A Direct Method for Solving Optimal Switching Problems of One-Dimensional Diffusions

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Abstract

In this paper, we propose a direct solution method for optimal switching problems of one-dimensional diffusions. This method is free from conjectures about the form of the value function and switching strategies, or does not require the proof of optimality through quasi-variational inequalities. The direct method uses a general theory of optimal stopping problems for one-dimensional diffusions and characterizes the value function as sets of the smallest linear majorants in their respective transformed spaces.

1 Introduction

Stochastic optimal switching problems (or starting and stopping problems) are important subjects both in mathematics and economics. Since there are numerous articles about real options in the economic and financial literature in recent years, the importance and applicability of control problems including optimal switching problems cannot be exaggerated.

A typical optimal switching problem is described as follows: The controller monitors the price of natural resources for optimizing (in some sense) the operation of an extraction facility. She can choose when to start extracting this resource and when to temporarily stop doing so, based upon price fluctuations she observes. The problem is concerned with finding an optimal switching policy and the corresponding value function. A number of papers on this topic are well worth mentioning: Brennan and Schwarz (1985) in conjunction with convenience yield in the energy market, Dixit (1989) for production facility problems, Brekke and Øksendal (1994) for resource extraction problems, Yushkevich (2001) for positive recurrent countable Markov chains, and Duckworth and Zervos (2001) for reversible investment problems. Hamdadène and Jeanblanc (2004) analyze a general adapted process for finite time horizon using reflected stochastic backward differential equations. Carmona and Ludkovski (2005) apply to energy tolling agreement in a finite time horizon using Monte-Carlo regressions.

A basic analytical tool for solving switching problems is quasi-variational inequalities. This method is indirect in the sense that one first conjectures the form of the value function and the switching policy and next verifies the optimality of the candidate function by proving that the candidate satisfies the variational inequalities. In finding the specific form of the candidate function, appropriate boundary conditions including the smooth-fit principle are employed. This formation shall lead to a system of non-linear equations that are often hard to solve and the existence of the solution to the system is also difficult to prove. Moreover,
this indirect solution method is specific to the underlying process and reward/cost structure of the problem. Hence a slight change in the original problem often causes a complete overhaul in the highly technical solution procedures.

Our solution method is direct in the sense that we first show a new mathematical characterization of the value functions and, based on the characterization, we shall directly find the value function and optimal switching policy. Therefore, it is free from any guesswork and applicable to a larger set of problems (where the underlying process is one-dimensional diffusions) than the conventional methods. Our approach here is similar to Dayanik and Karatzas (2003) and Dayanik and Egami (2005) that propose direct methods of solving optimal stopping problems and stochastic impulse control problems, respectively.

The paper is organized in the following way. In the next section, after we introduce our setup of one dimensional optimal switching problems, in section 2.1 we characterize the optimal switching times as exit times from certain intervals through sequential optimal stopping problems equivalent to the original switching problem. In section 2.2 we shall provide a new characterization of the value function, which leads to a direct solution method described in 2.3. We shall illustrate this method through examples in section 3, one of which is a new optimal switching problem. Section 4 concludes with comments on an extension to a further general problem.

2 Optimal Switching Problems

We consider the following optimal switching problems for one dimensional diffusions. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space with a standard Brownian motion \(W = \{W_t; t \geq 0\}\). Let \(Z_t\) be the indicator vector at time \(t\), \(Z_t \in \{z_1, z_2, \ldots, z_m\} \triangleq \mathcal{Z}\) where each vector \(z_i = (a_1, a_2, \ldots, a_k)\) with \(a\) is either 0 (closed) or 1 (open), so that \(m = 2^k\). In this section, we consider the case of \(k = 1\). That is, \(Z_t\) takes either 0 or 1. The admissible switching strategy is

\[
w = (\theta_0, \theta_1, \theta_2, \ldots, \theta_k; \zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_k, \ldots)
\]

with \(\theta_0 = 0\) where where where \(0 \leq \theta_1 < \theta_2 < \ldots\) are an increasing sequence of \(\mathcal{F}_t\)-stopping times and \(\zeta_1, \zeta_2\ldots\) are \(\mathcal{F}_{\theta_i}\)-measurable random variables representing the new value of \(Z_t\) at the corresponding switching times \(\theta_i\) (in this section, \(\zeta_i = 1\) or 0). The state process at time \(t\) is denoted by \((X_t)_{t\geq0}\) with state space \(I = (c, d) \subseteq \mathbb{R}\) and \(X_0 = x \in I\), and with the following dynamics:

If \(\zeta_0 = 1\) (starting in open state), we have, for \(m = 0, 1, 2, \ldots\),

\[
dX_t = \begin{cases} 
 dX_t^0 = \mu_0(X^0)dt + \sigma_0(X^0)dW_t, & \theta_{2m} \leq t < \theta_{2m+1}, \\
 dX_t^1 = \mu_1(X^1)dt + \sigma_1(X^1)dW_t, & \theta_{2m+1} \leq t < \theta_{2m+2},
\end{cases}
\]

and if \(\zeta_0 = 0\) (starting in closed state),

\[
dX_t = \begin{cases} 
 dX_t^0 = \mu_0(X^0)dt + \sigma_0(X^0)dW_t, & \theta_{2m} \leq t < \theta_{2m+1}, \\
 dX_t^1 = \mu_1(X^1)dt + \sigma_1(X^1)dW_t, & \theta_{2m+1} \leq t < \theta_{2m+2}.
\end{cases}
\]

We assume that \(\mu_i : \mathbb{R} \to \mathbb{R}\) and \(\sigma_i : \mathbb{R} \to \mathbb{R}\) are some Borel functions that ensure the existence and uniqueness of the solution of (2.1) for \(i = 1\) and (2.2) for \(i = 0\).
Our performance measure, corresponding to starting state \( i = 0, 1 \), is

\[
J^w_i(x) = \mathbb{E}^x \left[ \int_0^\infty e^{-\alpha s} f(X_s) ds - \sum_{j=1}^\infty e^{-\alpha \theta_j} H(X_{\theta_j-}, \zeta_j) \right]
\]  

(2.3)

where \( H : \mathbb{R} \times 
\mathcal{Z} \rightarrow \mathbb{R}_+ \) is the switching cost function and \( f : \mathbb{R} \rightarrow \mathbb{R} \) is a continuous function that satisfies

\[
\mathbb{E}^x \left[ \int_0^\infty e^{-\alpha s} |f(X_s)| ds \right] < \infty.
\]

(2.4)

In this section, the cost functions are of the form:

\[
H(X_{\theta-}, \zeta) = \begin{cases} 
H(X_{\theta-}, 1) & \text{opening cost}, \\
H(X_{\theta-}, 0) & \text{closing cost}.
\end{cases}
\]

The optimal switching problem is to optimize the performance measure for \( i = 0 \) (start in closed state) and \( i = 1 \) (start in open state). That is to find, for both \( i = 1 \) and \( i = 0 \),

\[
v_i(x) \triangleq \sup_{w \in \mathcal{W}} J^w_i(x) \quad \text{with} \quad X_0 = x
\]

(2.5)

where \( \mathcal{W} \) is the set of all the admissible strategies.

### 2.1 Characterization of switching times

For the remaining part of section 2, we assume that the state space \( X \) is \( \mathcal{I} = (c, d) \) where both \( c \) and \( d \) are natural boundaries of \( X \). But our characterization of the value function does not rely on this assumption. In fact, it is easily applied to other types of boundaries, for example, absorbing boundary.

The first task is to characterize the optimal switching times as exit times from intervals in \( \mathbb{R} \). For this purpose, we define two functions \( g_0 \) and \( g_1 : \mathbb{R}_+ \rightarrow \mathbb{R} \) with

\[
g_1(x) \triangleq \sup_{w \in W_0} J^w_1(x) \quad \text{and} \quad g_0(x) \triangleq \sup_{w \in W_0} J^w_0(x).
\]

(2.6)

where \( W_0 \triangleq \{ w \in \mathcal{W} : w = (\theta_0, \zeta_0, \theta_1 = +\infty) \} \). In other words, \( g_1(\cdot) \) is the discounted expected revenue by starting with \( \zeta_0 = 1 \) and making no switches. Similarly, \( g_0(\cdot) \) is the discounted expected revenue by staring with \( \zeta_0 = 0 \) and making no switches.

We set \( w_0 \triangleq g_1 \) and \( y_0 \triangleq g_0 \). We consider the following simultaneous sequential optimal stopping problems with \( w_n : \mathbb{R}_+ \rightarrow \mathbb{R} \) and \( y_n : \mathbb{R}_+ \rightarrow \mathbb{R} \) for \( n = 1, 2, \ldots \):

\[
w_n(x) \triangleq \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[ \int_0^\tau e^{-\alpha s} f(X_s) ds + e^{-\alpha \tau} (y_{n-1}(X_\tau) - H(X_{\tau-}, 1 - Z_{\tau-})) \right],
\]

(2.7)

and

\[
y_n(x) \triangleq \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[ \int_0^\tau e^{-\alpha s} f(X_s) ds + e^{-\alpha \tau} (w_{n-1}(X_\tau) - H(X_{\tau-}, 1 - Z_{\tau-})) \right],
\]

(2.8)

where \( \mathcal{S} \) is a set of \( \mathcal{F}_\tau \) stopping times. Note that for each \( n \), the sequential problem (2.7) (resp. (2.8)) starts in open (resp. closed) state.
On the other hand, we define \( n \)-time switching problems for \( \zeta_0 = 1 \):

\[
q^{(n)}(x) \triangleq \sup_{w \in W_n} J_w^n(x),
\]

where

\[
W_n \triangleq \{ w \in W; w = (\theta_1, \theta_2, \ldots \theta_{n+1}; \zeta_1, \zeta_2, \ldots \zeta_n); \theta_{n+1} = +\infty \}.
\]

In other words, we start with \( \zeta_0 = 1 \) (open) and are allowed to make at most \( n \) switches. Similarly, we define another \( n \)-time switching problems corresponding to \( \zeta_0 = 0 \):

\[
p^{(n)}(x) \triangleq \sup_{w \in W_0} J_w^n(x).
\]

