A class of concave Young functions possessing a positive fixed point

N. K. Agbeko
A CLASS OF CONCAVE YOUNG FUNCTIONS POSSESSING A POSITIVE FIXED POINT

N. K. AGBEKO

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Abstract. We obtained the class of all concave Young functions which possess a positive fixed point.

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1. INTRODUCTION

Let \( \varphi : (0, \infty) \to (0, \infty) \) be a right-continuous and decreasing function such that it is integrable on every finite interval \((0, x)\). It is easily seen that the function \( \Phi : [0, \infty) \to [0, \infty) \), defined by the equality

\[
\Phi(x) = \int_0^x \varphi(t) \, dt,
\]

is a nonnegative, increasing and concave function with \( \Phi(0) = 0 \). We further assume that \( \Phi(\infty) = \infty \). Function \( \Phi \) is thus referred to as a \emph{concave Young function} in the literature, and the set of all such functions will be denoted by \( Y_{\text{conc}} \). For more about these functions see, e.g., [1–3, 5].

In [3], we obtained the following results.

**Proposition 1.1.** Let \( \Phi \in Y_{\text{conc}} \) and \( s \in (0, \infty) \) be arbitrary. Then

\[ |\Phi(x) - \Phi(y)| \leq \varphi(s)|x - y| \]

for all numbers \( x, y \in (s, \infty) \).

We sought for all those positive numbers that can be a fixed point for a given concave Young function.

**Theorem 1.1.** Let \( \Phi \in Y_{\text{conc}} \) and \( c^* \) be any positive number. In order that the equality \( \Phi(c^*) = c^* \) hold, it is necessary and sufficient that the range of the function \( \Phi|_{[c^*, \infty)} : [c^*, \infty) \to [0, \infty) \), defined by \( \Phi|_{[c^*, \infty)}(x) = \Phi(x) \), should equal the interval \([c^*, \infty)\).
Let $\Phi \in \mathcal{Y}_{\text{conc}}$ with
\[
\int_1^\infty \frac{\varphi(t)}{t} \, dt < \infty. \tag{1.2}
\]
In [3], a number $c \in (0, \infty)$ was called the degree of contraction of $\Phi$ if
\[
\int_1^\infty \frac{\varphi(t)}{t} \, dt = 1
\]
and
\[
\int_c^{bc} \frac{\varphi(t)}{t} \, dt = \varphi(c)
\]
for some $b \in (1, \infty)$. We intend to extend this notion to other concave Young functions which do not possess property (1.2).

2. Main Result

**Theorem 2.1.** Let $\Phi \in \mathcal{Y}_{\text{conc}}$ be arbitrary with $\varphi$ denoting its derivative. In order that there be a constant $s > 0$ for which $\varphi(s) < 1$, it is necessary and sufficient that $\Phi$ admit a positive fixed point, i.e., $\Phi(x) = x$ for some number $x > 0$.

**Proof.** To prove the sufficiency, assume that there is a number $s > 0$ such that $\varphi(s) < 1$. Then by recalling Proposition 1.1 one can easily observe that $\Phi$ is a contraction in the interval $(s, \infty)$. Consequently, the Contraction Principle [6] yields $\Phi(x) = x$ for some $x \geq s$. Next, let us show the necessity. Assume that there exists some $x_0 > 0$ for which $\Phi(x_0) = x_0$, but in the contrary $\varphi(t) \geq 1$ for all $t > 0$. Then it is easy to check that $\Phi(x) \geq x$ for all $x > 0$. Since $\Phi$ is a strictly concave and increasing function, the graph of $\Phi$ must lie below that of the line $y = x$ on the interval $(x_0, \infty)$. This fact, however, contradicts the inequality $\Phi(x) \geq x$ for all $x > 0$. □

**Proposition 2.1.** Let $\Phi \in \mathcal{Y}_{\text{conc}}$ be arbitrary with $\varphi$ denoting its derivative. If $x_0 \in (0, \infty)$ is such that $\Phi(x_0) = x_0$, then $\varphi(x_0) < 1$.

**Proof.** It is not difficult to see that $\Phi(t) \geq t \varphi(t)$ whenever $t \in (0, \infty)$. Assume the existence of some $x_0 \in (0, \infty)$ for which $\Phi(x_0) = x_0$. Then, as noted above,
\[
x_0 = \Phi(x_0) \geq x_0 \varphi(x_0),
\]
and hence $\varphi(x_0) \leq 1$. Now, suppose that $\varphi(x_0) = 1$. Since $\varphi$ is a decreasing function on $(0, \infty)$, there must be some $\varepsilon \in (0, 1)$ such that $\varphi(x_0 + \varepsilon) < 1$, making $\Phi$ be a contraction on $(x_0 + \varepsilon, \infty)$, via Proposition 1.1. But then it would mean that there must be some $x^* \in (x_0 + \varepsilon, \infty)$ with $\Phi(x^*) = x^*$. Necessarily, it would ensue that $\Phi$ is not a concave function on the interval $(x_0, x^*)$, a contradiction. Therefore, $\varphi(x_0) < 1$. □

Now, we are in a position to reformulate the definition of the degree of contraction to cover a broader class of concave Young functions.
Definition 2.1. A number $s > 0$ is called the degree of contraction of a function $\Phi \in \mathcal{Y}_{\text{conc}}$ if $\varphi (s) = 1$, where $\varphi$ is the derivative of $\Phi$.

