Families of Artin-Schreier curves with Cartier-Manin matrix of constant rank

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Abstract

Let \( k \) be an algebraically closed field of characteristic \( p > 0 \). Every Artin-Schreier \( k \)-curve \( X \) has an equation of the form \( y^p - y = f(x) \) for some \( f(x) \in k(x) \) such that \( p \) does not divide the least common multiple \( L \) of the orders of the poles of \( f(x) \). Under the condition that \( p \equiv 1 \mod L \), Zhu proved that the Newton polygon of the \( L \)-function of \( X \) is determined by the Hodge polygon of \( f(x) \). In particular, the Newton polygon depends only on the orders of the poles of \( f(x) \) and not on the location of the poles or otherwise on the coefficients of \( f(x) \). In this paper, we prove an analogous result about the \( a \)-number of the \( p \)-torsion group scheme of the Jacobian of \( X \), providing the first non-trivial examples of families of Jacobians with constant \( a \)-number. Equivalently, we consider the semi-linear Cartier operator on the sheaf of regular 1-forms of \( X \) and provide the first non-trivial examples of families of curves whose Cartier-Manin matrix has constant rank.

Keywords: Cartier operator, Cartier-Manin matrix, Artin-Schreier curve, Jacobian, \( a \)-number.

MSC: 15A04, 15B33, 11G20, 14H40.

1 Introduction

Suppose \( k \) is an algebraically closed field of characteristic \( p > 0 \) and \( X \) is an Artin-Schreier \( k \)-curve, namely a smooth projective connected \( k \)-curve which is a \( \mathbb{Z}/p \)-Galois cover of the projective line. Studying the \( p \)-power torsion of the Jacobian of \( X \) is simultaneously feasible and challenging. For example, zeta functions of Artin-Schreier curves over finite fields are analyzed in [12, 13, 15, 18]. Newton polygons of Artin-Schreier curves are the focus of the papers [1, 2, 3, 19, 23].

Every Artin-Schreier \( k \)-curve \( X \) has an equation of the form \( y^p - y = f(x) \) for some non-constant rational function \( f(x) \in k(x) \) such that \( p \) does not divide the order of any of the poles of \( f(x) \). The genus of \( X \) depends only on the orders of the poles of \( f(x) \). Let \( m + 1 \) denote the number of poles of \( f(x) \) and let \( d_0, \ldots, d_m \) denote the orders of the poles. By the Riemann-Hurwitz formula, the genus of \( X \) is \( g_X = D(p - 1)/2 \) where \( D = \sum_{j=0}^{m}(d_j + 1) - 2 \). By definition, the \( p \)-rank of the Jacobian \( \text{Jac}(X) \) of \( X \) is the dimension \( s_X \) of \( \text{Hom}_{\mathbb{G}_m}(\mu_p, \text{Jac}(X)[p]) \) where \( \mu_p \) denotes the kernel of Frobenius morphism \( F \) on the multiplicative group scheme \( \mathbb{G}_m \). The \( p \)-rank also equals the length of the slope 0 portion of the Newton polygon. For an Artin-Schreier curve \( X \), the \( p \)-rank \( s_X \) equals \( m(p - 1) \) by the Deuring-Shafarevich formula, and thus depends only on the number of poles of \( f(x) \).

In most cases, the Newton polygon of \( X \) is not determined by the orders of the poles of \( f(x) \). One exception was found by Zhu: let \( L \) denote the least common multiple of the orders of the poles of \( f(x) \); under the condition that \( p \equiv 1 \mod L \), the Newton polygon of \( X \), shrunk by the factor \( p - 1 \) in the horizontal and vertical direction, equals the Hodge polygon of \( f(x) \) [25, Corollary 1.3], see Remark 3.1. In particular, this means that the Newton polygon depends only on the orders of
the poles of $f(x)$ and not on the location of the poles or otherwise on the coefficients of $f(x)$. In this paper, we prove an analogous result about the $a$-number of the Jacobian $\text{Jac}(X)$ or, equivalently, about the rank of the Cartier-Manin matrix of $X$.

