We study the generation of magnetic fields in the Higgs inflation model with the axial coupling in order to break the conformal invariance of the Maxwell action and produce strong enough magnetic fields for observed large-scale magnetic fields. This interaction breaks the parity and enables a production of only one of the polarization states of the electromagnetic field due to axion-like coupling of electromagnetic field to the inflation. Therefore, the produced magnetic fields are helical. In fact, calculations show the mode of one polarization undergoes amplification, while the other one diminishes. We consider radiatively corrected Higgs inflation potential. In comparison to the Starobinsky potential, we obtain an extra term as a one loop correction and determine the spectrum of generalized electromagnetic fields. The effect of quantum correction modifies potential so that in some certain conditions when back reaction is weak the observed large-scale magnetic field can be explained by our modified potential. We should emphasize in this model we only consider linear approximation for electromagnetic field so that the theory does not contain higher-order derivatives and the so-called ghost degrees of freedom. Therefore, the theory is consistent with cosmology. In addition, the magnetic field generated in this model has very small correlation length. It is impossible to explain within this model both the strength of magnetic field and its large coherence length. Due to the nontrivial helicity, the produced magnetic fields undergo the inverse cascade process in the turbulent plasma which can strongly increase their correlation length. We find that, for two values of coupling parameter $\chi_1 = 5 \times 10^5 M_p^{-2}$ and $\chi_1 = 7.5 \times 10^9 M_p^{-2}$, the back-reaction is weak and our analysis is valid.

**Keywords** Magnetogenesis · Axial coupling · Higgs inflation
1 Introduction

Recently magnetic fields were detected in the cosmic voids through the gamma-ray observations of distant blazars [15, 44, 56, 57] with very large coherence scale $\lambda_B \gtrsim 1 \text{ Mpc}$. The origin of these fields is a very intriguing problem which may shed light on the physical processes in the early Universe [19, 27, 29, 32, 36, 53, 62]. Combining these observations with the cosmic microwave background (CMB) data [1, 31, 54] constrains the strength of these magnetic fields to $10^{-17} \lesssim B_0 \lesssim 10^{-9} \text{ G}$. The extremely large correlation length of magnetic fields observed in the cosmic voids strongly suggests that they were produced during an inflationary stage of the evolution of the Universe because the inflationary magnetogenesis Refs. [46, 58] can easily attain very large coherence length. In addition, observations of distant blazars indirectly help to put a lower bound on the strength of magnetic fields present in the voids, and favoring the scenario that such magnetic fields are of primordial origin [59].

Since Maxwell’s action is conformally invariant, the fluctuations of the electromagnetic field are not enhanced in the conformally flat inflationary background [45]. A standard way to break the conformal invariance is to introduce the interaction with scalar field or curvature scalar [18, 25, 46, 58]. One of them is the kinetic coupling of the electromagnetic field to the scalar inflation field via the term $f^2(\phi) F_{\mu\nu} F^{\mu\nu}$, which was proposed by Ratra [46] and then studied in detail for different types of coupling functions in the literature [7, 17, 22, 23, 26, 33, 42, 60].

One of the essential features of the kinetic coupling model is the modification of the electromagnetic coupling constant. Indeed, since the standard electromagnetic Lagrangian is multiplied by $f^2$, one can rescale [17] the electromagnetic potential and absorb $f$. As a result, the electric charges of particles effectively will depend on $f^{-1}$. Obviously, for small $f$, this leads to a strong coupling problem. Therefore, one should require $f \geq 1$ in order to avoid this problem during inflation. Further, since the inflaton field and the scale factor change monotonously during inflation, it is natural to assume that the coupling function is a decreasing function during inflation which attains large values in the beginning (remember that in this scenario $f \propto a^\alpha$ and at the end of inflation $f = 1$ therefore when $f$ is a decreasing function the strong coupling problem does not occur during inflation. For more details see Ref. [17]). For a decreasing coupling function, it is also well known that the electric energy density dominates the magnetic one [17, 42, 60] and the energy density of the generated electric field may exceed that of the inflaton field leading to the back-reaction problem. In our
paper, we consider the axial coupling $RF_{\mu\nu}\tilde{F}^{\mu\nu}$ which violates parity and $\tilde{F}^{\mu\nu}$ is the dual of $F^{\mu\nu}$. Here $R$ is Ricci Scalar and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is electromagnetic field tensor. This term is presented in the Jordan frame. In the Einstein frame such term naturally produce non-trivial coupling between inflation and the electromagnetic field as we discuss in Sect. 3.

Inflation is a wonderful paradigmatic idea which naturally solves some very difficult problems of the hot Big Bang model [30,38,43,52]. Yet the nature of the inflaton field is an open question. Usually it is assumed to be a scalar or pseudo-scalar field. The Standard Model of elementary particles contains only one fundamental scalar field. This is the Higgs boson. Therefore, it is a natural minimalist idea to try to employ the Higgs boson as the inflaton field. Remarkably, this turned out to be a viable scenario if the Higgs field couples nonminimally to the curvature scalar $\xi R^2/2$ with sufficiently strong coupling constant $\xi$. It was shown in Ref. [8,50] that the non-minimal coupling $\xi$ ensures the flatness of the scalar potential in the Einstein frame at large values of the Higgs field. The successful inflation consistent with the amplitude of the scalar perturbations in the CMB takes place for very large values $\xi$ of order $10^4$. This simplest and most economical scenario provides the graceful exit from inflation and predicts the tilt of the spectrum of the scalar perturbations $n_s \simeq 0.97$ and a very small tensor-to-scalar ratio $r \simeq 0.003$. After inflationary period, the oscillations of the Higgs field about the Standard Model vacuum reheat the Universe producing the hot Big Bang with temperature $10^{13-14}$ GeV [9,24].

Certainly, the quantum radiative corrections can significantly modify the form of the effective potential. This issue was carefully studied in the literature [3–5,12,16] and it was found that the experimentally observed mass of the Higgs boson is less than the critical value. Interestingly, however, it was shown also in a recent paper [10] that the successful Higgs inflation can take place even if the Standard Model vacuum is metastable.

This paper is organized as follows. In Sect. 2 we consider the Higgs model where we use conformal transformation from the Jordan frame to the Einstein frame in which we consider a theory of canonical scalar field $\phi$ minimally coupled to the Einstein gravity. $\phi$ takes the role of inflation. In Sect. 3 we introduce axial coupling in the form of $f(R)F\tilde{F}$ and by using conformal transformation and solving Einstein equation we obtain $f(\phi)F\tilde{F}$ coupling form. In Sect. 4 the relations for determining power spectra are considered. In Sect. 5 we consider numerical calculations and finally in Sect. 6 we discuss about magnetic field in present-day. The summary of the obtained results is given in Sect. 7.

## 2 Radiatively corrected Higgs inflation

The Lagrangian in the Higgs inflation model is given by

$$L = \sqrt{-g} \left[ -f(h)R \frac{1}{2} (\partial h)^2 - U(h) - \Lambda \right], \quad (1)$$
where \( U(h) = \frac{1}{4} (h^2 - v^2)^2 \) is the Higgs potential of the Standard Model in unitary gauge, \( 2H^\dagger H = h^2 \), \( H = \frac{h}{\sqrt{2}} \) is Higgs field, \( v = (\sqrt{2}G_F)^{-1/2} \approx 246 GeV \) is vacuum expectation value of Higgs field, \( G_F \) is Fermi coupling constant and \( \Lambda \) is cosmological constant. Further, \( f(h) = \frac{M^2 + \xi_h h^2}{2} \). Here \( M \) is some mass parameter. If we choose \( 1 \ll \sqrt{\xi_h} \ll 10^{17} \) then \( M \approx M_p \) with very good accuracy. In this case this model has good particle physics phenomenology and of course good inflation. When we say good particle physics we want \( M \approx M_p \) in Lagrangian (1) otherwise the mass term in lagrangian is unknown and we should fix it with reduced Planck mass. Since \( \xi_h \) is an unknown constant that should be fixed in a manner in which the theory satisfies both particle physics and cosmology. For instance if we set \( \xi_h = 0 \) the coupling of the Higgs field to gravity is said to be “minimal”. Then \( M \) can be identified with Planck scale \( M_p \) related to the Newton constant as \( M_p = (8\pi G_N)^{-1/2} = 2.43 \times 10^{18} \) GeV. This model has good particle physics but bad cosmology. For more details see Ref. [8]. \( M_p \) is the reduced Planck mass and \( \xi_h \) is the non-minimal coupling constant. Note that the Lagrangian is written in the Jordan frame.

