Thermodynamics of the anisotropic two-channel Kondo problem

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(November 4, 2018)

We construct and solve numerically the thermodynamic Bethe Ansatz equations for the spin-anisotropic two-channel Kondo model in arbitrary external field $h$. At high temperatures the specific heat and the susceptibility show power law dependence. For $h \to 0$ and at temperatures below the Kondo temperature $T_K$, a two-channel Kondo effect develops characterized by a Wilson ratio $8/3$, and a logarithmic divergence of the susceptibility and the linear specific heat coefficient. Finite magnetic field, $h > 0$ drives the system to a Fermi liquid fixed point with an unusual Wilson ratio which depends sensitively on $h$.

PACS numbers: 75.20.Hr, 71.10.Hf, 71.27.+a, 72.15.Qm

I. INTRODUCTION

The two-channel Kondo model (2CKM) attracted a lot of interest during the past few years. This intense interest is mostly triggered by the rather unusual properties of this model: The existence of finite residual entropy in zero external field, the vanishing of single particle scattering amplitude at $T = 0$ temperature, and the logarithmic singularity of various thermodynamic quantities. All these unusual properties appear due to the presence of a hidden and conserved flavor quantum number of the electrons.

Several physical systems have been proposed to realize the 2CKM. An unambiguous realization of the 2CKM is provided by dilute Uranium and Cerium-based heavy fermion compounds. In these alloys the combination of strong spin-orbit interaction and crystal field symmetry effects leads to an effective 2CKM.

It has been also suggested that non-commutative tunneling centers may form two-level systems (TLS) and realize the 2CKM. In this case the localized spin is replaced by the position of the tunneling center, while the angular momenta of the conduction electrons play the role of the electron spin in the 2CKM. The electron spins in the TLS problem play the role of silent flavor indices. Though there have been several concerns raised concerning the microscopic structure of the TLS and the possibility of observing the two-channel Kondo behavior, many experiments may be consistently interpreted in terms of such dynamical two-channel Kondo impurities.

Another realization of the 2CKM is provided by quantum dots near to their charge degeneracy point. In this case the charging states of the dot replace the impurity spin states and they couple to the position variable of the conduction electrons. Again, the spin of the electrons acts as a flavor variable. Though it appears to be extremely difficult to observe experimentally the non-Fermi liquid behavior associated with the two-channel Kondo fixed point, some fingerprints of the two-channel Kondo effect have been observed in the capacitance of such semiconducting quantum dots.

The above physical realizations of the 2CKM have the common feature that the 2CKM that describes them is in all cases generically strongly spin-anisotropic, and breaks the full SU(2) spin symmetry. While this SU(2) symmetry breaking is known to be irrelevant at the two-channel Kondo fixed point, it affects both qualitatively and quantitatively the properties of the model at energies around and above the Kondo temperature, $T_K$, and has to be taken into account to make comparison with experiments.

In the present paper we generalize the Bethe Ansatz results of Ref. [11] to compute both analytically and numerically the thermodynamics of the anisotropic 2CKM at arbitrary temperatures and magnetic fields for various values of the anisotropy. Below the Kondo temperature, $T_K$, we find that the thermodynamic properties of the model are very similar to those of the isotropic 2CKM. This observation is in full agreement with the irrelevance of the anisotropy at the 2CKM. However, above $T_K$ the behavior of the model depends crucially on spin anisotropy. We find that above $T_K$ all thermodynamic quantities display a power law behavior with an exponent determined by the value of the anisotropy. Furthermore, we find, that at finite magnetic fields the system flows to an unusual Fermi liquid fixed point with an anisotropy-dependent Wilson ratio.

The paper is organized as follows. In Section II we introduce the 2CKM. In Section III we construct the thermodynamic Bethe Ansatz equations for the anisotropic 2CKM that we analyze in detail in Sec. IV. Some details of the computation are given in the Appendix.

II. MODEL

The anisotropic 2CKM consists of a spin $S_0 = 1/2$ impurity that couples dynamically to the spin of the conduction electrons through a strongly anisotropic exchange
interaction, and is described by the following first quantized Hamiltonian ($\hbar = k_B = \mu_B = 1)$:

$$
H = \sum_{j=1}^{N_e} \left\{ -i\partial/\partial x_j + \sum_{\alpha=x,y,z} \delta(x_j) J_\alpha S^\alpha_j S^\alpha_0 \right\} + h \left( g S^z_0 + g' \sum_j S^z_j \right).
$$

(1)

In this equation $x_j$ is the coordinate of the $j$'th conduction electron, $N_e = N \times f$ is the total number of electrons, and the $S^\alpha_j$'s ($j = 0, \ldots, N_e$) denote the spin 1/2 operators of the electrons. Electrons also have a conserved flavor quantum number, $m = \{1, \ldots, f\}$, implicit in Eq. (1). In the following we discuss the case $f \geq 2$ as well although for the physical realizations mentioned in the Introduction $f = 2$. (The $f = 1$ case has been analyzed in Refs. [12,11].)

The meaning of the various terms in this Hamiltonian depends on the particular physical realization. The external field $h$ couples to the local moment and the conduction electron spins with different $g$-factors. In the TLS problem and the quantum dot case, the external field $h$ represents the asymmetry of the TLS and the splitting between the two charging states of the quantum dot, respectively, and $g' = 0$. For heavy fermion systems, depending on the particular realization, $h$ can describe an external magnetic- or strain field, and may also couple to $\sum_j S^z_j$. In the present paper we mostly study the local susceptibility, corresponding to $g' = 0$.

