Regular and limiting normal cones to the graph of the subdifferential mapping of the nuclear norm

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Abstract

This paper focuses on the characterization for the regular and limiting normal cones to the graph of the subdifferential mapping of the nuclear norm, which is essential to derive optimality conditions for the equivalent MPEC (mathematical program with equilibrium constraints) reformulation of rank minimization problems.

Keywords: regular and limiting normal cone; subdifferential mapping; nuclear norm

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1 Introduction

Let $Z$ be a finite dimensional vector space, and $\mathbb{R}^{m \times n}$ be the vector space of all $m \times n$ real matrices equipped with the trace inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|_F$. Denote by $\|X\|_*$ the nuclear norm of a matrix $X \in \mathbb{R}^{m \times n}$, i.e., the sum of all singular values of $X$, and by $\|X\|$ the spectral norm of $X$. Consider the following optimization problem

$$
\min_{z \in Z} \left\{ f(z) : g(z) \in K, (G(z), H(z)) \in \text{gph} \partial \| \cdot \|_* \right\},
$$

(1)

where $f : Z \to \mathbb{R}$ is a locally Lipschitz function, $g : Z \to \mathbb{R}^l \times \mathbb{R}^{m \times n}$ and $G, H : Z \to \mathbb{R}^{m \times n}$ are continuously differentiable mappings, $K$ is a simple closed convex set of $\mathbb{R}^l \times \mathbb{R}^{m \times n}$, and $\text{gph} \partial \| \cdot \|_*$ denotes the graph of the subdifferential mapping of the nuclear norm.

By the proof of Lemma 2.1 below, we know that $(G(z), H(z)) \in \text{gph} \partial \| \cdot \|_*$ if and only if $H(z) \in \arg \max_{Y \in B} \langle G(z), Y \rangle$, where $B := \{ Z \in \mathbb{R}^{m \times n} \mid \|Z\| \leq 1 \}$ in this paper.

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This shows that the constraint \((G(z), H(z)) \in \text{gph} \partial \| \cdot \|_s\) represents a kind of optimality conditions. Therefore, problem (1) is a mathematical program with a matrix equilibrium constraint \((G(z), H(z)) \in \text{gph} \partial \| \cdot \|_s\), which extends the optimization problems with polyhedral variational inequality constraints \([19, 20]\), second-order cone complementarity constraints \([21, 22]\), or positive semidefinite (PSD) complementarity constraints \([4, 18]\) to those with general matrix equilibrium constraints.

Our interest in (1) comes from the fact that it covers an equivalent reformulation of low-rank optimization problems. Indeed, for the following rank minimization problem

\[
\min_{X \in \mathbb{R}^{m \times n}} \left\{ \text{rank}(X) : \|A(X) - b\| \leq \delta, X \in \Omega \right\},
\]

from [1, Section 3.1] and Lemma 2.1 in Section 2 we know that it can be reformulated as

\[
\min_{X, Y \in \mathbb{R}^{m \times n}} \left\{ \|Y\|_*: \|A(X) - b\| \leq \delta, X \in \Omega, (X, Y) \in \text{gph} \partial \| \cdot \|_s \right\},
\]

where \(A : \mathbb{R}^{m \times n} \to \mathbb{R}^N\) is a sampling operator, \(b \in \mathbb{R}^N\) is a noisy observation vector, \(\delta > 0\) is a constant related to the noise level, and \(\Omega \subseteq \mathbb{R}^{m \times n}\) is a closed convex set. Clearly, problem (3) is a special case of (1) with \(K = \{x \in \mathbb{R}^N : \|x\| \leq \delta\} \times \Omega\), and \(g(z) = (A(X) - b, X)\) and \((G(z), H(z)) = (X, Y)\) for \(z = (X, Y) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n}\).

As it is well known, low-rank optimization problems have wide applications in many fields such as statistics \([12]\), system identification and control \([6, 7]\), signal and image processing \([2]\), machine learning \([16]\), finance \([13]\), quantum tomography \([8]\), and so on. This motivates us to develop the optimality conditions and stability results for problem (1), especially the equivalent reformulation (3) of the rank minimization problem (2). To achieve this goal, an essential step is to provide the characterization for the regular and limiting normal cones to the graph of the subdifferential mapping of the nuclear norm. In this work, we shall resolve this critical problem and as a byproduct establish the (regular) coderivative of the projection operator onto the unit spectral norm ball.

Throughout this paper, we stipulate \(m \leq n\). Let \(S^m\) be the space of all \(m \times m\) real symmetric matrices and \(S^m_{++}\) be the cone of all PSD matrices from \(S^m\). Let \(O^{n_1 \times n_2}\) be the set of all \(n_1 \times n_2\) real matrices with orthonormal columns, and \(O^{n_1}\) be the set of all \(n_1 \times n_1\) real orthogonal matrices. For \(Z \in \mathbb{R}^{m \times n}\), \(Z_{\alpha \beta}\) denotes the submatrix consisting of those \(Z_{ij}\) with \((i, j) \in \alpha \times \beta\). Let \(e\) and \(E\) be the vector and the matrix of all ones whose dimensions are known from the context, and for a vector \(z\), \(\text{Diag}(z)\) denotes a diagonal matrix which may be square or rectangular. For a given set \(S\), \(\delta_S(\cdot)\) means the indicator function over \(S\); \(T_S(x)\) and \(T_S^\ominus(x)\) denote the tangent cone and the inner tangent cone to \(S\) at \(x\), respectively; and \(N^\ominus_S(x)\), \(\hat{N}_S(x)\) and \(N_S(x)\) denote the proximal normal cone, the regular normal cone and the limiting normal cone to \(S\) at \(x\), respectively (see \([15, 11]\)).

## 2 Preliminaries

This section includes three technical lemmas used for the subsequent analysis. The first one gives some characterizations for the graph of the subdifferential mapping \(\partial \| \cdot \|_s\).
Lemma 2.1 The graph of the subdifferential mapping \( \partial \| \cdot \|_s \) has the following forms

\[
gph \partial \| \cdot \|_s = \partial \| \cdot \|_s = \{ (X, Y) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n} \mid \| X \|_s - \langle X, Y \rangle = 0, \| Y \| \leq 1 \}.
\]

Proof: Notice that \((X, Y) \in \partial \| \cdot \|_s\) if and only if \(Y \in \partial \| \cdot \|_s\). Define \(\Pi\) as follows:

\[
\Pi = \{ (X, Y) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n} \mid \Pi_b(X + Y) = Y \}.
\]

This shows that \((X, Y) \in \partial \| \cdot \|_s\) if and only if \(X \in \partial \| \cdot \|_s\) and the first equality follows. Since \((X, Y) \in \partial \| \cdot \|_s\), the second equality follows.

Lemma 2.2 The projection map \(\Pi_b\) is calm \(B\)-differentiable at any given \(X \in \mathbb{R}^{m \times n}\), i.e., \(\Pi_b(X + h) - \Pi_b(X) - \Pi'_{\| \cdot \|_s}(X; h) = O(\| h \|^2)\) for any \(\mathbb{R}^{m \times n} \ni h \rightarrow 0\).

Proof: Define \(\psi(x) := \text{mid}(-e, x, e)\) for \(x \in \mathbb{R}^m\). The mapping \(\psi\) is Lipschitz continuous everywhere. For any given \(x \in \mathbb{R}^m\) and any \(h \in \mathbb{R}^m\), a simple calculation yields that

\[
\psi'(x; h) = \begin{cases} 0 & \text{if } |x_i| > 1, \\ \text{sign}(x_i) \min(0, \text{sign}(x_i)h_i) & \text{if } |x_i| = 1, \quad \text{for } i = 1, 2, \ldots, m. \end{cases}
\]

It is easy to check that \(\psi(x + h) - \psi(x) - \psi'(x; h) = 0\). Hence, \(\psi\) is calm \(B\)-differentiable in \(\mathbb{R}^m\). Since \(\psi\) is symmetric, i.e., \(\psi(x) = Q^T \psi(Qx)\) for any signed permutation matrix \(Q \in \mathbb{R}^{m \times m}\) and \(x \in \mathbb{R}^m\), the desired result follows by invoking [5, Theorem 5.5].

Next we give the expression of the directional derivative of \(\Pi_b\) by [5, Theorem 3.4].

Lemma 2.3 Let \(Z \in \mathbb{R}^{m \times n}\) have the SVD of the form \(Z = U [\text{Diag}(\sigma(Z)) \ 0] V^T\), where \(U \in \mathbb{O}^m\) and \(V = [V_1 \ V_2] \in \mathbb{O}^n\) with \(V_1 \in \mathbb{R}^{n \times m}\). Define the index sets

\[
\begin{align*}
(5a) & \quad \alpha := \{ i \in \{1, \ldots, m\} \mid \sigma_i(Z) > 1 \}, \quad \beta := \{ i \in \{1, \ldots, m\} \mid \sigma_i(Z) = 1 \}, \\
(5b) & \quad \gamma := \{ i \in \{1, \ldots, m\} \mid \sigma_i(Z) < 1 \}, \quad c := \{ m+1, m+2, \ldots, n \}.
\end{align*}
\]

