Conformal Scale Geometry of Spacetime – A lower bound for a total mass

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We devise a new approach for the analysis of issues of geometric pathologies and black holes of a spacetime, based on a new mass function defined on an ideal un-physical spacetime which models time-flow or time dilation. The mass function is interpreted as an ”extra” local energy density that encodes the rate at which time comes to a ”stop” (hardly visible) or it measures how quickly the (illusory) Event horizon forms. This latter is defined on the manifold with corners resulting from an appropriate conformal compactification of the original physical space-time, the concrete choice of compactification being tied to the geometric structure of collapsing spacetimes. We define the (illusory) Event horizon as the set of zero-mass function and provide conditions for which it stands as a ”black hole’s event horizon”. As a first main result owing to the new definitions here, we establish the existence of a lower bound for the ”total mass”, provided some conditions on the extrinsic curvature of the space-time are satisfied. The proof builds on geometric flow techniques. Namely, by flowing the ”black hole’s event horizon”, one is able to derive via a Lagrangian formulation, a Minimization Problem for the ”total mass” and which is addressed under Iso-perimetric constraints’ perspective. Prior to this main result, we first provide hypotheses which assure the trapped surface’s formation in this context.

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I. INTRODUCTION – MAIN RESULTS

One fundamental avenue in understanding the Einstein’s theory of general relativity and its implications is the study of issues of singularities and black holes, which are fascinating objects predicted by this theory\cite{11, 17, 43, 53, 73, 77}. Indeed, several efforts have been made in the analysis of the formation of singularities and black holes, the study of their nature and their stability\cite{11, 18, 43, 27, 36, 51, 75, 77, 88}. Some remarkable results for stability of blackholes include:

- the proof of global nonlinear stability of Minkowski spacetime\cite{4, 18, 46, 57, 58, 86}, - the proof of global nonlinear stability of Schwarzschild spacetime under polarized perturbations\cite{54}, - the proof of global nonlinear stability of the Kerr-de Sitter family of blackholes\cite{47}. However, many facets of the theory of general relativity remain questionable. As instance, the question of stability of the Kerr black holes, or the full proof of stability conjecture still open\cite{55}. Importantly, some open questions related to issues of singularities and black holes are: - the definition of black hole as a feature of spacetime, - the question of existence or definition of horizons\cite{13, 25, 60, 71, 72, 85, 78, 97}, - their nature and their stability—the fate of the Cauchy horizons - the deterministic character of the theory of general relativity\cite{2, 3, 10, 16, 22, 37, 88}. These problems harbor two conjectures known as weak and strong cosmic censorship conjectures formulated initially by Roger Penrose\cite{72}. These conjectures have received many formulations, have been to date at the core of many other scientific works in the field of general relativity and still power many research perspectives including, the question of definition of a general appropriate quasi-local mass in general relativity\cite{20, 21, 91, 96, 98}, the positivity of the total mass or the question of establishment of a positive lower bound for the total energy or mass of isolated systems in general resumed as Penrose’s conjecture. The Penroses’s Conjecture asserts that the total mass of a spacetime including Black holes of area $A$ is at least $\sqrt{\frac{A}{16\pi}}$. It dates to 1973\cite{71}. This beautiful lower bound inequality has been the source of many interesting research proposals in recent years leading to some important results such as: the proof of the positive mass theorem, the proof of the Riemannian Penrose-Inequality, and more recently the proof of the Null Penrose Conjecture\cite{4, 5, 9, 49, 50, 59, 61, 64, 80, 82, 84, 79, 96}. There are other questions raised by quantum gravity which make the satisfaction not yet within the reach for scientists of general relativity, these include in particular the definition of a horizon and its existence, and the hypothetical phenomenon of ”Firewall”. These reasons justify somewhat the emergence of theories of modified gravity.
Efforts in the comprehension of the general theory of relativity and its implication, and progress obtained for the issues mentioned above rest basically on the followings:

- mastery of advanced tools of differential geometry and in particular the study of Riemannian and Lorentzian structures\textsuperscript{11, 17, 19, 43, 53, 77, 87},

- study of exact solutions of the Einstein equations (Minkowski, Schwarzschild, Kerr, Reissner-Norström,...) notably their stability\textsuperscript{4, 18, 46, 57, 58, 86, 27, 36, 51, 76, 77, 88, 47, 54},

- systematic construction of spacetimes by solving Cauchy problems and the analysis of properties of the obtained solutions (initial data constraints problem, evolution problem, existence theorems for differential equations, asymptotic properties,...)\textsuperscript{11, 15, 38, 43, 52, 53, 67, 68},

- analysis of waves on fixed backgrounds (decay properties, boundedness,...)\textsuperscript{2, 3, 10, 16, 22, 37, 39, 41, 45, 48, 50, 56, 62, 66, 88, 90},

- and even numerical relativity.

Concerning methods of studying asymptotic questions in general relativity, many authors agree in the conformal treatment of infinity as conceived by Penrose\textsuperscript{70, 74}, and this substantiates also our present analysis.

This research paper which is a revised version of the first part of my preprint\textsuperscript{69}, aims at contributing to some of the issues above. For this purpose, we adopt a mass function modeling the time-flow or the time dilation, or which helps to encode time-visibility at every point, defined on an appropriate conformal manifold (with corners) associated to the original physical spacetime. This unphysical conformal manifold is obtained by conformally embedding the spacetime under consideration in a new manifold with boundaries via a specific gauge or coordinates system. Given that the current mass function determines positions in the road of time, we localize the ”blackhole’s event horizon” candidate $\mathcal{H}$ as the hypersurface of zero-mass meaning that time is ”stopped” (hardly visible), that is the intrinsic geometry of the Event horizon is required to be time independent, whereas the geometry outside may be dynamical and may admit gravitational or other radiations. Using the wave fronts sets, the Event horizon can be scended in an open subset $\mathcal{H}^+$ designed as the ”actual event horizon” and a subset $\mathcal{H}^-$ which we refer to as the ”Apparent horizon”. This new approach postulates a possible cohabitation of the (possible stationary) causal future of $\mathcal{H}^+$ (i.e. ”the Black hole’s region”) and the causal future of $\mathcal{H}^-$ expected to be the White hole and likely to be instable. It appears also in this setting that any mass (concentration or blow-up) singularity is ”preceded” by an illusory event horizon (which may coincide with
the black hole’s event horizon) giving an approach to the formulation of the Weak Cosmic Censorship Conjecture. We signal that the exact relation between the mass inflation here and singularities as incompleteness of causal’s geodesics remains a point to examine. We also emphasize as one can remark that the standard definition of black hole based on the idea of a global event horizon is abandoned here in favor of a new one which relies on ”time hardly visible in an appropriate gauge at later times”.

The main results of this paper are twofold and consist in a proof of a theorem relative to the formation of trapped surfaces, and a proof of the existence of a lower bound for the ”total mass” provided some hypotheses are given for the isoperimetric constraint. These results are achieved according to the new mass function, the related quantities and sets, and by formulating (thanks to geometric flow techniques and via a Lagrangian formulation) the lower bound problem for the ”total mass” as a Minimization problem under a dynamical Isoperimetric constraint. We emphasize that in the solution process the dynamical integrand in the ”total mass” satisfies the Euler-Lagrange equation, this offers rich perspective in terms of the choice of the Isoperimetric constraint – it might involve specific: energies, area or volume, ...; also the flow techniques might involve the flow of the metrics and/or the flow of a particular surface depending on the particular or desirable needs or ends.

**Theorem I.1** Let \((V, g)\) be a global in time spacetime solution of the Einstein field equations where \(g\) is of the following form in coordinates \((x^\alpha)\):

\[
g = -\frac{|\overline{g}|}{\Omega^2}(dx^0)^2 + \overline{g}_{ij}dx^idx^j, \quad \overline{g} = (\overline{g}_{ij}),
\]

\(|\overline{g}|\) is the determinant of \(\overline{g}\), \(\Omega \equiv \Omega(x^i)\) is an arbitrary positive given scalar density satisfying \(\Omega \to 0\) as \(r = |(x^i)| \to +\infty\).

Let denote \(h = \Omega^2g\) the conformal metric, \(G\) its inverse, and \(\hat{\nabla}\) its associated riemannian connection. Let be specified the gauge \((\omega^\alpha)\) for the spacetime \((V, g)\), with \(\omega^0 = e^{-\Omega x^0}\), \(\omega^1 = \Omega\), \(\omega^a = \arctan(\Omega x^a)\).

One suppose that \(\mathcal{H} = \{m \equiv G\left(\frac{d\omega^0}{\Omega}, \frac{d\omega^0}{\Omega}\right) = 0\} \neq \emptyset\) and defines a (smooth) hypersurface, and the attached geometric objects (in \((\hat{V}, h)\)), \(U, V\) and quantities \(a, b^\pm\) as described in \((53)-(54)\) satisfy:

\[
h \left(\hat{\nabla}_{\hat{\nabla} \frac{\partial \omega^0}{\partial \omega^0}} \frac{\partial \omega^0}{\partial \omega^0}, \Omega \frac{\partial \omega^0}{\partial \omega^0}\right) < 0, \quad h \left(\hat{\nabla}_{\hat{\nabla} \frac{\partial \omega^0}{\partial \omega^0}} \frac{\partial \omega^0}{\Omega \omega^0}\right) > 0;
\]

\[
tr V < 0, \quad tr U \in \left[-\frac{tr V}{b^\pm}, -\frac{tr V}{b^\pm}\right].
\]
Then the spacetime \((V, g)\) undergoes trapped surfaces’s formation.

**Theorem I.2** One supposes that a gauge \((\omega^a)\) is specified as in theorem I.1 for a spacetime \((V, g)\), \(\mathcal{H} = \{ m \equiv G \left( \frac{d\omega^0}{\Omega^0}, \frac{d\omega^0}{\Omega^0} \right) = 0 \} \neq \emptyset\) and defines a (smooth) hypersurface, and the attached geometric objects (in \((\tilde{V}, h)\)) \(U, V\), and quantities \(a, b\) as described in (52)-(54) satisfy:

\[
\begin{align*}
    h \left( \hat{\nabla}_{\omega^0}, \hat{\nabla}_{\omega^0} \right) < 0, \\
    h \left( \hat{\nabla}_{\omega^0}, \hat{\nabla}_{\omega^0} \right) > 0;
\end{align*}
\]

(4)

\[
\begin{align*}
    \text{tr } V < 0, \quad \text{tr } U \in \left[ -\frac{\text{tr } V}{b^-}, -\frac{\text{tr } V}{b^-} \right].
\end{align*}
\]

(5)

One supposes further that for a given smooth function \(P_0\) of three real variables, the fundamental equation (82) (with \(P\) described in (68))

\[
\frac{\partial P_0}{\partial \tilde{T}} - \frac{d}{ds} \left( \frac{\partial P_0}{\partial \tilde{T}'} \right) + \left[ \frac{1}{\lambda} \frac{\partial P_0}{\partial \tilde{T}'} (\tilde{T}(+\epsilon), \tilde{T}'(+\epsilon)) + \frac{\partial P_0}{\partial \tilde{T}'} (\tilde{T}(+\epsilon), \tilde{T}'(+\epsilon)) \right] \delta_{+\epsilon} - \left[ \frac{1}{\lambda} \frac{\partial P_0}{\partial \tilde{T}'} (\tilde{T}(-\epsilon), \tilde{T}'(-\epsilon)) + \frac{\partial P_0}{\partial \tilde{T}'} (\tilde{T}(-\epsilon), \tilde{T}'(-\epsilon)) \right] \delta_{-\epsilon} = 0
\]

(6)

admits a solution \(\tilde{T}_*(., \lambda)\) with \(\lambda\) satisfying \(I(\tilde{T}_*) = K\).

Then the total mass \(\mathfrak{M}\) corresponding to the mass function \(m = G \left( \frac{d\omega^0}{\Omega^0}, \frac{d\omega^0}{\Omega^0} \right) \equiv h \left( \hat{\nabla}_{\omega^0}, \hat{\nabla}_{\omega^0} \right)\) (with \(G\) the inverse of \(h \equiv \Omega^2 g\)) admits a lower bound.

The different steps of the paper comprise:

- the preliminaries related to the Cauchy problem for the Einstein equations and the role of gauges, the choice of temporal gauge offers the means of exhibiting a conformal factor for a partial (radial) compactification of spacetime,
- the compactification of spacetime using appropriate conformal factor,
- the definitions of geometric quantities and sets, and a derived geometric analysis, one establishes some hypotheses relative to trapped surfaces’s formation,
- the description and resolution of a lower bound problem for the total mass as a minimization problem with an associated Isoperimetric constraint, using a foliation of the compactified spacetime by a deformation of the “Black hole’s event horizon”, this yields one fundamental equation with important perspectives depending on the choice of the isoperimetric constraint,
- the conclusion and outlook.
II. PRELIMINARIES

The Einstein’s theory of general relativity postulates a spacetime \((\mathcal{M}, g)\) which must satisfy the Einstein’s equations (in units where \(c = G = 1\))

\[
Ric(g) - \frac{1}{2} R_{\text{scal}} g = 8\pi T,
\]

where \(Ric(g)\) is the Ricci curvature tensor of \(g\), \(R_{\text{scal}}\) the corresponding scalar curvature, and \(T\) the stress energy-momentum tensor of matter.