In the original formulation of the Bethe Ansatz, similar to the conformal field theory solution of the Kondo problem, the impurity spin is 'fused' with the spin degrees of freedom of the conduction electrons, and the external field couples to the total spin, corresponding to $g' = 1$. Thus the Bethe Ansatz calculates the impurity contribution to the global susceptibility with $g' = g$. Lowestin made a remarkable attempt to treat the $g' = 1$ case, however, his results were not fully conclusive.

In the present paper we use a different strategy to cope with the $g' = 0$ situation, and show that, similar to the case of the single channel anisotropic Kondo problem, the $g' = 0$ local susceptibility (i.e. the susceptibility arising when the field couples only to the impurity spin) is simply proportional to the global susceptibility. Therefore, after determining the appropriate normalization factor, we are able to extract the exact local susceptibility from the Bethe Ansatz calculation with $g = g'$.

In our analysis we shall assume that $J_z \neq J_x = J_y = J_\perp$ and thus the Hamiltonian has a $U(1)$ spin symmetry. This is true for the quantum dot and heavy fermion realizations, however, for the TLS problem $J_\perp \gg J_x > J_y = 0$. Fortunately, it is not necessary to treat the general Bethe Ansatz equations in order to determine the universal features of the TLS Hamiltonian in the TLS case either: Under scaling the terms $J_x$ and $J_y$, describing electron-assisted tunneling processes, become rapidly equal and therefore it suffices to consider an equivalent effective model with $J'^{\text{eff}} = J'^{\text{eff}}_z = J_z/2$ to determine the thermodynamics. This effective model has $U(1)$ orbital symmetry and can be much more easily analyzed by Bethe Ansatz techniques than the fully anisotropic one.

In constructing the wave function we implicitly have to introduce a further electron-electron interaction

$$
H^{\text{el-el}} = \sum_{1 \leq i < j \leq N_e} \sum_{\alpha=x,y,z} \delta(x_i - x_j) J_\alpha S^\alpha_i S^\alpha_j.
$$

(2)

Since electrons move with the same velocity and in the same direction, this interaction does not modify the thermodynamics in a crucial way for $J_z > J_\perp$. In particular, we have shown that the specific heat of the electrons is unaffected by this interaction. However, it does rescale the $g$-factor in Eq. (1): $g \rightarrow g^{\text{eff}}$. Fortunately, we shall be able to compensate this effect by the renormalization of $g$-factors in the magnetic field term.

For $J_z < J_\perp$ the artificial electron-electron interaction apparently generates a gap in the spin sector, and therefore it is impossible to capture the Kondo effect there within the algebraic Bethe Ansatz approach. Therefore, unfortunately, we have to restrict our discussion to the case $J_z > J_\perp$, and our calculations are not directly applicable to the quantum dot case.

In Eq. (1) only a term $\sim h S^z_0$ has been included, which acts as a local field in the $z$ direction. In reality, however, the field can also point into the ‘perpendicular’ direction corresponding to a term $\sim h_\perp S^z_0$. In the TLS case this term describes spontaneous tunneling between the TLS positions. The general effect of the latter term is very similar to that of $h S^z_0$, but quantitatively it behaves somewhat differently. For a TLS, e.g., in a typical situation $h$ is believed to be much larger than $h_\perp$ and furthermore, its effective value is orders of magnitude reduced by polaronic effects. Therefore, in most situations it is enough to include only the term $\sim h S^z_0$ in the Hamiltonian.

### III. BETHE-ANSATZ SOLUTION

The most important ingredients of the algebraic Bethe Ansatz (BA) are the various scattering matrices. The impurity-conduction electron S-matrix can be constructed directly from (3):

$$
R_{0j} = U_{0j}^{(\text{spin})}(\lambda_0 - \lambda_j)_{\sigma_0 \sigma_j; \sigma_j} \otimes I^{(\text{flavor})},
$$

(3)

where $\lambda_0 = -1$ and $\lambda_j$ are the “rapidities” of the impurity and conduction electron $j$, and $U$ is the $U(1)$ scattering matrix

$$
U_{0j}(\lambda) = a(\lambda) P_{\uparrow} + b(\lambda) P_{\downarrow} + \frac{1}{2} c(\lambda) (S^\uparrow_0 S^\downarrow_j + S^\downarrow_0 S^\uparrow_j),
$$

$$
a(\lambda) = \sinh(i \mu + \lambda \theta) / \sinh(i \mu), \quad b(\lambda) = \sinh(i \mu + \lambda \theta) / \sinh(i \theta),
$$

(4)
with \( P_{\uparrow \uparrow} \) and \( P_{\downarrow \downarrow} \) being the projection operators for parallel and opposite spins. Excepting the small coupling regime, the connection of the parameters \( J_2 \) and \( \theta \) with the bare couplings \( J_2 \) and \( J_1 \) is ambiguous and depends on the regularization procedure of the Dirac delta and the cutoff scheme used. Therefore, instead of \( J_2 \) and \( J_1 \), it is rather \( \mu \) and \( \theta \) that should be viewed as the basic parameters of the BA solution: \( \mu \) turns out to be connected to the renormalized phase shift while the ratio \( \mu/\theta \) determines the Kondo temperature, below which non-Fermi-liquid correlations appear:

\[
T_K = (1 - f \frac{\mu}{\pi}) 2D \exp\{-\pi \theta/\mu\},
\]

with \( D = N/L \).

