EMERGENT DYNAMICS OF A THERMODYNAMIC CUCKER-SMALE ENSEMBLE ON COMPLETE RIEMANNIAN MANIFOLDS

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Abstract. We study emergent collective behaviors of a thermodynamic Cucker-Smale (TCS) ensemble on complete smooth Riemannian manifolds. For this, we extend the TCS model on the Euclidean space to a complete smooth Riemannian manifold by adopting the work [30] for a CS ensemble, and provide a sufficient framework to achieve velocity alignment and thermal equilibrium. Compared to the model proposed in [30], our model has an extra thermodynamic observable denoted by temperature, which is assumed to be nonidentical for each particle. However, for isothermal case, our model reduces to the previous CS model in [30] on a manifold in small velocity regime. As a concrete example, we study emergent dynamics of the TCS model on the unit $d$-sphere $S^d$. We show that the asymptotic emergent dynamics of the proposed TCS model on the unit $d$-sphere exhibits a dichotomy, either convergence to zero velocity or asymptotic approach toward a common great circle. We also provide several numerical examples illustrating the aforementioned dichotomy on the asymptotic dynamics of the TCS particles on $S^2$.

1. Introduction. Collective behaviors of complex systems are often observed in our daily life, e.g., flashing of fireflies [5, 52], synchronization of biological cells [43], flocking of migratory birds [7, 50, 51], swarming of fish [49] and consensus of opinions [42], etc. We refer to survey articles and books [1, 2, 25, 43, 53, 55] for a detailed introduction on the subject. Among collective behaviors, we are

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concerned with flocking behaviors in which particles move with a common velocity using simple rules based on environmental information. After seminal works \cite{44, 51} on the flocking modeling by Reynolds and Vicsek et al., several mathematical models have been used in the study of flocking modeling. The CS model is one of the most successful one which has been extensively studied as a prototype mechanical model describing flocking phenomena in Euclidean space. It was first proposed in \cite{11}, and has been studied by researchers in applied mathematics community \cite{25, 31, 34, 39, 40} right after their work. Especially, in a recent work \cite{30}, the CS model was further generalized to a flocking model on complete Riemannian manifolds, and its emergent dynamics and large-time behavior were also studied. In literature \cite{3, 18, 20, 37, 38, 47, 48}, several first-order aggregation models on Riemannian manifolds were proposed, and their aggregation estimates were also investigated for specific Riemannian manifolds, e.g., the unit sphere \cite{19, 36, 38, 41, 54, 56}, matrix groups \cite{6, 12, 13, 28, 29, 36}, Lie group \cite{46}, hyperbolic space \cite{21, 45}, etc. We also refer to \cite{22, 23} for second-order models on the unit sphere and unitary matrix group.

As another direction of generalization, an adaptation of thermodynamics to the CS model was also considered by Ha and Ruggeri \cite{32} from the multi-temperature Eulerian fluid model under the assumptions of space homogeneity, Galilean invariance and entropy principle. In fact, the TCS model was first originated from the dynamics of mixture of $N$ fluids (ideal gases) in $\mathbb{R}^d$. More precisely, we consider the balanced equations of $N$ constituent fluids:

$$
\begin{align*}
&\frac{\partial \rho_\alpha}{\partial t} + \text{div}(\rho_\alpha v_\alpha) = \tau_\alpha, \quad (x, t) \in \mathbb{R}^d \times \mathbb{R}^+, \quad \alpha = 1, \ldots, N, \\
&\frac{\partial (\rho_\alpha v_\alpha)}{\partial t} + \text{div}(\rho_\alpha v_\alpha \otimes v_\alpha - t_\alpha) = m_\alpha, \\
&\frac{\partial (\frac{1}{2} \rho_\alpha |v_\alpha|^2 + \rho_\alpha \varepsilon_\alpha)}{\partial t} + \text{div} \left\{ \left( \frac{1}{2} \rho_\alpha |v_\alpha|^2 + \rho_\alpha \varepsilon_\alpha \right) v_\alpha - t_\alpha v_\alpha + q_\alpha \right\} = e_\alpha,
\end{align*}
$$

where $\varepsilon_\alpha, q_\alpha, t_\alpha$ are the internal energy, heat flux and stress tensor of the constituent $\alpha$, respectively. In particular, for spatially homogeneous mixture satisfying the relations:

$$
\rho_\alpha \equiv 1, \quad v_\alpha(x, t) \equiv v_\alpha(t), \quad \varepsilon_\alpha(x, t) = \varepsilon_\alpha(t),
$$

convection terms which are the second terms in the L.H.S. of (1) vanish. Then, we impose the following conditions to (1):

- (No chemical reactions and zero total momentum due to Galilean invariance):

  $$
  \tau_\alpha = 0, \quad \sum_\alpha \rho_\alpha v_\alpha = \sum_\alpha v_\alpha = 0.
  $$

- (Entropy principle): for $i = 1, \ldots, N - 1$,

  $$
  m_i = -\frac{1}{N} \sum_{j=1}^{N-1} \phi_{ij} \left( \frac{v_N}{T_N} - \frac{v_j}{T_j} \right), \quad e_i = -\frac{1}{N} \sum_{j=1}^{N-1} \zeta_{ij} \left( \frac{1}{T_j} - \frac{1}{T_N} \right),
  $$

- (Internal energy of ideal gas)

  $$
  \varepsilon_\alpha \propto T_\alpha,
  $$

to derive the TCS model on $\mathbb{R}^d$.

Inspired by the above works, we are mainly interested in the collective modeling of thermodynamic CS particles on Riemannian manifolds.
Next, we briefly describe the original TCS model in [32]. Let \( x_i = x_i(t), v_i = v_i(t) \) and \( T_i = T_i(t) \) be the position, velocity and temperature of the \( i \)-th particle in \( \mathbb{R}^d \) at time \( t \), respectively. Then, the TCS model on Euclidean space in the center of mass frame takes the following form:

\[
\begin{align*}
\frac{dx_i}{dt} &= v_i, \quad t > 0, \quad i = 1, \ldots, N, \\
\frac{dv_i}{dt} &= \frac{\kappa_1}{N} \sum_{j=1}^{N} \phi(x_i, x_j) \left( \frac{v_j}{T_j} - \frac{v_i}{T_i} \right), \\
\frac{dT_i}{dt} &= \frac{\kappa_2}{N} \sum_{j=1}^{N} \zeta(x_i, x_j) \left( 1\over T_i - 1\over T_j \right),
\end{align*}
\]

(2)

where \( \kappa_1 \) and \( \kappa_2 \) are positive coupling strengths. The TCS model (2) has been extensively investigated from various aspects. For example, emergence of flocking [15, 16, 28, 31] over various network topologies, interplay between time-delay and network topology [17], passage from discrete TCS to continuous TCS [14], singular interaction kernels [10], kinetic description and mean-field limit [24] and coupling with fluids [8, 9], etc.

In this paper, we propose an extension of the TCS model to a Riemannian manifold setting as in [30] and study its emergent dynamics. More precisely, we consider a connected, complete and smooth \( d \)-dimensional Riemannian manifold \( M \) without a boundary and its metric tensor \( g \). Then, our manifold extension of (2) on \( (M, g) \) is given as

\[
\begin{align*}
\dot{x}_i &= v_i, \quad t > 0, \quad i = 1, \ldots, N, \\
\nabla v_i &= \frac{\kappa_1}{N} \sum_{j=1}^{N} \phi(x_i, x_j) \left( \frac{P_{ij}v_j}{T_j} - \frac{v_i}{T_i} \right), \\
\frac{dT_i}{dt} &= \frac{\kappa_2}{N} \sum_{j=1}^{N} \zeta(x_i, x_j) \left( 1\over T_i - 1\over T_j \right),
\end{align*}
\]

(3)

where \( \dot{x}_i \) denotes the tangent vector of the curve \( t \mapsto x_i(t) \), \( \nabla \) is the Levi-Civita connection compatible with \( (M, g) \), \( \|v_i\|_x \) is the length of the vector \( v_i \in T_x M \) with respect to the metric tensor \( g(\cdot, \cdot) \), and \( P_{ij} \) is the parallel transport along the length minimizing geodesic from \( x_j \) to \( x_i \) (see Section 2.1 for basic terminology). For a well-posedness of system (3), we further assume that communication weight functions \( \phi \) and \( \zeta \) are nonnegative bounded symmetric smooth functions on \( M \times M \), and moreover the mapping

\[
\{(x_k, v_k)\}_{k=1}^{N} \mapsto (x_i, \phi(x_i, x_j) (P_{ij}v_j - v_i)) \in TM
\]

is smooth for all \( i, j = 1, \ldots, N \). To do this, it is necessary to impose the vanishing condition \( \phi(x, y) = 0 \) whenever there are more than one length minimizing geodesics between \( x \) and \( y \). Therefore, there exist two upper bounds \( \phi_M \) and \( \zeta_M \) such that

\[
0 \leq \phi(x, y) = \phi(y, x) \leq \phi_M, \quad 0 \leq \zeta(x, y) = \zeta(y, x) \leq \zeta_M, \quad \forall \ x, y \in M,
\]

and

\[
\phi(x, y) = 0, \quad \text{for every } x \in M \quad \text{and} \quad y \in \text{Cut}(x).
\]

Here, Cut(\( x \)) denotes the cut locus of the point \( x \) in \( M \), which is equivalent to the closure of the set of points \( y \) that has more than one length minimizing geodesics from \( y \) to \( x \). In this way, one can think (3) as an ODE of local coordinates: for each
i = 1, \ldots, N, consider a local coordinate chart \((U_i, \varphi_i)\) of \(N\) open sets \(U_1, \ldots, U_N\), and denote \(\varphi^j_i\) as the \(j\)-th coordinate of the function \(\varphi_i:\)

\[
\varphi^j_i : U_i \to \mathbb{R}, \quad \varphi_i(q) = (\varphi^1_i(q), \ldots, \varphi^d_i(q)) \quad \forall \ q \in U_i.
\]

Then, the local coordinate chart is given by

\[
\Phi = \Phi_1 \times \cdots \times \Phi_N : \pi^{-1}(U_1) \times \cdots \times \pi^{-1}(U_1) \to (\varphi_1(U_1) \times \mathbb{R}^d) \times \cdots \times (\varphi_N(U_N) \times \mathbb{R}^d),
\]

where each \(\Phi_i\) gives a local coordinates on \(TM:\)

\[
\Phi_i(q, u) := (\varphi^1_i(q), \ldots, \varphi^d_i(q), d\varphi_{iq}(u), \ldots, d\varphi_{iq}(u)), \quad \forall \ (q, u) \in \pi^{-1}(U_i).
\]

Then, we have

\[
\begin{align*}
\dot{x}^k_i &= v^k_i, \quad \forall \ 1 \leq i \leq N, \ 1 \leq k \leq d, \\
\dot{v}^k_i + \nabla^k \cdot v^m_i v^n_i &= dp^k_i (\nabla v_i) = dp^k_i \left[ \frac{\kappa_1^2}{N} \sum_{j=1}^{N} \phi(x_i, x_j) \left( \frac{P_{ij} v_j}{T_j} - \frac{v_i}{T_i} \right) \right], \\
T_i + g_{jk} v_i^j (\dot{v}^k_i + \Gamma_{mn} v^m_i v^n_i) &= \frac{\kappa_2}{N} \sum_{j=1}^{N} \zeta(x_i, x_j) \left( \frac{1}{T_i} - \frac{1}{T_j} \right),
\end{align*}
\]

where \((x^k_i)_{ik}\) and \((v^k_i)_{ik}\) are given as

\[
\Phi(x_1, v_1, \ldots, x_N, v_N) = \left( (x^1_1, \ldots, x^d_1, v^1_1, \ldots, v^d_1), \ldots, (x^1_N, \ldots, x^d_N, v^1_N, \ldots, v^d_N) \right).
\]

The local well-posedness of (4) can be obtained when all initial temperatures are strictly positive, and then the global well-posedness also follows, provided that all temperatures have a positive lower bound and all velocities have a finite upper bound (see Lemma 3.1 and Lemma 3.2).

