Can the Landau-Lifshitz equation explain the spin-wave instability in ferromagnetic thin films for parallel pumping?

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Abstract

Spin-wave instability is studied analytically in the case of parallel pumping for thin films under external field perpendicular to the film plane. It is examined whether the instability threshold derived from only the Landau-Lifshitz (LL) equation can explain experimental instability threshold without using the microscopically-derived spin-wave line width $\Delta H_k$, which is conventionally used. It is revealed that the butterfly curve cannot be explained from only the LL equation at least in an analytical way. By contrast, for the case of perpendicular pumping, the Suhl instability was well explained from the LL equation. The difference between the two cases comes from the nonlinear terms describing the relaxation of spin waves. It is suggested how the nonlinear terms in the LL equation should be related to $\Delta H_k$ for parallel pumping.

Key words: spin-wave instability, parallel pumping, thin film

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1 Introduction

Spin-wave instability is a nonlinear phenomena which is dominated by a balance between excitation and relaxation of spin waves. In early 1950s, anomalous phenomena incompatible with the conventional theory were observed in perpendicular-pumping ferromagnetic resonance (FMR) experiments [1,2]. The phenomena was successfully explained by Suhl’s theory [3]. In the theory, the coupling between the uniform mode ($k=0$) and other spin-wave modes

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\( (k \neq 0) \) plays a crucial role. In perpendicular pumping, the microwave magnetic field excites the uniform mode. Energy is fed from a \( k = 0 \) mode or a pair of \( k = 0 \) modes to a pair of spin waves \( k \) and \(-k\) (\( k \neq 0 \)). The pair of spin waves which is most strongly coupled to the uniform mode grows exponentially when the intensity of the pumping field exceeds a certain threshold. Such processes can be well described by using only the Landau-Lifshitz (LL) equation governing the magnetization dynamics.

On the other hand, the spin-wave instability for parallel pumping has been investigated in order to study relaxation phenomena of spin waves with definite wave vectors since the first experimental studies in 1960s [4,5]. In the parallel pumping experiments, the microwave magnetic field is applied parallel to the equilibrium magnetization. Therefore the longitudinal component of a standing spin-wave, which consists of a pair of traveling spin-waves \( k \) and \(-k\), interacts with the microwave magnetic field. When the power of the pumping field exceeds a certain threshold, the spin waves \( k \) and \(-k\) grow exponentially. The threshold depends on the balance between excitation and relaxation of spin waves. The relaxation of spin waves can be described by the spin-wave line width \( \Delta H_k \) which is derived from microscopic magnon-magnon scattering theories. To explain the experimentally-observed spin-wave instability threshold, i.e. so-called butterfly curve, \( \Delta H_k \) is used as a factor of damping in most of conventional theories. For example, Patton et al. proposed some trial \( \Delta H_k \) functions and successfully explained butterfly curves of the instability threshold [9]. In contrast to the Suhl instability in the perpendicular pumping, however, there has been no attempt to use only the LL equation for explaining the experimental butterfly curve.

In the 1980s, there occurred a renaissance on studies of FMR [6–8]. High-resolution experiments were carried out for spin-wave nonlinear dynamics in yttrium iron garnet (YIG) films and spheres in parallel and perpendicular pumping methods. For certain high power beyond the Suhl instability threshold, the interaction between excited spin-wave modes lead to various dynamical phenomena such as chaos and turbulence. Since then the studies on spin wave instabilities have acquired a renewed interest up to now [10–12]. Despite these pioneering experiments, the corresponding theoretical investigations for parallel pumping leave several crucial questions. In particular, no dynamical equation leading to auto-oscillations and chaos [13] and pattern formations [14] can touch on even a qualitative feature of the experimental butterfly curve for the instability threshold. In the case of parallel pumping, therefore, it is important to analyze the butterfly curve from a viewpoint of modern nonlinear dynamics.

In this paper, we try to derive analytically the spin-wave instability threshold in ferromagnetic thin films for parallel pumping without using the spin-wave line width \( \Delta H_k \) which comes from microscopic theories. By using only the
LL equation, besides the linear instability analysis, the nonlinear instability is examined for a simple case. It will be shown that the instability threshold obtained in that way cannot explain experimental butterfly curves. We will reveal why our nonlinear instability analysis fails and suggest what terms arising from the LL equation should correspond to $\Delta H_k$.

