INTRODUCTION TO W-ALGEBRAS AND THEIR REPRESENTATION THEORY

TOMOYUKI ARAKAWA

Abstract. These are lecture notes from author’s mini-course during Session 1: “Vertex algebras, W-algebras, and application” of INdAM Intensive research period “Perspectives in Lie Theory”, at the Centro di Ricerca Matematica Ennio De Giorgi, Pisa, Italy. December 9, 2014 – February 28, 2015.

1. Introduction

This note is based on lectures given at the Centro di Ricerca Matematica Ennio De Giorgi, Pisa, in Winter of 2014–2015. They are aimed as an introduction to W-algebras and their representation theory. Since W-algebras appear in many areas of mathematics and physics there are certainly many other important topics untouched in the note, partly due to the limitation of the space and partly due to the author’s incapability.

The W-algebras can be regarded as generalizations of affine Kac-Moody algebras and the Virasoro algebra. They appeared [Zam, FL, LF] in the study of the classification of two-dimensional rational conformal field theories. There are several ways to define W-algebras, but it was Feigin and Frenkel [FF1] who found the most conceptual definition of principal W-algebras that uses the quantized Drinfeld-Sokolov reduction, which is a version of Hamiltonian reduction. There are a lot of works on W-algebras (see [BS] and references therein) mostly by physicists in 1980’s and 1990’s, but they were mostly on principal W-algebras, that is, the W-algebras associated with principal nilpotent elements. It was quite recent that Kac, Roan and Wakimoto [KRW] defined the W-algebra $W^k(\mathfrak{g}, f)$ associated with a simple Lie algebra and its arbitrary nilpotent element $f$ by generalizing the method of quantized Drinfeld-Sokolov reduction.

The advantage of the method of quantized Drinfeld-Sokolov reduction is its functoriality, in the sense that it gives rise to a functor from the category of representations of affine Kac-Moody algebras and to the category of representations of W-algebras. Since it is difficult to study W-algebras directly (as no presentation by generators and relations (OPE’s) is known for a general W-algebra), in this note we spend the most of our efforts in understanding this functor.

Although our methods apply to much more general settings ([A1, A3, A6, A7, A9]) we focus on the W-algebras associated with Lie algebras $\mathfrak{g}$ of type $A$ and its principal nilpotent element that were originally defined by Fateev and Lykyanov [FL]. They can be regarded as affinization of the center of the universal enveloping algebra of $\mathfrak{g}$ via Konstant’s Whittaker model [Kos] and Kostant-Sternberg’s
description [KS] of Hamiltonian reduction via BRST cohomology, as explained in [FF1]. For this reason we start with a review of Kostant’s results and proceed to the construction of BRST complex in the finite-dimensional setting in §2. $W$-algebras are not Lie algebras, not even associated algebras in general, but vertex algebras. In many cases a vertex algebra can be considered as a quantization of arc spaces of an affine Poisson scheme. In §3 we study this viewpoint that is useful in understanding $W$-algebras and their representation theory. In §4 we study Zhu’s algebras of vertex algebras that connects $W$-algebras with finite $W$-algebras [dBT1, Pre]. In §5 we introduce $W$-algebras and study their basic properties. In §6 we start studying representation theory of $W$-algebras. In §7 we quickly review some fundamental results on irreducible representations of $W$-algebras obtained in [A2]. One of the fundamental problems (at least mathematically) on $W$-algebras was the conjecture of Frenkel, Kac and Wakimoto [FKW] on the existence and construction of so called the minimal models of $W$-algebras, which give rive to rational conformal field theories as in the case of the integrable representations of affine Kac-Moody algebras and the minimal models of the Virasoro algebra. In §8 we give an outline of the proof [A7] of this conjecture.

**Acknowledgments.** The author is grateful to the organizers of “Perspectives in Lie Theory”. He thanks Naoki Genra and Xiao He who wrote the first version of this note. He would also like to thank Anne Moreau and Alberto De Sole for useful comments on the preliminary version of this note. His research is supported by JSPS KAKENHI Grant Numbers 25287004 and 26610006.

2. Review of Kostant’s results

2.1. Companion matrices and invariant polynomials. Let $G = GL_n(\mathbb{C})$ be the general linear group, and let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ be the general linear Lie algebra consisting of $n \times n$ matrices. The group $G$ acts on $\mathfrak{g}$ by the adjoint action: $x \mapsto \text{Ad}(g)x = gxg^{-1}$, $g \in G$. Let $\mathbb{C}[\mathfrak{g}]^G$ be the subring of the ring $\mathbb{C}[\mathfrak{g}]$ of polynomial functions on $\mathfrak{g}$ consisting of $G$-invariant polynomials.

Recall that a matrix

\begin{align*}
A &= \begin{pmatrix}
0 & 0 & \cdots & 0 & -a_1 \\
1 & 0 & \cdots & 0 & -a_2 \\
0 & 1 & \cdots & 0 & -a_3 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & -a_n
\end{pmatrix}
\end{align*}

(2.1)

is called the companion matrix of the polynomial $a_1 + a_2t + a_3t^2 + \cdots + a_nt^n \in \mathbb{C}[t]$ since

\begin{align*}
\det(tI - A) &= a_1 + a_2t + a_3t^2 + \cdots + a_nt^n.
\end{align*}

(2.2)

Let $S$ be the affine subspace of $\mathfrak{g}$ consisting of companion matrices of the form (2.1).

