Comments on multiplicativity of
maximal $p$-norms when $p = 2$

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Abstract

We consider the maximal $p$-norm associated with a completely positive
map and the question of its multiplicativity under tensor products. We
give a condition under which this multiplicativity holds when $p = 2$, and
we describe some maps which satisfy our condition. This class includes
maps for which multiplicativity is known to fail for large $p$.

Our work raises some questions of independent interest in matrix the-
ory; these are discussed in two appendices.

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1 Introduction

In quantum information theory, noise is modeled by a completely positive and trace-preserving (CPT) map \( \Phi \) acting on the states of the quantum system. In general \( \Phi \) takes pure states into mixed states, and in order to assess the 'noisiness' of the map one is interested in knowing how close the image states may come to pure states. Amosov, Holevo and Werner (AHW) \[2\] observed that this could be measured by the quantity

\[
\nu_p(\Phi) = \sup \{ \| \Phi(\rho) \|_p : \rho > 0, \; \text{Tr} \; \rho = 1 \}
\]

where \( \| \gamma \|_p = \left[ \text{Tr} (\gamma^p) \right]^{1/p} \) and \( 1 \leq p \leq \infty \). For any density matrix \( \gamma \), \( \| \gamma \|_p \leq 1 \) with equality if and only if \( \gamma \) is pure. Hence \( \nu_p(\Phi) \leq 1 \) with equality if and only if there is a pure state \( \rho \) for which \( \Phi(\rho) \) is also pure (since by convexity the sup in (1) is achieved on a pure state).

The \( p \)-norm defined above can be extended to arbitrary matrices as \( \| A \|_p = \left[ \text{Tr} |A|^p \right]^{1/p} \) with \( |A| = \sqrt{A^*A} \). The following useful relationships, which were established in \[1\], can be readily verified.

\[
\nu_p(\Phi) = \sup_{\gamma > 0, \; \text{Tr} \; \gamma = 1} \| \Phi(\gamma) \|_p = \sup_{A > 0} \frac{\| \Phi(A) \|_p}{\text{Tr} A} \tag{2}
\]
\[
= \sup_{A=A^\dagger} \frac{\| \Phi(A) \|_p}{\text{Tr} |A|} = \sup_{A=A^\dagger} \frac{\| \Phi(A) \|_p}{\| A \|_1} \tag{3}
\]
\[
\leq \sup_{A} \frac{\| \Phi(A) \|_p}{\| A \|_1}. \tag{4}
\]

(The equivalence of (2) and (3) follows from the convexity of the \( p \)-norm and the fact that \( |\Phi(A_+ - A_-)| = \Phi(A_+) + \Phi(A_-) \) when \( A = A_+ - A_- \) is the decomposition of a self-adjoint operator into its positive and negative parts.) It follows immediately from (3) that for any self-adjoint \( A \)

\[
\| \Phi(A) \|_p \leq \nu_p(\Phi) \text{Tr} |A|. \tag{5}
\]

The representation (4) suggests viewing \( \Phi \) as a map between spaces of complex matrices with different \( p \)-norms. As suggested in \[1\] one can generalize this by defining

\[
\| \Phi \|_{q \rightarrow p} = \sup_{A} \frac{\| \Phi(A) \|_p}{\| A \|_q} \tag{6}
\]

and let \( \| \Phi \|_{q \rightarrow p}^R \) denote the same quantity when the supremum is restricted to the real vector space of self-adjoint operators. Then \( \nu_p(\Phi) \) is precisely \( \| \Phi \|_{1 \rightarrow p}^R. \)
In general, \( \|\Phi\|_{q\rightarrow p}^R \leq \|\Phi\|_{q\rightarrow p} \) and one would expect that the inequality could be strict for some \( \Phi \). However, the second part of Theorem 1 states that equality holds in all dimensions when \( p = q = 2 \); and in Appendix B.3 we show that equality holds for CPT maps on qubits when \( q = 1, p \geq 2 \). This raises the question of whether equality always holds and, if not, for what types of maps strict inequality is possible.

Amosov, Holevo and Werner conjectured [2] that \( \nu_p \) is multiplicative on tensor products, i.e., that
\[
\nu_p(\Phi \otimes \Omega) = \nu_p(\Phi) \nu_p(\Omega) \tag{7}
\]
This has been verified in a number of special cases, although it is now known to be false in general. Amosov and Holevo [1] proved (7) when both \( \Phi \) and \( \Omega \) are products of depolarizing CPT maps and \( p \) is integer. King proved (7) for all \( p \geq 1 \) and arbitrary \( \Omega \) under the additional assumption that \( \Phi \) is a unital qubit CPT map [11], \( \Phi \) is a depolarizing channel in any dimension [11], or \( \Phi \) is an entanglement breaking map [12]. However, Holevo and Werner [19] also showed that (7) need not hold in general by giving a set of explicit counterexamples for \( p > 4.79 \) and \( d \geq 3 \).

Amosov and Holevo conjectured [1] that the quantity in (6) should also be multiplicative for \( 1 \leq q \leq p \), i.e., that
\[
\|\Phi \otimes \Omega\|_{q \rightarrow p} = \|\Phi\|_{q \rightarrow p} \|\Omega\|_{q \rightarrow p} \tag{8}
\]
Beckner [5] established an analogous multiplicativity for commutative systems when \( 1 \leq q \leq p \). Curiously, Junge [8] proved (8) for completely positive (CP) maps with \( p \) and \( q \) in the opposite order, that is for the case \( 1 \leq p \leq q \). However, our main interest is the case \( q = 1 < p \), and Junge’s result does not seem to shed any direct light on this question.

