Sobolev Duals for Random Frames and ΣΔ Quantization of Compressed Sensing Measurements

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Abstract

Quantization of compressed sensing measurements is typically justified by the robust recovery results of Candès, Romberg and Tao, and of Donoho. These results guarantee that if a uniform quantizer of step size $\delta$ is used to quantize $m$ measurements $y = \Phi x$ of a $k$-sparse signal $x \in \mathbb{R}^N$, where $\Phi$ satisfies the restricted isometry property, then the approximate recovery $x^\#$ via $\ell_1$-minimization is within $O(\delta)$ of $x$. The simplest and commonly assumed approach is to quantize each measurement independently. In this paper, we show that if instead an $r$th order ΣΔ quantization scheme with the same output alphabet is used to quantize $y$, then there is an alternative recovery method via Sobolev dual frames which guarantees a reduction of the approximation error by a factor of $(m/k)^{(r-1/2)\alpha}$ for any $0 < \alpha < 1$, if $m \gtrsim k \log N^{1/(1-\alpha)}$. The result holds with high probability on the initial draw of the measurement matrix $\Phi$ from the Gaussian distribution, and uniformly for all $k$-sparse signals $x$ that satisfy a mild size condition on their supports.

1 Introduction

Compressed sensing is concerned with when and how sparse signals can be recovered exactly or approximately from few linear measurements [8, 10, 14]. Let $\Phi$ be an $m \times N$ matrix providing the measurements where $m \ll N$, and $\Sigma^N_k$ denote the space of $k$-sparse signals in $\mathbb{R}^N$, $k < m$. A standard objective, after a suitable change of basis, is that the mapping $x \mapsto y = \Phi x$ be injective on $\Sigma^N_k$. Minimal conditions on $\Phi$ that offer such a guarantee are well-known (see, e.g. [11]) and require at least that $m \geq 2k$. On the other hand, under stricter conditions on $\Phi$, such as the restricted isometry property (RIP), one can recover sparse vectors from their measurements by numerically efficient methods, such as $\ell^1$-minimization. Moreover, the recovery will also be robust when the measurements are

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corrupted \[9\], cf. \[15\]; if \(\hat{y} = \Phi x + \epsilon\) where \(\epsilon\) is any vector such that \(\|\epsilon\|_2 \leq \epsilon\), then the solution \(x^\#\) of the optimization problem

\[
\min \|z\|_1 \text{ subject to } \|\Phi z - \hat{y}\|_2 \leq \epsilon
\]

will satisfy \(\|x - x^\#\|_2 \lesssim \epsilon\).

The price paid for these stronger recovery guarantees is the somewhat smaller range of values available for the dimensional parameters \(m, k,\) and \(N\). While there are some explicit (deterministic) constructions of measurement matrices with stable recovery guarantees, best results (widest range of values) have been found via random families of matrices. For example, if the entries of \(\Phi\) are independently sampled from the Gaussian distribution \(\mathcal{N}(0, \frac{1}{m})\), then with high probability, \(\Phi\) will satisfy the RIP (with a suitable set of parameters) if \(m \sim k \log(\frac{N}{k})\). Significant effort has been put on understanding the phase transition behavior of the RIP parameters for other random families, e.g., Bernoulli matrices and random Fourier samplers.

**Quantization for compressed sensing measurements**

The robust recovery result mentioned above is essential to the practicality of compressed sensing, especially from an analog-to-digital conversion point of view. If a discrete alphabet \(\mathcal{A}\), such as \(\mathcal{A} = \delta \mathbb{Z}\) for some step size \(\delta > 0\), is to be employed to replace each measurement \(y_j\) with a quantized measurement \(q_j := \hat{y}_j \in \mathcal{A}\), then the temptation, in light of this result, would be to minimize \(\|e\|_2 = \|y - q\|_2\) over \(q \in \mathcal{A}^m\). This immediately reduces to minimizing \(|y_j - q_j|\) for each \(j\), i.e., quantizing each measurement separately to the nearest element of \(\mathcal{A}\), which is usually called Pulse Code Modulation (PCM).

Since \(\|y - q\|_2 \leq \frac{1}{2}\delta\sqrt{m}\), the robust recovery result guarantees that

\[
\|x - x^\#_{\text{PCM}}\|_2 \lesssim \delta\sqrt{m}.
\]

Note that \((2)\) is somewhat surprising as the reconstruction error bound does not improve by increasing the number of (quantized) measurements; on the contrary, it deteriorates. However, the \(\sqrt{m}\) term is an artifact of our choice of normalization for the measurement matrix \(\Phi\). In the compressed sensing literature, it is conventional to normalize a (random) measurement matrix \(\Phi\) so that it has unit-norm columns (in expectation). This is the necessary scaling to achieve isometry, and for random matrices it ensures that \(\mathbb{E}\|\Phi x\|^2 = \|x\|^2\) for any \(x\), which then leads to the RIP through concentration of measure and finally to the robust recovery result stated in \((1)\). On the other hand, this normalization imposes an \(m\)-dependent dynamic range for the measurements which scales as \(1/\sqrt{m}\), hence it is not fair to use the same value \(\delta\) for the quantizer resolution as \(m\) increases. In this paper, we investigate the dependence of the recovery error on the number of quantized measurements where \(\delta\) is independent of \(m\). A fair assessment of this dependence can be made only if the dynamic range of each measurement is kept constant while increasing the number of measurements. This suggests that the natural normalization in our setting should ensure that the entries of the measurement matrix \(\Phi\) are independent of \(m\). In the specific case of random matrices, we
can achieve this by choosing the entries of $\Phi$ standard i.i.d. random variables, e.g. according to $\mathcal{N}(0, 1)$. With this normalization of $\Phi$, the robust recovery result of [9], given above, can be modified as

$$\|\hat{y} - y\|_2 \leq \epsilon \implies \|x - x^\#\|_2 \lesssim \frac{1}{\sqrt{m}} \epsilon,$$

which also replaces (2) with

$$\|x - x^\#_{\text{PCM}}\|_2 \lesssim \delta. \tag{4}$$

As expected, this error bound does not deteriorate with $m$ anymore. In this paper, we will adopt this normalization convention and work with the standard Gaussian distribution $\mathcal{N}(0, 1)$ when quantization is involved, but also use the more typical normalization $\mathcal{N}(0, 1/m)$ for certain concentration estimates that will be derived in Section 3. The transition between these two conventions is of course trivial.

The above analysis of quantization error is based on PCM, which involves separate (independent) quantization of each measurement. The vast logarithmic reduction of the ambient dimension $N$ would seem to suggest that this strategy is essentially optimal since information appears to be squeezed (compressed) into few uncorrelated measurements. Perhaps for this reason, the existing literature on quantization of compressed sensing measurements focused mainly on alternative reconstruction methods from PCM-quantized measurements and variants thereof, e.g., [6, 12, 16, 20, 22, 29]. The only exception we are aware of is [7], which uses $\Sigma\Delta$ modulation to quantize $x$ before the random measurements are made.

On the other hand, it is clear that if (once) the support of the signal is known (recovered), then the $m$ measurements that have been taken are highly redundant compared to the maximum $k$ degrees of freedom that the signal has on its support. At this point, the signal may be considered oversampled. However, the error bound (4) does not offer an improvement of reconstruction accuracy, even if additional samples become available. (The RIP parameters of $\Phi$ are likely to improve as $m$ increases, but this does not seem to reflect on the implicit constant factor in (4) satisfactorily.) This is contrary to the conventional wisdom in the theory and practice of oversampled quantization in A/D conversion where reconstruction error decreases as the sampling rate increases, especially with the use of quantization algorithms specially geared for the reconstruction procedure. The main goal of this paper is to show how this can be done in the compressed sensing setting as well.

**Quantization for oversampled data**

Methods of quantization have long been studied for oversampled data conversion. Sigma-delta ($\Sigma\Delta$) quantization (modulation), for instance, is the dominant method of A/D conversion for audio signals and relies heavily on oversampling, see [13, 18, 24]. In this setting, oversampling is typically exploited to employ very coarse quantization (e.g., 1 bit/sample), however, the working principle of $\Sigma\Delta$ quantization is applicable to any quantization alphabet. In fact, it is more natural to consider $\Sigma\Delta$ quantization as a “noise shaping” method,

\footnote{The quantization error is often modeled as white noise in signal processing, hence the terminology. However our treatment of quantization error in this paper is entirely deterministic.}
for it seeks a quantized signal \((q_j)\) by a recursive procedure to push the quantization error signal \(y - q\) towards an unoccupied portion of the signal spectrum. In the case of bandlimited signals, this would correspond to high frequency bands.

As the canonical example, the standard first-order \(\Sigma\Delta\) quantizer computes a bounded solution \((u_j)\) to the difference equation

\[
(\Delta u)_j := u_j - u_{j-1} = y_j - q_j.
\]

(5)

This can be achieved recursively by choosing, for example,

\[
q_j = \arg \min_{p \in A} |u_{j-1} + y_j - p|.
\]

(6)

Since the reconstruction of oversampled bandlimited signals can be achieved with a low-pass filter \(\varphi\) that can also be arranged to be well-localized in time, the reconstruction error \(\varphi \ast (y - q) = \Delta \varphi \ast u\) becomes small due to the smoothness of \(\varphi\). It turns out that, with this procedure, the reconstruction error is reduced by a factor of the oversampling ratio \(\lambda\), defined to be the ratio of the actual sampling rate to the bandwidth of \(\varphi\).

This principle can be iterated to set up higher-order \(\Sigma\Delta\) quantization schemes. It is well-known that a reconstruction accuracy of order \(O(\lambda^{-r})\) can be achieved (in the supremum norm) if a bounded solution to the equation \(\Delta^r u = y - q\) can be found [13] (here, \(r \in \mathbb{N}\) is the order of the associated \(\Sigma\Delta\) scheme). The boundedness of \(u\) is important for practical implementation, but it is also important for the error bound. The implicit constant in this bound depends on \(r\) as well as \(\|u\|_\infty\). Fine analyses of carefully designed schemes have shown that optimizing the order can even yield exponential accuracy \(O(e^{-c\lambda})\) for fixed sized finite alphabets \(A\) (see [18]), which is optimal apart from the value of the constant \(c\). For infinite alphabets, there is no theoretical lower bound for the quantization error as \(\lambda\) increases. (However almost all practical coding schemes use some form of finite alphabet.)

The above formulation of noise-shaping for oversampled data conversion generalizes naturally to the problem of quantization of arbitrary frame expansions, e.g., [3]. Specifically, we will consider finite frames in \(\mathbb{R}^k\). Let \(E\) be a full-rank \(m \times k\) matrix and \(F\) be any left inverse of \(E\). In frame theory, one refers to the collection of the rows of \(E\) as the analysis frame and the columns of \(F\) as the synthesis (dual) frame. For any \(x \in \mathbb{R}^k\), let \(y = Ex\) be its frame coefficient vector, \(q \in A^m\) be its quantization, and let \(\hat{x} := Fq\) be its reconstruction using the dual frame. Typically \(A^m \cap y + \text{Ker}(F) = \emptyset\), so we have \(\hat{x} \neq x\). The reconstruction error is given by

\[
x - \hat{x} = F(y - q),
\]

(7)

and the goal of noise shaping amounts to arranging \(q\) in such a way that \(y - q\) is close to \(\text{Ker}(F)\).

