On groups of finite upper rank

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Rank and upper rank

For a finite group $G$ with Sylow $p$-subgroup $P$ the rank and the $p$-rank of $G$ are defined by

$$r(G) = \sup\{d(H) \mid H \leq G\},$$
$$r_p(G) = r(P),$$

where as usual $d(H)$ denotes the minimal size of a generating set for $H$. When $G$ is an arbitrary group, $\mathcal{F}(G)$ denotes the set of finite quotient groups of $G$, and we define the (‘local’ and ‘global’) upper ranks of $G$:

$$ur_p(G) = \sup\{r_p(Q) \mid Q \in \mathcal{F}(G)\}$$
$$ur(G) = \sup\{r(Q) \mid Q \in \mathcal{F}(G)\}.$$

A theorem of Lucchini [L], first proved for soluble groups by Kovács [K], asserts that for a finite group $G$,

$$\sup_p r_p(G) \leq r(G) \leq 1 + \sup_p r_p(G),$$

so the analogue holds for the upper ranks of an infinite group; in particular, $ur(G)$ is finite if and only if the local upper ranks $ur_p(G)$ are bounded as $p$ ranges over all primes.

Let us denote by $\mathcal{U}$ the class of all groups $G$ such that $ur_p(G)$ is finite for every prime $p$. One can describe $\mathcal{U}$ more colourfully as the class of groups whose profinite completion has a $p$-adic analytic Sylow pro-$p$ subgroup for every prime $p$ [DDMS].

Background

More than 20 years ago, Alex Lubotzky conjectured that there is a ‘subgroup growth gap’ for finitely generated soluble groups. We had recently established that a finitely generated (f.g.) residually finite group has polynomial subgroup growth if and only if it is virtually a soluble minimax group (see [LMS] or [LS], Chapter 5). I showed in [S3] that there exist f.g. groups of arbitrarily slow non-polynomial subgroup growth; the Lubotzky question amounts to: do there exist such groups that are soluble?
Now if a f.g. soluble group $G$ has subgroup growth of type strictly less than $n^{\log n/(\log \log n)^2}$ then $ur_p(G)$ is finite for every prime $p$ ([MS], Prop. 2.6, [S2], Proposition C). On the other hand, it is known that a finitely generated residually finite group has finite upper rank if and only if it is virtually a soluble minmax group [MS1]. So Lubotzky’s conjecture would follow from

**Conjecture A** [S2] Let $G$ be a f.g. soluble group. If $G \in \mathcal{U}$ then $G$ has finite upper rank.

Equivalently: if the upper $p$-ranks of $G$ are all finite, then they are bounded. If $G$ is assumed to be residually finite, this conclusion is equivalent to saying that $G$ is a minmax group.

In fact, Conjecture A would imply that a f.g. soluble group cannot have subgroup growth of type strictly between polynomial and $n^{\log n}$ ([S5], Proposition 5.1).

I am now doubtful about this conjecture, having spent over two decades failing to prove it. What follows is a survey of what is known on the topic.

**Olshanski-Osin groups**

In [MS] we raised the question: is Conjecture A true even without the solubility hypothesis? If $G$ is a group with $ur_2(G)$ finite then $G$ has a subgroup $H$ of finite index such that every finite quotient of $H$ is soluble ([LS], Theorem 5.5.1). This (at first sight surprising) consequence of the Odd Order Theorem suggests that the solubility hypothesis in Conjecture A may be redundant. Without that hypothesis, however, the conjecture is false, as was recently pointed out to me by Denis Osin. I am very grateful to him for allowing me to reproduce his argument here. It depends on

**Theorem 1** ([OO] Theorem 1.2) Let $P = (p_i)$ be an infinite sequence of primes. There exists an infinite 2-generator periodic group $G(P) = G_0$ having a descending chain of normal subgroups $(G_i)_{i \geq 0}$ with $\bigcap G_i = 1$ such that $G_{i-1}/G_i$ is abelian of exponent dividing $p_i$ for each $i \geq 1$.