According to Ref. [8], we can avoid \( \Lambda \) because its effect is negligible. It is possible to get rid of the non-minimal coupling to gravity by making the conformal transformation from the Jordan frame to the Einstein frame [8]. We will work in the Einstein frame because it is easier to calculate slow-roll parameters, spectral index, number of e-folds and tensor-to-scaler ratio. For this, we perform the conformal transformation [40,53]

\[
\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}. \tag{2}
\]

Here \( \bar{g} \) represents Einstein frame. This implies \( \sqrt{-\bar{g}} = \Omega^4 \sqrt{-g} \) and \( \bar{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu} \). In order to rewrite Eq. (1) in the Einstein frame, we have the following relations: [8,40]

\[
\frac{f(h)}{\Omega^2} = \frac{M_p^2}{2}. \tag{3}
\]

This implies the following relation between the conformal transformation and Higgs field [8,40]:

\[
\Omega^2 (h) = 1 + \frac{\xi_h h^2}{M_p^2} \tag{4}
\]

As a result, the Lagrangian density in the Einstein frame takes the form

\[
\bar{L} = \sqrt{-\bar{g}} \left[ -\frac{M_p^2}{2} \bar{R} + \frac{1}{2} \left( \frac{\Omega^2 + \frac{6\xi_h h^2}{M_p^2}}{\Omega^4} \right) \bar{g}^{\mu\nu} \partial_\mu h \partial_\nu h - \frac{U(h)}{\Omega^4} \right]. \tag{5}
\]

Note that \( \bar{R} = \Omega^{-2} \left[ R - \frac{6\Box}{\Omega^4} \right] \) where \( \Box = \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} \partial^\mu \right) \) [13,21,39,55,61]. This transformation [see Eqs. (2) and (4)] leads to non-minimal kinetic term for the Higgs

\( \mathbb{Q} \) Springer
field. If we introduce a new field $\phi$, then we can get canonically normalized kinetic term by the following redefinition [40]:

$$
\frac{d\phi}{dh} = \sqrt{\frac{\Omega^2 + 6\xi^2 h^2}{M_p^2 \Omega^4}}
$$

(6)

Finally, the Lagrangian in the Einstein frame is given by

$$
\tilde{L} = \sqrt{-\tilde{g}} \left[ -M_p^2 \tilde{R} + \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right], \quad V(\phi) = \frac{U(h)}{\Omega^4(\phi)}
$$

(7)

Let us discuss approximations which can be made. For small field values, $h \simeq \phi$ and $\Omega^2 \simeq 1$. Therefore, the potential for $\phi$ is the same as the initial potential for the Higgs field, i.e., $V(\phi) = U(\phi)$. For large values of the field $h \gg \sqrt{\xi_h}$, we have the following relation:

$$
h \simeq \frac{M_p}{\sqrt{\xi_h}} \exp \left( \frac{\phi}{\sqrt{6}M_p} \right)
$$

(8)

Equation (4) implies

$$
\frac{\phi}{M_p} = \sqrt{\frac{3}{2}} \ln \Omega^2.
$$

(9)

i.e.,

$$
\Omega^2 = \exp \left( \sqrt{\frac{2}{3}} \frac{\phi}{M_p} \right).
$$

(10)

Since $v \ll M_p$, we consider only the potential for the inflation. In fact, we can safely ignore vacuum expectation value $v$ of the Higgs field in term $1 + \xi_h \frac{v^2}{M_p^2} \approx 1$ (this term is appeared in potential after conformal transformation) for the evolution during inflation and even in preheating stage, and simply consider the potential only for inflation. From now on we ignore $v$ for all numerical calculations too.

Therefore, the inflationary potential is given by following [see Eqs. (5) and (7)] equation.

$$
V(\phi) = \frac{\lambda M_p^4}{4\xi_h^2} \left( 1 - \exp \left( -\sqrt{\frac{2}{3}} \frac{\phi}{M_p} \right) \right)^2
$$

(11)

Let us consider more complicated situation which is known as the Radiatively corrected Higgs Inflation (RHCI). In this case, we need to take into account corrections
to \( f (h) \) and \( U (h) \) in Lagrangian (1). We have [3,51]

\[
f (h) = \frac{M_p^2 + \xi_h h^2}{2} + \frac{h^2}{32\pi^2} C \ln \frac{h^2}{\mu^2}, \quad U (h) = \frac{\lambda}{4} \left( h^2 - v^2 \right)^2 + \frac{\lambda h^4}{128\pi^2 A} \ln \frac{h^2}{\mu^2}.
\]

(12)

where \( A \) is anomalous scaling, \( \mu \) is the normalization point and we use \( C = 3\xi_h \lambda \) as a one-loop correction to \( f (h) \) and \( \Omega_1 \) given by (see Ref. [3,40,51]).

\[
\Omega_1^2 (h) = 1 + \frac{\xi_h h^2}{M_p^2} + \frac{1}{M_p^2 16\pi^2} C \ln \frac{h^2}{\mu^2}.
\]

(13)

An essential point for inflation is the flatness of the scalar potential in the region of the field values \( h \sim 10M_p / \sqrt{\xi_h} \), what corresponds to the Einstein frame field \( \phi \sim 6M_p \) [8]. It is important that radiative corrections do not spoil this property (see Fig. 2). For more physics about \( C \) and \( A \) see Refs. [3,51]. Performing conformal transformation \( \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \), we obtain the following relation:

\[
\frac{d\phi}{dh} = \sqrt{-\tilde{g}} \left[ \Omega^2 + \frac{6\xi_h h^2}{M_p^2} + \frac{3}{4} \frac{\xi_h C h^2}{\pi^2 M_p^2} \left( 1 + \ln \frac{h^2}{\mu^2} \right) \right]^{-1/2} \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{U (h)}{\Omega_1^4}.
\]

(14)

As a result, we find the Lagrangian density in the following form:

\[
\tilde{L} = \sqrt{-\tilde{g}} \left[ \frac{M_p^2}{2} \tilde{R} + \frac{1}{2} \left( \Omega^2 + \frac{6\xi_h h^2}{M_p^2} + \frac{3}{4} \frac{\xi_h C h^2}{\pi^2 M_p^2} \left( 1 + \ln \frac{h^2}{\mu^2} \right) \right) \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{U (h)}{\Omega_1^4} \right].
\]

(15)

By using Eq. (14), we obtain canonical scalar field \( \phi \) coupled to the Einstein gravity in the following Lagrangian. Therefore, in the Einstein frame.

\[
\tilde{L} = \sqrt{-\tilde{g}} \left[ -\frac{M_p^2}{2} \tilde{R} + \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V (\phi) \right], \quad V (\phi) = \frac{U (h)}{\Omega_1^4 (\phi)}.
\]

(16)

Equations (12)–(14) determine \( V (\phi) \) and \( \phi \)

\[
V (\phi) \simeq \frac{\lambda M_p^4}{4\xi_h^2} \left[ 1 + \frac{A_l}{16\pi^2} \ln \frac{h}{\mu} - \frac{2M_p^2}{\xi_h h^2} \right].
\]

(17)

The above potential is the Einstein frame of effective potential in Ref. [51] where the inflationary anomalous scaling is \( A_l = A - 12\lambda \). The extra term \( \propto -12\lambda \) is due to the quantum Goldstone contributions in \( C \). See Ref. [51]
This mechanism can be interpreted as asymptotic freedom, because \( A^{4}_{64\pi^{2}} \) determines the strength of quantum corrections in inflationary dynamics \([6]\). We use \( C = 3 \xi_{h} \lambda \) as a one-loop correction to \( f (h) \) [see Eq. (12)] and also we assumed \( h > \frac{M_{p}}{\sqrt{\xi_{h}}} \ll v \) and \( \xi_{h} \gg 1 \) (these conditions are essential for inflation and correction should not spoil essential point for inflation). Using Eqs. (13) and (14) and neglecting second order of parameter \( C \), we find

\[
\frac{d\phi}{dh} \simeq \frac{M_{p} \sqrt{6} \xi_{h} h}{(M_{p}^{2} + \xi_{h} h^{2})} \left[ 1 + \frac{C}{16\pi^{2}\xi_{h}} + \frac{C}{8\pi^{2}\xi_{h}} \left( \frac{M_{p}^{2}}{M_{p}^{2} + \xi_{h} h^{2}} \right) \ln \frac{h}{\mu} \right].
\] (18)

Integrating the above equation, we obtain

\[
\phi = \frac{M_{p} \sqrt{6}}{2} \left[ \ln \left( 1 + \frac{\xi_{h} h^{2}}{M_{p}^{2}} \right) + \frac{C}{8\pi^{2}\xi_{h}} \left( \frac{\xi_{h} h^{2}}{M_{p}^{2}} \right) \ln \frac{h}{\mu} \right].
\] (19)

For large field assuming \( \xi_{h} h^{2} \gg M_{p}^{2} \), and setting \( \mu = \frac{M_{p}}{\sqrt{\xi_{h}}} \) we find

\[
\phi \simeq \frac{M_{p} \sqrt{6}}{2} \ln \left( \frac{\xi_{h} h^{2}}{M_{p}^{2}} \right) + \frac{C M_{p} \sqrt{6}}{32\pi^{2}\xi_{h}} \ln \left( \frac{\xi_{h} h^{2}}{M_{p}^{2}} \right).
\] (20)

Let us discuss about the values of normalization scale \( \mu \). In Ref. [11] the authors discussed two distinct values for normalization scale both in the Jordan frame and in the Einstein frame. In the Einstein frame \( \mu^{2} \propto M_{p}^{2} \) or \( \mu^{2} \propto M_{p}^{4}/(M_{p}^{2} + \xi_{h} h^{2}) \).

