New oscillation criteria of special type second-order non-linear dynamic equations on time scales

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Abstract
By the Riccati transformation technique, we study some new oscillatory properties for the second-order dynamic equation on an arbitrary time scale $\mathbb{T}$. We also establish the Kamenev-type and Philos-type oscillation criteria. At the end, we give examples which illustrate our main results.

Keywords Time scale · Oscillation · Delay dynamic equation · Riccati technique

Mathematical Subject Classification 34N05 · 34K11 · 39A10 · 39A99

Introduction
In [25], Kubyshkin and Moryakova considered a second-order differential–difference equation of delay type

$$x(t) + A x(t) + x(t) + K(x(t-h)) + W(x(t-h)) = 0,$$

(1.1)

where the real constants $A, h > 0$, and the functions $K, W : \mathbb{R} \to \mathbb{R}$ are defined by $K(x(t)) = k_1 x(t) + k_2 x^2(t) \cdots \cdots$ and $W(x(t)) = w_1 x(t) + w_2 x^2(t) \cdots \cdots \forall k_i, w_i \in \mathbb{R}$, respectively. The authors have investigated the oscillatory solution of Eq. (1.1). Such Eq. (1.1) occur in the modeling of electronic devices. In this paper, we modify and extend this work by considering the Eq. (1.1) on the arbitrary time scales $\mathbb{T}$,

$$y^{\Delta}(t) + a(t) y^\Delta(t) + y(t) + K(y(t-h)) + f(W(y(t-h))) = 0 \quad \forall t \in \mathbb{T},$$

(1.2)

where the functions $a : \mathbb{T} \to \mathbb{R}, f : \mathbb{R} \to \mathbb{R}$, and time scale $\mathbb{T}$ satisfies $t - h < t$ and $t - h \in \mathbb{T} \forall t \in \mathbb{T}$, for some positive real constant $h$. One can easily see that for some $h > 0$, Eq. (1.1) can be achieved by taking $\mathbb{T} = \mathbb{R}$, $a(t) = A > 0$ and $f(x) = x^2$ in Eq. (1.1). Equation (1.2) is very general in nature and techniques from time scale calculus to analyze it. Equation (1.2) covers not only differential equations (i.e., $\mathbb{T} = \mathbb{R}$) and difference equations (i.e., $\mathbb{T} = \mathbb{Z}$) but covers more general time scales $h\mathbb{Z} = \{hn : n \in \mathbb{Z}\}$ for $h > 0$, $\mathbb{T} = \bigcup_{k \in \mathbb{Z}} [k(a+b), k(a+b) + a]$ for $a, b > 0$, and $\mathbb{T} = \bigcup_{m \in \mathbb{Z}} \{m + \frac{1}{n} : n \in \mathbb{N}\} \cup \mathbb{Z}$, etc. Throughout this paper, we obtain the sufficient conditions of oscillation for the dynamic equation (1.2). To the best of our knowledge, no work has been done regarding the oscillatory behavior of (1.2) so far.

Following Stefan’s landmark [22], a rapidly expanding body of literature has sought to unify, extend, and generalize ideas from discrete, quantum and continuous calculus to an arbitrary time scale calculus. A time scale is an arbitrary non-empty closed subset of the real numbers which have the topology that inherits from the real numbers with the standard topology. It has applications in electrical engineering, quantum mechanics, population dynamic and economics etc [3, 11]. In particular, a time scale $q\mathbb{Z} \cup \{0\}, q > 1$ is used in quantum physics, see in [6, 7]. Many authors have worked on various aspects of new theory, see in [2, 6, 7, 14, 17, 19, 20, 21, 34] and the
references therein. These literatures summarize and organize much of time scale calculus.

Stefan’s theory has attracted the attention of many researchers on oscillation of second-order linear and nonlinear dynamic equation on time scales. In recent years, many researchers have focused on oscillation and nonoscillation criteria of second-order ordinary dynamic equations on time scales. Several authors have studied the oscillation criteria by employing the Riccati transformation technique as well as established the Kamenev-type and Philos-type oscillation criteria. For more details on such criteria, we refer the reader to the papers [1, 12, 13, 15, 20, 28, 30–33] and reference therein. To establish the oscillation criteria, in [24], Kamenev considered a second-order differential equation

\[ (p(t)x'(t))' + q(t)x(t) = 0 \]  \hspace{1cm} (1.3)

and investigated the following sufficient conditions of oscillations,

\[ \limsup_{t \to \infty} \frac{1}{n} \int_{t_0}^{t} (t - s)^{n} q(s) ds = \infty, \quad \text{for} \quad n > 1. \]  \hspace{1cm} (1.4)

Several authors (e.g., Sun [31], Philos [27]) have extended the Kamenev’s oscillation criteria to more general criteria by taking the kernel function \((t - s)^{n}\) as the general class of functions \(H(t, s)\), which satisfies some assumptions given below follow:

The kernel function \(H : \mathbb{D} = \{(t, s) : t_0 \leq s \leq t\} \to \mathbb{R}\) defined as continuous function such that

\[ H(t, t) = 0 \quad \text{for} \quad t_0 \leq t, H(t, s) > 0 \quad \text{for} \quad t_0 \leq s < t \quad \text{and} \quad \frac{\partial H(t, s)}{\partial s} < 0 \quad \text{on} \quad \mathbb{D}, \]

where \(\frac{\partial H(t, s)}{\partial s}\) is continuous on \(D\). Furthermore, define a continuous function \(h : \mathbb{D} \to \mathbb{R}\) such that

\[ \frac{\partial H(t, s)}{\partial s} = -h(t, s)\sqrt{H(t, s)}, \quad \text{for} \quad \text{all} \quad (t, s) \in \mathbb{D}. \]

In [27], Philos obtain the following sufficient conditions,

\[ \limsup_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \left( H(t, s)q(s) - \frac{1}{4} h^2(t, s) \right) ds = \infty, \quad \text{for} \quad t_0 \geq 0 \quad \text{and} \quad p(t) = 1. \]  \hspace{1cm} (1.5)

From the conditions (1.4) and (1.5), we conclude that the Philos-type is more general to Kamenev-type criteria. A discrete version of differential equation (1.3)

\[ \Delta(y_{n}x_{n}) + p_{n}x_{n+1} = 0 \]  \hspace{1cm} (1.6)

has been discussed by Chen and Erbe [9], for oscillation and non-oscillation. To harmonize the differential equation (1.3) and difference equation (1.6), in [12], Del and Kong, and in [16], Erbe, et al. have considered a following self-adjoint second-order scalar dynamic equation on time scale \(\mathbb{T}\)

\[ (p(t)x^2(t))' + q(t)x(\sigma(t)) = 0, \]  \hspace{1cm} (1.7)

and established the oscillation criteria. The problem (1.7) is not only the extension of results of [9, 24] but to the more general results on time scales. In addition, several authors have focused on study of both Kamenev-type and Philos-type oscillation criteria on the time scales. For example, in [4], Agwa et al. studied both Kamenev-type and Philos-type oscillation criteria of the following second-order nonlinear delay dynamic equation

\[ (r(t)g(x(t), x^\Delta(t)))^\Delta + p(t)f(x(\tau(t))) = 0, \quad t \in \mathbb{T}, t_0 \leq t. \]  \hspace{1cm} (1.8)

In [10], Chen et al. considered the second-order dynamic equation with damping on time scales

\[ ((x^\Delta(t))^\Delta + p(t)(x^\Delta(t))^\gamma + q(t)f(x^\delta(t))) = 0 \]  \hspace{1cm} (1.9)

and by employing Riccati transformation to established the sufficient conditions of Kamenev-type as well as Philos-type oscillation criteria. In [29], Saker have established the sufficient conditions of oscillation of the following second-order nonlinear neutral delay dynamic equations on time scales

\[ (r(t)[y(t) + p(t)y(t - \tau)]^\Delta)^\Delta + f(t, y(t - \eta)) = 0 \]  \hspace{1cm} (1.10)

and discussed both Kamenev-type and Philos-type oscillation criteria. In [8], Bohner and Saker have studied the oscillation criteria for the second-order perturbed dynamic equation on time scales:

\[ (r(t)(x^\Delta(t))^\Delta + F(t, x(\sigma(t))) = G(t, x(t), x^\Delta(t)). \]  \hspace{1cm} (1.11)

In [1], Agarwal et al. modified the dynamic equation (1.11) and considered the following equation

\[ (r(t)(x^\Delta(t))^\Delta + F(t, x(t))) = G(t, x(t), x^\Delta(t)), \]  \hspace{1cm} (1.12)

and studied the sufficient conditions for oscillation. They have assumed that there exist two positive rd-continuous functions \(p\) and \(q\) such that

