Some 2-adic conjectures concerning polyomino tilings of Aztec diamonds

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Dedicated to Michael Larsen on the occasion of his 60th birthday

Abstract

For various sets of tiles, we count the ways to tile an Aztec diamond of order $n$ using tiles from that set. The resulting function $f(n)$ often has interesting behavior when one looks at $n$ and $f(n)$ modulo powers of 2.

1 Introduction

I had a great time working on domino tilings of Aztec diamonds with Noam Elkies, Greg Kuperberg, and Michael Larsen back in the late 1980s, and the paper we wrote together [EKLP] had a huge impact on my career. So I'd like to honor Michael by proposing some new problems about tilings of Aztec diamonds (and other regions), many of which are more challenging than the one I shared with him thirty-something years ago and have a number-theoretic slant that I think he will enjoy. Ideally the solutions to these problems will involve interesting applications of algebra to combinatorics.

Here is some general background.

The main result of [EKLP] was that the number of domino-tilings of an Aztec diamond of order $n$ is $2^{n(n+1)/2}$ (A006125), where a domino is a rectangle in $\mathbb{R}^2$ of the form $[i, i+1] \times [j, j+1]$ or $[i, i+2] \times [j, j+1]$ (with $i, j \in \mathbb{Z}$) and the Aztec diamond of order $n$ is the union of the squares $[i, i+1] \times [j, j+1]$ lying entirely within the region $\{(x, y) : |x| + |y| \leq n+1\}$. Figure 1 shows one of the $2^{(4)(5)/2}$ domino tilings of the Aztec diamond of order 4.

Mihai Ciucu [Ci] proved combinatorially that the number of domino tilings of the $2n$-by-$2n$ square (A004003) can be written in the form $2^n f(n)^2$.
Figure 1: A domino tiling of the Aztec diamond of order 4.

where $f(n)$ is the number of domino tilings of the region exemplified for $n = 4$ in Figure 2 (A065072).

Figure 2: Ciucu’s way of halving the 8-by-8 square.

Henry Cohn [Co] proved that the function sending $n$ to $f(n)$ is uniformly continuous under the 2-adic metric and thus extends to a function defined on all of $\mathbb{Z}$ and indeed all of $\mathbb{Z}_2$; moreover, he showed that this extension satisfies

$$f(-1 - n) = \begin{cases} f(n) & \text{when } n \text{ is congruent to } 0 \text{ or } 3 \text{ (mod 4)}, \\ -f(n) & \text{when } n \text{ is congruent to } 1 \text{ or } 2 \text{ (mod 4)}. \end{cases} \quad (1)$$
Barkley and Liu [BL] have recently proved results about 2-divisibility for the number of perfect matchings of a graph, including as a special case the number of domino tilings of a rectangle, but there is more refined work still to be done along the lines of Cohn’s paper. For instance, the mod 8 residue of the number of domino tilings of the $2n$-by-$(2n + 2)$ rectangle appears to depend only on the mod 4 residue of $n$; the same goes for the number of domino tilings of the $2n$-by-$4n$ rectangle.

In this article we extend the discussion to other sorts of tiles, specifically, tetrominos. A tetromino is a connected subset of the grid that is a union of four grid-squares, just as a domino is a union of two grid-squares. Up to symmetry, there are five kinds of tetrominos: straight tetrominos, skew tetrominos, L-tetrominos, square tetrominos, and T-tetrominos. They are shown in Figure 3 preceded by the domino. These six tiles can be placed on a square grid in 2, 2, 4, 8, 1, and 4 translationally-inequivalent ways, respectively (where rotations and reflections are permitted). These are the sorts of tiles considered in this article. (*Trominos* – unions of three grid-squares – will be considered elsewhere.)

Figure 3: A domino, a straight tetromino, a skew tetromino, an L-tetromino, a square tetromino, and a T-tetromino.

## 2 Skew and straight tetrominos

I’ll start with a warm-up puzzle that’s roughly at the level of a math olympiad: Prove that an Aztec diamond of order $n$ can be tiled by skew and straight tetrominos (as shown in Figure 4 for $n = 3$) only if $n$ is congruent to 0 or 3 (mod 4).

