On a conjecture of Soundararajan

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Abstract
Building on recent work of Harper, and using various results of Chang and Iwaniec on the zero-free regions of $L$-functions $L(s, \chi)$ for characters $\chi$ with a smooth modulus $q$, we establish a conjecture of Soundararajan on the distribution of smooth numbers over reduced residue classes for such moduli $q$. A crucial ingredient in our argument is that, for such $q$, there is at most one ‘problem character’ for which $L(s, \chi)$ has a smaller zero-free region. Similarly, using the ‘Deuring–Heilbronn’ phenomenon on the repelling nature of zeros of $L$-functions close to one, we also show that Soundararajan’s conjecture holds for a family of moduli having Siegel zeros.

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1. Introduction

1.1. Set-up and background
Let $P(n)$ denote the largest prime factor of the natural number $n \geq 2$, and put $P(1) := 1$. A number $n$ is said to be $y$-smooth if $n$ has no prime factor exceeding $y$, that is, if $P(n) \leq y$.

Let $S(y)$ denote the set of all $y$-smooth numbers, and let $S(x, y)$ be the set of $y$-smooth numbers not exceeding $x$:

$$S(x, y) := S(y) \cap [1, x].$$

As usual, we use $\Psi(x, y)$ to denote the cardinality of $S(x, y)$.

In this note, we study the distribution of smooth numbers over arithmetic progressions $a \mod q$ with the coprimality condition $(a, q) = 1$. Defining

$$\Psi(x, y; q, a) := \sum_{\substack{n \in S(x, y) \atop n \equiv a \mod q}} 1 \quad \text{and} \quad \Psi_q(x, y) := \sum_{\substack{n \in S(x, y) \atop (n, q) = 1}} 1,$$

it is expected that the asymptotic relation

$$\Psi(x, y; q, a) \sim \frac{\Psi_q(x, y)}{\varphi(q)}$$

(1.1)
holds (with $\varphi$ the Euler function) under the condition $(a, q) = 1$, over a wide range in the parameters $x, y, q, a$.
Soundararajan [23, Conjecture I(A)] has proposed the following conjecture.

**Conjecture (Soundararajan).** For any fixed value of \( A > 0 \), if \( q \) is sufficiently large (depending only on \( A \)) with \( q \leq y^A \) and \((a,q) = 1\), then (1.1) holds as \( \log x/ \log q \to \infty \).

Earlier, Granville [7, 8] had established this result for \( A < 1 \), and he pointed out that the proof for arbitrarily large values of \( A \) must lie fairly deep, for it implies Vinogradov’s conjecture that the least quadratic nonresidue modulo \( q \) is of size \( q^{o(1)} \). Soundararajan [23] has shown that (1.1) holds for moduli

\[
q \leq y^{A \sqrt{\epsilon} - \epsilon}
\]

provided that

\[
y^{(\log \log y)^4} \leq x \leq \exp(y^{1-\epsilon}).
\]  

(1.2)

In [11] Harper demonstrates how to remove the conditions (1.2), thereby settling Soundararajan’s conjecture for all \( A < 4 \sqrt{\epsilon} \).

It must be very difficult to prove Soundararajan’s conjecture for larger values of \( A \), for if the conjecture is true, then one implication is that the least quadratic nonresidue modulo \( q \) lies below \( q^{1/A} \), and this requires a general improvement of the classical bound of Burgess [3] on character sums.

Short of such an improvement of the Burgess bound, one can consider the following variant of Soundararajan’s conjecture in which the moduli all belong to a prescribed subset \( Q \) of the natural numbers \( \mathbb{N} \).

**Hypothesis \( \text{SC}[Q] \).** Let \( Q \subseteq \mathbb{N} \). For every fixed \( A > 0 \), if \( q \) is sufficiently large (depending only on \( A \)) with

\[
q \leq y^A, \quad (a,q) = 1, \quad q \in Q,
\]

then (1.1) holds as \( \log x/ \log q \to \infty \).

Note that Soundararajan’s conjecture is nothing but \( \text{SC}[\mathbb{N}] \), and it is clear that \( \text{SC}[\mathbb{N}] \) holds if and only if \( \text{SC}[Q] \) is true for every set \( Q \subseteq \mathbb{N} \). We say that *Soundararajan’s Conjecture holds over \( Q \)* whenever \( \text{SC}[Q] \) is true.

### 1.2. New results

We start by establishing \( \text{SC}[Q] \) for a class of sufficiently smooth moduli.

**Theorem 1.1.** For every fixed value of \( A > 0 \), there is a number \( Q_A > 0 \) (depending only on \( A \)) for which the following holds. If

\[
P(q)^{Q_A} < q \leq y^A \quad \text{and} \quad (a,q) = 1,
\]  

(1.3)

then the asymptotic relation (1.1) holds as \( \log x/ \log q \to \infty \).

**Corollary 1.2.** Let \( Q \) be a set of natural numbers \( q \) with the property that

\[
\log P(q) = o(\log q) \quad (q \to \infty, \ q \in Q).
\]

Then *Soundararajan’s conjecture holds over \( Q \).*

An important special case of Corollary 1.2 is the set \( Q = p^\mathbb{N} \) consisting of all powers of a fixed prime \( p \). In fact, our work in the present paper has been initially motivated by a series of results on arithmetic problems involving progressions modulo large powers of a fixed prime. Such results include
bounds on the zero-free regions of $L$-functions, which leads to results on the distribution of primes in arithmetic progressions (see [1, 2, 6, 15]),

- asymptotic formulas in the Dirichlet problem on sums with the divisor function over arithmetic progressions modulo $p^n$ (see [17, 20]),

- asymptotic formulas for moments of $L$-functions (see [21, 22]).

It is clear that, for any given set $Q$, to establish $\text{SC} \llbracket Q, A \rrbracket$ one must show that the following hypothesis holds for all large $A$.

**Hypothesis $\text{SC} \llbracket Q, A \rrbracket$.** Fix $Q \subseteq \mathbb{N}$ and $A > 0$. There is a number $Q_A > 0$ such that if

$$Q_A < q \leq y^A, \quad (a, q) = 1, \quad q \in Q, \quad (1.4)$$

then (1.1) holds as $\log x / \log q \to \infty$.

Since the number of moduli $q \leq x$ for which $q \geq P(q)^{Q_A}$ is asymptotically $\Psi(x, x^{1/Q_A}) \sim \rho(Q_A)x$, where $\rho$ is the Dickman function, Theorem 1.1 implies that the following variant of Soundararajan’s conjecture (with $A$ arbitrary but fixed) holds over a set of positive asymptotic density.