A. Cauchy problem–Gauge

A systematic way to construct a spacetime without any symmetry assumption is by solving a Cauchy problem for the Einstein equations where initial data are prescribed on an appropriate initial hypersurface. The Einstein’s equations– geometric in nature, however read as a set of ill-posed complicated system of partial differential equations in arbitrary gauge or coordinates system for the unknown metric. It is usual to require that the coordinate system satisfy specific properties and this induces a choice of gauge. One known gauge is Harmonic gauge or wave gauge with its generalization the generalized wave map gauge used by P. T. Chruściel et al. (see and references in). Other known gauges are the Double null foliation gauge, and the Temporal gauge. The gauge has the role of splitting the Einstein’s equations to an evolution system and the set of equations known as the constraint’s equations. Two types of problems then arise which are the initial data constraints’s problem and the problem of evolution of initial data. The first consists of studying how to effectively prescribe the initial data satisfying the constraints’s equations on the initial manifold for the evolution system obtained according to the gauge, so that its solution yields the solution of the full system of Einstein’s equations. The initial manifold may be a spacelike hypersurface (spacelike Cauchy problem) or one or more null hypersurfaces (characteristic Cauchy problem). In the case of a spacelike Cauchy problem, the set of constraints are standard whereas in the case of characteristic hypersurfaces, they include standard constraints and other gauge-dependent constraints (see and references in). In the characteristic Cauchy problem setting, the constraints are a set of propagations equations along null geodesics generating the initial hypersurfaces, so that the main difficulties in the resolution of the initial data constraint’s problem in this setting reside rather in the
null geometric description and construction of such constraints (see \(21, 38, 67, 68\) and references in).

**B. Trouble with gauges**

The gauge has the merit of providing an evolution system of partial differential equations which can be solved by standard methods. However, there is a trouble in choosing the right one since the universe does not hand us with a preferred coordinates system which always works. There is in fact a possibility of drawing wrong conclusions about the properties of physical space because the coordinates system is inadequate. This problem is at the heart of issues of singularities and black holes (Measurement of Decay of waves, Stability,...), – new results of Klainermann–Szeftel\(^{54, 55}\); and Hintz–Vasy\(^{47}\), on the stability problem involve new techniques for finding the right coordinates system. Let’s recall that the equivalence principle in General Relativity (GR) is equivalent to an accelerating reference frame.

**C. Temporal gauge**

The gauge under consideration here is temporal gauge. If \((x^\alpha)\) is a system of coordinates of the spacetime \((V, g)\), the temporal gauge condition for this system reads

\[
\nabla^\lambda \nabla_\lambda x^0 = 0, \quad g_{0i} = 0;
\]

\(\nabla^\lambda \nabla_\lambda\) is the wave operator attached to \(g - \nabla\) is the covariant derivative operator attached to the metric \(g\); this induces a metric of the form

\[
g = -\tau^2 (dx^0)^2 + \mathcal{g}_{ij} dx^i dx^j, \quad \tau = c(x^i) \sqrt{|\mathcal{g}|},
\]

where the function \((x^i) \to c(x^i)\) is an arbitrary scalar density on the slices \(x^0 = t\), \(\mathcal{g} = (g_{ij})\) is the induced Riemannian metric on \(x^0 = t\), \(t \in \mathbb{R}^{11}\). The evolution system attached to the Einstein’s equations in temporal gauge reads\(^{11}\)

\[
\Omega_{ij} \equiv \partial_0 (R_{ij} - \Lambda_{ij}) - \nabla_i (R_{0j} - \Lambda_{0j}) - \nabla_j (R_{i0} - \Lambda_{i0}) = 0,
\]

where \(\Lambda_{\mu \nu} = T_{\mu \nu} + \frac{T^\Lambda}{1 + \sqrt{\mathcal{g}}} g_{\mu \nu}\). For \(c > 0\), and for fixed \(\mathcal{g}\), \(T\), the system \(\Omega_{ij} = 0\) is a quasidiagonal system of wave equations for the extrinsic curvature \(K = (K_{ij})\) of the hypersurface.
\( x^0 = t \). Together with the system

\[
\partial_0 \bar{g}_{ij} = -2\tau K_{ij},
\]

(11)

they form a hyperbolic Leray system for \( \bar{g} \) and \( K \) provided \( T \) is fixed, otherwise it is coupled with an appropriate system of equations describing the matter contents of the involved system. In reference\(^{38} \), by solving the characteristic initial data constraints problem related to such a system for Vlasov-scalar field matter, it appeared that the scalar density \( c \) is of specific form \( c(x^i) = \frac{1}{|\gamma|} \); is obtained actually by a null geometric description and resolution of the initial data constraints, and does not depend on the matter contents but only on a part of the free initial gravitational data. As our attention is focused here on the causal features of the expected global in time spacetime, this latter serves as a natural conformal factor for a partial (radial) ”bordification” of the spacetime; \( c \) is therefore the boundary defining function for infinity. Another reason in choosing the temporal gauge here corresponds to our intuition of mass function as measuring the effect of time which is a solution to the scalar wave equation in this setting. We mention also the fact that in temporal gauge the effect or variations of time \( t \) are visible by all observers in the spacetime \( (g(\nabla t, \nabla t) = g^{-1}(dt, dt) < 0) \), and our collapsing process includes the assignment of a coordinates system which assures: the compactification of the spacetime, that time effect ceases to be visible at later times, and time collapses to zero.

III. COMPACTIFICATION OF SPACETIME

We assume independently of the matter contents that a global in time spacetime \( (V, g) \) is established by solving the Einstein-field equations in temporal gauge, with \( g \) of the following form in coordinates \( (x^\alpha) \):

\[
g = -\frac{|\nabla|}{\Omega^2} (dx^0)^2 + \bar{g}_{ij} dx^i dx^j.
\]

(12)

\( \Omega \) is a \( x^0 \)-independent function,

\[
\Omega^2 g = -|\nabla|(dx^0)^2 + \Omega^2 \bar{g}_{ij} dx^i dx^j.
\]

(13)

We assume that \( \Omega > 0 \) is a bounded quantity tending to 0 as \( r := \sqrt{\sum_{i=1}^{n} (x^i)^2} \) tends to \(+\infty\), since it represents some initial scalar density in the spacetime. Furthermore, the gradient \( \nabla \Omega \) does not vanish and in particular we admit that \( \frac{\partial \Omega}{\partial x_i} \) does not vanish.
A. New gauge or observer in an accelerated reference frame

We proceed to a rescaling of the spacetime defining $\Phi$:

$$\hat{\Phi} : V \rightarrow \hat{V} = [0; 1] \times [0; \Omega_{\max}] \times (-\pi/2; +\pi/2)^{n-1}$$

$$x = (x^\alpha) \mapsto \omega := \left( e^{-\Omega x^0}, \Omega, \arctan(\Omega x^a) \right) := (\omega^\alpha). \tag{14}$$

One has the following expressions for $i, s = 1, \ldots, n; a, b = 2, \ldots, n$:

$$\frac{\partial \omega^0}{\partial x^0} = -\Omega \omega^0, \quad \frac{\partial \omega^s}{\partial x^0} = 0, \quad \frac{\partial \omega^0}{\partial x^i} = -x^0 \omega^0 \frac{\partial \Omega}{\partial x^i}, \quad \frac{\partial \omega^1}{\partial x^i} = \frac{\partial \Omega}{\partial x^i},$$

$$\frac{\partial \omega^a}{\partial x^1} = \frac{x^a \frac{\partial \Omega}{\partial x^1}}{1 + (\Omega x^a)^2}, \quad \frac{\partial \omega^a}{\partial x^b} = \frac{x^a \frac{\partial \Omega}{\partial x^b} + \Omega \delta^a_b}{1 + (\Omega x^a)^2}.$$

These expressions induce that the change of variables $\Phi : x^\alpha \rightarrow \omega^\delta$ observes the following jacobian:

$$J \equiv \left| \frac{D(\omega)}{D(x)} \right| = -\omega^0 \Omega^n \frac{\partial \omega^1}{\partial x^1} \prod_{a=2}^{n} \left( \frac{1}{1 + \tan^2 \omega^a} \right), \tag{15}$$

The inverse of the jacobian matrix has the following components:

$$\frac{\partial x^0}{\partial \omega^0} = -\Omega \omega^0, \quad \frac{\partial x^0}{\partial \omega^1} = -x^0 \omega^0 \frac{\partial \Omega}{\partial x^1}, \quad \frac{\partial x^0}{\partial \omega^a} = 0,$$

$$\frac{\partial x^1}{\partial \omega^0} = 0, \quad \frac{\partial x^1}{\partial \omega^1} = \Omega + \sum_{a=2}^{n} x^a \frac{\partial \omega^a}{\partial x^1}, \quad \frac{\partial x^1}{\partial \omega^a} = -x^a \omega^a \Omega \frac{\partial \Omega}{\partial x^1} + (1 + \tan^2 \omega^a) \Omega \frac{\partial \Omega}{\partial x^1},$$

$$\frac{\partial x^a}{\partial \omega^0} = 0, \quad \frac{\partial x^a}{\partial \omega^1} = -x^a \Omega, \quad \frac{\partial x^a}{\partial \omega^a} = 1 + \tan^2 \omega^a, \quad \frac{\partial x^a}{\partial \omega^b} = 0, \quad a \neq b.$$

Remark III.1 The manifold $\hat{V}$ has two specific boundary hypersurfaces $i^+ : \omega^0 = 0$ and $I : \omega^1 = 0$, correspondingly codimension 2 corners coexist.

B. b-Lorentzian metric – Scattering Lorentzian metric – Scattering dual metric

The change of coordinates from $V$ to a manifold with corners $\hat{V}$ induces the necessity to using the $b$–geometry on manifolds with corners$^{65}$, ”b” referring to boundary. We only mention that the manifold with corners $\hat{V}$ infers a Lie algebra $\mathcal{V}(\hat{V})$ of vector fields on $\hat{V}$ tangent to the boundary called $b$–vector fields as usual. $\mathcal{V}(\hat{V})$ is the space of sections of a natural vector bundle $\mathcal{T} \hat{V}$ over $\hat{V}$, the $b$–tangent bundle, which over the interior of $\hat{V}$ is naturally identified with $T \hat{V}$. $\mathcal{T} \hat{V}$ is spanned near the two-boundary hypersurfaces $\omega^0 = 0$ and $\omega^1 = 0$ by $\omega^0 \frac{\partial}{\partial \omega^0}, \omega^1 \frac{\partial}{\partial \omega^1}, \frac{\partial}{\partial x^a}$, and $\mathcal{V}(\hat{V})$ is spanned over $C^\infty(\hat{V})$ by these vector fields.
The bundle dual of $b^*V$ denoted $b^T\hat{V}$ and called $b-$cotangent bundle, is spanned locally near the two-boundary hypersurfaces $\omega^0 = 0$ and $\omega^1 = 0$ by $\frac{d\omega^0}{\omega^0}, \frac{d\omega^1}{\omega^1}$, $d\omega^a$.

Relatively to this dual basis, the unphysical metric $\Omega^2 g \equiv h$ has the form

$$h = \tilde{h}_{00} \left( \frac{d\omega^0}{\omega^0} \right)^2 + \tilde{h}_{01} \left( \frac{d\omega^0}{\omega^0} \otimes \frac{d\omega^1}{\omega^1} + \frac{d\omega^1}{\omega^1} \otimes \frac{d\omega^0}{\omega^0} \right) + \tilde{h}_{11} \left( \frac{d\omega^1}{\omega^1} \right)^2 +$$

$$\tilde{h}_{1a} \left( \frac{d\omega^1}{\omega^1} \otimes d\omega^a + d\omega^a \otimes \frac{d\omega^1}{\omega^1} \right) + \tilde{h}_{ab} d\omega^a \otimes d\omega^b, \quad (16)$$

which we refer to as the b-Lorentzian metric.

$$\tilde{h}_{00} = -\frac{|g|}{\Omega^2}; \quad \tilde{h}_{01} = \frac{|g| \ln \omega^0}{\Omega^2}; \quad (17)$$

$$\tilde{h}_{11} = -\frac{|g|}{\Omega} \left( \frac{\ln \omega^0}{\Omega} \right)^2 + \Omega^4 g_{11} \left( \frac{\Omega + \sum_{a=2}^{n} x^a \frac{\partial \Omega}{\partial x^a}}{\Omega \frac{\partial \Omega}{\partial x^1}} \right)^2 - 2 \Omega^2 \left( \frac{\Omega + \sum_{a=2}^{n} x^a \frac{\partial \Omega}{\partial x^a}}{\Omega \frac{\partial \Omega}{\partial x^1}} \right) \Omega \sum_{c=2}^{n} g_{1c} \tan \omega^c + g_{ab} \tan \omega^a \tan \omega^b; \quad (18)$$

$$\tilde{h}_{1a} = -\Omega^3 g_{11} \left( \frac{\Omega + \sum_{a=2}^{n} x^a \frac{\partial \Omega}{\partial x^a}}{\Omega \frac{\partial \Omega}{\partial x^1}} \right) \left( \frac{\frac{\partial \Omega}{\partial x^a}(1 + \tan^2 \omega^a)}{\Omega \frac{\partial \Omega}{\partial x^1}} \right) + \Omega^2 g_{1a} \left( \frac{\Omega + \sum_{c=2}^{n} x^c \frac{\partial \Omega}{\partial x^c}}{\Omega \frac{\partial \Omega}{\partial x^1}} \right)(1 + \tan^2 \omega^a) +$$

$$\Omega \sum_{c=2}^{n} g_{1c} \tan \omega^c \left( \frac{\frac{\partial \Omega}{\partial x^a}(1 + \tan^2 \omega^a)}{\Omega \frac{\partial \Omega}{\partial x^1}} \right) - n g_{ac}(1 + \tan^2 \omega^a) \tan \omega^c; \quad (19)$$

$$\tilde{h}_{ab} = g_{11} \left( \frac{1 + \tan^2 \omega^a}{\frac{\partial \Omega}{\partial x^1}} \right) \left( \frac{\partial \Omega}{\partial x^a} \right) \left( \frac{\partial \Omega}{\partial x^b} \right) +$$

$$2 g_{1a} \left( \frac{1 + \tan^2 \omega^a}{\frac{\partial \Omega}{\partial x^1}} \right) \left( \frac{\partial \Omega}{\partial x^b} \right) + g_{ab}(1 + \tan^2 \omega^a)(1 + \tan^2 \omega^b). \quad (20)$$

To this b-Lorentzian metric corresponds the volume density $dh$, a non-vanishing b-density and one can better consider b-density bundles in general which are locally spanned by $\frac{d\omega^0}{\omega^0}, \frac{d\omega^1}{\omega^1}, d\omega^2 ... d\omega^n$.

