Computing Masses from Effective Transfer Matrices

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Abstract

We study the use of effective transfer matrices for the numerical computation of masses (or correlation lengths) in lattice spin models. The effective transfer matrix has a strongly reduced number of components. Its definition is motivated by a renormalization group transformation of the full model onto a 1-dimensional spin model. The matrix elements of the effective transfer matrix can be determined by Monte Carlo simulation. We show that the mass gap can be recovered exactly from the spectrum of the effective transfer matrix. As a first step towards application we performed a Monte Carlo study for the 2-dimensional Ising model. For the simulations in the broken phase we employed a multimagnetical demon algorithm. The results for the tunnelling correlation length are particularly encouraging.
1 Introduction

The computation of the mass spectrum is one of the major goals of modern quantum field theory. In the Euclidean path integral formulation on the lattice \cite{1}, the mass spectrum can be recovered from the large-distance behaviour of suitably chosen correlation functions. The correlation functions can (in principle) be computed with the Monte Carlo method \cite{2}.

The bridge between the path integral and the Hamiltonian formulation is in the transfer matrix \cite{1}. The mass spectrum can be directly read off from the eigenvalues of the transfer matrix. The task is therefore to diagonalize the transfer matrix. An exact solution of the full problem is possible only in a restricted class of models. The most prominent example for a model that can be solved via the transfer matrix approach is the 2-dimensional Ising model \cite{3}. A direct numerical diagonalization of the transfer matrix is restricted to systems with very small spatial extension \cite{4}.

As the origin of the problem with the transfer matrix approach is in the huge size of the matrix to diagonalize, it is natural to reduce the number of degrees of freedom by some ‘coarse-graining procedure’. Inspired by block spin renormalization group ideas \cite{5}, we define an ‘effective transfer matrix’ \cite{6}. This effective transfer matrix has a drastically reduced size and can be diagonalized with standard numerical procedures.

In section 3 we motivate the concept of the effective transfer matrix, starting from a block spin renormalization group point of view. We arrive at an intuitive derivation of three different ‘rules’ for a definition of an effective transfer matrix. In section 3 we show (for one of the rules) that the mass of the first excited state can be exactly recovered from the effective transfer matrix. To learn more about the properties of the effective transfer matrix, we then study the 2-dimensional Ising model on small lattices. Here we can compute the spectrum of the effective transfer matrix without any approximation and compare it with the exactly known spectrum of the full transfer matrix. So we can study to what extent the low-lying spectrum of the full transfer matrix can be recovered from the effective transfer matrix. In the final section we present results for the low-lying spectrum of the 2-dimensional Ising model on lattices up to $64 \times 2048$ as obtained from effective transfer matrices computed by Monte Carlo simulations. We discuss results from simulations at the critical point and in the broken phase of the model. The multimagnetic demon algorithm we employed for the simulations in the broken phase provided us with high statistics for tunnelling events. The success is based on an increased probability for configurations with low magnetization and a high performance due to a very efficient implementation. The longest tunnelling correlation length we measured was $43500 \pm 1600$ on a $64 \times 128$ lattice, consistent with the exact value $44014.4 \ldots$.

\footnote{In the literature, the notion of an ‘effective transfer matrix’ is used in a variety of different contexts. The reader might be interested to compare our notion with that developed, e.g. in \cite{6}, \cite{7} and \cite{8}. Furthermore, for one of our definitions of the effective transfer matrix, our approach is very similar to a method that has been discussed in the context of glueball mass calculations \cite{9,10}. See also Appendix C.}
2 Effective 1-Dimensional Model

We consider a $(d+1)$-dimensional Euclidean quantum field theory on a cubic lattice $\Lambda$ of size $N_d^\sigma \times N_\tau$. Here, $N_\sigma$ denotes the spatial extension of the lattice, and $N_\tau$ is the extension in the time direction. Let us label the sites of the lattice by $(t, \vec{x})$, where $t$ runs from 1 to $N_\tau$, and the spatial coordinates $x_i$ cover the range $1 \ldots N^\sigma$. In the Euclidean path integral formulation one integrates over stochastic variables $\phi_{t,\vec{x}}$ that are attached to the sites of $\Lambda$. The model is defined by the partition function

$$Z = \sum_{\phi} \exp(-\mathcal{H}(\phi)) .$$

(1)

Let us denote the configurations of $\phi$ on ‘time slices’ $t$ by $\phi_t$. Assume that $\mathcal{H}(\phi)$ takes the following form:

$$\mathcal{H}(\phi) = \sum_t (K(\phi_t, \phi_{t+1}) + V(\phi_t)) .$$

(2)

Then for periodic boundary conditions in the $t$-direction the partition function may be written as

$$Z = \text{Tr}(T^{N_\tau}) .$$

(3)

The transfer matrix $T$ can be chosen to be symmetric:

$$T(\varphi, \varphi') = \exp(-K(\varphi, \varphi') - \frac{1}{2}[V(\varphi) + V(\varphi')])$$

(4)

$\varphi$ and $\varphi'$ are again time slice configurations, i.e. configurations on $d$-dimensional sub-lattices of size $N^d_\sigma$. We shall also use a ‘bra’ and ‘ket’ notation, e.g.

$$T(\varphi, \varphi') = \langle \varphi | T | \varphi' \rangle .$$

(5)

Let $\lambda_i$, $i = 0, 1 \ldots$ denote the eigenvalues of $T$, such that $\lambda_0 > \lambda_1 \geq \lambda_2 \geq \ldots$. The corresponding eigenvectors shall be denoted by $|i\rangle$. For eigenstates with zero momentum, masses are defined as

$$m_i = -\ln \frac{\lambda_i}{\lambda_0} \equiv \frac{1}{\xi_i} .$$

(6)

The $\xi_i$ denote the corresponding correlation lengths. We shall now consider block spin transformations that transform the statistical system from a $(d+1)$-dimensional one to a 1-dimensional one. Block spins are defined as averages over time slices:

$$\Phi_t \equiv N^{-d}_\sigma \sum_{\vec{x}} \phi_{t,\vec{x}} .$$

(7)

For integer $l \geq 1$ we define an ‘effective Hamiltonian’

$$\exp(-\mathcal{H}_{\text{eff}}^{(d)}(\Phi)) = \sum_{\phi} \exp(-\mathcal{H}(\phi)) \prod_{t \in G_l} \delta(\Phi_t - N^{-d}_\sigma \sum_{\vec{x}} \phi_{t,\vec{x}}) .$$

(8)
with $G_l = \{1, l + 1, 2l + 1, \ldots, N_r - l + 1\}$, and $Z$ can be rewritten as

$$Z = \left( \prod_{t \in G_l} \sum_{\Phi_t} \right) \exp \left( -\mathcal{H}_{\text{eff}}^{(l)}(\Phi) \right). \quad (9)$$

Let us now relabel the sites in $G_l$ with integers that run from 1 to $N'_r = N_r/l$. Let us furthermore assume that (after the relabelling) for some $l$ with good precision $\mathcal{H}_{\text{eff}}^{(l)}(\Phi)$ couples only nearest neighbours, i.e. is of the form

$$\mathcal{H}_{\text{eff}}^{(l)}(\Phi) = \sum_t \left( K_{\text{eff}}^{(l)}(\Phi_t, \Phi_{t+1}) + V_{\text{eff}}^{(l)}(\Phi_t) \right). \quad (10)$$

Note that, in general, the effective Hamiltonian will contain interaction terms of arbitrary range. However, one can generally assume that these terms decay exponentially with the distance and might be neglected for a first qualitative analysis.