The conjecture (7) is of greatest interest for \( p \) near 1 since taking the limit as \( p \rightarrow 1 \) yields the von Neumann entropy of \( \gamma \), another natural measure of purity, and the validity of (7) for \( p \) in an interval of the form \( [1, 1 + \epsilon] \) with \( \epsilon > 0 \) would imply additivity of the minimal entropy. Moreover, it has been shown [18] that additivity of minimal entropy is equivalent to several other important conjectures in quantum information theory, including additivity of Holevo capacity and additivity of entanglement of formation. Audenaert and Braunstein [3] have also observed a connection between multiplicativity for CP maps, and super-additivity of entanglement of formation.

In view of the Holevo-Werner example, it is natural to conjecture that (7) holds in the range \( 1 \leq p \leq 2 \). This is precisely the range of values of \( p \) for which the function \( f(x) = x^p \) is operator convex, and it is also the range for which a
number of convexity inequalities hold. Verifying (7) for the special case of $p = 2$ would suggest its validity for $1 \leq p \leq 2$. Unfortunately, even this seemingly simple case is not as straightforward as one might hope. In this note, we prove (7) when $p = 2$ for a special class of CP maps. Although this is a rather limited result, it gives some insight into the difficulties one encounters in the general case.

Note that the multiplicativity for CPT maps follows if it holds for all CP maps. We consider this more general case, as it does not seem more difficult. In fact, it is not hard to show that multiplicativity (7) holds for all $p \geq 1$ whenever $\Phi$ is an extreme CP map. This is because these are precisely the CP maps which can be written in the form $\Phi(\rho) = A^\dagger \rho A$, i.e., with one Kraus operator, and one can then assume without loss of generality that $A$ is diagonal. Thus, the extreme CP maps fall into the “diagonal” maps considered in Example 1 below, for which multiplicativity has been proved.

Although the first part of the following theorem is included in Junge’s result, we include an elementary argument here.

**Theorem 1** $\|\Phi\|_{2 \rightarrow 2}$ is multiplicative, i.e., $\|\Phi \otimes \Omega\|_{2 \rightarrow 2} = \|\Phi\|_{2 \rightarrow 2} \|\Omega\|_{2 \rightarrow 2}$. Moreover, $\|\Phi\|_{2 \rightarrow 2} = \|\Phi\|_{R_{2 \rightarrow 2}}^R$

**Proof:** First, recall that the complex $n \times n$ matrices form a Hilbert space with respect to the inner product $\langle A, B \rangle = \text{Tr} A^\dagger B$, and let $\hat{\Phi}$ denote the adjoint of the linear operator $\Phi$ with respect to this inner product. Since

$$\left(\|\Phi\|_{2 \rightarrow 2}\right)^2 = \sup_A \frac{\text{Tr} [\Phi(A)]^\dagger \Phi(A)}{\text{Tr} A^\dagger A} = \sup_A \frac{\langle A, (\hat{\Phi} \circ \Phi)(A) \rangle}{\langle A, A \rangle}, \quad (9)$$

it follows that $\|\Phi\|_{2 \rightarrow 2}$ is the usual operator sup norm on this Hilbert space or, equivalently, the largest singular value of $\Phi$. Thus, $\|\Phi\|_{2 \rightarrow 2}$ is the square root of the largest eigenvalue of $(\hat{\Phi} \circ \Phi)$. This is the same as the largest eigenvalue of $(\hat{\Phi} \circ \Phi) \otimes I$; therefore, $\|\Phi \otimes I\|_{2 \rightarrow 2} = \|\Phi\|_{2 \rightarrow 2}$. The main result then follows from the submultiplicativity of the Hilbert space operator norm under composition since

$$\|\Phi \otimes \Omega\|_{2 \rightarrow 2} = \|(\Phi \otimes I) \circ (I \otimes \Omega)\|_{2 \rightarrow 2} \leq \|\Phi \otimes I\| \|(I \otimes \Omega)\|_{2 \rightarrow 2} = \|\Phi\|_{2 \rightarrow 2} \|\Omega\|_{2 \rightarrow 2}.$$

Note that since $(\hat{\Phi} \circ \Phi)$ has real eigenvalues (in fact, they are non-negative), the solutions of the eigenvector equation $(\hat{\Phi} \circ \Phi)(B) = \mu B$ are self-adjoint (or can be so chosen if $\mu$ is degenerate). This implies that the supremum in (9) is achieved with a self-adjoint $A$, which implies the second statement in the Theorem.
2 Main Theorem

We now find it convenient to introduce some notation. When \( \{ e_j \} \) is an orthonormal basis for \( \mathbb{C}^d \), we will let \( E_{jk} = |e_j\rangle\langle e_k| \) denote the matrix with a 1 in the \( j \)-th row and \( k \)-th column and 0’s elsewhere. Then the set of operators \( \{ E_{jk} \} \) also form an orthonormal basis for the \( d \times d \) matrices with respect to the Hilbert-Schmidt inner product. Moreover, if \( \Gamma \) is a matrix on \( \mathbb{C}^d \otimes \mathbb{C}^{d'} \), we can write \( \Gamma = \sum_{jk} E_{jk} \otimes M_{jk} \) where \( M_{jk} = \text{Tr}_1(\Gamma E_{jk}^\dagger \otimes I) \). This is equivalent to saying that \( \Gamma \) is a block matrix with blocks \( M_{jk} \).

