Zeros of derivatives of strictly non-real meromorphic functions

J.K. Langley
February 20, 2020

Abstract
A number of results are proved concerning the existence of non-real zeros of derivatives of strictly non-real meromorphic functions in the plane. MSC 2000: 30D35.

1 Introduction
Let \( f \) be a meromorphic function in the plane and let \( \tilde{f}(z) = \overline{f(\overline{z})} \) (this notation will be used throughout). Here \( f \) is called real if \( \tilde{f} = f \), and strictly non-real if \( \tilde{f} \) is not a constant multiple of \( f \). There has been substantial research concerning non-real zeros of derivatives of real entire or real meromorphic functions \([1, 2, 4, 13, 14, 19, 22, 27, 28]\), but somewhat less in the strictly non-real case. The following theorem was proved in \([12]\).

**Theorem 1.1 ([12])** Let \( f \) be a strictly non-real meromorphic function in the plane with only real poles. Then \( f \), \( f' \) and \( f'' \) have only real zeros if and only if \( f \) has one of the following forms:

- \((I)\) \( f(z) = Ae^{Bz} \);
- \((II)\) \( f(z) = A(e^{i(cz+d)} - 1) \);
- \((III)\) \( f(z) = A \exp(\exp(i(cz+d))) \);
- \((IV)\) \( f(z) = A \exp\left[K(i(cz+d) - \exp(i(cz+d)))\right] \);
- \((V)\) \( f(z) = \frac{A \exp[-2i(cz+d) - 2 \exp(2i(cz+d))]\sin^2(cz+d)}{\sin^2(cz+d)} \);
- \((VI)\) \( f(z) = \frac{A}{e^{i(cz+d)} - 1} \).

Here \( A, B \in \mathbb{C} \), while \( c, d \) and \( K \) are real with \( cB \neq 0 \) and \( K \leq -1/4 \).

The first aim of the present paper is to prove a result in the spirit of Theorem 1.1, but with no assumption on the location of poles. In \([15, 16, 17]\) Hinkkanen determined all meromorphic functions \( f \) in the plane such that \( f \) and all its derivatives have only real zeros, using the fact that under these hypotheses \( f \) has at most two distinct poles, by the Pólya shire theorem \([10, \text{Theorem 3.6}]\). For strictly non-real functions, the following two theorems will be proved.
Theorem 1.2 Let $f$ be a strictly non-real meromorphic function in the plane such that all but finitely many zeros of $f^{(m)}$ are real for $m = 0, \ldots, 12$. Then either $f'/f$ is a rational function or

$$f(z) = B \left(1 - T(z)e^{iAz}\right), \quad A \in \mathbb{R}, \ B \in \mathbb{C}, \ AB \neq 0,$$

where $T$ is a rational function with $|T(x)| = 1$ for all $x \in \mathbb{R}$.

If, in addition, $f$, $f'$, $f''$ and $f'''$ have only real zeros, then $f$ is given by one of the following, in which $a, b, c, d \in \mathbb{C}$ and $\mu \in \mathbb{Z}$:

$$(i) \quad f(z) = (az + b)^\mu; \quad (ii) \quad f(z) = \frac{az + b}{cz + d}; \quad (iii) \quad f(z) = e^{az+b} - c.$$ (2)

Theorem 1.3 Let $f$ be a strictly non-real meromorphic function in the plane such that all zeros of $f^{(m)}$ are real for $m = 0, \ldots, 9$. Then $f$ is given by (2).

It is very unlikely that Theorems 1.2 and 1.3 are sharp in terms of the number of derivatives considered, but examples (III)-(VI) of Theorem 1.1 show that the absence of non-real zeros of $f$, $f'$ and $f''$ is not enough to imply (2).

The next result concerns the determination of all meromorphic functions $f$ in the plane such that $f$ and $f''$ have only real zeros and poles (thus discarding the hypothesis in Theorem 1.1 that $f'$ has only real zeros). Such a classification is not known in the real meromorphic case, except when $f$ has finitely many poles [2, 28], or finitely many zeros [23], but for strictly non-real functions of finite lower order the problem is solved by the following theorem, in which the terminology is from [10].

Theorem 1.4 Let $f$ be a strictly non-real meromorphic function in the plane, such that all but finitely many zeros and poles of $f$ and $f''$ are real. Then $f$ satisfies, as $r \to \infty$,

$$\overline{N}(r, f) + \overline{N}(r, 1/f) = O(r) \quad \text{and} \quad T(r, f'/f) = O(r \log r).$$ (3)

If, in addition, $f$ has finite lower order and all zeros and poles of $f$ and $f''$ are real then $f$ is given by one of

$$(a) \quad f(z) = e^{A_1z+B_1},$$

$$(b) \quad f(z) = e^{i\alpha_1 z}T_1(z)\sin(a_1z+b_1) - T_1(z)\cos(a_1z+b_1),$$

$$(c) \quad f(z) = \frac{T_1(z)}{e^{2i(a_1z+b_1)}-1}.$$ (4)

in which $A_1, B_1 \in \mathbb{C}$ and $\alpha_1, b_1 \in \mathbb{R}$, while $T_1$ is a polynomial of degree at most 1 such that $T_1(z) = 0$ implies $\sin(a_1z+b_1) = 0$.

If $T_1$ is a non-zero constant in (b) or (c) of (4) then $f$ reduces to (II) or (VI) of Theorem 1.1 and $f$, $f'$ and $f''$ all have only real zeros and poles. However, $T_1$ is non-constant in both of the following examples:

$$f_1(z) = e^{iz}(\sin z - \cos z), \quad f_1'(z) = e^{iz}(z \sin z + i(\sin z - \cos z)), \quad f_1''(z) = 2ze^{2iz};$$

$$f_2(z) = \frac{z}{e^{2iz} - 1}, \quad f_2'(z) = \frac{(1 - 2iz)e^{2iz} - 1}{(e^{2iz} - 1)^2}, \quad f_2''(z) = \frac{(4i - 4z)e^{2iz} - (4i + 4z)e^{4iz}}{(e^{2iz} - 1)^3}.$$
Here \( f'_1 \) and \( f'_2 \) each have infinitely many non-real zeros, but \( f''_2(z) = 0 \) forces
\[
e^{2iz} = \frac{4i - 4z}{4i + 4z}, \quad z = \tan z,
\]
all solutions of which are real (see Lemma 2.7), as are all zeros of \( f_1 \). Furthermore, writing
\[
\frac{f'_3(z)}{f_3(z)} = 1 + \frac{1}{i + z} + ie^{iz}, \quad \frac{f''_3(z)}{f_3(z)} = \left( \frac{i - z}{i + z} \right) e^{iz} - e^{2iz},
\]
defines a strictly non-real entire function \( f_3 \) of infinite order, with one zero, for which \( f'_3 \) has infinitely many non-real zeros, while all but finitely many zeros of \( f''_3 \) are real by [24, Lemma 2.3].

The author thanks John Rossi for helpful discussions, and the referee for a very careful reading of the manuscript and several helpful suggestions.

2 Preliminaries

The following theorem is a combination of results from [7, 8, 20] and uses notation from [10].

**Theorem 2.1 ([7, 8, 20])** Let \( h \) be a non-constant meromorphic function in the plane.

(i) For \( n \geq 3 \) there exists \( c_n > 0 \), depending only on \( n \), such that
\[
T(r, \frac{h'}{h}) \leq c_n \left( \frac{N(r, 1/h)}{h} + \frac{N(r, 1/h^{(n)})}{h} \right) + O(\log r) \quad \text{as } r \to \infty.
\]

(ii) If \( n \geq 2 \) and \( h \) and \( h^{(n)} \) have finitely many zeros, then \( h'/h \) is a rational function: equivalently, \( h = Se^{Q} \) with \( S \) a rational function and \( Q \) a polynomial.

Here part (i) follows from [8, Theorem 3] (which should be stated for functions which have transcendental logarithmic derivative, rather than merely being themselves transcendental), and part (ii) was proved in [7, 20].

**Theorem 2.2 ([3])** Let \( k \geq 2 \) and let \( \mathcal{H} \) be a family of functions meromorphic on a plane domain \( D \) such that \( hh^{(k)} \) has no zeros in \( D \), for each \( h \in \mathcal{F} \). Then the family \( \{ h'/h : h \in \mathcal{H} \} \) is normal on \( D \).

**Lemma 2.1 ([25])** Let \( h \) be a transcendental meromorphic function in the plane such that \( h'/h \) has finite lower order and \( h'/h \) and \( h''/h' \) have finitely many zeros. Then \( h''/h' \) is a rational function and \( h \) has finite order and finitely many poles.

The next two lemmas involve Tsuji’s analogue [29] for the upper half-plane of Nevanlinna’s characteristic function, which was developed further by Levin and Ostrovskii [26] (see also [2, 9]). The first is directly related to Theorem 2.1(i) and was deduced in [22] from Frank’s method [7].

**Lemma 2.2 ([22, 24])** Let \( f \) be a non-constant meromorphic function in the plane which satisfies at least one of the following two conditions: (a) \( f \) and \( f'' \) have finitely many non-real zeros and poles; (b) \( f \) and \( f^{(m)} \) have finitely many non-real zeros, for some \( m \geq 3 \). Then the Tsuji characteristic \( T_0(r, f'/f) \) in the upper half-plane satisfies \( T_0(r, f'/f) = O(\log r) \) as \( r \to \infty \).
The following lemma is due to Levin and Ostrovskii [26] (see also [2, 9] and [24, Lemma 2.2]).

**Lemma 2.3 ([26])** Let $H$ be a non-constant meromorphic function in the plane. If $H$ and $G = \tilde{H}$ satisfy, as $r \to \infty$,
\[ N(r, H) = O(r \log r) \quad \text{and} \quad T_0(r, H) + T_0(r, G) = O(\log r), \]
then $T(r, H) = O(r \log r)$ as $r \to \infty$.

**Lemma 2.4** Let $0 < \varepsilon < \pi/8$, $R > 0$ and $K > 1$. Let $(h_n)$ be a sequence of meromorphic functions on the domain $\{ z \in \mathbb{C} : |z| > R, 0 < \arg z < \pi \}$, each of them such that $h_n$, $h'_n$ and $h''_n$ have no zeros there. Suppose that there exists a positive sequence $(r_n)$ such that $r_n \to \infty$ and
\[
\min \left\{ \frac{h'_n(z)}{h_n(z)} : K^{-1}r_n \leq |z| \leq Kr_n, \varepsilon \leq \arg z \leq \pi - \varepsilon \right\} \to 0 \quad (5)
\]
as $n \to \infty$. Then
\[
\max \left\{ \frac{h'_n(z)}{h_n(z)} : K^{-1}r_n \leq |z| \leq Kr_n, \varepsilon \leq \arg z \leq \pi - \varepsilon \right\} \to 0
\]
as $n \to \infty$.

