Quantifying Causation

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I introduce an information-theoretic measure of causation, capturing how much a quantum system influences the evolution of another system. The measure discriminates among different causal relations that generate same-looking data, with no information about the quantum channel. In particular, it determines whether correlation implies causation, and when causation manifests without correlation. In the classical scenario, the quantity evaluates the strength of causal links between random variables. Also, the measure is generalized to identify rank concurrent sources of causal influence in many-body dynamics, enabling to reconstruct causal patterns in complex networks.

Introduction. – Causation manifests when a physical system influences the evolution of another system via interaction. Measures of correlations are powerful tools to describe statistical dependencies between data, but they are inadequate to evaluate causal relations. A vast literature spanning beyond STEM disciplines has discussed the problem for decades [1–5], as causal inference is crucial to explain the inner workings of many-body processes. However, there is no consensus about how to quantify causation. Many proposed quantities misinterpret causal links between classical random variables in simple case studies [6, 7]. Recently, the debate has percolated in the quantum information community. A stream of works have developed a set of techniques to evaluate the spatio-temporal causal order (if any) of quantum events, and extend the theory of causal networks to the quantum scenario [8–16]. While these results have reshaped the foundations of quantum theory, we have yet to build a quantitative characterization of causation. Here, I introduce a measure of causation, quantifying how much a quantum system affects the dynamics of an interacting system. A physically motivated notion of causal influence is exemplified by a measuring probe that updates its state when the evolution of two quantum systems resembles such a causal interaction. The quantity is computed without any knowledge of causation measure vanishes when no causal relation exists. Unlike correlation quantifiers, it unambiguously pinpoints the affecting (measured) and affected (measuring) system. It reliably describes the interplay between correlation and causation, capturing when correlation does imply causation, and when causation exists without correlation. In the classical scenario, it quantifies the strength of causal links between random variables. Notably, I show that a quantum computer, among its potential uses [17], can detect causal relations between classical systems that are untraceable by a classical device. Finally, the method is extended to quantify causation in multipartite systems. I build a measure of conditional causation that satisfies two important properties. First, it localizes the source of causation, i.e. the measured system(s), in three or more interacting parties. Second, it ranks multiple concurrent causes in terms of how much they affect a target system. Consequently, it makes possible to quantitatively describe causal patterns in many-body dynamics.

Results. – Suppose a bipartite quantum system $AB$ undergoes an evolution described by the unitary transformation $U_{AB}^{\text{in}} = U_{AB}^{\text{out}} \otimes \rho_B U_{AB}^{\text{out}}$. Assuming that the channel $U$ is unknown, the goal is to quantify how much the dynamics of system $A(B)$ influence the dynamics of $B(A)$. An instance of causation is embodied by a premeasurement coupling of an apparatus $B$, labeling measurement outcomes by $[i=0,1,\ldots]$, with a measured system $A$, $C_{A\rightarrow B}^{-i} \sum_i c_i |i0\rangle_{AB} = \sum_i c_i |ii\rangle_{AB}$, such that the state change in $B$ is determined by the state of $A$ [18]. Causation due to $U$ may be then quantified as how similar the input/output states are to the ones of a controlled gate. However, the notions of control and target systems are basis-dependent in the quantum case. For example, the two-qubit CNOT is equal to a controlled gate with swapped control and target qubits, and different measurement basis, $C_{A\rightarrow B}^{0,1\rightarrow 1,0} = C_{B\rightarrow A}^{1,1\rightarrow 0,1}$, $\pm = (|0\rangle \pm |1\rangle)/\sqrt{2}$ [19, 20]. The system $A$ therefore exerts maximal influence on $B$ via the controlled operation $C_{i\rightarrow j}$ only with respect to the bases $[i],[j]$. A second issue is that calculating the correlations in the input/output states, or the ability of a channel to generate correlations [21], is not sufficient for drawing conclusions on causation. This is indeed a transmission of information from a causing device to an affected system. If the roles of control and target system are inverted, the information arguably flows in the opposite direction, while the amount of correlations created is equivalent. Also, there can be a causal link with neither initial nor final correlations. For example, $c_{A\rightarrow B}^{0,1\rightarrow 0,1}|10\rangle_{AB} = |11\rangle_{AB}$ is a causal relation, conversely to the local bit flip $X_B|10\rangle_{AB} = |11\rangle_{AB}$. They are two different physical processes that generate the same output from the same input state [22].