Let \(\Omega_1, \Omega_2 \in \mathbb{S}^m\) and \(\Omega_3 \in \mathbb{R}^{m \times (n-m)}\) be the matrices associated with \(\sigma(Z)\), defined by

\[
\begin{align*}
(6) & \quad (\Omega_1)_{ij} := \begin{cases} \min(1, \sigma_i(Z)) - \min(1, \sigma_j(Z)) & \text{if } \sigma_i(Z) \neq \sigma_j(Z), \\ \sigma_i(Z) - \sigma_j(Z) & \text{if } \sigma_i(Z) = \sigma_j(Z), \\ 0 & \text{otherwise} \end{cases}, \quad i, j \in \{1, 2, \ldots, m\}, \\
(7) & \quad (\Omega_2)_{ij} := \begin{cases} \min(1, \sigma_i(Z)) + \min(1, \sigma_j(Z)) & \text{if } \sigma_i(Z) + \sigma_j(Z) \neq 0, \\ \sigma_i(Z) + \sigma_j(Z) & \text{if } \sigma_i(Z) + \sigma_j(Z) = 0, \\ 0 & \text{otherwise} \end{cases}, \quad i, j \in \{1, 2, \ldots, m\}, \\
(8) & \quad (\Omega_3)_{ij} := \begin{cases} \min(1, \sigma_i(Z)) & \text{if } \sigma_i(Z) \neq 0, \\ \sigma_i(Z) & \text{otherwise} \end{cases}, \quad i \in \{1, \ldots, m\}, \quad j \in \{1, \ldots, n-m\}.
\end{align*}
\]
Then, for any $H \in \mathbb{R}^{m \times n}$, with $\tilde{H}_1 = \overline{U}^T H \overline{V}_1$ and $\tilde{H} = [\overline{U}^T H \overline{V}_1 \overline{U}^T H \overline{V}_2]$ it holds that

$$
\Pi'_B(\overline{Z}; H) = \overline{U} \left[ (\Omega_2)_{\alpha \alpha} \circ (X(\tilde{H}_1))_{\alpha \alpha} (\Omega_2)_{\alpha \beta} \circ (X(\tilde{H}_1))_{\alpha \beta} \tilde{H}_{\alpha \gamma} (\Omega_3)_{\alpha \gamma} \circ \tilde{H}_{\alpha \gamma} \right] \overline{V}^T,
$$

where $\tilde{H}_{ij} = (\Omega_1)_{ij} S(\tilde{H}_1)_{ij} + (\Omega_2)_{ij} X(\tilde{H}_1)_{ij}$ for $(i, j) \in \alpha \times \gamma$ or $(i, j) \in \gamma \times \alpha$, and $S: \mathbb{R}^{m \times m} \to S^m$ and $X: \mathbb{R}^{m \times m} \to \mathbb{R}^{m \times m}$ are two linear mappings defined by $S(Z) := (Z + Z^T)/2$ and $X(Z) := (Z - Z^T)/2$ $\forall Z \in \mathbb{R}^{m \times m}$.

3 Regular and limiting normal cones to $\text{gph } \partial \| \cdot \|_*$

In this section we shall derive the exact formula for the regular and limiting normal cones to $\text{gph } \partial \| \cdot \|_*$ First, we focus on the formula of the regular normal cone to $\text{gph } \partial \| \cdot \|_*$.  

3.1 Regular normal cone

For the set $\text{gph } \partial \| \cdot \|_*$, we shall verify that its regular normal cone coincides with its proximal normal cone just as [18] did for $\text{gph } \mathcal{N}_T^*(\cdot)$. This requires the following two lemmas. Among others, Lemma 3.1 characterizes the tangent cone to $\text{gph } \partial \| \cdot \|_*$, while Lemma 3.2 provides the characterization for the proximal normal cone to $\text{gph } \partial \| \cdot \|_*$.

**Lemma 3.1** For any given $(X, Y) \in \text{gph } \partial \| \cdot \|_*$, the following equalities hold:

$$
T_{\text{gph } \partial \| \cdot \|_*}(X, Y) = T_{\text{gph } \partial \| \cdot \|_*}(X, Y) = \{(G, H) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n} | \Pi'_B(X + Y, G + H) = H \}.
$$

**Proof:** Let $(G, H)$ be an arbitrary point from $T_{\text{gph } \partial \| \cdot \|_*}(X, Y)$. By the definition of tangent cone, there exist $t_k \downarrow 0$ and $(G^k, H^k) \to (G, H)$ such that $(X, Y) + t_k(G^k, H^k) \in \text{gph } \partial \| \cdot \|_*$ for each $k$. By Lemma 2.1, $\Pi'_B(X + Y + t_k(G^k + H^k)) = Y + t_kH^k$. Notice that $Y = \Pi_B(X + Y)$ by virtue of $(X, Y) \in \text{gph } \partial \| \cdot \|_*$ and Lemma 2.1. Then we have

$$
\Pi'_B(X + Y, G + H) = \lim_{k \to \infty} \frac{1}{t_k}(\Pi'_B(X + Y + t_k(G^k + H^k)) - \Pi_B(X + Y)) = H.
$$

This, by the arbitrariness of $(G, H)$ in $T_{\text{gph } \partial \| \cdot \|_*}(X, Y)$, implies the following inclusion:

$$
T_{\text{gph } \partial \| \cdot \|_*}(X, Y) \subseteq \{(G, H) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n} | \Pi'_B(X + Y, G + H) = H \}.
$$

Since $T_{\text{gph } \partial \| \cdot \|_*}(X, Y) \subseteq T_{\text{gph } \partial \| \cdot \|_*}(X, Y)$, the rest only needs to establish the inclusion

$$
\{(G, H) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n} | \Pi'_B(X + Y, G + H) = H \} \subseteq T_{\text{gph } \partial \| \cdot \|_*}(X, Y).
$$

To this end, let $(G, H) \in \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n}$ with $\Pi'_B(X + Y, G + H) = H$. For any $t > 0$, write $Z_t := X + Y + t(G + H)$. By the definition of $\Pi_B(\cdot)$, we have $Z_t - \Pi_B(Z_t) \in \mathcal{N}_B(\Pi_B(Z_t))$. In addition, from $\Pi'_B(X + Y, G + H) = H$ and the definition of the directional derivative,

$$
\Pi_B(Z_t) = \Pi_B(X + Y) + tH + o(t) = Y + tH + o(t).
$$
This shows that \( X + tG + o(t) \in \mathcal{N}_B(\Pi_B(Z_t)) \), and then \( (X + tG + o(t), \Pi_B(Z_t)) \in \text{gph} \partial \| \cdot \|_s \) by Lemma 2.1. Along with the last equality, dist\((X + tG, Y + tH), \text{gph} \partial \| \cdot \|_s \) = 0(a(t)). This means that \((G, H) \in T^t_{\text{gph} \partial \| \cdot \|_s}(X, Y)\). So, the inclusion in (10) follows. □

**Lemma 3.2** For any given \((X, Y) \in \text{gph} \partial \| \cdot \|_s \), we have \((X^*, Y^*) \in \mathcal{N}^\pi_{\text{gph} \partial \| \cdot \|_s}(X, Y) \) iff

\[
\langle X^*, W - \Pi_B(X + Y; W) \rangle + \langle Y^*, \Pi_B'(X + Y; W) \rangle \leq 0 \quad \forall W \in \mathbb{R}^{m \times n}. \tag{11}
\]

**Proof:** Let \((X^*, Y^*) \in \mathcal{N}^\pi_{\text{gph} \partial \| \cdot \|_s}(X, Y) \). We prove that inequality (11) holds. For this purpose, let \( W \) be an arbitrary point from \( \mathbb{R}^{m \times n} \). For any \( t > 0 \), we write

\[
Y'_t := \Pi_B(X + Y + tW) \quad \text{and} \quad X'_t := X + Y + tW - \Pi_B(X + Y + tW). \tag{12}
\]

Clearly, \( X'_t \in \mathcal{N}_B(Y'_t) \). By Lemma 2.1, \( Y'_t \in \partial \| X'_t \|_s \). Since \((X^*, Y^*) \in \mathcal{N}^\pi_{\text{gph} \partial \| \cdot \|_s}(X, Y) \), by Part E of [15, Chapter 6] there exists \( \varepsilon > 0 \) such that for any \((X', Y') \in \text{gph} \partial \| \cdot \|_s \),

\[
\langle (X^*, Y^*), (X', Y') - (X, Y) \rangle \leq \varepsilon \| (X', Y') - (X, Y) \|_F^2.
\]

Take \((X', Y') = (X'_t, Y'_t) \). From this inequality, it follows that

\[
\langle (X^*, Y^*), (X'_t - X, Y'_t - Y) \rangle \leq \varepsilon \| (X'_t, Y'_t) - (X, Y) \|_F^2. \tag{13}
\]

Note that \( Y = \Pi_B(X + Y) \) since \( (X, Y) \in \text{gph} \partial \| \cdot \|_s \). From (12) and (13), we have that

\[
\begin{align*}
\langle X^*, W - \Pi_B'(X + Y; W) \rangle + \langle Y^*, \Pi_B(X + Y; W) \rangle & \\
\leq & \varepsilon \lim_{t \downarrow 0} \frac{1}{t} \left( \| X'_t - X \|_F^2 + \| Y'_t - Y \|_F^2 \right) \\
\leq & \varepsilon \lim_{t \downarrow 0} \frac{1}{t} \left( 3 \| Y - \Pi_B(X + Y + tW) \|_F^2 + 2t^2 \| W \|_F^2 \right) \leq \varepsilon \lim_{t \downarrow 0} \frac{1}{t} \left( 5t^2 \| W \|_F^2 \right) = 0,
\end{align*}
\]

where the last inequality is using \( Y = \Pi_B(X + Y) \) and the global Lipschitz continuity with modulus 1 of the projection operator \( \Pi_B(\cdot) \). This shows that (11) holds. Conversely, suppose that (11) holds. We shall prove \((X^*, Y^*) \in \mathcal{N}^\pi_{\text{gph} \partial \| \cdot \|_s}(X, Y) \). By Lemma 2.2, there exist \( \delta > 0 \) and a constant \( M > 0 \) such that for any \( Z' \) with \( \| Z' - (X + Y) \|_F \leq \delta \),

\[
\begin{align*}
\langle X^*, \Pi_B(Z') - \Pi_B(X + Y) \rangle & \leq \langle Y^*, \Pi_B(X + Y; Z' - X - Y) \rangle + M \| Z' - (X + Y) \|_F^2, \\
\langle Y^*, \Pi_B(Z') - \Pi_B(X + Y) \rangle & \leq \langle Y^*, \Pi_B(X + Y; Z' - X - Y) \rangle + M \| Z' - (X + Y) \|_F^2.
\end{align*}
\]

Thus, for any \((X', Y') \in \text{gph} \partial \| \cdot \|_s \) with \( \| (X', Y') - (X, Y) \|_F \leq \delta/2 \), we have that

\[
\begin{align*}
\langle X^*, \Pi_B(X' + Y') - \Pi_B(X + Y) \rangle & - \langle X^*, \Pi_B(X + Y; \Delta X + \Delta Y) \rangle \leq M \| \Delta X + \Delta Y \|_F^2, \tag{14} \\
\langle Y^*, \Pi_B(X' + Y') - \Pi_B(X + Y) \rangle & - \langle Y^*, \Pi_B(X + Y; \Delta X + \Delta Y) \rangle \leq M \| \Delta X + \Delta Y \|_F^2. \tag{15}
\end{align*}
\]
where $\Delta X = X' - X$ and $\Delta Y = Y' - Y$. Along with $\Pi_B(X' + Y') = Y'$ and $\Pi_B(X + Y) = Y$,

$$\langle (X', Y'), (X', Y') - (X, Y) \rangle = \langle X', \Delta X + \Delta Y - \Pi_B(X' + Y') + \Pi_B(X + Y) \rangle$$

$$+ \langle Y', \Pi_B(X' + Y') - \Pi_B(X + Y) \rangle$$

$$\leq \langle X', \Delta X + \Delta Y - \Pi_B(X + Y; \Delta X + \Delta Y) \rangle$$

$$+ \langle Y', \Pi_B(X + Y; \Delta X + \Delta Y) \rangle + 2M\|\Delta X + \Delta Y\|^2_F$$

$$\leq 4M\|\Delta X + \Delta Y\|^2_F,$$

where the first inequality is using (14) and (15), and the last one is by virtue of (11) with $W = \Delta X + \Delta Y$. Take $\varepsilon = \max\{4M, 2\|\langle X', Y' \rangle \|_F / \delta\}$. For any $(X', Y') \in \text{gph} \partial \| \cdot \|_*$, it holds that $\langle (X', Y' \rangle, (X', Y') - (X, Y) \rangle \leq \varepsilon \|\langle X', Y' \rangle - (X, Y)\|^2_F$. This, by Part E of [15, Chapter 6], shows that $(X', Y') \in N_{\text{gph} \partial \| \cdot \|_*}(X, Y)$. Thus, we finish the proof. $\square$

Now we are in a position to establish the coincidence between the regular normal cone to $\text{gph} \partial \| \cdot \|_*$ and the proximal normal cone to $\text{gph} \partial \| \cdot \|_*$. 

