A GRAPH THEORETIC PROOF OF THE TIGHT CUT LEMMA

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Abstract. In deriving their characterization of the perfect matchings polytope, Edmonds, Lovász, and Pulleyblank introduced the so-called Tight Cut Lemma as the most challenging aspect of their work. The Tight Cut Lemma in fact claims bricks as the fundamental building blocks that constitute a graph in studying the matching polytope and can be referred to as a key result in this field. Even though the Tight Cut Lemma is a matching (1-matching) theoretic statement that consists of purely graph theoretic concepts, the known proofs either employ a linear programming argument or are established upon results regarding a substantially wider notion than matchings. This paper presents a new proof of the Tight Cut Lemma, which attains both of the two reasonable features for the first time, namely, being purely graph theoretic as well as purely matching theory closed. Our proof uses, as the only preliminary result, the canonical decomposition recently introduced by Kita. By further developing this canonical decomposition, we acquire a new device of towers to analyze the structure of bricks, and thus prove the Tight Cut Lemma. We believe that our new proof of the Tight Cut Lemma provides a highly versatile example of how to handle bricks.

1. Introduction

Edmonds, Lovász, and Pulleyblank [6] introduced the Tight Cut Lemma as a key result in their paper characterizing the perfect matching polytope. They stated that proving the Tight Cut Lemma was the most difficult part.

Tight Cut Lemma. Any tight cut in a brick is trivial.

A graph is a brick if deleting any two vertices results in a connected graph with a perfect matching. A cut is tight if it shares exactly one edge with any perfect matching. A tight cut is trivial if it is a star cut.

The Tight Cut Lemma in fact characterizes the bricks as the fundamental building blocks that constitute a graph in the polyhedral study of matchings via the inductive operation the tight cut decomposition. As long as a given graph has a non-trivial tight cut, we can apply an operation that decomposes it into two smaller graphs that perfectly
inherit the matching theoretic property; this is the **tight cut decomposition**. In fact, we can view the Tight Cut Lemma as stating that the bricks are the irreducible class of the tight cut decomposition. Via the tight cut decomposition, Edmonds et al. [6] derived the dimension of the perfect matching polytope using as a parameter the number of bricks that constitute a given graph. Consequently, they determine the minimal set of inequalities that defines the perfect matching polytope.

Since Edmonds et al. [6], the study of bricks and the consequential results on the perfect matching polytope (and lattice) have flourished; see Lovász [16] and Carvalho, Lucchesi, and Murty [4, 2, 3, 5, 1]. Edmonds et al. [6] proves the Tight Cut Lemma via a linear programming argument, whereas the statement itself consists of purely graph theoretic notions only. This might be problematic as well as awkward because not knowing how to treat bricks and tight cuts combinatorially might limit our ability to investigate this field. Szigeti [19] later gives a purely graph theoretic proof using the theory of optimal ear-decomposition proposed by Frank [7].

In this paper, we give a new purely graph theoretic proof using the theory of *canonical decomposition* for general graphs with perfect matchings, which was recently proposed by Kita [9, 10]. As the term “canonical” conventionally means in the mathematical context, canonical decompositions are a standard tool to analyze graphs in matching theory. Several canonical decompositions are classically known such as the Gallai-Edmonds, the Kotzig-Lovász, and the Dulmage-Mendelsohn [17]. However, none of them target the general graphs with perfect matchings but rather more particular classes of graphs, until Kita [9, 10] introduced a new canonical decomposition. To prove the Tight Cut Lemma, we must assume that we are given a brick, a non-star cut, and a perfect matching that shares exactly one edge, say, \(e\), with the cut, and then find another perfect matching that shares more than one edge with the cut. Deleting \(e\) from the brick together with its ends results in a graph with perfect matchings. Hence, analyzing the structure of this graph with Kita’s canonical decomposition would appear to be more reasonable means of obtaining a new proof of the Tight Cut Lemma.

We further characterize our new proof as purely *matching (1-matching) theory closed* as well as purely graph theoretic. Our proof uses solely the canonical decomposition by Kita for known results, which is obtained from scratch via the most elementary discussion regarding 1-matchings. In contrast, Szigeti’s proof involves explicitly or implicitly a lot more things, some of which are not 1-matching closed; because, the optimal ear-decomposition theory is established upon not only many known results and notions in matching theory such as the Tutte-Berge formula and the notion of barriers, the Gallai-Edmonds decomposition,
and the theory of ear-decompositions of some classes of graphs, but also the theory of $T$-join, which is a substantially wider notion than 1-matchings. As the Tight Cut Lemma and the main applications are purely 1-matching theoretic, our proof has quite a reasonable nature. We also believe our proof to be significant in that it provides a highly versatile example of how to study bricks graph-theoretically.

The remainder of this paper is organized as follows. Section 2 presents preliminary definitions and results: Section 2.1 explains fundamental notation and definitions; Section 2.2 presents some elementary lemmas, and Section 2.3 introduces the canonical decomposition given by Kita [9, 10]. Section 3 introduces new results of us; here, we further develop a device to analyze the structure of graphs with perfect matchings, which will be used in Section 4. Section 4 gives the new proof of the Tight Cut Lemma.

2. Preliminaries

2.1. Notation and Definitions.

2.1.1. General Statements. For standard notations and definitions on sets and graphs, we mostly follow Shcrijver [18] in this paper. In this section, we list those that are exceptional or non-standard. We denote the vertex set and the edge set of a graph $G$ by $V(G)$ and $E(G)$, respectively. We sometimes refer to the vertex set of a graph $G$ simply as $G$. As usual, we often denote a singleton $\{x\}$ simply by $x$.

2.1.2. Operations of Graphs. Let $G$ be a graph, and let $X \subseteq V(G)$. The subgraph of $G$ induced by $X$ is denoted by $G[X]$. The notation $G - X$ denotes the graph $G[V(G) \setminus X]$. The contraction of $G$ by $X$ is denoted by $G/X$. Let $\hat{G}$ be a supergraph of $G$, and let $F \subseteq E(\hat{G})$. The notation $G + F$ and $G - F$ denotes the graphs obtained by adding and by deleting $F$ from $G$. Given another subgraph $H$ of $\hat{G}$, the graph $G + H$ denotes the union of $G$ and $H$. In referring to graphs obtained by these operations, we often identify their items such as vertices and edges with the naturally corresponding items of old graphs.

2.1.3. Paths and Circuits. We treat paths and circuits as graphs; i.e., a circuit is a connected graph in which every vertex is of degree two, and a path is a connected graph if every vertex is of degree no more than two and it is not a circuit. Given a path $P$ and two vertices $x$ and $y$ in $V(P)$, $xPy$ denotes the connected subgraph of $P$, which is of course a path, that has the ends $x$ and $y$.

2.1.4. Functions on Graphs. The set of neighbors of $X \subseteq V(G)$ in a graph $G$ is denoted by $N_G(X)$; namely, $N_G(X) := \{u \in V(G) \setminus X : \exists v \in X \text{ s.t. } uv \in E(G)\}$. Given $X, Y \subseteq V(G)$, $E_G[X, Y]$ denotes the set of edges of $G$ whose two ends are in $X$ and in $Y$. We denote...
Given a set of edges $M$, a circuit $C$ is $M$-alternating if $E(C) \cap M$ is a perfect matching of $C$. A path $P$ with two ends $x$ and $y$ is $M$-saturated if $E(P) \cap M$ is a perfect matching of $P$. A path $P$ with ends $x$ and $y$ is $M$-balanced from $x$ to $y$ if $E(P) \cap M$ is a matching of $P$ and, among the vertices in $V(P)$, only $y$ is disjoint from the edges in $E(P) \cap M$. We define a trivial graph, i.e., a graph with a single vertex and no edges, as an $M$-balanced path. In other words, if we trace an $M$-alternating circuit, or $M$-saturated, exposed, or balanced path from a vertex, then edges in $M$ and in $E(G) \setminus M$ appear alternately; in an $M$-saturated path, both edges adjacent to the ends are in $M$, whereas in an $M$-exposed path, neither of them are, and in an $M$-balanced path, one of them is in $M$ but the other is not.

