We consider the problem of finding paths of shortest transit time between two points (popularly known as Brachistochrone) for cylinders with off-centered center of mass, rolling down without slip, subject solely to the force of gravity. This problem is set up using principles of classical rigid body dynamics and the desired path function is solved for numerically using the method of discrete calculus of variations. We discover a distinct array of brachistochrone trajectories for off-centered cylinders, demonstrate a critical dependence of such paths on the initial location and orientation of cylinders’ centers of mass and bring new insights into the family of brachistochrone problems and solutions.

I. INTRODUCTION

The brachistochrone problem concerns the path of shortest transit time between two points for an object traversing solely under the force of gravity. This problem has a celebrated history, having attracted the best minds in the field of mathematics and physics for over three centuries. The solution to that problem for a simple point mass (bead) is a cycloid, which has been deduced by a variety of methods (Calculus of variations, Geometry, Laws of refraction, Optimal Control Theory). Several variations and generalizations of the problem have been proposed (with different force fields, types of object traversing the path) and solved. In all of the previous works, objects considered were beads, cylinders or other axisymmetric objects. In this work, we extend the literature of Brachistochrone to off-centered objects whose centers of mass (c.o.m) are not necessarily coincident with their centers of cylinder (c.o.c). Such objects may be realized with non-uniform spatial distribution of mass, an example of which is illustrated in FIG. 1.

II. PROBLEM FORMULATION

A. Governing Equations

Consider a cylinder of radius $r$, mass $m$, with its c.o.m located $r_m = \eta_m r$ (with $0 \leq \eta_m \leq 1$) away from its c.o.c and moment of inertia of $I_{cm} = kr^2$ about an axis through its c.o.m and parallel to axis of geometric symmetry. The mass distribution within the cylinder determines the value of $k$, whose range would be $0 \leq k \leq 1$. For example, a hollow cylinder, with all its mass distributed uniformly on the outer edge would have a value of $k = 1$ whereas a dense rod at the center connected to massless but rigid cylindrical shell with massless spokes would have a $k = 0$. Parameters $k$ and $\eta_m$ are not entirely independent as they both depend on the mass distribution within the circular frame of the cylinder. Considering the extremities of density distribution (for a given $m$ and $r$), it is deduced that for a given $\eta_m$, the maximum value $k$ can take is $1 - \eta_m^2$. Some example configurations of off-centered cylinders are shown in Appendix A.

Let the two points between which the cylinder would be rolling be defined as A (0, 0) and B ($L, H$) in cartesian coordinates system of ($X, Z$), with direction of gravity oriented along the +Z axis. Let the trajectory of the c.o.c of cylinder, as it rolls down from A to B, be described by $z_c(x)$. The desired brachistochrone curve (path on which the cylinder rolls) is related to, but different from, $z_c(x)$. This curve will be defined at the end of this section. Let the initial configuration of cylinder be such that its c.o.m makes an angle of $\theta_m$ with the horizontal axis as shown in FIG. 2.

Conservation of energy between potential and kinetic for such a system is given as follow:

$$\frac{1}{2}mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2 = mg(z_c + r_m(\sin \theta - \sin \theta_m))$$  (1)
Here, \(v_{cm}\) represents the linear speed of the c.o.m, \(\omega\) the angular speed, \(\theta\) the angular displacement of c.o.m and \(g\) acceleration due to gravity. Note that equation (1) is independent of the axial location of the c.o.m of the object (\(y_m\) - in the direction perpendicular to the plane of sketch in FIG. 2) and therefore the results obtained here hold true for all off-centered cylinders with a given \((r_m, \theta_m)\) regardless of \(y_m\).

