The quark mass dependence of the pion mass at infinite $N$

R. Narayanan  
*Department of Physics,*  
*Florida International University,*  
*Miami, FL 33199*  
rajamani.narayanan@fiu.edu

H. Neuberger  
*Rutgers University,*  
*Department of Physics and Astronomy,*  
*Piscataway, NJ 08855*  
neuberg@physics.rutgers.edu

Abstract

In planar QCD, in two space time dimensions, the meson eigenvalue equation has a nonlocal structure interpretable as resulting from hidden degrees of freedom. The nonlocality can be reconstructed from the functional form of the pion mass dependence on quark mass within an expansion starting from a special one dimensional Schrödinger problem. The one dimensional problem makes the pion mass depend on the quark mass through a simple quadratic relation which is shown to be compatible also with numerical data obtained in four dimensions.
INTRODUCTION

Intuitively, if strings indeed describe QCD, the best place to make this precise is in the meson sector, at infinite number of colors, $N$. The mesons do not influence the gauge field vacuum and are open string probes of the closed string background describing it. The mesons are free then, and, for fixed quark mass (assume all quarks have equal mass for simplicity) their masses squared should fall into some regular patterns, extending to infinite mass. An example is provided by $SU(N)$ gauge theory at infinite $N$ in two space-time dimensions. Although in two dimensions the gauge degrees of freedom are unlikely to provide a rich enough structure for a fully featured closed string background, a degenerate form, closer to topological string theory, might exist.

A speculation for four dimensions condenses all the unknown structure into one real function of one real variable, playing the role of a warp factor in a five dimensional metric. This function determines the closed string background and the meson mass dependence on quark mass. In simple examples the warp factor directly determines the masses of some particles by a local partial differential equation, but in the case of QCD it is unlikely that the equation giving the meson masses indeed is local in any set of variables. However, if one expands in energy, relative to a scale set by the string tension, the mass of the lightest mesons might be well described by a leading approximation, which does consist of a simple second order partial differential equation. Higher orders in the two dimensional field theory describing the four dimensional QCD string will induce further corrections in the meson-quark mass dependence. As we shall show below, in two dimensions, the function giving the mass of the pion, $m_\pi^2$, as a function of the mass of the quark, $m_q$, starts from a simple quadratic formula, structurally rooted in an ordinary second order eigenvalue problem. The full, exact dependence then determines all the higher order corrections in a low energy expansion which reproduces the full non-local meson equation. This leads us to a numerical test of a similar quadratic approximation in four dimensions.

The basic philosophy we adopt is to assume that mesonic spectral data could be used to reconstruct the unknown free string theory purportedly describing planar QCD, somewhat akin to an inverse scattering approach using spectral and scattering data to get the potential in a Schrödinger problem.

AdS/CFT motivated modeling seems to be relatively successful phenomenologically,
although for four dimensional planar QCD we only have ad-hoc motivated equations, and we even don’t yet have the numerical values of the masses with their dependence on quark mass we should compare these to, since the real world has only 3 colors. As already mentioned, it is unlikely that such equations, which work exactly in some AdS/CFT cases, at all exist as exact representations at infinite $N$, and it is much less likely that any such equations are exact for $N = 3$.

In this letter we take some steps to improve our understanding of mesonic planar QCD in two and four dimensions by focusing on the dependence of the lightest pseudo-Goldstone mesons (pions) on the quark mass at $N = \infty$.

Our work also provides a test of the numerical methodology we have been developing for dealing with planar QCD in the meson sector. Here, we shall skip most technical details, and concentrate on presenting the results, speculating on their possible meaning.

## TWO DIMENSIONS

We start in two dimensions, from ’t Hooft’s exact solution. The chiral condensate $\langle \bar{\psi}\psi \rangle$ and the meson masses $m^2$ are exactly known as functions of $m_q$, the quark mass. The ’t Hooft coupling $\lambda = \frac{g^2 N}{\pi}$ is used to set the scale. The physical dimensionless quantities are $\lambda^{-1/2} \langle \bar{\psi}\psi \rangle$, $\gamma = \lambda^{-1} m_q^2$ and $\mu^2 = \lambda^{-1} m^2$. Spontaneous symmetry breaking occurs when the order of limits is: $\lim_{m_q \to 0} \lim_{N \to \infty}$, leaving a non-zero condensate in the zero quark mass limit [6].