Given a set of vertices $X$, an $M$-exposed path is an $M$-ear relative to $X$ if the ends are in $X$ while the other vertices are disjoint from $X$; also, a circuit $C$ is an $M$-ear relative to $X$ if $V(C) \cap X = \{x\}$ holds and $C - x$ is an $M$-saturated path. In the first case, we say the $M$-ear is proper. Even in the second case, we call $x$ an end of the $M$-ear for convenience. An $M$-ear is trivial if it consists of only a single edge. We say an $M$-ear traverses a set of vertices $Y$ if it has a vertex other than the ends that is in $Y$.

### 2.2. Fundamental Properties

We now present elementary lemmas that will be used in later sections. They are easy to confirm.

**Lemma 2.1.** Let $G$ be a factorizable graph, and $M$ be a perfect matching of $G$. Given two distinct vertices $u, v \in V(G)$, $G - u - v$ is factorizable if and only if there is an $M$-saturated path between $u$ and $v$.

**Lemma 2.2.** Let $G$ be a factorizable graph and $M$ be a perfect matching of $G$. Let $C$ be an $M$-alternating circuit of $G$. Then $M \triangle E(C)$ is a perfect matching of $G$, and therefore the edges of $C$ are all allowed.

### 2.3. Canonical Decomposition for General Factorizable Graphs

We now introduce the canonical decomposition given by Kita [9, 10], which will be used in Sections 3 and 4 as the only preliminary result.

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1We denote the symmetric difference of two sets $A$ and $B$ by $A \triangle B$. 

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to derive the Tight Cut Lemma. The principal results that constitute the theory of this canonical decomposition are Theorems 2.5, 2.7, and 2.8. In this section, unless otherwise stated, $G$ denotes a factorizable graph.

**Definition 2.3.** Let $\hat{M}$ be the union of all perfect matchings of $G$. A factor-component of $G$ is the subgraph induced by $V(C)$, where $C$ is a connected component of the subgraph of $G$ determined by $\hat{M}$. The set of factor-components of $G$ is denoted by $\mathcal{G}(G)$. That is to say, a factorizable graph consists of factor-components and edges joining distinct factor-components. A separating set of $G$ is a set of vertices that is the union of the vertex sets of some factor-components of $G$. Note that if $X \subseteq V(G)$ is a separating set, then $\delta_G(X) \cap M = \emptyset$ for any perfect matching $M$ of $G$.

**Definition 2.4.** Given $G_1, G_2 \in \mathcal{G}(G)$, we say $G_1 \triangleleft_G G_2$ if there is a separating set $X \subseteq V(G)$ such that $V(G_1) \cup V(G_2) \subseteq X$ holds and $G[X]/V(G_1)$ is a factor-critical graph. We sometimes denote $\triangleleft_G$ simply by $\triangleleft$.

The next theorem is highly analogous to the known Dulmage-Mendelsohn decomposition for bipartite graphs [17], in that it describes a partial order over $\mathcal{G}(G)$:

**Theorem 2.5** (Kita [10, 9]). In any factorizable graph, $\triangleleft$ is a partial order over $\mathcal{G}(G)$.

Under Theorem 2.5, we denote the poset of $\triangleleft$ over $\mathcal{G}(G)$ by $\mathcal{O}(G)$. For $H \in \mathcal{G}(G)$, the set of upper bounds of $H$ in $\mathcal{O}(G)$ is denoted by $U_G(H)$. The union of vertex sets of all upper bounds of $H$ is denoted by $U_G^*(H)$. We denote $U_G^*(H) \setminus \{H\}$ by $U_G(H)$ and $U_G^*(H) \setminus V(H)$ by $U_G(H)$. We sometimes write them by omitting the subscripts “$G$”.

**Definition 2.6.** Given $u, v \in V(G)$, we say $u \sim_G v$ if $u$ and $v$ are contained in the same factor-component and $G - u - v$ has no perfect matching.

**Theorem 2.7** (Kita [10, 9]). In any factorizable graph $G$, $\sim_G$ is an equivalence relation on $V(G)$. Each equivalence class is contained in the vertex set of a factor-component.

Given $H \in \mathcal{G}(G)$, we denote by $\mathcal{P}_G(H)$ the family of equivalence classes of $\sim_G$ that are contained in $V(H)$. Note that $\mathcal{P}_G(H)$ gives a partition of $V(H)$. The structure given by Theorem 2.7 is called the generalized Kotzig-Lovász partition as it is a generalization of the results given by Kotzig [12, 14, 13] and Lovász [15].

Even though Theorems 2.5 and 2.7 were established independently, a natural relationship between the two is shown by the next theorem.
Theorem 2.8 (Kita \cite{10, 9}). Let $G$ be a factorizable graph, and let $H \in \mathcal{G}(G)$. Let $K$ be a connected component of $G[U_G(H)]$. Then, there exists $S \in \mathcal{P}_G(H)$ with $N_G(K) \cap V(H) \subseteq S$.

Intuitively, Theorem 2.8 states that each proper upper bound of a factor-component $H$ is tagged with a single member from $\mathcal{P}_G(H)$. As a result of Theorem 2.8, the two structures given by Theorems 2.5 and 2.8 are unified naturally to produce a new canonical decomposition that enables us to analyze a factorizable graph as a building-like structure in which each factor-component serves as a floor and each equivalence class serves as a foundation.

As given in Theorem 2.8, for $H \in \mathcal{G}(G)$ and $S \in \mathcal{P}_G(S)$, we define $\mathcal{U}_G(S) \subseteq \mathcal{U}_G(H)$ as follows: $I \in \mathcal{U}_G(H)$ is in $\mathcal{U}_G(S)$ if the connected component $K$ of $G[U_G(H)]$ with $V(I) \subseteq V(K)$ satisfies $N_G(K) \cap V(H) \subseteq S$. The union of vertex sets of factor-components in $\mathcal{U}_G(S)$ is denoted by $U_G^*(S)$. The sets $U_G^*(S) \setminus S$ and $U_G^*(H) \setminus U_G^*(S)$ are denoted by $U_G(S)$ and $U_G(H)$, respectively. Note that the family $\{U^*(S) : S \in \mathcal{P}_G(H)\}$ (resp. $\{U(S) : S \in \mathcal{P}_G(H)\}$) gives a partition of $U^*(H)$ (resp. $U(H)$). We sometimes omit the subscript "G" if the meaning is apparent from the context.

In the remainder of this section, we present some pertinent properties that will be used in later sections.

Lemma 2.9 (Kita \cite{11, 8}). Let $G$ be a factorizable graph and $M$ be a perfect matching of $G$, and let $H \in \mathcal{G}(G)$. Let $S \in \mathcal{P}_G(H)$, and let $T \in \mathcal{P}_G(H)$ be such with $S \neq T$.

(i) For any $x \in U^*(S)$, there exists $y \in S$ such that there is an $M$-balanced path from $x$ to $y$ whose vertices except for $y$ are in $U(S)$.

(ii) For any $x \in S$ and any $y \in T$, there is an $M$-saturated path between $x$ and $y$ whose vertices are in $U^*(H) \setminus U(S) \setminus U(T)$.

(iii) For any $x \in S$ and any $y \in U(S)$, there is an $M$-balanced path from $x$ to $y$ whose vertices are in $U^*(H) \setminus U(S)$.

