ON $\ell$-ADIC GALOIS POLYLOGARITHMS
AND TRIPLE $\ell$-TH POWER RESIDUE SYMBOLS

DENSUKE SHIRAISHI

ABSTRACT. The $\ell$-adic Galois polylogarithm is an arithmetic function on an absolute Galois group with values in $\ell$-adic numbers, which arises from Galois actions on $\ell$-adic étale paths on $\mathbb{P}^1\setminus\{0,1,\infty\}$. In the present paper, we discuss a relationship between $\ell$-adic Galois polylogarithms and triple $\ell$-th power residue symbols in some special cases studied by a work of Hirano-Morishita [HM]. We show that a functional equation of $\ell$-adic Galois polylogarithms by Nakamura-Wojtkowiak [NW2] implies a reciprocity law of triple $\ell$-th power residue symbols.

INTRODUCTION

Let $K$ be a number field, $\mathcal{K}$ its algebraic closure in the complex number field $\mathbb{C}$. For a prime number $\ell$, let $\zeta_\ell := \exp\left(\frac{2\pi i}{\ell}\right)$ a primitive $\ell$-th root of unity in $\mathcal{K}$.

Choose a $K$-rational point $z$ of $\mathbb{P}^1_K\setminus\{0,1,\infty\}$. For any prime number $\ell$, the absolute Galois group $G_K := \text{Gal}(\mathcal{K}/K)$ acts on the $\ell$-adic étale path space $\pi^1_\ell(\mathbb{P}^1_K\setminus\{0,1,\infty\}; \overline{01}, \overline{z})$ where $\overline{01}$ is the standard $K$-rational tangential base point. In [Wo], for a fixed $\ell$-adic étale path $\gamma \in \pi^1_\ell(\mathbb{P}^1_K\setminus\{0,1,\infty\}; \overline{01}, \overline{z})$, Z. Wojtkowiak introduced an arithmetic function

$$\ell^\ell_n(z, \gamma) : G_K \to \mathbb{Q}_\ell$$

(for $n = 2, 3, 4, \ldots$) valued in the $\ell$-adic number field $\mathbb{Q}_\ell$, called the $n$-th $\ell$-adic Galois polylogarithm, defined as a certain coefficient in the $\ell$-adic Magnus expansion of the loop

$$f^\ell_n(\sigma) := \gamma \cdot \sigma(\gamma)^{-1} \in \pi^1_\ell(\mathbb{P}^1_K\setminus\{0,1,\infty\}; \overline{01}) \quad (\sigma \in G_K).$$

On the other hand, following the analogy between knots and primes, M. Morishita introduced the mod $\ell$ Milnor invariant $\mu_\ell(123) \in \mathbb{Z}/\ell\mathbb{Z}$ for certain prime ideals $p_1, p_2, p_3$ of $\mathbb{Q}(\zeta_\ell)$ for $\ell = 2, 3$, as an arithmetic analog of the Milnor invariant of links ([Mo], [AMM]). As a result, the triple $\ell$-th power residue symbol is defined by

$$[p_1, p_2, p_3]_\ell := \zeta_\ell^\mu_\ell(123),$$

which controls the decomposition law of $p_3$ in a certain nilpotent extension $R^{(\ell)}_{p_1, p_2}/\mathbb{Q}(\zeta_\ell)$.

In the present paper, we relate $[p_1, p_2, p_3]_\ell$ to $\ell^\ell_n(z, \gamma)$ for $\ell = 2, 3$ as follows:

Main formula (Naive form). For $\ell \in \{2, 3\}$, we have

\begin{equation}
[p_1, p_2, p_3]_\ell = \pm \zeta_\ell^{-\ell^\ell_n(z, \gamma)(\sigma)},
\end{equation}

where $K, z, \gamma, \sigma$ are suitably chosen to satisfy certain conditions depending on the triple of primes $\{p_1, p_2, p_3\}$. (See Theorem 2.3 for more details.)

Moreover, as a consequence of (0.0.1), we derive a reciprocity law of the triple symbol $[p_1, p_2, p_3]_\ell$ due to Rédei [Rc], Amano-Mizusawa-Morishita [AMM] in the form

\begin{equation}
[p_1, p_2, p_3]_\ell \cdot [p_2, p_1, p_3]_\ell = 1 \quad (\ell = 2, 3)
\end{equation}

2010 Mathematics Subject Classification. 14H30; 11G55, 11R32, 11R99.

Key words and phrases. fundamental group, polylogarithm, triple power residue symbol.
from a functional equation between $\ell_2^{(\ell)}(z, \gamma)$ and $\ell_2^{(\ell)}(1 - z, \gamma')$ due to Nakamura-Wojtkowiak [NW2]. (See Corollary 2.9 for details.) Thus, by using a functional equation of $\ell$-adic Galois polylogarithms, we have another proof of a reciprocity law of triple $\ell$-th power residue symbols. This fact is an indication that the Galois action mentioned at the beginning of this introduction has abundant arithmetic information.

**Acknowledgements**

This work is inspired by the work of Hirano-Morishita [HM] and is based on my Master’s thesis [Sh]. I would like to express my deepest gratitude to my supervisor, Professor Hiroaki Nakamura for his helpful advice and warm encouragement. I am deeply grateful to Professor Masanori Morishita for inviting me to the Workshop “Low dimensional topology and number theory XI” and for providing useful comments on improving the expression of main theorems in the present paper during the workshop. I also wish to express my thanks to Professor Yasushi Mizusawa who told me how to compute the triple symbol $[p_1, p_2, p_3]_3$ by using PARI/GP.

1. Preliminaries

1.1. $\ell$-adic Galois polylogarithms. In this section, we recall the definition and some properties of $\ell$-adic Galois polylogarithms. Fix any prime number $\ell$. Let $K$ be a sub-field of $\mathbb{C}$, $\overline{K}$ a fixed algebraic closure of $K$, and $G_K := \text{Gal}(\overline{K}/K)$ the absolute Galois group of $K$ with respect to $\overline{K}$. Fix an embedding $\overline{K} \hookrightarrow \mathbb{C}$. For any positive integer $m$, let $\zeta_m := \exp(\frac{2\pi i}{m})$ a fixed primitive $\ell^m$-th root of unity in $\overline{K}$. Let

$$X := \mathbb{P}^1_K \setminus \{0, 1, \infty\}$$

be a projective line minus 3 points over $K$, $X_{\overline{K}} := X \times_K \overline{K}$ the base change of $X \to \text{Spec } K$ via $K \hookrightarrow \overline{K}$, and $X^{\text{an}} = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ the complex analytic space associated to the base change of $X_{\overline{K}}$ via $\overline{K} \hookrightarrow \mathbb{C}$.

First, we choose a $K$-rational point

$$z \in X(K).$$

As appropriate, we regard $z$ as a point on $X^{\text{an}}$ by the fixed embeddings $K \hookrightarrow \overline{K}$ and $\overline{K} \hookrightarrow \mathbb{C}$. We denote by $\pi_1^{\text{top}}(X^{\text{an}}; \overline{01}, z)$ the set of homotopy classes of piece-wise smooth topological paths on $X^{\text{an}}$ from the unit tangent vector $\overline{01}$ to $z$; by $\pi_1^{\text{top}}(X^{\text{an}}, \overline{01})$ the topological fundamental group of $X^{\text{an}}$ with the base point $\overline{01}$. The group $\pi_1^{\text{top}}(X^{\text{an}}, \overline{01})$ is a free group of rank 2 generated by the homotopy classes of $\{l_0, l_1\}$ in the following Figure:

$$\pi_1^{\text{top}}(X^{\text{an}}, \overline{01}) = \langle l_0, l_1 \rangle.$$

Fix a homotopy class

$$\gamma \in \pi_1^{\text{top}}(X^{\text{an}}, \overline{01}).$$

Let $\tilde{z} : \text{Spec } \overline{K} \to X_{\overline{K}}$ be the base change of $z : \text{Spec } K \to X$ via $K \hookrightarrow \overline{K}$. We denote by $\pi_1^\ell(X_{\overline{K}}; \overline{01}, \tilde{z})$ the profinite set of $\ell$-adic étale paths on $X_{\overline{K}}$ from the standard $K$-rational tangential base point $\overline{01}$ to $\tilde{z}$; by $\pi_1^\ell(X_{\overline{K}}; *)$ the pro-$\ell$ étale fundamental group of $X_{\overline{K}}$ with base point $* \in \{\overline{01}, \tilde{z}\}$. By using the comparison maps induced by the fixed embedding $\overline{K} \hookrightarrow \mathbb{C}$, we regard homotopy classes $l_0, l_1 \in \pi_1^{\text{top}}(X^{\text{an}}, \overline{01})$, $\gamma \in \pi_1^{\text{top}}(X^{\text{an}}, \overline{01}, z)$ as $\ell$-adic étale paths

$$l_0, l_1 \in \pi_1^\ell(X_{\overline{K}}, \overline{01}), \gamma \in \pi_1^\ell(X_{\overline{K}}, \overline{01}, \tilde{z}).$$
Then \( \pi_1^f(X_{\overline{\mathbb{Q}}}, \overline{01}) \) is a free pro-\( \ell \) group topologically generated by \( \{l_0, l_1\} \):
\[
\pi_1^f(X_{\overline{\mathbb{Q}}}, \overline{01}) = \langle l_0, l_1 \rangle.
\]

