TOPOLOGICAL GRAVITATION ON GRAPH MANIFOLDS

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A model of topological field theory is presented in which the vacuum coupling constants are topological invariants of the four-dimensional spacetime. Thus the coupling constants are theoretically computable, and they indicate the topological structure of our universe.

We construct an Abelian BF-type model in analogy with the ordinary four-dimensional topological field theory and with the low-energy effective $U(1)^r$-theory of Seiberg–Witten (SW)\(^2\), beginning with a $U(1)^r$-bundle $E$ over a four-dimensional topological space $X$ with a non-empty boundary $\partial X$, $E$ being a direct sum of linear bundles $L_1 \oplus \cdots \oplus L_r$. Let us define locally connection 1-forms $A^a$ $(a = 1, \ldots, r)$ on $E$ with values in the algebra $L$ of the group $U(1)$, and 2-forms $B_a$ with values in the dual algebra. Due to these analogies it is natural to write the action as $S = \int F^a \wedge B_a - \frac{1}{2} \Lambda^{ab} B_a \wedge B_b + \frac{i}{2} \Theta_{ab} F^a \wedge F^b$ where $F^a = dA^a$; $\Lambda^{ab}$ and $\Theta_{ab}$ are non-degenerate symmetric matrices called those of the coupling constants and theta angles matrices, respectively. Our action admits symmetry under dual conjugation similar to the electro–magnetic one (EM duality) of the SW theory, and theta angles matrices, respectively. These duality transformations carry $S$ into its equivalent, $S_D$, and they involve the strong–weak coupling duality.\(^2\) We call $\Lambda^{ab}$ and $\Lambda^{ab}$ strong and weak coupling constants matrices (in this sense), respectively.

Then we generalize Dirac’s quantization conditions: the flux through non-trivial 2-cycles $\Sigma_I$ must be $\int_{\Sigma_I} F^a = 2\pi m_I^a$, $m_I^a \in \mathbb{Z}$. Then from dynamical equations of our BF system $dB_a = 0$, $F^a = \Lambda^{ab} B_b$ together with the gauge symmetries it follows that the moduli space of this BF system is $H^2(X, \mathbb{Z}) \oplus H^2(X, \partial X, \mathbb{Z})$, meaning that $[\Sigma I(F^a)] \in H^2(X, \mathbb{Z})$ and $[\Sigma I(B_a)] \in H^2(X, \partial X, \mathbb{Z})$. Thus there are no local degrees of freedom and like in the case of the low energy effective SW action, our model describes the moduli space of vacua. The spacetime topology is non-trivial since we model the spacetime by the graph manifold.\(^3\) Each tree graph corresponds to a unique four-dimensional space $X$ with a boundary containing lens spaces and $\mathbb{Z}$-homology spheres. The latter ones are results of splicing of Seifert fibred homology (Sfh) spheres. The most important construction element is Sfh-sphere $\Sigma(\mathbf{a}) \equiv \Sigma(a_1, a_2, a_3)$ having three special orbits and being a three-dimensional manifold which is an intersection of the Brieskorn surface $z_1^{a_1} + z_2^{a_2} + z_3^{a_3} = 0$ ($z_i \in \mathbb{C}$) and a sphere $S^5$. Here $a_1$, $a_2$ and $a_3$ are mutually prime integers (Seifert invariants). To the end of constructing our cosmological model we need a specific family of Sfh-spheres to which we give the following definition:\(^3\) We take a succession of Sfh-spheres calling it the primary one: $\{\Sigma(q_{2n-1}, p_2, p_{2n+1})|n = 0, \ldots, 4\}$. Here $p_i$ is the $i$-th prime number in the natural series and $q_{2n+1}$ is the $n$-th prime number in the natural series. Then we define the “derivative” of a Sfh-sphere $\Sigma(\mathbf{a})$.\n
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Fig. 1. The graph associated with a four-dimensional manifold interpreted as a Euclidean region of spacetime $X$ with boundaries $\partial X = -\sum_{i=1}^{9} L(p_i, q_i) \Sigma M$. This graph describes a universe with five low energy interactions related to the first nine prime numbers as $(1,2), (3,5), (7,11), (13,17), (19,23)$ determining Seifert’s invariants of $S^4$-spheres being glued together. Vertices on horizontal lines represent (from left to right) successive “derivatives” of the primary $S^4$-spheres shown on the diagonal line. At the four-dimensional level splicing gives the plumbing operation.

as another $S^4$-sphere $\Sigma^{(1)}(a) := \Sigma(a_1, a_2, a_3, a + 1) \equiv \Sigma(a_1^{(1)}, a_2^{(1)}, a_3^{(1)})$. The spatial sections $M$ of our universe model we construct gluing together (by splicing) the $S^4$-spheres in agreement with tree-type graphs like that we give in Fig. 1.

