MODULI OF SEMISTABLE SHEAVES AS QUIVER MODULI

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Abstract. In the 1980s Drézet and Le Potier realized moduli spaces of Gieseker-semistable sheaves on $\P^2$ as what are now called quiver moduli spaces. We discuss how this construction can be understood using t-structures and exceptional collections on derived categories, and how it can be extended to a similar result on $\P^3 \times \P^1$.

1. Introduction

1.1. Monads and moduli of sheaves. A recurring theme in algebraic geometry is the study of moduli spaces, varieties whose points parameterize geometric objects of some kind. The first general construction of moduli spaces of vector bundles on a projective curve was given by Mumford using GIT, and then extended by Seshadri, Gieseker, Maruyama and Simpson among others to prove the existence, as projective schemes, of moduli spaces of semistable coherent sheaves on projective varieties of any dimension. We refer to the books [LP97, HL10] for comprehensive guides to the subject.

By the late 1970s, some people were studying an alternative and much more explicit way to construct moduli spaces of bundles over projective spaces: they were using monads, namely complexes $A \to B \to C$ of vector bundles with nonzero cohomology only at the middle term. This concept was first used by Horrocks [Hor64]: Barth [Bar77] showed that every stable bundle $\mathcal{E}$ of rank 2, degree 0 and $c_2 = k$ on the complex projective plane $\P^2 = \P(C(\mathbb{Z}))$ is isomorphic to the middle cohomology of a monad in which $A, B, C$ are fixed bundles and the maps between them only depend on a certain Kronecker module $\alpha \in \text{Hom}_C(C^k \otimes \mathbb{Z} \Gamma, C^k)$ constructed from $\mathcal{E}$. Moreover, this construction establishes a bijection between such bundles $\mathcal{E}$ up to isomorphism and elements of a subvariety $M \subset \text{Hom}_C(C^k \otimes \mathbb{Z} \Gamma, C^k)$ up to the action of $GL_k(C)$. This means that we have a surjective morphism $M \to M^\text{st}$ identifying the moduli space $M^\text{st}$ of stable bundles with the given numerical invariants as a $GL_k(C)$-quotient of $M$. By analyzing the variety $M$, Barth was then able to prove rationality and irreducibility of $M^\text{st}$. Then Barth and Hulek extended this construction first to all moduli spaces of rank 2 bundles [BH78, Hul79], and then to moduli of bundles with any rank and zero degree [Hul80]. These works were also fundamental to find explicit constructions of instantons, or anti self-dual Yang-Mills connections [AHD78, Don84].

These techniques were improved by Beilinson [Bei78], whose description of the bounded derived category of coherent sheaves on projective spaces gave a systematic way to produce monads for semistable sheaves, as explained e.g. in [OSS80] Ch. 2, §4. In this way, Drézet and Le Potier generalized in [DLP85] the works of Barth and Hulek to all Gieseker-semistable torsion-free sheaves on $\P^2$. They showed that, after imposing an analogue of Gieseker semistability, Kronecker complexes

\[
V_{-1} \otimes \O_{\P^2}(-1) \longrightarrow V_0 \otimes \O_{\P^2}(1) \longrightarrow V_1 \otimes \O_{\P^2}
\]

are forced to be monads, and taking their middle cohomology gives Gieseker-semistable sheaves. Moreover, this gives a bijective correspondence between isomorphism classes of semistable Kronecker complexes and isomorphism classes of semistable torsion-free sheaves, having fixed a class $v \in K_0(\P^2)$; thus the moduli space $\text{M}_{\text{ss}}^\text{st}(v)$ of such sheaves is a quotient of the semistable locus $R^\text{ss} \subset R$ in the vector space $R$ of all Kronecker complexes by the action of $G_V := \prod_i GL(C(V_i))$. Now we can observe that semistable Kronecker complexes can be seen as semistable representations of a quiver $B_3$ with relations, whose moduli spaces were constructed by King [Kin94] (see [2.4] for a short account); in particular, the set $R^\text{ss}$ becomes the semistable locus of a linearization of the action of $G_V$, so $\text{M}_{\text{ss}}^\text{st}(v)$ is the GIT quotient $R^\text{ss} // G_V$.

1 A similar interpretation of $R^\text{ss}$ via GIT was found in [LP94] without referring to quiver moduli; in fact, already in [Hul80] it was observed that the Kronecker modules $\alpha$ producing rank 2, degree 0 stable bundles can be seen as GIT-stable points.
as a projective variety independently from the general theory of Gieseker and Simpson. The same kind of construction was later carried out in [Kul97], where, for certain choices of the numerical invariants, moduli spaces of sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ were constructed as quiver moduli spaces.

The above-mentioned work [Bei78] was followed by many years of research on the structure of the bounded derived category $D^b(X)$ of a projective variety $X$. In particular, a theory of exceptional collections of objects of $D^b(X)$ was developed in the seminar [Kul90] for this purpose (we give a short introduction to this subject in §2.6). By using this machinery, it is natural to interpret the Abelian category of Kronecker complexes ([1]) as the heart of a bounded $t$-structure (Def. 2.1) on $D^b(\mathbb{P}^2)$ induced by an exceptional collection.

The aim of this paper is to understand the construction of Drézet and Le Potier from this point of view, extending it to a surface $X$ having an exceptional collection with similar properties. Basically, this is done by interpreting the arguments of [DLP09] in the language of $t$-structures; at some point we also use ideas from [FGIK16], where a similar analysis was carried out to construct moduli spaces of sheaves on a noncommutative projective plane. In particular, our result can be applied to $\mathbb{P}^1 \times \mathbb{P}^1$, providing thus a complete generalization of [Kul97].

Finally, we mention that there is a different way to relate moduli of sheaves and quiver moduli by using Bridgeland stability conditions [Bri07]: on a surface $X$ one can define a family of so-called geometric stability conditions containing a chamber where they are equivalent to Gieseker stability; on the other hand, a full strong exceptional collection on $X$ induces algebraic stability conditions, for which semistable objects are identified to semistable quiver representations. The isomorphisms between moduli of sheaves and quiver moduli then follow from the observation that when $X$ is $\mathbb{P}^2$, [Ohk10] or $\mathbb{P}^1 \times \mathbb{P}^1$ [AM17], the Gieseker chamber contains algebraic stability conditions. For $X = \mathbb{P}^2$, the author of [Ohk10] obtained with this method the explicit isomorphisms of our Theorems 5.1 and 5.3. The main difference in our approach is essentially that we use a weaker notion of stability structure, which includes Gieseker stability both for sheaves and Kronecker complexes. Then we can directly jump from one moduli space to the other, instead of moving through Bridgeland stability conditions.

1.2. Outline of the paper. In section 2 we briefly introduce the tools used in the rest of the paper: $t$-structures and exceptional collections on triangulated categories, stability structures, moduli spaces of sheaves and quiver representations.

Our goal is to identify some moduli spaces of semistable sheaves with quiver moduli spaces by using the above tools. The central idea is the following: take a smooth projective variety $X$ with a full strong exceptional collection $\mathcal{E}$ on $D^b(X)$, and let $M^s_{X,A}(v)$ be the moduli space of coherent sheaves on $X$ in a numerical class $v \in K_{num}(X)$ that are Gieseker-semistable with respect to an ample divisor $A \subset X$. When $\mathcal{E}$ is particularly well-behaved, $M^s_{X,A}(v)$ can be realized as a quiver moduli space: $\mathcal{E}$ induces a derived equivalence $\Psi$ between $D^b(X)$ and the bounded derived category $D^b(Q; J)$ of finite-dimensional representations of a certain quiver $Q$, possibly with relations $J$. The functor $\Psi$ is a non-standard bounded $t$-structure on $D^b(X)$, whose heart $\mathcal{K}$ consists of certain Kronecker complexes of sheaves, and Gieseker stability makes sense in a generalized way for objects of $\mathcal{K}$. The key observation is that, for suitable $v$, imposing Gieseker semistability forces the objects in the standard heart $\mathcal{C} \subset D^b(X)$ to be also semistable objects of $\mathcal{K}$, and the same is true with $\mathcal{C}$ and $\mathcal{K}$ exchanged. Moreover, semistable Kronecker complexes in $v$ are identified through $\Psi$ with $\theta$-semistable $d^v$-dimensional representations of $(Q, J)$, for some dimension vector $d^v$ and some weight $\theta$. All together, this gives an isomorphism between $M^s_{X,A}(v)$ and the moduli space $M_{Q,J,0}(d^v)$ of $\theta$-semistable representations of $(Q, J)$, which was constructed in [Kin94].

The simplest example of this phenomenon is discussed in §3 for sheaves on the projective line $\mathbb{P}^1$: in this case, the heart $\mathcal{K}$ can be also obtained by tilting $\mathcal{C}$ using the stability condition, and this description is used to easily recover via quiver representations the well-known classification of coherent sheaves on $\mathbb{P}^1$. When $X$ is a surface, however, this simple argument fails as the hearts $\mathcal{C}, \mathcal{K}$ are no longer related by a tilt. Nevertheless, to obtain an isomorphism $M^s_{X,A}(v) \simeq M_{Q,J,0}(d^v)$ it is enough to check that the hearts $\mathcal{C}, \mathcal{K}$ are somehow compatible with Gieseker stability, as explained in §2.3. In §4 we show that this compatibility holds under some hypotheses on the collection $\mathcal{E}$ and for appropriate choices of $v$; the precise statement is contained in Theorem 4.10.
In sections 2.2 and 6 we study the realizations of the isomorphism $M_{X,A}^{ss}(v) \simeq M_{Q,J,\theta,}(d^v)$ for the projective plane $\mathbb{P}^2$ and the smooth quadric $\mathbb{P}^1 \times \mathbb{P}^1$. Some consequences and some examples of these isomorphisms are also discussed.

2. Preliminaries

In this section we give short accounts of the notions used throughout the paper, mostly in order to fix notations and conventions. The material is almost all standard, except for some concepts and notations in [2.2, 2.3] and at the end of [2.3].

2.1. t-structures. Let $\mathcal{D}$ be a triangulated category. In this paragraph we recall the concept of t-structure on $\mathcal{D}$. All the details can be found in [BBD82].

Definition 2.1. A t-structure on $\mathcal{D}$ consists of a pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of strictly full subcategories of $\mathcal{D}$ such that, writing $\mathcal{D}^{\leq \ell} := \mathcal{D}^{\leq 0}[\ell]$ and $\mathcal{D}^{\geq \ell} := \mathcal{D}^{\geq 0}[\ell]$ for $\ell \in \mathbb{Z}$, we have:

1. $\text{Hom}_{\mathcal{D}}(X,Y) = 0 \forall X \in \mathcal{D}^{\leq 0}, \forall Y \in \mathcal{D}^{\geq 1};$
2. $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 0}[1]$ and $\mathcal{D}^{\geq 0} \subset \mathcal{D}^{\geq 0}[1];$
3. for all $E \in \mathcal{D}$ there is a distinguished triangle $X \to E \to Y \to X[1]$ for some $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$.

The intersection $\mathcal{A} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is called the heart of the t-structure. Finally, the t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is said to be bounded when for all $E \in \mathcal{D}$ there exists $\ell \in \mathbb{N}$ such that $E \in \mathcal{D}^{\leq \ell} \cap \mathcal{D}^{\geq -\ell}$.

It turns out that:

1. the heart $\mathcal{A}$ is an extension-closed Abelian category, and when the t-structure is bounded the inclusion $\mathcal{A} \hookrightarrow \mathcal{D}$ gives an isomorphism $K_0(\mathcal{A}) \cong K_0(\mathcal{D})$ between the Grothendieck groups;
2. a sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ in $\mathcal{A}$ is exact if and only if it can be completed to a distinguished triangle in $\mathcal{D};$
3. the inclusions $\mathcal{D}^{\leq \ell} \hookrightarrow \mathcal{D}, \mathcal{D}^{\geq \ell} \hookrightarrow \mathcal{D}$ have a right adjoint $\tau_{\leq \ell}$ and a left adjoint $\tau_{\geq \ell}$ respectively, and the functors

$$H^\ell_A := \tau_{\geq \ell} \circ H^\ell : \mathcal{D} \longrightarrow \mathcal{A}$$

are cohomological.

Examples 2.2. The t-structures that we will see in this paper will arise in three ways:

1. if $\mathcal{A}$ is an Abelian category, then the bounded derived category $D^b(\mathcal{A})$ has a standard bounded t-structure whose heart is $\mathcal{A};$
2. if $\Psi : \mathcal{D}_1 \to \mathcal{D}_2$ is an equivalence of triangulated categories, any t-structure on $\mathcal{D}_1$ induces a t-structure on $\mathcal{D}_2$ in the obvious way; in particular, when we are dealing with derived categories, the standard t-structures may be mapped to non-standard ones;
3. (see e.g. [Pol07] §1.1) if $(\mathcal{T}, \mathcal{F})$ is a torsion pair in the heart $\mathcal{A} \subset \mathcal{D}$ of a bounded t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$, then we can define a new t-structure $(\mathcal{D}^{\leq 0}_T, \mathcal{D}^{\geq 0}_T)$ on $\mathcal{D}$ via a tilt, i.e. by taking

$$\text{Ob}(\mathcal{D}^{\leq 0}_T) := \{ X \in \mathcal{D} \mid H^\ell_A(X) \in \mathcal{T}, \ H^\ell_A(X) = 0 \forall \ell > 0 \};$$
$$\text{Ob}(\mathcal{D}^{\geq 0}_T) := \{ X \in \mathcal{D} \mid H^\ell_A(X) \in \mathcal{F}, \ H^\ell_A(X) = 0 \forall \ell < -1 \};$$

moreover, we have that

$$(2.1) \quad \mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 0}_T \subset \mathcal{D}^{\leq 1},$$

and in fact this property characterizes all the t-structures which are obtained by tilting $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}).$

2.2. Stability structures. Let $\mathcal{A}$ be an Abelian category. We will discuss some notions of (semi)stability for the objects of $\mathcal{A}$. These are mostly standard techniques, although in some cases we introduce some new notations and definitions for later convenience.
2.2.1. **Weights and alternating forms.** The simplest notion we will use is that of stability with respect to a weight, that is a $\mathbb{Z}$-linear map $\nu : K_0(\mathcal{A}) \to R$ with values in an ordered Abelian group $(R, \leq)$ (which will typically be $\mathbb{Z}$, $\mathbb{R}$ or the polynomial ring $\mathbb{R}[t]$ with lexicographical order). This was introduced in [Knu94].

**Definition 2.3.** A nonzero object $A$ in $\mathcal{A}$ is said to be $\nu$-(semi)stable if $\nu(A) = 0$ and any strict subobject $0 \neq B \subset A$ satisfies $\nu(B) (\geq) > 0$. $A$ is $\nu$-polystable if it is a direct sum of $\nu$-stable objects.

It is easily checked that the semistable subobjects form (adding the zero object) a full Abelian subcategory $\mathcal{S}_\nu \subset \mathcal{A}$ closed under extensions. The (semi)simple objects of this category are the $\nu$-(poly)stable objects.

Second, we take an alternating $\mathbb{Z}$-bilinear form $\sigma : K_0(\mathcal{A}) \times K_0(\mathcal{A}) \to R$.

**Definition 2.4.** A nonzero object $A$ in $\mathcal{A}$ is said to be $\sigma$-(semi)stable if any strict subobject $0 \neq B \subset A$ satisfies $\sigma(B,A) (\leq) < 0$.

If we fix a class $v \in K_0(\mathcal{A})$, then we can define a weight $\nu_v := \sigma(v, \cdot)$ and observe that an object $A \in v$ is $\nu_v$-(semi)stable if and only if it is $\sigma$-(semi)stable.

We have given these two basic definitions of stability mostly for later notational convenience, and because they will be useful when used on different hearts in a triangulated category (see [2.2.4]). These definitions are very general and do not have particularly interesting properties, mainly because they are too weak to induce an order on the objects of $\mathcal{A}$. However, with $\sigma$ we can order the subobjects of a fixed object $A$, and we will use the following definition:

**Definition 2.5.** Let $A$ be a nonzero object. A nonzero subobject $S \subset A$ is said to be $\sigma$-(semi)maximal if for any subobject $S' \subset A$ we have $\sigma(S',A) \leq \sigma(S,A)$.

2.2.2. **Polynomial stabilities.** Take a $\mathbb{Z}$-linear map $P : K_0(\mathcal{A}) \to \mathbb{R}[t]$ and define an alternating form $\sigma_P : K_0(\mathcal{A}) \times K_0(\mathcal{A}) \to \mathbb{R}$ by

$$\sigma_P(v, w) := P_vP'_w - P_wP'_v,$$

where $P'_v(t) := \frac{d}{dt}P_v(t)$.

**Definition 2.6.** A nonzero object $A$ in $\mathcal{A}$ is said to be $P$-(semi)stable if it is $\sigma_P$-(semi)stable, that is if $P_vP'_w - P_wP'_v \leq 0$ for any $0 \neq B \subset A$.

As usual, polynomials are ordered lexicographically. This definition does not assume anything on the map $P$, but it turns out to be much more interesting when $P$ maps the classes of nonzero objects into the set $\mathbb{R}[t]_+ \subset \mathbb{R}[t]$ of polynomials with positive leading coefficient.

**Definition 2.7.** We call $P$ a polynomial stability on $\mathcal{A}$ if $P_A \in \mathbb{R}[t]_+$ for any nonzero object $A$ of $\mathcal{A}$.

