Reeb parallel or generalized Tanaka-Webster parallel, and so on (see [5], [8], [11], [15], etc.).

Furthermore, many geometricians considered such a parallelism for another tensor field of type $(1,1)$ on $M$, namely, the Jacobi operator $R_X$ defined $(R_X(Y))(p) = (R(Y,X))X(p)$, where $R$ denotes a Riemannian curvature tensor of type $(1,3)$ on $M$ and $X, Y$ denote tangent vector fields on $M$. Clearly, each tangent vector field $X$ to $M$ provides the Jacobi operator $R_X$ with respect to $X$. When it comes to $X = \xi$, the Jacobi operator $R_\xi$ is said to be a structure Jacobi operator. Related to the tensor field $R_\xi$ of type $(1,1)$ on $M$, Pérez, Jeong, and Suh [6] considered the parallelism, that is, $\nabla_X R_\xi = 0$ for any $X \in TM$ and obtained a non-existence property.

In this paper we consider a generalized notion for parallelism of tensor field of type $(1,1)$ on $M$ in $G_2(C^{m+2})$, namely, semi-parallelism. Actually, in [3] a tensor
field \( F \) of type \((1, s)\) on a Riemannian manifold is said to be *semi parallel* if \( R \cdot F = 0 \). It means that the Riemannian curvature tensor \( R \) of \( M \) acts as a derivation on \( F \). From this, it is natural that if a tensor field \( T \) of type \((1,1)\) is parallel, then \( T \) is said to be a *semi-parallel*. Geometricians have proved various results concerning the semi-parallelism conditions of real hypersurfaces in complex space form (see [4], [11], [13]). Recently, K. Panagiotidou and M.M. Tripathi suggested the notion of *semi-parallel normal Jacobi operator* for a real hypersurface in \( G_2(\mathbb{C}^{m+2}) \) (see [12]).

Motivated by these works, we consider semi-parallels of the shape operator and the structure Jacobi operator for real hypersurfaces in \( G_2(\mathbb{C}^{m+2}) \), and assert the following theorems, respectively:

**Theorem 1.** Let \( M \) be a connected real hypersurface in complex two-plane Grassmannians \( G_2(\mathbb{C}^{m+2}), m \geq 3 \). There does not exist Hopf hypersurfaces \( M \) with semi-parallel shape operator if the smooth function \( \alpha = g(A\xi, \xi) \) is constant along the direction of \( \xi \).

**Theorem 2.** Let \( M \) be a connected real hypersurface in complex two-plane Grassmannians \( G_2(\mathbb{C}^{m+2}), m \geq 3 \). There does not exist Hopf hypersurfaces \( M \) with semi-parallel structure Jacobi operator if the smooth function \( \alpha = g(A\xi, \xi) \) is constant along the direction of \( \xi \).

In [12], K. Panagiotidou and M.M. Tripathi proved the following

**Theorem C.** There does not exist any connected Hopf hypersurface in complex two-plane Grassmannians \( G_2(\mathbb{C}^{m+2}), m \geq 3 \), with semi-parallel normal Jacobi operator if the smooth function \( \alpha = g(A\xi, \xi) \neq 0 \) and \( Q^- \) or \( Q^+ \)-component of \( \xi \) is invariant by the shape operator.

From this we consider that \( M \) has a vanishing geodesic Reeb flow when it comes to normal Jacobi operator. Hence by virtue of [9] Lemma 3.1, it gives us a extended result with respect to Theorem C as follows.

**Theorem 3.** Let \( M \) be a connected real hypersurface in complex two-plane Grassmannians \( G_2(\mathbb{C}^{m+2}), m \geq 3 \). There does not exist Hopf hypersurfaces \( M \) with normal Jacobi operator if the smooth function \( \alpha = g(A\xi, \xi) \) is constant along the direction of \( \xi \).

In this paper, we refer [1], [2], [3], [10] and [7], [14], [15] for Riemannian geometric structures of \( G_2(\mathbb{C}^{m+2}) \) and its geometric quantities, respectively.

