Designing Metagratings Via Local Periodic Approximation: From Microwaves to Infrared

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Recently, metamaterials-inspired diffraction gratings (or metagratings) have demonstrated unprecedented efficiency in wavefront manipulation by means of relatively simple structures. Conventional one-dimensional (1D) gratings have a profile modulation in one direction and a translation symmetry in the other. In 1D metagratings, the translation invariant direction is engineered at a subwavelength scale what allows one to accurately control polarization line currents and, consequently, the scattering pattern. In bright contrast to metasurfaces, metagratings cannot be described by means of surface impedance densities (or local reflection and transmission coefficients). In this paper, we present a simulation-based design approach to construct metagratings in the “unit cell by unit cell” manner. It represents an analog of local periodic approximation (LPA) that has been used to design space modulated metasurfaces and allows one to overcome the limitations of straightforward numerical optimization and semi-analytical procedures that have been used up to date to design metagratings. Electric and magnetic metagrating structures responding to respectively transverse electric (TE) and transverse magnetic (TM) incident plane-waves are presented to validate the proposed design approach.

I. INTRODUCTION

In the last decade there has been tremendous interests in metasurfaces due to their amazing capabilities in manipulating electromagnetic fields [1–3] and broad range of potential applications [3–6]. Metasurfaces are represented by dense distributions of engineered subwavelength scatterers on a surface being planar analogs of metamaterials. When the characteristics of a metasurface are spatially modulated, it can perform wavefront transformations [5, 8]. Local periodic approximation (LPA) plays a crucial role in designing such metasurfaces and has been already used for long time [9, 10]. Essentially, it serves to estimate scattering properties of a unit cell embedded in a nonuniform array. To that end, a unit cell is placed in the corresponding uniform array whose reflection and transmission coefficients are then attributed to the unit cell in a nonuniform array. Scattering parameters of a uniform array are usually calculated from full-wave numerical simulations. However, there are particularly simple cases (e.g. metallic patches) that can be treated analytically [11–13].

Recently, metamaterials-inspired diffraction gratings have demonstrated unprecedented efficiency in manipulating scattering patterns with relatively simple structures [14–18]. Reflecting configuration of a metagrating represented by a one-dimensional periodic array of thin “wires” placed on top of a metal-backed dielectric substrate is illustrated in Fig. 1(a). Generally, each period of a metagrating consists of $N$ different “wires”. Note-worthy, the distance $d$ between adjacent “wires” always remains of the order of operating wavelength $\lambda$, which does not allow one to introduce neither averaged surface impedances nor local reflection coefficient in contrast to metasurfaces. Incident wave excites electric or magnetic polarization line currents in “wires” that can be controlled by judiciously adjusting the electromagnetic response of the “wires”. Consequently, it becomes possible to manipulate diffraction orders. It has been demonstrated that it is possible to achieve perfect nonspecular reflection and beam splitting with a single “wire” per period [14, 16, 18]. In a more general manner, it was shown that in order to perfectly control diffraction patterns one needs two degrees of freedom (represented by
reactively loaded “wires”) per each propagating diffraction order [19]. However, even having the number of reactive “wires” per period equal to the number of propagating diffraction orders enables to perform efficient multichannel reflection, as demonstrated in [17].

Practically, a “wire” is constituted from subwavelength meta-atoms arranged in a dense uniform 1D array. Geometrical parameters of meta-atoms determine electromagnetic response of a “wire”. Up to date metagratings have been designed either by performing 3D full-wave numerical optimization of a whole metagrating’s period [15] or semi-analytically [16, 17, 20, 21]. While the first approach can be very time consuming when it comes to designing metagratings having many “wires” per period, the second one allows one to consider only very simple meta-atoms such as printed capacitors [16] or dielectric cylinders [20]. In this paper, we develop local periodic approximation for designing metagratings with the help of 3D full-wave numerical simulations. In comparison to straightforward numerical optimization, it significantly reduces the time spent on the design of metagratings since within LPA one deals with a single unit cell at a time. It also allows one to deal with complex meta-atoms’ geometries and accurately account for interactions between adjacent “wires” which is not possible with simple analytical models.

