TODD GENUS AND $A_k$-GENUS OF UNITARY $S^1$-MANIFOLDS

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Abstract. Assume that $M$ is a compact connected unitary $2n$-dimensional manifold and admits a non-trivial circle action preserving the given complex structure. If the first Chern class of $M$ equals to $k_0x$ for a certain 2nd integral cohomology class $x$ with $|k_0| \geq n + 2$, and its first integral cohomology group is zero, this short paper shows that the Todd genus and $A_k$-genus of $M$ vanish.

1. Introduction

A smooth manifold $M$ is called a unitary manifold (sometimes being called stably almost complex manifold or weakly almost complex manifold), if the tangent bundle of $M$ admits a stably complex structure. Namely, there exists a bundle map

$$J : TM \oplus \mathbb{R}^l \to TM \oplus \mathbb{R}^l$$

such that $J^2 = -1$, where $\mathbb{R}^l$ denotes the trivial real $l$-plane bundle over $M$ for some $l$. A stable complex structure induces an orientation, obtained as the “difference” of the complex orientation on $TM \oplus \mathbb{R}^l$ and the standard orientation on $\mathbb{R}^l$. Hereafter a unitary manifold is always oriented in such a way. If $S^1$ acts smoothly on a unitary (respectively almost complex) manifold $M$ and if the differential of each element of $S^1$ preserves the given complex vector bundle structure then $M$ will be called unitary (respectively almost complex) $S^1$-manifold.

Given some assumptions, there are many results about the existence of $S^1$ actions on manifolds with the vanishing of Todd genus, $A_k$-genus or $\hat{A}$-genus. Let’s list some related results.

- Hattori proved that ([4, Proposition 3.21]): for a unitary $S^1$-manifold $M$ having only isolated fixed points, if the first Chern class is a torsion element then the Todd genus $\text{Td}(M)$ is zero.

- Kričever showed that ([7, Theorem 2.2]), for a unitary $S^1$-manifold $M$, if the first Chern class $c_1(M)$ is divisible by $k$, the $A_k$-genera of $M$ vanish, i.e., $A_k(M) = 0, k \geq 2$.

- Atiyah and Hirzebruch [1] gave the result that if $M$ is a connected $2n$-dimensional spin manifold and $S^1$ acts non-trivially on $M$, then the $\hat{A}$-genus $\hat{A}(M)$ vanishes.
Using Hattori’s result ([4, Proposition 3.21]) and the celebrated theorem of Atiyah and Hirzebruch in [1], Li ([8, Proposition 4.1]) showed that: for an almost complex manifold admitting a non-trivial $S^1$-action, if the first Chern class is a torsion element and is zero under mod 2 reduction, then the Todd genus is zero.

In [2, Theorem 1.1], Fang and Rong showed that: For a compact $2n$-manifold $M$ of finite fundamental group, if $M$ admits a fixed point free $S^1$-action, then for all $c \in H^2(M; \mathbb{Z})$ and $0 \leq m \leq \left\lfloor \frac{n}{2} \right\rfloor$, $(c^{n-2m} \cdot L_m)[M] = 0$, where $L = L_0 + L_1 + \cdots + L_{\left\lfloor \frac{n}{2} \right\rfloor}$ is the Hirzebruch $L$-polynomial of $M$. In particular, $(c^n)[M] = 0$.

A generalization of Fang-Rong’s result is the [9, Example 3.2] proved by Li: Let $M^{2m}$ be a manifold with $b_1(M) = 0$. If $M$ admits a fixed point free $S^1$-action, then for all $c \in H^2(M; \mathbb{Z})$ and $0 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor$, $(c^{m-2j} \cdot L_j)[M] = 0$, where $L_j$ is any polynomial of degree $j$ in the subalgebra of $H^*(M)$ generated by Pontrjagin classes $p_i(M)$ ($\deg(p_i) = i$). In particular, $(c^n)[M] = 0$.

As a supplement to the above mentioned results, we prove the following theorem, which is the main theorem of this paper.

