Nearly Linear-Time Approximation Schemes for Mixed Packing/Covering and Facility-Location Linear Programs

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Abstract

We describe nearly linear-time approximation algorithms for explicitly given mixed packing/covering and facility-location linear programs. The algorithms compute (1 + \( \epsilon \))-approximate solutions in time \( O(N \log(N)/\epsilon^2) \), where \( N \) is the number of non-zeros in the constraint matrix. We also describe parallel variants taking time \( O(\text{polylog}(1/\epsilon)^4) \) and requiring only near-linear total work, \( O(N \text{polylog}(1/\epsilon)^2) \). These are the first approximation schemes for these problems that have near-linear-time sequential implementations or near-linear-work polylog-time parallel implementations.

1 Introduction

Given non-negative matrices \( P \) and \( C \) and vectors \( p \) and \( c \), a mixed packing/covering linear program is of the form “find \( x \in \mathbb{R}^n_+ \) such that \( Cx \geq c \) and \( Px \leq p \).” If the linear program has a feasible solution, then a \( (1 + \epsilon) \)-approximate solution is an \( x \) such that \( Cx \geq c \) and \( Px \leq (1 + \epsilon)p \). Without loss of generality (by rescaling \( P \) and \( C \)) assume throughout that each \( p_i = 1 \) and \( c_i = 1 \). Define the input size of the instance, denoted \( N \), to be the number of non-zeros in \( P \) and \( C \).

Given a set \( C \) of customers, a set \( F \) of facilities, an opening cost \( f_j \geq 0 \) for each facility \( j \), and a cost \( c_{ij} \geq 0 \) for assigning customer \( i \) to facility \( j \), the standard facility-location linear program is

\[
\begin{align*}
\text{minimize} \quad & \text{cost}(x, y) = \sum_j f_j y_j + \sum_{ij} c_{ij} x_{ij} \\
\text{subject to} \quad & \sum_j x_{ij} \geq 1 \quad \text{for } i \in C, \\
& y_j \geq x_{ij} \geq 0 \quad \text{for } i \in C, j \in F.
\end{align*}
\]

Assume for notational convenience that \( c_{ij} = \infty \) if customer \( i \) may not be assigned to facility \( j \). Here a \( (1 + \epsilon) \)-approximate solution is a feasible pair \( (x, y) \) whose cost is at most \( 1 + \epsilon \) times the minimum. Define the input size of the instance, \( N \), to be the number of non-zeros in \( P \) and \( C \).

For mixed packing/covering linear programs, this paper gives an approximation algorithm that, given any feasible instance and \( \epsilon \in (0, 1/10) \), computes a \( (1 + O(\epsilon)) \)-approximate solution. The first contribution here is to show that the algorithm has a sequential implementation running in \( O(N \log(m)/\epsilon^2) \) time, where \( m \) is the number of constraints. This gives the first sequential approximation scheme to run in near-linear time. The second contribution is to show that the algorithm has a parallel implementation that runs in \( O(\log^2(m) \log(nm) \log(n \log(m)/\epsilon)/\epsilon^4) \) time and does only \( O(N \log(m) \log(n \log(m)/\epsilon)/\epsilon^2) \) work. This is the first (polylog-time) parallel approximation scheme that requires only near-linear work. These results are in Section 2.

The third contribution is an approximation scheme for facility-location linear programs whose sequential implementation runs in \( O(N \log(m)/\epsilon^2) \) time, where \( m \) is the number of customers. This is the first sequential approximation scheme for facility-location LPs to run in near-linear time. (Any facility-location LP can be reformulated as a covering LP by a standard reduction, but this
increases the number of non-zeros exponentially.) The fourth contribution is a parallel variant that takes time \(O(\log^2(m) \log(nm)/\epsilon^4)\) and does only \(O(N \log(m) \log(nm)/\epsilon^2)\) work. These results are in Section 3.

**Related work.** [8] Lemma 8, shows how to reduce mixed packing/covering problems of the more general form \(\min \{\lambda : \exists x \in \mathbb{R}_+^n : Cx \geq x; Px \leq \lambda p\}\) to (a small number of) mixed/packing problems of the form studied here.

[8] gives an approximation scheme for mixed packing/covering LPs, with a sequential implementation running in time \(O(md \log(m)/\epsilon^2)\), where \(m\) is the number of constraints and \(d\) is the maximum number of constraints that any variable appears in. The same paper gives a parallel implementation running in a time bound comparable to the one given here, but the total work done by the implementation is \(\Omega(md \log(m)/\epsilon^2)\). Note that for sparse problems, \(md\) can be much larger than \(N\), so (in general) that sequential algorithm is not near-linear time, and the parallel algorithm does not do only near-linear work.

For pure packing or covering linear programs (of the form \(\max \{w \cdot x : x \in \mathbb{R}_+^n, Px \leq p\}\) or \(\min \{w \cdot x : x \in \mathbb{R}_+^n, Cx \geq c\}\), [3, Thm. 3] gives a randomized approximation scheme running in expected time \(O(N + (n + m) \log(N)/\epsilon^2)\), where \(n + m\) is the number of variables plus constraints. That is faster than the algorithms here for dense problems, but that result does not apply to mixed packing/covering problems.

A recent paper by Allen-Zhu and Orecchia gives a parallel approximation scheme for pure packing or covering running in \(O(\log^2(N)/\epsilon^3)\) time and doing \(O(N \log^2(N)/\epsilon^3)\) work [1]. Notably, that is the first parallel approximation scheme whose running time is polylog with dependence on \(\epsilon\) less than \(1/\epsilon^4\). It is faster than the parallel algorithms here, but requires more total work and does not apply to mixed packing and covering.

[3] Thm. 3] gives an approximation scheme for facility-location LPs taking time \(O((mn)^{3/2}(1/\epsilon + \log \log m))\) for \(m\) customers and \(n\) facilities. [3] Thm. 4] gives an approximation scheme for pure packing LPs taking time \(O(n^2m \log(nm) + (n + m)\sqrt{n \log(m(1/\epsilon + \log \log n)})\) for \(n\) variables and \(m\) constraints. [3] Thm. 6] gives an approximation scheme for set-cover LPs taking time \(O((n + m)n^{3/2}\sqrt{\log m}/\epsilon) + O^*(n^2m)\) for \(n\) sets and \(m\) elements. [2] Thm. 12] gives an approximation scheme for mixed packing/covering taking time \(O(n^2K_p\max(K_p, K_c)K_p^m \log(m) \log(m/\epsilon)/\epsilon)\), where \(K_c\) (\(K_p\)) is the maximum number of nonzeros in any covering (packing) constraint, \(n\) is the number of variables, and \(m\) is the number of constraints. Many of the above results build on the work of Nesterov (e.g. [6]).

