SOME TYPE I SOLUTIONS OF RICCI FLOW WITH ROTATIONAL SYMMETRY

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Abstract. We prove that the Ricci flow on $\mathbb{CP}^n$ blown-up at one point starting with any rotationally symmetric Kähler metric must develop Type I singularities. In particular, if the total volume does not go to zero at the singular time, the parabolic blow-up limit of the Type I Ricci flow along the exceptional divisor is a complete non-flat shrinking gradient Kähler-Ricci soliton on a complete Kähler manifold homeomorphic to $\mathbb{C}^n$ blown-up at one point.

1. Introduction

In this paper, we study the Ricci flow on Kähler manifolds defined by

$$X_{n,k} = \mathbb{P}(\mathcal{O}_{\mathbb{CP}^{n-1}} \oplus \mathcal{O}_{\mathbb{CP}^{n-1}}(-k))$$

for $k, n \in \mathbb{N}^+$. Such manifolds are holomorphic $\mathbb{CP}^1$ bundle over the projective space $\mathbb{CP}^{n-1}$. They are called Hirzebruch surfaces when $n = 2$ and $X_{n,1}$ is exactly $\mathbb{CP}^n$ blown-up at one point. The maximal compact subgroup of the automorphism group of $X_{n,k}$ is given by $G_{n,k} = U(n)/\mathbb{Z}_k$ (2).

The unnormalized Ricci flow introduced by Hamilton [9] is defined on a Riemannian manifold $M$ starting with a Riemannian metric $g_0$ by

$$\frac{\partial g}{\partial t} = -\text{Ric}(g), \ g(0) = g_0.$$ (1.1)

We apply the Ricci flow (1.1) to $X_{n,k}$ with a $G_{n,k}$-invariant initial Kähler metric. In [18], it is shown that the Ricci flow (1.1) must develop finite time singularity and it either shrinks to a point, collapses to $\mathbb{CP}^{n-1}$ or contracts an exceptional divisor, in Gromov-Hausdorff topology.

When the flow shrinks to a point, $X_{n,k}$ is a Fano manifold and $1 \leq k < n$. It is shown by Zhu [28] that the flow must develop Type I singularities and the rescaled Ricci flow converges in Cheeger-Gromov-Hamilton sense to the unique compact Kähler-Ricci soliton on $X_{n,k}$ constructed in [8, 3, 24].

When the flow collapses to $\mathbb{CP}^{n-1}$, it is shown by Fong [7] that the flow must develop Type I singularities and the rescaled Ricci flow converges in Cheeger-Gromov-Hamilton sense to the ancient solution that splits isometrically as $\mathbb{C}^{n-1} \times \mathbb{CP}^1$.

Our main result is to show that the flow must also develop Type I singularities when it does not collapse and the blow-up limit is a nontrivial complete shrinking

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Let $X$ be $\mathbb{CP}^n$ blown-up at one point. Then the Ricci flow on $X$ must develop Type I singularities for any $U(n)$-invariant initial Kähler metric.

Let $g(t)$ be the smooth solution defined on $t \in [0, T)$, where $T \in (0, \infty)$ is the singular time. For every $K_j \to \infty$, we consider the rescaled Ricci flow $(X, g_j(t'))$ defined on $[-K_jT, 0)$ by

$$g_j(t') = K_jg(T + K_j^{-1}t').$$

Then one and only one of the following must occur.

1. If $\liminf_{t \to T} (T-t)^{-1}\text{Vol}(g(t)) = \infty$, then $(X, g_j(t'), p)$ subconverges in Cheeger-Gromov-Hamilton sense to a complete shrinking non-flat gradient Kähler-Ricci soliton on a complete Kähler manifold homeomorphic to $\mathbb{CP}^n$ blown-up at one point, for any $p$ in the exceptional divisor.

2. If $\liminf_{t \to T} (T-t)^{-1}\text{Vol}(g(t)) \in (0, \infty)$, then $(X, g_j(t'), p_j)$ subconverges in Cheeger-Gromov-Hamilton sense to $(\mathbb{C}^{n-1} \times \mathbb{CP}^1, g_{\mathbb{C}^{n-1}} \oplus (-t')g_{FS})$, where $g_{\mathbb{C}^{n-1}}$ is the standard flat metric on $\mathbb{C}^{n-1}$ and $g_{FS}$ the Fubini-Study metric on $\mathbb{CP}^1$ for any sequence of points $p_j$.

3. If $\liminf_{t \to T} (T-t)^{-1}\text{Vol}(g(t)) = 0$, then $(X, g_j(t'))$ converges in Cheeger-Gromov-Hamilton sense to the unique compact shrinking Kähler-Ricci soliton on $\mathbb{CP}^n$ blown-up at one point.