Next, we focus on the Galois action
\[
G_K \to \text{Aut}(\pi_1^f(X_{\overline{\mathbb{Q}}}, \overline{01}))
\]
defined by \( \sigma(p) := s_{\overline{01}}(\sigma) \cdot p \cdot s_{\overline{01}}(\sigma)^{-1} \) for \( \sigma \in G_K \) and \( p \in \pi_1^f(X_{\overline{\mathbb{Q}}}, \overline{01}, \overline{z}) \), where \( s_{\overline{01}} : G_K \to \pi_1^f(X_{\overline{\mathbb{Q}}}, *) \) is a canonical homomorphism induced by a geometric point \( * \in \{\overline{01}, \overline{z}\} \) on \( X_{\overline{\mathbb{Q}}} \) and paths are composed from left to right. Consider the continuous 1-cocycle
\[
\pi_1^f : G_K \to \pi_1^f(X_{\overline{\mathbb{Q}}}, \overline{01})
\]
defined by \( \pi_1^f(\sigma) := \gamma \cdot \sigma(\gamma)^{-1} \in \pi_1^f(X_{\overline{\mathbb{Q}}}, \overline{01}) \). To understand clearly the behavior of \( \pi_1^f \), we use the \( \ell \)-adic Magnus embedding
\[
E : \pi_1^f(X_{\overline{\mathbb{Q}}}, \overline{01}) \rightarrow \mathbb{Q}_\ell \left( \langle e_0, e_1 \rangle \right)
\]
defined by \( E(l_0) = \exp(s_{e_0}) \), \( E(l_1) = \exp(s_{e_1}) \) where \( \mathbb{Q}_\ell \left( \langle e_0, e_1 \rangle \right) \) is the \( \mathbb{Q}_\ell \)-algebra of formal power series over \( \mathbb{Q}_\ell \) in two non-commuting variables \( e_0 \) and \( e_1 \). The constant term of \( E(\pi_1^f(\sigma)) \in \mathbb{Q}_\ell \left( \langle e_0, e_1 \rangle \right) \) is equal to 1 for any \( \sigma \in G_K \), so we can consider the Lie formal power series \( \log(E(\pi_1^f(\sigma)))^{-1} \in \text{Lie}(\langle e_0, e_1 \rangle) \subset \mathbb{Q}_\ell \langle e_0, e_1 \rangle \) where \( \text{Lie}(\langle e_0, e_1 \rangle) \) is the complete free Lie algebra generated by \( e_0 \) and \( e_1 \).

Now we shall introduce a certain function on \( G_K \) which quantifies the loop \( \pi_1^f(\sigma) \in \pi_1^f(X_{\overline{\mathbb{Q}}}, \overline{01}) \) (\( \sigma \in G_K \)) as a “polylogarithm” with values in \( \ell \)-adic numbers. To introduce it, we need the following preparations. Let \( \delta \in \pi_1^{\text{top}}(X^{an}, \overline{01}, \overline{10}) \) be a homotopy class of the canonical path on \( X^{an} \) as in the above Figure, and
\[
\gamma' := \delta \cdot \varphi(\gamma) \in \pi_1^{\text{top}}(X^{an}, \overline{01}, 1 - z),
\]
where \( \varphi \in \text{Aut}(X^{an}) \) given by \( \varphi(\ast) = 1 - \ast \). We will choose \( z^{1/n}, (1 - z)^{1/n}, (1 - z^n)^{1/m} \) \((n, m \in \mathbb{N}, a \in \mathbb{Z})\) as the specific \( n \)-th power roots determined by \( \gamma \in \pi_1^{\text{top}}(X^{an}, \overline{01}, z) \) (See [NW1] for details). Let
\[
\rho_\delta \gamma' : G_K \rightarrow \mathbb{Z}_\ell, \quad \rho_{1 - z, \gamma'} : G_K \rightarrow \mathbb{Z}_\ell
\]
be the Kummer 1-cocycle along \( \gamma \) (resp. \( \gamma' \)) defined by \( \sigma(z^{1/n}) = \zeta_{\ell^n}^{\rho_\delta \gamma'(\sigma)} z^{1/n} \) (resp.
\[
\sigma((1 - z)^{1/m}) = \zeta_{\ell^n}^{\rho_{1 - z, \gamma'}(\sigma)} (1 - z)^{1/m} \]
for \( \sigma \in G_K \). Denote by \( \chi : G_K \rightarrow \mathbb{Z}_\ell^\times \) the \( \ell \)-adic cyclotomic character defined by \( \sigma(\zeta_{\ell^n}) = \zeta_{\ell^n}^{\chi(\sigma)} \) for \( \sigma \in G_K \).
Definition 1 ($\ell$-adic Galois polylogarithm function; [NW1], [Wo; §11]). We define a function $\ell_i(z, \gamma) : G_K \to \mathbb{Q}_\ell$ ($n \geq 2$) as a coefficient of $\text{ad}(e_0)^{n-1}(e_1)$ in the following Lie expression of $\log(E(\ell_i^{\ell}(\sigma)))^{-1}$ for any $\sigma \in G_K$:

$$\log(E(\ell_i^{\ell}(\sigma)))^{-1} \equiv \rho_{z, \gamma}(\sigma)e_0 + \rho_{1-z, -\gamma}(\sigma)e_1 + \sum_{n=2}^{\infty} \ell_i^{\ell}(z, \gamma)(\sigma)\text{ad}(e_0)^{n-1}(e_1) \mod I_{e_1},$$

where $I_{e_1}$ denotes the ideal generated by Lie monomials involving $e_1$ at least twice. This function $\ell_i^{\ell}(z, \gamma) : G_K \to \mathbb{Q}_\ell$ ($n \geq 2$) is called the $n$-th $\ell$-adic Galois polylogarithm function. We shall also define $\ell_i^{\ell}(z, \gamma) := \rho_{z, \gamma}$, $\ell_i^{\ell}(z, \gamma) := \rho_{1-z, -\gamma}$.

Here we shall introduce a certain character on $G_K$ which generalizes the so-called Soulé character.

Definition 2 ($\ell$-adic Galois polylogarithmic character; [NW1]). For any integer $m \geq 1$, we define $\tilde{\chi}^{z, \gamma}_m : G_K \to \mathbb{Z}_\ell$ by the following Kummer properties:

$$\zeta^{\tilde{\chi}^{z, \gamma}_m} = \sigma \left( \prod_{i=0}^{n-1} (1 - \zeta^{\ell_i}z^{1/\ell_i})^{-1/m} \right) / \left( \prod_{i=0}^{n-1} (1 - \zeta^{\ell_i + \rho_{z, \gamma}}z^{1/\ell_i})^{-1/m} \right) (n \geq 1).$$

This function $\tilde{\chi}^{z, \gamma}_m : G_K \to \mathbb{Z}_\ell$ ($m \geq 1$) valued in the ring $\mathbb{Z}_\ell$ of $\ell$-adic integers, is called the $m$-th $\ell$-adic Galois polylogarithmic character associated to $\gamma \in \pi_1^{\text{top}}(X_\text{an}; \overline{\mathbb{O}}, z)$.

In fact, $\ell$-adic Galois polylogarithmic characters describe values of the $\ell$-adic Galois polylogarithm function.

Theorem 1.1 (Explicit formula; [NW1; Corollary]). For each $\sigma \in G_K$, the quantity $\ell_i^{\ell}(z, \gamma)(\sigma)$ is explicitly described by $\ell$-adic Galois polylogarithmic characters as follows:

$$\ell_i^{\ell}(z, \gamma)(\sigma) = (-1)^{n+1} \sum_{k=0}^{n-1} \frac{B_k}{k!} (-\rho_{z, \gamma}(\sigma))^k \tilde{\chi}^{z, \gamma}_{n-k}(\sigma) (n \geq 1),$$

where $B_k$ denotes the $k$-th Bernoulli number.

One reason for the name “$\ell$-adic Galois polylogarithm” is that $\ell$-adic Galois polylogarithm functions/polylogarithmic characters satisfy some typical functional equations analogous to functional equations of the classical polylogarithm [NW2; Chapter 6]. The following functional equation is one example of them.

