THE STRUCTURES OF STANDARD \((\mathfrak{g}, K)\)-MODULES OF \(SL(3, \mathbb{R})\).

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ABSTRACT. We describe explicitly the structures of standard \((\mathfrak{g}, K)\)-modules of \(SL(3, \mathbb{R})\).

1. Introduction

As far as we know, for some ‘small’ semisimple Lie groups \(G\), the \((\mathfrak{g}, K)\)-module structures of standard representations are completely described. For example, the description of them for \(SL(2, \mathbb{R})\) is found in standard textbooks, and there are rather complete results for some groups of real rank 1, e.g. \(SU(n, 1)\) in [1] and \(Spin(1, 2n)\) in [6]. However, for Lie groups of higher rank, there are few references as far as the author knows. It seems to be difficult to describe the whole \((\mathfrak{g}, K)\)-module structures even for standard representations of classical groups of higher rank, since their \(K\)-types are not multiplicity free. In the papers [2] and [3], the \((\mathfrak{g}, K)\)-module structures of some standard representations of \(Sp(2, \mathbb{R})\) are described by T. Oda. In the former paper [3], we extend the result for principal series representations of \(Sp(2, \mathbb{R})\). The method in these papers is applicable to study of standard representations of another groups. In this paper, we use this method to study standard \((\mathfrak{g}, K)\)-modules of \(SL(3, \mathbb{R})\).

Before describing the case of \(SL(3, \mathbb{R})\), let us explain the problem in a more precise form for a general real semisimple Lie group \(G\) with its Lie algebra \(\mathfrak{g}\). Fix a maximal compact subgroup \(K\) of \(G\). Since any standard \((\mathfrak{g}, K)\)-modules are realized as subspaces of \(L^2(K)\) as \(K\)-modules, we investigate the \(K\)-module structure of standard \((\mathfrak{g}, K)\)-modules by the Peter-Weyl’s theorem. In order to describe the action of \(\mathfrak{g}\) or \(\mathfrak{g}_C = \mathfrak{g} \otimes _{\mathbb{R}} \mathbb{C}\), it suffices to investigate the action of \(\mathfrak{p}\) or \(\mathfrak{p}_C\), because of the Cartan decomposition \(\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}\). Therefore, the investigation of the action of \(\mathfrak{p}\) or \(\mathfrak{p}_C\) is essential to give the description of the \((\mathfrak{g}, K)\)-module structure of a standard representation. To study the action of \(\mathfrak{p}_C\), we compute the linear map \(\Gamma_{\tau, i}\) defined as follows. Let \((\pi, H_\pi)\) be a standard representation of \(G\) with its subspace \(H_{\pi, K}\) of \(K\)-finite vectors. For a \(K\)-type \((\tau, V_\tau)\) of \(\pi\), and a nonzero \(K\)-homomorphism \(\eta: V_\lambda \to H_{\pi, K}\), we define a linear map \(\tilde{\eta}: \mathfrak{p}_C \otimes _C V_\lambda \to H_{\pi, K}\) by \(X \otimes v \mapsto X \cdot \eta(v)\). Then \(\tilde{\eta}\) is a \(K\)-homomorphism with \(\mathfrak{p}_C\) endowed with the adjoint action \(Ad\) of \(K\). Let \(V_\tau \otimes _C \mathfrak{p}_C \simeq \bigoplus _{i \in I} V_{\tau_i}\) be the decomposition into a direct sum of irreducible \(K\)-modules and \(\iota_i\) an injective \(K\)-homomorphism from \(V_{\tau_i}\) to \(V_\tau \otimes _C \mathfrak{p}_C\) for each \(i\). We define a linear map \(\Gamma_{\tau, i}\): \(\text{Hom}\_K(V_\tau, H_{\pi, K}) \to \text{Hom}\_K(V_{\tau_i}, H_{\pi, K})\) by \(\eta \mapsto \tilde{\eta} \circ \iota_i\). These linear maps \(\Gamma_{\tau, i}\) \((i \in I)\) characterize the action of \(\mathfrak{p}_C\). Our purpose of this paper is to give explicit expressions of \(\iota_i\) and \(\Gamma_{\tau, i}\) when \(\pi\) is a \(P\)-principal series representation of \(G = SL(3, \mathbb{R})\) for each standard parabolic subgroup \(P\) of \(G\). As a result, we obtain infinite number of ‘contiguous relations’, a kind of system of differential-difference relations among vectors in \(H_{\pi}[\tau]\) and \(H_{\pi}[\tau_i]\). Here \(H_{\pi}[\tau]\) is \(\tau\)-isotypic component of \(H_{\pi}\). These are described in Proposition 4.2 Theorem 5.3 and 6.5.

As an application, we can utilize the contiguous relations to obtain the explicit formulae of some spherical functions. In the paper [2] H. Manabe, T. Ishii and T. Oda give the explicit formulae of Whittaker functions of principal series representations of \(SL(3, \mathbb{R})\) to solve the holonomic system of differential equations characterizing those functions, which is derived from the Capelli elements and the contiguous relations around minimal \(K\)-type. We can obtain the holonomic systems characterizing Whittaker functions of generalized principal series representations of \(SL(3, \mathbb{R})\) from the result of this paper. We hope that this interesting possibility will be considered in future work. On the other hand, if we have the explicit formula
of Whittaker function with a certain $K$-type, then we can give those with another $K$-type by using contiguous relations.

We give the contents of this paper. In Section 2, we recall the classical case $SL(2, \mathbb{R})$ shortly. In Section 3, we recall the structure of $SL(3, \mathbb{R})$ and define a standard representations obtained by a parabolic induction with respect to the standard parabolic subgroups. In Section 4, we introduce the standard basis of a finite dimensional irreducible representation of $K$ and give explicit expressions of $i_\tau : V_\tau \rightarrow V_\tau \otimes C\mathfrak{pc}$. In Section 5, we introduce the general setting of this paper and give matrix representations of $\Gamma_{\tau,i}$ for principal series representations in Theorem 5.5. In Section 6, we give the matrix representations of $\Gamma_{\tau,i}$ for generalized principal series representations in Theorem 6.5. In Section 7, we give explicit expressions of the action of $\mathfrak{pc}$ in Proposition 7.2.

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2. The standard $(\mathfrak{g}, K)$-modules of $SL(2, \mathbb{R})$

We start with a short review of the most classical case, i.e. the case of the group $SL(2, \mathbb{R})$.

2.1. The principal series representations of $SL(2, \mathbb{R})$. We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of rational integers, the real number field and the complex number field, respectively. Let $\mathbb{Z}_{\geq 0}$ be the set of non-negative integers, $1_\mathbb{R}$ be the unit matrix in the space $M_n(\mathbb{R})$ of real matrices of size $n$ and $O_{m,n}$ be the zero matrix of size $m \times n$. We denote by $\delta_{ij}$ the Kronecker delta, i.e.

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & \text{otherwise}. \end{cases}$$

For a Lie algebra $\mathfrak{l}$, we denote by $\mathfrak{l}_\mathbb{C} = \mathfrak{l} \otimes \mathbb{R} \mathbb{C}$ the complexification of $\mathfrak{l}$.

We put

$$G' = SL(2, \mathbb{R}), \quad M' = \{ m = \text{diag}(\varepsilon, \varepsilon^{-1}) \mid \varepsilon \in \{\pm 1\} \}, \quad \mathcal{A}' = \{ a(r) = \text{diag}(r, r^{-1}) \mid r \in \mathbb{R}_{>0} \},$$

$$N' = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\} \subset \mathbb{R}^2, \quad K' = SO(2) = \left\{ \kappa_t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \right\} \subset \mathbb{R}^2.$$

Let $\mathfrak{g}'$, $\mathfrak{k}'$, $\mathfrak{a}'$ and $\mathfrak{n}'$ be Lie algebras of $G'$, $K'$, $\mathcal{A}'$ and $N'$, respectively.

For $\nu \in \mathbb{C}$ and a character $\sigma$ of $M'$, the principal series representation $\pi_{(\nu, \sigma)}$ of $G'$ is defined as the right regular representation of $G'$ on the space $H_{(\nu, \sigma)}(\mathbb{R})$ which is the completion of

$$H_{(\nu, \sigma)}^\infty = \left\{ f : G' \rightarrow \mathbb{C} \text{ smooth} \mid f(namx) = r^{\nu+1} \sigma(m) f(x) \right\}$$

for $n \in N'$, $a = a(r) \in \mathcal{A}'$, $m \in M'$, $x \in G'$

with respect to the norm

$$\|f\|^2 = \int_{K'} |f(k)|^2 dk.$$

The restriction map $r_{K'} : H_{(\nu, \sigma)} \ni f \mapsto f|_{K'} \in L^2(K')$ is an injective $K'$-homomorphism when $L^2(K')$ is endowed with right regular action of $K'$. Then the image of $r_{K'}$ is the following subspace of $L^2(K')$:

$$L^2_{(\nu', \sigma)}(K') = \{ f \in L^2(K') \mid f(mx) = \sigma(m) f(x) \text{ for a.e. } m \in M', \ x \in K' \}.$$

We have an irreducible decomposition of the $K'$-module $L^2(K')$:

$$L^2(K') = \bigoplus_{p \in \mathbb{Z}} \mathbb{C} \cdot \tilde{\chi}_p,$$

where $\tilde{\chi}_p : K' \ni \kappa_t \mapsto e^{\pm pt} \in \mathbb{C}^\times$. 
Therefore we have an isomorphism
\[
H_{(\nu, \sigma)} \rightarrow L^2(M', \sigma)(K') = \begin{cases} \bigoplus_{p \in 2\mathbb{Z}} C : \hat{\chi}_p, & \text{if } \sigma(-1) = 1, \\ \bigoplus_{p \in 1+2\mathbb{Z}} C : \hat{\chi}_p, & \text{if } \sigma(-1) = -1. \end{cases}
\]
Let \(\chi_p \in H_{(\nu, \sigma)}\) be an inverse image of \(\hat{\chi}_p\) by this isomorphism.
Now we take a basis \(\{w, x_+, x_-\}\) of \(g'_C\) defined by
\[
w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad x_\pm = \begin{pmatrix} 1 & \pm \sqrt{-1} \\ \pm \sqrt{-1} & -1 \end{pmatrix}.
\]
Here we note that
\[g'_C = \mathfrak{t}'_C \oplus \mathfrak{p}'_C, \quad \mathfrak{t}'_C = C : w, \quad \mathfrak{p}'_C = C : x_+ \oplus C : x_-.
\]
is a complexification of a Cartan decomposition \(g' = \mathfrak{t}' \oplus \mathfrak{p}'\) with respect to a Cartan involution \(g' \ni X \mapsto {}^tX \in g'\) where \(^tX\) means transpose of \(X\).

Since \(w \in \mathfrak{t}'\), we see that
\[
(2.1) \quad \pi_{(\nu, \sigma)}(w)\chi_p = \sqrt{-1}p\chi_p
\]
from direct computation. Here we denote the differential of \(\pi_{(\nu, \sigma)}\) again by \(\pi_{(\nu, \sigma)}\). The action of \(\mathfrak{p}'_C\) is given in the following proposition.

**Proposition 2.1.** \(\pi_{(\nu, \sigma)}(x_\pm)\chi_p = (\nu + 1 \pm p)\chi_{p \pm 2}\).

**Proof.** By the relations
\[
[w, x_\pm] = \pm 2\sqrt{-1}x_\pm,
\]
we have
\[
(2.2) \quad \pi_{(\nu, \sigma)}(w)\pi_{(\nu, \sigma)}(x_\pm)\chi_p = \sqrt{-1}(p \pm 2)(\pi_{(\nu, \sigma)}(x_\pm)\chi_p).
\]
Here \([\cdot, \cdot]\) is the bracket product. From the equations (2.1) and (2.2), we see that \(\pi_{(\nu, \sigma)}(x_\pm)\chi_p \in C : \chi_{p \pm 2}\).

The elements \(x_\pm\) of \(\mathfrak{p}'_C\) have the following expressions according to Iwasawa decomposition \(g'_C = \mathfrak{n}'_C \oplus \mathfrak{a}'_C \oplus \mathfrak{t}'_C\):
\[
x_\pm = \pm 2\sqrt{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} E' + H' \mp \sqrt{-1}w
\]
where \(E' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{n}'_C\) and \(H' = \text{diag}(1, -1) \in \mathfrak{a}'_C\). From this expression and the definition of the space \(H_{(\nu, \sigma)}\), we have the value of \(\pi_{(\nu, \sigma)}(x_\pm)\chi_p\) at \(l_2 = k_0 \in K'\) as follows:
\[
\pi_{(\nu, \sigma)}(x_\pm)\chi_p(l_2) = \pm 2\sqrt{-1}\pi_{(\nu, \sigma)}(E')\chi_p(l_2) + \pi_{(\nu, \sigma)}(H')\chi_p(l_2) \mp \sqrt{-1}\pi_{(\nu, \sigma)}(w)\chi_p(l_2)
\]
\[
= 0 + (\nu + 1) \mp \sqrt{-1}(\sqrt{-1}p)
\]
\[
= \nu + 1 \pm p.
\]
Since \(\chi_{p \pm 2}(l_2) = 1\), we obtain \(\pi_{(\nu, \sigma)}(x_\pm)\chi_p = (\nu + 1 \pm p)\chi_{p \pm 2}\). \(\square\)

From this proposition, we obtain the following.