If the sequence \((f_j)_1^m\) of dual frame vectors were known to vary smoothly in \(j\) (including smooth termination into null vector), then \(\Sigma\Delta\) quantization could be employed without much alteration, e.g., [19][21]. However, this need not be the case for many examples of frames (together with their canonical duals) that are used in practice. For this reason, it has recently been proposed in [4] to use special alternative dual frames, called Sobolev dual
frames, that are naturally adapted to $\Sigma\Delta$ quantization. It is shown in [4] (see also Section 2) that for any frame $E$, if a standard $r$th order $\Sigma\Delta$ quantization algorithm with alphabet $A = \delta\mathbb{Z}$ is used to compute $q := q_{\Sigma\Delta}$, then with an $r$th order Sobolev dual frame $F := F_{\text{Sob},r}$ and $\hat{x}_{\Sigma\Delta} := F_{\text{Sob},r}q_{\Sigma\Delta}$, the reconstruction error obeys the bound
\[ \|x - \hat{x}_{\Sigma\Delta}\|_2 \lesssim_r \frac{\delta\sqrt{m}}{\sigma_{\min}(D^{-r}E)}, \] (8)
where $D$ is the $m \times m$ difference matrix defined by
\[ D_{ij} := \begin{cases} 1, & \text{if } i = j, \\ -1, & \text{if } i = j + 1, \\ 0, & \text{otherwise}, \end{cases} \] (9)
and $\sigma_{\min}(D^{-r}E)$ stands for the smallest singular value of $D^{-r}E$.

**Contributions**

For the compressed sensing application that is the subject of this paper, $E$ will simply be a sub-matrix of the measurement matrix $\Phi$, hence it may have been found by sampling an i.i.d. random variable. Minimum singular values of random matrices with i.i.d. entries have been studied extensively in the mathematical literature. For an $m \times k$ random matrix $E$ with i.i.d. entries sampled from a sub-Gaussian distribution with zero mean and unit variance\(^b\), one has
\[ \sigma_{\min}(E) \geq \sqrt{m} - \sqrt{k} \] (10)
with high probability \(^{26}\). Note that in general $D^{-r}E$ would not have i.i.d. entries. A naive lower bound for $\sigma_{\min}(D^{-r}E)$ would be $\sigma_{\min}(D^{-r})\sigma_{\min}(E)$. However (see Proposition 3.1), $\sigma_{\min}(D^{-r})$ satisfies
\[ \sigma_{\min}(D^{-r}) \asymp_r 1, \] (11)
and therefore this naive product bound yields no improvement on the reconstruction error for $\Sigma\Delta$-quantized measurements over the bound (4) for PCM-quantized ones. In fact, the true behavior of $\sigma_{\min}(D^{-r}E)$ turns out to be drastically different and is described in Theorem A, one of our main results (see also Theorem 3.7).

For simplicity, we shall work with standard i.i.d. Gaussian variables for the entries of $E$. In analogy with our earlier notation, we define the “oversampling ratio” $\lambda$ of the frame $E$ by
\[ \lambda := \frac{m}{k}. \] (12)

**Theorem A.** Let $E$ be an $m \times k$ random matrix whose entries are i.i.d. $\mathcal{N}(0,1)$. For any $\alpha \in (0,1)$, if $\lambda \geq c(\log m)^{1/(1-\alpha)}$, then with probability at least $1 - \exp(-c'm\lambda^{-\alpha})$,
\[ \sigma_{\min}(D^{-r}E) \gtrsim_r \lambda^{\alpha(r-\frac{1}{2})}\sqrt{m}, \] (13)
\(^b\) As mentioned earlier, we do not normalize the measurement matrix $\Phi$ in the quantization setting.
which yields the reconstruction error bound
\[ \|x - \hat{x}_{\Sigma\Delta}\|_2 \lesssim r \lambda^{-\alpha(r-\frac{1}{2})}\delta. \] (14)

While the kind of decay in this error bound is familiar to ΣΔ modulation, the domain of applicability of this result is rather surprising. Previously, the only setting in which this type of approximation accuracy could be achieved (with or without Sobolev duals) was the case of highly structured frames (e.g. when the frame vectors are found by sampling along a piecewise smooth frame path). Theorem A shows that such an accuracy is obtained even when the analysis frame is a random Gaussian matrix, provided the reconstruction is done via Sobolev duals.

In the compressed sensing setting, one needs (13) to be uniform for all the frames \( E \) that are found by selecting \( k \) columns of \( \Phi \) at a time. The proof of Theorem A extends in a straightforward manner using a standard “union bound” argument, provided \( \lambda \) is known to be slightly larger. More precisely, if \( \Phi \) is an \( m \times N \) matrix whose entries are i.i.d. according to \( \mathcal{N}(0,1) \), and if \( \lambda := m/k \geq c(\log N)^{1/(1-\alpha)} \), then (13) holds for all \( E = \Phi_T \) with \( \#T \leq k \) with the same type of probability bound (with new constants). This result can be utilized to improve the reconstruction accuracy of a sparse signal \( x \) from its ΣΔ-quantized compressed sensing measurements if the support \( T \) of \( x \) is known. This is because if \( T \) is known, \( \Phi_T \) is known, and its Sobolev dual can be found and used in the reconstruction. On the other hand, for most signals, recovering the exact or approximate support is already nearly guaranteed by the robust recovery result shown in (3) together with the stability of the associated ΣΔ quantizer. For example, a simple sufficient condition for full recovery of the support is that all the \( |x_j| \) for \( j \in T \) be larger than \( C\|y - q_{\Sigma\Delta}\|_2 \) for a suitable constant \( C \). A precise version of this condition is stated in Theorem B.

In light of all these results, we propose ΣΔ quantization as a more effective alternative of PCM (independent quantization) for compressed sensing. With high probability on the measurement matrix, a significant improvement of the reconstruction accuracy of sparse signals can be achieved through a two-stage recovery procedure:

1. **Coarse recovery:** \( \ell_1 \)-minimization (or any other robust recovery procedure) applied to \( q_{\Sigma\Delta} \) yields an initial, “coarse” approximation \( x# \) of \( x \), and in particular, the exact (or approximate) support \( T \) of \( x \).

2. **Fine recovery:** Sobolev dual of the frame \( \Phi_T \) applied to \( q_{\Sigma\Delta} \) yields a finer approximation \( \hat{x}_{\Sigma\Delta} \) of \( x \).

Combining all these, our second main theorem follows (also see Theorem 4.2):

**Theorem B.** Let \( \Phi \) be an \( m \times N \) matrix whose entries are i.i.d. according to \( \mathcal{N}(0,1) \). Suppose \( \alpha \in (0,1) \) and \( \lambda := m/k \geq c(\log N)^{1/(1-\alpha)} \) where \( c = c(r,\alpha) \). Then there are two constants \( c' \) and \( C' \) that depend only on \( r \) such that with probability at least \( 1 - \exp(-c'm\lambda^{-\alpha}) \) on the draw of \( \Phi \), the following holds: For every \( x \in \Sigma_k^N \) such that \( \min_{j \in \text{supp}(x)} |x_j| \geq C\delta \), the reconstruction \( \hat{x}_{\Sigma\Delta} \) satisfies
\[ \|x - \hat{x}_{\Sigma\Delta}\|_2 \lesssim r \lambda^{-\alpha(r-\frac{1}{2})}\delta. \] (15)
To put this result in perspective, note that the approximation error given in (15) decays as the “redundancy” $\lambda = \frac{m}{k}$ increases. In fact, by using an arbitrarily high order $\Sigma\Delta$ scheme, we can make this decay faster than any power law (albeit with higher constants). Note that such a decay is not observed in the reconstruction error bound for PCM given in (4). Of course, one could argue that these upper bounds may not reflect the actual behavior of the error. However, in the setting of frame quantization the performance of PCM is well investigated. In particular, let $E$ be an $m \times k$ real matrix, and let $K$ be a bounded set in $\mathbb{R}^k$. For $x \in K$, suppose we obtain $q_{PCM}(x)$ by quantizing the entries of $y = Ex$ using PCM with alphabet $A = \delta \mathbb{Z}$. Let $\Delta_{opt}$ be an optimal decoder. Then, Goyal et al. show in [17] that

$$\mathbb{E} \| x - \Delta_{opt}(q_{PCM}(x)) \|^2 \geq \frac{1}{\lambda} \delta$$

where $\lambda = m/k$ and the expectation is with respect a probability measure on $x$ that is, for example, absolutely continuous. This lower bound limits the extent to which one can improve the reconstruction by means of alternative reconstruction algorithms from PCM-quantized compressed sensing measurements. On the other hand, setting, for example, $\alpha = 3/4$ in Theorem 3 we observe that if we use a second-order $\Sigma\Delta$ scheme to quantize the measurements, and if we adopt the two-stage recovery procedure proposed above, the resulting approximation will be superior to that produced optimally from PCM-quantized measurements, provided $m/k$ is sufficiently large.

It is possible to imagine more sophisticated and more effective quantization and recovery algorithms for compressed sensing. However using $\Sigma\Delta$ quantization has a number of appealing features:

- It produces more accurate approximations than any known quantization scheme in this setting (even when sophisticated recovery algorithms are employed).
- It is modular in the sense that if the fine recovery stage is not available or practical to implement, then the standard (coarse) recovery procedure can still be applied as is.
- It is progressive in the sense that if new measurements arrive (in any given order), noise shaping can be continued on these measurements as long as the state of the system ($r$ real values for an $r$th order scheme) has been stored.
- It is universal in the sense that it uses no information about the measurement matrix or the signal.