Theorem 1.2 (a functional equation; [NW2; Chapter 6, (6.14)]). The 2nd $\ell$-adic Galois polylogarithm function holds the following functional equation. For any $\sigma \in G_K$,

$$\ell_{2, \ell}^{\ell}(z, \gamma)(\sigma) + \ell_{2, \ell}^{\ell}(1-z, -\gamma')(\sigma) = \ell_{2, \ell}^{\ell}(\overline{\mathbb{O}}, \delta)(\sigma).$$

By Theorem 1.1, this equation is equivalent to the following functional equation of the 2nd $\ell$-adic Galois polylogarithmic character. For any $\sigma \in G_K$,

$$\tilde{\chi}^{2, \gamma}(\sigma) + \tilde{\chi}^{1-z, -\gamma'}(\sigma) + \rho_{z, \gamma}(\sigma)\rho_{1-z, -\gamma'}(\sigma) = \frac{1}{24}(\chi(\sigma)^2 - 1).$$
Remark 3. The latter functional equation in Theorem 1.2 is an \( \ell \)-adic Galois analog of the functional equation
\[
Li_2(z) + Li_2(1 - z) + \log(z)\log(1 - z) = \frac{\pi^2}{6},
\]
where \( Li_2(z) \) denotes the classical dilogarithm function.

1.2. Triple \( \ell \)-th power residue symbols for \( \ell = 2, 3 \). The triple \( \ell \)-th power residue symbol is defined at present for \( \ell = 2, 3 \) in [Mo], [AMM]. In this section, following [HM; Section 4], [Mo], [AMM], we recall the definition and some properties of triple \( \ell \)-th power residue symbols for \( \ell = 2, 3 \).

1.2.1. Case of \( \ell = 2 \). Let \( p_1, p_2 \) be distinct prime numbers which satisfy
\[
(1.2.1) \quad p_i \equiv 1 \mod 4 \quad (i = 1, 2), \quad \left( \frac{p_i}{p_j} \right) = 1 \quad (1 \leq i \neq j \leq 2).
\]
By (1.2.1), there exist integers \( x, y, w \) satisfying the following conditions [Am; Lemma 1.1]:
\[
(1.2.2) \quad x^2 - p_1 y^2 - p_2 w^2 = 0, \quad \gcd(x, y, w) = 1, \quad y \equiv 0 \mod 2, \quad x - y \equiv 1 \mod 4.
\]
Note that the triple \( (x, y, w) \) is not unique. For such a pair \( (x, y) \), we let
\[
(1.2.3) \quad \theta^{(2)}_{p_1, p_2} := x + \sqrt{p_1} y.
\]
Moreover, we set
\[
(1.2.4) \quad R^{(2)} = R^{(2)}_{p_1, p_2} := \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{\theta^{(2)}_{p_1, p_2}}) \subset \mathbb{C},
\]
\[
(1.2.5) \quad K^{(2)} = K^{(2)}_{p_1, p_2} := \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}).
\]

Theorem 1.3 ([Am; Theorem 1.2, Corollary 1.5]). The field \( R^{(2)} \) is a finite Galois extension of \( \mathbb{Q} \) in \( \mathbb{C} \) which satisfies the following properties:
(i) The Galois group \( \text{Gal}(R^{(2)}/\mathbb{Q}) \) is isomorphic to the Heisenberg group
\[
H_3(\mathbb{Z}/2\mathbb{Z}) := \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \mid * \in \mathbb{Z}/2\mathbb{Z} \right\}
\]
(Note that this group \( H_3(\mathbb{Z}/2\mathbb{Z}) \) is isomorphic to the dihedral group \( D_3 \) of order 8);
(ii) Prime numbers ramified in \( R^{(2)}/\mathbb{Q} \) are only \( p_1, p_2 \) with ramification index 2;
(iii) The field \( R^{(2)} \) is independent of the choice of the triple \( (x, y, w) \). Hence, \( R^{(2)}/\mathbb{Q} \) depends only on the pair \( \{p_1, p_2\} \).

Theorem 1.4 (An arithmetic characterization of \( R^{(2)} \); [Am; Theorem 2.1]). Let \( p_1, p_2 \) be distinct prime numbers satisfying (1.2.1). For a number field \( L \subset \mathbb{C} \), the following conditions are equivalent:
(1) \( L \) is the field \( R^{(2)} \);
(2) \( L/\mathbb{Q} \) is a Galois extension in which only prime numbers \( p_1, p_2 \) are ramified with ramification index 2 and whose Galois group is isomorphic to the Heisenberg group \( H_3(\mathbb{Z}/2\mathbb{Z}) \).

Here we take another prime number \( p_3 \) satisfying
\[
(1.2.6) \quad p_3 \equiv 1 \mod 4, \quad \left( \frac{p_i}{p_j} \right) = 1 \quad (1 \leq i \neq j \leq 3).
\]
Note that $p_3$ is unramified in $R^{(2)}/\mathbb{Q}$ by Theorem 1.3 (ii). Then we introduce an arithmetic symbol which controls the decomposition of $p_3$ in the nilpotent extension $R^{(2)}/\mathbb{Q}$.

**Definition 4** (Triple quadratic residue symbol; [Mo; Section 8.4]). For prime ideals $(p_1), (p_2), (p_3)$ of $\mathbb{Z}$ where prime numbers $p_1, p_2, p_3$ satisfy (1.2.1) and (1.2.6), the triple quadratic residue symbol is defined by

$$[(p_1), (p_2), (p_3)]_2 := (-1)^{\mu_2(123)} \in \{1, -1\},$$

where $\mu_2(123) \in \mathbb{Z}/2\mathbb{Z}$ is the mod 2 Milnor invariant for the prime numbers $p_1, p_2, p_3$. See [Mo; Section 8.4] for the detailed account of $\mu_2(123)$.

Let $\tilde{p}_i$ be a prime ideal of $K^{(2)}$ above $p_i$. By (1.2.1) and (1.2.6), the primes $p_1, p_2, p_3$ are completely decomposed or ramified in $K^{(2)}/\mathbb{Q}$ as follows.

$$
\begin{array}{ccc}
R^{(2)} &=& K^{(2)}(\sqrt{\theta_{p_1,p_2}^{(2)}}) \\
K^{(2)} &=& \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}) \\
\mathbb{Q}(\sqrt{p_1}) &=& \tilde{p}_1 \cap \mathbb{Q}(\sqrt{p_1}) \quad \text{completely decomposed} \\
\mathbb{Q}(\sqrt{p_1}) &=& \tilde{p}_2 \cap \mathbb{Q}(\sqrt{p_1}) \quad \text{completely decomposed} \\
\mathbb{Q}(\sqrt{p_1}) &=& \tilde{p}_3 \cap \mathbb{Q}(\sqrt{p_1}) \quad \text{completely decomposed} \\
\mathbb{Q} &=& p_1 \quad \text{ramified} \\
\mathbb{Q} &=& p_2 \quad \text{completely decomposed} \\
\mathbb{Q} &=& p_3 \quad \text{completely decomposed}
\end{array}
$$

**Theorem 1.5** ([Mo; Section 8.4, Theorem 8.25]). Let $\sigma_{p_3} := \text{Frob}_{\tilde{p}_3} \in \text{Gal}(R^{(2)}/K^{(2)})$ be the Frobenius substitution of $\tilde{p}_3$ in $R^{(2)}/K^{(2)}$. Then we have

$$[(p_1), (p_2), (p_3)]_2 = \frac{\sigma_{p_3}(\sqrt{\theta_{p_1,p_2}^{(2)}})}{\sqrt{\theta_{p_1,p_2}^{(2)}}}.$$

In particular, $[(p_1), (p_2), (p_3)]_2 = 1$ if and only if $p_3$ is completely decomposed in $R^{(2)}/\mathbb{Q}$.

**Remark 1.6.** The right side of the equation in Theorem 1.5 is the Rédei symbol introduced by L. Rédei in [Ré]:

$$[p_1, p_2, p_3]_{\text{Rédei}} := \frac{\sigma_{p_3}(\sqrt{\theta_{p_1,p_2}^{(2)}})}{\sqrt{\theta_{p_1,p_2}^{(2)}}}.$$

That is, Theorem 1.5 means the triple quadratic residue symbol is equal to the Rédei symbol:

$$[(p_1), (p_2), (p_3)]_2 = [p_1, p_2, p_3]_{\text{Rédei}}.$$

In [Ré], L. Rédei proved the following reciprocity law of the triple symbol. In [Am], F. Aamano gave another simple proof of it.
Theorem 1.7 (Reciprocity law of triple quadratic residue symbols; [Rè], [Am]). Let \( \rho \in S_3 \) be any permutation of the set \( \{1, 2, 3\} \). Then

\[
[p_1], (p_2), (p_3)]_2 \cdot [(p_{\rho(1)}), (p_{\rho(2)}), (p_{\rho(3)})]_2 = 1,
\]
that is \( [(p_1), (p_2), (p_3)]_2 = [(p_{\rho(1)}), (p_{\rho(2)}), (p_{\rho(3)})]_2 \).