(iv) For any $x \in U^*(S)$ and any $y \in U^*(T)$ there is an $M$-saturated path between $x$ and $y$ whose vertices are in $U^*(H)$.

Lemma 2.10 (Kita \cite{10, 9}). Let $G$ be a factorizable graph, and let $M$ be a perfect matching of $G$. If there is an $M$-ear relative to $H_1 \in \mathcal{G}(G)$ and traversing $H_2 \in \mathcal{G}(G)$, then $H_1 \triangleright H_2$ holds.

From Lemma 2.10 the next lemma is easily derived.

Lemma 2.11 (Kita \cite{10, 9}). Let $G$ be a factorizable graph and $M$ be a perfect matching of $G$. Let $x \in V(G)$, and let $H \in \mathcal{G}(G)$ be such with $x \in V(H)$. If there is an $M$-ear $P$ relative to $\{x\}$, then the connected components of $P - E(H)$ are $M$-ears relative to $H$. Hence, if $I \in \mathcal{G}(G)$ has common vertices with $P$, then $H \triangleright I$ holds.
Lemma 2.12 (Kita [10, 9]). Let \( G \) be a factorizable graph, and \( M \) be a perfect matching of \( G \). If \( G_1 \in \mathcal{G}(G) \) is an immediate lower-bound of \( G_2 \in \mathcal{G}(G) \) with respect to \( \prec \), then there is an \( M \)-ear relative to \( G_1 \) and traversing \( G_2 \).

Remark 2.13. The results presented in this section are obtained without using any known results via a fundamental graph theoretic discussion on matchings.

3. Structure of Towers

The remainder of this paper introduces the new results. In this section, we further develop the theory of canonical decomposition in Section 2.3 to acquire lemmas for Section 4. We define and explore the notions of towers, arcs, and tower-sequences, and work towards assuring the existence of arcs and tower-sequences with certain maximality, spanning arcs and spanning tower-sequences. We aim at obtaining Lemmas 3.12 and 3.14; they will be the main tools in Section 4.2.

In this section, unless otherwise stated, let \( G \) be a factorizable graph and \( M \) be a perfect matching. The set of minimal elements in the poset \( \mathcal{O}(G) \) is denoted by \( \min \mathcal{O}(G) \).

Definition 3.1. Let \( H \in \mathcal{G}(G) \). A tower over \( H \) is the subgraph \( G[U^*(H)] \) and is denoted by \( T_G(H) \) or simply by \( T(H) \).

Given \( H_1, H_2 \in \mathcal{G}(G) \) such that neither \( H_1 \prec H_2 \) nor \( H_2 \prec H_1 \) hold, we say \( T(H_1) \) and \( T(H_2) \) are tower-adjacent or \( t \)-adjacent if \( U(H_1) \cap U(H_2) \neq \emptyset \) or \( E[U^*(H_1), U^*(H_2)] \neq \emptyset \) hold. Here, \( S_1 \in \mathcal{P}_G(H_1) \) is a part of this adjacency if \( U(S_1) \cap U(H_2) \neq \emptyset \) or \( E[U^*(S_1), U^*(H_2)] \neq \emptyset \) hold.

The next lemma will be used for Lemma 3.4 as well as for Lemma 4.11.

Lemma 3.2. Let \( G \) be a factorizable graph, and \( M \) be a perfect matching of \( G \). For any \( H \in \mathcal{G}(G) \), there is no non-trivial \( M \)-ear relative to \( T(H) \). Any trivial \( M \)-ear relative to \( V(T(H)) \) is an edge of \( T(H) \).

Proof. Let \( P \) be a non-trivial \( M \)-ear relative to \( T(H) \). Let \( x_1 \) and \( x_2 \) be the ends of \( P \), and let \( S_1, S_2 \in \mathcal{P}_G(H) \) be such with \( x_1 \in U^*(S_1) \) and \( x_2 \in U^*(S_2) \). By Lemma 2.9 [11], there is an \( M \)-balanced path \( Q_i \) from \( x_1 \) to a vertex \( t_i \in S_i \) with \( V(Q_i) \setminus \{t_i\} \subseteq U(S_i) \) for each \( i \in \{1, 2\} \). Trace \( Q_i \) from \( x_1 \), and let \( z_1 \) be the first encountered vertex that is in a factor-component \( I \) with \( V(I) \cap V(Q_2) \neq \emptyset \). Trace \( Q_2 \) from \( x_2 \), and let \( z_2 \) be the first encountered vertex in \( I \). Note that \( Q_i \) is an \( M \)-balanced path from \( x_i \) to \( t_i \) for each \( i \in \{1, 2\} \). Then, \( x_1Q_1z_1 + P + x_2Q_2z_2 \) is an \( M \)-ear relative to \( I \) and traversing the factor-components that \( P \) traverses. This implies by Lemma 2.10 that the factor-components traversed by \( P \) are upper bounds of \( I \in U^*(H) \) and accordingly, of \( H \), too. This is a contradiction. This proves the first statement. The remaining statement is obvious. \( \Box \)
Definition 3.3. Let \( H_1, H_2 \in \mathcal{G}(G) \) be such with \( H_1 \neq H_2 \). An \( M \)-exposed path \( P \) is an \( M \)-arc between \( H_1 \) and \( H_2 \) if the ends of \( P \) are in \( H_1 \) and \( H_2 \) whereas the other vertices are disjoint from \( H_1 \) and \( H_2 \).

The next two lemmas describe properties on \( M \)-arcs.

Lemma 3.4. Let \( G \) be a factorizable graph, and \( M \) be a perfect matching of \( G \). Let \( H_1, H_2 \in \mathcal{G}(G) \) be such that neither \( H_1 \triangleleft H_2 \) nor \( H_2 \triangleleft H_1 \) hold. If \( T(H_1) \) and \( T(H_2) \) are \( t \)-adjacent, with ports \( S_1 \in \mathcal{P}_G(H_1) \) and \( S_2 \in \mathcal{P}_G(H_2) \), then there is an \( M \)-arc between \( H_1 \) and \( H_2 \), whose ends are in \( S_1 \) and \( S_2 \) whereas the other vertices are contained in \( U(S_1) \cup U(S_2) \).

Proof. Let \( uv \in E[U^*(S_1), U^*(S_2) \setminus U^*(S_1)] \), where \( u \in U^*(S_1) \) and \( v \in U^*(S_2) \). By Lemma 2.9(i), there is an \( M \)-balanced path \( P_2 \) from \( v \) to a vertex \( w \in S_2 \) with \( V(P_2) \setminus \{w\} \subseteq U(S_2) \). Additionally, there is an \( M \)-balanced path \( P_1 \) from \( u \) to a vertex \( z \in S_2 \) with \( V(P_1) \setminus \{z\} \subseteq U(S_1) \).

Claim 3.5. The paths \( P_1 \) and \( P_2 \) are disjoint.

Proof. Suppose this claim fails. First, note that \( v \notin U^*(H_1) \); otherwise, \( v \in \mathcal{U}(S_1) \) holds, and this contradicts Theorem 2.8 or the assumption \( H_1 \neq H_2 \). Trace \( P_2 \) from \( v \), and let \( r \) be the first encountered vertex in \( U^*(H_1) \). Then, \( uv + vP_wr \) is a non-trivial \( M \)-ear relative to \( U^*(H_1) \), which contradicts Lemma 3.2.

By Claim 3.5, \( P_1 + uv + P_2 \) is a desired \( M \)-arc.

Lemma 3.6. Let \( G \) be a factorizable graph, and \( M \) be a perfect matching of \( G \). Let \( H_1, H_2 \in \mathcal{G}(G) \) be such that neither \( H_1 \triangleleft H_2 \) nor \( H_2 \triangleleft H_1 \) hold. Let \( P \) be an \( M \)-arc between \( H_1 \) and \( H_2 \), whose ends are \( u_1 \in S_1 \) and \( u_2 \in S_2 \), where \( S_1 \in \mathcal{P}_G(H_1) \) and \( S_2 \in \mathcal{P}_G(H_2) \). Then, \( P \) is disjoint from \( ^cU(S_1) \cup \ ^cU(S_2) \).

