Exact entanglement cost of quantum states and channels under PPT-preserving operations

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This paper establishes single-letter formulas for the exact entanglement cost of generating bipartite quantum states and simulating quantum channels under free quantum operations that completely preserve positivity of the partial transpose (PPT). First, we establish that the exact entanglement cost of any bipartite quantum state under PPT-preserving operations is given by a single-letter formula, here called the $\kappa$-entanglement of a quantum state. This formula is calculable by a semidefinite program, thus allowing for an efficiently computable solution for general quantum states. Notably, this is the first time that an entanglement measure for general bipartite states has been proven not only to possess a direct operational meaning but also to be efficiently computable, thus solving a question that has remained open since the inception of entanglement theory over two decades ago. Next, we introduce and solve the exact entanglement cost for simulating quantum channels in both the parallel and sequential settings, along with the assistance of free PPT-preserving operations. The entanglement cost in both cases is given by the same single-letter formula and is equal to the largest $\kappa$-entanglement that can be shared by the sender and receiver of the channel. It is also efficiently computable by a semi-definite program.

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I. INTRODUCTION

A. Background

Quantum entanglement, the most nonclassical manifestation of quantum mechanics, has found use in a variety of physical tasks in quantum information processing, quantum cryptography, thermodynamics, and quantum computing [HHHH09]. A natural and fundamental problem is to develop a theoretical framework to quantify and describe it. In spite of remarkable recent progress in the resource theory of entanglement (for reviews see, e.g., [PV07, HHHH09]), many fundamental challenges have remained open.
One of the most important aspects of the resource theory of entanglement consists of the interconversions of states, with respect to a class of free operations. In particular, the problem of entanglement dilution asks: given a target bipartite state $\rho_{AB}$ and a canonical unit of entanglement represented by the Bell state (or ebit) $|\Phi_2\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, what is the minimum rate at which we can produce copies of $\rho_{AB}$ from copies of $\Phi_2$ under a chosen set of free operations?

The entanglement cost [BDSW96] was introduced to quantify the minimal rate $R$ of converting $\Phi_2^\otimes nR$ to $\rho_{AB}^\otimes n$ with an arbitrarily high fidelity in the limit as $n$ becomes large. When local operations and classical communication (LOCC) are allowed for free, the authors of [HHT01] proved that the entanglement cost is equal to the regularized entanglement of formation [BDSW96]. When the free operations consist of quantum operations that completely preserve positivity of the partial transpose (the PPT-preserving operations [Rai99, Rai01]), it is known that the entanglement cost is not equal to the regularized entanglement of formation [APE03, Yur03, Hay06].

The exact entanglement cost [APE03] is an alternative and natural way to quantify the cost of entanglement dilution, being defined as the smallest asymptotic rate $R$ at which $\Phi_2^\otimes nR$ is required in order to reproduce $\rho_{AB}^\otimes n$ exactly. The exact entanglement cost under PPT-preserving operations (PPT entanglement cost) was introduced and solved for a large class of quantum states in [APE03], but it has hitherto remained unknown for general quantum states.

The above resource-theoretic problems can alternatively be phrased as simulation problems: How many copies of $\Phi_2$ are needed to simulate $n$ copies of a given bipartite state $\rho_{AB}$? As discussed above, the simulation can be either approximate, such that a verifier has little chance of distinguishing the simulation from the ideal case, while it can also be exact, such that a verifier has no chance at all for distinguishing the simulation from the ideal case.

With this perspective, it is natural to consider the simulation of a quantum channel, when allowing some set of operations for free and metering the entanglement cost of the simulation. The authors of [BBCW13] defined the entanglement cost of a channel to be the smallest rate $R$ at which $\Phi_2^\otimes nR$ is needed, along with the free assistance of LOCC, in order to simulate the channel $N^\otimes n$, in such a way that a verifier would have little chance of distinguishing the simulation from the ideal case of $N^\otimes n$. In [BBCW13], it was shown that the regularized entanglement of formation of the channel is equal to its entanglement cost, thus extending the result of [HHT01] in a natural way.

In a recent work [Wil18], it was observed that the channel simulation task defined in [BBCW13] is actually a particular kind of simulation, called a parallel channel simulation. The paper [Wil18] then defined an alternative notion of channel simulation, called sequential channel simulation, in which the goal is to simulate $n$ uses of the channel $N$ in such a way that the most general verification strategy would have little chance of distinguishing the simulation from the ideal $n$ uses of the channel. Although a general formula for the entanglement cost in this scenario was not found, it was determined for several key channel models, including erasure, dephasing, three-dimensional Holevo–Werner, and single-mode pure-loss and pure-amplifier bosonic Gaussian channels.

### B. Summary of results

In this paper, we solve significant questions in the resource theory of entanglement, one of which has remained open since the inception of entanglement theory over two decades
ago. Namely, we prove that the exact PPT-entanglement cost for both quantum states and channels have efficiently computable, single-letter formulas, reflecting the fundamental entanglement structures of bipartite quantum states and channels. Notably, this is the first time that an entanglement measure has been shown to be both efficiently computable and to possess a direct operational meaning. Furthermore, we prove that the exact parallel and sequential entanglement costs of quantum channels are given by the same efficiently computable, single-letter formula.

Our paper is structured as follows. In Section II, we first introduce the $\kappa$-entanglement measure of a bipartite state, and we prove that it satisfies several desirable properties, including monotonicity under completely-PPT-preserving channels, additivity, normalization, faithfulness, non-convexity, and non-monogamy. For finite-dimensional states, it is also efficiently computable by means of a semi-definite program.

Next, in Section III, we prove that the $\kappa$-entanglement is equal to the exact entanglement cost of a quantum state. This direct operational interpretation and the fact that both convexity and monogamy are violated for the $\kappa$-entanglement measure calls into question whether these properties are truly necessary for entanglement. We further evaluate the $\kappa$-entanglement (and the exact entanglement cost) for several bipartite states of interest in Section III B, including isotropic states, Werner states, maximally correlated states, some states supported on the $3 \times 3$ antisymmetric subspace, and all bosonic Gaussian states.

We then extend the $\kappa$-entanglement measure from bipartite states to point-to-point quantum channels in Section IV. We prove that it also satisfies several desirable properties, including non-increase under amortization, monotonicity under PPT superchannels, additivity, normalization, faithfulness, and non-convexity. For finite-dimensional channels, it is also efficiently computable by means of a semi-definite program.

In Section V, we show that the $\kappa$-entanglement of channels has a direct operational meaning as the entanglement cost of both parallel and sequential channel simulation. Thus, the theory of channel simulation significantly simplifies for the setting in which completely-PPT-preserving channels are allowed for free. In addition to all of the properties that it satisfies, this operational interpretation solidifies the $\kappa$-entanglement of a channel as a foundational measure of the entanglement of a quantum channel.

As the last contribution of this paper, we evaluate the $\kappa$-entanglement (and exact entanglement cost) of several important channel models in Section VI, including erasure, depolarizing, dephasing, and amplitude damping channels. In Section VII, we also leverage recent results in the literature [LSMGA17], regarding the teleportation simulation of bosonic Gaussian channels, in order to evaluate the $\kappa$-entanglement and exact entanglement cost for several fundamental bosonic Gaussian channels. We remark that these latter results provide a direct operational interpretation of the Holevo–Werner quantity [HW01] for these channels.

Finally, we conclude with a summary and some open questions.

II. $\kappa$-ENTANGLEMENT MEASURE FOR BIPARTITE STATES

We now introduce an entanglement measure for a bipartite state, here called the $\kappa$-entanglement measure:
Definition 1 ($\kappa$-entanglement measure) Let $\rho_{AB}$ be a bipartite state acting on a separable Hilbert space. The $\kappa$-entanglement measure is defined as follows:

$$E_\kappa(\rho_{AB}) := \inf \{ \log \text{Tr} S_{AB} : -S_{AB}^{T_B} \leq \rho_{AB}^{T_B} \leq S_{AB}^{T_B}, S_{AB} \geq 0 \}. \quad (1)$$

In the case that the state $\rho_{AB}$ acts on a finite-dimensional Hilbert space, then $E_\kappa(\rho_{AB})$ is calculable by a semi-definite program, and thus it is efficiently computable with respect to the dimension of the Hilbert space.

A. Monotonicity under completely-PPT-preserving channels

Throughout this paper, we consider completely-PPT-preserving operations [Rai99, Rai01], defined as a bipartite operation $\mathcal{P}_{AB \rightarrow A'B'}$ (completely positive map) such that the map $T_{B'} \circ \mathcal{P}_{AB \rightarrow A'B'} \circ T_B$ is also completely positive, where $T_B$ and $T_{B'}$ denote the partial transpose map acting on the input system $B$ and the output system $B'$, respectively. If $\mathcal{P}_{AB \rightarrow A'B'}$ is also trace preserving, such that it is a quantum channel, and $T_{B'} \circ \mathcal{P}_{AB \rightarrow A'B'} \circ T_B$ is also completely positive, then we say that $\mathcal{P}_{AB \rightarrow A'B'}$ is a completely-PPT-preserving channel.

The most important property of the $\kappa$-entanglement measure is that it does not increase under the action of a completely-PPT-preserving channel. Note that an LOCC channel [BDSW96, CLM+14], as considered in entanglement theory, is a special kind of completely-PPT-preserving channel, as observed in [Rai99, Rai01].

Theorem 1 (Monotonicity) Let $\rho_{AB}$ be a quantum state acting on a separable Hilbert space, and let $\{\mathcal{P}_{AB \rightarrow A'B'}^x\}_x$ be a set of completely positive, trace non-increasing maps that are each completely PPT-preserving, such that the sum map $\sum_x \mathcal{P}_{AB \rightarrow A'B'}^x$ is quantum channel. Then the following entanglement monotonicity inequality holds

$$E_\kappa(\rho_{AB}) \geq \sum_{x : p(x) > 0} p(x) E_\kappa(\mathcal{P}_{AB \rightarrow A'B'}^x(\rho_{AB})) \quad (2)$$

where $p(x) := \text{Tr} \mathcal{P}_{AB \rightarrow A'B'}^x(\rho_{AB})$. In particular, for a completely-PPT-preserving quantum channel $\mathcal{P}_{AB \rightarrow A'B'}$, the following inequality holds

$$E_\kappa(\rho_{AB}) \geq E_\kappa(\mathcal{P}_{AB \rightarrow A'B'}(\rho_{AB})). \quad (3)$$

Proof. Let $S_{AB}$ be such that

$$S_{AB} \geq 0, \quad -S_{AB}^{T_B} \leq \rho_{AB}^{T_B} \leq S_{AB}^{T_B}. \quad (4)$$

Since $\mathcal{P}_{AB \rightarrow A'B'}^x$ is completely-PPT-preserving, we have that

$$-(T_{B'} \circ \mathcal{P}_{AB \rightarrow A'B'}^x \circ T_B)(S_{AB}^{T_B}) \leq (T_{B'} \circ \mathcal{P}_{AB \rightarrow A'B'}^x \circ T_B)(\rho_{AB}^{T_B}) \leq (T_{B'} \circ \mathcal{P}_{AB \rightarrow A'B'}^x \circ T_B)(S_{AB}^{T_B}), \quad (5)$$

which reduces to the following for all $x$ such that $p(x) > 0$:

$$- \frac{[\mathcal{P}_{AB \rightarrow A'B'}^x(S_{AB})]^{T_{B'}}}{p(x)} \leq \frac{[\mathcal{P}_{AB \rightarrow A'B'}^x(\rho_{AB})]^{T_{B'}}}{p(x)} \leq \frac{[\mathcal{P}_{AB \rightarrow A'B'}(S_{AB})]^{T_{B'}}}{p(x)}. \quad (6)$$
Furthermore, since $S_{AB} \geq 0$ and $\mathcal{P}^{x}_{AB \rightarrow A'B'}(S_{AB})$ is completely positive, we conclude the following for all $x$ such that $p(x) > 0$:

$$\frac{\mathcal{P}^{x}_{AB \rightarrow A'B'}(S_{AB})}{p(x)} \geq 0. \quad (7)$$

Thus, the operator $\frac{\mathcal{P}^{x}_{AB \rightarrow A'B'}(S_{AB})}{p(x)}$ is feasible for $E_{\kappa}\left(\frac{\mathcal{P}^{x}_{AB \rightarrow A'B'}(\rho_{AB})}{p(x)}\right)$. Then we find that

$$\log \text{Tr}[S_{AB}] = \log \sum_{x} \text{Tr} \mathcal{P}^{x}_{AB \rightarrow A'B'}(S_{AB}) \quad (8)$$

$$= \log \sum_{x : p(x) > 0} p(x) \text{Tr} \frac{\mathcal{P}^{x}_{AB \rightarrow A'B'}(\rho_{AB})}{p(x)} \quad (9)$$

$$\geq \sum_{x : p(x) > 0} p(x) \log \text{Tr} \frac{\mathcal{P}^{x}_{AB \rightarrow A'B'}(\rho_{AB})}{p(x)} \quad (10)$$

$$\geq \sum_{x : p(x) > 0} p(x) E_{\kappa}\left(\frac{\mathcal{P}^{x}_{AB \rightarrow A'B'}(\rho_{AB})}{p(x)}\right). \quad (11)$$

The first equality follows from the assumption that the sum map $\sum_{x} \mathcal{P}^{x}_{AB \rightarrow A'B'}$ is trace preserving. The first inequality follows from concavity of the logarithm. The second inequality follows from the definition of $E_{\kappa}$ and the fact that $\frac{\mathcal{P}^{x}_{AB \rightarrow A'B'}(\rho_{AB})}{p(x)}$ satisfies (6) and (7). Since the inequality holds for an arbitrary $S_{AB} \geq 0$ satisfying (4), we conclude the inequality in (2).

The inequality in (3) is a special case of that in (2), in which the set $\{\mathcal{P}^{x}_{AB \rightarrow A'B'}\}_{x}$ is a singleton, consisting of a single completely-PPT-preserving quantum channel.

\[\blacksquare\]

### B. Dual representation and additivity

The optimization problem dual to $E_{\kappa}(\rho_{AB})$ in Definition 1 is as follows:

$$E_{\kappa}^{\text{dual}}(\rho_{AB}) := \sup \{ \log \text{Tr} \rho_{AB}(V_{AB} - W_{AB}) : V_{AB} + W_{AB} \leq I_{AB}, V^{T_{A}}_{AB}, W^{T_{B}}_{AB} \geq 0 \}, \quad (12)$$

which can be found by the Lagrange multiplier method (see, e.g., [Wat18, Section 1.2.2]). By weak duality [Wat18, Section 1.2.2], we have for any bipartite state $\rho_{AB}$ acting on a separable Hilbert space that

$$E_{\kappa}^{\text{dual}}(\rho_{AB}) \leq E_{\kappa}(\rho_{AB}). \quad (13)$$

For all finite-dimensional states $\rho_{AB}$, strong duality holds, so that

$$E_{\kappa}(\rho_{AB}) = E_{\kappa}^{\text{dual}}(\rho_{AB}). \quad (14)$$

This follows as a consequence of Slater’s theorem.

By employing the strong duality equality in (14) for the finite-dimensional case, along with the approach from [FAR11], we conclude that the following equality holds for all bipartite states $\rho_{AB}$ acting on a separable Hilbert space:

$$E_{\kappa}(\rho_{AB}) = E_{\kappa}^{\text{dual}}(\rho_{AB}). \quad (15)$$

We provide an explicit proof of (15) in Appendix A.

Both the primal and dual SDPs for $E_{\kappa}$ are important, as the combination of them allows for proving the following additivity of $E_{\kappa}$ with respect to tensor-product states.
Proposition 2 (Additivity) For any two bipartite states $\rho_{AB}$ and $\omega_{A'B'}$ acting on separable Hilbert spaces, the following additivity identity holds

$$E_\kappa(\rho_{AB} \otimes \omega_{A'B'}) = E_\kappa(\rho_{AB}) + E_\kappa(\omega_{A'B'}) .$$  \hspace{1cm} (16)

Proof. From Definition 1, we can write $E_\kappa(\rho_{AB})$ as

$$E_\kappa(\rho_{AB}) = \inf \{ \log \text{Tr} S_{AB} : -S^{T_B}_{AB} \leq \rho^{T_B}_{AB} \leq S^{T_B}_{AB}, S_{AB} \geq 0 \} .$$  \hspace{1cm} (17)

Let $S_{AB}$ be an arbitrary operator satisfying $-S^{T_B}_{AB} \leq \rho^{T_B}_{AB} \leq S^{T_B}_{AB}, S_{AB} \geq 0$, and let $R_{A'B'}$ be an arbitrary operator satisfying $-R^{T_{A'B'}}_{A'B'} \leq \omega^{T_{A'B'}}_{A'B'}, S_{AB} \geq 0$. Then it follows that

$$-(S_{AB} \otimes R_{A'B'})^{T_{BB'}} \leq (\rho_{AB} \otimes \omega_{A'B'})^{T_{BB'}} \leq (S_{AB} \otimes R_{A'B'})^{T_{BB'}}, S_{AB} \otimes R_{A'B'} \geq 0 ,$$  \hspace{1cm} (18)

so that

$$E_\kappa(\rho_{AB} \otimes \omega_{A'B'}) \leq \log \text{Tr} S_{AB} \otimes R_{A'B'} = \log \text{Tr} S_{AB} + \log \text{Tr} R_{A'B'} .$$  \hspace{1cm} (19)

Since the inequality holds for all $S_{AB}$ and $R_{A'B'}$ satisfying the constraints above, we conclude that

$$E_\kappa(\rho_{AB} \otimes \omega_{A'B'}) \leq E_\kappa(\rho_{AB}) + E_\kappa(\omega_{A'B'}) .$$  \hspace{1cm} (20)

To see the super-additivity of $E_\kappa$, i.e., the opposite inequality, let $\{V^1_{AB}, W^1_{AB}\}$ and $\{V^2_{A'B'}, W^2_{A'B'}\}$ be arbitrary operators satisfying the conditions in (12) for $\rho_{AB}$ and $\omega_{A'B'}$, respectively. Now we choose

$$R_{ABA'B'} = V^1_{AB} \otimes V^2_{A'B'} + W^1_{AB} \otimes W^2_{A'B'} ,$$  \hspace{1cm} (21)

$$S_{ABA'B'} = V^1_{AB} \otimes W^2_{A'B'} + W^1_{AB} \otimes V^2_{A'B'} .$$  \hspace{1cm} (22)

One can verify from (12) that

$$R^{T_{BB'}}_{ABA'B'}, S^{T_{BB'}}_{ABA'B'} \geq 0 ,$$  \hspace{1cm} (23)

$$R_{ABA'B'} + S_{ABA'B'} = (V^1_{AB} + W^1_{AB}) \otimes (V^2_{AB} + W^2_{AB}) \leq 1_{ABA'B'} ,$$  \hspace{1cm} (24)

which implies that $\{R_{ABA'B'}, S_{ABA'B'}\}$ is a feasible solution to (12) for $E_\kappa(\rho_{AB} \otimes \omega_{A'B'})$. Thus, we have that

$$E^{\text{dual}}_\kappa(\rho_{AB} \otimes \omega_{A'B'}) \geq \log \text{Tr}(\rho_{AB} \otimes \omega_{A'B'})(R_{ABA'B'} - S_{ABA'B'})$$

$$= \log[\text{Tr} \rho_{AB}(V^1_{AB} - W^1_{AB}) \cdot \text{Tr} \omega_{A'B'}(V^2_{A'B'} - W^2_{A'B'})]$$

$$= \log(\text{Tr} \rho_{AB}(V^1_{AB} - W^1_{AB})) + \log(\text{Tr} \omega_{A'B'}(V^2_{A'B'} - W^2_{A'B'})).$$  \hspace{1cm} (25)

Since the inequality has been shown for arbitrary $\{V^1_{AB}, W^1_{AB}\}$ and $\{V^2_{A'B'}, W^2_{A'B'}\}$ satisfying the conditions in (12) for $\rho_{AB}$ and $\omega_{A'B'}$, respectively, we conclude that

$$E^{\text{dual}}_\kappa(\rho_{AB} \otimes \omega_{A'B'}) \geq E^{\text{dual}}_\kappa(\rho_{AB}) + E^{\text{dual}}_\kappa(\omega_{A'B'}) .$$  \hspace{1cm} (26)

Applying (20), (28), and (15), we conclude (16).
C. Relation to logarithmic negativity

The following proposition establishes an inequality relating $E_\kappa$ to the logarithmic negativity [VW02, Ple05], defined as

$$E_N(\rho_{AB}) := \log \|\rho_{AB}^T\|_1.$$  \hspace{1cm} (29)

**Proposition 3**  Let $\rho_{AB}$ be a bipartite state acting on a separable Hilbert space. Then

$$E_\kappa(\rho_{AB}) \geq E_N(\rho_{AB}).$$  \hspace{1cm} (30)

If $\rho_{AB}$ satisfies the binegativity condition $|\rho_{AB}^T|^T \geq 0$, then

$$E_\kappa(\rho_{AB}) = E_N(\rho_{AB}).$$  \hspace{1cm} (31)

**Proof.** Consider from the dual formulation of $E_\kappa(\rho_{AB})$ in (12) that

$$E_\kappa^\text{dual}(\rho_{AB}) = \sup \log \text{Tr} \rho_{AB}(V_{AB} - W_{AB})$$

s.t. $V_{AB} + W_{AB} \leq 1_{AB}$, $V_{AB}^T$, $W_{AB}^T \geq 0$.  \hspace{1cm} (32)

Using the fact that the transpose map is its own adjoint, we have that

$$E_\kappa^\text{dual}(\rho_{AB}) = \sup \log \text{Tr} \rho_{AB}(V_{AB} - W_{AB})$$

s.t. $V_{AB}^T + W_{AB}^T \leq 1_{AB}$, $V_{AB}^T$, $W_{AB}^T \geq 0$.  \hspace{1cm} (33)

Then by a substitution, we can write this as

$$E_\kappa^\text{dual}(\rho_{AB}) = \sup \log \text{Tr} \rho_{AB}^T(V_{AB} - W_{AB})$$

s.t. $V_{AB}^T + W_{AB}^T \leq 1_{AB}$, $V_{AB}^T$, $W_{AB}^T \geq 0$.  \hspace{1cm} (34)

Consider a decomposition of $\rho_{AB}^T$ into its positive and negative part

$$\rho_{AB}^T = P_{AB} - N_{AB}.$$  \hspace{1cm} (35)

Let $\Pi_{AB}^P$ be the projection onto the positive part, and let $\Pi_{AB}^N$ be the projection onto the negative part. Consider that

$$|\rho_{AB}^T| = P_{AB} + N_{AB}.$$  \hspace{1cm} (36)

Then we can pick $V_{AB} = \Pi_{AB}^P \geq 0$ and $W_{AB} = \Pi_{AB}^N \geq 0$ in (34), to find that

$$\text{Tr} \rho_{AB}^T(\Pi_{AB}^P - \Pi_{AB}^N) = \text{Tr} (P_{AB} - N_{AB}) (\Pi_{AB}^P - \Pi_{AB}^N)$$

$$= \text{Tr} P_{AB} \Pi_{AB}^P + N_{AB} \Pi_{AB}^N$$

$$= \text{Tr} P_{AB} + N_{AB}$$

$$= \text{Tr} |\rho_{AB}^T|$$

$$= \|\rho_{AB}^T\|_1.$$  \hspace{1cm} (37)
Furthermore, we have for this choice that

\[ V_{AB}^{T_B} + W_{AB}^{T_B} = \left( \Pi_P^{T_B} + \Pi_N^{T_B} \right) \]

\[ = \left( \Pi_P^{T_B} + \Pi_N^{T_B} \right) \]

\[ = 1_{AB}^{T_B} \]

\[ = 1_{AB}. \]  

(42)

(43)

(44)

(45)

So this implies the inequality in (30), after combining with (13).

If \( \rho_{AB} \) satisfies the binegativity condition \( |\rho_{AB}^{T_B}| \geq 0 \), then we pick \( S_{AB} = |\rho_{AB}^{T_B}| \) in (1) and conclude that

\[ E_\kappa(\rho_{AB}) \leq E_N(\rho_{AB}). \]  

(46)

Combining with (30) gives (31) for this special case.

\[ \blacksquare \]

D. Normalization, faithfulness, no convexity, no monogamy

In this section, we prove that \( E_\kappa \) is normalized on maximally entangled states, and for finite-dimensional states, that it achieves its largest value on maximally entangled states. We also show that \( E_\kappa \) is faithful, in the sense that it is non-negative and equal to zero if and only if the state is a PPT state. Finally, we provide simple examples that demonstrate that \( E_\kappa \) is neither convex nor monogamous.

**Proposition 4 (Normalization)** Let \( \Phi_{AB}^M \) be a maximally entangled state of Schmidt rank \( M \). Then

\[ E_\kappa(\Phi_{AB}^M) = \log M. \]  

(47)

Furthermore, for any bipartite state \( \rho_{AB} \), the following bound holds

\[ E_\kappa(\rho_{AB}) \leq \log \min\{d_A, d_B\}, \]  

(48)

where \( d_A \) and \( d_B \) denote the dimensions of systems \( A \) and \( B \), respectively.

**Proof.** Consider that \( \Phi_{AB}^M \) satisfies the binegativity condition because

\[ \| (\Phi_{AB}^M)^{T_B} \|_{1} = \frac{1}{M} |F_{AB}|^{T_B} = \frac{1}{M} (1_{AB})^{T_B} = \frac{1}{M} 1_{AB} \geq 0, \]  

(49)

where \( F_{AB} \) is the unitary swap operator, such that \( F_{AB} = \Pi_S^{AB} - \Pi_A^{AB} \), with \( \Pi_S^{AB} \) and \( \Pi_A^{AB} \) the respective projectors onto the symmetric and antisymmetric subspaces. Thus, by Proposition 3, it follows that

\[ E_\kappa(\Phi_{AB}^M) = E_N(\Phi_{AB}^M) = \log \| (\Phi_{AB}^M)^{T_B} \|_1 = \log \text{Tr} \| (\Phi_{AB}^M)^{T_B} \| \]

\[ = \log \text{Tr} \frac{1}{M} |F_{AB}| = \frac{1}{M} \text{Tr} 1_{AB} = \log M, \]  

(50)

(51)

demonstrating (47).

To see (48), let us suppose without loss of generality that \( d_A \leq d_B \). Given the bipartite state \( \rho_{AB} \), Bob can first locally prepare a state \( \rho_{AB} \) and teleport the \( A \) system to Alice using a maximally entangled state \( \Phi_{AB}^{d_A} \) shared with Alice, which implies that there exists
a completely-PPT-preserving channel that converts $\Phi_{dA}$ to $\rho_{AB}$. Therefore, by the monotonicity of $E_\kappa$ with respect to completely-PPT-preserving channels (Theorem 1), we find that

$$\log d_A = E_\kappa(\Phi_{dA}) \geq E_\kappa(\rho_{AB}).$$

This concludes the proof.

**Proposition 5 (Faithfulness)** For a state $\rho_{AB}$ acting on a separable Hilbert space, we have that $E_\kappa(\rho_{AB}) \geq 0$ and $E_\kappa(\rho_{AB}) = 0$ if and only if $\rho_{AB}^{T_B} \geq 0$.

**Proof.** To see that $E_\kappa(\rho_{AB}) \geq 0$, take $V_{AB} = 1_{AB}$ and $W_{AB} = 0$ in (12), so that $E_\kappa^{\text{dual}}(\rho_{AB}) \geq 0$. Then we conclude that $E_\kappa(\rho_{AB}) \geq 0$ from the weak duality inequality in (13).

Now suppose that $\rho_{AB}^{T_B} \geq 0$. Then we can set $S_{AB} = \rho_{AB}$ in (1), so that the conditions $-S_{AB}^{T_B} \leq \rho_{AB}^{T_B} \leq S_{AB}^{T_B}$ and $S_{AB} \geq 0$ are satisfied. Then $\text{Tr} S_{AB} = 1$, so that $E_\kappa(\rho_{AB}) \leq 0$. Combining with the fact that $E_\kappa(\rho_{AB}) \geq 0$ for all states, we conclude that $E_\kappa(\rho_{AB}) = 0$ if $\rho_{AB}^{T_B} \geq 0$.

Finally, suppose that $E_\kappa(\rho_{AB}) = 0$. Then, by Proposition 3, $E_N(\rho_{AB}) = 0$, so that $\|\rho_{AB}^{T_B}\|_1 = 1$. Decomposing $\rho_{AB}^{T_B}$ into positive and negative parts as $\rho_{AB}^{T_B} = P - N$ (such that $P, N \geq 0$ and $PN = 0$), we have that $1 = \text{Tr} \rho_{AB} = \text{Tr} \rho_{AB}^{T_B} = \text{Tr} P - \text{Tr} N$. But we also have by assumption that $1 = \|\rho_{AB}^{T_B}\|_1 = \text{Tr} P + \text{Tr} N$. Subtracting these equations gives $\text{Tr} N = 0$, which implies that $N = 0$. From this, we conclude that $\rho_{AB}^{T_B} = P \geq 0$.

**Proposition 6 (No convexity)** The $\kappa$-entanglement measure is not generally convex.

**Proof.** Due to Proposition 3 and the fact that the binegativity is always positive for any two-qubit state [Ish04], the non-convexity of $E_\kappa$ boils down to finding a two-qubit example for which the logarithmic negativity is not convex. In particular, let us choose the two-qubit states $\rho_1 = \Phi_2$, $\rho_2 = \frac{1}{2}(|00\rangle\langle00| + |11\rangle\langle11|)$, and their average $\rho = \frac{1}{2}(\rho_1 + \rho_2)$. By direct calculation, we obtain

$$E_\kappa(\rho_1) = E_N(\rho_1) = 1,$$

$$E_\kappa(\rho_2) = E_N(\rho_2) = 0,$$

$$E_\kappa(\rho) = E_N(\rho) = \log \frac{3}{2}.$$

Therefore, we have

$$E_\kappa(\rho) > \frac{1}{2}(E_\kappa(\rho_1) + E_\kappa(\rho_2)),$$

which concludes the proof.

If an entanglement measure $E$ is monogamous [CKW00, Ter04, KW04], then the following inequality should be satisfied for all tripartite states $\rho_{ABC}$:

$$E(\rho_{AB}) + E(\rho_{AC}) \leq E(\rho_{A(BC)}),$$

where the entanglement in $E(\rho_{A(BC)})$ is understood to be with respect to the bipartite cut between systems $A$ and $BC$. It is known that some entanglement measures satisfy the monogamy inequality above [CKW00, KW04]. However, the $\kappa$-entanglement measure is not monogamous, as we show by example in what follows.
Proposition 7 (No monogamy) The $\kappa$-entanglement measure is not generally monogamous.

Proof. Consider a state $|\psi\rangle\langle\psi|_{ABC}$ of three qubits, where

$$|\psi\rangle_{ABC} = \frac{1}{2}(|000\rangle_{ABC} + |011\rangle_{ABC} + \sqrt{2}|110\rangle_{ABC}).$$

(58)

Due the fact that $|\psi\rangle_{ABC}$ can be written as

$$|\psi\rangle_{ABC} = \frac{1}{\sqrt{2}}(0\rangle_A \otimes |\Phi\rangle_{BC} + 1\rangle_A \otimes |10\rangle_{BC}),$$

(59)

where $|\Phi\rangle_{BC} = (|00\rangle_{BC} + |11\rangle_{BC})/\sqrt{2}$, this state is locally equivalent to $|\Phi\rangle_{AB} \otimes |0\rangle_C$ with respect to the bipartite cut $A|BC$. One then finds that $E_\kappa(\psi_{A|BC}) = E_\kappa(\Phi_{AB}) = E_N(\Phi_{AB}) = 1$. Furthermore, we have that $E_\kappa(\psi_{AB}) = E_N(\psi_{AB}) = \log \frac{3}{2}$, and $E_\kappa(\psi_{AC}) = E_N(\psi_{AC}) = \log \frac{3}{2}$, which implies that

$$E_\kappa(\psi_{AB}) + E_\kappa(\psi_{AC}) > E_\kappa(\psi_{A|BC}).$$

(60)

This concludes the proof.

III. EXACT ENTANGLEMENT COST OF QUANTUM STATES

In this section, we prove that the $\kappa$-entanglement of a bipartite state is equal to its exact entanglement cost, when completely-PPT-preserving channels are allowed for free. After doing so, we evaluate the exact entanglement cost of several key examples of interest: isotropic states, Werner states, maximally correlated states, some states supported on the $3 \times 3$ antisymmetric subspace, and bosonic Gaussian states. In particular, we conclude that the resource theory of entanglement (the exact PPT case) is irreversible by evaluating the max-Rains relative entropy of [WD16a] and $E_\kappa$ and showing that there is a gap between them.

A. $\kappa$-entanglement measure is equal to the exact PPT-entanglement cost

Let $\Omega$ represent a set of free operations, which can be either LOCC or PPT. The one-shot exact entanglement cost of a bipartite state $\rho_{AB}$, under the $\Omega$ operations, is defined as

$$E^{(1)}_{\Omega}(\rho_{AB}) = \inf_{\Lambda \in \Omega} \{ \log d : \rho_{AB} = \Lambda_{\hat{A}\hat{B} \rightarrow AB}(\Phi^d_{\hat{A}\hat{B}}) \},$$

(61)

where $\Phi^d_{\hat{A}\hat{B}} = [1/d] \sum_{i,j=1}^d |ii\rangle\langle jj|_{\hat{A}\hat{B}}$ represents the standard maximally entangled state of Schmidt rank $d$. The exact entanglement cost of a bipartite state $\rho_{AB}$, under the $\Omega$ operations, is defined as

$$E_{\Omega}(\rho_{AB}) = \liminf_{n \to \infty} \frac{1}{n} E^{(1)}_{\Omega}(\rho_{AB}^\otimes n).$$

(62)

The exact entanglement cost under LOCC operations was previously considered in [Nie99, TH00, Hay06, YC18], while the exact entanglement cost under PPT operations was considered in [APE03, MW08].
In [APE03], the following bounds were given for $E_{\text{PPT}}$:

$$E_N(\rho_{AB}) \leq E_{\text{PPT}}(\rho_{AB}) \leq \log Z(\rho_{AB}),$$

the lower bound being the logarithmic negativity recalled in (29), and the upper bound defined as

$$Z(\rho_{AB}) := \text{Tr} |\rho^T_B| + \text{dim}(\rho_{AB}) \max\{0, -\lambda_{\min}(|\rho^T_B| T_B)\}. \tag{64}$$

Due to the presence of the dimension factor dim($\rho_{AB}$), the upper bound in (63) clearly only applies in the case that $\rho_{AB}$ is finite-dimensional.

In what follows, we first recast $E^{(1)}_{\text{PPT}}(\rho_{AB})$ as an optimization problem, by building on previous developments in [APE03, MW08]. After that, we bound $E^{(1)}_{\text{PPT}}(\rho_{AB})$ in terms of $E_{\kappa}$, by observing that $E_{\kappa}$ is a relaxation of the optimization problem for $E^{(1)}_{\text{PPT}}(\rho_{AB})$. We then finally prove that $E_{\text{PPT}}(\rho_{AB})$ is equal to $E_{\kappa}$.

**Theorem 8** Let $\rho_{AB}$ be a bipartite state acting on a separable Hilbert space. Then the one-shot exact PPT-entanglement cost $E^{(1)}_{\text{PPT}}(\rho_{AB})$ is given by the following optimization:

$$E^{(1)}_{\text{PPT}}(\rho_{AB}) = \inf \left\{ \log_2 m : - (m - 1) G^T_B \leq \rho^T_B \leq (m + 1) G^T_B, \ G_{AB} \geq 0, \ \text{Tr} G_{AB} = 1 \right\}. \tag{65}$$

**Proof.** The achievability part features a construction of a completely-PPT-preserving channel $P_{\hat{A}\hat{B} \rightarrow AB}$ such that $P_{\hat{A}\hat{B} \rightarrow AB}(\Phi^m_{\hat{A}\hat{B}}) = \rho_{AB}$; and then the converse part demonstrates that the constructed channel is essentially the only form that is needed to consider for the one-shot exact PPT-entanglement cost task. The achievability part directly employs some insights of [APE03], while the converse part directly employs insights of [MW08]. In what follows, we give a proof for the sake of completeness.

