LOCAL PROBABILITIES FOR RANDOM WALKS CONDITIONED TO STAY POSITIVE

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Abstract. Let $S_0 = 0, \{S_n, n \geq 1\}$ be a random walk generated by a sequence of i.i.d. random variables $X_1, X_2, \ldots$ and let $\tau^- = \min\{n \geq 1 : S_n \leq 0\}$ and $\tau^+ = \min\{n \geq 1 : S_n > 0\}$. Assuming that the distribution of $X_1$ belongs to the domain of attraction of an $\alpha$-stable law we study the asymptotic behavior, as $n \to \infty$, of the local probabilities $P(\tau^= = n)$ and the conditional local probabilities $P(S_n \in [x, x+\Delta] | \tau^- > n)$ for fixed $\Delta$ and $x = x(n) \in (0, \infty)$.

1. Introduction and main result

Let $S_0 := 0, S_n := X_1 + \ldots + X_n, n \geq 1$, be a random walk where the $X_i$ are independent copies of a random variable $X$ and

$$\tau^- = \min\{n \geq 1 : S_n \leq 0\} \quad \text{and} \quad \tau^+ = \min\{n \geq 1 : S_n > 0\}$$

be the first weak descending and first strict ascending ladder epochs of $\{S_n, n \geq 0\}$. The aim of this paper is to study, as $n \to \infty$, the asymptotic behavior of the local probabilities $P(\tau^\pm = n)$ and conditional local probabilities $P(S_n \in [x, x+\Delta] | \tau^- > n)$ for fixed $\Delta > 0$ and $x = x(n) \in (0, \infty)$.

To formulate our results we let

$$A := \{0 < \alpha < 1; |\beta| < 1\} \cup \{1 < \alpha < 2; |\beta| \leq 1\} \cup \{\alpha = 1, \beta = 0\} \cup \{\alpha = 2, \beta = 0\}$$

be a subset in $\mathbb{R}^2$. For $(\alpha, \beta) \in A$ and a random variable $X$ write $X \in D(\alpha, \beta)$ if the distribution of $X$ belongs to the domain of attraction of a stable law with characteristic function

$$G_{\alpha,\beta}(t) := \exp\left\{-c|t|^\alpha \left(1 - i\beta \frac{t}{|t|} \tan \frac{\pi \alpha}{2}\right)\right\} = \int_{-\infty}^{+\infty} e^{itx} g_{\alpha,\beta}(u) du, \quad c > 0, \quad (1)$$

and, in addition, $\mathbb{E}X = 0$ if this moment exists.

Denote $\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}, \mathbb{Z}_+ := \{1, 2, \ldots\}$ and let $\{c_n, n \geq 1\}$ be a sequence of positive integers specified by the relation

$$c_n := \inf\left\{u \geq 0 : \mu(u) \leq n^{-1}\right\}, \quad (2)$$

where

$$\mu(u) := \frac{1}{u^2} \int_{-u}^{u} x^2 P(X \in dx).$$

It is known (see, for instance, [14] Ch. XVII, §5]) that for every $X \in D(\alpha, \beta)$ the function $\mu(u)$ is regularly varying with index $(-\alpha)$. This implies that $\{c_n, n \geq 1\}$...
is a regularly varying sequence with index $\alpha^{-1}$, i.e., there exists a function $l_1(n)$, slowly varying at infinity, such that

$$c_n = n^{1/\alpha}l_1(n).$$  \hspace{1cm} (3)

In addition, the scaled sequence $\{S_n/c_n, n \geq 1\}$ converges in distribution, as $n \to \infty$, to the stable law given by (1).

The following conditional limit theorem will be crucial for the rest of this article.

**Theorem 1.** If $X \in \mathcal{D}(\alpha, \beta)$, then there exists a nonnegative random variable $M_{\alpha,\beta}$ with density $p_{\alpha,\beta}(u)$ such that, for all $u_2 > u_1 \geq 0$,

$$\lim_{n \to \infty} P\left(\frac{S_n}{c_n} \in [u_1, u_2] \mid \tau^- > n\right) = P(M_{\alpha,\beta} \in [u_1, u_2]) = \int_{u_1}^{u_2} p_{\alpha,\beta}(v)dv. \hspace{1cm} (4)$$

The validity of the first equality in (4) was established by Durrett [12] and we believe that the absolutely continuity of the distribution of $M_{\alpha,\beta}$ is also known in the literature, but failed to find any reference. However, this fact will be a by-product of our arguments and we include it in (4) to simplify the statements of the main theorems of the present paper.

Our first result is an analog of the classical Stone local limit theorem.

**Theorem 2.** Suppose $X \in \mathcal{D}(\alpha, \beta)$ and the distribution of $X$ is non-lattice. Then, for every $\Delta > 0$,

$$c_nP(S_n \in [x, x + \Delta] \mid \tau^- > n) - \Delta p_{\alpha,\beta}(x/c_n) \to 0 \text{ as } n \to \infty$$  \hspace{1cm} (5)

uniformly in $x \in (0, \infty)$.

For the case when the distribution of $X$ belongs to the domain of attraction of the normal law, that is, when $X \in \mathcal{D}(2,0)$ relation (5) has been proved by Caravenna [5].

If the ratio $x/c_n$ varies with $n$ in such a way that $x/c_n \in (b_1, b_2)$ for some $0 < b_1 < b_2 < \infty$, we can rewrite (5) as

$$c_nP(S_n \in [x, x + \Delta] \mid \tau^- > n) \sim \Delta p_{\alpha,\beta}(x/c_n) \text{ as } n \to \infty.$$  \hspace{1cm} (6)

However, if $x/c_n \to 0$, then, in view of

$$\lim_{z \to 0} p_{\alpha,\beta}(z) = 0$$

(see (80) below), relation (6) gives only

$$c_nP(S_n \in [x, x + \Delta] \mid \tau^- > n) = o(1) \text{ as } n \to \infty.$$  \hspace{1cm} (7)

Our next theorem refines (6) in the mentioned domain of small deviations, i.e., when $x/c_n \to 0$. To formulate the desired statement we need some additional notation.

Set $\chi^+ := S_{\tau^+}$ and introduce the renewal function

$$H(u) := I\{u > 0\} + \sum_{k=1}^{\infty} P(\chi_1^+ + \ldots + \chi_k^+ < u).$$  \hspace{1cm} (7)

Clearly, $H$ is a left-continuous function.

**Theorem 3.** Suppose $X \in \mathcal{D}(\alpha, \beta)$ and the distribution of $X$ is non-lattice. Then

$$c_nP(S_n \in [x, x + \Delta] \mid \tau^- > n) \sim g_{\alpha,\beta}(0) \frac{\int_{x}^{x+\Delta} H(u)du}{nP(\tau^- > n)} \text{ as } n \to \infty$$  \hspace{1cm} (8)
uniformly in $x \in (0, \delta_n c_n]$, where $\delta_n \to 0$ as $n \to \infty$.

We continue by considering the lattice case and say that a random variable $X$ is $(h, a)$–lattice if the distribution of $X$ is lattice with span $h > 0$ and shift $a \in [0, h)$, i.e., the $h$ is the maximal number such that the support of the distribution of $X$ is contained in the set $\{a + kh, k = 0, \pm 1, \pm 2, \ldots\}$.

**Theorem 4.** Suppose $X \in \mathcal{D}(\alpha, \beta)$ and is $(h, a)$–lattice. Then
\[
c_n \mathbb{P}(S_n = an + x|\tau^- > n) \to h \rho_{\alpha, \beta}(an + x/c_n) / n \mathbb{P}(\tau^- > n) \quad \text{as} \quad n \to \infty
\]
uniformly in $x \in (-an, \infty) \cap h\mathbb{Z}$.

For $X \in \mathcal{D}(2, 0)$ and being $(h, 0)$–lattice relation (9) has been obtained by Bryn-Jones and Doney [4].

**Theorem 5.** Suppose $X \in \mathcal{D}(\alpha, \beta)$ and is $(h, a)$–lattice. Then
\[
c_n \mathbb{P}(S_n = an + x|\tau^- > n) \sim h \rho_{\alpha, \beta}(an + x) / n \mathbb{P}(\tau^- > n) \quad \text{as} \quad n \to \infty
\]
uniformly in $x \in (-an, -an + \delta_n c_n] \cap h\mathbb{Z}$, where $\delta_n \to 0$ as $n \to \infty$.

Note that Alili and Doney [1] established (10) under the assumptions $X$ is $(h, 0)$–lattice and $EX^2 < \infty$. Bryn-Jones and Doney [4] generalized their results to the $(h, 0)$–lattice $X \in \mathcal{D}(2, 0)$.

The next theorem describes the asymptotic behavior of the density function $p_{\alpha, \beta}$ at zero. The explicit form of $p_{\alpha, \beta}$ is known only for $\alpha = 2, \beta = 0$: $p_{2,0}(x) = xe^{-x^2/2}I(x > 0)$. For this reason we deduce an integral equation for $p_{\alpha, \beta}$ (see (79) below) and using Theorems 2–5 find the asymptotic behavior of $p_{\alpha, \beta}(z)$ at zero.

**Theorem 6.** For every $(\alpha, \beta) \in \mathcal{A}$ there exists a constant $C > 0$ such that
\[
p_{\alpha, \beta}(z) \sim Cz^{\alpha}\rho \quad \text{as} \quad z \downarrow 0,
\]
where $\rho := \int_{0,1} g_{\alpha, \beta}(u)du$.

One of our main motivations to be interested in the local probabilities of conditioned random walks is the question on the asymptotic behavior of the local probabilities of the ladder epochs $\tau^-$ and $\tau^+$. Before formulating the relevant results we recall some known facts concerning the properties of these random variables given
\[
\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}(S_n > 0) = \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}(S_n \leq 0) = \infty.
\]
The last means that $\{S_n, n \geq 0\}$ is an oscillating random walk, and, in particular, the stopping moments $\tau^-$ and $\tau^+$ are well-defined proper random variables. Moreover, it follows from the Wiener-Hopf factorization (see, for example, [3] Theorem 8.9.1, p. 376)) that for all $z \in [0, 1]$,
\[
1 - \mathbb{E}z^{\tau^-} = \exp \left\{ - \sum_{n=1}^{\infty} \frac{z^n}{n} \mathbb{P}(S_n \leq 0) \right\}
\]
and
\[
1 - \mathbb{E}z^{\tau^+} = \exp \left\{ - \sum_{n=1}^{\infty} \frac{z^n}{n} \mathbb{P}(S_n > 0) \right\}.
\]
Rogozin [18] investigated properties of $\tau^+$ and demonstrated that the Spitzer condition

$$n^{-1} \sum_{k=1}^{n} P(S_k > 0) \to \rho \in (0, 1) \quad \text{as } n \to \infty$$  \hspace{1cm} (13)

holds if and only if $\tau^+$ belongs to the domain of attraction of a positive stable law with parameter $\rho$. In particular, if $X \in D(\alpha, \beta)$ then (see, for instance, [22]) condition (13) holds with

$$\rho = \int_{0^+}^{\infty} g_{\alpha, \beta}(u) du = \begin{cases} \frac{1}{2}, & \alpha = 1, \\ \frac{1}{2} + \frac{\pi}{\alpha} \arctan \left( \beta \tan \frac{\pi \alpha}{2} \right), & \text{otherwise}. \end{cases}$$  \hspace{1cm} (14)

Since (11) and (12) imply

$$(1 - Ez^{\tau^+})(1 - Ez^{-\tau}) = 1 - z \quad \text{for all } z \in (0, 1),$$

one can deduce by Rogozin’s result that (13) holds if and only if there exists a function $l(n)$ slowly varying at infinity such that, as $n \to \infty$,

$$P(\tau^- > n) \sim \frac{l(n)}{n^{1-\rho}}, \quad P(\tau^+ > n) \sim \frac{1}{\Gamma(\rho)\Gamma(1-\rho)n^\rho l(n)}.$$  \hspace{1cm} (15)

We also would like to mention that, according to Doney [10], the Spitzer condition is equivalent to

$$P(S_n > 0) \to \rho \in (0, 1) \quad \text{as } n \to \infty.$$  \hspace{1cm} (16)

Therefore, both relations in (15) are valid under condition (16).

The asymptotic representations (15) include a slowly varying function $l(x)$ which is of interest as well. Unfortunately, to get a more detailed information about the asymptotic properties of $l(x)$ it is necessary to impose additional hypotheses on the distribution of $X$. Thus, Rogozin [18] has shown that $l(x)$ is asymptotically a constant if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n} \left( P(S_n > 0) - \rho \right) < \infty.$$  \hspace{1cm} (17)

It follows from the Spitzer-Rózsa theorem (see [3, Theorem 8.9.23, p. 382]) that if $EX^2 < \infty$, then (17) holds with $\rho = 1/2$, and, consequently,

$$P(\tau^\pm > n) \sim \frac{C^\pm}{n^{1/2}} \quad \text{as } n \to \infty,$$  \hspace{1cm} (18)

where $C^\pm$ are positive constants. Much less is known about the form of $l(x)$ if $EX^2 = \infty$. For instance, if the distribution of $X$ is symmetric, then, clearly,

$$\left| P(S_n > 0) - \frac{1}{2} \right| \leq \frac{1}{2} P(S_n = 0).$$  \hspace{1cm} (19)

Furthermore, according to [17] Theorem III.9, p. 49], there exists $C > 0$ such that for all $n \geq 1$,

$$P(S_n = 0) \leq \frac{C}{\sqrt{n}}.$$  

By this estimate and (19) we conclude that (14) holds with $\rho = 1/2$ and, therefore, (13) is valid for all symmetric random walks.