We investigate the relationship of these four problems:

**Lemma 2.1.** For any \( x \in \mathbb{R} \), \( w_n(x) = q^{(n)}(x) \) and \( y_n(x) = p^{(n)}(x) \).

**Proof.** We shall prove only the first assertion since the proof of the second is similar. We have set \( y_0(x) = g_0(x) \). Now we consider \( w_1 \) by using the strong Markov property of \( X \):

\[
w_1(x) = \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[ \int_0^\tau e^{-\alpha s}f(X_s)ds + e^{-\alpha \tau}(g_0(X_\tau) - H(X_{\tau-}, 0)) \right]
= \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[ \int_0^\infty e^{-\alpha s}f(X_s)ds - \int^\infty_\tau e^{-\alpha s}f(X_s)ds - e^{-\alpha \tau}(g_0(X_\tau) - H(X_{\tau-}, 0)) \right]
= \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[ e^{-\alpha \tau}(g_0(X_\tau) - g_1(X_\tau) - H(X_{\tau-}, 0)) \right] + g_1(x).
\]

On the other hand,

\[
q^{(1)}(x) = \sup_{w \in W_1} \mathbb{E}^x \left[ \int_0^\infty e^{-\alpha s}f(X_s)ds - e^{-\alpha \theta_1}H(X_{\theta_1-}, \zeta_1) \right]
= \sup_{w \in W_1} \mathbb{E}^x \left[ \int_0^{\theta_1} e^{-\alpha s}f(X_s)ds + \int_{\theta_1}^\infty e^{-\alpha s}f(X_s)ds - e^{-\alpha \theta_1}H(X_{\theta_1-}, 0) \right]
= \sup_{w \in W_1} \mathbb{E}^x \left[ (g_1(x) - e^{-\alpha \theta_1}g_1(X_{\theta_1})) - e^{-\alpha \theta_1}(g_0(X_{\theta_1}) - H(X_{\theta_1-}, 0)) \right]
= \sup_{w \in W_1} \mathbb{E}^x \left[ e^{-\alpha \theta_1}(g_0(X_{\theta_1}) - g_1(X_{\theta_1}) - H(X_{\theta_1-}, 0)) \right] + g_1(x).
\]

Since both \( \tau \) and \( \theta_1 \) are \( \mathcal{F}_\tau \) stopping times, we have \( w_1(x) = q^{(1)}(x) \) for all \( x \in \mathbb{R} \). Moreover, by the theory of the optimal stopping (see Appendix \[A\] especially Proposition \[A.4\]), \( \tau \) and hence \( \theta_1 \) are characterized as an exit time from an interval. Similarly, we can prove \( y_1(x) = p^{(1)}(x) \). Now we consider \( q^{(2)}(x) \) which is the value if we start in open state and make at most 2 switches (open → close → open). For this purpose, we consider the performance measure \( \bar{q}^{(2)} \) that starts in an open state and is allowed two switches: For arbitrary
switching times \( \theta_1, \theta_2 > \theta_1 \in S \), we have

\[
q^{(2)}(x) \triangleq \mathbb{E}^x \left[ \int_0^\infty e^{-\alpha s} f(X_s) ds - \sum_{j=1}^2 e^{-\alpha \theta_j} H(X_{\theta_j-}, \zeta_j) \right]
\]

\[
= \mathbb{E}^x \left[ \int_0^{\theta_1} e^{-\alpha s} f(X_s) ds + \int_{\theta_1}^{\theta_2} e^{-\alpha s} f(X_s) ds + \int_{\theta_2}^\infty e^{-\alpha s} f(X_s) ds 
- e^{-\alpha \theta_1} H(X_{\theta_1-}, 0) - e^{-\alpha \theta_2} H(X_{\theta_2-}, 1) \right]
\]

\[
= \left( g_1(x) - \mathbb{E}^x[e^{-\alpha \theta_1} g_1(X_{\theta_1})] \right) + \left( \mathbb{E}^x[e^{-\alpha \theta_1} g_0(X_{\theta_1}) - e^{-\alpha \theta_2} g_0(X_{\theta_2})] \right) + \mathbb{E}^x[e^{-\alpha \theta_2} g_1(X_{\theta_2})] 
- \mathbb{E}^x[e^{-\alpha \theta_1} H(X_{\theta_1-}, 0) + e^{-\alpha \theta_2} H(X_{\theta_2-}, 1)].
\]

Hence we have the following multiple optimal stopping problems:

\[
q^{(2)}(x) = \sup_{(\theta_1, \theta_2) \in S^2} \mathbb{E}^x \left[ e^{-\alpha \theta_1} \left( (g_0 - g_1)(X_{\theta_1}) - H(X_{\theta_1-}, 0) \right) + e^{-\alpha \theta_2} \left( (g_1 - g_0)(X_{\theta_2}) - H(X_{\theta_2-}, 1) \right) \right] 
+ g_1(x)
\]

where \( S^2 \triangleq \{ (\theta_1, \theta_2) ; \theta_1 \in S ; \theta_2 \in S \} \) and \( S_\sigma = \{ \tau \in S ; \tau \geq \sigma \} \) for every \( \sigma \in S \). Let us denote \( h_1(x) \triangleq g_1(x) - g_0(x) - H(x, 0) \), \( h_2(x) \triangleq g_0(x) - g_1(x) - H(x, 1) \),

\[
V_1(x) \triangleq \sup_{\tau \in S} \mathbb{E}^x[e^{-\alpha \tau} h_1(X_{\tau})] \quad \text{and} \quad V_2(x) \triangleq \sup_{\tau \in S} \mathbb{E}^x[e^{-\alpha \tau} (h_2(X_{\tau}) + V_1(X_{\tau}))].
\]

We also define

\[
\Gamma_1 \triangleq \{ x \in \mathcal{I} : V_1(x) = h_1(x) \} \quad \text{and} \quad \Gamma_2 \triangleq \{ x \in \mathcal{I} : V_2(x) = h_2(x) + V_1(x) \}
\]

with \( \sigma_n \triangleq \inf \{ t \geq 0 : X_t \in \Gamma_n \} \). By using Proposition 5.4. in Carmona and Dayanik (2003), we conclude that \( \theta_1 = \sigma_1 \) and \( \theta_2 = \theta_1 + \sigma_2 \circ s(\theta_1) \) is optimal strategy where \( s(\cdot) \) is the shift operator. Hence we only consider the maximization over the set of admissible strategy \( W^*_2 \) where

\[
W^*_2 \triangleq \{ w \in W_2 : \theta_1, \theta_2 \text{ are exit times from an interval in } \mathcal{I} \},
\]

and can use the relation \( \theta_2 - \theta_1 = \theta \circ s(\theta_1) \) with some exit time \( \theta \in S \).

\[
q^{(2)}(x) = \sup_{w \in W^*_2} \mathbb{E}^x \left[ \int_0^\infty e^{-\alpha s} f(X_s) ds - \sum_{j=1}^2 e^{-\alpha \theta_j} H(X_{\theta_j-}, \zeta_j) \right]
\]

\[
= \sup_{w \in W^*_2} \mathbb{E}^x \left[ \int_0^{\theta_1} e^{-\alpha s} f(X_s) ds + \int_{\theta_1}^{\theta_2} e^{-\alpha s} f(X_s) ds + \int_{\theta_2}^\infty e^{-\alpha s} f(X_s) ds 
- e^{-\alpha \theta_1} (H(X_{\theta_1-}, 0) + e^{-\alpha (\theta_2-\theta_1)} H(X_{\theta_2-}, 1)) \right]
\]

\[
= \sup_{w \in W^*_2} \mathbb{E}^x \left[ \int_0^{\theta_1} e^{-\alpha s} f(X_s) ds + e^{-\alpha \theta_1} \mathbb{E}^x \left[ \left( \int_0^\theta + \int_\theta^\infty \right) e^{-\alpha s} f(X_s) ds - e^{-\alpha \theta} H(X_{\theta-}, 1) \right] 
- e^{-\alpha \theta_1} H(X_{\theta_1-}, 0) \right].
\]
Now by using the result for $p^{(1)}$, we can conclude
\[
q^{(2)}(x) = \sup_{w \in W^2} \mathbb{E}^x \left[ \int_0^{\theta_1} e^{-\alpha \tau} f(X_\tau) d\tau + e^{-\alpha \theta_1} \left( p^{(1)}(X_{\theta_1}) - H(X_{\theta_1}, 0) \right) \right]
\]
\[
= \sup_{\theta_1 \in S} \mathbb{E}^x \left[ \int_0^{\theta_1} e^{-\alpha \tau} f(X_\tau) d\tau + e^{-\alpha \theta_1} \left( y_1(X_{\theta_1}) - H(X_{\theta_1}, 0) \right) \right] = w_2(x)
\]

Similarly, we can prove $y_2(x) = p^{(2)}(x)$ and we can continue this process inductively to conclude that $w_n(x) = q^{(n)}(x)$ and $y_n(x) = p^{(n)}(x)$ for all $x$ and $n$.

**Lemma 2.2.** For all $x \in \mathbb{R}$, $\lim_{n \to \infty} q^{(n)}(x) = v_1(x)$ and $\lim_{n \to \infty} p^{(n)}(x) = v_0(x)$.