However, they use \( \mu = m_{t} \) in which \( m_{t} \) is the top-quark mass. Of course the aim of the authors was estimating Higgs boson mass and producing successful Higgs inflation. In addition, they indicated that the requirement of successful Higgs-inflation allows to put an upper bound on the mass of the top quark. If t-quark mass were larger than 240–250 GeV, no choice of \( \xi_{h} \) and of the Higgs mass could lead to acceptable inflationary parameters. See Ref. [11].

We can find \( h (\phi) \) and substitute into Eq. (17) in order to express the effective potential in terms of \( \phi \). Equation (19) gives

\[
\frac{\xi_{h} h^{2}}{M_{p}^{2}} = \left( \exp \left( \frac{2}{3} \frac{\phi}{M_{p}} \right) - 1 \right) \left[ 1 - \frac{C}{16\pi^{2}\xi_{h}} \ln \left( \exp \left( \frac{2}{3} \frac{\phi}{M_{p}} \right) - 1 \right) / \left( \frac{\xi_{h} M_{p}^{2}}{M_{p}^{2}} \right) \right].
\] (21)

For large \( \phi \) from Eq. (20) we have

\[
h (\phi) = \frac{M_{p}}{\sqrt{\xi_{h}}} \left[ \exp \left( \frac{\phi}{M_{p} \sqrt{6} B} \right) \right].
\] (22)
where $B = 1 + \frac{C}{16\pi^2\xi_h}$ and $\phi = M_p\sqrt{6}\ln\left(\sqrt{\xi_h}h\right)B$. This is exactly Eq. (8) if we neglect the effect of quantum correction $C$.

We need an equation valid for the whole period of inflation. In order to obtain such equation one needs to look at Eq. (16) as mentioned earlier and to obtain potential to the linear order of $C$. We obtain [see Eq. (16)] the following relation for the potential:

$$V(\phi) = \Omega^{-4} \frac{\lambda M_p^4}{4 \xi_h^2} \left(e^{\sqrt{3} \frac{\phi}{M_p}} - 1\right)^2 \left[1 - \frac{C}{8\pi^2\xi_h} \ln\left(e^{\sqrt{3} \frac{\phi}{M_p}} - 1\right) + \frac{A_{I}}{32\pi^2} \ln\left(e^{\sqrt{3} \frac{\phi}{M_p}} - 1\right)\right], \Omega^{-4} = e^{-2\sqrt{3} \frac{\phi}{M_p}}.$$ (23)

The above equation shows potential to the first order in $C$, where $\Omega^{-4} = e^{-2\sqrt{3} \frac{\phi}{M_p}}$ and we used $\mu = \frac{M_p}{\sqrt{\xi_h}}$. To obtain the above equation, we used $\xi_h \gg 1$, $\xi_h^2h^2 \gg 1$ (This condition requires potential to be linear in $C$), and $v \ll h$. We neglected also all terms with $C \times A$ because $A$ is small. Using $\Omega^{-4} = e^{-2\sqrt{3} \frac{\phi}{M_p}}$ and $C = 3\xi_h\lambda$, we obtain the final expression for the potential

$$V(\phi) = \frac{\lambda M_p^4}{4 \xi_h^2} \left(1 - e^{-\sqrt{3} \frac{\phi}{M_p}}\right)^2 \left[1 + \frac{A_{I}}{32\pi^2} \ln\left(e^{\sqrt{3} \frac{\phi}{M_p}} - 1\right)\right].$$ (24)

Note that in Eq. (24) the potential is unbounded and one needs to be careful of negative values. We compare this potential with the Starobinsky potential [see Eq. (11)] provided that $3\mu^2 = \frac{\lambda M_p^2}{\xi_h^2}$. As a one loop correction we obtain an extra term. We estimate the numerical value of the constant $A_I$ and plot the two potentials in order to compare them. Then we can derive predictions for the scalar perturbations, spectral index, and tensor-to-scalar ratio. The equations of motion determine the spectrum of generated electromagnetic fields. If we assume $\frac{\xi_h^2h^2}{M_p^2} \gg 1$, then for large fields we find...
the following potential:

\[ V(\phi) \simeq \frac{\lambda}{4} \frac{M_p^4}{\xi_h^2} \left[ 1 + \frac{A_I}{32\pi^2} \sqrt{\frac{2}{3}} \frac{\phi}{M_p} - 2e^{-\sqrt{\frac{2}{3}} \frac{\phi}{M_p}} \right], \tag{25} \]

which coincides with the equation obtained in Ref. [41]. Potential (24) is applicable even at the end of inflation unlike the well-known potential of Ref. [41]. Let us calculate the numerical value of \( A_I \) in order to plot potentials both in the Starobinsky and our model. By using Refs. [41,51], we obtain the following relation [6,14]:

\[ A = \frac{3}{8\lambda} \left( 2g^4 + \left( g^2 + g'^2 \right) - 16y^4 \right) + 6\lambda + O \left( \xi_h^{-2} \right), \quad C = 3\xi_h\lambda + O \left( \xi_h^0 \right) \tag{26} \]

and then we plot potential (24).

We see from Figs. 1 and 2 that Radiatively Corrected Higgs Potential has a maximum and this maximum is due to allowed values of \( A_I \) and is independent of values of normalization scale \( \mu \). In obtaining potential (24) we have already set \( \mu = \frac{M_p}{\sqrt{\xi_h}} \). According to the left panel of Fig. 1, the potential of the Starobinsky model has asymptotic behavior.

Note that all curves in Figs. 1 and 2 are plotted for the same value of prefactor in Eqs. (11), (24). In this case we obtain \( \xi_h = 1.6 \times 10^4 \), which is in accordance with our assumptions for inflation and figures have no problem but for numerical calculations we just consider observational data i.e slow-roll parameters, number of e-folds, spectral index and tensor to scalar ratio in order to find lower and upper bound for \( A_I \). Therefore normalization scale \( \mu \) plays no roles in numerical calculations.

It is useful to calculate the slow-roll parameters for potential (24). We have [37]

\[ \epsilon = \frac{M_p^2}{2} \left( \frac{V_{\phi}}{V} \right)^2 = \frac{4}{3} \left( \frac{1 + A_I \frac{e^{-\sqrt{3/2} \frac{\phi}{M_p}}}{64\pi^2}}{e^{\sqrt{3/2} \frac{\phi}{M_p}} - 1} \right)^2, \]

\[ \eta = \frac{M_p^2 V_{\phi\phi}}{V} = \frac{4}{3} \left( \frac{2 - e^{\sqrt{3/2} \frac{\phi}{M_p}}} + \frac{A_I e^{\sqrt{3/2} \frac{\phi}{M_p}}}{16\pi^2} \right)^2 \tag{27} \]

