Abstract—The design of modulation schemes for the physical layer network-coded two-way MIMO relaying scenario is considered, with the denoise-and-forward (DNF) protocol which employs two phases: Multiple access (MA) phase and Broadcast (BC) phase. It is shown that for MIMO two-way relaying, deep fade occurs at the relay when the row space of the channel fade coefficient matrix is a subspace of a finite number of vector subspaces which are referred to as the singular fade subspaces. It is shown that proper choice of network coding map obtained by the completion of appropriate partially filled Latin Rectangle can remove most of the singular fade subspaces, referred to as the removable singular fade subspaces. For 2\(^{\lambda}\)-PSK signal set, the number of removable and non-removable singular fade subspaces are obtained analytically and it is shown that the number of non-removable singular fade subspaces is a small fraction of the total number of singular fade subspaces. The Latin Rectangles for the case when the end nodes use different number of antennas are shown to be obtainable from the Latin Squares for the case when they use the same number of antennas, irrespective of the number of antennas at the relay. For 2\(^{\lambda}\)-PSK signal set, the singular fade subspaces which are removed by the conventional XOR network code are identified. Also, using the notions of isotopic and transposed Latin Squares, the network coding maps which remove all the removable singular fade subspaces are shown to be obtainable from a small set of Latin Squares.

I. BACKGROUND AND PRELIMINARIES

We consider the two-way wireless relaying scenario shown in Fig.1 with \( n_A, n_B \) and \( n_R \) antennas respectively at nodes A, B and R (Fig. 1). Two-way data transfer takes place between the nodes A and B with the help of the relay R. It is assumed that all the three nodes operate in half-duplex mode, i.e., they cannot transmit and receive simultaneously in the same frequency band. We consider the denoise-and-forward (DNF) protocol originally introduced in [1], which consists of the following two phases: the multiple access (MA) phase, during which A and B simultaneously transmit to R and the broadcast (BC) phase during which R transmits to A and B. Network coding map, which is also referred to as the denoising map, is chosen at R in such a way that A (B) can decode the messages of B (A), given that A (B) knows its own messages.

The idea of physical layer network coding for the two way relay channel was first introduced in [2], where the multiple access interference occurring at the relay was exploited so that the communication between the end nodes can be done using a two phase protocol. The design principles governing the choice of modulation schemes to be used at the nodes for uncoded transmission were studied in [3].

It was observed in [3] that for uncoded transmission, the network coding map used at the relay needs to be changed adaptively according to the channel fade coefficient, in order to minimize the impact of the Multiple Access Interference (MAI). For the single antenna two-way relaying scenario, a computer search algorithm called the Closest-Neighbour Clustering (CNC) algorithm was proposed in [3] to obtain the adaptive network coding maps resulting in the best distance profile at R. An alternative procedure to obtain the network coding maps, based on the removal of deep channel fade conditions using Latin Squares was proposed in [4]. A quantization of the set of all possible channel realizations based on the network code used was obtained analytically in [5].

The CNC algorithm extended to the MIMO scenario in [6], is run for a given channel realization. The total number of network coding maps which would result is known only after the algorithm is run for all possible channel realizations which is uncountably infinite. Hence, the number of overhead bits required is not known beforehand. In this paper, it is shown that for the MIMO two-way relaying scenario, the MAI becomes severe when the row space of the channel fade coefficient matrix is a subspace of a finite number of vector subspaces of \( \mathbb{C}^{n_A+n_B} \) referred to as the singular fade subspaces. The proposed scheme is based on the removal of singular fade subspaces. Since the number of such vector subspaces is finite, the number of overhead bits required is known beforehand. Also, for the proposed scheme, obtaining the network coding maps reduces to completing a finite number of partially filled Latin Rectangles, which avoids the problem of performing exhaustive search for an uncountably infinite number of values.

For the SISO two-way relaying scenario, the effect of the MAI is totally captured by a single complex number which is the ratio of the channel fade coefficients [3], [4]. In contrast, for the MIMO two-way relaying scenario, the deep channel fade conditions, which cannot be described by a single complex number, have an interesting vector subspace interpretation in terms of the singular fade subspaces.