Meanwhile, if we assume small diffusion velocities \(\{v_i\}^N_{i=1}\) as in [26, 27], we may approximate (3) via the following simplified system:

\[
\begin{align*}
\dot{x}_i &= v_i, \quad t > 0, \quad i = 1, \ldots, N, \\
\nabla_n v_i &= \frac{\kappa_1}{N} \sum_{j=1}^{N} \phi(x_i, x_j) \left( \frac{P_{ij} v_j}{T_j} - \frac{v_i}{T_i} \right), \\
\frac{dT_i}{dt} &= \frac{\kappa_2}{N} \sum_{j=1}^{N} \zeta(x_i, x_j) \left( \frac{1}{T_i} - \frac{1}{T_j} \right).
\end{align*}
\]

Note that \(\zeta(\cdot, \cdot)\) in (3) or (5) does not have any vanishing condition as \(\phi\), which means that we may allow a positive lower bound of \(\zeta(\cdot, \cdot)\) without any violation of local well-posedness of system (5). For instance, it is even possible to set \(\zeta\) as a positive constant.

Throughout the paper, we call system (3) and (5) as a manifold TCS model and simplified manifold TCS model on \((M, g)\), respectively. Then, we define asymptotic states of TCS type models (3) - (5) on Riemannian manifold \((M, g)\) as follows.

**Definition 1.1.** Let \(C := \{(x_i, v_i, T_i)\}^N_{i=1}\) be a global smooth solution to (3) or (5).

1. (Velocity alignment): The configuration \(C\) exhibits (asymptotic) velocity alignment if

\[
\lim_{t \to \infty} \|P_{ij} v_j(t) - v_i(t)\|_{x_i(t)} = 0, \quad \forall \ i, j = 1, \ldots, N.
\]
2. (Thermal equilibrium): The configuration $C$ exhibits (asymptotic) thermal equilibrium if
\[\lim_{t \to \infty} |T_i(t) - T_j(t)| = 0, \quad \forall \, i, j = 1, \ldots, N.\]

The main results of this paper are three-fold. First, we derive a global well-posedness of (3) and (5). For this, we derive total energy conservation, entropy principle and find a positive lower bound for temperatures for (3) in terms of initial data. For the simplified manifold TCS model (5), we show the monotone decreasing and increasing properties of maximal and minimal temperatures along the flow, and then derive an exponential emergence of thermal equilibrium. Second, we show the emergence of asymptotic velocity alignment under a priori uniform continuity assumption or compactness assumption of ambient manifold. For the velocity alignment of (3), we use the convergence of entropy, whereas we use the convergence of entropy and kinetic energy for the simplified manifold TCS model (5) (see Section 2 for the definition of entropy and kinetic energy). Finally, we also show that the solution of (3) or (5) on $S^d$ exhibits either the zero convergence of kinetic energy or all particles approach toward a great circle (see Theorem 5.2).

The rest of this paper is organized as follows. In Section 2, we will briefly review basic background materials on Riemannian geometry, the TCS model on a Euclidean space $\mathbb{R}^d$ and its emergent behavior. Then, we also introduce an adaptation of the CS model on Riemannian manifolds and compare with our model (see Remark 1). In Section 3, we study emergent behavior of the TCS model on a Riemannian manifold (3). In Section 4, we study emergent behavior of simplified TCS model on a Riemannian manifold (5). In Section 5, we study detailed asymptotic dynamics of TCS particles on the unit sphere $S^d$. In Section 6, we provide several numerical simulations to check the analytical results in Section 5. Finally, Section 7 is devoted to a brief summary of our main results.

Notation: For simplicity, we use some handy notation:
\[
\max_i := \max_{1 \leq i \leq N}, \quad \max_{i,j} := \max_{1 \leq i,j \leq N}, \quad \sum_i := \sum_{i=1}^{N}, \quad \sum_{i,j} := \sum_{i=1}^{N} \sum_{j=1}^{N},
\]
and if there is no confusion, we also use the Einstein’s notation on the repeated indices from time to time.

2. Preliminaries. In this section, we review several terminologies and background materials related to Riemannian geometry, the TCS model on $\mathbb{R}^d$ and the manifold CS model.

2.1. Preparation on Riemannian geometry. Let $M$ be a $d$-dimensional smooth manifold. For each point $p \in M$, let $C^\infty(p)$ be the germs of $C^\infty$ functions at $p$, and the tangent space of $M$, denoted by $T_pM$, is the set of all $\mathbb{R}$-linear mappings $X_p : C^\infty(p) \to \mathbb{R}$ satisfying the Leibniz rule:
\[X_p(fg) = (X_p f) \cdot g(p) + f(p) \cdot (X_p g), \quad \forall f, g \in C^\infty(p).\]
If so, the set $T_pM$ can be regarded as a vector space for the addition $+$ and scalar multiplication:
\[(X_p + Y_p) := X_p + Y_p f, \quad (\alpha X_p)f := \alpha (X_p f), \quad \forall \alpha \in \mathbb{R}, \ f \in C^\infty(p),\]
and we say its dual $T_p^*M$ as a cotangent space of $M$ at $p$. Especially when $E$ is an open subset of Euclidean space $\mathbb{R}^d$ with usual inner product $\langle \cdot, \cdot \rangle$, we define $\frac{\partial}{\partial x^j} \big|_x \in T_x E$ as the unique linear mapping such that the evaluation of each smooth function $f$ is given by its $j$-th partial derivative:

$$\left( \frac{\partial}{\partial x^j} \big|_x \right) f = \frac{\partial f}{\partial x^j}(x), \quad \forall \ f \in C^\infty(x).$$

Then, $\left\{ \frac{\partial}{\partial x^j} \big|_x, \cdots, \frac{\partial}{\partial x^d} \big|_x \right\}$ indeed forms a basis of $T_x E$, and for any open neighborhood $U(\subset M)$ of $p$ with local coordinate chart $\varphi : U \rightarrow \varphi(U)$, we define a $j$-th derivation $\partial_j|_p$ as

$$\langle \partial_j|_p \rangle(f) := \frac{\partial (f \circ \varphi^{-1})}{\partial x^j}(\varphi(p)), \quad \forall \ f \in C^\infty(p),$$

so that $\{ \partial_1|_p, \cdots, \partial_d|_p \}$ forms a basis of $T_p M$. Moreover, we denote the unique dual basis of $\{ \partial_1|_p, \cdots, \partial_d|_p \}$ in $T_p^* M$ as $\{ dx^1|_p, \cdots, dx^d|_p \}$, which consists of $d$ independent 1-forms satisfying

$$dx^i(\partial_j) = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j). \end{cases}$$

If there is a smooth mapping $\gamma : I \rightarrow M$ for some nonempty connected set $I \subset \mathbb{R}$, we call $\gamma$ as a (smooth) curve on $M$. For every smooth curve $\gamma : (t - \varepsilon, t + \varepsilon) \rightarrow M$ with $\gamma(t) = p$, we call $\dot{\gamma}(t) \in T_p M$ as a tangent vector of $\gamma$ at $p$, where $\dot{\gamma}(t) : C^\infty(p) \rightarrow \mathbb{R}$ is a linear mapping satisfying

$$\langle \dot{\gamma}(t) \rangle f := \left. \frac{d}{ds} \right|_{s=0} f(\gamma(t + s)), \quad \forall f \in C^\infty(p).$$

Then, the tangent space $T_p M$ is indeed equivalent to the set of all such tangent vectors $\dot{\gamma}(t)$:

$$T_p M = \left\{ \dot{\gamma}(t) \mid \gamma : (t - \varepsilon, t + \varepsilon) \rightarrow M \text{ is smooth}, \quad \gamma(t) = p \right\}.$$

The tangent bundle $TM$ and cotangent bundle $T^* M$ are the union of all tangent and cotangent spaces, respectively:

$$TM := \{(p,v) \mid p \in M, \ v \in T_p M\} \quad \text{and} \quad T^* M := \{(p,df) \mid p \in M, \ df \in T_p^* M\}.$$

In fact, the tangent bundle $TM$ can also be regarded as a manifold by considering the following local coordinate charts: for each point $p \in M$ and local coordinate $(U, \varphi)$ of a neighborhood of $p$, the corresponding local coordinate chart of $TM$ is given by $(\pi^{-1}(U), \Phi)$, where

$$\pi^{-1}(U) = \{(q,u) \mid q \in U, \ u \in T_q M\}, \quad \Phi(q,u) = (\varphi(q), d\varphi_q(u)).$$

We say $X : M \rightarrow TM$ is a (smooth) tangent vector field if $p \mapsto X_p f$ is smooth for every smooth function $f$ on $M$, and denote $\mathcal{X}(M)$ as the set of all smooth tangent vector fields on $M$.

Now, for a given smooth manifold $M$, we denote a smooth affine connection $\nabla$ on $M$ as an $\mathbb{R}$-bilinear map:

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M)$$

$$(X,Y) \mapsto \nabla_X Y,$$

satisfying the following rules:

$$\nabla_{fX} Y = f \nabla_X Y, \quad \nabla_X (fY) = f \nabla_X Y + (Xf)Y, \quad \forall f \in C^\infty(M), \ X,Y \in \mathcal{X}(M).$$
Then for given local coordinates \((U, \varphi)\), the derivations \(\{\partial_j\}_{j=1}^d\) can be regarded as a locally defined tangent vector field which gives a basis for each tangent space, and we set 
\[
\nabla_{\partial_i} \partial_j =: \Gamma_{ij}^k \partial_k, \quad \forall \ i, j = 1, \cdots, d.
\]
The coefficients \(\Gamma_{ij}^k\) are called as the Christoffel symbols of the second kind.

With the help of Christoffel symbols, the vector \(\nabla_{\dot{\gamma}} Y\) can be defined on given smooth curve \(\gamma: (-\varepsilon, \varepsilon) \to M\) for a tangent vector field \(Y\) defined on \(\gamma\). If a vector field \(Y\) on a curve \(\gamma: [0, T] \to M\) satisfies
\[
\nabla_{\dot{\gamma}(t)} Y = 0, \quad t \in (0, T),
\]
then the vector field \(Y\) is called a parallel vector field. In local coordinates \((U, \varphi)\), this is equivalent to
\[
dy^k + \Gamma_{ij}^k \frac{d\gamma^i}{dt} y^j = 0 \quad \text{for all } k,
\]
where \(y^i\) and \(\gamma^i\) are the \(i\)-th coordinate of \(Y\) and \(\varphi \circ \gamma\), respectively:
\[
Y = y^1 \partial_1 + \cdots + y^d \partial_d, \quad \varphi \circ \gamma = (\gamma^1, \ldots, \gamma^d).
\]
Therefore, for each \((p, v) \in TM\) and a curve \(\gamma : [0, T] \to M\) with \(\gamma(0) = p\), there exists a unique parallel tangent vector field \(Y\) on \(\gamma\) satisfying \(Y(p) = v\), as a consequence of the well-posedness of first order linear ODE with smooth coefficients. Then, \(Y\) is called a parallel transport of \(v\) along \(\gamma\), and this parallel transport mapping \(P: v \mapsto P v := Y(\gamma(t))\) is linear. In particular, if a tangent vector field of the given curve \(\gamma\) is parallel to itself:
\[
\nabla_{\dot{\gamma}(t)} \dot{\gamma} = 0, \quad t \in (0, T), \tag{7}
\]
the curve \(\gamma\) is called an affine geodesic of a smooth manifold \(M\). Again, in a local coordinate chart \((U, \varphi)\), a curve \(\gamma : [a, b] \to U\) satisfies the affine geodesic condition (7) if the local coordinate \((\gamma^1(t), \ldots, \gamma^d(t)) := \varphi(\gamma(t))\) satisfies
\[
\frac{d^2 \gamma^k}{dt^2} + \Gamma_{ij}^k \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} = 0. \tag{8}
\]
The existence and uniqueness of affine geodesic for each \((p, v) \in TM\) (i.e., a geodesic \(\gamma\) satisfying \(\gamma(0) = p, \dot{\gamma}(0) = v\)) can be obtained locally from the local well-posedness of the nonlinear second-order ODE (8), though the existence may not be guaranteed globally in \(t\).