**Proof.** For $q = 1, 2$ let
\[
D_q = \{ z \in \mathbb{C} : K^{-q} < |z| < K^q, \varepsilon/q < \arg z < \pi - \varepsilon/q \}
\]
and let $E_1$ be the closure of $D_1$. Let $n_0 \in \mathbb{N}$ be large. By Theorem 2.2 the functions $p_n(z) = r_nh'_n(r_nz)/h_n(r_nz)$, $n \geq n_0$, form a normal family of zero-free meromorphic functions on $D_2$. Assuming that the assertion of the lemma is false gives, after passing to a subsequence if necessary,
\[
\liminf_{n \to \infty} (\sup \{ |p_n(z)| : z \in E_1 \}) > 0. \quad (6)
\]
On the other hand (5) implies that there exist $u_n \in E_1$ with $\lim_{n \to \infty} p_n(u_n) = 0$. After taking a further subsequence, if necessary, it may be assumed that, as $n \to \infty$, the points $u_n$ converge to some $u^* \in E_1 \subseteq D_2$ and the functions $p_n$ converge locally uniformly on $D_2$ to some $p$ with $p(u^*) = 0$. Thus $p$ is meromorphic on $D_2$ and $p \equiv 0$ by Hurwitz’ theorem, which contradicts (6).
\[ \square \]

**Lemma 2.5** Let $0 < \varepsilon < \pi/8$ and $K > 4$, and let $q$ be a positive integer. Then there exists $C_1 > 0$ with the following property.

Let $R \geq 1$ and let the function $h'$ be meromorphic on $D_R = \{ z \in \mathbb{C} : |z| > R, \Im z > 0 \}$, and assume that $h^{(q)}(z) \neq 0$ on $D_R$, and that $h'$ has no zeros in $D_R$. Let $(r_n)$, $(\rho_n)$ and $(S_n)$ be positive sequences such that $\lim_{n \to \infty} r_n = \infty$ and $\lim_{n \to \infty} \rho_n = 0$. For each $n$, assume that
\[
\max \left\{ \left| \frac{h^{(q+1)}(z)}{h^{(q)}(z)} \right| : z \in \Omega_n \right\} \leq \rho_n, \quad (7)
\]
where $\Omega_n = \{z \in \mathbb{C} : K^{-1}r_n \leq |z| \leq Kr_n, \varepsilon \leq \arg z \leq \pi - \varepsilon\}$, and that there exists $z_n$ with
\[
\frac{r_n}{2} \leq |z_n| \leq 2r_n, \quad 2\varepsilon \leq \arg z_n \leq \pi - 2\varepsilon, \quad \left| z_n \frac{h'(z_n)}{h(z_n)} \right| < S_n. \tag{8}
\]

Then, for all sufficiently large $n$, the set
\[
\left\{ \theta \in [\varepsilon, \pi - \varepsilon] : t_n \frac{h'(t_n e^{i\theta})}{h(t_n e^{i\theta})} < C_1 S_n \right\}, \quad t_n = K^{-1}r_n,
\]
has linear measure at least $\pi/2$.

**Proof.** By (7) there exists $c = c(n) \in \mathbb{C}$ such that integrating from $iKr_n$ to $z \in \Omega_n$ gives
\[
\log h^{(q)}(z) = c + o(1), \quad h^{(q)}(z) = e^c \left( 1 + \frac{\delta_q(z)}{q!} \right), \quad \delta_q(z) = o(1).
\]

It may be assumed that $e^c = q!$, since $h'/h$ and $h^{(q+1)}/h^{(q)}$ are unchanged if $h$ is replaced on $\Omega_n$ by $q! e^{-c(n)} h$. Thus repeated integration gives a monic polynomial $P = P_q, n$, of degree $q$, with the property that, for $j = 0, \ldots, q$ and for all $z$ in $\Omega_n$,
\[
h^{(j)}(z) = P^{(j)}(z) + \delta_j(z), \quad \delta_j(z) = o(|z|^{q-j}), \quad \delta'_j(z) = \delta_{j+1}(z) = o(|z|^{q-j-1});
\]
here all these estimates hold as $n \to \infty$, uniformly on $\Omega_n$, and the last estimate for $j = q$ follows from (7).

Let $n$ be large and denote by $c_j$ positive constants which are independent of $n$. Then (8) delivers a small $c_0$ such that the disc $|z - z_n| < c_0 r_n$ lies in $D_R$, and since $h'$ has no zeros in $D_R$ it follows from the minimum principle that
\[
\min \left\{ \left. \frac{w' (w)}{h'(w)} \right| : |w - z_n| = s \right\} \leq S_n \quad \text{for all} \quad s \in \left[c_0 r_n/4, c_0 r_n/2\right]. \tag{9}
\]

Now let $\{B_j\} = \{B_{j,n}\}$ denote the collection of all zeros of $P$ and $P'$. Let $Q_n$ be the closed set obtained by deleting from $\Omega_n$ the open discs $E_j$ of centre $B_j$ and radius $c_1 r_n$, where $c_1$ is assumed to be small. Then $z \in Q_n$ gives $|z - B_j| > c_2 |z|$ for every $j$, and hence
\[
|P(z)| > c_3 |z|^q, \quad |P'(z)| > c_3 |z|^{q-1}, \quad \left| \frac{P'(z)}{P(z)} \right| + \left| \frac{P''(z)}{P'(z)} \right| < \frac{c_4}{|z|}.
\]

For $z \in Q_n$ it follows that $\phi = h'/h$ satisfies
\[
\left| \frac{\phi'(z)}{\phi(z)} \right| = \left| \frac{P''(z) + \delta_2(z)}{P'(z) + \delta_1(z)} - \frac{P'(z) + \delta_1(z)}{P(z) + \delta_0(z)} \right| = \left| \frac{P''(z)}{P'(z)(1 + o(1))} - \frac{P'(z)}{P(z)(1 + o(1))} + \frac{o(1)}{|z|} \right| \leq \frac{c_5}{|z|}. \tag{10}
\]

Provided $c_1$ was chosen small enough, the following exist: a real number $s_n \in [c_0 r_n/4, c_0 r_n/2]$ such that the circle $|z - z_n| = s_n$ meets none of the discs $E_j$; a real number $u_n \in [r_n/3, 3r_n]$ such that the circle $|z| = u_n$ meets $|z - z_n| = s_n$ but none of the $E_j$; a set $T_n \subseteq [\varepsilon, \pi - \varepsilon]$, of
linear measure at least $\pi/2$, such that for $\theta \in T_n$ the line segment given by $K^{-1}r_n \leq |z| \leq Kr_n$, $\arg z = \theta$, lies in $Q_n$. Using (9), choose $w_n$ with

$$|w_n - z_n| = s_n, \quad |w_n h'(w_n)| \leq S_n.$$  

For $v = t_n e^{i\theta} = K^{-1}r_n e^{i\theta}$ with $\theta \in T_n$, there exists a path $\Gamma_v \subseteq Q_n$, joining $v$ to $w_n$, which consists of part of the ray $\arg z = \theta$ and arcs of the circles $|z| = u_n$ and $|z - z_n| = s_n$. The path $\Gamma_v$ has length at most $c_6 r_n$, and so integrating $\phi'/\phi$ along $\Gamma_v$ gives, using (10),

$$\left| \frac{h'(v)}{h(v)} \right| = |v \phi(v)| < c_7 |v \phi(w_n)| = c_7 \left| \frac{h'(w_n)}{h(w_n)} \right| < c_8 \left| w_n \frac{h'(w_n)}{h(w_n)} \right| \leq c_8 S_n.$$  

\[ \square \]

**Lemma 2.6** Let $B \in \mathbb{C}$ with $|B| = 1$ and $L \in \mathbb{Z} \setminus \{-1\}$, and let

$$F(z) = (e^z - 1)^L(e^z - B). \tag{11}$$

If all zeros of $F''$ lie on $i\mathbb{R} = \{ix : x \in \mathbb{R}\}$ then $F$ is given by one of the following:

(i) $F(z) = e^{2z} - 1$; (ii) $F(z) = e^z - B$; (iii) $F(z) = \frac{1}{e^z - 1}$.

**Proof.** Note first that $L = 0$ gives (ii) immediately, while if $L = 1$ then $F''(z) = 4e^{2z} - (1 + B)e^z$, which has zeros off $i\mathbb{R}$ unless $B = -1$, in which case $F$ is given by (i). Assume henceforth that $L \neq 0, \pm 1$, so that $LB + 1 \neq 0$, since $|B| = 1$. Now write $X = e^z$ and

$$F(z) = P(X) = (X - 1)^L(X - B),$$

$$F''(z) = X P'(X) + X^2 P''(X)$$

$$= X(L(X - 1)^{L-1}(X - B) + (X - 1)^L) +$$

$$+ X^2(L(L - 1)(X - 1)^{L-2}(X - B) + 2L(X - 1)^{L-1}),$$

from which it follows that

$$Q(X) = X^{-1}(X - 1)^{2-L}F''(z)$$

$$= X^2(L + 1)^2 + X(-3L - 2 - L^2B) + LB + 1$$

satisfies $Q(0) = LB + 1 \neq 0$. If $Q(C) = 0$ and $C \neq 1$ then $e^z = C$ implies $F''(z) = 0$. Hence the fact that all zeros of $F''$ lie on $i\mathbb{R}$ forces each root of $Q$ to have modulus 1, so that

$$|L + 1|^2 = |LB + 1| \leq |L| + 1,$$

which is impossible if $L \geq 2$. Now suppose that $L = -n \leq -2$. Then $(n - 1)^2 \leq n + 1$ and so $n \leq 3$, giving $L = -2, -3$. Now $L = -2$ forces $|-2B + 1| = 1$ and so $B = 1$, which leads to (iii). Finally, if $L = -3$ then $| -3B + 1| = 4$, from which it follows that $B = -1$ and

$$Q(X) = 4X^2 + 16X + 4,$$

which does not have two roots of modulus 1. \[ \square \]
Lemma 2.7 Let $A \in \mathbb{C}$. Then all solutions of $\tan z = z - A$ are real if and only if $A \in \pi \mathbb{Z} = \{n\pi : n \in \mathbb{Z}\}$.

Proof. This is proved in [18] (starting from formula (5) on p.73, with $B = 1$ in the notation there). However, since the method in [18] uses [14, Lemma 8], the proof of which is lengthy, it seems worth including the following self-contained argument. First, suppose that $p(z) = \tan z$ has a fixpoint $w$ in the open upper half-plane $H^+$. Since $p$ maps $H^+$ into itself, but not univalently, Schwarz’ lemma implies that $|p'(w)| < 1$ and that the iterates of $p$ converge to $w$ on $H^+$. Hence $w$ must lie on the positive imaginary axis $i\mathbb{R}^+$, since $p(i\mathbb{R}^+) \subseteq i\mathbb{R}^+$, which contradicts the fact that simple estimates give $\tanh y < y$ for $y > 0$. Hence all fixpoints of $\tan z$ are real and periodicity implies that so are all solutions of $\tan z = z - A$ for $A \in \pi \mathbb{Z}$.

Now suppose that all solutions of $\tan z = z - A$ are real. The real meromorphic function $g(z) = z - \tan z$ has no finite asymptotic values, and so no Picard values: thus $A \in \mathbb{R}$. Suppose that $n \in \mathbb{Z}$ and $n\pi < A < (n + 1)\pi$. Then $g(x)$ is decreasing on the interval $((n + 1/2)\pi,(n + 3/2)\pi)$ and has a fixpoint at $(n + 1)\pi$, which is a zero of $g - (n + 1)\pi$ of multiplicity 3. Hence there exists a level curve $\gamma$ on which $g(z)$ is real and decreasing, which starts at $(n + 1)\pi$ and enters the upper half-plane. Since $g$ has no finite asymptotic values, and all critical points of $g$ are fixpoints of $g$ in $\pi \mathbb{Z}$, the curve $\gamma$ must pass through a non-real $A$-point of $g$. \hfill \Box

3 Intermediate steps for Theorems 1.2 and 1.3

Throughout this section, let $M \geq 4$ be an integer and let $f$ be a strictly non-real meromorphic function in the plane such that $f, f', \ldots, f^{(M+3)}$ all have finitely many zeros in $\mathbb{C} \setminus \mathbb{R}$.