**Proposition 3.1** For any given $(X, Y) \in \text{gph} \partial \| \cdot \|_*$, $\tilde{N}_{\text{gph} \partial \| \cdot \|_*}(X, Y) = N_{\text{gph} \partial \| \cdot \|_*}(X, Y)$. 

**Proof:** Take an arbitrary point $(X', Y') \in \tilde{N}_{\text{gph} \partial \| \cdot \|_*}(X, Y)$. By [15, Proposition 6.5], $\langle (X', Y'), (G, H) \rangle \leq 0$ for any $(G, H) \in T_{\text{gph} \partial \| \cdot \|_*}(X, Y)$. From Lemma 3.1, clearly, $(W - \Pi_B(X + Y, W), \Pi_B(X + Y, W)) \in T_{\text{gph} \partial \| \cdot \|_*}(X, Y)$ for any $W \in \mathbb{R}^{m \times n}$, and then $\langle (X', W - \Pi_B(X + Y, W)), (Y', \Pi_B(X + Y, W)) \rangle \leq 0$. This, by Lemma 3.2, implies that $(X', Y') \in N_{\text{gph} \partial \| \cdot \|_*}(X, Y)$, and then $\tilde{N}_{\text{gph} \partial \| \cdot \|_*}(X, Y) \subseteq N_{\text{gph} \partial \| \cdot \|_*}(X, Y)$ follows. Next take an arbitrary point $(X', Y') \in N_{\text{gph} \partial \| \cdot \|_*}(X, Y)$. For any $(G, H) \in T_{\text{gph} \partial \| \cdot \|_*}(X, Y)$, by Lemma 3.1 it follows that $\Pi_B(X + Y, G + H) = H$. Using Lemma 3.2 with $W = G + H$ and noting that $\Pi_B(X + Y, W) = H$ yields that $\langle (X', Y'), (G, H) \rangle \leq 0$, i.e., $(X', Y') \in N_{\text{gph} \partial \| \cdot \|_*}(X, Y)$. So, $N_{\text{gph} \partial \| \cdot \|_*}(X, Y) \subseteq \tilde{N}_{\text{gph} \partial \| \cdot \|_*}(X, Y)$. The proof is completed. $\square$

Proposition 3.1 shows that, to characterize the regular normal cone to $\text{gph} \partial \| \cdot \|_*$, one only needs to characterize its proximal normal cone. Next we shall employ Lemma 3.2 and Lemma 2.3 to derive the expression of the proximal normal cone to $\text{gph} \partial \| \cdot \|_*$. 

**Theorem 3.1** For any given $(X, Y) \in \text{gph} \partial \| \cdot \|_*$, let $Z = X + Y$ have the SVD as given in Lemma 2.3. With $\Omega_1$ and $\Omega_2$ in (6)-(7), we define the following matrices

$$\Theta_1 := \begin{bmatrix}
0_{\alpha \alpha} & 0_{\alpha \beta} & (\Omega_1)_{\alpha \gamma} \\
0_{\beta \alpha} & 0_{\beta \beta} & E_{\beta \gamma} \\
(\Omega_1)_{\gamma \alpha} & E_{\gamma \beta} & E_{\gamma \gamma}
\end{bmatrix}, \quad \Theta_2 := \begin{bmatrix}
E_{\alpha \alpha} & E_{\alpha \beta} & E_{\alpha \gamma} - (\Omega_1)_{\alpha \gamma} \\
E_{\beta \alpha} & 0_{\beta \beta} & 0_{\beta \gamma} \\
E_{\gamma \alpha} - (\Omega_1)_{\gamma \alpha} & 0_{\gamma \beta} & 0_{\gamma \gamma}
\end{bmatrix},$$

$$\Sigma_1 := \begin{bmatrix}
(\Omega_2)_{\alpha \alpha} & (\Omega_2)_{\alpha \beta} & (\Omega_2)_{\alpha \gamma} \\
(\Omega_2)_{\beta \alpha} & 0_{\beta \beta} & E_{\beta \gamma} \\
(\Omega_2)_{\gamma \alpha} & E_{\gamma \beta} & E_{\gamma \gamma}
\end{bmatrix}, \quad \Sigma_2 := \begin{bmatrix}
E_{\alpha \alpha} - (\Omega_2)_{\alpha \alpha} & E_{\alpha \beta} - (\Omega_2)_{\alpha \beta} & E_{\alpha \gamma} - (\Omega_2)_{\alpha \gamma} \\
E_{\beta \alpha} - (\Omega_2)_{\beta \alpha} & 0_{\beta \beta} & 0_{\beta \gamma} \\
E_{\gamma \alpha} - (\Omega_2)_{\gamma \alpha} & 0_{\gamma \beta} & 0_{\gamma \gamma}
\end{bmatrix}.$$

Then $(X', Y') \in N_{\text{gph} \partial \| \cdot \|_*}(X, Y)$ if and only if $(X', Y')$ satisfies the following conditions...
where $\tilde{X}_1^*=\overline{U}_T^T X^*V_1$, $\tilde{Y}_1^*=\overline{U}_T^T Y^*V_1$, $\tilde{X}^*=\overline{U}_T^T X^*V$ and $\tilde{Y}^*=\overline{U}_T^T Y^*V$.

**Proof:** By Lemma 3.2 and Lemma 2.3, $(X^*, Y^*) \in \mathcal{N}_{\text{ph}}^\pi \partial || \cdot ||, (X, Y)$ iff for any $H \in \mathbb{R}^{m \times n}$,

$$\langle \tilde{X}^*, \tilde{H} - \begin{bmatrix} (\Omega_2)_{\alpha\beta} \circ (X(\tilde{H}_1))_{\alpha\beta} & (\Omega_3)_{\alpha\beta} \circ (X(\tilde{H}_1))_{\alpha\beta} & \tilde{H}_{\gamma\beta} \\ (\Omega_2)_{\alpha\beta} \circ (X(\tilde{H}_1))_{\alpha\beta} & (\Omega_3)_{\alpha\beta} \circ (X(\tilde{H}_1))_{\alpha\beta} & \tilde{H}_{\gamma\beta} \end{bmatrix} \rangle \leq 0 \tag{18}$$

where $\tilde{H}_1=\overline{U}_T^T H V_1$ and $\tilde{H}=\overline{U}_T^T H V$. Take $H=\overline{U}_\beta M_{\beta\gamma} \overline{V}_\gamma$ for any $M_{\beta\gamma} \in \mathbb{R}^{\beta \times \gamma}$. By the expressions of $\tilde{H}_1$ and $\tilde{H}$ and equation (18), it is easy to obtain that $\tilde{Y}^*_\beta = 0$. Using the similar arguments, we can achieve that $\tilde{Y}^*_\alpha = 0, \tilde{Y}^*_\gamma = 0, \tilde{Y}^*_c = 0$ and $\tilde{X}^*_\alpha \circ (E_{ac} - (\Omega_3)_{ac}) + \tilde{Y}^*_a \circ (\Omega_3)_{ac} = 0$. Taking $H=\overline{U}_\beta M_{\beta\gamma} \overline{V}_\gamma$ for any $M_{\beta\gamma} \in \mathbb{S}^{[\beta]}$, from (18) we have $\tilde{Y}^*_\beta \geq 0$; and by taking $H=\overline{U}_\beta M_{\beta\gamma} \overline{V}_\gamma$ for any $M_{\beta\gamma} \in \mathbb{S}^{[\beta]}$, we obtain that

$$0 \geq \langle \tilde{X}^*_\alpha + \tilde{Y}^*_a, \Pi_{[\alpha]}(S(\tilde{H}_\beta)) \rangle + \langle \tilde{Y}^*_\beta + \tilde{H}_\beta \tilde{H}_\gamma \rangle = \langle \tilde{X}^*_\alpha + \tilde{Y}^*_a, M_{\alpha\beta} \rangle,$$

which implies that $\tilde{X}^*_\alpha + \tilde{Y}^*_a \leq 0$. In addition, taking $H=\overline{U}_\alpha M_{\alpha\beta} \overline{V}_\beta$ for any $M_{\alpha\beta} \in \mathbb{R}^{[\alpha] \times [\alpha]}$ and observing that $\langle Z, X(M_{\alpha\beta}) \rangle = \langle X(Z), M_{\alpha\beta} \rangle$ for any $Z \in \mathbb{R}^{[\alpha] \times [\alpha]}$, from (18) we have

$$0 \geq \langle \tilde{X}^*_\alpha, M_{\alpha\beta} \rangle + \langle (\tilde{Y}^*_a - \tilde{X}^*_a) \circ (\Omega_2)_{\alpha\beta}, X(M_{\alpha\beta}) \rangle$$

$$= \langle \tilde{X}^*_\alpha, M_{\alpha\beta} \rangle + \langle X(\tilde{Y}^*_a - \tilde{X}^*_a) \circ (\Omega_2)_{\alpha\beta}, M_{\alpha\beta} \rangle,$$