**Proof.** Let us define $q(x) \triangleq \lim_{n \to \infty} q^{(n)}(x)$. Since $W_n \subset W$, $q^{(n)}(x) \leq v_1(x)$ and hence $q(x) \leq v_1(x)$. To show the reverse inequality, we define $W^+$ to be a set of admissible strategies such that

\[ W^+ = \{ w \in W : J^w_1(x) < \infty \text{ for all } x \in \mathbb{R} \}. \]

Let us assume that $v_1(x) < +\infty$ and consider a strategy $w^+ \in W^+$ and another strategy $w_n$ that coincides with $w^+$ up to and including time $\theta_n$ and then takes no further interventions.

\[ J^{w^+}_1(x) - J^w_1(x) = \mathbb{E}^x \left[ \int_{\theta_n}^{\infty} e^{-\alpha \tau} f(X_\tau) - f(X_{\theta_n}) \right] - \sum_{i \geq n+1} e^{-\alpha \theta_i} H(X_{\theta_i}, \zeta_i) \]

which implies

\[ |J^{w^+}_1(x) - J^w_1(x)| \leq \mathbb{E}^x \left[ \frac{2\|f\|}{\alpha} e^{-\alpha \theta_n} - \sum_{i \geq n+1} e^{-\alpha \theta_i} H(X_{\theta_i}, \zeta_i) \right]. \]

As $n \to +\infty$, the right hand side goes to zero by the dominated convergence theorem. Hence it is shown

\[ v_1(x) = \sup_{w \in W^+} J^w_1(x) = \sup_{w \in \bigcup_n W_n} J^w_1(x) \]

so that $v_1(x) \leq q(x)$. Next we consider $v_1(x) = +\infty$. Then we have some $m \in \mathbb{N}$ such that $w_m(x) = q^{(m)}(x) = \infty$. Hence $q^{(n)}(x) = \infty$ for all $n \geq m$. The second assertion is proved similarly.

We define an operator $\mathcal{L} : \mathcal{H} \to \mathcal{H}$ where $\mathcal{H}$ is a set of Borel functions

\[
\mathcal{L} u(x) \triangleq \sup_{\tau \in S} \mathbb{E}^x \left[ \int_0^\tau e^{-\alpha \tau} f(X_\tau) d\tau + e^{-\alpha \tau} (u(X_{\tau}) - H(X_{\tau}, 1 - Z_{\tau})) \right].
\]

**Lemma 2.3.** The function $w(x) \triangleq \lim_{n \to \infty} w_n(x)$ is the smallest solution, that majorizes $g_1(x)$, of the function equation $w = \mathcal{L}w$.

**Proof.** We renumber the sequence $(w_0, y_1, w_2, y_3, ...)$ as $(u_0, u_1, u_2, u_3, ...)$). Since $u_n$ is monotone increasing, the limit $u(x)$ exists. We have $u_{n+1}(x) = \mathcal{L} u_n(x)$ and apply the monotone convergence theorem
by taking $n \to \infty$, we have $u(x) = Lu(x)$. We assume that $u'(x)$ satisfies $u' = Lu'$ and majorizes $g(x) = u_0(x)$. Then $u' = Lu' \geq Lu_0 = u_1$. Let us assume, for induction argument that $u' \geq u_n$, then

$$u' = Lu' \geq Lu_n = u_{n+1}.$$  

Hence we have $u' \geq u_n$ for all $n$, leading to $u' \geq \lim_{n \to \infty} u_n = u$. Now we take the subsequence in $(w_0, y_1, w_2, y_3,...)$ to complete the proof.

**Proposition 2.1.** For each $x \in \mathbb{R}$, $\lim_{n \to \infty} w_n(x) = v_1(x)$ and $\lim_{n \to \infty} y_n(x) = v_0(x)$. Moreover, the optimal switching times, $\theta_i^*$ are exit times from an interval.

**Proof.** We can prove the first assertion by combining the first two lemmas above. Now we concentrate on the sequence of $w_n(x)$. For each $n$, finding $w_n(x)$ by solving (2.7) is an optimal stopping problem. By Proposition 2.4, the optimal stopping times are characterized as an exit time of $X$ from an interval for all $n$. This is also true in the limit: Indeed, by Lemma 2.3, in the limit, the value function of optimal switching problem $v_1(x) = w(x)$ satisfies $w = Lw$, implying that $v_1(x)$ is the solution of an optimal stopping problem. Hence the optimal switching times are characterized as exit time from an interval.

2.2 Characterization of the value functions

We go back to the original problem (2.3) to characterize the value function of the optimal switching problems. By the exit time characterization of the optimal switching times, $\theta_i^*$ are given by

$$\theta_i^* = \begin{cases} \inf\{t > \theta_{i-1}; X_i^1 \in \Gamma_1\} \\ \inf\{t > \theta_{i-1}; X_i^0 \in \Gamma_0\} \end{cases}$$  

(2.12)

where $\Gamma_1 = \mathbb{R} \setminus C_1$ and $\Gamma_0 = \mathbb{R} \setminus C_0$. We define here $C_i$ and $\Gamma_i$ to be continuation and stopping region for $X^i$, respectively. We can simplify the performance measure $J^w$ considerably. For $\zeta_0 = 1$, we have

$$J^w_1(x) = \mathbb{E}^x \left[ \int_0^\infty e^{-\alpha s} f(X_s)ds - \sum_{j=1}^\infty e^{-\alpha \theta_j} H(X_{\theta_j}, \zeta_j) \right]$$

$$= \mathbb{E}^x \left[ \int_0^{\theta_1} e^{-\alpha s} f(X_s)ds + \int_{\theta_1}^\infty e^{-\alpha s} f(X_s)ds - e^{-\alpha \theta_1} \left( H(X_{\theta_1}, 0) + \sum_{j=2}^\infty e^{-\alpha (\theta_j - \theta_1)} H(X_{\theta_j}, \zeta_j) \right) \right]$$

$$= \mathbb{E}^x \left[ \int_0^{\theta_1} e^{-\alpha s} f(X_s)ds + e^{-\alpha \theta_1} \mathbb{E}^x \left[ \int_0^\infty e^{-\alpha s} f(X_s)ds - \sum_{j=1}^\infty e^{-\alpha \theta_j} H(X_{\theta_j}, \zeta_j) \right] - e^{-\alpha \theta_1} H(X_{\theta_1}, 0) \right]$$
We notice that in the time interval \((0, \theta_1)\), the process \(X\) is not intervened. The inner expectation is just \(J_{0}^{w}(X_{\theta_1})\). Hence we further simplify

\[
J_{1}^{w}(x) = \mathbb{E}^x \left[ \int_{0}^{\theta_1} e^{-\alpha t} f(x) ds + e^{-\alpha \theta_1} (J_{0}^{w}(X_{\theta_1}) - H(x_{\theta_1}, 0)) \right]
\]

\[
= \mathbb{E}^x \left[ -e^{-\alpha \theta_1} g_1(X_{\theta_1}) + e^{-\alpha \theta_1} (J_{0}^{w}(X_{\theta_1}) - H(x_{\theta_1}, 0)) \right] + g_1(x)
\]

\[
= \mathbb{E}^x \left[ -e^{-\alpha \theta_1} g_1(X_{\theta_1}) + e^{-\alpha \theta_1} J_{1}^{w}(X_{\theta_1}) \right] + g_1(x).
\]

The third equality is a critical observation. Finally, we define \(u_1 \triangleq J_1 - g_1\) and obtain

\[
u_1(x) = J_{1}^{w}(x) - g_1(x) = \mathbb{E}^x \left[ e^{-\alpha \theta_1} u_1(X_{\theta_1}) \right]. \tag{2.13}
\]

Since the switching time \(\theta_1\) is characterized as a hitting time of a certain point in the state space, we can represent \(\theta_1 = \tau_a \triangleq \inf \{ t \geq 0 : X_t = a \} \) for some \(a \in \mathbb{R}\). Hence equation (2.13) is an optimal stopping problem that maximizes

\[
u_1(x) = J_{1}^{w}(x) - g_1(x) = \mathbb{E}^x \left[ e^{-\alpha \tau_a} u_1(X_{\tau_a}) \right]. \tag{2.14}
\]

among all the \(\tau_a \in S\). When \(\theta_1 = 0\) (i.e., \(x = X_{\theta_1}\)),

\[
u_1(x) = J_{1}^{w}(x) - g_1(x) = \mathbb{E}^x \left[ -g_1(x) + J_{0}^{w}(x) - H(x, 0) \right] + g_1(x)
\]

and hence

\[
u_1(x) = J_{0}^{w}(x) - H(x, 0) - g_1(x).
\]

In other words, we make a switch from open to closed immediately by paying the switching cost. Similarly, for \(\zeta_0 = 0\), we can simplify the performance measure \(J_{0}^{w}(\cdot)\) to obtain

\[
u_0(x) = J_{0}^{w}(x) - g_0(x) = \mathbb{E}^x \left[ e^{-\alpha \theta_1} u_0(X_{\theta_1}) \right].
\]

By defining \(u_0 \triangleq J_{0}^{w} - g_0\), we have

\[
u_0(x) = J_{0}^{w}(x) - g_0(x) = \mathbb{E}^x \left[ e^{-\alpha \theta_1} u_0(X_{\theta_1}) \right].
\]

Again, by using the characterization of switching times, we replace \(\theta_1\) with \(\tau_b\),

\[
u_0(x) = J_{0}^{w}(x) - g_0(x) = \mathbb{E}^x \left[ e^{-\alpha \tau_b} u_0(X_{\tau_b}) \right]. \tag{2.15}
\]

In summary, we have

\[
u_1(x) = \begin{cases} 
  u_0(x) + g_0(x) - H(x, 0) - g_1(x), & x \in \Gamma_1, \\
  \mathbb{E}^x [e^{-\alpha \tau_a} u_1(X_{\tau_a})] = \mathbb{E}^x [e^{-\alpha \tau_a} (u_0(X_{\tau_a}) + g_0(X_{\tau_a}) - g_1(X_{\tau_a}) - H(X_{\tau_a}, 0))], & x \in \mathcal{C}_1,
\end{cases}
\tag{2.16}
\]
Hence we should solve the following optimal stopping problems simultaneously:

\[
\begin{align*}
\mathbb{E}^x [e^{-\alpha \tau} u_0(X_{\tau})] &= \mathbb{E}^x [e^{-\alpha \tau} (u_1(X_{\tau}) + g_1(X_{\tau}) - g_0(X_{\tau}) - H(X_{\tau}, 1))], \quad x \in C_0, \\
u_1(x) + g_1(x) - H(x, 1) - g_0(x), \quad x \in \Gamma_0.
\end{align*}
\]

(2.17)