The outline of the paper is as follows. In the second section we describe a retrieval technique which is used to extract characteristics of a “wire” from scattering parameters of the corresponding uniform array. We consider “wires” possessing either electric or magnetic responses which allows one to deal with both TE and TM polarizations. In the same section, we outline a model that can be used in a full-wave simulation software to construct a look-up table connecting retrieved characteristics of “wires” with corresponding parameters of meta-atoms. The third section is devoted to validation examples where we demonstrate metagratings’ designs operating in microwave and infrared frequency domains. In the fourth section we discuss remained challenges and conclude the paper.

II. LOCAL PERIODIC APPROXIMATION FOR METAGRATINGS

We consider reflecting metagratings operating either under TE or TM incident wave polarization. Therefore, each “wire” composing a metagrating can be characterized by a scalar electric impedance density $Z_q$ (or scalar magnetic admittance density $Y_q$ in the case of a “wire” possessing magnetic response). However, the approach can be readily generalized to transmitting metagratings as it is discussed in the last section, polarization insensitive metagratings can be developed by designing “wires” having both electric and magnetic responses and applying the method developed further for each polarization separately. Previously, one has distinguished between load- and input-impedance densities [16, 17] which we do not separate in the present study dealing only with impedance density as the principal characteristic of a “wire”. To be more accurate, we assume that the impedance density represents the sum of the load-impedance density and reactive part of the input-impedance density. According to LPA, in order to find impedance density, a “wire” is placed in the corresponding uniform array (period $d < \lambda$) illuminated by a plane-wave incident at an angle $\theta$. (b) Schematic diagram of a uniform array of “wires” characterized by the same impedance density $Z_q$. The inset represents the different meta-atoms composing a “wire”. (c) Principal model used in numerical simulations to calculate the reflection coefficient from a uniform array of “wires” implemented with meta-atoms.

A. Electric response, TE polarization

An electric polarization line current $I$ (excited by TE plane-wave in a “wire” composing the uniform array) is linked to the complex amplitude $S_0^{E}$ of the electric field of the specularly reflected wave via the following formula (see Appendix A)

$$I = \frac{-2d (S_0^{E} - R_0^E e^{2j\beta_0 h})\beta_0}{k\eta (1 + R_0^E e^{2j\beta_0 h})}.$$  

(1)
Here, $k$ and $\eta$ are the wavenumber and characteristic impedance outside the substrate, $R_{0}^{TE}$ is the Fresnel's reflection coefficient from the metal-backed substrate of a TE plane-wave at incidence angle $\theta$ and $\beta_0 = k \cos(\theta)$. Since all “wires” are assumed to be very thin, they are modeled as lines represented mathematically by the Dirac delta function $\delta(y, z)$. Consequently, the interaction with the substrate and between adjacent “wires” can be taken into consideration analytically by means of the mutual-impedance density $Z_m$ (see Appendix A). It allows one to obtain the characteristic of a “wire” itself and not of the array. “Wire’s” electric impedance density is found by means of Ohm’s law leading to the following expression for $Z_q$

$$Z_q = \frac{E_0}{I} - \frac{k \eta}{4} - Z_m, \quad (2)$$

where $E_0 = (1 + R_{0}^{TE}) \exp[j \beta_0 h]$ represents the value of the external electric field (incident wave plus its reflection from the metal-backed substrate) at the the “wire” located at $y = 0$ and $z = -h$. The radiation resistance of a “wire” is equal to $k\eta/4$ being independent on its particular implementation as it follows from power conservation conditions [22]. It is important to note that mutual-impedance density depends only on the period of the uniform array and parameters of the metal-backed substrate, but does not depend on the current $I$. It means that the impedance density given by Eq. (2) accurately represents the characteristic of the corresponding “wire” in a nonuniform array as long as the distance between adjacent “wires” and the metal-backed substrate remain the same as those of the uniform array. However, we leave open the questions of accuracy when considering “wires” built up from finite size meta-atoms as infinitely thin and estimating mutual interactions by means of analytically calculated mutual-impedance density.