### 2 Mixed Packing and Covering

**Theorem 1.** (i) Algorithm [7] is a \((1 + O(\epsilon))\)-approximation algorithm for mixed packing/covering linear programs. That is, given an instance \((C, P) \in \mathbb{R}_+^{m_c \times n} \times \mathbb{R}_+^{m_p \times n}\), such that there exists an \(x \in \mathbb{R}_+^n\) with \(Cx \geq 1\) and \(Px \leq 1\), the algorithm returns an \(x\) such that \(Cx \geq 1\) and \(Px \leq 1 + O(\epsilon)\).

(ii) Algorithm [7] has a sequential implementation running in \(O(N \log(m)/\epsilon^2)\) time, where \(N\) is the number of non-zeros in \(P\) and \(C\), and \(m = m_c + m_p\) is the number of constraints.

(iii) Alg. [4] has a parallel implementation doing work \(O(N \log(n \log(m)/\epsilon) \log(m)/\epsilon^2),\) and taking time \(O\left(\log^2(m) \log(n \log(m)/\epsilon) \log(N)/\epsilon^4\right)\).

The rest of this section proves Thm. [1]. The first two lemmas use standard techniques.
function PACKING-COVERING(matrices $C$, $P$; initial solution $x = x^0 \in \mathbb{R}_+^n$, $\epsilon \in (0, 1/10)$) 

1. Define $U = (\max_i P_i x^0 + \ln m)/\epsilon^2$, 
2. $p_i(x) = (1 + \epsilon)^{P_i x}$, and $c_i(x) = (1 - \epsilon)^{C_i x}$ if $C_i x \leq U$, else $c_i(x) = 0$, 
3. $|p(x)| = |p(x)|_1$ and $|c(x)| = |c(x)|_1 = \sum_i c_i(x)$, 
4. $\lambda(x, j) = P_j^T p(x)/C_j^T c(x)$ and $\lambda^*(x) = \min_{j \in [n]} \lambda(x, j)$. 

5. Initialize $\lambda_0 \leftarrow |p(x)|/|c(x)|$. 
6. Repeatedly do either of the following two operations whose precondition is met: 

   a. precondition: $\lambda^*(x) \geq (1 + \epsilon)\lambda_0$ 
   i. Fix $(\epsilon \lambda_0)$ and $\lambda_0 \leftarrow (1 + \epsilon)\lambda_0$. 
   ii. Choose $\delta \in \mathbb{R}_+^n$ such that 
      (i) $\forall j \in [n], \delta_j > 0$ then $\lambda(x, j) \leq (1 + 4\epsilon)\lambda_0$, and 
      (ii) $\max\{\max_i P_i \delta, \max_{i : C_i x \leq U} C_i \delta\} = 1$ (the maximum increase in any $P_i x$ or active $C_i x$ is 1). 
   7. Let $x \leftarrow x + \delta$. 
   8. If $\min_i C_i x \geq U$ then return $x/U$. 

**Lemma 1.** In Algorithm 1, if $(P, C)$ is feasible, then, for any $x$, $\lambda^*(x) \leq |p(x)|/|c(x)|$. 

**Proof.** Let $x^*$ be a feasible solution. Since $x^*$ is feasible, it also satisfies $(Cx^*) \cdot c(x)/|c(x)| \geq 1 \geq (P x^*) \cdot p(x)/|p(x)|$, that is, $x^* \cdot (C^T c(x)/|c(x)| - P^T p(x)/|p(x)|) \geq 0$. Hence there exists a $j$ such that $C_j^T c(x)/|c(x)| - P_j^T p(x)/|p(x)| \geq 0$, which is equivalent to $\lambda(x, j) \leq |p(x)|/|c(x)|$. 

**Lemma 2.** Given any feasible instance $(P, C)$, Algorithm 1 (i) returns $x$ such that $Cx \geq 1$ and $Px \leq 1 + O(\epsilon)$, (ii) does step (a) at most $O(U)$ times, and (iii) does step (b) at most $O(mU)$ times, where $U = O(\log(m)/\epsilon^2 + \sum_i P_i x^0/\epsilon)$. 

**Proof.** (i) Fix $(P, C)$. First observe that the algorithm maintains the invariant $\lambda_0 \leq |p(x)|/|c(x)|$. The initial choice of $\lambda_0$ guarantees that the invariant holds initially. By inspection, Line 6 is executed only when $\lambda^*(x) = \min_j \lambda(x, j) \geq (1 + \epsilon)\lambda_0$. This and Lemma 1 ($\lambda^*(x) \leq |p(x)|/|c(x)|$) imply that $(1 + \epsilon)\lambda_0 \leq |p(x)|/|c(x)|$, so that the invariant is maintained.

Define $\lambda^c p(x) = \log_{1+\epsilon} \sum_i p_i(x)$ and $\lambda^c c(x) = \log_{1-\epsilon} \sum_i c_i(x)$ for $p$ and $c$ as defined in the algorithm. We show that the algorithm maintains the invariant 

$$(1 + O(\epsilon)) \lambda^c c(x) - \lambda^c x_0 \geq (1 - O(\epsilon)) (\lambda^c p(x) - \max_i P_i x^0/\epsilon - \lambda^c x_0),$$

where $x_0$ is the initial solution given to the algorithm. By inspection, the invariant is initially true. In a given execution of step (ii), let $x$ be as at the start of the step; let index $j$ be the one chosen for the step. The step increases $\lambda^c p(x)$ by at most $(1 + O(\epsilon)) \sum_j \delta_j P_j p(x)/|p(x)|$; it increases $\lambda^c c(x)$ by at least $(1 - O(\epsilon)) \sum_j \delta_j C_j c(x)/|c(x)|$ (see e.g. [7]). Since $\delta_j > 0$ only if $\lambda(x, j) \leq (1 + \epsilon)\lambda_0$, the first invariant $\lambda_0 \leq |p(x)|/|c(x)|$ implies that the invariant is maintained.

Consider the step when the algorithm returns $x/U$. Just before the step, at least one $i \in [m]$ had $C_i x < U$, so $\lambda^c c(x) < \log_{1-\epsilon} (1 - \epsilon) U = U$, and by the above invariant $\max_i P_i x \leq (1 + O(\epsilon)) U$. 