Let $m \geq 1$ be a positive integer and $G_{AB}$ a density operator such that the following inequalities hold

$$- (m - 1) G^T_B \leq \rho^T_B \leq (m + 1) G^T_B.$$ 

Then we take the completely-PPT-preserving channel $P_{\hat{A}\hat{B} \rightarrow AB}$ to be as follows:

$$P_{\hat{A}\hat{B} \rightarrow AB}(X_{\hat{A}\hat{B}}) = \rho_{AB} \text{Tr}[\Phi^m_{\hat{A}\hat{B}} X_{\hat{A}\hat{B}}] + G_{AB} \text{Tr}[(1_{\hat{A}\hat{B}} - \Phi^m_{\hat{A}\hat{B}}) X_{\hat{A}\hat{B}}]. \tag{67}$$

The action of $P_{\hat{A}\hat{B} \rightarrow AB}$ can be understood as a measure-prepare channel (and is thus a channel): first perform the measurement $\{\Phi^m_{\hat{A}\hat{B}} : 1_{\hat{A}\hat{B}} - \Phi^m_{\hat{A}\hat{B}}\}$, and if the outcome $\Phi^m_{\hat{A}\hat{B}}$ occurs, prepare the state $\rho_{AB}$, and otherwise, prepare the state $G_{AB}$. To see that the channel $P_{\hat{A}\hat{B} \rightarrow AB}$ is a completely-PPT-preserving channel, we now verify that the map $T_B \circ P_{\hat{A}\hat{B} \rightarrow AB} \circ T_B$ is completely positive. Let $Y_{R_A \hat{A}\hat{B} R_B}$ be a positive semi-definite operator with $R_A$ isomorphic
to $\hat{A}$ and $R_B$ isomorphic to $\hat{B}$. Then consider that

$$
(T_B \circ \mathcal{P}_{\hat{A}\hat{B} \to AB} \circ T_B)(Y_{R_A\hat{A}B_B}) = \rho_{AB}^{T_B} \text{Tr}_{\hat{AB}}[\Phi_{\hat{A}\hat{B}}^m Y_{R_A\hat{A}B_B}^{T_B}] + G_{AB}^{T_B} \text{Tr}_{\hat{AB}}[(\mathbb{1}_{\hat{A}B} - \Phi_{\hat{A}\hat{B}}^m)Y_{R_A\hat{A}B_B}^{T_B}],
$$

(68)

$$
= \rho_{AB}^{T_B} \text{Tr}_{\hat{AB}}[\Phi_{\hat{A}\hat{B}}^m Y_{R_A\hat{A}B_B}^{T_B}] + G_{AB}^{T_B} \text{Tr}_{\hat{AB}}[(\mathbb{1}_{\hat{A}B} - \Phi_{\hat{A}\hat{B}}^m)Y_{R_A\hat{A}B_B}^{T_B}],
$$

(69)

$$
= \frac{\rho_{AB}^{T_B}}{m} \text{Tr}_{\hat{AB}}[F_{\hat{A}B}Y_{R_A\hat{A}B_B}] + G_{AB}^{T_B} \text{Tr}_{\hat{AB}}[(\mathbb{1}_{\hat{A}B} - F_{\hat{A}B}/m)Y_{R_A\hat{A}B_B}],
$$

(70)

$$
= \frac{\rho_{AB}^{T_B}}{m} \text{Tr}_{\hat{AB}}[F_{\hat{A}B}Y_{R_A\hat{A}B_B}] + \frac{G_{AB}^{T_B}}{m} \text{Tr}_{\hat{AB}}[(m\mathbb{1}_{\hat{A}B} - F_{\hat{A}B})Y_{R_A\hat{A}B_B}],
$$

(71)

$$
= \frac{\rho_{AB}^{T_B}}{m} \text{Tr}_{\hat{AB}}[(\Pi_{AB}^S - \Pi_{AB}^A)Y_{R_A\hat{A}B_B}],
$$

(72)

$$
+ \frac{G_{AB}^{T_B}}{m} \text{Tr}_{\hat{AB}}[(m(\Pi_{AB}^S + \Pi_{AB}^A) - (\Pi_{AB}^S - \Pi_{AB}^A))Y_{R_A\hat{A}B_B}],
$$

(73)

The third equality follows because the partial transpose of $\Phi_{\hat{A}\hat{B}}^m$ is equal to the unitary flip or swap operator $F_{\hat{A}B}$. The fifth equality follows by recalling the definition of the projections onto the symmetric and antisymmetric subspaces respectively as

$$
\Pi_{AB}^S = \frac{\mathbb{1}_{\hat{A}B} + F_{\hat{A}B}}{2}, \quad \Pi_{AB}^A = \frac{\mathbb{1}_{\hat{A}B} - F_{\hat{A}B}}{2}.
$$

(74)

As a consequence of the condition in (66), it follows that $T_B \circ \mathcal{P}_{\hat{A}\hat{B} \to AB} \circ T_B$ is completely positive, so that $\mathcal{P}_{\hat{A}\hat{B} \to AB}$ is a completely-PPT-preserving channel as claimed. In fact, we can see that $T_B \circ \mathcal{P}_{\hat{A}\hat{B} \to AB} \circ T_B$ is a measure-prepare channel: first perform the measurement $\{\Pi_{AB}^S, \Pi_{AB}^A\}$ and if the outcome $\Pi_{AB}^S$ occurs, prepare the state $\frac{1}{m}[\rho_{AB}^{T_B} + (m - 1)G_{AB}^{T_B}]$, and otherwise, prepare the state $\frac{1}{m}[(m + 1)G_{AB}^{T_B} - \rho_{AB}^{T_B}]$. Thus, it follows that $\mathcal{P}_{\hat{A}\hat{B} \to AB}$ accomplishes the one-shot exact PPT-entanglement cost task, in the sense that

$$
\mathcal{P}_{\hat{A}\hat{B} \to AB}(\Phi_{\hat{A}\hat{B}}^m) = \rho_{AB}.
$$

(75)

By taking an infimum over all $m$ and density operators $G_{AB}$ such that (66) holds, it follows that the quantity on the right-hand side of (65) is greater than or equal to $E_{\text{PPT}}^{(1)}(\rho_{AB})$

Now we prove the opposite inequality. Let $\mathcal{P}_{\hat{A}\hat{B} \to AB}$ denote an arbitrary completely-PPT-preserving channel such that

$$
\mathcal{P}_{\hat{A}\hat{B} \to AB}(\Phi_{\hat{A}\hat{B}}^m) = \rho_{AB}.
$$

(76)

Let $T_{\hat{A}B}$ denote the following isotropic twirling channel [Wer89, HH99, Wat18]:

$$
T_{\hat{A}B}(X_{\hat{A}B}) = \int dU (U_{\hat{A}} \otimes U_{\hat{B}})X_{\hat{A}B}(U_{\hat{A}} \otimes U_{\hat{B}})^\dagger
$$

(77)

$$
= \Phi_{\hat{A}\hat{B}}^m \text{Tr}[\Phi_{\hat{A}\hat{B}}^m X_{\hat{A}B}] + \frac{1}{m^2 - 1} \text{Tr}[(\mathbb{1}_{\hat{A}B} - \Phi_{\hat{A}\hat{B}}^m)X_{\hat{A}B}].
$$

(78)
The channel $\mathcal{T}_{AB}$ is an LOCC channel, and thus is completely-PPT-preserving. Furthermore, due to the fact that $\mathcal{T}_{AB}(\Phi^m_{AB}) = \Phi^m_{AB}$, it follows that
\[
(\mathcal{P}_{AB \to AB} \circ \mathcal{T}_{AB})(\Phi^m_{AB}) = \rho_{AB}.
\] (79)

Thus, for any completely-PPT-preserving channel $\mathcal{P}_{AB \to AB}'$ such that (76) holds, there exists another channel $\mathcal{P}_{AB \to AB}' := \mathcal{P}_{AB \to AB} \circ \mathcal{T}_{AB}$ achieving the same performance, and so it suffices to focus on the channel $\mathcal{P}_{AB \to AB}'$ in order to establish an expression for the one-shot exact PPT-entanglement cost. Then, consider that, for any input state $\tau_{AB}$, we have that
\[
\mathcal{P}_{AB \to AB}'(\tau_{AB}) = \mathcal{P}_{AB \to AB}
\left(\Phi^m_{AB} \text{ Tr}[\Phi^m_{AB} \tau_{AB}] + \frac{1}{m^2 - 1} \text{ Tr}[(1 - \Phi^m_{AB})\tau_{AB}]\right)
\] (80)
\[
= \mathcal{P}_{AB \to AB}(\Phi^m_{AB} \text{ Tr}[\Phi^m_{AB} \tau_{AB}] + \mathcal{P}_{AB \to AB}
\left(\frac{1}{m^2 - 1} \text{ Tr}[(1 - \Phi^m_{AB})\tau_{AB}]\right)
\] (81)
\[
= \rho_{AB} \text{ Tr}[\Phi^m_{AB} \tau_{AB}] + G_{AB} \text{ Tr}[(1 - \Phi^m_{AB})\tau_{AB}],
\] (82)
where we have set
\[
G_{AB} = \mathcal{P}_{AB \to AB}
\left(\frac{1}{m^2 - 1} \text{ Tr}[(1 - \Phi^m_{AB})\tau_{AB}]\right).
\] (83)

In order for $\mathcal{P}_{AB \to AB}'$ to be completely-PPT-preserving, it is necessary that $T_B \circ \mathcal{P}_{AB \to AB}' \circ T_B$ is completely positive. Going through the same calculations as above, we see that it is necessary for the following operator to be positive semi-definite for an arbitrary positive semi-definite $Y_{RA\hat{A}BR_B}$:
\[
\frac{1}{m} \left[[\rho^T_{AB} + (m - 1) G^T_{AB}] \text{ Tr}_{AB} \Pi^S_{AB} Y_{RA\hat{A}BR_B} \right] + [(m + 1) G^T_{AB} - \rho^T_{AB}] \text{ Tr}_{AB} \Pi^A_{AB} Y_{RA\hat{A}BR_B}].
\] (84)

However, since $\Pi^S_{AB}$ and $\Pi^A_{AB}$ project onto orthogonal subspaces, this is possible only if the condition in (66) holds for $G_{AB}$ given in (83). Thus, it follows that the quantity on the right-hand side of (65) is less than or equal to $E^{(1)}_{\text{PPT}}(\rho_{AB})$.

![Image](image.png)

**Proposition 9** Let $\rho_{AB}$ be a bipartite state acting on a separable Hilbert space. Then
\[
\log(2^{E_{\text{PPT}}(\rho_{AB})} - 1) \leq E^{(1)}_{\text{PPT}}(\rho_{AB}) \leq \log(2^{E_{\text{PPT}}(\rho_{AB})} + 1).
\] (85)

**Proof.** The proof of this lemma utilizes basic properties of optimization theory. Let us first prove the first inequality. The key idea is to relax the bilinear optimization problem to a semidefinite optimization problem. By definition,
\[
E^{(1)}_{\text{PPT}}(\rho_{AB}) = \inf \{ \log m : -(m - 1) G^T_{AB} \leq \rho^T_{AB} \leq (m + 1) G^T_{AB}, G_{AB} \geq 0, \text{ Tr } G_{AB} = 1 \}.
\] (86)
\[
\geq \inf \{ \log m : -(m + 1) G^T_{AB} \leq \rho^T_{AB} \leq (m + 1) G^T_{AB}, G_{AB} \geq 0, \text{ Tr } G_{AB} = 1 \}
\] (87)
\[
= \inf \{ \log \text{ Tr } S_{AB} - 1 : -S^T_{AB} \leq \rho^T_{AB} \leq S^T_{AB}, S_{AB} \geq 0 \}
\] (88)
\[
= \log(2^{E_{\text{PPT}}(\rho_{AB})} - 1).
\] (89)
The first inequality follows by relaxing the constraint \(-(m - 1)G_{AB}^{T_B} \leq \rho_{AB}^{T_B}\) to \(-(m + 1)G_{AB}^{T_B} \leq \rho_{AB}^{T_B}\). The second-to-last equality follows by absorbing \(m\) into \(G_{AB}\) and setting \(S_{AB} = (m + 1)G_{AB}\). The last equality follows from the definition of \(E_{\kappa}(\rho_{AB})\).

By the same method, it is easy to prove that \(E^{(1)}_{\text{PPT}}(\rho_{AB}) \leq \log(2^{E_{\kappa}(\rho_{AB})} + 1)\).

**Theorem 10** Let \(\rho_{AB}\) be a bipartite state acting on a separable Hilbert space. Then the exact PPT-entanglement cost of \(\rho_{AB}\) is given by

\[
E_{\text{PPT}}(\rho_{AB}) = E_{\kappa}(\rho_{AB}).
\]

**Proof.** The main idea behind the proof is to employ the one-shot bound in Proposition 9 and then the additivity relation from Proposition 2. Consider that

\[
E_{\text{PPT}}(\rho_{AB}) = \liminf_{n \to \infty} \frac{1}{n} E_{\text{PPT}}^{(1)}(\rho_{AB}^\otimes n)
\]

\[
\leq \liminf_{n \to \infty} \frac{1}{n} \log(2^{E_{\kappa}(\rho_{AB}^\otimes n)} + 1)
\]

\[
= \liminf_{n \to \infty} \frac{1}{n} \log(2^n E_{\kappa}(\rho_{AB}) + 1)
\]

\[
= E_{\kappa}(\rho_{AB}).
\]

By a similar method, it is easy to show that \(E_{\text{PPT}}(\rho_{AB}) \geq E_{\kappa}(\rho_{AB})\).

As emphasized in the abstract of our paper, Theorem 10 constitutes a significant development for entanglement theory, representing the first time that it has been shown that an entanglement measure is not only efficiently computable but also possesses a direct operational meaning. In the work of [BP08, BP10], it was established that the regularized relative entropy of entanglement is equal to the entanglement cost and distillable entanglement of a bipartite quantum state, with the set of free operations being asymptotically non-entangling maps. However, in spite of the fact that the work of [BP08, BP10] gave a direct operational meaning to the regularized relative entropy of entanglement, this entanglement measure arguably has limited application beyond being a formal expression, due to the fact that there is no known efficient procedure for computing it.

Furthermore, in prior work, most discussions about the structure and properties of entanglement are based on entanglement measures. However, none of these measures, with the exception of the regularized relative entropy of entanglement, possesses a direct operational meaning. Thus, the connection made by Theorem 10 allows for the study of the structure of entanglement via an entanglement measure possessing a direct operational meaning. Given that \(E_{\kappa} = E_{\text{PPT}}\) is neither convex nor monogamous, this raises questions of whether these properties should really be required or necessary for measures of entanglement, in contrast to the discussions put forward in [Ter04, HHHH09] based on intuition. Furthermore, \(E_{\kappa}\) is additive (Proposition 2), so that Theorem 10 implies that \(E_{\text{PPT}}\) is additive as well:

\[
E_{\text{PPT}}(\rho_{AB} \otimes \omega_{A'B'}) = E_{\text{PPT}}(\rho_{AB}) + E_{\text{PPT}}(\omega_{A'B'}).
\]

Thus, \(E_{\text{PPT}}\) is the only known example of an operational quantity in entanglement theory for which the optimal rate is additive as a function of general quantum states.

Finally, we conclude this section with a few remarks:
• First, we have found an explicit example, given in Section III B, which demonstrates that both the upper and lower bounds on exact PPT-entanglement cost from [APE03] are not tight.

• Our result in Theorem 10 may shed light on the open question of whether distillable entanglement is convex [SST01]. In the multi-partite setting, it is known that a version of distillable entanglement is not convex [SST03].

• Finally, note that any quantity that is 1) monotone with respect to completely-PPT-preserving channels and 2) normalized on maximally entangled states is a lower bound on $E_{\text{PPT}}$, following from the development in [APE03]. Thus, since $E_{\text{PPT}} = E_\kappa$ as stated in Theorem 10, this means that $E_\kappa$ is the largest of all such entanglement measures. Examples of such entanglement measures are the logarithmic negativity [VW02, Ple05], the max-Rains relative entropy [WD16a, WFD17], and the Rains relative entropy [Rai99, Rai01].

B. Exact entanglement cost of particular bipartite states

In this subsection, we evaluate the exact entanglement cost for particular bipartite states, including isotropic states [HH99], Werner states [Wer89], maximally correlated states [Rai99, Rai01], some states supported on the $3 \times 3$ antisymmetric subspace, and bosonic Gaussian states [Ser17]. For the isotropic and Werner states, the exact PPT-entanglement cost was already determined [APE03, Hay06], and so we recall these developments here.

Let $A$ and $B$ be quantum systems, each of dimension $d$. For $t \in [0, 1]$ and $d \geq 2$, an isotropic state is defined as follows [HH99]:

$$\rho_{AB}^{(t,d)} := t \Phi_{AB}^d + (1 - t) \frac{1_{AB} - \Phi_{AB}^d}{d^2 - 1}.$$  \hspace{1cm} (96)

An isotropic state is PPT if and only if $t \leq 1/d$. It was shown in [Hay06, Exercise 8.73] that $\rho_{AB}^{(t,d)}$ satisfies the binegativity condition: $|\left(\rho_{AB}^{(t,d)}\right)^T_B|^T_B \geq 0$. By applying (63), this implies that

$$E_{\text{PPT}}(\rho_{AB}^{(t,d)}) = E_N(\rho_{AB}^{(t,d)}) = \begin{cases} \log dt & \text{if } t > \frac{1}{d} \\ 0 & \text{if } t \leq \frac{1}{d} \end{cases},$$  \hspace{1cm} (97)

with the second equality shown in [Hor01, Hay06].

Let $A$ and $B$ be quantum systems, each of dimension $d$. A Werner state is defined for $p \in [0, 1]$ as [Wer89]

$$W_{AB}^{(p,d)} := (1 - p) \frac{2}{d(d + 1)} \Pi_{AB}^S + p \frac{2}{d(d - 1)} \Pi_{AB}^A,$$  \hspace{1cm} (98)

where $\Pi_{AB}^S := (1_{AB} + F_{AB})/2$ and $\Pi_{AB}^A := (1_{AB} - F_{AB})/2$ are the projections onto the symmetric and antisymmetric subspaces of $A$ and $B$, respectively, with $F_{AB}$ denoting the swap operator. A Werner state is PPT if and only if $p \leq 1/2$. It was shown in [APE03] that $W_{AB}^{(p,d)}$ satisfies the binegativity condition: $|(W_{AB}^{(p,d)})^T_B|^T_B \geq 0$. By applying (63), this implies that [APE03]

$$E_{\text{PPT}}(W_{AB}^{(p,d)}) = E_N(W_{AB}^{(p,d)}) = \begin{cases} \log \left[\frac{2}{d}(2p - 1) + 1\right] & \text{if } p > 1/2 \\ 0 & \text{if } p \leq 1/2 \end{cases},$$  \hspace{1cm} (99)
with the second equality shown in [Hor01, Hay06].

A maximally correlated state is defined as [Rai99, Rai01]

$$\rho_{AB}^c := \sum_{i,j=0}^{d-1} c_{ij} |ii\rangle\langle jj|, \quad (100)$$

with the complex coefficients $c := \{c_{ij}\}_{i,j}$ being chosen such that $\sum_{i,j=0}^{d-1} c_{ij} |ij\rangle\langle ji|$ is a legitimate quantum state. Noting that $(\rho_{AB}^c)^T_B = \sum_{i,j=0}^{d-1} c_{ij} |ij\rangle\langle ji|$, a direct calculation reveals that

$$|(\rho_{AB}^c)^T_B| = \sum_{i,j=0}^{d-1} |c_{ij}| |ii\rangle\langle jj|. \quad (101)$$

Considering that $|(\rho_{AB}^c)^T_B| = |(\rho_{AB}^c)^T_B| \geq 0$, we have that

$$E_{PPT}(\rho_{AB}^c) = E_N(\rho_{AB}^c) = \log \left( \sum_{i,j} |c_{ij}| \right). \quad (102)$$

The maximally correlated state $\hat{\omega}_\alpha$ was considered recently in [YC18]:

$$\hat{\omega}_{AB}^\alpha := \alpha \Phi^2_{AB} + \frac{1-\alpha}{2} |11\rangle\langle 11|_{AB} \quad (103)$$

$$= \frac{\alpha}{2} |00\rangle\langle 11|_{AB} + \frac{\alpha}{2} |11\rangle\langle 00|_{AB} + \frac{1}{2} |00\rangle\langle 00|_{AB} + \frac{1}{2} |11\rangle\langle 11|_{AB}, \quad (104)$$

where $\alpha \in [0,1]$. The authors of [YC18] showed that the exact entanglement cost under LOCC is bounded as

$$\left[ \frac{1}{\log(\alpha + 1)} \right]^{-1} \geq E_{LOCC}(\hat{\omega}_{AB}^\alpha) \geq \log(\alpha + 1), \quad (105)$$

for $0 < \alpha < \sqrt{2} - 1$. However, under PPT-preserving operations, by (102), it holds that

$$E_{PPT}(\hat{\omega}_{AB}^\alpha) = \log(\alpha + 1). \quad (106)$$

for $\alpha \in [0,1]$. This demonstrates that the lower bound in (105) can be understood as arising from the fact that the inequality $E_{LOCC} \geq E_{PPT}$ generally holds for an arbitrary bipartite state.

The next example indicates the irreversibility of exact PPT entanglement manipulation, and it also implies that $E_{PPT}$ is generally not equal to the logarithmic negativity $E_N$. Consider the following rank-two state supported on the $3 \times 3$ antisymmetric subspace [WD17]:

$$\rho_v = \frac{1}{2} (|v_1\rangle\langle v_1| + |v_2\rangle\langle v_2|) \quad (107)$$

with $|v_1\rangle = 1/\sqrt{2}(|01\rangle - |10\rangle), |v_2\rangle = 1/\sqrt{2}(|02\rangle - |20\rangle)$. For the state $\rho_v$, it holds that

$$R_{\max}(\rho_v) = E_N(\rho_v) = \log \left( 1 + \frac{1}{\sqrt{2}} \right) < E_{PPT}(\rho_v) = 1 < \log Z(\rho) = \log \left( 1 + \frac{13}{4\sqrt{2}} \right), \quad (108)$$
where $R_{\text{max}}(\rho_v)$ denotes the max-Rains relative entropy [WD16a]. The strict inequalities in (108) also imply that both the lower and upper bounds from (63), i.e., from [APE03], are generally not tight.

The last examples that we consider are bosonic Gaussian states [Ser17]. As shown in [APE03], all bosonic Gaussian states $\rho_{AB}^G$ satisfy the binegativity condition $|(\rho_{AB}^G)^T_B|^T_B \geq 0$. Thus, as a consequence of Theorem 10 and Proposition 3, we conclude that

$$E_{\text{PPT}}(\rho_{AB}^G) = E_N(\rho_{AB}^G) \quad (109)$$

for all bosonic Gaussian states $\rho_{AB}^G$. Note that an explicit expression for the logarithmic negativity of a bosonic Gaussian state is available in [WEP03, Eq. (15)]. We stress again that it is not clear whether the equality in (109) follows from the upper bound in (63), given that the dimension of a bosonic Gaussian state is generally equal to infinity.

### IV. $\kappa$-ENTANGLEMENT MEASURE FOR QUANTUM CHANNELS

In this section, we extend the $\kappa$-entanglement measure from bipartite states to point-to-point quantum channels. We establish several properties of the $\kappa$-entanglement of quantum channels, including the fact that it does not increase under amortization, that it is monotone under the action of a PPT superchannel, that it is additive, normalized, faithful, and that it is generally not convex. The fact that it is monotone under the action of a PPT superchannel is a basic property that we would expect to hold for a good measure of the entanglement of a quantum channel.

In what follows, we consider a channel $\mathcal{N}_{A \rightarrow B}$ that takes density operators acting on a separable Hilbert space $\mathcal{H}_A$ to those acting on a separable Hilbert space $\mathcal{H}_B$. We refer to such channels simply as quantum channels, regardless of whether $\mathcal{H}_A$ or $\mathcal{H}_B$ is finite-dimensional. If the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ are both finite-dimensional, then we specifically refer to $\mathcal{N}_{A \rightarrow B}$ as a finite-dimensional channel.

We also make use of the Choi operator $J_{RB}^N$ [Hol11b, Hol11a] of the channel $\mathcal{N}_{A \rightarrow B}$, defined as

$$J_{RB}^N := N_{A \rightarrow B}(\Gamma_{RA}) := \sum_{i,j} |i\rangle_R \otimes \mathcal{N}_{A \rightarrow B}(|j\rangle_A), \quad (110)$$

where $R$ is isomorphic to the channel input $A$, $\Gamma_{RA} = |\Gamma\rangle\langle\Gamma|_{RA}$, and $|\Gamma\rangle_{RA}$ denotes the unnormalized maximally entangled vector:

$$|\Gamma\rangle_{RA} := \sum_i |i\rangle_R \otimes |i\rangle_A, \quad (111)$$

where $\{|i\rangle_R\}_i$ and $\{|i\rangle_A\}_i$ are orthonormal bases for the Hilbert spaces $\mathcal{H}_R$ and $\mathcal{H}_A$.

**Definition 2 ($\kappa$-entanglement of a channel)** Let $\mathcal{N}_{A \rightarrow B}$ be a quantum channel. Then the $\kappa$-entanglement of the channel $\mathcal{N}_{A \rightarrow B}$ is defined as

$$E_\kappa(\mathcal{N}_{A \rightarrow B}) := \inf \{ \log \| \text{Tr}_B Q_{AB} \|_\infty : -Q_{AB}^T_B \leq (J_{AB}^N)^T_B \leq Q_{AB}^T_B, Q_{AB} \geq 0 \}. \quad (112)$$

**Proposition 11** Let $\mathcal{N}_{A \rightarrow B}$ be a quantum channel. Then

$$E_\kappa(\mathcal{N}_{A \rightarrow B}) = \sup_{\rho_{RA}} E_\kappa(\mathcal{N}_{A \rightarrow B}(\rho_{RA})), \quad (113)$$

where the supremum is with respect to all states $\rho_{RA}$ with system $R$ arbitrary.
Proof. Due to Proposition 1, i.e., the fact that \(E_\kappa\) for states is monotone non-increasing with respect to completely-PPT-preserving channels (with one such channel being a local partial trace), it follows from purification, the Schmidt decomposition, and this local data processing, that it suffices to optimize with respect to pure states \(\rho_{RA}\) with system \(R\) isomorphic to system \(A\). Thus, we conclude that

\[
\sup_{\rho_{RA}} E_\kappa(\mathcal{N}_{A\to B}(\rho_{RA})) = \sup_{\phi_{RA}} E_\kappa(\mathcal{N}_{A\to B}(\phi_{RA})),
\]

where \(\phi_{RA}\) is pure and \(R \simeq A\).

By definition, and using the fact that every pure state \(\phi_{RA}\) of the form mentioned above can be represented as \(X_R\Gamma_{RA}X_R^\dagger\) with \(\|X_R\|_2 = 1\), we have that

\[
\sup_{\phi_{RA}} E_\kappa(\mathcal{N}_{A\to B}(\phi_{RA})) = \log \sup_{X_R: \|X_R\|_2 = 1, |X_R| > 0} \inf S_{RB} \left\{ \text{Tr} S_{RB} : -S_{RB}^{T_B} \leq X_R [J_{RB}^N]^{T_B} X_R^\dagger \leq S_{RB}^{T_B}, S_{RB} \geq 0 \right\},
\]

where the equality follows because the set of operators \(X_R\) satisfying \(\|X_R\|_2 = 1\) and \(|X_R| > 0\) is dense in the set of all operators satisfying \(\|X_R\|_2 = 1\). Now defining \(Q_{RB}\) in terms of \(S_{RB} = X_R Q_{RB} X_R^\dagger\), and using the facts that

\[
-S_{RB}^{T_B} \leq X_R [J_{RB}^N]^{T_B} X_R^\dagger \leq S_{RB}^{T_B} \iff -Q_{RB}^{T_B} \leq [J_{RB}^N]^{T_B} \leq Q_{RB}^{T_B},
\]

for operators \(X_R\) satisfying \(|X_R| > 0\), we find that

\[
\sup_{X_R: \|X_R\|_2 = 1, |X_R| > 0} \inf S_{RB} \left\{ \text{Tr} S_{RB} : -S_{RB}^{T_B} \leq X_R [J_{RB}^N]^{T_B} X_R^\dagger \leq S_{RB}^{T_B}, S_{RB} \geq 0 \right\}
\]

\[
= \sup_{X_R: \|X_R\|_2 = 1, |X_R| > 0} \inf Q_{RB} \left\{ \text{Tr} X_R Q_{RB} X_R^\dagger : -Q_{RB}^{T_B} \leq [J_{RB}^N]^{T_B} \leq Q_{RB}^{T_B}, Q_{RB} \geq 0 \right\}
\]

\[
= \sup_{\rho_R: |\rho_R| = 1, \rho_R > 0} \inf Q_{RB} \left\{ \text{Tr} [\rho_R^{T_B} Q_{RB}^{T_B}] : -Q_{RB}^{T_B} \leq [J_{RB}^N]^{T_B} \leq Q_{RB}^{T_B}, Q_{RB} \geq 0 \right\}
\]

\[
= \inf Q_{RB} \|\text{Tr} Q_{RB}\|_\infty : -Q_{RB}^{T_B} \leq [J_{RB}^N]^{T_B} \leq Q_{RB}^{T_B}, Q_{RB} \geq 0 \}
\]

The fourth equality follows from an application of the Sion minimax theorem \([Sio58]\), given that the set of operators satisfying \(\text{Tr} \rho_R = 1, \rho_R > 0\) is compact and both sets over which we are optimizing are convex. Putting everything together, we conclude (113). ■

A. Amortization collapse and monotonicity under PPT superchannels

In this subsection, we prove that the \(\kappa\)-entanglement of a quantum channel does not increase under amortization, which is a property that holds for the squashed entanglement of a channel \([TGW14a, TGW14b]\), a channel’s max-relative entropy of entanglement \([CMH17]\), and the max-Rains information of a channel \([BW18]\). We additionally prove that this property implies that the \(\kappa\)-entanglement of a quantum channel does not increase under the
Proposition 12 (Amortization inequality) Let $\rho_{A'AB'}$ be a quantum state acting on a separable Hilbert space and let $N_{A\rightarrow B}$ be a quantum channel. Then the following amortization inequality holds

$$E_\kappa(N_{A\rightarrow B}(\rho_{A'AB'})) - E_\kappa(\rho_{A'AB'}) \leq E_\kappa(N_{A\rightarrow B}).$$  \hfill (120)

Proof. A proof for this inequality follows similarly to the proof of [BW18, Proposition 1]. We first rewrite the desired inequality as

$$E_\kappa(N_{A\rightarrow B}(\rho_{A'AB'})) \leq E_\kappa(N_{A\rightarrow B}) + E_\kappa(\rho_{A'AB'}),$$  \hfill (121)

and then once again as

$$2E_\kappa(N_{A\rightarrow B}(\rho_{A'AB'})) \leq 2E_\kappa(N_{A\rightarrow B}) + 2E_\kappa(\rho_{A'AB'}).$$  \hfill (122)

Consider that

$$2E_\kappa(\rho_{A'AB'}) = \inf \left\{ \text{Tr} S_{A'AB'} : -S_{A'AB'}^T \leq T_{B'} \leq S_{A'AB'}^T, S_{A'AB'} \geq 0 \right\},$$  \hfill (123)

and let $Q_{RB}$ be an arbitrary operator satisfying

$$-Q_{RB}^T \leq J_{RB}^N \leq Q_{RB}^T, \quad Q_{RB} \geq 0.$$  \hfill (126)

Then let

$$F_{A'BB'} = \langle \Gamma |_{RA} (S_{A'AB'} \otimes Q_{RB}) |\Gamma \rangle |_{RA},$$  \hfill (127)

where $|\Gamma \rangle |_{RA}$ denotes the unnormalized maximally entangled vector. It follows that $F_{A'BB'} \geq 0$ because $S_{A'AB'} \geq 0$ and $Q_{RB} \geq 0$. Furthermore, we have from (125) and (126) that

$$F_{A'BB'}^{T_{B'}} = [\langle \Gamma |_{RA} (S_{A'AB'} \otimes Q_{RB}) |\Gamma \rangle |_{RA}]^{T_{B'}}$$  \hfill (128)

$$= \langle \Gamma |_{RA} (S_{A'AB'} \otimes Q_{RB}) |\Gamma \rangle |_{RA}$$  \hfill (129)

$$\geq \langle \Gamma |_{RA} (T_{B'} \otimes [J_{RB}^N]^T) |\Gamma \rangle |_{RA}$$  \hfill (130)

$$= [\langle \Gamma |_{RA} (S_{A'AB'} \otimes J_{RB}^N) |\Gamma \rangle |_{RA}]^{T_{B'}}$$  \hfill (131)

$$= [N_{A\rightarrow B}(\rho_{A'AB'})]^{T_{B'}}.$$  \hfill (132)
Similarly, we have that
\[ -F_{A'B'B'}^{T} \leq [N_{A\rightarrow B}(\rho_{A'AB'})]^{T_{BB'}}, \]  
by using \(-S_{A'AB'}^{T_{BB'}} \leq \rho_{A'AB'}^{T_{B'B'}}\) and \(-Q_{RB}^{T} \leq [J_{RB}]^{T_{B}}\). Thus, \(F_{A'B'B'}\) is feasible for 
\[ 2E_{\kappa}(N_{A \rightarrow B}(\rho_{A'AB'})). \]

Finally, consider that
\[ 2E_{\kappa}(N_{A \rightarrow B}(\rho_{A'AB'})) \leq \text{Tr} F_{A'B'B'}^{T} \]  
\[ = \text{Tr}(\Gamma|_{RA}(S_{A'AB'}^{T_{BB'}} \otimes Q_{RB})|_{RA}) \]  
\[ = \text{Tr} S_{A'AB'} Q_{AB}^{T_{A}} \]  
\[ = \text{Tr} [S_{A'AB'}^{T_{B'B'}} Q_{AB}^{T_{A}}] \]  
\[ \leq \text{Tr} S_{A'AB'}^{T_{B'B'}} \| \text{Tr} B Q_{AB}^{T_{A}} \|_{\infty} \]  
\[ = \text{Tr} S_{A'AB'}^{T_{B'B'}} \| \text{Tr} B Q_{AB}^{T_{A}} \|_{\infty}. \]  

The inequality above follows from Hölder’s inequality. The last equality follows because the spectrum of an operator remains invariant under the action of a transpose. Since the inequality above holds for all \(S_{A'AB'}\) and \(Q_{RB}\) satisfying (125) and (126), respectively, we conclude the inequality in (122).

\[ \begin{align*}
\text{Definition 3 (Amortized } \kappa\text{-entanglement of a channel) Following [KW18], we define the amortized } \kappa\text{-entanglement of a quantum channel } N_{A \rightarrow B} \text{ as} \\
E_{A}^{\kappa}(N_{A \rightarrow B}) & := \sup_{\rho_{A'AB'}} \left[ E_{\kappa}(N_{A\rightarrow B}(\rho_{A'AB'})) - E_{\kappa}(\rho_{A'AB'}) \right]. 
\end{align*} \]  

where the supremum is with respect to states \(\rho_{A'AB'}\) having arbitrary \(A'\) and \(B'\) systems.