**Corollary 1.3.** For any fixed $A > 0$, there is a set $Q \subseteq \mathbb{N}$ of positive asymptotic density for which $\text{SC} \llbracket Q, A \rrbracket$ holds.

Corollary 1.3 complements certain Bombieri–Vinogradov type results due to Granville and Shao [9] and Harper [12], which imply (1.1) for a set of moduli $q$ of asymptotic density one, but in more restrictive ranges of $y$. For example, none of those results apply to very smooth numbers with (say) $y$ of size $\log^q x$.

The next result asserts that the asymptotic relation (1.1) holds as $q$ varies over a set of exceptional moduli.

**Theorem 1.4.** For every fixed value of $A > 0$, there is a number $Q_A > 0$ (depending only on $A$) for which the following holds. If

$$Q_A < q \leq y^A \quad \text{and} \quad (a, q) = 1, \quad (1.5)$$

and there is a character $\chi$ modulo $q$ such that $L(s, \chi)$ has a zero $\beta + i\gamma$ of $L(s, \chi)$ satisfying

$$\beta > 1 - \frac{Q_A^{-1}}{\log q(|\gamma| + 3)}, \quad (1.6)$$

then the asymptotic relation (1.1) holds as $\log x / \log q \to \infty$.

**Corollary 1.5.** Let $Q$ be a set of natural numbers such that, for every $q \in Q$, there is a character $\chi$ modulo $q$ and a real zero $\beta_q$ of $L(s, \chi)$ satisfying

$$(1 - \beta_q) \log q = o(1) \quad (q \to \infty, \quad q \in Q).$$

Then Soundararajan’s conjecture holds over $Q$.

In particular, Corollary 1.5 shows that any future work on Soundararajan’s conjecture (over $\mathbb{N}$) can assume that Siegel zeros do not exist.

**Remark 1.6.** It also true that Soundararajan’s conjecture is true if one assumes the extended Riemann hypothesis. This is easily proved using Proposition 2.1 in §2.3.
Our proofs of Theorems 1.1 and 1.4 rely on the argument of Harper [11] (which in turn builds upon original ideas of Soundararajan [23]). The treatment of the so-called ‘problem range’ is the primary issue (see §2.2), thus a major part of the proof of Theorem 1.1 is devoted to elimination of this range. This is accomplished via a combination of results of Chang [4] and Iwaniec [15], which give bounds on certain character sums and on the zero-free regions of $L$-functions modulo highly composite integers. We remark that, for a slightly more restrictive class of moduli, some stronger bounds have been obtained by the authors (see [1, 2]), but these do not lead to better results on Soundararajan’s conjecture. Concerning Theorem 1.4, our proof exploits the ‘Deuring–Heilbronn’ phenomenon on the repelling nature of zeros of $L$-functions close to one.

2. Preliminaries

2.1. Notation

In what follows, given functions $F$ and $G > 0$ we use the equivalent notations $F = O(G)$ and $F \ll G$ to signify that the inequality $|F| \leq cG$ holds with some constant $c > 0$. Throughout the paper, any implied constants may depend on the parameters $A$ and $\Phi$ but are independent of other variables.

We also write $F \sim G$ or $F = \Theta(G)$ whenever $F, G > 0$ and we have both $F = O(G)$ and $G = O(F)$.

The notations $F \sim G$ and $F = o(G)$ are used to indicate that $F/G \to 1$ and $F/G \to 0$, respectively, as certain specified parameters tend to infinity.

2.2. Initial discussion

Harper’s theorem [11, Theorem 1] implies $\mathcal{S} \ll [Q, A]$ for any set $Q$ and any $A < 4\sqrt{e}$. Thus, in what follows we can assume that $A \geq 4\sqrt{e}$ and $y^{4\sqrt{e}} \leq q \leq y^A$. In particular, the parameters

$$u := \frac{\log x}{\log y} \quad \text{and} \quad v := \frac{\log x}{\log q}$$

are comparable in size (that is, $u \approx v$) since $A \geq u/v \geq 4\sqrt{e}$, and so $u \to \infty$ if and only if $v \to \infty$.

We remark that, in this section and the next, $y$ is sometimes required to exceed a large number that might depend on $A$. However, in view of (1.4), we can begin by taking $Q_A$ large enough to guarantee that $y$ meets these requirements.

For any character $\chi$ modulo $q$ we put

$$\Psi(x, y; \chi) := \sum_{n \in S(x, y)} \chi(n).$$

In particular, $\Psi_q(x, y) = \Psi(x, y; \chi_0)$, where $\chi_0$ is the principal character. Using Dirichlet orthogonality we see that (1.1) is equivalent to the assertion that

$$\sum_{\chi \neq \chi_0} \overline{\chi}(a) \Psi(x, y; \chi) = o(\Psi(x, y; \chi_0)) \quad (u \to \infty),$$

(2.1)

As in [11, 23] it suffices to establish a smooth variant of (2.1). More precisely, let $\Phi : [0, \infty) \to [0, 1]$ be a function that is supported on $[0, 2]$, equal to one on $[0, \frac{1}{2}]$, and such that $\Phi \in C^9$, that is, $\Phi$ is nine times continuously differentiable. For every character $\chi$ modulo $q$ we denote

$$\Psi(x, y; \chi, \Phi) := \sum_{n \in S(y)} \chi(n)\Phi(n/x).$$
Then, it is enough show that
\[
\sum_{\chi \neq \chi_0} \chi(a) \Psi(x, y; \chi, \Phi) = o\left(\Psi(x, y; \chi_0, \Phi)\right) \quad (u \to \infty)
\] (2.2)
holds, since the passage from (2.2) back to (2.1) can be accomplished using the unsmoothing method outlined by Harper [11, Appendix A].

