Lower bounds for eigenvalues of Laplacian operator and the clamped plate problem

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Abstract

We first investigate the lower bound for higher eigenvalues \( \lambda_i \) of the Laplace operator on a bounded domain and obtain a sharp lower bound. Then, we extend this estimate of the eigenvalues to general cases. Finally, we study the eigenvalues \( \Gamma_i \) for the clamped plate problem and deliver a sharp bound for the clamped plate problem for arbitrary dimension.

Mathematics Subject Classification 35P15 · 58G05

1 Introduction

Let \( \Omega \) be a bounded domain with piecewise smooth boundary \( \partial \Omega \) in an \( n \)-dimensional Euclidean space \( \mathbb{R}^n \). First of all, we focus on the following Dirichlet eigenvalue problem of Laplacian

\[
\begin{aligned}
\Delta u &= -\lambda u \quad \text{in } \Omega, \\
 u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\] (1.1)

It is well known that the spectrum of eigenvalue problem (1.1) is real and discrete (cf. [2, 6, 12, 15, 21])

\[
0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty,
\]

where each \( \lambda_i \) has finite multiplicity which is counted by its multiplicity.

Let \( V(\Omega) \) be the volume of \( \Omega \), and \( \omega_n \) the volume of the unit ball in \( \mathbb{R}^n \). Then the following well-known Weyl’s asymptotic formula holds

\[
\lambda_k \sim \frac{4\pi^2}{(\omega_n V(\Omega))^\frac{2}{n}} k^\frac{2}{n}, \quad k \to \infty,
\]
which implies that
\[
\frac{1}{k} \sum_{i=1}^{k} \lambda_i \sim \frac{n}{n+2} \frac{4\pi^2}{(\omega_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k \to \infty.
\] (1.2)

In 1961, Pólya [23] proved that, if \( n = 2 \) and \( \Omega \) is a tiling domain in \( R^2 \), then
\[
\lambda_k \geq \frac{4\pi^2}{(\omega_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k = 1, 2, \ldots.
\]

Based on the result above, he proposed the famous conjecture:

**Conjecture of Pólya.** If \( \Omega \) is a bounded domain in \( R^n \), then \( k \)-th eigenvalue \( \lambda_k \) of the eigenvalue problem (1.1) satisfies
\[
\lambda_k \geq \frac{4\pi^2}{(\omega_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k = 1, 2, \ldots.
\] (1.3)

It’s seen from the asymptotic formula (1.2), that Li-Yau’s inequality is the best possible in the sense of the average of eigenvalues. From (1.3), one can derive
\[
\lambda_k \geq \frac{n}{n+2} \frac{4\pi^2}{(\omega_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k = 1, 2, \ldots,
\]
which gives a partial solution to the Pólya conjecture with a factor \( \frac{n}{n+2} \). This conjecture is still open up to now.

During the past six decades, many mathematicians have focused on this problem and the related topics, there are a lot of important results on this aspect (cf. [4, 5, 7, 10, 11, 13, 14, 16, 18]) and we suggest that readers refer [25, 29] for more details. In 1983, Li and Yau [17] verified the famous Li-Yau inequality
\[
\frac{1}{k} \sum_{i=1}^{k} \lambda_i \geq \frac{n}{n+2} \frac{4\pi^2}{(\omega_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k = 1, 2, \ldots.
\] (1.4)

where \( c_n \) is a positive constant depending only on \( n \) and
\[
I(\Omega) = \min_{a \in R^n} \int_{\Omega} |x - a|^2 dx
\]
is called the moment of inertia of \( \Omega \). In fact \( c_n \leq \frac{1}{24(n+2)}. \) Obviously,
\[
I(\Omega) \geq \frac{n}{n+2} V(\Omega) \left( \frac{V(\Omega)}{\omega_n} \right)^{\frac{2}{n}}.
\]

In the formula (2.27) of [20], Males requires \( c \leq \min \{ \frac{1}{6}, (\frac{2\pi^2}{\omega_n})^{\frac{2}{n}} \}. \) According to \( \frac{\omega_n^4}{(2\pi)^2} \leq \frac{1}{2}, \) we get \( c \leq \frac{1}{6}. \) Putting \( c \leq \frac{1}{6} \) into the formula (2.27) of [20], we get \( c_n \leq \frac{1}{24(n+2)} \) in (1.4).
Afterwards, Kovařík, Vugalter and Weidl [13] improved this results when $n = 2$. They proved that

$$\sum_{i=1}^{k} \lambda_i \geq \frac{2\pi}{V(\Omega)} k^2 + C(a_0) V(\Omega)^{-\frac{3}{2}} k^{\frac{3}{2} - \varepsilon(k)} + (1 - a_0) \frac{V(\Omega)}{32 I(\Omega)} k, \quad (1.5)$$

where $C(a_0)$ is a positive constant depending on $a_0 \in [0, 1]$ and the length of the smooth parts of $\partial \Omega$, $\varepsilon(k) = \frac{2}{\log_2 \left( \frac{2\pi}{c} \right)}$ and $c = \sqrt{\frac{3\pi}{14}} 10^{-11}$.

The first purpose of this paper is to improve Melas’s estimate (1.4) by giving a sharper polynomial inequality, see Corollary 2.4. For more general cases, where $n \geq m \geq 2$ and $k \geq 1$, we obtain a lower bound for eigenvalues in Sect. 3, and we should mention that our result gives a sharp lower bounds by comparing Lemma 2.2 with the polynomial inequality in [20]. As a consequence of our result, we prove the Theorem 3.1. An interesting problem is to investigate the similar problem in a Cartan-Hadamard manifold and we recommend readers to refer to [27, 28] for details.

The second purpose of this paper is to estimate eigenvalues of the following clamped plate problem. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. We consider the following clamped plate problem, which describes characteristic vibrations of a clamped plate:

$$\begin{cases}
\Delta^2 u = \Gamma u, & \text{in } \Omega, \\
u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial \Omega,
\end{cases}$$

where $\Delta$ is the Laplacian operator and $\nu$ denotes the outward unit normal to the boundary $\partial \Omega$. As is known, this problem has a real and discrete spectrum (cf. [1])

$$0 < \Gamma_1 \leq \Gamma_2 \leq \Gamma_3 \leq \cdots \to \infty,$$

where each $\Gamma_i$ has finite multiplicity which is repeated according to its multiplicity.

For the eigenvalues of the clamped plate problem, Agmon [1] and Pleijel [22] gave the following asymptotic formula

$$\Gamma_k \sim \frac{16\pi^2}{(\omega_n V(\Omega))^\frac{4}{n}} k^\frac{4}{n}, \quad k \to \infty.$$

This implies that

$$\frac{1}{k} \sum_{i=1}^{k} \Gamma_i \sim \frac{n}{n + 4} \frac{16\pi^2}{(\omega_n V(\Omega))^\frac{4}{n}} k^\frac{4}{n}, \quad k \to \infty. \quad (1.6)$$

Furthermore, Levine and Protter [16] proved that the eigenvalues of the clamped plate problem satisfy

$$\frac{1}{k} \sum_{i=1}^{k} \Gamma_i \geq \frac{n}{n + 4} \frac{16\pi^4}{(\omega_n V(\Omega))^\frac{4}{n}} k^\frac{4}{n}.$$
where $n \geq 1$ and $k \geq 1$.

Recently, by using a different method, Cheng and Wei [9] got better lower bounds for eigenvalues of the clamped plate problem and proved that

$$
\frac{1}{k} \sum_{i=1}^{k} \Gamma_i \geq \frac{n}{n+4} \frac{16\pi^4}{(\omega_n V(\Omega))^{\frac{2}{n}}} \frac{1}{k^{\frac{4}{n}}} + \frac{n+2}{12n(n+4)} \frac{V(\Omega)}{I(\Omega)} \frac{n}{n+2} \frac{4\pi^2}{(\omega_n V(\Omega))^{\frac{2}{n}}} \frac{k^2}{\pi},
$$

where $n \geq 2$ and $k \geq 1$.

Furthermore, they gave upper bounds for the sum of $\Gamma_i$,

$$
\frac{1}{k} \sum_{i=1}^{k} \Gamma_i \leq \frac{1 + \frac{4(n+4)(n^2+2n+6)}{n^2} \frac{V(\Omega_0)}{V(\Omega)}}{1 - \frac{V(\Omega_0)}{V(\Omega)}} \frac{n}{n+4} \frac{16\pi^4}{(\omega_n V(\Omega))^{\frac{2}{n}}} \frac{1}{k^{\frac{4}{n}}},
$$

where $k \geq \frac{V(\Omega)}{r^0_n}$, and

$$
\Omega_r = \{ x \in \Omega \mid \text{dist}(x, \partial\Omega) < \frac{1}{r} \}.
$$

In [30], Yildirim and Yolcu improved Cheng and Wei’s estimates by replacing the last term in the right hand of (1.7) by a positive term of $k^{\frac{1}{n}}$. For any bounded open set $\Omega \subseteq \mathbb{R}^n$, where $n \geq 2$ and $k \geq 1$, Yildirim and Yolcu got the following inequality

$$
\sum_{i=1}^{k} \Gamma_i \geq \frac{n}{n+4} (\omega_n)^{-\frac{n}{2}} \alpha^{-\frac{n}{2}} k^{\frac{n+2}{2}} + \frac{1}{3(n+4)} (\omega_n)^{-\frac{2}{n}} \alpha^{\frac{2n-2}{n}} \frac{k^{\frac{n+2}{n}}}{\rho^2}
$$

where

$$
\alpha = \frac{V(\Omega)}{(2\pi)^n}, \quad \rho = 2(2\pi)^{-n} \sqrt{V(\Omega) I(\Omega)}.
$$

In Sect.4, we will improve Yildirim and Yolcu’s [30] estimate (1.8) by giving a shaper polynomial inequality when $n \geq 3$, see Corollary 4.4.