In spite of the possibility that amortization might increase \(E_{\kappa}\), a consequence of Proposition 12 is that in fact it does not:

\[ \text{Proposition 13 Let } N_{A \rightarrow B} \text{ be a quantum channel. Then the } \kappa\text{-entanglement of a channel does not increase under amortization:} \\
E_{A}^{\kappa}(N_{A \rightarrow B}) = E_{\kappa}(N_{A \rightarrow B}). \]  

\[ \text{Proof. The inequality } E_{A}^{\kappa}(N_{A \rightarrow B}) \geq E_{\kappa}(N_{A \rightarrow B}) \text{ follows from Proposition 11, by identifying } A' \text{ with } R, \text{ setting } B' \text{ to be a trivial system, and noting that } E_{\kappa}(\rho_{A'AB'}) \text{ vanishes for this choice. The opposite inequality is a direct consequence of Proposition 12.} \]

\[ \text{Theorem 14 (Monotonicity) Let } \mathcal{M}_{\tilde{A} \rightarrow \tilde{B}} \text{ be a quantum channel and } \Theta^{\text{PPT}} \text{ a completely-PPT-preserving superchannel of the form in (119). The channel measure } E_{\kappa} \text{ is monotone under the action of the superchannel } \Theta^{\text{PPT}}, \text{ in the sense that} \\
E_{\kappa}(\mathcal{M}_{\tilde{A} \rightarrow \tilde{B}}) \geq E_{\kappa}(\Theta^{\text{PPT}}(\mathcal{M}_{\tilde{A} \rightarrow \tilde{B}})). \]  

The first inequality follows because $E_{\kappa}(\mathcal{N}_{A\rightarrow B}(\rho_{A'AB'})) - E_{\kappa}(\rho_{A'AB'})$

\begin{align}
E_{\kappa}(\mathcal{N}_{A\rightarrow B}(\rho_{A'AB'})) - E_{\kappa}(\rho_{A'AB'})
&= E_{\kappa}(\mathcal{P}_{A\rightarrow AA_M B_M}^{\text{post}} \circ \mathcal{M}_{\tilde{A} \rightarrow \tilde{B}} \circ \mathcal{P}_{A\rightarrow AA_M B_M}^{\text{pre}}(\rho_{A'AB'})) - E_{\kappa}(\rho_{A'AB'}) \\
&\leq E_{\kappa}(\mathcal{P}_{A\rightarrow AA_M B_M}^{\text{post}} \circ \mathcal{M}_{\tilde{A} \rightarrow \tilde{B}} \circ \mathcal{P}_{A\rightarrow AA_M B_M}^{\text{pre}}(\rho_{A'AB'})) - E_{\kappa}(\mathcal{P}_{A\rightarrow AA_M B_M}^{\text{pre}}(\rho_{A'AB'})) \\
&\leq E_{\kappa}(\mathcal{M}_{\tilde{A} \rightarrow \tilde{B}}) - E_{\kappa}(\mathcal{P}_{A\rightarrow AA_M B_M}^{\text{pre}}(\rho_{A'AB'})) \\
&= E_{\kappa}(\mathcal{M}_{\tilde{A} \rightarrow \tilde{B}}).
\end{align}

The first inequality follows because $E_{\kappa}(\mathcal{P}_{A\rightarrow AA_M B_M}^{\text{pre}}(\rho_{A'AB'})) \leq E_{\kappa}(\rho_{A'AB'})$, given that $E_{\kappa}$ does not increase under the action of the completely PPT-preserving channel $\mathcal{P}_{A\rightarrow AA_M B_M}^{\text{pre}}$ (Proposition 1). The second inequality follows from a similar reasoning, but with respect to the completely-PPT-preserving channel $\mathcal{P}_{A\rightarrow AA_M B_M}^{\text{post}}$. The last inequality follows because $\mathcal{P}_{A\rightarrow AA_M B_M}^{\text{pre}}(\rho_{A'AB'})$ is a particular bipartite state to consider at the input of the channel $\mathcal{M}_{\tilde{A} \rightarrow \tilde{B}}$, but the quantity $E_{\kappa}(\mathcal{M}_{\tilde{A} \rightarrow \tilde{B}})$ involves an optimization over all such states. The final equality is a consequence of Proposition 13.

**Remark** We remark here that the same inequality holds for the max-Rains information of a channel $R_{\text{max}}(\mathcal{N})$, defined in [WD16b, WFD17] and considered further in [BW18] (see also [TWW17]). That is, for $\mathcal{M}_{\tilde{A} \rightarrow \tilde{B}}$ a quantum channel and $\Theta^{\text{PPT}}$ a completely-PPT-preserving superchannel of the form in (19), the following inequality holds

\begin{align}
R_{\text{max}}(\mathcal{M}_{\tilde{A} \rightarrow \tilde{B}}) \geq R_{\text{max}}(\Theta^{\text{PPT}}(\mathcal{M}_{\tilde{A} \rightarrow \tilde{B}})).
\end{align}

This follows because $R_{\text{max}}$ does not increase under amortization, as shown in [BW18], and because the max-Rains relative entropy does not increase under the action of a completely-PPT-preserving channel [WD16a].

Furthermore, a similar inequality holds for the squashed entanglement $E_{\text{sq}}$ of a channel and for a channel’s max-relative entropy of entanglement $E_{\text{max}}$. In particular, let $\Theta^{\text{LOCC}}$ denote an LOCC superchannel, which realizes the following transformation of a channel $\mathcal{M}_{\tilde{A} \rightarrow \tilde{B}}$ to a channel $\mathcal{N}_{A\rightarrow B}$ in terms of LOCC channels $\mathcal{L}_{A\rightarrow AA_M B_M}^{\text{pre}}$ and $\mathcal{L}_{A\rightarrow AA_M B_M}^{\text{post}}$:

\begin{align}
\mathcal{N}_{A\rightarrow B} = \Theta^{\text{LOCC}}(\mathcal{M}_{\tilde{A} \rightarrow \tilde{B}}) := \mathcal{L}_{A\rightarrow AA_M B_M}^{\text{post}} \circ \mathcal{M}_{\tilde{A} \rightarrow \tilde{B}} \circ \mathcal{L}_{A\rightarrow AA_M B_M}^{\text{pre}}.
\end{align}

Then the following inequalities hold:

\begin{align}
E_{\text{sq}}(\mathcal{M}_{\tilde{A} \rightarrow \tilde{B}}) \geq E_{\text{sq}}(\Theta^{\text{LOCC}}(\mathcal{M}_{\tilde{A} \rightarrow \tilde{B}})) \\
E_{\text{max}}(\mathcal{M}_{\tilde{A} \rightarrow \tilde{B}}) \geq E_{\text{max}}(\Theta^{\text{LOCC}}(\mathcal{M}_{\tilde{A} \rightarrow \tilde{B}}))
\end{align}

with both inequalities following because these measures do not increase under amortization, as shown in [TGW14a, TGW14b] and [CMH17], respectively, and the squashed entanglement [CW04] and max-relative entropy of entanglement of states [Dat09b, Dat09a] do not increase under LOCC channels.
B. Dual representation and additivity

The optimization that is dual to (112) is as follows:

\[ E^{\text{dual}}_\kappa(N_{A\to B}) := \sup \{ \log \text{Tr} J^N_{AB}(V_{AB} - W_{AB}) : V_{AB} + W_{AB} \leq \rho_A \otimes \mathbb{1}_B, V^T_{AB}, W^T_{AB} \geq 0, \rho_A \geq 0, \text{Tr} \rho_A = 1 \}. \]  

(152)

By weak duality, we have that

\[ E^{\text{dual}}_\kappa(N_{A\to B}) \leq E_\kappa(N_{A\to B}). \]  

(153)

If the channel \( N_{A\to B} \) is finite-dimensional, then strong duality holds, so that

\[ E^{\text{dual}}_\kappa(N_{A\to B}) = E_\kappa(N_{A\to B}). \]  

(154)

Furthermore, by employing the fact that \( E^{\text{dual}}_\kappa(N_{A\to B}) = \sup_{\rho_{RA}} E^{\text{dual}}_\kappa(N_{A\to B}(\rho_{RA})) \), Proposition 11, and (15), we conclude that the following equality holds for a quantum channel \( N_{A\to B} \):

\[ E^{\text{dual}}_\kappa(N_{A\to B}) = E_\kappa(N_{A\to B}). \]  

(155)

The additivity of \( E_\kappa \) with respect to tensor-product channels follows from both the primal and dual representations of \( E_\kappa(N) \).

**Proposition 15 (Additivity)** Given two quantum channels \( N_{A\to B} \) and \( M_{A'\to B'} \), it holds that

\[ E_\kappa(N_{A\to B} \otimes M_{A'\to B'}) = E_\kappa(N_{A\to B}) + E_\kappa(M_{A'\to B'}). \]  

(156)

**Proof.** The proof is similar to that of Proposition 2. To be self-contained, we show the details as follows. First, by definition, we can write \( E_\kappa(N_{A\to B}) \) as

\[ E_\kappa(N_{A\to B}) = \inf \{ \log \| \text{Tr}_B Q_{AB} \|_\infty : -Q^T_{AB} \leq (J^N_{AB})^T_{AB} \leq Q^T_{AB}, Q_{AB} \geq 0 \}. \]  

(157)

Let \( Q_{AB} \) be an arbitrary operator satisfying \(-Q^T_{AB} \leq (J^N_{AB})^T_{AB} \leq Q^T_{AB}, Q_{AB} \geq 0, \) and let \( P_{A'B'} \) be an arbitrary operator satisfying \(-P^T_{A'B'} \leq (J^M_{A'B'})^T_{A'B'} \leq P^T_{A'B'}, P_{A'B'} \geq 0. \) Then \( Q_{AB} \otimes P_{A'B'} \) satisfies

\[ -(Q_{AB} \otimes P_{A'B'})^T_{B'B'} \leq (J^N_{AB} \otimes J^M_{A'B'})^T_{B'B'} \leq (Q_{AB} \otimes P_{A'B'})^T_{B'B'}, Q_{AB} \otimes P_{A'B'} \geq 0, \]  

(158)

so that

\[ E_\kappa(N_{A\to B} \otimes M_{A'\to B'}) \leq \log \| \text{Tr}_{B'B'} Q_{AB} \otimes P_{A'B'} \|_\infty \]  

(159)

\[ = \log \| \text{Tr}_B Q_{AB} \|_\infty + \log \| \text{Tr}_{B'B'} P_{A'B'} \|_\infty. \]  

(160)

Since the inequality holds for all \( Q_{AB} \) and \( P_{A'B'} \) satisfying the above conditions, we conclude that

\[ E_\kappa(N \otimes M) \leq E_\kappa(N) + E_\kappa(M). \]  

(161)
To see the super-additivity of $E_\kappa$ for quantum channels, let us suppose that \{V_{AB}^1, W_{AB}^1, \rho_A^1\} and \{V_{AB}^2, W_{AB}^2, \rho_A^2\} are arbitrary operators satisfying the conditions in (152) for $\mathcal{N}_{A\to B}$ and $\mathcal{M}_{A'\to B'}$, respectively. Now we choose
\[ R_{ABA'B'} = V_{AB}^1 \otimes V_{ABA'}^2 + W_{AB}^1 \otimes W_{ABA'}^2, \quad (162) \]
\[ S_{ABA'B'} = V_{AB}^1 \otimes W_{ABA'}^2 + W_{AB}^1 \otimes V_{ABA'}^2. \quad (163) \]
One can verify from (152) that
\[ R_{ABA'B'}^{T_{BB'}} S_{ABA'B'}^{T_{BB'}} \geq 0, \quad (164) \]
\[ R_{ABA'B'} + S_{ABA'B'} = (V_{AB}^1 + W_{AB}^1) \otimes (V_{ABA'}^2 + W_{ABA'}^2) \leq \rho_A^1 \otimes \rho_A^2 \otimes \mathbb{1}_{BB'}, \quad (165) \]
which implies that \{R_{ABA'B'}, S_{ABA'B'}, \rho_A^1 \otimes \rho_A^2\} is feasible for $E_\kappa(\mathcal{N}_{A\to B} \otimes \mathcal{M}_{A'\to B'})$ in (152). Thus, we have that
\[ E_\kappa^{\text{dual}}(\mathcal{N}_{A\to B} \otimes \mathcal{M}_{A'\to B'}) \geq \log \text{Tr}(J_{ABA'}^N \otimes J_{ABA'}^M)(R_{ABA'B'} - S_{ABA'B'}) \]
\[ = \log[\text{Tr} J_{ABA'}^N(V_{AB}^1 - W_{AB}^1) \cdot \text{Tr} J_{ABA'}^M(V_{ABA'}^2 - W_{ABA'}^2)] \]
\[ = \log(\text{Tr} J_{ABA'}^N(V_{AB}^1 - W_{AB}^1)) + \log(\text{Tr} J_{ABA'}^M(V_{ABA'}^2 - W_{ABA'}^2)). \quad (166) \]
Since the inequality has been shown for arbitrary \{V_{AB}^1, W_{AB}^1, \rho_A^1\} and \{V_{ABA'}^2, W_{ABA'}^2, \rho_A^2\} satisfying the conditions in (152) for $\mathcal{N}_{A\to B}$ and $\mathcal{M}_{A'\to B'}$, respectively, we conclude that
\[ E_\kappa^{\text{dual}}(\mathcal{N}_{A\to B} \otimes \mathcal{M}_{A'\to B'}) \geq E_\kappa^{\text{dual}}(\mathcal{N}_{A\to B}) + E_\kappa^{\text{dual}}(\mathcal{M}_{A'\to B'}). \quad (169) \]
The proof is concluded by combining (161), (169), and (155).

### C. Normalization, faithfulness, and no convexity

In this subsection, we prove that the $\kappa$-entanglement of a quantum channel is normalized, faithful, and generally not convex.

**Proposition 16 (Normalization)** Let $\text{id}_{A\to B}^M$ be a noiseless quantum channel with dimension $d_A = d_B = M$. Then
\[ E_\kappa(\text{id}_{A\to B}^M) = \log M. \quad (170) \]
Moreover, for any finite-dimensional quantum channel $\mathcal{N}_{A\to B}$,
\[ E_\kappa(\mathcal{N}_{A\to B}) \leq \min\{\log d_A, \log d_B\}. \quad (171) \]

**Proof.** By Propositions 4 and 11, we have
\[ E_\kappa(\mathcal{N}_{A\to B}) = \sup_{\rho_{RA}} E_\kappa(\mathcal{N}_{A\to B}(\rho_{RA})) = \sup_{\psi_{RA}} E_\kappa(\mathcal{N}_{A\to B}(\psi_{RA})) \leq \log \min\{d_A, d_B\}, \quad (172) \]
where, in the second equality, the optimization is with respect to pure states with system $R$ isomorphic to the channel input system $A$.

This implies that $E_\kappa(\text{id}_{A\to B}^M) \leq \log M$. Furthermore,
\[ E_\kappa(\text{id}_{A\to B}^M) \geq E_\kappa(\text{id}_{A\to B}(\Phi_{RA}^M)) = \log M, \quad (173) \]
where $\Phi_{RA}^M$ denotes a maximally entangled state of Schmidt rank $M$ and the second equality follows from Proposition 4.\[ \square \]
Proposition 17 (Faithfulness) Let $\mathcal{N}_{A\rightarrow B}$ be a quantum channel. Then $E_\kappa(\mathcal{N}_{A\rightarrow B}) \geq 0$ and $E_\kappa(\mathcal{N}_{A\rightarrow B}) = 0$ if and only if $\mathcal{N}_{A\rightarrow B}$ is a PPT entanglement binding channel [HHH00].

**Proof.** To see that $E_\kappa(\mathcal{N}_{A\rightarrow B}) \geq 0$, we could utilize the dual representation in (152) and the equality in (155), or alternatively employ Propositions 5 and 11 to find that

$$E_\kappa(\mathcal{N}_{A\rightarrow B}) = \sup_{\rho_{RA}} E_\kappa(\mathcal{N}_{A\rightarrow B}(\rho_{RA})) \geq 0.$$ (174)

Now if $\mathcal{N}_{A\rightarrow B}$ is a PPT entanglement binding channel (as defined in [HHH00]), then the state $\mathcal{N}_{A\rightarrow B}(\rho_{RA})$ is PPT for any input state $\rho_{RA}$. Thus, $E_\kappa(\mathcal{N}_{A\rightarrow B}) = 0$. On the other hand, if $E_\kappa(\mathcal{N}_{A\rightarrow B}) = 0$, then for any $\rho_{RA}$ it holds that $E_\kappa(\mathcal{N}_{A\rightarrow B}(\rho_{RA})) = 0$. By Proposition 5, we conclude that $\mathcal{N}_{A\rightarrow B}(\rho_{RA})$ is PPT for any $\rho_{RA}$, and thus $\mathcal{N}_{A\rightarrow B}$ is a PPT entanglement binding channel. □

Proposition 18 (No convexity) The $\kappa$-entanglement of quantum channel is not generally convex.

**Proof.** To see this, we construct channels with Choi states given by the examples in Proposition 6. Let us choose the following qubit channels:

$$\mathcal{N}_1(\rho) = \rho,$$ (175)

$$\mathcal{N}_2(\rho) = |0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1|.$$ (176)

Since $\mathcal{N}_1$ is a qubit noiseless channel, Proposition 16 implies that $E_\kappa(\mathcal{N}_1) = 1$. Noting that $\mathcal{N}_2$ is a PPT entanglement binding channel, Proposition 17 implies that $E_\kappa(\mathcal{N}_2) = 0$.

Let $\mathcal{N} = \frac{1}{2}(\mathcal{N}_1 + \mathcal{N}_2)$ denote the uniform mixture of the two channels. The mixed channel $\mathcal{N}$ is actually a dephasing channel with dephasing parameter $1/2$. Then we have that $E_\kappa(\mathcal{N}) \geq \log \frac{3}{2}$, which follows by inputting one share of the maximally entangled state. Thus, we find that

$$E_\kappa(\mathcal{N}) > \frac{1}{2}(E_\kappa(\mathcal{N}_1) + E_\kappa(\mathcal{N}_1)).$$ (177)

This concludes the proof. □

V. EXACT ENTANGLEMENT COST OF QUANTUM CHANNELS

In this section, we introduce two channel simulation tasks. First, we consider the exact parallel simulation of a quantum channel, when completely-PPT-preserving channels are allowed for free and the goal is to meter the entanglement cost. We also consider the exact sequential simulation of a quantum channel. In both cases, the entanglement cost is equal to the $\kappa$-entanglement of the channel, thus endowing it with a direct operational meaning. After these results are established, we focus on PPT-simulable [KW18] and resource-seizable [Wil18] channels, demonstrating that the theory significantly simplifies for these kinds of channels.
A. Exact parallel simulation of quantum channels

Another fundamental problem is to quantify the entanglement required for an exact simulation of an arbitrary quantum channel, via free operations (LOCC or PPT) and by making use of an entangled resource state. Recall that $\Omega$ represents the set of free operations. Also, two quantum channels $N_{A \rightarrow B}$ and $M_{A \rightarrow B}$ are equal if for orthonormal bases $\{|i\rangle_A\}_i$ and $\{|k\rangle_B\}_k$, the following equalities hold for all $i,j,k,l \in \mathbb{N}$:

$$\langle k|_B N_{A \rightarrow B}(|i\rangle_A \langle j|_A)l\rangle_B = \langle k|_B M_{A \rightarrow B}(|i\rangle_A \langle j|_A)l\rangle_B.$$  \hfill (178)

This is equivalent to the Choi operators of the channels being equal:

$$N_{A \rightarrow B}(\Gamma_{RA}) = M_{A \rightarrow B}(\Gamma_{RA}).$$  \hfill (179)

Furthermore, the following identity holds for an arbitrary state $\rho_{CS}$ with $S \simeq R \simeq A$:

$$\langle \Gamma|_{SR}[\rho_{CS} \otimes N_{A \rightarrow B}(\Gamma_{RA})]|\Gamma\rangle_{SR} = N_{A \rightarrow B}(\rho_{CA}),$$  \hfill (180)

understood intuitively as a post-selected variant [Ben05, HM04] of quantum teleportation [BBC+93]. From the identity in (180), we conclude that if two channels are equal in the sense of (178) and (179), then there is no physical procedure that can distinguish them.

We define the one-shot exact entanglement cost of a quantum channel $N_{A \rightarrow B}$, under the $\Omega$ operations, as

$$E^{(1)}_{\Omega}(N_{A \rightarrow B}) = \inf_{\Lambda \in \Omega}\{\log d : N_{A \rightarrow B}(\Gamma_{RA}) = \Lambda_{\tilde{A}\tilde{B} \rightarrow AB}(\Gamma_{RA} \otimes \Phi^d_{\tilde{A}\tilde{B}})\}.$$  \hfill (181)

The exact parallel entanglement cost of quantum channel $N_{A \rightarrow B}$, under the $\Omega$ operations, is defined as

$$E^{(p)}_{\Omega}(N_{A \rightarrow B}) = \lim_{n \rightarrow \infty} \frac{1}{n} E^{(1)}_{\Omega}(N_{A \rightarrow B}^\otimes n).$$  \hfill (182)

**Theorem 19** The one-shot exact PPT-entanglement cost $E^{(1)}_{\text{PPT}}(N_{A \rightarrow B})$ of a quantum channel $N_{A \rightarrow B}$ is given by the following optimization:

$$E^{(1)}_{\text{PPT}}(N_{A \rightarrow B}) = \inf \{\log m : - (m - 1)Q_{AB}^{T_B} \leq (J_{AB}^N)^T_B \leq (m + 1)Q_{AB}^{T_B}, \ Q_{AB} \geq 0, \ Tr_B Q_{AB} = 1_A\}.$$  \hfill (183)
Proof. The proof is somewhat similar to the proof of Theorem 8. The achievability part features a construction of a completely-PPT-preserving channel $\mathcal{P}_{\hat{A}\hat{B} \rightarrow B}$ such that $\mathcal{P}_{\hat{A}\hat{B} \rightarrow B}(X_A \otimes \Phi_{\hat{A}\hat{B}}^m) = \mathcal{N}_{A \rightarrow B}(X_A)$ for all input operators $X_A$ (including density operators), and then the converse part demonstrates that the constructed channel is essentially the only form that is needed to consider for the one-shot exact PPT-entanglement cost task.

First, in order to have an exact simulation of a channel, it is only necessary to check the simulation on a single input, the maximally entangled vector $|\Gamma\rangle_{RA}$. So we require that

$$\mathcal{P}_{\hat{A}\hat{B} \rightarrow B}(\Gamma_{RA} \otimes \Phi_{\hat{A}\hat{B}}^m) = \mathcal{N}_{A \rightarrow B}(\Gamma_{RA}),$$  \hspace{1cm} (184)

where $\Gamma_{RA}$ is the unnormalized maximally entangled operator.

We now prove the achievability part. Let $m \geq 1$ be a positive integer and $Q_{AB}$ a Choi operator for a quantum channel (i.e., $Q_{AB} \geq 0$, $\text{Tr} Q_{AB} = 1_A$) such that the following inequalities hold

$$-(m-1)Q_{AB} \leq (J_{AB}^N)^{T_B} \leq (m+1)Q_{AB}. \hspace{1cm} (185)$$

Then we take the completely-PPT-preserving channel $\mathcal{P}_{\hat{A}\hat{B} \rightarrow B}$ to have a Choi operator given by

$$J_{\hat{A}\hat{B}B}^P = J_{AB}^N \otimes \Phi_{\hat{A}\hat{B}}^m + Q_{AB} \otimes (1_{\hat{A}\hat{B}} - \Phi_{\hat{A}\hat{B}}^m). \hspace{1cm} (186)$$

Observe that $J_{\hat{A}\hat{B}B}^P \geq 0$. Furthermore, we have that

$$\text{Tr}_B J_{\hat{A}\hat{B}B}^P = \text{Tr}_B J_{AB}^N \otimes \Phi_{\hat{A}\hat{B}}^m + \text{Tr}_B Q_{AB} \otimes (1_{\hat{A}\hat{B}} - \Phi_{\hat{A}\hat{B}}^m) = 1_A \otimes \Phi_{\hat{A}\hat{B}}^m + 1_A \otimes (1_{\hat{A}\hat{B}} - \Phi_{\hat{A}\hat{B}}^m). \hspace{1cm} (187)$$

Thus, $\mathcal{P}_{\hat{A}\hat{B} \rightarrow B}$ is a quantum channel. Setting $|\Gamma\rangle_{AA'\hat{A}'\hat{A}B'B'} \equiv |\Gamma\rangle_{AA'} \otimes |\Gamma\rangle_{\hat{A}\hat{A}'} \otimes |\Gamma\rangle_{BB'}$, its action on the input $\Gamma_{RA} \otimes \Phi_{\hat{A}\hat{B}}^m$ is given by

$$\langle \Gamma|_{AA'\hat{A}'\hat{A}B'B'} (\Gamma_{RA} \otimes \Phi_{\hat{A}\hat{B}}^m \otimes J_{\hat{A}\hat{B}B}^P) |\Gamma\rangle_{AA'\hat{A}'\hat{A}B'B'} \hspace{1cm} (188)$$

$$= \langle \Gamma|_{AA'\hat{A}'\hat{A}B'B'} (\Gamma_{RA} \otimes \Phi_{\hat{A}\hat{B}}^m \otimes J_{AB}^N \otimes \Phi_{\hat{A}\hat{B}}^m) |\Gamma\rangle_{AA'\hat{A}'\hat{A}B'B'} + \langle \Gamma|_{AA'\hat{A}'\hat{A}B'B'} (\Gamma_{RA} \otimes \Phi_{\hat{A}\hat{B}}^m \otimes Q_{AB} \otimes (1_{\hat{A}'\hat{B}'} - \Phi_{\hat{A}'\hat{B}'}^m)) |\Gamma\rangle_{AA'\hat{A}'\hat{A}B'B'} \hspace{1cm} (189)$$

The second equality follows because

$$\langle \Gamma|_{\hat{A}\hat{A}'} \otimes \langle \Gamma|_{\hat{B}\hat{B}'} \left( \Phi_{\hat{A}\hat{B}}^m \otimes \Phi_{\hat{A}'\hat{B}'}^m \right) (|\Gamma\rangle_{\hat{A}\hat{A}'} \otimes |\Gamma\rangle_{\hat{B}\hat{B}'}) = \text{Tr} \Phi_{\hat{A}\hat{B}}^m \Phi_{\hat{A}'\hat{B}'}^m = 1, \hspace{1cm} (190)$$

$$\langle \Gamma|_{\hat{A}\hat{A}'} \otimes \langle \Gamma|_{\hat{B}\hat{B}'} \left( \Phi_{\hat{A}\hat{B}}^m \otimes \Phi_{\hat{A}'\hat{B}'}^m \right) (|\Gamma\rangle_{\hat{A}\hat{A}'} \otimes |\Gamma\rangle_{\hat{B}\hat{B}'}) = \text{Tr} \Phi_{\hat{A}\hat{B}}^m = 1. \hspace{1cm} (191)$$

Thus, for the constructed channel, we have that (184) holds. Finally, we need to show that the constructed channel $\mathcal{P}_{\hat{A}\hat{B} \rightarrow B}$ is completely-PPT-preserving:

$$(J_{\hat{A}\hat{B}B}^P)_{\hat{B}\hat{B}'} \geq 0. \hspace{1cm} (192)$$
Consider that
\[
(J^P_{AABB})^T_{BB} = (J^N_{AB})^T_{BB} \otimes (\Phi^m_{AB})^T_{BB} + Q^T_{AB} \otimes (1_{AB} - \Phi^m_{AB})^T_{BB} \\
= \frac{1}{m}(J^N_{AB})^T_{BB} \otimes (F^T_{AB}) + Q^T_{AB} \otimes (1_{AB} - \frac{1}{m}F^T_{AB})
\] (196)

\[
= \frac{1}{m}(J^N_{AB})^T_{BB} \otimes (\Pi^S_{AB} - \Pi^A_{AB}) + Q^T_{AB} \otimes (\Pi^S_{AB} + \Pi^A_{AB} - \frac{1}{m}[\Pi^S_{AB} - \Pi^A_{AB}])
\] (197)

\[
= \left[ \frac{1}{m}(J^N_{AB})^T_{BB} + \left( 1 - \frac{1}{m} \right) Q^T_{AB} \right] \otimes \Pi^S_{AB} + \left[ \left( 1 + \frac{1}{m} \right) Q^T_{AB} - \frac{1}{m}(J^N_{AB})^T_{BB} \right] \otimes \Pi^A_{AB}
\] (198)

\[
= \frac{1}{m} \left[ (J^N_{AB})^T_{BB} + (m - 1)Q^T_{AB} \right] \otimes \Pi^S_{AB} + \frac{1}{m} \left[ (m + 1)Q^T_{AB} - (J^N_{AB})^T_{BB} \right] \otimes \Pi^A_{AB}.
\] (199)

Applying the condition in (185), we conclude (195). Thus, we have shown that for all \(m\) and \(Q_{AB}\) satisfying (185) and \(Q_{AB} \geq 0\), \(\text{Tr}_B Q_{AB} = 1_A\), there exists a completely-PPT-preserving channel \(\mathcal{P}_{AABB} \rightarrow B\) such that (184) holds. Now taking an infimum over all such \(m\) and \(Q_{AB}\), we conclude that the right-hand side of (183) is greater than or equal to \(E^{(1)}_{\text{PPT}}(\mathcal{N}_{A \rightarrow B})\).

To see the opposite inequality, let \(\mathcal{P}_{AABB \rightarrow B}\) be a completely-PPT-preserving channel such that (184) holds. Then preceding \(\mathcal{P}_{AABB \rightarrow B}\) by the isotropic twirling channel \(\mathcal{T}_{AB}\) results in a completely-PPT-preserving channel \(\mathcal{P}^{'AB \rightarrow B} = \mathcal{P}_{AABB \rightarrow B} \otimes \mathcal{T}_{AB}\) achieving the same simulation task, and so it suffices to focus on the channel \(\mathcal{P}^{'AABB \rightarrow B}\) in order to establish an expression for the one-shot exact PPT-entanglement cost. Consider that

\[
J^{'P}_{RABB'} = (\mathcal{P}^{'AABB \rightarrow B})(\Gamma_{RA} \otimes \Gamma_{A'\hat{A}} \otimes \Gamma_{B'B})
\] (200)

Considering that

\[
\mathcal{T}_{AB}(\Gamma_{A'\hat{A}} \otimes \Gamma_{B'B}) = \Phi^m_{AB} \otimes \text{Tr}_{AB}[\Phi^m_{AB}(\Gamma_{A'\hat{A}} \otimes \Gamma_{B'B})]
\]

\[
= \Phi^m_{AB} \otimes \text{Tr}_{AB}[(1_{AB} - \Phi^m_{AB}) (\Gamma_{A'\hat{A}} \otimes \Gamma_{B'B})]
\] (201)

\[
= \Phi^m_{AB} \otimes \Phi^m_{A'B'} + \frac{1_{AB} - \Phi^m_{AB}}{m^2 - 1} \otimes (1_{AB} - \Phi^m_{AB})
\] (202)

with the equalities understood in terms of entanglement swapping [BBC+93], we conclude that

\[
(\mathcal{P}_{AABB \rightarrow B} \circ \mathcal{T}_{AB})(\Gamma_{RA} \otimes \Gamma_{A'\hat{A}} \otimes \Gamma_{B'B})
\] (203)

\[
= (\mathcal{P}_{AABB \rightarrow B})(\Gamma_{RA} \otimes \Phi^m_{AB}) \otimes \Phi^m_{A'B'} + (\mathcal{P}_{AABB \rightarrow B})(\Gamma_{RA} \otimes \frac{1_{AB} - \Phi^m_{AB}}{m^2 - 1}) \otimes (1_{AB} - \Phi^m_{AB})
\] (204)

\[
= \mathcal{N}_{A \rightarrow B}(\Gamma_{RA}) \otimes \Phi^m_{A'B'} + \mathcal{P}_{AABB \rightarrow B}(\Gamma_{RA} \otimes \frac{1_{AB} - \Phi^m_{AB}}{m^2 - 1}) \otimes (1_{AB} - \Phi^m_{AB})
\] (205)

\[
= J^N_{RB} \otimes \Phi^m_{A'B'} + Q_{RB} \otimes (1_{A'B'} - \Phi^m_{A'B'}).
\] (206)

where we have used the assumption that (184) holds and set

\[
Q_{RB} = \mathcal{P}_{AABB \rightarrow B}(\Gamma_{RA} \otimes \frac{1_{AB} - \Phi^m_{AB}}{m^2 - 1}),
\] (207)
from which it follows that $Q_{RB} \geq 0$ and $\text{Tr}_B Q_{RB} = 1_R$. In order for the channel $\mathcal{P}'_{A\bar{A}B \rightarrow B}$ to be completely-PPT-preserving, it is necessary that

$$(J_{R\bar{A}B}'_{A\bar{A}B})_{T\bar{B}B} \geq 0. \quad (208)$$

Writing this out and using calculations given above, we find that it is necessary that the following operator is positive semi-definite

$$\frac{1}{m} [(J^N_{AB})_{T\bar{B}} + (m - 1) Q^T_{\bar{A}B}] \otimes \Pi^S_{\bar{A}B} + \frac{1}{m} [(m + 1) Q^T_{\bar{A}B} - (J^N_{AB})_{T\bar{B}}] \otimes \Pi^A_{AB}. \quad (209)$$

Since $\Pi^S_{\bar{A}B}$ and $\Pi^A_{AB}$ project onto orthogonal subspaces, we find that the condition (185) is necessary. Thus, it follows that the quantity on the right-hand side of (183) is less than or equal to $E_{\text{PPT}}^{(1)}(\mathcal{N}_{A\rightarrow B})$.

**Proposition 20** Let $\mathcal{N}_{A\rightarrow B}$ be a quantum channel. Then

$$\log(2^{E_\kappa(N)} - 1) \leq E_{\text{PPT}}^{(1)}(\mathcal{N}_{A\rightarrow B}) \leq \log(2^{E_\kappa(N)} + 1). \quad (210)$$

**Proof.** The proof follows the proof method in Proposition 9. Consider that

$$E_{\text{PPT}}^{(1)}(\mathcal{N}_{A \rightarrow B})$$

$$= \inf \left\{ \log m : -(m - 1) Q^T_{AB} \leq (J^N_{AB})_{T\bar{B}} \leq (m + 1) Q^T_{AB}, \text{Tr}_B Q_{AB} = 1_A \right\}$$

$$= \inf \left\{ \log m : -(m + 1) Q^T_{AB} \leq (J^N_{AB})_{T\bar{B}} \leq (m + 1) Q^T_{AB}, \text{Tr}_B Q_{AB} = 1_A \right\}$$

$$= \inf \left\{ \log m : -R^T_{\bar{A}B} \leq (J^N_{AB})_{T\bar{B}} \leq R^T_{\bar{A}B}, \text{Tr}_B R_{AB} = (m + 1) 1_A \right\}$$

$$= \inf \left\{ \log \left( \parallel \text{Tr}_B R_{AB} \parallel_{\infty} \right) - 1 \right\} : -R^T_{\bar{A}B} \leq (J^N_{AB})_{T\bar{B}} \leq R^T_{\bar{A}B}, \text{Tr}_B R_{AB} \geq 0 \right\}$$

$$= \log(2^{E_\kappa(N)} - 1). \quad (211)$$

The first inequality follows by relaxing the constraint $-(m - 1) Q^T_{AB} \leq (J^N_{AB})_{T\bar{B}}$ to $-(m + 1) Q^T_{AB} \leq (J^N_{AB})_{T\bar{B}}$. The second equality follows by absorbing $m$ into $Q_{AB}$ and setting $R_{AB} = (m + 1) Q_{AB}$. The last equality follows from the definition of $E_\kappa(N)$.

Similarly, we have that $E_{\text{PPT}}^{(1)}(\mathcal{N}_{A \rightarrow B}) \leq \log(2^{E_\kappa(N)} + 1)$. 

**Theorem 21** Let $\mathcal{N}_{A \rightarrow B}$ be a quantum channel. Then the exact parallel entanglement cost of $\mathcal{N}_{A \rightarrow B}$ is equal to its $\kappa$-entanglement:

$$E^{(p)}_{\text{PPT}}(\mathcal{N}_{A \rightarrow B}) = E_\kappa(\mathcal{N}_{A \rightarrow B}). \quad (212)$$

**Proof.** The main idea behind the proof is to employ the one-shot bound in Proposition 20 and then the additivity relation from Proposition 15. Consider that

$$E^{(p)}_{\text{PPT}}(\mathcal{N}_{A \rightarrow B}) = \lim_{n \rightarrow \infty} \frac{1}{n} E^{(1)}_{\text{PPT}}(\mathcal{N}^n_{A \rightarrow B}) \quad (213)$$

$$\leq \lim_{n \rightarrow \infty} \frac{1}{n} \log(2^{E_\kappa(\mathcal{N}^n_{A \rightarrow B})} + 1) \quad (214)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \log(2^{nE_\kappa(N)} + 1) \quad (215)$$

$$= E_\kappa(\mathcal{N}_{A \rightarrow B}). \quad (216)$$

Similarly, $E_{\text{PPT}}(\mathcal{N}_{A \rightarrow B}) \geq E_\kappa(\mathcal{N}_{A \rightarrow B})$. 


B. Exact sequential simulation of quantum channels

A more general notion of channel simulation, called sequential channel simulation, was recently proposed and studied in [Wil18]. In this section, we define and characterize exact sequential channel simulation, as opposed to the approximate sequential channel simulation focused on in [Wil18]. For concreteness, we set the free operations $\Omega$ to be completely-PPT-preserving channels. The main idea behind sequential channel simulation is to simulate $n$ uses of the channel $N_{A \rightarrow B}$ in such a way that they can be called in an arbitrary order, i.e., on demand when they are needed. An $(n, M)$ exact sequential channel simulation code consists of a maximally entangled resource state $\Phi^n_M$ of Schmidt rank $M$ and a set

$$\{P^{(i)}_{A_nB_{n-1}B_{n-2}\cdots B_1} \}_{i=1}^{n}$$

of completely-PPT-preserving channels. Note that the systems $A_nB_n$ of the final completely-PPT-preserving channel $P^{(n)}_{A_nB_{n-1}B_{n-2}\cdots B_1B_0}$ can be taken trivial without loss of generality. As before, Alice has access to all systems labeled by $A$, Bob has access to all systems labeled by $B$, and they are in distant laboratories. The structure of this simulation protocol is intended to be compatible with a discrimination strategy that can test the actual $n$ channels versus the above simulation in a sequential way, along the lines discussed in [CDP08, CDP09b] and [Gut12].

