Calculation of the vortex eigenfunctions of the finite propagation operator in the near-field diffraction

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Abstract. The propagation of vortex laser beams in the near diffraction (at a distance in the order of the wavelength) can be described by means of an expansion in plane waves, which after considering vortices reduces to an vortex propagation operator involving Fourier-Hankel transforms. The eigenfunctions of the operator, when eigenvalues are close to one, determine the characteristics of the signals (information) transmitted lossless (without distortion). The beam propagation distance, vortex order and the region of spatial frequency limitation are parameters of the operator and they essentially change the set of eigenvalues and functions. We calculate the vortex eigenfunctions of the finite propagation operator in the near diffraction zone and investigate their qualitative and quantitative characteristics depending on the propagation distance, the order of vortex and the constraints imposed in the object and spectral domains.

1. Introduction

Vortex optical beams with a phase singularity were first discussed in the work of Nye and Berry in 1974 [1]. In this paper vortex beams were identified with features of randomly scattered fields. Subsequent work on the study of optical and acoustic vortices also showed that vortex optical fields are generated by scattering of waves on rough surfaces [2, 3]. All optical vortices formed in this way are of the first order, since the vortex phase singularities of a higher order are unstable to random field perturbations and break up into the corresponding number of first-order optical vortices [4, 5].

It is important that vortex phase singularities can arise not only as a result of scattering in random media, but also be formed in laser resonators [6]. The Laguerre – Gaussian modes and the Bessel modes are well known examples of beams with helical phase dislocations, or phase vortices. The fundamental and applied aspects of this fact turned out to be so significant that they led to the emergence of a new scientific section, “singular optics” [7–9]. The field of application of vortex beams has a wide spectrum from optical manipulation of micro-objects to information transmission over long distances [10–20]. Of particular interest is the use of elements of singular optics in sharp focusing, when the combination of polarization and phase features leads to various effects, including overcoming the diffraction limit in the far-field diffraction [21, 22]. However, in this case, we need amplitude or phase apodization of the pupil of the focusing system [23-25], the use of special types of
polarization [26, 27] or the introduction of phase singularity into the beam [28], as well as combinations of all these factors [29, 30] with the purpose of optimization of the formed field [31, 32]. Another approach to overcoming the diffraction limit is the near-field optics [33–38], based on increasing the spatial frequency range, which ensures the conservation of the evanescent components of the source field. If a decrease in the size of the light spot outside the near-field zone, as a rule, is accompanied by a significant increase in side lobes [39, 40], in the region of the evanescent waves (at a distance less than the wavelength) there are no restrictions on the size of the light spot — the localization of a laser spot can be arbitrarily small, although it significantly depends on the size of the details of the focusing element [41, 42] or of the illuminating beams [43].

In this paper, the propagation of vortex laser beams in the near diffraction zone (a distance of the order of the wavelength) is considered using planewave expansion based on the Fourier – Hankel transform. The eigenfunctions of such operator, having eigenvalues close to one, determine the characteristics of signals (information) transmitted without loss (without distortion). The boundedness of the propagation operator in both the spatial and spectral regions leads to the necessity of numerical calculation [44–46] of eigenfunctions. The vortex eigenfunctions of a finite propagation operator in the near diffraction zone are calculated and their qualitative and quantitative characteristics are investigated depending on the propagation distance, the order of the vortex, and restrictions imposed in the object and spectral regions.

2. Theory
A scalar non-paraxial propagation operator using plane wave expansion is written in the form [21, 22]:

\[
E(u,v,z) = \int\int F(\xi,\eta)\exp\left(ikz\sqrt{1-\xi^2-\eta^2}\right)\exp\left[ik(\xi u + \eta v)\right]d\xi d\eta,
\]

(1)

where \( F(\xi,\eta) \) is the spatial spectrum of the plane wave expansion of the input field, and

\[
\Sigma_\sigma: \sigma_1 \leq \sqrt{\xi^2 + \eta^2} \leq \sigma_2 \quad \text{is the region of the spatial frequencies taken into account.}
\]

When \( \sigma_1 = 0, \sigma_2 = 1 \), only propagating waves are considered, and when \( \sigma_1 = 1, \sigma_2 > 1 \), only evanescent waves are considered.

In the case when the input field is a vortex with the \( m \)th vortex order:

\[
E_0(x,y) = E_0(r)\exp(im\phi),
\]

(2)

expression (1) can be simplified [43]:

\[
E(r,\theta,z) = k^2 \exp(im\theta) \int_0^{\sigma_0} \int_0^{\sigma_0} E_0(r)J_m(k\sigma r)rdr J_m(k\sigma)J_m(k\sigma)\sigma d\sigma,
\]

(3)

where \( r \) is the radial coordinate of the input field, \( \sigma \) is the radial coordinate in the frequency plane, \( \sigma_0 \) is the radius of the frequency plane domain, \( \rho \) is the radial coordinate of the output plane, and \( \theta \) is the angular coordinate in the output plane.