As in [11, 23] we start by writing
\[
\Psi(x, y; \chi, \Phi) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} L(s, \chi; y)x^{s} \bar{\Phi}(s) \, ds \quad (c > 0)
\] (2.3)
for any \(c > 0\), where
\[
L(s, \chi; y) := \prod_{p \leq y} (1 - \chi(p)p^{-s})^{-1} = \sum_{n \leq y} \chi(n)n^{-s}
\]
and
\[
\bar{\Phi}(s) := \int_{0}^{\infty} \Phi(t)t^{s-1} \, dt.
\]
Note that the bound
\[
\bar{\Phi}(s) \ll |s|^{-1}(|s| + 1)^{-8} \quad (2.4)
\]
follows from our smoothness assumption on \(\Phi\) (using integration by parts and the continuity of \(\Phi^{(9)}\)). We choose \(c = \alpha(x, y)\), the unique positive solution to the equation
\[
\sum_{p \leq y} \frac{\log p}{p^\alpha - 1} = \log x.
\]
The quantity \(\alpha\) is introduced in a saddle point argument of Hildebrand and Tenenbaum [13] for \(\Psi(x, y)\) (see also [14]), and it has been applied by de la Bretêche and Tenenbaum [5] to \(\Psi_q(x, y)\) for an arbitrary modulus \(q\). Using only the trivial bound \(\omega(q) \ll \log q\) on the number of distinct prime factors of \(q\), from (1.4) it follows immediately that \(\omega(q) \ll y^{1/2}/\log y\) (the implied constant is independent of \(A\) if \((\text{say})\ Q_A \geq A^{4/3}\)). Therefore, \(q\) satisfies one of the conditions \((C_1)\) or \((C_2)\) of [5, Corollary 2.2], and so an application of [5, Theorem 2.1] allows us to conclude that
\[
\Psi(x, y; \chi_0) \asymp \Psi(x, y) \prod_{p | q} (1 - p^{-\alpha}) \quad (2.5)
\]
provided both quantities \(y\) and \(u\) exceed a certain absolute constant; note that the implied constants in (2.5) are absolute. Combining (2.5) with [13, Theorem 1] it follows that
\[
\Psi(x, y; \chi_0) \asymp \Phi \cdot \frac{x^\alpha L(\alpha, \chi_0; y)}{\alpha \sqrt{1 + \log x/y}} \log x \log y. \quad (2.6)
\]
Since \(0 < \alpha \ll 1\) the quantities \(\Psi(\frac{1}{2}x, y; \chi_0)\) and \(\Psi(2x, y; \chi_0)\) are comparable in size; thus, as \(1_{[0, \frac{1}{2}]} \leq \Phi \leq 1_{[0, 2]}\) one finds that
\[
\Psi(x, y; \chi_0, \Phi) \asymp \Phi. \quad (2.7)
\]
Following [11, 23] we now denote
\[
\Xi_q(k) := X_q(k) \setminus X_q(k + 1) \quad (0 \leq k \leq \frac{1}{2} \log q)
\]
with
\[
X_q(k) := \{\chi \bmod q : \chi \neq \chi_0, L(\sigma + it, \chi) \neq 0 \text{ for } \sigma > 1 - k/\log q, \ |t| \leq q\}.
\]
Using (2.3) (with $c := \alpha$), the sum on the left side of (2.2) satisfies the bound
\[
\sum_{\chi \neq \chi_0} \chi(a)\Psi(x, y; \chi, \Phi) \ll \sum_{0 \leq k \leq \frac{1}{4}\log q} |\Xi_q(k)| \max_{\chi \in \Xi_q(k)} \left| \int_{\alpha - i\infty}^{\alpha + i\infty} L(s, \chi; y) x^s \Phi(s) \, ds \right|.
\]

For the specific moduli considered in Theorems 1.1 and in 1.4, we show that the $L$-functions $L(s, \chi)$ attached to characters $\chi$ modulo $q$ have no zeros close to one, with at most one exceptional ‘problem character’ $\chi_*$ (in the sense of [23]). More precisely, we need to know (see Proposition 2.1 in §2.3):
\[
\mathcal{A} := \bigcup_{k < k_0} \Xi_q(k) = \emptyset \text{ or } \{\chi_*\} \quad \text{with } k_0 := [4A\log A + D],
\]
where $D$ is the absolute constant described in [11, Rodosskiï Bound 1]. This means that the ‘problem range’ (in the sense of Harper [11]) can be handled easily (this is definitely not the case in situations where (2.7) fails). It is precisely for this reason that we have been able to remove the obstruction $A > 4\sqrt{e}$ encountered in the previous papers [11, 23] in the case that $Q := N$.

Assume from now on that $Q$ is a set of natural numbers such that (2.7) holds for every $q \in Q$.

Taking the above considerations into account, and introducing the notation
\[
I_q := \int_{\alpha - i\infty}^{\alpha + i\infty} L(s, \chi; y) x^s \Phi(s) \, ds \quad (\chi \neq \chi_0),
\]
to verify (2.2) (and establish $\mathcal{S}_Q[A]$) it suffices to show that
\[
|I_\chi| = o(W) \quad (u \to \infty)
\]
holds for a problem character $\chi_* \in \mathcal{A}$, and that
\[
\sum_{k > k_0} |\Xi_q(k)| \max_{\chi \in \Xi_q(k)} |I_\chi| = o(W) \quad (u \to \infty),
\]
where $W$ is defined by (2.6). For the most part, the results we need are already contained in [11]; these are briefly reviewed in §2.3.

In what follows, we use the terminology (cf. [11, p. 184]):

- $y$ is ‘large’ if $e^{(\log x)^{0.1}} < y \leq x$;
- $y$ is ‘small’ if $(\log \log x)^3 < y \leq e^{(\log x)^{0.1}}$;
- $y$ is ‘very small’ if $y < (\log \log x)^3$.

We also say (cf. [11, p. 186]) that $k$ lies in

- the ‘basic range’ if $\sqrt{\pi} \leq k \leq \frac{1}{2}\log q$;
- the ‘Rodosskiï range’ if $k_0 \leq k < \sqrt{\pi}$,
- the ‘problem range’ if $0 \leq k < k_0$,

where $k_0 := [4A\log A + D]$ as in (2.7). Again, we emphasize that our proofs of Theorems 1.1 and 1.4 essentially amount to showing that the problem range contains at most one character $\chi_*$ in each case.

2.3. Reduction to characters in the problem range

Both Theorems 1.1 and 1.4 follow from the following general statement (which does not assume anything about the arithmetic structure of the modulus $q$).
Proposition 2.1. Fix $Q \subseteq \mathbb{N}$ and $A > 0$. Suppose that, for every $q \in Q$, there is at most one nonprincipal character $\chi \star$ modulo $q$ for which the $L$-function $L(s, \chi \star)$ has a zero $\beta + i\gamma$ in the rectangle

$$
\beta > 1 - \frac{[4A \log A + \Delta]}{\log q}, \quad |\gamma| \leq q.
$$

Then $\text{SC} \llbracket Q, A \rrbracket$ is true.

