Gerstenhaber algebras and BV-algebras in Poisson geometry

PING XU *
Department of Mathematics
The Pennsylvania State University
University Park, PA 16802, USA
e-mail: ping@math.psu.edu

February, 1997

Abstract

The purpose of this paper is to establish an explicit correspondence between various geometric structures on a vector bundle with some well-known algebraic structures such as Gerstenhaber algebras and BV-algebras. Some applications are discussed. In particular, we found an explicit connection between the Koszul-Brylinski operator of a Poisson manifold and its modular class. As a consequence, we prove that Poisson homology is isomorphic to Poisson cohomology for unimodular Poisson structures.

1 Introduction

BV-algebras arise from the BRST theory of topological field theory [29]. Recently, there has been a great deal of interest in these algebras in connection with various subjects such as operads and string theory [7] [8] [17] [10] [21] [24] [25] [30].

Let us first recall various relevant definitions below. Here we will follow those as in [13].

A Gerstenhaber algebra consists of a triple \( (A = \bigoplus_{i \in \mathbb{Z}} A^i, \wedge, [\cdot, \cdot]) \) such that \((A, \wedge)\) is a graded commutative associative algebra, and \((A = \bigoplus_{i \in \mathbb{Z}} A^{(i)}, [\cdot, \cdot])\), with \(A^{(i)} = A^{i+1}\), is a graded Lie algebra, and \([a, \cdot], \) for each \(a \in A^{(i)}\) is a derivation with respect to \(\wedge \) with degree \(i\).

An operator \(D\) of degree \(-1\) is said to generate the Gerstenhaber algebra bracket if for every \(a \in A^{(a)}\) and \(b \in A\),

\[
[a, b] = (-1)^{|a|}(D(a \wedge b) - D a \wedge b - (-1)^{|a|} a \wedge D b).
\]

(1)

A Gerstenhaber algebra is said to be exact if there is an operator \(D\) of square zero generating the bracket. In this case, \(D\) is called a generating operator. An exact Gerstenhaber algebra is also called a Batalin-Vilkovisky algebra (or BV-algebra in short).

*Research partially supported by NSF grant DMS95-04913.
A differential Gerstenhaber algebra is a Gerstenhaber algebra equipped with a differential $d$, which is a derivation of degree 1 with respect to $\wedge$ and $d^2 = 0$. It is called a strong differential Gerstenhaber algebra if, in addition, $d$ is derivation of the graded Lie bracket.

Kosmann-Schwarzbach noted [12] that these algebra structures had also appeared in Koszul’s work in 1985 [16] in his study of Poisson manifolds. In fact, as pointed out in [12], these examples are connected with a certain differential structure on vector bundles, called Lie algebroids by Pradines [22]. Let us recall for the benefit of the reader the definition of a Lie algebroid [22] [23].

**Definition 1.1** A Lie algebroid $A$ over a manifold $M$ is a vector bundle $A$ over $M$ together with a Lie algebra structure on the space $\Gamma(A)$ of smooth sections of $A$, and a bundle map $\alpha : A \to TP$ (called the anchor), extended to a map between sections of these bundles, such that

(i) $\alpha([X,Y]) = [\alpha(X),\alpha(Y)]$; and

(ii) $[X,fY] = f[X,Y] + (\alpha(X)f)Y$

for any smooth sections $X$ and $Y$ of $A$ and any smooth function $f$ on $M$.

Among many examples of Lie algebroids are usual Lie algebras, the tangent bundle of a manifold, and an integrable distribution over a manifold (see [19]). Recently, Lie algebroids attract increasing interest in Poisson geometry. One of the main reason is due to the following example connected with Poisson manifolds.

Let $P$ be a Poisson manifold with Poisson tensor $\pi$. Then $T^*P$ inherits a natural Lie algebroid structure, called the cotangent Lie algebroid of the Poisson manifold $P$ [4]. The anchor map $\pi^\# : T^*P \to TP$ is defined by

$$\pi^\# : T^*_pP \to T_pP : \pi^\#(\xi)(\eta) = \pi(\xi,\eta), \quad \forall \xi, \eta \in T^*_pP$$

and the Lie bracket of 1-forms $\alpha$ and $\beta$ is given by

$$[\alpha,\beta] = -d\pi(\alpha,\beta) + L_{\pi^\#(\alpha)}\beta - L_{\pi^\#(\beta)}\alpha.$$

In [12], Kosmann-Schwarzbach constructed various examples of strong differential Gerstenhaber algebras and BV-algebras in connection with Lie algebroids. Motivated by [12], in this note, we will establish a more precise relation between these algebra structures and some of the well-known geometric structures in Poisson geometry.

More precisely, we will investigate the following question: Let $A$ be a vector bundle of rank $n$ over the base $M$, and let $\mathcal{A} = \oplus_{0 \leq k \leq n}\Gamma(\wedge^k A)$. With respect to the wedge product, $\mathcal{A}$ becomes a graded commutative associative algebra. Then the question is:

What additional structure on $\mathcal{A}$ will make $\mathcal{A}$ into a Gerstenhaber algebra, a strong differential Gerstenhaber algebra, or an exact Gerstenhaber algebra (or a BV-algebra)?

The answer is surprisingly simple. Gerstenhaber algebras and strong differential Gerstenhaber algebras, correspond, exactly to the structures of Lie algebroids and Lie bialgebroids (see Section 2 for the definition), respectively, as already indicated in [12]. And an exact Gerstenhaber algebra
structure corresponds to a Lie algebroid $A$ together with a flat $A$-connection on its canonical line bundle $\wedge^n A$. This fact was implicitly contained in Koszul’s work \cite{koszul} although he treated only the case of multivector fields. However, the formulas (9) and (14) establishing the explicit correspondence seem to be new.

Below is a table of the correspondence.

| Structures on algebra $A$ | Structures on the vector bundle $A$ |
|--------------------------|-----------------------------------|
| Gerstenhaber algebras    | Lie algebroids                    |
| strong differential Gerstenhaber algebras | Lie bialgebroids |
| exact Gerstenhaber algebras (BV-algebra) | Lie algebroids with a flat $A$-connection on $\wedge^n A$ |

The content above occupies Section 2 and Section 3. Section 4 is devoted to applications. In particular, we establish an explicit connection between the Koszul-Brylinski operator on a Poisson manifold with its modular class. As a consequence, we prove that Poisson homology is isomorphic to Poisson cohomology for unimodular Poisson structures (see \cite{27} \cite{3} for the definition).