1. \(r : \mathbb{T} \to \mathbb{R}\) is a positive and rd-continuous function and \(\gamma \in \mathbb{N}\) is odd,
2. \(p, q : \mathbb{T} \to \mathbb{R}\) are rd-continuous functions such that \(p(t) - q(t) > 0\) for \(t \in \mathbb{T}\),
3. \(F : \mathbb{T} \times \mathbb{R} \to \mathbb{R}\) and \(G : \mathbb{T} \times \mathbb{R} \to \mathbb{R}\) are functions such that \(uF(t, u) > 0\) and \(uG(t, u, v) > 0\) for all \(u \in \mathbb{R} \setminus \{0\}, v \in \mathbb{R}, t \in \mathbb{T}\),
4. \(|F(t, u)| \geq p(t)|u|^\gamma, |G(t, u, v)| \leq q(t)|u|^\gamma\forall u \in \mathbb{R} \setminus \{0\}, v \in \mathbb{R}, t \in \mathbb{T}\).
and established the sufficient conditions of non-oscillation. Recently, Negi et al. [26] considered the second-order nonlinear dynamic equations with integro forcing term and deviating argument on time scales

\[
\gamma(t) + \beta(t) y(t) = B(t) y(t) + H(t, y(t), \int_{-\infty}^{t} J(t-s) T(s, y(s)) \Delta s)
\]

(1.14)

and

\[
y(t) + \beta(t) y(t) = B(t) y(t) + W(t, y(t), y(w_1(t), y(t))),
\]

(1.15)

respectively, where \( \beta \geq 0, w_1(t, y(t)) = b_1(t, y(b_2(t, \ldots, y(b_m(t, y(t)) \ldots))) \). By the Riccati technique, Negi et al. have investigated the Kamenev-type oscillation criteria of both the Eqs. (1.14) and (1.15), respectively.

In this paper, we first deal with two functions \( K(y(t-h)) \) and \( W(y(t-h)) \), which play an important role in our analytical findings. As we see in the above assumption (4), the absolute value of functions \( F \) and \( G \) are related with the absolute value of the unknown function \( u(t) \), with the functions \( p(t) \) and \( q(t) \), respectively. In Eq. (1.10), Saker have assumed that the continuous function \( f : T \times R \rightarrow R \) such that \( uf(t, u) > 0 \forall u \neq 0 \) and \( |f(t, u)| \geq q(t)|u| \), where \( q(t) \) is a nonnegative function defined on \( T \). In Eq. (1.9), Chen have assumed function \( f \in C([0, R], R) \) such that \( x(t) > 0 \forall x \neq 0 \) and \( f(x) \geq Lx \), where \( L \) is a positive real constant. In Eq. (1.13), Graef considered the function \( f \in C([0, R], R) \) such that for \( \gamma > 0, \ |f(x(t))| \leq |x(t)| + B \) for all \( x(t), \forall t \in T \), where \( A, B \) are non-negative real constant. Motivated by the above literature, recently in [26] Negi et al. have assumed \( |H(t, \eta(t), \xi(t))| \geq p(t) |\eta(t)| \) and \( \eta(t) H(t, \eta(t), \xi(t)) < 0 \) for \( \eta(t) \neq 0 \) for the dynamic equation (1.14), and (1.15) by replacing \( H \) to \( W \).

To establish oscillation criteria for (1.2), we need \( |K(y(t))| \geq p(t)|y(t)| \) for \( y(t) \neq 0 \) such that \( y(t) K(y(t)) > 0 \). Moreover, there exist a function \( f \in C([0, R], R) \) such that \( |f(W(y(t)))| \geq M(t)|y(t)| \) as well as \( y(t) f(W(y(t))) > 0 \forall y(t) \neq 0 \) in \( R \), where \( M(t) \) is a non-negative rd-continuous function defined on \( T \). Now we choose the real coefficients \( w_j, k_j \) such that

\[
|K(y(t))| = |k_1 y(t) + k_2 y^2(t) + \cdots | \geq k_1 |y(t)|, \quad (1.16)
\]

such that \( y(t) K(y(t)) > 0 \forall y(t) \in R, \forall t \in T \). Similarly, we immediately obtain an inequality

\[
|W(y(t))| \geq w_1|y(t)|, \quad (1.17)
\]

Now let us consider a function \( f \in C([0, R], R) \) for \( y(t) \in R \) such that \( f(W(y(t))) = q(t) \text{sgn}(y(t))|W(y(t))|, \forall \in T \), then, from (1.17), we obtain,

\[
|f(W(y(t))| = |q(t) \text{sgn}(y(t))|W(y(t))| |y(t)|
\]

(1.18)

and \( y(t) f(W(y(t))) > 0 \), for \( y(t) \neq 0 \forall t \in T \) and \( q(t) \) is rd-continuous function defined on \( T \). In Eq. (1.18), absolute value of \( f \) related to the absolute value of \( y(t) \), \( \forall t \in T \).

For simplicity, throughout this paper, we denote \( |x(t)| = \{a(t) \cap T \} \). In addition, we also need the following assumptions, as follows:

1. \( (O_1) \) Assume \( a, p : [t_0, \infty) \rightarrow R \) are positive rd-continuous functions such that \( 0 < p(t) \leq k_1 < \infty \) and \( q(t) := p(t) \frac{u(t)}{w(t)} \).

2. \( (O_2) \) \( \int_{t_0}^{\infty} \frac{1}{e_{\tau(t)}(t, s_0)} \Delta t = \infty \), where

Let us recall that, a solution \( y(t) \) of (1.2) is a non-trivial or \( y(t) \neq 0 \), such that \( y(t) \in C_{\text{rd}}(\lbrack t_0, \infty), \) for certain \( t_0 \geq t_0 \). If \( y(t) \) is eventually positive or eventually negative, then it must be non-oscillatory, otherwise oscillatory, i.e., it is oscillatory if there exists a real sequence, say \( \{a_n\} \) such that \( a_n \rightarrow \infty \) as \( n \rightarrow \infty \), and \( a_n = 0 \forall n \in N \). Our attention is restricted to those solutions of (1.2) which exist on the half-line \( [t_0, \infty) \gamma \) and satisfy \( \{y(t) : t > t_1\} \neq 0 \) for any \( t_1 \leq t_0 \), and \( \sup \{y(t) : t > t_0\} \neq 0 \).

This paper is organized as follows: In Sect. 2, we give basic definitions and present some necessary Lemmas. In the next section, we establish the sufficient conditions of oscillation of our Eq. (1.2). We further establish the Kamenev-type and Philos-type oscillation criteria. Some remarks for the particular case are also discussed. At the end in Sect. 4, to validation our results, we give an example. We also discuss the cases when the time scale is of a particular form.
Preliminaries

In this section, we present some basic definitions, useful
Theorems and basic facts of time scales.

Definition 2.1 [6]. For \( t \in \mathbb{T} \), forward and backward jump operators \( \sigma, \rho : \mathbb{T} \to \mathbb{T} \) are defined by
\[
\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\},
\]
respectively. The classification of points of time scale \( \mathbb{T} \).

For \( t \in \mathbb{T} \), \( t \) is called right-scattered if \( t < \sigma(t) \), and right dense if for all \( t < \sup \mathbb{T} \) such that \( t = \sigma(t) \). Similarly, \( t \) is left-scattered if \( t > \rho(t) \), and left dense if for all \( t > \inf \mathbb{T} \) such that \( t = \rho(t) \). The graininess operator \( \mu : \mathbb{T} \to [0, \infty) \) is defined by \( \mu(t) = \sigma(t) - t \).

Remark 2.2 We put \( \inf \emptyset = \sup \mathbb{T} \) (i.e., \( \sigma(t) = t \) if \( \mathbb{T} \) has a maximum \( t \)), \( \sup \emptyset = \inf \mathbb{T} \) (i.e., \( \rho(t) = t \) if \( \mathbb{T} \) has a minimum \( t \)), where \( \emptyset \) is an empty set.

Definition 2.3 [6]. A function \( f : \mathbb{T} \to \mathbb{R} \) is called rd-continuous provided it is continuous at all right-dense points in \( \mathbb{T} \) and its left-sided limit exist (finite) at all left-dense points in \( \mathbb{T} \), which is denoted by \( C_{rd} = C_{rd}(\mathbb{T}, \mathbb{R}) \).

We define \( \mathbb{T}^\kappa = \mathbb{T} - \{\xi\} \), if \( \mathbb{T} \) has a left-scattered maximum \( \xi \), and \( \mathbb{T}^\kappa = \mathbb{T} \), otherwise.