which implies that $\tilde{X}^*_\alpha + X(\tilde{Y}^*_a - \tilde{X}^*_a) \circ (\Omega_2)_{\alpha\beta} = 0$. Similarly, taking $H=\overline{U}_\alpha M_{\alpha\beta} \overline{V}_\beta$ for any $M_{\alpha\beta} \in \mathbb{R}^{[\alpha] \times [\beta]}$, from equation (18) we obtain that

$$0 \geq \langle \tilde{X}^*_\alpha + \tilde{Y}^*_a, \Pi_{[\alpha]}(S(\tilde{H}_\beta)) \rangle + \langle \tilde{Y}^*_\beta + \tilde{H}_\beta \tilde{H}_\gamma \rangle = \langle \tilde{X}^*_\alpha + \tilde{Y}^*_a, M_{\alpha\beta} \rangle,$$

which shows that $\tilde{X}^*_\alpha + (X(\tilde{Y}^*_1 - \tilde{X}^*_1))_{\alpha\beta} \circ (\Omega_2)_{\alpha\beta} = 0$. Using the similar way, we have

$$\tilde{X}^*_\alpha + (X(\tilde{Y}^*_1 - \tilde{X}^*_1))_{\alpha\beta} \circ (\Omega_2)_{\alpha\beta} = 0, \tag{19a}$$

$$\tilde{X}^*_\alpha + (S(\tilde{Y}^*_1 - \tilde{X}^*_1))_{\alpha\gamma} \circ (\Omega_1)_{\alpha\gamma} + (X(\tilde{Y}^*_1 - \tilde{X}^*_1))_{\alpha\gamma} \circ (\Omega_2)_{\alpha\gamma} = 0, \tag{19b}$$

$$\tilde{X}^*_\alpha + (S(\tilde{Y}^*_1 - \tilde{X}^*_1))_{\gamma\alpha} \circ (\Omega_1)_{\gamma\alpha} + (X(\tilde{Y}^*_1 - \tilde{X}^*_1))_{\gamma\alpha} \circ (\Omega_2)_{\gamma\alpha} = 0. \tag{19c}$$
To sum up, the fact that inequality (18) holds for any $H \in \mathbb{R}^{m \times n}$ implies that
\[
\tilde{X}_{a\alpha} + X(Y_{a\alpha} - \tilde{X}_{a\alpha}) \circ (\Omega_2)_{a\alpha} = 0,
\tilde{X}_{\beta\beta} \leq 0, \quad \tilde{Y}_{\beta\beta} \geq 0,
\tilde{X}_{a\beta} + (X(Y_1^* - \tilde{X}_1^*))_{a\beta} \circ (\Omega_2)_{a\beta} = 0, \quad \tilde{X}_{\beta\alpha} + (X(Y_1^* - \tilde{X}_1^*))_{\beta\alpha} \circ (\Omega_2)_{\beta\alpha} = 0,
\tilde{X}_{a\gamma} + (S(Y_1^* - \tilde{X}_1^*))_{a\gamma} \circ (\Omega_1)_{a\gamma} + (X(Y_1^* - \tilde{X}_1^*))_{a\gamma} \circ (\Omega_2)_{a\gamma} = 0,
\tilde{Y}_{\gamma\alpha} = 0, \quad \tilde{Y}_{\gamma\beta} = 0, \quad \tilde{Y}_{\gamma\gamma} = 0, \quad \tilde{X}_{ac} \circ (E_{ac} - (\Omega_3)_{ac}) + \tilde{Y}_{ac} \circ (\Omega_3)_{ac} = 0.
\]

By the definitions of $\Theta_1, \Theta_2$ and $\Sigma_1, \Sigma_2$, equation (20) can be compactly written as (17a)-(17c). Conversely, it is easy to check that if $(X^*, Y^*)$ satisfies (20) or its compact form (17a)-(17c), then (18) holds for any $H \in \mathbb{R}^{m \times n}$, i.e., $(X^*, Y^*) \in \mathcal{N}^\pi_{\text{gph} \partial(\|\cdot\|, (X, Y)}$. □

**Remark 3.1** For any given $(X, Y) \in \text{gph} \partial\|\cdot\|$, let $\overline{Z} = X + Y$. By Theorem 3.1, if $\|\overline{Z}\| < 1$, then $(X^*, Y^*) \in \mathcal{N}^\pi_{\text{gph} \partial(\|\cdot\|, (X, Y)}$ if and only if $Y^* = 0$; if $\|\overline{Z}\| = 1$, then $(X^*, Y^*) \in \mathcal{N}^\pi_{\text{gph} \partial(\|\cdot\|, (X, Y)}$ if and only if $\tilde{X}^*$ and $\tilde{Y}^*$ take the following form
\[
\tilde{X}^* = \begin{bmatrix}
X_{\beta\beta}^* & X_{\gamma\gamma}^* & X_{\beta\gamma}^* \\
X_{\gamma\beta}^* & X_{\gamma\gamma}^* & X_{\gamma\gamma}^* \\
\end{bmatrix}
\text{ and } \tilde{Y}^* = \begin{bmatrix}
\tilde{Y}_{\beta\beta}^* & 0_{\beta\gamma} & 0_{\beta\gamma} \\
0_{\gamma\beta} & 0_{\gamma\gamma} & 0_{\gamma\gamma} \\
\end{bmatrix}
\text{ with } \tilde{X}_{\beta\beta} \leq 0, \tilde{Y}_{\beta\beta} \geq 0.
\]

### 3.2 Limiting normal cone

Let $\beta$ be a nonempty index set and denote the set of all partitions of $\beta$ by $\mathcal{P}(\beta)$. Write $\mathbb{R}_2^{[\beta]} := \{z \in \mathbb{R}^{[\beta]} : z_1 \geq \cdots \geq z_{|\beta|} > 0\}$. For any $z \in \mathbb{R}_2^{[\beta]}$, let $D(z) \in \mathcal{S}^{[\beta]}$ denote the generalized first divided difference matrix of $h(t) = \min(1, t)$ at $z$, which is defined as
\[
(D(z))_{ij} := \begin{cases}
\frac{\min(1, z_i) - \min(1, z_j)}{z_i - z_j} & \text{if } z_i \neq z_j, \\
0 & \text{if } z_i = z_j \geq 1, \\
1 & \text{otherwise}.
\end{cases}
\]

Write $U_{[\beta]} := \{\overline{\Omega} \in \mathcal{S}^{[\beta]} : \overline{\Omega} = \lim_{k \to \infty} D(z^k), z^k \to e_{[\beta]}, z^k \in \mathbb{R}_2^{[\beta]}\}$. For each $\Xi_1 \in U_{[\beta]}$, by equation (21) there exists a partition $(\beta_+, \beta_0, \beta_-) \in \mathcal{P}(\beta)$ such that
\[
\Xi_1 = \begin{bmatrix}
0_{\beta_+, \beta_+} & 0_{\beta_+, \beta_0} & (\Xi_1)_{\beta_+, \beta_-} \\
0_{\beta_0, \beta_+} & 0_{\beta_0, \beta_0} & E_{\beta_0, \beta_-} \\
(\Xi_1)_{\beta_+, \beta_-} & E_{\beta_+, \beta_-} & E_{\beta_-, \beta_-}
\end{bmatrix},
\]

where each entry of $(\Xi_1)_{\beta_+, \beta_-}$ belongs to $[0, 1]$. Let $\Xi_2$ be the matrix associated to $\Xi_1$:
\[
\Xi_2 = \begin{bmatrix}
E_{\beta_+, \beta_+} & E_{\beta_+, \beta_0} & E_{\beta_+, \beta_-} - (\Xi_1)_{\beta_+, \beta_-} \\
E_{\beta_0, \beta_+} & 0_{\beta_0, \beta_0} & 0_{\beta_0, \beta_-} \\
E_{\beta_-, \beta_+} - (\Xi_1)_{\beta_-, \beta_-} & 0_{\beta_-, \beta_0} & 0_{\beta_-, \beta_-}
\end{bmatrix}.
\]

With the above notations, we shall provide the exact formula for the limiting normal cone to $\text{gph} \partial\|\cdot\|_*$ in the following theorem, whose proof is included in Appendix.
Theorem 3.2 For any given \((X,Y)\in \text{gph} \partial \|\cdot\|_s\), let \(Z = X + Y\) have the SVD as in Lemma 2.3 with \(\alpha, \beta, \gamma\) and \(e\) defined by (5a)-(5b). Then, \((G,H) \in \mathcal{N}_{\text{gph} \partial \|\cdot\|_s}(X,Y)\) if and only if \((\tilde{G}, \tilde{H})\) with \(\tilde{G} = \tilde{U}^T \tilde{G} \tilde{V}\) and \(\tilde{H} = \tilde{U}^T \tilde{H} \tilde{V}\) satisfies the following conditions

\[
\begin{align*}
\Theta_1 \circ S(\tilde{H}_1) + \Theta_2 \circ S(\tilde{G}_1) + \Sigma_1 \circ \chi(\tilde{H}_1) + \Sigma_2 \circ \chi(\tilde{G}_1) &= 0, \\
\tilde{G}_{ac} + (\Omega_3)_{ac} \circ (\tilde{H}_{ac} - \tilde{G}_{ac}) &= 0, \\
\tilde{H}_{bc} &= 0, \\
\tilde{H}_{\gamma c} &= 0,
\end{align*}
\]

\[(24a)
\]

\[
\begin{align*}
(\tilde{G}_{\beta \beta}, \tilde{H}_{\beta \beta}) &\in \bigcup_{\substack{Q \in \mathbb{R}^{n \times n} \\ \Xi \in \mathbb{R}^{m \times m} \setminus \Xi_0}} \left\{ (M,N) \mid \Xi_1 \circ \tilde{N} + \Xi_2 \circ S(\tilde{M}) + \Xi_2 \circ \chi(\tilde{N}) = 0 \right\},
\end{align*}
\]

\[(24c)
\]

where \(\tilde{G}_1 = \tilde{U}^T \tilde{G} \tilde{V}_1, \tilde{H}_1 = \tilde{U}^T \tilde{H} \tilde{V}_1\), and \(\Theta_1, \Theta_2, \Sigma_1, \Sigma_2\) are the same as those before.