Proof. We outline what is needed to establish (2.8) and (2.9) for $y$ lying in various ranges; the proposition follows. The underlying ideas are due to Soundararajan [23], and subsequent refinements are due to Harper [11]. To begin, when $y$ is not very small, we express $I_p\chi q$ as a sum

$$
I_p\chi q = I_{0} p\chi q + I_{\pm} p\chi q + I_0 p\chi q,
$$

where the central integral is

$$
I_0 p\chi q := \int_{\alpha - i(y_0) \frac{1}{4}}^{\alpha + i(y_0) \frac{1}{4}} L(s, \chi; y)x^s \hat{\Phi}(s) \, ds,
$$

and the integral tails are given by

$$
I_{\pm} p\chi q := \pm \int_{\alpha \pm i(y_0) \frac{1}{4}}^{\alpha \pm i(\infty)} L(s, \chi; y)x^s \hat{\Phi}(s) \, ds
$$

for either choice of the sign $\pm$.

Case 1: Integral tails with $y$ not very small and $k$ arbitrary.

In view of (2.4), one has (cf. [11, p. 186])

$$
I_\pm(\chi) \ll \frac{x^{\alpha}L(\alpha, \chi_0; y)}{y^2q^2}.
$$

Since $\sum_{k} |\Xi_q(k)|$ is at most the total number of characters modulo $q$, that is,

$$
\sum_{k \geq 0} |\Xi_q(k)| \leq \varphi(q),
$$

(2.10)

and we have (cf. [11, p. 185])

$$
\alpha = \begin{cases} 
\Theta \left( \frac{y}{\log x \log y} \right) & \text{if } y \leq \log x, \\
1 - \frac{\log u \log u}{\log y} + O \left( \frac{1}{\log y} \right) & \text{if } y > \log x,
\end{cases}
$$

it is immediate that

$$
\sum_{k \geq 0} |\Xi_q(k)| \max_{\chi \in \Xi_q(k)} |I_\pm(\chi)| \ll \frac{W}{yq^2},
$$

(2.11)

where $W$ is defined in (2.6). Since $x \geq u$ and $y \geq (\log \log x)^3$, we have $y \to \infty$ as $u \to \infty$; this implies that the sums in (2.11) contribute an amount of size $o(W)$ to both (2.8) and (2.9).

Case 2: Central integral with $y$ large and $k$ in the basic range.

In addition to (2.10), one needs a strong individual bound on $|\Xi_q(k)|$ for smallish values of $k$. The papers [11, 23] use

$$
|\Xi_q(k)| \leq C_1 e^{C_2 k},
$$

(2.12)

where $C_1, C_2 > 0$ are certain absolute constants, which is a consequence of the log-free density estimate for Dirichlet $L$-functions; see, for example, Iwaniec and Kowalski [16, Chapter 18]. In
particular, in terms of the same constant $C_2$, Harper [11, p. 189] derives the bound
\[
\mathcal{I}_0(\chi) \ll \left\{ \frac{1}{yq^{0.99}} + e^{-(C_2+1)k} \right\} W
\]
for all sufficiently large $u$. We multiply this bound by $|\Xi_q(k)|$ and then sum over all $k$ in the basic range, taking into account (2.10) and (2.12), we get that
\[
\sum_{k \in \text{basic}} |\Xi_q(k)| \max_{\chi \in \Xi_q(k)} |\mathcal{I}_0(\chi)| \ll \left\{ \frac{1}{yq^{0.99}} + e^{-\sqrt{u}} \right\} W.
\]
As in Case 1, $y \to \infty$ as $u \to \infty$, so the sum in this bound contributes an amount of size $o(W)$ to the sum in (2.9).

CASE 3: Central integral with $y$ large and $k$ in the Rodosskiĭ range.

As an application of his Rodosskiĭ Bound 1 (which combines earlier results of Soundararajan [23, Lemmas 4.2 and 4.3]), Harper [11, p. 189] shows that
\[
\mathcal{I}_0(\chi) \ll e^{-\Theta(\sqrt{u \log u})} W
\]
holds in the present case (one requires that $y/(A+1)^2$ is sufficiently large, which we can assume). Using (2.12) we get that
\[
\sum_{k \in \text{Rodosskiĭ}} |\Xi_q(k)| \max_{\chi \in \Xi_q(k)} |\mathcal{I}_0(\chi)| \ll e^{-\Theta(\sqrt{u \log u})} W = o(W) \quad (u \to \infty).
\]

CASE 4: Central integral with $y$ large and $k$ in the problem range.

Following Harper [11, p. 191] we define $A$ as in (2.7) and put $B := |A| \leq 1$. When $B = 0$, there is nothing to do. When $B = 1$, using [11, Rodosskiĭ Bound 2] instead of [11, Rodosskiĭ Bound 1], and arguing as in [11, §2.4], we derive the individual bound
\[
\mathcal{I}_0(\chi) \ll e^{-\Theta(\sqrt{u \log u})} W,
\]
which suffices to establish (2.8).

REMARK 2.2. It is worth reiterating that our use of [11, Rodosskiĭ Bound 2] to derive (2.13) relies on the fact that the character $\chi_{*}$ (if it exists) has order two, which exceeds the cardinality $B$ of $\mathcal{A}$. In other words, the set $B$ defined in [11, §2.5] is empty.

CASE 5: Central integral with $y$ small and $k$ arbitrary.

Building on ideas of Soundararajan [23], Harper [11, §2.6] proves that
\[
\mathcal{I}_0(\chi) \ll W \begin{cases} 2^{2^{-y^{1/3}}} & \text{if } (\log \log x)^3 \leq y \leq \log x, \\ 2^{-(\log y)^4} & \text{if } \log x < y \leq e^{(\log x)^{0.1}}, \end{cases}
\]
holds for every $\chi \neq \chi_0$ when $k \geq k_0$. Again, using [11, Rodosskiĭ Bound 2] in place of [11, Rodosskiĭ Bound 1], it is further shown that the same individual bound holds for $\chi_{*}$. As in Case 1 we have $y \to \infty$ as $u \to \infty$, hence (2.8) follows.

Using (2.10) and (2.14) we also get that
\[
\sum_{k \geq k_0} |\Xi_q(k)| \max_{\chi \in \Xi_q(k)} |\mathcal{I}_0(\chi)| \ll W \begin{cases} q 2^{2^{-y^{1/3}}} & \text{if } (\log \log x)^3 \leq y \leq \log x, \\ q 2^{-(\log y)^4} & \text{if } \log x < y \leq e^{(\log x)^{0.1}}. \end{cases}
\]
Since $y \to \infty$ and $q \leq y^4 = o(2^{(\log y)^4})$, the above sum contributes $o(W)$ to (2.9).