Lemma 3.1 If $f$ is a polynomial then $f'/f$ is rational and $f$ has at least one non-real zero.

Proof. This follows at once from $f$ being strictly non-real. \hfill \Box

Assume henceforth that $f$ is not a polynomial, and let $g(z) = \tilde{f}(z) = \overline{f(\overline{z})}$. Then Lemma 2.2 shows that the Tsuji characteristics [2, 9, 29] of $f'/f$ and $g'/g$ satisfy

$$T_0(r, f'/f) + T_0(r, g'/g) = O(\log r) \quad \text{as} \quad r \to \infty. \quad (12)$$

The lemma of the logarithmic derivative for the Tsuji characteristic [26] and the formulas

$$\psi = \frac{\phi'}{\phi}, \quad \frac{\phi''}{\phi'} = \frac{\psi'}{\psi} + \psi, \quad (13)$$

then deliver, for all $m \geq 0$,

$$T_0(r, f^{(m+1)}/f^{(m)}) + T_0(r, g^{(m+1)}/g^{(m)}) = O(\log r) \quad \text{as} \quad r \to \infty. \quad (14)$$

For $0 \leq m \leq M + 1$ write

$$F_m(z) = z - \frac{f^{(m)}(z)}{f^{(m+1)}(z)}, \quad G_m(z) = z - \frac{g^{(m)}(z)}{g^{(m+1)}(z)} = \tilde{F}_m(z). \quad (15)$$
Lemma 3.2 Let $0 \leq m \leq M + 1$. Then the functions $F_m$ and $G_m$ are non-constant, and there exists a meromorphic function $K_m$, with finitely many zeros and poles, such that

$$F'_m = \frac{f^{(m)} f^{(m+2)}}{(f^{(m+1)})^2} = K_m \left( \frac{g^{(m)} g^{(m+2)}}{(g^{(m+1)})^2} \right) = K_m G'_m.$$  \hspace{1cm} (16)

The function $K_m$ satisfies $|K_m(x)| = 1$ for all $x \in \mathbb{R}$ and there exist a rational function $R_m$ and a real number $a_m$ such that

$$K_m(z) = R_m(z) e^{ia_m z}.$$  \hspace{1cm} (17)

Furthermore, if $f^{(m)}$, $f^{(m+1)}$ and $f^{(m+2)}$ have only real zeros, then $R_m$ is constant.

Proof. The first assertion holds since if $F_m$ is constant then $F'_m$ and $f^{(m+2)}$ vanish identically. Now $K_m$ has finitely many zeros and poles, since $f, \ldots, f^{(M+3)}$ have finitely many non-real zeros, and $K_m = 1/K_m$. Finally, (14) and Lemma 2.3 imply that (17) holds.

Lemma 3.3 For $0 \leq m \leq M + 1$:

(a) every real multiple zero of $f^{(m)}$ is a 1-point of $K_m$;
(b) if $K_m$ is constant in (16), then either $F_m = G_m$ or $f^{(m)}$ has at most one real zero, counting multiplicities;
(c) every real simple zero $a$ of $f^{(m+1)}$ either is a multiple zero of $f^{(m)}$ or satisfies $K'_m(a) = 0$.

Proof. To prove (a) and (b) take a real zero $x_0$ of $f^{(m)}$ of multiplicity $p$. Then $x_0$ is a zero of $g^{(m)}$ of the same multiplicity, and a common fixpoint of $F_m$ and $G_m$. If $p \geq 2$ then

$$F'_m(x_0) = G'_m(x_0) = \frac{p - 1}{p}, \hspace{1cm} K_m(x_0) = 1,$$

which proves (a). Next, if $K_m$ is constant but $F_m \neq G_m$ then there exists $c_m \in \mathbb{C}$ such that

$$G_m \neq F_m = K_m G_m + c_m,$$

so that $F_m$ and $G_m$ have at most one common fixpoint, and none at all if $K_m = 1$. In view of (a), this proves (b).

To prove (c) take a real simple zero $a$ of $f^{(m+1)}$ which is not a zero of $f^{(m)}$. Then $a$ is a simple pole of $F_m$, and there exists $b \in \mathbb{C} \setminus \{0\}$ such that, as $z \to a$,

$$F_m(z) = \frac{b}{z - a} + O(1), \hspace{1cm} F'_m(z) = \frac{-b}{(z-a)^2} + O(1), \hspace{1cm} G'_m(z) = \frac{-b}{(z-a)^2} + O(1).$$

This implies that $K'_m(a) = 0$. \hspace{1cm} \Box

The next three lemmas will treat a number of special cases.

Lemma 3.4 Assume that $0 \leq m \leq M + 1$ and at least one of the following holds:

(i) $f^{(m+1)}/f^{(m)}$ is real meromorphic;
(ii) $F_m = G_m$;
(iii) $g^{(m)} = c_m f^{(m)}$ for some $c_m \in \mathbb{C}$.

Then $f$ is a rational function with at least one non-real zero.
Proof. It is clear from (15) that (i) implies (ii) and (ii) implies (iii). Assume therefore that (iii) holds: then \(|c_m| = 1\), because \(g = \tilde{f}\) and \(f^{(n)} \neq 0\). Moreover, \(m \geq 1\), since \(f\) is strictly non-real, and \(f\) and \(g\) have the same poles with the same multiplicities. Hence there exists a non-constant meromorphic function \(H\) with finitely many zeros and poles such that, using (12),

\[
g = Hf, \quad \tilde{H} = \frac{1}{H}, \quad \frac{g'}{g} - \frac{f'}{f} = h = \frac{H'}{H}, \quad T_0(r, h) = O(\log r) \quad \text{as} \quad r \to \infty.
\]

Furthermore, integration gives a polynomial \(P \neq 0\), of degree at most \(m - 1 \leq M\), with

\[
g = Hf = P + c_m f, \quad \frac{f'}{f} = \frac{P'}{P} - \frac{H'}{H - c_m} = \frac{P'}{P} - \frac{h}{1 - c_m H^{-1}}.
\]

Hence \(T_0(r, H) = O(\log r)\) and \(T(r, H) = O(r \log r)\) as \(r \to \infty\), by (12), (18) and Lemma 2.3. Thus \(H(z) = T_1(z)e^{iA_1 z}\), where \(A_1 \in \mathbb{R}\) and \(T_1\) is a rational function with \(|T_1(x)| = 1\) on \(\mathbb{R}\).

If \(H\) is transcendental then (18) and (19) show that \(f\) satisfies the hypotheses of \([24, \text{Lemma 2.5}]\), and so \(f'''\) has infinitely many non-real zeros, contrary to assumption. Therefore \(H\) is a rational function and so is \(f\). Because \(H\) is non-constant and \(\tilde{H} = 1/H\), the function \(H\) has at least one pole and, since \(f\) and \(g\) have the same poles, \(f\) has at least one non-real zero.

**Lemma 3.5** Assume that \(f\) has finite order and finitely many poles. Then either \(f'/f\) is a rational function, or \(f\) satisfies (1).

**Proof.** The hypotheses imply that there exist meromorphic functions \(H\) and \(K\), each with finitely many zeros and poles, such that

\[
g = Hf, \quad g' = HKf', \quad \tilde{H} = \frac{1}{H}, \quad \tilde{K} = \frac{1}{K}.
\]

Since \(f\) is strictly non-real, \(H\) is non-constant. Write

\[
h = \frac{H'}{H}, \quad k = \frac{K'}{K}, \quad \tilde{h} = -h, \quad \tilde{k} = -k.
\]

Then \(h\) and \(k\) have finitely many poles and so are rational functions, since \(f\) has finite order. Moreover, \(h\) does not vanish identically, since \(H\) is non-constant, and \(h'/h\) is real.

Now (20) and (21) yield

\[
g' = hHf + Hf' = HKf'.
\]

Here \(K - 1\) cannot vanish identically because \(h\) does not. It follows that

\[
L = \frac{f'}{f} = \frac{h}{K - 1}.
\]

If \(K\) is a rational function, then so is \(f'/f\).

Assume henceforth that \(K\), which has finitely many zeros and poles, is transcendental; then \(k \neq 0\) in (21). Moreover, Lemma 2.3, (12), (20) and (22) imply that \(K(z) = T_1(z)e^{iA_1 z}\), where \(A_1 \in \mathbb{R} \setminus \{0\}\) and \(T_1\) is rational with \(|T_1(x)| = 1\) on \(\mathbb{R}\).
If \( h = \pm k \) then (22) shows that
\[
\frac{f'}{f} = \pm \frac{K'}{K(K - 1)}, \quad f = c(1 - 1/K)^\pm 1, \quad c \in \mathbb{C} \setminus \{0\},
\]
and so \( f \), which has finitely many poles, must satisfy (1).

Assume henceforth that \( h \neq \pm k \). Combining (13), (21) and (22) leads to
\[
\frac{f''}{f'} = L + \frac{L'}{L} = \frac{h}{K - 1} + \frac{h'}{h} - \frac{kK}{K - 1} = \frac{h - k}{K - 1} + \frac{h'}{h} - k. \tag{23}
\]
Observe that none of the functions \( k \pm h'/h, h \pm h'/h \) vanishes identically, since \( h'/h \) is real but \( h \) and \( k \) are not. If \( |z| \) is large, then (23) shows that \( z \) is a zero of \( f''/f' \) if and only if \( z \) is a solution of the following equations:
\[
\frac{h - k}{K - 1} = k - \frac{h'}{h}; \quad K - 1 = \frac{h - k}{k - h'/h}; \quad K = \frac{h - h'/h}{k - h'/h}.
\]
Thus \( f''/f' \) has infinitely many real zeros \( x \) which satisfy, by (20) and (21),
\[
\frac{k(x) - h'(x)/h(x)}{h(x) - h'(x)/h(x)} = \frac{1}{K(x)} = \frac{K(x)}{u(x)} = \frac{-h(x) - h'(x)/h(x)}{h(x) + h'(x)/h(x)} - \frac{k(x) - h'(x)/h(x)}{k(x) + h'(x)/h(x)}.
\]
Because \( k \) and \( h \) are rational functions, this forces
\[
h^2 - (h'/h)^2 = h^2 - (h'/h)^2, \quad h^2 = k^2,
\]
contradicting the assumption that \( h \neq \pm k \). \( \square \)

**Lemma 3.6** Assume that either \( f'/f \) is a rational function or \( f \) satisfies (1), and that \( f, f', f'' \) and \( f''' \) have only real zeros. Then \( f \) is given by (2).

**Proof.** Suppose first that \( f \) satisfies (1). Then \( f''/f' \) is a rational function, and so is \( F_1 \) in (15). Moreover, the function \( K_1 \) in (16) is rational and free of zeros and poles, and so is constant, but Lemma 3.4 implies that \( G_1 \neq F_1 \). Applying Lemma 3.3 shows that \( f' \) has at most one zero, and that any zero of \( f' \) is real and simple. Now (1) gives
\[
\frac{f'}{f - B} = \frac{T'}{T} + iA, \quad A \neq 0.
\]
If \( T \) is non-constant then \( T'/T \) has at least two poles in \( \mathbb{C} \), since \( \widetilde{T} = 1/T \), and so \( f' \) has at least two zeros in \( \mathbb{C} \), counting multiplicities. This is a contradiction, and so \( f \) is given by (2)(iii).

