STOCHASTIC DYNAMICS OF THE FITZHUGH-NAGUMO NEURON MODEL THROUGH A MODIFIED VAN DER POL EQUATION WITH FRACTIONAL-ORDER TERM AND GAUSSIAN WHITE NOISE EXCITATION

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Abstract. The stochastic response of the FitzHugh-Nagumo model is addressed using a modified Van der Pol (VDP) equation with fractional-order derivative and Gaussian white noise excitation. Via the generalized harmonic balance method, the term related to fractional derivative is splitted into the equivalent quasi-linear dissipative force and quasi-linear restoring force, leading to an equivalent VDP equation without fractional derivative. The analytical solutions for the equivalent stochastic equation are then investigated through

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the stochastic averaging method. This is thereafter compared to numerical solutions, where the stationary probability density function (PDF) of amplitude and joint PDF of displacement and velocity are used to characterized the dynamical behaviors of the system. A satisfactory agreement is found between the two approaches, which confirms the accuracy of the used analytical method. It is also found that changing the fractional-order parameter and the intensity of the Gaussian white noise induces P-bifurcation.

1. Introduction. Nonlinear systems in neurobiology have become a fascinating research direction over the past few years, the main goal being to understand and characterize various dynamical behaviors of brain activities. A general view, from experimental and theoretical studies, is that brain oscillations are excellent candidates in conveying information from the brain to organs and may adopt a broad range of features related to either normal or pathological behaviors. Particularly, exploring dynamical signatures of neuronal electrical activities might be of importance in understanding specific neural functions, and differentiating pathological from normal activities. Most of the mathematical formulations of the models capable of describing neural rhythms rely on dynamical systems, represented by coupled nonlinear differential equations (ODEs) \[22, 23\]. Seminal works in that context confirm the efficiency of the Hodgkin and Huxley (HH) model \[20\] and its simplified versions proposed by Hindmarsh and Rose \[18, 19\], and FitzHug \[15, 16\] and Nagumo (FHN) \[27\]. Particularly, the FHN model gives rise to a very interesting tradeoff between theory and experiment. Moreover, under the effect of external stimuli, neuronal activities may exhibit complex dynamical evolution, mainly related to variations in intra-membrane current. Combined external and parametric excitations were already introduced in the FHN model, where Tatchim et al. \[40\] detected different dynamical oscillatory behaviors related to the change in excitation frequencies. Along the same line, Guckenheimer and Kuehn \[17\] investigated homoclinic orbits of the FHN equations and further addressed the existence of canards explosions, including mixed-mode oscillations. Stochastically excited FHN equations have also been considered recently in the work of Tatchim et al. \[41\] and Bashkirtseva and Ryashko \[4\]. In the first contribution, a bistable modified FHN model was considered in the presence of two stable fixed points, separated by one unstable fixed point \[41\]. In the second work, computational method using the sensitivity function method was used to study the excitability of stochastically forced FHN equations \[4\]. More recently, Tabi \[38\] derived a modified Van der Pol (VDP) equation from a parametrically forced FHN system and proposed its generalized version by introducing a VDP fractional-order term. It was shown that the fractional-order derivative could importantly affect the various dynamical behaviors inherent to VDP oscillators. In the presence of fractional derivatives, depending on the type of excitation, authors have been using the averaging method \[38, 8, 9, 33, 32, 34\] and the stochastic averaging method (SAM) \[30\]. The averaging method is usually applied to model equations in the presence of periodic external or parametric excitations \[38, 8, 9, 33, 32, 34\]. In this case, the fractional-order derivative can be summarized in terms of additional damping and stiffness coefficients \[38, 8, 9, 33, 35\]. Shen et al. applied this method to a linear oscillator \[34\] and a Duffing oscillator \[35\] in the presence of fractional-order derivative. In Ref. \[38\], the same method was used to characterize the fractional-order effect on the dynamics of the VDP equation derived from the FHN model. On the other hand, the SAM can be useful in reducing the dimension of stochastic dynamical
systems. When Fractional-order derivatives are considered, the SAM takes advantage of the fact that the term related to the fractional derivative serves both as the classical damping and the restoring forces [8, 9]. Analytical solutions to various stochastic problems have been obtained using this technique. We may refer to Yang et al. [45, 46] who used the SAM to study the stationary response of a nonlinear system under Gaussian white noise excitation and Caputo-type fractional derivative [6]. Huang and Jin [21] made use of the SAM to study the stability and stochastic response of a single-degree-of-freedom (SDOF) stochastic model in the presence of both fractional derivative and Gaussian white noise excitation. Yang and Xu [46] also adopted the SAM to study the stochastic response of stochastic system endowed with Caputo-type fractional derivative damping under white noise excitation. The aim of the present work is to also make use of the SAM to characterize the dynamical comportments of the FHN model under external white noise excitation. We first of all reduce the FHN model to a SDOF equation and generalize it by including the VDP fractional term and external Gaussian white noise excitation. The fractional VDP equation is then reduced to its equivalent stochastic VDP version without fractional derivative term, via the generalized harmonic function method [8]. Analytical solutions are thereafter investigated using the SAM. Through the predictor-corrector algorithm, numerical solutions are also investigated and compared to the analytical ones, via the probability density functions (PDFs) of amplitude, displacement and velocity. Some concluding remarks are finally given.

