MONTE CARLO COMPUTATION OF MULTIPLE WEAK SINGULAR INTEGRALS OF SPHERICAL AND VOLTERRA’S TYPE.

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Abstract.

We offer a simple method Monte Carlo for computation of Volterra’s and spherical type multiple integrals with weak (integrable) singularities. An elimination of infinity of variance is achieved by incorporating singularities in the density, and we offer a highly effective way for generation of appeared multidimensional distribution.

We extend offered method onto multiple Volterra’s and spherical integrals with weak singularities containing parameter.

Keywords and phrases: Multivariate integrals of spherical and Volterra’s type, polygon and polygonal beta distribution, ball beta distribution, variance, Monte Carlo method, depending trial method, random variable and vector, marginal and conditional density, incorporating singularities in the density, random processes and fields (r.p.; r.f.), Central Limit Theorem (CLT) in Banach spaces.

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1 Introduction. Notations. Statement of problem.

Let us denote by $S_n$ the $n$ – dimensional polyhedron (simplex) of a form

$$S(n) = \{ \vec{s} = (s_1, s_2, \ldots, s_n) : 0 < s_1 < s_2 < \ldots < s_n < 1 \}.$$

It is easy to calculate that $\text{Vol}(S(n)) = 1/n!$.

Let also $\alpha = \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a constant vector such that $0 \leq \alpha_k < 1$, $k = 1, 2, \ldots, n$. 
We investigate in this article the problem of numerical computation by the Monte Carlo method of the following integral of Volterra’s type

\[ I_{\alpha,n}[z] := \int_0^1 ds_n \int_0^{s_n} ds_{n-1} \int_0^{s_{n-1}} ds_{n-2} \ldots \int_0^{s_2} ds_1 \frac{z(s_1, s_2, \ldots, s_n)}{s_1^{\alpha_1}(s_2 - s_1)^{\alpha_2}(s_3 - s_2)^{\alpha_3} \ldots (s_n - s_{n-1})^{\alpha_n}} ds_1 ds_2 \ldots ds_n = \]

\[ \int \int \ldots \int_{S(n)} \frac{z(s_1, s_2, \ldots, s_n)}{s_1^{\alpha_1}(s_2 - s_1)^{\alpha_2}(s_3 - s_2)^{\alpha_3} \ldots (s_n - s_{n-1})^{\alpha_n}} ds_1 ds_2 \ldots ds_n, \quad n = 2, 3, \ldots; \quad (1.1) \]

\[ I_{\alpha,1}[z] := \int_0^1 \frac{z(s_1)}{s_1^{\alpha_1}} ds_1. \quad (1.1a) \]

This kind of such integrals appear in the reliability theory [16]; in the theory of integral equations of Volterra’s type with weak singularity in kernel [15], [19], [20]; in the investigation of the spectres of some integral operators with singularities [11], p. 310; in the investigation of local times of random processes [10]; in the numerical analysis of the Navier - Stokes equation [1], [2], [3], [4], [12], [21] etc.

Let us discuss the last thesis in more detail. The so-called mild solution \( u = u(x, t) \) of the Navier-Stokes equation in the whole space \( x \in \mathbb{R}^d \) during its lifetime \( t \in [0, T], \quad 0 < T = \text{const} \leq \infty \) may be represented under some simple conditions as a limit as \( m \to \infty, m = 0, 1, 2, \ldots \) of the following recursion:

\[ u_{m+1}(x, t) = u_0(x, t) + G[u_m, u_m](x, t), \quad m = 1, 2, \ldots \]

where \( u_0(x, t) \) is the solution of heat equation with correspondent initial value and right-hand side and \( G[u, v] \) is bilinear unbounded pseudo-differential operator, [3].

The detail expression for each iteration starting at the value \( m = 2 \) contains multiple integrals of a form (1.1) with the values \( \alpha_k = 1/2, \quad k = 1, 2, \ldots, n \) and hence may be computed for example by means of the Monte - Carlo method, in addition to the deterministic methods, see [5], [6].

Note that the second iteration is considered in the article [7].

The case of spherical multiple integrals with weak singularities alike (1.1), i.e. when instead the polygon \( S(n) \) states (unit) ball \( B = B_1 \) will be considered further.

