CYLINDRIC PARTITIONS AND BRANES

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Abstract

We show that the generating function of the cylindric partitions is given by the open topological string partition function of a CY3-fold. Type IIA string theory compactified on this CY3-fold gives rise to $\mathcal{N} = 2$ U(1) gauge theory with an adjoint hypermultiplet. The two Kähler parameters of the geometry, which are quantized in units of the string coupling constant, and the representation of the branes are given by the profile of the cylindric partition. We relate the level-rank duality of the cylindric generating function to the exchange symmetry of the two Kähler parameters.
1 Introduction

The combinatorics of the topological string partition functions for toric Calabi-Yau threefolds has been an active area of research for the last few years. Considerable progress has been made in understanding the physical and mathematical reasons for the appearance of various combinatorial structures in the topological partition functions.

In an earlier paper [1] it was shown that the partition function of $\mathcal{N} = 2$ $U(1)$ gauge theory with an adjoint hypermultiplet compactified on a circle is given by generating function of the cylindric partitions with a certain profile. The mass of the adjoint hypermultiplet and the coupling constant of gauge theory were related to the profile of the cylindric partitions. In making this identification we used the refined topological vertex [2] to calculate the topological string partition function of the geometry which gives rise to the above mentioned gauge theory. One obvious question was whether there exists a topological string partition function which corresponds to the generating function of cylindric partitions with an arbitrary profile. In this paper we show that the generating function of the cylindric partitions with an arbitrary profile corresponds to the topological string partition function of the same geometry but with branes and quantized Kähler parameters. The presence of the branes modify the profile of the cylindric partitions.

The paper is organized as follows. In section 2 we review the definition and certain properties of the cylindric partitions and their generating function. In this section we mainly follow [3,4]. In section 3 we recall the geometry which gives rise to $\mathcal{N} = 2$ $U(1)$ gauge theory with adjoint hypermultiplet and show that the topological string partition function of this geometry with branes, calculated using the topological vertex, is given by the generating function of the cylindric partitions.

2 Cylindric Partitions

The 3D partitions or plane partitions are generalizations of Young diagrams. They are a weakly decreasing array of non-negative integers $\{\pi_{i,j} \mid i, j \geq 1\}$:

$$\pi_{i,j} \geq \pi_{i+r,j+s}, \quad r, s > 0.$$  \hspace{1cm} (2.1)

They have a very natural pictorial representation in terms of placing $\pi_{i,j}$ boxes at the $(i, j)$ position, similar to a Young diagram $\lambda$ for which we place column of height $\lambda_i$ at the $i^{th}$ place. We can also regard a 3D partition as a series of 2D partitions which satisfy the so-called interlacing condition. Let us first define what it means for a 2D partition to interlace another one and give the condition for the partitions obtained from slicing a 3D partition. For two 2D partitions $\mu$ and $\nu$ we say $\mu$
interlaces $\nu$, written as $\mu \succ \nu$, if the heights of the columns of partitions satisfy

$$\mu_1 \geq \nu_1 \geq \mu_2 \geq \nu_2 \geq \ldots .$$

(2.2)

We slice a 3D partition diagonally by looking each slice whose projection on the base is given by a set of linear equations parameterized by $a \in \mathbb{Z} : x - y = a$ such that

$$\eta(a) = \{ \pi_{i+a,i} \mid i \geq \max(1, -a + 1) \}.$$  

(2.3)

We have the following condition:

$$\eta(a + 1) \succ \eta(a), \quad a < 0 ,$$

$$\eta(a) \succ \eta(a + 1), \quad a \geq 0 .$$

(2.4)

A skew plane partition is defined as a weakly decreasing array of numbers with the bottom shape $\lambda/\nu$, assuming $\lambda \supset \nu$ for $\lambda/\nu$ to make sense. Note that the interlacing condition will depend on the $\nu$ for a skew plane partition with the base $\lambda/\nu$.

The cylindric partitions, first introduced in [5], are generalizations of the plane partitions. However, for our purposes, the reparameterization of them in [3] is more suitable, which we largely follow. The cylindric partitions naturally appear as the underlying combinatorial structure of the partition function of $\mathcal{N} = 2 \ U(1)$ gauge theory with as single adjoint hypermultiplet as shown in [1].

