The Hilbert scheme of space curves sitting on a smooth surface containing a line

Jan O. Kleppe

Abstract

We continue the study of maximal families $W$ of the Hilbert scheme, $H(d,g)_{sc}$, of smooth connected space curves whose general curve $C$ lies on a smooth degree-$s$ surface $S$ containing a line. For $s \geq 4$, we extend the two ranges where $W$ is a unique irreducible (resp. generically smooth) component of $H(d,g)_{sc}$. In another range, close to the border of the nef cone, we describe for $s=4$ and 5 components $W$ that are non-reduced, leaving open the non-reducedness of only 3 (resp. 2) families for $s \geq 6$ (resp. $s = 5$), thus making progress to recent results of Kleppe and Ottem in [28]. For $s=3$ we slightly extend previous results on a conjecture of non-reduced components, and in addition we show its existence in a subrange of the conjectured range.

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1 Introduction

Let $H(d,g)_{sc}$ be the Hilbert scheme of smooth connected curves in $\mathbb{P}^3$. In this paper we study irreducible, possibly non-reduced, components of $H(d,g)_{sc}$ whose general curve sits on a smooth surface containing a line. Since the first example of a non-reduced component was found by Mumford [34] there has been many geometers who were challenged by this phenomena and quite a lot of papers has appeared that consider the problem of non-reducedness and related questions, see e.g. [11,6,7,13,17,21,23,25,26,31,32,33,35,36] and the books [29] and [20].

A classical way of analyzing whether the closure of a family is a component, possibly non-reduced, is to take a general curve $C$ of a family and describe the curves in an open neighborhood of $(C) \in H(d,g)_{sc}$. More recently several authors has been able to sufficiently describe the obstruction of deforming $C$ and conclude similarly [13,26,32,33,35,36]. In the recent paper [28] we find non-reduced, as well as generically smooth, irreducible components of $H(d,g)_{sc}$, and we prove non-reducedness along the classical line. This works quite well if the genus is large and the minimal degree $s(C)$ of a surface containing a general curve $C$ is small (as in [17]).

The families we consider are $s$-maximal families. To define them let $W$ is an irreducible closed subset of $H(d,g)_{sc}$ and let $s(W):=s(C)$ where $C$ is a general curve of $W$. As in [21] we say $W$ is $s(W)$-maximal if it is maximal with respect to $s(W)$, i.e. $s(V) < s(W)$ for any closed irreducible subset $V$ properly containing $W$. If $d > s^2$, an $s$-maximal family containing $(C)$ is nothing but the image under the forgetful morphism $pr_1 : D(d,g;s)_{sc} \to H(d,g)_{sc}$, $(C,S) \mapsto (C)$ of an irreducible component of the Hilbert-flag scheme $D(d,g;s)_{sc}$ containing $(C,S)$, see Section 2 for details.

An $s$-maximal family $W$ needs not be a component of $H(d,g)_{sc}$. Indeed $4d \leq \dim W$ is obviously a necessary condition for $W$ to be a component, while $H^1(I_C(s)) = 0$, $I_C$ the sheaf ideal of $C \subset \mathbb{P}^3$, turns out to be a sufficient condition because $pr_1$ is smooth at $(C,S)$. When $H^1(I_C(s)) = 0$, $W$ is even generically smooth if the corresponding component of $D(d,g;s)_{sc}$ is, e.g. if $s \leq 4$. Since the dimension $\dim W$ of an $s$-maximal family is easy to compute for $s \leq 4$, we get in particular that $g \geq g_1 := 3d - 18$ is necessary (for $d > 9$ and $S$ smooth) while $g > g_2 := [(d^2 - 4)/8]$ is sufficient.
In this paper the ground field \( k \) is algebraically closed of characteristic zero (and equal to the complex numbers when the concept "very general" is used). A surface \( S \) in \( \mathbb{P}^3 \) is a hypersurface, and a curve \( C \) in \( \mathbb{P}^3 \) (resp. in \( S \)) is a pure one-dimensional subscheme of \( \mathbb{P} := \mathbb{P}^3 \) (resp. \( S \)) with ideal sheaf \( \mathcal{I}_C \) (resp. \( \mathcal{I}_{C|S} \)) and normal sheaf \( \mathcal{N}_C = \text{Hom}_{\mathcal{O}_S}(\mathcal{I}_C, \mathcal{O}_C) \) (resp. \( \mathcal{N}_{C|S} = \text{Hom}_{\mathcal{O}_S}(\mathcal{I}_{C|S}, \mathcal{O}_C) \)). If \( \mathcal{F} \) is a coherent \( \mathcal{O}_S \)-Module, we let \( H^i(\mathcal{F}) = H^i(\mathcal{F}, \mathcal{O}_S) \), \( h^i(\mathcal{F}) = \dim H^i(\mathcal{F}) \), \( \chi(\mathcal{F}) = \Sigma(-1)^i h^i(\mathcal{F}) \) and \( s(C) = \min \{ n \mid h^0(\mathcal{I}_C(n)) \neq 0 \} \). We denote by \( H(d, g) \) (resp. \( H(d, g)_{sc} \)) the Hilbert scheme of (resp. smooth connected) space curves of Hilbert polynomial \( \chi(\mathcal{O}_C(t)) = dt + 1 - g(C) \), and we let \( d := d(C) \) and \( g := g(C) \). A curve \( C \) is called unobstructed if \( H(d, g) \) is smooth at the corresponding point \( (C) \). A curve in a small enough open irreducible subset \( U \) of \( H(d, g) \) is called a general curve of

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**1.1 Notations and terminology**

In this paper the ground field \( k \) is algebraically closed of characteristic zero (and equal to the complex numbers when the concept "very general" is used). A surface \( S \) in \( \mathbb{P}^3 \) is a hypersurface, and a curve \( C \) in \( \mathbb{P}^3 \) (resp. in \( S \)) is a pure one-dimensional subscheme of \( \mathbb{P} := \mathbb{P}^3 \) (resp. \( S \)) with ideal sheaf \( \mathcal{I}_C \) (resp. \( \mathcal{I}_{C|S} \)) and normal sheaf \( \mathcal{N}_C = \text{Hom}_{\mathcal{O}_S}(\mathcal{I}_C, \mathcal{O}_C) \) (resp. \( \mathcal{N}_{C|S} = \text{Hom}_{\mathcal{O}_S}(\mathcal{I}_{C|S}, \mathcal{O}_C) \)). If \( \mathcal{F} \) is a coherent \( \mathcal{O}_S \)-Module, we let \( H^i(\mathcal{F}) = H^i(\mathcal{F}, \mathcal{O}_S) \), \( h^i(\mathcal{F}) = \dim H^i(\mathcal{F}) \), \( \chi(\mathcal{F}) = \Sigma(-1)^i h^i(\mathcal{F}) \) and \( s(C) = \min \{ n \mid h^0(\mathcal{I}_C(n)) \neq 0 \} \). We denote by \( H(d, g) \) (resp. \( H(d, g)_{sc} \)) the Hilbert scheme of (resp. smooth connected) space curves of Hilbert polynomial \( \chi(\mathcal{O}_C(t)) = dt + 1 - g(C) \), and we let \( d := d(C) \) and \( g := g(C) \). A curve \( C \) is called unobstructed if \( H(d, g) \) is smooth at the corresponding point \( (C) \). A curve in a small enough open irreducible subset \( U \) of \( H(d, g) \) is called a general curve of
H(d, g). A generalization $C' \subset \mathbb{P}^3$ of $C \subset \mathbb{P}^3$ in $H(d, g)$ is the general curve of some irreducible subset of $H(d, g)$ containing $(C)$. By an irreducible component of $H(d, g)$ we always mean a non-embedded irreducible component. A member of a closed irreducible subset $V$ of $H(s)$ or $H(d, g)_{sc}$ is called very general in $V$ if it is smooth and sits outside a countable union of proper closed subset of $V$.

2 Background

In the following we recall some results from the the background section of [28] that extend ideas and results appearing in [21], [25] and [24] and that use the deformation theory developed by Laudal in [29]; in particular the results rely on [29] Thm. 4.1.14]. Moreover ideas in [15] and [12] are central.