## 2 The essence of the method.

Let us denote for brevity for the values \( s = \tilde{s} \in S(n) \)

\[ R_{\alpha,n,S}(s) = R_{\alpha,S}(s) = s_1^{-\alpha_1}(s_2 - s_1)^{-\alpha_2}(s_3 - s_2)^{-\alpha_3} \ldots (s_n - s_{n-1})^{-\alpha_n}; \quad (2.1) \]

then
\[ I_{\alpha,n}[z] = \int_{S(n)} z(s) R_{\alpha,S}(s) \, ds. \] (2.2)

The "direct" probabilistic representation for \( I_{\alpha,n}[z] \), i.e. the expression of a form

\[ I_{\alpha,n}(z) = n! \cdot E_z(\eta) R_{\alpha,S}(\eta) \]

where the random vector \( \eta = \{ \eta_1, \eta_2, \ldots, \eta_n \} \) has an uniform distribution in the simplex \( S(n) \) leads in general case, more exactly, when \( \exists k : \alpha_k \geq 1/2 \), to the possibility of infinite variance in the integrand expression. And then, for the error estimate instead of the CLT will have to apply the Stable Limit Theorem (SLT), which drastically reduces the rate of convergence of the proposed method [11], chapter 5, section 5.14.

Recall that this is the case of the problem of the Navier-Stokes equation, in which all the values \( \alpha_k \) are equal exactly to 1/2.

We offer hence another probabilistic representation for \( I_{\alpha,n}[z] \).

Note first of all that

\[ \int \cdots \int_{S(n)} \frac{ds_1 ds_2 \cdots ds_n}{s_1^{\alpha_1}(s_2 - s_1)^{\alpha_2}(s_3 - s_2)^{\alpha_3} \cdots (s_n - s_{n-1})^{\alpha_n}} = K_{n,S}(\alpha) = K_{n,S}(\vec{\alpha}), \] (2.3)

where

\[ K_{n,S}(\alpha) = K_{n,S}(\vec{\alpha}) \overset{\text{def}}{=} \prod_{k=1}^{n} \frac{\Gamma(1 - \alpha_k)}{\Gamma(1 + \sum_{k=1}^{n} (1 - \alpha_k))}, \quad 0 \leq \alpha_k < 1, \] (2.4)

and \( \Gamma(\cdot) \) is ordinary Gamma function.

For example, let \( \alpha \) be arbitrary number such that \( 0 \leq \alpha < 1 \). Denote \( \beta = \beta(\alpha) = 1 - \alpha \). Then

\[ W_n(\beta) := K_{n,S}(\alpha, \alpha, \ldots, \alpha) = \int_{S(n)} R_{\alpha,\alpha,\ldots;\alpha;S}(s) \, ds = \] 

\[ \int \cdots \int_{S(n)} \frac{ds_1 ds_2 \cdots ds_n}{s_1^{\alpha_1}(s_2 - s_1)^{\alpha_2}(s_3 - s_2)^{\alpha_3} \cdots (s_n - s_{n-1})^{\alpha_n}} = \frac{\Gamma^n(\beta)}{\Gamma(1 + n \beta)}. \]

Evidently, \( \lim_{n \to \infty} W_n(\beta) = 0 \).

Note in addition the continuity and monotonicity of the function \( \beta \to W_n(\beta) : 0 < \beta_1 < \beta_2 \leq 1 \Rightarrow W_n(\beta_1) > W_n(\beta_2) \).

In particular,

\[ W_n(1/2) = \frac{\pi^{n/2}}{\Gamma(1 + n/2)}, \]

\[ \frac{1}{n!} = \frac{\Gamma^n(1)}{\Gamma(1 + n)} = W_n(1) < \frac{\Gamma^n(\beta)}{\Gamma(1 + n \beta)} = W_n(\beta), \quad 0 < \beta < 1. \]
The following function \( h_\alpha(s) = h_\beta(s), \ s \in S(n) \), could be chosen as a density of distribution with support on the simplex \( S(n) \):

\[
h_\alpha(s) = \frac{R_{\alpha,n}(s)}{K_{n,S}(\alpha)}, \tag{2.5}
\]

**Definition 2.1.** The random vector \( \kappa = \kappa_{\alpha,n} = \vec{\kappa} = \vec{\kappa}_{\alpha,n} \) with values in the polygon \( S(n) \) has by definition a **polygonal Beta distribution**, write: \( \text{Law}(\kappa) = PB(\alpha, n) \), iff it has a density \( h_\alpha(s), \ s \in S(n) \).

