Batched Bandits with Crowd Externalities

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Abstract
In Batched Multi-Armed Bandits (BMAB), the policy is not allowed to be updated at each time step. Usually, the setting asserts a maximum number of allowed policy updates and the algorithm schedules them so that to minimize the expected regret. In this paper, we describe a novel setting for BMAB, with the following twist: the timing of the policy update is not controlled by the BMAB algorithm, but instead the amount of data received during each batch, called crowd, is influenced by the past selection of arms. We first design a near-optimal policy with approximate knowledge of the parameters that we prove to have a regret in \(O(\sqrt{\ln x + \epsilon})\) where \(x\) is the size of the crowd and \(\epsilon\) is the parameter error. Next, we implement a UCB-inspired algorithm that guarantees an additional regret in \(O(\max(K \ln T, \sqrt{T \ln T}))\), where \(K\) is the number of arms and \(T\) is the horizon.

1. Introduction
This paper tackles a novel instance of Batched Multi-Armed Bandits (BMAB, Perchet et al., 2016; Gao et al., 2019), where the timing of updates is constrained by the environment, but the crowd, i.e. the number of samples collected in the next batch depends on the arms that have been pulled in past batches. While we believe that there are many more applications to this setting (see the broader impact section for some of them), we will use the following application example to illustrate and motivate our work.

Example 1 (Service in production). The service may only be updated once everyday over night. We have two (or more) options to deliver the service:

Arm 1. with advertisement: it yields income but low user satisfaction,

Arm 2. without advertisement: it yields costs but high user satisfaction.

Playing the first arm is profitable but decreases the crowd, and playing the second arm increases the crowd but is loss-making. In practice, the interplay between the users, their task success, the crowd dynamics may be extremely complex. While strongly motivated by real-world scenarios, in order to control the complexity of our study, which is the first of its kind, we will consider an idealized setting by making the following series of assumptions:

Assumption 1 (Idealized setting assumptions).

A1(i) The crowd size at next time step is the sum of individual growth: the number of samples to be collected at the next round induced by each arm pull. This sum is then capped by a known full capacity.

A1(ii) Individual growths and rewards are independent, identically distributed, and observable (even when the crowd has been capped).

A1(iii) Crowd size at time \(t\) is known beforehand.

Some of these assumptions could have been worked around, but we decided not to for clarity reasons, in order to remain in a pristine setting that is already sufficiently complex by itself. Under Assumptions 1, it may happen that the service is not sustainable: it is impossible to gain money while maintaining the crowd. In this case, the objective is to make the most of the initial crowd. In the other case, we will show that the optimal policy is to first invest to grow the crowd until reaching its full capacity and then to collect the return on investment while maintaining the full crowd.

While the environment is naturally a Markov Decision Process (MDP), we cannot use classic Reinforcement Learning (RL) algorithms to solve our setting, because only one trajectory is allowed, and exploration would lead it to the terminal state where no user remains in the crowd. Thus, our global objective is to design a bandit algorithm (Bubeck and Cesa-Bianchi, 2012) that deals with the exploration/exploitation trade-off when the crowd dynamics and the rewards are unknown. The exploration intends to reduce the model error. The exploitation intends to yield high rewards. The difficulty of our setting is that this trade-off must be performed under a survival effort: make sure that the crowd runs out only when the model is known to be unsustainable with high probability.
The effect of arm pulls on future rewards has been extensively studied in previous works. In restless bandit (Whittle, 1988), every time an arm is sampled, its state changes according to a transition matrix \( \tilde{q} \), while otherwise, its state changes according to another transition matrix \( q \). It has been shown that computing the optimal policy of restless bandits is PSPACE-hard (Papadimitriou and Tsitsiklis, 1999), and hence relaxation techniques are used for finding an approximation (Whittle, 1988; Guha et al., 2010). There exists a lot of variations of the standard multi-armed bandits, where the future rewards depends on the played actions and where finding the optimal policy is not intractable. In mortal bandits (Chakrabarti et al., 2009), each arm has a lifetime after which it disappears. In scratch games ( Féraud and Urvoy, 2013), an urn model is used for building the lifetime of arms. In Multi-Armed Bandits with known trend (Bouneffouf and Féraud, 2016), the future reward distribution depends on a known function of the number of times the arm has been sampled, while in recovering bandits the trend function is learnt (Pike-Burke and Grünewälder, 2019). Unlike this line of works, we consider here that sampling an arm does not modify the future rewards, but changes the arrival process of new subjects. To the best of our knowledge, the only work that considers the impact of the sampled actions to the arrival process of new subjects is bandit with positive externalities (Shah et al., 2018). While in the proposed problem setting the pull of arms influences the number of subjects that will arrive at the next time periods, in (Shah et al., 2018), the pull of arms influences the type of subjects that will arrive at the next time periods. Externalities have also been widely studied in economics (Cornes and Sandler, 1996; Klenow and Rodriguez-Claré, 2005).

The contributions of this article are the following: Section 2 formalizes the problem and casts it as a Markov Decision Process (MDP). Expressed this way, the policy optimization is intractable because of the stochasticity in the environment. We search for an approximate solution by solving the deterministic Reduced On-expectation MDP (ROeMDP). Section 3 develops the theory and proves the near optimality of the ROeMDP solution in \( O(\sqrt{\frac{\ln x}{x} + \epsilon}) \), where \( x \) is the current crowd size, and \( \epsilon \) is the error on the problem parameters (Theorem 2). Building on these findings, Section 4 introduces a novel UCB algorithm for the problem and proves it to have an additional regret in \( O(\max(K \ln T, \sqrt{T \ln T})) \), where \( K \) is the number of arms and \( T \) is the horizon, as compared to the ROeMDP approximate solution (Theorem 3). Section 5 runs some numerical simulations to validate the theoretical findings. We observe the unexpected result that the bandit algorithm often outperforms the ROeMDP solution with the true parameters. This phenomenon is explained by the bias induced by UCB’s optimism. Section 6 concludes the main document with perspectives for future work. Supplementary material includes all proofs and an overview of the broader impacts.

2. Problem formalization

2.1. Problem statement

In this paper we study a setting illustrated on Figure 1, where, at each time step \( t \), an agent independently and identically interacts with a set of subjects, called the crowd, which size is denoted by \( x_t \). For each subject, the agent selects its play among \( K \) arms and receives a reward as a result, similarly to what happens in any stochastic multi-armed bandit (MAB) setting. But contrary to standard MABs, we consider that the samples are received by batches, and that the number of subjects in the next batch depends on the past pulls. The agent decides an action \( a_t \): the number of pulls on each arm, inadverdently spread among subjects. Each pull on Arm \( k \) triggers an interaction \( \tau \) yielding a reward \( \tau_k \) sampled from distribution \( r_k \), and a growth \( \gamma_\tau \) sampled from distribution \( g_\tau \). \( \tau \in \mathbb{N} \) is the number of subjects being enrolled for next time step \( t + 1 \) stemming from interaction \( \tau \). The goal is therefore to optimize the selection of arms, accounting both for the immediate global reward \( r_t = \sum_{\tau=1}^{\tau} \tau_k \) and the future ones that are directly depending on the global crowd \( x_{t+1} = \min(\sum_{\tau=1}^{\tau} \gamma_\tau, x_t) \), where \( x_t \) is the known maximal population. We assume that \( \gamma_\tau \) and \( \tau \) are observed for each arm pull (Assumption A1(iii)).

In contrast with Shah et al. (2018) and Laroche and Féraud (2018), the externality of our setting is simpler, since it amounts to a factor effect over the rewards that is the same for all arms. However, it may be used as a controllable feature, and as such, may be regarded as a multi-state problem and therefore a Reinforcement Learning task (RL, Sutton and Barto, 1998). Still, contrary to classic RL tasks, the decision process involves a single trajectory with terminal states.

Notations: Let \( \Delta \subseteq \mathbb{S} \) denote a distribution over set \( S \). Let \([n]\) be the set of integers \( 1 \leq i \leq n \). We write \( \mathbb{E}_{\Delta}g_k = \mathbb{E}g_k \), which we call the expected growth of arm \( k \). \( r_k = \mathbb{E}r_k \) is similarly defined as its expected reward. \( \gamma_\tau \) (resp. \( \tau_\tau \)) is the minimum (resp. maximum) expected growth over the arms: \( \gamma_\tau = \min_{k \in [K]} \gamma_k \) and \( \tau_\tau = \max_{k \in [K]} \tau_k \), and the maximum expected reward is denoted by \( \tau_\tau = \max_{k \in [K]} \tau_k \).

We formalize the problem we intend to solve as follows.

Problem 1. Design and analyze an algorithm \( A \), that, at each time step \( t \), takes as a argument the history of past
we write the optimal values which we define below and therefore further rewards depend on past values of a policy.

2.2. Model of the environment as MDPs

The crowd, and therefore further rewards, depends on past actions. Hence, we need a Markov Decision Process to model the setting. We frame this type of dynamics as a Population MDP (PMDP).

**Definition 1** (Population MDP). A Population MDP (PMDP) is a stochastic MDP \( (X_p, A_p, P_p, R_p, \gamma) \), where \( X_p = \{x_i\} \) is the size of the population, \( A_p = \mathbb{N}^x \) is the action space, the stochastic transition function is \( P_p(x, a) = \min \{ \sum_{i=1}^{x} g_{a[i]}, x_i \} \), the stochastic reward function is the sum of individual stochastic rewards \( R_p(x, a) = \sum_{i=1}^{x} \tilde{r}_{a[i]} \), and \( \gamma \) is the discount factor.

\( V^\psi_p^{\gamma} \) and \( Q^\psi_p^{\gamma} \) denote the values of a policy \( \psi \) in the PMDP. We write the optimal values \( V^*_p^{\gamma} \) and \( Q^*_p^{\gamma} \), and \( \psi^* \) may refer to any optimal policy in the PMDP. Expressed this way, the policy optimization is intractable because of the stochasticity in the environment. We are going to search for an approximate solution by solving the following deterministic PMDP formulation, coined On-expectation MDP (OeMDP), which we define below:

**Definition 2** (On-expectation MDP). We define the On-expectation MDP (OeMDP) as the tuple \( (X_o, A_o, P_o, R_o, \gamma) \), where the state space is now continuous: \( X_o = \{0, x_i\} \subset \mathbb{R} \), the action space is a distribution over the arms: \( A_o = \Delta(K) \), the deterministic transition function is the expectation of growth: \( P_o(x, a) = \min \{ x \sum_{k=1}^{K} a_k \gamma_k, x_i \} \), and the deterministic reward function is the expectation of rewards: \( R_o(x, a) = x \sum_{k=1}^{K} a_k \gamma_k \).

We underline that the state space has to be defined on real numbers, since the expectation over a integer random variable lives in the real numbers. Consequently, the action space is a distribution over arms. \( V^\psi_o^{\gamma} \) and \( Q^\psi_o^{\gamma} \) denote the values of a policy \( \psi \) in the OeMDP. We write the optimal values \( V^*_o^{\gamma} \) and \( Q^*_o^{\gamma} \), and \( \psi^* \) refers to any optimal policy in the OeMDP. In the OeMDP, any action \( a \in \Delta(K) \) has an effective growth \( g_o = \sum_{k=1}^{K} a_k \gamma_k \). Conversely any value \( g \in [\gamma_1, \gamma_K] \) is achievable by an interpolation between two arms, and once \( g \) is selected, then an optimal policy must be

\[ V_{\alpha}(x_0) = \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t \sum_{\tau=1}^{x_t} \tilde{r}_{\tau} \right], \]

with \( \{ a_t = \mathbb{A}(h_t), k_{\tau} \in a_t, \tilde{r}_{\tau} \sim r_{k_{\tau}}, \tilde{g}_{\tau} \sim g_{k_{\tau}}, h_{t+1} = h_t \cup (k_{\tau}, \tau, \tilde{g}_{\tau}), x_{t+1} = \min(\sum_{\tau=1}^{x_t} \tilde{g}_{\tau}, x_t) \) and where history \( h_0 \) is initialized as \( \emptyset \).

**Definition 3** (Transformed action set and reward function). We define the transformed action set \( \Psi \) and the transformed reward function \( R \) as follows:

\[
\begin{align*}
\Psi(g) &= \arg\max_{a \in \Delta([G_1, G_2]), \sum_{k=1}^{K} a_k g_k = g} \sum_{k=1}^{K} a_k \gamma_k, \\
R(g) &= \max_{a \in \Delta([G_1, G_2]), \sum_{k=1}^{K} a_k g_k = g} \sum_{k=1}^{K} a_k \gamma_k.
\end{align*}
\]

Definition 3 allows us to simplify the OeMDP formalization into a Reduced On-expectation MDP (ROeMDP), which is defined below.

**Definition 4** (Reduced On-expectation MDP). We define the following deterministic Reduced On-expectation MDP (ROeMDP) as \( (X_o, \mathbb{G}, P_o, R_o, \gamma) \), where the actions are the growth \( \mathbb{G} = [G_1, G_2] \subset \mathbb{R}^2 \), the transition and reward functions are modified accordingly: \( P_o(x, g) = \min\{xg, x_\tau\} \) and \( R_o(x, g) = xR(g) \).

Similarly to notations of PMDP and OeMDP, \( V^{\psi}_{o}^{\gamma} \) and \( Q^{\psi}_{o}^{\gamma} \) denote the values of a policy \( \psi \) in the ROeMDP. We write the optimal values \( V^{\psi}_{o}^{\gamma} \) and \( Q^{\psi}_{o}^{\gamma} \), and \( \psi^* \) refers to any optimal policy in the ROeMDP.

3. Analysis

This section analyses the connections between our different MDP definitions 1, 2, and 4. We start with an analysis of the ROeMDP in the form of a series of properties on the transformed reward function, the optimal value function, and the optimal policy. For the sake of space constraint and clarity, most of the proofs have been moved to the supplementary material.

3.1. ROeMDP properties

Property 1 states properties of the transformed reward function. Figure 2 proposes some visual representations of parameter setting examples with their respective transformed reward function.

**Property 1** (Properties of \( R \)). \( R \) is a piece-wise linear concave function. It is the upper convex envelop of the arms parameters \( \{(\mathbb{G}_k \setminus K)\} \).

Property 2 states that both modelizations OeMDP and ROeMDP have the same optimal values. As a consequence,

\[ \text{For simplicity, we assume in all our proofs that } 1 \in \mathbb{G}, \text{ but the lemmas and theorems still stand when this is not the case with minor changes in their formulations, and in their consequent proofs.} \]
we may search for an optimal policy in the simpler ROeMDP and then, retrieve an optimal policy in the OeMDP with the transformed action set.

**Property 2 (ROeMDP/OeMDP optimality equivalence).** For every optimal policy \( \psi^* \) in the OeMDP, there exists an optimal policy \( \pi^* \) in the ROeMDP such that \( \psi^*(x) = \Psi(\pi^*(x)) \), and we have the optimal values equality:

\[
V^*_r(x) = V^*_g(x) \quad \text{for all states } x \in X_c.
\]

Properties 3 and 4 state remarkable characteristics of the optimal value functions in their MDPs, which are useful to the proofs of the main theorems. More precisely, Property 3 proves that the optimal value functions in the ROeMDP are continuous with respect to the state and the action, and Property 4 demonstrates that the optimal value functions in PMDP/OeMDP (and the ROeMDP by consequence) are monotonously increasing or decreasing depending on the MDP parameters.

**Property 3 (ROeMDP optimal value function continuity).** In the ROeMDP, the optimal value functions \( V^*_r \) and \( Q^*_r \) are continuous in \( x \) and \( g \).

**Property 4 (OeMDP/PMDP optimal value function monotonicity).** When there exists an arm with positive reward (resp. when all arms have a negative reward), the optimal value functions \( V^*_o \) or \( V^*_p \) are (i) positive (resp. negative), (ii) strictly monotonically increasing w.r.t. \( x \) (resp. decreasing), and (iii) concave w.r.t. \( x \) (resp. convex). When the highest reward among arms is equal to 0, then \( V^*_o = V^*_p = 0. \)

Property 5 proves that there exists an optimal policy \( \pi^* \) such that the ROeMDP actions taken over time are decreasing. However, it does not necessarily mean that \( \pi^* \) is a decreasing function of \( g \).