2.1 The Hilbert flag scheme and the relative Picard scheme

Let $D(d, g; s)$ be the Hilbert-flag scheme parameterizing pairs $(C, S)$ of curves $C$ contained in a degree $s$-surface $S \subset \mathbb{P}^3$ where $d$ and $g$ are the degree and genus of $C$. If $S$ is smooth then $\mathcal{N}_{C/S} \cong \omega_C \otimes \omega_S^{-1}$ and we have a connecting homomorphism $\delta : H^0(\mathcal{N}_{S|C}) \to H^1(\mathcal{N}_{C/S}) \cong H^0(\mathcal{O}_C(s - 4))^\vee$ induced by the sequence $0 \to \mathcal{N}_{C/S} \to \mathcal{N}_C \to \mathcal{N}_{S|C} \to 0$ of normal bundles. Let $\alpha_C := \delta \circ m$ be the composed map of the natural restriction $m : H^0(\mathcal{N}_S) \to H^0(\mathcal{N}_{S|C})$ with $\delta$, let $A^2 := \text{coker} \alpha$ and let $A^1$ be the tangent space of $D(d, g; s)$ at $(C, S)$. Then the tangent map $A^1 \to H^0(\mathcal{N}_C)$ of the $1^{st}$ projection,

$$pr_1 : D(d, g; s) \longrightarrow H(d, g), \quad \text{induced by} \quad pr_1((C_1, S_1)) = (C_1),$$

at $(C, S)$ fits into an exact sequence

$$0 \to H^0(\mathcal{I}_{C/S}(s)) \to A^1 \to H^0(\mathcal{N}_C) \to H^1(\mathcal{I}_C(s)) \to \text{coker} \alpha_C \to H^1(\mathcal{N}_C) \to H^1(\mathcal{O}_C(s)) \to 0 \quad (2)$$

from which we deduce $\dim A^1 - \dim A^2 = (4 - s)d + g + \binom{s + 3}{3} - 2$. Note that $pr_1$ is a projective morphism ([21] Thm. 24.7]). By [21] Lem. A10] $pr_1$ is smooth at $(C, S)$ if $H^1(\mathcal{I}_C(s)) = 0$. Moreover $A^2 = \text{coker} \alpha_C$ contains the obstructions of deforming the pair $(C, S)$ ([21] (2.6)]), and we have $A^2 = 0$ for $s \leq 4$ if $C$ is smooth and connected and not a complete intersection (c.i.) in $S$ (by the infinitesimal Noether-Lefschetz theorem if $s = 4$ and because $\delta = 0$ if $s \leq 3$). Let $d > s^2$. Restricting $pr_1$ to the open set $D(d, g; s)_{sc}$ where the curves are smooth and connected, we get that an $s$-maximal family $W$ of $H(d, g)_{sc}$ containing $(C)$ is nothing but the image under $pr_1$ of an irreducible component of $D(d, g; s)_{sc}$ containing $(C, S)$ ([23] Def. 1.24 and Cor. 1.26]).

We also need to consider the Hilbert scheme, $H(s) \simeq \mathbb{P}(\binom{s + 3}{3} - 1)$, of degree-$s$ surfaces in $\mathbb{P}^3$, the $2^{nd}$ projection $pr_2 : D(d, g; s) \longrightarrow H(s)$, induced by $pr_2((C_1, S_1)) = (S_1)$, and the relative Picard scheme, Pic, over the open set in $H(s)$ of smooth surfaces of degree $s$. There is a projection $p_2 : \text{Pic} \to H(s)$, forgetting the invertible sheaf, and a rational map:

$$\pi : D(d, g; s) \longrightarrow \text{Pic}, \quad \text{induced by} \quad \pi((C_1, S_1)) = (\mathcal{O}_{S_1}(C_1), S_1) \quad (3)$$

defined over the open subset $U \subset D(d, g; s)$ given by pairs $(C_1, S_1)$ where $C_1$ is Cartier on a smooth $S_1$. Obviously, if we restrict to $U$ we have $p_2 \circ \pi = pr_2$. If $H^1(S, \mathcal{O}_S(C)) = 0$ then $\pi$ is smooth at $(C, S)$ by [15] Rem. 4.5]. Indeed, $H^1(S, L) = 0$, $L := \mathcal{O}_S(C)$, implies a surjective map $A^1 \to T_{\text{Pic}, L}$ between the tangent spaces of $D(d, g; s)$ at $(C, S)$ and Pic at $(L)$ and an injection $\text{coker} \alpha_C \to \text{coker} \alpha_L$ of their obstruction spaces where $\alpha_L$ is the composition of $\alpha_C$ with the connecting homomorphism $H^1(\mathcal{N}_{C/S}) \to H^2(\mathcal{O}_S)$ induced from the exact sequence $0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{N}_{C/S} \to 0$ (cf. [12] Thm. 1], [25] Sect. 4] and [21]). Noticing that $H^1(S, \mathcal{O}_S(C)) \simeq H^1(\mathbb{P}^3, \mathcal{I}_C(s - 4))$, we have
Lemma 2.1. Let $S \subset \mathbb{P}^3$ be a smooth degree-$s$ surface, $H$ a hyperplane section, let $E$ and $C$ be curves on $S$ satisfying $C \equiv (i.e. linearly equivalent to) eE + fH$ for some $e \neq 0, f \in \mathbb{Z}$.

(i) Suppose $H^1(\mathcal{I}_E(s - 4)) = H^1(\mathcal{I}_C(s - 4)) = 0$. Then $D(d, g; s)$ is smooth at $(C, S)$ if and only if $D(d(E), g(E); s)$ is smooth at $(E, S)$, and we have
\[
\dim \operatorname{coker} \alpha_C + h^0(\mathcal{I}_{C/S}(s - 4)) = \dim \operatorname{coker} \alpha_E + h^0(\mathcal{I}_{E/S}(s - 4))
\]

(ii) If $H^1(\mathcal{I}_E(s)) = 0 = H^1(\mathcal{I}_C(s - 4)) = H^1(\mathcal{I}_C(s - 4))$ and $H(d(E), g(E))_{sc}$ is a smooth irreducible scheme and $(E) \in H(d(E), g(E))_{sc}$, then $D(d, g; s)$ is smooth at $(C, S)$ and every $(C', S') \in D(d, g; s)$ satisfying $C' \equiv eE' + fH'$ for some $(E', S') \in D(d(E), g(E); s)_{sc}$, $H'$ a hyperplane section of a smooth surface $S' \subset \mathbb{P}^3$, belongs to the unique irreducible component of $D(d, g; s)$ containing $(C, S)$.

Proof. The main points in the proof is that $\pi$ is smooth at $(C, S)$ and that $\alpha_{O_S(E)} = e \cdot \alpha_{O_S(E)}$ ( [12, Prop. 1.5 and Constr. 1.8]) leads to an isomorphism of their cokernels, see [28, Lem. 2.2].

Remark 2.2 (Example of a non-reduced irreducible component of the relative Picard scheme $\operatorname{Pic}$).

If both $H^1(\mathcal{I}_C(s))$ and $H^1(\mathcal{I}_C(s - 4))$ vanish, then the morphisms $\pi_1$ of (11) and $\pi$ of (3) are smooth at $(C, S)$. Using this we get that many properties of the Hilbert scheme $H(d, g)_{sc}$ at $(C)$ are transferred to the relative Picard scheme $\operatorname{Pic}$ at $(O_S(C), S)$, and vice versa. For instance if we take the general curve $C$ of the non-reduced component $V \subset H(14, 24)_{sc}$ of Mumford, and a smooth general surface $S$ of degree 6 containing $C$, then $(O_S(C), S)$ is the general point of a non-reduced irreducible component $P$ of $\operatorname{Pic}$. This follows from the fact that smooth morphisms take generic (resp. smooth) points onto generic (resp. smooth) points. Of course we have to verify the assumptions;

\[
H^1(\mathcal{I}_C(s)) = H^1(\mathcal{I}_C(s - 4)) = 0 \quad \text{for} \quad s = 6
\]

with $S$ smooth and $(C, S)$ a general point of $D(d, g; s)$. For the general curve in Mumford’s example it is known that the homogeneous ideal $I(C)$ allows 4 minimal generators of degree $\{3, 6, 6, 6\}$ that correspond to smooth surfaces by [23] and that $H^1(\mathcal{I}_C(v)) = 0$ for $v \notin \{3, 4, 5\}$. Hence we may take the general degree-6 surface containing $C$ to be smooth. Thus we conclude that $P$ is a non-reduced irreducible component of $\operatorname{Pic}$, see [4, 23, 34] for a comparison with the Noether-Lefschetz locus.

We will also need the following lemma (cf. [30, Cor. II 3.8] and see [30, Thm. II 3.1] for a proof).