A cylindric partition is a plane partition with additional symmetry imposed. We will take any 2D partition $\lambda$ of fixed $\lambda_1$ such that $\lambda$ is big enough to include another 2D partition $\mu_1$. In addition to the conditions defining a plane partition we want require periodicity. To this end, if we take the transposed partition $\lambda^t$ and place it on top of the first column of itself after a shift by $d$ boxes, the new plane partition should still be a plane partition as in Fig. [II]. As indicated in this figure, this modification is equivalent to the requirement that the partitions denoted by vertical red slices are identical, i.e., we can write these types of planes partitions on a cylinder.

The profile of the cylindric partition is its shape on the cylinder. It can be described by two sets of $N$ numbers where $N$ is the period of the partition [3]. It is easy to see that the period is equal to the sum of $\lambda_1$ and the shift $d$. Let $A = (a_1, a_2, \ldots, a_N)$ and $B = (b_1, b_2, \ldots, b_N)$ such that

$$a_i, b_i \in \{0, 1\}, \quad a_i + b_i = 1 \quad \text{for all} \quad i \in \{1, 2, \ldots, N\}$$

(2.5)

The profile of the partition is then given by $A$ and $B$ such that [4]

$$a_i = \begin{cases} 1, & \text{if the boundary slopes up and to the right on the } i\text{-th diagonal;} \\ 0, & \text{Otherwise.} \end{cases}$$

$$b_i = \begin{cases} 1, & \text{if the boundary slopes down and to the right on the } i\text{-th diagonal;} \\ 0, & \text{Otherwise.} \end{cases}$$

(2.6)

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1Our notation differs from most of the mathematics literature in the pictorial representation of the 2D partition. They are related by taking the transpose.
Figure 1: An example for a cylindric partition: $\lambda = (5, 5, \ldots, 5)$ and $\nu = (3, 2, 2, 1, 1)$ with the shift $d = 7$. The grey shaded region shows the excised region.

If we define $n$ and $\ell$ such that $\sum_{i=1}^{N} a_i = n$ and $\sum_{i=1}^{N} b_i = \ell$ then each profile is equivalent to a partition $\nu$ such that $\ell(\nu) \leq \ell$ and $\ell(\nu^t) \leq n$. Thus the partition can fit inside a $n \times \ell$ rectangle. This gives the shape of the boundary of the partition near the cut of the cylinder. For the example shown in Fig. 1 the profile is given by $A = (1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0)$ and $B = (0, 0, 1, 0, 1, 1, 0, 1, 0, 1, 1, 1)$.

As mentioned before, the interlacing condition is modified for a skew plane partition. There is a very direct relation to the sets $A$ and $B$. Let us first write the interlacing condition:

$$
\eta(5) \prec \eta(4) \prec \eta(3) \succ \eta(2) \prec \eta(1) \succ \eta(0) \\
\succ \eta(-1) \prec \eta(-2) \succ \eta(-3) \succ \eta(-4) \prec \eta(-5) \succ \eta(-6) \succ \eta(-7),
$$

where we need to keep in mind the periodicity condition we imposed, $\eta(5) = \eta(-7)$. A quick comparison shows that the 1’s in the set $A$ correspond to “$\prec$” and 0’s to “$\succ$”, and vice versa for the set $B$.

For our discussion of $U(1)$ theory with adjoint matter we will choose $\lambda$ to be a rectangular 2D partition of size $n \times \ell$ so that $N = n + \ell$. For $\nu$ trivial the profile is given by

$$
A = \begin{pmatrix} 1, 1, \ldots, 1, 0, 0, \ldots, 0 \end{pmatrix}_{n \times \ell}, \\
B = \begin{pmatrix} 0, 0, \ldots, 0, 1, 1, \ldots, 1 \end{pmatrix}_{n \times \ell}.
$$

Note that $\lambda_1 = n$ and $d = \ell$. 

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In terms of the partition $\nu$ the set $A$ and $B$ are
\begin{align*}
a_k &= 1 \text{ for } k = \nu_j^t - j + 1 + n, \quad j = 1, \ldots, n \\
b_k &= 1 \text{ for } k = n - \nu_i + i, \quad i = 1, \ldots, \ell.
\end{align*}