2. Model and equivalent modified VDP equation.

2.1. Model. The FHN equations have been intensively used to model a broad range of biological systems including the dynamics of the heart [1] and the propagation of calcium waves in living cells [36, 7, 43]. It is in fact the simplest formulation of the HH mathematical model which considers two variables: $\chi(t)$, the cell membrane potential (action potential), and $\eta(t)$, the cell membrane current. In the absence of any input current, it is given by the following set of coupled equations:

$$\frac{d\chi(t)}{dt} = f(\chi(t)) - \eta(t), \quad \frac{d\eta(t)}{dt} = \tau_1 \chi(t) - \tau_2 \eta(t), \quad (1)$$

where $\tau_1$ and $\tau_2$ are constant parameters. The nonlinear function $f(\chi) = \xi_1 \chi + \xi_2 \chi^2 + \xi_3 \chi^3$ models the ionic current, with $\xi_1$, $\xi_2$ and $\xi_3$ being constant parameters. The modified VDP equation can be obtained by eliminating the transmembrane currents $\eta$ from the system. This is then summarized by

$$\ddot{\chi} + \omega^2 \chi + \eta_0 (1 + \gamma_1 \chi + \gamma_2 \chi^2) \dot{\chi} + \delta_1 \chi^2 + \delta_2 \chi^3 = 0, \quad (2)$$

where $\eta_0$, $\gamma_1$, $\gamma_2$, $\delta_1$, $\delta_2$ are constant parameters, and $\omega^2$ the natural frequency of the action potential oscillations. As said so far, the external excitation that we consider in the present work is assumed to be stochastic, probably due to channel noise and other intrinsic noise sources. Denoted by $f(t)$, its statistical characteristics are such that $\langle f(t) \rangle = 0$ and $\langle f(t) f(t') \rangle = 2D \delta(t - t')$. Moreover, in Eq.(2), the VDP character of the model equation comes from the term $\eta_0 (1 + \gamma_1 \chi + \gamma_2 \chi^2) \dot{\chi}$, which in order to be generalized may contain a fractional-order derivative. All these considerations lead to the following generalized VDP equation

$$\ddot{\chi} + \omega^2 \chi + \eta_0 (1 + \gamma_1 \chi + \gamma_2 \chi^2) D^p[\chi(t)] + \delta_1 \chi^2 + \delta_2 \chi^3 = f(t), \quad (3)$$
where $D^\alpha[\chi(t)]$ is the $\alpha$-order fractional derivative of $\chi(t)$ with respect to time, $\eta_0$ is the fractional coefficient and $0(0 \leq \alpha \leq 1)$ is the fractional-order parameter. The fractional-order derivative has several definitions among which the Grunwald-Letnikov [29, 31], Riemann-Liouville [29], Caputo [29, 5] and Atangana-Baleanu [2] definitions. Moreover, recent works have developed new directions in fractional derivative operators and using Mittag-Leffler functions. For example, a new definition of fractional derivative, with a smooth kernel which considers two different representations for the temporal and spatial variable, was proposed [6]. Also, a new definition of fractional derivative called the new Riemann-Liouville fractional derivative without singular kernel was introduced by Goufo and Atangana [13]. The Korteweg-de Vries-Burgers was modified using the Caputo-Fabrizio fractional derivative with no singular kernel [12]. More recently, chaotic poles of attraction were studied in the Hindmarsh-Rose model in presence of fractional-order derivatives, using the Haar wavelet technique [14]. The one proposed by Caputo has been widely studied in the literature and summarizes fractional-order derivatives in the form [6, 3, 39]

$$D^\alpha[\chi(t)] = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\chi'(u)}{(t-u)^\alpha} du,$$

(4)

where $\Gamma(y)$ is the Gamma function satisfying $\Gamma(y+1) = y\Gamma(y)$. To remind, Eq.(3) has been derived from system (1) without any approximation, which justifies the other name of the VDP Bonhoeffer model. Treutlein and Schulten [42] were the first to study a stochastic version of the FHN model. This gave rise to other multiple contributions in the same context, with emphasis on stochastic resonance [26, 44, 10, 28]. However, this was in the absence of fractional-order derivative.

