CHIRALLY COSMETIC SURGERIES ON KINOSHITA-TERASAKA AND
CONWAY KNOT FAMILIES

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Abstract. In this note, we prove that a nontrivial Kinoshita-Terasaka or Conway knot does not admit chirally cosmetic surgeries, by calculating the finite type invariant of order 3.

1. Introduction

Let $K$ be a knot in $S^3$ and $r$ be a number in $\mathbb{Q} \cup \{\infty\}$, we denote by $S^3_r(K)$ the manifold obtained by the Dehn surgery along $K$ with slope $r$. Two surgeries along $K$ with distinct slopes $r$ and $s$ are called purely cosmetic if $S^3_r(K) \cong S^3_s(K)$, and called chirally cosmetic if $S^3_r(K) \cong -S^3_s(K)$. Here $M \cong N$ means that $M$ and $N$ are homeomorphic as oriented manifolds, and $-M$ represents the manifold $M$ with opposite orientation.

The (purely) Cosmetic Surgery Conjecture [17, Problem 1.81(A)], [4, Conjecture 6.1] asserts that if a knot $K$ is nontrivial, then it does not admit purely cosmetic surgeries. This conjecture has been studied in many cases using different obstructions. For instance, if $K$ admits purely cosmetic surgeries, then the surgery slope cannot be $\infty$ [5], the normalized Alexander polynomial $\Delta_K(t)$ of $K$ satisfies $\Delta''_K(1) = 0$ [2], the Jones polynomial $V_K(t)$ satisfies $V''_K(1) = V'''_K(1) = 0$ [8], and the finite type invariants satisfies $v^2(K) = v^3(K) = 0$ [8]. Some other constraints are given, for example, by using the LMO invariants [12], the quantum $SO(3)$-invariant [3]. Besides the above criteria, Heegaard Floer homology, as well as its immersed curve version, has been particularly effective for this conjecture. Combining the work of Ozsváth and Szabó [22], Ni and Wu [21], and Hanselman [6], we know that if two distinct slopes $r$ and $s$ are purely cosmetic, then $r = -s$, and the set $\{r, s\}$ can only be $\{\pm 2\}$ or $\{\pm 1/p\}$ for some integer $p$. This conjecture has been verified for many knots, including, Seifert genus one knots [28], cable knots [24], composite knots [25], 2-bridge knots [11], 3-braid knots [27], pretzel knots [23], and knots with at most 17 crossings [6, 3]. Indeed, the purely cosmetic surgeries are really rare. Specifically, given $b > 0$, there are only finitely many knots with braid index $b$ that possibly admit purely cosmetic surgeries [14].

On the other hand, the chirally case is rather complicated since there are two known families of chirally cosmetic surgeries for knots in $S^3$:

(A). For an amphicheiral knot $K$ and a slope $r$, we have $S^3_r(K) \cong -S^3_{-r}(K)$.

(B). For a $(2, k)$-torus knot $K$, we have $S^3_r(K) \cong -S^3_s(K)$, where $\{r, s\} = \{\frac{2k^2(2m+1)}{k(2m+1)+1}, \frac{2k^2(2m+1)}{k(2m+1)-1}\}$ for some integer $m$ [20].

With the exception of the above two cases, no knot was found to have chirally cosmetic surgeries. The conjecture for chirally case states as follows:

Conjecture 1. [10, Conjecture 1] Suppose $K$ is not amphichiral and is not a $(2, k)$-torus knot, then $K$ does not admit chirally cosmetic surgeries.
The conjecture has been verified for alternating genus one knots [9], alternating odd pretzel knots [27, 26], a certain family of the positive Whitehead doubles [26], and cable knots with some additional assumptions [13]. In this short note, we prove the following:

**Theorem 1.1.** Any nontrivial Kinoshita-Terasaka and Conway knot does not admit chirally cosmetic surgeries.

The purely cosmetic surgeries on knots of both two families are ruled out in [1], and we provide an alternative proof. Our proof is based on the calculation of the constraint $O(K)$ defined in [10]. Let $a_2(K)$ be the coefficient of the $z^2$-term of the Conway polynomial $\nabla_K(z)$, and $v_3(K)$ be the finite type invariant of order 3, $O(K)$ is defined as:

$$O(K) \triangleq \begin{cases} \frac{7a_2(K)^2 - a_2(K) - 10a_4(K)}{4v_3(K)}, & v_3(K) \neq 0; \\ \infty, & \text{otherwise}. \end{cases}$$

We appeals the following obstruction theorem:

**Theorem 1.2.** [10, Theorem 1.10] A knot $K$ has no chirally cosmetic surgeries if $O(K) \leq 2$.

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2. **Prove of the Main Result**

In [16], Kinoshita and Terasaka constructed a family of knots $KT_{r,n}$ parametrized by integers $r$ and $n$. These knots are obtained from a diagram of the four-stranded pretzel links $P(r+1, -r, r, -r-1)$ by introducing $2n$ twists, as shows in Figure 1. There are some redundancies in these knots. Specifically, $KT_{r,n}$ is isotopic to the unknot if and only if $r \in \{0, \pm 1, -2\}$ or $n = 0$. By turning the knot inside out, one can observe a symmetry which identifies $KT_{r,n}$ and $KT_{-r-1,n}$. Finally, we note that the mirror image of $KT_{r,n}$ is $KT_{r,-n}$.

![Figure 1](image-url)

**Figure 1.** The diagram of Kinoshita-Terasaka knots $KT_{r,n}$ (left) and Conway knots $C_{r,n}$ (right). The number $r$ in the boxes means $r$-times half twists.

The Conway knot $C_{r,n}$ shows in Figure 1, is obtained from $KT_{r,n}$ by a mutation. These knots also have a similar construction as $KT_{r,n}$, only using the four-stranded pretzel links $P(r+1, -r, -r-1, r)$ instead of $P(r+1, -r, r, -r-1)$. Therefore, Conway and Kinoshita-Terasaka knots...
satisfy many of the same relations. In particular, \(C_{r,n}\) is isotopic to the unknot if \(r \in \{0, \pm 1, -2\}\) or \(n = 0\), \(C_{r,n} = C_{-r-1,n}\), and \(C_{r,n}^* = C_{r,-n}\).

What makes these two families special is that they have trivial Alexander polynomials \(\Delta_K(t)\), as well as Conway polynomials \(\nabla_K(z)\), since \(\Delta_K(t) = \nabla_K(t^{1/2} - t^{-1/2})\). We note that their Conway polynomials can be computed directly by skein relation at 2\(n\)-twist part from those of pretzel link computed in [15, Theorem 3.2]. By definition, \(a_2(K) = a_4(K) = 0\). It remains to find \(v_3(K)\).

**Lemma 2.1.** \(v_3(K_{r,n}) = v_3(C_{r,n}) = -\frac{n(k+1)}{4}\), here \(k = \lfloor \frac{r}{2} \rfloor\).

**Proof.** By the symmetry identifying, there is no loss of generality in assuming \(r \geq 0\) and \(n \geq 0\). We compute for Conway knot \(C_{r,n}\) first.

Note that
\[
v_3(K) = -\frac{1}{144} V''_K(1) - \frac{1}{48} V''_K(1) = -\frac{1}{24} j_3(K),
\]
where \(V_K(t)\) is the Jones polynomial, and \(j_n(K)\) is the coefficient of \(t^n\) in \(V_K(t^h)\) of \(K\), by putting \(t = e^h\). Here we use another knot invariant \(w_3(K)\) defined by Lescop in [19]. The advantage is \(w_3\) satisfies a crossing change formula
\[
(1) \quad w_3(K_+) - w_3(K_-) = \frac{a_3(K') + a_3(K'')}{2} - \frac{a_3(K) + a_3(K) + \text{lk}^2(K', K'')}{4},
\]
where \((K_+, K_-, K' \cup K'')\) is a skein triple consisting of two knots \(K_\pm\) and a two-component link \(K' \cup K''\), cf. [19, Proposition 7.2] and [8]. On the other hand, a formula from Hoste [7, Theorem 1] states that
\[
(2) \quad \text{lk}(K', K'') = a_2(K_+) - a_2(K_-).
\]
In our case, by smoothing at 2\(n\)-twist part, \(K_+ \) is \(C_{r,n}\) and \(K_- \) is \(C_{r,n-1}\), both of which have trivial Conway polynomial; and \(K', K''\) are two components of pretzel link \(P(r+1, -r, -r-1, r)\). If \(r\) is even, the two components \(K'\) and \(K''\) are torus knots \(T_{2,r+1}\) and \(T_{2,-r-1}\), respectively; when \(r\) is odd, they are \(T_{2,r}\) and \(T_{2,-r}\). We compute \(a_2(T_{2,r})\) when \(T_{2,r}\) is a knot, i.e., \(r = 2k+1\) is odd. By equation (2) again,
\[
a_2(T_{2,2k+1}) - a_2(T_{2,2k-1}) = \text{lk}(K_1, K_2),
\]
here \(K_1\) and \(K_2\) are two components of the torus link \(T_{2,2k}\), thus with linking number \(k\). Notice that \(T_{2,1}\) is the unknot, so \(a_2(T_{2,1}) = 0\). Therefore, it is easy to see
\[
a_2(T_{2,2k+1}) = \frac{k(k+1)}{2}.
\]
Since \(\nabla_K(z) = \nabla_{K^*}(z)\) holds for knot \(K\) and its mirror \(K^*\), we can obtain
\[
w_3(C_{r,n}) - w_3(C_{r,n-1}) = \frac{a_3(K') + a_3(K'')}{2} = a_2(K') = \begin{cases} \frac{k(k+1)}{2}, & r = 2k; \\ \frac{k(k+1)}{2}, & r = 2k+1. \end{cases}
\]
Note that \(w_3(K) = \frac{1}{72} V''_K(1) + \frac{1}{24} V''_K(1) = -2v_3(K)\), cf. [8, Lemma 2.2]. When \(n = 0\), \(C_{r,0}\) is the unknot, so that \(w_3(C_{r,-1}) = -2v_3(\text{unknot}) = 0\). Therefore,
\[
v_3(C_{r,n}) = -\frac{1}{2} w_3(C_{r,n}) = -\frac{nk(k+1)}{4}.
\]

