The cosmological constant as an eigenvalue of $f(R)$-gravity Hamiltonian constraint

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Abstract
In the framework of ADM formalism, it is possible to find out eigenvalues of the WDW equation with the meaning of vacuum states, i.e. cosmological constants, for $f(R)$ theories of gravity, where $f(R)$ is a generic analytic function of the Ricci curvature scalar $R$. The explicit calculation is performed for a Schwarzschild metric where one-loop energy is derived by the zeta function regularization method and a renormalized running $\Lambda_0$ constant is obtained.

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1. Introduction

General relativity (GR), together with quantum field theory, is the major scientific achievement of last century. It is a theory of spacetime, gravity and matter unifying these concepts in a comprehensive scheme which gives rise to a new conception of the universe. However, in the last 30 years, several shortcomings came out in the Einstein scheme and people began to investigate if GR is the only theory able to explain the gravitational interaction. Such issues essentially spring up in cosmology and quantum field theory. In the first case, the presence of big-bang singularity, flatness and horizon problems [1] led to the result that standard cosmological model [2] is inadequate to describe the universe at extreme regimes. On the other hand, GR is a classical theory which does not work as a fundamental theory, when one wants to achieve a full quantum description of spacetime (and then of gravity). Due to these facts and, first of all, to the lack of a definitive quantum gravity theory, alternative theories of gravity have been pursued in order to attempt, at least, a semiclassical scheme where GR and its positive results could be recovered. A fruitful approach has been that of extended theories of gravity (ETG) which have become a sort of paradigm in the study of gravitational interaction based on corrections and enlargements of the Einstein scheme. The paradigm
consists, essentially, in adding higher-order curvature invariants and non-minimally coupled scalar fields into dynamics resulting from the effective action of quantum gravity [3, 4].

All these approaches are not the ‘full quantum gravity’ but are needed as working schemes towards it. In any case, they are going to furnish consistent and physically reliable results. Furthermore, every unification scheme, such as superstrings, supergravity or grand unified theories, takes into account effective actions where non-minimal couplings to the geometry or higher-order terms in the curvature invariants come out. Such contributions are due to one-loop or higher-loop corrections in the high-curvature regimes. Specifically, this scheme has been adopted in order to deal with quantization on curved spacetimes and the result has been that the interactions among quantum scalar fields and background geometry or the gravitational self-interactions yield corrective terms in the Hilbert–Einstein Lagrangian [5]. Moreover, it has been realized that such corrective terms are inescapable if we want to obtain the effective action of quantum gravity on scales closed to the Planck length [6].

Besides fundamental physics motivations, all these theories have acquired a huge interest in cosmology due to the fact that they ‘naturally’ exhibit inflationary behaviours able to overcome the shortcomings of standard cosmological model (based on GR). The related cosmological models seem very realistic and, several times, capable of matching with the observations [8–10]. Furthermore, it is possible to show that, via conformal transformations, the higher-order and non-minimally coupled terms always correspond to Einstein gravity plus one or more than one minimally coupled scalar fields [11–14]. This feature results very interesting if we want to obtain multiple inflationary events since a former early stage could select ‘very’ large-scale structures (clusters of galaxies today), while a latter stage could select ‘small’ large-scale structures (galaxies today) [15]. The philosophy is that each inflationary era is connected with the dynamics of a scalar field. Furthermore, these extended schemes naturally could solve the problem of ‘graceful exit’ bypassing the shortcomings of former inflationary models [9, 16].

Recently, ETG are also going to play an interesting role to describe the today observed universe. In fact, the amount of good quality data of last decade has made it possible to shed new light on the effective picture of the universe. Type Ia supernovae (SNeIa) [17], anisotropies in the cosmic microwave background radiation (CMBR) [18] and matter power spectrum inferred from large galaxy surveys [19] represent the strongest evidences for a radical revision of the cosmological standard model also at recent epochs. In particular, the concordance \( \Lambda \)CDM model predicts that baryons contribute only for \( \sim 4\% \) of the total matter–energy budget, while the exotic cold dark matter (CDM) represents the bulk of the matter content (\( \sim 25\% \)) and the cosmological constant \( \Lambda \) plays the role of the so-called dark energy (\( \sim 70\% \)) [20]. Although being the best fit to a wide range of data [21], the \( \Lambda \)CDM model is severely affected by strong theoretical shortcomings [22] that have motivated the search for alternative models [23]. Dark energy models mainly rely on the implicit assumption that Einstein’s general relativity is the correct theory of gravity indeed. Nevertheless, its validity on the larger astrophysical and cosmological scales has never been tested [24], and it is therefore conceivable that both cosmic speed up and dark matter represent signals of a breakdown in our understanding of gravitation law so that one should consider the possibility that the Hilbert–Einstein Lagrangian, linear in the Ricci scalar \( R \), should be generalized. Following this line of thinking, the choice of a generic function \( f(R) \) can be derived by matching the data and by the ‘economic’ requirement that no exotic ingredients have to be added\(^3\). This is the underlying philosophy of what are referred to as \( f(R) \) theories of gravity, see [25–31] and references therein. However, \( f(R) \) gravity can be encompassed in the ETG being a ‘minimal’ extension

\(^3\) Following the Occam razor prescriptions: ‘Entia non sunt multiplicanda praeter necessitatem’.
of GR where (analytical) functions of Ricci scalar are taken into account. Although higher-order gravity theories have received much attention in cosmology, since they are naturally able to give rise to the accelerating expansion (both in the late and in the early universe), it is possible to demonstrate that \( f(R) \) theories can also play a major role at astrophysical scales. In fact, modifying the gravity Lagrangian can affect the gravitational potential in the low energy limit. Provided that the modified potential reduces to the Newtonian one on the solar system scale, this implication could represent an intriguing opportunity rather than a shortcoming for \( f(R) \) theories. In fact, a corrected gravitational potential could fit galaxy rotation curves without the need of dark matter [32–34]. In addition, it is possible to work out a formal analogy between the corrections to the Newtonian potential and the usually adopted dark matter models. In general, any relativistic theory of gravitation can yield corrections to the Newton potential (see, for example, [36]) which, in the post-Newtonian (PPN) formalism, could furnish tests for the same theory [24, 37–39].

In this paper, we want to face the problem to study \( f(R) \) gravity at a fundamental level. In particular, in the framework of the Arnowitt–Deser–Misner (\( \text{ADM} \)) formalism [42], we want to investigate the possibility of finding out cosmological terms as eigenvalues of generalized \( f(R) \)-Hamiltonians in a Sturm–Liouville-like problem\(^4\). This issue is particularly relevant from several viewpoints. First of all, our aim is to show that vacuum energy of gravitational field is not a particular feature of GR where the cosmological constant has to be added by hand into dynamics. At a classical level, it is well known that \( f(R) \) gravity, for the Ricci scalar \( R \) equal to a constant, exhibits several de Sitter solutions [7] but a definite discussion, at a fundamental level, considering cosmological terms as eigenvalues of such theories is lacking. Besides, the computation of the Casimir energy, the seeking for zero-point energy in different backgrounds\(^5\) give a track to achieve one-loop energy regularization and renormalization for this kind of theories [40, 41]. On the other hand, these issues can be considered in a multigravity approach to spacetime foam if the \( N \) spacetimes constituting the foam are supposed to evolve, in general, with different curvature laws and ground states (cosmological constants) [44].

