The ellipsoidal universe in the Planck satellite era

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ABSTRACT

Recent Planck data confirm that the cosmic microwave background displays the quadrupole power suppression together with large-scale anomalies. Progressing from previous results, that focused on the quadrupole anomaly, we strengthen the proposal that the slightly anisotropic ellipsoidal universe may account for these anomalies. We solved at large scales the Boltzmann equation for the photon distribution functions by taking into account both the effects of the inflation produced primordial scalar perturbations and the anisotropy of the geometry in the ellipsoidal universe. We showed that the low quadrupole temperature correlations allowed us to fix the eccentricity at decoupling, $e_{\text{dec}} = (0.86 \pm 0.14) \times 10^{-2}$, and to constraint the direction of the symmetry axis. We found that the anisotropy of the geometry of the universe contributes only to the large-scale temperature anisotropies without affecting the higher multipoles of the angular power spectrum. Moreover, we showed that the ellipsoidal geometry of the universe induces sizeable polarization signal at large scales without invoking the reionization scenario. We explicitly evaluated the quadrupole TE and EE correlations. We found an average large-scale polarization $\Delta T_{\text{pol}} = (1.20 \pm 0.38) \mu K$. We point out that great care is needed in the experimental determination of the large-scale polarization correlations since the average temperature polarization could be misinterpreted as foreground emission leading, thereby, to a considerable underestimate of the cosmic microwave background polarization signal.

Key words: cosmic background radiation – cosmology: theory.

1 INTRODUCTION

The cosmic microwave background (CMB) anisotropy data produced by the final analysis of the Wilkinson Microwave Anisotropy Probe (WMAP; Bennett et al. 2013; Hinshaw et al. 2013) and, more recently, by the Planck satellite (Ade et al. 2013a,c,d) confirmed the standard cosmological Lambda cold dark matter ($\Lambda$CDM) model at an unprecedented level of accuracy. At large scales, however, several anomalous features have been reported: an unusual alignment of the preferred axes of the quadrupole and octopole (de Oliveira-Costa et al. 2004; Ralston & Jain 2004; Land & Magueijo 2005; Copi et al. 2006), non-Gaussian signatures due to a cold spot (Cruz et al. 2005), a hemispherical power asymmetry at large scales (Eriksen et al. 2004; Hansen, Banday & Gorski 2004). Nevertheless, we feel that one of the most important discrepancy resides in the low quadrupole moment, which signals an important suppression of power at large scales. In fact, the Planck Collaboration reported a statistical significant tension between the best-fitting $\Lambda$CDM model and the large-scale spectrum due to a systematic lack of power for $\ell \lesssim 40$ (Ade et al. 2013c) and to anomalies in the statistical isotropy of the sky maps (Ade et al. 2013e).

If these anomalies should turn out to have a cosmological origin, then it could have far reaching consequences for our present understanding of the universe.

Quite recently it has been suggested (Campanelli, Cea & Tedesco 2006, 2007; Cea 2010) that, if one admits that the large-scale spatial geometry of our universe is only plane-symmetric with eccentricity at decoupling of order $10^{-2}$, then the quadrupole amplitude can be drastically reduced without affecting higher multipoles of the angular power spectrum of the temperature anisotropy. As discussed in Campanelli et al. (2007), the anisotropic expansion described by a plane-symmetric metric can be generated by cosmological magnetic fields or topological defects, such as cosmic domain walls or cosmic strings. Indeed, topological cosmic defects are relic structures that are predicted to be produced in the course of symmetry breaking in the hot, early universe (e.g. see Vilenkin & Shellard 1994).

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1.1 Outline of the results

In an isotropic and homogeneous universe the most general metric is the Friedmann–Robertson–Walker (FRW) metric (see, for instance, Peebles 1993). In particular, the metric of standard cosmological model is given by

\[ ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j. \]  

(1)

If we assume that the large-scale spatial geometry of our universe is only plane-symmetric, then the metric equation (1) is replaced with the ellipsoidal universe metric:

\[ ds^2 = -dt^2 + a^2(t) (\delta_{ij} + h_{ij}) dx^i dx^j, \]

(2)

where \( h_{ij} \) is a metric perturbation which we assume to be of the form:

\[ h_{ij} = -\epsilon^2(t) n_i n_j. \]

(3)

In equation (3), \( \epsilon(t) = \sqrt{1 - (b(t)/a(t))^2} \) is the ellipticity and the unit vector \( n \) determines the direction of the symmetry axis.

In this paper, we shall further elaborate on the ellipsoidal universe proposal and extend previous investigations in several directions. For reader’s convenience, it is useful to summarize the main results of this paper.

First, we consider the Boltzmann equation for the photon distribution in the ellipsoidal universe, discussed for the first time in Cea (2010), by taking into account also the effects of the cosmological inflation produced primordial scalar perturbations. In the large-scale approximation, we explicitly show that the CMB temperature fluctuations can be written as

\[ \Delta T \simeq \Delta T^I + \Delta T^A, \]

(4)

where \( \Delta T^I \) and \( \Delta T^A \) are the temperature fluctuations induced by the cosmological scalar perturbations and by the spatial anisotropy of the metric of the universe, respectively. Since the temperature anisotropies caused by the inflation produced primordial scalar perturbations are discussed in several textbooks (Dodelson 2003; Mukhanov 2005), we focus on the temperature fluctuations induced by the anisotropy of the metric by solving the relevant Boltzmann equation. At large scales, we solve that equation and determine the solutions relevant for the CMB temperature and polarization fluctuations. Indeed, it is well known (Rees 1968; Basko & Polnarev 1980; Negroponte & Silk 1980) that anisotropic cosmological models give sizeable contributions to the large-scale polarization of the CMB radiation. In fact, polarization measurements could provide a unique signature of cosmological anisotropies.

We go beyond the approximations adopted in Cea (2010) and confirm that the main contributions to the CMB temperature fluctuations affect the quadrupole correlations. In addition, we also show that the effects of the spatial anisotropy of the metric of the universe extend to low-lying multipoles \( \ell \sim 10 \).

As is well known, the CMB temperature fluctuations are fully characterized by the power spectrum:

\[ (\Delta T_\ell)^2 = D_\ell = \frac{\ell(\ell + 1)}{2\pi} C_\ell, \quad C_\ell = \frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} |a_{\ell m}|^2. \]

(5)

In particular, the quadrupole anisotropy refers to the multipole \( \ell = 2 \). Remarkably, the Planck data (Ade et al. 2013c) confirmed that the observed quadrupole anisotropy:

\[ (\Delta T_2)^2 = D_2 \simeq 299.5 \, \mu K^2, \]

(6)

is much smaller than the quadrupole anisotropy expected according to the best-fitting \( \Lambda \)CDM model to the Planck data:

\[ \left( \Delta T_2^\Lambda \right)^2 = 1150 \pm 727 \, \mu K^2. \]

(7)

Note that in equation (6) we are neglecting the rather small measurement errors, while the uncertainties due to the so-called cosmic variance are included in the theoretical expectations, equation (7). In fact, using equation (4) we show that the quadrupole temperature anisotropy can be reconciled with observations in the ellipsoidal universe if the eccentricity at decoupling is

\[ \epsilon_{\text{dec}} = (0.86 \pm 0.14) \times 10^{-2}, \]

(8)

irrespective of the physical mechanism responsible for the generation of the spatial anisotropy in the early universe. Moreover, if we denote with \( b_n \) and \( l_n \) the galactic latitude and longitude of the symmetry axis, respectively, we also were able to show that the axis of symmetry were constrained to

\[ b_n \simeq \pm 17^\circ, \]

(9)

while the longitude \( b_n \) turns out to be poorly constrained, in qualitative agreement with Campanelli et al. (2007).