As another application, we introduce the notion of Lie algebroid homologies, which are the homology groups induced by generating operators $D : \Gamma(\wedge^* A) \longrightarrow \Gamma(\wedge^{*-1} A)$. Since a generating operator on a Lie algebroid depends on the choice of a flat $A$-connection on the canonical line bundle $\wedge^n A$, in general the homology depends on the choice of such a connection $\nabla$. When two connections are homotopic (see Section 4 for the precise definition), their corresponding homology groups are isomorphic. So for a given Lie algebroid, its homologies are in fact parameterized by the first Lie algebroid cohomology $H^1(A, \mathbb{R})$. When $A$ is a Lie algebra and $\nabla$ is the trivial connection, this reduces to the usual Lie algebra homology with trivial coefficients. On the other hand, Poisson homology can also be considered as a special case of Lie algebroid homology, when $A$ is taken as the cotangent Lie algebroid of a Poisson manifold.

We note that in a recent preprint \cite{5}, Evens, Lu and Weinstein have also established a connection between Poisson homology and the modular class of Poisson manifolds. Finally we also would like to refer the reader to a recent preprint \cite{9} of Huebschmann for its close connection with the present paper.

Acknowledgments. The author would like to thank Jean-Luc Brylinski, Jiang-hua Lu, and Alan Weinstein for useful discussions. He is especially grateful to Yvette Kosmann-Schwarzbach and Jim Stasheff for providing many helpful comments on the first draft of the manuscript. In addition to the funding sources mentioned in the first footnote, he would also like to thank IHES and Max-Planck Institut for their hospitality and financial support while part of this project was being done.
Gerstenhaber algebras and differential Gerstenhaber algebras

In this section, we will treat Gerstenhaber algebras and differential Gerstenhaber algebras arising from a vector bundle.

Again, let \( A \) be a vector bundle of rank \( n \) over \( M \), and let \( \mathcal{A} = \bigoplus_{0 \leq k \leq n} \Gamma(\wedge^k A) \). The following proposition establishes a one-one correspondence between Gerstenhaber algebra structures on \( \mathcal{A} \) and Lie algebroid structures on the vector bundle \( A \).

**Proposition 2.1** \( \mathcal{A} \) is a Gerstenhaber algebra iff \( A \) is a Lie algebroid.

This is a well-known result (see [6] [15] [20]). For completeness, we will sketch a proof below.

**Proof.** Suppose that there is a graded Lie bracket \( [\cdot, \cdot] \) that makes \( \mathcal{A} \) into a Gerstenhaber algebra. It is clear that \( (\Gamma(A), [\cdot, \cdot]) \) is a Lie algebra.

Second, for any \( X \in \Gamma(A) \) and \( f, g \in C^\infty(M) \), it follows from the derivation property that

\[
[X, fg] = [X, f]g + f[X, g].
\]

Hence, \( [X, \cdot] \) defines a vector field on \( M \), which will be denoted by \( a(X) \). It is easy to see that \( a \) is in fact induced by a bundle map from \( A \) to \( TP \). By applying the graded Jacobi identity, it follows that

\[
a([X, Y]) = [a(X), a(Y)].
\]

Finally, again from the derivation property, it follows that

\[
[X, fY] = (a(X)f)Y + f[X, Y].
\]

This shows that \( A \) is indeed a Lie algebroid.

Conversely, given a Lie algebroid \( A \), it is easy to check that \( \mathcal{A} = \bigoplus_{0 \leq k \leq n} \Gamma(\wedge^k A) \) forms a Gerstenhaber algebra (see [12] [20]).

\[\square\]

The following lemma gives an alternative way of characterizing a Lie algebroid, which should be of interest itself.

Recall that a *differential graded algebra* is a graded commutative associative algebra equipped with a differential \( d \), which is a derivation of degree 1 and of square zero.

**Lemma 2.2** [7] [14] Given a vector bundle \( A \) over \( M \). \( A \) is a Lie algebroid iff \( \wedge^* A^* \) is a differential graded algebra.
Proof. Given a Lie algebroid $A$, it is known that the space of sections $\Gamma(\wedge^* A^*)$ admits a differential $d$ that makes it into a differential graded algebra $[13]$. In this case, $d : \Gamma(\wedge^k A^*) \rightarrow \Gamma(\wedge^{k+1} A^*)$ is simply the differential defining the Lie algebroid cohomology given as below (see $[15]$ $[20]$ $[28]$):

$$d\omega(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} a(X_i)(\omega(X_1, \ldots, X_{k+1}))$$

$$+ \sum_{i < j} (-1)^{i+j}\omega([X_i, X_j], X_1, \ldots, X_k),$$

(4)

for $\omega \in \Gamma(\wedge^k A^*)$, $X_i \in \Gamma A$, $1 \leq i \leq k+1$.

Conversely, if $\Gamma(\wedge^* A^*)$ is a differential graded algebra with differential $d$, then the equations:

$$a(X)f = \langle df, X \rangle, \quad \forall f \in C^\infty(M) \text{ and } X \in \Gamma(A),$$

(5)

and

$$\langle [X, Y], \theta \rangle = a(X)(\theta \cdot Y) - a(Y)(\theta \cdot X) - (d\theta)(X, Y)$$

(6)

define the anchor map and the Lie bracket of a Lie algebroid structure on $A$.

\[\square\]

Remark. The lemma above is essentially Proposition 6.1 of $[15]$. Equation (3) is Formula (6.6) in $[15]$.

Recall that a *Lie bialgebroid* $[12]$ $[20]$ is a dual pair $(A, A^*)$ of vector bundles equipped with Lie algebroid structures such that the differential $d_*$, induced from the Lie algebroid structure on $A^*$ as defined by Equation (4), is a derivation of the Lie bracket on $\Gamma(A)$, i.e.,

$$d_*[X, Y] = [d_*X, Y] + [X, d_*Y], \quad \forall X, Y \in \Gamma(A).$$

(7)

The following result is due to Kosmann-Schwarzbach $[12]$.

Proposition 2.3 $A$ is a strong differential Gerstenhaber algebra iff $A$ is a Lie bialgebroid.

Proof. Assume that $A$ is a strong differential Gerstenhaber algebra. Then, $A^*$ is a Lie algebroid according to Lemma 2.2. Moreover, the derivation property of the differential with respect to the Lie bracket on $\Gamma(A)$ implies that $(A^*, A)$ is a Lie bialgebroid. This is equivalent to that $(A, A^*)$ is a Lie bialgebroid by duality $[20]$. Conversely, it is straightforward to see, for a given Lie bialgebroid $(A, A^*)$, that $A$ is a strong differential Gerstenhaber algebra (see $[12]$).

\[\square\]
Example 2.4 Let \( P \) be a Poisson manifold with Poisson tensor \( \pi \). Let \( A = TP \) with the tangent algebroid structure. It is well known that the space of multivector fields \( A = \oplus (\wedge^k TP) \) has a Gerstenhaber algebra structure, where the graded Lie bracket is called the Schouten bracket.