The layout of the paper is the following. In section 2, we recall the Hamiltonian formalism in the (\( \text{ADM} \)) approach of GR. It is developed for generic \( f(R) \) gravity in sections 3. Section 4 is devoted to find out the cosmological constant as the eigenvalue of a generalized \( f(R) \) Hamiltonian. We discuss the orthogonal decomposition of the wave functional and derive the total one-loop energy density for the transverse-traceless tensor component. In section 5, we give an example: the transverse-traceless spin-2 operator is calculated for the Schwarzschild metric and the energy density contributions to the cosmological constant are calculated in the WKB approximation. This is a realization of the above formal cosmological constant calculation. Section 6 is devoted to the one-loop energy calculation by the zeta function regularization method. The explicit value of the renormalized \( \Lambda_0 \) constant, considered as a running constant, is achieved. How it can be set to zero is explicitly derived for \( f(R) = \exp(-\alpha R) \). Summary and conclusions are drawn in section 7. In the appendix, details on zeta function regularization are given.

2. The Hamiltonian constraint of general relativity

Let us briefly report how to compute the Hamiltonian constraint for GR considering the standard Hilbert–Einstein theory \( f(R) = R \) and the Arnowitt–Deser–Misner (\( \text{ADM} \)) (3 + 1)

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\(^4\) See [43], for the application of the Sturm–Liouville problem in the simple case of \( f(R) = R \), even in the presence of a massive graviton.

\(^5\) For different \( f(R) \), we expect different zero-point energies and, obviously, different vacuum states.
decomposition [42]. In terms of these variables, the line element is
\[ ds^2 = g_{\mu\nu}(x) \, dx^\mu \, dx^\nu = (-N^2 + N_i N^i) \, dt^2 + 2N_j \, dt \, dx^j + g_{ij} \, dx^i \, dx^j, \]
where \( N \) is the \textit{lapse function}, while \( N_i \) the \textit{shift function}. In terms of these variables, the gravitational Lagrangian, with the boundary terms neglected, can be written as
\[ \int_\Sigma d^3x \sqrt{-g} R = \frac{1}{2\kappa} \int_\Sigma d^3x N \sqrt{\gamma} \left( K_{ij} K^{ij} - K^2 + \left( \frac{3}{\Lambda_1} \right) R - \frac{2}{\Lambda_1} \right), \] (1)
where \( K_{ij} \) is the second fundamental form, \( K = g^{ij} K_{ij} \) is the trace, \( ^3R \) is the three-dimensional scalar curvature and \( \sqrt{\gamma} \) is the three-dimensional determinant of the metric. The conjugate momentum is simply
\[ \pi^{ij} = \frac{\delta L}{\delta (\partial_t g^{ij})} = (^3g^{ij} K - K^{ij}) \frac{\sqrt{\gamma}}{2\kappa}. \] (2)
By a Legendre transformation, we calculate the Hamiltonian
\[ H = \int d^3x [N H + N_i H^i], \] (3)
where
\[ H = (2\kappa) G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{\gamma}}{2\kappa} \left( ^3R - \frac{2}{\Lambda_1} \right) \] (4)
and
\[ H^i = -2\nabla_j \pi^{ij}, \] (5)
where \( \Lambda_1 \) is the bare cosmological constant. The equations of motion lead to two classical constraints
\[ \begin{cases} H = 0 \\ H^i = 0, \end{cases} \] (6)
representing invariance under time re-parameterization and invariance under diffeomorphism, respectively. \( G_{ijkl} \) is the \textit{supermetric} defined as
\[ G_{ijkl} = \frac{1}{2 \sqrt{\gamma}} (g_{ik} g_{jl} + g_{il} g_{jk} - g_{ij} g_{kl}). \] (7)
When \( H \) and \( H^i \) are considered as operators acting on some wavefunction, we have
\[ H[\Psi[g_{ij}]] = 0 \] (8)
and
\[ H^i[\Psi[g_{ij}]] = 0. \] (9)
Equation (8) is the Wheeler–De Witt equation (WDW) [45]. Equations (8) and (9) describe the \textit{wavefunction of the universe} \( \Psi[g_{ij}] \). The WDW equation represents invariance under time re-parameterization in an operatorial form. This standard lore can be applied to a generic \( f(R) \) theory of gravity with the aim to achieve a cosmological term as an eigenvalue of the WDW equation.
3. The Hamiltonian constraint for a generic $f(R)$ theory of gravity

Let us consider now the Lagrangian density describing a generic $f(R)$ theory of gravity, namely
\[ \mathcal{L} = \sqrt{-g} (f(R) - 2\Lambda_c), \quad \text{with} \quad f'' \neq 0, \quad (10) \]
where $f(R)$ is an arbitrary smooth function of the scalar curvature and primes denote differentiation with respect to the scalar curvature. A cosmological term is also added in this case for the sake of generality. Obviously $f'' = 0$ corresponds to GR. The generalized Hamiltonian density for the $f(R)$ theory assumes the form\(^6\)
\[ H = \frac{1}{2\kappa} \left[ \frac{\mathcal{P}}{6} (3R - 2\Lambda_c - 3K_{ij}K^{ij} + K^2) + V(\mathcal{P}) - \frac{1}{3} g^{ij}\mathcal{P}_{ij} - 2p^{ij}K_{ij}\right], \quad (11) \]
where
\[ V(\mathcal{P}) = \sqrt{g} \left[ R f'(R) - f(R) \right]. \quad (12) \]
Henceforth, the superscript 3 indicating the spatial part of the metric will be omitted on the metric itself. When $f(R) = R$, $V(\mathcal{P}) = 0$ as it should be. Since
\[ \mathcal{P}^{ij} = -2\sqrt{g}g^{ij}f'(R), \quad \mathcal{P} = -6\sqrt{g}f'(R), \quad (13) \]
we have
\[ H = \frac{1}{2\kappa} \left[ -\sqrt{g}f'(R)(3R - 2\Lambda_c - 3K_{ij}K^{ij} + K^2) + V(\mathcal{P}) + 2g^{ij}(\sqrt{g}f'(R))_{ij} - 2p^{ij}K_{ij}\right]. \quad (14) \]

With the help of equation (2), equation (14) becomes
\[ H = f'(R) \left[ (2\kappa)G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{g}}{2\kappa} (3R - 2\Lambda_c) \right] + \frac{1}{2\kappa} \left[ \sqrt{g} f'(R) (2K_{ij}K^{ij}) \right. \
\quad + V(\mathcal{P}) + 2g^{ij}(\sqrt{g}f'(R))_{ij} - 2p^{ij}K_{ij}\right]. \quad (15) \]

However
\[ p^{ij} = \sqrt{g}K^{ij}, \quad (16) \]
then we obtain
\[ H = f'(R) \left[ (2\kappa)G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{g}}{2\kappa} (3R - 2\Lambda_c) \right] + \frac{1}{2\kappa} \left[ 2\sqrt{g}K_{ij}K^{ij} \right. \quad (f'(R) - 1) \quad \left. \right] + V(\mathcal{P}) + 2g^{ij}(\sqrt{g}f'(R))_{ij} \right]. \quad (17) \]

and transforming into canonical momenta, one gets
\[ H = f'(R) \left[ (2\kappa)G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{g}}{2\kappa} (3R - 2\Lambda_c) \right] + 2(2\kappa) \left[ G_{ijkl}\pi^{ij}\pi^{kl} + \frac{\pi^2}{4} \right] (f'(R) - 1) \quad \left. \right] + \frac{1}{2\kappa} [V(\mathcal{P}) + 2g^{ij}(\sqrt{g}f'(R))_{ij}]. \quad (18) \]