Since electrons move with the same velocity we have the liberty to define their scattering matrix in a way to maintain integrability:

\[
R_{ij} = U_{ij}(\lambda_i - \lambda_j) \otimes F_{ij}(\lambda_i - \lambda_j).
\]

Here \( U(\lambda) \) is given by Eq. (3) and \( F \) describes scattering in the flavor sector:

\[
F_{ij}(\lambda_i - \lambda_j) = \frac{\lambda_i - \lambda_j + ic X_{ij}}{\lambda_i - \lambda_j + ic}.
\]

with \( X_{ij} \) the flavor exchange operator of particle \( i \) and \( j \) and \( c \) an arbitrary constant to be defined later.

Starting from these scattering matrices we used the algebraic BA to determine the nested BA equations and then applied the dynamical fusion procedure of Ref. [17] to eliminate the flavor degrees of freedom. The fused equations considerably simplify with the choice \( c = \frac{\pi}{N} \).

Then the rapidities \( \{\lambda_\alpha; \alpha = 1, \ldots, M\} \) describing the spin sector of the wave function satisfy

\[
\frac{\sinh(\mu(\lambda_\alpha + \frac{i}{N}) + \theta)}{\sinh(\mu(\lambda_\alpha - \frac{i}{N}) + \theta)} = \left[ \frac{\sinh(\mu(\lambda_\alpha + i \frac{\pi}{N}))}{\sinh(\mu(\lambda_\alpha - i \frac{\pi}{N}))} \right]^N,
\]

\[
- \prod_{\beta=1}^{M} \frac{\sinh(\mu(\lambda_\alpha - \lambda_\beta + i))}{\sinh(\mu(\lambda_\alpha - \lambda_\beta - i))}.
\]

The momenta of the electrons and thus the total energy is determined by the periodic boundary conditions

\[
e^{ik_A L f} = \prod_{\alpha=1}^{M} \frac{\sinh(\mu(\lambda_\alpha + i \frac{\pi}{N}))}{\sinh(\mu(\lambda_\alpha - i \frac{\pi}{N}))}; \quad E = \sum_{A=1}^{N} f k_A,
\]

where \( f k_A \) denotes the total momentum of the fused \( f \)-electron composites, and \( L \) is the system size.

In the thermodynamic limit, \( L, N \to \infty, N/L = D \), the 'spin rapidities' \( \lambda_\alpha \) in Eq. (8) are organized into strings of length \( r \) and parity \( v = \pm \): \( \lambda \to \{\lambda_q^{(r,v)}; q = 1, \ldots, r\} \) with

\[
\lambda_q^{(r,v)} \leftrightarrow \lambda_q^{(r,v)} = \lambda_q^{(r,v)} + \left[ \frac{r+1}{2} - q \right] + i \frac{\pi}{\mu}(1 - v).
\]

We have verified that to obtain a stable solution for \( \mu < \pi/\nu, v \) and \( \nu \) must satisfy the same stability condition as for \( f = 1 \):

\[
v \sin(\mu q) \sin(\mu(r - q)) > 0, \quad q = 1, \ldots, \lfloor r/2 \rfloor.
\]

As shown by Takahashi and Susuki, the allowed \( (r,v) \) strings can be classified on the basis of an infinite (or finite) fraction expansion of \( \mu/\pi \). To be specific, here we only discuss the simplest case \( \mu = \pi/\nu \) and \( f < \nu \), where only \( \nu \) different stable string configurations exist: \( n = (r, v) = (1, +), (2, +), (\nu - 1, +) \) and \( (1, -) \). The case \( \mu = \pi/\nu \) represents a singular limit. For \( f = 1 \) it corresponds to the decoupling point \( T \), while for \( f = 2 \) it can be identified with the Emery-Kivelson point (see below).

### IV. THERMODYNAMICS

#### A. Thermodynamic Bethe-Ansatz equations

To derive the thermodynamic BA equations in the continuum limit \( L, N \to \infty \) and \( D \equiv N/L = \text{cst} \) we proceed in the usual way. We first defined the density of rapidities (rapidity holes), \( g_\nu(\lambda) (\tilde{g}_\nu(\lambda)) \). These are related to the `excitation energies' \( \varepsilon_n(\lambda) \) through \( \eta_n \equiv \tilde{g}_\nu/g_\nu \equiv \exp(\varepsilon_n)/T \). The functions \( \varepsilon_n(\lambda) \) are determined by the following integral equations for \( \nu > f \):

\[
\varepsilon_\nu/T = g h \nu/2T - s \ast \log(1 + \exp(\varepsilon_{\nu-2}/T)) + \delta_{\nu,f+1} \Theta(\lambda)
\]

\[
\varepsilon_{\nu-1}/T = g h \nu/2T + s \ast \log(1 + \exp(\varepsilon_{\nu-2}/T)) - \delta_{\nu,f+1} \Theta(\lambda)
\]

\[
\varepsilon_j/T = s \ast \log((1 + \exp(\varepsilon_{j+1}/T)) \ast (1 + \exp(\varepsilon_{j-1}/T))
\]

\[
+ \delta_{\nu-2} s \ast (1 + \exp(\varepsilon_{\nu-2}/T)) - \delta_{\nu,f} \Theta(\lambda) \quad (j < \nu - 1).
\]

where \( s \ast \) denotes convolution with the Kernel \( s(\lambda) = 1/cosh(\pi \lambda) \), the driving term is given by \( \Theta(\lambda) = 2D/T \arctg(\exp(\varepsilon_{\nu}/T)) \) and \( \varepsilon_0 \to -\infty \). The impurity contribution to the free energy is given by

\[
F_{\text{imp}} = -T \int_{-\infty}^{\infty} s(\lambda + \frac{i}{N}) \ln(1 + \exp(\varepsilon_1(\lambda)/T)) d\lambda,
\]

and, in principle, all thermodynamic quantities can be calculated by taking the derivatives of \( F_{\text{imp}} \).