The $a$-number is an invariant of the $p$-torsion group scheme $\text{Jac}(X)[p]$. Specifically, if $\alpha_p$ denotes the kernel of Frobenius on the additive group $\mathbb{G}_a$, then the $a$-number of (the Jacobian of) $X$ is $a_X = \dim_k \text{Hom}(\alpha_p, \text{Jac}(X)[p])$. It equals the dimension of the intersection of $\text{Ker}(F)$ and $\text{Ker}(V)$ on the Dieudonné module of $\text{Jac}(X)[p]$, where $V$ is the Verschiebung morphism. The $a$-number and the Newton polygon place constraints upon each other, but do not determine each other, see e.g., [10 [11].

The $a$-number is the co-rank of the Cartier-Manin matrix, which is the matrix for the modified Cartier operator on the sheaf of regular 1-forms of $X$. The modified Cartier operator is the $1/p$-linear map $\mathcal{C} : H^0(X, \Omega^1_X) \to H^0(X, \Omega^1_X)$ taking exact 1-forms to zero and satisfying $\mathcal{C}(f^{p-1} df) = df$. In other words, the $a$-number equals the dimension of the kernel of $\mathcal{C}$ on $H^0(X, \Omega^1_X)$.

In this paper, under the condition $p \equiv 1 \mod L$, we prove that the $a$-number of $X$ depends only on the orders of poles of $f(x)$ and not on the location of the poles or otherwise on the coefficients of $f(x)$ (see section 3.6).

**Theorem 1.1.** Let $X$ be an Artin-Schreier curve with equation $y^p - y = f(x)$, with $f(x) \in k(x)$. Suppose $f(x)$ has $m+1$ poles, with orders $d_0, \ldots, d_m$, and let $L = \text{LCM}(d_0, \ldots, d_m)$. If $p \equiv 1 \mod L$, then the $a$-number of $X$ is

$$a_X = \sum_{j=0}^m a_j, \quad \text{where} \quad a_j = \begin{cases} (p-1)d_j/4 & \text{if } d_j \text{ even}, \\ (p-1)(d_j - 1)(d_j + 1)/4d_j & \text{if } d_j \text{ odd}. \end{cases}$$

To our knowledge, Theorem 1.1 provides the first non-trivial examples of families of Jacobians with constant $a$-number when $p \geq 3$. When $p = 2$, the main result of [8] is that the Ekedahl-Oort type (and $a$-number) of an Artin-Schreier curve depend only on the orders of the poles of $f(x)$. For arbitrary $p$, it is easy to construct families of Jacobians with $a_X = 0$ (ordinary) or $a_X = 1$ (almost ordinary) and a family of Jacobians with $a_X = 2$ is constructed in [9 Corollary 4].

For fixed $p$, the families in Theorem 1.1 occur for every genus $g$ which is a multiple of $(p-1)/2$. The $a$-number of each curve in the family is roughly half of the genus. Using [10 Theorem 1.1 (2)], the dimension of the family can be computed to be $\sum_{i=0}^m (d_i + 1) - 3 = 2g/(p-1) - 1$.

Other results about $a$-numbers of curves can be found in [6 [7]. We end the paper with some open questions motivated from this work.

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## 2 Background

### 2.1 Artin-Schreier curves

Let $k$ be an algebraically closed field of characteristic $p > 0$. A *curve* in this paper is a smooth projective connected $k$-curve. An *Artin-Schreier curve* is a curve $X$ which admits a $\mathbb{Z}/p$-Galois cover of the projective line. Letting $x$ be a coordinate on the projective line, every Artin-Schreier curve has an equation of the form $y^p - y = f(x)$ for some non-constant rational function $f(x) \in k(x)$. By Artin-Schreier theory, after a change of variables, $f(x)$ can be chosen such that $p$ does not divide the order of any pole of $f(x)$. We assume that this is the case throughout the paper.

Let $\mathcal{B} \subset \mathbb{P}^1(k)$ be the set of poles of $f(x)$ and suppose $\# \mathcal{B} = m + 1$. We can assume that $\infty \in \mathcal{B}$ after a fractional linear transformation. We choose an ordering of the poles $\mathcal{B} = \{b_0, \ldots, b_m\}$ such
that $b_0 = \infty$. For $b_j \in \mathbb{B}$, let $d_j$ be the order of the pole of $f(x)$ at $b_j$. Let $x - e_j$ be a uniformizer at $b_j$ for $1 \leq j \leq m$. Let $x_0 = x$ and let $x_j = (x - e_j)^{-1}$ if $1 \leq j \leq m$. The partial fraction decomposition of $f(x)$ has the form:

$$f(x) = f_0(x) + \sum_{j=1}^{m} f_j \frac{1}{x - e_j} = \sum_{j=0}^{m} f_j(x_j),$$

where $f_j(x_j) \in k[x_j]$ is a polynomial of degree $d_j$ for $0 \leq j \leq m$ and $f_j(x)$ has no constant term for $1 \leq j \leq m$. Let $u_j \in k^\times$ be the leading coefficient of $f_j(x_j)$.

