Aharonov–Bohm effect with many vortices

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Abstract
The Aharonov–Bohm (A–B) effect is the prime example of a zero-field-strength configuration where a nontrivial vector potential acquires physical significance, a typical quantum mechanical effect. We consider an extension of the traditional A–B problem, by studying a two-dimensional medium filled with many point-like vortices. Systems like this might be present within a type II superconducting layer in the presence of a strong magnetic field perpendicular to the layer, and have been studied in different limits. We construct an explicit solution for the wave function of a scalar particle moving within one such layer when the vortices occupy the sites of a square lattice and have all the same strength, equal to half of the flux quantum. From this construction, we infer some general characteristics of the spectrum, including the conclusion that such a flux array produces a repulsive barrier to an incident low-energy charged particle, so that the penetration probability decays exponentially with distance from the edge.

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1. Introduction
In classical mechanics it is said that the vector potential has no physical meaning. Due to gauge invariance, only the electromagnetic field tensor has physical (measurable) effects. In quantum mechanics, however, the vector potential appears in gauge-invariant quantities that describe a new class of effects. In these cases, corresponding to topologically nontrivial configurations, we recognize the importance of the vector potential, even when the electromagnetic field vanishes everywhere in the regions accessible to a charged particle.

The standard example of this class is the Aharonov–Bohm (A–B) effect \cite{1}, in which a magnetic field is confined to a region of space, and electrically charged particles are only free to move outside this region. Although a particle cannot experience the field strength directly, the covariant momentum

\[ D_\mu = \partial_\mu - ieA_\mu \] (1)

is affected by this configuration, that is, the vector potential \( A_\mu \) carries a ‘memory’ of the presence of the magnetic field even outside the region where the field is localized. In this way, the particle is influenced by the field, through a shift in the phase of the wave function

\[ \frac{e}{\hbar} \oint A \, dx = \frac{e}{\hbar} \oint B \, ds = \frac{e}{\hbar} \Phi, \] (2)

where \( \Phi \) is the total magnetic flux inside the circuit (i.e. a closed path of the particle). This explains why the effect is called ‘topological’: the behavior of the particle is sensitive to the overall configuration of the system, even though there is no classical magnetic force at any point.

The extension of the A–B problem in the presence of many localized fluxes cannot be tackled exactly in general. There exists a simple argument \cite{2} due to Aharonov which shows, using the Bloch theorem, that an infinite line of equispaced point-like fluxes would constitute an impenetrable barrier to a particle of sufficiently low energy. The particle would not be able to pass through such an array because it could not satisfy simultaneously on both sides of the barrier the Bloch periodicity conditions on its phase, in the light of the A–B effect.
We are interested in exploring a possibly more realistic set-up by studying the propagation of a charged particle through a medium filled with point-like fluxes.

Experimentally, one might find a situation similar to this inside a type II superconducting layer in the presence of a large magnetic field perpendicular to the layer. Quasiparticles in the layer would encounter numerous vortices, each containing a superconductor flux quantum, and under some conditions might not penetrate the vortices (see, for instance, [3]). In the fractional quantum Hall effect, the strong magnetic field piercing a two-dimensional (2D) system is considered to be localized in flux tubes similar to those in the superconducting scenario [4].

Configurations of this type have been addressed by several authors in recent years, especially in connection with the Hall problem. Some authors [5] have considered the approximation in which the motion of electrons is restricted to a 2D lattice, and each plaquette is characterized by a different magnetic flux. Others have considered a 2D electron gas (2DEG) in a random distribution of vortices, in different regimes of both spatial and strength distribution (see [6–13] for instance).

There are many interesting aspects. Already the mathematical structure of the quantum mechanical problem and the existence of solutions and zero modes has fascinated many authors [10–13]. While the scattering of particles on a single vortex has an exact solution [1], it becomes very complex already with two vortices [14]. For an arbitrary number of fluxes there is no general solution, with the possible exception of Nambu’s approach [15]. The existence of zero modes has been addressed in [13] for the spinful case of a Pauli Hamiltonian.