Lemma 2.3. (A. Lopez) Let $E \subset \mathbb{P}^3$ be a smooth irreducible curve, let $n \geq 4$ be an integer and suppose the degree of every minimal generator of the homogeneous ideal of $E$ is at most $n - 1$. Let $S$ be a very general smooth degree of degree $n$ containing $E$ and let $H$ be a hyperplane section. Then $\operatorname{Pic}(S) \cong \mathbb{Z} \oplus \mathbb{Z}$ and we may take $\{O_S(H), O_S(E)\}$ as a $\mathbb{Z}$-basis for $\operatorname{Pic}(S)$.

2.2 On the maximum genus of space curves

Finally let us recall the definition of the maximum genus, $G(d, s)$, of smooth connected space curves of degree $d$ not contained in a surface of degree $s - 1$, cf. [10]. By definition,

\[
G(d, s) = \max \{ g(C) \mid (C) \in H(d, g)_{sc} \text{ and } H^0(\mathcal{I}_C(s - 1)) = 0 \}.
\]

In the C-range, i.e. when $d > s(s - 1)$, Gruson and Peskine showed in [16] that

\[
G(d, s) = 1 + \frac{d}{2} \left( \frac{d}{s} + s - 4 \right) - \frac{r(s - r)(s - 1)}{2s} \quad \text{where } d + r \equiv 0 \text{ mod } s \quad \text{for} \quad 0 \leq r < s,
\]
and that \( g(C) = G(d, s) \) if and only if \( C \) is directly linked to a plane curve \( E \) of degree \( r \) by a c.i. of type \( (s, f) \), \( f := (d + r)/s \). In the “extended \( C \)-range”: \( s(s - 1) \geq d \geq s^2 - 2s + 2 \) where the upper part of the \( B \)-range is included, letting \( \mu := d - (s^2 - 2s + 3) \), one knows that (13 Thm. A)

\[
G(d, s) = s^3 - 5s^2 + 9s - 6 + \frac{\mu(\mu + 2s - 3)}{2} \quad \text{for} \quad s - 3 \geq \mu \geq 0, \quad \text{and} \quad (6)
\]

\[
G(d, s) = 1 + d(s - 3) \quad \text{for} \quad d = s^2 - 2s + 2, \quad \text{i.e.} \quad \text{for} \quad \mu = -1. \quad (7)
\]

Moreover, if \( g(C) = G(d, s) \), then \( C \) is ACM with \( s(C) = s \) in (6), and \( C \) is a zero-section of the null correlation bundle twisted by \( s - 1 \) in (7).

3 Components of \( H(d, g)_{sc} \) for \( s \geq 4 \), the surface contains a line

Let \( S \) be a smooth surface of degree \( s \geq 4 \) in \( \mathbb{P}^3 \) defined by a form \( x_0P + x_1Q \), where \( P, Q \) are very general homogeneous polynomials of degree \( s - 1 \). Families of curves on such surfaces were studied in Theorems 1.1, 1.2 and 7.3 of [28]. Here we improve upon these results.

Let \( \Gamma_1 = \{ x_0 = x_1 = 0 \} \) and \( \Gamma_2 = \{ x_0 = Q = 0 \} \). The hyperplane section satisfies \( H \equiv \Gamma_1 + \Gamma_2, \)

\( H^2 = s \) and we may suppose \( \text{Pic}(S) \cong \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}H \) by Lemma [23] and that \( \Gamma_2 \) is a smooth connected curve. If \( C \equiv a\Gamma_1 + b\Gamma_2 \) then \( d = C \cdot H, K = (s - 4)H \) and the adjunction formula which implies that the intersection matrix is \( (\Gamma_i \cdot \Gamma_j) = (\frac{2-s}{s-1} \frac{1}{0}) \), leads also to

\[
d = a + (s - 1)b \quad \text{and} \quad g = 1 + (s - 1)ab + \frac{1}{2}((s - 4)a + (s - 4)(s - 1)b - (s - 2)a^2). \quad (8)
\]

Lemma 3.1. Let \( S \) be a smooth surface of degree \( s \) with \( \Gamma_1, \Gamma_2 \) as above. It holds:

(i) Any effective divisor on \( S \) is linearly equivalent to \( a\Gamma_1 + b\Gamma_2 \) where \( a, b \geq 0 \).

(ii) Every nef divisor is linearly equivalent to \( a\Gamma_1 + b\Gamma_2 \) where \( (s - 1)b \geq (s - 2)a \geq 0 \).

(iii) The divisor \( D = (s - 1)\Gamma_1 + (s - 2)\Gamma_2 \) is base-point free.

(iv) Any divisor \( C \) satisfying \( C \cdot \Gamma_1 \geq 0 \) and \( C \cdot \Gamma_2 \geq 0 \) is nef and base-point free.

(v) If a divisor \( C \) satisfies \( C \cdot \Gamma_1 \geq 0 \) and \( C \cdot \Gamma_2 > 0 \) then \( |C| \) contains a smooth irreducible curve.

See [28] for a proof. Now the Kawamata-Viehweg vanishing theorem, for \( D \equiv a\Gamma_1 + b\Gamma_2 \), implies:

\[
H^i(S, \mathcal{O}_S(D)) = 0 \quad \text{for} \quad i > 0 \quad \text{provided} \quad a > s - 4 \quad \text{and} \quad (s - 1)b \geq (s - 2)a + s - 4 \quad (9)
\]

because the assumptions on \( a, b \) imply that \( D - K \) is nef and big. From this, we got Theorem 7.3 of [28]. Here we generalize [9] leading to a significant improvement of that result, cf. [28] Lem. 2.5. Indeed we have

Lemma 3.2. Let \( S \) be a smooth surface of degree \( s \) with \( \Gamma_1, \Gamma_2 \) as above and let \( C \) be a divisor linearly equivalent to \( a\Gamma_1 + b\Gamma_2 \) where \( (s - 1)b - (s - 2)a = t \) with \( t \geq -2 \). Moreover suppose \( a > s - 4 \). Then

(i) \( H^i(S, \mathcal{O}_S(C)) = 0 \) for \( i > 0 \) provided \( t > -2 \) or \( t = -2 \) and \( a = s - 3 \), and

(ii) \( H^1(S, \mathcal{O}_S(C)) \simeq k \), \( H^2(S, \mathcal{O}_S(C)) = 0 \) provided \( t = -2 \) and \( a \neq s - 3 \).
Proof. Firstly we suppose \( a > s - 3 \). To apply \((\ref{9})\) onto \( D = C - \Gamma_1 \), we notice that
\[
(s - 1)b = (s - 2)a - (s - 2) + s - 2 + t \geq (s - 2)(a - 1) + s - 4
\]
by assumption. Thus \( H^i(S, O_S(C - \Gamma_1)) = 0 \) for \( i > 0 \) by \((\ref{9})\). Moreover since
\[
0 \rightarrow O_S(C - \Gamma_1) \rightarrow O_S(C) \rightarrow O_{\Gamma_1}(C \cdot \Gamma_1) \rightarrow 0
\]
is exact, we deduce the exact sequence
\[
\rightarrow H^1(O_S(C - \Gamma_1)) \rightarrow H^1(O_S(C)) \rightarrow H^1(O_{\Gamma_1}(C \cdot \Gamma_1)) \rightarrow H^2(O_S(C - \Gamma_1)) \rightarrow H^2(O_S(C)) \rightarrow 0
\]
where \( C \cdot \Gamma_1 = -a(s - 2) + b(s - 1) = t \) and we easily get the lemma in the case \( a \neq s - 3 \).

Finally if \( a = s - 3 \) we apply \((\ref{9})\) onto \( D := C \). Since we have \( a > s - 4 \) it suffices to show \((s - 1)b \geq (s - 2)(s - 3) + s - 4 \), or equivalently \( b \geq s - 3 - 1/(s - 1) \) which holds if \( b \geq s - 3 \). Since \( b \geq s - 4 \) follows from the assumption \((s - 1)b = (s - 2)a + t \geq (s - 2)(s - 3) + (-2) \) and the curve \( \Gamma_1 + (s - 4)H \equiv (s - 3, s - 4) \) is ACM, we get the lemma.