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Algorithm 2 Sequential implementation of packing/covering Algorithm

1. **function** `SEQUENTIAL-PACKING-COVERING(P, C, ε)`
2. Initialize $x_j ← 0$ for $j ∈ [n]$, $λ_0 ← |p(x)|/|c(x)| = m_p/m_c$, and $U = \ln(m)/ε^2$.
3. For each $i$, maintain estimates $\hat{P}_i$ and $\hat{C}_i$ of $P_i x$ and $C_i x$, respectively, to satisfy invariant
   
   $\text{for all } i, \quad \hat{P}_i ≤ P_i x < \hat{P}_i + 1 \quad \text{and} \quad \hat{C}_i ≤ C_i x < \hat{C}_i + 1$. \hfill (3)

4. Maintain vectors $\hat{p}$ and $\hat{c}$ defined by $\hat{p}_i = (1 + ε)\hat{P}_i$ and $\hat{c}_i = (1 - ε)\hat{C}_i$ if $\hat{C}_i ≤ U$, else $\hat{c}_i = 0$.
5. repeat
6.   for each $j ∈ [n]$ do \hfill \text{▷ do a run of operation (b)’s for } j$
7.     Compute values of $P_j^T \hat{p}$ and $C_j^T \hat{c}$ from $\hat{p}$ and $\hat{c}$. Define $\hat{λ}_j = P_j^T \hat{p} / C_j^T \hat{c}$.
8.     while $\hat{λ}_j ≤ (1 + ε)^2λ_0/(1 - ε)$ do
9.         Let $x_j ← x_j + z$, choosing $z$ such that $\max\{ \max_i P_{ij} z, \max_i C_{ij} x ≤ U C_{ij} z \} = 1$.
10.        If $\min_i \hat{C}_i ≥ U$ then return $x/U$.
11.    end while
12.     As described in text, to maintain invariant (3):
13.        For selected $i$, update $\hat{P}_i$, $\hat{p}_i$, and $P_j^T \hat{p}$. For selected $i$, update $\hat{C}_i$, $\hat{c}_i$, and $C_j^T \hat{c}$.
14.     Let $λ_0 ← (1 + ε)λ_0$. \hfill \text{▷ operation (a)}

During the step $\max_i P_i x$ increases by at most $1 = O(εU)$, so after the step $\max_i P_i x ≤ (1 + O(ε))U$ still holds. Part (i) follows.

(ii) The algorithm maintains $λ_0 ≤ |p(x)|/|c(x)|$, with equality at the start. Each time $λ_0$ increases, it does so by a $1 + ε$ factor, but throughout, $P_i x = O(U)$ and $\min_i C_i x = O(U)$, so $|p(x)|/|c(x)|$ is always at most $m(1 + ε)^O(U)/(1 - ε)^O(U)$. Part (ii) follows.

(iii) Each execution of operation (b) either increases some $P_i x$ by at least 1, or increases some $C_i x$ by at least 1 where $C_i x ≤ U$. Since $P_i x = O(U)$ throughout, part (iii) follows.

Lemma 3 (i) implies part (i) of Thm. 1. Next we prove part (ii).

**Lemma 3.** Algorithm 1 has a sequential implementation running in $O(N \log(m)/ε^2)$ time.

**Proof.** The implementation is Alg. 2. It groups iterations of the main loop into rounds. Within a given round, it iterates once over the indices $j ∈ [n]$. While focusing on a particular $j ∈ [n]$, it does operation (b) for that $j$ (taking $δ_j = 1$ and $δ_{j'} = 0$ for $j' ≠ j$) as long as the condition $\hat{λ}_j ≤ (1 + ε)λ_0/(1 - ε)^2$ is met, where $\hat{λ}_j$ is an estimate of $λ(x, j)$ that is guaranteed to satisfy $(1 - ε)λ(x, j) ≤ \hat{λ}_j ≤ (1 + ε)λ(x, j)$. Each sequence of operation (b)’s done for a given $j$ is called a run (for $j$).

Before we describe how to maintain each $\hat{λ}_j$, we observe that, as long as the guarantee on $\hat{λ}_j$ is met during each run for $j$, the implementation is indeed a valid implementation of Alg. 1. Indeed, Alg. 2 only does operation (b) for a given $j$ when $\hat{λ}_j ≤ (1 + ε)^2λ_0/(1 - ε)$, which (with the guarantee on $\hat{λ}_j$ and $ε ≤ 1/10$) ensures $λ(x, j) ≤ (1 + 4ε)λ_0$. Likewise, Alg. 2 only does operation (a) when $\min_j \hat{λ}_j > (1 + ε)^2λ_0/(1 - ε)$, which (with the guarantee) ensures $\min_j λ(x, j) ≥ (1 + ε)λ_0$. Thus, the precondition of each operation is appropriately met. (Alg. 2’s termination condition is slightly different than that of Alg. 1 but this does not affect correctness.) Hence, provided it the guarantee for $\hat{λ}_j$ is met during each run for $j$, Alg. 2 is a valid implementation of Alg. 1.

To maintain the guarantee, Alg. 2 maintains $\hat{p}$ and $\hat{c}$ to satisfy Invariant (3). It also maintains $\hat{p}$ and $\hat{c}$ as defined in line 4 and, during a run for a given $j$, maintains $P_j^T \hat{p}$, $C_j^T \hat{c}$, and...
their ratio $\tilde{\lambda}_j$ as defined in line \[4.\] By inspection, as long as all $\hat{P}_i$ and $\hat{C}_i$ satisfy \[3.\], and the above quantities are maintained correctly, the guarantee on $\tilde{\lambda}_j$ is met during each run for $j$.

To finish, we describe how Alg. 2 maintains each $\hat{P}_i$ and $\hat{C}_i$. At the start of each run for a given $j$, Alg. 2 guarantees that all estimates $\hat{P}_i$ and $\hat{C}_i$ are exactly accurate (that is, $\hat{P}_i = P_i x$ and $\hat{C}_i = C_i x$). It initializes $P_i^T \hat{p}$ and $C_i^T \hat{c}$ from $\hat{p}$ and $\hat{c}$ by direct computation (summing over the non-zero entries of $P_i^T$ and $C_i^T$, respectively). Then, after an operation (b) is done for $j$, in line \[12.\] it updates only those $\hat{P}_i$ and $\hat{C}_i$ for which \[3.\] is in danger of violation, as follows.

**Selecting the $\hat{P}_i$'s and $\hat{C}_i$'s to update.** Define the top of any number $y > 0$ to be the smallest power of 2 greater than or equal to $y$. Within each column $C_i^T$ and $P_i^T$ separately, partition the non-zero entries $C_{ij}$ and $P_{ij}$ into equivalence classes according to their tops, and order the groups by decreasing top (presort the entries within each column to do so). After an operation (b) is done within a run for a given $j$, say that a group $G$ in $P_i^T$ with top $2^t$ is eligible for update if the increase $\delta_G$ in $x_j$ since the last update of group $G$ during the run (or since the start of the run) is at least $1/2^{t+1}$.

Starting with the group $G$ in $P_i^T$ with largest top, Alg. 2 checks the group to see if it’s eligible for update ($\delta_G \geq 1/2^{t+1}$). If it is, then, for each $i$ in the group $G$, Alg. 2 increases $\hat{P}_i$ to $P_i x$ in constant time by adding $P_{ij} \delta_G$ to $\hat{P}_i$. Likewise, it updates all dependents of $\hat{P}_i$ ($\hat{p}_i$, $P_i^T \hat{p}$, $\lambda_j$) in constant time. It then continues with the next group in $P_i^T$ (the one with next smaller top). It stops processing the groups in $P_i^T$ as soon as it reaches a group that is not eligible for update — it doesn’t process any subsequent groups with smaller tops, regardless of eligibility.

Alg. 2 then processes the groups within $C_i^T$ analogously. In addition, when it updates $\hat{C}_i$ for $i$ in a group within $C_i^T$, it checks whether $\hat{C}_i \geq U$, and if so, the algorithm deletes row $i$ from $C$ and associated data structures. When the last row of $C$ is deleted, the algorithm stops, returning $x/U$.