Here we want to concentrate on a spinless problem. Moreover, we will take the vortices to be pointlike. This sort of configuration has been addressed by Ouvry and co-workers in several papers [9]. After some analytical preliminaries, they resorted to numerical methods to compute the density of states of electrons in a medium filled with point-like fluxes. They found that for small to moderate flux strengths, the singularities are smothered and the 2DEG essentially sees an average effective magnetic field, which develops standard Landau levels, broadened by some disorder. In the strong field limit, the picture changes drastically, the topological nature of the problem cannot be avoided anymore, and one should observe a depletion of states for zero energy.

The former observation is consistent with the analysis of [8]. In this work, Kiers and Weiss investigated the validity of a mean-field treatment of the problem. The question is how accurately one can replace the fluxes with an extended magnetic field, while in fact the particles are never subject to a Lorentz force. They find that the unsurprising condition of validity for this approximation is that the flux strength has to be sufficiently small in units of a quantum of flux. In our work, we consider the opposite limit of the largest possible observable flux, a half-quantum.

Our aim is to consider a 2D layer, punctured by magnetic fluxes, and to study the wave function of a single scalar particle entering this medium. For simplicity, we take the vortices as point-like, so that the space available for the particles is a punctured plane. These are the same conditions Nambu applied in his work [15]. We shall consider a somewhat less general setting that allows us a more explicit solution, and then compare our results to his work.

We are going to show that a lattice of impenetrable magnetic fluxes (vortices), such as the one described above, constitutes a barrier to a low-energy charged particle trying to pass through the medium. That is, the distribution of the vortices creates a configuration whose topological constraints on the wave function are comparable to an effective repulsive potential. Qualitatively, there are a number of ways to see this:

- The presence of the fluxes generates a nonzero vector potential inside the medium, raising the minimum energy (that is the square of the covariant momentum, equation (2)) required for an electrically charged particle to exist in the medium.
- Particles are repelled by the vortices, as their wave functions must vanish on the vortex sites. Therefore, the larger the typical amplitude of the wave function in the flux-containing region, the bigger the energy due to the sharp spatial variation. This means that for low-energy states the wave function will not be able to reach a value appreciably different from zero in the presence of fluxes.
- The analysis of Nambu in [15] indicates that the medium constitutes a barrier even from the point of view of angular momentum. In his aforementioned paper, he argues that the angular momentum of a particle should be greater than the magnetic flux present in the medium if the particle wave function is to satisfy the boundary conditions. In other words, the lower angular momentum levels are missing and are not part of the spectrum.

These arguments are corroborated by the aforementioned numerical simulations [9] showing a Lifschitz tail in the density of states at low energies for a random distribution of vortices. From a physical point of view, it seems quite clear that a charged particle approaching the medium with sufficiently low energy will be repelled, that is, its penetration will be exponentially damped. In the same way, if we localize a particle in its ground state in a region without vortices, the particle will not be able to escape outside that region through one containing vortices except by tunneling, and we should be able to construct a bound state of topological character (actually a very long-lived resonance), even though there is no classical force. The fact that a bound state can be topological in nature, already heralded by Aharonov’s ‘wall’ [2], also was suggested in the work of Nambu [15].

2. Mathematical preliminaries

We concentrate on the case in which all the $N$ fluxes have equal strength $\Phi = \Phi_0/2$, where $\Phi_0 = 2\pi h e$ is the quantum unit of flux. In this case it can be shown (see, for instance, [16]) that the problem is invariant under time-reversal, meaning that there exists a gauge in which we can choose the wave functions to be real.