Definition 2.4 [6]. For a function \( f : \mathbb{T} \to \mathbb{R} \) and \( t \in \mathbb{T}^\kappa \), we define \( f^\Delta(t) \), to be a number (provided it exists) with the property that for any \( \epsilon > 0 \), there exists a neighborhood \( Z = (t - \delta, t + \delta) \cap \mathbb{T} \) for some \( \delta > 0 \) such that
\[
|f(\sigma(t)) - f(r)| - f^\Delta(t)(\sigma(t) - r)| \leq \epsilon|\sigma(t) - r| \quad \forall r \in Z.
\]
Thus, we call \( f^\Delta(t) \) the \( \Delta \) or Hilger derivative of \( f \) at \( t \).

Theorem 2.5 [6]. For the functions \( g, f : \mathbb{T} \to \mathbb{R} \) and \( t \in \mathbb{T}^\kappa \), the following statements are true:

1. If \( f \) is differentiable at \( t \), then \( f \) is continuous at \( t \);
2. If \( f \) is continuous at \( t \) and \( t \) is right-scattered, then \( f \) is \( \Delta \)-derivative at \( t \) and \( f^\Delta(t) = \lim_{r \to t, r \neq t} \frac{f(t) - f(r)}{\mu(t)} \) exists and finite value;
3. If \( t \) is right-dense, then \( f \) is differentiable at \( t \) iff \( f^\Delta(t) = \lim_{t \to \sigma(t), t \neq \sigma(t)} \frac{f(t) - f(\sigma(t))}{\mu(t)} \exists \text{ and finite value} \);
4. If \( f \) is differentiable at \( t \), then \( f^\sigma(t) = f(\sigma(t)) = f(t) + \rho(t)f^\Delta(t) \);
5. If \( f \) and \( g \) both are differentiable at \( t \), then a product \( fg : \mathbb{T} \to \mathbb{R} \) is differentiable at \( t \) and
\[
(fg)^\Delta(t) = f^\Delta(t)g(t) + f(t)g^\Delta(t)
\]
\[
= f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)),
\]
hence, for \( t \in \mathbb{T} \) such that \( a \leq t \leq b, \forall a, b \in \mathbb{T} \), we have the following facts
\[
\int_a^b f^\Delta(s)g^\Delta(s)ds = f(b)g(b) - f(a)g(a) - \int_a^b f^\Delta(s)g(s)ds
\]
\[
\int_a^b f^\Delta(s)g(s)ds = f(b)g(b) - f(a)g(a) - \int_a^b f^\Delta(s)g^\sigma(s)ds;
\]
6. If \( g(t)g(\sigma(t)) \neq 0 \), then \( \frac{f(t)g(t) - f(t)g(\sigma(t))}{g(t)g(\sigma(t))} \) is differentiable at \( t \) and
\[
\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)(g(t) - g(\sigma(t))}{g(t)g(\sigma(t))}.
\]
Before going to our main section, we first introduce some necessary lemmas which are crucial for our proofs.

**Lemma 2.11** Let \( y(t) \) be a non-oscillate solution of (1.2) and assume that \( O_1, O_2 \), relations (1.16) and (1.18) hold, then there exists \( s_0 \leq 0 : s_0 > t_0 \) such that
\[
y(t) > 0, y^\Delta(t) > 0 \quad \text{and} \quad y^\Delta\Delta(t) < 0
\]
and
\[
y(t - h) > 0, y^\Delta(t - h) > 0 \quad \text{and} \quad y^\Delta\Delta(t - h) < 0, \quad \text{on } [s_0, \infty)_T.
\]
\[
(2.5) \quad \text{and} \quad (2.6)
\]

**Proof** Let \( y(t) \) be an eventually positive solution of (1.2). Then there exists \( t_1 \in [t_0, \infty)_T \) such that \( y(t) > 0 \) on \( [t_1, \infty)_T \). Let us take \( t_2 \in [t_1, \infty)_T \) such that \( t_2 \geq t_1 + h \), then we must have \( y(t) > 0 \) and \( y(t - h) > 0 \) on \( [t_2, \infty)_T \). From equation (1.2) and \( O_1 - O_2 \), (1.16) and (1.18), we obtain
\[
y^\Delta(t) + a(t)y^\Delta(t) < - (y(t) + p(t)y(t - h) + p(t)\mu(t)y^\Delta(t - h))
\]
\[
\leq - (y(t) + p(t)y(t - h) + p(t)\mu(t)y^\Delta(t - h))
\]
\[
\leq - (y(t) + p(t)y^\sigma(t - h)) < 0.
\]
(2.7)

Since \( a(t)\mu(t) < 1 \) and using the above relation, we obtain
\[
y^\Delta\Delta(t) - \frac{a(t)}{1 - a(t)\mu(t)}y^\Delta(t) < 0, \quad \text{on } [t_2, \infty)_T.
\]
(2.8)

which is equivalent to
\[
\left( e_{-\frac{\mu(t)}{1-a(t)}(t,s_0)} \right)^\Delta y^\Delta(t) < 0, \quad \forall t \in [t_2, \infty)_T.
\]
(2.9)

Then \( e_{-\frac{\mu(t)}{1-a(t)}(t,s_0)}y^\Delta(t) \) is an eventually decreasing function and thus it is eventually one sign. We claim that it is eventually non-negative. Let us assume that it is eventually negative, i.e., \( y^\Delta(t) \) is eventually negative because \( e_w(t,s_0) > 0 \) as \( w > 0 \), for all \( w \in \mathbb{R}^+ \), then there exist \( t_2 \leq t_3 \) and a constant \( C > 0 \) such that
\[
e_{-\frac{\mu(t)}{1-a(t)}(t,s_0)}y^\Delta(t) \leq C < 0, \quad \text{for all } t_3 + h \leq t_4 \leq t.
\]
(2.10)

Integrating Eq. (2.10) from \( t_4 \) to \( t \), we obtain
\[
y(t) \leq y(t_4) + C \int_{t_4}^{t} e_{-\frac{\mu(t)}{1-a(t)}(s,s_0)}\Delta s.
\]

For sufficiently large \( t \), we obtain a contradiction because of \( O_2 \). Therefore, we have
\[
y(t) > 0, y^\Delta(t) > 0 \quad \text{and} \quad y^\Delta\Delta(t) < 0,
\]
and
\[
y(t - h) > 0, y^\Delta(t - h) > 0 \quad \text{and} \quad y^\Delta\Delta(t - h) < 0, \quad \text{on } [s_0, \infty)_T.
\]
\[
\square
\]

**Lemma 2.12** If (2.5) holds, then for \( t \neq s_0 \), we have
\[
0 < G(t) \leq \frac{y(t)}{y^\sigma(t)} \leq 1,
\]
(2.11)

where \( G(t) := \frac{\I_{s_0}^{t-s_0}}{\I_{s_0}^{t-s_0 + \mu(t)}} \).

**Proof** For \( t \neq s_0 \), and from (2.5), we obtain
\[
y(t) > y(t) - y(s_0) = \int_{s_0}^{t} y^\Delta(s)\Delta s > y^\Delta(t)(t - s_0).
\]
(2.12)

From (2.12) and using (4) of Theorem 2.5, we obtain
\[
0 < \frac{t - s_0}{t - s_0 + \mu(t)} := G(t) \leq \frac{y(t)}{y^\sigma(t)} \leq 1.
\]

Now with the help of Lemmas 2.11 and 2.12, we obtain a new Lemma as follows,

**Lemma 2.13** If Eqs. (2.5) and (2.11) hold, then for \( 2s_0 \leq t \), we have
\[
\frac{t}{2} \frac{G(t)w(t)}{\delta(t)} \leq \frac{(t - s_0)}{\delta(t)} \frac{G(t)w(t)}{\delta(t)} \leq \frac{y(t - h)}{y^\sigma(t)} \leq 1,
\]
(2.13)

where \( w(t) = \delta(t) \frac{y^\Delta(t)}{y^\sigma(t)} \) is Riccati transformation function.

**Proof** The relation \( 2s_0 \leq t \) can be rewritten as \( t - s_0 \geq \frac{t}{2} \).

Using (2.6), we obtain
\[
y(t - h) - y(s_0 - h) = \int_{s_0 - h}^{t-h} y^\Delta(s)\Delta s > y^\Delta(t - h)(t - s_0)
\]
\[
> y^\Delta(t)(t - s_0),
\]
which implies
\[
\frac{t}{2} y^\Delta(t) < y(t - h).
\]
(2.14)

From Eqs. (2.14), (2.11) and \( w(t) \). Hence, we obtain
\[
\frac{t}{2} \frac{G(t)w(t)}{\delta(t)} \leq \frac{y(t - h)}{y^\sigma(t)} \leq 1.
\]
\[
\square
\]

**Oscillation criteria**

In this section, we establish some sufficient conditions of oscillation for Eq. (1.2). For \( s_0 \in T \).