At the end of inflation, \( \epsilon = 1 \). Therefore, for end of inflation scalar field obeys the following relation:

\[ \phi_e = \sqrt{\frac{3}{2}} M_p \ln \left[ \frac{1 + \left( 2/\sqrt{3} \right)}{1 - \frac{A_I}{\sqrt{332\pi^2}}} \right] \tag{28} \]
and the number of e-folds is given by

\[ N = -\frac{3}{4} \sqrt{\frac{2}{3}} x |_{x_e} + \frac{3}{4} \ln \left( 1 + \frac{A_I e^{\sqrt{2/3} x}}{32\pi^2} \right) |_{x_e} - \frac{3}{4} \left( e^{-\sqrt{2/3} x} \left( 1 + \frac{A_I e^{\sqrt{2/3} x}}{64\pi^2} \right)^{-1} \right) |_{x_e}, \quad (29) \]

where \( x = \frac{\phi}{M_p} \) and \( x_e = \frac{\phi_e}{M_p} \). In view of the shape of our potential [see Eq. (24)] and Fig. 2, we are not able to use relations (27)–(29). They are obtained in first order of \( A_I \). The requirement of slow-rolling of the inflation field imposes us to obtain more accurate relations

\[ \epsilon = -\frac{4}{3} \left( \frac{A_I e^{\sqrt{2/3} \phi / M_p}}{1 + \frac{A_I}{32\pi^2} \ln \left( e^{\sqrt{2/3} \phi / M_p} - 1 \right)} \right)^2 \]

\[ \eta = \frac{4}{3} \left( 2 - e^{\sqrt{2/3} \phi / M_p} \right) + \frac{A_I}{16\pi^2} e^{\sqrt{2/3} \phi / M_p} \left[ 1 + \frac{A_I}{32\pi^2} \ln \left( e^{\sqrt{2/3} \phi / M_p} - 1 \right) \right]^{-1} \]

\[ \left( e^{\sqrt{2/3} \phi / M_p} - 1 \right)^2 \]

(30)

and the number of e-folds is given by

\[ N = -\int_x^{x_e} \frac{1}{2\sqrt{2}} \left( 1 - e^{-\sqrt{2/3} x} \right) e^{\sqrt{2/3} x} \left[ 1 + \frac{A_I}{64\pi^2} e^{\sqrt{2/3} x} \left( 1 + \frac{A_I}{32\pi^2} \ln \left( e^{\sqrt{2/3} x} - 1 \right) \right)^{-1} \right]^{-1} dx, \quad (31) \]

where \( x = \frac{\phi}{M_p} \) and \( x_e = \frac{\phi_e}{M_p} \). The spectral index and the tensor-to-scalar ratio can be expressed in terms of slow-roll parameters as follows [28,41]:

\[ n_s = 1 - 6\epsilon_* + 2\eta_*, \quad (32) \]

\[ r = 16\epsilon_* \quad (33) \]

where the quantities with * means that the corresponding quantities are calculated when the pivot scale \( k_* \) crosses the horizon. One needs to solve Eqs. (30) and (31) numerically and to insert into Eqs. (32) and (33) in order to compare the obtained values with observations.

Calculations show only for \( -4 < A_I < 10 \) the predictions for slow-roll parameters are compatible with the Planck data [2]. For instance, \( r = 0.00190286, n_s = 0.960691, A_I = -3, N_* = 60. \)
Fig. 2 Left and right: Potential of Eq. (24) for $A_I = -3$ and $A_I = -4$ (Green), $+4$ (Blue), $N_a = 60$ and $\mu = 1.3 \times 10^{-5}$. Note that $V_\infty = \frac{3\mu^2}{4}$.

If we look at Eq. (28) and more accurately Eq. (30) when inflation ends it is impossible to get negative value for inflaton field. Therefore, there is no need to be worried about negative values. In fact, inflation ends before it reaches to negative value according to the equations and values that we mention above. For instance setting $A_I = -3$ and $\phi_0 \sim 5 M_p$ do not have any problems and these values are acceptable. In this case calculations show $\phi_e = 0.933471 M_p$.

3 Axial coupling

The Maxwell action $S = -\frac{1}{4} \int d^4x \sqrt{-g} F_{\mu\nu} F^{\mu\nu}$ does not change from the Jordan to the Einstein frame. Let us consider this by using conformal transformation $\tilde{g}_{\mu\nu} = \frac{1}{\Omega^2} g_{\mu\nu}$, i.e. $\sqrt{-g} L_{\text{in}} = -\frac{1}{4} \sqrt{-\tilde{g}} \tilde{g}^{\mu\alpha} \tilde{g}^{\nu\beta} \tilde{F}_{\alpha\beta} \tilde{F}_{\mu\nu}$.

We decompose $A_\mu$ into its transverse and longitudinal parts. We have $A_\mu = (A_0, A_i)$ with $A_i = A_i^T + \partial_i \chi$, where $\partial_i A_i^T = 0$. Then the Maxwell action $S = -\frac{1}{4} \int d^4x \sqrt{-\tilde{g}} F_{\mu\nu} F^{\mu\nu}$ reads as

$$S_0 = \frac{1}{2} \int d^4x \left( A_i^{T'} A_i^{T'} + A_i^T \Delta A_i^T \right), \quad (34)$$

where $\Delta$ is the Laplacian and $t$ denotes derivative with respect to the conformal time $d\eta = \frac{dt}{a}$ and $d^4x = d\eta d^3x$.

Before we start introducing axial coupling it is useful to consider similarities and differences of Starobinsky model and Higgs model because we use the same form of coupling function for Higgs model. The $R^2$ term in the Starobinsky inflationary model $S_{GR} = \int d^4x \sqrt{-g} f(R), \quad f(R) = -\frac{M_p^2}{2} \left( R - \frac{R^2}{6\mu^2} \right)$ can be regarded as a leading quantum correction to the gravitational effective action [60]. Where $\mu = 1.3 \times 10^{-5} M_p$ and $M_p = 2.4 \times 10^{18}$ Gev is the reduced Planck mass. A conformal transformation (Weyl transformation) with $g_{\mu\nu} \rightarrow g_{\mu\nu} \exp\left[-\sqrt{\frac{2}{3}} \frac{\phi}{M_p} \right]$ transforms the action of the theory to the usual Einstein frame with a new spatially uniform scalar field $\phi$ whose potential is given by $V(\phi) = \frac{3\mu^2 M_p^2}{4} \left( 1 - \exp\left[-\sqrt{\frac{2}{3}} \frac{\phi}{M_p} \right] \right)^2$. 

$\Sigma$ Springer
When we set \(3\mu^2 = \frac{\lambda M_p^2}{\sqrt{8}}\), then it seems satrobinsky potential and Higgs inflation are dynamically the same. See Eq. (11).

Let us discuss about Higgs inflation and Starobinsky models. It seems because we obtain the same potentials [see Eq. (11)] therefore the inflationary dynamics in the Higgs inflation and the Starobinsky inflation are the same but we need to consider interactions too. In the Higgs model the lagrangian is linear with respect to space-time curvature \(R\) which this term non-minimally coupled to Higgs field whereas in Starobinsky model we have \(R^2\) term as a quantum correction to Einstein gravity. We have done conformal transformations for both cases but we need to bear in mind in Starobinsky model at first we do not have any scalar field from the beginning but only the curvature, the field appears when we amend the \(R^2\) term by conformal transformation. But from the very beginning we have Higgs field which its interaction with curvature is not in canonical form as we mentioned earlier. However, one can speak about two different representations of two models at least at classical level. The main idea for conformal transformation and discussions come from Maeda [40], Ketov [34] and Ketov and Starobinsky [35].

In order to break conformal invariance of electromagnetic field one needs to consider coupling functions like \(f(\phi) FF\) or axial coupling \(f(\phi) F\tilde{F}\) as well as their combinations \(f(\phi, R) FF\) and \(f(\phi, R) F\tilde{F}\) and other coupling functions that are mentioned in the literatures (for an overview see [19,53]).

We assume that parity-violating non-minimal couplings between curvature and electromagnetic field are present in the effective action. In the case of kinetic coupling \(\int d^4x \sqrt{-g} \left[ k_1 RF_{\mu\nu}F^{\mu\nu} \right]\) it is not possible to produce significant magnetic fields.

