GLOBAL EXISTENCE AND BLOW-UP OF SOLUTIONS TO A SINGULAR NON-NEWTON POLYTROPIC FILTRATION EQUATION WITH CRITICAL AND SUPERCRITICAL INITIAL ENERGY

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(Communicated by Junping Shi)

Abstract. In this paper, we revisit the singular Non-Newton polytropic filtration equation, which was studied extensively in the recent years. However, all the studies are mostly concerned with subcritical initial energy, i.e., $E(u_0) < d$, where $E(u_0)$ is the initial energy and $d$ is the mountain-pass level. The main purpose of this paper is to study the behaviors of the solution with $E(u_0) \geq d$ by potential well method and some differential inequality techniques.

1. Introduction. In this paper, we study the following Non-Newton polytropic filtration equation

$$
\begin{cases}
|x|^{-s} u_t - \nabla \cdot (|\nabla u|^{p-2} \nabla u^m) = u^q, & (x, t) \in \Omega \times (0, T), \\
 u(x, t) = 0, & (x, t) \in \partial \Omega \times (0, T), \\
 u(x, 0) = u_0(x), & x \in \Omega,
\end{cases}
$$

(1)

where $u_0(x)$ is a nonnegative and nontrivial function, $T \in (0, +\infty]$ is the maximal existence time of solution, $\Omega$ is a bounded domain in $\mathbb{R}^N$ ($N > p$) with smooth boundary $\partial \Omega$, and the parameters in problem (1) satisfy

$$
m \geq 1, \quad p \geq 2, \quad 0 \leq s \leq 1 + \frac{1}{m},
$$

$$
mp - m < q \leq \frac{m(Np - N + p)}{N - p}.
$$

(2)

When $m = 1$, problem (1) was considered in [14, 15]. Later on, the authors in [9, 16] studied problem (1) with $p = 2$. For $m \geq 1$ and $p \geq 2$, in [17], the
authors tackled the solution of problem (1) by potential well method and Hardy-Sobolev’s inequality. In order to introduce the main results of these papers, we firstly introduce some functions, sets and definitions, which were given in [17] and also be used throughout this paper.

**Definition 1.1.** A nonnegative function \( u(t) \) is called a weak solution of problem (1) on \( Q_T := \Omega \times (0, T) \) if

\[
\begin{align*}
&u^m \in L^\infty \left(0, T; W_0^{1,p}(\Omega)\right), u \in L^{m+q}(Q_T), \\
&\int_0^T \|x|^{-\frac{m+1}{2}} \left(u^{m+1}\right)\,dt < \infty,
\end{align*}
\]

and \( u(x,t) \) satisfies (1) in the distribution sense.

In this paper, we denote the norm of Banach space \( L^r(\Omega) \) and Sobolev space \( W_1^{1,p}_0(\Omega) \) by \( \|\cdot\|_r \) and \( \|\nabla(\cdot)\|_p \) respectively. We define

\[
Q := \{ u : u^m \in W_0^{1,p}(\Omega) \}.
\]

Due to (2), for any \( u \in Q \), we have \( u^m \in L^{m+q}(\Omega) \), then the energy functional can be defined by

\[
E(u) := \frac{1}{mp} \|\nabla u^m\|_p^p - \frac{1}{m+q} \|u^m\|_{m+q}^{m+q},
\] (3)

The corresponding Nehari functional and Nehari manifold can be denoted by

\[
H(u) := \|\nabla u^m\|_p^p - \|u^m\|_{m+q}^{m+q},
\]

\[
K := \{ u \in Q, H(u) = 0 \} \backslash \{0\},
\] (4)

respectively. Further, we define the potential depth by

\[
d := \inf_{u \in K} E(u) = \frac{m+q-mp}{mp} \frac{M^{-\frac{p(m+q)}{mp}}}{M^{-\frac{p(m+q)}{mp}}},
\] (5)

where \( M \) is the optimal constant of the Sobolev embedding for \( W_0^{1,p}(\Omega) \hookrightarrow L^{\frac{m+q}{m}}(\Omega) \), i.e.,

\[
\|u^m\|_{\frac{m+q}{m}} \leq M \|\nabla u^m\|_p, \quad \forall u \in Q.
\] (6)

We define the sets related to global existence and blow-up as follows:

\[
\Sigma_1 := \{ u \in Q : 0 < E(u) < d, H(u) > 0 \} \cup \{0\},
\]

\[
\Sigma_2 := \{ u \in Q : 0 < E(u) < d, H(u) < 0 \},
\]

\[
S := \left\{ u : u^m \in H_0^1(\Omega), \|\nabla u^m\|_2 < \left(\frac{2m(m+p-1)}{p-1-m}d\right)^\frac{1}{2} \right\},
\]

\[
B := \left\{ u : u^m \in H_0^1(\Omega), \|\nabla u^m\|_2 > \left(\frac{2m(m+p-1)}{p-1-m}d\right)^\frac{1}{2} \right\},
\]

\[
\partial S = \partial B := \left\{ u : u^m \in H_0^1(\Omega), \|\nabla u^m\|_2 = \left(\frac{2m(m+p-1)}{p-1-m}d\right)^\frac{1}{2} \right\}.
\]

With the above preparations, the main results of [14, 15, 16, 17] can be stated as the following theorem.
Theorem 1.2. Let \( u(t) \) be the weak solution of problem (1), then we have the following conclusions:

(i) If \( u_0 \in \Sigma_1 \), then \( u(t) \) exists globally. Moreover, \( u^m(t) \) decays exponentially in \( W_0^{1,p}(\Omega) \), namely, there exist two positive constants \( \alpha \) and \( C \) such that
\[
\lim_{t \to +\infty} e^{\alpha t} \| \nabla u^m \|_p^p \leq C.
\]

(ii) If \( u_0 \in \Sigma_2 \) or \( E(u_0) \leq 0 \), then \( u(t) \) blows up in finite time.

Corollary 1. For any \( t_0 \in [0,T) \), if \( E(u(t_0)) \leq 0 \), then the weak solution of problem (1) blows up at some finite time \( T \).

Proof. If \( E(u(t_0)) \leq 0 \), then taking \( t_0 \) as initial time, by Theorem 1.2 (ii), we see the solution blows up at some finite time \( T \). \qed

The assumptions in Theorem 1.2 imply the initial energy is subcritical, i.e., \( E(u_0) < d \). For the critical initial case, i.e., \( E(u_0) = d \), in [9], the authors get the following result.