### 2 Lower bounds for sums of Dirichlet eigenvalues

In this section we prove the following theorem.
Theorem 2.1  For any bounded domain $\Omega \subseteq R^n$, $n \geq 2$ we have
\[
\sum_{j=1}^{k} \lambda_j(\Omega) \geq \omega_n^{-\frac{n-2}{2}} \alpha^{-\frac{n+2}{2}} k^{\frac{n+2}{n}} - \frac{s_3^3 \alpha^2}{(n+2) \rho^3} k
+ c_1 \omega_n^\frac{1}{2} s_4^{\frac{3n+1}{n}} k^{\frac{n-1}{2n}} \rho \frac{s_3}{(n+2) \rho^3},
\]
where
\[
c_1 \leq \min\left\{ 1, \max\left\{ \frac{4 \sqrt{2} n s_3^k}{(3n+1) s_4^4}, \frac{4 \sqrt{2} (n+2) k}{(3n+1) s_4^4} \right\} \right\},
\]
\[
s_l l = (a + 1)^l - a^l,
\]
$\alpha$, $\rho$ are defined by (1.9) and $a$ is defined by (2.16).

Firstly, we introduce some notations and definitions. For a bounded domain $\Omega$, the moment of inertia of $\Omega$ is defined by
\[
I(\Omega) = \min_{a \in R^n} \int_{\Omega} |x - a|^2 dx.
\]
By a translation of the origin and a suitable rotation of axes, we can assume that the center of mass is the origin and
\[
I(\Omega) = \int_{\Omega} |x|^2 dx.
\]
We now fix a $k \geq 1$ and let $u_1, \ldots, u_k$ denote an orthonormal set of eigenfunctions of (1.1) corresponding to the set of eigenvalues $\lambda_1(\Omega), \ldots, \lambda_k(\Omega)$. We consider the Fourier transform of each eigenfunction
\[
f_j(\xi) = \hat{u}_j(\xi) = (2\pi)^{-n/2} \int_{\Omega} u_j(x) e^{ix \cdot \xi} dx.
\]
It seems from Plancherel’s Theorem that $f_1, \ldots, f_k$ is an orthonormal set in $R^n$. Since these eigenfunctions $u_1, \ldots, u_k$ are also orthonormal in $L_2(\Omega)$, Bessel’s inequality implies that for every $\xi \in R^n$
\[
\sum_{j=1}^{k} |f_j(\xi)|^2 \leq (2\pi)^{-n} \int_{\Omega} |e^{ix \cdot \xi}|^2 dx = (2\pi)^{-n} V(\Omega). \tag{2.1}
\]
Since
\[
\nabla f_j(\xi) = (2\pi)^{-n/2} \int_{\Omega} i x u_j(x) e^{ix \cdot \xi} dx,
\]
we have
\[
\sum_{j=1}^{k} |\nabla f_j(\xi)|^2 \leq (2\pi)^{-n/2} \int_{\Omega} |i x e^{ix \cdot \xi}|^2 dx = (2\pi)^{-n} I(\Omega).
\]
By the boundary condition, we get
\[
\int_{R^n} |\xi|^2 |f_j(\xi)|^2 d\xi = \int_{\Omega} |\nabla u_j(x)|^2 dx = \lambda_j(\Omega)
for each $1 \leq j \leq k$. Set

$$F(\xi) = \sum_{j=1}^{k} |f_j(\xi)|^2.$$  

From (2.1), we have

$$0 \leq F(\xi) \leq (2\pi)^{-n} V(\Omega),$$  

(2.2)

$$|\nabla F(\xi)| \leq 2 \left( \sum_{j=1}^{k} |f_j(\xi)|^2 \right)^{1/2} \left( \sum_{j=1}^{k} |\nabla f_j(\xi)|^2 \right)^{1/2} \leq 2(2\pi)^{-n} \sqrt{V(\Omega) I(\Omega)}$$  

(2.3)

for each $\xi \in \mathbb{R}^n$. We also get

$$\int_{\mathbb{R}^n} F(\xi) d\xi = k,$$  

(2.4)

$$\int_{\mathbb{R}^n} |\xi|^2 F(\xi) d\xi = \sum_{j=1}^{k} \lambda_j(\Omega).$$  

(2.5)

Assume (by approximating $F$) that the decreasing function $\phi : [0, +\infty) \rightarrow [0, (2\pi)^{-n} V(\Omega)]$ is absolutely continuous. Let $F^*(\xi) = \phi(|\xi|)$ denote the decreasing radial rearrangement of $F$. Put $\mu(t) = |\{F^* > t\}| = |\{F > t\}|$. It follows from the coarea formula that

$$\mu(t) = \int_0^{(2\pi)^{-n} V(\Omega)} \int_{\{F=\phi(s)\}} \frac{1}{|\nabla F|} d\sigma_s ds.$$  

Since $F^*$ is radial, we have $\mu(\phi(s)) = |\{F^* > \phi(s)\}| = \omega_n s^n$. Differentiating both side of the above equality, we have $n\omega_n s^{n-1} = \mu'(\phi(s))\phi'(s)$ for almost all $s$. This together with (2.3), $\rho = 2(2\pi)^{-n} \sqrt{V(\Omega) I(\Omega)}$ and the isoperimetric inequality implies

$$-\mu'(\phi(s)) = \int_{\{F=\phi(s)\}} |\nabla F|^{-1} d\sigma_{\phi(s)} \geq \rho^{-1} \text{Vol}_{n-1}(\{F = \phi(s)\}) \geq \rho^{-1} n\omega_n s^{n-1}.$$  

For almost all $s$, we have

$$-\rho \leq \phi'(s) \leq 0.$$  

(2.6)

Since the map $\xi \mapsto |\xi|^2$ is radial and increasing, applying (2.5), we get

$$k = \int_{\mathbb{R}^n} F(\xi) d\xi = \int_{\mathbb{R}^n} F^*(\xi) d\xi = n\omega_n \int_{0}^{\infty} s^{n-1} \phi(s) ds$$  

(2.7)

and

$$\sum_{j=1}^{k} \lambda_j(\Omega) = \int_{\mathbb{R}^n} |\xi|^2 F(\xi) d\xi \geq \int_{\mathbb{R}^n} |\xi|^2 F^*(\xi) d\xi = n\omega_n \int_{0}^{\infty} s^{n+1} \phi(s) ds.$$  

(2.8)

The following lemma will be used in the proof of Theorem 2.1.
Lemma 2.2  Let $n \geq 2$, $\rho > 0$, $A > 0$. If $\psi : [0, +\infty) \to [0, +\infty)$ is a decreasing function (and absolutely continuous) satisfying

$$-\rho \leq -\psi'(s) \leq 0$$  \hspace{1cm} (2.9)

and

$$\int_0^\infty s^{n-1} \psi(s) ds = A.$$

Then

$$\int_0^\infty s^{n+1} \psi(s) ds \geq \left(\frac{nA}{n+2}\right)^{\frac{n+2}{n}} - \frac{s_3^3(nA)\psi(0)^2}{n(n+2)\rho^2} + \frac{s_4^4(nA)^{n-1}\psi(0)^{\frac{3n+1}{n}}}{n(n+2)\rho^2},$$

where

$$s_l = (a + 1)^l - a^l \geq 1.$$

**Proof**  We choose the function $\alpha \psi(\beta t)$ for appropriate $\alpha, \beta > 0$, such that $\rho = 1$ and $\psi(0) = 1$. By [20] we can also assume that $B = \int_0^\infty s^{n+1} \psi(s) ds < \infty$. If we let $q(s) = -\psi'(s)$ for $s \geq 0$, we have $0 \leq q(s) \leq 1$ and $\int_0^\infty q(s) = \psi(0) = 1$. Moreover, integration by parts implies that

$$\int_0^\infty s^n q(s) ds = n \int_0^\infty s^{n-1} \psi(s) ds = nA$$

and

$$\int_0^\infty s^{n+2} q(s) ds \leq (n + 2)B.$$

Next, let $0 \leq a < +\infty$ satisfies that

$$\int_a^{a+1} s^n ds = \int_0^\infty s^n q(s) ds = nA. \hspace{1cm} (2.10)$$

By the same argument as in Lemma 1 of [17], such real number $a$ exists. From [20], we have

$$(n + 2)B \geq \int_0^\infty s^{n+2} q(s) ds \geq \int_a^{a+1} s^{n+2} ds. \hspace{1cm} (2.11)$$

To estimate the last integral we take $\tau > 0$ to be chosen later. Applying (2.11) and integrating the both sides of the following inequality

$$ns^{n+2} - (n + 2)\tau^2 s^n + 2\tau^{n+2} \geq 2\tau^n(s - \tau)^2 + 4s\tau^{n-1}(s - \tau)^2, \ s \in [a, a + 1], \hspace{1cm} (2.12)$$
we get
\[
\begin{align*}
n(n + 2)B - (n + 2)\tau^2nA + 2\tau^{n+2} \\
\geq 2\tau^n \int_a^{a+1} (s - \tau)^2 + 4\tau^{n-1} \int_a^{a+1} s(s - \tau)^2 ds \\
\geq 2\tau^n \left( \frac{s^3}{3} - s^2\tau + s\tau^2 \right) \bigg|_a^{a+1} \\
+ 4\tau^{n-1} \left( \frac{s^4}{4} - \frac{2s^3\tau}{3} + \frac{s^2\tau^2}{2} \right) \bigg|_a^{a+1} \\
= 2s\tau^{n+2} + 2s^2\tau^{n+1} - 2s^3\tau^2 - 2s^2\tau^{n+1} + s^4\tau^{n-1} \bigg|_a^{a+1} \\
= 2\tau^{n+2} - 2s^3\tau + s^4\tau^{n-1}.
\end{align*}
\]

where
\[
s_j^l = (a + 1)^j - a^j \geq 1.
\]

Putting, \(\tau = (nA)^{1/n}\) we get
\[
B \geq \frac{1}{n}(nA)^{\frac{n+2}{n}} - \frac{s^3}{n(n + 2)}(nA) + \frac{s^4}{n(n + 2)}(nA)^{\frac{n-1}{n}}.
\]

This proves Lemma 2.2.