Assume henceforth that \( R = f'/f \) is a rational function. Then so are \( F_0 \) and \( F_1 \) in (15), and the same argument as in the previous paragraph shows that \( K_0 \) and \( K_1 \) are constant. However, Lemmas 3.3 and 3.4 and the fact that \( f \) is strictly non-real imply the following: \( G_0 \neq F_0 \) and \( G_1 \neq F_1 \); neither \( f'/f \) nor \( f''/f' \) is real; any zero of \( f \) is real, simple and unique, and the same applies to zeros of \( f' \) and \( R \).
Suppose first that \( R(\infty) = \infty \). Then, since it has at most one zero, must have form \( R(z) = \alpha(z - x_0) \) with \( x_0 \in \mathbb{R} \) and \( 0 \neq \alpha \in \mathbb{C} \), so that \( f''(z)/f(z) = \alpha + \alpha^2(z - x_0)^2 \). Because \( f'' \) has only real zeros, \( \alpha \) is real and so is \( f''f \), a contradiction.

If \( R \) is a non-zero constant, then \( f \) satisfies (2)(iii). Suppose next that \( R \) is non-constant, with \( R(\infty) \neq 0, \infty \). Then \( R \) is a Möbius transformation, since it has at most one zero. Applying a change of variables \( w = a_1 z + b_1 \) with \( a_1, b_1 \in \mathbb{R} \) makes it possible to assume that the unique zero of \( R \) is at the origin, and that

\[
\frac{f'(z)}{f(z)} = R(z) = \frac{az}{z - z_0} = a + \frac{az_0}{z - z_0}, \quad \frac{f''(z)}{f(z)} = \frac{a^2z^2 - az_0}{(z - z_0)^2},
\]

where \( a, z_0 \in \mathbb{C} \setminus \{0\} \). Here \( b = az_0 \) is an integer and \( z_0 \notin \mathbb{R} \), since otherwise \( a \) and \( f'/f \) are real. Thus \( b \) must be negative and \( z_0 \) is a pole of \( f \) and a double pole of \( f''/f \). Next, \( z_0/a \) must be real and positive, since \( f'' \) has only real zeros, and so must \( -z_0^2 = -bz_0/a \). Now write

\[
\frac{f''(z)}{f(z)} = \frac{az(z^2 - az_0)}{(z - z_0)^3} + \frac{2a^2z}{(z - z_0)^2} - \frac{2(a^2z^2 - az_0)}{(z - z_0)^3} = \frac{(a^2z^3 - 3az_0 z + 2z_0)}{(z - z_0)^3}.
\]

Because \( z_0^2 \) and \( az_0 \) are real, so is \( a^2z \). Since \( z_0 \) is not real, \( f'' \) must have at least one non-real zero, contradicting the fact that \( R = f'/f \) is not real. So both \( r_j \) are real and negative, as are \( z_1/z_2 \) and \( a \), and \( f(z_1) = f(z_2) = \infty \). Now

\[
\frac{f''(z)}{f(z)} = a^2\frac{z^2}{(z - z_1)^2(z - z_2)^2} + \frac{a}{(z - z_1)(z - z_2)} - \frac{az}{(z - z_1)^2(z - z_2)} - \frac{az}{(z - z_1)(z - z_2)^2} = \frac{(a^2 - a)z^2 + az_2 z}{(z - z_1)^2(z - z_2)^2}.
\]

Since \( a < 0 \) this forces \( z_1 z_2 \) to be real and positive, and so \( z_1^2 \) and \( z_2^2 \) are real and negative. Next,

\[
\frac{f''(z)}{f(z)} = \frac{a(z((a^2 - a)z^2 + az_2 z) + (a^2 - a)2z}{(z - z_1)^3(z - z_2)^3} + \frac{(a^2 - a)2z}{(z - z_1)^2(z - z_2)^2} + \frac{2((a^2 - a)z^2 + az_2 z)}{(z - z_1)^3(z - z_2)^2} - \frac{2((a^2 - a)z^2 + az_2 z)}{(z - z_1)^2(z - z_2)^3} = \frac{(a^2 - a)z^2 + az_2 z)(az - 4z + 2(z_1 + z_2)) + (a^2 - a)2z(z - z_1)(z - z_2)}{(z - z_1)^3(z - z_2)^3} = \frac{a(a - 1)(a - 2)z^3 + z_1 z_2(3a^2 - 6a)z + 2az_1 z_2(z_1 + z_2)}{(z - z_1)^3(z - z_2)^3}.
\]
But \( a < 0 \), and \( f''/f \) has triple poles at \( z_1 \) and \( z_2 \). Hence \( f''/f \) has three zeros in \( \mathbb{C} \), counting multiplicities, all of them real. Because \( z_1 z_2 \) is real, \( z_1 + z_2 \) must be real, and so 0. But then (24) implies that \( f'/f \) is real, a contradiction.

Finally, suppose that \( R \) has a zero at \( \infty \) of multiplicity at least two. Then integration of \( R \) around a circle \( |z| = r \) with \( r \) large shows that \( f \) has in \( \mathbb{C} \) the same number of zeros as poles, counting multiplicities, and so exactly one of each. Hence \( f \) satisfies (2)(ii).

\[ \square \]

Assume for the remainder of this section that \( f \) has either infinite order of growth or infinitely many poles. Then \( f^{(m+1)}/f^{(m)} \) is transcendental, for each \( m \geq 0 \).

**Lemma 3.7** The following statements all hold.

(i) If \( 0 \leq m \leq M + 1 \) and \( K_m \) is constant then \( f^{(m)} \) has finitely many zeros.

(ii) If \( 0 \leq m \leq M \) and \( K_m \) and \( K_{m+1} \) are both non-constant, then \( \overline{N}(r, 1/f^{(m+1)}) = O(r) \) as \( r \to \infty \).

(iii) If \( 0 \leq m \leq M \) and \( K_m \) and \( K_{m+1} \) are both non-constant rational functions, then \( f^{(m+1)} \) has finitely many zeros;

(iv) If \( 0 \leq m \leq M \) and \( K_m \) and \( K_{m+1} \) are both rational functions, then \( f^{(m)} \) or \( f^{(m+1)} \) has finitely many zeros.

**Proof.** Since \( f \) is transcendental by assumption, Lemma 3.4 shows that \( F_m \neq G_m \) for \( 0 \leq m \leq M + 1 \). Thus (i) follows from Lemma 3.3(b).

Next, assume the hypotheses of (ii), and let \( x_0 \) be a real zero of \( f^{(m+1)} \). By Lemma 3.3, either \( x_0 \) is a multiple zero of \( f^{(m)} \) or \( f^{(m+1)} \), and hence a 1-point of \( K_m \) or \( K_{m+1} \), or \( x_0 \) is a zero of \( K_m' \). Now (ii) and (iii) follow, by (17), and combining (i) and (iii) gives (iv). \[ \square \]

**Lemma 3.8** There exists \( \alpha > 0 \) such that, for \( 1 \leq m \leq M + 2 \),

\[ T(r, f^{(m+1)}/f^{(m)}) + T(r, g^{(m+1)}/g^{(m)}) < \alpha r \quad \text{as} \quad r \to \infty. \]  

(25)

**Proof.** If \( K_0 \) or \( K_1 \) is constant, then \( f \) or \( f' \) has finitely many zeros, by Lemma 3.7. If \( K_0 \) and \( K_1 \) are both non-constant then \( \overline{N}(r, 1/f') = O(r) \) as \( r \to \infty \). This implies that

\[ \overline{N}(r, 1/f^{(m)}) = O(r) \quad \text{as} \quad r \to \infty \]  

(26)

holds for \( m = 0 \) or \( m = 1 \). Since \( M \geq 4 \), the same argument may be applied to \( K_4 \) and \( K_5 \) to show that (26) holds for \( m = 4 \) or \( m = 5 \). This delivers \( p \in \{0, 1\} \) and \( q \in \{3, 4, 5\} \) such that (26) holds for \( m = p \) and \( m = p + q \). Now Theorem 2.1 implies that there exists \( d_1 > 0 \) with

\[ T(r, f^{(p+1)}/f^{(p)}) \leq d_1 \left( \overline{N}(r, 1/f^{(p)}) + \overline{N}(r, 1/f^{(p+q)}) \right) + O(\log r) = O(r) \]

as \( r \to \infty \) outside a set of finite measure. This gives (25) for some \( m \in \{0, 1\} \) and positive \( \alpha \). The existence of \( \alpha > 0 \) such that (25) holds for \( 1 \leq m \leq M + 2 \) then follows from (13). \[ \square \]

**Lemma 3.9** Call an integer \( m \) exceptional if \( m \geq 0 \) and \( f^{(m+1)} \) has finitely many zeros. Then there exists at most one exceptional \( m \).
Proof. Suppose that there exist \( m \) and \( m' \) with \( 0 \leq m < m' \) such that \( f^{(m+1)} \) and \( f^{(m'+1)} \) have finitely many zeros. If \( m' \geq m + 2 \) then \( f^{(m+1)} \) has finite order and finitely many poles, by Theorem 2.1, a contradiction. If \( m' = m + 1 \) then the same contradiction is obtained by applying Lemma 2.1 to \( f^{(m)} \), using the fact that \( f^{(m+1)}/f^{(m)} \) has finitely many zeros and hence finite order by (14) and Lemma 2.3.

\[ \alpha^* = \liminf_{r \to \infty} \frac{T(r, f^{(m+1)}/f^{(m)})}{r} \in (0, +\infty). \] (27)

Proof. Assume that \( m \) is as in the statement but (27) fails. In view of (25) it must therefore be the case that \( \alpha^* = 0 \) in (27). Then there exists a sequence \( s_n \to \infty \) with \( T(s_n, f^{(m+1)}/f^{(m)}) = o(s_n) \) and so, by (13), (16), (25) and the lemma of the logarithmic derivative,

\[ \sum_{j=m}^{m+4} T(s_n, f^{(j+1)}/f^{(j)}) = o(s_n), \quad T(s_n, F_{m+2}) + T(s_n, F'_{m+3}) = o(s_n). \]

Since \( m + 2 \leq M \), it follows that \( a_{m+2} = a_{m+3} = 0 \) in (17) and hence, by Lemma 3.7, that \( f^{(m+2)} \) or \( f^{(m+3)} \) has finitely many zeros, which contradicts Lemma 3.9.

\[ \sum_{j=m}^{m+4} T(s_n, f^{(j+1)}/f^{(j)}) = o(s_n), \quad T(s_n, F_{m+2}) + T(s_n, F'_{m+3}) = o(s_n). \]

Since\( m + 2 \leq M \), it follows that \( a_{m+2} = a_{m+3} = 0 \) in (17) and hence, by Lemma 3.7, that \( f^{(m+2)} \) or \( f^{(m+3)} \) has finitely many zeros, which contradicts Lemma 3.9.

\[ \sum_{j=m}^{m+4} T(s_n, f^{(j+1)}/f^{(j)}) = o(s_n), \quad T(s_n, F_{m+2}) + T(s_n, F'_{m+3}) = o(s_n). \]

Since \( m + 2 \leq M \), it follows that \( a_{m+2} = a_{m+3} = 0 \) in (17) and hence, by Lemma 3.7, that \( f^{(m+2)} \) or \( f^{(m+3)} \) has finitely many zeros, which contradicts Lemma 3.9.