In 1977, Lichnerowicz introduced a differential operator \( d = [\pi, \cdot] \), which he used to define the Poisson cohomology \([18]\). It is obvious that \( A \) becomes a strong differential Gerstenhaber algebra, so it should correspond to a Lie bialgebroid structure on \((TP, T^*P)\) according to Proposition 2.3. It is, however, quite amazing that the Lie algebroid structure on \( T^*P \) was not known until the middle of 1980’s (see \([14]\) for the references) and the Lie bialgebroid structure comes much later! For the Lie algebroid \( T^*P \), the associated differential operator on \( \Gamma(\wedge^* T^*P) \) is the Lichnerowicz differential \( d = [\pi, \cdot] \). This property was proved, independently by Bhaskara and Viswanath \([1]\), and Kosmann-Schwarzbach and Magri \([15]\).

3 Exact Gerstenhaber algebra structures on the exterior algebra of a vector bundle

In this section, we will move to exact Gerstenhaber algebras arising from a vector bundle.

Let \( A \rightarrow M \) be a Lie algebroid with anchor \( a \) and \( E \rightarrow M \) a vector bundle over \( M \). By an \( A \)-connection on \( E \), we mean a \( \mathbb{R} \)-linear map:

\[
\Gamma(A) \otimes \Gamma(E) \rightarrow \Gamma(E)
\]

\[
X \otimes s \rightarrow \nabla_X s
\]

satisfying the axioms resembling those of the usual linear connections, i.e., \( \forall f \in C^\infty(M), X \in \Gamma(A), s \in \Gamma(E), \)

\[
\nabla_{fX}s = f \nabla_X s; \quad \nabla_X(fs) = (a(X)f)s + f \nabla_X s.
\]

Similarly, the curvature \( R \) of an \( A \)-connection \( \nabla \) is the element in \( \Gamma(\wedge^2 A^*) \otimes \text{End}(E) \) defined by

\[
R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}, \quad \forall X,Y \in \Gamma(A). \tag{8}
\]

Given a Lie algebroid \( A \) and an \( A \)-connection \( \nabla \) on the canonical line bundle \( E = \wedge^n A \), define a differential operator \( D : \Gamma(\wedge^k A) \rightarrow \Gamma(\wedge^{k-1} A) \) as follows. Let \( U \) be any section in \( \Gamma(\wedge^k A) \) and write, locally, \( U = \omega \mathbf{J} \Lambda \), where \( \omega \in \Gamma(\wedge^{n-k} A^*) \) and \( \Lambda \in \Gamma(\wedge^n A) \). At each point \( m \in M \), set

\[
DU|_m = (-1)^{\left| \omega \right|} (d\omega \mathbf{J} \Lambda + \sum_{i=1}^n (\alpha_i \wedge \omega) \mathbf{J} \nabla_X \Lambda), \tag{9}
\]

where \( X_1, \ldots, X_n \) is a basis of \( A|_m \) and \( \alpha_1, \ldots, \alpha_n \) its dual basis in \( A^*|_m \). Clearly, this definition is independent of the choice of the basis.
Remark We would like to make a remark on the notation. Let $E$ be a vector bundle over $M$. Assume that $V \in \Gamma(\wedge^k E)$ and $\theta \in (\wedge^l E^*)$ with $k \geq l$. Then, by $\theta \llcorner V$ we denote the section of $\wedge^{k-l} E$ given by

$$(\theta \llcorner V)(\omega) = V(\theta \wedge \omega), \quad \forall \omega \in \Gamma(\wedge^{k-l} E^*).$$

We will stick to this notation in the sequel no matter whether $E$ is the Lie algebroid $A$ itself or its dual $A^*$.

**Proposition 3.1** $D$ is a well-defined operator and

$$D^2 U = -R \llcorner U,$$

where $R \in \Gamma(\wedge^2 A^*)$ is the curvature of the connection $\nabla$ (note that $\text{End} E$ is a trivial line bundle).

**Proof.** Suppose that $f$ is a locally nonzero function on $M$, and $U = f \omega \llcorner \frac{1}{f} \Lambda$. Then,

$$d(f \omega) \llcorner \frac{1}{f} \Lambda + \sum_i (\alpha_i \wedge f \omega) \llcorner \nabla_{X_i} \left( \frac{1}{f} \Lambda \right)$$

$$= \frac{1}{f} (df \wedge \omega) \llcorner \Lambda + d\omega \llcorner \Lambda$$

$$+ \sum_i f((aX_i)\frac{1}{f})(\alpha_i \wedge \omega) \llcorner \Lambda + \sum_i (\alpha_i \wedge \omega) \llcorner \nabla_{X_i} \Lambda$$

$$= \frac{1}{f} (df \wedge \omega) \llcorner \Lambda + f(\frac{1}{f}) \wedge \omega \llcorner \Lambda + d\omega \llcorner \Lambda + \sum_i (\alpha_i \wedge \omega) \llcorner \nabla_{X_i} \Lambda$$

$$= d\omega \llcorner \Lambda + \sum_i (\alpha_i \wedge \omega) \llcorner \nabla_{X_i} \Lambda.$$

Therefore, $D$ is well-defined.

For the second part, we have

$$D^2 U = -(-1)^{|\omega|} D(d\omega \llcorner \Lambda + \sum_{i=1}^n (\alpha_i \wedge \omega) \llcorner \nabla X_i \Lambda)$$

$$= - \left( \sum_i (\alpha_i \wedge d\omega) \llcorner \nabla X_i \Lambda + \sum_i d(\alpha_i \wedge \omega) \llcorner \nabla X_i \Lambda + \sum_{j,i} (\alpha_j \wedge \alpha_i \wedge \omega) \llcorner \nabla X_i \nabla X_j \Lambda \right)$$

$$= - \left[ \sum_i \omega \llcorner (d\alpha_i \llcorner \nabla X_i \Lambda) + \sum_{j,i} \omega \llcorner (\alpha_j \wedge \alpha_i \llcorner \nabla X_i \nabla X_j \Lambda) \right]$$

The conclusion thus follows from the following

**Lemma 3.2**

$$\sum_i d\alpha_i \llcorner \nabla X_i \Lambda + \sum_{j,i} (\alpha_i \wedge \alpha_j) \llcorner \nabla X_i \nabla X_j \Lambda = -R \llcorner \Lambda.$$
Proof. It is a straightforward verification, and is left to the readers. \hfill \blacksquare

Proposition 3.3 Let $D : \Gamma(\wedge^k A) \rightarrow \Gamma(\wedge^{k-1} A)$ be the operator defined in Equation (9). Then, $D$ generates the Gerstenhaber algebra bracket on $\Gamma(\wedge^* A)$, i.e., for any $U \in \Gamma(\wedge^u A)$ and $V \in \Gamma(\wedge^v A)$,

$$[U, V] = (-1)^{u}(D(U \wedge V) - DU \wedge V - (-1)^{u}U \wedge DV).$$

(10)

We need a couple of lemmas before proving this proposition.