Indicating with $(x_i, y_i), i = 1, \ldots, N$, the coordinates of the vortices, we can write the vector potential in the standard
circular gauge as
\[
(A_x, A_y) = \Phi \left( \sum_{i=1}^{N} \frac{y - y_i}{(x - x_i)^2 + (y - y_i)^2} - \sum_{i=1}^{N} \frac{x - x_i}{(x - x_i)^2 + (y - y_i)^2} \right)
\]
\[
= \Phi \nabla \sum_{i=1}^{N} \tan^{-1} \left( \frac{y - y_i}{x - x_i} \right)
\]
\[
= i \Phi \nabla \sum_{j=1}^{N} \ln \left( \frac{(x - x_j) + i(y - y_j)}{(x - x_j) - i(y - y_j)} \right).
\]
\[
\nabla \times A = 2\pi \Phi \sum_{i=1}^{N} \delta^2 (x - x_i, y - y_i).
\]

The equation of motion for a particle in this medium is given by the Schrödinger equation (in units \(\hbar = m = 1\))
\[
\frac{1}{2m} (\nabla - iA)^2 \Psi + E \Psi = 0
\]
and, in these units, integer values of \(\Phi\) correspond to an unobservable, quantized flux (in our case \(\Phi = 1/2\), i.e. half a quantum of flux). Thus, \(\Psi\) is a single-valued wave function.

Following Nambu's idea \([15]\), we implement a singular gauge transformation \(G\) to remove the vector potential:
\[
\Psi = G \psi,
\]
\[
G = \prod_{j=1}^{N} \left( \frac{(x - x_j) - i(y - y_j)}{(x - x_j) + i(y - y_j)} \right)^{1/2}.
\]
In this way, we reduce our problem to a 'free-field' case
\[
-\frac{1}{2m} \nabla^2 \psi = E \psi,
\]
but now with nontrivial (topological) boundary conditions on the wave functions \(\psi\) in the region surrounding each vortex.

In constructing our solutions, we must require that the wave functions vanish on the vortex sites
\[
\psi(x = x_i, y = y_i) = 0, \quad i = 1, \ldots, N
\]
and that they acquire the A–B phase \(e^{2\pi \Phi} = -1\) each time a particle completes a turn around a vortex. More precisely stated, in this singular gauge the effect of the vector potential is represented by a phase-matching condition on the wave function
\[
\psi(\theta) = -\psi(\theta + 2\pi),
\]
where \(\theta\) is the azimuthal angle about the vortex.

We know from standard complex analysis that this condition implies the existence in the 2D plane of a cut connecting two distinguished Riemann sheets. For a real wave function this last condition implies that there exists at least one line exiting each vortex site on which the function has to vanish in order to change its sign. Thus it is the wave function \(\psi\) which may be chosen explicitly real.
Figure 2. The vortices are paired and connected by segments on which the wave function has to vanish in order to satisfy the topological conditions. The gray lines indicate the real periodicity of the lattice and identify the fundamental region over which we shall work.

paths going to spatial infinity), and therefore to exhibit a larger value for the integrated square of the gradient.

To construct the lowest energy solutions let us consider the vortices in pairs, connecting nearest neighbors with line segments along which the solution has to vanish. For definiteness, we connect fluxes in the horizontal direction, requiring the wave function to change sign when it crosses these segments (see figure 2).

Along these segments the wave function possesses odd parity. If we are interested in the low energy modes, this means that along the continuation of these segments, the function will be even and so its derivative must vanish there. To conclude our analysis on the boundary conditions, we notice that our system is clearly periodic. To ensure periodicity of the wave function, we require its derivative to vanish identically along the sides of each square centered on a flux (see figure 3).

Bearing these considerations in mind, we now have to solve a problem with mixed Dirichlet and Neumann boundary conditions. We can further reduce the system under study and concentrate on two of the quadrants around a flux site, because the rest of the lattice can be covered by mirroring and flipping this unit (figure 4).

In summary, we now have to solve the problem of a free particle in a rectangular box with sides of length 2 and 1 (in units of half of a lattice spacing). We impose Neumann boundary conditions everywhere, except on half of one of the long sides, where we require the Dirichlet boundary condition.

