Dark energy and Chern-Simons like gravity from a dynamical four-form

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Abstract

We consider the dynamics of a four-form field $\tilde{w}$, treating it as a distinct physical degree of freedom, independent of the metric. The equations of motion are derived from an action which, besides having the standard Hilbert-Einstein term and the matter part, consists of a new action for $\tilde{w}$. The evolution of this four-form in the framework of a flat FRW model is studied, and it is shown that the parameters of the theory admit solutions wherein it is possible to have an equation of state $p_{\phi} \approx -\epsilon_{\phi}$ for $\tilde{w}$, so that it leads to an accelerating universe. Taking cue from the paper by Jackiw and Pi (2003), we also put forward electromagnetic as well as gravitational ‘Chern-Simons’ like terms using $\tilde{w}$ that arise naturally in 4D without any loss of Lorentz invariance. This entails on one hand a modified Einstein-Maxwell equation, having the potential to be conrained observationally by CMBR and other astrophysical data, and an enlarged system of Einstein equation, on the other, involving a ‘Cotton’ like tensor. It is shown that the presence of gravitational Chern-Simons like term in the theory does not affect the flat FRW model analysis of the evolution of $\tilde{w}$. We also demonstrate that the scalar-density associated with $\tilde{w}$ can be employed to construct a generalized exterior derivative that converts a p-form density to a (p+1)-form density of identical weight.
There is growing evidence over almost a decade and a half that the rate of expansion of the universe is increasing with time [1, 2, 3]. A recent analysis with 414 Type Ia supernovae suggests that the equation of state parameter \( w \) for the hypothesized dark energy (DE), responsible for the acceleration of the expansion, is -0.969 with an error of about 10% [4]. The observed value of \( w \) is tantalizingly close to -1 suggesting a non-zero cosmological constant as the panacea. The cosmological constant scenario, however, is replete with problems of fine-tuning and cosmic-coincidences [5, 6]. Alternate popular models of DE involve scalars like quintessence and phantom fields [7, 8, 9, 10, 11, 12]. In this paper, we explore the possibility of generating an accelerating universe by means of a dynamical four-form field \( \tilde{w} \) evolving in the standard 4D spacetime.

Four-forms derived from three-form gauge potentials have been invoked to examine the origin of cosmological constant from a different perspective [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. Related approaches have also been extended, in recent times, to studies pertaining to DE and the cosmological constant in the realm of string and M-theories where coupling of four-form fields with branes is generic [6, 24, 25, 26, 27, 28, 29, 30, 31]. A dynamical volume-form independent of the metric has also been discussed in the literature wherein the new degree of freedom is associated with a massless dilatonic scalar field that leads to a scalar-tensor theory of gravitation [32, 33]. Guendelman and Kaganovich (2008) have proposed a two-measures field theory in which besides the standard volume-measure \( \sqrt{-gd^4x} \) there is a dynamical volume-form made up of four one-forms, each associated with an independent scalar field [34]. Possibility of generating an inflationary phase by means of n-forms has also been studied recently [35]. Contrary to the previous investigations, in the present analysis we adopt a different approach treating the scalar-density of weight +1 corresponding to the four-form \( \tilde{w} \) as fundamental, whose dynamics is determined by a Lagrangian density constructed out of its covariant derivatives.

In special relativity, there are two nontrivial tensors that are invariant under proper Lorentz transformations - the Minkowski metric \( \eta_{\mu \nu} \) and the totally antisymmetric Levi-Civita tensor \( \epsilon_{\mu \nu \rho \sigma} \) [36]. When one makes a transition to classical general relativity, the metric encoding the geometry of the spacetime, attains a dynamical status \( g_{\mu \nu}(x) \) whose time evolution is determined by the Einstein equation. Taking the analogy of \( \eta_{\mu \nu} \) meta-
morphosising into a dynamical $g_{\mu\nu}$ one step lateral, we put to scrutiny the hypothesis of
Levi-Civita tensor transforming into a dynamical field $w_{\mu\nu\rho\sigma}(x)$.