**Theorem 3.1** Assume that \( O_1, O_2 \), relations (1.16) and (1.18) hold. Moreover, if there exist a \( \Delta \)-derivative function
\(\delta(t) > 0\) and \(s_0 \geq 0; t_0 \leq 2s_0 < t\) respectively and satisfying the following

\[
\limsup_{t \to \infty} \int_{2s_0}^{t} \mathcal{F}(s) \Delta s = \infty,
\]

where

\[
\mathcal{F}(t) := \left( \frac{\delta^2(t) - \left( \frac{\delta^2(t) - \mathcal{G}(t) \mathcal{G}(t)}{4 \mathcal{G}(t) \mathcal{G}(t)} \right)^2}{4 \mathcal{G}(t) \mathcal{G}(t)} \right).
\]

then, Eq. (1.2) oscillates on \([t_0, \infty)\).  

**Proof** Assume to the contrary that (1.2) has a non-oscillatory solution. Let \(y(t)\) be a nonoscillatory solution of (1.2). Then, without loss of generality, we assume that \(y(t)\) is an eventually positive solution, i.e., there exists \(s_0 \geq t_0\) such that \(y(t) > 0 \quad \forall t \in [s_0, \infty)\). A similar argument holds for the case when \(y(t)\) is an eventually negative solution. We define a Riccati transformation function such that

\[w(t) = \delta(t) \frac{y^A(t)}{y(t)}, \quad t_0 \leq 2s_0 < t.\]

\(\Delta\)-derivative of Eq. (3.3) w.r.t \(t\), we have

\[
w^A(t) = \delta^2(t) \frac{y^A(t)}{y(t)} + \delta^2(t) \left( \frac{y^A(t)y(t) - (y^A(t))^2}{y(t)y^A(t)} \right).
\]

From (2.11) and (3.4), we obtain

\[
w^A(t) \leq w(t) \delta^2(t) \frac{y^A(t)}{y(t)} - w^2(t) \mathcal{G}(t) \delta^2(t) + \frac{\delta^2(t)}{y^2(t)} y^{\Delta A}(t).
\]

To solve the right-hand side of Eq. (3.5), we use Eqs. (2.7), (2.11) and a relation \(y^A(t) > 0\), we obtain

\[
\frac{\delta^2(t)}{y^2(t)} y^{\Delta A}(t) \leq - \delta^2(t) \left( \frac{y(t)}{y^2(t)} + p(t) \frac{y^A(t)}{y^2(t)} - a(t) \frac{y^A(t)}{y^2(t)} \right)
\]

\[
\leq - \delta^2(t) \mathcal{G}(t) - \delta^2(t) a(t) \frac{\mathcal{G}(t)}{\delta(t)} w(t) - \delta^2(t) p(t) \frac{y(t-h)}{y^2(t)}.
\]

From (3.6) and (2.13), we obtain

\[
\frac{\delta^2(t)}{y^2(t)} y^{\Delta A}(t) \leq - \delta^2(t) \mathcal{G}(t) - \delta^2(t) a(t) \frac{\mathcal{G}(t)}{\delta(t)} w(t) - \delta^2(t) p(t) \frac{\mathcal{G}(t)}{\delta(t)} \frac{t \mathcal{G}(t)}{2 \delta(t)} w(t).
\]

Substituting (3.7) into (3.5), we arrive at

\[
w^A(t) \leq w(t) \delta^2(t) \frac{\mathcal{G}(t)}{\delta(t)} - w^2(t) \mathcal{G}(t) \delta^2(t) - \delta^2(t) \mathcal{G}(t)
\]

\[
- \delta^2(t) a(t) \frac{\mathcal{G}(t)}{\delta(t)} w(t) - p(t) \delta^2(t) \frac{t \mathcal{G}(t)}{2 \delta(t)} w(t).
\]

Integrating Eq. (3.9) from \(2s_0\) to \(t\), we have

\[
\int_{2s_0}^{t} \mathcal{F}(s) \Delta s \leq w(2s_0) < \infty.
\]

For sufficient large \(t\), we derive a contradiction to (3.1), as the left-hand side of (3.10) finite, which completes the proof of our theorem.

From Theorem 3.1, we may also obtain some results concerning the oscillation behavior of solutions of Eq. (1.2).

**Corollary 3.2** Assume that \(O_1, O_2,\) relations (1.16) and (1.18) hold. Moreover, if there exist a \(\Delta\)-derivative function \(\delta(t) > 0\) and \(s_0 \geq 0, t_0 \leq 2s_0 < t\), respectively, and satisfying the following conditions as follows,

\[
\limsup_{t \to \infty} \int_{2s_0}^{t} \delta^2(s) \mathcal{G}(s) \Delta s = \infty
\]

and

\[
\limsup_{t \to \infty} \int_{2s_0}^{t} \left( \frac{\delta^2(s) - \mathcal{G}(s) \delta(s) (a(s) + (p(s))^{\frac{1}{2}})}{4 \delta^2(s) \mathcal{G}(s)} \right) \Delta s < \infty,
\]

then, Eq. (1.2) oscillates on \([t_0, \infty)\).  

The next result immediately follows from Theorem 3.1 by different choices of \(\delta(t)\). In particular, we take \(\delta(t)\) as positive constant (say \(C > 0\), we establish the following corollary.
Corollary 3.3 Assume that $O_1, O_2$, relations (1.16) and (1.18) hold. Moreover, if there exists $s_0 \geq 0$ such that $t_0 \leq 2s_0 < t$, and satisfying the following condition below
\[ \limsup_{t \to \infty} \int_{s_0}^{t} \frac{G(s)}{4} \left( 4 - \left( a(s) + p(s) \frac{s^2}{2} \right)^2 \right) \Delta s = \infty, \]  
(3.13)
then, Eq. (1.2) oscillates on $[t_0, \infty) \setminus$. We introduce one more condition $\mu(t) \frac{\delta^2(t)}{\sigma^2(t)} < 1$ to obtain a new oscillations criteria for (1.2).

Theorem 3.4 Assume that $O_1, O_2$, relations (1.16) and (1.18) hold. Moreover, if there exist a $\Delta$-derivative function $\delta(t) > 0$ and $s_0 \geq 0$ such that $\mu(t) \frac{\delta(t)}{\sigma(t)} < 1$ and $t_0 \leq 2s_0 < t$, respectively, and satisfying the condition
\[ \limsup_{t \to \infty} \int_{s_0}^{t} \frac{1}{1 - \frac{\sigma(t)}{\delta(t)} \mu(s)} \left[ G(s) \delta(s) - \frac{\delta(s) A^2(s)}{4G(s)} \right] \Delta s = \infty, \]  
(3.14)
where
\[ A(t) := \frac{\delta^3(t)}{\sigma(t)} - G(t) \left( a(t) + p(t) \frac{t^2}{2} \right). \]  
(3.15)
then, Eq. (1.2) oscillates on $[t_0, \infty) \setminus$.

Proof Assume to the contrary that (1.2) has a non-oscillatory solution. Let $y(t)$ be a non-oscillatory solution of (1.2). Then, without loss of generality, we assume that $y(t)$ is an eventually positive solution of (1.2), i.e., there exists $t_0 \leq s_0$ such that $y(t) > 0 \forall t \in [s_0, \infty) \setminus$. A similar argument holds for the case when $y(t)$ is eventually negative. Now $\Delta$-differentiate equation (3.3) w.r.t $t$, we have
\[ w^\Delta(t) = \delta(t) \left( \frac{\delta(t)}{y(t)} \right)^\Delta + \delta^3(t) \left( \frac{\delta(t)}{y(t)} \right)^\sigma. \]  
(3.16)

From Eqs. (3.3), (2.11) and (3.16), we obtain
\[ w^\Delta(t) \leq \frac{\delta(t)}{\sigma(t)} w^\sigma(t) - \frac{G(t)}{\delta(t)} w^2(t) + \delta(t) \frac{\delta^3(t)}{y^3(t)} \]  
(3.17)
\[ \left( \delta(s) G(s) - \frac{(\delta(s) - \delta(s) G(s)(a(s) + p(s) \frac{s^2}{2}))}{4G(s)} \right) \Delta s = \infty, \]  
(3.21)
then, Eq. (1.2) oscillates on $[t_0, \infty) \setminus$.