**Theorem 1.1.** Let $M$ be a compact connected unitary manifold of dimension $2n > 2$ with $H^1(M; \mathbb{Z}) = 0$. Suppose that $M$ admits a non-trivial $S^1$-action which preserves the complex structure of the stable tangent bundle of $M$. If the first Chern class $c_1(M) = k_0x$ for some $x \in H^2(M; \mathbb{Z})$ and $|k_0| \geq n + 2$, then

1. $x^{n-2j} \hat{A}_j = 0$, where $\hat{A}_j := \hat{A}_j(p_1, \ldots, p_j)$ are the $\hat{A}$ polynomials, $0 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor$.
2. $x^{n-i}T_i = 0$, where $T_i := T_i(c_1, \ldots, c_i)$ are the Todd polynomials, $0 \leq i \leq n$. In particular, the Todd genus $\text{Td}(M) = T_n[M] = 0$.
3. The $A_k$-genus $A_k(M) = 0$, $k \geq 2$.

The proof of Theorem 1.1 is mainly based on the following theorem of Hattori [3, Theorem 3.13].

**Theorem 1.2** (Hattori). Let $M$ be a closed connected weakly almost complex manifold of dimension $2n > 2$ with $H^1(X; \mathbb{Z}) = 0$. Suppose that $M$ admits a non-trivial $S^1$-action which preserves the given complex structure of the stable tangent bundle of $M$. If $c_1(M) = k_0x, x \in H^2(M; \mathbb{Z})$, then we have

$$\left\{ \exp \left( \frac{kx}{2} \right) \cdot \hat{A}(TM) \right\} [M] = 0$$

for each integer $k$ such that $k \equiv k_0 \pmod{2}$ and $|k| < |k_0|$.

**Remark 1.3.** (1) In [10, Corollary 4.2], for an almost complex $S^1$-manifold $M$ of dimension $2n$ with only isolated fixed points, Sabatini showed that if $c_1(M) = k_0x$ for some $x \in H^2(M; \mathbb{Z})$ and $k_0 \geq n + 2$, then $x^{n-k}T_k = 0, 0 \leq k \leq n$. Although Sabatini’s conclusion is similar to our Theorem 1.1 (2), the condition of isolated fixed points is not necessary in Theorem 1.1.
(2) The vanishing of $A_k$-genera in Theorem 1.1 (3) hold for any $k \geq 2$, and the vanishing of $A_k$-genera in [7, Theorem 2.2] hold for those $k$ satisfying that the first Chern class is divisible by $k$.

2. Preliminaries

Firstly, let’s briefly introduce the Todd genus, $A_k$-genus and $\hat{A}$-genus.

Let $M$ be a connected, closed $2n$-dimensional unitary manifold. Since the Chern classes are stable invariants, which means that they are unchanged if we add a trivial (complex) bundle, the structure one needs for defining the Todd genus is a stable almost complex structure. For abbreviation, the Chern classes of the unitary manifold $M$ are denoted as $c_i := c_i(M) := c_i(TM \oplus \mathbb{R}^l)$. The Todd genus $Td(M)$ of $M$ is the evaluation of the Todd polynomial $T_n(c_1, c_2, \ldots, c_n)$ in the Chern classes on the fundamental class $[M]$ of $M$. The multiplicative sequence \( \{ T_k(c_1, \ldots, c_k) \} \) is associated to the power series $Q(x) = \frac{x}{1 - \exp(-x)}$ ([5, §1.7]). Define the Todd class of $M$ by

$$\text{td}(TM \oplus \mathbb{R}^l) = \sum_{k=0}^{\infty} T_k(c_1, \ldots, c_k),$$

then the Todd genus of $M$ is the evaluation

$$Td(M) = \text{td}(TM \oplus \mathbb{R}^l)[M] = T_n(c_1, \ldots, c_n)[M].$$

Associated with the characteristic power series $Q(x) = \frac{kx \exp(x)}{\exp(kx) - 1}$, $k \geq 2$, there is a multiplicative $A_k$-class of $M$ ([7]), which we denote it as $A_k(TM \oplus \mathbb{R}^l)$, and the $A_k$-genus $A_k(M)$ is the evaluation