### 2.2 The genus and $p$-rank of an Artin-Schreier curve

The genus of a curve $X$ is the dimension of the vector space $H^0(X, \Omega_X^1)$ of regular 1-forms. By the Riemann-Hurwitz formula [20, Proposition VI.4.1], the genus of an Artin-Schreier curve $X : y^p - y = f(x)$ where $f(x)$ has $m+1$ poles with prime-to-$p$ orders $d_0, \ldots, d_j$ as described in Section 2.1 is

$$g_X = D(p - 1)/2$$

where $D = -2 + \sum_{j=0}^{m} (d_j + 1)$.

Given a smooth projective $k$-curve $X$ of genus $g$, let $\text{Jac}(X)[p]$ denote the $p$-torsion group scheme of the Jacobian of $X$. Let $\mu_p$ be the kernel of Frobenius on the multiplicative group $\mathbb{G}_m$. The $p$-rank of $X$ is $s_X = \dim_{\mathbb{F}_p} \text{Hom}(\mu_p, \text{Jac}(X)[p])$. The number of $p$-torsion points of $\text{Jac}(X)(k)$ satisfies $\# \text{Jac}(X)[p](k) = p^{s_X}$. The $p$-rank of a curve satisfies the inequality $0 \leq s_X \leq g$. By a special case of the Deuring-Shafarevich formula, see [21, Theorem 4.2] or [5], if $X$ is an Artin-Schreier curve with equation $y^p - y = f(x)$ as described above, then the $p$-rank of $X$ is $s_X = m(p - 1)$.

### 2.3 The $a$-number

Let $\alpha_p$ be the kernel of Frobenius on the additive group $\mathbb{G}_a$. The $a$-number of $X$ is $a_X = \dim_{\mathbb{F}_p} \text{Hom}(\alpha_p, \text{Jac}(X)[p])$. Equivalently, the $a$-number is the dimension of $\text{Ker}(F) \cap \text{Ker}(V)$ on the Dieudonné module of $\text{Jac}(X)[p]$. The $a$-number also equals the dimension of $\text{Ker}(V)$ on $H^0(X, \Omega_X^1)$ [14, 5.2.8]. By definition, $0 \leq a_X + s_X \leq g$.

The $a$-number is an invariant of the $p$-torsion group scheme $\text{Jac}(X)[p]$. In some cases, it gives information about $\text{Jac}(X)$ as well. If $a_X = g$, then $\text{Jac}(X)$ is isomorphic to a product of supersingular elliptic curves. If $s_X < g$, then $a_X > 0$. This can be used to show that the number of factors appearing in the decomposition of $\text{Jac}(X)$ into simple principally polarized abelian varieties is at most $s_X + a_X$.

**Remark 2.1.** In [17], formulas are given for the $a$-number of an Artin-Schreier curve when $f(x)$ is a monomial $x^d$ with $p \nmid d$. If $p \equiv 1 \mod d$, then the main result of this paper extends [17, Corollary 3.3] to all Artin-Schreier curves $X : y^p - y = f(x)$ having the property that the orders of the poles of $f(x)$ divide $p - 1$. If $p \not\equiv 1 \mod d$, let $h_b \in [0, p-1]$ be the integer such that $h_b \equiv (-1 - b)d^{-1} \mod p$. By [17, Remark 3.4], the $a$-number of $X : y^p - y = x^d$ is given by

$$a_X = \sum_{b=0}^{d-2} \min(h_b, p - [(p + 1 + bp)/d]).$$
2.4 The Cartier operator and the $a$-number

The (modified) Cartier operator $C$ is the semi-linear map $C : H^0(X, \Omega_X^1) \to H^0(X, \Omega_X^1)$ with the following properties: $C(\omega_1 + \omega_2) = C(\omega_1) + C(\omega_2); C(f^p \omega) = fC(\omega);$ and

$$C(f^{n-1} df) = \begin{cases} df & \text{if } n = p, \\ 0 & \text{if } 1 \leq n < p. \end{cases}$$