The scattering Lorentzian metric in this context in turn corresponds to the writing of $g$ and its dual with respect to the bases \( \left( \frac{d\omega^0}{\Omega_x}, \frac{d\Omega}{\Omega_x}, \frac{d\omega^a}{\Omega_x} \right) \), \( (\omega^0 \Omega_x, \Omega_x \partial \omega^a, \Omega_x \partial \Omega_x, \Omega_x \omega^a) \).
The scattering dual metric reads:
\[
g^{-1} = \left( g^{00} + \frac{(\ln \omega^0)^2}{\Omega^4} g^{ij} \frac{\partial \Omega}{\partial x^i} \frac{\partial \Omega}{\partial x^j} \right) \left( \omega^0 \partial_{\omega^0} \right)^2 + 2 \frac{\ln \omega^0}{\Omega} g^{ij} \frac{\partial \Omega}{\partial x^i} \frac{\partial \Omega}{\partial x^j} \Omega \omega^0 \omega^b \otimes \Omega^2 \partial_{\Omega} + \\
2 \frac{\partial \Omega}{\partial x^i} \left( \omega^0 \partial_{\omega^0} \right) \left( \omega^0 \partial_{\omega^0} \right) \left( \omega^0 \partial_{\omega^0} \right) + \Omega \partial_{\omega^0} + \\
1 \left( \frac{\partial \Omega}{\partial x^i} \frac{\partial \Omega}{\partial x^j} \right) \left( \omega^b \partial_{\omega^b} \right) \left( \omega^b \partial_{\omega^b} \right) \Omega \partial_{\omega^b} \otimes \Omega \partial_{\omega^b}. \tag{21}
\]

**Remark III.2** The b-Lorentzian metric \([16]\) together with its dual and also the scattering Lorentzian metric and its dual are very useful in studying asymptotic questions to the two boundary hypersurfaces \(\omega^0 = 0\), and \(\omega^1 = 0\), notably the analysis of the structure of the null-geodesics flow, that is the flow of the Hamilton function within the characteristic set\([14]\). In the gauge here it appears from \((21)\) that the regularity of infinity depends on the estimate of \(g^{ij} \partial_{\omega^i} \partial_{\omega^j}\).

We concentrate in this article on a lower bound inequality for the "total mass" assuming that the considered system collapses to a black hole.

For what follows, some definitions are necessary. We signal that some terminologies used below might not coincide with the usual one in the literature.

**IV. MASS FUNCTION AND DERIVED GEOMETRIC ANALYSIS**

A. Motivations

The conformal geometry here in order to fit to collapse requires that relative time effect passes from visible at early times to invisible at later ones. That is \(\frac{\partial \omega^0}{\partial x^i}\) passes from the interior of the dual light cone to its exterior.

The dual of the conformal metric \(h \equiv \Omega^2 g = -|\mathbf{g}|(dx^0)^2 + \Omega^2 \mathbf{g}_{ij} dx^i dx^j\) in coordinates \((x^\alpha)\) reads:
\[
G \equiv h^{-1} = -\frac{1}{|\mathbf{g}|} \left( \frac{\partial}{\partial x^0} \right)^2 + \Omega^{-2} \mathbf{g}^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j}. \tag{22}
\]

In coordinates \((\omega^\alpha)\), components of tensors are decorated with a hat "\(\hat{\cdot}\)". One has in particular:
\[
G(d\omega^0, d\omega^0) = \hat{G}^{\alpha\alpha} = (\omega^0)^2 \Omega^2 \left( -\frac{\Omega^2}{|\mathbf{g}|} + \frac{(\ln \omega^0)^2}{\Omega^4} \mathbf{g}^{ij} \frac{\partial \Omega}{\partial x^i} \frac{\partial \Omega}{\partial x^j} \right). \tag{23}
\]
On the hypersurface $\omega^0 = 1$, $\hat{G}^{00} = -\frac{\Omega^4}{\Omega^2} |_{\omega^0 = 1} < 0$, provided $\frac{1}{\Omega^4} g^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} |_{\omega^0 = 1} \not\rightarrow +\infty$; the covariant vector $\frac{d\omega^0}{d\omega^0}$ is timelike near $\omega^0 = 1$. Now, conceptually it is desirable that this covariant vector $\frac{d\omega^0}{d\omega^0}$ is spacelike near the hypersurface $\omega^0 = 0$, and that the equation $G\left(\frac{d\omega^0}{d\omega^0}, \frac{d\omega^0}{d\omega^0}\right) = 0$ defines a smooth hypersurface. This induces certainly some estimates for the term $(\ln \omega^0)^2 \frac{\Omega^4}{\Omega^2} g^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j}$ that would eventually be elucidated according to the desired end structure at infinity. Let’s recall that in the orthogonal splitting of the metric $g = -\alpha^2 (dt)^2 + \hat{g}$, the lapse $\alpha = \sqrt{|\hat{g}|}$ can be chosen as bounded (see O. Müller, M. Sanchez, 2009, and references in) so that the behavior of $\hat{G}^{00}(\omega \omega^0)$ at the boundaries hypersurfaces is given by the one of $(\ln \omega^0)^2 \hat{g}^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j}$. On the other hand $\hat{G}^{11} = (\omega^1)^{-2} \hat{g}^{ij} \frac{\partial \omega^1}{\partial x^i} \frac{\partial \omega^1}{\partial x^j}$ and since $\nabla \omega \neq (0)$ one has $\hat{G}^{11} > 0$ and then $\frac{d\omega^1}{d\omega^0}$ is spacelike for $\omega^1 \neq 0$. Depending on the chosen end structure, $\omega^1 = 0$ might be a null hypersurface.

According to what precedes and assuming some smoothness assumptions, there exists a hypersurface beyond which $\frac{d\omega^0}{d\omega^0}$ ceases to be timelike. It seems that time ceases to exist on such hypersurface. As indicated by the theory of Relativity due to Sir Albert Einstein, the speed of light is the same everywhere whether one is under the influence of very high or low gravity. So it is reasonable that something decreases as one approaches the speed of light, and it is the time or its effect. The same situation is valid for a Black hole since light can not escape from it so the time nearby it is hardly visible and felt. It is difficult to notice the change in the Black hole’s surroundings. We then resume that the manifold with corners $\hat{V}$ is split in two (exterior and interior) regions as $\frac{d\omega^0}{d\omega^0}$ moves from the interior of the dual characteristic cone to its exterior. We therefore adopt the following definitions.

**Definition IV.1** A system of coordinates $(\omega^\alpha)$ of the manifold with corners $(\hat{V}, h)$ is called gravitational collapse-adapted if one of the coordinates $\omega^\alpha$, say $\omega^0$, satisfies:
- the vector field $\frac{\partial}{\partial \omega^0}$ is timelike at least beyond some hypersurface (or asymptotically timelike, i.e. timelike in the neighborhood of the final state $\omega^0 = 0$),
- the dual vector $d\omega^0$ passes from the timelike character in some exterior region to the spacelike character in the neighborhood of the final state.

**B. Heuristics**

To motivate more our present understanding of the event horizon, let’s recall that the local energy density in a spacetime equipped with a system of coordinates $(x^\alpha)$ is defined
as $\mu = \frac{1}{8\pi}E_{00}$, where $E$ is the Einstein tensor. In a spacetime solution of the Einstein’s equations with matter, this local energy equals $T_{00}$. $T$ is the stress energy-momentum tensor of matter. One can then write:

$$\frac{R_{00} - 8\pi T_{00}}{R} = \frac{1}{2}g^{00}.$$ 

In relation with the gauge $\omega^\alpha$, one has

$$G\left(\frac{d\omega^0}{\Omega\omega^0}, \frac{d\omega^0}{\Omega\omega^0}\right) = g^{-1}(dx^0,dx^0) + \frac{(\ln\omega^0)^2}{\Omega^4} \Gamma^j_{ij} \frac{\partial\Omega}{\partial x^i} \frac{\partial\Omega}{\partial x^j}$$

$$= \frac{2}{R}(R_{00} - 8\pi T_{00}) + \left(\frac{x^0}{\Omega}\right)^2 \Gamma^j_{ij} \frac{\partial\Omega}{\partial x^i} \frac{\partial\Omega}{\partial x^j}$$

$$= h\left(\frac{\nabla\omega^0}{\Omega\omega^0}, \frac{\nabla\omega^0}{\Omega\omega^0}\right).$$

These expressions are useful in various respects. The first equality could be used in vacuum. They suggest that the region where time effects are hidden is described locally by $G\left(\frac{d\omega^0}{\Omega\omega^0}, \frac{d\omega^0}{\Omega\omega^0}\right) > 0$ and corresponds to some interior region whereas the region $G\left(\frac{d\omega^0}{\Omega\omega^0}, \frac{d\omega^0}{\Omega\omega^0}\right) < 0$ is an exterior region where time remains visible. One can also claim that in vacuum every gravitational collapse is dictated by the time effect. In the case where $G\left(\frac{d\omega^0}{\Omega\omega^0}, \frac{d\omega^0}{\Omega\omega^0}\right)$ tends to $+\infty$ as $\omega^0$ tends to zero, we call $\omega^0 = 0$ a mass singularity. Finally the total time effect is expected to be responsible to the final state of the system.

The hypersurface $G\left(\frac{d\omega^0}{\Omega\omega^0}, \frac{d\omega^0}{\Omega\omega^0}\right) = 0$ if smooth defines consequently an (illusory) event horizon which might comprise a part of the blackhole’s event horizon and a part equals to the apparent horizon.

C. Geometric objects and properties

**Definition IV.2** The mass function at a point $(\omega^\alpha)$ in the manifold with corners $(\tilde{V}, h \equiv \Omega^2 g)$ is the quantity $m(\omega^0, \omega^1, ..., \omega^n) \equiv G(\frac{d\omega^0}{\Omega\omega^0}, \frac{d\omega^0}{\Omega\omega^0})(\omega^0, \omega^1, ..., \omega^n)$.

We remark that this definition of the mass is valid up to the boundary of the considered manifold since $\frac{d\omega^0}{\omega^0}$ is non-singular (and non-trivial) as a b-covector at $\omega^0 = 0$, and the same holds for $\frac{d\omega^1}{\omega^1}$ at $\omega^1 = 0$. This mass-function measures some extra local energy density since it may exist even in vacuum. At every point $(\omega^\alpha)$ it encodes the rate at which the dual vector $d\omega^0$ passes from the interior of the dual characteristic cone at $(\omega^\alpha)$ to its exterior.

**Notation IV.1** In the rest of the paper we adopt the notation $d\omega = d\omega^2...d\omega^n$. 

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Definition IV.3 The (squared) mass of a region $D$ in the the manifold with corners $(\hat{V}, h)$ is the quantity
\[ \int_D m(\omega^0, \omega^1, ..., \omega^n) d\omega^0 d\omega^1 d\varpi. \] This quantity may be negative meaning that its squared root is a complex number.

Definition IV.4 The (squared) total mass of the spacetime is the quantity
\[ \int_{\hat{V}} m(\omega^0, \omega^1, ..., \omega^n) d\omega^0 d\omega^1 d\varpi. \]

Definition IV.5 The (illusory) Event horizon in the spacetime $(V, g)$ is the hypersurface $H$ of points $(\omega^\alpha)$ in the manifold with corners $(\hat{V}, h)$ which satisfy the equation
\[ m(\omega^0, \omega^1, ..., \omega^n) = 0. \] (25)

Provided a monotonicity argument for $m$ is established as described in the sequel of the paper, the existence of the (illusory) event horizon (assumed smooth for simplicity) induces the following definitions.

Definition IV.6 The interior region in the spacetime $(V, g)$ denoted $\text{Int}(V)$ is:
\[ \text{Int}(V) = \{ \omega \in \hat{V} / m(\omega^0, \omega^1, ..., \omega^n) \geq 0 \}. \] (26)

Definition IV.7 The exterior region in the spacetime $(V, g)$ denoted $\text{Ext}(V)$ is:
\[ \text{Ext}(V) = \{ \omega \in \hat{V} / m(\omega^0, \omega^1, ..., \omega^n) < 0 \}. \] (27)

Remark IV.1 The wave fronts (WF) for a system of partial differential equations are submanifolds $\{ f = \text{cste} \}$ of the spacetime where the scalar function $f$ is solution of the eikonal equation (21, P. 270). A wave front is generated by the bicharacteristics or rays (the characteristics of the eikonal equation).

In this setting (i.e. Temporal gauge), the principal part in the considered evolution system is the operator $\Box \partial_0$, the characteristic determinant w.r.t. the coordinates $(x^\alpha)$ is the hyperbolic polynomial
\[ P(x, \zeta) = (g^{00}(\zeta_0)^2 + g^{ij}(\zeta_0)^2 \zeta_i \zeta_j) \zeta_0, \]
and the corresponding eikonal equation is
\[ \frac{\partial f}{\partial x^0} G^{\alpha \beta} \frac{\partial f}{\partial x^\alpha} \frac{\partial f}{\partial x^\beta} = 0. \]

Furthermore the change of coordinates \((x \to \omega)\) induces that
\[ \Box \partial_0 \simeq -\Omega^\omega \partial \partial_0, \]
correspondingly
\[ P(\omega, \zeta) = -\Omega^\omega \hat{G}^{\alpha \beta} \zeta^\alpha \zeta^\beta \zeta_0. \]

Two types of wave fronts are then concerned here:
- \( f = \text{cst} \) s.t. \( \frac{\partial f}{\partial x^0} = 0 \),
- \( f = \text{cst} \) s.t. \( \hat{G}^{\alpha \beta} \frac{\partial f}{\partial x^\alpha} \frac{\partial f}{\partial x^\beta} = 0. \)

Given a point \( \omega \in \hat{V} \), the tangents to the rays issued from \( \omega \) generate a cone in the tangent space to the spacetime at \( \omega \), called the wave cone, dual to the characteristic cone. This wave cone is the envelope of the hyperplanes whose normals belong to the characteristic cone. We denote by \( WF(\omega) \) the wave front at the point \( \omega \). \( WF(\omega) \) is tangent to the wave cone at \( \omega \) along the direction of the bicharacteristic or ray.