By imposing the Hamiltonian constraint, we obtain
\[ f'(R) \left[ (2\kappa)G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{g}}{2\kappa} (3R) \right] + 2(2\kappa) \left[ G_{ijkl}\pi^{ij}\pi^{kl} + \frac{\pi^2}{4} \right] (f'(R) - 1) \quad \left. \right] + \frac{1}{2\kappa} [V(\mathcal{P}) + 2g^{ij}(\sqrt{g}f'(R))_{ij}] = -f'(R)\sqrt{\frac{\Lambda_c}{\kappa}}. \quad (19) \]

\(^6\) See also [46] for technical details.
If we assume that $f'(R) \neq 0$ the previous expression becomes
\[
\left[ (2\kappa) G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{g}}{2\kappa} (3) R \right] + (2\kappa) \left[ G_{ijkl} \pi^{ij} \pi^{kl} + \frac{\pi^2}{4} \right] \frac{2(f'(R) - 1)}{f'(R)} \\
+ \frac{1}{2\kappa f'(R)} [V(P) + 2g^{ij}(\sqrt{g} f'(R))_{ij}] = -\sqrt{g} \frac{\Lambda_1}{\kappa}.
\] (20)

Now, we integrate over the hypersurface $\Sigma$ to obtain
\[
\int_\Sigma d^3 x \left\{ \left[ (2\kappa) G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{g}}{2\kappa} (3) R \right] + (2\kappa) \left[ G_{ijkl} \pi^{ij} \pi^{kl} + \frac{\pi^2}{4} \right] \frac{2(f'(R) - 1)}{f'(R)} \right\} \\
+ \int_\Sigma d^3 x \frac{1}{2\kappa f'(R)} [V(P) + 2g^{ij}(\sqrt{g} f'(R))_{ij}] = -\frac{\Lambda_1}{\kappa} \int_\Sigma d^3 x \sqrt{g}.
\] (21)

The term
\[
\frac{1}{\kappa} \int_\Sigma d^3 x \frac{1}{f'(R)} g^{ij}(\sqrt{g} f'(R))_{ij}
\] (22)
appears to be a three-divergence and therefore will not contribute to the computation. The remaining equation simplifies into
\[
\int_\Sigma d^3 x \left\{ \left[ (2\kappa) G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{g}}{2\kappa} (3) R \right] + (2\kappa) \left[ G_{ijkl} \pi^{ij} \pi^{kl} + \frac{\pi^2}{4} \right] \frac{2(f'(R) - 1)}{f'(R)} \right\} \\
\times \frac{V(P)}{f'(R)} + \frac{V(P)}{2\kappa f'(R)} = -\frac{\Lambda_1}{\kappa} \int_\Sigma d^3 x \sqrt{g}.
\] (23)

By a canonical procedure of quantization, we want to obtain the vacuum state of a generic $f(R)$ theory.

4. The cosmological constant as an eigenvalue for the generalized $f(R)$ Hamiltonian

The standard WDW equation (8) can be cast into the form of an eigenvalue equation
\[
\hat{\Lambda}_\Sigma \Psi[g_{ij}] = \Lambda(\tilde{x}) \Psi[g_{ij}],
\] (24)
where
\[
\hat{\Lambda}_\Sigma = (2\kappa) G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{g}}{2\kappa} (3) R.
\] (25)

If we multiply equation (24) by $\Psi^*[g_{ij}]$ and we functionally integrate over the three spatial metric $g_{ij}$, we get
\[
\int D[g_{ij}] \Psi^*[g_{ij}] \hat{\Lambda}_\Sigma \Psi[g_{ij}] = \int D[g_{ij}] \Lambda(\tilde{x}) \Psi^*[g_{ij}] \Psi[g_{ij}] = \int D[g_{ij}] \Lambda(\tilde{x}) \Psi^*[g_{ij}] \Psi[g_{ij}] = \int D[g_{ij}] \Lambda(\tilde{x}) \Psi^*[g_{ij}] \Psi[g_{ij}] = -\frac{\Lambda_1}{\kappa},
\] (26)

and after integrating over the hypersurface $\Sigma$, one can formally rewrite the modified WDW equation as
\[
\frac{1}{V} \int D[g_{ij}] \Psi^*[g_{ij}] \int_\Sigma d^3 x \hat{\Lambda}_\Sigma \Psi[g_{ij}] = \frac{1}{V} \frac{\langle \Psi \mid \int_\Sigma d^3 x \hat{\Lambda}_\Sigma \mid \Psi \rangle}{\langle \Psi \mid \Psi \rangle} = -\frac{\Lambda_1}{\kappa},
\] (27)

where the explicit expression of $\Lambda(\tilde{x})$ has been used and we have defined the volume of the hypersurface $\Sigma$ as
\[
V = \int_\Sigma d^3 x \sqrt{g}.
\] (28)
The formal eigenvalue equation (27) is a simple manipulation of equation (8). We can gain more information considering a separation of the spatial part of the metric into a background term, \( \bar{g}_{ij} \), and a quantum fluctuation, \( h_{ij} \),

\[
g_{ij} = \bar{g}_{ij} + h_{ij}.
\]

Thus, equation (27) becomes

\[
\langle \Psi | \int_{\Sigma} d^3x \left[ \hat{\Lambda}^{(0)}_\Sigma + \hat{\Lambda}^{(1)}_\Sigma + \hat{\Lambda}^{(2)}_\Sigma + \cdots \right] |\Psi \rangle = -\frac{\Lambda_c}{\kappa} \Psi[g_{ij}],
\]

where \( \hat{\Lambda}^{(i)}_\Sigma \) represents the \( i \)th order of perturbation in \( h_{ij} \). By observing that the kinetic part of \( \hat{\Lambda}_\Sigma \) is quadratic in the momenta, we only need to expand the three-scalar curvature \( f d^3x \sqrt{g} R^{(3)} \) up to the quadratic order and we get

\[
\int_{\Sigma} d^3x \sqrt{g} \left[ -\frac{1}{4} h \nabla h + \frac{1}{4} h h_{ii} - \frac{1}{4} h_{ij} \nabla_i h_j + \right. \\
\left. + \frac{1}{2} h \nabla_i h_j + \frac{1}{2} h_{ij} R_{ii}^j h^i + \frac{1}{2} h R_{ij} h^j + \frac{1}{4} h (R^{(0)} - h) \right],
\]

where \( h \) is the trace of \( h_{ij} \) and \( R^{(0)} \) is the three-dimensional scalar curvature. By repeating the same procedure for the generalized WDW equation, equation (23), we obtain

\[
\frac{1}{V} \langle \Psi | \int_{\Sigma} d^3x \left[ \hat{\Lambda}^{(2)}_\Sigma \right] |\Psi \rangle + \frac{2\kappa}{V} \frac{2(f'(R) - 1)}{f'(R)} \langle \Psi | \int_{\Sigma} d^3x \left[ G_{ijkl} \pi^{ij} \pi^{kl} + \frac{1}{\kappa^2} \right] |\Psi \rangle \\
+ \frac{1}{V} \langle \Psi | \int_{\Sigma} d^3x \sqrt{g} (P) |\Psi \rangle = \frac{\Lambda_c}{\kappa},
\]