The meson spectrum and its dependence on quark mass (here the quarks have been taken of equal mass for simplicity) is governed by ’t Hooft’s equation [7]:

$$\gamma \left( \frac{1}{x} + \frac{1}{1-x} \right) \phi(x) - P \int_0^1 \frac{\phi(y) - \phi(x)}{(y-x)^2} dy = \mu^2 \phi(x) \quad (1)$$

$x$ is the fraction of meson light-cone momentum carried by one of the quarks and varies between 0 and 1.

It is natural to ask whether ’t Hooft’s equation contains any structural hints that it might admit a geometrical interpretation tied to self-consisted string propagation, which also could employ extra dimensions for the string to propagate in. More precisely, as a first step, we wish to see whether the addition of some auxiliary continuous arguments can turn ’t Hooft’s equation into a local differential equation of second order. The answer is positive.
While the first term in (1) is local in $x$, the second is highly non-local. One can easily expand it in derivatives of $\phi$. To deal with the combinatorics of the coefficients in a more efficient way, we seek a set of functions that diagonalize this term. This set is easily found, using the integral:

$$P \int_a^b dx \frac{(x-a)^{\nu-1}(b-x)^{-\nu}}{x-c} = -\frac{\pi(c-a)^{\nu-1}}{(b-c)^\nu} \cot(\nu\pi),$$

where $a < c < b$ and $0 < \Re\mu < 1$; we shall use the formula also at $\Re\nu = 0, 1$. It is now convenient to introduce $s = \log \frac{x}{1-x}$, measuring the rapidity difference between the quark and anti-quark, and the equation gets recast into the following form:

$$\left[1 - \pi \frac{d}{ds} \cot \left( \pi \frac{d}{ds} \right) \right] \Psi(s) + \frac{\mu^2}{4 \cosh^2 \frac{s}{2}} \Psi(s) = \gamma \Psi(s)$$

In this form of the equation, $\mu^2$ is more naturally viewed as part of the potential term, while the eigenvalue role is more naturally taken up by $\gamma$, measuring the quark mass. Using

$$\pi \rho \coth \pi \rho - 1 = \sum_{n \neq 0} \frac{p^2}{p^2 + n^2}$$

it becomes obvious that we could localize the equation by adding a new dimension corresponding to the internal coordinate along and open string, where the modes are labeled by nonzero integers $n$.

Explicitly, we introduce a real field $\chi(s, \sigma)$, where $\sigma \in [0, \pi]$, in addition to the field $\Psi(s)$, which plays the role of the meson wave function, but is taken as real. The field $\chi$ obeys Neumann boundary conditions in $\sigma$ and is further constrained to contain no zero mode.

$$\frac{\partial \chi(s, 0)}{\partial \sigma} = \frac{\partial \chi(s, \pi)}{\partial \sigma} = 0, \quad \int_0^\pi \chi(s, \sigma) = 0$$

We now introduce a local action functional $S[\Psi, \chi]$ whose extremization subject to the above boundary condition produces the meson wave equation.

$$S[\phi, \chi] = \frac{1}{2} \int_{-\infty}^{\infty} ds \int_0^\pi \frac{d\sigma}{\pi} \chi(s, \sigma) \left[ \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial \sigma^2} \right] \chi(s, \sigma) - \frac{\mu^2}{4 \cosh^2 \frac{s}{2}} \Psi(s)$$

$$\chi(s, \sigma) = \sum_{n \neq 0} \cos(n\sigma) \chi_n(s) \quad \chi_n \equiv \chi_{-n}$$

Eliminating the variables $\chi_n(s)$ reproduces the meson wave equation. We have ended up with an open bosonic string that moves in one dimension but whose center of mass is
stuck. The nonlocality of \[1\] reflects the integration over the stringy modes labelled by \(n\) and the equations resemble those of \[8\]. There likely are other ways to localize the eigenvalue equation, and it is unclear whether these various localizations would be in some sense geometrically equivalent.