Now, for a smooth manifold \(M\), we call \(g = \{g_p\}_{p \in M}\) as a metric tensor of \(M\) if 
\[
g_p : T_p M \times T_p M \to \mathbb{R}
\]
is a symmetric, positive definite and bilinear map for each \(p \in M\), and \(p \mapsto g_p(X_p, Y_p)\) is smooth for every \(X, Y \in \mathcal{X}(M)\). When a smooth manifold \(M\) attains such a metric tensor \(g\), we call \((M, g)\) as a Riemannian manifold. For simplicity, we also write 
\[
\begin{cases}
\langle v, w \rangle_p := g_p(v, w), & \|v\|_p := \sqrt{g_p(v, v)}, \quad \forall \ p \in M \ \text{and} \ v, w \in T_p M, \\
g(X, Y) := \text{A map } p \mapsto g_p(X_p, Y_p), & g_{ij} := g(\partial_i, \partial_j), \quad \forall \ i, j = 1, \cdots, d.
\end{cases}
\]
Then, if given affine connection \(\nabla\) on \((M, g)\) satisfies 
\[
\begin{align*}
(1) \ X(g(Y, Z)) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z), \\
(2) \ \nabla_X Y - \nabla_Y X &= XY - YX, \tag{9}
\end{align*}
\]
we say $\nabla$ is a Levi-Civita connection of Riemannian manifold $(M, g)$. The Levi-Civita connection $\nabla$ of given Riemannian manifold $(M, g)$ is known to be unique, and the Christoffel symbols of $\nabla$ can be completely determined from the metric tensor $g$ as below:

$$\Gamma^i_{kl} := \frac{1}{2} g^{im} (\partial_l g_{mk} + \partial_k g_{ml} - \partial_m g_{kl}), \quad \forall i, k, l = 1, \ldots, d.$$ 

Here, $(g^{ij})$ is the inverse of the positive definite matrix $(g_{ij})$.

According to (9)(1), the parallel transport map preserves the angles between two tangent vectors:

$$\nabla_{\dot{\gamma}} X = \nabla_{\dot{\gamma}} Y = 0 \implies (\dot{\gamma})(g(X, Y)) = 0 \implies g(X, Y) \text{ is constant along } \gamma,$$

and in particular $g(X, X)$ is constant along $\gamma$, whenever $X$ is parallel vector field on $\gamma$. Moreover, whenever the corresponding connection $\nabla$ satisfies (9)(1), the affine geodesic $\gamma$ is locally length-minimizing, where the length of a curve $\omega : [\alpha, \beta] \to M$ is defined as

$$\ell(\omega) := \int_{\alpha}^{\beta} \sqrt{g_{\omega(t)}(\dot{\omega}(t), \dot{\omega}(t))} dt.$$

In fact, for any Riemannian manifold $(M, g)$, one can define a metric $d(\cdot, \cdot)$ as a minimal curve length connecting two points:

$$d(x, y) := \inf \left\{ \ell(\omega) : \omega(\alpha) = x, \omega(\beta) = y, \omega \text{ piecewise smooth} \right\}$$

gives a metric structure on $(M, g)$, and this metric topology coincides with the topology of $M$ itself. If an affine geodesic $\gamma_v$ of $(M, g)$ with $\gamma_v(0) = p, \gamma_v(0) = v$ is well defined at least $0 \leq t \leq 1$, we denote $\exp_p v := \gamma_v(1)$ as an exponential map.

Finally, we state the Hopf-Rinow theorem, which provides a sufficient condition to guarantee that the exponential map is well-defined on the whole domain $TM$.

**Proposition 1.** (Hopf-Rinow) [35] A connected, smooth Riemannian manifold $(M, g)$ is topologically complete if and only if the exponential map $\exp_p v$ is well-defined for any $(p, v) \in TM$, and this implies the existence of geodesics (possibly not unique) connecting any two points $x, y$ on $(M, g)$.

As a corollary, for a connected, smooth and complete Riemannian manifold $(M, g)$, any two points $x, y$ on $(M, g)$ admit at least one minimizer $\gamma$ of length $\ell(\gamma)$ among the set

$$\{ \omega(\alpha) = x, \omega(\beta) = y, \omega \text{ is piecewise smooth curve on } M, \alpha, \beta \in \mathbb{R} \},$$

and this $\gamma$ is one of the geodesics joining $x$ and $y$, which is therefore smooth. We call this $\gamma$ as a length-minimizing geodesic joining $x$ and $y$.

### 2.2. The Euclidean TCS model

In this subsection, we briefly review the TCS model on $\mathbb{R}^d$ introduced in [26, 32]. Let $x_i, v_i$ and $T_i$ be the position, velocity and temperature of the $i$-th TCS particle, respectively. Then, the dynamics of TCS
particles is given by the following first-order ODE system:

\[
\begin{aligned}
\frac{dx_i}{dt} &= v_i, \quad x_i \in \mathbb{R}^d, \quad t > 0, \quad i = 1, \ldots, N, \\
\frac{dv_i}{dt} &= \frac{\kappa_1}{N} \sum_j \phi(\|x_i - x_j\|) \left( \frac{v_j - v}{T_j} - \frac{v_i - v}{T_i} \right), \\
\frac{dT_i}{dt} &= \frac{\kappa_2}{N} \sum_j \zeta(\|x_i - x_j\|) \left( \frac{1}{T_i} - \frac{1}{T_j} \right) \\
&\quad + \frac{\kappa_1}{N} \sum_j \phi(\|x_i - x_j\|) \left( \frac{v_j - v}{T_j} - \frac{v_i - v}{T_i} \right) \cdot v,
\end{aligned}
\]

where \( v := \frac{1}{N} \sum_j v_j \) is the average velocity. As a thermomechanically consistent model, an entropy functional \( S \) for (2) is also introduced in [32]:

\[
S(t) := \sum_i \ln T_i(t).
\]

In the following proposition, we list several basic properties of (2) such as Galilean invariance, conservation of total energy and entropy principle.

**Proposition 2.** [32] Let \( \{(x_i, v_i, T_i)\}_{i=1}^N \) be a global smooth solution to (2). Then, the following three assertions hold.

1. **(Galilean invariance):** For any constant vector \( c \in \mathbb{R}^d \), system (2) is invariant under Galilean transformation:

\[
(x_i, v_i, T_i) \implies (x_i + tc, v_i + c, T_i).
\]

2. **(Global conservation laws):** The total momentum and energy are conserved along the flow (2):

\[
\frac{d}{dt} \sum_i v_i = 0 \quad \text{and} \quad \frac{d}{dt} \sum_i \left( T_i + \frac{1}{2} \|v_i\|^2 \right) = 0, \quad t > 0.
\]

3. **(Entropy principle):** The total entropy is non-decreasing:

\[
\frac{dS}{dt} \geq 0, \quad t > 0.
\]

2.3. **The manifold CS model.** In this subsection, we introduce some previous results in [30] on the velocity alignment of CS model on complete Riemannian manifolds. First, let \( (M, g) \) be a connected, complete and smooth Riemannian manifold. Then, the CS model on \( (M, g) \) reads as:

\[
\begin{aligned}
\dot{x}_i &= v_i, \quad t > 0, \quad i = 1, \ldots, N, \\
\nabla v_i &= \frac{\kappa}{N} \sum_{j=1}^N \phi(x_i, x_j) (P_{ij} v_j - v_i), \\
(x_i(0), v_i(0)) &= (x_i^0, v_i^0) \in TM, \quad i = 1, \ldots, N,
\end{aligned}
\]

where the communication weight \( \phi \) is assumed to be a nonnegative symmetric smooth function that makes the mapping

\[
\{(x_k, v_k, T_k)\}_{k=1}^N \mapsto \phi(x_i, x_j) (P_{ij} v_j - v_i)
\]

smooth for all \( (i, j) \). Moreover, we further assume that the coupling strength \( \kappa \) is strictly positive, and denote \( P_{ij} \) as the parallel transport operator along the length-minimizing geodesic from \( x_j \) to \( x_i \).
Note that system (10) does not conserve the total momentum anymore. In fact, it might be problematic to define the total momentum of system (10) properly, since each velocity vector \( v_i \) lies in different tangent space \( T_{x_i} M \) and thus the summation of those \( v_i \)'s are not well defined. Although we may consider the summation of those tangent vectors via parallel transport or isometric embedding to higher dimensional Euclidean space, the total momentum is still not conserved. Nevertheless, we have a dissipation of total kinetic energy \( E := \sum \|v_i\|_{x_i}^2 \) as in the original CS model. We here slightly abused the language to denote \( \sum \|v_i\|_{x_i}^2 \) as a kinetic energy, since the constant multiplication does not affect our analysis.

**Proposition 3.** [30] Let \( \{ (x_i, v_i) \}_{i=1}^N \) be a global smooth solution to (10) with the initial data \( \{(x_0^i, v_0^i)\}_{i=1}^N \). Then, the total (kinetic) energy \( E \) is non-increasing in time. More precisely, we have

\[
\frac{dE}{dt} = -\frac{\kappa}{N} \sum_{i,k} \phi(x_i, x_k) \|P_{ik} v_k - v_i\|_{x_i}^2 \leq 0, \quad t > 0.
\]

Although the total kinetic energy \( E \) is a nonincreasing nonnegative function, it does not immediately imply the convergence of its derivative to zero. However, once we know that the derivative has a uniform continuity, one can indeed conclude the desired zero convergence.

**Lemma 2.1.** [4] (Barbalat’s lemma) Let \( f : (0, \infty) \to \mathbb{R} \) be a real-valued \( C^1 \) function with uniformly continuous derivative \( f' \). If \( f \) converges to a limit \( \alpha \) as \( t \to \infty \), we have

\[
\lim_{t \to \infty} f'(t) = 0.
\]

Then, as a consequence of Proposition 3 and Lemma 2.1, one can deduce the emergence of velocity alignment (6) under proper a priori conditions.

**Proposition 4.** [30] Let \( \{ (x_i, v_i) \}_{i=1}^N \) be a global smooth solution to (10) with the initial data \( \{(x_0^i, v_0^i)\}_{i=1}^N \) satisfying the following a priori conditions:

\[
\inf_{0 \leq t < \infty} \min_{i,j} \phi(x_i(t), x_j(t)) =: \phi_m > 0, \quad \sup_{0 \leq t < \infty} \max_{i,j} \frac{d}{dt} \|v_j - v_i\|_{x_i}^2 < \infty. \tag{11}
\]

Then, velocity alignment emerges:

\[
\lim_{t \to \infty} \|v_j - v_i\|_{x_i} = 0, \quad \forall i,j \in \{1, \cdots, N\}.
\]

**Proof.** Although a proof is analogous to [30], we here present its proof for the completeness of the paper. First, it follows from Proposition 3 that

\[
\int_0^T \left( \sum_{i,k} \phi(x_i, x_k) \|P_{ik} v_k - v_i\|_{x_i}^2 \right) dt = \frac{N}{\kappa} \left( E(0) - E(T) \right) \leq \frac{N E(0)}{\kappa} < \infty, \quad \forall T > 0.
\]

Then, a priori assumption (11) implies that the improper integral of \( \|v_j - v_i\|_{x_i}^2 \) is also finite:

\[
\int_0^\infty \|P_{ik} v_k - v_i\|_{x_i}^2 dt \leq \frac{N E(0)}{\kappa \phi_m} < \infty, \quad \forall i,k \in \{1, \cdots, N\}.
\]

Now, since an arbitrary primitive of \( \sum_{ik} \|P_{ik} v_k(t) - v_i(t)\|_{x_i(t)}^2 \) satisfies the conditions in Lemma 2.1, we can conclude the desired velocity alignment. \(\Box\)
Remark 1. 1. The manifold TCS model (3) cannot be reduced to manifold CS model (10) even if all initial temperatures are the same, because the isothermal ansatz is not consistent with asymptotic velocity alignment taking account the equation (3). On the other hand, the simplified manifold TCS model (5) is equivalent to (10) if \( T_0^1 = \cdots = T_0^N \).