CASE 6: Full integral with $y$ very small and $k$ arbitrary.
Assuming \( y/(A+1)^2 \) is large enough, Harper [11, §2.6] shows that the bound \( \mathcal{I}_0(\chi) \ll W/\log x \) holds for \( y \leq (\log \log x)^3 \), and thus

\[
\sum_{k \geq 0} |\Xi_q(k)| \max_{\chi \in \Xi_q(k)} |\mathcal{I}_0(\chi)| \ll \frac{q}{\log x} W.
\]

Since \( q \leq (\log \log x)^{3/4} \) in this case, the sum here contributes an amount of size \( o(W) \) to the sum in (2.9). In view of (2.11) we complete the proof of (2.8) and (2.9) in this situation provided that \( yq \to \infty \) as \( u \to \infty \), for example, whenever \( q\sqrt{y} > (\log \log x)^{1/3} \).

When \( q\sqrt{y} \leq (\log \log x)^{1/3} \), we express \( \mathcal{I}(\chi) \) as a sum \( \mathcal{J}_x(\chi) + \mathcal{J}_0(\chi) \) with

\[
\mathcal{J}_x(\chi) = \int_{|t| \geq 1} L(\alpha + it, \chi; y)x^{\alpha + it} \tilde{\Phi}(\alpha + it) \, dt,
\]

\[
\mathcal{J}_0(\chi) = \int_{|t| < 1} L(\alpha + it, \chi; y)x^{\alpha + it} \tilde{\Phi}(\alpha + it) \, dt,
\]

and we apply the method of Harper in [11, §2.7] with \( \varepsilon = 1 \) in his notation. Harper shows that

\[
\mathcal{J}_x(\chi) \ll \frac{\sqrt{y}}{(\log \log x)^{2/5}} W,
\]

hence by (2.10) one has

\[
\sum_{k \geq 0} |\Xi_q(k)| \max_{\chi \in \Xi_q(k)} |\mathcal{J}_x(\chi)| \ll \frac{q\sqrt{y}}{(\log \log x)^{2/5}} W \leq \frac{W}{(\log \log x)^{1/15}}.
\]

Thus, this sum contributes an amount of size \( o(W) \) to both (2.8) and (2.9). On the other hand, arguing as in [11, §2.7] and applying [11, Rodosskiĭ Bound 1, Rodosskiĭ Bound 2] as appropriate, for some small constant \( c \in (0, 1) \) we have

\[
\left| \frac{L(\alpha + it, \chi; y)}{L(\alpha, \chi_0; y)} \right| \ll \left( 1 + \frac{\log x}{y} \right)^{-0.2c\sqrt{y}}.
\]

Therefore, for \( 100/c^2 \leq y < (\log \log x)^3 \), and keeping in mind the definition (2.6), we get that

\[
\mathcal{J}_0(\chi) \ll \frac{(\log \log x)^{3/2}}{\log x} W.
\]

Using (2.10) again, it follows that

\[
\sum_{k \geq k_0} |\Xi_q(k)| \max_{\chi \in \Xi_q(k)} |\mathcal{J}_0(\chi)| \ll \frac{q(\log \log x)^{3/2}}{\log x} W.
\]

This sum also contributes an amount of size \( o(W) \) to both (2.8) and (2.9), and we are done. □

3. Characters in the problem range

3.1. Exceptional zeros

We apply two familiar principles that are commonly used in treatments of Linnik’s theorem. The first is the zero-free region for Dirichlet \( L \)-functions (see Gronwall [10], Landau [18] and Titchmarsh [24]).
Lemma 3.1. There is an absolute constant \( c_1 > 0 \) such that, for every \( q \in \mathbb{N} \), the function
\[
\prod_{\chi \mod q} L(s, \chi)
\] (3.1)
has at most one zero \( \beta + i\gamma \) satisfying
\[
\beta > 1 - \frac{c_1}{\ell} \quad \text{with} \quad \ell := \log q(|\gamma| + 3).
\]
Such a zero, if one exists, is simple and real, and corresponds to a nonprincipal real character.

The second principle, which is due to Linnik [19], is often referred to as the ‘Deuring–Heilbronn’ phenomenon.

Lemma 3.2. There is an absolute constant \( c_2 > 0 \) for which the following holds. Suppose the exceptional zero in Lemma 3.1 exists and is (say) \( \beta = 1 - \varepsilon / \log q \). Then the function (3.1) does not vanish in the region
\[
\sigma > 1 - \frac{c_2 \log(\varepsilon^{-1})}{\ell} \quad \text{with} \quad \ell := \log q(|t| + 3).
\]

3.2. Bounds on character sums

The proof of Theorem 1.1 relies heavily on the following result of Chang [4, Corollary 9], which bounds short character sums over intervals for certain primitive characters with a smooth conductor.

Lemma 3.3. There are absolute constants \( c_1, c_2 > 0 \) for which the following holds. Let \( \chi \) be a primitive character modulo \( q \), let \( T \geq 1 \), and let \( I \) be an arbitrary interval of length \( N \), where \( q > N > P(q)^{1000} \) and
\[
\log N \geq (\log qT)^{1-c_1} + c_2 \log \left( \frac{2 \log q}{\log q_2} \right) \frac{\log q_2}{\log \log q} \quad \text{with} \quad q_2 := \prod_{p \mid q} p.
\]
Then
\[
\left| \sum_{n \in I} \chi(n)n^{-it} \right| \leq N e^{-\sqrt{\log N}} \quad (|t| \leq T). \tag{3.2}
\]

We apply the following corollary of Lemma 3.3, which provides a weaker bound but has the advantage that it can be applied to longer intervals.

Corollary 3.4. Fix \( \nu, \tau > 0 \). There is a constant \( c_3(\nu, \tau) > 0 \), which depends only on \( \nu \) and \( \tau \), such that the following holds. Put
\[
q_5 := P(q)^{1000} + \exp \left( \frac{c_3(\nu, \tau) \log q}{\log \log q} \right) \quad \text{and} \quad \xi := \min\{1, \frac{1}{3\nu}\}.
\]
For any primitive character \( \chi \) modulo \( q \), if \( q_5 < N < M \leq 2N \) and \( N \leq q^\nu \), then
\[
\left| \sum_{N < n \leq M} \chi(n)n^{-\sigma-it} \right| \leq 4N^{1-\sigma} e^{-\xi \sqrt{\log N}} \quad (1 < \sigma < 1, \ |t| \leq 3q^{\tau}). \tag{3.3}
\]
Proof. By partial summation, it suffices to show that if \( I \) is an arbitrary interval whose length \( N \) lies in \( (q_b, q^\gamma) \), then
\[
\sum_{n \in I} |\chi(n)n^{-it}| \leq N e^{-\xi\sqrt{\log N}} \quad (|t| \leq 3q^\gamma).
\] (3.4)