If \( \Omega \) is a CP map, then \((I \otimes \Omega)(\Gamma) = \sum_{jk} E_{jk} \otimes \Omega(M_{jk}) > 0 \) which implies that any \( 2 \times 2 \) submatrix \( \begin{pmatrix} \Omega(M_{jj}) & \Omega(M_{jk}) \\ \Omega(M_{kj}) & \Omega(M_{kk}) \end{pmatrix} \) is positive semi-definite. This implies in turn that

\[
\Omega(M_{jk}) = \Omega(M_{jj})^{1/2} R_{jk} \Omega(M_{kk})^{1/2}
\]

where \( R_{jk} \) is a contraction. Hence

\[
\text{Tr} \Omega(M_{jk})^\dagger \Omega(M_{jk}) \leq \|\Omega(M_{jj})\|_2 \|\Omega(M_{kk})\|_2.
\] (11)

**Theorem 2** Let \( \Phi \) and \( \Omega \) be CP maps one of which (say \( \Phi \)) satisfies the condition

\[
\text{Tr} \Phi(E_{ik})^\dagger \Phi(E_{j\ell}) \geq 0 \quad \forall \ i, j, k, \ell.
\]

Then \( \nu_2(\Phi \otimes \Omega) = \nu_2(\Phi)\nu_2(\Omega) \).

**Proof:** Writing an arbitrary density matrix \( \Gamma \) as above, one finds

\[
(\Phi \otimes \Omega)(\Gamma) = \sum_{jk} \Phi(E_{jk}) \otimes \Omega(M_{jk})
\] (13)

Thus

\[
\text{Tr} [(\Phi \otimes \Omega)(\Gamma)]^\dagger (\Phi \otimes \Omega)(\Gamma)
\]

\[
= \sum_{ik} \sum_{j\ell} \text{Tr} \Phi(E_{ik})^\dagger \Phi(E_{j\ell}) \text{Tr} \Omega(M_{ik})^\dagger \Omega(M_{j\ell})
\] (14)

\[
\leq \sum_{ik} \sum_{j\ell} |\text{Tr} \Phi(E_{ik})^\dagger \Phi(E_{j\ell})| \|\Omega(M_{ik})\|_2 \|\Omega(M_{j\ell})\|_2
\]

\[
\leq \sum_{ik} \sum_{j\ell} |\text{Tr} \Phi(E_{ik})^\dagger \Phi(E_{j\ell})| \sqrt{\|\Omega(M_{ii})\|_2 \|\Omega(M_{jj})\|_2 \|\Omega(M_{kk})\|_2 \|\Omega(M_{\ell\ell})\|_2}
\]

\[
\leq [\nu_2(\Omega)]^2 \sum_{ik} \sum_{j\ell} |\text{Tr} \Phi(E_{ik})^\dagger \Phi(E_{j\ell})| \sqrt{\text{Tr} M_{ii} M_{jj} M_{kk} M_{\ell\ell}}
\] (15)
where we have used (11) and (5). Now, note that the matrix

\[ N = \sum_{ik} \sqrt{\text{Tr } M_{ii} \text{Tr } M_{kk}} E_{ik} \]  \hspace{1cm} (16)

= \begin{pmatrix} \sqrt{\text{Tr } M_{11}} \\ \sqrt{\text{Tr } M_{22}} \\ \vdots \\ \sqrt{\text{Tr } M_{dd}} \end{pmatrix} \begin{pmatrix} \sqrt{\text{Tr } M_{11}} & \sqrt{\text{Tr } M_{22}} & \ldots & \sqrt{\text{Tr } M_{dd}} \end{pmatrix} \] \hspace{1cm} (17)

is positive semi-definite and

\[ \sum_{ik} \sum_{j\ell} \text{Tr } \Phi(E_{ik})^\dagger \Phi(E_{j\ell}) \sqrt{\text{Tr } M_{ii} \text{Tr } M_{jj} \text{Tr } M_{kk} \text{Tr } M_{\ell\ell}} \]

\[ = \text{Tr } \Phi(N)^\dagger \Phi(N) \] \hspace{1cm} (18)

\[ \leq [\nu_2(\Phi)]^2 (\text{Tr } N)^2 \] \hspace{1cm} (19)

\[ = [\nu_2(\Phi)]^2 \left( \sum_i \text{Tr } M_{ii} \right)^2 \]

\[ = [\nu_2(\Phi)]^2 (\text{Tr } \Gamma)^2. \] \hspace{1cm} (20)

When (12) holds the absolute value bars are redundant in (15). One can then substitute (20) in (15) to yield

\[ \text{Tr } \left[ (\Phi \otimes \Omega)(\Gamma) \right]^\dagger (\Phi \otimes \Omega)(\Gamma) \leq [\nu_2(\Omega)]^2 [\nu_2(\Phi)]^2 (\text{Tr } \Gamma)^2. \] \hspace{1cm} (21)

Taking the square root and dividing both sides by $\text{Tr } \Gamma$, one finds

\[ \frac{\| (\Phi \otimes \Omega)(\Gamma) \|_2}{\text{Tr } \Gamma} \leq \nu_2(\Omega) \nu_2(\Phi) \quad \forall \ \Gamma \geq 0. \] \hspace{1cm} (22)

Taking the supremum over $\Gamma$ gives the desired result.

Although condition (12) is simple, it is basis dependent. In order that $\Phi$ be multiplicative, it suffices that (12) holds for the matrices $E_{jk} = |e_j \rangle \langle e_k|$ associated with some basis for $C^d$. The question of when such a basis can be found gives rise to some interesting questions in matrix theory which we remark on in Appendix A. We remark here only that, although we do not expect (12) to hold for all CP maps, we also do not have a counter-example. Thus, one cannot exclude the possibility that the hypothesis of Theorem 2 is actually satisfied by all CP, or by all CPT, maps.
3 Special cases

It order to show that our results are not vacuous, we now give some examples of maps which satisfy (12).