Lemma 3.4 For any $U \in \Gamma(\wedge^u A)$, $V \in \Gamma(\wedge^v A)$ and $\theta \in \Gamma(\wedge^{u+v-1} A^*)$,

$$[U, V] \lhd \theta = (-1)^{(u-1)(v-1)}U \lhd d(V \lhd \theta) - V \lhd d(U \lhd \theta) - (-1)^{u+1}U \wedge V \lhd d\theta.$$  

(11)

Proof. See Equation (1.16) in [26].

Lemma 3.5 For any $U \in \Gamma(\wedge^u A)$ and $\theta \in \Gamma(\wedge^{u-1} A^*)$,

$$\theta \lhd DU = (-1)^{|\theta|}D(\theta \lhd U) + d\theta \lhd U.$$  

(12)

Proof. Assume that $U = \omega \lhd \Lambda$. Then, $\theta \lhd U = (\omega \wedge \theta) \lhd \Lambda$, and therefore,

$$D(\theta \lhd U) = (-1)^{|\omega|+|\theta|}(d(\omega \wedge \theta) \lhd \Lambda + \sum_{i} (\alpha_i \wedge \omega \wedge \theta) \lhd \nabla_{X_i} \Lambda)$$

$$= (-1)^{|\omega|+|\theta|}(d\omega \wedge \Lambda + (-1)^{|\omega|}(\omega \wedge d\theta) \lhd \Lambda + \sum_{i} (\alpha_i \wedge \omega \wedge \theta) \lhd \nabla_{X_i} \Lambda)$$

$$= (-1)^{|\theta|}(\theta \lhd DU) - (-1)^{|\theta|}d\theta \lhd U.$$  

\hfill \Box

Proof of Proposition 3.3 For any $U \in \Gamma(\wedge^u A)$, $V \in \Gamma(\wedge^v A)$ and $\theta \in \Gamma(\wedge^{u+v-1} A^*)$, using Equation (12), we have

$$\theta \lhd D(U \wedge V) = (-1)^{|\theta|}D(\theta \lhd (U \wedge V)) + d\theta \lhd (U \wedge V).$$

On the other hand, we have

$$\theta \lhd (U \wedge DV) = (U \lhd \theta) \lhd DV$$

$$= (-1)^{|\theta|-u}D((U \lhd \theta) \lhd V) + d(U \lhd \theta) \lhd V,$$

and
\[
\theta \downarrow (DU \wedge V) \\
= (-1)^{(u-1)v} \theta \downarrow (V \wedge DU) \\
= (-1)^{(u-1)v} ((-1)^{|[\theta]| - v} D((V \downarrow \theta) \downarrow U) + d(V \downarrow \theta) \downarrow U) \\
= (-1)^{uv + |\theta|} D((V \downarrow \theta) \downarrow U) + (-1)^{(u-1)v} d(V \downarrow \theta) \downarrow U).
\]

It thus follows that
\[
\theta \downarrow ((-1)^u D(U \wedge V) - (-1)^u DU \wedge V - U \wedge DV) \\
= (-1)^{|[\theta]| + u} D(\theta \downarrow (U \wedge V)) - (-1)^{u + |\theta| + u} D((V \downarrow \theta) \downarrow U) - (-1)^{|[\theta]| - u} D((U \downarrow \theta) \downarrow V) \\
+ (-1)^u d\theta \downarrow (U \wedge V) + (-1)^{(u-1)(v-1)} d(V \downarrow \theta) \downarrow U - d(U \downarrow \theta) \downarrow V.
\]

The conclusion thus follows from Equation (11) and the following formula:
\[
\theta \downarrow (U \wedge V) = (U \downarrow \theta) \downarrow V + (-1)^{uv} (V \downarrow \theta) \downarrow U.
\] (13)

\[\Box\]

Conversely, the connection \(\nabla\) can be readily recovered from the operator \(D\). More precisely, we have

**Proposition 3.6** Suppose that \(D : \Gamma(\wedge^k A) \rightarrow \Gamma(\wedge^{k-1} A)\) is the operator corresponding to an \(A\)-connection \(\nabla\) on \(\wedge^n A\). Then, for any \(X \in \Gamma(A)\) and \(\Lambda \in \Gamma(\wedge^n A)\),
\[
\nabla_X \Lambda = -X \wedge DA.
\] (14)

**Proof.** By definition, \(DA = -\alpha_i \downarrow \nabla X_i \Lambda\). Hence,
\[
-X \wedge DA = \sum_i X \wedge (\alpha_i \downarrow \nabla X_i \Lambda) \\
= \sum_i \alpha_i(X) \nabla X_i \Lambda \\
= \nabla_X \Lambda,
\]

where the last equality uses the identity: \(X = \sum_i \alpha_i(X)X_i\), and the second equality follows from the following simple fact in linear algebra:

**Lemma 3.7** Let \(V\) be any vector space, \(X \in V\), \(\alpha \in V^*\) and \(\Lambda \in \wedge^n V\). Then,
\[
X \wedge (\alpha \downarrow \Lambda) = \alpha(X) \Lambda.
\] \[\Box\]
Theorem 3.8 Let $A$ be a Lie algebroid with anchor $a$, and $A = \oplus_i \Gamma(\Lambda^i A)$ its corresponding Gerstenhaber algebra. There is a one-to-one correspondence between algebroid $A$-connections on $E \cong \Lambda^n A$ and linear operators $D$ generating the Gerstenhaber algebra bracket on $A$. Under this correspondence, flat connections correspond to operators of square zero.

Proof. It remains to prove that Equation (P4) indeed defines an $A$-connection on $\Lambda^n A$ if $D$ is an operator generating the Gerstenhaber algebra bracket.

First, it is clear that, with this definition, $\nabla fX\Lambda = f\nabla X\Lambda$ for any $f \in C^\infty(M)$.