**Lemma 2.1.** For $A \in \mathfrak{g}$ the following conditions are equivalent.

1. $A \in G \cdot S$. 

There exists a vector \( v \in \mathbb{C}^n \) such that \( v, Av, A^2v, \ldots, A^{n-1}v \) are linearly independent.

**Theorem 2.2.** The restriction map gives the isomorphism

\[
\mathbb{C}[g]^G \cong \mathbb{C}[S].
\]

**Proof.** Let \( f \in \mathbb{C}[g] \) be a \( G \)-invariant polynomial such that \( f|_S = 0 \). Then clearly \( f|_{G.S} = 0 \). On the other hand it follows from Lemma 2.1 that \( G.S \) is a Zariski open subset in \( g \). Therefore \( f = 0 \). To see the surjectiveness define \( p_1, \ldots, p_n \in \mathbb{C}[g]^G \) by

\[
\det(tI - A) = t^n + p_1(A)t^{n-1} - \cdots + p_n(A), \quad A \in g.
\]

By (2.2), we have \( \mathbb{C}[S] = \mathbb{C}[p_1|_S, \ldots, p_n|_S] \). This completes the proof. \( \square \)

Put

\[
f := \begin{pmatrix}
0 \\
1 & \ddots \\
& \ddots & \ddots \\
& & 1 & 0
\end{pmatrix} \in S.
\]

(2.3)

Note that \( f \) is a nilpotent element of \( g \), that is, \((\text{ad} f)^r = 0 \) for a sufficiently large \( r \). We have

\[
S = f + a,
\]

where

\[
a = \begin{cases}
\begin{pmatrix}
0 & \cdots & 0 & * \\
& \ddots & \cdots & * \\
& & \ddots & \cdots & * \\
& & & \cdots & * \\
0 & \cdots & & & 0 & *
\end{pmatrix}
\end{cases}.
\]

Let \( b, n \) be the subalgebras of \( g \) defined by

\[
b = \begin{cases}
\begin{pmatrix}
* & \cdots & * \\
& \ddots & \cdots & * \\
& & 0 & \cdots & * \\
& & & \ddots & \cdots & * \\
& & & & \cdots & * 
\end{pmatrix}
\end{cases}, \quad n = \begin{cases}
\begin{pmatrix}
0 & \cdots & * \\
& \ddots & \cdots & * \\
& & 0 & \ddots & * \\
& & & \ddots & \cdots & * \\
& & & & \cdots & 0 
\end{pmatrix}
\end{cases} \subset b,
\]

and let \( N \) be the unipotent subgroup of \( G \) corresponding to \( n \), i.e.,

\[
(2.4) \quad N = \begin{cases}
\begin{pmatrix}
1 & \cdots & * \\
& \ddots & \cdots & * \\
& & 0 & \cdots & * \\
& & & \ddots & \cdots & * \\
& & & & \cdots & 1 
\end{pmatrix}
\end{cases}.
\]

Let ( \( | \) ) be the invariant inner product of \( g \) defined by \( (x|y) = \text{tr}(xy) \). This gives a \( G \)-equivariant isomorphism \( g \cong g^* \).
Define $\chi \in \mathfrak{n}^*$ by

$$\chi(x) = (f|x) \quad \text{for} \ x \in \mathfrak{n}. $$

Note that $\chi$ is a character of $\mathfrak{n}$, that is, $\chi([\mathfrak{n},\mathfrak{n}]) = 0$. Hence $\chi$ defines an one-dimensional representation of $N$.

Consider the restriction map $\mu : \mathfrak{g}^* \to \mathfrak{n}^*$. Then

$$\mu^{-1}(\chi) = \chi + \mathfrak{n}^\perp \simeq f + \mathfrak{b}. $$

Here $\mathfrak{g}$ is identified with $\mathfrak{g}^*$ via $(|)$. Since $\mu$ is $N$-equivariant and $\chi$ is a one-point orbit of $N$, it follows that $f + \mathfrak{b}$ is stable under the action of $N$.

**Theorem 2.3** (Kostant [Kos]). The adjoint action gives the isomorphism

$$N \times \mathcal{S} \to f + \mathfrak{b}, \quad (g, x) \mapsto \text{Ad}(g)x$$

of affine varieties.

**Proof.** It is not difficult to see that the adjoint action gives the bijection $N \times \mathcal{S} \to f + \mathfrak{b}$. Since it is a morphism of irreducible varieties and $f + \mathfrak{b}$ is normal, the assertion follows from Zariski’s Main Theorem (see e.g., [TY, Corollary 17.4.8]). \qed

**Corollary 2.4.** The restriction map gives the isomorphisms

$$\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[f + \mathfrak{b}]^N \to \mathbb{C}[\mathcal{S}].$$

**Proof.** By Theorem 2.3, we have

$$\mathbb{C}[f + \mathfrak{b}]^N \cong \mathbb{C}[N]^N \otimes \mathbb{C}[\mathcal{S}] \cong \mathbb{C}[\mathcal{S}].$$

Hence the assertion follows from Theorem 2.2. \qed

2.2. Transversality of $\mathcal{S}$ to $G$-orbits.

**Lemma 2.5.** The affine spaces $\mathcal{S}$ and $f + \mathfrak{b}$ intersect transversely at $f$ to $\text{Ad}G \cdot f$.