\footnote{Note that through the paper we shall use units in which \( c = 1, \, h = 1 \) and \( k_B = 1 \).}
As concern the CMB polarization, we confirm our previous result (Cea 2010) that the ellipsoidal geometry of the universe induces sizeable polarization signal at large scales without invoking the reionization scenario. In particular, we find an average large-scale polarization:

$$\Delta T_{\text{pol}} = \frac{1}{4\pi} \int \Delta T^E(\theta, \phi) \, d\Omega = (1.20 \pm 0.38) \mu K,$$

(10)

where $\Delta T^E(\theta, \phi)$ is the polarization of the CMB temperature fluctuations. Moreover, we evaluate the quadrupole temperature–polarization cross-correlation (TE) and polarization–polarization (EE) correlation. We find

$$\Delta T_2^{\text{TE}} = 3.14 \pm 0.76 \mu K,$$

(11)

and

$$\Delta T_2^{\text{EE}} = 0.83 \pm 0.27 \mu K.$$

(12)

These values should be compared with the available observational data. Since the Planck Collaboration do not yet make public the large-scale polarization data, we must rely on the final analysis of the Wilkinson Microwave Anisotropy Probe Collaboration. The WMAP nine-year full-sky maps of the polarization detected at large scales in the foreground corrected maps an average E-mode polarization power (Bennett et al. 2013; Hinshaw et al. 2013). In particular, for the quadrupole correlations we have (including only the statistical uncertainties)

$$\frac{l(l+1)}{2\pi} C_{l=2}^{\text{TE}} = 2.4439 \pm 2.2831 \mu K^2, \quad \text{WMAP nine-years}$$

(13)

and

$$\frac{l(l+1)}{2\pi} C_{l=2}^{\text{EE}} = -0.0860 \pm 0.0247 \mu K^2, \quad \text{WMAP nine-years.}$$

(14)

Using the definition in equation (5), we estimate

$$\Delta T_2^{\text{TE}} = 1.56 \pm 0.73 \mu K, \quad \text{WMAP nine-years},$$

(15)

which within two standard deviations agrees with our result equation (11). On the other hand, as concern the quadrupole EE correlation, equation (14) at best gives an upper bound which, however, is not consistent with our result equation (12). We believe that this discrepancy could be due to the fact that in the ellipsoidal universe model, at variance of the standard reionization scenario, there is a non-zero average temperature polarization. In fact, the eventual presence of an average temperature polarization could be misinterpreted as foreground emission leading to an underestimate of the CMB polarization signal.

The plan of the paper is as follows. In Section 2, we discuss the Boltzmann equation of the cosmic background radiation in the ellipsoidal universe. In Section 3, we determine the solutions of the Boltzmann equation at large scales. Section 4 is devoted to the problem of the quadrupole anomaly in the temperature–temperature fluctuation correlations. In Section 5, we discuss the large-scale polarization. In particular, we determine the quadrupole TE and EE correlations. Finally, our conclusions are drawn in Section 6. Some technical details are relegated in Appendix A, while in Appendix B we discuss the multipole expansion of the large-scale temperature anisotropies.

2 THE BOLTZMANN EQUATION IN THE ELLIPSoidal UNIVERSE

We are interested in the temperature fluctuations of the cosmic background radiation induced by eccentricity of the universe and by the inflation produced primordial cosmological perturbations. We assume that the photon distribution function $f(x, t)$ is an isotropically radiating blackbody at a sufficiently early epoch. The subsequent evolution of $f(x, t)$ is determined by the Boltzmann equation (Dodelson 2003; Mukhanov 2005):

$$\frac{df}{dt} = \left( \frac{\delta f}{\delta t} \right)_\text{coll},$$

(16)

where $\left( \frac{\delta f}{\delta t} \right)_\text{coll}$ is the collision integral which takes care of Thomson scatterings between matter and radiation.

The metric of the standard FRW universe is

$$ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j.$$

(17)

Here, we are interested in primordial scalar perturbations induced by the inflation. In the conformal Newtonian, or longitudinal gauge (Mukhanov 2005), the metric equation (17) can be written as

$$ds^2 = -[1 + 2\Phi(x, t)] dt^2 + a^2(t) \delta_{ij} [1 + 2\Phi(x, t)] dx^i dx^j.$$

(18)

In this gauge, the perturbations to the metric are determined by the functions $\Phi(x, t)$ and $\delta(x, t)$, which correspond to the Newtonian potential and the perturbation to the spatial curvature, respectively. In the ellipsoidal universe, the metric would be

$$ds^2 = -[1 + 2\Phi(x, t)] dt^2 + a^2(t) \delta_{ij} [\delta_{ij} + h_{ij}] [1 + 2\Phi(x, t)] dx^i dx^j.$$

(19)
where \( h_{ij} \) is given by equation (3). However, both the primordial perturbations \( \Psi(x, t) \), \( \Phi(x, t) \) and the ellipticity are to be considered small at the times and scales of interest. Therefore, in the following we shall neglect all terms quadratic in them. Accordingly, instead of equation (19), we have

\[
dx^2 = -[1 + 2\Psi(x, t)] \, dt^2 + a^2(t) \, [\delta_{ij} + h_{ij}] \, dx^i dx^j. \tag{20}
\]

We are interested in the anisotropies in the cosmic distribution of photons. To this end, we need to evaluate the photon distribution function \( f(x, t) \) which satisfies the Boltzmann equation (16). Actually, the distribution function depends on the space–time point \( x^\mu \) and the momentum vector \( p^\mu \) defined by

\[
p^\mu = \frac{d x^\mu}{d \lambda}, \tag{21}
\]

where \( \lambda \) parametrizes the particle’s path. For massless particles we, obviously, have

\[
P^2 = g_{\mu \nu} \, p^\mu \, p^\nu = 0. \tag{22}
\]

Using the metric in equation (20) and defining

\[
\lambda_{ij} \equiv g_{ij} \, p^i \, p^j, \tag{23}
\]

from equation (22), we easily obtain

\[
P^0 \simeq p \, [1 - \Psi]. \tag{24}
\]

It is convenient to consider the distribution function as a function of the magnitude of momentum \( p \) and momentum direction \( \lambda^i \), \( \delta_{ij} \, \lambda^i \lambda^j = 1 \). Therefore, we have

\[
\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i} \frac{dx^i}{dt} + \frac{\partial f}{\partial p^i} \frac{dp^i}{dt} + \frac{\partial f}{\partial \lambda^i} \frac{d\lambda^i}{dt}. \tag{25}
\]

Now we note that

\[
\frac{dx^i}{dt} = \frac{dx^i}{d\lambda} \frac{d\lambda}{dt} = \frac{p^i}{P^0}. \tag{26}
\]

Let us write

\[
p^i = C \, \lambda^i, \tag{27}
\]

then it is easy to find

\[
C \simeq \frac{p}{a(t)} \, \left[ 1 - \Phi - \frac{1}{2} h_{ij} \, \lambda^i \lambda^j \right]. \tag{28}
\]

So that we have

\[
\frac{dx^i}{dt} \simeq \frac{\dot{\lambda}^i}{a(t)} \, \left[ 1 - \Phi - \Psi - \frac{1}{2} h_{ij} \, \lambda^i \lambda^j \right]. \tag{29}
\]

Thus, we obtain

\[
\frac{df}{dt} \simeq \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i} \frac{\dot{\lambda}^i}{a(t)} \left[ 1 - \Phi - \Psi - \frac{1}{2} h_{ij} \, \lambda^i \lambda^j \right] + \frac{\partial f}{\partial p^i} \frac{dp^i}{dt} + \frac{\partial f}{\partial \lambda^i} \frac{d\lambda^i}{dt} \simeq \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i} \frac{\dot{\lambda}^i}{a(t)} + \frac{\partial f}{\partial p^i} \frac{dp^i}{dt} + \frac{\partial f}{\partial \lambda^i} \frac{d\lambda^i}{dt}. \tag{30}
\]

since \( \frac{\delta f}{\delta \lambda^i} \) is already a first-order term. To evaluate \( \frac{dp^i}{dt} \), we note that the time component of the geodesic equations gives

\[
\frac{dp^0}{d\lambda} = -\Gamma_{a0b}^0 \, p^a \, p^b. \tag{31}
\]

Since

\[
\frac{dp^0}{d\lambda} = \frac{dp^0}{dt} \frac{dt}{d\lambda} = p^0 \frac{dp^0}{dt}, \tag{32}
\]

after using equation (24) we obtain

\[
\frac{dp}{dt} \simeq p \left( \frac{\partial \Psi}{\partial t} + \frac{\dot{\lambda}^i}{a(t)} \frac{\partial \Psi}{\partial x^i} \right) + \Gamma_{a0b}^0 \, \frac{p^a \, p^b}{p} \, [1 + 2\Psi] = p \left( \frac{\partial \Psi}{\partial t} + \frac{\dot{\lambda}^i}{a(t)} \frac{\partial \Psi}{\partial x^i} \right) - \Gamma_{a0b}^0 \, \frac{p^a \, p^b}{p} \, [1 + 2\Psi]. \tag{33}
\]

Moreover, a standard calculation (Dodelson 2003) shows that

\[
\Gamma_{a0b}^0 \, \frac{p^a \, p^b}{p} \simeq p (1 - 2\Psi) \left[ \frac{\partial \Psi}{\partial t} + 2 \frac{\dot{\lambda}^i}{a(t)} \frac{\partial \Psi}{\partial x^i} + \frac{\partial \Phi}{\partial t} + \frac{1}{2} \frac{\dot{\lambda}^i}{a(t)} \frac{\partial h_{ij}}{\partial t} + H \right]. \tag{34}
\]
where $H = \dot{a}/a$ is the Hubble rate. Finally, inserting equations (33) and (34) into equation (30) and collecting terms we obtain
\[
\frac{df}{dt} = \frac{df}{a(t)} + \frac{\dot{\rho}}{\rho} \frac{df}{d\chi} - \frac{df}{\rho} \left[ H(t) + \frac{\dot{\rho}}{\rho} \frac{\partial \Phi}{\partial \chi} + \frac{\dot{\rho}}{\rho} \frac{\partial \Psi}{\partial \chi} + \frac{1}{2} \frac{\dot{\rho}}{\rho} \frac{\partial \Phi_i}{\partial t} \right].
\] (35)