**Definition IV.8** The Apparent event horizon of the spacetime \((V, g)\) is the surface denoted \( \mathcal{H}^- \) and defined by:
\[ \mathcal{H}^- = \{ \omega \in \mathcal{H} / WF(\omega) \cap Ext(V) \neq \emptyset \}. \] (28)

**Definition IV.9** The actual event horizon of the spacetime is the surface denoted \( \mathcal{H}^+ \) and defined by:
\[ \mathcal{H}^+ = \{ \omega \in \mathcal{H} / WF(\omega) \subset Int(V) \}. \] (29)

It is obvious that \( \mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^- \).

**Definition IV.10** The Black hole region of the spacetime \( V \) denoted \( B \) is defined as:
\[ B = \{ \omega \in Int(V) / WF(\omega) \subset Int(V) \}. \] (30)

**Definition IV.11** The Blackhole’s event horizon is the boundary of the Black hole region \( B \). We denote it \( \partial B \).
Definition IV.12  The White hole region of the spacetime denoted $W$ is:

$$W = \text{Int}(V) \setminus B.$$  \hfill (31)

Its boundary is denoted $\partial W$

Remark IV.2  Under the decreasing property for $m$ w.r.t. $\omega^0$:

a- the illusory event horizon observes obviously the following disjoint union:

$$\mathcal{H} = \{ \omega = (\omega^0, \omega^i) \in \mathcal{H} / \frac{\partial m}{\partial \omega^0}(\omega) < 0 \} \cup \{ \omega = (\omega^0, \omega^i) \in \mathcal{H} / \frac{\partial m}{\partial \omega^0}(\omega) = 0 \}; \hfill (32)$$

b- the subset $\{ m(\omega^0, \omega^i) = 0, \frac{\partial m}{\partial \omega^0}(\omega^0, \omega^i) < 0 \}$ of $\mathcal{H}$ has an equation of the form $\omega^0 = X(\omega^i)$, by the implicit function theorem.

c- the condition for the event horizon $\mathcal{H}$ to be a wave front is that $m$ satisfies the eikonal equation, i.e. $\frac{\partial m}{\partial \omega^0} \hat{G}^{ij} \frac{\partial m}{\partial \omega^0} \hat{G}^{ij} = 0$. This equation reduces if $\mathcal{H}$ has an equation $\omega^0 = X(\omega^i)$ to

$$\left[ \frac{\partial m}{\partial \omega^0} \right]^3 \left( -2 \hat{G}^{0i} \frac{\partial X(\omega^i)}{\partial \omega^i} + \hat{G}^{ij} \frac{\partial X(\omega^i)}{\partial \omega^i} \frac{\partial X(\omega^i)}{\partial \omega^j} \right) = 0; \hfill (33)$$

d- It is reasonable to assume that any blackhole’s event horizon admits an equation of the form $\omega^0 - \tilde{X}(\omega^i) = 0$ since it must form a ”hoop”. Intuitively the singularity here is pushed off to infinity, indeed, $G(d\omega^0, d\omega^0) = 0$ if and only if $\omega^0 = 0$ or $\Omega = 0$ or $m = 0$.

Proposition IV.1  a-The (illusory) event horizon $\mathcal{H}$ is not necessarily a null surface.

b- At each point $\omega$ of the actual event horizon $\mathcal{H}^+$, one has $n$-dimensional wave fronts $\{ \omega^0 = \text{cste} \}$.

c- The apparent horizon $\mathcal{H}^-$ is a wavefront, it has always a null character.

d- If the (illusory) event horizon $\mathcal{H}$ admits an equation of the form $\omega^0 = \text{cste}$, then it is a null hypersurface.

Proof

Obvious since the (illusory) event horizon is of equation $m \equiv \frac{\hat{G}^{00}}{(\omega^0)^2} = 0$ and the eikonal equation reads $\frac{\partial f}{\partial \omega^0} \hat{G}^{ij} \frac{\partial f}{\partial \omega^0} \frac{\partial f}{\partial \omega^0} = 0$. Furthermore, $\mathcal{H}^- = \{ m = 0, \frac{\partial m}{\partial \omega^0} = 0 \}$.

D. Monotonicity property for the mass function

Theorem IV.1  The mass function $m$ is decreasing relatively to the time $\omega^0$ if and only if the extrinsic curvature $K$ of the spacetime satisfies the inequality

$$\text{tr}K + \frac{2|\hat{g}| \sqrt{|\hat{g}|} \ln \omega^0}{\Omega^4} g^{ij} \frac{\partial \Omega}{\partial x^i} \frac{\partial \Omega}{\partial x^j} - \frac{2|\hat{g}|^2 (\ln \omega^0)^2}{\Omega^6} K^{ij} \frac{\partial \Omega}{\partial x^i} \frac{\partial \Omega}{\partial x^j} \leq 0. \hfill (34)$$
Proof
The proof of this theorem is based merely on straightforward computations. We recall for this purpose that \( m \) is given by:

\[
m(\omega^0) = \left( \frac{\Omega^2}{|g|} + \frac{(\ln \omega^0)^2}{\Omega^4} g^{ij} \frac{\partial \Omega}{\partial x^i} \frac{\partial \Omega}{\partial x^j} \right).
\]

In order to address the expression of \( \frac{\partial m}{\partial \omega^0} \), we collect the following useful expressions:

\[
\frac{\partial x^0}{\partial \omega^0} = \frac{1}{\Omega^2 \omega^0}, \quad \frac{\partial |g|}{\partial \omega^0} = \frac{1}{\Omega^2 \omega^0}, \quad \frac{\partial g^{ij}}{\partial \omega^0} = -\frac{1}{\Omega^2 \omega^0} \frac{\partial g^{ij}}{\partial x^0}.
\]  

(35)

Under the temporal gauge condition, one has

\[
\frac{\partial g^{ij}}{\partial x^0} = -\frac{2}{\sqrt{|g|}} K_{ij}, \quad \frac{\partial |g|}{\partial x^0} = \frac{2}{\Omega^2 \omega^0} \sqrt{|g|} K_{ij}, \quad \frac{\partial |g|}{\partial x^0} = \frac{1}{\Omega^4 \omega^0} \frac{\partial g^{ij}}{\partial x^0}.
\]

where \( K = (K_{ij}) \) is the extrinsic curvature of the hypersurfaces \( x^0 = t \). It follows that:

\[
\frac{\partial |g|}{\partial \omega^0} = \frac{1}{\Omega^2 \omega^0} \sqrt{|g|} \text{tr} K, \quad \frac{\partial g^{ij}}{\partial \omega^0} = -\frac{2}{\Omega^4 \omega^0} \sqrt{|g|} K^{ij}.
\]  

(36)

Combining all these expressions one has:

\[
\frac{\partial m}{\partial \omega^0} = \frac{1}{\omega^0} \left( \frac{\sqrt{|g|}}{|g|^2} \text{tr} K + \frac{2 \ln \omega^0}{\Omega^4} g^{ij} \frac{\partial \Omega}{\partial x^i} \frac{\partial \Omega}{\partial x^j} - \frac{2 \sqrt{|g|} (\ln \omega^0)^2}{\Omega^6} K^{ij} \frac{\partial \Omega}{\partial x^i} \frac{\partial \Omega}{\partial x^j} \right).
\]  

(37)

The conclusion is then obvious according to this expression.

Corollary IV.1 For the mass function to be strictly decreasing w.r.t. to the time \( \omega^0 \), it suffices that the extrinsic curvature \( K \) of the spacetime satisfies the inequalities

\[
\text{tr} K < 0, \quad K^{ij} \frac{\partial \Omega}{\partial x^i} \frac{\partial \Omega}{\partial x^j} \geq 0.
\]  

(38)

Proof
It is obvious according to the expression (37) of \( \frac{\partial m}{\partial \omega^0} \) above.

Corollary IV.2 For the mass function to be strictly decreasing w.r.t. to the time \( \omega^0 \), it suffices that the extrinsic curvature \( K \) of the spacetime satisfies the inequalities

\[
(lapse)^2 \left( g^{ij} \frac{\partial \Omega}{\partial x^i} \frac{\partial \Omega}{\partial x^j} \right)^2 + 2\text{tr} K K^{ij} \frac{\partial \Omega}{\partial x^i} \frac{\partial \Omega}{\partial x^j} < 0, \quad K^{ij} \frac{\partial \Omega}{\partial x^i} \frac{\partial \Omega}{\partial x^j} > 0.
\]  

(39)

Proof
The proof is obvious according to the expression (37) of \( \frac{\partial m}{\partial \omega^0} \) above.

Remark IV.3 In the rest of this work, it would be interesting to see how \( \frac{\partial m}{\partial s} \) is related to \( \frac{dm}{ds} \), where \( s \) is the black hole’s horizon perturbation parameter.
E. Monotonicity property: Another view

In the following lines, we are interested in another possible geometrical analysis and physical interpretation of the monotonicity property (34) above, i.e.:

$$\frac{\partial m}{\partial \omega^0} \leq 0; \quad (40)$$

and eventually the condition (48) used in the sequel of this work, i.e.:

$$\hat{h}^{0i} \frac{\partial m}{\partial \omega^i} > 0, \text{ on } H^+.$$

(41)

We address the following computations where covariant derivatives are carried according to the unphysical metric $h$. Given then $\hat{\nabla}$ the riemannian connection of $(\hat{V}, h)$, one has:

$$\frac{\partial m}{\partial \omega^0} = \frac{\partial}{\partial \omega^0} \left( h \left( \frac{\hat{\nabla} \omega^0}{\Omega^0}, \frac{\hat{\nabla} \omega^0}{\Omega^0} \right) \right) = 2 h \left( \frac{\hat{\nabla} \omega^0}{\Omega^0}, \frac{\hat{\nabla} \omega^0}{\Omega^0} \right)$$

(42)

From this expression, it results that $\frac{\partial m}{\partial \omega^0} < 0$, if and only if

$$h \left( \frac{\hat{\nabla} \omega^0}{\Omega^0}, \frac{\hat{\nabla} \omega^0}{\Omega^0} \right) < 0.$$

(43)

On the other hand and similarly with the above computations, one has:

$$\frac{\partial m}{\partial \omega^i} = 2 h \left( \frac{\hat{\nabla} \omega^i}{\Omega^0}, \frac{\hat{\nabla} \omega^0}{\Omega^0} \right),$$

(44)

this implies that:

$$\hat{h}^{0i} \frac{\partial m}{\partial \omega^i} = 2 h \left( \frac{\hat{\nabla} \omega^0}{\Omega^0}, \frac{\hat{\nabla} \omega^0}{\Omega^0} \right).$$

(45)

Proposition IV.2  

i- The condition $\frac{\partial m}{\partial \omega^0} < 0$ is equivalent to

$$h \left( \frac{\hat{\nabla} \omega^0}{\Omega^0}, \frac{\hat{\nabla} \omega^0}{\Omega^0} \right) < 0.$$

(46)

\(ii\)- The condition $\hat{h}^{0i} \frac{\partial m}{\partial \omega^i} > 0$ is equivalent to

\(h \left( \frac{\hat{\nabla} \omega^0}{\Omega^0}, \frac{\hat{\nabla} \omega^0}{\Omega^0} \right) > 0.$$

(47)
F. On trapped surfaces

Since the definition of black hole given here is not a priori the standard one, it appears obvious to question if there is a formation of trapped surfaces in this setting. We therefore present here what can be viewed as an attempt to answer the question since the relation between the hypotheses formulated below and Cauchy data remains another important task to investigate.

