How the work of Gian Carlo Rota had influenced my group research and life

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Abstract: One outlines here in a brief overview how the work of Gian Carlo Rota had influenced my research and life, starting from the end of the last century up to present time state of The Internet Gian Carlo Rota Polish Seminar.

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http://ii.uwb.edu.pl/akk/sem/sem_rota.htm

1 How did I had come over

How and when I did came across the work of Gian Carlo Rota - this I do not remember. May be it was only in 1997 because of streams of thousands of references on the so called $q$-deformations (extensions) that I was slightly involved in. Then I started to agnize, to be more fully aware of the Gian Carlo Rota’s and his friends’ and disciples’ outstanding importance with his and theirs mathematical culture main stream inherited ideas, language and goals especially there, where both analysis and combinatorics meet to enjoy the join into the alloy ore - the crystalline formation of Mathemagics. May it be then in December or so in 1998 at Białystok - when I was much impressed by a series of Professor Oleg Viktorovich Viskov from Steklov Institute lectures on umbral calculus and all that. Since that time in almost all my “umbra” articles I frequently refer to Professor Viskov contributions [1-4] and others - for more see [5-7]. These [5-7] references are examples of my first contributions (including “upside down notation”) to the extended umbral calculus. What is this “upside down notation” from [5-7] ? It is just this: $k_F \equiv F_k$, where $F$ is a natural numbers valued sequence. This notation inspired by Gauss and in the spirit of Knuth via the reasoning just repeated with ”$k_F$” numbers replacing $k$ - natural numbers leads one to transparent clean results in a lot of cases as for example in the recent acyclic digraph’s articles [8-10]. For this notation see also Appendix in [11].

2 Graves-Heisenberg-Weyl algebra

The ingenious ideas of differential and dual graded posets that we owe to Stanley and Fomin (see [10]) bring together combinatorics, representation theory, topology, geometry and many more specific branches of mathematics and mathemat-
ical physics thanks to intrinsic ingredient of these mathematical descriptions which is the Graves-Heisenberg-Weyl (GHW) algebra usually attributed to Heisenberg by physicists and to Herman Weyl by mathematicians and sometimes to both of them.

As noticed by Oleg Viktorovich Viskov in [4] the formula

\[ [f(a), b] = cf'(a) \]

where

\[ [a, b] = c, [a, c] = [b, c] = 0 \]

pertains to Charles Graves from Dublin [12]. Then it was re-discovered by Paul Adrien Maurice Dirac and others in the next century.

Let us then note that the picture that emerges in [5-7] discloses the fact that any umbral representation of finite (extended) operator calculus or equivalently - any umbral representation of GHW algebra makes up an example of the algebraization of the analysis with generalized differential operators of Markowsky acting on the algebra of polynomials or other algebras as for example formal series algebras.

3 Cobweb posets and DAGs named KoDAGs

KoDAGs are Hasse diagrams -hence directed acyclic graphs of cobweb partially ordered sets which are secluded in a natural way from multi-ary relations chains' digraphs. The family of these so called cobweb posets has been invented by the author at the dawn of this century (for earlier references see [13,14]- for the recent ones see [8-11]) . These structures are such a generalization of the Fibonacci tree growth that allows joint combinatorial interpretation [13,14] for all of them under the combinatorial admissibility condition.

Let \( \{F_n\}_{n \geq 0} \) be a natural numbers valued sequence with \( F_0 = 1 \) (or \( F_0! \equiv 0! \) being exceptional as in case of Fibonacci numbers). Any such sequence uniquely designates both \( F \)-nomial coefficients of an \( F \)-extended umbral calculus as well as \( F \)-cobweb poset defined in [13]. If these \( F \)-nomial coefficients are natural numbers or zero then we call the sequence \( F \) - the \( F \)-cobweb admissible sequence.