It is yet possible to construct a measure of causal influence. Note that for any four systems $A',B,B'$, one has
FIG. 1. Quantifying causation. The causal influence $C_C(A \rightarrow B)$ exerted by $A$ on $B$ during the joint evolution $U_{AB}$ is quantified by implementing the depicted scheme, and computing the measure defined in Eq. 5.

As a minimal working example, consider the unitary to be the CNOT gate $C^{0,1}_{A \rightarrow B}$, and the input two-qubit state to be $|+0\rangle_{AB}$:

$$|+0\rangle_{AB} = 1/ \sqrt{2}(00 + |11\rangle)_{A} 1/ \sqrt{2}(|+\rangle + |\rangle)_{B} = \{(00) + (11)\}_A(00) + (11)\}_B / 2.$$  

STEP 2 – Let the system $AB$ evolve,

$$\rho^{i1U}_{ABB'} := (I_A \otimes U_{AB} \otimes I_B) \rho^{i1}_{ABB}, \quad (I_A \otimes U^\dagger_{AB} \otimes I_B).$$  

That is,

$$C^{0,1}_{A \rightarrow B} (00) + (11)_{A} |(00) + (11)\}_B / 2 = \{(00) + (11)\}_A(00) + (11)\}_B / 2.$$

STEP 3 (final) – Apply a second pair of local controlled operations with respect to the reference bases $\{i\}_A, \{j\}_B$, but swapping the roles of control and target systems:

$$\rho^{i1U}_{ABB'} := C^{j+i}_{A \rightarrow B} \rho^{i1U}_{ABB} C^{j+i}_{B \rightarrow B'} A \rightarrow A',$$

in which the minus sign for the target bases is redundant for qubits, but crucial in dimension $d > 2$. As expected, the output state in the example displays correlations between $A'$ and $B$,

$$C^{0,1}_{A \rightarrow B} (00) + (11)_{A} |(00) + (11)\}_B / 2 = \{(00) + (11)\}_A(00) + (11)\}_B / 2.$$  

The result in Eq. 1 implies that the role of control and target system can be inferred from the correlations in the state $\rho^{i1U}_{ABB'}$. The statistical dependence between two systems $\alpha, \beta$ is quantified by the mutual information $I(\alpha : \beta) := S(\alpha) + S(\beta) - S(\alpha \beta)$, where $S(\alpha) := -tr[\rho \log_2 \rho]$ is the von Neumann entropy of the state $\rho$. For any third system $\gamma$, the conditional mutual information reads $I(\alpha : \gamma|\beta) := I(\alpha : \beta|\gamma) = I(\alpha : \beta) [23]$. I propose to measure the causal influence that $A$ has on $B$ due to $U$ by

$$C_U(A \rightarrow B) := I(B : A'A|B'),$$

which is computed on the final state $\rho^{i1U}_{ABB'}$. Consequently, the influence of $B$ on $A$ during the interaction under study is given by $C_U(B \rightarrow A) = I(B : BB'|A')$. To justify the proposal, I discuss how the measure meets several desirable properties, which I clarify by explicit calculations in a number of instructive cases, reported in Tables I, II.

Information-theoretic consistency. There is no causation without interaction. For local unitaries $U_{AB} = U_A \otimes U_B$, one has $C_U(U_A U_B) = B(AB) = 0$. Two systems can influence each other by a symmetric two-way information flow, e.g. $V_{AB} = |+\rangle_A |+\rangle_B$, with $V_{AB} = C_{A \rightarrow B, B \rightarrow A}^{0,1} = C_{A \rightarrow B, B \rightarrow A}^{0,1}$. In such a case, $C_{V_{AB}}(A \rightarrow B) = C_{V_{AB}}(B \rightarrow A) = 2$. Yet,
Quantifying causation in the global evolution of two-dimensional systems with respect to the reference bases \( |i_i\rangle \otimes |j_j\rangle = 0, 1, \ldots, d\). Note that the information flow is reversed in \( A_{\rightarrow B} \), a controlled gate with respect to mutually unbiased bases.

