REAL HYPERSURFACES IN COMPLEX HYPERBOLIC TWO-PLANE GRASSMANNIANS WITH COMMUTING STRUCTURE JACOBI OPERATORS

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Abstract. In this paper, we introduce a new commuting condition between the structure Jacobi operator and symmetric \((1,1)\)-type tensor field \(T\), that is, \(R\xi \phi T = TR\xi \phi\), where \(T = A\) or \(T = S\) for Hopf hypersurfaces in complex hyperbolic two-plane Grassmannians. By using simultaneous diagonalization for commuting symmetric operators, we give a complete classification of real hypersurfaces in complex hyperbolic two-plane Grassmannians with commuting condition respectively.

Introduction

It is one of the main topics in submanifold geometry to investigate immersed real hypersurfaces of homogeneous type in Hermitian symmetric spaces of rank 2 (HSS2) with certain geometric conditions. Understanding and classifying real hypersurfaces in HSS2 is one of important problems in differential geometry. One of these spaces is the complex two-plane Grassmannian \(G_2(\mathbb{C}^{m+2}) = SU_{2+m}/S(U_2\cdot U_m)\) defined by the set of all complex two-dimensional linear subspaces in \(\mathbb{C}^{m+2}\). Another one is the complex hyperbolic two-plane Grassmannian \(G^*_2(\mathbb{C}^{m+2}) = SU_{2,m}/S(U_2\cdot U_m)\) defined by the set of all complex two-dimensional linear subspaces in indefinite complex Euclidean space \(\mathbb{C}^{m+2}_2\).

These are typical examples of HSS2. Characterizing typical model spaces of real hypersurfaces under certain geometric conditions is one of our main interests in the classification theory in \(G_2(\mathbb{C}^{m+2})\) or \(SU_{2,m}/S(U_2\cdot U_m)\) (see \[13\] and \[14\]).

Our recent interest is the study by applying geometric conditions used in submanifolds in \(G_2(\mathbb{C}^{m+2})\) to submanifolds in \(SU_{2,m}/S(U_2\cdot U_m)\).

\(G_2(\mathbb{C}^{m+2}) = SU_{2+m}/S(U_2\cdot U_m)\) has compact transitive group \(SU_{2+m}\), however \(SU_{2,m}/S(U_2\cdot U_m)\) has noncompact indefinite transitive group \(SU_{2,m}\). This distinction gives various remarkable results.

The complex hyperbolic two-plane Grassmannian \(SU_{2,m}/S(U_2\cdot U_m)\) is the unique noncompact, irreducible, Kähler and quaternionic Kähler manifold which is not a hyperkähler manifold.

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Let $M$ be a real hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2U_m)$. Let $N$ be a local unit normal vector field on $M$. Since the complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2U_m)$ has the Kähler structure $J$, we may define a Reeb vector field $\xi = -JN$ and a 1-dimensional distribution $C^\perp = \text{Span}\{\xi\}$.

Let $\mathcal{C}$ be the orthogonal complement of distribution $\mathcal{C}^\perp$ in $T_pM$ at $p \in M$. It is the complex maximal subbundle of $T_pM$. Thus the tangent space of $M$ consists of the direct sum of $\mathcal{C}$ and $\mathcal{C}^\perp$ as follows: $T_pM = \mathcal{C} \oplus \mathcal{C}^\perp$. The real hypersurface $M$ is said to be Hopf if $AC \subset \mathcal{C}$, or equivalently, the Reeb vector field $\xi$ is principal with principal curvature $\alpha = g(\xi, \xi)$, where $g$ denotes the metric. In this case, the principal curvature $\alpha$ is said to be a Reeb curvature of $M$.

From the quaternionic Kähler structure $\overline{J} = \text{Span}\{J_1, J_2, J_3\}$ of $SU_{2,m}/S(U_2U_m)$, there naturally exist almost contact 3-structure vector fields $\xi_\nu = -J_\nu N, \nu = 1, 2, 3$. Let $Q^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$. It is a 3-dimensional distribution in the tangent space $T_pM$ of $M$ at $p \in M$. In addition, $Q$ stands for the orthogonal complement of $Q^\perp$ in $T_pM$. It is the quaternionic maximal subbundle of $T_pM$. Thus the tangent space of $M$ can be splitted into $Q$ and $Q^\perp$ as follows: $T_pM = Q \oplus Q^\perp$.

Thus, we have considered two natural geometric conditions for real hypersurfaces in $SU_{2,m}/S(U_2U_m)$ such that the subbundles $\mathcal{C}$ and $Q$ of $TM$ are both invariant under the shape operator. By using these geometric conditions, we will use the results in Suh [13] Theorem 1).

On the other hand, a Jacobi field along geodesics of a given Riemannian manifold $(\bar{M}, \bar{g})$ plays an important role in the study of differential geometry. It satisfies a well-known differential equation which inspires Jacobi operators. It is defined by $(\bar{R}_X(Y))(p) = (\bar{R}(Y, X)X)(p)$, where $\bar{R}$ denotes the curvature tensor of $\bar{M}$ and $X, Y$ denote any vector fields on $\bar{M}$. It is known to be a self-adjoint endomorphism on the tangent space $T_p\bar{M}, p \in \bar{M}$. Clearly, each tangent vector field $X$ to $M$ provides a Jacobi operator with respect to $X$. Thus the Jacobi operator on a real hypersurface $M$ of $\bar{M}$ with respect to $\xi$ is said to be a structure Jacobi operator and will be denoted by $R_\xi$. The Riemannian curvature tensor of $M$ (resp., $\bar{M}$) is denoted by $R$ (resp., $\bar{R}$).

For a commuting problem concerned with the structure Jacobi operator $R_\xi$ and the structure tensor $\phi$ of Hopf hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$, that is, $R_\xi \phi A = A R_\xi \phi$, Lee, Suh and Woo [5] proved that a Hopf hypersurface $M$ with $R_\xi \phi A = A R_\xi \phi$ and $\xi \alpha = 0$ is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. Motivated by this result, we consider the same condition in the different ambient space, that is,

$$R_\xi \phi AX = AR_\xi \phi X$$

for any tangent vector field $X$ on $M$ in $SU_{2,m}/S(U_2U_m)$. The geometric meaning of $R_\xi \phi AX = AR_\xi \phi X$ can be explained in such a way that any eigenspace of $R_\xi$ on the distribution $\mathcal{C} = \{X \in T_pM \mid X \perp \xi\}, p \in M$, is invariant under the shape operator $A$ of $M$ in $SU_{2,m}/S(U_2U_m)$. Then by using [13] Theorem 1, we give a complete classification of Hopf hypersurfaces in $SU_{2,m}/S(U_2U_m)$ with $R_\xi \phi AX = AR_\xi \phi X$ as follows:

**Theorem 1.** Let $M$ be a Hopf hypersurface in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2U_m)$, $m \geq 3$ with $R_\xi \phi A = AR_\xi \phi$. If the Reeb curvature
\[ \alpha = g(A\xi, \xi) \text{ is constant along the Reeb direction of the structure vector field } \xi, \]
then \( M \) is locally congruent to one of the following:

(i) a tube over a totally geodesic \( SU_{2,m-1}/S(U_2 U_{m-1}) \) in \( SU_{2,m}/S(U_2 U_m) \) or

(ii) a horosphere in \( SU_{2,m}/S(U_2 U_m) \) whose center at infinity is singular and of type \( JX \in \mathfrak{J}X \).