Now let $G = G(P)$ where $P$ consists of distinct primes. Each quotient $G/G_n$ is finite. Given $m \in \mathbb{N}$ there exists $n$ such that $p_i \nmid m$ for all $i \geq n$. It is easy to see that each element of $G_n$ has order coprime to $m$, whence $G_n \leq G^m$. It follows that for each prime $p$,

$$ur_p(G) = \sup \{ ur_p(G/G^m) \mid m \in \mathbb{N} \}$$

$$= \sup \{ r_p(G/G_n) \mid n \in \mathbb{N} \} = \left\{ \begin{array}{ll} r_p(G/G_k) & \text{if } p = p_k \\ 0 & \text{if } p \neq p_i \forall i \end{array} \right\} < \infty.$$

Thus $G \in \mathcal{U}$. On the other hand, $G$ is residually finite and not virtually soluble (as it is infinite, f.g. and periodic), and so $G$ has infinite upper rank by the theorem from [MS1] quoted above.

Whether Conjecture A holds with ‘soluble’ replaced by ‘torsion-free’ is still an open problem.
The groups of slow subgroup growth constructed in [S3] are built out of finite simple groups. The groups $G(P)$, in contrast, have all their finite quotients soluble: I call such groups of prosoluble type (because their profinite completions are prosoluble). As far as I know, these provide the first such examples with arbitrarily slow non-polynomial subgroup growth; they show that Lubotzky’s conjecture becomes false if ‘soluble’ is replaced by ‘of prosoluble type’:

**Proposition 2** let $f : \mathbb{N} \to \mathbb{R}_{>0}$ be an unbounded non-decreasing function. Then there exists a sequence $P$ of primes such that the group $G = G(P)$ satisfies

$$s_n(G) \leq n^{f(n)}$$

for all large $n$, but $G$ does not have polynomial subgroup growth.

Here, $s_n(G)$ denotes the number of subgroups of index at most $n$ in $G$.

**Proof.** Suppose that $P = (p_i)$ is a strictly increasing sequence of primes. Let $H$ be a proper subgroup of index $\leq n$ in $G$. Then $G_0 > H \geq G_{n!} \geq G_k$ for some $k$. Let $k$ be minimal such. Then $G_k \leq H \cap G_{k-1} < G_{k-1}$, so

$$p_k \mid |G_{k-1} : H \cap G_{k-1}| \leq n.$$

It follows that

$$s_n(G) = s_n(G_{k(n)})$$

where $k(n)$ is the largest $k$ such that $p_k \leq n$.

Put $Q_n = G/G_{k(n)}$. According to [LS], Corollary 1.7.2,

$$s_n(Q_n) \leq n^{2+r(n)}$$

where $r(n) = \max_p r_p(Q_n)$. Write $m_j = |G : G_{j-1}|$ for each $j \geq 1$. Since $G$ is a 2-generator group, $G_{j-1}$ can be generated by $1 + m_j$ elements, and so $r_{p_j}(Q_n) \leq 1 + m_j$ for $j \leq k(n)$, while $r_p(Q_n) = 0$ if $p \notin \{p_1, \ldots, p_{k(n)}\}$.

Now we can choose the sequence $P$ recursively as follows: $p_1$ is arbitrary. Set $\mu_1 = 1$. Given $p_i$ and $\mu_i$ for $i \leq t$, set

$$\mu_{t+1} = \mu_t \cdot p_t^{1+\mu_t}$$

and let $p_{t+1} > p_t$ be a prime so large that

$$f(p_{t+1}) \geq 3 + \mu_{t+1}.$$

Note that $|G_{j-1} : G_j| \leq p_j^{1+m_j}$ for each $j$, so $m_{j+1} \leq m_j \cdot p_j^{1+m_j}$. It follows that $m_j \leq \mu_j$ for all $j$. Then

$$r(n) \leq \max\{1 + m_j \mid j \leq k(n)\} \leq \max\{1 + \mu_j \mid j \leq k(n)\} = 1 + \mu_{k(n)} \leq f(p_{k(n)}) - 2 \leq f(n) - 2.$$
Thus
\[ s_n(G) = s_n(Q_n) \leq n^{2+r(n)} \leq n^{f(n)}. \]

Of course, \( G \) does not have polynomial subgroup growth because it has infinite upper rank, as observed above. 