We propose that a four-form field $\tilde{w}$ which, in a coordinate basis, can be expressed as,

$$\tilde{w} = \frac{1}{4!} w_{\mu\nu\rho\sigma} \tilde{dx}^\mu \wedge \tilde{dx}^\nu \wedge \tilde{dx}^\rho \wedge \tilde{dx}^\sigma$$

represents a new physical degree of freedom in the gravitational theory, completely independent of the metric, that couples universally to all fields. Since $w_{\mu\nu\rho\sigma}(x)$ is totally antisymmetric, it can be equated in a 4D spacetime manifold to $\phi(x) \epsilon_{\mu\nu\rho\sigma}$, where $\phi(x)$ is a scalar-density of weight +1, with $\phi \rightarrow \phi/J$, under a general coordinate transformation with Jacobian $J$. A p-vector $w$ with totally antisymmetric components $w^{\mu\nu\rho\sigma}$ corresponding to the four-form $\tilde{w}$ can be constructed by demanding that $w^{\mu\nu\rho\sigma} w_{\mu\nu\rho\sigma} = -4! [37]$. Then, it follows that $w^{\mu\nu\rho\sigma} = \epsilon^{\mu\nu\rho\sigma}/\phi(x)$. We assume $\phi = w_{0123}$ to be dimensionless.

The metric has not entered the picture thus far, but it does so as soon as one switches on the gravitational interaction. The covariant derivatives of the four-form field and its corresponding p-vector are given by,

$$w_{\mu\nu\alpha\beta \lambda} = [(\ln \phi)_{,\lambda} - \Gamma_{\sigma\lambda}^\sigma] w_{\mu\nu\alpha\beta}$$

and

$$w^{\mu\nu\alpha\beta \lambda} = -[(\ln \phi)_{,\lambda} - \Gamma_{\sigma\lambda}^\sigma] w^{\mu\nu\alpha\beta}$$

which, by virtue of equivalence principle, guide us to write down below, an action $S$ that is invariant under general coordinate transformations,

$$S = \frac{m_{Pl}^2}{16\pi} \int R \sqrt{-g} d^4x + \int L \sqrt{-g} d^4x +$$

$$+ \frac{A}{4!} \int \phi w^{\mu\nu\alpha\beta} w_{\mu\nu\alpha\beta} \phi \, d^4x + B \int \phi(x) d^4x,$$

where $m_{Pl}$ and $L$ are the Planck mass and the Lagrangian density of the matter fields, respectively. $A$ and $B$ are real parameters of the theory with dimensions (mass)$^2$ and (mass)$^4$, respectively. The portion of the action in Eq.(3) pertaining to the four-form is by no means unique. For instance, we could add a term $\propto \int R \phi d^4x$ to the above action, but at present we restrict ourselves to gravitational minimal coupling. Later we shall discuss a gravitational Chern-Simons like term involving $\phi$.

Since, $\tilde{w}$ is determined completely by the scalar density $\phi$ in a (3+1)-dimensional manifold, we will often refer to it as $\phi$ from now on. We mention in passing that we could have
raised the indices of $w_{\mu\nu\rho\sigma}$ using $g_{\mu\nu}$ to get a totally antisymmetric contravariant tensor
$\equiv W^{\mu\nu\rho\sigma} = (\sqrt{\gamma})^2 w^{\mu\nu\rho\sigma}$ which, however, is different from the p-vector components $u^{\mu\nu\rho\sigma}$.

By extremizing $S$ with respect to $g_{\mu\nu}$ and $\phi$, respectively, we obtain the following equations of motion,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi}{m_{Pl}^2}[T_{\mu\nu} + \Theta_{\mu\nu}],$$

(4)

$$\psi^{\alpha} \equiv \frac{1}{\sqrt{-g}}(\sqrt{-g}g^{\alpha\beta}\psi)_{,\beta} = \frac{1}{2}\left[g^{\mu\nu}\frac{\psi_{,\mu}\psi_{,\nu}}{\psi} + \frac{B}{A}\psi\right],$$