Notations: \( \mathcal{CN}(0, \sigma^2 I_n) \) denotes the circularly symmetric complex Gaussian random vector of length \( n \) with covariance matrix \( \sigma^2 I_n \), where \( I_n \) is the identity matrix of order \( n \). The binary field is denoted by \( \mathbb{F}_2 \). All the vector spaces and vector subspaces considered in this paper are over the complex field.
C. Let \(\text{span}(c_1, c_2, \ldots, c_L)\) denote the vector space over \(\mathbb{C}\) spanned by the complex vectors \(c_1, c_2, \ldots, c_L \in \mathbb{C}^n\). For a vector subspace \(V\) of \(\mathbb{C}^n\), \(V^\perp\) denotes the vector subspace \(\{x : x^Hv = 0 \forall v \in V\}\). For a matrix \(H \in \mathbb{C}^{m \times n}\), \(R(H)\) denotes the row space of \(H\). The all zero vector of length \(n\) is denoted by \(0_n\). For a vector \(x\) of length \(n\), \(x(i)\), \(1 \leq i \leq n\) denotes the \(i\)th component of \(x\). For vector subspaces \(V_1\) and \(V_2\) of \(\mathbb{C}^n\), \(V_1 \subseteq V_2\) means that \(V_1\) is a subspace of \(V_2\) and \(V_1 \not\subseteq V_2\) means that \(V_1\) is not a subspace of \(V_2\). By \(n_A \times n_B\) system, we refer to the two-way relaying system with \(n_A\) and \(n_B\) antennas at the nodes A and B respectively, with no restriction placed on the number of antennas at the relay \(n_R\).

For a vector \(x\), \(x[j], 1 \leq i \leq j \leq n\) denotes the vector obtained by taking only the \(i\)th to \(j\)th components of \(x\). For a matrix \(A\), \(A^T\) denotes its transpose.

**A. Signal Model**

It is assumed that the complex numbers transmitted in each one of the antennas at the end nodes belong to the signal set \(S\) of size \(M = 2^\lambda\), where \(\lambda\) is an integer. Assume that A (B) wants to transmit a \(\lambda\)A-bit (\(\lambda\)B-bit) binary tuple to B (A). At node A (B), the \(\lambda\)A (\(\lambda\)B) bits are spatially multiplexed into \(n_A\) (\(n_B\)) streams with each one of the streams consisting of \(\lambda\) bits. The \(\lambda\) bits in each one of the streams are mapped to the signal set \(S\) using the map \(\mu : \mathbb{F}_2^\lambda \rightarrow S\) and are transmitted.

**Multiple Access (MA) phase:** Throughout, we assume a block fading scenario with the Channel State Information (CSI) available only at the receivers. Let \(x_A = [\mu(s_A(1)), \mu(s_A(2)), \ldots, \mu(s_A(n_A))]^T \in S^{n_A}, x_B = [\mu(s_B(1)), \mu(s_B(2)), \ldots, \mu(s_B(n_B))]^T \in S^{n_B}\) denote the complex vectors transmitted by A and B respectively, where \(s_A(1), s_A(2), \ldots, s_A(n_A), s_B(1), s_B(2), \ldots, s_B(n_B) \in \mathbb{F}_2^\lambda\). The received signal at R is given by \(Y_R = H_Ax_A + H_Bx_B + Z_R\), where \(H_A\) of size \(n_R \times n_A\) and \(H_B\) of size \(n_R \times n_B\) are the channel matrices associated with the A-R and B-R links respectively. The additive noise vector \(Z_R\) is assumed to be \(\mathcal{CN}(0, \sigma^2 I_{n_R})\).

Let \(S_R(H_A, H_B) \subset \mathbb{C}^{n_R}\) denote the effective constellation seen at the relay during the MA phase, i.e.,

\[
S_R(H_A, H_B) = \{H_Ax_A + H_Bx_B | x_A \in S^{n_A}, x_B \in S^{n_B}\}.
\]

Let \(d_{\min}(H_A, H_B)\) denote the minimum distance between the points in the effective constellation \(S_R(H_A, H_B)\), i.e.,

\[
d_{\min}(H_A, H_B) = \min_{(x_A, x_B) \in S_R(H_A, H_B)} \|H_A(x_A-x_A^*) + H_B(x_B-x_B^*)\|,
\]

From (1), it is clear that there exists values of \(H_A, H_B\) for which \(d_{\min}(H_A, H_B) = 0\).