To prove that it satisfies the second axiom of a linear connection, we observe that for any $f \in C^\infty(M)$, and $\Lambda \in \Gamma(\Lambda^n A)$,

$$D(f\Lambda) = (Df)\Lambda + fD\Lambda + [f,\Lambda] = fD\Lambda + [f,\Lambda].$$

Hence,

$$\nabla_X(f\Lambda) = -X \wedge D(f\Lambda) = -X \wedge (fD\Lambda + [f,\Lambda]) = f\nabla_X\Lambda - X \wedge [f,\Lambda].$$

On the other hand, using the property of Gerstenhaber algebras,

$$[f, X \wedge \Lambda] = [f, X] \wedge \Lambda + (-1)X \wedge [f,\Lambda] = -(a(X)f)\Lambda - X \wedge [f,\Lambda].$$

Thus, $X \wedge [f,\Lambda] = -(a(X)f)\Lambda$. Hence, $\nabla_X(f\Lambda) = f\nabla_X\Lambda + (a(X)f)\Lambda$.

\[\square\]

A flat $A$-connection always exists on the line bundle $E = \Lambda^n A$. To see this, note that $E \otimes E$ is a trivial line bundle, which always admits a flat connection. So the “square root” of this connection (see Proposition 4.3 in [5]) is a flat connection we need. Therefore, for a given Lie algebroid, there always exists an operator of degree $-1$ and of square zero generating the corresponding Gerstenhaber algebra. Such an operator is called a generating operator.

Any $A$-connection $\nabla$ on the Lie algebroid $A$ itself induces an $A$-connection on the line bundle $E = \Lambda^n A$. Therefore, it corresponds to a linear operator $D$ generating the Gerstenhaber algebra $A$. In particular, if it is torsion free, i.e.,

$$\nabla_X Y - \nabla_Y X = [X, Y], \quad \forall X, Y \in \Gamma(A),$$

$D$ possesses a simpler expression. Note that $\nabla$ induces an $A$-connection on the dual bundle $A^*$, and its exterior powers, which is denoted by the same symbol.
Proposition 3.9 Suppose that $\nabla$ is a torsion free $A$-connection on $A$. Let $D : \Gamma(\wedge^* A) \to \Gamma(\wedge^{*-1} A)$ be the induced operator. Then, for any $U \in \Gamma(\wedge^u A)$,
\[
DU|_m = - \sum_i \alpha_i \mathcal{D}X_i U,
\]
where $X_1, \ldots, X_n$ is a basis of $A|_m$ and $\alpha_1, \ldots, \alpha_n$ the dual basis of $A^*_|_m$.

**Proof.** Assume that $U = \omega \mathcal{D}\Lambda$ for some $\Lambda \in \Gamma(\wedge^n A)$ and $\omega \in \Gamma(\wedge^{n-u} A^*)$. Then,
\[
\sum_i \alpha_i \mathcal{D}X_i (\omega \mathcal{D}\Lambda) = \sum_i \alpha_i \mathcal{D}[\mathcal{D}X_i \omega \mathcal{D}\Lambda + \omega \mathcal{D}X_i \Lambda]
\]
\[
= \sum_i [(\mathcal{D}X_i \omega \wedge \alpha_i) \mathcal{D}\Lambda + (\omega \wedge \alpha_i) \mathcal{D}X_i \Lambda]
\]
\[
= (-1)^{|\omega|} (\sum_i (\alpha_i \wedge \mathcal{D}X_i \omega) \mathcal{D}\Lambda + \sum_i (\alpha_i \wedge \omega) \mathcal{D}X_i \Lambda).
\]

The conclusion thus follows from the following

**Lemma 3.10** For any $\omega \in \Gamma(\wedge^{1,0} A^*)$,
\[
d\omega = \sum_i \alpha_i \wedge \mathcal{D}X_i \omega.
\]

**Proof.** Define an operator $\delta : \Gamma(\wedge^k A^*) \to \Gamma(\wedge^{k+1} A^*)$, for all $0 \leq k \leq n$, by
\[
\delta \omega = \sum_i \alpha_i \wedge \mathcal{D}X_i \omega.
\]
It is simple to check that $\delta$ is a derivation with respect to the wedge product, i.e.,
\[
\delta (\omega \wedge \theta) = \delta \omega \wedge \theta + (-1)^{|\omega|} \omega \wedge \delta \theta.
\]

For any $f \in C^\infty(M)$,
\[
\delta f = \sum_i \alpha_i \mathcal{D}X_i f = \sum_i [a(X_i)f]\alpha_i = df.
\]

For any $\theta \in \Gamma(A^*)$ and $X, Y \in \Gamma(A)$,
\[
(\delta \theta)(X, Y) = \sum_i (\alpha_i \wedge X_i \theta)(X, Y)
\]
\[
= \sum_i \alpha_i(X)(\mathcal{D}X_i \theta)(Y) - \alpha_i(Y)(\mathcal{D}X_i \theta)(X)
\]
\[
= \sum_i \alpha_i(X)(\mathcal{D}X_i (\theta \cdot Y) - \theta \cdot \mathcal{D}X_i Y) - \sum_i \alpha_i(Y)(\mathcal{D}X_i (\theta \cdot X) - \theta \cdot \mathcal{D}X_i X)
\]
\[
= \sum_i \alpha_i(X)(a(X_i)(\theta \cdot Y) - \theta \cdot \mathcal{D}X_i Y) - \sum_i \alpha_i(Y)(a(X_i)(\theta \cdot X) - \theta \cdot \mathcal{D}X_i X)
\]
\[
= a(X)(\theta \cdot Y) - \theta \cdot \mathcal{D}X Y - a(Y)(\theta \cdot X) + \theta \cdot \mathcal{D}Y X
\]
\[
= a(X)(\theta \cdot Y) - a(Y)(\theta \cdot X) - \theta \cdot (\mathcal{D}X Y - \mathcal{D}Y X)
\]
\[
= a(X)(\theta \cdot Y) - a(Y)(\theta \cdot X) - \theta \cdot [X, Y]
\]
\[
= d\theta(X, Y).
\]
Therefore, $\delta$ coincides with the exterior derivative $d$, since $\Gamma(\wedge^*A)$ is generated by $\Gamma(A^*)$ over the module $\mathcal{C}^\infty(M)$.

\[\square\]

**Remark** (1) Theorem 3.8 was proved by Koszul for the case of the tangent bundle Lie algebroid $TP$ [16]. In fact, his result was the main motivation of our work here. However, Koszul used an indirect argument instead of using Equations (9) and (14). We will see more applications of these equations in the next section.

(2) A flat $A$-connection on a vector bundle $E$ is also called a representation of the Lie algebroid by Mackenzie [19].

We end this section by introducing the notion of generalized divergence. Let $\nabla$ be a flat $A$-connection on $\wedge^nA$, and $D$ its corresponding generating operator. For any section $X \in \Gamma(A)$, we use $\text{div}_\nabla X$ to denote the function $DX$. When $A = TP$ with the usual Lie algebroid structure, and $\nabla$ is the flat connection induced by a volume, $DX$ is the divergence in the ordinary sense. So $DX$ can be indeed considered as a generalized divergence.

The following proposition gives a simple geometric characterization for the divergence of a section $X$ of $A$.