The generalization of Theorem 1.1 for $X_{n,k}$ is given in section 6. In order to exclude Type II singularities, we first prove a lower bound for the holomorphic bisectional curvature and then we apply Cao’s splitting theorem for the Kähler Ricci flow with nonnegative holomorphic bisectional curvature [4]. Theorem 1.1 gives evidence that the Kähler-Ricci flow can only develop Type I singularities for Kähler surfaces and if the flow does not collapse in finite time. Combined with the results of [18, 19], Theorem 1.1 verifies that the flow indeed performs a geometric canonical surgery with minimal singularities in the Kähler case. We also remark that the shrinking soliton as the pointed blow-up limit is trivial if the parabolic rescaling takes place at a fixed base point outside the exceptional divisor $D_0$. We believe that the blow-up limit should be the unique homothetically rotationally symmetric complete shrinking soliton on $\mathbb{C}^2$ blown-up at one point constructed by Feldman-Ilmanen-Knopf in [6]. Unfortunately, we are unable to show that that limiting complete Kähler manifold is biholomorphic to $\mathbb{CP}^n$ blown-up at one point, although it has the same topological structure with the unitary group $U(n)$ lying in the isometry group of the limiting soliton.

The organization of the paper is as follows. In section 2, we introduce the Calabi ansatz. In section 3, we obtain a lower bound for the holomorphic bisectional curvature. In section 4, we prove the flow must develop Type I singularities if non-collapsing. In section 5, we construct the blow-up limit. In section 6, we discuss some generalizations of Theorem 1.1.

We would also like to mention that we have been informed by Davi Maximo that he has a different approach to understand the singularity formation in similar settings [13].
2. Calabi symmetry

In this section, we introduce the Calabi ansatz on $\mathbb{C}P^n$ blown-up at one point introduced by Calabi [2] (also see [3, 6, 18]). From now on, we let $X$ be $\mathbb{C}P^n$ blown-up at one point and it is in fact a $\mathbb{C}P^1$ bundle over $\mathbb{C}P^{n-1}$ given by

$$X = \mathbb{P}(O_{\mathbb{C}P^{n-1}} \oplus O_{\mathbb{C}P^{n-1}}(-1)).$$

Let $D_0$ be the exceptional divisor of $X$ defined by the image of the section $(1,0)$ of $O_{\mathbb{C}P^{n-1}} \oplus O_{\mathbb{C}P^{n-1}}(-1)$ and $D_\infty$ be the divisor of $X$ defined by the image of the section $(0,1)$ of $O_{\mathbb{C}P^{n-1}} \oplus O_{\mathbb{C}P^{n-1}}(-1)$. Both the 0-section $D_0$ and the $\infty$-section are complex hypersurfaces in $X$ isomorphic to $\mathbb{C}P^{n-1}$. The Kähler cone on $X$ is given by

$$\mathcal{K} = \{-a[D_0] + b[D_\infty] \mid 0 < a < b\}.$$

In particular, when $n = 2$, $D_0$ is a holomorphic $S^2$ with self-intersection number $-1$.

Let $z = (z_1, \ldots, z_n)$ be the standard holomorphic coordinates on $\mathbb{C}^n$. Let $\rho = \log |z|^2 = \log(|z_1|^2 + |z_2|^2 + \ldots + |z_n|^2)$. We consider a smooth convex function $u = u(\rho)$ for $\rho \in (-\infty, \infty)$ satisfying the following conditions.

1. $u'' > 0$ for $\rho \in (-\infty, \infty)$.
2. There exist $0 < a < b$ and smooth function $u_0, u_\infty : [0, \infty) \rightarrow \mathbb{R}$ such that
   $$u_0'(0) > 0, \quad u_\infty'(0) > 0,$$
   $$u_0(e^\rho) = u(\rho) - a\rho, \quad u_\infty(e^{-\rho}) = u(\rho) - b\rho.$$

For any $u$ satisfying the above conditions, $\omega = \sqrt{-1} \partial \bar{\partial} u$ defines a smooth Kähler metric on $\mathbb{C}^n \setminus \{0\}$ and it extends to a smooth global Kähler metric on $\mathbb{C}P^n$ blown-up at one point in the Kähler class $-a[D_0] + b[D_\infty]$.

On $\mathbb{C}^n \setminus \{0\}$, the Kähler metric $g$ induced by $u$ is given by

$$g_{ij} = e^{-\rho} u' \delta_{ij} + e^{-2\rho} \bar{z}_i z_j (u'' - u').$$

Obviously, the Kähler metric $g$ induced by $u$ is invariant under the standard unitary $U(n)$ transformations on $\mathbb{C}^n$.