the measure is not additive. Given \( U_{AB} = V_{AB}W_{AB} \), in general \( C_U(A \rightarrow B) \neq C_V(A \rightarrow B) + C_W(A \rightarrow B) \). Indeed, a controlled operation with control \( A \) and target \( B \) can be transformed in one with control \( B \) and target \( A \) by local unitaries, e.g. \( C^{0,1}_{A \rightarrow B} = H_A \otimes H_B C^{0,1}_{B \rightarrow A} H_A \otimes H_B \), where \( H \) is the Hadamard gate.

The measure is maximized by a controlled operation with respect to the reference bases and pure input states, \( C_U(A \rightarrow B) = 2 \log_2 d \). The unitary creates \( 2 \log_2 d \) bits of classical correlations between \( A' \) and \( B' \), and \( 2 \log_2 d \) bits of quantum correlations generated by consuming local coherence with respect to the local \( A' \) basis \( |i_i\rangle \otimes |j_j\rangle \). For the maximally mixed input state, one has \( C_U(A \rightarrow B) = 2 \log_2 d \), because only \( 2 \log_2 d \) bits of classical correlations are created. A further important point is that, while I focus on unitary dynamics for the sake of clarity, the method in Fig. 1 also enables to quantify causation in noisy dynamics, described by completely-positive, trace-preserving maps \( \Phi_{AB} \).

Asymmetry. The measure, unlike correlation quantifiers, distinguishes between causes and effects, \( C_U(A \rightarrow B) \neq C_U(B \rightarrow A) \). Consider \( C^{0,1}_{A \rightarrow B} |0\rangle \otimes |1\rangle_{AB} \). Evaluating causation with respect to \( i_i = j_j = 0 \), one has \( C^{0,1}_{A \rightarrow B} (A \rightarrow B) = 2 \), and \( C^{0,1}_{A \rightarrow B} (B \rightarrow A) = 0 \). On the other hand, reminding that \( C^{0,1}_{A \rightarrow B} = C^{0,1}_{B \rightarrow A} \), the causation with respect to \( i_i = j_j = + \) is \( -C^{0,1}_{A \rightarrow B} (A \rightarrow B) = 0 \), and \( C^{0,1}_{A \rightarrow B} (B \rightarrow A) = 2 \). The measure correctly identifies control (the source of causation) and target (the affected system), which are determined by the reference bases.

Quantifying causation with and without correlations. One of the main challenges in evaluating causation is discriminating between causal links and correlations. The measure defined in Eq. 5 takes zero value for systems \( A, B \) that do not exchange information, regardless of the presence of correlations. That is, two correlated systems \( AB \) are left correlated by local unitary channels, but there is no causation. The proposed method crucially ignores the initial correlations, being the input state \( \rho_A \otimes \rho_B \) rather than the full state \( \rho_{AB} \), detecting no causation in such a case.

A more elusive manifestation of causation is when there is influence without correlations, e.g. \( C^{0,1}_{A \rightarrow B} |1\rangle_{AB} = |1\rangle_{AB} \). The measure is able to detect such causal relations, discriminating between the controlled operation and the very same input/output transformation when due to a local unitary, \( X_B |1\rangle_{AB} = |1\rangle_{AB} \). Note that there is causation even when the state of the target system does not change, i.e. \( C_{A \rightarrow B}^{0,1} |00\rangle_{AB} = |00\rangle_{AB} \), as the system \( B \) still receives the instruction “do nothing” that determines its output state. That is, the controlled gate, while generating no state change, is a distinct physical process from the identity.

Quantifying classical causation. A measure of causal influence between random variables \( A,B \) is obtained by restricting the study to classically correlated states \( \sum_{kl} p_{kl} |kl\rangle \langle l|_{AB} \).

Remarkably, the celebrated Granger causality \( [1] \), widely employed in econometrics to infer causation between random variables, is confined to detect linear causal relations. On the same hand, the transfer entropy and the causation entropy \( [26, 27] \), which have found many uses in network theory, fail to detect causation generated by logical gates, such as the CNOT (XOR) gate applied to time series, \( A_t = A_{t-1} \otimes B_t = A_{t-1} \otimes B_t \), and the Swap, \( A_t = B_{t-1} \otimes A_t = A_{t-1} \) \([6, 7]\).