Indeed, we can give $\mathbb{R}[t]_+$ an alternative total preorder $\preceq_G$ by setting

$$p \preceq_G q \iff pq' - p'q \leq 0$$

for $p, q \in \mathbb{R}[t]_+$; we also write $p \equiv_G q$ when $p \preceq_G q$ and $q \preceq_G p$. We have the following equivalent characterizations of $\preceq_G$, which show that it is indeed a preorder (that is, a total, reflexive and transitive relation) and that it coincides with the preorder considered in [Knu97] §2:

**Lemma 2.8.** Take two polynomials $p, q \in \mathbb{R}[t]_+$ and write them as $p(t) = \sum_{i=0}^{\deg p} a_it^i$ and $q(t) = \sum_{j=0}^{\deg q} b_jt^j$. Then the following statements are equivalent:

(i) $p(\preceq_G) q$;
(ii) we have $\deg p > \deg q$ or $d \begin{cases} \deg p = \deg q =: d, \\ \frac{p(t)}{a_d} (\leq) < \frac{q(t)}{b_d} \end{cases}$;
(iii) $b_{\deg p} p(t)(\leq) < a_{\deg p} q(t)$.

Moreover we have $p \equiv_G q$ if and only if $p$ and $q$ are proportional.
(2.4)
\[ \sigma_Z(v, w) := -\Re Z(v)\Im Z(w) + \Re Z(w)\Im Z(v). \]

**Definition 2.11.** A nonzero object \( A \) in \( \mathcal{A} \) is said to be \( Z \)-\((semi)stable if it is \( \sigma_Z \)-(semi)stable.

Equivalently, we are asking that \( A \) is \((semi)stable with respect to the polynomial map \( P_t(z) := t^3 Z(v) - \Re Z(v) \). Again, this notion of stability is most useful when the positive cone in \( K_0(\mathcal{A}) \) is mapped to a proper subcone of \( \mathbb{C} \), as this allows to order the objects of \( \mathcal{A} \) according to the phases of their images under \( Z \). Commonly, one requires that the positive cone is mapped by \( Z \) inside the semi-closed upper half-plane \( \mathbb{H} \cup \mathbb{C}_{<0} \), which is as saying that \( P \) is a polynomial stability:

**Definition 2.12.** \( Z \) is called a stability function, or central charge, when for any nonzero object \( A \) we have \( \Im Z(A) \geq 0 \), and we have \( \Im Z(A) = 0 \) only if \( \Re Z(A) < 0 \). \( Z \) has the HN property if the polynomial stability \( P \) has.

In this case we denote by \( \phi_Z(A) := \arg Z(A) / \pi \in (0, 1] \) the phase of a nonzero object \( A \). Note that now we have \( P_A \preceq P_B \) if and only if \( \phi_Z(A) \leq \phi_Z(B) \), and similarly if we replace phases by slopes \( \mu_Z(A) := -\cot \phi_Z(A) = -\Re Z(A) / \Im Z(A) \in (-\infty, +\infty] \).

Thus, objects are ordered by their slopes, and the above definitions of stability and HN filtrations take now the usual forms. We write \( S^{(Z)}_\bullet \subset \mathcal{A} \) for the Abelian subcategory of \( Z \)-semistable objects of fixed phase \( \phi \).

2.2.4. Stability in triangulated categories. Finally, we extend the previous notions of stability to a triangulated category \( D \): by a stability structure (of any of the above types) on \( D \) we mean a stability structure on the heart \( \mathcal{A} \subset D \) of a bounded \( t \)-structure. Notice that fixing e.g. a bilinear form \( \sigma : K_0(D) \times K_0(D) \to \mathbb{R} \) gives a stability structure on any heart in \( D \); when an object \( D \in D \) lies in different hearts, it is necessary to specify with respect to which of them we are considering it being (semi)stable or not (being a subobject is a notion that depends on the heart).

A situation in which a stability structure behaves well when changing the heart is when a \( t \)-structure is built via a tilt with respect to a stability function:

\[ \text{let } Z : K_0(\mathcal{A}) \to \mathbb{C} \text{ be a stability} \]

2The same argument works using a polynomial stability instead of \( Z \), but we will not need this level of generality.
function with the HN property on a heart $\mathcal{A}$, and take $\phi \in (0, 1]$. Then, as in \cite[Lemma 6.1]{Lu08}, $Z$ induces a torsion pair $(\mathcal{T}_Z^Z, \mathcal{P}_Z^Z)$ in $\mathcal{A}$, given by
\begin{align}
\text{Ob}(\mathcal{T}_Z^Z) &= \{ A \in \mathcal{A} \text{ with all the HN phases } \geq \phi \}, \\
\text{Ob}(\mathcal{P}_Z^Z) &= \{ A \in \mathcal{A} \text{ with all the HN phases } < \phi \},
\end{align}
where by HN phases we mean the phases $\phi_Z(A_i/A_{i-1})$ of the quotients in the HN filtration of $A$.

Thus we can consider $Z$-stability with respect to either $\mathcal{A}$ or the heart $\mathcal{A}^#$ of the tilted $t$-structure (although $Z$ does not map the positive cone of $K_0(\mathcal{A}^#)$ in the upper half plane, so typically one rotates $Z$ accordingly; we do not perform this operation), and for objects in the intersection $\mathcal{A} \cap \mathcal{A}^# = T^Z_{Z^Z}$ the two notions coincide.

2.3. Compatibility of hearts under a stability structure. Take a triangulated category $\mathcal{D}$, an alternating $\mathbb{Z}$-bilinear form $\sigma : K_0(\mathcal{D}) \times K_0(\mathcal{D}) \to \mathbb{Z}[t]$, the hearts $\mathcal{A}, \mathcal{B} \subset \mathcal{D}$ of two bounded $t$-structures, and $v \in K_0(\mathcal{D})$.

To relate $\sigma$-(semi)stable objects with respect to one heart or the other, we would like the tuple $(\sigma, \mathcal{A}, \mathcal{B}, v)$ to satisfy the following compatibility conditions. First, we want the $\sigma$-(semi)stable objects in one heart to belong also to the other:

(C1) For any object $D \in \mathcal{D}$ belonging to the class $v$, the following conditions hold:

(a) if $D$ is a $\sigma$-(semi)stable object of $\mathcal{A}$, then it also belongs to $\mathcal{B}$;
(b) if $D$ is a $\sigma$-(semi)stable object of $\mathcal{B}$, then it also belongs to $\mathcal{A}$.

Second, we want that $\sigma$-(semi)stability can be equivalently checked in one heart or the other:

(C2) For any object $D \in \mathcal{A} \cap \mathcal{B}$ belonging to the class $v$, we have that $D$ is $\sigma$-(semi)stable in $\mathcal{A}$ if and only if it is $\sigma$-(semi)stable in $\mathcal{B}$.

Example 2.13. These conditions are satisfied by the 4-tuple $(\sigma_Z, \mathcal{A}, \mathcal{A}^#, v)$, where $Z : K_0(\mathcal{A}) \to \mathbb{C}$ is a stability function on $\mathcal{A}$ with the HN property, $\mathcal{A}^#$ is the tilted heart at the torsion pair $(2.5)$ for some some $\phi \in (0, 1]$ and $v$ has phase $\phi_Z(v) \in (\phi, 1]$.

Typically there is some notion of families of objects of $\mathcal{D}$, so that we have moduli stacks (or even moduli spaces) $\mathcal{M}_{\mathcal{A}, \sigma}(v), \mathcal{M}_{\mathcal{B}, \sigma}(v)$ of $\sigma$-(semi)stable objects in $\mathcal{A}, \mathcal{B}$ respectively, and belonging to the class $v$. Then we have $\mathcal{M}_{\mathcal{A}, \sigma}(v) = \mathcal{M}_{\mathcal{B}, \sigma}(v)$ if conditions (C1), (C2) are verified for $(\sigma, \mathcal{A}, \mathcal{B}, v)$.

2.4. Quiver moduli. Here we briefly recall the main aspects of the geometric representation theory of quivers introduced in \cite{Kin94}: this summary is essentially based on that paper and on the notes \cite{Rei98}.

Our notations are as follows: a quiver $Q = (I, \Omega)$, consists of a set $I$ of vertices, a collection $\Omega$ of arrows between them and source and target maps $s, t : \Omega \to I$. We only consider finite and acyclic (that is, without oriented loops) quivers. We denote by $\text{Rep}_C(Q)$ the Abelian category of finite-dimensional complex representations of $Q$, which are identified with left modules of finite dimension over the path algebra $CQ$ (we adopt the convention in which arrows are composed like functions). When a $I$-graded $C$-vector space $V = \oplus_{i \in I} V_i$ is fixed, we write
\[ R_V := \oplus_{h \in \Omega} \text{Hom}_C(V_{s(h)}, V_{t(h)}) \]
for the vector space of representations of $Q$ on $V$, whose elements are collections $f = \{ f_h \}_{h \in \Omega}$ of linear maps. The isomorphism classes of such representations are the orbits of the obvious action of $G_V := \prod_{i \in I} \text{GL}_C(V_i)$ on $R_V$. The subgroup $\Delta := \{ (\lambda \text{Id}_i)_{i \in I} : \lambda \in \mathbb{C}^\times \}$ acts trivially, so the action descends to $PG_V := G_V/\Delta$.

Under our assumptions of finiteness and acyclicity of $Q$, the category $\text{Rep}_C(Q)$ is of finite length and hereditary, and its simple objects are the representations $S(i)$ with $C$ at the $i$th vertex and zeroes elsewhere; in particular, the classes of these objects form a basis of the Grothendieck group $K_0(Q) := K_0(\text{Rep}_C(Q))$, which is then identified with the lattice $\mathbb{Z}^I$ by taking the dimension vector $\dim V = (\dim_C V_i)_{i \in I}$ of a representation $V$. Hence, giving a $\mathbb{Z}$-valued weight as in \cite[2.2.1]{Lu08} on the category $\text{Rep}_C(Q)$ is the same as giving an array $\theta \in \mathbb{Z}^I$: this defines $\nu_\theta : \mathbb{Z}^I \cong K_0(\text{Rep}_C(Q)) \to \mathbb{Z}$ by
\[ \nu_\theta(d) := \theta \cdot d := \sum_{i \in I} \theta^i d_i. \]
Similarly, we can consider $\mathbb{R}$-valued or even polynomial-valued arrays $\theta$ and weights $\nu_\theta$. 
Definition 2.14. Fix \( \theta \in \mathbb{R}^I \) or \( \theta \in \mathbb{R}[t]^I \). We call a representation \((V,f)\) \(\theta\)-(semi)stable when it is \(\nu_v\)-(semi)stable according to Def. 2.3, namely when we have \(\theta \cdot \dim V = 0\) and \(\theta \cdot \dim W(\geq 0)\) for any subrepresentation \(0 \neq W \subseteq V\). We denote by \(R^s_{V,\theta} \subset R^s_{V,\theta} \subset RV\) the subsets of \(\theta\)-stable and \(\theta\)-semistable representations on \(V\).

Remark 2.15. Suppose that a dimension vector \(d \in \mathbb{N}^I\) such that \(\theta \cdot d = 0\) is \(\theta\)-coprime, meaning that we have \(\theta \cdot d' \neq 0\) for any \(0 \neq d' < d\) (which means that \(0 \neq d' \neq d\) for all \(i \in I\)). Then a \(d\)-dimensional representation cannot be strictly \(\theta\)-semistable. Note also that if \(d\) is \(\theta\)-coprime, then it is a primitive vector of \(\mathbb{Z}^I\). Conversely, if \(d\) is a primitive vector, \(\theta \cdot d = 0\) and the components \(\theta^0, ..., \theta^l \in \mathbb{R}^I\) of \(\theta\) span a subspace of dimension at least equal to \(\#I - 1\), then \(d\) is \(\theta\)-coprime.

Now we fix a \(I\)-graded vector space \(V\) of dimension vector \(v\), and we want to consider a quotient space of \(RV\) by the reductive group \(GV\) to parameterize geometrically the representations of \(Q\). The set-theoretical quotient often is not a variety, while the classical invariant theory quotient \(RV/PGV\) is just a point, so we consider a GIT quotient with respect to a linearization: given an integral array \(\theta \in \mathbb{Z}^I\) such that \(v \cdot \theta = 0\), we construct a character \(\chi_{\theta} : PGV \to \mathbb{C}^\times\) by \(\chi_{\theta}(g) := \prod_{i \in I} (\det g_i)^{\theta_i}\); this induces a linearization of the trivial line bundle on \(RV\) and then a notion of (semi)stability which is exactly the same as \(\theta\)-(semi)stability, and GIT quotients which we denote by

\[ M^s_{Q,\theta}(d) := R^s_{V,\theta}/\chi_{\theta}PGV, \quad M^s_{Q,\theta}(d) := R^s_{V,\theta}/\chi_{\theta}PGV \]

the latter being the stable quotient. These varieties corepresent the quotient stacks \(\mathcal{M}^s_{Q,\theta}(d) = [R^s_{V,\theta}/GV]\) and \(\mathcal{M}^s_{Q,\theta}(d) = [R^s_{V,\theta}/GV]\), which can be also defined as moduli stacks of families of representations (which are defined e.g. in [Kin94] §5]). This is why it is meaningful to call \(M^s_{Q,\theta}(d)\) and \(M^s_{Q,\theta}(d)\) the moduli spaces of semistable and stable representations.

Remark 2.16. Here we list the main properties of these moduli spaces:

1. \(M^s_{Q,\theta}(d)\) is a projective variety, and \(M^s_{Q,\theta}(d)\) is an open set in it;
2. \(M^s_{Q,\theta}(d)\) is smooth of dimension

\[ \dim M^s_{Q,\theta}(d) = 1 - \chi(d,d) = \sum_{h \in \Omega} d_h \ell(h) - \sum_{i \in I} d_i^2 + 1, \]

where \(\chi\) is the Euler form on \(K_0(\text{Rep}_C(Q)) \simeq \mathbb{Z}^I\); the stacks \(\mathcal{M}^s_{Q,\theta}(d)\) and \(\mathcal{M}^s_{Q,\theta}(d)\) are smooth of dimension \(-\chi(d,d)\);
3. \(M^s_{Q,\theta}(d)\) is a coarse moduli space for \(S\)-equivalence classes\(^3\) of \(\theta\)-semistable representations on \(V\), while the points of \(M^s_{Q,\theta}(d)\) correspond to isomorphism classes of \(\theta\)-stable representations;
4. [Rei08] §5.4] if \(d\) is primitive (that is, \(\gcd(d_i)_{i \in I} = 1\)), then \(M^s_{Q,\theta}(d)\) admits a universal family.
5. if \(d\) is \(\theta\)-coprime (see Remark 2.15), then there are no strictly semistable representations, so \(M^s_{Q,\theta}(d) = M^s_{Q,\theta}(d)\) is smooth and projective, and it admits a universal family.

After we have fixed a dimension vector \(d = \dim V\), we can partition the hyperplane \(d^\perp \subset \mathbb{R}^I\) into finitely many locally closed subsets where different \(\theta\) give the same \(\theta\)-(semi)stable representations: call \(\theta_1, \theta_2 \in d^\perp\) numerically equivalent when for any \(d' \leq d\) (which means that \(d'_i \leq d_i\) for all \(i \in I\)) \(\theta_1 \cdot d'\) and \(\theta_2 \cdot d'\) have the same sign (±1 or 0). Then we have a finite collection \(\{W_j\}_{j \in J}\) of rational hyperplanes in \(d^\perp\), called (numerical) walls, of the form

\[ W(d') = \{ \theta \in d^\perp \mid \theta \cdot d' = 0 \} \]

where \(d' \in \mathbb{N}^I\) is such that \(d' \leq d\) but does not divide \(d\). The numerical equivalence classes in \(d^\perp\) are the connected components of the locally closed subsets \(\cap_{j \in J} W_j \setminus \cup_{j \in J} W_j\), for some partition \(J = J_1 \sqcup J_2\) (for \(J = J_2\) these are called (numerical) chambers). By construction, the

\[^3\text{This is the definition used in [Kin94]; [Rei08] uses instead the opposite convention, and in fact he defines a slope \(\mu_{\theta}\) to order the representations and to have a HN property.}\]

\[^4\text{Two representations are S-equivalent when the closures of their \(PGV\)-orbits in \(R^s_{V,\theta}\) intersect, or equivalently when they have the same composition factors as elements of the subcategory \(S_{Q} \subset \text{Rep}_C(Q)\) of \(\theta\)-semistable representations.}\]
subsets $R^{ss}_{v,\theta}$ and $R^{s}_{v,\theta}$ do not change when $\theta$ moves inside a numerical equivalence class, and any such a class contains integral arrays, because it is a cone and the walls are rational.

This means that also for a real or polynomial array $\theta$ orthogonal to $d$ the moduli spaces $M_{Q,\theta}(d)$ and $M_{Q,\theta}^{ss}(d)$ make sense and are constructed as GIT quotients after choosing a numerically equivalent integral weight $\theta^\prime \in \mathbb{Z}^2$: for example, if $\theta = t\theta_0 + \theta_0 \in \mathbb{R}[t]^2$, then we can choose $\epsilon > 0$ small enough so that $\theta^\prime$-(semi)stability is equivalent to $\theta^\prime$-(semi)stability, where $\theta^\prime \in \mathbb{Z}^2$ is some integral array lying in the same numerical equivalence class as $\theta_1 + \epsilon \theta_0$.