**Minimax groups: a reminder**

Let us denote by \( S \) the class of all residually finite virtually soluble minimax groups. The following known results will be used without special mention:

- If \( G \in S \) then \( G \) is virtually nilpotent-by-abelian.
- If \( G \in S \) then \( G \) is virtually residually (finite nilpotent).
- A minimax group is in \( S \) if and only if it is virtually torsion-free.
- The class \( S \) is extension-closed.
- If \( G \) has a nilpotent normal subgroup \( N \) such that \( G/N' \) is (a) minimax resp. (b) of finite upper rank, then \( G \) is (a) minimax, resp. (b) of finite upper rank.
- Let \( G \) be f.g. and residually finite. Then \( \text{ur}(G) \) is finite if and only if \( G \in S \).

For most of these, see [LR], Chapter 5 and Chapter 1. The penultimate claim is an easy consequence of [LR], 1.2.11. The final claim is [MS1], Theorem A.

**Some known cases**

**Proposition 3** Let \( G \) be a f.g. nilpotent-by-polycyclic group. If \( G \in U \) then \( G \) is a minimax group.

**Proof.** Let \( N \) be a nilpotent normal subgroup of \( G \) with \( G/N \) polycyclic. It will suffice to show that \( G/N' \) is minimax, so replacing \( G \) by this quotient we may assume that \( N \) is abelian. Then \( N \) is Noetherian as a \( G/N \)-module, so the torsion subgroup \( T \) of \( N \) has finite exponent, \( e \) say. Let \( \sigma \) be the set of prime divisors of \( e \).

By P Hall’s ‘generic freeness lemma’ (cf. [LR], 7.1.6) \( N/T \) has a free abelian subgroup \( F_1/T \) such that \( N/F_1 \) is a \( \pi \)-group for some finite set of primes \( \pi \). Then \( F_1 = T \times F \) where \( F \) is free abelian, and \( N/F \) is a \( \pi \cup \sigma \)-group.

Let \( p \not\in \pi \cup \sigma \) be a prime. Then \( N = FN^p \) and \( N^p \cap F = F^p \), so \( F/F^p \cong N/N^p \). Now \( G/N^p \) is residually finite and the image of \( N/N^p \) in any finite quotient of \( G/N^p \) has rank at most \( \text{ur}_p(G) = r_p \); it follows that \( |F/F^p| = |N/N^p| \leq p^{r_p} \). Therefore \( F \) has rank at most \( r_p \). Hence for each prime \( q \not\in \pi \cup \sigma \) we have
\[ \text{ur}_q(G) \leq \text{ur}(G/N) + \text{ur}(F) \leq \text{ur}(G/N) + r_p. \]
It follows that \( ur(G) \) is finite, since \( G/N \) is polycyclic and \( \pi \cup \sigma \) is finite.

As \( G \) is residually finite it follows that \( G \) is a minimax group. (For the quoted properties of f.g. abelian-by-polycyclic groups, see for example \([LR]\), Chapters 4 and 7.)

The upper \( p \)-rank of a group \( G \) can equivalently be defined as the rank of a Sylow pro-\( p \) subgroup \( P \) of \( \hat{G} \), the profinite completion of \( G \), where for a profinite group \( P \), the rank of \( P \) is

\[
\text{r}(P) = \sup \{ \text{r}(P/N) \mid N \triangleleft P, \ N \text{ open} \}.
\]

The pro-\( p \) groups of finite rank are well understood (see \([DDMS]\)); in particular, they are linear groups in characteristic 0.

**Proposition 4** ([LS], Window 8, Lemma 9) Let \( K \) be a f.g. residually nilpotent group. Suppose that the pro-\( p \) completion \( \hat{K}_p \) of \( K \) has finite rank for some prime \( p \). Then there exists a finite set of primes \( \pi \) such that the natural map

\[
K \rightarrow \prod_{q \in \pi} \hat{K}_q
\]

is injective.

This is the key to

**Theorem 5** ([S5], Theorem 5) Let \( G \) be a f.g. group that is virtually residually nilpotent. If \( G \in \mathcal{U} \) then \( G \) has finite upper rank.