Finally, at the end of the run for $j$, for each group $G$ within column $P_i^T$, for each $i$ in $G$, Alg. 2 updates $\hat{P}_i$ to the exact value of $P_i x$ by increasing $\hat{P}_i$ by $P_{ij} \delta_G$ (for $\delta_G$ as defined above); it updates the dependent $\hat{p}_i$ accordingly. Likewise it updates $\hat{C}_i$ (and $\hat{c}_i$) for every $i$ with $C_{ij} \neq 0$ to their exact values.

**Correctness.** To verify that invariant \[3.\] is met, we claim that, if a given group $G$ with top $2^t$ is not updated after a given operation (b) for $j$, then $\delta_G \leq 1/2^t$. (Invariant \[3.\] for $\hat{P}_i$ follows, because, for $i \in G$, the increase $P_{ij} \delta_G$ in $P_i x$ since the last update of $\hat{P}_i$ is less than $2^t/2^t - 1$). Suppose for contradiction that the claim fails. Consider the first operation (b) for which it fails, and the group $G$ with largest top $2^t$ for which $\delta_G > 1/2^t$ after that operation. Group $G$ cannot be the group with maximum top in its column, because the algorithm considered that group after the operation (b). Let $G'$ be the group with next larger top $2^{t'} > 2^t$. Note that $G'$ was not updated after the operation (b), because if it had been $G$ would have been considered and updated. Let $x_j$ denote the current value of $x_j$, and let $x_j' < x_j$ denote the value at the most recent update of $G'$.

When group $G'$ was last updated, group $G$ was considered but not updated (for, if $G$ had been updated then, we would now have $\delta_G = \delta_{G'} \leq 1/2^{t'} < 1/2^t$). Thus, letting $x_j''$ be the value of $x_j$ at the most recent update of $G$, we have $x_j' - x_j'' < 1/2^{t+1}$. Since group $G'$ was not updated after the current operation (b), we have (by the choice of $G$) that $x_j - x_j' = \delta_{G'} \leq 1/2^{t'} \leq 1/2^{t+1}$. Summing gives $x_j - x_j'' < 2/2^{t+1} = 1/2^t$, violating the supposition $\delta_G > 1/2^t$. 5
Algorithm 3  Parallel implementation of packing/covering Algorithm

function PARALLEL-PACKING-COVERING(\(P, C, \epsilon\))
Initialize \(x_j \leftarrow 1 - n^{-1}/\max_j P_{ij}\) for \(j \in [n]\), \(\lambda_0 \leftarrow |p(x)|/|c(x)| = m_p/m_c\).
Maintain \(U_i, P_{ij}, C_{ij}, p_i(x), c_i(x), \lambda(x, j)\), etc. as defined in Alg. 1.
repeat
while \(\lambda^*(x) \leq (1 + \epsilon)\lambda_0\) do
Define \(J = \{j \in [n] : \lambda(x, j) \leq (1 + \epsilon)\lambda_0\}\), and, for \(j \in J\),
\(I_j^p = \{i : P_{ij} \neq 0\}\) and \(I_j^c = \{i : C_{ij} \neq 0\}\) and \(C_{ij} x \leq U\}.
For \(j \in J\), let \(\delta_j = x x_j + \delta_j\).
For \(i \in \bigcup_{j \in J} I_j^p\), update \(P_{ij} x\) and \(p_i(x)\) for \(i \in \bigcup_{j \in J} I_j^c\), update \(C_{ij} x\) and \(c_i(x)\).
For \(j \in J\), update \(C_{ij}^x c(x)\), \(P_{ij}^c p(x)\), and \(\lambda(x, j)\).
If \(\min_i C_{ij} x \geq U\) then return \(x/U\).
Let \(\lambda_0 \leftarrow (1 + \epsilon)\lambda_0\).
end while
end repeat

The same argument shows that Invariant [3] holds for \(\tilde{C}_i\) (for \(i\) with \(\tilde{C}_i \leq U\)).

**Time.** At the start of each run for a given \(j\), the time it spends in line [11] is proportional to the number of non-zeroes in columns \(j\) of \(P\) and \(C\), as is the time it spends updating all \(\tilde{P}_i\) (for \(P_{ij} \neq 0\)) and \(\tilde{C}_i\) (for \(C_{ij} \neq 0\)) at the end of the run. Thus, the time spent on these actions during a given round (a single iteration of the repeat loop), is \(O(N)\). By Lemma 2 (ii), Alg. 2 does \(O(U)\) rounds (iterations of its outer loop), so the total time for the actions outside of the while loop is \(O(NU)\), as desired.

Within the while loop, the time spent is proportional to the number of changes in the \(\tilde{P}_i\)’s and \(\tilde{C}_i\)’s within the runs. Since \(\tilde{P}_i\) only changes within a run when its group \(G\) is updated, the change in \(\tilde{P}_i\) is \(P_{ij} \delta G \geq P_{ij}/2^{t+1} > 2^{t-1}/2^{t+1} = 1/4\). But \(\tilde{P}_i \leq (1 + O(\epsilon))U\) throughout, so the total time for updates to a given \(\tilde{P}_i\) is \(O(U)\). Likewise (using that \(\tilde{C}_i\) is tracked only while \(\tilde{C}_i \leq U\)), the total time for updates to a given \(\tilde{C}_i\) is \(O(U)\). Hence, the total time spent within the runs is \(O(mU)\), where \(m\) is the total number of constraints. This is \(O(NU)\), as desired.

Next we prove part (iii) of Thm. 1.

**Lemma 4.** Alg. [7] has a parallel implementation doing work \(O(N \log(m) \log(n \log(m)/\epsilon)/\epsilon^2)\), and taking time \(O((\log^2(m) \log(N) \log(n \log(m)/\epsilon)/\epsilon^4))\), where \(n = |F|\) and \(m = |C|\) are, respectively, the number of facilities and customers.

**Proof.** The implementation is Alg. [8]. By inspection Alg. [8] is a valid implementation of Alg. 1. Also, it has max_i P_{ij} x^0 \leq 1, so U = O(log(m)/\epsilon^2). By Lemma 2 (i), it computes a (1 + O(\epsilon))-approximate solution.

Call each iteration of the repeat loop a phase. By Lemma 2 (ii), Alg. 3 uses \(O(U)\) phases. Within each phase, during the first iteration of the while loop Alg. 3 computes all quantities directly. This takes \(O(\log(n + m))\) time and \(O(N)\) total work. In each subsequent iteration of the while loop within the phase, it computes all quantities incrementally, in time \(O(\log(n + m))\) and doing total work linear in the sizes of the sets \(A_p\) and \(A_c\) of active pairs: \(A_p = \{(i, j) : j \in J, i \in I_j^c\}\) and
$A_c = \{(i, j) : j \in J, i \in P_c^\prime\}$. (For example: update each $P_j x$ by noting that the iteration increases $P_j x$ by $\Delta_j^\prime = \sum_{j : i \in P_j^\prime} p_{ij}\delta_j$; update $P_j^\prime p(x)$ by noting that the iteration increases it by $\sum_{i \in P_j^\prime} P_{ij} \Delta_j^\prime$. Update $J$ by noting that $\lambda(x, j)$ only increases within the phase, so $J$ only shrinks, so it suffices to delete a given $j$ from $J$ the first iteration when $\lambda(x, j)$ exceeds $(1 + \epsilon)\lambda_0$.)