Now we let the infinitesimal generators of \(X^1\) and \(X^0\) be \(A_1\) and \(A_0\), respectively. We consider \((A_i - \alpha)\psi(x) = 0\) for \(i = 0, 1\). This ODE has two fundamental solutions, \(\psi_i(\cdot)\) and \(\phi_i(\cdot)\). We set \(\psi_i(\cdot)\) is an increasing and \(\phi_i(\cdot)\) is a decreasing function. Note that \(\psi_i(c+)=0, \phi_i(c+)=\infty\) and \(\psi_i(d-) = \infty, \phi_i(d-) = 0\). We define

\[
F_i(x) \triangleq \frac{\psi_i(x)}{\phi_i(x)} \quad \text{and} \quad G_i(x) \triangleq -\frac{\phi_i(x)}{\psi_i(x)} \quad \text{for } i = 0, 1.
\]

By referring to Dayanik and Karatzas (2003), we have the following representation

\[
\mathbb{E}^x[e^{-\alpha \tau} \mathbf{1}_{\{\tau < \tau_1\}}] = \frac{\psi(l)\phi(x) - \psi(x)\phi(l)}{\psi(l)\phi(r) - \psi(r)\phi(l)}, \quad \mathbb{E}^x[e^{-\alpha \tau} \mathbf{1}_{\{\tau > \tau_1\}}] = \frac{\psi(x)\phi(r) - \psi(r)\phi(x)}{\psi(l)\phi(r) - \psi(r)\phi(l)},
\]

for \(x \in [l, r]\) where \(\tau_1 \triangleq \inf\{t > 0; X_t = l\}\) and \(\tau_r \triangleq \inf\{t > 0; X_t = r\}\).

By defining

\[
W_1 = (u_1/\psi_1) \circ G_1^{-1} \quad \text{and} \quad W_0 = (u_0/\phi_0) \circ F_0^{-1},
\]

the second equation in (2.16) and the first equation in (2.17) become

\[
W_1(G_1(x)) = W_1(G_1(a)) \frac{G_1(d) - G_1(x)}{G_1(d) - G_1(a)} + W_1(G_1(a)) \frac{G_1(x) - G_1(a)}{G_1(d) - G_1(a)} \quad x \in [a, d],
\]

(2.19)

and

\[
W_0(F_0(x)) = W_0(F_0(c)) \frac{F_0(b) - F_0(x)}{F_0(b) - F_0(c)} + W_0(F_0(b)) \frac{F_0(x) - F_0(c)}{F_0(b) - F_0(c)}, \quad x \in (c, b),
\]

(2.20)

respectively. We should understand that \(F_0(c) \triangleq F_0(c+) = \psi_0(c+)/\phi_0(c+) = 0\) and that \(G_1(d) \triangleq G_1(d-) = -\phi_1(d-)/\psi_1(d-) = 0\). In the next subsection, we shall explain \(W_1(G_1(d-))\) and \(W_0(F_0(c+))\) in details. Both \(W_1\) and \(W_0\) are a linear function in their respective transformed spaces. Hence under the appropriate transformations, the two value functions are linear functions in the continuation region.

2.3 Direct Method for a Solution

We have established a mathematical characterization of the value functions of optimal switching problems. We shall investigate, by using the characterization, a direct solution method that does not require the recursive optimal stopping schemes described in section 2.1. Since the two optimal stopping problems (2.18)
have to be solved simultaneously, finding \( u_0 \) in \( x \in C_0 \), for example, requires that we find the smallest \( F_0 \)-concave majorant of \((u_1(x) + g_1(x) - g_0(x) - H(x, 1))/\varphi_0(x)\) as in (2.17) that involves \( u_1 \).

There are two cases, depending on whether \( x \in C_1 \cap C_0 \) or \( x \in \Gamma_1 \cap C_0 \), as to what \( u_1(\cdot) \) represents. In the region \( x \in \Gamma_1 \cap C_0 \), \( u_1(\cdot) \) that shows up in the equation of \( u_0(x) \) is of the form \( u_1(x) = u_0(x) + g_0(x) - H(x, 1, 0) - g_1(x) \). In this case, the “obstacle” that should be majorized is in the form

\[
\begin{align*}
\limsup_x \frac{(u_1(x) + g_1(x) - g_0(x) - H(x, 1))}{\varphi_0(x)} & = (u_0(x) + g_0(x) - H(x, 0) - g_1(x) + g_1(x) - g_0(x) - H(x, 1)) \\
& = u_0(x) - H(x, 0) - H(x, 1) < u_0(x).
\end{align*}
\]

(2.21)

This implies that in \( x \in \Gamma_1 \cap C_0 \), the \( u_0(x) \) function always majorizes the obstacle. Similarly, in \( x \in \Gamma_0 \cap C_1 \), the \( u_1(x) \) function always majorizes the obstacle.

Next, we consider the region \( x \in C_0 \cap C_1 \). The \( u_0(\cdot) \) term in (2.16) is represented, due to its linear characterization, as

\[
W_0(F_0(x)) = \beta_0(F_0(x)) + d_0
\]

with some \( \beta_0 \in \mathbb{R} \) and \( d_0 \in \mathbb{R}_+ \) in the transformed space. (The nonnegativity of \( d_0 \) will be shown.) In the original space, it has the form of \( \varphi_0(x)(\beta_0 F_0(x) + d_0) \). Hence by the transformation \((u_1/\psi_1) \circ G^{-1}\), \( W_1(G_1(x)) \) is the smallest linear majorant of

\[
\frac{K_1(x) + \varphi_0(x)(\beta_0 F_0(x) + d_0)}{\psi_1(x)} = \frac{K_1(x) + \beta_0 \psi_0(x) + d_0 \varphi_0(x)}{\psi_1(x)}
\]

on \((G_1(d-), G_1(a^*))\) where

\[
K_1(x) \triangleq g_0(x) - g_1(x) - H(x, 0).
\]

(2.22)

This linear function passes a point \((G_1(d-), l_d)\) where \( G_1(d-) = 0 \) and

\[
l_d = \limsup_{x \uparrow d} \frac{(K_1(x) + \beta_0 \psi_0(x) + d_0 \varphi_0(x))^+}{\psi_1(x)}.
\]

Let us consider further the quantity \( l_d \geq 0 \). By noting

\[
\limsup_{x \uparrow d} \frac{(K_1(x) + \beta_0 \psi_0(x))^+}{\psi_1(x)} \leq \limsup_{x \uparrow d} \frac{(K_1(x) + \beta_0 \psi_0(x) + d_0 \varphi_0(x))^+}{\psi_1(x)} \leq \limsup_{x \uparrow d} \frac{(K_1(x) + \beta_0 \psi_0(x))^+}{\psi_1(x)} + \limsup_{x \uparrow d} \frac{d_0 \varphi_0(x)}{\psi_1(x)}
\]

and \( \limsup_{x \uparrow d} \frac{d_0 \varphi_0(x)}{\psi_1(x)} = 0 \), we can redefine \( l_d \) by

\[
l_d \triangleq \limsup_{x \uparrow d} \frac{(K_1(x) + \beta_0 \psi_0(x))^+}{\psi_1(x)}
\]

(2.23)

to determine the finiteness of the value function of the optimal switching problem, \( v_1(x) \), based upon Proposition [A.5][A.7] Let us concentrate on the case \( l_d = 0 \).
Similar analysis applies to (2.17). $u_1(x)$ in (2.17) is represented as

$$W_1(G_1(x)) = \beta_1 G_1(x) + d_1$$

with some $\beta_1 \in \mathbb{R}$ and $d_1 \in \mathbb{R}_+$. Note that $d_1 = l_d \geq 0$. In the original space, it has the form of $\psi_1(x)(\beta_1 G_1(x) + d_1)$. Hence by the transformation $(u_0/\varphi_0(x)) \circ F^{-1}$, $W_0(F_0(x))$ is the smallest linear majorant of

$$\frac{K_0(x) + \psi_1(x)(\beta_1 G_1(x) + d_1)}{\varphi_0(x)} = \frac{K_0(x) - \beta_1 \varphi_1(x) + d_1 \psi_1(x)}{\varphi_0(x)}$$

on $(F_0(c^+), F_0(b^*))$ where

$$K_0(x) \triangleq g_1(x) - g_0(x) - H(x, 1). \tag{2.24}$$

This linear function passes a point $(F_0(c^+), l_c)$ where $F_0(c^+) = 0$ and

$$l_c = \limsup_{x \uparrow c} \frac{(K_0(x) - \beta_1 \varphi_1(x) + d_1 \psi_1(x))^+}{\varphi_0(x)}.$$

Hence we have $l_c = d_0 \geq 0$. By the same argument as for $l_d$, we can redefine

$$l_c \triangleq \limsup_{x \uparrow c} \frac{(K_0(x) - \beta_1 \varphi_1(x))^+}{\varphi_0(x)}. \tag{2.25}$$

**Remark 2.1.** (a) Evaluation of $l_d$ or $l_c$ does not require knowledge of $\beta_0$ or $\beta_1$, respectively unless the orders of $\max(K_1(x), \psi_1(x))$ and $\psi_0(x)$ are equal, for example. (For this event, see Proposition 2.4.)

Otherwise, we just compare the order of the positive leading terms of the numerator in (2.23) and (2.24) with that of the denominator.