We define the Ricci potential of $\omega = \sqrt{-1} \partial \bar{\partial} u$ by

$$v = -\log \det g = n\rho - (n - 1) \log u'(\rho) - \log u''(\rho).$$

and the Ricci tensor of $g$ is given by

$$R_{ij} = e^{-\rho} u' \delta_{ij} + e^{-2\rho} \bar{z}_i z_j (v'' - v').$$

After applying a unitary transformation, we can assume $z = (z_1, 0, \ldots, 0)$ and then

$$\{g_{ij}\} = e^{-\rho} \text{diag}\{u'', u', \ldots, u'\}$$

$$R_{ij} = \sqrt{-1} e^{-\rho} \text{diag}\{v'', v', \ldots, v'\}.$$
where

\[ c_t = - \log u''(0, t) - (n - 1)u'(0, t) \]

and \[ u'(\rho, t) = \frac{\partial}{\partial \rho} u(\rho, t) \]. The evolving Kähler form \( \omega(t) \) is then given by

\[ \omega(t) = \sqrt{-1} \partial \bar{\partial} u(\rho, t). \]

It is also shown in [18] that if the initial Kähler class is given by \( -a_0[D_0] + b_0[D_{\infty}] \), the evolving Kähler class is given by

\[ [\omega(t)] = -a_t [D_0] + b_t [D_{\infty}], \quad a_t = a_0 - (n - 1)t, \quad b_t = b_0 - (n + 1)t. \]

In particular, we have an immediate bound for \( u'(\rho, t) \)

(2.5) \[ \lim_{\rho \to -\infty} u'(\rho, t) = a_t, \quad \lim_{\rho \to \infty} u'(\rho, t) = b_t. \]

3. A lower bound for the holomorphic bisectional curvature

In this section, we will obtain a lower bound for the holomorphic bisectional curvature. We consider the Ricci flow (1.1) on \( X \) with a \( U(n) \)-invariant initial Kähler metric in the Kähler class \( -a_0[D_0] + b_0[D_{\infty}] \). For our purpose, it suffices to consider the case

\[ 0 < a_0(n + 1) < b_0(n - 1). \]

This assumption is shown in [18] to be equivalent to the condition

\[ \lim_{t \to T} \inf Vol(g(t)) > 0, \quad \text{or,} \quad \lim_{t \to T} (T - t)^{-1} Vol(g(t)) = \infty \]

and then the Kähler-Ricci flow will contract the exceptional divisor \( D_0 \) at the singular time

\[ T = \frac{a_0}{n - 1}. \]

We will assume throughout this section that the initial Kähler class lies in \( -a_0[D_0] + b_0[D_{\infty}] \) with \( 0 < a_0(n + 1) < b_0(n - 1) \).

The following theorem is proved in [22].

Theorem 3.1. For any relatively compact set \( K \) of \( X \setminus D_0 \) and \( k > 0 \), there exists \( C_{K,k} > 0 \) such that for all \( t \in [0, T) \),

\[ \|g(t)\|_{C^k(K, g_0)} \leq C_{K,k}. \]

It immediately implies that the Ricci flow converges in local \( C^\infty \) topology outside the exceptional divisor \( D_0 \) as \( t \to T \).

The evolution equations for \( u', u'', u''' \) are derived in [18] as below.
\[
\frac{\partial}{\partial t} u' = \frac{u'''}{w'} + \frac{(n-1)u''}{w'} - n \tag{3.6}
\]
\[
\frac{\partial}{\partial t} u'' = \frac{u(4)}{w''} - \frac{(u'''^2 + (n-1)u'' - (n-1)(u')^2}{w'} \tag{3.7}
\]
\[
\frac{\partial}{\partial t} u''' = \frac{u(5) - 3u'''u(4)}{(u'')^2} + \frac{2(u'''^3)}{(u')^3} + \frac{(n-1)u(4)}{u'} - \frac{3(n-1)u''u'''^2}{(u')^2} + \frac{2(n-1)(u'')^3}{(u')^3}. \tag{3.8}
\]

The following lemma is proved in [18] for the collapsing case when \(a_0(n+1) > b_0(n-1)\) and the same proof can be applied here. We include the proof for the sake of completeness.

**Lemma 3.1.** There exists \(C > 0\) such that for all \(t \in [0, T)\) and \(\rho \in (-\infty, \infty)\),

\[
(n-1)(T-t) \leq u' \leq C
\]

and

\[
0 \leq \frac{u''}{u'} \leq C, \quad -C \leq \frac{u'''}{u''} \leq C. \tag{3.10}
\]

**Proof.** The estimate (3.9) follows from the monotonicity of \(u'\) with \(a_t < u' < b_t\) and \(a_t = (n-1)(T-t)\).

We apply the maximum principle to prove (3.10). It is straightforward to verify that for all \(t \in [0, T)\),

\[
\lim_{\rho \to -\infty} \frac{u''(\rho, t)}{u'(\rho, t)} = \lim_{\rho \to \infty} \frac{u''(\rho, t)}{u'(\rho, t)} = 0
\]

\[
\lim_{\rho \to -\infty} \frac{u'''(\rho, t)}{u''(\rho, t)} = 1, \quad \lim_{\rho \to \infty} \frac{u'''(\rho, t)}{u''(\rho, t)} = -1.
\]