**Property 5 (Existence of decreasing optimal policy).** In the ROeMDP, if \( \max_{g \geq 1} R(g) \leq 0 \), or if \( \gamma \) is chosen such that \( \max_{g \geq \frac{1}{\gamma}} R(g) > 0 \), there exists an optimal policy \( \pi^* \) that is monotonically decreasing with time: \( \forall x, \pi^*(x) \geq \pi^*(\min\{x, \pi^*(x), x^1\}) \).

From those properties, depending on the parameters of the ROeMDP, we may classify the setting into three different cases:

- **Case (a)** \( \max_{g \geq 0} R(g) \leq 0 \) which is equivalent to \( \max_{k \in [K]} \tau_k \leq 0 \): It means that all rewards are negative and the goal is therefore to diminish the crowd at the least cost. Figure 2(a) illustrates this case.

- **Case (b)** \( \max_{g \geq 0} R(g) > 0 \) but \( \max_{g \geq 1} R(g) \leq 0 \): it means that it is possible to get a positive return, but impossible to do it in a sustainable way. Figure 2(b) illustrates this case.

- **Case (c)** \( \max_{g \geq 1} R(g) > 0 \): it means that it is possible to get a positive return in a sustainable way. Figure 2(c) illustrates this case.

For Cases (a-b), there exists an analytical solution: Theorem 1 proves that there exists a constant optimal policy \( \pi^* \) in the ROeMDP and that there exists \( \psi^* \in \Psi(\pi^*) \) that is deterministic, i.e. selects a single arm with probability 1.

**Theorem 1 (ROeMDP solution in Cases (a-b)).** When \( \max_{g \geq 1} R(g) \leq 0 \), no sustainable positive reward is possible, consequently, if \( \tau_\gamma \geq 0 \), or if \( \gamma \) is chosen close enough to 1:

\[
\gamma \geq \max_{k \in P_{<1}} \frac{\tau_k - \tau_{\gamma}}{g_k \tau_{\gamma} - \tau_k},
\]

where \( P_{<1} \) is the set of arms \( k \) such that \( g_k < 1 \), then under the optimal policy, the crowd decreases geometrically with time. Furthermore, the optimal policy is to constantly and deterministically play the same arm maximizing the value function:

\[
V^*_r(x) = x \max_{k \in P_{<1}} \frac{\tau_k}{1 - \gamma g_k}.
\]

Thus, Cases (a-b) are similar, and the optimal arm may be geometrically interpreted by letting a half-line anchored on
the critical point \( (\frac{1}{\gamma}, 0) \) fall on the transformed reward \( R \) curve. The optimal arm is the one that is in contact with the half-line. This is illustrated on Figures 2(a-b) with the dashed green line (here \( \gamma \) is set to 1). In Case (c), we set \( \gamma \) sufficiently close to 1 to ensure that \( \max_{g \geq 1} R(g) > 0 \). Property 5 proves the existence of a decreasing optimal policy that first ensures the crowd growth, possibly at some cost, until reaching \( x_* \), where it selects transformed action \( g_* = \arg\max_{g \geq 1} R(g) \) (\( g_* = 1 \) on Figure 2(c)).

### 3.2. PMDP near-optimality of the ROeMDP optimal policy with model errors

This theorem states that the ROeMDP formalization allows to find a policy that is near optimal in the true PMDP environment, even with an imperfect model of the ROeMDP environment. Its proof has been kept in the main document because the most technical parts are abstracted into lemmas and corollaries that the interested reader may find in the supplementary material. For clarity, the theoretical results are presented below in order of magnitude. The multiplicative constants may be retrieved by looking at the lemmas, corollaries, and properties the theorem relies on.

**Theorem 2** (approximate model error on the optimal PMDP value). In the real PMDP environment, the difference between its optimal value and the value of the ROeMDP-optimal policy with estimated parameters obeys the following order of magnitude:

\[
\mathcal{O}\left(\frac{\hat{g}_r V^*_r}{1 - \gamma} \sqrt{\frac{\ln x}{x}} + \epsilon V^*_r \right),
\]

where \( x \) is the current crowd, \( \hat{g}_r \) is the maximal growth, \( V^*_r \) is the maximal value, and \( \epsilon \) is the maximal error of the arms played by the true PMDP optimal policy and the estimated-ROeMDP optimal policy.

**Proof.** The error of control is the difference between the optimal value in the true PMDP environment \( V^*_p \) and the value of \( \hat{\psi} \) in the true PMDP environment \( V^*_p \hat{\psi} \), where \( \hat{\psi} = \Psi(\hat{\pi}) \), and \( \hat{\pi} \) is the policy that is optimal in the ROeMDP built from the imperfect model of the environment. \( V^*_p - V^*_p \hat{\psi} \) may be broken down into five terms:

\[
V^*_p - V^*_p \hat{\psi} + V^*_p \hat{\psi} - \hat{\psi} \hat{\pi} + \hat{\psi} \hat{\pi} - \hat{\pi} \hat{\pi} + \hat{\pi} \hat{\pi} - V^*_p \hat{\psi}.
\]

\((\text{I})\) Corollary 2 states that this term is non-positive and may therefore be upper bounded by 0.

\(\hat{g}_r \) is the maximal growth. \( V^*_r \) is the maximal value and \( \epsilon \) is the maximal error of the arms played by the true PMDP optimal policy and the estimated-ROeMDP optimal policy.

**Theorem 2** (approximate model error on the optimal PMDP value). In the real PMDP environment, the difference between its optimal value and the value of the ROeMDP-optimal policy with estimated parameters obeys the following order of magnitude:

\[
\mathcal{O}\left(\frac{\hat{g}_r V^*_r}{1 - \gamma} \sqrt{\frac{\ln x}{x}} + \epsilon V^*_r \right),
\]

where \( x \) is the current crowd, \( \hat{g}_r \) is the maximal growth, \( V^*_r \) is the maximal value, and \( \epsilon \) is the maximal error of the arms played by the true PMDP optimal policy and the estimated-ROeMDP optimal policy.

**Proof.** The error of control is the difference between the optimal value in the true PMDP environment \( V^*_p \) and the value of \( \hat{\psi} \) in the true PMDP environment \( V^*_p \hat{\psi} \), where \( \hat{\psi} = \Psi(\hat{\pi}) \), and \( \hat{\pi} \) is the policy that is optimal in the ROeMDP built from the imperfect model of the environment. \( V^*_p - V^*_p \hat{\psi} \) may be broken down into five terms:

\[
V^*_p - V^*_p \hat{\psi} + V^*_p \hat{\psi} - \hat{\psi} \hat{\pi} + \hat{\psi} \hat{\pi} - \hat{\pi} \hat{\pi} + \hat{\pi} \hat{\pi} - V^*_p \hat{\psi}.
\]

\((\text{I})\) Corollary 2 states that this term is non-positive and may therefore be upper bounded by 0.

\(\hat{g}_r \) is the maximal growth. \( V^*_r \) is the maximal value and \( \epsilon \) is the maximal error of the arms played by the true PMDP optimal policy and the estimated-ROeMDP optimal policy.

**Theorem 2** (approximate model error on the optimal PMDP value). In the real PMDP environment, the difference between its optimal value and the value of the ROeMDP-optimal policy with estimated parameters obeys the following order of magnitude:

\[
\mathcal{O}\left(\frac{\hat{g}_r V^*_r}{1 - \gamma} \sqrt{\frac{\ln x}{x}} + \epsilon V^*_r \right),
\]

where \( x \) is the current crowd, \( \hat{g}_r \) is the maximal growth, \( V^*_r \) is the maximal value, and \( \epsilon \) is the maximal error of the arms played by the true PMDP optimal policy and the estimated-ROeMDP optimal policy.

**Proof.** The error of control is the difference between the optimal value in the true PMDP environment \( V^*_p \) and the value of \( \hat{\psi} \) in the true PMDP environment \( V^*_p \hat{\psi} \), where \( \hat{\psi} = \Psi(\hat{\pi}) \), and \( \hat{\pi} \) is the policy that is optimal in the ROeMDP built from the imperfect model of the environment. \( V^*_p - V^*_p \hat{\psi} \) may be broken down into five terms:

\[
V^*_p - V^*_p \hat{\psi} + V^*_p \hat{\psi} - \hat{\psi} \hat{\pi} + \hat{\psi} \hat{\pi} - \hat{\pi} \hat{\pi} + \hat{\pi} \hat{\pi} - V^*_p \hat{\psi}.
\]

\((\text{I})\) Corollary 2 states that this term is non-positive and may therefore be upper bounded by 0.

**II** This term is the error induced by the misplacement of the upper convex envelop of the arms parameters in the ROeMDP. It is further broken down as follows:

\[
V^*_r - \hat{V}^*_r = V^*_r - \hat{V}^*_r + \hat{V}^*_r - \hat{V}^*_r + \hat{V}^*_r - \hat{V}^*_r + \hat{V}^*_r - V^*_r,
\]

where \( \hat{\pi} \) is an optimal policy in the true ROeMDP, under the constraint that \( \forall x, \hat{\pi}(x) \in \mathcal{G} \cap \hat{\mathcal{G}} \).

\((i)\) Lemma 6 proves that this error is linearly decreasing with the growth domain error \( \mathcal{G} \cap \hat{\mathcal{G}} \).

\((ii)\) Lemma 5 proves that this error is linearly decreasing with the upper convex envelop reward error \( \| R(g) - \hat{R}(g) \|\).

\((iii)\) This term has to be non-positive since \( \hat{\pi} \) is optimal in the estimated ROeMDP.

As a consequence, the error \( V^*_r - \hat{V}^*_r \) linearly depends on the model error \( \epsilon \) on played arms, either by the true optimal policy or by the target policy.

**III** Property 2 states that the values are equal, hence, their difference is 0.

**IV** This term is the reduction error. It accounts for the error between the estimated dynamics and the true dynamics in the OeMDP. Lemma 7 proves that the reduction error is bounded as a function of the error on the dynamics estimates over the arms in the image \( \hat{\psi}[x_*] \) of the trained policy \( \hat{\psi} \).

More precisely, \( \hat{V}^*_o - V^*_o \) is upper bounded by:

\[
\frac{x \max_{k \in \hat{\psi}[x_*]} |\hat{\pi}_k - \pi_k| + \gamma V^*_o \max_{k \in \hat{\psi}[x_*]} |\tilde{g}_k - \hat{g}_k|}{1 - \gamma},
\]

where \( \tilde{g}_k \) (resp. \( \hat{\pi}_k \)) is the expected growth (resp. reward) estimate of Arm \( k \).

**V** This term is the OeMDP error: the error made by planning in a deterministic on-expectation environment instead of the real stochastic PMDP environment. Lemma 1 states that the error decreases exponentially with \( x_r - x \) in Case (a-b) and Lemma 3 deals with Case (c) to demonstrate an overall upper bound of this error in \( \mathcal{O}\left(\frac{\hat{g}_r V^*_r}{(1 - \gamma) \sqrt{x}}\right) \), where \( x \) is the current crowd, \( \hat{g}_r \) is the maximal growth, and \( V^*_r \) is the maximal value in the OeMDP.

The main result displayed in the abstract and the introduction is retrieved when dependencies in \( \gamma, V^*_r \), and \( \hat{g}_r \) are omitted. The first term may be interpreted as the amplitude of error due to the PMDP-suboptimality of the on-expectation optimal policy, and the second term as the error due to the model error.
we propose a fully online Has opposed to batched algorithm similarly to classic ucb Hauer et al. RPPR; auer and irreversible as a consequence horizon be collected it means that the decision to refute case is important difference once case is refuted, the crowd optimality is either ruled out or confirmed. However, there is an important difference: once Case (c) is refuted, the crowd geometrically decreases and only little more samples are to be collected. It means that the decision to refute Case (c) is irreversible. As a consequence, horizon $T$ has to be known in advance to select the high probability hyperparameter $\delta$.

Since our bandit algorithms intend to retrieve the ROeMDP-optimal policy, we are going to use the concept of instantaneous expected regret relative to the policy that is obtained by solving the ROeMDP with the true parameters:

$$\rho_{ins}(A, t) = E_{\pi^*} \left[ \sum_{\tau=1}^{t} \hat{r}_\tau \right] - E_{A^*} \left[ \sum_{\tau=1}^{t} \hat{r}_\tau \right],$$

where $\pi^*$ is the optimal policy in the ROeMDP with the true parameters $\{\hat{g}_k, \hat{r}_k\}_{k \in [K]}$. $x_t$ is a random variable denoting the size of the population at time $t$, and $\hat{r}_\tau$ is a random variable denoting the reward received from individual $\tau$ at time $t$. The cumulative regret is the discounted sum of instantaneous regret over time. Below, Theorem 3 provides an upper bound to the asymptotic regret of Algorithm 1a, that guides us, without knowledge on the encountered case, to set $\delta \in O\left(\frac{1}{T}\right)$, so that the overall expected regret due to parameter estimation would follow an asymptotic regret in $O(\sqrt{T \ln T})$ in the worst setting (Case (c) with growth of the max-reward arm smaller than 1) and $O(K \ln T)$ otherwise.

**Theorem 3** (Algorithm 1a expected regret). The cumulative regret of Algorithm 1a is upper bounded by an error term
that decays with the following order of magnitude as a function of the number of arms $K$, the horizon $T$, and the high probability hyper-parameter $\delta$:

$$
\text{Case (a-b)} \left\{ \begin{array}{ll}
O(K \ln \frac{1}{\delta} + K\delta T) & \text{if } g_{k_*} > 1, \\
O(K \ln \frac{1}{\delta} + K\delta T) & \text{otherwise}.
\end{array} \right.
$$

$$
\text{Case (c)} \left\{ \begin{array}{ll}
O\left(K \ln \frac{1}{\delta} + K\delta T \right) & \text{if } g_{k_*} = \arg\max_{k \in [K]} r_k > 1, \\
O\left(K \ln \frac{1}{\delta} + K\delta T + \sqrt{T \ln \frac{1}{\delta}} \right) & \text{otherwise}.
\end{array} \right.
$$

The batched version is formalized in Algorithm 1b. The conversion is simple: whereas the outcomes of the arm selections are observed by batches, the selection of arms itself is known and the corresponding confidence interval may be updated. Another difference has to be noted: the number of samples is dependent of the size of the batch and not only the horizon $T$. Still, the nature of the Case (c) policy is such that the maximum crowd is quickly reached (the crowd grows geometrically), point from which the maximal number of samples remaining to be collected until the horizon is easy to upper bound: $(T - t)x_r$.

### 5. Numerical analysis

In order to validate our theoretical findings, we use a generator of problems described in Algorithm 5 in Supplementary Material with maximal crowd $x_r = 10,000$ and horizon $T = 1,000$. This generator has been tuned in order to generate an interesting distribution of problems, i.e.:

- that have approximately 10% chance to be of Case (a), 40% of Case (b), and 50% of Case (c),
- that have transformed reward functions composed of several segments,
- and that have a maximal reward that is obtained for growth below 1.

Figures 2(a-c) show settings created with this generator. During the design of the task generator, it quickly appeared that some are easier than others. In order to account for this and to analyze to performance correlation with the difficulty, we define hereafter the decidability: a generalization of the notion of gap classically used in MAB.

**Definition 5** (Decidability). Given a task characterized by its arms parameters, we define the decidability as the distance between its transformed reward function and point $(1, 0)$.

Indeed, the further it is from this point, the easier it is to determine whether the problem is of Cases (a-b) or Case (c).