Location of cylinder’s c.o.m is related to that of its c.o.c as follows

\[
x_{cm} = x_c + r_m \cos \theta \\
z_{cm} = z_c + r_m \sin \theta
\]  

(2)

(3)

Time derivatives (denoted by \(\cdot\) on variables) of above equations yield the below relationships for speeds of cylinder’s c.o.m and c.o.c.

\[
v_{cm} = \sqrt{x_{cm}^2 + z_{cm}^2} \\
v_c = \sqrt{x_c^2 + z_c^2}
\]  

(4)

(5)

No-slip condition between the cylinder and path establishes the following relationship\[\] between \(v_c\) and \(\omega\).

\[
v_c = r \omega = r \dot{\theta}
\]  

(6)

Combining equations (1) - (6), time taken for the cylinder to roll down from A to B is obtained as follows (details of derivation are given in Appendix B).

\[
T[z_c(x_c)] = \int_0^L \frac{\sqrt{1 + \eta_m^2 - 2 \eta_m \sin(\theta - \beta) + k}}{2g(z_c + r_m(\sin \theta - \sin \theta_m))} \sqrt{1 + z_c'^2} dx_c
\]

Here, \(z_c'\) indicate the derivatives of \(z_c(x_c)\) with respect to \(x_c\) and \(\beta\) is related to the instantaneous slope of the curve as follows.

\[
\beta = \tan^{-1}(z_c')
\]

(8)

The no-slip condition is re-written in terms \(z_c(x_c)\) and \(\theta(x_c)\) to obtain the following relation.

\[
\theta(x_c) = \theta_m + \frac{1}{r} \int_0^{x_c} \sqrt{1 + z_c'^2} dx_c
\]

(9)

The desired path of least time on which the cylinder rolls down \(z_c(x_c)\) is obtained by minimizing the functional \(T[z_c(x_c)]\).

\[
\delta T[z_c(x_c)] = 0
\]

(10)

With trajectory of c.o.c solved for, the desired brachistochrone path (on which the cylinder rolls) is then obtained by tracking the point at a distance \(r\) away and normal to \(z_c(x_c)\). Such a path, defined as \(z_b(x_b)\), is given by the following equations.

\[
x_b = x_c - r \sin \beta \\
z_b = z_c + r \cos \beta
\]

(11)

(12)

From equation (11), it can be inferred that the brachistochrone path only depends on the location of c.o.m \((\eta_m\) and \(\theta_m)\) but also on the moment of inertia factor \((k)\). This is in contrast with axisymmetric objects where their brachistochrone paths are independent of their moments of inertia \[\] . It can also be verified that, for the case of \(\eta_m = 0\), the above equations simplify to the familiar system for a uniform cylinder\[\].

**B. Solution Methodology**

The manner of coupling between the equations (1) - (12) does not lend them to analytical solutions using classical calculus of variations (barring the limiting case of \(\eta_m = 0\)). Therefore, a discretized and numerical approach\[\] is taken here. Additionally, steep gradients in the solution for \(z_c(x_c)\) prohibit a direct numeric solution. Therefore, the system is recast in parametric coordinates as \(x_c(\xi), z_c(\xi)\), where \(\xi\) has a range of \(0 \leq \xi \leq 1\) and represents dimensionless path length along the curve \(z_c(x_c)\), as measured from A. This parameterized curve is discretized into \(n\) points \(\{x_c(\xi_j) = x_j, z_c(\xi_j) = z_j\}_{j = 1, 2, \ldots, n}\). These points along the curve are equally spaced by defining \(\xi_j := (j - 1)/(n - 1)\). This method is akin to arc length formulation commonly used to solve moving boundary problems with complex domains in fluid mechanics\[\].

Governing equations for \(x_j\)'s are given by equations (13) - (15), where, equations (13) and (14) enforce the boundary conditions and equation set (14) solve for the x-grid points.

\[
x_1 - \frac{r_c' z_c'}{\sqrt{1 + z_c'^2}} = 0 \quad \text{for} \ j = 1
\]

(13)

\[
\sqrt{(x_j - x_{j-1})^2 + (z_j - z_{j-1})^2} - S \Delta \xi = 0 \quad \text{for} \ j = 2, 3, \ldots, n
\]

(14)

\[
x_n - \frac{r_c' z_c'}{\sqrt{1 + z_c'^2}} = L \quad \text{for} \ S
\]

(15)

Here, \(S\) represents the total arc length of the curve, which is defined and computed by equation (16).

\[
S = \sum_{j=2}^{n} \sqrt{(x_j - x_{j-1})^2 + (z_j - z_{j-1})^2}
\]

(16)