Assuming a nearest neighbour interaction effective Hamiltonian, we can again rewrite the partition function in terms of a transfer matrix

$$Z = \text{Tr} \left( [T_{\text{eff}}^{(l)}]_{N'_r} \right), \quad (11)$$

and

$$T_{\text{eff}}^{(l)}(\Phi, \Phi') = \exp \left( -K_{\text{eff}}^{(l)}(\Phi, \Phi') - \frac{1}{2}[V_{\text{eff}}^{(l)}(\Phi) + V_{\text{eff}}^{(l)}(\Phi')] \right). \quad (12)$$

What have we gained? The number of degrees of freedom has been considerably reduced. Take for example the Ising model. Here the original transfer matrix is $2^{N^d}$ by $2^{N^d}$. If we choose as block spin the time slice magnetization then $T_{\text{eff}}^{(l)}$ is a matrix with $N^d + 1$ by $N^d + 1$ components. So there is a drastic simplification in the eigenvalue problem. The crucial question is, of course, to which extent the spectrum of $T_{\text{eff}}^{(l)}$ reflects properties of the original system. This question will be studied in detail in the next section.

Let us first discuss how one can express the elements of $T_{\text{eff}}^{(l)}$ as expectation values in the statistical system. This is necessary if we want to compute them, e.g. by Monte Carlo simulation. To simplify the discussion, we shall assume in the following that the $\Phi_t$ take discrete values only. Let us denote the discrete values of $\Phi_t$ by $M, N, \ldots$. They correspond to ‘improper states’ $|M\rangle$, $|N\rangle$, etc. We shall use throughout ‘double brackets’ to denote states in the space that $T_{\text{eff}}^{(l)}$ acts on. We now introduce the operator $\delta_{\Phi_t, M}$ that takes the value 1 if $\Phi_t = M$, and $\delta_{\Phi_t, M} = 0$ elsewhere. Assuming periodic boundary conditions in the $t$-direction, the correlator of two such operators at distance one can be written as follows:

$$\langle \delta_{\Phi_1, M} \delta_{\Phi_{t+1}, N} \rangle = Z^{-1} \langle [M | T_{\text{eff}}^{(l)} | N] \rangle \langle [N | T_{\text{eff}}^{(l)} | M] \rangle \quad (13)$$

In the following we shall sometimes omit the extra factor $Z^{-1}$, because it leads only to an irrelevant shift of the ground-state energy. Masses and correlation lengths are unaffected. Equation (13) has to be resolved with respect to the effective transfer matrix elements. This is particularly easy for the case $N'_r = 2$,

$$\langle [M | T_{\text{eff}}^{(l)} | N] \rangle = \sqrt{\langle \delta_{\Phi_1, M} \delta_{\Phi_2, N} \rangle}. \quad (14)$$
We say that the effective transfer matrix is defined according to the symmetric periodic lattice rule. For larger $N' \tau$ (asymmetric periodic lattice rule), the solution for $T_{\text{eff}}^{(l)}$ can be found by iteration (cf. Appendix A). The situation becomes simple again in the limit $N' \tau \rightarrow \infty$.

\begin{equation}
< \delta_{\Phi_t, M} \delta_{\Phi_{t+1}, N} > = \langle \langle 0 | N \rangle \langle N | T_{\text{eff}}^{(l)} | M \rangle \langle M | 0 \rangle \rangle . \tag{15}
\end{equation}

Using the fact that $< \delta_{\Phi_t, M} > = \langle \langle 0 | M \rangle \langle M | 0 \rangle \rangle$, one finds

\begin{equation}
\langle \langle M | T_{\text{eff}}^{(l)} | N \rangle \rangle = \frac{< \delta_{\Phi_t, M} \delta_{\Phi_{t+1}, N} >}{\sqrt{< \delta_{\Phi_t, M} > < \delta_{\Phi_{t+1}, N} >}} . \tag{16}
\end{equation}

We say that $T_{\text{eff}}^{(l)}$ is defined here via the $N \tau = \infty$ rule.

3 How the Mass Gap is Recovered

In this section we want to discuss how the effective transfer matrix $T_{\text{eff}}^{(l)}$ is related to the transfer matrix of the basic system, $T$, and how much of the spectrum of this matrix can be recovered from the spectrum of the effective transfer matrix. We will concentrate on the discussion of the $N \tau = \infty$ rule.

We define ‘magnetization pieces of the ground-state’ by

\begin{equation}
|M\rangle = N_M^{-1} \sum_{\varphi} \delta (\bar{\varphi} = M) |\varphi\rangle \langle \varphi | 0 \rangle . \tag{17}
\end{equation}

Here, $|\varphi\rangle$ are time slice configuration states, and

\begin{equation}
\bar{\varphi} = N_\sigma^{-d} \sum_{\bar{x}} \varphi_{\bar{x}} . \tag{18}
\end{equation}

The constants $N_M$ are chosen such that $\langle M | M \rangle = 1$. One can convince oneself that $\langle N | M \rangle = \delta_{N,M}$. However, the $|M\rangle$'s do not, of course, span the complete Hilbert space. Let us denote by $P$ the projector onto the subspace spanned by the $|M\rangle$'s,

\begin{equation}
P = \sum_M |M\rangle \langle M | . \tag{19}
\end{equation}

The effective transfer matrix for the $N \tau = \infty$ rule is given by

\begin{equation}
\langle \langle M | T_{\text{eff}}^{(l)} | N \rangle \rangle = \langle M | T^l | N \rangle . \tag{20}
\end{equation}

Let us denote the eigenstates of $T_{\text{eff}}^{(l)}$ by $|i\rangle$, and the corresponding eigenvalues by $\Lambda_i^{(l)}$. We now claim that $\Lambda_0^{(l)} = \lambda_0^l$. The proof is as follows:

\begin{equation}
\langle \langle M | T_{\text{eff}}^{(l)} | 0 \rangle \rangle = \sum_N \langle \langle M | T_{\text{eff}}^{(l)} | N \rangle \rangle \langle \langle N | 0 \rangle \rangle = \sum_N \langle M | T^l | N \rangle \langle N | 0 \rangle = \langle M | T^l P | 0 \rangle = \lambda_0^l \langle \langle M | 0 \rangle \rangle . \tag{21}
\end{equation}
Here we have used the fact that $\mathcal{P}|0\rangle = |0\rangle$.

What about the first excited state? Let us try the following wave function:

$$\langle M|1'\rangle = c_1^{-1/2} \langle M|1\rangle. \quad (22)$$

The constant $c_1$ is chosen such that this state is normalized to 1:

$$c_1 = \sum_M \langle M|1\rangle^2 = \langle 1|\mathcal{P}|1\rangle. \quad (23)$$

The action of the effective transfer matrix on this state is easily found to be

$$\langle M|T_{\text{eff}}^{(l)}|1'\rangle = c_1^{-1/2} \langle M|T\mathcal{P}|1\rangle. \quad (24)$$

Now consider the first excited state $|1\rangle$ of the original transfer matrix $T$ with eigenvalue $\lambda_1$ and rewrite it as follows:

$$|1\rangle = c_1^{-1} \mathcal{P}|1\rangle + \sum_{i>1} f_{1,i} |i\rangle. \quad (25)$$

That only states with $i>1$ contribute in the sum is due to the fact that $\langle 0|\mathcal{P}|0\rangle = 1$, and $\langle 0|\mathcal{P}|1\rangle = 0$. Solving this equation for $\mathcal{P}|1\rangle$ and inserting this in eq. (24), one obtains

$$\langle M|T_{\text{eff}}^{(l)}|1'\rangle = c_1 \lambda_1^l \left( \langle M|1'\rangle - \sum_{i>1} f_{1,i} \left( \frac{\lambda_i}{\lambda_1} \right)^l \langle M|i\rangle \right). \quad (26)$$

For large $l$ one therefore has

$$T_{\text{eff}}^{(l)}|1'\rangle = c_1 \lambda_1^l |1'\rangle + O \left( \left( \frac{\lambda_2}{\lambda_1} \right)^l \right). \quad (27)$$