To prove (2.12), we need to show that for any \(\tau > 0\) we have
\[
ns^{n+2} - (n + 2)\tau^2s^n + 2\tau^{n+2} - 2\tau^n(s - \tau)^2 - 4s\tau^{n-1}(\tau - s)^2 \geq 0.
\]

Taking \(t = \frac{s}{\tau}\), we define \(f(t)\) (for \(t > 0\)) by
\[
f(t) = nt^{n+2} - (n + 2)t^n + 2 - 2(t - 1)^2 - 4t(t - 1)^2.
\]

Differentiating, \(f(t)\) we have
\[
f'(t) = n(n + 2)t^{n+1} - (n + 2)nt^{n-1} - 4(t - 1)^2 - 8t(t - 1)
\]
\[
= [n(n + 2)t^{n-2}(t + 1) - 12]t(t - 1).
\]

It follows from the above formula that if \(n \geq 2\), then \(t = 1\) is the minimum point of \(f\) and \(f \geq \min\{f(1) = 0, f(0) = 0\}\). This implies
\[
f(t)\tau^{n+2} = ns^{n+2} - (n + 2)\tau^2s^n + 2\tau^{n+2} - 2\tau^n(s - \tau)^2 - 4s\tau^{n-1}(\tau - s)^2 \geq 0.
\]

\(\square\)

Next we will give the proof of Theorem 2.1.

**Proof of Theorem 2.1** Applying Lemma 2.2 to the function \(\phi\) with \(A = (n\omega_n)^{-1}k, \rho = 2(2\pi)^{-n}\sqrt{V(\Omega)}I(\Omega)\) and submitting it to (2.8), we obtain
\[
\begin{align*}
\sum_{j=1}^k \lambda_j(\Omega) \geq & \omega_n^{\frac{2}{n}}\psi(0)^{\frac{2}{n}} k^{\frac{n+2}{n}} \left( \frac{s^4\psi(0)^2}{(n + 2)\rho^2} + c_1\omega_n^{\frac{4}{n}} \frac{s^{4n+1}}{(n + 2)^{n-1}} \right) \\
& \frac{s^4\psi(0)^{\frac{2n+1}{n}} k^{\frac{n-1}{n}}}{(n + 2)^{\frac{n-1}{n}}}. 
\end{align*}
\]
where $0 < c_1 \leq 1$ is a constant and $a$ is defined by

$$
\int_0^{a+1} \xi^n d\xi = \int_0^\infty -\xi^n \phi'(\xi) d\xi.
$$

(2.16)

We observe the following facts

(i) $0 < \psi(0) \leq (2\pi)^{-n} V(\Omega)$,

(ii) if $R$ is a positive constant such that $\omega_n R^n = V(\Omega)$, then

$$
I(\Omega) \geq \int_{B(R)} |x|^2 dx = \frac{n\omega_n R^{n+2}}{n+2}.
$$

(2.17)

It follows from the above properties

$$
\rho \geq (2\pi)^{-n} \omega_n^{-\frac{1}{n}} V(\Omega)^{\frac{n+1}{n}}.
$$

(2.18)

On the other hand, we consider the following function

$$
g(t) = g_1(t) + g_2(t),
$$

for $t \in (0, (2\pi)^{-n} V(\Omega)]$, where

$$
g_1(t) = \omega_n^{-\frac{2}{n}} t^{-2} k^{\frac{n+2}{n}}
$$

and

$$
g_2(t) = -\frac{s_3^3 t^2}{(n+2) \rho^2} k + c_1 \omega_n^{-\frac{1}{n}} s_4^4 k^{3n+1} \frac{k^{n-1}}{n}.
$$

Then we have

$$(n+2) \rho^2 g_2(t) = -2s_3^3 k t + c_1 \omega_n^{-\frac{1}{n}} s_4^4 k^{3n+1} \frac{k^{n-1}}{n}.
$$

By a direct calculation, we see from $\omega_n = \frac{2\pi \frac{\pi}{2}}{\Gamma(\frac{\pi}{2})}$ that

$$
\frac{\omega_n^{\frac{4}{n}}}{(2\pi)^2} \leq \frac{1}{2},
$$

where $\Gamma(t)$ is the Gamma function.

Therefore, in view of (2.18), if

$$
c_1 \leq \min \left\{ 1, \frac{4\sqrt{2n} \frac{3}{4} k^3}{(3n+1) s_4^4} \right\},
$$

then $g_2(t)$ is decreasing on $(0, (2\pi)^{-n} V(\Omega)]$. Now we consider another estimate. Setting

$$
G(t) = G_1(t) + G_2(t),
$$

where

$$
G_1(t) = \omega_n^{-\frac{2}{n}} \psi(0) - \frac{2}{n} k^{\frac{n+2}{n}} + c_1 \omega_n^{-\frac{1}{n}} s_4^4 \psi(0) \frac{k^{3n+1}}{n} \frac{k^{n-1}}{n} (n+2) \rho^3.
$$
and
\[ G_2(t) = -\frac{s_3^2 \psi(0)^2}{(n + 2)\rho^2} k, \]
we have
\[ G_1'(t) \rho^2 = -\frac{2}{n} \omega_n \frac{1}{n} \frac{2}{\pi} k^{n+2} \rho^2 + \frac{c_1 (3n + 1) \omega_n}{n} \frac{1}{n} \frac{s_4^4 t^{2n+1}}{(n + 2)\rho^2} T_1^2(T) \frac{n+1}{n^1}. \]

Therefore, we conclude that if
\[ c_1 \leq \frac{4\sqrt{2}(n + 2)k^3}{(3n + 1)s_4^3}, \]
then \( G(t) \) is decreasing on \((0, (2\pi)^{-n} V(\Omega))\). Finally, we obtain
\[
\sum_{j=1}^{k} \lambda_j(\Omega) \geq \omega_n \frac{2}{\pi} \alpha^{-\frac{2}{3}} k^{\frac{n+2}{n}} - \frac{s_3^3 \alpha^2}{(n + 2)\rho^2} k + c_1 \omega_n \frac{1}{n} s_4^4 \alpha^{\frac{3n+1}{n}} \frac{k^{n-1}}{\rho^2},
\]
where \( \alpha, \rho \) are defined in (1.9) and
\[ c_1 \leq \min \left\{ 1, \max \left\{ \frac{4\sqrt{2}n s_3^3 k^{\frac{1}{2}}}{(3n + 1)s_4^3}, \frac{4\sqrt{2}(n + 2)k^\frac{3}{2}}{(3n + 1)s_4^3} \right\} \right\}. \]

Note that \( \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \). This together with the above lemma implies the following estimate for higher eigenvalues.

**Corollary 2.3** For any bounded domain \( \Omega \subseteq \mathbb{R}^n, n \geq 2 \) and any \( k \geq 1 \) we have
\[
\lambda_k(\Omega) \geq \omega_n \frac{2}{\pi} \alpha^{-\frac{2}{3}} k^{\frac{n+2}{n}} - \frac{s_3^3 \alpha^2}{(n + 2)\rho^2} k + c_1 \omega_n \frac{1}{n} s_4^4 \alpha^{\frac{3n+1}{n}} \frac{k^{n-1}}{\rho^2},
\]
where
\[ c_1 \leq \min \left\{ 1, \max \left\{ \frac{4\sqrt{2}n s_3^3 k^{\frac{1}{2}}}{(3n + 1)s_4^3}, \frac{4\sqrt{2}(n + 2)k^\frac{3}{2}}{(3n + 1)s_4^3} \right\} \right\}, \]
\[ s_1^l = (a + 1)^l - a^l, \]
\( \alpha, \rho \) are defined by (1.9).

In fact, if we choose special \( a \) in (2.13), we also have the following result.
Corollary 2.4  For any bounded domain \( \Omega \subseteq \mathbb{R}^n \), \( n \geq 2 \) and any \( k \geq 1 \) we have

\[
\sum_{i=1}^{k} \lambda_i \geq \frac{n \omega_n^{-\frac{2}{n}} (2\pi)^{\frac{2}{n}} V(\Omega)^{-\frac{2}{n}}}{n+2} k^{\frac{n+2}{n}} + \frac{1}{24(n+2)} \left( \frac{V(\Omega)}{I(\Omega)} \right)^k \frac{-\frac{1}{2} \alpha^{-\frac{3n+1}{n}}}{9(n+2) \rho^{\frac{3}{2}}} k^{\frac{n-1}{n}}. 
\]

(2.20)

Proof  Combining with the formula (2.25) in [20] and (57) in [30], we have

\[
n(n+2) B - (n+2) \tau^2 n A + 2 \tau^{n+2} \geq 2 \tau^n \int_{a}^{a+1} (s-\tau)^2 + 4 \tau^{n-1} \int_{a}^{a+1} s(s-\tau)^2 ds \\
\geq \frac{\tau^n}{6} + \frac{\tau^{n-1}}{9}.
\]

By using similar discussion in the proof of Theorem 2.1, we get

\[
\sum_{i=1}^{k} \lambda_i \geq \frac{n \omega_n^{-\frac{2}{n}} \phi(0)^{-\frac{2}{n}}}{n+2} k^{\frac{n+2}{n}} + \frac{C_1 \phi(0)^2}{(n+2) \rho^2} k + \frac{C_2 \omega_n^{-\frac{3n+1}{n}} \phi(0)}{(n+2) \rho^3} k^{\frac{n-1}{n}},
\]