One more situation was analyzed by Doney [7]. Assuming that $P(X > x) = (x^\alpha l_0(x))^{-1}$, $x > 0$, with $1 < \alpha < 2$ and $l_0(x)$ slowly varying at infinity, he
established some relationships between the asymptotic behavior of \(I_0(x)\) and \(I(x)\) at infinity for a number of cases.

Thus, up to now there is a group of results describing the behavior of the probabilities \(P(\tau^+ > n)\) as \(n \to \infty\) and the functions involved in their asymptotic representations. We complement the mentioned statements by the following two theorems describing the behavior of the local probabilities \(P(\tau^\pm = n)\) as \(n \to \infty\).

**Theorem 7.** If \(X \in \mathcal{D}(\alpha, \beta)\) then there exists a sequence \(\{Q_n^-, n \geq 1\}\) such that

\[
P(\tau^- = n) = Q_n^- \frac{l(n)}{n^{2-\rho}}(1 + o(1)) \quad \text{as } n \to \infty.
\]  

(20)

The sequence \(\{Q_n^-, n \geq 1\}\) is bounded from above, and there exists a positive constant \(Q^*_-\) such that \(Q_n^- l(Q_n^- > 0) \geq Q^*_-\) for all \(n \geq 1\). Moreover, we may choose \(Q_n^- \equiv 1 - \rho\) if and only if one of the following conditions holds:

(a) \(E(-S_{\tau^-}) = \infty\),
(b) \(E(-S_{\tau^-}) < \infty\) and the distribution of \(X\) is \((h, 0)\)-lattice,
(c) \(E(-S_{\tau^-}) < \infty\) and the distribution of \(X\) is non-lattice.

**Remark 8.** The statement of the theorem includes the quantity \(E(-S_{\tau^-})\), which depends on \(\tau^-\), a random variable being the objective of the theorem. This is done only to simplify the form of the theorem. In fact, Chow [6] has shown that \(E(-S_{\tau^-})\) is finite if and only if

\[
\int_0^{\infty} \frac{x^2}{\int_0^{\infty} y \min\{x, y\} P(X^+ \in dy)} P(X^- \in dx) < \infty,
\]

where \(X^+ := \max\{0, X\}\) and \(X^- := -\min\{0, X\}\).

**Remark 9.** The simple random walk in which \(P(X = \pm 1) = 1/2\) is the most natural example with \(Q_n^- \not\equiv 1 - \rho\). Here \(P(\tau^- = 2k + 1) = 0\) and, consequently, \(Q_n^+ = 0\) for all \(k \geq 1\). On the other hand, \(\lim_{k \to \infty} Q_{2k}^-\) exists and is strictly positive. This result is in complete agreement with Theorem [4] the step-distribution of the simple random walk is \((2, 1)\)-lattice.

For the stopping time \(\tau^+\) we have a similar statement:

**Theorem 10.** If \(X \in \mathcal{D}(\alpha, \beta)\) then there exists a sequence \(\{Q_n^+, n \geq 1\}\) such that

\[
P(\tau^+ = n) = Q_n^+ \frac{l(n)}{n^{1+\rho}}(1 + o(1)) \quad \text{as } n \to \infty.
\]  

(21)

The sequence \(\{Q_n^+, n \geq 1\}\) is bounded from above, and there exists a positive constant \(Q^*_+\) such that \(Q_n^+ l(Q_n^+ > 0) \geq Q^*_+\) for all \(n \geq 1\). Moreover, we may choose \(Q_n^+ \equiv \rho/\Gamma(\rho)\Gamma(1 - \rho)\) if and only if one of the following conditions holds:

(a) \(E_{\tau^+} = \infty\),
(b) \(E_{\tau^+} < \infty\) and the distribution of \(X\) is \((h, 0)\)-lattice,
(c) \(E_{\tau^+} < \infty\) and the distribution of \(X\) is non-lattice.

In some special cases the asymptotic behavior of \(P(\tau^\pm = n)\) is already known from the literature. Eppel [13] proved that if \(EX = 0\), \(EX^2\) is finite, and the distribution of \(X\) is non-lattice, then

\[
P(\tau^\pm = n) \sim \frac{C^\pm}{n^{3/2}} \quad \text{as } n \to \infty.
\]  

(22)
Clearly, $X \in \mathcal{D}(2, 0)$ in this case. For aperiodic random walks on integers with $EX = 0$ and $EX^2 < \infty$ representations (22) were obtained by Alili and Doney [1].

Asymptotic relation (22) is valid for all continuous symmetric (implying $\rho = 1/2$ in (16)) random walks (see [14, Chapter XII, Section 7]). Note that the restriction $X \in \mathcal{D}(\alpha, \beta)$ is superfluous in this situation.

Recently A.Borovkov [2] has shown that if (13) is valid and

$$n^{1-\rho} \left( P(S_n > 0) - \rho \right) \to const \in (-\infty, \infty) \quad as \ n \to \infty, \quad (23)$$

then (20) holds with $l(n) \equiv const \in (0, \infty)$. Proving the mentioned result Borovkov does not assume that the distribution of $X$ is taken from the domain of attraction of a stable law. However, he gives no explanations how one can check the validity of (23) in the general situation.

Further, Alili and Doney [1, Remark 1, p. 98] have demonstrated that if $X$ is $(h, 0)$-lattice and $E|X|^3 < \infty$ then (21) holds with $Q_n^+ \sim \rho/(\Gamma(\rho)\Gamma(1-\rho))$.

Finally, Mogulski and Rogozin [16] established (20) for $X$ satisfying the conditions $EX = 0$ and $E|X|^3 < \infty$. Moreover, they proved that $Q_n^+ \sim const$ if and only if the distribution of $X$ is either non-lattice or $(h, 0)$-lattice. Observe that $E(\!-\!S_{\tau^-}) \!<\! \infty$ under their conditions.

2. Auxiliary results

2.1. **Notation.** In what follows we denote by $C, C_1, C_2, \ldots$ finite positive constants which may be different from formula to formula and by $l(x), l_0(x), l_1(x), l_2(x), \ldots$ functions slowly varying at infinity which are, as a rule, fixed once and forever.

It is known that if $X \in \mathcal{D}(\alpha, \beta)$ with $\alpha \in (0, 2)$, and $F(x) := P(X < x)$, then

$$1 - F(x) + F(-x) \sim \frac{1}{x^{\alpha}l_0(x)} \quad as \ x \to \infty, \quad (24)$$

where $l_0(x)$ is a function slowly varying at infinity. Besides, for $\alpha \in (0, 2)$,

$$\frac{F(-x)}{1 - F(x) + F(-x)} \to q, \quad \frac{1 - F(x)}{1 - F(x) + F(-x)} \to p \quad as \ x \to \infty, \quad (25)$$

with $p + q = 1$ and $\beta = p - q$ in (1). It is easy to see that (24) implies

$$\mu(u) \sim \frac{\alpha}{2 - \alpha} P(|X| > u) \quad as \ u \to \infty. \quad (26)$$

By this relation and the definition of $c_n$ we deduce

$$P(|X| > c_n) \sim \frac{2 - \alpha}{\alpha} \frac{1}{n} \quad as \ n \to \infty. \quad (27)$$

2.2. **Some results from the fluctuation theory.** Now we formulate a number of statements concerning the distributions of the random variables $\tau^-, \tau^+$ and $\chi^+$. Recall that a random variable $\zeta$ is called relatively stable if there exists a non-random sequence $d_n \to \infty$ as $n \to \infty$ such that

$$\frac{1}{d_n} \sum_{k=1}^{n} \zeta_k \sim \frac{\alpha}{\alpha} \quad as \ n \to \infty,$$

where $\zeta_k \doteq \zeta, \ k = 1, 2, \ldots$, and are independent.
Lemma 11. (see [18, Theorem 9]) Assume $X \in D(\alpha, \beta)$. Then, as $x \to \infty$,
\begin{equation}
\mathbf{P}(\chi^+ > x) \sim \frac{1}{x^{\alpha \rho} l_2(x)} \text{ if } \alpha \rho < 1,
\end{equation}
and $\chi^+$ is relatively stable if $\alpha \rho = 1$.

Lemma 12. Suppose $X \in D(\alpha, \beta)$. Then, as $x \to \infty$,
\begin{equation}
H(x) \sim \frac{x^{\alpha \rho} l_2(x)}{\Gamma(1 - \alpha \rho) \Gamma(1 + \alpha \rho)}
\end{equation}
if $\alpha \rho < 1$, and
\begin{equation}
H(x) \sim x l_3(x)
\end{equation}
if $\alpha \rho = 1$, where
\begin{equation}
l_3(x) := \left( \int_0^x \mathbf{P}(\chi^+ > y) \, dy \right)^{-1}, \quad x > 0.
\end{equation}

In addition, there exists a constant $C > 0$ such that, in both cases
\begin{equation}
H(c_n) \sim C n \mathbf{P}(\tau^+ > n) \quad \text{as } n \to \infty.
\end{equation}

Proof. If $\alpha \rho < 1$, then by [14, Chapter XIV, formula (3.4)]
\begin{equation}
H(x) \sim \frac{1}{\Gamma(1 - \alpha \rho) \Gamma(1 + \alpha \rho)} \frac{1}{\mathbf{P}(\chi^+ > x)} \text{ as } x \to \infty.
\end{equation}
Hence, recalling (28), we obtain (29).

If $\alpha \rho = 1$, then (30) follows from Theorem 2 in [18].

Let us demonstrate the validity of (31). We know from [18] (see also [15]) that $\tau^+ \in D(\rho, 1)$ under the conditions of the lemma and, in addition, $\chi^+ \in D(\alpha \rho, 1)$ if $\alpha \rho < 1$. This means, in particular, that for sequences \{a_n, n \geq 1\} and \{b_n, n \geq 1\} specified by
\begin{equation}
\mathbf{P}(\tau^+ > a_n) \sim \frac{1}{n} \quad \text{and} \quad \mathbf{P}(\chi^+ > b_n) \sim \frac{1}{n} \quad \text{as } n \to \infty,
\end{equation}
and vectors $(\tau_k^+, \chi_k^+), k = 1, 2, ..., \text{ being independent copies of } (\tau^+, \chi^+)$, we have
\begin{equation}
\frac{1}{a_n} \sum_{k=1}^n \tau_k^+ \overset{d}{\to} Y_\rho \quad \text{and} \quad \frac{1}{b_n} \sum_{k=1}^n \chi_k^+ \overset{d}{\to} Y_{\alpha \rho} \quad \text{as } n \to \infty.
\end{equation}

Moreover, it was established by Doney (see Lemma in [9, p. 358]) that
\begin{equation}
b_n \sim C c_{[a_n]} \text{ as } n \to \infty,
\end{equation}
where $[x]$ stands for the integer part of $x$. Therefore, $c_n \sim C b_{[a^{-1}(n)]}$, where, with a slight abuse of notation, $a^{-1}(n)$ is the inverse function to $a_n$. Hence, on account of (32),
\begin{equation}
\mathbf{P}(\chi^+ > c_n) \sim C_1 \mathbf{P}(\chi^+ > b_{[a^{-1}(n)]}) \sim \frac{C_1}{a^{-1}(n)}
\end{equation}
\begin{equation}
\sim C_2 \mathbf{P}(\tau^+ > a_{[a^{-1}(n)]}) \sim C_3 \mathbf{P}(\tau^+ > n).
\end{equation}

If $\alpha \rho = 1$, then, instead of the second equivalence in (32), one should define $b_n$ by
\begin{equation}
\frac{1}{b_n} \int_0^{b_n} \mathbf{P}(\chi^+ > y) \, dy \sim \frac{1}{n} \quad \text{as } n \to \infty.
\end{equation}
In this case the second convergence in (33) transforms to
\[ \frac{1}{b_n} \sum_{k=1}^{n} \chi_k^+ \xrightarrow{p} 1 \quad \text{as} \quad n \to \infty, \]
while (35) should be changed to
\[ \frac{1}{c_n} \int_0^{c_n} P(\chi^+ > y)dy \sim \frac{C_1}{b_{[a^{-1}(n)]}} \int_0^{b_{[a^{-1}(n)]}} P(\chi^+ > y)dy \sim \frac{C_1}{a^{-1}(n)} \]
\[ \sim C_1 P(\tau^+ > a_{[a^{-1}(n)]}) \sim C_2 P(\tau^+ > n). \] (36)