(b) A sufficient condition for $l_d = l_c = 0$: since we have

$$0 \leq l_d \leq \limsup_{x \uparrow d} \frac{(K_1(x))^+}{\psi_1(x)} + \limsup_{x \uparrow d} \frac{\beta_0 \psi_0(x))^+}{\psi_1(x)}.$$

a sufficient condition for $l_d = 0$ is

$$\limsup_{x \uparrow d} \frac{(K_1(x))^+}{\psi_1(x)} = 0 \text{ and } \limsup_{x \uparrow d} \frac{\psi_0(x)}{\psi_1(x)} = 0. \tag{2.26}$$

Similarly,

$$0 \leq l_c \leq \limsup_{x \uparrow c} \frac{(K_0(x))^+}{\varphi_0(x)} + \limsup_{x \uparrow c} \frac{-\beta_1 \varphi_1(x))^+}{\varphi_0(x)}.$$

Hence a sufficient condition for $l_c = 0$ is

$$\limsup_{x \uparrow c} \frac{(K_0(x))^+}{\varphi_0(x)} = 0 \text{ and } \limsup_{x \uparrow c} \frac{\varphi_1(x)}{\varphi_0(x)} = 0. \tag{2.27}$$

Moreover, it is obvious $\beta_1 < 0$ and $\beta_0 > 0$ since the linear majorant passes the origin of each transformed space. Recall a points in the interval $(c, d) \in \mathbb{R}_+$ will be transformed by $G(\cdot)$ to $(G(c), G(d-)) \in \mathbb{R}_-$.\n
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We summarize the case of \( l_c = l_d = 0 \):

**Proposition 2.2.** Suppose that \( l_d = l_c = 0 \), the quantities being defined by (2.23) and by (2.25), respectively. The value functions in the transformed space are the smallest linear majorants of

\[
R_1(\cdot) = \frac{r_1(G_1^{-1}(\cdot))}{\psi_1(G_1^{-1}(\cdot))} \quad \text{and} \quad R_0(\cdot) = \frac{r_0(F_0^{-1}(\cdot))}{\varphi_0(F_0^{-1}(\cdot))}
\]

where

\[
r_1(x) \triangleq g_0(x) - g_1(x) + \beta_0 \psi_0(x) - H(x, 0)
\]

and

\[
r_0(x) \triangleq g_1(x) - g_0(x) - \beta_1 \varphi_1(x) - H(x, 1)
\]

for

\[
\beta_0 > 0 \quad \text{and} \quad \beta_1 < 0.
\]

Furthermore, \( \Gamma_1 \) and \( \Gamma_0 \) in (2.16) and (2.17) are given by

\[
\Gamma_1 \triangleq \{ x \in (c, d) : W_1(G_1(x)) = R_1(G_1(x)) \}, \quad \text{and} \quad \Gamma_0 \triangleq \{ x \in (c, d) : W_0(F_0(x)) = R_0(F_0(x)) \}
\]

**Corollary 2.1.** If either of the boundary points \( c \) or \( d \) is absorbing, then \((F_0(c), W_0(F_0(c))) \) or \((G_1(d), W_1(G_1(d)))\) is obtained directly. We can entirely omit the analysis of \( l_c \) or \( l_d \). The characterization of the value function (2.19) and (2.20) remains exactly the same.

**Remark 2.2.** An algorithm to find \((a^*, b^*, \beta_0^*, \beta_1^*)\) can be described as follows:

1. Start with some \( \beta_1' \in \mathbb{R} \).
2. Calculate \( r_0 \) and then \( R_0 \) by the transformation \( R_0(\cdot) = \frac{r_0(F_0^{-1}(\cdot))}{\varphi_0(F_0^{-1}(\cdot))} \).
3. Find the linear majorant of \( R_0 \) passing the origin of the transformed space. Call the slope of the linear majorant, \( \beta_0 \) and the point, \( F_0(b) \), where \( R_0 \) and the linear majorant meet.
4. Plug \( b \) and \( \beta_0 \) in the equation for \( r_1 \) and calculate \( R_1 \) by the transformation \( R_1(\cdot) = \frac{r_1(G_1^{-1}(\cdot))}{\psi_1(G_1^{-1}(\cdot))} \).
5. Find the linear majorant of \( R_1 \) passing the origin of the transformed space. Call the slope of the linear majorant, \( \beta_1 \) and the point, \( G_1(a) \), where \( R_1 \) and the linear majorant meet.
6. Iterate step 1 to 5 until \( \beta_1 = \beta_1' \).

If both \( R_1 \) and \( R_0 \) are differentiable functions with their respective arguments, we can find \((a^*, b^*)\) analytically. Namely, we solve the following system for \( a \) and \( b \):

\[
\begin{align*}
\frac{dR_0(y)}{dy} & \bigg|_{y = F_0(b)} (F_0(b) - F_0(c)) = R_0(F_0(b)) \\
\frac{dR_1(y)}{dy} & \bigg|_{y = G_1(a)} (G_1(a) - G_1(d)) = R_1(G_1(a)) \\
\end{align*}
\]

(2.29)

where \( \frac{dR_0(y)}{dy} \bigg|_{y = F_0(b^*)} = \beta_0^* \) and \( \frac{dR_1(y)}{dy} \bigg|_{y = G_1(a^*)} = \beta_1^* \).
Indeed, since we have continuous, it follows \( v \) is impossible. Hence if the value functions exist, then we must necessarily have

**Proposition 2.3.** If the optimal continuation regions for both of the value functions are connected and if \( l_c = l_d = 0 \), then the pair of the value functions \( v_1(x) \) and \( v_0(x) \) are represented as

\[
v_1(x) = \begin{cases} 
\hat{v}_0(x) - H(x, 0), & x \leq a^*, \\
\hat{v}_1(x) \triangleq \psi_1(x)W_1(G_1(x)) + g_1(x), & a^* < x,
\end{cases}
\]

and

\[
v_0(x) = \begin{cases} 
\hat{v}_0(x) \triangleq \varphi_0(x)W_0(F_0(x)) + g_0(x), & x < b^*, \\
\hat{v}_1(x) - H(x, 1), & b^* \leq x,
\end{cases}
\]

for some \( a^*, b^* \in \mathbb{R} \) with \( a^* < b^* \).

**Proof.** If the optimal continuation regions for both of the value functions are connected and if \( l_d = l_c = 0 \), then the optimal intervention times (2.30) have the following form:

\[
\theta^*_i = \begin{cases} 
\inf \{t > \theta_{i-1}; X_t \notin (a^*, d)\}, & Z = 1, \\
\inf \{t > \theta_{i-1}; X_t \notin (c, b^*)\}, & Z = 0.
\end{cases}
\]

(2.30)

Indeed, since we have \( l_c = l_d = 0 \), the linear majorants \( W_1(\cdot) \) and \( W_0(\cdot) \) pass the origins in their respective transformed coordinates. Hence the continuation regions shall necessarily of the form of (2.30).

By our construction, both \( v_1(x) \) and \( v_0(x) \) are continuous in \( x \in \mathbb{R} \). Suppose we have \( a^* > b^* \). In this case, by the form of the value functions, \( v_0(b-) - H(b, 1, 0) = v_1(b) \). Since the cost function \( H(\cdot) > 0 \) and continuous, it follows \( v_0(b-) > v_1(b) \). On the other hand, \( v_0(b+) = v_1(b) - H(b, 0, 1) \) implying \( v_0(b+) < v_1(b) \). This contradicts the continuity of \( v_0(x) \). Also, \( a^* = b^* \) will lead to \( v_1(x) = v_1(x) - H(x, 1, 0) \) which is impossible. Hence if the value functions exist, then we must necessarily have \( a^* < b^* \). \( \square \)

In relation to Proposition 2.3 we have the following observations:

**Remark 2.3.**

(a) It is obvious that

\[
v_0(x) = \hat{v}_0(x) > \hat{v}_0(x) - H(x, 0) = v_1(x), \quad x \in (c, a^*),
\]

and

\[
v_1(x) = \hat{v}_1(x) > \hat{v}_1(x) - H(x, 1) = v_0(x), \quad x \in (b^*, d).
\]

(b) Since \( u_1(x) \) is continuous in \( (c, d) \), the “obstacle” \( u_1(x) + g_1(x) - g_0(x) - H(x, 1) \) to be majorized by \( u_0(x) \) on \( x \in C_0 = (c, b^*) \) is also continuous, in particular at \( x = a^* \). We proved that \( u_0(x) \) always majorizes the obstacle on \( (c, a^*) \). Hence \( F(a^*) \in \{y : W_0(y) > R_0(y)\} \) if there exists a linear majorant of \( R_0(y) \) in an interval of the form \( (F_0(q), F_0(d)) \) with some \( q \in (c, d) \); otherwise, the continuity of \( u_1(x) + g_1(x) - g_0(x) - H(x, 1) \) does not hold. Similarly, we have \( F(b^*) \in \{y : W_1(y) > R_1(y)\} \) if there exists a linear majorant of \( R_0(y) \) in an interval of the form \( (G_1(c), G_1(q)) \).
Finally, we summarize other cases than \( l_c = l_d = 0 \):

**Proposition 2.4.**

(a) If either \( l_d = +\infty \) or \( l_c = +\infty \), then \( v_1(x) = v_0(x) \equiv +\infty \).

(b) If both \( l_d \) and \( l_c \) are finite, then \( l_d = l_c = 0 \).

**Proof.** (a) The proof is immediate by invoking Proposition A.5. (b) When \( l_{c} \) is finite, we know by Proposition A.5 that the value function \( v_0(x) \) is finite. On \( x \in (c, a^*) \), \( u_1(x) + g_1(x) - g_0(x) - H(x, 1) < u_0(x) < +\infty \) is finite (see (2.21)) and thereby

\[
    l_c = \limsup_{x \downarrow c} \frac{u_1(x) + g_1(x) - g_0(x) - H(x, 1)}{\phi_0(x)} = 0.
\]

The same argument for \( l_d = 0 \).

Therefore, we can conclude that \( l_d = 0 \) for the situation where the orders of \( \max(K_1(x), \psi_1(x)) \) and \( \psi_0(x) \) are equal (\( \Rightarrow l_d \) is finite) as described in Remark 2.1(a).