The puzzle can be solved using a valuation argument (sometimes called a generalized coloring argument): one can construct a mapping from the grid-cells to an appropriate abelian group (a “weight function”) and show that when $n$ is 1 or 2 (mod 4), the sum of the weights of the tiles can’t equal the sum of the weights of the region being tiled, where the weight of a tile or a
region being tiled is the sum of the weights of the constituent cells. Readers who are already familiar with this technique might enjoy the challenge of attempting to solve the problem purely mentally.

3 Dominos and square tetrominos

In this section we use dominos and square tetrominos. Thus an Aztec diamond of order 1 (better known as the 2-by-2 square) can be tiled in 3 ways: with two horizontal dominos, two vertical dominos, or a single square tetromino. The Aztec diamond of order 2 can be tiled in $2^2(3)/2 = 8$ ways using dominos, and can be tiled in an additional 11 ways if one or more square tetrominos are included, as shown in Figure 5. Thus there are a total of $8 + 11 = 19$ ways to tile an Aztec diamond of order 2 using dominos and square tetrominos.

Define $M(n)$ (with $n \geq 0$) as the number of tilings of the Aztec diamond of order $n$ using dominos and square tetrominos. This is [A356512]. Trivially we have $M(0) = 1$ (since the Aztec diamond of order 0 is empty) and we have already seen that $M(1) = 3$ and $M(2) = 19$. Figure 6 shows the terms of the sequence $M(n)$ for $n$ ranging from 0 to 12, computed using a program written by David desJardins.

The reader may wish to pause here to consider the problem of showing that $M(n)$ is always odd; a solution will be given in section 5.
These numbers grow quadratic-exponentially as a function of \( n \), and I have no conjectural formula for the \( n \)th term, nor a conjectural recurrence relation for the sequence, nor any efficient method of computing terms. Nonetheless, something systematic is going on. I have already mentioned that all the terms are odd. Taking this observation further, one notices that the numbers’ residues mod 4 are

\[
1, 3, 3, 1, 1, 3, 3, 1, 3, 3, 1, 1,
\]

the residues mod 8 are

\[
1, 3, 3, 5, 5, 7, 7, 1, 1, 3, 3, 5, 5,
\]

and the residues mod 16 are

\[
1, 3, 3, 5, 5, 7, 7, 9, 9, 11, 11, 13, 13.
\]

**Conjecture 1:** For all \( k \geq 1 \), the mod \( 2^k \) residue of \( M(n) \) is periodic with period dividing \( 2^k \). That is, \( 2^k \) divides \( M(n + 2^k) - M(n) \) for all \( k, n \).
I tried to prove this conjecture by reducing it to an assertion about alternating-sign matrices but I was unsuccessful.

Note that if the conjecture is true then \( M(n) \equiv n + 1 + (1 + (-1)^{n+1})/2 \pmod{8} \). This congruence might also hold mod 16 but it certainly cannot hold mod \( 2^k \) for all \( k \), since that would require that \( M(n) \) actually equals \( n + 1 + (1 + (-1)^{n+1})/2 \), which is clearly not the case for \( n \geq 2 \). And indeed \( M(2) = 19 \not\equiv 3 \pmod{32} \).

A deeper consequence of Conjecture 1 is that the function sending \( n \) to \( M(n) \) is 2-adically continuous. Moreover, the function appears to satisfy a kind of symmetry analogous to the functional equation (1) mentioned at the end of section 1.

**Conjecture 2:** For all \( k \geq 1 \), if \( n + n' \equiv -3 \pmod{2^k} \) then \( M(n) + M(n') \equiv 0 \pmod{2^k} \).

That is, if one extends \( M : \mathbb{N} \to \mathbb{N} \) to the 2-adic function \( \widehat{M} : \mathbb{Z}_2 \to \mathbb{Z}_2 \), one has \( \widehat{M}(-3 - n) = -\widehat{M}(n) \).

Although in this article I am limiting myself to discussion of tilings of Aztec diamonds, I have also looked at tilings of other regions using dominos and square tetrominos, and the same phenomenon of 2-adic continuity arises.
fairly broadly there. For instance, for the $2n$-by-$2n$ square, the $2n$-by-$2n+2$ rectangle, and the $2n$-by-$4n$ rectangle, the number of tilings with dominos and square tetrominos always seems to be congruent to $2n + 1 \mod 8$.