To go further, we expand the photon distribution about its zero-order Bose–Einstein value:
\[
f_0(p, t) = \frac{1}{e^{\beta p} - 1}.
\] (36)

We write
\[
f(x, t, p, \hat{p}) = \frac{1}{e^{\beta p} - 1},
\] (37)
and expand to the first order in the perturbation $\Theta(x, t, p, \hat{p})$:
\[
f(x, t, p, \hat{p}) \simeq f_0(p, t) - p \frac{\partial f_0}{\partial p} \Theta(x, t, p, \hat{p}).
\] (38)

Using the relation $\nu n = \lambda n_p \simeq -1$ which is valid in the Rayleigh–Jeans region, we can rewrite equation (38) as
\[
f(x, t, p, \hat{p}) \simeq f_0(p, t) [1 + \Theta(x, t, p, \hat{p})].
\] (39)

If we neglect the perturbations, it is easy to see that the zero-order Boltzmann equation is satisfied by the Planck distribution equation (36) with $T(t) \sim a(t)$. To determine the perturbed distribution $\Theta(x, t, p, \hat{p})$, we need to evaluate the Boltzmann equation to the first order. From equations (35) and (37), it follows that
\[
\left( \frac{df}{dt} \right)_{\text{first order}} \simeq -p \frac{\partial f_0}{\partial p} \left\{ \frac{\partial \Theta}{\partial t} + \frac{\dot{\rho}}{\rho} \frac{\partial \Theta}{\partial \chi} + \frac{\partial \Phi}{\rho a(t)} \frac{\partial \chi}{\partial t} + \frac{\dot{\rho}}{\rho} \frac{\partial \Psi}{\rho a(t)} \frac{\partial \chi}{\partial t} + \frac{1}{2} \frac{\dot{\rho}}{\rho} \frac{\partial \Phi_i}{\partial t} \right\}.
\] (40)

Thus, the first-order Boltzmann equation becomes
\[
\frac{\partial \Theta}{\partial t} + \frac{\dot{\rho}}{\rho} \frac{\partial \Theta}{\partial \chi} + \frac{\partial \Phi}{\rho a(t)} \frac{\partial \chi}{\partial t} + \frac{\dot{\rho}}{\rho} \frac{\partial \Psi}{\rho a(t)} \frac{\partial \chi}{\partial t} + \frac{1}{2} \frac{\dot{\rho}}{\rho} \frac{\partial \Phi_i}{\partial t} \simeq \frac{1}{f_0} \left( \frac{df}{dt} \right)_{\text{coll}}.
\] (41)

The collision integral is in general a non-linear functional of the distribution function. However, in the first-order approximation it is a linear functional of $\Theta(x, t, p, \hat{p})$. Moreover, since we are interested in the solutions of the Boltzmann equation at large scales, we may neglect the effects due to the bulk velocity of the electrons which participate to the photon Compton scatterings. In this case, the collision integral can be considered a linear homogeneous functional of the distribution function $\Theta(x, t, p, \hat{p})$. As a consequence, if we write
\[
\Theta(x, t, p, \hat{p}) \simeq \Theta^4(x, t, p, \hat{p}) + \Theta^1(x, t, p, \hat{p}),
\] (42)
then we also have
\[
\left( \frac{df}{dt} \right)_{\text{coll}} [\Theta] \simeq \left( \frac{df}{dt} \right)_{\text{coll}} [\Theta^4] + \left( \frac{df}{dt} \right)_{\text{coll}} [\Theta^1].
\] (43)

In fact, equation (41) suggests that we may associate $\Theta^4$ and $\Theta^1$ with the temperature fluctuations induced by the spatial anisotropy of the geometry of the universe and by the scalar perturbations generated during the inflation, respectively. Accordingly, we set
\[
\frac{\partial \Theta^4}{\partial t} + \frac{\dot{\rho}}{\rho} \frac{\partial \Theta^4}{\partial \chi} + \frac{\partial \Phi}{\rho a(t)} \frac{\partial \chi}{\partial t} + \frac{1}{2} \frac{\dot{\rho}}{\rho} \frac{\partial \Phi_i}{\partial t} \simeq \frac{1}{f_0} \left( \frac{df}{dt} \right)_{\text{coll}} [\Theta^4],
\] (44)
and
\[
\frac{\partial \Theta^1}{\partial t} + \frac{\dot{\rho}}{\rho} \frac{\partial \Theta^1}{\partial \chi} + \frac{1}{2} \frac{\dot{\rho}}{\rho} \frac{\partial \Phi_i}{\partial t} \simeq \frac{1}{f_0} \left( \frac{df}{dt} \right)_{\text{coll}} [\Theta^1].
\] (45)

We note that from equation (37) it follows that the distribution function $\Theta(x, t, p, \hat{p})$ is identified with the temperature contrast function:
\[
\Theta(x, t, p, \hat{p}) = \frac{\Delta T(x, t, p, \hat{p})}{T(t)}.
\] (46)

Therefore, at large scales equation (42) implies that
\[
\Delta T(x, t, p, \hat{p}) \simeq \Delta T^4(x, t, p, \hat{p}) + \Delta T^1(x, t, p, \hat{p}).
\] (47)

Note that equation (47) was assumed in Campanelli et al. (2006, 2007) and Cea (2010). Even though this hypothesis was considered reasonable, an explicit proof was lacking. Our discussion shows that equation (47) arises as a natural consequence of the Boltzmann equation which, however, is valid only at large distances.

We may conclude that to determine the CMB temperature fluctuations at large scales we need to solve the Boltzmann equations equations (44) and (45). Equation (44) is the Boltzmann equation of the standard $\Lambda$CDM cosmological model, and it has been extensively discussed in several textbooks (Dodelson 2003; Mukhanov 2005). Therefore, in the following we focus on the Boltzmann equation (45), derived for the first time in Cea (2010), which allows us to find the CMB temperature fluctuations caused by the anisotropy of the geometry of the universe.
3 LARGE-SCALE SOLUTIONS OF THE BOLTZMANN EQUATION

In this section, we discuss the Boltzmann equation in the ellipsoidal universe equation (45):

$$\frac{\partial \Theta(x, t, p, \hat{p})}{\partial t} + \frac{\hat{p}^i}{a(t)} \frac{\partial \Theta(x, t, p, \hat{p})}{\partial x^i} + \frac{1}{2} \hat{p}^i \hat{p}^j \frac{\partial h_{ij}}{\partial t} \simeq \frac{1}{f_0} \left( \frac{\partial f}{\partial t} \right)_{\text{coll}}.$$  \hspace{1cm} (48)

where, for simplicity, the superscript \( A \) has been dropped. In Cea (2010), we have discussed the solutions of equation (48) by neglecting the spatial dependence of the temperature contrast function \( \Theta(x, t, p, \hat{p}) \). Here, we try to solve equation (48) in general. To do this, we introduce the Fourier transform of the temperature contrast function:

$$\Theta(x, t, p, \hat{p}) = \int \frac{d^3k}{(2\pi)^3} e^{ik \cdot x} \Theta(k, t, p, \hat{p}).$$  \hspace{1cm} (49)

Taking into account that the collision integral depends linearly on \( \Theta \), we easily obtain

$$\frac{\partial \Theta(k, t, p, \hat{p})}{\partial t} + \frac{i \hat{k} \cdot \hat{p}}{a(t)} \Theta(k, t, p, \hat{p}) + \frac{1}{2} \hat{p}^i \hat{p}^j \frac{\partial h_{ij}}{\partial t} \simeq \frac{1}{f_0} \left( \frac{\partial f}{\partial t} \right)_{\text{coll}} [\Theta(k, t, p, \hat{p})].$$  \hspace{1cm} (50)

To determine the polarization of the CMB, we need the polarized distribution function which, in general, is represented by a column vector whose components are the four Stokes parameters (Chandrasekhar 1960). In fact, due to the axial symmetry of the metric only two Stokes parameters need to be considered, namely the two intensities of radiation with electric vectors in the plane containing \( p \) and \( n \) and perpendicular to this plane, respectively. As a consequence, instead of equation (39) we have

$$f(x, t, p, \hat{p}) \simeq f_0(p, t) \left[ \begin{array}{c} 1 \\ \Theta(x, t, p, \hat{p}) \end{array} \right],$$  \hspace{1cm} (51)

where \( \Theta(x, t, p, \hat{p}) \) is a two component column vector. Using equation (3) and defining

$$\mu = \cos \theta_{\rho n}, \quad \cos \theta_{\rho n} = \frac{k \cdot \hat{p}}{k},$$  \hspace{1cm} (52)

we obtain from equation (50):

$$\frac{\partial \Theta(k, t, \mu)}{\partial t} + \frac{i k}{a(t)} \cos \theta_{\rho n} \Theta(k, t, \mu) \simeq \frac{1}{2} \left[ \frac{d}{dt} \epsilon^2(t) \right] \mu^2 \left[ \begin{array}{c} 1 \\ 1 \end{array} \right]$$

$$- \sigma_T n_e \left[ \Theta(k, t, \mu) - \frac{3}{8} \int_{\mu=1}^{1} \frac{2(1-\mu^2)(1-\mu^2) + \mu^2 \mu^2}{\mu^2} \mu^2 \right] \Theta(k, t, \mu') d\mu'.$$  \hspace{1cm} (53)

where \( \sigma_T \) is the Thomson cross-section and \( n_e(t) \) the electron number density (Chandrasekhar 1960).