The assumption \( t \geq -2 \) in Lemma \((\ref{3.2})\) may be weakened. Indeed we have

Lemma 3.3. Let \( S \) be a smooth surface of degree \( s \) with \( \Gamma_1, \Gamma_2 \) as above, let \( C \equiv a\Gamma_1 + b\Gamma_2 \) and \( t := (s - 1)b - (s - 2)a \) and suppose \(-1 \geq t \geq -4 \), \( a > s - 2 \), \( s \geq 4 \) and \((t, s) \neq (-4, 4) \) (resp. \((t, s) = (-4, 4) \)). Then
\[
dim H^1(S, O_S(C)) = -t - 1 \quad (\text{resp. } 4 \text{ if }(t, s) = (-4, 4)).
\]

Proof. Suppose \((t, s) \neq (-4, 4) \). Then we have \( H^i(S, O_S(C - \Gamma_1)) = 0 \) for \( i > 0 \) by Lemma \((\ref{3.2})\) since \( a - 1 > s - 3 \) and \((s - 1)b - (s - 2)(a - 1) = t + s - 2 > -2 \), i.e. the assumptions of the lemma are fulfilled for \((a - 1)\Gamma_1 + b\Gamma_2 \). By \((\ref{10})\), we get \( h^1(O_S(C)) = h^1(O_{\Gamma_1}(C \cdot \Gamma_1)) = -t - 1 \) because \( C \cdot \Gamma_1 = t \). If \((t, s) = (-4, 4) \), then Lemma \((\ref{3.2})\) yields \( H^1(S, O_S(C - \Gamma_1)) \simeq k \) and we conclude by \((\ref{10})\).

We are now ready to generalize \((\ref{28})\) Thms. 1.1, 1.2, 7.3. Below we often write \((a, b)\) for \( a\Gamma_1 + b\Gamma_2 \).

Theorem 3.4. Let \( S \subset \mathbb{P}^3 \) be a smooth degree-\( s \) surface containing a line \( \Gamma_1 \), let \( \Gamma_2 \equiv H - \Gamma_1 \) be a smooth curve and suppose \( \text{Pic}(S) \simeq \mathbb{Z}\Gamma_1 \oplus \mathbb{Z}\Gamma_2 \) (e.g. \( S \) is very general) and \( s \geq 4 \). Let \( C \equiv a\Gamma_1 + b\Gamma_2 \) be a smooth connected curve of degree \( d > s^2 \) with \( a \neq b \).

(i) Suppose \( a > s - 4 \). Then \( C \) belongs to a unique \( s \)-maximal family \( W \subset H(d, g)_{\text{sc}} \). Moreover if \( S \) is a degree-\( s \) surface containing a very general member of \( W \), then \( \text{Pic}(\tilde{S}) \) is freely generated by the classes of a line and a smooth plane degree-\((s - 1)\) curve, and every \( C \equiv a\Gamma_1 + b\Gamma_2 \) contained in some surface \( S \) as above belongs to \( W \). Furthermore
\[
\dim W = (4 - s)d + g + \binom{s + 3}{3} + \binom{s - 1}{3} - s + 1 \quad \text{with } d, g \text{ as in } (\ref{5}),
\]
and if \( (s, a, b) \notin \{(4, 6, 4), (4, 9, 6)\} \) then \( W \) is an irreducible component of \( H(d, g)_{\text{sc}} \).

(ii) Suppose \( s < a < \frac{(s - 1)b - 2}{s - 2} \) or \((a, b) = (s + 1, s) \). Then \( W \) is moreover a generically smooth irreducible component of \( H(d, g)_{\text{sc}} \).
(iii) Suppose $(s-1)b-2 \leq a \leq \frac{(s-1)b}{s-2}$, $(a,b) \neq (s+1,s)$ and $(s,a,b) \notin \{(4,6,4),(4,9,6)\}$. Then $W$ is a non-reduced irreducible component of $H(d,g)_{sc}$ provided $h^{0}(N_{C}) > \dim W$, or equivalently, provided the map $H^{0}(N_{C}) \to H^{1}(I_{C}(s))$ appearing in (2) is non-zero for the general curve $C$ of $W$. In particular $W$ is a non-reduced irreducible component of $H(d,g)_{sc}$ if $s = 4$, or $s = 5$ and $(a,b) = (4k,3k)$, $k \geq 2$.

**Remark 3.5.** (A) Note that for any $C \equiv (a,b)$ of the theorem we have $a \leq \frac{(s-1)b}{s-2}$ because $C$ is nef.

(B) Since the assumption in (iii) and $d > s^{2}$ imply $a > s$, there are exactly three families, 

$$(a,b) = ((s-1)n - \mu, (s-2)n - \mu + 1), \quad n \geq 3$$

corresponding to (a): $\mu = s-3$, (b): $\mu = s-2$ and (c): $\mu = s-1$ respectively, that are covered by (iii) of Theorem 3.4 (where (c) is on the border of the nef cone). Thanks to Lemma 3.3 we have $h^{1}(I_{C}(s)) = 1$ (resp. $h^{1}(I_{C}(s)) = 2$, $h^{1}(I_{C}(s)) = 3$) for the family (a) (resp. (b), (c)) if $s \geq 5$ for family (c). We expect that the corresponding irreducible components of $H(d,g)_{sc}$ are non-reduced. Indeed they are non-reduced for the family (c) when $s = 5$ by Theorem 3.4.

(C) If $s = 4$ then Theorem 3.4 implies that the three families described in (B) form non-reduced components (with $n \geq 5$ in (c)). Thus we slightly generalize [28] Thm. 1.2 (II) by including two more cases of non-reduced components. Note that for family (c), $h^{1}(I_{C}(s)) = 4$ in this case.

(D) For $s = 5$, Theorem 3.4 (ii), resp. (iii) slightly generalize (I), resp. (II) of [28] Thm. 1.2 by including one more infinite family (resp. one more case of a non-reduced component).

(E) For $s \geq 6$ the generalization in Theorem 3.4 of [28] Thm. 7.3] is more substantial because Theorem 3.4 (ii), resp. (iii) includes, as $s$ increases, an increasing number of infinite families for which Theorem 7.3 holds (compared with what holds by [28] Thm. 7.3).

**Proof.** The assumptions on $a,b$ in (i), resp. (ii), imply that $H^{1}(O_{S}(C)) = 0$, resp. $H^{1}(I_{C}(s)) \cong H^{1}(O_{S}(C - 4H)) = 0$ by Lemma 3.2. We therefore get the stated properties of $W$ in (i), resp. (ii) by the same proof is that in [28] Thm. 7.3] (except for $W$ being a component in (i)); the whole point is only that Lemma 3.2 imply the vanishing of $H^{1}(O_{S}(C + vH))$ under weaker assumptions than those used in [28] Thm. 7.3], which lead to corresponding improvements of Theorem 3.4 (i) and (ii).

In (i) it only remains to see that $W$ is an irreducible component. Observe that $H^{1}(I_{C}(s)) = 0$ does not only prove the generic smoothness of $W$, but it also implies that $W$ is an irreducible component of $H(d,g)_{sc}$. There are, however, cases (i.e. the 3 families mentioned in (B) above) not covered by (ii) of the theorem, and for these we prove that $W$ is an irreducible component as in [28] Thm. 7.3] by e.g. showing $g \geq G(d,s+1)$, except when $(s,a,b) \in \{(4,7,5),(4,12,8),(5,8,6),(6,10,8)\}$. In these four cases we were not able to show that $W$ was an irreducible component in [28] Thms. 1.1, 7.3] because $g < G(d,s+1)$. To get (i) and (ii) as stated above it remains to consider them now.

In these cases we suppose there is an irreducible component $V$ of $H(d,g)_{sc}$ satisfying $W \subset V$ and $\dim W < \dim V$. Since $W$ is $s$-maximal, we may suppose that the general curve $X$ of $V$ satisfies $s(X) > s$.

**The case** $(a,b) = (12,8)$ and $s = 4$. For this class we compute the following numbers: $(d,g) = (36,145)$, $G(d,5) = 147$ and $G(d,6) = 145$, cf. (4). We first consider the option $s(X) = 5$. To see that $X$, whence $V$ does not exist we use the theory on Halphen’s gaps given in Ellia’s paper [8]. If such a curve exists, we compute $r$ in $d(X) + r = 0 \mod 5$, and we get $r = 4 = s(X) - 1$. Using [8] Prop. IV.3] it follows that there are no such $X$ with maximal numerical character, and by [8] Lem. VI.2] that the genus of $X$ is equal to the genus of the numerical character of $X$. This implies that $X$ is arithmetically Cohen-Macaulay (ACM).
By Riemann-Roch $\chi(I_X(5)) = 20$, whence $h^1(O_X(5)) = 19$ and since $\chi(I_X(8)) = 21$, we get $h^1(O_X(8)) \leq 1$. It follows that $\dim \text{Hom}(I_X, H^1(I_X)) = 19$. Using

$$h^0(N_X) = 4d + \dim \text{Hom}(I_X, H^1(I_X)) + \dim \text{Hom}(I_X, H^1(O_X))$$

which holds for curves of maximal rank (cf. Remark 3.6), in particular for ACM curves (where we have $\text{Hom}(I_X, H^1(I_X)) = 0$), we conclude that $h^0(N_X) = 163 < g + 33 = 178$ which contradicts the assumption $W < \dim V$.