We define the simulation to be exact if the following equalities hold for orthonormal bases $\{|i\rangle_A\}_{\mathcal{A}}$ and $\{|k\rangle_B\}_{\mathcal{B}}$ and for all $i_1, j_1, k_1, l_1, \ldots, i_n, j_n, k_n, l_n \in \mathbb{N}$:

$$p^{(i_r,j_r,k_r,l_r)}_{i_1,j_1,k_1,l_1} = \prod_{r=1}^{n} \langle k_r | B_r N_{A_r \rightarrow B_r} (|i_r\rangle \langle j_r| A_r) | l_r \rangle_{B_r},$$

where

$$\begin{align*}
P^{i_1,j_1,k_1,l_1}_{A_1B_1} &:= \langle k_1 | B_1 \left[ P^{(1)}_{A_1A_0B_0 \rightarrow B_1A_1B_1} (|i_1\rangle \langle j_1| A_1 \otimes \Phi^M_{A_0B_0}) \right] | l_1 \rangle_{B_1}, \\
P^{i_2,j_2,k_2,l_2}_{A_2B_2} &:= \langle k_2 | B_2 \left[ P^{(2)}_{A_2A_1B_1 \rightarrow B_2A_2B_2} (|i_2\rangle \langle j_2| A_2 \otimes P^{i_1,j_1,k_1,l_1}_{A_1B_1}) \right] | l_2 \rangle_{B_2}, \\
&\vdots \\
P^{i_n,j_n,k_n,l_n}_{\mathcal{A}_{n-1}B_{n-1}} &:= \langle k_n | B_n \left[ P^{(n)}_{A_nA_{n-1}B_{n-1} \rightarrow B_n} (|i_n\rangle \langle j_n| A_n \otimes P^{i_{n-1},j_{n-1},k_{n-1},l_{n-1}}_{A_{n-2}B_{n-2}}) \right] | l_n \rangle_{B_n}.
\end{align*}$$

Figure 2 depicts the channel simulation and the exact simulation condition in (218).

By defining the completely-PPT-preserving quantum channel $P^{n}_{A_nA_{n-1}B_{n-1}B_{n-2}\cdots B_1B_0}$ as the serial composition of the individual channels in (217) (depicted in Figure 3)

$$P^{n}_{A_nA_{n-1}B_{n-1}B_{n-2}\cdots B_1B_0} := (P^{(n)}_{A_nA_{n-1}B_{n-1}B_{n-2}\cdots B_1B_0} \circ \cdots \circ P^{(2)}_{A_2A_1B_1 \rightarrow B_2A_2B_2} \circ P^{(1)}_{A_1A_0B_0 \rightarrow B_1A_1B_1}),$$

we can write the exact simulation as

$$P^{n}_{A_nA_{n-1}B_{n-1}B_{n-2}\cdots B_1B_0} = P^{n}_{A_nA_{n-1}B_{n-1}B_{n-2}\cdots B_1B_0}.$$
we conclude that the condition in (218) is equivalent to the following condition:

$$(\mathcal{N}_{A\rightarrow B})^\otimes n(\Gamma_{R^n A^n}) = \mathcal{P}_{A^n A_0 B_0 \rightarrow B^n}(\Gamma_{R^n A^n} \otimes \Phi^M_{A_0 B_0})$$

(224)

where $\Gamma_{R^n A^n} := \otimes_{i=1}^n \Gamma_{R_i A_i}$. This latter condition is depicted in Figure 4.

The $n$-shot exact sequential simulation cost of the channel $\mathcal{N}_{A\rightarrow B}$ is then defined as

$$E_{\text{PPT}}(\mathcal{N}_{A\rightarrow B}, n) := \inf \left\{ \log M : (\mathcal{N}_{A\rightarrow B})^\otimes n(\Gamma_{R^n A^n}) = \mathcal{P}_{A^n A_0 B_0 \rightarrow B^n}(\Gamma_{R^n A^n} \otimes \Phi^M_{A_0 B_0}) \right\}$$

(225)

where the optimization is with respect to sequential protocols of the form in (217) and the channel $\mathcal{P}_{A^n A_0 B_0 \rightarrow B^n}$ is defined as in (223). The exact (sequential) simulation cost of the
The exact channel simulation condition in (218) is equivalent to the condition that the Choi operators as depicted above are equal, as written in (224).

channel $\mathcal{N}_{A\rightarrow B}$ is defined as

$$E_{\text{PPT}}(\mathcal{N}_{A\rightarrow B}) := \liminf_{n \to \infty} \frac{1}{n} E_{\text{PPT}}(\mathcal{N}_{A\rightarrow B}, n).$$

(226)

The condition in (224) illustrates that a sequential simulation is a particular kind of parallel simulation, but with more constraints. That is, in a parallel simulation, the channel $\mathcal{P}_{A^n R_0 B_0 \rightarrow B^n}$ can be arbitrary, whereas in a sequential simulation, it is constrained to have the form in (217). For this reason, we can immediately conclude the following bound for all integer $n \geq 1$:

$$E_{\text{PPT}}^{(1)}((\mathcal{N}_{A\rightarrow B})^{\otimes n}) \leq E_{\text{PPT}}(\mathcal{N}_{A\rightarrow B}, n),$$

(227)

which in turn implies that

$$E_{\text{PPT}}^{(q)}(\mathcal{N}_{A\rightarrow B}) \leq E_{\text{PPT}}(\mathcal{N}_{A\rightarrow B}).$$

(228)

### C. Physical justification for definition of exact sequential channel simulation

The most general method for distinguishing the $n$ channel uses from its simulation is with an adaptive discrimination strategy. Such a strategy was described in [Wil18] and consists of an initial state $\rho_{R_1 A_1}$, a set $\{\mathcal{A}_{R_i B_i \rightarrow R_{i+1} A_{i+1}}^{(i)}\}_{i=0}^{n-1}$ of adaptive channels, and a quantum measurement $\{Q_{R_n B_n}, 1_{R_n B_n} - Q_{R_n B_n}\}$. Let us employ the shorthand $\{\rho, \mathcal{A}, Q\}$ to abbreviate such a discrimination strategy. Note that, in performing a discrimination strategy, the discriminator has a full description of the channel $\mathcal{N}_{A\rightarrow B}$ and the simulation protocol, which consists of $\Phi_{R_0 B_0}$ and the set in (217). If this discrimination strategy is performed on the $n$ uses of the actual channel $\mathcal{N}_{A\rightarrow B}$, the relevant states involved are

$$\rho_{R_{i+1} A_{i+1}} \equiv \mathcal{A}_{R_i B_i \rightarrow R_{i+1} A_{i+1}}^{(i)}(\rho_{R_i B_i}),$$

(229)
FIG. 5: An adaptive protocol for discriminating the original channels (top) from their simulation (bottom).

for $i \in \{1, \ldots, n-1\}$ and

$$\rho_{R_iB_i} \equiv \mathcal{N}_{A_i\rightarrow B_i}(\rho_{R_iA_i}),$$

(230)

for $i \in \{1, \ldots, n\}$. If this discrimination strategy is performed on the simulation protocol discussed above, then the relevant states involved are

$$\tau_{R_iB_iA_iB_i} \equiv \mathcal{D}^{(i)}_{A_iA_iB_i\rightarrow B_iA_iB_i}(\tau_{R_iA_i} \otimes \Phi_{A_iB_i}),$$

$$\tau_{R_{i+1}A_{i+1}A_iB_i} \equiv \mathcal{A}^{(i)}_{R_iB_i\rightarrow R_{i+1}A_{i+1}}(\tau_{R_iA_i\bar{A}_iB_i}),$$

(231)

for $i \in \{1, \ldots, n-1\}$, where $\tau_{R_1A_1} = \rho_{R_1A_1}$, and

$$\tau_{R_iB_iA_iB_i} \equiv \mathcal{D}^{(i)}_{A_iA_iB_i\rightarrow B_iA_iB_i}(\tau_{R_iA_iA_iB_i}),$$

(232)

for $i \in \{2, \ldots, n\}$. The discriminator then performs the measurement $\{Q_{R_nB_n}, \mathds{1}_{R_nB_n} - Q_{R_nB_n}\}$ and guesses “actual channel” if the outcome is $Q_{R_nB_n}$ and “simulation” if the outcome is $\mathds{1}_{R_nB_n} - Q_{R_nB_n}$. Figure 5 depicts the discrimination strategy in the case that the actual channel is called $n = 3$ times and in the case that the simulation is performed.

From the physical point of view, the $n$ channel uses of $\mathcal{N}_{A\rightarrow B}$ are perfectly indistinguishable from the simulation if every possible discrimination strategy as described above leads to the exact same final decision probabilities. That is, for all possible discrimination strategies, the original channels and their simulation are indistinguishable if the following equality holds

$$\text{Tr} \ Q_{R_nB_n} \rho_{R_nB_n} = \text{Tr} \ Q_{R_nB_n} \tau_{R_nB_n}.$$  

(233)

We now prove that this physical notion of exact channel simulation is equivalent to the more mathematical notion of exact channel simulation described in the previous section.
First, suppose that the physical notion of exact channel simulation holds; i.e., the equality in (233) holds for all possible discrimination strategies. Then this means that \( \rho_{R_n B_n} = \tau_{R_n B_n} \) for all possible discrimination strategies. One possible strategy could be to pick the input state for each system \( A_i \) as one of the following states

\[
\rho_{x,y}^{A} = \begin{cases} 
| x \rangle \langle x |_{A} & \text{if } x = y \\
\frac{1}{2} (| x \rangle + | y \rangle) \langle x | + \langle y |) & \text{if } x < y \\
\frac{1}{2} (| x \rangle + i| y \rangle) \langle x | - i\langle y |) & \text{if } x > y
\end{cases}
\]

and the output system \( B_i \) could be measured in the same way, but with respect to an orthonormal basis for the output system. Then all input state choices and measurement outcomes could be stored in auxiliary classical registers. Consider that for all \( x, y \) such that \( x < y \), the following holds

\[
| x \rangle \langle y |_{A} = \left( \rho_{A}^{x,y} - \frac{1}{2} \rho_{A}^{x,x} - \frac{1}{2} \rho_{A}^{y,y} \right) - i \left( \rho_{A}^{y,x} - \frac{1}{2} \rho_{A}^{x,x} - \frac{1}{2} \rho_{A}^{y,y} \right),
\]

\[
| y \rangle \langle x |_{A} = \left( \rho_{A}^{x,y} - \frac{1}{2} \rho_{A}^{x,x} - \frac{1}{2} \rho_{A}^{y,y} \right) + i \left( \rho_{A}^{y,x} - \frac{1}{2} \rho_{A}^{x,x} - \frac{1}{2} \rho_{A}^{y,y} \right),
\]

so that linear combinations of all the outcomes realize the operator basis discussed in the mathematical definition of equivalence. Since the equivalence holds for all possible discrimination strategies, we can collect the data from them in the auxiliary registers, and then finally conclude that the condition in (218) holds.

To see that the mathematical notion of exact sequential simulation implies the physical one, we use the method of post-selected teleportation, essentially the same idea as what was used in the proof of [BSW11, Theorem 4]. Consider the channel defined by the serial composition of the channels in the discrimination strategy \( \{ \rho, \mathcal{A}, Q \} \):

\[
\mathcal{A}_{B^n \rightarrow A^n R_n} = \mathcal{A}_{R_{n-1} B_{n-1} \rightarrow R_n A_n}^{(n-1)} \circ \cdots \circ \mathcal{A}_{R_2 B_2 \rightarrow R_3 A_3}^{(2)} \circ \mathcal{A}_{R_1 B_1 \rightarrow R_2 A_2}^{(1)} \circ \rho_{R_1 A_1},
\]

where the notation \( \rho_{R_1 A_1} \) indicates a preparation channel that tensors in the state \( \rho_{R_1 A_1} \). Figure 6 depicts this channel. By acting on both sides of the exact simulation condition...
FIG. 7: This figure depicts the operator \( A_{B^n A^n} \circ \mathcal{P}_{A^n A_0 B_0} \circ \Gamma_{S^n A^n} \otimes \Phi_{A_0 B_0}^M \) in order to help visualize the argument in (238)–(242). By projecting the systems \( S_1 A_1, S_2 A_2 \) onto \( \langle \Gamma | S_1 A_1 \rangle \), and \( S_3 A_3 \) onto \( \langle \Gamma | S_3 A_3 \rangle \), the method of post-selected teleportation guarantees that the remaining state is \( \tau_{R_3 B_3} \), which is the final state of the bottom part of Figure 5.

with the channel and then the projection onto \( |\Gamma\rangle_{A^n S^n} \), with \( S \simeq R \), we find that

\[
\langle \Gamma | A^n S^n \left[ A_{B^n A^n} \circ \left( \mathcal{N}_{A \rightarrow B} \right)^{\otimes n} \left( \Gamma_{S^n A^n} \right) \right] |\Gamma\rangle_{A^n S^n} = \langle \Gamma | A^n S^n \left[ A_{B^n A^n} \circ \mathcal{P}_{A^n A_0 B_0} \circ \Gamma_{S^n A^n} \otimes \Phi_{A_0 B_0}^M \right] |\Gamma\rangle_{A^n S^n}. \tag{238}
\]

where

\[
|\Gamma\rangle_{A^n S^n} = |\Gamma\rangle_{A_1 S_1} \otimes |\Gamma\rangle_{A_2 S_2} \otimes \cdots \otimes |\Gamma\rangle_{A_n S_n}. \tag{239}
\]

From the method of post-selected teleportation, we conclude that

\[
\langle \Gamma | A^n S^n \left[ A_{B^n A^n} \circ \left( \mathcal{N}_{A \rightarrow B} \right)^{\otimes n} \left( \Gamma_{S^n A^n} \right) \right] |\Gamma\rangle_{A^n S^n} = \rho_{R_n B_n}, \tag{240}
\]

\[
\langle \Gamma | A^n S^n \left[ A_{B^n A^n} \circ \mathcal{P}_{A^n A_0 B_0} \circ \Gamma_{S^n A^n} \otimes \Phi_{A_0 B_0}^M \right] |\Gamma\rangle_{A^n S^n} = \tau_{R_n B_n}. \tag{241}
\]

Putting these together, we finally conclude that

\[
\rho_{R_n B_n} = \tau_{R_n B_n}. \tag{242}
\]

Thus, no physical discrimination strategy can distinguish the original channels from their simulation if the exact simulation condition in (224) holds. Figure 7 depicts the operator \( A_{B^n A^n} \circ \mathcal{P}_{A^n A_0 B_0} \circ \Gamma_{S^n A^n} \otimes \Phi_{A_0 B_0}^M \) in order to help visualize the above argument.

**D. Exact sequential channel simulation cost**

We first establish the following bounds on the \( n \)-shot exact sequential simulation cost:
Proposition 22 Let $\mathcal{N}_{A \rightarrow B}$ be a quantum channel such that $E_{\kappa}(\mathcal{N}) > 0$. Then the $n$-shot exact sequential simulation cost is bounded as

$$\log \left[ 2^{nE_{\kappa}(\mathcal{N})} - 1 \right] \leq E_{\text{PPT}}(\mathcal{N}_{A \rightarrow B}, n) \leq \log \left[ \frac{2^{(n+1)E_{\kappa}(\mathcal{N})} - 1}{2^{E_{\kappa}(\mathcal{N})} - 1} \right].$$

(243)

If $E_{\kappa}(\mathcal{N}) = 0$, then $E_{\text{PPT}}(\mathcal{N}_{A \rightarrow B}, n) = 0$.

Proof. Suppose that $E_{\kappa}(\mathcal{N}) > 0$. The inequality

$$\log \left[ 2^{nE_{\kappa}(\mathcal{N})} - 1 \right] \leq E_{\text{PPT}}(\mathcal{N}_{A \rightarrow B}, n)$$

(244)

is a direct consequence of (227), Proposition 20, and Proposition 15.

So we now prove the other inequality. The main idea behind the construction is for the $i$th completely-PPT-preserving channel to perform the following exact simulation:

$$\mathcal{P}^{(i)}_{A_i A_{i-1} B_{i-1} \rightarrow B_i A_i B_i} (\rho_{A_i} \otimes \Phi^{M_{i-1}}_{A_{i-1} B_{i-1}}) = \mathcal{N}_{A \rightarrow B}(\rho_{A_i}) \otimes \Phi^{M_i}_{A_i B_i},$$

(245)

for $i \in \{1, \ldots, n-1\}$ and for the $n$th completely-PPT-preserving channel to perform the following exact simulation:

$$\mathcal{P}^{(n)}_{A_n A_{n-1} B_{n-1} \rightarrow B_n} (\rho_{A_n} \otimes \Phi^{M_{n-1}}_{A_{n-1} B_{n-1}}) = \mathcal{N}_{A \rightarrow B}(\rho_{A_n}).$$

(246)

Note that, in order to perform the simulation in (245), we could actually simulate the channel $\mathcal{N}_{A \rightarrow B} \otimes \text{id}^{M_i}$, and then send one share of the maximally entangled state $\Phi^{M_i}_{A_i B_i}$ through the exactly simulated identity channel $\text{id}^{M_i}$ to produce the output in (245).

Thus, we should now determine an upper bound on the simulation cost when using this construction. The most effective way to do so is to start from the final ($n$th) simulation. By the one-shot bound from Proposition 20, its cost $\log M_{n-1}$ is bounded as

$$\log M_{n-1} \leq \log \left[ 2^{E_{\kappa}(\mathcal{N})} + 1 \right].$$

(247)

The cost $\log M_{n-2}$ of the $n-1$ simulation is then bounded as

$$\log M_{n-2} \leq \log \left[ 2^{E_{\kappa}(\mathcal{N} \otimes \text{id}^{M_{n-1}})} + 1 \right] \leq \log \left[ 2^{E_{\kappa}(\mathcal{N}) + \log M_{n-1}} + 1 \right] \leq \log \left[ 2^{E_{\kappa}(\mathcal{N})} M_{n-1} + 1 \right] \leq \log \left[ 2^{E_{\kappa}(\mathcal{N})} (2^{E_{\kappa}(\mathcal{N})} + 1) + 1 \right] = \log \left[ \sum_{\ell=0}^{2} 2^{\ell E_{\kappa}(\mathcal{N})} \right],$$

(248)

(249)

(250)

(251)

(252)

where we made use of the subadditivity inequality from Proposition 15. Performing this kind of reasoning iteratively, going backward until the first simulation, we find the following bound:

$$\log M_0 \leq \log \left[ \sum_{\ell=0}^{n} 2^{\ell E_{\kappa}(\mathcal{N})} \right] = \log \left[ \frac{2^{(n+1)E_{\kappa}(\mathcal{N})} - 1}{2^{E_{\kappa}(\mathcal{N})} - 1} \right].$$

(253)

If $E_{\kappa}(\mathcal{N}) = 0$, then the channel $\mathcal{N}$ is PPT entanglement binding by Proposition 17 and thus can be simulated at no cost, so that $E_{\text{PPT}}(\mathcal{N}_{A \rightarrow B}, n) = 0$. This concludes the proof. \qed
Theorem 23 Let $\mathcal{N}_{A \to B}$ be a quantum channel. Then the exact sequential channel simulation cost of $\mathcal{N}_{A \to B}$ is equal to its $\kappa$-entanglement:

$$E_{\text{PPT}}(\mathcal{N}_{A \to B}) = E_\kappa(\mathcal{N}_{A \to B}).$$

(254)

Proof. First suppose that $E_\kappa(\mathcal{N}) > 0$. The lower bound follows from Proposition 22 and Theorem 21. The upper bound follows from Proposition 22:

$$\liminf_{n \to \infty} \frac{1}{n} E_{\text{PPT}}(\mathcal{N}_{A \to B}, n) \leq \liminf_{n \to \infty} \frac{1}{n} \log \left[ \frac{2^{(n+1)E_\kappa(\mathcal{N}) - 1}}{2E_\kappa(\mathcal{N}) - 1} \right]$$

(255)

$$= \liminf_{n \to \infty} \frac{1}{n} \log \left[ \frac{2^{nE_\kappa(\mathcal{N}) - 2 - E_\kappa(\mathcal{N})}}{1 - 2 - E_\kappa(\mathcal{N})} \right]$$

(256)

$$= E_\kappa(\mathcal{N}).$$

(257)

If $E_\kappa(\mathcal{N}) = 0$, then the channel $\mathcal{N}$ is PPT entanglement binding by Proposition 17 and thus can be simulated at no cost. This concludes the proof. ■

By combining Theorems 21 and 23, we reach the conclusion that the exact entanglement cost of parallel and sequential simulation of quantum channels are in fact equal and given by the $\kappa$-entanglement of the channel. Thus, the $\kappa$-entanglement is a fundamental measure of the entanglement of a quantum channel. Not only is it efficiently computable by means of a semi-definite program (for finite-dimensional channels), but it also possesses a direct operational meaning in terms of these channel simulation tasks. It is the only known channel entanglement measure possessing these properties, and from this perspective, it can be helpful in understanding the fundamental structure of entanglement of quantum channels.

E. PPT-simulable channels

Although the theory of exact simulation of quantum channels under PPT operations simplifies significantly due to Theorems 21 and 23, there is a class of channels for which the theory is even simpler. These channels were defined in [KW18] and are known as PPT-simulable channels. In this section, we recall their definition and show how the theory of exact entanglement cost is quite simple for certain PPT-simulable channels.

Definition 4 (PPT-simulable channel [KW18]) A channel $\mathcal{N}_{A \to B}$ is PPT-simulable with associated resource state $\omega_{A'B'}$ if there exists a completely PPT-preserving channel $\mathcal{P}_{AA'B'B'}$ such that, for all input states $\rho_A$

$$\mathcal{N}_{A \to B}(\rho_A) = \mathcal{P}_{AA'B'B'}(\rho_A \otimes \omega_{A'B'}).$$

(258)

A particular kind of PPT-simulable channel is one that is resource-seizable, as defined in [Wil18, Section VI]:

Definition 5 (Resource-seizable [Wil18]) Let $\mathcal{N}_{A \to B}$ be a PPT-simulable channel with associated resource state $\omega_{A'B'}$. The channel $\mathcal{N}_{A \to B}$ is resource-seizable if there exists a PPT state $\tau_{A_MAB_M}$ and a completely PPT-preserving post-processing channel $\mathcal{D}_{A_MBB_M \to A'B'}$ such that

$$\mathcal{D}_{A_MBB_M \to A'B'}(\mathcal{N}_{A \to B}(\tau_{A_MAB_M})) = \omega_{A'B'}. $$

(259)
For PPT-simulable channels, it follows that the exact entanglement cost of sequential channel simulation is bounded from above by the exact entanglement cost of the underlying resource state:

**Theorem 24** Let $\mathcal{N}_{A \rightarrow B}$ be a PPT-simulable channel with associated resource state $\omega_{A'B'}$. Then the PPT-assisted entanglement cost of a channel is bounded from above as

$$E_{\text{PPT}}(N_{A \rightarrow B}) \leq E_{\text{PPT}}(\omega_{A'B'}) = E_\kappa(\omega_{A'B'}).$$  \hfill (260)

**Proof.** The proof for this inequality follows the same reasoning given in [Wil18, Corollary 1]. First simulate a large number of copies of the resource state $\omega_{A'B'}$ and then use the PPT-preserving channel $P_{AA'B' \rightarrow B}$ from (258) to simulate the channel $\mathcal{N}_{A \rightarrow B}$. The equality follows from Proposition 10. ■

If a PPT-simulable channel is additionally resource-seizable, then its exact entanglement cost is given by the $\kappa$-entanglement of the underlying resource state:

**Theorem 25** Let $\mathcal{N}_{A \rightarrow B}$ be a PPT-simulable channel with associated resource state $\omega_{A'B'}$. Suppose furthermore that it is resource-seizable, as given in Definition 5. Then

$$E_{\text{PPT}}(N_{A \rightarrow B}) = E_{\text{PPT}}^{(p)}(N_{A \rightarrow B}) = E_\kappa(N_{A \rightarrow B}) = E_{\text{PPT}}(\omega_{A'B'}) = E_\kappa(\omega_{A'B'}).$$  \hfill (261)

**Proof.** The following inequality

$$E_{\text{PPT}}(N_{A \rightarrow B}) \leq E_{\text{PPT}}(\omega_{A'B'}) = E_\kappa(\omega_{A'B'}).$$  \hfill (262)

is a consequence of Theorem 24. To establish the opposite inequality, consider that we always have that

$$E_{\text{PPT}}^{(p)}(N_{A \rightarrow B}) \geq E_{\text{PPT}}(N_{A \rightarrow B}),$$  \hfill (263)

where $E_{\text{PPT}}^{(p)}$ denotes the exact parallel simulation entanglement cost. From Theorem 21, we have that

$$E_{\text{PPT}}^{(p)}(N_{A \rightarrow B}) = E_\kappa(N_{A \rightarrow B}).$$  \hfill (264)

So it suffices to prove that

$$E_\kappa(N_{A \rightarrow B}) = E_\kappa(\omega_{A'B'}).$$  \hfill (265)

Letting $\rho_{RA}$ be an arbitrary input state, we have that

$$E_\kappa(N_{A \rightarrow B}(\rho_{RA})) = E_\kappa(P_{AA'B' \rightarrow B}(\rho_{RA} \otimes \omega_{A'B'})) \leq E_\kappa(\rho_{RA} \otimes \omega_{A'B'}) \leq E_\kappa(\omega_{A'B'}),$$  \hfill (266)

where the inequality follows from the monotonicity of $E_\kappa$ under PPT-preserving channels and the final equality follows because the bipartite cut is taken as $RAA' |B'$. Since this holds for an arbitrary input state $\rho_{RA}$, we conclude that

$$E_\kappa(\omega_{A'B'}) \geq E_\kappa(N_{A \rightarrow B}).$$  \hfill (269)
Now we prove the opposite inequality, by using the fact that $\mathcal{N}_{A \rightarrow B}$ is resource-seizable. Let $\tau_{ABM}$ be the input PPT state from Definition 5. Consider that

$$E_\kappa(\omega_{A'B'}) = E_\kappa(D_{A'MBBM} \mapsto A'B' \mathcal{N}_{A \rightarrow B}(\tau_{ABM}))$$

(270)

$$\leq E_\kappa(\mathcal{N}_{A \rightarrow B}(\tau_{ABM}))$$

(271)

$$= E_\kappa(\mathcal{N}_{A \rightarrow B}(\tau_{ABM})) - E_\kappa(\tau_{ABM})$$

(272)

$$\leq E_\kappa(\mathcal{N}_{A \rightarrow B}).$$

(273)

The first inequality follows because $E_\kappa$ does not increase under the action of the completely PPT-preserving channel $D_{A'MBBM} \mapsto A'B'$ (Theorem 1). The second equality follows because $\tau_{ABM}$ is a PPT state, so that $E_\kappa(\tau_{ABM}) = 0$. The final inequality is a consequence of the amortization inequality in Proposition 12. ■

F. Relationship to other quantities

A previously known efficiently computable upper bound for quantum capacity is the partial transposition bound [HW01]:

$$Q_\Theta(N) := \log \|N_{A \rightarrow B} \circ T_{B \rightarrow B}\|_{\Diamond},$$

(274)

where $T_{B \rightarrow B}$ is the transpose map and $\| \cdot \|_{\Diamond}$ is the completely bounded trace norm or diamond norm. Note that $\| \cdot \|_{\Diamond}$ for finite-dimensional channels is efficiently computable via semidefinite programming [Wat13].

**Proposition 26** For any quantum channel $\mathcal{N}_{A \rightarrow B}$, we have that

$$Q_\Theta(N_{A \rightarrow B}) \leq E_\kappa(N_{A \rightarrow B}).$$

(275)

**Proof.** Given any quantum channel $\mathcal{N}_{A \rightarrow B}$, it holds that

$$E_\kappa(N_{A \rightarrow B}) = \sup_{\phi_{RA}} E_\kappa(N_{A \rightarrow B}(\phi_{RA}))$$

(276)

$$\geq \sup_{\phi_{RA}} E_N(N_{A \rightarrow B}(\phi_{RA}))$$

(277)

$$= \sup_{\phi_{RA}} \log \|N_{A \rightarrow B}(\phi_{RA})^{T_B}\|_1$$

(278)

$$= \log \|N_{A \rightarrow B} \circ T_{B \rightarrow B}\|_{\Diamond}.$$

(279)

The equality in (276) follows from Proposition 11. The inequality in (277) follows from the property of $E_\kappa$ in Proposition 3. The last equality follows due to the definition of the completely bounded trace norm. ■

**Remark 2** For qubit-input qubit-output channels, we have that

$$E_\kappa(N_{A \rightarrow B}) = Q_\Theta(N_{A \rightarrow B}).$$

(280)

This follows because it suffices to optimize $E_\kappa(N_{A \rightarrow B})$ with respect to two-qubit input states $\phi_{RA}$, and then the output state consists of two qubits, so that the result of [Ish04] applies. That is, for this case,

$$E_\kappa(N_{A \rightarrow B}) = \sup_{\phi_{RA}} E_\kappa(N_{A \rightarrow B}(\phi_{RA})) = \sup_{\phi_{RA}} E_N(N_{A \rightarrow B}(\phi_{RA})) = Q_\Theta(N_{A \rightarrow B}).$$

(281)
VI. EXACT ENTANGLEMENT COST OF FUNDAMENTAL CHANNELS

Theorem 25 provides a formula for the exact PPT-entanglement cost of any resource-seizable, PPT-simulable channel, given in terms of the entanglement cost of the underlying resource state \( \omega_{A'B'} \). We detail some simple examples here for which this simplified formula applies. We also consider amplitude damping channels, for which it is necessary to invoke Theorems 21 and 23 in order to determine their exact entanglement costs.

Let us begin by recalling the notion of a covariant channel \( N_{A\rightarrow B} \) [Hol02]. For a group \( G \) with unitary channel representations \( \{U^g_A\}_g \) and \( \{V^g_B\}_g \) acting on the input system \( A \) and output system \( B \) of the channel \( N_{A\rightarrow B} \), the channel \( N_{A\rightarrow B} \) is covariant with respect to the group \( G \) if the following equality holds

\[
N_{A\rightarrow B} \circ U^g_A = V^g_B \circ N_{A\rightarrow B}.
\]

(282)

If the averaging channel is such that \( \frac{1}{|G|} \sum_g U^g_A(X) = \text{Tr}[X]/|A| \), then we simply say that the channel \( N_{A\rightarrow B} \) is covariant.

Then from [CDP09a, Section 7], we conclude that any covariant channel is PPT-simulable with associated resource state given by the Choi state of the channel, i.e., \( \omega_{A'B'} = N_{A\rightarrow B}(\Phi_{A'A}) \). As such, covariant channels are resource-seizable, so that the equality in Theorem 25 applies to all covariant channels. Thus, the exact entanglement cost of a covariant channel is equal to the exact entanglement cost of its Choi state.

A. Erasure channel

The quantum erasure channel is denoted by

\[
E_p(\rho) = (1 - p)\rho + p|e\rangle\langle e|,
\]

(283)

where \( \rho \) is a \( d \)-dimensional input state, \( p \in [0, 1] \) is the erasure probability, and \( |e\rangle\langle e| \) is a pure erasure state orthogonal to any input state, so that the output state has \( d + 1 \) dimensions. This channel is covariant.

The Choi matrix of \( E_p \) is given by

\[
J_{E_p} = (1 - p) \sum_{i,j=0}^{d-1} |ii\rangle\langle jj| + p \sum_{i=0}^{d-1} |i\rangle\langle i| \otimes |e\rangle\langle e|.
\]

(284)

By direct calculation, we find that

\[
E_{\text{PPT}}(E_p) = E_{\text{PPT}}(J_{E_p}/d) = E_N(J_{E_p}/d) = \log(d[1 - p] + p).
\]

(285)

B. Depolarizing channel

Consider the qudit depolarizing channel:

\[
N_{D,p}(\rho) = (1 - p)\rho + \frac{p}{d^2 - 1} \sum_{0 \leq i,j \leq d-1 \atop (i,j) \neq (0,0)} X^i Z^j \rho (X^i Z^j)^{\dagger},
\]

(286)
where \( p \in [0, 1] \) and \( X, Z \) are the generalized Pauli operators. This channel is covariant.

The Choi matrix of \( \mathcal{N}_{D,p} \) is

\[
J_{\mathcal{N}_{D,p}} = d \left[ (1 - p)\Phi_{AB} + \frac{p}{d^2 - 1} (\mathbb{1}_{AB} - \Phi_{AB}) \right],
\]

where \( \Phi = \frac{1}{d} \sum_{i,j=0}^{d-1} |ii\rangle\langle jj| \). Observe that the state \( J_{\mathcal{N}_{D,p}} \) is an isotropic state. Applying the previous result from (97), we conclude that

\[
E_{\text{PPT}}(\mathcal{N}_{D,p}) = \begin{cases} 
\log d(1-p) & \text{if } 1 - p \geq \frac{1}{d} \\
0 & \text{if } 1 - p < \frac{1}{d}
\end{cases}
\]

(288)

C. Dephasing channel

The qubit dephasing channel is given as

\[
\mathcal{D}_q(\rho) = (1 - q)\rho + qZ\rho Z.
\]

(289)

Note that this channel is covariant with respect to the Heisenberg–Weyl group of unitaries. The Choi matrix of \( \mathcal{D}_q \) is as follows:

\[
J_{\mathcal{D}_q} = 2[(1 - q)\psi_1 + q\psi_2],
\]

(290)

where

\[
|\psi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),
\]

(291)

\[
|\psi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle).
\]

(292)

By direct calculation, we find that

\[
E_{\text{PPT}}(\mathcal{D}_q) = E_{\text{PPT}}(J_{\mathcal{D}_q}/2) = E_N(J_{\mathcal{D}_q}/2) = \log(1 + 2|q - 1/2|).
\]

(293)

We note that this approach also works for a \( d \)-dimensional dephasing channel.

D. Amplitude damping channel

An amplitude damping channel corresponds to the process of asymmetric relaxation in a quantum system, which is a key noise process in quantum information science. The qubit amplitude damping channel is given as \( \mathcal{N}_{AD,r} = \sum_{i=0}^1 E_i \cdot E_i^\dagger \) with

\[
E_0 = |0\rangle\langle 0| + \sqrt{1 - r}|1\rangle\langle 1|,
\]

(294)

\[
E_1 = \sqrt{r}|0\rangle\langle 1|.
\]

(295)

This channel is covariant with respect to \( \{I, Z\} \), but not with respect to a one-design. So Theorem 25 does not apply, and we instead need to evaluate the exact entanglement cost of this channel by applying Theorems 21 and 23.

We plot \( E_{\text{PPT}}(\mathcal{N}_{AD,r}) \) in Figure 8 and compare it with the max-Rains information of [WD16b, WFD17]. The fact that there is a gap between these two quantities demonstrates that the resource theory of entanglement (exact PPT case) is irreversible, given that the max-Rains information is an upper bound on the exact distillable entanglement of an arbitrary channel [BW18].
VII. EXACT ENTANGLEMENT COST OF QUANTUM GAUSSIAN CHANNELS

In this subsection, we determine formulas for the exact entanglement cost of particular quantum Gaussian channels, which include all single-mode bosonic Gaussian channels with the exception of the pure-loss and pure-amplifier channels. In this sense, the results found here are complementary to those found recently in [Wil18, Theorem 2]. The presentation and background given in this section largely follows that given recently in [Wil18].

A. Preliminary observations about the exact entanglement cost of single-mode bosonic Gaussian channels

The starting point for our analysis of single-mode bosonic Gaussian channels is the Holevo classification from [Hol07], in which canonical forms for all single-mode bosonic Gaussian channels have been given, classifying them up to local Gaussian unitaries acting on the input and output of the channel. It then suffices for us to focus our attention on the canonical forms, as it is self-evident from definitions that local unitaries do not alter the exact entanglement cost of a quantum channel. The thermal and amplifier channels form the class C discussed in [Hol07], and the additive-noise channels form the class $B_2$ discussed in the same work. The classes that remain are labeled $A$, $B_1$, and $D$ in [Hol07]. The channels in $A$ and $D$ are entanglement-breaking [Hol08], and are thus entanglement-binding, and as a consequence of Proposition 17 and Theorems 21 and 23, they have zero exact entanglement cost. Channels in the class $B_1$ are perhaps not interesting for practical applications, and as it turns out, they have infinite quantum capacity [Hol07]. Thus, their exact entanglement cost is also infinite, because a channel’s quantum capacity is a lower bound on its distillable entanglement, which is in turn a lower bound on its partial transposition bound. The partial transposition bound is finally a lower bound on its $\kappa$-entanglement, as shown in Proposition 26. For the same reason, the exact entanglement cost of the bosonic identity
channel is also infinite.