From the spectrum of the effective transfer matrix, we obtain estimates for the mass of the first excited state

$$m_1^{(l)} \equiv -\frac{1}{l} \ln \left( \frac{\Lambda_1^{(l)}}{\Lambda_0^{(l)}} \right) \equiv \frac{1}{\xi_1^{(l)}}. \quad (28)$$

It is easy to see that $m_1^{(l)}$ behaves like

$$m_1^{(l)} = m_1 - \frac{\ln c_1}{l} + \text{exponentially small corrections}. \quad (29)$$

One can get rid of the $1/l$ corrections by combining pairs of results with different $l$'s. For the correlation length of the first excited state one gets

$$\xi_1 = \frac{l_1 \xi_1^{(l_1)} - l_2 \xi_1^{(l_2)}}{l_1 - l_2} + \text{exponentially small corrections}. \quad (30)$$

What about the higher states? Let us first see what happens with the second excitation. Proceeding naively, we would start with

$$\langle M|2'\rangle = c_2^{-1/2} \langle M|2\rangle, \quad (31)$$

and go along the same lines as in the proof above. We would then find that everything would work just the same as for the first excited state, provided that $\langle 1|\mathcal{P}|2\rangle$ vanishes. (This can happen for symmetry reasons, see the discussion of the Ising model below.) Generally, when for higher states the overlaps $\langle i|\mathcal{P}|j\rangle$ are very small for $i<j$, then the state $|i'\rangle$ allows the reconstruction of $\lambda_i$ with good precision.
4 Exact Results on Small Lattices

As a first study of the effective transfer matrices defined above, we made numerical calculations for the 2-dimensional Ising model on lattices with small spatial extension. These calculations are exact in the sense that only standard numerical techniques were used. No Monte Carlo simulations were involved.

The model is defined through its partition function

\[ Z = \sum_{\sigma_x = \pm 1} \exp \left( \beta \sum_{(x,y)} \sigma_x \sigma_y \right). \]  

(32)

Here we have denoted lattice sites by \( x = (t,i) \), and \( \langle xy \rangle \) denotes a nearest neighbour pair. The (symmetrically chosen) transfer matrix of this model is given by

\[ T(\phi_t, \phi_{t+1}) = \exp \left( \frac{\beta N_\sigma}{2} \sum_{i=1}^{N_\sigma} \sigma_{t,i} \sigma_{t,i+1} + \frac{\beta}{2} \sum_{i=1}^{N_\sigma} \sigma_{t,i} \sigma_{t+1,i} + \beta \sum_{i=1}^{N_\sigma} \sigma_{t+1,i} \sigma_{t+1,i+1} \right); \]  

(33)

\( \phi \) here denotes time slice configurations of the Ising spins. We assume periodic boundary conditions in the space direction, i.e. the site with \( i = N_\sigma + 1 \) is identified with the site \( i = 1 \).

The 2-dimensional Ising model was first solved by Onsager in 1944 [3]. The eigenvalues of the transfer matrix can be read off easily from [11]. See also the reviews in ref. [12].

As a consequence of the fact that the transfer matrix commutes with the spatial translation operator, the eigenstates can be chosen as simultaneous eigenstates of energy and momentum. They can further be classified in states that are symmetric/antisymmetric with respect to reversal of all spins. We shall denote the corresponding masses or correlation lengths with a subscript \( s \) or \( a \), respectively.

In the infinite volume limit, i.e. when \( N_\sigma \to \infty \), the spectrum is twofold degenerate in the broken phase (\( \beta > \beta_c \)). For finite \( N_\sigma \) the degeneracy in the low-temperature regime is lifted and level splitting occurs because of tunnelling.

We will consider the time slice magnetization as effective spin in the following. In this case only the zero-momentum eigenvalues can be recovered from the effective transfer matrix. [2]

We shall give results for the \( N_\tau = \infty \) rule and for the symmetric periodic lattice rule. We have results for \( N_\sigma \leq 8 \). The basis for our computations is an accurate determination of the eigenvalues and eigenfunctions of the transfer matrix defined in eq. (33). In principle the eigenfunctions of \( T \) are given in the work by Kaufmann [11]. However, the expressions seem too complicated to be of practical use for our goals, so we decided to use the computer.

Let us first discuss the case \( N_\tau = \infty \). We numerically determined the eigenstates \( |i\rangle \) and the ‘magnetization pieces’ \( |M\rangle \) defined in eq. (17). Together with the eigenvalues of \( T \), they enter the matrix elements of \( T_{eff}^{(l)} \) as follows:

\[ \langle M|T_{eff}^{(l)}|N\rangle = \sum_i \lambda_i^l \langle M|i\rangle \langle i|N\rangle. \]  

(34)

Note that effective states that couple to nonzero momentum could also be employed.
The |M⟩’s are zero momentum states. Hence we only needed to calculate the translational invariant eigenstates of $T$.

In the case of the periodic system with $N_r = 2l$ (symmetric periodic lattice rule) one finds

$$\langle \langle M | T_{\text{eff}}^{(l)} | N \rangle \rangle = \sqrt{\sum_{i,j} \lambda_l \lambda_{j} \langle i | M \rangle \langle M | j \rangle \langle j | N \rangle \langle N | i \rangle}. \quad (35)$$

For both rules we computed the spectrum of $T_{\text{eff}}^{(l)}$ for various $l$ and determined the $\xi^{(l)}$ defined in eq. (28) for the lowest-lying states. Our results for the $N_r = \infty$ rule are listed in tables 1 and 2. The results for the symmetric periodic lattice rule are quoted in tables 3 and 4. We always quote the results for $l = 1, 2, 4$ and 8. In addition, we give an estimate for the ‘true’ $\xi$ obtained by combining $\xi^{(4)}$ and $\xi^{(8)}$ according to eq. (30). The exact results are always quoted with an ‘e’ in the second column.

Let us first look at the results for the $N_r = \infty$ rule. As we proved in section 3, the $\xi^{(l)}_{0,a}$ should converge towards the exact limit $\xi_{0,a}$. This can indeed be observed. The second largest correlation length $\xi^{(l)}_{1,s}$ also nicely converges. This is a consequence of the fact that the corresponding state is symmetric in the magnetization while the first excited state is antisymmetric. Therefore the matrix element $\langle 1s | P | 0a \rangle$ vanishes, and the proof that the correct correlation length can be recovered goes the same way as for the first excited state. The convergence for $\xi^{(l)}_{1,a}$ is also quite good. This is anticipated from the discussion at the end of section 3, since $c_{0a}$ is very close to 1. Following the discussion of section 3, the constant $c_i = \sum_M \langle M | i \rangle^2$ generalizing eq. (23) should indicate the goodness of the effective eigenstate corresponding to $|i\rangle$; $c_i = 1$ indicates a perfect representation of the eigenfunction $|i\rangle$ by the corresponding effective eigenfunction $|i'\rangle$.

For $L = 4, 6, 8$ and 10 we calculated this quantity explicitly, starting from the exact eigenfunctions with labels $i = (0, a), (1, s)$ and $(1, a)$. The results for the $\beta$-range $0.35 \rightarrow 0.55$ are given in fig. 1. We make the following observations: The overlap $c_{0a}$ is very close to 1. It becomes better for $\beta \rightarrow 0$ and for $\beta \rightarrow \infty$. The overlaps $c_{1s}$ and $c_{1a}$ are worse. The deviation from 1 is in the range of several percent; $c_{1s}$ and $c_{1a}$ are ‘best’ in the critical domain. All $c$’s become worse with increasing $N_\sigma$. It is an open question whether they have a finite limit in the $N_\sigma \rightarrow \infty$ limit.

The $\xi_{2,s}^{(l)}$ do not converge. Here the states of the original system seem to mix somehow to the effective states.