The paper is organized as follows. We review the basics of $\Sigma\Delta$ quantization and Sobolev duals in frame theory in Section 2, followed by the reconstruction error bounds for random Gaussian frames in Section 3. We then present the specifics of our proposed quantization and recovery algorithm for compressed sensing in Section 4. We present our numerical experiments in Section 5 and conclude with extensions to more general settings in Section 6.
2 Background on ΣΔ quantization of frame expansions

ΣΔ quantization

The governing equation of a standard \( r \)th order ΣΔ quantization scheme with input \( y = (y_j) \) and output \( q = (q_j) \) is

\[
(\Delta^r u)_j = y_j - q_j, \quad j = 1, 2, \ldots, \quad (16)
\]

where the \( q_j \in \mathcal{A} \) are chosen according to some quantization rule given by

\[
q_j = Q(u_{j-1}, \ldots, u_{j-r}, y_j, \ldots, y_{j-s}). \quad (17)
\]

Not all ΣΔ quantization schemes are presented (or implemented) in this canonical form, but they all can be rewritten as such for an appropriate choice of \( r \) and \( u \). We shall not be concerned with the specifics of the mapping \( Q \), except that we need \( u \) to be bounded. The smaller the size of the alphabet \( \mathcal{A} \) gets relative to \( r \), the harder it is to guarantee this property. The extreme case is 1-bit quantization, i.e., \(|\mathcal{A}| = 2\), which is typically the most challenging setting. We will not be working in this case. In fact, for our purposes, \( \mathcal{A} \) will in general have to be sufficiently fine to allow for the recovery of the support of sparse signals. In order to avoid technical difficulties, we shall work with the infinite alphabet \( \mathcal{A} = \delta \mathbb{Z} \), but also note that only a finite portion of this alphabet will be used for bounded signals. A standard quantization rule that has this “boundedness” property is given by the greedy rule which minimizes \(|u_j|\) given \( u_{j-1}, \ldots, u_{j-r} \) and \( y_j \), i.e.,

\[
q_j = \arg\min_{a \in \mathcal{A}} \left| \sum_{i=1}^{r} (-1)^{i-1} \binom{r}{i} u_{j-i} + y_j - a \right|. \quad (18)
\]

It is easy to check that with this rule, one has \(|u_j| \leq 2^{-1} \delta \) and \(|y_j - q_j| \leq 2^{r-1} \delta \). In turn, if \( \|y\|_{\infty} < C \), then one needs only \( L := 2 \lceil \frac{C}{\delta} \rceil + 2^r + 1 \) levels. In this case, the associated quantizer is said to be \( \log_2 L \)-bit, and we have

\[
\|u\|_{\infty} \lesssim \delta \text{ and } \|y - q\|_{\infty} \lesssim_r \delta. \quad (19)
\]

With more stringent quantization rules, the first inequality would also have an \( r \)-dependent constant. In fact, it is known that for quantization rules with a 1-bit alphabet, this constant will be as large as \( O(r^r) \), e.g., see [13][18]. In this paper, unless otherwise stated, we shall be working with the greedy quantization rule of \( (18) \).

The initial condition of the recursion in \( (16) \) can be set arbitrarily, but it will be convenient for us to set them equal to zero for finite frames. With \( u_{-r+1} = \cdots = u_0 = 0 \), and \( j = 1, \ldots, m \), the difference equation \( (16) \) can be rewritten as a matrix equation

\[
D^r u = y - q, \quad (20)
\]

where \( D \) is as in \( (9) \).

As before, we assume \( E \) is an \( m \times k \) matrix whose rows form the analysis frame and \( F \) is a \( k \times m \) left inverse of \( E \) whose columns form the dual (synthesis) frame. Given any
we set $y = Ex$, and define its $r$th order $\Sigma\Delta$ quantization $q_{\Sigma\Delta}$ and its reconstruction $\hat{x}_{\Sigma\Delta} := Fq_{\Sigma\Delta}$. Substituting (20) into (7), we obtain the error expression

$$x - \hat{x} = FD^r u. \quad (21)$$

With this expression, $\|x - \hat{x}\|$ can be bounded for any norm $\|\cdot\|$ simply as

$$\|x - \hat{x}\| \leq u_\infty \sum_{j=1}^m \|(FD^r)_j\|. \quad (22)$$

Here $(FD^r)_j$ is the $j$th column of $FD^r$. This bound is also valid in infinite dimensions, and in fact has been used extensively in the mathematical treatment of oversampled A/D conversion of bandlimited functions.

For $r = 1$, and the $\ell_2$ norm, the sum term on the right hand side motivated the study of the so-called frame variation defined by

$$V(F) := \sum_{j=1}^m \|f_j - f_{j+1}\|_2, \quad (23)$$

where $(f_j)$ are the columns of $F$, and one defines $f_{m+1} = 0$. Higher-order frame variations to be used with higher-order $\Sigma\Delta$ schemes are defined similarly, see [2,3]. Frames (analysis as well as synthesis) that are obtained via uniform sampling a smooth curve in $\mathbb{R}^k$ (so-called frame path) are typical in many settings. However, the “frame variation bound” is useful in finite dimensions when the frame path terminates smoothly. Otherwise, it does not provide higher-order reconstruction accuracy. Designing smoothly terminating frames can be technically challenging, e.g., [5].

**Sobolev duals**

Recently, a more straightforward approach was proposed in [21] for the design of (alternate) duals of finite frames for $\Sigma\Delta$ quantization. Here, one instead considers the operator norm of $FD^r$ on $\ell_2$ and the corresponding bound

$$\|x - \hat{x}\|_2 \leq \|FD^r\|_{op}\|u\|_2. \quad (24)$$

Note that this bound is not available in the infinite dimensional setting of bandlimited functions due to the fact that $u$ is typically not in $\ell_2$. It is now natural to minimize $\|FD^r\|_{op}$ over all dual frames of a given analysis frame $E$. These frames, introduced in [4], have been called Sobolev duals, in analogy with $\ell_2$-type Sobolev (semi)norms.

$\Sigma\Delta$ quantization algorithms are normally designed for analog circuit operation, so they control $\|u\|_\infty$, which would control $\|u\|_2$ only in a suboptimal way. However, it turns out that there are important advantages in working with the $\ell_2$ norm in the analysis. The first advantage is that Sobolev duals are readily available by an explicit formula. The solution $F_{\text{sub},r}$ of the optimization problem

$$\min_F \|FD^r\|_{op} \text{ subject to } FE = I \quad (25)$$

...
is given by the matrix equation

\[ F_{\text{sob},r} D^r = (D^{-r}E)^\dagger, \]  

(26)

where \( ^\dagger \) stands for the Moore-Penrose inversion operator, which, in our case, is given by \( E^\dagger := (E^*E)^{-1}E^* \). Note that for \( r = 0 \) (i.e., no noise-shaping, or PCM), one simply obtains \( F = E^\dagger \), the canonical dual frame of \( E \).

The second advantage of this approach is that highly developed methods are present for spectral norms of matrices, especially in the random setting. Plugging (26) into (24), it immediately follows that

\[ \|x - \hat{x}\|_2 \leq \|(D^{-r}E)^\dagger\|_{\text{op}} \|u\|_2 = \frac{1}{\sigma_{\text{min}}(D^{-r}E)} \|u\|_2, \]  

(27)

where \( \sigma_{\text{min}}(D^{-r}E) \) stands for the smallest singular value of \( D^{-r}E \).

3 Reconstruction error bound for random frames

In what follows, \( \sigma_j(A) \) will denote the \( j \)th largest singular value of the matrix \( A \). Similarly, \( \lambda_j(B) \) will denote the \( j \)th largest eigenvalue of the Hermitian matrix \( B \). Hence, we have \( \sigma_j(A) = \sqrt{\lambda_j(A^*A)} \). We will also use the notation \( \Sigma(A) \) for the diagonal matrix of singular values of \( A \), with the convention \( (\Sigma(A))_{jj} = \sigma_j(A) \). All matrices in our discussion will be real valued and the Hermitian conjugate reduces to the transpose.

We have seen that the main object of interest for the reconstruction error bound is \( \sigma_{\text{min}}(D^{-r}E) \) for a random frame \( E \). Let \( H \) be a square matrix. The first observation we make is that when \( E \) is i.i.d. Gaussian, the distribution of \( \Sigma(HE) \) is the same as the distribution of \( \Sigma(\Sigma(H)E) \). To see this, let \( U\Sigma(H)V^* \) be the singular value decomposition of \( H \) where \( U \) and \( V \) are unitary matrices. Then \( HE = U\Sigma(H)V^*E \). Since the unitary transformation \( U \) does not alter singular values, we have \( \Sigma(HE) = \Sigma(\Sigma(H)V^*E) \), and because of the unitary invariance of the i.i.d. Gaussian measure, the matrix \( \tilde{E} := V^*E \) has the same distribution as \( E \), hence the claim. Therefore it suffices to study the singular values of \( \Sigma(H)E \). In our case, \( H = D^{-r} \) and we first need information on the deterministic object \( \Sigma(D^{-r}) \). The following result will be sufficient for our purposes:

**Proposition 3.1.** Let \( r \) be any positive integer and \( D \) be as in (9). There are positive numerical constants \( c_1(r) \) and \( c_2(r) \), independent of \( m \), such that

\[ c_1(r)\left(\frac{m}{j}\right)^r \leq \sigma_j(D^{-r}) \leq c_2(r)\left(\frac{m}{j}\right)^r, \quad j = 1, \ldots, m. \]  

(28)

The proof of this result is rather standard in the study of Toeplitz matrices, and is given in Appendix A.
3.1 Lower bound for $\sigma_{\text{min}}(D^{-r}E)$

In light of the above discussion, the distribution of $\sigma_{\text{min}}(D^{-r}E)$ is the same as that of

$$\inf_{\|x\|_2=1} \|\Sigma(D^{-r})Ex\|_2. \tag{29}$$

We replace $\Sigma(D^{-r})$ with an arbitrary diagonal matrix $S$ with $S_{jj} =: s_j > 0$. The first two results will concern upper bounds for the norm of independent but non-identically distributed Gaussian vectors. They are rather standard, but we include them for the definiteness of our discussion when they will be used later.

**Proposition 3.2.** Let $\xi \sim \mathcal{N}(0, \frac{1}{m}I_m)$. For any $\Theta > 1$,

$$\Pr \left( \sum_{j=1}^m s_j^2 \xi_j^2 > \Theta \|s\|_\infty^2 \right) \leq \Theta^{m/2} e^{-(\Theta-1)m/2}. \tag{30}$$

**Proof.** Since $s_j \leq \|s\|_\infty$ for all $j$, we have

$$\Pr \left( \sum_{j=1}^m s_j^2 \xi_j^2 > \Theta \|s\|_\infty^2 \right) \leq \Pr \left( \sum_{j=1}^m \xi_j^2 > \Theta \right). \tag{31}$$

This bound is the (standard) Gaussian measure of the complement of a sphere of radius $\sqrt{m\Theta}$ and can be estimated very accurately. We use a simple approach via

$$\Pr \left( \sum_{j=1}^m \xi_j^2 > \Theta \right) \leq \min_{\lambda \geq 0} \int_{\mathbb{R}^m} e^{-\left(\Theta - \sum_{j=1}^m x_j^2\right) \lambda/2} \prod_{j=1}^m e^{-mx_j^2/2} \frac{dx_j}{\sqrt{2\pi/m}}$$

$$= \min_{\lambda \geq 0} e^{-\lambda\Theta/2 (1 - \lambda/m)^{-m/2}}$$

$$= \Theta^{m/2} e^{-(\Theta-1)m/2}, \tag{32}$$

where in the last step we set $\lambda = m(1 - \Theta^{-1})$. □

**Lemma 3.3.** Let $E$ be an $m \times k$ random matrix whose entries are i.i.d. $\mathcal{N}(0, \frac{1}{m})$. For any $\Theta > 1$, consider the event

$$\mathcal{E} := \left\{ \|SE\|_{\ell^2_\Theta \rightarrow \ell^2_t} \leq 2\sqrt{\Theta}\|s\|_\infty \right\}.$$

Then

$$\Pr(\mathcal{E}^c) \leq 5^k \Theta^{m/2} e^{-(\Theta-1)m/2}.$$

**Proof.** We follow the same approach as in [1]. The maximum number of $\rho$-distinguishable points on the unit sphere in $\mathbb{R}^k$ is at most $(\frac{\rho}{2} + 1)^k$. (This follows by a volume argument.)