**Proposition 2.2.** (i) Let \(k\) be an integer such that \(k \geq 2\). If \(\nu = k - 1\) and \(\sigma(-1) = (-1)^k\), there is an injective homomorphism from \(D'^{\mp}_k\) to \(\pi_{(\nu, \sigma)}\). Here \(D'^+_k\) and \(D'^-_k\) are discrete series representations of \(SL(2, \mathbb{R})\) with the Blattner parameter \(k\) and \(-k \in \mathbb{Z}\), respectively. Moreover the quotient \((g', K')\)-modules \(\pi_{(\nu, \sigma)}/(D'^+_k \oplus D'^-_k)\) is of dimension \(k - 1\).

(ii) Let \(k\) be an integer such that \(k \geq 2\). If \(\nu = -k + 1\) and \(\sigma(-1) = (-1)^k\), the \((k - 1)\)-dimensional subspace \(F_{k-2}\) of \(H_{(\nu, \sigma)}\) generated by
\[
\{\chi_p \mid p = -k + 2, -k + 4, \ldots, k - 2\}
\]
is \(G'\)-invariant and is isomorphic to the symmetric tensor representation of degree \(k - 2\). Moreover the quotient \(\pi_{(\nu, \sigma)}/F_{k-2}\) is isomorphic to \(D'^+_k \oplus D'^-_k\).
(iii) If \( \nu = 0 \) and \( \sigma(1_2) = -1, \pi_{(\nu, \sigma)} \) is a direct sum of two irreducible representations, called limit of discrete series representations.

(iv) If \( (\nu, \sigma) \) is not in the cases of (i), (ii) and (iii), \( \pi_{(\nu, \sigma)} \) is irreducible.

We are going to show the analogue of Proposition \([2.1]\) for \( SL(3, \mathbb{R}) \) in Theorem \([5.3]\) and \([6.5]\).

### 3. Preliminaries

#### 3.1. Groups and algebras.
Let \( G \) be the special linear group \( SL(3, \mathbb{R}) \) of degree three and \( \mathfrak{g} \) be its Lie algebra. We define a Cartan involution \( \theta \) of \( G \) by \( G \ni g \mapsto ^t g^{-1} \in G \). Here \( g^{-1} \) means the inverse of \( g \). Then a maximal compact subgroup of \( G \) is given by

\[
K = \{ g \in G \mid \theta(g) = g \} = SO(3).
\]

If we denote the differential of \( \theta \) again by \( \theta \), then we have \( \theta(X) = -^t X \) for \( X \in \mathfrak{g} \). Let \( \mathfrak{k} \) and \( \mathfrak{p} \) be the +1 and the −1 eigenspaces of \( \theta \) in \( \mathfrak{g} \), respectively, that is,

\[
\mathfrak{k} = \{ X \in \mathfrak{g} \mid ^t X = -X \} = \mathfrak{so}(3), \quad \mathfrak{p} = \{ X \in \mathfrak{g} \mid ^t X = X \}.
\]

Then \( \mathfrak{k} \) is the Lie algebra of \( K \) and \( \mathfrak{g} \) has the Cartan decomposition \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \).

Put \( \mathfrak{a}_0 = \{ \text{diag}(t_1, t_2, t_3) \mid t_i \in \mathbb{R}, 1 \leq i \leq 3, t_1 + t_2 + t_3 = 0 \} \). Then \( \mathfrak{a}_0 \) is a maximal abelian subalgebra of \( \mathfrak{p} \). For each \( 1 \leq i \leq 3 \), we define a linear form \( e_i \) on \( \mathfrak{a}_0 \) by \( \mathfrak{a}_0 \ni \text{diag}(t_1, t_2, t_3) \mapsto t_i \in \mathbb{C} \). The set \( \Sigma \) of the restricted roots for \( (\mathfrak{a}_0, \mathfrak{g}) \) is given by \( \Sigma = \Sigma(\mathfrak{a}_0, \mathfrak{g}) = \{ e_i - e_j \mid 1 \leq i < j \leq 3 \} \) forms a positive root system. For each \( \alpha \in \Sigma \), we denote the restricted root space by \( \mathfrak{g}_\alpha \) and choose a restricted root vector \( E_\alpha \) in \( \mathfrak{g}_\alpha \) as follows:

\[
E_{e_1-e_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{e_1-e_3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{e_2-e_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},
\]

and \( E_{-\alpha} = ^t E_\alpha \) for \( \alpha \in \Sigma^+ \). If we put \( \mathfrak{n}_0 = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha \), then \( \mathfrak{g} \) has an Iwasawa decomposition \( \mathfrak{g} = \mathfrak{n}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{k} \). Also we have \( G = N_0 A_0 K \), where \( N_0 = \exp(\mathfrak{n}_0) \) and \( A_0 = \exp(\mathfrak{a}_0) \).

The group \( G \) has three non-trivial standard parabolic subgroups \( P_0, P_1, P_2 \) with

\[
P_0 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in G \right\}, \quad P_1 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in G \right\}, \quad P_2 = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \in G \right\}.
\]

Let \( \mathfrak{n}_1, \mathfrak{n}_2 \) be subalgebras of \( \mathfrak{n}_0 \) defined by \( \mathfrak{n}_1 = \mathfrak{g}_{e_1-e_2} \oplus \mathfrak{g}_{e_1-e_3}, \mathfrak{n}_2 = \mathfrak{g}_{e_1-e_3} \oplus \mathfrak{g}_{e_2-e_3} \). We take a basis \( \{ H_1, H_2 \} \) of \( \mathfrak{a}_0 \) defined by

\[
H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},
\]

and set \( H^{(1)} = 2H_1 - H_2, \ H^{(2)} = H_1 + H_2 \). We define subalgebras \( \mathfrak{a}_1, \mathfrak{a}_2 \) of \( \mathfrak{a}_0 \) by \( \mathfrak{a}_1 = \mathbb{R} \cdot H^{(1)}, \mathfrak{a}_2 = \mathbb{R} \cdot H^{(2)} \). We specify Langland decompositions of \( P_i = N_i A_i M_i \) \( (0 \leq i \leq 2) \) by

\[
M_0 = \{ \text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_1 \varepsilon_2) \mid \varepsilon_i \in \{ \pm 1 \} \mid 1 \leq i \leq 2 \} \},
\]

\[
M_1 = \left\{ \begin{pmatrix} \det(h)^{-1} & O_{1,2} \\ O_{2,1} & h \end{pmatrix} \right\} h \in SL^\pm(2, \mathbb{R}), \quad A_1 = \exp(\mathfrak{a}_1), \quad N_1 = \exp(\mathfrak{n}_1),
\]

\[
M_2 = \left\{ \begin{pmatrix} h & O_{2,1} \\ O_{1,2} & \det(h)^{-1} \end{pmatrix} \right\} h \in SL^\pm(2, \mathbb{R}), \quad A_2 = \exp(\mathfrak{a}_2), \quad N_2 = \exp(\mathfrak{n}_2).
\]

Here \( SL^\pm(2, \mathbb{R}) = \{ g \in GL(2, \mathbb{R}) \mid \det(g) = \pm 1 \} \). For \( i = 1, 2 \), let \( \mathfrak{m}_i \) be a Lie algebra of \( M_i \).
3.2. Definition of the $P_i$-principal series representations of $G$. For $0 \leq i \leq 2$, in order to define the $P_i$-principal series representation of $G$, we prepare the data $(\nu_i, \sigma_i)$ as follows.

For $\nu_i \in \text{Hom}_R(a_i, C)$, we define a coordinate $(\nu_{0,1}, \nu_{0,2}) \in C^2$ by $\nu_{0,i} = \nu_0(H_i)$ $(i = 1, 2)$. Then the half sum $\rho_0 = \frac{1}{2} \left( \sum_{\alpha \in \Sigma_+} \alpha \right) = e_1 - e_3$ of the positive roots has coordinate $(\rho_{0,1}, \rho_{0,2}) = (2, 1)$. We define a quasicharacter $e^{\rho_0} \colon A_0 \to C^\times$ by

$$e^{\rho_0}(a) = a_{1,0}^{\rho_{0,1}} a_{2,0}^{\rho_{0,2}}, \quad a = \text{diag}(a_1, a_2, a_3) \in A_0.$$  

We fix a character $\sigma_0$ of $M_0$. $\sigma_0$ is realized by $(\sigma_{0,1}, \sigma_{0,2}) \in \{0, 1\}^\oplus 2$ such that

$$\sigma_0(\text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_1 \varepsilon_2)) = \varepsilon_1^{\sigma_{0,1}} \varepsilon_2^{\sigma_{0,2}}, \quad \varepsilon_1, \varepsilon_2 \in \{\pm 1\}.$$  

For each $i = 1, 2$, we identify $\nu_i \in \text{Hom}_R(a_i, C)$ with a complex number $\nu_i(H_i) \in C$. Let $\rho_i$ $(i = 1, 2)$ be the half sums of positive roots whose root spaces are contained in $n_i$, i.e. $\rho_1 = \frac{1}{3}(2e_1 - e_2 - e_3)$, $\rho_2 = \frac{1}{3}(e_1 + e_2 - 2e_3)$. Then both $\rho_1$ and $\rho_2$ are identified with 3. We identify $M_i$ $(i = 1, 2)$ with $SL^\pm(2, R)$ by natural isomorphisms $m_i \colon SL^\pm(2, R) \to M_i$ $(i = 1, 2)$ defined by

$$m_1(h) = \left( \begin{array}{cc} \det(h)^{-1} & 0_{1,2} \\ O_{2,1} & h \end{array} \right), \quad m_2(h) = \left( \begin{array}{ccc} h & O_{2,1} \\ O_{1,2} & \det(h)^{-1} \end{array} \right) \quad (h \in SL^\pm(2, R)).$$  

Then we fix a discrete series representation $\sigma_i = D_k = \text{Ind}^{SL^\pm(2, R)}_{SL(2, R)}(D^+_k)$ of $M_i \simeq SL^\pm(2, R)$, where $D^+_k$ is a discrete series representation of $SL(2, R)$ with the Blattner parameter $k \geq 2$.

**Definition 3.1.** For $0 \leq i \leq 2$, we define the $P_i$-principal series representation $\pi_{(\nu_i, \sigma_i)}$ of $G$ by

$$\pi_{(\nu_i, \sigma_i)} = \text{Ind}_{P_i}^G(1_{N_i} \otimes e^{\nu_i + \rho_i} \otimes \sigma_i),$$  

i.e. $\pi_{(\nu_i, \sigma_i)}$ is the right regular representation of $G$ on the space $H_{(\nu_i, \sigma_i)}$ which is the completion of

$$H_{(\nu_i, \sigma_i)}^\infty = \left\{ f \colon G \to V_{\sigma_i} \text{ smooth} \mid f(namx) = e^{\nu_i + \rho_i}(a)\sigma_i(m)f(x) \right\}.$$  

with respect to the norm

$$||f||^2 = \int_K ||f(k)||_{\sigma_i}^2 dk.$$  

Here $V_{\sigma_i}$ is a representation space of $\sigma_i$ and $|| \cdot ||_{\sigma_i}$ is its norm.

4. Representations of $K = SO(3)$

4.1. The spinor covering. To describe the finite dimensional representations of $SO(3)$, the simplest way seems to be the one utilizing the double covering $\varphi \colon SU(2) = Spin(3) \to SO(3)$. We use the following realization of the double covering $\varphi$, which is introduced in [2].

The Hamilton quaternion algebra $H$ is realized in $M_2(C)$ by

$$H = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in M_2(C) \mid a, b \in C \right\}.$$  

Then $SU(2)$ is the subgroup of the multiplicative group consisting of quaternions with reduced norm 1, i.e. $SU(2) = \{ x \in H \mid \text{det } x = 1 \}$. Let $P = \{ x \in H \mid \text{tr } x = 0 \}$ be the 3-dimensional real Euclidean space consisting of pure quaternions. Then for each $x \in SU(2)$, the map $P \ni p \mapsto x \cdot p \cdot x^{-1} \in P$ preserve the Euclidean norm $p \mapsto \text{det } p$ and the orientation, hence we have the homomorphism

$$\varphi : SU(2) \to SO(P, \det) \simeq SO(3),$$
which is surjective, since the range is a connected group. The kernel of this homomorphism is given by \{±1\}. An explicit expression of the covering map \(\varphi\) is given by
\[
\varphi(x) = \begin{pmatrix}
p^2 + q^2 - r^2 - s^2 & -2(ps - qr) & 2(pr + qs) \\
2(ps + qr) & p^2 - q^2 + r^2 - s^2 & -2(pq - rs) \\
-2(pr - qs) & 2(pq + rs) & p^2 - q^2 - r^2 + s^2
\end{pmatrix}
\]
for \(x = \begin{pmatrix} p + \sqrt{-1}q & r + \sqrt{-1}s \\
-r + \sqrt{-1}s & p - \sqrt{-1}q \end{pmatrix} \in SU(2)\ (p, q, r, s \in \mathbb{R})\).