Combining (35) and (36) with (29) and (30) gives
\[ H(c_n) \sim C P(\tau^+ > n) \quad \text{as} \quad n \to \infty \]
for all \( X \in D(\alpha, \beta). \) Using (15) finishes the proof of the lemma. \( \square \)

**Lemma 13.** If \( E(-S_{\tau^-}) < \infty, \) then there exists a positive constant \( C_0 \) such that
\[ c_n \sim C_0 \frac{n^{1-\rho}}{\xi(n)}. \] (37)

**Proof.** Let \( T^- := \min\{k \geq 1 : -S_k > 0\} \) and \( \chi^- = -S_{T^-} \) be the first strict ladder height for the random walk \( \{-S_n, n \geq 0\}. \) Applying (36) to \( \{-S_n, n \geq 0\}, \) we have
\[ \frac{1}{c_n} \int_0^{c_n} P(\chi^- > y)dy \sim C P(T^- > n). \] (38)

Obviously, \( E(-S_{\tau^-}) < \infty \) yields \( E \chi^- < \infty. \) Therefore \( \int_0^{c_n} P(\chi^- > y)dy \to E \chi^- \) as \( n \to \infty. \) Combining this with (38), and recalling that \( P(T^- > n) \sim CP(\tau^- > n) \) in view of the equality
\[ \sum_{n=1}^{\infty} P(T^- > n)z^n = \sum_{n=1}^{\infty} P(\tau^- > n)z^n \exp \left\{ \sum_{k=1}^{\infty} \frac{z^k}{k} P(S_k = 0) \right\}, \]
asymptotic representation (15), and the estimate
\[ \sum_{k=1}^{\infty} \frac{1}{k} P(S_k = 0) < \infty, \]
we obtain
\[ \lim_{n \to \infty} c_n P(\tau^- > n) =: C_0 \in (0, \infty). \]
On account of (15) this proves (37). \( \square \)

**2.3. Upper estimates for local probabilities.** For \( x \geq 0 \) and \( n = 0, 1, 2, ..., \) let
\[ B_n(x) := P(S_n \in (0, x); \tau^- > n), \]
\[ b_n(x) := B_n(x+1) - B_n(x) = P(S_n \in [x, x+1); \tau^- > n). \]
Note that by the duality principle for random walks

\[
1 + \sum_{j=1}^{\infty} B_j(x) = 1 + \sum_{j=1}^{\infty} P(S_j \in (0, x); \tau^- > j)
\]

\[
= 1 + \sum_{j=1}^{\infty} P(S_j \in (0, x); S_j > S_0, S_j > S_1, ..., S_j > S_{j-1})
\]

\[
= H(x), \quad x > 0.
\]  

(39)

**Lemma 14.** The sequence of functions \( \{B_n(x), n \geq 1\} \) satisfies the recurrence equations

\[
n B_n(x) = P(S_n \in (0, x)) + \sum_{k=1}^{n-1} \int_0^x P(S_k \in (0, x-y)) dB_{n-k}(y)
\]  

(40)

and

\[
n B_n(x) = P(S_n \in (0, x)) + \sum_{k=1}^{n-1} \int_0^x B_{n-k}(x-y)P(S_k \in dy).
\]  

(41)

**Remark 15.** The proof of (41) is contained in Eppel [13] (see formula (5) there). Representation (40) is not given by Eppel. However, it can be easily obtained by the same method. Here we demonstrate the mentioned relations only for the completeness of the presentation.

**Proof.** Let

\[
B_n(t) := E[e^{i t S_n}; \tau^- > n] = \int_0^\infty e^{i t x} P(S_n \in dx; \tau^- > n), \quad t \in (-\infty, \infty),
\]

be the Fourier transform of the measure \( B_n \). It is known (see, for instance, [20], Chapter 4, Section 17) that

\[
1 + \sum_{n=1}^{\infty} z^n B_n(t) = \exp \left\{ \sum_{k=1}^{\infty} \frac{z^k}{k} S_k(t) \right\}, \quad |z| < 1,
\]

where \( S_k(t) := E[e^{i t S_k}; S_k > 0] \). Differentiation with respect to \( z \) gives

\[
\sum_{n=1}^{\infty} n z^{n-1} B_n(t) = \left( 1 + \sum_{n=1}^{\infty} z^n B_n(t) \right) \sum_{k=1}^{\infty} z^{k-1} S_k(t).
\]

Comparing the coefficients of \( z^{n-1} \) in the both sides of this equality, we get

\[
n B_n(t) = S_n(t) + \sum_{k=1}^{n-1} B_{n-k}(t) S_k(t).
\]  

(42)

Going back to the distributions, we obtain the desired representations. \( \square \)

From now on we assume without loss of generality that \( h = 1 \) in the lattice case and, to study the asymptotic behavior of the probabilities of small deviations when \( X \) is \((1,a)\)-lattice, introduce a shifted sequence \( \tilde{S}_n := S_n - an \) and probabilities \( \tilde{b}_n(x) := P(\tilde{S}_n = x) = b_n(an + x) \). Further, for fixed \( x \in \mathbb{Z} \) and \( 1 \leq k \leq n-1 \) set

\[
I_x(k,n) := (-a(n-k), ak + x) \cap \mathbb{Z}.
\]
Lemma 16. The sequence of functions \( \{\bar{b}_n(x), n \geq 1\} \) satisfies the recurrence equation

\[
n\bar{b}_n(x) = \mathbb{P}(\bar{S}_n = x) + \sum_{k=1}^{n-1} \sum_{y \in I_{x(k,n)}} \bar{b}_k(x-y) \mathbb{P}(\bar{S}_{n-k} = y).
\]

(43)

Remark 17. Alili and Doney [1] obtained this representation in the case when \( X \) is \((h,0)\)-lattice.

Proof. It follows from (42) that

\[
n\bar{b}_n(x) = \mathbb{P}(\bar{S}_n = x) + \sum_{k=1}^{n-1} \sum_{\bar{S}_n = y} \bar{b}_k(x-y) \mathbb{P}(\bar{S}_{n-k} = y),
\]

where the second sum is taken over all \( y \in \mathbb{Z} \) satisfying the conditions \( ak + x - y > 0, a(n - k) + y > 0 \). This proves the lemma. \( \square \)

Lemma 18. Assume \( X \in \mathcal{D}(\alpha, \beta) \). Then there exists \( C > 0 \) such that, for all \( y > 0 \) and all \( n \geq 1 \),

\[
b_n(y) \leq \frac{C}{c_n n^{1-\rho}} \frac{l(n)}{n^{1-\rho}}
\]

(44) and

\[
B_n(y) \leq \frac{C}{c_n n^{1-\rho}} \frac{(y + 1) l(n)}{n^{1-\rho}}.
\]

(45)

Proof. For \( n = 1 \) the statement of the lemma is obvious. Let \( \{S^*_n, n \geq 0\} \) be a random walk distributed as \( \{S_n, n \geq 0\} \) and independent of it. One can easily check that for each \( n \geq 2 \),

\[
b_n(y) = \mathbb{P}(y \leq S_n < y + 1; \tau^- > n)
\]

\[
= \int_0^\infty \mathbb{P}(y - S_{[n/2]} \leq S_{[n/2]} - S_{[n/2]} < y + 1 - S_{[n/2]}; S_{[n/2]} \in dz; \tau^- > n)
\]

\[
\leq \int_0^\infty \mathbb{P}(y - z \leq S^*_n - S_{[n/2]} < y + 1 - z; S_{[n/2]} \in dz; \tau^- > [n/2])
\]

\[
\leq \mathbb{P}(\tau^- > [n/2]) \sup_z \mathbb{P}(z \leq S^*_n - S_{[n/2]} < z + 1).
\]

(46)

Since the density of any \( \alpha \)-stable law is bounded, it follows from the Gnedenko and Stone local limit theorems that there exists a constant \( C > 0 \) such that for all \( n \geq 1 \) and all \( z \geq 0 \),

\[
\mathbb{P}(S_n \in [z, z + \Delta]) \leq \frac{C\Delta}{c_n}.
\]

(47)

Hence it follows, in particular, that, for any \( z > 0 \),

\[
\mathbb{P}(S_n \in [0, z]) \leq \frac{C(z + 1)}{c_n}.
\]

(48)

Substituting (47) into (46), and recalling (3) and properties of regularly varying functions, we get (44). Estimate (45) follows from (44) by summation. \( \square \)

Lemma 19. If \( X \in \mathcal{D}(\alpha, \beta) \) then there exists a constant \( C \in (0, \infty) \) such that

\[
b_n(x) \leq \frac{C \cdot H(x)}{n e_n}
\]

(49)
and
\[ B_n(x) \leq C \frac{xH(x)}{nc_n} \] (50)
for all \( n \geq 1 \) and all \( x \in (0, c_n] \).

**Remark 20.** Comparing (49) and (10) (to be proved later), we see that, in the domain of small deviations, the estimates given by the lemma are optimal up to a constant factor.

**Proof.** By (41) we get
\[
nb_n(x) = P(S_n \in [x, x+1]) + \sum_{k=1}^{n-1} \int_0^x b_{n-k}(x-y) P(S_k \in dy)
\]
\[ + \sum_{k=1}^{n-1} \int_x^{x+1} B_{n-k}(x+1-y) P(S_k \in dy). \] (51)

Using (44), (48) and properties of slowly varying functions, we deduce
\[
\sum_{k=1}^{\lfloor n/2 \rfloor} \int_0^x b_{n-k}(x-y) P(S_k \in dy) \leq C \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{l(n-k)}{c_{n-k} (n-k)^1-\rho} P(S_k \in [0, x])
\]
\[ \leq C_1 \frac{l(n)}{c_n n^{1-\rho}} \sum_{k=1}^{\lfloor n/2 \rfloor} P(S_k \in [0, x)). \] (52)

On the other hand, in view of (47) and monotonicity of \( B_k(x) \) in \( x \) we conclude (assuming that \( x \) is integer without loss of generality and letting \( B_k(-1) = 0 \) and \( H(-1) = 0 \) that
\[
\sum_{k=\lfloor n/2 \rfloor + 1}^{n} \int_0^x b_{n-k}(x-y) P(S_k \in dy)
\]
\[ \leq \sum_{k=\lfloor n/2 \rfloor + 1}^{n} \sum_{j=0}^{x} (B_{n-k}(x-j+1) - B_{n-k}(x-j-1)) P(S_k \in [j, j+1])
\]
\[ \leq \sum_{k=\lfloor n/2 \rfloor + 1}^{n} \sum_{j=0}^{x} (B_{n-k}(x-j+1) - B_{n-k}(x-j-1)) \frac{C}{c_k}
\]
\[ \leq \frac{C}{c_n} \sum_{j=0}^{x} \sum_{k=0}^{\infty} (B_k(x-j+1) - B_k(x-j-1))
\]
\[ = \frac{C}{c_n} \sum_{j=0}^{x} (H(x-j+1) - H(x-j-1))
\]
\[ \leq \frac{C}{c_n} (H(x) + H(x+1)) \leq \frac{2C}{c_n} H(x+1), \]
where for the intermediate equality we have used (59). This gives
\[
\sum_{k=\lfloor n/2 \rfloor + 1}^{n} \int_0^x b_{n-k}(x-y) P(S_k \in dy) \leq \frac{C}{c_n} H(x+1). \] (53)
Since $x \mapsto B_n(x)$ increases for every $n$,

$$\sum_{k=1}^{n-1} \int_k^{x+1} B_{n-k}(x+1-y) \mathbf{P}(S_k \in dy) \leq \sum_{k=1}^{n-1} B_{n-k}(1) \mathbf{P}(S_k \in [x,x+1]). \quad (54)$$

Further, in view of (45) and (47) we have

$$\sum_{k=1}^{[n/2]} B_{n-k}(1) \mathbf{P}(S_k \in [x,x+1]) \leq C \frac{C_1}{c_n} \frac{l(n)}{n^{1-\rho}} \sum_{k=1}^{[n/2]} \mathbf{P}(S_k \in [x,x+1]). \quad (55)$$

Applying (47) once again yields

$$\sum_{k=[n/2]+1}^{n-1} B_{n-k}(1) \mathbf{P}(S_k \in [x,x+1]) \leq \frac{C}{c_n} \sum_{k=[n/2]+1}^{n-1} B_{n-k}(1) \leq \frac{C}{c_n} H(1). \quad (56)$$

Combining (51)-(56) and using the monotonicity of $H(x)$, we obtain the estimate

$$n b_n(x) \leq \frac{C}{c_n} \left( H(x+1) + \frac{l(n)}{n^{1-\rho}} \sum_{k=1}^{[n/2]} \mathbf{P}(S_k \in [0,x+1]) \right).$$

Therefore, to complete the proof of (49) it remains to show that

$$\frac{l(n)}{n^{1-\rho}} \sum_{k=1}^{[n/2]} \mathbf{P}(S_k \in [x,x+1]) \leq CH(x+1). \quad (57)$$

This will be done separately for the cases $\alpha \in (1,2], \alpha \in (0,1)$, and $\alpha = 1$.