4 Skew tetrominos and square tetrominos

In [Pr] I considered tilings of Aztec diamonds by skew tetrominos and square tetrominos. If we require that all skew tetrominos be horizontal, interesting numerical patterns appear. (Of course we would get the same result if we required that all skew tetrominos be vertical.) In this section we allow horizontal skew tetrominos and square tetrominos as seen in Figure 7, which depicts all six tilings of the Aztec diamond of order 3 using square tetrominos and horizontal skew tetrominos.

![Figure 7: Tiling the Aztec diamond of order 3 with horizontal skew tetrominos and square tetrominos.](image)

Define $L(n)$ (with $n \geq 0$) as the number of tilings of the Aztec diamond of order $n$ using horizontal skew tetrominos and square tetrominos. This is [A356513](https://oeis.org/A356513). Trivially we have $L(0) = 1$ and $L(1) = 1$. Figure 8 shows the terms of the sequence $L(n)$ for $n$ ranging from 0 to 15, again computed using the program written by David desJardins.
| $n$ | $L(n)$          |
|-----|----------------|
| 0   | 1              |
| 1   | 1              |
| 2   | 2              |
| 3   | 6              |
| 4   | 40             |
| 5   | 364            |
| 6   | 7904           |
| 7   | 226152         |
| 8   | 15835008       |
| 9   | 1439900880     |
| 10  | 324189571584   |
| 11  | 94080051207136 |
| 12  | 68041472016287744 |
| 13  | 6314592712713361600 |
| 14  | 146637148542938673930240 |
| 15  | 435697213021432661980535936 |

Figure 8: Enumeration of tilings of Aztec diamonds using horizontal skew tetrominos and square tetrominos.

The sequence grows quadratic-exponentially, and once again, I have no conjectural formula, but as before there are patterns that call out for explanation. Noticing that all but the first two terms are even, one might naturally think to look at the multiplicity of 2 in the prime factorization of $L(n)$, obtaining the sequence 0,0,1,1,3,2,5,3,7,4,9,5,11,6,13,7,…, which (once we throw out the initial 0) we recognize as an interspersal of the arithmetic progressions 0,1,2,3,4,5,6,7,… and 1,3,5,7,9,11,13,….

**Conjecture 3:** For $n \geq 1$, the multiplicity of 2 in the prime factorization of $L(n)$ is $n - 1$ if $n$ is even and $(n - 1)/2$ if $n$ is odd.

Going further, let $L_0(m) = L(2m)/2^{2m-1}$ and $L_1(m) = L(2m - 1)/2^{m-1}$, so that (if Conjecture 3 holds) $L_0(m)$ and $L_1(m)$ are odd integers for all $m$. These two new sequences are shown in Figure 9.

The mod 4 residues of the $L_0$ sequence go 1, 1, 3, 3, 1, 1, 3, 1 while those of the $L_1$ sequence go 1, 3, 3, 1, 1, 3, 3, 1. That’s not much evidence to go on, so perhaps it would be prudent not to make a conjecture, but I choose to be
Figure 9: Values of $L_0(m)$ and $L_1(m)$.

Conjecture 4: For all $k \geq 0$, the mod $2^k$ residue of $L_0(m)$ is periodic with period dividing $2^k$. Likewise for $L_1(m)$.

We do not observe such patterns in the numbers of tilings when both horizontal and vertical skew tetrominos are allowed along with square tetrominos as in [Pr]. More specifically, if we count tilings of Aztec diamonds in which we are permitted to use all four kinds of skew tetrominos as well as square tetrominos, the resulting sequence, taken mod 4, goes 1, 0, 0, 0, 0, 0, 2, 2, 0, 0, 0, 0, . . . ; if there is a period here, and it is a power of 2, it must be at least 16.

The prime $p = 2$ appears to be special for the enumerative problems I described above; looking at the $M$ and $L$ sequences mod 3 or mod 5 yields no discernible patterns.

5 Assorted congruential problems

For each of the sixty-three nonempty subsets of the set of six tiles shown in Figure 3, we can ask in how many ways it is possible to tile the Aztec diamond of order $n$ using only tiles from that set, allowing translations, rotations, and reflections of tiles. These are the enumerative problems considered in this section.

(One could expand the set of tiling problems by distinguishing between different orientations of the tiles, as was done in the preceding section where
we permitted horizontal skew tetrominos but forbade vertical skew tetrominos; since there are \(2 + 2 + 4 + 8 + 1 + 4 = 21\) different tiles up to translation, we would obtain over two million different problems, and even if we mod out the \(2^{21} - 1\) problems by a dihedral action of order 8, that is still too many problems to consider exhaustively. One that appears to be interesting is discussed at the end of this section.