Governing equations for \(z_j\)'s are given by equations (17) - (19), where equations (17) and (19) enforce the boundary conditions and equation set (15) correspond to the discretized version of functional minimization, equation (10).
numerical approach employed and code written: formed to evaluate the validity of the equations developed, equations with 2

C. Benchmarking and Validation Tests

Equations (13) - (19) form a system of coupled non-linear equations with 2n + 1 variables, which are solved using MATLAB f‐solve function. Four independent tests were performed to evaluate the validity of the equations developed, numerical approach employed and code written:

- Code developed is used to simulate rolling dynamics of off-centered cylinder down an inclined plane (by forcing the curve between the end points to be a straight line) for previously studied systems. FIG. 3 shows the transients of instantaneous speed of cylinder's c.o.m and how it matches the results reported in prior studies.

- Code developed is used to determine the brachistochrone for uniform cylinder (by setting \( \eta_m = 0 \)) traversing from A(0, 0) to B (\( r + \pi, r + 2 \)). The shortest transit time was computed to be 1.229 s, which matches the analytical solution of \( \pi \sqrt{3/2g} \) reported in prior work within a tolerance of 0.05%.

- The brachistochrone results for off-centered cylinders \((L = 3, r = 1/3, \eta_m = 2/3, \theta_m = 0, k = 0.5)\) are verified to be grid independent by comparing the paths \( z_c(x_c) \) for various values of \( n \). Additionally, the dimensionless time of transit, \( \tau = T(z_c(x_c)) \sqrt{2g/H} \), corresponding to \( n = 25, 50, 100 \) and 200 are seen to converge as 4.97, 4.99, 4.99 and 4.99 respectively. For the remainder of the solutions presented here, \( n \) is set to 100.

- Accuracy of numerical solutions are ensured by keeping the absolute value of norm of residual errors and solution updates below \( 10^{-6} \) for all results reported here.

Without loss of generality, the vertical distance \((H)\) between points A and B is set to 1 length unit, making it the reference length scale for this problem. For each of the parameter values studied, the system of equations (13) - (19) is solved. The brachistochrone path \( z_b(x_b) \) and corresponding shortest transit time are then obtained using equations (11) - (12) and (7), respectively.

III. RESULTS

FIG. 4 shows the brachistochrone path for a representative off-centered cylinder \((r = 1/3, \eta_m = 2/3, \theta_m = 0, k = 0.5)\) over a domain size of \( L = 3 \). The transit time for the cylinder through this path is 4.99. For comparison, few other standard curves between the same two points and corresponding transit times for this cylinder over the same domain, are shown.
As a second feasibility check, the curvature of the path $\kappa_b(z_b)$, is computed using equation (23) and compared against that of the cylinder ($\kappa_0 = 1/r$). For the brachistochrone curve in FIG. 4, $\kappa_0$ varied between (-1.34, 1.52) staying well below $\kappa_0 = 3$. This confirms that this brachistochrone path does not have sharp enough turns to physically prevent the cylinder from rolling on those turns.

$$\kappa_b = \frac{x_b\ddot{z}_b - \dot{x}_b\dot{z}_b}{(\dot{x}_b^2 + \dot{z}_b^2)^{3/2}}$$

(23)

For all of the cases studied in the following subsections, the normal reaction forces were verified to remain positive ($R_n > 0$) throughout the cylinders’ motions on their respective brachistochrone paths and the curvatures of those paths were verified to be less than that of the cylinder ($\kappa_b < \kappa_0$), confirming the feasibility of the solutions presented.