### 3 Examples

We recall some useful observations. If \( h(\cdot) \) is twice-differentiable at \( x \in I \) and \( y \equiv F(x) \), then we define \( H(y) \equiv h(F^{-1}(y))/\varphi(F^{-1}(y)) \) and we obtain \( H'(y) = m(x) \) and \( H''(y) = m'(x)/F'(x) \) with

\[
    m(x) = \frac{1}{F'(x)} \left( \frac{h}{\varphi} \right)'(x), \quad \text{and} \quad H''(y)(A - \alpha)h(x) \geq 0, \quad y = F(x) \tag{3.1}
\]

with strict inequality if \( H''(y) \neq 0 \). These identities are of practical use in identifying the concavities of \( H(\cdot) \) when it is hard to calculate its derivatives explicitly. Using these representations, we can modify (2.29) to

\[
\left\{ \begin{array}{l}
\frac{1}{F_0(b)} \left( \frac{m_1}{\psi_0} \right)'(b)(F_0(b) - F_0(c)) = \frac{r_1(b)}{\psi_0(b)} \\
\frac{1}{G_1(a)} \left( \frac{m_1}{\psi_1} \right)'(a)(G_1(a) - G_1(d)) = \frac{r_1(a)}{\psi_1(a)}
\end{array} \right. \tag{3.2}
\]

**Example 3.1. Brekke and Øksendal (1994):** We first illustrate our solution method by using a resource extraction problem solved by Brekke and Øksendal (1994). The price \( P_t \) at time \( t \) per unit of the resource follows a geometric Brownian motion. \( Q_t \) denotes the stock of remaining resources in the field that decays exponentially. Hence we have

\[
dP_t = \alpha P_t dt + \beta P_t dW_t \quad \text{and} \quad dQ_t = -\lambda Q_t dt
\]

where \( \alpha, \beta, \lambda > 0 \) (extraction rate) are constants. The objective of the problem is to find the optimal switching times of resource extraction:

\[
v(x) = \sup_{w \in W} J^w(x) = \sup_{w \in W} \mathbb{E}^x \left[ \int_0^\infty e^{-\rho t}(\lambda P_t Q_t - K)Z_t dt - \sum_i e^{-\rho \theta_i} H(X_{\theta_i -}, Z_{\theta_i}) \right]
\]
where \( \rho \in \mathbb{R}_+ \) is a discount factor with \( \rho > \alpha \), \( K \in \mathbb{R}_+ \) is the operating cost and \( H(x, 0) = C \in \mathbb{R}_+ \) and \( H(x, 1) = L \in \mathbb{R}_+ \) are constant closing and opening costs. Since \( P \) and \( Q \) always show up in the form of \( PQ \), we reduce the dimension by defining \( X_t = P_t Q_t \) with the dynamics:

\[
dX_t = (\alpha - \lambda Z_t) X_t dt + \beta X_t dW_t.
\]

**Solution:** (1) We shall calculate all the necessary functions. For \( Z_t = 1 \) (open state), we solve \( (A_1 - \rho)v(x) = 0 \) where \( A_1 = (\alpha - \lambda) xv''(x) + \frac{1}{2} \beta^2 x^2 v'''(x) \) to obtain \( \psi_1(x) = x^{\nu_+} \) and \( \varphi_1(x) = x^{\nu_-} \) where \( \nu_{+, -} = \beta^{-2} \left(-\alpha + \lambda + \frac{1}{2} \beta^2 \pm \sqrt{(\alpha - \lambda - \frac{1}{2} \beta^2)^2 + 2 \rho \beta^2} \right) \). Similarly, for \( Z_t = 0 \) (closed state), we solve \( (A_0 - \rho)v(x) = 0 \) where \( A_0 = \alpha xv'(x) + \frac{1}{2} \beta^2 x^2 v''(x) \) to obtain \( \psi_0(x) = x^{\mu_+} \) and \( \varphi_0(x) = x^{\mu_-} \) where \( \mu_{+, -} = \beta^{-2} \left(-\alpha + \frac{1}{2} \beta^2 \pm \sqrt{(\alpha - \frac{1}{2} \beta^2)^2 + 2 \rho \beta^2} \right) \). Note that under the assumption \( \rho > \alpha \), we have \( \nu_{+, \mu_+} > 1 \) and \( \nu_{-, \mu_-} < 0 \).

By setting \( \Delta_1 = \sqrt{(\alpha - \lambda - \frac{1}{2} \beta^2)^2 + 2 \rho \beta^2} \) and \( \Delta_0 = \sqrt{(\alpha - \frac{1}{2} \beta^2)^2 + 2 \rho \beta^2} \), we have \( G_1(x) = -\varphi_1(x)/\psi_1(x) = -x^{-2 \Delta_1 / \beta^2} \) and \( F_0(x) = \psi_0(x)/\varphi_0(x) = x^{2 \Delta_0 / \beta^2} \). It follows that \( G_1^{-1}(y) = (-y)^{-2 \beta^2 / \Delta_1} \) and \( F_0^{-1}(y) = y^{\beta^2 / 2 \Delta_0} \). In this problem, we can calculate \( g_1(x), g_0(x) \) explicitly:

\[
g_1(x) = \mathbb{E}^x \left[ \int_0^\infty e^{-\beta s} (\lambda X_s - K) ds \right] = x \frac{1}{\rho + \lambda - \alpha} - \frac{K}{\rho} - C \quad \text{and} \quad K_0(x) = g_1(x) - g_0(x) - H(x, 0) = - \left( \frac{x}{\rho + \lambda - \alpha} - \frac{K}{\rho} \right) - C \quad \text{and} \quad K_0(x) =
\]

and \( g(x) = 0 \). Lastly, \( K_1(x) = g_0(x) - g_1(x) - H(x, 0) = - \left( \frac{x}{\rho + \lambda - \alpha} - \frac{K}{\rho} \right) - C \) and \( K_0(x) =
\]

(2) The state space of \( X \) is \( (c, d) = (0, \infty) \) and we evaluate \( l_c \) and \( l_d \). Let us first note that \( \Delta_0 - \Delta_1 + \lambda > 0 \).

Since \( \lim_{x \to 0} \frac{\psi_1(x)}{\psi_0(x)} = \lim_{x \to 0} x^{-\Delta_0 / \beta^2} = 0 \) and \( \lim_{x \to 0} \frac{K_0(x)}{\varphi_0(x)} = 0 \), we have \( l_c = l_0 = 0 \) by (2.27). Similarly, by noting \( \lim_{x \to +\infty} \frac{\psi_1(x)}{\varphi_0(x)} = \lim_{x \to +\infty} x^{-\Delta_1 / \beta^2} = 0 \) and \( \lim_{x \to +\infty} \frac{K_1(x)}{\varphi_0(x)} = 0 \), we have \( l_d = l_+ = 0 \) by (2.26).

(3) To find the value functions together with continuation regions, we set

\[
r_1(x) = - \left( \frac{x}{\rho + \lambda - \alpha} - \frac{K}{\rho} \right) - C + \beta_0 \psi_0(x) \quad \text{and} \quad r_0(x) = \left( \frac{x}{\rho + \lambda - \alpha} - \frac{K}{\rho} \right) - L - \beta_1 \varphi_1(x)
\]

and make transformations \( R_1(y) = r_1(F^{-1}(y))/\psi_1(F^{-1}(y)) \) and \( R_0(y) = r_0(F^{-1}(y))/\varphi_0(F^{-1}(y)) \), respectively. We examine the shape and behavior of the two functions \( R_1(\cdot) \) and \( R_0(\cdot) \) with an aid of (3.1).

By calculating \( (r_0/\varphi_0)'(x) \) explicitly to examine the derivative of \( R_0(y) \), we can find a critical point \( x = q \), at which \( R_0(F(x)) \) attains a local minimum and from which \( R_0(F(x)) \) is increasing monotonically on \( (F_0(q), \infty) \). Moreover, we can confirm that \( \lim_{y \to -\infty} R_0(y) = \lim_{x \to -\infty} \frac{r_0(x)/\varphi_0(x)}{\varphi_0(x)} = 0 \), which shows that there exists a finite linear majorant of \( R_0(y) \). We define

\[
p(x) = \beta_1 \omega x^{\nu_-} - (\rho - \alpha) \left( \frac{x}{\rho + \lambda - \alpha} \right) + (K + \rho L)
\]

such that \( (A_0 - \rho)r_0(x) = p(x) \) where \( \omega \triangleq (\rho - \frac{1}{2} \beta^2) \nu_-(\nu_- - 1) - \alpha \nu_- = \frac{1}{2} \beta^2 (\Delta_0 - \Delta_1 + \lambda)(\Delta_0 + \Delta_1 - \lambda) > 0 \). By the second identity in (3.1), the sign of the second derivative \( R_0''(y) \) is the same as the sign

\[15\]
of $p(x)$. It is easy to see that $p(x)$ has only one critical point. For any $\beta_1 < 0$, the first term is dominant as $x \to 0$, so that $\lim_{x \to 0} p(x) < 0$. As $x$ gets larger, for $|\beta_1|$ sufficiently small, $p(x)$ can take positive values, providing two positive roots, say $x = k_1, k_2$ with $k_1 < k_2$. We also have $\lim_{x \to +\infty} p(x) = -\infty$. In this case, $R_0(y)$ is concave on $(0, F(k_1) \cup (F(k_2), +\infty)$ and convex on $(F(k_1), F(k_2))$. Since we know that $R_0(y)$ attains a local minimum at $y = F(q)$, we have $q < k_2$, and it implies that there is one and only on tangency point of the linear majorant $W(y)$ and $R_0(y)$ on $(F(q), \infty)$, so that the continuation region is of the form $(0, b^*)$.

From this analysis of the derivatives of $R_0(y)$, there is only one tangency point of the linear majorant $W_0(y)$ and $R_0(y)$. (See Figure 3.1(a)). A similar analysis shows that there is only one tangency point of the linear majorant $W_1(y)$ and $R_1(y)$. (See Figure 3.1(b)).

**Figure 1:** A numerical example of resource extraction problem. with parameters $(\alpha, \beta, \lambda, \rho, K, L, C) = (0.01, 0.25, 0.01, 0.05, 0.4, 2, 2)$(a) The smallest linear majorant $W_0(F_0(x))$ and $R_0(F_0(x))$ with $b^* = 1.15042$ and $\beta_1^* = 10.8125$. (b)The smallest linear majorant $W_1(G_1(x))$ and $R_1(G_1(x))$ with $a^* = 0.18300$ and $\beta_1^* = -0.695324$. (c) The value function $v_0(x)$. (d) The value function $v_1(x)$.