2. In [30], the authors also verified a priori conditions (11) for several specific cases. For example, for \( M = \mathbb{S}^2 \) embedded isometrically in \( \mathbb{R}^3 \) with usual Euclidean metric, one can show the uniform continuity of \( \| P_{ij} v_j - v_i \|_{x_i}^2 \) under (11), and the assumption (11) can be shown, when \( N = 2 \) and the communication weight \( \phi \) is given as

\[
\phi(x_i, x_j) = \phi(d(x_i, x_j)).
\]

On the other hand, for the Poincaré half plane model on \( \mathbb{H} \), a smooth communication weight \( \phi \) satisfying

\[
0 < \phi_m \leq \phi(x) \leq \phi_M < \infty, \quad \forall \ x \in \mathbb{H}
\]

can be considered, since each pair \( (x, y) \in M \times M \) has a unique length minimizing geodesic. Then, a global well-posedness and the assumptions (11) for \( \mathbb{H} \) can also be verified.

In next two sections, we study emergent dynamics of the TCS type models (3) and (5) on Riemannian manifolds separately.

3. The manifold TCS model. In this section, we study emergent dynamics of the manifold TCS model (3) on a complete Riemannian manifold.

3.1. Basic estimates. In this subsection, we study basic estimates on energy conservation and entropy principle. Note that the total momentum conservation is not true for (3), and therefore we cannot explicitly determine the asymptotic limit of \( \{(x_i, v_i, T_i)\}_{i=1}^N \) a priori. However, as in the Manifold CS model and TCS model in \( \mathbb{R}^d \), we have the total energy conservation and entropy principle.

Lemma 3.1. Let \( \{(x_i, v_i, T_i)\}_{i=1}^N \) be a smooth solution to (3) for \( t \in [0, \tau) \), \( \tau \leq \infty \). Then, the following two assertions hold.

1. (Conservation of total energy): The total energy is conserved along the flow (3):

\[
\frac{d}{dt} \sum_i \left( T_i + \frac{1}{2} |v_i|_{x_i}^2 \right) = 0, \quad \forall \ t \in (0, \tau).
\]

2. (Entropy principle): The entropy functional \( S := \sum_i \ln T_i \) is nondecreasing along the flow (3):

\[
\frac{dS}{dt} \geq 0, \quad \forall \ t \in (0, \tau).
\]

Proof. (i) We use the symmetric property of \( \zeta \) to get

\[
\frac{d}{dt} \sum_i \left( T_i + \frac{1}{2} |v_i|_{x_i}^2 \right) = \kappa_2 \sum_{i=1}^N \sum_{j=1}^N \zeta(x_i, x_j) \left( \frac{1}{T_i^2} - \frac{1}{T_j^2} \right) = 0.
\]
Lemma 3.2. Let \( T \) and convergence of total entropy. Then, as the total sum of \( \ln T \)

\[
\frac{dS}{dt} = \sum_i \frac{1}{T_i} \frac{dT_i}{dt} = \sum_i \frac{1}{T_i} \left( T_i + \frac{1}{2} \left\| v_i \right\|_x^2 \right) - \sum_i \frac{1}{T_i} \left( v_i, \nabla v_i \right) x_i
\]

\[
= \frac{\kappa_2}{N} \sum_{i,j} \zeta(x_i, x_j) \left( \frac{1}{T_i} - \frac{1}{T_j} \right)
\]

\[
- \frac{\kappa_1}{N} \sum_{i,j} \phi(x_i, x_j) \left( \frac{\langle P_{ij} v_i, v_i \rangle_{x_i}}{T_j} - \left\| \frac{v_i}{T_i} \right\|_{x_i}^2 \right)
\]

\[
= \frac{\kappa_2}{2N} \sum_{i,j} \zeta(x_i, x_j) \left( \frac{1}{T_i} - \frac{1}{T_j} \right)^2 + \frac{\kappa_1}{2N} \sum_{i,j} \phi(x_i, x_j) \left\| \frac{P_{ij} v_j}{T_j} - \frac{v_i}{T_i} \right\|_{x_i}^2 \geq 0.
\]

Now, as a result of Lemma 3.1, one can show the strict positivity of temperatures and convergence of total entropy.

**Lemma 3.2.** Let \( \{(x_i^0, v_i^0, T_i^0)\}_{i=1}^N \) be a collection of \( N \) position-velocity-temperature configuration in \( TM \times \mathbb{R} \), where each initial temperature \( T_i^0 \) is strictly positive. Then, the following assertions hold.

1. There is a unique global smooth solution \( \{(x_i, v_i, T_i)\}_{i=1}^N \) to (3) with the initial data \( \{(x_i^0, v_i^0, T_i^0)\}_{i=1}^N \) for all time \( t \geq 0 \). Moreover, there exist positive constants \( T_m^\infty \) and \( T_M^\infty \) such that

\[
0 < T_m^\infty \leq T_i(t) \leq T_M^\infty < \infty, \quad \forall \ t \geq 0, \ i = 1, \ldots, N.
\]

2. There exists a constant \( S_\infty \) such that

\[
\lim_{t \to \infty} S(t) = S_\infty.
\]

**Proof.**

(i) Suppose that at least one of \( T_i \) converges to zero in finite time, since otherwise we get a global existence immediately. Then, there is a maximal time \( \tau_* \) such that all \( T_i \)'s stay positive during \( t \in [0, \tau_*] \), i.e.,

\[
\tau_* := \sup \left\{ \tau \geq 0 : \inf_{0 \leq t \leq \tau} \min_i T_i(t) > 0 \right\} \in (0, \infty),
\]

since \( \min_i T_i^0 > 0 \) and the zero convergence of \( \min_i T_i \) in finite time. In particular, we have

\[
\lim_{t \to \tau_*} \min_i T_i(t) = 0,
\]

as a consequence of the maximality of \( \tau_* \). Now, it follows from the total energy conservation in Lemma 3.1 that

\[
T_i(t) \leq \sum_i \left( T_i^0 + \frac{1}{2} \| v_i^0 \|_{x_i}^2 \right), \quad \forall \ t \in [0, \tau_*), \ i = 1, \ldots, N.
\]

Then, as the total sum of \( \ln T_i \) increases monotonically, we also obtain a positive lower bound of \( T_i \) in \( [0, \tau_*) \): for each \( i = 1, \ldots, N \),

\[
T_i(t) = \frac{e^{S(t)}}{\prod_{j \neq i} T_j(t)} \geq \frac{e^{S(0)}}{\prod_{j \neq i} T_j(0)} \geq \frac{\prod_{j=1}^N T_j^0}{\left( \sum_j \left( T_j^0 + \frac{1}{2} \| v_j^0 \|_{x_j}^2 \right) \right)^{N-1}}, \quad \forall \ t \in [0, \tau_*).
\]

However, this contradicts to the zero convergence of \( \min_i T_i(t) \) as \( t \to \tau_* \), and thus \( \tau_* = \infty \).
Finally, since the boundedness of velocities comes from the conservation of total energy, one can conclude the global well-posedness of (3).

(ii) Since the total entropy $S$ is nondecreasing and bounded above, we have the convergence of $S$, as $t$ tends to infinity.  

\begin{remark}
\label{rem}
According to Lemma 3.2, one can verify the following remarks:
\begin{enumerate}
\item The assertion \eqref{eq13} is satisfied for $T_\infty^m$ and $T_\infty^M$ as below:
\begin{equation}
T_\infty^m := \frac{\prod_{j=1}^{N} T_0^0}{\left( \sum_{j} \left( T_0^0 + \frac{1}{2} \| v_0^0 \|_x^2 \right) \right)^{N-1}}, \quad T_\infty^M := \sum_{j} \left( T_0^0 + \frac{1}{2} \| v_0^0 \|_x^2 \right).
\end{equation}
\item Since initial temperatures are strictly positive, local well-posedness of (3) can be guaranteed by the standard Cauchy-Lipschitz theory. On the other hand, it follows from the strict positivity of temperatures in Lemma 3.2 that the R.H.S. of (3) is bounded and Lipschitz continuous, and we have a global well-posedness of (3).
\end{enumerate}
\end{remark}

\subsection{Emergent dynamics}

In this subsection, we state and verify our main results on the collective behaviors of the manifold TCS model (3).

\begin{theorem}
\label{thm3.3}
Let $\{(x_i, v_i, T_i)\}_{i=1}^{N}$ be a global smooth solution to (3) with the initial data $\{(x_0^i, v_0^i, T_0^0)\}_{i=1}^{N}$, where each initial temperature $T_0^0$ is strictly positive. Moreover, assume that the temperature interaction kernel $\zeta$ has a positive lower bound:
\begin{equation}
\inf_{x, y \in M} \zeta(x, y) =: \zeta_m > 0.
\end{equation}
Then, the solution $\{(x_i, v_i, T_i)\}_{i=1}^{N}$ exhibits the following asymptotic behaviors.
\begin{enumerate}
\item The asymptotic uniform thermal equilibrium emerges:
\[ \exists \ T^\infty > 0 \text{ such that } \lim_{t \to \infty} T_i(t) = T^\infty, \quad \forall \ i \in \{1, \cdots, N\}. \]
\item The kinetic energy
\[ E = \sum_{i=1}^{N} \| v_i \|^2_{x_i} \]
converges as $t \to \infty$.
\item If we further assume the following a priori conditions:
\begin{equation}
\inf_{0 \leq t < \infty} \min_{i, j} \phi(x_i, x_j) =: \phi_m > 0, \quad \sup_{0 \leq t < \infty} \max_{i, j} \left| \frac{d}{dt} \left| \frac{P_{ij} v_j}{T_j} - \frac{v_i}{T_i} \right|^2 \right| < \infty,
\end{equation}
then we have the asymptotic velocity alignment:
\[ \lim_{t \to \infty} \| P_{ij} v_j - v_i \|_{x_i} = 0. \]
\end{enumerate}
\end{theorem}

\begin{proof}
(i) It follows from \eqref{eq12} that
\begin{align*}
\kappa_2 \zeta_m \sum_{i, j} \int_{0}^{\infty} \left( \frac{1}{T_i} - \frac{1}{T_j} \right)^2 dt & \leq \kappa_2 \sum_{i, j} \int_{0}^{\infty} \zeta(x_i, x_j) \left( \frac{1}{T_i} - \frac{1}{T_j} \right)^2 dt \\
& \leq \lim_{t \to \infty} S(t) - S(0) < \infty,
\end{align*}
\end{proof}
where the second inequality follows from (12).