By the derivation \(d\varphi: \mathfrak{su}(2) \to \mathfrak{so}(3)\) of \(\varphi\), the standard generators:
\[
u_1 = \begin{pmatrix} \sqrt{-1} & 0 \\
0 & -\sqrt{-1} \end{pmatrix}, \quad \nu_2 = \begin{pmatrix} 0 & 1 \\
-1 & 0 \end{pmatrix}, \quad \nu_3 = \begin{pmatrix} 0 & \sqrt{-1} \\
-\sqrt{-1} & 0 \end{pmatrix}
\]
are mapped to \(-2K_{23}, 2K_{13}, -2K_{12}\) with
\[
K_{23} = \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0 \end{pmatrix}, \quad K_{13} = \begin{pmatrix} 0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0 \end{pmatrix}, \quad K_{12} = \begin{pmatrix} 0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(3),
\]
respectively.

4.2. **Representations of** \(SU(2)\). The set of equivalence classes of the finite dimensional continuous representations of \(SU(2)\) is exhausted by the symmetric tensor product \(\tau_l \ (l \in \mathbb{Z}_{\geq 0})\) of the representation \(SU(2) \ni g \mapsto (v \mapsto g \cdot v) \in GL(\mathbb{C}^2)\). We use the following realizations of those which are introduced in [2].

Let \(V_l\) be the subspace consisting of degree \(l\) homogeneous polynomials of two variables \(x, y\) in the polynomial ring \(\mathbb{C}[x, y]\). For \(g \in SU(2)\) with \(g^{-1} = \begin{pmatrix} a & b \\
-b & a \end{pmatrix}\) and \(f(x, y) \in V_l\) we set
\[
\tau_l(g)f(x, y) = f(ax + by, -bx + ay).
\]

Passing to the Lie algebra \(\mathfrak{su}(2)\), the derivation of \(\tau_l\), denoted by same symbol, is described as follows by using the standard basis \(\{v_k = x^k y^{l-k} \mid 0 \leq k \leq l\}\) and the standard generators \(\{\nu_1, \nu_2, \nu_3\}\). Namely we have
\[
\tau_l(H)v_k = (l - 2k)v_k, \quad \tau_l(E)v_k = -kv_{k-1}, \quad \tau_l(F)v_k = (k - l)v_{k+1}.
\]

Here \(\{E, H, F\}\) is \(\mathfrak{sl}_2\)-triple defined by
\[
H = -\sqrt{-1}\nu_1, \quad E = \frac{1}{2}(\nu_2 - \sqrt{-1}\nu_3), \quad F = \frac{1}{2}(\nu_2 + \sqrt{-1}\nu_3) \in \mathfrak{su}(2)_\mathbb{C} = \mathfrak{sl}(2, \mathbb{C}).
\]

The condition that \(\tau_l\) defines a representation of \(SO(3)\) by passing to the quotient with respect to \(\varphi: SU(2) \to SO(3)\) is that \(\tau_l(-1_2) = (-1)^l = 1\), i.e. \(l\) is even. For \(l \in \mathbb{Z}_{\geq 0}\), we denote the irreducible representation of \(SO(3)\) induced from \((\tau_{2l}, V_{2l})\) again by \((\tau_{2l}, V_{2l})\).

4.3. **The adjoint representation of** \(K\) on \(\mathfrak{p}_C\). It is known that \(\mathfrak{p}_C\) becomes a \(K\)-module via the adjoint action of \(K\). Concerning this, we have the following lemma.
Lemma 4.1. Let \( \{w_j \mid 0 \leq j \leq 4\} \) be the standard basis of \((\tau_4, V_4)\) and \(\{X_j \mid 0 \leq j \leq 4\}\) be a basis of \(p_C\) defined as follows:

\[
X_0 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & -\sqrt{-1} \\
0 & -\sqrt{-1} & -1
\end{pmatrix},
X_1 = -\frac{1}{2} \begin{pmatrix}
0 & \sqrt{-1} & 1 \\
\sqrt{-1} & 0 & 0 \\
1 & 0 & 0
\end{pmatrix},
X_2 = \frac{1}{3} \begin{pmatrix}
2 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix},
X_3 = \frac{1}{2} \begin{pmatrix}
0 & \sqrt{-1} & 1 \\
\sqrt{-1} & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix},
X_4 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & \sqrt{-1} \\
0 & \sqrt{-1} & -1
\end{pmatrix}.
\]

Then via the unique isomorphism \(V_4\) and \(p_C\) as \(K\)-modules we have the identification \(w_j = X_j\) \((0 \leq j \leq 4)\).

Proof. By direct computation, we have the following table of the adjoint actions of the basis \(\{d\varphi(E), d\varphi(H), d\varphi(F)\}\) of \(t_C\) on the basis \(\{X_j \mid 0 \leq j \leq 4\}\) of \(p_C\).

| \(d\varphi(H)\) | \(X_0\) | \(X_1\) | \(X_2\) | \(X_3\) | \(X_4\) |
|-----------------|--------|--------|--------|--------|--------|
| 4\(X_0\) | 2\(X_1\) | 0 | -2\(X_3\) | -4\(X_4\) |
| 0 | -\(X_0\) | -2\(X_1\) | -3\(X_2\) | -4\(X_3\) |
| -4\(X_1\) | -3\(X_2\) | -2\(X_3\) | -1\(X_4\) | 0 |

TABLE. The adjoint actions of \(t_C\) on the basis \(\{X_j \mid 0 \leq j \leq 4\}\) of \(p_C\).

Comparing the actions in the above table with the actions in Subsection 4.2 we have the assertion. \(\square\)

4.4. Clebsch-Gordan coefficients for the representations of \(sl(2, C)\) with respect to standard basis. In the later sections, we need irreducible decomposition of the tensor product \(V \otimes_C p_C\) as \(K\)-modules for each \(K\)-type \((\tau, V)\) of \(\pi_{(\nu, \sigma)}\). From the previous arguments, it suffices to consider the irreducible decomposition of \(V_l \otimes_C V_4\) as \(sl(2, C) = su(2)_C\)-modules for arbitrary non-negative integer \(l\).

Generically, the tensor product \(V_l \otimes_C V_4\) has five irreducible components \(V_{l+4}, V_{l+2}, V_l, V_{l-2}\) and \(V_{l-4}\). Here some components may vanish. We give an explicit expression of a nonzero \(sl(2, C)\)-homomorphism from each irreducible component to \(V_l \otimes_C V_4\) as follows.

Proposition 4.2. Let \(\{v_k^{(l)} \mid 0 \leq k \leq l\}\) be the standard basis of \(V_l\) for \(l \in \mathbb{Z}_{\geq 0}\). We put \(v_k^{(l)} = 0\) when \(k < 0\) or \(k > l\).

If \(V_{l+2m}\)-component of \(V_l \otimes_C V_4\) does not vanish, then we define linear maps \(I_{2m}^l : V_{l+2m} \rightarrow V_l \otimes_C V_4\) \((-2 \leq m \leq 2)\) by

\[
I_{2m}^l(v_k^{(l+2m)}) = \sum_{i=0}^4 A_{[l,2m;k,i]} \cdot v_k^{(l+2m-i)} \otimes w_i.
\]

Here the coefficients \(A_{[l,2m;k,i]} = a(l,2m;k,i)/d(l,2m)\) are defined by following formulae.
Formula 1: The coefficients of $I_4^t: V_{l+4} \to V_l \otimes_C V_4$ are given as follows:

\[
\begin{align*}
  a(l, 4; k, 0) &= (l + 4 - k)(l + 3 - k)(l + 2 - k)(l + 1 - k), \\
  a(l, 4; k, 1) &= 4(l + 4 - k)(l + 3 - k)(l + 2 - k)k, \\
  a(l, 4; k, 2) &= 6(l + 4 - k)(l + 3 - k)k(k - 1), \\
  a(l, 4; k, 3) &= 4(l + 4 - k)k(k - 1)(k - 2), \\
  a(l, 4; k, 4) &= k(k - 1)(k - 2)(k - 3), \\
  d(l, 4) &= (l + 4)(l + 3)(l + 2)(l + 1).
\end{align*}
\]

Formula 2: The coefficients of $I_2^t: V_{l+2} \to V_l \otimes_C V_4$ are given as follows:

\[
\begin{align*}
  a(l, 2; k, 0) &= (l + 2 - k)(l + 1 - k)(l - k), \\
  a(l, 2; k, 1) &= 4(l + 2 - k)(l + 1 - k)(l - 4k), \\
  a(l, 2; k, 2) &= 3(l + 2 - k)(l + k - 2k + 2k), \\
  a(l, 2; k, 3) &= (3l - 4k + 8)k(k - 1), \\
  a(l, 2; k, 4) &= k(k - 1)(k - 2), \\
  d(l, 2) &= (l + 2)(l + 1)l.
\end{align*}
\]

Formula 3: The coefficients of $I_0^t: V_l \to V_l \otimes_C V_4$ are given as follows:

\[
\begin{align*}
  a(l, 0; k, 0) &= (l - k)(l - 1 - k), \\
  a(l, 0; k, 1) &= 2(l - k)(l - 2k - 1), \\
  a(l, 0; k, 2) &= (l^2 - 6kl + 6k^2 - l), \\
  a(l, 0; k, 3) &= 2(l - 2k + 1)k, \\
  a(l, 0; k, 4) &= k(k - 1), \\
  d(l, 0) &= l(l - 1).
\end{align*}
\]

Formula 4: The coefficients of $I_{-2}^t: V_{l-2} \to V_l \otimes_C V_4$ are given as follows:

\[
\begin{align*}
  a(l, -2; k, 0) &= (l - k - 2), \\
  a(l, -2; k, 1) &= -(3l - 4k - 6), \\
  a(l, -2; k, 2) &= 3(l - 2k - 2), \\
  a(l, -2; k, 3) &= -(l - 4k - 2), \\
  a(l, -2; k, 4) &= -k, \\
  d(l, -2) &= l - 2.
\end{align*}
\]

Formula 5: The coefficients of $I_{-4}^t: V_{l-4} \to V_l \otimes_C V_4$ are given as follows:

\[
\begin{align*}
  a(l, -4; k, 0) &= 1, \\
  a(l, -4; k, 1) &= -4, \\
  a(l, -4; k, 2) &= 6, \\
  a(l, -4; k, 3) &= -4, \\
  a(l, -4; k, 4) &= 1, \\
  d(l, -4) &= 1.
\end{align*}
\]

Then $I_{2m}^1$ is a generator of $\operatorname{Hom}_{sl(2, C)}(V_{l+2m}, V_l \otimes_C V_4)$, which is unique up to scalar multiple.

Proof. We have

\[
(\tau_l \otimes \tau_4)(E) \circ I_{2m}^1(v_0^{(l+2m)}) = \sum_{i=0}^4 A_{[l, 2m; 0, i]} \cdot (\tau_l(E)v_0^{(l)}_{2^{-2m-i}}) \otimes w_i + \sum_{i=0}^4 A_{[l, 2m; 0, i]} \cdot v_0^{(l)}_{2^{-2m-i}} \otimes (\tau_4(E)w_i)
\]

\[
= \sum_{i=0}^4 A_{[l, 2m; 0, i]} \cdot ((2 - m - i)v_1^{(l)}_{1-m-i}) \otimes w_i + \sum_{i=1}^4 A_{[l, 2m; 0, i]} \cdot v_2^{(l)}_{2-m-i} \otimes (-iw_{i-1})
\]

\[
= -\sum_{i=0}^4 ((2 - m - i)A_{[l, 2m; 0, i]} + (i + 1)A_{[l, 2m; 0, i+1]}) \cdot v_1^{(l)}_{1-m-i} \otimes w_i.
\]

Here we put $A_{[l, 2m; 0, 5]} = 0$. By direct computation, we confirm

\[(2 - m - i)A_{[l, 2m; 0, i]} + (i + 1)A_{[l, 2m; 0, i+1]} = 0\]

for $-2 \leq m \leq 2$ and $0 \leq i \leq 4$. Hence

\[(\tau_l \otimes \tau_4)(E) \circ I_{2m}^1(v_0^{(l+2m)}) = 0.\]

Moreover, we have

\[(\tau_l \otimes \tau_4)(H) \circ I_{2m}^1(v_0^{(l+2m)}) = (l + 2m)I_{2m}^1(v_0^{(l+2m)}),\]
Then we define a homomorphism $\Phi_r$ as the irreducible decomposition of $K$.

Proof. These are obtained by direct computation.

Via the unique isomorphism between $V_i$, we have

$$0 \leq k \leq l + 2m.$$ 

We confirm these equations by direct computation.

The dual representation of $\tau^V$ is equivalent to $V_i$ as $SU(2)$-modules, since irreducible $l + 1$-dimensional representation of $SU(2)$ is unique up to isomorphism.

Lemma 4.4. Let $\{v_k(l)\} \leq k \leq l\}$ is the dual basis of the standard basis $\{v_k(l) \leq k \leq l\}$.

Via the unique isomorphism between $V_i$ and $V_i^*$ as $K$-modules we have the identification

$$v_k(l) = (-1)^k \frac{(l - k)!}{l!} v_{l-k}.$$ 

for $0 \leq k \leq l$.