2 Equations of motion for the spin-wave amplitudes

The LL equation is given by

$$\frac{1}{\gamma} \partial_t M = M \times H_{\text{eff}} - \frac{\lambda}{M_0} M \times (M \times H_{\text{eff}}).$$

(1)

Here, $M(r)$ is a magnetization field and $M_0$ is its value in thermal equilibrium; $\gamma$ is the gyromagnetic ratio ($\gamma < 0$ for spins); $\lambda$ is a damping constant parameter ($\lambda < 0$ in this case); $H_{\text{eff}}$ is the effective magnetic field:

$$H_{\text{eff}} = D \nabla^2 M + H^d + H^a + H_0 + h \cos \omega t.$$ 

(2)

The first term comes from the exchange interaction; the second term is the demagnetization term; the third term is the anisotropy field, which is omitted below for convenience; the fourth and fifth terms are the external static and pumping fields, respectively. We assume that $H_0//h$ and these fields are parallel to the $z$ axis. The demagnetization field is given as the gradient of the magneto-static potential $\phi$:

$$H^d = -\nabla \phi.$$ 

(3)

The magneto-static potential obeys the Poisson equation:

$$\nabla^2 \phi = \begin{cases} 
4\pi \nabla \cdot M, \text{ inside the sample;} \\
0, \text{ outside the sample.}
\end{cases}$$

(4)

Throughout the text, we call a set of Eqs. (1)-(4) as the LL equation. Now we introduce the normalized magnetization $\vec{S} = M/M_0$, considering the length of $M$ is invariant: $M_0 = |M|$. It is convenient to project $\vec{S}$ stereographically onto a complex variable $\psi(r, t)$ [15]:

$$\psi = \frac{S_x + iS_y}{1 + S_z},$$ 

(5)
where

\[ S_x = \frac{\psi + \psi^*}{1 + \psi \psi^*}, \quad S_y = \frac{i(\psi^* - \psi)}{1 + \psi \psi^*}, \quad S_z = \frac{1 - \psi \psi^*}{1 + \psi \psi^*}. \quad (6) \]

In terms of \( \psi \), Eq. (1) is rewritten as

\[
\partial_t \psi - i(1 - i\lambda) \left\{ D M_0 \left[ \nabla^2 \psi - \frac{2 \psi^* (\nabla \psi)^2}{1 + \psi \psi^*} \right] - \frac{1}{2} (1 - \psi^2) \partial_z \phi \\
- \frac{i}{2} (1 + \psi^* \partial_y \psi + (\partial_z \phi - H_0 - h \cos \omega t) \psi) \right\} = 0. \quad (7)
\]

Inside the sample, \( \phi \) satisfies

\[
\nabla^2 \phi = \frac{4\pi M_0}{(1 + \psi \psi^*)^2} \left\{ (1 - \psi^2) \partial_x \psi + (1 - \psi^2) \partial_x \psi^* \\
+ i[(1 + \psi^2) \partial_y \psi^* - (1 + \psi^* \partial_y \psi] - 2(\psi \partial_z \psi + \psi^* \partial_z \psi) \right\}. \quad (8)
\]

The demagnetizing field is affected by boundary conditions. We consider a film of thickness \( d \) infinitely extended in the \( x-y \) plane under the external field applied parallel to \( z \) axis. We assume unpinned surface spins, which satisfy Neumann-like boundary conditions,

\[
\partial_z S \big|_{z = \pm d/2} = 0, \quad (9)
\]

namely,

\[
\partial_z \psi \big|_{z = \pm d/2} = \partial_z \psi^* \big|_{z = \pm d/2} = 0. \quad (10)
\]

Introducing the dimensionless time and space units [14],

\[
t \to \frac{t}{4\pi |\gamma|M_0}, \quad \mathbf{r} \to \mathbf{r}d, \quad (11)
\]

we obtain the linearized equations of motion corresponding to Eqs. (7) and (8):

\[
\partial_t \psi + i(1 - i\lambda) \left[ \frac{i^2 \nabla^2 \psi - \frac{1}{2} (\partial_x + i\partial_y) \Phi + (\partial_z \Phi - \omega_H - \omega h \cos \omega pt) \psi}{2} \right] = 0, \quad (12)
\]

\[
\nabla^2 \Phi = (\partial_x - i\partial_y) \psi + (\partial_z + i\partial_y) \psi^* \quad \text{when} \quad -\frac{1}{2} < z < \frac{1}{2}, \quad (13)
\]
\[
\Phi = \frac{\phi}{4\pi M_0 d}, \quad \omega_H = \frac{H_0}{4\pi M_0}, \quad \omega_h = \frac{h}{4\pi M_0}, \quad \omega_p = \frac{\omega}{4\pi M_0 |\gamma|} \tag{14}
\]