Let us define $G^\ell,n_\nu(s)$ to be the generating function of cylindric plane partitions,
\[ G^\ell,n_\nu(s) = \sum_{\text{cylindric partitions } \pi \text{ of profile } \nu} s^{\vert \pi \vert}, \]
where $\nu$ is such that $\ell(\nu) \leq \ell$ and $\ell(\nu^t) \leq n$. The above generating function was determined in [3] and is given by
\[ G^\ell,n_\nu(s) = \prod_{k=1}^\infty \left(1 - s^{kN}\right)^{-1} \prod_{i,j,A[i]=1,B[j]=1} \left(1 - s^{(i-j)(N)+(k-1)N}\right)^{-1}, \]
where
\begin{align*}
(i-j)(N) &= \begin{cases} i-j, & \text{if } i-j > 0 \\
i-j+N, & \text{if } i-j < 0.
\end{cases}
\end{align*}

Using Eq. 2.7 we can write $G^\ell,n_\nu(s)$ as
\[ G^\ell,n_\nu(s) = \prod_{k=1}^\infty \left(1 - s^{kN}\right)^{-1} \prod_{(i,j) \in \nu} \left(1 - s^{h_\nu(i,j)+(k-1)N}\right)^{-1} \prod_{(i,j) \notin \nu} \left(1 - s^{h_\nu(i,j)+kN}\right)^{-1}, \]
where $h_\nu(i,j)$ is the hook length given by
\[ h_\nu(i,j) = \nu_i + \nu_j^t - i - j + 1. \]
3 Elliptic Calabi-Yau Threefold and Branes

The $\mathcal{N} = 2$ abelian supersymmetric theory with one adjoint hypermultiplet can be geometrically engineered in type IIA string theory using a non-compact elliptic Calabi-Yau threefold $X_H$. This elliptic Calabi-Yau threefold is an elliptic fibration and can also be obtained by partial compactification of the resolved conifold, $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \mapsto \mathbb{P}^1$ [6]. In the toric/web picture this partial compactification is achieved by identification of the two external legs as shown in Fig. 2. In this section we will show that the topological string partition function of $X_H$ with a stack of branes, in the fiber over the elliptic curve, is equal to the cylindric partition function.

![Figure 2: The web diagram of $X_H$ (a) and web diagram of $X_H$ with branes (b).](image)

It was shown in [7] that the partition function of the abelian theory with the adjoint hypermultiplet can be computed using the topological vertex [8, 9]. Recently, the corresponding more refined partition function was computed using the refined topological vertex [2] and shown to be exactly the same as the refined generating function of cylindric partitions of a special profile. The parameters defining this special profile are identified with the Kähler parameters of $X_H$ [1]. In the geometry considered in [1] there were no branes present. In what follows we are going to argue that the correspondence between the cylindric partitions and the geometry $X_H$ also holds in presence of the branes.

The refined partition function for this geometry can be easily computed using the refined topological vertex. The refined topological vertex has a preferred direction [2] and we need to make a choice for it. For simplicity, we glue the vertices along the un-preferred direction and put the

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3The refined partition function of the $\mathcal{N} = 2$ gauge theory is a function of the equivariant parameters $\epsilon_1$ and $\epsilon_2$ corresponding to the $U(1) \times U(1)$ action on $\mathbb{C}^2$ $(z_1, z_2) \mapsto (e^{\epsilon_1} z_1, e^{\epsilon_2} z_2)$. The usual partition function corresponds to $\epsilon_1 + \epsilon_2 = 0$. 

5
branes with representation $\nu$ along the preferred one as shown in Fig. 2(b).  \[ \]

\[
Z_\nu(Q_1, Q_2, t, q) = \sum_{\lambda, \mu} (-Q_1)^{\lambda} (-Q_2)^{\mu} C_{\lambda \mu \nu} (t^{-1}, q^{-1}) C_{\lambda \mu \nu} (q^{-1}, t^{-1}) \\
= q^{-\|\nu\|^2} Z_\nu (t^{-1}, q^{-1}) \sum_{\lambda, \mu, n_1, n_2} Q_1^{\lambda} Q_2^{\mu} \left( \frac{t}{q} \right)^{\frac{1}{2} |n_1| - |n_2|} s_{\lambda/n_1} (t^0 q^\nu) s_{\mu/n_2} (t^{\nu'} q^0) \\
\times s_{\lambda'/n_1} (q^{-\rho}) s_{\mu'/n_2} (t^{-\rho}) \\
= M(t, q)^{-1} q^{-\|\nu\|^2} Z_\nu (t^{-1}, q^{-1}) \\
\times \sum_{\lambda, \mu} Q_1^{\lambda} s_{\lambda/\mu} \left( q^{\nu+\frac{\rho}{2}}, Q_2^{-1} t^{-\rho} \right) s_{\lambda/\mu} \left( q^\rho t^{\nu'}, \sqrt{\frac{t}{q}} Q_2^{-1} q^{-\rho} \right), \quad (3.1)
\]