Proof Assume to the contrary that (1.2) has a non-oscillatory solution. Let $y(t)$ be a non-oscillatory solution of (1.2). Then, without loss of generality, we assume that $y(t)$ is an eventually positive solution of (1.2) i.e., there exists $t_0 \leq s_0$ such that $y(t) > 0 \forall t \in [s_0, \infty) \setminus$. A similar argument holds for the case when $y(t)$ is eventually negative. Now from equations (2.7), (3.3) and (3.16), we obtain
\[ \begin{align*}
\frac{d^3(t)}{\sigma(t)} - G(t) \delta(t) & - a(t) G(t) w(t) - p(t) \frac{t^2}{2} G(t) w(t), \\
\frac{d^3(t)}{\sigma(t)} & - G(t) \delta(t) - a(t) G(t) w(t) - p(t) \frac{t^2}{2} G(t) w(t),
\end{align*} \]  
(3.18)
From (3.18), we have
\[
\delta(t) \frac{\gamma(t)}{\gamma(t)} \leq - \mathcal{G}(t) \delta(t) - a(t) \mathcal{G}(t) w(t) - p(t) \frac{t}{2} \mathcal{G}(t) w(t).
\]  
(3.22)

Substituting (3.23) into (3.22), we arrive at
\[
w^A(t) \leq - \delta(t) \mathcal{G}(t) + \frac{\delta(t)}{\delta(t)} w^\sigma(t) - \frac{\delta(t)}{\delta(t)} (w^\sigma(t))^2 - \frac{\delta(t)}{\delta(t)} \mathcal{G}(t) \left( a(t) + p(t) \frac{t}{2} \right) w(t) - \frac{\delta(t)}{\delta(t)} \mathcal{G}(t) (a(t) + p(t) \frac{t}{2}) w(t).
\]  
(3.23)

Substituting (3.25) into (3.24), which yields
\[
w^A(t) \leq - \delta(t) \mathcal{G}(t) + \frac{\delta(t)}{\delta(t)} w^\sigma(t) - \frac{\delta(t)}{\delta(t)} \mathcal{G}(t) \left( a(t) + p(t) \frac{t}{2} \right) w(t).
\]  
(3.25)

Substituting (3.25) into (3.24), which yields
\[
w^A(t) \leq - \delta(t) \mathcal{G}(t) + \frac{\delta(t)}{\delta(t)} w^\sigma(t) - \frac{\delta(t)}{\delta(t)} \mathcal{G}(t) \left( a(t) + p(t) \frac{t}{2} \right) w(t).
\]  
(3.26)

From (3.3), (2.11) and \(y^\Delta(t) < 0\), we obtain a relation below,
\[
w(t) \geq \frac{\delta(t)}{\delta(t)} w^\sigma(t).
\]  
(3.27)

By following the similar steps of Eqs. (3.8) and (3.9), Eq. (3.27) become
\[
w^A(t) \leq - \delta(t) \mathcal{G}(t) + \frac{\delta(t)}{\delta(t)} \mathcal{G}(t) \left( a(t) + p(t) \frac{t}{2} \right) w^\sigma(t).
\]  
(3.28)

Integrating Eq. (3.28) from \(2s_0\) to \(t\). Therefore, we have
\[
\int_{2s_0}^{t} \left[ \delta(s) \mathcal{G}(s) - \frac{\delta(s)}{\delta(s)} \mathcal{G}(s) (a(s) + p(s) \frac{t}{2}) \right] \Delta s \leq w(2s_0) < \infty.
\]  
(3.29)

For sufficiently large \(t\), we derive a contradiction to (3.21), as the left-hand side finite, which completes the proof of our theorem. \(\square\)

In view of Theorem 3.5, we immediately obtain the following corollary.

**Corollary 3.6** Assume that \(O_1, O_2\), relations (1.16) and (1.18) hold. Moreover, if there exist \(\Delta\)-derivative function \(\delta(t) > 0\) and \(s_0 \geq 0\), \(t_0 \leq 2s_0 < t\), respectively, and satisfying the following conditions,
\[
\lim_{t \to \infty} \int_{2s_0}^{t} \delta(s) \mathcal{G}(s) \Delta s = \infty
\]  
(3.30)
and
\[
\lim_{t \to \infty} \int_{2s_0}^{t} \frac{(\delta(s) - \delta(t) \mathcal{G}(s) (a(s) + p(s) \frac{t}{2}))^2}{4\delta(s)} \Delta s < \infty,
\]  
(3.31)
then, Eq. (1.2) oscillates on \([t_0, \infty)\).

**Remark 3.7** In view of Theorems 3.4 and 3.5, we obtain various sufficient conditions of oscillation of Eq. (1.2) by taking different choice of \(\delta(t)\).

To present our next theorems, we first introduce Saker’s result [29] as follows
\[
((t - s) \mathcal{N})^N \leq - \mathcal{N}(t - \sigma(s))^N - 1 \leq 0 \quad \text{for} \ N > 1 \quad \text{and} \quad \sigma(s) \leq t.
\]  
(3.32)

By using an integral averaging technique of Kamenev-type, we present some new oscillation criteria of (1.2).

**Theorem 3.8** Assume that \(O_1, O_2\), relations (1.16) and (1.18) hold. Moreover, if there exist \(\Delta\)-derivative function \(\delta(t) > 0\) and there exists \(\mathcal{N} > 1\) and all sufficiently large \(s_0 \geq 0\), \(t_0 \leq 2s_0 < t\) such that
\[
\lim_{t \to \infty} \frac{1}{\mathcal{N}} \int_{2s_0}^{t} (t - s)^N \mathcal{F}(s) \Delta s = \infty,
\]  
(3.33)
where
\[
\mathcal{F}(s) := \left( \delta(s) \mathcal{G}(s) - \frac{(\delta(s) - \delta(t) \mathcal{G}(s) (a(s) + p(s) \frac{t}{2}))^2}{4\delta(s)} \right).
\]  
(3.34)

Then, Eq. (1.2) oscillates on \([t_0, \infty)\).

**Proof** Assume to the contrary that (1.2) has a non-oscillatory solution. Let \(y(t)\) be a non-oscillatory solution of (1.2). Then, without loss of generality, we assume that \(y(t)\) is an eventually positive function, i.e., there exists \(t_0\) such that \(y(t) > 0\) for \(t \in [t_0, \infty)\). A similar argument holds.
also for the case when \( y(t) \) is eventually negative. From Eq. (3.9), we have
\[
\mathcal{F}(t) \leq -w^A(t), \quad \text{for } 2s_0 \leq t.
\]
Multiplying above relation by \((t-s)^N\) and then integrating from \(2s_0\) to \(t\) we obtain
\[
\int_{2s_0}^{t} (t-s)^N \mathcal{F}(s) \Delta s \leq -\int_{2s_0}^{t} (t-s)^N w^A(s) \Delta s.
\] (3.35)
By comparing the right-hand side of (3.35) with Eq. (2.2), we have
\[
-\int_{2s_0}^{t} (t-s)^N w^A(s) \Delta s = (t-2s_0)^N w(2s_0)
\] (3.36)
and
\[
\int_{2s_0}^{t} ((t-s)^N) \Delta w^\sigma(s) \Delta s.
\]
From Eqs. (3.32), (3.35) and (3.36), we arrive at
\[
\int_{2s_0}^{t} (t-s)^N \mathcal{F}(s) \Delta s \leq (t-2s_0)^N w(2s_0).
\] (3.37)
Thus
\[
\frac{1}{t^N} \int_{2s_0}^{t} (t-s)^N \mathcal{F}(s) \Delta s \leq (t-2s_0)^N w(2s_0),
\] (3.38)
for \( 2s_0 \leq t \).
Taking lim sup as \( t \to \infty \) both sides of Eq. (3.38), we have
\[
\limsup_{t \to \infty} \frac{1}{t^N} \int_{2s_0}^{t} (t-s)^N \mathcal{F}(s) \Delta s < \infty.
\] (3.39)
Thus, we derive a contradiction to (3.33), which completes the proof of our theorem.

**Corollary 3.9** Assume that \( O_1, O_2 \), relations (1.16) and (1.18) hold. Moreover, if there exists \( \Delta \)-derivative function \( \delta(t) > 0 \), and for \( N > 1 \), and all sufficiently large \( s_0 \geq 0 \), such that \( t_0 \leq 2s_0 < t \) and satisfying the conditions below,
\[
\limsup_{t \to \infty} \frac{1}{t^N} \int_{2s_0}^{t} (t-s)^N \delta(t) \mathcal{G}(t) \Delta s = \infty,
\] (3.40)
and
\[
\limsup_{t \to \infty} \frac{1}{t^N} \int_{2s_0}^{t} (t-s)^N \left( \delta(t) - \delta(t)^\sigma(t) \right) \Delta s < \infty,
\] (3.41)
then, Eq. (1.2) oscillates on \([t_0, \infty)\).