1.2.2. Case of \( \ell = 3 \). In this section, we essentially follow [HM, Section 4.2] for various assumptions. Let \( k := \mathbb{Q}((\sqrt{-3}) \) be the Eisenstein field where \( \zeta_3 := \exp(\frac{2\pi \sqrt{-3}}{3}) = -1 + \sqrt{3} \). Let \( p_i = (p_i) (i = 1, 2) \) be distinct prime ideals of \( k \) which satisfy

\[
Np_i \equiv 1 \mod 9 \quad (\text{for } i = 1, 2), \quad \left(\frac{p_i}{p_j}\right)_3 = 1 \quad (1 \leq i \neq j \leq 2).
\]

Following [AMM, Corollary 5.9], [HM, Section 4.2], we assume that

\[
each p_i \text{ is an associate of a rational prime number in } k.
\]

There is an ambiguity of the choice of \( p_i \) up to units \( \mathbb{Z}[\zeta_3]^\times = \{\pm \zeta_3^m | m = 0, 1, 2\} \), but we can take it uniquely by posing the following condition (cf. [AMM; Lemma 1.1]):

\[
p_i \equiv 1 \mod (3\sqrt{-3}).
\]

We fix such a prime element \( p_i \in \mathbb{Z}[\zeta_3] \). We set \( K_1 := k(\sqrt[p_i]{3}) \). The field \( K_1 \) is a cyclic extension of degree 3 over \( k \) in which only \( p_i \) is ramified (cf. [AMM; Theorem 3.5]). Let \( \phi \) be a generator of \( \text{Gal}(K_1/k) \) determined by \( \phi(\sqrt[p_i]{3}) = \zeta_3 \sqrt[p_i]{3} \). By (1.2.7) and (1.2.9), there exist algebraic integers

\[
(1.2.10)
\alpha_{p_1, p_2} \in \mathcal{O}_{K_1}, \quad w \in \mathbb{Z}[\zeta_3]
\]
together with prime ideals \( \mathfrak{p}, \mathfrak{b} \) of \( K_1 \) which satisfy the following conditions [AMM; Proposition 5.6]:

\[
(1.2.11)
N_{K_1/k}(\alpha_{p_1, p_2}) = p_2 w^3,
\]

\[
(\alpha_{p_1, p_2}) = \mathfrak{p}^e \mathfrak{b}^f, \quad (e, 3) = 1, \ (\mathfrak{b}, 3) = 1, \ f \equiv 0 \mod 3.
\]

Note that \( \alpha_{p_1, p_2} \) is not unique. For such an \( \alpha_{p_1, p_2} \in \mathcal{O}_{K_1} \), we let

\[
(1.2.12)
\theta_{p_1, p_2} := \phi(\alpha_{p_1, p_2})(\phi^2(\alpha_{p_1, p_2}))^2.
\]

Moreover, we set

\[
(1.2.13)
R^{(3)} := R^{(3)}_{p_1, p_2} := k(\sqrt[p_i]{3}, \sqrt[p_i]{2}, \sqrt[p_i]{3^{(3)}}) \subset \mathbb{C},
\]

\[
(1.2.14)
K^{(3)} := K^{(3)}_{p_1, p_2} := k(\sqrt[p_i]{3}, \sqrt[p_i]{2}).
\]

By using the assumption (1.2.8), we obtain the following theorem.

Theorem 1.8 ([AMM; Theorem 5.11, Corollary 5.12]). The field \( R^{(3)} \) is a finite Galois extension of \( k \) in \( \mathbb{C} \) which holds the following properties:

(i) The Galois group \( \text{Gal}(R^{(3)}/k) \) is isomorphic to the Heisenberg group

\[
H_3(\mathbb{Z}/3\mathbb{Z}) := \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} | * \in \mathbb{Z}/3\mathbb{Z} \right\};
\]

(ii) Prime ideals ramified in \( R^{(3)}/k \) are only \( p_1, p_2 \) with ramification index 3;

(iii) The field \( R^{(3)} \) is independent of the choice of \( \alpha_{p_1, p_2} \in \mathcal{O}_{K_1} \). Hence, \( R^{(3)}/k \) depends only on the pair \( \{p_1, p_2\} \).
Theorem 1.9 (An arithmetic characterization of $R^{(3)}$; cf. [AMM; Corollary 5.12]). Let $p_1 = (p_1), p_2 = (p_2)$ be distinct prime ideals of $k$ satisfying (1.2.7), (1.2.8), (1.2.9) and (1.2.15). For a finite extension field $L$ of $k$ in $C$, the following conditions are equivalent:

1. $L$ is the field $R^{(3)}$;
2. $L/k$ is a Galois extension in which only primes $p_1, p_2$ are ramified with ramification index 3 and whose Galois group is isomorphic to the Heisenberg group $H_3(\mathbb{Z}/3\mathbb{Z})$.

Here we take another prime ideal $p_3 = (p_3)$ of $k$ satisfying (1.2.15) $N_{p_3} \equiv 1 \mod 9$,

\[(p_i/p_j)_3 = 1 \quad (1 \leq i \neq j \leq 3).\]

Note that $p_3$ is unramified in $R^{(3)}/k$ by Theorem 1.8 (ii). Then we introduce an arithmetic symbol which controls the decomposition of $p_3$ in the nilpotent extension $R^{(3)}/k$.

Definition 5 (Triple cubic residue symbol; [AMM; Definition 6.2]). For a triple of primes $(p_1, p_2, p_3)$ of $k$ satisfying (1.2.7) and (1.2.15), the triple cubic residue symbol is defined by

$$[p_1, p_2, p_3]_3 := \zeta_3^\mu_3(123) \in \{1, \zeta_3, \zeta_3^{-1}\},$$

where $\mu_3(123) \in \mathbb{Z}/3\mathbb{Z}$ is the mod 3 Milnor invariant for the primes $p_1, p_2, p_3$. See [AMM; (2.3) of Chapter 2, Theorem 4.4] for the detailed account of $\mu_3(123)$.

Let $\tilde{p}_i$ be a prime ideal of $K^{(3)}$ above $p_i$. By (1.2.7) and (1.2.15), the primes $p_1, p_2, p_3$ are completely decomposed or ramified in $K^{(3)}/k$ as follows.

\[
\begin{array}{c}
R^{(3)} = K^{(3)}(\sqrt[3]{\theta_{p_1, p_2}^{(3)}}) \\
\bigg| \\
K^{(2)} = k(\sqrt[3]{p_1}, \sqrt[3]{p_2}) \\
\bigg| \\
K_1 = k(\sqrt[3]{p_1}) \\
\bigg| \\
k = \mathbb{Q}(\zeta_3) \\
\end{array}
\]

\[
\begin{array}{c}
\tilde{p}_1 \\
\tilde{p}_2 \\
\tilde{p}_3 \\
\tilde{p}_1 \cap K_1 \\
\tilde{p}_2 \cap K_1 \\
p_1 \\
p_2 \\
p_3 \\
\end{array}
\]

completely decomposed

completely decomposed

ramified

completely decomposed

Theorem 1.10 ([AMM; Theorem 6.3]). Let $\sigma_{p_3} := \text{Frob}_{p_3} \in \text{Gal}(R^{(3)}/K^{(3)})$ be the Frobenius substitution of $\tilde{p}_3$ in $R^{(3)}/K^{(3)}$. Then we have

$$[p_1, p_2, p_3]_3 = \frac{\sigma_{p_3}(\sqrt{\theta_{p_1, p_2}^{(3)}})}{\sqrt[3]{\theta_{p_1, p_2}^{(3)}}}.$$ In particular, $[p_1, p_2, p_3]_3 = 1$ if and only if $p_3$ is completely decomposed in $R^{(3)}/k$. 

Theorem 1.11 (a reciprocity law of triple cubic residue symbols; [AMM; Proposition 6.5]). We have
\[ [p_1, p_2, p_3] \cdot [p_2, p_1, p_3] = 1, \]
that is \([p_2, p_1, p_3] = [p_1, p_2, p_3]^{-1}\).

Hereafter, following Hirano-Morishita [HM, Section 4.2], we shall restrict ourselves to the case with Assumption (A): The pair \((p_1, p_2)\) has \(\alpha_{p_1, p_2} \in \mathcal{O}_K\) in (1.2.10) and (1.2.11) which can be given in the form \(\alpha_{p_1, p_2} = x + y\sqrt[p_1]{1} \quad (x, y \in k)\).

Under the above assumption (A), the conditions (1.2.11), (1.2.12) are equivalent to
\[(1.2.16) \quad x^3 + p_1 y^3 = p_2 w^3, \]
\[(1.2.17) \quad \theta^{(3)}_{p_1, p_2} = (x + \zeta_3 y^{\sqrt[p_1]{1}})(x + \zeta_3^2 y^{\sqrt[p_1]{1}})^2. \]
These equations will play an important role in the next section.

2. Triple \(\ell\)-th power residue symbols and \(\ell\)-adic Galois polylogarithms

In this section, we interpret triple \(\ell\)-th power residue symbols in terms of \(\ell\)-adic Galois polylogarithms for \(\ell = 2, 3\). As a result, we derive a reciprocity law of triple \(\ell\)-th power residue symbols from a functional equation of \(\ell\)-adic Galois polylogarithms.