Introducing the conformal time:

$$\eta(t) = \int_0^t \frac{dt'}{a(t')},$$  \hspace{1cm} (54)

we rewrite equation (53) as

$$\frac{\partial \Theta(k, \eta, \mu)}{\partial \eta} + i k \cos \theta_{\rho n} \Theta(k, \eta, \mu) \simeq \frac{1}{2} \left[ \frac{d}{d\eta} \epsilon^2(\eta) \right] \mu^2 \left[ \begin{array}{c} 1 \\ 1 \end{array} \right]$$

$$- a(\eta) \sigma_T n_e \left[ \Theta(k, \eta, \mu) - \frac{3}{8} \int_{\mu=1}^{1} \frac{2(1-\mu^2)(1-\mu^2) + \mu^2 \mu^2}{\mu^2} \mu^2 \right] \Theta(k, \eta, \mu') d\mu'.$$  \hspace{1cm} (55)

with a suitable overall normalization of the blackbody intensity. To determine the general solutions of equation (55), we write (Basko & Polnarev 1980; Cea 2010)

$$\Theta(k, \eta, \mu) = \theta_\phi(k, \eta) \left( \mu^2 - \frac{1}{3} \right) \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] + \theta_p(k, \eta) (1-\mu^2) \left[ \begin{array}{c} 1 \\ -1 \end{array} \right].$$  \hspace{1cm} (56)

From equation (56), it is evident that \( \theta_\phi \) measures the degree of anisotropy, while \( \theta_p \) gives the polarization of the primordial radiation. With the aid of equation (56), we rewrite equation (55) as

$$\frac{\partial \theta_\phi(k, \eta)}{\partial \eta} + i k \cos \theta_{\rho n} \theta_\phi(k, \eta) \simeq \Delta H(\eta) - a(\eta) \sigma_T n_e \left[ \frac{9}{10} \theta_\phi(k, \eta) + \frac{3}{5} \theta_p(k, \eta) \right]$$

$$\frac{\partial \theta_p(k, \eta)}{\partial \eta} + i k \cos \theta_{\rho n} \theta_p(k, \eta) \simeq - a(\eta) \sigma_T n_e \left[ \frac{1}{10} \theta_\phi(k, \eta) + \frac{2}{5} \theta_p(k, \eta) \right].$$  \hspace{1cm} (57)
where we introduced the cosmic shear (Negroponte & Silk 1980; Cea 2010):

$$\Delta H(\eta) = \frac{1}{2} \frac{d}{d \eta} e^i(\eta) .$$

(58)

The solution of the linear differential system equation (57) is the sum of the general solution of the homogeneous system and a particular solution. The solution of the homogeneous system (i.e. $\Delta H(\eta) = 0$) is

$$\theta_0(k, \eta) = \theta_p(k, \eta) = 0 ,$$

(59)

for the are no anisotropies without cosmological perturbations. To determine the particular solution of equation (57), we note that the linear combination:

$$\tilde{\theta}(k, \eta) = \theta_s(k, \eta) + \theta_p(k, \eta)$$

(60)

satisfies the following equation:

$$\frac{d}{d \eta} \tilde{\theta}(k, \eta) + i k \cos \theta_p \tilde{\theta}(k, \eta) \simeq \Delta H(\eta) - a(\eta) \sigma_T n_e \tilde{\theta}(k, \eta) .$$

(61)

Introducing the optical depth:

$$\tau(\eta, \eta') = \int_0^\eta \sigma_T n_e a(\eta') d\eta' ,$$

(62)

it is easy to verify that the solution of equation (61) is given by

$$\tilde{\theta}(k, \eta) = \int_{\eta_0}^\eta \Delta H(\eta') e^{-i \tau(\eta', \eta')} e^{ik \cos \theta_p (\eta' - \eta)} d\eta' ,$$

(63)

where $\eta_0$ is an early conformal time such that $\tilde{\theta}(k, \eta_0) = 0$. It is now easy to determine $\theta_s$ and $\theta_p$. We obtain

$$\theta_s(k, \eta) = \frac{1}{7} \int_{\eta_0}^\eta \Delta H(\eta') [6e^{-\tau(0, \eta')} + e^{-\frac{4}{7} \tau(0, \eta')} - e^{-\frac{3}{7} \tau(0, \eta')} ] e^{ik \cos \theta_p (\eta' - \eta)} d\eta' ,$$

(64)

$$\theta_p(k, \eta) = \frac{1}{7} \int_{\eta_0}^\eta \Delta H(\eta') [6e^{-\tau(0, \eta')} + e^{-\frac{4}{7} \tau(0, \eta')} - e^{-\frac{3}{7} \tau(0, \eta')} ] e^{ik \cos \theta_p (\eta' - \eta)} d\eta' .$$

(65)

In summary, we have found that the temperature fluctuations induced by the spatial anisotropy of the geometry of the universe at large scales is given by equations (56), (64) and (65). Obviously, we are interested in the temperature anisotropies for $\eta = \eta_0$ ($\eta_0$ is the conformal time now). As will be evident later on, the main contributions to the integrals in equations (64) and (65) come from conformal times near the decoupling conformal time $\eta_0$. Moreover, observing that $\eta_0 \ll \eta_0$ we may write

$$\Theta(k, \eta_0, \mu, \hat{p}) \simeq \Theta_s \left( \mu^2 - \frac{1}{3} \right) e^{-ik \cos \theta_p \eta_0} \left( \frac{1}{1} \right) + \Theta_p (1 - \mu^2) e^{-ik \cos \theta_p \eta_0} \left( \frac{1}{1} \right) ,$$

(66)

where

$$\Theta_s \simeq \frac{1}{7} \int_{\eta_0}^{\eta_0} \Delta H(\eta') [6e^{-\tau(0, \eta')} + e^{-\frac{4}{7} \tau(0, \eta')} ] d\eta' ,$$

(67)

$$\Theta_p \simeq \frac{1}{7} \int_{\eta_0}^{\eta_0} \Delta H(\eta') [6e^{-\tau(0, \eta')} + e^{-\frac{4}{7} \tau(0, \eta')} - e^{-\frac{3}{7} \tau(0, \eta')} ] d\eta' .$$

(68)

In Appendix A, we evaluate the two parameters $\theta_s$ and $\theta_p$. We find (see equations A15 and A8)

$$\theta_s \simeq - \frac{1}{2} \times 0.944 \epsilon_{\text{dec}}^2 ,$$

(69)

$$\theta_p \simeq 8.92 \times 10^{-3} \epsilon_{\text{dec}}^2 .$$

(70)

4 THE QUADRUPOLE ANOMALY

We are, now, in position to discuss the low quadrupole anomaly in the CMB temperature anisotropies detected by WMAP and recently confirmed by Planck. The temperature anisotropies of the cosmic background depend on the polar angle $\theta, \phi$, so that one usually expands in terms of spherical harmonics:

$$\frac{\Delta T(\theta, \phi)}{T_0} = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\theta, \phi) .$$

(71)
where \( T_0 \simeq 2.7255 \, K \) (Fixsen 2009) is the actual (average) temperature of the CMB radiation. Note that the \( a_{lm} \)'s in equation (71) are dimensionless and are obtained from the corresponding coefficients in equation (5) by dividing by \( T_0 \). After that, one introduces the power spectrum:

\[
\left( \frac{\Delta T_{\ell}}{T_0} \right)^2 = \frac{1}{2\ell+1} \sum |a_{lm}|^2,
\]

that fully characterizes the properties of the CMB temperature anisotropy. In particular, we focus on the quadrupole anisotropy \( \ell = 2 \):

\[
Q^2 = \left( \frac{\Delta T_{\ell}}{T_0} \right)^2.
\]

In the standard model, the CMB temperature fluctuations are induced by the cosmological perturbations of the FRW homogeneous and isotropic background metric generated by the inflation-produced potentials. In the ellipsoidal universe, we must also consider the effects on the CMB anisotropies induced by the anisotropic expansion of the universe. In fact, as discussed in Section 2, at large scales the observed anisotropies in the CMB temperature are due to the linear superposition of the two contributions according to equation (47). Therefore, we may write

\[
a_{lm} = a_{lm}^A + a_{lm}^I.
\]

In the previous section, we have determined the contributions to temperature contrast function induced by the anisotropic expansion of the universe:

\[
\Theta^A(k, \theta_0, \mu, \theta) \simeq \theta_0 \left( \mu^2 - \frac{1}{3} \right) e^{-ik \cos \theta \theta_0}, \quad \mu = \cos \theta \mu_0.
\]

In Appendix B, starting from equation (75) we perform the multipole expansion of the temperature fluctuation correlations and obtain the multipole coefficients \( a_{lm}^A \). However, it is evident from equation (75) that the main contribution to the temperature fluctuations is for \( k \simeq 0 \). It is easy to see that this corresponds to solve the Boltzmann equation (48) by neglecting the spatial dependence on the temperature contrast function. In this case, we obtain at once

\[
\Delta T_{\ell}^A(\theta, \phi) \simeq \theta_0 \left( \cos^2 \theta_{\mu_0} \theta - \frac{1}{3} \right),
\]

where \( \theta_0 \) is given by equation (69) and \( \theta, \phi \) are the polar angles of the photon momentum \( p \).