To close this paper, we point out that Theorem 3.2 also provides the characterization for the coderivative of \(\Pi_{\mathbb{R}^n}\). Indeed, by Lemma 2.1, \(\text{gph} \Pi_{\mathbb{R}^n} = \mathcal{L}^{-1}(\text{gph} \partial \|\cdot\|_s)\) with

\[
\mathcal{L}(X,Y) := (X - Y, Y) \quad \text{for} \quad (X,Y) \in \mathbb{R}^{m \times n \times n}.
\]

Since the linear map \(\mathcal{L} : \mathbb{R}^{m \times n \times n} \to \mathbb{R}^{m \times n \times n}\) is onto, from [15, Exercise 6.7]

\[
\mathcal{N}_{\text{gph} \Pi_{\mathbb{R}^n}}(X,Y) = \mathcal{L}^*(\mathcal{N}_{\text{gph} \partial \|\cdot\|_s}(\mathcal{L}(X,Y))),
\]

where \(\mathcal{L}^*\) is the adjoint of \(\mathcal{L}\). By the definition of coderivative (see [15, Definition 8.33]),

\[
W \in D^* \Pi_{\mathbb{R}^n}(X,Y)(S) \iff (W, W - S) \in \mathcal{N}_{\text{gph} \partial \|\cdot\|_s}(X - Y, Y).
\]

Similarly, Theorem 3.1 also provides the characterization for the regular coderivative:

\[
W \in D^* \Pi_{\mathbb{R}^n}(X,Y)(S) \iff (W, W - S) \in \mathcal{N}_{\text{gph} \partial \|\cdot\|_s}(X - Y, Y).
\]

In addition, with the help of Theorem 3.2 and the Mordukhovich criterion [10, Proposition 3.5] on the Aubin property of a multifunction, one may easily obtain the practical conditions for the Aubin property of \(\partial \|\cdot\|_s\). In our future work, we shall use Theorem 3.1 and 3.2 to derive the optimality conditions of the rank minimization problem (2).

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Appendix

Proof of Theorem 3.2: Throughout the proof, let $\nu_1 > \nu_2 > \cdots > \nu_r$ denote the nonzero distinct singular values of $\overline{Z}$, and $a, b$ and $a_k$ be the index sets defined by

$$a := \{i \in \{1, \ldots, m\} \mid \sigma_i(\overline{Z}) > 0\}, \quad b := \{i \in \{1, \ldots, m\} \mid \sigma_i(\overline{Z}) = 0\}, \quad (27a)$$

$$a_k := \{i \in \{1, \ldots, m\} \mid \sigma_i(\overline{Z}) = \pi_k\} \quad \text{for} \quad k = 1, 2, \ldots, r. \quad (27b)$$

In addition, we write $\alpha = \bigcup_{i=1}^{r-1} a_i$, $\beta = a_l$ and $\gamma_1 := \{i \in \gamma \mid \sigma_i(\overline{Z}) > 0\} = \bigcup_{i=l+1}^r a_i$.

$\implies$. Let $(G, H) \in N_{gh}(\overline{Z}, X, Y)$. By Proposition 3.1 and the definition of limiting normal cones, there exist sequences $(X^k, Y^k) \rightarrow (X, Y)$ and $(G^k, H^k) \rightarrow (G, H)$ with $(G^k, H^k) \in N_{gh}(\overline{Z}^k, X^k, Y^k)$ for each $k$. For each $k$, we write $Z^k = X^k + Y^k$ and let $Z^k$ have the SVD as $U^k \text{Diag}(\sigma(Z^k))(V^k)^T$ where $U^k \in \mathbb{O}^m$ and $V^k = [V_1^k \ V_2^k] \in \mathbb{O}^n$ with $V_1^k \in \mathbb{O}^{n \times m}$. Since $\{(U^k, V^k)\}$ is uniformly bounded, by taking a subsequence if necessary, we may assume that $\lim_{k \rightarrow \infty} (U^k, V^k) = (\hat{U}, \hat{V})$. Clearly, $\overline{Z} = \hat{U} [\text{Diag}(\sigma(\overline{Z}))] 0 \hat{V}^T$.

By [5, Proposition 2.5], there exist orthogonal matrices $Q' \in \mathbb{O}^{[b]}$ and $Q'' \in \mathbb{O}^{[n - [a]}}$ and a block diagonal orthogonal matrix $Q = \text{Diag}(Q_1, Q_2, \ldots, Q_r)$ with $Q_k \in \mathbb{O}^{[a_k]}$ such that

$$\hat{U} = U \begin{bmatrix} Q & 0 \\ 0 & Q' \end{bmatrix} \quad \text{and} \quad \hat{V} = V \begin{bmatrix} Q & 0 \\ 0 & Q'' \end{bmatrix}. \quad (28)$$

Since $\sigma(\overline{Z}) = \lim_{k \rightarrow \infty} \sigma(Z^k)$, for all sufficiently large $k$, we have $\sigma_i(Z^k) > 1$ if $i \in \alpha$ and $\sigma_i(Z^k) < 1$ if $i \in \gamma$. Since $\lim_{k \rightarrow \infty} \sigma_i(Z^k) = 1$ for $i \in \beta$, we assume (if necessary taking a subsequence) that there exists a partition $(\beta_+, \beta_0, \beta_-)$ of $\beta$ such that for each $k$,

$$\sigma_i(Z^k) > 1 \ \forall i \in \beta_+, \quad \sigma_i(Z^k) = 1 \ \forall i \in \beta_0 \quad \text{and} \quad \sigma_i(Z^k) < 1 \ \forall i \in \beta_-. \quad (28)$$
Since \((G^k, H^k) \in \mathcal{N}_{g^k, \partial ||\cdot ||}^\Xi (X^k, Y^k)\) for each \(k\), by Theorem 3.1 there exist the matrices

\[
\Theta_1^k = \begin{bmatrix}
0_{\alpha \alpha} & 0_{\alpha \beta_+} & 0_{\alpha \beta_0} & (\Omega_1^k)_{\alpha \beta_-} & (\Omega_1^k)_{\alpha \gamma} \\
0_{\beta_+ \alpha} & 0_{\beta_+ \beta_+} & 0_{\beta_+ \beta_0} & (\Omega_1^k)_{\beta_+ \beta_-} & (\Omega_1^k)_{\beta_+ \gamma} \\
0_{\beta_0 \alpha} & 0_{\beta_0 \beta_+} & 0_{\beta_0 \beta_0} & E_{\beta_0 \beta_-} & E_{\beta_0 \gamma} \\
(\Omega_1^k)_{\beta_0 \alpha} & (\Omega_1^k)_{\beta_0 \beta_+} & E_{\beta_0 \beta_0} & E_{\beta_0 \beta_-} & E_{\beta_0 \gamma} \\
(\Omega_1^k)_{\gamma \alpha} & (\Omega_1^k)_{\gamma \beta_+} & E_{\gamma \beta_0} & E_{\gamma \beta_-} & E_{\gamma \gamma} \\
\end{bmatrix},
\]

(29)

\[
\Theta_2^k = \begin{bmatrix}
E_{\alpha \alpha} & E_{\alpha \beta_+} & E_{\alpha \beta_0} & (\tilde{\Omega}_1^k)_{\alpha \beta_-} & (\tilde{\Omega}_1^k)_{\alpha \gamma} \\
E_{\beta_+ \alpha} & E_{\beta_+ \beta_+} & E_{\beta_+ \beta_0} & (\Omega_2^k)_{\beta_+ \beta_-} & (\Omega_2^k)_{\beta_+ \gamma} \\
E_{\beta_0 \alpha} & E_{\beta_0 \beta_+} & 0_{\beta_0 \beta_0} & 0_{\beta_0 \beta_-} & 0_{\beta_0 \gamma} \\
(\Omega_2^k)_{\beta_0 \alpha} & (\Omega_2^k)_{\beta_0 \beta_+} & E_{\beta_0 \beta_0} & E_{\beta_0 \beta_-} & E_{\beta_0 \gamma} \\
(\Omega_2^k)_{\gamma \alpha} & (\Omega_2^k)_{\gamma \beta_+} & E_{\gamma \beta_0} & E_{\gamma \beta_-} & E_{\gamma \gamma} \\
\end{bmatrix}
\]

(30)

and

\[
\Sigma_1^k := \begin{bmatrix}
(\Omega_1^k)_{\alpha \alpha} & (\Omega_1^k)_{\alpha \beta_+} & (\Omega_1^k)_{\alpha \beta_0} & (\Omega_1^k)_{\alpha \beta_-} & (\Omega_1^k)_{\alpha \gamma} \\
(\Omega_1^k)_{\beta_+ \alpha} & (\Omega_1^k)_{\beta_+ \beta_+} & (\Omega_1^k)_{\beta_+ \beta_0} & (\Omega_1^k)_{\beta_+ \beta_-} & (\Omega_1^k)_{\beta_+ \gamma} \\
(\Omega_1^k)_{\beta_0 \alpha} & (\Omega_1^k)_{\beta_0 \beta_+} & 0_{\beta_0 \beta_0} & E_{\beta_0 \beta_-} & E_{\beta_0 \gamma} \\
(\Omega_1^k)_{\gamma \alpha} & (\Omega_1^k)_{\gamma \beta_+} & 0_{\gamma \beta_0} & 0_{\gamma \beta_-} & 0_{\gamma \gamma} \\
\end{bmatrix},
\]

(31)

\[
\Sigma_2^k := \begin{bmatrix}
(\Omega_2^k)_{\alpha \alpha} & (\Omega_2^k)_{\alpha \beta_+} & (\tilde{\Omega}_2^k)_{\alpha \beta_0} & (\tilde{\Omega}_2^k)_{\alpha \beta_-} & (\tilde{\Omega}_2^k)_{\alpha \gamma} \\
(\tilde{\Omega}_2^k)_{\beta_+ \alpha} & (\tilde{\Omega}_2^k)_{\beta_+ \beta_+} & (\Omega_2^k)_{\beta_+ \beta_0} & (\Omega_2^k)_{\beta_+ \beta_-} & (\Omega_2^k)_{\beta_+ \gamma} \\
(\Omega_2^k)_{\beta_0 \alpha} & (\Omega_2^k)_{\beta_0 \beta_+} & 0_{\beta_0 \beta_0} & 0_{\beta_0 \beta_-} & 0_{\beta_0 \gamma} \\
(\Omega_2^k)_{\gamma \alpha} & (\Omega_2^k)_{\gamma \beta_+} & 0_{\gamma \beta_0} & 0_{\gamma \beta_-} & 0_{\gamma \gamma} \\
\end{bmatrix}
\]