This is a non-standard problem; as we are not aware of any previous study on a system with these boundary conditions, we shall proceed in constructing the solution starting from a basis compatible with the conditions. In region I of figure 4, we identify a convenient basis in the set \( \{ \cosh[k_n(1+x)]\cos(n\pi y)\}_{n=0}^{\infty} \), while in region II, we expand the solution on \( \{ \cosh[K_\ell(1-x)]\sin[\ell + \frac{1}{2} \pi y]\}_{\ell=0}^{\infty} \) with the condition \( n^2\pi^2 - k_n^2 = (\ell + \frac{1}{2})^2\pi^2 - K_\ell^2 = 2mE \). Note that the reality of \( \psi \) implies that the \( k_n \) and \( K_\ell \) cannot be complex, and therefore must be either pure real or pure imaginary. We expect that the minimum possible value of \( E \) is positive, and this implies that, at least for \( n = 0, k_n \) must be imaginary (meaning a cosine function instead of a hyperbolic cosine). Note that the eigenvalue \( E \) is a constant, the same for all \( n \) and \( \ell \).

By matching the wave function and its derivative across the line \( x = 0 \), we may seek the values of \( E = 2mE \) for which the system admits a solution. In principle, this would involve the calculation of the determinant of an infinite matrix. To obtain an approximate solution, we truncated the system to a finite size, and found the first energy eigenvalue \( E_0 = 2mE_0 \).
and we find the first energy eigenvalue $\varepsilon_0 = 2mE_0$ corresponding to this shortened system. This is the plot of $N$ versus $\varepsilon_0$ and its fit with a polynomial in inverse powers of $N$ up to the third order (higher orders do not contribute appreciably).

Figure 5. We truncate the infinite-dimensional matrix to a size $N$ and we find the first energy eigenvalue $\varepsilon_0 = 2mE_0$ corresponding to this shortened system. This is the plot of $N$ versus $\varepsilon_0$ and its fit with a polynomial in inverse powers of $N$ up to the third order (higher orders do not contribute appreciably).

The continuous black line indicates where its derivative is zero (Neumann condition). We require periodicity on the vertical axis and exponential decay in the horizontal direction. The horizontal line at $\varepsilon_0 = 2mE_0$ shows that the spectrum for a particle in such a medium has to vanish. Depending on the direction of motion, the particle entering the medium with zero energy. This problem depends on the direction in which the particle is traveling, or at least ‘attempting’ to travel, in that it is connected with the choice of the ray/segment over which the solution has to vanish. Depending on the direction of motion, the wave function may ‘choose’ different configurations for these segments.

We solve the problem for a particle moving along the $x$-direction. That is, we construct a solution which exhibits periodic behavior in the $y$-direction and real decay in $x$ (figure 6). Again, we expand the wave function in appropriate bases: in region I and III of figure 6 we use $[e^{i\pi x}\cos(\pi y)]_{n=0}^\infty$ for right-moving and $[e^{-i\pi x}\cos(\pi y)]_{n=0}^\infty$ for left-moving modes. In region II, we expand on $\{e^{i(\ell+1/2)\pi x}\sin(\ell+1/2)\pi y]\}_{\ell=0}^\infty$ for right-moving and $\{e^{-(\ell+1/2)\pi x}\sin(\ell+1/2)\pi y]\}_{\ell=0}^\infty$ for left-moving modes.

We impose matching of the wave function and its derivative across the lines $x = -1$ and $x = 1$ and we write the damping of the solution by requiring an exponential suppression:

$$\psi(x = -2, y) = e^{4K}\psi(x = 2, y),$$

$$\frac{d\psi}{dx}(x = -2, y) = e^{4K}\frac{d\psi}{dx}(x = 2, y). \quad (12)$$

We look for the lowest value of $K$ for which the system admits a solution.

As before, the system of equations is infinite-dimensional, so we found the lowest value for $K$ as a function of the order $N$ of the matrix and performed a fit with inverse powers of $N$ to retain the zeroth order of the polynomial as the solution (see figure 7).

In this way, we find a decay factor for a particle moving along the horizontal direction:

$$K = (0.88 \pm 0.01) \times \frac{2}{L} = (1.76 \pm 0.02)L^{-1} \quad (13)$$

and the same $K$ holds for a particle moving in the vertical direction because we have the freedom to rotate the system by 90 degrees and rearrange the segments connecting the vortices in the new direction.

Because our choice of direction and of cut lines, however appealing, is not deduced from some fundamental principle, we should consider the minimum value of $K$ that we found as an upper bound on the lowest possible $K$.