Let \(\Delta S\) denote the difference constellation of the signal set \(S\), i.e., \(\Delta S = \{s - s' | s, s' \in S\}\). Let us define \(\Delta x = [(x_A-x_A^*)^T, (x_B-x_B^*)^T]^T \in \Delta S^{n_A+n_B}\), where \(x_A = x_A^* \in \Delta S^{n_A}\) and \(x_B = x_B^* \in \Delta S^{n_B}\), and \(H_R = [H_A, H_B]\). Also let \(H_R = [h_1 h_2 \ldots h_{n_R}]^T\), where the row vector \(h_k\), \(1 \leq k \leq n_R\) of length \(n_A + n_B\) is the \(k\)th row of \(H_R\). From (1), it follows that,

\[
d^2_{\min}(H_A, H_B) = \min_{\Delta x \in \Delta S^{n_A+n_B}, \Delta x \neq 0_{n_A+n_B}} \|H_R \Delta x\|^2,
\]

\[
= \min_{\Delta x \in \Delta S^{n_A+n_B}, \Delta x \neq 0_{n_A+n_B}} \sum_{k=1}^{n_R} |h_k^T \Delta x|^2.
\]

(2)

From (2) it follows that \(d^2_{\min}(H_A, H_B) = 0\) if \(H_R\) is such that \(h_k^T \Delta x = 0, 1 \leq k \leq n_R\), for some \(\Delta x \in \Delta S^{n_A+n_B}\). Equivalently, for \(d^2_{\min}(H_A, H_B) = 0\), the vectors \(h_k\), \(1 \leq k \leq n_R\) should belong to the subspace \(\text{span}(\Delta x)^\perp\) for some \(\Delta x \in \Delta S^{n_A+n_B}\). In other words, the row space of \(H_R\) should be a subspace of the vector subspace \(\text{span}(\Delta x)^\perp\), for some \(\Delta x \in \Delta S^{n_A+n_B}\), for \(d^2_{\min}(H_A, H_B)\) to become zero.

The channel fade coefficient matrix \(H_R\) is said to be a deep fade matrix, if \(d_{\min}(H_A, H_B) = 0\). The row space of the deep fade matrices are referred to as the deep fade spaces. The deep fade spaces are subspaces of the vector subspaces \(\text{span}(\Delta x)^\perp\), \(\Delta x \in \Delta S^{n_A+n_B}\), which are referred to as the singular fade subspaces. Let \(\mathcal{F}\) denote the set of all singular fade subspaces, i.e., \(\mathcal{F} = \{\text{span}(\Delta x)^\perp | \Delta x \in \Delta S^{n_A+n_B}\}\).

Note that singular fade subspaces depend only on the set \(\{\Delta x : \Delta x \in \Delta S^{n_A+n_B}\}\), i.e., they are independent of \(n_R\).

**Broadcast (BC) phase:** Let \((\hat{x}_A, \hat{x}_B)\) denote the Maximum Likelihood (ML) estimate of \((x_A, x_B)\) at R based on the received complex vector \(Y_R\), i.e.,

\[
(x_A, x_B) = \arg \min_{(x_A', x_B') \in \Delta S^{n_A+n_B}} \|Y_R - H_Ax_A' - H_Bx_B'\|.
\]

Depending on \(H_R\), R chooses a many-to-one map \(M_{HR} : \Delta S^{n_A+n_B} \rightarrow S',\) where \(S' \subset \mathbb{C}^{n_R}\) is the signal set of size between \(\max\{M_{n_A}, M_{n_B}\}\) and \(M_{n_A+n_B}\) used by R during the BC phase. Note that \(|S'|\) should be at least \(\max\{M_{n_A}, M_{n_B}\}\) to transmit \(\max\{\lambda n_A, \lambda n_B\}\) information bits. The elements in \(\Delta S^{n_A+n_B}\) which are mapped on to the same complex vector in \(S'\) by the map \(M_{HR}\) are said to form a cluster. Let \(\{L_0, L_2, \ldots, L_{t-1}\}\) denote the set of all such clusters. The formation of clusters for \(H_R\) is called clustering, and is denoted by \(C_{HR}\). For simplicity, in the rest of the paper, the cluster \(L_k\) is denoted by the subscript \(k\), where \(0 \leq k \leq t - 1\).