B. Comparison against other objects

FIG. 6 shows brachistochrone paths (indicated by solid lines) for a bead, uniform cylinder and off-centered cylinder for a representative set of system parameters. Shape of bead’s brachistochrone is a cycloid\textsuperscript{[11]} Shape of uniform cylinder’s brachistochrone is a curve traced by its outer edge as its $c.o.m$ traverses a cycloid\textsuperscript{[11]}. The off-centered cylinder’s brachistochrone presents a richer curve with more features, referred to here as “valleys and peaks”. Dashed lines in FIG. 6 indicate trajectories of $c.o.m$ of the two cylinders - uniform (black) and off-centered (blue). As the off-centered cylinder is rolling from A to B, the brachistochrone path is “pulled-in” when the $c.o.m$ is at the bottom and “pushed-out” when the $c.o.m$ is at the top. Such a path facilitates the $c.o.m$ to not fall too deep and lose too much potential energy which it will then need to gain back to reach B. In terms of path lengths, bead has the shortest path (of 3.57) among the three, with uniform cylinder and off-centered cylinder having near equal path lengths (of 3.66). In terms of transit times, the bead is the quickest ($\tau = 4.52$). However, the off-centered cylinder comes second (4.99) ahead of uniform cylinder (5.12), despite having near equal path lengths. Similar trends (of off-centered cylinder having quicker transit times compared to that of uniform cylinder) have been reported for standard inclined plane\textsuperscript{[16]}

Compared to beads, uniform cylinder is known to take a longer time due to the additional inertia from its rotational motion\textsuperscript{[8]}. In the case of off-centered cylinders, this exchange of energy between kinetic and gravitational forms is influenced by the initial location of $c.o.m$ and its subsequent trajectory. FIG. 7 compares the kinetic energy evolution for a bead, uniform cylinder and off-centered cylinder. The trends of instantaneous linear and angular speeds of the cylinders will follow that of the total kinetic energy as they are directly related by $\omega \propto \sqrt{KE}$ and equation (\textsuperscript{[6]})

In the following subsections, the effects of various parameters - initial locations of $c.o.m$ in the angular ($\theta_{in}$) and radial ($\eta_{in}$) directions, moment of inertia ($k$), radius ($r$) of cylinder and length of domain ($L$) - on the brachistochrone paths and
FIG. 6. Brachistochrones of bead, uniform cylinder and off-centered cylinder \((r = 1/3, \eta_m = 2/3, \theta_m = 0, k = 0.5)\) for a domain size of \(L = 3\).

FIG. 7. Evolution of total kinetic energy (non-dimensionalized with \(mgH\)) for bead, uniform cylinder and off-centered cylinder (parameters as in FIG. 6) as they traverse through their brachistochrones. Corresponding shortest transit times are systematically studied.

C. Effect of radial location of \(c.o.m\)

FIG. 8 shows brachistochrone paths for a family of cylinders when radial \(c.o.m\) \(\eta_m\) is varied from 0 (axis-symmetric configurations) to \(\sqrt{1-k}\) (lumps of mass distributed along the rim of a mass-less rigid shell), with all other parameters held constant \((L = 3, r = 1/3, \theta_m = 0, k = 0.5)\). This result illustrates the effect of off-centered \(c.o.m\) in gradually inducing the peaks and valleys into its brachistochrone, and making them more pronounced as \(\eta_m\) increases.

D. Effect of initial angular location of \(c.o.m\)

FIG. 9 shows brachistochrone paths for a family of off-centered cylinders when initial location of angular \(c.o.m\) \(\theta_m\) varies from \(-\pi/2\) (\(c.o.m\) is above \(z = 0\)) to \(\pi/2\) (\(c.o.m\) is below \(z = 0\)), with all other parameters held constant \((\eta_m = 2/3, r = 1/3, L = 3, k = 0.5)\). While \(\eta_m\) dictated the extent of peaks and valleys, this result shows that \(\theta_m\) determines the location of those peaks and valleys.

E. Effect of moment of inertia

FIG. 10 shows brachistochrone paths for a family of cylinders when \(k\) is varied from 0.1 (mass mostly concentrated at \(\eta_m\)) to \(1 - \eta_m^2\) (lumps of mass distributed on the rim of a mass-less rigid shell), with all other parameters held constant.
FIG. 10. Brachistochrone paths and c.o.m trajectories for the system of $L = 3, r = 1/3, \eta_m = 2/3, \theta_m = 0$ with $0.1 \leq k \leq 1 - \eta_m^2 = 5/9$.

FIG. 11. Brachistochrone paths and c.o.m trajectories for the system of $L = 3, \eta_m = 2/3, \theta_m = 0, k = 0.5$ with $2 \leq L \leq 4$.

FIG. 12. Brachistochrone paths and c.o.m trajectories for the system of $r = 1/3, \eta_m = 2/3, \theta_m = 0, k = 0.5$ with $2 \leq L \leq 4$.