From the Riemannian curvature tensor \( R \) of \( M \), we can define the Ricci tensor \( S \) of \( M \) in such a way that

\[ g(SX,Y) = \sum_{i=1}^{4m-1} g(R(e_i, X)Y, e_i), \]

where \( \{ e_1, \ldots, e_{4m-1} \} \) denotes a basis of the tangent space \( T_p M \), \( p \in M \), in \( SU_{2,m}/S(U_2 U_m) \) (see [15]). Then we can consider another new commuting condition

\[ R_\xi \phi SX = SR_\xi \phi X \]

for any tangent vector field \( X \) on \( M \). That is, the operator \( R_\xi \phi \) commutes with the Ricci tensor \( S \).

Then by [13] Theorem 1, we also give another classification related to the Ricci tensor \( S \) of \( M \) in \( SU_{2,m}/S(U_2 U_m) \) as follows:

**Theorem 2.** Let \( M \) be a Hopf hypersurface in complex hyperbolic two-plane Grassmannians \( SU_{2,m}/S(U_2 U_m) \), \( m \geq 3 \) with \( R_\xi \phi S = SR_\xi \phi \). If the smooth function \( \alpha = g(A\xi, \xi) \) is constant along the direction of \( \xi \), then \( M \) is locally congruent to one of the following:

(i) a tube over a totally geodesic \( SU_{2,m-1}/S(U_2 U_{m-1}) \) in \( SU_{2,m}/S(U_2 U_m) \) or

(ii) a horosphere in \( SU_{2,m}/S(U_2 U_m) \) whose center at infinity is singular and of type \( JX \in \mathfrak{J}X \).

In this paper, we refer [10], [13], [14] and [15] for Riemannian geometric structures of complex hyperbolic two-plane Grassmannians \( SU_{2,m}/S(U_2 U_m) \), \( m \geq 3 \).

1. **The Complex Hyperbolic Two-Plane Grassmannian \( SU_{2,m}/S(U_2 U_m) \)**

In this section we summarize basic material about complex hyperbolic two-plane Grassmann manifolds \( SU_{2,m}/S(U_2 U_m) \), for details we refer to [9], [11], [13] and [15]. The Riemannian symmetric space \( SU_{2,m}/S(U_2 U_m) \), which consists of all complex two-dimensional linear subspaces in indefinite complex Euclidean space \( \mathbb{C}^{m+2} \) is a connected, simply connected, irreducible Riemannian symmetric space of noncompact type and with rank two. Let \( G = SU_{2,m} \) and \( K = S(U_2 U_m) \), and denote by \( \mathfrak{g} \) and \( \mathfrak{t} \) the corresponding Lie algebra of the Lie group \( G \) and \( K \) respectively. Let \( B \) be the Killing form of \( \mathfrak{g} \) and denote by \( \mathfrak{p} \) the orthogonal complement of \( \mathfrak{t} \) in \( \mathfrak{g} \) with respect to \( B \). The resulting decomposition \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p} \) is a Cartan decomposition of \( \mathfrak{g} \). The Cartan involution \( \theta \in Aut(\mathfrak{g}) \) on \( su_{2,m} \) is given by \( \theta(A) = I_{2,m} AI_{2,m} \), where

\[ I_{2,m} = \begin{pmatrix} -I_2 & 0_{2,m} \\ 0_{m,2} & I_m \end{pmatrix}, \]

\( I_2 \) and \( I_m \) denote the identity \( 2 \times 2 \)-matrix and \( m \times m \)-matrix respectively. Then \( < X, Y > = -B(X, \theta Y) \) becomes a positive definite \( Ad(K) \)-invariant inner product
on \( g \). Its restriction to \( p \) induces a metric \( g \) on \( SU_{2,m}/S(U_2U_m) \), which is also known as the Killing metric on \( SU_{2,m}/S(U_2U_m) \). Throughout this paper we consider \( SU_{2,m}/S(U_2U_m) \) together with this particular Riemannian metric \( g \).

The Lie algebra \( \mathfrak{k} \) decomposes orthogonally into \( \mathfrak{k} = \mathfrak{su}_2 \oplus \mathfrak{su}_m \oplus \mathfrak{u}_1 \), where \( \mathfrak{u}_1 \) is the one-dimensional center of \( \mathfrak{k} \). The adjoint action of \( \mathfrak{su}_2 \) on \( p \) induces the quaternionic Kähler structure \( \mathfrak{j} \) on \( SU_{2,m}/S(U_2U_m) \), and the adjoint action of

\[
Z = \begin{pmatrix}
\frac{m+2}{m+2} I_2 & 0_{2,m} & 0_{2,m} \\
0_{m,2} & -\frac{2i}{m+2} I_m
\end{pmatrix} \in \mathfrak{u}_1
\]

induces the Kähler structure \( J \) on \( SU_{2,m}/S(U_2U_m) \). By construction, \( J \) commutes with each almost Hermitian structure \( J_\nu \) in \( \mathfrak{j} \) for \( \nu = 1, 2, 3 \). Recall that a canonical local basis \( \{ J_1, J_2, J_3 \} \) of a quaternionic Kähler structure \( \mathfrak{j} \) consists of three almost Hermitian structures \( J_1, J_2, J_3 \) in \( \mathfrak{j} \) such that \( J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu \), where the index \( \nu \) is to be taken modulo 3. The tensor field \( JJ_\nu \), which is locally defined on \( SU_{2,m}/S(U_2U_m) \), is self-adjoint and satisfies \( (JJ_\nu)^2 = I \) and \( \text{tr} (JJ_\nu) = 0 \), where \( I \) is the identity transformation. For a nonzero tangent vector \( X \), we define \( \mathbb{R}X = \{ \lambda X | \lambda \in \mathbb{R} \} \), \( \mathbb{C}X = \mathbb{R}X \oplus \mathbb{R}JX \), and \( \mathbb{H}X = \mathbb{R}X \oplus \mathbb{J}X \).

We identify the tangent space \( T_oSU_{2,m}/S(U_2U_m) \) of \( SU_{2,m}/S(U_2U_m) \) at \( o \) with \( p \) in the usual way. Let \( a \) be a maximal abelian subspace of \( p \). Since \( SU_{2,m}/S(U_2U_m) \) has rank two, the dimension of any such subspace is two. Every nonzero tangent vector \( X \in T_oSU_{2,m}/S(U_2U_m) \cong p \) is contained in some maximal abelian subspace of \( p \). Generically this subspace is uniquely determined by \( X \), in which case \( X \) is called regular. If there exist more than one maximal abelian subspaces of \( p \) containing \( X \), then \( X \) is called singular. There is a simple and useful characterization of the singular tangent vectors: A nonzero tangent vector \( X \in p \) is singular if and only if \( JX \in \mathbb{J}X \) or \( JX \perp \mathbb{J}X \).

Up to scaling there exists a unique \( SU_{2,m} \)-invariant Riemannian metric \( g \) on \( SU_{2,m}/S(U_2U_m) \). Equipped with this metric, \( SU_{2,m}/S(U_2U_m) \) is a Riemannian symmetric space of rank two which is both Kähler and quaternionic Kähler. For computational reasons we normalize \( g \) such that the minimal sectional curvature of \( (SU_{2,m}/S(U_2U_m),g) \) is \(-4\). The sectional curvature \( K \) of the noncompact symmetric space \( SU_{2,m}/S(U_2U_m) \) equipped with the Killing metric \( g \) is bounded by \(-4\leq K \leq 0\). The sectional curvature \(-4\) is obtained for all two-planes \( \mathbb{C}X \) when \( X \) is a non-zero vector with \( JX \in \mathbb{J}X \).

When \( m = 1 \), \( G_2^*(\mathbb{C}^3) = SU_{1,2}/S(U_1U_2) \) is isometric to the two-dimensional complex hyperbolic space \( \mathbb{C}H^2 \) with constant holomorphic sectional curvature \(-4\).