Proof. Suppose the statement fails, and let \( x \) be the first vertex in, e.g., \( ^cU(S_1) \) that is encountered when we trace \( P \) from \( u_1 \); \( P \) is an \( M \)-ear relative to \( ^cU(S_1) \cup S_1 \).

By Lemma 2.9(iii), there is an \( M \)-saturated path \( Q \) between \( u_1 \) and \( x \) with \( V(Q) \subseteq \ ^cU(S_1) \cup S_1 \). Then, \( P + Q \) is an \( M \)-saturated path that contains non-allowed edges in \( \delta(\ ^cU(S_1) \cup S_1) \). By Lemma 2.2, this is a contradiction. Hence \( P \) is disjoint from \( ^cU(S_1) \), and also, by symmetry, from \( ^cU(S_2) \).

Definition 3.7. Let \( H_1, \ldots, H_k \in \mathcal{G}(G) \), where \( k \geq 1 \). For each \( i \in \{1, \ldots, k\} \), let \( S_i^+, S_i^- \in \mathcal{P}_G(H_i) \) be such with \( S_i^+ \neq S_i^- \). We say \( H_1, \ldots, H_k \) is a tower-sequence, from \( H_1 \) to \( H_k \), if \( k = 1 \) holds or if \( k > 1 \) holds and for each \( i \in \{1, \ldots, k-1\} \), \( T(H_i) \) and \( T(H_{i+1}) \) are \( t \)-adjacent with ports \( S_i^+ \) and \( S_{i+1}^- \).

The next lemma states that no repetition occurs in a tower-sequence.
Lemma 3.8. Let $G$ be a factorizable graph, and $M$ be a perfect matching of $G$. Let $H_1, \ldots, H_k \in \text{minO}(G)$, where $k > 1$, be a tower-sequence with ports $S_i^+, S_i^- \in \mathcal{P}_G(H_i)$ for $i \in \{1, \ldots, k\}$. Then,

(i) $H_i \neq H_j$ holds for any $i, j \in \{1, \ldots, k\}$ with $i \neq j$, and

(ii) there is an $M$-arc between $H_1$ and $H_k$ whose ends are in $S_1^+$ and $S_k^-$ and which, if $k \geq 3$ holds, traverses each $H_2, \ldots, H_{k-1}$.

Proof. We proceed by induction on $k$. If $k = 2$, then (i) and (ii) hold by the definition of tower-sequences and by Lemma 3.4. Let $k > 2$, and suppose (i) and (ii) hold for $1, \ldots, k-1$. By applying induction hypothesis to the substructures $H_1, \ldots, H_{k-1}$ and $H_2, \ldots, H_k$, we obtain that $H_i \neq H_j$ holds for any $i, j \in \{1, \ldots, k\}$ with $i \neq j$ and $\{i, j\} \neq \{1, k\}$. Consider the subsequence $H_1, \ldots, H_{k-1}$. There is an $M$-arc $P$ between $H_1$ and $H_{k-1}$ that satisfies (ii). Let $s \in S_1^+$ and $t \in S_{k-1}^-$ be the ends of $\hat{P}$. By Lemma 3.4, there is an $M$-arc $P$ between $H_{k-1}$ and $H_k$, whose ends are $s \in S_{k-1}^+$ and $t \in S_k^-$, such that its vertices except for $s$ and $t$ are in $U(S_{k-1}^+) \cup U(S_k^-)$. By Lemma 2.10(ii), there is an $M$-saturated path $Q$ between $s$ and $t$ with $V(Q) \subseteq U^*(H_{k-1}) \cup U(S_{k-1}^-) \cup U(S_k^+)$. Let $\hat{Q} := P + Q$; then, $\hat{Q}$ is an $M$-balanced path from $s$ to $t$ that traverses $H_{k-1}$.

Claim 3.9. The paths $\hat{P}$ and $\hat{Q}$ have only $t$ as a common vertex.

Proof. Suppose this claim fails, and let $x$ be the first vertex in $\hat{P}$ that is encountered if we trace $\hat{Q}$ from $t$. If $x \hat{P}t$ has an even number of edges, then $x \hat{P}t + \hat{Q}x$ is an $M$-alternating circuit that contains non-allowed edges in $\delta(H_{k-1})$. This is a contradiction by Lemma 2.2. Otherwise, if $x \hat{P}t$ has an odd number of edges, then $x \hat{P}t + \hat{Q}x$ is an $M$-ear relative to $x$ and traversing $H_{k-1}$. This implies by Lemma 2.10 that $H_{k-1}$ has a lower-bound in $\text{O}(G)$ that is distinct from itself. This is again a contradiction.

By Claim 3.9, $\hat{P} + \hat{Q}$ is an $M$-exposed path that traverses $H_2, \ldots, H_{k-1}$. If $H_1 = H_k$ holds, then $\hat{P} + \hat{Q}$ is an $M$-ear relative to $H_1$. This is a contradiction by Lemma 2.10 because $H_2, \ldots, H_{k-1} \in \text{minO}(G)$. Hence, we obtain $H_1 \neq H_k$, and so $H_1, \ldots, H_k$ are all mutually distinct. Accordingly, $\hat{P} + \hat{Q}$ is an $M$-arc satisfying the statement.

Definition 3.10. A factor-component $H \in \text{minO}(G)$ is a border of $G$ if $T(H)$ is $t$-adjacent with no other tower or if exactly one member $S$ from $\mathcal{P}_G(H)$ can be a port by which $T(H)$ is $t$-adjacent with other towers, i.e., $E[U^*(S), V(G) \setminus U^*(H)] = \emptyset$ and $E[U^*(T), V(G) \setminus U^*(H)] = \emptyset$ for any $T \in \mathcal{P}_G(H) \setminus \{S\}$ hold. Here, $S$ is the port of the border $H$. We denote the set of borders of $G$ by $\partial \text{O}(G)$.

Definition 3.11. We say a tower-sequence $H_1, \ldots, H_k \in \text{minO}(G)$, where $k \geq 1$, is spanning if $H_1$ and $H_k$ are borders of $G$. An $M$-arc
between $H \in \mathcal{G}(G)$ and $I \in \mathcal{G}(G)$ is spanning if $H$ and $I$ are borders of $G$.

Finally, we can derive the lemmas on spanning tower-sequences and spanning $M$-arcs. From Lemma 3.8, the next lemma is obtained rather easily.

**Lemma 3.12.** Let $G$ be a factorizable graph. For a tower-sequence $H_1, \ldots, H_k \in \min \mathcal{O}(G)$, there is a spanning tower-sequence $I_1, \ldots, I_l \in \min \mathcal{O}(G)$ with $l \geq k$ and $I_i = H_1, \ldots, I_{i+k} = H_k$ for some $i \in \{1, \ldots, k-l\}$.

**Proof.** We first prove the following claim:

**Claim 3.13.** If $H_1, \ldots, H_k \in \min \mathcal{O}(G)$ is not spanning, there exists a tower-sequence $H_1, \ldots, H_k, H_{k+1} \in \min \mathcal{O}(G)$ or $H_0, H_1, \ldots, H_k \in \min \mathcal{O}(G)$.