First of all, we consider the undriven case (i.e. \(\omega_h = 0\)) and expand \(\psi(r, t)\) and \(\psi^*(r, t)\) so that they fulfills the boundary conditions (10). For even modes, \(k_z = 2m\pi\) (\(m\) integer),

\[
\psi(r, t) = \sum_k a_k(t) e^{i(k_x x + k_y y)} \cos k_z z
\]
\[
\psi^*(r, t) = \sum_k a^*_k(t) e^{i(k_x x + k_y y)} \cos k_z z.
\tag{15}
\]

For odd modes, \(k_z = (2m + 1)\pi\) (\(m\) integer),

\[
\psi(r, t) = \sum_k a_k(t) e^{i(k_x x + k_y y)} \sin k_z z
\]
\[
\psi^*(r, t) = -\sum_k a^*_k(t) e^{i(k_x x + k_y y)} \sin k_z z.
\tag{16}
\]

Using the expansions (15) and (16), we obtain solutions of Eq. (13):

\[
\Phi(r) = -i \sum_k \frac{e^{i(k_x x + k_y y)}}{k^2} (k_- a_k + k_+ a^*_k) f_k \cos k_z z \quad \text{for even modes};
\]
\[
\Phi(r) = -i \sum_k \frac{e^{i(k_x x + k_y y)}}{k^2} (k_- a_k - k_+ a^*_k) f_k \sin k_z z \quad \text{for odd modes}. \tag{17}
\]

Here \(k_+ = k_x + ik_y\), \(k_- = k_x - ik_y\), \(k_\perp = \sqrt{k_x^2 + k_y^2}\) and

\[
f_k = 1 - \left(1 - e^{-k_\perp}\right) \frac{k_\perp}{k^2} \quad \text{when } k_z = 0;
\]
\[
f_k = 1 - 2 \left(1 - e^{-k_\perp}\right) \frac{k_\perp}{k^2} \quad \text{when } k_z \neq 0.
\tag{18}
\]

The detailed derivation of these solutions is shown in Appendix A and Ref. [16]. The derivatives \(\partial_x \Phi\) and \(\partial_y \Phi\) are calculated from Eq. (17). However, \(\partial_x \Phi = -H_z^d / 4\pi M_0\) is approximated with the value of uniform magnetization, \(k = 0\): \(H_z^d = -4\pi N_z M_z\), where \(N_z\) is a demagnetizing factor. In this case, \(N_z = 1\). Considering \(M_z \simeq M_0\), we have \(\partial_x \Phi = 1\). Then Eq. (12) for \(\omega_h = 0\) is rewritten
as
\[ \partial_t \psi + i(1 - i\lambda) \left[ i^2 \nabla^2 \psi - \frac{1}{2} (\partial_x + i\partial_y) \Phi + (1 - \omega_H) \psi \right] = 0. \tag{19} \]

From Eqs. (15)-(19), the linearized equation of motion of \( a_k \) is found to be
\[ \partial_t a_k + i(1 - i\lambda) A_k a_k + i(1 - i\lambda) B_k a^*_k = 0, \tag{20} \]
where
\[
A_k = 1 - l^2 k^2 - \omega_H - \frac{1}{2} f_k \sin^2 \theta_k,
\]
\[
B_k = -\frac{(-1)^n}{2} e^{3i\varphi_k} f_k \sin^2 \theta_k,
\]
\[
\sin \theta_k = \frac{k_{\perp}}{k}, \quad \exp(i\varphi_k) = \cos \varphi_k + i \sin \varphi_k = \frac{k_x}{k_{\perp}} + i \frac{k_y}{k_{\perp}}. \tag{21} \]

Here, \( n \) is an integer and \( k_z = \pi n \).

Equation (20) represents two coupled harmonic oscillators \( a_k, a^*_k \) and can be diagonalized by means of the Holstein-Primakoff transformation:
\[
a_k = \nu_k b_k - \mu_k b^*_{-k}, \\
a^*_k = \nu_k b^*_{-k} - \mu_k b_{-k}, \tag{22} \]

where
\[
\nu_k = \cosh \frac{\chi_k}{2}, \quad \mu_k = e^{i\beta_k} \sinh \frac{\chi_k}{2},
\]
\[
\cosh \chi_k = \frac{|A_k|}{\left[ A_k^2 - (1 + \lambda^2) |B_k|^2 \right]^{1/2}}. \tag{23} \]

After the transformation (22), Eq. (20) becomes
\[ \partial_t b_k - i(\omega_k + i\eta_k) b_k = 0, \tag{24} \]
where
\[
\omega_k^2 = A_k^2 - (1 + \lambda^2) |B_k|^2, \tag{25} \\
\eta_k = \lambda A_k \tag{26} \]