To bound the total work, note that in each iteration, if a given $j$ is in $J$, then the iteration increases $x_j$, and when that happens the parameter $z$ is at least $\Theta(1/U)$ (using $P_j x = O(U)$ and $C_i x = O(U)$) so $x_j$ increases by at least a factor of $1 + \Theta(1/U)$. The value of $x_j$ is initially at least $n^{-1}/\max_i P_{ij}$ and finally $O(U\max_i P_{ij})$. It follows that $j$ is in the set $J$ during at most $O(U\log(nU))$ iterations. Thus, for any given non-zero $P_j$, the pair $(i, j)$ is in $A_p$ in at most $O(U\log(nU))$ iterations. Likewise, for any given non-zero $C_{ij}$, the pair $(i, j)$ is in $A_c$ in at most $O(U\log(nU))$ iterations. Hence, the total work for Alg. $A$ is $O(NU\log(nU))$, as desired.

To bound the total time, note that, within each of the $O(U)$ phases, some $j$ remains in $J$ throughout the phase. As noted above, no $j$ is in $J$ for more than $O(U\log(nU))$ iterations. Hence, each phase has $O(U\log(nU))$ iterations. To finish, recall that each iteration takes $O(\log(n + m))$ time.

\end{proof}

\section{Facility Location}

\begin{theorem}
The facility-location LP has a $(1 + O(\epsilon))$-approximation algorithm that (i) has a sequential implementation running in time $O(N\log(m)/\epsilon^2)$, where $m = |C|$ is the number of customers and $N = \{(i, j) \in F \times C : c_{ij} < \infty\}$ is the size of the input, and (ii) has a parallel implementation doing work $O(N\log(N/\epsilon)\log(m)/\epsilon^2)$ and taking time $O(\log^2(m)\log(N/\epsilon)\log(N)/\epsilon^4)$.
\end{theorem}

The standard reduction of Facility Location to Set Cover. To prove the theorem we start with the standard reduction of facility location to covering. Given a facility-location instance $(F, C, f, c)$, the reduction gives the following (set) covering LP $(A, w)$. For each facility $j \in F$ and subset $S \subseteq C$ of customers, the covering LP has a variable $x_j^\prime$ with cost $w_j^\prime = f_j + \sum_{i \in S} c_{ij}$, where $j^\prime = j^\prime(j, S)$ is unique to the pair $(j, S)$. For each customer $i \in C$, the covering LP has a constraint $A_ix \geq 1$, where $A_{ij^\prime} = 1$ if $i \in S$ and 0 otherwise (where $j^\prime = j^\prime(j, S)$). The covering LP is then

\begin{equation}
\min \{w \cdot x' : x' \in \mathbb{R}_+^n, Ax \geq 1\},
\end{equation}

where $n$ is the number of facility/subset pairs $j, S$.

This LP is equivalent to the facility-location LP in the following sense: each feasible solution $x'$ to the covering LP yields a corresponding feasible solution $(x, y)$ of the facility-location LP (where $x_{ij} = \sum_{S \ni i} x_j^\prime(i, S)$ and $y_j = \max_i x_{ij}$) of the same or lesser cost, and the optimal costs of the two LP’s are the same. Because the covering LP is exponentially large, the algorithm won’t construct the covering LP explicitly; rather, it will work with it implicitly.

Generic approximation algorithm for covering. We will apply the generic covering Algorithm $A$ to the covering LP. We will use the next lemma to help prove the performance guarantee for Algorithm $A$ applied to any covering LP $\min \{w \cdot x : x \in \mathbb{R}_+^n, Cx \geq 1\}$.

\begin{lemma}
In Alg. $A$ for any $x$, $\text{OPT}(A, w) \geq |a(x)| \lambda^*(x)$, where $\lambda^*(x) = \min_{j \in [n]} \lambda(x, j)$.
\end{lemma}

\begin{proof}
Let $x^*$ be a solution of cost $w \cdot x^* = \text{OPT}(A, w)$. Draw a single $j \in [n]$ at random from the distribution $x^*/|x^*|$. By calculation the expectation of the quantity $A_j^\prime a(x)/|a(x)| - w_j/(w \cdot x^*)$ is proportional to $(Ax^*)^\top a(x)/|a(x)| - x^* \cdot w/(w \cdot x^*)$, which is non-negative (as $Ax \geq 1$), so with positive probability the quantity is non-negative, implying $w \cdot x^* \geq |a(x)| \lambda(x, j)$.
\end{proof}
Algorithm 4 Generic approximation algorithm for covering linear programs.

1 function COVERING(matrix $A$, cost $w$, initial solution $x = x^0 \in \mathbb{R}_+^n$, $\epsilon \in (0, 1/10)$)
2 Define $U = \ln(m)/\epsilon^2$, where $m$ is the number of constraints,
3 $a_i(x) = (1 - \epsilon)A_i^T x$ if $A_i^T x \leq U$, else $a_i(x) = 0$,
4 $|a(x)| = \sum_i a_i(x)$,
5 $\lambda(x, j) = w_j/(A_j^T a(x))$ ($A_j^T$ is column $j$ of $A$).
6 while $\min_i A_i^T x \leq U$ do
7     Choose vector $\delta \in \mathbb{R}_+^n$ such that
8     (i) $\forall j \in [n]$ if $\delta_j > 0$ then $\lambda(x, j) \leq (1 + O(\epsilon)) \text{OPT}(A, w)/|a(x)|$
9     (ii) $\max\{A_i\delta : i \in [m], A_i^T x \leq U\} = 1$.
10    Let $x \leftarrow x + \delta$.
11 Return $x/U$. 

Here is the performance guarantee for Alg. 4 applied to any covering instance $(A, w)$.

Lemma 6. Alg. 4 returns a solution $x$ such that $w \cdot x \leq (1 + O(\epsilon)) \text{OPT}(A, w) + w \cdot x^0$, where $x^0$ is the initial solution given to the algorithm.

Proof. First we observe that the algorithm is well-defined. In each iteration, by definition of $\lambda^*(x)$, there exists a $j \in [n]$ such that $\lambda(x, j) = \lambda^*(x)$. By Lemma 5 for this $j$, $\lambda(x, j) \leq \text{OPT}(A, w)/|a(x)|$. So, in each iteration there exists a suitable vector $\delta \in \mathbb{R}_+^n$. Next we prove the approximation ratio.