From equation (32), we can define a ’modified’ \( \hat{\Lambda}^{(2)}_\Sigma \) operator which includes \( f'(R) \). Thus, we obtain

\[
\langle \Psi | \int_{\Sigma} d^3x \left[ \hat{\Lambda}^{(2)}_{\Sigma,f(R)} \right] |\Psi \rangle + \frac{2\kappa}{V} \frac{2(f'(R) - 1)}{f'(R)} \langle \Psi | \int_{\Sigma} d^3x \left[ \frac{1}{\kappa} \right] |\Psi \rangle \\
+ \frac{1}{V} \langle \Psi | \int_{\Sigma} d^3x \sqrt{g} (P) |\Psi \rangle = \frac{\Lambda_c}{\kappa},
\]

where

\[
\hat{\Lambda}^{(2)}_{\Sigma,f(R)} = (2\kappa) h(R) G_{ijkl} \pi^{ij} \pi^{kl} - \sqrt{\frac{3}{2\kappa}} 3 \text{R}_{\text{lin}}
\]

with

\[
h(R) = 1 + \frac{2[f'(R) - 1]}{f'(R)}
\]

and where \( \text{R}_{\text{lin}} \) is the linearized scalar curvature whose expression is shown in square brackets of equation (31). Note that when \( f(R) = R \), consistently it is \( h(R) = 1 \). From equation (33), we redefine \( \Lambda_c' \):

\[
\Lambda_c' = \Lambda_c + \frac{1}{2V} \frac{\langle \Psi | \int_{\Sigma} d^3x \sqrt{g} (P) |\Psi \rangle}{\langle \Psi |\Psi \rangle} = \Lambda_c + \frac{1}{2V} \int_{\Sigma} d^3x \sqrt{g} \frac{R f'(R) - f(R)}{f'(R)},
\]

where we have explicitly used the definition of \( V(P) \). In order to make explicit calculations, we need an orthogonal decomposition for both \( \pi_{ij} \) and \( h_{ij} \) to disentangle gauge modes from physical deformations. We define the inner product

\[
\langle h, k \rangle := \int_{\Sigma} \sqrt{g} G^{ijkl} h_{ij}(x) k_{kl}(x) d^3x,
\]
by means of the inverse WDW metric \( G^{ijkl} \), to have a metric on the space of deformations, i.e. a quadratic form on the tangent space at \( h_{ij} \), with
\[
G^{ijkl} = \frac{1}{2} (g^{ik} g^{jl} + g^{il} g^{jk} - 2 g^{ij} g^{kl}).
\] (38)

The inverse metric is defined on cotangent space and it assumes the form
\[
\langle p, q \rangle := \int_{\Sigma} \sqrt{g} G^{ijkl} p^{ij}(x) q^{kl}(x) \, d^3 x,
\] (39)
so that
\[
G^{ijnm} G^{nmkl} = \frac{1}{2} (\delta^i_k \delta^j_l + \delta^i_l \delta^j_k).
\] (40)

Note that in this scheme the ‘inverse metric’ is actually the WDW metric defined on phase space. The desired decomposition on the tangent space of 3-metric deformations \([47–50]\) is
\[
h_{ij} = \frac{1}{3} h g_{ij} + (L \xi)_{ij} + h^\perp_{ij}
\] (41)
where the operator \( L \) maps \( \xi_i \) into symmetric tracefree tensors
\[
(L \xi)_{ij} = \nabla_i \xi_j + \nabla_j \xi_i - \frac{2}{3} g_{ij} (\nabla \cdot \xi).
\] (42)

Thus, the inner product between three-geometries becomes
\[
\langle h, h \rangle := \int_{\Sigma} \sqrt{g} G^{ijkl} h_{ij}(x) h_{kl}(x) \, d^3 x = \int_{\Sigma} \sqrt{g} \left[ -\frac{2}{3} h^2 + (L \xi)^{ij} (L \xi)_{ij} + h^\perp_{ij} h^\perp_{ij} \right].
\] (43)

With the orthogonal decomposition in hand we can define the trial wave functional as
\[
\Psi[h_{ij}(\vec{x})] = \mathcal{N} \Psi[h^\perp_{ij}(\vec{x})] \Psi[h^\parallel_{ij}(\vec{x})] \Psi[h^{\text{trace}}_{ij}(\vec{x})],
\] (44)
where
\[
\Psi[h^\perp_{ij}(\vec{x})] = \exp \left\{ -\frac{1}{2} \langle h K^{-1} h \rangle^\perp_{x,y} \right\},
\]
\[
\Psi[h^\parallel_{ij}(\vec{x})] = \exp \left\{ -\frac{1}{2} \langle (L \xi) K^{-1} (L \xi) \rangle^\parallel_{x,y} \right\},
\]
\[
\Psi[h^{\text{trace}}_{ij}(\vec{x})] = \exp \left\{ -\frac{1}{2} \langle h K^{-1} h \rangle^{\text{trace}}_{x,y} \right\}.
\] (45)

The symbol ‘\( \perp \)’ denotes the transverse-traceless tensor (TT) (spin 2) of the perturbation, while the symbol ‘\( \parallel \)’ denotes the longitudinal part (spin 1) of the perturbation. Finally, the symbol ‘\( \text{trace} \)’ denotes the scalar part of the perturbation. \( \mathcal{N} \) is a normalization factor, \( \langle \cdot, \cdot \rangle_{x,y} \) denotes space integration and \( K^{-1} \) is the inverse ‘propagator’. We will fix our attention to the TT tensor sector of the perturbation representing the graviton. Therefore, representation (44) reduces to
\[
\Psi[h_{ij}(\vec{x})] = \mathcal{N} \exp \left\{ -\frac{1}{2} \langle h K^{-1} h \rangle^\perp_{x,y} \right\}.
\] (46)

Actually, there is no reason to neglect longitudinal and trace perturbations. However, following the analysis of \([49, 51, 52]\) on the perturbation decomposition, we can discover that the relevant components can be restricted to the TT modes and to the trace modes. Moreover, for certain backgrounds, TT tensors can be a source of instability as shown in \([51–53]\). Even the trace part can be regarded as a source of instability. Indeed this is usually termed conformal instability. The appearance of an instability on the TT modes is known as non-conformal instability. This means that there does not exist a gauge choice that can eliminate negative modes. To proceed with equation (30), we need to know the action of some basic operators on \( \Psi[h_{ij}] \). The action of the operator \( h_{ij} \) on \( \langle \Psi \rangle = \Psi[h_{ij}] \) is realized by \([54]\)
\[
h_{ij}(x) \Psi = h_{ij}(x) \Psi[h_{ij}].
\] (47)
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The action of the operator \( \pi_{ij} \) on \( |\Psi\rangle \), in general, is

\[
\pi_{ij}(x)|\Psi\rangle = -i \frac{\delta}{\delta h_{ij}(x)} |\Psi\rangle,
\]

while the inner product is defined by the functional integration:

\[
\langle \Psi_1 | \Psi_2 \rangle = \int [Dh_{ij}] \Psi_1^* [h_{ij}] \Psi_2 [h_{ij}].
\]