(5)

where $\psi(x)$, a scalar field variable, is defined to be $\psi \equiv \frac{\phi}{\sqrt{-g}}$, while $T_{\mu\nu}$ is the standard matter energy-momentum tensor and $\Theta_{\mu\nu}$ is the energy-momentum tensor for $\phi$ given by,

$$\Theta_{\mu\nu} = 2A\left[\frac{\psi_{,\mu}\psi_{,\nu}}{\psi} - g_{\mu\nu}\psi^{\alpha}\psi_{,\alpha}\right].$$

(6)

When $\phi$ satisfies the equation of motion given by Eq.(5), its energy-momentum tensor (Eq.(6)) takes the form,

$$\Theta_{\mu\nu} = 2A\left[\frac{\psi_{,\mu}\psi_{,\nu}}{\psi} - g_{\mu\nu}\psi^{\alpha}\psi_{,\alpha}\right].$$

(7)

The action for $\tilde{w}$ takes a simpler shape if one uses the variable $\psi$ instead of $\phi$ in Eq.(3),

$$S_{\phi} = \int \left[\frac{A}{\psi}g^{\mu\nu}\psi_{,\mu}\psi_{,\nu} + B\psi\right]\sqrt{-g}d^4x.$$  

(8)

A point to be emphasized here is that although $g_{\mu\nu}$ and $\phi$ are mutually independent, the variable $\psi$ depends on both, so that when $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$, it induces $\psi \rightarrow \psi + \delta \psi$, where $\delta \psi = \frac{1}{2}\psi g_{\mu\nu}\delta g^{\mu\nu}$. Of course, Eq.(4) then follows from $\delta S/\delta g^{\mu\nu} = 0$.

We now proceed to get a handle on the unknown parameters $A$ and $B$ by invoking the current cosmological scenario. For a flat Friedmann-Robertson-Walker (FRW) model, Eq.(4) reduces to,

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi}{3m_{Pl}^2}[T_0^0 + \Theta_0^0],$$

(9)

and,

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = \frac{8\pi}{m_{Pl}^2}[T_1^1 + \Theta_1^1],$$

(10)

where $T_0^0 = \epsilon$ and $\Theta_0^0 = A\frac{\dot{\psi}^2}{\psi} - B\psi = \epsilon_{\phi}$ are the energy densities for matter and $\phi$, respectively, while $-T_1^1 = p$ and $-\Theta_1^1 = A\frac{\dot{\psi}^2}{\psi} + B\psi = p_{\phi}$ are the pressures for matter and the four-form, respectively. For a flat FRW model, the equation of motion for $\phi$ given by Eq.(5) assumes the form,

$$\ddot{\psi} + 3\frac{\dot{a}}{a}\dot{\psi} + \frac{B_0}{2}\psi - \frac{1}{2}\frac{\dot{\psi}^2}{\psi} = 0,$$

(11)
where $B_0 \equiv -\frac{B}{A}$. Eq.(11) can be turned into a second order linear differential equation by setting $\psi = f^2$ so that,
\[ \ddot{f} + 3\frac{\dot{a}}{a}\dot{f} + \frac{B_0}{4}f = 0. \] (12)

In terms of $f$, the energy density and pressure for $\phi$ are given by,
\[ \epsilon_\phi = 4A\dot{f}^2 - Bf^2 = 4A\left[\dot{f}^2 + \frac{B_0}{4}f^2\right] \] (13)
and,
\[ p_\phi = 4A\dot{f}^2 + Bf^2 = 4A\left[\dot{f}^2 - \frac{B_0}{4}f^2\right] \] (14)
respectively, and hence the expression for the equation of state parameter is simply,
\[ w_\phi \equiv \frac{p_\phi}{\epsilon_\phi} = -(1 - \frac{4\dot{f}^2}{B_0f^2})/(1 + \frac{4\dot{f}^2}{B_0f^2}). \] (15)

It is obvious from the above equation that if $\dot{f}^2$ happens to be sufficiently small over a long stretch of time, the value of $w_\phi$ is $\approx -1$, a required condition for an accelerated expansion of the universe. This raises the possibility of the scalar-density being a source of DE. According to Eqs.(13) and (14), $A > 0$ and $B < 0$ make $f$ mimic a quintessence field yielding $w_\phi > -1$, while $A < 0$ and $B < 0$ reproduce the case of a phantom field with $w_\phi < -1$ [7, 8, 9, 10, 12]. In the present work, we assume that the kinetic energy and the mass terms for $f$ are positive definite so that $A > 0$ and $B < 0$, implying $B_0 > 0$. Bearing these points in mind, we study the dynamics of $f$ after the universe has ceased to be radiation-dominated.