**Proof.** It follows from Proposition 4 that \( G \) has a subgroup \( K \) of finite index such that \( K \) is residually (finite nilpotent of rank at most \( r \)); here \( r = \max_{q \in \pi} ur_q(G) \). By a result mentioned above, we may also take it that every finite quotient of \( K \) is soluble. The main result of [S1] now shows that \( K \) is virtually nilpotent-by-abelian (see also \([LS]\), Window 8, Corollary 5), and the result follows by Proposition 3.

Let \( H \) denote the class of all groups \( G \) with the property: every virtually residually nilpotent quotient of \( G \) is a minimax group.

Theorem 5 shows that finitely generated groups in \( \mathcal{U} \) belong to \( H \). It is not true that every f.g. soluble residually finite group in \( H \) has finite upper rank:

**Proposition 6** ([PS], Proposition 10.1) Let \( p \) be a prime, let

\[
H = \langle x_n \mid n \in \mathbb{Z} \rangle; \ x_p^n = x_{n-1}
\]

be the additive group of \( \mathbb{Z}[1/p] \) written multiplicatively, and let \( \tau \) be the automorphism of \( H \) sending \( x_n \) to \( x_{n+1} \) for each \( n \). Extend \( \tau \) to an automorphism of the group algebra \( \mathbb{F}_pH \) and then to an automorphism of \( \mathbb{W} = \mathbb{F}_pH \rtimes H = C_p \rtimes H \). Set \( G = \mathbb{W} \rtimes \langle \tau \rangle \). Then
• $G$ is a 3-generator residually finite abelian-by-minimax group
• $G \in \mathcal{H}$
• $ur_q(G) = 2$ for every prime $q \neq p$
• $ur_p(G)$ is infinite.

This shows, also, that the hypothesis of Conjecture A can’t be weakened by omitting finitely many primes.

Still, the strongest result so far obtained towards Conjecture A rests on a consideration of certain groups in $\mathcal{H}$. It seems clear that the trouble with the last example is due to the presence of ‘bad’ torsion; if we exclude this we obtain the following:

**Theorem 7** (PS, Theorem 3.2) Let $G \in \mathcal{H}$ be f.g. and residually finite. Suppose that $G$ has a metabelian normal subgroup $N$ with $G/N$ polycyclic. Then $G/N$ is minimax. If $N'$ has no $\pi$-torsion where $\pi = \text{spec}(G/N')$ then $G$ is minimax.

(For a minimax group $H$, $\text{spec}(H)$ denotes the (finite) set of primes $p$ such $C_{p^\infty}$ is a section of $H$.)

From this, it is relatively straightforward to deduce

**Theorem 8** (cf. PS, Corollary 3.3) Let $G$ be a finitely generated group that is nilpotent-by-abelian-by-polycyclic. If $G \in \mathcal{U}$ then $G$ has finite upper rank.

**Proof.** We may assume that $G$ satisfies the hypotheses of Theorem 7. Keeping the notation there, put $A = N'$, an abelian normal subgroup of $G$. For a prime $p$ and $K \triangleleft_f G$ let $D_p(K)/(A \cap K)$ be the $p'$-component of the finite abelian group $A/(A \cap K)$. Then

$$r_p(G/K) = r_p(G/KD_p(K)).$$

So if we set

$$D = \bigcap_{K \triangleleft_f G} K D_p(K),$$

we have $ur_p(G) = ur_p(G/D)$ for all $p \notin \pi$.

Now $AD/D \cong A/(A \cap D)$ has no $\pi$-torsion, since each $A/(A \cap KD_p(K))$ is a $p$-group. Clearly $G/D$ is residually finite, so Theorem 7 applies to show that $G/D$ is a minimax group. Hence

$$ur_p(G) = ur_p(G/D) \leq ur(G/D) < \infty$$

for every $p \notin \pi$, and as $\pi$ is finite it follows that $ur_p(G)$ is bounded over all primes $p$. ■
The hypotheses in Theorem 8 seem rather restrictive. However, if we could only replace ‘nilpotent-by-abelian’ with ‘abelian-by-nilpotent’ then we could deduce the full force of Conjecture A; this is explained below.