Theorem 1.3. Consider problem (1) with \( p = 2 \). If \( E(u_0) = d, u_0 \in \mathcal{S} \), then problem (1) admits a global weak solution \( u(t) \) such that \( u^m(t) \in L^\infty(0, +\infty; H_0^1(\Omega)) \) and \( u(t) \in \overline{\mathcal{S}} = \mathcal{S} \cup \partial \mathcal{S} \) for \( 0 \leq t < +\infty \); if \( u_0 \in \mathcal{B} \), then the solution of problem (1) blows up in finite time.

In view of the above results, a natural question is that whether the global solution problem (1) exists when \( E(u_0) = d \) or \( E(u_0) > d \), which is the main task of this paper. More specifically, for any solution of problem (1) with \( E(u_0) = d \), we will prove that there exist a sufficient small \( t_1 \) such that \( E(u(t_1)) < d \) by some differential inequality techniques, then take \( t_1 \) as the initial time, we can deduce our results with the help of Theorem 1.2. For \( E(u_0) > d \), we will construct two sets \( \Psi_\alpha \) and \( \Phi_\alpha \), and prove that the solution will blow up in finite or infinite time if the initial value belongs to \( \Psi_\alpha \), while the solution of problem (1) will exist globally and tends to zero as time \( t \to +\infty \) when the initial value belongs to \( \Phi_\alpha \).

Furthermore, it worthwhile to point out that the models related to problem (1) were studied extensively in recent years (see for example [4, 5, 6, 7, 10, 11, 12]), and the conditions about global existence and blow-up were got in these papers for the subcritical initial energy case (i.e., \( E(u_0) < d \)). We remark that the methods used in this paper can also be used to study the solutions of these models with the critical and supercritical initial energy cases (i.e., \( E(u_0) \geq d \)).

In order to give the main results of the present paper, we introduce some necessary definitions as follows:

\[
\begin{align*}
K_- & := \{ u \in Q, H(u) < 0 \}, \\
K_+ & := \{ u \in Q, H(u) > 0 \}, \\
E^\alpha & := \{ u \in Q, E(u) < \alpha \}, \\
\end{align*}
\]

where \( \alpha \) is a constant. For all \( \alpha > d \), it is easy to see that
\[
K_\alpha := K \cap E^\alpha = \left\{ u \in K \left| \| \nabla u^m \|_p \leq \sqrt{\frac{mp(m+q)\alpha}{m+q - mp}} \right\} \right. \neq \emptyset.
\]
Corollary 2. If generalize the blow-up part of Theorem 1.2 and Theorem 1.5 to following corollary:

Thus, it follows from the definition of 1808 GUANGYU XU AND JUN ZHOU

By (13), we know that λ

If the last inequality of (2) is strict, i.e., the parameters satisfy

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Finally, we give the results about the solution with E(u₀) > d.

Theorem 1.4. If E(u₀) = d, H(u₀) ≥ 0, then (1) admits a global weak solution u(t) such that 0 < E(u(t)) ≤ d, H(u(t)) ≥ 0 for 0 ≤ t < +∞. Moreover, if H(u(t)) > 0 for 0 < t < +∞, the solution does not extinct in finite time. If not, the solution extinct in finite time.

Theorem 1.5. If E(u₀) = d and H(u₀) < 0, then the solution of (1) blows up in finite time.

If E(u₀) ≤ 0, we can prove that H(u₀) < 0. In fact, by the definition of E(u) in (3) and mp < m + q, we have

Thus, it follows from the definition of H(u) in (4) that H(u₀) < 0. Then we can generalize the blow-up part of Theorem 1.2 and Theorem 1.5 to following corollary:

Corollary 2. If E(u₀) ≤ d and H(u₀) < 0, then the solution of problem (1) blows up in finite time.

Finally, we give the results about the solution with E(u₀) > d.

Theorem 1.6. For any α ∈ (d, +∞), the following conclusions hold.

(i) If u₀ ∈ Φ⁺, then the solution u(t) of problem (1) exists globally and u(t) → 0 as t → +∞;

(ii) If u₀ ∈ Ψ⁻, then the solution u(t) of problem (1) blows up in finite or infinite time,

where

Φ⁺ = K⁺ ∩ \{ φ ∈ W₁,⁺₀ \ | \ \frac{1}{m + 1} \int_{Ω} |x|^{-s} |φ|^{m+1} dx < λ⁺, \ d < E(φ) ≤ α \},

Ψ⁻ = K⁻ ∩ \{ φ ∈ W₁,⁻₀ \ | \ \frac{1}{m + 1} \int_{Ω} |x|^{-s} |φ|^{m+1} dx > Λ⁻, \ d < E(φ) ≤ α \},

λ⁺, Λ⁻ are two constants defined in (9).

Remark 1. If the last inequality of (2) is strict, i.e., the parameters satisfy

m ≥ 1, \ p ≥ 2, \ 0 ≤ s < 1 + \frac{1}{m},

mp - m < q < \frac{m(Np - N + p)}{N - p}.

By (13), we know that λ⁺ admits a positive lower bound and Λ⁺ admits a finite positive upper bound, then the two sets Φ⁺, Ψ⁻ are well-defined, which ensure the statements in Theorem 1.6 make sense.
The rest of this paper is organized as follows. In Section 2, we give some useful lemmas, which will be used in the proof of the main results. In Section 3, we give the proof of our main results.

2. Preliminaries. We begin this section with the following lemma.

Lemma 2.1 ([1]). Let $\mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^{N-k}$, $2 \leq k \leq N$ and $x = (y, z) \in \mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^{N-k}$. For given $n, \beta$ satisfying $1 < n < N, 0 \leq \beta \leq n$ and $\beta < k$, let

$$\gamma(\beta, N, n) := \frac{n(N - \beta)}{N - n}.$$ 

Then there exists a positive constant $C$ depending on $\beta, n, N$ and $k$ such that for any $u \in W^{1,n}_{0}(\mathbb{R}^N)$, it holds

$$\int_{\mathbb{R}^N} \frac{|u(x)|^\gamma}{|y|^\beta} \leq C \left( \int_{\mathbb{R}^N} |\nabla u|^n dx \right)^{\frac{N-\beta}{N-n}}.$$

Lemma 2.2. Let (12) hold, then the constants $\lambda_\alpha, \Lambda_\alpha$ defined in (9) satisfy

$$0 < C_3 \leq \lambda_\alpha \leq \Lambda_\alpha \leq C_1 \left( \frac{mp(m + q)\alpha}{m + q - mp} \right)^{\frac{m+1}{2m}} < +\infty,$$

where $C_1, C_3$ are two positive constants given in (14) and (18) respectively.