We demand that

\[
\frac{1}{V} \langle \Psi | h_{ij}(x) | \Psi \rangle = 0
\]

and

\[
\frac{1}{V} \langle \Psi | h_{ij}(x) h_{kl}(y) | \Psi \rangle = K_{ijkl}(x, y).
\]

Extracting the TT tensor contribution, we get

\[
\hat{\Lambda}_{\Sigma, f(R)}^{(2)\perp} = \frac{1}{4V} \int_{\Sigma} d^3x \sqrt{\bar{g}} G^{ijkl} \left[ (2\kappa) h(R) K^{ij\perp}(x, x)_{ijkl} + \frac{1}{(2\kappa)} (\triangle_2)^a_{ij} K_{ijkl}^{\perp}(x, x) \right].
\]

The propagator \( K^{\perp}(x, x)_{ijkl} \) can be represented as

\[
K^{\perp}(x, y)_{ijkl} := \sum_{\tau} h^{(1)\perp}_{ia}(x) h^{(1)\perp}_{k\ell}(y),
\]

where \( h^{(1)\perp}_{ia}(x) \) are the eigenfunctions of \( \triangle_2 \). \( \tau \) denotes a complete set of indices and \( \lambda(\tau) \) are a set of variational parameters to be determined by the minimization of equation (53). The expectation value of \( \hat{\Lambda}_{\Sigma}^{\perp} \) is easily obtained by inserting the form of the propagator into equation (53):

\[
-\frac{\Lambda'(\lambda_i)}{\kappa} = \frac{1}{4} \sum_{\tau} \sum_{j=1}^2 \left[ (2\kappa) h(R) \lambda_i(\tau) + \frac{\omega^2_2(\tau)}{(2\kappa)\lambda_i(\tau)} \right].
\]

By minimizing with respect to the variational function \( \lambda_i(\tau) \), we obtain the total one-loop energy density for TT tensors

\[
\Lambda'(\lambda_i) = -\kappa \sqrt{h(R)} \frac{1}{4} \sum_{\tau} \left[ \sqrt{\omega^2_1(\tau)} + \sqrt{\omega^2_2(\tau)} \right],
\]

where \( \Lambda' \) is expressed by equation (36). The above expression makes sense only for \( \omega_1^2(\tau) > 0 \). It is the main formal result of this paper. It is true for generic \( f(R) \) functions since \( h(R) \) explicitly appears in it.
5. The transverse-traceless (TT) spin-2 operator for the Schwarzschild metric and the WKB approximation

The above considerations can be specified choosing a given metric. For example, the quantity \( \Lambda'_c \) can be calculated for a Schwarzschild metric in the WKB approximation. Apparently, there is no strong motivation to consider a Schwarzschild metric as a probe for a cosmological problem. Nevertheless, every quantum field induces a ‘cosmological term’ by means of vacuum expectation values and the variational approach we have considered is particularly easy to use for a spherically symmetric metric. The Schwarzschild metric is the simplest sourceless solution of the Einstein field equations which can be used to compute a cosmological constant spectrum. Of course, also Minkowski space can be put in the form of a spherically symmetric metric, but in that case there is no gravity at all. The other solution needs a source which is not considered in the present paper. In this sense, the computation is a real vacuum contribution to the cosmological term. The spin-2 operator for the Schwarzschild metric is defined by

\[
\left( \Delta^2 h^{TT} \right)^j_i := - \left( \Delta_T h^{TT} \right)^j_i + 2(Rh^{TT})^j_i,
\]

(57)

where the transverse-traceless (TT) tensor for the quantum fluctuation is obtained by the following decomposition:

\[
h^j_i = h^j_i - \frac{1}{3} \delta^j_i h + \frac{1}{3} \delta^j_i h = (h^T)^j_i + \frac{1}{3} \delta^j_i h.
\]

(58)

This implies that \( (h^T)^j_i \delta^i_j = 0 \). The transversality condition is applied on \( (h^T)^j_i \) and becomes \( \nabla_j (h^T)^j_i = 0 \). Thus,

\[-(\Delta_T h^{TT})^j_i = -\Delta_S (h^{TT})^j_i + \frac{6}{r^2} \left( 1 - \frac{2MG}{r} \right),\]

(59)

where \( \Delta_S \) is the scalar curved Laplacian, whose form is

\[
\Delta_S = \left( 1 - \frac{2MG}{r} \right) \frac{d^2}{dr^2} + \left( \frac{2r - 3MG}{r^2} \right) \frac{d}{dr} - \frac{L^2}{r^2}
\]

(60)

and \( R^a_i \) is the mixed Ricci tensor whose components are

\[
R^a_i = \left\{ -\frac{2MG}{r^3}, \frac{MG}{r^3}, \frac{MG}{r^3} \right\}.
\]

(61)

This implies that the scalar curvature is traceless. We are therefore led to study the following eigenvalue equation:

\[
\left( \Delta^2 h^{TT} \right)^j_i = \omega^2 h^j_i
\]

(62)

where \( \omega^2 \) is the eigenvalue of the corresponding equation. In doing so, we follow Regge and Wheeler in analysing the equation as modes of definite frequency, angular momentum and parity [55]. In particular, our choice for the three-dimensional gravitational perturbation is represented by its even-parity form

\[
(h^{even})^j_i (r, \vartheta, \phi) = \text{diag}( H(r), K(r), L(r)) Y_{lm}(\vartheta, \phi),
\]

(63)

with

\[
\begin{align*}
H(r) &= h^1_1(r) - \frac{1}{3} h(r) \\
K(r) &= h^2_2(r) - \frac{1}{3} h(r) \\
L(r) &= h^3_3(r) - \frac{1}{3} h(r).
\end{align*}
\]

(64)
From the transversality condition, we obtain $h_2^2(r) = h_3^2(r)$. Then $K(r) = L(r)$. For a generic value of the angular momentum $L$, representation (63) joined to equation (59) lead to the following system of PDEs:

$$
\begin{cases}
-\Delta_S + \frac{6}{r^2} \left( 1 - \frac{2MG}{r} \right) - \frac{4MG}{r^3} H(r) = \omega_{1,l}^2 H(r) \\
-\Delta_S + \frac{6}{r^2} \left( 1 - \frac{2MG}{r} \right) + \frac{2MG}{r^3} K(r) = \omega_{2,l}^2 K(r).
\end{cases}
$$

Defining the ‘reduced’ fields

$$
H(r) = \frac{f_1(r)}{r}, \quad K(r) = \frac{f_2(r)}{r},
$$

and passing to the proper geodesic distance from the throat of the bridge

$$
dx = \pm \frac{dr}{\sqrt{1 - \frac{2MG}{r}}},
$$

the system (65) becomes

$$
\begin{cases}
\left[ -\frac{d^2}{dx^2} + V_1(x) \right] f_1(x) = \omega_{1,l}^2 f_1(x) \\
\left[ -\frac{d^2}{dx^2} + V_2(x) \right] f_2(x) = \omega_{2,l}^2 f_2(x)
\end{cases}
$$

with

$$
V_1(r) = \frac{l(l+1)}{r^2} + U_1(r) + m_1^2(r) \\
V_2(r) = \frac{l(l+1)}{r^2} + U_2(r) + m_2^2(r),
$$

where we have defined $r = r(x)$ and

$$
\begin{cases}
U_1(r) = \left[ \frac{6}{r^2} \left( 1 - \frac{2MG}{r} \right) - \frac{3MG}{r^3} \right] \\
U_2(r) = \left[ \frac{6}{r^2} \left( 1 - \frac{2MG}{r} \right) + \frac{3MG}{r^3} \right].
\end{cases}
$$