**Lemma 3.10** Assume that \( m \in \{1, \ldots, M-2\} \) is exceptional. Then

\[ \alpha^* = \liminf_{r \to \infty} \frac{T(r, f^{(m+1)}/f^{(m)})}{r} \in (0, +\infty). \] (27)

Proof. Assume that \( m \) is as in the statement but (27) fails. In view of (25) it must therefore be the case that \( \alpha^* = 0 \) in (27). Then there exists a sequence \( s_n \to \infty \) with \( T(s_n, f^{(m+1)}/f^{(m)}) = o(s_n) \) and so, by (13), (16), (25) and the lemma of the logarithmic derivative,

\[ \sum_{j=m}^{m+4} T(s_n, f^{(j+1)}/f^{(j)}) = o(s_n), \quad T(s_n, F_{m+2}) + T(s_n, F'_{m+3}) = o(s_n). \]

Since \( m + 2 \leq M \), it follows that \( a_{m+2} = a_{m+3} = 0 \) in (17) and hence, by Lemma 3.7, that \( f^{(m+2)} \) or \( f^{(m+3)} \) has finitely many zeros, which contradicts Lemma 3.9.

**Lemma 3.11** There exist a real number \( M_1 > 1 \) and an increasing positive sequence \( (r_n) \) with limit \( \infty \) such that, for all large \( n \) and all \( m \in \{1, \ldots, M\} \),

\[ T(2r_n, f^{(m+1)}/f^{(m)}) \leq M_1 T(r_n, f^{(m+1)}/f^{(m)}). \]

Proof. Let \( M_2 > 1 \). By (25) and a growth lemma of Hayman [11, Lemma 4], each set

\[ X_m = \{ r \geq 1 : T(2r, f^{(m+1)}/f^{(m)}) > M_2 T(r, f^{(m+1)}/f^{(m)}) \} \]

has upper logarithmic density at most \( d_0 = -\frac{\log 2}{\log M_2} \). Hence it suffices to take \( M_2 \) so large that \( Md_0 < 1 \), and choose a sequence \( r_n \to \infty \) in the complement of the union of the \( X_m \).

**Lemma 3.12** Let \( m \in \{1, \ldots, M\} \) and let \( \varepsilon > 0 \).

(A) If \( m \) is exceptional and \( \varepsilon \) is sufficiently small, then for each large \( n \) there exist \( I_{m,n} \in \{ f^{(m)}, g^{(m)} \} \) and \( v_n \) satisfying

\[ |v_n| = r_n, \quad 2\varepsilon \leq \arg v_n \leq \pi - 2\varepsilon, \quad \left| v_n \frac{L'_{m,n}(v_n)}{L_{m,n}(v_n)} \right| < \exp((-1/4)T(r_n, f^{(m+1)}/f^{(m)})) \] (28)

(B) If \( a_m \neq 0 \) in (17) then for each large \( n \) there exist \( I_{m,n} \in \{ f^{(m)}, f^{(m+1)}, g^{(m)}, g^{(m+1)} \} \) and \( v_n \) satisfying

\[ |v_n - ir_n| < 4, \quad \left| v_n \frac{L'_{m,n}(v_n)}{L_{m,n}(v_n)} \right| < e^{-|a_m|r_n/2}. \] (29)
Proof. To prove (A) assume that \( m \) is exceptional and let \( n \) be large. Since \( f^{(m+1)}/f^{(m)} \) is transcendental and has finitely many zeros, combining Lemma 3.11 with a well known estimate due to Edrei and Fuchs [6, p.322] shows that, provided \( \epsilon \) is small enough, the set
\[
\{ \theta \in [0, 2\pi] : \left| \frac{f^{(m+1)}(r_n e^{i\theta})}{f^{(m)}(r_n e^{i\theta})} \right| < \exp\left( -\frac{1}{2} T(r_n, f^{(m+1)}/f^{(m)}) \right) \}
\]
has measure at least \( 16\epsilon \). Hence there exist \( L_{m,n} \in \{ f^{(m)}, g^{(m)} \} \) and \( v_n \) such that (28) holds.

To prove (B), assume that \( a_m \neq 0 \) in (17) and again let \( n \) be large. By interchanging \( f \) and \( g \) it may be assumed that \( a_m > 0 \). This implies that
\[
|K_m(z)| < e^{-|a_m|r_n/2} \quad \text{for} \quad |z - ir_n| < 2. \tag{30}
\]
It follows immediately from (16) and (30) that, for each \( z \) with \( |z - ir_n| < 2 \),
\[
either (a) \quad |G_m'(z)| > e^{|a_m|r_n/4} \quad \text{or} \quad (b) \quad |F_m'(z)| < e^{-|a_m|r_n/4}. \tag{31}
\]
Suppose first that case (a) holds in (31), for some \( z \) with \( |z - ir_n| < 2 \). Because \( n \) is large and \( G_m \) has finitely many non-real poles, by (15), Cauchy’s estimate for derivatives implies that there exists \( v_n \) with \( |v_n - ir_n| < 4 \) such that
\[
|G_m(v_n)| > e^{a_m|r_n/6}, \quad \left| v_n \frac{g^{(m+1)}(v_n)}{g^{(m)}(v_n)} \right| < e^{-|a_m|r_n/8}.
\]
This gives (29) with \( L_{m,n} = g^{(m)} \).

Now suppose that case (b) holds in (31), for all \( z \) with \( |z - ir_n| < 2 \), in which case
\[
|F_m(z) - F_m(ir_n)| < 2e^{-|a_m|r_n/4} \quad \text{for} \quad |z - ir_n| < 2. \tag{32}
\]
Choose \( v_n \) with \( |v_n - ir_n| < 2 \) such that \( |v_n - F(ir_n)| \geq 1 \). Then (15) and (32) imply that
\[
\left| \frac{f^{(m)}(v_n)}{f^{(m+1)}(v_n)} \right| = |(v_n - F_m(ir_n)) - (F_m(v_n) - F_m(ir_n))| \geq \frac{1}{2}.
\]

It now follows from (31) that
\[
\left| v_n \frac{f^{(m+2)}(v_n)}{f^{(m+1)}(v_n)} \right| = \left| v_n \cdot \frac{f^{(m+1)}(v_n)}{f^{(m)}(v_n)} \right| < e^{-|a_m|r_n/8}.
\]
Thus (29) holds with \( L_{m,n} = f^{(m+1)} \). \qed

Lemma 3.13 Let \( Y \) be the set of integers \( m \in \{1, \ldots, M\} \) such that either \( m \) is exceptional or \( a_m \neq 0 \) in (17). Then there do not exist integers \( m_1, m_2, m_3 \in Y \) satisfying
\[
m_2 \geq m_1 + 2, \quad m_3 \geq m_2 + 2. \tag{33}
\]
Proof. Assume that \( m_1, m_2, m_3 \in Y \) satisfy (33). If any \( m_\nu \) is exceptional and \( m_\nu \leq M - 2 \) then it is unique, by Lemma 3.9: in this case let \( \alpha^* \) be as in (27), so that \( \alpha^* > 0 \). On the other hand, if no such \( m_\nu \) exists let \( \alpha^* = 0 \). In either case the set \( \{\alpha^*, |a_{m_1}|, |a_{m_2}|, |a_{m_3}|\} \) has at least one positive member, which will be denoted by \( \beta \).

Define \( S_n \) and \( K \) by

\[
S_n = e^{-\beta r_n / 8}, \quad K = 4 + \frac{128 \alpha}{3} 
\]

where \( \alpha \) is as in (25), and let \( \varepsilon \) be small and positive.

Apply Lemma 3.12 with \( m = m_\nu \) and \( \nu = 1, 2, 3 \); this is possible since if \( a_{m_\nu} = 0 \) then \( m_\nu \) is exceptional, by the definition of \( Y \). Passing to a subsequence then allows the following assumptions to be made for \( \nu = 1, 2, 3 \) and all sufficiently large \( n \): first, if \( m_\nu \) is exceptional then (28) holds for \( m = m_\nu \), while if \( m_\nu \) is not exceptional then \( a_{m_\nu} \neq 0 \) and (29) holds for \( m = m_\nu \); second, \( H'_\nu = L_{m_\nu,n} \) is for each \( n \) the same element of the set \( \{f^{(m_\nu)}, f^{(m_\nu+1)}, g^{(m_\nu)}, g^{(m_\nu+1)}\} \), with \( H'_\nu \in \{f^{(m_\nu)}, g^{(m_\nu)}\} \) if \( m_\nu \) is exceptional. It is then possible to choose \( j, k \in \{1, 2, 3\} \) with \( j < k \) such that \( H_j \) and \( H_k \) are both derivatives of \( f \), or both derivatives of \( g \). It follows from (33) that \( H_k = H_j^{(q)} \) for some \( q \geq 1 \).

Suppose that \( m_j \) is exceptional. Then \( H_j = f^{(m_j)} \) or \( g^{(m_j)} \) and, because \( 1 \leq m_j \leq m_k - 2 \leq M - 2 \), the choice of \( \alpha^* \) and \( \beta \) forces \( \alpha^* > 0 \) and \( \alpha^* \geq \beta \). Therefore, in this case, (27) yields

\[
\frac{1}{4} T(r_n, H'_j / H_j) \geq \frac{\alpha^* r_n}{8} \geq \frac{\beta r_n}{8} \quad \text{for large } n. 
\]

Thus, whether or not \( m_j \) is exceptional, (34) and (28) or (29) give \( z_n \) such that (8) holds with \( h = H_j \). Moreover, since \( m_k + 1 \leq M + 1 \) and \( f, \ldots, f^{(M+3)} \) have finitely many non-real zeros, combining Lemma 2.4 with (28) or (29) for \( m = m_k \) gives (7), for some sequence \( \rho_n \to 0 \).

Lemma 2.5 now implies that for large \( n \) the set

\[
\left\{ \theta \in [\varepsilon, \pi - \varepsilon] : \left| \frac{H'_j(t_n e^{i\theta})}{H_j(t_n e^{i\theta})} \right| < C_1 S_n \right\}, \quad t_n = K^{-1} r_n, 
\]

has linear measure at least \( \pi / 2 \). On combination with (34) this yields, as \( n \to \infty \),

\[
m(t_n, H'_j / H_j) \geq \frac{1}{4} \log \frac{1}{S_n} - O(1) = \frac{\beta r_n}{32} - O(1) = \frac{\beta K t_n}{32} - O(1) \geq 2 \alpha t_n, 
\]

which contradicts (25). \( \square \)

4 Proof of Theorem 1.2

Assume the hypotheses of Theorem 1.2. Then the results of Section 3 hold, with \( M = 9 \). If \( f \) has finite order and finitely many poles then both conclusions of Theorem 1.2 follow from Lemmas 3.1, 3.5 and 3.6. Assume henceforth that \( f \) has infinite order or infinitely many poles and let \( Y \) be as in Lemma 3.13. It will be shown that there exist integers \( m_1, m_2, m_3 \in Y \) satisfying (33), contradicting Lemma 3.13.

Suppose first that some \( m \in \{1, \ldots, 8\} \) is exceptional. Then \( f^{(m'+1)} \) has infinitely many zeros for \( 0 \leq m' \neq m \), by Lemma 3.9. If \( m \leq 3 \) then at least one of \( a_5 \) and \( a_6 \) is non-zero
in (17), by Lemma 3.7, as is at least one of $a_8$ and $a_9$: thus $m_1 = m$, while $m_2 \in \{5,6\}$ and $m_3 \in \{8,9\}$. Similarly, if $4 \leq m \leq 6$ then at least one of $a_1$ and $a_2$ is non-zero, as is at least one of $a_3$ and $a_4$. Furthermore, if $m \geq 7$ then at least one of $a_5$ and $a_6$ is non-constant, but finitely many are real zeros or poles of $H_\beta$. 

Suppose finally that $f^{(m+1)}$ has infinitely many zeros, for $m = 1, \ldots, 8$. Then Lemma 3.7 implies that at least one of $a_2$ and $a_3$ is non-zero in (17), as are at least one of $a_5$ and $a_6$ and at least one of $a_8$ and $a_9$. 