2.1. Main formula. Let \(\ell \in \{2, 3\}\). Let \(k := \begin{cases} \mathbb{Q} & \text{(if } \ell = 2\text{),} \\ \mathbb{Q}(\zeta_3) & \text{(if } \ell = 3\text{)} \end{cases}\) and
\[(2.1.1) \quad K := \mathbb{Q}(\zeta_\ell)(\sqrt[p_1]{1}, \sqrt[p_2]{2}), \]
where \(\zeta_\ell = \exp(\frac{2\pi i}{\ell})\) is a fixed primitive \(\ell\)-th root of unity in \(\overline{K} \subset \mathbb{C}\). We set \(p_i \in \mathbb{Z}[\zeta_\ell] \quad (i = 1, 2, 3)\), \(x, y, w \in k\),
\[(2.1.2) \quad x^\ell - (-y)^\ell p_1 = w^\ell p_2, \]
\[(2.1.3) \quad \theta^{(\ell)}_{p_1, p_2} = \prod_{i=0}^{\ell-1} (x + \zeta_{\ell}^i y^{\sqrt[p_1]{1}})^i. \]
For the prime element \(p_i \in \mathbb{Z}[\zeta_\ell] \quad (i = 1, 2, 3)\), we denote by
\[p_i = (p_i)\]
the prime ideal of \(k\) generated by \(p_i\). For the triple \((p_1, p_2, p_3)\) of primes of \(k\), the triple \(\ell\)-th power residue symbol \([p_1, p_2, p_3]_\ell\) is defined as discussed in Section 1.2.

Moreover, we choose
\[(2.1.4) \quad z := p_1 \left( \frac{-y}{x} \right)^\ell. \]
Since \(z \in K \setminus \{0, 1\}\), we regard \(z\) as a \(K\)-rational point of \(\mathbb{P}^1_K \setminus \{0, 1, \infty\}\). Let
\[(2.1.5) \quad \delta_{p_3} \in \text{Gal}(\overline{K}/K)\]
be an extension of the Frobenius substitution \( \sigma_{\mathfrak{p}_3} := \text{Frob}_{\mathfrak{p}_3} \in \text{Gal}(R_{\mathfrak{p}_3}/K) \) where \( \mathfrak{p}_3 \) is a prime ideal of \( K \) above \( \mathfrak{p}_3 \). Let \( \tilde{z} : \text{Spec } K \to \mathbb{P}_K^1 \setminus \{0, 1, \infty\} \) be the base change of \( z \) via Spec \( K \to \text{Spec } K \). Fix a homotopy class 
\[
\gamma \in \pi_1^{\text{top}}(\mathbb{P}_1(C) \setminus \{0, 1, \infty\}; \overrightarrow{01}, z)
\]
of a piece-wise smooth topological path on \( \mathbb{P}_1(C) \setminus \{0, 1, \infty\} \) from \( \overrightarrow{01} \) to \( z \). Then, the 2nd \( \ell \)-adic Galois polylogarithms \( \ell_2^\gamma(z, \gamma) : G_K \to \mathbb{Q}_\ell, \overline{\chi}_2^\gamma : G_K \to \mathbb{Z}_\ell \) are defined as discussed in Section 1.1.

**Proposition 2.1.** Let the notations and assumptions be as above. For any \( \tau \in G_K \), the value \( \ell_2^\gamma(z, \gamma)(\tau) \) mod \( \ell \), together with \( \overline{\chi}_2^\gamma(\tau) \) mod \( \ell \), is independent of the choice of \( \gamma \).

**Proof.** Let \( \tau \in G_K \). By (2.1.1), we have \( \chi(\tau) \equiv 1, \rho_{z, \gamma}(\tau) \equiv 0 \) mod \( \ell \). Hence, it follows from Definition 2 that
\[
(2.1.6) \quad \chi_2^\gamma(\tau) = \tau \left( \prod_{i=0}^{\ell-1} (1 - \zeta_\ell^i z^{1/\ell})^{1/\ell} \right) / \prod_{i=0}^{\ell-1} (1 - \zeta_\ell^i z^{1/\ell})^{\frac{1}{\ell}}.
\]

Let \( \gamma_0, \gamma_1 \in \pi_1^{\text{top}}(\mathbb{P}_1(C) \setminus \{0, 1, \infty\}; \overrightarrow{01}, z) \). For \( \epsilon \in \{0, 1\} \), we choose \( z_{\epsilon}^{1/\ell}, (1 - \zeta_\ell^{i-\epsilon} z_{\epsilon}^{1/\ell}) \) as the specific \( \ell \)-th power roots determined by \( \gamma_\epsilon \) (cf. [NW1]).

First, in order to prove that \( \overline{\chi}_2^\gamma(\tau) \) mod \( \ell \) is independent of the choice of \( \gamma \), it suffices to show
\[
(2.1.7) \quad \chi_2^{\gamma_0}(\tau) = \chi_2^{\gamma_1}(\tau)
\]
by comparing the right hand side of (2.1.6) for \( \gamma = \gamma_0, \gamma_1 \). We now show (2.1.7). Let
\[
A_\epsilon := \prod_{i=0}^{\ell-1} (1 - \zeta_\ell^i z_{\epsilon}^{1/\ell})^{\frac{1}{\ell}} (\epsilon \in \{0, 1\}).
\]

Take \( s \in \mathbb{Z}/\ell \mathbb{Z} \) such that \( z_{1/\ell}^1 = \zeta_{\ell}^{-s} \cdot z_{0}^{1/\ell} \). For each \( i \in \mathbb{Z}/\ell \mathbb{Z} \), there exists \( t_i \in \mathbb{Z}/\ell \mathbb{Z} \) such that \( (1 - \zeta_\ell^i z_{1/\ell}^{1/\ell})^{1/\ell} = \zeta_\ell^{-s} (1 - \zeta_\ell^{-i+s} z_{0}^{1/\ell})^{1/\ell} \) since \( (1 - \zeta_\ell^{i-1} z_{1/\ell}^{1/\ell}) = (1 - \zeta_\ell^{-i+s} z_{0}^{1/\ell}) \). Then, we compute \( A_1/A_0 \) as follows:
\[
\frac{A_1}{A_0} = \frac{\prod_{i=0}^{\ell-1} (1 - \zeta_\ell^i z_{0}^{1/\ell})^{\frac{1}{\ell}} \cdot \sum_{i=0}^{\ell-1} u_i}{\prod_{i=0}^{\ell-1} (1 - \zeta_\ell^i z_{0}^{1/\ell})^{\frac{1}{\ell}}} = \frac{\prod_{j=0}^{\ell-1} (1 - \zeta_\ell^j z_{0}^{1/\ell})^{1/\ell} \cdot \sum_{i=0}^{\ell-1} u_i}{\prod_{i=0}^{\ell-1} (1 - \zeta_\ell^i z_{0}^{1/\ell})^{\frac{1}{\ell}}} = \left( \prod_{j=0}^{\ell-1} (1 - \zeta_\ell^j z_{0}^{1/\ell})^{1/\ell} \right)^s \cdot \zeta_\ell^{s-1} \cdot \prod_{i=0}^{\ell-1} u_i \cdot \zeta_\ell^{-s} \cdot \zeta_\ell^{s-1} u_i.
\]
Let the notations and assumptions be as above. For Theorem 2.3.

Galois polylogarithm.

Let the notations and assumptions be as above. Based on Proposition 2.1, (2.1.4), we obtain

\[ [\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3]_\ell = \frac{\sigma_{\mathcal{P}_3}}{\sigma_{\mathcal{P}_1}} \left( \prod_{i=0}^{\ell-1} \left( x + \zeta^{i}_\ell \sqrt[\ell]{\mathcal{P}_1} \right) \right)^{\frac{1}{\ell}} \left( \prod_{i=0}^{\ell-1} \left( 1 + \zeta^{i}_\ell \frac{\sqrt[\ell]{\mathcal{P}_1}}{x} \right) \right)^{\frac{1}{\ell}} .
\]

