Chebyshev polynomial coefficient bounds for a subclass of bi-univalent functions

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February 10, 2017

Abstract
In this work, considering a general subclass of bi-univalent functions and using the Chebyshev polynomials, we obtain coefficient expansions for functions in this class.

Keywords: Chebyshev polynomials; bi-univalent functions; coefficient bounds; subordination.

Mathematics Subject Classification, 2010: 30C45, 30C50.

1 Introduction and Definitions

Let $D$ be the unit disk $\{z : z \in \mathbb{C} \text{ and } |z| < 1\}$, $A$ be the class of functions analytic in $D$, satisfying the conditions

$$f(0) = 0 \text{ and } f'(0) = 1.$$  

Then each function $f$ in $A$ has the Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$  \hspace{1cm} (1)

Further, by $S$ we shall denote the class of all functions in $A$ which are univalent in $D$.

The Koebe one-quarter theorem \textsuperscript{5} ensures that the image of $D$ under every function $f$ from $S$ contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse $f^{-1}$ which satisfies

$$f^{-1}(f(z)) = z, \quad (z \in D)$$

and

$$f\left(f^{-1}(w)\right) = w, \quad \left(|w| < r_0(f), \quad r_0(f) \geq \frac{1}{4}\right),$$

1
where

\[ f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2a_3 + a_4) w^4 + \cdots. \]

A function \( f \in A \) is said to be bi-univalent in \( D \) if both \( f \) and \( f^{-1} \) are univalent in \( D \). Let \( \Sigma \) represent the class of bi-univalent functions in \( D \) given by (1). For some intriguing examples of functions and characterization of the class \( \Sigma \), one could refer Srivastava et al. [14], and the references stated therein (see also, [12]). Recently there has been triggering interest to study the bi-univalent functions class \( \Sigma \) (see [1], [3], [9], [10], [13]) and obtain non-sharp estimates on the first two Taylor-Maclaurin coefficients |\( a_2 \)| and |\( a_3 \)|. Not much is known about the bounds on the general coefficient |\( a_n \)| for \( n \geq 4 \). In the literature, there are only a few works determining the general coefficient bounds |\( a_n \)| for the analytic bi-univalent functions ([2], [7], [8]). The coefficient estimate problem for each of |\( a_n \)| (\( n \in \mathbb{N} \setminus \{1, 2\} \); \( \mathbb{N} = \{1, 2, 3, \ldots\} \)) is still an open problem.

Chebyshev polynomials have become increasingly important in numerical analysis, from both theoretical and practical points of view. There are four kinds of Chebyshev polynomials. The majority of books and research papers dealing with specific orthogonal polynomials of Chebyshev family, contain mainly results of Chebyshev polynomials of first and second kinds \( T_n(t) \) and \( U_n(t) \) and their numerous uses in different applications, see for example, Doha [6] and Mason [11].

The Chebyshev polynomials of the first and second kinds are well known. In the case of a real variable \( x \) on \((-1, 1)\), they are defined by

\[ T_n(t) = \cos n\theta, \]

\[ U_n(t) = \frac{\sin(n+1)\theta}{\sin \theta}, \]

where the subscript \( n \) denotes the polynomial degree and where \( t = \cos \theta \).

If the functions \( f \) and \( g \) are analytic in \( D \), then \( f \) is said to be subordinate to \( g \), written as

\[ f (z) \prec g (z), \quad (z \in D) \]

if there exists a Schwarz function \( w (z) \), analytic in \( D \), with

\[ w (0) = 0 \quad \text{and} \quad |w (z)| < 1 \quad (z \in D) \]

such that

\[ f (z) = g (w(z)) \quad (z \in D). \]

**Definition 1** A function \( f \in \Sigma \) is said to be in the class \( H_\Sigma (\lambda, t) \), \( \lambda \geq 0 \) and \( t \in \left( \frac{\sqrt{2}}{2}, 1 \right] \), if the following subordination hold

\[ (1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left( 1 + \frac{zf''(z)}{f'(z)} \right) \prec H(z, t) = \frac{1}{1 - 2tz + z^2} \quad (z \in D) \quad (2) \]

\[ \text{with} \quad H(z, 0) = 1. \]
and

$$(1 - \lambda) \frac{w g'(w)}{g(w)} + \lambda \left(1 + \frac{w g''(w)}{g'(w)}\right) < H(w, t) = \frac{1}{1 - 2tw + w^2} \quad (w \in D)$$

where $g(w) = f^{-1}(w)$.