**Modules of finite upper rank**

Let $G$ be a counterexample to Conjecture A of least possible derived length, $l$; we may assume that $G$ is residually finite. Let $A$ be maximal among abelian normal subgroups of $G$ that contain $G^{(l-1)}$. Then $G/A$ is residually finite (by an elementary lemma) and has finite upper rank, so $G/A$ is a minimax group. In particular, $G/A$ is virtually nilpotent-by-abelian and so $G$ is abelian-by-nilpotent-by-polycyclic: this is the point of the final remark in the preceding section.

Putting $\Gamma = G/A$ we consider $A$ as a $\Gamma$-module, written additively as $A\Gamma$. If $B$ is a submodule of finite index in $A\Gamma$ then $G/B$ is residually finite (because $\mathcal{S}$ is extension-closed), whence

$$r_p(A/B) \leq ur_p(G)$$

for each prime $p$; and it is clear that

$$ur(G/B) \leq r(A/B) + ur(G/A).$$

Let us define the upper rank of a $\Gamma$-module $M$ by $ur(M) = \sup \{r(M/B) \mid B \leq \Gamma \ M, \ M/B \text{ finite}\}$, and set

$$ur_p(M) = \sup \{r(M/B) \mid pM \leq B \leq \Gamma \ M, \ M/B \text{ finite}\}
= ur(M/pM).$$

I will say that $M$ is a quasi-f.g. $\Gamma$-module if there exists a f.g. group $G$ that is an extension of $M$ by $\Gamma$. The preceding observations now show that $A\Gamma$ is a counterexample to

**Conjecture B** Let $\Gamma$ be a f.g. residually finite soluble minimax group and let $M$ be a quasi-f.g. $\Gamma$-module. If $ur_p(M)$ is finite for every prime $p$ then $M$ has finite upper rank.

Conversely, it is easy to see that if $M$ is a counterexample to Conjecture B then the corresponding extension $G$ is a counterexample to Conjecture A. So the two conjectures are equivalent.

Theorem 8 establishes Conjecture B for the special case where $\Gamma$ is abelian-by-polycyclic. A reduction step in the proof is Proposition 5.2 of [PS], which shows that $M$ contains a finitely generated $\Gamma$-submodule $B$ such that the finite module quotients of $B$ are ‘nearly all’ isomorphic to finite quotients of $M$, and conversely. The hypothesis that $\Gamma$ is abelian-by-polycyclic is used in the proof of this reduction, but can be dispensed with; this is explained in the next section. The main part of the proof, however, does depend on $\Gamma$ having an abelian normal subgroup $A$ such that $\Gamma/A$ is polycyclic. Following a strategy devised by P. Hall
and further developed by Roseblade [R], one examines the structure of $B$ as a module for the group ring $\mathbb{Z}A$, with $\Gamma/A$ as a group of operators. The necessary module theory is developed in [S4] and [S2].

For the general case of Conjecture B, it would seem necessary to generalize this machinery in one of two directions: either allow $A$ to be nilpotent (rather than abelian), or allow $\Gamma/A$ to be minimax (rather than polycyclic – while still assuming $\Gamma/C\Gamma(A)$ to be polycyclic, if one takes $A$ inside the centre of the Fitting subgroup of $\Gamma$). Whether either of these approaches is feasible remains unclear. Machinery relevant to the first approach has been developed by Tushev [T]. A major difficulty with the second approach is the fact that the ‘generic freeness’ property mentioned above definitely fails when $\Gamma/A$ is not polycyclic, as observed by Kropholler and Lorensen in [KL], Cor. 5.6. Other aspects of the Hall-Roseblade theory have been usefully generalized by Brookes [B].

On the other hand, if one is seeking a counterexample to conjecture B, the simplest candidate would seem to be the following group: Let $K$ be the Heisenberg group over $\mathbb{Z}[1/2]$ and take $\Gamma = K \rtimes \langle t \rangle$ where $t$ acts on a matrix by doubling the off-diagonal entries (and multiplying the top right corner entry by 4). Then $M$ could be the quotient $\mathbb{Z}\Gamma/J$ where $J$ is a carefully constructed right ideal: generators of $J$ should be chosen to ensure that $\mathbb{Z}\Gamma/J$ has finite upper $p$-rank for each prime $p$, but in such a way that these ranks are unbounded.