Let \(H = \frac{u''}{u'}\). \(H\) is strictly positive for all \(\rho \in (-\infty, \infty)\) and \(t \in [0, T)\). The evolution for \(H\) is given by

\[
\frac{\partial H}{\partial t} = \frac{H''}{u''} + \frac{2H'}{u'} - \frac{2H^2 - H}{u'}.
\]

Therefore \(\sup_{\rho \in (-\infty, \infty), t \in [0, T]} H \leq C\) for some uniform constant \(C > 0\) by applying the maximum principle.

Let \(G = \frac{u'''}{u''}\). Then the evolution for \(G\) is given by

\[
\frac{\partial}{\partial t} G = \frac{1}{u''} G'' + \left(\frac{n-1}{u'} - \frac{u'''}{(u'')^2}\right) G' - \frac{2(n-1)u''}{(u')^2} \left( G - \frac{u''}{u'} \right).
\]

Therefore \(\sup_{\rho \in (-\infty, \infty), t \in [0, T]} |G| \leq C\) for some uniform constant \(C > 0\) by combining the maximum principle and the uniform upper bound for \(H\).

\[\square\]
By taking the trace, we obtain an explicit expression for the scalar curvature (3.11)
\[ R = -\frac{\partial u''}{\partial t} - \frac{(n - 1)2u'}{u'} - \frac{u^{(4)}}{(u'')^2} + \frac{(u'')^2}{(u'')^3} - \frac{2(n - 1)u''}{u'u''} - \frac{(n - 1)(n - 2)u''}{(u')^2} + \frac{n(n - 1)}{u'} . \]

**Corollary 3.1.** There exists \( C > 0 \) such that for all \( \rho \in (-\infty, \infty) \) and \( t \in [0, T) \),
\[ -\frac{u^{(4)}}{(u'')^2} + \frac{(u'')^2}{(u'')^3} \geq \frac{C}{T - t} . \]

**Proof.** Since the scalar curvature \( R \) is uniformly bounded below, there exists \( C_1 > 0 \) such for all \( t \in [0, T) \) and \( \rho \in (-\infty, \infty) \),
\[ -\frac{u^{(4)}}{(u'')^2} + \frac{(u'')^2}{(u'')^3} - \frac{2(n - 1)u''}{u'u''} - \frac{(n - 1)(n - 2)u''}{(u')^2} + \frac{n(n - 1)}{u'} \geq -C_1 . \]

There also exist \( C_2, C_3 > 0 \) such that
\[ u' \geq C_2(T - t) \]
and
\[ \left| \frac{u''}{u'} \right| + \left| \frac{u''}{u''} \right| \leq C_3 . \]

The estimate (3.12) immediately follows from the above estimates.

The holomorphic bisectional curvature \( R_{ijkl} \) is computed in [3] and is given by
\[ R_{ijkl} = e^{-2\rho}(u' - u'')(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) + e^{-2\rho}(3u'' - 2u' - u'')(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{kl}\delta_{ij} + \delta_{kj}\delta_{il}) + e^{-2\rho}\left(6u'' - 11u'' - u^{(4)} + 6u' + \frac{(u'' - u'')^2}{u''}\right)\delta_{ijkl} + e^{-2\rho}\frac{(u' - u'')^2}{u'}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{kl}\delta_{ij} + \delta_{kj}\delta_{il}) \]

Here \( \delta_{ij} \) and \( \delta_{ijkl} \) vanish unless all the indices are 1, while \( \delta_{ij} \) vanishes unless \( i = j \neq 1 \).

For any point \( p \) on \( \mathbb{C}^n \setminus \{0\} \), we can assume the coordinates at \( p \) are given by \( z(p) = (z_1, ..., z_n) = (z_1, 0, ..., 0) \) after a unitary transformation.

Then all the nonvanishing terms of the holomorphic bisectional curvature are given by
\[ R_{1111} = e^{-2\rho}\left(-u^{(4)} + \frac{(u'')^2}{u'}\right) \]
\[ R_{kkkk} = 2e^{-2\rho}(u' - u''), \ k > 1 \]
\[ R_{11kk} = e^{-2\rho}\left(-u'' + \frac{(u'')^2}{u'}\right), \ k > 1 \]
\[ R_{kkll} = e^{-2\rho}(u' - u''), \ k > 1, l > 1, \ k \neq l. \]
Lemma 3.2. There exists $C > 0$ such that on for all $t \in [0, T)$, $p = (z_1, 0, ..., 0)$ and $i, j, k, l$, we have at $(p, t)$,

$$R_{ijkl} \geq -\frac{C}{T-t}(g_{ij}g_{kl} + g_{il}g_{kj}).$$

Furthermore,

$$|R_{ijkl}| \leq \frac{C}{T-t}(g_{ij}g_{kl} + g_{il}g_{kj})$$

unless $i = j = k = l = 1$.