When \( m = 2 \), we note that the isomorphism \( SO(4,2) \simeq SU_{2,2} \) yields an isometry between \( G_2^*(\mathbb{C}^4) = SU_{2,2}/S(U_2U_2) \) and the indefinite real Grassmann manifold \( G_2^*(\mathbb{R}^6) \) of oriented two-dimensional linear subspaces of an indefinite Euclidean space \( \mathbb{R}^6 \). For this reason we assume \( m \geq 3 \) from now on, although many of the subsequent results also hold for \( m = 1, 2 \).

From now on, hereafter \( X, Y \) and \( Z \) always stand for any tangent vector fields on \( M \).
The Riemannian curvature tensor $\bar{R}$ of $SU_{2,m}/S(U_2U_m)$ is locally given by

$$-2\bar{R}(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX$$
$$- g(JX,Z)JY - 2g(JX,Y)JZ$$
$$+ \sum_{\nu=1}^{3} \{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\}$$
$$+ \sum_{\nu=1}^{3} \{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\},$$

where $\{J_1, J_2, J_3\}$ is any canonical local basis of $\mathcal{J}$.

2. Fundamental formulas in $SU_{2,m}/S(U_2U_m)$

In this section, we derive some basic formulas and the Codazzi equation for a real hypersurface in $SU_{2,m}/S(U_2U_m)$ (see \cite{13}, \cite{14} and \cite{15}).

Let $M$ be a real hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2U_m)$, that is, a hypersurface in $SU_{2,m}/S(U_2U_m)$ with real codimension one. The induced Riemannian metric on $M$ will also be denoted by $g$, and $\nabla$ denotes the Levi Civita covariant derivative of $(M, g)$. We denote by $\mathcal{C}$ and $\mathcal{Q}$ the maximal complex and quaternionic subbundle of the tangent bundle $TM$ of $M$, respectively. Now let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu (X)N$$

for any tangent vector field $X$ of a real hypersurface $M$ in $SU_{2,m}/S(U_2U_m)$, where $\phi X$ denotes the tangential component of $JX$ and $N$ a unit normal vector field of $M$ in $SU_{2,m}/S(U_2U_m)$.

From the Kähler structure $J$ of $SU_{2,m}/S(U_2U_m)$ there exists an almost contact metric structure $(\phi, \xi, \eta, g)$ induced on $M$ in such a way that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi \xi = 0, \quad \eta(X) = g(X, \xi)$$

for any vector field $X$ on $M$. Furthermore, let $\{J_1, J_2, J_3\}$ be a canonical local basis of $\mathcal{J}$. Then the quaternionic Kähler structure $J_\nu$ of $SU_{2,m}/S(U_2U_m)$, together with the condition $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, in section 1, induces an almost contact metric 3-structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on $M$ as follows:

$$\phi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu, \quad \eta_\nu(\xi_\nu) = 1, \quad \phi_\nu \xi_\nu = 0,$$
$$\phi_{\nu+1} \xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu \xi_{\nu+1} = \xi_{\nu+2},$$
$$\phi_\nu \phi_{\nu+1} X = \phi_{\nu+2} X + \eta_{\nu+1}(X)\xi_\nu,$$
$$\phi_{\nu+1} \phi_\nu X = -\phi_{\nu+2} X + \eta_\nu(X)\xi_{\nu+1}$$

for any vector field $X$ tangent to $M$. Moreover, from the commuting property of $J_\nu J = J J_\nu$, $\nu = 1, 2, 3$ in section 1 and $2.1$, the relation between these two contact metric structures $(\phi, \xi, \eta, g)$ and $(\phi_\nu, \xi_\nu, \eta_\nu, g)$, $\nu = 1, 2, 3$, can be given by

$$\phi_\nu (\phi X) = \phi_\nu \phi X + \eta_\nu(X)\xi_\nu,$$
$$\eta_\nu(\phi X) = \eta(\phi_\nu X), \quad \phi_\nu = \phi_\nu \xi.$$
On the other hand, from the parallelism of Kähler structure $J$, that is, $\tilde{\nabla}J = 0$ and the quaternionic Kähler structure $\mathfrak{J}$, together with Gauss and Weingarten formulas, it follows that

\begin{align}
(\nabla_X \phi) Y &= \eta(Y) AX - g(A X, Y) \xi, \quad \nabla_X \xi = \phi AX, \\
(\nabla_X \xi)_\nu &= q_{\nu+2}(X) \xi_{\nu+1} - q_{\nu+1}(X) \xi_{\nu+2} + \phi_\nu AX, \\
(\nabla_X \phi_\nu) Y &= -q_{\nu+1}(X) \phi_{\nu+2} Y + q_{\nu+2}(X) \phi_{\nu+1} Y + \eta_\nu(Y) AX
\end{align}

(2.5) (2.6) (2.7)

Combining these formulas, we find the following:

\begin{align}
\nabla_X (\phi_\nu \xi) &= \nabla_X (\phi_\nu) \\
&= (\nabla_X \phi) \xi_\nu + \phi(\nabla_X \xi_\nu) \\
&= q_{\nu+2}(X) \phi_\nu \xi_\nu - q_{\nu+1}(X) \phi_\nu \xi_{\nu+2} + \phi_\nu \phi AX \\
&= g(AX, \xi_\nu) + \eta(\xi_\nu) AX.
\end{align}

(2.8)

Finally, using the explicit expression for the Riemannian curvature tensor $R$ of $SU_{2,m}/S(U_2 U_m)$ in [14], the Codazzi equation takes the form

\begin{align}
-2(\nabla_X A) Y + 2(\nabla_Y A) X &= \eta(X) \phi Y - \eta(Y) \phi X - 2g(\phi X, Y) \xi \\
&+ \sum_{\nu=1}^3 \left\{ \eta_\nu(X) \phi_\nu Y - \eta_\nu(Y) \phi_\nu X - 2g(\phi_\nu X, Y) \xi_\nu \right\} \\
&+ \sum_{\nu=1}^3 \left\{ \eta_\nu(\phi X) \phi_\nu Y - \eta_\nu(\phi Y) \phi_\nu X \right\} \\
&+ \sum_{\nu=1}^3 \left\{ \eta_\nu(X) \eta_\nu(\phi Y) - \eta_\nu(Y) \eta_\nu(\phi X) \right\} \xi_\nu,
\end{align}

(2.9)

for any vector fields $X$ and $Y$ on $M$.

On the other hand, by differentiating $AX = \alpha \xi$ and using (2.4), we get the following

\begin{align}
g(\phi X, Y) &= \sum_{\nu=1}^3 \left\{ \eta_\nu(X) \eta_\nu(\phi Y) - \eta_\nu(Y) \eta_\nu(\phi X) - g(\phi_\nu X, Y) \eta_\nu(\xi) \right\} \\
&= g(AX, Y) - (AX, Y) \\
&= g(AX, 2X) - g((\nabla_X A) \xi, Y) - g((\nabla_Y A) \xi, X) \\
&= (X \alpha) \eta(Y) - (Y \alpha) \eta(X) + \alpha g((A \phi + \phi A) X, Y) - 2g(A \phi AX, Y).
\end{align}

(2.10)

Putting $X = \xi$ gives

\begin{align}
Y \alpha &= \eta(Y) + 2 \sum_{\nu=1}^3 \eta_\nu(\xi) \eta_\nu(\phi Y).
\end{align}

(2.11)