(32)

such that

\[
\Theta_1^k \circ S(\tilde{H}_1^k) + \Theta_2^k \circ S(\tilde{G}_1^k) + \Sigma_1^k \circ \mathcal{X}(\tilde{H}_1^k) + \Sigma_2^k \circ \mathcal{X}(\tilde{G}_1^k) = 0,
\]

(33a)

\[
\tilde{G}_{ac}^k \circ (E_{ac} - (\Omega_3^k)_{ac}) + \tilde{H}_{ac}^k \circ (\Omega_3^k)_{ac} = 0, \quad \tilde{H}_{bc}^k = 0, \quad \tilde{H}_{bc}^k = 0,
\]

(33b)

\[
\tilde{G}_{\beta_+ c}^k \circ [E_{\beta_+ c} - (\Omega_3^k)_{\beta_+ c}] + \tilde{H}_{\beta_+ c}^k \circ (\Omega_3^k)_{\beta_+ c} = 0, \quad \tilde{H}_{\gamma c}^k = 0,
\]

(33c)

\[
\tilde{G}_{\beta_0 \beta_0}^k \leq 0, \quad \tilde{H}_{\beta_0 \beta_0}^k \geq 0
\]

(33d)

where \(\tilde{\Omega}_1^k = E - \Omega_1^k\) and \(\tilde{\Omega}_2^k = E - \Omega_2^k\) with \(\Omega_1^k, \Omega_2^k\) and \(\Omega_3^k\) defined by (6)-(8) with \(\sigma(Z^k)\), \(\tilde{G}_1^k = (U^k)^\top G^k V_1^k, \tilde{H}_1^k = (U^k)^\top H^k V_1^k\) and \(\tilde{G}^k = (U^k)^\top G^k V^k, \tilde{H}^k = (U^k)^\top H^k V^k\). By the definition of \(\Omega_1^k\), we calculate that \(\lim_{k \to \infty} (\Omega_1^k)_{\alpha \beta_-} = 0_{\alpha \beta_-}\), \(\lim_{k \to \infty} (\Omega_1^k)_{\alpha \gamma} = (\Omega_1)_{\alpha \gamma}\) and \(\lim_{k \to \infty} (\Omega_2^k)_{\beta_+ \gamma} = E_{\beta_+ \gamma}\). Together with the definitions of \(\Theta_1^k\) and \(\Theta_2^k\), there exist \(\Xi_1 \in \mathcal{U}_{\beta}\) and the corresponding \(\Xi_2\) defined by (22) and (23), respectively, such that \(\lim_{k \to \infty} \Theta_1^k = \hat{\Theta}_1\) and \(\lim_{k \to \infty} \Theta_2^k = \hat{\Theta}_2\) with

\[
\hat{\Theta}_1 = \Theta_1 + \begin{bmatrix}
0_{\alpha \alpha} & 0_{\alpha \beta} & 0_{\alpha \gamma} \\
0_{\beta_+ \alpha} & \Xi_1 & 0_{\beta_-} \\
0_{\gamma \alpha} & 0_{\gamma \beta} & 0_{\gamma \gamma} \\
\end{bmatrix} \quad \text{and} \quad \hat{\Theta}_2 = \Theta_2 + \begin{bmatrix}
0_{\alpha \alpha} & 0_{\alpha \beta} & 0_{\alpha \gamma} \\
0_{\beta_+ \alpha} & \Xi_2 & 0_{\beta_-} \\
0_{\gamma \alpha} & 0_{\gamma \beta} & 0_{\gamma \gamma} \\
\end{bmatrix}.
\]
By the definition of $\Omega_k^k$, $\lim_{k \to \infty} (\Omega_k^k)_{\alpha,\alpha+\beta} = (\Omega_2)_{\alpha,\alpha+\beta}$, $\lim_{k \to \infty} (\Omega_k^k)_{\beta+\delta,\beta+\delta} = E_{\beta+\delta,\beta+\delta}$, and $\lim_{k \to \infty} (\Omega_k^k)_{\beta+\delta,\delta} = E_{\beta+\delta,\delta}$. Then, we have that

$$\lim_{k \to \infty} \Sigma_k^k = \Sigma_2 \quad \text{and} \quad \lim_{k \to \infty} \Sigma_1^k = \Sigma_1 + \begin{bmatrix} 0_{\alpha\alpha} & 0_{\alpha\beta} & 0_{\alpha\gamma} \\ 0_{\beta\alpha} & \Xi_1 + \Xi_2 & 0_{\beta\gamma} \\ 0_{\gamma\alpha} & 0_{\gamma\beta} & 0_{\gamma\gamma} \end{bmatrix} := \hat{\Sigma}_1.$$

Let $\hat{G}_1 = \hat{U}^T\hat{G}_1$ and $\hat{H}_1 = \hat{U}^T\hat{H}_1$, where $\hat{V}_1 \in \mathbb{C}^{n \times m}$ is the matrix consisting of the first $m$ columns of $\hat{V}$. Now taking the limit $k \to \infty$ to equation (33a) yields that

$$\hat{\Theta}_1 \circ \mathcal{S}(\hat{H}_1) + \hat{\Theta}_2 \circ \mathcal{S}(\hat{G}_1) + \hat{\Sigma}_1 \circ \mathcal{X}(\hat{H}_1) + \hat{\Sigma}_2 \circ \mathcal{X}(\hat{G}_1) = 0. \quad (34)$$

By the definitions of $\hat{G}_1$ and $\hat{H}_1$ and equation (28), one may calculate that

$$\hat{G}_1 = \begin{bmatrix} Q_{\alpha}^2 \tilde{G}_{\alpha\alpha} Q_{\alpha} & Q_{\alpha}^2 \tilde{G}_{\alpha\beta} Q_{\beta} & Q_{\alpha}^2 \tilde{G}_{\alpha\gamma} Q_{\gamma} \\ Q_{\beta}^2 \tilde{G}_{\beta\alpha} Q_{\alpha} & Q_{\beta}^2 \tilde{G}_{\beta\beta} Q_{\beta} & Q_{\beta}^2 \tilde{G}_{\beta\gamma} Q_{\gamma} \\ Q_{\gamma}^2 \tilde{G}_{\gamma\alpha} Q_{\alpha} & Q_{\gamma}^2 \tilde{G}_{\gamma\beta} Q_{\beta} & Q_{\gamma}^2 \tilde{G}_{\gamma\gamma} Q_{\gamma} \\ (Q')^T \tilde{G}_{bb} Q_{\alpha} & (Q')^T \tilde{G}_{bb} Q_{\beta} & (Q')^T \tilde{G}_{bb} Q_{\gamma} \end{bmatrix},$$

$$\hat{H}_1 = \begin{bmatrix} Q_{\alpha}^2 \tilde{H}_{\alpha\alpha} Q_{\alpha} & Q_{\alpha}^2 \tilde{H}_{\alpha\beta} Q_{\beta} & Q_{\alpha}^2 \tilde{H}_{\alpha\gamma} Q_{\gamma} \\ Q_{\beta}^2 \tilde{H}_{\beta\alpha} Q_{\alpha} & Q_{\beta}^2 \tilde{H}_{\beta\beta} Q_{\beta} & Q_{\beta}^2 \tilde{H}_{\beta\gamma} Q_{\gamma} \\ Q_{\gamma}^2 \tilde{H}_{\gamma\alpha} Q_{\alpha} & Q_{\gamma}^2 \tilde{H}_{\gamma\beta} Q_{\beta} & Q_{\gamma}^2 \tilde{H}_{\gamma\gamma} Q_{\gamma} \\ (Q')^T \tilde{H}_{bb} Q_{\alpha} & (Q')^T \tilde{H}_{bb} Q_{\beta} & (Q')^T \tilde{H}_{bb} Q_{\gamma} \end{bmatrix},$$

where $Q_\alpha = \text{Diag}(Q_1, \ldots, Q_{l-1})$ and $Q_{\gamma} = \text{Diag}(Q_{l+1}, \ldots, Q_r)$ are the block diagonal orthogonal matrices. By the definitions of $\hat{\Theta}_1, \hat{\Theta}_2, \hat{\Sigma}_1, \hat{\Sigma}_2$, we write (34) equivalently as

$$\tilde{G}_{\alpha\alpha} + (\Omega_2)_{\alpha\alpha} \circ (\mathcal{X}(\hat{H} - \tilde{G}_1))_{\alpha\alpha} = 0,$$

$$\tilde{G}_{\alpha\beta} + (\Omega_2)_{\alpha\beta} \circ (\mathcal{X}(\hat{H} - \tilde{G}_1))_{\alpha\beta} = 0, \quad \tilde{G}_{\alpha\gamma} + (\Omega_2)_{\alpha\gamma} \circ (\mathcal{X}(\hat{H} - \tilde{G}_1))_{\alpha\gamma} = 0,$$

$$\tilde{G}_{\beta\alpha} + (\Omega_2)_{\beta\alpha} \circ (\mathcal{X}(\hat{H} - \tilde{G}_1))_{\beta\alpha} = 0, \quad \tilde{G}_{\beta\beta} + (\Omega_2)_{\beta\beta} \circ (\mathcal{X}(\hat{H} - \tilde{G}_1))_{\beta\beta} = 0, \quad \tilde{G}_{\beta\gamma} + (\Omega_2)_{\beta\gamma} \circ (\mathcal{X}(\hat{H} - \tilde{G}_1))_{\beta\gamma} = 0,$$

and

$$\tilde{G}_{\alpha\beta} + (\Omega_2)_{\alpha\beta} \circ (\tilde{H}_{bb} - \tilde{G}_{bb}) = 0, \quad \tilde{G}_{\alpha\gamma} + (\Omega_2)_{\alpha\gamma} \circ (\tilde{H}_{bb} - \tilde{G}_{bb}) = 0, \quad \tilde{G}_{\beta\alpha} + (\Omega_2)_{\beta\alpha} \circ (\tilde{H}_{bb} - \tilde{G}_{bb}) = 0,$$

$$\Xi_1 \circ (S(Q_1^T \tilde{H}_{\beta\beta} Q_1)) + (\Xi_1 + \Xi_2) \circ (S(Q_1^T \tilde{H}_{\beta\beta} Q_1)) + \Xi_2 \circ (S(Q_1^T \tilde{H}_{\beta\beta} Q_1)) = 0, \quad \tilde{H}_{\beta\gamma} = 0, \quad \tilde{H}_{\gamma\beta} = 0, \quad \tilde{H}_{\gamma\gamma} = 0, \quad \tilde{H}_{\beta\gamma} = 0, \quad \tilde{H}_{\gamma\beta} = 0, \quad \tilde{H}_{\gamma\gamma} = 0,$$