The non-locality we have analyzed seems intrinsic, in the sense that no representation of the equation is known which would make it local without the addition of extra independent variables. There is one case when the non-locality can be neglected to a good approximation: When there is one very light particle, as a result of \(\gamma << 1\), the nonlocality amounts to determining a parameter in a local description of the light particle, as one would expect. We can approximate then \[4\] by assuming \(p^2 << 1\) and then, in the light quark limit, one gets the usual form

\[
\mu_{\pi}^2(\gamma) = \frac{2\pi}{\sqrt{3}} \sqrt{\gamma} + O(\gamma) \tag{7}
\]

The equation producing this result is a well known case of an explicitly factorizable Schrödinger equation, with a shape invariant potential \[9\]:

\[
\frac{\pi^2}{3} \frac{d^2}{ds^2} \Psi(s) + \frac{\mu^2}{4 \cosh^2 \frac{s}{2}} \Psi(s) = \gamma \Psi(s) \tag{8}
\]

This equation is an approximation, valid for the lightest meson, in the limit of light quark mass. The roles of parameter and eigenvalue of meson mass and quark mass got reversed. The equation can be thought of as an analogue of an AdS/CFT motivated partial differential equation determining the pion mass for given quark mass, but only at leading order in an energy scale given by the string tension.

A systematic expansion in \((\frac{d}{ds})^2\) (this is not a chiral Lagrangian expansion) allows the calculation of higher order terms in \(\sqrt{\gamma}\). If we knew by some other means the entire series in \(\sqrt{\gamma}\) expressing \(\mu_{\pi}^2(\sqrt{\gamma})\) we could iteratively invert this to determine the nonlocality of the differential equation, once the lowest order \[8\] is assumed as given. The exact function \(\mu_{\pi}^2(\sqrt{\gamma})\) reproduces all the sigma model corrections to the leading form which here is given by a simple potential problem in one dimension.

If we only know \(\mu_{\pi}^2(\sqrt{\gamma})\) numerically, we are limited in the extend to which we can make inferences on the equation determining the meson masses. This is the situation we find ourselves in four dimensions, where we don’t have a general formula for the lowest order, starting point, equation either.
All is not lost though, and one can try something much less ambitious than determining the full equation for the mesons, something which nevertheless carries some nontrivial information. Let us see what the approximate equations tells us about the function $\mu_\pi^2(\sqrt{\gamma})$.

That $\mu_\pi^2(\sqrt{\gamma})$ vanishes linearly in $\sqrt{\gamma}$ as $\gamma \to 0$ follows from general field theoretical considerations. Because we are at infinite $N$, all chiral logarithms are suppressed. By a simple calculation we obtain:

$$\mu_\pi^2(\gamma) = \frac{2\pi}{\sqrt{3}}\sqrt{\gamma} + 4\left(1 - \frac{\pi^2}{90}\right)\gamma + ...$$  \hspace{1cm} (9)

Demanding the subleading correction to be smaller than 10% of the leading term gives

$$\frac{2\sqrt{\gamma}}{\mu_\pi} < \frac{1}{3}$$  \hspace{1cm} (10)

However, the more natural expansion of the full 't Hooft equation is in $\left(\frac{d}{ds}\right)^2$. The correction coming from the subleading term $\frac{d^4}{ds^4}$ is the $\frac{\pi^2}{90}$ term in (9), which is only ten percent of the order $\gamma$ correction. Had we used this as a correction when estimating the accuracy of (9) there would have been no restriction, since the right hand side would have become unity, and this is about as high as the ratio can ever get. Thus, one would say that (9) is uniformly valid to ten percent accuracy. The factorization of (8) into $-A^\dagger A$, with $A$ of first order in $\frac{d}{ds}$, induces one to replace $\mu^2$ by a variable $\Delta$:

$$\frac{1}{4}\mu^2 = \Delta(\Delta + \frac{\pi}{2\sqrt{3}})$$  \hspace{1cm} (11)

or,

$$\Delta = \frac{1}{2}\left[\sqrt{\mu^2 + \left(\frac{\pi}{2\sqrt{3}}\right)^2} - \frac{\pi}{2\sqrt{3}}\right]$$  \hspace{1cm} (12)