Let us now turn to the discussion of the results for the periodic symmetric rule. Table 3 is encouraging: when we combine the $l = 4$ and $l = 8$ data, all correlation lengths are reproduced within the given numerical accuracy, with the exception of $\xi_{2,s}$. However, this is only so on very small lattices. A careful look at table 4 reveals that only the leading correlation lengths $\xi_{0,a}^{(l)}$ converge to the exact value $\xi_{0,a}$. The estimates for the other $\xi$’s are not entirely off. They might be regarded as unsystematic approximations. However, we do not know what happens if the spatial extension of the lattice is further increased. See also the discussion of Monte Carlo results on larger lattices in section 3.
5 Monte Carlo Results for the 2D Ising Model

5.1 The Critical Model, $N_\tau = \infty$ rule

At the critical coupling $\beta_c = 0.4406868...$ of the 2-dimensional Ising model, we performed Monte Carlo simulations on $N_\sigma \times N_\tau = 16 \times 512, 32 \times 1024$ and $64 \times 2048$ lattices. We always measured and stored the magnetizations of all time slices after five Swendsen-Wang updates \[13\] of the entire lattice.

We made 5000 measurements for $N_\sigma = 16$, 3900 measurements for $N_\sigma = 32$, and 3000 measurements for $N_\sigma = 64$.

For the simulations at criticality we used the $N_\tau = \infty$ rule to get estimates for the effective transfer matrix elements. ‘Infinitely long’ lattice here means, of course, that $N_\tau$ is much larger than the largest correlation length involved, so that the influence of a finite lattice extension in the $t$-direction can be neglected. In all three cases, $N_\tau/\xi_0, a$ is about 25. This is certainly on the safe side.

The eigenvalues and eigenstates of the effective transfer matrix were then calculated using standard numerical procedures. From the eigenvalues of the effective transfer matrix we determined the $\xi(l)$ for various distances $l$.

In order to estimate the statistical errors of our results we divided the whole sample in 5, 10, 20 and 40 bins and computed the effective transfer matrix and its eigenvalues separately within each of the bins. The statistical errors quoted in the tables were obtained from the mean square fluctuations over the outcomes of the various bins. We considered the error estimate as reliable when it was approximately independent of the number of bins.

This binning analysis was also used to check for a bias due to a too small statistics. We averaged the results from the bins and compared the outcome of various bin sizes. Only those results can be trusted where there is an agreement of the results over the various bin sizes. The quantities for which we obtained stable estimates are quoted in tables 3, 4 and 5.

With increasing $l$, at least the largest two correlation lengths $\xi_{0,a}$ and $\xi_{1,s}$ converge towards the corresponding exact value. Similar to the exact results on small lattices we find the estimates for $\xi_{1,a}$ converging within the statistical accuracy towards the exact results.

The tables also contain our final estimates for the correlation lengths obtained by combining pairs of effective correlation lengths with different $l$ according to eq. (30). The convergence and stability of these combined estimates shows that the $l$ dependence is indeed of the form of eq. (29).

In fig. 2 we present plots of the effective wave functions (eigenfunctions of $T^{(l)}_{\text{eff}}$) for $N_\sigma = 64$ and for $l = 5, 10$ and 20. One can observe that the effective wave functions obtained from different $l$ are almost identical. For $l = 20$ the effective wave function for $|1a\rangle$ becomes noisy because of insufficient statistics.

In order to compare the effective transfer matrix approach with a ‘traditional’ determination of masses, we present here the results of a conventional evaluation of our Monte Carlo data for the 2-dimensional Ising model at criticality. As an
example we choose $N_\tau = 64$ and $N_\tau = 2048$. We define

\[ 
G_{0,a}(t) = \langle \phi_0 \phi_t \rangle, \\
G_{1,s}(t) = \langle \phi_0^2 \phi_t^2 \rangle - \langle \phi_0^2 \rangle \langle \phi_t^2 \rangle. 
\] (36)

These correlation functions behave like $G_i(t) = g_i \exp(-t/\xi_i) + \ldots$. We define ‘effective correlation lengths’ as

\[ \xi_{i,t}^{\text{eff}} = 1/(\ln(G_i(t)) - \ln(G_i(t + 1))). \] (37)

With increasing $t$ these quantities should converge to the true correlation lengths. Table 8 shows our results for the $\xi_{i,t}^{\text{eff}}$, for $i = (0, a)$ and $i = (1, s)$. We used exactly the same Monte Carlo data as for the effective transfer matrix. The comparison of the results obtained with the two different methods shows that the transfer matrix results have statistical errors that are roughly a factor of two smaller than the ‘conventional’ ones.

We evaluated the same Monte Carlo data with a third technique: Based on the results for the eigenfunctions of the effective transfer matrix $T_{\text{eff}}(l)$, we determined observables that are expected to have improved overlap with the eigenstates $|0, a\rangle$ and $|1, s\rangle$, respectively. For details see Appendix C. With the improved observables, we again computed the ‘effective correlation lengths’ $\xi_{i,t}^{\text{eff}}$, for $i = (0, a)$ and $i = (1, s)$. The comparison of these quantities, ‘standard’ and ‘improved’ is shown in figs. 3 and 4. At least for $\xi_{0,a}$, the improvement is striking.

\section*{5.2 Monte Carlo Results for the Broken Phase}

We also did simulations in the broken phase, at $\beta = 0.47$. This value is low enough so that the tunnelling correlation length becomes very large even with modest $N_\sigma$, but is close enough to $\beta_c$ so that the bulk correlation length (4.349 . . . when $N_\sigma = \infty$) is still substantially larger than one.

In order to fight the supercritical slowing down due to exponentially suppressed tunnelling rates, we employed a multimagnetical algorithm. Thanks to multispin coding and the usage of demons to implement the nonlocal magnetization-dependent interaction terms in the multimagnetical ensemble, the algorithm performed very well on a CRAY X-MP. For more details, see Appendix B. We performed simulations for lattices with spatial extensions $N_\sigma = 16, 32$ and 64. For an overview of the runs and the statistics see table 9.

For the analysis in the broken phase, we used the asymmetric periodic lattice rule throughout. Our results are quoted in tables 11, 12 and 13. We performed the simulations with $N_\tau = N_\sigma/2$, $N_\sigma$ and $2N_\sigma$. Generally, we observe two trends: the larger $N_\tau$ is, or the closer one is to the symmetric periodic lattice rule, the smaller the systematic error in the tunnelling correlation length measurements becomes. When $N_\tau = 2N_\sigma$, we find an impressive reproduction of the tunnelling correlation length that gets as big as 44014 on the $N_\sigma = 64$ lattice. Note that the tables also include results for the symmetric periodic lattice rule: the last row in each block of
the table quotes the results for \( l = N_{\tau}/2 \). We also present the tunnelling correlation length data for the \( N_\sigma = 64 \) lattices in fig. 5.

We also tried to reproduce the large tunnelling correlation length on the \( N_\sigma = 64 \) lattice using the standard technique of fitting the correlation functions of time slice magnetizations. However, this did not lead to any sensible result. To us this seems to be a situation where our effective transfer matrix technique is completely superior. We therefore consider our method well suited for the study of the interface tension \([17]\).

The other correlation lengths besides the tunnelling length are only approximately correct. It seems that they do not converge. Here certainly further investigation is needed.

The effective wave functions for \( \beta = 0.47 \) and \( N_\sigma = 64 \) are plotted in fig. 6, for \( l = 8, 16, \) and \( 32 \). That the tunnelling length is so large corresponds to the fact that the square of the ground-state wave function and the square of the first excited state differ only in a small neighbourhood of zero magnetization.