**Example 2.17.** A recurring example in this paper will be the *Kronecker quiver*

$$K_n : \quad -1 \quad \longrightarrow \quad 0$$

with $n$ arrows. Its representations can be seen as linear maps $f : V_0 \otimes Z \to V_0$, where $Z$ is a $n$-dimensional vector space with a fixed basis, or with left modules over the *Kronecker algebra* $\mathbb{C}K_n = \left( \begin{array}{cc} \mathbb{C} & \mathbb{C} \\ 0 & 0 \end{array} \right)$. Notice that the only arrays $\theta \in \mathbb{Z}^{(-1,0)}$ giving nontrivial stability weights $\nu_\theta$ on the representations of $K_n$ are those with $\theta^0 > 0$, and these are all in the same chamber: a Kronecker module $f$ is, accordingly, (semi)stable if and only if for any subrepresentation $W \subset V$ with $W_0 \neq 0$ we have

$$\frac{\dim W_0}{\dim W_{-1}} (\geq) \frac{\dim V_0}{\dim V_{-1}}.$$  

This is the usual notion of (semi)stability for Kronecker modules (see e.g. [Dre87, Prop. 15]). We denote by

$$K(n;d_{-1},d_0) := M_{K_n,(-d_{-1},d_{-1})}^{ss}(d)$$

the moduli space of semistable Kronecker modules of dimension vector $d$, and by $K_{st}(n;d_{-1},d_0) \subset K(n;d_{-1},d_0)$ the stable locus. Some useful facts on these spaces are:

1. if $n d_{-1} < d_0$ or $d_{-1} > n d_0$, then $K(n;d_{-1},d_0) = \emptyset$;
2. $K(n;d_{-1},n d_{-1}) = K(n;nd_0,d_0) = pt$;
3. $\dim K_{st}(n;d_{-1},d_0) = n d_{-1} d_0 + 1 - d_{-1}^2 - d_0^2$;
4. (Dre87) Prop. 21-22 we have isomorphisms $K_{st}(n;d_{-1},d_0) \cong K_{st}(n;nd_0,d_0)$ and $K_{st}(n;d_{-1},d_0) \cong K_{st}(n;nd_{-1},d_{-1})$;
5. $K(n;1,k) \cong K(n;k,1) \cong G_k(n)$, the Grassmannian of $k$-dimensional subspaces of $\mathbb{C}^n$;
6. (Dre87) Lemme 25 $K(3;2,2) \cong \mathbb{P}^5$.

Often we will consider representations of $Q$ subject to certain *relations*, that is combinations of arrows of length $\geq 2$ generating an ideal $J \subset Q$. These form an Abelian subcategory $\text{Rep}_C^J(Q) \subset \text{Rep}_{C}^J(Q)$ equivalent to left $\mathbb{C}Q/J$-modules of finite dimension. Given $V$ and $\theta$ as above, the representations on $V$ subject to the relations makes a $G_V$-invariant closed subvariety $X_V \subset R_V$, and thus the $\theta$-(semi)stable ones are parameterized by moduli stacks $M_{Q,\theta}^{ss/\text{st}}(J) = [R_{V,\theta}^{ss/\text{st}} \cap G_V]$ or by moduli spaces obtained as closed subvarieties $M_{Q,\theta}^{ss/\text{st}}(d) \subset M_{Q,\theta}^{ss/\text{st}}(d)$.

### 2.5. Moduli spaces of semistable sheaves

Let $X$ be a smooth projective irreducible complex variety polarized by an ample divisor $A \subset X$. $\text{Coh}_{\mathcal{O}_X}$ denotes the Abelian category of coherent $\mathcal{O}_X$-modules, and $K_0(X)$ its Grothendieck group.

Given a sheaf $\mathcal{E} \in \text{Coh}_{\mathcal{O}_X}$, we denote by $\text{rk} \mathcal{E}$, $\text{deg} A \mathcal{E} := c_1(\mathcal{E}) \cdot A$ its rank and degree (where $c(\mathcal{E}) = 1 + \sum_{i \geq 1} c_i(\mathcal{E})$ is the Chern class), by $ch \mathcal{E}$ its Chern character, by

$$P_{\mathcal{E},A}(t) = \sum_{i=0}^{\dim \mathcal{E}} \frac{\alpha_i(\mathcal{E})}{i!} t^i := \chi(X; \mathcal{E}(tA))$$

its Hilbert polynomial and by $\chi(\mathcal{E}) := P_{\mathcal{E},A}(0)$ its Euler characteristic. Note that these quantities are additive on short exact sequences: given a class $v \in K_0(X)$, it makes thus sense to write $\text{rk} v$, $\text{deg} A v$, $ch v$ and $P_{\mathcal{E},A}$. We also write $\mu_A(\mathcal{E}) := \text{deg} A \mathcal{E} / \text{rk} \mathcal{E}$ for the slope of $\mathcal{E}$, $dim \mathcal{E}$ for the dimension of its support, and $p_{\mathcal{E},A}(t) := P_{\mathcal{E},A}(t)/\alpha_{\dim \mathcal{E}}(\mathcal{E})$ for the reduced Hilbert polynomial. When $\mathcal{E}$ is torsion-free, we have $dim \mathcal{E} = dim X$ and $\text{rk} \mathcal{E} = \alpha_{\dim X}(\mathcal{E}) / A^{\dim X}$. Finally, the Hilbert polynomial $P_{v,A}$ can be computed by the Hirzebruch-Riemann-Roch Theorem:
(1) if \( \dim X = 1 \) and \( g(X) \) is the genus of \( X \), then
\[
P_{v,A}(t) = t \, \text{rk} \, v \, \deg(A) + \deg v + \text{rk} v (1 - g(X)) ;
\]
(2) if \( \dim X = 2 \), then
\[
P_{v,A}(t) = t^2 \, \text{rk} \, v \, A^2 + t \left( \deg_A v - \text{rk} \, v \frac{A \cdot K_X}{2} \right) + \chi(v),
\]
where \( \chi(v) = \text{rk} \, v \chi(X; \mathcal{O}_X) + (\text{ch}_2 v + c_1(v) c_1(X)/2) \).

Now we recall the main aspects of moduli spaces of semistable coherent sheaves, mainly following [HL10].

**Definition 2.18.** \( E \in \text{Coh}_{\mathcal{O}_X} \) is said to be Gieseker-(semi)stable with respect to \( A \) if it is \( P_{v,A} \)-\((semi)stable according to Def. 2.6.

Here we are seeing the Hilbert polynomial as a polynomial stability \( P_{v,A} : K_0(X) \rightarrow \mathbb{R}[t] \). So \( E \) is Gieseker-(semi)stable if and only if for any coherent subsheaf \( 0 \neq \mathcal{F} \subsetneq E \) we have the inequality \( P_{\mathcal{F},A} \preceq_G P_{E,A} \), where \( \preceq_G \) is the preorder introduced in eq. (2.3). Lemma 2.8 says that this inequality is equivalent to (where as usual \( \leq \) is the lexicographical order)
\[
\alpha_{\dim E} (\mathcal{E}) P_{\mathcal{F},A}(t) (\leq) < \alpha_{\dim E} (\mathcal{F}) P_{E,A}(t) ,
\]
so our definition agrees with the standard one given in [HL10] 1.2. This reformulation of Gieseker stability will turn out to be useful in the rest of the paper.

**Definition 2.19.** A torsion-free sheaf \( E \in \text{Coh}_{\mathcal{O}_X} \) is slope-(semi)stable if for any coherent subsheaf \( \mathcal{F} \subsetneq E \) with \( 0 < \text{rk} \, \mathcal{F} < \text{rk} \, E \), we have \( \mu_A(\mathcal{F}) (\leq) < \mu_A(E) \).

Some remarks on the notion of Gieseker (semi)stability:

(1) If \( E \) is Gieseker-semistable, then it is automatically pure (that is, all its subsheaves have the same dimension), and in particular it is torsion-free if and only if \( \dim E = \dim X \).

(2) The category \( \text{Coh}_{\mathcal{O}_X} \) is Noetherian and Hilbert polynomials are numerical; then, as discussed after Def. 2.10 any coherent sheaf \( E \) has a unique Harder-Narasimhan filtration
\[
0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{\ell} = E
\]
with Gieseker-semistable quotients \( E_i/E_{i-1} \) of \( \preceq_G \)-decreasing Hilbert polynomials (when \( E \) is pure this simply means that \( p_{E_{i-1},A} > p_{E_{i-1}/E_i,A} > \cdots > p_{E/E_{\ell-1},A} \)). We write
\[
P_{E,A,\text{max}} := P_{E_1,A}, \quad P_{E,A,\text{min}} := P_{E/E_{\ell-1},A}.
\]

Moreover, Gieseker-semistable sheaves with fixed reduced Hilbert polynomial \( p \in \mathbb{Q}[t] \) form an Abelian subcategory \( S_p \subset \text{Coh}_{\mathcal{O}_X} \) of finite length and closed under extensions; two sheaves in \( S_p \) are called \( S \)-equivalent if they have the same composition factors.

(3) Suppose that \( \dim X = 1 \): any \( E \in \text{Coh}_{\mathcal{O}_X} \) is the direct sum of its torsion-free and torsion parts, so it is pure if and only if they are not both nonzero; a torsion-free \( E \) (which is also a vector bundle) is Gieseker-(semi)stable if and only if it is slope-(semi)stable, and the slope condition can be checked on vector subbundles only; on the other hand, any torsion sheaf is Gieseker-semistable, and it is Gieseker-stable if and only if it is a simple object in \( \text{Coh}_{\mathcal{O}_X} \).

The main reason to introduce semistability was the construction of moduli spaces:

**Theorem 2.20.** Fix a numerical class \( v \in K_{\text{num}}(X) \). There exists a projective \( \mathbb{C} \)-scheme \( M^\text{st}_{X,A}(v) \) which is a coarse moduli space for \( S \)-equivalence classes of coherent \( \mathcal{O}_X \)-modules in \( v \) which are Gieseker-semistable with respect to \( A \). It also has an open subscheme \( M^\text{st}_{X,A}(v) \) parameterizing isomorphism classes of Gieseker-stable sheaves.

Again, these are moduli spaces in that they corepresent moduli stacks \( \mathcal{M}^\text{st}_{X,A}(v) \) and \( \mathcal{M}^\text{st}_{X,A}(v) \) of families of sheaves. See [HL10] Neu09 for the details.

**Remark 2.21.** When \( \dim X = 2 \), some properties of these spaces proven from their construction are:

(1) [HL10] Theorem 3.4.1 if \( M^\text{st}_{X,A}(v) \neq \emptyset \), then the Bogomolov inequality holds:
\[
\Delta(v) := c_1(v)^2 - 2 \, \text{rk} \, v \, \text{ch}_2(v) > 0;
\]

(2.10)
(2) if \( \deg_H \omega_X < 0 \), then for any stable \( F \) the obstruction space \( \text{Ext}^2(F,F) \cong \text{Hom}(F,F \otimes \omega_X)^{\vee} \) vanishes and \( \text{End}(F) \cong \mathbb{C} \), thus the tangent space \( \text{Ext}^1(F,F) \) has dimension \( 1 - \chi(F,F) \); hence, by [HL10] Corollary 4.5.2, \( M_{X,A}^\text{iten} \) is smooth of dimension
\[
\text{dim } M_{X,A}^\text{iten}(v) = 1 - \chi(v,v) = 1 - (\text{rk } v)^2 \chi(O_X) + \Delta(v)
\]
(3) [HL10] Corollary 4.6.7 if \( \gcd(\text{rk } v, \text{deg } A, \chi(v)) = 1 \), then \( M_{X,A}^\text{iten} \) is equal to \( M_{X,A}^\text{iten} \) and it has a universal family.

Finally, to simplify the computations we introduce the alternating forms \( \sigma_M, \sigma_X : K_0(X) \times K_0(X) \to \mathbb{Z} \) given by
\[
\sigma_M(v,w) := \text{deg}_A v \text{rk } w - \text{deg}_A w \text{rk } v,
\]
and also the \( \mathbb{Z}[t] \)-valued form \( \sigma_G := t \sigma_M + \sigma_X \). Now we can express Gieseker stability on curves and surfaces as stability with respect to these forms, in the sense of Def. [2.4]

Lemma 2.22.

(1) If \( \dim X = 1 \), then \( \sigma_M = \sigma_X \), and Gieseker (semi)stability and \( \sigma_M \)-stability are equivalent for objects of positive rank these are also equivalent to slope-stability.

(2) if \( \dim X = 2 \), then for sheaves of positive rank Gieseker (semi)stability is equivalent to \( \sigma_G \)-stability; for torsion-free sheaves, slope semistability is equivalent to \( \sigma_M \)-semistability.

For \( \dim X = 2 \), the restriction to positive rank is necessary as \( \sigma_G \) vanishes identically on sheaves supported on points. Note also that \( O_X \) is slope-stable but not \( \sigma_M \)-stable, as \( \sigma_M(I_x, O_X) = 0 \), where \( I_x \subset O_X \) is the ideal sheaf of a point.

Proof.

(1) The first statement is just the observation that the alternating form induced by the Hilbert polynomial as in eq. \([2.2]\) is \( \sigma_P = \text{deg } A \sigma_M = \text{deg } A \sigma_X \). The second statement is also obvious.

(2) In this case we have
\[
\sigma_P = \frac{t^2}{2} A^2 \sigma_M + \left( \frac{t A^2 - A \cdot K_X}{2} \right) \sigma_X + \sigma_0,
\]
where \( \sigma_0(v,w) := \chi(v) \text{deg } w - \chi(w) \text{deg } v \). But if \( \text{rk } w \neq 0 \), then \( \sigma_0 \) is irrelevant as \( \sigma_M(v,w) = \sigma_X(v,w) = 0 \) implies \( \sigma_0(v,w) = 0 \), so \( \sigma_P \) can be replaced by \( \sigma_G = t \sigma_M + \sigma_X \). The final claim follows from the equality
\[
\mu_A(v) = \text{deg}(F/E) = -1 - \text{deg}(F/E) \leq 0.
\]

Remark 2.23. In fact, the same arguments apply to any heart \( A \subset D^b(X) \) of a bounded t-structure: if \( \dim X = 1 \), then \( P_{\cdot,A} \)-stability and \( \sigma_M \)-stability in \( A \) are equivalent; if \( \dim X = 2 \), then \( P_{\cdot,A} \)-stability and \( \sigma_G \)-stability are equivalent for objects of nonzero rank in \( A \).

2.6. Exceptional collections. Let \( D \) be a \( C \)-linear triangulated category of finite type.

Definition 2.24. An object \( E \in \text{Ob}(D) \) is called exceptional when, for all \( \ell \in \mathbb{Z} \),
\[
\text{Hom}_D(E,E[\ell]) = \begin{cases} \mathbb{C} & \text{if } \ell = 0 \\ 0 & \text{if } \ell \neq 0. \end{cases}
\]
A sequence \( \mathcal{E} = (E_0, \ldots, E_n) \) of exceptional objects is called an exceptional collection if
\[
\text{Hom}_D(E_i,E_j[\ell]) = 0
\]
for all \( i > j \) and all \( \ell \in \mathbb{Z} \). The exceptional collection is said to be strong if in addition \( \text{Hom}_D(E_i,E_j[\ell]) = 0 \) for all \( i,j \) and all \( \ell \in \mathbb{Z} \setminus \{0\} \); it is said to be full if the smallest triangulated subcategory containing \( E_0, \ldots, E_n \) is \( D \).
Finally, exceptional collections $\mathcal{E} = (\mathcal{E}_n, \ldots, \mathcal{E}_0)$ and $\mathcal{E}^\vee = (\mathcal{E}_n^\vee, \ldots, \mathcal{E}_0^\vee)$ are respectively called left dual and right dual to $\mathcal{E}$ if

$$\Hom_D(\mathcal{E}_i, \mathcal{E}_j) = \begin{cases} \mathbb{C} & \text{if } i = j = n - \ell \\ 0 & \text{otherwise} \end{cases}, \quad \Hom_D(\mathcal{E}_i, \mathcal{E}_j^\vee) = \begin{cases} \mathbb{C} & \text{if } i = j = \ell \\ 0 & \text{otherwise} \end{cases}.$$  

Given an exceptional collection $\mathcal{E}$, its left and right dual always exist and are unique, and they can be realized by repeated mutations $[\text{GK01} \S 2]$. If $\mathcal{E}$ is full, then so are $\mathcal{E}$ and $\mathcal{E}^\vee$. Notice also that if a full exceptional collection exists, then the Euler form $\chi$ is nondegenerate, and $K_0(D) = K_{\text{num}}(D)$ is freely generated by the elements of the collection.

**Examples 2.25.**

1. $D^b(\mathbb{P}^2)$ has a full exceptional collection $\mathcal{E} = (\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1))$ which has left dual $\mathcal{E} = (\mathcal{O}_{\mathbb{P}^2}(1), \tau_{\mathbb{P}^2}, \wedge^2 \tau_{\mathbb{P}^2}(-1))$ and right dual $\mathcal{E}^\vee = (\mathcal{O}_{\mathbb{P}^2}(1), \tau_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1))$. All these collections are strong.

2. $D^b(\mathbb{P}^1 \times \mathbb{P}^1)$ has a full exceptional collection $(\mathcal{O}_{\mathbb{X}}(0, -1)[-1], \mathcal{O}_{\mathbb{X}}[-1], \mathcal{O}_{\mathbb{X}}(1, -1), \mathcal{O}_{\mathbb{X}}(1, 0))$ with left dual given by $(\mathcal{O}_{\mathbb{X}}(1,0), \tau_{\mathbb{P}^1} \boxtimes \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(-1), \tau_{\mathbb{P}^1} \boxtimes \tau_{\mathbb{P}^2}(-1))$, or by $(\mathcal{O}_{\mathbb{X}}(1,0), \mathcal{O}_{\mathbb{X}}(2,0), \mathcal{O}_{\mathbb{X}}(1,1), \mathcal{O}_{\mathbb{X}}(2,1))$. The latter collection is strong, while the former is not.