Next, we show the uniform continuity of \( \left( \frac{1}{T_i} - \frac{1}{T_j} \right)^2 \). For this, we use \( T_m^\infty \) and \( T_M^\infty \) in (14) to control the derivatives

\[
\left| \frac{dT_i}{dt} \right| = \left| \frac{d}{dt} \left( T_i + \frac{1}{2} \| v_i \|_{x_i}^2 \right) - g_{x_i}(v_i, \nabla v_i) \right|
\]

\[
= \frac{\kappa_2}{N} \sum_j \sum_j \zeta(x_i, x_j) \left( \frac{1}{T_i} - \frac{1}{T_j} \right) - \frac{\kappa_1}{N} \sum_j \phi(x_i, x_j) \left( \frac{g_{x_i}(P_{ij}v_j, v_i)}{T_j} - \| v_i \|_{x_i}^2 \right) \]

\[
\leq \frac{2\kappa_2 \zeta_M}{T_m^\infty} + \frac{\kappa_1 \phi_M}{N} \sum_j \left( \| v_j \|_{x_j} \| v_i \|_{x_i} + \| v_i \|_{x_i}^2 \right)
\]

\[
\leq \frac{2\kappa_2 \zeta_M}{T_m^\infty} + \frac{\kappa_1 \phi_M}{N} \sum_j \left( \frac{2T_M^\infty}{T_m^\infty} + \frac{2T_M^\infty}{T_m^\infty} \right) = \frac{2\kappa_2 \zeta_M + 4\kappa_1 \phi_M T_M^\infty}{T_m^\infty}, \quad i = 1, \ldots, N,
\]

and

\[
\left| \frac{d}{dt} \left( \frac{1}{T_i} - \frac{1}{T_j} \right)^2 \right| = 2 \left| \left( \frac{1}{T_i} - \frac{1}{T_j} \right) \left( \frac{T_i}{T_i^2} - \frac{T_j}{T_j^2} \right) \right| \leq \frac{16\kappa_2 \zeta_M + 32\kappa_1 \phi_M T_M^\infty}{T_m^\infty} < \infty,
\]

where we used \( \| v_i \|_{x_i}^2 \leq 2T_M^\infty \) in the above inequality. Therefore, we can apply Lemma 2.1 to the primitive of \( \left( \frac{1}{T_i} - \frac{1}{T_j} \right)^2 \) to obtain the emergence of asymptotic thermal equilibrium. (ii) Since total entropy \( S \) has a limit \( S^\infty \) and asymptotic thermal equilibrium (i) emerges as \( t \to \infty \), we know that each temperature \( T_i \) converges to \( (e^{S^\infty})^\frac{1}{2} \). On the other hand, the total energy

\[
\sum_i \left( T_i + \frac{1}{2} \| v_i \|_{x_i}^2 \right) = \sum_{i=1}^N T_i + \frac{1}{2} \mathcal{E}
\]

is conserved from Lemma 3.1. Therefore, we have

\[
\lim_{t \to \infty} \mathcal{E}(t) = 2 \sum_i \left( T_i^0 + \frac{1}{2} \| v_i^0 \|_{x_i}^2 \right) - 2N \left( e^{S^\infty} \right)^\frac{1}{2}.
\]

(iii) Similar to the argument in (i), we have

\[
\phi_M \sum_{i,j} \int_0^\infty \left\| \frac{P_{ij}v_j}{T_j} - \frac{v_i}{T_i} \right\|_{x_i}^2 dt \leq \sum_{i,j} \int_0^\infty \phi(x_i, x_j) \left\| \frac{P_{ij}v_j}{T_j} - \frac{v_i}{T_i} \right\|_{x_i}^2 dt < \infty,
\]

where the second inequality again comes from the entropy convergence and (12). Therefore, under the a priori assumption (16), we apply Lemma 2.1 to any primitive of \( \left\| \frac{P_{ij}v_j}{T_j} - \frac{v_i}{T_i} \right\|_{x_i}^2 \) and combine with the first assertion to obtain the emergence of asymptotic velocity alignment.

Now, we consider a case when \((M, g)\) is a compact Riemannian manifold without boundary. Since every compact metric space is complete, the completeness condition of \((M, g)\) is redundant in this case. First, we recall a notion of \( \omega \)-limit set and its basic property from the classical dynamical systems theory.
Definition 3.4. Let \((M, g)\) be a d-dimensional Riemannian manifold and \(\varphi_x : \mathbb{R} \to M\) be the flow of the vector field \(\xi \in \mathcal{X}(M)\) through \(x \in M\). That is, \(\{\varphi_x(t)\}_{t \in \mathbb{R}}\) satisfies
\[
\dot{\varphi}_x(t) = \xi(\varphi_x(t)) \quad \text{and} \quad \varphi_x(0) = x \quad \text{for every} \quad t \in \mathbb{R}.
\]
Then, the \(\omega\)-limit set of \(x\), denoted by \(\omega(x)\), is defined as
\[
\omega(x) := \left\{ y \in M : \exists \{t_n\}_{n \geq 1} \text{ such that } \lim_{n \to \infty} t_n = \infty \quad \text{and} \quad \lim_{n \to \infty} \varphi_x(t_n) = y \right\}.
\]

Then, it is well known in [33] that the \(\omega\)-limit set is closed and positive invariant (i.e., \(\{\varphi_y(t)\}_{t \geq 0}\) is contained in \(\omega(x)\) for every \(y \in \omega(x)\)), when \(M\) is an Euclidean space. In particular, \(\omega(x)\) is even nonempty when a priori boundedness of \(\varphi_x(\cdot)\) is provided, as a consequence of Bolzano-Weierstrass. Since \((3)\) and \((5)\) can be regarded as first order ODEs in \((TM)^N \times \mathbb{R}^N\), the \(\omega\)-limit set for TCS flow (and simplified TCS flow) can also be considered, where the closedness and positive invariance can be also obtained by a simple adaptation of the arguments in [33] to a Riemannian manifold setting. Moreover, the non-emptiness of the \(\omega\)-limit set can be shown by the following argument.

Lemma 3.5. Let \((M, g)\) be a compact Riemannian manifold without boundary, and let \(\{(x(n), v(n), T(n))\}_{n \geq 1}\) be a sequence of points on \(TM \times \mathbb{R}\). If we further assume
\[
\sup_{1 \leq n < \infty} \|v(n)\|_{x(n)} < \infty, \quad \sup_{1 \leq n < \infty} |T(n)| < \infty,
\]
then there exists a convergent subsequence \(\{(x(n_k), v(n_k), T(n_k))\}_{k \geq 1}\).

Proof. First, since \(M\) is compact and \(T\) is bounded, we may assume \(x(n)\) and \(T(n)\) converge to certain point \(x(\infty)\) and \(T(\infty)\) by taking a subsequence if necessary. Then, we consider a coordinate chart \((U, \varphi)\) with \(x(\infty) \in U\). Since the smallest eigenvalue of the matrix \(\{g_{ij}\}\) is continuous in \(U\), it has a positive lower bound in any compact neighborhood \(C(\subseteq U)\) of \(x(\infty)\). Therefore, the boundedness of \(\|v(n)\|_{x(n)}\) induces the boundedness of \(v^i(n)\) for each \(i\), where \((v^1, \cdots, v^N)\) is the coordinate expression of \(v\) via \((U, \varphi)\). Therefore, from the Bolzano-Weierstrass theorem, one can find a subsequence \(\{n_k\}_{k \geq 1}\) such that \(\{v^i(n_k)\}_{k \geq 1}\) converges for every \(i = 1, \cdots, N\), and this is equivalent to the convergence of \(\{v(n_k)\}_{k \geq 1}\).

As a consequence, one can deduce the following corollary.

Corollary 1. Let \((M, g)\) be a compact Riemannian manifold without boundary, and let \(\{(x_i, v_i, T_i)\}_{i=1}^N\) be a global smooth solution to \((3)\) with the initial data \(\{(x_i^0, v_i^0, T_i^0)\}_{i=1}^N\), where each initial temperature \(T_i^0\) is strictly positive. Then, we have
\[
\zeta(x_i, x_j)(T_i - T_j)^2 \to 0, \quad \phi(x_i, x_j) \left| \frac{P_{ij}v_j}{T_j} - \frac{v_i}{T_i} \right|^2 \to 0, \quad \forall \ i, j = 1, \cdots, N.
\]

Proof. We set
\[
x_i^0 := (x_i^0, v_i^0, T_i^0), \quad i = 1, \cdots, N, \quad z^0 := (z_{ij}^0, \cdots, z_{ij}^0).
\]
Let \(\omega(Z^0)\) be the \(\omega\)-limit set of given system \((3)\) with the initial data \(Z^0\). Then by using Lemma 3.5, one can see that the solution \(\{(x_i, v_i, T_i)\}_{i=1}^N\) approaches \(\omega(Z^0)\) as \(t \to \infty\), since otherwise we can find a sequence \(\{t_n\}_{n \geq 1}\) with \(t_n \to \infty\) and \(\{(x_i(t_n), v_i(t_n), T_i(t_n))\}_{i=1}^N\) does not meet to a certain open neighborhood of \(\omega(Z^0)\). This is a contradiction to Lemma 3.5. Now, it follows from Lemma 3.2 that the
entropy $S$ converges to a real number, say $S^\infty$. Then, by definition of $\omega(Z^0)$, the entropy functional $S$ is indeed equal to the constant function $S \equiv S^\infty$ in $\omega(Z^0)$. Since $\omega(Z^0)$ is positively invariant, this means that $\frac{dS}{dt}$ is identically zero in $\omega(Z^0)$.

By using (12) and the existence of positive lower bound for temperatures in Lemma 3.2, we conclude the desired result.

Remark 3. In [30], authors employed Barbalat’s lemma (Lemma 2.1) to derive the velocity alignment of (10), which requires the uniform continuity of $\|P_{ij}v_j - v_i\|_2^2$ and a priori positive lower bound for $\phi(x_i, x_j)$. However, this can only be done for several specific cases of $(M, g)$ such as $d$-sphere and $d$-dimensional hyperbolic space that allow the explicit formula of parallel transport $P_{ij}$. On the other hand, this LaSalle type argument enables us to guarantee a relaxed version of asymptotic thermal equilibrium and velocity alignment for the TCS model on compact Riemannian manifolds.

4. Simplified manifold TCS model. In this section, we study the emergent dynamics of the simplified manifold TCS model (5) on a complete Riemannian manifold $(M, g)$. In this case, we have an additional conservation of total internal energy $\sum_i T_i$ compared to the full model (3) in which the total energy conservation holds, while the entropy principle is also satisfied as in Lemma 3.1. The main difficulty to analyze emergent dynamics of the simplified model is that we cannot obtain the boundedness of velocities $\{v_i\}_{i=1}^d$ from the energy conservation law as before. Moreover, the convergence of entropy $S = \sum_i \ln T_i$ only gives the integrability of $\zeta(x_i, x_j) \left( \frac{1}{T_i} - \frac{1}{T_j} \right)^2$, and therefore it is necessary to find another way to guarantee asymptotic velocity alignment.

4.1. Basic estimates. We first state internal energy conservation law and entropy principle for the simplified manifold TCS model (5).

Lemma 4.1. Let $\{(x_i, v_i, T_i)\}_{i=1}^N$ be a smooth solution to (5) for $t \in [0, \tau)$, $\tau \leq \infty$. Then, the following two assertions hold.

1. (Conservation of total internal energy): The total internal energy is conserved along the flow (5):

$$\sum_i T_i(t) = \sum_i T_i(0), \quad \forall \ t \in (0, \tau).$$

2. (Entropy principle): The entropy functional $S = \sum_i \ln T_i$ is nondecreasing:

$$\frac{dS(t)}{dt} \geq 0, \quad \forall \ t \in (0, \tau).$$

Proof. Since the proof is similar to that of Lemma 3.1, we omit a detailed proof for the total internal energy conservation. For the entropy principle, we differentiate $S$ to get

$$\frac{dS}{dt} = \frac{\kappa_2}{2N} \sum_{i,j} \zeta(x_i, x_j) \left( \frac{1}{T_i} - \frac{1}{T_j} \right)^2 \geq 0,$$

which yields the desired result. \qed
Although the total energy is not conserved in (5), we can also find positive upper and lower bounds for temperatures \( \{T_i\}_{i=1}^N \) and eventually guarantee global well-posedness by using a similar argument as in Lemma 3.2, provided that all the initial temperatures \( T_i^0 \) are positive. In fact, one can also show the following result for (5).