6 Summary and Conclusion

We have studied the use of effective transfer matrices for the computation of masses of a Euclidean quantum field theory, or, equivalently, the correlation lengths of a statistical mechanical model. Many questions are left open and deserve further theoretical and numerical investigation. The theoretical study of the other rules besides the \( N_\tau = \infty \) rule is not yet complete. Furthermore, one could consider the usage of blocks with finite extension in the time direction. In principle, one can employ any sort of ‘effective spin’ (not just the magnetization or its absolute value). Application of the method to systems with continuous degrees of freedom would also be interesting.

A report on a study of the 3-dimensional Ising model will be published elsewhere \([14]\).

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Appendix A: Matrix Equation for Asymmetric Rule

The problem of solving the equation

\[
\langle \delta \Phi_t, M | \delta \Phi_{t+1}, N \rangle = \langle M | T_{\text{eff}}^{(l)} | N \rangle \langle N | T_{\text{eff}}^{(l)} N_{\tau-1} | M \rangle
\]

(38)
with respect to the components of the effective transfer matrix (given the left-hand side, e.g. as an outcome of a Monte Carlo simulation), may be stated as follows: For a given symmetric matrix $B$ with matrix elements $B_{ij}$, find a matrix $A$ with matrix elements $A_{ij}$ such that the following equation holds:

$$B_{ij} = A_{ij} (A^n)_{ij}, \quad \text{(39)}$$

where $A^n$ denotes the $n^{th}$ matrix power of $A$. We do not know of a closed solution of this problem, but, inspired by the famous iteration prescription for the problem of finding the square root of a number $a$, $x \rightarrow \frac{1}{2}(x + \frac{a}{x})$, we try the following iteration prescription for the solution of eq. (39):

$$A_{ij} \rightarrow \frac{1}{1 + \zeta} (A_{ij} + \zeta \frac{B_{ij}}{(A^n)_{ij}}), \quad \text{(40)}$$

where $\zeta$ denotes a parameter that can be tuned in order to optimize the convergence. With the exception of a few special cases the algorithm converged, although the convergence rates were sometimes very slow.

## Appendix B: Multimagnetical Demon Algorithm

We here present a brief description of the multimagnetical demon algorithm used for the simulations at $\beta = 0.47$.

When $\beta > \beta_c$, the tunnelling rate between the ordered states becomes exponentially smaller as the size of the system is increased. However, in order to extract the tunnelling correlation length we need a good statistics of the tunnelling events – i.e. the configurations with the magnetization between the bulk expectation values $\pm M_B$. The multimagnetical method \cite{15} solves this problem by artificially enhancing the probability of these states. In the standard (no demons) approach, this is achieved by modifying the probability of the spin configuration $\bar{\sigma}$ with an extra weight function $G(M)$:

$$P_{\text{mm}}(\bar{\sigma}) \propto e^{-\beta H(\bar{\sigma})} G(M), \quad \text{(41)}$$

where $M_{\bar{\sigma}}$ is the magnetization of the configuration $\bar{\sigma}$. The probability of the magnetization $M$ becomes

$$P_{\text{mm}}(M) \propto P_{\text{can}}(M) G(M) \propto \left( \sum_{\bar{\sigma}} e^{-\beta H(\bar{\sigma})} \delta(M_{\bar{\sigma}} - M) \right) G(M). \quad \text{(42)}$$

We use a notation where the $\beta$-dependence of $G$ and $P$ is suppressed. Usually one aims at a constant probability distribution between $\pm M_B$, implying that $G(M) \propto 1/P_{\text{can}}(M)$. However, since $P_{\text{can}}$ is unknown, one has to use an approximate form instead, which can be obtained, for example, by scaling up the function $G$ used in the simulations performed with smaller volumes. This is often further refined by performing test runs and adjusting $G(M)$ until satisfactory $P_{\text{mm}}(M)$ is obtained. For further details, see refs. \cite{13, 14}.
The function $G(M)$ depends on the global magnetization, rendering the update non-local and thus preventing straightforward vectorization and parallelization. In this work modified the standard multimagnetical algorithm by utilizing demons to change the magnetization; this approach enables us to use powerful multispin coding and is highly vectorizable. This method is a magnetic analogue of the multicanonical demon algorithm presented in [16]. The actual update becomes a two-stage process: first, the demon magnetization $M_D$ is changed by coupling the demons to a “multimagnetical heat bath”, and second, the spin system magnetization $M_S$ is changed by coupling the spins to demons and performing the spin update while preserving the total magnetization $M_T = M_D + M_S$. More precisely, we perform the simulation according to the joint probability distribution

$$P(M_S, M_D) \propto P_{\text{can}}(M_S) n_D(M_D) W(M_T), \quad (43)$$

where $n_D(M_D)$ is the number of demon states with magnetization $M_D$, and $W$ is a new weight function that depends only on $M_T$. The canonical expectation value of an observable $O$ can now be obtained by reweighting:

$$<O> = \frac{\sum_i O_i W^{-1}(M_{T,i})}{\sum_i W^{-1}(M_{T,i})}, \quad (44)$$

where $O_i$ and $M_{T,i}$ refer to the individual measurements of the corresponding quantities. Summing over $M_D$ in eq. (43) and comparing it with eq. (42), we note that $G$ and $W$ are related: if $G(M_S) \propto \sum M_D n_D(M_D) W(M_T)$, we obtain a similar probability distribution for $M_S$ in the two cases. In this work we are using demons carrying $\pm 1$ units of magnetization; with $N_D$ demons, the demon density of states becomes

$$n_D(M_D) = \frac{N_D!}{[(N_D + M_D)/2]! [(N_D - M_D)/2]!}. \quad (45)$$

Let us look closer at the individual update steps.

- In the multimagnetical demon refresh step the new $M_D$ is chosen with probability

$$P(M_D) \propto n_D(M_D) W(M_S + M_D). \quad (46)$$

While one could use this probability to refresh each demon individually, this is not very efficient ($\propto N_D$ steps). We used eq. (46) to directly choose new random $M_D$. Depending on the old value of $M_D$, we then either added or subtracted magnetization from randomly selected demons, until the right demon magnetization was reached. This whole process takes, on the average, only $\propto \sqrt{N_D}$ steps, and yields a new demon magnetization independent of the old one. In our simulations we used four times as many demons as spins ($N_D = 4 N_S$).

- In order to ensure the canonical energy distribution the spin system is connected to a heat bath. To make the multispin coding easier, we used a second set of demons, this time carrying energy. Individual demon energies vary in units of 4: 0, 4, 8, ... Before each update sweep through the spin system, every spin is connected to a magnetic demon and an energy demon; the demons are chosen with random order to ensure fast mixing. A spin flip is accepted if and only if the
demons can absorb the change in energy and magnetization. The energy demons are periodically refreshed with a heat bath. Note that one could also perform a normal Metropolis or heat bath update without the energy demons; the update is accepted/rejected with the magnetic demon.

Because the spin update is very fast, we interleaved 5 sweeps through the lattice for one demon update. For the largest lattice \((128 \times 64)\), an individual spin update took 16 ns, whereas the total time divided by the number of spin updates was 28 ns on a Cray X-MP.

**Appendix C: Observables with Improved Overlap**

In a ‘conventional’ correlation length measurement one studies the exponential decay of correlators \(\langle A_t A_{t+\tau} \rangle\), where \(A_t\) is an observable that depends on the configuration of a single time slice \(t\). On an infinitely long lattice, the correlator can be written as

\[
\langle A_t A_{t+\tau} \rangle = \frac{\langle 0 | A T^{\tau} A | 0 \rangle}{\langle 0 | T^{\tau} | 0 \rangle}.
\]  

(47)

Expanding in terms of eigenfunctions of the transfer matrix one obtains

\[
A | 0 \rangle = \sum_i a_i | i \rangle,
\]  

(48)

where \(a_i = \langle i | A | 0 \rangle\). Inserting this in eq. (47), one obtains

\[
\langle A_t A_{t+\tau} \rangle = \sum_i \frac{a_i^2 \lambda_i^{\tau}}{\lambda_0^{\tau}} = \sum_i a_i^2 \exp(-m_i \tau),
\]  

(49)

where the masses are defined by \(m_i = -\ln(\lambda_i/\lambda_0)\). For a general observable \(A\) the \(a_i\) will be nonzero for all \(i\).