Proof. For any $u \in K_\alpha$, using the Hardy-Sobolev’s inequality given in Lemma 2.1, we have

$$\frac{1}{m + 1} \int_{\Omega} |x|^{-s}|u|^m \frac{m+1}{m} dx \leq \frac{C}{m + 1} \left( \int_{\Omega} |\nabla u|^m \frac{N(m+1)}{mN + m + 1 - ms} dx \right)^{\frac{m+1}{m}},$$

where $C$ is a positive constant depending only on $N, m, s$. Since $m \geq 1, p \geq 2$ and $0 \leq s \leq 1/m$, we get

$$p \geq \frac{N(m+1)}{mN + m + 1 - ms}. $$

Then it follows from the H"{o}lder’s inequality and (8) that

$$\frac{C}{m + 1} \left( \int_{\Omega} |\nabla u|^m \frac{N(m+1)}{mN + m + 1 - ms} dx \right)^{\frac{m+1}{m}} \leq C_1 \left( \int_{\Omega} |\nabla u|^p dx \right)^{\frac{m+1}{mp}} < C_1 \frac{mp(m + q)\alpha}{m + q - mp} \right)^{\frac{m+1}{2m}},$$

where

$$C_1 = \begin{cases} \frac{C}{m + 1}, & \text{if } p = \frac{N(m+1)}{mN + m + 1 - ms}; \\ \frac{C}{m + 1} |\Omega|^{\frac{mN + m + 1 - ms}{mp} - \frac{m+1}{mp}}, & \text{if } p > \frac{N(m+1)}{mN + m + 1 - ms}, \end{cases}$$

which, together with the definition of $\Lambda_\alpha$, leads to

$$\Lambda_\alpha = \sup_{u \in K_\alpha} \frac{1}{m + 1} \int_{\Omega} |x|^{-s}|u|^m dx \leq C_1 \left( \frac{mp(m + q)\alpha}{m + q - mp} \right)^{\frac{m+1}{2m}}.$$

On the other hand, since $\Omega$ is a bounded domain in $\mathbb{R}^N$, there exists a positive constant $\rho$ such that

$$|x| = \sqrt{x_1^2 + \cdots + x_N^2} \leq \rho, \quad \forall x \in \bar{\Omega}.$$
Then we have
\[
\int_{\Omega} |x|^{-s} |u|^{m+1} \, dx \geq \rho^{-s} \int_{\Omega} |u|^{m+1} \, dx.
\] (15)

In view of \( p < N \), \( q > 1 \) and
\[
q < \frac{m(Np - N + p)}{N - p},
\]
we get
\[
\frac{N - p}{Np} < \frac{m}{m + q} < \frac{m}{m + 1}.
\]

Then by the Gagliardo-Nirenberg’s inequality [2], we have
\[
\|u^m\|_{m+q}^{m+q} \leq C_2 \|\nabla u^m\|_p^{\frac{m+q}{m}} \cdot \|u^m\|_{m+q}^{1-\theta},
\]
where \( C_2 \) is a positive constant depending only on \( \Omega \) and
\[
\theta = \frac{m}{m+1} - \frac{m}{m+q} \left( \frac{N-p}{Np} \right) \in (0, 1).
\]

For any \( u \in K \), we get from the definition of \( K \) and the above inequality that
\[
\|\nabla u^m\|_p = \|u^m\|_{m+q}^{\frac{m+q}{m}} \leq C_2 \|\nabla u^m\|_p^{\frac{m+q}{m}} \cdot \|u^m\|_{m+q}^{1-\theta},
\]
i.e.,
\[
\left( \frac{1}{C_2} \right)^{\frac{m+q}{m}} \|\nabla u^m\|_p^{\frac{(m+1)(m-p-(m+q)\theta)}{m(m+q)(1-\theta)}} \leq \|u^m\|_{m+1}.
\] (16)

Since \( m + q > mp \), it follows from the definition of \( K \) and (5) that
\[
\inf_{u \in K} \|\nabla u^m\|_p = \left( \frac{mp(m+q)d}{m+q-mp} \right)^{\frac{1}{p}} > 0.
\] (17)

If \( \frac{(m+1)(mp-(m+q)\theta)}{m(m+q)(1-\theta)} \geq 0 \), by the definition of \( \lambda_\alpha \), (15), (16) and (17), for any \( u \in K_\alpha \subset K \), we get
\[
\lambda_\alpha = \frac{1}{m+1} \inf_{u \in K_\alpha} \int_\Omega |x|^{-s} |u|^{m+1} \, dx
\]
\[
\geq \frac{\rho^{-s}}{m+1} \inf_{u \in K_\alpha} \|u^m\|_{m+1}^{\frac{m+1}{m}}
\]
\[
\geq \frac{\rho^{-s}}{m+1} \left( \frac{1}{C_2} \right)^{\frac{m+q}{m}} \left( \inf_{u \in K} \|\nabla u^m\|_p^{\frac{(m+1)(mp-(m+q)\theta)}{m(m+q)(1-\theta)}} \right)
\]
\[
= \frac{\rho^{-s}}{m+1} \left( \frac{1}{C_2} \right)^{\frac{m+q}{m}} \left( \frac{mp(m+q)d}{m+q-mp} \right)^{\frac{(m+1)(mp-(m+q)\theta)}{m(m+q)(1-\theta)}}
\]
\[
> 0.
\]
If \( \frac{(m+1)|mp-(m+q)\theta|}{m(m+q)(1-\theta)} < 0 \), by the definition of \( \lambda_\alpha \), (15), (16) and (8), for any \( u \in K_\alpha \subset K \), we have
\[
\lambda_\alpha = \frac{\rho^{-s}}{m+1} \left( \frac{1}{C_2} \right)^{\frac{m+1}{m+q}(1-\theta)} \inf_{u \in K_\alpha} \left\| \nabla u^m \right\|_p \frac{(m+1)|mp-(m+q)\theta|}{m(m+q)(1-\theta)}.
\]
So for any \( u \in K_\alpha \), we get \( \lambda_\alpha \geq C_3 \), where
\[
C_3 := \begin{cases} 
\frac{\rho^{-s}}{m+1} \left( \frac{1}{C_2} \right)^{\frac{m+1}{m+q}(1-\theta)} \left( \frac{mp(m+q)\alpha}{m+q-mp} \right) & \text{if } \frac{(m+1)|mp-(m+q)\theta|}{m(m+q)(1-\theta)} \geq 0; \\
\frac{\rho^{-s}}{m+1} \left( \frac{1}{C_2} \right)^{\frac{m+1}{m+q}(1-\theta)} \left( \frac{mp(m+q)\alpha}{m+q-mp} \right) & \text{if } \frac{(m+1)|mp-(m+q)\theta|}{m(m+q)(1-\theta)} < 0.
\end{cases}
\]
Finally, by the definition of \( \lambda_\alpha \) and \( \Lambda_\alpha \), it is easy to see that \( \lambda_\alpha \leq \Lambda_\alpha \), so Lemma 2.2 is proved.