Suppose $\beta = \{\omega_1, \ldots, \omega_g\}$ is a basis for $H^0(X, \Omega_X^1)$. For each $\omega_j$, let $m_{i,j} \in k$ be such that

$$C(\omega_j) = \sum_{i=1}^{g} m_{i,j} \omega_i.$$

The $g \times g$-matrix $M = (m_{i,j})$ is the (modified) Cartier-Manin matrix and it gives the action of the (modified) Cartier operator. The Cartier-Manin matrix is $\tilde{M} = (m_{i,j}^p)$; it is the matrix for the (unmodified) Cartier operator, see [24]. The action of $V$ is the same as the action of the (unmodified) Cartier operator on $H^0(X, \Omega^1)$, see [4], and so the $a$-number satisfies $a_X = g_X - \text{rank}(\tilde{M}) = g_X - \text{rank}(M)$. At the risk of confusion, we drop the word modified in the rest of the paper.

3 The $a$-number of a family of Artin-Schreier curves

3.1 Regular 1-forms on an Artin-Schreier curve

Let $X$ be an Artin-Schreier curve as described in Section 2.1. By [22, Lemma 1], a basis for $\Omega^1_X$ is given by $W = \cup_{j=0}^m W_j$ where

$$W_0 = \left\{x^b y^r dx \mid r, b \geq 0 \text{ and } rd_0 + bp \leq (p-1)(d_0 - 1) - 2 \right\}, \text{ and}$$

$$W_j = \left\{x_j^b y^r dx \mid r \geq 0, b \geq 1, \text{ and } rd_j + bp \leq (p-1)(d_j + 1) \right\} \text{ if } 1 \leq j \leq m.$$

There is a slight difference between the cases $j = 0$ and $1 \leq j \leq m$. This is in some way unavoidable as can be seen from the formula for the $p$-rank. To shorten the exposition, we let $\epsilon_j = -1$ if $j = 0$ and $\epsilon_j = 1$ if $1 \leq j \leq m$. Note that $\# W_j = (d_j + \epsilon_j)(p-1)/2.$

We define an ordering $<$ on the basis $W$. Define $x_i^{b_1} y^{r_1} dx < x_j^{b_2} y^{r_2} dx$ if $r_1 < r_2$, or if $r_1 = r_2$ and $i < j$, or if $r_1 = r_2$, $i = j$ and $b_1 < b_2$.

3.2 Action of the Cartier operator

Consider the action of the Cartier operator on $H^0(X, \Omega_X^1)$. In general,

$$C \left( x_j^b y^r dx \right) = C \left( x_j^b (y^p - f(x))^r dx \right).$$

To simplify notation, let $\tau = (\tau_{-1}, \ldots, \tau_m)$ denote a tuple of length $m + 2$ whose entries are non-negative integers and let $|\tau| = \sum_{j=-1}^m \tau_j$. Using the extended binomial theorem, we see that

$$(y^p - f(x))^r = \sum_{\tau, |\tau| = r} c_\tau y^{p \tau_0} f_0^{\tau_1} (x) f_1^{\tau_1} (x) \cdots f_m^{\tau_m} (x_m),$$
where
\[ c_r = (-1)^{r-r_1} \binom{r}{\tau_1, \ldots, \tau_m}. \]

So,
\[ C \left( x^b y^r dx \right) = \sum_{\tau, |\tau|=r} c_\tau y^{r-1} C \left( x^b \frac{f_0^\tau(x)f_1^\tau_1(x_1) \cdots f_m^\tau_m(x_m) dx}{r} \right). \quad (1) \]

One can check that
\[ C \left( x^{ap+\epsilon_j} dx \right) = x^{a+\epsilon_j} dx. \quad (2) \]

### 3.3 An assumption on the orders of the poles

Let \( L = \text{LCM}(d_0, \ldots, d_m) \). From now on, we assume that \( p \equiv 1 \mod L \); in other words, the order \( d_j \) of the \( j \)th pole of \( f(x) \) divides \( p - 1 \) and we define \( \gamma_j = (p - 1)/d_j \) for \( 0 \leq j \leq m \). Under this condition, we prove a result about the \( a \)-number of the Jacobian of \( X \) which is analogous to the following result of Zhu:

**Remark 3.1.** Suppose \( f(x) \in \mathbb{F}_q(x) \) for some power \( q \) of \( p \) and let \( N_s = \#X(\mathbb{F}_{q^s}) \) for \( s \in \mathbb{N} \). Since \( X \) is a smooth projective curve, the zeta function of \( X \) is a rational function of the form:

\[ Z_X(u) := \exp(\sum_{s=1}^{\infty} N_s u^s) = \frac{L_X(u)}{(1 - u)(1 - qu)}, \]

where the \( L \)-function \( L_X(u) \in \mathbb{Z}[u] \) is a polynomial of degree \( 2g \). Under the condition \( p \equiv 1 \mod L \), Zhu proved that the Newton polygon of \( L_X(u) \) (shrunk by a factor of \( p - 1 \) in the horizontal and vertical direction) equals the Hodge polygon of \( f(x) \) \([25] \text{ Corollary } 1.3\). The Hodge polygon has slopes of 0 and 1 each occurring with multiplicity \( m \) and slopes \( \{1/d_j, \ldots, (d_j - 1)/d_j\} \) for \( 0 \leq j \leq m \). In particular, this means that the Newton polygon depends only on the orders of the poles of \( f(x) \) and not on the location of the poles or otherwise on the coefficients of \( f(x) \).

Under the condition \( p \equiv 1 \mod L \), for \( 0 \leq j \leq m \), the 1-forms \( x^b y^r dx \in W_j \) are in bijection with ordered pairs \((b, r)\) of integers in the closed triangle bounded by \( r = 0, b = (1 + \epsilon_j)/2 \), and \( r = (p - 2 + \epsilon_j \gamma_j) - \gamma_j b \).

### 3.4 Linearly independent columns of the Cartier-Manin matrix

In this section, we define a subset \( H \subset W \) and show that the columns of the Cartier-Manin matrix associated with elements of \( H \) are linearly independent. This gives a lower bound on the rank of the Cartier-Manin matrix, and thus an upper bound on the \( a \)-number.

Recall that \( \epsilon_j = -1 \) if \( j = 0 \) and \( \epsilon_j = 1 \) if \( 1 \leq j \leq m \). We partition the 1-forms in \( W_j \) into two subsets:

\[ H_j = \left\{ x^b y^r dx \in W_j \mid r \geq (b - \epsilon_j)\gamma_j \right\}, \]

and the set-theoretic complement
\[ A_j = W_j - H_j. \]

Let \( H = \bigcup_{j=0}^m H_j \) and \( A = \bigcup_{j=0}^m A_j \).

**Definition 3.2.** If \( \omega = x^b y^r dx \in H_j \), the key term \( \kappa(C(\omega)) \) of \( C(\omega) \) is the 1-form \( x^b y^{r-(b-\epsilon_j)\gamma_j} dx \).

**Lemma 3.3.** If \( \omega \in H \), the coefficient of \( \kappa(C(\omega)) \) is non-zero in \( C(\omega) \).
Figure 1: The subsets $H_j$ and $A_j$ of $W_j$.

Proof. Suppose $\omega \in H_j$ for some $0 \leq j \leq m$. The claim is that, if $r \geq (b - \epsilon_j)\gamma_j$, then the coefficient of the 1-form $x^by^{r-(b-\epsilon_j)\gamma_j}dx$ in $\mathcal{C}(x^by^rdx)$ is non-zero. Consider the tuple $\tau$ given by $\tau_1 = r - (b - \epsilon_j)\gamma_j$, $\tau_j = (b - \epsilon_j)\gamma_j$, and $\tau_i = 0$ for all $i \not\in \{-1, j\}$. If $r \geq (b - \epsilon_j)\gamma_j$, by Equation (1), the following term appears in $\mathcal{C}(x^by^rdx)$:

$$c_\tau y^{r-(b-\epsilon_j)\gamma_j}C \left(x_j^by^j(f^{(b-\epsilon_j)\gamma_j}_j(x_j)) dx \right).$$

Because $\deg_{x_j}(x_j^by^{(b-\epsilon_j)\gamma_j}(x_j)) = (b - \epsilon_j)p + \epsilon_j$, we see from (2) that $c_\tau u_j^{(b-\epsilon_j)\gamma_j/p}x_j^by^{r-(b-\epsilon_j)\gamma_j}dx$ appears in (3).