We apply Lemma 3.3 with \( T = 3q^\gamma \). With this choice, the inequality
\[
(\log qT)^{1-c_1} + c_2 \log \left( \frac{2\log q}{\log \chi} \right) \log \frac{q^\gamma}{\log q} \leq \frac{c_3(\nu, \tau) \log q}{\log \log q}
\]
clearly holds if \( c_3(\nu, \tau) \) is large enough. In the case that \( q_b < N < \frac{1}{2} q \) we obtain (3.2), which clearly implies (3.4). When \( \frac{1}{2} q \leq N \leq q^\nu \), we put \( k := [2N/q] \), \( L := N/k \), and split \( I \) into a sum of \( k \) disjoint subintervals of length \( L \). Since \( q > L \geq \frac{1}{2} q \) we can use (3.2) to bound the sum over each subinterval; this gives
\[
\left| \sum_{n \in I} \chi(n)n^{-it} \right| \leq k L e^{-\sqrt{\log L}}.
\]

Since \( kL = N \) and
\[
\log L \geq \log(q/2) \geq \frac{1}{3\nu} \log q^\nu \geq \frac{1}{3\nu} \log N,
\]
we obtain (3.4) in this case as well. \( \square \)

3.3. \( L \)-function bounds and zero-free regions

Corollary 3.5. Fix \( \tau \geq 1 \). There is a constant \( c_4(\tau) > 0 \), which depends only on \( \tau \), such that the following holds. Put \( \nu = 8\tau \), and let \( c_3(\nu, \tau) > 0 \) have the property described in Corollary 3.4. For every primitive character \( \chi \) of modulus \( q > c_4(\tau) \) we have
\[
|L(s, \chi)| \leq \eta^{-1} q^\eta \quad (\sigma > 1 - \eta, \ |t| \leq 3q^\gamma),
\]
where
\[
\eta := \ell^{-1/2}(\log 2\ell)^{-3/4}, \quad \text{with} \ \ell := \log q(|t| + 3),
\]
and
\[
q_b := P(q)^{1000} + \exp\left( \frac{c_3(\nu, \tau) \log q}{\log \log q} \right).
\]

Proof. Since \( |t| \leq 3q^\gamma \) and \( \tau \geq 1 \), it is easy to check that
\[
2\ell = 2 \log q(|t| + 3) \leq 8\tau = \nu,
\]
and thus \( q^\nu \geq e^{2\ell} \) holds throughout the region \( \{\sigma > 1 - \eta, \ |t| \leq 3q^\gamma\} \). As in the proof of [15, Lemma 8] we get that
\[
\left| \sum_{n \leq q_b} \chi(n)n^{-s} \right| < \eta^{-1} (q_b^\eta - 1) + 1 \quad \text{and} \quad \left| \sum_{n \geq 2} \chi(n)n^{-s} \right| < 1 \quad (Z \geq q^\nu).
\]

If \( q^\nu \leq q_b \) then we are done. Otherwise, we split the interval \( (q_b, q^\nu) \) into dyadic subintervals and apply Corollary 3.4 to bound the sum over each subinterval. For \( q_b < N < M \leq 2N \) and \( N \leq q^\nu \), we have by (3.3):
\[
\left| \sum_{N < n \leq M} \chi(n)n^{-s} \right| \leq b(N) := 4N\eta e^{-\xi\sqrt{\log N}}.
\]

By calculus, \( b(N) \) is decreasing for
\[
N < \Omega := \exp(2\eta^{-2}\xi^2) = \exp\left( \frac{1}{4}\xi^2\ell(\log 2\ell)^{3/2} \right),
\]
hence at all points $N \in (q_b, q^\nu]$ provided that $c_4(\tau)$ is large enough. For any such $N$ we have
\[
\log b(N) \leq \log b(q_b) = \log 4 + \eta \log q_b - \xi \log q_b < \log 4 - \frac{1}{2} \xi \sqrt{\log q_b},
\]
where the last inequality follows from $q_b < \Omega$, and we conclude that
\[
b(N) \leq \frac{1}{10\nu \log q} \quad (N \in (q_b, q^\nu])
\]
holds provided that
\[
\log(40\nu \log q) \leq \frac{1}{2} \xi \sqrt{\log q_b}.
\]
Since by definition
\[
\log q_b > \frac{c_3(\nu, \tau) \log q}{\log \log q},
\]
the latter condition is verified if $c_4(\tau)$ is large enough. Finally, summing the contributions from all subintervals, we find that
\[
\left| \sum_{q_b \leq n \leq q^\nu} \chi(n) n^{-s} \right| \leq \frac{\nu \log q}{\log 2} \cdot \frac{1}{10\nu \log q} < 1,
\]
and the result follows. \hfill \Box

**Lemma 3.6.** Let $\eta \in (0, \frac{1}{3})$, $M \geq e$, and put
\[
\Theta = \eta^{-1} \log M, \quad \vartheta := \frac{1}{400\Theta}.
\]
Let $q \geq 3$, and suppose that
\[
8 \log(5 \log 3q) + 24\eta^{-1} \log(160\Theta) \leq \frac{2}{3} \Theta.
\]
There is at most one nonprincipal character $\chi$ modulo $q$ such that simultaneously
\begin{enumerate}[(i)]
  \item $|L(s, \chi)| \leq M$ holds for all $s = \sigma + it$ with $\sigma > 1 - \eta$ and $|t| \leq 3T$;
  \item $L(s, \chi)$ has a zero $\beta + i\gamma$ with $\beta > 1 - \vartheta$ and $|\gamma| \leq T$.
\end{enumerate}
Such a zero, if it exists, is unique, simple and real.

**Proof.** The inequality (3.5) is equivalent to
\[
8 \log(5 \log 3q) + \frac{24}{\eta} \log(2M/5\vartheta) \leq \frac{1}{15\vartheta}.
\]
The first part of the proof of [15, Lemma 11] shows that $L(s, \chi) \neq 0$ throughout the region
\[
\Gamma := \begin{cases}
  \{\sigma + it : \sigma > 1 - \vartheta, \ |t| \leq T\} & \text{if } \chi^2 \neq \chi_0, \\
  \{\sigma + it : \sigma > 1 - \vartheta, \ \eta/4 < |t| \leq T\} & \text{if } \chi^2 = \chi_0,
\end{cases}
\]
provided that
\[
6 \log(5 \log 3q) + \frac{16}{\eta} \log(M/5\vartheta) + \frac{8}{\eta} \log(2M/5\vartheta) \leq \frac{1}{15\vartheta}.
\]
The second part of the proof of [15, Lemma 11] shows that $L(s, \chi)$ has at most one zero in the region
\[
\Delta := \{\sigma + it : \sigma > 1 - \vartheta, \ |t| \leq \eta/4\},
\]
and any such zero is simple and real provided that
\[ 8 \log(5 \log 3q) + \frac{16}{\eta} \log(M/5\vartheta) \leq \frac{1}{15\vartheta}. \]