**Theorem 3.10** Assume that \( O_1, O_2 \), relations (1.16) and (1.18) hold. Moreover, if there exists \( \Delta \)-derivative function \( \delta(t) > 0 \) such that \( \mu(t) \left( \frac{\delta(t)}{\sigma(t)} \right)^\sigma(t) < 1 \). For \( N > 1 \) and all sufficiently large \( s_0 \geq 0 \), \( t_0 \leq 2s_0 < t \) and satisfying the condition below,
\[
\limsup_{t \to \infty} \frac{1}{t^N} \int_{2s_0}^{t} \left( \frac{(t-s)^N}{1 - \frac{\delta(t)}{\sigma(t)} \mu(t)} \right) \left[ \mathcal{G}(s) \delta(s) - \frac{\delta(s)A^2(s)}{4\mathcal{G}(s)} \right] \Delta s = \infty,
\] (3.42)
where \( A(t) \) is given by (3.15). Then, Eq. (1.2) oscillates on \([t_0, \infty)\).

**Proof** Assume to the contrary that (1.2) has a non-oscillatory solution. Let \( y(t) \) be a non-oscillatory solution of (1.2). Then, without loss of generality, we assume that \( y(t) \) is an eventually positive function, i.e., there exists \( t_0 \) such that \( y(t) > 0 \) for \( t > t_0 \). A similar argument holds for the case when \( y(t) \) is eventually negative. From Eq. (3.20), we have
\[
\frac{1}{t^N} \int_{2s_0}^{t} \left( \frac{(t-s)^N}{1 - \frac{\delta(t)}{\sigma(t)} \mu(t)} \right) \left[ \mathcal{G}(s) \delta(s) - \frac{\delta(s)A^2(s)}{4\mathcal{G}(s)} \right] \Delta s \leq \frac{1}{t^N} \int_{2s_0}^{t} (t-s)^N w^A(s) \Delta s.
\] (3.43)
By following the similar steps of Eqs. (3.35)–(3.37), Eq. (3.43) become
\[
\int_{2s_0}^{t} \left( \frac{(t-s)^N}{1 - \frac{\delta(t)}{\sigma(t)} \mu(t)} \right) \left[ \mathcal{G}(s) \delta(s) - \frac{\delta(s)A^2(s)}{4\mathcal{G}(s)} \right] \Delta s \leq (t-2s_0)^N w(2s_0).
\] (3.44)
Thus, we have
\[
\frac{1}{t^N} \int_{2s_0}^{t} \left( \frac{(t-s)^N}{1 - \frac{\delta(t)}{\sigma(t)} \mu(t)} \right) \left[ \mathcal{G}(s) \delta(s) - \frac{\delta(s)A^2(s)}{4\mathcal{G}(s)} \right] \Delta s \leq \frac{(t-2s_0)^N}{t^N} w(2s_0).
\] (3.45)
Taking lim sup as \( t \to \infty \) in Eq. (3.45), we obtain
\[
\limsup_{t \to \infty} \frac{1}{t^N} \int_{2s_0}^{t} \left( \frac{(t-s)^N}{1 - \frac{\delta(t)}{\sigma(t)} \mu(t)} \right) \left[ \mathcal{G}(s) \delta(s) - \frac{\delta(s)A^2(s)}{4\mathcal{G}(s)} \right] \Delta s < \infty.
\] (3.46)
Thus, we derive a contradiction to (3.42), which completes the proof of our theorem. \hfill \Box

**Theorem 3.11** Assume that $O_1, O_2$, relations (1.16) and (1.18) hold. Moreover, if there exists a $\Delta$-derivative function $\delta(t) > 0$. For $N > 1$ and all sufficiently large $s_0 \geq 0$, $t_0 \leq s_0 < t$, and satisfying the condition below

$$
\limsup_{t \to -\infty} \frac{1}{N} \int_{2s_0}^{t} (t-s)^N \left( \delta(s)G(s) - \frac{(\delta(s) - \delta(s)G(s)(a(s) + p(s)\frac{1}{2}))^2}{4\delta(s)} \right) \Delta s = \infty,
$$

(3.47)

Then, Eq. (1.2) oscillates on $[t_0, \infty)_\gamma$.

**Proof** Assume to the contrary that (1.2) has a non-oscillatory solution. Let $\gamma(t)$ be a non-oscillatory solution of (1.2). Then, without loss of generality, we assume that $\gamma(t)$ is an eventually positive function, i.e., there exists $t_0$ such that $\gamma(t) > 0 \forall t \in [t_0, \infty)_\gamma$. From Eq. (3.28) and following the similar steps of Eqs. (3.35)–(3.39), we easily obtain a following relation

$$
\frac{1}{N} \int_{2s_0}^{t} (t-s)^N \left( \delta(s)G(s) - \frac{(\delta(s) - \delta(s)G(s)(a(s) + p(s)\frac{1}{2}))^2}{4\delta(s)} \right) \Delta s \leq \left( 1 - \frac{2s_0}{t} \right)^N w(2s_0) < \infty.
$$

(3.48)

For all sufficiently large $t$, we derive a contradiction to (3.47). \hfill \Box

**Corollary 3.12** Assume that $O_1, O_2$, relations (1.16) and (1.18) hold. Moreover, if there exists a $\Delta$-derivative function $\delta(t) > 0$. For $N > 1$ and all sufficiently large $s_0 \geq 0$, $t_0 \leq s_0 < t$ and satisfying the following conditions below,

$$
\limsup_{t \to -\infty} \frac{1}{N} \int_{2s_0}^{t} (t-s)^N \delta(s)G(s) \Delta s = \infty,
$$

(3.49)

and

$$
\limsup_{t \to -\infty} \frac{1}{N} \int_{2s_0}^{t} (t-s)^N \left( \delta(s)G(s)(a(s) + p(s)\frac{1}{2}) \right)^2 \Delta s < \infty.
$$

(3.50)

Then, Eq. (1.2) oscillates on $[t_0, \infty)_\gamma$.

Our next aim to establish the Philos-type oscillation criteria for (1.2). We define some elementary assumptions as follows:

For any number $\eta \in \mathbb{R}$, we define positive and negative parts, $\eta_{+}$ and $\eta_{-}$, respectively, of $\eta$ by

$$
\eta_{+} := \max\{0, \eta\} \quad \text{and} \quad \eta_{-} := \max\{0, -\eta\}.
$$

Assume that the rd-continuous functions $H, h : \mathbb{R} \to \mathbb{R}$, where $\mathbb{R} = \{ (t,s); t_0 \leq s_0 \leq t \}$ such that

$$
H(t, t) = 0, H(t, s) > 0 \quad \text{and} \quad H^\Delta(t, s) < 0, \quad t_0 \leq s < t
$$

(3.51)

and $H^\Delta(t, s)$ ($\Delta$-derivative w.r.t second variable) is rd-continuous function.

**Theorem 3.13** Assume that $O_1, O_2$, relations (1.16), (1.18) and Eq. (3.51) hold. Moreover, if there exist $\Delta$-derivative function $\delta(t) > 0$ and $s_0 \geq 0$, $t_0 \leq s_0 < t$, respectively, and satisfying the following conditions below,

$$
H^\Delta(s, t) + \frac{H^\sigma(s, t)}{\delta(t)} \left( \delta^\Delta(t) - \delta^\sigma(t) \left( a(t) + \frac{p(t)}{2} \right) \right) = - h(t, s) \frac{H^\sigma(s, t)}{\delta(t)} \sqrt{H^\sigma(s, t)}
$$

(3.52)

and

$$
\limsup_{t \to -\infty} \frac{1}{H^\sigma (s, t)} \int_{2s_0}^{t} H^\sigma(s, t) \Delta s = \infty,
$$

(3.53)

then, Eq. (1.2) oscillates on $[t_0, \infty)_\gamma$.