We first provide conditions which guarantee the existence of some particular codimension 2 spacelike hypersurfaces in the region \( \{ m \geq 0 \} \). Namely, we assume that the vector field \( \nabla m \) is timelike in \( \{ m \geq 0 \} \), this induces, under the condition \( \partial m / \partial \omega < 0 \) that

\[
\hat{h}^{0i} \partial m / \partial \omega^i > 0, \quad \hat{h}^{ij} \partial m / \partial \omega^i \partial m / \partial \omega^j > 0; \quad (48)
\]

and

\[
\left( \hat{h}^{0i} \partial m / \partial \omega^i \right)^2 - \hat{h}^{00} \hat{h}^{ij} \partial m / \partial \omega^i \partial m / \partial \omega^j > 0; \quad (49)
\]

the hypersurfaces of constant \( m \), \( \{ m = \nu, \nu > 0 \} \) are spacelike. On the other hand, in \( m > 0 \) the hypersurfaces of constant \( \omega^0 + m \), \( \{ \omega^0 + m = \mu \} \), are also spacelike. We consider the codimension 2 spacelike surfaces in \( \{ m > 0 \} \):

\[
H_{\mu \nu} := \{ (\omega^0) \in \hat{V}, \omega^0 + m = \mu, \ m = \nu, \ \mu > 0, \ \nu > 0 \};
\]

and suppose that they have a compact feeling (this is possible if \( \{ m = \nu, \nu > 0 \} \) is a Cauchy hypersurface and \( H_{\mu \nu} \) is the boundary of a compact region of it). At every point \( \omega \in H_{\mu \nu} \), \( (T_\omega H_{\mu \nu})^\perp \) has dimension 2 and is timelike. Such a two plane cuts the future light cone at \( \omega \) along two future null directions. We choose two future null vectors \( l^+, l^- \) supported by these directions up to a scaling by a positive factor \( -h(l^+, l^-) < 0 \). One can choose the scaling factor \( a > 0 \) such that \( h(l^+, l^-) = -2 \). Taking \( (e_a) \) a reference frame on \( H_{\mu \nu} \), hence orthogonal to \( l^+ \) and \( l^- \), we define the two null fundamental forms of \( H_{\mu \nu} \) by:

\[
\chi_{ab} = h(\hat{\nabla}_e l^+, e_b), \quad \chi_{ab} = h(\hat{\nabla}_e l^-, e_b). \quad (50)
\]

The null mean curvatures of \( H_{\mu \nu} \) denoted \( tr\chi \) and \( tr\chi \) are defined by:

\[
tr\chi = \hat{h}^{ab} \chi_{ab}, \quad tr\chi = \hat{h}^{ab} \chi_{ab}. \quad (51)
\]
Now, at every point $\omega \in H_{\mu\nu}$, the vector fields $\hat{\nabla} \omega^0$ and $\hat{\nabla} m$ belong to $(T_\omega H_{\mu\nu})^\perp$, and we assume $\hat{\nabla} m$ to be future pointing, consequently

$$l^+ = a(b^+ \hat{\nabla} \omega^0 + \hat{\nabla} m), \quad l^- = a(b^- \hat{\nabla} \omega^0 + \hat{\nabla} m);$$  

$$b^\pm = \frac{-h(\hat{\nabla} \omega^0, \hat{\nabla} m) \pm \sqrt{(h(\hat{\nabla} \omega^0, \hat{\nabla} m))^2 - h(\hat{\nabla} \omega^0, \hat{\nabla} \omega^0) h(\hat{\nabla} m, \hat{\nabla} m)}}{h(\hat{\nabla} \omega^0, \hat{\nabla} \omega^0)}.$$  

The two null forms read in terms of the components in the basis $(e_a)$ of the covariant derivatives (w.r.t. $e_a$) of the vector fields $\hat{\nabla} \omega^0$ and $\hat{\nabla} m$ as:

$$\chi_{ab} = ab^+ h(\hat{\nabla}_{e_a} \omega^0, e_b) + h(\hat{\nabla}_{e_a} \omega^0, e_b) \equiv ab^+ U_{ab} + aV_{ab},$$

$$\chi_{ab} = ab^- h(\hat{\nabla}_{e_a} \omega^0, e_b) + h(\hat{\nabla}_{e_a} \omega^0, e_b) \equiv ab^- U_{ab} + aV_{ab}.\quad (53)$$

**G. On hypotheses on which trapped surfaces’s formation is based**

From the expressions (54) above it follows:

$$tr \chi = ab^+ tr U + a tr V, \quad tr \overline{\chi} = ab^- tr U + a tr V;$$

one deduces:

$$tr V = \frac{b^+ tr \chi - b^- tr \overline{\chi}}{a(b^+ - b^-)},$$

and hence to have $tr \chi < 0$, $tr \overline{\chi} < 0$, it is necessary that $tr V < 0$. Further, the condition $tr \chi tr \overline{\chi} > 0$ requires that:

$$tr U \in \left[ -\frac{tr V}{b^-}, -\frac{tr V}{b^+} \right].$$

Conversely if

$$tr V < 0, \quad tr U \in \left[ -\frac{tr V}{b^-}, -\frac{tr V}{b^+} \right],$$

then

$$tr \chi < 0, \quad tr \overline{\chi} < 0.$$  

This previous analysis and the results of corollary [IV.1] and proposition [IV.2] induce the following theorem and its corollary.
H. Theorem – Corollary

Theorem IV.2 Let \((V, g)\) be a global in time spacetime solution of the Einstein field equations where \(g\) is of the following form in coordinates \((x^\alpha)\):

\[
g = -\frac{\Omega^2}{g} (dx^0)^2 + \mathcal{G}_{ij} dx^i dx^j, \quad \mathcal{G} = (\mathcal{G}_{ij}),
\]

\(\Omega \equiv \Omega(x^i)\) is an arbitrary positive given scalar density satisfying \(\Omega \rightarrow 0\) as \(r = |(x^i)| \rightarrow +\infty\).

Let denote \(h = \Omega^2 \mathcal{G}\) the conformal metric, \(G\) its inverse, and \(\tilde{\nabla}\) its associated riemannian connection. Let be specified the gauge \((\omega^\alpha)\) for the spacetime \((V, g)\), with \(\omega^0 = e^{-\Omega x^0}, \omega^1 = \Omega, \omega^a = \arctan(\Omega x^a)\).

One suppose that \(\mathcal{H} = \{m \equiv G \left( \frac{\partial \omega^0}{\partial x^0}, \frac{\partial \omega^0}{\partial x^1} \right) = 0 \} \neq \emptyset\) and defines a (smooth) hypersurface, and the attached geometric objects (in \((\tilde{V}, \tilde{h})\)), \(U, V\) and quantities \(a, b^\pm\) as described in (52)-(54) satisfy:

\[
\text{tr} V < 0, \text{tr} U \in \left[ -\frac{\text{tr} V}{b^-}, -\frac{\text{tr} V}{b^+} \right].
\]

Then the spacetime \((V, g)\) undergoes trapped surfaces's formation.

Corollary IV.3 One supposes that a gauge \((\omega^\alpha)\) is specified as above for a spacetime \((V, g)\), and the extrinsic curvature \(K\) of the spacetime together with the attached geometric objects (in \((\tilde{V}, \tilde{h})\)), \(U, V\) and quantities \(a, b^\pm\) as described in (52)-(54) satisfy:

\[
tr K < 0, K^{ij} \frac{\partial \omega^1}{\partial x^i} \frac{\partial \omega^1}{\partial x^j} \geq 0, \quad h \left( \hat{\nabla}^\frac{\omega^0}{\omega^0}, \hat{\nabla}^\frac{\omega^0}{\omega^1} \right) > 0;
\]

\[
\text{tr} V < 0, \text{tr} U \in \left[ -\frac{\text{tr} V}{b^-}, -\frac{\text{tr} V}{b^+} \right].
\]

Then the spacetime \((V, g)\) undergoes trapped surfaces’s formation.

Remark IV.4 To appreciate the physical interpretation of the various hypotheses above, one should first remember that \(\hat{\nabla} \omega^0\) is viewed as time effect or time-flux, \(m\) is the relative length of such flux, \(\hat{\nabla} m\) measures variation of such length. Furthermore, the conditions required in the statements of the results above rest on components of the covariant derivatives of \(\hat{\nabla} \omega^0\) and \(\hat{\nabla} m\) in specific directions.
V. A LOWER BOUND PROBLEM FOR THE TOTAL MASS AS A MINIMIZATION PROBLEM VIA A LAGRANGIAN FORMULATION

The concept of mass or quasi-local mass is very important in general relativity. In [S.-T. Yau, Seminar on Differential Geometry (1982)], Penrose listed the search for a definition of such quasi-local mass as his number one problem in classical general relativity. Clearly, many important statements in general relativity require a good definition of quasi-local mass. This latter might help to control the dynamics of the gravitational field. It might also be used for energy methods for the analysis of hyperbolic equations in spacetimes. The positivity of the total mass is a matter of particular interest as soon as such a quantity is defined. This latter would be obtained if a positive lower bound inequality for the total mass is established. Let’s recall that a lower bound for the total mass of isolated systems could also guarantee the stability of such systems. In this section, according to the mass function adopted in this work, we analyze such a problem as a minimization one under some prospective isoperimetric constraints whose nature (energy, area or volume,...) should depend on the desirable ends. For this purpose we use a foliation of the conformal spacetime by the deformations of the black hole’s event horizon as follow.

First of all let’s remark that the actual event horizon $H^+$ is defined by an equation of the form $\omega^0 = X(\omega^1, ..., \omega^n)$, and has an obvious parametrization

$$H : (\omega^1, \omega^n) \rightarrow (X(\omega^1, ..., \omega^n), \omega^1, ..., \omega^n).$$

The Black hole’s event horizon correspondingly is defined by an equation of the form

$$\partial B : \omega^0 = \tilde{X}(\omega^i);$$

where

$$(X(\omega^1, ..., \omega^n), \omega^1, ..., \omega^n) = (\tilde{X}(\omega^1, ..., \omega^n), \omega^1, ..., \omega^n) \text{ on } H^+. \quad (64)$$

We denote by $H_s$, $s \in [-\varepsilon, +\varepsilon]$, a deformation of the ”Black hole’s event horizon” $\partial B$ such that $H_0 \equiv \partial B$, adopted as follows:

$$H_0 \ni (\omega^a) \mapsto (\tilde{T}(s)(\omega^i), \omega^j) \equiv ((T \circ s(\omega^a), \omega^1, ..., \omega^n)) \in H_s.$$

Let denote $H = [0, \Omega_{\text{max}}] \times (]-\frac{\pi}{2}, +\frac{\pi}{2}[)^{n-1}$, one has for $s \in [-\varepsilon, +\varepsilon]$:

$$H_s = \{ (\omega^0, \omega^i) \in \tilde{V}/\omega^0 = \tilde{T}(s)(\omega^i), (\omega^i) \in H \}, \quad H_0 = \{ (\omega^0, \omega^i) \in \tilde{V}/\omega^0 = \tilde{X}(\omega^i), (\omega^i) \in H \}. \quad (65)$$
The total mass (squared) \( M = 2M^2 \) of the spacetime is written according to the deformation of the Black hole’s event horizon as:

\[
M = \int_{-\epsilon}^{+\epsilon} \int_{H} \frac{m^*(\tilde{T}(s)(\omega^i), \omega^j)}{\tilde{T}(s)} d\omega^1 d\omega^2 ... d\omega^n ds,
\]

(66)

with \( m^* = \omega^0 \omega^1 m \). We remark that \( m^* \) is integrable up to the boundary.

Now set \( \tilde{T}(s) = T \circ s(.) \), then \( \tilde{T}(s) \) is a distribution on \( H \) — and define the functionals \( J \):

\[
J(\tilde{T}) = \int_{-\epsilon}^{+\epsilon} \int_{H} \frac{m^*(\tilde{T}(s)(\omega^i), \omega^j)}{\tilde{T}(s)} d\omega^1 d\omega^2 ... d\omega^n ds,
\]

(67)

which can then be rewritten as

\[
J(\tilde{T}) = \int_{-\epsilon}^{+\epsilon} P(s, \tilde{T}(s), \frac{d\tilde{T}(s)}{ds}) ds;
\]

(68)

and and associate to \( J \) another functional \( I \):

\[
I(\tilde{T}) = \int_{-\epsilon}^{+\epsilon} P_0(s, \tilde{T}(s), \frac{d\tilde{T}(s)}{ds}) ds;
\]

(69)

where the exact form of \( P_0 \) depends on the specific needed end. This induces a minimization problem:

\[
(P) \begin{cases} 
\text{Find } \inf_{\tilde{T} \in H^1[-\epsilon;+\epsilon]} J(\tilde{T}), \\
I(\tilde{T}) = K, \quad K \text{ is a constant.}
\end{cases}
\]

(70)

where \( H^1[-\epsilon;+\epsilon] \) is the classical Sobolev space. As one can notice we have given a Lagrangian formulation to the lower bound problem for the here considered total mass where the Lagrangian is the function \( P(., \tilde{T}(.), \frac{d\tilde{T}(.)}{ds}) \).

**Remark V.1** In the geometric flow technique here, we have not considered the flow of the metric since at this stage we are not interested in a specific end for the spacetime. The natural metric on any subspace here is the induced one.