Then, substituting (2.11) into (2.10) the above equation, we have the following

\begin{align}
A \phi AX &= \frac{\alpha}{2} (A \phi + \phi A) Y + \sum_{\nu=1}^3 \left\{ \eta_\nu(Y) \eta_\nu(\xi) \phi_\nu Y + \eta_\nu(\xi) \eta_\nu(\phi Y) \phi_\nu Y \right\} \\
&- \frac{1}{2} \phi Y - \frac{1}{2} \sum_{\nu=1}^3 \left\{ \eta_\nu(Y) \phi_\nu Y + \eta_\nu(\phi Y) \phi_\nu Y + \eta_\nu(\xi) \phi_\nu Y \right\}.
\end{align}

(2.12)
By differentiating and using (2.4), (2.5) and (2.6), we have
\[ \nabla_X (\text{grad} \alpha) = X(\xi_\alpha) \phi AX \]
\[ - 2 \sum_{\nu=1}^{3} \left\{ q_{\nu+2}(X) \eta_{\nu+1}(\xi) - q_{\nu+1}(X) \eta_{\nu+2}(\xi) + 2 \eta_{\nu}(\phi AX) \right\} \phi \xi_\nu \]
\[ - 2 \sum_{\nu=1}^{3} \eta_{\nu}(\xi) \left\{ - q_{\nu+1}(X) \phi_{\nu+2} \xi + q_{\nu+2}(X) \phi_{\nu+1} \xi + \eta_{\nu}(\xi) AX \right\} \]
\[ - g(AX, \xi) \xi_\nu + \phi_\nu \phi AX \]
\[ = X(\xi_\alpha) \phi AX - 4 \sum_{\nu=1}^{3} \eta_{\nu}(\phi AX) \phi \xi_\nu \]
\[ - 2 \sum_{\nu=1}^{3} \eta_{\nu}(\xi) \left\{ \eta_{\nu}(\xi) AX - g(AX, \xi) \xi_\nu + \phi_\nu \phi AX \right\}. \]

By taking the skew-symmetric part to the above equation, we have
\[ 0 = X(\xi_\alpha) \eta(Y) - Y(\xi_\alpha) \eta(X) + (\xi_\alpha) g((A \phi + \phi A)X, Y) \]
\[ - 4 \sum_{\nu=1}^{3} \left\{ \eta_{\nu}(\phi AX) g(\phi \xi_\nu, Y) - \eta_{\nu}(\phi AY) g(\phi \xi_\nu, X) \right\} \]
\[ + 2 \alpha \sum_{\nu=1}^{3} \eta_{\nu}(\xi) \left\{ \eta(X) \eta_{\nu}(Y) - \eta(Y) \eta_{\nu}(X) \right\} \]
\[ - 2 \sum_{\nu=1}^{3} \eta_{\nu}(\xi) \left\{ g(\phi_\nu \phi AX, Y) - g(\phi_\nu \phi AY, X) \right\}. \]

From this, by putting \( X = \xi \) we have the following
\[ (2.13) \quad Y(\xi_\alpha) = \xi(\xi_\alpha) \eta(Y) + 2 \alpha \sum_{\nu=1}^{3} \eta_{\nu}(\xi) \eta_{\nu}(Y) - 2 \sum_{\nu=1}^{3} \eta_{\nu}(\xi) \eta_{\nu}(AY). \]

From this, if we assume that \( \xi_\alpha = 0 \), then it follows that
\[ \sum_{\nu=1}^{3} \eta_{\nu}(\xi) \eta_{\nu}(AX) = \alpha \sum_{\nu=1}^{3} \eta_{\nu}(\xi) \eta_{\nu}(X). \]

**Lemma 2.1.** Let \( M \) be a Hopf real hypersurface in \( SU_{2,m}/S(U_2 U_m) \). If the principal curvature \( \alpha \) is constant along the direction of \( \xi \), then the distribution \( \mathcal{Q} \) or \( \mathcal{Q}^\perp \) component of the structure vector field \( \xi \) is invariant by the shape operator.

3. **Proof of Theorem 1**

Let \( M \) be a Hopf hypersurface in \( SU_{2,m}/S(U_2 U_m) \) with
\[ (C-1) \quad R_\xi \phi AX = A R_\xi \phi X. \]

The structure Jacobi operator \( R_\xi \) of \( M \) is defined by \( R_\xi X = R(X, \xi) \xi \) for any tangent vector \( X \in T_p M, p \in M \) (see [1] and [7]). Then for any tangent vector
field $X$ on $M$ in $SU_{2,m}/S(U_2U_m)$, we calculate the structure Jacobi operator $R_{\xi}$
\[ 2R_{\xi}(X) = 2R(X, \xi)\xi \]
\[ (3.1) \]
\[
- \eta + \eta(X)\xi + \sum_{\nu=1}^3 \{ \eta_{\nu}(X)\xi_{\nu} - \eta(X)\eta_{\nu}(\xi)\xi_{\nu} \\
+ 3\eta_{\nu}(\phi X)\phi_{\nu}\xi + \eta_{\nu}(\xi)\phi_{\nu}\phi X \} + 2\alpha AX - 2\eta(AX)A\xi, 
\]
where $\alpha$ denotes the Reeb curvature defined by $g(A\xi, \xi)$.

**Lemma 3.1.** Let $M$ be a Hopf hypersurface in $SU_{2,m}/S(U_2U_m)$ with the commuting condition $R_{\xi}\phi AX = AR_{\xi}\phi X$. If the smooth function $\alpha$ is constant along the direction of $\xi$ on $M$, then the Reeb vector field $\xi$ belongs to either the distribution $Q$ or the distribution $Q^\perp$.

**Proof.** To prove this lemma, without loss of generality, $\xi$ may be written as
\[ \xi = \eta(X_0)X_0 + \eta(\xi)\xi_1 \]
where $X_0$ (resp., $\xi_1$) is a unit vector in $Q$ (resp., $Q^\perp$) and $\eta(X_0)\eta(\xi_1) \neq 0$.

From $\phi$ and $\phi \xi = 0$, we have
\[
\phi X_0 = -\eta(\xi_1)\phi X_0, \\
\phi \xi_1 = \phi_1 \xi = \eta(X_0)\phi X_0, \\
\phi_1 \phi X_0 = \eta(\xi)X_0. 
\]
(3.2)

Let $\mathcal{U} = \{ p \in M \mid \alpha(p) \neq 0 \}$ be an open subset of $M$. From now on, we discuss our arguments on $\mathcal{U}$. By virtue of Lemma 2.11, $\xi\alpha = 0$ gives $AX_0 = \alpha X_0$ and $A\xi_1 = \alpha \xi_1$.

The equation (2.12) yields $\alpha A\phi X_0 = (\alpha^2 - 2\eta^2(X_0))\phi X_0$ by substituting $X = X_0$.

Since $\alpha$ is non-vanishing on $\mathcal{U}$, it becomes
\[ A\phi X_0 = \sigma\phi X_0, \]
where $\sigma = \alpha^2 - 2\eta^2(X_0)$.

From (3.2) and (3.3), we have
\[
R_{\xi}(X_0) = \alpha^2 X_0 - \alpha^2 \eta(X_0)\xi, \\
R_{\xi}(\xi_1) = \alpha^2 \xi_1 - \alpha^2 \eta(\xi_1)\xi, \\
R_{\xi}(\phi X_0) = (\alpha^2 - 4\eta^2(X_0))\phi X_0. 
\]
(3.4)

On $\mathcal{U}$, substituting $X$ by $\phi X_0$ into (C-1), we have
\[ X_0 - \eta(X_0)\xi = 0, \]
which is a contradiction. Therefore, $\mathcal{U} = \emptyset$, and thus it must be $p \in M - \mathcal{U}$. Since the set $M - \mathcal{U} = \text{Int}(M - \mathcal{U}) \cup \partial(M - \mathcal{U})$, we consider the following two cases. Here $\text{Int}$ (resp., $\partial$) denotes an interior (resp., the boundary) of $(M - \mathcal{U})$.