The coefficient $c_\tau$ in Equation (3) is nonzero because $r \leq p - 2$ for all $\omega \in H$. Also, $u_j \neq 0$ as it is the leading coefficient of $f_j(x_j)$. This term is canceled by no others. To see this, notice that the coefficient of $x_j^by^{r-(b-\epsilon_j)\gamma_j}dx$ in Equation (1) is zero unless $\tau_1 = r - (b - \epsilon_j)\gamma_j$ and $\tau_j \geq (b - \epsilon_j)\gamma_j$.

The next lemma shows that the coefficient of $\kappa(\mathcal{C}(\omega))$ is zero in $\mathcal{C}(\omega')$ for any 1-form $\omega' \in W$ which is smaller than $\omega$.

Lemma 3.4. If $\omega \in H$ and $\omega' \in W$ with $\omega' \prec \omega$, then the coefficient of $\kappa(\mathcal{C}(\omega))$ is zero in $\mathcal{C}(\omega')$.

Proof. Write $\omega' = x_k^by^Rdx$ and recall the calculation:

$$\mathcal{C}(x_k^by^Rdx) = \sum_{\tau, |\tau| = R} c_\tau y^{r-1}C \left(x_k^bf_0^\tau_0(x)f_1^\tau_1(x_1) \cdots f_m^\tau_m(x_m) \right).$$

Case 1: Suppose $\omega = x^by^rdx \in H_0$. The claim is that the coefficient $c_\omega$ of $\kappa(\mathcal{C}(\omega)) = x^by^{r-(b+1)\gamma_0}dx$ in Equation (1) is zero for any $\omega' \prec x^by^rdx$. The coefficient $c_\omega$ will be zero unless $\tau_1 = r - (b + 1)\gamma_0$. This gives the restriction that $\tau_0 \leq R - (r - (b + 1)\gamma_0)$.

If $k = 0$, $c_\omega$ will be zero unless $\tau_0d_0 + B \geq (b + 1)p - 1$. Combining these inequalities yields that $R - r \geq (b - B)/d_0$. 

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Because both \(b\) and \(B\) are less than \(d_0 - 2\), \(c_\omega\) is non-zero only if \(R > r\) or if \(R = r\) and \(B \geq b\).

If \(k \neq 0\), the coefficient \(c_\omega\) of \(x^b y^{r-(b+1)} dx\) in Equation (1) will be zero unless \(\tau_0 d_0 - B \geq (b+1)p - 1\). Combining the given inequalities shows that

\[
R - r \geq (b + B)/d_0.
\]

As \(B > 0\), this shows that \(c_\omega\) is non-zero only if \(R > r\). In both cases, \(\omega' = x^b y^r dx \neq \omega = x^b y^r dx\).

**Case 2:** Suppose \(\omega \in H_j\) for some \(1 \leq j \leq m\). The claim is that the coefficient \(c_\omega\) of the 1-form \(x^b_j y^{r-(b+1)} dx\) in \(C(\omega')\) is zero for any \(\omega' < x^b_j y^r dx\). The coefficient \(c_\omega\) is non-zero only if \(\tau_{-1} = r - (b-1)\gamma_j\). This gives the restriction that \(\tau_j \leq R - (r - (b-1)\gamma_j)\).

If \(k \neq j\), then \(c_\omega\) is non-zero only if \(\tau_j d_j \geq (b-1)p + 1\) and so

\[
R - r \geq b/d_j.
\]

As \(b > 0\), \(c_\omega\) is non-zero only if \(R > r\).

If \(k = j\), the coefficient \(c_\omega\) is non-zero only if \(\tau_j d_j + B \geq (b-1)p + 1\) which yields that

\[
R - r \geq (b - B)/d_j.
\]

Since \(b\) and \(B\) are both bounded by \(d_j\), this is only satisfied if \(R > r\) or if \(R = r\) and \(B \geq b\), in other words, only if \(\omega' = x^b y^r dx \neq \omega = x^b y^r dx\).

**Proposition 3.5.** The columns of the Cartier-Manin matrix \(M\) corresponding to the 1-forms in \(H\) are linearly independent.

**Proof.** This follows from Lemmas 3.3 and 3.4 since the key terms \(\kappa(C(\omega))\) yield pivots of \(M\) for \(\omega \in H\).  

\[
\square
\]

### 3.5 Linearly dependent columns of the Cartier-Manin matrix

In this section, we prove that the columns of the Cartier-Manin matrix associated with the 1-forms in \(A\) do not contribute to the rank of the Cartier-Manin matrix, because they are linearly dependent on the columns associated with the 1-forms in \(H\).