Particles with high Lorentz gamma factors play an insignificant role in the evolution of the scale factor for redshifts less than $\approx 10^3$ [36]. For such later epochs, if $\epsilon_\phi$ is negligible compared to the energy-density of non-relativistic matter, universe is matter-dominated with scale factor $a \propto t^{2/3}$. This in turn implies $3\frac{\dot{a}}{a} = \frac{2}{t}$, leading to an exact solution of Eq.(12) during the matter-dominated era,
\[ f(t) = \frac{K_0}{t} \sin\left(\frac{\sqrt{B_0}t}{2} + K_1\right), \] (16)
where $K_0$ and $K_1$ are constants of integration.

However, from times around the epoch of matter-$\phi$ equality $t_\phi \equiv 0.655(2/3H_0)$, the four-form energy density $\epsilon_\phi$, far from being insignificant, plays a major role in the dynamics of $a(t)$, since the density parameter $\Omega_{\phi0}$ today is $\approx 0.7$ [4, 38, 39]. Therefore, Eqs.(9)-(11) need
to be solved in a self-consistent manner, with the exact solution (Eq.(16)) acting as a rough
guide. In order to solve these coupled set of differential equations, we adopt the procedure
of Dutta and Scherrer (2008).

Introducing a new function $g$ through

$$f(t) = a^{-3/2}g(t)$$  \hspace{1cm} (17)$$

and substituting it in Eq.(12), we arrive at the following equation after using Eq.(10),

$$\ddot{g} + \left[ \frac{B_0}{4} + \frac{6\pi p_\phi}{m_p^2} \right] g = 0.$$  \hspace{1cm} (18)$$

In deriving the above equation, we have set the matter-pressure $p$ to zero in Eq.(10), as we
are dealing with a ‘non-relativistic matter+ $\phi$’-dominated phase.

During the early part of matter-dominated phase, $f$ is expected to trace the solution
given by Eq. (16), displaying an oscillatory behaviour. With a suitable choice of the phase
$K_1$, one can arrange $f$ to reach a local maximum around the epoch $t_\phi$, and hover there till
the present epoch. This essentially amounts to making a ‘slow-rolling’ assumption,

$$\frac{4\dot{f}^2}{B_0f^2} \ll 1,$$  \hspace{1cm} (19)$$

and, hence when it is combined with Eq.(15), the outcome is,

$$p_\phi \approx -\epsilon_\phi$$  \hspace{1cm} (20)$$

Slow roll of $f$ (Eq.(19)) leads to $f$ and $\epsilon_\phi$ being roughly constant. Then, from Eq.(9),
with matter energy density going as $a^{-3}$, we have the standard solution,

$$a(t) = a_0 \left( \frac{\Omega_{m0}}{\Omega_{\phi0}} \right)^{1/3} \sinh^2 \left( \frac{3}{2} H_0 \sqrt{\Omega_{\phi0}} t \right),$$  \hspace{1cm} (21)$$

with,

$$\Omega_{\phi0} \equiv \frac{8\pi}{3m_p^2H_0^2} \epsilon_{\phi0} \approx \frac{8\pi AB_0 f^2}{3m_p^2H_0^2},$$  \hspace{1cm} (22)$$

and $\Omega_{m0} + \Omega_{\phi0} = 1$, for a flat FRW model.