3. Let $Q$ be an ordered quiver with relations $J$, whose vertices are labeled by $0, 1, \ldots, n$ (this means that there are no arrows from $i$ to $j$ if $j \leq i$). Then we have full exceptional collections $\mathcal{E}, \mathcal{E}^\vee$ on the bounded derived category $D^b(Q, J) := D^b(\text{Rep}_C(Q; J))$ made by the objects $E_i = S(i)[i - n], \quad \mathcal{E}_i^\vee = P(i),$

where $S(i)$ and $P(i)$ denote the standard simple and projective representations associated to each vertex $i$. Moreover, the collection $\mathcal{E}^\vee$ is obviously strong and it is left dual to $\mathcal{E}$ because of the formula

$$\text{Ext}^\ell(P(i), S(j)) = \begin{cases} \mathbb{C} & \text{if } i = j, \ell = 0 \\ 0 & \text{otherwise} \end{cases}.$$  

In the last example, the full strong collection made by the projective representations is somehow prototypical: suppose that $\mathcal{E} = (\mathcal{E}_n, \ldots, \mathcal{E}_0)$ is a full and strong exceptional collection on $D$, and let $T := \oplus_{i=0}^n \mathcal{E}_i$; the endomorphism algebra

$$A := \text{End}_D(T) = \begin{pmatrix} \text{Hom}(\mathcal{E}_n, \mathcal{E}_n) \\ \vdots \\ \text{Hom}(\mathcal{E}_0, \mathcal{E}_0) \end{pmatrix}$$

is basic, and hence it can be identified with $(CQ/J)^{\text{op}}$ for some ordered quiver $Q$ with vertices $I = \{0, 1, \ldots, n\}$ and relations $J \subset CQ$ in particular, we identify right $A$-modules of finite dimension with representations of $(Q, J)$. Then we have (under some additional hypotheses on $D$ which are satisfied e.g. when $D = D^b(X)$ for a smooth projective variety $X$):

**Theorem 2.26.** $[\text{B08}]$ Thm 6.2. $\mathcal{E}$ induces a triangulated equivalence

$$\Phi_{\mathcal{E}} = R\text{Hom}_D(T, \cdot) : D \longrightarrow D^b(Q; J).$$

More explicitly, $\Phi_{\mathcal{E}}$ maps an object $D$ of $D$ to a complex of representations which at the vertex $i \in \{0, \ldots, n\}$ of $Q$ has the graded vector space $R\text{Hom}(\mathcal{E}_i, D)$.

**Remark 2.27.** Notice that $\Phi_{\mathcal{E}}$ maps each $\mathcal{E}_i$ to the projective representation $P(i)$ of $Q$ and each dual $E_i$ to the simple $S(i)[i - n]$, and the standard heart $\text{Rep}_C(Q; J) \subset D^b(Q; J)$ is the extension closure of the simple modules $S(i)$. Hence, $\mathcal{E}$ induces a bounded $t$-structure on $D$ whose heart is the extension closure of the objects $E_i[n - i], i = 0, \ldots, n.$
2.7. Families of objects in the derived category. Let $X$ be a smooth projective irreducible complex variety polarized by an ample divisor $A \subset X$.

Take a $\mathbb{C}$-scheme $S$ of finite type. By a family over $S$ of objects of $D^b(X)$ having a common property $P$ we mean an object $F$ of $D^b(X \times S)$ such that, for any (closed) point $s \in S$, the object $F_s := L_s^* F$ has the property $P$, where $L_s: X \to X \times S$ maps $x$ to $(x,s)$.

We are mostly interested in two kinds of families of objects:

(1) Denote by $\mathcal{C} = \text{Coh}_{X,S} \subset D^b(X)$ the heart of the standard t-structure. If $F$ is a family of objects concentrated in degree zero, then by [Huy06, Lemma 3.31] it is isomorphic to a coherent $\mathcal{O}_{X \times S}$-module flat over $S$. In particular, the moduli space $M^X_{\mathcal{C}}(v)$ of $\mathcal{O}_{X,S}$-modules in the sense of [Kin94, Def. 5.1].

(2) Suppose that $D^b(X)$ has a full strong exceptional collection $\mathcal{E}$, and consider the equivalence $\Phi_{\mathcal{E}}: D^b(X) \to D^b(Q;J)$ of Theorem 2.26 together with the induced isomorphism $\phi: K_0(X) \to K_0(Q;J)$. If $S$ is affine, then a family of objects in the induced heart $\Phi_{\mathcal{E}}^{-1}(\text{Rep}_C(Q;J))$ is mapped by $\Phi_{\mathcal{E}}$ to a family of $\mathbb{C} Q/J$-modules in the sense of [Ohk10, Prop. 4.4] for details.

3. Sheaves on $\mathbb{P}^1$ and Kronecker modules

In this section the well-known classification of coherent sheaves on $\mathbb{P}^1$ is deduced via the representation theory of the Kronecker quiver $K_2$, as an easy anticipation of the ideas introduced in the next sections.

3.1. Representations of $K_2$ and Kronecker complexes on $\mathbb{P}^1$. Let $Z$ be a 2-dimensional $\mathbb{C}$-vector space with a basis $\{e_0, e_1\}$, and consider the complex projective line $\mathbb{P}^1 := \mathbb{P}_\mathbb{C}(Z)$. Fix also an integer $k \in \mathbb{Z}$.

We are interested in the finite-dimensional representations of the Kronecker quiver

$$ K_2 : \begin{array}{c} -1 \cr \end{array} \begin{array}{c} 0 \end{array} , $$

that is Kronecker modules $f \in \text{Hom}_\mathbb{C}(V_{-1} \otimes Z, V_0)$ (see Example 2.17), and their relations with sheaves on $\mathbb{P}^1$. The couple $\mathcal{E}_k = (E_{-1}, E_0) := (\mathcal{O}_{\mathbb{P}^1}(k-1), \mathcal{O}_{\mathbb{P}^1}(k))$ is a full strong exceptional collection in $D^b(\mathbb{P}^1)$, and so is its dual collection $\mathcal{E}_k^\vee = (\mathcal{O}_{\mathbb{P}^1}(k), \mathcal{O}_{\mathbb{P}^1}(k-1))$ induces by Theorem 2.26 a derived equivalence

$$ \Psi_k := \Phi_{\mathcal{E}_k} : D^b(\mathbb{P}^1) \to D^b(K_2), $$

as $\text{End}_{\mathcal{O}_{\mathbb{P}^1}}(T_k)$ may be identified with $\mathbb{C} K_2^{\text{opp}}$ via the isomorphism $H^0(\mathbb{P}^1; \tau_{\mathbb{P}^1}(-1)) \cong Z$. $\Psi_k$ sends a complex $\mathcal{F}^\bullet$ of coherent sheaves to the complex of representations

$$ R \text{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(\tau_{\mathbb{P}^1}(k-1), \mathcal{F}^\bullet) \cong R \text{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{O}_{\mathbb{P}^1}(k), \mathcal{F}^\bullet) . $$

We denote by $\mathcal{C} \subset D^b(\mathbb{P}^1)$ the heart of the standard t-structure and by $K_k \subset D^b(\mathbb{P}^1)$ the heart of the t-structure induced from the standard one in $D^b(K_2)$ via the equivalence $\Psi_k$.

**Lemma 3.1.** The objects of $K_k$ are, up to isomorphism in $D^b(\mathbb{P}^1)$, the Kronecker complexes

$$ V_{-1} \otimes \mathcal{O}_{\mathbb{P}^1}(k-1) \longrightarrow V_0 \otimes \mathcal{O}_{\mathbb{P}^1}(k) . $$

**Proof.** Let $A := \text{End}_{\mathcal{O}_{\mathbb{P}^1}}(T_k)$. $\Psi_k$ maps the exceptional objects $\mathcal{E}_i$, $i = 0, -1$, to the standard projective right $A$-modules $Id_{\mathcal{E}_i} A$, which correspond to the Kronecker modules $P_0 = (0 \to \mathcal{O})$ and $P_{-1} = (\mathcal{O} \otimes \mathbb{Z} \to \mathcal{O})$; now the heart $\text{Rep}_C(K_2)$, which is the extension closure of the simple modules $S_{-1}, S_0$, is mapped to the extension closure $K_k$ of $E_{-1}[1], E_0$ (see Remark 2.27), whose objects are Kronecker complexes. □
Ψ_k induces an isomorphism ψ_k : K_0(ℙ^1) → K_0(K_2) between the Grothendieck groups, which are free of rank 2. Hence, coordinates of an element v ∈ K_0(ℙ^1) are provided either by the couple (rk v, deg v) or by the dimension vector
d'' = (d''_1, d''_0) := \text{dim}(ψ_k(v)).

The simple representations S(−1) and S(0), whose dimension vectors are (1, 0) and (0, 1) respectively, correspond to the complexes O_{ℙ^1}(k−1)[1], with (rk, deg) = (−1, 1−k), and O_{ℙ^1}(k), with (rk, deg) = (1, k). So we deduce that the linear transformation between the two sets of coordinates is given by

\[
\begin{pmatrix}
\text{rk } v \\
\text{deg } v
\end{pmatrix}
= \begin{pmatrix}
-1 & 1 \\
1-k & k
\end{pmatrix}
\begin{pmatrix}
d''_1 \\
d''_0
\end{pmatrix},
\begin{pmatrix}
d''_1 \\
d''_0
\end{pmatrix}
= \begin{pmatrix}
-k & 1 \\
1-k & 1
\end{pmatrix}
\begin{pmatrix}
\text{rk } v \\
\text{deg } v
\end{pmatrix}.
\]

3.2. Semistable sheaves and Kronecker complexes. As in the end of [2.5] we consider the alternating form σ_M : K_0(ℙ^1) × K_0(ℙ^1) → ℤ given by

\[
\sigma_M(v, w) := \text{deg } v \text{rk } w − \text{deg } w \text{rk } v.
\]

This is also the alternating form σ_Z induced by the central charge Z = −deg + i rk as in equation [2.4]. We have seen in Lemma 2.22 that, on the standard heart C = CohO_{ℙ^1}, σ_M reproduces Gieseker-stability. Now we also consider σ_M-stability on the heart K_k:

Definition 3.2. A Kronecker complex K_V is said to be (semi-)stable when it is σ_M-(semi)stable in K_k (Def. 2.4), that is when for any nonzero Kronecker subcomplex K_W ⊂ K_V we have

\[
\text{deg } K_V \text{rk } K_W − \text{deg } K_W \text{rk } K_V (\leq 0).
\]

If we fix v ∈ K_0(ℙ^1), then we can write

\[
\nu_{M,v}(w) := \sigma_M(v, w) = -d_0''d''_{w,1} + d''_{1,1}d''_{w,0} = \theta_{M,v} \cdot d''_w,
\]

where the dot is the standard scalar product in ℤ^{(-1,0)} and

\[
\theta_{M,v} := -d''_0 \begin{pmatrix}
 k-1 \\
 k−1
\end{pmatrix}
\begin{pmatrix}
\text{rk } v − \text{deg } v \\
\text{deg } v
\end{pmatrix}.
\]

So, via the equivalence Ψ_k, (semi)stability of a Kronecker complex K_V ∈ v is equivalent to θ_{M,v}-(semi)stability of the corresponding representation V of K_2; being θ_{M,v} = d''_{1,1} positive, this is the usual definition of (semi)stable Kronecker module (see Ex. 2.17).

Consider an object in the intersection of the hearts K_k and C in D^b(ℙ^1): this can be seen either as an injective Kronecker complex or as the sheaf given by its cokernel. The following observation shows that for such an object the two notions of stability coincide:

Proposition 3.3. K_k is the heart obtained by tilting the standard heart C with respect to the central charge Z = −deg + i rk at phase φ_k := arg(−k+i)/π, as in [2.2,4]. In particular, for any φ ∈ [φ_k, 1] the categories of Z-semistable objects with phase φ in the two hearts coincide: S_{φ}^{(C,Z)} = S_{φ}^{(K_k,Z)}.

We denote by R_k ⊂ K_0(ℙ^1) the cone spanned by the objects of C ∩ K_k, that is

\[
R_k := \{ v ∈ K_0(ℙ^1) \mid \text{rk } v \geq 0 \text{ and } \text{deg } v \geq k \text{rk } v \}
\]

\[
= \{ v ∈ K_0(ℙ^1) \mid d''_0 \geq d''_{1,1} \geq 0 \}.
\]

The proposition implies that, if we fix a class v ∈ R_k, then the tuple (v, C, K, σ_C) verifies conditions (C1), (C2) of [2.3]. Namely, we have:

(C1) a slope-(semi)stable sheaf F ∈ C with [F] = v belongs to K_k, that is it is isomorphic to the cokernel of an injective Kronecker complex K_V ∈ K_k; similarly, a (semi)stable Kronecker complex K_V ∈ K_k with [K_V] = v belongs to C, which means that it is injective;

(C2) an object K_V ≃ F of class v in C ∩ K_k is (semi)-stable as a Kronecker complex if and only if it is (semi)-stable as a sheaf.

Proof. The heart K_k lies in ⟨C, C[1]⟩_{ext} and then by [P07, Lemma 1.1.2] it is obtained by tilting C at the torsion pair (T_k, F_k) given by T_k := C ∩ K_k and F_k := C ∩ K_k[−1]. Now fix k ∈ ℤ, take the phase φ_k = arg(−k+i)/π of O_{ℙ^1}(k), and consider the torsion pair (T^{Z}_{≥ φ_k}, F^{Z}_{≥ φ_k}) induced by Z, eqn. (2.2). Using the explicit form (3.1) of Ψ_k, we observe that T^{Z}_{≥ φ_k} ⊂ T_k and F^{Z}_{≥ φ_k} ⊂ F_k, which implies that the two torsion pairs must coincide: indeed, a sheaf T ∈ T^{Z}_{≥ φ_k} satisfies
Figure 1. The hearts $\mathcal{C}, \mathcal{K}_k \subset D^b(\mathbb{P}^1)$

Figure 2. The Grothendieck group $K_0(\mathbb{P}^1)$

$\text{Ext}^1_{\mathcal{O}_{\mathbb{P}^1}}(\tau_{\mathbb{P}^1}(k-1), \mathcal{T}) = \text{Ext}^1_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{O}_{\mathbb{P}^1}(k), \mathcal{T}) = 0$ by Serre duality, and thus it belongs to $\mathcal{K}_k$; on the other hand, for a sheaf $\mathcal{F} \in \mathcal{F}_{\leq \phi_k}$ we have $\text{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(\tau_{\mathbb{P}^1}(k-1), \mathcal{F}) = \text{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{O}_{\mathbb{P}^1}(k), \mathcal{F}) = 0$, which means that it belongs to $\mathcal{K}_k[-1]$. □

Corollary 3.4. (Birkhoff-Grothendieck Theorem) Every coherent sheaf $\mathcal{F} \in \text{Coh}_{\mathcal{O}_{\mathbb{P}^1}}$ is a direct sum of line bundles $\mathcal{O}_{\mathbb{P}^1}(\ell)$ and structure sheaves of fat points.

Proof. For an object in $\mathcal{T}_{\geq \phi_k}$, being indecomposable is the same when considered in $\mathcal{C}$ or $\mathcal{K}_k$. All the indecomposable representations of $K_2$ are listed below for $n \geq 1$ (see e.g. [Ben98, Theorem 4.3.2]):

\[
\begin{align*}
\mathbb{C}^n &\xrightarrow{\varphi_n} \mathbb{C}^n, \\
\mathbb{C}^n &\xrightarrow{J_n(\lambda)^{t}} \mathbb{C}^n, \\
\mathbb{C}^n &\xrightarrow{I_n} \mathbb{C}^{n+1}, \\
\mathbb{C}^{n+1} &\xrightarrow{(0 I_n)} \mathbb{C}^n,
\end{align*}
\]

where $J_n(\lambda)$ is the $n$-dimensional Jordan matrix with eigenvalue $\lambda \in \mathbb{C}$. The first three representations correspond to injective Kronecker complexes whose cokernels are, respectively, a torsion sheaf with length $n$ support at the point $[-\lambda : 1]$, a torsion sheaf with length $n$ support at $[1 : 0]$ and the line bundle $\mathcal{O}_{\mathbb{P}^1}(k+n)$. The last representation gives a Kronecker complex which is not in $\mathcal{C}$.

Now take any $\mathcal{F} \in \text{Coh}_{\mathcal{O}_{\mathbb{P}^1}}$ and choose $k \in \mathbb{Z}$ such that the minimum HN phase of $\mathcal{F}$ is at least $\phi_k = \text{arg}(-k+i)/\pi$. If $\mathcal{F} = \bigoplus_i \mathcal{F}_i$ is the decomposition of $\mathcal{F}$ in indecomposables, then every $\mathcal{F}_i$ has HN phases $\geq \phi_k$, so $\mathcal{F}_i \in \mathcal{T}_{\geq \phi_k}$, and then it is also an indecomposable object in $\mathcal{K}_k$, which means that it is isomorphic to one of the three sheaves listed above. □

3.3. Moduli spaces. Fix $k \in \mathbb{Z}$ and a class $v \in \mathbb{R}_k$ (see eq. (3.4)). By Proposition 3.3 (see also [2.7]), the moduli spaces $M_{\mathbb{P}^1}^a(v)$ and $M_{K_2, \partial M, v}^a = K(2; d_{-1}^v, d_0^v)$ are isomorphic, as well as the subspaces of stable objects:

$M_{\mathbb{P}^1}^a(v) \simeq K(2; d_{-1}^v, d_0^v), \\
M_{\mathbb{P}^1}^a(v) \simeq K_{st}(2; d_{-1}^v, d_0^v)$. 

In this subsection we will describe explicitly these moduli spaces for all values of \( v \in \mathfrak{R}_k \), that is for all \( d \in \mathbb{Z}^{(-1,0)} \) with \( d_0 \geq d_{-1} \geq 0 \).

First of all, as already mentioned in Ex. 2.17 we have:

**Lemma 3.5.** \[\text{[Drô87]} \text{Prop. 21-22] There are isomorphisms } K(2; d_-, d_0) \simeq K(2; 2d_--d_0, d_-1) \simeq K(2; d_0, 2d_0-d_-1) \text{ and } K(2; d_-1, d_0) \simeq K(2; d_0, d_-1), \text{ restricting to isomorphisms of the stable loci.} \]

We can visualize these isomorphisms as follows: consider the linear transformation \( M = \left( \begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix} \right) \) acting in the \((d_-, d_0)\) plane; the orbits of \( M \) are on lines of slope 1. The region \( \mathfrak{R} = \{ d_0 \geq d_-1 > 0 \} \) and the diagonal \( d_-1 = d_0 \) are invariant under \( M \). The lemma says that integral points in \( \mathfrak{R} \) lying in the same \( M \)-orbit, as well as symmetric points with respect to the diagonal \( d_-1 = d_0 \), give isomorphic moduli spaces.