**A possible reduction: quasi-f.g. modules.**

Let $\Gamma$ be a f.g. residually finite soluble minimax group. Then $\Gamma$ has a nilpotent normal subgroup $K$ such that $\Gamma/K$ is virtually abelian. We fix a normal subgroup $Z$ of $\Gamma$ with $Z \leq Z(\Gamma)$, and let $R = \mathbb{Z}Z$ denote its group ring. For a multiplicatively closed subset $\Lambda$ of $R$, an $R$-module $M$ is said to be $\Lambda$-torsion if every element of $M$ is annihilated by some element of $\Lambda$.

**Proposition 9** Let $A$ be a quasi-f.g. $\Gamma$-module. Then $A$ has a finitely generated $\Gamma$-submodule $B$ such that $A/B$ is $\Lambda$-torsion for each $\Lambda$ of the form $R \setminus L$ where $L$ is a maximal ideal of finite index in $R$ not containing the augmentation ideal $(Z - 1)R$.

Before giving the proof we note a corollary. For a $\Gamma$-module $M$, let $\mathcal{F}(M)$ denote the set of isomorphism types of finite quotient $\Gamma$-modules of $M$.

**Corollary 10** For $A$ and $B$ as above, we have

$$\mathcal{F}(A) \setminus S = \mathcal{F}(B) \setminus S$$

where $S$ consists of the finite $\Gamma$-modules that have a composition factor on which $Z$ acts trivially.

This is essentially a formal consequence of the stated condition on $\Lambda$-torsion, which implies that

$$AJ + B = A, \quad AJ \cap B = BJ$$
whenever $J$ is the annihilator in $R$ of some finite $\Gamma$-module not in $S$. Thus questions about the upper ranks of $A$ might be reduced to questions about the upper ranks of the finitely generated module $B$, if - by some subsidiary argument - one could leave aside the quotients lying in $S$ (this is in principle the approach taken in [PS, §§5, 6]).

To establish the Proposition, we consider a f.g. group $E$ with an abelian normal subgroup $A$ such that $E/A = \Gamma$. In $E$ there is a series of normal subgroups

$$E > K_1 \geq Z_1 \geq A \geq \gamma_{c+1}(K_1)[Z_1, K_1]$$

where $K_1/A = K$ is nilpotent of class $c$, say, and $Z_1/A = Z$. Now $Z$ is an abelian minimax group, hence contains a finite subset $Y_1$ such that $Z/\langle Y_1 \rangle$ is divisible. Since $E/K_1 \cong \Gamma/K$ is virtually abelian and $E$ is f.g., $K_1$ is finitely generated as a normal subgroup of $E$; we choose a finite set $X = X - 1$ of normal generators for $K_1$ and assume that $X$ contains a set $Y$ of representatives for the elements of $Y_1$. Finally, let $S = S^{-1}$ be a finite set of generators for $E$.

**Lemma 11** Let $L$ be a maximal ideal of finite index in $R = \mathbb{Z}Z$ not containing $Z - 1$. Then $\Lambda = R \setminus L$ satisfies

$$(\Lambda^g + 1) \cap Y_1 \neq \emptyset$$

for every $g \in E$.

**Proof.** Write $D = Z \cap (L + 1)$. If (1) fails for $g$ then $D^g \supseteq Y_1$ which implies $D^g = Z$ since $Z/\langle Y_1 \rangle$ is divisible while $|Z : D^g|$ is finite. Hence $D = Z$ and so $L \supseteq Z - 1$. ■

Now we define $B$ to be the $E$-submodule of $A$ generated by the finite set

$$\{[x, y], [x^s, y] \mid x \in X, y \in Y, s \in S\}.$$

We aim to show that if $\Lambda$ is a multiplicatively closed subset of $R$ satisfying (1) for every $g \in E$, then the $R$-module $A/B$ is $\Lambda$-torsion; with Lemma 11 this will complete the proof of Proposition 9.