The received signals at A and B during the BC phase are respectively given by,

\[
Y_A = H'_A X_R + Z_A, \quad Y_B = H'_B X_R + Z_B,
\]

where \(X_R = M_{HR}(\hat{x}_A, \hat{x}_B) \in S'\) is the complex vector transmitted by R. The fading matrices of size \(n_A \times n_R\) and \(n_B \times n_R\) corresponding to the R-A and R-B links are denoted by \(H'_A\) and \(H'_B\) respectively and the additive noises \(Z_A\) and \(Z_B\) are \(\mathcal{CN}(0, \sigma^2)\).

In order to ensure that A (B) is able to decode B’s (A’s) messages, the clustering \(C_{HR}\) should satisfy the exclusive law [3], i.e.,

\[
M_{HR}(x_A, x_B) \neq M_{HR}(x'_A, x'_B), \forall x_A \neq x'_A \in S^{n_A}, x_B \in S^{n_B}, M_{HR}(x_A, x_B) \neq M_{HR}(x'_A, x'_B), \forall x'_B \neq x_B \in S^{n_B}, x_A \in S^{n_A}.
\]

*Definition 1:* The cluster distance between a pair of clusters \(L_i\) and \(L_j\) is the minimum among all the distances calculated between the points \(H_Ax_A + H_Bx_B\) and \(H_Ax'_A + H_Bx'_B\) in \(S_R(H_A, H_B)\), where \((x_A, x_B) \in L_i\) and \((x'_A, x'_B) \in L_j\).
The minimum cluster distance of the clustering $C^{HR}$ is the minimum among all the cluster distances, i.e.,

$$d_{\text{min}}(C^{HR}) = \min_{(x_A,x_B) \in S^{n_A+n_B}} \| H_A(x_A - x_B) \| + d_{\text{min}}(H_R)$$

The minimum cluster distance determines the performance during the MA phase of relaying. The performance during the BC phase is determined by the minimum of the distance between all the singular fade subspaces, a clustering needs to be chosen such that the minimum cluster distance is non-zero. For a clustering $C$ which removes that singular fade subspace.

For a singular fade subspace $f \in F$, let $d_{\text{min}}(C^f, H_R)$ be defined as,

$$d_{\text{min}}(C^f, H_R) = \min_{(x_A,x_B) \in S^{n_A+n_B}} \| H_A(x_A - x_B) \| + d_{\text{min}}(H_R)$$

where $M^f$ is the map associated with the clustering $C^f$.

A clustering $C^f$ is said to remove a singular fade subspace $f \in F$, if the minimum cluster distance $d_{\text{min}}(C^f, H_R)$ is greater than zero, for every $H_R$ such that $R(H_R) \leq f$.

It is important to note that certain singular fade subspaces cannot be removed. These are precisely the singular fade subspaces which are of the form $[\text{span}(\Delta x)]'$, for which

$$\Delta x = [0_{n_A}^T \Delta x_B^T]^T, \Delta x_B \in \Delta S^{n_B}, \Delta x = [\Delta x_A^T 0_{n_B}^T]^T, \Delta x_A \in \Delta S^{n_A}$$

and are referred to as the non-removable singular fade subspaces. The reason for this is as follows: The pair $(x_A, x_B)$ and $(x_A', x_B')$ result in $\Delta x = [0_{n_A}^T \Delta x_B^T]^T$. But $(x_A, x_B)$ and $(x_A', x_B')$ cannot be placed in the same cluster since exclusive law will be violated.

Let $C_F = \{ C^f : f \in F \}$ denote the set of all clusterings, which remove a removable singular fade subspace. For every removable singular fade subspace, the set $C_F$ contains exactly one clustering which removes that singular fade subspace.

For a given realization of $H_R$, the clustering $C^{HR}$ is chosen to be $C^f$, which satisfies $d_{\text{min}}(C^f, H_R) \geq d_{\text{min}}(C^{{f'}}, H_R), \forall f' \neq f' \in F$. The choice of $C^{HR}$ is indicated by $R$ to A and B using overhead bits.

The contributions and organization of this paper are as follows.