**Proof.** We need to show that

$$(2.5) \quad T_f \mathfrak{g} = T_f \mathcal{S} + T_f(\text{Ad}G \cdot f)$$

But $T_f \mathfrak{g} \cong \mathfrak{g}$, $T_f \mathcal{S} \cong \mathfrak{a}$, $T_f(\text{Ad}G \cdot f) \cong [\mathfrak{g}, f]$. The assertion follows since $\mathfrak{g} = \mathfrak{a} + [\mathfrak{g}, f]$. \qed

Using the Jacobson-Morozov theorem, we can embed $f$ into an $\mathfrak{sl}_2$-triple $\{e, f, h\}$ in $\mathfrak{g}$. Explicitly, we can choose the following elements for $e$ and $h$:

$$(2.6) \quad e = \sum_{i=1}^{n-1} i(n-i)e_{i,i+1}, \quad h = \sum_{i=1}^{n} (n+1-2i)e_{i,i},$$

where $e_{i,j}$ denotes the standard basis element of $\mathfrak{g} = \text{Mat}_n(\mathbb{C})$. 

The embedding $\mathfrak{sl}_2 = \text{span}_C \{e, h, f\} \to \mathfrak{g}$ exponents to a homomorphism $SL_2 \to G$. Restricting it to the torus $C^*$ consisting of diagonal matrices we obtain a one-parameter subgroup $\gamma : C^* \to G$. Set

(2.7) $\rho : C^* \ni t \to t^2 \text{Ad}(t) \in GL(\mathfrak{g})$.

Then

$\rho(t)(f + \sum_{i \leq j} c_{ij} e_{i,j}) = f + \sum_{i \leq j} t^{2(i-j+1)} c_{ij} e_{i,j}$. 

Thus it define a $C^*$-action on $\mathfrak{g}$ that preserves $f + b$ and $S$. This action on $f + b$ and $S$ contracts to $f$, that is, $\rho(t)x \to f$ when $t \to 0$.

**Proposition 2.6.** The affine space $f + b$ (resp. $S$) intersects $\text{Ad} G \cdot x$ transversely at any point $x \in f + b$ (resp. $x \in S$).

**Proof.** By Lemma 2.5 the intersection of $f + b$ with $\text{Ad} G$-orbits is trasversal at each point in some open neighborhood of $f$ in $f + b$. By the contracting $C^*$-action $\rho$, it follows that the same is true for all points of $f + b$. \qed

2.3. The trasversal slice $S$ as a reduced Poisson variety. The affine variety $\mathfrak{g}^*$ is equipped with the Kirillov-Kostant Poisson structure: the Poisson algebra structure of $C[\mathfrak{g}^*]$ is given by

$$\{x, y\} = [x, y] \quad \text{for } x, y \in \mathfrak{g} \subset C[\mathfrak{g}^*].$$

Consider the restriction map $\mu : \mathfrak{g}^* \to \mathfrak{n}^*$, which is a moment map for the $N$-action on $\mathfrak{g}^*$. That is, $\mu$ is a regular $N$-equivariant morphism that gives the following commutative diagram of Lie algebras:

$$
\begin{array}{ccc}
\mathbb{C}[\mathfrak{g}^*] & \xrightarrow{\mu^*} & \mathfrak{n}^* \\
\downarrow & & \downarrow \\
\mathbb{C}[\mathfrak{g}^*] & \xrightarrow{\text{Der}} & \text{Der}\mathbb{C}[\mathfrak{g}^*] \\
\end{array}
$$

Here $\mu^* : \mathfrak{n} \to \mathfrak{g} \subset \mathbb{C}[\mathfrak{g}^*]$ is the pullback map, the map $\mathbb{C}[\mathfrak{g}^*] \to \text{Der}\mathbb{C}[\mathfrak{g}^*]$ is given by $\phi \to \{\phi, ?\}$, and $\mu : \mathfrak{n} \to \text{Der}\mathbb{C}[\mathfrak{g}^*]$ is the Lie algebra homomorphism induced by the coadjoint action of $G$ on $\mathfrak{g}^*$.

The transversality statement of Proposition 2.6 for $f + b$ is equivalent to that $\chi$ is a regular value of $\mu$. By Theorem 2.3 the action of $N$ on $\mu^{-1}(\chi) = \chi + \mathfrak{n}^\perp$ is free and

$$S \cong \mu^{-1}(\chi)/N.$$

Therefore $S$ has the structure of the reduced Poisson variety, obtained from $\mathfrak{g}^*$ by the Hamiltonian reduction.