where $M(t, q) = \prod_{i,j=1}^\infty (1 - t^{-i+1} q^{-j})^{-1}$ is a refinement of the MacMahon function and $Q_\nu = Q_1 Q_2$. $Z_\nu (t, q)$ is related to the Macdonald function $P_\nu(x; q, t)$,

\[
P_\nu(t^{-\rho}; q, t) = t^{\|\nu\|^2} \tilde{Z}_\nu (t, q) \\
= t^{\|\nu\|^2} \prod_{(i,j) \in \nu} \left( 1 - t^{\ell(i,j)} q^{\ell(i,j)} \right)^{-1}, \quad a(i, j) = \nu_j - i, \quad \ell(i, j) = \nu_i - j. \quad (3.2)
\]

Using the following identity

\[
\sum_{\lambda, \mu} \rho^{\lambda} s_{\lambda/\mu} (x) s_{\lambda/\mu} (y) = \prod_{k=1}^\infty \left( 1 - \rho^k \right)^{-1} \prod_{i,j=1}^\infty \left( 1 - \rho^k x_i y_j \right)^{-1}, \quad (3.3)
\]

we get

\[
Z_\nu (Q_1, Q_2, t, q) = q^{-\|\nu\|^2} Z_\nu (t^{-1}, q^{-1}) \\
\times \prod_{k=1}^\infty \left[ (1 - Q_k)^{-1} \prod_{i,j=1} \frac{(1 - Q_k Q_1^{-1} t^{\nu'_j - i + \frac{1}{2}} q^{-j + \frac{1}{2}})(1 - Q_k Q_2^{-1} q^{\nu'_j - i + \frac{1}{2}} t^{-i + \frac{1}{2}})}{(1 - Q_k Q_2^{-1} q^{-\nu'_j + \frac{1}{2}} t^{i - \frac{1}{2}})} \right]. \quad (3.4)
\]

The closed string partition function is obtained by taking $\nu = \emptyset$,

\[
Z_\emptyset (Q_1, Q_2, t, q) = \prod_{k=1}^\infty \left[ (1 - Q_k)^{-1} \prod_{i,j=1} \frac{(1 - Q_k Q_1^{-1} t^{\nu'_j - i + \frac{1}{2}} q^{-j + \frac{1}{2}})(1 - Q_k Q_2^{-1} q^{\nu'_j - i + \frac{1}{2}} t^{-i + \frac{1}{2}})}{(1 - Q_k q^{-\nu'_j + \frac{1}{2}} t^{i - \frac{1}{2}})} \right]. \quad (3.5)
\]

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4 $C_{\lambda \mu \nu} (t, q) = \left( \frac{\rho}{t} \right)^{\|\nu\|^2 + \frac{1}{2} |\nu|} t^{\frac{1}{2} |\nu|} P_{\nu} (t^{-\rho}; q, t) \sum_{\lambda, \mu} \rho^{\lambda} s_{\lambda/\mu} (t^{-\rho} q^{-\nu}) s_{\mu/\eta} (t^{-\nu'} q^\rho) \\
\rho = \left\{ -\frac{1}{2}, -\frac{3}{2}, \ldots \right\} \\
\|\nu\|^2 = \sum_i \nu_i^2, \quad \kappa(\nu) = \|\nu\|^2 - \|\nu'\|^2.$

5 After analytic continuation to $t^{-1}$ and $q^{-1}$. 

6
The above partition function can also be obtained directly using the refined Gopakumar-Vafa (GV) invariants of the curves in this geometry [10]. Recall that the topological string partition function in terms of refined GV invariants is given by [7]

\[
Z = \prod_{\beta \in H_2(X,\mathbb{Z})} Z_\beta \quad (3.6)
\]

\[
Z_\beta := \prod_{j_L,j_R \ k_L = -j_L \ k_R = -j_R} \prod_{m_1,m_2=1}^{+j_L} \prod_{-j_L}^{+j_R} \prod_{-j_R}^{+j_R} \left(1 - q_1^{k_L + k_R + m_1 - \frac{j}{2}} q_2^{k_L - k_R + m_2 + \frac{j}{2}} Q_\beta \right)^{(-1)^{2(j_L + j_R)} N_{j_L,j_R}^{j_L,j_R}},
\]