1. Maps with only diagonal Kraus operators. In this case \( \Phi(E_{jk}) = a_{jk}E_{jk} \) for some positive matrix \( A = (a_{jk}) \), and the condition (12) follows from the orthonormality of the \( \{E_{jk}\} \). This class of CP maps was studied by Landau and Streater who named them the diagonal maps. In fact a more complicated analysis using the Lieb-Thirring inequality can be used to show multiplicativity for all \( p \geq 1 \) for these maps [14].

2. Maps for which \( (I \otimes \Phi)(M) = \sum_{jk} E_{jk} \otimes \Phi(E_{jk}) \) has non-negative elements, where \( M = \sum_{jk} E_{jk} \otimes E_{jk} \) is the maximally entangled state. (This is the block matrix with blocks \( \Phi(E_{jk}) \); it is sometimes called the Choi matrix or Jamiolkowski state representative of \( \Phi \).) Although this condition is clearly sufficient to satisfy (12), it is not necessary. For example, let \( A \) be a positive semi-definite matrix with some \( a_{jk} < 0 \). Then for that particular \( j, k \) the corresponding map in Example 1 has \( \Phi(E_{jk}) = a_{jk}E_{jk} \) with one strictly negative element.

3. Multiplicativity at \( p = 2 \) has been proven for all qubit CPT maps [9]. However, we can verify the condition (12) only for a subset of qubit CPT maps. This subset is described using the parametrization of qubit maps that was derived in [15] and summarized in Appendix B.1.

In terms of that notation, the condition (12) is satisfied when \( t_1 = t_2 = 0 \) (since in this case \( \Phi(E_{jk}) \) is diagonal when \( j = k \) and skew diagonal when \( j \neq k \)). It is interesting to note that multiplicativity is known to hold for all \( p \geq 1 \) under the stronger condition \( t_1 = t_2 = t_3 = 0 \) [11], so this result suggests that it may hold for \( p \geq 1 \) for a larger class of CPT maps.

Another class of qubit maps which satisfy (12) are those with \( \lambda_1 \geq \pm \lambda_2, t_2 = 0, \) and \( t_1 \geq 0 \) (again using the notation in Appendix B.1). These maps belong to Example 2 since \( \Phi(E_{jk}) \) has non-negative elements for all \( j, k \). Furthermore, in Theorem 3 of [9], King proved that multiplicativity holds for these channels for all integer \( p \geq 1 \), and later [13] extended this to all \( p \geq 2 \).

4. The special class of maps satisfying (23) and discussed below. This class includes maps for which multiplicativity does not hold for some \( p > 2 \).
Let $M$ denote a $d \times d$ Hermitian matrix with elements $m_{jk} = x_{jk} + iy_{jk}$ with $x_{jk}, y_{jk}$ real. Let $\Phi : M \mapsto \Phi(M)$ denote a linear map with the following very special properties:

$$[\Phi(M)]_{jk} = \begin{cases} \sum_{\ell} d_{j\ell}m_{\ell\ell} & \text{when } j = k, \\ a_{jk}(x_{jk} + i\epsilon_{jk}y_{jk}) & \text{when } j \neq k \end{cases}$$

where $d_{j\ell} \geq 0$, $a_{jk}$ are the off-diagonal elements of a fixed Hermitian matrix and $\epsilon_{jk} = \epsilon_{kj} = \pm 1$. The map $\Phi$ is trace-preserving if and only if the matrix $D$ with elements $d_{j\ell}$ is column stochastic. Not every map of the form (23) will necessarily be CP. However, certain special subclasses can be identified.

a) $a_{jk} = 0 \ \forall j \neq k$. In this case, $\Phi$ is a QC map consisting of the projection onto the diagonal part of $M$ followed by the action of a column stochastic matrix on the classical probability vector corresponding to the diagonal.

b) $A > 0$ is a fixed positive semi-definite matrix, $d_{j\ell} = \delta_{j\ell}a_{jj}$ and $\epsilon_{jk} = +1$. This is exactly the diagonal class described above.

c) $d_{j\ell} = 1 - \delta_{j\ell}$, $a_{jk} = -1 \ \forall j \neq k$ and $\epsilon_{jk} = -1$. In this case, $\Phi(M) = (\text{Tr } M)I - M^T$ and $\frac{1}{d-1}\Phi(M)$ is the CPT map for which Holevo and Werner showed that (7) does not hold for large $p$.

Since King showed that multiplicativity holds for all $p \geq 1$ for maps of type (a) and (b), multiplicativity at $p = 2$ may not seem very significant. However, maps of type (c) are precisely those used to establish that multiplicativity does not hold for sufficiently large $p$. Moreover, the full class includes convex combinations of maps of type (a) with one of type (b) or type (c), and King’s results do not apply to this class. Thus this class of maps is neither trivial nor uninteresting.

Although one can verify that CP maps satisfying (23) always satisfy the hypothesis of Theorem 2, we state and prove their multiplicativity as a separate result. Inequality (11) again plays a key role in the proof.

**Theorem 3** Let $\Phi$ be a CP map satisfying (23) and let $\Omega$ be an arbitrary CP map. Then $\nu_2(\Phi \otimes \Omega) = \nu_2(\Phi)\nu_2(\Omega)$.