As mentioned above, we consider in this paper the following axial interaction (see [47]):

\[
S_{\text{int}}^a = -\int d^4x \sqrt{-g} \chi_1 R F_{\mu\nu} \tilde{F}^{\mu\nu},
\]

where \(\chi_1\) is dimensional coupling constant and \(\tilde{F}^{\mu\nu} = \frac{1}{2} \eta^{\mu\nu\rho\sigma} F_{\rho\sigma}\) is the dual of the electromagnetic field tensor and \(\eta^{\mu\nu\rho\sigma} = \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}\). Note that \(\epsilon^{\mu\nu\rho\sigma}\) is the totally antisymmetric Levi–Civita symbol with \(\epsilon^{0123} = 1\). This interaction breaks the parity and enables a production of only one of the polarization states of the electromagnetic field due to axion-like coupling of electromagnetic (EM) field to the inflation. Therefore the produced magnetic fields are helical. Using conformal transformation \(\tilde{g}_{\mu\nu} = \Omega^{-2} g_{\mu\nu}\), we obtain action in the Einstein frame

\[
S_{\text{int}}^a = -\chi_1 \int d^4x \sqrt{-\tilde{g}} \left( \tilde{\nabla} \phi \right)^2 + \frac{\sqrt{6} M_p^2}{\Omega^2} \Box \phi \right) F_{\mu\nu} \tilde{F}^{\mu\nu}. \tag{36}
\]

If we consider only the linear approximation for electromagnetic field, then the Einstein equation gives

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{M_p^2} \left[ \nabla_{\mu} \phi \nabla_{\nu} \phi - g_{\mu\nu} \left( \frac{1}{2} \left( \nabla \phi \right)^2 - V(\phi) \right) \right]. \tag{37}
\]
The equation of motion for the scalar field reads

\[ \Box \phi + \frac{dV(\phi)}{d\phi} = 0. \] (38)

Using Eqs. (37) and (38), we find

\[ S_{int}^a = 3\chi_1 \int d^4x \sqrt{-g} e^{\left(\sqrt{\frac{2}{3}} \frac{\phi}{M_p}\right)} \left[ \frac{1}{3M_p^2} (4V(\phi)) + \frac{\sqrt{2}}{\sqrt{3}M_p} \left( \frac{dV}{d\phi} \right) \right] F_{\mu\nu} \tilde{F}^{\mu\nu}. \] (39)

In the Coulomb gauge \( A_0 = 0, \partial_j A^j = 0 \). Then Eq. (39) can be written as follows:

\[ S_{int}^a = 12\chi_1 \int d^4x e^{\left(\sqrt{\frac{2}{3}} \frac{\phi}{M_p}\right)} \left[ \frac{1}{3M_p^2} (4V(\phi)) + \frac{\sqrt{2}}{\sqrt{3}M_p} \left( \frac{dV}{d\phi} \right) \right] \epsilon_{ijk} A_i^{T'} \partial_j A_k^T, \] (40)

where \( \sqrt{-g} F_{\mu\nu} \tilde{F}^{\mu\nu} = 4\epsilon_{ijk} A_i^{T'} \partial_j A_k^T \) and \( ' \) indicates derivative with respect to conformal time.

If we use Eqs. (34) and (40) so that \( S = S_0 + S_{int}^a \). Then we find the equation of motion

\[ A_i'' - \nabla^2 A_i^T + \omega' \epsilon_{ijk} \partial_j A_k^T = 0, \] (41)

where

\[ \omega = 12\chi_1 e^{\left(\sqrt{\frac{2}{3}} \frac{\phi}{M_p}\right)} \left[ \frac{1}{3M_p^2} (4V(\phi)) + \frac{\sqrt{2}}{\sqrt{3}M_p} \left( \frac{dV}{d\phi} \right) \right]. \] (42)

The quantization of the electromagnetic field is achieved by imposing the canonical commutation relations

\[ [A_i(t, x), \pi_j(t, y)] = i \int \frac{d^3k}{(2\pi)^3} \exp(i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})) \Delta_{ij}, \] (43)

where \( \Delta_{ij} = \delta_{ij} - k_i k_j / k^2 \) and \( \pi_j = \dot{A}_j \). The electromagnetic field can be expanded in terms of the creation and annihilation operators \( a_{+\lambda}(\mathbf{k}) \) and \( a_{\lambda}(\mathbf{k}) \)

\[ A_i(t, x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2k}} \sum_{\lambda=1}^2 \epsilon_{i\lambda}(\mathbf{k}) \left[ a_{\lambda}(\mathbf{k}) A(t, \mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{x}) + a_{+\lambda}^{\dagger}(\mathbf{k}) A^*(t, \mathbf{k}) \exp(-i\mathbf{k} \cdot \mathbf{x}) \right]. \] (44)
We introduce the orthogonal spatial basis as

\[ ( \varepsilon_k^1, \varepsilon_k^2, \hat{k} ), |\varepsilon_k^i|^2 = 1, \hat{k} = \frac{k}{|k|} \varepsilon_k^3 = \frac{1}{\sqrt{2}} (\varepsilon_k^1 \pm i \varepsilon_k^2) \]  

(45)

Therefore, the Fourier modes of the vector potential take the form

\[ A_i^T (\eta, k) = A_+ \varepsilon_+ + A_- \varepsilon_. \]  

(46)

We arrive at the following equation for helicity modes:

\[ A_h'' + \left( k^2 + h k \omega \right) A_h = 0, \]  

(47)

where \( h = \pm \) denotes the helicity. In terms of cosmic time, Eq. (47) takes the form

\[ \ddot{A}_h (t, k) + H \dot{A}_h (t, k) + \left( k^2 - \frac{h \omega(t)}{a} \right) A_h (t, k) = 0, \]  

(48)

where

\[ \dot{\omega} = 12 \chi_1 \phi e^{\left( \sqrt{\frac{2}{3}} \frac{\phi}{M_p} \right)} \left[ \sqrt{\frac{2}{3}} \frac{1}{3 M_p^2} (4 V (\phi)) + \frac{2}{M_p^2} \left( \frac{dV}{d\phi} \right) + \frac{\sqrt{2}}{\sqrt{3} M_p} \frac{d^2 V}{d\phi^2} \right]. \]  

(49)

Taking the first and second derivative of Eq. (24) and substitute them into Eq. (49), we obtain

\[ \dot{\omega} = V_0 \sqrt{\frac{2}{3}} \frac{1}{M_p} \chi_1 \phi e^{\left( \sqrt{\frac{2}{3}} \frac{\phi}{M_p} \right)} \left[ 1 + \frac{A_f}{32 \pi^2} \ln \left( e^{\sqrt{\frac{2}{3}} \frac{\phi}{M_p}} - 1 \right) + \frac{3 A_f}{64 \pi^2} \right], \]  

(50)

where \( V_0 = \frac{\chi_1}{4 \varepsilon_n} \). All remains to be done is to numerically solve Eq. (48) and obtain spectrum of electromagnetic field.

### 4 Power spectra of electric and magnetic field

The power spectrum of magnetic fields is defined by

\[ \frac{d \rho_B}{d \ln k} = \frac{k^3}{(2 \pi)^2} P_S, \]  

(51)

where

\[ P_{S/A} = \frac{k^2}{a^4} \left( |A_+(t, k)|^2 \pm |A_-(t, k)|^2 \right) \]  

(52)
and the upper sign corresponds to $P_S$ and the lower sign to $P_A$. For the electric field, the spectrum is given by

$$
\frac{d\rho_E}{d \ln k} = \frac{k^3}{(2\pi)^2} \frac{1}{a^2} \left( \left| \frac{\partial A_+(t, k)}{\partial t} \right|^2 + \left| \frac{\partial A_-(t, k)}{\partial t} \right|^2 \right)^{1/2}.
$$

In the case of maximally helical magnetic field, $|A_+| = |A|$ with $|A_-| = 0$, we have

$$P_S = P_A = \frac{k^2}{a^4} |A|^2. \quad (54)$$

### 5 Numerical calculations

We use the Bunch–Davies vacuum initial condition for Eqs. (47) and (48).

$$A_h(t, k) = \frac{1}{\sqrt{2k}} \exp(-i k \eta), \quad k \eta \to \infty \quad (55)$$

As we mentioned in Sect. 2 one needs to solve Eqs. (30) and (31) numerically and to insert into Eqs. (32) and (33) in order to compare the obtained values with observations.

Calculations show only for $-4 < A_I < 10$ the predictions for slow-roll parameters are compatible with the Planck data [2]. For spectral index $n_s$ and tensor to scalar ratio $r$, see Fig. 3. For instance, $r = 0.00190286$, $n_s = 0.960691$, $A_I = -3$, $N_* = 60$, $\phi_0 = 5.20568 M_p$, $\phi_e = 0.933471 M_p$.

![Fig. 3](image-url) Theoretical prediction for $n_s$ and $r$ in RCHI model with different values of $-4 < A_I < 10$ for $50 < N_* < 60$ in comparison with Planck 2018.
Fig. 4 The time dependence of the modulus of the mode function $|A_s(k,t)|$ multiplied by $\sqrt{2k}$ for modes with momentum $k = 10^{18} M_p$ and positive (blue solid line) or negative (red dashed line) helicity (color figure online)

Accordingly, We set $A_I = -3, V_0 = 7.04914 \times 10^{-11}, \phi_0 = 5.20568 M_p$, and $N = 60$. For relevant values of $\chi_1$ and $k$ we solve Eq. (48) in order to plot the spectrum of magnetic field [see Eq. (51)].