Consider first the case $\alpha \in (1,2]$. It follows from (48) that

$$\sum_{k=1}^{[n/2]} \mathbf{P}(0 \leq S_k < x+1) \leq C(x+1) \sum_{k=1}^{n} \frac{1}{c_k} \leq C(x+1) \frac{n}{c_n}, \quad (58)$$

where at the last step we have used the relation

$$\sum_{k=1}^{n} \frac{1}{c_k} \sim \frac{\alpha}{\alpha-1} \frac{n}{c_n} \quad \text{as } n \to \infty. \quad (59)$$

By Lemma 12 and properties of regularly varying functions we conclude that there exists a non-decreasing function $\phi(u)$ such that $u/H(u) \sim \phi(u)$ as $u \to \infty$. Therefore, for any $\varepsilon \in (0,1/2)$ there exists a $u_0 = u_0(\varepsilon)$ such that, for all $u \geq u_0$,

$$(1-\varepsilon)\phi(u) \leq \frac{u}{H(u)} \leq (1+\varepsilon)\phi(u).$$

From this estimate it is not difficult to conclude that there exists a constant $C$ such that, for all $n \geq 1$ and all $x \in (0,c_n]$,

$$\frac{x}{H(x)} \leq C \frac{c_n}{H(c_n)}.$$ 

Hence we see that the right-hand side of (55) is bounded from above by

$$C \frac{n H(x+1)}{H(c_n)}.$$
Recalling that \( H(x) \) is regularly varying as \( x \to \infty \), and applying (31) and (15), we finally arrive at the inequality
\[
\sum_{k=1}^{[n/2]} P(0 \leq S_k < x + 1) \leq C H(x + 1) \frac{n^{1-\rho}}{l(n)}.
\]
This justifies (57) for \( \alpha \in (1, 2] \).

Now we turn to the case \( \alpha \in (0, 1) \). Letting \( N_x := \max\{k \geq 1 : c_k \leq x + 1\} \) and applying (47), we get
\[
\sum_{k=1}^{[n/2]} P(0 \leq S_k < x + 1) \leq N_x + C(x + 1) \sum_{k=N_x+1}^{n} \frac{1}{c_k}
\]
\[
\leq N_x + C(x + 1) \frac{N_x}{c_{N_x+1}}.
\]

Here we have used the asymptotic representation
\[
\sum_{k=n+1}^{\infty} \frac{1}{c_k} \sim \frac{\alpha}{1 - \alpha c_{n+1}} n \quad \text{as} \quad n \to \infty.
\]

If \( \alpha = 1 \), then, in view of (3),
\[
\sum_{k=N_x+1}^{n} \frac{1}{c_k} = \frac{N_x + 1}{c_{N_x+1}} \sum_{k=N_x+1}^{n} \frac{l_1(N_x + 1)}{l_1(k)} c_k.
\]

From the Karamata representation for slowly varying functions (see [19], Theorem 1.2) we conclude that for every slowly varying function \( l^*(x) \) and every \( \gamma > 0 \) there exists a constant \( C = C(\gamma) \) such that
\[
\frac{l^*(x)}{l^*(y)} \leq C \max \left\{ \left( \frac{x}{y} \right)^\gamma, \left( \frac{x}{y} \right)^{-\gamma} \right\} \quad \text{for all} \quad x, y > 0.
\]

Applying this inequality to \( l_1(x) \), we obtain
\[
\sum_{k=N_x+1}^{n} \frac{1}{c_k} \leq C \frac{N_x + 1}{c_{N_x+1}} \left( \frac{n}{N_x + 1} \right)^\gamma \log \left( \frac{n}{N_x + 1} \right).
\]

Combining this bound with (60), and using the inequality \( c_{N_x+1} \geq x + 1 \), we conclude that
\[
\sum_{k=1}^{[n/2]} P(0 \leq S_k < x + 1) \leq C_1 N_x \left( \frac{n}{N_x} \right)^{2\gamma}
\]
for all \( \alpha \in (0, 1] \). Consequently,
\[
\frac{l(n)}{n^{1-\rho}} \sum_{k=1}^{[n/2]} P(0 \leq S_k < x + 1) \leq C_1 H(x + 1) \left( \frac{n}{N_x} \right)^{2\gamma} \frac{l(n)N_x}{n^{1-\rho}H(x + 1)}.
\]

The definition of \( N_x \), (51), and (15) imply
\[
H(x + 1) \geq H(c_{N_x}) \geq C l(N_x) N_x^\rho.
\]

Therefore,
\[
\frac{l(n)}{n^{1-\rho}} \sum_{k=1}^{[n/2]} P(0 \leq S_k < x + 1) \leq C_1 H(x + 1) \left( \frac{N_x}{n} \right)^{1-\rho+2\gamma} \frac{l(n)}{l(N_x)}.
\]
Applying (61) to \(l(x)\) and choosing \(\gamma = (1 - \rho)/4\), we finally arrive at the inequality

\[
\frac{l(n)}{n^{1-\rho}} \sum_{k=1}^{[n/2]} P(0 \leq S_k < x + 1) \leq CH(x + 1) \bigg( \frac{N_x}{n} \bigg)^{(1-\rho)/4} \leq CH(x + 1).
\]

(63)

establishing (57) for \(\alpha \in (0, 1]\). Thus, (57) is justified for all \(X \in \mathcal{D}(\alpha, \beta)\), and, consequently, (49) is proved.

The second statement of the lemma follows by summation. \(\square\)

Later on we need the following refined version of Lemma 19:

**Corollary 21.** Suppose \(X \in \mathcal{D}(\alpha, \beta)\). Then there exists a constant \(C \in (0, \infty)\) such that, for all \(n \geq 1\),

\[
b_n(x) \leq C \frac{H(\min(c_n, x))}{nc_n}
\]

and

\[
B_n(x) \leq C \frac{\min(c_n, x)H(\min(c_n, x))}{nc_n}.
\]

(64)

(65)

**Proof.** The desired estimates follow from (44), (45) and Lemma 19. \(\square\)

**Lemma 22.** There exists a constant \(C \in (0, \infty)\) such that, for all \(z \in [0, \infty)\),

\[
\limsup_{\varepsilon \downarrow 0} \varepsilon^{-1} P(M_{\alpha, \beta} \in [z, z + \varepsilon]) \leq C \min\{1, z^{\alpha\rho}\}.
\]

In particular,

\[
\lim_{z \downarrow 0} \limsup_{\varepsilon \downarrow 0} \varepsilon^{-1} P(M_{\alpha, \beta} \in [z, z + \varepsilon]) = 0.
\]

**Proof.** For all \(z \geq 0\) and all \(\varepsilon > 0\) we have

\[
P(M_{\alpha, \beta} \in [z, z + \varepsilon]) \leq \limsup_{n \to \infty} P(S_n \in [c_n z, c_n (z + \varepsilon)] | \tau^- > n).
\]

Applying (64) gives

\[
P(S_n \in [c_n z, c_n (z + \varepsilon)] | \tau^- > n) \leq C \frac{H(\min(c_n, (z + \varepsilon)c_n))}{nc_n P(\tau^- > n)} \varepsilon c_n.
\]

Recalling that \(H(x)\) is regularly varying with index \(\alpha\rho\) by Lemma 12 and taking into account (31), we get

\[
P(S_n \in [c_n z, c_n (z + \varepsilon)] | \tau^- > n) \leq C \varepsilon \min\{1, (z + \varepsilon)^{\alpha\rho}\} \frac{H(c_n)}{n P(\tau^- > n)} \leq C \varepsilon \min\{1, (z + \varepsilon)^{\alpha\rho}\}.
\]

Consequently,

\[
P(M_{\alpha, \beta} \in [z, z + \varepsilon]) \leq C \varepsilon \min\{1, (z + \varepsilon)^{\alpha\rho}\}.
\]

(66)

This inequality shows that there exists a constant \(C \in (0, \infty)\) such that

\[
\limsup_{\varepsilon \downarrow 0} \varepsilon^{-1} P(M_{\alpha, \beta} \in [z, z + \varepsilon]) \leq C \min\{1, z^{\alpha\rho}\} \text{ for all } z \geq 0
\]

as desired. \(\square\)
3. Probabilities of Normal deviations: Proofs of Theorems 2 and 4

The first part of the proof to follow is one and the same for non-lattice (Theorem 2) and lattice (Theorem 4) cases.

It follows from (40) that

\[
\begin{align*}
nb_n(x) &= P(S_n \in [x, x+1]) + \sum_{k=1}^{n-1} \int_0^x P(S_k \in [x-y, x-y+1]) dB_{n-k}(y) \\
&\quad + \sum_{k=1}^{n-1} \int_x^{x+1} P(S_k \in (0, x-y+1)) dB_{n-k}(y) \\
&=: R^{(1)}_x(x) + R^{(2)}_x(x) + R^{(3)}_x(x) + R^{(0)}_x(x),
\end{align*}
\]

where, for any fix \( \varepsilon \in (0, 1/2) \) and with a slight abuse of notation

\[
\begin{align*}
R^{(1)}_x(x) &= \sum_{k=1}^{\lfloor \varepsilon n \rfloor} P(S_k \in [x-y, x-y+1]) dB_{n-k}(y), \\
R^{(2)}_x(x) &= \sum_{k=\lfloor (1-\varepsilon) n \rfloor+1}^{\lfloor \varepsilon n \rfloor} P(S_k \in [x-y, x-y+1]) dB_{n-k}(y), \\
R^{(3)}_x(x) &= P(S_n \in [x, x+1]) + \sum_{k=\lfloor (1-\varepsilon) n \rfloor+1} P(S_k \in [x-y, x-y+1]) dB_{n-k}(y), \\
R^{(0)}_x(x) &= \sum_{k=1}^{n-1} \int_x^{x+1} P(S_k \in (0, x-y+1)) dB_{n-k}(y).
\end{align*}
\]

First observe that

\[
R^{(0)}_x(x) \leq \sum_{k=1}^{n-1} P(S_k \in (0, 1)) b_{n-k}(x).
\]

Applying Corollary 21 we may simplify the estimate above to

\[
R^{(0)}_x(x) \leq C \sum_{k=1}^{n-1} P(S_k \in (0, 1)) H_{n-k}(c_{n-k}) (n-k)c_{n-k}
\]

\[
\leq C \frac{H(c_n)}{nc_n} \sum_{k=1}^{[n/2]} P(S_k \in (0, 1)) + C \frac{[n/2]}{c_n} \sum_{k=1}^{H(c_k)} \frac{H(c_k)}{k c_k},
\]

where at the last step we have used the properties of \( c_n \) and inequality (47).

Since \( H(x) \leq Cx \), we have

\[
\sum_{k=1}^{[n/2]} \frac{H(c_k)}{k c_k} \leq C \sum_{k=1}^{[n/2]} \frac{1}{k} \leq C \log n.
\]

Further, by (58) and (62) with \( x = 0 \) we know that

\[
\sum_{k=1}^{[n/2]} P(S_k \in (0, 1)) \leq C \left( \frac{n}{c_n} I(\alpha \in (1, 2]) + n^\gamma I(\alpha \in (0, 1]) \right) \leq C \left( \frac{n}{c_n} + n^\gamma \right).
\]
Substituting these estimates into (68) leads to the inequalities

\[ R^{(0)}(x) \leq \frac{C}{c_n} \left( \frac{H(c_n)}{c_n} + \frac{H(c_n)}{n^{1-\gamma}} + \log n \right). \]

By these relations, recalling that \( P(\tau^- > n) \) is regularly varying with index \( \rho - 1 > -1 \) (see (15)) and using (31), we obtain

\[ \limsup_{n \to \infty} \frac{c_n}{n} P(\tau^- > n) R^{(0)}(x) = 0. \]

Now we evaluate the remaining terms in (67). In view of (47)

\[ R^{(3)}(x) \leq \frac{C}{c_n} \left( 1 + \sum_{k=1}^{[\varepsilon n]} B_k(x) \right) \leq \frac{C}{c_n} \sum_{k=0}^{[\varepsilon n]} P(\tau^- > k) \]

for all \( x > 0 \). Further, by (15)

\[ \sum_{k=0}^{[\varepsilon n]} P(\tau^- > k) \sim \rho^{-1} \varepsilon^n n P(\tau^- > n) \quad \text{as} \quad n \to \infty. \]

As a result we obtain

\[ \limsup_{n \to \infty} \frac{c_n}{n} P(\tau^- > n) \sup_{x > 0} R^{(3)}(x) \leq C \varepsilon^\rho. \]