### A. Frechet derivative \( J'(\tilde{T}) \)

The function \( P(., ., .) \) is smooth and we find the Frechet-derivative of \( J \). The Isoperimetric constraint \( I(\tilde{T}) = K \) is combined with the boundary conditions that induce the set of constraints (we search for distribution \( \tilde{T} \in H^1([-\epsilon;+\epsilon]) \))

\[
C = \{ \tilde{T} \in H^1([-\epsilon;+\epsilon]), \tilde{T}(+\epsilon) \equiv 0, \tilde{T}(-\epsilon) \equiv \tilde{Z} \}.
\]

(71)
$\tilde{Z}$ is such that the 0-set of $\omega^0 - \tilde{Z}$ defines the initial hypersurface. Obviously $\tilde{Z}$ is given by:

$$
\tilde{Z} : \hat{V} \to \mathbb{R} \\
(\omega^0, \omega^i) \mapsto \tilde{Z}(\omega^0, \omega^i) = e^{-\omega^i \tilde{Z}(\omega^i)} \quad Z(\omega^i) \equiv x^0(\omega^1, \ldots, \omega^n).
$$

For $u \in C$, the set of admissible directions of $u$ in the sense of Frechet denoted $K(u)$ is:

$$
K(u) =: \{ w \in H^1(\gamma - \epsilon; +\epsilon) / \text{there exists a sequence } (w_n) \in H^1(\gamma - \epsilon; +\epsilon) \}.
$$

(72)

If $u + e_n w_n \in C$ then one can deduce that $w_n(+) = 0$, $w_n(-\epsilon) = 0$, and since functions of $H^1(\gamma - \epsilon; +\epsilon)$ are continuous at the boundary, and the application trace is also continuous, one concludes that $w(-\epsilon) = 0$, $w(+) = 0$. Conversely if $w(-\epsilon) = 0$, $w(+\epsilon) = 0$, one constructs $u + \frac{1}{n} w$ which satisfies the constraints. It follows that

$$
K(u) = H^1_0(\gamma - \epsilon; +\epsilon).
$$

(73)

For $w \in H^1_0(\gamma - \epsilon; +\epsilon)$,

$$
(J'(u), w) = \lim_{\lambda \to 0} \frac{J(u + \lambda w) - J(u)}{\lambda} = \lim_{\lambda \to 0} \frac{1}{\lambda} \int_{-\epsilon}^{+\epsilon} \{ P(u + \lambda w, u'T + \lambda w') - P(u, u') \} ds.
$$

(74)

(75)

Since the application $(\tilde{T}, \tilde{T}') \mapsto P((\tilde{T}, \tilde{T}'))$ is differentiable, one can write:

$$
P(u + \lambda w, u'T + \lambda w') = P(u, u') + DP(u, u').\lambda(w, w') + |\lambda||P(w, w')|O(\lambda(w, w')),
$$

where $DP$ denotes the differential of $P$ — this implies that

$$
(J'(u), w) = \int_{-\epsilon}^{+\epsilon} DP(u, u').(w, w') ds
$$

$$
= \int_{-\epsilon}^{+\epsilon} \left[ \frac{\partial P}{\partial T}(u, u') w + \frac{\partial P}{\partial T'}(u, u') w' \right] ds
$$

$$
= \int_{-\epsilon}^{+\epsilon} \left[ \frac{\partial P}{\partial T}(u, u'). w \right] ds + \int_{-\epsilon}^{+\epsilon} \left[ \frac{d}{ds} \left( \frac{\partial P}{\partial T'}(u, u') \right) w \right] ds
$$

$$
= \int_{-\epsilon}^{+\epsilon} \left[ \frac{\partial P}{\partial T}(u, u') - \frac{d}{ds} \left( \frac{\partial P}{\partial T'}(u, u') \right) \right] . w ds + \left[ \frac{\partial P}{\partial T'}(u, u') \right]_{-\epsilon}^{+\epsilon}
$$

$$
= \int_{-\epsilon}^{+\epsilon} \left[ \frac{\partial P}{\partial T}(u, u') - \frac{d}{ds} \left( \frac{\partial P}{\partial T'}(u, u') \right) \right] . w ds + \left[ \frac{\partial P}{\partial T'}(u, u') \right]_{-\epsilon}^{+\epsilon}
$$

$$
\frac{\partial P}{\partial T'}(u(+\epsilon), u'(+\epsilon)) \delta_{+\epsilon}(w) - \frac{\partial P}{\partial T'}(u(-\epsilon), u'(-\epsilon)) \delta_{-\epsilon}(w);
$$

24
\( \delta_{\pm \epsilon} \) denotes Dirac-distribution at \( \pm \epsilon \), one deduces that:

\[
J'(u) = \frac{\partial P}{\partial T}(u, u') - \frac{d}{ds} \left( \frac{\partial P}{\partial T'}(u, u') \right) + \frac{\partial P}{\partial T'}(u(+\epsilon), u'(+\epsilon))\delta_{+\epsilon} - \frac{\partial P}{\partial T'}(u(-\epsilon), u'(-\epsilon))\delta_{-\epsilon}.
\]

(76)

Let’s recall that:

\[
P(\widetilde{T}, \frac{d\widetilde{T}}{ds}) = \int_H \frac{m^*(\widetilde{T}(\omega^i), \omega^i)}{\widetilde{T}^2(\omega^i)} \frac{d\widetilde{T}(\omega^i)}{ds} \frac{d\omega^1}{\omega^1} d\omega^2 ... d\omega^n;
\]

therefore

\[
\frac{\partial P}{\partial T} = \int_H \frac{\widetilde{T}(\omega^i) \frac{\partial m^*}{\partial \omega^i}(\widetilde{T}(\omega^i))}{\widetilde{T}^2(\omega^i)} \frac{d\widetilde{T}(\omega^i)}{ds} \frac{d\omega^1}{\omega^1} d\omega^2 ... d\omega^n,
\]

(77)

\[
\frac{\partial P}{\partial T'} = \int_H \frac{m^*(\widetilde{T}(\omega^i), \omega^i)}{\widetilde{T}(\omega^i)} \frac{d\omega^1}{\omega^1} d\omega^2 ... d\omega^n,
\]

(78)

\[
\frac{d}{ds} \left( \frac{\partial P}{\partial T'} \right) = \int_H \frac{d}{ds} \left( \frac{m^*(\widetilde{T}(\omega^i), \omega^i)}{\widetilde{T}(\omega^i)} \right) \frac{d\omega^1}{\omega^1} d\omega^2 ... d\omega^n
\]

(79)

It appears that \( P \) satisfies the Lagrange’s equation

\[
\frac{\partial P}{\partial T} - \frac{d}{ds} \left( \frac{\partial P}{\partial T'} \right) = 0, \forall \widetilde{T}.
\]

(80)

**B. Euler’s condition for \( R \equiv P + \lambda P_0 - A \) fundamental equation**

We recall here that if \( J(\widetilde{T}) \) is an extremum for the functional \( J(\cdot) \) subject to the isoperimetric constraint \( I(\cdot) = K \), then \( \widetilde{T} \) is an extremal of \( \int R(\cdot) = \int (P + \lambda P_0)(\cdot) \) for some constant \( \lambda \). The initially unknown multiplier \( \lambda \) must be determined at the end using the isoperimetric constraint \( \int P_0 = K \)

The Euler’s equation for \( R = P + \lambda P_0 \) reads:

\[
\frac{\partial P}{\partial T} - \frac{d}{ds} \left( \frac{\partial P}{\partial T'} \right) + \lambda \left[ \frac{\partial P_0}{\partial T} - \frac{d}{ds} \left( \frac{\partial P_0}{\partial T'} \right) \right] + \frac{\partial P}{\partial T'}(\widetilde{T}(+\epsilon), \widetilde{T}'(+\epsilon))\delta_{+\epsilon} - \frac{\partial P}{\partial T'}(\widetilde{T}(-\epsilon), \widetilde{T}'(-\epsilon))\delta_{-\epsilon} +
\]
\[
\lambda \frac{\partial P_0}{\partial T^r}(\tilde{T}(+\epsilon), \tilde{T}'(+\epsilon))\delta_{+\epsilon} - \lambda \frac{\partial P_0}{\partial T^r}(\tilde{T}(-\epsilon), \tilde{T}'(-\epsilon))\delta_{-\epsilon} = 0. \quad (81)
\]

Consider that \( P \) satisfies the equation \( \frac{\partial P}{\partial T} - \frac{d}{ds}\left( \frac{\partial P}{\partial T^r} \right) = 0, \forall \tilde{T} \), this latter equation reduces then for \( \lambda \neq 0 \) to the following that we consider as a fundamental equation:

\[
\frac{\partial P_0}{\partial T} - \frac{d}{ds}\left( \frac{\partial P_0}{\partial T^r} \right) + \left[ \frac{1}{\lambda} \frac{\partial P}{\partial T^r}(\tilde{T}(+\epsilon), \tilde{T}'(+\epsilon)) + \frac{\partial P_0}{\partial T^r}(\tilde{T}(+\epsilon), \tilde{T}'(+\epsilon)) \right] \delta_{+\epsilon} - \\
\left[ \frac{1}{\lambda} \frac{\partial P}{\partial T^r}(\tilde{T}(-\epsilon), \tilde{T}'(-\epsilon)) + \frac{\partial P_0}{\partial T^r}(\tilde{T}(-\epsilon), \tilde{T}'(-\epsilon)) \right] \delta_{-\epsilon} = 0. \quad (82)
\]

C. Example of \( P_0 \) – A Sturm- Liouville operator

Here, we consider a situation where the perturbation of the black hole is due to a forced term \( f \) and the interested functional is given by

\[
I(\tilde{T}) = \int_{-\epsilon}^{+\epsilon} P_0(s, \tilde{T}(s), \tilde{T}'(s))ds = \int_{-\epsilon}^{+\epsilon} \left( \frac{1}{2}(\tilde{T}'(s))^2 + \frac{1}{2}(\tilde{T}(s))^2 - f(s)\tilde{T}(s) \right) ds.
\]

A minimizer of \( J(\tilde{T}) \) subject to the isoperimetric constraint \( I(\tilde{T}) = K \) must be a function \( \tilde{T}_*(s, \lambda) \) satisfying the differential equation

\[-\tilde{T}'' + \tilde{T} - f + \left[ \frac{1}{\lambda} \frac{\partial P}{\partial T^r}(\tilde{T}(+\epsilon), \tilde{T}'(+\epsilon)) + \frac{\partial P_0}{\partial T^r}(\tilde{T}(+\epsilon), \tilde{T}'(+\epsilon)) \right] \delta_{+\epsilon} - \\
\left[ \frac{1}{\lambda} \frac{\partial P}{\partial T^r}(\tilde{T}(-\epsilon), \tilde{T}'(-\epsilon)) + \frac{\partial P_0}{\partial T^r}(\tilde{T}(-\epsilon), \tilde{T}'(-\epsilon)) \right] \delta_{-\epsilon} = 0, \quad (83)
\]

and where the parameter \( \lambda \) enjoys \( I(\tilde{T}_*) = K \). This equation presents a Sturm-Liouville operator and one might expect quantization in its resolution process.

D. Second derivative \( J''(\tilde{T})\tilde{V} \)

In order to characterize the minimality condition we are now studying

\[
\frac{J'(\tilde{T} + \sigma \tilde{V}, w) - J'(\tilde{T}, w)}{\sigma}.
\]

Let’s recall that:

\[
J'(\tilde{T}) = \frac{\partial P}{\partial T}(\tilde{T}, \tilde{T}') - \frac{d}{ds}\left( \frac{\partial P}{\partial T^r}(\tilde{T}, \tilde{T}') \right) + \frac{\partial P}{\partial T^r}(\tilde{T}(+\epsilon), \tilde{T}'(+\epsilon))\delta_{+\epsilon} - \frac{\partial P}{\partial T^r}(\tilde{T}(-\epsilon), \tilde{T}'(-\epsilon))\delta_{-\epsilon}. \quad (84)
\]
We denote by $S$ the application 

$$S : (\tilde{T}, \tilde{T}') \rightarrow \frac{\partial P}{\partial T'} = \int_H \frac{m^*(\tilde{T})}{\tilde{T}} \frac{d\omega^1}{\omega^1} d\omega^2 ... d\omega^n,$$  
(85)

which is independent of $\tilde{T}'$ and is differentiable for every $(\tilde{T}, \tilde{T}')$.

$$J'(\tilde{T} + \sigma \tilde{V}, w) =$$ 

$$\left\langle \int_H \frac{m^*(\tilde{T}(+\epsilon) + \sigma \tilde{V}(+\epsilon))}{\tilde{T}(+\epsilon) + \sigma \tilde{V}(+\epsilon)} \frac{d\omega^1}{\omega^1} d\omega^2 ... d\omega^n \delta_{+\epsilon}, w \right\rangle -$$ 

$$\left\langle \int_H \frac{m^*(\tilde{T}(-\epsilon) + \sigma \tilde{V}(-\epsilon))}{\tilde{T}(-\epsilon) + \sigma \tilde{V}(-\epsilon)} \frac{d\omega^1}{\omega^1} d\omega^2 ... d\omega^n \delta_{-\epsilon}, w \right\rangle.$$

One has $\frac{\partial S}{\partial T'} \equiv 0$ and:

$$\frac{\partial S}{\partial T} = \int_H \frac{\frac{\partial m^*}{\partial \omega^1}(\tilde{T}) \times \tilde{T} - m^*(\tilde{T})}{\tilde{T}^2} \frac{d\omega^1}{\omega^1} ... d\omega^n,$$

and

$$DS(\tilde{T}, \tilde{T}')(U, W) = \frac{\partial S}{\partial T} . U + 0.W \equiv \frac{\partial S}{\partial T} . U.$$

One can then write:

$$\left\langle S \left( (\tilde{T}(+\epsilon) + \sigma \tilde{V}(+\epsilon), \tilde{T}'(+\epsilon) + \sigma \tilde{V}'(+\epsilon)) \right) \delta_{+\epsilon}, w \right\rangle$$

$$= \left\langle \left\{ S(\tilde{T}(\epsilon), \tilde{T}'(\epsilon)) + \sigma DS(\tilde{T}(\epsilon), \tilde{T}'(\epsilon))(\tilde{V}(\epsilon), \tilde{V}'(\epsilon)) \right\} \delta_{+\epsilon}, w \right\rangle + \langle \sigma O(\sigma), w \rangle,$$

The same holds substituting $+\epsilon$ by $-\epsilon$. One then deduces that:

$$\left\langle J'(\tilde{T} + \sigma \tilde{V}), w \right\rangle - \left\langle J'(\tilde{T}), w \right\rangle$$

$$= \left\langle \left\{ DS(\tilde{T}(\epsilon), \tilde{T}'(\epsilon))(\tilde{V}(\epsilon), \tilde{V}'(\epsilon)) \right\} \delta_{+\epsilon}, w \right\rangle -$$

$$\left\langle \left\{ DS(\tilde{T}(-\epsilon), \tilde{T}'(-\epsilon))(\tilde{V}(-\epsilon), \tilde{V}'(-\epsilon)) \right\} \delta_{-\epsilon}, w \right\rangle + \langle O(\sigma), w \rangle,$$

and

$$\left\langle J''(\tilde{T})\tilde{V}, w \right\rangle = \left\langle \left\{ \frac{\partial S}{\partial T}(\tilde{T}(+\epsilon), \tilde{T}'(+\epsilon)) \cdot \tilde{V}(+\epsilon) \right\} \delta_{+\epsilon}, w \right\rangle$$

$$\left\langle \left\{ \frac{\partial S}{\partial T}(\tilde{T}(-\epsilon), \tilde{T}'(-\epsilon)) \cdot \tilde{V}(-\epsilon) \right\} \delta_{-\epsilon}, w \right\rangle;$$
using the expression of $\frac{\partial S}{\partial t}$ above, one has finally:

$$\left\langle J''(\widetilde{T})\widetilde{V}, w\right\rangle =$$

$$\left\langle \int_{\mathcal{H}} \frac{\partial m^*}{\partial \omega^0}(\widetilde{T}(+\epsilon)).\widetilde{T}(+\epsilon) - m^*(\widetilde{T}(+\epsilon)) \frac{d\omega^1}{\omega^1} d\omega \widetilde{V}(+\epsilon)\delta_{+\epsilon}, w\right\rangle$$

$$- \left\langle \int_{\mathcal{H}} \frac{\partial m^*}{\partial \omega^0}(\widetilde{T}(-\epsilon)).\widetilde{T}(-\epsilon) - m^*(\widetilde{T}(-\epsilon)) \frac{d\omega^1}{\omega^1} d\omega \widetilde{V}(-\epsilon)\delta_{-\epsilon}, w\right\rangle.$$  (86)