**Definition 1** Let \( n \in N \cup \{0\} \cup \{\infty\} \). Let \( r, s \in N \cup \{0\} \). Let \( \Pi_n \) be the graded partial ordered set (poset) i.e. \( \Pi_n = (\Phi_n, \leq) = (\bigcup_{k=0}^{n} \Phi_k, \leq) \) and \( (\Phi_k)_{k=0}^{n} \) constitutes ordered partition of \( \Pi_n \). A graded poset \( \Pi_n \) with finite set of minimal elements is called cobweb poset iff

\[ \forall x, y \in \Phi \text{ i.e. } x \in \Phi_r \text{ and } y \in \Phi_s \text{ } r \neq s \Rightarrow x \leq y \text{ or } y \leq x, \]

\[ \Pi_\infty \equiv \Pi. \]

See Fig.1.
Definition 2 Let any $F$-cobweb admissible sequence be given then $F$-nomial coefficients are defined as follows

\[
\binom{n}{k}_F = \frac{n_F!}{k_F!(n-k)_F!} = \frac{n_F \cdot (n-1)_F \cdot \ldots \cdot (n-k+1)_F}{1_F \cdot 2_F \cdot \ldots \cdot k_F} = \frac{n^k}{k_F!}
\]

while $n, k \in \mathbb{N}$ and $0_F! = n^0_F = 1$ with $\frac{n^k}{k_F!}$ staying for falling factorial.

Definition 3 $C_{\text{max}}(\Pi_n) \equiv \{c = <x_k, x_{k+1}, \ldots, x_n>, x_s \in \Phi_s, s = k, \ldots, n\}$ i.e. $C_{\text{max}}(\Pi_n)$ is the set of all maximal chains of $\Pi_n$.

Definition 4 Let

\[
C_{\text{max}}(\Phi_k \rightarrow \Phi_n) \equiv \{c = <x_k, x_{k+1}, \ldots, x_n>, x_s \in \Phi_s, s = k, \ldots, n\}.
\]

Then the $C(\Phi_k \rightarrow \Phi_n)$ set of Hasse sub-diagram corresponding maximal chains defines biunivoquely the layer $\langle \Phi_k \rightarrow \Phi_n \rangle = \bigcup_{s=k}^{n} \Phi_s$ as the set of maximal chains’ nodes and vice versa - for these graded DAGs (KoDAGs included).

The equivalent to that of [13,14] formulation of combinatorial interpretation of cobweb posets via their cover relation digraphs (Hasse diagrams) is the following.

Theorem (Kwaśniewski) For $F$-cobweb admissible sequences $F$-nomial coefficient $\binom{n}{k}_F$ is the cardinality of the family of equipotent to $C_{\text{max}}(P_m)$ mutually disjoint maximal chains sets, all together partitioning the set of maximal chains $C_{\text{max}}(\Phi_k+1 \rightarrow \Phi_n)$ of the layer $\langle \Phi_k+1 \rightarrow \Phi_n \rangle$, where $m = n - k$.

For environment needed and then simple combinatorial proof see [14,13] easily accessible via Arxiv.

One uses for that to proof the graded structure of Hasse diagram and the notion of the layer.

Comment 1. For the above Kwaśniewski combinatorial interpretation of $F$-nominals’ array the diagram being directed or not does not matter of course, as this combinatorial interpretation is equally valid for partitions of the family of $\text{SimplePath}_{\text{max}}(\Phi_k - \Phi_n)$ in comparability graph of the Hasse digraph with self-explanatory notation used on the way. And to this end recall: a poset is graded if and only if every connected component of its comparability graph is graded. We are concerned here with connected graded graphs and digraphs.

If one imposes further requirements with respect $F$-sequences denominating both $F$-extended Umbral (Finite Operator) Calculus and cover relation diagrams (Hasse) of the corresponding cobweb poset then further specific problems, their solutions and specific digraph-combinatorial interpretations are arrived at.

For fresh results of the Student participant of The Internet Gian Carlo Rota Polish Seminar see [17]. For his recent discoveries see [16,17].
Figure 1: Display of the layer $\langle \Phi_1 \rightarrow \Phi_4 \rangle = \text{the subposet } P_4 \text{ of the } F = \text{Gaussian integers sequence } (q = 2) \text{ F-cobweb poset and } \sigma P_4 \text{ subposet of the } \sigma \text{ permuted Gaussian } (q = 2) \text{ F-cobweb poset.}$

4 How all that had influenced my and my research group life?

The Gian Carlo Rota Polish Seminar has been transformed in 2008 and is active now as The Internet Gian Carlo Rota Polish Seminar: http://ii.uwb.edu.pl/akk/sem/sem_rota.htm. We are continuing the research.

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