$$A_k(M) = A_k(TM \oplus \mathbb{R}^l)[M].$$

Note that, if $k = 1$ is allowed in [7], $A_1(M)$ is exactly the Todd genus $Td(M)$. If we consider a formal factorization of the total Chern class as $c(TM \oplus \mathbb{R}^l) = (1 + x_1) \cdots (1 + x_n)$, then by [6, §1.8],

$$Td(M) = \left( \prod_{i=1}^{n} \frac{x_i}{1 - \exp(-x_i)} \right) [M],$$

$$A_k(M) = \left( \prod_{i=1}^{n} \frac{kx_i \exp(x_i)}{\exp(kx_i) - 1} \right) [M].$$

For a compact oriented differentiable $2n$-manifold $M$, Hirzebruch ([5, page 197]) defined the $\hat{A}$-sequence of $M$ as a certain polynomials in the Pontrjagin classes of $M$. More concretely, the even power series

$$Q(x) = \frac{\frac{1}{2}x}{\sinh \left( \frac{1}{2}x \right)} = 1 - \frac{x^2}{24} + \frac{7x^4}{5760} - \frac{31x^6}{967680} + \frac{127x^8}{154828800} + \cdots$$
defines a multiplicative sequence \( \{ \hat{A}_j(p_1, \ldots, p_j) \} \) (for abbreviation, \( p_j := p_j(TM) \)), where \( \hat{A}_j(p_1, \ldots, p_j) \) is a rational homogeneous polynomial of degree \( 4j \) in the Pontrjagin classes. The \( \hat{A} \)-class \( \hat{A}(TM) \) is defined as follows:

\[
\hat{A}(TM) = \sum_{j=0}^{\infty} \hat{A}_j(p_1, \ldots, p_j).
\]

The \( \hat{A} \)-genus of \( M \) is the evaluation

\[
\hat{A}(M) = \hat{A}(TM)[M].
\]

Let \( p(TM) = (1 + x_1^2) \cdots (1 + x_n^2) \) be the formal factorization of the total Pontrjagin class, then by [6, §1.6],

\[
\hat{A}(M) = \left( \prod_{i=1}^{n} \frac{1}{2} \frac{x_i}{\sinh \left( \frac{1}{2} x_i \right)} \right) [M].
\]

3. Proof of the main Theorem 1.1

Proof of Theorem 1.1. (1) Under the assumptions of Theorem 1.1, by Theorem 1.2,

\[
\left\{ \exp \left( \frac{kx}{2} \right) \cdot \hat{A}(TM) \right\} [M] = 0
\]

for each integer \( k \) such that \(|k| < |k_0|, k \equiv k_0 \mod 2\). Let \( u^{(2n)} \) be the term of degree \( 2n \) of an element \( u \in \bigoplus_{k=0}^{2n} H^k(M; \mathbb{Z}) \). By the Poincaré duality, from (3.1) we can obtain

\[
\left( \exp \left( \frac{kx}{2} \right) \cdot \hat{A}(TM) \right)^{(2n)} = 0 \in H^{2n}(M; \mathbb{Z}).
\]

**Case 1:** \( n \) is even.

We have

\[
\left( \exp \left( \frac{kx}{2} \right) \cdot \hat{A}(TM) \right)^{(2n)} = \frac{1}{n!} \left( \frac{kx}{2} \right)^n + \frac{1}{(n-2)!} \left( \frac{kx}{2} \right)^{n-2} \hat{A}_1 + \cdots
\]

\[
+ \frac{1}{2!} \left( \frac{kx}{2} \right)^2 \hat{A}_{\frac{n}{2} - 1} + \hat{A}_{\frac{n}{2}}.
\]