- The structure and the exact number of non-removable and removable singular fade subspaces for $2^k$-PSK signal set are obtained analytically (Section II).

- It is shown that the requirement of satisfying the exclusive law is same as the clustering being represented by a partially filled Latin Rectangle (PFLR) and can be used to get the clustering which removes singular fade subspaces, by completing the PFLR (Section III A).

- It is shown that the Latin Rectangles which remove the singular fade subspaces for the case when end nodes have unequal number of antennas, i.e., $n_A \neq n_B$ can be obtained from the Latin Squares which remove the singular fade subspaces for the case when the nodes have equal number of antennas $n = \max\{n_A, n_B\}$ (Section III B).

- For the $n \times n$ system, the singular fade subspaces which are removed by the conventional Exclusive-OR map are identified and it is shown that finding the network coding maps which remove all the singular fade subspaces reduces to finding a small set of maps. (Section III C).

- It is shown that most of the Latin Squares which remove the singular fade subspaces for the $n \times n$ system, $n \geq 2$, are obtainable from Latin Squares of lower size (Section IV).

The proofs of Lemmas and other claims are omitted due to lack of space but are available in [8], with several illustrative examples.

II. SINGULAR FADE SUBSPACES FOR $2^k$-PSK SIGNAL SET

The following lemma gives the total number of non-removable and removable singular fade subspaces for $2^k$-PSK signal set.

**Lemma 1:** For $M$-PSK signal set ($M$ any power of 2),

- The total number of non-removable singular fade subspaces is given by,

$$\sum_{k=1}^{n_A} \binom{n_A}{k} \left( \binom{M}{2}^k \frac{M}{2} + 1 \right) M^{k-1}$$

- The total number of removable singular fade subspaces is given by,

$$\sum_{k=2}^{n_A+n_B} \binom{n_A+n_B}{k} \left( \binom{M}{2}^k \frac{M}{2} + 1 \right) M^{k-1} - \binom{n}{k}$$

where $\binom{n}{k}$ is defined to be zero if $b > a$.

**Example 1:** Consider the MIMO two-way relaying system with $n_A = 2$ and $n_B = 2$. From Lemma 1, the non-removable singular fade subspaces for BPSK and QPSK signal sets are respectively 4 and 28 in number. For BPSK and QPSK signal sets, the total number of removable singular fade subspaces are 32 and 1456 respectively. For examples of removable and non-removable singular fade subspaces, see [8].

From Lemma 1, it can be seen that the number of non-removable singular fade subspaces is $O(M^{2\max\{n_A, n_B\}-1})$, while the number of removable singular fade subspaces is $O(M^{2(\max\{n_A, n_B\})-1})$. Hence, the number of non-removable singular fade subspaces is a small fraction of the total number of singular fade subspaces and the fraction tends to zero for increasing values of $M$. 

III. THE EXCLUSIVE LAW AND LATIN RECTANGLES

In this section, we establish the connection between Latin Rectangles and network coding maps satisfying the exclusive law, for the MIMO two-way relaying scenario.
Definition 4: [7] A Latin Rectangle $L$ of order $N_1 \times N_2$ on the symbols from the set $Z_t = \{0,1, \ldots, t-1\}$ is an $N_1 \times N_2$ array, in which each cell contains one symbol and each symbol occurs at most once in each row and column. A Latin Rectangle of order $N \times N$ is called a Latin Square of order $N$.

Let the points in the $M$-point signal set used for transmission at the nodes be indexed by the elements of the set $Z_M = \{0,1,2, \ldots, M-1\}$. Consider an $M^{nA} \times M^{nB}$ array at the relay with the rows (columns) indexed by the $nA$-tuple($nB$-tuple) $[x_A(1), x_A(2), \ldots, x_A(nA)] \ (\{x_B(1), x_B(2), \ldots, x_B(nB)\})$ denoting the complex vector transmitted by node A (B). Our aim is to form clusters from the slots corresponding to a Latin Square obtained by taking the bit-wise XOR of the individual components of the two vectors, after decimal to binary conversion. Every cell in the Latin Square corresponding to the bit-wise XOR mapping is filled with the decimal equivalent of the bit-wise XOR of the row index and the column index.