Thus it is enough to consider the wedge \( d_0 \geq 2d_-1 \) and the diagonal \( d_-1 = d_0 \). We start by analyzing the diagonal:

**Lemma 3.6.** \( K(2; 1, 1) = K_{\text{st}}(2; 1, 1) \simeq \mathbb{P}^1 \) and \( K(2; m, m) \simeq \mathbb{P}^m, K_{\text{st}}(2; m, m) = \emptyset \) for \( m \geq 2 \).

Given a Kronecker module \( f \in \text{Hom}_C(V_{-1} \otimes Z, V_0) \), we often write \( f_z := f(\cdot \otimes z) \in \text{Hom}_C(V_{-1}, V_0) \) for \( z \in Z \), and \( f_j := f(e_j) \) for \( j = 0, 1 \) \((\{e_0, e_1\} \text{ is the basis of } Z \text{ that we fixed from the beginning})\); the index \( j \) is tacitly summed when repeated.

**Proof.** Clearly, \( f : \mathbb{C} \otimes Z \rightarrow \mathbb{C} \) is semistable if and only if it is stable if and only if \( f \neq 0 \); we can identify thus \( R^{\text{ss}} \simeq \mathbb{C}^2 \setminus \{0\} \); \( PGV \simeq \mathbb{C}^2 \) acts by scalar multiplication, hence the quotient is \( \mathbb{P}^1 \). Now let \( m \geq 2 \): first observe that the semistable and stable loci in the representation space \( R = R_{C^m \otimes C^m} = \text{Hom}_C(C^m \otimes Z, C^m) \) are

\[
R^{\text{ss}} = \left\{ f \in R \mid \max_{z \in Z} \text{rk } f_z = m \right\}, \quad R^{\text{st}} = \emptyset.
\]

Indeed, the set \( U := \{ f \in R \mid \max_{z \in Z} \text{rk } f_z = m \} \) is open and \( G_V \)-invariant, and it is contained in \( R^{\text{ss}} \) because there are no subrepresentations of dimension \((d_-1, d_0)\) with \( d_-1 > d_0 \), being the generic \( f_z \) an isomorphism. Moreover, any polystable representation \( f \in R \) can be written, up to isomorphism, as \( f_z = \text{diag}(a^1_j, \ldots, a^m_j) \) for \([a^1], \ldots, [a^m] \in \mathbb{P}^1 \text{ (unique up to permutations)}\), and thus \( f_z = z^j f_j \) has nonvanishing determinant for general \( z \in Z \). So \( U \) contains the polystable locus \( R^{\text{ps}} \), which implies that \( U = R^{\text{ss}} \).

Now let \( \mathbb{C}[Z]_m \) be the vector space of homogeneous polynomial functions \( Z \rightarrow \mathbb{C} \) of degree \( m \), and \( \mathbb{P}_C(\mathbb{C}[Z]_m) \) be the projective space of lines in it. We can consider the \( G_V \)-invariant morphism \( \phi : R^{\text{ps}} \rightarrow \mathbb{P}_C(\mathbb{C}[Z]_m) \) sending a module \( f \) to the class of the polynomial function \( z \mapsto \det f_z \). \( \phi \) sends the polystable representation \( f_j = \text{diag}(a^1_j, \ldots, a^m_j) \) to the class \([\prod_{\ell=1}^m a^\ell_j e^{\ast \ell}]\), where \([e^{\ast 0}, e^{\ast 1}] \) is the dual basis of \([e_0, e_1] \); this shows that \( \phi \) maps non-isomorphic polystable representations to distinct classes and that \( \phi \) is surjective, because every element \( h \in \mathbb{C}[Z]_m \) can be factored as \( h = \prod_{\ell=1}^m a^\ell_j e^{\ast \ell} \). This is enough to conclude that \( \phi \) is the categorical quotient map. \( \square \)

Now we turn to dimension vectors with \( d_0 \geq 2d_-1 \):

**Lemma 3.7.** If \( d_0 > 2d_-1 \), then \( K(2; d_-1, d_0) = \emptyset \). Moreover, \( K(2; 1, 2) = K_{\text{st}}(2; 1, 2) = \text{pt} \) while for \( m > 1 \) we have \( K(2; m, 2m) = \text{pt} \) and \( K_{\text{st}}(2; m, 2m) = \emptyset \).

**Proof.** Take a Kronecker module \( f : V_{-1} \otimes Z \rightarrow V_0 \), and consider the submodule \( W \) with \( W_{-1} = V_{-1} \) and \( W_0 = \text{im } f_0 + \text{im } f_1 \). If \( d_0 > 2d_-1 \), then \( f \) is always unstable, as \( W \) destabilizes it.

So we assume now on that \( d_0 = 2d_-1 \). \( f \) is semistable if and only if \( f_0, f_1 \) are both injective and have complementary images, that is \( V_0 = \text{im } f_1 \oplus \text{im } f_2 \) (otherwise \( W \) is again destabilizing); moreover, all semistable representations are always isomorphic, as each of them is completely described by the images \( f_j(e_k) \) of the basis vectors of \( V_{-1} \), and these form a basis of \( V_0 \). Finally, a semistable \( f \) is stable if and only if \( (d_-, d_0) = (1, 2) \).

Collecting the last three lemmas we can now describe explicitly all the moduli spaces \( K(2; d_-1, d_0) \):

For \( p, q \in \mathbb{Z} \) we define lines \( \ell_p, r_q \) in the \((d_-1, d_0)\)-plane as follows: \( \ell_p \) is the line \( \{pd_0 = (p+1)d_-1\} \) if \( p > 0 \), the diagonal \( \{d_0 = d_-1\} \) for \( p = 0 \) and the line \( \{pd_-1 = (p-1)d_0\} \) for \( p < 0 \), while \( r_q := \{d_0 = d_-1 + q\} \).

**Theorem 3.8.** We assume that \( d_-1 > 0 \) and \( d_0 > 0 \).

1. \( K(2; d_-1, d_0) \) is empty unless \( (d_-1, d_0) \) lies on a line \( \ell_p \).
Figure 3. The moduli spaces \( K(2; d_{-1}, d_0) \) for all values of \( d_{-1}, d_0 \in \mathbb{N} \).

Figure 4. The moduli spaces \( M_{\mathbb{P}^1}(v) \) for all values of \( v \in K_0(\mathbb{P}^1) \) with \( \text{rk} \ v \geq 0 \).

(2) if \( (d_{-1}, d_0) \in \ell_p \cap r_q \) for some \( p, q \in \mathbb{Z} \) with \( q \neq 0, \pm 1 \), then \( K(2; d_{-1}, d_0) = \text{pt} \) while \( K_{\text{st}}(2; d_{-1}, d_0) = \emptyset \);
(3) if \( (d_{-1}, d_0) \in \ell_p \cap r_1 \) or \( (d_{-1}, d_0) \in \ell_p \cap r_{-1} \) for some \( p \in \mathbb{Z} \), then \( K(2; d_{-1}, d_0) = K_{\text{st}}(2; d_{-1}, d_0) = \text{pt} \);
(4) if \( (d_{-1}, d_0) \in \ell_0 = r_0 \), then \( K(2; d_{-1}, d_0) \simeq \mathbb{P}^{d_0} \); moreover \( K_{\text{st}}(2; 1, 1) \simeq \mathbb{P}^1 \), while \( K_{\text{st}}(2; m, m) = \emptyset \) for \( m \geq 2 \).

This is summarized in Figure 3. Now we can translate this into a classification of moduli of sheaves on \( \mathbb{P}^1 \) (depicted in Figure 4):

**Corollary 3.9.** Fix \( v \in K_0(\mathbb{P}^1) \).

(1) Suppose that \( \text{rk} \ v > 0 \) and \( \deg v \) is a multiple of \( \text{rk} \ v \); then \( M^\text{ss}_{\mathbb{P}^1}(v) \) is a point, while \( M^\text{st}_{\mathbb{P}^1}(v) \) is a point if \( \text{rk} \ v = 1 \) and empty otherwise;
(2) if \( \text{rk} \ v = 0 \) and \( \deg v \geq 0 \), then \( M^\text{ss}_{\mathbb{P}^1}(v) \simeq \mathbb{P}^{\deg v} \); moreover \( M^\text{st}_{\mathbb{P}^1}(v) \simeq \mathbb{P}^1 \) for \( \deg v = 1 \), while \( M^\text{st}_{\mathbb{P}^1}(v) = \emptyset \) if \( \deg v \geq 2 \);
(3) in all the other cases \( M^\text{st}_{\mathbb{P}^1}(v) \) is empty.

**Proof.** Choose \( k \in \mathbb{Z} \) so that \( v \in \mathbb{R}_k \). The statements immediately follow from the theorem, by noticing that the transformation (3.3) maps the lines \( \ell_p \) with \( p > 0 \) and the lines \( r_q \) respectively to the lines \( (p + k) \text{rk} \ v = \deg v \) and the horizontal lines \( \text{rk} \ v = q \) in the \((- \deg v, \text{rk} \ v)\) plane. \( \square \)
Remark 3.10. The statements of the corollary can be easily explained in sheaf-theoretic terms via Birkhoff-Grothendieck Theorem:

1. A semistable sheaf of rank $r > 0$ must have a direct sum of $r$ copies of the same line bundle $\mathcal{O}_{\mathbb{P}^1}(\ell)$, so it is of degree $r\ell$; it is stable if and only if $\ell = 1$.

2. The polystable sheaves of rank 0 and degree $d$ are direct sums $\mathcal{O}_{\mathbb{P}^1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}$ of skyscraper sheaves and are such that they are in 1-1 correspondence with points of the $d$th symmetric product $\mathbb{P}^d$ of $\mathbb{P}^1$; in particular, they can be stable if and only if $d = 1$. The structure sheaves of fat points are also semistable, but they degenerate to direct sums of skyscraper sheaves on the reduced points of their support.

4. Gieseker stability and quiver stability on surfaces

In this section we discuss how to relate Gieseker-semistable sheaves on a surface $X$ whose bounded derived category satisfies certain assumptions to semistable representations of a quiver associated to $X$. The idea is analogous to what we saw in the previous section, but the situation becomes now more involved and requires a different analysis. In the next sections we will apply this discussion to the surfaces $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

Let $X$ be a smooth irreducible projective complex surface with an ample divisor $A \subset X$.

First of all we assume that $X$ has a strong full exceptional collection $\mathcal{E} = (\mathcal{E}_n, \ldots, \mathcal{E}_0)$, so that we get an equivalence (see Theorem 2.26 and subsequent remark; for convenience we include now a shift)

$$
\Psi := \Phi_{\mathcal{E}}[1] : D^b(X) \to D^b(Q; J).
$$

Recall that $\Psi$ maps a complex $\mathcal{F}^\bullet$ in $D^b(X)$ to a complex of representations of $Q$ given, at a vertex $i \in \{0, \ldots, n\}$ of $Q$, by the graded vector space

$$
R \text{Hom}_D(\mathcal{E}_i, \mathcal{F}^\bullet)[1].
$$

$\Psi$ induces in particular an isomorphism $\psi : K_0(X) \to K_0(Q; J)$, and a t-structure on $D^b(X)$ whose heart is denoted by $\mathcal{K} := \Psi^{-1}(\mathcal{Rep}_C^L(Q; J))$ and equals the extension closure of the objects $E_i[n - i - 1]$, where $\mathcal{E} = (E_0, \ldots, E_n)$ is the right dual collection to $\mathcal{E}$. The alternating form $\sigma_G$, defined after eq. (2.12) and reproducing Gieseker stability (Lemma 2.22), defines a stability structure on $\mathcal{K}$, and hence on $\mathcal{Rep}_C^L(Q; J)$; in particular, a representation $V$ in a given class $\psi(v) \in K_0(Q; J)$ is $\sigma_G$-(semi)stable if and only if it is $\theta_{G, v}$-(semi)stable in the sense of Def. 2.14 where $\theta_{G, v} = \theta_{\mathcal{F}(\mathcal{V}), v} + \theta_{\mathcal{F}(\mathcal{V}), v} \in \mathbb{Z}[t]^I$ is defined by (the dot denotes the standard scalar product in $K_0(Q; J) \cong \mathbb{Z}^I$)

$$
\nu_{\mathcal{F}(\mathcal{V}), v}(w) = \sigma_{\mathcal{F}(\mathcal{V}), v}(w, v) = \theta_{\mathcal{F}(\mathcal{V}), v} \cdot \dim \mathcal{V}(w),
\nu_{\mathcal{F}(\mathcal{V}), v}(w) = \sigma_{\mathcal{F}(\mathcal{V}), v}(w, v) = \theta_{\mathcal{F}(\mathcal{V}), v} \cdot \dim \mathcal{V}(w).
$$

Unlike what happened for $\mathbb{P}^1$, now $\mathcal{K}$ is not obtained as a tilt of the standard heart $\mathcal{C}$ with respect to a stability condition (it never satisfies eq. (2.1) because it intersects three shifts of $\mathcal{C}$, see Figure 4). So there seems to be no reason to expect a priori any relation between stability on one heart and on the other. Nevertheless, we will see that under certain hypotheses this kind of compatibility exists; more precisely, we discuss under which hypotheses the 4-ple $(\sigma_G, \mathcal{C}, \mathcal{V}, v)$ satisfies conditions (C1), (C2) of [2,3]. Doing this requires some extra assumptions on the collections $\mathcal{E}, \mathcal{V}$Eq. In this section we will suppose that:

(A1) the objects $\mathcal{E}_i$ are coherent sheaves which are Gieseker-semistable with respect to $A$;

(A2) every element of $\mathcal{K}$ is isomorphic to a complex $K_V$ of locally free sheaves concentrated in degrees $-1, 0, 1$;

(A3) there is a line bundle $\mathcal{L}_0 \in \text{Pic}(X)$ such that any such complex satisfies $K_V \otimes \mathcal{L}_0 \in \mathcal{K}$, where $K_V^\vee$ denotes the dual cochain complex.

In the specific cases that will be examined, the second assumption will follow from the fact that the objects $E_i[n - i - 1]$ generating $\mathcal{K}$ turn out to group into three blocks, where the objects in each block are orthogonal to each other, and they are all isomorphic to vector bundles shifted to degree $-1, 0$ or $1$, depending on the block.[6] Because of this, the complex $K_V \in \mathcal{K}$ corresponding to some representation $V$ of $(Q, J)$ consists, in each degree $\ell = -1, 0, 1$, of a direct sum of vector bundles of the form $V_i \otimes E_i[n - i - 1]$, where $E_i$ ranges in the block of objects in degree $\ell$. This means in

---

[6] Three-blocks collections on Del Pezzo surfaces were studied in [2,3].
particular that we can write down explicitly the cohomological functors $H^\ell_K$ of the non-standard t-structure as functors mapping a complex $F^\bullet \in D^b(X)$ to a complex $K^V \in K$ with

$$V_i = R^{\ell+1} \text{Hom}(\vee E_i, F^\bullet).$$

The third assumption requires that taking the duals of the above vector bundles and twisting them by $L_0$, blocks -1 and 1 are exchanged, while block 0 is fixed by this operation.

4.1. Condition (C1). In this subsection we will study condition (C1) of § 2.3. First of all, we want to show that a semistable sheaf in a class $v$ also belongs to the heart $K$, that is it is isomorphic to the middle cohomology of a certain monad $K^V$ (recall that a monad is a complex with zero cohomology in degrees $\ell \neq 0$). This amounts to checking the vanishing of the cohomological functors $H^\ell_K$ for $\ell \neq 0$, which in turn reduces, by eq. (4.1), to verify the vanishing of some Ext spaces. For this to work we need to choose $v$ appropriately: we denote by $R_A \subset R^G_A \subset K_0(X)$ the regions

$$R_A := \{ v \in K_0(X) \mid \text{rk } v > 0, \ \max_i \mu_A(\vee E_i \otimes \omega_X) < \mu_A(v) < \min_i \mu_A(\vee E_i) \},$$

$$R^G_A := \{ v \in K_0(X) \mid \text{rk } v > 0, \ \max_i p^{\vee E_i \otimes \omega_X, A} < p_{v, A} < \min_i p^{\vee E_i, A} \}.$$

For these regions to be nonempty, the exceptional sheaves $\vee E_i$ must have their slopes concentrated in a sufficiently narrow region, and the anticanonical bundle $\omega_X$ must be sufficiently positive (Figure 6).

**Remark 4.1.** We could also twist the collection $E$ by a line bundle, to shift the regions $R_A, R^G_A$ accordingly: if these are wide enough and the line bundle has small but nonzero degree, then with such twists we can cover the whole region $\text{rk } v > 0$. When this is the case, like in the examples that we will consider, we are thus free to start with any class $v \in K_0(X)$ with positive rank, provided that we choose $E$ appropriately.

**Lemma 4.2.** Suppose that $v \in R^G_A$ (resp. $v \in R_A$). Then any Gieseker-semistable (resp. slope-semistable) sheaf $F \in v$ belongs to the heart $K$. 

![Figure 5. The hearts $C, K \subset D^b(X)$](image1.png)

![Figure 6. The region $R_A$.](image2.png)
Proof. Since each \( \mathcal{V}_{E_i} \) is Gieseker-semistable, we have \( \text{Hom}_{\mathcal{D}(X)}(\mathcal{V}_{E_i}, \mathcal{F}) = 0 \) because of the inequality \( p_F,A < p_{E_i,A} \); on the other hand, the inequality \( p_{E_i,\omega,A} < p_{F,A} \) and Serre duality give \( \text{Ext}^2_{\mathcal{D}(X)}(\mathcal{V}_{E_i}, \mathcal{F}) = \text{Hom}_{\mathcal{D}(X)}(\mathcal{F}, \mathcal{V}_{E_i} \otimes \omega_X) = 0 \). So \( H^2_{\mathcal{C}}(\mathcal{F}) = H^2_{\mathcal{K}}(\mathcal{F}) = 0 \) by eq. (1.1). For the case of slope semistability the proof is the same.