Also the case $s(X) \geq 6$ must be considered. Since $g = G(d, 6) = 145$ and $d(X) \equiv 0 \mod 6$, $X$ is a c.i. of type $(6, 6)$ by [5]. Since $\chi(I_X(6)) = 12$ we get $h^1(O_X(6)) = 10$ and $\dim I_X \text{Hom}(I_X, H^1(O_X)) = 4d + 2h^1(O_X(6)) = 164$, a contradiction. This shows that $W$ is a component.

The case $(a, b) = (7, 5)$ and $s = 4$. We compute the following numbers: $(d, g) = (22, 57)$, $\chi(I_X(5)) = 2$, $\chi(I_X(6)) = 8$ and $G(d, 5) = 58$. Note that $g = G(d, 5) - 1$; whence we suppose a minimal resolution of $I_X$ that the genus of $X$ is equal to the genus of the numerical character of $X$. The conjecture in [19, Conj. 3.5] is true for $s(X) \leq 5$, and [19, Cor. 3.6] to compute the dimension of the bilinked family. Let us instead use that the minimal resolution of $I_X$ is known (the resolution of the skew lines have the same Betti numbers) and the mapping cone construction used twice yields:

$$0 \to R(-8) \to R(-9) \oplus R(-8) \oplus R(-7)^4 \to R(-8) \oplus R(-6)^4 \oplus R(-5) \to I_X \to 0.$$

Thus $X$ is of maximal rank and applying (11), we get $h^0(N_X) \leq 4d + 2 = 90$ because $\dim I_X \text{Hom}(I_X, H^1(I_X)) = h^1(I_X(4)) = 1$ and $h^1(O_X(5)) = 1$. Since $\dim W = g + 33 = 90$, we get a contradiction.

Thirdly if the numerical character is not maximal we can again use [8] Lem. VI.1.2 (or [8]) to see that the genus of $X$ is equal to the genus of the numerical character of $X$. This implies that $X$ is ACM and using (11), we get $h^0(N_X) \leq 4d + 1 = 89$, i.e. a contradiction.

The case $(a, b) = (8, 6)$ and $s = 5$. We compute the following numbers: $(d, g) = (32, 113)$, and $G(d, 6) = 115$. Note that $g = G(d, 6) - 2$. Firstly suppose $s(X) \geq 7$. Then $X$ belongs to the $B$-range in the classification of curves of maximal genus. The conjecture in [19, Conj. 3.5] is true for $s(X) \leq 9$ by [19, Rem. 3.8.1]. To find the conjectured value of $G(d, 7)$ we compute $A(7, f)$ for several $f$ to find the largest $f$ satisfying $A(7, f) \leq d$ (see [8] or [19, p. 530] for the definition of $A(s, f)$ and $B(s, f)$). We find $A(7, 7) = 33$, $A(7, 6) = 28$ and $B(7, 6) = 31$, whence $G(d, 7) = 111$ by [19, Conj. 3.5] and such a generization $X$ of $C$ does not exist because $g = 113$.

Therefore we suppose $s(X) = 6$. Since $d(X) + r \equiv 0 \mod s(X)$ allows $r = 4 = s(X) - 2$, it follows from Elia’s paper [8] Prop. IV.4] that $X$ is bilinked via c.i.’s of type $(6, 10)$ and $(6, 5)$ to a degree-2 curve $Z$ satisfying $g(Z) = -2$ provided the numerical character is maximal. One may use [8] Prop. V.2 to see $h^0(N_Z) = \dim I_Z H^1(d, g) = 9$ and [23] Cor. 3.6] to compute the dimension of the bilinked family. Let us instead use that the minimal resolution of $I(Z)$ is known ([8 Lem. V.1]). Then the mapping cone construction used twice yields a resolution:

$$0 \to R(-10) \to R(-9)^2 \oplus R(-8)^2 \oplus R(-11) \to R(-8) \oplus R(-7)^3 \oplus R(-6) \to I_X \to 0.$$
where we have skipped a redundant term \( R(-10) \) “in the middle”) since we may use the same surface of degree 6 in both linkages. This is a curve of maximal rank. To apply (III), we compute the following numbers using well known liaison formulas (e.g. [23]) \( \dim(M) = 3 \): \( h^1(\mathcal{I}_X(4)) = h^0(\mathcal{I}_Z(-1)) = 1 \), \( h^1(\mathcal{O}_X(6)) = 4 + h^1(\mathcal{O}_Z(1)) = 4 \), \( h^1(\mathcal{O}_X(7)) = 1 + h^1(\mathcal{O}_Z(2)) = 1 \) and that \( H^1(\mathcal{I}(v)) = 0 \) for \( v \notin \{4,5,6\} \). Hence \( \dim \text{Hom}_R(\mathcal{I}(X), H^1_s(\mathcal{O}_X)) \leq 7 \) and then (III) implies \( h^0(N_X) \leq 4d + 7 + 1 = 135 \). Since \( \dim W = -d + g + 56 = 137 \) by the proven part of Theorem [3.4] we get a contradiction.

If the numerical character is not maximal we can again use [8] Lem. VI.2 (or [5]) to see that the genus of \( X \) is equal to the genus of the numerical character of \( X \). This implies that \( X \) is smooth and ACM. By Riemann-Roch, \( \chi(\mathcal{I}_X(6)) = 4 \) and \( \chi(\mathcal{I}_X(7)) = 8 \). If \( h^0(\mathcal{I}_X(6)) = 1 \), then \( h^1(\mathcal{O}_X(6)) = 0 \) and we get at least \( \dim \text{Hom}_R(\mathcal{I}(X), H^1_s(\mathcal{O}_X)) \leq 7 \) (by looking at the options given by \( h^0(\mathcal{I}_X/S(7)) := q \), \( h^1(\mathcal{O}_X(7)) = 4 - q \), \( 0 < q \leq 4 \) where \( S \) is defined by a degree-6 polynomial), and if \( h^0(\mathcal{I}_X(6)) = 2 \), then a linkage via a c.i. of type \((6,6)\) yields an ACM curve of degree 4 and genus 1, and one shows \( \dim \text{Hom}_R(\mathcal{I}(X), H^1_s(\mathcal{O}_X)) = 4 \). In any case we get \( h^0(N_X) \leq 135 \) by (III), i.e. a contradiction, and the proof of this case is complete.

The case \((a,b) = (10,8)\) and \( s = 6 \). We compute the following numbers: \((d,g) = (50,251)\), and \( G(d,7) = 252 \). cf. [4]. Note that \( g = G(d,7) - 1 \). Let \( X \) be a genericization of \( C \) satisfying \( s(X) > 6 \). We first consider the option \( s(X) = 7 \). If such a curve exists, we have \( d(X) + r \equiv 0 \mod 7 \), i.e. \( r = 6 \). It is tempting to say that \((d,g)\) is a known Halphen’s gap (cf. [5]), but before we can do that we have to compute \( G(d,8) \). We have \( t^2 - 2t + 2 = d \) if \( t = 8 \), and in this part of the \( B \)-range (or extended \( C \)-range) it is known that \( G(d,8) = 1 + d(t - 3) = 251 \) by (IV). So strictly speaking, since \( g = G(d,s(X) + 1) \), it is not an Halphen’s gap. We can, however, still use Ellia’s results in [8] to show that \( X \) does not exists. Indeed by [8] Prop. IV.3 it follows that there are no such \( X \) with maximal numerical character. Since [8] Lem. VI.2 implies that the numerical character of \( X \) had to be maximal, such an \( X \) does not exist.