The Poisson structure of $S$ is described as follows. Let

$$I_\chi = \mathbb{C}[\mathfrak{g}^*] \sum_{x \in \mathfrak{n}} (x - \chi(x)),$$

so that

$$\mathbb{C}[\mu^{-1}(\chi)] = \mathbb{C}[\mathfrak{g}^*]/I_\chi.$$
Then \( \mathbb{C}[S] \) can be identified as the subspace of \( \mathbb{C}[g^*]/I_\chi \) consisting of all cosets \( \phi + \mathbb{C}[g^*]I_\chi \) such that \( \{ x, \phi \} \in \mathbb{C}[g^*]I_\chi \) for all \( x \in \mathfrak{n} \). In this realization, the Poisson structure on \( \mathbb{C}[S] \) is defined by the formula

\[
\{ \phi + \mathbb{C}[g^*]I_\chi, \phi' + \mathbb{C}[g^*]I_\chi \} = \{ \phi, \phi' \} + \mathbb{C}[g^*]I_\chi
\]

for \( \phi, \phi' \) such that \( \{ x, \phi \}, \{ x, \phi' \} \in \mathbb{C}[g^*]I_\chi \) for all \( x \in \mathfrak{n} \).

**Proposition 2.7.** We have the isomorphism \( \mathbb{C}[g^*]^G \cong \mathbb{C}[S] \) as Poisson algebras. In particular the Poisson structure of \( S \) is trivial.

**Proof.** The restriction map \( \mathbb{C}[g^*]^G \cong \mathbb{C}[S] \) (see Corollary 2.4) is obviously a homomorphism of Poisson algebras.

In the next subsection we shall describe the above Hamiltonian reduction in more factorial way, in terms of the BRST cohomology (where BRST refers to the physicists Becchi, Rouet, Stora and Tyutin) for later purpose.

### 2.4. BRST reduction

Let \( Cl \) be the Clifford algebra associated with the vector space \( \mathfrak{n} \oplus \mathfrak{n}^* \) and its non-degenerate bilinear form \( \langle \cdot, \cdot \rangle \) defined by \( (f + x|g + y) = f(y) + g(x) \) for \( f, g \in \mathfrak{n}^*, x, y \in \mathfrak{n} \). Namely, \( Cl \) is the unital \( \mathbb{C} \)-superalgebra that is isomorphic to \( \Lambda(\mathfrak{n}) \otimes \Lambda(\mathfrak{n}^*) \) as \( \mathbb{C} \)-vector spaces, the natural embeddings \( \Lambda(\mathfrak{n}) \hookrightarrow Cl \), \( \Lambda(\mathfrak{n}^*) \hookrightarrow Cl \) are homogeneous homomorphism of superalgebras, and

\[
[x, f] = f(x) \quad x \in \mathfrak{n} \subset \Lambda(\mathfrak{n}), \quad f \in \mathfrak{n}^* \subset \Lambda(\mathfrak{n}^*)
\]

(Note that \( [x, f] = xf - fx \) since \( x, f \) are odd.)

Let \( \{ x_\alpha \}_{\alpha \in \Delta_+} \) be a basis of \( \mathfrak{n} \), \( \{ x_\alpha^* \}_{\alpha \in \Delta_+} \) the dual basis of \( \mathfrak{n}^* \), and \( c^\gamma_{\alpha \beta} \) the structure constants of \( \mathfrak{n} \), that is, \( [x_\alpha, x_\beta] = \sum_{\gamma} c^\gamma_{\alpha \beta} x_\gamma \).

**Lemma 2.8.** The following map gives a Lie algebra homomorphism.

\[
\rho : \mathfrak{n} \longrightarrow Cl
\]

\[
x_\alpha \longrightarrow \sum_{\beta, \gamma \in \Delta_+} c^\gamma_{\alpha \beta} x_\gamma x_\beta^*
\]

We have

\[
[\rho(x), y] = [x, y] \in \mathfrak{n} \subset Cl \quad \text{for } x, y \in \mathfrak{n}
\]

Define an increasing filtration on \( Cl \) by setting \( Cl_p := \Lambda^{\leq p}(\mathfrak{n}) \otimes \Lambda(\mathfrak{n}^*) \). We have

\[
0 = Cl_{-1} \subset Cl_0 \subset Cl_1 \cdots \subset Cl_N = Cl,
\]

where \( N = \dim \mathfrak{n} = \frac{n(n-1)}{2} \), and

\[
(2.8) \quad Cl_p \cdot Cl_q \subset Cl_{p+q}, \quad [Cl_p, Cl_q] \subset Cl_{p+q-1}.
\]

Let \( \overline{Cl} \) be its associated graded algebra:

\[
\overline{Cl} := \text{gr } Cl = \bigoplus_{p \geq 0} \frac{Cl_p}{Cl_{p-1}}
\]

By (2.8), \( \overline{Cl} \) is naturally a graded Poisson superalgebra, called the classical Clifford algebra.
Lemma 2.9. We have\(\overline{Cl} = \Lambda(n) \otimes \Lambda(n^*)\) as a commutative superalgebra. Its Poisson (super)bracket is given by
\[
\{x, f\} = f(x), \quad x \in n \subset \Lambda(n), \quad f \in n^* \subset \Lambda(n^*),
\]
\[
\{x, y\} = 0, \quad x, y \in n \subset \Lambda(n), \quad \{f, g\} = 0, \quad f, g \in n^* \subset \Lambda(n^*).
\]

Lemma 2.10. We have \(\overline{Cl}^0 = \Lambda(n)\), where \(\overline{Cl}^0 := \{w \in \overline{Cl} \mid \{x, w\} = 0, \forall x \in n\}\).

