MULTI-STEP ONLINE UNSUPERVISED DOMAIN ADAPTATION

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ABSTRACT

In this paper, we address the Online Unsupervised Domain Adaptation (OUDA) problem, where the target data are unlabelled and arriving sequentially. The traditional methods on the OUDA problem mainly focus on transforming each arriving target data to the source domain, and they do not sufficiently consider the temporal coherency and accumulative statistics among the arriving target data. We propose a multi-step framework for the OUDA problem, which institutes a novel method to compute the mean-target subspace inspired by the geometrical interpretation on the Euclidean space. This mean-target subspace contains accumulative temporal information among the arrived target data. Moreover, the transformation matrix computed from the mean-target subspace is applied to the next target data as a preprocessing step, aligning the target data closer to the source domain. Experiments on four datasets demonstrated the contribution of each step in our proposed multi-step OUDA framework and its performance over previous approaches.

Index Terms— Unsupervised domain adaptation, online domain adaptation, mean subspace, Grassmann manifold.

1. INTRODUCTION

Domain Adaptation (DA) [1] aims to reduce the discrepancy between different distributions of the source and the target domains. In particular, the Unsupervised Domain Adaptation (UDA) problem focuses on the study that the target data are completely unlabelled, which is more plausible assumption for the recognition tasks in the real world.

There have been many studies on the UDA problem. For one branch of the studies, Gong et al. [2] and Fernando et al. [3] assumed that the source and the target domains share the common low-dimensional subspace. For another branch of studies on the UDA problem, Long et al. [4] and Sun et al. [5] directly minimized the discrepancy between the source and the target domains. Furthermore, Zhang et al. [6] and Wang et al. [7] combined the techniques in both branches. Vascon et al. [8] and Wulfmeier et al. [9] suggested a new technique for the UDA problem using Nash equilibrium [10] and Generative Adversarial Networks (GAN) [11], respectively.

We notice that only a few work has been conducted on the Online Unsupervised Domain Adaptation (OUDA) problem, which assumes that the target data are arriving sequentially as a small batch. Mancini et al. [12] adopted a batch normalization technique [13] for online domain adaptation, which was restricted to the kitting task only. Wulfmeier et al. [9] expanded his previous work on GANs to the online case. Bitarafan et al. proposed Incremental Evolving Domain Adaptation (IEDA) [14] algorithm, which computes the target data transformation using Geodesic Flow Kernel (GFK) [2] followed by updating the source subspace using Incremental Partial Least Square (IPLS) [15]. This approach is vulnerable when the target data are predicted incorrectly because the ill-labelled target data would be merged with the source-domain data, leading to worse prediction of future target data. Hoffman et al. [16] proposed an OUDA method using Continuous Manifold-based Adaptation (CMA), which formulated the OUDA problem as a non-convex optimization problem. However, this method merely considered the coherency among the adjacent target-data batches.

To overcome the drawbacks of the above methods – contamination of the source domain and lack of temporal coherency, we propose a multi-step framework for the OUDA problem, which institutes a novel method of computing the mean-target subspace inspired by the geometrical interpretation in the Euclidean space. Previous subspace-based methods on the OUDA problem merely compute the transformation matrix between the source subspace and each target subspace. Our method instead computes the transformation matrix between the source subspace and the mean-target subspace, which is incrementally obtained on the Grassmann manifold. Since a subspace is represented as a single point on the Grassmann manifold, the mean-target subspace is regarded as the mean point of multiple points that represent target subspaces of target-data batches. Although Karcher mean [17] is a well-known method for computing the mean point on the Grassmann manifold, it is not suitable for the OUDA problem since the Karcher mean is computed with an iterative process. Instead of the Karcher mean, we propose to compute the mean-target subspace by a geometrical process, which resembles the process of incremental computation for the mean point of a given multiple points on the Euclidean space. The transformation matrix computed with our proposed method is robust to the abrupt change of arriving target batches, leading to a stable domain transfer. We also feed the transformation matrix back to the next target batch, which moves it closer to the source domain. This preprocessing step of the next target batch leads to a more precise computation of the mean-target subspace. Experiments on four datasets demonstrated that our proposed method outperforms the traditional methods in terms of performance and computation speed.