The quantity in Eq. 5 instead correctly describes causal relations implemented by classical logical gates (see Table 1). For instance, for maximally mixed inputs one has \( C_{\text{Swapp}_{AB}}(A \rightarrow B) = C_{\text{Swapp}_{AB}}(B \rightarrow A) = \log_2 d \), regardless of the chosen implementation in terms of three controlled gates with alternated information flow in \( d = 2 \) (indeed, causation is not additive), or the more complex gate synthesis required in \( d > 2 \) \([28]\).

More generally, a surprising result is obtained. The proposed method detects causal relations between classical systems that are untraceable via a fully classical analysis. Consider for instance the process \( C^{0,1}_{A \rightarrow B} |0\rangle_{A} |0\rangle_{B} \otimes |0\rangle_{A} |0\rangle_{B} \). If \( A', B' \) are classical systems, after STEP 1 their state is \( |0000\rangle |0000\rangle_{AB} \), as the four-bit register admits a uniquely defined basis \( |01\rangle^d \). Consequently, no causation can be detected. If \( A', B', A, B \) are quantum systems, as in Fig. 1, quantum correlations can be created via STEP 1/2, obtaining after STEP 3 the very same final state of the working example \( |++\rangle_{AB} \), and then \( C^{0,1}_{A \rightarrow B} (A \rightarrow B) = 2 \). Finally, if \( A, B \) are two classical random variables, but their information can be stored into the states \( \rho_A, \rho_B \) of two qubits, with quantum ancillae \( A', B' \), one can run the full-quantum protocol. Then, since \( A, B \) are classical, an additional STEP 4 is included to offset quantum correlations: projecting the final state in the reference basis, the unique basis in the classical case, one obtains

\[
\sum_{i_j=0,1} |i\rangle \langle j|_{A'B'B'B} = \frac{1}{2} [0000 | 0000]_{A'B'B'B} \otimes I_B / 2.
\]

Computing the measure defined in Eq. 5 on the projected state gives \( C^{0,1}_{A \rightarrow B} (A \rightarrow B) = 1 \). The result is expected as 1 bit of correlations is created between \( A' \) and \( B' \).

**Scalarity:** Localizing and ranking multiple causes. The proposed measure of causation extends to many-body systems. Defining a third system \( E \), the global evolution of the tripar-

| Process \( U_{AB} U_{AB}^\dagger \) | Corr. \( C_U(A \rightarrow B) \) | Interdependence \( C_U(B \rightarrow A) \) |
|-----------------------------|------------------|------------------|
| \( V_A \otimes W_{B} \rho_{AB} \) \( V_A \otimes W_{B} \rho_{AB} \) | \( 0, \forall V, W, \rho \) | \( 0, \forall V, W, \rho \) |
| \( C^{0,1}_{A \rightarrow B} |0\rangle \otimes |0\rangle_{AB} \) | \( 2 \log_2 d, \forall \psi, \phi \) | \( 0, \forall \psi, \phi \) |
| \( C^{0,1}_{B \rightarrow A} |0\rangle \otimes |0\rangle_{AB} \) | \( 0, \forall \psi, \phi \) | \( 2 \log_2 d, \forall \psi, \phi \) |
| \( C^{0,1}_{C \rightarrow A} |0\rangle \otimes |0\rangle_{AB} \) | \( 0, \forall \psi, \phi \) | \( 0, \forall \psi, \phi \) |
| \( C^{0,1}_{C \rightarrow B} |0\rangle \otimes |0\rangle_{AB} \) | \( 2, \forall \psi, \phi \) | \( 2 \log_2 d, \forall \psi, \phi \) |
| \( \text{Swap}_{AB} |0\rangle \otimes |0\rangle_{AB} \) | \( 2 \log_2 d, \forall \psi, \phi \) | \( 2 \log_2 d, \forall \psi, \phi \) |
TABLE II. Quantifying causation in tripartite interactions of quantum and classical systems with respect to the reference bases \([i_1,i_2,i_3] = 0,1,\ldots,d\). The measure discriminates between, for example, a CNOT between \(A\) and \(B\) with \(E\) uncorrelated, and one with correlated \(E\). Also, the two control systems \(EA\) of the Toffoli gate are more affecting \(B\) when in the state \((|00⟩ + |11⟩)/\sqrt{2}\) than in, for instance, the uncorrelated state \(\sum_{j=0}^{d} |j⟩/d\), as in the former any element of the superposition corresponds to a different instruction to \(B\).