**Proof** Assume to the contrary that (1.2) has a non-oscillatory solution. Let $\gamma(t)$ be a non-oscillatory solution of (1.2). Then, without loss of generality, we assume that $\gamma(t)$ is an eventually positive function, i.e., there exists $t_0$ such that $\gamma(t) > 0 \forall t \in [t_0, \infty)_\gamma$. A similar argument holds for the case when $\gamma(t)$ is eventually negative. We have defined Riccati transformation function in (3.3). Now from (3.8), we have

$$
w^\Delta(t) \leq - \delta^\sigma(t)G(t) + \frac{1}{\delta(t)} \left( \delta^\Delta(t) - \delta^\sigma(t) \left( a(t) + \frac{p(t)}{2} \right) \right) w(t)
$$

(3.54)

and

$$
- \delta^\sigma(t) \frac{G(t)w^2(t)}{\delta^2(t)} \text{ for } 2s_0 \leq t.
$$

Multiplying Eq. (3.54) by $H^\sigma(s, t)$ i.e., $H(s, t, s)$, and then integrating from $2s_0$ to $s(t)$, we obtain
\[
\int_{2s_0}^{\sigma(t)} H^\sigma(\sigma(t), s) \delta^\sigma(s) G(s) \Delta s \leq \int_{2s_0}^{\sigma(t)} H^\sigma(\sigma(t), s) w^A(s) \Delta s
\]
\[+ \int_{2s_0}^{\sigma(t)} H^\sigma(\sigma(t), s) \delta^\sigma(s) G(s) \Delta s \leq H(\sigma(t), 2s_0))w(2s_0) + \int_{2s_0}^{\sigma(t)} (h_-(t, s))^2 \Delta s.
\]
(3.59)

Dividing (3.59) by \(H(\sigma(t), 2s_0)\), we obtain
\[
\frac{1}{H(\sigma(t), 2s_0)} \int_{2s_0}^{\sigma(t)} \left( \left( H^\sigma(\sigma(t), s) \delta^\sigma(s) G(s) - \frac{(h_-(t, s))^2}{4\sigma^\sigma(s) G(s)} \right) \Delta s \right) \leq w(2s_0) < \infty.
\]
(3.60)

for sufficiently large \(t\). Thus, we derive a contradiction to (3.53).

\[\square\]

**Corollary 3.14** Assume that \(O_1, O_2\), relations (1.16), (1.18) and Eq. (3.51) hold. Moreover, if there exist \(t\)-derivative function \(\delta(t) > 0\) and \(s_0 \geq 0\), \(t_0 \leq 2s_0 < t\), respectively, and satisfy the following conditions below
\[
H^\delta(\sigma(t), s) + \frac{H^\sigma(\sigma(t), s)}{\delta(t)} \left( \delta^\delta(t) - G(t) \sigma^\sigma(s) \left( a(s) + p(t) \frac{s}{2} \right) \right) = \frac{h(t, s)}{\delta(t)} \sqrt{H^\sigma(\sigma(t), s)}.
\]
(3.61)

\[
\limsup_{t \to \infty} \frac{1}{H(\sigma(t), 2s_0)} \int_{2s_0}^{\sigma(t)} H^\sigma(\sigma(t), s) \delta^\sigma(s) G(s) \Delta s = \infty \quad \text{and} \quad \limsup_{t \to \infty} \frac{1}{H(\sigma(t), 2s_0)} \int_{2s_0}^{\sigma(t)} (h_-(t, s))^2 \Delta s < \infty.
\]
(3.62)

Then, Eq. (1.2) oscillates on \([t_0, \infty)\).

**Theorem 3.15** Assume that \(O_1, O_2\), relations (1.16), (1.18) and Eq. (3.51) hold. Moreover, if there exist \(t\)-derivative function \(\delta(t) > 0\) and \(s_0 \geq 0\), such that \(\mu(t) \frac{\delta^\delta(t)}{\sigma(t)} < 1\) and \(t_0 \leq 2s_0 < t\), respectively, and satisfy the conditions below
\[
H^\delta(\sigma(t), s) + \frac{H^\sigma(\sigma(t), s)}{\delta(t)} \left( 1 - \frac{\delta^\delta(t)}{\sigma(t)} \mu(t) \right) A(t) = \frac{h(t, s)}{\sigma(t)} \delta^\delta(t) \sqrt{H^\sigma(\sigma(t), s)},
\]
(3.63)

and
\[
\int_{2s_0}^{\sigma(t)} H^\sigma(\sigma(t), s) \delta^\sigma(s) G(s) \Delta s \leq H(\sigma(t), 2s_0))w(2s_0) + \int_{2s_0}^{\sigma(t)} (h_-(t, s))^2 \Delta s.
\]
\[
\limsup_{t \to \infty} \frac{1}{H(\sigma(t), 2s_0)} \int_{2s_0}^{\sigma(t)} \left( H^\sigma(\sigma(t), s)G(s) - \frac{(h_-(t, s))^2}{4G(s)} \right) \Delta s \leq H(\sigma(t), 2s_0)w(2s_0)
\]

\[
+ \int_{2s_0}^{\sigma(t)} \frac{\delta(s)(h_-(t, s))^2}{4\left(1 - \frac{\delta(s)}{\delta^\sigma(t)} \mu(s)\right)\Delta s}.
\]

(3.64)

where \(A(t)\) is given by (3.15). Then, Eq. (1.2) oscillates on \([t_0 \infty)_\gamma\).

**Proof** Assume to the contrary that (1.2) has a non-oscillatory solution. Let \(y(t)\) be a non-oscillatory solution of (1.2). Then, without loss of generality, we assume that \(y(t)\) is an eventually positive function, i.e., there exists \(t_0\) such that \(y(t) > 0\) for \(t \in [t_0 \infty)_\gamma\). A similar argument holds for the case when \(y(t)\) is eventually negative. We have defined Riccati transformation function in (3.3). Now from (3.19), we have

\[
\left(1 - \frac{\delta^\Lambda(t)}{\delta^\sigma(t)} \mu(t) \right)w^\Lambda(t) \leq -G(t)\delta(t) + A(t)w(t) - \frac{H(t)}{\delta^\sigma(t)} \Delta s < w(2s_0) < \infty.
\]

(3.65)

Multiplying Eq. (3.65) by \(H^\sigma(\sigma(t), s)\) and then integrating from \(2s_0\) to \(\sigma(t)\), we have

\[
\int_{2s_0}^{\sigma(t)} H^\sigma(\sigma(t), s)G(s)\Delta s \leq \left(1 - \frac{\delta^\Lambda(t)}{\delta^\sigma(t)} \mu(t)\right)w(s) \Delta s
\]

\[
+ \int_{2s_0}^{\sigma(t)} H^\sigma(\sigma(t), s)A(s)w(s) \Delta s
\]

\[
+ \int_{2s_0}^{\sigma(t)} \frac{H^\sigma(\sigma(t), s)G(s)}{\delta(s)\left(1 - \frac{\delta^\Lambda(t)}{\delta^\sigma(t)} \mu(s)\right)} w^2(s) \Delta s.
\]

(3.66)

From (3.63), (3.66) and by following the similar steps of Eqs. (3.56)–(3.59), we obtain

\[
\int_{2s_0}^{\sigma(t)} H^\sigma(\sigma(t), s)G(s)\delta(s) \Delta s \leq H(\sigma(t), 2s_0)w(2s_0)
\]

(3.67)

Dividing Eq. (3.67) by \(H(\sigma(t), 2s_0)\), we obtain

\[
\frac{1}{H(\sigma(t), 2s_0)} \int_{2s_0}^{\sigma(t)} \left( H^\sigma(\sigma(t), s)G(s) - \frac{(h_-(t, s))^2}{4G(s)} \right) \delta(s) \Delta s < w(2s_0) < \infty.
\]

(3.68)

For sufficiently large \(t\), we derive a contradiction to (3.64), which completes the proof of our theorem.

**Theorem 3.16** Assume that \(O_1, O_2, \text{relations } (1.16), (1.18) \text{ and Eq. (3.51) hold. Moreover, if there exist } \Delta\text{-derivative function } \delta(t) > 0 \text{ and } s_0 \geq 0 \text{, } t_0 \leq 2s_0 < t, \text{ respectively, and satisfying the following conditions below:}

\[
H^\lambda(t, s) + \frac{H(t, s)}{\delta^\sigma(t)} \left(\delta^\Lambda(t) - G(t)\delta(t)(a(t) + p(t)\frac{t}{2})\right)
\]

\[
= -h(t, s)\delta^\sigma(t)\sqrt{H(t, s)},
\]

(3.69)

and

\[
\limsup_{t \to \infty} \frac{1}{H(t, 2s_0)} \int_{2s_0}^{t} \left( H(t, s)\delta(s)G(s) - \frac{(h_-(t, s))^2}{4\delta(s)} \right) \Delta s = \infty,
\]

(3.70)

then, Eq. (1.2) oscillates on \([t_0 \infty)_\gamma\).