(2.21)

where \( 0 < C_1 \leq \frac{1}{6} \) and \( 0 < C_2 \leq \frac{1}{9} \) are two constants which will be determined later. We consider the following function

\[
g(t) = \frac{n \omega_n^{-\frac{2}{n}} t^{-\frac{2}{n}}}{n+2} k^{\frac{n+2}{n}} + \frac{C_1 t^2}{(n+2) \rho^2} k + \frac{C_2 \omega_n^{-\frac{3n+1}{n}} t}{(n+2) \rho^3} k^{\frac{n-1}{n}},
\]

which would be decreasing on \((0, (2\pi)^{-n} V(\Omega)]\) if \( g'( (2\pi)^{-n} V(\Omega) ) \leq 0 \). In view of (2.18), the depression of \( g(t) \) is equal to the following inequality

\[
2k^{\frac{2}{n}} \geq 2C_1 \frac{\omega_n^{\frac{2}{n}}}{(2\pi)^{\frac{2}{n}}} + C_2 \frac{3n+1}{n} \frac{\omega_n^{\frac{2}{n}}}{(2\pi)^{\frac{3}{n}}}.
\]

Since \( k \geq 1 \) and \( \frac{\omega_n^{\frac{4}{n}}}{(2\pi)^{2}} \leq \frac{1}{2} \), we can choose \( C_1 = \frac{1}{6} \). Therefore, \( C_2 \) satisfies

\[
C_2 \leq \min\{ \frac{1}{9}, \tilde{C}_2 \},
\]

where

\[
\tilde{C}_2 = \frac{\sqrt{2} (2\pi)^{4}}{3.5} \left( 2 - \frac{1}{6} \right).
\]

Obviously, \( \tilde{C}_2 \geq \frac{1}{9} \). Hence, we complete our proof. \( \square \)

3 Lower bounds for Dirichlet eigenvalues in higher dimensions

In this section we will give a universal lower bound on the sum of eigenvalues for \( n \geq m + 1 \), where \( m \geq 2 \).
Theorem 3.1 For any bounded domain $\Omega \subseteq \mathbb{R}^n$, $n \geq m + 1 \geq 3$ and $k \geq 1$, we have
\[
\sum_{i=1}^k \lambda_i \geq \omega_n \frac{2}{\pi} \alpha \frac{2}{n} \pi k \frac{n+2}{n} - \frac{2\omega_n^{m+1}}{(n+2)\rho^{m+1}} k \frac{n-m+1}{n} + c_2 \frac{2\omega_n^m (m+1) S_{m+3} \alpha}{(n+2)(m+3)\rho^{m+2}} k \frac{n-m}{n},
\]
where
\[
c_2 \leq \min \left\{ 1, \frac{(m+1)n + m - 1}{m+2n+m} S_{m+2} \frac{1}{m+1} \frac{1}{\pi} \right\},
\]
\[
S_l = (a + 1)\tau - a',
\]
\[
\alpha = \frac{V(\Omega)}{(2\pi)^n}, \quad \rho = 2(2\pi)^{-n} \sqrt{V(\Omega)I(\Omega)} \quad \text{and} \quad a \quad \text{is defined by (2.16)}.
\]

The following lemma will be used in the proof of Theorem 3.1.

Lemma 3.2 For an integer $n \geq m + 1 \geq 0$ and positive real numbers $s$ and $\tau$ we have the following inequality:
\[
n s^{n+2} - (n+2)\tau s^n + 2\tau^{n+2} - \sum_{k=1}^{m+1} 2k s^{k-1} \tau^{n-k+1} (\tau - s)^2 \geq 0.
\]

Proof Setting $t = \frac{s}{\tau}$, and putting
\[
f(t) = nt^{n+2} - (n+2)t^n + 2 - \sum_{k=1}^{m+1} 2kt^{k-1}(t - 1)^2,
\]
for $t \geq 0$, we get
\[
f'(t) = n(n+2)t^{n+1} - n(n+2)t^{n-1}
\]
\[
= n(n+2)t^{n+1} - n(n+2)t^{n-1}
\]
\[
- (t - 1) \left[ 4 + \sum_{k=1}^{m} [2k(n+1)t^{k-1}(t - 1) + 4(k+1)t^{k}] \right]
\]
\[
= n(n+2)t^{n+1} - n(n+2)t^{n-1}
\]
\[
- (t - 1) \left[ 2(m+2)(m+1)t^m + \sum_{k=1}^{m-1} [2k(n+1)t^{k} - 2k(n+1)t^{k-1} + 4(k+1)t^{k}] \right]
\]
\[
= n(n+2)t^{n+1} - n(n+2)t^{n-1} - 2(m+2)(m+1)t^m (t - 1)
\]
\[
= t^n (t - 1) \left[ n(n+2)t^{n-m-1}(t + 1) - 2(m+2)(m+1) \right].
\]

(3.1)
It follows from the above formula that if \( n \geq m + 1 \), then \( t = 1 \) is the minimum point of \( f(t) \) and \( f \geq \min\{ f(1) = 0, f(0) = 0 \} \). So, we get

\[
\tau^{n+2} f(t) = n s^{n+2} - (n + 2) \tau^2 s^n - \sum_{k=1}^{m+1} 2k s^{k-1} \tau^{n-k+1} (\tau - s)^2 \geq 0.
\]

\[\square\]

Next we will give the proof of Theorem 3.1.

**Proof of Theorem 3.1** For \( l \geq 0, \tau \geq \frac{1}{2} \) and \( a \geq 0 \), we have

\[
\int_a^{a+1} s^{l} (\tau - s)^2 \, ds = \left. \frac{s^{l+3}}{l + 3} - \frac{2s^{l+2}}{l + 2} \tau + \frac{s^{l+1}}{l + 1} \tau^2 \right|_a^{a+1} = \frac{S_{l+3}}{l + 3} - \frac{2S_{l+2}}{l + 2} \tau + \frac{S_{l+1}}{l + 1} \tau^2,
\]

where

\[
S_j = (a + 1)^j - a^j \geq 1.
\]

Therefore, we get

\[
n(n + 2) B - (n + 2) \tau^2 n A + 2 \tau^{n+2} \geq \sum_{k=1}^{m+1} 2k \tau^{n-k+1} \left( \frac{S_k + 2}{k + 2} - \frac{2S_k + 1}{k + 1} \tau + \frac{S_k}{k} \tau^2 \right).
\]

From

\[
\sum_{k=1}^{m+1} 2k \tau^{n-k+1} \left( \frac{S_k}{k} \tau^2 - \frac{2S_k + 1}{k + 1} \tau + \frac{S_k + 2}{k + 2} \right)
= \sum_{k=1}^{m+1} 2 \tau^{n+2} + 2 \frac{S_{k+1}}{k+2} \tau^{n-k+2} - 2 \frac{2S_k + 1}{k + 1} \tau^{n-k+2} + 2 \frac{S_k + 2}{k + 2} \tau^{n-k+1}
= \sum_{k=1}^{m+1} 2 \tau^{n+2} + 2 \frac{S_{k+1}}{m+2} \tau^{n-k+1} - 4(1 + \frac{k - 1}{k + 1} - \frac{2k}{k + 1}) S_{k+1} \tau^{n-k+2}
= \sum_{k=1}^{m+1} 2 \tau^{n+2} + 2 \frac{S_{k+1}}{m+2} \tau^{n-k+1} - \frac{4(m+1)S_{m+2}}{m+2} \tau^{n-m+1}
+ \frac{2(m + 1)S_{m+3}}{m + 3} \tau^{n-m} - 2 S_{2} \tau^{n+1} - \frac{4(m+1)S_{m+2}}{m+2} \tau^{n-m+1}
+ \frac{2(m + 1)S_{m+3}}{m + 3} \tau^{n-m} - 2 S_{2} \tau^{n+1} - \frac{4(m+1)S_{m+2}}{m+2} \tau^{n-m+1}
+ \frac{2(m + 1)S_{m+3}}{m + 3} \tau^{n-m},
\]

and

\[
\sum_{k=2}^{m} \left( 1 + \frac{k - 1}{k + 1} - \frac{2k}{k + 1} \right) S_{k+1} \tau^{n-k+2} = 0.
\]
we obtain
\[
n(n + 2)B - (n + 2)\tau^2 n A + 2\tau^{n+2} \geq 2\tau^{n+2} - 2S_{m+2}\tau^{n-m+1} + \frac{2(m + 1)S_{m+3}}{m + 3} \tau^{n-m}.
\]

Choosing \(\tau = (nA)^\frac{1}{2}\), we get
\[
B \geq \frac{(nA)^{\frac{n+2}{n}}}{n} - \frac{2S_{m+2}(nA)^{\frac{n-m+1}{n}}}{n(n + 2)} + \frac{2(m + 1)S_{m+3}(nA)^{\frac{n-m}{n}}}{n(n + 2)(m + 3)}.
\]

(3.3)

It follows from (3.3) that
\[
\int_0^\infty s^{n+1}\psi(s)ds \geq \frac{(nA)^{\frac{n+2}{n}}}{n} - \frac{2S_{m+2}(nA)^{\frac{n-m+1}{n}}}{n(n + 2)} - \frac{2(m + 1)S_{m+3}(nA)^{\frac{n-m}{n}}}{n(n + 2)(m + 3)} \rho^{m+1}.
\]

(3.4)

From (2.8), we know
\[
\sum_{i=1}^k \lambda_i \geq \omega_n \int_0^\infty s^{n+1}\psi(s)ds
\]
\[
\geq \omega_n(nA)^{\frac{n+2}{n}}\psi(0)^{-\frac{2}{n}} - \frac{2\omega_n S_{m+2}(nA)^{\frac{n-m+1}{n}}}{(n + 2)\rho^{m+1}} - \frac{2\omega_n(m + 1)S_{m+3}(nA)^{\frac{n-m}{n}}}{(n + 2)(m + 3)\rho^{m+2}}.
\]

In view of \(A = \frac{k}{n\omega_n}\), we have
\[
\sum_{i=1}^k \lambda_i \geq \omega_n^{-\frac{2}{n}}\psi(0)^{-\frac{2}{n}} k^{\frac{n+2}{n}} - \frac{2\omega_n^{\frac{n-1}{n}} S_{m+2}\psi(0)}{(n + 2)\rho^{m+1}} - \frac{2\omega_n^{\frac{n}{n}}(m + 1)S_{m+3}\psi(0)}{(n + 2)(m + 3)\rho^{m+2}}
\]
\[
+ c_2 \frac{2\omega_n^{\frac{m}{n}}(m + 1)S_{m+3}\psi(0)}{(n + 2)(m + 3)\rho^{m+2}} k^{\frac{n-m}{n}},
\]

(3.5)

where \(0 < c_2 \leq 1\) is a constant.