Using the inequalities

\[ \int_j^{j+1} P(S_k \in [x-y, x-y+1]) dB_{n-k}(y) \leq P(S_k \in [x-j-1, x-j+1]) b_{n-k}(j) \] \hspace{1cm} (71)

and

\[ \int_{[x]}^{x} P(S_k \in [x-y, x-y+1]) dB_{n-k}(y) \leq P(S_k \in [0, 2]) b_{n-k}([x]), \] \hspace{1cm} (72)

and applying Corollary 21 we get

\[ R^{(1)}_\varepsilon(x) \leq C \frac{H(c_n)}{nc_n} \sum_{k=1}^{[\varepsilon n]} P(0 < S_k < x) \leq C \frac{H(c_n)}{c_n}. \]

From this estimate and (31) we deduce

\[ \limsup_{n \to \infty} \frac{c_n}{n} P(\tau^- > n) R^{(1)}_\varepsilon(x) \leq C \varepsilon. \]

Evaluating \( R^{(2)}_\varepsilon(x) \) we have to distinguish the non-lattice (Theorem 2) and lattice (Theorem 4) cases. Detailed estimates are given for the non-lattice case only. To deduce the respective estimates for the lattice case one should use the Gnedenko local limit theorem instead of the Stone local limit theorem.
Thus, in the non-lattice case we combine the Stone local limit theorem with the first equality in (4) and obtain, uniformly in \( x > 0 \), as \( n \to \infty \),

\[
R^{(2)}(\varepsilon)(x) = \sum_{k=[\varepsilon n]+1}^{[1-\varepsilon]n} \frac{1}{c_{n-k}} \int_0^x g_{\alpha,\beta} \left( \frac{x-y}{c_{n-k}} \right) dB_{n-k}(y) + o \left( \frac{1}{c_{n\varepsilon}} \sum_{k=1}^n B_k(x) \right)
\]

\[
= \sum_{k=[\varepsilon n]+1}^{[1-\varepsilon]n} \frac{\mathbb{P}(\tau > k)}{c_{n-k}} \int_0^{x/c_k} g_{\alpha,\beta} \left( \frac{x-c_k u}{c_{n-k}} \right) \mathbb{P}(M_{\alpha,\beta} \in du) + o \left( \frac{1}{c_{n\varepsilon}} \sum_{k=1}^n B_k(x) + \sum_{k=1}^{n-1} \mathbb{P}(\tau > k) \right).
\]

According to (15)

\[
\sum_{k=1}^n B_k(x) \leq \sum_{k=1}^n \mathbb{P}(\tau > k) \leq Cn \mathbb{P}(\tau > n).
\]

Hence it follows that

\[
R^{(2)}(\varepsilon)(x) = \sum_{k=[\varepsilon n]+1}^{[1-\varepsilon]n} \frac{\mathbb{P}(\tau > k)}{c_{n-k}} \int_0^{x/c_k} g_{\alpha,\beta} \left( \frac{x-c_k u}{c_{n-k}} \right) \mathbb{P}(M_{\alpha,\beta} \in du) + o \left( \frac{1}{c_{n\varepsilon}} \sum_{k=1}^n \mathbb{P}(\tau > k) \right).
\]

Since \( c_k \) and \( \mathbb{P}(\tau > k) \) are regularly varying and \( g_{\alpha,\beta}(x) \) is uniformly continuous in \(( -\infty, \infty )\), we let, for brevity, \( v = x/c_n \) and continue the previous estimates for \( R^{(2)}(\varepsilon)(x) \) with

\[
= \frac{\mathbb{P}(\tau > n)}{c_n} \sum_{k=[\varepsilon n]+1}^{[1-\varepsilon]n} \left( \frac{k/n}{1-k/n} \right)^{\alpha-1} \int_0^{v/(k/n)^{1/\alpha}} \mathbb{P}(M_{\alpha,\beta} \in du) + o \left( \frac{n \mathbb{P}(\tau > n)}{c_{n\varepsilon}} \right)
\]

\[
= \frac{n \mathbb{P}(\tau > n)}{c_n} f(\varepsilon, 1-\varepsilon; v) + o \left( \frac{n \mathbb{P}(\tau > n)}{c_{n\varepsilon}} \right)
\]

where, for \( 0 \leq w_1 \leq w_2 \leq 1 \),

\[
f(w_1, w_2; v) := \int_{w_1}^{w_2} t^{\alpha-1} dt \int_0^{v/t^{1/\alpha}} \mathbb{P}(M_{\alpha,\beta} \in du)
\]

Observe that by boundness of \( g_{\alpha,\beta}(y) \)

\[
f(0, \varepsilon; v) \leq C \int_0^\varepsilon t^{\alpha-1} dt \leq C \varepsilon^\alpha.
\]
Further, it follows from (66) that
\[
\int \varphi(u) P(M_{\alpha,\beta} \in du) \leq C \int \varphi(u) du
\]
for every non-negative integrable function \( \varphi \). Therefore,
\[
f(1 - \varepsilon, 1; v) \leq C \int_{1-\varepsilon}^{1} t^{\rho-1} dt \int_{0}^{v/1/\alpha} g_{\alpha,\beta} \left( \frac{v - t^{1/\alpha} u}{1 - t^{1/\alpha}} \right) du = \left( z = \frac{v - t^{1/\alpha} u}{1 - t^{1/\alpha}} \right)
\]
As a result we have
\[
\limsup_{n \to \infty} \sup_{x > 0} \left| \frac{c_n}{nP(\tau^- > n)} R_n^{(2)}(x) - f(0, 1; x/c_n) \right| \leq C \varepsilon. \tag{75}
\]
Combining (69) – (75) with representation (67) leads to
\[
\limsup_{n \to \infty} \sup_{x > 0} \left| \frac{c_n}{nP(\tau^- > n)} b_n(x) - f(0, 1; x/c_n) \right| \leq C \varepsilon. \tag{76}
\]
Since \( \varepsilon > 0 \) is arbitrary, it follows that, as \( n \to \infty \)
\[
\frac{c_n}{nP(\tau^- > n)} b_n(x) - f(0, 1; x/c_n) \to 0 \tag{77}
\]
uniformly in \( x > 0 \). Recalling (41), we deduce by integration of (77) and evident transformations that
\[
\int_{u_1}^{u_2} f(0, 1; z) dz = \mathbf{P}(M_{\alpha,\beta} \in [u_1, u_2]) \tag{78}
\]
for all \( 0 < u_1 < u_2 < \infty \). This means, in particular, that the distribution of \( M_{\alpha,\beta} \) is absolutely continuous. Furthermore, it is not difficult to see that \( z \mapsto f(0, 1; z) \) as a continuous version of the density of the distribution of \( M_{\alpha,\beta} \) and let \( p_{\alpha,\beta}(z) := f(0, 1; z) \). This and (77) imply the statement of Theorem 2 for \( \Delta = 1 \). To establish the desired result for arbitrary \( \Delta > 0 \) it suffices to consider the random walk \( S_n/\Delta \) and to observe that
\[
c_n^\Delta := \inf \left\{ u \geq 0 : \frac{1}{u^2} \int_{-u}^{u} x^2 P \left( \frac{X}{\Delta} \in dx \right) \right\} = c_n/\Delta.
\]
Note that (72) gives an interesting representation for \( p_{\alpha,\beta}(v) \):
\[
p_{\alpha,\beta}(z) = \int_{0}^{1} t^{\rho-1} dt \int_{0}^{z/1/\alpha} g_{\alpha,\beta} \left( \frac{z - t^{1/\alpha} u}{1 - t^{1/\alpha}} \right) p_{\alpha,\beta}(u) du. \tag{79}
\]
Besides, it follows from Lemma 22 that
\[
p_{\alpha,\beta}(z) \leq C \min \{1, z^{\alpha \rho} \}
\]
and, that is not surprising,
\[
\lim_{z \to 0} p_{\alpha,\beta}(z) = 0. \tag{80}
\]
In section 4.3 we refine these statements.
4. Probabilities of Small deviations

4.1. Lattice case: Proof of Theorem 5 Recall that the span $h = 1$ according to our agreement. Fix any $\varepsilon \in (0, 1/2)$ and, using Lemma 16, write

$$n\bar{b}_n(x) = R_{\varepsilon n}(x) + \bar{R}_{\varepsilon n}(x), \quad (81)$$

where

$$R_{\varepsilon n}(x) := \Pr(\bar{S}_n = x) + \sum_{k=1}^{[\varepsilon n]} \sum_{y \in I_x(k,n)} \bar{b}_k(x - y) \Pr(\bar{S}_{n-k} = y)$$

and

$$\bar{R}_{\varepsilon n}(x) := \sum_{k=[\varepsilon n]+1}^{n-1} \sum_{y \in I_x(k,n)} \bar{b}_k(x - y) \Pr(\bar{S}_{n-k} = y).$$

In view of Lemma 19,

$$\bar{R}_{\varepsilon n}(x) \leq C \sum_{k=[\varepsilon n]+1}^{n-1} \sum_{y \in I_x(k,n)} \frac{H(ak + x - y)}{kc_k} \Pr(\bar{S}_{n-k} = y)$$

$$\leq C(\varepsilon) \frac{H(an + x)}{nc_n} \sum_{k=1}^{n-[\varepsilon n]} \Pr(0 \leq S_k < an + x).$$

Introduce the set

$$G_n := (-an, -an + \delta_n c_n] \cap \mathbb{Z}.$$ 

Taking into account estimate (58) (with $[n/2]$ replaced by $n - \lfloor n\varepsilon \rfloor$), we see that for $\alpha \in (1, 2)$

$$\limsup_{n \to \infty} \sup_{x \in G_n} c_n \bar{R}_{\varepsilon n}(x) \leq C(\varepsilon) \limsup_{n \to \infty} \sup_{x \in G_n} \frac{an + x}{cn} = C(\varepsilon) \limsup_{n \to \infty} \delta_n = 0. \quad (82)$$

Similarly, writing $c^{-1}(n)$ for the inverse function of $c_n$ we conclude by (62) (with $[n/2]$ replaced by $n - \lfloor n\varepsilon \rfloor$) that for $\alpha \in (0, 1)$ and every $\gamma < 1/2$.

$$\limsup_{n \to \infty} \sup_{x \in G_n} c_n \bar{R}_{\varepsilon n}(x) \leq C(\varepsilon) \limsup_{n \to \infty} \left(\frac{Na_n c_n}{n}\right)^{1-2\gamma}$$

$$= C(\varepsilon) \limsup_{n \to \infty} \left(\frac{e^{-1}(\delta_n c_n)}{e^{-1}(c_n)}\right)^{1-2\gamma} = 0 \quad (83)$$

According to the Gnedenko local limit theorem

$$\sup_{k \in [1, n(1-\varepsilon)]} \sup_{y \in I_x(k,n)} |c_{n-k} \Pr(\bar{S}_{n-k} = y) - g_{\alpha,\beta}(0)| \to 0 \quad \text{as } n \to \infty.$$ 

Therefore,

$$\sum_{y \in I_x(k,n)} \bar{b}_k(x - y) \Pr(\bar{S}_{n-k} = y)$$

$$= g_{\alpha,\beta}(0) + \Delta_{1}(x, n - k) \sum_{y \in I_x(k,n)} \bar{b}_k(x - y),$$
where $\Delta_1(x, n-k) \to 0$ as $n \to \infty$ uniformly in $x \in \mathcal{G}_n$ and $k \in [1, n(1 - \varepsilon)]$.
Hence, by the identity
\[
\sum_{y \in \mathcal{I}_s(k, n)} \bar{b}_k(x - y) = B_k(a(n-k) + x),
\]
we see that
\[
\begin{aligned}
R_{\varepsilon n}(x) &= (g_{\alpha, \beta}(0) + \Delta_2(x, n)) \left( \frac{1}{c_n} + \sum_{k=1}^{[\varepsilon n]} \frac{1}{c_{n-k}} B_k(a(n-k) + x) \right), 
\end{aligned}
\]  
(84)
where $\Delta_2(x, n) \to 0$ as $n \to \infty$ uniformly in $x \in \mathcal{G}_n$. Since the sequence $\{c_n, n \geq 1\}$ is non-decreasing and varies regularly with index $1/\alpha$ as $n \to \infty$, we have
\[
\sum_{k=1}^{[\varepsilon n]} B_k(a(n-k) + x) \leq c_n \sum_{k=n-[\varepsilon n]}^{n-1} \frac{1}{c_k} B_{n-k}(ak + x)
\]
\[
\leq \left( (1 - \varepsilon)^{-1/\alpha} + \Delta_3(x, n) \right) \sum_{k=1}^{[\varepsilon n]} B_k(a(n-k) + x) 
\]  
(85)
where $\Delta_3(x, n) \to 0$ as $n \to \infty$ uniformly in $x \in \mathcal{G}_n$. On the other hand, for all $x > -an$,
\[
H(an + x) - \sum_{k=[\varepsilon n]+1}^{\infty} B_k(a(n-k)+x) \leq 1 + \sum_{k=1}^{[\varepsilon n]} B_k(a(n-k)+x) \leq H(an + x). 
\]  
(86)
Applying (85) gives for some constant $C_1 = C_1(\varepsilon)$
\[
\sum_{k=[\varepsilon n]+1}^{\infty} B_k(a(n-k)+x) \leq (an + x)H(an + x) \sum_{k=[\varepsilon n]+1}^{\infty} \frac{C}{kc_k}
\]
\[
\leq C_1 \frac{(an + x)H(an + x)}{c_n} \leq C_1 \delta_n H(an + x) 
\]  
(87)
for all $x \in \mathcal{G}_n$. From (86) and (87) we conclude that
\[
\frac{1}{H(an + x)} \left( 1 + \sum_{k=1}^{[\varepsilon n]} B_k(a(n-k)+x) \right) - 1 \to 0 
\]  
(88)
uniformly in $x \in \mathcal{G}_n$. Combining (84), (85), and (88) leads to
\[
\limsup_{n \to \infty} \sup_{x \in \mathcal{G}_n} \left| \frac{c_n R_{\varepsilon n}(x)}{H(an + x)} - g_{\alpha, \beta}(0) \right| \leq r(\varepsilon),
\]
where $r(\varepsilon) \to 0$ as $\varepsilon \to 0$. This estimate, (82), and (83) show that
\[
\limsup_{n \to \infty} \sup_{x \in \mathcal{G}_n} \left| \frac{c_n n}{H(an + x)} \bar{b}_n(x) - g_{\alpha, \beta}(0) \right| \leq r(\varepsilon).
\]
Letting $\varepsilon \to 0$ and recalling that
\[
\bar{b}_n(x) = P(S_n = an + x \mid \tau^- > n)P(\tau^- > n)
\]
we finish the proof of Theorem 5.
4.2. Non-lattice case: Proof of Theorem [3] As in the proof of Theorem [2] we restrict our attention to the case \( \Delta = 1 \). Some of our subsequent arguments are similar to those used in the proof of Theorem [5] and we skip the respective details.