In each of the sixty-three cases, I used the aforementioned program to count the tilings of the Aztec diamond of order \(n\), with \(n\) going from 1 to 8, using the allowed tiles. Although no 2-adic continuity phenomena arose from these experiments, there were definite patterns in the parity, and in a few cases there were congruence patterns modulo higher powers of 2. Here I will adopt a six-bit code to represent the sixty-three tiling problems, in which the six successive bits (from left to right) equal 1 or 0 according to whether or not dominos, straight tetrominos, skew tetrominos, L-tetrominos, square tetrominos, and T-tetrominos are allowed. For instance, the case treated in section 2, in which only straight tetrominos and skew tetrominos are allowed (see Figure 4), would be assigned the code 011000; the case treated in section 3, in which only dominos and square tetrominos are allowed (see Figure 5), would be assigned the code 100010; and the case of unconstrained skew and square tetrominos (briefly discussed in section 4) would be assigned the code 001010.

In one-third of the 63 cases, I observed that for all \(n\) between 1 and 8, the number of tilings of the Aztec diamond of order \(n\) is even. These were the cases associated with the six-bit codes 001001, 001100, 001101, 011001, 011100, 011101, 100010, 100100, 100101, 101000, 101001, 101100, 101101, 110000, 110001, 110100, 110101, 111000, 111001, 111100, and 111101.

Presumably some (perhaps all) of these examples can be resolved by showing that there are no tilings that are invariant under the full dihedral group, since in that case all orbits would contain an even number of tilings.

Three of the 21 cases were especially interesting. In case 011100, all terms were divisible by 8; in case 100010, all terms after the first were divisible by 8; and in case 110001, all terms were congruent to 2 (mod 4).

There were also four cases in which I observed that the number of tilings of the Aztec diamond of order \(n\) is even for all \(n\) between 2 and 8 (with the number of tilings being the odd number 1 in the case \(n = 1\)). These were the cases associated with the six-bit codes 001010, 001110, 011010, and 011110.

In the cases 001101, 100001, 100011, and 111000 it appears that the exponent of 2 in the number of tilings may be going to infinity with \(n\),
though with such scant evidence it would be rash to place too much faith in this guess.

Additionally, there is one case in which the number of tilings of the Aztec diamond of order \( n \) is always odd, namely, tilings using only dominos and square tetrominos. Indeed, if we assign each tiling weight \((-1)^s\) where \( s \) is the number of square tetrominos, I claim that the sum of the weights is 1. We can prove this using a sign-reversing involution that scans through the tiling in some fashion in search of a 2-by-2 block that is tiled either with a square tetromino or with two vertical dominos and switches between the two possibilities. The fixed points of this involution are tilings that use only horizontal dominos, and there is just one of those. [NOTE: A referee pointed out that the preceding proof is incorrect and suggested a way to fix it. This change was implemented in the final, published version of the article at https://math.colgate.edu/~integers/x30/x30.pdf.] Finally, leaving the small world of the \( 2^n - 1 \) problems and dipping our toe into the big world of the \( 2^{21} - 1 \) problems, we consider tilings of the Aztec diamond of order \( n \) by dominos and horizontal straight tetrominos. This is [A356523] and begins 1, 2, 11, 209, 12748, 2432209, 1473519065, \ldots. It appears that the number of tilings is even when \( n \equiv 1 \pmod{3} \) and odd otherwise; this has been verified for \( 1 \leq n \leq 16 \).

6 Reduction to perfect matchings

The \( L \) sequence from section 4 has an interpretation in terms of perfect matchings. To see why, suppose we have a tiling of the Aztec diamond of order \( n \) using horizontal skew tetrominos and square tetrominos. Dividing each tetromino into two horizontal dominos gives us a tiling of the Aztec diamond by horizontal dominos, but it is easy to see that there is exactly one such tiling (call it \( T \)). Hence each tetromino is obtained by gluing together two dominos in \( T \). That is, the tetromino tilings correspond to perfect matchings in the graph whose vertices correspond to the dominos in \( T \) with an edge joining two vertices if the corresponding dominos form a horizontal skew tetromino or square tetromino. It is not hard to see that this graph is similar to the \( n \)-by-\( n \) square except that the diagonal has been “doubled”; for instance, the right panel of Figure 10 shows the graph for \( n = 4 \).