**Case 1.** $p \in \text{Int}(M - \mathcal{U})$.

If $p \in \text{Int}(M - \mathcal{U})$, then $\alpha = 0$. For this case, it was proved by the equation (2.11).

**Case 2.** $p \in \partial(M - \mathcal{U})$.

Since $p \in \partial(M - \mathcal{U})$, there exists a sequence of points $p_n$ such that $p_n \to p$ with $\alpha(p) = 0$ and $\alpha(p_n) \neq 0$. Such a sequence will have an infinite subsequence where $\eta(\xi_1) = 0$ (in which case $\xi \in Q$ at $p$, by the continuity) or an infinite subsequence where $\eta(X_0) = 0$ (in which case $\xi \in Q^\perp$ at $p$).
Accordingly, we get a complete proof of our lemma. \qed

From Lemma 3.1, we consider the case that $\xi$ belongs to the distribution $Q^\perp$. Thus without loss of generality, we may put $\xi = \xi_1$. Differentiating $\xi = \xi_1$ along any direction $X \in TM$ and using (2.5) and (2.6), it gives us

\[2\eta_3(AX)\xi_2 - 2\eta_2(AX)\xi_3 + \phi_1 AX - \phi AX = 0.\]

Then, by using the symmetric (resp., skew-symmetric) property of the shape operator $A$ (resp., the structure tensor field $\phi$), we also obtain

\[2\eta_3(X)A\xi_2 - 2\eta_2(X)A\xi_3 + A\phi_1 X - A\phi X = 0.\]

Applying $\phi_1$ to (3.6), it implies

\[2\eta_3(X)A\xi_2 + 2\eta_2(X)\xi_2 - AX + \alpha\eta(X)\xi - \phi_1 \phi AX = 0.\]

On the other hand, replacing $X = \phi X$ into (3.6), we have

\[-2\eta_2(X)A\xi_2 - 2\eta_3(X)A\xi_3 + A\phi_1 \phi X - AX - \alpha\eta(X)\xi = 0.\]

**Lemma 3.2.** Let $M$ be a Hopf hypersurface in $SU_{2,m}/S(U_2U_m)$, $m \geq 3$, with $R_\xi \phi A = AR_\xi \phi$. If the Reeb vector field $\xi$ belongs to the distribution $Q^\perp$, then the shape operator $A$ commutes with the structure tensor field $\phi$.

**Proof.** Applying $\xi = \xi_1$ into right hand side (resp., left hand side) of (C-1), we get

\[2R_\xi \phi AX = -A\phi X + 2\alpha A^2 \phi X - 2\eta_3(X)A\xi_2 + 2\eta_2(X)A\xi_3 - A\phi_1 X,
\]

\[2AR_\xi \phi X = -\phi AX + 2\alpha A\phi AX - 2\eta_3(AX)\xi_2 + 2\eta_2(AX)\xi_3 - \phi_1 AX.\]

Combining (3.6) and (3.9), the above equations become

\[R_\xi \phi AX = -A\phi X + \alpha A^2 \phi X,
\]

\[AR_\xi \phi X = -\phi AX + \alpha A\phi AX.\]

Hence, (C-1) is equivalent to

\[A\phi - \phi A = \alpha A(A\phi - \phi A)\]

Taking the symmetric part of (3.10), we have

\[A\phi - \phi A = \alpha (A\phi - \phi A) A.\]

From this, we can divide into the following three cases:

First, let us consider an open subset $\Omega = \{p \in M \mid \alpha(p) \neq 0\}$ of $M$. Naturally we can apply (3.10) and (3.11) on the open subset $\Omega$.

\[(A\phi - \phi A)AX = A(A\phi - \phi A)X.\]

Since the shape operator $A$ and the tensor $A\phi - \phi A$ are both symmetric operators and commute with each other, there exists a common orthonormal basis $\{E_i\}_{i=1,\ldots,4m-1}$ which gives a simultaneous diagonalization. Specifically, we have

\[AE_i = \lambda_i E_i,\]

\[(A\phi - \phi A)E_i = \beta_i E_i,\]

where $\lambda_i$ and $\beta_i$ are scalars for all $i = 1, 2, \ldots, 4m - 1$.

Taking the inner product with $E_i$ into (3.13), we have

\[\beta_i g(E_i, E_i) = g((A\phi - \phi A)E_i, E_i) = 2\lambda_i g(\phi E_i, E_i) = 0.\]
Since $g(E_i, E_i) = 1, \beta_i = 0$ for all $i = 1, 2, \ldots, 4m - 1$. Hence $A\phi X = \phi AX$ for any tangent vector field $X$ on $\mathcal{M}$.

Next, if $p \in \text{Int}(M - \mathcal{M})$, then $\alpha(p) = 0$. From this, the equation (3.11) gives $(A\phi - \phi A)X(p) = 0$.

Finally, let us assume that $p \in \partial(M - \mathcal{M})$, where $\partial(M - \mathcal{M})$ is the boundary of $M - \mathcal{M}$. Then there exists a subsequence $\{p_n\} \subseteq \mathcal{M}$ such that $p_n \to p$. Since $(A\phi - \phi A)X(p_n) = 0$ on the open subset $\mathcal{M}$ in $M$, by the continuity we also get $(A\phi - \phi A)X(p) = 0$.

Summing up these observations, it is natural that the shape operator $A$ commutes with the structure tensor field $\phi$ under our assumption. □

By [11] we assert $M$ with the assumptions given in lemma 3.2 is locally congruent to one of the following hypersurfaces:

$(T_A)$ a tube over a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1})$ in $SU_{2,m}/S(U_2U_m)$ or,

$(H_A)$ a horosphere in $SU_{2,m}/S(U_2U_m)$ whose center at infinity is singular and of type $JX \in \mathfrak{J}X$.

In a paper due to [11], Suh gave some information related to the shape operator $A$ of $T_A$ and $H_A$ as follows:

**Proposition A.** Let $M$ be a connected real hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2U_m)$, $m \geq 3$. Assume that the maximal complex subbundle $\mathcal{C}$ of $TM$ and the maximal quaternionic subbundle $\mathcal{Q}$ of $TM$ are both invariant under the shape operator of $M$. If $JN \in \mathfrak{J}N$, then one of the following statements holds:

$(T_A)$ $M$ has exactly four distinct constant principal curvatures

$$\alpha = 2 \coth(2r), \ \beta = \coth(r), \ \lambda_1 = \tanh(r), \ \lambda_2 = 0,$$

and the corresponding principal curvature spaces are

$$T_n = TM \ominus \mathcal{C}, \ T_\beta = \mathcal{C} \ominus \mathcal{Q}, \ T_{\lambda_1} = E_{-1}, \ T_{\lambda_2} = E_{+1}.$$  

The principal curvature spaces $T_{\lambda_1}$ and $T_{\lambda_2}$ are complex (with respect to $J$) and totally complex (with respect to $\mathfrak{J}$).

$(H_A)$ $M$ has exactly three distinct constant principal curvatures

$$\alpha = 2, \ \beta = 1, \ \lambda = 0$$

with corresponding principal curvature spaces

$$T_n = TM \ominus \mathcal{C}, \ T_\beta = (\mathcal{C} \ominus \mathcal{Q}) \oplus E_{-1}, \ T_\lambda = E_{+1}.$$  

Here, $E_{+1}$ and $E_{-1}$ are the eigenbundles of $\phi\phi_1|_{\mathcal{Q}}$ with respect to the eigenvalues $+1$ and $-1$, respectively.

Since the symmetric tensor $A\phi - \phi A$ vanishes identically on $T_A$ (resp. $H_A$), it trivially satisfies (6.11). Hence we assert that $T_A$ (resp., $H_A$) in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2U_m)$ has the our commuting condition (C-1) (see [11]).