Lemma 2.3. Let \( u(t) \) be a weak solution of problem (1) and
\[
f(t) := \frac{1}{m+1} \int_{\Omega} |x|^{-s} |u(t)|^{m+1} dx, \quad \forall t \geq 0,
\]
then it holds
\[
f'(t) = -H(u),
\]
where \( H(u) \) is defined in (4).

Proof. Multiplying the first equation of problem (1) with \( u^m \) and integrating over \( \Omega \), then by the definition of \( H(u) \) in (4) we get (20).

Lemma 2.4. Let \( u(t) \) be a weak solution of problem (1), then the energy functional \( E(u(t)) \) defined by (3) is nonincreasing with respect to \( t \). Moreover,
\[
\frac{4}{(m+1)^2} \int_t^{t'} \left\| x^{-s} \left( u^{\frac{m+1}{m}}(x, \tau) \right) \right\|_2^2 d\tau + E(u(t)) = E(u_0).
\]

Proof. Multiplying the first equation of problem (1) with \( \frac{1}{m} \left( u^m \right) \), and integrating over \( \Omega \times (0, t) \), we get (21), then it is easy to see that \( E(u(t)) \) is nonincreasing with respect to \( t \).

Lemma 2.5. We have \( E(u) > 0 \) for any \( u \in K_+ \). Moreover, for any \( \alpha > 0 \) and \( u \in E^\alpha \cap K_+ \), it holds
\[
\left\| \nabla u^m \right\|_p < \left( \frac{mp(m+q)}{m+q-mp} \right)^{\frac{m}{p}} \alpha.
\]
Proof. By the definition of $K_+$ in (7), for any $u \in K_+$, it holds $H(u) > 0$, i.e.,
\[
\|\nabla u^m\|_p > \frac{1}{1-p} \|u^m\|_p^{\frac{m+q}{1-p}}.
\]
Since $mp - m < q$, we get
\[
\frac{1}{mp} \|\nabla u^m\|_p > \frac{1}{m+q} \|u^m\|_p^{\frac{m+q}{m+q}},
\]
then it follows from the definition of $E(u)$ that $E(u) > 0$.

On the other hand, for any $u \in E^\alpha \cap K_+$, i.e., $E(u) < \alpha$ and $H(u) > 0$, we get
\[
\alpha > E(u) = \frac{m+q-qmp}{mp(m+q)} \|\nabla u^m\|_p + \frac{1}{m+q} H(u) \geq \frac{m+q-qmp}{mp(m+q)} \|\nabla u^m\|_p,
\]
which yields (22).

Lemma 2.6. Let $K_-$ be defined in (7), then we have dist$(0, K_-) > 0$.

Proof. For any $u \in K_-$, by the definition of $K_-$ we have $H(u) < 0$, i.e.,
\[
\|\nabla u^m\|_p < \|u^m\|_p^{\frac{m+q}{1-p}},
\]
which combining with (6) leads to
\[
\|\nabla u^m\|_p < M^{\frac{m+q}{m+q}} \|\nabla u^m\|_p^{\frac{m+q}{m+q}}.
\]
Since $m+q > mp$, then it follows from the above inequality that
\[
\|\nabla u^m\|_p > M^{-\frac{m+q}{m+q} m+q} > 0,
\]
which implies that
\[
\text{dist}(0, K_-) = \min_{u \in K_-} \|\nabla u^m\|_p \geq M^{-\frac{m+q}{m+q} m+q} > 0.
\]

Next, we denote by $S(t)$ the nonlinear semigroup associated to problem (1). Instead of $u = u(t)$ we will also write $u = S(t)u_0$ for all $t < T$. If $T = +\infty$, we define the $\omega$-lim set of $u_0$ as follow:
\[
\omega(u_0) := \bigcap_{t \geq 0} \{ u_m(s) : s \geq t \}^{W^{1,p}_0}.
\]
and we have the following lemma, which shows that $\omega(u_0) \neq \emptyset$ if $u^m(t)$ is uniformly bounded in $W^{1,p}_0$.

Lemma 2.7. Let $u(t) = S(t)u_0$ be a global weak solution of problem (1) such that $u^m(t)$ is uniformly bounded in $W^{1,p}_0$, then $\omega(u_0)$ contains stationary solutions of problem (1).

Proof. The method in the following proof is similar to [15, Theorem 1.4], for reader’s convenience, we give a complete proof here.

We choose a monotone increasing sequence $\{t_n\}_{n=1}^{+\infty}$ such that $t_n \to +\infty$ $(n \to +\infty)$, and let $u_n = u(t_n)$. By the uniform boundedness of $\{u^m_n\}_{n=1}^{+\infty}$ in $W^{1,p}_0$, there exist a subsequence of $\{u^m_n\}_{n=1}^{+\infty}$ which is still denoted by $\{u^m_n\}_{n=1}^{+\infty}$ and a function $\omega$ such that $u^m_n \rightharpoonup \omega$ weakly in $W^{1,p}_0$ and $u^m_n \to \omega$ a.e. in $\Omega$. (24)
Next we introduce some suitable test functions. For any \( \bar{T} < +\infty \), we take two functions \( \psi \) and \( \varrho \) satisfying

\[
\psi^m \in W^{1,p}_0(\Omega), \quad \varrho \in C^2_0(0, \bar{T}), \quad \varrho \geq 0, \quad \int_0^{\bar{T}} \varrho(s) \, ds = 1,
\]

and let

\[
\phi(x, t) := \begin{cases} \varrho(t - t_n) \psi^m(x), & (x, t) \in \Omega \times (t_n, +\infty), \\ 0, & (x, t) \in \Omega \times [0, t_n]. \end{cases}
\]