When \(m = 1\), we complete the proof of Theorem 2.1 in Sect. 2. We assume that \(m \geq 2\).

Putting
\[
g(t) = g_1(t) + g_2(t),
\]
where
\[
g_1(t) = \omega_n^{-\frac{2}{n}} t^{-\frac{2}{n}} k^{\frac{n+2}{n}}
\]
and
\[
g_2(t) = -\frac{2\omega_n^{\frac{m}{n}} S_{m+2}t^{\frac{(m+1)n-m-1}{n}}}{(n + 2)\rho^{m+1}} k^{\frac{n-m+1}{n}}
\]
\[
+ c_2 \frac{2\omega_n^{\frac{m}{n}}(m + 1)S_{m+3}t^{\frac{(m+2)n+m}{n}}}{(n + 2)(m + 3)\rho^{m+2}} k^{\frac{n-m}{n}}.
\]
we have
\[
\frac{(n + 2)\rho^{m+1} \omega_n^m g_2'(t)}{2k \frac{n-m}{n}} = - \frac{(m + 1)n + m - 1}{\omega_n^{\frac{1}{n}}} \frac{1}{S_{m+2} t^{\frac{m+1}{n}}} \frac{1}{k^{\frac{1}{n}}}
+ c_2 \frac{(m + 2)n + m (m + 1)S_{m+3} t^{\frac{(m+1)n+m}{n}}}{(m + 3)\rho t^{\frac{n-m}{n}}}. 
\]

When
\[
c_2 \leq \frac{(m + 1)n + m - 1}{(m + 2)n + m} \frac{1}{\sqrt{2} S_{m+2} m + 3} \frac{k^{\frac{1}{n}}}{k},
\]
we get that \(g_2(t)\) is decreasing on \((0, (2\pi)^{-n} V(\Omega))\] by using the following formulas
\[
\rho \geq (2\pi)^{-n} \omega_n^{\frac{1}{n}} \frac{V(\Omega)}{n^{\frac{n+1}{n}}}. 
\]

Hence \(g(t)\) is also decreasing on \((0, (2\pi)^{-n} V(\Omega))\]. This implies
\[
\sum_{i=1}^{k} \lambda_i \geq \omega_n^{\frac{n^2}{2}} \psi(0) - \frac{2\omega_n^{\frac{n-1}{n}}}{(n + 2)\rho^{m+1}} \frac{S_{m+2} \psi(0)}{n} \frac{(m+1)n+m-1}{n} \frac{k^{\frac{1}{n}}}{k^{\frac{n-m}{n}}}
+ c_2 \frac{2\omega_n^{\frac{n}{n}} (m + 1)S_{m+3} \psi(0)}{(n + 2)(m + 3)\rho^{m+2}} \frac{k^{\frac{n-m}{n}}}{k^{\frac{1}{n}}},
\]
where
\[
\psi(0) = \frac{V(\Omega)}{(2\pi)^n},
\]
and
\[
\rho = \frac{V(\Omega)}{(2\pi)^n} \frac{\omega_n^{\frac{1}{n}}}{\rho^{m+1}}.
\]

From the above lemma, we have the following universal lower bounds for higher eigenvalues.

**Corollary 3.3** For any bounded domain \(\Omega \subseteq \mathbb{R}^n\), \(n \geq m + 1 \geq 3\) and \(k \geq 1\) we have
\[
\lambda_k \geq \omega_n^{\frac{n^2}{2}} \alpha^{\frac{n}{2}} k^{\frac{2}{n}}
- \frac{2\omega_n^{\frac{n-1}{n}}}{(n + 2)\rho^{m+1}} \frac{S_{m+2} \alpha^{(m+1)n+m-1}}{n} \frac{k^{\frac{m+1}{n}}}{k^{\frac{n-m}{n}}}
+ c_2 \frac{2\omega_n^{\frac{n}{n}} (m + 1)S_{m+3} \alpha^{(m+2)n+m}}{(n + 2)(m + 3)\rho^{m+2}} \frac{k^{\frac{n-m}{n}}}{k^{\frac{1}{n}}},
\]
where
\[
c_2 \leq \min \left\{ 1, \frac{(m + 1)n + m - 1}{(m + 2)n + m} \frac{\sqrt{2} S_{m+2} m + 3}{S_{m+3}} \frac{k^{\frac{1}{n}}}{k^{\frac{1}{n}}} \right\},
\]
satisfying \(S_l = (a + 1)^l - a^l\),
\[ \alpha = \frac{V(\Omega)}{(2\pi)^n}, \quad \rho = 2(2\pi)^{-n} \sqrt{V(\Omega)I(\Omega)} \] and \( a \) is defined by (2.16).

Due to the similar discussion to Corollary 2.4, we have

**Corollary 3.4** For any bounded domain \( \Omega \subseteq \mathbb{R}^n, n \geq 3 \) and any \( k \geq 1 \) we have

\[
\begin{align*}
\sum_{i=1}^{k} \lambda_i & \geq \frac{n\omega_n}{n+2} \frac{2(n+2) V(\Omega) k^{n+2}}{\pi} + \frac{1}{24(n+2)} \left( \frac{V(\Omega)}{I(\Omega)} \right)^k k \\
& + \frac{\omega_n}{9(n+2) \rho^4} k^{n-1} + \frac{3\omega_n}{80(n+2) \rho^4} k^{n-2}.
\end{align*}
\]

**Proof** According to Lemma 3.2, we have

\[
\begin{align*}
n(n+2)B - (n+2)\tau^2nA + 2\tau^{n+2} & \geq \frac{\tau^5}{6} + \frac{\tau^{n-1}}{9} + \frac{3\tau^{n-2}}{80}.
\end{align*}
\]

By using similar discussion in the proof of Theorem 2.1, we get

\[
\begin{align*}
\sum_{i=1}^{k} \lambda_i & \geq \frac{n\omega_n}{n+2} \frac{2(n+2) \phi(0) k^{n+2}}{\pi} + \frac{\phi(0)^2}{6(n+2) \rho^2} k + \frac{\omega_n}{9(n+2) \rho^4} k^{n-1} \\
& + \frac{C_3\omega_n}{(n+2) \rho^4} \frac{4n+2}{n} k^{n-2},
\end{align*}
\]

where \( 0 < C_3 \leq \frac{3}{80} \) is a constant which will be chosen. By using the similar discussion in the proof of Corollary 4.4, one can choose \( C_3 = \frac{3}{80} \). Hence, we complete our proof. \( \Box \)

### 4 A universal lower bound on eigenvalues of the clamped plate problem

In this section, let \( \phi(z) \) be the decreasing radial rearrangement of \( h(z) \) where \( h(z) \) is defined as (4.9). Then, \( a \) is defined by

\[
\int_a^{a+1} z^{n+3} dz = \int_0^\infty -z^{n+3} \phi'(z) dz.
\]

We will give a universal lower bounds on the sum of eigenvalues for \( n \geq m \), where \( m \geq 1 \).

**Theorem 4.1** For any bounded domain \( \Omega \subseteq \mathbb{R}^n, n \geq m \geq 1 \) and \( k \geq 1 \) we have

1. When \( n = 1 \) and

\[
\frac{2\sqrt{2}S_3}{5} \leq k,
\]

we have

\[
\begin{align*}
\sum_{i=1}^{k} \Gamma_i & \geq \omega_n \frac{2}{\pi} k^{1+\frac{4}{n}} - \omega_n \frac{4S_m+2}{(n+4) \rho^m} \frac{mn+4}{n} k^{n-m+4} \\
& + \omega_n \frac{m-3}{(n+4)(m+2) \rho^{m+1}} \frac{4mS_m+2}{\alpha} k^{n-m+3}.
\end{align*}
\]
where $\alpha$, $\rho$ are defined by (1.9) and
\[
S_l = (a + 1)^l - a^l.
\]

(2) When $m \geq 2$, we have
\[
\sum_{i=1}^{k} \Gamma_i \geq \omega_n^{-\frac{4}{n}} \alpha^{-\frac{4}{n}} k^{1+\frac{3}{n}} - \omega_n^{-\frac{4}{n}} \frac{4S_{m+2}}{(n+4)\rho^m \alpha^{\frac{m+1-3}{n}}} \frac{a^{m+1-4}}{n} k^{n-\frac{4}{n}} + c_3 \omega_n^{-\frac{4}{n}} \frac{4mS_{m+2}}{(n+4)(m+2)\rho^{m+1} \alpha^{\frac{m+1-3}{n}}} k^{n-\frac{4}{n}},
\]
where
\[
c_3 \leq \min \left\{ 1, \frac{2^{m+1}}{5^{m+2}} (m+2)(m+1) + \frac{m+1}{n} \right\}.
\]

Next, we recall the definition and several properties of the symmetric decreasing rearrangements. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Its symmetric rearrangement $\Omega^*$ is the open ball with the same volume as $\Omega$,
\[
\Omega^* = \left\{ x \in \mathbb{R}^n \mid |x| < \left( \frac{V(\Omega)}{\omega_n} \right)^{\frac{1}{n}} \right\}.
\]

By using a symmetric rearrangement of $\Omega$, we have
\[
I(\Omega) = \int_{\Omega} |x|^2 \, dx \geq \int_{\Omega^*} |x|^2 \, dx = \frac{n}{n+2} V(\Omega) \left( \frac{V(\Omega)}{\omega_n} \right)^{\frac{2}{n}}. \tag{4.3}
\]
Then we have
\[
\int_{\mathbb{R}^n} |x|^4 F(x) \, dx \geq \int_{\mathbb{R}^n} |x|^4 F^*(x) \, dx = n\omega_n \int_0^{\infty} s^{n+3} \phi(s) \, ds. \tag{4.4}
\]

The following lemma is useful in the proof of Theorem 4.1.