Now we deal with the same problem with the two hearts \( \mathcal{C}, \mathcal{K} \) exchanged: we want a \( \sigma_G \)-semistable object \( K \in \mathcal{V} \) to be in \( \mathcal{K} \), that is to belong to \( \mathcal{C} \). To obtain this, we observe that when \( K \) is not a monad, we can construct a destabilizing subcomplex or quotient complex using the following idea from [FGIK10]: consider the skyscraper sheaf \( \mathcal{O}_x \) over some point \( x \in X \). Clearly \( H^f_{\mathcal{C}}(\mathcal{O}_x) = 0 \) for all \( f \neq -1 \), which means that there is a \( K_x \in \mathcal{C} \) which has cohomology \( \mathcal{O}_x \) in degree 1, and zero elsewhere, that is to say that \( K_x \simeq \mathcal{O}_x([-1]) \) in \( D^b(X) \). Furthermore, by assumption (A3), there is another Kronecker complex \( K_z \) such that \( K_z \otimes \mathcal{L}_0 \simeq K_x \).

**Proposition 4.3.** If the second map in a Kronecker complex \( K \) is not surjective at some point \( x \in X \), then there is a nonzero morphism \( K \to K_x \). If the first map in \( K \) is not injective at \( x \), then there is a nonzero morphism \( K_x \to K \).

Proof. Suppose that the second map \( m \) in \( K \) is not surjective at some \( x \in X \): we have then a surjective morphism \( c : K^1 \to \mathcal{O}_x \) such that \( c \circ m = 0 \), and this gives a cochain map \( K \to \mathcal{O}_x[-1] \), and thus a nonzero morphism \( K \to K_x \) in \( \mathcal{K} \).

Now suppose that the first map is not injective at \( x \): then we can apply the previous argument to the complex \( K^0 \otimes \mathcal{L}_0 \) to get a nonzero map \( K^0 \otimes \mathcal{L}_0 \to K_x \), hence a nonzero \( K_x \to K^0 \otimes \mathcal{L}_0 \to K \).

In the following two lemmas we prove that for any complex \( K \in \mathcal{C} \) of vector bundles of class \( v \) and any \( \sigma_G \)-maximal subobject \( K \subset K \) in \( \mathcal{C} \) (see Def. 2.5), we have some vanishings in the cohomologies of \( K \) and \( K/\mathcal{K} \), provided that the class \( v \in \mathcal{C}(X) \) chosen satisfies some constraints imposed by the complexes \( K \) and \( K_z \). Notice that when \( K \) is \( \sigma_G \)-semistable, then it is a \( \sigma_G \)-maximal subobject of itself, and thus \( K \) will turn out in Cor. 4.7 to be a monad.

**Lemma 4.4.** Take \( K \in \mathcal{C} \) of class \( v \). Suppose that for any \( x \in X \) and any nonzero subobject \( S \subset K_x \) in \( \mathcal{K} \) we have \( \nu_{\mathcal{G},v}(S) := \sigma_G(v,S) > 0 \). Then any \( \sigma_G \)-maximal subobject \( K \subset K \) satisfies \( H^1_{\mathcal{C}}(K) = 0 \).

Proof. If \( H^1_{\mathcal{C}}(K) \neq 0 \), which means that the second map in \( K \) is not surjective at some point \( x \in X \), then there is a nonzero morphism \( f : K \to K_x \) by Prop. 4.3. So we have \( \nu_{\mathcal{G},v}(K) = \nu_{\mathcal{G},v}(K_x) + \nu_{\mathcal{G},v}(\text{im } f) \) and, by hypothesis, \( \nu_{\mathcal{G},v}(\text{im } f) > 0 \). If \( \text{ker } f = 0 \) then \( \nu_{\mathcal{G},v}(K) > 0 \), while if \( \ker f \neq 0 \) then \( \nu_{\mathcal{G},v}(\ker f) = \nu_{\mathcal{G},v}(K) - \nu_{\mathcal{G},v}(\text{im } f) < \nu_{\mathcal{G},v}(K) \); in both cases, \( \sigma_G(K) \neq K_x \) is not maximal.

**Lemma 4.5.** Take \( K \in \mathcal{C} \) of class \( v \). Suppose that, for any \( x \in X \), \( K_x \) is \( \nu_{M,v} \)-semistable and every quotient \( Q \) of \( K_x \) with \( \nu_{M,v}(Q) < 0 \) satisfies \( H^{-1}_{\mathcal{C}}(Q) = 0 \). Then for any \( \nu_{M,v} \)-maximal subobject \( K \subset K \) we have \( H^{-1}_{\mathcal{C}}(K/\mathcal{K}) = 0 \).

Notice that if \( K \subset K \) is \( \sigma_G \)-maximal then it is also \( \sigma_M \)-maximal.

Proof. Let \( K \subset K \) be a \( \sigma_M \)-maximal subobject, which means that the quotient \( K := K/K \) maximizes \( \nu_M = \sigma_M(v,v) \). We have to prove that the first map in \( K \) is injective. This is clearly true if such a map is injective at every point of \( X \); thus suppose now that it is not injective at some point \( x \in X \), so that we have a nonzero morphism \( g : K \to K_x \) by Prop. 4.3. Now \( \nu_{M,v}(K) = \nu_{M,v}(K_x) + \nu_{M,v}(\ker g) \) and \( \nu_{M,v}(K_x) = 0 \) by hypothesis. If \( \ker g = 0 \), then \( 0 \leq \nu_{M,v}(K) = \nu_{M,v}(K_x) \leq 0 \), implying \( H^{-1}_{\mathcal{C}}(K) = 0 \). If \( \ker g \neq 0 \), then we have \( \nu_{M,v}(\ker g) = 0 \) (otherwise \( \nu_{M,v}(K) < \nu_{M,v}(\ker g) \) would contradict maximality of \( \nu_{M,v}(K) \)); moreover, in this case \( K \to K_x^{(1)} := \text{coker } g \) is also a quotient of \( K \) of maximal \( \nu_{M,v} \), and \( H^{-1}_{\mathcal{C}}(\ker g) = H^{-1}_{\mathcal{C}}(K_x/\ker g) = 0 \).

By applying the same argument to \( K_x^{(1)} \) we see that either we can immediately conclude that \( H_{\mathcal{C}}^{-1}(K_x^{(1)}) = 0 \), in which case we stop here, or we can construct a further quotient \( K_x^{(2)} \) with maximal \( \nu_{M,v} \) and such that \( H^{-1}_{\mathcal{C}}(\ker g^{(1)}) = 0 \). After finitely many steps we end up with a chain

\[
K = K^{(0)} \to K^{(1)} \to K^{(2)} \to \cdots \to K^{(\ell)}
\]
of surjections with $H^{-1}_c(\ker c^{(i)}) = 0$ for all $i \geq 0$ and $H^{-1}_c(K^{(l)}_U) = 0$. This implies that $H^{-1}_c(K_U) = 0$. \hfill \Box

**Remark 4.6.** Notice that the hypotheses of Lemmas 4.4 and 4.5 are verified under the stronger assumptions that $\text{rk} \, v > 0$ and for all $x \in X$, $K_x$ and $\tilde{K}_x$ are $\nu_{M,v}$-stable and $H^{-1}_c(K_x) = 0$.\footnote{As for the hypothesis of Lemma 4.4, note that $\nu_{G,v}(K_V) = \text{rk} \, v > 0$.}

It is convenient to gather the conditions on $v$ imposed by the hypotheses of Lemmas 4.4 and 4.5 or by Remark 4.6 in the definition of the two regions $S^*_A \subset S_A \subset K_0(X)$:

\[
S^*_A := \left\{ v \in K_0(X) \mid \text{rk} \, v > 0; \text{ for all } x \in X, K_x \text{ and } \tilde{K}_x \text{ are } \nu_{M,v} \text{-stable and } H^{-1}_c(\tilde{K}_x) = 0 \right\},
\]

(4.5)

\[
S_A := \left\{ v \in K_0(X) \mid \text{for any } x \in X \text{ and any } 0 \neq S \subset K_x, \nu_{G,v}(S) > 0; \text{ for any } x \in X, \tilde{K}_x \text{ is } \nu_{M,v} \text{-semistable, and for any quotient } K_x \to Q \text{ with } \nu_{M,v}(Q) = 0 \text{ we have } H^{-1}_c(Q) = 0. \right\}.
\]

Again, we will see in the examples that it is often enough to twist the collection $\mathfrak{C}$ by a line bundle to have any $v \in K_0(X)$ of positive rank inside such a region.

**Corollary 4.7.** Take $K_V \in K$ of class $v \in S_A$. If $K_V$ is $\sigma_G$-semistable then it is a monad, that is $K_V \in \mathcal{C}$.

Proof. If a nonzero $K_V$ is $\sigma_G$-semistable (hence $\nu_{G,v}$-semistable), then it has minimal $\nu_{G,v}$ between its subobjects, and maximal $\nu_{M,v}$ between its quotients. So we can apply Lemmas 4.4 and 4.5 to deduce that $H^{-1}_c(K_V) = H^0_c(K_V) = 0$. \hfill \Box

Summing up, Lemma 4.2 and Corollary 4.7 say that if $v \in R^G_{\mathfrak{A}} \cap S_A$, then condition (C1) of 2.3 is verified for Gieseker stability $\sigma_G$ and the hearts $\mathcal{C}$.

4.2. **Condition (C2).** Now we turn to the analysis of condition (C2) of 2.3: we want to show that a monad $K_V \in K$ of class $v$ is $\sigma_G$-(semi)stable as an object of $K$ if and only if its middle cohomology is $\sigma_G$-(semi)stable as an object of $\mathcal{C}$, that is, a Gieseker-(semi)stable sheaf. First we prove the “only if” direction:

**Lemma 4.8.** Suppose that $v \in R^G_{\mathfrak{A}}$, and let $K_V \in v$ be monad which is a $\sigma_G$-(semi)stable object of $K$. Then its middle cohomology $H^0_c(K_V)$ is a Gieseker-(semi)stable sheaf.

Proof. Suppose that $F := H^0_c(K_V)$ is not Gieseker-semistable. Let $F_1 \subset F$ be the maximally destabilizing subsheaf (i.e. the first nonzero term in the TN filtration of $F$), which is semistable and satisfies $P_{F_1,A} = G P_{F,A} \supset P_{F/F_1,A,max}$ (notations as in eq. (2.9)). Then, as in the proof of Lemma 4.2, we deduce that $\text{Hom}_{\mathcal{C}}(\tau^F_{\leq 0}E_1,F/F_1) = 0$ and $\text{Ext}^2_{\mathcal{O}_X}(\tau^F_{\leq 0}E_1,F_1) = \text{Hom}_{\mathcal{C}}(F_1,\tau^F_{\leq 0}E_1 \otimes \omega_X) = 0$ for all $t$. These vanishing mean that $H^t_c(F/F_1) = 0$ for all $t$, $1$ and $H^t_c(F_1) = 0$ for all $t \neq -1,0$, so we get a long exact sequence

\[
0 \to H^{-1}_c(F_1) \to 0 \to 0 \to H^0_c(F_1) \to H^0_c(F) \to H^0_c(F/F_1) \to 0 \to 0 \to H^1_c(F/F_1) \to 0,
\]

(4.6)

showing that $H^{-1}_c(F_1) = H^0_c(F/F_1) = 0$, that is $F,F/F_1 \in \mathcal{C}$, and the remaining short exact sequence means that $K_V = H^0_c(F)$ is not $\sigma_G$-semistable.

Finally, if $F$ is strictly $\sigma_G$-semistable, then we take $F_1 \subsetneq F$ with $P_{F_1,A} = G P_{F,A} = G P_{F/F_1,A}$ (hence $F_1$ and $F/F_1$ are semistable) and again we get a short exact sequence as in eq. (4.6), showing that $K_V$ is not $\sigma_G$-stable.

Now we prove the “if” direction:

**Lemma 4.9.** Suppose that $K_V \in K$ is a monad of class $v \in S_A$ whose middle cohomology $H^0_c(K_V)$ is a Gieseker-(semi)stable sheaf. Then $K_V$ is $\sigma_G$-(semi)stable as an object of $K$.

Proof. Suppose that $K_V$ is $\sigma_G$-stable, and take a $\sigma_G$-maximal subobject $0 \neq K_W \subseteq K_V$ in $K$ (this exists as the subobjects of $K_V$ can only belong to finitely many classes in $K_0(X)$) and apply Lemmas 4.4 and 4.5 to get the vanishing $H^1_c(K_W) = H^1_c(K_V/K_W) = 0$ and then an exact sequence

\[
0 \to H^{-1}_c(K_W) \to 0 \to 0 \to H^0_c(K_W) \to H^0_c(K_V) \to H^0_c(K_V/K_W) \to 0 \to 0 \to H^1_c(K_V/K_W) \to 0
\]

(4.7)
showing that $K_W, K_V/K_W \in \mathcal{C}$ and that $\mathcal{F} := H^0_b(K_V)$ is also $\sigma_G$-unstable as an object of $\mathcal{C}$. Now suppose that $K_V$ is strictly $\sigma_G$-semistable: we have again a $0 \neq K_W \subseteq K_V$ maximizing $\nu_G, v$, so that the lemmas apply and we end up with a short exact sequence as in (4.7) showing that $\mathcal{F}$ is not $\sigma_G$-stable.

\section{Conclusions}

We recall that $X$ is a smooth projective complex variety, $A$ is an ample divisor, and we are supposing that $D^b(X)$ has a full strong exceptional collection $\mathcal{E}$ for which the assumptions (A1), (A2), (A3) at the beginning of this section hold.

\begin{theorem}
For all $v \in \mathcal{M}^\text{ss}_{X,A}(\mathcal{E})$ (see equations (4.4) and (4.5)), the 4-ple $(v, \mathcal{C}, \mathcal{K}, \sigma_G)$ satisfies conditions (C1) and (C2) of §4.2.
\end{theorem}

As already observed, this theorem implies that the moduli stack $\mathcal{M}^\text{ss}_{X,A}(\mathcal{E})$ of $\sigma_G$-semistable objects in $\mathcal{C}$ with class $v$ coincides with the moduli stack of $\sigma_G$-semistable objects in $\mathcal{K}$ with class $v$, which (recall the discussion of (2.7) is isomorphic to the quiver moduli stack $\mathcal{M}^\text{ss}_{Q,J,\theta_G,v}(\psi(v))$, where $\theta_G, v = t\theta_M, v + \chi, v \in \mathbb{Z}[t]^3$ was defined in eq. (4.2); similar arguments apply to the stable loci:

\begin{corollary}
For all $v \in \mathcal{M}^\text{ss}_{X,A}(\mathcal{E})$ we have isomorphisms $\mathcal{M}^\text{ss}_{X,A}(\mathcal{E}) \simeq \mathcal{M}^\text{ss}_{Q,J,\theta_G,v}(\psi(v))$ and $\mathcal{M}^\text{ss}_{X,A}(\mathcal{E}) \simeq \mathcal{M}^\text{ss}_{Q,J,\theta_G,v}(\psi(v))$. In particular, we have isomorphisms $\mathcal{M}^\text{ss}_{X,A}(\mathcal{E}) \simeq \mathcal{M}^\text{ss}_{Q,J,\theta_G,v}(\psi(v))$ and $\mathcal{M}^\text{st}_{X,A}(\mathcal{E}) \simeq \mathcal{M}^\text{st}_{Q,J,\theta_G,v}(\psi(v))$ between the coarse moduli spaces.
\end{corollary}

Recall that the construction of the moduli space $\mathcal{M}^\text{ss}_{Q,J,\theta_G,v}(\psi(v))$ for a polynomial array $\theta_G, v \in \mathbb{Z}[t]^3$ was explained in §2.4 just before Example 2.17.

\section{Application to $\mathbb{P}^2$}

In this section we apply the previous results taking $X$ to be the projective plane $\mathbb{P}^2 = \mathbb{P}_\mathbb{C}(Z)$, where $Z$ is a 3-dimensional $\mathbb{C}$-vector space. We choose the ample divisor as $A = H$, the divisor of a line, and we write $\deg := \deg_H$ for simplicity.

Take $v \in K_0(\mathbb{P}^2) = K_{\text{num}}(\mathbb{P}^2)$. By the Hirzebruch-Riemann-Roch formula (2.8) we have
\[
\chi(v) = P_{v,H}(t) = (t^2 + 3t) \frac{\deg v}{2} + t \deg v + \chi(v),
\]
with $\chi(v) = P_v(0) = \deg v + (3/2) \deg v + \chi_2(v).