**Proof** Assume to the contrary that (1.2) has a non-oscillatory solution. Let \(y(t)\) be a non-oscillatory solution of (1.2). Then, without loss of generality, we assume that \(y(t)\) is an eventually positive function, i.e., there exists \(t_0\) such that \(y(t) > 0\) for \(t \in [t_0 \infty)_\gamma\). We have defined Riccati transformation function in (3.3). Now multiplying Eq. (3.28) by \(H(t, s)\) and integrating from \(2s_0\) to \(t\), we have following relation

\[
\int_{2s_0}^{t} H(t, s)\delta(t)G(s)\Delta s \leq -\int_{2s_0}^{t} H(t, s)w^\Lambda(s)\Delta s
\]

\[
+ \int_{2s_0}^{t} H(t, s)\left(\delta^\Lambda(s) - \delta(s)G(s)(a(s) + p(s)\frac{s}{2})\right)\Delta s.
\]
From (2.2) and (3.51), we obtain
\[ \int_{2s_0}^{t} H(t, s)\delta(s) G(s)\Delta s \leq H(t, 2s_0) w(2s_0) \]
\[ + \int_{2s_0}^{t} \left( H(t, s) + H(t, s) \frac{(\delta^b(s) - \delta(s)) G(s)(a(s) + p(s) \frac{t}{2})}{\delta^a(s)} \right) w^2(s) \Delta s \]
\[ - \int_{2s_0}^{t} H(t, s)\delta(s) G(s)\Delta s \leq H(t, 2s_0) w(2s_0) \]
\[ - \int_{2s_0}^{t} H(t, s)\delta(s) G(s)\Delta s \leq H(t, 2s_0) w(2s_0) \]
\[ - \int_{2s_0}^{t} H(t, s)\delta(s) G(s)\Delta s \leq H(t, 2s_0) w(2s_0) \]
\[ (3.71) \]

From Eqs. (3.69), (3.72) and by following the similar steps of Eqs. (3.35)–(3.39), we obtain a new relation
\[ \frac{1}{H(t, 2s_0)} \int_{2s_0}^{t} \left( H(t, s)\delta(s) G(s) - \frac{(h_-(t, s))^2}{4\delta(s)} \right) \Delta s \leq w(2s_0) < \infty, \]
for sufficiently large \( t \). Thus, we derive a contradiction to (3.70), which completes the proof of our theorem. \( \square \)

**Corollary 3.17** Assume that \( O_1, O_2, \) relations (1.16), (1.18) and Eq. (3.51) hold. Moreover, if there exist \( \Delta \)-derivative function \( \delta(t) > 0 \) and \( s_0 \geq 0, t_0 \leq 2s_0 < t, \) respectively, and satisfying the following conditions below,
\[ H^b(t, s) + H(t, s) \frac{\delta^b(t) - \delta(t)}{\delta^a(t)} \left( a(t) + p(t) \frac{t}{2} \right) \]
\[ = - \frac{h(t, s)}{\delta^a(t)} \sqrt{H(t, s)}, \]
\[ (3.74) \]
and
\[ \limsup_{t \to \infty} \frac{1}{H(t, 2s_0)} \int_{2s_0}^{t} H(t, s)\delta(s) G(s)\Delta s \]
\[ = \infty \text{ and } \limsup_{t \to \infty} \frac{1}{H(t, 2s_0)} \int_{2s_0}^{t} \frac{(h_-(t, s))^2}{4\delta(s)} \Delta s < \infty, \]
\[ (3.75) \]
then, Eq. (1.2) oscillates on \([t_0, \infty)\). We can easily see that Eq. (1.2) in its general form involves differential equation and different types of difference equations depending on the choice of the time scale \( T \).

**Remark 3.18** When \( T = \mathbb{R} \), Eq. (1.2) become second-order linear delay difference equation:
\[ y''(t) + a(t)y'(t) + y(t) + K(y(t - h)) + f(W(y(t - h))) = 0, \]
\[ (3.76) \]
where we have \( \sigma(t) = t, \mu(t) = 0, g^\Delta(t) = g'(t) \) and
\[ \int_{a}^{b} g(t)\Delta t = \int_{a}^{b} g(t) \Delta t \]

**Remark 3.19** When \( T = \mathbb{Z} \), Eq. (1.2) become second-order linear delay difference equation:
\[ \Delta^2 y(t) + a(t)\Delta y(t) + y(t) + K(y(t - h)) + f(W(\Delta y(t - h))) = 0, \]
\[ (3.77) \]
where we have \( \sigma(t) = t + 1, \mu(t) = 1, g^\Delta(t) = \Delta g(t) \) and
\[ \int_{a}^{b} g(t)\Delta t = \sum_{t=a}^{b-1} g(t). \]

**Remark 3.20** When \( T = h'h, h' > 0, \) Eq. (1.2) become second-order linear delay difference equation with step size \( h' \)
\[ \Delta_{h'}^2 y(t) + a(t)\Delta_{h'} y(t) + y(t) + K(y(t - h)) + f(W(\Delta_{h'} y(t - h))) = 0, \]
\[ (3.78) \]
where we have \( \sigma(t) = t + h', \mu(t) = h', g^\Delta(t) = \Delta_{h'} f(t) \) and
\[ \int_{a}^{b} f(t)\Delta t = \sum_{t=a}^{b-1} f(h'_t) h'. \]

For a function
\[ f(W(x^A(t))) = q(t) \text{sgn}(x(t)) |W(x^A(t))| \quad \forall x \in \mathbb{R}, \]
where \( q(t) = \frac{w(t)}{w(t) \mu(t)} \forall t \in T, \) we would like to illustrate our results by means of the following examples.

**Example**

Let us consider a second-order non-linear dynamic equation on time scale \( T \).
\[ y''(t) + \frac{1}{t} y'(t) + y(t) + \sum_{n=1}^{\infty} k(2n-1)(y(t - h))(2n-1) \]
\[ + \frac{1}{2} \mu(t) \sum_{m=1}^{\infty} w(2m-1)(y(t - h))(2m-1)^2 \]
\[ = 0. \]
\[ (4.1) \]
In view of Corollary 3.3, we choose the parameters as follows: \( \sigma(t) = 1, a(t) = \frac{t}{t}, p(t) = \frac{t}{t} \), \( \forall k_n, w_m \in \mathbb{R}^+ \) such that \( 0 < p(t) \leq k_1 < \infty, k_{2n+1} > 0, k_{2n} := 0 \) and \( w_{2m} := 0, w_{(2m-1)} > 0 \forall n, m \in \mathbb{N} \) respectively. Also \( 0 < \mu(t) < t - s_0 < t \) for all \( t > 2s_0, s_0 \geq 2. \) Thus, we have \( \frac{t}{t} \leq G(t) < 1. \) Therefore, it is easy to verify that the conditions \( O_1, O_2, (1.16), (1.18) \) and (3.13) hold. The parameters also satisfying the following conditions below:
\[ \int_{2s_0}^{t} \left( G(s) - G(s) \frac{(a(s) + p(s) \frac{t}{2})}{4} \right) \Delta s \geq \int_{2s_0}^{t} \left( \frac{1}{2} \frac{1}{s^2} \right) \Delta s \]
\[ \geq \frac{7}{16} (t - 2s_0) \rightarrow \infty \text{ as } t \rightarrow \infty. \]
\[ (4.2) \]
When $\mu(t) = 0$, we have
\[
\int_{2s_0}^t \left( G(s) - \frac{G(s)}{4} \left( \frac{a(s) + p(s) \frac{t}{2}}{4} \right)^2 \right) ds = \int_{2s_0}^t \left( 1 - \frac{1}{s^2} \right) ds
> \frac{3}{4} (t - 2s_0) \to \infty \text{ as } t \to \infty.
\]

Therefore, from Eq. (4.3) and in view of corollary 3.3, we conclude that (4.5) oscillates on $[2, \infty)$.

**Conclusion**

The oscillation of a function is a number that quantifies how much a function varies between its extreme values as it approaches infinity or some other point. It is very important qualitative property of a function. There are many criteria for oscillations, out of which two important are Kamenev and Philos type. In this present work, we have established such criteria for Eq. (1.2). First we have proved Lemmas 2.11–2.13 and then used Riccati transformation technique to establish the main result. We have obtained some new conditions for the oscillations. It is also important to note that the results presented in this paper are not valid for $\mathbb{T} = q^\mathbb{Z}$. The reason is given in the Remark 2.9. We have also constructed few examples for some time scale $T$, i.e., $T = \mathbb{R}, T = \mathbb{Z}$ and $T = h'\mathbb{Z}$ to illustrate the results.

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