We note in this case that $\varphi (s + \delta) < 1$ for any positive number $\delta$, which makes $\Phi$ be a contraction on the interval $(s + \delta, \infty)$ for some suitable $\delta$.

Example 1. The degree of contraction of $\Phi (x) = 4\sqrt{x} + 1 - 4$, $x \in [0, \infty)$, equals 3.

Example 2. For any fixed number $p \in (0, 1)$, the degree of contraction of the function $\Phi_p (x) = x^p$, $x \in [0, \infty)$ is equal to $p^{1/(1-p)}$.

Example 3. The function $\Phi (x) = \log (x + 1)$, $x \in [0, \infty)$, has no degree of contraction.

Example 4. The degree of contraction of function $\Phi (x) = 2 \log (x + 1)$ exists and equals 1.

Example 5. The concave Young function $\Phi$ defined by $\Phi (x) = \frac{x}{2} + \sqrt{x}$ does not meet condition (1.2). Yet its degree of contraction exists and equals 1.

An algorithm for finding positive fixed points for concave Young functions:

Step 1: Input $\Phi (x)$ a concave Young function, $c_0$ a positive number.

Step 2: Compute the derivative $\varphi (x)$ of $\Phi (x)$.

Step 3: Starting from $c_0$ find an approximate root of the equation $\varphi (x) - 1 = 0$ and put the result into $c$.

Step 4: If $c = 0$ then STOP else GOTO Step 5.

Step 5: Starting from $c$ apply the Fixed Point algorithm, i. e.,

$x_0 := c; x_{k+1} := \Phi (x_k); k = k + 1$.

3. CONCLUDING REMARKS

In dynamic models, stationary equilibrium is typically described as a solution of the equation $x = f (x)$, where $f$ is a mapping which determines the current state as a function of the previous state, or as a function of the expected future state. In many cases $x$ is a finite dimensional vector, and in general positive solutions (i. e., fixed points of $f$) are rather sought for. Problems of this kind have been investigated for decades, and often for concave functions. Alfred Tarski in [7] obtained, in particular, the following result.

**Theorem 3.1** (Tarski). Suppose $f$ is an increasing function from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $f (a) > a$ for some positive vector $a$, and $f (b) < b$ for some vector $b > a$. Then $f$ has a positive fixed point.

For the proof we refer the reader, e. g., to [4]. In [4], J. Kennan obtained the result stated below by using Tarski’s theorem and [4, Theorem 3.1]. He observed that it gave simple sufficient conditions for the existence and uniqueness of a positive fixed point.
Theorem 3.2 ([4, Theorem 3.3]). Suppose that $f$ is an increasing and strictly concave function from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $f(0) \geq 0$, $f(a) > a$ for some positive vector $a$, and $f(b) < b$ for some vector $b > a$. Then $f$ has a unique positive fixed point.

We note that the concavity and increasing property of $f$ mean that every component $f_k$ ($k = 1, \ldots, n$) of $f$, considered as a function from $\mathbb{R}^n$ to $\mathbb{R}$, is increasing and strictly concave in every argument $x_j \in \mathbb{R}$, $j = 1, \ldots, n$.

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References

[1] N. K. Agbeko, “Concave function inequalities for sub- (super-) martingales,” Ann. Univ. Sci. Budapest. Eötvös Sect. Math., vol. 29, pp. 9–17 (1987), 1986.
[2] N. K. Agbeko, “Necessary and sufficient condition for the maximal inequality of concave Young-functions,” Ann. Univ. Sci. Budapest. Eötvös Sect. Math., vol. 32, pp. 267–270 (1990), 1989.
[3] N. K. Agbeko, “Studies on concave Young-functions,” Miskolc Math. Notes, vol. 6, no. 1, pp. 3–18, 2005.
[4] J. Kennan, “Uniqueness of positive fixed points for increasing concave functions on $\mathbb{R}^n$: An elementary result,” Review of Economic Dynamics, vol. 4, no. 4, pp. 893–899, 2001.
[5] J. Mogyoródi, “On a concave function inequality for martingales,” Ann. Univ. Sci. Budapest. Eötvös Sect. Math., vol. 24, pp. 265–271, 1981.
[6] M. C. Reed, Fundamental ideas of analysis. New York: John Wiley & Sons Inc., 1998.
[7] A. Tarski, “A lattice-theoretical fixpoint theorem and its applications,” Pacific J. Math., vol. 5, pp. 285–309, 1955.

Author’s address

N. K. Agbeko
Institute of Mathematics, University of Miskolc, H-3515 Miskolc–Egyetemváros, Hungary
E-mail address: matagbek@uni-miskolc.hu