In order to improve this situation one could think of constructing observables where only one \(a_i\) is non-zero. First recall that observables correspond to operators which are diagonal in the basis of configuration states \(|\phi\rangle\). This means that

\[
\langle \phi | A | \phi' \rangle = \langle \phi | A | \phi \rangle \delta_{\phi,\phi'} \equiv A_{\phi} \delta_{\phi,\phi'}.
\]  

(50)

The condition that \(a_i\) should be the only non-vanishing vacuum overlap is equivalent to \(A | 0 \rangle = | i \rangle\), or

\[
A_{\phi} = \frac{\langle \phi | i \rangle}{\langle \phi | 0 \rangle}.
\]  

(51)

The problem with this equation is that we do not know the wave functions \(\langle \phi | i \rangle\) and \(\langle \phi | 0 \rangle\) exactly.

However, we know the eigenfunctions \(|i\rangle\) of the effective transfer matrix. Let us embed them into the Hilbert space of the full model. We define

\[
|i'\rangle \equiv \sum_M |M\rangle \langle M | i \rangle.
\]  

(52)
Note that (compare section 3) this projection leaves the vacuum invariant, i.e. 
\[ \langle i'|0 \rangle = \langle i|0 \rangle \]. As improved observable we now consider the ratio
\[
A'_\phi = \frac{\langle \phi|i' \rangle}{\langle \phi|0 \rangle}.
\] (53)

One can easily convince oneself that \( A'_\phi \) can be expressed in terms of the effective wavefunctions \( |i \rangle \rangle \) as
\[
A'_\phi = \frac{\langle \langle M(\phi)|i \rangle \rangle}{\langle \langle M(\phi)|0 \rangle \rangle}.
\] (54)

The relation of our approach with that of [10] becomes apparent when one identifies:
1) Our \( \delta_{\Phi_t,M} \) with Kronfeld’s \( \Phi^{(i)}_r(t) \) in his eq. (4.1), where the label \( M \) corresponds to \( (i) \). Note that our correlation matrix is already diagonal for distance 0.
2) Our \( A'_\phi \) corresponds to Kronfeld’s \( z^{(i)}_n \) defined in eq. (4.6).
3) The diagonalization of the effective transfer matrix has its counterpart in the variation of \( C_{n,r}(t) \).
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Figure Captions

**Figure 1** Overlaps $c_i = \sum_M \langle M | i \rangle^2$ for the three lowest states of the 2-dimensional Ising model with $N_\sigma = 6, 8$ and 10.

**Figure 2** Eigenstates of the effective transfer matrix $T^{(l)}_{\text{eff}}$ for $l = 5, 10$ and 20. $N_\sigma$ is 64, and $\beta = \beta_c$. The full line is the ground-state, the dotted line is $|0, a\rangle$, the dashed line shows $|1, s\rangle$, and the dash-dotted line gives $|1, a\rangle$.

**Figure 3** Comparison of the convergence of the ‘effective correlation lengths’ $\xi^{\text{eff}, t}_{0, a}$, ‘improved’ and ‘standard’.

**Figure 4** Comparison of the convergence of the ‘effective correlation lengths’ $\xi^{\text{eff}, t}_{1, s}$, ‘improved’ and ‘standard’.

**Figure 5** The tunnelling correlation length $\xi_{0, a}$, measured from the $N_\sigma = 64$ lattices.

**Figure 6** Eigenstates of the effective transfer matrix $T^{(l)}_{\text{eff}}$ for $l = 5, 10$ and 20. $N_\sigma$ is 64, and $\beta = 0.47$. The full line is the ground-state, the dotted line is $|0, a\rangle$, the dashed line shows $|1, s\rangle$, and the dash-dotted line gives $|1, a\rangle$. 
Table 1: as obtained from the exact effective transfer matrix for $N_\sigma = 4$, using the $N_\tau = \infty$ rule

| $\beta$ | $l$ | $\xi_{0,a}$ | $\xi_{1,s}$ | $\xi_{1,a}$ | $\xi_{2,s}$ |
|--------|-----|-------------|-------------|-------------|-------------|
| 0.30   | 1   | 1.50263     | 0.51838     | 0.28378     | 0.21878     |
| 0.30   | 2   | 1.50263     | 0.51880     | 0.28378     | 0.23855     |
| 0.30   | 4   | 1.50263     | 0.51908     | 0.28378     | 0.26000     |
| 0.30   | 8   | 1.50263     | 0.51924     | 0.28379     | 0.28450     |
| 0.30   | 4,8 | 1.50263     | 0.51940     | 0.28379     | 0.30900     |
| 0.30   | e   | 1.50263     | 0.51940     | 0.28378     | 0.28868     |
| 0.45   | 1   | 5.41722     | 0.66675     | 0.35470     | 0.27231     |
| 0.45   | 2   | 5.41722     | 0.66726     | 0.35470     | 0.28238     |
| 0.45   | 4   | 5.41722     | 0.66756     | 0.35470     | 0.29141     |
| 0.45   | 8   | 5.41722     | 0.66771     | 0.35470     | 0.29822     |
| 0.45   | 4,8 | 5.41722     | 0.66786     | 0.35470     | 0.30503     |
| 0.45   | e   | 5.41722     | 0.66786     | 0.35470     | 0.30243     |
| 0.60   | 1   | 26.11653    | 0.53669     | 0.34962     | 0.28156     |
| 0.60   | 2   | 26.11653    | 0.53705     | 0.34962     | 0.28537     |
| 0.60   | 4   | 26.11653    | 0.53726     | 0.34962     | 0.28802     |
| 0.60   | 8   | 26.11653    | 0.53736     | 0.34962     | 0.28907     |
| 0.60   | 4,8 | 26.11653    | 0.53746     | 0.34962     | 0.29012     |
| 0.60   | e   | 26.11653    | 0.53747     | 0.34962     | 0.29087     |
Table 2: Correlation lengths $\xi^{(l)}$ as obtained from the exact effective transfer matrix for $N_\sigma = 8$, using the $N_\tau = \infty$ rule

| $\beta$ | $l$ | $\xi_{0,a}$ | $\xi_{1,s}$ | $\xi_{1,a}$ | $\xi_{2,s}$ |
|---------|-----|-------------|-------------|-------------|-------------|
| 0.30    | 1   | 1.57471     | 0.67038     | 0.38527     | 0.26235     |
| 0.30    | 2   | 1.57479     | 0.67318     | 0.38709     | 0.27564     |
| 0.30    | 4   | 1.57484     | 0.67574     | 0.38861     | 0.31248     |
| 0.30    | 8   | 1.57486     | 0.67740     | 0.38958     | 0.35499     |
| 0.30    | 4,8 | 1.57488     | 0.67906     | 0.39055     | 0.39750     |
| 0.30    | e   | 1.57488     | 0.67913     | 0.39058     | 0.41348     |
| 0.45    | 1   | 12.25453    | 1.25922     | 0.62386     | 0.40659     |
| 0.45    | 2   | 12.25990    | 1.26925     | 0.62784     | 0.41963     |
| 0.45    | 4   | 12.26276    | 1.27622     | 0.63047     | 0.43690     |
| 0.45    | 8   | 12.26420    | 1.28000     | 0.63191     | 0.45234     |
| 0.45    | 4,8 | 12.26564    | 1.28378     | 0.63335     | 0.46796     |
| 0.45    | e   | 12.26565    | 1.28381     | 0.63338     | 0.47114     |
| 0.60    | 1   | 418.95213   | 0.70382     | 0.52749     | 0.39890     |
| 0.60    | 2   | 419.01927   | 0.71339     | 0.53172     | 0.40984     |
| 0.60    | 4   | 419.05376   | 0.71861     | 0.53417     | 0.41564     |
| 0.60    | 8   | 419.07105   | 0.72131     | 0.53546     | 0.41864     |
| 0.60    | 4,8 | 419.08834   | 0.72401     | 0.53675     | 0.42164     |
| 0.60    | e   | 419.08835   | 0.72404     | 0.53678     | 0.42169     |
Table 3: Correlation lengths $\xi^{(l)}$ as obtained from the exact effective transfer matrix for $N_\sigma = 4$, using the symmetric periodic lattice rule