On the other word, \( P(\kappa \in G) = \int_G h_\alpha(s) \, ds \overset{\text{def}}{=} \mu_{\alpha,n}(G), \ G \subset S(n). \) \( \tag{2.6} \)

Evidently, \( \mu_{\alpha,n}(\cdot) \) is the probabilistic Borelian measure on the set \( S(n) \).

The expression for the source integral (1.1) may be represented as follows.

\[
I_{\alpha,n}[z] = K_{n,S}(\alpha) \cdot \int_{S(n)} \frac{z(s) R_{\alpha,n}(s)}{K_{n,S}(\alpha)} \, ds = K_{n,S}(\alpha) \cdot \int_{S(n)} z(s) h_\alpha(s) \, ds = K_{n,S}(\alpha) \cdot E z(\kappa), \tag{2.7}
\]

where the random vector \( \kappa \) has the polygonal Beta distribution: \( \text{Law}(\kappa) = PB(\alpha, n) \).

Let us estimate the second moment of the variable \( K_{n,S}(\alpha) \cdot z(\kappa) \):

\[
E [K_{n,S}(\alpha) \cdot z(\kappa)]^2 = K_{n,S}(\alpha) \cdot ||z||^2 L_2(\mu_{\alpha,n}). \tag{2.8}
\]

The last expression is finite if for instance the function \( z(\cdot) \) is bounded.

Moreover, it is interest to note that the variance of the r.v. \( K_{n,S}(\alpha) \cdot z(\kappa) \), i.e.

the expression

\[
\sigma^2_{n,S}[z] := \text{Var}\{K_{n,S}(\alpha) \cdot z(\kappa)\} = K_{n,S}(\alpha) \cdot ||z||^2 L_2(\mu_{\alpha,n}) - I_{\alpha,n}[z]
\]

tends very rapidly to zero as \( n \to \infty \), if for example \( \sup_k \alpha_k < 1 \).

Thus, we can use by computation of this integral and by error estimation the classical Monte Carlo method with application of Central Limit Theorem (CLT):

\[
I_{\alpha,n}^{(N)}[z] := N^{-1} \sum_{i=1}^{N} K_{n,S}(\alpha) \cdot z(\kappa_i), \tag{2.9}
\]

where \( \kappa_i \) are independent random vectors with polygonal beta distribution. The variance \( \sigma^2_{n,S}[z] \) may be estimated as well as the integral \( I_{\alpha,n}[z] \).

Note that the ”regular” case, i.e. when \( \vec{\alpha} = 0 \) may be obtained by substitution \( \beta = 1 \). For example,

\[
W_n(\beta(0)) = W_n(1) = 1/n!,
\]

\[
E [K_{n,S}(0) \cdot z(\kappa)]^2 = (n!)^{-2} \cdot ||z||^2 L_2(\mu_{0,n}).
\]
3 Generation of used random vectors.

Let us dwell briefly on the issue of the generation of this distribution on the basis of the standard uniform \( U(0, 1) \) generator. Our purpose in this section is to prove that the generation of multidimensional polygonal beta distribution is not much harder as in one dimensional case.

1. The one-dimensional case \( n = 1 \) is very simple, as long as the r.v. \( \kappa_1 \) has a power distribution with the density:

\[
f_{\kappa_1}(x) = (1 - \alpha_1)x^{-\alpha_1}, \ x \in (0, 1).
\]

The (cumulative) distribution function \( F_{\kappa_1}(x) \) has a view:

\[
F_{\kappa_1}(x) = (1 - \alpha_1) \int_0^x z^{-\alpha_1} \, dz = x^{1-\alpha_1}, \ x \in [0, 1];
\]

hence

\[
F_{\kappa_1}^{-1}(x) = x^{1/(1-\alpha_1)}, \ x \in [0, 1].
\]

Thus, the power distribution may be generated very simple by means of the inverse function method.

2. A two-dimensional case \( n = 2 \).

Let the two-dimensional random vector \((\xi, \eta)\) be polygonal beta distributed:

\[
f_{\xi,\eta}(x, y) = \frac{1}{K_{2,S(2)}(\alpha_1, \alpha_2)} \frac{I(0 < x < y < 1)}{x^{\alpha_1} \ (y - x)^{\alpha_2}},
\]

where

\[
0 \leq \alpha_1, \alpha_2 < 1; \quad I(A) = 1, (x, y) \in A, \ I(A) = 0, (x, y) \notin A.
\]

We calculate the marginal density \( f_\eta(y), \ y \in (0, 1): \)

\[
f_\eta(y) = \frac{1}{K_{2,S(2)}(\alpha_1, \alpha_2)} \int_0^y \frac{dx}{x^{\alpha_1} \ (y - x)^{\alpha_2}} = (2 - \alpha_1 - \alpha_2) \ y^{1-\alpha_1-\alpha_2},
\]

i.e. the r.v. \( \eta \) has the power distribution with parameter \( 1 - \alpha_1 - \alpha_2 \).