**Proposition 3.11** For any $X \in \Gamma(A)$ and $\Lambda \in \Gamma(\wedge^nA)$,

$$L_X \Lambda - \nabla_X \Lambda = (\text{div}_\nabla X)\Lambda.$$  

In other words, $\text{div}_\nabla X$ is the function on $M$ defining the bundle map $L_X - \nabla_X$ of the line bundle $\wedge^nA$.

**Proof.** Assume that $X = \omega \lrcorner \Lambda$ for some $\omega \in \Gamma(\wedge^{n-1}A^*)$. Then,

$$DX = -(-1)^{|\omega|}(d\omega \lrcorner \Lambda + \sum_{i=1}^n (\alpha_i \wedge \omega) \lrcorner \nabla X_i \Lambda).$$

Now

\[
\sum_{i=1}^n ((\alpha_i \wedge \omega) \lrcorner \nabla X_i \Lambda)\Lambda = \sum_{i=1}^n ((\alpha_i \wedge \omega) \lrcorner \Lambda) \nabla X_i \Lambda \\
= \sum_{i} (-1)^{|\omega|}(\omega \wedge \alpha_i) \lrcorner \nabla X_i \Lambda \\
= \sum_{i} (-1)^{|\omega|}(\alpha_i \lrcorner X) \nabla X_i \Lambda \\
= \sum_{i} (-1)^{|\omega|}X(\alpha_i) \nabla X_i \Lambda \\
= (-1)^{|\omega|} \nabla X \Lambda.
\]
On the other hand, it follows from Equation (11) that
\[ [X, \Lambda] \downarrow \theta = -\Lambda \downarrow d(X \downarrow \theta), \]
for any \( \theta \in \Gamma(\wedge^n A^*) \). It is simple to see that \( X \downarrow \theta = (\omega \downarrow \Lambda) \downarrow \theta = (-1)^{|\omega|} \omega \downarrow (n-|\omega|) \omega. \) Since \( n = |\omega| - 1 \), then \( X \downarrow \theta = (-1)^{|\omega|} \omega \), and \( [X, \Lambda] \downarrow \theta = (-1)^{|\omega|} d\omega \downarrow \Lambda. \)

Hence, \( (DX)\Lambda = [X, \Lambda] - \nabla_X \Lambda = L_X \Lambda - \nabla_X \Lambda. \)

\[ \square \]

4 Homology of Lie algebroids

Suppose that \( A \) is a Lie algebroid, and \( \nabla \) a flat \( A \)-connection on the line bundle \( E = \wedge^n A \). Let \( D \) be the corresponding generating operator and \( \partial = (-1)^{n-k} D : \Gamma(\wedge^k A) \rightarrow \Gamma(\wedge^{k-1} A). \) Then, \( \partial^2 = 0. \) The reason for choosing this sign in the definition of \( \partial \) will become clear later (see Equation (17)).

As usual, define the homology by
\[ H_*(A, \nabla) = \ker \partial / \text{Im} \partial. \]

Since \( D \) is a derivation with respect to \([\cdot, \cdot]\), immediately we have

**Proposition 4.1** The Schouten bracket passes to the homology \( H_*(A, \nabla) \).

Since this homology depends on the choice of the connection \( \nabla \), it is natural to ask how \( H_*(A, \nabla) \) changes according to the connection \( \nabla \).

**Proposition 4.2** Let \( \tilde{D} \) and \( D \) be two generating operators. Then \( D - \tilde{D} = i_\alpha \), where \( \alpha \in \Gamma(A^*) \). In this case,
\[ \tilde{D}^2 - D^2 = -i_{d\alpha}. \]
In particular, if \( \tilde{D}^2 = D^2 = 0 \), then \( \alpha \in \Gamma(A^*) \) is closed.

**Proof.** Let \( \tilde{\nabla} \) and \( \nabla \) be any two \( A \)-connections on \( E = \wedge^n A \). Then there is \( \alpha \in \Gamma(A^*) \) such that
\[ \tilde{\nabla}_X s = \nabla_X s + \langle \alpha, X \rangle s, \ \forall s \in \Gamma(\wedge^n A). \]

Let \( \tilde{D} \) and \( D \) be their corresponding generating operators. It follows from a direct verification that
\[ \tilde{D} = D - i_{\alpha}. \]
According to Proposition 3.1, \( \tilde{D}^2 U - D^2 U = -(\tilde{R} - R) \downarrow U \), where \( \tilde{R} \) and \( R \) are the curvatures of \( \tilde{\nabla} \) and \( \nabla \), respectively. Finally, it is routine to check that \( \tilde{R} - R = d\alpha. \)

\[ \square \]
**Definition 4.3** A-connections $\nabla_1$ and $\nabla_2$ are said to be homotopic if they differ by an exact form in $\Gamma(A^*)$.

Similarly two generating operators $D_1$ and $D_2$ are said to be homotopic if they differ by an exact form, i.e., $D_1 - D_2 = i_\alpha$ for some exact form $\alpha \in \Gamma(A^*)$.

The following result is thus immediate.

**Proposition 4.4** Let $\nabla_1$ and $\nabla_2$ be two flat $A$-connections on the canonical line bundle $E = \wedge^n A$, and $D_1$ and $D_2$ the corresponding generating operators. If $\nabla_1$ and $\nabla_2$ are homotopic (or equivalently $D_1$ and $D_2$ are homotopic), then,

$$H_*(A, \nabla_1) \cong H_*(A, \nabla_2). \quad (16)$$

Now assume that $\wedge^n A$ is a trivial bundle, so there exists a nowhere vanishing volume $\Lambda \in \Gamma(\wedge^n A)$. This volume induces a flat $A$-connection $\nabla_0$ on $\wedge^n A$ simply by $(\nabla_0)_X \Lambda = 0$ for all $X \in \Gamma(A)$. Let $D_0$ be its corresponding generating operator. Note that $\Lambda$ being horizontal is equivalent to the condition:

$$D_0 \Lambda = 0.$$

Suppose that $\Lambda'$ is another nonvanishing volume, and $\nabla'$ its corresponding flat connection on $E$. Assume that $\Lambda' = f \Lambda$ for some positive $f \in C^\infty(M)$. Then, it is easy to see that

$$\nabla'_X s = (\nabla_0)_X s - d\ln f, X > s.$$

In other words, their corresponding generating operators are homotopic.

Let us now fix such a volume $\Lambda \in \Gamma(\wedge^n A)$. Define a $*$-operator from $\Gamma(\wedge^k A^*)$ to $\Gamma(\wedge^{n-k} A)$ by

$$* \omega = \omega \wedge \Lambda.$$

Clearly $*$ is an isomorphism.

The following proposition follows immediately from definition.