\section{5 Proof of Theorem 1.3}

Assume the hypotheses of Theorem 1.3. Again the results of Section 3 hold, this time with $M = 6$, and by Lemmas 3.1, 3.5 and 3.6 it suffices to consider the case where $f^{(m)}$ has infinite order or infinitely many poles, for each $m \geq 0$. Lemmas 3.2 and 3.7 imply that if $0 \leq m \leq 7$ and $a_m = 0$ in (17) then $K_m$ in (16) is constant and $f^{(m)}$ has finitely many zeros.

The following argument gives integers $m_1, m_2, m_3 \in Y$ satisfying (33), where $Y$ is as in Lemma 3.13, and so delivers a contradiction. Suppose first that some $m \in \{1, \ldots, 5\}$ is exceptional, so that $f^{(m+1)}$ has infinitely many zeros for $0 \leq m' \neq m$, by Lemma 3.9. This implies that if $m \leq 2$ then $a_4a_6 \neq 0$, by Lemma 3.7, while if $3 \leq m \leq 4$ then $a_1a_6 \neq 0$, and if $m = 5$ then $a_1a_3 \neq 0$. On the other hand, if no $m \in \{1, \ldots, 5\}$ is exceptional, then $f^{(m+1)}$ has infinitely many zeros, for $m = 1, \ldots, 5$, and $a_2a_4a_6 \neq 0$. 

\section{6 Proof of Theorem 1.4}

Let $f$ be a strictly non-real meromorphic function in the plane such that all but finitely many zeros and poles of $f$ and $f''$ are real. Write

$$g = \overline{f}, \quad \frac{f'}{f} = \alpha + i\beta, \quad \frac{g'}{g} = \alpha - i\beta, \quad 2\alpha = \frac{f'}{f} + \frac{g'}{g}, \quad 2i\beta = \frac{f'}{f} - \frac{g'}{g},$$

in which $\alpha$ and $\beta$ are real meromorphic functions. Here $\beta$ is not identically zero, since $f/g$ is non-constant, but $\beta$ has finitely many poles. Furthermore, all poles of $\alpha$ are simple, and all but finitely many are real zeros or poles of $f$. Since $f''/f$ and $g''/g$ have, with finitely many exceptions, the same zeros and poles there exists a meromorphic function $H$ with finitely many zeros and poles such that

$$\frac{f''}{f} = \alpha' + \alpha^2 - \beta^2 + i(\beta' + 2\alpha\beta) = \frac{Hg''}{g} \quad \Rightarrow \quad H = 1_{H'}.$$ (36)

In view of Lemmas 2.2 and 2.3, as well as standard properties of the Tsuji characteristic,

$$T_0(r, f'/f) + T_0(r, g'/g) + T_0(r, \beta) = O(\log r) \quad \text{and} \quad T(r, \beta) + T(r, H) = O(r \log r)$$ (37)

as $r \to \infty$. If $f$ has finite lower order then $\beta$ is a rational function.
Now $H \equiv 1$ implies that $f''/f$ is real meromorphic and $f'/f$ is a rational function, by [24, Theorem 1.3], and so (3) evidently holds: moreover, the same result shows that if, in addition, $f$ and $f''$ have only real zeros and poles then $f$ satisfies (4)(a).

Assume henceforth that $H \not\equiv 1$. Then rearranging (36) delivers

$$\alpha' + \alpha^2 - \beta^2 = C(\beta' + 2\alpha\beta), \quad C = i \left(\frac{H + 1}{H - 1}\right),$$

in which $C$ is a real meromorphic function.

**Lemma 6.1** If $z_0 \in \mathbb{C}$ is a pole of $\alpha$ but not of $\beta$, and if $\text{Res}(\alpha, z_0) \neq 1$, then $C(z_0) = \infty$. This holds in particular if $|z_0|$ is large and $z_0$ is a pole or multiple zero of $f$.

**Proof.** The residue condition implies that $z_0$ is a double pole of $\alpha' + \alpha^2$, and hence a pole of $C$, by (38). The second assertion follows from (35). \hfill $\square$

Now (38) yields

$$0 = \alpha' - C\beta' - C'\beta + \alpha^2 - 2\alpha C\beta + C^2\beta^2 + C'\beta - (1 + C^2)\beta^2$$

and so

$$0 = \gamma' + \gamma^2 + C'\beta - (1 + C^2)\beta^2, \quad \gamma = \alpha - C\beta. \tag{39}$$

**Lemma 6.2** Assume that $H$ is a rational function in (36). Then $f$ satisfies (3).

If, in addition, $f$ has finite lower order and all zeros and poles of $f$ and $f''$ are real, then $\beta$, $\gamma$, $\alpha$ and $f'/f$ are all constants, and $f$ satisfies the first equation of (4).

**Proof.** Since $H$ is a rational function, so is $C$. By (35) and Lemma 6.1, all but finitely many poles of $\alpha$ are real and simple with residue 1, and the same is true of $\gamma$ by (39). Let $x_0$ be large and positive, and choose $x_1 > x_0$ such that $\gamma(x_1) \neq \infty$. The Riccati equation (39) may be linearised by writing

$$U(x_1) = 1, \quad \frac{U'}{U} = \gamma, \quad U'' + (C'\beta - (1 + C^2)\beta^2)U = 0. \tag{40}$$

Then $U$ extends to be analytic in the half-plane $H_0$ given by $\text{Re } z > x_0$, and $U$ is real on $(x_0, \infty)$. For $x > x_0$, write $C'(x) = \rho(x)C(x)$, where $\rho(x)$ is small and real, so that

$$1 \geq \frac{\rho^2}{4} - \left(C\beta - \frac{\rho}{2}\right)^2 = \rho C\beta - C^2\beta^2 \geq \rho C\beta - (1 + C^2)\beta^2 = C'\beta - (1 + C^2)\beta^2.$$

Thus the Sturm comparison theorem [5, p.355] applied to $U(x)$ and $V(x) = \sin x$ implies that the number of zeros of $U$ in the interval $[x_0, x]$ is $O(x)$ as $x \to +\infty$, and the same is true for the number of poles of $\gamma$, and hence of $\alpha$ and $f'/f$, by (35) and (39). Applying a similar argument on the negative real axis proves the first estimate of (3), and the second follows using (37) and Lemma 2.3.

Suppose in addition that $f$ has finite lower order and all zeros and poles of $f$ and $f''$ are real. Then $\beta$ is a polynomial in (35), and the rational function $H$ is free of zeros and poles, and so is constant, as is $C$. Moreover, all poles of $\gamma$ are real and simple with residue 1, so that
$U$ is now a real entire function, with only real zeros, of finite order by (40). Furthermore, $U$ has at most one zero, by the Sturm comparison theorem applied to $U(x)$ and $V(x) = 1$. Thus $\gamma = \alpha - C\beta = U'/U$ has at most one pole, and so is a rational function. Hence there exist a polynomial $K = \beta \sqrt{1 + C^2} \neq 0$ and a constant $\eta = \pm 1$ such that, as $z \to \infty$, (39) delivers $\gamma(z) = O(|K(z)|)$ and

$$K(z)^2 = \gamma(z)^2 + \gamma(z) \cdot \frac{O(1)}{z} = \gamma(z)^2 + K(z) \cdot \frac{O(1)}{z},$$

$$\gamma(z) = \eta K(z) + X(z) = \eta K(z) + \frac{O(1)}{z},$$

$$0 = \eta K'(z) + X'(z) + 2\eta K(z)X(z) + X(z)^2 = \eta K'(z) + 2\eta K(z)X(z) + \frac{O(1)}{z^2},$$

as well as

$$\frac{U''(z)}{U(z)} + \frac{K'(z)}{2K(z)} = \gamma(z) + \frac{K'(z)}{2K(z)} = \eta K(z) + X(z) + \frac{K'(z)}{2K(z)} = \eta K(z) + \frac{O(1)}{z^2} K(z).$$

The argument principle now shows that $U$ and $K$ have no zeros, and hence $K$ and $\beta$ are constant, while $\gamma$ is a polynomial and is also constant, as are $\alpha$ and $f'/f$.

Assume henceforth that $H$ is transcendental in (36). The next lemma follows immediately from (37).

**Lemma 6.3** There exist $a \in \mathbb{R} \setminus \{0\}$ and a rational function $T$ with $|T(x)| = 1$ for all $x \in \mathbb{R}$, such that $H(z) = T(z)e^{iaz}$.

It may be assumed that $a = 2$ and $T(\infty) = 1$ in Lemma 6.3, so that (38) gives

$$H(z) = e^{2i\zeta(z)}, \quad \zeta(z) = z + \frac{\log T(z)}{2i}, \quad C(z) = i\left(\frac{H(z) + 1}{H(z) - 1}\right) = \cot \zeta(z), \quad \zeta \neq \zeta(z)$$

in which the logarithm is the principal branch, while $\zeta(z)$ is analytic near infinity with $\tilde{\zeta} = \zeta$ there. Thus (39) becomes

$$0 = \gamma' + \gamma^2 - (1 + C^2)(\beta\zeta' + \beta^2) = \gamma' + \gamma^2 - (\beta\zeta' + \beta^2)S^2, \quad S = \frac{1}{\sin \zeta}. \quad (42)$$

**Lemma 6.4** Let $x_0$ be large and positive and let $I \subseteq \mathbb{R} \setminus [-x_0, x_0]$ be an open interval containing no poles of $S(z)$. Then $I$ contains at most one pole of $f'/f$.

**Proof.** Choose $x_1 \in I$ such that $\gamma(x_1) \neq \infty$ and linearise (42) near $x_1$ by writing

$$u(x_1) = 1, \quad \frac{u'}{u} = \gamma, \quad u'' + Au = 0, \quad A = -(\beta\zeta' + \beta^2)S^2.$$

Thus $u$ extends to be analytic on a domain containing $I$, and $u$ is real-valued on $I$. Define a zero-free comparison function $v$ on $I$ by $v(x_1) = 1$ and

$$\frac{v'}{v} = \frac{\zeta' \cot \zeta}{2} - \frac{\zeta''}{2\zeta'} = \frac{\zeta C}{2} - \frac{\zeta''}{2\zeta'},$$

which
so that
\[
\frac{v''}{v} = \frac{\zeta''C}{2} - \frac{(\zeta')^2(1 + C^2)}{2} - \frac{\zeta''}{2\zeta'} + \frac{(\zeta')^2/4}{2} - \frac{\zeta''C}{2} + \frac{(\zeta')^2}{4(\zeta')^2}
\]
\[
= -\frac{(\zeta')^2(1 + C^2)}{2} + \frac{(\zeta')^2(1 + C^2 - 1)}{4} - \frac{\zeta''}{2\zeta'} + \frac{3(\zeta')^2}{2\zeta'^3} + \frac{3(\zeta'')^2}{4(\zeta')^2}.
\]

Since \(\zeta'\) is a real rational function with \(\zeta'(\infty) = 1\) and \(x_0\) is large, this gives
\[
A = -(\beta\zeta' + \beta^2)S^2 = -\left(\left(\beta + \frac{\zeta'}{2}\right)^2 - \frac{(\zeta')^2}{4}\right)S^2 \leq \frac{(\zeta')^2S^2}{4} \leq -\frac{v''}{v}
\]
on \(I\). The Sturm comparison theorem [5] now implies that \(u\) has at most one zero in \(I\), so that \(\gamma\) has at most one pole there, as have \(\alpha\) and \(f'/f\), by (35) and (39).