**B. Magnetic response, TM polarization**

The case of TM polarization and “wires” possessing magnetic response can be treated with the help of duality relations [22]: $E \rightarrow H, H \rightarrow -E, I \rightarrow V$ and $\eta \rightarrow 1/\eta$. Since the metal-backed dielectric substrate is not replaced by the corresponding dual equivalent, we have to additionally make the following substitution $R_{0}^{TM} \rightarrow R_{0}^{TM}$. Thus, from Eq. (1) one can arrive at the formula for retrieving the magnetic current $V$ from the complex amplitude of the magnetic field of the specularly reflected plane-wave (see Appendix B)

$$V = -\frac{2d\eta}{k} \frac{(S_0^{TM} - R_{0}^{TM} e^{2j \beta_0 h})\beta_0}{(1 + R_{0}^{TM}) e^{j \beta_0 h}}, \quad (3)$$

where $R_{0}^{TM}$ is the Fresnel’s reflection coefficient from the metal-backed substrate of a TM-polarized plane-wave at incidence angle $\theta$. As previously, the interaction with the substrate and between adjacent “wires” can be taken into account by means of the mutual-admittance density $Y_m$ calculated analytically (see Appendix B). Then, the magnetic admittance density $Y_q$ can be found as

$$Y_q = \frac{H_0}{V} - \frac{k}{4\eta} - Y_m. \quad (4)$$

Here $H_0 = (1 + R_{0}^{TM}) \exp[j \beta_0 h]$ is the value of the external magnetic field at the “wire” located at $y = 0$ and $z = -h$, $k/(4\eta)$ represents the radiation conductance.
periodic boundary conditions are imposed on the side faces (as shown in Fig. 1(c)). The model is excited by a periodic port assigned to the face of the air region in opposite to the unit cell, as highlighted in red color in Fig. 1(c). The periodic Port creates a plane-wave incident at angle \( \theta \). It is important to take into account \( \theta \) since meta-atoms are usually spatially dispersive (impedance density depends on the incidence angle, see for instance Ref. [10]).

C. Look-up table

As it is stated above, in practice a “wire” would be implemented with subwavelength meta-atoms arranged in a line. The ultimate goal of the developing approach is to construct a look-up table linking geometrical parameters of meta-atoms with corresponding impedance (admittance) densities. To that end, we harness 3D full-wave numerical simulations (in our examples COMSOL MULTIPHYSICS is used). The geometry of the model consists of two principal parts: a tested unit cell (illustrated by a printed inductance on a metal-backed dielectric substrate in Fig. 1(c)) and air region. Periodic boundary conditions are imposed on the side faces (as shown in Fig. 1(c)). The model is excited by a periodic port assigned to the face of the air region in opposite to the unit cell, as highlighted in red color in Fig. 1(c). The periodic Port creates a plane-wave incident at angle \( \theta \). It is important to take into account \( \theta \) since meta-atoms are usually spatially dispersive (impedance density depends on the incidence angle, see for instance Ref. [10]).

III. EXAMPLES

A. Microwave frequency range

We start by considering simple meta-atoms represented by printed capacitance and inductance (illustrated in Fig. 2) that have already been used to implement metagratings at microwave frequencies by means of printed circuit board (PCB) technology [18, 19]. In the simulations, we take into account practical aspects of the design such as finite thickness of the copper traces \( t_m = 35 \mu m \) and dielectric losses introduced by the substrate (F4BM220 in our examples, \( \varepsilon_r = 2.2 \) with loss tangent \( 10^{-5} \)). Figure 2 depicts the impedance densities calculated by means of the developed LPA at 10 GHz (vacuum wavelength \( \lambda = 30 \text{ mm} \)). It is seen that having printed capacitance and inductance one is able to cover a broad range of impedance densities (imaginary part) that is normally enough to realize any diffraction pattern for TE polarization. It is important to note, that “wires” can exhibit significant resistive response (see Fig. 2(f) at the resonance) which should be kept in mind while designing metagratings. When comparing the numerical results with analytical models used in Refs. [16, 19], one would see very good agreement for the imaginary part of the impedance density at small values of parameter \( \alpha \). The resonance observed in Fig. 2(f) appears when one decreases the parameter \( C \) and it cannot be found by simple analytical formula used in Ref. [19].