**Proof:** As before, let $\Gamma = \sum_{jk} E_{jk} \otimes M_{jk}$ be the matrix with blocks $M_{jk}$. Then $(\Phi \otimes \Omega)(\Gamma) > 0$ has blocks

$$[(\Phi \otimes \Omega)(\Gamma)]_{jk} = \begin{cases} \sum_{\ell} d_{j\ell}\Omega(M_{\ell\ell}) & \text{when } j = k, \\ a_{jk}(\Omega(\Re M_{jk}) + i\epsilon_{jk}\Omega(\Im M_{jk})) & \text{when } j \neq k \end{cases}$$

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where \( \Re(M_{jk}) = \frac{1}{2}(M_{jk} + M_{jk}^\dagger) \) and \( \Im(M_{jk}) = \frac{-i}{2}(M_{jk} - M_{jk}^\dagger) \). A straightforward calculation gives

\[
\text{Tr} \left[ (\Phi \otimes \Omega)(\Gamma)^\dagger (\Phi \otimes \Omega)(\Gamma) \right] = \sum_j \sum_{\ell} \sum_n d_{\ell j} d_{jn} \text{Tr} \Omega(M_{\ell\ell}) \text{Tr} \Omega(M_{nn}) + \sum_{j \neq k} |a_{jk}|^2 \text{Tr} \Omega(M_{jk}^\dagger) \text{Tr} \Omega(M_{jk})
\]

\[
\leq \sum_j \sum_{\ell} \sum_n d_{\ell j} d_{jn} \| \Omega(M_{\ell\ell}) \|_2 \| \Omega(M_{nn}) \|_2 + \sum_{j \neq k} |a_{jk}|^2 \| \Omega(M_{jj}) \|_2 \| \Omega(M_{kk}) \|_2
\]

\[
\leq [\nu_2(\Omega)]^2 \left( \sum_j \sum_{\ell} \sum_n d_{\ell j} d_{jn} \text{Tr} M_{\ell\ell} \text{Tr} M_{nn} + \sum_{j \neq k} |a_{jk}|^2 \text{Tr} M_{jj} \text{Tr} M_{kk} \right)
\]

where we have used the Schwarz inequality for the Hilbert-Schmidt inner product and (11). Now, the term in parentheses in (26) is precisely \( \text{Tr} \Phi(N)^\dagger \Phi(N) \) where \( N \) is the matrix with elements \( N_{jk} = (\text{Tr} M_{jj} \text{Tr} M_{kk})^{-\frac{1}{2}} \). The desired result then follows as in the proof of Theorem 2 since

\[
\text{Tr} \left[ (\Phi \otimes \Omega)(\Gamma)^\dagger (\Phi \otimes \Omega)(\Gamma) \right] \leq [\nu_2(\Omega)]^2 \nu_2(\Phi)^2 \text{Tr} N^2 = [\nu_2(\Omega)]^2 [\nu_2(\Phi)]^2 (\text{Tr} \Gamma)^2.
\]

### 4 Concluding remarks

If one could replace the operator basis \( \{ E_{jk} \} \) by a more general orthonormal operator basis \( \{ G_m \} \) for \( C^{d \times d} \), one could always satisfy the analogue of (12). One need only choose \( \{ G_m \} \) to be the basis which diagonalizes the positive semi-definite operator \( \hat{\Phi} \Phi \), i.e., for which

\[
(\hat{\Phi} \circ \Phi)(G_m) = \mu_m^2 G_m.
\]

where \( \mu_m \) are the singular values of \( \Phi \). Then \( \text{Tr} \Phi(G_m)^\dagger \Phi(G_n) = \mu_m^2 \delta_{mn} \geq 0 \). Moreover, as noted at the end of the proof of Theorem 1, one can always choose the basis so that each \( G_m = G_m^\dagger \) is self-adjoint.

Using the orthogonality condition \( \text{Tr} G_m^\dagger G_n = \delta_{mn} \), one can show that a density matrix \( \Gamma \) on a tensor product space can be written in the form

\[
\Gamma = \sum_m G_m \otimes W_m \quad \text{with} \quad W_m = \text{Tr}_1 (G_m^\dagger \otimes I) \Gamma.
\]

Note \( G_m \) self-adjoint implies that \( W_m \) is also self-adjoint.
We now try to imitate the proof of Theorem 2. Since \((\Phi \otimes \Omega)(\Gamma) = \sum_m \Phi(G_m) \otimes \Omega(W_m)\),

\[
\text{Tr} [(\Phi \otimes \Omega)(\Gamma)]^\dagger (\Phi \otimes \Omega)(\Gamma) = \sum_{mn} \text{Tr} \Phi(G_m)^\dagger \Phi(G_n) \text{Tr} \Omega(W_m)^\dagger \Omega(W_n)
\]

\[
= \sum_n \mu_n^2 \text{Tr} \Omega(W_n)^\dagger \Omega(W_n) \quad (29)
\]

\[
\leq [\nu_2(\Omega)]^2 \sum_n \mu_n^2 (\text{Tr} |W_n|)^2 \quad (30)
\]

\[
= [\nu_2(\Omega)]^2 \text{Tr} \Phi(N)^\dagger \Phi(N)
\]

\[
\leq [\nu_2(\Omega)]^2 [\nu_2(\Phi)]^2 (\text{Tr} |N|)^2 \quad (31)
\]

where \(N = \sum_m G_m \text{Tr} |W_m|\) (and the first inequality implicitly used the assumption that \(G_m\) is self-adjoint so that \(W_m\) is).