When \(|k_0| \geq n + 2\), there exist integers \( k_1, k_2, \ldots, k_{\frac{n}{2} + 1} \) such that

\[
|k_1| < |k_2| < \cdots < |k_{\frac{n}{2}}| < |k_{\frac{n}{2} + 1}| < |k_0|,
\]

\[
k_1 \equiv k_2 \equiv \cdots \equiv k_{\frac{n}{2}} \equiv k_{\frac{n}{2} + 1} \equiv k_0 \mod 2,
\]

and

\[
\left( \exp \left( \frac{k_ix}{2} \right) \cdot \hat{A}(TM) \right)^{(2n)} = 0, \quad i = 1, 2, \ldots, \frac{n}{2} + 1.
\]
By (3.2), we get the following equation of matrices:

\[
\begin{bmatrix}
  k_1^n & k_1^{n-2} & \cdots & k_1^2 & 1 \\
  k_2^n & k_2^{n-2} & \cdots & k_2^2 & 1 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  k_{n/2}^n & k_{n/2}^{n-2} & \cdots & k_{n/2}^2 & 1 \\
  k_{n/2+1}^n & k_{n/2+1}^{n-2} & \cdots & k_{n/2+1}^2 & 1 \\
\end{bmatrix}
\begin{bmatrix}
  \frac{1}{n!}x^n \\
  \frac{1}{(n-2)!}x^{n-2} \hat{A}_1 \\
  \vdots \\
  \frac{1}{2!}x^2 \hat{A}_{n/2-1} \\
  \frac{1}{x} \hat{A}_{n/2} \\
\end{bmatrix} = 0.
\]

Since \(k_1, k_2, \ldots, k_{n/2+1}\) are mutually distinct, which means the determinant of left matrix in the above equation is that of the Vandermonde and this matrix is invertible, so we get

\[x^n = 0, \ x^{n-2} \hat{A}_1 = 0, \ldots, \ x^2 \hat{A}_{n/2-1} = 0, \ \hat{A}_{n/2} = 0. \quad (3.3)\]

**Case 2:** \(n\) is odd.

We have

\[
\left(\exp\left(\frac{kx}{2}\right) \cdot \hat{A}(TM)\right)^{(2n)} = \frac{1}{n!} \left(\frac{kx}{2}\right)^n + \frac{1}{(n-2)!} \left(\frac{kx}{2}\right)^{n-2} \hat{A}_1 + \cdots + \frac{1}{3!} \left(\frac{kx}{2}\right)^3 \hat{A}_{n-1} + \frac{1}{1!} \left(\frac{kx}{2}\right) \hat{A}_{n-1}.
\]

When \(|k_0| \geq n+2\), there exist integers \(k_1, k_2, \ldots, k_{n+1/2}\) such that

\[
0 < |k_1| < |k_2| < \cdots < |k_{n+1/2}| < |k_{n+1}| < |k_0|, \quad k_1 \equiv k_2 \equiv \cdots \equiv k_{n/2} \equiv k_{n+1/2} \equiv k_0 \mod 2,
\]

and

\[
\left(\exp\left(\frac{k_ix}{2}\right) \cdot \hat{A}(TM)\right)^{(2n)} = 0, \ i = 1, 2, \ldots, \frac{n+1}{2}. \quad (3.4)
\]

By (3.4), we get the following equation of matrices:

\[
\begin{bmatrix}
  k_1^n & k_1^{n-2} & \cdots & k_1^3 & k_1 \\
  k_2^n & k_2^{n-2} & \cdots & k_2^3 & k_2 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  k_{n/2}^n & k_{n/2}^{n-2} & \cdots & k_{n/2}^3 & k_{n/2} \\
  k_{n/2+1}^n & k_{n/2+1}^{n-2} & \cdots & k_{n/2+1}^3 & k_{n/2+1} \\
\end{bmatrix}
\begin{bmatrix}
  \frac{1}{n!}x^n \\
  \frac{1}{(n-2)!}x^{n-2} \hat{A}_1 \\
  \vdots \\
  \frac{1}{3!}x^3 \hat{A}_{n/2} \\
  \frac{1}{x} \hat{A}_{n/2} \\
\end{bmatrix} = 0.
\]