#### B. Analytical results

Many of the thermodynamic properties can be determined from the asymptotic form of the \( \eta_n \). Using the Ansatz \( \eta_n(\lambda \to \pm \infty) \approx \eta_{n,0}^\pm + b_n^\pm \exp(\varepsilon_{\pm \lambda}) \) one obtains a set of algebraic equations for the \( n,0 \)'s, \( b_n^\pm \)'s and the exponents \( \varepsilon_{\pm \lambda} \). The latter exponents govern the scaling of the free energy in the vicinity of the low- and high-energy fixed points and are given by \( \tau_+ = 4/(2+f) \) and
\( \tau_- = 2\mu/\pi \). The crossover between the two regimes occurs at the Kondo scale Eq. (3), which emerges naturally if one rewrites the thermodynamic BA equations above in a 'universal' form by removing the cutoff \( D \) \cite{22,23}. The asymptotic form of the impurity free energy for \( h \ll T \ll T_K \) is given by

\[
-F^{\text{imp}}/T \sim \left\{ \begin{array}{ll}
S^{\text{imp}} + (a + b(T_h/2)) (T_h/T)^{4/2 + f} & f > 2, \\
S^{\text{imp}} + (a + b(T_h/2)) (T_h/T)^{2 + \ln(T/T_K)} & f = 2,
\end{array} \right.
\]

implying the divergence of the linear specific heat coefficient and the bulk magnetization \( g \) in the absence of the impurity spin (but with electron-electron interaction of Eq. (2) by rescaling constants \( a \) and \( b \) above depend on the specific value of \( f \) and \( \mu \), and the residual entropy \( S^{\text{imp}} \) is the same as in the isotropic case \cite{12}.

\[
S^{\text{imp}} = \ln \left[ \frac{\sin(f/2)}{\sin(f/2 + 1)} \right] \cdot (12)
\]

To determine the renormalization of the \( g \)-factor and the Wilson ratio, we calculated the linear specific heat coefficient and the bulk magnetization \( M_{\text{tot}}^{\text{2}} = -\partial F^{\text{tot}}/\partial h \) in the absence of the impurity spin (but with \( g' = g \)). Similar to the case \( f = 1 \) \cite{23} the linear specific heat coefficient agrees with that of the spin sector of non-interacting free electrons. However, the magnetization does not. Following similar lines as in Ref. \cite{12}, the total spin can be related to \( \eta^{+} \) and \( \eta^{-} \) and thus the magnetization is simply given by

\[
M^2 = -\frac{\partial}{\partial h} F^{\text{tot}} = g \sum_{i=1}^{N_v} S_i^z = g^2 \frac{fL}{4\pi} \frac{h}{1 - f \mu/\pi} \cdot (13)
\]

with the term, \( g^2 fL/4\pi \), the Pauli susceptibility of free electrons.

From this equation it immediately follows by integration that at zero temperature

\[
F^{\text{tot}} = -\frac{1}{2} g^2 \frac{fL}{4\pi} \frac{h^2}{1 - f \mu/\pi} \cdot (14)
\]

Thus the \( g \)-factor is renormalized due to the electron-electron interaction as \( g \rightarrow g/(1 - \mu f/\pi)^{1/2} \), and to compensate the effect of Eq. (3), we have to chose

\[
g \equiv (1 - \mu f/\pi)^{1/2}.
\]

Having thus compensated the effect of the artificial electron-electron interaction of Eq. (3) by rescaling \( g \), we can proceed to calculate the impurity contribution to the global susceptibility (defined with \( g' = 1 \), \( g = 1 \) but no electron-electron interaction). We find indeed that with the choice \( g \equiv (1 - \mu f/\pi)^{1/2} \) the low temperature (global) Wilson ratio, defined in terms of this global impurity susceptibility, takes on a universal value

\[
R_{\text{glob}}^{\text{imp}} = \lim_{T \rightarrow 0} \lim_{h \rightarrow 0} \frac{c^{\text{bulk}}}{\chi^{\text{imp}}_{\text{glob}}} = \frac{8}{3} \cdot (15)
\]

as in the isotropic case \cite{22} proving again that exchange anisotropy is irrelevant at the two-channel Kondo fixed point \cite{22,23}. In the following we shall always denote quantities that were calculated without (i.e. compensating) the artificial electron-electron interaction by the super-script 'imp'.

To capture the meaning of the parameter \( \mu \) we also determined the impurity contribution to the global susceptibility in the high temperature regime:

\[
\chi_{\text{glob}}^{\text{imp}} = \frac{g^2}{4T} = \frac{1 - f \mu/\pi}{4T} \cdot (16)
\]

Using Abelian bosonization techniques we were also able to prove analytically that for \( J_\perp \ll J_z \) at high temperatures

\[
\chi_{\text{glob}}^{\text{imp}} (T \rightarrow \infty) = (1 - 2f \delta^2)^2 \frac{1}{4T} \cdot (17)
\]

with \( \delta \) the phase shift generated by \( J_z \) (see Appendix A for the derivation and the precise definition of the phase shift) \cite{23,22}.

This immediately implies the important relation

\[
\frac{\mu}{\pi} = 4 \frac{\delta}{\pi} - 4f \delta^2 \cdot (18)
\]

Comparing this expression with the results of Ref. \cite{27} we notice immediately that \( \mu/\pi \) is nothing but the scaling dimension of \( J_\perp \), which satisfies the following scaling equation at energy scales \( \omega \) well above \( T_K \) \cite{22,23}:

\[
\frac{d \ln J_\perp}{d \ln (\omega_0/\omega)} = \frac{\mu}{\pi}, \quad (\omega \gg T_K) \cdot (19)
\]

Here \( \omega_0 \) is a high energy cutoff. For a TLS it is of the order of the Debye temperature, while for heavy fermions it is usually of the order of the Fermi energy. For a quantum dot the cutoff is the charging energy, \( \omega_0 \sim E_c \). The effective perpendicular coupling \( J_\perp \) at energy scale \( \omega \) can be obtained by simply integrating this equation.