Note that

$$
\begin{align*}
U_1(r) &\geq 0 \quad \text{when} \quad r \geq \frac{5MG}{2} \\
U_1(r) &< 0 \quad \text{when} \quad 2MG < r < \frac{5MG}{2} \\
U_2(r) &> 0 \quad \forall r \in [2MG, +\infty).
\end{align*}
$$

In order to use the WKB approximation, we define two $r$-dependent radial wave numbers $k_1(l, r, \omega_{1,nl})$ and $k_2(l, r, \omega_{2,nl})$:

$$
\begin{cases}
k_1^2(l, r, \omega_{1,nl}) = \omega_{1,nl}^2 - \frac{l(l+1)}{r^2} - m_1^2(r) \\
k_2^2(l, r, \omega_{2,nl}) = \omega_{2,nl}^2 - \frac{l(l+1)}{r^2} - m_2^2(r),
\end{cases}
$$

where we have defined two $r$-dependent effective masses $m_1^2(r)$ and $m_2^2(r)$. The WKB approximation we will use to evaluate equation (56) is equivalent to the scattering phase shift
method and to the entropy computation in the brick wall model. We begin by counting the number of modes with frequency less than $\omega_i$, $i = 1, 2$. This is given approximately by

$$\tilde{g}(\omega_i) = \int \nu_i(l, \omega_i) (2l + 1),$$  

(73)

where $\nu_i(l, \omega_i)$, $i = 1, 2$, is the number of nodes in the mode with $(l, \omega_i)$, such that $(r \equiv r(x))$

$$\nu_i(l, \omega_i) = \frac{1}{2} \pi \int_{-\infty}^{+\infty} dx \sqrt{k^2_i(r, l, \omega_i)}.$$

(74)

Here, it is understood that the integration with respect to $x$ and $l$ is taken over those values which satisfy $k^2_i(r, l, \omega_i) \geq 0$, $i = 1, 2$. With the help of equations (73) and (74), we obtain the one-loop total energy for TT tensors which is

$$\frac{1}{8} \pi^2 \sum_{i = 1}^{2} \int_{-\infty}^{+\infty} dx \left[ \int_{0}^{+\infty} d\omega_i \frac{d\tilde{g}(\omega_i)}{d\omega_i} d\omega_i \right].$$

(75)

By extracting the energy density contributing to the cosmological constant, we get

$$\Lambda'_c = \Lambda'_{c,1} + \Lambda'_{c,2} = \rho_1 + \rho_2 = -\sqrt{h(R)} \frac{\kappa}{16\pi^2}$$

$$\times \left\{ \int_{0}^{+\infty} \omega_1^2 \sqrt{\omega_1^2 - m_1^2(r)} d\omega_1 + \int_{0}^{+\infty} \omega_2^2 \sqrt{\omega_2^2 - m_2^2(r)} d\omega_2 \right\},$$

(76)

where we have included an additional $4\pi$ coming from the angular integration.

### 6. One-loop energy regularization and renormalization

In this section, we will use the zeta function regularization method to compute the energy densities $\rho_1$ and $\rho_2$. Note that this procedure is completely equivalent to the subtraction procedure of the Casimir energy computation where the zero-point energy (ZPE) in different backgrounds with the same asymptotic properties is involved. To this purpose, we introduce the additional mass parameter $\mu$ in order to restore the correct dimension for the regularized quantities. Such an arbitrary mass scale emerges unavoidably in any regularization scheme. Then we have

$$\rho_i(\varepsilon) = -\sqrt{h(R)} \frac{\kappa}{16\pi^2} \mu^2 \int_{0}^{+\infty} d\omega_i \frac{\omega_i^2}{(\omega_i^2 - m_i^2(r))^{\varepsilon - 1}},$$

(77)

where

$$\begin{cases} 
\rho_1(\varepsilon) = -\sqrt{h(R)} \frac{\kappa}{16\pi^2} \int_{0}^{+\infty} \omega_1^2 \sqrt{\omega_1^2 - m_1^2(r)} d\omega_1 \\
\rho_2(\varepsilon) = -\sqrt{h(R)} \frac{\kappa}{16\pi^2} \int_{0}^{+\infty} \omega_2^2 \sqrt{\omega_2^2 - m_2^2(r)} d\omega_2.
\end{cases}$$

(78)

The integration has to be meant in the range where $\omega_i^2 - m_i^2(r) \geq 0^7$. One gets

$$\rho_i(\varepsilon) = \sqrt{h(R)} \kappa \frac{m_i^2(r)}{256\pi^2} \left[ \frac{1}{\varepsilon} + \ln \left( \frac{\mu^2}{m_i^2(r)} \right) + 2 \ln 2 - \frac{1}{2} \right],$$

(79)

$i = 1, 2$. In order to renormalize the divergent ZPE, we write

$$\Lambda'_c = 8\pi G [\rho_1(\varepsilon) + \rho_2(\varepsilon) + \rho_1(\mu) + \rho_2(\mu)].$$

(80)

Details of the calculation can be found in the appendix.
where we have separated the divergent part from the finite part. For practical purposes, it is useful to divide $\Lambda'_c$ with the factor $\sqrt{h(R)}$. To this aim, we define

$$\frac{\Lambda'_c}{\sqrt{h(R)}} = \left[ \Lambda_c + \frac{1}{2V} \int_{\Sigma} \frac{d^3x}{\sqrt{g}} \frac{Rf'(R) - f(R)}{f'(R)} \right] \frac{1}{\sqrt{h(R)}}$$

(81)

and we extract the divergent part of $\Lambda_c$ in the limit $\epsilon \to 0$, by setting

$$\Lambda^\text{div} = 8\pi G \left[ \rho_1(\epsilon) + \rho_2(\epsilon) \right] = \frac{G}{32\pi \epsilon} \left[ m_1^2(r) + m_2^2(r) \right].$$

(82)

Thus, the renormalization is performed via the absorption of the divergent part into the redefinition of the bare classical cosmological constant $\Lambda_c$, that is

$$\Lambda_c \to \Lambda_0 + \sqrt{h(R)} \Lambda^\text{div}.$$

(83)

The remaining finite value for the cosmological constant reads

$$\Lambda'_0(\mu) = \rho_1(\mu) + \rho_2(\mu) = \frac{1}{256\pi^2} \left[ m_1^4(r) \left[ \ln \left( \frac{\mu^2}{m_1^2(r)} \right) + 2 \ln 2 - \frac{1}{2} \right] + m_2^4(r) \left[ \ln \left( \frac{\mu^2}{m_2^2(r)} \right) + 2 \ln 2 - \frac{1}{2} \right] \right] = \rho_{\text{eff}}^{TT}(\mu, r),$$

(84)

where

$$\Lambda'_0(\mu) = \frac{1}{\sqrt{h(R)}} \left[ \Lambda_0(\mu) + \frac{1}{2V} \int_{\Sigma} \frac{d^3x}{\sqrt{g}} \frac{Rf'(R) - f(R)}{f'(R)} \right]$$