The benchmark of our experiments only includes our UCB-based algorithm because, to the best of our knowledge, no algorithm in the literature is able to take into account the dual growth-reward feedback. As a consequence, classic MAB/BMAB would eventually select the arm with the highest reward, which is unlikely to have a growth higher than 1, and therefore would deplete the crowd. RL algorithm, as well, are not designed to deal with settings such as ours, where only one trajectory is allowed and some states are final. Consequently, RL algorithms would explore states with low crowds and risk crowd depletion. Finally, we considered a Thompson Sampling (TS) formulation of the ROeMDP parameter exploration/exploitation (Chapelle and Li, 2011; Kaufmann et al., 2012). However, the results were so poor that we opted to not report them and to provide instead the following qualitative explanation on why it cannot work as well as UCB in our setting. UCB is by nature an optimistic algorithm. Its optimism may sometimes be detrimental, but is actually virtuous in our setting because it naturally urges the algorithm to pull arms with high growths, which mitigates the risk of depleting the crowd. In contrast, TS indifferently accounts for uncertainty, sometimes optimistically, but also sometimes pessimistically, which implies in our setting to commit to Case (a-b) too early, and therefore to deplete the crowd. We do not claim that there does not exist any efficient TS-based algorithm for our setting but the adaptation is not
Figure 4: Batched experiments. (left) instantaneous regret averaged on all Case (c), (centre) instantaneous regret averaged on Case (c) when no run lost all crowd, and (right) cumulative regret on a log scale as a function of the decidability (negative for Cases (a-b) and positive for Case (c)), where a cross means that the cumulative regret is negative.

As straightforward as for UCB, at the very least.

Figure 3 shows the results for the online experiments. On the left, we have the averaged instantaneous regret as a function of time for all cases and 4 values of $\xi$ ($\xi = 0.5$ is off chart). Indeed, it is more practical to directly perform a hyper-parameter search on $\xi$ (defined on Line 2 of Algorithm 1a), rather than $\delta$. It is interesting to observe that the instantaneous regret as defined in Equation 7 gets negative: the bandit algorithm does better in practice than the optimal point it is looking for. In the middle, we isolate Case (c), which reveals to be responsible for such an unexpected result. This is explained as follows: the bandit uses optimistic values for the unknown parameters of the problem and chooses the best trade-off on it such that $g = 1$ when the population is already maximal. It means that in practice, it aims at growth that are a little smaller than required, meaning that it hits the $x_7$ ceiling less frequently and remains more likely at a safe distance from it. In contrast the true optimal will hit $x_7$ more often and lose some expected benefit from it. On the right, we observe the $\log_{10}$ averaged total regret per decidability level. We observe that smaller $\xi$ benefits to Cases (a-b) while this is the opposite for Case (c). This was expected since a higher $\xi$ means that Case (c) strategy has to be followed for a longer time.

Figure 4 shows the results for the batched experiments. Once again, we once again notice on the left figure that the instantaneous regret gets negative. However, contrarily to the online setting, the instantaneous regret does not tend to 0 asymptotically. This negative regret is explained by the fact that, in some settings, the optimal growth is very small and incurs a risk of crowd depletion. In contrast, the bandit algorithm is reward optimistic, which makes it select much higher growth targets. The middle figure represents the results when none of the runs suffered a crowd depletion, where the instantaneous regret is positive, and tends to 0 asymptotically. The right figure displays the total reward as a function of the decidability, with similar results as in the online experiments.

6. Conclusion

We tackled the problem of Batched Multi-Armed Bandits in an environment where the future affluence depends on the past arm selections. We built an approximate formulation of the problem in order to make it tractable. We proved the near-optimality of the approximate solution, and expressed its sensitivity to errors in the parameters. We identified three cases: (a) all arm rewards are negative, (b) it is impossible to maintain the crowd and get positive expected rewards, and (c) it is possible to get positive expected rewards while maintaining the crowd. We designed a novel UCB algorithm that allows to grow the crowd as long as the case is undefined and show that this algorithm suffers a regret in $\mathcal{O}(\max(K \ln T, \sqrt{T \ln T}))$ as compared to the approximate solution. We ran experiments that reveal that the bandit often performs better than the approximate solution in the . This unexpected phenomenon is explained by the fact that UCB’s optimism has a positive effect on the setting.

Perspectives: We studied asymptotic regret in the general case, but focusing on the regret in Cases (a-b) may have a greater impact. For instance, management of public health strategies facing epidemic could be cast into our model (Libin et al., 2018), but our analysis and algorithms would be inefficient at dealing with such settings where the case is known. More generally, we believe that in practice, information about arms is often known: for instance, Arm $k$ is known to yield more reward than Arm $k'$, but generates less growth, and it would be more practical to be able to design algorithm that could take advantage of such prior knowledge. Finally, we have empirical evidence that the ROeMDP solution could be improved by staying at safe distance from $x_7$. Formal analysis would be welcome.
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A. Broader Impact

Our work focuses on discovering optimal mitigation between immediate rewards/costs and future spread/containment of the popularity of a system. Our initial motivation for this work is centered around service delivery popularization in a sustainable way, i.e. while making it profitable. We endeavour below to enumerate the potential positive (+) and negative (−) societal impacts:

(+) **Stronger economy**: this was our primary objective and we prove that our algorithm makes sure that services without beneficial margins are no longer sustained.

(+) **Faster response to pests**: further work in the same direction but a stronger focus on the Case (a) could be applied to public response to epidemics, locust, or organic pollution (e.g. spreading algae).

(−) **Unfairness**: our algorithm chooses arms regardless individual identity, and decides global policies for the best of all, which may, and almost certainly will induce discrimination: e.g. Arm 1 would be preferred to Arm 2 because it offers a better service to a majority, even though it is worse for a minority (Thomas et al., 2019).

(−) **Poorly designed reward/cost**: our algorithm optimizes the behaviour in order to maximize a reward function. In practice, the design of the reward function is often an inextricable task: how to mix heterogeneous objectives such as monetary expenditure/income, human casualties (diseased, wounded, deaths), environmental debts/benefits, etc. Some of these effects may only be measured years after, and often partially (Orseau and Armstrong, 2016).

(−) **Ill-intentioned objectives**: our algorithm could be used for ideological purposes: e.g. optimize the spreading of fake news or corrupted ideas, to assist an agenda. Like any tool, it may be used for wrong purposes.

B. Generator of problems

**Input**: $K$.

1. $\alpha \sim \mathcal{U}(0, 1)$.
2. $\forall k, \bar{y}_k \sim \mathcal{U}(0, 2)$.
3. $\forall k, g_k = \mathcal{G}\left(\frac{\bar{y}_k}{\bar{y}_k + 1}\right)$.
4. $\forall k, r_k \sim (0.6 + 0.7\alpha)(\mathcal{U}(0, 1) - |2\bar{y}_k - 1|) - 0.5 + 1.47\alpha$
5. $\forall k, r_k = 4\mathcal{B}\left(\frac{\bar{y}_k + 2}{4}\right) - 2$

**Algorithm 5**: Generator of problems

with $\mathcal{U}$, $\mathcal{G}$, and $\mathcal{B}$, respectively denote the uniform, geometric and Bernoulli distributions.
C. Proofs

**Property 1** (Properties of $\mathcal{R}$). $\mathcal{R}$ is a piece-wise linear concave function. It is the upper convex envelop of the arms parameters $\{(y_k, r_k)\}_{k \in [K]}$.

*Proof.* $\mathcal{R}(g)$ may be interpreted as the upper convex envelop of the arms parameters: the expected growth versus the expected reward. Since, there are a finite number of arms, the upper convex envelop must be piece-wise linear and concave. \hfill \Box

**Property 2** (ROeMDP/OeMDP optimality equivalence). For every optimal policy $\psi^*$ in the OeMDP, there exists an optimal policy $\pi^*$ in the ROeMDP such that $\psi^*(x) \in \Psi(\pi^*(x))$, and we have the optimal values equality: $V^*_{\psi}(x) = V^*_{\pi}(x)$ for all states $x \in \mathcal{X}_o$.

*Proof.* $\mathcal{R}$ is the upper convex envelop of the points $(y_k, r_k)$ formed by the $K$ arms. The transition function being entirely determined by the choice of $g$, all optimal action $a$ must belong to $\Psi(g)$, meaning that any optimal policy $\psi^*$ of the OeMDP implements a policy $\pi$ in the ROeMDP. We may infer that $V^*_{\psi} = V^*_{\pi}$.

Conversely, any optimal policy $\pi^*$ in the ROeMDP may be implemented in the OeMDP by some policy $\psi$ that is a mixture of the two arms surrounding it on the upper convex envelop of the arms parameters. We may infer that $V^*_{\pi} = V^*_{\psi}$.

From both inequalities, we may conclude that $V^*_{\pi} = V^*_{\psi}$. \hfill \Box

**Property 3** (ROeMDP optimal value function continuity). In the ROeMDP, the optimal value functions $V^*_{\pi}$ and $Q^*_{\pi}$ are continuous in $x$ and $g$.

*Proof.* We first prove the continuity of the optimal value functions $V$ and $Q$ with respect to $x$:

\[
\lim_{\epsilon \to 0} [V(x) - V(x - \epsilon)] \leq \lim_{\epsilon \to 0} [V(x) - Q(x - \epsilon, \pi(x))]
\]

\[
= \lim_{\epsilon \to 0} \left[ x\mathcal{R}(\pi(x)) - (x - \epsilon)\mathcal{R}(\pi(x)) + \gamma V(\min(x\pi(x), x)) - \gamma V(\min((x - \epsilon)\pi(x), x)) \right],
\]

where inequality 8 is obtained because $V$ is optimal. We further upper bound and expand $V(x\pi(x)) - V((x - \epsilon)\pi(x))$ iteratively and obtain:

\[
\lim_{\epsilon \to 0} [V(x) - V(x - \epsilon)] \leq \lim_{\epsilon \to 0} \epsilon \left[ \sum_{t=0}^{\infty} \gamma^t \mathcal{R}(\pi(x^{(t)})) \prod_{t'=0}^{t-1} \pi(x^{(t')}) \right]
\]

with $x^{(t+1)} = \min \left( x^{(t)} \pi(x^{(t)}), x^* \right)$, and $x^{(0)} = x$.

We now prove that the term inside the brackets is finite:

\[
\mathcal{R}(\pi(x^{(t)})) \leq \max_{k \in [K]} r_k
\]

\[
\prod_{t'=0}^{t-1} \pi(x^{(t')}) \leq \frac{x^*}{x}
\]

We conclude with an upper bound of the limit:

\[
\lim_{\epsilon \to 0} [V(x) - V(x - \epsilon)] \leq 0.
\]

We can similarly prove the mirrored inequality:

\[
\lim_{\epsilon \to 0} [V(x - \epsilon) - V(x)] \leq 0,
\]

and conclude the proof of the continuity of the optimal value function with respect to $x$. 

---

**Batched Bandits with Crowd Externalities**
The continuity of the action-state value function with respect to \( x \) and \( a \) follows directly from the following expansion, only containing functions that are continuous in \( x \) and \( a \):

\[
Q(x, a) = x \mathcal{R}(a) + \gamma V(\min(xa, x_r))
\]

(16)

**Property 4** (OeMDP/PMDP optimal value function monotonicity). When there exists an arm with positive reward (resp. when all arms have a negative reward), the optimal value functions \( V_o^* \) or \( V_p^* \) are (i) positive (resp. negative), (ii) strictly monotonically increasing w.r.t. \( x \) (resp. decreasing), and (iii) concave w.r.t. \( x \) (resp. convex). When the highest reward among arms is equal to 0, then \( V_o^* = V_p^* = 0 \).

**Proof.** When the highest reward among arms is equal to 0, it is impossible by design to yield positive rewards. The values of always selecting a 0-reward arm are trivially 0, which is therefore optimal.

(i) The optimal policy has a value that is larger than the one always selecting the positive arm, which must have a positive value. Conversely, if there is no positive reward, then the value cannot be positive.

(ii) For brevity, we use the same notations for \( V_o^* \) and \( V_p^* \): \( V \). When there exists a positive arm, the optimal value in \( x + \delta x \) has to be larger than the value of the policy copying \( \pi(x) \) for the a subcrowd of size \( x \), and takes the maximum immediate reward for the remaining of the crowd \( \delta x \):

\[
V(x + \delta x) \geq V(x) + \delta x \max_{k \in [K]} \tau_k.
\]

(17)

\( \max_{k \in [K]} \tau_k \) has to be positive since we assume that there exists an arm with a positive expected reward.

Conversely, when there is no positive arm, we have:

\[
V(x + \delta x) \leq V(x) + \delta x \max_{k \in [K]} \tau_k + \gamma \max_{x'} V(x').
\]

(18)

We know that the second term is negative and the last term is non-positive, therefore the strict decreasing property is proven.

(iii) When there exists a positive arm, let us consider the decomposition of the value into \( n \) units as follows:

\[
V(x) = \sum_{i=1}^{n} \left\{ V \left( \frac{i}{n} x \right) - V \left( \frac{i-1}{n} x \right) \right\} + V(0)
\]

\[
= \sum_{i=1}^{n} U_i \quad \text{with} \quad U_i = V \left( \frac{i}{n} x \right) - V \left( \frac{i-1}{n} x \right)
\]

(19)

(20)

\((U_i)\) has to be a decreasing sequence, otherwise, a reordering \( \sigma \) of the sequence \( (U_{\sigma(i)}) \) would yield higher values, which is inconsistent with the optimality assumption made on the values. As a consequence, for any \( \delta x \), we have:

\[
V(x + \delta x) - V(x) \leq V(x) - V(x - \delta x)
\]

\[
\Leftrightarrow \quad V(x) \geq \frac{1}{2} \left( V(x + \delta x) + V(x - \delta x) \right)
\]

(22)

(23)

which is sufficient to conclude that \( V(x) \) is concave.

Conversely, when there is no positive arm, the same proof may be developed, but this time the constraint in size has a positive effect on the value: it prevents the crowd from growing more than what it should and therefore saves subsequent negative reward. As a consequence, the value functions are convex in this case.

**Property 5** (Existence of decreasing optimal policy). In the ROeMDP, if \( \max_{g \geq 1} \mathcal{R}(g) \leq 0 \), or if \( \gamma \) is chosen such that \( \max_{g \geq 1} \mathcal{R}(g) > 0 \), there exists an optimal policy \( \pi^* \) that is monotonically decreasing with time: \( \forall x, \pi^*(x) \geq \pi^*(\min\{x, \pi^*(x), x_r\}) \).
Proof. When \( \max_{g \geq 1} \mathcal{R}(g) \leq 0 \), the problem is trivially solved by Theorem 1. The policy is constant, therefore decreasing in its broad sense.

In the complementary case, \( \max_{g \geq 1} \mathcal{R}(g) > 0 \), we may choose \( \gamma \) such that: \( \max_{g \geq 1} \mathcal{R}(g) > 0 \). Now, we look at the situation in

\[
g_* = \arg\max_{g' \geq \frac{1}{2}} \mathcal{R}(g').
\]

(24)

In a first step, we prove that the optimal policy satisfies \( \pi^*(x) \geq g^* \) for all \( x \) (I). In a second step, we prove that, if \( x'' \geq xg_1^2 \), then the optimal way to reach \( x'' \) after two action \( g_1 \) and \( g_2 \) implies that \( g_1 \geq g_2 \) (II).

(I) We prove here that for all \( x \), the optimal policy \( \pi^*(x) \) is necessarily larger than \( g^* \). To do so, we assume \( g < g_* \) and prove \( Q_\pi^N(x, g) \leq Q_\pi^N(x, g_*) \). We write that \( \mathcal{R}(g') = \alpha g' + \beta \) in the lower vicinity of \( g_* \). Since \( g_* \) is maximal, \( \alpha \geq 0 \):

\[
Q_\pi^N(x, g) - Q_\pi^N(x, g_*) = x(\mathcal{R}(g) - \mathcal{R}(g_*)) + \gamma(V_\pi^N(xg) - V_\pi^N(xg_*))
\]

\[
\leq x\alpha (g - g_*) + \gamma \left( \frac{g}{g_*} V_\pi^N(xg_*) - V_\pi^N(xg_*) \right),
\]

\[
= x\alpha (g - g_*) + \frac{\gamma}{g_*} (g - g_*) V_\pi^N(xg_*) \quad \text{by concavity } \mathcal{R}(g) \leq \alpha g + \beta \text{ and } V_\pi^N(xg) \leq \frac{\gamma}{g_*} V_\pi^N(xg_*) \text{ by Corollary 3}
\]

\[
= (g - g_*) \left( \frac{x\alpha}{\alpha g + \beta} + \frac{\gamma}{g_*} V_\pi^N(xg_*) \right). \quad \text{by assumption } \frac{x\alpha}{\alpha g + \beta} \geq 0 \quad \text{and } \gamma \geq 0
\]

(27)

(28)

This concludes the proof that the optimal value of \( g_* \) is always greater than the optimal value of \( g < g_* \), and therefore that \( \pi^*(x) \geq g^* \) for all \( x \).