---

*Ball with radii $\rho/2$ and centers at a $\rho$-distinguishable set of points on the unit sphere are mutually disjoint and are all contained in the ball of radius $1 + \rho/2$ centered at the origin. Hence there can be at most $(1 + \rho/2)^k/(\rho/2)^k$ of them.*
as in e.g., [23, p.487].) Fix a maximal set $Q$ of $\frac{1}{2}$-distinguishable points of the unit sphere in $\mathbb{R}^k$ with $\#Q \leq 5^k$. Since $Q$ is maximal, it is a $\frac{1}{2}$-net for the unit sphere. For each $q \in Q$, consider $\xi_j = (Eq)_j$, $j = 1, \ldots, m$. Then $\xi \sim \mathcal{N}(0, \frac{1}{m}I_m)$. As before, we have $$\|SEq\|_2^2 = \sum_{j=1}^{m} s_j^2 \xi_j^2.$$ Let $\mathcal{E}(Q)$ be the event $\left\{ \|SEq\|_2 \leq \sqrt{\Theta} \|s\|_\infty, \ \forall q \in Q \right\}$. Then, by Proposition 3.2, we have the union bound

$$P(\mathcal{E}(Q)^c) \leq 5^k \Theta^{m/2} e^{-(\Theta-1)m/2}. \quad (33)$$

Assume the event $\mathcal{E}(Q)$, and let $M = \|SE\|_{\ell_2^k \rightarrow \ell_2^m}$. For each $\|x\|_2 = 1$, there is $q \in Q$ with $\|q - x\|_2 \leq 1/2$, hence $$\|SEx\|_2 \leq \|SEq\|_2 + \|SE(x - q)\|_2 \leq \sqrt{\Theta} \|s\|_\infty + \frac{M}{2}.$$ Taking the supremum over all $x$ on the unit sphere, we obtain $$M \leq \sqrt{\Theta} \|s\|_\infty + \frac{M}{2},$$ i.e., $\|SE\|_{\ell_2^k \rightarrow \ell_2^m} \leq 2\sqrt{\Theta} \|s\|_\infty$. Therefore $\mathcal{E}(Q) \subset \mathcal{E}$, and the result follows.

The following estimate concerns a lower bound for the Euclidean norm of $(s_1 \xi_1, \ldots, s_m \xi_m)$. It is not sharp when the $s_j$ are identical, but it will be useful for our problem where $s_j = \sigma_j(D^{-r})$ obey a power law (see Corollary 3.5).

**Proposition 3.4.** Let $\xi \sim \mathcal{N}(0, \frac{1}{m}I_m)$. For any $\gamma > 0$,

$$P \left( \sum_{j=1}^{m} s_j^2 \xi_j^2 < \gamma \right) \leq \min_{1 \leq L \leq m} \left( \frac{e^{\gamma m}}{L} \right)^{L/2} \left( s_1 s_2 \cdots s_L \right)^{-1}. \quad (34)$$

**Proof.** For any $t \geq 0$ and any integer $L \in \{1, \ldots, m\}$, we have

$$P \left( \sum_{j=1}^{m} s_j^2 \xi_j^2 < \gamma \right) \leq \int_{\mathbb{R}^m} e^{(\gamma - \sum_{j=1}^{m} s_j^2 \xi_j^2) t/2} \prod_{j=1}^{m} e^{-m \xi_j^2 / 2} \frac{dx_j}{\sqrt{2\pi/m}}$$

$$= e^{\gamma t/2} \prod_{j=1}^{m} \int_{\mathbb{R}} e^{-x_j^2 / 2} \frac{dx_j}{\sqrt{2\pi/m}}$$

$$= e^{\gamma t/2} \prod_{j=1}^{m} (1 + ts_j^2 / m)^{-1/2}$$

$$\leq e^{\gamma t/2} \prod_{j=1}^{L} (ts_j^2 / m)^{-1/2}$$
\[ \leq e^{\frac{\gamma}{2}} (m/t)^{L/2} (s_1 s_2 \cdots s_L)^{-1}. \]  

(35)

For any \( L \), we can set \( t = L/\gamma \), which is the critical point of the function \( t \mapsto e^{\gamma t} t^{-L} \). Since \( L \) is arbitrary, the result follows.

\[ \square \]

**Corollary 3.5.** Let \( \xi \sim \mathcal{N}(0, \frac{1}{m}1_m) \), \( r \) be a positive integer, and \( c_1 > 0 \) be such that

\[ s_j \geq c_1 \left( \frac{m}{j} \right)^r, \quad j = 1, \ldots, m. \]

(36)

Then for any \( \Lambda \geq 1 \) and \( m \geq \Lambda \),

\[ \mathbb{P} \left( \sum_{j=1}^{m} s_j^2 \xi_j^2 < c_1^2 \Lambda^{r-1} \right) < (60m/\Lambda)^{r/2} e^{-m(r-1/2)/\Lambda}. \]

(37)

**Proof.** By rescaling \( s_j \), we can assume \( c_1 = 1 \). For any \( L \in \{1, \ldots, m\} \), we have

\[ (s_1 s_2 \cdots s_L)^{-1} \leq (L)^r \left( \frac{m}{e^r m^r} \right)^{L}, \]

where we have used the coarse estimate \( L! < e^{1/2} (2\pi L)^{1/2} (L/e)^L < (8L)^{1/2} (L/e)^L \). Setting \( \gamma = \Lambda^{2r-1} \) in Proposition 3.4, we obtain

\[ \mathbb{P} \left( \sum_{j=1}^{m} s_j^2 \xi_j^2 < \Lambda^{2r-1} \right) < (8L)^{r/2} \left( \frac{\Lambda L}{em} \right)^{r-1/2}. \]

(38)

We set \( L = \lfloor \frac{m}{\Lambda} \rfloor \). Since \( 1 \leq \Lambda \leq m \), it is guaranteed that \( 1 \leq L \leq m \). Since \( \Lambda L \leq m \), we get

\[ \left( \frac{\Lambda L}{em} \right)^L \leq e^{-L} < e^{1 - \frac{m}{\Lambda}}. \]

Plugging this in (38) and using \( 8e^2 < 60 \), we find

\[ \mathbb{P} \left( \sum_{j=1}^{m} s_j^2 \xi_j^2 < \Lambda^{2r-1} \right) < (60m/\Lambda)^{r/2} e^{-m(r-1/2)/\Lambda}. \]

(39)

\[ \square \]

**Theorem 3.6.** Let \( E \) be an \( m \times k \) random matrix whose entries are i.i.d. \( \mathcal{N}(0, \frac{1}{m}) \), \( r \) be a positive integer, and assume that the entries \( s_j \) of the diagonal matrix \( S \) satisfy

\[ c_1 \left( \frac{m}{j} \right)^r \leq s_j \leq c_2 m^r, \quad j = 1, \ldots, m. \]

(40)

Let \( \Lambda \geq 1 \) be any number and assume \( m \geq \Lambda \). Consider the event

\[ \mathcal{F} := \left\{ \|SEx\|_2 \geq \frac{1}{2} c_1 \Lambda^{-1/2} \|x\|_2, \forall x \in \mathbb{R}^k \right\}. \]

Then

\[ \mathbb{P} (\mathcal{F}^c) \leq 5^k e^{-m/2} + 8^r (17c_2/c_1)^k \Lambda^{k/2} \left( \frac{m}{\Lambda} \right)^{r(k+1/2)} e^{-m(r-1/2)\Lambda}. \]

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Proof. Consider a \( \rho \)-net \( \tilde{Q} \) of the unit sphere of \( \mathbb{R}^k \) with \( \# \tilde{Q} \leq \left( \frac{2}{\rho} + 1 \right)^k \) where the value of \( \rho < 1 \) will be chosen later. Let \( \tilde{E}(\tilde{Q}) \) be the event \( \{ \| SEq \|_2 \geq c_1 \Lambda^{-1/2}, \ \forall q \in \tilde{Q} \} \). By Corollary 3.5 we know that

\[
\mathbb{P}\left( \tilde{E}(\tilde{Q})^c \right) \leq \left( \frac{2}{\rho} + 1 \right)^k \left( \frac{60m}{\Lambda} \right)^{r/2} e^{-m(r-1)/\Lambda}. \tag{41}
\]

Let \( E \) be the event in Lemma 3.3 with \( \Theta = 4 \). Let \( E \) be any given matrix in the event \( E \cap \tilde{E}(\tilde{Q}) \). For each \( \| x \|_2 = 1 \), there is \( q \in \tilde{Q} \) with \( \| q - x \|_2 \leq \rho \), hence by Lemma 3.3 we have

\[
\| SE(x - q) \|_2 \leq 4 \| s \|_\infty \| x - q \|_2 \leq 4c_2 \rho.
\]

Choose

\[
\rho = \frac{c_1 \Lambda^{-1/2}}{8c_2} = \frac{c_1 \Lambda^r}{8c_2 \sqrt{m}}.
\]

Hence

\[
\| SEx \|_2 \geq \| SEq \|_2 - \| SE(x - q) \|_2 \geq c_1 \Lambda^{-1/2} - 4c_2 \rho = \frac{1}{2} c_1 \Lambda^{-1/2}.
\]

This shows that \( E \cap \tilde{E}(\tilde{Q}) \subset F \). Clearly, \( \rho \leq 1/8 \) by our choice of parameters and hence \( \frac{2}{\rho} + 1 \leq \frac{17}{8} \). Using the probability bounds of Lemma 3.3 and (41), we have

\[
\mathbb{P}(F^c) \leq 5^k 4^{m/2} e^{-3m/2} + \left( \frac{17}{8\rho} \right)^k \left( \frac{60m}{\Lambda} \right)^{r/2} e^{-m(r-1)/\Lambda}
\]

\[
\leq 5^k e^{-m/2} + 8^r (17c_2/c_1)^k \Lambda^{k/2} \left( \frac{m}{\Lambda} \right)^{r(k+1/2)} e^{-m(r-1)/\Lambda}, \tag{42}
\]

where we have used \( 2 < e \) and \( \sqrt{60} < 8 \) for simplification.

The following theorem is now a direct corollary of the above estimate.