**Lemma 4.2.** Let \( \{(x_i,v_i,T_i)\}_{i=1}^N \) be a smooth solution to (5) in a nonempty interval \( t \in [0,\tau) \) with the initial data \( \{(x_i^0,v_i^0,T_i^0)\}_{i=1}^N \), where each initial temperature \( T_i^0 \) is strictly positive. Then, if \( \zeta(\cdot,\cdot) \) does not vanish in \( M \times M \):

\[
\zeta(x,y) > 0, \quad \forall \, x,y \in M,
\]

one has

\[
0 < \min_k T_k^0 \leq T_i(t) \leq \max_k T_k^0 < \infty, \quad \forall \, t \in (0,\tau), \quad i = 1,\cdots,N. \quad (17)
\]

**Proof.** For every \( \varepsilon \in (0,\min_k T_k^0) \), consider a set \( S_\varepsilon \) denoted by

\[
S_\varepsilon := \left\{ t \in (0,\tau) : \min_k T_k(t) \leq \min_k T_k^0 - \varepsilon \right\},
\]

and assume that \( S_\varepsilon \) is nonempty. Then, since \( \min_k T_k \) is continuous, the minimum \( t_\varepsilon := \min S_\varepsilon \) has to be finite and strictly positive.

Now, let \( I \subset \{1,\cdots,N\} \) be the index set that minimizes \( T_i(t_\varepsilon) \), i.e.,

\[
T_i(t_\varepsilon) = \min_k T_k(t_\varepsilon), \quad \forall \, i \in I.
\]

If \( I = \{1,\cdots,N\} \), the uniqueness of the solution \( \{(x_i,v_i,T_i)\}_{i=1}^N \) implies

\[
T_i^0 = T_j^0 \quad \text{and} \quad T_i(t) \equiv T_j^0 \quad \forall \, i,j = 1,\cdots,N, \quad t > 0,
\]

which gives a contradiction to \( S_\varepsilon \neq \emptyset \).

On the other hand, if \( I \neq \{1,\cdots,N\} \), then for any index \( i \in I \), we have

\[
\dot{T}_i(t_\varepsilon) = \frac{\kappa_2}{N} \sum_{j=1}^N \zeta(x_i(t_\varepsilon),x_j(t_\varepsilon)) \left( \frac{1}{T_i(t_\varepsilon)} - \frac{1}{T_j(t_\varepsilon)} \right) > 0,
\]

so that \( t_\varepsilon - \delta \in S_\varepsilon \) for some \( \delta \in (0,t_\varepsilon) \). Therefore, we deduce \( S_\varepsilon = \emptyset \) for each \( \varepsilon \in (0,\min_k T_k^0) \), which is equivalent to

\[
\min_k T_k^0 \leq T_i(t), \quad \forall \, t \geq 0, \quad i = 1,\cdots,N.
\]

Finally, by using similar argument, one can also show

\[
T_i(t) \leq \max_k T_k^0 < \infty, \quad \forall \, t \geq 0, \quad i = 1,\cdots,N,
\]

to conclude the desired result. \( \square \)

Note that the above lemma also says that \( \min_k T_k \) is monotonically increasing and \( \max_k T_k \) is monotonically decreasing in time \( t \), and hence the convergence of both functions. If both limits are different, one can consider a time evolution of the difference

\[
\mathcal{D}(T) := \max_k T_k - \min_k T_k,
\]

and expect a negative upper bound of its derivative while the maximal, minimal indices are not changed, provided that \( \zeta(\cdot,\cdot) \) has a positive lower bound as in (15). However, this negative upper bound of \( \frac{d\mathcal{D}}{dt}(T) \) contradicts to \( \mathcal{D}(T) \geq 0 \), which means that \( \mathcal{D}(T) \) has to converge to zero. We justify the above argument rigorously in
next subsection.

4.2. Emergent dynamics. In this subsection, we present collective estimates of the simplified manifold TCS model (5). First, we study an exponential thermal equilibrium by the following proposition.

**Proposition 5.** Let \( \{ (x_i, v_i, T_i) \}_{i=1}^N \) be a global smooth solution to (5) with the initial data \( \{ (x_i^0, v_i^0, T_i^0) \}_{i=1}^N \) satisfying \( \min_i T_i^0 > 0 \). If we assume the a priori condition

\[
\inf_{x,y \in M} \zeta(x,y) =: \zeta_m > 0,
\]

one has

\[
\mathcal{D}(T(t)) \leq \mathcal{D}(T(0)) e^{-\frac{k_2 \zeta_m}{(\max_k T_k)^2} t}, \quad t \geq 0.
\]

**Proof.** For simplicity, we write

\[
T_m := \min_k T_k, \quad T_M := \max_k T_k,
\]

and

\[
I_m(t) := \{ i : T_i(t) = T_m(t) \}, \quad I_M(t) := \{ i : T_i(t) = T_M(t) \}.
\]

Then, we define \( \Lambda_0 := \frac{k_2 \zeta_m}{(T_M(0))^2} \) and

\[
h(t) := \mathcal{D}(T(t)) e^{\Lambda_0 t}, \quad h_{ij}(t) := (T_j(t) - T_i(t)) e^{\Lambda_0 t}, \quad \forall \ t \geq 0, \ i, j = 1, \ldots, N.
\]

If \( h(0) = 0 \), one can use the uniqueness of solution of (5) to deduce

\[
T_i^0 = T_j^0 \quad \text{and} \quad T_i(t) \equiv T_i^0 \quad \forall \ i, j = 1, \ldots, N, \quad t > 0,
\]

and hence \( \mathcal{D}(T(t)) \equiv 0 \).

For the case \( h(0) > 0 \), we consider an arbitrary positive number \( \varepsilon > 0 \) and a set \( \mathcal{G}_\varepsilon \) denoted by

\[
\mathcal{G}_\varepsilon := \{ t : h(t) \geq h(0) + \varepsilon \}.
\]

and assume that \( \mathcal{G}_\varepsilon \) is nonempty. Then, since \( h(t) \) is continuous, the minimum \( \tau_\varepsilon := \min \mathcal{G}_\varepsilon \) has to be finite and strictly positive.

Now, for any \((i, j)\) in \( I_m(\tau_\varepsilon) \times I_M(\tau_\varepsilon) \), we have

\[
\dot{T}_j(\tau_\varepsilon) - \dot{T}_i(\tau_\varepsilon) = \frac{k_2}{N} \sum_{k=1}^N \zeta(x_j(\tau_\varepsilon), x_k(\tau_\varepsilon)) \left( \frac{1}{T_j(\tau_\varepsilon)} - \frac{1}{T_k(\tau_\varepsilon)} \right)
\]

\[
- \frac{k_2}{N} \sum_{k=1}^N \zeta(x_i(\tau_\varepsilon), x_k(\tau_\varepsilon)) \left( \frac{1}{T_i(\tau_\varepsilon)} - \frac{1}{T_k(\tau_\varepsilon)} \right)
\]

\[
\leq \frac{k_2}{N} \sum_{k=1}^N \zeta_m \left( \frac{1}{T_j(\tau_\varepsilon)} - \frac{1}{T_k(\tau_\varepsilon)} \right) - \frac{k_2}{N} \sum_{k=1}^N \zeta_m \left( \frac{1}{T_i(\tau_\varepsilon)} - \frac{1}{T_k(\tau_\varepsilon)} \right)
\]

\[
= -\frac{k_2 \zeta_m}{T_i(\tau_\varepsilon)} - \frac{k_2 \zeta_m}{T_j(\tau_\varepsilon)} < -\frac{k_2 \zeta_m}{(T_M(0))^2} (T_j(\tau_\varepsilon) - T_i(\tau_\varepsilon)),
\]

where we used Proposition 5 and \( h(0) > 0 \) in the last inequality. Therefore, at time \( \tau_\varepsilon \), we have

\[
\frac{dh_{ij}}{dt}(\tau_\varepsilon) < 0,
\]
so that
\[ h(\tau - \delta) \geq h_{ij}(\tau - \delta) > h_{ij}(\tau) = h(\tau) \geq h(0) + \varepsilon \]
for some \( \delta \in (0, \tau_\varepsilon) \). This contradicts to the minimality of \( \tau_\varepsilon \), and we conclude \( \mathcal{G}_\varepsilon = 0 \) for all \( \varepsilon > 0 \), which is the desired result.

**Remark 4.** Since the solution \( \{(x_i, v_i, T_i)\}_{i=1}^N \) exhibits an asymptotic thermal equilibrium and total internal energy \( \sum T_i \) is conserved, the limit of each temperature is just the average of initial temperatures:
\[ \lim_{t \to \infty} T_i(t) = \frac{1}{N} \sum_{i=1}^N T_i^0, \quad \forall i = 1, \cdots, N. \]

Then, one can apply Lemma 4.3 after any sufficiently long time and deduce
\[ \mathcal{D}(T(t)) \lesssim e^{-At}, \quad \forall 0 < A < \frac{\kappa_2 \zeta_m}{\left( \frac{1}{N} \sum_{i=1}^N T_i^0 \right)^2}. \]

In Lemma 4.1–Lemma 4.2, we proved the emergence of asymptotic thermal equilibrium under the a priori condition (15) as in Theorem 3.3. However, since the total entropy does not imply the integrability of \( \| P_{ij} v_j - v_i \|^2 \) as in (3), we consider the asymptotic behavior of total kinetic energy
\[ \mathcal{E} = \sum_{i=1}^N \| v_i \|^2_{x_i}, \]
as in the manifold C-S model (10) to find an analogous result to Proposition 4. Below, we first estimate the temporal derivative of \( \mathcal{E} \).

**Lemma 4.3.** Let \( \{(x_i, v_i, T_i)\}_{i=1}^N \) be a smooth solution to (5) with the initial data \( \{(x_i^0, v_i^0, T_i^0)\}_{i=1}^N \) satisfying \( \min_i T_i^0 > 0 \). Then, we have
\[ \frac{d\mathcal{E}}{dt} = -\frac{\kappa_1}{2N} \sum_{i,j} \phi_{ij} \left\{ \left( \frac{1}{T_i} + \frac{1}{T_j} \right) \| P_{ij} v_j - v_i \|^2_{x_i} + \left( \frac{1}{T_i} - \frac{1}{T_j} \right) \left( \| v_i \|^2_{x_i} - \| v_j \|^2_{x_j} \right) \right\}, \]
where \( \phi_{ij} \) is an abbreviation of \( \phi(x_i, x_j) \).

**Proof.** By direct calculation, one has
\[ \frac{d\mathcal{E}}{dt} = 2 \sum_{i=1}^N \langle v_i, \nabla_{x_i} v_i \rangle_{x_i} = \frac{2\kappa_1}{N} \sum_{i,j} \phi(x_i, x_j) \left( \frac{\langle P_{ij} v_j, v_i \rangle_{x_i}}{T_j} - \frac{\| v_i \|^2_{x_i}}{T_i} \right) \]
\[ = \frac{\kappa_1}{N} \sum_{i,j} \phi(x_i, x_j) \left( \frac{\langle P_{ij} v_j, v_i \rangle_{x_i}}{T_j} + \frac{\langle P_{ij} v_j, v_j \rangle_{x_i}}{T_i} - \frac{\| v_i \|^2_{x_i}}{T_i} - \frac{\| v_j \|^2_{x_j}}{T_j} \right) \]
\[ = -\frac{\kappa_1}{2N} \sum_{i,j} \phi_{ij} \left\{ \left( \frac{1}{T_i} + \frac{1}{T_j} \right) \| P_{ij} v_j - v_i \|^2_{x_i} + \left( \frac{1}{T_i} - \frac{1}{T_j} \right) \left( \| v_i \|^2_{x_i} - \| v_j \|^2_{x_j} \right) \right\}. \]

Note that if \( \mathcal{D}(T(0)) = 0 \), the energy \( \mathcal{E} \) of solution \( \{(x_i, v_i, T_i)\}_{i=1}^N \) can be regarded as an energy of manifold C-S model (10), and we can apply Proposition 5 to deduce the asymptotic velocity alignment. Although this cannot explain the velocity alignment rigorously, the exponential decay of \( \mathcal{D}(T(t)) \) in Proposition 5
guarantees a similar result. We here present our second main result on the emergence of asymptotic velocity alignment.