We write operator (3) in the form:

\[
E(r,\theta,z) = k^2 \exp(im\theta) \int_0^{\sigma_0} E_0(r)K_m(r,\rho,\theta)\rho d\rho,
\]

(4)

where

\[
K_m(r,\rho,\theta) = \int_0^{\sigma_0} \exp\left(ikz\sqrt{1-\sigma^2}\right)J_m\left(\frac{2\pi}{k}r\sigma\right)J_m\left(\frac{2\pi}{k}\rho\sigma\right)\sigma d\sigma.
\]

(5)

Then the problem of calculating the radial part of the vortex eigenfunctions in the near diffraction zone reduces to the search for the eigenfunctions of the bounded operator (5) in the form:

\[
E(r,\theta,z) = k^2 \exp(im\theta) \int_0^{\sigma_0} E_0(r)K_m(r,\rho,\theta)\rho d\rho,
\]

(4)

where

\[
K_m(r,\rho,\theta) = \int_0^{\sigma_0} \exp\left(ikz\sqrt{1-\sigma^2}\right)J_m\left(\frac{2\pi}{k}r\sigma\right)J_m\left(\frac{2\pi}{k}\rho\sigma\right)\sigma d\sigma.
\]

(5)
\[ b_{n,m}(z) = \int_0^1 \psi_{n,m}(r,z) K_m(r,p,z) r dr , \]

where \( z \) is the distance, \( b_{n,m}(z) \) are the eigenvalues, \( \psi_{n,m}(r,z) \) is the radial part of the eigenfunctions.

Obviously, the characteristics of the eigenfunctions will depend not only on the propagation distance \( z \) and order \( m \), but also on the restrictions imposed on the field in the object and spectral regions.

3. Calculation of the vortex eigenfunctions of a bounded propagation operator in the near diffraction zone

The calculation of eigenvalues and eigenfunctions was performed for various values of parameters at the test wavelength of laser radiation \( \lambda = 1 \mu m \).

The calculation of eigenfunctions in the region of evanescent waves, i.e. at a distance less than the wavelength \( z = 0.5 \lambda \) at \( \sigma_0 = 5 \), the radius of the input region is chosen equal \( r_0 = 10 \lambda \).

Fig. 1 shows the form of matrices (5), which are the kernels of the transformation (4) for \( m = 0, 1, 5, 9 \). As can be seen from Fig. 1, with an increase of the number \( m \), the region of zero values in the lower left part of the matrix increases. This is due to the structure of high order Bessel functions.

![Figure 1. Amplitude of matrices (5), which are the core of the transformation (4) with \( z = 0.5 \lambda \), \( \sigma_0 = 5 \), \( r_0 = 10 \lambda \) for \( m=0 \) (a), \( m=1 \) (b), \( m=5 \) (c), \( m=9 \) (d).](image)

Fig. 2 shows the graphs of the modules of eigenvalues \( b_{n,m}(z) \). Eigenvalue modules determine the "survival" possibility of the corresponding eigenmode at a given distance. If the eigenvalue is close to unity, then the mode will propagate without distortion and energy loss. Thus, the number of eigenvalues close to unity determines the number of degrees of freedom of the considered optical system and the possibility of its resolution.

As follows from fig. 2, the number of degrees of freedom decreases with increasing \( m \).

![Figure 2. Graphs of the eigenvalues' modules \( b_{n,m}(z) \) with \( z = 0.5 \lambda \), \( \sigma_0 = 5 \), \( r_0 = 10 \lambda \) for \( m=0 \) (red color), \( m=1 \) (green color), \( m=5 \) (blue color), \( m=9 \) (black color).](image)

Figure 3 shows the normalized graphs of the calculated eigenfunctions \( \psi_{n,m}(r,z) \). Since the functions in the general case are complex, only the real part is shown. The number of transitions of functions through zero corresponds to the index \( n \). It is evident from Fig. 3 that the eigenfunctions are zero at \( p=0 \) for \( m=0 \), and, with increasing \( m \), the central region of zero values of the functions increases. Note that simultaneously with increasing \( m \) the width of the oscillations in the peripheral
area of the function decreases, which corresponds to smaller details of the optical signal, which can be described by eigenfunctions.

![Graphs of the eigenfunctions](image)

**Figure 3.** Graphs of the eigenfunctions’ real part \( \psi_{n,m}(\rho, z) \) at \( z = 0.5\lambda, \sigma_0 = 5, r_0 = 10\lambda \) for \( n=3 \) (a) and \( n=8 \) (b), \( m=0 \) (red color), \( m=1 \) (green color), \( m=5 \) (blue color), \( m=9 \) (black color).