Proof. We denote by $\langle , \rangle$ the canonical pairing on $V_i^* \otimes_C V_i$.

Since

$$\langle \tau^V_i(H)v_k(l), v_m(l) \rangle = -\langle v_k(l), \tau^V_i(H)v_m(l) \rangle = (2m - l)\delta_{km} = (2k - l)\delta_{km},$$

we have $\tau^V_i(H)v_k(l) = (2k - l)v_k(l)^*$. Similarly, we obtain

$$\tau^V_i(E)v_k(l) = (k + 1)v_{k+1}(l)^*, \quad \tau^V_i(F)v_k(l) = (l - k + 1)v_{k-1}(l)^*.$$ 

From these equations, we obtain the assertion.

5. The $(\mathfrak{g}, K)$-module structures of principal series representations

5.1. Irreducible decomposition of $(\pi(\nu_0, \sigma_0)|_K, F(\nu_0, \sigma_0))$ as $K$-modules. We set

$$L^2_{(M_0, \sigma_0)}(K) = \{f \in L^2(K) \mid f(mx) = \sigma_0(m)f(x) \text{ for a.e. } m \in M, \ x \in K\}$$

and give a $K$-module structure by the right regular action of $K$. Then the restriction map $r_K : H(\nu_0, \sigma_0) \ni f \mapsto f|_K \in L^2_{(M_0, \sigma_0)}(K)$ is an isomorphism of $K$-modules.

$L^2(K)$ has a $K \times K$-bimodule structure by the two sided regular action:

$$(k_1, k_2)f(x) = f(k_1^{-1}xk_2), \quad x \in K, \ f \in L^2(K), \ (k_1, k_2) \in K \times K.$$ 

Then we define a homomorphism $\Phi_l : V_{2l}^* \otimes_C V_{2l} \to L^2(K)$ of $K \times K$-bimodules by

$$w \otimes v \mapsto (x \mapsto \langle w, \tau_{2l}(x)v \rangle).$$
Then the Peter-Weyl’s theorem tells that
\[ \bigoplus_{l \in \mathbb{Z}_{\geq 0}} \Phi_l : \bigoplus_{l \in \mathbb{Z}_{\geq 0}} V_{2l}^* \otimes \mathbb{C} V_{2l} \to L^2(K) \]
is an isomorphism as $K \times K$-bimodules. Here $\bigoplus$ means a Hilbert space direct sum.
Since $L^2(M_0, \sigma_0)(K) \subset L^2(K)$, we have an irreducible decomposition of $L^2(M_0, \sigma_0)(K)$:
\[ L^2(M_0, \sigma_0)(K) \simeq \bigoplus_{l \in \mathbb{Z}_{\geq 0}} (V_{2l}[\sigma_0]) \otimes \mathbb{C} V_{2l}. \]
Here $V[\sigma_0]$ means the $\sigma_0$-isotypic component in $(\tau|_{M_0}, V)$ for a $K$-module $(\tau, V)$. Therefore we obtain an isomorphism
\[ r_{K}^{-1} \circ \bigoplus_{l \in \mathbb{Z}_{\geq 0}} \Phi_l : \bigoplus_{l \in \mathbb{Z}_{\geq 0}} (V_{2l}[\sigma_0]) \otimes \mathbb{C} V_{2l} \to H(\nu_0, \sigma_0). \]
Since $M_0$ is generated by the two elements
\[ m_{0,1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad m_{0,2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in M_0, \]
we note that $v \in V_{2l}[\sigma_0]$ if and only if
\[ \tau_{2l}(m_{0,i})v = \sigma_0(m_{0,i})v = (-1)^{\sigma_0,i}v \quad (i = 1, 2) \]
for $v \in V_{2l}$. From the definition of $(\tau_{2l}, V_{2l})$ and
\[ \varphi_1^{-1}(m_{0,1}) = \left\{ \pm \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 \end{pmatrix} \right\}, \quad \varphi_1^{-1}(m_{0,2}) = \left\{ \pm \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} \right\}, \]
we have $\tau_{2l}(m_{0,1})v_k^{(2l)} = (-1)^{k}v_{2l-k}^{(2l)}$ and $\tau_{2l}(m_{0,2})v_k^{(2l)} = (-1)^{l-k}v_k^{(2l)}$. Hence we have
\[ V_{2l}[\sigma_0] = \bigoplus_{k \in Z(\sigma_0;l)} \mathbb{C} \cdot (v_{2l-k}^{(2l)} + (-1)^{\epsilon(\sigma_0;l)}v_k^{(2l)}), \]
where $\epsilon(\sigma_0;l) \in \{0, 1\}$ such that $\epsilon(\sigma_0;l) \equiv l - \sigma_1 - \sigma_2 \text{ mod } 2$ and
\[ Z(\sigma_0;l) = \begin{cases} \{ k \in \mathbb{Z} \mid 0 \leq k \leq l, \ k \equiv l - \sigma_0 \text{ mod } 2 \} & \text{if } \epsilon(\sigma_0;l) = 0, \\ \{ k \in \mathbb{Z} \mid 0 \leq k \leq l - 1, \ k \equiv l - \sigma_0 \text{ mod } 2 \} & \text{if } \epsilon(\sigma_0;l) = 1. \end{cases} \]
By the identification $V_{2l}^* = V_{2l}$ in Lemma 4.24, we note that \{ $v_{2l-k}^{(2l)} + (-1)^{\epsilon(\sigma_0;l)}v_k^{(2l)} \mid k \in Z(\sigma_0;l)$ \} is the basis of $V_{2l}^*[\sigma_0]$.
Now we define the elementary function $s(l; p, q) \in H(\nu_0, \sigma_0)$ by
\[ s(l; p, q) = r_K^{-1} \circ \Phi_l^{(j)}((v_{2l-k}^{(2l)} + (-1)^{\epsilon(\sigma_0;l)}v_k^{(2l)})) \otimes \nu_q^{(2l)}) \]
for $l \in \mathbb{Z}_{\geq 0}$, $p \in Z(\sigma_0;l)$ and $0 \leq q \leq 2l$.
For each $p \in Z(\sigma_0;l)$, we put $S(l; p)$ a column vector of degree $2l + 1$ whose $q + 1$-th component is $s(l; p, q)$, i.e. $\{ s(l; p, 0), \ s(l; p, 1), \cdots, s(l; p, 2l) \}$.
Moreover we denote by $\langle S(l; p) \rangle$ the subspace of $H(\nu_0, \sigma_0)$ generated by the entries of the vector $S(l; p)$, i.e. $\langle S(l; p) \rangle = \bigoplus_{q=0}^{2l} \mathbb{C} \cdot s(l; p, q) \simeq V_{2l}$. Via the unique isomorphism between $\langle S(l; p) \rangle$ and $V_{2l}$, we identify $\{ s(l; p, q) \mid 0 \leq q \leq 2l \}$ with the standard basis.
From above arguments, we obtain the following.
Proposition 5.1. As an unitary representation of $K$, it has an irreducible decomposition:

$$H_{(v_{0},v_{0})} \simeq \bigoplus_{l \in \mathbb{Z}_{\geq 0}} (V_{2l}[\sigma_{0}]) \otimes_{\mathbb{C}} V_{2l}.$$  

Then the $\tau_{2l}$-isotypic component of $\pi_{(v_{0},v_{0})}$ is given by

$$\bigoplus_{p \in \mathbb{Z}(\sigma_{0};l)} \langle S(l; p) \rangle.$$  

Corollary 5.2. Let $d(\sigma_{0}; l)$ be the dimension of the space $\text{Hom}_{K}(V_{2l}, H_{(v_{0},v_{0})}, K)$ of intertwining operators. Then

$$d(\sigma_{0}; l) = \begin{cases} \frac{l+2}{2} & \text{if } (\sigma_{0,1}, \sigma_{0,2}) = (0, 0) \text{ and } l \text{ is even}, \\ \frac{l-1}{2} & \text{if } (\sigma_{0,1}, \sigma_{0,2}) = (0, 0) \text{ and } l \text{ is odd}, \\ \frac{l}{2} & \text{if } (\sigma_{0,1}, \sigma_{0,2}) \neq (0, 0) \text{ and } l \text{ is even}, \\ \frac{l+1}{2} & \text{if } (\sigma_{0,1}, \sigma_{0,2}) \neq (0, 0) \text{ and } l \text{ is odd}. \end{cases}$$

5.2. General setting. Let $H_{(\nu_{i},\sigma_{i}),K}$ be the $K$-finite part of $H_{(\nu_{i},\sigma_{i})}$. In order to describe the action of $\mathfrak{g}$ or $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, it suffices to investigate the action of $\mathfrak{p}$ or $\mathfrak{p}_{\mathbb{C}}$, because of the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

For a $K$-type $(\tau_{2l}, V_{2l})$ of $\pi_{(\nu_{i},\sigma_{i})}$ and a nonzero $K$-homomorphism $\eta: V_{2l} \rightarrow H_{(\nu_{i},\sigma_{i}),K}$, we define a linear map

$$\tilde{\eta}: \mathfrak{p}_{\mathbb{C}} \otimes_{\mathbb{C}} V_{2l} \rightarrow H_{(\nu_{i},\sigma_{i}),K}$$

by $X \otimes v \mapsto \pi_{(\nu_{i},\sigma_{i})}(X)\eta(v)$. Here we denote differential of $\pi_{(\nu_{i},\sigma_{i})}$ again by $\pi_{(\nu_{i},\sigma_{i})}$. Then $\tilde{\eta}$ is $K$-homomorphism with $\mathfrak{p}_{\mathbb{C}}$ endowed with the adjoint action $\text{Ad}$ of $K$.

Since

$$V_{2l} \otimes_{\mathbb{C}} \mathfrak{p}_{\mathbb{C}} \simeq V_{2l} \otimes_{\mathbb{C}} V_{4} \simeq \bigoplus_{-2 \leq m \leq 2} V_{2l+m},$$

there are five injective $K$-homomorphisms

$$I_{2m}^{2l}: V_{2l+m} \rightarrow V_{2l} \otimes_{\mathbb{C}} \mathfrak{p}_{\mathbb{C}}, \quad -2 \leq m \leq 2$$

for general $l \in \mathbb{Z}_{\geq 0}$. Then we define $\mathbb{C}$-linear maps

$$\Gamma_{l,m}^{i}: \text{Hom}_{K}(V_{2l}, H_{(\nu_{i},\sigma_{i}),K}) \rightarrow \text{Hom}_{K}(V_{2l+m}, H_{(\nu_{i},\sigma_{i}),K}), \quad -2 \leq m \leq 2$$

by $\eta \mapsto \tilde{\eta} \circ I_{2m}^{2l}$.

Now we settle two purposes of this paper:

(i): Describe the injective $K$-homomorphism $I_{2m}^{2l}$ in terms of the standard basis.

(ii): Determine the matrix representations of the linear homomorphisms $\Gamma_{l,m}^{i}$ with respect to the induced basis defined in the next subsection.

We have already accomplished the first purpose in Proposition 4.2. We accomplish the second purpose in Theorems 5.3 and 5.5. As a result, we obtain infinite number of ‘contiguous relations’, a kind of system of differential-difference relations among vectors in $H_{(\nu_{i},\sigma_{i})}[\tau_{2l}]$ and $H_{(\nu_{i},\sigma_{i})}[\tau_{2l+m}]$. Here $H_{(\nu_{i},\sigma_{i})}[\tau_{l}]$ is $\tau$-isotypic component of $H_{(\nu_{i},\sigma_{i})}$.

5.3. The canonical blocks of elementary functions. Let $\eta: V_{2l} \rightarrow H_{(\nu_{i},\sigma_{i}),K}$ be a non-zero $K$-homomorphism. Then we identify $\eta$ with the column vector of degree $2l+1$ whose $q+1$-th component is $\eta(v_{2l}^{(q+1)})$ for $0 \leq q \leq 2l$, i.e. $\{ \eta(v_{0}^{(0)}), \eta(v_{1}^{(2l)}), \eta(v_{1}^{(2l)}), \ldots, \eta(v_{2l}^{(2l)}) \}$.

By this identification, we identify $S(l;p)$ with the injective $K$-homomorphism

$$V_{2l} \ni v_{q}^{(2l)} \mapsto s(l;p,q) \in H_{(\nu_{0},v_{0}),K}, \quad 0 \leq q \leq 2l$$

for $p \in \mathbb{Z}(\sigma_{0};l)$. We note that $\{ S(l;p) \mid p \in \mathbb{Z}(\sigma_{0};l) \}$ is a basis of $\text{Hom}_{K}(V_{2l}, H_{(\nu_{0},v_{0}),K})$ and we call it the induced basis from the standard basis.
We define a certain matrix of elementary functions corresponding to the induced basis \( \{ S(l; p) \mid p \in Z(\sigma_0; l) \} \) of \( \Hom_K(V_{2l}, H_{(\nu_0, \sigma_0), K}) \) for each \( K \)-type \( \tau_{2l} \) of our principal series representation \( \pi_{(\nu_0, \sigma_0)} \).