(II) We consider two states \( x \) and \( x'' \geq xg_2^2 \) and the optimal way to go from \( x \) to \( x'' \) in two steps. In particular, \( x'' \) could be chosen to be equal to \( x\pi^*(x) \pi^*(x\pi^*(x)) \), and infer properties over \( \pi^* \) but this study is not limited to it. We will use the notation of the double-action value function \( Q_\pi^N(x, g_1, g_2) \) which is the optimal value of taking action \( g_1 \) followed with action \( g_2 \) from state \( x \). Its Bellman equation is:

\[
Q_\pi^N(x, g_1, g_2) = x\mathcal{R}(g_1) + \gamma xg_1 \mathcal{R}(g_2) + \gamma^2 V_\pi^N(xg_1g_2)
\]

(29)

\[
Q_\pi^N(x, g, \frac{x''}{g}) = x\mathcal{R}(g) + \gamma xg \mathcal{R} \left( \frac{x''}{g} \right) + \gamma^2 V(x'')
\]

(30)

\[
\frac{\partial Q_\pi^N}{\partial g} \left( x, g, \frac{x''}{g} \right) = x \frac{\partial \mathcal{R}}{\partial g}(g) + \gamma x \frac{\partial \mathcal{R}}{\partial g} \left( \frac{x''}{g} \right) + \gamma xg \frac{\partial \mathcal{R}}{\partial g} \left( \frac{x''}{g} \right)
\]

\[
= x \alpha + \gamma x \left( \alpha' \frac{x''}{g} + \beta' + g\alpha' \frac{\partial \left( \frac{x''}{g} \right)}{\partial g} \right). \quad \text{by } \mathcal{R}(g) = \alpha g + \beta
\]

\[
= x \alpha + \gamma x \left( \alpha' \frac{x''}{g} + \beta' - g\alpha' \frac{x''}{g^2} \right) \quad \text{by } \frac{\partial \left( \frac{x''}{g} \right)}{\partial g} = \frac{-x'^2}{g^3}
\]

\[
= x(\alpha + \gamma \beta'),
\]

(31)

(32)

(33)

(34)

where \( \mathcal{R}(g) = \alpha g + \beta \) in the vicinity of \( g \), and \( \mathcal{R}(g) = \alpha' g + \beta' \) in the vicinity of \( \frac{x''}{g} \). Because we know that the upper convex envelop is concave in addition of being piece-wise linear, we know that:

\[
\text{sign} \left( g - \frac{x''}{g} \right) = -\text{sign} (\alpha - \alpha') = \text{sign} (\beta - \beta'),
\]

(35)

and we can conclude that the partial derivative of \( Q_\pi^N(x, g, \frac{x''}{g}) \) with respect to \( g \) is decreasing and piece-wise constant at each change of segment in the upper convex envelop. It means that that \( Q_\pi^N(x, \frac{x''}{g}) \) is continuous, piece-wise linear and concave as a function of \( g \).
Since \( g^* \) is maximal, and since \( \mathcal{R} \) is concave, \( \mathcal{R} \) must be decreasing in \( \sqrt{\frac{x'}{x}} \geq g^* \) (by assumption). If we write \( \mathcal{R}(g) = \alpha g + \beta \) in the vicinity of \( \sqrt{\frac{x'}{x}} \), then we observe that \( \alpha \) is non-positive, and again by concavity, that \( \alpha g^* + \beta \) must be positive. As a consequence of both observation, we can infer that \( \alpha + \beta \) must be positive, and therefore that \( \alpha + \beta > 0 \), which in turn implies that the partial derivative is positive in \( \sqrt{\frac{x'}{x}} \). Thus, we proved that, if \( \sqrt{\frac{x'}{x}} \geq g^* \), \( Q^\star_r(x, g, \frac{x'}{x}) \) reaches its maximum for a value higher than \( \sqrt{\frac{x'}{x}} \).

\[ \square \]

**Theorem 1** (ROeMDP solution in Cases (a-b)). When \( \max_{g \geq 1} \mathcal{R}(g) \leq 0 \), no sustainable positive reward is possible, consequently, if \( \mathcal{T}_r \geq 0 \), or if \( \gamma \) is chosen close enough to 1:

\[
\gamma \geq \max_{k \in \mathcal{P}_{<1}} \frac{\mathcal{T}_k - \mathcal{T}_\ell}{\mathcal{R}_k \mathcal{T}_\ell - \mathcal{T}_k},
\]

where \( \mathcal{P}_{<1} \) is the set of arms \( k \) such that \( \mathcal{R}_k < 1 \), then under the optimal policy, the crowd decreases geometrically with time. Furthermore, the optimal policy is to constantly and deterministically play the same arm maximizing the value function:

\[
V^\star_r(x) = x \max_{k \in \mathcal{P}_{<1}} \frac{\mathcal{T}_k}{1 - \gamma \mathcal{R}_k}.
\]

**Proof.** We compute the value \( V_a \) of constantly repeating action \( a < 1 \):

\[
V_a(x) = x \mathcal{R}(a) + \gamma V_a(xa)
\]

\[
= x \sum_{t=0}^{\infty} (a \gamma)^t \mathcal{R}(a)
\]

\[
= x \frac{\mathcal{R}(a)}{1 - a \gamma}
\]

Since \( \mathcal{R}(a) \) is piece-wise linear, we may search for the optimal point on one of its segments defined on \([a_1, a_2]\), such that \( \mathcal{R}(a) = \alpha a + \beta \). The derivative of \( \mathcal{R}(a) \) on \([a_1, a_2]\) is therefore:

\[
\frac{\partial V_a}{\partial a}(x) = x \gamma \frac{\alpha + \beta}{(1 - a \gamma)^2}.
\]

On the considered segment \( \alpha + \beta \) is constant, and we may conclude that \( V_a \) takes its minimal value either in \( a_1 \) or \( a_2 \). To minimize \( V_a \) on the full domain of \( a \), one just has to look at the singular points of the upper convex envelop, which are the coordinate \((\mathcal{T}_k, \mathcal{R}_k)\) of the arms \( k \) that are on it with \( \mathcal{R}_k < 1 \), which we concisely write \( k \in \mathcal{P}_{<1} \). We call \( V_{g^*} \) the value of the optimal constant policy playing repeatedly \( g^* = \mathcal{T}_{k^*} \), with \( r^* = \mathcal{R}(g^*) \). Consequently, \( V_{g^*} \) may be written as follows:

\[
k^* = \arg\max_{k \in \mathcal{P}_{<1}} \frac{\mathcal{T}_k}{1 - \gamma \mathcal{R}_k}
\]

\[
V_{g^*}(x) = x \frac{r^*}{1 - \gamma g^*}.
\]

Now we prove by contradiction that there is no possible policy improvement over \( V_{g^*} \):

\[
0 < Q^\star_r(x, g) - V_{g^*}(x) = x \mathcal{R}(g) + \gamma V_{g^*}(\min(gx, x_r)) - x \frac{r^*}{1 - \gamma g^*}
\]

\[
= x \mathcal{R}(g) + \min(gx, x_r) \gamma \frac{r^*}{1 - \gamma g^*} - x \frac{r^*}{1 - \gamma g^*}
\]

\[
= \frac{1}{1 - \gamma g^*} (x \mathcal{R}(g) - x g^* \mathcal{R}(g) + \min(gx, x_r) \gamma r^* - xr^*),
\]
which has the sign of \( R(g) - \gamma g^* R(g) + g\gamma r^* - r^* \) if \( g < 1 \). It implies that:

\[
\frac{R(g)}{1 - \gamma g} \geq \frac{r^*}{1 - \gamma g^*},
\]

which is contradictory with the optimality in \( k^* \).

If \( g \geq 1 \), we distinguish two cases: \( R(g^*) < 0 \) and \( R(g^*) \geq 0 \). If \( R(g^*) < 0 \), we get:

\[
\frac{xR(g) - g^* x \gamma R(g) + \min(gx, x \gamma) r^* - x r^*}{x} \leq R(g) - g^* \gamma R(g) + g\gamma r^* - r^* \\
\leq (1 - g^* \gamma)R(g) - (1 - \gamma)R(g^*) \\
\leq (1 - g^*) \max_{g \geq 1} R(g) - (1 - \gamma) r^*,
\]

which is always negative when \( \max_{g \geq 1} R(g) \leq R(g^*) \), or under the assumption we made on \( \gamma \):

\[
\gamma \geq \max_{k \in \mathbb{P}_1} \max_{g \geq 1} \frac{R(g) - \tau_k}{g - \tau_k}.
\]

This condition ensures that \( \gamma \) is close enough to 1 so that the optimal policy would not be to keep up a population of \( x \), and lose rewards, rather than investing to reduce the population.

If \( g \geq 1 \) and \( r^* \geq 0 \), then we quickly consider the case where \( g \leq \frac{1}{\gamma} \), and observe that, in this case \( 1 - g^* \gamma \geq 0 \) and \( R(g) < 0 \), therefore the first term of Equation 44 is negative, and \( \min(gx) \gamma r^* - x r^* \leq x r^* (\gamma g - 1) \) is also negative.

In the remaining case, when \( g \geq \frac{1}{\gamma} \) and \( r^* \geq 0 \), let us upper bound the upper convex envelop by its local linear expression around \( \frac{1}{\gamma} \): \( R(g) = \alpha g^* + \beta \). We know by the concavity of the upper convex envelop that \( r^* \leq \alpha g^* + \beta \), and \( R(g) \leq \alpha g + \beta \), which gives us:

\[
\frac{xR(g) - g^* x \gamma R(g) + \min(gx, x \gamma) r^* - x r^*}{x} \leq R(g) - g^* \gamma R(g) + g\gamma r^* - r^* \\
\leq (1 - g^* \gamma)(\alpha g + \beta) + (g\gamma - 1)(\alpha g^* + \beta) \\
= \alpha g + \beta - \alpha g^* \gamma - \beta g^* \gamma + \alpha g\gamma + \beta g\gamma - \alpha g^* - \beta \\
= (\alpha + \beta \gamma)(g - g^*),
\]

which has to be non positive, because \( g \geq 1 > g^* \) and \( \alpha + \beta \gamma = \gamma R(\frac{1}{\gamma}) \leq \gamma \max_{g \geq 1} R(g) \leq 0 \).

We may therefore conclude that, under those conditions, \( V_{\gamma}^* \) is the optimal value, and constantly performing \( g^* \) is an optimal policy, and since \( g^* < 1 \), it will geometrically deplete the crowd.

\[\square\]

**Corollary 1** (PMDP-optimal value comparison). \( V_{\gamma}^* \geq V_p^* \).

**Proof.** We use the concavity of the value functions demonstrated in Property 4 in conjunction with the Jensen’s inequality:

\[
V_p^*(x) = \sum_{k \in [K]} (\pi_k^* (x) \bar{r}_k) + \gamma E_{x' \sim P_k(x, \pi^*_k(x))} V_p^*(x') \\
\leq \sum_{k \in [K]} (\pi_k^* (x) \bar{r}_k) + \gamma V_p(\bar{E}_{x' \sim P_k(x, \pi^*_k(x))} x')
\]

\[
V_p^*(x_t) \leq \sum_{k \in [K]} (\pi_k^* (x_t) \bar{r}_k) + \gamma V(x_{t+1}) \quad \text{with} \ x_{t+1} = \bar{E}_{x' \sim P_k(x_t, \pi^*_k(x_t))} x' \\
V_p^*(x_0) \leq \sum_{t=0}^{\infty} \gamma^t \sum_{k \in [K]} (\pi_k^* (x_t) \bar{r}_k) \quad \text{with} \ x_{t+1} = \bar{E}_{x' \sim P_k(x_t, \pi^*_k(x_t))} x' \\
= V_{\gamma}^*(x_0)
\]

\[\square\]
Corollary 2 (cross-optimal values comparison). \( V^*_p \geq V^*_r \).

**Proof.** By construction, we have \( V^*_p(x) \geq V^*_r(x) \), and from Corollary 1, we have \( V^*_p(x) \geq V^*_p(x) \). This concludes the proof. \( \square \)

**Corollary 3 (Pseudo-lipschitzness).**

\[
\forall x > x' \geq 0, \quad V^*_p(x) - V^*_p(x') \leq \frac{x - x'}{x} V^*_p(x),
\]

(57)

**Proof.** This is a direct consequence of the concavity property demonstrated in Property 4 and the trivial fact that \( V^*_p(0) = 0 \):

\[
\forall 0 \leq \lambda \leq 1, \quad V^*_p(\lambda x) \geq \lambda V^*_p(x) + (1 - \lambda)V^*_p(0)
\]

(58)

\[
\geq \lambda V^*_p(x)
\]

(59)

\[
\Leftrightarrow \forall 0 \leq x' \leq x, \quad V^*_p(x') \geq \frac{x'}{x} V^*_p(x)
\]

(60)

\[
\Leftrightarrow V^*_p(x) - V^*_p(x') \leq V^*_p(x) - \frac{x'}{x} V^*_p(x)
\]

(61)

\[
= \frac{x - x'}{x} V^*_p(x),
\]

(62)

which concludes the proof. \( \square \)

In order to give a value to \( \pi^* \) in the ROeMDP, we extend \( \pi^* \) to the domain of definition of the ROeMDP as the interpolation to its closest integer values:

\[
\pi^*(x) = (\lceil x \rceil - x)\pi^*(\lfloor x \rfloor) + (x - \lfloor x \rfloor)\pi^*(\lceil x \rceil).
\]

(63)

**Lemma 1 (Optimal value error upper bound in the PMDP when \( \max_{g \geq 1} \mathcal{R}(g) < 0 \)).**

If \( \max_{g \geq 1} \mathcal{R}(g) < 0 \),

\[
V^*_p(x) - V^*_p(x) \leq e^{-s_0(x_r-x_0)} (x_r + g_r) \max_{k \in [K], s.t. \pi_k < 1} \frac{\pi_k}{1 - \theta_k},
\]

(64)

where \( s_0 \) is a constant, as defined in Lemma 2.