Eq. (13) is further confirmed by noticing that for \( f = 2 \) at the Emery-Kivelson line, \( \delta = \pi/4 \), the global susceptibility Eq. (10) identically vanishes, in complete agreement with the results of Refs. \cite{28}. The point \( \delta = \pi/2f \) corresponding to \( \mu = \pi/f \) is highly singular, and needs special care \cite{12}. At this particular point the amplitude of the leading irrelevant operator, responsible for the divergence of the susceptibility, and the linear specific heat coefficient, becomes zero \cite{12}.

The global susceptibility, \( \chi_{\text{glob}}^{\text{imp}} \), and the associated global Wilson ratio, \( R_{\text{glob}}^{\text{imp}} \), defined in Eq. (15), are useful for magnetic Kondo systems. However, for quantum dots and TLS's it is rather the local impurity susceptibility that is of interest, i.e., the response of the system to an external field coupled only to the impurity spin. Hence, we studied the impurity contribution to the susceptibility when \( g = 1 \) and \( g' = 0 \) (and in the absence of the artificial electron-electron interaction). In order to determine
this we generalized the path integral derivation of Ref. [4] to show that
\[ F^{imp}(h, T, g, g' = g) = F^{imp}(\tilde{h}, T, g, g' = 0) , \]
where \( \tilde{h} = h(1 - 2f\delta/\pi) = h(1 - \mu f/\pi)^{1/2} \). To establish the second equality we used Eq. (20). Thus we can calculate the local impurity properties (those for \( g = 1, g' = 0 \) and no electron-electron interaction) from the BA equations with \( g' = g = (1 - f\mu/\pi)^{1/2} \) by simply rescaling the field \( h \) according to Eq. (20).

Henceforth, unless otherwise stated, we shall denote the local impurity susceptibility obtained from the BA solution in combination with Eq. (22) by \( \chi^{imp} \) and the associated local Wilson ratio by \( R^{imp} \) (see below).

Following the above procedure we find at high temperature
\[ \chi^{imp} = \frac{1}{4T} \left[ 1 - B \left( \frac{T_K}{T} \right)^{2\mu/\pi} \right], \]
\[ C^{imp} \sim \left( \frac{T_K}{T} \right)^{2\mu/\pi} \]
Note that at high temperatures the specific heat exhibits a power law behavior which crosses over to a logarithmic behavior in the isotropic case \( \mu \rightarrow 0 \). Similarly, the corrections to the susceptibility about the free behavior are power law like and these power laws give way to logarithmic corrections in the isotropic limit \( \mu \rightarrow 0 \). We note that the above exponent \( 2\mu/\pi \) is formally the same as found for the \( f = 1 \) case [4]. However, the relation between \( \mu \) and the bare couplings is quite different in the two cases and involves the channel number \( f \) (see Eq. (13)).

The power-law corrections can be very easily understood from the scaling picture. Expanding the free energy in terms of \( J_\perp \) one finds that the leading correction is second order in \( J_\perp \). Making use of the scaling equation Eq. (13) it immediately follows that the leading corrections to the free energy behave as \( T^{1 - 2\mu/\pi} \), implying Eq. (22). Similar arguments lead to the conclusion that the impurity-induced resistivity correction behaves at high temperatures as
\[ \rho^{imp} = A + B \left( \frac{T_K}{T} \right)^{2\mu/\pi} \]
for \( T \gg T_K \).

At low temperatures, \( T \ll T_K \), some care is required in discussing thermodynamic properties. In contrast to the case \( f = 1 \), where \( F \) is analytical around \( T = h = 0 \), here the \( h = 0, T = 0 \) point is an essential singularity and the two limits \( T \rightarrow 0 \) and \( h \rightarrow 0 \) are not interchangeable. Taking the limit \( h \rightarrow 0 \) first, we find the following non-Fermi liquid behavior
\[ \chi^{imp} \sim \frac{1}{T_K} \ln \left( \frac{T}{T_K} \right), \]
\[ C^{imp} \sim \frac{T}{T_K} \ln \left( \frac{T}{T_K} \right), \]

\[ R^{imp} = \lim_{T \rightarrow 0} \lim_{h \rightarrow 0} \frac{\chi^{bulk}}{\chi^{imp}} \sim \frac{1}{1 - f\mu/\pi} \frac{8}{3}. \]

Here, we have defined a local Wilson ratio, \( R^{imp} \). It is expressed in terms of the local impurity susceptibility \( \chi^{imp} \), and the impurity contribution to the specific heat \( c^{imp} \). It differs from the Wilson ratio defined in Eq. (15) (which is the usual definition for this quantity in the magnetic Kondo problem) by having \( g' = 0 \). We see that this local Wilson ratio depends on anisotropy \( \mu \) and thus the phase shift \( \delta \), and is therefore not universal.