where \(Q_\beta = e^{-\int_\beta \omega}\) is given by the complexified Kähler form \(\omega\) and \(N_{j_L,j_R}^{j_L,j_R}\) are the refined GV invariants which are the number of cohomology classes, with spin \((j_L, j_R)\), of the moduli space of D2-branes wrapped on the holomorphic curve in the class \(\beta\). The moduli space of the D2-brane is not just the moduli space of the holomorphic curve on which it is wrapped because the D2-brane has a \(U(1)\) gauge field living on its worldvolume and therefore the moduli space of D2-brane includes the moduli of the flat connections on the curve coming from the gauge field as well as the moduli of the curve. The moduli space of the D2-brane is therefore a \(T^{2g}\) fibration over the moduli space of the curve since the moduli of the flat connection over a smooth genus \(g\) curve is \(T^{2g}\). Since the moduli space of the holomorphic curve in a Calabi-Yau threefold is a Kähler manifold, the total moduli space of D2-brane is also Kähler manifold such that the Lefshetz action by the Kähler class is the diagonal of the \(SU(2)_L \times SU(2)_R\) action on the moduli space. The \(SU(2)_L\) acts on the fiber and the \(SU(2)_R\) acts on the moduli space of curve, the base.

In the example we are considering \(H_2(X_H)\) is two dimensional generated by the two genus zero curves \(C_1\) and \(C_2\). The curve \(E := C_1 + C_2\) has genus one. The holomorphic curve in this geometry are given by (Appendix A)

\[
(k + 1) E, \ k E + C_1, \ k E + C_2, \ k \geq 0.
\]

(3.7)

The curves \(k E + C_1\) and \(k E + C_2\) are genus zero and are rigid therefore the moduli space of D2-brane wrapped on these curves is just a point,

\[
N_{kE+C_1}^{j_L,j_R} = N_{kE+C_2}^{j_L,j_R} = \delta_{jL,0} \delta_{jR,0}. \quad (3.8)
\]

The curve \((k + 1)E\) is of genus one but is also rigid therefore the only moduli of the D2-brane wrapped on these curves are the ones coming from the gauge field \(i.e.,\) the moduli space of the D-brane wrapped on \((k + 1)E\) is \(T^2\),

\[
N_{(k+1)E}^{j_L,j_R} = \delta_{jR,0} \delta_{jL,\frac{1}{2}}. \quad (3.9)
\]
Using these invariants in Eq. 3.6 we get \((a = 1, 2)\)

\[
Z_{kE + C_a} = \prod_{i,j=1}^{\infty} \left(1 - q_1^{i-\frac{1}{2}} q_2^{j-\frac{1}{2}} Q^k Q_a\right)
\]

\[
Z_{(k+1)E} = \prod_{i,j=1}^{\infty} \left[\left(1 - q_1^i q_2^j Q^{k+1}\right) \left(1 - q_1^{i-1} q_2^{j-1} Q^{k+1}\right)\right]^{-1}.
\]

The full partition function given by

\[
Z = \prod_{k=0}^{\infty} \left(Z_{kE + C_1} Z_{kE + C_2} Z_{(k+1)E}\right),
\]

is exactly the same as Eq. 3.5 for \((q_1, q_2) = (t^{-1}, q^{-1})\).

### 3.1 Cylindric Partitions and Quantized Kähler Parameters

In this section we will see that the open topological string partition function given by Eq. 3.1 acquires a combinatorial interpretation as the generating function of the cylindric partitions once we quantize the Kähler parameters \(Q_1\) and \(Q_2\). The quantization of the Kähler parameters appears naturally in the geometric transition which relates the Chern-Simons theory on the conifold with the topological string theory on the \(O(-1) \oplus O(-1) \mapsto \mathbb{P}^1\) [11].