The Lie algebra homomorphism \(\rho : n \rightarrow Cl_1 \subset Cl\) induces a Lie algebra homomorphism
\[
(2.9) \quad \overline{\rho} := \sigma_1 \circ \rho : n \rightarrow \overline{Cl},
\]
where \(\sigma_1\) is the projection \(Cl_1 \rightarrow Cl_1/Cl_0 \subset \text{gr} Cl\). We have
\[
\{\overline{\rho}(x), y\} = [x, y] \quad \text{for } x, y \in n.
\]

Set
\[
\bar{C}(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*] \otimes \overline{Cl}.
\]
Since it is a tensor product of Poisson superalgebras, \(\bar{C}(\mathfrak{g})\) is naturally a Poisson superalgebra.

Lemma 2.11 (\cite{BDT} Lemma 7.13.3). There exists a unique element \(\bar{Q} \in \bar{C}^1(\mathfrak{g})\) such that
\[
\{\bar{Q}, 1 \otimes x\} = \bar{\theta}_\chi(x) \quad \text{for } x \in n.
\]
We have \(\{\bar{Q}, \bar{Q}\} = 0\).

Proof. Existence. It is straightforward to see that the element
\[
\bar{Q} = \sum_\alpha (x_\alpha - \chi(x_\alpha)) \otimes x^*_\alpha - 1 \otimes \frac{1}{2} \sum_{\alpha, \beta, \gamma} c_{\alpha, \beta, \gamma} a^*_\alpha a_\beta x_\gamma
\]
satisfies the condition.

Uniqueness. Suppose that \(\bar{Q}_1, \bar{Q}_2 \in \bar{C}^1(\mathfrak{g})\) satisfy the condition. Set \(R = Q_1 - Q_2 \in \bar{C}^1(\mathfrak{g})\). Then \(\{R, 1 \otimes x\} = 0\), and so, \(R \in \mathbb{C}[\mathfrak{g}^*] \otimes \overline{Cl}^0\). But by Lemma 2.9, \(\overline{Cl} \cap \overline{Cl}^0 = 0\). Thus \(R = 0\) as required.

To show that \(\{\bar{Q}, \bar{Q}\} = 0\), observe that
\[
\{1 \otimes x, \{1 \otimes y, \{\bar{Q}, \bar{Q}\}\}\} = 0, \quad \forall x, y \in n
\]
(note that \(\bar{Q}\) is odd). Applying Lemma 2.9 twice, we get that \(\{\bar{Q}, \bar{Q}\} = 0\). \(\square\)
Since $Q$ is odd, Lemma 2.11 implies that
\[ \{ Q, \{ Q, a \} \} = \frac{1}{2} \{ \{ Q, Q \}, a \} = 0 \]
for any $a \in \bar{C}(\mathfrak{g})$. That is, $\text{ad} \bar{Q} := \{ \bar{Q}, \cdot \}$ satisfies that
\[ (\text{ad} \bar{Q})^2 = 0. \]
Thus, $(\bar{C}(\mathfrak{g}), \text{ad} \bar{Q})$ is a \textit{differential graded Poisson superalgebra}. Its cohomology $H^\bullet(\bar{C}(\mathfrak{g}), \text{ad} \bar{Q}) = \bigoplus_{i \in \mathbb{Z}} H^i(\bar{C}(\mathfrak{g}), \text{ad} \bar{Q})$ inherits a graded Poisson superalgebra structure from $\bar{C}(\mathfrak{g})$.

According to Kostant and Sternberg [KS] the Poisson structure of $\mathbb{C}[S]$ may be described through the following isomorphism:

**Theorem 2.12 (KS).** We have $H^i(\bar{C}(\mathfrak{g}), \text{ad} \bar{Q}) = 0$ for $i \neq 0$ and $H^0(\bar{C}(\mathfrak{g}), \text{ad} \bar{Q}) \cong \mathbb{C}[S]$ as Poisson algebras.

**Proof.** Give a bigrading on $\bar{C} := \bar{C}(\mathfrak{g})$ by setting
\[ \bar{C} = \bigoplus_{i \geq 0, j \leq 0} \bar{C}^{i,j} = \mathbb{C}[\mathfrak{g}^*] \otimes \Lambda^i(n^*) \otimes \Lambda^{-j}(n), \]
so that $\bar{C} = \bigoplus_{i \geq 0, j \leq 0} \bigoplus_{n \geq 0} \bar{C}^{i,j,n}$.