As for the conditional density of distribution \( f_\xi(x/\eta = y) \), we have:

\[
f_\xi(x/\eta = y) = C(\alpha_1, \alpha_2) \frac{I(0 < x < y < 1)}{x^{\alpha_1} \ (y - x)^{\alpha_2}}.
\]
The last relation implies that the r.v. $\xi$ has under condition $\eta = y$ the well-known one-dimensional beta distribution on the interval $(0, y)$ with parameters $(\alpha_1, \alpha_2)$.

Note that in the computer system MATLAB there is a command $R = \text{betarnd}(A, B)$ which generated a (sequence) of the independent one-dimensional beta distributed random (pseudo-random) variables on the set $(0, 1)$ with parameters $A, B$.

n. Multidimensional case.

Let the random vector $\kappa = \vec{\kappa} = (\kappa_1, \kappa_2, \ldots, \kappa_n)$ has a polygonal beta distribution. It is easy to calculate

$$f_{\kappa_2, \kappa_3, \ldots, \kappa_n}(x_2, x_3, \ldots, x_n) = \int f_{\kappa_1, \kappa_2, \kappa_3, \ldots, \kappa_n}(x_1, x_2, x_3, \ldots, x_n)dx_1 =$$

$$C \cdot \frac{I(0 < x_2 < x_3 \ldots < x_n < 1)}{x_2^{1-\alpha_1-\alpha_2}(x_3 - x_2)^{\alpha_3} \ldots (x_n - x_{n-1})^{\alpha_n}},$$

i.e. the $(n - 1)$ dimensional subvector $(\kappa_2, \kappa_3, \ldots, \kappa_n)$ has also polygonal beta distribution.

Thus, the problem of $n$ dimensional polygonal beta distribution random generating may be easy reduced to the $(n - 1)$ dimensional.

**Example.** Let us consider an important two-dimensional case $n = 2$ with $\alpha_1 = \alpha_2 = 1/2$; then the correspondent one-dimensional function of distribution of a second component $\kappa_2$ has a form

$$F_{\kappa_2}(z) = \pi^{-1} \int_0^z \frac{dx}{\sqrt{x(1 - x)}} = \pi^{-1}[\arcsin(2z - 1) + \pi/2], \ z \in [0, 1].$$

The inversion function has an explicit view:

$$F_{\kappa_2}^{-1}(z) = 0.5 + 0.5 \sin(\pi(z - 1/2)), \ z \in [0, 1].$$

4 Spherical case

Let us consider the following example. Let $A = (A_1, A_2, \ldots, A_n)$ be a $n$ - dimensional numerical vector such that $-1 < A_i \leq 0$ and $D := \sum_{k=1}^{n} A_k + n > 0$.

For all the $n$ - dimensional numerical vector $x = (x_1, x_2, \ldots, x_n) \in R^n$ we define as usually the monomial

$$x^A = |x_1|^{A_1}|x_2|^{A_2} \ldots |x_n|^{A_n}.$$

The unit ball with the center in origin in the classical Euclidean distance will be denoted by $B = B_1$:

$$B = B_1 = \{x, \ x \in R^n, \ |x| \overset{def}{=} \sqrt{(x, x)} \leq 1\}.$$ 

It is known, see e.g. [25]
\[
\int_B x^A dx = K_{n,B}(\alpha) = K_{n,B}(\vec{\alpha}) \overset{\text{def}}{=} \frac{\prod_{k=1}^n \Gamma((A_k + 1)/2)}{\Gamma(D/2 + 1)}. \tag{4.1}
\]

For example, if \(\alpha_1 = \alpha_2 = \ldots = \alpha_n = 2\gamma - 1, \gamma = \text{const} > 0\), then

\[
K_{n,B}(\alpha, \alpha, \ldots, \alpha) = \frac{\Gamma^n(\gamma)}{\Gamma(1 + n\gamma)} = W_n(\gamma).
\]

The last expression tends again to zero as \(n \to \infty\) very rapidly, as well as \(W_n(\beta)\).