**Lemma 4.2** For integers $n \geq m \geq 1$ and positive real numbers $s$ and $\tau$, we have the following inequality:
\[
ns^{n+4} - (n+4)s^n + 4\tau^{n+4} - \sum_{k=1}^{m} 4ks^{k-1}\tau^{n-k+3}(\tau - s)^2 \geq 0. \tag{4.5}
\]

**Proof** Taking $t = \frac{s}{\tau}$, and putting $f(t)$
\[
f(t) = nt^{n+4} - (n+4)t^n + 4 - 4(t - 1)^2 - \sum_{k=2}^{m} 4kt^{k-1}(t - 1)^2.
\]
for \( t \geq 0 \), we get
\[
f'(t) = n(n + 4)t^{n+3} - n(n + 4)t^{n-1}
\]
\[
- \left[ 8(t - 1) + \sum_{k=2}^{m} 4k(k - 1)t^{k-2}(t - 1)^2 + \sum_{k=2}^{m} 8kt^{k-1}(t - 1) \right]
\]
\[
= n(n + 4)t^{n+3} - n(n + 4)t^{n-1}
\]
\[
- (t - 1) \left[ 8 + \sum_{k=2}^{m} 4k(k - 1)t^{k-2}(t - 1) + \sum_{k=2}^{m} 8kt^{k-1} \right]
\]
\[
= n(n + 4)t^{n+3} - n(n + 4)t^{n-1}
\]
\[
- (t - 1) \left[ 8 + \sum_{k=2}^{m} 4k(k - 1)t^{k-1} - \sum_{k=2}^{m} 4k(k - 1)t^{k-2} + \sum_{k=2}^{m} 8kt^{k-1} \right]
\]
\[
= n(n + 4)t^{n+3} - n(n + 4)t^{n-1}
\]
\[
- (t - 1) \left[ 4m(m + 1)t^{m-1} + \sum_{k=3}^{m} (4(k - 1)(k - 2) - 4k(k - 1) + 8(k - 1))t^{k-2} \right]
\]
\[
= n(n + 4)t^{n+3} - n(n + 4)t^{n-1} - 4m(m + 1)t^{m-1}(t - 1)
\]
\[
= [n(n + 4)t^{m-1}(t^2 + 1)(t + 1) - 4m(m + 1)]t^{m-1}(t - 1).
\]

From the above formula, it is clear that when \( n \geq m \), we have \( t = 1 \) is the minimum point of \( f(t) \) and then \( f \geq \min \{ f(1) = 0, f(0) = 0 \} \). We get
\[
\tau^{n+4}f(t) = ns^{n+4} - (n + 4)\tau^4s^n - \sum_{k=1}^{m} 4ks^{k-1}\tau^{n-k+3}(\tau - s)^2 \geq 0.
\]

\[\square\]

Now, we will give the proof of Theorem 4.1.

**Proof of Theorem 4.1** Let \( \{ u_j \}_{j=1}^{\infty} \) be the eigenfunction corresponding to the eigenvalue \( \Gamma_j \), \( j = 1, 2, \ldots \) which satisfy
\[
\begin{aligned}
\Delta^2 u_j &= \Gamma_j u_j, \quad \text{in } \Omega, \\
u_j &= \frac{\partial u_j}{\partial v} = 0, \quad \text{on } \partial \Omega, \\
\int_{\Omega} u_i(x)u_j(x)dx &= \delta_{ij}, \quad \text{for any } i, j.
\end{aligned}
\]

Thus, \( \{ u_j \}_{j=1}^{\infty} \) forms an orthonormal basis of \( L^2(\Omega) \). We define a function \( \varphi_j \) by
\[
\varphi_j(x) = \begin{cases} 
  u_j(x), & x \in \Omega, \\
  0, & x \in \mathbf{R}^n \setminus \Omega.
\end{cases}
\]

Denote by \( \hat{\varphi}_j(z) \) the Fourier transform of \( \varphi_j(x) \). For any \( z \in \mathbf{R}^n \), we have
\[
\hat{\varphi}_j(z) = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} \varphi_j(x)e^{i(x,z)}dx = (2\pi)^{-\frac{n}{2}} \int_{\Omega} u_j(x)e^{i(x,z)}dx.
\]

By the Plancherel formula, we have
\[
\int_{\mathbf{R}^n} \hat{\varphi}_i(z)\hat{\varphi}_j(z) = \delta_{ij} \quad \text{(4.6)}
\]
for any \(i, j\). Since \(\{u_j\}_{j=1}^\infty\) is an orthonormal basis in \(L^2(\Omega)\), the Bessel inequality implies that

\[
\sum_{j=1}^k |\hat{\varphi}_j(z)|^2 \leq (2\pi)^{-n} \int_\Omega |e^{i(x,z)}|^2 \, dx = (2\pi)^{-n} V(\Omega).
\]

For each \(j = 1, \ldots, k\), we deduce from the divergence theorem and \(u_j|_{\partial \Omega} = \frac{\partial u_j}{\partial \nu}|_{\partial \Omega} = 0\) that

\[
z_p^2 \hat{\varphi}_j(z) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi_j(x)(-i)^2 \frac{\partial^2 e^{i(x,z)}}{\partial x_p^2} \, dx
\]

\[
= -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \frac{\partial^2 \varphi_j(x)}{\partial x_p^2} e^{i(x,z)} \, dx
\]

\[
= -\frac{\partial^2 \varphi_j}{\partial x_p^2}(z).
\]

It follows from the Parseval’s identity that

\[
\int_{\mathbb{R}^n} |z|^4 |\hat{\varphi}_j(z)|^2 \, dz = \int_{\mathbb{R}^n} (|z|^2 |\varphi_j(z)|)^2 \, dz
\]

\[
= \int_{\Omega} |\Delta u_j(x)|^2 \, dx
\]

\[
= \Gamma_j.
\]

(4.7)

Since

\[
\nabla \hat{\varphi}_j(z) = (2\pi)^{-\frac{n}{2}} \int_{\Omega} i x u_j(x)e^{i(x,z)} \, dx,
\]

we obtain

\[
\sum_{j=1}^k |\nabla \hat{\varphi}_j(z)|^2 \leq (2\pi)^{-n} \int_{\Omega} |ix e^{i(x,z)}|^2 \, dx = (2\pi)^{-n} I(\Omega).
\]

(4.8)

Putting

\[
h(z) := \sum_{j=1}^k |\hat{\varphi}_j(z)|^2,
\]

(4.9)

one derives from (4.6) that \(0 \leq h(z) \leq (2\pi)^{-n} V(\Omega)\). It follows from (4.8) and the Cauchy-Schwarz inequality that

\[
|\nabla h(z)| \leq 2 \left( \sum_{j=1}^k |\varphi_j(z)|^2 \right)^{\frac{1}{2}} \left( \sum_{j=1}^k |\nabla \varphi_j(z)|^2 \right)^{\frac{1}{2}}
\]

\[
\leq 2(2\pi)^{-n} \sqrt{V(\Omega)I(\Omega)}
\]

for every \(z \in \mathbb{R}^n\). From the Parseval’s identity, we derive

\[
\int_{\mathbb{R}^n} h(z) dz = \sum_{j=1}^k \int_{\Omega} |u_j(x)|^2 \, dx = k.
\]

(4.10)
Applying the symmetric decreasing rearrangement to \( h(z) \) and noting that \( \zeta = \sup |\nabla h| \leq 2(2\pi)^{-n} \sqrt{V(\Omega)} I(\Omega) := \eta \), we see from (2.6)

\[-\eta \leq -\zeta \leq \phi'(s) \leq 0\]

for almost every \( s \). According to (4.4) and (4.7), we infer

\[
\sum_{i=1}^{k} \Gamma_i = \int_{\mathbb{R}^n} |z|^4 h(z) dz \\
\geq \int_{\mathbb{R}^n} |z|^4 h^*(z) dz \\
= n\omega_n \int_{0}^{\infty} s^{n+3} \phi'(s) ds.
\]

(4.11)

In order to apply Lemma 4.2, from (4.4) and the definition of \( A \), we take

\[
\psi(s) = \phi(s), \quad A = \frac{k}{n\omega_n}, \quad \eta = 2(2\pi)^{-n} \sqrt{V(\Omega)} I(\Omega).
\]

(4.12)

From (4.3), we deduce that

\[
\rho \geq 2(2\pi)^{-n} \left( \frac{n}{n+2} \right)^{\frac{1}{2}} \omega_n^{-\frac{1}{2}} n V(\Omega)^{-\frac{n+1}{n}}.
\]

(4.13)

On the other hand, \( 0 < \phi(0) \leq \sup h^*(z) = \sup h(z) \leq (2\pi)^{-n} V(\Omega) \).