Unfortunately, we can not conclude that \(\text{Tr} |N| \leq \text{Tr} \Gamma\) as needed to complete the proof. (If we had instead \(N = \sum_m G_m \text{Tr} W_n\), then we would have \(N = \text{Tr} \Gamma > 0\) and \(\text{Tr} N = \text{Tr} |N| = \text{Tr} \Gamma\).) This is a real problem. Using the Pauli basis for qubits, for which \(G_k = 2^{-1/2} \sigma_k\), consider the maximally entangled Bell state \(\Gamma = G_0 \otimes G_0 + G_1 \otimes G_1 - G_2 \otimes G_2 + G_3 \otimes G_3\). Then \(\text{Tr} \Gamma = \frac{1}{2} I\), but \(N = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & 2 + i \\ 2 - i & 0 \end{pmatrix}\) is not positive semi-definite and \(\text{Tr} |N| > 1\).

Although our results do not prove it, we conjecture that multiplicativity does hold for all CP maps at \(p = 2\). If this conjecture turns out to be false, then it seems unlikely that any other value of \(p\) between 1 and 2 would play a special role, and there would probably be counterexamples to multiplicativity all the way down to \(p = 1\). In this case additivity of minimal entropy would be an isolated result and the attempt to prove it using \(p\)-norms would probably be futile.

**A  Comments on positivity condition (I2):**

To find conditions under which (I2) holds, note that it is equivalent to the requirement that

\[
x_{ik,j\ell} = \text{Tr} E_{ik}^\dagger (\widehat{\Phi} \circ \Phi)(E_{j\ell}) \geq 0 \quad \forall \ i, j, k, \ell
\]

which is precisely the condition that the matrix \(X\) representing the positive semi-definite linear operator \(\widehat{\Phi} \circ \Phi\) in the orthonormal operator basis \(\{E_{jk}\}\) also has non-negative elements \(x_{ik,j\ell}\).
If \( |f_j⟩ = U|e_j⟩ \) denotes another O.N. basis for \( \mathbb{C}^d \), then \( F_{jk} = |f_j⟩⟨f_k| = U E_{jk} U^\dagger \) is an orthonormal operator basis for \( \mathbb{C}^{d×d} \), and the corresponding matrix representative of \( \Phi \) is \( (U^T \otimes U^\dagger) X (U \otimes U) \). In the proof of Theorem 2 above we require only that \( E_{jk} \) have the form \( |e_j⟩⟨e_k| \) for some orthonormal basis \( \{e_j\} \) for \( \mathbb{C}^d \) so that we can use (11).

Therefore, multiplicativity of \( \nu_2(\Phi) \) will hold if there is a unitary operator \( U \) on \( \mathbb{C}^d \) such that the matrix \( (U^T \otimes U^\dagger) X (U \otimes U) \) has non-negative elements [where \( X \) is the matrix with elements defined by (32)]. Unfortunately, this is not a very tractable condition.

One way to find an example of a CP map satisfying (12) on \( \mathbb{C}^d \) is to find a \( d^2 \times d^2 \) positive semi-definite matrix \( X \) with non-negative elements. Regarding the \( d \times d \) blocks of \( X \) as \( \Phi(E_{jk}) \) defines a CP map of the type considered in Example 2. However, as noted before, CP maps satisfying (12) need not have this form.

A \( d^2 \times d^2 \) matrix \( X \) with elements \( x_{ik,j\ell} ≥ 0 \) also defines a positive semi-definite linear map \( \Omega \) which one can write as \( \Omega = \hat{\Phi} \circ \Phi \). The map \( \Phi \) then satisfies (12), but it need not be CP. We only know that \( \Phi = \Lambda_U \circ \sqrt{\Omega} \) for some linear operator \( \Lambda_U \) on \( \mathbb{C}^{d\times d} \) which is unitary in the sense \( \text{Tr} \Lambda_U(A)\Lambda_U(B) = \text{Tr} A^\dagger B \) for all \( d \times d \) matrices \( A, B \). This implies that \( \Phi \) must have the form \( \Phi(A) = U^\dagger \sqrt{\Omega}(A)U \) for some unitary matrix \( U \).\(^1\) Hence, \( \Phi \) is CP if and only if \( \sqrt{\Omega} \) is; however, this does not seem easy to check.

### B Qubit maps

#### B.1 Notation:

It will be useful to summarize some basic facts about the representation of matrices and CP maps on qubits using the identity and Pauli matrices as bases. One can write an arbitrary matrix as \( A = z_0 I + z \cdot \sigma \) with \( z_0 \in \mathbb{C}, z = w + iu \) and \( w, u \) vectors in \( \mathbb{R}^3 \). When \( z_0 \neq 0 \), any norm satisfies \( \|z_0 I + z \cdot \sigma\| = |z_0|\|I + \frac{1}{z_0}z \cdot \sigma\| \).

Therefore, we will present most results only for \( z_0 = 1 \); results for \( z_0 = 0 \) are generally straightforward. The most general CP map has the form

\[
\Phi(I + z \cdot \sigma) = (1 + s \cdot w + i s \cdot u)I + (t + Tw + iTu) \cdot \sigma \quad (33a)
\]
\[
\Phi(z \cdot \sigma) = s \cdot zI + (Tz) \cdot \sigma \quad (33b)
\]

\(^1\)The fact that \( \Lambda_U \) must have the form \( \Lambda_U(A) = U^\dagger A U \) is probably well-known, but was first brought to the attention of MBR by Nicolas Boulant, who includes a proof in his paper [6]. If one writes \( \Lambda_U \) in Kraus form, one can then use the fact that \( \hat{\Lambda}_U \Lambda_U = I \) to show that the Kraus operators can be chosen to be a single unitary.
where \( \mathbf{s}, \mathbf{t} \) are vectors in \( \mathbb{R}^3 \) and \( T \) is a real \( 3 \times 3 \) matrix. \( \Phi \) is TP if and only if \( \mathbf{s} = 0 \); and \( \Phi \) is unital if and only if \( \mathbf{t} = 0 \).