Figure 4 shows how the modes with certain value of momentum $k = 10^{18} M_p$ and different helicity evolve in time. Before the horizon crossing both polarizations oscillate in time with constant amplitude representing the Bunch–Davies mode function. In this regime, $\sqrt{2k}|A_s(t,k)| \approx 1$. The situation changes drastically after the mode exits the horizon.

The mode of one polarization ($A_+$ in our case, see the blue solid line in Fig. 4) undergoes amplification, while the other one ($A_-$, see the red dashed line in Fig. 4) diminishes. This is a consequence of the axion-like coupling of the electromagnetic (EM) field to the inflation. As a result, an electromagnetic field with nontrivial helicity is generated.

The power spectra of the generated electric and magnetic fields at the end of inflation are shown in Fig. 5 for three values of the coupling parameter $\chi_1$.

According to this figure, we conclude that the spectra are blue with the spectral index very close to the unperturbed value $n_B \simeq 4$. The amplitude of the spectrum strongly depends on the value of the coupling parameter. For example, increasing $\chi_1$ only by 2 times, we obtain 10 orders of magnitude of amplification. It is important to note that for large values of the coupling parameter (e.g., for $\chi_1 = 10^{10} M_p^{-2}$) the energy density of the electromagnetic field exceeds that of the inflation. This means that the back-reaction of electromagnetic fields on the background evolution cannot be neglected and has to be self-consistently taken into account. For two other values, $\chi_1 = 5 \times 10^9 M_p^{-2}$ and $\chi_1 = 7.5 \times 10^9 M_p^{-2}$, the back-reaction is weak and our analysis is valid. The maximum in the spectrum is observed for the mode $k_{\text{max}} \sim 10^{21} M_p$ or $k_{\text{max}} \sim 8.8 \times 10^{-6} M_p = 3.3 \times 10^{51} Mpc^{-1}$. This corresponds to the following correlation length of the magnetic field at the end of inflation.
Fig. 5 The spectral densities of the electric (a) and magnetic (b) energy densities at the end of inflation stage as functions of the momentum for three values of the coupling parameter: $\chi_1 = 5 \times 10^9 M_p^{-2}$ (red dashed-dotted lines), $\chi_1 = 7.5 \times 10^9 M_p^{-2}$ (green dashed lines), and $\chi_1 = 1 \times 10^{10} M_p^{-2}$ (blue solid lines). The spectral densities of ordinary vacuum fluctuations (without amplification) are shown by the thin black dashed lines. The energy density of the inflation is shown by the purple dotted line. The maximum in the spectrum is observed for the mode $k_{\text{max}} \sim 10^{21} M_p$ or $k_{\text{max}} a_e = 8.8 \times 10^{-6} M_p = 3.3 \times 10^{51} \text{Mpc}^{-1}$ at the end of inflation (color figure online).

$$\lambda_B(t_e) = \frac{2 \pi a(t_e)}{k_{\text{max}}} \sim 10^6 M_p^{-1},$$

where $a(t_e) \approx e^{60}$ is the scale factor at the end of inflation.

Note that the value of $V_0$ comes from the correct value of the scalar perturbations amplitude.

$$A_s = P_s(k) = \left(\frac{H^2}{2 \pi |\dot{\phi}|}\right)^2 |N_s| = \frac{1}{12 \pi^2 M_p^4 V_{\phi}^2} V_{,\phi} = \frac{dV}{d\phi}$$

Here $N_s$ indicates number of e-folds at horizon-crossing. From Ref. [2] $P_s = 2.33361 \times 10^{-9}$. Therefore, from the above relation we find $V_0$ then we can find $\xi_h$.

As we mentioned before for $P_s = 2.33361 \times 10^{-9}$ and $A_I = 3$ and $\phi_0 = 5.20568 M_p$ we obtain $V_0 = 7.04914 \times 10^{-11}$. Therefore, $\xi_h = 2.1 \times 10^4$ and normalization scale $\mu = \frac{M_p}{\sqrt{\xi_h}} = 6.9 \times 10^{-3} M_p$.

Finally, integrating the spectral densities over the range of modes which exit the horizon from the beginning of inflation until a given moment of time, we calculate the total electric and magnetic energy densities. Their time dependences are shown in Fig. 6 for three values of the coupling parameter: $\chi_1 = 5 \times 10^9 M_p^{-2}$ (red lines), $\chi_1 = 7.5 \times 10^9 M_p^{-2}$ (green lines), and $\chi_1 = 1 \times 10^{10} M_p^{-2}$ (blue lines).

First, we see that the electric energy density is always greater than the magnetic one. Therefore, strong electric fields are generated during inflation together with magnetic ones and the Schwinger pair production may be important for magnetogenesis. This issue, however, deserves a separate investigation and has to be addressed elsewhere. At second, our approach, which does not take into account the back-reaction of generated fields, is applicable only when the electromagnetic energy density is much less than that of the inflation (shown in Fig. 6 by the purple dashed-dotted line).

For $\chi_1 = 1 \times 10^{10} M_p^{-2}$, and e.g., at $t \approx 9.7 \times 10^6 M_p^{-1}$ the electric energy density becomes equal to that of the inflation and the back-reaction regime occurs. The curves...
The time dependences of the electric (solid lines) and magnetic (dashed lines) energy densities during the inflation stage for three values of the coupling parameter: \( \chi_1 = 5 \times 10^9 M_p^{-2} \) (red lines), \( \chi_1 = 7.5 \times 10^9 M_p^{-2} \) (green lines), and \( \chi_1 = 1 \times 10^{10} M_p^{-2} \) (blue lines). The curves for \( \chi_1 = 1 \times 10^{10} M_p^{-2} \) in the back-reaction regime are shown in pale blue color. The energy density of the inflation is shown by the purple dashed-dotted line (color figure online).

after this moment of time are shown in pale blue color and do not describe the correct time dependences any more. Previous studies of the back-reaction regime showed that at this moment of time the generation of electric and magnetic fields should stop and the energy densities should remain almost constant until the end of inflation. Thus, the maximal possible energy density of the magnetic field generated during inflation can be estimated by the value at the time when the back-reaction occurs, \( \rho_{B,\text{max}} \approx 5 \times 10^{-12} M_p^4 \).

The important question is the post-inflationary evolution of the generated electromagnetic fields. The electric field quickly dissipates in the highly conducting medium produced during reheating. However, the magnetic fields can survive until the present time. Moreover, due to the nontrivial helicity, they undergo the inverse cascade process in the turbulent plasma which can strongly increase their correlation length. It has been shown in Ref. [48] that the maximally helical magnetic fields with the energy density \( \rho_B \sim 10^{-12} M_p^4 \) and the correlation length \( \lambda_B \sim 10^6 M_p^{-1} \) at the end of inflation are transformed into the present-day magnetic fields with the strength \( B_0 \sim 10^{-15} \) G and the correlation length \( \lambda_{B,0} \sim 1 \) pc. Below we briefly explain it.

### 6 Present-day magnetic field

If we want to study the evolution of magnetic field through the sequence of cosmological epochs until the present, then we should follow the steps of Ref. [48] in which the authors give full details of estimating magnetic field in present-day. However, this procedure is quite different from that of the authors of Ref. [49] because the authors use kinetic coupling which is quite different from axial coupling. However, we should briefly explain.
After the end of inflation, the preheating takes place when the inflation field oscillates in the potential minimum and decays into different particles. The EM field could be amplified during the preheating stage due to the interaction with the rapidly oscillating inflation, e.g., through the mechanism of parametric resonance. The back-reaction regime, which is favorable for obtaining large magnetic fields, slows down the inflation evolution decreasing its oscillation amplitude and reducing the efficiency of the parametric resonance.

The authors of Ref. [48] assume that the magnetic field is not enhanced during the preheating stage. In addition, during reheating, the plasma of created particles thermalizes and the Universe enters its hot radiation dominated phase. They also assume that at preheating stage the turbulent regime does not occur and the magnetic field and its correlation length evolve adiabatically, i.e., the corresponding co-moving quantities remain constant [48].

According to Fig. 6 we estimate $\rho_{\text{inf}} \sim 10^{-10} M_p^4$ and energy density of magnetic field is $\rho_B \sim 10^{-12} M_p^4$. The present-day magnetic fields is $B_0 \sim 10^{-15}$ G. As we discussed earlier, we would like to emphasis once more, it has been shown in Ref. [48] that the maximally helical magnetic fields with the energy density $\rho_B \sim 10^{-12} M_p^4$ and the correlation length $\lambda_B \sim 10^6 M_p^{-1}$ at the end of inflation are transformed into the present-day magnetic fields with the strength $B_0 \sim 10^{-15}$ G and the correlation length $\lambda_{B,0} \sim 1$ pc. Therefore, our estimation for present-day magnetic field is the same because we have the same magnetic energy density $\rho_B \sim 10^{-12} M_p^4$ at the end of inflation.