Finally we suppose \( s(X) \geq 8 \). Since \( g = G(d,8) \), it follows from [18], cf. (IV) and [9], that \( X \) is a zero section of the null correlation bundle \( \mathcal{E} \); more precisely there is an exact sequence

\[
0 \rightarrow \mathcal{O}_p \rightarrow \mathcal{E} \rightarrow \mathcal{I}_X(14) \rightarrow 0
\]  

where \( \mathcal{F} = \mathcal{E}(7) \). To compute the dimension of the component \( V \) of \( H(d,g) \) to which \( X \) belongs we will use the formula appearing in [27] Cor. 2.3] (see also Ellia’s joint work with Fiorentini [10]), stating that if we have (II) and \( H^1(\mathcal{I}_X(c_1)) = 0 = H^1(\mathcal{I}_X(c_1 - 4)) \), \( c_i := c_i(\mathcal{F}) \) the \( i \)-th Chern class, then \( \mathcal{F} \) is a smooth point of its moduli space \( M_{p^3}(c_1,c_2,c_3) \) if and only if \( X \) is unobstructed and

\[
\dim(\mathcal{F}) M_{p^3}(c_1,c_2,c_3) + h^0(\mathcal{F}) = \dim(\mathcal{F}) H(d,g) + h^0(\omega_X(-c_1 + 4)).
\]  

Using this formula for \( \mathcal{F} = \mathcal{E}(7) \), remarking that \( H^1(\mathcal{E}(v)) = 0 \) for \( v \neq -1 \) is well known (since a section of \( \mathcal{E}(1) \) corresponds to two skew lines), we see that the assumptions for (III) to hold are fulfilled. Since \( \dim M_{p^3}(0,1,0) = 5 \) and we have \( h^0(\omega_X(-10)) = 1 \) and \( h^0(\mathcal{F}) = h^0(\mathcal{E}(7)) = 231 \) by Riemann-Roch, we get \( \dim V = \dim_X H(50,251) = 235 \). Finally using that \( \dim W = -2d + g + 84 + 10 - 5 = 240 > \dim V \), we get a contradiction and the proof of (i) is complete.

(iii) The component \( W \) is non-reduced if we can show \( \dim W < h^0(N_C) \) for \( C \) general. Since \( \dim W = \dim A \), \( h^0(\mathcal{I}_C/S(5)) = 0 \) and \( \dim \text{coker} \alpha_C = 2 \) it suffices by (II) to prove \( h^3(\mathcal{I}_C(5)) \geq 3 \). This follows from Lemma [3.3] and proof of Theorem [3.4] is complete.

**Remark 3.6.** Suppose that \( X \) has maximal rank, or more generally that \( \text{Ext}^1_R(\mathcal{I}(X), H^1_s(\mathcal{I})) = 0 \) for \( i = 0, 1 \) and \( 2 \). Then the formula (III) follows easily from the exact sequence [26] (2.4) because (2.1) implies \( \text{Ext}^2_R(\mathcal{I}(X), I(I)) = 0 \) and \( \text{Ext}^3_R(\mathcal{I}(X), I(I)) \simeq \text{Hom}_R(\mathcal{I}(X), H^1_s(\mathcal{O}_X)) \) and the duality in [26] (2.2) implies \( \text{Ext}^3_R(\mathcal{I}(X), I(I))^\vee \simeq \text{Hom}_R(\mathcal{I}(X), H^1_s(\mathcal{I})) \).

\[ \square \]
If $X$ is ACM, or more generally if $\text{Ext}^2_R(I(X), I(X)) = 0$ and $X$ is of maximal rank, then $X$ is unobstructed, and we may replace $h^0(N_X)$ by $\dim_X H(d,g)$ in (11) (cf. [26, Thm. 2.6], [17]).

**Remark 3.7.** If $s = 4$ then the necessary condition $g+33 \geq 4d$ for $W$ to be an irreducible component implies $a \geq 4$ for $d > 16$ because $g = ad - 2a^2 + 1$ in Theorem [3,4]. By [23, Rem. 5.3], if $D := C - 4H = x\Gamma_1 + y\Gamma_2$ is effective and $h^1(S, O_S(D)) > 0$, then either $\Gamma_1$ is a fixed component of $|D|$ or $D$ is composed with a pencil. In the latter case we have $x = 0$, i.e. $C = 4\Gamma_1 + r\Gamma_2$ for $r \geq 6$.

In a recent preprint ([36]) Nasu shows that these curves $C$ are obstructed for every $r \geq 6$ and in the case $r = 6$ where $h^1(\mathcal{I}_C(4)) = 1$, he shows that $W$ is a non-reduced component of $H(22, 57)$. More recently we independently finished the case $(a, b) = (7, 5)$, $s = 4$ in the proof above for which $h^1(\mathcal{I}_C(4)) = 2$ and $(d, g)$ attains the value $(d, g) = (22, 57)$. Indeed our analysis also applies to show that $W$ is a non-reduced component in the case $r = 6$. So there are at least 2 non-reduced components of $H(22, 57)$ with $s = 4$ (and by Theorem [4,7] one more non-reduced component for which $s = 3$).

## 4 Non-reduced components of $H(d, g)_{sc}$ for $s = 3$

Motivated by the Mumford’s ([34]) example of a non-reduced component, we showed in [22] the existence of 3-maximal families that form non-reduced components of $H(d, [(d^2 - 4)/8])_{sc}$ for every $d \geq 14$. In [22] we also made a conjecture about non-reduced components when $s = 3$. A rough motivation for the conjecture is that the dimension of $s$-maximal families $W(s) := W$ often seems to decrease with increasing $s$, thus making the inclusion $W(s) \subset W(s')$ for $s' > s$ rare without having a particular reason for such an inclusion to exist.

**Conjecture 4.1.** Let $W$ be a 3-maximal family of smooth connected, linearly normal space curves of degree $d > 9$ and genus $g$, whose general member $C$ is contained in a smooth cubic surface. Then $W$ is a non-reduced irreducible component of $H(d,g)_{sc}$ if and only if

$$d \geq 14, \quad 3d - 18 \leq g \leq (d^2 - 4)/8 \quad \text{and} \quad H^1(\mathcal{I}_C(3)) \neq 0.$$

Note that a 3-maximal family $W$ is closed and irreducible by our definition, and that $\dim W = d + g + 18$ holds for $d > 9$. The above conjecture, originating in [22], is here presented by modifications proposed by Ellia [7] (see also [6] by Dolcetti, Pareschi), because they found counterexamples which heavily depended on the fact the general curves $C$ were not linearly normal (i.e. $H^1(\mathcal{I}_C(1)) = 0$).

The conjecture is known to be true in many cases. Indeed Mumford’s example of a non-reduced component is in the range of Conjecture [4,1] (minimal with respect to both degree and genus). Also the main result by the author in [21] shows the conjecture provided

$$g > 7 + (d - 2)^2 / 8, \quad d \geq 18.$$  \hfill (14)

Ellia shows in [7] that Conjecture [4,1] holds in the larger range $g > G(d, 5)$, $d \geq 21$ by first proving

**Proposition 4.2.** (Ellia) Let $d \geq 21$ and $g \geq 3d - 18$, let $W$ be a 3-maximal family of smooth connected space curves whose general curve $C$ sits on a smooth surface and suppose that $H^1(\mathcal{I}_C(1)) = 0$. If $X$ is a generalization of $C$ in $H(d,g)_{sc}$ satisfying $H^0(\mathcal{I}_X(3)) = 0$, then $H^0(\mathcal{I}_X(4)) = 0$.

See [7, Prop. VI.2] for a proof. More recently Nasu proves (and re-proves) a part of the conjecture by showing that the cup-product (i.e. the primary obstruction) is nonzero if $h^1(\mathcal{I}_C(3)) = 1$ ([35]).

When we try to show that generalizations $X$ of a curve $C$ with $s(C) = s - 1$ do not exist, the hard part is usually the case $s(X) = s$. The non-existence of such $X$ are taken care of by Proposition [4,2] for $s = 4$. Therefore combining Ellia’s result with semi-continuity arguments when $s(X) > s$, we
can extend the range where Conjecture 4.1 holds in a good number of cases. This is what we do in the Theorems 4.4 and 4.5. Recalling that we can associate a curve C on S and its corresponding invertible sheaf \( \mathcal{O}_S(C) \) with a 7-tuple of integers \((\delta, m_1, \ldots, m_6)\) satisfying (15) below (by blowing up \( \mathbb{P}^2 \) in six general points in the usual way, cf. [17]), we first remark (cf. [21] Lem. 16 and Cor. 17)

**Lemma 4.3.** With notations as above it holds

(i) \( d = 3\delta - \sum_{i=1}^{6} m_i \), \( g = \binom{\delta - 1}{2} - \sum_{i=1}^{6} \binom{m_i}{2} \).