In order to deal with TM polarization, we harness split ring resonators (SRRs) excited by the magnetic field and, thus, having effective magnetic response. Figure 3(a) illustrates the schematics of the unit cell which at close look consists of two SRRs separated by a short distance as seen from Fig. 3(b). The unit cell has an inversion center allowing to eliminate the bianisotropic response attributed to single and double SRRs [24, 25]. In order to adjust the response of a “wire” represented by a 1D
absorption

FIG. 4. (a) Schematic of a metagrating having period 7λ/2 (λ is the operating vacuum wavelength) and exciting seven propagating diffraction orders under normally incident plane-wave. The red and green beams represent suppressed and equally excited orders, respectively. (b), (c) Simulated frequency response (normalized power scattered in propagating diffraction orders versus frequency) of the metagrating operating under (b) TE and (c) TM polarizations and establishing the diffraction pattern illustrated by figure (a). In accordance with Ref. [17], to that end we need the number of reactive “wires” per period equal to the number of propagating diffraction orders, i.e. seven. Although Ref. [17] deals only with electric line currents and TE polarization, it is straightforward to generalize the approach to magnetic currents and TM polarization (See Appendix B). Figures 3(b) and (c) demonstrate the frequency response of electric and magnetic metagratings designed for 10 GHz. Overall, one can see that despite all practical limitations, the designed metagratings almost perfectly perform the desired splitting of the incident wave. It is important to note that the response of the metagrating operating under TE polarization (Fig. 4(b)) shows a broader response than the one operating under TM polarization (Fig. 4(c)). It is naturally explained by the resonant behavior of the considered SRR-based unit cell (see Fig. 3(b)). Indeed, printed capacitance and inductance (illustrated, respectively, by Figs. 2(a) and (c)) used for the metagrating dealing with TE polarization do not exhibit resonances (see Figs. 2(b) and (d)). The other feature of the magnetic metagrating is the significant absorption (compared to the case of TE polarization) which, however, does not deteriorate the overall performance.

B. Infrared frequency range

In this subsection we give an example of possible designs of unit cells that can be used as building blocks for metagratings operating at infrared frequencies. In what follows, the operation frequency is set to 75 THz corresponding to the vacuum wavelength of 4 μm. In order to implement capacitive and inductive unit cells for infrared domain we consider metallic (gold) patches and wires. Gold elements are placed on a dielectric substrate (silicon dioxide) backed with gold as illustrated in Figs. 5(a) and (b). The capacitive response is attributed to the gap between two patches. By changing the size of the gap or the size of patches one is able to adjust the capacitance. The inductance of a metallic wire is determined only by the cross section area. Although the design of the unit cells is relatively simple, it allows one to obtain impedance densities in a quite wide range of values as shown in Figs. 5(c) and (d). Interestingly, the real part of the impedance density remains very small due to non-resonant response of the unit cells even though we take into account absorption in gold and silicon dioxide.

It is worthwhile to note that a single straight metallic wire may not be enough in case strong inductive re-
FIG. 5. (a), (b) Schematic diagrams of (a) two gold patches and (b) a gold “wire” exhibiting capacitive and inductive responses, respectively. The gold elements are placed on a silicon dioxide layer backed by a gold plating. (c), (d) Impedance densities calculated by means of LPA as functions of geometrical parameters of the unit cells and corresponding, respectively, to “wires” built up from gold patches shown in figure (a) and gold wires in figure (b). Other parameters are fixed: silicon dioxide layer has permittivity $\varepsilon_s \approx 1.93$ and thickness $h = 700 \text{ nm}$, $d = 2 \mu\text{m}$, $w = 200 \text{ nm}$, $t_m = 200 \text{ nm}$ ($t_m = w$ in case of figure (d)), $g = 350 \text{ nm}$. Working frequency is set to 75 THz (vacuum wavelength of 4 $\mu\text{m}$). Normally incident plane-wave is assumed where electric field is oriented along the $B$ dimension.

IV. DISCUSSION AND CONCLUSION

In this work we have presented a simulation-based design approach to construct metragratings in the “unit cell by unit cell” manner. It represents an analog of local periodic approximation that has been used to design space modulated metasurfaces. The developed approach has been validated via 3D full-wave numerical simulations by demonstrating designs of metagratings controlling scattering patterns at microwave and infrared frequencies for both TE and TM polarizations. Simple and accurate analytical model describing metagratings allows one to take into account the impact of the metal-backed dielectric substrate and the interaction between “wires” composing the metagratings. It makes local periodic approximation a rigorous approach to design metagratings represented by nonuniform arrays of “wires” (in bright contrast to metasurfaces).