**Proof.** Because it is not a spanning tower-sequence, either $H_1$ or $H_k$ is a non-border; say, let $H_k \notin \partial \mathcal{O}(G)$. Let $S_i^+, S_i^- \in \mathcal{P}_G(H_i)$ be the ports of $H_1, \ldots, H_k$ for $i \in \{1, \ldots, k\}$. There exists $T \in \mathcal{P}_G(H_i)$ with $T \neq S_k^-$ such that $T(H_k)$ is $t$-adjacent with another tower over $I \in \min \mathcal{O}(G)$ with $T$ being a port. Then, $H_1, \ldots, H_k, H_{k+1}$, where $H_{k+1} = I$, is a tower-sequence. □

According to Claim 3.13, given a non-spanning tower-sequence, we can repeat extending it by adding an element. By Lemma 3.8 (i), this repetition ends at some point, and a spanning tower-sequence is obtained. □

The next lemma follows as an easy consequence of Lemmas 3.12 and 3.8.

**Lemma 3.14.** Let $G$ be a factorizable graph and $M$ be a perfect matching of $G$, and let $H \in \min \mathcal{O}(G)$.

(i) There exists a spanning tower-sequence $H_1, \ldots, H_k \in \min \mathcal{O}(G)$ with $H = H_i$ for some $i \in \{1, \ldots, k\}$.

(ii) There exists a spanning $M$-arc that has common vertices with $H$.

**Proof.** Consider the tower-sequence that consists solely of $H$. By Lemma 3.12 Statement (i) is obtained. Moreover, by Lemma 3.8 (ii) Statement (ii) is obtained. □

**Remark 3.15.** From Lemma 3.14, the following is implied: if the poset $\mathcal{O}(G)$ of a factorizable graph $G$ has more than one minimal element, then $G$ has at least two distinct borders.

The next lemma is about the nature of borders, but can be derived without other results in this section. This lemma will also be used in Section 4.2.
Lemma 3.16. Let $G$ be a factorizable graph. Let $H \in \partial \mathcal{O}(G)$, and let $S$ be the port of $H$. Then, the set of vertices that can be reached from $S$ by an $M$-saturated path is $\hat{e}^U(S)$.

Proof. According to Lemma 2.9 (iii) there is an $M$-saturated path between each vertex in $S$ and each vertex in $\hat{e}^U(S)$. Hence, it suffices to show that there is no $M$-saturated path between any vertex in $S$ and any vertex in $V(G) \setminus \hat{e}^U(S)$. Suppose that for vertices $x \in S$ and $z \in V(G) \setminus \hat{e}^U(S)$, there is an $M$-saturated path $Q$ between $x$ and $z$. By Lemma 2.1 $z \not\in S$ holds. Trace $Q$ from $x$. Obviously, the vertex that we encounter immediately after $x$ is in $V(H) \setminus S$. Keep tracing $Q$, and let $w$ be the first encountered vertex in $V(G) \setminus \hat{e}^U(S) \setminus S$, and let $r$ be the vertex immediately before $w$; $r$ is in $S$, and the edge $rw$ is not in $M$. Hence, the path $xQr$ is an $M$-saturated path between $x \in S$ and $r \in S$, which is a contradiction by Lemma 2.1. □

4. A NEW PROOF OF THE TIGHT CUT LEMMA

4.1. General Statements. In this Section 4 we introduce our new proof of the Tight Cut Lemma. Here, in Section 4.1 we present definitions and assumptions that will be used throughout the remainder of the paper. We also explain the organization of the new proof and provide some lemmas.

Formal Statement of the Tight Cut Lemma. Let $\hat{G}$ be a brick, and $\hat{S} \subseteq V(\hat{G})$ be such with $1 < |\hat{S}| < |V(\hat{G})| - 1$. Then, there is a perfect matching with more than one edge in $\delta_{\hat{G}}(\hat{S})$.

Let $\hat{G}$ and $\hat{S}$ be as given above. We need to prove that $\delta_{\hat{G}}(\hat{S})$ is not a tight cut. Let $\hat{M}$ be a perfect matching of $\hat{G}$. If $|\delta_{\hat{G}}(\hat{S}) \cap \hat{M}| > 1$ holds, then we have nothing to do. Hence, in the following, we assume $|\delta_{\hat{G}}(\hat{S}) \cap \hat{M}| = 1$ and prove $\hat{S}$ is not a tight cut by finding a $\hat{S}$-fat perfect matching, i.e., a perfect matching with more than one edge in $\delta_{\hat{G}}(\hat{S})$.

Let $\hat{S}^c = V(\hat{G}) \setminus \hat{S}$. Let $u \in \hat{S}$ and $v \in \hat{S}^c$ be such that $\delta_{\hat{G}}(\hat{S}) \cap \hat{M} = \{uv\}$. We denote $\hat{G} - u - v$ by $G$, $\hat{S} - u$ by $S$, $\hat{S}^c - v$ by $S^c$, and $\hat{M} - uv$ by $M$.

Note that $G$ is connected and has a perfect matching $M$. Additionally, $\delta_G(S) \cap M = \emptyset$ holds in $G$. If $S$ is not a separating set, then of course $\delta_{\hat{G}}(\hat{S})$ is not a tight cut in $G$, and we are done. Therefore, in the following, we assume that $S$ is a separating set of $G$ and prove the Tight Cut Lemma for this case.

Without loss of generality, we also assume in the following that $G$ has a border whose vertices are contained in $S$.

According to Lemma 2.2 if we find an $\hat{M}$-alternating circuit $C$ of $\hat{G}$ with more than one edges in $\delta_{\hat{G}}(\hat{S}) \setminus \hat{M}$, then a $\hat{S}$-fat perfect matching
is obtained by taking $E(C) \triangle \hat{M}$. We find such an $\hat{M}$-alternating circuit by analyzing the matching structure of $G$ using the canonical decomposition in Section 2.3. The succeeding Sections 4.2 and 4.3 correspond to proofs of the respective case analyses:

- In Section 4.2 a proof is given for the case where $\text{min}O(G)$ also has a factor-component whose vertex set is contained in $S^c$;
- Section 4.3 is the counterpart to Section 4.2 and gives a proof for the case where every factor-component in $\text{min}O(G)$ has the vertex set contained in $S$, which completes the new proof of the Tight Cut Lemma.

In the following, we present lemmas that will be used by Sections 4.2 and 4.3 when we find a cut vertex in $G$.

**Lemma 4.1.** Let $x$ be a cut vertex of $G$, and let $C$ be one of the connected components of $G - x$. Then, $N_{\hat{G}}(w) \cap V(C) \neq \emptyset$ holds for each $w \in \{u, v\}$.

**Proof.** Suppose that the claim fails, i.e., suppose $N_{\hat{G}}(w) \cap V(C) = \emptyset$, where $w \in \{u, v\}$. It follows that $\{z, x\}$, where $z \in \{u, v\} \setminus \{w\}$, is a vertex-cut of $\hat{G}$, which leaves $C$ as one of the connected components of $\hat{G} - \{z, x\}$. This is a contradiction, because $\hat{G}$ is 3-connected. □

**Lemma 4.2.** Let $x$ be a cut vertex of $G$, and let $C$ be one of the connected components of $G - x$. If $V(C) \cup \{x\}$ is a separating set of $G$, then, for each $w \in \{u, v\}$, there exists $y \in V(C) \cap N_{\hat{G}}(w)$ such that $G$ has an $M$-saturated path between $x$ and $y$.

**Proof.** Let $z \in \{u, v\} \setminus \{w\}$. As $\hat{G}$ is a brick, there is an $\hat{M}$-saturated path, $P$, between $x$ and $z$. If we trace $P$ from $z$, then the second vertex on $P$ is $w$ and the third vertex, $y$, is such with $y \in N_{\hat{G}}(w) \cap V(C)$ and that $xPy$ is an $\hat{M}$-saturated path, for which $V(xPy) \subseteq V(C)$ holds. Therefore, $xPy$ gives a desired path. □

### 4.2. When there exists a factor-component in $\text{min}O(G)$ whose vertices are in $S^c$.

In this Section 4.2 we assume that $\text{min}O(G)$ has a factor-component, which may be or may not be a border, whose vertex set is contained in $S^c$. We prove the Tight Cut Lemma for this case, using mainly the results obtained in Section 3. The next lemma is obtained from Lemmas 3.16 and 4.1 and will be used in the proof of Lemma 4.3.