We now consider the limit of \( T \rightarrow 0 \), with \( h \) remaining either finite, or taken to zero subsequently. In this case, and for \( h \ll T_K \), we find from the numerical results of the next section, that there is a low energy scale \( T_{FL} = h^2/T_K \), below which the thermodynamics is Fermi liquid like. In particular for \( T \ll T_{FL} \) and small magnetic fields \( h \ll T_K \) we find
\[ \chi^{imp}(h, T \ll T_{FL}) \sim -\ln(h/T_K), \]
\[ C^{imp}(h, T \ll T_{FL}) \sim \frac{T}{T_F}, \]
with \( T_{FL} \sim h^2/T_K \) (for \( h \ll T_K \)). A local Wilson ratio for arbitrary local magnetic field \( h \) can be defined as
\[ R^{imp}(h) = \lim_{T \rightarrow 0} \frac{c^{bulk}(h)}{\chi^{imp}(h)} \sim R^{imp}. \]

In contrast to the \( f = 1 \) case, for which this quantity is independent of \( h \), but dependent on anisotropy (being given by \( R^{imp,f=1}(h) = 2/(1 - \mu/\pi) \)), for the present \( f = 2 \) case it depends explicitly on both \( h \) and anisotropy (Fig. 1). This important result, which is consistent with the result for the isotropic case [2], will be discussed in the following section on the numerical solution. We note here, however, that the local Wilson ratio for the \( f = 2 \) case agrees with the \( f = 1 \) local Wilson ratio in the case of asymptotically large magnetic fields, \( h \gg T_K \), i.e. in this case we have
\[ R^{imp}(h \gg T_K) = \frac{2}{1 - \frac{\mu}{\pi}}, \]
although the meaning of \( \mu \) is different in the two cases (see Eq. (15)). The detailed dependence of this local Wilson ratio on \( h \) will be discussed in the next section.

C. Numerical solution

In order to obtain the thermodynamics at all temperatures, it was necessary to solve the thermodynamic BA equations of Sec.IVA numerically. A procedure for doing this, which is valid for arbitrary values of the magnetic field \( h \) and temperature has been developed in Ref. [4] for \( f = 1 \). With small modifications, the same procedure applies also to the present case. We considered
FIG. 1. The impurity contribution to the entropy, $S^{\text{imp}}$, specific heat, $C^{\text{imp}}$, and (local) susceptibility, $\chi^{\text{imp}}$, as functions of $T/T_K$ for magnetic fields $h/T_K = 2^{-6}$ (solid), $h/T_K = 2^{-4}$ (dotted), $h/T_K = 1$ (dashed) and $h/T_K = 2^4$ (long-dashed) for the case $\nu = 3$ corresponding to the largest anisotropy studied.

FIG. 2. The anisotropy dependence of the entropy, specific heat and susceptibility at small magnetic fields ($h/T_K = 2^{-4}$), (note that (b) is scaled by a factor 4 for comparison with the corresponding case of large magnetic fields shown in Fig. 3). The anisotropies shown are for $\mu/\pi = 1/\nu$ with $\nu = 3$ (solid), $\nu = 4$ (dotted), $\nu = 5$ (dashed) and $\nu = 6$ (long-dashed) anisotropies given by $\mu/\pi = 1/\nu$ with $\nu = 3, 4, 5, 6$. In Fig. 3a-c we show the thermodynamics of the anisotropic 2CKM for a large anisotropy ($\nu = 3$) as one might have in a realistic system. The characteristic non-Fermi liquid behavior, in particular the $\ln(2)/2$ entropy and the logarithmically divergent $\chi^{\text{imp}}(T)$ and $C^{\text{imp}}(T)/T$, are found at zero field. A finite field, $h > 0$, restores Fermi liquid behavior at temperatures below a low energy scale $T_{\text{FL}} = h^2/T_K$, as found for the isotropic, $\nu = \infty$, case. The non-Fermi liquid behavior for $0 < h < T_K$ is therefore restricted to an intermediate range of temperatures, $T_{\text{FL}} < T < T_K$, and we see that a clear signature of such behavior (such as the two peaks in the specific heat with each peak having only $\ln(2)/2$ entropy, or a $\ln(T)$ behavior of $\chi(T)$ for a temperature range below $T_K$) is possible, even at moderate magnetic fields, $h \sim T_K/16$.

In Fig. 3a-c and Fig. 3a-c we show the effect of different anisotropies on the thermodynamics for both a small external fields ($h \ll T_K$, Fig. 3) and a large magnetic fields ($h \gg T_K$, Fig. 3). At $h \ll T_K$, the main effect of anisotropy is to modify the thermodynamics at intermediate, $T_{\text{FL}} < T < T_K$, and high temperatures, $T > T_K$. For large magnetic field, $h \gg T_K$, the anisotropy modifies the thermodynamic properties at temperatures, $T \gtrsim T_K$ (Fig. 3).

A characteristic feature of the present model ($f = 2$), is
The impurity susceptibility at $T = 3\nu \log 4$ indicates by symbols and are well reproduced by the numerical results.

The anisotropies shown are for $\mu/\pi = 1/\nu$ with $\nu = 3$ (solid), $\nu = 4$ (dotted), $\nu = 5$ (dashed) and $\nu = 6$ (long-dashed).

The susceptibility at high temperatures, $T \gg T_K$, shows power-law behavior $C_{\text{imp}}(T) \sim (T_K/T)^{2\mu/\pi}$ with $\mu/\pi = 1/\nu$. The logarithmic derivative of this, $d\log C_{\text{imp}}(T)/d\log T$, (solid line) approaches $-2\mu/\pi$ at $T \gg T_K$ and is shown here for $\nu = 5$ ($\mu/\pi = 1/5$). It is seen to approach $-2\mu/\pi = -2/5$ (dashed line) at high temperatures.