Let \( \theta_0, \phi_0 \) be the polar angles of the direction of the axis of symmetry \( n \), then

\[
\frac{\Delta T_{\ell}^A(\theta, \phi)}{T_0} \simeq \frac{2}{3} \theta_0 P_2(\cos \theta_{\mu_0}) = \frac{2}{3} \theta_0 \frac{4\pi}{5} \sum_{m=-2}^{+2} Y_{2m}(\theta, \phi) Y_{2m}^*(\theta_0, \phi_0).
\]

Since from equation (71) it follows that

\[
a_{2m}^A = \int d\Omega \frac{\Delta T_{\ell}^A(\theta, \phi)}{T_0} Y_{2m}^*(\theta, \phi),
\]

we obtain immediately

\[
a_{2m}^A \simeq -\frac{4\pi}{15} \epsilon^2 Y_{2m}^*(\theta_0, \phi_0), \quad \epsilon^2 \equiv 0.944 \epsilon_{\text{dec}}^2,
\]

while \( a_{lm}^I = 0 \) for \( \ell \neq 2 \). In other words, at large scales the anisotropy of the metric contributes mainly to the quadrupole CMB temperature anisotropies. From equation (79), we find

\[
a_{20}^A \simeq -\frac{4\pi}{15} \epsilon^2 \sqrt{\frac{5}{16\pi}} \left[ 1 - 3 \cos^2 \theta_0 \right],
\]

\[
a_{21}^A = (a_{2,-1}^A) \simeq +i \frac{4\pi}{15} \epsilon^2 \sqrt{\frac{15}{8\pi}} e^{-i\theta_0} \sin \theta_0 \cos \theta_0,
\]

\[
a_{22}^A = (a_{2,-2}^A) \simeq \frac{4\pi}{15} \epsilon^2 \sqrt{\frac{15}{32\pi}} e^{-2i\theta_0} \sin^2 \theta_0.
\]

After a little algebra, we rewrite equation (80) as

\[
a_{20}^A \simeq +\frac{1}{6} \epsilon^2 \sqrt{\frac{\pi}{5}} \left[ 1 + 3 \cos^2 (2\theta_0) \right],
\]

\[
a_{21}^A = (a_{2,-1}^A) \simeq +i \sqrt{\frac{\pi}{30}} \epsilon^2 e^{-i\theta_0} \sin (2\theta_0),
\]

\[
a_{22}^A = (a_{2,-2}^A) \simeq +i \sqrt{\frac{\pi}{30}} \epsilon^2 e^{-2i\theta_0} \sin^2 \theta_0.
\]
Defining the quadrupole anisotropy:

\[
Q_i^2 = \left( \frac{\Delta T_{i}^2}{T_0} \right)^2,
\]

we find

\[
Q_i \simeq \frac{2}{5\sqrt{3}} \epsilon^2.
\]

To determine the coefficients \(a_{i,m}\), equation (74), we need to know the \(a_{i,m}^l\). First, we observe that the temperature anisotropies are real functions, so that we must have \(a_{i,-m} = (-1)^m(a_{i,m})^*\). Observing that \(a_{20}^l = (-1)^m(a_{2,0}^l)^*\) [see equation (81)], we have the constraints \(a_{20}^l = (-1)^m(a_{2,0}^l)^*\). Moreover, because the standard inflation-produced temperature fluctuations are statistically isotropic, we reasonably assume that the \(a_{i,m}^l\) coefficients are equal up to a phase factor. Therefore, we can write

\[
a_{20}^l \simeq \sqrt{\frac{\pi}{3}} Q_j,
\]

\[
a_{21}^l = -\left(a_{2,-1}^l\right)^* \simeq +i \sqrt{\frac{\pi}{3}} e^{i\phi_1} Q_j,
\]

\[
a_{22}^l = \left(a_{2,2}^l\right)^* \simeq \sqrt{\frac{\pi}{3}} e^{i\phi_2} Q_j,
\]

where \(0 \leq \phi_1, \phi_2 \leq 2\pi\) are unknown phases. It is easy to check that

\[
Q_i^2 = \left( \frac{\Delta T_i^2}{T_0} \right)^2.
\]

Using equation (7), we obtain the estimate

\[
Q_i \simeq (12.44 \pm 3.93) \times 10^{-6}.
\]

Taking into account equations (73), (74), (81) and (84), we obtain for the total quadrupole:

\[
Q^2 = Q_A^2 + Q_i^2 + 2 f (\theta_0, \phi_0, \phi_1, \phi_2) Q_A Q_i,
\]

where

\[
f (\theta_0, \phi_0, \phi_1, \phi_2) = \frac{1}{4\sqrt{3}} \left[ 1 + 3 \cos (2\theta_0) \right] + \frac{3}{10} \sin (2\theta_0) \cos (\phi_1 + \phi_2) + \frac{3}{10} \sin^2 \theta_0 \cos (2\phi_0).
\]

Equations (87) and (88) show that, indeed, if the space–time background metric is not isotropic, the quadruple anisotropy may become smaller than the one expected in the standard isotropic \(\Lambda\)CDM cosmological model of temperature fluctuations. In fact, from equation (74) and using equations (81) and (84), we obtain

\[
a_{20} \simeq +i \sqrt{\frac{\pi}{3}} e^{i\phi_1} Q_j + \frac{1}{6} \epsilon^2 \sqrt{\frac{\pi}{3}} \left[ 1 + 3 \cos (2\theta_0) \right],
\]

\[
a_{21} \simeq +i \sqrt{\frac{\pi}{3}} e^{i\phi_2} Q_j + i \frac{\pi\epsilon^2}{30} \sin (2\theta_0),
\]

\[
a_{22} \simeq +i \sqrt{\frac{\pi}{3}} e^{i\phi_2} Q_j + \frac{\pi\epsilon^2}{30} \sin (2\theta_0).
\]

Equations (89)–(91) give a system of five equations which can be solved to obtain the five unknown parameters \(\epsilon^2, \theta_0, \phi_0, \phi_1, \phi_2\). To do this, however, we need the observed values of the \(a_{i,m}\). In fact, Campanelli et al. (2007) used the cleaned CMB temperature fluctuation maps of the WMAP data obtained using the internal linear combination with galactic foreground subtraction. In particular, these authors used three different maps (de Oliveira-Costa & Tegmark 2006; Hinshaw et al. 2007; Park, Park & Gott 2007). Actually, the same procedure can be applied to the foreground-cleaned CMB maps obtained from the Planck data as detailed in Ade et al. (2013b). However, irrespective from the adopted CMB cleaned map the quadrupole anomalies detected by WMAP and confirmed by Planck are accounted for if

\[
a_{21} \approx 0,
\]