Since poles of \(S\) are poles of \(C\) and zeros of \(H - 1\), Lemmas 6.1, 6.3 and 6.4 imply that \(f\) satisfies the first estimate of (3), from which the second follows using (37) and Lemma 2.3.

To complete the proof of Theorem 1.4, assume henceforth that \(f\) has finite lower order, all zeros and poles of \(f\) and \(f''\) are real and \(H\) is transcendental. Then \(\beta\) is a polynomial, of degree \(d\) say. Furthermore, \(H\) is free of zeros and poles, so that it may be assumed that \(H(z) = e^{2iz}\), while \(\zeta(z) = z\) and \(C(z) = \cot z\). Since \(\zeta'' \equiv 0\), the next lemma follows from (38), (39), Lemma 6.1 and an argument identical to that in Lemma 6.4.

**Lemma 6.5**

(i) Any pole of \(f'/f\) in \(\mathbb{C} \setminus \pi\mathbb{Z}\) is a simple zero of \(f\).

(ii) If \(z_0 \in \pi\mathbb{Z}\) is a pole of \(f'/f\) then \(\text{Res}(f'/f, z_0) = 2\beta(z_0) + 1\).

(iii) If \(n \in \mathbb{Z}\) then \(f'/f\) has in \(I_n = (n\pi, (n + 1)\pi) \subseteq \mathbb{R}\) at most one pole.

(iv) \(f\) satisfies
\[
N(r, f) + N(r, 1/f) = O(r^{d+1}) \quad \text{as } r \to \infty.
\]

Now fix \(x_1 \in I_0 = (0, \pi)\) with \(\gamma(x_1) \neq \infty\) and linearise (42) via \(u(x_1) = 1\) and \(u'/u = \gamma\), so that \(u\) solves
\[
u'' + Au = 0, \quad A(z) = -\frac{\beta(z)(\beta(z) + 1)}{\sin^2 z}. \tag{44}
\]
Then \(u\) extends to be analytic in \(\Omega = \mathbb{C} \setminus \{n\pi - it : n \in \mathbb{Z}, t \in [0, +\infty)\}\), with \(u\) real on \(I_0\).

**Lemma 6.6**

Let \(0 < \varepsilon < \pi/4\) and denote by \(E_0(z)\) any term which satisfies \(\log^+|E_0(z)| = o(|z|)\) as \(z \to \infty\) with \(\varepsilon < \arg z < \pi - \varepsilon\). Then there exists a polynomial \(P \neq 0\) of degree at most 1 such that
\[
\frac{u''(z)}{u(z)} = E_0(z)e^{2iz}, \quad u(z) = P(z) + E_0(z)e^{2iz}, \quad \gamma(z) = \frac{P'(z)}{P(z)} + E_0(z)e^{2iz}. \tag{45}
\]

**Proof.** The first estimate follows from (44) and the remaining two are proved by the method of Gronwall’s lemma, exactly as in [24, Lemma 4.3].

\[\square\]
Lemma 6.7 The order of $f$ is at most $d + 1$.

Proof. (43) makes it possible to write $f = \Pi e^Q$ where $\Pi$ is a meromorphic function with real zeros and poles and order at most $d + 1$, while $Q$ must be a polynomial. It follows from (35), (39), (45) and standard estimates for logarithmic derivatives that, as $z \to \infty$ with $\varepsilon < \arg z < \pi - \varepsilon$,

$$Q'(z) = \frac{f'(z)}{f(z)} - \frac{\Pi'(z)}{\Pi(z)} = \gamma(z) + (\cot z + i\beta(z)) - \frac{\Pi'(z)}{\Pi(z)} = O(|z|^{d+1/2}),$$

so that $Q$ has degree at most $d + 1$.

Lemma 6.8 If the degree $d$ of $\beta$ is positive then, as $x \to +\infty$ with $x \in \mathbb{R}$,

$$\left|\left(\frac{f'}{f}\right)'(x + i) + |a'(x + i)| + |\gamma'(x + i)| = o(|(x + i)\beta(x + i)|) = o(|\beta(x + i)|^2).$$

Proof. It suffices by (35) and (39) to prove that $(f'/f)'(x + i) = o(|(x + i)\beta(x + i)|)$. Let $x \in (0, +\infty)$ be large, set $w = x + i$ and take $R \in [2|w|, 2|w| + 1]$ such that $f(z) \neq 0$, $\infty$ on $|z| = R$. Denote by $a_j$ the zeros and poles of $f$ in $|z| < R$, repeated according to multiplicity. Applying the twice differentiated Poisson-Jensen formula [10, (1.17)] to $f$ in the disc $|z| < R$ gives

$$\left|\left(\frac{f'}{f}\right)'(w)\right| \leq \frac{2}{\pi} \int_0^{2\pi} R |\log |f(Re^{it})|| dt + \sum \left(\frac{1}{|a_j - w|^2} + \frac{|a_j|^2}{|R^2 - a_jw|^2}\right),$$

in which $|Re^{it} - w| \geq R/2$, while $|R^2 - a_jw| \geq (1/2)R^2$ and $|a_j - w| \geq 1$. Lemma 6.5 implies that the number of distinct zeros and poles of $f$ in the interval $[x - R/\log R, x + R/\log R]$ is $O(R/\log R)$, and that each of these has multiplicity at most $4M(R, \beta)$. It now follows from Lemma 6.7 that

$$\left|\left(\frac{f'}{f}\right)'(w)\right| \leq \frac{32}{R^2} \left(m(R, f) + m(R, 1/f)\right) + O\left(\frac{RM(R, \beta)}{\log R}\right) + +n(R, f) + n(R, 1/f)) \left(\frac{(\log R)^2}{R^2} + \frac{4}{R^2}\right)

\leq O(R^d) + O\left(\frac{RM(R, \beta)}{\log R}\right) = o\left(RM(R, \beta)\right) = o(|w\beta(w)|).$$

Lemma 6.9 The polynomial $\beta$ has degree $d = 0$ and, without loss of generality, there exists a real meromorphic function $W$ on $\mathbb{C}$ of order at most 1 such that

$$f(z) = W(z)e^{i\beta z}, \quad \frac{W'}{W} = \alpha = \gamma + \beta C = \frac{u'}{u} + \beta C.$$
Proof. Assume that $\beta$ has positive degree $d$ and let $\varepsilon$ be small and positive. The equations (35) and (39) and the fact that $f$ has finite order give $M_2 > 0$ and arbitrarily large positive $R$ with $\gamma(z) = O\left(R^{M_2}\right)$ on $|z| = R$. Now Lemma 6.6 shows that

$$\frac{(\gamma(z) - P'(z)/P(z))\sin z}{\beta(z)} \to 0$$

as $z \to \infty$ with $\arg z = 2\varepsilon$, whereas (44) and Lemma 6.8 imply that

$$\gamma(x + i) \sim \frac{\beta(x + i)}{\sin(x + i)}, \quad \frac{(\gamma(x + i) - P'(x + i)/P(x + i))\sin(x + i)}{\beta(x + i)} \to \pm 1,$$

as $x \to +\infty$ with $x \in \mathbb{R}$. Since $\gamma$ has only real poles, this contradicts the Phragmén-Lindelöf principle. The remaining assertions follow from (35), (39) and Lemma 6.7. \hfill \Box

Lemma 6.10 If $u(z)$ and $u(z + \pi)$ are linearly dependent on $\Omega$ then $f$ satisfies (4).

Proof. The hypotheses imply that $\gamma = u'/u$ has period $\pi$ and so have the sequences of poles and zeros of $f$, by (35) and (39). Thus, by Lemma 6.5, either $f$ has in each interval $I_n = (n\pi, (n + 1)\pi), n \in \mathbb{Z}$, exactly one simple zero and no poles, or $f$ has no zeros and poles in the $I_n$. Moreover, the residue of $f'/f$ at each zero of $\sin z$ is a fixed integer $m$, possibly 0. It follows that $f$ has a representation

$$f(z) = (e^{2iz} - 1)^L(e^{2iz} - E)e^{pz+q}, \quad L \in \mathbb{Z}, \quad E, p, q \in \mathbb{C}, \quad |E| = 1,$$  \hspace{1cm} (47)

in which $E = 1$ is not excluded. This implies in view of (35) and (39) that, as $z \to \infty$ in $\varepsilon < \arg z < \pi - \varepsilon$,

$$\frac{f'(z)}{f(z)} = p + o(1), \quad \alpha(z) = p - i\beta + o(1), \quad \gamma(z) = \alpha(z) - \beta \cot z = p + o(1),$$

so that $p = 0$ by Lemma 6.6. Now $L \neq -1$ in (47), since $f$ is strictly non-real, and $f$ is determined by applying Lemma 2.6 to $F(z) = e^{-q}f(z/2i)$. \hfill \Box

Assume henceforth that $u(z)$ and $u(z + \pi)$ are linearly independent solutions on $\Omega$ of (44). The proof of Theorem 1.4 will be completed by first considering certain values of $\beta$ with $|\beta|$ small, following which the remaining possibilities for $\beta$ will be disposed of together.

Lemma 6.11 If $\beta \in \{-2, -1, 1\}$ then $f$ satisfies (4).

Proof. Suppose first that $\beta = -1$: then (44) shows that $u'' = 0$. By (46) and the fact that $u(z)$ and $u(z + \pi)$ are linearly independent, there exists a polynomial $T_1$, of degree 1, such that

$$\gamma = \frac{u'}{u} = \frac{T'_1}{T_1}, \quad f(z) = W(z)e^{-iz} = \frac{T_1(z)}{e^{2iz} - 1}.$$  

Now (36) and (39) lead to

$$\frac{f''}{f} = \gamma' + (1 + C^2) + \gamma^2 - 2C]\gamma + C^2 - 1 - 2i(\gamma - C)$$

$$= -2C\gamma + 2C^2 - 2i(\gamma - C) = 2(C + i)(C - \gamma).$$

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Since $f''$ has only real zeros, all zeros of $C - \gamma$ must be real. Thus the zero of $T_1$ belongs to $\pi\mathbb{Z}$; if this is not the case then Lemma 2.7 gives a non-real zero $z^*$ of $\tan z - T_1(z)/T_1'(z)$, with $\tan(z^*) \neq 0, \infty$ and so $T_1(z^*) \neq 0$, which implies that $z^*$ is a non-real zero of $C - \gamma$, a contradiction. It follows that $f$ is given by (4)(c).

Now suppose that $\beta \in \{-2, 1\}$. Then $\beta(\beta + 1) = 2$ and (44) solves explicitly to give $A_1, B_1 \in \mathbb{C}$ with

$$u(z) = A_1 \cot z + B_1(1 - z \cot z),$$

in which $B_1 \neq 0$ since $u(z)$ and $u(z + \pi)$ are linearly independent. Hence there exists a polynomial $T_1$ of degree 1 such that, in view of (46),

$$f(z) = (T_1'(z) - T_1(z) \cot z)(\sin z)^\beta e^{i\beta z}. \quad (48)$$

If $\beta = 1$ this gives (4)(b), and again the zero of $T_1$ must belong to $\pi\mathbb{Z}$ by Lemma 2.7.