Proof. Let \( \ell \in \{2, 3\} \). We compute the triple symbol \([\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3]_\ell \) as follows:

\[ [\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3]_\ell = \frac{\sigma_{\mathcal{P}_3}}{\sigma_{\mathcal{P}_1}} \left( \prod_{i=0}^{\ell-1} \left( x + \zeta^{i}_\ell \sqrt[\ell]{\mathcal{P}_1} \right) \right)^{\frac{1}{\ell}} \left( \prod_{i=0}^{\ell-1} \left( 1 + \zeta^{i}_\ell \frac{\sqrt[\ell]{\mathcal{P}_1}}{x} \right) \right)^{\frac{1}{\ell}} .
\]
Since \( z = p_1 \left( -\frac{y}{x} \right) \) by (2.1.4), the second factor of the above last side is equal to
\[
\frac{\tilde{\sigma}_{p_3} \left( \prod_{i=0}^{\ell-1} (1 - \zeta_{\ell}^i z^{1/\ell})^\frac{1}{\ell} \right)}{\prod_{i=0}^{\ell-1} (1 - \zeta_{\ell}^i z^{1/\ell})^\frac{1}{\ell}} = \zeta_{\ell}^{\frac{1}{2}(\tilde{\sigma}_{p_3})} \quad \text{(by (2.1.6))}.
\]
Therefore, by combining above formulas and (2.1.8), we obtain
\[
[p_1, p_2, p_3]_{\ell} = \frac{\tilde{\sigma}_{p_3}(x^{\frac{1}{2}(\ell-1)})}{x^{\frac{1}{2}(\ell-1)}} \cdot \zeta_{\ell}^{\frac{1}{3}(\tilde{\sigma}_{p_3})} = \frac{\tilde{\sigma}_{p_3}(x^{\frac{1}{2}(\ell-1)})}{x^{\frac{1}{2}(\ell-1)}} \cdot \zeta_{\ell}^{\frac{1}{3}(\tilde{\sigma}_{p_3})},
\]
\[\square\]

**Corollary 2.4** (Case of \( \ell = 2 \)). Let the notations and assumptions be as above. Then we have
\[
[p_1, p_2, p_3]_2 = (-1)^{\rho(z)(\tilde{\sigma}_{p_3}) - \ell_2^{(2)}(z)(\tilde{\sigma}_{p_3})},
\]
where the value \( \rho_2(\tilde{\sigma}_{p_3}) \in \mathbb{Z}/2\mathbb{Z} \) is defined by \( \tilde{\sigma}_{p_3}(\sqrt{x})/\sqrt{x} = (-1)^{\rho_2(\tilde{\sigma}_{p_3})}. \) Hence, we obtain
\[
\mu_2(123) = \rho_2(\tilde{\sigma}_{p_3}) - \ell_2^{(2)}(z)(\tilde{\sigma}_{p_3}) \mod 2.
\]
**Proof.** The assertion follows from Theorem 2.3 and Definition 4. \[\square\]

**Corollary 2.5** (Case of \( \ell = 3 \)). Let the notations and assumptions be as above. Then we have
\[
[p_1, p_2, p_3]_3 = \zeta_{\ell}^{\frac{1}{3}(\tilde{\sigma}_{p_3})}.
\]
Hence, we obtain
\[
\mu_3(123) = -\ell_2^{(3)}(z)(\tilde{\sigma}_{p_3}) \mod 3.
\]
**Proof.** The assertion follows from Theorem 2.3 and Definition 5. \[\square\]

2.2. Deriving a reciprocity law. Let the notations and assumptions be as in previous section. Note that \( \gamma' = \delta \cdot \varphi(\gamma) \in \pi_1^{top}(\mathbb{P}^1(\mathbb{C})\setminus\{0, 1, \infty\}; \overline{1}, 1 - z) \) is as in Section 1.1.

**Proposition 2.6.** For any \( \tau \in G_K \), the value \( \ell_2^{(\ell)}(1 - z, \gamma')(\tau) \mod \ell, \) together with \( \tilde{\chi}_{\ell}^{1-z, \gamma'}(\tau) \mod \ell, \) is independent of the choice of \( \gamma \).
**Proof.** The proof can be done in the same way as the proof of Proposition 2.1. \[\square\]

**Definition 2.7.** Based on Proposition 2.6, for \( \tau \in G_K \), we let
\[
\ell_2^{(\ell)}(1 - z)(\tau) \mod \ell := \ell_2^{(\ell)}(1 - z, \gamma')(\tau) \mod \ell,
\]
\[
\tilde{\chi}_{\ell}^{1-z}(\tau) \mod \ell := \chi_{\ell}^{1-z, \gamma'}(\tau) \mod \ell,
\]
that is
\[
\zeta_{\ell}^{\ell_2^{(\ell)}(1 - z)(\tau)} := \zeta_{\ell}^{\ell_2^{(\ell)}(1 - z, \gamma')(\tau)}, \quad \zeta_{\ell}^{\ell_2^{(\ell)}(1 - z, \gamma')(\tau)'} := \zeta_{\ell}^{\ell_2^{(\ell)}(1 - z, \gamma')(\tau)}.
\]
Firstly, to derive a reciprocity law of triple \( \ell \)-th power residue symbols, we describe the triple symbol \([p_2, p_1, p_3]_{\ell}\) by the 2nd \( \ell \)-adic Galois polylogarithmic character.

**Theorem 2.8.** For \( \ell \in \{2, 3\}, \) we have
\[
[p_2, p_1, p_3]_{\ell} = \frac{\tilde{\sigma}_{p_3}(x^{\frac{1}{2}(\ell-1)})}{x^{\frac{1}{2}(\ell-1)}} \cdot \zeta_{\ell}^{\ell_2^{(\ell)}(\tilde{\sigma}_{p_3})}.
\]
Proof. Let \( \ell \in \{2, 3\} \). Since \( x^\ell - (-y)^\ell p_1 = w^\ell p_2 \iff x^\ell - w^\ell p_2 = (-y)^\ell p_1 \) by (2.1.2), we can take

\[
\theta_{p_2, p_1}^{(\ell)} = \prod_{i=0}^{\ell-1} (x - \zeta_i^\ell w\sqrt[p_2]{y})^i
\]

by replacing \( p_1, p_2, \) and \( y \) in (2.1.3) with \( p_2, p_1, \) and \(-w\). As with Theorem 2.3 we compute the triple symbol \([p_2, p_1, p_3]_{\ell}\) as follows:

\[
[p_2, p_1, p_3]_{\ell} = \sigma_{p_3} \left( \frac{\sqrt[\ell]{\theta_{p_2, p_1}^{(\ell)}}}{\sqrt[\ell]{\theta_{p_2, p_1}^{(\ell)}}} \right) \quad \text{(by Theorem 1.5, Theorem 1.10)}
\]

\[
\begin{aligned}
&= \sigma_{p_3} \left( \prod_{i=0}^{\ell-1} (x - \zeta_i^\ell w\sqrt[p_2]{y})^i \right) / \sqrt[\ell]{\prod_{i=0}^{\ell-1} (x - \zeta_i^\ell w\sqrt[p_2]{y})^i} \quad \text{(by (2.2.1))}

&= \sigma_{p_3} \left( \prod_{i=0}^{\ell-1} (x - \zeta_i^\ell w\sqrt[p_2]{y})^i \right) / \prod_{i=0}^{\ell-1} (x - \zeta_i^\ell w\sqrt[p_2]{y})^i

&= \sigma_{p_3} \left( \prod_{i=0}^{\ell-1} (x - \zeta_i^\ell w\sqrt[p_2]{y})^i \right) / \prod_{i=0}^{\ell-1} (1 - \zeta_i^\ell w/x_{p_2}^{1/\ell})^i

&= \sigma_{p_3} \left( \prod_{i=0}^{\ell-1} (1 - \zeta_i^\ell w/x_{p_2}^{1/\ell})^i \right)

&= \sigma_{p_3} \left( \prod_{i=0}^{\ell-1} (1 - \zeta_i^\ell w/x_{p_2}^{1/\ell})^i \right).
\end{aligned}
\]

Since \( 1 - z = \frac{x^\ell - (-y)^\ell p_1}{x^\ell} = \frac{w^\ell}{x^\ell} p_2 \) by (2.1.2), the second factor of the above last side is equal to

\[
\frac{\sigma_{p_3} \left( \prod_{i=0}^{\ell-1} (1 - \zeta_i^\ell (1 - z)^{1/\ell})^i \right)}{\prod_{i=0}^{\ell-1} (1 - \zeta_i^\ell (1 - z)^{1/\ell})^i} = \zeta_\ell^{1-\hat{z}(\sigma_{p_3})} \quad \text{(by Definition 2.1.1, (2.1.5)).}
\]

Therefore we obtain the assertion of the theorem. \(\square\)

Now, we derive a reciprocity law of triple \(\ell\)-th power residue symbols from the functional equation of \(\ell\)-adic Galois polylogarithms introduced in Theorem 1.2.