To see the effect of quantized Kähler parameters consider Eq. (3.1),

\[
Z_\nu(Q_1, Q_2, t, q) = M(t, q)^{-1} q^{-\frac{\|\nu\|^2}{2}} \tilde{Z}_\nu(t^{-1}, q^{-1})
\]

\[
\times \sum_{\lambda, \mu} Q_1^{\lambda} Q_2^{\mu} \left(t^{\rho \frac{1}{2}} q^{\nu \frac{1}{2}}, Q_1^{-1} t^{-\rho}\right) s_{\lambda/\mu}(q^{\rho} t^{\rho}, \sqrt{\frac{q}{t} Q_2^{-1} q^{-\rho}}).
\]

The arguments of the above skew-Schur functions are two infinite set of variables which reduce to finite sets when we specialize to certain values of \(Q_1\) and \(Q_2\) (Appendix B),

\[
\{t^{i-\frac{1}{2}} q^{i+\frac{1}{2}}, Q_1^{-1} t^{-\rho}\} \xrightarrow{Q_1=\sqrt{\frac{t}{q}} \ell} \{t^{-i} q^{i\frac{1}{2}}, i = 1, \ldots, \ell\}
\]

\[
\{q^{\rho} t^{\rho}, \sqrt{\frac{q}{t} Q_2^{-1} q^{-\rho}}\} \xrightarrow{Q_2=\sqrt{\frac{q}{t}} n} \{q^{-j\frac{1}{2}} t^{\nu \frac{1}{2}}, j = 1, \ldots, n\},
\]

with \(\ell(\nu) \leq \ell, \ell(\nu') \leq n\) and \(Q_\bullet = Q_1 Q_2 = t^\ell q^n \left(\frac{t}{q}\right)^\ell\). After the above identification of the
Kähler parameters Eq(3.11) takes the following form:

\[
Z_{\nu}(Q_1, Q_2, t, q) = M(t, q)^{-1} q^{-\frac{1}{2} \| \nu \|^2} \tilde{Z}_{\nu}(t^{-1}, q^{-1}) \times \sum_{\lambda, \mu} Q_{\lambda} | s_{\lambda/\mu} \left( t^{-1} q^{\nu_1 + \frac{1}{2}}, \ldots, t^{-\ell} q^{\nu_{\ell} + \frac{1}{2}} \right) s_{\lambda/\mu} \left( q^{-\frac{1}{2}} t^{\nu_1}, \ldots, q^{-n+\frac{1}{2}} t^{\nu_n} \right) \]

\[
= M(t, q)^{-1} q^{-\frac{1}{2} \| \nu \|^2} \tilde{Z}_{\nu}(t^{-1}, q^{-1}) \prod_{k=1}^{\infty} \left( (1 - Q_k)^{-1} \prod_{i,j=1}^{\ell,n} (1 - Q_{i,j} t^{-i+j+\nu_i+1})^{-1} \right),
\]

According to the crystal picture of the topological vertex, the open topological string partition function of \( \mathbb{C}^3 \) with three stack of branes on the three legs (\( U(1) \) invariant locus) is the same as the generating function of the 3D partition (with boundaries specified by the stack of branes) after factoring out the MacMahon function [1, 12–14]. This factor is associated with the vacuum contribution [12]. In other words, we need to multiply the partition function first with the MacMahon function before comparing it with the generating function of cylindric partitions. To this end, let us define

\[
Z_{\ell,n}^{\nu}(t, q) = M(t, q) Z_{\nu}(Q_1, Q_2, t, q) \bigg|_{Q_1=\sqrt{q} t^\ell, Q_2=\sqrt{q} q^n} = q^{-\frac{1}{2} \| \nu \|^2} \tilde{Z}_{\nu}(t^{-1}, q^{-1}) \prod_{k=1}^{\infty} \left( (1 - Q_k)^{-1} \prod_{i,j=1}^{\ell,n} (1 - Q_{i,j} t^{-i+j+\nu_i+1})^{-1} \right).
\]

In the previous section, we have computed the generating function for the cylindric partitions and a quick comparison shows that

\[
Z_{\ell,n}^{\nu}(s^{-1}, s^{-1}) = s^{-\frac{1}{2} \| \nu \|^2} G_{\ell,n}^{\nu}(s)
\]

The more general partition function \( Z_{\ell,n}^{\nu}(t, q) \) can also be interpreted in terms of cylindric plane partitions. Given a cylindric plane partition \( \pi \) let \( \eta(a) \) be the 2D partitions obtained by diagonal slicing of \( \pi \), \( \| \pi \| = \sum_{a=1}^{N} | \eta(a) | \),

\[
Z_{\ell,n}^{\nu}(t, q) = q^{-\frac{1}{2} \| \nu \|^2} \sum_{\substack{\text{cylindric partition } \pi \\ \text{of profile } \nu}} (t^{-1})^{\| \pi \|} (q^{-1})^{\| \eta(1) \|} (q^{-1})^{\| \eta(2) \|} \ldots (q^{-1})^{\| \eta(N) \|} \]