For any \( k \geq 1 \) and \( a \geq 0 \), we have

\[
\int_{a}^{a+1} s^{k-1}(\tau - s)^2 ds = \left. \frac{s^{k+2}}{k+2} - \frac{2s^{k+1}}{k+1} \tau + \frac{s^k}{k} \tau^2 \right|_a^{a+1} \\
= \frac{S_{k+2}}{k+2} - \frac{2S_{k+1}}{k+1} \tau + \frac{S_k}{k} \tau^2,
\]

(4.14)

where

\[
S_l = (a + 1)^l - a^l.
\]

Let \( D' = \int_{a}^{a+1} s^{n+4} ds \), from the above lemma, integrating the both sides of (4.5) over \([a, a+1]\), we get

\[
n(n+4)D' - (n+4)\tau^4 nA + 4\tau^{n+4} \geq \sum_{k=1}^{m} 4k\tau^{n-k+3} \left( \frac{S_k}{k} \tau^2 - \frac{2S_{k+1}}{k+1} \tau + \frac{S_{k+2}}{k+2} \right).
\]

(4.15)
\[-4S_2 \tau^{n+3} - \frac{8m S_{m+2}}{m+1} \tau^{n-m+4} + 4 \sum_{k=2}^{m-1} \left(1 - \frac{k-1}{k+1} - \frac{2k}{k+1}\right) S_{k+1} \tau^{n-k+4}\]
\[= 4\tau^{n+4} + 4S_2 \tau^{n+3} + \frac{4m S_{m+2}}{m+2} \tau^{n-m+3} + \frac{4(m-1)S_{m+1}}{m+1} \tau^{n-m+4}\]
\[-4S_2 \tau^{n+3} - \frac{8m S_{m+2}}{m+1} \tau^{n-m+4}\]
\[= 4\tau^{n+4} - 4S_{m+2} \tau^{n-m+4} + \frac{4m S_{m+2}}{m+2} \tau^{n-m+3},\]

and
\[\begin{align*}
4 \sum_{k=2}^{m-1} \left(1 - \frac{k-1}{k+1} - \frac{2k}{k+1}\right) S_{k+1} \tau^{n-k+4} &= 0,
\end{align*}\]

we get
\[n(n + 4)D' - (n + 4)\tau^4 nA + 4\tau^{n+4} \geq 4\tau^{n+4} - 4S_{m+2} \tau^{n-m+4} + \frac{4m S_{m+2}}{m+2} \tau^{n-m+3} .\]

This implies that
\[n(n + 4)D' \geq (n + 4)\tau^4 (nA) - 4S_{m+2} \tau^{n-m+4} + \frac{4m S_{m+2}}{m+2} \tau^{n-m+3} .\]

Taking \(\tau = (nA)^{\frac{1}{n}}\), we get
\[D' \geq \frac{(nA)}{n} \tau^4 - \frac{4S_{m+2}}{n(n + 4)} \tau^{n-m+4} + \frac{4m S_{m+2}}{n(n + 4)(m + 2)} \tau^{n-m+3} \geq \frac{(nA)}{n} \tau^4 - \frac{4S_{m+2}(nA)^{n-m+4}}{n(n + 4)} + \frac{4m S_{m+2}(nA)^{n-m+3}}{n(n + 4)(m + 2)} .\]

Then, we get
\[\int_{0}^{\infty} s^{n+3} \psi(s) ds \geq \frac{(nA)\frac{1}{n}}{n} \psi(0) - \frac{4}{n} - \frac{4S_{m+2}(nA)^{n-m+4}}{n(n + 4)\rho^m} \psi(0) \frac{mn+m-4}{n} + \frac{4m S_{m+2}(nA)^{n-m+3}}{n(n + 4)(m + 2)\rho^m} \rho^{(m+1)n+m-3} .\]
According to (4.4), (4.7) and the above inequality, we conclude

\[
\sum_{i=1}^{k} \Gamma_i = \int_{\mathbb{R}^n} |z|^4 h(z) \, dz \\
\geq \int_{\mathbb{R}^n} |z|^4 h^+(z) \, dz \\
= n \omega_n \int_{0}^{\infty} s^{n+3} \phi(s) \, ds \\
\geq n \omega_n \frac{(nA)^{1+\frac{4}{n}}}{n} \psi(0)^{-\frac{4}{n}} - n \omega_n \frac{4S_{m+2}(nA)^{\frac{n-m+4}{n}}}{n(n+4)\rho^m} \psi(0)^{\frac{m-n+4}{n}} \\
+ n \omega_n \frac{4mS_{m+2}(nA)^{\frac{n-m+3}{n}}}{n(n+4)(m+2)\rho^{m+1}} \psi(0)^{(m+1)n-m-3} \\
= \omega_n (nA)^{1+\frac{4}{n}} \psi(0)^{-\frac{4}{n}} - \omega_n \frac{4S_{m+2}(nA)^{\frac{n-m+4}{n}}}{n(n+4)\rho^m} \psi(0)^{\frac{m-n+4}{n}} \\
+ \omega_n \frac{4mS_{m+2}(nA)^{\frac{n-m+3}{n}}}{n(n+4)(m+2)\rho^{m+1}} \psi(0)^{(m+1)n-m-3}.
\]

For \( m = 1 \) and \( n = 1 \), we define \( f(t) \) as follows

\[
f(t) = f_1(t) + f_2(t),
\]
on \((0, (2\pi)^{-n} V(\Omega)]\), where

\[
f_1(t) = \xi \omega_n (nA)^{1+\frac{4}{n}} t^{-\frac{4}{n}} - \omega_n \frac{4S_{m+2}(nA)^{\frac{n-m+4}{n}}}{n(n+4)\rho^m} t^{-\frac{3}{n}},
\]
and

\[
f_2(t) = (1 - \xi) \omega_n (nA)^{1+\frac{4}{n}} t^{-\frac{4}{n}} + \omega_n \frac{4mS_{m+2}(nA)^{\frac{n-m+3}{n}}}{n(n+4)(m+2)\rho^{m+1}} t^{-\frac{2n-2}{n}},
\]
for \( 0 < \xi \leq 1 \). Then

\[
\frac{nf_1'(t)}{4\omega_n(nA)^{\frac{4}{n}}} = -\xi (nA) t^{-\frac{n+4}{n}} + \frac{2S_{n+2}}{(n+4)\rho} t^{-\frac{n+2}{n}}.
\]

When

\[
\frac{2\sqrt{2}S_3}{5k} \leq \xi \leq 1,
\]
we prove that \( f(t) \) decreases on \((0, (2\pi)^{-n} V(\Omega)]\) by using

\[
\frac{\omega_n^{\frac{4}{n}}}{(2\pi)^2} \leq \frac{1}{2},
\]
and

\[
\rho \geq (2\pi)^{-n} \omega_n^{-\frac{1}{n}} V(\Omega)^{\frac{n+1}{n}}.
\]
Therefore, if
\[ \frac{2\sqrt{2}S_3}{5} \leq k, \]
we get
\[
\sum_{i=1}^{k} \Gamma_i \geq \omega_n(nA)^{1 + \frac{4}{n} - \frac{1}{2} \alpha} - \omega_n \frac{4S_{m+2}(nA)^{n-m+4} \rho^{m+1}}{(n+4)(m+2)} \alpha^{\frac{n}{n-1}} - \omega_n \frac{4mS_{m+2}(nA)^{n-m+3} \rho^{m+1}}{(n+4)(m+2)} \alpha^{\frac{n}{n-1}},
\]
where
\[ \alpha = \frac{V(\Omega)}{(2\pi)^n}, \]
and
\[ \rho = \frac{V(\Omega)^{\frac{n+1}{n}}}{(2\pi)^n \omega_n^{\frac{1}{n}}}. \]

Noting that \( A = \frac{k}{n\omega_n} \), we obtain the following inequality
\[
\sum_{i=1}^{k} \Gamma_i \geq \omega_n(nA)^{1 + \frac{4}{n} - \frac{1}{2} \alpha} - \omega_n \frac{4S_{m+2}(nA)^{n-m+4} \rho^{m+1}}{(n+4)(m+2)} \alpha^{\frac{n}{n-1}} - \omega_n \frac{4mS_{m+2}(nA)^{n-m+3} \rho^{m+1}}{(n+4)(m+2)} \alpha^{\frac{n}{n-1}},
\]
(4.16)

When \( m \geq 2 \), \( F(t) \) is defined by
\[ F(t) = F_1(t) + F_2(t) \]
for \( t \in (0, (2\pi)^{-n} V(\Omega)] \), where
\[ F_1(t) = \omega_n(nA)^{1 + \frac{4}{n} - \frac{1}{2} \alpha} - \omega_n \frac{4S_{m+2}(nA)^{n-m+4} \rho^{m+1}}{(n+4)(m+2)} \alpha^{\frac{n}{n-1}} - \omega_n \frac{4mS_{m+2}(nA)^{n-m+3} \rho^{m+1}}{(n+4)(m+2)} \alpha^{\frac{n}{n-1}}, \]
for \( 0 < c_3 \leq 1 \) and
\[ F_2(t) = -\omega_n \frac{4S_{m+2}(nA)^{n-m+4} \rho^{m+1}}{(n+4)(m+2)} \alpha^{\frac{n}{n-1}}. \]

This implies
\[
\frac{F_1'(t)}{4\omega_n} = -\frac{1}{n(nA)^{1 + \frac{4}{n} - \frac{1}{2} \alpha} t^{-\frac{n+4}{n}}} + c_3 \frac{(m+1)n + m - 3}{n} \omega_n(nA)^{n-m+3} \alpha^{\frac{n}{n-1}} - \omega_n \frac{4mS_{m+2}(nA)^{n-m+3} \rho^{m+1}}{(n+4)(m+2)} \alpha^{\frac{n}{n-1}}.
\]