**Definition 4.1.** The \(n\) - dimensional random vector \(\zeta = \zeta_{\alpha,n}\) with values in the unit ball \(B\) has a ball beta distribution \(BB(\vec{\alpha}) = BB(\alpha)\), if it has a density \(g_\alpha(x)\), \(x \in B\) of a form

\[
g_\alpha(x) \overset{\text{def}}{=} \frac{1}{K_{n,B}(\alpha)} \cdot x^A : P(\zeta \in G) = \int_G g_\alpha(x) \, dx \overset{\text{def}}{=} \nu_{\alpha,n}(G), \; G \subset B. \tag{4.2}
\]

Obviously, \(\nu_{\alpha,n}(\cdot)\) is probabilistic Borelian measure on the set \(B\).

We consider in this section the problem of Monte Carlo computation of the multiple integral

\[
J_{\alpha,n}[z] \overset{\text{def}}{=} \int_B |x|^A z(x) \, dx. \tag{4.3}
\]

As before, the ”direct” simulation leads in general case to the infinity of variance, therefore we need to transform this integral:

\[
J_{\alpha,n}[z] = K_{n,B}(\alpha) \cdot \int_B z(x) \, g_\alpha(x) \, dx = E[K_{n,B}(\alpha) \cdot z(\zeta)], \tag{4.4}
\]

where the random vector \(\zeta\) has the ball beta distribution \(BB(\alpha)\).

Let us estimate the second moment of the variable \(K_{n,B}(\alpha) \cdot z(\zeta)\):

\[
E [K_{n,B}(\alpha) \cdot z(\zeta)]^2 = K_{n,B}^2(\alpha) \cdot ||z||^2 L_2(\nu_{\alpha,n}). \tag{4.5}
\]

The last expression is finite iff the function \(z(\cdot)\) belongs to the space \(L_2(B, \nu_{\alpha,n})\), for instance, if it is bounded.

Moreover, it is interest to note that the variance of the r.v. \(K_{n,B}(\alpha) \cdot z(\zeta)\), i.e. the expression

\[
\sigma_{n,B}[z] := \text{Var} \{K_{n,B}(\alpha) \cdot z(\zeta)\} = K_{n,B}^2(\alpha) \cdot ||z||^2 L_2(\nu_{\alpha,n}) - J_{\alpha,n}^2[z]
\]

tends very rapidly to zero as \(n \to \infty\), if for example \(\sup_k \alpha_k < 1\).

Thus, we can use as in the second section by computation of this integral and by error estimation the classical Monte Carlo method with application of Central Limit Theorem (CLT).

There are not difficulties also to generate the multivariate ball beta distribution as well as the polygonal beta distribution generating.
5 Parametric Volterra’s integrals with weak singularities

Let \( z(\cdot) \) be numerical function which dependent not only on the variable \( s, s \in S(n) \) but on some variable \( \theta, \theta \in \Theta \), where \( \Theta \) is any compact metrizable topological space: \( z = z(s, \theta) \). We consider the following multidimensional parametric integral

\[
Q(\theta) = \int_{S(n)} z(s, \theta) R_{\alpha,S}(s) \, ds = K_{n,S}(\alpha) \cdot \int_{S(n)} z(s, \theta) h_{\alpha}(s) \, ds = K_{n,S}(\alpha) \cdot E z(\kappa, \theta),
\]  

(5.1)

where as before the r.v. \( \kappa \) has the polygonal beta distribution with parameters \( \alpha, n \).

We offer for the parametric integral (5.1) computation the so-called ”depending trial method”, see [13], [15], [11], chapter 5, section 11:

\[
Q_N(\theta) := \sup_{i=1}^N z(\kappa_i, \theta),
\]

(5.2)

where \( \{\kappa_i\} \) are independent polygonal distributed random vectors.

In order to estimate a random uniform norm error for approximation \( \sup_{\theta} |Q_N(\theta) - Q(\theta)| \), we need to use the Central Limit Theorem CLT on the Banach space of continuous functions \( C(\Theta) \) with ordinary uniform norm

\[
||f|| = \sup_{\theta \in \Theta} |f(\theta)|,
\]

see [8], [9], [11], [18], [22], [23], [24] etc. In detail, assume that the CLT in the space \( C(\Theta) \) for the sequence of the random fields \( \{Q_N(\theta) - Q(\theta)\} \) there holds; then

\[
\lim_{N \to \infty} P \left( \sqrt{N} \sup_{\theta \in \Theta} |Q_N(\theta) - Q(\theta)| > u \right) = P \left( \sup_{\theta \in \Theta} |\xi(\theta)| > u \right), \quad u > 0,
\]