E. On a second order condition for a minimum

As a result of the expression of $\left\langle J''(\widetilde{T})\widetilde{V}, w\right\rangle$ above, it follows that:

$$\left\langle J''(\widetilde{T})\widetilde{V}, \widetilde{V}\right\rangle =$$

$$\int_{\mathcal{H}} \frac{\partial m^*}{\partial \omega^0}(\widetilde{T}(+\epsilon)).\widetilde{T}(+\epsilon) - m^*(\widetilde{T}(+\epsilon)) \frac{d\omega^1}{\omega^1} d\omega \widetilde{V}(+\epsilon)^2$$

$$- \int_{\mathcal{H}} \frac{\partial m^*}{\partial \omega^0}(\widetilde{T}(-\epsilon)).\widetilde{T}(-\epsilon) - m^*(\widetilde{T}(-\epsilon)) \frac{d\omega^1}{\omega^1} d\omega \widetilde{V}(-\epsilon)^2.$$  (87)

Since $\widetilde{V}(+\epsilon)$ in any case is required to satisfy $\widetilde{V}(+\epsilon) = 0$, it follows that the consistent term is

$$\left\langle J''(\widetilde{T})\widetilde{V}, \widetilde{V}\right\rangle = - \int_{\mathcal{H}} \frac{\partial m^*}{\partial \omega^0}(\widetilde{T}(-\epsilon)).\widetilde{T}(-\epsilon) - m^*(\widetilde{T}(-\epsilon)) \frac{d\omega^1}{\omega^1} d\omega \widetilde{V}(-\epsilon)^2.$$  (87)

Now one wants to know if

$$\left\langle J''(\widetilde{T}), \widetilde{V}, \widetilde{V}\right\rangle \geq 0,$$

independently of the value of $\widetilde{V}(-\epsilon)$. It appears therefore necessary to study the term

$$A(\widetilde{T}(s), \omega^i) = \frac{\partial m^*}{\partial \omega^0}(\widetilde{T}(s)(\omega^i), \omega^i).\widetilde{T}(s)(\omega^i) - m^*(\widetilde{T}(s)(\omega^i), \omega^i)$$

since $\left\langle J''(\widetilde{T})\widetilde{V}, \widetilde{V}\right\rangle \geq 0$ if

$$A(\widetilde{T}(-\epsilon), \omega^i) \leq 0.$$  (88)

We proceed to an analysis of the function $A(\omega^0, \omega^i) = \omega^0 \frac{\partial m^*}{\partial \omega^0}(\omega^0, \omega^i) - m^*(\omega^0, \omega^i)$. First of all, we have that

$$\omega^0 \frac{\partial m^*}{\partial \omega^0} - m^* = \omega^1(\omega^0)^2 \frac{\partial m}{\partial \omega^0}.$$  (88)

The sign of $A(\omega^0, \omega^i)$ is then the one of $\frac{\partial m}{\partial \omega^0}$ whose expression is given by the relation (87) above. It results the following proposition.
Proposition V.1 The function 

\[-A(\omega^0, \omega^i) = m^* - \omega^0 \frac{\partial m^*}{\partial \omega^0}\]

is non negative if the extrinsic curvature $K$ of the spacetime satisfies the inequalities

\[tr K < 0, \ K^{ij} \frac{\partial \omega^1}{\partial x^i} \frac{\partial \omega^1}{\partial x^j} \geq 0.\]  (89)

\[\text{Proof}\]

The proof is an obvious consequence of the sign of 

\[-A(\omega^\alpha) \equiv -\omega^1 \omega^0 \frac{\partial m^*}{\partial \omega^0}.\]

This proposition induces that independently of the value of $\tilde{V}(\epsilon)$, $\langle J^* (\tilde{T}) \tilde{V} \rangle \geq 0$ under the hypotheses (89).

F. Theorem – Corollary

Theorem V.1 One supposes that a gauge $(\omega^\alpha)$ is specified as in theorem IV.2 for a spacetime $(V, g)$, $\mathcal{H} = \{ m \equiv G(\frac{d\omega^0}{\Omega \omega^0}, \frac{d\omega^0}{\Omega \omega^0}) = 0 \} \neq \emptyset$ and defines a (smooth) hypersurface, and the attached geometric objects (in $(\tilde{V}, h)$) $U, V$, and quantities $a, b^\pm$ as described in (52)-(54) satisfy:

\[h \left( \nabla \frac{\partial \omega^0}{\partial \omega^0}, \nabla \omega^0 \right) < 0, \ h \left( \nabla \frac{\partial \omega^0}{\partial \omega^0}, \Omega \omega^0 \right) > 0;\]

\[tr V < 0, \ tr U \in \left[ -\frac{tr V}{b}, -\frac{tr V}{b^+} \right].\]  (90)

One supposes further that for a given smooth function $P_0$ of three real variables, the fundamental equation (82) (with $P$ described in (68))

\[\frac{\partial P_0}{\partial \tilde{T}} - \frac{d}{ds} \left( \frac{\partial P_0}{\partial \tilde{T}'} \right) + \left[ \frac{1}{\lambda} \frac{\partial P}{\partial \tilde{T}'}(\tilde{T}(-\epsilon), \tilde{T}'(-\epsilon)) + \frac{\partial P_0}{\partial \tilde{T}'}(\tilde{T}(+\epsilon), \tilde{T}'(+\epsilon)) \right] \delta_{+\epsilon} - \left[ \frac{1}{\lambda} \frac{\partial P}{\partial \tilde{T}'}(\tilde{T}(\epsilon), \tilde{T}'(\epsilon)) + \frac{\partial P_0}{\partial \tilde{T}'}(\tilde{T}(-\epsilon), \tilde{T}'(-\epsilon)) \right] \delta_{-\epsilon} = 0\]  (92)

admits a solution $\tilde{T}_*(\cdot, \lambda)$ with $\lambda$ satisfying $I(\tilde{T}_*) = K$.

Then the total mass $\mathfrak{M}$ corresponding to the mass function $m = G(\frac{d\omega^0}{\partial \omega^0}, \frac{d\omega^0}{\partial \omega^0}) \equiv h(u \frac{\tilde{\omega}^0}{u \frac{\tilde{\omega}^0}{\omega^0}}, \tilde{\omega}^0)$(with $G$ the inverse of $h \equiv \Omega^2 g$) admits a lower bound.

\[\text{Proof}\]

The proof is already done since $\tilde{T}_*(\cdot, \lambda)$ is an extremal of $J(\tilde{T})$ and the assumptions in the theorem imply that $\langle J^* (\tilde{T}) \tilde{V} \rangle \geq 0$ as established above.
Corollary V.1 One supposes that a gauge \((\omega^\alpha)\) is specified as above for a spacetime \((V, g)\), and the extrinsic curvature \(K\) of the spacetime together with the attached geometric objects \((\hat{V}, h)\) \(U, V\), and quantities \(a, b^\pm\) as described in (52)-(54) satisfy:

\[
trK < 0, \ K_{ij} \frac{\partial \omega^1}{\partial x^i} \frac{\partial \omega^1}{\partial x^j} \geq 0, \ h \left( \nabla_{\omega}^{\omega_0} \omega_0^0, \ n_0^0 \right) > 0; \tag{93}
\]

\[
tr \ V < 0, \ tr \ U \in \left[ -\frac{tr \ V}{b^+}, -\frac{tr \ V}{b^-} \right]. \tag{94}
\]

One supposes further that for a given smooth function \(P_0\) of three real variables, the equation (92) admits a solution \(\tilde{T}_*\) with \(\lambda\) satisfying \(I(\tilde{T}_*) = K\). Then total mass \(M\) corresponding to the mass function \(m = G \left( \frac{d\omega^0}{\omega^0}, \frac{d\omega^0}{\omega^0} \right) \equiv h (\nabla_{\omega}^{\omega_0}, \nabla_{\omega}^{\omega_0})\) admits a lower bound.

VI. CONCLUSION AND OUTLOOK

There are many theorems and conjectures about black holes difficult to prove and even often to state in a precise way\(^{11, 17, 24, 31, 36, 37, 43, 46, 72, 77, 92}\). In this paper, we have proposed a framework with the hope that some aspects of issues of singularities and black holes might have a chance to be understood though the relation between the current approach and standard ones should be examined carefully. Indeed, considering the crucial and central question of mass or quasi-local mass in general relativity, we have proposed a new mass function and for this mass we have established a possibility of the existence of a lower bound for the corresponding ”total mass” for a system collapsing to a black hole thanks to some hypotheses on the extrinsic curvature of the spacetime. Such inequality in general is likely to guarantee the stability of isolated systems, furthermore, it can help as a tool in the analysis of partial differential equations (energy estimates,...) in black holes’s geometry. Concerning the question of existence or the nature of horizons, some information were gained through the analysis above, however, a profound analysis of the relation between the horizon as described here with the standard definition or the one attributed to Hayward Sean A.\(^{42}\) would be interesting, probably in relation with the detection of horizons using curvature invariants. For the question of singularities and related problems (the fate of the Cauchy horizons when they exist, cosmic censorship conjectures,...\(^{2, 3, 10, 16, 22, 27, 48}\)), it should be analyzed according to the approach here in a subsequent work starting from standard black holes. Importantly,
the characterization of trapped surfaces in this setting or analysis of the topology of the obtained black hole in general should be done. The framework of this paper offers also new possibilities for the analysis of waves on fixed backgrounds, conformal scattering is one such possibilities. Let’s recall that the choice of an appropriate gauge is at the heart of measurement of decays of waves and of the stability problem. On the other hand, the conditions that guarantee the existence of black holes’s region in this context should be compared if possible to other energy conditions involving the energy momentum tensor.

All the results here are based on a mass function $m$ whose relations with other mass-functions if established would provide new insights to the understanding of implications of general relativity. This mass-function is not always positive, but the positivity of the "total mass" of the system is obtainable provided some conditions are given on the isoperimetric constraint; this offers really many perspectives as one can appreciate through the example. However, it is worth noting that there are unified theory projects in the literature based on Einstein-Yang-Mills-Dirac equations where the system has negative energy and hence does not satisfy the positivity conditions in the Penrose-Hawking singularity theorem.

The results obtained here assume the existence of a global in time space-time, it is essential to study the global solvability of the wave equations and quasi-linear wave equations (Einstein equations) together with the analysis of the properties of solutions and their decays. As illustration, there is a well known great interest to enquire what happens to solutions of waves equations when Cauchy horizons occur. Novel approaches to the global study of nonlinear hyperbolic equations based on microlocal analysis have been proved successful in the case of cosmological black holes. At least the approach of this paper is based on the conformal embedding of the physical spacetime into a manifold with corners, this permits to envisage using of methods of microlocal analysis of Hintz and Vasy for various problems under consideration.

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CONFLICT OF INTEREST

The author declares that he has no conflict of interest.

REFERENCES

1 Alexakis, S.: The Penrose inequality on perturbations of the Schwarzschild exterior. arXiv:1506.06400 (2015)
2 Aretakis, S.: Stability and instability of extreme Reissner-Norström Black holes for linear scalar perturbations I. Commun. Math. Phys. 307(1), 17-63 (2011)
3 Aretakis, S.: Stability and instability of extreme Reissner-Norström Black holes for linear scalar perturbations II. Ann. Henri Poincaré, 12(8), 1491-1508 (2011)
4 Bieri, L., Zipser, N.: Extensions of the stability theorem of the Minkowski space in general relativity, AMS/IP Studies in Adv. Math. 45, (2009)
5 Bray, H. L.: Proof of the Riemannian Penrose inequality using the positive mass theorem. J. Differ. Geom. 59, 177-267 (2001)
6 Bray, H. L.: Black holes, geometric flows, and the Penrose inequality in general relativity. Notices of the AMS. 49(11), 1372-1381 (2002)
7 Bray, H. L., Chruściel, P. T.: The Penrose inequality. Arxiv: 0312047v2 [gr-qc](2004)
8 Bray, H. L., Lee, D. A.: On the riemannian Penrose inequality in dimensions less than eight. Duke Math. J. 148(1), 81-106 (2009)
9 Bray, H. L., Roesch, H. P.: Null geometry and the Penrose conjecture. Arxiv: 1708.00941V1 [gr-qc] (2017)
10 Cardoso V., Costa, J. L., Destounis, K., Hintz, P., Jansen, A.: Quasinormal modes and strong cosmic consorship. Phys. Rev. Lett. 120, 031103 (2018)
11 Choquet-Bruhat, Y.: General relativity and the Einstein equations, Oxford Mathematical Monographs, Oxford University Press, Oxford. (2009)

12 Christodoulou, D.: A mathematical theory of gravitational collapse, Commun. Math. Phys. 109, 613-647 (1987)

13 Christodoulou, D.: Examples of naked singularity formation in the gravitational collapse of a scalar field. Ann. Math. 140, 607-653 (1994)

14 Christodoulou, D.: The formation of black holes and singularities in spherically symmetric gravitational collapse. Commun. Pure Appl. Math. 44(3), 339-373 (1991)

15 Christodoulou, D.: On the global initial value problem and the issue of singularities. Class. Quantum Grav. 16, A23- A35 (1999)

16 Christodoulou, D.: The instability of naked singularities in the gravitational collapse of a scalar field. Ann. Math. 149, 183-217 (1999)

17 Christodoulou, D.: The Formation of Black Holes in General Relativity. ArXiv:[gr-qc] 0805.3880, (2008) x+589., MR MR2488976 (2009k:83010).