**Proposition 4.5** The operator $\partial_0 = (-1)^{n-k} D_0$ equals to $- * \circ d \circ *^{-1}$. That is,

$$\partial_0 = - * \circ d \circ *^{-1}. \quad (17)$$

Thus, as a consequence, we have

**Theorem 4.6** Let $\nabla_0$ be an $A$-connection on $\wedge^n A$ such that there exists a nowhere vanishing horizontal volume $\Lambda \in \Gamma(\wedge^n A)$. Then

$$H_*(A, \nabla_0) \cong H^{n-*}(A, \mathbb{R}).$$
Remark We see, from the discussion above, that there is a family of Lie algebroid homologies, which are in some sense parameterized by the first Lie algebroid cohomology. In the case that the canonical line bundle $\wedge^nA$ is trivial, one of them is isomorphic to the Lie algebroid cohomology with trivial coefficients, and all the others can be considered as Lie algebroid cohomology with some twisted coefficients. In general, the Lie algebroid homology introduced above is a special case of Lie algebroid cohomology with general coefficients in a line bundle (see [19] [5]).

Below are two special interesting cases.

Let $\mathfrak{g}$ be an $n$-dimensional Lie algebra. Then $\wedge^n\mathfrak{g}$ is one-dimensional and obviously has a trivial $\mathfrak{g}$-connection. This induces an operator $D_0: \wedge^*\mathfrak{g} \rightarrow \wedge^{*-1}\mathfrak{g}$ which is of square zero and generates the Schouten bracket on $\wedge^*\mathfrak{g}$. On the other hand, there exists another operator $D: \wedge^*\mathfrak{g} \rightarrow \wedge^{*-1}\mathfrak{g}$, which is dual to the differential operator of the Lie algebra cohomology. In general, $D$ is different from $D_0$ and in fact it is easy to check that $D - D_0 = i_\alpha$, where $\alpha$ is the modular character of the Lie algebra. In particular, when $\mathfrak{g}$ is a unimodular Lie algebra, the Lie algebra homology is isomorphic to Lie algebra cohomology, a well-known result.

Another interesting case, which does not seem trivial, is the one when $A$ is the cotangent Lie algebroid $T^*P$ of a Poisson manifold $P$ (see Equations (2) and (3)). In this case, $\Gamma(\wedge^kT^*P) = \Omega^k(P)$. There exists an operator $D: \Omega^k(P) \rightarrow \Omega^{k-1}(P)$ introduced by Koszul [16] and studied by Brylinski [2]. It is given by

$$D = [i_\pi, d].$$

The corresponding homology is called Poisson homology by Brylinski, and denoted by $H_*(P, \pi)$. It was shown in [16] that the operator $D$ indeed generates the Gerstenhaber bracket on $\Omega^*(P)$ induced from the cotangent Lie algebroid of $P$.

Therefore, $D$ corresponds to a flat Lie algebroid connection on $\wedge^n T^*P$, which is given by

$$\nabla_\theta \Omega = -\theta \wedge D\Omega = \theta \wedge d(\pi \lrcorner \Omega),$$

for any $\theta \in \Omega^1(P)$ and $\Omega \in \Omega^n(P)$, according to Equation (14). A similar formula was also discovered independently, by Evens-Lu-Weinstein [5].

The Koszul-Brylinski operator $D$ is intimately related to the so called modular class of the Poisson manifold, a classical analogue of the modular form of a von Neumann algebra, which was introduced recently by Weinstein [27], and independently by Brylinski and Zuckerman [3].

For simplicity, let us assume that $P$ is orientable, and $\Omega$ is a volume form. The modular vector field $\nu_\Omega$ is the vector field defined by

$$f \mapsto (L_Xf\Omega)/\Omega, \quad \forall f \in C^\infty(P).$$

It can be shown that the above map is a derivation on the space of functions $C^\infty(P)$, so it indeed defines a vector field. It also can be shown that $\nu_\Omega$ is a Poisson vector field. When the volume $\Omega$ changes, the corresponding modular vector fields differ by a hamiltonian vector field. So its class is a well-defined element in the first Poisson cohomology $H^1_\pi(P)$, which is called the modular class of the Poisson manifold. A Poisson manifold is called unimodular if its modular class vanishes. In fact, the modular class can be defined for any Poisson manifold by just replacing the volume by a positive density. We refer the interested reader to [27] for more detail.
Now let \( P \) be an orientable Poisson manifold with volume form \( \Omega \), and let \( D_0 \) be its corresponding generating operator as in the observation preceding Theorem 4.6.

The following proposition follows immediately from a direct verification.

**Proposition 4.7** Let \( D \) be the Koszul-Brylinski operator of a Poisson manifold \( P \). Then \( D - D_0 = i_{\nu_\Omega} \), where \( \nu_\Omega \) is the modular vector field corresponding to the volume \( \Omega \).

As an immediate consequence, we have

**Theorem 4.8** If \( P \) is an orientable unimodular Poisson manifold, then

\[
H_\ast(P, \pi) \cong H^{n-\ast}_\pi(P).
\]

In particular, this holds for any symplectic manifold, which was first proved by Brylinski [2].

**Remark** The above situation can be generalized to the case of triangular Lie bialgebroids. Let \( A \) be a Lie algebroid with anchor \( a \). A triangular \( r \)-matrix is a section \( \pi \) in \( \Gamma(\wedge^2 A) \) satisfying the condition \([\pi, \pi] = 0\). One may think that this is a sort of generalized “Poisson structure” on the generalized manifold \( A \). In this case, \( A^* \) is equipped with a Lie algebroid structure with the anchor \( a \circ \pi^\# \) and the Lie bracket defined by an equation identical to the one defining the bracket on one-forms of a Poisson manifold.

Similarly, \( D = [i_\pi, d] \) is an operator \( \Gamma(\wedge^k A^*) \rightarrow \Gamma(\wedge^{k-1} A^*) \) of square zero and generates the bracket \([\cdot, \cdot]\) on \( \Gamma(\wedge^* A^*) \). A form of top degree \( \Omega \in \Gamma(\wedge^n A^*) \) satisfies the condition \( D \Omega = 0 \) iff \( \pi_\wedge \Omega \in \Gamma(\wedge^{n-2} A^*) \) is closed. If there exists such a nowhere vanishing form, the homology \( H_\ast(A, \nabla) \) is then isomorphic to the cohomology \( H^{n-\ast}(A, \mathbb{R}) \).

## 5 Discussion

We end this paper by some open questions.

**Question 1:** As in the remark at the end of the last section, is the condition that \( \pi_\wedge \Omega \in \Gamma(\wedge^{n-2} A^*) \) is closed equivalent to that the Lie algebroid \( A^* \) is unimodular?