2. PROPOSED APPROACH

2.1. Problem Description

We assume that the data in the source domain \( \mathbf{X}_S \in \mathbb{R}^{N_S \times d} \) are static and labelled as \( \mathbf{Y}_S \in \mathbb{R}^{N_S \times c} \), where \( N_S, d \) and \( c \) indicate the numbers of source data, dimension of the data and the number of the class categories, respectively. Data in the target domain are unlabelled and arriving as one batch in each timestep as \( \mathbf{X}_T = \{ \mathbf{X}_{T,1}, \mathbf{X}_{T,2}, \ldots, \mathbf{X}_{T,B} \} \), which are assumed to be sequential and...
Fig. 1: The proposed OUDA method consists of four steps: 1) Subspace representation, 2) Averaging mean-target subspaces, 3) Domain adaptation, and 4) Recursive feedback.

temporally correlated. We use the term mini-batch for the $n^{th}$ target-data batch $X_{T,n} \in \mathbb{R}^{N_T \times d}$ and $B$ indicates the number of mini-batches and $N_T$ indicates the number of data in each mini-batch. $N_T$ is assumed to be constant for $n = 1, 2, \ldots, B$ and very small compared to $N_S$. In our notation, the subscripts $S$ and $T$ indicate the source and the target domains, respectively. Furthermore, subscript $(T, n)$ represents the $n^{th}$ mini-batch in the target domain.

Our goal is to align the target-data batch $X_{T,n}$ to the source domain at $n = 1, 2, \ldots, B$ in an online manner so that the transformed target data $X'_{T,n}$ can be recognized correctly as $X_{T,n}$ with the classifier pre-trained in the source domain. Using the notation of [2], we denote the subspace with its basis $P_S$ and $P_{T,n} \in \mathbb{R}^{d \times k}$, where $d$ is the dimension of the original data and $k$ is the dimension of the subspace. For instance, $P_T = \{P_{T,1}, P_{T,2}, \ldots, P_{T,B}\}$ is the set of target subspaces composed of entire mini-batches, whereas $P_{T,n}$ is the target subspace for the $n^{th}$ mini-batch. For example, for Principal Component Analysis (PCA) [18], this subspace represents the projection matrix from the original space to the subspace.

2.2. Proposed OUDA Method

As shown in Fig. 1, our proposed OUDA framework consists of four steps for the incoming $n^{th}$ mini-batch: 1) Subspace representation, 2) Averaging mean-target subspace, 3) Domain adaptation, and 4) Recursive feedback. Step one computes the low-dimensional subspace, $P_{T,n}$, of the target domain using PCA. Step two computes the mean of the target subspaces $P_{T,n}$ embedded in the Grassmann manifold using our novel technique, Incremental Computation of Mean Subspace (ICMS). Step three is the domain adaptation and it computes the transformation matrix $G_n$ from the target domain to the source domain based on the approach of Bitarafan et al. [14] which adopts the GFK method [2], a manifold alignment technique. Step four provides recursive feedback by feeding $G_n$ back to the next mini-batch $X_{T,n+1}$. Each step is described next in detail.

2.2.1. Subspace Representation

The ultimate goal of our proposed OUDA method is to find the transformation matrix $G = \{G_1, G_2, \ldots, G_B\}$ that transforms the set of target mini-batches $X_T = \{X_{T,1}, X_{T,2}, \ldots, X_{T,B}\}$ to $X'_T = \{X'_{T,1}, X'_{T,2}, \ldots, X'_{T,B}\}$ so that these transformed target data are well aligned to the source domain, where $G_n \in \mathbb{R}^{d \times d}$ indicates the transformation matrix from $X_{T,n}$ to $X'_{T,n}$. However, we prefer not to use the methods that compute $G_n$ directly on the original data space with high dimension $d$. For example, raw input image features have dimension $d = 4096$, and the technique, which directly computes the transformation matrix by Correlation Alignment (CORAL) [5], requires to compute $4096 \times 4096$ matrix. Since our technique is desired to be conducted in online manner, we embed the source $X_S$ and the target data $X_T = \{X_{T,1}, X_{T,2}, \ldots, X_{T,B}\}$ to low-dimensional spaces as $P_S$ and $P_T = \{P_{T,1}, P_{T,2}, \ldots, P_{T,B}\}$, respectively, which preserve the meaningful information of the original data space. We adopt PCA to obtain $P_S$ and $P_T$ since PCA algorithm is simple and fast for online DA, and it is available for both labelled and unlabelled data unlike other dimension-reduction techniques such as Linear Discriminant Analysis (LDA) [19].