Multiplying the first term of problem (1) with \( \phi \) and integrating by parts, then, by the definition of the above functions, we have

\[
\int_{t_n}^{t_n + \bar{T}} \int_\Omega (|x|^{-s} u_t \varrho) \, dx \, dt = \int_{t_n}^{t_n + \bar{T}} \int_\Omega (|x|^{-s} u \varrho(t - t_n) \psi^m) \, dx \, dt
\]

\[
= \int_\Omega \left[ |x|^{-s} u(t_n + \bar{T}) \varrho(\bar{T}) \psi^m - |x|^{-s} u(t_n) \varrho(0) \psi^m \right] dx
\]

\[- \int_{t_n}^{t_n + \bar{T}} \int_\Omega |x|^{-s} u(t - t_n) \psi^m \, dx \, dt
\]

\[- = \int_{t_n}^{t_n + \bar{T}} \int_\Omega |x|^{-s} u\varrho'(t - t_n) \psi^m \, dx \, dt.
\]

Hence, for problem (1), it is easy to see that

\[
\int_{t_n}^{t_n + \bar{T}} \int_\Omega \left[ |x|^{-s} u \varrho'(t - t_n) \psi^m - \varrho(t - t_n) \nabla u^m \nabla \psi^m + u^q \varrho(t - t_n) \psi^m \right] \, dx \, dt = 0.
\]

Making a transformation \( \check{s} = t - t_n \) in above inequality, we get

\[
\int_0^{\bar{T}} \int_\Omega \left[ |x|^{-s} u(t_n + \check{s}) \varrho'(\check{s}) \psi^m - \varrho(\check{s}) \nabla u^m (t_n + \check{s}) \nabla \psi^m + u^q (t_n + \check{s}) \varrho(\check{s}) \psi^m \right] \, dx \, d\check{s} = 0.
\]

Due to (2), we know the embedding \( W^{1,p}_0(\Omega) \rightarrow L^{\frac{m+q}{m}}(\Omega) \) is continuous, so for any \( \check{s} \in [0, \bar{T}] \), it follows from (24) that there exists a subsequence of \( \{ u_n \}_{n=1}^{+\infty} \) which is still denoted by \( \{ u_n \}_{n=1}^{+\infty} \) and \( \check{\omega} \in L^{m+q}(\Omega) \) such that

\[
u^m(t_n + \check{s}) \rightarrow \check{\omega}^m \text{ weakly in } L^{\frac{m+q}{m}}(\Omega) \text{ and } u^m(t_n) \rightarrow \omega^m \text{ weakly in } L^{\frac{m+q}{m}}(\Omega).
\]

We claim that \( \check{\omega} = \omega \) a.e. in \( \Omega \). In fact, since the solution is global, by Corollary 1 we know that \( E(u(t)) > 0 \) for all \( t \in [0, +\infty) \), then by (21) we get

\[
\frac{4}{(m + 1)^2} \int_0^{+\infty} \left\| x^{-\frac{m}{2}} \left( u^{\frac{m+q}{m}}(x, t) \right) \right\|^2 \, dt \leq E(u_0) < +\infty.
\]
Let $\rho$ be the positive constant given in (15), by Hölder’s inequality and the above inequality, we obtain

\[
\rho^{-s} \int_{\Omega} \left| u^{m+1}(t_n + \tilde{s}) - u^{m+1}(t_n) \right|^2 dx \leq \int_{\Omega} |x|^{-s} \left| u^{m+1}(t_n + \tilde{s}) - u^{m+1}(t_n) \right|^2 dx
\]

\[
= \int_{\Omega} |x|^{-s} \left[ \int_{t_n}^{t_n + \tilde{s}} \left( u^{m+1}(x,t) \right)_t \right]^2 dx
\]

\[
\leq \tilde{s} \int_{t_n}^{t_n + \tilde{s}} \int_{\Omega} |x|^{-s} \left[ \left( u^{m+1}(x,t) \right)_t \right]^2 dx dt
\]

\[
\leq \tilde{T} \int_{t_n}^{t_n + \tilde{T}} \left\| |x|^{-\frac{s}{2}} \left( u^{m+1}(x,t) \right)_t \right\|^2 dt
\]

\[ \rightarrow 0 \]

as $t_n \rightarrow +\infty$, i.e.,

\[
u^{m+1}(t_n + \tilde{s}) - u^{m+1}(t_n) \rightarrow 0 \text{ strongly in } L^2(\Omega) \text{ as } t_n \rightarrow +\infty
\]

uniform in $\tilde{s} \in [0, \tilde{T}]$ for any fixed $\tilde{T} < +\infty$. Hence, $\tilde{\omega} = \omega$ a.e. in $\Omega$, which proves the claim.

Letting $n \rightarrow +\infty$ in (25), it follows from the dominated convergence theorem, (24), (26) and the choice of $\varrho$ that

\[
\int_{0}^{\tilde{T}} \int_{\Omega} \left| x \right|^{-s} \omega g'(\tilde{s}) \psi^m - \varrho(\tilde{s}) |\nabla \omega^m|^{p-2} \nabla \omega^m \nabla \psi^m + \varrho(\tilde{s}) \omega^q \psi^m \right] \ dx \ dt = 0. \tag{27}
\]

Since $\varrho(0) = \varrho(\tilde{T}) = 0$, we have

\[
\int_{0}^{\tilde{T}} \int_{\Omega} \left| x \right|^{-s} \omega g'(\tilde{T}) \psi^m \ dx \ dt = \int_{\Omega} \left| x \right|^{-s} \omega g(\tilde{T}) \psi^m - \left| x \right|^{-s} \omega g(0) \psi^m \right] \ dx = 0.
\]

Then by (27), we have

\[
\int_{\Omega} \left[ |\nabla \omega^m|^{p-2} |\nabla \omega^m \nabla \psi^m - \omega^q \psi^m \right] \ dx
\]

\[
= \frac{1}{\tilde{T}} \int_{0}^{\tilde{T}} \int_{\Omega} \left[ |\nabla \omega^m|^{p-2} |\nabla \omega^m \nabla \psi^m - \omega^q \psi^m \right] \ dx \ dt = 0,
\]

which implies $\omega$ is a stationary solution of problem (1).}

The proof of the following lemma is very similar to that of [8, 13], so we omit the proof here.