### 1. Semi-parallel shape operator

In this section, let \( M \) represent a Hopf real hypersurface in \( G_2(\mathbb{C}^{m+2}), m \geq 3 \), and \( R \) denote the Riemannian curvature tensor of \( M \). Hereafter unless otherwise stated, we consider that \( X, Y, \) and \( Z \) are any tangent vector field on \( M \). Let \( W \) be any tangent vector field on \( Q \).

We first give the fundamental equation for the semi-parallelism of a tensor field \( T \) of type \((1,1)\) on \( M \) and prove our Theorem 1.
As mentioned in the introduction, a tensor field $T$ on $M$ is said to be semi-parallel, if $T$ satisfies $R \cdot T = 0$. It is equal to

\begin{equation}
(R(X, Y)T)Z = 0.
\end{equation}

Since $(R(X, Y)T)Z = R(X, Y)(TZ) - T(R(X, Y)Z)$, the equation (1) is equivalent to the following

\begin{equation}
R(X, Y)(TZ) = T(R(X, Y)Z).
\end{equation}

Using this discussion, let us prove our Theorem 1 given in Introduction. In order to do this, suppose that $M$ has the semi-parallel shape operator, that is, the shape operator $A$ of $M$ satisfies the condition $(R(X, Y)A)Z = 0$. From the relation between (1) and (2), we see that the given condition is equivalent to

\begin{equation}
R(X, Y)(AZ) = A(R(X, Y)Z).
\end{equation}

Therefore from (14) The equation of Gauss, it becomes

\begin{align*}
g(Y, AZ)X - g(X, AZ)Y + g(\phi Y, AZ)\phi X - g(\phi X, AZ)\phi Y \\
- 2g(\phi X, Y)\phi AZ + g(AY, AZ)AX - g(AX, AZ)AY + \\
\sum_{\nu} \left\{ g(\phi_{\nu} Y, AZ)\phi_{\nu} X - g(\phi_{\nu} X, AZ)\phi_{\nu} Y - 2g(\phi_{\nu} X, Y)\phi_{\nu} A\right\} \\
+ \sum_{\nu} \left\{ g(\phi_{\nu} Y, AZ)\phi_{\nu} X - g(\phi_{\nu} X, AZ)\phi_{\nu} Y \right\} \\
- \sum_{\nu} \left\{ \eta(Y)\eta_{\nu}(AZ)\phi_{\nu} X - \eta(X)\eta_{\nu}(AZ)\phi_{\nu} Y \right\} \\
- \sum_{\nu} \left\{ \eta(X)g(\phi_{\nu} Y, AZ) - \eta(Y)g(\phi_{\nu} X, AZ) \right\} \xi_{\nu}.
\end{align*}

(1.2)

\begin{align*}
= g(Y, Z)AX - g(X, Z)AY + g(\phi Y, Z)A\phi X - g(\phi X, Z)A\phi Y \\
- 2g(\phi X, Y)A\phi Z + g(AY, Z)A^{2}X - g(AX, Z)A^{2}Y + \\
\sum_{\nu} \left\{ g(\phi_{\nu} Y, Z)A\phi_{\nu} X - g(\phi_{\nu} X, Z)A\phi_{\nu} Y - 2g(\phi_{\nu} X, Y)A\phi_{\nu} Z \right\} \\
+ \sum_{\nu} \left\{ g(\phi_{\nu} Y, Z)A\phi_{\nu} X - g(\phi_{\nu} X, Z)A\phi_{\nu} Y \right\} \\
- \sum_{\nu} \left\{ \eta(Y)\eta_{\nu}(Z)A\phi_{\nu} X - \eta(X)\eta_{\nu}(Z)A\phi_{\nu} Y \right\} \\
- \sum_{\nu} \left\{ \eta(X)g(\phi_{\nu} Y, Z) - \eta(Y)g(\phi_{\nu} X, Z) \right\} A\xi_{\nu},
\end{align*}

where $\sum_{\nu}$ moves from $\nu = 1$ to $\nu = 3$.