Assume now that $\beta = -2$. Then Lemma 6.5 implies that $f$ has no multiple zeros. Suppose that $x_0 \in \mathbb{R}$ is a simple zero of $f$, and so a simple pole with residue 1 of the real meromorphic function $\alpha$. Then there exists $D_0 \in \mathbb{R}$ such that, as $z \to x_0$,

$$f'(z) = \alpha(z) + i\beta = \frac{1}{z - x_0} + D_0 - 2i + O(|z - x_0|), \quad \frac{f''(z)}{f(z)} = \frac{2(D_0 - 2i)}{z - x_0} + O(1).$$

This shows that $x_0$ is a pole of $f''/f$, and so not a zero of $f''$. Thus every zero of $f''$ must be a real zero of $f''/f$ and so of $\alpha$, by (36). But (48) leads to

$$\alpha = \frac{f'}{f} - i\beta = \frac{-T_1'C + T_1(1 + C^2)}{T_1' - T_1C} - 2C = \frac{-3T_1'C + T_1 + 3T_1C^2}{T_1' - T_1C}.$$ 

Hence if $|z|$ is large and $\alpha(z) = 0$ then $C \neq \infty$ and $3C^2 + 1 = o(1)C$, so that $C$ is non-real and so is $z$. Therefore $f''$ has finitely many zeros and, by the main result of [21], $f$ has finitely many poles, contradicting (48). \qed

**Lemma 6.12** Let $n \in \mathbb{Z}$. Then near $n\pi$ there exist linearly independent local solutions $u_1, u_2$ of (44) of form

$$u_1(z) = (z - n\pi)^{-\beta}h_1(z), \quad u_2(z) = (z - n\pi)^{\beta + 1}h_2(z), \quad h_j(z) = 1 + \sum_{k=1}^{\infty} a_{j,k}(z - n\pi)^k, \quad (49)$$

in which the $h_j$ are analytic on $|z - n\pi| < \pi$ and the coefficients $a_{j,k}$ are independent of $n$. Moreover, $2\beta + 1$ is an integer, and $\beta \neq \pm 1/2$ and $\beta \neq -3/2$. Finally, if $u_3, u_4$ are non-trivial solutions on $\Omega$ of (44), then $u_3^2, u_4^2$ and $u_3/u_4$ all extend to be meromorphic in the plane.

**Proof.** Choose some $n \in \mathbb{Z}$ and observe first that, near the regular singular point $n\pi$, there exists $\delta \in \{-\beta, \beta + 1\}$ such that (44) has a solution of form

$$U_1(z) = (z - n\pi)^\delta H_1(z), \quad H_1(z) = 1 + \sum_{k=1}^{\infty} b_k(z - n\pi)^k, \quad (50)$$

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with $H_1$ analytic on $|z - n\pi| < \pi$. Since $U_1(z + \pi)$ solves (44), for $z$ near $(n-1)\pi$, such a solution exists for any $n$, with the same choice of $b_k$. To obtain a further solution near $n\pi$ write

$$U_2(z) = U_1(z) \int U_1(z)^{-2} dz = U_1(z) \int (z - n\pi)^{-2\delta} (1 - 2b_1(z - n\pi) + \ldots) dz = U_1(z) (c_1 \log(z - n\pi) + (z - n\pi)^{1-2\delta}(d_0 + d_1(z - n\pi) + \ldots)),$$

in which the series $\sum_{k=0}^{\infty} d_k(z - n\pi)^k$ is obtained by formal integration but has positive radius of convergence. Suppose first that $c_1 \neq 0$. Then $-2\delta \in \mathbb{Z}$, and so $U_1^{-2} = (U_2/U_1)'$ has a meromorphic extension to a neighbourhood of $n\pi$, as have $\gamma = u'/u$ and $U'_1/U_1$. Write the solution $u$ of (44) locally in the form $u = \alpha_1 U_1 + \alpha_2 U_2$ near $n\pi$, with the $\alpha_j \in \mathbb{C}$. Then the logarithmic derivative of $\alpha_1 + \alpha_2 U_2/U_1$ extends meromorphically to a neighbourhood of $n\pi$, and if $\alpha_2 \neq 0$ so does $U_2/U_1$, a contradiction. Hence $u$ must locally be a constant multiple of $U_1$ only, so that $u(z)$ and $u(z + \pi)$ are linearly dependent, contrary to assumption.

Thus a logarithm cannot arise in (51), which forces $\beta \neq -1/2$ and $-\beta \neq \beta + 1$, and there exist local solutions $u_1, u_2$ as in (49), obtained via (50) and (51), with the coefficients $a_j, k$ independent of $n$. It follows that near $n\pi$ the meromorphic function $W$ in (46) is a linear combination of

$$v_1(z) = k_1(z), \quad v_2(z) = (z - n\pi)^{2\beta + 1} k_2(z),$$

where the $k_j$ are analytic on $|z - n\pi| < \pi$, with $k_j(n\pi) \neq 0$. But then, if $2\beta + 1 \notin \mathbb{Z}$, it must be the case that $W$ is a constant multiple of $v_1$ only, so that $u(z)$ and $u(z + \pi)$ are linearly dependent, again contrary to assumption.

Next, suppose that $\beta = 1/2$ or $\beta = -3/2$. Then one of $-\beta$ and $\beta + 1$ is $-1/2$ and by (49) there exists, near 0, a solution of (44) of form $U_3(z) = z^{-1/2}(1 + e_1 z + e_2 z^2 + \ldots)$, so that

$$\beta(\beta + 1)U_3(z) = \frac{3}{4} z^{-1/2} (1 + e_1 z + e_2 z^2 + \ldots) = U_3''(z) \sin^2 z$$

$$= \left( \frac{3}{4} z^{-5/2} - \frac{1}{4} e_1 z^{-3/2} + \frac{3}{4} e_2 z^{-1/2} + \ldots \right) \left( z^2 - \frac{z^4}{3} + \ldots \right)$$

$$= \frac{3}{4} z^{-1/2} - \frac{1}{4} e_1 z^{1/2} + z^{3/2} \left( \frac{3}{4} e_2 - \frac{1}{4} \right) + \ldots.$$ 

Comparing the coefficients of $z^{3/2}$ yields a contradiction.

To complete the proof observe that, because $2\beta + 1 \in \mathbb{Z}$, the $u_j$ in (49) are such that $u^2_1, u^2_2, u_1 u_2$ and $u_1/u_2$ extend to be meromorphic on a neighbourhood of $n\pi \in \pi\mathbb{Z}$. 

In view of Lemmas 6.11 and 6.12, as well as the fact that $f$ is strictly non-real, it remains only to consider the case where $2\beta + 1 \in \mathbb{Z}$ but

$$\beta \notin \{-2, -3/2, -1, -1/2, 0, 1/2, 1\}, \quad \beta + 1 \notin \{-1, -1/2, 0, 1/2, 1, 3/2, 2\}.$$ (52)

\[ \square \]

**Lemma 6.13** If $n \in \mathbb{Z}$ then $u^2$ has at $n\pi$ a zero or pole of multiplicity at least 3.

Furthermore, there exist infinitely many $n \in \mathbb{Z}$ such that $n\pi$ is a pole of $u^2$, and infinitely many $n \in \mathbb{Z}$ such that $n\pi$ is a zero of $u^2$. 

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Proof. The equation (44) has local solutions \( u_j \) as in (49), in which \(-\beta\) and \(\beta+1\) have opposite signs, and \( u^2 \) has a zero or pole at \( n\pi \) of multiplicity \( 2|\beta| \geq 3 \) or \( 2|\beta+1| \geq 3 \), by (52).

To prove the last assertion, assume that \( u^2 \) has a pole at all but finitely many \( n\pi, n \in \mathbb{Z} \), or that \( u^2 \) has a zero at all but finitely many of these points. In the first case set \( V = u^2 \), and in the second set \( V = u^{-2} \). Then \( V \) satisfies, as \( r \to \infty \),

\[
\frac{6r}{\pi} - O(1) \leq n(r, V), \quad \frac{6r}{\pi} - O(\log r) \leq N(r, V).
\]

On the other hand, Lemma 6.5 shows that if \( n \in \mathbb{Z} \) then in the interval \( I_n = (n\pi, (n+1)\pi) \) the function \( f''/f \) has at most one pole, and any such pole has residue 1. The same is true of \( \alpha \) and \( \gamma = u'/u \), by (35) and (39), and so \( u^2 \) has no poles and at most two zeros in \( I_n \). This implies that, as \( r \to \infty \), by (53) and Jensen’s formula,

\[
N(r, 1/V) \leq 4r - O(\log r) \leq \left( \frac{2}{3} + o(1) \right) N(r, V) \leq \frac{3}{4} T(r, 1/V).
\]

Since \( f \) has finite order, applying [11, Lemma 4] gives \( C_0 > 1 \) and a set \( E_1 \subseteq [1, \infty) \), of positive lower logarithmic density, such that \( T(2r, 1/V) \leq C_0 T(r, 1/V) \) for all \( r \in E_1 \). Choose a positive constant \( \varepsilon \), so small that

\[
88C_0 \varepsilon \left( 1 + \log^+ \frac{1}{4\varepsilon} \right) < \frac{1}{16}.
\]

Then Lemma 6.6, the fact that \( u^2 \) is real meromorphic and an inequality of Edrei and Fuchs [6, p.322] together deliver, for large \( r \in E_1 \),

\[
m(r, 1/V) \leq O(\log r) + 11 \left( \frac{2r}{2r - r} \right) 4\varepsilon \left( 1 + \log^+ \frac{1}{4\varepsilon} \right) T(2r, 1/V)
\]

\[
\leq O(\log r) + 88C_0 \varepsilon \left( 1 + \log^+ \frac{1}{4\varepsilon} \right) T(r, 1/V) \leq \frac{1}{8} T(r, 1/V),
\]

which contradicts (54).

\[\square\]

Lemma 6.14 The function

\[
G(z) = \frac{u(z + \pi) - u(z)}{\pi}
\]

is a non-trivial solution of (44) with period \( \pi \) on \( \Omega \).

Proof. Lemma 6.6 shows that \( u(z) \) is asymptotic to a polynomial \( P \neq 0 \) of degree at most 1 as \( z \to \infty \) in \( \varepsilon < \arg z < \pi - \varepsilon \). The Wronskian \( W_u \) of \( u(z) \) and \( u(z + \pi) \) is constant, by Abel’s identity and (44). If \( P \) is constant then \( W_u \) tends to 0 in a sector and so must vanish identically, forcing \( u(z) \) and \( u(z + \pi) \) to be linearly dependent, contrary to assumption.

Thus \( P \) must be non-constant, and \( G(z) \) and \( G(z + \pi) \) both solve (44) and are asymptotic to the same non-zero constant as \( z \to \infty \) in \( \varepsilon < \arg z < \pi - \varepsilon \). The argument of the previous paragraph now shows that \( G(z) \) and \( G(z + \pi) \) are linearly dependent and must be equal. \[\square\]
It is now possible to write
\[ u(z) = zG(z) + K(z), \quad \frac{K(z)}{G(z)} = \frac{u(z)}{G(z)} - z, \]
where \( K \) also has period \( \pi \) on \( \Omega \). Moreover, \( G^2 \) and \( K/G \) are meromorphic in the plane, by Lemma 6.12, and have period \( \pi \). Lemma 6.13 implies that \( G^2 \) has at least one pole in \( \pi \mathbb{Z} \), and so a pole at every point of \( \pi \mathbb{Z} \). If \( n \in \mathbb{Z} \) and \( n\pi \) is not a pole of \( u^2 \) then, as \( z \to n\pi \) with \( z \in \Omega \),
\[ u(z) = zG(z) + K(z) = O(1), \quad \frac{K(z)}{G(z)} = \frac{u(z)}{G(z)} - z \to -n\pi, \]
which cannot hold for more than one such \( n\pi \), since \( K/G \) is periodic. Thus \( u^2 \) has a pole at all but at most one \( n\pi \in \pi \mathbb{Z} \), contradicting Lemma 6.13.

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