Next, due to Lemma 3.1, let us suppose that $\xi \in \mathcal{Q}$ (i.e., $JN \perp \mathfrak{J}N$).
By virtue of the result in [13], we assert that a Hopf hypersurface $M$ in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2U_m)$ satisfying the hypotheses in Theorem 1 is locally congruent to
(T_B) $M$ is an open part of a tube around a totally geodesic quaternionic hyperbolic space $\mathbb{H}^n$ in $SU_{2,2m}/S(U_2U_m)$, $m = 2n$.

(\mathcal{H}_B) $M$ is an open part of a horosphere in $SU_{2,m}/S(U_2U_m)$ whose center at infinity is singular and of type $JN \perp 3N$, or

(\mathcal{E}) The normal bundle $\nu M$ of $M$ consists of singular tangent vectors of type $JX \perp 3X$.

when $\xi \in Q$. Hereafter, the model spaces of $T_B$, $\mathcal{H}_B$ or $\mathcal{E}$ is denoted by $M_B$. Let us check whether the shape operator $A$ of model spaces of $M_B$ satisfy our conditions, conversely. In order to do this, let us introduce the following proposition given by Suh [13].

**Proposition B.** Let $M$ be a connected hypersurface in $SU_{2,m}/S(U_2U_m)$, $m \geq 3$. Assume that the maximal complex subbundle $\mathcal{C}$ of $TM$ and the maximal quaternionic subbundle $Q$ of $TM$ are both invariant under the shape operator of $M$. If $JN \perp 3N$, then one of the following statements holds:

(T_B) $M$ has five (four for $r = \sqrt{2}\tanh^{-1}(1/\sqrt{3})$ in which case $\alpha = \lambda_2$) distinct constant principal curvatures

\[ \alpha = \sqrt{2}\tanh(\sqrt{2}r), \quad \beta = \sqrt{2}\coth(\sqrt{2}r), \quad \gamma = 0, \]

\[ \lambda_1 = \frac{1}{\sqrt{2}}\tanh\left(\frac{1}{\sqrt{2}}r\right), \quad \lambda_2 = \frac{1}{\sqrt{2}}\coth\left(\frac{1}{\sqrt{2}}r\right), \]

and the corresponding principal curvature spaces are

\[ T_\alpha = TM \oplus \mathcal{C}, \quad T_\beta = TM \ominus \mathcal{C}, \quad T_\gamma = J(TM \ominus \mathcal{C}) = JT_\beta. \]

The principal curvature spaces $T_\lambda_1$ and $T_\lambda_2$ are invariant under $3$ and are mapped onto each other by $J$. In particular, the quaternionic dimension of $SU_{2,m}/S(U_2U_m)$ must be even.

(\mathcal{H}_B) $M$ has exactly three distinct constant principal curvatures

\[ \alpha = \beta = \sqrt{2}, \quad \gamma = 0, \quad \lambda = \frac{1}{\sqrt{2}} \]

with corresponding principal curvature spaces

\[ T_\alpha = TM \ominus (\mathcal{C} \cap Q), \quad T_\gamma = J(TM \ominus Q), \quad T_\lambda = \mathcal{C} \cap Q \cap JQ. \]

(\mathcal{E}) $M$ has at least four distinct principal curvatures, three of which are given by

\[ \alpha = \beta = \sqrt{2}, \quad \gamma = 0, \quad \lambda = \frac{1}{\sqrt{2}} \]

with corresponding principal curvature spaces

\[ T_\alpha = TM \ominus (\mathcal{C} \cap Q), \quad T_\gamma = J(TM \ominus Q), \quad T_\lambda \subset \mathcal{C} \cap Q \cap JQ. \]

If $\mu$ is another (possibly nonconstant) principal curvature function, then $JT_\mu \subset T_\lambda$ and $JT_\mu \subset T_\lambda$. Thus, the corresponding multiplicities are

\[ m(\alpha) = 4, \quad m(\gamma) = 3, \quad m(\lambda), \quad m(\mu). \]

Let us assume that the structure Jacobi operator $R_\xi$ of $M_B$ satisfies the property (C9). The tangent space of $M_B$ can be split into

\[ TM = T_{a_1} \oplus T_{a_2} \oplus T_{a_3} \oplus T_{a_4} \oplus T_{a_5}, \]

where $T_{a_1} = [\xi]$, $T_{a_2} = \text{span}\{\xi_1, \xi_2, \xi_3\}$, $T_{a_3} = \text{span}\{\phi \xi_1, \phi \xi_2, \phi \xi_3\}$ and $T_{a_4} \oplus T_{a_5}$ is the orthogonal complement of $T_{a_1} \oplus T_{a_2} \oplus T_{a_3}$ in $TM$. Since $\xi \in Q$ and $\phi \phi_0 \xi =
In Proposition B, for $T_B$ we give a complete proof of Theorem 2. To prove it, we assume that $M$.

It implies that the eigenvalue $\alpha_2$ vanishes, since $\phi \xi_2$ is a unit tangent vector field. But in Proposition B, for $T_B$ (resp. $H_B$ or $E$) we see that the eigenvalue $\alpha_2 = \beta = \sqrt{2} \coth(\sqrt{2} r)$ (resp. $\alpha_2 = \alpha = \frac{1}{\sqrt{2}}$) is non-vanishing. This gives us a contradiction.

4. PROOF OF THEOREM 2

In this section, by using geometric quantities in [3, 4, 5, 13, 14, and 15], we give a complete proof of Theorem 2. To prove it, we assume that $M$ is a Hopf hypersurface in $SU_{2,m}/S(U_2 U_m)$ with commuting structure Jacobi operator and Ricci tensor, that is,

\[(C-2) \quad (R \phi) SX = S(R \phi) X.\]

From the definition of the Ricci tensor and the fundamental formulas in [15, Section 2], the Ricci tensor $S$ of $M$ in $SU_{2,m}/S(U_2 U_m)$ is given by

\[2SX = -(4m + 7)X + 3\eta(X) \xi + 2hAX - 2A^2X\]
\[\quad + \sum_{\nu=1}^{3} \{3\eta_\nu(X) \xi_\nu - \eta_\nu(\phi \xi_\nu) \phi X + \eta_\nu(\phi X) \phi \xi_\nu + \eta(X) \eta_\nu(\xi_\nu) \xi_\nu\},\]

where $h$ denotes the trace of the shape operator $A$.

Using equations (C-2) and (4.1), we prove that the Reeb vector field $\xi$ of $M$ belongs to either the distribution $Q$ or the distribution $Q^\perp$.

**Lemma 4.1.** Let $M$ be a Hopf hypersurface in $SU_{2,m}/S(U_2 U_m)$, $m \geq 3$, with (C-2). If the principal curvature $\alpha = g(\xi, \xi)$ is constant along the direction of $\xi$, then $\xi$ belongs to either the distribution $Q$ or the distribution $Q^\perp$.

**Proof.** In order to prove this lemma, for some unit vectors $X_0 \in Q$, $\xi_1 \in Q^\perp$, we put

\[\xi = \eta(X_0) X_0 + \eta(\xi_1) \xi_1,\]

where $\eta(X_0) \eta(\xi_1) \neq 0$ is the assumption we will disprove in this proof by contradiction.

Let $\Omega = \{p \in M \mid \alpha(p) \neq 0\}$ be the open subset of $M$. From now on, we discuss our arguments on $\Omega$.