**Lemma 2.8.** For any $u \in Q$, we have

(i) $\lim_{\lambda \to 0} E(\lambda u) = 0$, $\lim_{\lambda \to +\infty} E(\lambda u) = -\infty$;

(ii) there exists a unique $\lambda^* > 0$, namely,

\[
\lambda^* = \frac{1}{m+q-mp} \left( \frac{\|u_m\|_{L^p(\Omega)}}{\|u^{m+1}\|_{m+q}^{m+1}} \right)
\]

such that

\[
\frac{d}{d\lambda} E(\lambda u) \bigg|_{\lambda=\lambda^*} = 0,
\]
3. Proofs of the main results. In this section, we aim to prove our main results. Firstly, we prove Theorem 1.4.

Proof of Theorem 1.4. The proof is divided into two steps.

Step 1. Global existence

By $E(u_0) = d$, we get $u_0 \neq 0$. Let the sequence $\{\lambda_n\}_{n=1}^{+\infty}$ satisfy $0 < \lambda_n < 1$, $n = 1, 2, \ldots$, $\lambda_n \to 1$ as $n \to +\infty$ and $u_{0n}(x) = \lambda_n u_0(x)$. Consider problem (1) corresponding to the initial condition

$$u(x, 0) = u_{0n}(x), \quad x \in \Omega. \quad (28)$$

From $H(u_0) \geq 0$ and Lemma 2.8, we have $\lambda^* = \lambda^*(u_0) \geq 1$. Thus, we get

$$H(u_{0n}) = H(\lambda_n u_0) > 0$$

and

$$E(u_{0n}) = E(\lambda_n u_0) < E(u_0) = d.$$

On the other hand, by $H(u_{0n}) > 0$ and Lemma 2.5, we have $E(u_{0n}) > 0$. So we transformed the case $E(u_0) = d, H(u_0) \geq 0$ into $0 < E(u_{0n}) < d, H(u_{0n}) > 0$ under the initial condition (28), i.e., $u_{0n} \in \Sigma_1$ with the initial condition (28). Then similar to the proof of Theorem 1.2, for each $n$, problem (1) with initial condition (28) gets a global weak solution $u_n \in \Sigma_1$ for $0 < t < +\infty$, and

$$\frac{4}{(m + 1)^2} \int_{Q_T} \varpi_n(x) \left( \left( \frac{u_n^{m+1}}{u_n^{m+1}} \right)_{\tau} \right)^2 \, dx \, dt + E(u_n) < d,$$

$$\frac{4}{(m + 1)^2} \int_{Q_T} \varpi_n(x) \left( \left( \frac{u_n^{m+1}}{u_n^{m+1}} \right)_{\tau} \right)^2 \, dx \, dt + \frac{m + q - mp}{mp(m + q)} \| \nabla u_n^m \|_p < d,$$

where $\varpi_n(x) = \min \{ |x|^{-s}, n \}$. Thus,

$$\int_{Q_T} \varpi_n(x) \left( \left( \frac{u_n^{m+1}}{u_n^{m+1}} \right)_{\tau} \right)^2 \, dx \, dt < \frac{d(m + 1)^2}{4},$$

$$\| \nabla u_n^m \|_p < \frac{dmp(m + q)}{m + q - mp},$$

$$\| u_n^m \|_{m+2} \leq M \| \nabla u_n^m \|_p < M \left( \frac{dmp(m + q)}{m + q - mp} \right)^{1/2}.$$

Then there exists a subsequence (denoted by $\{u_n\}_{n \geq 1}$ again) and a function $u$ such that for all $T > 0$,

$$\sqrt{\varpi_n(x)} \left( \left( \frac{u_n^{m+1}}{u_n^{m+1}} \right)_{\tau} \right) \to |x|^{-s/2} \left( \frac{u^{m+1}}{u^{m+1}} \right), \quad \text{weakly in } L^2(Q_T),$$

$$u_n \to u, \quad u_n^m \to u^m,$$

$$u_n^m \to u^m, \quad \text{weakly star in } L^\infty(0, +\infty; W_0^{1,p}(\Omega)).$$

So we pass to the limit to derive problem (1) admits a global solution satisfies $I(u(t)) \geq 0$ and $E(u(t)) \leq d$ for $0 \leq t < +\infty$. Moreover, we can get $E(u(t)) > 0$ for $0 \leq t < +\infty$. If not, then there exists some $t_0$ such that $E(u(t_0)) = 0$. We take $t = t_0$ as the initial time, by Theorem 1.2, we know the solution blows up in finite time, which contradicts the fact that the solution is global.
Step 2. Extinction and non-extinction

Firstly, we suppose that \( H(u(t)) > 0 \) for \( 0 < t < +\infty \), then the solution \( u(x, t) \) does not extinct in finite time. If not, there exists a time \( t_0 > 0 \) such that \( u(x, t_0) \equiv 0 \), then we have \( H(u(x, t_0)) = 0 \), which contradicts the fact that \( H(u(t)) > 0 \) for \( 0 < t < +\infty \).

Next, we suppose \( H(u(t)) > 0 \) for \( 0 < t < t_0 \) and \( H(u(t_0)) = 0 \). Let \( f(t) \) be the function defined by (19), by Lemma 2.3, we get

\[
f'(t) = -H(u(t)) < 0, \quad 0 < t < t_0. \tag{29}
\]

On the other hand, we have

\[
f'(t) = \int_{\Omega} |x|^{-s}|u|^{m-1}uu_t \, dx,
\]

which combining with (29) implies

\[
\int_{\Omega} |x|^{-s}|u|^{m-1}uu_t \, dx < 0, \quad 0 < t < t_0. \tag{30}
\]

By Lemma 2.4 and \( E(u_0) = d \), we have

\[
E(u(t_0)) = d - \frac{4}{(m+1)^2} \int_0^{t_0} \left\| |x|^{-\frac{s}{2}} \left( u^{\frac{m+1}{2}}(x, \tau) \right) \right\|_2^2 \, d\tau. \tag{31}
\]