It depends only on the anisotropy $\mu$ (that corresponds to the dissipation strength $\alpha_3$ in the equivalent dissipative two-state system). In contrast, for the present case ($f = 2$), the local Wilson ratio depends both on the anisotropy $\mu$ and on the magnetic field $h$. Thus, even though the regime $T \ll T_{FL}$ describes a Fermi liquid, in the sense of Eq. 22, the Fermi liquid state appears different to that for the $f = 1$ case. We note that the local Wilson ratio for $f = 2$ deviates increasingly from the usual $f = 1$ Fermi-liquid Wilson ratio with decreasing $h$, i.e., as the range over which non-Fermi liquid behavior dominates increases. The numerical results for $R_{\text{imp}}(h)$ in Fig. 3 indicate that $R_{\text{imp}}(h)$ vanishes as $h \to 0$. This result is consistent with the numerical analysis of the
h → 0 estimates of the susceptibility and specific heat in
Eq. [28] In contrast, from Eq. [24] \( \tilde{R}^{\text{imp}}(T \to 0, h = 0) \) is finite, so we see again that the two limits \( T \to 0, h = 0 \)
and \( T = 0, h \to 0 \) cannot be exchanged. The inset to
Fig. 3 shows that \( \tilde{R}^{\text{imp}}(h) \) exhibits power law behavior
at large magnetic fields. For \( h \gg T_K \) we find
\[
\alpha_1 \tilde{R}^{\text{imp}} = 2 - a(T_K/h)^{2\mu/\pi}.
\]

Finally, for completeness, we show in Fig. 7 the impurity
magnetization (or polarizability) in the presence of
a local field, \( M_z(h) = \langle S_z \rangle(h) \). For each anisotropy, \( \mu \),
\( M_z(h) \) is a universal function of \( h/T_K \) (the same holds for
\( \tilde{R}^{\text{imp}}(h) \)). The approach of \( M_z(h) \) to the free value for
\( h \gg T_K \) depends on anisotropy and is found numerically
to behave like
\[
M_z(h) = 1/2 - b(T_K/h)^{2\mu/\pi}.
\]

We expect that in the isotropic limit \( \mu \to 0 \) the above
power laws will give way to logarithmic corrections in
the same way that the power law corrections to the high
temperature thermodynamics gave rise to logarithmic corrections in this limit.

V. CONCLUSIONS

In summary, we presented a detailed analysis of the
thermodynamics of spin-anisotropic 2-channel Kondo
model by using the Bethe Ansatz technique combined
with bosonization and renormalization group arguments,
and discussed quantitatively the role of the anisotropy
and the magnetic field.

We showed that at high temperatures the thermodynamics is very different from that of the isotropic model: The local impurity susceptibility is essentially free impurity-like, however, the coefficient of the global susceptibility is non-universal, and is related to the phase shift \( \delta \) generated by the coupling \( J_z \). In particular, the global susceptibility vanishes at the Emery-Kivelson line, \( \delta = \pi/2f \). More interestingly, the impurity specific heat (and the corrections to the susceptibility about the free behavior) exhibits a power law behavior at high temperature
\[
C^{\text{imp}}(T \gg T_K, h) \sim \left( \frac{T_K}{T} \right)^{2\mu/\pi}. \tag{31}
\]

The anomalous exponent \( \mu \) is the anisotropy parameter in the Bethe Ansatz. We have shown that \( \mu/\pi \) is just
the anomalous scaling exponent of the spin flip term \( J_z \)
at high temperature and that \( \mu/\pi \) is related to the phase shifts generated by \( J_z \) through [22]
\[
\frac{\mu}{\pi} = 4 \frac{\delta}{\pi} - 4f \frac{\delta^2}{\pi^2}. \tag{32}
\]

On general scaling arguments, [23] a similar power-law
dependence is expected to appear in the impurity resistivity. For \( f = 2 \) we find
\[
\rho^{\text{imp}}(T \gg T_K) \sim \left( \frac{T_K}{T} \right)^{2\mu/\pi}. \tag{33}
\]
rather then a simple logarithmic scaling.

For \( h = 0 \), and for temperatures below the Kondo
temperature, the thermodynamics is governed by the
isotropic 2CK fixed point and most of the thermodynamic properties resemble very much those of the fully isotropic model, even for strong anisotropies.

For finite \( h \) we showed that the non-Fermi liquid behavior found for \( h = 0 \) persists for an intermediate region of temperatures \( T_{FL} < T < T_K \), provided \( h < T_K \) so that the new scale \( T_{FL} = h^2/T_K \) is well below \( T_K \), just as in the isotropic case [2]. We also showed that the Fermi liquid behavior below \( T_{FL} \) is unusual in that the Wilson ratio, \( R^{\text{supp}}(h) \), depended very sensitively on the magnetic field \( h \). (in contrast to the \( f = 1 \) case, which is completely independent of \( h \)) and we calculated the detailed dependence of this quantity for several anisotropies.

While here we focused our attention to specific values of the anisotropy, our calculations can be easily generalized to other anisotropy values and serve as a basis for any interpolation necessary for cases where there is wide distribution of anisotropies present.

The authors are grateful to A.M. Tsvelik and A. Zawadowski for useful discussions. This research has been supported by the Hungarian Grants OTKA T026327, OTKA F030041, and OTKA T029813, and NSF grant No. DMR 99-81283. The hospitality of the Institut-Laue Langevin during a visit (GZ) is gratefully acknowledged.