Finally, [15, Lemma 12] asserts that there is at most one character \( \chi \neq \chi_0 \) such that \( L(s, \chi) \) has a real zero \( \beta > 1 - \vartheta \) provided that
\[ 2 \log(5 \log 3q) + \frac{12}{\eta} \log(M/5\vartheta) \leq \frac{2}{15\vartheta}. \]

In view of (3.6) the above three inequalities hold, and since for any \( \chi \neq \chi_0 \) we have
\[ \Gamma \cup \Delta = \{ \sigma + it : \sigma > 1 - \vartheta, |t| \leq T \}, \]
the result follows. \( \square \)

Finally, we use the following statement, which is an immediate consequence of a result of Iwaniec [15, Theorem 2].

**Lemma 3.7.** For any \( q \geq 3 \), there is no primitive character \( \chi \) modulo \( q \) for which \( L(s, \chi) \) has a zero \( \beta + i\gamma \) satisfying
\[ \beta > 1 - \frac{1}{40000(\log q + (\ell \log 2\ell)^{3/4})} \quad \text{and} \quad \gamma \neq 0, \]
where \( \ell := \log q(|\gamma| + 3) \).

4. **Proofs of the main results**

4.1. **Proof of Theorem 1.1**

Let \( Q \) be the set of numbers that satisfy the conditions of Theorem 1.1 with some large \( Q_A > 0 \). Let \( q \in Q \) with \( q > Q_A \), and observe that the condition (1.3) of Theorem 1.1 implies the condition (1.4) of Hypothesis SC \( \llbracket Q, A \rrbracket \). Thus, by Proposition 2.1, to prove the desired result it suffices to establish (2.7); that is, we need to show that
\[ \mathcal{A} := \{ \chi \neq \chi_0 : L(s, \chi) = 0 \text{ has a zero in } \{ \sigma > 1 - k_0/\log q, |t| \leq q \} \}. \]
has cardinality at most one.

First, we claim that there is a sufficiently large constant \( \tau_A > 0 \) (depending only on \( A \)) with the following property. For every character \( \chi \) modulo \( q \), let \( \check{\chi} \) be the conductor of \( \chi \), and let \( \check{\chi} \) be the character modulo \( \check{\chi} \) that induces \( \chi \). Put
\[ \mathcal{R}_\chi := \{ \sigma + it : \sigma > 1 - k_0/\log q, |t| \leq \min\{q, \check{\chi}^\tau\A\} \}. \]
Then
\[ \mathcal{A} = \{ \chi \neq \chi_0 : L(s, \chi) = 0 \text{ has a zero in } \mathcal{R}_\chi \}. \quad (4.1) \]

To prove the claim, suppose on the contrary that there is a character \( \chi \neq \chi_0 \) such that \( L(s, \chi) \) has a zero \( \beta + i\gamma \) satisfying
\[ \beta > 1 - \frac{k_0}{\log q}, \quad \min\{q, \check{\chi}^\tau\A\} < |\gamma| \leq q. \quad (4.2) \]

Clearly, this is not possible unless \( \check{\chi}^\tau < q \), which we assume. Put \( \ell := q(|\gamma| + 3) \) and \( \check{\ell} := \check{\chi}(|\gamma| + 3) \), and note that
\[ 40000(\log \check{\chi} + (\check{\ell} \log 2\check{\ell})^{3/4}) \leq 40000(\tau_A^{-1} \log q + (\ell \log 2\ell)^{3/4}) \leq k_0^{-1} \log q. \]
if $\tau_A$ is large enough, since $\ell \ll \log q$ and $q > \tilde{q}^{\tau_A} > 2^\tau_A$ (we remind the reader that, in the definition (2.7) of $k_0$, the constant $D$ is absolute). Therefore, (4.2) implies
\[
\beta > 1 - \frac{1}{40,000 (\log \tilde{q} + (\ell \log 2\tilde{\ell})^{3/4})} \quad \text{and} \quad \gamma \neq 0.
\]
As this contradicts Lemma 3.7 (with $\tilde{\chi}, \tilde{\ell}, \tilde{q}$ replacing $\chi, \ell, q$, respectively), we conclude that (4.1) is a correct description of the set $\mathcal{A}$.

Next, we eliminate from $\mathcal{A}$ certain characters with a bounded conductor. Let $c_4(\tau_A) > 0$ be the constant described in Corollary 3.5 for $\tau = \tau_A$. Let $\chi \in \mathcal{A}$, and suppose that $\tilde{q} \leq c_4(\tau_A)$. Since $L(s, \chi)$ has a zero $\beta + i\gamma \in \mathcal{R}_\chi$, using the inequalities
\[
\log \tilde{q} \geq \log Q_A
\]
and
\[
\log \tilde{q}(|\gamma| + 3) \leq \log \tilde{q}((\tau_A + 3) \log \tilde{q} \leq (\tau_A + 3) \log c_4(\tau_A),
\]
we derive that
\[
\beta > 1 - \frac{k_0}{\log \tilde{q}} \geq 1 - \frac{k_0}{\log Q_A}
\]
\[
\geq 1 - \frac{k_0}{\log Q_A} \frac{\log c_4(\tau_A)}{\log \tilde{q}(|\gamma| + 3)} > 1 - \frac{k_0(\tau_A + 3) \log c_4(\tau_A)}{\log \tilde{q}(|\gamma| + 3)}.
\]
Since $Q_A$ can be chosen after both $\tau_A$ and $c_4(\tau_A)$ are defined, and $\beta + i\gamma$ is a zero of $L(s, \chi)$, taking $Q_A$ large enough and applying Lemma 3.1 we deduce that $\gamma = 0$. Consequently, the real zero $\beta$ of $L(s, \chi)$ satisfies
\[
\beta > 1 - \frac{k_0(\tau_A + 3) \log c_4(\tau_A)}{\log 9}.
\]
However, this situation is untenable if $Q_A$ is sufficiently large, for there are only finitely many characters $\chi$ modulo $q$ with a conductor $\tilde{q} \leq c_4(\tau_A)$, and the $L$-function attached to any one of these characters has at most finitely many zeros in the real interval $[0, 1]$; such zeros must lie in $(0, 1)$, hence they are bounded away from one by a constant that depends only on $\tau$. In summary, if $Q_A$ is large enough, then every $\chi \in \mathcal{A}$ has $\tilde{q} > c_4(\tau_A)$, and so (4.1) transforms to
\[
\mathcal{A} = \{ \chi \neq \chi_0 : \tilde{q} > c_4(\tau_A) \text{ and } L(s, \chi) = 0 \text{ has a zero in } \mathcal{R}_\chi \}. \tag{4.3}
\]
It remains to show that $\mathcal{A}$ defined by (4.3) has cardinality at most one:
\[
|\mathcal{A}| \leq 1. \tag{4.4}
\]
To this end, let $\chi \in \mathcal{A}$. Applying Corollary 3.5 with $\tau = \tau_A$ we have
\[
|L(s, \tilde{\chi})| \leq \tilde{\eta}^{-1} q_0^{\nu} \quad (\sigma > 1 - \tilde{\eta}, |t| \leq 3q^{\tau_A}),
\]
where
\[
\tilde{\eta} := \tilde{\ell}^{-1/2} (\log 2\tilde{\ell})^{-3/4} \quad \text{with} \quad \tilde{\ell} := \log \tilde{q}(|t| + 3),
\]
and
\[
q_0 := P(\tilde{q})^{1000} + \exp \left( \frac{c_3(\nu, \tau) \log \tilde{q}}{\log \log \tilde{q}} \right). \tag{4.5}
\]
We apply Lemma 3.6 with
\[
M := \tilde{\eta}^{-1} q_0^{\tilde{\eta}}, \quad \Theta := \tilde{\eta}^{-1} \log M, \quad T := \tilde{q}^{\tau_A}.
\]
To establish (the analogue of) the bound (3.5), it suffices to show