**Theorem 3.7.** Let \( E \) be an \( m \times k \) random matrix whose entries are i.i.d. \( \mathcal{N}(0, \frac{1}{m}) \), \( r \) be a positive integer, \( D \) be the difference matrix defined in (9), and the constant \( c_1 = c_1(r) \) be as in Proposition 3.1. Let \( 0 < \alpha < 1 \) be any number. Assume that

\[
\lambda := \frac{m}{k} \geq c_3 (\log m)^{1/(1-\alpha)}, \tag{43}
\]

where \( c_3 = c_3(r) \) is an appropriate constant. Then

\[
\mathbb{P}\left( \sigma_{\min}(D^{-r} E) \geq c_1 \lambda^{\alpha(r-1/2)} \right) \geq 1 - 2e^{-c_4 m^{1-\alpha} k^\alpha}, \tag{44}
\]

for some constant \( c_4 = c_4(r) > 0 \).
Proof. Set $\Lambda = \lambda^\alpha$ in Lemma 3.6. We only need to show that
\[
\max \left[ 5^k e^{-m/2}, 8\sigma (17c_2/c_1)^k \Lambda^{k/2} \left( \frac{m}{\Lambda} \right)^{r(k+1/2)} e^{-m(r-1/2)/\Lambda} \right] \leq e^{-c_4 m^{1-\alpha} k^\alpha}.
\]
It suffices to show that
\[
k \log 5 - m/2 \leq -c_4 m^{1-\alpha} k^\alpha
\]
and
\[
r \log 8 + k \log (17c_2/c_1) + \frac{1}{2} k \log \Lambda + r(k + \frac{1}{2}) \log (m/\Lambda) - (r - \frac{1}{2}) \frac{m}{\Lambda} \leq -c_4 m^{1-\alpha} k^\alpha.
\]
The first inequality is easily seen to hold if $\lambda \geq \frac{\log 5}{2c_4}$. For the second inequality, first notice that $m/\Lambda = m^{1-\alpha} k^\alpha$. Since $k + 1/2 \approx k$, and $r - 1/2 \approx r$, it is easily seen that we only need to check that
\[
k \log m \leq c_5 \frac{m}{\Lambda}
\]
for a sufficiently small $c_5$. This follows from our assumption on $\lambda$ by setting $c_5 = 1/c_3^{1-\alpha}$. □

Remark. By replacing $E$ in Theorem 3.7 with $\sqrt{m} E$, we obtain Theorem A.

3.2 Implication for compressed sensing matrices

Theorem 3.8. Let $r, D, c_1(r)$ be as in Theorem 3.7 and $\Phi$ be an $m \times N$ random matrix whose entries are i.i.d. $\mathcal{N}(0, 1/m)$. Let $0 < \alpha < 1$ be any number and assume that
\[
\lambda := \frac{m}{k} \geq c_6 (\log N)^{1/(1-\alpha)}, \tag{45}
\]
where $c_6 = c_6(r)$ is an appropriate constant. Then with probability at least $1 - 2e^{-c_7 m \lambda^{-\alpha}}$ for some $c_7 = c_7(r) > 0$, every $m \times k$ submatrix $E$ of $\Phi$ satisfies
\[
\sigma_{\min}(D^{-r} E) \geq c_1 \lambda^{\alpha(r-1/2)}. \tag{46}
\]

Proof. We will choose $c_7 = c_4/2$, where $c_4$ is as in Theorem 3.7. The proof will follow immediately by a union bound once we show that
\[
\binom{N}{k} \leq e^{\frac{1}{2} c_4 m^{1-\alpha} k^\alpha}.
\]
Since $\binom{N}{k} \leq N^k$, it suffices to show that
\[
k \log N \leq \frac{c_4}{2} m^{1-\alpha} k^\alpha.
\]
Both this condition and the hypothesis of Theorem 3.7 will be satisfied if we choose
\[
c_6 = \max(c_3, (2/c_4)^{1/(1-\alpha)}).
\]
Remark. If $\Phi$ is a Gaussian matrix with entries i.i.d. $\mathcal{N}(0, 1)$ rather than $\mathcal{N}(0, \frac{1}{m})$, Theorem 3.8 applied to $\frac{1}{\sqrt{m}}\Phi$ implies that every $m \times k$ submatrix $E$ of $\Phi$ satisfies
\[ \sigma_{\min}(D^{-1}E) \geq c_1 \lambda^\alpha(r-1/2) \sqrt{m}. \quad (47) \]

4 $\Sigma\Delta$ quantization of compressed sensing measurements

In this section we will assume that the conditions of Theorem 3.8 are satisfied for some $0 < \alpha < 1$ and $r$, and the measurement matrix $\Phi$ that is drawn from $\mathcal{N}(0, 1)$ yields (47). For definiteness, we also assume that $\Phi$ admits the robust recovery constant $C_1 = 10$, i.e., the solution $x^\#$ of the program (1) satisfies
\[ \|\hat{y} - y\|_2 \leq \epsilon \implies \|x - x^\#\|_2 \leq 10 \frac{1}{\sqrt{m}} \epsilon. \]

Note again that our choice of normalization for the measurement matrix $\Phi$ is different from the compressed sensing convention. As mentioned in the Introduction, it is more appropriate to work with a measurement matrix $\Phi \sim \mathcal{N}(0, 1)$ in order to be able to use a quantizer alphabet that does not depend on $m$. For this reason, in the remainder of the paper, $\Phi$ shall denote an $m \times N$ matrix whose entries are i.i.d. from $\mathcal{N}(0, 1)$.

Let $q := q_{\Sigma\Delta}$ be output of the standard greedy $r$th order $\Sigma\Delta$ quantizer with the alphabet $A = \delta \mathbb{Z}$ and input $y$. As stated in Section 2, we know that $\|y - q\|_\infty \leq 2^{r-1}\delta$ and therefore $\|y - q\|_2 \leq 2^{r-1}\delta \sqrt{m}$.

Coarse recovery and recovery of support

Our first goal is to recover the support $T$ of $x$. For this purpose we shall use a coarse approximation of $x$. Let
\[ x' := \arg\min \|z\|_1 \text{ subject to } \|\Phi z - q\|_2 \leq \epsilon := 2^{r-1}\delta \sqrt{m}. \quad (48) \]

By the robust recovery result (for our choice of normalization for $\Phi$), we know that
\[ \|x - x'\|_2 \leq \eta := 5 \cdot 2^r \delta. \]

The simplest attempt to recover $T$ from $x'$ is to pick the positions of its $k$ largest entries. This attempt can fail if some entry of $x_j$ on $T$ is smaller than $\eta$ for then it is possible that $x'_j = 0$ and therefore $j$ is not picked. On the other hand, it is easy to see that if the smallest nonzero entry of $x$ is strictly bigger than $2\eta$ in magnitude, then this method always succeeds. (Since $\|x - x'\|_\infty \leq \eta$, the entries of $x'$ are bigger than $\eta$ on $T$ and less than $\eta$ on $T^c$.) The constant 2 can be replaced with $\sqrt{2}$ by a more careful analysis, and can be pushed arbitrarily close to 1 by picking more than $k$ positions. The proposition below gives a precise condition on how well this can be done. We also provide a bound on how much of $x$ can potentially be missed if no lower bound on $|x_j|$ is available for $j \in T$. 

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Proposition 4.1. Let $\|x - x'\|_F^N \leq \eta$, $T = \text{supp } x$ and $k = |T|$. For any $k' \in \{k, \ldots, N-1\}$, let $T'$ be the support of (any of) the $k'$ largest entries of $x'$.

(i) $\|x_{T \setminus T'}\|_2 \leq \beta \eta$ where $\beta \leq \left(1 + \frac{k}{k'}\right)^{1/2}$.

(ii) If $|x_j| > \gamma \eta$ for all $j \in T$, where $\gamma := \left(1 + \frac{1}{k'-k+1}\right)^{1/2}$, then $T' \supset T$.

Proof. (i) We have
\[ \sum_{j \in T} |x_j - x_j'|^2 + \sum_{j \in T^c} |x_j'|^2 = \|x - x'\|_2^2 \leq \eta^2. \tag{49} \]
In particular, this implies
\[ \sum_{j \in T \setminus T'} |x_j - x_j'|^2 + \sum_{j \in T' \setminus T} |x_j'|^2 \leq \eta^2. \tag{50} \]
Suppose $T \setminus T' \neq \emptyset$. Then $T' \setminus T$ is also nonempty. In fact, we have
\[ |T' \setminus T| = |T \setminus T'| + k' - k. \]
Now, observe that
\[ \frac{1}{|T \setminus T'|} \sum_{j \in T \setminus T'} |x_j'|^2 \leq \max_{j \in T \setminus T'} |x_j'|^2 \leq \min_{j \in T \setminus T'} |x_j'|^2 \leq \frac{1}{|T' \setminus T|} \sum_{j \in T' \setminus T} |x_j'|^2, \]
which, together with (50) implies
\[ \|x_{T \setminus T'}\|_2 \leq \|x'_{T \setminus T'}\|_2 + \|(x - x')_{T \setminus T'}\|_2 \leq \|x'_{T \setminus T'}\|_2 + \sqrt{\eta^2 - \frac{|T' \setminus T|}{|T \setminus T'|} \|x'_{T \setminus T'}\|_2^2}. \]
It is easy to check that for any $A > 0$, and any $0 \leq t \leq \eta/\sqrt{A},$
\[ t + \sqrt{\eta^2 - At^2} \leq \left(1 + \frac{1}{A}\right)^{1/2} \eta. \tag{51} \]
The result follows by setting $A = |T' \setminus T|/|T \setminus T'|$ and noticing that $A \geq k'/k$.

(ii) Let $z_1 \geq \cdots \geq z_N$ be the decreasing rearrangement of $|x_1'|, \ldots, |x_N'|$. We have
\[ \sum_{j \in T} |x_j'|^2 \leq \sum_{i=1}^k z_i^2 \]
so
\[ \sum_{j \in T^c} |x_j'|^2 \geq \sum_{i=k+1}^N z_i^2 \geq \sum_{i=k+1}^{k'+1} z_i^2 \geq (k' - k + 1) z_{k'+1}^2. \]
Hence by (49) we have
\[ \max_{j \in T} |x_j - x_j'|^2 + (k' - k + 1)z_{k'+1}^2 \leq \eta^2. \]

Since \( |x_j'| \geq |x_j| - |x_j - x_j'| \), the above inequality now implies
\[ \min_{j \in T} |x_j'| \geq \min_{j \in T} |x_j| - \max_{j \in T} |x_j - x_j'| \geq \min_{j \in T} |x_j| - \sqrt{\eta^2 - (k' - k + 1)z_{k'+1}^2}. \]

Now, another application of (51) with \( A = k' - k + 1 \) yields
\[ -\sqrt{\eta^2 - (k' - k + 1)z_{k'+1}^2} \geq z_{k'+1} - \gamma \eta \]
and therefore
\[ \min_{j \in T} |x_j'| \geq \min_{j \in T} |x_j| + z_{k'+1} - \gamma \eta \geq z_{k'+1} = \max_{j \in T} |x_j'|. \]

It is then clear that \( T \subset T' \) because if \( T^c \cap T \neq \emptyset \), the inequality
\[ \max_{j \in T^c \setminus T} |x_j'| \geq \max_{j \in T^c \cap T} |x_j'| \geq \min_{j \in T} |x_j'| \]
would give us a contradiction. \( \square \)

Note that if the \( k' \) largest entries of \( x' \) are picked with \( k' > k \), then one would need to work with \( T' \) for the fine recovery stage, and therefore the starting assumptions on \( \Phi \) have to be modified for \( k' \). For simplicity we shall stick to \( k' = k \) and consequently \( \gamma = \sqrt{2} \).

**Fine recovery**

Once \( T \) is found, the \( r \)th order Sobolev dual frame \( F := F_{\text{Sob},r} \) of \( E = \Phi T \) is computed and we set \( \hat{x}_{\Sigma \Delta} = Fq \). We now restate and prove Theorem B.