2.2. Equivalent modified stochastic VDP equation. The term associated with fractional derivatives brings about classical damping force and also includes restoring forces. It was shown recently, using the generalized harmonic function method, that it can be replaced by [8, 9, 35]

$$D^\alpha \chi(t) = C(\alpha)\dot{\chi}(t) + K(\alpha)\chi(t),$$

(5)

where $C(\alpha)$ and $K(\alpha)$ can be obtained using the general formulae

$$\lim_{T \to \infty} \int_0^T \frac{\sin(\omega t)}{t^\alpha} dt = \omega^{\alpha-1}\Gamma(1-\alpha) \cos \left(\frac{\alpha\pi}{2}\right),$$

$$\lim_{T \to \infty} \int_0^T \frac{\cos(\omega t)}{t^\alpha} dt = \omega^{\alpha-1}\Gamma(1-\alpha) \sin \left(\frac{\alpha\pi}{2}\right).$$

(6)

Trial solutions for Eq. (5) are considered to be

$$X = \chi(t) = A(t) \cos \phi'(t) \quad \text{and} \quad Y = \dot{\chi}(t) = -A(t)\omega \sin \phi'(t),$$

(7)

with $\phi'(t) = \omega t + \theta'(t)$. We then obtain that $D^\alpha[A \cos \phi'] = -C(\alpha)A\omega \sin \phi' + K(\alpha)A \cos \phi'$. The terms $\sin \phi'$ and $\cos \phi'$ are called the first and second harmonics.
functions. We have to find $C(\alpha)$ in the form

$$C(\alpha) = -\frac{1}{\pi A\omega} \int_0^{2\pi} D^\alpha (A \cos \phi') \sin \phi' d\phi'$$

$$= -\frac{1}{\pi A\omega} \lim_{T \to \infty} \frac{2}{T} \int_0^T D^\alpha (A \cos \phi') \sin \phi' d\phi'$$

$$= -\frac{2}{A\omega \Gamma(1-\alpha)} \lim_{T \to \infty} \frac{1}{T} \int_0^T \left\{ \left[ \int_0^t -A\omega \sin(\omega u + \theta') \frac{du}{(t-u)^\alpha} \right] \sin(\omega t + \theta') \right\} dt$$

$$= \frac{2}{\Gamma(1-\alpha)} \lim_{T \to \infty} \frac{1}{T} \int_0^T \left\{ \left[ \int_0^t \sin(\omega u + \theta') \frac{du}{(t-u)^\alpha} \right] \sin(\omega t + \theta') \right\} dt \quad (8)$$

which, by making use of the change of coordinates $s = t-u$ and $ds = -du$, becomes

$$C(\alpha) = \frac{2}{\Gamma(1-\alpha)} \lim_{T \to \infty} \frac{1}{T} \int_0^T \left\{ \left[ \int_0^t \frac{\sin(\omega s + \theta') \cos(\omega t + \theta')}{s^\alpha} ds \right] \sin(\omega t + \theta') \right\} dt$$

$$= \frac{2}{\Gamma(1-\alpha)} \lim_{T \to \infty} \frac{1}{T} \int_0^T \left\{ \left[ \int_0^t \frac{\cos(\omega s) ds}{s^\alpha} \right] \sin^2(\omega t + \theta') \right\} dt$$

$$- \frac{2}{\Gamma(1-\alpha)} \lim_{T \to \infty} \frac{1}{T} \int_0^T \left\{ \left[ \int_0^t \frac{\sin(\omega s) ds}{s^\alpha} \right] \sin(\omega t + \theta') \cos(\omega t + \theta') \right\} dt$$

$$= C_1 + C_2. \quad (9)$$

Thanks to integration by part, the first term of the above integral $C_1$ is obtained in the form