Consider the set of singular fade subspaces $\{\text{span}([\Delta x_A^T \Delta x_B^T])^\perp\}$, denoted by $F_{\pm}$, which satisfy the condition that $\Delta x_A(i) = \pm \Delta x_B(i)$, $\forall i \leq n$.

Lemma 3: When the user nodes use $2^\lambda$-PSK constellations, all the singular fade subspaces which belong to the set $F_{\pm}$ are removed by bit-wise XOR mapping, for all $\lambda$.

Example 4: For the $2 \times 2$ system with BPSK signal set, $\{[-2, -2, 2, 2]^T\}^\perp \in F_{\pm}$ and is removed by the bit-wise XOR mapping given in Fig. 3.

Definition 2: A Latin Square $L^T$ is said to be the Transpose of a Latin Square $L$, if $L^T(i,j) = L(j,i)$ for all $i,j \in \{0,1,2, \ldots, M-1\}$.

Lemma 4: If the Latin Square $L$ removes $\{\text{span}([\Delta x_A^T \Delta x_B^T])^\perp\}$, then the Latin Square $L^T$ removes $\{\text{span}([\Delta x_B^T \Delta x_A^T])^\perp\}$.

Definition 3: [7] Two Latin Squares $L$ and $L'$ (using the same symbol set) are isomorphic if there is a triple $(f,g,h)$, where $f$ is a row permutation, $g$ is a column permutation and $h$ is a symbol permutation, such that applying these permutations on $L$ gives $L'$.

Consider a vector $\Delta \tilde{x} = [\Delta x_A^T \Delta x_B^T]^T$, where $\Delta x_A$ and $\Delta x_B$ are obtained by applying the permutations $\sigma_A$ and $\sigma_B$ on the components of $\Delta x_A$ and $\Delta x_B$ respectively. Equivalently, this can be viewed as applying the permutations $\sigma_A$ and $\sigma_B$ on the indices of the transmitting antennas at $A$ and $B$ respectively. Hence, we have the following lemma.

Lemma 5: If a Latin Square $L$ removes $\{\text{span}(\Delta x)^\perp\}$, the isotopic Latin Square $L'$ obtained by applying the permutation $\sigma_A$ on the components of the row indices and the permutation $\sigma_B$ on the components of the column indices of $L$ removes $\{\text{span}(\Delta \tilde{x})^\perp\}$. 

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Fig. 2. Latin Square that removes $\{\text{span}([0, -2, -2, 2]^T)^\perp\}$.

Fig. 3. The bit-wise XOR map $\{\text{span}([0, -2, -2, 2]^T)^\perp\}$. 

A. Removing singular fade subspaces, Singularity-removal Constraints and Constrained Latin Rectangles

Consider a singular fade subspace $f \in F$. Let $(k,l)(k',l') \in Z_M^{nA} \times Z_M^{nB}$ be the pairs which result in $\Delta x$ such that $\{\text{span}(\Delta x)^\perp\} = f$. If $(k,l)$ and $(k',l')$ are not clustered together, the minimum cluster distance will be zero, for all $H_R$ such that $R(H_R) \leq f$. To avoid this, those pairs should be in the same cluster. This requirement is termed as singularity-removal constraint.

So, we need to obtain Latin Rectangles which can remove singular fade subspaces and with minimum value for $t$. Therefore, initially we will fill the slots in the $M^{nA} \times M^{nB}$ array such that for the slots corresponding to a singularity-removal constraint the same element will be used to fill slots. This removes that particular singular fade subspace. Such a partially filled Latin Rectangle is called a Constrained Partial Latin Rectangle (CPLR). After this, to make this a Latin Rectangle, we will try to fill the other slots of the CPLR with minimum number of symbols from $Z_t$.

Example 2: Consider the $2 \times 2$ system with BPSK signal set $\{\pm 1\}$ used at the end nodes. For the singular fade subspace $\{\text{span}([0, -2, -2, 2]^T)^\perp\}$, the singularity removal constraints are $\{\{(0,0), (0,1), (1,0)\}, \{(1,0), (0,1), (1,1), (1,0)\}\}$. The filled Latin Square which removes this singular fade subspace is shown in Fig. 2.