**Theorem 4.2.** Let \( \Phi \) be an \( m \times N \) matrix whose entries are i.i.d. according to \( \mathcal{N}(0,1) \). Suppose \( \alpha \in (0,1) \) and \( \lambda := m/k \geq c(\log N)^{1/(1-\alpha)} \) where \( c = c(r, \alpha) \). Then there are two constants \( c' \) and \( C \) that depend only on \( r \) such that with probability at least \( 1 - \exp(-c'm\lambda^{-\alpha}) \) on the draw of \( \Phi \), the following holds: For every \( x \in \Sigma^N_k \) such that \( \min_{j \in \text{supp}(x)} |x_j| \geq C\delta \), the reconstruction \( \hat{x}_{\Sigma \Delta} \) satisfies
\[ \|x - \hat{x}_{\Sigma \Delta}\|_2 \lesssim_r \lambda^{-\alpha(r-\frac{1}{2})} \delta. \] (52)

**Proof.** Suppose that \( \lambda \geq c(\log N)^{1/(1-\alpha)} \) with \( c = c_0 \) as in the proof of Theorem 3.8. Let \( q_{\Sigma \Delta} \) be obtained by quantizing \( y := \Phi x \) via an \( r \)th order \( \Sigma \Delta \) scheme with alphabet \( A = \delta \mathbb{Z} \) and with the quantization rule as in (18), and let \( u \) be the associated state sequence as in (16). Define \( x\# \) as the solution of the program
\[ \min \|z\|_1 \text{ subject to } \|\Phi z - q_{\Sigma \Delta}\|_2 \leq \epsilon. \]
Suppose that $\Phi$ admits the robust recovery constant $C_1$, i.e., the solution $x^\#$ of the program \([3]\) satisfies $\|x - x^\#\|_2 \leq C_1 \epsilon / \sqrt{m}$ for every $x$ in $\Sigma^N_k$ provided that $\|y - q\Sigma\Delta\| \leq \epsilon$. Note that $C_1$, as given for example in [9], only depends on the RIP constants of $\Phi$ and is well-behaved if $m$ and $N$ satisfy the hypothesis of the theorem. As discussed in Section 2, in this case we have $\|y - q\Sigma\Delta\|_2 \leq 2^{r-1} \delta \sqrt{m}$ which implies

$$\|x - x^\#\|_2 \leq C_1 2^{r-1} \delta.$$ 

Assume that

$$\min_{j \in T} |x_j| \geq C_1 \cdot 2^{r-1/2} \delta =: C\delta. \quad (53)$$

Then, Proposition 4.1 (with $\gamma = \sqrt{2}$ and $\eta = C_1 2^{r-1}$) shows that $T'$, the support of the $k$ largest entries of $x^\#$, is identical to the support $T$ of $x$. Finally, set

$$\hat{x}\Sigma\Delta = F_{sob,r} q\Sigma\Delta$$

where $F_{sob,r}$ is the $r$th order Sobolev dual of $\Phi_T$. Using the fact that $\|u\|_2 \leq 2^{-1} \delta \sqrt{m}$ (see Section 2) together with the conclusion of Theorem 3.8 and the error bound (27), we conclude that

$$\|x - \hat{x}\Sigma\Delta\|_2 \leq \frac{\|u\|_2}{\sqrt{m} \sigma_{\min}(D^{-r}E)} \leq \frac{\lambda^{-\alpha(r-1/2)}}{2c_1} \delta. \quad (54)$$

Note that the RIP and therefore the robust recovery will hold with probability $1 - \exp(c'' m)$, and our Sobolev dual reconstruction error bound will hold with probability $1 - \exp(-c_7 m \lambda^{-\alpha})$. Here $c_1$ and $c_7$ are as in the proof of Theorem 3.8.

**Remark.** To interpret the size condition in a concrete case, assume that $\Phi$ admits the robust recovery constant $C_1 = 10$, and that we have

$$\min_{j \in T} |x_j| \geq \sqrt{2} \eta = 5 \cdot 2^{r+1/2} \delta. \quad (55)$$

If PCM is used as the quantization method, then the best error guarantee we have that holds uniformly on $T$ would be

$$\|x - x^\#_{PCM}\|_\infty \leq \|x - x^\#_{PCM}\|_2 \leq 5 \delta.$$ 

It can be argued that the approximately recovered entries of $x^\#_{PCM}$ are meaningful only when the minimum nonzero entry of $x$ is at least as large as the maximum uncertainty in $x^\#_{PCM}$, which is only known to be bounded by $5 \delta$. Hence, in some sense the size condition (55) is natural (modulo the factor $2^{r+1/2}$).
Quantizer choice and rate-distortion issues

So far we have not made any assumptions on the step size $\delta$ of the uniform infinite quantizer $\mathcal{A} = \delta \mathcal{Z}$. An important question concerns how large $\delta$ should be for the most effective use of resources. This question is motivated by the fact that infinite quantizers are not practical and have to be replaced by finite ones. In the same vein, an alternative question is to determine the minimum number of bits that the quantizer needs to incorporate as well as the resulting approximation error. First, let us assume that

$$A \leq |x_j| \leq \rho := 2^b A \quad \text{for all } j \in T. \quad (56)$$

For usefulness of our results, one would be interested in the regime $A \ll \rho$. Thus, we introduce $2^b = \rho/A$ to represent the number of dyadic scales over which the input is allowed to range. Clearly, $\delta_r$, the quantization step size used by an $r$th order $\Sigma\Delta$ scheme for our support recovery results to hold must satisfy

$$\delta_r \leq \frac{A}{5^{r+1/2}} \quad \text{(as before, we assume } C_1 = 10).$$

Let us for the moment use the largest allowable step-size, i.e., set

$$\delta_r := \frac{A}{5^{r+1/2}}. \quad (57)$$

Next, let us assume that a $B_r$-bit uniform quantizer of step size $\delta_r$ is to replace $\mathcal{A} = \delta \mathcal{Z}$. We know that $\|q\|_\infty$ could be as large as $2^{r-1}\delta_r + \|y\|_\infty$, therefore we need to bound $\|y\|_\infty$ efficiently. If we use the RIP, then $\Phi$ does not expand the $\ell_2$-norm of $k$-sparse vectors by more than a factor of $2\sqrt{m}$ (note our choice of normalization for $\Phi$), and therefore it follows that

$$\|y\|_\infty \leq \|y\|_2 \leq 2\sqrt{m}\|x\|_2 \leq 2\rho\sqrt{mk},$$

which is a restatement of the inequality

$$\|E\|_{\ell_\infty^k \rightarrow \ell_\infty^m} \leq \sqrt{k}\|E\|_{\ell_2^k \rightarrow \ell_2^m}$$

that holds for any $m \times k$ matrix $E$. However, it can be argued that the $(\infty, \infty)$-norm of a random matrix should typically be smaller. In fact, if $E$ were drawn from the Bernoulli model, i.e., $E_{ij} \sim \pm 1$, then we would have

$$\|E\|_{\ell_\infty^k \rightarrow \ell_\infty^m} = k = \lambda^{-1/2}\sqrt{mk},$$

as can easily be seen from the general formula

$$\|E\|_{\ell_\infty^k \rightarrow \ell_\infty^m} = \max_{1 \leq i \leq m} \sum_{j=1}^k |E_{ij}|. \quad (58)$$

Using simple concentration inequalities for Gaussian random variables, it turns out that for the range of aspect ratio $\lambda = m/k$ and probability of encountering a matrix $\Phi$ that we are interested in, we have $\|E\|_{\ell_\infty^k \rightarrow \ell_\infty^m} \leq \lambda^{-a/2}\sqrt{mk}$ for every $m \times k$ submatrix $E$ of $\Phi$. We start with the following estimate:
Proposition 4.3. Let $\xi_1, \ldots, \xi_k$ i.i.d. standard Gaussian variables. Then, for any $\Theta > 1$,

$$\mathbb{P} \left( \sum_{j=1}^{k} |\xi_j| > \Theta \right) \leq 2^k e^{-\Theta^2/(2k)}. \quad (59)$$

Proof.

$$\mathbb{P} \left( \sum_{j=1}^{k} |\xi_j| > \Theta \right) \leq \min_{t \geq 0} \int_{\mathbb{R}^k} e^{-\Theta t - \sum_{j=1}^{k} |x_j|} t \prod_{j=1}^{k} e^{-x_j^2/2} \frac{dx_j}{\sqrt{2\pi}}$$

$$= \min_{t \geq 0} e^{-\Theta t} \left( e^{t^2/2} \int_{\mathbb{R}} e^{-\frac{1}{2} (|x|-t)^2} \frac{dx}{\sqrt{2\pi}} \right)^k$$

$$= \min_{t \geq 0} e^{-\Theta t} \left( 2e^{t^2/2} \int_{0}^{\infty} e^{-\frac{1}{2} (x-t)^2} \frac{dx}{\sqrt{2\pi}} \right)^k$$

$$\leq 2^k \min_{t \geq 0} e^{-\Theta t + kt^2/2}$$

$$= 2^k e^{-\Theta^2/(2k)}. \quad (60)$$

where in the last step we set $t = \Theta/k$. \hfill \Box

Proposition 4.4. Let $\Phi$ be an $m \times N$ random matrix whose entries are i.i.d. $\mathcal{N}(0,1)$. Let $0 < \alpha < 1$ be any number and assume that

$$\lambda := \frac{m}{k} \geq c_1 (\log N)^{1/(1-\alpha)}, \quad (61)$$

where $c_1$ is an appropriate constant. Then with probability at least $1 - e^{-c_2m^{1-\alpha}k^\alpha}$ for some $c_2 > 0$, every $m \times k$ submatrix $E$ of $\Phi$ satisfies

$$\|E\|_{\ell_\infty \rightarrow \ell_\infty} \leq \lambda^{-\alpha/2} \sqrt{mk}. \quad (62)$$

Proof. Proposition 4.3 straightforwardly implies that

$$\mathbb{P} \left( \{ \exists T \text{ such that } |T| = k \text{ and } \|\Phi_T\|_{\ell_\infty \rightarrow \ell_\infty} > \Theta \} \right) \leq \binom{N}{k} m^k e^{-\Theta^2/(2k)}. \quad (63)$$

Let $\Theta = \lambda^{-\alpha/2} \sqrt{mk}$. It remains to show that

$$k \log N + k \log 2 + \log m + c_2 m^{1-\alpha}k^\alpha \leq \frac{\Theta^2}{2k}.$$

If $c_1$ in (61) is sufficiently large and $c_2$ is sufficiently small, then the expression on the left hand side is bounded by $k\lambda^{1-\alpha}/2 = \Theta^2/(2k)$. \hfill \Box