As observed in [15], one can use the singular value decomposition to assume without loss of generality that \( T \) is diagonal with real (but not necessarily positive) elements \( \lambda_k \). This leads to the canonical form

\[
\Phi(I + w \cdot \sigma) = I + \sum_k (t_k + \lambda_k w_k) \sigma_k
\]

for CPT maps introduced in [15]. Conditions on the parameters \( t_k, \lambda_k \) which guarantee that \( \Phi \) is CPT are given in [17]; some special cases were considered earlier in [4].

### B.2 Useful formulas

We now restrict attention to CPT maps acting on \( A = I + z \cdot \sigma \) for which \( \Phi(A) = I + (t + Tw + iTu) \cdot \sigma \). Then

\[
A^\dagger A = (1 + |z|^2)I + 2(w + u \times w) \cdot \sigma
\]

with

\[
|w + u \times w| = |w|^2 + |w|^2 |u|^2 - (u \cdot w)^2.
\]

Therefore, the eigenvalues of \( A^\dagger A \) are \( 1 + |z|^2 \pm 2|w + u \times w| \) or, equivalently,

\[
1 + |z|^2 \pm 2\sqrt{|w|^2 + |w|^2 |u|^2 - (u \cdot w)^2}.
\]

and those of \( \Phi(A)^\dagger \Phi(A) \) are

\[
\phi_{\pm} = 1 + |t + Tw|^2 + |Tu|^2
\pm 2\sqrt{|t + Tw|^2(1 + |Tu|^2) - |(t + Tw) \cdot (Tu)|^2}.
\]

When \( (t + Tw) \cdot (Tu) = 0 \), (38) becomes \( (|t + Tw|^2 + \sqrt{1 + |Tu|^2})^2 \).

We now wish to evaluate and bound \( \|A\|_2^2 = (\text{Tr } |A|^2) \) Note that (37) implies that the eigenvalues of \( |A| = \sqrt{A^\dagger A} \) are

\[
\sqrt{1 + |z|^2 \pm 2\sqrt{|w|^2 + |w|^2 |u|^2 - (u \cdot w)^2},}
\]

and observe that their product can be written as

\[
(1 + |z|^2)^2 - 4(|w|^2 + |w|^2 |u|^2 - (u \cdot w)^2) = (1 - |w|^2 + |u|^2)^2 + 4(u \cdot w)^2.
\]
Therefore,

$$(\text{Tr} |A|)^2 = 2 \left( 1 + |z|^2 + \sqrt{(1 - |w|^2 + |u|^2)^2 + 4(u \cdot w)^2} \right)$$  \hspace{1cm} (40)$$

$$\geq 2 \left( 1 + |w|^2 + |u|^2 + |1 - |w|^2 + |u|^2| \right)$$  \hspace{1cm} (41)$$

$$\geq \begin{cases} 
4(1 + |u|^2) & \text{if } |w|^2 \leq 1 + |u|^2 \\
4|w|^2 & \text{if } |w|^2 > 1 + |u|^2 . 
\end{cases}$$  \hspace{1cm} (42)$$

**B.3 Equality for CPT maps when $p \geq 2$:**

We now show that $\|\Phi\|_{1\rightarrow p}^{R} = \|\Phi\|_{1\rightarrow p}$ for CPT maps on qubits when $p \geq 2$. For $A = I + z \cdot \sigma$ we prove the somewhat stronger result that

$$\frac{\|\Phi[I + (w + iu) \cdot \sigma]\|_p^2}{\|I + (w + iu) \cdot \sigma\|_1^2} \leq \frac{\|\Phi[I + \hat{w} \cdot \sigma]\|_p^2}{\|I + \hat{w} \cdot \sigma\|_1^2}$$  \hspace{1cm} (43)$$

where $\hat{w}$ is the unit vector defined by $w = |w| \hat{w}$. Our argument will use the following easily verified results. When $a \geq 0$ and $m \geq 1$,

$$f(x) = |x + a|^m + |x - a|^m \quad \text{is increasing for } x > 0$$  \hspace{1cm} (44)$$

$$g(x) = \frac{[f(x)]^{2/m}}{x^2} \quad \text{is decreasing for } x > 0.$$  \hspace{1cm} (45)$$

Note that $f(x)$ is symmetric in $x$ and $a$ from which it follows that the expression on the right in (44) is also increasing in $a$.

It follows from (42) and (38) that

$$\frac{\|\Phi(A)\|_p^2}{\|A\|_1^2} \leq \left( \frac{(\phi_+)^{p/2} + |\phi_-|^{p/2}}{4 \max(|w|^2, 1 + |u|^2)} \right)^{2/p}$$  \hspace{1cm} (46)$$

Since the numerator has the form of $f$ in (44) with $m = p/2$, the monotonicity in $a$ implies that dropping the dot product terms in (38) increases the right side of (46). Hence

$$\frac{\|\Phi(A)\|_p^2}{\|A\|_1^2} \leq \left( \frac{|t + Tw| + \sqrt{1 + |Tu|^2}}{4 \max(|w|^2, 1 + |u|^2)} \right)^{2/p}$$  \hspace{1cm} (47)$$