(4) By solving the system of equations (2.29), we can find $(a^*, b^*, \beta_0^*, \beta_1^*)$. We transform back to the original space to find

$$\hat{v}_1(x) = \psi_1(x)W_1(G_1(x)) + g_1(x) = \psi_1(x)\beta_1^*G_1(x) + g_1(x)$$

$$= -\beta_1^* \varphi_1(x) + g_1(x) = -\beta_1^* x^{\nu_-} + \left(\frac{x}{\rho + \lambda - \alpha} - \frac{K}{\rho}\right),$$

and

$$\hat{v}_0(x) = \varphi_0(x)W_0(F_0(x)) + g_0(x) = \varphi_0(x)\beta_0^*F_0(x) + g_0(x) = \beta_0^* \psi_0(x) + g_0(x) = \beta_0^* x^{\mu_+}.$$
Without loss of generality, we set revenue generated by renting the facility until the time \( \lambda \) (see for example, Lebedev(1972, pp 284, 290)). Since in terms of the Hermite function \( g(2) \) the state space of the Uhrenbeck process. Consider a firm whose revenue solely depends on the price of one product. Due to its cyclical nature of the prices, the firm does not want to have a large production facility and decides to rent additional production facility when the price is favorable. The revenue process to the firm is

\[
\text{d}X_t = \delta(m - X_t - \lambda Z_t) \text{d}t + \sigma \text{d}W_t,
\]

where \( \lambda = r/\delta \) with \( r \) being a rent per unit of time. The firm’s objective is to maximize the incremental revenue generated by renting the facility until the time \( \tau_0 \) when the price is at an intolerably low level. Without loss of generality, we set \( \tau_0 = \inf \{ t > 0 : X_t = 0 \} \). We keep assuming constant operating cost \( K \), opening cost, \( L \) and closing cost \( C \). Now the value function is defined as

\[
v(x) = \sup_{w \in W} J^w(x) = \sup_{w \in W} \mathbb{E}^x \left[ \int_{\tau_0}^{\infty} e^{-\alpha t}(X_t - K)Z_t \text{d}t - \sum_{\theta_i < \tau_0} e^{-\alpha \theta_i} H(X_{\theta_i}, Z_{\theta_i}) \right].
\]

**Solution:**

1. We denote, by \( \tilde{\psi}(\cdot) \) and \( \tilde{\varphi}(\cdot) \), the functions of the fundamental solutions for the auxiliary process \( P_t \triangleq (X_t - m + \lambda)/\sigma, t \geq 0 \), which satisfies \( dP_t = -\delta P_t \text{d}t + \text{d}W_t \). For every \( x \in \mathbb{R} \),

\[
\tilde{\psi}(x) = e^{\beta x^2/2D_{-\alpha/\delta}}(-x\sqrt{2\beta}) \quad \text{and} \quad \tilde{\varphi}(x) = e^{\beta x^2/2D_{-\alpha/\delta}}(x\sqrt{2\beta}),
\]

which leads to \( \psi_1(x) = \tilde{\psi}((x - m + \lambda)/\sigma), \varphi_1(x) = \tilde{\varphi}((x - m + \lambda)/\sigma), \psi_0(x) = \tilde{\psi}((x - m)/\sigma), \) and \( \varphi_0(x) = \tilde{\varphi}((x - m)/\sigma) \) where \( D_{\nu}(\cdot) \) is the parabolic cylinder function; (see Borodin and Salminen (2002, Appendices 1.24 and 2.9) and Carmona and Dayanik (2003, Section 6.3)). By using the relation

\[
D_{\nu}(z) = 2^{-\nu/2}e^{-z^2/4}H_{\nu}(z/\sqrt{2}), \quad z \in \mathbb{R}
\]

in terms of the Hermite function \( H_{\nu} \) of degree \( \nu \) and its integral representation

\[
H_{\nu}(z) = \frac{1}{\Gamma(\nu)} \int_0^{\infty} e^{-t^2 - 2tz - t^{-\nu - 1}} \text{d}t, \quad \text{Re}(\nu) < 0,
\]

(see for example, Lebedev(1972, pp 284, 290)). Since \( \mathbb{E}^x[X_t] = e^{-\delta t}x + (1 - e^{-\delta t})(m - \lambda) \), we have \( g_0(x) = 0 \) and \( g_1(x) = \frac{x - (m - \lambda)}{\delta + \alpha} + \frac{m - \lambda - K}{\alpha} \).

2. The state space of \( X \) is \( (c, d) = (0, +\infty) \). Since the left boundary 0 is the absorbing, the linear majorant passes \( (0, F_0(0)) \). Since \( \lim_{x \to +\infty} \psi_0(x)/\psi_1(x) = 0 \), we have \( l_d = 0 \).
(3) We formulate
\[ r_1(x) = -\left(\frac{x - (m - \lambda)}{\delta + \alpha} + \frac{m - \lambda - K}{\alpha}\right) - C + \beta_0 \psi_0(x) \]
and
\[ r_0(x) = \left(\frac{x - (m - \lambda)}{\delta + \alpha} + \frac{m - \lambda - K}{\alpha}\right) - L - \beta_1 \varphi_1(x) \]
and make transformations: \( R_1(y) = r_1(F^{-1}(y))/\psi_1(F^{-1}(y)) \) and \( R_0(y) = r_0(F^{-1}(y))/\varphi_0(F^{-1}(y)) \), respectively. We examine the shape and behavior of the two functions \( R_1(\cdot) \) and \( R_0(\cdot) \) with an aid of (3.1). First we check the sign of \( R_0'(y) \) and find a critical point \( x = q \), at which \( R_0(F(x)) \) attains a local minimum and from which \( R_0(F(x)) \) is increasing monotonically on \((F_0(q), \infty)\). It can be shown that \( R_0'(+\infty) = 0 \) by using (3.3) and (3.4) and the identity \( \mathcal{H}'_\nu(z) = 2
\nu \mathcal{H}_{\nu-1}(z), z \in \mathbb{R} \) (see Lebedev (1972, p.289), for example.) This shows that there must exist a (finite) linear majorant of \( R_0(y) \) on \((F(q), \infty)\). To check convexity of \( R_0(y) \), we define
\[ p(x) = -\frac{\sigma^2 \beta_1}{2} \varphi_1''(x) + \delta(m - x - \lambda) \left( \frac{1}{\delta + \alpha} - \beta_1 \varphi_1(x) \right) - \alpha r_0(x) \]
such that \( (A_0 - \alpha)r_0(x) = p(x) \). We can show easily \( \lim_{x \to +\infty} p(x) = -\infty \) since \( \varphi_1(+\infty) = \varphi_1'(+\infty) = \varphi_1''(+\infty) = 0 \). Due to the monotonicity of \( \varphi_1(x) \) and its derivatives, \( p(x) \) can have at most one critical point and \( p(x) = 0 \) can have one or two positive roots depending on the value of \( \beta_1 \). In either case, let us call the largest positive root \( x = k_2 \). We also have \( \lim_{x \to -\infty} p(x) = -\infty \). Since we know that \( R_0(y) \) attains a local minimum at \( y = F(q) \) and is increasing thereafter, we have \( q < k_2 \). It follows that there is one and only on tangency point of the linear majorant \( W(y) \) and \( R_0(y) \) on \((F(q), \infty)\), so that the continuation region is of the form \((0, b^*)\). A similar analysis shows that there is only one tangency point of the linear majorant \( W_1(y) \) and \( R_1(y) \).

(4) Solving (3.2), we can find \((a^*, b^*, \beta_0^*, \beta_1^*)\). We transform back to the original space to find
\[ \hat{v}_1(x) = \psi_1(x)W_1(G_1(x)) + g_1(x) = \psi_1(x)\beta_1^* G_1(x) + g_1(x) = -\beta_1^* \varphi_1(x) + g_1(x) \]
\[ = -\beta^*_1 e^{\frac{\delta (x-m+\lambda)}{2\sigma^2}} D_{-\alpha/\delta} \left( \frac{(x-m+\lambda) \sqrt{2\delta}}{\sigma} \right) + \frac{x-(m-\lambda)}{\delta+\alpha} + \frac{m-\lambda}{\alpha} \]
and
\[ \hat{v}_0(x) = \varphi_0(x)W_0(F_0(x)) + g_0(x) = \varphi_0(x)\beta_0^* (F_0(x) - F_0(0)) + g_0(x) \]
\[ = \beta_0^* \{ \psi_0(x) - F_0(0) \varphi_0(x) \} + g_0(x) \]
\[ = \beta_0^* e^{\frac{\delta (x-m+\lambda)}{2\sigma^2}} \left\{ D_{-\alpha/\delta} \left( -\frac{(x-m+\lambda)}{\sigma} \sqrt{2\delta} \right) - F(0) D_{-\alpha/\delta} \left( \frac{(x-m)}{\sigma} \sqrt{2\delta} \right) \right\} . \]
Hence the solution is, using the above functions,
\[ v_1(x) = \begin{cases} \hat{v}_0(x) - C, & x \leq a^*, \\ \hat{v}_1(x), & x > a^* \end{cases}, \quad v_0(x) = \begin{cases} \hat{v}_0(x), & x \leq b^*, \\ \hat{v}_1(x) - L, & x > b^* \end{cases} \]
See Figure 3.2 for a numerical example.
Figure 2: A numerical example of leasing production facility problem with parameters $(m, \alpha, \sigma, \delta, \lambda, K, L, C) = (5, 0.105, 0.35, 0.05, 4, 0.4, 0.2, 0.2)$: (a) The value function $v_0(x)$ with $b^* = 1.66182$ and $\beta_0^* = 144.313$. (b) The value function $v_1(x)$ with $a^* = 0.781797$ and $\beta_1^* = -2.16941$.

4 Extensions and conclusions

4.1 An extension to the case of $k \geq 2$

It is not difficult to extend to a general case of $k \geq 2$ where more than one switching opportunities are available. But we put a condition that $z \in \mathcal{Z}$ is of the form $z = (a_1, a_2, ..., a_k)$ where only one element of this vector is 1 with the rest being zero, i.e., $z = (0, 0, 0, ..., 1, 0)$ for example.