Putting $Y = Z = \xi$ and using the condition of Hopf, the equation (1.2) can be reduced to

\begin{equation}
AX + \alpha A^{2}X
\end{equation}

\begin{align*}
- \sum_{\nu} \left\{ \eta_{\nu}(X) - \eta(X)\eta_{\nu}(\xi) \right\} A\xi_{\nu} + 3\eta_{\nu}(\phi X)A\phi_{\nu} X + \eta_{\nu}(\xi)A\phi_{\nu} X
\end{align*}

\begin{equation}
= \alpha X + \alpha^{2}AX
\end{equation}
\[ -\alpha \sum_{\nu} \left\{ \eta_\nu(X) - \eta(X)\eta_\nu(\xi) \right\} \xi_\nu + 3\eta_\nu(\phi X)\phi_\nu \xi + \eta_\nu(\xi)\phi_\nu \phi X. \]

Our first purpose is to show that \( \xi \) belongs to either \( Q \) or \( Q^\perp \).

**Lemma 1.1.** Let \( M \) be a Hopf hypersurface with semi-parallel shape operator in \( G_2(\mathbb{C}^{m+2}) \), \( m \geq 3 \). If the principal curvature \( \alpha = g(A\xi, \xi) \) is constant along the direction of Reeb vector field \( \xi \), then \( \xi \) belongs to either the distribution \( Q \) or the distribution \( Q^\perp \).

**Proof.** We consider that \( \xi \) satisfies
\[
(*) \quad \xi = \eta(X_0)X_0 + \eta(\xi)\xi_1
\]
for some unit vectors \( X_0 \in Q, \xi_1 \in Q^\perp \), and \( \eta(X_0)\eta(\xi_1) \neq 0 \).

By virtue of [7, Equation (2.10)] and the assumption of \( \xi \alpha = 0 \), we get \( AX_0 = \alpha X_0 \) and \( A\xi_1 = \alpha \xi_1 \).

In the case of \( \alpha \neq 0 \), using the equation in [2, Lemma 1],
\[
Y \alpha = (\xi \alpha)\eta(Y) - 4\sum_{\nu=1}^{3} \eta_\nu(\xi)\eta_\nu(\phi Y),
\]
we obtain that \( \xi \) belongs to either \( Q \) or \( Q^\perp \). We next consider the case \( \alpha \neq 0 \).

Substituting \( X = \phi X_0 \) in (1.3) and using basic formulas including (4), we get
\[
A\phi X_0 - 3\eta(X_0)\eta(\xi)A\phi_1 \xi + \eta(\xi)A\phi_1 X_0 - \eta(\xi)\eta(X_0)A\phi_1 \xi + \alpha A^2 \phi X_0
\]
\[
= \alpha \phi X_0 - 3\alpha \eta(X_0)\eta(\xi)\phi_1 \xi + \alpha \eta(\xi)\phi_1 X_0 - \alpha \eta(\xi)\eta(X_0)\phi_1 \xi + \alpha A^2 \phi X_0.
\]
From (4) and \( \phi \xi = 0 \), we obtain that \( \phi_1 \xi = \eta(X_0)\phi_1 X_0 \) and \( \phi X_0 = -\eta(\xi_1)\phi_1 X_0 \).

In addition, substituting \( X \) by \( X_0 \) into [7, Lemma 2.2] and applying \( AX_0 = \alpha X_0 \), we see that both vector fields \( \phi X_0 \) and \( \phi_1 X_0 \) are principal with same corresponding principal curvature \( k = \frac{\alpha^2 + 4\eta^2(X_0)}{\alpha} \).

From this, (1.5) gives
\[
-4k\eta^2(X_0)\phi X_0 + \alpha k^2 \phi X_0 - 4\alpha \eta^2(X_0)\phi X_0 - \alpha^2 k \phi X_0 = 0.
\]
Since \( \alpha \neq 0 \), multiplying \( \alpha \) to this equation, we obtain
\[
4k\eta^2(X_0)(8\eta^2(X_0) + \alpha^2)\phi X_0 = 0.
\]
By our assumptions, we get \( \eta(X_0)\eta(\xi_1) \neq 0 \) which means \( \phi X_0 = 0 \). This makes a contradiction. Accordingly, we get a complete proof of our Lemma. \( \square \)

From Lemma 1.1, we only have two cases, \( \xi \in Q \) or \( \xi \in Q^\perp \), under our assumptions. Next we further study the case \( \xi \in Q^\perp \).