Note that $\gamma_{i+1}(K_1)$ is generated by the elements $v_i(x, w)^g$ for $g \in E$ and

$$v_i(x, w) = [x_0, x_1^{w_1}, \ldots, x_i^{w_i}],$$

$x_j \in X$, $w_j \in E$. Put

$$A_i = \langle [v_i(x, w), z]^g \mid x_j \in X, z \in Y, g, w_j \in E \rangle,$$

$$B_i = \langle [x^v, y]^g \mid x \in X, y \in Y, g, v \in E, l(v) \leq i \rangle$$

where $l(v)$ denotes the least $n$ such that $v = s_1 \ldots s_n$ ($s_j \in S$). Note that $B_1 = B$.

Claim: $A/A_c$ is $\Lambda$-torsion.
To see this, choose \( y \in Y \) with \( \overline{y} - 1 \in \Lambda \) where \( \overline{y} = Ay \). Then (mixing additive and multiplicative notation)

\[
A(\overline{y} - 1)^c = [A, y] \subseteq \gamma_{c+1}(K_1) \subseteq A.
\]

Given a generator \( v_c(x, w)^g \) of \( \gamma_{c+1}(K_1) \), choose \( z \in Y \) such that \( \overline{z} - 1 \in \Lambda \). Then

\[
v_c(x, w)^g(\overline{z} - 1) = [v_c(x, w), z] \in A_c.
\]

**Claim:** For \( i > 1 \), \( B_i/B_{i-1} \) is \( \Lambda \)-torsion.

To see this, say \( b = [x^{\gamma_{u}}, y] \) is a generator of \( B_i \) where \( l(u) \leq i - 1 \). Choose \( z \in Y \) such that \( \overline{z} - 1 \in \Lambda \). Then

\[
(b(\overline{z} - 1))^{-g-1}y = [x^{\gamma_{u}}, y, z^{-u}]^{-y^{-1}} = [x^{u}, x^{-\gamma_{u}}, y^{-1}]z^{\gamma_{u}} + [y^{-1}, z^{-u}, x^{\gamma_{u}}]z^{u}.
\]

The first term lies in \( B_1 \leq B_{i-1} \) and the second term lies in \( A_{i-1} \). The claim follows since each of these modules is \( E \)-invariant.

**Claim:** Write \( B_{\infty} = \bigcup_j B_j \). Then for \( i > 1 \), \( A_i \leq B_{\infty} + A_{i-1} \).

To see this, let \( x = (x', x) \) and \( w = (1, w_1, \ldots) = (w', w) \) be \( (i + 1) \)-tuples in \( X, E \) respectively, and let \( z \in Y \). Then

\[
[v_i(x, w), z]^{-x^{-w}} = [v_{i-1}(x', w'), x^{w}, z]^{-x^{-w}} = [x^{w}, z^{-1}, v_{i-1}(x', w')][z + [z, v_{i-1}(x', w')^{-1}, x^{w}]v_{i-1}(x', w')].
\]

The first term lies in \( B_{\infty} \) and the second term lies in \( A_{i-1} \). The claim follows since each of these modules is \( E \)-invariant.

The three claims together now imply that \( A/B \) is \( \Lambda \)-torsion, and the proof is complete.

**Further reductions**

Suppose that the pair \( (\Gamma, M) \) furnishes a counterexample to Conjecture B, where \( M \) is finitely generated as a \( \Gamma \)-module. With quite a lot of extra work, generalizing some ideas from [S4], one can establish

**Proposition 12** The module \( M \) has a torsion-free residually finite quotient \( \tilde{M} \) of infinite upper rank such that every proper, \( \pi \)-torsion-free residually finite quotient of \( \tilde{M} \) has finite rank, where \( \pi = \text{spec}(\Gamma) \).

(Here \( \text{spec}(\Gamma) \) denotes the (finite) set of primes \( p \) such that \( \Gamma \) has a section \( C_{p^{\infty}} \).

This reduces the problem to consideration of a ‘minimal counterexample’, in a rather weak sense. Whether this is any help is not clear, and there seems little point in including the proof here.
Further results that may be relevant are obtained in [KL1]; these can be used to show that a module like our putative counterexample has many finite-rank quotients that split as direct sums.

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