Next, one expands $\Delta$ in $\sqrt{\gamma}$:

$$\Delta = \sqrt{\gamma} - \frac{\pi\sqrt{3}}{45}\gamma...$$  \hspace{1cm} (13)

Both expansions require the inclusion of the same amount of extra $\left(\frac{d}{ds}\right)^{2n}$ terms at any fixed order, but the derivative expansion delivers reliable information about its accuracy while the series expansion does not. Taking for example $\gamma = 1$, and comparing (9) with the exact result, the approximation is seen to be very good numerically, much better than the relative magnitude of the order $\gamma$ term would have indicated, but in agreement with the order $\frac{d^4}{ds^4}$ correction.
The quadratic relation between the meson mass and the Δ variable is reminiscent of calculations done in the AdS/CFT context, but the meaning of Δ differs from the conformal case. In the AdS/CFT case, the quadratic nature of the relationship reflects its origin from an equation that has no derivatives higher than second. This local structure of the equation is a direct expression of the relevance of the extra (fifth for four dimensional space-time) coordinate of AdS.

The approximation in which we introduce Δ to replace $\mu^2$ and set $\Delta = \sqrt{\gamma}$ includes the correct leading asymptotic behavior of the quark mass dependence of the pion both at very small and very large quark masses. However, when we start expanding Δ in a power series in $\sqrt{\gamma}$, we are only reproducing correctly subleading terms at the low quark mass end. Using the exact equation in the nonlocal form given in (11), it is possible to set up an expansion of $\mu^2$ in $\gamma$ which is valid as both are very large. To this end one needs to carry out a canonical change of variables, from $(s, d_s^2)$ to $(d_q^2, -q)$. After that one can scale variables so that the heavy quark limit is smooth and one obtains a leading form of the wave equation that has a potential proportional to $|q|$ and a kinetic energy of normal form. The equation describes a point particle in a linear potential and the fact that this holds in $q$, a variable conjugate to $s$, perhaps makes it easier to accept the previous picture of an open string with Neumann boundary conditions as describing a meson made out of massive quarks. From the new equation one extracts the following fact:

$$\lim_{\gamma \to \infty} \left[ \gamma - \frac{\mu^2}{4} - 1 - \left( \frac{\mu \pi}{4} \right)^{2/3} \alpha \right] = 0$$ (14)

$\alpha$ is the first zero of the derivative of the Airy function, $\alpha \approx -1.0188$. As a result [11],

$$\mu \approx 2\sqrt{\gamma} \left[ 1 - \frac{\alpha}{2} \left( \frac{\pi}{2} \right)^{2/3} \gamma^{-2/3} \right],$$ (15)

a behavior that cannot be represented by a truncated power expansion of $\gamma$ in $\Delta$. The expansion in the derivative terms $\left( \frac{d^2}{ds^2} \right)^n$ is inappropriate at $m_q \to \infty$. At heavy quark masses, a different expansion, which produces the above result at leading order, can be used to generate a series of subleading terms.

From this discussion we extract the message that a parametrization of the pion mass in terms of a quadratically related $\Delta$ produces an expansion of the functional dependence of the pion mass on light quark mass which converges better. While the chiral expansion controls the structure at small quark masses, what happens in the regime of intermediate quark
masses already depends on the specific dynamics of the model. Because of he dynamics of this specific model, as opposed to other models with exactly the same chiral Lagrangian to order $m_q^2$, a quadartic formula for $m_\pi^2$ in terms of $m_q$ has a larger domain of applicability than can be argued on the basis of the chiral Lagrangian alone.