In this work, we have considered a reflecting configuration of a 1D metagrating but the developed design approach can be generalized to deal also with transmitting and 2D metagratings. Up to date there are two methods to control transmission with metagratings: either by means of an asymmetric three-layer array of electrically only “wires” or a single-layer array of bianisotropic particles. In order to deal with 2D metagratings one
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Appendix A: Retrieval of electric current and impedance density from specular reflection

In this section, we derive Eqs. (A1) and (A2). To that end we use analytical formulas for the electric field scattered by a N-cells metagrating given in Ref. [17]. Since in LPA we simulate only a single “wire” per period, the electric field of the wave reflected from the periodic array depicted in Fig. 1(b) can be represented via plane-wave expansion as follows

$$E_x(y, z) = -j \frac{k\eta}{2d} \sum_{m=1}^{+\infty} \left(1 + \frac{R_{m}^{TE}}{\beta_m} \right) e^{-j(\xi_m y + j\beta_m(z+h))} + \frac{R_0^{TE}}{\beta_0} e^{-j\eta_0 y + j\beta_0(z+2h)} ,$$

(A1)

where \( \xi_m = k \sin \theta + 2\pi m/d \) and \( \beta_m = \sqrt{k^2 - \xi_m^2} \) are, respectively, tangential and normal components of the wave vector of the \( m \)th diffraction order. The Fresnel’s reflection coefficient from the substrate backed by perfect electric conductor (PEC) is given by the following formula

$$R_m^{TE} = \frac{j\xi_m \tan[\beta_m h]}{j\xi_m \tan[\beta_m h] + 1} , \quad \beta_m = \frac{k \eta_0 \beta_0}{k \eta_0 \beta_0} ,$$

(A2)

where \( \eta = \sqrt{\mu/\varepsilon} \) and \( \eta_s = \sqrt{\mu_s/\varepsilon} \). When a metal backing the dielectric substrate cannot be modeled as PEC, one has to correspondingly modify the reflection coefficient. The period \( d \) and incidence angle \( \theta \) are such that the only propagating diffraction order corresponds to the specular reflection (\( m = 0 \)). Then, the amplitude \( S_0^{TE} \) of the zeroth diffraction order can be found from Eq. (A1) to be

$$S_0^{TE} = R_0^{TE} e^{2j\beta_0h} - j \frac{k\eta_0 \beta_0}{2d} e^{j\beta_0h} .$$

(A3)

This formula is then used to express the current \( I \) leading to Eq. (1).

In order to calculate the impedance density of a “wire” one needs to find the ratio between the total electric field \( E_{(loc)} \) at the position of a “wire” and the current \( I \) in the “wire” (and then subtract the radiation resistance). \( E_{(loc)} \) can be represented by the sum of the external electric field \( E_0 \), the electric field created by all other line currents, and the field resulted from the reflection from the metal-backed substrate. The last two can be united and expressed as \(-Z_m I \) with the mutual-impedance density \( Z_m \) given by the following formula

$$Z_m = \frac{k\eta}{2} \sum_{n=1}^{+\infty} \cos[k \sin \theta n L] H_0^{(2)}(knL)$$

$$+ \frac{k\eta}{2d} \sum_{m=+\infty}^{+\infty} \frac{R_{m}^{TE}}{\beta_m} .$$

(A4)

Thus one arrives at the Eq. (2).
Appendix B: Magnetic metagratings

A reflecting metagrating possessing magnetic-only response is modeled as a periodic array of magnetic line currents placed on top of a metal-backed dielectric substrate. In the first two subsections we derive analytical formulas (following Ref. [17]) for the field radiated by such a system and describe the way towards control of diffraction patterns with magnetic metagratings. In the last subsection we derive Eqs. (3) and (4).

1. Radiation of an array of magnetic line currents

A single magnetic line current \( \mathbf{M}(r) = V \delta(y, z) \delta_x \) radiates a TM-polarized wave with the magnetic field being along the \( x \)-direction (see Ref. [23])

\[
H_x(y, z) = -\frac{k}{4\eta} VH_0^{(2)} \left[ k \sqrt{y^2 + z^2} \right],
\]

where \( H_0^{(2)} \left[ k \sqrt{y^2 + z^2} \right] \) is the Hankel function of the second kind. Consequently, the magnetic field radiated by a periodic array of \( N \) magnetic line currents per period \( \mathbf{M}_{nq}(r) = V_q \exp[-jk \sin(\theta)L] \delta(y - y_{nq}, z) \delta_x \) (see Fig. [1] (a)) is given by the series of Hankel functions

\[
H_x(y, z) = -\frac{k}{4\eta} \sum_{q=1}^{N} \sum_{n=-\infty}^{\infty} V_q e^{-jk \sin(\theta)L} e^{-jI_m y} e^{-jI_m |z|}.
\]