**Proof.** We know from Theorem 1 that \( \pi^* \) is constantly selecting a single arm when \( \max_{g \geq 1} \mathcal{R}(g) \leq 0 \). As a consequence, and since there is no discounting (\( \gamma = 1 \)), we may consider that the batches are of size 1. Let \( k_\theta \) be this arm, \( \theta_\theta \) its expected growth and \( \theta_\theta \) its expected reward (we set \( \gamma = 1 \)). Let \( G_{x_0,x_r} \) be the random variable of the sum of rewards \( R_t \) collected during the process starting from crowd \( x_0 \) with maximal crowd \( x_r \geq x_0 \). If \( \pi_\theta > 0 \), then we have\(^4\):

\[
V^*_p(x) - V^*_p(x) = \mathbb{E}[G_{x_0,\infty}] - \mathbb{E}[G_{x_0,x_r}]
\]

(65)

\[
= \mathbb{E}[G_{x_0,\infty}] - \mathbb{E}[G_{x_0,x_r}]
\]

(66)

\[
= \mathbb{P}(\exists t, s.t. X_{T_r} > x_r | X_{0} = x_0) \mathbb{E} \left[ \sum_{t=0}^{T_r-1} R_t + G_{X_{T_r},\infty} - \sum_{t=0}^{T_r-1} R_t + G_{x_r,x_r} \right]
\]

(67)

\[
= \mathbb{P}(\exists t, s.t. X_{T_r} > x_r | X_{0} = x_0) \left( \mathbb{E} [G_{X_{T_r},\infty}] - \mathbb{E} [G_{x_r,x_r}] \right)
\]

(68)

\[
\leq e^{-s_0(x_r-x_0)} \mathbb{E}[X_{T_r}] \frac{\pi_\theta}{1 - \theta_\theta}
\]

(69)

which concludes the proof. \( \square \)

\(^4\)The inequalities are reversed if \( \pi_\theta < 0 \), and \( V^*_p(x) - V^*_p(x) < 0 \).
Lemma 2 (Probability to exceed crowd under decreasing regime). Let \((\xi_{t,i})_{t \geq 0, i \geq 1}\) be a family of iid copies of a random variable \(\xi\) taking values in \(\mathbb{N}\), and not concentrated on \(\{0, 1\}\). Let \((X_t)_{t \geq 0}\) be such that

\[X_{t+1} = \sum_{i=1}^{X_t} \xi_{t,i}.
\]

Suppose that \(\mathbb{E}[\xi] = m < 1\) and that \(\mathbb{E}[e^{s\xi}] < \infty\) for every \(s \geq 0\). Then, there exists a unique \(s_0 > 0\) such that \(\mathbb{E}[e^{s_0 \xi}] = e^{s_0}\), and we have

\[
P(\exists t, \text{ s.t. } X_t > x_\tau | X_0 = x_0) \leq e^{-s_0(x_\tau - x_0)}.
\]

In particular, if \(\xi\) follows a geometric distribution, which is a commonly used law for modeling propagation of disease/information, we find that \(s_0 = -\ln m\), and therefore that:

\[
P(\exists t, \text{ s.t. } X_t > x_\tau | X_0 = x_0) \leq m^{x_\tau - x_0}.
\]

Proof. Let \(\Lambda\) denote the cumulant generating function of \(\xi\), that is \(\Lambda(s) = \ln \mathbb{E}[e^{s\xi}]\), which exists and is finite for each \(s \geq 0\) by assumption. Then \(\Lambda(0) = 0\), \(\Lambda\) is continuous and convex. Note also that a Taylor expansion at \(s \to 0\) yields \(\Lambda'(0) = \mathbb{E}[\xi] = m\). Furthermore, since \(\mathbb{P}(\xi \geq 2) \geq \epsilon > 0\), we have \(\Lambda(s) \geq \ln(\epsilon e^{2s})\) and therefore

\[
\Delta(s) - s \geq \ln (\epsilon e^{2s}) - s = s + \ln \epsilon,
\]

which implies that \(\Delta(s) - s \to +\infty\) as \(s \to \infty\). It follows that the equation \(\Lambda(s) = s\) has aside from the trivial solution \(s = 0\), a unique positive solution \(s_0\), as claimed.

In particular, we have \(\mathbb{E}[e^{s_0\xi}] = e^{s_0}\), which allows us to construct a martingale as follows. For \(t \in \mathbb{N}\), let \(M_t = e^{s_0 X_t}\). Then, for each \(t\), \(M_t\) is integrable and

\[
\mathbb{E}[M_{t+1} | X_0, \ldots, X_t] = \mathbb{E}
\left[
\frac{e^{s_0 \sum_{k=0}^{X_t} \xi_{t,k}}}{\mathbb{E}[e^{s_0 \xi_{t,k}}]} | X_0, \ldots, X_t
\right] = \prod_{k=1}^{X_t} \mathbb{E}[e^{s_0 \xi_{t,k}}] = \prod_{k=1}^{X_t} e^{s_0} = M_t,
\]

so that \((M_t)_{t \geq 0}\) is a martingale.

Let now \(T_\tau = \inf\{n, \text{ s.t. } X_n > x_\tau\}\). Then \(T_\tau\) is a stopping time, so the stopped process \((M_{\min(T_\tau, t)})_{t \geq 0}\) is also a martingale. On the one hand, we have \(\mathbb{E}[M_{\min(T_\tau, 0)}] = e^{s_0 x_0}\), and on the other hand, by the martingale property, for any \(t \geq 0\):

\[
e^{s_0 x_0} = \mathbb{E}[M_{\min(T_\tau, 0)}] = \mathbb{E}[M_{\min(T_\tau, t)}] = \mathbb{E}[M_{\min(T_\tau, t)} 1(T_\tau = +\infty)] + \mathbb{E}[M_{\min(T_\tau, t)} 1(T_\tau < +\infty)].
\]

The process \(M_{\min(T_\tau, t)}\) is bounded (by \(e^{s_0 X_\tau}\)) and therefore, since \(X_t \to 0\) almost surely as \(t \to \infty\) since the branching process is subcritical (\(\mathbb{E}[\xi] = m < 1\), see (Athreya and Ney, 1972)), it follows that

\[
\mathbb{E}[M_{\min(T_\tau, t)} 1(T_\tau = +\infty)] = \mathbb{E}[M_t 1(T_\tau = +\infty)] \xrightarrow{t \to \infty} \mathbb{P}(T_\tau = +\infty).
\]

Moreover since \(M_{\min(T_\tau, t)} = e^{s_0 X_\tau}\) for \(t \geq T_\tau\), we have

\[
\lim_{t \to +\infty} \mathbb{E}[M_{\min(T_\tau, t)} 1(T_\tau < +\infty)] \geq \mathbb{P}(T_\tau < +\infty) e^{s_0 X_\tau}.
\]

Injecting the above two inequalities in (70) yields:

\[
P_{x_0}(\exists t : X_t > x_\tau) = \mathbb{P}(T_\tau < +\infty) \leq \frac{e^{s_0 x_0} - 1}{e^{s_0 x_\tau} - 1} \leq e^{s_0(x_0 - x_\tau)},
\]

since \(1 \leq x_0 \leq x_\tau\). \(\square\)
Lemma 3 (Value error upper bound in the OeMDP when \( \max_g R(g) > 0 \)). Let \( \pi \) be a policy such that \( V^\pi_o(x) \) increases with \( x \in [x_1] \), then, if \( \max_g R(g) > 0 \), we have the following upper bound on the error:

\[
V^\pi_o(x) - V^\pi_P(x) \leq \frac{\gamma V^\pi_o}{1 - \gamma} \left( \frac{\gamma \zeta^x}{(1 - \gamma)^2} + \frac{2 + \dot{g} + \sqrt{\log(x)}}{2\sqrt{\gamma}} \right).
\]

Proof.

\[
\begin{align*}
V^\pi_o(x) - V^\pi_P(x) &= \gamma (V^\pi_o P_o - V^\pi_P P_p) \pi(x) \\
&= \gamma (V^\pi_o (P_o - P_p) + (V^\pi_o - V^\pi_P) P_p) \pi(x) \\
&= \gamma V^\pi_o (P_o - P_p) \pi (\mathbb{I} - \gamma P_p \pi)^{-1} (x) \\
&= \frac{\gamma}{1 - \gamma} \sum_{x' = 0}^{x_1} d^\pi_o(x, x') V^\pi_o (P_o - P_p) \pi(x'),
\end{align*}
\]

where, for brevity, \( \pi \) above returns the state-action couple resulting from the application of the policy to a given state, where \( d^\pi_o(x, \cdot) \) is the normalized discounted sum of visited states, starting from \( x \), under policy \( \pi \) in the real stochastic environment. Line 76 is obtained by moving the right-hand term to the left side of the equality and then the terms are factorized with \( V^\pi_o - V^\pi_P \), and inverted. \( \mathbb{I} - \gamma P_p \pi \) is always invertible because \( \gamma < 1 \). Line 77 is simply a rewriting of \( (\mathbb{I} - \gamma P_p \pi)^{-1} \) which sums to \( \frac{1}{1 - \gamma} \) with the discounted visitation density \( d^\pi_o(x, \cdot) \), which sums to 1.

Now, we are interested in estimating an upper bound of the term inside the sum:

\[
\begin{align*}
V^\pi_o (P_o - P_p) (x, a) &= \mathbb{E}_{x'_o \sim P_p(x, a, \cdot)} \left[ V^\pi_o (x'_o) - V^\pi_o (x'_p) \right] \\
&= \sum_{x'_o = 0}^{x_1} P_p(x, a, x'_o) (V^\pi_o (x'_o) - V^\pi_o (x'_p)),
\end{align*}
\]

where \( x'_o \) is the deterministic successor of \( x \) after executing \( a \) in the OeMDP, and where \( x'_p \sim P_p(x, a, \cdot) \) is the stochastic successor of \( x \) after executing \( a \) in the PMDP. Since the value is monotonically increasing with \( x \in [x_1] \) (by assumption), we may upper bound the error on the transitions that are under \( x'_o \):

\[
\begin{align*}
V^\pi_o (P_o - P_p) (x, a) &\leq \sum_{x'_o = 0}^{x_1} P_p(x, a, x'_o) (V^\pi_o (x'_o) - V^\pi_o (x'_p)) \\
&= \sum_{x'_o = 0}^{x_1} P_p(x, a, x'_o) (V^\pi_o (x'_o) - V^\pi_o (x'_p)) + \sum_{x'_o = x'_o - CI_1}^{x_1} P_p(x, a, x'_o) (V^\pi_o (x'_o) - V^\pi_o (x'_p)) \\
&\leq V^\pi_o (x'_o - CI_1) \exp \left( -\frac{2 CI_1^2}{x' g'^2} \right) + \sum_{x'_o = x'_o - CI_1}^{x_1} P_p(x, a, x'_o) \frac{V^\pi_o (x'_o)}{x'_o} \\
&\leq V^\pi_o (x'_o) \exp \left( -\frac{2 CI_1^2}{x' g'^2} \right) + CI_1 \sum_{x'_o = x'_o - CI_1}^{x_1} P_p(x, a, x'_o) \\
&\leq V^\pi_o (x'_o) \left( \exp \left( -\frac{2 CI_1^2}{x' g'^2} \right) + \frac{CI_1}{x} \right),
\end{align*}
\]

\footnote{For the sake of simplicity, we do not deal with the rounding errors. \( \sum_{i=x}^{y} \) with \( x \) and \( y \) real numbers will mean the sum for all \( i \in \mathbb{N} \cup [x, y] \).}
where line 81 is obtained by decomposing the sum in parts at a cutting point $x'_o - CI_1$ that is going to be determined later. Line 82 is obtained by applying Hoeffding’s bound on the first sum and Corollary 3 on the second term. Line 83 is obtained by upper bounding $x'_o - x'_p$ with $CI_1$. Line 84 is obtained because the transition kernel sums to 1. Finally, Line 85 is a simple factorization and a lower bound of $x'_o$ as $x$, since we assumed that $R(x'_o) > 0$, and Property 5 states that $\pi(x) \geq 1$ for all $x$ under this assumption. In particular, if we choose $CI_1 = \frac{\gamma}{2} \sqrt{x \ln x}$, we get:

$$V^\pi_o (P_o - P_p) (x, a) \leq V^\pi_o (x'_o) \left( \exp \left( -\frac{\ln x}{2} \right) + \frac{\dot{g} \sqrt{\ln x}}{2\sqrt{x}} \right) \tag{86}$$

$$= V^\pi_o (x'_o) \left( \frac{1}{\sqrt{x}} + \frac{\dot{g} \sqrt{\ln x}}{2\sqrt{x}} \right) \tag{87}$$

Starting back from Equation 77:

$$V^\pi_o (x) - V^\pi_p (x) = \frac{\gamma}{1 - \gamma} \int_{x_p}^{x} d^\pi_p (x, x') V^\pi_o (P_o - P_p) (x', \pi(x')) dx' \tag{88}$$

$$= \frac{\gamma}{1 - \gamma} \sum_{x' = 0}^{x - 1} d^\pi_p (x, x') V^\pi_o (P_o - P_p) (x', \pi(x')) + \frac{\gamma}{1 - \gamma} \sum_{x' = x}^{x'} d^\pi_p (x, x') V^\pi_o (P_o - P_p) (x', \pi(x'))' \tag{89}$$

$$\leq \frac{\gamma}{1 - \gamma} \sum_{x' = x}^{x} d^\pi_p (x, x') V^\pi_o (x') \left( \frac{1}{\sqrt{x}} + \frac{\dot{g} \sqrt{\ln x}}{2\sqrt{x}} \right) \tag{90}$$

$$\leq \frac{\gamma V^\pi}{1 - (1 - \gamma)^2} + \frac{\gamma V^\pi}{1 - \gamma} \left( \frac{1}{\sqrt{x}} + \frac{\dot{g} \sqrt{\ln x}}{2\sqrt{x}} \right) \sum_{x' = x}^{x} d^\pi_p (x, x') \tag{91}$$

$$\leq \frac{\gamma V^\pi}{1 - \gamma} \left( \frac{\gamma \zeta x}{(1 - \gamma)^2} + \frac{2 + \dot{g} \sqrt{\ln x}}{2\sqrt{x}} \right), \tag{92}$$

where line 89 is once more a decomposition of the sum in two parts. Line 90 replaces the result of Lemma 4 with constant $\zeta < 1$, and by injecting the result of Equation 87 inside the second sum. In line 91, we upper bound the expression by replacing $x'$ with the value that maximizes it. Finally, line 92 is a simple refactorization that concludes the proof. See below for details about $\zeta$.

$\zeta$ is a constant related to the problem and the policy resulting from solving the ROeMDP associated with it. The choice of $x'_o \leq x$; used in Lemma 4 is balance between choosing it high and such that $m = \min_{x \in [x'_o]} \pi(x)$ is large$^6$:

$$\sum_{x' = 0}^{x - 1} d^\pi_p (x, x') = \sum_{t = 0}^{\infty} g^t P(X_t < x | \pi = \pi) \tag{93}$$

$$\leq \sum_{t = 0}^{\infty} g^t P(X_t < x | m = m) \tag{94}$$

$$\leq \min_{x'_o \in [x, x_T]} \left\{ \exp \left( -x (\pi(x'_o) - 1)^3 \right) \right\} \tag{95}$$

$$\leq \min_{x'_o \in [x, x_T]} \left\{ \exp \left( -x (\pi(x'_o) - 1)^3 \right) \right\} \tag{96}$$

$$\leq \max_{x'_o \in [x, x_T]} \left\{ \exp \left( -x (\pi(x'_o) - 1)^3 \right) \right\} \tag{97}$$

$^6$Actually $m$ is the expectation the minimum over the random variables, and not their mean.
If \( x \) gets too close from \( x_{\tau} \), it may happen that \( x' \) gets constrained by it being larger than \( x \). In this case, one may choose \( x' \) smaller and replace \( x \) with \( x' \) in the exponentiation of \( \zeta \). This detail is omitted in the main result for the sake of conciseness.