(ii) If \( g > (d^2 - 4)/8 \), then \( H^1(I_C(3)) = 0 \), whence \( C \) is unobstructed.

(iii) If \( d \geq 14 \) and \( g \geq 3d - 18 \), then \( H^1(I_C(3)) \neq 0 \) and \( H^1(I_C(1)) = 0 \iff 1 \leq m_6 \leq 2 \).

By Lemma 4.3 (ii) the conditions of Conjecture 4.1 are necessary for \( W \) to be a non-reduced component, and for the converse, and we may suppose \( m_6 = 1 \) or 2 by (iii). If \( m_6 = 1 \) we recall

**Theorem 4.4.** Let \( W \) be a 3-maximal family of smooth connected space curves, whose general member sits on a smooth cubic surface and corresponds to the 7-tuple \((\delta, m_1, \ldots, m_6)\) satisfying

\[
\delta \geq m_1 \geq \ldots \geq m_6 \quad \text{and} \quad \delta \geq m_1 + m_2 + m_3.
\]  

Then \( W \) is a non-reduced irreducible component of \( \text{H}(d,g)_{sc} \) provided

a) \( m_6 = 1, \; m_5 \geq 6, \; d \geq 35 \) and \((\delta, m_1, \ldots, m_6) \neq (\lambda + 18, \lambda + 6, 6, \ldots, 1) \) for any \( \lambda \geq 2 \), or

b) \( m_6 = 1, m_5 = 5, m_4 \geq 7, \; d \geq 35 \) and \((\delta, m_1, \ldots, m_6) \neq (\lambda + 21, \lambda + 7, 7, \ldots, 7, 5, 1) \) for \( \lambda \geq 2 \).

For a proof, see the appendix to [28, Thm. A.3] by the first author. For \( m_6 = 2, \; m_5 \geq 3 \) see [35].

It is also possible to use the idea of [21, Sect. 4] to determine bounds for \( \dim V \) where \( V \supseteq W \) and \( \dim V > \dim W \). This is done in [28, Sect. 4], leading to [28, Prop. A.7] which we will generalize and completely prove. Note that by Proposition 4.2 we can skip considering components with \( s(V) = 4 \).

**Theorem 4.5.** Let \( W \) be a 3-maximal family of smooth connected space curves, whose general member is linearly normal and sits on a smooth cubic surface. If

\[
g > \max \left\{ \frac{d^2}{10} - \frac{d}{2} + 18, \; G(d,t) \right\}, \; d \geq t^2 - 2t + 2, \tag{16}
\]

for some \( t \) satisfying \( 6 \leq t \leq 8 \), then \( W \) is an irreducible component of \( \text{H}(d,g)_{sc} \). Moreover, \( W \) is non-reduced if and only if \( H^1(I_C(3)) \neq 0 \). In particular Conjecture 4.1 holds in the range (16), e.g. if

\[
g > \frac{d^2}{10} - \frac{d}{2} + 18, \; d \geq 54. \tag{17}
\]

We have \( G(d,6) \geq \frac{d^2}{10} - \frac{d}{2} + 18 \) if and only if \( d \leq 74 \). This is one of the reasons for proving (16) not only for \( t = 6 \), but also \( t = 7 \) and 8 since it enlarges the range where the conjecture holds. Note also that Ellia proves the conjecture for \( g > G(d,5) = \frac{d^2}{10} + \frac{d}{2} + 1 - \frac{2r(5-r)}{5} \), so the improvement in Theorem 4.5 is only of order \( d \). The improvement in Theorem 4.4 is more substantial.

In the proof of Theorem 4.5 we will need (see [28, Prop. 4.4] for a proof):

**Proposition 4.6.** Let \( V \) be an irreducible component of \( \text{H}(d,g)_{sc} \) whose general curve \( C \) sits on some integral surface \( F \) of degree \( s \geq 4 \). If \( d > s^2 \), then

\[
\dim V \leq \left( \frac{s + 3}{3} \right) - 1 + \max \left\{ \frac{d^2}{s} - g, \frac{d^2}{2s}, \; (4 - s)d + g - 1 + h^0(\mathcal{O}_C(s - 4)) \right\}.
\]

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Proof (of Theorem 4.5). To see that $W$ is an irreducible component, we suppose there exists a component $V$ of $H(d, g)_{sc}$ satisfying $W \subset V$ and $\dim W < \dim V$. Then $s(V) \geq 4$ by the definition of a 3-maximal family, whence $s(V) \geq 5$ by Proposition 4.2. Firstly suppose $g > G(d, 6)$ and $d \geq 26$. It follows that the general curve of $V$ satisfies $s := s(X) = 5$. To get a contradiction we will use Proposition 4.6 for $s = 5$ and the fact $\dim W = d + g + 18$. Indeed since $X$ is a generization of a smooth connected linearly normal curve, it follows that a surface containing $X$ of the least possible degree is integral and moreover that $X$ is smooth, connected and linearly normal. We have

$$d + g + 18 < 55 + \max \{ \left\lfloor \frac{d^2}{5} \right\rfloor - g, \left\lfloor \frac{d^2}{10} \right\rfloor, -d + g - 1 + 4 + h^1(\mathcal{I}_X(1)) \}.$$ 

Suppose the maximum to the right is attained by $55 - d + g - 1 + 4$. Then $d + g + 18 < 55 - d + g + 3$ which is absurd since we have assumed $d > 25$. Similarly if the maximum is attained by $55 + \left\lfloor \frac{d^2}{5} \right\rfloor - g$ we get $d + g + 18 < 55 + \left\lfloor \frac{d^2}{5} \right\rfloor - g$ or equivalently $2g \leq 36 + \left\lfloor \frac{d^2}{5} \right\rfloor - d$ which contradicts the assumption $g > \frac{d^2}{10} - \frac{d}{2} + 18$. Also $d + g + 18 < 55 + \left\lfloor \frac{d^2}{10} \right\rfloor$ leads to a contradiction because $36 + \frac{d^2}{10} - d \leq \frac{d^2}{10} - \frac{d}{2} + 18$ for $d \geq 36$ and $36 + \frac{d^2}{10} - d \leq G(d, 6)$ for $26 \leq d < 36$. For the latter inequality, we remark that we, for $26 \leq d < 31$, have to compute $G(d, 6)$ in the $B$-range (or “extended $C$-range”) using (4) and (7). Thus we have proved that $W$ is an irreducible component of $H(d, g)_{sc}$.

Secondly we suppose $g > G(d, 7)$ and $d \geq 37$. It follows from the previous paragraph that the assumptions $s(X) = 5$ and $g > \frac{d^2}{10} - \frac{d}{2} + 18$ yield a contradiction. Hence we may assume $s(X) = 6$. Now if we use Proposition 4.6 for $s := s(X) = 6$, which requires $d > 36$, we get

$$d + g + 19 \leq 83 + \max \{ \left\lfloor \frac{d^2}{6} \right\rfloor - g, \left\lfloor \frac{d^2}{12} \right\rfloor, -2d + g - 1 + h^0(\mathcal{O}_X(2)) \}.$$ 

To estimate $h^0(\mathcal{O}_X(2))$ we use Clifford’s theorem to see $h^0(\mathcal{O}_X(2)) - 1 \leq \max \{ 2d - g, d \}$. Since it is easy to see

$$\max \{ 32 + \frac{1}{2} \left\lfloor \frac{d^2}{6} \right\rfloor - \frac{d}{2}, 64 + \left\lfloor \frac{d^2}{12} \right\rfloor - d \} \leq \frac{d^2}{10} - \frac{d}{2} + 18$$

for $d \geq 40$, we have a contradiction except for $37 \leq d \leq 39$. For these exceptions in the extended $C$-range we compute $G(d, 7)$ by using (4) and (7) and we get a contradiction since $g > G(d, 7)$ for $37 \leq d \leq 39$. Again we can conclude that $W$ is an irreducible component of $H(d, g)_{sc}$.