$$C_1 = \frac{2}{\Gamma(1-\alpha)} \lim_{T \to \infty} \frac{1}{T} \int_0^T \left\{ \left[ \int_0^t \frac{\cos(\omega s) ds}{s^\alpha} \right] \sin^2(\omega t + \theta') \right\} dt$$

$$= \frac{2}{\Gamma(1-\alpha)} \lim_{T \to \infty} \left[ \left( \frac{2\omega t - \sin(2\omega t + 2\theta')}{4\omega T} \right) \left( \int_0^t \frac{\cos(\omega s) ds}{s^\alpha} \right) \right]_0^T$$

$$- \frac{2}{\Gamma(1-\alpha)} \lim_{T \to \infty} \frac{1}{T} \int_0^T \left( \frac{2\omega t - \sin(2\omega t + 2\theta')}{4\omega} \right) \left( \frac{\cos(\omega t)}{t^\alpha} \right) dt$$

$$= C_{11} + C_{12}. \quad (10)$$

where $C_{11}$ is given by

$$C_{11} = \frac{2}{\Gamma(1-\alpha)} \lim_{T \to \infty} \left[ \left( \frac{2\omega t - \sin(2\omega t + 2\theta')}{4\omega T} \right) \left( \int_0^t \frac{\cos(\omega s) ds}{s^\alpha} \right) \right]_0^T$$

$$= \frac{2}{\Gamma(1-\alpha)} \omega^{\alpha-1} \Gamma(1-\alpha) \sin \left( \frac{\alpha \pi}{2} \right) \lim_{T \to \infty} \left( \frac{2\omega t - \sin(2\omega t + 2\theta')}{4\omega T} \right)$$

$$= 2\omega^{\alpha-1} \sin \left( \frac{\alpha \pi}{2} \right) \lim_{T \to \infty} \left[ \frac{1}{2} \left( 1 - \sin(2\omega t + 2\theta') \right) \right]$$

$$= \omega^{\alpha-1} \sin \left( \frac{\alpha \pi}{2} \right) \quad (11)$$

and the second part, $C_{12}$, given by

$$C_{12} = -\frac{2}{\Gamma(1-\alpha)} \lim_{T \to \infty} \frac{1}{T} \int_0^T \left( \frac{2\omega t - \sin(2\omega t + 2\theta')}{4\omega} \right) \left( \frac{\cos(\omega t)}{t^\alpha} \right) dt = 0 \quad (12)$$
The second part of integral (9), \( C_2 \), is such that
\[
C_2 = -\frac{2}{\Gamma(1-\alpha)} \lim_{T \to \infty} \frac{1}{T} \int_0^T \left\{ \int_0^t \frac{\sin(\omega s)}{s^\alpha} ds \right\} \sin(\omega t + \theta') \cos(\omega t + \theta') \, dt
\]
\[
= -\frac{1}{\Gamma(1-\alpha)} \lim_{T \to \infty} \frac{1}{T} \int_0^T \sin(2\omega t + 2\theta') \int_0^t \frac{\sin(\omega s)}{s^\alpha} ds \, dt
\]
\[
= -\frac{1}{2\omega \Gamma(1-\alpha)} \lim_{T \to \infty} \frac{1}{T} \int_0^T \left\{ \int_0^t \frac{\sin(\omega s)}{s^\alpha} ds \int_0^T \sin(2\omega t + 2\theta') \, dt \right\} \sin(\omega t) \, dt
\]
\[
= C_{21} + C_{22},
\]
(13)
where \( C_{21} \), the first part of (13), is given by
\[
C_{21} = -\frac{1}{2\omega \Gamma(1-\alpha)} \lim_{T \to \infty} \frac{1}{T} \int_0^T \left[ \cos(2\omega t + 2\theta') \int_0^t \frac{\sin(\omega s)}{s^\alpha} ds \right]^T_0
\]
\[
= -\frac{1}{2\omega \Gamma(1-\alpha)} \lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{\cos(2\omega t + 2\theta')}{T} \int_0^T \frac{\sin(\omega s)}{s^\alpha} ds \, dt
\]
\[
= -\frac{1}{\Gamma(1-\alpha)} \lim_{T \to \infty} \frac{1}{2\omega T} \int_0^T \frac{\cos(2\omega t + 2\theta')}{T} \int_0^T \frac{\sin(\omega s)}{s^\alpha} ds = 0
\]
(14)
Along the same line, the second part of (14), \( C_{22} \), is such that
\[
C_{22} = -\frac{1}{2\omega \Gamma(1-\alpha)} \lim_{T \to \infty} \frac{1}{T} \int_0^T \left[ \int_0^T \cos(2\omega t + 2\theta') \frac{\sin(\omega t)}{t^\alpha} \right] \, dt = 0.
\]
(15)
One finally obtains
\[
C(\alpha) = C_1 + C_2 = \omega^{\alpha-1} \sin \left( \frac{\alpha \pi}{2} \right).
\]
(16)
The coefficient \( K(\alpha) \) of Eq.(5) is given by
\[
K(\alpha) = \frac{1}{\pi A} \int_0^{2\pi} D^\alpha (A \cos \phi') \cos \phi' \, d\phi'
\]
\[
= \frac{1}{A} \lim_{T \to \infty} \frac{2}{T} \int_0^T D^\alpha (A \cos \phi') \cos \phi' \, d\phi'
\]
\[
= \frac{2}{\alpha T \Gamma(1-\alpha)} \lim_{T \to \infty} \frac{1}{T} \int_0^T \left\{ \int_0^t \frac{\sin(\omega u + \theta')}{(t-u)^\alpha} du \right\} \cos(\omega t + \theta') \, dt
\]
\[
= \frac{2\omega}{\alpha T \Gamma(1-\alpha)} \lim_{T \to \infty} \frac{1}{T} \int_0^T \left\{ \int_0^t \frac{\sin(\omega u + \theta')}{(t-u)^\alpha} du \right\} \cos(\omega t + \theta') \, dt.
\]
(17)
The above procedure, used to calculate \( C(\alpha) \), can be adopted here, which leads to
\[
K(\alpha) = \omega^\alpha \cos \left( \frac{\alpha \pi}{2} \right).
\]
(18)
The fractional derivative in Eq.(5) then writes
\[
D^\alpha \chi(t) = C(\alpha) \chi(t) + K(\alpha) \chi(t)
\]
\[
= \omega^{\alpha-1} \sin \left( \frac{\alpha \pi}{2} \right) \chi(t) + \omega^\alpha \cos \left( \frac{\alpha \pi}{2} \right) \chi(t).
\]
(19)
Reporting this into (3) leads to the following stochastic equation without fractional derivative