B. Thermal, amplifier, and additive-noise bosonic Gaussian channels

In light of the previous discussion, for the remainder of this section, let us focus our attention on the thermal, amplifier, and additive-noise channels. Each of these are defined respectively by the following Heisenberg input-output relations:

\[
\hat{b} = \sqrt{\eta} \hat{a} + \sqrt{1 - \eta} \hat{e}, \quad (296)
\]

\[
\hat{b} = \sqrt{G} \hat{a} + \sqrt{G - 1} \hat{e}^\dagger, \quad (297)
\]

\[
\hat{b} = \hat{a} + (x + ip) / \sqrt{2}, \quad (298)
\]

where \(\hat{a}, \hat{b},\) and \(\hat{e}\) are the field-mode annihilation operators for the sender’s input, the receiver’s output, and the environment’s input of these channels, respectively.

The channel in (296) is a thermalizing channel, in which the environmental mode is prepared in a thermal state \(\theta(N_B)\) of mean photon number \(N_B \geq 0\), defined as

\[
\theta(N_B) \equiv \frac{1}{N_B + 1} \sum_{n=0}^{\infty} \left( \frac{N_B}{N_B + 1} \right)^n |n\rangle \langle n|, \quad (299)
\]

where \(\{|n\rangle\}_{n=0}^{\infty}\) is the orthonormal, photonic number-state basis. When \(N_B = 0\), \(\theta(N_B)\) reduces to the vacuum state, in which case the resulting channel in (296) is called the pure-loss channel—it is said to be quantum-limited in this case because the environment is injecting the minimum amount of noise allowed by quantum mechanics. The parameter \(\eta \in (0, 1)\) is the transmissivity of the channel, representing the average fraction of photons making it from the input to the output of the channel. Let \(\mathcal{L}_{\eta,N_B}\) denote this channel, and we make the further abbreviation \(\mathcal{L}_\eta \equiv \mathcal{L}_{\eta,N_B=0}\) when it is the pure-loss channel. The channel in (296) is entanglement-breaking when \((1 - \eta) N_B \geq \eta\) \cite{Hol08}, and is thus entanglement-binding, and as a consequence of Proposition 17 and Theorems 21 and 23, it has zero exact entanglement cost for these values.

The channel in (297) is an amplifier channel, and the parameter \(G > 1\) is its gain. For this channel, the environment is prepared in the thermal state \(\theta(N_B)\). If \(N_B = 0\), the amplifier channel is called the pure-amplifier channel—it is said to be quantum-limited for a similar reason as stated above. Let \(\mathcal{A}_{G,N_B}\) denote this channel, and we make the further abbreviation \(\mathcal{A}_G \equiv \mathcal{A}_{G,N_B=0}\) when it is the quantum-limited amplifier channel. The channel in (297) is entanglement-breaking when \((G - 1) N_B \geq 1\) \cite{Hol08}, and is thus entanglement-binding, and as a consequence of Proposition 17 and Theorems 21 and 23, it has zero exact entanglement cost for these values.

Finally, the channel in (298) is an additive-noise channel, representing a quantum generalization of the classical additive white Gaussian noise channel. In (298), \(x\) and \(p\) are zero-mean, independent Gaussian random variables each having variance \(\xi \geq 0\). Let \(\mathcal{T}_\xi\) denote this channel. The channel in (298) is entanglement-breaking when \(\xi \geq 1\) \cite{Hol08}, and is thus entanglement-binding, and as a consequence of Proposition 17 and Theorems 21 and 23, it has zero exact entanglement cost for these values.

Kraus representations for the channels in (296)–(298) are available in \cite{ISS11}, which can be helpful for further understanding their action on input quantum states.
Due to the entanglement-breaking regions discussed above, we are left with a limited range of single-mode bosonic Gaussian channels to consider, which is delineated by the white strip in Figure 1 of [GGPCH14].

C. Exact entanglement cost of thermal, amplifier, and additive-noise bosonic Gaussian channels

We can now state our main result for this section, which applies to all thermal, amplifier, and additive-noise channels that are neither entanglement-breaking nor quantum-limited:

**Theorem 27** For a thermal channel $L_{\eta,N_B}$ with transmissivity $\eta \in (0,1)$ and thermal photon number $N_B \in (0,\eta/[1-\eta])$, an amplifier channel $A_{G,N_B}$ with gain $G > 1$ and thermal photon number $N_B \in (0,1/[G-1])$, and an additive-noise channel $T_\xi$ with noise variance $\xi \in (0,1)$, the following formulas characterize the exact entanglement costs of these channels:

$$E_{\text{PPT}}(L_{\eta,N_B}) = E_{\text{PPT}}^{(p)}(L_{\eta,N_B}) = \log\left(\frac{1+\eta}{(1-\eta)(2N_B+1)}\right),$$  

$$E_{\text{PPT}}(A_{G,N_B}) = E_{\text{PPT}}^{(p)}(A_{G,N_B}) = \log\left(\frac{G+1}{(G-1)(2N_B+1)}\right),$$  

$$E_{\text{PPT}}(T_\xi) = E_{\text{PPT}}^{(p)}(T_\xi) = \log(1/\xi).$$

**Proof.** To arrive at the following inequalities:

$$E_{\text{PPT}}(L_{\eta,N_B}) \leq \log\left(\frac{1+\eta}{(1-\eta)(2N_B+1)}\right),$$  

$$E_{\text{PPT}}(A_{G,N_B}) \leq \log\left(\frac{G+1}{(G-1)(2N_B+1)}\right),$$  

$$E_{\text{PPT}}(T_\xi) \leq \log(1/\xi),$$

we apply Proposition 24, along with some recent developments, to the single-mode thermal, amplifier, and additive-noise channels that are neither entanglement-breaking nor quantum-limited. Some recent papers [LSMGA17, KW17, TDR18] have shown how to simulate each of these channels by using a bosonic Gaussian resource state along with variations of the continuous-variable quantum teleportation protocol [BK98]. Of these works, the one most relevant for us is the original one [LSMGA17], because these authors proved that the logarithmic negativity of the underlying resource state is equal to the logarithmic negativity that results from transmitting through the channel one share of a two-mode squeezed vacuum state with arbitrarily large squeezing strength. That is, let $\mathcal{N}_{A\rightarrow B}$ denote a single-mode thermal, amplifier, or additive-noise channel. Then one of the main results of [LSMGA17] is that, associated to this channel, there is a bosonic Gaussian resource state $\omega_{A'B'}$ and a Gaussian LOCC channel $G_{A'A'B'\rightarrow B}$ such that

$$E_N(\omega_{A'B'}) = \sup_{N_S \geq 0} E_N(\sigma_{R'B}^{N_S}) = \lim_{N_S \rightarrow \infty} E_N(\sigma_{R'B}^{N_S}),$$
where
\[ \sigma_{RB}^{N_S} \equiv \mathcal{N}_{A\to B}(\phi_{RA}^{N_S}), \quad (308) \]
\[ \phi_{RA}^{N_S} \equiv |\phi^{N_S}\rangle \langle \phi^{N_S}|_{RA}, \quad (309) \]
\[ |\phi^{N_S}\rangle_{RA} \equiv \frac{1}{\sqrt{N_S + 1}} \sum_{n=0}^{\infty} \sqrt{\left( \frac{N_S}{N_S + 1} \right)^n} |n\rangle_R |n\rangle_A, \quad (310) \]
and for all input states \( \rho_A \),
\[ \mathcal{N}_{A\to B}(\rho_A) = G_{AA'B'\to B}(\rho_A \otimes \omega_{A'B'}). \quad (311) \]

In the above, \( \phi_{RA}^{N_S} \) is the two-mode squeezed vacuum state [Ser17]. Note that the equality in (307) holds because one can always produce \( \phi_{RA}^{N_S} \) from \( \phi_{RA}^{N_S'} \) such that \( N_S' \geq N_S \), by using Gaussian LOCC and the local displacements involved in the Gaussian LOCC commute with the channel \( \mathcal{N}_{A\to B} \) [GEC03] (whether it be thermal, amplifier, or additive-noise). Furthermore, the logarithmic negativity does not increase under the action of an LOCC channel.

Thus, applying the above observations and Proposition 24, it follows that there exist bosonic Gaussian resource states \( \omega_{\eta,N_S}^{A'B'} \), \( \omega_{G,N_S}^{A'B'} \), and \( \omega_{\xi}^{A'B'} \) associated to the respective thermal, amplifier, and additive-noise channels in (296)–(298), such that the following inequalities hold
\[ E_{\text{PPT}}(\mathcal{L}_{\eta,N_B}) \leq E_{\kappa}(\omega_{\eta,A'}^{N_B}) = E_{\kappa}(\omega_{A'}^{\eta,N_B}) = \log \left( \frac{1 + \eta}{(1 - \eta)(2N_B + 1)} \right), \quad (312) \]
\[ E_{\text{PPT}}(\mathcal{A}_{G,N_B}) \leq E_{\kappa}(\omega_{G,A'}^{N_B}) = E_{\kappa}(\omega_{A'}^{G,N_B}) = \log \left( \frac{G + 1}{(G - 1)(2N_B + 1)} \right), \quad (313) \]
\[ E_{\text{PPT}}(\mathcal{T}_{\xi}) \leq E_{\kappa}(\omega_{\xi}^{A'B'}) = E_{\kappa}(\omega_{A'}^{\xi,N_B}) = \log(1/\xi). \quad (314) \]
where the first equalities in each line follow because \( E_{\kappa} = E_N \) for bosonic Gaussian states (see (31) and [APE03]), and the explicit formulas on the right-hand side are found in [HW01, LSMGA17].

On the other hand, Theorems 21 and 23 imply that
\[ E_{\text{PPT}}(\mathcal{L}_{\eta,N_B}) = E_{\text{PPT}}^{(\rho)}(\mathcal{L}_{\eta,N_B}) \quad (315) \]
\[ \geq \lim_{N_S \to \infty} E_N(\sigma_{\eta,N_B}^{N_S}(N_S)_{RB}) \quad (316) \]
\[ = \log \left( \frac{1 + \eta}{(1 - \eta)(2N_B + 1)} \right), \quad (317) \]
\[ E_{\text{PPT}}(\mathcal{A}_{G,N_B}) = E_{\text{PPT}}^{(\rho)}(\mathcal{A}_{G,N_B}) \quad (318) \]
\[ \geq \lim_{N_S \to \infty} E_N(\sigma_{G,N_B}^{N_S}(N_S)_{RB}) \quad (319) \]
\[ = \log \left( \frac{G + 1}{(G - 1)(2N_B + 1)} \right), \quad (320) \]
\[ E_{\text{PPT}}(\mathcal{T}_{\xi}) = E_{\text{PPT}}^{(\rho)}(\mathcal{T}_{\xi}) \quad (321) \]
\[ \geq \lim_{N_S \to \infty} E_N(\sigma_{\xi}^{N_S}(N_S)_{RB}) \quad (322) \]
\[ = \log(1/\xi). \quad (323) \]
Combining the inequalities above, we conclude the statement of the theorem.

The significance of Theorem 27 above is that it establishes a clear operational meaning of the Holevo–Werner quantity \([\text{HW01}]\) (partial transposition bound) for the basic bosonic channels that are not quantum limited. This quantity has been used for a variety of purposes in prior work, as an upper bound on unassisted quantum capacity \([\text{HW01}]\), as an upper bound on LOCC-assisted quantum capacity \([\text{MHRW16}]\), as a tool in arriving at a no-go theorem for Gaussian quantum error correction \([\text{NFC09}]\), and as a tool in the teleportation simulation of bosonic Gaussian channels \([\text{LSMGA17}]\). Finally, Theorem 27 solves the long-standing open problem of giving the Holevo–Werner quantity a direct operational meaning for the basic bosonic channels, in terms of exact entanglement cost of parallel and sequential channel simulation.

In light of the results stated in Theorem 27, it is quite natural to conjecture that the following formulas hold for the pure-loss and pure-amplifier channels with \(\eta \in (0, 1)\) and \(G > 1\), respectively:

\[
E_{\text{PPT}}(L_\eta) = E_{\text{PPT}}^{(p)}(L_\eta) \overset{?}{=} \log \left( \frac{1 + \eta}{1 - \eta} \right),
\]

\[
E_{\text{PPT}}(A_G) = E_{\text{PPT}}^{(p)}(A_G) \overset{?}{=} \log \left( \frac{G + 1}{G - 1} \right).
\]  

(324)  

(325)

Theorems 21 and 23 imply that the following inequalities hold

\[
E_{\text{PPT}}(L_\eta) = E_{\text{PPT}}^{(p)}(L_\eta) \geq \log \left( \frac{1 + \eta}{1 - \eta} \right),
\]

\[
E_{\text{PPT}}(A_G) = E_{\text{PPT}}^{(p)}(A_G) \geq \log \left( \frac{G + 1}{G - 1} \right).
\]  

(326)  

(327)

However, what excludes us from making a rigorous statement about the opposite inequalities is the lack of a legitimate quantum state that can be used to simulate these channels exactly, as was the case for the channels considered in Theorem 27. For example, it is not clear that we could simply “plug in” the “EPR state” (i.e., the limiting object \(\lim_{N_S \to \infty} \phi_{RA}^{N_S}\)) and use the teleportation simulation argument as before. There are several issues: the limiting object is not actually a state, any finite squeezing leads to a slight error or inexact simulation, and the logarithmic negativity is not known to be continuous. In spite of these obstacles, we think that it is highly plausible that the equalities in (324)–(325) hold. More generally, based on the results of \([\text{NFC09}]\), we suspect that the following equality holds for an arbitrary Gaussian channel \(\mathcal{N}\) described by a scaling matrix \(X\) and a noise matrix \(Y\) \([\text{Ser17}]\):

\[
E_{\text{PPT}}(\mathcal{N}) \overset{?}{=} Q_{\Theta}(\mathcal{N}) \overset{?}{=} \frac{1}{2} \log \min \left\{ \frac{(1 + \det X)^2}{\det Y}, 1 \right\}.
\]  

(328)

VIII. CONCLUSION

In the zoo of entanglement measures \([\text{HHHH09, Chr06, PV07}]\), the \(\kappa\)-entanglement of a bipartite state is the first entanglement measure that is efficiently computable while possessing a direct operational meaning for general bipartite states. This unique feature of \(E_\kappa\) may help us better understand the structure and power of quantum entanglement. As
a generalization of this notion, the $\kappa$-entanglement of a quantum channel is also efficiently computable while possessing a direct operational meaning as the entanglement cost for exact parallel and sequential simulation of a quantum channel.

Going forward from here, the most pressing open question is to determine whether the formula in (328) holds, for the exact entanglement cost of quantum Gaussian channels. One could potentially require new methods beyond the scope of this paper in order to establish (328).

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Appendix A: Equality of $E_\kappa$ and $E^{\text{dual}}_\kappa$ for states acting on separable Hilbert spaces

In this appendix, we prove that

$$E_\kappa(\rho_{AB}) = E^{\text{dual}}_\kappa(\rho_{AB}),$$

(A1)

for a state $\rho_{AB}$ acting on a separable Hilbert space. To begin with, let us recall that the following inequality always holds from weak duality

$$E_\kappa(\rho_{AB}) \geq E^{\text{dual}}_\kappa(\rho_{AB}).$$

(A2)

So our goal is to prove the opposite inequality. We suppose throughout that $E^{\text{dual}}_\kappa(\rho_{AB}) < \infty$. Otherwise, the desired equality in (A1) is trivially true. We also suppose that $\rho_{AB}$ has full support. Otherwise, it is finite-dimensional and the desired equality in (A1) is trivially true.

To this end, consider sequences $\{\Pi_{A}^k\}_k$ and $\{\Pi_{B}^k\}_k$ of projectors weakly converging to the identities $1_A$ and $1_B$ and such that $\Pi_{A}^k \leq \Pi_{A}^{k'}$ and $\Pi_{B}^k \leq \Pi_{B}^{k'}$ for $k' \geq k$. Furthermore, we suppose that $[\Pi_{B}^k]^T_B = \Pi_{B}^k$ for all $k$. Then define

$$\rho_{AB}^k := (\Pi_{A}^k \otimes \Pi_{B}^k) \rho_{AB} (\Pi_{A}^k \otimes \Pi_{B}^k).$$

(A3)

It follows that [Del67]

$$\lim_{k \to \infty} \|\rho_{AB} - \rho_{AB}^k\|_1 = 0.$$  

(A4)

We now prove that

$$E^{\text{dual}}_\kappa(\rho_{AB}) \geq E^{\text{dual}}_\kappa(\rho_{AB}^k),$$

(A5)

for all $k$. Let $A^k$ and $B^k$ denote the subspaces onto which $\Pi_{A}^k$ and $\Pi_{B}^k$ project. Let $V_{A^kB^k}$ and $W_{A^kB^k}$ be arbitrary operators satisfying $V_{A^kB^k}^k + W_{A^kB^k}^k \leq 1_{A^kB^k} = (\Pi_{A}^k \otimes \Pi_{B}^k)$, $[V_{A^kB^k}^k]^T_B, [W_{A^kB^k}^k]^T_B \geq 0$. Set

$$\overline{V}_{AB}^k := (\Pi_{A}^k \otimes \Pi_{B}^k) V_{A^kB^k}^k (\Pi_{A}^k \otimes \Pi_{B}^k),$$

(A6)

$$\overline{W}_{AB}^k := (\Pi_{A}^k \otimes \Pi_{B}^k) W_{A^kB^k}^k (\Pi_{A}^k \otimes \Pi_{B}^k).$$

(A7)
and note that
\[ V_{AB}^k + W_{AB}^k \leq 1_{AB}, \] (A8)
\[ [V_{AB}^k]_{TB}^T, [W_{AB}^k]_{TB}^T \geq 0. \] (A9)

Then
\[
\begin{align*}
\text{Tr} \rho_{AB}^k (V_{AB}^k - W_{AB}^k) &= \text{Tr} (\Pi_A^k \otimes \Pi_B^k) \rho_{AB} (\Pi_A^k \otimes \Pi_B^k) (V_{AB}^k - W_{AB}^k) \\
&= \text{Tr} \rho_{AB} (\Pi_A^k \otimes \Pi_B^k) (V_{AB}^k - W_{AB}^k) (\Pi_A^k \otimes \Pi_B^k) \\
&\leq E_{\kappa}^{\text{dual}}(\rho_{AB}).
\end{align*}
\] (A10)

Since the inequality holds for arbitrary \( V_{AB}^k \) and \( W_{AB}^k \) satisfying the conditions above, we conclude the inequality in (A5).

Thus, we conclude that
\[
E_{\kappa}^{\text{dual}}(\rho_{AB}) \geq \limsup_{k \to \infty} E_{\kappa}^{\text{dual}}(\rho_{AB}^k).
\] (A11)

Now let us suppose that \( E_{\kappa}^{\text{dual}}(\rho_{AB}) < \infty \). Then for all \( V_{AB} \) and \( W_{AB} \) satisfying \( V_{AB} + W_{AB} \leq 1_{AB}, [V_{AB}]_{TB}^T, [W_{AB}]_{TB}^T \geq 0 \), as well as \( \text{Tr} \rho_{AB}(V_{AB} - W_{AB}) \geq 0 \), we have that
\[
\text{Tr} \rho_{AB}(V_{AB} - W_{AB}) < \infty.
\] (A12)

Since \( \rho_{AB} \) has full support, this means that
\[
\|V_{AB} - W_{AB}\|_{\infty} < \infty.
\] (A13)

Considering that from Hölder’s inequality
\[
\left| \text{Tr}(\rho_{AB} - \rho_{AB}^k)(V_{AB} - W_{AB}) \right| \leq \|\rho_{AB} - \rho_{AB}^k\|_1 \|V_{AB} - W_{AB}\|_{\infty},
\] (A14)
and setting
\[
V_{AB}^k := (\Pi_A^k \otimes \Pi_B^k) V_{AB} (\Pi_A^k \otimes \Pi_B^k),
\] (A15)
\[
W_{AB}^k := (\Pi_A^k \otimes \Pi_B^k) W_{AB} (\Pi_A^k \otimes \Pi_B^k),
\] (A16)
we conclude that
\[
\text{Tr} \rho_{AB}(V_{AB} - W_{AB}) \leq \liminf_{k \to \infty} \text{Tr} \rho_{AB}^k(V_{AB} - W_{AB})
\] (A17)
\[
= \liminf_{k \to \infty} \text{Tr} \rho_{AB}^k(V_{AB}^k - W_{AB}^k)
\] (A18)
\[
\leq \liminf_{k \to \infty} \sup_{V_{AB}^k, W_{AB}^k} \text{Tr} \rho_{AB}^k(V_{AB}^k - W_{AB}^k)
\] (A19)
\[
= \liminf_{k \to \infty} E_{\kappa}^{\text{dual}}(\rho_{AB}).
\] (A20)

Since the inequality holds for arbitrary \( V_{AB} \) and \( W_{AB} \) satisfying the above conditions, we conclude that
\[
E_{\kappa}^{\text{dual}}(\rho_{AB}) \leq \liminf_{k \to \infty} E_{\kappa}^{\text{dual}}(\rho_{AB}^k).
\] (A21)
Putting together (A14) and (A24), we conclude that
\[ E^\text{dual}_\kappa(\rho_{AB}) = \lim_{k \to \infty} E^\text{dual}_\kappa(\rho^k_{AB}). \] (A25)

From strong duality for the finite-dimensional case, we have for all \( k \) that
\[ E^\text{dual}_\kappa(\rho^k_{AB}) = E_\kappa(\rho^k_{AB}), \] (A26)
and thus that
\[ \lim_{k \to \infty} E^\text{dual}_\kappa(\rho^k_{AB}) = \lim_{k \to \infty} E_\kappa(\rho^k_{AB}). \] (A27)

It thus remains to prove that
\[ \lim_{k \to \infty} E_\kappa(\rho^k_{AB}) = E_\kappa(\rho_{AB}). \] (A28)

We first prove that
\[ E_\kappa(\rho_{AB}) \geq \limsup_{k \to \infty} E_\kappa(\rho^k_{AB}). \] (A29)

Let \( S_{AB} \) be an arbitrary operator satisfying
\[ S_{AB} \geq 0, \quad -S_{AB}^{T_B} \leq \rho_{AB}^{T_B} \leq S_{AB}^{T_B}. \] (A30)

Then, defining \( S^k_{AB} = (\Pi^k_A \otimes \Pi^k_B) S_{AB} (\Pi^k_A \otimes \Pi^k_B) \), we have that
\[ S^k_{AB} \geq 0, \quad -[S^k_{AB}]^{T_B} \leq [\rho^k_{AB}]^{T_B} \leq [S^k_{AB}]^{T_B}. \] (A31)

Then
\[ \log \text{Tr } S_{AB} \geq \log \text{Tr } S^k_{AB} \geq E_\kappa(\rho^k_{AB}). \] (A32)

Since the inequality holds for all \( S_{AB} \) satisfying (A30), we conclude that
\[ E_\kappa(\rho_{AB}) \geq E_\kappa(\rho^k_{AB}) \] (A33)
for all \( k \), and thus (A29) holds.

The rest of the proof follows [FAR11] closely. Since the condition \( \Pi^k_A \leq \Pi^{k'}_A \) and \( \Pi^k_B \leq \Pi^{k'}_B \)
for \( k' \geq k \) holds, in fact the same sequence of steps as above allows for concluding that
\[ E_\kappa(\rho^k_{AB}) \geq E_\kappa(\rho^k_{AB}), \] (A34)
meaning that the sequence is monotone non-decreasing with \( k \). Thus, we can define
\[ \mu := \lim_{k \to \infty} E_\kappa(\rho^k_{AB}) \in \mathbb{R}^+, \] (A35)
and note from the above that
\[ \mu \leq E_\kappa(\rho_{AB}). \] (A36)

For each \( k \), let \( S^k_{AB} \) denote an optimal operator such that \( E_\kappa(\rho^k_{AB}) = \log \text{Tr } S^k_{AB} \). From the fact that \( S^k_{AB} \geq 0 \), and \( \text{Tr } S^k_{AB} \leq 2^n \), we conclude that \( \{S^k_{AB}\}_{k} \) is a bounded sequence in the trace class operators. Since the trace class operators form the dual space of the compact operators \( \mathcal{K}(\mathcal{H}_{AB}) \) [RS78], we can apply the Banach–Alaoglu theorem [RS78] to find a subsequence \( \{S^k_{AB}\}_{k \in \Gamma} \) with a weak* limit \( \tilde{S}_{AB} \) in the trace class operators such that \( \tilde{S}_{AB} \geq 0 \).
and \( \text{Tr}[\tilde{S}_{AB}] \leq 2^n \). Furthermore, the sequences \( [\tilde{\rho}^A_{AB}]^T_B + [\tilde{S}^A_{AB}]^T_B \) and \( [S^A_{AB}]^T_B - [\tilde{\rho}^A_{AB}]^T_B \) converge in the weak operator topology to \( \rho^T_B + \tilde{S}^T_B \) and \( \tilde{S}^T_B - \rho^T_B \), respectively, and we can then conclude that \( \rho^T_B + \tilde{S}^T_B, \tilde{S}^T_B - \rho^T_B \geq 0 \). But this means that

\[
E_\kappa(\rho_{AB}) \leq \log \text{Tr} \tilde{S}_{AB} \leq \mu,
\]

which implies that

\[
E_\kappa(\rho_{AB}) \leq \liminf_{k \to \infty} E_\kappa(\rho_{AB}^k).
\]

Putting together (A29) and (A38), we conclude that

\[
E_\kappa(\rho_{AB}) = \lim_{k \to \infty} E_\kappa(\rho_{AB}^k).
\]

Finally, putting together (A25), (A27), and (A39), we conclude (A1).
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I. INTRODUCTION

The resource theory of entanglement [BDSW96] has been one of the richest contributions to quantum information theory [Hol12, Hay06, Wil17, Wat18], and these days, the seminal ideas coming from it are influencing diverse areas of physics [CG18]. A fundamental question in entanglement theory is to determine the smallest rate at which Bell states (or ebits) are needed, along with the assistance of free classical communication, in order to generate \( n \) copies of an arbitrary bipartite state \( \rho_{AB} \) reliably (in this introduction, \( n \) should be understood to be an arbitrarily large number) [BDSW96]. The optimal rate is known as the entanglement cost of \( \rho_{AB} \) [BDSW96], and a formal expression is known for this quantity in terms of a regularization of the entanglement of formation [HHT01]. An upper bound in terms of entanglement of formation has been known for some time [BDSW96, HHT01], while a lower bound has been determined recently [WD17]. Conversely, a related fundamental question is to determine the largest rate at which one can distill ebits reliably from \( n \) copies of \( \rho_{AB} \), again with the assistance of free classical communication [BDSW96]. This optimal rate is known as the distillable entanglement, and various lower bounds [DW05] and upper bounds [Rai99, Rai01, CW04, WD16] are known for it.

The above resource theory is quite rich and interesting, but soon after learning about it, one might immediately question its operational significance. How are the bipartite states \( \rho_{AB} \) established in the first place? Of course, a quantum communication channel, such as a fiber-optic or free-space link, is required. Consequently, in the same paper that introduced the resource theory of entanglement [BDSW96], the authors there appreciated the relevance of this point and proposed that the distillation question could be extended to quantum channels. The distillation question for channels is then as follows: given \( n \) uses of a quantum channel \( \mathcal{N}_{AB} \) connecting a sender Alice to a receiver Bob, along with the assistance of free classical communication, what is the optimal rate at which these channels can produce ebits reliably [BDSW96]? By invoking the teleportation protocol [BBC+93] and the fact that free classical communication is allowed, this rate is also equal to the rate at which arbitrary qubits can be reliably communicated by using the channel \( n \) times [BDSW96]. The optimal rate is known as the distillable entanglement of the channel [BDSW96], and various lower bounds [DW05] and upper bounds [TGW14a, TGW14b, Wil16, BW18] are now known for it, strongly related to the bounds for distillable entanglement of states, as given above.

Some years after the distillable entanglement of a channel was proposed in [BDSW96], the question converse to it was proposed and addressed in [BBCW13]. The authors of [BBCW13] defined the entanglement cost of a quantum channel \( \mathcal{N}_{AB} \) as the smallest rate at which entanglement is required, in addition to the assistance of free classical communication, in order to simulate \( n \) uses of \( \mathcal{N}_{AB} \). Key to their definition of entanglement cost is the particular notion of simulation considered. In particular, the goal of their simulation protocol is to simulate \( n \) parallel uses of the channel, written as \( (\mathcal{N}_{AB})^n \). Furthermore, they considered a simulation protocol \( \mathcal{P}_{A^n \rightarrow B^n} \) to have the following form:

\[
\mathcal{P}_{A^n \rightarrow B^n}(\omega_{A^n}) \equiv \mathcal{L}_{A^n \rightarrow B^n}(\omega_{A^n} \otimes \Phi_{\mathcal{N}^n_{AB}}),
\]
where $\omega_{A^n}$ is an arbitrary input state, $L_{A^n}\rightarrow B^n$ is a free channel, whose implementation is restricted to consist of local operations and classical communication (LOCC) [BDSW96, CLM+14], and $\Phi_{A^n}\rightarrow B^n$, is a maximally entangled resource state. For $\varepsilon \in [0, 1]$, the simulation is then considered $\varepsilon$-distinguishable from $(N_{A\rightarrow B})^n$ if the following condition holds

$$
\frac{1}{2} \| (N_{A\rightarrow B})^n - P_{A^n\rightarrow B^n} \|_0 \leq \varepsilon,
$$

where $\|\cdot\|_0$ denotes the diamond norm [Kit97]. The physical meaning of the above inequality is that it places a limitation on how well any discriminator can distinguish the channel $(N_{A\rightarrow B})^n$ from the simulation $P_{A^n\rightarrow B^n}$ in a guessing game. Such a guessing game consists of the discriminator preparing a quantum state $\rho_{RA^n}$, the referee picking $(N_{A\rightarrow B})^n$ or $P_{A^n\rightarrow B^n}$ at random and then applying it to the $A^n$ systems of $\rho_{RA^n}$, and the discriminator finally performing a quantum measurement on the systems $RB^n$. If the inequality in (2) holds, then the probability that the discriminator can correctly distinguish the channel from its simulation is bounded from above by $\frac{1}{2} (1 + \varepsilon)$, regardless of the particular state $\rho_{RA^n}$ and final measurement chosen for his distinguishing strategy [Kit97,Hel69,Hel73,Hel76]. Thus, if $\varepsilon$ is close to zero, then this probability is not much better than random guessing, and in this case, the channels are considered nearly indistinguishable and the simulation thus reliable.

In parallel to the above developments in entanglement theory, there have undoubtably been many advances in the theory of quantum channel discrimination [CDP08a, CDP09b, DFY09, HHLW10, CMW16] and related developments in the theory of quantum interactive proof systems [GW07,Gut09,Gut12,GRS18]. Notably, the most general method for distinguishing a quantum memory channel from another one consists of a quantum-memory-assisted discrimination protocol [CDP08a, CDP09b]. In the language of quantum interactive proof systems, memory channels are called strategies and memory-assisted discrimination protocols are called co-strategies [GW07,Gut09,Gut12]. For a visual illustration of the physical setup, please consult [CDP08a, Figure 2] or [GW07, Figure 2]. In subsequent work after [GW07, CDP08a], a number of theoretical results listed above have been derived related to memory channel discrimination or quantum strategies.

The aforementioned developments in the theory of quantum channel discrimination indicate that the notion of channel simulation proposed in [BBCW13] is not the most general notion that could be considered. In particular, if a simulator is claiming to have simulated $n$ uses of the channel $N_{A\rightarrow B}$, then the discriminator should be able to test this assertion in the most general way possible, as given in [GW07, CDP08a, CDP09b]. That is, we would like for the simulation to pass the strongest possible test that could be performed to distinguish it from the $n$ uses of $N_{A\rightarrow B}$. Such a test allows for the discriminator to prepare an arbitrary state $\rho_{RA^n}$, call the first channel use $N_{A\rightarrow B}$, or its simulation, apply an arbitrary channel $A_{R^n\rightarrow B^n}^{(1)}$ call the second channel use or its simulation, etc. After the $n$th call is made, the discriminator then performs a joint measurement on the remaining quantum systems. See Figure 1 for a visual depiction. If the simulation is good, then the probability for the discriminator to distinguish the $n$ channels from the simulation should be no larger than $\frac{1}{2} (1 + \varepsilon)$, for small $\varepsilon$.

In this paper, I propose a new definition for the entanglement cost of a channel $N_{A\rightarrow B}$, such that it is the smallest rate at which ebits are needed, along with the assistance of free classical communication, in order to simulate $n$ uses of $N_{A\rightarrow B}$, in such a way that a discriminator performing the most stringent test, as described above, cannot distinguish the simulation from $n$ actual calls of $N_{A\rightarrow B}$ (Section II B). Here I denote the optimal rate by $E_C(N)$, and the prior quantity defined in [BBCW13] by $E_C^{(p)}(N)$, given that the simulation there was only required to pass a less stringent parallel discrimination test, as discussed above. Due to the fact that it is more difficult to pass the simulation test as specified by the new definition, it follows that $E_C(N) \geq E_C^{(p)}(N)$ (discussed in more detail in what follows). After establishing definitions, I then prove a general upper bound on the entanglement cost of a quantum channel, using the notion of teleportation simulation (Section III A). I prove that the entanglement cost of certain “resource-seizable,” teleportation-simulable channels takes on a particularly simple form (Section III B), which allows for concluding single-letter formulas for the entanglement cost of dephasing, erasure, three-dimensional Werner–Holevo channels, and depolarizing channels (complements of depolarizing channels), as detailed in Section IV. Note that the result about entanglement cost of dephasing channels solves an open question from [BBCW13]. I then extend the results to the case of bosonic Gaussian channels (Section V), proving single-letter formulas for the entanglement cost of fundamental channel models, including pure-loss and pure-amplifier channels (Theorem 2 in Section V G). These examples lead to the conclusion that the resource theory of entanglement for quantum channels is not reversible. I also prove that the entanglement cost of thermal, amplifier, and additive-noise bosonic Gaussian channels is bounded from below by the entanglement cost of their “Choi states.” In Section VI, I discuss how to generalize the basic notions to other resource theories. Finally, Section VII concludes with a summary and some open questions.

II. NOTIONS OF QUANTUM CHANNEL SIMULATION

In this section, I review the definition of entanglement cost of a quantum channel, as detailed in [BBCW13], and I also review the main theorem from [BBCW13].
partite states where the optimization is with respect to all pure bi-
pletely positive and trace preserving) [CLM14]. Let us now review the notion of entanglement cost from [BBCW13]. Fix $n, M \in \mathbb{N}$, $\varepsilon \in [0,1]$, and a quantum channel $\mathcal{N}_{A\rightarrow B}$. According to [BBCW13], an $(n,M,\varepsilon)$ (parallel) LOCC-assisted channel simulation code consists of an LOCC channel $\mathcal{L}_{A^mB^m\rightarrow AB}$ and a maximally entangled resource state $\Phi_{\mathcal{A}_0\mathcal{B}_0}$ of Schmidt rank $M$, such that together they implement a simulation channel $\mathcal{P}_{A^n\rightarrow B^n}$, as defined in (1). In this model, to be clear, we assume that Alice has access to all systems labeled by $A$, Bob has access to all systems labeled by $B$, and they are in distant laboratories. The simulation $\mathcal{P}_{A^n\rightarrow B^n}$ is considered $\varepsilon$-distinguishable from $n$ parallel calls $(\mathcal{N}_{A\rightarrow B})^{\otimes n}$ of the actual channel $\mathcal{N}_{A\rightarrow B}$ if the condition in (2) holds. Note here again that the condition in (2) corresponds to a discriminator who is restricted to performing only a parallel test to distinguish the $n$ calls of $\mathcal{N}_{A\rightarrow B}$ from its simulation. Let us also note here that the condition in (2) can be understood as the simulation $\mathcal{P}_{A^n\rightarrow B^n}$ providing an approximate teleportation simulation of $(\mathcal{N}_{A\rightarrow B})^{\otimes n}$, in the language of the later work of [KW18].