Equations (25) and (26) express the dispersion relation and a damping rate for the spin wave, respectively.
3 Linear instability

We will first analyze the linear instability for the spin wave under the driving field. When the pumping field \( h \cos \omega t \) is applied, the equation of motion of \( a_k \) corresponding to Eq. (20) becomes

\[
\partial_t a_k + i(1 - i\lambda)A_k a_k + i(1 - i\lambda)B_k a_k^\ast - i(1 - i\lambda)\omega_h \cos(\omega_p t) a_k = 0. \tag{27}
\]

After the the transformation (22), we substitute the following equations into Eq. (27),

\[
b_k(t) = b_0^\ast(t) \exp[i(\omega_p/2)t - \eta_k t], \\
b_k^\ast(t) = b_0(t) \exp[-i(\omega_p/2)t - \eta_k t],
\]

(28)

since the resonance occurs at \( \omega_k = \omega_p/2 \). When only terms contributing to the resonance are left, the variable \( b_k^0 \) satisfies

\[
\partial_t^2 b_k^0 + \left[ (\omega_k - \omega_p/2)^2 - |\rho_k|^2 \right] b_k^0 = 0,
\]

(29)

where

\[
|\rho_k| = \omega_h \sqrt{1 + \lambda^2} \frac{|B_k|}{2\omega_k}.
\]

(30)

Therefore, the exponentially increasing solution for \( b_k \) is

\[
b_k \propto \exp \left[ (|\rho_k| - \eta_k) t + i(\omega_p/2)t \right],
\]

(31)

where

\[
|\rho_k| > \eta_k.
\]

(32)

From Eqs. (30) and (31), the instability threshold \( \omega_h^{\text{crit}} \) is given as

\[
\omega_h^{\text{crit}} = \frac{\omega_p}{\sqrt{1 + \lambda^2}} \min_k \left\{ \eta_k \frac{|B_k|}{|B_k|} \right\}.
\]

(33)

Let us rewrite Eq. (33) with use of Eqs. (25) and (26):

\[
\omega_h^{\text{crit}} = \frac{\omega_p \lambda}{\sqrt{1 + \lambda^2}} \min_k \left[ \frac{\omega_p^2}{4|B_k|^2} + 1 + \lambda^2 \right]^{1/2}.
\]

(34)
By using Eqs. (25) and (34), we might see a theoretical butterfly curve, which is an instability threshold curve plotted against the static field. However, the butterfly curve calculated in that way is found to be totally different from experimental ones. Typical butterfly curves observed in experiments have a cusp at a certain static field: as the static field increasing, $\omega_{crit}^h$ first decreases up to the cusp point and then increases above that point. Here let us focus on the case for static fields below the cusp point. For those static field, the minimum threshold mode corresponds to a spin wave propagating with $\theta_{k} = \pi/2$, i.e. parallel to the film plane. As the static field increases, the wave vector $k$ of the threshold modes decreases, and $k \approx 0$ at the cusp. On the contrary, from Eq. (21), one notices that $\omega_{crit}^h$ in Eq. (34) depends only on $f_k$ when $\theta_{k}$ is fixed and that it grows as the static field increases up to the cusp point. In fact, for $\theta_{k} = \pi/2$,

$$f_k = 1 - (1 - e^{-k})\frac{1}{k}$$

(35)

because $k_z = 0$ and $k_\perp = k$. As $k \to 0$, $f_k \to 0$ and then $\omega_{crit}^h \to \infty$. Such a divergence of $\omega_{crit}^h \to \infty$ has not been observed in experiments. Namely, the theoretical threshold $\omega_{crit}^h$ in Eq. (34) fails to explain the experimental butterfly curve.

One may say that the linear analysis is not sufficient to discuss the spin-wave instability threshold. In the next section, several nonlinear terms are included in equation of motion, and a nonlinear instability is analyzed with expectation to overcome the above problem.

4 Nonlinear instability

Let us rewrite Eqs. (7) and (8) up to third order of $\psi$. Then we have

$$\partial_t \psi + i(1 - i\lambda) \left\{ t^2 \left[ \nabla^2 \psi - 2\psi^*(\nabla\psi)^2 \right] - \frac{1}{2}(\partial_x + i\partial_y)\Phi + \frac{1}{2}\psi^2(\partial_x - i\partial_y)\Phi 
+ (\partial_z\Phi - \omega_H - \omega_h \cos \omega_p t)\psi \right\},$$