\subsection{The first equivalence}

Take, as in Ex. 2.25(1), the full strong collections $\mathcal{E} = (E_{-1}, E_0, E_1) = (O_{\mathbb{P}^2}(-1), O_{\mathbb{P}^2}, O_{\mathbb{P}^2}(1)), \quad \mathcal{E} = (E_{-1}, E_0, E_{-1}) = (O_{\mathbb{P}^2}(1), \tau_{\mathbb{P}^2}, \lambda^2 \tau_{\mathbb{P}^2})$ (note that $\lambda^2 \tau_{\mathbb{P}^2}(-1) \simeq O_{\mathbb{P}^2}(2)$). We apply Theorem 2.26 to the collection $\mathcal{E}$: the tilting sheaf $T = O_{\mathbb{P}^2}(1) \oplus \tau_{\mathbb{P}^2} \oplus \lambda^2 \tau_{\mathbb{P}^2}(-1)$ has endomorphism algebra
\[
\text{End}_{\mathcal{O}_{\mathbb{P}^2}}(T) = \begin{pmatrix}
\mathbb{C} & Z & \mathbb{C} \\
Z & \mathbb{C} & \lambda^2 Z \\
\lambda^2 Z & Z & \mathbb{C}
\end{pmatrix}
\]
which is identified, after fixing a $\mathbb{C}$-basis $e_0, e_1, e_2$ of $Z$, to the opposite of the bound quiver algebra $\mathbb{C}B_3/J$ of the Beilinson quiver
\[
B_3 : \quad \begin{tikzpicture}
\node (a1) at (-1,0) {$a_1$};
\node (a2) at (0,0) {$a_2$};
\node (a3) at (1,0) {$a_3$};
\node (b1) at (-1,1) {$b_1$};
\node (b2) at (0,1) {$b_2$};
\node (b3) at (1,1) {$b_3$};
\draw[->] (a1) -- (a2) node[midway, below] {0};
\draw[->] (a2) -- (a3) node[midway, below] {1};
\draw[->] (b1) -- (b2) node[midway, above] {a};
\draw[->] (b2) -- (b3) node[midway, above] {b};
\end{tikzpicture}
\]
with quadratic relations $J = (b_ia_j + b_ja_i, i, j = 1, 2, 3)$. So we get a triangulated equivalence
\[
\Psi := \Phi_{\mathcal{E}}[1] : D^b(\mathbb{P}^2) \to D^b(B_3; J).
\]
This maps a complex $\mathcal{F}^\bullet \in D^b(\mathbb{P}^2)$ to the complex of resolutions
\[
\text{R Hom}_{\mathcal{O}_{\mathbb{P}^2}}(\lambda^2 \tau_{\mathbb{P}^2}(-1), \mathcal{F}^\bullet)[1] \cong \text{R Hom}_{\mathcal{O}_{\mathbb{P}^2}}(\tau_{\mathbb{P}^2}, \mathcal{F}^\bullet)[1] \cong \text{R Hom}_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{F}^\bullet)[1].
\]
The standard heart of $D^b(B_3; J)$ is sent to the heart
\[
\mathcal{K} := (\mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1)[-1])_{\text{ext}}
\]
whose objects are \textit{Kronecker complexes}

$$K_v : V_{-1} \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \longrightarrow V_0 \otimes \mathcal{O}_{\mathbb{P}^2} \longrightarrow V_1 \otimes \mathcal{O}_{\mathbb{P}^2}(1) ,$$

where the middle sheaf is in degree 0. Moreover, the dual of such a complex is again an object of $\mathcal{K}$, and the objects of $\mathcal{E}$ are semistable bundles. Thus the assumptions (A1), (A2), (A3) on $\mathcal{E}$ made at the beginning of § 2 are satisfied with $L_0 = \mathcal{O}_{\mathbb{P}^2}$.

The equivalence $\Psi$ also gives an isomorphism $\psi : K_0(\mathbb{P}^2) \rightarrow K_0(B_3; J)$; coordinates on the Grothendieck groups are provided by the isomorphisms

$$K_0(\mathbb{P}^2) \overset{(\text{rk,deg})}{\longrightarrow} \mathbb{Z}^3 , \quad K_0(B_3; J) \overset{\dim}{\longrightarrow} \mathbb{Z}^3 ,$$

and we denote by

$$(d^e_{-1}, d^e_0, d^e_1) = d^e := \dim \psi(v)$$

the coordinates of $\psi(v) \in K_0(B_3; J)$ with respect to the basis of simple representations $S(-1), S(0), S(1)$; using the fact that these are mapped to $\mathcal{O}_{\mathbb{P}^2}(-1)[1], \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1)[-1]$, we find that the base-change matrices between the two coordinate sets are

\begin{equation}
\begin{pmatrix}
d^e_{-1} \\
d^e_0 \\
d^e_1 
\end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 3 & -2 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \text{rk v} \\ \text{deg v} \\ \chi(v) \end{pmatrix} , \quad \begin{pmatrix}
\text{rk v} \\
\text{deg v} \\
\chi(v)
\end{pmatrix} = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} d^e_{-1} \\
d^e_0 \\
d^e_1 
\end{pmatrix} .
\end{equation}

So, given $v \in K_0(\mathbb{P}^2)$, the arrays $\theta_{M,v}, \theta_{X,v} \in \mathbb{Z}^{(-1,0,1)}$ associated to the alternating forms $\sigma_M, \sigma_X$ as in equation (4.2) are given by

$$\theta_{M,v} = \begin{pmatrix} -\text{rk v} - \text{deg v} \\ \text{deg v} \\ \text{rk v} - \text{deg v} \end{pmatrix} = \begin{pmatrix} -d^e_0 + 2d^e_1 \\ d^e_{-1} - d^e_1 \\ -2d^e_0 + d^e_1 \end{pmatrix} ,$$

$$\theta_{X,v} = \begin{pmatrix} -\chi(v) \\ -\text{rk v} + \chi(v) \\ 3\text{rk v} - \chi(v) \end{pmatrix} = \begin{pmatrix} -d^e_0 + 3d^e_1 \\ -d^e_{-1} - 2d^e_1 \\ -3d^e_0 + 2d^e_1 \end{pmatrix} .$$

The regions $\mathcal{R}_H, \mathcal{S}_{\mathcal{H}} \subset K_0(\mathbb{P}^2)$ of equations (4.4) and (4.5) read now

$$\mathcal{R}_H = \mathcal{S}_{\mathcal{H}} = \{ |\text{deg v}| < \text{rk v} \} = \{ d^e_0 > 2d^e_{-1} \text{ and } d^e_0 > 2d^e_1 \} .$$

Indeed, take $x \in \mathbb{P}^2$ and let $p, q \in \mathbb{Z}^\vee$ be linear forms whose common zero is $x$, and notice that the Kronecker complex

$$K_x : \mathcal{O}_{\mathbb{P}^2}(-1) \overset{\psi}{\longrightarrow} \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1)$$

is quasi-isomorphic to $\mathcal{O}_x[-1]$, as well as its dual $K_x^\vee = K_x$, and its only nontrivial subcomplexes have dimension vectors $w$ equal to $(0, 2, 1), (0, 1, 1)$ and $(0, 0, 1)$; the inequalities $\theta_{M,v} \cdot w > 0$ give the above formula for $\mathcal{S}_{\mathcal{H}}$.

Notice that, after twisting by a line bundle, every sheaf of positive rank can be brought inside the region $\mathcal{R}_H$. Hence it is enough to consider this region to describe all the moduli spaces $M_{p,q}^{\text{st},H}(v)$ with $\text{rk v} > 0$.

We can now apply Corollary 4.11.

\textbf{Theorem 5.1.} For any $v \in \mathcal{R}_H$ we have isomorphisms $M_{p,q}^{\text{st},H}(v) \simeq M_{p,q}^{\text{st},J, \theta_{G,v}}(d^e)$ and $M_{p,q}^{\text{st},H}(v) \simeq M_{p,q}^{\text{st},J, \theta_{G,v}}(d^e)$.

Many of the known properties of $M_{p,q}^{\text{st},H}(v)$ can be recovered from these isomorphisms. We observe for example that:

1. $v \in \mathcal{R}_H$ is primitive if and only if $\gcd(\text{rk v}, \text{deg v}, \chi(v)) = 1$. In this case $M_{p,q}^{\text{st},H}(v) = M_{p,q}^{\text{st},H}(v)$ and there is a universal family, either by Remark 2.22 or Remark 2.16.\footnote{Notice that for $v \in \mathcal{R}_H$ the arrays $\theta_{M,v}, \theta_{X,v}$ are linearly independent, so $d^e$ is $\theta_{G,v}$-coprime by Remark 2.15.}

2. We can compute the dimension of $M_{p,q}^{\text{st},H}(v)$ as $M_{p,q}^{\text{st},J, \theta_{G,v}}(d^e)$ as the dimension of the quotient $R_{p,q}^{\text{st},H}(B_3)/PG_v$ and subtracting the number $6d^e_{-1}d^e_1$ of relations imposed; the result is

$$\dim M_{p,q}^{\text{st},H}(v) = 1 - \text{rk v}^2 + \Delta(v) ,$$

in agreement with eq. (2.11).
(3) We observe that if $\theta^{-1}_{M,v} > 0$ or $\theta^1_{M,v} < 0$ then every $d^v$-dimensional representation is $\theta_{G,v}$-unstable, so $\text{M}^{as}_{B_3,J,\theta_{G,v}}(d^v)$ is empty. But for all $v \in \mathcal{R}_H$ we have $\theta^{-1}_{M,v} = -\text{rk} v - \deg v < 0$ and $\theta^1_{M,v} = \text{rk} v - \deg v > 0$.

Notice also that the existence of a semistable sheaf $F$ in $v \in \mathcal{R}_H$ implies that all the components of the array $\dim v$ are nonnegative. Thus for example we have $2\text{ch}_2 v = -d^v_{-1} - d^v_1 \leq 0$, with the equality holding only when $F$ is trivial. From this simple observation we can easily deduce the Bogomolov inequality (2.10):

**Proposition 5.2.** If $\text{M}^{as}_{B_3,J}(v) \neq \emptyset$ for some $v \in K_0(\mathbb{P}^2)$, then $\Delta(v) := (\deg v)^2 - 2\text{rk} v \text{ch}_2(v) \geq 0$.

**Proof.** For $\text{rk} v = 0$ the statement is obvious. If $\text{rk} v > 0$, then after twisting by a line bundle (which does not change the discriminant $\Delta$) we can reduce to the case $v \in \mathcal{R}_H$: for such $v$ we have just observed that $\text{ch}_2 v \leq 0$, and hence $\Delta(v) \geq 0$. \hfill $\square$

5.2. **The second equivalence.** Now we will use instead the full strong collections

$$\mathcal{C}' = (E'_1, E'_0, E'_1) = (\Omega_{B_3}(2), \Omega_{B_3}(1), \Omega_{B_3})$$

$$\forall \mathcal{C}' = (v' E'_1, v' E'_0, v' E'_1) = (\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2)).$$

The tilting sheaf $T' = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2)$ has endomorphism algebra

$$\text{End}_{\mathcal{O}_{\mathbb{P}^2}}(T') = \left( \begin{array}{ccc} \mathbb{C} & \mathbb{C} & \mathbb{C} \\ \mathbb{S}^2 \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{array} \right)$$

which is identified, after fixing a $\mathbb{C}$-basis $e_0, e_1, e_2$ of $\mathbb{Z}$, to the opposite of the bound quiver algebra $\mathcal{C}B_3/J'$, where now $J' = (b_i a_j - b_j a_i, i, j = 1, 2, 3)$. The new equivalence

$$\Psi' := \Phi_{v,\mathcal{C}'}^{[1]} : D_{B_3}^b(\mathbb{P}^2) \rightarrow D_{B_3}^b(J')$$

sends a complex $\mathcal{F}^\bullet \in D_{B_3}(\mathbb{P}^2)$ to the complex of representations

$$R\text{Hom}_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{\mathbb{P}^2}(2), \mathcal{F}^\bullet)[1] \cong R\text{Hom}_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{F}^\bullet)[1] \cong R\text{Hom}_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{F}^\bullet)[1]$$

and the standard heart of $D_{B_3}^b(J')$ is now sent to the heart

$$K' := (\mathcal{O}_{\mathbb{P}^2}(2)[1], \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}[-1])_{\text{ext}}$$

whose objects are complexes

$$K'_V : V_1 \otimes \mathcal{O}_{\mathbb{P}^2}(2) \rightarrow V_0 \otimes \mathcal{O}_{\mathbb{P}^2}(1) \rightarrow V_1 \otimes \mathcal{O}_{\mathbb{P}^2}$$

with the middle term in degree 0. These are the Kronecker complexes originally used in [DLPS85], and we see that the conditions (A1), (A2), (A3) are satisfied with the choice $L_0 = \mathcal{O}_{\mathbb{P}^2}(-1)$.

$\Psi'$ induces a different isomorphism $\psi' : K_0(\mathbb{P}^2) \rightarrow K_0(\mathcal{O}_{\mathbb{P}^2}(2))$. Given $v \in K_0(\mathbb{P}^2)$, we write now

$$(d^v_{-1}, d^v_0, d^v_1) = d^{\psi'} := \text{dim} \psi'(v)$$

for the coordinates with respect to the basis of simple representations $S(-1), S(0), S(1)$; these are mapped to the objects $\mathcal{O}_{\mathbb{P}^2}(2)[1] \simeq \mathcal{O}_{\mathbb{P}^2}(-1)[1], \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}$, for which the triple $(\text{rk}, \deg, \chi)$ is equal to $(-1, 1, 0), (2, -1, 0)$ and $(-1, 0, -1)$ respectively. This gives the linear transformations

$$\begin{pmatrix} d^{\psi'}_{-1} \\ d^{\psi'}_0 \\ d^{\psi'}_1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \text{rk} v \\ \deg v \\ \chi(v) \end{pmatrix}, \quad \begin{pmatrix} \text{rk} v \\ \deg v \\ \chi(v) \end{pmatrix} = \begin{pmatrix} -1 & 2 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} d^{\psi'}_{-1} \\ d^{\psi'}_0 \\ d^{\psi'}_1 \end{pmatrix}.$$  

(5.2)

For $v \in K_0(\mathbb{P}^2)$ define now $\theta_{G,v}' = \theta_{M,v}' + \theta_{\chi,v}'$ by

$$\nu_{M,v} = \sigma_M(v, w) = \theta_{M,v}' \cdot d^{\psi'}, \quad \nu_{\chi,v} = \sigma_{\chi}(v, w) = \theta_{\chi,v}' \cdot d^{\psi'},$$

where the new arrays $\theta_{M,v}', \theta_{\chi,v}' \in \mathbb{Z}(-1,0,1)$ are given by

$$\theta_{M,v}' = \begin{pmatrix} -\text{rk} v - \deg v \\ 2 \deg v + \text{rk} v \\ -\deg v \end{pmatrix} = \begin{pmatrix} -d^{\psi'}_0 + d^{\psi'}_1 \\ d^{\psi'}_{-1} - d^{\psi'}_1 \\ d^{\psi'}_{-1} + d^{\psi'}_0 \end{pmatrix},$$

$$\theta_{\chi,v}' = \begin{pmatrix} -\chi(v) \\ 2\chi(v) \\ \text{rk} v - \chi(v) \end{pmatrix} = \begin{pmatrix} -2d^{\psi'}_1 \\ -d^{\psi'}_{-1} + 2d^{\psi'}_0 \end{pmatrix}.$$  

(5.3)
The regions of interest are now
\[ \mathcal{R}_H'' = \mathcal{S}_H'' = \{ v \in K_0(\mathbb{P}^2) \mid 0 < -deg v < rk v \} = \{ v \in K_0(\mathbb{P}^2) \mid d_i^v > d_i^{v-1} \text{ and } d_0^v > d_0^{v+1} \}, \]
\[ \mathcal{R}_H^{GI} = \{ v \in K_0(\mathbb{P}^2) \mid -t \\text{rk} v < t \text{deg} v + \chi(v) < \text{rk} v \}, \]
\[ \mathcal{S}_H'' = \{ v \in K_0(\mathbb{P}^2) \mid -(t+1) \text{rk} v < \text{deg} v + \chi(v) < \text{rk} v, \ \text{deg} v \neq -\text{rk} v \}. \]
To find these expressions for \( \mathcal{S}_H'' \) and \( \mathcal{S}_H' \) we observe that for any \( x \in \mathbb{P}^2 \) we can take a section \( s \in H^0(\Omega^1_{\mathbb{P}^2}(1)) \) whose zero locus is \( x \), and define the Kronecker complex
\[ K'_x : \mathbb{P}^2 \rightarrow O_{\mathbb{P}^2} \]
so that \( K'_x \cong \mathcal{O}_x \rightarrow O_{\mathbb{P}^2} \rightarrow : K'_x \). Then we notice that the nontrivial quotients \( Q \) of \( K'_x \) have dimensions \( (1,1,0) \) and \( (1,0,0) \), and only for the latter we have \( H^{-1}_C(Q) \neq 0 \).

This time the cone \( \mathcal{R}_H'' \) is not wide enough to describe all moduli spaces for positive rank: if a torsion-free sheaf \( F \) has \( \mu_H(F) \in \mathbb{Z} \), then no twist of it is in \( \mathcal{R}_H'' \). However, if \( F \) is non-trivial and Gieseker-semistable, then it has a twist \( F(k) \) of zero slope and \( \chi(F(k)) = \text{rk} F + 3 \text{deg}_H F/2 + \text{ch}_2 F < \text{rk} F \) (because \( \text{ch}_2 F < 0 \), as observed just before Prop. 5.2), so that it is contained in
\[ \mathcal{R}_H^{GI} \cap \mathcal{S}_H'' = \{ v \in K_0(\mathbb{P}^2) \mid -t \text{rk} v < t \text{deg} v + \chi(v) < \text{rk} v, \ \text{deg} v \neq -\text{rk} v \}. \]

So we can apply Corollary 5.11 to the collection \( \mathcal{C} \).

**Theorem 5.3.** Let \( v \in \mathcal{R}_H^{GI} \cap \mathcal{S}_H'' \). Then we have isomorphisms \( M^{ss}_{2,H}(v) \cong M^{ss}_{B_3,v,\theta_G,s}(d^v) \) and \( M^{st}_{2,H}(v) \cong M^{st}_{B_3,v,\theta_G,s}(d_i^v) \).

Remarks analogous to those after Theorem 5.2 apply to this situation.

**5.3. Examples.** Now we will see some examples in which \( M^{ss}_{2,H}(v) \) can be determined more or less explicitly using the isomorphisms of Theorems 5.1 and 5.3.

In the examples we often choose \( v \in K_0(\mathbb{P}^2) \) so that at least one of the invariants \( d_i^v, d_i^{v-1} \) vanishes and via equations 5.1 and 5.2 each of these conditions turns into a linear relation on \( \text{rk} v, \text{deg} v \) and \( \chi(v) \).

In these cases, the representations of \( B_3 \) under consideration reduce to representations of the Kronecker quiver \( K \), the relations \( J \) and \( J' \) are trivially satisfied and in any case the stability conditions reduce to the standard one for Kronecker modules. This means that \( M^{ss}_{2,H}(v) \) is isomorphic to some Kronecker moduli space \( K(3;m,n) \), for which we can use the properties listed in Ex. 2.17.