18 Christodoulou, D., Klainerman, S.: The global nonlinear stability of the Minkowski space. Princeton Math. Ser. 41, (1993)

19 Chruściel, P. T.: Conformal boundary extensions of Lorentzian manifolds. ArXiv:[gr-qc] 0606101 v1, (2006)

20 Chruściel, P. T., Jezierski, J., Leski, S.: The Trautman-Bondi mass of hyperboloidal initial data sets. Adv. Theor. Math. Phys. 8, 83-139 (2004)

21 Chruściel, P. T. and Paetz, T. T.: The mass of the light cones: Class. Quantum Grav. 31 102001 (2014)

22 Dafermos, M.: Stability and instability of the Cauchy horizon for the spherically symmetric Einstein-Maxwell scalar field equations. Ann. Math. 158, 875-928 (2003)

23 Dafermos, M.: Stability and instability of the Reissner-Norsøm Cauchy horizon and the problem of uniqueness in general relativity. Contemp. Math. 350, 99-113 (2004)

24 Dafermos, M.: The interior of a charged black hole and the problem of uniqueness in general relativity. Commun. Pure Appl. Math. LVIII, 0445-0504 (2005)

25 Dafermos, M.: On naked singularities and the collapse of self-gravitating Higgs fields. Adv. Theor. Math. Phys. 9(4), 575-591 (2005)

26 Dafermos, M.: Spherically symmetric spacetimes with a trapped surface. Class. Quantum Grav. 22, 2221-2232 (2005)

33
27 Dafermos, M., Holzegel, G.: On the nonlinear stability of higher dimensional triaxial Bianchi IX black holes. Adv. Theor. Math. Phys. **10**, 503-523 (2006)
28 Dafermos, M., Rendall, A.: An extension principle for the Einstein-Vlasov system in spherical symmetry. Ann. Henri Poincaré **6**, 1137-1155 (2006)
29 Dafermos, M., Rendall, A.: Inextendibility for expanding cosmological models with symmetry. Class. Quantum Grav. **22**, L143-L147 (2005)
30 Dafermos, M., Rendall, A.: Strong cosmic censorship for surface-symmetric cosmological spacetimes with collisionless matter. Arxiv. 0701034v1 [gr-qc] (2007)
31 Dafermos, M., Rodniansky, I.: A proof of Price’s law for the collapse of a self-gravitating scalar field. Invent. Math. **162**, 381-457 (2005)
32 Dafermos, M., Rodniansky, I.: Lectures on black holes and linear waves. Evolution equations, Clay Mathematics proceedings **17**, 97-205 (2008)
33 Dafermos, M., Rodniansky, I.: The red-shift effect and radiation decay on black hole spacetimes. Commun. Pure Appl. Math. **62**(7), 859-919 (2009)
34 Dafermos, M., Rodniansky, I.: Decay for solutions of the wave equation on Kerr exterior spacetimes I-II: The cases $|a| \ll M$ or axisymmetry. arXiv:1010.5132 (2010)
35 Dafermos, M., Rodniansky, I.: A proof of the uniform boundedness of solutions to the wave equation on slowly rotating Kerr backgrounds. Invent. Math. **185**(3), 467-559 (2011)
36 Dafermos, M., Holzegel, G., Rodniansky, I.: The linear stability of the Schwarzschild solutions to gravitational perturbations. Arxiv. 1601. 06467v1 [gr-qc] (2016)
37 Dafermos, M., Rodniansky, I., Shlapentokh-Rothman, Y.: Decay for solutions of the wave equation on Kerr exterior spacetimes III: The full subextremal case $|a| < M$. Annal. of Math. **183**(3), 787-913 (2016)
38 Dossa, M., Patenou, J. B.: Cauchy problem on two characteristic hypersurfaces for the Einstein-Vlasov-Scalar field equations in temporal gauge. C.R. Math. Rep. Acad. Sci. Canada **39**(2), 45-59 (2017)
39 Dyatlov , S.: Asymptotics of linear waves and resonances with applications to black holes. Commun. Math. Phys. **335**(3), 1445-1485 (2015)
40 Frauendiner, J.: On the Penrose inequality. Phys. Rev. Lett. **87**(10), 101101 (2001)
41 Frieldander, F. G.: The wave equation on a curved space time, Cambridge Univ. Press. (1975)
42 Hayward, S. A.: General laws of black-hole dynamics. Phys. Rev. D **49**, 6467 (1994)
43 Hawking, S. W., Ellis, G. F. R.: The large scale structure of space-time. Cambridge Univ. Press, Cambridge. (1973)

44 Hintz, P.: Global analysis of linear and nonlinear waves equations on cosmological spacetimes. PhD thesis, Stanford University. (2015)

45 Hintz, P.: Boundedness and decay of scalar waves at the Cauchy horizon of the Kerr spacetime. Arxiv: [math.AP] 1512.08003v1 (2015)

46 Hintz, P., Vazy, A.: A global analysis proof of the stability of Minkowski space and the polyhomogeneity of the metric. arXiv:1711.00195v1 [math.AP] (2017).

47 Hintz, P., Vazy, A.: The global non-linear stability of the Kerr-de Sitter family of black holes. Acta Math., 220, 1-206 (2018).

48 Hod, S.: Quasinormal modes and strong cosmic censorship in near-extremal Kerr-Newman-de Sitter black hole spacetimes. Arxiv: 1803.05443 [gr-qc](2018)

49 Huisken, G., Ilmanen, T.: The inverse mean curvature flow and the riemannian Penrose inequality. J. Differ. Geom. 59(3), 353-457 (2001)

50 Jang, P. S., Wald, R. M.: The positive energy conjecture and the cosmic censor hypothesis. J. Math. Phys. 18(1), 41-44 (1977)

51 Kay, B. S., Wald, R. M.: Linear stability of Schwarzschild under perturbations which are non-vanishing on the bifurcation 2-sphere. Class. Quantum Grav. 4(4), 893-898 (1987)

52 Klainerman, S., Nicolò, F.: On local and global aspects of the Cauchy problem in General Relativity. Class. Quantum Grav. 16, R73-R157 (1999)

53 Klainerman, S., Nicolò, F.: The evolution Problem in General Relativity. Progress in Mathematical physics, Birkhauser. (2003)

54 Klainerman, S., Szeftel, J.: Global Nonlinear Stability of Schwarzschild spacetime under Polarized conditions. arXiv:1711.07597v2 [gr-qc], (2018)

55 Klainerman, S., Szeftel, J.: Construction of GCM spheres in perturbations of Kerr. arXiv:1911.00697v1 [math.AP], (2019)

56 Komekini, J.: The global structure of spherically symmetric charged scalar field spacetimes. Commun. Math. Phys. 323, 35-106 (2013)

57 LeFloch, P. G., Ma, Y.: The global nonlinear stability of Minkowski space for the Einstein equations in presence of a massive field. C. R. Acad. Sci. Paris, Ser. I. 354, 948-953 (2016)

58 Lindblad, H., Rodnianski, I.: The global stability of Minkowski spacetime in harmonic gauge, Ann. Math. 171, 1401–1477 (2010)
Liu, C.-C. M., Yau, S.-T.: Positivity of the quasilocal mass II. J. Amer. Math. Soc. 19(1), 181-204 (2006)

Lu, W., Kumar, P., Narayan, R.: Stellar disruption events support the existence of the black hole event horizon. MNRSA 468(1), 910-919 (2017)

Ludvigsen, M., Vickers, J.: An inequality relating total mass and the area of a trapped surface in general relativity. J. Phys. A: Math. Gen. 16(14), 3349-53 (1983)

Luk, J., Oh, S.-J.: Proof of linear instability of the Reissner-Norström Cauchy horizon under scalar perturbations. Duke. Math. J. 166(3), 437-493 (2017)

Luk, J., Sbierski, J.: Instability results for the wave equation in the interior of Kerr black holes. J. Funct. Anal. 271(7), 1948-1995 (2016)

Mars, M., Soria, A.: On the Penrose inequality along null hypersurfaces. Class. Quantum Grav. 33 (11):115019 (2016)

Melrose R. B.: Differential analysis on manifolds with corners. Book, in preparation, available online, 1996

Melrose R. B., Barreto, S. A., Vazy, A.: Asymptotics of solutions of the wave equation on de Sitter-Schwarzschild space. Commun. Part. Diff. Eq. 39(3), 512-529 (2014)

Patenou, J. B.: Cauchy problem on a characteristic cone for the Einstein-Vlasov system: (I) The initial data constraints. C.R. Acad. Sci. Paris, Ser. I. 355, 187-192 (2017)

Patenou, J. B.: On the Initial Data Constraints on the Light-cone for the Einstein-Vlasov system. ArXiv: 2062316 [math.AP] (2017)

Patenou, J. B.: Conformal Scale Geometry of Spacetime. [arXiv:1811.05787v1 [math-AP], (2018)

Penrose, R.: Asymptotic properties of fields and spacetimes. Phys. Rev. Lett. 10(2), 66-68 (1963)

Penrose, R.: Naked singularities. Annals New YorK Acad. Sci. 224(1), 125-134 (1973)

Penrose, R.: The question of cosmic censorship. J. Astrophys. Astron. 20, 233-248 (1999)

Penrose, R.: Gravitational collapse: The role of general relativity. Gen. Relat. Gravit. 34(7), 1141-1165 (2002)

Penrose, R.: Republication of: Conformal treatment of infinity. Gen. relativ. Grav. 43, 901-922 (2011)

Poisson, E., Israel, W.: Internal structure of black holes. Phys. Rev. D. 3(6), 1796-1809 (1990)
76 Ringström, H.: Future stability of the Einstein-non-linear scalar field system. Invent. Math. 173 (1), 123-208 (2008)
77 Ringström, H.: On the topology and future stability of the universe. Oxford Mathematical Monographs. (2013)
78 Roberts, M. D.: Scalar field counterexamples to cosmic censorship hypothesis. Gen. Rel. Grav. 21(9), 907-939 (1989)
79 Roesch, H. P.: Proof of a null Penrose conjecture using a new quasi-local mass. PhD thesis, Duke University. (2017)
80 Schoen, R., Yau, S.-T.: On the proof of the positive mass conjecture in general relativity. Commun. Math. Phys. 65(1), 45-76 (1979)
81 Schoen, R., Yau, S.-T.: Proof of the positive mass theorem. ii. Commun. Math. Phys. 79(2) 231-260 (1981)
82 Schoen, R., Yau, S.-T.: Proof that the Bondi mass is positive. Phys. Rev. Lett. 48, 369 (1982)
83 Schoen, R., Yau, S.-T.: The existence of black hole due to condensation of matter. Commun. Math. Phys. 90 575-579 (1983)
84 Shi, Y., Tam, L.-F.: Positive mass theorem and the boundary behaviors of compact manifolds with nonnegative scalar curvature. J. Differ. Geom. 62, 79-125 (2002)
85 Shi, Y., Tam, L.-F.: Quasi-local mass and the existence of horizons. Commun. Math. Phys. 274, 277-295 (2007)
86 Speck, J.: The global stability of the Minkowski spacetime solution to the Einstein nonlinear system in wave coordinates. Anal. PDE 7(4), 771-901 (2014)
87 Spivak, M.: A comprehensive introduction to Differential Geometry. Wilmington: Publish or Perish, Inc. (1970)
88 Van de Moortel, M.: Stability and instability of the sub-extremal Reissner-Norström Black hole interior for the Einstein-Maxwell-Klein-Gordon Equations in spherically symmetry. Commun. Math. Phys. 360, 103-168 (2018)
89 Vazy, A.: The wave equation on asymptotically de Sitter-like spaces. Adv. Math. 223(1), 49-97 (2010)
90 Vazy, A.: The wave equation on asymptotically anti de Sitter spaces. Anal. PDE 5(1), 81-144 (2012)
91 Wang, M.-T., Yau, S.-T.: A generalization of Liu-Yau’s quasi-local mass. Commun. Anal.
92 Wang, M.-T., Yau, S.-T.: Quasilocal mass in general relativity. Phys. Rev. Lett. 102(2), 021101 (2009)

93 Wang, M.-T., Yau, S.-T.: Isometric embeddings into the Minkowski space and new quasi-local mass. Commun. Math. Phys. 288, 919-942 (2009)

94 Wang, M.-T., Yau, S.-T.: Limit of quasi-local mass at spatial infinity. Commun. Math. Phys. 296, 271-283 (2010)

95 Wang, X.: Mass for asymptotically hyperbolic manifolds. J. Differ. Geom. 57, 273-299 (2001)

96 Witten, E.: A new proof of the positive energy theorem. Commun. Math. Phys. 80(3), 381-402 (1981)

97 Yodzis, P., Seifert, H.-J., Müller Zum Hagen, H.: On the occurrence of naked singularities in general relativity. Commun. Math. Phys. 34, 135-148 (1973)

98 Zhang, X. A definition of total energy-momenta and positive mass theorem on asymptotically hyperbolic 3 manifolds I. Commun. Math. Phys. 249, 529-548 (2004)