For the case that \(r < 0\) or \(n < 0\), the formula can also be verified due to the facts that \(C_{r,n} = C_{-r-1,n}\), \(C_{r,n}^* = C_{r,-n}\) and \(v_3(K) = -v_3(K^*)\).
For the knot $KT_{r,n}$, since $v_3(K)$ is determined by its Jones polynomial, which is invariant under mutation, $v_3(KT_{r,n}) = v_3(C_{r,n})$ for all $r,n$. □

**Remark 2.2.** One can also compute $w_3(KT_{r,n})$ directly in the same way. The only difference is the two components of pretzel link $P(r + 1, −r, r, −r − 1)$ are a unknot and a connected sum $T_{2,r+1}#T_{2,−r−1}$ or $T_{2,r}#T_{2,−r}$, depending on the parity of $r$. And then, $a_2(K#K') = a_2(K) + a_2(K')$ follows the fact that $\nabla_{K#K'}(z) = \nabla_K(z) \cdot \nabla_{K'}(z)$.

**Example 2.1.** The knot $K11n34$ in Hoste-Thistlethwaite table [18] is the mirror of the original Conway knot $C_{2,1}$, that is, $K11n34 = C_{2,−1}$. The Jones polynomial is

$$V(q) = −q^4 + 2q^3 − 2q^2 + 2q + q^−2 − 2q−3 + 2q−4 − 2q−5 + q−6.$$ 

Direct calculation of the derivatives of $V$ at $q = 1$ gives that $V''(1) = 0$ and $V'''(1) = −72$. So, we have

$$v_3 = −\frac{1}{48} V''(1) = \frac{1}{144} V'''(1) = \frac{1}{2}.$$ 

Note that $\lfloor \frac{5}{2} \rfloor = 0$ iff $r = 0$ or 1, and $\lfloor \frac{5}{2} \rfloor = −1$ iff $r = −1$ or $−2$. We have the following corollary immediately.

**Corollary 2.3.** Let $K$ belong to one of the families $KT_{r,n}$ and $C_{r,n}$. $v_3(K) = 0$ if and only if $K$ is isotopic to the unknot, i.e., $r \in \{0, ±1, −2\}$ or $n = 0$.

Now we can verify the purely and chirally cosmetic surgeries on these knots.

**Corollary 2.4.** [1, Theorem 2.] The purely cosmetic surgery conjecture is true for all nontrivial Kinoshita-Terasaka and Conway knots.

**Proof.** This is a direct consequence of Corollary 2.3 and the following theorem. □

**Theorem 2.5.** [8, Theorem 3.5.] If a knot $K$ has the finite type invariant $v_2(K) \neq 0$ or $v_3(K) \neq 0$, then $S^3_3(K) \neq S^3_3(K)$ when $r \neq s$.

**Proof of Theorem 1.1.** Suppose $K$ be a nontrivial member of $KT_{r,n}$ or $C_{r,n}$. From Corollary 2.3, we have $v_3(K) \neq 0$. By definition,

$$O(K) = \left| \frac{7a_2(K)^2 − a_2(K) − 10a_4(K)}{4v_3(K)} \right| = 0.$$ 

As a consequence of Theorem 1.2 ([10, Theorem 1.10.]), $K$ has no chirally cosmetic surgeries. □

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