Without loss of generality, we may now assume that $\Phi$ also satisfies the conclusion of Proposition 4.4. Hence we have an improved bound on the range of $y$ given by

$$\|y\|_{\infty} \leq \rho \lambda^{-\alpha/2} \sqrt{mk} = \rho \lambda^{(1-\alpha)/2} k. \quad (64)$$
We assume $B_r$ is chosen to satisfy
\[ 2^{B_r - 1} \delta_r = 2^{r-1} \delta_r + \rho \lambda^{(1-\alpha)/2} k, \]  
so that the quantizer is not overloaded. Since $\rho/\delta_r \approx 2^{r+1/2+b}$ by (56) and (57), we see that the second term on the right hand side of (65) is significantly larger than the first, which implies
\[ 2^{B_r - 1} \delta_r \approx 2^b A \lambda^{(1-\alpha)/2} k. \]  
Hence, using (57) again, $B_r$ must satisfy
\[ 2^{B_r - 1} \approx 5 \ 2^{b+r+1/2} \lambda^{(1-\alpha)/2} k. \]  

Based on Theorem 4.2, the approximation error (the distortion) $\mathcal{D}_{\Sigma \Delta}$ incurred after the fine recovery stage via Sobolev duals satisfies the bound
\[ \mathcal{D}_{\Sigma \Delta} \lesssim \lambda^{-\alpha(r-1/2)} \delta_r \approx \frac{\lambda^{-\alpha(r-1/2)} A}{2^{r+1/2}}. \]  

A similar calculation for the PCM encoder with the same step size $\delta_r$ and the standard $\ell_1$ decoder results in the necessity for roughly the same number of bits $B_r$ as the $\Sigma \Delta$ encoder (because of the approximation (66)), but provides only the distortion bound
\[ \mathcal{D}_{\text{PCM}} \lesssim \delta_r \approx \frac{A}{2^{r+1/2}}. \]  

Note that the analysis above requires that both PCM and $\Sigma \Delta$ encoders utilize high-resolution quantizers, however the benefit of using $\Sigma \Delta$ encoders is obvious upon comparing (68) and (69).

5 Numerical experiments

In order to test the accuracy of Theorem 3.7, our first numerical experiment concerns the minimum singular value of $D^{-r} E$ as a function of $\lambda = m/k$. In Figure 1, we plot the worst case (the largest) value, among 1000 realizations, of $1/\sigma_{\min}(D^{-r} E)$ for the range $1 \leq \lambda \leq 25$, where we have kept $k = 50$. As predicted by this theorem, we find that the negative slope in the log-log scale is roughly equal to $r - 1/2$, albeit slightly less, which seems in agreement with the presence of our control parameter $\alpha$. As for the size of the $r$-dependent constants, the function $5^r \lambda^{-r+1/2}$ seems to be a reasonably close numerical fit, which also explains why we observe the separation of the individual curves after $\lambda > 5$.

Our next experiment involves the full quantization algorithm for compressed sensing including the “recovery of support” and “fine recovery” stages. To that end, we first generate a $1000 \times 2000$ matrix $\Phi$, where the entries of $\Phi$ are drawn i.i.d. according to $\mathcal{N}(0, 1)$. To examine the performance of the proposed scheme as the redundancy $\lambda$ increases in comparison to the performance of the standard PCM quantization, we run a set of experiments: In
For each experiment we fix the sparsity $k \in \{5, 10, 20, 40\}$, and we generate $k$-sparse signals $x$ with the non-zero entries of each signal supported on a random set $T$, but with magnitude $1/\sqrt{k}$. This ensures that $\|x\|_2 = 1$. Next, for $m \in \{100, 200, \ldots, 1000\}$ we generate the measurements $y = \Phi^{(m)} x$, where $\Phi^{(m)}$ is comprised of the first $m$ rows of $\Phi$. We then quantize $y$ using PCM, as well as the 1st and 2nd order $\Sigma\Delta$ quantizers, defined via (16) and (18) (in all cases the quantizer step size is $\delta = 10^{-2}$). For each of these quantized measurements $q$, we perform the coarse recovery stage, i.e., we solve the associated $\ell_1$ minimization problem to recover a coarse estimate of $x$ as well as an estimate $\tilde{T}$ of the support $T$. The approximation error obtained using the coarse estimate (with PCM quantization) is displayed in Figures 2 and 3 (see the dotted curve). Next, we implement the fine recovery stage of our algorithm. In particular, we use the estimated support set $\tilde{T}$ and generate the associated dual $F_{\text{sob},r}$. Defining $F_{\text{sob},0} := (\Phi^{(m)}_{\tilde{T}})\dagger$, in each case, our final estimate of the signal is obtained via the fine recovery stage as $\hat{x}_T = F_{\text{sob},r} q$, $\hat{x}_{\tilde{T}} = 0$. Note that this way, we obtain an alternative reconstruction also in the case of PCM. We repeat this experiment 100 times for each $(k, m)$ pair and plot the average of the resulting errors $\|x - \hat{x}\|_2$ as a function of $\lambda$ in Figure 2 as well as the maximum of $\|x - \hat{x}\|_2$ in Figure 3. For our final experiment, we choose the entries of $x_T$ i.i.d. from $\mathcal{N}(0,1)$, and use a quantizer step size $\delta = 10^{-4}$. Otherwise, the experimental setup is identical to the previous one. The average of the resulting errors $\|x - \hat{x}\|_2$ as a function of $\lambda$ is reported in Figure 4 and the maximum of $\|x - \hat{x}\|_2$ in Figure 5.

The main observations that we obtain from these experiments are as follows:

- $\Sigma\Delta$ schemes outperform the coarse reconstruction obtained from PCM quantized measurements significantly even when $r = 1$ and even for small values of $\lambda$.

- For the $\Sigma\Delta$ reconstruction error, the negative slope in the log-log scale is roughly equal to $r$. This outperforms the (best case) predictions of Theorem 13 which are obtained through the operator norm bound and suggests the presence of further cancellation due to the statistical nature of the $\Sigma\Delta$ state variable $u$, similar to the white noise hypothesis.

- When a fine recovery stage is employed in the case of PCM (using the Moore-Penrose pseudoinverse of the submatrix of $\Phi$ that corresponds to the estimated support of $x$), the approximation is consistently improved (when compared to the coarse recovery). Moreover, the associated approximation error is observed to be of order $O(\lambda^{-1/2})$, in contrast with the error corresponding to the coarse recovery from PCM quantized measurements (with the $\ell_1$ decoder only) where the approximation error does not seem to depend on $\lambda$. A rigorous analysis of this behaviour will be given in a separate manuscript.
6 Remarks on extensions

6.1 Other noise shaping matrices

In the above approach, the particular quantization scheme that we use can be identified with its “noise-shaping matrix”, which is $D^r$ in the case of an $r$th order $\Sigma\Delta$ scheme and the identity matrix in the case of PCM.

The results we obtained above are valid for the aforementioned noise-shaping matrices. However, our techniques are fairly general and our estimates can be modified to investigate the accuracy obtained using an arbitrary quantization scheme with the associated invertible noise-shaping matrix $H$. In particular, the estimates depend solely on the distribution of the singular values of $H$. Of course, in this case, we also need change our “fine recovery” stage and use the “$H$-dual” of the corresponding frame $E$, which we define via

$$F_H = (HE)^\dagger.$$  

As an example, consider an $r$th order high-pass $\Sigma\Delta$ scheme whose noise shaping matrix is $H^r$ where $H$ is defined via

$$H_{ij} := \begin{cases} 1, & \text{if } i = j \text{ or } i = j + 1, \\ 0, & \text{otherwise}. \end{cases}$$

It is easy to check that the singular values of $H$ are identical to those of $D$. It follows that all the results presented in this paper are valid also if the compressed measurements are quantized via an an $r$th order high-pass $\Sigma\Delta$ scheme, provided the reconstruction is done using the $H^r$-duals instead of the $r$th order Sobolev duals. Note that such a result for high-pass $\Sigma\Delta$ schemes is not known to hold in the case of structured frames.

6.2 Measurement noise and compressible signals

One of the natural questions is whether the quantization methods developed in this paper are effective in the presence of measurement noise in addition to the error introduced during the quantization process. Another natural question is how to extend this theory to include the case when the underlying signals are not necessarily strictly sparse, but nevertheless still “compressible”.

Suppose $x \in \mathbb{R}^N$ is not sparse, but compressible in the usual sense (e.g. as in [9]), and let $y = \Phi x + e$, where $e$ stands for additive measurement noise. The coarse recovery stage inherits the stability and robustness properties of $\ell_1$ decoding for compressed sensing, therefore the accuracy of this first reconstruction depends on the best $k$-term approximation error for $x$, and the deviation of $\Phi x$ from the quantized signal $q$ (which comprises of the measurement noise $e$ and the quantization error $y - q$). Up to constant factors, the quantization error for any (stable) $\Sigma\Delta$ quantizer is comparable to that of PCM, hence the reconstruction error at the coarse recovery stage would also be comparable. In the fine recovery stage, however, the difference between $\sigma_{\max}(F_H^r)$ and $\sigma_{\max}(F_H)$ plays a critical role. In the particular case of
$H = D^r$ and $F_H = F_{\text{soh},r}$, the Sobolev duals we use in the reconstruction are tailored to reduce the effect of the quantization error introduced by an $r$th order $\Sigma\Delta$ quantizer. This is reflected in the fact that as $\lambda$ increases, the kernel of the reconstruction operator $F_{\text{soh},r}$ contains a larger portion of high-pass sequences (like the quantization error of $\Sigma\Delta$ modulation), and is quantified by the bound $\sigma_{\max}(F_{\text{soh},r}D^r) \lesssim \lambda^{-(r-1/2)m^{-1/2}}$ (see Theorem A (26) and (27)). Consequently, obtaining more measurements increases $\lambda$, and even though $\|y-q\|_2$ increases as well, the reconstruction error due to quantization decreases. At the same time, obtaining more measurements would also increase the size of the external noise $e$, as well as the “aliasing error” that is the result of the “off-support” entries of $x$. However, this noise+error term is not counteracted by the action of $F_{\text{soh},r}$. In fact, for any dual $F$, the relation $FE = I$ implies $\sigma_{\max}(F) \geq 1/\sigma_{\max}(E) \gtrsim m^{-1/2}$ already and in the case of measurement noise, it is not possible to do better than the canonical dual $E^\dagger$ on average. In this case, depending on the size of the noise term, the fine recovery stage may not improve the total reconstruction error even though the “quantizer error” is still reduced.

One possible remedy for this problem is to construct alternative quantization schemes with associated noise-shaping matrices that balance the above discussed trade-off between the quantization error and the error that is introduced by other factors. This is a delicate procedure, and it will be investigated thoroughly in future work. However, a first such construction can be made by using “leaky” $\Sigma\Delta$ schemes with $H$ given by

$$H_{ij} := \begin{cases} 1, & \text{if } i = j, \\ -\mu & \text{if } i = j + 1, \\ 0, & \text{otherwise}, \end{cases}$$

where $\mu \in (0,1)$. Our preliminary numerical experiments (see Figures 6 and 7) suggest that this approach can be used to improve the accuracy of the approximation further in the fine recovery stage in this more general setting. We note that the parameter $\mu$ above can be adjusted based on how compressible the signals of interest are and what the expected noise level is.