\[ \dddot{\chi} + \left[ \eta_0 \omega^{\alpha - 1} (1 + \gamma_1 \chi + \gamma_2 \chi^2) \sin \left( \frac{\alpha \pi}{2} \right) \right] \ddot{\chi} + \omega_0^2 \chi + \eta_1 \chi^2 + \eta_2 \chi^3 = f(t), \quad (20) \]

with

\[ \omega_0^2 = \left[ \omega^2 + \eta_0 \omega^\alpha \cos \left( \frac{\alpha \pi}{2} \right) \right], \quad \eta_1 = \left[ \delta_1 + \eta_0 \gamma_1 \omega^\alpha \cos \left( \frac{\alpha \pi}{2} \right) \right], \quad \eta_2 = \left[ \delta_2 + \eta_0 \gamma_2 \omega^\alpha \cos \left( \frac{\alpha \pi}{2} \right) \right]. \quad (21) \]

Eq. (20) represents a generalized form of the stochastic VDP equations that have already been obtained in the literature. However, it includes higher-order nonlinear terms which, to the best of our knowledge, have not been considered in previous contributions. This equation also includes the fractional-order parameter \( \alpha \) whose effect will be studied in the rest of this paper.

![Figure 1](image.png)

**Figure 1.** The panel shows the stationary probability density function (PDF) of the amplitude for different values of the fractional-order parameter \( \alpha \). Solid lines correspond to analytical results, while symbol (\( \triangle \)) corresponds to results from numerical calculations.

3. **Stochastic averaging method and the response of the system.** The application of the SAM to Eq. (20) requires that the joint response process \((\chi, \dot{\chi})\) be transformed into a pair slowly varying processes. Consequently, solutions for Eq. (20) are assumed as

\[ X = \chi(t) = A(t) \cos \phi(t), \]
\[ Y = \dot{\chi}(t) = -A(t) \omega_0 \sin \phi(t), \]
\[ \phi(t) = \omega_0 t + \theta, \]
\[ U(\chi) = \int_0^\chi g(u)du = \frac{1}{2} \omega_0^2 \chi^2 + \frac{1}{3} \eta_1 \chi^3 + \frac{1}{4} \eta_2 \chi^4, \quad (22) \]
with \( A, \phi \) and \( \theta \) being random processes. Differentiating the above with respect to time leads to

\[
\dot{A} \cos \phi - A \dot{\theta} \sin \phi = 0. \tag{23}
\]