A rate $R$ is said to be achievable for (parallel) channel simulation of $\mathcal{N}_{A\rightarrow B}$ if for all $\varepsilon \in (0,1]$, $\delta > 0$, and sufficiently large $n$, there exists an $(n,2^n[R+\delta],\varepsilon)$ LOCC-assisted channel simulation code. The (parallel) entan-
glement cost $E_c^{(p)}(N)$ of the channel $N$ is equal to the infimum of all achievable rates, with the superscript $(p)$ indicating that the test of the simulation is restricted to be a parallel discrimination test.

The main result of [BBCW13] is that the channel's entanglement cost $E_c^{(p)}(N)$ is equal to the regularization of its entanglement of formation. To state this result precisely, recall that the entanglement of formation for a bipartite state $\rho_{AB}$ is defined as [BDSW96]

$$E_F(A; B|\rho) = \inf \left\{ \sum_x p(x) H(A|\psi^x) : \rho_{AB} = \sum_x p(x) \psi^x_{AB} \right\}, \quad (7)$$

where the infimum is with respect to all convex decompositions of $\rho_{AB}$ into pure states $\psi^x_{AB}$ and

$$H(A|\psi^x) \equiv -\text{Tr}\{\psi^x_A \log_2 \psi^x_A\} \quad (8)$$
is the quantum entropy of the marginal state $\psi^x_A = \text{Tr}_B\{\psi^x_{AB}\}$. The entanglement of formation does not increase under the action of an LOCC channel [BDSW96]. A channel's entanglement of formation $E_F(N)$ is then defined as

$$E_F(N) \equiv \sup_{\psi_{RA}} E_F(R; B|\omega), \quad (9)$$

where $\omega_{RB} \equiv N_{A\to B}(\psi_{RA})$, and it suffices to take the optimization with respect to a pure state input $\psi_{RA}$, with system $R$ isomorphic to system $A$, due to purification, the Schmidt decomposition theorem, and the LOCC monotonicity of entanglement of formation [BDSW96]. We can now state the main result of [BBCW13] described above:

$$E_c^{(p)}(N) = \lim_{n \to \infty} \frac{1}{n} E_F(N^\otimes n). \quad (10)$$

The regularized formula on the right-hand side may be difficult to evaluate in general, and thus can only be considered a formal expression, but if the additivity relation $\frac{1}{n} E_F(N^\otimes n) = E_F(N)$ holds for a given channel $N$ for all $n \geq 1$, then it simplifies significantly as $E_c^{(p)}(N) = E_F(N)$.

**B. Proposal for a revised notion of entanglement cost of a channel**

Now I propose the new or revised definition for entanglement cost of a channel. As motivated in the introduction, a parallel test of channel simulation is not the most general kind of test that can be considered. Thus, the new definition proposes that the entanglement cost of a channel should incorporate the most stringent test possible.

To begin with, let us fix $n, M \in \mathbb{N}, \varepsilon \in [0, 1]$, and a quantum channel $N_{A\to B}$. We define an $(n, M, \varepsilon)$ (sequential) LOCC-assisted channel simulation code to consist of a maximally entangled resource state $\Phi_{\tau_0, \tau_0}$ of Schmidt rank $M$ and a set

$$\{L_{A_i, \tau_{i-1}, \tau_i, i \to B_i}\}_{i=1}^n \quad (11)$$
of LOCC channels. Note that the systems $\tau_0, \tau_0$ of the final LOCC channel $L_{A_i, \tau_{i-1}, \tau_i, i \to B_i}$ can be taken trivial without loss of generality. As before, Alice has access to all systems labeled by $A$, Bob has access to all systems labeled by $B$, and they are in distant laboratories. The structure of this simulation protocol is intended to be compatible with a discrimination strategy that can test the actual $n$ channels versus the above simulation in a sequential way, along the lines discussed in [CDP08a, CDP09b] and [Gut12]. I later show how this encompasses the parallel tests discussed in the previous section.

A discrimination strategy consists of an initial state $\rho_{R_1 A_1}$, a set $\{A_{R_1, B_1 \to R_{i+1}, A_{i+1}}\}_{i=1}^{n-1}$ of adaptive channels, and a quantum measurement $\{Q_{R_n B_n}, I_{R_n B_n} \equiv Q_{R_n B_n}\}$. Let us employ the shorthand $\{\rho, A, Q\}$ to abbreviate such a discrimination strategy. Note that, in performing a discrimination strategy, the discriminator has a full description of the channel $N_{A\to B}$ and the simulation protocol, which consists of $\Phi_{\tau_0, \tau_0}$ and the set in (11). If this discrimination strategy is performed on the $n$ uses of the actual channel $N_{A\to B}$, the relevant states involved are

$$\rho_{R_{i+1} A_{i+1}} \equiv A_{R_i B_i \to R_{i+1}, A_{i+1}}(\rho_{R_i B_i}), \quad (12)$$

for $i \in \{1, \ldots, n - 1\}$ and

$$\rho_{R_n B_n} \equiv N_{A_n \to B_n}(\rho_{R_n A_n}), \quad (13)$$

for $i \in \{1, \ldots, n\}$. If this discrimination strategy is performed on the simulation protocol discussed above, then the relevant states involved are

$$\tau_{R_i B_i \to \tau_{i+1}, \tau_i, i \to B_i} \equiv L_{A_i, \tau_{i-1}, \tau_i, i \to B_i}(\tau_{R_i A_i} \otimes \Phi_{\tau_0, \tau_0}), \quad (14)$$

for $i \in \{1, \ldots, n - 1\}$, where $\tau_{R_i A_i} \equiv \rho_{R_i A_i}$, and

$$\tau_{R_n B_n} \equiv L_{A_n, \tau_{n-1}, \tau_n, n \to B_n}(\tau_{R_n A_n} \otimes \tau_{R_n B_n}). \quad (15)$$

for $i \in \{2, \ldots, n\}$. The discriminator then performs the measurement $\{Q_{R_n B_n}, I_{R_n B_n} \equiv Q_{R_n B_n}\}$ and guesses “actual channel” if the outcome is $Q_{R_n B_n}$ and “simulation” if the outcome is $I_{R_n B_n} \equiv Q_{R_n B_n}$. Figure 1 depicts the discrimination strategy in the case that the actual channel is called $n = 3$ times and in the case that the simulation is performed.

If the *a priori* probabilities for the actual channel or simulation are equal, then the success probability of the discriminator in distinguishing the channels is given by

$$\frac{1}{2} \left[ \text{Tr}\{Q_{R_n B_n} \rho_{R_n B_n}\} + \text{Tr}\{(I_{R_n B_n} \equiv Q_{R_n B_n}) \tau_{R_n B_n}\} \right] \leq \frac{1}{2} \left( 1 + \frac{1}{2} \left\| \rho_{R_n B_n} - \tau_{R_n B_n} \right\|_1 \right), \quad (16)$$
where the latter inequality is well known from the theory of quantum state discrimination [Hel69, Hol73, Hel76]. For this reason, we say that the $n$ calls to the actual channel $\mathcal{N}_{A\to B}$ are $\varepsilon$-distinguishable from the simulation if the following condition holds for the respective final states

$$\frac{1}{2} \| \rho_{R_{n}B_{n}} - \tau_{R_{n}B_{n}} \|_1 \leq \varepsilon. \quad (17)$$

If this condition holds for all possible discrimination strategies $\{\rho, A, Q\}$, i.e., if

$$\frac{1}{2} \sup_{\{\rho, A\}} \| \rho_{R_{n}B_{n}} - \tau_{R_{n}B_{n}} \|_1 \leq \varepsilon, \quad (18)$$

then the simulation protocol constitutes an $(n, M, \varepsilon)$ channel simulation code. It is worthwhile to remark: If we ascribe the shorthand $(N)^n$ for the $n$ uses of the channel and the shorthand $(L)^n$ for the simulation, then the condition in (18) can be understood in terms of the $n$-round strategy norm of [CDP08a, CDP09b, Gut12]:

$$\frac{1}{2} \| (N)^n - (L)^n \|_{0,n} \leq \varepsilon. \quad (19)$$

As before, a rate $R$ is achievable for (sequential) channel simulation of $N$ if for all $\varepsilon \in (0, 1]$, $\delta > 0$, and sufficiently large $n$, there exists an $(n, 2^n[R+\delta], \varepsilon)$ (sequential) channel simulation code for $N$. We define the (sequential) entanglement cost $E_C(N)$ of the channel $N$ to be the infimum of all achievable rates. Due to the fact that this notion is more general, we sometimes simply refer to $E_C(N)$ as the entanglement cost of the channel $N$ in what follows.

C. LOCC monotonicity of the entanglement cost

Let us note here that if a channel $\mathcal{N}_{A\to B}$ can be realized from another channel $\mathcal{M}_{A'\to B'}$ via a preprocessing LOCC channel $L_{A'\to A'MB_M}^{\text{pre}}$ and a postprocessing LOCC channel $L_{B'MB_M\to B}^{\text{post}}$, then it follows that any $(n, M, \varepsilon)$ protocol for sequential channel simulation of $\mathcal{M}_{A'\to B'}$ realizes an $(n, M, \varepsilon)$ protocol for sequential channel simulation of $\mathcal{N}_{A\to B}$. This is an immediate consequence of the fact that the best strategy for discriminating $\mathcal{N}_{A\to B}$ from its simulation can be understood as a particular discrimination strategy for $\mathcal{M}_{A'\to B'}$, due to the structural decomposition in (20). Following definitions, a simple consequence is the following LOCC monotonicity inequality for the entanglement cost of these channels:

$$E_C(N) \leq E_C(M). \quad (21)$$

Thus, it takes more or the same entanglement to simulate the channel $M$ than it does to simulate $N$. Furthermore, the decomposition in (20) and the bound in

\[ (21) \]

D. Parallel tests as a special case of sequential tests

A parallel test of the form described in Section II A is a special case of the sequential test outlined above. One can see this in two seemingly different ways. First, we can think of the sequential strategy taking a particular form. The state $\xi_{RA_1A_2\cdots A_n}$ is prepared, and here we identify systems $RA_2\cdots A_n$ with system $R_1$ of $\rho_{R_1A_1}$ in an adaptive protocol and system $A_1$ of $\xi_{RA_1A_2\cdots A_n}$ with system $A_1$ of $\rho_{R_1A_1}$. Then the channel $\mathcal{N}_{A_1\to B_1}$ or its simulation is called. After that, the action of the first adaptive channel is simply to swap in system $A_2$ of $\xi_{RA_1A_2\cdots A_n}$ to the second call of the channel $\mathcal{N}_{A_2\to B_2}$ or its simulation, while keeping systems $RB_1A_3\cdots A_n$ as part of the reference $R_2$ of the state $\rho_{R_2A_2}$. Then this iterates and the final measurement is performed on all of the remaining systems.

The other way to see how a parallel test is a special kind of sequential test is to rearrange the simulation protocol as has been done in Figure 2. Here, we see that the simulation protocol has a memory structure, and it is clear that the simulation protocol can accept as input a state $\xi_{RA_1A_2\cdots A_n}$ and outputs a state on systems $RB_1A_3\cdots A_n$, which can subsequently be measured.

As a consequence of this reduction, any $(n, M, \varepsilon)$ parallel channel simulation protocol can serve as an $(n, M, \varepsilon)$ parallel channel simulation protocol. Furthermore, if $R$ is an achievable rate for sequential channel simulation, then it is also an achievable rate for parallel channel simulation. Finally, these reductions imply the following inequality:

$$E_C(N) \geq E_C(N)^{(p)} \quad (22)$$

\[ (22) \]
Intuitively, one might sometimes require more entanglement in order to pass the more stringent test that occurs in sequential channel simulation. As a consequence of (10) and (22), we have that
\[ E_C(N) \geq \lim_{n \to \infty} \frac{1}{n} E_F(N^{\otimes n}). \quad (23) \]

It is an interesting question (not addressed here) to determine if there exists a channel such that the inequality in (22) is strict.

If desired, it is certainly possible to obtain a non-asymptotic, weak-converse bound that implies the above bound after taking limits. Let us state this bound as follows:

**Proposition 1** Let \( N_{A\to B} \) be a quantum channel, and let \( n, M \in \mathbb{N} \) and \( \varepsilon \in [0, 1] \). Set \( d = \min\{|A|, |B|\} \), i.e., the minimum of the input and output dimensions of the channel \( N_{A\to B} \). Then the following bound holds for any \((n, M, \varepsilon)\) sequential channel simulation code:
\[ \frac{1}{n} \log_2 M \geq \frac{1}{n} E_F(N^{\otimes n}) - \sqrt{\varepsilon} \log d - \frac{1}{n} g_2(\sqrt{\varepsilon}), \quad (24) \]
where \( \frac{1}{n} \log_2 M \) is understood as the non-asymptotic entanglement cost of the protocol and the bosonic entropy function \( g_2(x) \) is defined for \( x \geq 0 \) as
\[ g_2(x) = (x + 1) \log_2(x + 1) - x \log_2 x. \quad (25) \]

**Proof.** To see this, suppose that there exists an \((n, M, \varepsilon)\) protocol for sequential channel simulation. Then by the above reasoning (also see Figure 2), it can be thought of as a parallel channel simulation protocol, such that the criterion in (2) holds. Suppose that \( \psi_{RA_1\cdots A_n} \) is a test input state, with \(|R| = |A|^n\), leading to \( \omega_{RB_1\cdots B_n} = (N_{A\to B})^{\otimes n}(\psi_{RA_1\cdots A_n}) \) when the actual channels are applied and \( \omega_{RB_1\cdots B_n} \) when the simulation is applied. Then we have that
\[
\begin{align*}
E_F(R; B_1 \cdots B_n) &\leq E_F(R; B_1 \cdots B_n) + n\sqrt{\varepsilon} \log d + g_2(\sqrt{\varepsilon}) \\
&\leq E_F(RA_1 \cdots A_n A_0; B_0)_{\psi_{B_0}} + n\sqrt{\varepsilon} \log d + g_2(\sqrt{\varepsilon}) \\
&= E_F(A_0; B_0) + n\sqrt{\varepsilon} \log d + g_2(\sqrt{\varepsilon}) \\
&= \log_2 M + n\sqrt{\varepsilon} \log d + g_2(\sqrt{\varepsilon}).
\end{align*}
\]
(26)
The first inequality follows from the condition in (18), as well as from the continuity bound for entanglement of formation from [Win16b, Corollary 4]. The second inequality follows from the LOCC monotonicity of the entanglement of formation [BDSW96], here thinking of the person who possesses systems \( RA_1 \cdots A_n \) to be in the same laboratory as the one possessing the systems \( A_i \), while the person who possesses the \( B_i \) systems is in a different laboratory. The first equality follows from the fact that \( \psi_{RA_1\cdots A_n} \) is in tensor product with \( \Phi_{A_0 B_0} \), so that by a local channel, one may remove \( \psi_{RA_1\cdots A_n} \) or append it for free. The final equality follows because the entanglement of formation of the maximally entangled state is equal to the logarithm of its Schmidt rank. Since the bound holds uniformly regardless of the input state \( \psi_{RA_1\cdots A_n} \), after an optimization and a rearrangement we conclude the stated lower bound on the non-asymptotic entanglement cost \( \frac{1}{n} \log_2 M \) of the protocol. ■

**Remark 1** Let us note here that the entanglement cost of a quantum channel is equal to zero if and only if the channel is entanglement-breaking [HSR03, Hol08]. The “if-part” follows as a straightforward consequence of definitions and the fact that these channels can be implemented as a measurement followed by a preparation [HSR03, Hol08], given that this measure-prepare procedure is a particular kind of LOCC and thus allowed for free (without any cost) in the above model. The “only-if” part follows from (22) and [BBCW13, Corollary 18], the latter of which depends on the result from [YHHSR05].

### III. Bounds for the Entanglement Cost of Teleportation-Simulable Channels

**A. Upper bound on the entanglement cost of teleportation-simulable channels**

The most trivial method for simulating a channel is to employ the teleportation protocol [BBC+93] directly. In this method, Alice and Bob could use the teleportation protocol so that Alice could transmit the input of the channel to Bob, who could then apply the channel. Repeating this \( n \) times, this trivial method would implement an \((n, |A|^n, 0)\) simulation protocol in either the parallel or sequential model. Alternatively, Alice could apply the channel first and then teleport the output to Bob, and repeating this \( n \) times would implement an \((n, |B|^n, 0)\) simulation protocol in either the parallel or sequential model. Thus, they could always achieve a rate of \( \log_2(\min\{|A|, |B|\}) \) using this approach, and this reasoning establishes a simple dimension upper bound on the entanglement cost of a channel:
\[ E_C(N_{A\to B}) \leq \log_2(\min\{|A|, |B|\}). \quad (27) \]
In this context, also see [KW18, Proposition 9].

A less trivial approach is to exploit the fact that some channels of interest could be teleportation-simulable with associated resource state \( \omega_{A'B'} \), in which the resource state need not be a maximally entangled state (see [BDSW96, Section V] and [HHH99, Eq. (11)]). Recall from these references that a channel \( N_{A\to B} \) is teleportation-simulable with associated resource state \( \omega_{A'B'} \) if there exists an LOCC channel \( L_{AA'B'} \to B \) such that the following equality holds for all input states \( \rho_A \):
\[ N_{A\to B}(\rho_A) = L_{AA'B'}(\rho_A \otimes \omega_{A'B'}). \quad (28) \]
If a channel possesses this structure, then we arrive at the following upper bound on the entanglement cost:
Proposition 2 Let $\mathcal{N}_{A\to B}$ be a quantum channel that is teleportation-simulable with associated resource state $\omega_{A'B'}$, as defined in (28). Let $n, M \in \mathbb{N}$ and $\varepsilon \in (0,1)$. Then there exists an $(n, M, \sqrt{\varepsilon})$ sequential channel simulation code satisfying the following bound

$$\frac{1}{n} \log_2 M \leq \frac{1}{n} E_{F,0}^{\varepsilon/2}(A^n; B^n)_{\omega^\otimes n},$$

where $\frac{1}{n} \log_2 M$ is understood as the non-asymptotic entanglement cost of the protocol, and $E_{F,0}^{\varepsilon/2}(A^n; B^n)_{\omega^\otimes n}$ is the $\varepsilon/2$-smooth entanglement of formation (EOF) [BD11] recalled in Definition 1 below.

Definition 1 (Smooth EOF [BD11]) Let $\delta \in (0,1)$ and $\tau_{CD}$ be a bipartite state. Let $\mathcal{E} = \{p_X(x), \phi^\partial_{CD}\}$ denote a pure-state ensemble decomposition of $\tau_{CD}$, meaning that $\tau_{CD} = \sum_x p_X(x) \phi^\partial_{CD}$, where $\phi^\partial_{CD}$ is a pure state and $p_X$ is a probability distribution. Define the conditional entropy of order zero $H_0(K|L)_{\omega}$ of a bipartite state $\omega_{KL}$ as

$$H_0(K|L)_{\omega} = \max_{\sigma_L} \left[ - \log_2 \text{Tr}(\Pi^\partial_{KL}(I_K \otimes \sigma_L)) \right],$$

where $\Pi^\partial_{KL}$ denotes the projection onto the support of $\omega_{KL}$ and $\sigma_L$ is a density operator. Then the $\delta$-smooth entanglement of formation of $\tau_{CD}$ is given by

$$E_{F,0}^\delta(C; D)_\tau = \min_{\mathcal{E} : H_0(C|X)_{\tau}} \frac{1}{n} H_0(C|X)_\tau,$$

where the minimization is with respect to all pure-state ensemble decompositions $\mathcal{E}$ of $\tau_{CD}$, $\tau_{CD} = \sum_x p_X(x) x \otimes \phi^\partial_{CD}$ is a labeled pure-state extension of $\tau_{CD}$, and the $\delta$-ball $B^\delta_{\phi}(\tau_{XC})$ of $\phi$-states for a $\phi$ state $\tau_{XC}$ is defined as

$$B^\delta_{\phi}(\tau_{XC}) = \left\{ \omega_{XC} : \omega_{XC} \geq 0, \omega_{XC} = \sum_x |x\rangle \otimes \omega^\varepsilon_x, \|\omega_{XC} - \tau_{XC}\|_1 \leq \delta \right\}.$$

The $\delta$-smooth entanglement of formation has the property that, for a tensor-power state $\tau_{C^n}$, the following limit holds [BD11, Theorem 2]

$$\lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} E_{F,0}^\delta(C^n; D^n)_{\tau_{C^n}} = \lim_{n \to \infty} \frac{1}{n} E_F(C; D)_\tau = E_C(\tau_{CD}).$$

Corollary 1 Let $\mathcal{N}_{A\to B}$ be a quantum channel that is teleportation-simulable with associated resource state $\omega_{A'B'}$, as defined in (28). Then the entanglement cost of the channel $\mathcal{N}$ is never larger than the entanglement cost of the resource state $\omega_{A'B'}$:

$$E_C(\mathcal{N}) \leq E_C(\omega_{A'B'}).$$

The above corollary captures the intuitive idea that if a single instance of the channel $\mathcal{N}$ can be simulated via LOCC starting from a resource state $\omega_{A'B'}$, then the entanglement cost of the channel should not exceed the entanglement cost of the resource state. The idea of the above proof is simply to prepare a large number $n$ of copies of $\omega_{A'B'}$ and then use these to simulate $n$ uses of the channel $\mathcal{N}$, such that the simulation could not be distinguished from $n$ uses of the channel $\mathcal{N}$ in any sequential test.
B. The entanglement cost of resource-seizable, teleportation-simulable channels

In this section, I define teleportation-simulable channels that are resource-seizable, meaning that one can seize the channel’s underlying resource state by the following procedure:

1. prepare a free, separable state,
2. input one of its systems to the channel, and then
3. post-process with a free, LOCC channel.

This procedure is indeed related to the channel processing described earlier in (20). After that, I prove that the entanglement cost of a resource-seizable channel is equal to the entanglement cost of its underlying resource state.

**Definition 2 (Resource-seizable channel)** Let \( \mathcal{N}_{A \rightarrow B} \) be a teleportation-simulable channel with associated resource state \( \omega_{A'B'} \), as defined in (28). Suppose that there exists a separable input state \( \rho_{AMABM} \) to the channel and a postprocessing LOCC channel \( \mathcal{D}_{AMBBM \rightarrow A'B'} \) such that the resource state \( \omega_{A'B'} \) can be seized from the channel \( \mathcal{N}_{A \rightarrow B} \) as follows:

\[
\mathcal{D}_{AMBBM \rightarrow A'B'}(\mathcal{N}_{A \rightarrow B}(\rho_{AMABM})) = \omega_{A'B'}. \tag{40}
\]

Then we say that the channel is a resource-seizable, teleportation-simulable channel.

In Appendix A, I discuss how resource-seizable channels are related to those that are “implementable from their image,” as defined in [CMH17, Appendix A]. In Section VI, I also discuss how to generalize the notion of a resource-seizable channel to an arbitrary resource theory.

The main result of this section is the following simplifying form for the entanglement cost of a resource-seizable channel (as defined above), establishing that its entanglement cost in the asymptotic regime is the same as the entanglement cost of the underlying resource state. Furthermore, for these channels, the entanglement cost is not increased by the need to pass a more stringent test for channel simulation as required in a sequential test.

**Theorem 1** Let \( \mathcal{N}_{A \rightarrow B} \) be a resource-seizable, teleportation-simulable channel with associated resource state \( \omega_{A'B'} \), as given in Definition 2. Then the entanglement cost of the channel \( \mathcal{N}_{A \rightarrow B} \) is equal to its parallel entanglement cost, which in turn is equal to the entanglement cost of the resource state \( \omega_{A'B'} \):

\[
E_C(\mathcal{N}) = E_C^{(p)}(\mathcal{N}) = E_C(\omega_{A'B'}). \tag{41}
\]

**Proof.** Consider from (22) that

\[
E_C(\mathcal{N}) \geq E_C^{(p)}(\mathcal{N}) = \lim_{n \to \infty} \frac{1}{n} E_F(\mathcal{N}^\otimes n). \tag{42}
\]

Let \( \psi_{RA^n} \equiv \psi_{RA_1 \cdots A_n} \) be an arbitrary pure input state to consider at the input of the tensor-power channel \( (\mathcal{N}_{A \rightarrow B})^\otimes n \), leading to the state

\[
\sigma_{RB^n} \equiv \left( \mathcal{N}_{A \rightarrow B} \right)^\otimes n(\psi_{RA_1 \cdots A_n}). \tag{43}
\]

From the assumption that the channel is teleportation-simulable with associated resource state \( \omega_{A'B'} \), we have from (28) that

\[
\sigma_{RB^n} = (\mathcal{L}_{AA'B' \rightarrow B})^\otimes n(\psi_{RA^n} \otimes \omega_{A'B'}^\otimes n) \tag{44}
\]

Then

\[
E_F(R; B^n) \leq E_F(RA^n; B^n)_{\psi_{RA^n} \otimes \omega_{A'B'}^\otimes n} = E_F(A^n; B^n)_{\omega_{A'B'}^\otimes n}, \tag{45}
\]

where the inequality follows from LOCC monotonicity of the entanglement of formation. Since the bound holds for an arbitrary input state, we conclude that the following inequality holds for all \( n \in \mathbb{N} \):

\[
\frac{1}{n} E_F(\mathcal{N}^\otimes n) \leq \frac{1}{n} E_F(A^n; B^n)_{\omega_{A'B'}^\otimes n}. \tag{47}
\]

Now taking the limit \( n \to \infty \), we conclude that

\[
E_C^{(p)}(\mathcal{N}) \leq E_C(\omega_{A'B'}). \tag{48}
\]

To see the other inequality, let a decomposition of the separable input state \( \rho_{AMABM} \) be given by

\[
\rho_{AMABM} = \sum_x p_x (x) \psi_{A_M A}^x \otimes \phi_{B_M}^x. \tag{49}
\]

Considering that \( [\psi_{A_M A}^x]_{\otimes n} \) is a particular input to the tensor-power channel \( (\mathcal{N}_{A \rightarrow B})^\otimes n \), we conclude that

\[
E_F(\mathcal{N}^\otimes n) \geq E_F(A^n_M; B^n)_{[\psi_{A_M A}^x]_{\otimes n}}. \tag{50}
\]

Since this holds for all \( x \), we have that

\[
E_F(\mathcal{N}^\otimes n) \geq \sum_x p_x (x) E_F(A^n_M; B^n)_{[\psi_{A_M A}^x]_{\otimes n}} = \sum x p_x (x) E_F(A_M^n; B^n)_{[\psi_{A_M A}^x]_{\otimes n}} \geq E_F(A^n_M; B^n)_{\omega_{A'B'}^\otimes n}, \tag{51}
\]

where the equality follows because introducing a product state locally does not change the entanglement, the second inequality follows from convexity of entanglement of formation \([BDSW96] \), and the last inequality follows from the assumption in (40) and the LOCC monotonicity of the entanglement of formation. Since the inequality holds for all \( n \in \mathbb{N} \), we can divide by \( n \) and take the limit \( n \to \infty \) to conclude that

\[
E_C^{(p)}(\mathcal{N}) \geq E_C(\omega_{A'B'}), \tag{52}
\]

and in turn, from (48), that

\[
E_C^{(p)}(\mathcal{N}) = E_C(\omega_{A'B'}). \tag{53}
\]

Combining this equality with the inequalities in (39) and (42) leads to the statement of the theorem. ■
IV. EXAMPLES

The equality in Theorem 1 provides a formal expression for the entanglement cost of any resource-seizable, teleportation-simulable channel, given in terms of the entanglement cost of the underlying resource state \( \omega_{A'B'} \). Due to the fact that the entanglement cost of a state is generally not equal to its entanglement of formation [Has09], it could still be a significant challenge to compute the entanglement cost of these special channels. However, for some special states, the equality \( E_C(\omega_{A'B'}) = E_F(\Phi_{A'B'}^q) \) does hold, and I discuss several of these examples and related channels here.

Let us begin by recalling the notion of a covariant channel \( \mathcal{N}_{A\rightarrow B} \) [Hol02]. For a group \( G \) with unitary channel representations \( \{ U_A^g \} \) and \( \{ V_B^g \} \) acting on the input system \( A \) and output system \( B \) of the channel \( \mathcal{N}_{A\rightarrow B} \), the channel \( \mathcal{N}_{A\rightarrow B} \) is covariant with respect to the group \( G \) if the following equality holds

\[
\mathcal{N}_{A\rightarrow B} \circ U_A^g = V_B^g \circ \mathcal{N}_{A\rightarrow B}.
\]  

(54)

If the averaging channel is such that \( \frac{1}{|G|} \sum_g U_A^g(X) = Tr[X]I/|A| \) (implementing a unitary one-design), then we simply say that the channel \( \mathcal{N}_{A\rightarrow B} \) is covariant.

Then from [CDP09a, Section 7] (see also [WTB17, Appendix A]), we conclude that any covariant channel is teleportation-simulable with associated resource state given by the Choi state of the channel, i.e., \( \omega_{A'B'} = \mathcal{N}_{A\rightarrow B}(\Phi_{A'A}) \). As such, covariant channels are resource-seizable, so that the equality in Theorem 1 applies to all covariant channels. Thus, the entanglement cost of a covariant channel is equal to the entanglement cost of its Choi state. In spite of this reduction, it could still be a great challenge to compute formulas for the entanglement cost of these channels, due to the fact that the entanglement of formation is not necessarily equal to the entanglement cost for the Choi states of these channels. For example, the entanglement cost of an isotropic state [Wer89], which is the Choi state of a depolarizing channel, is not known. In the next few subsections, I detail some example channels for which it is possible to characterize their entanglement cost.

A. Erasure channels

A simple example of a channel that is covariant is the quantum erasure channel, defined as [GBP97]

\[
\mathcal{E}^q(\rho) = (1 - q)\rho + q|e\rangle\langle e|,
\]  

(55)

where \( \rho \) is a \( d \)-dimensional input state, \( q \in [0,1] \) is the erasure probability, and \( |e\rangle\langle e| \) is a pure erasure state orthogonal to any input state, so that the output state has \( d + 1 \) dimensions. By the remark above, we conclude that \( E_C(\mathcal{E}^q) = E_C(\mathcal{E}_{A\rightarrow B}^q(\Phi_{RA})) \), and so determining the entanglement cost boils down to determining the entanglement cost of the Choi state

\[
\mathcal{E}_{A\rightarrow B}^q(\Phi_{RA}) = (1 - q)\Phi_{RA} + \frac{I_R}{d} \otimes |e\rangle\langle e|.
\]  

(56)

An obvious pure-state decomposition for \( \mathcal{E}_{A\rightarrow B}^q(\Phi_{RA}) \) (see [BBCW13, Eqs. (93)–(95)]) leads to

\[
E_C(\mathcal{E}_{A\rightarrow B}^q(\Phi_{RA})) \leq E_F(\mathcal{E}_{A\rightarrow B}^q(\Phi_{RA})) \leq (1 - q) \log_2 d.
\]  

(57)

As it turns out, these inequalities are tight, due to an operational argument. In particular, the distillable entanglement of \( \mathcal{E}_{A\rightarrow B}^q(\Phi_{RA}) \) is exactly equal to \( (1 - q) \log_2 d \) [BDS97], and due to the operational fact that the distillable entanglement of a state cannot exceed its entanglement cost [BDSW96], we conclude that

\[
E_C(\mathcal{E}_{A\rightarrow B}^q(\Phi_{RA})) = (1 - q) \log_2 d, \text{ and in turn that}
\]  

\[
E_C(\mathcal{E}^q) = E_C(\mathcal{E}_{A\rightarrow B}^q(\Phi_{RA})) = (1 - q) \log_2 d.
\]  

(58)

This result generalizes the finding from [BBCW13], which is that \( E_C(\mathcal{E}^q) = (1 - q) \log_2 d \), and so we conclude that for erasure channels, the entanglement cost of these channels is not increased by the need to pass a more stringent test for channel simulation, as posed by a sequential test. Note also that the distillable entanglement of the erasure channel is given by \( E_D(\mathcal{E}^q) = (1 - q) \log_2 d \), due to [BDS97].

The fact that the distillable entanglement of an erasure channel is equal to its entanglement cost, implies that, if we restrict the resource theory of entanglement for quantum channels to consist solely of erasure channels, then it is reversible. By this, we mean that, in the limit of many channel uses, if one begins with an erasure channel of parameter \( q \) and distills ebits from it at a rate \( (1 - q) \log_2 d \), then one can subsequently use these distilled ebits to simulate the same erasure channel again.

As we see below, this reversibility breaks down when considering other channels.

B. Dephasing channels

A \( d \)-dimensional dephasing channel has the following action:

\[
\mathcal{D}^q(\rho) = \sum_{i=0}^{d-1} q_i Z_i \rho Z_i^\dagger,
\]  

(60)

where \( q \) is a vector containing the probabilities \( q_i \) and \( Z \) has the following action on the computational basis \( Z|x\rangle = e^{2\pi i x/d} |x\rangle \). This channel is covariant with respect to the Heisenberg–Weyl group of unitaries, which are well known to be a unitary one-design. Furthermore, as remarked previously (e.g., in [TWW17]), the Choi state
$D_{A \rightarrow B}^{\Phi_{RA}}$ of this channel is a maximally correlated state [Rai99, Rai01], which has the form

$$
\sum_{i,j} \alpha_{i,j} |i\rangle_R \otimes |j\rangle_B.
$$

(61)

As such, Theorem 1 applies to these channels, implying that

$$
E_C(D^q) = E_C^{(p)}(D^q) = E_C(D_{A \rightarrow B}^{\Phi_{RA}}),
$$

(62)

$$
E_F(D_{A \rightarrow B}^{\Phi_{RA}}),
$$

(63)

with the final equality resulting from the fact that the entanglement cost is equal to the entanglement of formation for maximally correlated states [VDC02, HSS03]. In [HSS03, Section VI-A], an optimization procedure is given for calculating the entanglement of formation of maximally correlated states, which is simpler than that needed from the definition of entanglement of formation.

A qubit dephasing channel with a single dephasing parameter $q \in [0, 1]$ is defined as

$$
D^q(\rho) = (1 - q) \rho + q Z \rho Z.
$$

(64)

For the Choi state of this channel, there is an explicit formula for its entanglement of formation [Woo98], from which we can conclude that

$$
E_C(D^q) = E_C^{(p)}(D^q) = h_2(1/2 + \sqrt{q(1 - q)}),
$$

(65)

where

$$
h_2(x) \equiv -x \log_2 x - (1 - x) \log_2 (1 - x)
$$

(66)

is the binary entropy. The equality in (65) solves an open question from [BBCW13], where it had only been shown that $E_C^{(p)}(D^q) \leq h_2(1/2 + \sqrt{q(1 - q)})$.

The results of [BDSW96, Eq. (57)] and [Hay06, Eq. (8.114)] gave a simple formula for the distillable entanglement of the qubit dephasing channel:

$$
E_D(D^q) = 1 - h_2(q).
$$

(67)

Thus, this formula and the formula in (65) demonstrate that the resource theory of entanglement for these channels is irreversible. That is, if one started from a qubit dephasing channel with parameter $q \in (0, 1)$ and distilled ebits from it at the ideal rate of $1 - h_2(q)$, and then subsequently wanted to use these ebits to simulate a qubit dephasing channel with the same parameter, this is not possible, because the rate at which ebits are distilled is not sufficient to simulate the channel again.

Figure 3 compares the formulas for entanglement cost and distillable entanglement of the qubit dephasing channel, demonstrating that there is a noticeable gap between them. At $q = 1/2$, the qubit dephasing channel is a completely dephasing, classical channel, so that $E_C(D^{1/2}) = E_F(D^{1/2}) = 0$. Thus, a reasonable approximation to the difference is given by a Taylor expansion about $q = 1/2$:

$$
E_C(D^q) - E_D(D^q) = \frac{1}{\ln 2} \left[2 \ln \left(\frac{1}{q - \frac{1}{2}}\right) - 1\right] (q - \frac{1}{2})^2 + O((q - \frac{1}{2})^4).
$$

(68)

C. Werner–Holevo channels

A particular kind of Werner–Holevo channel performs the following transformation on a $d$-dimensional input state $\rho$ [WH02]:

$$
\mathcal{W}^{(d)}(\rho) = \frac{1}{d - 1} (\text{Tr} \{\rho\} I - T(\rho))
$$

(69)

where $T$ denotes the transpose map $T(\cdot) = \sum_{i,j} |i\rangle \langle j| T(\cdot) |i\rangle \langle j|$.