Observe that $\text{ad} \bar{Q}$ decomposes as $\text{ad} \bar{Q} = d_+ + d_-$ such that
\[ (2.10) \quad d_-(\bar{C}^{i,j}) \subset \bar{C}^{i,j+1}, \quad d_+(\bar{C}^{i,j}) \subset \bar{C}^{i+1,j}. \]
Explicitly, we have
\[ d_- = \sum_i (x_i - \chi(x_i)) \otimes \text{ad} x_i^*, \]
\[ d_+ = \sum_i \text{ad} x_i \otimes x_i^* - 1 \otimes \frac{1}{2} \sum_{k,j} c_{i,j,k}^k x_i^* x_j^* \text{ad} x_k + \sum_i 1 \otimes \bar{p}(x_i) \text{ad} x_i^*. \]
Since $\text{ad} \bar{Q}^2 = 0$, (2.10) implies that
\[ d_-^2 = d_+^2 = [d_-, d_+] = 0. \]
It follows that there exists a spectral sequence
\[ E_r \Rightarrow H^\bullet(\bar{C}(\mathfrak{g}), \text{ad} \bar{Q}) \]
such that
\[ E_1^{p,q} = H^q(\bar{C}(\mathfrak{g}), d_-) = H^q(\mathbb{C}[\mathfrak{g}^*] \otimes \Lambda(n), d_-) \otimes \Lambda^p(n^*), \]
\[ E_2^{p,q} = H^p(H^q(\bar{C}(\mathfrak{g}), d_-), d_+). \]
Observe that $(\bar{C}(\mathfrak{g}), d_-)$ is identical to the Koszul complex $\mathbb{C}[\mathfrak{g}^*]$ associated with the sequence $x_1 - \chi(x_1), x_2 - \chi(x_2), \ldots, x_N - \chi(x_N)$ tensorized with $\Lambda(n^*)$. Since $\mathbb{C}[\mu^{-1}(\chi)] = \mathbb{C}[\mathfrak{g}^*]/\sum C[\mathfrak{g}^*](x_i - \chi(x_i))$, we get that
\[ H^i(\bar{C}(\mathfrak{g}), d_-) = \begin{cases} \mathbb{C}[\mu^{-1}(\chi)] \otimes \Lambda(n^*), & \text{if } i = 0 \\ 0, & \text{if } i \neq 0. \end{cases} \]
Next, notice that \((H^0(C(\mathfrak{g}),d_-),d_+)\) is identical to the Chevalley complex for the Lie algebra cohomology \(H^\bullet(n,\mathbb{C}[\mu^{-1}(\chi)])\). Therefore Theorem 2.3 gives that
\[
H^i(H^\bullet(C(\mathfrak{g}),d_-),d_+) = \begin{cases} \mathbb{C}[S], & i = 0 \\ 0, & i \neq 0. \end{cases}
\]
Hence the spectral sequence collapses at \(E_2 = E_\infty\) and we get that \(H^i(\bar{C}(\mathfrak{g}), ad \bar{Q}) = 0\) for \(i \neq 0\). Moreover, there is an isomorphism
\[
H^0(\bar{C}(\mathfrak{g}), ad \bar{Q}) \cong H^0(H^0(\bar{C}(\mathfrak{g}), d_-), d_+ = \mathbb{C}[S], \quad [c] \mapsto [c].
\]
This completes the proof. \(\square\)

**Theorem 2.13.** The natural map \(\mathbb{C}[\mathfrak{g}^*]^G \to H^0(\bar{C}(\mathfrak{g}), ad \bar{Q})\) defined by sending \(p\) to \(p \otimes 1\) is an isomorphism of Poisson algebras.

**Proof.** It is clear that the map is a well-defined homomorphism of Poisson algebras since \(\mathbb{C}[\mathfrak{g}^*]^G\) is the Poisson center of \(\mathbb{C}[\mathfrak{g}^*]\). The assertion follows from the commutativity of the following diagram.

\[
\begin{array}{ccc}
\mathbb{C}[\mathfrak{g}^*]^G & \cong & H^0(\bar{C}(\mathfrak{g}), ad \bar{Q}) \\
\downarrow & & \downarrow \\
\mathbb{C}[S] & \cong & H^0(\bar{C}(\mathfrak{g}), ad \bar{Q}).
\end{array}
\]

\(\square\)

2.5. **Quantized Hamiltonian reduction.** We shall now quantize the above construction following [KS].

Let \(\{U_i(\mathfrak{g})\}\) be the PBW filtration of the universal enveloping algebra \(U(\mathfrak{g})\) of \(\mathfrak{g}\), that is, \(U_i(\mathfrak{g})\) is the subspace of \(U(\mathfrak{g})\) spanned by the products of at most \(i\) elements of \(\mathfrak{g}\). Then
\[
0 = U_{-1}(\mathfrak{g}) \subset U_0(\mathfrak{g}) \subset U_1(\mathfrak{g}) \subset \ldots, \quad U(\mathfrak{g}) = \bigcup_i U_i(\mathfrak{g}),
\]
\[
U_i(\mathfrak{g}) \cdot U_j(\mathfrak{g}) \subset U_{i+j}(\mathfrak{g}), \quad [U_i(\mathfrak{g}), U_j(\mathfrak{g})] \subset U_{i+j-1}(\mathfrak{g}).
\]
The associated graded space \(\text{gr} U(\mathfrak{g}) = \bigoplus_{i \geq 0} U_i(\mathfrak{g})/U_{i-1}(\mathfrak{g})\) is naturally a Poisson algebra, and the PBW Theorem states that
\[
\text{gr} U(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}^*]
\]
as Poisson algebras. Thus, \(U(\mathfrak{g})\) is a quantization of \(\mathbb{C}[\mathfrak{g}^*]\).