**Lemma 1.2.** Let \( M \) be a Hopf hypersurface with semi-parallel shape operator in \( G_2(\mathbb{C}^{m+2}) \), \( m \geq 3 \). If the Reeb vector field \( \xi \) belongs to the distribution \( Q^\perp \), then \( M \) must be a \( Q^\perp \)-invariant hypersurface.

**Proof.** Since \( \xi \in Q^\perp \), we may put \( \xi = \xi_1 \in Q^\perp \) for the sake of our convenience. Differentiating \( \xi = \xi_1 \) along any direction \( X \in TM \) and using fundamental formulae in [10, Section 2], it gives us
\[
\phi AX = 2\eta_3(AX)\xi_2 - 2\eta_2(AX)\xi_3 + \phi_1 AX.
\]
Taking the inner product of (1.6) with \( W \in Q \) and taking symmetric part, we also have
\[
A\phi W = A\phi_1 W.
\]
Putting $X = \xi_2$ and $X = \xi_3$ into (1.3), we get, respectively,
\[
\begin{align*}
2A\xi_2 + \alpha A^2\xi_2 &= 2\alpha\xi_2 + \alpha^2 A\xi_2, \\
2A\xi_3 + \alpha A^2\xi_3 &= 2\alpha\xi_3 + \alpha^2 A\xi_3.
\end{align*}
\]

For $\alpha = 0$, clearly $Q^\perp$ is invariant under the shape operator, i.e., $AQ^\perp \subset Q^\perp$. Thus, let us consider $\alpha \neq 0$. Then the previous equations imply that
\[
\begin{align*}
A^2\xi_2 &= \frac{\alpha^2 - 2}{\alpha} A\xi_2 + 2\xi_2, \\
A^2\xi_3 &= \frac{\alpha^2 - 2}{\alpha} A\xi_3 + 2\xi_3.
\end{align*}
\]

Moreover, restricting $X = \xi_2$, $Y = \xi_3$ and putting $Z = W \in Q$, the equation (1.2) becomes
\[
4\eta_3(AW)\xi_2 - 4\eta_2(AW)\xi_3 + 2\phi AW - 2\phi_1 AW + \eta_3(A^2W)A\xi_2 - \eta_2(A^2W)A\xi_3 = 2A\phi W - 2A\phi_1 W + \eta_3(AW)A^2\xi_2 - \eta_2(AW)A^2\xi_3.
\]

Applying (1.6), (1.7) and (1.8) to this equation, it follows $\eta_3(AW)\xi_2 = \eta_2(AW)\xi_3$. This means $\eta_3(AW) = \eta_2(AW) = 0$ for any tangent $W \in Q$. It completes the proof.

From this lemma, we see that $M$ satisfying the assumptions in Lemma 1.2 is locally congruent to a model space of Type $(A)$ in $G_2(C^{m+2})$. Now, if we assume $\xi \in Q$, then $M$ with semi-parallel shape operator is locally congruent to one of Type $(B)$ by virtue of Theorem B.

Summing up these discussions, we conclude: let $M$ be a Hopf hypersurface in $G_2(C^{m+2})$, $m \geq 3$. If $M$ satisfies (1.1) and $\alpha = 0$, then $M$ must be a model space of Type $(A)$ or $(B)$.

Hereafter, let us check whether the shape operator of a model space of Type $(A)$ (or one of Type $(B)$) satisfies the semi-parallel condition (1.1) by [2] Proposition 3 (or [2] Proposition 2), respectively.