2.2.2. Averaging Mean-target Subspace

Throughout this paper, we utilize Grassmann manifold $G(k, d)$ [20], a space that parameterizes all $k$ dimensional linear subspaces of $d$ dimensional vector space. Since a subspace is represented as a single point on the Grassmann manifold, $P_S$ and $P_{T,1}, P_{T,2}, \ldots, P_{T,B}$ are represented as $(B + 1)$ points on $G(k, d)$.

For solving the offline OUDA problem (i.e., $B = 1$), Gong et al. [2] utilized the geodesic flow from $P_S$ to $P_T$ on $G(k, d)$. Previous methods for the OUDA problem directly compute the transformation matrix based on the source and the target subspaces of each mini-batch. We propose a novel technique, called Incremental Computation of Mean Subspace (ICMS), which computes the mean subspace in the target domain inspired by the geometrical interpretation on the Euclidean space. Then we compute the geodesic flow from $P_S$ to $P_{T,n}$. Formally, when the $n^{th}$ mini-batch $X_{T,n}$ arrives and is represented as the subspace $P_{T,n}$, we incrementally compute the mean-target subspace $P_{T,n}$ using $P_{T,n-1}$ and $P_{T,n-1}$, where $P_{T,n-1}$ is the mean subspace of $(n-1)$ target subspaces $P_{T,1}, P_{T,2}, \ldots, P_{T,n-1}$.

As shown in Fig. 2(a), the mean point $\bar{X}_n$ can be computed in an incremental way when $n$ points $X_1, X_2, \ldots, X_n$ are on the Euclidean space. If the mean point $\bar{X}_{n-1}$ of $n-1$ points $X_1, X_2, \ldots, X_{n-1}$ and the $n^{th}$ point $X_n$ are given, the updated mean point $\bar{X}_n$ is computed as $\bar{X}_n = \{(n-1)\bar{X}_{n-1} + X_n\}/n$. From a geometrical perspective, $\bar{X}_n$ is the internal point where the distances from $X_n$ to $\bar{X}_{n-1}$ and to $\bar{X}_n$ have the ratio of $1:(n-1)$:

$$|\bar{X}_{n-1} - \bar{X}_n| = \frac{|X_{n-1} - \bar{X}_n|}{n}.$$ (1)
We adopt this ratio concept to the Grassmann manifold from a geometrical perspective. As shown in Fig. 2(b), we update the mean-target subspace \( \overline{P}_{T,n} \) of \( n \) target subspaces when the previous mean subspace \( \overline{P}_{T,n-1} \) of \((n-1)\) target subspaces and \( n^{th} \) subspace \( P_{T,n} \) are given. Using the geodesic parameterization [21] with a single parameter \( t \), the geodesic flow from \( \overline{P}_{T,n-1} \) to \( P_{T,n} \) is parameterized as \( \Psi_n(t) \in G(k,d) \):

\[
\Psi_n(t) = \overline{P}_{T,n-1} U_{1,n} \Gamma_n(t) - \overline{R}_{T,n-1} U_{2,n} \Sigma_n(t) \quad (2)
\]

under the constraints \( \Psi_n(0) = \overline{P}_{T,n-1} \) and \( \Psi_n(1) = P_{T,n} \). It is valid to apply this ratio concept on the Euclidean space to the geodesic flow on the Grassmann manifold since \( t \) is parameterized proportionally to the arc length of \( \Psi_n(t) \) [22]. \( \overline{R}_{T,n-1} \in \mathbb{R}^{d \times (d-k)} \) denotes the orthogonal complement to \( \overline{P}_{T,n-1} \); that is, \( \overline{R}_{T,n-1}^T \overline{P}_{T,n-1} = 0 \). Two orthonormal matrices \( U_{1,n} \in \mathbb{R}^{k \times k} \) and \( U_{2,n} \in \mathbb{R}^{(d-k) \times (d-k)} \) are given by the following pair of singular-value decompositions (SVDs),