Define
\[
C(\mathfrak{g}) = U(\mathfrak{g}) \otimes \text{Cl}.
\]
It is naturally a \(\mathbb{C}\)-superalgebra, where \(U(\mathfrak{g})\) is considered as a purely even sub-superalgebra. The filtration of \(U(\mathfrak{g})\) and \(\text{Cl}\) induces the filtration of \(C(\mathfrak{g})\): \(C_p(\mathfrak{g}) = \sum_{i+j \leq p} U_i(\mathfrak{g}) \otimes \text{Cl}_j\), and we have
\[
\text{gr} C(\mathfrak{g}) \cong \bar{C}(\mathfrak{g})
\]
as Poisson superalgebras. Therefore, \(C(\mathfrak{g})\) is a quantization of \(\bar{C}(\mathfrak{g})\).
Define the $\mathbb{Z}$-grading $C(g) = \bigoplus_{n \in \mathbb{Z}} \mathfrak{C}^n(g)$ by setting $\deg(u \otimes 1) = 0$ ($u \in U(g)$), $\deg(1 \otimes f) = 1$ ($f \in \mathfrak{n}^*$), $\deg(1 \otimes x) = -1$ ($x \in \mathfrak{n}$). Then

$$C^n(g) = U(g) \otimes (\bigoplus_{j-i=n} \Lambda^i(n) \otimes \Lambda^j(n^*)).$$

**Lemma 2.14.** The following map defines a Lie algebra homomorphism.

$$\theta : \mathfrak{n} \rightarrow C(g)$$

$$x \mapsto (x - \chi(x)) \otimes 1 + 1 \otimes \rho(x)$$

**Lemma 2.15** ([BDT Lemma 7.13.7]). There exists a unique element $Q \in C^1(g)$ such that

$$[Q, 1 \otimes x] = \theta_\chi(x), \quad \forall x \in \mathfrak{n}.$$ We have $Q^2 = 0$.

**Proof.** The proof is similar to that of Lemma 2.11. In fact the element $Q$ is explicitly given by the same formula as $\overline{Q}$:

$$Q = \sum_{\alpha} (x_\alpha - \chi(x_\alpha)) \otimes x^*_\alpha - 1 \otimes \frac{1}{2} \sum_{\alpha,\beta,\gamma} c^\gamma_{\alpha,\beta} x^*_\alpha x^*_\beta x_\gamma.$$ Since $Q$ is odd, Lemma 2.15 implies that

$$(\text{ad } Q)^2 = 0.$$ Thus, $(C(g), \text{ad } Q)$ is a differential graded algebra, and its cohomology $H^\bullet(C(g), \text{ad } Q)$ is a graded superalgebra.

However the operator on $\text{gr} C(g) = \tilde{C}(g)$ induced by $\text{ad } Q$ does not coincide with $\text{ad } \tilde{Q}$. To remedy this, we introduce the Kazhdan filtration $K_\bullet C(g)$ of $C(g)$ as follows: Defined a $\mathbb{Z}$-grading on $\mathfrak{g}$ by

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j, \quad \mathfrak{g}_j = \{x \in \mathfrak{g} : [h, x] = 2jx\}$$

where $h$ is defined in (2.6). Then $\mathfrak{n} = \bigoplus_{j > 0} \mathfrak{g}_j \subset \mathfrak{b} = \bigoplus_{j \geq 0} \mathfrak{g}_j$, and

$$\mathfrak{h} := \mathfrak{g}_0$$

is the Cartan subalgebra of $\mathfrak{g}$ consisting of diagonal matrices. Extend the basis $\{x_\alpha\}_{\alpha \in \Delta^+}$ of $\mathfrak{n}$ to the basis $\{x_\alpha\}_{\alpha \in \Delta^+ \cup I}$ of $\mathfrak{b}$ by adding a basis $\{x_i\}_{i \in I}$ of $\mathfrak{h}$. Let $c^d_{a,b}$ denote the structure constant of $\mathfrak{b}$ with respect to this basis.

**Lemma 2.16.** The map $\rho : \mathfrak{n} \rightarrow Cl$ extends to the Lie algebra homomorphism

$$\rho : \mathfrak{b} \rightarrow Cl, \quad x_a \mapsto \sum_{\beta, \gamma \in \Delta^+} c^\gamma_{a,\beta} x^*_\beta.$$

Define the Lie algebra homomorphism

$$\theta_0 : \mathfrak{b} \rightarrow C(g), \quad x_i \mapsto x_i \otimes 1 + 1 \otimes \rho(x_i),$$
and define a $\mathbb{Z}$-grading on $C(\mathfrak{g})$ by

$$C(\mathfrak{g}) = \bigoplus_{j \in \mathbb{Z}} C(\mathfrak{g})[j], \quad C(\mathfrak{g})[j] = \{ c \in C(\mathfrak{g}) \mid [\theta_0(h), c] = 2jx \}.$$ 