(36)

and

$$\nabla^2 \Phi = (1 - 2\psi\psi^*) \left[ (\partial_x - i\partial_y)\psi + (\partial_x + i\partial_y)\psi^* \right] 
- 2(\psi\partial_x\psi^* + \psi^*\partial_x\psi) - \psi^2(\partial_x - i\partial_y)\psi - \psi^2(\partial_x - i\partial_y)\psi^*.$$  

(37)

Our interest lies in the region of static fields below the cusp point, where $\theta_{k} = \pi/2$. Then the expansion of $\psi$ is
\[ \psi(r, t) = \sum_k a_k(t) e^{i(k_xx + k_yy)}; \]
\[ \psi^*(r, t) = \sum_k a^*_k(t) e^{i(k_xx + k_yy)}. \]

Substituting Eq. (38) into Eq. (37), we have

\[ \nabla^2 \Phi = i \sum_k e^{i(k_xx + k_yy)} (k_- a_k + k_+ a^*_k) \]
\[ - i \sum_{k_1, k_2, k_3} e^{i[(k_1 x + k_2 x + k_3 x) + (k_1 y + k_2 y + k_3 y)]} \]
\[ \times \left[ (k_{3+} + 2k_{1+}) a^*_{-k_1} a^*_{-k_2} a_{k_3} + (k_{3-} + 2k_{2-}) a_{k_1} a_{k_2} a^*_{-k_3} \right]. \]

It is convenient to restrict wave vectors for solving this equation. When \( k_1 + k_2 + k_3 = k \), the solution of Eq. (39) is

\[ \Phi = -i \sum_k e^{i(k_xx + k_yy)} \frac{(k_- a_k + k_+ a^*_k)}{k^2} \]
\[ + \sum_{k_1, k_2, k_3} \left[ (k_{3+} + 2k_{1+}) a^*_{-k_1} a^*_{-k_2} a_{k_3} + (k_{3-} + 2k_{2-}) a_{k_1} a_{k_2} a^*_{-k_3} \right] f_k, \]

where

\[ f_k = 1 - \frac{1 - e^{-k_\perp}}{k}. \]

The derivation of these equations is similar to that of the linearized equations.

The nonlinear terms contributing to the resonance have spin waves whose wave vector is \( k \) or \(-k\). When the linear terms have \( k \), a possible combination of \( k_1, k_2 \) and \( k_3 \) is \((k, k', -k')\). Here, however, we assume \( k_1, k_2, k_3 = \pm k \). This assumption, which forbids the multi-mode couplings of spin waves, incorporates the substantial nonlinear terms. Then, from Eqs. (36), (38) and (40), we obtain

\[ \partial_t a_k + i(1 - i\lambda) \left\{ A_k a_k + B_k a^*_{-k} - \omega_l \cos(\omega_p t) a_k \right\} \]
\[ + 2(C_k + f_k) a_k a_{-k} a^*_{-k} + (-C_k + f_k) a_{-k} a_k a^*_{-k} \]
\[ + \frac{2}{3} f_k e^{-2ik} a_k a_{-k} a_{-k} + \frac{2}{3} f_k e^{2ik} (2a^*_{-k} a_k a_{-k} + a^*_{-k} a_{-k} a_{-k}) \right\} = \Phi, \]

where

\[ A_k = 1 - l^2 k^2 - \omega_H - \frac{1}{2} f_k. \]
\[ B_k = -\frac{1}{2} e^{2i\varphi_k} f_k, \quad C_k = -2\ell^2 k^2. \]  

(43)

As in the case of linear instability, we perform the Holstein-Primakoff transformation (22) and substitute Eq. (28). When only terms contributing to the resonance are left, the equations of motion for \( b^o_k \) and \( b^{o*}\_k \) are

\[
\begin{align*}
\partial_t b^o_k - i \left( \omega_k - \frac{\omega_b}{2} + i\eta_k \right) b^o_k + i\xi_k |b^o_k|^2 b^o_k + 2i\zeta_k |b^{o*}\_k|^2 b^o_k - \rho_k b^{o*}\_k &= 0, \\
\partial_t b^{o*}\_k + i \left( \omega_k - \frac{\omega_b}{2} - i\eta_k \right) b^{o*}\_k - i\xi_k |b^{o*}\_k|^2 b^{o*}\_k - 2i\zeta_k |b^o_k|^2 b^{o*}\_k - \rho_k^* b^o_k &= 0 \quad (44)
\end{align*}
\]