(85)

is the modified cosmological constant. The quantity in equation (84) depends on the arbitrary mass scale $\mu$. It is appropriate to use the renormalization group equation to eliminate such a dependence. To this aim, we impose that [56]

$$\frac{1}{8\pi G} \frac{\partial \Lambda'_0(\mu)}{\partial \mu} = \mu \frac{d \rho_{\text{eff}}^{TT}(\mu, r)}{d\mu}.$$  

(86)

Solving it, we find that the renormalized constant $\Lambda_0$ should be treated as a running one in the sense that it varies, provided that the scale $\mu$ is changing.

$$\Lambda'_0(\mu, r) = \Lambda'_0(\mu_0, r) + \frac{G}{16\pi} \left[ m_1^4(r) + m_2^4(r) \right] \ln \frac{\mu}{\mu_0}.$$  

(87)

Substituting equation (87) into equation (84) we find

$$\frac{\Lambda'_0(\mu_0, r)}{8\pi G} = -\frac{1}{256\pi^2} \left[ m_1^4(r) \left[ \ln \left( \frac{m_1^2(r)}{\mu_0^2} \right) - 2 \ln 2 + \frac{1}{2} \right] + m_2^4(r) \left[ \ln \left( \frac{m_2^2(r)}{\mu_0^2} \right) - 2 \ln 2 + \frac{1}{2} \right] \right].$$

(88)

It is worth remarking that while $m_2^2(r)$ is constant in sign, $m_1^2(r)$ is not. Indeed, for the critical value $\bar{r} = 5MG/2$, $m_1^2(\bar{r}) = m_2^2$ and in the range $(2MG, 5MG/2)$ for some values of $m_2^2$, $m_1^2(\bar{r})$ can be negative. It is interesting therefore to concentrate in this range. To further proceed, we observe that $m_1^2(r)$ and $m_2^2(r)$ can be recast into a more suggestive and useful form, namely

$$\left\{ \begin{align*}
m_1^2(r) &= U_1(r) = m_1^2(r, M) - m_2^2(r, M) \\
m_2^2(r) &= U_2(r) = m_1^2(r, M) + m_2^2(r, M),
\end{align*} \right.$$  

(89)

Since $m_1^2(r)$ can change in sign, when we integrate over $\alpha_1$ we can use either $I_1$ or $I_2$. This leads to the appearance of the absolute value.
where \( m_1^2(r, M) \to 0 \) when \( r \to \infty \) or \( r \to 2MG \) and \( m_2^2(r, M) = 3MG/r^3 \). Nevertheless, in the above-mentioned range \( m_1^2(r, M) \) is negligible when compared with \( m_2^2(r, M) \). So, in a first approximation we can write

\[
\begin{align*}
& m_1^2(r) \approx -m_2^2(r_0, M) = -m_2^2(M) \\
& m_2^2(r) \approx m_2^2(r_0, M) = m_2^2(M),
\end{align*}
\]

where we have defined a parameter \( r_0 > 2MG \) and \( m_2^2(M) = 3MG/r_0^3 \). The main reason for introducing a new parameter resides in the fluctuation of the horizon that forbids any kind of approach. Of course, the quantum fluctuation must obey the uncertainty relations. Thus, equation (88) becomes

\[
\frac{\Lambda_0'(\mu_0, r)}{8\pi G} = -m_2^4(M) \frac{\ln \left( \frac{m_2^2(M)}{4\mu_0^2} \right) + \frac{1}{2}}{128\pi^2}.
\]

Now, we compute the maximum of \( \Lambda_0' \), by setting

\[
x = \frac{m_0^2(M)}{4\mu_0^2}.
\]

Thus, \( \Lambda_0' \) becomes

\[
\Lambda_0'(\mu_0, x) = -\frac{G\mu_0^4}{\pi} x^2 \left[ \ln(x) + \frac{1}{2} \right].
\]

As a function of \( x \), \( \Lambda_0(\mu_0, x) \) vanishes for \( x = 0 \) and \( x = \exp(-\frac{1}{2}) \) and when \( x \in \left[ 0, \exp(-\frac{1}{2}) \right], \Lambda_0(\mu_0, x) \geq 0 \). It has a maximum for

\[
\bar{x} = \frac{1}{e} \iff m_0^2(M) = \frac{4\mu_0^2}{e}
\]

and its value is

\[
\Lambda_0(\mu_0, \bar{x}) = \frac{G\mu_0^4}{2\pi e^2} \text{ or } \frac{1}{\sqrt{h(R)}}
\]

\[
\times \left[ \Lambda_0(\mu_0, \bar{x}) + \frac{1}{2V} \int \Sigma d^3x \sqrt{g} \frac{Rf'(R) - f(R)}{f'(R)} \right] = \frac{G\mu_0^4}{2\pi e^2}.
\]

Isolating \( \Lambda_0(\mu_0, \bar{x}) \), we get

\[
\Lambda_0(\mu_0, \bar{x}) = \sqrt{h(R)} \frac{G\mu_0^4}{2\pi e^2} - \frac{1}{2V} \int \Sigma d^3x \sqrt{g} \frac{Rf'(R) - f(R)}{f'(R)}.
\]

Note that \( \Lambda_0(\mu_0, \bar{x}) \) can be set to zero when

\[
\sqrt{h(R)} \frac{G\mu_0^4}{2\pi e^2} = \frac{1}{2V} \int \Sigma d^3x \sqrt{g} \frac{Rf'(R) - f(R)}{f'(R)}.
\]

Let us see what happens when \( f(R) = \exp(-\alpha R) \). This choice is simply suggested by the regularity of the function at every scale and by the fact that any power of \( R \), considered as a correction to GR, is included. In this case, equation (97) becomes

\[
\sqrt{3\alpha \exp(-\alpha R) + 2\frac{G\mu_0^4}{\alpha \exp(-\alpha R)}} = \frac{1}{\alpha V} \int \Sigma d^3x \sqrt{g}(1+\alpha R).
\]

For Schwarzschild, it is \( R = 0 \), then

\[
\frac{G\mu_0^4}{\alpha \exp(-\alpha R)} = \sqrt{\frac{1}{(3\alpha + 2\alpha)}}.\]
By setting $\alpha = G$, we have the relation

$$\mu_0^4 = \pi e^2 \frac{1}{G \sqrt{(3G + 2)G}}.$$  \hfill (100)