Since the determinant of left matrix in the above equation is a multiple of that of the Vandermonde, we get

\[x^n = 0, \ x^{n-2} \hat{A}_1 = 0, \ldots, \ x^3 \hat{A}_{n/2} = 0, \ x \hat{A}_{n/2} = 0. \quad (3.5)\]
(2) By [5, (12) in page 13] and $A_s = 2^s \hat{A}_s$ in [5, page 197], we have

$$T_k = \sum_{r+2s=k} \frac{1}{r!} \cdot 2^r c_1^r \hat{A}_s$$

$$= \sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k_0^{k-2s}}{(k-2s)!} \cdot 2^{k-2s} x^{k-2s} \hat{A}_s,$$

where $r$ and $s$ are non-negative integers. Then for any $0 \leq k \leq n$,

$$x^{n-k} T_k = x^{n-k} \cdot \sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k_0^{k-2s}}{(k-2s)!} \cdot 2^{k-2s} x^{n-2s} \hat{A}_s$$

$$= \sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k_0^{k-2s}}{(k-2s)!} \cdot 2^{k-2s} x^{n-2s} \hat{A}_s$$

$$= 0.$$

(3) Let

$$A_k(TM \oplus \mathbb{R}^l), \ A_{\frac{1}{2}}(TM \oplus \mathbb{R}^l)$$

be the multiplicative characteristic classes of $M$ associated to the characteristic power series

$$\frac{k x \exp(x)}{\exp(kx) - 1}, \quad \frac{x \exp\left(\frac{x}{k}\right)}{\exp(x) - 1}, \quad k \geq 2,$$

and $A_k(M), A_{\frac{1}{2}}(M)$ denote the corresponding genus of $M$ respectively, then

$$A_k(M) = k^n \cdot A_{\frac{1}{2}}(M). \quad \text{(3.6)}$$

The identity

$$\frac{x \exp\left(\frac{x}{k}\right)}{\exp(x) - 1} = \exp\left(\left(\frac{1}{k} - \frac{1}{2}\right) x\right) \cdot \frac{\frac{x}{k}}{\sinh\left(\frac{x}{2}\right)}$$

implies that

$$A_{\frac{1}{2}}(TM \oplus \mathbb{R}^l) = \exp\left(\left(\frac{1}{k} - \frac{1}{2}\right) c_1(M)\right) \cdot \hat{A}(TM).$$

Thus, as the proof of of Theorem 1.1 (1), using (3.3) and (3.5), we also get $A_{\frac{1}{2}}(TM \oplus \mathbb{R}^l)$ are zero, then the $A_{\frac{1}{2}}$-genera and $A_k$-genera of $M$ vanish, $k \geq 2$. \hfill \Box

**Remark 3.1.** In fact, for $A_2$-genus, the corresponding power series $\frac{2x \cdot \exp(x)}{\exp(2x) - 1}$ is an even power series, then $A_2$-genus is expressible in Pontrjagin numbers and hence defined for an oriented smooth manifold ([6, §1.6]). By [6, Appendix III], $A_{\frac{1}{2}}(M) = \chi(M, K^{1/2}) = \hat{A}(M)$,
where, \( \chi(M, K^{1/2}) \) is the genus with respect to the characteristic power series \( \frac{x \exp \left( \frac{x}{2} \right)}{\exp(x) - 1} \).

So by (3.6), we have

\[
A_2(M) = 2^n \cdot \hat{A}(M).
\]

**Corollary 3.2.** Under the assumptions in Theorem 1.1, if \( c := c_1(M) \in H^2(M; \mathbb{Z}) \) is a torsion element, then

\[
(c^{n-i}T_i)[M] = 0, 0 \leq i \leq n;
\]

\[
(c^{n-2j} \hat{A}_j)[M] = 0, 0 \leq j \leq \left[ \frac{n}{2} \right];
\]

\[
A_k(M) = 0, k \geq 2.
\]

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