B. Obtaining Latin Rectangles from Latin Squares

Without loss of generality, assume $n_B > n_A$. If $\{\text{span}([\Delta x_A^T \Delta x_B^T])^\perp\}$ is a singular fade subspace for the $n_A \times n_B$ system, then $\{\text{span}([0_{n_B-n_A}^T \Delta x_A^T \Delta x_B^T])^\perp\}$ is a singular fade subspace for the $n_B \times n_B$ system. For a Latin Square $L$ of order $n$, let $L_{[1:M^{nA}]}$ denote the Latin Rectangle of order $r \times n$ obtained by taking only the first $r$ rows of $L$.

Lemma 2: For $n_B > n_A$, if the Latin Square $L$ removes $\{\text{span}([0_{n_B-n_A}^T \Delta x_A^T \Delta x_B^T])^\perp\}$ for the $n_B \times n_B$ system, the Latin Rectangle $L_{[1:M^{nA}]}$ removes $\{\text{span}([\Delta x_A^T \Delta x_B^T])^\perp\}$ for the $n_A \times n_B$ system.

Example 3: The Latin Rectangle obtained by taking only the first two rows of the Latin Square in Fig. 2 which removes $\{\text{span}([0, -2, -2, 2]^T)^\perp\}$ for the $2 \times 2$ system, removes $\{\text{span}([0, -2, -2, 2]^T)^\perp\}$ for the $1 \times 2$ system, with BPSK signal set.

Since the Latin Rectangles for the $n_A \times n_B$ system are obtainable from Latin Squares, in the rest of the paper it is assumed that $n_A = n_B = n$.

C. Some Special Constructions of Latin Squares

Recall that the rows and columns of the Latin Squares are indexed by vectors which belong to $Z_M^n$. By bit-wise XOR of two such vectors, it is meant to be the vector obtained by taking the bit-wise XOR of the individual components of the two vectors, after decimal to binary conversion. Every cell in the Latin Square corresponding to the bit-wise XOR mapping is filled with the decimal equivalent of the bit-wise XOR of the row index and the column index.

Consider the set of singular fade subspaces $\{\text{span}([\Delta x_A^T \Delta x_B^T])^\perp\}$, denoted by $F_{\pm}$, which satisfy the condition that $\Delta x_A(i) = \pm \Delta x_B(i)$, $\forall i \leq n$.

Lemma 3: When the user nodes use $2^\lambda$-PSK constellations, all the singular fade subspaces which belong to the set $F_{\pm}$ are removed by bit-wise XOR mapping, for all $\lambda$.

Example 4: For the $2 \times 2$ system with BPSK signal set, $\{[2, -2, 2, 2]^T\}^\perp \in F_{\pm}$ and is removed by the bit-wise XOR mapping given in Fig. 3.

Definition 2: A Latin Square $L^T$ is said to be the Transpose of a Latin Square $L$, if $L^T(i,j) = L(j,i)$ for all $i,j \in \{0,1,2, \ldots, M-1\}$.

Lemma 4: If the Latin Square $L$ removes $\{\text{span}([\Delta x_A^T \Delta x_B^T])^\perp\}$, then the Latin Square $L^T$ removes $\{\text{span}([\Delta x_B^T \Delta x_A^T])^\perp\}$.

Definition 3: [7] Two Latin Squares $L$ and $L'$ (using the same symbol set) are isomorphic if there is a triple $(f,g,h)$, where $f$ is a row permutation, $g$ is a column permutation and $h$ is a symbol permutation, such that applying these permutations on $L$ gives $L'$.

Consider a vector $\Delta \tilde{x} = [\Delta x_A^T \Delta x_B^T]^T$, where $\Delta x_A$ and $\Delta x_B$ are obtained by applying the permutations $\sigma_A$ and $\sigma_B$ on the components of $\Delta x_A$ and $\Delta x_B$ respectively. Equivalently, this can be viewed as applying the permutations $\sigma_A$ and $\sigma_B$ on the indices of the transmitting antennas at $A$ and $B$ respectively. Hence, we have the following lemma.

Lemma 5: If a Latin Square $L$ removes $\{\text{span}(\Delta x)^\perp\}$, the isotopic Latin Square $L'$ obtained by applying the permutation $\sigma_A$ on the components of the row indices and the permutation $\sigma_B$ on the components of the column indices of $L$ removes $\{\text{span}(\Delta \tilde{x})^\perp\}$. 