\[
\begin{align*}
\overline{P}_{T,n-1}^T P_{T,n} &= U_{1,n} \Gamma_n V_n^T \quad (3) \\
\overline{R}_{T,n-1}^T P_{T,n} &= -U_{2,n} \Sigma_n V_n^T \quad (4)
\end{align*}
\]

where \( \Gamma_n \in \mathbb{R}^{k \times k} \) and \( \Sigma_n = [(\Sigma_{1,n}^T O^T)^T] \in \mathbb{R}^{(d-k) \times k} \) are diagonal and block diagonal matrices, respectively, and \( \Sigma_{1,n} \in \mathbb{R}^{k \times k} \) and \( O \in \mathbb{R}^{(d-2k) \times k} \). Since the dimension of \( O \) should be positive, \((d - 2k)\) should be greater than 0. We assume that the dimension of the subspace \( k \) is much smaller than the dimension of the original space \( d \) so that \( k \ll d/2 \). The diagonal elements of \( \Gamma_n \) and \( \Sigma_{1,n} \) are \( \cos \theta_{i,n} \) and \( \sin \theta_{i,n} \) for \( i = 1, 2, \ldots, k \). These \( \theta_{i,n} \)'s are the principal angles [23] between \( \overline{P}_{T,n-1} \) and \( P_{T,n} \). \( \Gamma_n(t) \) and \( \Sigma_n(t) = [\Sigma_{1,n}(t)^T O(t)^T]^T \) are diagonal and block diagonal matrices whose elements are \( \cos(\theta_{i,n}(t)) \) and \( \sin(\theta_{i,n}(t)) \), respectively.

Finally, we adopt the ratio concept from Eq. (1) to \( \Psi_n(t) \) and obtain \( \overline{P}_{T,n} = \Psi_n \left( \frac{1}{n} \right) \). Hence, we can incrementally compute the mean-target subspace as follow:

\[
\overline{P}_{T,n} = \overline{P}_{T,n-1} U_{1,n} \Gamma_n \left( \frac{1}{n} \right) - \overline{R}_{T,n-1} U_{2,n} \Sigma_n \left( \frac{1}{n} \right). \quad (5)
\]

Note that \( n \) refers to the \( n^{th} \) mini-batch in the target domain. Since \( 0 \leq \frac{1}{n} \leq 1 \), \( \Gamma_n \left( \frac{1}{n} \right) \) and \( \Sigma_n \left( \frac{1}{n} \right) \) are well defined.

### 2.2.3. Domain Adaptation

After computing the mean-target subspace \( \overline{P}_{T,n} \), we parameterize another geodesic flow from \( P_S \) to \( \overline{P}_{T,n} \) as \( \Phi_n(t) \in G(k,d) \):

\[
\Phi_n(t) = P_S U_{3,n} \Lambda_n(t) - R_S U_{4,n} \Omega_n(t) \quad (6)
\]

under the constraints \( \Phi_n(0) = P_S \) and \( \Phi_n(1) = P_{T,n} \). \( R_S \in \mathbb{R}^{d \times (d-k)} \) denotes the orthogonal complement to \( P_S \); that is, \( R_S^T P_S = 0 \). Two orthonormal matrices \( U_{3,n} \in \mathbb{R}^{k \times k} \) and \( U_{4,n} \in \mathbb{R}^{(d-k) \times (d-k)} \) are given by the following pair of SVDs,

\[
\begin{align*}
P_S^T P_{T,n} &= U_{3,n} \Lambda_n V_n^T \quad (7) \\
R_S^T P_{T,n} &= -U_{4,n} \Omega_n W_n^T \quad (8)
\end{align*}
\]

Based on the GFK, the transformation matrix \( G_n \) from the target domain to the source domain is found by projecting and integrating over the infinite set of all intermediate subspaces between them:

\[
\int_0^1 \Phi_n(\alpha)^T x_i(\alpha) d\alpha = x_i G_n x_j. \quad (9)
\]

From the above equation, we can derive the closed form of \( G_n \) as:

\[
G_n = \int_0^1 \Phi_n(\alpha) \Phi_n(\alpha)^T d\alpha. \quad (10)
\]

We adopt this \( G_n \) as the transformation matrix to the preprocessed target data as \( X'_{T,n} = X_{T,n}^r G_n \), which better aligns the target data to the source domain. \( X_{T,n}^r \) is the target data fed back from the previous mini-batch, which is described in the next section.