Thus \( Z_{\ell,n}^{\nu}(t, q) \) is obtained if we count each slice of \( \pi \) with parameter \( t \) or \( q \) depending on the shape of \( \nu \).
**Level-Rank Duality:** The generating function $G^{\ell,n}_{\mu}(s)$ has a symmetry:

$$G^{\ell,n}_{\mu}(s) = G^{n,\ell}_{\mu}(s). \quad (3.16)$$

This symmetry corresponds to the reflection of $n \times \ell$ rectangle across the diagonal. In [4] it was shown that this symmetry is a manifestation of level-rank duality and the function $G^{\ell,n}_{\mu}(s)$ is the character of a level $\ell$ irreducible representation of $\hat{gl}_n$. In the topological string partition function this level-rank duality corresponds to the symmetry between the Kähler parameter $t_1$ and $t_2$, which in turn is due to the exchange symmetry between the two $\mathbb{P}^1$'s in the geometry.

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Appendix A: $X_H$ and Holomorphic Curves

In this section we will argue that the only holomorphic curves in the geometry $X_H$ are the ones given by Eq(3.7). The geometry $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1$ is toric and has the Newton polygon shown in Fig. 3(a). The geometry $X_H$ has a non-planar Newton polygon given by identifying the two vertical edges of the Newton polygon given in Fig. 3(a). Thus for this geometry the Newton polygon lives on a cylinder. On the covering space of this cylinder we can represent the Newton polygon of $X_H$ by Fig. 3(b).

![Figure 3: The toric diagram of the resolved conifold and the infinite cover of the toric diagram of $X_H$.](image)

The web diagram corresponding to Fig. 3(b) is shown in Fig. 4(b).

![Figure 4: The web diagram of $X_H$ (a) and its cover (b).](image)

The compact part of this geometry is an infinite chain of $\mathbb{P}^1$’s. Since this corresponds to the cover of $X_H$ there are only two distinct curves $C_1$ and $C_2$. The covering geometry is actually the maximal blowup of the resolution of $\mathbb{C}^2/\mathbb{Z}_N \times \mathbb{C}$ in the limit $N \rightarrow \infty$. The holomorphic curves in this geometry satisfy $C \cdot C = 2g - 2$ for $g \geq 0$, where $g$ is the genus of the curve $C$. The only curves in this geometry are rational curves given by connected chains of $\mathbb{P}^1$’s. If $A$ and $B$ are two rational curve with $A \cdot B = 0$ then $(A + B) \cdot (A + B) = -4$ hence $A + B$ is not holomorphic unless
they form a connected chain. It is easy to see that connected chains belong to one of the following curve classes:

\[(k + 1)(C_1 + C_2), \quad \text{If the first and last cycle in the chain are not the same,} \quad (3.17)\]
\[k(C_1 + C_2) + C_1, \quad \text{If the first and last cycle in the chain are both } C_1, \]
\[k(C_1 + C_2) + C_2, \quad \text{If the first and last cycle in the chain are both } C_2. \]

**Appendix B: Refined Topological Vertex Computation**

In this section, we want to show the details of the refined vertex computation and argue that the reduction of the arguments of the skew-Schur function happens in Eq. 3.12. According to the regular gluing rules of the refined topological vertex, the partition function takes the following form:

\[
Z_\nu(Q_1, Q_2, t, q) = \sum_{\lambda, \mu} (-Q_1)^{\lambda|}(-Q_2)^{\mu|}C_{\lambda \mu \nu}(t^{-1}, q^{-1})C_{\lambda t \mu \emptyset}(q^{-1}, t^{-1}) \\
= q^{-\frac{||\mu||^2}{2}} Z_\nu(t^{-1}, q^{-1}) \sum_{\lambda, \mu, \eta_1, \eta_2} (-Q_1)^{\lambda|}(-Q_2)^{\mu|} \left( \frac{t}{q} \right)^{\frac{|\eta_1| - |\eta_2|}{2}} s_{\lambda \mu / \eta_1}(t^\rho q^\nu) s_{\mu / \eta_1}(t^\mu t^\rho) \\
\times s_{\lambda / \eta_2}(q^\rho) s_{\mu / \eta_2}(t^\rho) \\
= q^{-\frac{||\mu||^2}{2}} Z_\nu(t^{-1}, q^{-1}) \sum_{\lambda, \mu, \eta_1, \eta_2} Q_1^{\lambda|}Q_2^{\mu|} \left( \frac{t}{q} \right)^{\frac{|\eta_1| - |\eta_2|}{2}} s_{\lambda \mu / \eta_1}(t^\rho q^\nu) s_{\mu / \eta_1}(t^\mu t^\rho) \\
\times s_{\lambda t / \eta_2}(q^{-\rho}) s_{\mu / \eta_2}(t^{-\rho}),
\]