We note that if $t = \cos \alpha, \alpha \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$, then

$$H(z, t) = \frac{1}{1 - 2tz + z^2} = 1 + \sum_{n=1}^{\infty} \frac{\sin((n+1)\alpha)}{\sin \alpha} z^n \quad (z \in D).$$

Thus

$$H(z, t) = 1 + 2\cos \alpha z + (3\cos^2 \alpha - \sin^2 \alpha)z^2 + \cdots \quad (z \in D).$$

Following see, we write

$$H(z, t) = 1 + U_1(t)z + U_2(t)z^2 + \cdots \quad (z \in D, \ t \in (-1, 1)),$$

where $U_{n-1} = \frac{\sin(n \arccos t)}{\sqrt{1 - t^2}} \quad (n \in \mathbb{N})$ are the Chebyshev polynomials of the second kind. Also it is known that

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t),$$

and

$$U_1(t) = 2t,$$

$$U_2(t) = 4t^2 - 1,$$

$$U_3(t) = 8t^3 - 4t,$$

$$\vdots$$

(4)

The Chebyshev polynomials $T_n(t), \ t \in [-1, 1]$, of the first kind have the generating function of the form

$$\sum_{n=0}^{\infty} T_n(t)z^n = \frac{1 - tz}{1 - 2tz + z^2} \quad (z \in D).$$

However, the Chebyshev polynomials of the first kind $T_n(t)$ and the second kind $U_n(t)$ are well connected by the following relationships

$$\frac{dT_n(t)}{dt} = nU_{n-1}(t),$$

$$T_n(t) = U_n(t) - tU_{n-1}(t),$$

$$2T_n(t) = U_n(t) - U_{n-2}(t).$$
In this paper, motivated by the earlier work of Dziok et al. \cite{4}, we use the Chebyshev polynomial expansions to provide estimates for the initial coefficients of bi-univalent functions in $H_{\Sigma}(\lambda, t)$. We also solve Fekete-Szegő problem for functions in this class.

2 Coefficient bounds for the function class $H_{\Sigma}(\lambda, t)$

**Theorem 2** Let the function $f(z)$ given by (1) be in the class $H_{\Sigma}(\lambda, t)$. Then

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{(1 + \lambda)^2 - 4(\lambda + \lambda^2)t^2}}$$

and

$$|a_3| \leq \frac{4t^2}{(1 + \lambda)^2} + \frac{t}{1 + 2\lambda}$$

**Proof.** Let $f \in H_{\Sigma}(\lambda, t)$. From \eqref{2} and \eqref{3}, we have

$$(1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + U_1(t)w(z) + U_2(t)w^2(z) + \cdots, \quad (5)$$

and

$$(1 - \lambda) \frac{wg'(w)}{g(w)} + \lambda \left(1 + \frac{wg''(w)}{g'(w)}\right) = 1 + U_1(t)v(w) + U_2(t)v^2(w) + \cdots, \quad (6)$$

for some analytic functions $w, v$ such that $w(0) = v(0) = 0$ and $|w(z)| < 1, |v(w)| < 1$ for all $z \in D$. From the equalities \eqref{5} and \eqref{6}, we obtain that

$$(1 - \lambda) \frac{zf'(z)}{f(z)} + \lambda \left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + U_1(t)c_1z + \left[U_1(t)c_2 + U_2(t)c_1^2\right] z^2 + \cdots, \quad (7)$$

and

$$(1 - \lambda) \frac{wg'(w)}{g(w)} + \lambda \left(1 + \frac{wg''(w)}{g'(w)}\right) = 1 + U_1(t)d_1w + \left[U_1(t)d_2 + U_2(t)d_1^2\right] w^2 + \cdots. \quad (8)$$