**Definition 5.3.** The following \((2l + 1) \times d(\sigma_0; l)\) matrix \( S(\sigma_0; l) \) is called the canonical block of elementary functions for \( \tau_{2l} \)-isotypic component: When \((\sigma_{0,1}, \sigma_{0,2}) = (0, 0)\), we consider the matrix

\[
S(\sigma_0; l) = \begin{cases} 
( S(l; 0), S(l; 2), S(l; 4), \cdots, S(l; l) ) & \text{if } l \text{ is even}, \\
( S(l; 1), S(l; 3), S(l; 5), \cdots, S(l; l - 2) ) & \text{if } l \text{ is odd}.
\end{cases}
\]

When \((\sigma_{0,1}, \sigma_{0,2}) = (1, 0)\), we consider the matrix

\[
S(\sigma_0; l) = \begin{cases} 
( S(l; 0), S(l; 2), S(l; 4), \cdots, S(l; l - 2) ) & \text{if } l \text{ is even}, \\
( S(l; 1), S(l; 3), S(l; 5), \cdots, S(l; l) ) & \text{if } l \text{ is odd}.
\end{cases}
\]

When \(\sigma_{0,2} = 1\), we consider the matrix

\[
S(\sigma_0; l) = \begin{cases} 
( S(l; 1), S(l; 3), S(l; 5), \cdots, S(l; l - 1) ) & \text{if } l \text{ is even}, \\
( S(l; 0), S(l; 2), S(l; 4), \cdots, S(l; l - 1) ) & \text{if } l \text{ is odd}.
\end{cases}
\]

5.4. The \(\mathbf{pC}\)-matrix corresponding to \( I_{2m}^l \). For two integers \( c_0, c_1 \) such that \( c_0 \leq c_1 \) and a rational function \( f(x) \) in the variable \( x \), we denote by

\[
\text{Diag} \ (f(n))_{c_0 \leq n \leq c_1}
\]

the diagonal matrix of size \( c_1 - c_0 + 1 \) with an entry \( f(n) \) at the \((n - c_0 + 1, n - c_0 + 1)\)-th component. Let \( e_i^{(l)} \) \((0 \leq i \leq l)\) be the column unit vector of degree \( l + 1 \) with its \( i + 1 \)-th component \( 1 \) and the remaining components \( 0 \). Moreover, let \( e_i^{(0)} \) be the column zero vector of degree \( l + 1 \) when \( i < 0 \) or \( l < i \).

In this subsection, we define \(\mathbf{pC}\)-matrix \( \mathbf{c}_{l,m} \) of size \((2(l + m) + 1) \times (2l + 1)\) corresponding to \( I_{2m}^l \) with respect to the standard basis.

Let \( \sum_{i=0}^{4} \iota_i^{(l,m)} \otimes X_i \) be the image of \( I_{2m}^l \) by the composite of natural linear maps

\[
\Hom_K(V_{2(l+m)}, V_2 \otimes \mathbf{C} \mathbf{pC}) \rightarrow \Hom_C(V_{2(l+m)}, V_2 \otimes \mathbf{C} \mathbf{pC}) \simeq \Hom_C(V_{2(l+m)}, V_2) \otimes \mathbf{C} \mathbf{pC}.
\]

Then we define \(\mathbf{pC}\)-matrix \( \mathbf{c}_{l,m} = \sum_{i=0}^{4} R(\iota_i^{(l,m)}) \otimes X_i \) where \( R(\iota_i^{(l,m)}) \) is the matrix representation of \( \iota_i^{(l,m)} \) with respect to the standard basis. Explicit expression of the matrix \( R(\iota_i^{(l,m)}) \) of size \( (2(l + m) + 1) \) \times \((2l + 1)\) is given by

\[
\begin{pmatrix}
O_{2(l+m)+1,m+2}, R(\iota_0^{(l,m)}), O_{2(l+m)+1,m+2} \\
O_{2(l+m)+1,4} - i & \text{Diag}_{0 \leq k \leq 2(l+m)}\ A_{[2l,2m;k,i],1}
\end{pmatrix}
\]

for \(-2 \leq m \leq 2\) and \(0 \leq i \leq 4\). Here we erase the symbol \( O_{m,n} \) when \( m = 0 \) or \( n = 0 \).

For a column vector \( \mathbf{v} = \{ v_0, v_1, \cdots, v_{2l} \} \) \(\in (H_{(\nu_0, \sigma_0), K})^{\otimes 2l+1} \) which is identified with an element of \( \Hom_K(V_{2l}, H_{(\nu_0, \sigma_0), K}) \), we define \( \mathbf{c}_{l,m} \mathbf{v} \in (H_{(\nu_0, \sigma_0), K})^{\otimes 2(l+m)+1} \simeq \mathbf{C}^{2(l+m)+1} \otimes \mathbf{C} H_{(\nu_0, \sigma_0), K} \) by

\[
\mathbf{c}_{l,m} \mathbf{v} = \sum_{0 \leq i \leq 4} \sum_{0 \leq q \leq 2l} R(\iota_i^{(l,m)}) \cdot e_q^{(2l)} \otimes (\pi_{(\nu_0, \sigma_0)}(X_i)v_q).
\]

Here \( R(\iota_i^{(l,m)}) \cdot e_q^{(2l)} \) is the ordinal product of matrices \( R(\iota_i^{(l,m)}) \) and \( e_q^{(2l)} \).

From the definition of \( \mathbf{c}_{l,m} \), we note that the vector \( \mathbf{c}_{l,m} \mathbf{v} \) is identified with the image of \( \mathbf{v} \) by \( \Gamma_{l,m}^i \).
5.5. The contiguous relations.

**Lemma 5.4.** The standard basis $X_i$ ($0 \leq i \leq 4$) in $\mathfrak{p}_C$ have the following expressions according to the Iwasawa decomposition $\mathfrak{g}_C = \mathfrak{n}_C \oplus \mathfrak{a}_C \oplus \mathfrak{f}_C$:

\[
\begin{align*}
X_0 &= -2\sqrt{-1}E_{e_2-e_3} + H_2 + \sqrt{-1}K_{23}, \\
X_1 &= -(E_{e_1-e_3} + \sqrt{-1}E_{e_1-e_2}) + \frac{1}{2}(K_{13} + \sqrt{-1}K_{12}), \\
X_2 &= -\frac{1}{3}(2H_1 - H_2), \\
X_3 &= (E_{e_1-e_3} - \sqrt{-1}E_{e_1-e_2}) - \frac{1}{2}(K_{13} - \sqrt{-1}K_{12}), \\
X_4 &= 2\sqrt{-1}E_{e_2-e_3} + H_2 - \sqrt{-1}K_{23}.
\end{align*}
\]

**Proof.** We obtain the assertion immediately from Lemma 4.1. □

We give the matrix representation of $\Gamma^0_{l,m}$ with respect to the induced basis as follows.

**Theorem 5.5.** For $l \in \mathbb{Z}_{\geq 0}$, $-2 \leq m \leq 2$ such that $d(\sigma_0;l) > 0$ and $d(\sigma_0;l+m) > 0$, we have

\[
\mathcal{E}_{l,m} \mathbf{S}(\sigma_0;l) = \mathbf{S}(\sigma_0;l+m) \cdot R(\Gamma^0_{l,m})
\]

with the matrix representation $R(\Gamma^0_{l,m}) \in M_{d(\sigma_0;l+m),d(\sigma_0;l)}(\mathbb{C})$ of $\Gamma^0_{l,m}$ with respect to the induced basis $\{S(l;p) \mid p \in \mathbb{Z}(\sigma_0;l)\}$.

Explicit expressions of the matrix $R(\Gamma^0_{l,m})$ of size $d(\sigma_0;l+m) \times d(\sigma_0;l)$ is given as follows:

When $\sigma_{0,2} = 0$ and $(m,\sigma_{0,1}+l) \in \{0, \pm 2\} \times (2\mathbb{Z})$, the matrix $R(\Gamma^0_{l,m})$ is given by

\[
\begin{pmatrix}
O_{n(\sigma_0;l),d(\sigma_0;l)} \\
R(\Gamma^0_{l,m})
\end{pmatrix} = \begin{pmatrix}
\text{Diag} & O_{1,d(\sigma_0;l)} \\
\gamma_{l,m:2k+\delta(\sigma_0;l),-1}^{(0)} & \text{Diag} & O_{2,d(\sigma_0;l)} \\
& \text{Diag} & O_{2,d(\sigma_0;l)} \\
& & O_{1,d(\sigma_0;l)} \\
& & \gamma_{l,m:2k+\delta(\sigma_0;l),0}^{(0)}
\end{pmatrix}
\]

When $\sigma_{0,2} = 0$ and $(m,\sigma_{0,1}+l) \in \{0, \pm 2\} \times (1+2\mathbb{Z})$, the matrix $R(\Gamma^0_{l,m})$ is given by

\[
\begin{pmatrix}
O_{n(\sigma_0;l),d(\sigma_0;l)} \\
R(\Gamma^0_{l,m})
\end{pmatrix} = \begin{pmatrix}
\text{Diag} & O_{1,d(\sigma_0;l)} \\
\gamma_{l,m:2k+\delta(\sigma_0;l),-1}^{(0)} & \text{Diag} & O_{2,d(\sigma_0;l)} \\
& \text{Diag} & O_{2,d(\sigma_0;l)} \\
& & O_{1,d(\sigma_0;l)} \\
& & \gamma_{l,m:2k+\delta(\sigma_0;l),0}^{(0)}
\end{pmatrix}
\]

When $\sigma_{0,2} = 0$, $(m,\sigma_{0,1}+l) \in \{\pm 1\} \times (2\mathbb{Z})$ and $d(\sigma_0;l) = 1$, the matrix $R(\Gamma^0_{l,m})$ is given by

\[
R(\Gamma^0_{l,m}) = \begin{pmatrix}
\gamma_{l,m:\delta(\sigma_0;l),-1}^{(0)}
\end{pmatrix}
\]
When $\sigma_{0,2} = 0$, $(m, \sigma_{0,1} + l) \in \{\pm 1\} \times (2\mathbb{Z})$ and $d(\sigma_{0}; l) > 1$, the matrix $R(\Gamma_{l,m}^0)$ is given by

$$
\begin{pmatrix}
O_{n(\sigma_{0},l,m),d(\sigma_{0}; l)} & \sum_{0 \leq k \leq d(\sigma_{0}; l)-1} \left( \begin{array}{c}
\operatorname{Diag} \left( \gamma_{[l,m;2k+\delta(\sigma_{0}; l),-1]}^{(0)} \right)
\end{array} \right)
\end{pmatrix} = \begin{pmatrix}
O_{1,d(\sigma_{0}; l)} & -1 & 0 \\
O_{2,d(\sigma_{0}; l)} & 0 & 2,1 \\
\operatorname{Diag} \left( \gamma_{[l,m;2k+\delta(\sigma_{0}; l),1]}^{(0)} \right) & O_{d(\sigma_{0}; l)-2,1} & -\gamma_{[l,m;1]}^{(0)} e_{d(\sigma_{0}; l)-3}
\end{pmatrix}.
$$

When $\sigma_{0,2} = 0$ and $(m, \sigma_{0,1} + l) \in \{\pm 1\} \times (1 + 2\mathbb{Z})$, the matrix $R(\Gamma_{l,m}^0)$ is given by

$$
\begin{pmatrix}
O_{n(\sigma_{0},l,m),d(\sigma_{0}; l)} & \sum_{0 \leq k \leq d(\sigma_{0}; l)-1} \left( \begin{array}{c}
\operatorname{Diag} \left( \gamma_{[l,m;2k+\delta(\sigma_{0}; l),-1]}^{(0)} \right)
\end{array} \right)
\end{pmatrix} = \begin{pmatrix}
O_{1,d(\sigma_{0}; l)} & -1 & 0 \\
O_{2,d(\sigma_{0}; l)} & 0 & 2,1 \\
\operatorname{Diag} \left( \gamma_{[l,m;2k+\delta(\sigma_{0}; l),1]}^{(0)} \right) & O_{d(\sigma_{0}; l)-2,1} & -\gamma_{[l,m;1]}^{(0)} e_{d(\sigma_{0}; l)-2}
\end{pmatrix}.
$$

When $\sigma_{0,2} = 1$, the matrix $R(\Gamma_{l,m}^0)$ is given by

$$
\begin{pmatrix}
O_{n(\sigma_{0},l,m),d(\sigma_{0}; l)} & \sum_{0 \leq k \leq d(\sigma_{0}; l)-1} \left( \begin{array}{c}
\operatorname{Diag} \left( \gamma_{[l,m;2k+\delta(\sigma_{0}; l),-1]}^{(0)} \right)
\end{array} \right)
\end{pmatrix} = \begin{pmatrix}
O_{1,d(\sigma_{0}; l)} & -1 & 0 \\
O_{2,d(\sigma_{0}; l)} & 0 & 2,1 \\
\operatorname{Diag} \left( \gamma_{[l,m;2k+\delta(\sigma_{0}; l),1]}^{(0)} \right) & (-1)^{\varepsilon(\sigma_{0}; l+m)} \gamma_{[l,m;1]}^{(0)} e_{d(\sigma_{0}; l)-2}
\end{pmatrix}.
$$

Here

$$
\gamma_{[l,m,p;1]}^{(0)} = (\nu_{0,2} + \rho_{0,2} - l + p) A_{[2,2m;2l-p+m-2,0]},
$$

$$
\gamma_{[l,m,p;0]}^{(0)} = -\frac{1}{3} \left( 2\nu_{0,1} - \nu_{0,2} + 2\rho_{0,1} - \rho_{0,2} + lm - 3 + \frac{m(m+1)}{2} \right) A_{[2,2m;2l-p+m-2]},
$$

$$
\gamma_{[l,m,p;-1]}^{(0)} = (\nu_{0,2} + \rho_{0,2} + l - p) A_{[2,2m;2l-p+m+2,4]},
$$

$$
n(\sigma_{0}; l, m) = \begin{cases} 
(2-m)/2 & \text{if } m \in \{0, \pm 2\}, \\
(3-m)/2 & \text{if } (m, l + \sigma_{0,2}) \in \{\pm 1\} \times (2\mathbb{Z}), \\
(1-m)/2 & \text{if } (m, l + \sigma_{0,2}) \in \{\pm 1\} \times (1 + 2\mathbb{Z}),
\end{cases}
$$

and $\delta(\sigma_{0}; l) \equiv l - \sigma_{0,2}$ mod 2.