Next, we claim that

\[
\frac{4}{(m+1)^2} \int_0^{t_0} \left\| |x|^{-\frac{s}{2}} \left( u^{\frac{m+1}{2}}(x, \tau) \right) \right\|_2^2 \, d\tau = \int_0^{t_0} \int_{\Omega} |x|^{-s}|u|^{m-1}u_{tt}^2 \, dx \, d\tau > 0. \tag{32}
\]

Arguing by contradiction, if the above inequality does not hold, then we have

\[
(uu_t)(x, t) = 0
\]

for a.e. \( (x, t) \in \Omega \times (0, t_0) \), which contradicts (30). Hence, we get (32), and then it follows from (31) that

\[
E(u(t_0)) < d. \tag{33}
\]

Combine \( H(u(t_0)) = 0 \) and (33), we can prove \( u(t_0) = 0 \). In fact, if \( u(t_0) \neq 0 \), then \( u(t_0) \in K \), and then by the definition of \( d \), we get \( E(u(t_0)) \geq d \), which contradicts (33). The proof is complete. \( \square \)

Proof of Theorem 1.5. By \( E(u_0) = d > 0, H(u_0) < 0 \) and the continuity of \( E(u(t)) \) and \( H(u(t)) \) with respect to \( t \), we know that there exists a sufficiently small \( t_1 > 0 \) such that \( E(u(t_1)) > 0 \) and \( H(u(t)) < 0 \) for \( 0 \leq t \leq t_1 \). Let \( f(t) \) be the function defined by (19), by Lemma 2.3, we get

\[
f'(t) = -H(u(t)) > 0, \quad 0 \leq t \leq t_1.
\]

On the other hand, we have

\[
f'(t) = \int_{\Omega} |x|^{-s}|u|^{m-1}uu_t \, dx,
\]

which combining with the above inequality implies

\[
\int_{\Omega} |x|^{-s}|u|^{m-1}uu_t \, dx > 0, \quad 0 \leq t \leq t_1.
\]

By using the above inequality, similar to the proof of (32), we can get

\[
\frac{4}{(m+1)^2} \int_0^{t_1} \left\| |x|^{-\frac{s}{2}} \left( u^{\frac{m+1}{2}}(x, \tau) \right) \right\|_2^2 \, d\tau > 0, \quad 0 \leq t \leq t_1.
\]
Then it follows from (21) that
\[ E(u(t_1)) = d - \frac{4}{(m+1)^2} \int_0^{t_1} \left\| |x|^{-\frac{q}{2}} \left( u^{\frac{m+1}{q}}(x,\tau) \right) \right\|_2^2 \, d\tau < d, \quad 0 \leq t \leq t_1, \]
which combining with \( H(u(t_1)) < 0 \) and \( E(u(t_1)) > 0 \) implies that \( u(t_1) \in \Sigma_2 \), so by Theorem 1.2, we know that the weak solution blows up in finite time. \( \square \)

**Proof of Theorem 1.6.** The idea of the proof comes from \([3]\). For all \( t \in [0, T) \), let \( u(t) = S(t)u_0 \) be a weak solution of problem (1). By Lemma 2.3, we have
\[
\frac{d}{dt} \left( \frac{1}{m+1} \int |x|^{-s}|u|^{m+1} \, dx \right) = -H(u(t)). \tag{34} \]
On the other hand, from Lemma 2.4, we know that \( E(u(t)) \) is nonincreasing with respect to \( t \). Thus, we get
\[
E(u(t)) \leq E(u_0) \quad \text{for all } t \in [0, T). \tag{35} \]

(i). If \( u_0 \in \Phi_\alpha \), then by the definition of \( \Phi_\alpha \) in (11) and the monotonicity property of \( \lambda_\alpha \) in (10), we have \( d < E(u_0) \leq \alpha \) and
\[
u_0 \in K_+, \quad \frac{1}{m+1} \int \Omega |x|^{-s}|u_0|^{m+1} \, dx < \lambda_\alpha \leq \lambda_{E(u_0)}. \tag{36} \]
We claim that \( u(t) \in K_+ \) for all \( t \in [0, T) \). Arguing by contradiction, if the claim is not true, then there is a \( t_0 \in (0, T) \) such that \( u(t) \in K_- \) for \( 0 \leq t < t_0 \) and \( u(t_0) \in K \). Then by the definition of \( K_+ \) and (34), we know that \( \int \Omega |x|^{-s}|u(t)|^{m+1} \, dx \) is strictly decreasing on \([0, t_0]\). So, it follows from (35) and (36) that
\[
E(u(t_0)) \leq E(u_0) \tag{37} \]
and
\[
\frac{1}{m+1} \int \Omega |x|^{-s}|u(x,t_0)|^{m+1} \, dx \leq \frac{1}{m+1} \int \Omega |x|^{-s}|u_0|^{m+1} \, dx < \lambda_{E(u_0)}. \tag{38} \]
By \( u(t_0) \in K \) and (37), we get \( u(x,t_0) \in K_{E(u_0)} \). Thus, it follows from the definition of \( \lambda_{E(u_0)} \) that
\[
\lambda_{E(u_0)} \leq \frac{1}{m+1} \int \Omega |x|^{-s}|u(x,t_0)|^{m+1} \, dx, \]
which contradicts (38). So the claim is true. Then it follows from (35) that \( u(t) \in E^{E(u_0)} \cap K_+ \). Hence, by (22), we obtain
\[
\|\nabla u^m\|_p < \left( \frac{mp(m+q)}{m+q-mp} E(u_0) \right)^{\frac{1}{p}}, \quad \forall t \in [0, T). \tag{39} \]
Since the right-hand of (39) is independent of \( T \), then we can extend the solution to infinite, i.e., \( T = +\infty \), and we further have (39) holds for \( 0 \leq t < +\infty \). Moreover, by (39) we know that \( u^m \) is bounded uniformly in \( W_0^{1,p}(\Omega) \), then it follows from Lemma 2.7 that the \( \omega \)-limit set defined in (23) is not a empty set.