It is interesting to compute the intersection matrices for these manifolds. But let us first recall that due to the Poincaré–Lefschetz duality $\omega_Z : H^2(X, \partial X, \mathbb{Z}) \otimes H^2(X, \mathbb{Z}) \to \mathbb{Z}$ the integer intersection form can be determined as a cup product $\omega_Z(b, f) = \langle b \cup f, [X, \partial X] \rangle$ for any $b \in H^2(X, \partial X, \mathbb{Z})$ and $f \in H^2(X, \mathbb{Z})$. This is a generalization of the intersection form in the de Rham representation, $\int_X b \wedge f$.

It is well known that, if one takes some bases $b_i$ and $f_i$ of groups $H^2(X, \partial X, \mathbb{Z})$ and $H^2(X, \mathbb{Z})$ which are mutually dual in the sense that $\omega_Z(b_i, f_j) = \delta_i^j$, the integer intersection matrix $\omega_Z(b_i, b_j)$ will be inverse to the rational one $\omega_Q(f_i, f_j)$. These intersection matrices represent basic topological invariants of any graph manifold. We consider solutions of the dynamical equations $dB = 0$, $F = \Lambda^{ab} B_b$ as mutually dual bases of the respective cohomology groups, so that $f^a = \frac{1}{2\pi} F^a$ and $b_a = \frac{1}{2\pi} B_a$. The second equation yields $\omega_Q(f^a, f^b) = \Lambda^{ab}$, $\omega_Q(b_a, b_b) = \Lambda_{ab}$. Thus the coupling constants matrices $\Lambda^{ab}$ and $\Lambda_{ab}$ should be identified with the rational and integer intersection matrices, respectively. Now we return to the graph which describes spacetime of our model. Each free vertex of this graph (i.e., each $S^4$-sphere) corresponds to an element $b_a$ of the basis and to an element $f^a$ of the dual basis. Then the intersection matrix elements correspond to the respective $S^4$-spheres forming the graph. We associate with any
Sfh-sphere a certain interaction (inclusion of one more Sfh-sphere into our universe model results in switching on one new interaction). We associate with the Sfh-spheres in the extreme right-hand “column” low-energy (LE) interactions (strong, electromagnetic, weak, gravitational and cosmological). Elimination of these Sfh-spheres yields a graph describing the earlier stage of the cosmological evolution and a family of higher energy interactions, thus our model involves an interactions unification scheme. Each Sfh-sphere has three edges corresponding to three special orbits, so one may glue it into the graph in three different manners which yields $3^{15}$ different universes. A hypothesis that the “real universe” is found in a mixed state suggests that the physical meaning belongs to the average intersection matrix:

$$
\begin{pmatrix}
10^4 & 0 & 0 & 10^{-3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0

0 & 10^{-20} & 10^{-13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0

0 & 10^{-13} & 10^{-6} & 10^{-7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0

0 & 0 & 10^{-7} & 10^{-4} & 10^{-4} & 0 & 10^{-6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0

10^{-9} & 0 & 0 & 10^{-4} & 10^{-2} & 10^{-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0

0 & 0 & 0 & 0 & 10^{-2} & 10^{-2} & 10^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0

0 & 0 & 0 & 10^{-6} & 0 & 0 & 10^{-5} & 10^{-12} & 0 & 0 & 10^{-10} & 0 & 0 & 0 & 0 & 0

0 & 0 & 0 & 0 & 0 & 0 & 10^{-12} & 10^{-11} & 10^{-23} & 0 & 0 & 0 & 0 & 0 & 0 & 0

0 & 0 & 0 & 0 & 0 & 0 & 0 & 10^{-21} & 10^{-21} & 10^{-44} & 10^{-44} & 10^{-68} & 0 & 0 & 0 & 0

0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10^{-10} & 0 & 0 & 0 & 10^{-9} & 10^{-17} & 0 & 0 & 0

0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10^{-17} & 10^{-16} & 10^{-33} & 0 & 0 & 0 & 0 & 0

0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10^{-33} & 10^{-31} & 10^{-65} & 0 & 0 & 0 & 0

0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10^{-65} & 10^{-61} & 10^{-130} & 0 & 0 & 0

0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10^{-130} & 10^{-197} & 0 & 0 & 0
\end{pmatrix}
$$

In this matrix, the (red) boldface non-diagonal elements represent the dimensionless coupling constants of LE interactions. These elements really reproduce the hierarchy experimentally known for coupling constants of the fundamental interactions. It is remarkable that in our model the coupling constants of which $\Lambda^{ab}$ is built, are basic topological invariants of the four-dimensional spacetime $X$. In fact, we theoretically reproduced the vacuum coupling constants hierarchy existing in the real universe if the spacetime is a graph manifold. In our scheme the five LE interactions are related to the first nine prime numbers. To obtain any new interaction, one has to attach a new pair of prime numbers to the preceding set. With the next pair (29,31), the same algorithm yields a new coupling constant of the order of magnitude $\alpha_6 \approx 10^{-361}$. Thus our model answers the question: How many fundamental interactions may exist in the universe? To the infinite succession of prime numbers should correspond infinite number of interactions. We simply cannot detect too weak interactions beginning with $\alpha_6$, and all subsequent are even much weaker.

References

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