We would like to explain how the kind of coupling i.e. either kinetic coupling of the form $f(\phi) F_{\mu\nu} F^{\mu\nu}$ or axial coupling of the form $f(\phi) F_{\mu\nu} \tilde{F}^{\mu\nu}$ can make influence of creating magnetic field. In kinetic coupling because energy density of electric field at the beginning of inflation is much stronger than energy density of magnetic field one expects the generated magnetic field and its evolution should be quite different from axial coupling where both energy densities in this case are close to each other.

In addition, the form of coupling function itself is very important too. For instance we see in Refs. [49,60] the authors use Starobinsky potential with kinetic coupling method but with different coupling functions, and they obtain different results. In our work if we use kinetic coupling with the same coupling function [see Eq. (42)] the generated magnetic field will be negligibly small.

One may ask that we can change our method and use another coupling function in order to produce strong enough magnetic field for observed large-scale magnetic field but the problem is, the form of coupling function in this work comes from conformal transformation we do [see Eqs. (35), (36)]. In addition, the most important reason is, due to allowed values of $A_I$ and form of kinetic coupling our potential appears to be very close to Starobinsky potential which does not produce strong electromagnetic field. Therefore, what is new here, is using modified potential i.e. Eq. (24) and axial coupling with coupling parameter $\chi_1$ which all together is producing strong enough magnetic field for observed large-scale magnetic field with the same coupling function but with different coupling parameter $\chi_1$ through axial coupling.

Let us discuses about Fig. 6 and why in comparison to Refs. [23,26,46,58,60] and of course other works in the literature, our work is new? To the best of our knowledge
this potential and of course Higgs field [see Eq. (24)] have not been studied in the literature before for magnetogenesis. This potential [Eq. (24)] is applicable [unlike the well-known potential of Ref. [41], i.e. Eq. (25)] for whole period of inflation and it is more accurate potential due to one loop correction especially when one uses axial coupling. In addition, the method is quite different. We use axial coupling whereas they use kinetic coupling, and most of them use power law coupling function whereas we use Eq. (42) in which we obtain from conformal transformation due to Higgs model. None of above authors use Higgs field in order to produce electromagnetic fields.

7 Conclusions

In this work we studied the generation of magnetic fields in radiatively Higgs inflation model. We used one loop quantum correction to the potential of Eq. (11). In comparison to the potential in Ref. [41] [see Eq. (25)]. We obtained more accurate potential which is applicable for whole period of inflation. The well known potential [Eq. (25)] is not applicable at the end of inflation whereas our potential is applicable. We used the axial coupling in order to break the conformal invariance of the Maxwell action and produce a strong magnetic field. We estimated the values $-4 < A_I < 10$ for which our results are compatible with the Planck data [2]. We numerically solved Eq. (48).

Due to the axial coupling of the electromagnetic field with the inflation the mode of one polarization ($A_+\,$ in our case, see the blue solid line in Fig. 4) undergoes amplification, while the other one ($A_-\,$, see the red dashed line in Fig. 4) diminishes. Therefore, the electromagnetic field with nontrivial helicity is generated.

We found that the power spectra of the generated electric and magnetic fields at the end of inflation are blue with the spectral index very close to the unperturbed value $n_B \simeq 4$. However, the amplitude of power spectra strongly depends on the value of coupling parameter $\chi_1$. For large values of $\chi_1$, we found that the back-reaction occurs. However, for two values, $\chi_1 = 5 \times 10^9 M_p^{-2}$ and $\chi_1 = 7.5 \times 10^9 M_p^{-2}$, the back-reaction is weak and our analysis is valid.

One may argue that in Ref. [47] the authors show magnetogenesis is not possible because of the strong back reaction of the produced fields on the inflation.

The reason for back-reaction problem in Ref. [47] is, the form of potential due to flatness of the Starobinsky potential. Due to flatness of Starobinsky potential, the coupling function changes very slowly during inflation so that the form of the power spectrum of electromagnetic field in the amplification domain is not essentially modified with respect to to the vacuum case [47].

For this reason the observed large-scale magnetic field cannot be explained. However, in this paper, the effect of quantum correction modifies potential so that in some certain condition when back reaction is weak the observed large-scale magnetic field can be explained by potential of Eq. (24). Therefore, one loop correction can change dynamic of magnetogenesis in this model. In fact, the potential of Eq. (24) has a maximum and due to allowed values of $A_I$ and shape of potential the field slow-rolls from left of maximum value and goes to its true minimum. Therefore, in some certain conditions as shown in Figs. 5 and 6 strong magnetic field is produced. This is the effect of quantum correction on Eq. (11).
In addition, we found that, first of all, the electric energy density is always greater than the magnetic one. Therefore, strong electric fields are generated during inflation together with magnetic ones and the Schwinger pair production may be important for magnetogenesis. Second, since we avoided the back-reaction problem, our calculations and the method are applicable only when the electromagnetic energy density is much less than that of the inflation. At $t \approx 9.7 \times 10^6 M_p^{-1}$ the electric energy density becomes equal to that of the inflation and the back-reaction regime occurs.

We found that the maximal possible energy density of magnetic field generated during inflation can be estimated as $\rho_{B,\text{max}} \approx 5 \times 10^{-12} M_p^4$. This estimate is done at the time when the back-reaction occurs. Finally when we set $A_I = 0$ our potential [Eq. (24)] reduces to potential [Eq. (11)] and amplification of magnetic field is negligible. Finally, we should emphasize that axial coupling does not contradict observations now and there is no principal requirement for parity to be conserved. Because $F_{\mu\nu} \tilde{F}^{\mu\nu} = -4 (E \cdot B)$ vanishes when $E$ disappears after reheating. In addition, this theory does not contain so-called ghost degrees of freedom because quantum correction is small enough to create new dynamic for generating magnetic field (ghost degrees of freedom is important when EM field is very strong). Therefore, we solved Einstein equation (37) for linear approximation of electromagnetic fields.

In summary, in this work the potential is new and valid for whole stage of inflation [see Eq. (24)]. New geometry comes from conformal transformation so that the curvature $\mathcal{R}$ in interaction term [see Eqs. (35), (36)] can efficiently excite the gauge field. Moreover, the form of the coupling to the gauge field is different. In this model, direct coupling to photon field is impossible.

Acknowledgements The author is thankful to S. Vilchinskii, E.V. Gorbar, and O. Sobol for critical comments and useful discussions during the preparation of manuscript. The author is also thankful to O. Sobol for his assistance in plotting figures.