Now for any \( \omega \in \omega(u_0) \), by the above discussions, we get
\[
E(\omega) \leq E(u_0) \quad \text{and} \quad \frac{1}{m+1} \int \Omega |x|^{-s}|\omega|^{m+1} \, dx < \lambda_{E(u_0)}. \]
By the first inequality, we know that \( \omega \in E^{E(u_0)} \). By the second inequality and the definition of \( \lambda_{E(u_0)} \), we know that \( \omega \notin K_{E(u_0)} \). Since \( K_{E(u_0)} = K \cap E^{E(u_0)} \), we then obtain \( \omega \notin K \).
Similarly, we get $\omega(u_0) = \{0\}$. (40) In fact, it follows from $u(t) \in K_+$ and the definitions of $E(u)$ and $K_+$ that

$$E(u) = \frac{1}{mp} \|\nabla u^m\|^p_p - \frac{1}{m + q} \|u^m\|^{m+q}_{\frac{m+q}{m}} > \frac{m + q - mp}{mp(m + q)} \|\nabla u^m\|^p_p,$$

i.e., $E(u(t))$ is bounded below. Then combine the fact that $E(u(t))$ is nonincreasing with respect to $t$ we know there is a constant $c$ such that

$$\lim_{t \to +\infty} E(u(t)) = c.$$  

So for any $\omega \in \omega(u_0)$, we have $E(u_\omega(t)) = c$ for all $t \geq 0$, where $u_\omega(t)$ is the solution of problem (1) with initial value $\omega$. Then combining (21) we get $u_\omega(t) \equiv \omega$, and then it follows from (34) that

$$H(\omega) = 0, \quad \forall \omega \in \omega(u_0). \quad (41)$$

Combining (41), $\omega \not\in K$ and the definition of $K$ we know $\omega = 0$, then by the arbitrariness of $\omega$, we get (40). In other words, the solution $u(t) \to 0$ as $t \to +\infty$.

(ii). Similar to the proof of the first part, if $u_0 \in \Psi_\alpha$, then by the definition of $\Psi_\alpha$ and the monotonicity property of $\Lambda_\alpha$, we have $d < E(u_0) \leq \alpha$ and

$$u_0 \in K_-, \quad \frac{1}{m + 1} \int_\Omega |x|^{-s}|u_0|^{m+1}dx > \Lambda_\alpha \geq \Lambda_{E(u_0)}.$$  

We claim that $u(t) \in K_-$ for all $t \in [0, T)$. By contradiction, if the claim is not true, then there is a $t_1 > 0$ such that $u(t) \in K_-$ for $0 \leq t < t_1$ and $u(t_1) \in K$. (34) and (35) imply that

$$\frac{1}{m + 1} \int_\Omega |x|^{-s}|u(x, t_1)|^{m+1}dx > \frac{1}{m + 1} \int_\Omega |x|^{-s}|u_0|^{m+1}dx > \Lambda_{E(u_0)}, \quad (42)$$

$$E(u(t_1)) \leq E(u_0).$$

Similarly, we get $u(x, t_1) \in K_{E(u_0)}$. Thus, it follows from the definition of $\Lambda_{E(u_0)}$ that

$$\Lambda_{E(u_0)} \geq \frac{1}{m + 1} \int_\Omega |x|^{-s}|u(x, t_1)|^{m+1}dx,$$

which contradicts (42). So the claim is true.

If the solution blows up in infinite time, then the proof is complete. So in the following proof we assume that the solution does not blow up in infinite time and prove the solution blows up in finite by contradiction. Assuming $u(t)$ exists globally, then

$$u(t) \in E_{E(u_0)} \cap K_-, \quad \forall t \in [0, +\infty),$$

and $\int_\Omega |x|^{-s}|u(t)|^{m+1}dx$ is strictly increasing on $[0, +\infty)$ (by Lemma 2.3). Furthermore, we claim that $u^m(t)$ is uniformly bounded in $W_0^{1,p}(\Omega)$, i.e., there exists a positive constant $\Theta$ such that

$$\|\nabla u^m(t)\|^p_p \leq \Theta, \quad \forall t \in [0, +\infty).$$

In fact, if the claim is not true, then there exists a monotone increasing sequence $\{t_n\}_{n=1}^{+\infty}$ such that

$$\|\nabla u^m(t_n)\|^p_p > n, \quad n = 1, 2, \cdots \quad (43)$$

By the monotone property of $\{t_n\}_{n=1}^{+\infty}$, we know that $t_n \to +\infty$ ($n \to +\infty$) or there exists a positive constant $t^*$ such that $t_n \to t^*$ ($n \to +\infty$). If the first case happens, by (43), we know that $u(t)$ blows up in infinite time, which contradicts
the assumption that solution does not blow up in infinite time; if the second case happen, by (43), we know that $u(t)$ blows up in infinite time, which contradicts the assumption that $u(t)$ exists globally. So the claim is true, namely, the assumptions in Lemma 2.7 holds, so we get $\omega(u_0) \neq \emptyset$.

For every $\omega \in \omega(u_0)$, have
\[
\int_{\Omega} |x|^{-s}|\omega|^{m+1} \, dx > \int_{\Omega} |x|^{-s}|u_0|^{m+1} \, dx > \Lambda_{E(u_0)}(\omega) \quad \text{and} \quad E(\omega) \leq E(u_0),
\]
so we get $\omega \in E^{E(u_0)}$ and $\omega \notin K_{E(u_0)}$, then it follows from the definition of $K_{E(u_0)}$ that $\omega(u_0) \cap K = \emptyset$.

Since $E(u(t))$ is nonincreasing with respect to $t$, then we have following two cases:

(a) there is a constant $c$ such that $\lim_{t \to +\infty} E(u(t)) = c$;

(b) $\lim_{t \to +\infty} E(u(t)) = -\infty$.

Next we will prove both the above cases contradict $T = +\infty$, then we get that the solution $u(t)$ blows up in finite time.

We firstly consider the case (a). If $\lim_{t \to +\infty} E(u(t)) = c$, then by the similar discussions as in the proof of (i), we obtain (41). Combine $\omega(u_0) \cap K = \emptyset$ and the definition of $K$ we get $\omega(u_0) = \{0\}$. However, it follows from Lemma 2.6 that $\operatorname{dist}(0, K) > 0$, which implies $0 \notin \omega(u_0)$, so we get a contradiction.

Finally, we consider the case (b). If $\lim_{t \to +\infty} E(u(t)) = -\infty$, then there must exist a time $t_1$ such that $E(u(t_1)) \leq 0$. Then take $u(t_1)$ as the initial value, by Theorem 1.2 we know that the corresponding solution $U(t) = u(t + t_1)$ blows up in finite time, which contradicts $T = +\infty$, thus Theorem 1.6 is proved. \qed

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Received June 2017; revised September 2017.

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