Let $M_A$ be a model space of Type $(A)$ in $G_2(C^{m+2})$. To show our purpose, we suppose that $M_A$ has the semi-parallel shape operator. From (1.3), (2) Proposition 3, and $\xi \in Q^\perp$, we have
\[
(\lambda - \alpha)(2 + \alpha\lambda)X = 0
\]
for any tangent vector $X \in T_\lambda = \{X \in T_\lambda M | X \perp \xi_\nu, \phi X = \phi_1 X, x \in M\}$. Since $\alpha = \sqrt{8}\cot\sqrt{8}r$ and $\lambda = -\sqrt{2}\tan\sqrt{2}r$ where $r \in (0, \pi/\sqrt{8})$, it implies that every $X \in T_\lambda$ is a zero vector. This gives rise to a contradiction. In fact, the dimension of the eigenspace $T_\lambda$ is $2m - 2$ where $m \geq 3$.

Now let us consider our problem for a model space of Type $(B)$ denoted by $M_B$. Similarly, we assume that the shape operator of $M_B$ is semi-parallel. By virtue of [2] Proposition 2, we see that $\xi$ of $M_B$ belongs to $Q$. Therefore we obtain $\alpha\beta(\alpha - \beta)\xi_1 = 0$, if we put $X$ as a unit vector field $\xi_1 \in T_\beta$ into (1.3). As we know $\alpha = -2\tan(2r), \beta = 2\cot(2r)$ where $r \in (0, \pi/4)$ on $M_B$, we get a contradiction. This completes the proof of our Theorem 1.

Therefore we assert:

**Remark 1.3.** The shape operator $A$ of a model space of Type $(A)$ nor Type $(B)$ in $G_2(C^{m+2})$ does not satisfy the semi-parallelism condition.
Summing up these discussions, we complete the proof of our Theorem 1 given in the introduction. \qed

2. Semi-parallel structure Jacobi operator

In this section, we give a complete prove our Theorem 2. Suppose the structure Jacobi operator of $M$ has semi-parallelism, that is, $M$ satisfies the condition $(R(X, Y)R_{\xi}Z) = 0$. Besides, from the relation between \ref{1} and \ref{2} we see that the given condition is equivalent to
\begin{equation}
R(X, Y)(R_{\xi}Z) = R_{\xi}(R(X, Y)Z).
\end{equation}
The structure Jacobi operator $R_{\xi}$ is defined by $R_{\xi}(X) = R(X, \xi\xi)$, where $R$ denotes the Riemannian curvature tensor on $M$. Then from the Gauss equation, it can be written as
\begin{equation}
R_{\xi}X = X - \eta(X)\xi + \eta(A\xi)AX - \eta(AX)A\xi
\end{equation}
where $\sum_{\nu}$ denotes from $\nu = 1$ to $\nu = 3$. From this, we see that $R_{\xi}\xi = 0$.

Put $Y = Z = \xi$ into (2.1), due to $R_{\xi}\xi = 0$, we get:
\begin{equation}
R_{\xi}(R_{\xi}X) = 0.
\end{equation}
Using these observation from now on we show that $\xi$ belongs to either $Q$ or its orthogonal complement $Q^\perp$ such that $TM = Q \oplus Q^\perp$.

**Lemma 2.1.** Let $M$ be a Hopf hypersurface in $G_2(C^{m+2})$, $m \geq 3$, with semi-parallel structure Jacobi operator. If the principal curvature $\alpha = g(A\xi, \xi)$ is constant along the direction of $\xi$, then $\xi$ belongs to either the distribution $Q$ or the distribution $Q^\perp$.