Proof. Since $p = (z_1, 0, ..., 0)$, it suffices to verify the estimates for $R_{1111}$, $R_{11kk}$ and $R_{kkkl}$ for $k, l = 2, ..., n$.

Let $Q_{ijkl} = g_{ij}g_{kl} + g_{il}g_{kj}$. Then

$$Q_{1111} = 2e^{-2\rho(u')}^2,$$
$$Q_{kkkk} = 2e^{-2\rho(u')}^2, \quad k > 1,$$
$$Q_{11kk} = e^{-2\rho}u'u'', \quad k > 1,$$
$$Q_{kkll} = e^{-2\rho}u'^2, \quad k > 1, l > 1, \quad k \neq l.$$

Comparing $R_{ijkl}$ and $Q_{ijkl}$, the lemma follows immediately. \hfill \Box

Proposition 3.1. The holomorphic bisectional curvature is uniformly bounded below by $-C(T-t)^{-1}$ on $X \times [0, T)$ for some fixed constant $C > 0$.

Proof. It suffices to calculate the lower bound of the holomorphic bisectional curvature at a point $p = (z_1, 0, ..., 0)$ and $t \in [0, T)$. Let $V = V^i \frac{\partial}{\partial z^i}$ and $W = W^i \frac{\partial}{\partial z^i}$ be two vectors in $TX_p$. Then there exists $C > 0$ such that

\[
R_{ijkl}V^iV^jW^kW^l
= R_{1111}V^1V^1W^1W^1 + (1 - \delta_{ijkl})R_{ijkl}W^kW^l
\geq -\frac{2C}{T-t}g_{11}g_{11} |V^1|^2 |W^1|^2 - \frac{C}{T-t}(g_{ij}g_{kl} + g_{il}g_{kj}) |V^i| |V^j| |W^k| |W^l|
\geq -\frac{4C}{T-t} |V|^2 |W|^2.
\]

Definition 3.1. Let $g$ be a Kähler metric on a Kähler manifold $M$. At each point $p \in X$, we can choose the normal coordinates at $p$ such that for $i, j = 1, ..., n$, $g_{ij}(p) = \delta_{ij}$ is the identity matrix and

$$R_{ij}(p) = \delta_{ij}\lambda_j.$$

We define the $k^{th}$ symmetric polynomial of Ricci curvature of $g$ at $p$ by

$$\sigma_k = \sigma_k(\text{Ric}(g)) = \sum_{j_1 < j_2 < ... < j_k} \lambda_{j_1}\lambda_{j_2}...\lambda_{j_k}$$

for $1 \leq k \leq n$. 
The next proposition gives a uniform bound for $\sigma_k$ in terms of the curvature tensor $R_{1\bar{1}1\bar{1}}$ at each point $z = (z_1, 0, ..., 0)$.

**Proposition 3.2.** There exists $C > 0$ such that for all $(p, t) \in X \times [0, \infty)$,

$$|\sigma_k(p, t)| \leq \frac{C |Rm(p, t)|}{(T - t)^{k-1}}.$$  

**Proof.** For any point $p \in \mathbb{C}^n \setminus \{0\}$, we can assume that $p = (z_1, 0, ..., 0)$. Then the eigenvalues of $Ric(g)$ at $p$ with respect to $g$ are given by

$$\lambda_1 = -\frac{\partial u''}{u''} = -\frac{1}{2} \frac{u'''}{u''}^2 + \frac{(n-1)u''}{(u'')^2} + \frac{n-1}{u''}$$

$$\lambda_2 = ... = \lambda_n = -\frac{\partial u''}{u'} = -\frac{u''}{u''' - \frac{(n-1)u''}{(u'')^2} + \frac{n}{u''}.$$ 

Then $(T - t)|\lambda_j|$ is uniformly bounded for $j = 2, ..., n$ and

$$|\sigma_k|(p, t) = \sum_{j_1 < j_2 < ... < j_k} |\lambda_{j_1} \lambda_{j_2} ... \lambda_{j_k}| \leq C(T-t)^{-(k-1)}|\lambda_1| \leq C(T-t)^{-(k-1)}|Rm|_{g(p, t)}.$$ 

□

**Lemma 3.3.** For any $p \in D_0$, we have

$$|Ric(p, t)|_{g(t)} \geq \frac{1}{T - t}.$$ 

**Proof.** It suffices to compute $e^{-\rho v'}$ which is one of the eigenvalues in the Ricci tensors since $D_0 = \{\rho = -\infty\}$.