$$\tilde{H}_{bb}^T \tilde{G}_{bb} + \tilde{H}_{bc}^T \tilde{G}_{bc} + \tilde{H}_{cc}^T \tilde{G}_{cc} + (\Omega_2)_{\alpha\beta} \circ (\tilde{H}_{bb} - \tilde{G}_{bb}) = 0,$$

where equalities (36a) and (36b) are using $(\Omega_1)_{bb} = (\Omega_2)_{bb}$ and the fact that the entries in each column of $(\Omega_1)_{bb}$ are the same. Taking the limit $k \to \infty$ to (33b)-(33c), we get

$$\tilde{G}_{\alpha\beta} Q_{bc} + \tilde{G}_{\alpha\gamma} Q_{cc} + (\Omega_2)_{\alpha\alpha} \circ (\tilde{H}_{bb} - \tilde{G}_{bb}) Q_{bc} + (\tilde{H}_{bc} - \tilde{G}_{bc}) Q_{cc} = 0, \quad \tilde{H}_{\beta\gamma} Q_{bc} + \tilde{H}_{\gamma\beta} Q_{bc} = 0,$$

$$\tilde{H}_{\beta\alpha} Q_{bc} + \tilde{H}_{\alpha\gamma} Q_{cc} = 0, \quad \tilde{H}_{\beta\gammac} Q_{cc} = 0, \quad \tilde{H}_{\gamma\beta} Q_{bc} + \tilde{H}_{\gamma\gamma} Q_{cc} = 0, \quad \tilde{H}_{\gamma\beta} Q_{bc} + \tilde{H}_{\gamma\gamma} Q_{cc} = 0.$$

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Notice that \((\Omega_1)_{ab} = (\Omega_4)_{ab}\) and they have the same entries in each row. Hence, equations (36b) and (37a) are equivalent to saying that \(\tilde{G}_{\alpha,b,\cdot} + (\Omega_1)_{a,b,\cdot} \circ (H_{a,b,\cdot} - \tilde{G}_{a,b,\cdot}) = 0\). The equalities in (36c) and (37b)-(37c) are equivalent to saying that \(\tilde{H}_{\beta,b,\cdot} = 0\) and \(\tilde{H}_{\gamma,b,\cdot} = 0\). The three equalities, along with the equalities in (35), (36a) and (36d), can be compactly written as (24a) and (24b). Finally, by the partition \((\beta_+, \beta_0, \beta_-)\) of \(\beta\), we may write \(Q_1 = [Q_{\beta_+} \ Q_{\beta_0} \ Q_{\beta_-}] \in \mathcal{O}^{[\beta]}\) with \(Q_{\beta_0} \in \mathcal{O}^{[\beta] \times [\beta_0]}\). Then, taking the limit to (33d) yields that \(Q_{\beta_0}^T \tilde{G}_{\beta \gamma} Q_{\beta_0} \preceq 0\) and \(Q_{\beta_0}^T \tilde{H}_{\beta \gamma} Q_{\beta_0} \succeq 0\). Together with (36c), it follows that \(\tilde{G}_{\beta \gamma}\) and \(\tilde{H}_{\beta \gamma}\) satisfy (24c). To sum up, we achieve the desired result in (24a)-(24c).

“\(\Leftarrow\)” Write \(\tilde{\sigma} := \sigma(\tilde{Z})\). Notice that \(\Pi_B(X + Y) = Y\). From the SVD of \(\tilde{Z} = X + Y\),

\[
X = \overline{U}_\alpha \text{Diag}(\overline{\sigma}_\alpha - e_\alpha) \overline{V}_\alpha^T \quad \text{and} \quad Y = \overline{U} \begin{bmatrix}
\text{Diag}(e_{\alpha}) & 0_{\alpha \beta} & 0_{\alpha \gamma} & 0_{\alpha c} \\
0_{\beta \alpha} & \text{Diag}(e_{\beta}) & 0_{\beta \gamma} & 0_{\beta c} \\
0_{\gamma \alpha} & 0_{\gamma \beta} & \text{Diag}(\overline{\sigma}_\gamma) & 0_{\gamma c}
\end{bmatrix} \overline{V}_\gamma^T.
\]  

(38)

Let \((G, H)\) satisfy (24a)-(24c). Then there exist \(Q \in \mathcal{O}^{[\beta]}\), \(\Xi_1 \in \mathcal{U}_{[\beta]}\) and a partition \((\beta_+, \beta_0, \beta_-) \in \mathcal{P}(\beta)\) such that \(\Xi_1\) and the associated \(\Xi_2\) take the form (22)-(23), and

\[
\Xi_1 \circ (Q^T \tilde{H}_{\beta \gamma} Q) + \Xi_2 \circ S(Q^T \tilde{G}_{\beta \gamma} Q) + \Xi_2 \circ X(Q^T \tilde{H}_{\beta \gamma} Q) = 0,
\]  

(39a)

\[
Q_{\beta_0}^T \tilde{G}_{\beta \gamma} Q_{\beta_0} \preceq 0 \quad \text{and} \quad Q_{\beta_0}^T \tilde{H}_{\beta \gamma} Q_{\beta_0} \succeq 0.
\]  

(39b)

Since \(\Xi_1 \in \mathcal{U}_{[\beta]}\), there exists a sequence \(\{z^k\} \subseteq \mathbb{R}_{++}^{[\beta]}\) converging to \(e_\beta\) such that \(\Xi_1 = \lim_{k \to \infty} D(z^k)\). Without loss of generality, we may assume that for all \(k\),

\[
z^k_i > 1 \quad \forall i \in \beta_+ \quad z^k_i = 1 \quad \forall i \in \beta_0 \quad \text{and} \quad 0 < z^k_i < 1 \quad \forall i \in \beta_-.
\]

For each \(k\), we construct the matrices \(X^k\) and \(Y^k\) as follows:

\[
X^k = \hat{U} \begin{bmatrix}
\text{Diag}(\overline{\sigma}_\alpha - e_\alpha) & 0_{\alpha \beta_+} & 0_{\alpha \beta_0} & 0_{\alpha \beta_-} & 0_{\alpha \gamma} & 0_{\alpha c} \\
0_{\beta_+ \alpha} & \text{Diag}(z^k_{\beta_+} - e_{\beta_+}) & 0_{\beta_+ \beta_0} & 0_{\beta_+ \beta_-} & 0_{\beta_+ \gamma} & 0_{\beta_+ c} \\
0_{\beta_0 \alpha} & 0_{\beta_0 \beta_+} & 0_{\beta_0 \beta_0} & 0_{\beta_0 \beta_-} & 0_{\beta_0 \gamma} & 0_{\beta_0 c} \\
0_{\beta_- \alpha} & 0_{\beta_- \beta_+} & 0_{\beta_- \beta_0} & \text{Diag}(e_{\beta_-}) & 0_{\beta_- \beta_-} & 0_{\beta_- \gamma} & 0_{\beta_- c} \\
0_{\gamma \alpha} & 0_{\gamma \beta_+} & 0_{\gamma \beta_0} & 0_{\gamma \beta_-} & \text{Diag}(\overline{\sigma}_\gamma) & 0_{\gamma \gamma} & 0_{\gamma c}
\end{bmatrix} \hat{V}^T,
\]

\[
Y^k = \hat{U} \begin{bmatrix}
\text{Diag}(e_{\alpha}) & 0_{\alpha \beta_+} & 0_{\alpha \beta_0} & 0_{\alpha \beta_-} & 0_{\alpha \gamma} & 0_{\alpha c} \\
0_{\beta_+ \alpha} & \text{Diag}(e_{\beta}) & 0_{\beta_+ \beta_0} & 0_{\beta_+ \beta_-} & 0_{\beta_+ \gamma} & 0_{\beta_+ c} \\
0_{\beta_0 \alpha} & 0_{\beta_0 \beta_+} & \text{Diag}(e_{\beta_0}) & 0_{\beta_0 \beta_-} & 0_{\beta_0 \gamma} & 0_{\beta_0 c} \\
0_{\beta_- \alpha} & 0_{\beta_- \beta_+} & 0_{\beta_- \beta_0} & \text{Diag}(e_{\beta_-}) & 0_{\beta_- \beta_-} & 0_{\beta_- \gamma} & 0_{\beta_- c} \\
0_{\gamma \alpha} & 0_{\gamma \beta_+} & 0_{\gamma \beta_0} & 0_{\gamma \beta_-} & \text{Diag}(\overline{\sigma}_\gamma) & 0_{\gamma \gamma} & 0_{\gamma c}
\end{bmatrix} \hat{V}^T
\]

where \(\hat{U} = [\overline{U}_\alpha \ \overline{U}_\beta \ Q \ \overline{U}_\gamma]\) and \(\hat{V} = [\overline{V}_\alpha \ \overline{V}_\beta \ Q \ \overline{V}_\gamma \ \overline{V}_c]\). It is immediate to see that \(\Pi_B(X^k + Y^k) = Y^k\) for each \(k\), which by Lemma 2.1 shows that \((X^k, Y^k) \in \text{gph} \Theta\|\cdot\|_*\). Also, comparing with (38), we have that \((X^k, Y^k)\) converges to \((X, Y)\). For each \(k\), we write \(Z^k = X^k + Y^k\) and define the matrices \(\Theta_1^k, \Theta_2^k, \Sigma_1^k, \Sigma_2^k \in \mathbb{S}^m\) as in (29)-(32) with
σ(Z^k). Observe that \( \sigma(Z^k) = (\sigma_{\alpha}; z^k_{\beta_+}; \sigma_{\beta_0}; z^k_{\beta_-};\sigma_{\gamma}) \). Then, we have that

\[
(\Omega^k_1)_{ij} = (\Omega_1)_{ij} \quad \text{and} \quad (\Omega^k_2)_{ij} = (\Omega_2)_{ij} \quad \text{for} \quad (i, j) \in (\alpha \cup \beta_0 \cup \gamma) \times (\alpha \cup \beta_0 \cup \gamma), \quad (41)
\]

\[
\lim_{k \to \infty} (\Omega^k_1)_{ij} = (\Omega_1)_{ij}, \quad \lim_{k \to \infty} (\Omega^k_2)_{ij} = (\Omega_2)_{ij} \quad \text{for} \quad (i, j) \in \{1, 2, \ldots, m\} \times (\beta_+ \cup \beta_-). \quad (42)
\]