(5.3)

where \( \xi(\theta) \) is mean zero continuous Gaussian random process (field) with at the same covariation function as the random field \( z(\kappa_1, \theta) - E z(\kappa_1, \theta) : \)

\[
E \xi(\theta_1) \xi(\theta_2) = K^2_{n,S}(\alpha) \int_{S(n)} z(s, \theta_1) z(s, \theta_2) \mu_{\alpha,n}(ds) - Q(\theta_1)Q(\theta_2).
\]

(5.4)

The equality (5.3) allow us to construct the confidence interval in the uniform norm for calculated function \( Q(\theta) \), see e.g. [11], chapter 5, section 11; we investigate in the rest of this report the CLT in the considered case.

Some notations.

\[
Y(s) := \sup_{\theta \in \Theta} |z(s, \theta)|, \quad \rho(\theta_1, \theta_2) := \sup_{s \in S(n)} \left[ \frac{|z(s, \theta_1) - z(s, \theta_2)|}{Y(s)} \right].
\]

The function \( (\theta_1, \theta_2) \to \rho(\theta_1, \theta_2) \) is bounded: \( \rho(\theta_1, \theta_2) \leq 2 \) continuous pseudo-metric on the set \( \Theta \). We denote as usually by \( H(\Theta, \rho, \epsilon) \) an entropy of the set \( \Theta \)
relative the semi-distance $\rho$ at the point $\epsilon$, i.e. the natural logarithm the minimal numbers of $\rho$ – balls of radii $\epsilon$, $\epsilon > 0$ which cover all the set $\Theta$.

**Theorem.**

**A.** Suppose

$$E|Y(\kappa)| = \int_{S(n)} \sup_{\theta \in \Theta} |z(s, \theta)| \mu_{a,n}(ds) < \infty. \quad (5.5)$$

Then the sequence $Q_N(\theta)$ converges to $Q(\theta)$ as $N \to \infty$ uniformly with probability one:

$$P \left( \lim_{N \to \infty} \sup_{\theta \in \Theta} |Q_N(\theta) - Q(\theta)| \to 0 \right) = 1. \quad (5.6)$$

**B.** If

$$\int_{S(n)} \sup_{\theta \in \Theta} |z(s, \theta)|^2 \mu_{a,n}(ds) < \infty \quad (5.7)$$

and

$$\int_0^1 H^{1/2}(\Theta, \rho, \epsilon) \, d\epsilon < \infty, \quad (5.8)$$

then the sequence of r.f. $\sqrt{N} [Q_N(\theta) - Q(\theta)]$ satisfies the CLT in the space $C(\Theta, \rho)$.

**Remark 5.1.** The last condition (5.8) is satisfied if for example $\Theta$ is closure of bounded open set in the space $R^d$ and $\rho(\theta_1, \theta_2) \leq C \, |\theta_1 - \theta_2|^\gamma$, $\gamma = \text{const} \in (0, 1]$; in this case

$$H(\Theta, \rho, \epsilon) \leq C_2 + (d/\gamma) \, |\ln \epsilon|, \, \epsilon \in (0, 1).$$

Moreover, this condition is satisfied if $\dim_{\rho} \Theta < \infty$.

**Proof.** Let us consider the centered random field

$$\lambda(\theta) = z(\kappa, \theta) - Q(\theta).$$

This field belongs to the (separable) Banach space $C(\Theta, \rho)$ with probability one and

$$E||\lambda|| \leq 2 \int_{S(n)} \sup_{\theta \in \Theta} |z(s, \theta)| \mu_{a,n}(ds) < \infty.$$

The first proposition of theorem follows from the well-known LLN in Banach spaces, theorem of Forte-Mourier.

In order to prove the second assertion **B**, we need to use the famous result belonging to Jain and Marcus [8] about CLT in the space of continuous functions. We have based of definition of the distance $\rho$

$$|\lambda(\theta_1) - \lambda(\theta_2)| \leq 2Y(\kappa)\rho(\theta_1, \theta_2).$$

Since $EY^2(\kappa) < \infty$ and $\int_0^1 H^{1/2}(\Theta, \rho, \epsilon) \, d\epsilon < \infty$, we deduce needed for us CLT in the space $C(\Theta, \rho)$. 

9
Analogously may be considered the multiple parametric weak singular integral relative the ball beta distribution.

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