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A Singular values of $D^{-r}$

It will be more convenient to work with the singular values of $D'$. Note that because of our convention of descending ordering of singular values, we have

$$
\sigma_j(D^{-r}) = \frac{1}{\sigma_{m+1-j}(D')}, \quad j = 1, \ldots, m.
$$

(73)

For $r = 1$, an explicit formula is available [27,28]. Indeed, we have

$$
\sigma_j(D) = 2 \cos \left( \frac{\pi j}{2m+1} \right), \quad j = 1, \ldots, m,
$$

(74)

which implies

$$
\sigma_j(D^{-1}) = \frac{1}{2 \sin \left( \frac{\pi(j-1/2)}{2(m+1/2)} \right)}, \quad j = 1, \ldots, m.
$$

(75)

The first observation is that $\sigma_j(D^r)$ and $(\sigma_j(D))^r$ are different, because $D$ and $D^*$ do not commute. However, this becomes insignificant as $m \to \infty$. In fact, the asymptotic distribution of $(\sigma_j(D^r))_{j=1}^m$ as $m \to \infty$ is rather easy to find using standard results in the theory of Toeplitz matrices: $D$ is a banded Toeplitz matrix whose symbol is $f(\theta) = 1 - e^{i\theta}$, hence the symbol of $D^r$ is $(1 - e^{i\theta})^r$. It then follows by Parter’s extension of Szegő’s theorem [25] that for any continuous function $\psi$, we have

$$
\lim_{m \to \infty} \frac{1}{m} \sum_{j=1}^m \psi(\sigma_j(D^r)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(|f(\theta)|^r) \, d\theta.
$$

(76)

We have $|f(\theta)| = 2 \sin |\theta| / 2$ for $|\theta| \leq \pi$, hence the distribution of $(\sigma_j(D^r))_{j=1}^m$ is asymptotically the same as that of $2^r \sin^r(\pi j / 2m)$, and consequently, we can think of $\sigma_j(D^{-r})$ roughly as $(2^r \sin^r(\pi j / 2m))^{-1}$. Moreover, we know that $\|D^r\|_{\text{op}} \leq \|D\|^r_{\text{op}} \leq 2^r$, hence $\sigma_{\min}(D^{-r}) \geq 2^{-r}$.

When combined with known results on the rate of convergence to the limiting distribution in Szegő’s theorem, the above asymptotics could be turned into an estimate of the kind given in Proposition 3.1, perhaps with some loss of precision. Here we shall provide a more direct approach which is not asymptotic, and works for all $m \geq 4r$. The underlying observation is that $D$ and $D^*$ almost commute: $D^*D - DD^*$ has only two nonzero entries, at $(1, 1)$ and $(m, m)$. Based on this observation, we show below that $D^*D^r$ is then a perturbation of $(D^rD)^r$ of rank at most $2r$.

**Proposition A.1.** Let $C^{(r)} = D^*D^r - (D^rD)^r$ where we assume $m \geq 2r$. Define

$$
I_r := \{1, \ldots, r\} \times \{1, \ldots, r\} \cup \{m - r + 1, \ldots, m\} \times \{m - r + 1, \ldots, m\}.
$$

Then $C^{(r)}_{i,j} = 0$ for all $(i, j) \notin I_r$. Therefore, rank$(C^{(r)}) \leq 2r$. 

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Proof. Define the set \( \mathcal{C}_r \) of all “\( r \)-cornered” matrices as
\[
\mathcal{C}_r = \{ M : M_{i,j} = 0 \text{ if } (i,j) \in I_r^c \},
\]
and the set \( \mathcal{B}_r \) of all “\( r \)-banded” matrices as
\[
\mathcal{B}_r = \{ M : M_{i,j} = 0 \text{ if } |i - j| > r \}.
\]
Both sets are closed under matrix addition. It is also easy to check the following facts (for the admissible range of values for \( r \) and \( s \)):

(i) If \( B \in \mathcal{B}_r \) and \( C \in \mathcal{C}_s \), then \( BC \in \mathcal{C}_{r+s} \) and \( CB \in \mathcal{C}_{r+s} \).

(ii) If \( B \in \mathcal{B}_r \) and \( \tilde{B} \in \mathcal{B}_s \), then \( B\tilde{B} \in \mathcal{B}_{r+s} \).

(iii) If \( C \in \mathcal{C}_r \) and \( \tilde{C} \in \mathcal{C}_s \), then \( C\tilde{C} \in \mathcal{C}_{\max(r,s)} \).

(iv) If \( C \in \mathcal{C}_r \), then \( D^*CD \in \mathcal{C}_{r+1} \).

Note that \( DD^*, D^*D \in \mathcal{B}_1 \) and the commutator \([D^*, D] =: \Gamma_1 \in \mathcal{C}_1 \). Define
\[
\Gamma_r := (D^*D)^r - (DD^*)^r = (DD^* + \Gamma_1)^r - (DD^*)^r.
\]
We expand out the first term (noting the non-commutativity), cancel \((DD^*)^r\) and see that every term that remains is a product of \( r \) terms (counting each \( DD^* \) as one term) each of which is either in \( \mathcal{B}_1 \) or in \( \mathcal{C}_1 \). Repeated applications of (i), (ii), and (iii) yield \( \Gamma_r \in \mathcal{C}_r \).

We will now show by induction on \( r \) that \( C^{(r)} \in \mathcal{C}_r \) for all \( r \) such that \( 2r \leq m \). The cases \( r = 0 \) and \( r = 1 \) hold trivially. Assume the statement holds for a given value of \( r \). Since \( C^{(r+1)} = D^*(C^{(r)} + \Gamma_r)D \) and \( \Gamma_r \in \mathcal{C}_r \), property (iv) above now shows that \( C^{(r+1)} \in \mathcal{C}_{r+1} \). \( \square \)

The next result, originally due to Weyl (see, e.g., [19, Thm 4.3.6]), will now allow us to estimate the eigenvalues of \( D^*D^r \) using the eigenvalues of \( (D^*D)^r \):

**Theorem A.2** (Weyl). Let \( B \) and \( C \) be \( m \times m \) Hermitian matrices where \( C \) has rank at most \( p \). Then
\[
\lambda_{j+p}(B) \leq \lambda_j(B + C) \leq \lambda_{j-p}(B), \quad j = p + 1, \ldots, m - p,
\]
where we assume eigenvalues are in descending order.

We are now fully equipped to prove Proposition 3.1.
Proof of Proposition 3.1. We set $p = 2r$, $B = (D^*D)^r$, and $C = C^{(r)} = D^{*r}D^r - (D^*D)^r$ in Weyl’s theorem. By Proposition A.1, $C$ has rank at most $2r$. Hence, we have the relation

$$\lambda_{j+2r}((D^*D)^r) \leq \lambda_j(D^{*r}D^r) \leq \lambda_{j-2r}((D^*D)^r), \quad j = 2r + 1, \ldots, m - 2r. \quad (78)$$

Since $\lambda_j((D^*D)^r) = \lambda_j(D^*D)^r$, this corresponds to

$$\sigma_{j+2r}(D^r) \leq \sigma_j(D^r) \leq \sigma_{j-2r}(D^r), \quad j = 2r + 1, \ldots, m - 2r. \quad (79)$$

For the remaining values of $j$, we will simply use the largest and smallest singular values of $D^r$ as upper and lower bounds. However, note that

$$\sigma_1(D^r) = \|D^r\|_{op} \leq \|D\|_{op}^r = (\sigma_1(D))^r$$

and similarly

$$\sigma_m(D^r) = \|D^{-r}\|_{op}^{-1} \geq \|D^{-1}\|_{op}^{-r} = (\sigma_m(D))^r.$$

Hence (79) can be rewritten as

$$\sigma_{\min(j+2r,m)}(D^r) \leq \sigma_j(D^r) \leq \sigma_{\max(j-2r,1)}(D)^r, \quad j = 1, \ldots, m. \quad (80)$$

Inverting these relations via (73), we obtain

$$\sigma_{\min(j+2r,m)}(D^{-1}) \leq \sigma_j(D^{-r}) \leq \sigma_{\max(j-2r,1)}(D^{-1})^r, \quad j = 1, \ldots, m. \quad (81)$$

Finally, to demonstrate the desired bounds of Proposition 3.1 we rewrite (75) via the inequality $2x/\pi \leq \sin x \leq x$ for $0 \leq x \leq \pi/2$ as

$$\frac{m + 1/2}{\pi(j - 1/2)} \leq \sigma_j(D^{-1}) \leq \frac{m + 1/2}{2(j - 1/2)}, \quad (82)$$

and observe that $\min(j + 2r, m) \asymp_r j$ and $\max(j - 2r, 1) \asymp_r j$ for $j = 1, \ldots, m$. \qed

Remark. The constants $c_1(r)$ and $c_2(r)$ that one obtains from the above argument would be significantly exaggerated. This is primarily due to the fact that Proposition 3.1 is not stated in the tightest possible form. The advantage of this form is the simplicity of the subsequent analysis in Section 3.1. Our estimates of $\sigma_{\min}(D^{-r}E)$ would become significantly more accurate if the asymptotic distribution of $\sigma_j(D^{-r})$ is incorporated into our proofs in Section 3.1. However, the main disadvantage would be that the estimates would then hold only for all sufficiently large $m$.

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Figure 1: Numerical behavior (in log-log scale) of $1/\sigma_{\min}(D^{-r}E)$ as a function of $\lambda = m/k$, for $r = 0, 1, 2, 3, 4$. In this figure, $k = 50$ and $1 \leq \lambda \leq 25$. For each problem size, the largest value of $1/\sigma_{\min}(D^{-r}E)$ among 1000 realizations of a random $m \times k$ matrix $E$ sampled from the Gaussian ensemble $\mathcal{N}(0, \frac{1}{m}I_m)$ was recorded.
Figure 2: The average performance of the proposed ΣΔ quantization and reconstruction schemes for various values of $k$. For this experiment the non-zero entries of $x$ are constant and $\delta = 0.01$. 
Figure 3: The worst case performance of the proposed ΣΔ quantization and reconstruction schemes for various values of $k$. For this experiment the non-zero entries of $x$ are constant and $\delta = 0.01$. 
Figure 4: The average performance of the proposed ΣΔ quantization and reconstruction schemes for various values of $k$. For this experiment the non-zero entries of $x$ are i.i.d. $\mathcal{N}(0, 1)$ and $\delta = 10^{-4}$. 

(a) $k = 5$

(b) $k = 10$

(c) $k = 20$

(d) $k = 40$
Figure 5: The worst case performance of the proposed ΣΔ quantization and reconstruction schemes for various values of $k$. For this experiment the non-zero entries of $x$ are i.i.d. $\mathcal{N}(0, 1)$ and $\delta = 10^{-4}$. 

- (a) $k = 5$
- (b) $k = 10$
- (c) $k = 20$
- (d) $k = 40$
Figure 6: The average case performance of the proposed Σ∆ quantization and reconstruction schemes (with general duals) for various values of $k$. For this experiment the non-zero entries of $x$ are constant and $\delta = 0.01$. 
Figure 7: The worst case performance of the proposed Σ∆ quantization and reconstruction schemes (with general duals) for various values of $k$. For this experiment the non-zero entries of $x$ are constant and $\delta = 0.01$. 