Based on this, the equations for the amplitude \( A \) and the phase angle \( \theta \) are obtained from Eq. (20) as

\[
\begin{align*}
\frac{dA}{dt} & = F_{11}(A, \theta) + F_{12}(A, \theta) + G_1(A, \theta) f(t), \\
\frac{d\theta}{dt} & = F_{21}(A, \theta) + F_{22}(A, \theta) + G_2(A, \theta) f(t),
\end{align*} \tag{24}
\]

where

\[
\begin{align*}
F_{11}(A, \theta) & = -A \sin^2 \phi \left[ \eta_0 \omega - 1 + \gamma_1 (A \cos \phi) + \gamma_2 (A \cos \phi)^2 \right] \sin \left( \frac{\alpha \pi}{2} \right), \\
F_{12}(A, \theta) & = \frac{\sin \phi (1 + \gamma_1 (A \cos \phi)^2 + \gamma_2 (A \cos \phi)^3)}{\xi_0}, \\
F_{21}(A, \theta) & = -\sin \phi \cos \phi \left[ \eta_0 \omega - 1 + \gamma_1 (A \cos \phi) + \gamma_2 (A \cos \phi)^2 \right] \sin \left( \frac{\alpha \pi}{2} \right), \\
F_{22}(A, \theta) & = \frac{\cos \phi (1 + \gamma_1 (A \cos \phi)^2 + \gamma_2 (A \cos \phi)^3)}{\xi_0}, \\
G_1(A, \theta) & = -\frac{\sin \phi}{\xi_0}, \quad G_2(A, \theta) = -\frac{\cos \phi}{\xi_0}. \tag{25}
\end{align*}
\]

Since the average Itô equation for \( A(t) \) is independent of \( \theta(t) \), the limiting process \( A(t) \) is one-dimensional diffusion process governed by the following Itô equation

\[
dA = m(A) dt + \sigma(A) dB(t), \tag{26}
\]

in which \( m(A) \) and \( \sigma(A) \) are the drift and diffusion coefficients, respectively given by

\[
m(A) = \left\langle F_{11} + F_{12} + D \frac{\partial G_1}{\partial A} G_1 + D \frac{\partial G_1}{\partial \theta} G_2 \right\rangle \phi \quad \text{and} \quad \sigma^2(A) = \left\langle 2DG_1^2 \right\rangle \phi. \tag{27}
\]

Here, \( \langle \cdot \rangle \phi \) denotes the averaging operation with respect to \( \phi \). Some few calculations give expressions for Eqs. (27) as

\[
m(A) = -\eta_0 \omega - 1 \sin \left( \frac{\alpha \pi}{2} \right) \left[ \frac{A^2}{2} + \frac{A^3 \gamma_2}{8} \right] + \frac{D}{2 \omega_0^2 A} \quad \text{and} \quad \sigma^2(A) = \frac{D}{\omega_0}. \tag{28}
\]

where

\[
\omega_0^2 = \left[ \omega^2 + \eta_0 \omega - 1 \sin \left( \frac{\alpha \pi}{2} \right) \right]. \tag{29}
\]

Knowing the drift and diffusion coefficients, the Fokker-Planck-Kolmogorov (FPK) equation related to Eq. (22) is written as

\[
\frac{\partial p}{\partial t} = -\frac{\partial}{\partial A} [m(A)p] + \frac{1}{2} \frac{\partial^2}{\partial A^2} [\sigma^2(A)p]. \tag{30}
\]

The boundary conditions of FPK equation with respect to \( A \) are \( p = p_0 \) for \( A = 0 \), and \( p \to 0, \partial p/\partial A \to 0 \), when \( A \to \infty \). Using such boundary conditions, one finds
Figure 2. The panels show the joint PDF of the displacement $X$ and velocity $Y$, corresponding to Fig. 1, for different values of the fractional-order parameter $\alpha$. Panels (aj)$_{j=1,2,3}$ correspond to our analytical calculations, while their corresponding panels (bj)$_{j=1,2,3}$ are obtained from numerical simulations: (a1)-(b1) $\alpha = 0.8$, (a2)-(b2) $\alpha = 0.6$ and (a3)-(b3) $\alpha = 0.3$.