For fixed \(j\) and \(r\), let \(B\) vary and consider the ordered pair \((B, R)\) of exponents in \(\kappa(C(x^b_j y^r dx))\). The points \((B, R)\) lie on a line of slope \(-\gamma_j\), specifically the line \(R = r + \epsilon_j \gamma_j - \gamma_j B\), where \(\epsilon_j = -1\) if \(j = 0\) and \(\epsilon_j = 1\) if \(1 \leq j \leq m\). For \(0 \leq j \leq m\) and \(r \leq (p-2)/2\), let

\[
Z_{j,r} = \left\{ x^b_j y^r dx \in W_j \mid R = r + \epsilon_j \gamma_j - \gamma_j B \right\}.
\]

Note that \(Z_{0,r}\) is empty if \(0 \leq r < \gamma_0\). Let

\[
Y_{j,r} = \begin{cases} 
\cup_{r=0}^r Z_{0,t} & \text{if } j = 0, \\
\cup_{r=0}^r Z_{j,t} & \text{if } 1 \leq j \leq m.
\end{cases}
\]

**Lemma 3.6.** Suppose \(\eta = x^b_j y^r dx \in W_j\) for some \(0 \leq j \leq m\) with \(r \leq (p-2)/2\). Then \(C(\eta) \in \text{span}(Y_{i,r} \mid 0 \leq i \leq m)\).

**Proof.** Fix \(\sigma \in W_i\) with \(0 \leq i \leq m\) and let \(c_\sigma\) denote the coefficient of \(\sigma\) in \(C(\eta)\). It suffices to show that \(\sigma \in Y_{i,r}\) whenever \(c_\sigma \neq 0\). Write \(\sigma = x^b_i y^r dx\). By Equation (1), \(c_\sigma = 0\) unless \(\tau_{-1} = R\). This
Suppose the largest value of $b$ in $x_j^{b}f_i^{\tau_i}(x_i)$ is $B$. By Proposition 3.5 and Lemma 3.8, the rank of the Cartier-Manin matrix is equal to $\sum_{j=0}^{m} \#H_j$. Therefore, $\deg x_j^{b}f_i^{\tau_i}(x_i) \leq b + \tau_i d_i$. If $i = 0$ then $b < d_0 - 2$ and $B \geq 0$, and if $1 \leq i \leq m$ then $b \leq d_i$ and $B \geq 1$. So, $\deg x_j^{b}f_i^{\tau_i}(x_i) < (B - \epsilon_i)p + \epsilon_i$. Thus, $c_\sigma = 0$ when $R > r + \epsilon_i \gamma_i - \gamma_i B$. \hfill \square

**Lemma 3.7.** Suppose $r \leq (p-2)/2$ and $0 \leq i \leq m$. Every element of $Y_{i,r}$ is a key term of $C(\omega)$ for some $\omega \in H_i$.

**Proof.** Let $x_j^{B}y^R dx \in Y_{i,r}$. Define $\omega = x_j^{B}y^R dx$ where $\rho = R - \epsilon_i \gamma_i + \gamma_i B$. It suffices to show that $\omega \in H_i$, since $\kappa(C(\omega)) = x_j^{B}y^R dx$. If $x_j^{B}y^R dx \in Y_{i,r}$ then $R \leq r + \epsilon_i \gamma_i - \gamma_i B$, so $\rho \leq r$. The 1-form $x_j^{B}y^R dx$ is in $H_i$ because $B \geq 0$, and $-\epsilon_i \gamma_i + \gamma_i B \leq \rho \leq (p-2)/2$. \hfill \square