Finally, this is something we can look for by numerical means. It is not very restrictive, nor very solidly formulated, but testable in planar QCD in four space-time dimensions.

**NUMERICAL COMPUTATION OF THE PION MASS**

Recent work \[12\] has established that planar QCD on an Euclidean torus of size and shape $l^4$ is $l$-independent so long as $l > l_c$. This is a string–like property. One refers to this property as “continuum reduction”, on account of the elimination of the infinite volume factor from the total number of degrees of freedom. For a fixed bare gauge coupling, $g_0^2 N = \lambda_0$, this means that computations in the large $N$ limit of four dimensional QCD can be performed on an $L^4$ lattice of relatively small size, with $L > L_c(b)$, where $b = \frac{1}{\lambda_0}$. It is sufficient to pick $L$ just slightly above $L_c(b)$, since continuum reduction implies that there are no finite volume effects, once $N$ is large enough. One should think about $L$ as setting the minimal length scale in the problem, in this case given by $\frac{l_c}{T}$.

The lattice gauge field configuration consists of a collection of $SU(N)$ matrices, associated with a link in the direction $\hat{\mu}$ emanating from a site $x$ and denoted by $U_{\mu}(x)$. These matrices are generated with a probability given by Wilson’s plaquette action, with coupling $b$, and the configuration is probed by a lattice fermion propagator that has exact chiral symmetry at zero quark mass, known as the “overlap” \[13\] propagator. Given the lattice gauge field $U_{\mu}(x)$ on an $L^4$ lattice at some coupling $b$, the lattice quark propagator using overlap fermions is denoted by $G(U_{\mu}, m_o)$ where

$$G(U_{\mu}, m_o) = \frac{1}{1 - m_o} \left[ \frac{2}{1 + m_o + (1 - m_o)\gamma_5 \epsilon[H_w(U_{\mu})]} - 1 \right]$$  \hspace{1cm} (16)$$

and $m_o$ is the bare overlap quark mass parameter. $H_w$ is the Wilson lattice Dirac operator at mass $m_0$ with a so called $r$-parameter set to unity. The meson momentum is implemented by changing the gauge fields felt by the constituent quarks by independent $U(1)$ phase factors; this is the so called “quenched momentum prescription”. Meson propagators are computed
using this quenched momentum prescription and are given by

\[ \mathcal{M}_\Gamma(p, m_o) = \text{Tr} \left[ STG(U_\mu e^{ip_\mu/2}, m_o)ST^\dagger G(U_\mu e^{-ip_\mu/2}, m_o) \right] \]  

(17)

The definitions of \( S \) and \( \Gamma \) follow below: We choose \( \Gamma = \gamma_5 \) for a pseudoscalar meson and \( \Gamma = 1 \) for a scalar meson. The two quark propagators see gauge fields that differ by a \( U(1) \) phase and this difference is the momentum that is carried by the meson. The meson momentum is taken along a lattice axis:

\[ p_\mu = \begin{cases} 
0 & \text{if } \mu = 1, \ldots, 3 \\
\frac{2\pi n}{NL} & \text{if } \mu = 4; \quad 0 < n < N
\end{cases} \]  

(18)

\( S \) smears the operator in the remaining, perpendicular, \( \mu = 1, \ldots, d - 1 \) directions. This “smearing” creates an extended object that more closely approximates the true constituent quark structure of the meson. The smear operator is chosen to be the inverse of the gauged laplacian,

\[ S^{-1} = \frac{1}{2} \sum_{\mu=1}^{d-1} (2 - T_\mu - T_\mu^\dagger), \]  

(19)

where \( T_\mu \) is the gauge covariant translation operator defined by the action \((T_\mu \psi)(x) = U_\mu(x)\psi(x + \hat{\mu})\). Because of fluctuations, \( S \) actually has a sizable gap and should be thought of as a lattice version of the covariant Laplacian with a mass of the order of the inverse lattice spacing. The lattice theory is in the confining, \( l \)-independent phase, when \( L > L_c(b) \); there the \( Z_4^4 \) symmetries associated with the Polyakov loops in the four directions are unbroken.