**Table 1.** Pictures and characteristics of eigenfunctions \( \Psi_{n,m,L}(\rho, \theta) \) for \( L = 0.5\lambda, \sigma_0 = 5, r_0 = 10\lambda \).

| Eigenfunction at \( z=0 \) (amplitude and phase) | Graph of a spatial spectrum | Eigenfunction at \( z=10\lambda \) (amplitude and phase) |
|--------------------------------------------------|-----------------------------|--------------------------------------------------|
| \((n,m)=(3,0)\) \( b_{3,0,L} = 1 \) | ![Image](image) | ![Image](image) |
| \((n,m)=(8,0)\) \( b_{8,0,L} = 0.99 \) | ![Image](image) | ![Image](image) |
| \((n,m)=(8,5)\) \( b_{8,5,L} = 0.98 \) | ![Image](image) | ![Image](image) |
| \((n,m)=(15,5)\) \( b_{15,5,L} = 0.90 \) | ![Image](image) | ![Image](image) |
| \((n,m)=(15,9)\) \( b_{15,9,L} = 0.72 \) | ![Image](image) | ![Image](image) |
4. Analysis of the vortex eigenfunctions properties

Table 1 shows examples of calculated eigenfunctions $\Psi_{n,m,L}(\rho,\theta)$, their spatial spectra and the results of propagation in free space for a distance outside the region of evanescent waves.

As can be seen from the shown results, with an increase in the index $n$, which corresponds to the number of rings in the distribution of the function, the size of the rings narrows and this corresponds to smaller details of the optical signal, which can be prescribed using the corresponding eigenfunctions (second column of Table 1). At a fixed value of the index $n$, an increase in the number of the vortex component $m$ leads to the formation of zero values in the central part of the region, and the width of the peripheral rings also decreases.

Obviously, a decrease in the size of the rings with an increase in the indices $n$ and $m$ corresponds to a shift of the spatial spectrum (the third column of Table 1) to the region of higher spatial frequencies. As the larger part of the energy of the spatial spectrum is in the region of evanescent waves ($\sigma > 1$), the more strongly the eigenfunction is distorted during propagation in free space. It is possible to estimate the fraction of energy concentrated in the region of propagating waves ($0 < \sigma < 1$) by the modulus of the eigenvalue. If it is equal to unity, then the entire spectrum of spatial frequencies is in the region of propagating waves, and the eigenfunction will not change with propagation up to the distance $z = L$.

Note that although the eigenfunctions shown in Tab. 1 were calculated for $L = 0.5\lambda$, they retain their appearance for much longer, up to $z = 10\lambda$ (fourth column of Table 1), if their eigenvalues are close to unity (first column of Table 1).

The calculated functions can be used to form an approximation of a given signal passing through the considered system without distortion and energy loss [46], as well as for solving the inverse diffraction problem, including the problem of diffraction limit overcoming [47].

5. Conclusion

In this paper, a parametric method is proposed for calculating the vortex eigenfunctions of free space in the near diffraction zone, while limiting the spatial frequency range. In this case, the beam propagation distance (of the order of several wavelengths) and the boundary of spatial frequency region are system parameters and significantly change the set of eigenvalues and eigenfunctions, determining the number of degrees of freedom for approximating a given field. The vortex eigenfunctions of a restricted propagation operator in the near diffraction zone are calculated. It is shown that the number of eigenfunctions with eigenvalues close to unity, i.e. the number of degrees of freedom of the optical system and the possibility of its resolution decreases with increasing order of the vortex singularity. However, simultaneously with an increase in $m$, the width of the rings in the peripheral region of the eigenfunction decreases, which corresponds to a decrease in the details of the optical signal, which can be prescribed using eigenfunctions.

6. References

[1] Nye J F and Berry M V 1974 Dislocations in wave trains Proceedings of the Royal Society A 336 165-190
[2] Berry M V 1981 Singularities in waves and rays Physics of Defects 35 453-543
[3] Baranova N B, Mamaev A V, Pilipetskii N F, Shkukov V V and Zel’dovich B Ya 1983 Wavefront dislocations: topological limitations for adaptive systems with phase conjugation Journal of the Optical Society of America 73 525-528
[4] Bazhenov V Yu, Soskin M S and Vasnetsov M V 1992 Screw dislocations in light wavefronts Journal of Modern Optics 39 985-990
[5] Gbur G and R K Tyson 2008 Vortex beam propagation through atmospheric turbulence and topological charge conservation Journal of the Optical Society of America A 25 225
[6] Siegman A E 1986 Lasers (Sausalito, California: University Science Books) p 1283
[7] Vasnetsov M V and Staliunas K 1999 Optical Vortices (New York: Nova Science) p 218
[8] Soskin M S and Vasnetsov M V 2001 Singular optics Progress in Optics (Amsterdam: North Holland) p 42
[9] Gbur G J 2017 Singular Optics (Boca Raton, FL: CRC Press) p 545
[10] He H, Heckenberg N R and Rubinsztein-Dunlop H 1995 Optical particle trapping with higher-order doughnut beams produced using high efficiency computer-generated holograms J. Mod. Opt. 42 217-223