$$\lim_{\rho \to -\infty} e^{-\rho v'}(\rho) = -\lim_{\rho \to -\infty} \frac{u''}{u'} - \frac{(n-1)u''}{(u')^2} + \lim_{\rho \to -\infty} \frac{n}{u'}$$

$$= (n-1) \lim_{\rho \to -\infty} (u')^{-1}$$

$$= \frac{1}{T - t}.$$ 

Therefore $|Ric|_g$ is uniformly bounded below by $(T-t)^{-1}$ along the exceptional divisor $D_0$. □

4. **Type I singularities**

In this section, we prove that the Ricci flow must develop Type I singularities with the same assumptions in section 4.

Let’s first recall the definition for a Type I singularity of the Ricci flow.

**Definition 4.1.** Let $(M, g(t))$ be a smooth solution of the Ricci flow $(1.1)$ for $t \in [0, T)$ with $T < \infty$. It is said to develop a Type I singularity at $T$ if it cannot be smoothly extended past $T$ and there exists $C > 0$ such that for all $t \in [0, T)$,

$$\sup_M |Rm(g(t))|_{g(t)} \leq \frac{C}{T - t}.$$ 


The following splitting theorem is proved in [4] as a complex analogue of Hamilton’s splitting theorem on Riemannian manifolds with nonnegative curvature operator [10].

**Theorem 4.1.** Let \( g \) be a complete solution of the Kähler-Ricci flow on a noncompact simply connected Kähler manifold \( M \) of dimension \( n \) for \( t \in (-\infty, \infty) \) with bounded and nonnegative holomorphic bisectional curvature. Then either \( g \) is of positive Ricci curvature for all \( p \in M \) and all \( t \in (-\infty, \infty) \), or \((M, g)\) splits holomorphically isometrically into a product \( \mathbb{C}^k \times N^{n-k} \) \((k \geq 1)\) flat in \( \mathbb{C}^k \) direction and \( N \) being of nonnegative holomorphic bisectional curvature and positive Ricci curvature.

We are now able to exclude Type II singularities.

**Theorem 4.2.** Let \( X = \mathbb{CP}^n \) blown-up at one point and \( g(t) \) be the solution of the Kähler-Ricci flow on \( X \) starting with a \( U(n) \)-invariant Kähler metric \( g_0 \). If \( g_0 \) lies in the Kähler class \(-a_0[D_0] + b_0[D_\infty]\) for \( 0 < a_0(n+1) < b_0(n-1) \). Then the flow develops Type I singularities at \( T = a_0/(n-1) \).

**Proof.** Suppose the flow develops Type II singularities. Let \( t_j \) be an increasing sequence converging to \( T = (n-1)a_0 > 0 \) and \( p_j \) a sequence of points on \( X \) such that

\[
K_j = |Rm(p_j, t_j)|_{g(t_j)} = \sup_X |Rm|_{g(t_j)}
\]

and

\[
\lim_{j \to \infty} (T - t_j)^{-1} K_j^{-1} = 0.
\]

Applying the standard parabolic rescaling, we define

\[
g_j(t) = K_j g(t_j + K_j^{-1} t).
\]

After extracting a convergent subsequence, \((X, g_j(t), p_j)\) converges in pointed Cheeger-Gromov-Hamilton sense to a complete eternal solution \((X_\infty, g_\infty(t), p_\infty)\) on a complete Kähler manifold \( X_\infty \) of dimension \( n \). Furthermore, by the lower bound of the holomorphic bisectional curvature of \( g(t) \) by Proposition 3.1, the limiting Kähler metric \( g_\infty(t) \) has nonnegative holomorphic bisectional curvature everywhere on \( X_\infty \). On the other hand, the symmetric product of the Ricci curvature \( g_\infty \) vanishes everywhere in \( X_\infty \),

\[
\sigma_k(Ric(g_\infty)) = 0
\]

for \( 2 \leq k \leq n \). This implies that the Ricci curvature of \( g_\infty \) is not positive at each point of \( X_\infty \). By applying the splitting theorem [4] for \((n-1)\) times, \((X_\infty, g_\infty, p_\infty)\), the eternal solution on the universal cover of \((X_\infty, g_\infty, p_\infty)\), splits holomorphically isometrically into \( \mathbb{C}^{n-1} \times N \), where \( N \) is a compact or complete Riemann surface with positive scalar curvature. By the classification of eternal solutions of real dimension 2 by Hamilton [11], \((N, \tilde{g}_\infty(t)|_N)\) is a steady gradient soliton and hence it must be the cigar soliton. However, it violates Peralman’s local non-collapsing [15], so does \((X_\infty, g_\infty)\). It then leads to a contradiction.
5. Blow-up limits

In this section, we will prove that the blow-up limit of the Ricci flow near the singular time $T$ along the exceptional divisor is a nontrivial complete shrinking gradient Kähler-Ricci soliton.