FIG. 13. Variation of shortest transit times for off-centered cylinders ($r = 1/3, k = 0.5$) with varying initial locations of c.o.m. ($L = 3$).

F. Effects of cylinder’s radius

FIG. 11 shows brachistochrone paths for a family of off-centered cylinders of increasing radii, $r$ from 0.2 to 0.5, with all other parameters held constant ($\eta_m = 1/3, \theta_m = 0, L = 3, k = 0.5$). While $\eta_m, \theta_m$ and $k$ dictated the size and location of peaks and valleys in the brachistochrone, $r$ influences the number of peaks and valleys with smaller cylinders generally having more of them. There is dual effect of $r$ on the solutions as it increases both the size of the cylinder and the radial c.o.m $\eta_m = \eta_m r$. The transit times are also seen to vary with $r$, unlike their axisymmetric counterparts whose transit times are independent of $r$.

G. Effects of domain length

FIG. 12 shows brachistochrone paths for an off-centered cylinder over different domain lengths ($L$), with all other parameters held constant ($\eta_m = 2/3, \theta_m = 0, r = 1/3, k = 0.5$). As expected, longer domains result in longer paths for brachistochrones and longer transit times (3.96, 4.99 and 6.26 for $L = 2, 3$ and 4 respectively).

H. Effect of c.o.m locations on shortest transit times

Following specific case studies in prior subsections, a wider parametric space is explored in this subsection. The domain length $L$ and radius of the cylinder $r$ are fixed to 3 and 1/3 respectively. The moment of inertia $k$ is set to 0.5 (same as that for an uniform cylinder). The initial locations of c.o.m in
radial ($\eta_m$) and angular ($\theta_m$) directions are varied. For $\eta_m$, the 
physically allowable range is $[0, \sqrt{1-k}]$, considering all ex-
tremities of mass distribution. For $\theta_m$, the full range of $[-\pi, \pi]$ 
is physically allowable but there are other restrictions. When 
$\theta_m \in [-\pi/2, \pi/2]$, initial motion of both c.o.c and c.o.m with 
respect to c.o.c is downwards. This ensures that the net ini-
tial motion of c.o.m is downwards, in the direction of gravity. When $\theta_m \in [-\pi, -\pi/2] \cup [\pi/2, \pi]$, initial motion of c.o.c would 
still be downwards but c.o.m would be moving upwards with 
respect to c.o.c. Depending on the value of $\eta_m$ and the 
path taken, these cases may result in aphysical scenarios (net 
upward initial motion of c.o.m, against gravity) or violation of 
no-slip assumption, equations (6) and (9). Therefore, param-
eter range of $\theta_m$ is restricted to $[-\pi/2, \pi/2]$ in this analysis.

For each pair of ($\eta_m, \theta_m$) in the parametric space of $0 \leq 
\eta_m \leq 1/\sqrt{2}$ and $-\pi/2 \leq \theta_m \leq \pi/2$, brachistochrone paths 
are obtained and corresponding shortest transit times are com-
pared. FIG. [13] present a summary of shortest transit times. 
The first inference from this plot is that when the cylinder’s 
c.o.m coincides with its c.o.c ($\eta_m = 0$), transit time $\tau$ is inde-
pendent of initial angular location $\theta_m$ as expected and equals 
that of an uniform cylinder, 5.12). Secondly, a more interest-
ing result is that as $\eta_m$ increases, the effect of $\theta_m$ becomes 
more important. While lower $\theta_m$ indicates higher initial po-
tential energy which can translate to higher kinetic energy and 
consequently shorter transit times, $\tau$ does not monotonically vary 
with $\theta_m$ but has a minimum. Thirdly, the location of 
this minimum does not vary with $\eta_m$ but remains constant at 
$\theta_m = -\pi/6$. These trends demonstrate that when a cylinder’s 
mass is non-uniformly distributed, its initial orientation plays 
a critical role in determining its brachistochrone path and cor-
responding shortest transit time.