**Lemma 4.3.** Let $H \in \partial O(G)$, and let $S \in \mathcal{P}_G(H)$ be the port of $H$. Then, $\mathcal{U}_G(S) \cap N_{\hat{G}}(w) \neq \emptyset$ holds for each $w \in \{u, v\}$.

**Proof.** First, consider the case where $S$ is a singleton, which consists of $x \in V(H)$. Then, $x$ is a cut vertex of $G$, and for some connected components $C_1, \ldots, C_k$ of $G - x$ (in fact $k = 1$ holds), $V(C_1) \cup \cdots \cup$
Without loss of generality, that \( x \neq y \). As \( \hat{G} \) is a brick, it has a \( M \)-saturated path \( P \) between \( x \) and \( y \). By Lemma 2.1, \( P \) is not a path of \( G \), which implies \( uv \in E(P) \). Let \( z_1 \in N_G(u) \cap V(P) \setminus \{v\} \) and \( z_2 \in N_G(v) \cap V(P) \setminus \{u\} \), and assume, without loss of generality, that \( x, z_1, z_2, y \) appear in this order if we trace \( P \) from \( x \). Then, \( xPz_1 \) and \( yPz_2 \) are \( M \)-saturated paths of \( G \).

From Lemma 3.16, we obtain \( z_1, z_2 \in ^cU_G(S) \). The lemma is proven.

\( \square \)

Lemma 4.3 derives the next lemma, which provides the main strategy to find a desired \( \hat{M} \)-alternating circuit.

**Lemma 4.4.** If \( G \) has a spanning \( M \)-arc with an edge in \( E_G[S, S^c] \), then \( \hat{G} \) has a \( \hat{S} \)-fat perfect matching.

**Proof.** Let \( P \) be a spanning \( M \)-arc with \( E(P) \cap E_G[S, S^c] \neq \emptyset \), between two borders \( H_1 \) and \( H_2 \). Let \( s_1 \) and \( s_2 \) be the ends of \( P \), and let \( S_1 \in P_G(H_1) \) and \( S_2 \in P_G(H_2) \) be such with \( s_1 \in S_1 \) and \( s_2 \in S_2 \). Without loss of generality, we can assume \( V(H_1) \subseteq S \); if \( V(H_1) \subseteq S^c \) holds, then we can exchange the roles of \( S \) and \( S^c \) without contradicting the assumption on \( \partial \mathcal{O}(G) \). According to Lemma 4.3, there exist \( t_1 \in N_G(v) \cap ^cU_G(S_1) \) and \( t_2 \in N_G(u) \cap ^cU_G(S_2) \). By Lemma 2.9 (iii), there exists an \( M \)-saturated path \( Q_i \), \( t_i \) to \( s_i \), with \( V(Q_i) \setminus \{s_i\} \subseteq ^cU_G(S_i) \) for each \( i \in \{1, 2\} \). According to Lemma 3.6, \( P \) is disjoint from \( Q_1 \) and \( Q_2 \) except for the ends. Therefore, \( C \) is an \( \hat{M} \)-alternating circuit of \( \hat{G} \), where \( \hat{C} := Q_1 + t_1v + uv + vt_2 + Q_2 + P \).

If \( t_1 \in S \) holds, then \( t_1v \in E_G[\hat{S}, S^c] \setminus \hat{M} \) holds and therefore \( |E(C) \cap E_G[\hat{S}, S^c] \setminus \hat{M}| \geq 2 \) follows. Hence, from Lemma 2.2, \( \hat{M} \triangle E(C) \) is a \( \hat{S} \)-fat perfect matching. Otherwise, if \( t_1 \in S^c \) holds, then \( Q_1 \) has an edge in \( E_G[S, S^c] \) because the other end \( s_1 \) is in \( S \). Hence, again, \( |E(C) \cap E_G[\hat{S}, S^c] \setminus \hat{M}| \geq 2 \) follows, and the statement is proven for this case, too. This completes the proof of this lemma.

\( \square \)

As Lemma 4.3 is obtained, we give the following two lemmas to find such a spanning \( M \)-arc.

**Lemma 4.5.** If \( G \) also has a border whose vertices are in \( S^c \), then there is a spanning \( M \)-arc that has an edge in \( E_G[S, S^c] \).

**Proof.** Define \( \mathcal{H}_1 \subseteq \min \mathcal{O}(G) \) (resp. \( \mathcal{H}_2 \subseteq \min \mathcal{O}(G) \)) as follows: \( H \in \min \mathcal{O}(G) \) is in \( \mathcal{H}_1 \) (resp. \( \mathcal{H}_2 \)) if there is a tower-sequence from a border whose vertex set is contained in \( S \) (resp. \( S^c \)) to \( H \).

By Lemma 3.14 (i), \( \mathcal{H}_1 \cup \mathcal{H}_2 = \min \mathcal{O}(G) \).

**Claim 4.6.** The two sets \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) intersect.
Proof. Suppose that this claim fails, namely, that $\mathcal{H}_1 \cap \mathcal{H}_2 = \emptyset$ holds. As $G$ is connected, the two sets of vertices $\bigcup_{H \in \mathcal{H}_1} U^*_G(H)$ and $\bigcup_{H \in \mathcal{H}_2} U^*_G(H)$ either intersect, or are disjoint with some edges joining them. This implies that there exist $H_1 \in \mathcal{H}_1$ and $H_2 \in \mathcal{H}_2$ such that $\mathcal{T}_G(H_1)$ and $\mathcal{T}_G(H_2)$ are t-adjacent. By Lemma 3.14(i), there is a spanning tower-sequence $I_1, \ldots, I_k \in \text{min}\mathcal{O}(G)$ with $k \geq 2$ and $H_1 = I_i$ for some $i \in \{1, \ldots, k\}$. If $V(I_1) \subseteq S^c$ or $V(I_k) \subseteq S^c$ hold, then $H_1 \in \mathcal{H}_1 \cap \mathcal{H}_2$, which is a contradiction. Otherwise, if $V(I_1) \subseteq S$ and $V(I_k) \subseteq S$, then either $I_1, \ldots, I_i, H_2$ or $H_2, I_i, \ldots, I_k$ is a tower-sequence. Thus, $H_2 \in \mathcal{H}_1 \cap \mathcal{H}_2$ holds, which is again a contradiction. □

Claim 4.7. There is a spanning tower-sequence from a border whose vertex set is contained in $S$ to a border whose vertex set is contained in $S^c$.

Proof. By Claim 4.6, there exists $H \in \mathcal{H}_1 \cap \mathcal{H}_2$. By $H \in \mathcal{H}_1$, there is a tower-sequence from $H_1 \in \partial\mathcal{O}(G)$ to $H$ with $V(H_1) \subseteq S$. Hence, by Lemma 3.12, there is a spanning tower-sequence $H_1, \ldots, H_k \in \text{min}\mathcal{O}(G)$ with $k \geq 2$ and $H = H_i$ for some $i \in \{1, \ldots, k\}$. If $V(H_k) \subseteq S^c$ holds, we are done; thus, let $V(H_k) \subseteq S$. By $H \in \mathcal{H}_2$, there is a tower-sequence $I_1, \ldots, I_l \in \text{min}\mathcal{O}(G)$ with $l \geq 1$, $I_1 \in \partial\mathcal{O}(G)$, $V(I_1) \subseteq S^c$, and $I_l = H$. Either $H_1, \ldots, H_i = H = I_i, \ldots, I_1$ or $H_k, \ldots, H_l = H = I_l, \ldots, I_1$ forms a spanning tower-sequence, satisfying the statement of this claim. □

By Lemma 3.8(ii) and Claim 4.7, we obtain a desired spanning $M$-arc. □

The next lemma treats the counterpart case to Lemma 4.5.