**Proof.** Put $\xi$ satisfies \ref{1} for some unit vectors $X_0 \in Q$ and $\xi_1 \in Q^\perp$.
Substituting $X = \xi_1$ in (2.2), we have $R_{\xi}(\xi_1) = \alpha^2\xi_1 - \alpha^2\eta(\xi_1)\xi$. This gives that
\begin{equation}
R_{\xi}(R_{\xi}\xi_1) = R_{\xi}(\alpha^2\xi_1 - \alpha^2\eta(\xi_1)\xi)
\end{equation}
\begin{equation}
= \alpha^2R_{\xi}\xi_1 - \alpha^2\eta(\xi_1)R_{\xi}\xi
\end{equation}
\begin{equation}
= \alpha^4\xi_1 - \alpha^4\eta(\xi_1)\xi.
\end{equation}
So, the condition of semi-parallel structure Jacobi operator implies
\begin{equation}
\alpha^4\xi_1 - \alpha^4\eta(\xi_1)\xi = 0.
\end{equation}
From this, taking the inner product with $X_0 \in Q$, it gives $\alpha^4\eta(\xi_1)\eta(X_0) = 0$. So we obtain the following three cases: $\alpha = 0$, $\eta(X_0) = 0$ or $\eta(\xi_1) = 0$. When $\alpha$ is identically vanishing, by virtue of \ref{1} we conclude that $\xi$ belongs to either $Q$ or $Q^\perp$. For $\eta(\xi_1) = 0$, then $\xi$ belongs to $Q$ because of our notation \ref{4}. Moreover, $\xi$ belongs to $Q^\perp$ if $\eta(X_0) = 0$. Accordingly, it completes the proof of our Lemma. \qed

According to Lemma 2.1 we consider the case $\xi \in Q^\perp$.

**Lemma 2.2.** Let $M$ be a Hopf hypersurface with semi-parallel structure Jacobi operator in $G_2(C^{m+2})$, $m \geq 3$. If the Reeb vector field $\xi$ belongs to the distribution $Q^\perp$, then $g(AQ, Q^\perp) = 0$. 


From this and (2.6), we obtain
\[ \eta \text{ induces that } \]
\[ (2.6) \]
respectively, it becomes
\[ \text{Again taking the inner product with } W \]
\[ (2.5) \]
\[ \alpha \eta \]
for any tangent vector field \( W \)
\[ (2.4) \]
function \( \text{vanishing.} \)

Proof. We may put \( \xi = \xi_1 \), because \( \xi \in Q^\perp \). Differentiating \( \xi = \xi_1 \) for any direction \( X \) on \( M \), we obtain
\[
\begin{align*}
q_2(X) &= 2 g(AX, \xi_2), \quad q_3(X) = 2 g(AX, \xi_3) \quad \text{and} \\
AX &= \eta(AX) \xi + 2 g(AX, \xi_2) \xi_2 + 2 g(AX, \xi_3) \xi_3 - \phi \phi_1 AX
\end{align*}
\]
(2.4)

(or \( AX = \eta(X) A \xi + 2 \eta_2(X) A \xi_2 + 2 \eta_3(X) A \xi_3 - A \phi \phi_1 X \)).

Putting \( X = \xi_2 \) into (2.2), it follows that \( R_\xi(\xi_2) = 2 \xi_2 + \alpha A \xi_2 \). If the smooth function \( \alpha \) vanishes, it makes a contradiction. In fact, from (2.2) we see that \( R_\xi(R_\xi \xi_2) = 4 \xi_2 = 0 \). Thus we may consider that the smooth function \( \alpha \) is non-vanishing.

On the other hand, it follows that for any \( W \in Q \) the equation (2.2) becomes
\[ R_\xi(W) = W + \phi_1 \phi W + \alpha AW, \]
from this, together with the semi-parallelism of \( R_\xi \), it follows that
\[
0 = R_\xi(R_\xi W)
= 2 \alpha AW + 2 \alpha \eta_3(\alpha W) \xi_3 + 2 \alpha \eta_2(\alpha W) \xi_2 - \alpha \phi_1 \phi AW
+ \alpha^2 A^2 W + \alpha A \phi_1 \phi W.
\]
From (2.4) and \( \alpha \neq 0 \), it follows that \( 2 AW + \alpha A^2 W = 0 \), where \( AW = -A \phi_1 \phi W \) for any tangent vector field \( W \in Q \). Taking the inner product with \( \xi_2 \) and \( \xi_3 \), respectively, it becomes
\[ \alpha \eta_2(A^2 W) = -2 \eta_2(AW), \quad \alpha \eta_3(A^2 W) = -2 \eta_3(AW). \]
Moreover, according to (2.2), we also have \( R_\xi(A \xi_2) = 2 A \xi_2 + \alpha A^2 \xi_2 \), which induces that
\[
0 = R_\xi(R_\xi \xi_2) = R_\xi(2 \xi_2 + \alpha A \xi_2)
= 2 R_\xi(\xi_2) + \alpha R_\xi(A \xi_2)
= 4 \xi_2 + 4 \alpha A \xi_2 + \alpha^2 A^2 \xi_2.
\]
Again taking the inner product with \( W \in Q \) and using the fact \( \alpha \neq 0 \), we have
\[
\alpha \eta_2(A^2 W) = -4 \eta_2(AW).
\]
From this and (2.6), we obtain \( \eta_2(AW) = 0 \) for any tangent vector field \( W \in Q \).