In the above equations, we put $A_{[2,2m;2l-k,0]} = 0$ for $k < 0$ or $k > 2(l + m)$, and erase the symbols $e_{n \leq k \leq -1}$.

Proof. Since

$$
s(l; p, q)(13) = \left( (v_{2l-p}^{(2l)})^* + (-1)^{\varepsilon(\sigma_{0}; l)} v_{p}^{(2l)} \right) = \delta_{2l-p,q} + (-1)^{\varepsilon(\sigma_{0}; l)} \delta_{pq},
$$

we have

$$
S(l; p)(13) = e_{2l-p}^{(2l)} + (-1)^{\varepsilon(\sigma_{0}; l)} e_{p}^{(2l)}.
$$
Hence \(S(l; p)(1_3)\) \((p \in Z(\sigma_0; l))\) are linearly independent over \(\mathbf{C}\). Thus we note that it suffices to evaluate the both side of the equation (5.11) at \(1_3 \in G\).

First, we compute \(\{\pi_{(\nu_0, \sigma_0)}(X_i) s(l; p, q)\}(1_3)\) for \(0 \leq i \leq 4, \ p \in Z(\sigma_0; l)\) and \(0 \leq q \leq 2l\). Since \(\{s(l; q) \mid 0 \leq q \leq 2l\}\) is the standard basis of \(\langle S(l; p) \rangle\), we obtain

\[
\begin{align*}
\{\pi_{(\nu_0, \sigma_0)}(\sqrt{-1} K_{23}) s(l; p, q)\}(1_3) &= (l - q)s(l; p, q)(1_3) \\
&= (l - q)(\delta_{2l-p} + (-1)^\varepsilon(\sigma_0; l)\delta_{pq}),
\end{align*}
\]
\[
\begin{align*}
\{\pi_{(\nu_0, \sigma_0)}(K_{13} + \sqrt{-1} K_{12}) s(l; p, q)\}(1_3) &= -q \cdot s(l; p, q - 1)(1_3) \\
&= -q(\delta_{2l-p+1} + (-1)^\varepsilon(\sigma_0; l)\delta_{p+1q}),
\end{align*}
\]
\[
\begin{align*}
\{\pi_{(\nu_0, \sigma_0)}(K_{13} - \sqrt{-1} K_{12}) s(l; p, q)\}(1_3) &= (2l - q)s(l; p, q + 1)(1_3) \\
&= (2l - q)(\delta_{2l-p-1} + (-1)^\varepsilon(\sigma_0; l)\delta_{p-1q}).
\end{align*}
\]

Moreover, we obtained

\[
\begin{align*}
\{\pi_{(\nu_0, \sigma_0)}(E_\alpha) s(l; p, q)\}(1_3) &= 0, \quad \alpha \in \Sigma^+, \\
\{\pi_{(\nu_0, \sigma_0)}(H_i) s(l; p, q)\}(1_3) &= (\nu_0, i + \rho_0, i)s(l; p, q)(1_3) \\
&= (\nu_0, i + \rho_0, i)(\delta_{2l-p} + (-1)^\varepsilon(\sigma_0; l)\delta_{pq}), \quad i = 1, 2,
\end{align*}
\]

from the definition of principal series representation. From these computations and Iwasawa decomposition in Lemma 5.4, we obtain

\[
\begin{align*}
\{\pi_{(\nu_0, \sigma_0)}(X_0) s(l; p, q)\}(1_3) &= (\nu_0, 2 + \rho_0, 2 + l - q)(\delta_{2l-p} + (-1)^\varepsilon(\sigma_0; l)\delta_{pq}), \\
\{\pi_{(\nu_0, \sigma_0)}(X_1) s(l; p, q)\}(1_3) &= -\frac{q}{2}(\delta_{2l-p+1} + (-1)^\varepsilon(\sigma_0; l)\delta_{p+1q}), \\
\{\pi_{(\nu_0, \sigma_0)}(X_2) s(l; p, q)\}(1_3) &= -\frac{1}{3}(2\nu_0, 1 - \nu_0, 2 + 2\rho_0, 1 - \rho_0, 2)(\delta_{2l-p} + (-1)^\varepsilon(\sigma_0; l)\delta_{pq}), \\
\{\pi_{(\nu_0, \sigma_0)}(X_3) s(l; p, q)\}(1_3) &= -\frac{2l - q}{2}(\delta_{2l-p-1} + (-1)^\varepsilon(\sigma_0; l)\delta_{p-1q}), \\
\{\pi_{(\nu_0, \sigma_0)}(X_4) s(l; p, q)\}(1_3) &= (\nu_0, 2 + \rho_0, 2 - l + q)(\delta_{2l-p} + (-1)^\varepsilon(\sigma_0; l)\delta_{pq}).
\end{align*}
\]

We set

\[
\pi_{(\nu_0, \sigma_0)}(X_i) S(l; p) = \sum_{0 \leq q \leq 2l} e_{l(q)}^{(2l)} \otimes (\pi_{(\nu_0, \sigma_0)}(X_i) s(l; p, q)).
\]

Then we obtain

\[
\begin{align*}
\{\pi_{(\nu_0, \sigma_0)}(X_0) S(l; p)\}(1_3) &= (\nu_0, 2 + \rho_0, 2 - l + p)e_{2l-p}^{(2l)} + (-1)^\varepsilon(\sigma_0; l)(\nu_0, 2 + \rho_0, 2 + l - p)e_{p}^{(2l)}, \\
\{\pi_{(\nu_0, \sigma_0)}(X_1) S(l; p)\}(1_3) &= -\frac{2l - p + 1}{2}e_{2l-p-1}^{(2l)} + (-1)^\varepsilon(\sigma_0; l)p + \frac{1}{2}e_{p}^{(2l)}, \\
\{\pi_{(\nu_0, \sigma_0)}(X_2) S(l; p)\}(1_3) &= -\frac{1}{3}(2\nu_0, 1 - \nu_0, 2 + 2\rho_0, 1 - \rho_0, 2)e_{2l-p}^{(2l)} + (-1)^\varepsilon(\sigma_0; l)e_{p}^{(2l)}, \\
\{\pi_{(\nu_0, \sigma_0)}(X_3) S(l; p)\}(1_3) &= -\frac{p + 1}{2}e_{2l-p-1}^{(2l)} + (-1)^\varepsilon(\sigma_0; l)\frac{2l - p + 1}{2}e_{p}^{(2l)}, \\
\{\pi_{(\nu_0, \sigma_0)}(X_4) S(l; p)\}(1_3) &= (\nu_0, 2 + \rho_0, 2 + l - p)e_{2l-p}^{(2l)} + (-1)^\varepsilon(\sigma_0; l)(\nu_0, 2 + \rho_0, 2 - l + p)e_{p}^{(2l)}.
\end{align*}
\]
Let us compute \( \{c_{l,m}S(l;p)\}(1_3) \). By the above equations, we have

\[
\{c_{l,m}S(l;p)\}(1_3) = \sum_{0 \leq i \leq 1} R(\xi^{(l,m)}_i) \cdot e^{(2l)}_q \otimes \{(\pi_{(\nu_0,\sigma_0)}(X_i)s(l;p,q))\}(1_3)
\]

\[
= \sum_{0 \leq i \leq 1} R(\xi^{(l,m)}_i) \cdot \{(\pi_{(\nu_0,\sigma_0)}(X_i)S(l;p))\}(1_3)
\]

\[
= R(i^{(l,m)}_0) \cdot \{(\nu_{0,2} + \rho_{0,2} - l + p)e^{(2l)}_p - (1)^{(\sigma_0,l)}(\nu_{0,2} + \rho_{0,2} + l - p)e^{(2l)}_p \}
\]

\[
+ R(i^{(l,m)}_1) \cdot \{- \frac{2l - p + 1}{2} e^{(2l)}_{2l-p+1} - (1)^{(\sigma_0,l)} p + \frac{1}{2} e^{(2l)}_p \}
\]

\[
+ R(i^{(l,m)}_2) \cdot \{- \frac{1}{3} (2\nu_{0,1} - \nu_{0,2} + 2\rho_{0,1} - \rho_{0,2})(e^{(2l)}_{2l-p} + (1)^{(\sigma_0,l)} e^{(2l)}_p \}
\]

\[
+ R(i^{(l,m)}_3) \cdot \{- \frac{p + 1}{2} e^{(2l)}_{2l-p-1} - (1)^{(\sigma_0,l)} \frac{2l - p + 1}{2} e^{(2l)}_p \}
\]

\[
+ R(i^{(l,m)}_4) \cdot \{(\nu_{0,2} + \rho_{0,2} + l - p)e^{(2l)}_{2l-p} + (1)^{(\sigma_0,l)}(\nu_{0,2} + \rho_{0,2} - l + p)e^{(2l)}_p \}.
\]

Since

\[
R(i^{(l,m)}_i)e^{(2l)}_q = A_{2l,2m;q+m+2i+1} e^{(2l+m)}_{q+m+2i}, \quad -2 \leq m \leq 2,
\]

we obtain

\[
\{c_{l,m}S(l;p)\}(1_3) = \sum_{-1 \leq i \leq 1} \{\alpha_{(l,m;p,i)} e^{(2l+m)}_{2l+m-(p+m+2i)} + (1)^{(\sigma_0,l)} \beta_{(l,m;p,i)} e^{(2l+m)}_{p+m+2i} \},
\]

where

\[
\alpha_{(l,m;p,1)} = (\nu_{0,2} + \rho_{0,2} - l + p)A_{2l,2m;2l-p+m-2,0},
\]

\[
\alpha_{(l,m;p,0)} = - \frac{1}{3} (2\nu_{0,1} - \nu_{0,2} + 2\rho_{0,1} - \rho_{0,2})A_{2l,2m;2l-p+m,2}
\]

\[
- \frac{2l - p + 1}{2} A_{2l,2m;2l-p+m,1} - \frac{p + 1}{2} A_{2l,2m;2l-p+m,3},
\]

\[
\alpha_{(l,m;p,-1)} = (\nu_{0,2} + \rho_{0,2} + l - p)A_{2l,2m;2l-p+m+2,4},
\]

\[
\beta_{(l,m;p,1)} = (\nu_{0,2} + \rho_{0,2} - l + p)A_{2l,2m;2l-p+m+2,4},
\]

\[
\beta_{(l,m;p,0)} = - \frac{1}{3} (2\nu_{0,1} - \nu_{0,2} + 2\rho_{0,1} - \rho_{0,2})A_{2l,2m;2l-p+m,2}
\]

\[
- \frac{p + 1}{2} A_{2l,2m;2l-p+m,1} - \frac{2l - p + 1}{2} A_{2l,2m;2l-p+m,3},
\]

\[
\beta_{(l,m;p,-1)} = (\nu_{0,2} + \rho_{0,2} + l - p)A_{2l,2m;2l-p+m-2,0}.
\]

By the relations of the coefficients \( A_{2l,2m;k,i} \) in Lemma 4.3 we see that

\[
\alpha_{(l,m;p,i)} = (-1)^{i} \beta_{(l,m;p,i)} = \gamma_{(l,m;p,i)}^{(0)}, \quad -1 \leq i \leq 1.
\]

Therefore, \( \{c_{l,m}S(l;p)\}(1_3) \) become

\[
\{c_{l,m}S(l;p)\}(1_3) = \sum_{-1 \leq i \leq 1} \gamma_{(l,m;p,i)}^{(0)} \{e^{(2l+m)}_{2l+m-(p+m+2i)} + (1)^{(\sigma_0,l+m)} e^{(2l+m)}_{p+m+2i} \}.
\]

From the equations 4.3.2, 5.1.4 and

\[
(\sigma_0;l) + m \equiv (\sigma_0;l + m) \mod 2,
\]

we obtain the assertion. \( \square \)
6. The $(g, K)$-module structures of the generalized principal series representations

In this section, we set $i = 1$ or 2.