Along with the gauge symmetry, one can use this to show

\[ \mathcal{M}_\Gamma(p, m_o) = \text{Tr} \left[ STG(U_\mu e^{iq_\mu + \frac{in\pi}{2}}, m_o)ST^\dagger G(U_\mu e^{iq_\mu - \frac{in\pi}{2}}, m_o) \right]; \]  

(20)

\[ q_\mu = \begin{cases} 
0 & \text{if } \mu = 1, \ldots, 3 \\
\frac{2\pi n}{NL} & \text{if } \mu = 4; \quad 0 < n < N
\end{cases} \]  

(21)

making it explicit that the meson propagator depends only on the difference between the two phases seen by the two valence quarks.

Both the pseudoscalar meson and the scalar meson propagator computed above are given by sums over an infinite number of poles. Smearing reduces the residues associated with the higher poles. We can further reduce their contribution by working with the sum,

\[ F(p, m_o) = \frac{1}{2} \left[ M_1(p, m_o) + M_\gamma_5(p, m_o) \right]. \]  

(22)
Excited pseudoscalar mesons and scalar mesons get closer in mass as one looks at high excited levels in QCD [14] and one gets cancellations between the higher states contributing to $F(p, m_0)$. In practice, a single pole fit to the gauge field average of $F(p, m_0)$ works well.

$$\langle F(p, m_0) \rangle = \frac{r^2_\pi(m_0)}{p^2 + m_\pi^2(m_0)} \quad (23)$$

We compute the meson propagator using a stochastic estimate of the trace in (17). We start by picking one Gaussian chiral source, $|q\rangle$, such that $\gamma_5|q\rangle = |q\rangle$. Then we compute $SG(e^{ip\mu/2}, m_0)|q\rangle$ and $G(e^{-ip\mu/2}, m_0)S|q\rangle$. The action of the overlap propagator on a vector is computed using a standard multiple mass conjugate gradient algorithm. The hermitian Wilson-Dirac operator, $H_w$, has a substantial spectral gap for large $N$ gauge fields due to the gap in the eigenvalue distribution of the single plaquette parallel transporter. This eliminates the need to project out approximate zero modes from $H_w$, making the calculation straightforward. A satisfactorily accurate action of $\epsilon(H_w)$ is achieved by just using the 21st order Zolotarev approximation. The stochastic estimate of $F(p, m_0)$ is then given by

$$\bar{F}(p, m_o) = \langle q | G^\dagger(e^{ip\mu/2}, m_o)S^1 + \frac{\gamma_5}{2}G(e^{-ip\mu/2}, m_o)S|q\rangle \quad (24)$$

A multiple mass conjugate gradient algorithm for the calculation of $G$ easily traces the dependence on quark mass. Statistical errors are reduced by using the same Gaussian chiral source all all momenta. This numerical procedure results in an estimate for $\langle F(p, m_0) \rangle$ where the values at all $p$ and $m_0$ are correlated.

The stochastic estimate of $F(p, m_0)$ is then fitted as a function of $p$ at a fixed quark mass to give $m^2_\pi(m_0)$. Minimization fits are performed using the full correlation matrix and errors are estimated using the jackknife method. We obtain in this way the pion mass as a function of the bare overlap quark mass parameter at various gauge couplings. This result is free of finite volume effects since we use large enough $N$, leaving only finite lattice spacing effects to worry about.

In order to take the continuum limit, we first need to convert the bare overlap quark mass parameter to a physical quantity. As is well known, the quark mass itself can be defined to renormalize multiplicatively, and with the help of the overlap propagator this can be done also on the lattice. To eliminate this renormalization effect we consider instead the quantity $m_o\Sigma(b)$ where $\Sigma(b)$ is the bare chiral condensate at a coupling of $b$ and zero quark mass. We can then convert both the pion mass, $m_\pi$, and $m_o\Sigma(b)$ into dimensionless quantities
by forming $m_\pi L_c(b)$ and $m_\sigma \Sigma(b)L_c^4(b)$. Plots of the functions relating these dimensionless quantities obtained at different lattice spacings should approximately fall on a common curve that ceases to depend on the lattice spacing as one approaches the continuum limit.