Define $\text{lmin} a(x) = \log_{1-\epsilon}|a(x)|$ for $a$ as defined in the algorithm. We show that the algorithm maintains the invariant

$$(1 + O(\epsilon))(\text{lmin} a(x) - \log_{1-\epsilon} m) \geq \frac{w \cdot x - w \cdot x^0}{\text{OPT}(A, w)}.$$  \hspace{1cm} (4)

The invariant is initially true by inspection. In a given iteration of the algorithm, let $x$ be as at the start of the iteration, let vector $\delta$ be the one chosen in that iteration. The iteration increases $w \cdot x/\text{OPT}(A, w)$ by $\sum_j \delta_j w_j/\text{OPT}(A, w)$. It increases $\text{lmin} a(x)$ by at least $(1 - O(\epsilon)) \sum_j \delta_j A_j^T a(x)/|a(x)|$ (see e.g. [7]). By the choice of $\delta$, the definition of $\lambda^*$, and Lemma 5 if $\delta_j > 0$ then $w_j/\text{OPT}(A, w) \leq (1 + O(\epsilon)) A_j^T a(x)/|a(x)|$, so the invariant is maintained.

Before the last iteration, at least one $i$ has $A_i^T x \leq U$, so $\text{lmin} \text{cov}(x) \leq \log_{1-\epsilon}(1 - \epsilon) U = U$. This and the invariant imply that finally $(w \cdot x - w \cdot x^0)/\text{OPT}(A, w) \leq 1 + (1 + O(\epsilon)) U + \log_{1-\epsilon} m$. By the choice of $U$ this is $(1 + O(\epsilon)) U$.

Next we analyze Alg 5, an implementation of Alg. 4 suited to sequential implementation.

Lemma 7. (i) Algorithm 5 is a valid implementation of the generic covering Algorithm 4, and (ii) Algorithm 5 does Operation (a) $O(U) = O(\log(m)/\epsilon^2)$ times.

Proof. (i) Observe that the algorithm maintains the invariant $\lambda_0 \leq \text{OPT}(A, w)/|a(x)|$. The invariant is true for the initial choice of $\lambda_0$ because the minimum cost to satisfy just a single constraint $A_i^T x \geq 1$ is $\min_{j \in [n]} w_j/A_{ij}$. Operation (a) only decreases $|a(x)|$, so maintains the invariant. Operation (b) is done only when $(1 + \epsilon)\lambda_0 \leq \lambda^*(x)$, which by Lemma 5 is at most $\text{OPT}(A, w)/|a(x)|$, so Operation
Algorithm 5 Sequential implementation of covering Algorithm 4.

1: function SEQUENTIAL-COVERING($A, w, \epsilon$)
2: Define $U, a_i, \lambda(x, j)$, etc. as in Alg. 4
3: Initialize $x_j \leftarrow 0$ for $j \in [n]$ and $\lambda_0 \leftarrow \max_i \min_j \sum_{i\in[n]} w_j / (A_{ij} a(x))$.
4: Repeatedly do one of the following two operations whose precondition is met:
5: a. precondition: $\lambda^*(x) \geq (1 + \epsilon) \lambda_0$
6: Let $\lambda_0 \leftarrow (1 + \epsilon) \lambda_0$.
7: b. precondition: $\lambda^*(x) \leq (1 + 4\epsilon) \lambda_0$ \hspace{1cm} $\triangleright \lambda^*(x) = \min_{j \in [n]} \lambda(x, j)$
8: Choose $j \in [n]$ such that $\lambda(x, j) \leq (1 + 4\epsilon) \lambda_0$.
9: Let $x_j \leftarrow x_j + \min\{1/A_{ij} : A_i x \leq U\}$.
10: If $\min_i A_i x \geq U$, then return $x/U$.

Algorithm 6 Sequential approximation algorithm for facility-location linear program.

1: function SEQUENTIAL-FACILITY-LOCATION($F$, customers $C$, costs $f, c, \epsilon$)
2: Define $U = \ln(m)/\epsilon^2$,
3: $A_i x = \sum_{j \in F} x_{ij}$ (for $i \in C$),
4: $a_i(x) = (1 - \epsilon) A_i x$ if $A_i x \leq U$, else $a_i(x) = 0$,
5: $\lambda(x, j, S) = (f_j + \sum_{c_{ij} \epsilon S} a_i(x)) / \sum_{i \epsilon S} a_i(x)$ (for $j \in F$, $S \subseteq C$).
6: Initialize $y_j, x_{ij} \leftarrow 0$ for $j \in F, i \in C$ and $\lambda_0 \leftarrow (\max_{i \epsilon C} \min_{j \epsilon F} f_j + d_{ij})/|a(x)|$.
7: repeat
8: for each $j \in F$ do
9: while $\lambda(x, j, S_j) \leq (1 + \epsilon) \lambda_0$ where $S_j = \{i \in C : c_{ij} < (1 + \epsilon) \lambda_0 a_j(x)\}$ do
10: Let $y_j \leftarrow y_j + 1$, and, for $i \in S_j$, let $x_{ij} \leftarrow x_{ij} + 1$. \hspace{1cm} $\triangleright$ Operation (b)
11: If $\min_{i \epsilon C} A_i x \geq U$, then return $(x/U, y/U)$. \hspace{1cm} $\triangleright$ Operation (a)
12: until...

(b) also preserves the invariant. Since the algorithm maintains this invariant, it is a special case of Alg. 4 with cost($x^0$) = 0.

(ii) One way to satisfy every constraint $A_i x \geq 1$ is as follows: for every $i$, choose $j$ minimizing $w_j/A_{ij}$ to $x_j$. The cost of this solution is at most $m \lambda_0$. Hence, the initial value of $\lambda_0$ is at least $m^{-1} \text{OPT}(A, w)/|a(x)|$. At termination (by the invariant from part (i) above) $\lambda_0$ is at most $\text{OPT}(A, w)/|a(x)|$. Also, $|a(x)|$ decreases by at most a factor of $m/(1 - \epsilon)^U$ during the course of the algorithm, while each Operation (a) increases $\lambda_0$ by a factor of $1 - \epsilon$. It follows that the number of Operation (a)’s is at most $\log_{1+\epsilon} m^2/(1 - \epsilon)^U = O(U)$.

Returning to Facility Location. Algorithm 6 is a sequential approximation algorithm for facility-location linear programs. To prove its correctness, we show that it is a valid implementation of the sequential covering Algorithm 4 as applied to the covering LP that comes from the standard reduction. Because of the $2^{\lceil |C| \rceil}$ subsets of customers, that reduction gives an exponentially large LP. Algorithm 6 uses the following lemma to deal with that. During the course of Algorithm 6, the lemma allows the algorithm to restrict attention to a single canonical set $S_j$ of customers for any
given facility \( j \in F \).

**Lemma 8.** Consider any \( x, \lambda_0, \) and \( i \in F \) during the execution of Alg. \( \mathcal{G} \). Let \( \lambda(x, j, S) \) be as defined there and \( \lambda_0 = (1 + \epsilon)\lambda_0 \). Then \( \min_{S \subseteq C} \lambda(x, j, S) \leq \lambda_0 \) iff \( \lambda(x, j, S_j) \leq \lambda_0 \) where \( S_j = \{ i \in C : c_{ij} < \lambda_0 a_i(x) \} \).