While brachistochrone transit times of axisymmetric ob-
jects are known to be always higher than that of a bead the 
same need not hold true for off-centered circular objects. 
For example, for the case of $k = 0.1$, $\eta_m = 4/5$, $\theta_m = -\pi/6$, 
$r = 1/3$, the off-centered cylinder has transit time of $\tau = 4.06$ 
which is quicker than that of bead (4.52) over the same do-
main ($L = 3$).

IV. CONCLUSION

Brachistochrones of off-centered cylinders, which hitherto 
have been an unopened and untapped system, present a trea-
sure trove of problems rich in non-linear rigid body dynamics. 
Trajectories and transit times of off-centered cylinders have 
been shown to differ significantly from their axisymmetric 
counterparts. Systematic exploration of this multi-parameter 
problem demonstrated the sensitivity of its solutions to those 
parameters, in particular to that of c.o.m location. Behavior of 
such systems in more complex force field prove to be chal-
 lenging extensions. Off-centered cylinders also raise other 
interesting questions such as ‘what is the quickest brachisto-
chrone ever achievable by any object between two given 
points?’ The method presented in section [11] is general 
and extensible to tackle a spectrum of problems in this genre.

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Appendix A: Types of off-centered cylinders

Example configurations of cylinders, all with same mass $m$ 
and radius $r$, but varying moments of inertia ($k$) and center of mass 
locations ($\eta_m$)

Appendix B: Derivation of equation (7)

Differentiating equations (2) and (3) with time gives

$$\dot{x}_{cm} = \dot{x}_c - r_m \dot{\theta} \sin \theta$$
$$\dot{z}_{cm} = \dot{z}_c + r_m \dot{\theta} \cos \theta$$
Plugging the above expressions in equation (4) gives

\[ v_{cm}^2 = x_c^2 + r_m^2 \dot{\theta}_c^2 \sin^2 \theta - 2r_m \dot{\theta}_c \sin \theta \]

\[ + \dot{z_c}^2 + r_m^2 \dot{\theta}_c^2 \cos^2 \theta + 2r_m \dot{\theta}_c \dot{z_c} \cos \theta \]

\[ = v_c^2 + r_m^2 \dot{\theta}_c^2 - 2r_m \dot{\theta}_c (\dot{x}_c \sin \theta - \dot{z}_c \cos \theta) \quad \text{(B1)} \]

The relationship that slope of the curve \( z'_c = \dot{z}_c / \dot{x}_c \) and equation (5) for \( v_c \) yield following equations for \( \dot{x}_c \) and \( \dot{z}_c \).

\[ \dot{x}_c = v_c / \sqrt{1 + z_c^2} \]

\[ \dot{z}_c = v_c \dot{z}_c' / \sqrt{1 + z_c^2} \]

Plugging the above expressions along with equation (6) for \( \dot{\theta} \) in equation (B1) and replacing \( r_m / r \) with \( \eta_m \) gives

\[ v_{cm}^2 = v_c^2 + \eta_m v_c^2 (\sin \theta - z_c' \cos \theta) / \sqrt{1 + z_c^2} \]

Defining \( \beta \) using equation (8) simplifies the above equation to

\[ v_{cm}^2 = v_c^2 (1 + \eta_m^2 - 2 \eta_m \sin(\theta - \beta)) \]

Plugging the above expression in equation (1) gives

\[ \frac{1}{2} m v_c^2 (1 + \eta_m^2 - 2 \eta_m \sin(\theta - \beta)) + \frac{1}{2} k m r^2 \omega^2 \]

\[ = mg(z_c + r_m (\sin \theta - \sin \theta_m)) \]

Using equation (6) for \( \omega \) and rewriting the above equation for \( v_c \) gives

\[ v_c = \left( \frac{2g(z_c + r_m (\sin \theta - \sin \theta_m))}{1 + \eta_m^2 - 2 \eta_m \sin(\theta - \beta) + k} \right)^{1/2} \quad \text{(B2)} \]

The total time of transit for the cylinder between the two points is then given by

\[ T = \int_0^S \frac{ds_c}{v_c} \]

where \( ds_c \) represents the incremental distance traversed by the c.o.c. Using \( ds_c = \sqrt{1 + z_c^2} dx_c \) and equation (B2) for \( v_c \), we obtain equation (7).

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