**NUMERICAL RESULTS**

We work with four different couplings as shown in Table I. The associated values for $L$ and $N$ used in the numerical simulation are shown in the same table. The critical sizes, $L_c(b)$, are known from our previous work on continuum reduction \[12\] and is given by

\[
b_I = b\epsilon(b) \quad e(b) = \frac{1}{N}\langle TrU_{\mu,\nu}(x) \rangle \quad L_c(b) = 0.26 \left( \frac{11}{48\pi^2 b_I} \right)^{\frac{31}{11}} e^{\frac{24\pi^2 b_I}{11}}. \tag{25}\]

We also have a good estimate of the bare chiral condensate at $b = 0.350$ from previous work \[15\], but the estimates at other couplings, like $b = 0.355$, are not as accurate. Therefore, we use the known value at $b = 0.350$ and adjust the values at the other couplings such that all plots of $m_\pi^2 L_c^2(b)$ as a function of $m_\sigma \Sigma(b)L_c^4(b)$, for different values of $b$, fall on one curve. This did not have to happen, but it does, indicating that lattice spacing effects are below our statistical errors. The values of $\Sigma(b)$ so obtained are shown in Table I. The value at $b = 0.355$ in Table I is higher by less than 5% compared to the numbers in [15] but this is well within the error. The value at $b = 0.345$ used here is compatible with the value found at $b = 0.346$ in [15]. The value at $b = 0.360$ used here indicates that the value at $b = 0.3585$ found in [15] probably could go up a little bit if one does a careful analysis.

There are potential order $a^2$ differences between the condensate values listed in Table I and those determined using chRMT in [15]. The running of $\Sigma(b)$ is given up to one-loop by [16]

\[
\Sigma(b)L_c^2(b) = C \left[ \ln(L_c(b)\Lambda) \right]^{\frac{2\pi}{11}}. \tag{26}\]
We find that $C = 0.828$ and $\Lambda = 0.268$ fits the numbers in Table I quite well for $b \neq 0.345$ and we find a 8% deviation at $b = 0.345$.

The plot of $m_\pi^2 L_c^2(b)$ as a function of $m_\sigma \Sigma(b)L_c^4(b)$ is shown in Fig. 1. The data is then fitted to

$$\Delta = \frac{1}{2} \sqrt{m_\pi^2 L_c^2(b) + \Lambda_\pi^2} - \Lambda_\pi; \quad \frac{1}{4} m_\pi^2 L_c^2(b) = \Delta (\Delta + \Lambda_\pi)$$

(27)
with
\[\Delta = m_o \Sigma(b) L_c^4(b) + \frac{1}{\Lambda_q} m_o^2 \Sigma^2(b) L_c^8(b) + \ldots. \]  
(28)

\(\Lambda_\pi = 6.91\) and \(\Lambda_q = 1.03\) fit the data over the whole range as shown by the solid black curve in Fig. 1. A plot of a truncated chiral expansion,
\[m_o^2 L_c^2(b) = 4 \Lambda_\pi m_o \Sigma(b) L_c^4(b) + 4 \left(1 + \frac{\Lambda_\pi}{\Lambda_q}\right) m_o^2 \Sigma^2(b) L_c^8(b),\]  
(29)
shows that the second term in the above equation makes a large contribution for \(m_o \Sigma(b) L_c^4(b) > 0.2\). Keeping two orders in the chiral expansion also does not agree with the data as well as the \(\Delta\) parametrization. One should note that \(\Delta \gg \Lambda_\pi\) in the large quark mass limit and this is not the case for the range plotted in Fig. 1.