Using (71), (72) and Lemma [19] gives (in the notation introduced after formula (67))
\[
R^{(1)}(x) + R^{(2)}(x) = \sum_{k=1}^{[(1-\varepsilon)n]} \int_0^x P(S_k \in [x-y, x-y+1])dB_{n-k}(y)
\leq C(\varepsilon) \frac{H(x)^{[(1-\varepsilon)n]}}{n^c_n} \sum_{k=1}^{[(1-\varepsilon)n]} P(0 \leq S_k \leq x + 1).
\]

By the arguments mimicing those used in the lattice case one can easily show that
\[
\lim_{n \to \infty} \sup_{0 < x \leq \delta_n e_n} \frac{c_n}{H(x)} \left( R^{(1)}(x) + R^{(2)}(x) \right) = 0. \tag{89}
\]
Further, by the Stone local limit theorem
\[
\int_0^x P(S_k \in [x-y, x-y+1])dB_{n-k}(y) = \frac{g_{\alpha,\beta}(0) + \Delta_1(k, x)B_{n-k}(x)}{c_k},
\]
where \( \Delta_1(k, x) \to 0 \) uniformly in \( x \in (0, \delta_n e_n) \) and \( k \in [(1-\varepsilon)n, n] \). Therefore,
\[
R^{(3)}(x) = P(S_n \in [x, x+1]) + \sum_{k=1}^{[(1-\varepsilon)n]+1} \int_0^x P(S_k \in [x-y, x-y+1])dB_{n-k}(y)
= (g_{\alpha,\beta}(0) + \Delta_2(n, x)) \left( \frac{1}{c_n} + \sum_{k=1}^{[cn]} \frac{1}{c_{n-k}}B_k(x) \right),
\]
where \( \Delta_2(n, x) \to 0 \) uniformly in \( x \in (0, \delta_n e_n) \). Therefore, as in the lattice case,
\[
\lim_{n \to \infty} \sup_{0 < x \leq \delta_n e_n} \left| \frac{c_n}{H(x)} R^{(3)}(x) - g_{\alpha,\beta}(0) \right| \leq r(\varepsilon), \tag{90}
\]
where \( r(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \). Combining (89) and (90), we get
\[
\lim_{n \to \infty} \sup_{0 < x \leq \delta_n e_n} \left| \frac{c_n}{H(x)} \left( R^{(1)}(x) + R^{(2)}(x) + R^{(3)}(x) \right) - g_{\alpha,\beta}(0) \right| = 0. \tag{91}
\]

Now using definition (67) we write \( R^{(0)}(x) = R^{(4)}(x) + R^{(5)}(x) \), where
\[
R^{(4)}(x) := \sum_{k=1}^{[(1-\varepsilon)n]} \int_x^{x+1} P(S_k \in (0, x-y+1))dB_{n-k}(y)
\]
and
\[
R^{(5)}(x) := \sum_{k=([(1-\varepsilon)n]+1}^{n-1} \int_x^{x+1} P(S_k \in (0, x-y+1))dB_{n-k}(y).
\]
Evidently,
\[
R^{(4)}(x) \leq \sum_{k=1}^{[(1-\varepsilon)n]} P(S_k \in (0, 1))b_{n-k}(x).
\]
Applying (47) and (49), we see that
\[ R(4) \epsilon(x) \leq H(x) \sum_{k=1}^{1-\epsilon n} \frac{1}{c_k (n-k)c_{n-k}} \leq \frac{C(\epsilon)}{n} \sum_{k=1}^{n} \frac{1}{c_k} \]

Observing that \( \sum_{k=1}^{n} c_k^{-1} \leq C(1 + n/c_n) \), we conclude that
\[ \lim_{n \to \infty} \sup_{0 < x \leq \delta_n c_n} \frac{c_n}{H(x)} R(4) \epsilon(x) = 0. \] (92)

Further, by the Stone local limit theorem,
\[ \int_x^{x+1} P(S_k \in [0, x-y+1]) dB_{n-k}(y) \]
\[ = \frac{(g_{\alpha,\beta}(0) + \Delta_3(k, x))}{c_k} \int_x^{x+1} (x-y+1) dB_{n-k}(y), \]
where \( \Delta_3(k, x) \to 0 \) uniformly in \( x \in (0, \delta_n c_n] \) and \( k \in [(1 - \epsilon)n, n] \). Integration by parts gives
\[ \int_x^{x+1} (x-y+1) dB_{n-k}(y) = -B_{n-k}(x) + \int_x^{x+1} B_{n-k}(y) dy. \]

Consequently,
\[ R(5) \epsilon(x) = (g_{\alpha,\beta}(0) + \Delta_4(n, x)) \sum_{k=1}^{cn} \frac{1}{c_{n-k}} \left( \int_x^{x+1} B_k(y) dy - B_k(x) \right), \] (93)
where \( \Delta_4(n, x) \to 0 \) uniformly in \( x \in (0, \delta_n c_n] \).

Setting
\[ I(x) := \int_x^{x+1} H(y) dy - H(x) \]
we see, similarly to the proof in the lattice case, that
\[ \lim_{n \to \infty} \sup_{0 < x \leq \delta_n c_n} \left| \frac{c_n}{I(x)} \sum_{k=1}^{cn} \frac{1}{c_{n-k}} \left( \int_x^{x+1} B_k(y) dy - B_k(x) \right) - g_{\alpha,\beta}(0) \right| \leq r(\epsilon), \] (94)
where \( r(\epsilon) \to 0 \) as \( \epsilon \to 0 \). By (92) - (94) we deduce
\[ \lim_{n \to \infty} \sup_{0 < x \leq \delta_n c_n} \left| \frac{c_n}{I(x)} R(0)(x) - g_{\alpha,\beta}(0) \right| = 0. \] (95)

Substituting (91) and (95) into (67) finishes the proof.

4.3. Proof of Theorem 6. It is sufficient to show that there exists a constant \( C > 0 \) such that
\[ p_{\alpha,\beta}(\varepsilon_m) \sim C\varepsilon_m^{\alpha\rho} \text{ as } m \to \infty \] (96)
for every sequence \( \varepsilon_m \to 0 \). Since \( H(x) \) is regularly varying with index \( \alpha \rho \), there exists a sequence \( n_1(m) \to \infty \) as \( m \to \infty \) such that
\[ \sup_{n \geq n_1(m)} \frac{\varepsilon_m H(c_n)}{H(\varepsilon_m c_n)} - 1 \to 0 \text{ as } m \to \infty. \]
From this fact and Theorem 3 we deduce:

\[ c_n P(S_n \in [\varepsilon_m c_n, \varepsilon_m c_n + 1]|\tau > n) = g_{\alpha, \beta}(0) \frac{H(\varepsilon_m c_n)}{n P(\tau > n)}(1 + \varphi^{(1)}_{n,m}) \]

\[ = g_{\alpha, \beta}(0) \frac{\varepsilon_m^\alpha H(c_n)}{n P(\tau > n)}(1 + \varphi^{(2)}_{n,m}) \tag{97} \]

where, for \( i = 1, 2 \),

\[ \sup_{n \geq n_1(m)} |\varphi^{(i)}_{n,m}| \to 0 \text{ as } m \to \infty. \]

Further, according to Theorem 2,

\[ c_n P(S_n \in [\varepsilon_m c_n, \varepsilon_m c_n + 1]|\tau > n) = p_{\alpha, \beta}(\varepsilon_m) + \varphi_{n,m} \tag{98} \]

where \( \varphi_{n,m} = \varphi_{n,m}(\varepsilon_m) \to 0 \text{ as } n \to \infty \) uniformly for all possible choices of \( \varepsilon_m \), that is,

\[ \sup_{\varepsilon_m} |\varphi_{n,m}| \leq \Phi_n \text{ and } \lim_{n \to \infty} \Phi_n = 0. \tag{99} \]

Comparing (97) and (98) gives

\[ p_{\alpha, \beta}(\varepsilon_m) = g_{\alpha, \beta}(0) \frac{\varepsilon_m^\alpha H(c_n(m))}{n(m) P(\tau > n(m))}(1 + \varphi^{(2)}_{n(m),m}) - \varphi_{n(m),m} \tag{100} \]

where \( n(m) \) is any sequence satisfying \( n(m) \geq n_1(m) \) for all \( m \geq 1 \). Let \( n_2(m) \) be defined by the relation

\[ n_2(m) := \min\{n \geq 1 : \sup_{k \geq n} \Phi_n < \varepsilon_m^{\alpha + 1}\}. \]

Now, if \( n(m) \geq \max\{n_1(m), n_2(m)\} \), then from the definition of \( n_2(m) \) and (100) we have

\[ p_{\alpha, \beta}(\varepsilon_m) = g_{\alpha, \beta}(0)\varepsilon_m^\alpha \frac{H(c_n(m))}{n(m) P(\tau > n(m))}(1 + \varphi^{(2)}_{n(m),m}) + O(\varepsilon_m^{\alpha + 1}). \]

Taking into account (31), we obtain (96). The theorem is proved.

5. Proof of Theorem 7

We start with the following technical lemma which may be known from the literature.

**Lemma 23.** Let \( w(n) \) be a monotone increasing function. If, for some \( \gamma > 0 \), there exist slowly varying functions \( l^*(n) \) and \( l^{**}(n) \) such that, as \( n \to \infty \),

\[ \sum_{k=n}^\infty \frac{w(k)}{k^{\gamma + 1} l^*(k)} \sim \frac{1}{n^{\gamma l^{**}(n)}}, \]

then, as \( n \to \infty \),

\[ w(n) \sim \frac{l^*(n)}{l^{**}(n)}. \]
Proof. Let, for this lemma only, \( r_i(n), n = 1, 2, \ldots; i = 1, 2, 3, 4 \) be sequences of real numbers vanishing as \( n \to \infty \). For \( \delta \in (0, 1) \) we have by monotonicity of \( w(n) \) and properties of slowly varying functions

\[
w([\delta n]) \sum_{k=\lfloor \delta n \rfloor}^{n} \frac{1}{k^{\gamma+1}l^*(k)} = w([\delta n]) \frac{1 + r_2(n)}{\gamma n^{\gamma l^*(n)}} (\delta^{-\gamma} - 1) \\
\leq \sum_{k=\lfloor \delta n \rfloor}^{n} \frac{w(k)}{k^{\gamma+1}l^*(k)} = \frac{1 + r_1(n)}{n^{\gamma l^*(n)}} (\delta^{-\gamma} - 1) \\
\leq w(n) \sum_{k=\lfloor \delta n \rfloor}^{n} \frac{1}{k^{\gamma+1}l^*(k)} \\
= w(n) \frac{1 + r_2(n)}{\gamma n^{\gamma l^*(n)}} (\delta^{-\gamma} - 1).
\]

Hence it follows that

\[
w([\delta n]) \leq \frac{1 + r_1(n)}{1 + r_2(n)} \frac{\gamma l^*(n)}{l^{**}(n)} \leq w(n)
\]

and, therefore,

\[
\frac{1 + r_1(n)}{1 + r_2(n)} \frac{\gamma l^*(n)}{l^{**}(n)} \leq w(n) \leq \frac{1 + r_3([n\delta^{-1}])}{1 + r_4([n\delta^{-1}])} \frac{\gamma l^*([n\delta^{-1}])}{l^{**}([n\delta^{-1}])}.
\]

Since \( l^* \) and \( l^{**} \) are slowly varying functions, we get

\[
\lim_{n \to \infty} \frac{w(n)}{\gamma l^*(n)} = 1,
\]

as desired. \( \square \)

**Remark 24.** By the same arguments one can show that if \( w(x) \) is a monotone increasing function and, for some \( \gamma > 0 \), there exist slowly varying functions \( l^*(x) \) and \( l^{**}(x) \) such that, as \( x \to \infty \),

\[
\int_{x}^{\infty} \frac{w(y)dy}{y^{\gamma+1}l^*(y)} \sim \frac{1}{x^{\gamma l^*(x)}},
\]

then, as \( x \to \infty \),

\[
w(x) \sim \frac{l^*(x)}{l^{**}(x)}.
\]

Note also that this statement for the case \( l^*(x) \equiv \text{Const} \) can be found in [14] Chapter VIII, Section 9.