4. Conclusions and outlook

Considering a lattice of point-like magnetic vortices, we showed that the spectrum for a particle in such a medium is discrete, and that, for a finite lattice spacing, the lowest energy eigenvalue is greater than zero, by explicitly constructing the wave function with energy $(E_0 = (2.0682 \pm 0.0002)m^{-1}L^{-2})$. 

Figure 6. Decay of the zero-energy solution moving horizontally. We require periodicity on the vertical axis and exponential decay in the horizontal direction. The continuous black line indicates where the wave function must vanish (Dirichlet condition) and the dashed lines where its derivative is zero (Neumann condition).

Figure 7. Decay of the zero-energy solution moving horizontally. We require periodicity on the vertical axis and exponential decay in the horizontal direction. The continuous black line indicates where the wave function must vanish (Dirichlet condition) and the dashed lines where its derivative is zero (Neumann condition).
This contrasts with what was predicted by Nambu in [15]. In his paper, the author argues that a solution of the Schrödinger equation in our gauge would have to be either holomorphic or anti-holomorphic. His argument goes as follows: let us switch to complex coordinates to describe the plane. The free particle equation now reads:

\[ \partial_z \partial_{\bar{z}} \psi = E \psi \]  

and therefore the solution for zero energy is either analytical or anti-analytical. Nambu argues that, by continuity, this property should persist at higher energies as well. However, in the preceding section we constructed a nonzero-energy solution which clearly is neither holomorphic, nor anti-holomorphic, nor a linear combination of the two.

The analyticity or anti-analyticity of the solutions is an important point of Nambu’s construction that leads him eventually to conclude that the states with lower angular momentum are not admissible in the spectrum. This would imply that a particle entering the medium with zero energy would undergo a suppression which is not merely exponential, but at least Gaussian. For that reason, we argue that our approximation comes closer to the true behavior, because by allowing more penetration it reduces uncertainty-principle energy. This statement applies even for zero energy, where Nambu’s argument appears rigorous at first sight from (14).

The loophole, we believe, is that for strong vortices not all of them have the wave function rotating in phase in the same direction (this reduces the net variation of the wave function, and clearly lowers the energy, which of course never can be less than zero). In other words, at some of the vortices the wave function is analytic, and at some it is anti-analytic. Therefore the wave function as a whole is neither analytic nor anti-analytic. This is especially clear at half-flux, the case we have taken: the reality of \( \Psi \) implies that close to each vortex the wave function is a linear combination of analytic and anti-analytic pieces, and so maximally violates Nambu’s either–or assumption.

We computed the decay factor for a zero-energy particle moving along one of the lattice directions to be \( K = (1.76 \pm 0.02) L^{-1} \), and showed that this decay is purely exponential. The magnitude of this suppression depends on the direction of travel. To compute the decay factor in other directions it would be necessary to modify \( ad \ hoc \) the boundary condition (the positioning of the ray where the wave function vanishes). The condition we worked with is the one that minimizes the extension of such rays and therefore seems to pose the minimal constraint on the solution. Any other choice would have a greater impact on the shape of the wave function and would change the effective decay length. The directional dependence is easy to understand, because the coupling between charge and vortex is strong, so that the lattice length scale and the decay length are comparable: in the limit of vanishing lattice constant the decay length also vanishes. A quantitative analysis for generic directions would require a different formalism from the one implemented here.

Nonetheless, we believe that the order of magnitude of the effect has been established, in that the lowest energy eigenvalue and the decay rate \( K \) for zero energy agree quite well, especially if one takes into account that the wave function still has a periodic variation along with the exponential decay. Such a solution is characterized by a real and an (orthogonal) imaginary wave-vector, equal in magnitude, to guarantee zero energy. It seems sensible that the real wave vector should be larger in magnitude than for the lowest-energy solution, because orthogonality to the imaginary vector is an extra constraint. The real decay rate \( K \) might be viewed as arising from an effective potential inside the medium. For instance, if we think about it from a WKB point of view we have

\[ K = \frac{1}{\hbar} \int \sqrt{V(x)} \, dx. \]  

This argument implies that the topological constraints imposed by the configuration of vortices act as an effective repulsive potential of order unity (in units with mass \( m = 1 \)). This potential is clearly not constant, and in principle its precise value can be calculated from a detailed knowledge of \( K \). It is more meaningful, however, to consider the average potential over a unit cell. As we just argued that \( K \) depends on the direction of travel of the particle, we see that this average effective potential is direction-dependent as well.