The following stationary solution for (30)

\[ p(A) = \frac{C_0}{\sigma^2(A)} \exp \left[ \int_0^A \frac{2m(u)}{\sigma^2(u)} \, du \right] \]

\[ = \frac{C_0 \omega_0^2 A}{D} \exp \left[ -2\omega_0^2 \eta_0 \frac{A^2}{D\omega_1^{1-\alpha}} \sin \left( \frac{\alpha \pi}{2} \right) \left( \frac{A^2}{4} + \frac{\gamma_2 A^4}{32} \right) \right], \quad (31) \]

where $C_0$ is the normalization constant. The stationary density of the system Hamiltonian $H = U(A) = \frac{1}{2} \omega_0^2 A^2 + \frac{1}{3} \eta_1 A^3 + \frac{1}{4} \eta_2 A^4$ can be obtained from Eq.(31) as

\[ p(H) = p(A) \left| \frac{dA}{dH} \right| = \frac{C_0 \omega_0^2}{D(\omega_0^2 + \eta_1 A + \eta_2 A^2)} \]

\[ \times \exp \left[ -2\omega_0^2 \eta_0 \frac{A^2}{D\omega_1^{1-\alpha}} \sin \left( \frac{\alpha \pi}{2} \right) \left( \frac{A^2}{4} + \frac{\gamma_2 A^4}{32} \right) \right] \bigg|_{A=U^{-1}(H)} \quad (32) \]

The joint stationary probability density of displacement and velocity is finally
Figure 3. The panel shows the stationary probability density function (PDF) of the amplitude for different values of the fractional-order parameter $\alpha$. Solid lines correspond to analytical results, while symbol ($\triangle$) corresponds to results from numerical calculations.

derived from Eq.(32) using the relation

$$p(X, Y) = \frac{p(H)}{T(H)} \bigg|_{H = \frac{1}{2} Y^2 + U(\chi)},$$

with $T(H) = 2\pi/\omega_0$. We must precise that the results presented in this part are purely analytical and their accuracy needs to be verified using direct numerical simulations of Eq.(3).

4. Results and discussion. In order to confirm our analytical predictions, Eq.(3) should be integrated numerically using a suitable algorithm that applies to fractional-order differential equations with suitable initial condition [11]. In so doing, the equation

$$D^\alpha X(t) = f(t, X(t)), \quad \text{with} \quad X(0) = X_0,$$

(34)

can be solved numerically using the predictor-corrector iteration formula given by

$$X_h(t_{i+1}) = X_0 + \frac{h^\alpha}{\Gamma(\alpha + 2)} f(t_{i+1}, X_h^p(t_{i+1})) + \frac{h^\alpha}{\Gamma(\alpha + 2)} \sum_{j=0}^{i} a_{j,i+1} f(t_j, X_h(t_j)),$$

(35)

where the weight of the corrector $a_{j,i+1}$ is defined as

$$a_{j,i+1} = \begin{cases} 
\frac{1^{\alpha+1} - (i - \alpha)(i + 1)^\alpha}{\Gamma(\alpha + 2)}, & j = 0, \\
\frac{(i - j + 2)^{\alpha+1} + (i - j)^{\alpha+1} - 2(i - j + 1)^{\alpha+1}}{\Gamma(\alpha + 2)}, & 1 \leq j \leq i, \\
1, & j = i + 1
\end{cases}$$

(36)

In Eq.(35), the quantity $X_h^p(t_{i+1})$ represents the predicted value, governed by the equation

$$X_h^p(t_{i+1}) = X_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{i} b_{j,i+1} f(t_j, X_h(t_j)),$$

(37)
with $b_{j,i+1}$ being the weight of the predictor that is expressed as

$$b_{j,i+1} = \frac{h^\alpha}{\alpha}[(i-j+1)^\alpha - (i-j)^\alpha].$$

(38)