where we made use of \( s_{\lambda / \mu}(q^\rho) = (-1)^{|\lambda| - |\mu|} s_{\lambda t / \mu}(q^{-\rho}) \) in the second equality above. We first perform the sum over \( \eta_2 \):

\[
\sum_{\eta_2} \left( \frac{t}{q} \right)^{\frac{|\eta_1| - |\eta_2|}{2}} s_{\lambda t / \eta_2}(q^{-\rho}) s_{\mu / \eta_2}(t^{-\rho}) = \left( \frac{t}{q} \right)^{-\frac{||\mu||}{2}} \prod_{i,j=1}^{\infty} (1 - t^i q^j)^{-1} \\
\times \sum_{\tau} s_{\tau / \lambda t}(t^{-\rho}) s_{\tau / \mu}(q^{-\rho - 1/2} t^{1/2}).
\]

At this point we can sum over the partition \( \mu \) to combine two skew-Schur functions into one:

\[
\sum_{\mu} Q_2^{\mu |} s_{\tau / \mu}(q^{-\rho - 1/2} t^{1/2}) s_{\mu / \eta_1}(t^\mu t^\rho) = Q_2^{\eta_1 |} s_{\tau / \eta_1}(q^{-\rho - 1/2} t^{1/2}, Q_2 t^\mu t^\rho).
\]

A similar sum should be performed over \( \lambda \) to get the other skew-Schur function in the final result Eq. 3.1 Let us now show how the infinite number of arguments of the Schur function reduce to...
a finite number of arguments when we specialize the Kähler parameters. First note that the Schur functions can be written in terms of power sums:

\[ s_\lambda(x) = \sum_{\mu} m_{\lambda\mu} p_\mu(x), \quad (3.21) \]

where \( p_\mu(x) = p_{\mu_1}(x) p_{\mu_2}(x) \ldots \) with \( p_k(x) \)'s are the power sums,

\[ p_k(x) = \sum_{i=1}^{\infty} x_i^k. \quad (3.22) \]

Hence, we can conclude that our assertion is correct for the Schur functions once we show it for the power sums. The generalization to skew Schur functions is straightforward. For the power sum symmetric functions

\[ p_k\left(t^{\rho - \frac{1}{2}} q^{\nu + \frac{1}{2}}, Q_1^{-1} t^{-\rho}\right) = \sum_{i=1}^{\infty} t^{-ki} q^{(\nu_i + \frac{1}{2})} + \sum_{i=1}^{\infty} t^{k(i - \frac{1}{2})}, \quad Q_1 = \sqrt{\frac{t}{q}} \quad (3.23) \]

\[ = \sum_{i=1}^{\ell} t^{-ki} q^{(\nu_i + \frac{1}{2})} = p_k(t^{-1} q^{\nu_1 + \frac{1}{2}}, t^{-2} q^{\nu_2 + \frac{1}{2}}, \ldots, t^{-\ell} q^{\nu_\ell + \frac{1}{2}}) \]

\[ p_k(q^\rho t^{\nu}, Q_2^{-1} q^{-\rho - \frac{1}{2} t^{\frac{1}{2}}}) = \sum_{j=1}^{\infty} q^{k(-j + \frac{1}{2})} t^{k\nu_j} + \sqrt{\frac{t}{q}} Q_2^{-k} \sum_{j=1}^{\infty} q^{k(i - \frac{1}{2})}, \quad Q_2 = \sqrt{\frac{t}{q^n}} \]

\[ = \sum_{j=1}^{n} q^{k(-j + \frac{1}{2})} t^{k\nu_j} = p_k(q^{-\frac{1}{2}} t^{\nu_1}, q^{-\frac{3}{2}} t^{\nu_2}, \ldots, q^{-n + \frac{1}{2}} t^{\nu_n}), \]

where we have used analytic continuation to write the infinite sum over \( t^i \) and \( q^j \) in terms of \( t^{-1} \) and \( q^{-1} \).

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