and

\[
|a_{20}|^2 \ll 2 |a_{22}|^2.
\]
In fact, it is easy to check that these equations imply both the almost planarity and the suppression of power of the quadrupole moment. Remarkably, it turns out that equations (92) and (93) allow us to determine the eccentricity at decoupling and constraint the polar angles of the symmetry axis. Inserting equation (92) into equation (90), we readily obtain
\[ \epsilon^2 \simeq \frac{\sqrt{10} Q_I}{|\sin(2\theta_n)|} \cdot \tag{94} \]
where \( \phi_n + \phi_1 \simeq 0^\circ, 360^\circ \) if \( \sin(2\theta_n) < 0 \), or \( \phi_n + \phi_1 \simeq 180^\circ, 540^\circ \) if \( \sin(2\theta_n) > 0 \). Moreover, from equation (89) and taking into account equation (93), we find
\[ \cos(2\theta_n) \simeq -\frac{1}{3} - 2\sqrt{\frac{5}{3}} \frac{Q_I}{\epsilon^2}. \tag{95} \]
Combining equations (94) and (95), we obtain
\[ \theta_n \simeq \arctan\left(\pm \sqrt{\frac{5}{2}} + 2\right) \simeq 73^\circ, 107^\circ. \tag{96} \]
This last equation together with equations (86) and (94) gives the eccentricity at decoupling:
\[ \epsilon_{\text{dec}} \simeq (0.86 \pm 0.14) \times 10^{-2}. \tag{97} \]
Finally, using equations (73), (92) and (93), we obtain
\[ Q^2 \simeq \frac{6}{25\pi} [a_{22}]^2 \simeq \frac{6}{25\pi} Q_I^2 \left[ 1 + \frac{\sin^4 \theta_n}{\sin^2(2\theta_n)} + \frac{2\sin^2 \theta_n}{|\sin(2\theta_n)|} \cos(\phi_2 + 2\phi_n) \right]. \tag{98} \]
Using the observed value of the quadrupole temperature anisotropy equation (6), we estimate from equation (98)
\[ \cos(\phi_2 + 2\phi_n) \simeq -0.92 \pm 0.12. \tag{99} \]
To summarize, our almost model-independent analysis allowed to fix the eccentricity at decoupling, equation (97). As concern the symmetry axis, using the galactic coordinates \( b_n, l_n \), we found
\[ b_n \simeq \pm 17^\circ, \tag{100} \]
while the longitude \( l_n \) turned out to be poorly constrained in qualitative agreement with Campanelli et al. (2007).

5 THE LARGE-SCALE POLARIZATION

In this section, we discuss the large-scale polarization in the primordial cosmic background. In our previous work (Cea 2010), we argued that the ellipsoidal geometry of the universe induces sizeable polarization signal at large scale without invoking the CMB reionization mechanism. If we assume that early CMB reionization is negligible, then it is well known that at large scale the primordial inflation induced cosmological perturbations do not produce sizeable polarization signal (Dodelson 2003; Mukhanov 2005). In this case, the polarization of the temperature fluctuations are fully given by the anisotropic expansion of the universe. According to our discussion in Section 3, we may write
\[ \Theta^E(k, \eta_0, \mu, \hat{\nu}) \simeq \Theta_p (1 - \cos^2 \theta_{\text{pm}}) e^{-ik \cos \theta_{\text{pm}}}, \tag{101} \]
where the superscript \( E \) indicates that the temperature polarized signal contributes only to the so-called E-modes. In fact, equation (66) shows that the anisotropy of the metric of the universe gives rise only to a linear polarization of the cosmic background radiation. In Appendix B, we discuss the multipole expansion of the temperature polarization correlations. As we have already observed, the main contribution to the polarization temperature contrast functions is for \( k \simeq 0 \), which corresponds to neglect the spatial dependence of the solutions of the Boltzmann equation. Thus, we have
\[ \left. \frac{\Delta T^E(\Theta, \phi)}{T_0} \right|_{\ell = 0} \simeq \Theta_p \left(1 - \cos^2 \theta_{\text{pm}}\right) = \frac{2}{3} \Theta_p - \frac{2}{3} \Theta_p P_2(\cos \theta_{\text{pm}}). \tag{102} \]
We may, now, expand in terms of spherical harmonics as in equation (71). It is evident from equation (102) that the non-zero multipole coefficients \( a^E_{\ell m} \) are for the monopole \( \ell = 0 \) and the quadrupole \( \ell = 2 \). The monopole term determines the average large-scale polarization of the CMB:
\[ \left. \frac{\Delta T_{\text{pol}}}{T_0} \right|_{\ell = 0} \equiv \frac{1}{4\pi} \int d\Omega \left. \frac{\Delta T^E(\Theta, \phi)}{T_0} \right|_{\ell = 0} \simeq \frac{2}{3} \Theta_p. \tag{103} \]
Using equations (70) and (97), we obtain
\[ \Delta T_{\text{pol}} \simeq (1.20 \pm 0.38) \mu K. \tag{104} \]
in qualitative agreement with our previous estimate (Cea 2010).
(105)

which implies

\[ a_{l0}^E \simeq - \frac{8\pi}{15} \theta_p \sqrt{\frac{5}{16\pi}} \left[ 1 - 3 \cos^2 \theta_0 \right]. \]

(106)

or better:

\[ a_{l0}^E \simeq + \frac{1}{3} \theta_p \sqrt{\frac{\pi}{5}} \left[ 1 + 3 \cos^2(2\theta_0) \right]. \]

(107)

Equation (107) allows us to evaluate the quadrupole EE correlation:

\[ \left( \frac{\Delta T_{EE}^2}{T_0} \right)^2 = \frac{3}{5\pi} \sum_{m=-2}^{m=2} |a_{lm}^E|^2. \]

In fact, a straightforward calculation gives

\[ \left( \frac{\Delta T_{EE}^2}{T_0} \right)^2 \simeq \frac{16}{75} \theta_p^2. \]

Using again equations (70) and (97), we find

\[ \Delta T_{EE}^2 \simeq 0.83 \pm 0.27 \mu K. \]

We may, also, estimate the quadrupole TE correlation:

\[ \left( \frac{\Delta T_{TE}^2}{T_0} \right)^2 = \frac{3}{5\pi} \sum_{m=-2}^{m=2} a_{lm}^T \left( a_{lm}^E \right)^* = \frac{3}{5\pi} \left\{ a_{l0}^T a_{l0}^E + 2 \Re \left[ a_{l0}^E \right] + 2 \Re \left[ a_{l1}^E \right] \right\}. \]

(111)

where the \( a_{lm}^T \) are given by equations (89)–(91). Using equations (92) and (93), we may simplify equation (111) as

\[ \left( \frac{\Delta T_{TE}^2}{T_0} \right)^2 \simeq 4 \frac{6}{5\pi} \theta_p \sin^2 \theta_0 \left[ Q_j \cos(\phi_2 + 2\phi_0) + \frac{1}{\sqrt{10}} \epsilon^2 \sin^2 \theta_0 \right]. \]

(112)

After using equation (94), this last equation can be rewritten as

\[ \left( \frac{\Delta T_{TE}^2}{T_0} \right)^2 \simeq 4 \frac{4}{5\sqrt{10}} \theta_p Q_j \sin^2 \theta_0 \left[ \cos(\phi_2 + 2\phi_0) \sin^2 \theta_0 \right]. \]

(113)

Finally, using our previous estimates equations (96) and (99) we find

\[ \Delta T_{TE}^2 \simeq 3.14 \pm 0.76 \mu K. \]

6 CONCLUSIONS

In this paper, we solved at large scales the Boltzmann equation for the CMB photon distribution function by considering the effects of the inflation primordial scalar perturbations and the anisotropy of the geometry in the ellipsoidal universe model. We showed explicitly that the CMB temperature fluctuations are obtained by the linear superposition of the temperature fluctuations induced by the cosmological scalar perturbations and by the spatial anisotropy of the metric. We found that the anisotropic expansion of the universe, the so-called cosmic shear, affects mainly the quadrupole correlation functions. Moreover, we showed that these effects extend also to the low-lying multipoles \( \ell \sim 10 \).
We confirmed previous results that the low quadrupole temperature correlation, detected by WMAP and by the Planck satellite, could be accounted for if the geometry of the universe is plane-symmetric with eccentricity at decoupling of the order of $10^{-2}$. We showed that the ellipsoidal geometry of the universe produces sizeable polarization signal at large scales. We found that our estimate of the quadrupole TE correlation were in agreement both in sign and magnitude with observations. On the other hand, regarding the quadrupole EE correlation our result did not compare well with the final analysis of the Wilkinson Microwave Anisotropy Probe Collaboration. However, we feel that the rather low polarization signal detected by WMAP at large scales could be due to an overestimation of the foreground polarization signal. In fact, in the standard reionization scenario the large-scale polarization in the temperature fluctuations is produced by the fraction of the rescattered photons on the scales corresponding to the reionization horizon. As a consequence in this usually adopted scenario the polarization anisotropies are present for $\ell \geq 2$. That means, in particular, that there is no average polarization. On the other hand, we have shown that the anisotropic expansion in the ellipsoidal universe model implies the presence of large-scale polarization in the temperature fluctuations without invoking reionization processes. In fact, at variance with the usually accepted scenario, in the ellipsoidal universe we have a sizeable average polarization signal at level $\sim \mu$K. If this average polarization in the temperature fluctuations of the cosmic background is misinterpreted as foreground polarization signal, then it could result in a considerable underestimate of the CMB polarization signal at large scales. Therefore a careful characterization of foreground polarization is certainly crucial for polarization measurements.

In conclusion, we are reinforcing the proposal that the ellipsoidal universe cosmological model is a viable alternative that could account for the detected large-scale anomalies in the cosmic microwave anisotropies.