Using the above scheme and the analytical formulae of the stationary probability densities of amplitude $p(A)$, the effect of the fractional-order parameter on the later is, for example, displayed in Fig. 1. The solid lines show $p(A)$ as obtained from formula (31), while curves with $(\Delta)$ result from the Monte Carlo simulations of Eq.(3). Interestingly, the collected results are in agreement with each other, which shows the accuracy of the adopted analytical method of calculation. Moreover, one notices that $p(A)$ is very sensitive to the change in $\alpha$, since the probability of large amplitude oscillations increases with decreasing $\alpha$. However, the accuracy of the results depends on the fractional parameter $\alpha$. In other terms, the agreement of analytical results with those obtained from simulations is much better for small $\alpha$, i.e., $\alpha = 0.3$, for example. This implies that small values for the fractional parameter increase the probability of the action potential emission. The same accuracy between numerical and analytical results is obvious in Fig. 2, where we have plotted $p(X,Y)$, with panels $a_{j})_{j=1,2,3}$ showing analytical results and $b_{j})_{j=1,2,3}$ displaying numerical calculation. We should stress that the results displayed in Fig. 2 correspond to what has been plotted in Fig. 1, which explains why the effect of $\alpha$ remains the same. Such results are also in agreement with what was obtained by Yang et al. [45] in the case of the VDP equation with fractional-order derivative and stochastic excitation. However, the dynamics might be enriched due to the strong nonlinear terms in Eq.(3), as well as its modified fractional VDP term.

The joint effect of the intensity of Gaussian white noise and fractional-order parameter is studied in Figs. 3 and 4. First, in Fig. 3(a), we have fixed $D = 0.1$ and, beyond the perfect agreement between analytical and numerical results, we see how the probability increases with decreasing $\alpha$. The same is obvious for Fig. 3(b) and (c), where we took respectively $D = 0.05$ and $D = 0.01$. In the meantime, one observes that smaller random intensity reduces the probability of oscillations with large amplitude. This satisfactorily agrees with the results of Fig. 4, where we have numerically plotted $p(X,Y)$. Nevertheless, the change in $D$ importantly affects the shape of the joint PDF here. Compared to the PDF of Fig. 2, where $D = 0.15$, the results of Fig. 4 indicate that when $D$ decreases the joint PDF migrates from a singular peak to a crater. This otherwise supports the occurrence of the so-called $P$–bifurcation. This also picture the change in action potential oscillation which gets modified when $D$ changes, as already predicted by Treutlein and Schulten [42] who simulated the stationary probability density of the FHN model. They concluded that crater-like shapes of the density, as those observed in Fig. 4, may correspond to oscillatory behaviors due to medium-sized noise intensity. Along the same line, Kostur et al. [24] proposed the numerical solution of the FPK equation for the FHN model with external noise excitation and proposed that crater-like shapes of the PDF may better be used to explain the occurrence of coherent resonance. This takes place above limit cycles of self-sustained oscillations, and was designated as stochastic limit cycles by Lindner et al. [25]. In the meantime, there appear more interesting behaviors related to the change in fractional-order parameter $\alpha$. The diameter of the crater-like shapes increases when $\alpha$ decreases. Indeed such a behavior has been observed in contributions directly related to the VDP oscillators, not related to biological systems, but applied to engineering. One
Figure 4. The panels show numerical results for the joint PDF displacement $X$ and velocity $Y$. The lines form top to bottom correspond to different values of $D$, the Gaussian white noise intensity: (aj) $j=1,2,3$, $D = 0.1$, (bj) $j=1,2,3$, $D = 0.05$ and (cj) $j=1,2,3$, $D = 0.01$. The columns from left to right respectively correspond to $\alpha = 0.8$, $\alpha = 0.6$ and $\alpha = 0.3$.

should for example refer to works by Shen et al. [33, 32, 34], including the references therein.

5. Conclusion. The paper was devoted to the study of the stochastic response of a generalized FHN model, in the presence of fractional-order derivative and Gaussian white noise excitation. We have first shown that the model, made of two coupled nonlinear ODEs, can be fully reduced to a modified VDP equation. Using the harmonic balance method, we have shown that the fractional-order derivative can be simplified into the equivalent quasi-linear dissipative force and quasi-linear restoring force. Based on this, an equivalent stochastic VDP equation without fractional
derivative has been derived, followed by the investigation of its analytical solutions with the help of the stochastic averaging method. Specifically, expressions for the stationary PDF of amplitude and joint PDF for displacement and velocity have been derived. In order to confirm the accuracy of the analytical predictions, numerical calculations, using the Monte-Carlo method, have been performed. Satisfactory agreement between the two approaches has been obtained. The results reveal in fact that probability densities are very sensitive to the change in both the fractional-order parameter and the noise intensity. For the first parameter, the stationary probability densities $p(A)$ and $p(X,Y)$ increases with decreasing $\alpha$. Decreasing $D$ tends to have the same effect, but the right choice of values for $\alpha$ increases the probability densities. More importantly, higher values of $D$ give rise to the so-called $P-$bifurcation. This proves the existence of different modes of oscillations and further confirms the strong relationship between the modified VDP and the FHN model.

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