**Corollary 2.9** (a reciprocity law). Let the notations and assumptions be as above. For \( \ell \in \{2, 3\} \), we have

\[
[p_1, p_2, p_3]_{\ell} \cdot [p_2, p_1, p_3]_{\ell} = 1.
\]
Proof. By combining Theorem 2.3 and Theorem 2.8

\[
[p_1, p_2, p_3] \ell \cdot [p_2, p_1, p_3] \ell = \begin{cases} 
\frac{\sigma_{p_3}(\sqrt[3]{\zeta})}{\sqrt[3]{\zeta}} (-1) \zeta_2^{\frac{1}{2}}(\sigma_{p_3}) \cdot \frac{\sigma_{p_3}(\sqrt[3]{\zeta})}{\sqrt[3]{\zeta}} (-1) \zeta_2^{1-\varepsilon}(\sigma_{p_3}) & \text{(if } \ell = 2), \\
\zeta_3 \zeta_2^1(\sigma_{p_3}) \cdot \zeta_4 \zeta_2^{1-\varepsilon}(\sigma_{p_3}) & \text{(if } \ell = 3)
\end{cases}
\]

This completes the proof. \qed

Appendix A. Examples - Case of \( \ell = 3 \) with the assumption (A)

In this appendix, we present examples of \([p_1, p_2, p_3] \mod 3 \) for some pairs \((p_1, p_2)\) which have \( \alpha_{p_1, p_2} \) satisfying the assumption (A) in Section 1.2.2. The rational primes \( p \) which satisfy \( p \equiv 1 \mod 9 \) and \( 1 \leq -p \leq 1000 \) are the following 29 numbers:

\[
L := \{ -359, -431, -449, -467, -503, -521, -557, -593, -647, -683, -701, -719, -773, -809, -827, -863, -881, -953, -971 \}.
\]

For any pair \((p_1, p_2)\) of distinct rational primes in \( L \), prime ideals \( p_1 = (p_1), p_2 = (p_2) \) of \( \mathbb{Q}(\zeta_3) \) satisfy the conditions (1.2.7), (1.2.8) and (1.2.9).

In [AMM; Example 6.4], F. Amano showed that one can take \( \alpha_{-17, -53} = 8 - 3 \sqrt[3]{17} \) satisfying (A) in the case where \( (p_1, p_2) = (-17, -53) \), and gave values of \([p_1, p_2, p_3] \mod 3\) for \( p_3 = -71, -89, -107, -179, -197, -233, -251, -269, -359, -431, -449, -467, -503, -521, -557, -593, -647, -683, -701, -719, -773, -809, -827, -863, -881, -953, -971 \).

According to [HM; Example 4.2.15], Y. Mizusawa found other pairs \((p_1, p_2) = (-17, -467), (-107, -449), (-431, -233)\) which have \( \alpha_{p_1, p_2} \in O_{k_1} \) with the assumption (A). For example, one can take

\[
\alpha_{-17, -469} = 6 - 9 \sqrt[3]{17}, \\
\alpha_{-107, -449} = -24 - 5 \sqrt[3]{-17}, \\
\alpha_{-431, -233} = -68 - 9 \sqrt[3]{-431}
\]

respectively for these cases.

Let us now give new examples. Consider the case where \((p_1, p_2) = (-17, -593)\). In this case, we can take

\[
x = 9, y = 2, w = -1
\]

as a solution of (1.2.16) and

\[
\alpha_{p_1, p_2} = \alpha_{-17, -593} = 9 + 2 \sqrt[3]{17}
\]

satisfying (A). Hence

\[
\theta_{p_1, p_2}^{(3)} = \theta_{-17, -593}^{(3)} = (9 + 2 \zeta_3 \sqrt[3]{17})(9 + 2 \zeta_3^2 \sqrt[3]{17})^2.
\]
Moreover, let \( p_3 = (p_3) \) be a prime ideal of \( \mathbb{Q}(\zeta_3) \) which satisfies (1.2.15). Then,

\[
[p_1, p_2, p_3]_3 = \frac{\sigma_{p_3}(\sqrt[3]{o_{p_3}})}{\sqrt[3]{o_{p_3}}} \quad \text{(by Theorem 1.10)}
\]

\[
\equiv \theta_{p_1, p_2}^{\frac{p_3^2 - 1}{3}} \pmod{p_3},
\]

where \( \tilde{p}_3 \) is a prime ideal of \( K_{p_1, p_2}^{(3)} \) above \( p_3 \). Since \( \theta_{p_1, p_2}^{(3)} \equiv \frac{p_3^2 - 1}{3} \pmod{\tilde{p}_3} \),

\[
[p_1, p_2, p_3]_3 \equiv \theta_{p_1, p_2}^{\frac{p_3^2 - 1}{3}} \pmod{p_3 \cap K_1}.
\]

Therefore, we obtain the following test: for \( c = 0, 1, -1, \)

\[
(A.0.3) \quad [p_1, p_2, p_3]_3 = \zeta_3^c \iff N_{K_1/\mathbb{Q}}(\theta_{p_1, p_2}^{(3)} \frac{p_3^2 - 1}{3} - \zeta_3^c) \equiv 0 \pmod{p_3}.
\]

On the other hand, we can take \( \alpha_{p_2, p_1} = \alpha_{-593, -17} = 9 + \sqrt[3]{-593} \) satisfying (A). Hence, \( \theta_{p_2, p_1}^{(3)} \equiv \theta_{p_1, p_2}^{(3)} \), \( (9 + \zeta_3 \sqrt[3]{-593}) \times (9 + \zeta_3^2 \sqrt[3]{-593})^2 \). By replacing \( \theta_{p_1, p_2}^{(3)} \) (resp. \( K_1 \)) with \( \theta_{p_2, p_1}^{(3)} \) (resp. \( K_2 = \mathbb{Q}(\zeta_3, \sqrt[3]{p_2}) \) in (A.0.3), we obtain the following test: for \( c = 0, 1, -1, \)

\[
(A.0.4) \quad [p_2, p_1, p_3]_3 = \zeta_3^c \iff N_{K_2/\mathbb{Q}}(\theta_{p_2, p_1}^{(3)} \frac{p_3^2 - 1}{3} - \zeta_3^c) \equiv 0 \pmod{p_3}.
\]

Checking the right hand condition of (A.0.3) and (A.0.4) by PARI/GP, we can compute \( [p_1, p_2, p_3]_3 \) and \( [p_2, p_1, p_3]_3 \). Furthermore, by combining with Theorem 2.3 and Theorem 2.8, we can also compute \( \ell_{(2)}(z) \) \( \pmod{3} \) and \( \ell_{(2)}(1 - z) \) \( \pmod{3} \) where

\[
\left( -p_1 y^3 \right. = \frac{136}{729}.
\]

Consequently, for \( p_3 \in \mathbf{L} \setminus \{-17, -593\}, \) we get TABLE 1. Thus, we can be assured that the reciprocity law \( [p_1, p_2, p_3]_3 \cdot [p_2, p_1, p_3]_3 = 1 \) of Theorem 1.11 holds.

Let us examine more general behaviors of

\[
[p_{\rho(1)}, p_{\rho(2)}, p_{\rho(3)}]_3
\]

where \( \rho \in S_3 \) is any permutation of the set \( \{1, 2, 3\} \), in the cases

\[
\{p_1, p_2, p_3\} = \{-17, -53, -431\}, \{17, -557, -773\}, \{-17, -593, -773\}.
\]

Finding a solution of (1.2.16) and checking the test (A.0.3) for each case, we can compute \( [p_{\rho(1)}, p_{\rho(2)}, p_{\rho(3)}]_3 \). Furthermore, by combining with Corollary 2.5, we can also compute \( \ell_{(2)}(z) \) \( \pmod{3} \). Consequently, we get TABLE 2. Based on TABLE 2, it may be plausible to expect that

\[
(A.0.5) \quad [p_{\rho(1)}, p_{\rho(2)}, p_{\rho(3)}]_3 = [p_1, p_2, p_3]_3^{\sgn(\rho)},
\]

where \( \sgn(\rho) \in \{1, -1\} \) is the signature of \( \rho \in S_3 \), for \( (p_1, p_2, p_3) \) satisfying the conditions (1.2.7), (1.2.8), (1.2.9) and (1.2.15).
Table 1. Table of \([p_1, p_2, p_3]_3\), \([p_2, p_1, p_3]_3\), \(\ell_i^{(3)}(z)(\tilde{\sigma}_{p_3}) \mod 3\) and 
\(\ell_i^{(3)}(1 - z)(\tilde{\sigma}_{p_3}) \mod 3\) for \((p_1, p_2) = (-17, -593), p_3 \in \mathbf{L}\{p_1, p_2\}\)