**Proof.** \( \lambda(x, j, S) \leq \lambda_0 \) iff \( \sum_{i \in S} (c_{ij} - \lambda_0 a_i(x)) \leq 0 \), so \( S = S_j \) is the best set to take given \( j \). \qed

Next is the analysis of Alg. 6 proving Thm. 2.

**Lemma 9.** (i) Algorithm 6 is a valid implementation of generic covering Algorithm 4 on the instance given by the reduction to covering. (ii) Algorithm 6 has a sequential implementation running in time \( O(N \log(\log(N/m)/\epsilon^2)) \).

**Proof.** (i) Keeping in mind the reduction from facility location to covering, doing an Operation (b) in Alg. 5 corresponds to choosing a facility \( j \in F \) and setting \( S_j \) as defined there and \( \lambda \) as defined there and \( \lambda_0 \) as defined there. When Alg. 6 does Operation (b) for \( j, S_j \), this precondition is met because \( \lambda(x, j, S_j) \leq (1 + \epsilon)\lambda_0 \).

Similarly, when Alg. 6 does an Operation (a), the precondition for doing the operation (in Alg. 5) translates to \( \lambda(x, j, S) \geq (1 - \epsilon)\lambda_0 \) for all \( j \in F \) and \( S \subseteq C \). By inspection, Alg. 6 guarantees that \( \lambda(x, j, S_j) \geq (1 - \epsilon)\lambda_0 \) for all \( j \in F \). This suffices by Lemma 8.

(ii) To implement the algorithm, maintain \( x, y, A_i, c_i \) for each \( i \in C \). Within each iteration of the for loop for a given \( j \in F \), call the iterations of the outer loop a phase. To do the first iteration of each phase, compute \( S_j \) and \( \lambda(x, j, S_j) \) directly (from scratch). Then, in each subsequent iteration within the phase, update the relevant quantities incrementally: e.g., after doing Operation (b) for \( j, S_j \), update \( A_i, c_i \) for \( i \in S_j \) with \( A_i, c_i \leq U \) by noting that the operation increases \( A_i, c_i \) by 1 and decreases \( a_i(x) \) by a factor of \( 1 - \epsilon \); delete from \( S_j \) any \( i \)'s that cease to satisfy \( c_{ij} < \lambda_0 a_i(x) \).

Then, the total time spent for the phase is proportional to \( (A) \ |\{ i \in C : c_{ij} < \infty \}| \) (the number of non-zeros associated with facility \( j \)), plus \( (B) \) the number of increments to the \( y_j \)'s and the \( x_{ij} \)'s during the phase. By Thm. 2 (ii), there are \( O(U) \) iterations of the outer loop, so the total time spent for (A) is \( O(UN) \). Since each \( y_i \) and each \( x_{ij} \) never exceeds \( U \), the total time spent for (B) is also \( O(UN) \). \qed

**Parallel facility location.** Next we prove part (iii) of Thm. 2 (the parallel implementation).

**Lemma 10.** (i) Algorithm 7 is a \((1 + O(\epsilon))\)-approximation algorithm for facility-location LPs. (ii) Algorithm 7 has a parallel implementation running in time \( O((\log^2(m) \log(N/\epsilon) \log(N)/\epsilon^4)) \) and doing work \( O(N \log(N/\epsilon) \log(m)/\epsilon^2) \).

**Proof.** The function \( \top-up(x) \) in line 7 does the following for each customer \( i \in C \) independently. Consider the facilities \( j \in F \) such that \( c_{ij} < \lambda_0 a_i(x) \) and \( x_{ij} < y_j \), in order of increasing \( c_{ij} \). When considering facility \( j \), increase \( x_{ij} \) just until either \( x_{ij} = y_j \) or \( c_{ij} = \lambda_0 a_i(x) \) (recall \( a_i(x) = (1 - \epsilon)^A_i x \)). (Do this in parallel in \( O(\log(m+n)) \) time and \( O(N) \) work as follows. For each customer \( i \), assume its facilities are presorted by increasing \( c_{ij} \). Raise each \( x_{ij} \) by \( \delta_{ij} \), computed as follows. Compute the prefix sums \( d_j = \sum_{j' < j} y_{j'} - x_{ij'} \) where \( j' < j \) if \( j' \) is before \( j \) in the ordering. Check for each \( j \) whether \( c_{ij} < (1 - \epsilon)^A_i x + d_j \). If so, then let \( \delta_{ij} = y_j - x_{ij} \); otherwise, if \( c_{ij} < \lambda_0 (1 - \epsilon)^A_i x + d_j \),
Algorithm 7 Return \((1 + \epsilon)-\)approximate facility-location LP solution

1. `function PARALLEL-FACILITY-LOCATION(facilities \(F\), customers \(C\), costs \(f, c\))`
2. `Define \(U, A_i x, a_i(x)\), etc. as in Alg. 4 and \(\ell = \max_{i \in C} \min_{j \in F} f_j + c_{ij}\)`
3. `Initialize \(x_{ij} = \ell / (f_j + c_{ij})|F||C| \) for \(j \in F, i \in C\), and \(y_j = \sum_{i \in C} x_{ij} \) for \(j \in F\).
4. `Initialize \(\lambda_0 = \ell / |a(x)|\).
5. **repeat**
6.   `Let \(\lambda_0 \leftarrow (1 + \epsilon)\lambda_0\).
7.   `Let \(x \leftarrow \text{TOP-UP}(x)\) as described in text. **Guarantees** \(\forall i, j. c_{ij} < \lambda_0 a_i(x)\) only if \(x_{ij} = y_j\).
8. **repeat**
9.   `Define \(S_j = \{ i \in C : c_{ij} < \lambda_0 a_i(x) \}\) and \(J = \{ j \in F : \lambda(x, j, S_j) \leq \lambda_0 \}\).
10.   `For each \(j \in J\), increase \(y_j\) by \(z y_j\), and increase \(x_{ij}\) by \(z y_j\) for \(i \in S_j\),
11.   `choosing \(z\) s.t. \(z \max_{i \in C} \sum_{j \in J, i \in S_j} y_j = 1\) (the max. increase in any \(A_i x\) is 1).
12. **if** \(\min_{i \in C} A_i x \geq U\) **then return** \(x/U\).
13. **until** \(J = \emptyset\)

where \(j’\) is the facility preceding \(j\), then choose \(\delta_{ij}\) to make \(c_{ij} = \lambda_0 (1 - \epsilon)A_i x + d_j + \delta_{ij}\); otherwise, take \(\delta_{ij} = 0\).

(i) We first observe that, except for the call to \text{TOP-UP} in line 7, Alg. 7 is an implementation of algorithm Alg. 4 (as it applies to the covering instance produced by the reduction).