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For $2^λ$-PSK signal set, consider two singular fade subspaces $[\text{span}(\Delta x)]^\perp$ and $[\text{span}(\Delta \bar{x})]^\perp$ which are such that the absolute values of the components of $\Delta x$ and $\Delta \bar{x}$ are equal. Let $[\phi_A^T \phi_B^T]^T$ and $[\phi_A^T \phi_B^T]^T$ be the vectors consisting of the phases of individual components of $\Delta x$ and $\Delta \bar{x}$ respectively. Let $\Delta \phi_A = \phi_A - \phi_B$ and $\Delta \phi_B = \phi_B - \phi_B$. Also let $\Delta \phi_A(i) = \frac{k_a 2^n}{M}$ and $\Delta \phi_B(i) = \frac{k_b 2^n}{M}$.

Lemma 6: For $2^λ$-PSK signal set, let $L$ denote the Latin Square which removes the singular fade subspace $[\text{span}(\Delta x)]^\perp$. The Latin Square $L'$ which removes the singular fade subspace $[\text{span}(\Delta \bar{x})]^\perp$ can be obtained from $L$ as follows: To the $i^{th}$ component of all the row indices of $L$ add $k_A$, modulo $M$ $\forall 1 \leq i \leq n$ and to the $i^{th}$ component of all the column indices of $L$ add $k_B$, modulo $M \forall 1 \leq i \leq n$, to obtain the Latin Square $L'$.

The usefulness of Lemmas 4-6 is that the set of all Latin Squares which remove all the singular fade subspaces can be obtained from a small set of Latin Squares, as illustrated for the $2 \times 2$ system in [8].

![Latin Square Diagram](image)

IV. OBTAINING LATIN SQUARES FOR THE $n \times n$ SYSTEM FROM LATIN SQUARES OF LOWER ORDER

In this section, it is shown that most of the Latin Squares which remove the singular fade subspaces of the $n \times n$ system, $n \geq 2$, are obtainable from the Latin Squares which remove the singular fade subspaces of $m \times m$ system, where $m < n$.

Definition 4: For two vectors $y$ and $z$ of length $2a$ and $2b$ respectively, the compound vector of $y$ and $z$, denoted as $\text{comp}(y,z)$, is the vector of length $2a + 2b$ given by $[y^T | z^T] = \begin{bmatrix} y_{1,a} & z_{1,b} & y_{2,a} & z_{2,b+1} \end{bmatrix}^T$.

For a Latin Square $L$, let $L_{[i,j,k,l]}$ denote the $(j-i+1) \times (l-k+1)$ array obtained by taking only the $i^{th}$ to $j^{th}$ rows and $k^{th}$ to $l^{th}$ columns of $L$. Let $L+c$ denote the Latin Square obtained by adding integer $c$ to all the cells of $L$. Let $\text{max}(L)$ denote the maximum among all the integers filled in the cells of the Latin Square $L$.

Definition 5: The Cartesian product of the two Latin Squares $L_1$ of order $M^a$ and $L_2$ of order $M^b$, denoted as $(L_1 \times L_2)$, is the Latin Square of order $M^{a+b}$ for which $((L_1 \times L_2)|_{(i-1)M^{a}+1 \leq i \leq M^{a}+(j-1)M^{b}+1}) = L_1 + L_2(i,j)(\max(L_1) + 1)$, where $1 \leq i, j \leq M^b$.

For example illustrating the notion of Cartesian product of Latin Squares, see [8].

Lemma 7: Let $L_1$ and $L_2$ respectively denote the Latin Squares of order $M^a \times M^a$ and $M^b \times M^b$, which remove the singular fade subspaces $[\text{span}(\Delta x_1)]^\perp$ of the $a \times a$ system, and $[\text{span}(\Delta x_2)]^\perp$ of the $b \times b$ system. The Latin Square $L_1 \times L_2$ removes all the singular fade subspaces of the form $[\text{span}(\text{comp}(\Delta x_1, k \Delta x_2))]^\perp$, $k \in \mathbb{C}$, for the $(a+b) \times (a+b)$ system.

V. SIMULATION RESULTS

Simulation results presented are for QPSK and $n_A = n_B = n_R = 2$. As a reference scheme, we consider the case when bit-wise XOR network code is used at $R$, irrespective of the channel conditions.