| Process | \(\rho_{EABB}U_{EABB}^†\) | \(C_U(\rho_{EA} \rightarrow B)\) | \(C_U(\rho_{AE} \rightarrow B)\) | \(C_U(\rho_{A} \rightarrow B)\) |
|---------|-----------------|-----------------|-----------------|-----------------|
| \(V_{EABB}\) | 0, \(\forall V, W, \rho\) | 0, \(\forall V, W, \rho\) | 0, \(\forall V, W, \rho\) |
| \(C_{EABB}^{(1),i,j=0,\ldots,d} \rho_{EABB} = \sum_{\phi} C_{EABB}^{(1),i,j=0,\ldots,d} \phi_{EABB} \) | \(2 \log_2 d, \forall \psi, \phi\) | \(\log_2 d, \forall \psi, \phi\) | \(\log_2 d, \forall \psi, \phi\) |
| \(C_{EABB}^{(2),i,j=0,\ldots,d} \rho_{EABB} = \sum_{\phi} C_{EABB}^{(2),i,j=0,\ldots,d} \phi_{EABB} \) | \(2 \log_2 d, \forall \psi, \phi\) | \(2 \log_2 d, \forall \psi, \phi\) | \(2 \log_2 d, \forall \psi, \phi\) |
| \(C_{EABB}^{(3),i,j=0,\ldots,d} \rho_{EABB} = \sum_{\phi} C_{EABB}^{(3),i,j=0,\ldots,d} \phi_{EABB} \) | \(\log_2 d, \forall \psi, \phi\) | 0 | \(\log_2 d\) |
| Toffoli\(EABB \rightarrow \rho(00⟩ + |11⟩)/\sqrt{2} \) | \(2, \forall \psi\) | \(1, \forall \psi\) | \(1, \forall \psi\) |
| Toffoli\(EABB \rightarrow \rho \) | \(3/2 \log_4 4 + 3/2 < 1, \forall \psi, \phi\) | \(3/4 \log_4 4 + 3/2 < 1, \forall \psi, \phi\) | \(3/4 \log_4 4 + 3/2 < 1, \forall \psi, \phi\) |
| Toffoli\(EABB \rightarrow \rho \) | \(3/4 \log_4 4 + 3/2 < 1, \forall \psi, \phi\) | \(3/2 \log_2 d, \forall \psi, \phi\) | \(3/2 \log_2 d, \forall \psi, \phi\) |

The causal influence due to \(A\) on \(B\) given full information about \(E\), with respect to the reference bases \([i_1,i_2,i_3]\), then quantified by the difference between the causal influence of \(A\) and the contribution of \(E\) computed by ignoring \(A\):

\[
C_U(A|E \rightarrow B) = C_U(EA \rightarrow B) - C_U(E \rightarrow B) = I(B: A')E'EB').
\]

The chain rule of the conditional mutual information implies that the total causal influence on a subsystem is decomposable as the sum of conditional causalities.

**Conclusion.** – I introduced an information-theoretic measure of causation for quantum systems, Eq. 5, which is computed via the method illustrated in Fig. 1. Among the desirable properties, the quantity evaluates the strength of causal relations between classical random variables, and it has been extended to quantify causation in many-body systems. The study paves the way for a resource theory of causation [29], a mathematical framework studying causal effects as a computational resource [30]. Causation, rather than correlations, could indeed be the key resource for tasks in which the different parts of a composite system play different roles, e.g. control [31], metrology [32], and learning [33]. Being the causation measure extracted by correlation dynamics, it may be possible to distinguish between genuine quantum and causal effects, as it happens for correlation quantifiers [34].
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