**Remark.** Note that in any case, the maximum of $\Lambda$ corresponds to the minimum of the energy density.

### 7. Summary and conclusions

Despite of the successes of general relativity, such a theory can only be considered as a step towards a much more complete and comprehensive structure due to a large number of weaknesses. Among them, the issue to find out the fundamental gravitational vacuum state is one of the main problems to achieve a definite quantum gravity theory which till now is lacking. However, several semiclassical approaches have been proposed and, from several points of view, it is clear that the former Hilbert–Einstein scheme has to be enlarged. The $f(R)$ theories of gravity are a minimal but well-founded extension of GR where the form of the function $f(R)$ is not supposed ‘a priori’ but is reconstructed by the observed dynamics at galactic and cosmological scales [31, 34]. Also if they seem a viable scheme from cosmology and astrophysics viewpoints, their theoretical foundation has to be sought at a fundamental level. In particular, one has to face the possibility of encompassing the $f(R)$ gravity in the general framework of quantum field theory on curved spacetime. In this paper, we have dealt with the problem to find out vacuum states for $f(R)$ gravity via the $(3+1)$ ADM formalism. Analogously to GR, we have constructed the Hamiltonian constraint of a generic $f(R)$ theory and then achieved a canonical quantization giving the $f(R)$-WDW equation. In this context, the cosmological constant (vacuum state) emerges as a WDW eigenvalue. The related wave functional can be split by an orthogonal decomposition and then, constructing the transverse-traceless propagator, it is possible to obtain, after a variational minimization, the total one-loop energy density for the TT tensors. Such a quantity explicitly depends on the form of $f(R)$. As an application, we derive the energy density contributions to the cosmological constant for a TT spin-2 operator in the Schwarzschild metric and in the WKB approximation. The one-loop energy regularization and renormalization are achieved by the zeta function regularization method. The resulting renormalized $\Lambda_0$ is a running constant which can be set to zero depending on the value of an arbitrary mass scale parameter $\mu$. As an explicit calculation, we find out the value of such a parameter for a theory of the form $f(R) = \exp(-\alpha R)$ in the Schwarzschild metric. This case can be used for several applications at cosmological and astrophysical scales. In particular, truncated versions of such an exponential function, power law $f(R)$, have been used for galactic dynamics [32, 34, 35]. In those cases, a corrected Newtonian potential, derived from the $f(R)$ Schwarzschild solution, has been used to fit, with great accuracy, data from low surface brightness galaxies without using dark matter haloes. This approach has allowed to fix a suitable mass scale comparable with the core size of galactic systems. Such a mass can be directly related to the above parameter $\alpha$ depending on the core radius $r_c$ (see also [35]).

In summary, the application of quantum field theory methods to $f(R)$ gravity seems a viable scheme and gives positive results towards the issue to select vacuum states (eigenvalues) which can be interpreted as the cosmological constant. However, further studies are needed in order to generalize such results to other metrics and other extended theories of gravity.
Appendix. Zeta function regularization

In this appendix, we report details on computation leading to expression (77). We begin with the following integral:

\[ \rho(\varepsilon) = \begin{cases} 
I_+ = \mu^{2\varepsilon} \int_0^{+\infty} d\omega \frac{\omega^2}{(\omega^2 + m^2(r))^{\varepsilon - \frac{1}{2}}} \\
I_- = \mu^{2\varepsilon} \int_0^{+\infty} d\omega \frac{\omega^2}{(\omega^2 - m^2(r))^{\varepsilon - \frac{1}{2}}} 
\end{cases} \]  

(A.1)

with \( m^2(r) > 0 \).

A.1. \( I_+ \) computation

If we define \( t = \omega/\sqrt{m^2(r)} \), the integral \( I_+ \) in equation (A.1) becomes

\[ \rho(\varepsilon) = \mu^{2\varepsilon} m^{4-2\varepsilon}(r) \frac{1}{2} \mu^{2\varepsilon} m^{4-2\varepsilon}(r) B\left(\frac{3}{2}, \varepsilon - 2\right) \]

(A.2)

\[ \frac{1}{2} \mu^{2\varepsilon} m^{4-2\varepsilon}(r) \frac{\Gamma\left(\frac{1}{2}\right) \Gamma(\varepsilon - 2)}{\Gamma(\varepsilon - \frac{1}{2})} = \frac{\sqrt{\pi}}{4} m^4(r) \left( \frac{\mu^2}{m^2(r)} \right)^\varepsilon \frac{\Gamma(\varepsilon - 2)}{\Gamma(\varepsilon - \frac{1}{2})}, \]

where we have used the following identities involving the beta function

\[ B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)} \quad \text{Re } x > 0, \text{ Re } y > 0 \]  

(A.3)

related to the gamma function by means of

\[ B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)}. \]  

(A.4)

Taking into account the following relations for the \( \Gamma \)-function

\[ \Gamma(\varepsilon - 2) = \frac{\Gamma(1 + \varepsilon)}{\varepsilon(\varepsilon - 1)(\varepsilon - 2)} \]

\[ \Gamma\left(\varepsilon - \frac{1}{2}\right) = \frac{\Gamma\left(\varepsilon + \frac{1}{2}\right)}{\varepsilon - \frac{1}{2}}, \]  

(A.5)

and the expansion for small \( \varepsilon \)

\[ \Gamma(1 + \varepsilon) = 1 - \gamma \varepsilon + O(\varepsilon^2) \]

\[ \Gamma\left(\varepsilon + \frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right) - \varepsilon \Gamma\left(\frac{1}{2}\right) (\gamma + 2 \ln 2) + O(\varepsilon^2) \]  

(A.6)

\[ x^\varepsilon = 1 + \varepsilon \ln x + O(\varepsilon^2), \]

where \( \gamma \) is the Euler’s constant, we find

\[ \rho(\varepsilon) = -\frac{m^4(r)}{16} \left[ \frac{1}{\varepsilon} + \ln \left( \frac{\mu^2}{m^2(r)} \right) + 2 \ln 2 - \frac{1}{2} \right]. \]  

(A.7)
A.2. $I_-$ computation

If we define $t = \omega/\sqrt{m^2(r)}$, the integral $I_-$ in equation (A.1) becomes

$$\rho(\varepsilon) = \mu^2 m^{4-2\varepsilon}(r) \int_0^{\infty} \frac{dt}{(t^2 - 1)^{\varepsilon - \frac{1}{2}}} = \frac{1}{2} \mu^2 m^{4-2\varepsilon}(r) B \left( \varepsilon - 2, \frac{3}{2} - \varepsilon \right)$$

$$\frac{1}{2} \mu^2 m^{4-2\varepsilon}(r) \frac{\Gamma \left( \frac{1}{2} - \varepsilon \right) \Gamma(\varepsilon - 2)}{\Gamma(-\frac{1}{2})} = -\frac{1}{4 \sqrt{\pi}} m^4(r) \left( \frac{\mu^2}{m^2(r)} \right)^{\varepsilon} \left( \frac{3}{2} - \varepsilon \right) \Gamma(\varepsilon - 2),$$

where we have used the following identity involving the beta function

$$\frac{1}{p} B \left( 1 - v - \frac{\mu}{p}, v \right) = \int_1^{\infty} \frac{dt}{t^{\mu-1} \left( t^p - 1 \right)^{v-1}}$$

$$p > 0, \quad \Re v > 0, \quad \Re \mu < p - p \Re v$$

and the reflection formula

$$\Gamma(z) \Gamma(1 - z) = -\pi \Gamma(-z) \Gamma(z).$$

From the first of equation (A.5) and from the expansion for small $\varepsilon$

$$\Gamma \left( \frac{3}{2} - \varepsilon \right) = \Gamma \left( \frac{3}{2} \right) (1 - \varepsilon(-\gamma - 2 \ln 2 + 2)) + O(\varepsilon^2)$$

$$x^\varepsilon = 1 + \varepsilon \ln x + O(\varepsilon^2),$$

we find

$$\rho(\varepsilon) = -\frac{m^4(r)}{16} \left[ \frac{1}{\varepsilon} + \ln \left( \frac{\mu^2}{m^2(r)} \right) + 2 \ln 2 - \frac{1}{2} \right].$$

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