Making use of Eqs.(20) and (22) in Eq.(18), one obtains,

$$\ddot{g} + \frac{1}{4} (B_0 - B_1) g \approx 0.$$  \hspace{1cm} (23)$$
where $B_1 \equiv 9H_0^2\Omega_{\phi_0}$. The preceding differential equation can be trivially solved. Defining $A_0 \equiv a_0^{-3/2} \sqrt{\Omega_{\phi_0}}$, and combining Eqs. (17) with (21), one arrives at,

$$f(t) = A_0 \frac{A_1 \sin(\Delta t + A_2)}{\sinh(\frac{\sqrt{B_1}}{2})}, \quad \text{for } B_0 > B_1$$

$$= A_0 \frac{A_3 \sinh(\gamma t) + A_4 \cosh(\gamma t)}{\sinh(\frac{\sqrt{B_1}}{2})}, \quad \text{for } B_0 < B_1,$$

where $\Delta \equiv \frac{1}{2} \sqrt{B_0 - B_1}$, $\gamma \equiv \frac{1}{2} \sqrt{B_1 - B_0}$ and $A_1, ..., A_4$ are constants of integration. Comparison of Eqs.(24a) and (24b) with Eq.(16) along with the asymptotic behaviour of $f$ as $t \to \infty$, leads us to the conclusion that the solution described by Eq.(24a) is physically more meaningful, imposing the restriction $B_0 > B_1$. Having narrowed the choice for $B_0$, we turn our attention to the self-consistency of Eq.(24a) vis-a-vis the condition given by Eq.(19).

To study ‘slow rolling’, we use Eq.(24a) to get the slope of $f$ so that,

$$\frac{2 \dot{f}}{\sqrt{B_0} f} = \sqrt{1 - \frac{1}{r} \cot \left( \frac{\sqrt{(r-1)B_1}}{2} t + A_2 \right) - \frac{\coth(\frac{\sqrt{B_1}}{2} t)}{r^{1/2}}},$$

with $r \equiv \frac{B_0}{B_1}$. If $A_2 = 0$ and $r = 1 + \epsilon^2$, we have from the above equation,

$$\frac{2 \dot{f}}{\sqrt{B_0} f} \approx (\frac{\sqrt{B_1}}{2} t)^{-1} - \coth(\frac{\sqrt{B_1}}{2} t),$$

for $\epsilon^2$ infinitesmally small. Then, for $\epsilon \leq 10^{-1}$, as $\frac{\sqrt{B_1}}{2} t$ goes from 0.1 to 1 we find that $\frac{\dot{f}^2}{B_0 f^2}$ varies from $10^{-3}$ to 0.1 satisfying thereby the condition given in Eq.(19). This is also evident from Eq.(24a) since for $A_2 = 0$ and for very small values of $\Delta$ (i.e. $B_0 \approx B_1$), $f$ is effectively flat over a long period of time. We therefore conclude that if the parameters $A$ and $B$ appearing in Eq.(3) are such that $-B$ is marginally in excess of $\approx 9AH_0^2\Omega_{\phi_0}$, the solution provided by Eq.(24a) is reasonably good, with the upshot being that $w_{\phi} \approx -1$ during $t = (0.18 - 1.8)(2/3H_0)$. Although a specific range of values for the parameters does reproduce the conditions necessary for the observed acceleration of the expansion factor, the refrain of fine-tuning haunts this scenario too.

It is interesting to observe that a Chern-Simons (CS) like term mirroring the coupling between electromagnetic field and the four-form $\tilde{w}$ arises very naturally in our model,

$$S_{CS} = J \int w^{\mu\nu\alpha\beta} F_{\mu\nu} A_\alpha \phi_{\beta} \, d^4x$$

$$= J \int \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} A_\alpha (\ln \psi)_\beta \, d^4x,$$

$$= J \int \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} A_\alpha (\ln \psi)_\beta \, d^4x,$$
where \( J \) is a dimensionless constant, \( F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \) and,

\[
\phi_{,\beta} = \phi_{,\beta} - \Gamma^{\alpha}_{\alpha\beta}\phi
\]  