It is fairly well-known that if $|w(z)| = |c_1z + c_2z^2 + c_3z^3 + \cdots| < 1$ and $|v(w)| = |d_1w + d_2w^2 + d_3w^3 + \cdots| < 1$, $z, w \in D$, then

$$|c_j| \leq 1, \quad \forall j \in \mathbb{N}.$$  

It follows from \eqref{7} and \eqref{8} that

$$1 + \lambda a_2 = U_1(t)c_1, \quad (9)$$

$$2(1 + 2\lambda) a_3 - (1 + 3\lambda) a_2^2 = U_1(t)c_2 + U_2(t)c_1^2, \quad (10)$$
and
\[- (1 + \lambda) a_2 = U_1(t)d_1, \quad (11)\]
\[2 (1 + 2\lambda) (2a_2^2 - a_3) - (1 + 3\lambda) a_2^2 = U_1(t)d_2 + U_2(t)d_1^2. \quad (12)\]
From (9) and (11) we obtain
\[c_1 = -d_1 \quad (13)\]
and
\[2 (1 + \lambda)^2 a_2^2 = U_1(t) \left( c_1^2 + d_1^2 \right). \quad (14)\]
By adding (10) to (12), we get
\[[4 (1 + 2\lambda) - 2 (1 + 3\lambda)] a_2^2 = U_1(t) (c_2 + d_2) + U_2(t) (c_1^2 + d_1^2). \quad (15)\]
By using (14) in equality (15), we have
\[
\left[ 2 (1 + \lambda) - \frac{2U_2(t)}{U_1(t)} (1 + \lambda)^2 \right] a_2^2 = U_1(t) (c_2 + d_2). \quad (16)\]
From (4) and (16) we get
\[|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{\left(1 + \lambda)^2 - 4 \left( \lambda + \lambda^2 \right) t^2}}. \]
Next, in order to find the bound on $|a_3|$, by subtracting (12) from (10), we obtain
\[4 (1 + 2\lambda) a_3 - 4 (1 + 2\lambda) a_2^2 = U_1(t) (c_2 - d_2) + U_2(t) (c_1^2 - d_1^2). \quad (17)\]
Then, in view of (13) and (14), we have from (17)
\[a_3 = \frac{U_2(t)}{2 (1 + \lambda)^2} \left( c_1^2 + d_1^2 \right) + \frac{U_1(t)}{4 (1 + 2\lambda)} (c_2 - d_2). \]
Notice that (4), we get
\[|a_3| \leq \frac{4t^2}{(1 + \lambda)^2} + \frac{t}{1 + 2\lambda}. \]
3 Fekete-Szegö inequalities for the function class $H_{\Sigma}(\lambda, t)$

**Theorem 3** Let $f$ given by (1) be in the class $H_{\Sigma}(\lambda, t)$ and $\mu \in \mathbb{R}$. Then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{t}{1 + 2\lambda}; \\ for |\mu - 1| \leq \frac{1}{8(1+2\lambda)} \left| \frac{(1+\lambda)^2}{t^2} - 4\lambda(1 + \lambda) \right| \\ 8|1 - \mu| t^3; \\ 4(1 + \lambda^2) - (1 + \lambda)^2 (4t^2 - 1); \\ for |\mu - 1| \geq \frac{1}{8(1+2\lambda)} \left| \frac{(1+\lambda)^2}{t^2} - 4\lambda(1 + \lambda) \right| \end{cases}.$$  

**Proof.** From (10) and (11)

$$a_3 - \mu a_2^2 = (1 - \mu) \frac{U_3(t) (c_2 + d_2)}{2(1 + \lambda) U_1^2(t) - 2U_2(t)(1 + \lambda)^2} + \frac{U_3(t)}{4(1 + 2\lambda)} (c_2 - d_2)$$

$$= U_1(t) \left[ \left( h(\mu) + \frac{1}{4(1+2\lambda)} \right) c_2 + \left( h(\mu) - \frac{1}{4(1+2\lambda)} \right) d_2 \right]$$

where

$$h(\mu) = \frac{U_1^2(t) (1 - \mu)}{2 \left[ (1 + \lambda) U_1^2(t) - U_2(t)(1 + \lambda)^2 \right]}.$$  

Then, in view of (11), we conclude that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{t}{1 + 2\lambda}; \\ 0 \leq |h(\mu)| \leq \frac{1}{4(1+2\lambda)} \\ 4t |h(\mu)| |h(\mu)| \geq \frac{1}{4(1+2\lambda)}. \end{cases}$$  

Taking $\mu = 1$ we get

**Corollary 4** If $f \in H_{\Sigma}(\lambda, t)$, then

$$|a_3 - a_2^2| \leq \frac{t}{1 + 2\lambda}.$$  

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