Similarly, from (2.2) we get \( R_\xi \xi_3 = 2 \xi_3 + \alpha A \xi_3 \) and \( R_\xi(A \xi_3) = 2 A \xi_3 + \alpha A^2 \xi_3 \), which gives
\[
0 = R_\xi(R_\xi \xi_3) = R_\xi(2 \xi_3 + \alpha A \xi_3)
= 4 \xi_3 + 4 \alpha A \xi_3 + \alpha^2 A^2 \xi_3.
\]
From this, taking the inner product with \( W \in Q \) and using \( \alpha \neq 0 \), we have \( 4 \eta_3(AX) + \alpha \eta_3(A^2 X) = 0 \). Combining this and (2.6), we get also \( \eta_3(AW) = 0 \) for any \( W \in Q \). Until now, we have proven if \( M \) satisfies our assumptions, then the distribution \( Q^\perp \) is invariant under the shape operator, that is, \( g(A Q, Q^\perp) = 0 \). This gives a complete proof of our lemma. \( \square \)

From this lemma and Theorem A given by Berndt and Suh [2], we see that a Hopf hypersurface \( M \) satisfying the assumptions in Lemma (2.2) is locally congruent to a model space of Type (A). Now, if \( \xi \) belongs to \( Q \), then by virtue of Theorem B a Hopf hypersurface \( M \) with semi-parallel structure Jacobi operator is locally congruent to a real hypersurface of Type (B) in \( G_2(C^{m+2}) \). Hence we conclude that let
Let $M$ be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$. If $M$ satisfies (2.1) and $\xi\alpha = 0$, then $M$ is a model space of Type (A) or (B).

From such a point of view, let us consider the converse problem. More precisely, we check whether the structure Jacobi operator $R_\xi$ of a model space of Type (A) (or of Type (B), resp.) satisfies the semi-parallel condition (2.1).

In order to check our problem for a model space $M_A$, we suppose that $M_A$ has the semi-parallel structure Jacobi operator. By virtue of Proposition 3 in [2], we see that $\xi = \xi_1 \in T_\alpha$ and $\xi_j \in T_\beta$ for $j = 2, 3$. From this, the semi-parallel condition for $R_\xi$ becomes

$$R_\xi(R_\xi \xi_2) = 4\xi_2 + 4\alpha\beta\xi_2 + \alpha^2\beta^2\xi_2$$

(1.3)

when we put $X = \xi_2$ in (2.3). It implies $(\alpha\beta + 2) = 0$. But since $\alpha = \sqrt{2}\cot(\sqrt{2}r)$ and $\beta = \sqrt{2}\cot(\sqrt{2}r)$, we obtain $(\alpha\beta + 2) = 2\cot^2(\sqrt{2}r) \neq 0$ for $r \in (0, \pi/2\sqrt{2})$. Thus it gives us a contradiction.