A similar analysis can be applied to tilings of Aztec diamonds using horizontal skew tetrominos and horizontal straight tetrominos. In this case the
Aztec diamond splits into two non-interacting halves (top half and bottom half), each of which can be tiled independently of the other, and the tilings of either half correspond to perfect matchings of a triangle graph as shown in Figure 11. Thus the number of such tetromino tilings of the Aztec diamond of order \( n \) is equal to the square of the \( n \)th term of sequence \( A071093 \).

Studying the first 25 terms, I find that the sequence seems to have 2-adic properties of its own. The largest power of 2 dividing the \( n \)th term of the sequence \( A071093 \) appears to be \( \lfloor n/2 \rfloor \), and the 2-free part appears to satisfy 2-adic continuity: for instance, its value mod 16 seems to be determined by \( n \) mod 16.

What if we superimpose the two graphs, obtaining the graph shown at the right half of Figure 12? This is equivalent to tiling an Aztec diamond using horizontal skew tetrominos, horizontal straight tetrominos, and square tetrominos. Then, counting the tilings, we obtain the integer sequence 1, 2, 10, 116, 3212, 209152, 32133552, 11631456480, 9922509270288, 19946786274879008, 9449287410363897152, 105486519875214776174448, \ldots. This is \( A356514 \). It appears that the number of tilings is divisible by \( 2^{\lfloor n/2 \rfloor} \).

7 Some thoughts

The articles of Lovasz [Lo], Ciucu [Ci], Pachter [Pa], and Barkley and Liu [BL] give ways to find the largest power of 2 that divides the number of perfect matchings of a graph. This should provide traction for Conjecture 3, since
we saw in section 6 that the $L$ sequence has an interpretation in terms of perfect matchings of certain graphs. Graphs of this kind appear in the paper of Ciucu [Ci]; in particular, his Lemma 1.1 shows that the number of perfect matchings is divisible by $2^{\lfloor n/2 \rfloor}$. By bringing ideas from Pachter [Pa], one might be able to prove Conjecture 3, as well as some of the other 2-divisibility conjectures from this article.

The only work I know of that provides detailed 2-adic information about the 2-free part of numbers that count tilings is the work of Cohn [Co]. Cohn’s approach presupposes the existence of an exact formula (in Cohn’s case, an explicit product of algebraic integers); perhaps something similar can be done for perfect matchings of the square graph with doubled diagonal, yielding a proof of Conjecture 4.

Conjectures 1 and 2 seem harder. The product formula exploited by Cohn was discovered by Temperley and Fischer [TF] and independently by Kasteleyn [Ka] at about the same time; those researchers made use of the fact that, just as determinants and Pfaffians of matrices can be expressed as sums of terms associated with perfect matchings of the set of rows and columns, one can conversely express the number of perfect matchings of a planar graph in terms of the determinant or Pfaffian of an associated matrix. I know of no way of recast the $M$ sequence as enumerating perfect matchings of graphs. However, it is easy to recast the $M$ sequence as enumerating perfect matchings of certain hypergraphs. Can any of the existing notions of hyperdeterminants be brought to bear? Perhaps a reading of [GKZ] would
suggest possible approaches.

Kuperberg’s elegant solution \cite{Ku} to the alternating sign matrix conjecture exploits the power of the Yang-Baxter equation in statistical mechanics. It’s possible that tools for analyzing the new problems described in this article will be found in the existing literature at the interface between algebra and statistical mechanics.

In any case, inasmuch as Conjectures 1 and 2 are reminiscent of Cohn’s work, and inasmuch as Cohn’s argument hinges on an exact product formula, one might hope that an exact formula of some kind can be found for the $M$ sequence. Such an exact formula would have other uses. In \cite{CEP} and \cite{CLP}, Henry Cohn, Noam Elkies, Michael Larsen and I used exact enumeration results to prove concentration theorems for random tilings. One might hope that the curious 2-adic phenomena discussed in this article hint at the existence of algebraic machinery that could be applied to the task of showing us what random tilings associated with Conjecture 1 look like in the limit as size goes to infinity. Preliminary experiments suggest that there is a “frozen region” near the boundary, but I have no idea how far into the interior it extends.

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