$$\log(5 \log 3q) \leq \frac{1}{6} \Theta$$  
and  
$$3\eta^{-1} \log(160\Theta) \leq \frac{1}{6} \Theta,$$

(4.6)

where

$$\Theta := \eta^{-1} \log M = \log \tilde{q}_b - \eta^{-1} \log \eta.$$ 

In $\mathcal{R}_\chi$ we have $|t| \leq 3\eta^{-a}$, so that

$$\tilde{\ell} = \log \tilde{q}$$  
and  
$$\tilde{\eta} = (\log \tilde{q})^{-1/2}(\log \log \tilde{q})^{-3/4},$$

(4.7)

where the implied constants can be made explicit and depend only on $A$. In view of (4.5) we deduce that

$$\Theta = \log \tilde{q}_b + O((\log \tilde{q})^{1/2}(\log \log \tilde{q})^{7/4}) = \log \tilde{q}_b + O((\log \tilde{q}_b)^{2/3}).$$

Increasing the value of $c_4(\tau_A)$ if necessary, the first inequality in (4.6) follows from

$$\log(5 \log 3q) \leq \frac{1}{10} \log \tilde{q}_b,$$

which is clear if $c_4(\tau_A)$ is large enough in view of (4.5). Indeed, it suffices that $c_3(\nu, \tau) \geq 1$ and that $c_4(\tau_A)$ exceeds a certain absolute constant, and this can all be arranged before the value of $Q_A$ is chosen. Next, taking into account the second estimate of (4.7), we see that there is a constant $C_A > 0$ (depending only on $A$) such that the second inequality in (4.6) follows from

$$(\log q)^{1/2}(\log \log q)^{7/4} \leq C_A \log \tilde{q}_b \quad (\tilde{q} > c_4(\tau_A)),$$

and this inequality is also clear (for large $c_4(\tau))$ by (4.5).

The preceding argument shows that every character $\chi \in \mathcal{A}$ satisfies (3.5) and the condition (i) of Lemma 3.6.

We claim that the condition (ii) also holds when

$$P(q)^{Q_A} < q$$

(4.8)

holds with some suitably large number $Q_A > 0$. To prove the claim, suppose that (4.8) holds. Since $\chi \in \mathcal{A}$ we see that $L(s, \chi)$ has a zero $\beta + i\gamma$ satisfying

$$\beta > 1 - \frac{k_0}{\log q}, \quad |\gamma| \leq \min\{q, \tilde{q}^{-A}\}.$$ 

Since $T := \tilde{q}^{-A}$, it follows that

$$\beta > 1 - \vartheta, \quad |\gamma| \leq T,$$

and so the condition (ii) is satisfied, provided that

$$k_0 \leq \vartheta \log q = \frac{\log q}{400\Theta} = (1 + o(1)) - \frac{\log q}{400 \log \tilde{q}_b} \quad (c_4(\tau_A) \to \infty).$$

Since $\tilde{q}_b \leq q_b$, for large enough $c_4(\tau_A)$ it suffices to have $q > q_b^{500k_0}$. Recalling the definition of $q_b$, we see that a value

$$Q_A \geq 500 000 k_0 = 500 000 (4A \log A + D)$$

in (4.8) ensures that the condition (ii) of Lemma 3.6 holds.

Above, we have shown that any characters in $\mathcal{A}$ satisfy all of the conditions of Lemma 3.6. By the lemma, there is at most one nonprincipal character meeting these conditions, thus we obtain (4.4). This completes the proof of Theorem 1.1.
4.2. Proof of Theorem 1.4

Let $Q$ be the set of numbers that satisfy the conditions of Theorem 1.4 with some large $Q_A > 0$. Let $q \in Q$ with $q > Q_A$, and observe that the condition $(1.5)$ of Theorem 1.4 agrees with the condition $(1.4)$ of Hypothesis $\mathcal{SC}\llbracket Q, A \rrbracket$.

If $Q_A^{-1} \leq c_1$, then Lemma 3.1 asserts that there is at most one nonprincipal character $\chi$ modulo $q$ such that $L(s, \chi)$ has a zero $\beta + i\gamma$ satisfying $(1.6)$, and in this case $\beta > 1 - Q_A^{-1}/\log q$ and $\gamma = 0$. Let $\chi_*$ be such a character. Taking $\varepsilon := Q_A^{-1}$, Lemma 3.2 asserts that for any nonprincipal character $\chi \neq \chi_*$ the $L$-function $L(s, \chi)$ does not vanish in the region defined by

$$\sigma > 1 - \frac{c_2 \log Q_A}{\log q(|t| + 3)}.$$ 

In particular, if $Q_A$ is large enough, then $L(s, \chi)$ cannot have a zero $\beta + i\gamma$ with

$$\beta > 1 - \frac{[4A \log A + D]}{\log q}, \quad |\gamma| \leq q;$$

therefore $\mathcal{SC}\llbracket Q, A \rrbracket$ is true by Proposition 2.1.

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