The simple expectation is that the lowest energy eigenvalue has to be equal to the average potential. However, the contribution of the relatively large orthogonal real vector mentioned above can make the imaginary wavenumber bigger than implied by equating the potential energy to the energy of the lowest solution. In our calculation, we found a good, but only approximate agreement. Different directions of travel would feel a different potential and, conceivably, generate a better agreement. The important result here is that the existence of an exponential decay, together with its magnitude, has been established and it can be interpreted as the effect of an average effective potential. Such a potential could be used to trap a particle in a region, just by surrounding that region with a medium of localized fluxes. Conceivably this could be a new form of trapping.

The above discussion may be related to a ‘generalized Bloch theorem’ which appears in various forms in the literature (for instance, see [18, 19]). The simplest version is that for a periodic potential the lowest positive energy also gives the imaginary wavenumbers of lower energy solutions, as if they were moving in a constant potential equal to the lowest positive energy. In our case, we find the imaginary wave vector at zero energy is bigger than this consideration would suggest. It is possible that some other ansatz would lower the wave vector magnitude, but for reasons discussed above we suspect that it still would be above the value that the naive generalized Bloch theorem would yield. This might mean that in the magnetic context there is a further generalization of the generalized Bloch theorem. This could be an interesting topic for further study.

In [15], Nambu argues that the proper description of the system would need to treat the vortices as dynamical objects themselves. Our formalism does not contemplate such an extension, and in the example of superconductor flux the inertia of the fluxes would be much greater than that of an electron. Thus the static-flux approximation makes physical sense. In [9], the authors consider a random distribution of fluxes, but they are not interested in calculating the single-particle lowest energy level. However, it seems...
plausible to us that the order of magnitude of the decay length and the qualitative characteristics of the problem would not be very different from the ones found with our model. Our reason for saying this is that one could replace the random vortex distribution with a random distribution of short line segments on which the wave function vanishes, and this array surely would be equivalent to a repulsive potential of characteristic magnitude, leading to exponential, not Gaussian decay.

The qualitative behavior we find is anyway in agreement with the analysis in [9], where it is established that for a flux strength of around $\Phi_c \sim 0.3–0.4$ a transition happens from a density of states in qualitative agreement with a Landau level picture to one characterized by a Lifschitz tail and a strong depletion of states at the bottom of the band, like the one we observe. In agreement with [8] as well, for $\Phi > \Phi_c$ a mean-field approximation fails, and the behavior of a particle in a medium of point-like vortices is completely different from the one we would observe if the particle moved through an extended average magnetic field. In fact, if we calculate the lowest Landau level for our system, as if the particle were actually subjected to a Lorenz force, we would find:

$$E_{\text{Landau}}^2 = \frac{\omega_c}{2} = \frac{B}{2L^2m} = 0.25 m^{-1} L^{-2}. \quad (16)$$

This is clearly a very different value from the one we found in our work (equation (11)), almost an order of magnitude smaller, showing that in our regime of large flux strength the mean field approximation is not valid.

In finishing, we notice that the picture changes drastically if one introduces spin into the problem. In fact, as shown in many works including [13] for a Pauli Hamiltonian system, in certain cases particles with magnetic moment parallel to the magnetic field could occupy zero-modes.