As observed in [WH02, Section II] and [LM15, Section VII], this channel is covariant, and so an immediate consequence of [CDP09a, Section 7] is that these channels are teleportation simulable with associated resource state given by their Choi state. The latter fact was explicitly observed in [LM15, Sections VI and VII], as well as [CMH17, Appendix A]. Furthermore, its Choi state is given by

$$
\mathcal{W}_{A \rightarrow B}^{(d)}(\Phi_{RA}) = \alpha_d \equiv \frac{1}{d(d - 1)} (I_{RB} - F_{RB}),
$$

(70)

where $\alpha_d$ is the antisymmetric state, i.e., the maximally mixed state on the antisymmetric subspace of a $d \times d$ quantum system and $F_{RB} \equiv \sum_{i,j} |i\rangle \langle j| R \otimes |j\rangle \langle i|_B$ denotes the unitary swap operator. Theorem 1 thus applies to
these channels, and we find that
\[
E_C(W^{(d)}) = E_C^{(p)}(W^{(d)}) = E_C(\alpha_d) \geq \log_2(4/3) \approx 0.415,
\]
with the inequality following from [CSW12, Theorem 2]. We also have that
\[
E_C(W^{(d)}) = E_C^{(p)}(W^{(d)}) = E_C(\alpha_d) \leq E_F(\alpha_d) = 1,
\]
with the last equality following from the result stated in [VW01, Section IV-C]. For \(d = 3\), the entanglement cost \(E_C(\alpha_3)\) is known to be equal to exactly one ebit [Yur03]:
\[
E_C(W^{(3)}) = E_C^{(p)}(W^{(3)}) = 1. \tag{75}
\]

It was observed in [CMH17, Appendix A] (as well as [Win16a]) that the distillable entanglement of the Werner–Holevo channel \(W^{(d)}\) is equal to the distillable entanglement of its Choi state:
\[
E_D(W^{(d)}) = E_D(\alpha_d). \tag{76}
\]
Thus, an immediate consequence of [CSW12, Theorem 1 and Eq. (5)] is that
\[
E_D(W^{(d)}) \leq \begin{cases} 
\log_2 \frac{d^2 + 2}{d^2} & \text{if } d \text{ is even} \\
\log_2 \frac{d^2 + 3}{d^2} & \text{if } d \text{ is odd}
\end{cases} \tag{77}
\]
\[
= \frac{2}{d} \log_2 \left( 1 - \frac{1}{d} \right) + O\left( \frac{1}{d^3} \right). \tag{78}
\]

We can now observe that the resource theory of entanglement is generally not reversible when restricted to Werner–Holevo channels. The case \(d = 2\) is somewhat trivial: in this case, one can verify that the channel \(W^{(2)}\) is a unitary channel, equivalent to acting on the input state with the Pauli \(Y\) unitary. Thus, for \(d = 2\), the channel is a noiseless qubit channel, and we trivially have that
\[
E_D(W^{(2)}) = E_C(W^{(2)}) = 1, \tag{79}
\]
so that the resource theory of entanglement is clearly reversible in this case. For \(d = 3\), the upper bound on distillable entanglement in (77) evaluates to \(\frac{1}{2} \log_2(3) \approx 0.793\), while the entanglement cost is equal to one, as stated in (75), so that
\[
E_D(W^{(3)}) \leq 0.793 < 1 = E_C(W^{(3)}). \tag{80}
\]
Thus, the resource theory of entanglement is not reversible for \(W^{(3)}\). For \(d \in \{4, 5, 6\}\), the upper bound in (77) and the lower bound in (73) are not strong enough to make a definitive statement (interestingly, the bounds in (77) and (73) are actually equal for \(d = 6\)). Then for \(d \geq 7\), the upper bound in (77) and the lower bound in (73) are strong enough to conclude that
\[
E_D(W^{(d)}) < E_C(W^{(d)}), \tag{81}
\]
so that the resource theory is not reversible for \(W^{(d)}\). Figure 4 summarizes these observations.

### FIG. 4. Lower bound on the entanglement cost \(E_C(W^{(d)})\) from (73) and upper bound on distillable entanglement \(E_D(W^{(d)})\) from (77) for the Werner–Holevo channel \(W^{(d)}\) as a function of the parameter \(d \geq 4\), with the lines connecting the dots demonstrating the gap between them. For \(d = 2\), the points are exact due to (79), and reversibility holds. For \(d = 3\), the entanglement cost \(E_C(W^{(3)})\) is exactly equal to one, as recalled in (75), while (77) applies to \(E_D(W^{(3)})\), and the resource theory is irreversible. For \(d \in \{4, 5, 6\}\), the bounds are not strong enough to reach a conclusion about reversibility. For \(d \geq 7\), the resource theory is irreversible, and the gap \(E_C(W^{(d)}) - E_D(W^{(d)})\) grows at least as large as the difference of (73) and (78).

## D. Epolarizing channels (complements of depolarizing channels)

The \(d\)-dimensional depolarizing channel is a common model of noise in quantum information, transmitting the input state with probability \(1 - q \in [0, 1]\) and replacing it with the maximally mixed state \(\pi \equiv \frac{1}{d} \mathbb{I}\) with probability \(q\):
\[
\Delta^q(\rho) = (1 - q) \rho + q \pi. \tag{82}
\]
According to Stinespring’s theorem [Sti55], every quantum channel \(\mathcal{N}_{A \to B}\) can be realized by the action of some isometric channel \(\mathcal{U}_{A \to BE}\) followed by a partial trace:
\[
\mathcal{N}_{A \to B}(\rho_A) = \text{Tr}_E(\mathcal{U}_{A \to BE}(\rho_A)). \tag{83}
\]
Due to the partial trace and its invariance with respect to isometric channels acting exclusively on the \(E\) system, the extending channel \(\mathcal{U}_{A \to BE}\) is not unique in general, but it is unique up to this freedom. Then given an isometric channel \(\mathcal{U}_{A \to BE}\) extending \(\mathcal{N}_{A \to B}\) as in (83), the complementary channel \(\mathcal{N}_{A \to E}^{\perp}(\rho_A)\) is defined by a partial trace over the system \(B\) and is interpreted physically as the channel from the input to the environment:
\[
\mathcal{N}_{A \to E}^{\perp}(\rho_A) = \text{Tr}_B(\mathcal{U}_{A \to BE}(\rho_A)). \tag{84}
\]
Due to the fact that properties of the original channel are related to properties of its complementary channel...
there has been significant interest in understanding complementary channels. In this spirit, and due to the prominent role of the depolarizing channel, researchers have studied its complementary channels [DFH06, LW17]. In [DFH06, Eq. (3.6)], the following form was given for a complementary channel of $\Delta^q$:  
\begin{equation}
    \rho \rightarrow S_{AF}^q (\rho_A \otimes I_F) S_{AF}^q, \tag{85}
\end{equation}

where $I_F$ is a $d$-dimensional identity operator and  
\begin{equation}
    S_{AF}^q = \frac{q}{d} I_{AF} + \sqrt{d} \left( -\frac{\sqrt{q}}{d} + \sqrt{1 - q \left( \frac{d^2 - 1}{d^2} \right)} \right) \Phi_{AF}. \tag{86}
\end{equation}

A channel complementary to $\Delta^q$ has been called an “epolarizing channel” in [LW17].

An alternative complementary channel, related to the above one by an isometry acting on the output systems $AF$, but perhaps more intuitive, is realized in the following way. Consider the isometry $U_{A\rightarrow SG_1G_2A}$ defined as  
\begin{equation}
    U_{A\rightarrow SG_1G_2A}|\psi\rangle_A \equiv C\text{-SWAP}_{SG_1A} \left( |\phi^s\rangle_S \otimes |\Phi\rangle_{G_1G_2} \otimes |\psi\rangle_A \right), \tag{87}
\end{equation}

where the control qubit $|\phi^s\rangle_S \equiv \sqrt{1 - q}|0\rangle_S + \sqrt{q}|1\rangle_S$, $|\Phi\rangle_{G_1G_2}$ is a maximally entangled state of Schmidt rank $d$, and the controlled-SWAP unitary is given by  
\begin{equation}
    C\text{-SWAP}_{SG_1A} \equiv |0\rangle\langle 0| \otimes I_{G_1A} + |1\rangle\langle 1| \otimes \text{SWAP}_{G_1A}, \tag{88}
\end{equation}

with SWAP$_{G_1A}$ denoting a unitary swap operation. By tracing over the systems $SG_1G_2$, we recover the original depolarizing channel  
\begin{equation}
    \Delta^q(\rho_A) = \text{Tr}_{SG_1G_2} \{U_{A} U_A^\dagger\}. \tag{89}
\end{equation}

Thus, by definition, a channel complementary to $\Delta^q$ is realized by  
\begin{equation}
    \Lambda_{A\rightarrow SG_1G_2}^q(\rho_A) \equiv \text{Tr}_A \{U_{A} U_A^\dagger\}, \tag{90}
\end{equation}

and in what follows, let us refer to $\Lambda_{A\rightarrow SG_1G_2}^q$ as the epolarizing channel.

The isometry $U_{A\rightarrow SG_1G_2A}$ in (87) is unitarily covariant, in the sense that for an arbitrary unitary $V_A$ acting on the identity, we have that  
\begin{equation}
    U_{A\rightarrow SG_1G_2A} V_A = (V_G \otimes \overline{V}_G) U_{A\rightarrow SG_1G_2A}, \tag{91}
\end{equation}

where $\overline{V}$ denotes the complex conjugate of $V$. The identity in (91) follows because  
\begin{align}
    U_{A\rightarrow SG_1G_2A} V_A |\psi\rangle_A \\
    = C\text{-SWAP}_{SG_1A} \left( |\phi^s\rangle_S \otimes |\Phi\rangle_{G_1G_2} V_A |\psi\rangle_A \right) \\
    = C\text{-SWAP}_{SG_1A} \left( |\phi^s\rangle_S \left( V_G \otimes \overline{V}_G \right) |\Phi\rangle_{G_1G_2} V_A |\psi\rangle_A \right) \\
    = (V_G \otimes \overline{V}_G) C\text{-SWAP}_{SG_1A} \left( |\phi^s\rangle_S |\Phi\rangle_{G_1G_2} |\psi\rangle_A \right) \\
    = (V_G \otimes \overline{V}_G) U_{A\rightarrow SG_1G_2A} |\psi\rangle_A. \tag{92}
\end{align}

The above analysis omits tensor-product symbols for brevity. The third equality uses the well known fact that $|\Phi\rangle_{G_1G_2} = (V_G \otimes \overline{V}_G) |\Phi\rangle_{G_1G_2}$. In the fourth equality, we have exploited the facts that $\overline{V}_G$ commutes with C-SWAP$_{SG_1A}$ and that  
\begin{equation}
    \text{SWAP}_{G_1A} (V_G \otimes V_A) = (V_G \otimes V_A) \text{SWAP}_{G_1A}. \tag{93}
\end{equation}

The covariance in (91) then implies that the epolarizing channel is covariant in the following sense:  
\begin{equation}
    (\Lambda_{A\rightarrow SG_1G_2}^q \circ V_A)(\rho_A) = ((V_G \otimes \overline{V}_G) \circ \Lambda_{A\rightarrow SG_1G_2}^q)(\rho_A), \tag{94}
\end{equation}

where $V$ denotes the unitary channel realized by the unitary operator $V$. As such, by the discussion after (54), the epolarizing channel is a resource-seizable, teleportation-simulable channel with associated resource state given by $\Lambda_{A\rightarrow SG_1G_2}^q(\Phi_{RA})$. Thus, Theorem 1 applies to these channels, implying that the first two of the following equalities hold  
\begin{equation}
    E_C(\Lambda^q) = E_C[^p](\Lambda^q) = E_C(\Lambda^q(\Phi_{RA})) \tag{95}
\end{equation}
\begin{equation}
    = E_F(\Lambda^q(\Phi_{RA})) \tag{96}
\end{equation}
\begin{equation}
    = \left( 1 - q + \frac{q^2}{d} \right) \log_2 \left( 1 - q \frac{q}{d} \right) - (d - 1) \frac{q^2}{d} \log_2 \frac{q^2}{d}. \tag{97}
\end{equation}

Let us now justify the final two equalities, which give a simple formula for the entanglement cost of epolarizing channels. First, consider that the Choi state $\Lambda_{A'\rightarrow SG_1G_2}^q(\Phi_{A'A})$ of the epolarizing channel is equal to the state resulting from sending in the maximally mixed state to the isometric channel $U_{A\rightarrow SG_1G_2A}$, defined from (87):  
\begin{equation}
    \Lambda_{A'\rightarrow SG_1G_2}^q(\Phi_{A'A}) = U_{A\rightarrow SG_1G_2A}(\pi_A), \tag{98}
\end{equation}

and system $A'$ is isomorphic to $A$. This equality is shown in Appendix B. As such, then [MSW04, Theorem 3] applies, as discussed in Example 6 therein, and as a consequence, we can conclude the second and fourth equalities in the following, with the bipartite cut of systems taken as $SG_1G_2|A$:  
\begin{equation}
    E_C(\Lambda_{A\rightarrow SG_1G_2}^q(\Phi_{A'A})) = E_C(U_{A\rightarrow SG_1G_2A}(\pi_A)) \tag{99}
\end{equation}
\begin{equation}
    = E_F(U_{A\rightarrow SG_1G_2A}(\pi_A)) \tag{100}
\end{equation}
\begin{equation}
    = H_{\text{min}}(\Delta^q). \tag{101}
\end{equation}

The last line features the minimum output entropy of the depolarizing channel, which was identified in [Kin03] and shown to be equal to (97). As discussed in previous examples, it is worthwhile to consider the reversibility of the resource theory of entanglement for epolarizing channels. In this spirit, by invoking the covariance of $\Lambda^q$, the discussion after (54),
there is a gap for every value of \( q \in (0, 1) \), demonstrating that the resource theory of entanglement is irreversible for epolarizing channels.

\[ E_D(\Lambda^q) \leq R(T; SG_1G_2)_{\Lambda^q(\Phi)}, \quad (102) \]

where \( R(T; SG_1G_2)_{\Lambda^q(\Phi)} \) denotes the Rains relative entropy of the state \( \Lambda^q_{\Lambda^q(\Phi)}(\Psi_{TA}) \). Recall that the Rains relative entropy for an arbitrary state \( \rho_{AB} \) is defined as \([\text{Rai01}]\)

\[ R(A; B)_{\rho} \equiv \min_{\tau_{AB} \in \text{PPT}'(A; B)} D(\rho_{AB} || \tau_{AB}), \quad (103) \]

where the quantum relative entropy is defined as \([\text{Ume62}]\)

\[ D(\rho || \tau) \equiv \text{Tr}\{\rho \log_2 \rho - \log_2 \tau \} \quad (104) \]

and the Rains set \( \text{PPT}'(A; B) \) is given by

\[ \text{PPT}'(A; B) \equiv \{ \tau_{AB} : \tau_{AB} \geq 0 \land \| T_B(\tau_{AB}) \|_1 \leq 1 \}, \quad (105) \]

with \( T_B \) denoting the partial transpose \([\text{ADMVW02}]\). Appendix C details a Matlab program taking advantage of recent advances in [FSP18, FF18], in order to compute the Rains relative entropy of any bipartite state.

Figure 5 plots the entanglement cost of the epolaring channel for \( d = 2 \) (qubit input), and it also plots the Rains bound on distillable entanglement in (102). There is a gap for every value of \( q \in (0, 1) \), demonstrating that the resource theory of entanglement is irreversible for epolarizing channels. The figure also plots the coherent information of the state \( \Lambda^q_{\Lambda^q(\Phi)}(\Psi_{TA}) \), optimized with respect to \( |\Psi^s\rangle_{TA} \equiv \sqrt{s}|0\rangle_{TA} + \sqrt{1-s}|1\rangle_{TA} \) for \( s \in [0, 1] \), which is known to be a lower bound on the distillable entanglement of \( \Lambda^q \) \([\text{DW05}]\). Note that the coherent information plot is not in contradiction with the recent result of [LW17], which states that the coherent information is strictly greater than zero for all \( q \in (0, 1) \). It is simply that the coherent information is so small for \( q \lesssim 0.18 \), that it is difficult to witness its strict positivity numerically. Matlab files to generate Figure 5 are available with the arXiv posting of this paper.

V. BOSONIC GAUSSIAN CHANNELS

In this section, I extend the main ideas of the paper in order to characterize the entanglement cost of all single-mode bosonic Gaussian channels \([\text{Ser17}]\). From a practical perspective, we should be most interested in the single-mode thermal, amplifier, and additive-noise channels, as these are of the greatest interest in applications, as stressed in \([\text{Hol12}, \text{Section 12.6.3}]\) and \([\text{HG12}, \text{Section 3.5}]\). However, it also turns out that these are the only non-trivial cases to consider among all single-mode bosonic Gaussian channels, as discussed below.

A. On the definition of entanglement cost for infinite-dimensional channels

Before beginning, let us note that there are some subtleties involved when dealing with quantum information theory in infinite-dimensional Hilbert spaces \([\text{Hol12}]\). For example, as advised in \([\text{SH08}]\), the direct use of the diamond norm in infinite-dimensional Hilbert spaces could be too strong for applications, and this observation has motivated some recent work \([\text{Shi18, Win17}]\) on modifications of the diamond norm that take into account physical constraints such as energy limitations. On the other hand, the recent findings in \([\text{Wil18}]\) suggest that the direct use of the diamond norm is reasonable when considering single-mode thermal, amplifier, and additive-noise channels, as well as some multi-mode bosonic Gaussian channels. As it turns out, we can indeed directly employ the diamond norm when analyzing the entanglement cost of these channels. In fact, one of the main contributions of \([\text{Wil18}]\) was to consider uniform convergence issues in the teleportation simulation of bosonic Gaussian channels, and due to the fact that the operational framework of entanglement cost is directly related to the approximate teleportation simulation of a channel, one should expect that the findings of \([\text{Wil18}]\) would be related to the issues involved in the entanglement cost of bosonic Gaussian channels.

With this in mind, let us define the entanglement cost for an infinite-dimensional channel almost exactly as it has been defined in Section II.B, with the exception that we allow for LOCC channels that have a continuous classical index \( e.g., \) as considered in \([\text{Shi10, Section 4}]\), thus going beyond the LOCC channels considered in (4). Specifically, let us define an \((n, M, \varepsilon)\) sequential channel
simulation code as it has been defined in Section II B, noting that the $\varepsilon$-error criterion is given by (18), representing the direct generalization of the strategy norm of [CDP08a, CDP09b, Gut12] to infinite-dimensional systems. Achievable rates and the entanglement cost are then defined in the same way.

B. Preliminary observations about the entanglement cost of single-mode bosonic Gaussian channels

The starting point for our analysis of single-mode bosonic Gaussian channels is the Holevo classification from [Hol07], in which canonical forms for all single-mode bosonic Gaussian channels have been given, classifying them up to local Gaussian unitaries acting on the input and output of the channel. It then suffices for us to focus our attention on the canonical forms, as it is self-evident from definitions that local unitaries do not alter the entanglement cost of a quantum channel. The thermal and amplifier channels form the class C discussed in [Hol07], and the additive-noise channels form the class B2 discussed in the same work. The classes that remain are labeled A, B1, and D in [Hol07]. The channels in A and D are entanglement-breaking [Hol08], and as a consequence of the “if-part” of Remark 1, they have zero entanglement cost. Channels in the class B1 are perhaps not interesting for practical applications, and as it turns out, they have infinite quantum capacity [Hol07]. Thus, their entanglement cost is also infinite, because a channel’s quantum capacity is a lower bound on its distillable entanglement, which is in turn a lower bound on its entanglement cost—these relationships are a direct consequence of the definitions of the underlying quantities. For the same reason, the entanglement cost of the bosonic identity channel is also infinite.

C. Thermal, amplifier, and additive-noise channels

In light of the previous discussion, for the remainder of the paper, let us focus our attention on the thermal, amplifier, and additive-noise channels. Each of these are defined respectively by the following Heisenberg input-output relations:

\begin{align*}
\hat{b} &= \sqrt{\eta} \hat{a} + \sqrt{1-\eta} \hat{e}, \\
\hat{b} &= \sqrt{G} \hat{a} + \sqrt{G-1} \hat{e}, \\
\hat{b} &= \hat{a} + (x + ip) / \sqrt{2},
\end{align*}

where $\hat{a}$, $\hat{b}$, and $\hat{e}$ are the field-mode annihilation operators for the sender’s input, the receiver’s output, and the environment’s input of these channels, respectively.

The channel in (106) is a thermalizing channel, in which the environmental mode is prepared in a thermal state $\theta(N_B)$ of mean photon number $N_B \geq 0$, defined as

$$
\theta(N_B) \equiv \frac{1}{N_B + 1} \sum_{n=0}^{\infty} \left( \frac{N_B}{N_B + 1} \right)^n |n\rangle\langle n|, 
$$

where $\{ |n\rangle \}_{n=0}^{\infty}$ is the orthonormal, photonic number-state basis. When $N_B = 0$, $\theta(N_B)$ reduces to the vacuum state, in which case the resulting channel in (106) is called the pure-loss channel—it is said to be quantum-limited in this case because the environment is injecting the minimum amount of noise allowed by quantum mechanics. The parameter $\eta \in (0, 1)$ is the transmissivity of the channel, representing the average fraction of photons making it from the input to the output of the channel. Let $\mathcal{L}_{\eta,N_B}$ denote this channel, and we make the further abbreviation $\mathcal{L}_{\eta} \equiv \mathcal{L}_{\eta,N_B=0}$ when it is the pure-loss channel. The channel in (106) is entanglement-breaking when $(1-\eta) N_B \geq \eta$ [Hol08], and by Remark 1, the entanglement cost is equal to zero for these values.

The channel in (107) is an amplifier channel, and the parameter $G > 1$ is its gain. For this channel, the environment is prepared in the thermal state $\theta(N_B)$. If $N_B = 0$, the amplifier channel is called the pure-amplifier channel—it is said to be quantum-limited for a similar reason as stated above. Let $\mathcal{A}_{G,N_B}$ denote this channel, and we make the further abbreviation $\mathcal{A}_G \equiv \mathcal{A}_{G,N_B=0}$ when it is the quantum-limited amplifier channel. The channel in (107) is entanglement-breaking when $(G-1) N_B \geq 1$ [Hol08], and by Remark 1, the entanglement cost is equal to zero for these values.

Finally, the channel in (108) is an additive-noise channel, representing a quantum generalization of the classical additive white Gaussian noise channel. In (108), $x$ and $p$ are zero-mean, independent Gaussian random variables each having variance $\xi \geq 0$. Let $\mathcal{T}_\xi$ denote this channel. The channel in (108) is entanglement-breaking when $\xi \geq 1$ [Hol08], and by Remark 1, the entanglement cost is equal to zero for these values.

Kraus representations for the channels in (106)–(108) are available in [ISS11], which can be helpful for further understanding their action on input quantum states.

Due to the entanglement-breaking regions discussed above, we are left with a limited range of single-mode bosonic Gaussian channels to consider, which is delineated by the white strip in Figure 1 of [GGPCH14].

D. Upper bound on the entanglement cost of teleportation-simulable channels with bosonic Gaussian resource states

In this section, I determine an upper bound on the entanglement cost of any channel $N_{A \rightarrow B}$ that is teleportation simulable with associated resource state given by a bosonic Gaussian state. Related bosonic teleportation channels have been considered previously [BK98, BST02, TBS02, GIC02, WPGG07, NFC09], in the case that the LOCC channel associated to $N_{A \rightarrow B}$ is a Gaussian LOCC
channel. Proposition 3 below states that the entanglement cost of these channels is bounded from above by the Gaussian entanglement of formation [WGK+04] of the underlying bosonic Gaussian resource state, and as such, this proposition represents a counterpart to Proposition 2. Before stating it, let us note that the Gaussian entanglement of formation $E_{\rho}^G(A;B)$ of a bipartite state $\rho_{AB}$ [WGK+04] is given by the same formula as in (7), with the exception that the pure states $|\psi_{AB}\rangle$ in the ensemble decomposition are required to be Gaussian. Note that continuous probability measures are allowed for the decomposition (for an explicit definition, see [WGK+04, Section III]). Let us note here that the first part of the proof outlines a procedure for the formation of $n$ approximate copies of a bipartite state, and even though this kind of protocol has been implicit in prior literature, I have included explicit steps for clarity. After proving Proposition 2, I discuss its application to thermal, amplifier, and additive-noise bosonic Gaussian channels.

**Proposition 3** Let $\mathcal{N}_{A\rightarrow B}$ be a channel that is teleportation simulable as defined in (28), where the resource state $\omega_{A'B'}$ is a bosonic Gaussian state composed of $k$ modes for system $A'$ and $\ell$ modes for system $B'$, with $k, \ell \geq 1$. Then the entanglement cost of $\mathcal{N}_{A\rightarrow B}$ is never larger than the Gaussian entanglement of formation of the bosonic Gaussian resource state $\omega_{A'B'}$:

$$E_C(\mathcal{N}) \leq E_{\rho}^G(A';B')_\omega.$$  \hfill (110)

**Proof.** The main idea of the proof is to first form $n$ approximate copies of the bosonic Gaussian resource state $\omega_{A'B'}$, by using entanglement and LOCC as related to the approach from [BBPS96], and then after that, simulate $n$ uses of the channel $\mathcal{N}_{A\rightarrow B}$ by employing the structure of the channel $\mathcal{N}_{A\rightarrow B}$ from (28). Indispensable to the proof is the analysis in [WGK+04, Sections II and III], where it is shown that every bosonic Gaussian state can be decomposed as a Gaussian mixture of local displacements acting on a fixed Gaussian pure state and that such a decomposition is optimal for the Gaussian entanglement of formation [WGK+04, Proposition 1]. The Gaussian mixture of local displacements can be understood as an LOCC channel $\mathcal{G}_{A'B'}$, and let $\psi_{A'B'}^\omega$ denote the aforementioned fixed Gaussian pure state such that $\psi_{A'B'}^\omega(\psi_{A'B'}^\omega) = \omega_{A'B'}$.

Since $\psi_{A'B'}^\omega$ is Gaussian, the marginal state $\psi_{A'}^\omega$ is Gaussian, and thus it has finite entropy $H(B')_\omega$, as well as finite entropy variance, i.e.,

$$V(B')_\omega \equiv \text{Tr}(\psi_{A'B'}^\omega[-\log_2 \psi_{A'B'}^\omega - H(B')_\omega]^2) < \infty,$$  \hfill (111)

the latter statement following from the Williamson decomposition [Wil36] for Gaussian states as well as the formula for the entropy variance of a bosonic thermal state [WRG16]. For $\delta > 0$, recall that the entropy-typical projector $\Pi_\delta^{B'n}$ [OP93, Sdi95] of the state $\psi_{B'}^\omega$ is defined as the projection onto

$$\text{span}\{ |\xi_{zn}\rangle : |-n^{-1} \log_2 (p_{Z^n}(z^n)) - H(B')_\omega| \leq \delta \},$$  \hfill (112)

where a countable spectral decomposition of $\psi_{B'}^\omega$ is given by

$$\psi_{B'}^\omega = \sum_z p_Z(z)|\xi_z\rangle\langle \xi_z|,$$  \hfill (113)

and

$$|\xi_{zn}\rangle \equiv |\xi_{z_1}\rangle \otimes \cdots \otimes |\xi_{z_n}\rangle,$$  \hfill (114)

$$p_{Z^n}(z^n) \equiv p_Z(z_1) \cdots p_Z(z_n).$$  \hfill (115)

The entropy-typical projector $\Pi_\delta^{B'n}$ projects onto a finite-dimensional subspace of $[\psi_{B'}^\omega]^\otimes n$, and satisfies the conditions $[\Pi_\delta^{B'n}, [\psi_{B'}^\omega]^\otimes n] = 0$ and

$$2^{-n[H(B')_\omega + \delta]} \Pi_\delta^{B'n} \leq \Pi_\delta^{B'n}[\psi_{B'}^\omega]^\otimes n \Pi_\delta^{B'n} \leq 2^{-n[H(B')_\omega - \delta]} \Pi_\delta^{B'n}.$$  \hfill (116)

It then follows that $\text{Tr}(\Pi_\delta^{B'n}) \leq 2^{n[H(B')_\omega + \delta]}$. Furthermore, consider that the entropy-typical projector $\Pi_\delta^{B'n}$ for the state $[\psi_{B'}^\omega]^\otimes n$ satisfies

$$\text{Tr}\{(I_{A'n} \otimes \Pi_\delta^{B'n})[\psi_{A'B'}^\omega]^\otimes n \} = \text{Tr}(\Pi_\delta^{B'n}[\psi_{B'}^\omega]^\otimes n)$$

$$\geq 1 - \frac{V(B')_\omega}{\delta^2 n},$$  \hfill (117)

with the inequality following from the definition of the entropy-typical projector and an application of the Chebyshev inequality. By the gentle measurement lemma [Win99, ON07] (see [Wil17, Lemma 9.4.1] for the version employed here), we conclude that

$$\frac{1}{2} \left\| [\psi_{A'B'}^\omega]^\otimes n - \psi_{A'B'}^\omega \right\| < \sqrt{\frac{V(B')_\omega}{\delta^2 n}},$$  \hfill (118)

where

$$\psi_{A'B'}^\omega \equiv \frac{(I_{A'n} \otimes \Pi_\delta^{B'n})[\psi_{A'B'}^\omega]^\otimes n (I_{A'n} \otimes \Pi_\delta^{B'n})}{\text{Tr}(I_{A'n} \otimes \Pi_\delta^{B'n})[\psi_{A'B'}^\omega]^\otimes n}.$$  \hfill (119)

Observe that the system $B'n$ of $\psi_{A'B'}^\omega$ is supported on a finite-dimensional subspace of $B'n$.

Now, the idea of forming $n$ approximate copies $\tilde{\psi}_{A'B'}^\omega$ is then the same as it is in [BBPS96]: Alice prepares the state $\tilde{\psi}_{A'B'}^\omega$ locally, Alice and Bob require beforehand a maximally entangled state of Schmidt rank no larger than $2^{n[H(B')_\omega + \delta]}$, and then they perform quantum teleportation [BBC+93] to teleport the $B'n$ system to Bob. At this point, they share exactly the state $\tilde{\psi}_{A'B'}^\omega$, which becomes less and less distinguishable from $[\psi_{A'B'}^\omega]^\otimes n$ as $n$ grows large, due to (118). Now applying the Gaussian LOCC channel $(\mathcal{G}_{A'B'}^\omega)^\otimes n$, the data processing inequality to (118), and the fact that $\mathcal{G}_{A'B'}(\tilde{\psi}_{A'B'}^\omega) = \omega_{A'B'}$, we conclude that

$$\frac{1}{2} \left\| \psi_{A'B'}^\otimes n - (\mathcal{G}_{A'B'})^\otimes n(\tilde{\psi}_{A'B'}^\omega) \right\| \leq \sqrt{\frac{V(B')_\omega}{\delta^2 n}}.$$  \hfill (120)
Thus, to see that $H(B')_ω$ is an achievable rate for forming $\omega_{A'B'}^n$, fix $\varepsilon \in (0, 1]$ and $\delta > 0$. Then choose $n$ large enough so that $\sqrt{\frac{\varepsilon}{\delta + \delta}} \leq \varepsilon$. Apply the above procedure, using LOCC and a maximally entangled state of Schmidt rank no larger than $2^n[H(B')_ω + \delta]$. Then the rate of entanglement consumption to produce $n$ approximate copies of $\omega_{A'B'}^n$ satisfying (120) is $H(B')_ω + \delta$. Since this is possible for $\varepsilon \in (0, 1]$, $\delta > 0$, and sufficiently large $n$, we conclude that $H(B')_ω$ is an achievable rate for the formation of $\omega_{A'B'}^n$. Now, since achieving this rate is possible for any pure state $\psi_{A'B'}^\omega$ such that $\omega_{A'B'} = \mathcal{G}_{A'B'}(\psi_{A'B'}^\omega)$, we conclude that the infimum of $H(B')_ω$ with respect to all such pure states is an achievable rate. But this latter quantity is exactly the Gaussian entanglement of formation according to [WGK+04, Proposition 1].

The idea for simulating $n$ uses of the channel $\mathcal{N}_{A\rightarrow B}$ is then the same as the idea used in the proof of Proposition 2. First form $n$ approximate copies of $\omega_{A'B'}^n$ according to the procedure described above. Then, when the $i$th call to the channel $\mathcal{N}_{A\rightarrow B}$ is made, use the LOCC channel $\mathcal{L}_{A'A'B'B'}$ from the definition in (28) along with the $i$th $A'$ and $B'$ systems of the state approximating $\omega_{A'B'}^n$ to simulate it. By the same reasoning that led to (38), the distinguishability of the final states of any sequential test is limited by the distinguishability of the state $\omega_{A'B'}^n$ from its approximation, which I argued in (120) can be made arbitrarily small with increasing $n$. Thus, the Gaussian entanglement of formation $\omega_{A'B'}$ is an achievable rate for sequential channel simulation of $n$ uses of $\mathcal{N}_{A\rightarrow B}$.

1. Upper bound for the entanglement cost of thermal, amplifier, and additive-noise bosonic Gaussian channels

I now discuss how to apply Proposition 2 to single-mode thermal, amplifier, and additive-noise channels. Some recent papers [LSMGA17, KW17, TDR18] have shown how to simulate each of these channels by using a bosonic Gaussian resource state along with variations of the continuous-variable quantum teleportation protocol [BK98]. Of these works, the one most relevant for us is the latest one [TDR18], because these authors proved that the entanglement of formation of the underlying resource state is equal to the entanglement of formation that results from transmitting through the channel one share of a two-mode squeezed vacuum state with arbitrarily large squeezing strength. That is, let $\mathcal{N}_{A\rightarrow B}$ denote a single-mode thermal, amplifier, or additive-noise channel. Then one of the main results of [TDR18] is that, associated to this channel, there is a bosonic Gaussian resource state $\omega_{A'B'}$ and a Gaussian LOCC channel $\mathcal{G}_{A'A'B'\rightarrow B}$ such that

$$E_F(A'; B')_ω = \sup_{\sigma(N_S)} E_F(R; B)_σ(N_S)$$

(121)

$$= \lim_{N_S \rightarrow \infty} E_F(R; B)_σ(N_S),$$

(122)

where

$$\sigma(N_S) \equiv \mathcal{N}_{A\rightarrow B}(\psi_{RA}^N),$$

(123)

$$\phi_{RA}^N \equiv |\phi_{RA}^N⟩⟨\phi_{RA}^N|,$$

(124)

$$|\phi_{RA}^N⟩ RA = \frac{1}{\sqrt{N_S + 1}} \sum_{n=0}^{N_S} \sqrt{\frac{N_S}{N_S + 1}} |n⟩_R |n⟩_A,$$

(125)

and for all input states $\rho_A$,

$$\mathcal{N}_{A\rightarrow B}(\rho_A) = \mathcal{G}_{A'A'B'\rightarrow B}(\rho_A \otimes \omega_{A'B'}).$$

(126)

In the above, $\phi_{RA}^N$ is the two-mode squeezed vacuum state $[\text{Ser17}].$ Note that the equality in (122) holds because one can always produce $\phi_{RA}^N$ from $\phi_{RA}^N$ such that $N_S \geq N_S$, by using Gaussian LOCC and the local displacements involved in the Gaussian LOCC commute with the channel $\mathcal{N}_{A\rightarrow B}$ [GEC03] (whether it be thermal, amplifier, or additive-noise). Furthermore, the entanglement of formation does not increase under the action of an LOCC channel.

Thus, applying the above observations and Proposition 3, it follows that there exist bosonic Gaussian resource states $\omega_{A'B'}$, $\omega_{A'B'}^{N_S}$, and $\omega_{A'B'}^n$ associated to the respective thermal, amplifier, and additive-noise channels in (106)–(108), such that the following inequalities hold

$$E(C(\eta_{N_S})) \leq E_F(A'; B')_{ω', N_B}$$

(127)

$$E(C(A_{G, N_S})) \leq E_F(A'; B')_{ω', N_B},$$

(128)

$$E(C(T_ε)) \leq E_F(A'; B')_{ω', ε}.$$

(129)

Analytical formulas for the upper bounds on the right can be found in [TDR18, Eqs. (4)–(6)].

E. Lower bound on the entanglement cost of bosonic Gaussian channels

In this section, I establish a lower bound on the non-asymptotic entanglement cost of thermal, amplifier, or additive-noise bosonic Gaussian channels. After that, I show how this bound implies a lower bound on the entanglement cost. Finally, by proving that the state resulting from sending one share of a two-mode squeezed vacuum through a pure-loss or pure-amplifier channel has entanglement cost equal to entanglement of formation, I establish the exact entanglement cost of these channels by combining with the results from the previous section.