where

\[
\begin{align*}
\xi_k &= C_k \left\{ \frac{|B_k|^2(1 + \lambda^2)}{2\omega_k^2} - 1 - \frac{i\eta_k}{\omega_k} \right\} + f_k \left\{ 1 + \frac{3|B_k|^2(1 + \lambda^2)}{2\omega_k^2} + \frac{i\eta_k}{\omega_k} - \frac{3|A_k|f_k\lambda^2}{4\omega_k^2} - \frac{3i\lambda f_k}{4\omega_k} \right\}, \\
\zeta_k &= C_k \left\{ \frac{|B_k|^2(1 + \lambda^2)}{2\omega_k^2} + 1 + \frac{i\eta_k}{\omega_k} \right\} + f_k \left\{ 1 + \frac{3|B_k|^2(1 + \lambda^2)}{2\omega_k^2} + \frac{i\eta_k}{\omega_k} - \frac{3|A_k|f_k\lambda^2}{4\omega_k^2} - \frac{3i\lambda f_k}{4\omega_k} \right\}. \quad (45)
\end{align*}
\]

The detailed derivation is given in Appendix B.

The time-independent solutions (fixed points) of Eq. (44), \( b^o_k = \overline{W} \) and \( b^{o*}\_k = \overline{W}^* \), are given by

\[
|\overline{W}|^2 = \frac{\eta_k \text{Im}(\xi_k + 2\zeta_k) \pm \left[ \eta_k^2 \text{Im}(\xi_k + 2\zeta_k)^2 + |\xi_k + 2\zeta_k|^2 \left( |\rho_k|^2 - \eta_k^2 \right) \right]^{1/2}}{|\xi_k + 2\zeta_k|^2}. \quad (46)
\]

Since the inside of the square root in Eq. (46) should be positive,

\[
|\rho_k|^2 \geq \frac{\text{Re}(\xi_k + 2\zeta_k)^2}{|\xi_k + 2\zeta_k|^2 - \eta_k^2}. \quad (47)
\]

When \( |\rho_k|^2 > \eta_k^2 \), the double sign changes into the plus sign because \( |\overline{W}|^2 \geq 0 \).

By using \( b^o_k = \overline{W} + b_1 \) and \( b^{o*}\_k = \overline{W}^* + b_2 \) in Eq. (44), the linear equations of \( b_1 \) and \( b_2 \) are written as

\[
\frac{d}{dt} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = - \begin{pmatrix} \eta_k + 2i(\xi_k + 2\zeta_k) |\overline{W}|^2 & i(\xi_k + 2\zeta_k) \overline{W}^2 - \rho_k \\ -i(\xi_k^* + 2\zeta_k^*) \overline{W}^2 - \rho_k^* & \eta_k - 2i(\xi_k^* + 2\zeta_k^*) |\overline{W}|^2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \quad (48)
\]
where we assumed $b_1^* = b_2$ and $b_2^* = b_1$. The eigenvalues of the matrix are given as

$$
\Lambda_+ = -\eta_k + 2\text{Im}(\xi_k + 2\zeta_k) |W|^2 + \left[\eta_k^2 - 4\{\text{Re}(\xi_k + 2\zeta_k)^2 |W|^4\}^{1/2},
\Lambda_- = -\eta_k + 2\text{Im}(\xi_k + 2\zeta_k) |W|^2 - \left[\eta_k^2 - 4\{\text{Re}(\xi_k + 2\zeta_k)^2 |W|^4\}^{1/2}. \quad (49)
$$

Since $\Lambda_+ > \Lambda_-$, the instability condition is $\Lambda_+ > 0$. However, this condition does not lead to any nontrivial condition. Then the actual instability condition is Eq. (47). The instability threshold now becomes

$$
\omega_{h\text{crit}} = \frac{\omega_p \eta_k |\text{Re}(\xi_k + 2\zeta_k)|}{|B_k| \sqrt{1 + \lambda^2} |\xi_k + 2\zeta_k|}. \quad (50)
$$

However, this equation cannot explain the experimental butterfly curve of the instability threshold, either. The factor $|\text{Re}(\xi_k + 2\zeta_k)|/|\xi_k + 2\zeta_k|$ is almost constant for the region where our calculation is performed, and we cannot overcome the problem of divergence $\omega_{h\text{crit}} \rightarrow \infty$ as $k \rightarrow 0$.

5 Discussion

We have tried to derive the spin-wave instability threshold for parallel pumping from the LL equation. Neither the linear nor nonlinear analysis has proved to explain the feature of experimental butterfly curves of the threshold. By contrast, the Suhl instability for perpendicular pumping can be well explained from the LL equation [3]. The successful explanation of Suhl instability is based on the fact that the couplings between the uniform mode and other spin-wave modes are dominant for perpendicular pumping. In parallel pumping, however, the uniform mode is not excited. That is the reason why the Suhl instability cannot be applied to the present case.