By virtue of Lemma 2.1, $\xi \alpha = 0$ gives $AX_0 = \alpha X_0$ and $A\xi_1 = \alpha \xi_1$. From (4.1), we have

\[\begin{align*}
S\phi X_0 &= \kappa \phi X_0, \\
SX_0 &= -(2m - 4 + h\alpha - \sigma^2) X_0 + 2\eta(X_0) \xi, \\
S\xi_1 &= -(2m - 2 + h\alpha - \sigma^2) \xi_1 + 2\eta_1(\xi) \xi_1, \\
S\xi &= -(2m - 2 + h\alpha - \sigma^2) \xi + 2\eta_1(\xi) \xi_1,
\end{align*}\]

where $\kappa := -2m - 4 + h\sigma - \sigma^2$ and $\sigma = \frac{\alpha^2 - 2\sigma^2(X_0)}{\alpha}$ on $\Omega$.\]
Put $X = \phi X_0$ into (C-2), we have
\begin{equation}
\kappa R_\xi (X_0) = S R_\xi (X_0).
\end{equation}
Taking the inner product of (4.3) with $\xi$ and using (3.4) and (4.2), we have
\begin{equation}
-2\alpha^2 \eta^2 (\xi_1) \eta (X_0) = 0.
\end{equation}
It implies that $U = \emptyset$. Thus it must be $p \in M - U$. The set $M - U = \text{Int}(M - U) \cup \partial (M - U)$, where $\text{Int}$ (resp., $\partial$) denotes the interior (resp., the boundary) of $M - U$, we consider the following two cases:

- **Case 1.** $p \in \text{Int}(M - U)$

  If $p \in \text{Int}(M - U)$, then $\alpha = 0$. Our lemma was proved on $\text{Int}(M - U)$ by the equation (2.11) and (*).

- **Case 2.** $p \in \partial(M - U)$

  Since $p \in \partial(M - U)$, there exists a sequence of points $p_n \in U$ such that $p_n \rightarrow p$ with $\alpha(p) = 0$ and $\alpha(p_n) \neq 0$. Such a sequence will have an infinite subsequence where $\eta(\xi_1) = 0$ (in which case $\xi \in Q$ at $p$, by the continuity) or an infinite subsequence where $\eta(X_0) = 0$ (in which case $\xi \in Q^\perp$ at $p$). Accordingly, we get a complete proof of the Lemma.

Now, we shall divide our consideration into two cases that $\xi$ belongs to either the distribution $Q$ or the distribution $Q^\perp$, respectively. Let us consider the case $\xi \in Q^\perp$. We may put $\xi = \xi_1 \in Q^\perp$ for the sake of convenience. Then, (4.1) is simplified:
\begin{equation}
2SX = -(4m + 7)X + 7\eta(X)\xi + 2\eta_2(X)\xi_2
+ 2\eta_3(X)\xi_3 - \phi_1 \phi X + 2hA^2 X - 2A^3 X.
\end{equation}

By replacing $X$ as $AX$ into (4.4) and using (3.8), we obtain
\begin{equation}
2SAX = -(4m + 6)AX + 6\alpha \eta(X)\xi + 2hA^2 X - 2A^3 X.
\end{equation}

Applying the shape operator $A$ to (4.4) and using (3.9), we get
\begin{equation}
2ASX = -(4m + 6)AX + 6\alpha \eta(X)\xi + 2hA^2 X - 2A^3 X.
\end{equation}

From (4.5) and (4.6), we see that the Ricci tensor $S$ commutes with the shape operator $A$, that is,
\begin{equation}
SA = AS.
\end{equation}

On the other hand, the equations (4.5) and (4.6) give us
\begin{equation}
2\eta_3(SX)\xi_2 - 2\eta_2(SX)\xi_3 + \phi_1 SX - \phi SX
= (2m + 4)\{2\eta_3(X)\xi_2 - 2\eta_2(X)\xi_3 + \phi X - \phi_1 X\}
:= \text{Rem}(X).
\end{equation}

Taking the symmetric part of (4.8), we obtain
\begin{equation}
2\eta_3(SX)\xi_2 - 2\eta_2(SX)\xi_3 + \phi_1 X - S\phi X = \text{Rem}(X).
\end{equation}

**Lemma 4.2.** Let $M$ be a Hopf hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$ with (C-2). If $\xi \in Q^\perp$, then $S\phi = \phi S$. 

\[ \]
Proof. By virtue of equation (4.8) and (4.9), we obtain the left and right sides of (C-2), respectively, as follows:

\[
2R_\xi \phi SX = -\phi SX + 2\alpha A\phi SX - 2\eta_3(SX)\xi_2 + 2\eta_2(SX)\xi_3 - \phi_1 SX \\
= -2\phi SX + 2\alpha A\phi SX - \text{Rem}(X),
\]
and

\[
2SR_\xi \phi X = -S\phi X + 2\alpha SA\phi X - 2\eta_3(X)S\xi_2 + 2\eta_2(X)S\xi_3 - S\phi_1 X \\
= -2S\phi X + 2\alpha SA\phi X - \text{Rem}(X).
\]

That is,

\[
(4.10) \quad R_\xi \phi SX = -\phi SX + 2\alpha A\phi SX - \frac{1}{2}\text{Rem}(X)
\]
and

\[
(4.11) \quad SR_\xi \phi X = -S\phi X + 2\alpha SA\phi X - \frac{1}{2}\text{Rem}(X).
\]

From these two equations, the condition (C-2) is equivalent to

\[
(S\phi - \phi S)X = \alpha(SA\phi - A\phi S)X \\
= \alpha A(S\phi - \phi S)X,
\]

by virtue of our assertion that the shape operator $A$ commutes the Ricci tensor $S$ with each other given in (4.7).

Taking the symmetric part of (4.12), we have

\[
(4.13) \quad (S\phi - \phi S)E_i = \alpha(S\phi - \phi S)A E_i
\]
for all tangent vector fields $X$ on $M$.

From (4.12) and (4.13), we know

\[
(4.14) \quad \alpha A(S\phi - \phi S) = \alpha(S\phi - \phi S)A.
\]

Let $\mathcal{U} = \{p \in M \mid \alpha(p) \neq 0\}$ be an open subset of $M$. Then (4.14) implies the shape operator $A$ and the symmetric tensor $S\phi - \phi S$ commute with each other on $\mathcal{U}$. Hence they are simultaneous diagonalizable, there exists a common orthonormal basis $\{E_1, E_2, ..., E_{4m-1}\}$ such that the shape operator $A$ and the tensor $S\phi - \phi S$ both can be diagonalizable. In other words,

\[
(4.15) \quad AE_i = \lambda_i E_i,
\]
\[
(4.16) \quad (S\phi - \phi S)E_i = \beta_i E_i,
\]
where $\lambda_i$ and $\beta_i$ are scalars for all $i = 1, 2, ..., 4m - 1$.