| $\beta$ | $l$ | $\xi_{0,a}$ | $\xi_{1,s}$ | $\xi_{1,a}$ | $\xi_{2,s}$ |
|---------|-----|-------------|-------------|-------------|-------------|
| 0.30    | 1   | 1.50167     | 0.51944     | 0.28375     | 0.22264     |
| 0.30    | 2   | 1.50263     | 0.51880     | 0.28378     | 0.23901     |
| 0.30    | 4   | 1.50263     | 0.51908     | 0.28378     | 0.26007     |
| 0.30    | 8   | 1.50263     | 0.51924     | 0.28378     | 0.28203     |
| 0.30    | 4,8 | 1.50263     | 0.51940     | 0.28378     | 0.30399     |
| 0.30    | e   | 1.50263     | 0.51940     | 0.28378     | 0.28868     |
| 0.45    | 1   | 5.41132     | 0.66909     | 0.35467     | 0.27595     |
| 0.45    | 2   | 5.41713     | 0.66737     | 0.35470     | 0.28348     |
| 0.45    | 4   | 5.41722     | 0.66756     | 0.35470     | 0.29205     |
| 0.45    | 8   | 5.41722     | 0.66771     | 0.35470     | 0.29564     |
| 0.45    | 4,8 | 5.41722     | 0.66786     | 0.35470     | 0.29923     |
| 0.45    | e   | 5.41722     | 0.66786     | 0.35470     | 0.30243     |
| 0.60    | 1   | 26.10328    | 0.53833     | 0.34962     | 0.28395     |
| 0.60    | 2   | 26.11621    | 0.53726     | 0.34962     | 0.28641     |
| 0.60    | 4   | 26.11653    | 0.53726     | 0.34962     | 0.28834     |
| 0.60    | 8   | 26.11653    | 0.53736     | 0.34962     | 0.29040     |
| 0.60    | 4,8 | 26.11653    | 0.53746     | 0.34962     | 0.29246     |
| 0.60    | e   | 26.11653    | 0.53747     | 0.34962     | 0.29087     |
Table 4: Correlation lengths $\xi^{(l)}$ as obtained from the exact effective transfer matrix for $N_\sigma = 8$, using the symmetric periodic lattice rule

| $\beta$ | $l$ | $\xi_{0,a}$ | $\xi_{1,s}$ | $\xi_{1,a}$ | $\xi_{2,s}$ |
|---------|-----|-------------|-------------|-------------|-------------|
| 0.30    | 1   | 1.56250     | 0.67186     | 0.26386     | 0.38557     |
| 0.30    | 2   | 1.57475     | 0.67310     | 0.27658     | 0.38685     |
| 0.30    | 4   | 1.57484     | 0.67573     | 0.31295     | 0.38815     |
| 0.30    | 8   | 1.57486     | 0.67739     | 0.35092     | 0.38818     |
| 0.30    | 4,8 | 1.57488     | 0.67905     | 0.38889     | 0.38821     |
| 0.30    | e   | 1.57488     | 0.67913     | 0.39058     | 0.41348     |
| 0.45    | 1   | 11.68811    | 1.29494     | 0.41267     | 0.63080     |
| 0.45    | 2   | 12.22444    | 1.27367     | 0.42339     | 0.62794     |
| 0.45    | 4   | 12.25978    | 1.27677     | 0.44527     | 0.62700     |
| 0.45    | 8   | 12.26354    | 1.27856     | 0.52313     | 0.65118     |
| 0.45    | 4,8 | 12.26730    | 1.28035     | 0.60099     | 0.67536     |
| 0.45    | e   | 12.26565    | 1.28381     | 0.63338     | 0.47114     |
| 0.60    | 1   | 404.80434   | 0.73313     | 0.53976     | 0.40926     |
| 0.60    | 2   | 417.52452   | 0.72285     | 0.53477     | 0.41495     |
| 0.60    | 4   | 418.82046   | 0.72058     | 0.53259     | 0.42259     |
| 0.60    | 8   | 418.96946   | 0.71637     | 0.58641     | 0.53098     |
| 0.60    | 4,8 | 419.11846   | 0.71216     | 0.64023     | 0.63937     |
| 0.60    | e   | 419.0835    | 0.72404     | 0.53678     | 0.42169     |

Table 5: Estimates $\xi^{(l)}$ for $N_\sigma = 16$, $N_\tau = 512$. The effective transfer matrix was determined by Monte Carlo, using the $N_\tau = \infty$ rule. The coupling is $\beta = \beta_c$

| $l$ | $\xi_{0,a}$ | $\xi_{1,s}$ | $\xi_{1,a}$ | $\xi_{2,s}$ |
|-----|-------------|-------------|-------------|-------------|
| 1   | 20.10(8)    | 2.402(4)    | 1.147(2)    | 0.704(1)    |
| 2   | 20.17(7)    | 2.451(4)    | 1.171(3)    | 0.726(2)    |
| 3   | 20.22(7)    | 2.480(6)    | 1.181(5)    | 0.756(8)    |
| 4   | 20.25(7)    | 2.499(7)    | 1.193(7)    | 0.76(1)     |
| 5   | 20.26(7)    | 2.509(10)   | 1.190(15)   |             |
| 6   | 20.27(7)    | 2.511(12)   |             |             |
| 7   | 20.28(7)    | 2.518(15)   |             |             |
| 8   | 20.29(8)    | 2.515(20)   |             |             |
| e   | 20.339      | 2.555       | 1.213       | 0.8730      |
| 1,2 | 20.25(7)    | 2.502(6)    | 1.195(5)    | 0.749(6)    |
| 2,4 | 20.32(8)    | 2.55(1)     | 1.215(15)   |             |
| 3,6 | 20.32(9)    | 2.54(2)     |             |             |
| 4,8 | 20.34(9)    |             |             |             |
Table 6: Estimates $\xi^{(l)}$ for $N_{\sigma} = 32$, $N_{\tau} = 1024$. The effective transfer matrix was determined by Monte Carlo, using the $N_{\tau} = \infty$ rule. The coupling is $\beta = \beta_c$.