We first prove a diameter bound of the exceptional divisor $D_0$.

**Lemma 5.1.** For all $t \in [0, T)$,

\begin{equation}
    g(t)|_S = a_0(n - 1)(T - t)g_{FS}.
\end{equation}

and so

\begin{equation}
    \text{diam}(S, g(t)|_{D_0}) = \alpha_n(a_0(n - 1)(T - t))^{1/2}
\end{equation}

where $g_{FS}$ is a Fubini-Study metric on $\mathbb{CP}^{n-1}$ and $\alpha_n$ is the diameter of $(\mathbb{CP}^{n-1}, g_{FS})$.

**Proof.** The Kähler metric $g(t)$ is the metric completion of the following metric on $\mathbb{C}^n \setminus \{0\}$

\[ \omega(t) = a_0(n - 1)(T - t)\sqrt{-1}\partial\bar{\partial}\rho + \sqrt{-1}\partial\bar{\partial}u_0(e^\rho, t), \]

where $u_0(\cdot, t)$ is smooth and for each $t \in [0, T)$ with $u'(0, t) > 0$. Note that after extending $\sqrt{-1}\partial\bar{\partial}\rho = \sqrt{-1}\partial\bar{\partial}\log |z|^2$ to $\mathbb{CP}^n$ blown-up at one point, its restriction on $D_0$ is exactly a Fubini-Study metric. The lemma then follows immediately. \(\square\)

Now we can complete the proof of Theorem 1.1 by identifying the blow-up limit of the Ricci flow at the singular time.

**Proposition 5.1.** Fix any $p \in D_0$. Then for every $K_j \to \infty$, the rescaled Ricci flows $(X, g_j(t), p)$ defined on $[-K_jT, 0)$ by

\[ g_j(t) = K_jg(T + K_j^{-1}t) \]

subconverges in Cheeger-Gromov-Hamilton sense to a complete shrinking gradient Kähler-Ricci soliton on a complete Kähler manifold homeomorphic to $\mathbb{C}^n$ blown-up at one point.

**Proof.** We first show that the blow-up limit is a nontrivial complete shrinking soliton. Fix any point $p \in D_0$ in the exceptional divisor. Since $(X, g(t))$ is a Type I Ricci flow, the rescaled Ricci flow $(X, g_j(t), p)$ always subconverges to a shrinking gradient soliton $(X_\infty, g_\infty(t), p_\infty)$ in pointed Cheeger-Gromov-Hamilton sense, by the compactness result of Naber [14]. Such a limiting soliton cannot be flat because of Lemma 3.3. In particular, $(X_\infty, g_\infty, p_\infty)$ is a complete shrinking gradient Kähler-Ricci soliton on a complete Kähler manifold $X_\infty$.

We now show that $X_\infty$ is in fact homeomorphic to $\mathbb{C}^n$ blown-up at one point. Fix a closed interval $[a, b] \subset (-\infty, 0)$, the rescaled Ricci flow $g_j(t)$ restricted to $D_0$ is uniformly equivalent to a fixed standard Fubini-Study metric on $\mathbb{CP}^{n-1}$ for all $j$ and $t \in [a, b]$ by Lemma 5.1 and so there exist $d, D > 0$ such that the diameter of $D_0$ with respect to $g_j(t)$ is uniformly bounded between $d$ and $D$ for all $j$ and $t \in [a, b]$. We denote by

\[ B_g(p, R) \]
the geodesic ball with respect to \( g \) centered at \( p \) with radius \( R \). We then consider
\[
B_{j,t}(D_0, R) = \bigcup_{p \in D_0} B_{g_j(t)}(p, R)
\]
for each \( t \in [a, b] \). By choosing \( R \) sufficiently large, we have
\[
B_{g_j(t)}(p, R) \subset B_{j,t}(D_0, R) \subset B_{g_j(t)}(p, 2R)
\]
for any point \( p \in D_0 \) because \( g_j(t) \) is \( U(n) \)-invariant. By definition, for all \( t \in [a, b] \), \( B_{g_j(t)}(p, R) \) subconverges to \( B_{g_{\infty}(t)}(p_{\infty}, R) \) in Cheeger-Gromov-Hamilton sense and so \( B_{g_{\infty}(t)}(p_{\infty}, R) \) is homeomorphic to \( B_{g_j(t)}(p, R) \) for sufficiently large \( j \). We then obtain an exhaustion \( B_{g_{\infty}(t)}(p, R_k) \) with each \( R_k \) sufficiently large and \( R_k \to \infty \). Each of them is homeomorphic to \( \mathbb{C}^n \) blown-up at one point. Therefore \( X_{\infty} \) is homeomorphic to \( \mathbb{C}^n \) blown-up at one point.