**Lemma 4** (Discounted time under threshold). We consider the following process:

\[
\begin{align*}
X_0 &= x_0 \\
X_t &= \left\{ \sum_{i=1}^{X_t} \xi_{t,i} \right\} \land x_{\tau},
\end{align*}
\]

where \((\xi_{t,i})_{t \geq 0, i \geq 1}\) are iid copies of a random variable \( \xi \) with expected value \( \mathbb{E}[\xi] = m > 1 \) and finite variance \( \sigma^2 \). Then, for any \( x_1 \in \{0, \ldots, x_0\} \) we have

\[
\sum_{t=0}^{\infty} \gamma^t \mathbb{P}(X_t < x_1) \leq \frac{\gamma}{(1-\gamma)^2} \left\{ \exp \left( -\frac{x_0(m-1)^3}{2m(m-1)+\sigma^2} \right) + \exp \left( -\frac{x_1(m-1)^2}{4(\sigma^2+m^2)} \right) \right\}.
\]

**Proof.** The following version of the process where the upper bound has been dropped will be useful:

\[
\begin{align*}
X_0' &= x_0 \\
X_t' &= \left\{ \sum_{i=1}^{X_t'} \xi_{t,i} \right\} \land x_{\tau} \quad \text{with} \quad \mathbb{E}[\xi] = m > 1
\end{align*}
\]

One can couple \((X_t)\) and \((X_t')\) in such a way that \( X_t = X_t' \) as long as \( X_t' \leq x_{\tau} \), that is for every \( t \leq T := \inf \{ t' \geq 0 : X_t' > x_{\tau} \} \). In the following, we write \( \mathbb{P}_x(\cdot) = \mathbb{P}(\cdot | X_0 = x) \). Straightforward computation and the strong Markov property used at time \( T \) yields

\[
\begin{align*}
\sum_{t=0}^{\infty} \gamma^t \mathbb{P}_x(X_t < x_1) &= \sum_{t=0}^{\infty} \mathbb{E}_{x_0} \left[ \gamma^t \mathbbm{1}_{X_t < x_1} \right] \\
&= \mathbb{E}_{x_0} \left[ \sum_{t=0}^{\infty} \gamma^t \mathbbm{1}_{X_t < x_1} \right] \\
&= \mathbb{E}_{x_0} \left[ \sum_{t=0}^{T-1} \gamma^t \mathbbm{1}_{X_t < x_1} + \sum_{t=T}^{\infty} \gamma^t \mathbbm{1}_{X_t < x_1} \right] \\
&= \mathbb{E}_{x_0} \left[ \sum_{t=0}^{T-1} \gamma^t \mathbbm{1}_{X_t < x_1} + \sum_{t=T}^{\infty} \gamma^t \mathbbm{1}_{X_t < x_1} \right] \\
&\leq \mathbb{E}_{x_0} \left[ \sum_{t=0}^{\infty} \gamma^t \mathbbm{1}_{X_t < x_1} \right] + \mathbb{E}_{x_0} \left[ \sum_{t=0}^{\infty} \gamma^t \mathbbm{1}_{X_t < x_1} \right] \\
&\leq \sum_{t=0}^{\infty} \gamma^t \mathbb{P}_x(X_t' < x_1) + \mathbb{E}_{x_0} \left[ \gamma^T \mathbb{E}_{x_1} \left[ \sum_{t=0}^{\infty} \gamma^t \mathbbm{1}_{X_t < x_1} \right] \right] \\
&\leq \sum_{t=0}^{\infty} \gamma^t \mathbb{P}_x(X_t' < x_1) + \mathbb{E}_{x_1} \left[ \sum_{t=0}^{\infty} \gamma^t \mathbbm{1}_{X_t < x_1} \right].
\end{align*}
\]

First, we deal with the first term where we have a sum of \( x_0 \) independent random variables \( Z_t \) defined as follows:

\[
\begin{align*}
Z_0 &= 1 \\
Z_{t+1} &= \sum_{i=1}^{Z_t} \xi_{t,i}.
\end{align*}
\]

In particular, for each \( t \geq 1 \), we have \( \mathbb{E}Z_t = m \mathbb{E}Z_{t-1} = m^t \), and \( Z_t \geq 0 \). Then, writing \((Z^{(j)}_t)_{1 \leq j \leq x_0}\) for \( x_0 \) iid copies of \((Z_t)_{t \geq 0}\), we have

\[
X_t' = \sum_{j=1}^{x_0} Z^{(j)}_t.
\]
For summands distributed like $Z_t$, we have one-sided Bernstein concentration inequality:

\[
P_{x_0}(X'_t < x_\gamma) = P \left( \sum_{j=1}^{x_0} Z^{(j)}_t < x_\gamma \right)
\]

\[
= P \left( \sum_{j=1}^{x_0} \left( Z^{(j)}_t - \mathbb{E} Z_t \right) < x_\gamma - x_0 m^t \right)
\]

\[
\leq \exp \left( -\frac{(x_0 m^t - x_\gamma)^2}{2 x_0 v_t} \right),
\]

where $v_t = \mathbb{E}[Z^2_t] = \text{Var}(Z_t) + \mathbb{E}[Z_t]^2$. It is standard (Athreya and Ney, 1972) that, when $m \neq 1$ and writing $\sigma^2 = \text{Var}(\xi)$, we have

\[
v_t = \frac{\sigma^2}{m-1} + m^2
\]

\[
= m^2 \left( 1 + \frac{\sigma^2}{m(m-1)} (1 - m^{-t}) \right)
\]

\[
\leq m^2 \left( 1 + \frac{\sigma^2}{m(m-1)} \right).
\]

It follows that

\[
P_{x_0}(X'_t < x_\gamma) \leq \exp \left( -\frac{(x_0 (m^t - 1) + x_0 - x_\gamma)^2}{2 x_0 v_t} \right)
\]

\[
= \exp \left( -x_0 \frac{(m^t - 1)^2}{2 v_t} \right) \times \exp \left( -\frac{(m^t - 1)(x_0 - x_\gamma)}{v_t} \right) \times \exp \left( -\frac{(x_0 - x_\gamma)^2}{2 x_0 v_t} \right).
\]

Note that, the bound in (112) implies that

\[
\exp \left( -x_0 \frac{(m^t - 1)^2}{2 v_t} \right) \leq \exp \left( -x_0 \frac{m(m-1)(1-m^{-t})^2}{2(m(m-1) + \sigma^2)} \right)
\]

\[
\leq \exp \left( -x_0 \frac{(m-1)^3}{2 m(m-1) + \sigma^2} \right),
\]

for all $t \geq 1$, where the second line is obtained because $1 - m^{-t} > 1 - m^{-1} = \frac{m-1}{m}$. The second and third factors in (115) are smaller than 1 and dropped. We deduce that

\[
\sum_{t \geq 0} \gamma^t P_{x_0}(X'_t < x_\gamma) \leq \frac{\gamma}{1 - \gamma} \exp \left( -x_0 \frac{(m-1)^3}{2 m(m-1) + \sigma^2} \right).
\]

We now move on to the bound on the second term of (106). In order to deal with the push down resulting from the upper bound at $x_\gamma$, we proceed as follows. We assume here for simplicity that $x_\gamma > 2 x_\gamma$; otherwise, we always have $x_\gamma > \kappa x_\gamma$ for some $\kappa > 1$ and we replace 2 by $\kappa$ in the following definitions. Let $x_m := \lfloor x_\gamma / 2 \rfloor$. Define the auxiliary process $X''_0 = X_0$ and

\[
X''_{t+1} = \sum_{i=1}^{x_m} \xi_t, i.
\]

Let $T_{m} := \inf \{ t \geq 0 : X''_t < x_m \}$. Then, since dropping some individuals only decreases the population, for each $t < T_{m}$, we have $X_t \geq X''_t \land x_\gamma$. Furthermore, the random variables $(X''_t)_{t \geq 1}$ are actually independent and identically distributed. Note that

\[
P(X_t < x_\gamma) \leq P(X_t < x_m)
\]

\[
\leq P(X_t < x_m, X'_k \geq x_m \forall k \leq t) + P(\exists k \leq t : X'_k < x_m)
\]

\[
\leq P(\exists k \leq t : X''_k < x_m).
\]
It follows easily by the union bound and Bernstein’s one-sided inequality that
\[ P(X_t < x) \leq t \cdot P \left( x_m \sum_{i=1}^{m} \xi_t < x_m \right) \]
\[ = t \cdot P \left( x_m \sum_{i=1}^{m} (\xi_t - m) < x_m (1 - m) \right) \]  
\[ \leq t \exp \left( -\frac{x_m^2 (m - 1)^2}{2v} \right) \]
\[ = t \exp \left( -\frac{x_m (m - 1)^2}{2v} \right) . \]

where \( v = \mathbb{E}[\xi^2] = \sigma^2 + m^2 \). The second term of the right-hand side of (106) is therefore such that
\[ E_{x_t} \left[ \sum_{t \geq 0} \gamma^t I_{X_t < x} \right] = \sum_{t \geq 0} \gamma^t P(X_t < x) \leq \exp \left( -\frac{x_m (m - 1)^2}{2v} \right) \sum_{t \geq 0} t \gamma^t \]
\[ = \frac{\gamma}{(1 - \gamma)^2} \exp \left( -\frac{x_m (m - 1)^2}{2v} \right) . \]

Putting (118) and (127) together and rejoining the expressions for \( v_t \) and \( v \) yields (99). \( \square \)

**Lemma 5** (Upper convex envelop reward divergence). Let \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) be two reduced reward functions defined on the same interval \([\bar{g}, \bar{g}^*] \). Let \( V_1^\pi \) and \( V_2^\pi \) be the values of some policy \( \pi \) in the ROeMDPs respectively induced by \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \). Then, we have:
\[ \| V_1^\pi - V_2^\pi \|_\infty \leq \frac{x^\pi}{1 - \gamma} \| \mathcal{R}_1 - \mathcal{R}_2 \|_\infty \]

**Proof.** The dynamics are not affected by the upper convex envelop reward divergence: \( \forall t, x_{t+1} = \min(x_t \pi(x_t), x_t) \). As a consequence, the value error is the discounted sum of errors made on the rewards:
\[ V_1^\pi(x_0) - V_2^\pi(x_0) = x_0 \mathcal{R}_1(\pi(x_0)) - x_0 \mathcal{R}_2(\pi(x_0)) + \gamma V_1^{\pi}(x_1) - \gamma V_2^{\pi}(x_1) \]
\[ = \sum_{t=0}^{\infty} x_t \gamma^t (\mathcal{R}_1(\pi(x_t)) - \mathcal{R}_2(\pi(x_t))) \]
\[ \leq \sum_{t=0}^{\infty} x_t \gamma^t \| \mathcal{R}_1 - \mathcal{R}_2 \|_\infty \]
\[ = \frac{x^\pi}{1 - \gamma} \| \mathcal{R}_1 - \mathcal{R}_2 \|_\infty \]

**Lemma 6** (Upper convex envelop domain divergence). Let \( \mathcal{R} \) be a reduced reward function defined on \([\bar{g}, \bar{g}^*] \). Let \( \mathcal{M} \) be the ROeMDP induced by \( \mathcal{R} \) on action set \( \mathcal{G} = [\bar{g}, \bar{g}^*] \) and \( \widetilde{\mathcal{M}} \) be the ROeMDP induced by \( \mathcal{R} \) on action set \( \tilde{\mathcal{G}} = [\tilde{g}, \tilde{g}^*] \subset \mathcal{G} \). Let \( \tilde{V}_1^\pi \) and \( \tilde{V}_2^\pi \) be the respective optimal values in \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \). Then, we have the optimal value error \( \tilde{V}_1^\pi(x) - \tilde{V}_2^\pi(x) \) that decreases linearly with upper convex envelop domain divergence: \( \tilde{g} - \bar{g} \) and \( \tilde{g} - \bar{g}^* \).

**Proof.** Note that, by convexity assumption, we know that:
\[ \mathcal{R}(\bar{g}) - \mathcal{R}(\tilde{g}) \leq \alpha_+ (\bar{g} - \tilde{g}) \quad \text{where} \quad \mathcal{R}(g) = \alpha_+ g + \beta_+ \quad \text{in} \quad \text{the upper vicinity of} \quad \tilde{g} \]
\[ \mathcal{R}(\bar{g}) - \mathcal{R}(\tilde{g}) \leq \alpha_- (\bar{g} - \tilde{g}) \quad \text{where} \quad \mathcal{R}(g) = \alpha_- g + \beta_- \quad \text{in} \quad \text{the lower vicinity of} \quad \tilde{g} \]
We split the proof in two cases: (I) when $\max_{g \geq 1} R(g) \leq 0$, and (II) when $\max_{g \geq 1} R(g) > 0$.

(I) When $\max_{g \geq 1} R(g) \leq 0$, according to Theorem 1, the optimal values are:

$$V^*_r(x) = x \frac{R(g_*)}{1 - \gamma g_*} \quad \text{with} \quad g_* = \arg\max_{g \in \mathcal{G} \cap [0,1]} \frac{R(g)}{1 - \gamma g}$$

$$\hat{V}^*_r(x) = x \max_{g \in \mathcal{G} \cap [0,1]} \frac{R(g)}{1 - \gamma g}$$

$$V^*_r(x) - \hat{V}^*_r(x) \leq x \max \left\{ \begin{array}{ll}
\frac{R(g_*)}{1 - \gamma g_*} - \frac{R(\hat{g}_*)}{1 - \gamma \hat{g}_*} & \text{case: } g_* < \hat{g}_* \\
0 & \text{case: } g_* = \hat{g}_* \\
\frac{R(g_*)}{1 - \gamma g_*} - \frac{R(\hat{g}_*)}{1 - \gamma \hat{g}_*} & \text{case: } g_* > \hat{g}_*
\end{array} \right\}$$

Below, we unfold for the case $g_* > \hat{g}_*$, but the same may be identically done for the case $g_* < \hat{g}_*$:

$$\frac{R(g_*)}{1 - \gamma g_*} - \frac{R(\hat{g}_*)}{1 - \gamma \hat{g}_*} = \frac{(1 - \gamma g_*)R(g_*) - (1 - \gamma \hat{g}_*)R(\hat{g}_*)}{(1 - \gamma g_*)(1 - \gamma \hat{g}_*)}$$

$$= \frac{R(g_*) - R(\hat{g}_*) + \gamma (g_* - \hat{g}_*) R(\hat{g}_*) - \gamma g_* R(g_*)}{(1 - \gamma g_*)(1 - \gamma \hat{g}_*)}$$

$$= \frac{R(g_*) - R(\hat{g}_*) + \gamma (g_* - \hat{g}_*) R(\hat{g}_*) + \gamma \hat{g}_* (R(\hat{g}_*) - R(g_*))}{(1 - \gamma g_*)(1 - \gamma \hat{g}_*)}$$

$$= \frac{(1 - \gamma \hat{g}_*) (R(\hat{g}_*) - R(g_*)) + \gamma (g_* - \hat{g}_*) R(\hat{g}_*)}{(1 - \gamma g_*)(1 - \gamma \hat{g}_*)}$$

$$\leq \frac{(1 - \gamma \hat{g}_*) \alpha_* (g_* - \hat{g}_*) + \gamma (g_* - \hat{g}_*) R(\hat{g}_*)}{(1 - \gamma g_*)(1 - \gamma \hat{g}_*)}$$

$$= \frac{(1 - \gamma \hat{g}_*) \alpha_* + \gamma R(\hat{g}_*) (g_* - \hat{g}_*)}{(1 - \gamma g_*)(1 - \gamma \hat{g}_*)}$$

Reinjecting in Equation 139, we have the following upper bound for $V^*_r(x) - \hat{V}^*_r(x)$:

$$x \max \left\{ 0, \frac{(1 - \gamma \hat{g}_*) \alpha_* + \gamma R(\hat{g}_*) (g_* - \hat{g}_*)}{(1 - \gamma g_*)(1 - \gamma \hat{g}_*)}, \frac{(1 - \gamma \hat{g}_*) \alpha_* + \gamma R(\hat{g}_*) (g_* - \hat{g}_*)}{(1 - \gamma g_*)(1 - \gamma \hat{g}_*)}, \frac{(1 - \gamma g_*)}{1 - \gamma g} + \frac{\gamma \epsilon}{(1 - \gamma g)^2} + o(\epsilon^2) \right\}.$$

We observe that it is linear with $g_* - \hat{g}_*$, but with a constant that is not really one, since it depends on both $\mathcal{G}_{\leq 1}$ and $\hat{M}_r$.

We can further make replacement of either $\mathcal{G}_{\leq 1}$ or $\hat{M}_r$ with $\epsilon_{r+1} = \mathcal{G}_{\leq 1} - \hat{M}_r$, and use the following Taylor expansion to prove that the expression remains linear in $\epsilon_{r+1}$:

$$\frac{1}{1 - \gamma (g - \epsilon)} = \frac{1}{1 - \gamma g} + \frac{\gamma \epsilon}{(1 - \gamma g)^2} + o(\epsilon^2),$$

which concludes the first part of the Lemma.