Thirdly we suppose $g > G(d, 8)$ and $d \geq 50$. It follows from the previous paragraph that the assumptions $s(X) = 6$ (or 5) and $g > \frac{d^2}{10} - \frac{d}{2} + 18$ yield a contradiction. We may therefore assume $s(X) = 7$. Now using Proposition 4.6 for $s := s(X) = 7$, which requires $d > 49$, we get

$$d + g + 19 \leq 119 + \max \{ \left\lfloor \frac{d^2}{7} \right\rfloor - g, \left\lfloor \frac{d^2}{14} \right\rfloor, -3d + g - 1 + h^0(\mathcal{O}_X(3)) \}.$$ 

We use Clifford’s theorem to get $h^0(\mathcal{O}_X(3)) - 1 \leq \max \{ 3d - g, 3d/2 \}$. Since it is rather easy to see

$$\max \{ 50 + \frac{1}{2} \left\lfloor \frac{d^2}{7} \right\rfloor - \frac{d}{2}, 100 + \left\lfloor \frac{d^2}{14} \right\rfloor - d \} \leq \frac{d^2}{10} - \frac{d}{2} + 18$$

for $d \geq 46$, we have a contradiction, i.e. $W$ is an irreducible component of $H(d, g)_{sc}$.

Since $G(d, 8) \leq \frac{d^2}{10} - \frac{d}{2} + 18$ for $d \geq 58$ and $d \in \{ 54, 55, 56 \}$, then $W$ is an irreducible component (again we need to work in the extended $C$-range).

Finally for $d = 57, g = G(d, 8) = 315$ we check $\dim W \geq \dim V$ directly. Indeed the general curve $X$ of $V$ is by (5) directly linked via a c.i. of type $(8, 8)$ to a plane curve $X'$ of degree $r = 7$ since $d + r \equiv 0 \mod 8$. The mapping cone construction yields a minimal resolution $0 \to R(-15) \oplus R(-9) \to R(-8)^3 \to I(X) \to 0$, and since well known linkage formulas imply $h^1(\mathcal{O}_X(8)) = h^0(\mathcal{I}_X(4)) = 20$ and $h^1(\mathcal{O}_X(9)) = 10$, we get $\dim V = 4d + 3h^1(\mathcal{O}_X(8)) - h^1(\mathcal{O}_X(9)) = 285$ by (11) while $\dim W = d + g + 18 = 390$, a contradiction. Thus (17) imply that $W$ is an irreducible component of $H(d, g)_{sc}$.
Theorem 4.7. For every $d > 4$ and genus $g$ provided

$$(d+12)\sqrt{d+9}-11d/2-35 \leq g \leq 1+(d^2-4d)/8.$$ (18)

Indeed Rathmann shows the existence of a 6-tuple $(\delta, m_1, \ldots, m_6)$ satisfying $\mathcal{O}_S(C)^2 > 0$ and $\delta \geq m_1 \geq \cdots \geq m_5 \geq 0$ and $\delta \geq m_1 + m_2 + m_3$ for every $(d,g)$ in the range (18). Letting $m_6 = 0$ this leads to a 7-tuple $(\delta, m_1, \ldots, m_6)$ satisfying (15) for every $(d,g)$ given as in Lemma 4.3 (i) in the range (18), whence we get the existence of a smooth connected curve $C$ sitting on our smooth cubic surface $S \subset \mathbb{P}^3$. If we now add a 7-tuple corresponding to $n$ hyperplanes, $(3n,n, \ldots, n)$, to $(\delta, m_1, \ldots, m_5, 0)$, the corresponding linear system will contain smooth connected curves $X \in |C+nH|$ satisfying $m_6 = n$ and $H^1(I_X(v)) = 0$ for every non-negative integer $v \leq n$. Using the adjunction formula for the genus we easily see that the degree $d'$ and genus $g'$ of $X$ satisfy

$$d = d' - 3n \text{ and } g = -nd' + g' + 3(n^2 + n)/2.$$ Inserting these formulas into (18) we get for every $d' > 3n + 4$, $g'$ in the range

$$(d'+12-3n)\sqrt{d'+9-3n+d'(n-11/2)}-35+\frac{3(10n-n^2)}{2} \leq g' \leq 1+\frac{d'^2+2n-4d'-3n^2}{8}$$ (19)

the existence of such a curve $X$. Using this for $n = 1$ and $n = 2$ we get the following result.

Theorem 4.7. For every $d \geq 14$ and $g$, $(d,g) \neq (14,22)$ in the range

$$(d+9)\sqrt{d+6}-\frac{9d}{2}-\frac{43}{2} \leq g \leq \frac{d^2-4}{8}$$ (20)

there exists a smooth connected curve $C$ of degree $d$ and genus $g$, contained in a smooth cubic surface in $\mathbb{P}^3$, whose corresponding 7-tuple $(\delta, m_1, \ldots, m_6)$ has $m_6$ equal to 1 or 2 and such that $(\delta, m_1, \ldots, m_6) \neq (\lambda + 3m_6, \lambda + m_6, m_6, \ldots, m_6)$ for every $\lambda \geq 2$. If also $g \geq 3d-18$ holds, then $C$ satisfies

$$H^1(I_C(3)) \neq 0 \text{ and } H^1(I_C(1)) = 0.$$ In particular for every $d, g \leq \frac{d^2-4}{8}$ satisfying either (14), (16), (17) or $g > G(d,5), d \geq 21$ there exists a non-reduced component $W$ of $H(d,g)_{sc}$ whose general curve sits on a smooth cubic hypersurface.

Proof. Let $G_1 := \frac{d^2-2d+5}{8}$, $G_2 := \frac{d^2-4}{8}$, and let

$$g_1 := (d+9)\sqrt{d+6}-\frac{9d}{2}-\frac{43}{2}, \quad g_2 := (d+6)\sqrt{d+3}-\frac{7d}{2}-11.$$ Using (19) with $n = 1$ (resp. $n = 2$) we get the existence of a smooth connected curve $C$ for every $(d,g)$ in the range $g_1 \leq g \leq G_1$ (resp. $g_2 \leq g \leq G_2$) with a 7-tuple with the desired properties. One verifies that $g_1 \leq g_2 \leq G_2$ for $d \geq 14$ and that $g_2 < G_1 < G_2$ for $d \geq 17$. Since the only integer $k$ satisfying $G_1 < k < g_2$ for $14 \leq d \leq 16$ is $k = 22$ in which case $d = 14$, we get the first conclusion of the theorem. Moreover invoking Lemma 4.3 (iii) we get the next conclusion. Finally looking at the statements accompanying (14), (16), (17) or $g > G(d,5), d \geq 21$, we see that the conjecture holds in these ranges, and we get the final conclusion of the theorem.
Remark 4.8. (i) For \( d \geq 17 \) we have \( g_1 \leq g_2 \leq G_1 \leq G_2 \), with notations as in the proof. If \( d, g \) satisfy \( g_2 \leq g \leq G_1 \) it follows from the proof that there exists two 7-tuples, one with \( m_6 = 1 \) and one with \( m_6 = 2 \) with \( d, g \) satisfying (20). If we in addition are in a range of the proven part of the conjecture, we have two different 3-maximal families that form non-reduced components of \( \text{H}(d,g)_{sc} \). 

(ii) If we use (19) for \( n = 3 \) we get the existence of a smooth connected curve \( C \) for every \( d \geq 14 \) and \( g \) in the range

\[
(d + 3)\sqrt{d} - \frac{5d}{2} - \frac{7}{2} \leq g \leq \frac{d^2 + 2d - 19}{8},
\]

with \( m_6 = 3 \) and \((\delta, m_1, \ldots, m_6) \neq (\lambda + 9, \lambda + 3, 3, \ldots, 3)\) for any \( \lambda \geq 2 \). Such \( C \) belongs to a generically smooth, irreducible component of \( \text{H}(d,g)_{sc} \) because \( H^1(I_C(3)) = 0 \). This improves upon the lower bound of [25, Prop. 3.1] for \( d \gg 0 \). Indeed [25, Prop. 3.1], which states that for \( d > 9 \) and \( (d,g) \)

\[
3d - 17 + \frac{(d - 9)(d - 18)}{18} \leq g \leq 1 + \frac{d(d - 3)}{6},
\]

\((d,g) \notin \{(30,91),(33,103),(34,109)\}\) there exists an unobstructed curve \( C \) of \( \text{H}(d,g)_{sc} \) contained in a smooth cubic surface, gives a better lower bound for \( 9 < d \leq 161 \). The proof of [25, Prop. 3.1] which makes use of Gruson and Peskine’s existence result [17, Prop. 2.10], produces a curve \( C \) with 7-tuple satisfying \( m_6 \geq 3 \) and such that \( H^1(I_C(v)) = 0 \) for \( v \leq 3 \) for every \( (d,g) \) in the mentioned range, thus \( C \) belongs a 3-maximal family \( W \) which is a generically smooth component of \( \text{H}(d,g)_{sc} \).

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