We should introduce the switching operator $M_0$ on $h \in \mathcal{H}$,

$$M_0 h(u, z) = \max_{\zeta \in \mathcal{Z} \setminus \{z\}} \{ h(u, \zeta) - H(u, z; \zeta) \}. \quad (4.1)$$

In words, this operator would calculate which production mode should be chosen by moving from the current production mode $z$. Now the recursive optimal stopping (2.7) becomes

$$w_{n+1}(x) \triangleq \sup_{\tau \in \mathcal{S}} \mathbb{E}^x \left[ \int_0^\tau e^{-\alpha s} f(X_s) ds + e^{-\alpha \tau} M w_n(X_\tau) \right].$$

Accordingly, the optimization procedure will become two-stage. To illustrate this, we suppose $k = 2$ so that $i = 0, 1, 2$. By eliminating the integral in (4.1), we redefine the switching operator,

$$M h_z(x) \triangleq \max_{\zeta \in \mathcal{Z} \setminus \{z\}} \{ h_\zeta(x) + g_\zeta(x) - g_z(x) - H(x, z; \zeta) \}, \quad (4.2)$$

where

$$g_z(x) \triangleq \sup_{w \in \mathcal{W}_0} J_w^z(x) = \mathbb{E}^x \left[ \int_0^\infty e^{-\alpha s} f(X_s) ds \right].$$

Hence (2.13) will be modified to $u_z(x) = \mathbb{E}^x[e^{-\alpha \tau} M u_z(X_\tau)]$. It follows that our system of equations (2.18) is now

$$
\begin{cases}
\bar{v}_2(x) \triangleq \sup_{\tau \in \mathcal{S}} \mathbb{E}^x [e^{-\alpha \tau} M \bar{v}_2(X_\tau)] \\
\bar{v}_1(x) \triangleq \sup_{\tau \in \mathcal{S}} \mathbb{E}^x [e^{-\alpha \tau} M \bar{v}_1(X_\tau)] \\
\bar{v}_0(x) \triangleq \sup_{\tau \in \mathcal{S}} \mathbb{E}^x [e^{-\alpha \tau} M \bar{v}_0(X_\tau)]
\end{cases} \quad (4.3)
$$
The first stage is optimal stopping problem. One possibility of switching production modes is \((0 \rightarrow 1, 1 \rightarrow 2, 2 \rightarrow 0)\). First, we fix this switching scheme, say \(c\), and solve the system of equations \((4.3)\) as three optimal stopping problems. All the arguments in Section 2.3 hold. This first-stage optimization will give \((x_0^c(c), x_1^c(c), x_2^c(c), \beta_1^c(c), \beta_2^c(c))\), where \(x_i\)'s are switching boundaries, depending on this switching scheme \(c\).

Now we move to another switching scheme \(c'\) and solve the system of optimal stopping problems until we find the optimal scheme.

### 4.2 Conclusions

We have studied optimal switching problems for one-dimensional diffusions. We characterize the value function as linear functions in their respective spaces, and provide a direct method to find the value functions and the opening and switching boundaries at the same time. Using the techniques we developed here as well as the ones in Dayanik and Karazas (2003) and Dayanik and Egami (2005), we solved two specific problems, one of which involves a mean-reverting process. This problem might be hard to solve with just the HJB equation and the related quasi-variational inequalities. Finally, an extension to more general cases is suggested. We believe that this direct method and the new characterization will expand the coverage of solvable problems in the financial engineering and economic analysis.

### A Summary of Optimal Stopping Theory

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space with a standard Brownian motion \(W = \{W_t; t \geq 0\}\) and consider the diffusion process \(X^0\) with state pace \(\mathcal{I} \subseteq \mathbb{R}\) and dynamics

\[
dX^0_t = \mu(X^0_t)dt + \sigma(X^0_t)dW_t
\]

for some Borel functions \(\mu : \mathcal{I} \rightarrow \mathbb{R}\) and \(\sigma : \mathcal{I} \rightarrow (0, \infty)\). We emphasize here that \(X^0\) is an uncontrolled process. We assume that \(\mathcal{I}\) is an interval with endpoints \(-\infty \leq a < b \leq +\infty\), and that \(X^0\) is regular in \((a, b)\); in other words, \(X^0\) reaches \(y\) with positive probability starting at \(x\) for every \(x\) and \(y\) in \((a, b)\). We shall denote by \(\mathbb{F} = \{\mathcal{F}_t\}_t\) the natural filtration generated by \(X^0\).

Let \(\alpha \geq 0\) be a real constant and \(h(\cdot)\) a Borel function such that \(\mathbb{E}^x[e^{-\alpha \tau h(X^0_\tau)}]\) is well-defined for every \(\mathbb{F}\)-stopping time \(\tau\) and \(x \in \mathcal{I}\). Let \(\tau_y\) be the first hitting time of \(y \in \mathcal{I}\) by \(X^0\), and let \(c \in \mathcal{I}\) be a fixed point of the state space. We set:

\[
\psi(x) = \begin{cases} 
\mathbb{E}^x[e^{-\alpha \tau_c} \mathbb{1}_{\{\tau_c < \infty\}}], & x \leq c, \\
1/\mathbb{E}^c[e^{-\alpha \tau_c} \mathbb{1}_{\{\tau_c < \infty\}}], & x > c,
\end{cases}
\]

and

\[
\varphi(x) = \begin{cases} 
1/\mathbb{E}^c[e^{-\alpha \tau_c} \mathbb{1}_{\{\tau_c < \infty\}}], & x \leq c, \\
\mathbb{E}^x[e^{-\alpha \tau_c} \mathbb{1}_{\{\tau_c < \infty\}}], & x > c,
\end{cases}
\]

and

\[
F(x) \triangleq \frac{\psi(x)}{\varphi(x)}, \quad x \in \mathcal{I}.
\]

Then \(F(\cdot)\) is continuous and strictly increasing. It should be noted that \(\psi(\cdot)\) and \(\varphi(\cdot)\) consist of an increasing and a decreasing solution of the second-order differential equation \((\mathcal{A} - \alpha)u = 0\) in \(\mathcal{I}\) where \(\mathcal{A}\) is the
infinitesimal generator of $X^0$. They are linearly independent positive solutions and uniquely determined up to multiplication. For the complete characterization of $\psi(\cdot)$ and $\varphi(\cdot)$ corresponding to various types of boundary behavior, refer to Itô and McKean (1974).

Let $F : [c, d] \to \mathbb{R}$ be a strictly increasing function. A real valued function $u$ is called $F$-concave on $[c, d]$ if, for every $a \leq l < r \leq b$ and $x \in [l, r]$,

$$
    u(x) \geq u(l) \frac{F(r) - F(x)}{F(r) - F(l)} + u(r) \frac{F(x) - F(l)}{F(r) - F(l)}.
$$

We denote by

$$
    V(x) \triangleq \sup_{\tau \in \mathcal{S}} \mathbb{E}^x[e^{-\alpha \tau} h(X^0_{\tau})], \quad x \in [c, d]
$$

the value function of the optimal stopping problem with the reward function $h(\cdot)$ where the supremum is taken over the class $\mathcal{S}$ of all $\mathbb{F}$-stopping times. Then we have the following results, the proofs of which we refer to Dayanik and Karatzas (2003).

**Proposition A.1.** For a given function $U : [c, d] \to [0, +\infty)$ the quotient $U(\cdot)/\varphi(\cdot)$ is an $F$-concave function if and only if $U(\cdot)$ is $\alpha$-excessive, i.e.,

$$
    U(x) \geq \mathbb{E}^x[e^{-\alpha \tau} U(X^0_{\tau})], \forall \tau \in \mathcal{S}, \forall x \in [c, d].
$$

**Proposition A.2.** The value function $V(\cdot)$ of (A.3) is the smallest nonnegative majorant of $h(\cdot)$ such that $V(\cdot)/\varphi(\cdot)$ is $F$-concave on $[c, d]$.

**Proposition A.3.** Let $W(\cdot)$ be the smallest nonnegative concave majorant of $H \triangleq (h/\varphi) \circ F^{-1}$ on $[F(c), F(d)]$, where $F^{-1}(\cdot)$ is the inverse of the strictly increasing function $F(\cdot)$ in (A.2). Then $V(x) = \varphi(x) W(F(x))$ for every $x \in [c, d]$.

**Proposition A.4.** Define

$$
    \mathcal{S} \triangleq \{x \in [c, d] : V(x) = h(x)\}, \quad \text{and} \quad \tau^+ \triangleq \inf\{t \geq 0 : X^0_t \in \mathcal{S}\}. \quad (A.5)
$$

If $h(\cdot)$ is continuous on $[c, d]$, then $\tau^+$ is an optimal stopping rule.

When both boundaries are natural, we have the following results:

**Proposition A.5.** We have either $V \equiv 0$ in $(c, d)$ or $V(x) < +\infty$ for all $(c, d)$. Moreover, $V(x) < +\infty$ for every $x \in (c, d)$ if and only if

$$
    l_c \triangleq \limsup_{x \downarrow c} \frac{h^+(x)}{\varphi(x)} \quad \text{and} \quad l_d \triangleq \limsup_{x \uparrow d} \frac{h^+(x)}{\psi(x)} \quad (A.6)
$$

are both finite.

In the finite case, furthermore,

**Proposition A.6.** The value function $V(\cdot)$ is continuous on $(c, d)$. If $h : (c, d) \to \mathbb{R}$ is continuous and $l_c = l_d = 0$, then $\tau^+$ of (A.5) is an optimal stopping time.
Proposition A.7. Suppose that $l_c$ and $l_d$ are finite and one of them is strictly positive, and $h(\cdot)$ is continuous. Define the continuation region $C \triangleq (c, d) \setminus \Gamma$. Then $\tau^*$ of (A.5) is an optimal stopping time, if and only if

- there is no $r \in (c, d)$ such that $(c, r) \subset C$ if $l_c > 0$ and
- there is no $l \in (c, d)$ such that $(l, d) \subset C$ if $l_d > 0$.

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