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References

[1] Aharonov Y and Bohm D 1959 Significance of electromagnetic potentials in the quantum theory Phys. Rev. 115 485
[2] Aharonov Y private communication unpublished
[3] Saint-James D, Sarma G and Thomas E J 1969 Type II Superconductivity (Oxford: Pergamon)
[4] Yoshioka D 2002 The Quantum Hall Effect (Berlin: Springer)
[5] Pryor C and Zee A 1992 Electron hopping in the presence of random flux Phys. Rev. B 46 3116
Lusakowski A and Turski L A 1993 Motion of a quantum particle in a random-flux field Phys. Rev. B 48 3835
[6] Gavazzi G, Wheatley J M and Schofield A J 1993 Single-particle motion in a random magnetic flux Phys. Rev. B 47 15170
[7] Brey L and Fertig H A 1993 Hall resistance of a two-dimensional electron gas in the presence of magnetic-flux tubes Phys. Rev. B 47 15961
Nielsen M and Hedegård P 1995 Two-dimensional electron transport in the presence of magnetic flux vortices Phys. Rev. B 51 7679
Reijniers J, Peeters F M and Matulis A 2001 Electron scattering on circular symmetric magnetic profiles in a two-dimensional electron gas Phys. Rev. B 64 245314
[8] Kiers K and Weiss N 1994 Scattering from a two-dimensional array of flux tubes: A study of the validity of mean field theory Phys. Rev. D 49 2081
[9] Ouvry A 1994 δ perturbative interactions in the Aharonov–Bohm and anyon models Phys. Rev. D 50 5296
Desbois J, Furtlehner C and Ouvry S 1995 Random magnetic impurities and the Landau problem Nucl. Phys. B 453 759
Desbois J, Furtlehner C and Ouvry S 1996 Random magnetic impurities and the delta-impurity problem J. Phys. I 6 641
Desbois J, Ouvry S and Texier C 1999 Hall conductivity in the presence of repulsive magnetic impurities Eur. Phys. J. B 7 527
[10] Ouvry S 2005 Random Aharonov–Bohm vortices and some exactly solvable families of integrals J. Stat. Mech. P09004
Mashkevich S and Ouvry S 2008 Random Aharonov–Bohm vortices and some exact families of integrals: Part II arXiv:0801.4818
[11] Borg J L and Pule J V 2004 Lifshits tails for random smooth magnetic vortices J. Math. Phys. 45 4493
[12] Mine T and Nomura Y 2006 Periodic Aharonov–Bohm solenoids in a constant magnetic field Rev. Math. Phys. 18 913
[13] Geyer V A and Šlovíček 2004 Zero modes in a system of Aharonov–Bohm fluxes Rev. Math. Phys. 16 851
Rozenblum G and Shirokov N 2006 Infiniteness of zero modes for the Pauli operator with singular magnetic field J. Funct. Anal. 233 135
[14] Mashkevich S, Myrheim J and Ouvry S 2004 Quantum mechanics of a particle with two magnetic impurities Phys. Lett. A 330 41
[15] Nambu Y 2000 The Aharonov–Bohm problem revisited Nucl. Phys. B 579 590
[16] Aharonov Y, Coleman S, Goldhaber A S, Nussinov S, Popescu S, Reznik B, Rohrlich D and Vaidman L 1994 AB and Berry phases for a quantum cloud of charge Phys. Rev. Lett. 73 918
[17] Guhevi F V, Stodolsky L and Zakkharov V I 2001 On the significance of the quantity $A^2$ Phys. Rev. Lett. 86 2220
[18] Karpeshina Y E 1983 Spectrum and eigenfunctions of the Schrödinger operator with the zero-range potential of the homogeneous two-dimensional lattice type in three-dimensional space Teor. Mat. Fiz. 57 414 (Engl. Transl.)
Karpeshina Y E 1983 Theor. Math. Phys. 57 1231
[19] Helffer B, Hoffmann-Ostenhof M, Hoffmann-Ostenhof T and Owen M P 1999 Nodal sets for the groundstate of the Schrödinger operator with zero magnetic field in a non simply connected domain Commun. Math. Phys. 202 629
Helffer B, Hoffmann-Ostenhof M, Hoffmann-Ostenhof T and Owen M 1999 Nodal sets, multiplicity and superstability in non simply connected domains Connectivity and Superstability (Springer Lecture Notes in Physics m62) ed J Berger and J Rubinstein p 63–86