Lemma 4.8. Assume every border of $G$ has the vertex set that is contained in $S$. If there exists a non-border element of $\text{min}\mathcal{O}(G)$ whose vertex set is contained in $S^c$, then there is a spanning $M$-arc that has some edges in $E_G[S, S^c]$.

Proof. Let $H \in \text{min}\mathcal{O}(G)$ be such with $V(H) \subseteq S^c$. As given in Lemma 3.14(ii), take a spanning $M$-arc $P$ with $V(P) \cap V(H) \neq \emptyset$. The ends of $P$ are in $S$, so $P$ has at least two edges in $E_G[S, S^c]$. □

Regardless of whether there is a border that has the vertex set in $S^c$ or not, Lemmas 4.5 and 4.8 assure that $G$ has an $M$-arc with an edge in $\delta_G(S)$. Hence, from Lemma 4.4, we conclude that $\hat{G}$ has a $\hat{S}$-perfect matching, and the Tight Cut Lemma is proven for the case of Section 4.2

4.3. When every factor-component in $\text{min}\mathcal{O}(G)$ has the vertex set contained in $S$. 
4.3.1. Shared Assumptions and Lemmas. Here in Section 4.3 we assume that the vertex set of any factor-component in min$\mathcal{O}(G)$ is contained in $S$ and prove the Tight Cut Lemma under this assumption.

Section 4.3.1 explains the assumptions, definitions, and lemmas that will be used throughout Section 4.3. Let $S_0 \subseteq S$ be the inclusion-wise maximal separating subset of $S$ such that $\{H_1, \ldots, H_p\}$ is a lower-ideal of $\mathcal{O}(G - S)$, where $S_0 = V(H_1) \cup \cdots \cup V(H_p)$. Arbitrarily choose a connected component $C$ of $G - S_0$. Sections 4.3.2 and 4.3.3 give the proofs of the Tight Cut Lemma for the cases where $|N_G(C) \cap S_0| = 1$ holds and does not hold, respectively. Note that $V(C)$ is a separating set in $G$ and $C$ is factorizable. In addition, note that for each $H \in \text{min}\mathcal{O}(C)$, $V(H) \subseteq S^c$ holds by the definition of $S_0$. The following two lemmas will be used in both of the succeeding case analyses.

**Lemma 4.9.** For each $H \in \text{min}\mathcal{O}(C)$, $G$ has an $M$-ear, $P_H$, relative to $S_0$ and traversing $H$.

**Proof.** Because $H \not\in \text{min}\mathcal{O}(G)$ holds here, $\mathcal{O}(G)$ has an immediate lower-bound element $I \in \mathcal{G}(G)$ of $H$. By Lemma 2.12 there is an $M$-ear $P$ relative to $I$ and traversing $H$.

**Claim 4.10.** The vertex set of $I$ is contained in $S_0$.

**Proof.** Suppose this claim fails, i.e., suppose $V(I) \subseteq V(G) \setminus S_0$. If there exists $H' \in \mathcal{G}(G)$ with $V(H') \subseteq S_0$ such that $P$ traverses $H'$, then $I \prec_G H'$ holds by Lemma 2.10 which contradicts the definition of $S_0$. Hence, $V(I) \cup V(P) \cup V(H) \subseteq V(G) \setminus S_0$ holds, and accordingly, $V(I) \cup V(P) \cup V(H) \subseteq V(C)$ holds. This implies $I \prec_C H$, which contradicts $H \in \text{min}\mathcal{O}(C)$. Hence, $V(I) \subseteq S_0$ follows. □

Under Claim 4.10, the connected components of $P - E(G[S_0])$ are $M$-ears relative to $S_0$, and one of them, $P_H$, traverses $H$. □

Under Lemma 4.9 for each $H \in \text{min}\mathcal{O}(C)$, arbitrarily choose and fix an $M$-ear relative to $S_0$ and traversing $H$; in the remainder of this paper, we denote it by $P_H$.

**Lemma 4.11.** Let $y \in V(C)$, and let $H \in \text{min}\mathcal{O}(C)$ be such that $y \in U^*_C(H)$. Then, there is an $M$-balanced path $Q^y_H$ from $y$ to $x_H$, one of the ends of the $M$-ear $P_H$, with $V(Q^y_H) \setminus \{x_H\} \subseteq V(C)$.

**Proof.** Consider the possibly identical connected components $R_1$ and $R_2$ of $P_H - E(C[U^*_C(H)])$ that contain the ends of $P_H$; if $P_H$ is proper, let us denote it by $R_1$, and let $R_2$ be an empty graph. Let $z_1, z_2 \in U^*_C(H)$ be the ends of $R_1$ and $R_2$ that are distinct from the ends of $P_H$. Let $R' := P_H - E(R_1 + R_2)$. We have $E(R') \subseteq E(C[U^*_C(H)])$; otherwise, the connected components of $R' - E(C[U^*_C(H)])$ are non-trivial $M$-ears relative to $U^*_C(H)$ or are trivial $M$-ears that contradict Lemma 3.2. Accordingly, $T_1 \neq T_2$ holds; otherwise, the connected components of
R' − UC(T1) are M-saturated paths whose ends are all in T1, which contradicts Lemma 2.1.

Let T3 ∈ PC(H) be such with y ∈ UC(T3). Either T1 or T2 is not identical to T3; without loss of generality, let T1 ≠ T3. Let x_H be the end of P_H that is in V(R1). By Lemma 2.9(iv), there is an M-saturated path L between y and z1 with V(L) ⊆ UC(H). The path L + z1R1x_H is a desired path Q_H^y. □

Following the above, in the remainder of this paper, for each H ∈ minO(C) and each y ∈ UC(H), let Q_H^y be the path as given in Lemma 4.11 and let x_H be the end of the M-ear P_H that is also an end of the path Q_H^y.

4.3.2. Case with |NG(C) ∩ S0| = 1. Here in Section 4.3.2, we assume that there exists x_0 ∈ S_0 with NG(C) ∩ S_0 = {x_0}, and prove the Tight Cut Lemma under this assumption. In this case, of course P_H is a non-proper M-ear with the unique end x_H, which is accordingly equal to x_0 for each H ∈ minO(C).

**Lemma 4.12.** If |NG(C) ∩ S0| = 1 holds, then ˆG has a ˆS-fat perfect matching.

*Proof.* Note that x_0 is a cut vertex of G such that C is a connected component of G − x_0. Therefore, by Lemma 4.2, there exists z ∈ NG(v) such that there is an M-saturated path R between x_0 and z with V(R) ⊆ S_0. Note zv ∈ E_G[ˆS, ˆS^c] \ M.

By Lemma 4.1, there exists y ∈ V(C) ∩ NG(u). Let H ∈ minO(C) be such with y ∈ UC(H), and take a path Q_H^y as given in Lemma 4.11. Let K := R + zv + uv + vy + Q_H^y. Note that K is an M-alternating circuit of ˆG.

If y ∈ S^c holds, then yu ∈ E_G[ˆS, ˆS^c] \ M holds. Otherwise, if y ∈ S holds, then Q_H^y has edges in E_G[S, S^c] and, accordingly, in E_G[ˆS, ˆS^c] \ M; this is because Lemma 4.11 assures that Q_H^y traverses S^c while the ends of Q_H^y are in S. Therefore, in each case, K has at least two edges in E_G[ˆS, ˆS^c] \ M. Hence, by Lemma 2.2, ˆM ∆ E(K) is a ˆS-fat perfect matching. □

From Lemma 4.12, the proof of the Tight Cut Lemma for the case analysis of Section 4.3.2 is completed.