References

1. Ade, P.A.R., et al.: (Planck Collaboration): Planck 2015 results. XIX. Constraints on primordial magnetic fields. Astron. Astrophys. 594, A19 (2016)
2. Aghanim, N. et al. (Planck Collaboration): Planck 2018 results. VI. Cosmological parameters. arXiv:1807.06209v1
3. Barvinsky, A., Kamenshchik, A.Y., Starobinsky, A.: Inflation scenario via the standard model Higgs boson and LHC. JCAP 0811, 021 (2008)
4. Barvinsky, A.O., Kamenshchik, A.Y., Kiefer, C., Starobinsky, A.A., Steinwachs, C.: Asymptotic freedom in inflationary cosmology with a non-minimal coupled Higgs field. JCAP 0912, 003 (2009)
5. Barvinsky, A.O., Shaposhnikov, M.: Standard Model Higgs boson mass from inflation: two loop analysis. JHEP 07, 089 (2009)
6. Barvinsky, A.O., Kamenshchik, A.Y., Kiefer, C., Starobinsky, A.A., Steinwachs, C.F.: Higgs boson, renormalization group, and naturalness in cosmology. arXiv:0910.1041v3 [hep-ph] (2012)
7. Bamba, K., Yokoyama, J.: Large scale magnetic fields from inflation in dilaton electromagnetism. Phys. Rev. D 69, 043507 (2004)
8. Bezrukov, F.L., Shaposhnikov, M.: The standard model Higgs boson as the inflation. Phys. Lett. B 659, 703706 (2008)
9. Bezrukov, F., Gorbunov, D., Shaposhnikov, M.: On the initial conditions for the Hot Big Bang. JCAP 0906, 029 (2009)
10. Bezrukov, F., Rubio, J., Shaposhnikov, M.: Living beyond the edge: Higgs inflation and vacuum metastability. Phys. Rev. D 92, 88 (2009)
11. Bezrukov, F., Shaposhnikov, M.: Standard model Higgs boson mass from inflation: two loop analysis. arXiv:0904.1537v2 [hep-ph] (2009)
12. Bezrukov, F.L., Magnin, A., Shaposhnikov, M.: Standard model Higgs boson mass from inflation. Phys. Lett. B 675, 703 (2009)
13. Birrell, N.D., Davies, P.C.W.: Quantum Fields in Curved Space. Cambridge University Press, Cambridge (1982)
14. Buttazzo, D., Degrassi, G., Giardino, P.P., Giudice, G.F., Sala, F., Salvio, A., Strumia, A.: Investigating the near-criticality of the Higgs boson. JHEP 12(089), 089 (2013)
15. Caprini, C., Gabici, S.: Gamma-ray observations of blazars and the intergalactic magnetic field spectrum. Phys. Rev. D 91, 123514 (2015)
16. de Simone, A., Hertzberg, M.P., Wilczek, F.: Running inflation in the standard model. Phys. Lett. B 678, 1 (2008)
17. Demozi, V., Mukhanov, V.M., Rubinstein, H.: Magnetic fields from inflation? J. Cosmol. Astropart. Phys. 08, 025 (2009)
18. Dolgov, A.D.: Breaking of conformal invariance and electromagnetic field generation in the universe. Phys. Rev. D 48, 2499 (1993)
19. Durrer, R., Neronov, A.: Cosmological magnetic fields: their generation, evolution and observation. Astron. Astrophys. Rev. 21, 62 (2013)
20. Durrer, R., Hollenstein, L., Kumar Jain, R.: Can slow roll inflation induce relevant helical magnetic fields. JCAP 03(037), 037 (2011)
21. Faraoni, V., Gunzig, E., Nardone, P.: Conformal transformations in classical gravitational theories and in cosmology. arXiv:gr-qc/9811047v1
22. Ferreira, R.J.Z., Jain, R.K., Sloth, M.S.: Inflationary magnetogenesis without the strong coupling problem. J. Cosmol. Astropart. Phys. 10, 004 (2013)
23. Ferreira, R.J.Z., Jain, R.K., Sloth, M.S.: Inflationary magnetogenesis without the strong coupling problem II: Constraints from CMB anisotropies and B-modes. J. Cosmol. Astropart. Phys. 06, 053 (2014)
24. Garcia-Bellido, J., Figueroa, D.G., Rubio, J.: Preheating in the standard model with the Higgs-inflation coupled to gravity. Phys. Rev. D 79, 063531 (2009)
25. Garretson, W.D., Field, G.B., Carroll, S.M.: Primordial magnetic fields from pseudo Goldstone bosons. Phys. Rev. D 46, 5346 (1992)
26. Giovannini, M.: On the variation of the gauge couplings during inflation. Phys. Rev. D 64, 061301 (2001)
27. Giovannini, M.: The magnetized universe. Int. J. Mod. Phys. D 13, 391 (2004)
28. Gorbunov, D.S., Rubakov, V.A.: Introduction to the Theory of the Early Universe: Cosmological Perturbations and Inflationary Theory. World Scientific Publishing, Singapore (2011)
29. Grasso, D., Rubinstein, H.R.: Magnetic fields in the early universe. Phys. Rep. 348, 163 (2001)
30. Guth, A.: The Inflationary Universe: The Quest for a New Theory of Cosmic Origins. Perseus Books (1997)
31. Jedamzik, K., Saveliev, A.: A stringent limit on primordial magnetic fields from the cosmic microwave background radiation. arXiv:1804.06115 [astro-ph.CO]
32. Kandus, A., Kunze, K.E., Tsagas, C.G.: Primordial magnetogenesis. Phys. Rep. 505, 1 (2011)
33. Kanno, S., Soda, J., Watanabe, M.: Cosmological magnetic fields from inflation and backreaction. J. Cosmol. Astropart. Phys. 12, 009 (2009)
34. Ketov, S.V.: Modified supergravity and early universe: the meeting point of cosmology and high-energy physics. arXiv:1201.2239v3 [hep-th]
35. Ketov, S.V., Starobinsky, A.A.: Embedding $R + R^2$ inflation in supergravity. Phys. Rev. D 83, 063512 (2011)
36. Kronberg, P.P.: Extragalactic magnetic fields. Rep. Prog. Phys. 57, 325 (1994)
37. Liddle, A.R., Parsons, P., Barrow, J.D.: Formalizing the slow roll approximation in inflation. Phys. Rev. D 50, 7222 (1994)
38. Linde, A.: Particle physics and inflationary cosmology. Contemp. Concepts Phys. 5, 1 (2005)
39. Lyth, D., Liddle, A.: The Primordial Density Perturbation. Cambridge University Press, Cambridge (2009)
40. Maeda, K.I.: Towards the Einstein–Hilbert action via conformal transformation. Phys. Rev. D 39, 3159 (1989)
41. Martin, J., Ringeval, C., Vennin, V.: Encyclopedia inflationaris. Phys. Dark Univ. 75, 5–6 (2014)
42. Martin, J., Yokoyama, J.: Generation of large-scale magnetic fields in single-field inflation. JCAP 01, 025 (2008)
43. Mukhanov, V.: Physical Foundations of Cosmology. Cambridge University Press, Cambridge (2005)
44. Neronov, A., Vovk, I.: Evidence for strong extragalactic magnetic fields from Fermi observations of TeV blazars. Science 328, 73 (2010)
45. Parker, L.: Particle creation in expanding universes. Phys. Rev. Lett. 21, 562 (1968)
46. Ratra, B.: Cosmological ‘seed’ magnetic field from inflation. Astrophys. J. 391, L1 (1992)
47. Savchenko, O., Shtanov, Y.: Magnetogenesis by non-minimal coupling to gravity in the Starobinsky inflationary model. arXiv:1808.06193v1
48. Sobol, O.O., Gorbar, E.V., Vilchinskii, S.I.: Backreaction of electromagnetic fields and the Schwinger effect in pseudoscalar magnetogenesis. Phys. Rev. D 100, 063523 (2019)
49. Sobol, O.O., Gorbar, E.V., Kamarpour, M., Vilchinskii, S.I.: Influence of back-reaction of electric fields and the Schwinger effect on inflationary magnetogenesis. Phys. Rev. D 98, 063534 (2018)
50. Spokoiny, B.L.: Inflation and generation of perturbations in broken-symmetric theory of gravity. Phys. Lett. B 147, 39 (1984)
51. Steinwachs, C.F., Kamenshchik, A.Y.: Non-minimal Higgs inflation and frame dependence in cosmology. arXiv:1301.5543
52. Starobinsky, A.A.: A new type of isotropic cosmological models without singularity. Phys. Lett. B 91, 99–102 (1980)
53. Subramanian, K.: The origin, evolution and signatures of primordial magnetic fields. Rep. Prog. Phys. 79, 076901 (2016)
54. Sutton, D.R., Feng, C., Reichardt, C.L.: Current and future constraints on primordial magnetic fields. Astrophys. J. 846, 164 (2017)
55. Synge, J.L.: Relativity: The General Theory. North Holland, Amsterdam (1955)
56. Tavecchio, F., Ghisellini, G., Foschini, L., Bonnoli, G., Ghirlanda, G., Coppi, P.: The intergalactic magnetic field constrained by Fermi/LAT observations of the TeV blazar 1ES 0229+200. Mon. Not. R. Astron. Soc. 406, L70 (2010)
57. Taylor, A.M., Vovk, I., Neronov, A.: Extragalactic magnetic fields constraints from simultaneous GeV-TeV observations of blazars. Astron. Astrophys. 529, A144 (2011)
58. Turner, M.S., Widrow, L.M.: Inflation-produced, large-scale magnetic fields. Phys. Rev. D 37, 2743 (1988)
59. Vachaspati, T.: Progress on cosmological magnetic fields. arXiv:2010.10525v1 [astro-ph.CO] (2020)
60. Vilchinskii, S., Sobol, O., Gorbar, E.V., Rudennok, I.: Magnetogenesis during inflation and preheating in the Starobinsky model. Phys. Rev. D 95, 083509 (2017)
61. Wald, R.M.: General Relativity. Chicago University Press, Chicago (1984)
62. Widrow, L.M.: Origin of galactic and extragalactic magnetic fields. Rev. Mod. Phys. 74, 775 (2002)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.