**Question 2:** For a general Lie algebroid \( A \), does there exist any canonical generating operator corresponding to the modular class of the Lie algebroid just as in the case of cotangent algebroid of a Poisson manifold (see Proposition 4.7)?

**Question 3:** For a Poisson manifold, there is a family of homologies parameterized by the first Poisson cohomology \( H^1_\pi(P) \). What is the meaning of the homologies other than the Poisson homology?

**Question 4:** Suppose that \( (A, A^*) \) is a Lie bialgebroid and \( \nabla \) a flat \( A \)-connection on \( \wedge^n A \). Then \( (\Gamma(\wedge^* A), \wedge, d_\ast, [\cdot, \cdot], D) \) is a strong differential BV-algebra. It is clear that \( d_\ast D + Dd_\ast \) is a derivation with respect to both \( \wedge \) and \([\cdot, \cdot]\). When is \( d_\ast D + Dd_\ast \) inner and in particular, when is \( d_\ast D + Dd_\ast = 0 \)?
When $A = T^*P$ is the cotangent Lie algebroid of a Poisson manifold, $A^* = TP$ the usual Lie algebroid on the tangent bundle, and the connection $\nabla$ is as in Equation (18), then $d_\ast$ is the usual de-Rham differential and $D$ is the Koszul-Brylinski operator. Thus, $d_\ast D + Dd_\ast$ is automatically zero, which gives rise to the Brylinski double complex. On the other hand, if we switch the order and consider $A = TP$ and $A^* = T^*P$ for a Poisson manifold $P$ equipped with a volume generating the connection on the line bundle $\wedge^n TP$, then $A = \oplus \Gamma(\wedge^i A)$ is the space of multivector fields. In this case, $d_\ast = [\pi, \cdot]$ is the Lichnerowicz differential defining the Poisson cohomology, and $D = -(-1)^{n-k} * \circ d \circ *^{-1}$, where $*$ is the isomorphism between multivector fields and differential forms induced by the volume element. Then $d_\ast D + Dd_\ast = LX$, where $X$ is the modular vector field of the Poisson manifold (see P. 265 of [16]). It vanishes iff $P$ is unimodular. It would be interesting to explore this in general.

References

[1] Bhaskara, K., and Viswanath, K., Calculus on Poisson manifolds, Bull. London Math. Soc. 20 (1988), 68-72.
[2] Brylinski, J.-L., A differential complex for Poisson manifolds, J. Diff. Geom. 28 (1988), 93-114.
[3] Brylinski, J.-L., Zuckerman, G., The outer derivation of a complex Poisson manifold, preprint, 1996.
[4] Coste, A., Dazord, P. and Weinstein, A., Groupoïdes symplectiques, Publications du Département de Mathématiques de l’Université de Lyon, I, 2/A (1987), 1-65.
[5] Evens, S., Lu, J.-H., and Weinstein, A., Transverse measures, the modular class, and a cohomology pairing for Lie algebroids, preprint, 1996.
[6] Gerstenhaber, M. and Schack, S. D., Algebras, bialgebras, quantum groups and algebraic deformations Contemp. Math. 134, AMS, Providence (1992), 51-92.
[7] Getzler, E., Batalin-Vilkovisky algebras and two-dimensional topological field theories, Comm. Math. Phys. 159 (1994), 265-285.
[8] Getzler, E., and Kapranov, M. M., Cyclic operads and cyclic homology, Geometry, Topology and Physics for Raoul Bott, S.-T. Yau (eds.), (1995), 167-201.
[9] Huebschmann, J., Duality for Lie-Rinehart algebras and the modular class, preprint, 1997.
[10] Kimura, T., Voronov, A. and Stasheff, J. On operad structures of moduli spaces and string theory, Comm. Math. Phys. 171 (1995), 1-25.
[11] Kontsevich, M., Course on deformation theory, UC Berkeley, 1994.
[12] Kosmann-Schwarzbach, Y., Exact Gerstenhaber algebras and Lie bialgebroids, Acta Appl. Math. 41 (1995), 153-165.
[13] Kosmann-Schwarzbach, Y., Graded Poisson brackets and field theory, Modern group theoretical methods in physics, J. Berttrand et al. (eds.), (1995), 189-196.
[14] Kosmann-Schwarzbach, Y., The Lie bialgebroid of a PN-maniofold, *Lett. Math. Phys.*, to appear.

[15] Kosmann-Schwarzbach, Y., and Magri, F., Poisson-Nijenhuis structures, *Ann. Inst. H. Poincaré Phys. Théor.* (53), 1990, 35–81.

[16] Koszul, J.-L., Crochet de Schouten-Nijenhuis et cohomologie, *Astérisque, numéro hors série* (1985), 257–271.

[17] Lian, B.H. and Zukerman, G.J., New perspectives on the BRST-algebraic structure of string theory, *Comm. Math. Phys.* **154** (1993), 613-646.

[18] Lichnerowicz, A., Les variétés de Poisson et leurs algèbres de Lie associées, *J. Diff. Geom.* **12** (1977), 253-300.

[19] Mackenzie, K., *Lie Groupoids and Lie Algebroids in Differential Geometry*, LMS Lecture Notes Series, **124**, Cambridge Univ. Press, 1987.

[20] Mackenzie, K.C.H. and Xu, P., Lie bialgebroids and Poisson groupoids, *Duke Math. J.* **73** (1994), 415-452.

[21] Penkava, M. and Schwarz, A., On some algebraic structures arising in string theory, preprint, 1991.

[22] J. Pradines, *Théorie de Lie par les groupoïdes différentiables*, C. R. Acad. Sci. Paris, Série A **267** (1968), 245–248.

[23] J. Pradines, *Troisième théorème de Lie pour les groupoïdes différentiables*, C. R. Acad. Sci. Paris, Série A **267** (1968), 21–23.

[24] Stasheff, J., Differential graded Lie algebras, quasi-Hopf algebras and higher homotopy algebras, Lecture Notes in Math. **1510**, 120-137.

[25] Stasheff, J., From operads to ‘physically’ inspired theories: An operad-chik looks at configuration spaces, moduli spaces and mathematical physics, *Contemporary Mathematics AMS* to appear.

[26] Vaisman, I., Lectures on the geometry of Poisson manifolds, PM **118**, Basel; Boston; Berlin; Birkhäuser 1994.

[27] Weinstein, A., The modular automorphism group of a Poisson manifold, preprint, 1996.

[28] Weinstein, A. and Xu, P., Extensions of symplectic groupoids and quantization, *J. Reine Angew. Math.* **417** (1991), 159-189.

[29] Witten, E., A note on antibracket formalism, *Modern Physics Letters, A* **5**, 487-494.

[30] Zwiebach, B., Closed string theory: quantum action and the BV master equation, *Nucl. Phys.*, **B 390**, 33-152.