(II) When $\max_{g \geq 1} R(g) > 0$, we may choose $\gamma$ such that $\max_{g \geq 1} \frac{1}{2} R(g) > 0$. In this case, we can observe that the worst case scenario happens when $\alpha_* > 0^*$ and $R(\mathcal{G}) - R(\hat{g}_*) = \alpha_* (\mathcal{G} - \hat{g}_*)$. This worst case scenario is easy to solve since Property 4 states that when $\max_{g \geq 1} R(g) > 0$, the optimal policy is decreasing with time and until reaching $x_\gamma$ when the optimal is to play $\arg\max_{g \geq 1} \frac{1}{2} R(g)$, which in our worst case scenario equals $\mathcal{G}_\gamma$ in $M$ and $\hat{g}_\gamma$ in $\hat{M}$. We may conclude that $\forall x, \pi^*(x) = \mathcal{G}_\gamma$ and $\hat{\pi}^*(x) = \hat{g}_\gamma$. With this information, we can compute the difference in value $V^*_r(x) - \hat{V}^*_r(x)$:

$$\sum_{t=0}^{t-1} x(\mathcal{G}_\gamma)^t R(\mathcal{G}_\gamma) + \sum_{t=t+1}^{\infty} x(\gamma \mathcal{G}_\gamma)^t R(\mathcal{G}_\gamma) - \sum_{t=0}^{t-1} x(\gamma \hat{g}_\gamma)^t R(\hat{g}_\gamma) - \sum_{t=t+1}^{\infty} x(\gamma \hat{g}_\gamma)^t R(\hat{g}_\gamma),$$

We could break down various cases to improve the constants of the bounds depending on each specific case, but we considered that it complicates the proof while the interesting part of the theorem is that the value decays linearly with the upper convex envelop domain divergence.
where \( t_i \) (resp. \( \hat{t}_i \)) is the time to reach the maximal state \( x_i \):

\[
t_i = \frac{\ln \frac{x_i}{\gamma}}{\ln \gamma} \quad \text{and} \quad \hat{t}_i = \frac{\ln \frac{x_i}{\gamma}}{\ln \gamma},
\]

(149)

We proceed as follows to estimate \( V_r^*(x) - \hat{V}_r^*(x) \):

\[
\begin{align*}
&\leq \sum_{t=0}^{\hat{t}_i} x(\gamma g_i)^t R(\gamma g_i) + \sum_{t=0}^{\hat{t}_i-1} x(\gamma g_i)^t R(\gamma g_i) - \sum_{t=0}^{\hat{t}_i-1} x(\gamma g_i)^t R(\gamma g_i) - \sum_{t=0}^{\hat{t}_i-1} x(\gamma g_i)^t R(\gamma g_i) \\
&= \sum_{t=0}^{\hat{t}_i-1} x(\gamma g_i)^t (R(\gamma g_i) - \hat{g}_i^t R(\gamma g_i)) + \sum_{t=0}^{\hat{t}_i-1} x(\gamma g_i)^t (R(\gamma g_i) - \hat{g}_i^t R(\gamma g_i)) \\
&= \sum_{t=0}^{\hat{t}_i-1} x(\gamma g_i)^t (R(\gamma g_i) - \hat{g}_i^t R(\gamma g_i)) + \sum_{t=0}^{\hat{t}_i-1} x(\gamma g_i)^t (R(\gamma g_i) - \hat{g}_i^t R(\gamma g_i)) + \frac{x_i \alpha_i \gamma^{\hat{t}_i}}{1 - \gamma} (\gamma g_i - \hat{g}_i) \\
&= \sum_{t=0}^{\hat{t}_i-1} x(\gamma g_i)^t (R(\gamma g_i) - \hat{g}_i^t R(\gamma g_i)) + \frac{x_i \alpha_i \gamma^{\hat{t}_i}}{1 - \gamma} (\gamma g_i - \hat{g}_i) \\
&= \sum_{t=0}^{\hat{t}_i-1} x(\gamma g_i)^t (R(\gamma g_i) - \hat{g}_i^t R(\gamma g_i)) + \frac{x_i \alpha_i \gamma^{\hat{t}_i}}{1 - \gamma} (\gamma g_i - \hat{g}_i).
\end{align*}
\]

(150)

The sum inside the first term requires a bit of work:

\[
\begin{align*}
&\sum_{t=0}^{\hat{t}_i-1} \gamma^{t-1} (\hat{g}_i^t - \hat{g}_i^t) = \sum_{i=0}^{\hat{t}_i-1} \gamma^{t-1} (\hat{g}_i^t - \hat{g}_i^t) \\
&\leq \sum_{i=0}^{\hat{t}_i-1} \gamma^{t-1} (\hat{g}_i^t - \hat{g}_i^t) \\
&= (\gamma g_i - \hat{g}_i) \sum_{t=0}^{\hat{t}_i-1} t(\gamma g_i)^{t-1} \\
&= (\gamma g_i - \hat{g}_i) \frac{\partial (\gamma g_i)^t}{\partial (\gamma g_i)} \\
&= (\gamma g_i - \hat{g}_i) \frac{\partial \left( \sum_{i=0}^{\hat{t}_i-1} (\gamma g_i)^t \right)}{\partial (\gamma g_i)} \\
&= (\gamma g_i - \hat{g}_i) \frac{\partial \left( \frac{(\gamma g_i)^{\hat{t}_i-1}}{\gamma g_i - 1} \right)}{\partial (\gamma g_i)} \\
&= (\gamma g_i - \hat{g}_i) \frac{\partial \left( \frac{(\gamma g_i)^{\hat{t}_i-1}}{\gamma g_i - 1} \right)}{\partial (\gamma g_i)} \\
&= (\gamma g_i - \hat{g}_i) \frac{\gamma^{\hat{t}_i-1} \gamma g_i - \gamma g_i - \hat{g}_i}{(\gamma g_i - 1)^2} \\
&\leq (\gamma g_i - \hat{g}_i) \frac{\gamma^{\hat{t}_i-1} (\gamma g_i - 1)^2}{(\gamma g_i - 1)^2}.
\end{align*}
\]

(155)

(156)

(157)

(158)

(159)

(160)

(161)

(162)

which allows us to conclude the second part of the proof by reinjecting this expression into the optimal value error (Equation 154). Also, we may notice that \( \hat{g}_i^{\hat{t}_i} = \frac{x_i}{\gamma} \):

\[
V_r^*(x) - \hat{V}_r^*(x) \leq \left( x(\gamma g_i)^{\hat{t}_i} (\gamma g_i)^{\hat{t}_i} + \frac{x_i \gamma^{\hat{t}_i-1}}{\gamma g_i - 1} \alpha_i + \frac{x_i \alpha_i \gamma^{\hat{t}_i}}{1 - \gamma} \right) (\gamma g_i - \hat{g}_i).
\]

(163)

For the sake of simplicity, we treat them as integer.
We observe that it is linear with $\bar{g}_r - \hat{g}_r$, but with a constant that is not really one, since it depends on both $\bar{g}_r$ and $\hat{g}_r$. We can still further make replacement of either $\bar{g}_r$ or $\hat{g}_r$ with $\epsilon = |\bar{g}_r - \hat{g}_r|$, and use the following Taylor expansion to prove that the expression remains linear in $\epsilon$:

\[
\frac{1}{\gamma \bar{g}_r - 1} = \frac{1}{\gamma (\bar{g}_r - \epsilon) - 1} = \frac{1}{\gamma \bar{g}_r - 1} + \frac{\gamma \epsilon}{(\gamma \bar{g}_r - 1)^2} + O(\epsilon^2).
\]  

(164)

Only $\hat{t}_r$ remains an uncontrolled variable for the moment. From 149, it is direct that:

\[
\hat{t}_r = \frac{\ln \bar{g}_r}{\ln \hat{g}_r} t_r = t_r \left(1 + \frac{\ln \left(1 + \frac{\epsilon}{\bar{g}_r} \right)}{\ln \hat{g}_r} \right)
\]

(165)

\[
\leq t_r \left(1 + \frac{\epsilon}{\bar{g}_r \ln \hat{g}_r} \right).
\]

(166)

\[
\leq t_r \left(1 + \frac{\epsilon}{\bar{g}_r \ln \hat{g}_r} + O(\epsilon^2) \right).
\]

(167)

Injecting it back to Equation 163, we obtain for $\frac{1}{\epsilon} \left(V^*_r(x) - \hat{V}^*_r(x)\right)$:

\[
\leq x \gamma R(\bar{g}_r) \frac{t_r \left(1 + \frac{\epsilon}{\bar{g}_r \ln \hat{g}_r} + O(\epsilon^2) \right) (\gamma \bar{g}_r)^\alpha \left(\gamma \hat{g}_r\right)^{\gamma \epsilon t_r \ln \hat{g}_r} + O(\epsilon^2)}{(\gamma \bar{g}_r - 1)^2} + x \frac{\gamma^{\alpha t_r \frac{x \epsilon}{\bar{g}_r}} \frac{\alpha t_r \frac{x \epsilon}{\bar{g}_r} - 1}{\gamma \bar{g}_r - 1}}{1 - \gamma} + x \frac{\alpha t_r \frac{x \epsilon}{\bar{g}_r}}{1 - \gamma} + O(\epsilon).
\]

(169)

\[
\leq x \gamma R(\bar{g}_r) \frac{t_r \left(1 + O(\epsilon) \right) (\gamma \bar{g}_r)^\alpha \left(\gamma \hat{g}_r\right)^{\gamma \epsilon t_r \ln \hat{g}_r} + O(\epsilon^2)}{(\gamma \bar{g}_r - 1)^2} + x \frac{\gamma^{\alpha t_r \frac{x \epsilon}{\bar{g}_r}} \frac{\alpha t_r \frac{x \epsilon}{\bar{g}_r} - 1}{\gamma \bar{g}_r - 1}}{1 - \gamma} + x \frac{\alpha t_r \frac{x \epsilon}{\bar{g}_r}}{1 - \gamma} + O(\epsilon).
\]

(170)

\[
\leq x \gamma R(\bar{g}_r) \frac{t_r \left(\gamma \bar{g}_r\right)^\alpha \left(\gamma \hat{g}_r\right)^{\gamma \epsilon t_r \ln \hat{g}_r} + O(\epsilon^2)}{(\gamma \bar{g}_r - 1)^2} + x \frac{\gamma^{\alpha t_r \frac{x \epsilon}{\bar{g}_r}} \frac{\alpha t_r \frac{x \epsilon}{\bar{g}_r} - 1}{\gamma \bar{g}_r - 1}}{1 - \gamma} + x \frac{\alpha t_r \frac{x \epsilon}{\bar{g}_r}}{1 - \gamma} + O(\epsilon).
\]

(171)

\[
\leq x \gamma R(\bar{g}_r) \frac{t_r \left(\gamma \bar{g}_r\right)^\alpha \left(\gamma \hat{g}_r\right)^{\gamma \epsilon t_r \ln \hat{g}_r} + O(\epsilon^2)}{(\gamma \bar{g}_r - 1)^2} + x \frac{\gamma^{\alpha t_r \frac{x \epsilon}{\bar{g}_r}} \frac{\alpha t_r \frac{x \epsilon}{\bar{g}_r} - 1}{\gamma \bar{g}_r - 1}}{1 - \gamma} + x \frac{\alpha t_r \frac{x \epsilon}{\bar{g}_r}}{1 - \gamma} + O(\epsilon),
\]

(172)

which concludes the second part of the Lemma. \qed

Lemma 7 (OeMDP model error). Let $M$ and $M'$ be two OeMDPs induced by $K$ arms of respective parameters \{(\tau_{k}, \bar{g}_k)\}_{k \in [K]} and \{(\tau'_{k}, \bar{g}'_k)\}_{k \in [K]}. Then, for any optimal policy $\psi$ in $M$, we have the following upper bound on the value error $V^*_M(x) - V^*_{M'}(x)$ that decreases linearly with their model distance: $\max_{k \in [K]} |\tau_k - \tau'_k|$ and $\max_{k \in [K]} |\bar{g}_k - \bar{g}'_k|.$

Proof. This proof is very similar to that of Lemma 6. We split the proof in two cases: (I) when $\forall k \in [K]$, such that $\bar{g}_k \geq 1$, $\tau_k \leq 0$, and (II) otherwise, when $\exists k \in [K]$, such that $\bar{g}_k \geq 1$ and $\tau_k > 0$.

(I) When $\forall k \in [K]$, such that $\bar{g}_k \geq 1$, $\tau_k \leq 0$, we know that the optimal policy is constant for all $x$:

\[
\psi(x) = k^* \in \arg\max_{k \in [K], \tau_k < 0} \left(\frac{\tau_k}{1 - \gamma \bar{g}_k}\right).
\]

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Batched Bandits with Crowd Externalities

Then, under the assumption that \( \bar{g}_{k^*} \leq 1 \), which is mild since \( \bar{g}_{k^*} < 1 \), the error in value is direct:

\[
V_{M}^{\psi}(x) - V_{M^*}^{\psi}(x) = \frac{r_{k^*} - r'_{k^*}}{1 - \gamma \bar{g}_{k^*}} - \frac{r'_{k^*}}{1 - \gamma \bar{g}'_{k^*}}
\]

which concludes the first part of the proof.

\((\Pi)\) When \( \exists k \in [K] \), such that \( \bar{g}_k \geq 1 \) and \( \tau_k > 0 \), we may choose \( \gamma \) such that \( \exists k \in [K] \), such that \( \bar{g}_k \geq \frac{1}{\gamma} \) and \( \tau_k > 0 \). In this case, we can observe that the worst case scenario happens when:

\[
k_* \in \arg\max \bar{g}_k \cap \arg\max \tau_k \cap \arg\max (\bar{g}_k - \bar{g}'_k) \cap \arg\max (\tau_k - \tau'_k)
\]

It is direct to notice that in this worst case scenario, the optimal policy also happens to be constant: \( \forall x, \psi(x) = k_* \). From now on, the proof is identical to that of Lemma 6 (\(\Pi\)) with the following result:

\[
V_{M}^{\psi}(x) - V_{M^*}^{\psi}(x) \leq x \gamma \tau'_{k_*} \left( \frac{t\gamma(\bar{g}'_{k^*})^{t\gamma}}{(\gamma \bar{g}'_{k^*} - 1)^2} (\bar{g}_{k^*} - \bar{g}'_{k^*}) + \frac{x}{\gamma \bar{g}_{k^*} - 1} (r_{k^*}^* - r'_{k^*}) + \frac{x}{1 - \gamma} (r'_{k^*} - r_{k^*}) \right), \text{ with } t = \frac{\ln \frac{\epsilon}{\delta}}{\ln \bar{g}_{k^*}}
\]

**Theorem 3** (Algorithm 1a expected regret). The cumulative regret of Algorithm 1a is upper bounded by an error term that decays with the following order of magnitude as a function of the number of arms \( K \), the horizon \( T \), and the high probability hyper-parameter \( \delta \):

\[
\text{Case (a-b) } O\left(K \ln \frac{1}{\delta} + K \delta T\right).
\]

\[
\text{Case (c) } \begin{cases} 
O\left(K \ln \frac{1}{\delta} + K \delta T\right) & \text{if } g_{k_*} > 1, \\
O\left(K \ln \frac{1}{\delta} + K \delta T + \sqrt{T \ln \frac{1}{\delta}}\right) & \text{otherwise}.
\end{cases}
\]

**Proof.** Cases (a-b) result is directly stems from Lemma 8, where only the dependencies in \( K, \delta, \) and \( T \) are retained. Case (c) result, first part, is proven in Lemma 9. And finally, Case (c) result, second part, is demonstrated in Lemma 10.

**Lemma 8.** If the problem is Case (a-b), then the regret of Algorithm 1a is \( \text{Regret}(K, \delta, T, \epsilon) \in O\left(\frac{K \ln \frac{1}{\epsilon}}{\epsilon} + K \delta T\right) \), as a function of \( K \) the number of arms, \( \delta \) a concentration probability hyperparameter for the algorithm, \( T \) the total number of pulls, and \( \epsilon \) the decidability of the setting.

**Proof.** **Disclaimer:** The proof of this lemma is kept as a sketch for the sake of simplicity.

We start by expressing the optimistic parameters for the reward and the growth of arms (in orange on Figure 6):

\[
r'_{k} = \hat{r}_{k} + \frac{\xi}{\sqrt{n_k}} \quad \text{and} \quad g_{k} = \hat{g}_{k} + \frac{\xi}{\sqrt{n_k}},
\]

\[\text{(179)}\]
where $\hat{r}_k$ and $\hat{g}_k$ are the empirical means (in yellow) of, respectively, reward and growth for Arm $k$, and where $n_k$ is the number of times Arm $k$ has been pulled. For each arm $k$, let $e_k$ be the maximal difference between the true parameters $(g_k, r_k)$ (in blue) and their empirical means $(\hat{g}_k, \hat{r}_k)$ with high probability $1 - \delta$, obtained thanks to 2-sided Hoeffding:

$$
\hat{g}_k \in [\bar{g}_k - e_k, \bar{g}_k + e_k] \quad \text{and} \quad \hat{r}_k \in [\bar{r}_k - e_k, \bar{r}_k + e_k]
$$

(180)

with

$$
e_k = \frac{\max(\hat{g}_r, \hat{r}_r - \hat{r}_r) \sqrt{\ln \frac{2}{\delta}}}{\sqrt{2n_k}}.
$$

(181)

The empirical estimate has therefore to be in the yellow area with high probability $1 - \delta$. If we choose $e_k = \frac{\xi}{\sqrt{n_k}}$ and therefore:

$$
\xi = \frac{\max(\hat{g}_r, \hat{r}_r - \hat{r}_r) \sqrt{\ln \frac{2}{\delta}}}{\sqrt{2}}
$$

(182)

then we have:

$$
g_k^* \in [\bar{g}_k, \bar{g}_k + 2e_k] \quad \text{and} \quad g_k^* \in [\bar{r}_k, \bar{r}_k + 2e_k],
$$

(183)

i.e. the optimistic estimate has to be in the orange area with high probability $1 - \delta$. Graphically, we observe that the worst situation happens when the optimistic parameters hit their upper bound: $g_k^* = \bar{g}_k + 2e_k$ and $r_k^* = \bar{r}_k + 2e_k$ (in green).