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APPENDIX A: EVALUATION OF THE PARAMETERS $\theta_3$ AND $\theta_P$

In this appendix, we evaluate the parameters $\theta_3$ and $\theta_P$ given by equations (67) and (68). In fact, these two parameters have been estimate in Cea (2010) by assuming that the plane-symmetric geometry is induced by a cosmological magnetic field. Presently, we would like to present a slightly better estimate which is valid irrespective of the physical mechanism responsible for the generation of the spatial anisotropy in the early universe.
Let us consider, first, the parameter $\theta_p$. Defining $\tau(\eta) = \tau(\eta_0, \eta)$, it is easy to verify that $\tau(\eta', \eta) = \tau(\eta') - \tau(\eta)$. Observing that $\tau(\eta_0) \simeq 0$, we rewrite equation (68) as

$$\theta_p \simeq \frac{1}{7} \int_0^{\eta} \Delta H(\eta') \left[ e^{-\tau(\eta')} - e^\frac{\Lambda}{3} \tau(\eta') \right] d\eta'.$$

(A1)

It is convenient to rewrite the integral in equation (A1) in terms of the cosmic time:

$$\theta_p \simeq \frac{1}{7} \int_0^{\eta} \frac{1}{2} t'(t') \left[ e^{-\tau(t')} - e^\frac{\Lambda}{3} \tau(t') \right] dt',$$

(A2)

where $t_0$ is the age of the universe and we used equation (58). To evaluate the derivative in equation (A2), we note that in general (Campanelli et al. 2007) $e^2(t) \sim a(t)^{-3/2}$. Thus, in the matter-dominated era we may write near decoupling:

$$\frac{1}{2} \frac{d}{dt} e^2(t) \simeq - \frac{3}{4} e^2(t) H(t).$$

(A3)

After changing the integration variable by using instead of the cosmic time $t$ the red-shift $z$, we obtain

$$\theta_p \simeq - \frac{3}{28} \int_0^\infty e^2(z') \left[ \frac{1 + z}{1 + z_d} \right]^{3/2} \left[ e^{-\tau(z')} - e^\frac{\Lambda}{3} \tau(z') \right] dz'.

$$

(A4)

Since near decoupling we may write

$$e^2(z) \simeq e^{2}_{\text{dec}} \left[ \frac{1 + z}{1 + z_d} \right]^{3/2}, \quad e^{2}_{\text{dec}} = e^2(z_d),$$

(A5)

where $z_d \simeq 1090$ is the redshift at decoupling, we obtain

$$\theta_p \simeq - \frac{3}{28} e^{2}_{\text{dec}} \int_0^\infty \left( \frac{1 + z}{1 + z_d} \right) \frac{1}{1 + z} \left[ e^{-\tau(z)} - e^\frac{\Lambda}{3} \tau(z) \right] dz.$$

(A6)

Finally, it is known that near decoupling to a good approximation one can write (Jones & Wyse 1985)

$$\tau(z) \simeq 0.37 \left( \frac{z}{1000} \right)^{1.25}, \quad 500 < z < 1400.$$  

(A7)

This allows us to evaluate numerically the integral in equation (A6). We obtain

$$\theta_p \simeq 8.92 \times 10^{-3} e^{2}_{\text{dec}}.$$  

(A8)

To evaluate the parameter $\theta_\alpha$, we note that

$$\theta_\alpha \simeq \frac{1}{7} \int_0^{\eta} \Delta H(\eta') \left[ 6e^{-\tau(\eta')} + e^\frac{\Lambda}{3} \tau(\eta') \right] d\eta' + \int_0^{\eta} \Delta H(\eta') d\eta',$$

(A9)

where $\eta^*$ is a conformal time such that $\tau(\eta_0) = 0$ for $\eta \geq \eta^*$. The second integral in the right hand in equation (A9) is elementary:

$$\int_0^{\eta^*} \Delta H(\eta') d\eta' = \int_0^{\eta^*} \frac{1}{2} \frac{d}{d\eta'} e^2(\eta') d\eta' = \int_0^{\eta^*} \frac{1}{2} \frac{d}{d\eta} e^2(t') d\eta' = \frac{1}{2} e^2(t_0) - \frac{1}{2} e^2(t^*) = - \frac{1}{2} e^2(t^*),$$

(A10)

since $e^2(t_0) = 0$. On the other hand, using equation (A3) we have

$$\frac{1}{7} \int_0^{\eta} \Delta H(\eta)[6e^{-\tau(\eta')} + e^\frac{\Lambda}{3} \tau(\eta')] d\eta' \simeq - \frac{3}{4} \int_0^{t^*} e^2(t') H(t') \left[ \frac{6}{7} e^{-\tau(t')} - \frac{1}{7} e^\frac{\Lambda}{3} \tau(t') \right] dt'.$$

(A11)

After using equation (A5), we obtain

$$\theta_\alpha \simeq - \frac{1}{2} e^{2}_{\text{dec}} f(z^*),$$

(A12)

where

$$f(z^*) = \left( \frac{1 + z^*}{1 + z_d} \right)^{3/2} + \frac{3}{2} \int_{z^*}^{\infty} \left( \frac{1 + z}{1 + z_d} \right)^{3/2} \frac{1}{1 + z} \left[ \frac{6}{7} e^{-\tau(z)} - \frac{1}{7} e^\frac{\Lambda}{3} \tau(z) \right] dz.$$  

(A13)

The integral in equation (A13) can be evaluated numerically. In fact, we find that $f(z^*)$ is almost independent of $z^*$:

$$f(z^*) \simeq 0.944, \quad 200 \leq z^* \leq 900.$$  

(A14)

Thus, our final result is

$$\theta_\alpha \simeq - \frac{1}{2} \times 0.944 e^{2}_{\text{dec}}.$$  

(A15)
APPENDIX B: MULTIPOLe EXPANSiON OF THE LARGE-SCALE TEMPERATURE ANISOTROPiES

In this appendix, we would like to discuss the multipole expansion of the temperature fluctuation correlation functions. According to the results in Section 3, we have (omitting the superscript $A$):

$$\Theta^T(\mathbf{k}, \mathbf{n}) \simeq \theta_n \left( \cos^2 \theta_{np} - \frac{1}{3} \right) e^{-i k \cos \theta_{np} \theta_n},$$  \hspace{1cm} (B1)

$$\Theta^E(\mathbf{k}, \mathbf{n}) \simeq \theta_p \left( 1 - \cos^2 \theta_{np} \right) e^{-i k \cos \theta_{np} \theta_n},$$  \hspace{1cm} (B2)

corresponding to the temperature and polarization contrast functions, respectively. For definiteness, let us discuss first the temperature–temperature correlations. As is well known (Dodelson 2003), we need to evaluate

$$\langle \Theta^T(\mathbf{p} \cdot \mathbf{n}, \mathbf{p}) \Theta^T(\mathbf{x}, \mathbf{n}, \mathbf{p}) \rangle = \int \frac{d^3k}{(2\pi)^3} \Theta^T(\mathbf{k}, \mathbf{n}, \mathbf{p}) [\Theta^T(\mathbf{k}, \mathbf{n}, \mathbf{p})]^*.$$  \hspace{1cm} (B3)

After expanding $\Theta^T(\mathbf{x}, \mathbf{n}, \mathbf{p})$ in spherical harmonics, one obtains

$$\langle \Theta^T(\mathbf{x}, \mathbf{n}, \mathbf{p}) \Theta^T(\mathbf{x}, \mathbf{n}, \mathbf{p}) \rangle = \sum_{\ell} C_{\ell}, \quad C_{\ell} = \frac{1}{2\ell + 1} \sum_{m=-\ell}^{+\ell} C_{\ell m},$$  \hspace{1cm} (B4)

with

$$C_{\ell m} = \int \frac{d^3k}{(2\pi)^3} \int d\Omega_{\mathbf{p}} \int d\Omega_{\mathbf{p}'} Y^*_{\ell m}(\mathbf{p}) \Theta^T(\mathbf{k}, \mathbf{n}, \mathbf{p}) Y_{\ell m}(\mathbf{p}') [\Theta^T(\mathbf{k}, \mathbf{n}, \mathbf{p})]^*.$$  \hspace{1cm} (B5)