In order to explain the instability threshold for parallel pumping, it is necessary to deal with the spin-wave relaxation in a proper way. Conventionally, the spin-wave line width $\Delta H_k$, which is calculated from the microscopic theory, is used to describe the spin-wave relaxation. In Ref. [9], the $k$-dependence of $\Delta H_k$ plays a vital role in obtaining the butterfly curve. Such a $k$-dependence of $\Delta H_k$ should correspond to the multi-mode spin-wave coupling, which is not considered in the present treatment of the LL equation. However, the equation of motion with multi-mode couplings is too complex to be solved analytically. That may be the reason why the instability threshold for parallel pumping...
has never been derived in the context of nonlinear dynamics by using only the LL equation.

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A Demagnetization field

Substituting Eqs. (15) and (16) into the Poisson equation, we have

\[
\nabla^2 \Phi = i \sum_k e^{i(k_xx+k_yy)} (k_- a_k + k_+ a_k^*) \cos k_z \quad \text{for even modes;}
\]

\[
\nabla^2 \Phi = i \sum_k e^{i(k_xx+k_yy)} (k_- a_k - k_+ a_k^*) \sin k_z \quad \text{for odd modes.}
\] (A.1)

Generally, for the Poisson equation given as

\[
\nabla^2 \phi(r) = -4\pi \rho(r),
\] (A.2)

the solution is

\[
\phi(r) = \int \frac{\rho(r') \, dV'}{|r' - r|}.
\] (A.3)

By using Eq. (A.3), the solutions of Eq. (A.1) are written as

\[
\Phi(r) = -\frac{i}{4\pi} \sum_k (k_- a_k + k_+ a_k^*) \int \frac{dV'}{|r' - r|} e^{i(k_xx'y') \cos k_z z'}
\]

for even mode;

\[
\Phi(r) = -\frac{i}{4\pi} \sum_k (k_- a_k - k_+ a_k^*) \int \frac{dV'}{|r' - r|} e^{i(k_xx'y') \sin k_z z'}
\] (A.4)

for odd mode.

After integration, Eq. (A.4) becomes
\( \Phi(r) = -i \sum_{k} \frac{e^{i(k_xx + k_yy)}}{k^2} (k_- a_k + k_+ a^*_k) \left[ \cos k_z z - (-1)^m e^{-k_{\perp}/2} \cosh k_{\perp} z \right] \)

for even mode;

\( \Phi(r) = -i \sum_{k} \frac{e^{i(k_xx + k_yy)}}{k^2} (k_- a_k - k_+ a^*_k) \left[ \sin k_z z - (-1)^m e^{-k_{\perp}/2} \sinh k_{\perp} z \right] \)

for odd mode. \( \text{(A.5)} \)

Now let us introduce the projection \( F(z) \) of an arbitrary function \( f(z) \) onto \( \cos k_z z \):

\[ F(z) = \frac{1}{\int_{-1/2}^{1/2} dz f(z) \cos k_z z} / \int_{-1/2}^{1/2} dz \cos^2 k_z z , \]

(A.6)

for \(-1/2 < z < 1/2\). The projection onto \( \sin k_z z \) is also defined in the same way. Applying the above projection, we obtain

\( \Phi(r) = -i \sum_{k} \frac{e^{i(k_xx + k_yy)}}{k^2} (k_- a_k + k_+ a^*_k) f_k \cos k_z z \) for even modes;

\( \Phi(r) = -i \sum_{k} \frac{e^{i(k_xx + k_yy)}}{k^2} (k_- a_k - k_+ a^*_k) f_k \sin k_z z \) for odd modes. \( \text{(A.7)} \)

Here,

\[ f_k = 1 - \left(1 - e^{-k_{\perp}} \right) \frac{k_{\perp}}{k^2} \text{ when } k_z = 0; \]

\[ f_k = 1 - 2 \left(1 - e^{-k_{\perp}} \right) \frac{k_{\perp}}{k^2} \text{ when } k_z \neq 0. \]

(A.8)

**B Nonlinear equation of motion for \( b^0_k \)**

First of all, let the terms of Eq. (42) be represented by \( P_{1,k} \), \( P_{2,k} \), \( P_{3,k} \), \( P_{4,k} \) and \( P_{5,k} \):

\[ P_{1,k} \equiv \partial_t a_k + i(1 - i\lambda) \left( A_k a_k + B_k a_{-k}^* - \omega_h \cos(\omega_p t) a_k \right) \]

\[ P_{2,k} \equiv 2i(1 - i\lambda)(C_k + f_k) a_k a_{-k} a_{-k}^* \]

\[ P_{3,k} \equiv i(1 - i\lambda)(-C_k + f_k) a_k a_k a_k^* \]

\[ P_{4,k} \equiv \frac{3}{2} i(1 - i\lambda) f_k e^{-2i\varphi_k} a_k a_k a_{-k} \]

\[ P_{5,k} \equiv \frac{1}{2} i(1 - i\lambda) f_k e^{2i\varphi_k} (2a_{-k}^* a_k a_k + a_{-k}^* a_{-k} a_{-k}). \]