**Theorem 4.4.** Let \( \{(x_i, v_i, T_i)\}_{i=1}^N \) be a global smooth solution to (5) with the initial data \( \{(x_0^i, v_0^i, T_0^i)\}_{i=1}^N \)

\[
\begin{align*}
\min_i T_0^i > 0, \quad & \inf_{x,y \in M} \zeta(x,y) = \zeta_m > 0. \tag{18}
\end{align*}
\]

Then, one has

1. the kinetic energy \( E \) converges as \( t \to \infty \).
2. If we further assume the following a priori conditions:

\[
\inf_{0 \leq t < \infty} \min_{i,j} \phi(x_i, x_j) = \phi_m > 0, \quad & \sup_{0 \leq t < \infty} \max_{i,j} \left| \frac{d}{dt} \| P_{ij} v_j - v_i \|_{x_i}^2 \right| < \infty,
\]

then the solution \( \{(x_i, v_i, T_i)\}_{i=1}^N \) exhibits an asymptotic velocity alignment.

**Proof.** (1) We first denote

\[
T_m := \min_k T_k, \quad T_M := \max_k T_k, \quad \Lambda_0 := \frac{\kappa_2 \zeta_m}{T_M(0)^2}.
\]

Then it follows from Proposition 5 and Lemma 4.3 that

\[
\begin{align*}
\frac{d}{dt} E & \leq -\frac{\kappa_1}{2N} \sum_{i,j} \phi(x_i, x_j) \left( \frac{1}{T_i} - \frac{1}{T_j} \right) \left( \| v_i \|_{x_i}^2 - \| v_j \|_{x_j}^2 \right) \\
& \leq \frac{\kappa_1 \phi_M}{2N} \sum_{i,j} \left| \frac{1}{T_i} - \frac{1}{T_j} \right| \left( \| v_i \|_{x_i}^2 + \| v_j \|_{x_j}^2 \right) \tag{19}
\end{align*}
\]

which again implies

\[
\frac{d}{dt} \left( \ln E + \frac{\kappa_1 \phi_M D(T(0)) e^{-\Lambda_0 t}}{\Lambda_0 T_m(0)^2} \right) \leq 0.
\]

Therefore, we first obtain the uniform boundedness of \( E \): for every \( t > 0 \),

\[
\ln E(t) \leq \ln E(0) + \frac{\kappa_1 \phi_M D(T(0)) (1 - e^{-\Lambda_0 t})}{\Lambda_0 T_m(0)^2} \leq \ln E(0) + \frac{\kappa_1 \phi_M D(T(0))}{\Lambda_0 T_m(0)^2}. \tag{20}
\]

Now, we set

\[
C_1 := \frac{\kappa_1 \phi_M D(T(0))}{T_m(0)^2},
\]

and substitute (20) to (19) to get

\[
\frac{d}{dt} \left( E + C_1 e^{\frac{C_1}{\Lambda_0}} E(0) e^{-\Lambda_0 t} \right) \leq 0, \quad \forall \ t > 0.
\]

Therefore, the nonnegative nonincreasing functional

\[
E + C_1 e^{\frac{C_1}{\Lambda_0}} E(0) e^{-\Lambda_0 t}
\]

converges, which implies the convergence of \( E \).
By using Lemma 4.3, one has
\[\frac{d}{dt} \left( E + \frac{C_1}{\Lambda_0} e^{\frac{C_1}{\Lambda_0} E(0)} e^{-\Lambda_0 t} \right) \leq -\frac{\kappa_1}{2N} \sum_{i,j} \phi(x_i, x_j) \left( \frac{1}{T_i} + \frac{1}{T_j} \right) \| P_{ij} v_j - v_i \|_{x_i}^2, \]
and this implies the finiteness of the following integral:
\[\int_0^\infty \| P_{ij} v_j - v_i \|_{x_i}^2 dt.\]
Therefore, we apply Barbalat’s lemma to conclude the desired asymptotic velocity alignment.

**Remark 5.** According to a proof of Theorem 4.4, the uniform bound
\[\sup_{0 \leq t < \infty} E(t) \leq E(0) e^{\frac{C_1}{\Lambda_0}} \]
for the energy can be obtained without the assumption on the positive lower bound of \( \phi(\cdot, \cdot) \). This means that the kinetic energy remains small if initially it does, so that the approximation \( v_i \sim 0 \) from (3) does not contradict to itself. Moreover, this uniform bound implies the global well-posedness of (5) and becomes smaller if \( \frac{\kappa_1 \phi_M}{\eta^2 \zeta_m} \) and \( D(T(0)) \) are small.

Now, the compactness assumption on \((M, g)\) again guarantees a relaxed version of velocity alignment for (5) with minimal assumptions.

**Corollary 2.** Let \((M, g)\) be a compact Riemannian manifold without boundary, and let \( \{(x_i, v_i, T_i)\}_{i=1}^N \) be a global smooth solution to (5) with the initial data \( \{(x_i^0, v_i^0, T_i^0)\}_{i=1}^N \), where each initial temperature \( T_i^0 \) is strictly positive. If we further assume
\[\min_{x, y \in M} \zeta(x, y) > 0,\]
the solution exhibits the asymptotic thermal equilibrium and velocity alignment:
\[T_i - T_j \to 0, \quad \phi(x_i, x_j) \| P_{ij} v_j - v_i \|_{x_i}^2 \to 0, \quad \forall i, j = 1, \ldots, N.\]

**Proof.** Similar to the proof of Corollary 1, we consider the \( \omega \)-limit set \( \omega(Z^0) \) for (5). Then the solution \( \{(x_i, v_i, T_i)\}_{i=1}^N \) approaches \( \omega(Z^0) \) by the same reasoning, and Lemma 4.1 and Theorem 4.4 implies that both the entropy \( S \) and kinetic energy \( E \) converge as \( t \) tends to infinity. Then, \( S \) and \( E \) are two constant functions in \( \omega(Z^0) \), and since \( \omega(Z^0) \) is positive invariant, this means that \( \frac{dS}{dt} \) and \( \frac{dE}{dt} \) are identically zero in \( \omega(Z^0) \). By using Lemma 4.1, Lemma 4.3, and the positive assumption of \( \zeta \), we have \( T_i - T_j = 0 \) and \( \phi(x_i, x_j) \| P_{ij} v_j - v_i \|_{x_i} = 0 \) in \( \omega(Z^0) \), and conclude the desired result.

**Remark 6.** Recall that the positive minimum of \( \zeta \) was not required in Corollary 1. Once we assume \( \zeta \) has a positive minimum, then both Corollary 1 and 2 says that the asymptotic thermal equilibrium emerges and all trajectories become more similar to geodesics as \( t \) tends to infinity.
5. The TCS model on the unit d-sphere. In this section, we study emergent dynamics of manifold TCS models (3) and (5) on the unit d-sphere $S^d$ with the induced metric $g$ by the natural immersion $S^d \hookrightarrow \mathbb{R}^{d+1}$.

Recall that in [30], the explicit representation of the CS model on the unit sphere $S^d$ is only provided for $d = 2$ using Rodrigues’ rotation formula, which is valid only for $S^2$. We here generalize the system in [30] to the $d$-sphere with arbitrary dimension, and characterize their asymptotic behaviors. We first present formulas for covariant derivative and parallel transport, write the explicit equations for the TCS models (3)–(5) on $S^d$ and describe the detailed asymptotic patterns of configuration $(x_1, \cdots, x_N, v_1, \cdots, v_N)$.

**Proposition 6.** Let $S^d$ be the unit d-sphere with a usual inner product $\langle \cdot, \cdot \rangle$, and $\nabla$ be the Levi-Civita connection of $(S^d, \langle \cdot, \cdot \rangle)$.

1. For any smooth curve $x : \mathbb{R} \to S^d (\subset \mathbb{R}^{d+1})$, we have
   \[ \nabla_{\dot{x}} \dot{x} = \ddot{x} - \langle \ddot{x}, x \rangle x = \ddot{x} + \| \dot{x} \|^2 x. \]

2. For any $x, y \in S^d (\subset \mathbb{R}^{d+1})$ satisfying $x \perp v$, $x + y \neq 0$, the parallel transport $v'$ of $v$ along the length minimizing geodesic of $S^d$ from $x$ to $y$ is given by
   \[ v' = v - \frac{\langle v, y \rangle}{1 + \langle x, y \rangle} (x + y). \]

*Proof.* (i) We can easily verify that covariant derivative is $\frac{dv_i}{dt} + \|v_i\|^2 x_i$, since $S^d$ is embedded in $\mathbb{R}^{d+1}$ and the covariant derivative $\nabla v_i$ is the orthogonal projection (i.e., the intrinsic part) of $\dot{v}_i$ to the tangent plane at $x_i$:
   \[ \nabla v_i = \dot{v}_i - \langle \dot{v}_i, x_i \rangle x_i = \dot{v}_i + \|v_i\|^2 x_i. \]

(ii) We set $\xi := y - \langle x, y \rangle x$ and $\ell := \arccos \langle x, y \rangle$.

Then, the length-minimizing geodesic from $x$ to $y$ is given by
   \[ \gamma : [0, \ell] \to S^d, \quad \gamma(s) = x \cos s + \frac{\xi}{\| \xi \|} \sin s. \]

Moreover, the covariant derivative of tangent vector field
   \[ s \mapsto v - \frac{\langle v, \gamma(s) \rangle}{1 + \langle x, \gamma(s) \rangle} (x + \gamma(s)) (\in T_{\gamma(s)} S^d) \]

is zero, since
   \[
   \frac{d}{ds} \left( v - \frac{\langle v, \xi \rangle}{\| \xi \|} \sin s \right) (x + x \cos s + \frac{\xi}{\| \xi \|} \sin s) = - \left( v, \frac{\xi}{\| \xi \|} \right) \frac{d}{ds} \left( x \sin s + (1 - \cos s) \frac{\xi}{\| \xi \|} \right) = - \langle v, \xi \rangle \left( x \cos s + \frac{\xi}{\| \xi \|} \sin s \right) = - \left( v, \frac{\xi}{\| \xi \|} \right) \gamma(s)
   \]

is orthogonal to the tangent space at $\gamma(s)$. Therefore, the vector field $v'$ in Proposition 6 is the parallel transport along $\gamma$. \qed
Now, we use Proposition 6 to rewrite the manifold TCS model (3) on \((\mathbb{S}^d, \langle \cdot, \cdot \rangle)\) embedded in \(\mathbb{R}^{d+1}\):

\[
\begin{align*}
\dot{x}_i &= v_i, \quad t > 0, \quad i = 1, \cdots, N, \\
\dot{v}_i &= -\|v_i\|^2 x_i + \frac{\kappa_1}{N} \sum_{j=1}^{N} \phi_{ij} \left( \frac{v_j}{T_j} - \frac{v_i}{T_i} - \frac{\langle x_i, v_j \rangle}{T_j(1 + \langle x_i, x_j \rangle)} (x_i + x_j) \right), \\
\dot{T}_i &= \frac{\kappa_2}{N} \sum_{j=1}^{N} \xi_{ij} \left( \frac{1}{T_i} - \frac{1}{T_j} \right),
\end{align*}
\]

\(i,j\)

and the simplified manifold TCS model (5):

\[
\begin{align*}
\dot{x}_i &= v_i, \quad t > 0, \quad i = 1, \cdots, N, \\
\dot{v}_i &= -\|v_i\|^2 x_i + \frac{\kappa_1}{N} \sum_{j=1}^{N} \phi_{ij} \left( \frac{v_j}{T_j} - \frac{v_i}{T_i} - \frac{\langle x_i, v_j \rangle}{T_j(1 + \langle x_i, x_j \rangle)} (x_i + x_j) \right), \\
\dot{T}_i &= \frac{\kappa_2}{N} \sum_{j=1}^{N} \xi_{ij} \left( \frac{1}{T_i} - \frac{1}{T_j} \right),
\end{align*}
\]

Moreover, if \(\phi(x, y)\) is given as a smooth function of the form

\[
\phi(x, y) = (1 + \langle x, y \rangle) \psi(x, y), \quad \psi: \text{nonnegative, symmetric, smooth},
\]

a global well-posedness of (21) and (22) are guaranteed and \(\phi(x, y) = 0\) for all antipodal pair \((x, y)\).