**Proposition 4** Let $\mathcal{N}_{A\rightarrow B}$ be a thermal, amplifier, or additive-noise channel, as defined in (106)–(108). Let
Then the following bound holds for any \((n,M,\varepsilon)\) sequential or parallel channel simulation code for \(N_{A\to B}\):

\[
\frac{1}{n} \log_2 M \geq \frac{1}{n} E_F(R^n; B^n)_{\omega^{\otimes n}} - (\varepsilon' + 2\delta) H(\phi_R^{N_S/\delta}) - \frac{1}{n} [2(1+\varepsilon') g_2(\varepsilon') + 2h_2(\delta)], \\
\tag{130}
\]

where \(\omega_{RB} \equiv N_{A\to B}(\phi_{RA}^{N_S})\) and \(\frac{1}{n} \log_2 M\) is understood as the non-asymptotic entanglement cost of the protocol.

**Proof.** The reasoning here is very similar to that given in the proof of Proposition 1, but we can instead make use of the continuity bound for the entanglement of formation of energy-constrained states [Shi16, Proposition 5]. To begin, suppose that there exists an \((n,M,\varepsilon)\) protocol for sequential channel simulation. Then by previous reasoning (also see Figure 2), it can be thought of as a parallel channel simulation protocol, such that the criterion in (2) holds. Let us take \((\phi_{RA}^{N_S})^{\otimes n}\) to be a test input state, leading to \(\omega_{RN} = N_{A\to B}(\phi_{RA}^{N_S})^{\otimes n}\) when the actual channels are applied and \(\sigma_{R_1\cdots R_nB_1\cdots B_n}\), when the simulation is applied. Set

\[
f(n,\varepsilon,\varepsilon',N_S) \equiv n (\varepsilon' + 2\delta) H(\phi_R^{N_S/\delta}) + 2(1+\varepsilon') g_2(\varepsilon') + 2h_2(\delta). \tag{131}\]

Then we have that

\[
E_F(R^n; B^n)_{\omega^{\otimes n}} \\
\leq E_F(R^n; B^n)_{\sigma} + f(n,\varepsilon,\varepsilon',N_S) \\
\leq E_F(R^nA^n; B^n)_{\psi^\otimes N} + f(n,\varepsilon,\varepsilon',N_S) \\
= E_F(\bar{A}_0B_0)_{\psi^\otimes N} + f(n,\varepsilon,\varepsilon',N_S) \\
= \log_2 M + f(n,\varepsilon,\varepsilon',N_S). \tag{132}\]

The first inequality follows from the condition in (18), as well as from the continuity bound for entanglement of formation from [Shi16, Proposition 5], noting that the total photon number of the reduced (thermal) state on systems \(R^n\) is equal to \(nN_S\). The second inequality follows from the LOCC monotonicity of the entanglement of formation, here thinking of the person who possesses systems \(R^n\) to be in the same laboratory as the one possessing the systems \(A_i\), while the person who possesses the \(B_i\) systems is in a different laboratory. The first equality follows from the fact that \((\phi_{RA}^{N_S})^{\otimes n}\) is in tensor product with \(\Phi_{\bar{A}_0,\bar{B}_0}\), so that by a local channel, one may remove \((\phi_{RA}^{N_S})^{\otimes n}\) or append it for free. The final equality follows because the entanglement of formation of the maximally entangled state is equal to the logarithm of its Schmidt rank.

A direct consequence of Proposition 4 is the following lower bound on the entanglement cost of the thermal, amplifier, and additive-noise channels:

**Proposition 5.** Let \(N_{A\to B}\) be a thermal, amplifier, or additive-noise channel, as defined in (106)–(108). Then the entanglement costs \(E_C(N)\) and \(E_C^{(p)}(N)\) are bounded from below by the entanglement of the state \(N_{A\to B}(\phi_{RA}^{N_S})\), where the two-mode squeezed vacuum state \(\phi_{RA}^{N_S}\) has arbitrarily large squeezing strength:

\[
E_C(N) \geq E_C^{(p)}(N) \tag{133}
\]

\[
\geq \sup_{N_S \geq 0} E_C(N_{A\to B}(\phi_{RA}^{N_S})) \tag{134}
\]

\[
= \lim_{N_S \to \infty} E_C(N_{A\to B}(\phi_{RA}^{N_S})). \tag{135}
\]

**Proof.** The first inequality follows from definitions, as argued previously in (22). To arrive at the second inequality, in Proposition 4, set \(\varepsilon' = \sqrt{2\varepsilon}\), and take the limit as \(n \to \infty\) and then as \(\varepsilon \to 0\). Employing the fact that \(\lim_{\varepsilon \to 0} \xi H(\phi_{RA}^{N_S/\delta}) = 0\) [Shi06, Proposition 1] and applying definitions, we find for all \(N_S \geq 0\) that

\[
E_C(N) \geq E_C^{(p)}(N) \tag{136}
\]

\[
\geq \lim_{n \to \infty} \frac{1}{n} E_F(N_{A\to B}(\phi_{RA}^{N_S}))^{\otimes n} \tag{137}
\]

\[
= E_C(N_{A\to B}(\phi_{RA}^{N_S})). \tag{138}
\]

Since the above bound holds for all \(N_S \geq 0\), we conclude the bound in the statement of the proposition. The equality in (135) follows for the same reason as given for the equality in (22), and due to the fact that entanglement cost is non-increasing with respect to an LOCC channel by definition.

**F. Additivity of entanglement of formation for pure-loss and pure-amplifier channels**

The bound in Proposition 5 is really only a formal statement, as it is not clear how to evaluate the lower bound explicitly. If it would however be possible to prove that

\[
\frac{1}{n} E_F(N_{A\to B}(\phi_{RA}^{N_S}))^{\otimes n} \geq E_F(N_{A\to B}(\phi_{RA}^{N_S})) \tag{139}
\]

for all integer \(n \geq 1\) and all \(N_S \geq 0\), then we could conclude the following

\[
E_C(N) \geq \lim_{N_S \to \infty} E_F(N_{A\to B}(\phi_{RA}^{N_S})), \tag{140}
\]

implying that this lower bound coincides with the upper bound from (127)–(129), due to the recent result of [TDR18] recalled in (121)–(122).

In Proposition 6 below, I prove that the additivity relation in (139) indeed holds whenever the channel \(N_{A\to B}\) is a pure-loss channel \(L_\eta\) or pure-amplifier channel \(A_G\). The linchpin of the proof is the multi-mode bosonic minimum output entropy theorem from [GHGP15] and [GHM15, Theorem 1].
Proposition 6 For $\mathcal{N}_{A\rightarrow B}$ a pure-loss channel $\mathcal{L}_\eta$ with transmissivity $\eta \in (0,1)$ or a pure-amplifier channel $\mathcal{A}_G$ with gain $G > 1$, the following additivity relation holds for all integer $n \geq 1$ and $N_S \in [0, \infty)$:

$$
\frac{1}{n} E_F([\mathcal{N}_{A\rightarrow B}(\phi^{N_S}_{RA})]^{\otimes n}) = E_F(\mathcal{N}_{A\rightarrow B}(\phi^{N_S}_{RA})) = E^\eta_F(\mathcal{N}_{A\rightarrow B}(\phi^{N_S}_{RA})),
$$

(141)

where $\phi^{N_S}_{RA}$ is the two-mode squeezed vacuum state from (125) and $E^\eta_F$ denotes the Gaussian entanglement of formation. Thus, the entanglement cost of $\mathcal{N}_{A\rightarrow B}(\phi^{N_S}_{RA})$ is equal to its entanglement of formation:

$$
EC(\mathcal{N}_{A\rightarrow B}(\phi^{N_S}_{RA})) = E_F(\mathcal{N}_{A\rightarrow B}(\phi^{N_S}_{RA})).
$$

(143)

Proof. The proof of this proposition relies on three key prior results:

1. The main result of [KW04] is that the entanglement of formation $E_F(A; B)_{\psi}$ is equal to the classically-conditioned entropy $H(A|E)_{\psi}$ for a tripartite pure state $\psi_{ABE}$:

$$
E_F(A; B)_{\psi} = H(A|E)_{\psi},
$$

(144)

where

$$
H(A|E)_{\psi} = \inf_{\{A_E\}} \sum_x p_X(x) H(A)_{\sigma^x},
$$

(145)

with the optimization taken with respect to a positive operator-valued measure $\{A_E\}_x$ and

$$
p_X(x) \equiv \text{Tr}\{A^x_E \psi_E\},
$$

(146)

and

$$
\sigma^x_A \equiv \frac{1}{p_X(x)} \text{Tr}_E\{(I_A \otimes A^x_E) \psi_{AE}\}.
$$

(147)

The sum in (145) can be replaced with an integral for continuous-outcome measurements. The equality in (144) can be understood as being a consequence of the quantum steering effect [Sch35].

2. The determination of and method of proof for the classically-conditioned entropy $H(A|E)_{\rho}$ of an arbitrary two-mode Gaussian state $\rho_{AE}$ with covariance matrix in certain standard forms [PSB+14]. (As remarked below, there is in fact a significant strengthening of the main result of [PSB+14], which relies on item 3 below.)

3. The multi-mode bosonic minimum output entropy theorem from [GHGP15] and [HGP15, Theorem 1] (see the related work in [MGH14, GGP14] also), which implies that the following identity holds for a phase-insensitive, single-mode bosonic Gaussian channel $\mathcal{G}$ and for all integer $n \geq 1$:

$$
\inf_{\rho^{\otimes n}} H(G^{\otimes n}(\rho^{(n)})) = H(G^{\otimes n}([0|0]^{\otimes n})) = nH(G([0|0])),
$$

(148)

where the optimization is with respect to an arbitrary $n$-mode input state $\rho^{(n)}$ and $[0|0]$ denotes the bosonic vacuum state.

Indeed, these three key ingredients, with the third being the linchpin, lead to the statement of the proposition after making a few observations. Consider that a purification of the state $\rho_{AB} = (id_{R\rightarrow A} \otimes \mathcal{L}_\eta)(\phi^{N_S}_{RA})$ is given by

$$
\psi_{ABE} = (id_{R\rightarrow A} \otimes \mathcal{B}^\eta_{AE\rightarrow BE})(\phi^{N_S}_{RA} \otimes |0\rangle|0\rangle_E),
$$

(149)

where $\mathcal{B}^\eta_{AE\rightarrow BE}$ represents the unitary for a two-mode squeezer [Ser17] and $|0\rangle|0\rangle_E$ again denotes the vacuum state. Tracing over the system $B$ gives the state $\psi_{AB} = (id_{R\rightarrow A} \otimes \mathcal{C}_{1-\eta})(\phi^{N_S}_{RA})$, where $\mathcal{C}_{1-\eta}$ is a pure-loss channel of transmissivity $1-\eta$. The state $\psi_{AE}$ is well known to have its covariance matrix in standard form [Ser17] (see discussion surrounding [PSB+14, Eq. (5)]) as

$$
\begin{pmatrix}
 a & c & 0 \\
 0 & a & -c \\
 c & 0 & b \\
 0 & -c & b
\end{pmatrix}
$$

(150)

and is also known as a two-mode squeezed thermal state [Ser17]. As such, the main result of [PSB+14] applies, and we can conclude that heterodyne detection is the optimal measurement in (145), which in turn implies from (144) that the entanglement of formation of $\rho_{AB}$ is equal to the Gaussian entanglement of formation.

However, what we require is that the same results hold for the multi-copy state $\psi_{AE}^{\otimes n}$. Inspecting Eqs. (9)–(14) of [PSB+14], it is clear that the same steps hold, except that we replace Eq. (12) therein with (148). Thus, it follows that $n$ individual heterodyne detections on each $E$ mode of $\psi_{AE}^{\otimes n}$ is the optimal measurement, so that

$$
\frac{1}{n} H(A|E)_{\psi^{\otimes n}} = H(A|E)_{\psi}. 
$$

(151)

By applying (144) (as applied to the states $\rho_{AB}^{\otimes n}$ and $\psi_{AE}^{\otimes n}$), we conclude that

$$
\frac{1}{n} E_F(A^n; B^n)_{\rho^{\otimes n}} = E_F(A; B)_{\rho}. 
$$

(152)

Furthermore, since the optimal measurement is given by heterodyne detection, performing it on mode $E$ of $\psi_{ABE}$ induces a Gaussian ensemble of pure states $\{p_X(x), \psi_{AB}^x\}$, which is the optimal decomposition of $\psi_{AB} = \rho_{AB}$, and thus we conclude that $E_F(A; B)_{\rho} = E^\eta_F(A; B)_{\rho}$.

A similar analysis applies for the quantum-limited amplifier channel. I give the argument for completeness. Consider that a purification of the state $\sigma_{AB} = (id_{R\rightarrow A} \otimes \mathcal{A}_G)(\phi^{N_S}_{RA})$ is given by

$$
\varphi_{ABE} = (id_{R\rightarrow A} \otimes \mathcal{S}^G_{AE\rightarrow BE})(\phi^{N_S}_{RA} \otimes |0\rangle|0\rangle_E),
$$

(153)

where $\mathcal{S}^G_{AE\rightarrow BE}$ represents the unitary for a two-mode squeezer [Ser17] and $|0\rangle|0\rangle_E$ again denotes the vacuum state. Tracing over the system $B$ gives the state $\varphi_{AE} = (id_{R\rightarrow A} \otimes \mathcal{A}_G)(\phi^{N_S}_{RA})$, where $\mathcal{A}_G$ denotes the channel conjugate to the quantum-limited amplifier. The state $\varphi_{AE}$
has its covariance matrix in the form (see Mathematica files included with the arXiv posting or alternatively [Ser17, Appendix D.4])
\[
\begin{bmatrix}
ap & 0 & c 
0 & ap & c 
cp & 0 & b 
0 & cp & b 
\end{bmatrix},
\]
and so the same proof approach to get (151) can be used to conclude that
\[
\frac{1}{n} H(A^n|E^n)_{\varphi^n} = H(A|E)_\varphi.
\] (155)
Indeed, this additionally follows from the discussion after [PSB+14, Eqs. (17)–(19)]. As such, we conclude in the same way that
\[
\frac{1}{n} E_F(A^n; B^n)_{\varphi^n} = E_F(A; B)\sigma = E_F(A; B)_\sigma.
\] (156)

The final statement about entanglement cost in (143) follows from the fact that it is equal to the regularized entanglement of formation. ■

**Remark 2** As can be seen from the proof above, the multi-mode minimum output entropy theorem recalled in (148) provides a significant strengthening of the results from [PSB+14]. Indeed, for a two-mode Gaussian state considered in [PSB+14], the following equality holds
\[
\frac{1}{n} H(A^n|E^n)_{\varphi^n} = H(A|E)_\rho,
\] (157)
implying that the measurement \(\{A^{n}_E\}_x\) optimal for the right-hand side leads to a measurement \(\{A^{n}_E\} \times \cdots \times A^{n}_E\}_{x_1, \ldots, x_n}\) that is optimal for the left-hand side. Furthermore, by the relation in (144), for any purification \(\psi_{ABE}\) of the state \(\rho_{AE}\) mentioned above, we conclude that
\[
\frac{1}{n} E_F(A^n; B^n)_{\psi^n} = E_F(A; B)\psi,
\] (158)
for all integer \(n \geq 1\), thus giving a whole host of two-mode Gaussian states for which their entanglement cost is equal to their entanglement of formation: \(E_F(A; B)_\rho = E_F(A; B)\sigma = E_F(A; B)\sigma\). As far as I am aware, these are the first examples of two-mode Gaussian states for which the additivity relation in (158) has been explicitly shown.

**Remark 3** One might wonder whether the same method of proof as given in Proposition 6 could be used to establish the equalities in (141)–(142) for general thermal, amplifier, and additive-noise channels. At the moment, it is not clear how to do so. The issue is that the state \((|id_R \otimes \mathcal{L}_B\rangle\langle \mathcal{L}_B|_N)^N\) for \(N > 0\) is a faithful state, meaning that it is positive definite and thus has two symplectic eigenvalues > 1. This means that any purification of it requires at least four modes [HW01, Section III-D]. Then tracing over the B system leaves a three-mode state, of which we should be measuring two of them, and so it is not clear how to apply the methods of [PSB+14] to such a state. The same issues apply to the states \((|id_R \otimes \mathcal{A}_G\rangle\langle \mathcal{A}_G|_N)^N\) for \(N > 0\) and \((|id_R \otimes \mathcal{T}_B\rangle\langle \mathcal{T}_B|_\xi)^N\) for \(\xi > 0\), which are the states resulting from the amplifier and additive-noise channels, respectively.

**G. Entanglement cost of pure-loss and pure-amplifier channels**

Based on the results in the previous sections, we conclude the following theorem, which gives simple formulas for the entanglement cost of two fundamental bosonic Gaussian channels:

**Theorem 2** For a pure-loss channel \(\mathcal{L}_\eta\) with transmissivity \(\eta \in (0, 1)\) or a pure-amplifier channel \(\mathcal{A}_G\) with gain \(G > 1\), the following formulas characterize the entanglement costs of these channels:

\[
E_C(\mathcal{L}_\eta) = E^{(p)}_C(\mathcal{L}_\eta) = \frac{h_2(1 - \eta)}{1 - \eta},
\]
(159)
\[
E_C(\mathcal{A}_G) = E^{(p)}_C(\mathcal{A}_G) = \frac{g_2(G - 1)}{G - 1},
\]
(160)
where \(h_2(\cdot)\) is the binary entropy defined in (66) and \(g_2(\cdot)\) is the bosonic entropy function defined in (25).

**Proof.** Recalling the discussion in Section V.D.1, for a pure-loss and pure-amplifier channel, there exist respective resource states \(\omega^{A'B'}_a\) and \(\omega^{A'B'}_b\) such that

\[
E_C(\mathcal{L}_\eta) \leq E_C(A'; B')_{\omega^{A'B'}_a},
\]
(161)
\[
= \lim_{N_S \to \infty} E_F(R; B)_{\sigma^{N_S}(N_S)},
\]
(162)
\[
E_C(\mathcal{A}_G) \leq E_C(A'; B')_{\omega^{A'B'}_b},
\]
(163)
\[
= \lim_{N_S \to \infty} E_F(R; B)_{\sigma^{N_S}(N_S)},
\]
(164)
where

\[
\sigma^{\eta}(N_S)_{RB} \equiv (|id_R \otimes \mathcal{L}_\eta\rangle\langle \mathcal{L}_\eta|_N)^N_{RA},
\]
(165)
\[
\sigma^{G}(N_S)_{RB} \equiv (|id_R \otimes \mathcal{A}_G\rangle\langle \mathcal{A}_G|_N)^N_{RA},
\]
(166)
with the equalities in (162) and (164) being one of the main results of [TDR18]. Furthermore, explicit formulas for \(E_F(A'; B')_{\omega^{A'B'}_a}\) and \(E_F(A'; B')_{\omega^{A'B'}_b}\) have been given in [TDR18, Eqs. (4)–(6)], and evaluating these formulas leads to the expressions in (159)–(160) (supplementary Mathematica files that automate these calculations are available with the arXiv posting of this paper).
On the other hand, Propositions 5 and 6 imply that
\[
E_C(\mathcal{L}_\eta) \geq E_C(\mathcal{L}_\eta') (167)
\]
\[
\geq \lim_{N_S \to \infty} E_C(\sigma^\eta(N_S)_{RB}) (168)
\]
\[
= \lim_{N_S \to \infty} E_F(R; B)_{\sigma^\eta(N_S)}, (169)
\]
\[
E_C(A_G) \geq E_C(A_G') (170)
\]
\[
\geq \lim_{N_S \to \infty} E_C(\sigma^G(N_S)_{RB}) (171)
\]
\[
= \lim_{N_S \to \infty} E_F(R; B)_{\sigma^G(N_S)}. (172)
\]

Combining the inequalities above, we conclude the statement of the theorem.

It is interesting to consider various limits of the formulas in (159)–(160):
\[
\lim_{\eta \to 1} \frac{h_2(1-\eta)}{1-\eta} = \lim_{G \to 1} \frac{g_2(G-1)}{G-1} = \infty, (173)
\]
\[
\lim_{\eta \to 0} \frac{h_2(1-\eta)}{1-\eta} = \lim_{G \to \infty} \frac{g_2(G-1)}{G-1} = 0. (174)
\]

We expect these to hold because the channels approach the ideal channel in the limits \(\eta, G \to 1\), which we previously argued has infinite entanglement cost, while they both approach the completely depolarizing (useless) channel in the no-transmission limit \(\eta \to 0\) and infinite-amplification limit \(G \to \infty\). Furthermore, these formulas obey the symmetry
\[
\frac{h_2(1-\eta)}{1-\eta} = \frac{g_2(1/\eta - 1)}{1/\eta - 1}, (175)
\]
which is consistent with the idea that the transformation \(\eta \to 1/\eta\) takes a channel of transmissivity \(\eta \in [0,1]\) and produces a channel of gain \(1/\eta\). Finally, we have the Taylor expansions:
\[
\frac{h_2(1-\eta)}{1-\eta} = \frac{\eta}{\ln 2}(1-\ln(\eta)) + O(\eta^2), (176)
\]
\[
\frac{g_2(G-1)}{G-1} = \frac{1+\ln(G)}{G\ln 2} + O(1/G^2), (177)
\]
which are relevant in the low-transmissivity and high-gain regimes.

In [Pir17], simple formulas for the distillable entanglement of these channels were determined and given by
\[
E_D(\mathcal{L}_\eta) = -\log_2(1-\eta), (178)
\]
\[
E_D(A_G) = -\log_2(1-1/G). (179)
\]

Thus, the prior results and the formulas in Theorem 2 demonstrate that the resource theory of entanglement for these channels is irreversible. That is, if one started from a pure-loss channel of transmissivity \(\eta\) and distilled ebits from it at the ideal rate of \(-\log_2(1-\eta)\), and then subsequently wanted to use these ebits to simulate a pure-loss channel with the same transmissivity, this is not possible, because the rate at which ebits are distilled is not sufficient to simulate the channel again. The same statement applies to the pure-amplifier channel. Figures 6 and 7 compare the formulas for entanglement cost and distillable entanglement of these channels, demonstrating that there is a noticeable gap between them. I note here that the differences are given by
\[
E_C(\mathcal{L}_\eta) - E_D(\mathcal{L}_\eta) = -\frac{\eta \log_2 \eta}{1-\eta}, (180)
\]
\[
E_C(A_G) - E_D(A_G) = \frac{\log_2 G}{G-1}, (181)
\]

implying that these differences are strictly greater than zero for all the relevant channel parameter values \(\eta \in (0,1)\) and \(G > 1\).

\[
\text{VI. EXTENSION TO OTHER RESOURCE THEORIES}
\]

Let us now consider how to extend many of the concepts in this paper to other resource theories (see [CG18] for a review of quantum resource theories). In fact, this can be accomplished on a simple conceptual level by replacing “LOCC channel” with “free channel,” “separable state” with “free state,” and (roughly) “maximally entangled state” with resource state throughout the paper. To be precise, let \(F\) denote the set of free states for a given resource theory, and let \(F\) be a free channel, which takes a free state to a free state. In [KW18, Section 7], a general notion of distillation of a resource from \(n\) uses of a channel was given (see Figure 4 therein). In particular, one interleaves \(n\) uses of a given channel by free channels, and the goal is to distill resource from the \(n\) channels. As
Then for the same reasons given there, the simulation cost of \( \mathcal{N} \) should never exceed the simulation cost of \( \mathcal{M} \).

Finally, let us note that some discussions about channel simulation for the resource theory of coherence have appeared in the last paragraph of \cite{BDGDWM17}, as well as the last paragraphs of \cite{DFW18}. It is clear from the findings of the present paper that identifying interesting resource-seizable channels could be a useful first step for understanding interconversion costs of simulating one channel from another in the resource theory of coherence. It could also be helpful in further understanding channel simulation in the resource theory of thermodynamics \cite{FBB18}.

**VII. CONCLUSION**

In summary, this paper has provided a new definition for the entanglement cost of a channel, in terms of the most general strategy that a discriminator could use to distinguish \( n \) uses of the channel from its simulation. I established an upper bound on the entanglement cost of a teleportation-simulable channel in terms of the entanglement cost of the underlying resource state, and I proved that the bound is saturated in the case that the channel is resource-seizable (Definition 2). I then established single-letter formulas for the entanglement cost of erasure, dephasing, three-dimensional Werner–Holevo channels, and epolarizing channels (complements of depolarizing channels), by leveraging existing results about the entanglement cost of their Choi states. I finally considered single-mode bosonic Gaussian channels, establishing bounds on the entanglement cost of the thermal, amplifier, and additive-noise channels, while giving simple formulas for the entanglement cost of pure-loss and pure-amplifier channels. By relating to prior work on the distillable entanglement of these channels, it became clear that the resource theory of entanglement for quantum channels is irreversible.

Going forward from here, there are many directions to pursue. The discrimination protocols considered in Section II B do not impose any realistic energy constraint on the states that can be used in discriminating the actual \( n \) uses of the channel from the simulation. We could certainly do so by imposing that the average energy of all the states input to the actual channel or its simulation should be less than a threshold, and the result is to demand only that the energy-constrained strategy norm \cite{CDP08a, CDP09b, Gut12} is less than \( \varepsilon \in (0, 1) \). To be specific, let \( H_A \) be a (positive semi-definite) Hamiltonian acting on the input of the channel \( \mathcal{N}_{A \rightarrow B} \) and let \( E \in [0, \infty) \) be an energy constraint. Then,
demanding that the supremum in (18) is taken over all strategies such that
\[ \frac{1}{n} \sum_{i=1}^{n} \text{Tr} \{ H_i \rho_{A_i} \}, \frac{1}{n} \sum_{i=1}^{n} \text{Tr} \{ H_i \tau_{A_i} \} \leq E, \tag{185} \]
the resulting quantity is an energy-constrained strategy norm. With an energy constraint in place, one would expect that less entanglement is required to simulate the channel than if there is no constraint at all, and the resulting entanglement cost would depend on the given energy constraint. For example, Proposition 4 leads to a lower bound on entanglement cost for an energy-constrained sequential simulation, but it remains open to determine if there is a matching upper bound.

Similar to how measures like squashed entanglement [CW04] and relative entropy of entanglement [VP98] allow for obtaining converse bounds or fundamental limitations on the distillation rates of quantum states or channels, simply by making a clever choice of a squashings channel or separable state, it would be useful to have a measure like this for bounding entanglement cost from below. That is, it would be desirable for the measure to involve a supremum over a given set of test states or channels rather than an infimum as is the case for squashed entanglement and relative entropy of entanglement. For example, it would be useful to be able to bound the entanglement cost of thermal, amplifier, and additive-noise channels from below, in order to determine how tight are the upper bounds in (127)–(129). Progress on this front is available in [WD17], but more results in this area would be beneficial.

One of the main tools used in the analysis of the (parallel) entanglement cost of channels from [BBCW13] is a de-Finetti-style approach, consisting of the post-selection technique [CKR09]. In particular, the problem of asymptotic (parallel) channel simulation was reduced to simulating the channel on a single state, called the universal de Finetti state. For the asymptotic theory of (sequential) entanglement cost of channels, could there be a single universal adaptive channel discrimination protocol to consider, such that simulating the channel well for such a protocol would imply that it has been simulated well for all protocols?

For the task of entanglement cost, one could modify the set of free channels to be either those that completely preserve the positivity of partial transpose [Rai99, Rai01] or are $k$-extendible in the sense of [KDW18]. Could we find simpler lower bounds on entanglement cost of channels in this way? The semi-definite programming quantity from [WD17] could be helpful here also.

Another way to think about quantum channel simulation is to allow the entanglement to be free but count the cost of classical communication. This was the approach taken for the reverse Shannon theorem [BDH14, BCR11], and these works also considered only parallel channel simulation. How are the results there affected if the goal is sequential channel simulation instead? Is the previous answer from [BDH14, BCR11], the mutual information of the channel, robust under this change? How do prior results on simulation of quantum measurements [Win04, WHBH12, BRW14] hold up under this change? A comprehensive summary of results on parallel simulation of quantum channels, including the quantum reverse Shannon theorem, measurement simulation, and entanglement cost, is available in [Ber13].

Finally, is there an example of a channel for which its sequential entanglement cost is strictly greater than its parallel entanglement cost? The examples discussed here are those for which either there are equalities or no conclusion could be drawn. Evidence from quantum channel discrimination [HHLW10] and related evidence from [DW17] suggests the possibility. One concrete example to examine in this context is the channel presented in [CMH17, Appendix A], given that it is not implementable from its image, as discussed there.

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### Appendix A: Relation between resource-seizable channels and those that are implementable from their image

Definition 2 introduced the notion of a resource-seizable channel, and Section VI discussed how this notion can play a role in an arbitrary resource theory. In [CMH17, Appendix A], a channel $\mathcal{N}_{A \rightarrow B}$ was defined to be implementable from its image if there exists a state $\sigma_{RA}$ and an LOCC channel $\mathcal{L}_{A'^{n} \rightarrow B}$ such that the following equality holds for all input states $\rho_{A}$:

\[
\mathcal{N}_{A \rightarrow B}(\rho_{A}) = \mathcal{L}_{A'^{n} \rightarrow B}(\rho_{A} \otimes \mathcal{N}_{A'^{n} \rightarrow B}(\sigma_{A'A''})), \tag{A1}
\]

where system $A''$ is isomorphic to system $A$ and system $B'$ is isomorphic to system $B$. An example of a channel that is not implementable from its image was discussed at length in [CMH17, Appendix A].

Here, I prove that a channel is resource-seizable in the resource theory of entanglement if and only if it is implementable from its image. To see this, suppose that a channel is implementable from its image. Then, given the above structure in (A1), it is clear that $\mathcal{N}_{A \rightarrow B}$ is teleportation simulable with associated resource state given by $\omega_{A'B'} = \mathcal{N}_{A'' \rightarrow B}(\sigma_{A'A'})$. Thus, one can trivially seize the resource state $\omega_{A'B'}$ by sending in the input state $\sigma_{A'A''}$, which is clearly separable between Alice and Bob, given that Bob’s “system” here is trivial.
Now suppose that a teleportation-simulable channel is resource-seizable, as in Definition 2. This means that

\[ \mathcal{N}_{A\rightarrow B}(\rho_A) = \mathcal{M}_{A'\rightarrow B'}(\rho_A \otimes \omega_{A'B'}) , \]  

(A2)

where \( \omega_{A'B'} \) is the resource state and \( \mathcal{M}_{A'\rightarrow B'} \) is an LOCC channel. Furthermore, since it is resource-seizable, this means that there exists a separable state \( \rho_{AB} \) and a postprocessing LOCC channel \( \mathcal{D}_{AB \rightarrow A'B'} \) such that

\[ \mathcal{D}_{AB \rightarrow A'B'}(\mathcal{N}_{A\rightarrow B}(\rho_{AB})) = \omega_{A'B'} . \]  

(A3)

To see that the channel is implementable from its image, consider that \( \rho_{AB} \) has a decomposition as follows, given that it is separable:

\[ \sum_x p_x(x) \psi^x_{A_x} \otimes \phi^x_{B_x} , \]  

(A4)

for \( p_x \) a probability distribution and \( \{ \psi^x_{A_x} \}_x \) and \( \{ \phi^x_{B_x} \}_x \) sets of pure states. Now define the input state \( \sigma_{AX} \) as

\[ \sigma_{AX} = \sum_x p_x(x) \psi^x_{A_x} \otimes |x_x\rangle \langle x_x| , \]  

(A5)

and note that this is the state we can use for implementing the channel’s image. Define the LOCC measure-prepare channel \( \mathcal{P}_{XA\rightarrow B} \) as

\[ \mathcal{P}_{XA\rightarrow B}(\cdot) \equiv \sum_x |x_x\rangle \langle x_x| \mathcal{P}_{A\rightarrow B}(\cdot) , \]  

(A6)

which is understood to be implemented via LOCC by measuring Alice’s system \( X \), communicating the outcome \( x \) to Bob, who then prepares the state \( \phi^x_{B_x} \) based on the outcome. We find that

\[ (\mathcal{D}_{AB} \circ \mathcal{P}_{XA\rightarrow B} \circ \mathcal{N}_{A\rightarrow B})(\sigma_{AX}) = \omega_{A'B'} , \]  

(A7)

We finally conclude that

\[ \mathcal{N}_{A\rightarrow B}(\rho_A) = \mathcal{M}_{A'\rightarrow B' \otimes A'\rightarrow B'}(\sigma_{AX}) , \]  

(A8)

where

\[ \mathcal{L}_{A'A\rightarrow B} \equiv \mathcal{M}_{A'\rightarrow B} \circ \mathcal{D}_{AB} \circ \mathcal{P}_{XA \rightarrow B} \circ \mathcal{N}_{A\rightarrow B} \]  

(A9)

so that the channel is implementable from its image by inputting the state \( \sigma_{AX} \) and postprocessing with the LOCC channel \( \mathcal{M}_{A'\rightarrow B} \circ \mathcal{D}_{AB} \circ \mathcal{P}_{XA \rightarrow B} \).

**Appendix B: Relation between Choi state of a complementary channel and maximally mixed state sent through isometric extension**

The purpose of this appendix is to prove the equality in (98). Consider a d-dimensional depolarizing channel

\[ \rho \rightarrow (1-p)\rho + \frac{p}{d}I \]  

(B1)

As noted in [DFH06, Eq. (3.2)], a Kraus representation for this channel is as follows:

\[ \{ \sqrt{1-p}I, \sqrt{p/d}|i\rangle \langle j|_{i,j} \} \]  

(B2)

This is because

\[ \left[ \sqrt{1-p}I \right] \rho \left[ \sqrt{1-p}I \right]^\dagger + \sum_{i,j} \left[ \sqrt{p/d}|i\rangle \langle j| \right] \rho \left[ \sqrt{p/d}|j\rangle \langle i| \right]^\dagger = (1-p)\rho + \frac{p}{d} \text{Tr}(\rho)I . \]  

(B3)

Now consider a generic channel \( \mathcal{N}_{A\rightarrow B} \) with Kraus operators \( \{ N_i \} \), so that an isometric extension is given by \( \sum_i N_i \otimes |i\rangle_E \). Send the maximally mixed state \( \pi = I/d \) through the isometric extension \( \sum_i N_i \otimes |i\rangle_E \). This leads to the state

\[ \frac{1}{d} \sum_{i,j} N_i N_j^\dagger \otimes |i\rangle \langle j|_E . \]  

(B5)

Furthermore, a complementary channel of the original channel, resulting from the isometric extension \( \sum_i N_i \otimes |i\rangle_E \), is then

\[ \rho \rightarrow \mathcal{N}_{\tilde{A}\rightarrow E}(\rho) = \sum_{i,j} \text{Tr}(N_i \rho N_j^\dagger) |i\rangle \langle j|_E . \]  

(B6)

The Choi state for this complementary channel is given by

\[ \Phi_{RA} = \frac{1}{d} \sum_{k,l,i,j} |k\rangle \langle l|_R \otimes \text{Tr}(N_i^k |l\rangle \langle l| A N_j^l \rangle J) |i\rangle \langle j|_{E} \]  

(B7)

where \( T(N_j^l N^i) \) denotes the transpose of \( N^j N^i \). If it holds that \( N^i N^j = T(N^j N^i) \), then we conclude that the state resulting from sending in the maximally mixed
state to the isometric extension of the channel is the same as the Choi state of the complementary channel. This is the case for the depolarizing channel with the Kraus operators in (B2). Since all complementary channels and isometric extensions of a channel are related by an isometry acting on the environment system, we are arrive at the same conclusion for any isometric extension and the corresponding complementary channel to which it leads.

Appendix C: Matlab code for computing Rains relative entropy

This appendix provides a brief listing of Matlab code that can be used to compute the Rains relative entropy of a bipartite state $\rho_{AB}$ [Rai01, ADMVW02]. The code requires the QuantInf package in order to generate a random state [Cub], the CVX package for semi-definite programming optimization [GB14], and the CVXQuad package [Faw] for relative entropy optimization [FSP18, FF18].

Listing 1. Matlab code for calculating the Rains relative entropy of a random bipartite state $\rho_{AB}$.

```matlab
na = 2; nb = 2;
rho = randRho(na*nb); % Generate a random bipartite state rho
cvx_begin sdp
    variable tau(na*nb,na*nb) hermitian;
    minimize ( quantum.rel.entr(rho, tau)/ log(2) );
    tau >= 0;
    norm_nuc(Tx(tau, 2, [ na nb ])) <= 1;
cvx_end

rains.rel.entr = cvx_optval;
```

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