Now we use the well-known identities (Abramowitz & Stegun 1970):

$$P_t(\hat{x} \cdot \hat{x}) = \frac{4\pi}{2\ell + 1} \sum_{m=-\ell}^{+\ell} Y^*_{\ell m}(\hat{x}) Y_{\ell m}(\hat{x}),$$  \hspace{1cm} (B6)

and

$$e^{-ik \cdot x} = \sum_{l=0}^{+\infty} i^l (2\ell + 1) j_{\ell}(k x) P_t(\hat{k} \cdot \hat{x}) = 4\pi \sum_{\ell m} i^\ell j_{\ell}(k x) Y_{\ell m}(\hat{k}) Y^*_{\ell m}(\hat{x}),$$  \hspace{1cm} (B7)

to obtain

$$C_{\ell m} = \frac{8}{9\pi} \theta_n^2 \sum_{\ell_1 m_1 \ell_2 m_2 = -\ell}^{\ell} \int_0^\infty dk k^2 j_{\ell_1}^2(k \eta_0) \int d\Omega_{\mathbf{p}} Y^*_{\ell_1 m_1}(\mathbf{p}) Y_{\ell m}(\mathbf{p}) P_2(\hat{p} \cdot \hat{n}) \int d\Omega_{\mathbf{p}'} Y_{\ell m}(\mathbf{p}) Y_{\ell m}(\mathbf{p}') P_2(\hat{p} \cdot \hat{n}).$$  \hspace{1cm} (B8)

Using again equation (B6), we rewrite equation (B8) as

$$C_{\ell m} = \frac{8}{9\pi} \left( \frac{4\pi}{5} \right)^2 \theta_n^2 \sum_{\ell_1 m_1 \ell_2 m_2 = -\ell}^{\ell} \int_0^\infty dk k^2 j_{\ell_1}^2(k \eta_0) Y_{2\ell_1}(\hat{n}) Y^*_{2\ell m}(\hat{n}) \left| \int d\Omega_{\mathbf{p}} Y_{\ell m}(\mathbf{p}) Y_{\ell_1 m_1}(\mathbf{p}) Y_{2\ell m}(\hat{p}) \right|^2.$$

The angular integral can be expressed in terms of the Wigner 3j symbols (Messiah 1961):

$$\int d\Omega_{\mathbf{p}} Y_{\ell_1 m_1}(\mathbf{p}) Y_{2\ell_2 m_2}(\mathbf{p}) Y_{2\ell_3 m_3}(\mathbf{p}) = \sqrt{\frac{(2\ell_1 + 1)(2\ell_2 + 1)(2\ell_3 + 1)}{4\pi}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}.$$

(B10)

In our case, using the well-known properties of the 3j symbols, we have the constraints:

$$\ell_1 = \ell, \quad \ell \pm 2.$$  \hspace{1cm} (B11)

Actually, we are interested in the limit of large $\ell$. To this end, we use the estimate for the asymptotic limit of the average 3j symbols (Borodin, Kroshilin & Tolmachev 1978):

$$\left \langle \left( \begin{array}{c} \ell_1 \\ m_1 \\ m_2 \\ m_3 \end{array} \right) \right \rangle \sim \frac{1}{2\pi \ell^2}, \quad \ell_1 \sim \ell, \quad m_1 \sim m.$$  \hspace{1cm} (B12)

to obtain

$$C_{\ell m} \sim \frac{8}{9\pi} \theta_n^2 \frac{5}{4\pi} \frac{1}{\ell^2} \int_0^\infty \frac{dk}{\ell^2} j_{\ell_1}^2(k \eta_0) \left( \frac{4\pi}{5} \right)^2 \sum_{m_2 = -\ell}^{\ell} Y_{2\ell_1}(\hat{n}) Y^*_{2\ell m}(\hat{n}).$$  \hspace{1cm} (B13)

Since

$$\frac{4\pi}{5} \sum_{m_2 = -\ell}^{\ell} Y_{2\ell_1}(\hat{n}) = P_2(1) = 1,$$  \hspace{1cm} (B14)
we obtain

\[ C_\ell = \frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} C_{\ell m} \simeq \frac{8}{9 \pi^2} \theta_p^2 \int_0^\infty dk \, k^2 \, j^2_\ell(k \eta_0). \]  

To evaluate the integral over \( k \), we note that

\[ j_\ell(x) = \sqrt{\frac{\pi}{2x}} \, J_{\ell + \frac{1}{2}}(x). \]

So that we are left with the following integrals:

\[ I_\ell = \int_0^\infty dk \, k^2 \, j^2_\ell(k \eta_0) = \frac{\pi}{2} \int_0^\infty dk \, \frac{k}{\eta_0} \, J^2_{\ell + \frac{1}{2}}(k \eta_0). \]  

It is easy to see that the integrals in equation (B17) are divergent in the ultraviolet region \( k \to \infty \). This divergence is an artefact of our approximations. To overcome this problem, we must cut-off the spectrum for high wavenumbers. To our purpose, it is enough to assume a power-law cut-off function \( k^{-\alpha} \), \( 0 < \alpha < 1 \). Thus, we obtain

\[ I_\ell = \frac{\pi}{2 \eta_0} \int_0^\infty dx \, x^{1-\alpha} \, J^2_{\ell + \frac{1}{2}}(x). \]

Using (Gradsteyn & Ryzhik 1983)

\[ \int_0^\infty dt \, t^{-\lambda} \, J_\nu(t) \, J_\mu(t) = \frac{\Gamma(\lambda) \, \Gamma\left( \frac{\mu+\nu+2+1}{2} \right)}{2^{\lambda} \, \Gamma\left( \frac{\mu+\nu+1}{2} \right) \, \Gamma\left( \frac{\mu+2+1}{2} \right) \, \Gamma\left( \frac{\nu+2+1}{2} \right)}, \]

we obtain

\[ I_\ell = \frac{\pi}{2^{\nu-1} \eta_0^{1-\alpha}} \, \frac{\Gamma(\alpha-1)}{\Gamma\left( \frac{\nu}{2} \right)^2} \, \ell^{-1+\alpha}. \]

For large \( \ell \), we use the estimate (Gradsteyn & Ryzhik 1983):

\[ \lim_{|z| \to \infty} \frac{\Gamma(z+\alpha)}{\Gamma(z)} = e^{-a \ln |z|} = |z|^{-\alpha}, \]

to obtain

\[ I_\ell \sim \frac{\pi}{2^{\nu} \eta_0^{1-\alpha}} \, \frac{\Gamma(\alpha-1)}{\Gamma\left( \frac{\nu}{2} \right)^2} \, \ell^{-1+\alpha}, \quad 0 < \alpha < 1. \]

This last equation shows, indeed, that the anisotropy of the metric contributes mainly at large scales affecting only the low-lying multipoles, at least for the temperature–temperature anisotropy correlations.

For the polarization correlations, we rewrite equation (B2) as

\[ \Theta^\ell(k, \mathbf{n}, \hat{p}) \simeq \frac{2}{3} \theta_p e^{-ik \cos \theta_p \eta_0} - \frac{2}{3} \theta_p P_2(\cos \theta_{\bar{n}p}) \, e^{-ik \cos \theta_{\bar{n}p} \eta_0}. \]

Therefore, we have two contributions to the EE correlations. The second term on the right hand of equation (B24) is analogous to the temperature contrast function equation (B1), while the first term would contribute to the coefficient \( C_\ell \) with

\[ C_\ell \simeq \frac{8}{9 \pi^2} \theta_p^2 \int_0^\infty dk \, k^2 \, j^2_\ell(k \eta_0). \]

Using equations (B19)–(B21), we find

\[ \ell(\ell + 1) \, C_\ell \sim \frac{8}{9} \theta_p^2 \frac{\Gamma(\alpha-1)}{2^{\alpha} \eta_0^{1-\alpha}} \, \Gamma\left( \frac{\nu}{2} \right)^2 \, \ell^{\ell+\alpha}, \quad 0 < \alpha < 1. \]

Equation (B26) would imply that the EE correlation functions due to the anisotropy of the metric are sizeable not only for the low-lying multipoles, but also for higher multipoles. However, from equation (65) we see that the polarization of the CMB (without reionization) at the present time is essentially that produced around the time of recombination, since much later the free electron density is negligible, while much earlier the optical depth is very large. Then, the present polarization is the result of Thomson scattering around the time of decoupling of
matter and radiation, which occurs after the free electron density starts to drop significantly (Peebles 1993). Moreover, to obtain equation (66) we assumed that $k\Delta \eta_d \ll 1$, where $\Delta \eta_d$ is the conformal time duration of the decoupling process (the thickness of the last scattering surface). In fact, for $k\Delta \eta_d \gg 1$ the oscillations in the integrand produce a cancellation of the temperature anisotropy polarization. In other words, the finite thickness of the last scattering surface damps the final temperature polarization on these scales. Thus, for wavelengths comparable or smaller than the width of the last scattering surface, the polarization should fall off very rapidly. Indeed, the polarization signal should be confined up to multipoles such that $\ell \sim \frac{\Delta \eta_d}{dz} \sim 10^{-1}$. Actually, more precise statements can be only obtained by solving numerically the radiative transfer equation for the CMB including polarization in anisotropic universes. Remarkably, quite recently Pontzen & Challinor (2007) have derived the radiative transfer equation in the nearly FRW limit of homogeneous, but anisotropic, universes classified via their Bianchi type. In fact, these authors argued that the polarization signal is mostly confined to multipoles $\ell \lesssim 10$.

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