### 2.2.4. Recursive Feedback

Previous work on the OUDA problem does not evidently consider the temporal dependency between the subspace of adjacent target mini-batches. Unlike traditional methods, our proposed OUDA method feeds \( G_n \) back to the next target mini-batch as \( X'_{T,n+1} = X_{T,n+1} G_n \) at the next timestep \((n+1)\), which imposes the temporal dependency between \( X_{T,n} \) and \( X_{T,n+1} \) by moving \( P_{T,n} \) closer to \( \overline{P}_{T,n} \) on the Grassmann manifold. PCA is conducted from this \( X_{T,n+1}^r \) to represent the \((n+1)^{th}\) target subspace \( P_{T,n+1} \).

### 3. EXPERIMENTAL RESULTS

#### 3.1. Datasets

To evaluate our proposed OUDA method in data classification, we performed experiments on four datasets [14]– the Traffic dataset, the Car dataset, the Waveform21 dataset, and the Waveform40 dataset. These datasets provided a large variety of time-variant images and signals to test upon. The Traffic dataset includes images captured from a fixed traffic camera observing a road over a 2-week period. It consists of 5412 instances of \( d = 512 \) dimensional features with two classes as either heavy traffic or light traffic. Figure 3 depicts the image samples of the Traffic dataset. The Car dataset contains images of automobiles manufactured between 1950 and 1999 acquired from online database. It includes 1770 instances of \( d = 4096 \) dimensional features with two classes as sedans or trucks. The Waveform21 dataset is composed of 5000 wave instances of \( d = 21 \) dimensional features with three classes. The Waveform40 dataset is the second version of the Waveform21 with additional features. This dataset consists of \( d = 40 \) dimensional features.
the accuracy by incrementally including each step to the process of OUDA. Except for the EDA method, which adopted Incremental Semi-Supervised Learning (ISSL) technique for classifying the unlabelled target data, all other approaches adopted the basic K-Nearest-Neighbors [24] or Support-Vector-Machine [25] classifiers for target-label prediction. Table 1 shows that averaging the mean-target subspace (Gmean) and recursive feedback (FB) steps improved the performance the most. Gmean and FB steps improved the performance at 4.27% and 4.08% respectively, compared to EDA. These results indicated that computing the mean-target subspace leads to stable computation of the transformation matrix $G_n$. Furthermore, feeding $G_n$ back to the $(n+1)^{th}$ target mini-batch shifted it closer to the source domain.

### 3.3. Ablation Study

In order to understand which step of our proposed method contributes to the improvement of the accuracy performance, we also measured the accuracy for the different variants of our proposed OUDA method and compared their performance. We compared the accuracy by incrementally including each step to the process of OUDA. For the EDA method, which adopted Incremental Semi-Supervised Learning (ISSL) technique for classifying the unlabelled target data, all other approaches adopted the basic K-Nearest-Neighbors [24] or Support-Vector-Machine [25] classifiers for target-label prediction.

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### 3.4. Computation Time

We evaluated the computation time of our proposed OUDA method as compared to the previous methods in the same datasets above. As shown in Table 2, our proposed OUDA method was significantly faster (i.e., 1.84 to 7.00 times) for all the datasets except the Car dataset, which indicated that our proposed method was more suitable for online DA. Since the Car dataset consists of $d = 4096$ dimensional features, it consumed more time to compute the mean-target subspace as well as the geodesic curve from the source subspace to the mean-target subspace.

### 4. CONCLUSIONS

We have described a multi-step framework for tackling the OUDA problem for classification problem when target data are arriving in mini-batches. Inspired by the geometrical interpretation of computing mean point on the Euclidean space, we proposed computing the mean-target subspace on the Grassmann manifold incrementally for mini-batches. Inspired by the geometrical interpretation of computing mean point on the Euclidean space, we proposed computing the mean-target subspace on the Grassmann manifold incrementally for mini-batches. The transformation matrix computed from the source subspace and the mean-target subspace aligned the target data closer to the source domain. Recursive feedback of domain adaptation increases the robustness of the recognition system for abrupt change of target data. Fast computation time due to the usage of low-dimensional space enables our proposed method to be applied to OUDA in real-time.
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