The effect of the metal-backed dielectric substrate on the field radiated by the array \( \mathbf{M}_{nq}(r) = V_q \exp[-jk \sin(\theta)L] \delta(y - y_{nq}, z + h) \) can be derived following Ref. [21]. After some algebra, one would arrive at the following expressions for the magnetic field profile outside the substrate (\( z < -h \))

\[
H_x(y, z < -h) = \frac{k}{2\eta L} \sum_{m=-\infty}^{\infty} \rho_m^{(V)} e^{-jI_m y} e^{-jI_m (z+h)},
\]

Here \( \rho_m^{(V)} \) is the Fresnel’s reflection coefficient of a plane-wave (having tangential component of wave vector equal to \( \xi_m \)) from the metal-backed dielectric substrate

\[
\rho_m^{(V)} = \frac{jI_m \tan(\beta_m h)}{jI_m \tan(\beta_m h) + 1} \quad \gamma_m = \frac{k \eta \beta_m}{k \eta \beta_m + 1}.
\]

2. Controlling diffraction patterns with magnetic metagratings

When a magnetic metagrating is illuminated by a TM-polarized plane-wave incident at angle \( \theta \), the scattered magnetic field can be represented as a superposition of plane-waves \( \sum_{m=-\infty}^{\infty} \mathbf{H}_m^{TM} e^{-j\xi_m y + j\beta_m z} \). Amplitudes of the plane-waves are found analytically (see previous subsection) and can be expressed as follows

\[
S_m^{TM} = -\frac{k}{2\eta L} \left[ 1 + R_m^{TM} \right] e^{j\beta_m h} + \delta_{m0} R_m^{TM} e^{2j\beta_m h},
\]

where \( \delta_{m0} \) is the Kronecker delta representing the reflection of the incident wave from the metal-backed substrate. As in case of TE polarization described in Ref. [17], Eq. (B6) demonstrates that magnetic line currents contribute to the scattered plane-waves via the parameter \( \rho_m^{(V)} \). Magnetic line currents \( V_q \) can be used to control plane-waves scattered in the far-field \( (|\xi_m| < k) \). Since in practice it is more straightforward to deal with passive structures and also the excitation of magnetic line currents can be of particular difficulty, we are interested in the case when magnetic line currents are polarization currents excited by the incident plane wave in thin “wires” characterized by admittance densities \( Y_q \).

Then, necessary currents \( V_q \) can be obtained by choosing admittance densities \( Y_q \) from the following equation

\[
\left( Y_q + \frac{k}{2\eta} \right) V_q = H_q - \sum_{p=1}^{N} Y_{qp}^{(m)} V_p.
\]

The right-hand side of Eq. (B7) represents the total magnetic field at the location of the \( q \)-th “wire”, where \( H_q = (1 + R_0^{TM}) \exp[j\beta_0 h - j\xi_0 (q-1)d] \) is the external magnetic field, \( Y_{qp}^{(m)} \) are the mutual-impedance densities which account for the interaction with the substrate and adjacent “wires”. The magnetic field created by \( q \)-th line current from all periods (except the zeroth one) is given by the following series

\[
-\frac{k}{2\eta} \sum_{n=1}^{\infty} \cos[k \sin(\theta)L] H_0^{(2)}[knL].
\]

The magnetic field created by all other line currents can be accounted for as follows

\[
-\frac{k}{4\eta} \sum_{p=1,p\neq q}^{N} \sum_{n=-\infty}^{\infty} e^{-jksin(\theta)L} H_0^{(2)}[k(q-p)d - nL].
\]

Eventually, the waves reflected from the metal-backed substrate create the following magnetic field at the location of the \( q \)-th “wire” (zeroth period)

\[
-\frac{k}{2\eta L} \sum_{p=1}^{N} V_p \sum_{m=-\infty}^{\infty} e^{j\xi_m (p-q)d} R_m^{TM} e^{2j\beta_m h}. 
\]
In contrast to the case of TE polarization discussed in Refs. [17, 21], the series in Eq. (B10) does not converge. Indeed, \( R^{TM}_m \) has \((\varepsilon_3 - 1)/(\varepsilon_3 + 1)\) as limit when \( m \) goes to infinity and \( \beta_m \sim -jm \) for large \( m \). Meanwhile, it is well known that the harmonic series is divergent. However, the divergence in Eq. (B10) is artificial and was brought when using the Poisson’s formula (see Eq. (B3)). Thus, we can avoid divergence by using the Poisson’s formula backwards. To that end, we perform the following transformation of the series in Eq. (B10)