Recall also that twisting by \( O_{\mathbb{P}^2}(1) \) gives isomorphic moduli spaces. In the examples we will only consider classes \( v \) normalized as before, that is belonging to the regions \( \mathcal{R}_H \) or \( \mathcal{R}_H^{GI} \cap \mathcal{S}_H'' \).

Since we have an isomorphism \( K_0(\mathbb{P}^2) \rightarrow \mathbb{Z}^3, \) we will often use the notations
\[ M^{ss}_{2,H}(\text{rk} v, \text{deg} v, \chi(v)), \quad M^{st}_{2,H}(\text{rk} v, \text{deg} v, \chi(v)) \]
to indicate \( M^{ss}_{2,H}(v) \) and \( M^{st}_{2,H}(v) \).

**Examples 5.4.**

1. Let \( r \) be a positive integer, and let \( (d_i^v, d_0^v, d_1^v) = (0, r, 0) \), so that \( (\text{rk} v, \text{deg} v, \chi(v)) = (r, 0, r) \). For this choice there is a unique representation of \( B_3 \), which is always semistable, and stable only for \( r = 1 \). \( M^{ss}_{2,H}(r, 0, r) \) is a point, and \( M^{st}_{2,H}(r, 0, r) \) is a point for \( r = 1 \) and empty for \( r > 1 \).

2. Again, \( M^{ss}_{2,H}(2m, -m, 0) \) is a point, and \( M^{st}_{2,H}(2m, -m, 0) \) is a point for \( m = 1 \) and empty for \( m > 1 \).

3. Let \( m \) be a positive integer. We have \( M^{ss}_{2,H}(5m, -2m, 0) \cong K(3; m, 3m) \cong \mathbb{P}^3 \), with empty stable locus for \( m > 1 \).

4. \( M^{st}_{2,H}(2, 0, 0) \cong K(3; 2, 2) \cong \mathbb{P}^3 \), having used Theorem 5.3 and Example 2.17 (see also [USS80] Ch.2, §4.3) for a sheaf-theoretical proof of this isomorphism.

5. Since \( \text{Pic}^0(\mathbb{P}^2) \) is trivial, by sending a 0-dimensional subscheme \( Z \subset X \) of length \( \ell \) to its ideal sheaf \( \mathcal{I}_Z \subset \mathcal{O}_X \) we get an isomorphism \( \text{Hilb}_0(\mathbb{P}^2) \cong M_{2,H}(1, 0, 1 - \ell) \). Where \( \text{Hilb}_0(\mathbb{P}^2) \) is the Hilbert scheme of \( \ell \) points in \( \mathbb{P}^2 \). In particular, \( \text{Hilb}_0(\mathbb{P}^2) \cong \mathbb{P}^2 \) must be isomorphic (by using the above formulas to compute \( d_i^v, \theta_G, s^v, \theta_G, s^v \)) to the moduli spaces \( M_{B_3,J,(t+3,-2,-t+3)}(1, 3, 1) \) and \( M_{B_3,J,-(-t,t,1)}(1, 1, 0) \), and also to their stable loci.
We can obtain these isomorphisms directly from the representation theory of $B_3$: for the second isomorphism we just observe that
\[ M_{B_3, J'}^\text{ss}(t-t,1,1,0) = K(3;1,1) = K_{Gt}(3;1,1,1) \simeq G_t(3) \simeq \mathbb{P}^2. \]
To see the isomorphism $M_{B_3, J}(t+3,-2,-t+3)(3,3,1) \simeq \mathbb{P}^2$, first note that $\theta_{G,v} = (t + 3, -2, -t + 3)$ is equivalent to $\bar{\theta} = (-1, -1, 4)$ by looking at the walls in $(1,3,1)^\perp$ (see Figure 7). Then by the symmetry $B_3 \simeq B_3^{op}$ we also see that $M_{B_3, J}(-1, -1, 4)(1,3,1) \simeq M_{B_3, J}^\text{ss}(4,1,1,1)(1,3,1)$. So we are interested in understanding $(-4,1,1)$-stability for representations

\[
\begin{array}{ccccc}
C & \overset{a_1}{\longrightarrow} & C^3 & \overset{b_1}{\longrightarrow} & C
\end{array}
\]

We also write $a = (a_1, a_2, a_3), b^t = (b_1^t, b_2^t, b_3^t) \in M_3(C)$. Such a representation $(a,b)$ is $(-4,1,1)$-unstable if and only if it admits a subrepresentation of dimension $(1,2,1)$ or $(1,1,2)$ for some $w_0 \in \{0,1,2,3\}$, and this happens if and only if $rk a \leq 2$ or $rk b \leq 2$. Hence note also that $(1,3,1)$ is $(-4,1,1)$-corime the $(-4,1,1)$-(semi)stable locus in $R := \mathbb{P}C\oplus\mathbb{P}C(B_3) \cong M_3(C)^{\oplus 2}$ is
\[ R_{ss} =R^{ss} = \{(a, b) \in M_3(C)^{\oplus 2} \mid rk a = 3 \text{ and } b \neq 0\}. \]
The map $R \rightarrow M_3(C)$ given by $(a, b) \mapsto ba = (b_j a_i)_{i,j=1,2,3}$ descends to an isomorphism
\[ M_{B_3, J}^\text{ss}(1,3,1) = R^{ss}/PG(1,3,1) \rightarrow \mathbb{P}(M_3(C)) \simeq \mathbb{P}^5. \]
Finally, the relations $J$ cut down the subvariety $X_J = \{(a, b) \in R \mid a_i b_j + a_j b_i = 0\}$, thus the previous isomorphism restricts to
\[ M_{B_3, J}^\text{ss}(1,3,1) = (X_J \cap R^{ss})/PG(1,3,1) \simeq \mathbb{P}(\text{Ant}_3(C)) \simeq \mathbb{P}^2, \]
where $\text{Ant}_3(C) \subset M_3(C)$ is the subspace of antisymmetric matrices.

(6) For $(d_{-1}^t, d_0^t, d_1^t) = (1,3,1)$ we have $\theta_{G,v} = (-2t + 1, -2, 2t + 5)$, which is also equivalent to $\bar{\theta} = (-1, -1, 4)$ (see Figure 7). Imposing the symmetric relations $J'$ instead, the isomorphism 5.4 restricts to $M_{B_3, J'}^\text{ss}(1,3,1) \simeq \mathbb{P}(\text{Sym}_3(C)) \simeq \mathbb{P}^5$, where $\text{Sym}_3(C) \subset M_3(C)$ is the subspace of symmetric matrices. Hence
\[ M_{B_3, J'}^\text{ss}(4,-5,1) \simeq M_{B_3, J'}^\text{ss}(-2t+1,-2,2t+5)(1,3,1) \simeq \mathbb{P}^5. \]

6. Application to $\mathbb{P}^1 \times \mathbb{P}^1$

Let $Z$ be a 2-dimensional $C$-vector space and set $X := \mathbb{P}_C(Z) \times \mathbb{P}_C(Z)$. Recall that $\text{Pic}(X) = \mathbb{Z}H \oplus \mathbb{Z}F$, where $H, F$ are inverse images of a point under the first and second projections $X \rightarrow \mathbb{P}^1$ respectively. Take a divisor $A = aH + bF$. $A$ is ample if and only if $a, b$ are both positive, and by the Hirzebruch-Riemann-Roch formula 2.8 we have
\[ P_{v,A}(t) = t^2ab \text{rk } v + t(a \deg_H v + b \deg_F v + \text{rk } v(a+b)) + \chi(v) \]
and $\chi(v) = P_{v,A}(0) = \text{rk } v + \deg_H v + \deg_F v + ch_2 v$.

Consider the exceptional collections
\[ \mathcal{E} = (E_{0,0,0}, E_{0,0,1}, E_{0,1,0}, E_{1,0,0}) = (\mathcal{O}_X(0,1)|_{\{0\}}, \mathcal{O}_X[-1]|_{\{0\}}, \mathcal{O}_X(1,0)|_{\{1\}}, \mathcal{O}_X(0,1)|_{\{1\}}, \mathcal{O}_X(1,0)|_{\{0\}}), \]
\[ \mathcal{E}^a = (\mathcal{O}_X(1,0), \mathcal{O}_X(0,1), \mathcal{O}_X(1,1), \mathcal{O}_X(1,1)) \]
seen in Example 2.25(2) (note that the objects of $\mathcal{E}$ are isomorphic to $\mathcal{O}_X(1,0), \mathcal{O}_X(2,0), \mathcal{O}_X(1,1)$, and $\mathcal{O}_X(2,1)$). We apply Theorem 2.26 to the full strong collection $\mathcal{E}$. We have now the tilting bundle $T := \oplus t \mathcal{E}_t$ (here $I = \{(0,1),(1,0),(1,1)\}$) and its endomorphism algebra
\[ \text{End}_{\mathcal{O}_X}(T) = \begin{pmatrix} C & C \\ C \otimes Z & C \\ Z \otimes Z & 0 & C \otimes C \end{pmatrix}. \]
Figure 7. The plane $(1, 3, 1) \perp$ in $\mathcal{K}_0(B_3) \simeq \mathbb{Z}^3$, represented with respect to the basis $\{(-1,0,1), (-3,1,0)\}$. The lines are the numerical walls, while the dots are the points $\theta_{G,(1,3,1)} = (-1,0,1) + \epsilon(3,-2,3)$, $\theta'_{G,(1,3,1)} = (-2,0,2) + \epsilon(1,-2,5)$ and $\bar{\theta} := (-1,-1,4)$, for $\epsilon = 0.1$.

Choosing a basis $\{e_1, e_2\}$ of $\mathcal{Z}$, $\text{End}_{\mathcal{O}_X}(T)$ identifies with the opposite of the bound quiver algebra $\mathbb{C}Q_4/J$, where

$$Q_4 : \begin{array}{c}
(0, -1) \\
(1, -1)
\end{array} \xrightarrow{a_1^1} (0, 0) \xrightarrow{b_1^1} (1, 0) \xrightarrow{a_1^2} \begin{array}{c}
(0, -1) \\
(1, -1)
\end{array} \xrightarrow{b_1^2} \begin{array}{c}
(0, 0) \\
(1, 0)
\end{array}$$

and $J = (b_1^1a_1^2 + b_1^2a_1^1, i = 1, 2)$. So we have again an equivalence

$$\Psi := \Phi \circ [1] : D^b(X) \rightarrow D^b(Q_4; J)$$

which sends a complex $\mathcal{F} \in D^b(X)$ to the complex of representations

$$\xrightarrow{R\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(1, 1), \mathcal{F}^*)[1]}$$

$$\xrightarrow{R\text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(2, 1), \mathcal{F}^*)[1]}$$

and the standard heart in $D^b(Q_4; J)$ corresponds to the heart

$$\mathcal{K} := \langle \mathcal{O}_X(0, -1)[1], \mathcal{O}_X, \mathcal{O}_X(1, -1), \mathcal{O}_X(1, 0)[-1]\rangle_{\text{ext}},$$

whose objects are Kronecker complexes

$$K_V : V_{0,-1} \otimes \mathcal{O}_X(0, -1) \rightarrow V_{0,0} \otimes \mathcal{O}_X \otimes V_{1,-1} \otimes \mathcal{O}_X(1, -1) \rightarrow V_{1,0} \otimes \mathcal{O}_X(1, 0)$$

with the middle bundle in degree 0. Also in this case we see immediately that the assumptions (A1), (A2), (A3) on $\mathfrak{E}$ made at the beginning of §4 are satisfied, with the choice $\mathcal{L}_0 = \mathcal{O}_X(1, -1)$.
Let $\psi : K_0(X) \to K_0(Q_4; J)$ be the isomorphism induced by the equivalence $\Psi$; we have coordinates on these Grothendieck groups given by the isomorphisms

$$K_0(X) \xrightarrow{(\text{rk}, \deg_H, \deg_F, \chi)} \mathbb{Z}^4, \quad K_0(Q_4; J) \xrightarrow{\dim} \mathbb{Z}^4,$$

and as usual we write

$$(d_{-1}^v, d_{0,0}^v, d_{1,0}^v, d_{1,1}^v) = d^v := \dim \psi(v)$$

for the coordinates of $\psi(v) \in K_0(Q_4; J)$ with respect to the basis of simple representations $S(i)$, where $i \in I = \{(0, -1), (0, 0), (1, -1), (1, 0)\}$; these are mapped to the objects $\mathcal{O}_X(0, -1)[1], \mathcal{O}_X, \mathcal{O}_X(1, -1)$, and $\mathcal{O}_X(1, 0)[-1]$, so we find the transformations

$$
\begin{pmatrix}
    d_{-1}^v & d_{0,0}^v & d_{1,0}^v & d_{1,1}^v \\
    1 & 1 & 2 & -1 \\
    2 & 0 & 2 & -1 \\
    1 & 1 & 1 & -1 \\
    1 & 0 & 1 & -1
\end{pmatrix}
\begin{pmatrix}
    \text{rk} v \\
    \deg_H v \\
    \deg_F v \\
    \chi(v)
\end{pmatrix}
= 
\begin{pmatrix}
    \text{rk} v \\
    \deg_H v \\
    \deg_F v \\
    \chi(v)
\end{pmatrix}
= 
\begin{pmatrix}
    -\chi(v) \\
    -\text{rk} v + \chi(v) \\
    2\text{rk} v - \chi(v)
\end{pmatrix}.
$$

The arrays $\theta_{M,v}, \theta_{X,v} \in \mathbb{Z}^I$ of equation 4.2, are given by

$$
\theta_{M,v} = 
\begin{pmatrix}
    -\deg_A v - b\text{rk} v \\
    \deg_A v \\
    -\deg_A v - (a - b)\text{rk} v \\
    -\deg_A v + a\text{rk} v
\end{pmatrix},
\theta_{X,v} = 
\begin{pmatrix}
    -\chi(v) \\
    -\text{rk} v + \chi(v) \\
    \chi(v)
\end{pmatrix}.
$$

The region $\mathcal{R}_A \subset K_0(X)$ of eq. (4.4) reads

$$\mathcal{R}_A = \{ v \in K_0(X) \mid \text{rk} v > 0, -b\text{rk} v < \deg_A v < a\text{rk} v \}.$$ 

Given $x = ([z_1], [z_2]) \in X$, take $p_1, p_2 \in \mathbb{Z}^v$ vanishing on $z_1$ and $z_2$ respectively. We have $K_x \simeq \mathcal{O}_x[-1] \simeq K_x^v \otimes \mathcal{O}_X(1, -1) = K_x$, where now

$$K_x : \mathcal{O}_X(0, -1) \xrightarrow{(p_1)} \mathcal{O}_X \oplus \mathcal{O}_X(1, -1) \xrightarrow{(p_2)} \mathcal{O}_X(1, 0).$$

Then we can check that $\mathcal{S}^s_A = \mathcal{R}_H$, and twisting by line bundles we can bring any sheaf of positive rank inside this region. Hence Corollary 4.11 describes again all moduli spaces of semistable sheaves of positive rank:

**Theorem 6.1.** Let $v \in \mathcal{R}_A$. We have isomorphisms $M^s_{X,A}(v) \simeq M^s_{Q_4, J, \theta_{G,v}}(\psi(v))$ and $M^{st}_{X,A}(v) \simeq M^{st}_{Q_4, J, \theta_{G,v}}(\psi(v))$.

Like after Theorem 5.1 we have some immediate remarks:

1. If for $v \in \mathcal{R}_A$ the dimension vector $d^v$ is $\theta_{G,v}$-coprime, then $\gcd(\text{rk} v, \deg_A v, \chi(v)) = 1$. In this case $M^{\text{st}}_{X,A}(v) = M^{\text{st}}_{X,A}(v)$ and there is a universal family (by Remarks 2.21 and 2.16).
2. The dimension of $M^{\text{st}}_{X,A}(v) \simeq M^{\text{st}}_{Q_4, J, \theta_{G,v}}(\psi(v))$, is given by the dimension of $M^{\text{st}}_{Q_4, \theta_{G,v}}(\psi(v))$ minus the number $4d_{0,-1}^v d_{1,0}^v$ of relations imposed, which gives

$$\dim M^{\text{st}}_{X,A}(v) = 1 - \text{rk} v^2 + \Delta(v),$$

in agreement with eq. (2.11).

3. For all $v \in \mathcal{R}_A$ we have $\theta_{M,v}^{(0,-1)} = -\text{rk} v - \deg_A v < 0$ and $\theta_{M,v}^{(1,0)} = a\text{rk} v - \deg_A v > 0$.

**Examples 6.2.** We use the notation $M^{\text{ss}/\text{st}}_{X,A}(\text{rk} v, \deg_H v, \deg_F v, \chi(v)) := M^{\text{ss}/\text{st}}_{X,A}(v)$.

1. Let $r$ be a positive integer. Taking $\dim \psi(v) = (0, r, 0, 0)$ we get $M^{\text{ss}}_{X,A}(r, 0, 0, r) = \{\mathcal{O}_X^{\oplus r}\}$, while for $\dim \psi(v) = (0, 0, r, 0)$ we find $M^{\text{ss}}_{X,A}(r, r, -r, 0) = \{\mathcal{O}_X(1, -1)^{\oplus r}\}$.

2. Let $\ell$ be a positive integer. The choice $\dim \psi(v) = (\ell, \ell + 1, \ell, \ell)$ gives the Hilbert scheme of points:

$$\text{Hilb}^\ell(X) = M^{\text{ss}}_{X,A}(v) \simeq M^{\text{ss}}_{Q_4, J, \theta_{G,v}}(\ell, \ell + 1, \ell, \ell),$$

where $\theta_{G,v} = (-b + (\ell - 1), -\ell, \ell(b - a) + (1 - \ell), ta + (\ell + 1))$.

3. If we choose $v \in K_0(X)$ with at least one between $d_{0,-1}^v$ and $d_{1,0}^v$ vanishing, then the representations we are considering reduce to representations of the quivers.
respectively, and the relations $J$ are trivially satisfied. These are the cases considered in [Kul97].

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