**Lemma 3.8.** If $\eta \in A$, then $C(\eta)$ is contained in $\operatorname{span} \{ C(\omega) \mid \omega \in H \}$.

**Proof.** Write $\eta = x_j^{B}y^R dx$ for some $0 \leq j \leq m$. Since $\eta \in A$, $r \leq (p-2)/2$. By Lemma 3.6, $C(\eta) \in \operatorname{span}(Y_{i,r} \mid 0 \leq i \leq m)$. By Lemma 3.7, $C(\eta) \in \operatorname{span}(\kappa(C(\omega)) \mid \omega \in H)$. Let $\omega^* = x_j^{B}y^R dx$ be the largest 1-form in $H$ for which the coefficient of $\kappa(C(\omega^*))$ in $C(\eta)$ is non-zero. From the proof of Lemma 3.7, we see that $R < (p-2)/2$. Let $\nu \in k^\times$ be such that the coefficient of $\kappa(C(\omega^*))$ is zero in $C(\eta) - \nu C(\omega^*)$. If $\tau = \kappa(C(\omega^*))$ for some $\omega^* \in H$, Lemma 3.4 implies that $\omega^* \prec \omega^\dagger$. Therefore, the terms in $C(\eta) - \nu C(\omega^*)$ are key terms of $C(\omega^*)$ for $\omega^* \prec \omega^\dagger$. Repeating this process shows that $C(\eta)$ can be written as a linear combination $\sum_{\omega \in H} \nu(\omega) C(\omega)$. \hfill \square

### 3.6 Main result

**Theorem 3.9.** Let $X$ be an Artin-Schreier curve with equation $y^p - y = f(x)$, with $f(x) \in k(x)$. Suppose $f(x)$ has $m+1$ poles, with orders $d_0, \ldots, d_m$ and let $L = \operatorname{LCM}(d_0, \ldots, d_m)$. If $p \equiv 1 \mod L$, then the $a$-number of $X$ is

$$a_X = \sum_{j=0}^{m} a_j, \quad \text{where} \quad a_j = \begin{cases} (p-1)d_j/4 & \text{if } d_j \text{ even}, \\ (p-1)(d_j-1)(d_j+1)/4d_j & \text{if } d_j \text{ odd}. \end{cases}$$

**Proof.** By Proposition 3.5 and Lemma 3.8, the rank of the Cartier-Manin matrix is equal to $\sum_{j=0}^{m} \#H_j$. Since $a = g - \operatorname{rank}(M)$ and $g = \#W$, this implies $a = \sum_{j=0}^{m} (\#W_j - \#H_j)$. It thus suffices to show that $\#W_j - \#H_j = a_j$ for the value of $a_j$ as stated for $0 \leq j \leq m$.

Recall that $\#W_j = (p-1)(d_j + \epsilon_j)/2$. We will count the ordered pairs $(b, r)$ corresponding to $x_j^{b}y^R dx \in H_j$. The lines $r = p-2 + \epsilon_j \gamma_j - \gamma_j b$ and $r = \gamma_j b - \epsilon_j \gamma_j$ intersect at $b = d_j/2 + \epsilon_j - 1/2 \gamma_j$. The largest value of $b$ appearing in $H_j$ is

$$y' = \begin{cases} d_j/2 + \epsilon_j - 1 & \text{if } d_j \text{ is even}, \\ d_j/2 + \epsilon_j - 1/2 & \text{if } d_j \text{ is odd}. \end{cases}$$
Let $b_j = 0$ if $j = 0$ and $b_j = 1$ if $j \neq 0$. Then

$$a_j = \#W_j - \#H_j = (p-1)(d_j + \epsilon_j)/2 - \sum_{b_j} (p - 1 + 2\epsilon_j \gamma_j - 2\gamma_j b) = (p-1)(d_j + \epsilon_j)/2 - (p - 1 + 2\epsilon_j \gamma_j)(b' - b_j + 1) + 2\gamma_j b' (b' + 1)/2 = \begin{cases} (p-1)d_j/4 & \text{if } d_j \text{ even,} \\ (p-1)(d_j - 1)(d_j + 1)/4d_j & \text{if } d_j \text{ odd.} \end{cases}$$

\[\square\]

### 3.7 Open questions

Here are two questions that emerge from this work:

**Question 1:** Under the condition $p \equiv 1 \mod L$, are the Ekedahl-Oort type and the Dieudonné module of the Jacobian of the Artin-Schreier curve $X : y^p - y = f(x)$ determined by the orders of the poles of $f(x)$?

**Question 2:** What are other families of curves for which the $p$-rank, Newton polygon, $a$-number, and Ekedahl-Oort type of the fibres of the family are constant?

For example, when $p = 2$, the Ekedahl-Oort type (and 2-rank and $a$-number) of an Artin-Schreier (hyperelliptic) curve depend only on the orders of the poles of $f(x)$ \[8\].

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