In the sequel, we check whether $R_\xi$ of a model space $M_B$ of Type (B) is semi-parallel. To do this, we assume that $R_\xi$ of $M_B$ satisfies the condition (2.1). On a tangent vector space $T_x M_B$ at any point $x \in M_B$, the Reeb vector $\xi$ belongs to $Q$. From this and (2.2), the condition of (2.1) implies that for $X = \xi_2 \in T_\beta$

$$R_\xi(R_\xi \xi_2) = \alpha^2\beta^2\xi_2 = 0.$$

On the other hand, from [2, Proposition 2], since $\alpha = -2\tan(2r)$ and $\beta = 2\cot(2r)$ where $r \in (0, \pi/4)$ on $M_B$, we get $(\alpha\beta)^2 = 16$. So, we consequently see that the tangent vector $\xi_2$ must be zero, which gives a contradiction.

Therefore we assert:

**Remark 2.3.** The structure Jacobi operator $R_\xi$ of a model space of Type (A) nor Type (B) in $G_2(\mathbb{C}^{m+2})$ does not satisfy the semi-parallelism condition.

Summing up these discussions, we complete the proof of our Theorem 2 given in the introduction.

\[ \square \]

3. **Semi-parallel normal Jacobi operator**

Now, we observe a Hopf hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with semi-parallel normal Jacobi operator, that is, the normal Jacobi operator $\bar{R}_N$ of $M$ satisfies

$$(R(X, Y)\bar{R}_N)Z = 0$$

for all tangent vector fields $X, Y, Z$ on $M$.

In order to prove Theorem 3 mentioned in Introduction, let us consider the case that $M$ has vanishing geodesic Reeb flow.

**Lemma 3.1.** Let $M$ be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with vanishing geodesic Reeb flow. If the normal Jacobi operator $\bar{R}_N$ of $M$ is semi-parallel, then $M$ is locally congruent to a model space of Type (A) or Type (B).

**Proof.** When the function $\alpha = g(A\xi, \xi)$ identically vanishes, it can be seen directly by (1.4) that $\xi$ can be divided into $\xi \in Q$ or $\xi \in Q^\perp$. Then we first consider the case
that $\xi$ belongs to $Q$. By virtue of Theorem B, we get that $M$ is locally congruent to a model space of Type (B).

Next, we consider the case $\xi \in Q^\perp$. Substitution of the previous two relations in [12, (4.17)] gives

$$7W + 7\alpha AW - 6\phi_1 \phi W = 2\alpha \eta_2 (AW)\xi_2 + 2\alpha \eta_3 (AW)\xi_3 + \phi_1 \phi (\phi_1 \phi W) - \alpha \phi_1 \phi AW.$$ (3.1)

Since $\alpha = 0$, it follows that $7W - 6\phi_1 \phi W = \phi_1 \phi (\phi_1 \phi W)$ for any $W \in Q$. Moreover, from $\phi \phi \nu X = \phi \nu (X) - \eta (X)\xi$, $\nu = 1, 2, 3$, we obtain $\phi \phi (\phi_1 \phi W) = W$. Thus (3.1) implies $\phi_1 \phi W = W$. It implies $AW = 0$ for any $W \in Q$, together with (2.4). It gives us a complete proof for $\alpha = 0$. □

It remains to be checked if the normal Jacobi operator $\bar{R}_N$ of a model space $M_A$ or $M_B$ satisfy the semi-parallelism condition. For $\xi \in Q^\perp$, we easily get $2\xi = 0$ from [12 Equations (5.2) and (5.3)]. For $\xi \in Q$, as we know $\alpha = -2 \tan(2r)$ with $r \in (0, \pi/4)$ on a real hypersurface of Type (B), $\alpha$ never vanishes (see [2, Proposition 2]). So, neither the normal Jacobi operator $\bar{R}_N$ of $M_A$ nor $M_B$ does not satisfy the semi-parallelism condition. Thus we get the following:

**Corollary 3.2.** Let $M$ be a real hypersurface in $G_2(C^{m+2})$, $m \geq 3$, with vanishing geodesic Reeb flow. Then there does not exist any Hopf hypersurface if the normal Jacobi operator $\bar{R}_N$ of $M$ satisfies the condition of semi-parallelism.

Combining Theorem C and Corollary 3.2, we give a complete proof of Theorem 3 in the introduction. □

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