(28)
since \( \phi \) is a scalar-density of weight +1. It is straightforward to establish that the above action is invariant under diffeomorphisms as well as gauge transformations, \( A_{\mu} \rightarrow A_{\mu} + \partial_{\mu}\chi \), modulo boundary terms. The outcome of adding \( S_{CS} \) (Eq.(27)) to the standard action \( S_{EM} \) for an electromagnetic field interacting with charge particles in the presence of gravitation is that, upon extremizing the full action \( S_{EM} + S_{CS} \) with respect to \( A_{\mu} \), one arrives at the following modified Einstein-Maxwell equation,

\[
F^{\alpha\beta}_{,;\beta} = -4\pi j^\alpha + 8\pi J w^{\mu\alpha\beta} F_{\mu\nu} \psi_{,\beta}
\]  

(29)
with \( j^\alpha \) being the 4-current density associated with charge particles. Applying the covariant derivative to the above equation with respect to \( x^\alpha \) leads to the usual continuity equation implying conservation of electric charge,

\[
(\sqrt{-g} j^\alpha)_{,\alpha} = 0.
\]  

(30)

Eq.(29) opens up the possibility of constraining the CS parameter \( J \) when used in conjunction with WMAP and other astrophysical data. This requires further examination, and will form the subject of a separate paper.

The CS-like analysis described above is similar to the one carried out by Jackiw and Pi (2003), except that they employed an external fixed four-vector \( v_{\mu} \) instead of a dynamical \( \phi_{,\mu} \), entailing violation of Lorentz invariance in their model [40]. In our case, both general as well as Lorentz covariance is maintained all through. Inspired by the seminal work of Jackiw and Pi, we undertake the exercise of studying the following gravitational CS term that involves \( \tilde{w} \) and the Christoffel symbols,

\[
S_{GCS} = H \int w^{\mu\alpha\beta} [\Gamma^\sigma_{\nu\tau} \partial_{\alpha} \Gamma^\tau_{\beta\sigma} + \frac{2}{3} \Gamma^\sigma_{\nu\tau} \Gamma^\tau_{\alpha\eta} \Gamma^\eta_{\beta\sigma}] \phi_{,\mu} \, d^4x
= H \int \epsilon^{\mu\alpha\beta} [\Gamma^\sigma_{\nu\tau} \partial_{\alpha} \Gamma^\tau_{\beta\sigma} + \frac{2}{3} \Gamma^\sigma_{\nu\tau} \Gamma^\tau_{\alpha\eta} \Gamma^\eta_{\beta\sigma}] (\ln \psi)_{,\mu} \, d^4x,
\]  

(31)

\( H \) being a dimensionless constant. After integrating by parts once, Eq.(31) can be expressed in terms of the Riemann tensor as,

\[
S_{GCS} = -\frac{H}{2} \int \ln \psi \ast RRd^4x,
\]  

(32)
where,

\[ *RR \equiv \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} R^\tau_{\alpha\beta} R^\sigma_{\tau\mu\nu} = 8 \left[ R^\tau_{\sigma 01} R_{\sigma 23} + R^\tau_{\sigma 12} R_{\sigma 03} + R^\tau_{\sigma 13} R_{\sigma 20} \right] \]  

(33a)

and,

\[ *R^\tau_{\rho\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} R^\tau_{\alpha\beta} \]  

(33b)

with,

\[ R^\tau_{\sigma \alpha \beta} = \partial_\alpha \Gamma^\tau_{\sigma \beta} - \partial_\beta \Gamma^\tau_{\sigma \alpha} + \Gamma^\tau_{\alpha \eta} \Gamma^\eta_{\beta \sigma} - \Gamma^\tau_{\beta \eta} \Gamma^\eta_{\alpha \sigma} \]

and \( R_{\alpha \beta} = R^\tau_{\sigma \tau \beta} \).