APPENDIX A: DERIVATION OF EQ. (17)

To prove Eq. (17) let us consider the limit \( J_z \rightarrow 0 \) of Eq. (10). In this case the interaction part of the Hamiltonian becomes:

\[
H_{\text{int}} = J_z \sum_{\alpha,j=1}^{f} (\psi^\dagger_{\alpha,j}(0)\psi_{\alpha,j}(0) - \psi^\dagger_{\alpha,j}(0)\psi_{\alpha,j}(0))S^2_{\alpha,j},
\]

where the electronic field operator \( \psi^\dagger_{\alpha,j} \) creates chiral (right-moving) electron with spin \( \alpha = \{+,-\} = \{\uparrow,\downarrow\} \) and channel index \( j = \{1,\ldots,f\} \). Thus the interaction simply produces a \( \phi \)-dependent potential scattering, and gives rise to a phase shift \( \delta \)

\[
\psi_{\alpha,j}(x = 0^+) = \psi_{\alpha,j}(x = 0^-)e^{-4\phi_0\delta S^2_{\alpha,j}}.
\]

In general, the connection between \( J_z \) and \( \delta \) depends on the particular cutoff scheme used except for the small coupling limit, \( J_z \ll 1 \). To be specific, here we shall use Abelian bosonization on a system of finite size \( L \) and the cutoff-scheme associated with it. In the bosonization procedure we rewrite the Hamiltonian as \((g = g' = 1)\):

\[
H = \sum_{\alpha,j} \int \frac{dx}{4\pi} (\partial_x \Phi_{\alpha,j})^2 + \frac{\pi}{L} \sum_{j,\alpha} N^2_{\alpha,j} + \frac{J_z}{2} S^2_{\alpha,j} \sum_{\alpha,j} \frac{\alpha}{L} \partial_x \Phi_{\alpha,j} + \frac{\pi}{L} \alpha N_{\alpha,j})
\]

\[
+ h\left( \sum_{\alpha,j} \frac{\alpha}{L} N_{\alpha,j} + S^2_{\alpha,j} \right),
\]

where the external field \( h \) couples to the total pseudospin of the system, \( N_{\alpha,j} \) denotes the total number of electrons with respect to the ground state with spin \( \alpha \) in channel \( j \), and the free bosonic fields satisfy:

\[
[\partial_x \Phi_{\alpha,j}(x), \Phi_{\alpha',j}(x')] = -i 2\pi \delta_{j,j'} \delta_{\alpha,\alpha'} \delta(x - x').
\]

The original fermion fields can be represented as

\[
\psi_{\alpha,j}(x) = \frac{1}{\sqrt{a}} F_{\alpha,j} e^{-i\phi_{\alpha,j}(x)},
\]

where \( F_{\alpha,j} \) denotes the Klein factor, and \( a \) is a small distance cut-off of the order of the lattice spacing.

The phase shift can be most easily calculated by introducing charge and spin fields and quantum numbers:

\[
\begin{align*}
\left( \Phi_{c,j}, \Phi_{s,j} \right) &= \left( \frac{1}{\sqrt{2}} \left( 1 + 1 \right) \Phi_{\uparrow,j}, \Phi_{\downarrow,j} \right), \\
N_{c,j} &= N_{\uparrow,j} + N_{\downarrow,j}, \\
N_{s,j} &= \frac{1}{2} (N_{\uparrow,j} - N_{\downarrow,j}),
\end{align*}
\]

and performing a unitary transformation on the Hamiltonian by \( U = e^{i\sum J_z \Phi_{\uparrow,j}(0)S^z_{\uparrow,j}/(2\pi)^2} \), resulting in

\[
\partial_x \Phi_{\alpha,j}(x) \rightarrow \partial_x \Phi_{\alpha,j}(x) - \frac{J_z}{2} S^z_{\alpha,j}(x),
\]

and the ‘noninteracting’ Hamiltonian:

\[
H = H_0 + H_s.
\]

\[
H_0 = \sum_{j=1}^{f} \sum_{\mu = c,s} \int \frac{dx}{4\pi} (\partial_x \Phi_{\mu,j})^2 + \frac{2\pi}{2L} \sum_{j} N^2_{\alpha,j},
\]

\[
H_s = h S^z_0 + \sum_{j} (h N_{s,j} + \frac{2\pi}{L} N^2_{s,j} + J_z N_{c,j}/L S^z_0).
\]

From Eqs. (A9), (A10), (A11), and (A12) immediately follows that in the bosonization cut-off scheme simply

\[
\delta = \frac{J_z}{8}.
\]

To prove Eq. (17) we observe that the external field only appears in (A12). Therefore the partition function factorizes as

\[
Z(\beta) = Z_0(\beta) \times Z_s(\beta, h),
\]

\[
Z_s(\beta, h) = \sum_{N_s} \sum_{S^z_0 = \pm 1/2} e^{-\beta H_s}.
\]

It is easy to evaluate the sum above in the \( L \rightarrow \infty \) limit giving:

\[
Z_s \sim e^{\beta f \frac{\pi \hbar^2}{8L}} \times \cosh \left[ \frac{\beta h}{2} \left( 1 - J_z \frac{\pi}{4L} \right) \right].
\]
spin coupled to a renormalized magnetic field and gives a Curie susceptibility:

$$\chi_{\text{TLS}}^{\text{glob}} = \frac{(1 - fJ_z/4\pi)^2}{4T}.$$  \hfill (A15)

Together with Eq. (A13), this yields Eq. (17).

As a further test, one can compare this result with the exact relation, Eq. (16) in the small coupling limit. Within the Bethe Ansatz cutoff scheme

\[
\cos(\mu) = \frac{\cos(J_z/2)}{\cos(J_\perp/2)},
\]

which in the appropriate small coupling limit gives

\[
\mu = J_z/2 + O(J_\perp^2) = 4\delta + O(\delta^2).
\]

Substituting this expression into Eq. (16) we indeed recover the exact relation Eq. (17) in linear order in $\delta$.

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