So, if
\[ c_3 \leq \frac{2^{m+1}(n+2)(m+2)}{S_{m+2}((m+1)n + m - 3)} k^{m+1}, \]
we obtain that $F(t)$ decreases on $(0, (2\pi)^{-n} V(\Omega))$, which yields that
\[
\sum_{i=1}^{k} \Gamma_i \geq \omega_n \frac{4}{n} \alpha^{-\frac{4}{n}} k^{1+\frac{4}{n}} \omega_n \frac{m-4}{n} \frac{4S_{m+2}}{(n+4) \rho^m k^{n-m+4}} \alpha^{\frac{mn+m-4}{n}} k^{n-m+4} + \frac{4mS_{m+2}}{(n+4)(m+2) \rho^{m+1}} \alpha^{\frac{(m+1)n-m-3}{n}} k^{n-m+4},
\]
where
\[
c_3 \leq \min \left\{ 1, \frac{2^{m+1} (n+2)(m+2)}{S_{m+2}[(m+1)n+m-3]} k^{m+1} \right\}.
\]

For higher eigenvalues, we have the following universal lower bounds

**Corollary 4.3** For any bounded domain $\Omega \subseteq \mathbb{R}^n$, $n \geq m \geq 1$ and any $k \geq 1$ we have

(1) When $n = 1$ and
\[
\frac{2\sqrt{2} S_3}{5} \leq k,
\]
we have
\[
\Gamma_k \geq \omega_n \frac{4}{n} \alpha^{-\frac{4}{n}} k^{1+\frac{4}{n}} \omega_n \frac{m-4}{n} \frac{4S_{m+2}}{(n+4) \rho^m \alpha^{\frac{mn+m-4}{n}} k^{n-m+4}} + \frac{4mS_{m+2}}{(n+4)(m+2) \rho^{m+1}} \alpha^{\frac{(m+1)n-m-3}{n}} k^{n-m+4},
\]
where $\alpha, \rho$ are defined by (1.9) and
\[
S_l = (a + 1)^l - a^l.
\]

(2) When $m \geq 2$, we have
\[
\Gamma_k \geq \omega_n \frac{4}{n} \alpha^{-\frac{4}{n}} k^{1+\frac{4}{n}} \omega_n \frac{m-4}{n} \frac{4S_{m+2}}{(n+4) \rho^m \alpha^{\frac{mn+m-4}{n}} k^{n-m+4}} + c_3 \omega_n \frac{m-3}{n} \frac{4mS_{m+2}}{(n+4)(m+2) \rho^{m+1}} \alpha^{\frac{(m+1)n-m-3}{n}} k^{n-m+4},
\]
where
\[
c_3 \leq \min \left\{ 1, \frac{2^{m+1} (n+2)(m+2)}{S_{m+2}[(m+1)n+m-3]} k^{m+1} \right\}.
\]

According to Lemma 4.2 and the proof of Theorem 4.1, we also have the following result.

**Corollary 4.4** For any bounded domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$ and any $k \geq 1$ we have
\[
\sum_{i=1}^{k} \Gamma_i \geq \frac{n}{n+4} (\omega_n)^{-\frac{4}{n}} \alpha^{-\frac{4}{n}} k^{\frac{4+n}{n}} + \frac{1}{3(n+4)} (\omega_n)^{-\frac{2}{n}} \alpha^{\frac{2n-2}{n}} k^{\frac{n+2}{n}} + \frac{2}{9(n+4)} (\omega_n)^{-\frac{1}{n}} \alpha^{-\frac{3n-1}{n}} k^{\frac{n+1}{n}} + \frac{3\alpha^4}{40(n+4) \rho^3 k}.
\]
**Proof** By using Lemma 4.2, we get

\[
s^{n+4} - (n + 4)\tau^4 s^n + 4\tau^{n+4} \geq 4\tau^{n+2}(\tau - s)^2 + 8s\tau^{n+1}(\tau - s)^2 + 12s^2\tau^n(\tau - s)^2.
\]

In view of (4.15) and (57) in [30], integrating the both sides of the above inequality over \([a, a + 1]\), we have

\[
n(n + 4)D' - (n + 4)\tau^4 nA + 4\tau^{n+4} \geq \frac{\tau^{n+2}}{3} + \frac{2\tau^{n+1}}{9} + 12\tau^n \int_0^1 s^2(\tau - s)^2 \, ds \\
\geq \frac{\tau^{n+2}}{3} + \frac{2\tau^{n+1}}{9} + 12\tau^n \text{min}_{\tau \geq \frac{1}{4}} \int_0^{\frac{1}{2}} s^2(\tau - s)^2 \\
\geq \frac{\tau^{n+2}}{3} + \frac{2\tau^{n+1}}{9} + \frac{3\tau^n}{40}.
\]

By using similar discussion in the proof of Theorem 4.1 and taking \(\tau = (nA)^{\frac{1}{n}}\), we get

\[
\int_0^{\infty} s^{n+3}\psi(s) \, ds \geq \frac{1}{n + 4} s^{n+4} + \frac{\tau^{n+2}}{3(n + 4)} + \frac{2\tau^{n+1}}{9(n + 4)} + \frac{3\tau^n}{40(n + 4)}.
\]

Hence, we arrive at

\[
\sum_{i=1}^k \Gamma_i \geq \frac{n\omega_n}{n + 4} \left(nA\right)^{\frac{n+4}{n}} \psi(0)^{-\frac{4}{n}} + \frac{\omega_n(nA)^{\frac{n+2}{n}}}{3(n + 4)\rho^2} \psi(0)^{\frac{2n-2}{n}} \\
+ \frac{2\omega_n(nA)^{\frac{n+1}{n}}}{9(n + 4)\rho^3} \psi(0)^{-\frac{3n-1}{n}} + d_1 \frac{3\omega_n(nA)}{40(n + 4)\rho^4} \psi(0)^4 \\
= \frac{n\omega_n}{n + 4} \left(\frac{k}{\omega_n}\right)^{\frac{n+4}{n}} \psi(0)^{-\frac{4}{n}} + \frac{\omega_n \left(\frac{k}{\omega_n}\right)^{\frac{n+2}{n}}}{3(n + 4)\rho^2} \psi(0)^{\frac{2n-2}{n}} \\
+ \frac{2\omega_n \left(\frac{k}{\omega_n}\right)^{\frac{n+1}{n}}}{9(n + 4)\rho^3} \psi(0)^{-\frac{3n-1}{n}} + d_1 \frac{3\omega_n \left(\frac{k}{\omega_n}\right)}{40(n + 4)\rho^4} \psi(0)^4,
\]

where \(0 < d_1 \leq 1\) is a constant to be determined. Let \(t \in (0, (2\pi)^{-n} V(\Omega))\), we define

\[
Q(t) = \frac{n}{n + 4} \left(\frac{k}{\omega_n}\right)^{\frac{n+4}{n}} t^{-\frac{4}{n}} + \frac{\left(\frac{k}{\omega_n}\right)^{\frac{n+2}{n}}}{3(n + 4)\rho^2} t^{\frac{2n-2}{n}} \\
+ \frac{2\left(\frac{k}{\omega_n}\right)^{\frac{n+1}{n}}}{9(n + 4)\rho^3} t^{-\frac{3n-1}{n}} + d_1 \frac{3\left(\frac{k}{\omega_n}\right)}{40(n + 4)\rho^4} t^4.
\]
which would be decreasing on \((0, (2\pi)^{-n} V(\Omega))\] if \(Q'(2\pi)^{-n} V(\Omega)) \leq 0\). Obviously, \(Q'(2\pi)^{-n} V(\Omega)) \leq 0\) is equal to
\[
4 \left(\frac{k}{\omega_n}\right)^{\frac{4}{n}} \left(\frac{(2\pi)^{n}}{V(\Omega)}\right)^{1+\frac{4}{n}} \geq \frac{(2n-2)}{n} \left(\frac{k}{\omega_n}\right)^{\frac{2}{n}} \frac{1}{\rho^2} \left(\frac{V(\Omega)}{(2\pi)^{n}}\right)^{\frac{n+2}{n}} + \frac{3n-1}{n} \left(\frac{k}{\omega_n}\right)^{\frac{1}{n}} \frac{2}{9(n+4)\rho^3} \left(\frac{V(\Omega)}{(2\pi)^{n}}\right)^{\frac{2n-1}{n}} + 4d_1 \frac{3}{40(n+4)\rho^4} \left(\frac{V(\Omega)}{(2\pi)^{n}}\right)^{3}.
\]

Due to (2.18) and \(\frac{\omega_n}{(2\pi)^{n}} \leq \frac{1}{2}\), if
\[d_1 \leq \min\{1, d_0\},\]
we have \(Q'(2\pi)^{-n} V(\Omega)) \leq 0\), where
\[d_0 = \frac{140(2\pi)^{2}}{3} \left(4 \left(\frac{k}{\omega_n}\right)^{\frac{4}{n}} - \frac{1}{\sqrt{2\pi}} \left(\frac{k}{\omega_n}\right)^{\frac{2}{n}} - \frac{2}{21(2\pi)^{3}} \left(\frac{1}{2}\right)^{\frac{4}{n}} \left(\frac{k}{\omega_n}\right)^{\frac{1}{n}}\right).\]

By direct computation, one has \(d_0 > 1\). Therefore, we obtain the following eigenvalue inequality
\[
\sum_{i=1}^{k} \Gamma_i \geq \frac{n \omega_n}{n+4} \left(\frac{k}{\omega_n}\right)^{\frac{n+4}{n}} \alpha^{-\frac{4}{n}} + \frac{\omega_n}{3(n+4)\rho^2} \alpha^{\frac{n+2}{n}} + \frac{2\omega_n}{9(n+4)\rho^3} \alpha^{\frac{3n-1}{n}} + \frac{3\omega_n}{20(n+4)\rho^4} \alpha^{4}.
\]

**Acknowledgements**  This work was supported by the National Natural Science Foundation of China, Grant No. 12071424, 11531012. The first author was also supported by the Postdoctoral Fund of Zhejiang Province, China, Grant No. ZJ2022004. Ji would like to thank Professor Kefeng Liu for his continued support, advice and encouragement.

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