Let \( \Omega^k_3 \) be defined by (8) with \( \sigma(Z^k) \). Clearly, \( (\Omega^k_3)_{ij} = (\Omega_3)_{ij} \) for \( (i, j) \in \alpha \times c. \) The rest is to construct a sequence \{\( (G^k, H^k) \)\} converging to \( (G, H) \) such that \( (G^k, H^k) \in \mathcal{N}^\infty_{\text{gr}h, \theta|\|} (X^k, Y^k) \) for each \( k. \) For this purpose, we shall define \( \hat{G}^k, \hat{H}^k \in \mathbb{R}^{m \times n}. \) Let

\[
(\hat{G}^k)_{ij} := \tilde{G}_{ij} \quad \text{and} \quad (\hat{H}^k)_{ij} := \tilde{H}_{ij} \quad \text{for} \quad (i, j) \in (\alpha \cup \beta_0 \cup \gamma) \times (\alpha \cup \beta_0 \cup \gamma) \text{ or } \alpha \times c. \quad (43)
\]

For \( (i, j) \notin (\alpha \cup \beta_0 \cup \gamma) \times (\alpha \cup \beta_0 \cup \gamma) \) or \( \alpha \times c, \) we define \( (\hat{G}^k)_{ij} \) and \( (\hat{H}^k)_{ij} \) as below, where \( \tilde{G}_1 \) and \( \tilde{H}_1 \) are the matrices consisting of the first \( m \) columns of \( \hat{G}^k \) and \( \hat{H}^k. \)

**Case 1:** \((i, j) \) or \((j, i) \in \alpha \times \beta_+. \) In this case, we let \( \hat{H}^k_{ij} := \tilde{H}_{ij} \) for each \( k \) and define

\[
\hat{G}^k_{ij} := \frac{(\Omega^k_2)_{ij}}{(\Omega^k_3)_{ij}} - 1 \left[ X(\tilde{H}_1) \right]_{ij},
\]

Notice that \( (\Omega^k_3)_{ij} = (\Omega_3)_{ij}. \) Then we have \( \hat{G}^k_{ij} = -\tilde{G}^k_{ji} \) for each \( k, \) which implies that

\[
\hat{G}^k_{ij} + (\Omega^k_3)_{ij} [X(\tilde{H}_1) - \tilde{G}^k_{ij}]_{ij} = 0. \quad (44)
\]

**Case 2:** \((i, j) \) or \((j, i) \in \alpha \times \beta_- . \) In this case, we let \( \hat{H}^k_{ij} := \tilde{H}_{ij} \) for each \( k \) and define

\[
\hat{G}^k_{ij} := \frac{(\Omega^k_3)_{ij}}{(\Omega^k_2)_{ij}} - 1 \left[ X(\tilde{H}_1) \right]_{ij} + \frac{(\Omega^k_1)_{ij}}{(\Omega^k_2)_{ij}} - 1 \left[ S(\tilde{H}_1) \right]_{ij},
\]

Then \( [S(\hat{G}^k_1)]_{ij} = \frac{(\Omega^k_3)_{ij}}{(\Omega^k_2)_{ij}} [S(\tilde{H}^k_1)]_{ij} \) and \( [X(\hat{G}^k_1)]_{ij} = \frac{(\Omega^k_1)_{ij}}{(\Omega^k_2)_{ij}} [X(\tilde{H}^k_1)]_{ij} \) by using the symmetry of \( \Omega^k_2 \) and \( \Omega^k_1. \) Consequently, for each \( k, \) it holds that

\[
(\Omega^k_1)_{ij} [S(\tilde{H}^k_1)]_{ij} + (\Omega^k_2)_{ij} [S(\hat{G}^k)]_{ij} = 0, \quad (\Omega^k_2)_{ij} [X(\tilde{H}^k_1)]_{ij} + (\Omega^k_1)_{ij} [X(\hat{G}^k_1)]_{ij} = 0. \quad (45)
\]

**Case 3:** \((i, j) \) or \((j, i) \in (\beta_+ \cup \beta_0) \times \beta_+ . \) For each \( k, \) we let \( \hat{G}^k_{ij} := Q^k_{ij} \tilde{G}_{ij} + Q^k_{ij} Q_j \) and define

\[
\hat{H}^k_{ij} := \frac{(\Omega^k_3)_{ij}}{(\Omega^k_2)_{ij}} - 1 \left[ X(\tilde{G}^k_1) \right]_{ij},
\]

Notice that \( \hat{G}^k_{ij} = -\tilde{G}^k_{ji} \) implied by equation (39a). Then, we immediately have that

\[
(\hat{G}^k)_{ij} - (\Omega^k_3)_{ij} [X(\tilde{G}^k_1)]_{ij} + (\Omega^k_2)_{ij} [X(\tilde{H}^k_1)]_{ij} = 0.
\]

**Case 4:** \((i, j) \) or \((j, i) \in \beta_+ \times \beta_- . \) For each \( k, \) we let \( \hat{G}^k_{ij} := Q^k_{ij} \tilde{G}_{ij} + Q^k_{ij} Q_j \) and define

\[
\hat{H}^k_{ij} := \frac{(\Omega^k_3)_{ij}}{(\Omega^k_2)_{ij}} - 1 \left[ X(\tilde{G}^k_1) \right]_{ij} + \frac{(\Omega^k_1)_{ij}}{(\Omega^k_2)_{ij}} - 1 \left[ S(\tilde{G}^k_1) \right]_{ij}.
\]
Then $[S(\hat{H}^k)]_{ij} = - \frac{\Omega_{ij}}{||\omega||} [S(G^k)]_{ij}$ and $[X(\hat{H}^k)]_{ij} = - \frac{\Omega_{ij}}{||\omega||} [X(G^k)]_{ij}$, and hence

$$(\Omega_{ij})_{ij} [S(\hat{H}^k)]_{ij} + (\Omega_{ij})_{ij} [S(G^k)]_{ij} = 0, \ (\Omega_{ij})_{ij} [X(\hat{H}^k)]_{ij} + (\Omega_{ij})_{ij} [X(G^k)]_{ij} = 0.$$  \hspace{1cm} (46)

**Case 5:** $(i,j)$ or $(j,i) \in \beta_+ \times \gamma$. In this case, we let $\hat{G}_{ij} := \tilde{G}_{ij}$ for each $k$ and define

$$\hat{H}_{ij} := \frac{(\Omega_{ij})_{ij} - 1}{(\Omega_{ij})_{ij}} [X(G^k)]_{ij} + \frac{(\Omega_{ij})_{ij} - 1}{(\Omega_{ij})_{ij}} [S(G^k)]_{ij}.$$  

Then, using the same arguments as those for Case 4, we obtain that

$$(\Omega_{ij})_{ij} [S(\hat{H}^k)]_{ij} + (\Omega_{ij})_{ij} [S(G^k)]_{ij} = 0, \ (\Omega_{ij})_{ij} [X(\hat{H}^k)]_{ij} + (\Omega_{ij})_{ij} [X(G^k)]_{ij} = 0.$$  \hspace{1cm} (47)

**Case 6:** $(i,j) \in \beta_+ \times c$. For each $k$, let $\hat{G}_{ij} := \tilde{G}_{ij}$ and $\hat{H}_{ij} := \frac{(\Omega_{ij})_{ij} - 1}{(\Omega_{ij})_{ij}} \tilde{G}_{ij}$. Then, \[ (E_{\beta_+} - (\Omega_{ij})_{ij} - 1)(\hat{G}^k)_{\beta_+} + (\Omega_{ij})_{ij} - 1)(\hat{G}^k)_{\beta_+} = 0. \]  \hspace{1cm} (48)

**Case 7:** $(i,j) = \beta_+ \cup \gamma$ or $(i,j) \in \beta_+ \times (\gamma \cup \beta_0)$. Now for each $k$, let $\hat{G}_{ij} := \tilde{G}_{ij}$ and $\hat{H}_{ij} := 0$.

For each $k$, let $G^k := \hat{G}^k \hat{V}^\top$ and $H^k := \hat{H}^k \hat{V}^\top$. By the construction of $\hat{G}^k$ and $\hat{H}^k$, \[ \Theta^k \circ S(\hat{U}^T H^k \hat{V}^1) + \Theta^k \circ S(\hat{U}^T G^k \hat{V}^1) + \Sigma^k \circ X(\hat{U}^T H^k \hat{V}^1) + \Sigma^k \circ X(\hat{U}^T G^k \hat{V}^1) = 0, \] \hspace{1cm} (49)

$$(\hat{U}^T G^k \hat{V})_{\beta_+} \circ (E_{\beta_+} - (\Omega_{ij})_{ij} - 1)(\hat{G}^k)_{\beta_+} + (\Omega_{ij})_{ij} - 1)(\hat{G}^k)_{\beta_+} = 0, \] \hspace{1cm} (49)

$$\tilde{U}^T H^k \hat{V})_{\beta_+} = 0, \ \tilde{U}^T H^k \hat{V})_{\beta_+} = 0.$$

In addition, from equations (43) and (39b), for each $k$ we have that

$$(\hat{U}^T G^k \hat{V})_{\beta_+} = Q_{\beta_0} \hat{G}^k Q_{\beta_0} \leq 0 \quad \text{and} \quad (\hat{U}^T H^k \hat{V})_{\beta_+} = Q_{\beta_0} \hat{H}^k Q_{\beta_0} \geq 0.$$  \hspace{1cm} (50)

From (49)-(50) and Theorem 3.1, $(G^k, H^k) \in \mathcal{N}_{\gamma_0, L_\gamma}^\pi (X^k, Y^k)$ for each $k$. Since $(G, H)$ satisfies (24a)-(24b), from the construction of $(\hat{G}^k, \hat{H}^k)$ and equation (41)-(42), we have

$$\lim_{k \to \infty} \hat{U}^T \hat{G}^k \hat{V} = \lim_{k \to \infty} \hat{G}^k = \hat{U}^T \hat{G}^k \hat{V} \quad \text{and} \quad \lim_{k \to \infty} \hat{U}^T \hat{H}^k \hat{V} = \lim_{k \to \infty} \hat{H}^k = \hat{U}^T \hat{H}^k \hat{V}.$$

This implies that $\lim_{k \to \infty} (G^k, H^k) = (G, H)$. Together with $\lim_{k \to \infty} (X^k, Y^k) = (X, Y)$ and $(G^k, H^k) \in \mathcal{N}_{\gamma_0, L_\gamma}^\pi (X^k, Y^k)$, the converse conclusion follows. \qed