Note that \(\ell \leq \text{OPT}(F, C, f, c) \leq |C|\ell\), because the minimum cost to serve any single customer \(j\) is \(\ell = \min_{j \in F} f_j + c_{ij}\), and each customer can be served at cost at most \(\ell\).

Following the proof of correctness of Algorithms 5 and 6, observe that Alg. 7 maintains the invariant \(\lambda_0 \leq (1 + \epsilon)\text{OPT}(F, C, f, c)/|a(x)|\). (Indeed, initially \(\lambda_0 = \ell / |a(x)| \leq \text{OPT}(F, C, f, c)/|a(x)|\), because \(\ell \leq \text{OPT}(F, C, f, c)\). Increasing \(x_{ij}\)’s and \(y_j\)’s only decreases \(|a(x)|\), so preserves the invariant. When Alg. 7 increases \(\lambda_0\) to \((1 + \epsilon)\lambda_0\), by inspection \(\min_{j \in F} \lambda(x, j, S_j) > \lambda_0\). By Lemma 8, this ensures \(\min_{j \in F, S \subseteq C} \lambda(x, j, S) \geq \lambda_0\), which by Lemma 5 implies \(\lambda_0 \leq \text{OPT}(F, C, f, c)/|a(x)|\), so the invariant is preserved.

Since \(\lambda_0 \leq (1 + \epsilon)\text{OPT}(F, C, f, c)/|a(x)|\), by inspection of the definition of \(J\) in Alg. 7, the increment to \(x\) and \(y\) corresponds to a valid increment in Alg. 4.

Hence, except for the call to \text{TOP-UP} in line 7, Alg. 7 is an implementation of algorithm Alg. 4.

Regarding the call to \text{TOP-UP}, we claim that it preserves Invariant 4 in the proof of correctness of Alg. 4. To verify this, consider any \(x_{ij}\) with \(x_{ij} < y_j\) and \(c_{ij} < \lambda_0 a_i(x)\). Increasing \(x_{ij}\) increases \(c \cdot x/\text{OPT}\) at rate \(c_{ij}/\text{OPT}\) which by the assumption on \(c_{ij}\) and the invariant on \(\lambda_0\) is at most \((1 + \epsilon)a_i(x)/|a(x)|\). On the other hand, increasing \(x_{ij}\) increases \(\text{min}(a_i(x))\) at rate at least \((1 - O(\epsilon))a_i(x)/|a(x)|\) (see e.g. 7). Hence, invariant 4 is preserved, and the performance guarantee from Lemma 5 (i) holds here.

Since \(\ell \leq \text{OPT}(F, C, f, c)\), the initial solution \((a^0, y^0)\) costs at most \(\epsilon\text{OPT}(F, C, f, c)\), so, by that performance guarantee, Alg. 7 returns a cover of cost \((1 + O(\epsilon))\text{OPT}(F, C, f, c)\). This shows part (i) of Lemma 10.

(ii) Call each iteration of the outer loop a phase. By Lemma 9 (ii), there are \(O(U)\) phases. Consider any phase. In the first iteration of the inner loop, compute all quantities \(J, A_i x, a_i(x)\) for each \(i \in C, \lambda(x, j, S_j)\) etc. directly, from scratch, in \(O(N)\) work and \(O(\log(n + m))\) time. In each subsequent iteration within the phase, update each changing quantity incrementally (similarly to
Alg. 3). In this way, the work during each subsequent iteration is proportional to $\sum_{i \in F} |S_j|$, the number of pairs $i, j$ where $i \in S_j$.

At the start of the phase, the call to top-up ensures that $x_{ij} = y_j$ for $i \in S_j$. Because each $S_j$ decreases monotonically during a phase, this property is preserved throughout the phase. In the choice of $z$, each sum $\sum_{j \in J : i \in S_j} y_j$ is therefore equal to $\sum_{j \in J : i \in S_j} x_{ij}$, which is less than $U$ (as $a_i(x) = 0$ if the sum exceeds $U$). Therefore, $z$ is at least $1/U$. Hence, for each $i \in S_j$, the variable $x_{ij}$ increases by at least a $1 + 1/U$ factor in the iteration. On the other hand, at the start of the algorithm $x_{ij} = \epsilon \ell / (f_i + c_{ij})$, while at the end (by the performance guarantee (i) and $\ell \leq \text{OPT}(f, c)$), $x_{ij} \leq \ell / (f_i + c_{ij})$. Hence, $x_{ij}$ increases by at most a factor of $|F| |C| / \epsilon$ throughout. Hence, the number of iterations in which $i$ occurs in any $S_j$ is $O(\log_U |F| |C| / \epsilon) = O(U \log(|F| |C| / \epsilon)) = O(U \log mn/\epsilon)$. In each such iteration, $i$ contributes to at most $|\{ j | c_{ij} < \infty \}|$ pairs. Hence, the total work is $O(NU \log(mn/\epsilon))$.

To bound the time, note that within each of the $O(U)$ phases, there is some pair $i', j'$ such that $i'$ is in $S_{j'}$ in the next-to-last iteration of the phase, and (since the sets $J$ and $S_{j'}$ monotonically decrease during the phase) $i'$ is in $S_{j'}$ in every iteration of the phase. As observed above, $i'$ occurs in $\cup_j S_j$ in at most $O(U \log(mn/\epsilon))$ iterations. Hence, the number of iterations of the inner loop within each phase is $O(U \log(mn/\epsilon))$. Since each iteration can be implemented in $O(\log mn)$ time, (ii) follows.

References

[1] Z. Allen-Zhu and L. Orecchia. Using optimization to break the epsilon barrier: A faster and simpler width-independent algorithm for solving positive linear programs in parallel. arXiv preprint arXiv:1407.1925, 2014.

[2] D. Bienstock and G. Iyengar. Approximating fractional packings and coverings in $O(1/\epsilon)$ iterations. SIAM Journal on Computing, 35(4):825–854, 2006.

[3] F. A. Chudak and V. Eleutério. Improved approximation schemes for linear programming relaxations of combinatorial optimization problems. In Integer Programming and Combinatorial Optimization, pages 81–96. Springer, 2005.

[4] C. Koufogiannakis and N. E. Young. Beating Simplex for fractional packing and covering linear programs. In Foundations of Computer Science, 2007. FOCS’07. 48th Annual IEEE Symposium on, pages 494–504. IEEE, 2007.

[5] C. Koufogiannakis and N. E. Young. A nearly linear-time PTAS for explicit fractional packing and covering linear programs. Algorithmica, accepted, 2013.

[6] Y. Nesterov. Rounding of convex sets and efficient gradient methods for linear programming problems. Optimisation Methods and Software, 23(1):109–128, 2008.

[7] N. E. Young. K-medians, facility location, and the Chernoff-Wald bound. In Proceedings of the eleventh annual ACM-SIAM symposium on Discrete algorithms, pages 86–95. Society for Industrial and Applied Mathematics, 2000.

[8] N. E. Young. Sequential and parallel algorithms for mixed packing and covering. In Proceedings of IEEE Symposium on Foundations of Computer Science, pages 538–546, 2001.