6.1. Discrete series representations of $SL^+(2, \mathbb{R})$. We set $y_0 = \text{diag}(1, -1) \in O(2)$. Then a discrete series representation $(D_k, V_{D_k})$ of $SL^+(2, \mathbb{R})$ is uniquely determined by specifying the $G' = SL(2, \mathbb{R})$-module structure together with the action of $y_0$. Since $D_k|G' = D_k^+ \oplus D_k^-$ and $D_k^+ \oplus D_k^-$ is identified with $G'$-submodule of the principal series representation $(\pi_{(\nu, \sigma)}, H_{(\nu, \sigma)})$ of $G'$ by Proposition 2.2, we obtain the following realization of $(D_k, V_{D_k})$:

$$V_{D_k, O(2)} = \bigoplus_{\alpha \in \mathbb{Z}_{\geq 0}} W_{k+2\alpha}$$

and

$$D_k(w)\chi_p = \sqrt{1-p}\chi_p, \quad D_k(x_+)\chi_p = (k+p)\chi_{p+2}, \quad D_k(x_-)\chi_p = (k-p)\chi_{p-2},$$

$$D_k(\kappa_t)\chi_p = e^{\sqrt{1-p}t}\chi_p \quad (t \in \mathbb{R}), \quad D_k(y_0)\chi_p = \chi_p.$$  

Here we denote differential of $D_k$ again by $D_k$ and the $O(2)$-finite part of $V_{D_k}$ by $V_{D_k, O(2)}$.

6.2. Irreducible decompositions of $(\pi_{(\nu_1, \sigma_1)}|K, H_{(\nu_1, \sigma_1)})$ and $(\pi_{(\nu_2, \sigma_2)}|K, H_{(\nu_2, \sigma_2)})$ as $K$-modules. We analyzes the $K$-type of the representation space $H_{(\nu, \sigma)}$ of the $P_1$-principal series representation. the target space $V_{\sigma}$ of functions $f$ in $H_{(\nu, \sigma)}$ has a decomposition:

$$V_{\sigma} = V_{D_k} = \bigoplus_{\alpha \in \mathbb{Z}_{\geq 0}} W_{k+2\alpha}.$$  

Denote the corresponding decomposition of $f$ by

$$f(x) = \sum_{\alpha = 0}^{\infty} (f_{k+2\alpha}(x) \otimes \chi_{k+2\alpha} + f_{-(k+2\alpha)}(x) \otimes \chi_{-(k+2\alpha)}).$$

From the definition of the space $H_{(\nu, \sigma)}$, we have

$$f|_K(mx) = \sigma_i(m)f|_K(x) \quad \text{(a.e. } x \in K, \ m \in K_i = M_i \cap K \simeq O(2)).$$

For $m = m_i(\kappa_t), m_i(y_0)$, comparing the coefficients of $\chi_p$ in the left hand side with those in the right hand side, we have the equations

$$f_p|_K(m_i(\kappa_t)x) = e^{\sqrt{1-p}t}f_p|_K(x), \quad f_p|_K(m_i(y_0)x) = f_{-p}|_K(x).$$

Moreover, from the equality of inner products

$$\int_K ||f|_K(x)||^2_{\nu, \sigma} dx = \sum_{\epsilon \in \{\pm 1\}, \alpha \in \mathbb{Z}_{\geq 0}} \left\{ \int_K |f_{\epsilon(k+2\alpha)}|_K(x) | dx \right\} ||\chi_{\epsilon(k+2\alpha)}||^2_{\nu, \sigma},$$

we have $f_p|_K \in L^2(K)$. Therefore $f|_K$ belongs to

$$\bigoplus_{\alpha \in \mathbb{Z}_{\geq 0}} L^2_i(K; W_{k+2\alpha})$$

where

$L^2_i(K; W_p) = \{ f : K \to W_p \mid f(x) = f(x) \otimes \chi_p + f(m_i(y_0)x) \otimes \chi_{-p}, \ f \in L^2(K_i, \chi_p)(K), x \in K \}$,

$L^2_i(K_i, \chi_p)(K) = \{ f \in L^2(K) \mid f(m_i(\kappa_t)x) = e^{\sqrt{1-p}t}f(x), \ m_i(\kappa_t) \in K_i, \ x \in K \}$. 

Here $K_1^\circ$ means the connected component of $K$, which is isomorphic to $SO(2)$. We easily see that the restriction map

$$r_K^{(i)} : H_{(\nu, \sigma)} \ni f \mapsto f|_K \in \bigoplus_{\alpha \in \mathbb{Z}_{\geq 0}} L_1^2(K; W_{k+2\alpha})$$

is a $K$-isomorphism.

By the Peter-Weyl's theorem, we have an irreducible decomposition of $L_2^2(K; \chi_\nu)(K)$:

$$L_2^2(K; \chi_\nu)(K) \simeq \bigoplus_{l \in \mathbb{Z}_{\geq 0}} (V_{2l}^*[\xi_{(i;l)}]) \otimes \mathbb{C} V_{2l}.$$ 

Here

$$\xi_{(i;l)} : K \ni m_i(\kappa_t) \mapsto e^{\sqrt{-1}pt} \in \mathbb{C}^\times$$

and $V[\xi_{(i;l)}]$ means the $\xi_{(i;l)}$-isotypic component in $(\tau|_{K_1^\circ}, V)$ for a $K$-module $(\tau, V)$.

In this section, we denote by \{v_{1,q}^{(2l)} \mid 0 \leq q \leq 2l\} the standard basis of $V_{2l}$. We define another basis \{v_{2,q}^{(2l)} \mid 0 \leq q \leq 2l\} of $V_{2l}$ by

$$v_{2,q}^{(2l)} = \tau_{2l}(u_c)v_{1,q}^{(2l)} = \frac{1}{2l}(x+y)^q(-x+y)^{2l-q} \quad (0 \leq q \leq 2l)$$

where

$$u_c = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in SO(3).$$

We note that $v \in V_{2l}[\xi_{(i;l)}]$ if and only if

$$\tau_{2l}(m_i(\kappa_t))v = \xi_{(i;l)}(m_i(\kappa_t))v = e^{-\sqrt{-1}pt}v \quad (t \in \mathbb{R})$$

for $v \in V_{2l}$. From the definition of $(\tau_{2l}, V_{2l})$ and

$$\varphi^{-1}(m_1(\kappa_t)) = \varphi^{-1}(u_c^{-1}m_2(\kappa_t)u_c) = \{ \pm \text{diag}(e^{-\sqrt{-1}t/2}, e^{\sqrt{-1}t/2}) \},$$

we have $\tau_{2l}(m_i(\kappa_t))v_{i,q}^{(2l)} = e^{\sqrt{-1}(q-t)}v_{i,q}^{(2l)}$. Hence we have

$$V_{2l}[\xi_{(i;l)}] = \begin{cases} \mathbb{C} \cdot v_{i, l-p}^{(2l)} & \text{if } -l \leq p \leq l, \\ 0 & \text{otherwise}. \end{cases}$$

By the identification $V_{2l}^* = V_{2l}$ in Lemma 4.3, we obtain

$$L_2^2(K; \chi_\nu)(K) \simeq \bigoplus_{l \in \mathbb{Z}_{\geq 0}} \left( \mathbb{C} \cdot v_{i, l+p}^{(2l)^*} \right) \otimes \mathbb{C} V_{2l}.$$ 

Moreover, since

$$\varphi^{-1}(m_1(y_0)) = \left\{ \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}, \quad \varphi^{-1}(u_c^{-1}m_2(y_0)u_c) = \left\{ \pm \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} \right\},$$

we have

$$\tau_{2l}^*(m_1(y_0)^{-1})v_{1,l+p}^{(2l)^*} = (-1)^{l-p}v_{1,l+p}^{(2l)^*}, \quad \tau_{2l}^*(m_2(y_0)^{-1})v_{2,l+p}^{(2l)^*} = (-1)^{l}v_{2,l+p}^{(2l)^*}.$$

For $0 \leq p \leq l - k$ such that $p \equiv l - k \mod 2$, we define the elementary function $t_i(l; p, q) \in H_{(\nu, \sigma)}$ by

$$t_i(l; p, q) = r_K^{(i)}(\tilde{t}_i(l; p, q))$$

where

$$\tilde{t}_i(l; p, q) = r_K^{(i)}(t_i(l; p, q))$$
Definition 6.3. For our $P$ basis $l = T_l$ Subsection 5.3, we identify for $0 < i$ the entries of the vector isomorphism between $\langle t \rangle - \kappa$ we define a certain matrix of elementary functions corresponding to the induced basis From above arguments, we obtain the following.

Proposition 6.1. As an unitary representation of $K$, it has an irreducible decomposition:

$$H_{(\nu_i, \sigma_i)} = \bigoplus_{l \in \mathbb{Z}_{\geq 0}, 0 \leq p < l - k \atop p \equiv l - k \text{ mod } 2} \langle T_i(l; p) \rangle$$

for $i = 1, 2$. Then the $\tau_2l$-isotypic component of $\pi_{(\nu_i, \sigma_i)}$ is given by

$$\bigoplus_{0 \leq p < l - k \atop p \equiv l - k \text{ mod } 2} \langle T_i(l; p) \rangle.$$

Corollary 6.2. Let $d(\sigma; l)$ be the dimension of the space $\text{Hom}_K(V_{2l}, H_{(\nu_i, \sigma_i)}, K)$ of intertwining operators. Then

$$d(\sigma; l) = \begin{cases} (l - k + 2)/2 & \text{if } k \leq l \text{ and } l - k \text{ is even,} \\ (l - k + 1)/2 & \text{if } k \leq l \text{ and } l - k \text{ is odd,} \\ 0 & \text{if } k > l. \end{cases}$$

6.3. The canonical blocks of elementary functions. By the identification introduced in Subsection 5.3 we identify $T_i(l; p)$ with the injective $K$-homomorphism

$$V_{2l} \ni v_{1,q}^{(2l)} \mapsto t_i(l; p, q) \in H_{(\nu_i, \sigma_i), K}, \quad 0 \leq q \leq 2l$$

for $0 \leq p \leq l - k$ such that $p \equiv l - k \text{ mod } 2$. We note that $\{T_i(l; p) \mid 0 \leq p \leq l - k, \ p \equiv l - k \text{ mod } 2\}$ is a basis of $\text{Hom}_K(V_{2l}, H_{(\nu_i, \sigma_i), K})$ and we call it the induced basis from the standard basis.

We define a certain matrix of elementary functions corresponding to the induced basis $\{T_i(l; p) \mid 0 \leq p \leq l - k, \ p \equiv l - k \text{ mod } 2\}$ of $\text{Hom}_K(V_{2l}, H_{(\nu_i, \sigma_i), K})$ for each $K$-type $\tau_2l$ of our $P_i$-principal series representation $\pi_{(\nu_i, \sigma_i)}$.

Definition 6.3. For $l \in \mathbb{Z}_{\geq 0}$ such that $d(\sigma; l) > 0$, the following $(2l + 1) \times d(\sigma; l)$ matrix $T_i(\sigma; l)$ is called the canonical block of elementary functions for $\tau_2l$-isotypic component: When $l - k$ is even, we consider the matrix

$$T_i(\sigma; l) = (T_i(l; 0), T_i(l; 2), T_i(l; 4), \cdots, T_i(l; l - k)).$$

When $l - k$ is odd, we consider the matrix

$$T_i(\sigma; l) = (T_i(l; 1), T_i(l; 3), T_i(l; 5), \cdots, T_i(l; l - k)).$$
6.4. The contiguous relations.

**Lemma 6.4.** (i) The standard basis \( \{ X_j \mid 0 \leq j \leq 4 \} \) of \( p_C \) have the following expressions according to the decomposition \( g_C = n_{1,C} \oplus a_{1,C} \oplus m_{1,C} \oplus t_C \):

\[
X_0 = m_1(x_-), \quad X_1 = -(E_{e_1-e_3} + \sqrt{-1}E_{e_1-e_2}) + \frac{1}{2}(K_{13} + \sqrt{-1}K_{12}),
\]
\[
X_2 = -\frac{1}{3}H^{(1)}, \quad X_3 = (E_{e_1-e_3} - \sqrt{-1}E_{e_1-e_2}) - \frac{1}{2}(K_{13} - \sqrt{-1}K_{12}), \quad X_4 = m_1(x_+).
\]

(ii) The basis \( \{ X'_j = u_cX_ju_c^{-1} \mid 0 \leq j \leq 4 \} \) of \( p_C \) have the following expressions according to the decomposition \( g_C = n_{2,C} \oplus a_{2,C} \oplus m_{2,C} \oplus t_C \):

\[
X'_0 = -m_2(x_-), \quad X'_1 = (E_{e_1-e_3} - \sqrt{-1}E_{e_1-e_2}) - \frac{1}{2}(K_{13} - \sqrt{-1}K_{23}),
\]
\[
X'_2 = \frac{1}{3}H^{(2)}, \quad X'_3 = -(E_{e_1-e_3} + \sqrt{-1}E_{e_1-e_2}) + \frac{1}{2}(K_{13} + \sqrt{-1}K_{23}), \quad X'_4 = -m_2(x_+).
\]

**Proof.** We obtain the assertion immediately from Lemma 4.1. \( \square \)

We give the matrix representation of \( \Gamma^i_{l,m} \) with respect to the induced basis as follows.