In the sequel, we further analyze the \(\omega\)-limit set of (21) and (22) in detail.

**Lemma 5.1.** Let \(\{x_i\}_{i=1}^{N}\) be a collection of \(N\) constant speed geodesic curves on \(\mathbb{S}^d\) satisfying

\[
(1 + \langle x_i, x_j \rangle)(P_j v_j - v_i) = 0, \quad v_i := \dot{x}_i, \quad \forall \ i, j = 1, \cdots, N.
\]

Then, we have

\[\|v_1\| = \cdots = \|v_N\| =: v \geq 0,\]

and the following dichotomy holds:

either \(\{v = 0\}\) or \(\{v > 0\}\) and \(\exists t_1, \cdots, t_N, \ x_1(t_1 + t) = \cdots = x_N(t_N + t)\).

**Proof.** First of all, if \(\|v_i\| \neq \|v_j\|\) for some \(i, j\), then the length of relative velocity \(P_j v_j - v_i\) has a positive lower bound, and therefore \(1 + \langle x_i, x_j \rangle\) is identically zero, i.e.,

\[x_i(t) + x_j(t) = 0, \quad \forall \ t \in \mathbb{R}.\]

However, this cannot be achieved under the condition \(\|v_i\| \neq \|v_j\|\), since

\[\|\ddot{x}_i + \ddot{x}_j\| = \|\langle (\|v_i\|^2 - \|v_j\|^2) x_i \rangle\| \neq 0.\]
Theorem 5.2. Let \( \{(x_i, v_i, T_i)\}_{i=1}^N \) be a global smooth solution to (21) or (22) with the initial data \( \{(x_i^0, v_i^0, T_i^0)\}_{i=1}^N \) satisfying
\[
\min_i T_i^0 > 0, \quad \min_{x,y \in \mathbb{S}^d} \zeta(x, y) = \zeta_m > 0,
\]

Therefore, one can find a nonnegative constant \( v \) satisfying \( \|v_1\| = \cdots = \|v_N\| = v \) as above.

Now, we assume that \( v \) is strictly positive. Then, each \( x_i \) can be represented by
\[
 x_i(t) = p_i \cos(\theta_i + vt) + q_i \sin(\theta_i + vt), \quad \forall t \in \mathbb{R},
\]
where \( p_i \perp q_i \in \mathbb{S}^d \) and \( \theta_i \in \mathbb{R} \) are constant vectors and real number, respectively. Then, the condition (24) indicates
\[
(1 + \langle x_i, x_j \rangle)(v_j - v_i) \equiv \langle x_i, v_j \rangle(x_i + x_j), \quad \forall i, j = 1, \cdots, N. \tag{25}
\]
Moreover, we can consider an index change \( i \leftrightarrow j \) of (25) and add it to (25) itself to obtain the following: for \( i, j = 1, \cdots, N, \)
\[
\Rightarrow \left( \frac{d}{dt}\|x_i + x_j\|^2 \right) \cdot \|x_i + x_j\| = 0 \quad \Rightarrow \quad \frac{d}{dt}\|x_i + x_j\|^2 = 0. \tag{26}
\]
If \( p_i, q_i, p_j, q_j \) are linearly independent for some \( i, j \), the coefficients for \( p_i \) and \( q_i \) in (25) yield
\[
(1 + \langle x_i, x_j \rangle)v_i + \langle x_i, v_j \rangle x_i \equiv 0,
\]
and therefore
\[
1 + \langle x_i, x_j \rangle = -\frac{\langle x_i, v_j \rangle}{v^2} = 0.
\]
This contradicts to the independence assumption.

If \( a_ip_i + b_iq_i + a_jp_j + b_jq_j = 0 \) for some nonzero vector \((a_i, b_i, a_j, b_j)\), we have
\[
a_i^2 + b_i^2 = a_j^2 + b_j^2 > 0.
\]
Therefore, one can find a pair of real numbers \((\alpha_i, \alpha_j)\) satisfying the relation
\[
p_i \cos \alpha_i + q_i \sin \alpha_i = p_j \cos \alpha_j + q_j \sin \alpha_j =: \tilde{p},
\]
so that \( x_i \) and \( x_j \) can be written as:
\[
x_{k}(t) = \tilde{p} \cos(\tilde{\theta}_k + vt) + \tilde{q}_k \sin(\tilde{\theta}_k + vt), \quad \tilde{\theta}_k := \theta_k - \alpha_k,
\]
\[
\tilde{q}_k := -p_i \sin \alpha_i + q_i \cos \alpha_i, \quad k = i, j.
\]
If \( \tilde{p}, \tilde{q}_i \) and \( \tilde{q}_j \) are linearly independent, then the coefficients for \( \tilde{q}_i \) and \( \tilde{q}_j \) in (25) yield
\[
-(1 + \langle x_i, x_j \rangle)v \cos(\tilde{\theta}_i + vt) = \langle x_i, v_j \rangle \sin(\tilde{\theta}_i + vt),
\]
\[
(1 + \langle x_i, x_j \rangle)v \cos(\tilde{\theta}_j + vt) = \langle x_i, v_j \rangle \sin(\tilde{\theta}_j + vt).
\]
In particular, when \( x_i(t) = \tilde{p} \), the above relations can be satisfied only if \( 1 + \langle x_i(t), x_j(t) \rangle = 0 \), which means that \( \tilde{\theta}_i - \tilde{\theta}_j \equiv \pi \) modulo \( 2\pi \). But then (26) implies \( x_i + x_j \equiv 0 \), which is again a contradictory as we have seen in the beginning of the proof.

Therefore, for every \( i, j = 1, \cdots, N \), we have either \( \tilde{q}_i = \tilde{q}_j \) or \( \tilde{q}_i = -\tilde{q}_j \). However, since (26) clearly excludes the latter case, we conclude the desired result.

Now, we are ready to present the main result of this section.

\begin{flushright}
\Box
\end{flushright}
and assume that there exists a strictly positive smooth function $\psi : S^d \times S^d \to \mathbb{R}$ satisfying (23). Then, we have the following dichotomy for the asymptotic dynamics of $\{(x_i, v_i, T_i)\}_{i=1}^N$:

1. either kinetic energy converges to zero:
   $$\lim_{t \to \infty} E(t) = 0.$$

2. or kinetic energy converges to a nonzero positive value and all positions approach to a common great circle asymptotically: for every $i, j, k = 1, \cdots, N$ and $a, b, c = 1, \cdots, d + 1$, we have
   $$\lim_{t \to \infty} E(t) > 0, \quad \lim_{t \to \infty} \det \begin{pmatrix} x_i^a(t) & x_i^b(t) & x_i^c(t) \\ x_j^a(t) & x_j^b(t) & x_j^c(t) \\ x_k^a(t) & x_k^b(t) & x_k^c(t) \end{pmatrix} = 0,$$
   where $x^j$ denotes the $j$-th coordinate of the vector $x \in S^d$.

Proof. According to proofs of Corollary 1 and Corollary 2, each point in the $\omega$-limit set $\omega(Z^0)$ satisfies
   $$T_1 = \cdots = T_N, \quad \phi(x_i, x_j)(P_{ij} v_j - v_i) = 0, \quad i, j = 1, \cdots, N.$$
   Moreover, since $\omega(Z^0)$ is positively invariant, each curve started from $\omega(Z^0)$ satisfies (21)(resp. (22)), and therefore it is indeed a constant speed geodesic. Now, by Lemma 5.1, each point in $\omega(Z^0)$ satisfies either $\|v_1\| = \cdots = \|v_N\| = 0$ which means $E = 0$, or all position vectors $\{x_i\}_{i=1}^N$ lie on a common great circle, i.e.,
   $$\det \begin{pmatrix} x_i^a(t) & x_i^b(t) & x_i^c(t) \\ x_j^a(t) & x_j^b(t) & x_j^c(t) \\ x_k^a(t) & x_k^b(t) & x_k^c(t) \end{pmatrix} = 0 \quad \forall \ i, j, k = 1, \cdots, N \text{ and } a, b, c = 1, \cdots, d + 1.$$

Since the asymptotic dynamics of $\{(x_i, v_i, T_i)\}_{i=1}^N$ is completely determined by $\omega(Z^0)$, we obtain the desired dichotomy. \hfill $\square$

Remark 7. Since the manifold CS model (10) is a special case of the simplified manifold TCS model (5), the above dichotomy also holds for (10).

6. Numeric simulations. In this section, we provide several numerical results of the simplified manifold TCS model (22) on $S^2$ for $N = 15$, and compare them with our analytical results. Clearly, our numerical results show the asymptotic thermal equilibrium and asymptotic velocity alignment, respectively. In all simulations, we take communication weight function and system parameters as
   $$\phi(x, y) = \zeta(x, y) = 1 + \langle x, y \rangle, \quad \kappa_1 = 20, \quad \kappa_2 = 10, \quad dt = 0.01,$$
   and perform simulations by using the fourth-order Runge-Kutta scheme in $\mathbb{R}^3$ until $t = 200s$. Below, we verify three analytical results obtained in previous sections numerically. Namely, convergence of $E$, asymptotic emergence of thermal equilibrium and velocity alignment.

Note that the TCS particles approach to a certain fixed great circle as in Figure 1. The corresponding graph of the kinetic energy $E$ and temperatures are provided in Figure 2. In Figure 2,(A), we can observe the asymptotic thermal equilibrium, and Figure 2,(B) shows the convergence of kinetic energy to a nonzero positive value. Note that the functional $E$ is not monotonically decreasing as in Proposition 3 for the manifold CS model (10), but still shows the convergence, as $t$ goes to infinity. Although Theorem 5.2 does not imply the formation of fixed great circle by $\{x_i\}_{i=1}^N$
asymptotically, we could not numerically find a case in which the configuration $\{x_i\}_{i=1}^N$ approaches to a moving great circle that is not convergent as $t \to \infty$.

However, this is not a unique asymptotic behavior observed from numerical simulations, since there exists another case showing that the kinetic energy $E$ converges to zero.

In Figure 3 and Figure 4, we show the other case of dichotomy in Theorem 5.2. More precisely, in this case, the kinetic energy $E$ converges to zero exponentially fast and each particle approaches to a certain fixed point which may not lie in a common great circle. This is the other case in our dichotomy, but we could not prove the exponential convergence of $E$ and the convergence of $\{x_i\}_{i=1}^N$. Although Theorem 5.2 does not exclude the zero convergence of $E$ without an exponential decay, we could not find such case in our simulations.

7. Conclusion. In this paper, we have proposed a manifold counterpart of the TCS model to a connected, complete Riemannian manifold without a boundary and analyzed its detailed emergent dynamics under some sufficient frameworks. First, we studied a global well-posedness of the TCS model on complete Riemannian manifolds. For this, we have verified that entropy principle and conservation of total energy also hold for the manifold TCS model, and found a positive lower bound of temperatures in terms of initial energy and entropy. Then, under proper a priori
condition on $\phi$ or compactness assumption of given manifold, one can still obtain the emergence of velocity alignment and thermal equilibrium. As a definite example, we considered the unit sphere $S^d$ and provide the TCS and simplified TCS models on $S^d$ explicitly and show that the solution exhibits a relax version of asymptotic thermal equilibrium and velocity alignment. We also provided several numerical examples for the simplified manifold TCS model on $S^2$ to compare with our analytical results. For the TCS model on $S^d$, one can show that the solution exhibits either $v_i \to 0$ for all $i$ or all particles approach to a great circle. Unfortunately, we cannot identify admissible initial configurations and system parameters leading to one of dichotomy asymptotically, i.e., identification of basin of attractions. Also, we are not able to bound the geodesic distance between TCS particles, which were easily guaranteed in Euclidean space. These interesting issues will be addressed in a future work.

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