4.3.3. Case with |NG(C) ∩ S0| > 1. Here, in Section 4.3.3, we treat the counterpart case to Section 4.3.2; namely, we assume |NG(C) ∩ S0| > 1. We use the next lemma as the main strategy to obtain a desired perfect matching:

**Lemma 4.13.** If G has a proper M-ear relative to S_0 and traversing S^c, then ˆG has an ˆS-fat perfect matching.
TIGHT CUT LEMMA

Proof. Let \( P \) be a proper \( M \)-ear relative to \( S_0 \) and traversing \( S^c \), and let \( x \) and \( y \) be the ends of \( P \), with \( x \neq y \). As \( \hat{G} \) is a brick, Lemma 2.1 implies that it has an \( \hat{M} \)-saturated path \( Q \) between \( x \) and \( y \). This \( Q \) is not a path in \( G \), otherwise \( P + Q \) is an \( M \)-alternating circuit of \( G \) containing non-allowed edges in \( \delta_G(S_0) \), which is a contradiction by Lemma 2.2. Hence, \( uv \in E(Q) \) holds. Let \( Q_1 \) and \( Q_2 \) be the connected components of \( Q - u - v \); note that they are \( M \)-saturated paths.

Claim 4.14. The paths \( Q_1 \) and \( Q_2 \) are disjoint from \( P \), except for the ends \( x \) and \( y \).

Proof. Without loss of generality, let \( x \) be one of the ends of \( Q_1 \). Suppose the claim fails, and let \( z \) be the first encountered vertex in \( P - x \) if we trace \( Q_1 \) from \( x \).

If \( xPz \) has an even number of edges, then \( xQ_1z + xPz \) is an \( M \)-alternating circuit of \( G \) containing non-allowed edges in \( \delta_G(S_0) \). This contradicts Lemma 2.2. Otherwise, if \( xPz \) has an odd number of edges, then \( xQ_1z + xPz \) is an \( M \)-ear relative to \( z \). From Lemma 2.11, this implies that there exist \( H_z \in G(G) \) with \( V(H_z) \subseteq V(G) \setminus S_0 \) and \( H_0 \in G(G) \) with \( V(H_0) \subseteq S_0 \) such that \( H_z \triangleright G H_0 \) holds. This contradicts the definition of \( S_0 \). Hence, the statement is obtained for \( Q_1 \). With the symmetrical argument, the statement also holds for \( Q_2 \) \( \square \)

By Claim 4.14, \( P + Q \) forms an \( M \)-alternating circuit, and it has at least two edges in \( E_G[S, S^c] \), because \( P \) has. Hence, \( \hat{M} \triangle E(P + Q) \) is a desired \( \hat{S} \)-perfect matching. \( \square \)

As given Lemma 4.13, we aim at finding such a proper \( M \)-ear. If the \( M \)-ear \( P_H \) is proper for some \( H \in \min\mathcal{O}(C) \), then Lemma 4.13 gives a \( \hat{S} \)-fat matching of \( \hat{G} \). Hence, in the remainder of this proof, we assume that

\( P_H \) is not proper, having the unique end \( x_H \), for each \( H \in \min\mathcal{O}(C) \).

The next two lemmas find desired \( M \)-ears and therefore \( \hat{S} \)-fat perfect matchings under the assumptions that are the counterparts to each other.

Lemma 4.15. Let \( H \in \min\mathcal{O}(C) \). If \( N_G(U^*_C(H)) \cap S_0 \) contains a vertex other than \( x_H \), then \( \hat{G} \) has a \( \hat{S} \)-fat matching.

Proof. Let \( z \) be a vertex in \( N_G(U^*_C(H)) \cap S_0 \) that is distinct from \( x_H \), and let \( y \in U^*_C(H) \) be such with \( zy \in E(G) \). Take a path \( Q'_H \) as in Lemma 4.11. Then \( Q'_H + zy \) is an \( M \)-ear relative to \( S_0 \), with the two distinct vertices \( x_H \) and \( y \), and traversing \( V(H) \subseteq S^c \). Hence, by Lemma 4.13 this lemma is now proven. \( \square \)
As the counterpart of Lemma 4.12, the next lemma treats the case where $N_G(U_C^c(H)) \cap S_0 = \{x_H\}$ for any $H \in \text{min}\mathcal{O}(C)$. Note that according to the assumption of Section 4.3.3 there exist $H, I \in \text{min}\mathcal{O}(C)$ with $x_H \neq x_I$.

**Lemma 4.16.** If $N_G(U_C^c(H)) \cap S_0 = \{x_H\}$ holds for any $H \in \text{min}\mathcal{O}(C)$, then $G$ has a $S$-fat perfect matching.

**Proof.** As $C$ is connected, there exist $H_1, H_2 \in \text{min}\mathcal{O}(C)$ with $x_{H_1} \neq x_{H_2}$ such that $T_C(H_1)$ and $T_C(H_2)$ are $t$-adjacent. Let $S_i \in P_C(H_i)$ be the ports of this adjacency for $i \in \{1, 2\}$. From Lemma 3.4 we obtain an $M$-arc $R$ whose vertices except for the ends are in $U_C(S_1) \cup U_C(S_2)$. Let $s_1 \in S_1$ and $s_2 \in S_2$ be the ends of $R$. Take an $M$-balanced path $Q_{H_i}^s$ from $s_i$ to $x_{H_i}$ as stated in Lemma 4.11 for each $i \in \{1, 2\}$.

**Claim 4.17.** The path $Q_{H_1}^s$ is disjoint from $R$ for each $i \in \{1, 2\}$.

**Proof.** Suppose this claim fails. Trace $Q_{H_i}^s$ from $s_i$, and let $t$ be the first encountered vertex in $R$. If $s_i R t$ has an even number of edges, then $s_i Q_{H_i}^s t + t R s_i$ is an $M$-alternating circuit of $G$ containing non-allowed edges in $\delta_C(S_i)$. This contradicts Lemma 2.2. If $s_i R t$ has an odd number of edges, then $s_i Q_{H_i}^s t + t R s_i$ is an $M$-ear relative to $t$ and traversing $H_i$. Under Lemma 2.11 this implies that $H_i$ is not minimal in $\mathcal{O}(C)$, which is a contradiction. The claim is now proven. □

**Claim 4.18.** The paths $Q_{H_1}^s$ and $Q_{H_2}^s$ are disjoint.

**Proof.** Suppose the claim fails. If we trace $Q_{H_1}^s$ and let $t$ be the first encountered vertex in $Q_{H_2}^s$, we have $K := R + s_1 Q_{H_1}^s t + t Q_{H_2}^s s_2$ is a circuit.

If $t Q_{H_2}^s s_2$ has an even number of edges, then $K$ is an $M$-ear relative to $t$ and traversing $H_1$. By Lemma 2.11 this implies $H_1 \in \text{min}\mathcal{O}(C)$. Otherwise, if $t Q_{H_2}^s s_2$ has an odd number of edges, then $K$ is an $M$-alternating circuit containing non-allowed edges in $\delta_C(H_1)$. By Lemma 2.2 this is a contradiction.

From Claims 4.17 and 4.18 $R + Q_{H_1}^s + Q_{H_2}^s$ forms an $M$-ear relative to $S_0$, possessing the distinct ends $x_{H_1}$ and $x_{H_2}$, and traversing $H_1$ and $H_2$, which are contained in $S^c$. Therefore, the proof is now completed by Lemma 4.13 □

This completes the proof for the case of Section 4.3.3 Therefore, the whole proof of the Tight Cut Lemma is completed.

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