We remark that the convergence in the above proof is \( U(n) \)-equivariant and the limiting shrinking soliton \( (X_{\infty}, g_{\infty}, p_{\infty}) \) is invariant under a free action of the unitary group \( U(n) \). We also remark that the Type I blow-up limit is a trivial shrinking soliton if one chooses a fixed base point outside the exceptional divisor \( D_0 \). This is because the flow converges in local \( C^\infty \) topology outside \( D_0 \) to a smooth Kähler metric on \( X \setminus D_0 \) by Theorem \textbf{[3.1]} \textbf{[22]}

Combing Theorem \textbf{4.2} and Proposition \textbf{5.1} we complete the proof of Theorem \textbf{[1.1]}

6. Some generalizations

In this section, we discuss some generalizations of Theorem \textbf{[1.1]}. First, Theorem \textbf{[1.1]} can be easily generalized to \( X_{n,k} \) defined in section 1 by the same argument in the previous sections.

\textbf{Theorem 6.1.} The Ricci flow on \( X_{n,k} \) must develop Type I singularities for any \( G_{n,k} \)-invariant initial Kähler metric.

Let \( g(t) \) be the smooth solution defined on \( t \in [0, T) \), where \( T \in (0, \infty) \) is the singular time. For every \( K_j \to \infty \), we consider the rescaled Ricci flow \( (X, g_j(t')) \) defined on \( [-K_j T, 0) \) by
\[
g_j(t') = K_j g(T + K_j^{-1} t').
\]
Then one and only one of the following must occur.

1. If \( \lim \inf_{t \to T} (T-t)^{-1} \text{Vol}(g(t)) = \infty \), then \( (X, g_j(t'), p) \) subconverges in Cheeger-Gromov-Hamilton sense to a complete nontrivial shrinking gradient Kähler-Ricci soliton on a complete Kähler manifold homeomorphic to the total space of \( L^{-k} = O_{\mathbb{C}^{n-1}}(-k) \), for any \( p \) in the exceptional divisor.

2. If \( \lim \inf_{t \to T} (T-t)^{-1} \text{Vol}(g(t)) \in (0, \infty) \), then \( (X, g_j(t'), p_j) \) subconverges in Cheeger-Gromov-Hamilton sense to \( (\mathbb{C}^{n-1} \times \mathbb{C}^1, g_{\mathbb{C}^{n-1}} \oplus (-t') g_{FS}) \), where \( g_{\mathbb{C}^{n-1}} \) is the standard flat metric on \( \mathbb{C}^{n-1} \) and \( g_{FS} \) the Fubini-Study metric on \( \mathbb{C}^1 \) for any sequence of points \( p_j \) \textbf{[7]}

3. If \( \lim \inf_{t \to T} (T-t)^{-1} \text{Vol}(g(t)) = 0 \), then \( (X, g_j(t')) \) converges in Cheeger-Gromov-Hamilton sense to the unique compact shrinking Kähler-Ricci soliton on \( X_{n,k} \) blown-up at one point \textbf{[24]}.
We can also consider the Calabi symmetry introduced by Calabi [2] for projective bundles over a Kähler-Einstein manifold (also see [12, 20]). In particular, we can consider the Ricci flow on generalizations of $X_{n,k}$

$$X_{m,n,k} = \mathbb{P} (\mathcal{O}_{\mathbb{C}P^n} \oplus \mathcal{O}_{\mathbb{C}P^n} (-k)^{\oplus (m+1)}), \ k = 1, 2, \ldots.$$  

Similar results are obtained for $X_{m,n,k}$ in [20] for global Gromov-Hausdorff convergence at the singular time, as those for $X_{n,k}$ in [18]. Furthermore, one can obtain the same lower bound for the holomorphic bisectional curvature as in Proposition 3.1.

**Proposition 6.1.** Let $g(t)$ be the solution of the Ricci flow on $X_{m,n,k}$ for an initial Kähler metric with Calabi symmetry. Then if $1 \leq m \leq n$ and if

$$\liminf_{t \to T} \text{Vol}(g(t)) > 0$$

where $T > 0$ is the singular time, then the holomorphic bisectional curvature of $g(t)$ is uniformly bounded below by $-\frac{C}{t}$ for some constant $C > 0$.

Although we are unable to exclude Type II singularities, one can show by the same argument in section 4, that the universal cover of the blow-up limit is an eternal solution of the Ricci flow which splits into $\mathbb{C}^n \times N^{m+1}$ flat in $\mathbb{C}^n$ and $N^{m+1}$ of nonnegative holomorphic bisectional curvature, if the flow develops Type II singularities. Of course, a Type I bound for the scalar curvature suffices to prove a similar theorem as Theorem 1.1.

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