Since $|Tu| \leq |u|$, (47) again implies that the right side of (47) increases when $|Tu|$ is replaced by $|u|$ in the numerator. Also we can only increase the ratio
on the right side of (47) by choosing $t \cdot T \mathbf{w}$ to be positive. Therefore, we can conclude from (44) that this ratio is increasing in $|w|$ for $|w|^2 \leq 1 + |u|^2$, and from (45) that it is decreasing in $|w|$ for $|w|^2 \geq 1 + |u|^2$. Hence this ratio is maximized when $|w|^2 = 1 + |u|^2$. Therefore the ratio in (47) is less than

$$\frac{\left( (|t + T \mathbf{w}| + |w|)^p + |t + T \mathbf{w}| - |w|^p \right)^{2/p}}{4|w|^2},$$

(48)

which we want to show is smaller than the RHS of (43). Since $|w|^2 = 1 + |u|^2 \geq 1,$

$$|t + T \mathbf{w}| \leq |w| |t + T \mathbf{w}| = |w| |t + T \mathbf{w}|.$$  

(49)

Using (44) again to replace the $|t + T \mathbf{w}|$ term in (48), we find

$$\frac{\|\Phi(A)\|_p^2}{\|A\|_1^2} \leq \frac{\left( (|t + T \mathbf{w}| + |w|)^p + |t + T \mathbf{w}| - 1|^p \right)^{2/p}}{4}$$

(50)

$$= \frac{\|\Phi(I + \hat{\mathbf{w}} \cdot \sigma)\|_p^2}{\|I + \hat{\mathbf{w}} \cdot \sigma\|_1^2}$$

(51)

$$\leq (\|\Phi\|_{\mathcal{R}}^2 \nu_p(\Phi)^2).$$

(52)

We next consider the case $z_0 = 0$, for which $A = \mathbf{z} \cdot \sigma$, $|A| = |z|I$ and $\Phi(A) = (T \mathbf{z}) \cdot \sigma$. Then $\|\Phi(A)\|_p = |T \mathbf{z}|$ for all $p$ so that

$$\frac{\|\Phi(A)\|_p}{\|A\|_1} = \frac{\|\Phi(\mathbf{z} \cdot \sigma)\|_p}{\|\mathbf{z} \cdot \sigma\|_1} = \frac{|T \mathbf{z}|}{|\mathbf{z}|} \leq \max_k \lambda_k \leq \nu_p(\Phi)$$

(53)

where the last inequality follows from the fact that $[\nu_2(\Phi)]^2 \geq \frac{1}{2} (1 + |t + T \mathbf{w}|)^2 \geq |T \mathbf{w}|^2$ which can be made equal to $\max_k \lambda_k^2$ for some $\mathbf{w}$ with $|\mathbf{w}| = 1$.

To complete the proof, recall that for $z_0 \neq 0$, $\|z_0 I + \mathbf{z} \cdot \sigma\|_p = |z_0| \|I + \frac{1}{z_0} \mathbf{z} \cdot \sigma\|_p$ and note that the factor $|z_0|$ will cancel in any ratio of norms. Therefore, if we take the supremum over all complex matrices $A$, we can use (52) and (53) to conclude that $\|\Phi\|_{\mathcal{R}} \leq \nu_p(\Phi) = \|\Phi\|_{\mathcal{R}}$ when $p \geq 2$. The reverse inequality $\|\Phi\|_{\mathcal{R}} \leq \|\Phi\|_{\mathcal{R}}$ always holds; therefore, we must have equality for $p \geq 2$.

### B.4 Remarks

Suppose that both $\nu_p(\Phi) = \|\Phi\|_{\mathcal{R}}$ and $\|\Phi\|_{\mathcal{R}}$ are multiplicative for some $p, \Phi$. Suppose also that $\|\Phi\|_{\mathcal{R}} = \|\Phi\|_{\mathcal{R}}$ in dimension $d$ (e.g., $d = 2$.) Then in dimension $d^2$ (e.g., $d = 4$),

$$\|\Phi \otimes \Phi\|_{\mathcal{R}} = \left( \|\Phi\|_{\mathcal{R}} \right)^2 = \left( \|\Phi\|_{\mathcal{R}} \right)^2 = \|\Phi \otimes \Phi\|_{\mathcal{R}}.$$  

(54)
Thus, equality also holds in dimension $d^2$ for maps of the form $\Phi \otimes \Phi$. The argument in Section B.3 breaks down for $1 \leq p < 2$. Although one does not expect (43) to hold, the weaker inequality $\|\Phi(A)\|_p \leq \|A\|_1 \nu_1(\Phi)$ might still hold, and this is all that is needed to show $\|\Phi\|_{1 \to p} \leq \|\Phi\|_{R_{1 \to p}}$. However, even for $p = 1$, we have been unable to verify (or find a counter-example to) this.

For the general CP form (33a) with $s \neq 0$, $\nu_1(\Phi) = 1 + |s| > 1$ is achieved with $w = \frac{s}{|s|}$, and the eigenvalues of $\Phi(A)\Phi(A)^\dagger$ are

$$S^2 + |t + Tw|^2 + |Tu|^2 \pm 2\sqrt{|t + Tw|^2[(S^2 + |Tu|^2) - |(t + Tw) \cdot (Tu)|]^2}$$

with $S^2 = (1 + s \cdot w)^2 + (s \cdot u)^2$. If one tries to use the argument in the previous section, the RHS of (47) becomes

$$\left.\frac{((t + Tw) + \sqrt{S^2 + |Tu|^2})^p + |t + Tw| - \sqrt{S^2 + |Tu|^2}}{4 \max(|w|^2, 1 + |u|^2)}\right|^{2/p}.$$ (55)

This does not have the form $|x + a|^m + |x - a|^m$ because $a = \sqrt{S^2 + |Tu|^2}$ and $S^2$ depends on $w$.

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