5.1. **Proof of Theorem** For \( \{0 < \alpha < 2, \beta < 1\} \). For a fixed \( \varepsilon \in (0, 1) \) write

\[
P(\tau^- = n) = P(S_n \leq 0; \tau^- > n - 1) =: J_1(\varepsilon c_n) + J_2(\varepsilon c_n)
\]

where

\[
J_1(\varepsilon c_n) := \int_{\varepsilon}^{\infty} P(X \leq -yc_n) P(S_{n-1} \in c_n dy; \tau^- > n - 1).
\]

and

\[
J_2(\varepsilon c_n) := \int_{0}^{\varepsilon c_n} P(X \leq -y) P(S_{n-1} \in dy; \tau^- > n - 1)
\]

First we study properties of \( J_1(\varepsilon c_n) \).
We know from (24) and (25) that if $X \in D(\alpha, \beta)$ with $0 < \alpha < 2$ and $\beta < 1$, then, for a $q \in (0, 1]$,
\[
P(X \leq -y) \sim \frac{q}{y^\alpha l_0(y)} \text{ as } y \to \infty,
\]
and, according to (27),
\[
P(X \leq -c_n) \sim \frac{q(2 - \alpha)}{\alpha n} \text{ as } n \to \infty.
\]
Moreover, for any $\varepsilon > 0$,
\[
P(X \leq -\varepsilon c_n) \to y^{-\alpha} \text{ as } n \to \infty,
\]
uniformly in $y \in (\varepsilon, \infty)$.

It easily follows from (102) and (4) that, as $n \to \infty$,
\[
J_1(\varepsilon c_n) = \int_{\varepsilon}^{\infty} \frac{P(M_{\alpha, \beta} \in dy)}{y^\alpha} \leq C \int_{0}^{1} y^{-\alpha + \rho} dy + P(M_{\alpha, \beta} > 1).
\]

From Theorem 6 follows that $p_{\alpha, \beta}(y) \leq Cy^{\alpha \rho}$ for some positive constant $C$ and all $y \in (0, 1]$. Consequently,
\[
\int_{0}^{\infty} \frac{P(M_{\alpha, \beta} \in dy)}{y^\alpha} < \infty.
\]

Therefore,
\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{\alpha n^{2-\rho}}{q(2 - \alpha)l(n)} J_1(\varepsilon c_n) = \int_{0}^{\infty} \frac{P(M_{\alpha, \beta} \in dy)}{y^\alpha}.
\]

Now to complete the proof of Theorem 7 in the case $\{0 < \alpha < 2, \beta < 1\}$ it remains to demonstrate that
\[
\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{q^{2-\rho}}{l(n)} J_2(\varepsilon c_n) = 0.
\]

To this aim we observe that
\[
J_2(\varepsilon c_n) \leq \sum_{j=0}^{\lfloor \varepsilon c_n \rfloor + 1} P(X \leq -j) b_{n-1}(j) =: R(\varepsilon c_n)
\]
and evaluate $R(\varepsilon c_n)$ separately for the following two cases:
(i) $\beta \in (-1, 1)$;
(ii) $\beta = -1$. 

(i). In view of (19), equivalences (29) and (24), we have

\[
R(\varepsilon c_n) \leq C \sum_{j=1}^{[\varepsilon c_n]+1} \frac{j^{\alpha} l_2(j)}{n c_n} \leq C_2 \frac{1}{n c_n} (\varepsilon c_n)^{1-\alpha(1-\rho)} \frac{l_2(\varepsilon c_n)}{l_0(\varepsilon c_n)} \leq C_3 \varepsilon^{1-\alpha(1-\rho) - \gamma} c_n \leq C_4 \varepsilon^{1-\alpha(1-\rho) - \gamma} H(c_n) \frac{P(|X| > n)}{n}
\]

for any fixed \( \gamma \in (0, 1 - \alpha(1 - \rho)) \) and all sufficiently large \( n \). At the third step we have applied the inequalities \( 2 \varepsilon^{1-\alpha(1-\rho) - \gamma} c_n \leq C_4 \varepsilon^{1-\alpha(1-\rho) - \gamma} H(c_n) \frac{P(|X| > n)}{n} \)

for any fixed \( \gamma \in (0, 1 - \alpha(1 - \rho)) \) and all sufficiently large \( n \). At the third step we have applied (61) to the function \( l_2(x)/l_0(x) \). Using (27) and (31), we get

\[
R(\varepsilon c_n) \leq C \varepsilon^{1-\alpha(1-\rho) - \gamma} \frac{P(\tau^+ > n)}{n}.
\]

Hence on account of (15) we conclude that

\[
R(\varepsilon c_n) \leq C \frac{l(n)}{n^{2-\rho}} \varepsilon^{1-\alpha(1-\rho) - \gamma}.
\]

(ii). It follows from (13) that if \( \beta = -1 \), then \( \alpha \rho = 1 \). By Lemma (12) \( H(x) \leq C x l_3(x) \). Combining this estimate with (49) yields

\[
b_n(j) \leq C \frac{j^{2} l_3(j)}{n c_n}.
\]

Recalling (101) and using (61), we obtain for any fixed \( \gamma \in (0, 2 - \alpha) \) and all \( n \geq n(\gamma) \),

\[
R(\varepsilon c_n) \leq C \sum_{j=0}^{[\varepsilon c_n]+1} \frac{P(X = j)}{n c_n} \frac{j^{2} l_3(j)}{n c_n} \leq C_1 (\varepsilon c_n)^{2-\alpha} \frac{l_3(\varepsilon c_n)}{n c_n l_0(\varepsilon c_n)} \leq C_2 \varepsilon^{2-\alpha - \gamma} \frac{l(n)}{n^{2-\rho}},
\]

where at the last step we have applied the inequalities \( H(c_n) \leq C c_n l_3(c_n) \leq C n^\alpha l(n) \), following from (30), (31), and (15), and the relation

\[
1 \sim n \frac{\alpha}{2 - \alpha} \frac{1}{c_n^2 l_0(c_n)}
\]

being a corollary of (26).

Estimates (106) and (107) imply (105). Combining (104) with (105) leads to

\[
P(\tau^+ = n) \sim \frac{q(2 - \alpha) l(n)}{\alpha n^{2-\rho}} \int_0^\infty \frac{P(M_{\alpha, \beta} \in dy)}{y^{\alpha}} = \frac{q(2 - \alpha) l(n)}{\alpha n^{2-\rho}} E(M_{\alpha, \beta})^{-\alpha}.
\]

Summation over \( n \) gives

\[
P(\tau^+ > n) = \sum_{k=n+1}^{\infty} P(\tau^+ = k) \sim \frac{q(2 - \alpha)}{\alpha (1 - \rho)} \frac{l(n)}{n^{1-\rho}} E(M_{\alpha, \beta})^{-\alpha}.
\]

Comparing this with (15), we get an interesting identity

\[
E(M_{\alpha, \beta})^{-\alpha} = \alpha(1 - \rho)/q(2 - \alpha)
\]

which, in view of (108), completes the proof of Theorem 7 for \( 0 < \alpha < 2, \beta < 1 \).
5.2. **Proof of Theorem 7** for \( \{1 < \alpha < 2, \beta = 1\} \cup \{\alpha = 2, \beta = 0\} \). We consider only the lattice random walks with \( a \in (0, 1) \) and \( h = 1 \). The non-lattice case requires only minor changes. The main reason for the choice of the lattice situation is the fact that only in this case we can get oscillating sequences \( Q_n^- \).

By the total probability formula,

\[
P(\tau^- = n + 1) = \sum_{k > -an} P(S_n = an + k; \tau^- > n)P(X \leq -an - k). \tag{110}
\]

One can easily verify that under the conditions imposed on the distribution of \( X \) there exists a sequence \( \delta_n \to 0 \) such that \( \delta_n c_n \to \infty \) and

\[
P(X \leq -\delta_n c_n) = o(n^{-1}) \text{ as } n \to \infty. \tag{111}
\]

Using, as earlier, the notation \( G_n = (-an, -an + \delta_n c_n) \cap \mathbb{Z} \), and combining (110) with (111), we obtain

\[
P(\tau^- = n + 1) = \sum_{k \in G_n} P(S_n = an + k; \tau^- > n)P(X \leq -an - k) + o\left(\frac{l(n)}{n^{2-\rho}}\right). \tag{112}
\]

Let \( \{an\} \) be the fractional part of \( an \). By Theorem 5

\[
P(\tau^- = n + 1) = \frac{g_{\alpha, \beta}(0) + o(1)}{nc_n} \sum_{k \in G_n} H(an + k)P(X \leq -an - k) + o\left(\frac{l(n)}{n^{2-\rho}}\right)
\]

\[
= \frac{g_{\alpha, \beta}(0) + o(1)}{nc_n} \sum_{j=0}^{\delta_n c_n} H(\{an\} + j)P(X \leq -\{an\} - j) + o\left(\frac{l(n)}{n^{2-\rho}}\right). \tag{112}
\]

For \( z \geq 0 \) set

\[
\omega(z; n) := \sum_{j=0}^{\delta_n c_n} H(z + j)P(X \leq -z - j), \quad \omega(n) := \omega(0; n),
\]

and using the equality

\[
E(-S_{\tau^-}) = \int_0^\infty H(x)P(X \leq -x)dx \tag{113}
\]

(see Doney [8]) consider the "if" part of Theorem 7 under the hypotheses of points (a), (b), and (c) separately.

(a) Condition \( E(-S_{\tau^-}) = \infty \) implies

\[
\omega(n) \to \infty \text{ as } n \to \infty. \tag{114}
\]

Since \( H(u) \) is a renewal function, there exists a constant \( C \) such that

\[
H(u + v) - H(u) \leq C(v + 1) \text{ for all } u, v \geq 0. \tag{115}
\]

By (115) and monotonicity of \( H(u) \) and \( P(X \leq -u) \) we conclude that

\[
\omega(\{an\}; n) \leq \sum_{j=0}^{\delta_n c_n} H(j + 1)P(X \leq -j) \leq \omega(n) + C \sum_{j=0}^{\delta_n c_n} P(X \leq -j)
\]
Lemma 23 implies
and

\[ \omega(\{an\}; n) \geq \sum_{j=0}^{\delta_n c_n} H(j)P(X \leq -j - 1) \geq \omega(1; n) - C \sum_{j=0}^{\delta_n c_n} P(X \leq -j) \]

\[ \geq \omega(n) - C \sum_{j=0}^{\delta_n c_n} P(X \leq -j). \]

From (114) and the fact that \( H(x) \to \infty \) as \( x \to \infty \) we deduce that

\[ \sum_{j=0}^{\delta_n c_n} P(X \leq -j) = o(\omega(n)) \text{ as } n \to \infty. \]

This yields \( \omega(\{an\}; n) \sim \omega(n) \) as \( n \to \infty \) which, combined with (112), gives

\[ P(\tau^- = n + 1) = \frac{g_{\alpha,\beta}(0) + o(1)}{nc_n} \omega(n) + o\left(\frac{l(n)}{n^{2-\rho}}\right), \quad n \to \infty. \]  

(116)

Summing over \( n \geq k \), we get, as \( k \to \infty \),

\[ \frac{l(k)}{k^{1-\rho}} \sim P(\tau^- > k) = (g_{\alpha,\beta}(0) + o(1)) \sum_{n=k}^{\infty} \frac{\omega(n)}{nc_n} + o\left(\frac{l(k)}{k^{1-\rho}}\right). \]

We know from (14) that \( \rho = 1 - 1/\alpha \) if \( \{1 < \alpha < 2, \beta = 1\} \) or \( \{\alpha = 2, \beta = 0\} \).