From Eq.(32) we find that \( \ln \psi \) acts like the parameter \( \theta \) of Jackiw and Pi’s paper in which the external vector \( v_\mu = \theta_\mu \). Adding \( S_{CS} + S_{GCS} \) to \( S \) of Eq.(3) and then extremizing the total action with respect to \( \phi \) and \( g_{\mu\nu} \) leads to,

\[ \psi^{;\alpha}_{;\alpha} = \frac{1}{2} \left[ g^{\mu\nu} \psi^{,\mu}_{;\nu} \psi \right] + B \psi + \frac{1}{2A} \psi w^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} - \frac{H}{4A} \psi w^{\mu\nu\alpha\beta} R^\tau_{\alpha\beta} R^\sigma_{\tau\mu\nu} \]  

(34)

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = - \frac{8\pi}{m_{Pl}^2} \left[ T_{\mu\nu} + \Theta_{\mu\nu} + C_{\mu\nu} \right] \]  

(35)

where the modified Cotton tensor \( C^{\mu\nu} \) is defined as,

\[ C^{\mu\nu} \equiv -\frac{1}{\sqrt{-g}} \left[ \frac{1}{4} *RR g^{\mu\nu} - (\ln \psi)^{;\alpha}_{;\alpha} \left( *R^\beta_{\mu\nu} + *R^\alpha_{\mu\nu} \right) + (\ln \psi)^{,\alpha}_{,\alpha} \left( \epsilon^{\alpha\mu\sigma\tau} R^\nu_{\sigma\tau} + \epsilon^{\alpha\nu\sigma\tau} R^\mu_{\sigma\tau} \right) \right] \]  

(36)

The first term in Eq.(36) is new and is not present in the expression for the Cotton tensor as delineated by Jackiw and Pi. Here it appears because under an infinitesimal variation \( g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu} \), the change in \((\ln \psi)^{,\mu}_{,\mu}\) occurring in Eq.(31) is given by,

\[ \delta(\ln \psi)^{,\mu}_{,\mu} = \delta(\phi_\mu / \phi) = -\delta \Gamma^\alpha_{\alpha\mu} = -\frac{1}{2} \delta(g^{\alpha\beta} g_{\alpha\beta,\mu}) \]  

(37)

which follows after making use of Eq.(28).

The natural question to ask is with the entry of \( S_{GCS} \) can the dynamics of \( \phi \), now governed by Eqs.(34) and (35), still lead to an accelerating universe? We show that the answer is in the affirmative. In the context of a flat FRW model, all the non-zero components of the Riemann tensor are obtained from,

\[ R_{0101} = a \ddot{a} \quad R_{0202} = r^2 a \ddot{a} \quad R_{0303} = (r \sin \theta)^2 a \ddot{a} \]

\[ R_{1212} = -r^2 (a \ddot{a})^2 \quad R_{1313} = \sin^2 \theta R_{1212} \quad R_{2323} = r^2 R_{1313} \]  

(38)

Using Eq.(38) in Eq.(33a), one readily verifies that \( *RR = 0 \) so that the first term in Eq.(36) vanishes. Hence, in the matter dominated era the equation of motion for \( \phi \) (Eq.(34)) simply
reduces to Eq.(5) as the last term of Eq.(34) is proportional to \( \ast RR \). Since in the case of FRW models, only the diagonal components of Ricci tensor are non-zero with \( R^1_1 = R^2_2 = R^3_3 \) and that they depend on time alone, we find after tedious algebra involving Eqs.(38) and (33b) that even the second and third terms in Eq.(36) vanish. This explicitly demonstrates that the modified Cotton tensor is zero for a flat FRW universe implying that the inclusion of \( S_{GCS} \) does not alter our solution for \( f(t) \) given by Eq.(24a) that is responsible for an accelerating universe, so long as ‘slow rolling’ condition is met. We could have reached this conclusion straightaway without labourious calculations from the fact that \( \ast RR = 0 \) implies \( S_{GCS} = 0 \) from Eq.(32).

Apart from the possibility of adding ‘Chern-Simons’ like terms described above, and a natural coupling that may arise between \( \tilde{w} \) and branes in string/M-theories \([24, 25, 26, 27, 28, 29, 30, 31]\), there may also be a differential geometric significance of the scalar-density \( \phi \), in the sense that it can be used to generate an antiderivation on antisymmetric tensor-densities of arbitrary weights. If \( \tilde{\alpha} \) is a p-form density with weight \( w \) such that its components \( \alpha_{\nu_1\nu_2...\nu_p} \) transform to \( J^{-w}\alpha_{\nu_1\nu_2...\nu_p} \), under a general coordinate transformation, \( J \) being the Jacobian, then we define a generalized exterior derivative \( \tilde{d}_w \) in the following manner,

\[
\tilde{d}_w\tilde{\alpha} = \frac{1}{p!} \partial_\mu \alpha_{\nu_1\nu_2...\nu_p} \tilde{dx}^\mu \wedge \tilde{dx}^{\nu_1} \wedge ... \wedge \tilde{dx}^{\nu_p} = -w \partial_\mu (\ln \phi) \tilde{dx}^\mu \wedge \tilde{\alpha} - w \tilde{d}(\ln \phi) \wedge \tilde{\alpha}.
\]