| $l$ | $\xi_{0,a}$ | $\xi_{1,a}$ | $\xi_{1,s}$ | $\xi_{2,s}$ |
|-----|-------------|-------------|-------------|-------------|
| 1   | 39.28(12)   | 4.44(1)     | 2.099(2)    | 1.266(2)    |
| 2   | 39.86(13)   | 4.61(1)     | 2.176(3)    | 1.316(2)    |
| 3   | 40.15(14)   | 4.71(1)     | 2.225(5)    | 1.350(4)    |
| 4   | 40.32(15)   | 4.77(1)     | 2.259(6)    | 1.385(5)    |
| 5   | 40.45(16)   | 4.83(1)     | 2.283(8)    | 1.414(6)    |
| 6   | 40.56(17)   | 4.87(1)     | 2.31(1)     | 1.445(9)    |
| 8   | 40.68(18)   | 4.92(2)     | 2.34(2)     |              |
| 10  | 40.75(18)   | 4.96(2)     | 2.34(3)     |              |
| e   | 40.727      | 5.097       | 2.404       | 1.7014      |
| 1,2 | 40.45(15)   | 4.78(1)     | 2.258(6)    | 1.371(4)    |
| 2,4 | 40.79(17)   | 4.96(2)     | 2.348(11)   | 1.461(9)    |
| 3,6 | 40.97(20)   | 5.05(2)     | 2.395(20)   | 1.55(2)     |
| 4,8 | 41.05(22)   | 5.08(2)     | 2.42(4)     |              |
| 5,10| 41.05(22)   | 5.10(3)     |              |              |
Table 7: Estimates $\xi^{(l)}$ for $N_\sigma = 64$, $N_\tau = 2048$. The effective transfer matrix was determined by Monte Carlo, using the $N_\tau = \infty$ rule. The coupling is $\beta = \beta_c$

| $l$ | $\xi_{0,a}$ | $\xi_{1,s}$ | $\xi_{1,a}$ | $\xi_{2,s}$ |
|-----|-------------|-------------|-------------|-------------|
| 1   | 73.62(31)   | 8.13(1)     | 3.783(5)    | 2.271(3)    |
| 2   | 76.00(31)   | 8.54(2)     | 3.985(4)    | 2.396(5)    |
| 3   | 77.32(32)   | 8.82(2)     | 4.119(6)    | 2.483(4)    |
| 4   | 78.15(31)   | 9.01(2)     | 4.215(7)    | 2.547(5)    |
| 5   | 78.74(33)   | 9.15(2)     | 4.291(8)    | 2.598(7)    |
| 6   | 79.17(34)   | 9.27(2)     | 4.346(9)    | 2.641(9)    |
| 7   | 79.50(34)   | 9.36(2)     | 4.391(1)    | 2.671(1)    |
| 8   | 79.77(34)   | 9.44(2)     | 4.431(1)    | 2.701(1)    |
| 9   | 79.98(35)   | 9.50(2)     | 4.461(1)    | 2.731(1)    |
| 10  | 80.13(36)   | 9.56(2)     | 4.492(2)    | 2.762(2)    |
| 12  | 80.37(36)   | 9.64(2)     | 4.542(2)    |              |
| 14  | 80.54(37)   | 9.70(3)     | 4.572(2)    |              |
| 16  | 80.67(38)   | 9.75(3)     | 4.613(3)    |              |
| 20  | 80.88(40)   | 9.81(4)     |              |              |
|     | 81.479      | 10.188      | 4.797       | 3.4014      |

Table 8: ‘Conventional’ correlation length estimates $\xi_{0,a}^{eff,t}$ and $\xi_{1,s}^{eff,t}$ at $\beta = \beta_c$ from two-slice correlation functions. The lattice is $N_\sigma = 64$ by $N_\tau = 2048$

| $t$ | 10   | 20   | 40   | 80   |
|-----|------|------|------|------|
| $\xi_{0,a}^{eff,t}$ | 78.9(5) | 81.4(6) | 82.0(8) | 83.4(14) |

| $t$ | 1   | 3   | 5   | 9   | 11  | 13  | 15 |
|-----|-----|-----|-----|-----|-----|-----|----|
| $\xi_{1,s}^{eff,t}$ | 6.95(2) | 8.33(2) | 9.05(4) | 9.68(6) | 9.81(7) | 9.95(8) | 9.95(11) |

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Table 9: Statistics $\text{stat}$ for the multimagnetical simulations at $\beta = 0.47$, given in units of one million lattice sweeps

| $N_x$ | 16 | 16 | 32 | 32 | 32 | 64 | 64 | 64 |
|------|----|----|----|----|----|----|----|----|
| $N_t$ | 16 | 32 | 16 | 32 | 64 | 32 | 64 | 128 |
| $\text{stat}$ | 20 | 40 | 40 | 50 | 40 | 50 | 50 | 60 |

Table 10: Estimates $\xi(l)$ for $\beta = 0.47$ and $N_\sigma = 16$, obtained from the effective transfer matrix with the asymmetric rule

| $N_x$ | $l$ | $\xi_{0,a}$ | $\xi_{1,s}$ | $\xi_{1,a}$ |
|------|-----|-------------|-------------|-------------|
| 16   | 1   | 76.3(7)     | 2.041(5)    | 1.131(2)    |
| 16   | 2   | 76.5(8)     | 2.104(5)    | 1.158(2)    |
| 16   | 4   | 76.7(8)     | 2.163(6)    | 1.173(6)    |
| 16   | 8   | 76.6(8)     | 2.21(1)     |             |
| 32   | 1   | 78.5(4)     | 2.013(4)    | 1.126(1)    |
| 32   | 2   | 78.8(4)     | 2.076(4)    | 1.155(2)    |
| 32   | 4   | 78.9(5)     | 2.136(5)    | 1.179(4)    |
| 32   | 8   | 79.0(5)     | 2.18(1)     |             |
| 32   | 16  | 79.1(5)     |             |             |
| $\infty$ | 78.159 | 2.205 | 1.218 |

Table 11: Estimates $\xi(l)$ for $\beta = 0.47$ and $N_\sigma = 32$, obtained from the effective transfer matrix with the asymmetric rule

| $N_x$ | $l$ | $\xi_{0,a}$ | $\xi_{1,s}$ | $\xi_{1,a}$ |
|------|-----|-------------|-------------|-------------|
| 16   | 1   | 558(9)      | 3.42(2)     | 1.914(6)    |
| 16   | 2   | 607(10)     | 3.53(2)     | 2.034(6)    |
| 16   | 4   | 644(11)     | 3.65(2)     | 2.172(8)    |
| 16   | 8   | 656(13)     | 3.72(2)     | 2.27(1)     |
| 32   | 1   | 718(9)      | 2.68(1)     | 1.779(4)    |
| 32   | 2   | 728(10)     | 2.84(1)     | 1.879(5)    |
| 32   | 4   | 730(10)     | 3.02(1)     | 1.982(5)    |
| 32   | 8   | 731(10)     | 3.18(1)     | 2.03(1)     |
| 32   | 16  | 732(11)     | 3.30(2)     | 2.15(1)     |
| 64   | 1   | 730(7)      | 2.65(1)     | 1.776(3)    |
| 64   | 2   | 740(7)      | 2.80(1)     | 1.875(3)    |
| 64   | 4   | 746(8)      | 2.98(1)     | 1.980(4)    |
| 64   | 8   | 751(8)      | 3.13(1)     | 2.06(1)     |
| 64   | 16  | 757(9)      |             |             |
| 64   | 32  | 760(10)     |             |             |
| $\infty$ | 753.48 | 3.311 | 2.198 |

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Table 12: Estimates $\xi^{(l)}$ for $\beta = 0.47$ and $N_\sigma = 64$, obtained from the effective transfer matrix with the asymmetric rule

| $N_\sigma$ | $l$  | $\xi_0,a$       | $\xi_{1,s}$ | $\xi_{1,a}$ |
|-----------|------|-----------------|--------------|-------------|
| 32        | 2    | 25800(900)      | 4.00(4)      | 2.52(2)     |
| 32        | 4    | 30600(900)      | 4.15(4)      | 2.80(2)     |
| 32        | 8    | 33800(1100)     | 4.29(4)      | 3.12(2)     |
| 32        | 16   | 35400(1900)     | 4.38(4)      | 3.35(3)     |
| 64        | 4    | 38700(2100)     | 3.15(6)      | 2.60(3)     |
| 64        | 8    | 38600(2000)     | 3.46(5)      | 2.84(4)     |
| 64        | 16   | 39500(2000)     | 3.72(5)      |             |
| 64        | 32   | 41000(2200)     |             |             |
| 128       | 16   | 41500(1400)     |             |             |
| 128       | 32   | 42800(1500)     |             |             |
| 128       | 64   | 43500(1600)     |             |             |
| $e^\infty$ | 44014.4 |     | 4.002 | 3.311 |
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