(B.1)
Next, we introduce a new quantity $R_{i,k} \equiv \nu_k P_{i,k} + \mu_k P^*_{i,-k}$ for $i = 1, \ldots, 5$. After the Holstein-Primakoff transformation (22), we use Eq. (28) and remove the terms which do not contribute to the resonance. Then, we have

\[
R_{1,k} = \partial_t b_k^0 - i \left( \omega_k - \frac{\omega}{2} + i \eta_k \right) b_k^0 - \rho_k b_{-k}^0
\]

\[
R_{2,k} = 2i (C_k + f_k) \left\{ \left( \frac{A_k^2}{\omega_k^2} + \frac{i \eta_k}{\omega_k} \right) |b_{-k}^0|^2 + \frac{|B_k|^2 (1 + \lambda^2)}{2 \omega_k^2} |b_k^0|^2 \right\} \, b_k^0
\]

\[
R_{3,k} = i (\gamma C_k + \gamma f_k) \left\{ \left( 1 + \frac{|B_k|^2 (1 + \lambda^2)}{2 \omega_k^2} + \frac{i \eta_k}{\omega_k} \right) |b_k^0|^2 + \frac{|B_k|^2 (1 + \lambda^2)}{2 \omega_k^2} |b_{-k}^0|^2 \right\} \, b_k^0
\]

\[
R_{4,k} = -\frac{3i f_k^2}{8 \omega_k^2} \left\{ - (1 - \lambda^2) \frac{|A_k|}{\omega_k} + 2i \lambda \right\} \left( 2 |b_{-k}^0|^2 + |b_k^0|^2 \right) \, b_k^0
\]

\[
R_{5,k} = -\frac{3i |A_k| f_k (1 + \lambda^2)}{8 \omega_k^2} \left( 2 |b_{-k}^0|^2 + |b_k^0|^2 \right) \, b_k^0
\]  

(B.2)

where

\[
\rho_k = i (1 - i \lambda) \omega_k \frac{B_k}{2 \omega_k}.
\]  

(B.3)

The terms in Eq. (B.2) satisfy the following:

\[
R_{1,k} + R_{2,k} + R_{3,k} + R_{4,k} + R_{5,k} = 0.
\]  

(B.4)

Equation (44) is obtained from Eqs. (B.2) and (B.4).

References

[1] R. W. Damon, Rev. Mod. Phys. 25, 239 (1953).
[2] N. Bloembergen and S. Wang, Phys. Rev. 93, 72 (1954).
[3] H. Suhl, J. Phys. Chem. Solids 1, 209 (1957).
[4] E. Schlömann, J. J. Green and U. Milano, J. Appl. Phys. 31, 386S (1960).
[5] F. R. Morgenthaler, J. Appl. Phys. 31, 95S (1960).
[6] F. M. de Aguiar and S. M. Rezende, Phys. Rev. Lett. 56, 1070 (1986).
[7] P. Bryant, C. D. Jeffries and K. Nakamura, Phys. Rev. Lett. 60, 1185 (1988); Phys. Rev. A 38, 4223 (1988).
[8] T. L. Carroll, M. L. Pecora and F. J. Rachford, J. Appl. Phys. 64, 5396 (1988).
[9] See, for example, M. Chen and C. E. Patton, in *Nonlinear Phenomena and Chaos in Magnetic Materials*, edited by P. E. Wigen (World Scientific, Singapore, 1994), pp. 33-82.

[10] J. Becker, F. Rödelsperger, Th. Weyrauch, H. Benner, W. Just and A. Čenys, Phys. Rev. E **59**, 1622 (1999).

[11] I. Laulicht and P. E. Wigen, J. Magn. Magn. Mater. **207**, 103 (1999).

[12] I. Laulicht and P. E. Wigen, Prog. Theor. Phys. Suppl. **139**, 375 (2000).

[13] V. S. L’vov, *Wave Turbulence Under Parametric Excitation* (Springer-Verlag, Berlin, 1994).

[14] F. J. Elmer, Phys. Rev. B **53**, 14323 (1996).

[15] M. Lakshmanan and K. Nakamura, Phys. Rev. Lett. **53**, 2497 (1984).

[16] K. Kudo and K. Nakamura, cond-mat/0604329.