It is easy to see that \( \tilde{d}_w\tilde{\alpha} \) is a (p+1)-form density of weight \( w \).

If \( \chi_1 \) and \( \chi_2 \) are scalar-densities of weights \( w_1 \) and \( w_2 \), respectively, then the above equation leads to,

\[
\tilde{d}_w\chi_i = \partial_\mu \chi_i \tilde{dx}^\mu - w_i \chi_i \partial_\mu (\ln \phi) \tilde{dx}^\mu, \quad i = 1, 2 \tag{39a}
\]

being one-form densities and, furthermore, one can show that,

\[
\tilde{d}_w(\chi_1 \tilde{d}_w\chi_2) = \tilde{d}_w\chi_1 \wedge \tilde{d}_w\chi_2. \tag{39b}
\]

The generalized exterior derivative also satisfies (a) \( \tilde{d}_w\tilde{d}_w = 0 \) and (b) \( \tilde{d}_w(\tilde{\alpha} \wedge \tilde{\beta}) = \tilde{d}_w\tilde{\alpha} \wedge \tilde{\beta} + (-1)^p \tilde{\alpha} \wedge \tilde{d}_w\tilde{\beta} \), where \( \tilde{\alpha} \) and \( \tilde{\beta} \) are p- and q-form densities of weights \( w_1 \) and \( w_2 \), respectively. These properties are sufficient to qualify \( \tilde{d}_w \) to the role of a well-defined antiderivation on differential form-densities \([41]\). For instance, from Eq.(39a) it follows that,

\[
\tilde{d}_w \sqrt{-g} = -\sqrt{-g} \tilde{d} \ln(\sqrt{-g}), \tag{40}
\]
since $\sqrt{-g}$ is a scalar-density of weight +1. There are other physically meaningful antisymmetric tensor-densities, e.g. dual of $F^{\mu\nu}$, on which we may apply $\tilde{d}_w$\[42, 43\]. It is interesting to note that $\tilde{d}_w\phi = 0$. This is analogous to the vanishing of $g_{\mu\nu;\lambda}$.

To summarize in a nutshell, what we have demonstrated in this paper is that if Einstein’s geometrical theory of gravitation is extended by including a new degree of freedom $\tilde{w}$ that is independent of the metric $g_{\mu\nu}$, there exists a long band of allowed region in the parameter-plane constituted by $A$ and $B$, such that the dynamics of $\phi$ does give rise to an accelerating universe in the context of a flat FRW model. We concede here that even this model cannot shake off the fine-tuning problem that plagues other DE models, as it too relies on the ‘slow rolling’ condition to attain $w_\phi \approx -1$.

A Chern-Simons like coupling between electromagnetic fields and $\phi$ comes about naturally, causing a modification of Einstein-Maxwell equation. What ensues from a gravitational Chern-Simons like term is that a modified Cotton tensor appears in the Einstein equation, although this has no effect on the dynamics in a flat FRW universe. The scalar-density associated with $\tilde{w}$ leads to a well-defined exterior derivative that turns a differential p-form density into a $(p+1)$-form density of same weight. Since the notion of an n-form and exterior derivative in a differential manifold does not require either an affine connection or a metric, further studies are required to investigate the role of $\tilde{w}$ in situations where metric is ill-defined and in the possibility of its causing transitions in manifold-orientability.

I thank N. Mukunda and Joseph Samuel for drawing my attention to some key references. I also acknowledge technical assistance extended to me by Ujjwal Dasgupta. This research has made use of NASA’s Astrophysics Data System.

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