The algorithm will only play arms that are on the optimistic convex hull, the orange broken line on Figure 7. With time, it will take the form of a line, that is pushed down-left, as more pulls are performed. Once, it is pushed down below the critical point $(1,0)$ (or more rigorously the red semi-line), the case is identified as being Case (a-b), and then the corresponding strategy is applied. The orange line depends on the random pulls outcomes and it is more convenient to consider the green line which dominates the orange line with high probability $1 - K\delta$, and we are going to measure the regret until getting the green line under the critical point $(1,0)$.

When that happens, the slope of the orange line depends on the parameters of the setting and may also differ from one run to another, but in any configuration (see Remark 1 for more), if $e_k$ is the $\ell_\infty$ distance between point $(\bar{g}_k, \bar{r}_k)$ and the straight orange line that the optimist estimates converge to, with high probability $1 - K\delta$, the total number of pulls on each arm is of the order of $\frac{\xi^2}{e_k}$, which occurs a total regret in the order of $\sum_k \frac{\xi^2}{e_k} \leq \frac{K\xi^2}{\xi}$ before starting the crowd decrease. The crowd...
Figure 7: We keep the same color code as for Figure 6: the true parameters are in blue, the empirical mean in yellow, the optimistic estimate in orange and the upper bound of the optimistic estimate in green. Additionally, the critical semi-line is shown in red. Since the green Pareto front always dominates the orange one, and since the orange Pareto position with respect to the critical semi-line determines the decision apply Case (a-b) policy, we may show that this decision will be taken at the latest when the green Pareto goes under the critical semi-line.

decrease induces a constant regret close enough to \( V^*_o(x_1) - V^*_o(x_0) \), which is positive in Case (b) (so, to be substracted from the regret), and negative in Case (a) (so, to be added to the regret). Please also note that, in Case (a), it is probable that \( \epsilon_k \) is large for every Arm \( k \) and that the maximum is never to be reached. With complementary probability \( K\delta \), nothing can be said about the algorithm expect that the regret is linear with \( T \) (the regret is smaller on expectation that the expected reward of the worst arm times the number of pulls). The total regret is therefore of order:

\[
\text{Regret}(K, \delta, T, \epsilon) \in \mathcal{O} \left( \frac{K \ln \frac{1}{\epsilon}}{\epsilon} + K\delta T \right).
\]

Remark 1 (Discussion around an upper bound on the number of pulls). In the end, the process amounts to having sufficient precision to make the straight line formed with the optimistic estimates go below critical point \((1,0)\) for a pair of arms, and make sure that all optimistic estimates of other arms are below that straight line. It also has to be noted that, once the maximal crowd reached, the chosen interpolation between arms is 1 (or it means that the optimistic estimate of the arms considers the maximal reward arm to have a growth larger than 1, which means that it will be pulled deterministically until this is not the case anymore), which implies an actual expected growth lower than one, but then the next chosen interpolation will ultimately compensate for the crowd loss by choosing a higher crowd. Over time, the overall growth will be at least the one aimed at: 1. The amount of pulls may get significantly larger if the orange line converges very close to one specific arm, because it means that this arm would need to be pulled many times to be sufficiently precise. Further, we study from a single pair of arms how many pulls are required to reach a sufficient precision. For simplicity, let us assume that these two arms are \( k = 1, 2 \). We know that the straight line \((d_{1,2})\) passing through \((\bar{g}_1, r_1)\) and \((\bar{g}_2, r_2)\) is at distance \( \epsilon_{1,2} \) below the critical
point (1, 0), which gives us that:

\[
(d_{1,2}) : \quad y = \frac{r_2 - r_1}{g_2 - g_1} x + \frac{r_1 g_2 - r_2 g_1}{g_2 - g_1}
\]

\[
(185)
\]

\[
(\tau_2 - \tau_1 + r_1 g_2 - r_2 g_1) = -\epsilon_{1,2} \sqrt{(\tau_2 - \tau_1)^2 + (g_2 - g_1)^2}
\]

\[
(186)
\]

\[
(\tau_2(1 - g_1) + r_1(g_2 - 1)) = -\epsilon_{1,2} \sqrt{(\tau_2 - \tau_1)^2 + (g_2 - g_1)^2} := \rho_{1,2}
\]

\[
(187)
\]

We are interested in finding the conditions for the straight line \((d_{1,2}^*)\) passing through \((g_1^*, r_1^*)\) and \((g_2^*, r_2^*)\) to be below the critical point \((1, 0)\), which gives:

\[
(d_{1,2}^*) : \quad y = \frac{r_2^* - r_1^*}{g_2^* - g_1^*} x + \frac{r_1^* g_2^* - r_2^* g_1^*}{g_2^* - g_1^*}
\]

\[
(188)
\]

\[
(\tau_2^*(1 - g_1^*) + r_1^*(g_2^* - 1)) \leq 0
\]

\[
(189)
\]

\[
(\rho_{1,2} = \frac{2\epsilon}{\sqrt{n_2}}(1 - g_1^* + r_1^*) + \frac{2\epsilon}{\sqrt{n_1}}(g_2^* - 1 - r_2^*)
\]

\[
(190)
\]

However, \(n_1\) and \(n_2\) are related:

\[
n_1 \approx \alpha_{1,2} N_{1,2} \quad \text{and} \quad n_2 \approx (1 - \alpha_{1,2}) N_{1,2} \quad \text{with} \quad \alpha_{1,2} = \frac{g_2 - 1}{g_2 - g_1}.
\]

\[
(192)
\]

\[
n_1 \approx \frac{n_2}{1 - \alpha_{1,2}} = \frac{g_2 - 1}{1 - g_1}
\]

\[
(193)
\]

Injecting this, we get that having:

\[
\sqrt{N_{1,2}} \geq \frac{2\epsilon}{\rho_{1,2}} \left( \frac{g_2 - 1 - \tau_2}{\sqrt{\alpha_{1,2}}} + \frac{1 - \tau_1 + \tau_2}{\sqrt{1 - \alpha_{1,2}}} \right)
\]

\[
(194)
\]

\[
\geq \frac{2\epsilon}{\rho_{1,2}} \left( \frac{\alpha_{1,2}(g_2 - g_1) - \tau_2}{\sqrt{\alpha_{1,2}}} + \frac{(1 - \alpha_{1,2})(g_2 - g_1) + \tau_1}{\sqrt{1 - \alpha_{1,2}}} \right)
\]

\[
(195)
\]

\[
\geq \frac{2\epsilon}{\rho_{1,2}} \left( \sqrt{\alpha_{1,2}}(g_2 - g_1) - \frac{\tau_2}{\sqrt{\alpha_{1,2}}} + \sqrt{1 - \alpha_{1,2}}(g_2 - g_1) + \frac{\tau_1}{\sqrt{1 - \alpha_{1,2}}} \right) := T_{1,2}
\]

\[
(196)
\]

guarantees that the optimistic convex envelop is below the critical semi-line \([d_0]\). Moreover:

\[
T_{1,2} = 2\epsilon \left( \sqrt{\alpha_{1,2}} + \frac{1 - \alpha_{1,2}}{\sqrt{1 - \alpha_{1,2}}} \right) \left( \frac{g_2 - g_1}{\rho_{1,2}} - \frac{\tau_2}{\sqrt{\alpha_{1,2}}} + \frac{\tau_1}{\sqrt{1 - \alpha_{1,2}}} \right)
\]

\[
(197)
\]

\[
\leq \frac{2\epsilon}{\epsilon_{1,2}} \left( \sqrt{2} + \frac{\tau_1 - \tau_2}{\sqrt{(\tau_2 - \tau_1)^2 + (g_2 - g_1)^2}} \left( \frac{1}{\sqrt{\alpha_{1,2}}} + \frac{1}{\sqrt{1 - \alpha_{1,2}}} \right) \right)
\]

\[
(198)
\]

\[
\leq \frac{2\epsilon}{\epsilon_{1,2}} \left( \sqrt{2} + \frac{1}{\sqrt{\alpha_{1,2}}} + \frac{1}{\sqrt{1 - \alpha_{1,2}}} \right)
\]

\[
(199)
\]

We observe that \(N_{1,2}\) might get large when either \(g_1^* (\alpha \text{ is close to 1})\) or \(g_2^* (\alpha \text{ is close to 0})\) are close to 1. If \(g_1^*\) is very close to 1, then it means that \(r_1^*\) will soon be negative and Arm 1 will not be selected anymore. If \(g_2^*\) is very close to 1, then after selecting Arm 2 often times, a pair of \(k = 1\) and some \(k = 3\) such that \(g_3^*\) should be larger than \(g_2^*\). The worst case would consist of \(k = 2\) being the only arm with \(g_2^* > 1\) (and still \(g_2^*\) very close to 1).
Lemma 9. If there exists $k_*$, such that $\bar{g}_* \geq 1$ and $\tau_* = \max_k \tau_k \geq 0$, the regret of Algorithm 1a is:

$$\text{Regret}(K, \delta, T, \epsilon) \in \mathcal{O} \left( K\delta T + K \ln \frac{1}{\delta} \right),$$

(200)
as a function of $K$ the number of arms, $\delta$ a concentration probability hyperparameter for the algorithm, $T$ the total number of pulls, and $\epsilon$ the decidability of the setting.

Proof. We assume here, that there exists $k_*$, such that $\bar{g}_* \geq 1$ and $\tau_* = \max_k \tau_k \geq 0$. At each time step, Algorithm 1a plays an arm $k$ such that $r_k^* \geq r_*$. With high probability $1 - K\delta$, we know that, for all $k$, $r_k^* \in [\tau_k, r_k^1]$. As a consequence, each arm $k \neq k_*$ may be pulled only if $r_k^1 \geq \tau_*$, which may happen a maximum $n_k$ times:

$$n_k = \frac{(\hat{r}_* - \hat{r}_k)^2 \ln \frac{1}{\delta}}{\Delta_k^2},$$

(201)
which yields an expected regret of:

$$\Delta_k n_k = \frac{(\hat{r}_* - \hat{r}_k)^2 \ln \frac{1}{\delta}}{\Delta_k},$$

(202)
and therefore a total regret\(^9\) of:

$$\mathcal{O} \left( K \ln \frac{1}{\delta} \right).$$

(203)

With complementary probability $K\delta$, we are in the concentration failure mode and we suffer a linear regret as a function of $T$. The overall regret is therefore:

$$\mathcal{O} \left( K\delta T + K \ln \frac{1}{\delta} \right).$$

(204)

Lemma 10. If the problem in Case (c), and there does not exist $k_*$, such that $\bar{g}_* \geq 1$ and $\tau_* = \max_k \tau_k \geq 0$, the regret of Algorithm 1a is:

$$\text{Regret}(K, \delta, T, \epsilon) \in \mathcal{O} \left( K \delta T + K \ln \frac{1}{\delta} + \sqrt{T \ln \frac{1}{\delta}} \right),$$

(205)
as a function of $K$ the number of arms, $\delta$ a concentration probability hyperparameter for the algorithm, $T$ the total number of pulls, and $\epsilon$ the decidability of the setting.

Proof. We assume here, that there does not exist $k_*$, such that $\bar{g}_* \geq 1$ and $\tau_* = \max_k \tau_k \geq 0$. It means that, there is either an optimal pair of arms $(k_1, k_2)$ that should be played with an interpolation parameter $\alpha = \frac{\bar{g}_2 - 1}{\bar{g}_2 - \bar{g}_1}$. With high probability $1 - K\delta$, we know that, for all $k$, $r_k^* \in [\tau_k, r_k^1]$. As a consequence, each arm $k \notin \{k_1, k_2\}$ may be pulled only if the point $(g_k^*, r_k^1)$ is over the line $(d_{1,2})$ passing through $(\bar{g}_1, \tau_1)$ and $(\bar{g}_2, \tau_2)$, which may happen a maximum $n_k$ times:

$$n_k = \frac{\max(\hat{g}_*, \hat{r}_* - \hat{r}_k)^2 \ln \frac{1}{\delta}}{\Delta_k^2},$$

(206)
and therefore a total regret\(^10\) of:

$$\mathcal{O} \left( K \ln \frac{1}{\delta} \right).$$

(207)

\(^9\)We replace $\sum_{k \in [K]} \frac{1}{\Delta_k}$ with $K$ in the order of magnitude.

\(^10\)We replace $\sum_{k \in [K]} \frac{1}{\Delta_k}$ with $K$ in the order of magnitude.
It may also happen that the wrong ratio $\hat{\alpha} > \alpha$ is used. This means that the played growth $\hat{g}$ is actually lower than one, and the regret at each time step is of order $1 - \hat{g}$:

\begin{align*}
\hat{g} &= \hat{\alpha} \bar{g}_1 + (1 - \hat{\alpha}) \bar{g}_2 \\
&= \frac{g_2 - 1}{g_2 - g_1} \bar{g}_1 + \frac{1 - g_1^2}{g_2 - g_1} \bar{g}_2 \\
&= \frac{\bar{g}_2 - \bar{g}_1 + \bar{g}_1 g_2 - g_1 \bar{g}_2}{g_2 - g_1}
\end{align*}

\begin{align*}
1 - \hat{g} &= \frac{\bar{g}_1 - \bar{g}_2 + (1 - \bar{g}_1) g_2 - g_1 (\bar{g}_2 - 1)}{g_2 - g_1}
\leq \frac{\bar{g}_1 - \bar{g}_2 + (1 - \bar{g}_1) (g_2 + 2e_2) + (\bar{g}_1 + 2e_1) (\bar{g}_2 - 1)}{g_2 - g_1}
\end{align*}

where $e_k = \frac{\max(\hat{g}_r, \hat{r}_r - \hat{r}_\perp) \sqrt{\ln \frac{2}{\delta}}}{\sqrt{2n_k}}$.

We use here the same trick as in Lemma 8: since we know that Algorithm 1a will ultimately maintain crowd, we will experience an overall growth of 1, and therefore we know that $n_1$ and $n_2$ are tied together with the true ratio $\alpha$: $n_1 \approx \alpha N_{1,2}$ and $n_2 \approx (1 - \alpha) N_{1,2}$. We therefore get:

\begin{align*}
1 - \hat{g} &\leq \frac{\max(\hat{g}_r, \hat{r}_r - \hat{r}_\perp) \sqrt{2 \ln \frac{2}{\delta}}}{\sqrt{N_{1,2}}} \frac{1 - \bar{g}_1}{\sqrt{\alpha}} + \frac{\bar{g}_2 - 1}{\sqrt{1 - \alpha}}
\leq \mathcal{O} \left( \sqrt{\frac{\ln \frac{1}{\delta}}{T}} \right).
\end{align*}

If we sum over $T$ timesteps, we get a cumulative regret in $\mathcal{O} \left( \sqrt{T \ln \frac{1}{\delta}} \right)$.

With complementary probability $K\delta$, we are in the concentration failure mode and we suffer a linear regret as a function of $T$. The overall regret is therefore:

\begin{align*}
\mathcal{O} \left( K\delta T + K \ln \frac{1}{\delta} + \sqrt{T \ln \frac{1}{\delta}} \right),
\end{align*}

which concludes the proof.