\[
-k \frac{R^{TM}_m}{2\eta L} = -k \frac{R^{TM}_m}{2\eta L} e^{j\varepsilon_n(p-q)d} - k \frac{\epsilon_3 - 1}{\epsilon_3 + 1} \sum e^{j\varepsilon_n(p-q)d} \frac{1}{\beta_m} (R^{TM}_m - \frac{\epsilon_3 - 1}{\epsilon_3 + 1}) - k \frac{\epsilon_3 - 1}{\epsilon_3 + 1} \sum e^{j\varepsilon_n(p-q)d} \frac{1}{\beta_m}
\]

The first series on the right hand side of Eq. (B11) is now converging while the second one contains the singularity and should be transformed by means of the Poisson’s formula in the following way

\[
-k \frac{\epsilon_3 - 1}{\epsilon_3 + 1} \sum e^{j\varepsilon_n(p-q)d} - k \frac{\epsilon_3 - 1}{\epsilon_3 + 1} \sum e^{-j\varepsilon_n(p-q)d} \frac{1}{\beta_m} (R^{TM}_m - \frac{\epsilon_3 - 1}{\epsilon_3 + 1}) - k \frac{\epsilon_3 - 1}{\epsilon_3 + 1} \sum e^{-j\varepsilon_n(p-q)d} \frac{1}{\beta_m}
\]

Summarizing Eqs. (B8), (B9), (B10) and (B12) one arrives at the explicit expression for the mutual-admittance density

\[
\gamma_{qp}^{(m)} = \left(1 + \frac{\epsilon_3 - 1}{\epsilon_3 + 1}\right) k \frac{\epsilon_3 - 1}{\epsilon_3 + 1} \sum e^{-j\varepsilon_n(p-q)d} \frac{1}{\beta_m} (R^{TM}_m - \frac{\epsilon_3 - 1}{\epsilon_3 + 1}) + k \frac{\epsilon_3 - 1}{\epsilon_3 + 1} \sum e^{-j\varepsilon_n(p-q)d} \frac{1}{\beta_m}
\]

3. Retrieval of magnetic current and admittance density from specular reflection

In this subsection, we derive Eqs. (3) and (4) of the main text following the same order as in Appendix A. The magnetic field of the wave reflected from the uniform array of magnetic line currents illustrated in Fig. 1 (b) can be represented via the plane-wave expansion as follows

\[
H_x(y, z) = \frac{k}{2\eta d} V \sum \frac{1 + R^{TM}_m}{\beta_m} e^{-j\varepsilon_n(p-q)d} + R^{TE}_m e^{-j\varepsilon_n(p-q)d}
\]

The amplitude of specularly reflected plane-wave (the only propagating diffraction order, \( d < \lambda \)) is then found
to be (see also Eq. \[B10\])

\[
S_0^{TM} = \frac{k}{2d_I} \frac{(1 + R_0^{TM}) e^{j\beta_0 h}}{\beta_0} V + R_0^{TM} e^{2j\beta_0 h}. \quad \text{\(B15\)}
\]

From this equation it is straightforward to arrive at Eq. \[3\]

The other equation (Eq. \[4\]) is readily obtained from Eq. \[B17\] when \(N\) equals 1. The mutual-admittance density in this case is given by the following expression (see Eq. \[B13\])

\[
Y_m = \left(1 + \frac{\varepsilon_s - 1}{\varepsilon_s + 1}\right) \frac{k}{2\eta} \sum_{n=1}^{\infty} \cos[k \sin(\theta)nL] H_0^{(2)}(knd)
+ \frac{k}{2d_I} \sum_{m=-\infty}^{\infty} \frac{1}{\beta_m} \left( R_m^{TM} - \frac{\varepsilon_s - 1}{\varepsilon_s + 1} + \frac{k}{\varepsilon_s + 14\eta} \right). \quad \text{\(B16\)}
\]

Table \[I\] provides impedance (admittance) densities as well as geometrical parameters of the “wires” composing the metagratings demonstrated as examples in Section III.

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**Appendix C: Parameters of metagratings shown in examples**

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