[11] Kotlyar V V, Soifer V A and Khonina S N 1997 An algorithm for the generation of laser beams with longitudinal periodicity: rotating images J. Modern Opt 44 1409-1416

[12] Török P and Munro P R T 2004 The use of Gauss-Laguerre vector beams in STED microscopy Opt. Express 12 3605-3617

[13] Khonina S N and Golub I 2012 How low can STED go? Comparison of different write-erase beam combinations for stimulated emission depletion microscopy J. Opt. Soc. Am. A 29 2242-2246

[14] Wang J, Yang J-Y, Fazal I M, Ahmed N, Yan Y, Huang H, Ren Y, Yue Y, Dolinar S, Tur M and Willner A E 2012 Terabit free-space data transmission employing orbit angular momentum multiplexing Nature Photonics 6 488-496

[15] Soifer V A, Korotkova O, Khonina S N and Shchepakina E A 2016 Vortex beams in turbulent media: review Computer Optics 40(5) 605-624 DOI: 10.18287/2412-6179-2016-40-5-605-624

[16] Porfirov A P, Kirilenko M S, Khonina S N, Skidanov R V and Soifer V A 2017 Study of propagation of vortex beams in aerosol optical medium Applied Optics 56 E8-E15

[17] Khonina, S N, Karpeev S V and Paranin V D 2018 A technique for simultaneous detection of individual vortex states of Laguerre–Gaussian beams transmitted through an aqueous suspension of microparticles Optics and Lasers in Engineering 105 68-74

[18] Lochab P, Senthilkumaran P and Khare K 2018 Designer vector beams maintaining a robust intensity profile on propagation through turbulence Physical Review A 98 023831

[19] Cheng M, Guo L, Li J, Li J and Yan X 2019 Enhanced vortex beams resistance to turbulence with polarization modulation J. of Quant. Spect. and Rad. Tran. 227 219-225

[20] Liu Y, Zhang K, Chen Z and Pu J 2019 Scintillation index of double vortex beams in turbulent atmosphere Optik 181 571-574

[21] Khonina S N and Golub I 2011 Optimization of focusing of linearly polarized light Optics Letters 36 352-354

[22] Khonina S N 2013 Simple phase optical elements for narrowing of a focal spot in high-numerical-aperture conditions Optical Engineering 52 091711

[23] Quabis S, Dorn R, Eberler M, Glockl O and Leuchs G 2000 Focusing light to a tighter spot Opt. Commun. 179 1-7

[24] Wang H, Shi L, Lukyanchuk B, Sheppard C and Chong Ch T 2008 Creation of a needle of longitudinally polarized light in vacuum using binary optics Nature Photonics 2 501-505

[25] Khonina S N, Ustinov A V and Pelevina E A 2011 Analysis of wave aberration influence on reducing focal spot size in a high-aperture focusing system J. Opt. 13 095702

[26] Dorn R, Quabis S and Leuchs G 2003 Sharper focus for a radially polarized light beam Phys. Rev. Lett. 91 233901

[27] Kozawa Y and Sato S 2007 Sharper focal spot formed by higher-order radially polarized laser beams J. Opt. Soc. Am. A 24 1793-1798

[28] Helseth L E 2004 Optical vortices in focal regions Opt. Commun 229 85-91

[29] Pereira S F and Van de Nes A S 2004 Superresolution by means of polarisation, phase and amplitude pupil masks Opt. Commun. 234 119-124

[30] Beversluis M R, Novotny L and Stranick S J 2006 Programmable vector point-spread function engineering Opt. Express 14 2650-2656

[31] Rao L, Pu J, Chen Zh and Ye P Focus shaping of cylindrically polarized vortex beams by a high numerical-aperture lens Opt. & Las. Techn. 41 241-246

[32] Khonina S N and Golub I 2012 Enlightening darkness to diffraction limit and beyond: comparison and optimization of different polarizations for dark spot generation J. Opt. Soc. Am. A 29 1470-1474

[33] Betzig E and Trautman J K 1992 Near-field optics: microscopy, spectroscopy, and surface modification beyond the diffraction limit Science 257 189-194

[34] Girard C and Dereux A 1996 Near-field optics theories Rep. Prog. Phys. 59 657-699
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