**Theorem 6.5.** For \( i = 1, 2 \) and \(-2 \leq m \leq 2 \), we have an following equation with the matrix representation \( R(\Gamma^i_{l,m}) \in M_{d(\sigma;i+l+m),d(\sigma;i)}(C) \) of \( \Gamma^i_{l,m} \) with respect to the induced basis \( \{ T_i(l;p) \mid 0 \leq p \leq l-k, \ p \equiv l-k \mod 2 \} \):

\[
(6.1) \quad e_{l,m}T_i(\sigma;i;l) = T_i(\sigma;i+l+m) \cdot R(\Gamma^i_{l,m}).
\]

Explicit expressions of the matrix \( R(\Gamma^i_{l,m}) \) of size \( d(\sigma;i+l+m) \times d(\sigma;i;l) \) is given as follows:

The matrix \( R(\Gamma^i_{l,m}) \) is given by

\[
R(\Gamma^i_{l,m}) = \begin{pmatrix}
\text{Diag} & 0_{1,d(\sigma;i)} - 1 & 0_{1,d(\sigma;i)} - 1 \\
0_{0 \leq j \leq d(\sigma;i) - 1} & \gamma_{l,m;2j+\delta(\sigma;i),1}^{(i)} & O_{2,1} \\
O_{1,d(\sigma;i)} - 1 & 0_{0 \leq j \leq d(\sigma;i) - 2} & \gamma_{l,m;2j+\delta(\sigma;i),1}^{(i)} & O_{d(\sigma;i),l} - 1
\end{pmatrix}.
\]

Here

\[
\gamma_{l,m;p,1}^{(i)} = (-1)^{i+1}(k + l + p)A_{[2l,2m;2l-p+m-2,0]},
\]
\[
\gamma_{l,m;p,0}^{(i)} = \frac{(-1)^i}{3}(\nu_l + \rho_l + lm - 3 + \frac{m(m+1)}{2})A_{[2l,2m;2l-p+m,2]},
\]
\[
\gamma_{l,m;p,-1}^{(i)} = (-1)^{i+1}(k + l - p)A_{[2l,2m;2l-p+m+2,4]},
\]
\[
n(\sigma;i;l,m) = \begin{cases}
(2 - m)/2 & \text{if } m \in \{0, \pm 2\}, \\
(3 - m)/2 & \text{if } (m, l-k) \in \{\pm 1\} \times (2\mathbb{Z}), \\
(1 - m)/2 & \text{if } (m, l-k) \in \{\pm 1\} \times (1+2\mathbb{Z}),
\end{cases}
\]

and \( \delta(\sigma;i;l) \in \{0,1\} \) such that \( \delta(\sigma;i;l) \equiv l-k \mod 2 \).

In the above equations, we put \( A_{[2l,2m;p,j]} = 0 \) for \( p < 0 \) or \( p > 2(l+m) \), and erase the symbols \( \text{Diag} \ (f(n)) \) \( (c_0 > c_1) \), \( O_{m,n} \) \( (m \leq 0 \ or \ n \leq 0) \).

**Proof.** By the similarly computation in the proof of Theorem 6.4 using Lemma 6.4 (i), we obtain the assertion in the case of \( i = 1 \). However, in the case of \( i = 2 \), It is difficult to prove the assertion by the same method since the value of \( T_2(l;p) \) at \( 1_3 \in G \) is not simple. We avoid this problem as follows.
We put

\[ t_2'(l; p, j) = \pi_{(\sigma_2)}(u_\nu) t_2(l; p, j) \quad (0 \leq j \leq 2l), \]

\[ T'_2(l; p) = t'(\ t'_l(l; p, 0), \ t'_2(l; p, 1), \ldots, \ t'_2(l; p, 2l)). \]

\[ T'_2(\sigma_2; l) = \begin{cases} (T'_2(l; 0), \ T'_4(l; 2), \ T'_2(l; 4), \ldots, \ T'_2(l; l - k)) & \text{if } l - k \text{ is even}, \\ (T'_2(l; 1), \ T'_2(l; 3), \ T'_2(l; 5), \ldots, \ T'_2(l; l - k)) & \text{if } l - k \text{ is odd}, \end{cases} \]

\[ c'_{i,m} = \sum_{j=0}^{4} R(t_{j}^{(l,m)}) \otimes X_{j}. \]

Then we see that

\[ (6.2) \quad c'_{i,m} T'_2(\sigma_2; l) = T'_2(\sigma_2; l + m) \cdot R(\Gamma_{i,m}^2), \]

and

\[ T'_2(l; p)(1_3) = e_{2l-p}^{(2l)} \otimes \chi_{l-p} + (-1)^{l} e_{p}^{(2l)} \otimes \chi_{p-l}. \]

Thus, by the similar computation as in Lemma 6.3 (ii), we also obtain the assertion in the case of \( i = 2 \) evaluating the both side of the equation (6.2) at \( 1_3 \in G \). \( \square \)

7. The action of \( p_C \)

The linear map \( \Gamma_{i,m}^l \) characterize the action of \( p_C \). In this section, we give a explicit description of the action of \( p_C \) on the elementary functions.

7.1. The projectors for \( V_i \otimes_C V_4 \). For \(-2 \leq m \leq 2\), we describe a surjective \( sl(2, C)\)-homomorphism \( P_{2m}^l \) from \( V_i \otimes_C V_4 \) to \( V_{l+2m} \) in terms of the standard basis as follows.

Lemma 7.1. Let \( \{ v_q^{(l)} \mid 0 \leq q \leq l \} \) be the standard basis of \( V_i \) for \( l \in \mathbb{Z}_{\geq 0} \). We put \( v_q^{(l)} = 0 \) when \( q < 0 \) or \( q > l \).

We define linear maps \( P_{2m}^l : V_i \otimes_C V_4 \to V_{l+2m} \) \((-2 \leq m \leq 2\) by

\[ P_{2m}^l(v_q^{(l)} \otimes w_r) = B_{[l, 2m; q, r]} v_{q+r+m-2}^{(l+2m)}, \]

when \( V_{l+2m} \)-component of \( V_i \otimes_C V_4 \) does not vanish.

Here the coefficients \( B_{[l, 2m; q, r]} = b(l, 2m; q, r)/d'(l, 2m) \) are defined by following formulae.

Formula 1: The coefficients of \( P_{1}^l : V_i \otimes_C V_4 \to V_{l+4} \) are given as follows:

\[ b(l, 4; q, r) = \begin{cases} 1 & (0 \leq r \leq 4), \\ 0 & (r = 5). \end{cases} \]

Formula 2: The coefficients of \( P_{2}^l : V_i \otimes_C V_4 \to V_{l+2} \) are given as follows:

\[ b(l, 2; q, 0) = 4q, \quad b(l, 2; q, 1) = -(l - 4q), \quad b(l, 2; q, 2) = -2(l - 2q), \]
\[ b(l, 2; q, 3) = -(3l - 4q), \quad b(l, 2; q, 4) = -4(l - q), \quad d'(l, 2) = l + 4. \]

Formula 3: The coefficients of \( P_{3}^l : V_i \otimes_C V_4 \to V_{l} \) are given as follows:

\[ b(l, 0; q, 0) = 6q - q - 1, \quad b(l, 0; q, 1) = -3q - (l - 2q + 1), \]
\[ b(l, 0; q, 2) = l^2 - 6q + 6q^2 - l, \quad b(l, 0; q, 3) = 3(l - 2q - 1)(l - q), \]
\[ b(l, 0; q, 4) = 6(l - q)(l - q - 1), \quad d'(l, 0) = (l + 3)(l + 2). \]

Formula 4: The coefficients of \( l_{-2}^l : V_{l-2} \to V_i \otimes_C V_4 \) are given as follows:

\[ b(l, -2; q, 0) = 4q - (q - 1)(q - 2), \quad b(l, -2; q, 1) = -q - (q - 1)(3l - 4q + 2), \]
\[ b(l, -2; q, 2) = 2q(l - 2q)(l - q), \quad b(l, -2; q, 3) = -(l - 4q - 2)(l - q)(l - q - 1), \]
\[ b(l, -2; q, 4) = -4(l - q)(l - q - 1)(l - q - 2), \quad d'(l, -2) = (l + 2)(l + 1). \]
Formula 5: The coefficients of $I_{-4}^l$: $V_{i-4} \to V_i \otimes C V_4$ are given as follows:

- $b(l, -4; q, 0) = q(q - 1)(q - 2)(q - 3)$,
- $b(l, -4; q, 1) = -q(q - 1)(q - 2)(l - q)$,
- $b(l, -4; q, 2) = q(q - 1)(l - q)(l - q - 1)$,
- $b(l, -4; q, 3) = -q(l - q)(l - q - 1)(l - q - 2)$,
- $b(l, -4; q, 4) = (l - q)(l - q - 1)(l - q - 2)(l - q - 3)$,
- $\gamma(l, -4) = (l + 1)(l - 1)(l - 2)$.

Then $P_{2m}^l$ is the generator of $\text{Hom}_{s(2, C)}(V_i \otimes C V_4, V_{i+2m})$ such that $P_{2m}^l \circ I_{2m}^l = \text{id}_{V_{i+2m}}$.

Proof. The composite

$$V_i \otimes C V_4 \simeq V_i^* \otimes C V_4^* \simeq (V_i \otimes C V_4)^* \ni f \mapsto f \circ I_{2m}^l \in V_{i+2m}^* \simeq V_{i+2m}$$

is a surjective $s(2, C)$-homomorphism from $V_i \otimes C V_4$ to $V_{i+2m}$, which is unique up to scalar multiple. Therefore we obtain the assertion from Proposition 1.2 and Lemma 2.3 \hfill \Box

7.2. The action of $p_C$ on the elementary functions.

Proposition 7.2. (i) An explicit expression of the action of $p_C$ on the basis $\{s(l; p, q) \mid l \geq 0, \ p \in Z(\sigma_0; l), \ 0 \leq q \leq 2l\}$ of $H_{(\nu_0, \sigma_0), K}$ is given by following equation:

$$\pi_{(\nu_0, \sigma_0)}(X_r)s(l; p, q) = \sum_{-1 \leq j \leq 1} \sum_{-2 \leq m \leq 2} \gamma_{[l, m, p, j]}^{(0)} B_{[2l, 2m; q, r]} s(l + m; p + m + 2j, q + m + r - 2).$$

Here we put

$$\gamma_{[0, m, 0, j]}^{(0)} = B_{[0, 2m; 0, r]} = 0 \text{ for } m < 2, \quad \gamma_{[1, m, p, j]}^{(0)} = B_{[2m, q, r]} = 0 \text{ for } m < 0,$n

$s(l; p, q) = 0$ whenever $p \leq l$ such that $p \not\in Z(\sigma_0; l)$ or $q < 0$ or $q > 2l$,

$s(l; p, q) = (-1)^{q(\sigma_0 l)} s(l; 2l - p, q)$ for $p > l$.

(ii) For $i = 1, 2$, the explicit expression of the action of $p_C$ on the basis $\{t_i(l; p, q) \mid l \geq k, \ 0 \leq p \leq l - k, \ p \equiv l - k \mod 2, \ 0 \leq q \leq 2l\}$ of $H_{(\nu_i, \sigma_i), K}$ is given by following equation:

$$\pi_{(\nu_i, \sigma_i)}(X_r)t_i(l; p, q) = \sum_{-1 \leq j \leq 1} \sum_{-2 \leq m \leq 2} \gamma_{[l, m, p, j]}^{(i)} B_{[2l, 2m; q, r]} t_i(l + m; p + m + 2j, q + m + r - 2).$$

Here we put $t_i(l; p, q) = 0$ unless $0 \leq p \leq l - k, \ p \equiv l - k \mod 2$ and $0 \leq q \leq 2l$.

Proof. Since

$$\pi_{(\nu_0, \sigma_0)}(X_r)s(l; p, q) = \sum_{-2 \leq m \leq 2} \Gamma_{l, m}^0 (S(l; p)) \circ P_{2m}^l (v_{2l}^{(2)}) \otimes X_r),$$

$$\pi_{(\nu_i, \sigma_i)}(X_r)t_i(l; p, q) = \sum_{-2 \leq m \leq 2} \Gamma_{l, m}^i (T_i(l; p)) \circ P_{2m}^l (v_{2l}^{(2)}) \otimes X_r) \quad (i = 1, 2),$$

we obtain the assertion from Theorem 5.5, 6.5 and Lemma 7.1 \hfill \Box

References

[1] Hrvoje Kraljević. Representations of the universal curve over the group $SU(n, 1)$. Glasnik Mat. Ser. III, Vol. 8(28), pp. 23–72, 1973.
[2] Hiroyuki Manabe, Taku Ishii, and Takayuki Oda. Principal series Whittaker functions on $SL(3, R)$. Japan J. Math. (N.S.), Vol. 30, No. 1, pp. 183–226, 2004.
[3] Tadashi Miyazaki. The $(g, k)$-module structures of principal series representations of $Sp(2, R)$, preprint.
[4] Takayuki Oda. The standard $(g, k)$-modules of $Sp(2, R)$ I, preprint.
[5] Takayuki Oda. The standard $(g, k)$-modules of $Sp(2, R)$ II, preprint.
[6] Ernest Thiekeker. On the integrable and square-integrable representations of $Spin(1, 2m)$. Trans. Amer. Math. Soc., Vol. 230, pp. 1–40, 1977.

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