Since \( \omega(n) \) is non-decreasing and, by (13), \( c_n \) is regularly varying with index \( 1/\alpha \), Lemma 23 implies

\[ \frac{\omega(n)}{nc_n} \sim \frac{1 - \rho}{\alpha} \frac{l(n)}{g_{\alpha,\beta}(0) n^{2-\rho}} \text{ as } n \to \infty. \]

Consequently,

\[ P(\tau^- = n) = (1 - \rho) \frac{l(n)}{n^{2-\rho}} (1 + o(1)), \quad n \to \infty. \]

This finishes the proof of (20), given \( E(-S_{-\rho}) = \infty. \)

(b) The assumption \( E(-S_{-\rho}) < \infty \) and relations (29), (30), and (113) imply

\[ \sum_{j>\delta_n c_n} H(\{an\} + j)P(X \leq -\{an\} - j) \to 0 \text{ as } n \to \infty \]

and, consequently,

\[ \omega(\{an\}; n) = \Omega(\{an\}) + o(1) \text{ as } n \to \infty \]

where

\[ \Omega(\{an\}) := \sum_{j=0}^{\infty} H(\{an\} + j)P(X \leq -\{an\} - j) \]

Combining this representation with (112), observing that \( \Omega(\{an\}) < C < \infty \) if \( E(-S_{-\rho}) < \infty \), and recalling Lemma 13 we see that

\[ P(\tau^- = n + 1) = \frac{g_{\alpha,\beta}(0)}{nc_n} \Omega(\{an\}) + o\left(\frac{l(n)}{n^{2-\rho}}\right), \quad n \to \infty. \]  

(117)

Denote

\[ \tilde{\Omega}(\{an\}) := P(X \leq -\{an\})I(\{an\} > 0) + P(X \leq -1)I(\{an\} = 0). \]

Since \( X \) is \((1, a)-\)lattice, the quantity \( \tilde{\Omega}(\{an\}) \) is either 0 or not less than some positive number \( \tilde{\Omega}_*. \) Furthermore, one can easily verify that \( \Omega(\{an\}) \geq \tilde{\Omega}(\{an\}) \) and \( \Omega(\{an\}) = 0 \) if and only if \( \tilde{\Omega}(\{an\}) = 0. \) Consequently, \( \Omega(\{an\}) \) is either zero or
not less than $\tilde{\Omega}$. Finally, in view of $\Omega(\{an\}) = 0$ implies $P(\tau^- = n + 1) = 0$. Therefore, we can rewrite (117) in the form

$$P(\tau^- = n + 1) = \frac{g_{a,\beta}(0)}{\chi n} \Omega(\{an\})(1 + o(1)).$$

(118)

Now (118) and (37) give (20) with

$$Q^-_n := C_0 g_{a,\beta}(0) \Omega(\{an \cdot (n - 1)\}).$$

(119)

If $a = 0$, then, evidently,

$$Q_n^- = C_0 g_{a,\beta}(0) \Omega(0) = C_0 g_{a,\beta}(0) \chi(-S_{\tau^-}) := Q,$$

and, consequently,

$$P(\tau^- = n) = Q \frac{i(n)}{n^{2-\rho}} (1 + o(1)).$$

Comparing this asymptotic equality with the known tail behavior of the distribution of $\tau^-$, we infer that $Q$ should be equal to $1 - \rho$.

This finishes the proof of (20) under the conditions of point (b).

To demonstrate the validity of (20) under the conditions of point (c) one should made only evident minor changes of the just used arguments and we omit the respective details.

To justify the "only if" part of Theorem 7 we need to show that the sequence $\Omega(\{an\}, n \geq 1)$ defined in (119) does not converge if $E(-S_{\tau^-}) < \infty$ and $X$ is $(1, a)$-lattice with some $a \in (0, 1)$.

Assume first that $a$ is rational, i.e. $a = i/j$ for some $1 \leq i < j < \infty$ with g.c.d. $(i, j) = 1$. Let $b = b(a)$ be the smallest natural number satisfying $\{ab\} = 1 - a$. Then $\{a(kj + b)\} = 1 - a$ for all $k \geq 1$. Consequently,

$$\Omega(\{a(kj + b)\}) = \sum_{m=0}^{\infty} H((1-a)+m)P(X \leq -(1-a)-m)$$

and

$$\Omega(\{akj\}) = \sum_{m=0}^{\infty} H(m)P(X \leq -m).$$

Observing that $P(X \leq -m) = P(X \leq -(1-a)-m)$, we obtain

$$\Omega(\{a(kj + b)\}) - \Omega(\{akj\}) = \sum_{m=0}^{\infty} (H((1-a)+m) - H(m))P(X \leq -(1-a)-m)$$

$$\geq (H(1-a) - H(0))P(X \leq -(1-a))$$

$$= H(1-a)P(X < 0)$$

$$> P(X < 0).$$

From this inequality it follows that the sequence $\{\Omega(\{an\}), n \geq 1\}$, does not converge.

Assume now that $a$ is irrational. Define $\mathcal{N}_1 := \{n : \{an\} < (1-a)/3\}$ and $\mathcal{N}_2 := \{n : \{an\} \in 2(1-a)/3, (1-a)\}$. The cardinality of each of the sets is infinite. In addition, one can easily verify that

$$\Omega(\{an_2\}) - \Omega(\{an_2\}) \geq (H(2(1-a)/3) - H((1-a)/3))P(X < 0)$$

$$\geq P(X < 0)P(X \in ((1-a)/3, 2(1-a)/3)) > 0$$
for all \( n_1 \in \mathcal{N}_1 \) and \( n_2 \in \mathcal{N}_2 \). Therefore, in the case of irrational shift the sequence \( \Omega(\{an\}) \), \( n \geq 1 \), is oscillating as well.

Theorem 7 is proved.

**Remark 25.** Analysing the proof of Theorem 7 one can see that the sequence \( \{Q_n, n \geq 1\} \) in (20) may be written in the form

\[ Q_n := D(\{a(n - 1)\}), \]

where \( D(x), 0 \leq x < 1 \), is a nonnegative function and where we agree to take \( a = 0 \) for non-lattice distributions.

6. DISCUSSION AND CONCLUDING REMARKS

We see by (11) that the distribution of \( \tau^- \) is completely specified by the sequence \( \{\mathbf{P}(S_n > 0), n \geq 1\} \). As we have mentioned in the introduction, the validity of condition (16) is sufficient to reveal the asymptotic behavior of \( \mathbf{P}(\tau^- > n) \) as \( n \to \infty \). Thus, in view of (15), informal arguments based on the plausible smoothness of \( l(n) \) immediately give the desired answer

\[
\mathbf{P}(\tau^- = n) = \mathbf{P}(\tau^- > n - 1) - \mathbf{P}(\tau^- > n)
\]

\[
= \frac{l(n - 1)}{(n - 1)^{1-\rho}} - \frac{l(n)}{n^{1-\rho}} \approx l(n) \left( \frac{1}{(n - 1)^{1-\rho}} - \frac{1}{n^{1-\rho}} \right)
\]

\[
\approx \frac{(1-\rho)l(n)}{n^{2-\rho}} \sim \frac{1-\rho}{n} \mathbf{P}(\tau^- > n)
\]

under the Doney condition only. In the present paper we failed to achieve such a generality. However, it is worth mentioning that the Doney condition, being formally weaker than the conditions of Theorem 7, requires in the general case the knowledge of the behavior of the whole sequence \( \{\mathbf{P}(S_n > 0), n \geq 1\} \), while the assumptions of Theorem 7 concern a single summand only. Of course, imposing a stronger condition makes our life easier and allows us to give, in a sense, a constructive proof showing what happens in reality at the distant moment \( \tau^- \) of the first jump of the random walk in question below zero. Indeed, our arguments for the case \( \{0 < \alpha < 2, \beta < 1\} \) demonstrate (compare (101), (102), and (103)) that for any \( x_2 > x_1 > 0 \),

\[
\lim_{n \to \infty} \mathbf{P}(S_{n-1} \in (c_n x_1, c_n x_2) | \tau^- = n)
\]

\[
= \lim_{n \to \infty} \frac{\mathbf{P}(\tau^- > n - 1)}{\mathbf{P}(\tau^- = n)} \int_{x_1}^{x_2} \mathbf{P}(X < -yc_n) \mathbf{P}(S_{n-1} \in c_n dy | \tau^- > n - 1)
\]

\[
= \lim_{n \to \infty} \frac{\mathbf{P}(\tau^- > n - 1)}{\mathbf{P}(\tau^- = n)} \frac{q(2-\alpha)}{\alpha(1-\rho)} \int_{x_1}^{x_2} \mathbf{P}(X < -yc_n) \mathbf{P}(S_{n-1} \in c_n dy | \tau^- > n - 1)
\]

\[
= \frac{q(2-\alpha)}{\alpha(1-\rho)} \int_{x_1}^{x_2} \frac{\mathbf{P}(M_{\alpha,\beta} \in dy)}{y^\alpha}.
\]

In view of (109) this means that the contribution of the trajectories of the random walk satisfying \( S_{n-1}c_n^{-1} \to 0 \) or \( S_{n-1}c_n^{-1} \to \infty \) as \( n \to \infty \) to the event \( \{\tau^- = n\} \) is negligibly small in probability. A "typical" trajectory looks in this case as follows: it is located over the level zero up to moment \( n - 1 \) with \( S_{n-1} \in (\varepsilon c_n, \varepsilon^{-1} c_n) \) for sufficiently small \( \varepsilon > 0 \) and at moment \( \tau^- = n \) the trajectory makes a big negative jump \( X_n < -S_{n-1} \) of order \( O(c_n) \).
On the other hand, if \( \{1 < \alpha < 2, \beta = 1\} \) and \( \mathbb{E}(-S_{\tau^{-}}) < \infty \), then, in the \((1, a)\)-lattice case, for all \( i \geq 0 \),
\[
P(S_{n-1} = \{a(n-1)\} + i | \tau^{-} = n) = \frac{H(\{a(n-1)\} + i)P(X \leq -a(n-1) - i)}{\Omega(\{a(n-1)\})} (1 + o(1))
\]
promised that \( \Omega(\{a(n-1)\}) > 0 \). Since
\[
\sum_{i=0}^{\infty} H(\{a(n-1)\} + i)P(X \leq -a(n-1) - i) = \Omega(\{a(n-1)\}),
\]
the main contribution to \( P(\tau^{-} = n) \) is given in this case by the trajectories located over the level zero up to moment \( n - 1 \) with \( S_{n-1} \in [0, N] \) for sufficiently big \( N \) and with not "too big" jump \( X_n < -S_{n-1} \) of order \( O(1) \).

Unfortunately, our approach to investigate the behavior of \( P(\tau^{-} = n) \) in the case when \( \mathbb{E}(-S_{\tau^{-}}) = \infty \) and \( \{1 < \alpha < 2, \beta = 1\} \cup \{\alpha = 2, \beta = 0\} \) is pure analytical and does not allow us to extract typical trajectories without further restrictions on the distribution of \( X \). However, we can still deduce from our proof some properties of the random walk conditioned on \( \{\tau^{-} = n\} \). Observe that, for any fixed \( \varepsilon > 0 \), the trajectories with \( S_{n-1} > \varepsilon c_n \) give no essential contribution to \( P(\tau^{-} = n) \). More precisely, there exists a sequence \( \delta_n \to 0 \) such that
\[
P(S_{n-1} > \delta_n c_n | \tau^{-} = n) = o(1).
\]
Furthermore, one can easily verify that if \( \sum_{j=1}^{\infty} H(j)P(X \leq -j) = \infty \), then for every \( N \geq 1 \),
\[
\sum_{j=1}^{N} P(S_{n-1} = j; \tau^{-} > n - 1)P(X \leq -j) = o\left(\frac{l(n)}{n^{3/2}}\right) \quad \text{as} \quad n \to \infty,
\]
i.e., the contribution of the trajectories with \( S_{n-1} = O(1) \) to \( P(\tau^{-} = n) \) is negligible small. As a result we see that \( S_{n-1} \to \infty \) but \( S_{n-1} = o(c_n) \) for all "typical" trajectories meeting the condition \( \{\tau^{-} = n\} \). Thus, in the case \( \{1 < \alpha < 2, \beta = 1\} \cap \{\mathbb{E}(-S_{\tau^{-}}) = \infty\} \) we have a kind of "continuous transition" between the different strategies for \( \{\beta < 1\} \) and \( \{1 < \alpha < 2, \beta = 1\} \cap \{\mathbb{E}(-S_{\tau^{-}}) < \infty\} \).

Acknowledgement. The first version of the paper was based on the preprint [21]. We are thankful to an anonymous referee who attracted our attention to the fact that by our methods one can prove not only local theorems [7] and [10] but the Gnedenko and Stone type conditional local theorems [25] as well. V.W. is thankful to Anatoly Mogulskii for simulating discussions on ladder epochs.

This project was started during the visits of the first author to the Weierstrass Institute in Berlin and the second author to the Steklov Mathematical Institute in Moscow. The hospitality of the both institutes is greatly acknowledged.

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