The parameter \(\Lambda_\pi\) governs the low quark mass regime through the relation \(f_\pi = \frac{1}{\sqrt{2\Lambda_\pi l_c}}\). Using, \(1/l_c = T_c = 264\) MeV, we get \(f_\pi = 71\) MeV. This translates to \(f_\pi = 123\) MeV for SU(3) at zero quark mass, significantly higher than the conventional value of 86 MeV. Apparently, \(1/N\) effects on the pion decay constant are larger than on the glueball mass or the finite temperature phase transition, all measured in units of string tension.

**SUMMARY**

For simple quantum mechanical problems the spectrum and scattering data can be used to determine the potential by the inverse scattering method. Could we use spectral meson data at infinite \(N\) to determine the string descriptions of mesons?

In this paper we took a very primitive first step with this philosophy in mind. We were led to the speculation that the structure of QCD in the planar limit, where chiral logarithms are suppressed, explains why mass formulae of the type \(m_\pi^2 = 2B m_q + c m_q^2\) seem to work even when \(\frac{c m_q^2}{2B}\) is not small. Of course, this may be just saying that the heavy quark regime, where \(m_\pi \sim 2m_q\) is smoothly connected to the light quark regime. We observed that this could also reflect the approximate validity of an equation similar to (3).

**ACKNOWLEDGEMENTS**

R. N. acknowledges partial support by the NSF under grant number PHY-0300065 and also partial support from Jefferson Lab. The Thomas Jefferson National Accelerator Facility.
(Jefferson Lab) is operated by the Southeastern Universities Research Association (SURA) under DOE contract DE-AC05-84ER40150. H. N. acknowledges partial support by the DOE under grant number DE-FG02-01ER41165 at Rutgers University, discussions with R. Brower, M. Einhorn, D. Gross, A. Jevicki, J. Polchinski, S-J Rey and A. Zhitnitsky, and thanks the KITP for hospitality.

[1] D. Gross, W. Taylor, Nucl. Phys. B403 (1993) 395; Nucl. Phys. B400 (1993) 181.
[2] A. M. Polyakov, [hep-th/0407209](http://arxiv.org/abs/hep-th/0407209).
[3] Clifford V. Johnson, D-Branes, Cambridge University Press, 2003.
[4] H. B. Thacker, C. Quigg, J. L. Rosner, Phys. Rev. D18 (1978) 274.
[5] J. Polchinski, M. Strassler, Phys. Rev. Lett. 88 (2002) 031601; G. F. de Teramond and S. J. Brodsky, [hep-th/0501022](http://arxiv.org/abs/hep-th/0501022).
[6] A. R. Zhitnisky, Phys. Lett. B165 (1985) 405.
[7] G. ’t Hooft, Nucl. Phys. B117 (1976) 519; G. ’t Hooft in New Phenomena in Subnuclear Physics, PART A, Proceedings of the International School of Subnuclear Physics, Erice, Sicily, July 11 - Aug. 1, 1975, Volume 1, Ed. A. Zichichi, Plenum Press, 1977.
[8] D. Bak, S-J Rey, Nucl. Phys. B572 (2000) 151.
[9] F. Cooper, A. Khare, U. Sukhatme, Phys. Rep. 251 (1995) 267.
[10] J. Kiskis, R. Narayanan, H. Neuberger, Phys. Rev. D66 (2002) 025019.
[11] S. Huang, J. Negele, J. Polonyi, Nucl. Phys. B307 (1988) 669.
[12] J. Kiskis, R. Narayanan, H. Neuberger, Phys. Lett. B574 (65) 2003.
[13] H. Neuberger, Phys. Lett. B417 (1998) 1411, R. Narayanan, H. Neuberger, Nucl. Phys. B443 (1995) 305.
[14] L. Glozman, Phys. Lett. B541 (2002) 115.
[15] R. Narayanan, H. Neuberger, Nucl. Phys. B696 (2004) 107.
[16] M.E. Peskin, D.V. Schroeder, An introduction to Quantum Field Theory, Westview Press (1995).