Combining equations in (4.11), we get

\[
(4.17) \quad S\phi X - \phi SX = hA\phi X - A^2\phi X - h\phi AX + \phi A^2 X.
\]

Using (4.15), (4.16) and (4.17), we obtain

\[
(4.18) \quad (S\phi - \phi S)E_i = hA\phi E_i - A^2\phi E_i - h\lambda_i \phi E_i + \lambda_i^2 \phi E_i.
\]

Taking the inner product with $E_i$ into (4.18), we have

\[
\beta_i g(E_i, E_i) = h\lambda_i g(\phi E_i, E_i) - \lambda_i^2 g(\phi E_i, E_i) - h\lambda_i g(\phi E_i, E_i) + \lambda_i^2 g(\phi E_i, E_i) = 0.
\]

Since $g(E_i, E_i) = 1$, we get $\beta_i = 0$ for all $i = 1, 2, ..., 4m - 1$. This is equivalent to

\[
(S\phi - \phi S)E_i = 0 \text{ for all } i = 1, 2, ..., 4m - 1.
\]

It follows that $S\phi X = \phi SX$ for any
we assert that $SU$ congruent to a hypersurface of $M$. Moreover, when $\xi \in \mathbb{Q}^1$, we obtain the following hypersurfaces:

$$\text{(} T_A \text{)} \text{ a tube over a totally geodesic } SU_{2,m-1}/S(U_2U_m-1) \text{ in } SU_{2,m}/S(U_2U_m)$$

or,

$$\text{(} H_A \text{)} \text{ a horosphere in } SU_{2,m}/S(U_2U_m) \text{ whose center at infinity is singular and of type } JX \in \mathfrak{J}X.$$

By virtue of the result given by Suh in [14], we assert that if $\xi \in \mathbb{Q}^1$, then a Hopf hypersurface $M$ in $SU_{2,m}/S(U_2U_m)$ with (C-2) is locally congruent to one of the following hypersurfaces:

- $(T_A)$ a tube over a totally geodesic $SU_{2,m-1}/S(U_2U_m-1)$ in $SU_{2,m}/S(U_2U_m)$
- $(H_A)$ a horosphere in $SU_{2,m}/S(U_2U_m)$ whose center at infinity is singular and of type $JX \in \mathfrak{J}X$.

Moreover, when $\xi \in \mathbb{Q}^1$, (C-2) is equivalent to (1.12). Since the symmetric tensor $(S\phi - \phi S)$ vanishes identically on $T_A$ (resp. $H_A$), it trivially satisfies (1.12). Hence we assert that $T_A$ (resp., $H_A$) in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2U_m)$ has the our commuting condition (C-2) (see [14]).

When $\xi \in \mathbb{Q}$, a Hopf hypersurface $M$ in $SU_{2,m}/S(U_2U_m)$ with (C-2) is locally congruent to a hypersurface of $M_B$ by [13]. From now on, let us show whether model spaces of $M_B$ satisfy the condition (C-2) or not. Then the tangent space of $M_B$ can be splitted into

$$TM_B = T_{\alpha_1} \oplus T_{\alpha_2} \oplus T_{\alpha_3} \oplus T_{\alpha_4} \oplus T_{\alpha_5},$$

where $T_{\alpha_1} = \{\xi\}, T_{\alpha_2} = \text{span}\{\xi_1, \xi_2, \xi_3\}, T_{\alpha_3} = \text{span}\{\phi \xi_1, \phi \xi_2, \phi \xi_3\}$ and $T_{\alpha_4} \oplus T_{\alpha_5}$ is the orthogonal complement of $T_{\alpha_1} \oplus T_{\alpha_2} \oplus T_{\alpha_3}$ in $TM$ such that $JT_{\alpha_5} \subset T_{\alpha_4}$ (see [14]).

On $T_pM_B, p \in M_B$, the equations (4.1) and (3.1) are reduced to the following equations, respectively:

$$2SX = -(4m + 7)X + 3\eta(X)\xi + 2hAX - 2A^2X$$

and

$$2R_{\xi}(X) = -X + \eta(X)\xi + 2\alpha AX - 2\alpha^2\eta(X)\xi$$

$$+ \sum_{\nu=1}^{3}\{\eta_{\nu}(X)\xi_{\nu} + 3\eta_{\nu}(\phi X)\phi_{\nu}\xi\}.$$

From [14] Proposition 5.1, we obtain the following

$$(4.19) \quad SX = \left\{ \begin{array}{ll}
-2m - 2 + h\alpha_1 - \alpha_2^2 \xi & \text{if } X = \xi \in T_{\alpha_1} \\
-2m - 2 + h\alpha_2 - \alpha_3^2 \xi & \text{if } X = \xi \in T_{\alpha_2} \\
-2m - 4 \phi \xi & \text{if } X = \phi \xi \in T_{\alpha_3} \\
-2m - 7 + h\alpha_4 - \lambda_2^2 X & \text{if } X \in T_{\alpha_4} \\
-2m - \frac{7}{2} + h\alpha_5 - \alpha_3^2 X & \text{if } X \in T_{\alpha_5}
\end{array} \right.$$
In order to check whether $T_B$, $H_B$ or $E$ model spaces satisfy the (C-2) or not, we should verify the following equations vanishes for all cases.

\begin{equation}
G(X) := (R_\xi \phi)SX - S(R_\xi \phi)X.
\end{equation}

Putting $X = \xi_1 \in T_{\alpha_1}$ into (4.21), we have $G(\xi_1) = -2(2 + \alpha_2 h - \alpha_2^2)\phi\xi_1$ which derives

\begin{equation}
2 + \alpha_2 h - \alpha_2^2 = 0.
\end{equation}

- **Case 1.** Tube $T_B$

In this case, we get $\alpha_1 = \alpha$, $\alpha_2 = \beta$, $\alpha_3 = \gamma = 0$, $\alpha_4 = \lambda$ and $\alpha_5 = \mu$.

By calculation, we have $\lambda + \mu = \beta$ on $T_B$. Thus we obtain $h = \alpha + 3\beta + (4n - 4)(\lambda + \mu) = \alpha + (2m - 1)\beta$. Then (4.22) is $4 + 2(2 - m - 1)\beta^2 > 0$, which is a contradiction.

- **Case 2.** Horosphere $H_B$

On $H_B$, $\alpha_1 = \sqrt{2}$, $\alpha_2 = \sqrt{2}$, $\alpha_3 = \gamma = 0$, $\alpha_4 = \frac{1}{\sqrt{2}}$ and $\alpha_5 = \frac{1}{\sqrt{2}}$. Thus (4.22) gives $h = 0$. Since $h = \alpha_1 + 3\alpha_2 + 3\alpha_3 + (4n - 4)(\alpha_4 + \alpha_5)$, we have $2\sqrt{2}m = 0$ which is a contradiction.

- **Case 3.** Exceptional case $\mathcal{E}$

For $X \in T_{\alpha_1} \subset T_\mathcal{E}$, $G(X) = -\frac{1}{4}(\alpha_5 - \alpha_4)(\alpha_5 + \alpha_4)\phi X$. On $T_\mathcal{E}$ we have $\alpha_1 = \alpha = \sqrt{2}$, $\alpha_4 = \lambda = \frac{1}{\sqrt{2}}$ and $\alpha_5 = \mu = \pm \frac{\sqrt{2}}{2}$. Because $\mu \neq \lambda$, it should be $\mu = -\frac{\sqrt{2}}{2}$. Moreover, since $J\mathcal{T}_\mu \subset T_\mathcal{E}$ and $3\mathcal{T}_\mu \subset T_\mathcal{E}$, we see that the corresponding multiplicities of the eigenvalues $\mu$ and $\lambda$ satisfy $m(\mu) \geq m(\lambda)$. Since $m(\alpha) = 4$, $m(\gamma) = 3$ and $m(\lambda) + m(\mu) = 4m - 8$ on $\mathcal{E}$, the trace of the shape operator $A$ denoted by $h$ becomes $h = 4\alpha + 3\gamma + m(\lambda)\lambda + m(\mu)\mu = 4\sqrt{2} + \frac{\sqrt{2}}{2}(m(\lambda) - m(\mu))$, which makes a contradiction. In fact, since we obtained $h = 0$ on $\mathcal{T}_\mu \subset T_\mathcal{E}$, it yields $(m(\lambda) - m(\mu)) = -8 < 0$. Thus, this case does not occur.

This shows that hypersurfaces of $T_B$, $H_B$ or $E$ cannot satisfy the condition (C-2), and therefore in the situation of Theorem 2, the case $X \in \mathcal{Q}$ cannot occur. This completes the proof of Theorem 2.

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