| \(p_3\) | \([p_1, p_2, p_3]_3\) | \([p_2, p_1, p_3]_3\) | \(\ell_i^{(3)}(z)(\tilde{\sigma}_{p_3}) \mod 3\) | \(\ell_i^{(3)}(1 - z)(\tilde{\sigma}_{p_3}) \mod 3\) |
|---|---|---|---|---|
| -53 | \(\zeta_3\) | \(\zeta_3^{-1}\) | -1 | 1 |
| -71 | \(\zeta_3^{-1}\) | \(\zeta_3\) | 1 | -1 |
| -89 | \(\zeta_3\) | \(\zeta_3^{-1}\) | -1 | 1 |
| -107 | \(\zeta_3^{-1}\) | \(\zeta_3\) | 1 | -1 |
| -179 | \(\zeta_3\) | \(\zeta_3^{-1}\) | -1 | 1 |
| -197 | \(\zeta_3\) | \(\zeta_3^{-1}\) | -1 | 1 |
| -233 | 1 | 1 | 0 | 0 |
| -251 | \(\zeta_3\) | \(\zeta_3^{-1}\) | -1 | 1 |
| -269 | \(\zeta_3\) | \(\zeta_3^{-1}\) | -1 | 1 |
| -359 | \(\zeta_3^{-1}\) | \(\zeta_3\) | 1 | -1 |
| -431 | \(\zeta_3^{-1}\) | \(\zeta_3\) | 1 | -1 |
| -449 | 1 | 1 | 0 | 0 |
| -467 | \(\zeta_3\) | \(\zeta_3^{-1}\) | -1 | 1 |
| -503 | \(\zeta_3^{-1}\) | \(\zeta_3\) | 1 | -1 |
| -521 | \(\zeta_3\) | \(\zeta_3^{-1}\) | -1 | 1 |
| -557 | \(\zeta_3\) | \(\zeta_3^{-1}\) | -1 | 1 |
| -647 | \(\zeta_3^{-1}\) | \(\zeta_3\) | 1 | -1 |
| -683 | \(\zeta_3^{-1}\) | \(\zeta_3\) | 1 | -1 |
| -701 | \(\zeta_3^{-1}\) | \(\zeta_3\) | 1 | -1 |
| -719 | \(\zeta_3^{-1}\) | \(\zeta_3\) | 1 | -1 |
| -773 | \(\zeta_3^{-1}\) | \(\zeta_3\) | 1 | -1 |
| -809 | \(\zeta_3^{-1}\) | \(\zeta_3\) | 1 | -1 |
| -827 | 1 | 1 | 0 | 0 |
| -863 | \(\zeta_3\) | \(\zeta_3^{-1}\) | -1 | 1 |
| -881 | \(\zeta_3^{-1}\) | \(\zeta_3\) | 1 | -1 |
| -953 | \(\zeta_3\) | \(\zeta_3^{-1}\) | -1 | 1 |
| -971 | 1 | 1 | 0 | 0 |
Table 2. Table of \([p_1, p_2, p_3]_3\) and \(\ell_2^{(3)}(z)(\tilde{\sigma}_{p_3})\) mod 3 for the cases of \(\{p_1, p_2, p_3\} = \{-17, -53, -431\}, \{-17, -557, -773\}, \{-17, -593, -773\}\)

| \((p_1, p_2)\) | \(\alpha_{p_1, p_2} = x + y \sqrt[3]{p_1}\) | \(z = -p_1 \frac{y^3}{x^2}\) | \(p_3\) | \([p_1, p_2, p_3]_3\) | \(\ell_2^{(3)}(z)(\tilde{\sigma}_{p_3})\) mod 3 |
|-----------------|-----------------|-----------------|------|-----------------|-----------------|
| \((-17, -53)\)  | \(8 + 3\sqrt[3]{-17}\)  | \(-\frac{459}{512}\) | \(-431\) | \(1\) | \(0\) |
| \((-53, -17)\)  | \(8 + \sqrt[3]{-53}\)  | \(-\frac{53}{272}\) | \(-431\) | \(1\) | \(0\) |
| \((-17, -431)\) | \(31 + 15\sqrt[3]{-17}\)  | \(-\frac{53775}{29791}\) | \(-51\) | \(1\) | \(0\) |
| \((-431, -53)\) | \(10 + 3\sqrt[3]{-431}\)  | \(-\frac{11637}{1000}\) | \(-17\) | \(1\) | \(0\) |
| \((-53, -431)\) | \(10 - \sqrt[3]{-53}\)  | \(-\frac{53}{1000}\) | \(-17\) | \(1\) | \(0\) |
| \((-431, -17)\) | \(31 - 4\sqrt[3]{-431}\)  | \(-\frac{27584}{29791}\) | \(-53\) | \(1\) | \(0\) |
| \((-17, -557)\) | \(-42 - 16\sqrt[3]{-17}\)  | \(-\frac{8704}{9261}\) | \(-773\) | \(\zeta_3\) | \(-1\) |
| \((-557, -17)\) | \(-42 - 2\sqrt[3]{-557}\)  | \(-\frac{557}{9261}\) | \(-773\) | \(\zeta_3^{-1}\) | \(1\) |
| \((-17, -773)\) | \(-23 + 8\sqrt[3]{-17}\)  | \(-\frac{8704}{12167}\) | \(-557\) | \(\zeta_3^{-1}\) | \(1\) |
| \((-773, -557)\) | \(-6 - \sqrt[3]{-773}\)  | \(-\frac{557}{226}\) | \(-17\) | \(\zeta_3^{-1}\) | \(1\) |
| \((-557, -773)\) | \(-6 + \sqrt[3]{-557}\)  | \(-\frac{557}{226}\) | \(-17\) | \(\zeta_3\) | \(-1\) |
| \((-773, -17)\) | \(-23 - 3\sqrt[3]{-773}\)  | \(-\frac{20871}{12167}\) | \(-557\) | \(\zeta_3\) | \(-1\) |
| \((-17, -593)\) | \(9 + 2\sqrt[3]{-17}\)  | \(-\frac{136}{129}\) | \(-773\) | \(\zeta_3^{-1}\) | \(1\) |
| \((-593, -17)\) | \(9 + \sqrt[3]{-593}\)  | \(-\frac{903}{129}\) | \(-773\) | \(\zeta_3\) | \(-1\) |
| \((-17, -773)\) | \(-23 + 8\sqrt[3]{-17}\)  | \(-\frac{8704}{12167}\) | \(-593\) | \(\zeta_3\) | \(-1\) |
| \((-773, -593)\) | \(-55 - 6\sqrt[3]{-773}\)  | \(\frac{16668}{166375}\) | \(-17\) | \(\zeta_3\) | \(-1\) |
| \((-593, -773)\) | \(-55 + \sqrt[3]{-593}\)  | \(-\frac{503}{166375}\) | \(-17\) | \(\zeta_3^{-1}\) | \(1\) |
| \((-773, -17)\) | \(-23 - 3\sqrt[3]{-773}\)  | \(-\frac{20871}{12167}\) | \(-593\) | \(\zeta_3^{-1}\) | \(1\) |
References

[Am] F. Amano. On Rédei’s dihedral extension and triple reciprocity law. Proc. Japan Acad. Ser. A Math. Sci. 90 (2014), no. 1, 1–5.

[AMM] F. Amano, Y. Mizusawa, M. Morishita. On mod 3 triple Milnor invariants and triple cubic residue symbols in the Eisenstein number field. Res. Number Theory 4 (2018), no. 1, Art. 7, 29 pp.

[De] P. Deligne. Le groupe fondamental de la droite projective moins trois points. in “Galois groups over $\mathbb{Q}$” (Berkeley, CA, 1987), 79–297, Math. Sci. Res. Inst. Publ., 16, Springer, New York, 1989.

[HM] H. Hirano, M. Morishita. Arithmetic topology in Ihara theory II: Milnor invariants, dilogarithmic Heisenberg coverings and triple power residue symbols. J. Number Theory 198 (2019), 211–238.

[Ih] Y. Ihara. Braids, Galois groups, and Some Arithmetic Functions. Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), 99–120, Math. Soc. Japan, Tokyo, 1991.

[Mo] M. Morishita. Knots and Primes – An Introduction to Arithmetic Topology. Universitext. Springer, London, 2012.

[Na] H. Nakamura. Tangential base points and Eisenstein power series. in “Aspects of Galois Theory” (Gainesville, FL, 1996), 202–217, London Math. Soc. Lecture Note Ser., 256, Cambridge Univ. Press, Cambridge, 1999.

[NW1] H. Nakamura, Z. Wojtkowiak. On explicit formulae for $l$-adic polylogarithms. in “Arithmetic fundamental groups and noncommutative algebra” (Berkeley, CA, 1999), 285–294, Proc. Sympos. Pure Math., 70, Amer. Math. Soc., Providence, RI, 2002.

[NW2] H. Nakamura, Z. Wojtkowiak. Tensor and homotopy criteria for functional equations of $l$-adic and classical iterated integrals. in “Non-abelian Fundamental Groups and Iwasawa Theory”, 258–310, London Math. Soc. Lecture Note Ser., 393, Cambridge Univ. Press, Cambridge, 2012.

[Ré] L. Rédei. Ein neues zahlentheoretisches Symbol mit Anwendungen auf die Theorie der quadratischen Zahlkörper. J. J. Reine Angew. Math. 180 (1939), 1–43.

[Sh] D. Shiraishi. Galois actions on fundamental groups of $\mathbb{P}^1\setminus\{0, 1, \infty\}$ and multiple $l$-th power residue symbols (in Japanese). Master’s thesis, Department of Mathematics, Osaka University, February 2019.

[Woo] Z. Wojtkowiak. On $l$-adic iterated integrals, II – Functional equations and $l$-adic polylogarithms. Nagoya Math. J. 177 (2005), 117–153.