Set

$$K_pC(\mathfrak{g}) = \sum_{i-j \leq p} C_i(\mathfrak{g})[j], \quad \text{where } C_i(\mathfrak{g})[j] = C_i(\mathfrak{g}) \cap C(\mathfrak{g})[j].$$

Then $K_\bullet C(\mathfrak{g})$ defines an increasing, exhaustive, separated filtration of $C(\mathfrak{g})$ such that $K_pC(\mathfrak{g}) \cdot K_qC(\mathfrak{g}) \subset K_{p+q}C(\mathfrak{g})$, $[K_pC(\mathfrak{g}), K_qC(\mathfrak{g})] \subset K_{p+q-1}C(\mathfrak{g})$, and $gr_K C(\mathfrak{g}) = \bigoplus_p K_pC(\mathfrak{g})/K_{p-1}C(\mathfrak{g})$ is isomorphic to $\bar{C}(\mathfrak{g})$ as Poisson superalgebras. Moreover, the complex $(gr_K C(\mathfrak{g}), \text{ad } Q)$ is identical to $(\bar{C}(\mathfrak{g}), \text{ad } \bar{Q})$.

Let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$.

**Theorem 2.17** ([Kos]). We have $H^i(C(\mathfrak{g}), \text{ad } Q) = 0$ for $i \neq 0$ and the map $Z(\mathfrak{g}) \rightarrow H^0(C(\mathfrak{g}), \text{ad } Q)$ defined by sending $z$ to $[z \otimes 1]$ is an isomorphism of algebras. Here $[z \otimes 1]$ denotes the cohomology class of $z \otimes 1$.

**Proof.** We have the spectral sequence

$$E_r \Rightarrow H^\bullet(C(\mathfrak{g}), \text{ad } Q)$$

such that

$$E_1^{i,i} = H^i(gr_K C(\mathfrak{g}), \text{ad } \bar{Q}) \cong \begin{cases} \mathbb{C}[\mathfrak{g}^{\ast}]^G, & \text{if } i = 0 \\ 0, & \text{if } i \neq 0. \end{cases}$$

Therefore the spectral sequence collapses at $E_1 = E_\infty$, so we get

$$gr H^0(C(\mathfrak{g}), \text{ad } Q) \cong \mathbb{C}[\mathfrak{g}^{\ast}]^G.$$ 

Since the homomorphism $Z(\mathfrak{g}) \rightarrow H^0(C(\mathfrak{g}), \text{ad } Q), z \mapsto [z \otimes 1]$, respects the filtration $Z_\bullet(\mathfrak{g})$ and $K_\bullet H^0(C(\mathfrak{g}), \text{ad } Q)$, where $Z_p(\mathfrak{g}) = Z(\mathfrak{g}) \cap U_p(\mathfrak{g})$, $K_pH^\bullet(C(\mathfrak{g}), \text{ad } Q) = \text{im}(H^0(K_pC(\mathfrak{g}), \text{ad } Q) \rightarrow H^0(C(\mathfrak{g}), \text{ad } Q))$, we get the desired isomorphism. \hfill \square

**Remark 2.18** (see [AD] §2 for the details). As in the case of $\bar{C}(\mathfrak{g})$, $C(\mathfrak{g})$ is also bigraded, we can also write $\text{ad } Q = d_+ + d_-$ such that $d_+(C^{i,j}) \subset C^{i+1,j}, d_-(C^{i,j}) \subset C^{i,j+1}$ and get a spectral sequence

$$E_r \Rightarrow H^\bullet(C(\mathfrak{g}), \text{ad } Q)$$

such that

$$E_2^{p,q} = H^p(H^q(C(\mathfrak{g}), d_-), d_+) \cong \delta_{q,0} H^p(n, U(\mathfrak{g}) \otimes U(n) \mathbb{C}_\chi)$$

$$\cong \delta_{p,0} \delta_{q,0} H^0(n, U(\mathfrak{g}) \otimes U(n) \mathbb{C}_\chi) \cong \text{End}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes U(n) \mathbb{C}_\chi)^{op}.$$ 

Where $\mathbb{C}_\chi$ is the one-dimensional representation of $n$ defined by the character $\chi$. Thus we get the Whittaker model isomorphism [Kos]

$$Z(\mathfrak{g}) \cong H^0(C(\mathfrak{g}), \text{ad } Q) \cong \text{End}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes U(n) \mathbb{C}_\chi)^{op}.$$
2.6. Classical Miura map. Let $n_\alpha = \bigoplus_{j<0} g_j$ be the subalgebra of $g$ consisting of lower triangular matrices, and set $b_\alpha = \bigoplus_{j\leq 0} g_j = n_\alpha \oplus b$. We have

$$
(2.11)
$$

$$
\mathfrak{g} = b_\alpha \oplus n_\alpha.
$$

Extend the basis $\{x_\alpha\}_{\alpha \in \Delta_+ \cup I}$ to the basis $\{x_\alpha\}_{\alpha \in \Delta_+ \cup I \cup \Delta_-}$ by adding a basis $\{x_\alpha\}_{\alpha \in \Delta_-}$ of $n_-$. Let $c_{a,b}^\gamma$ be the structure constant with respect to this basis. Extend $\theta_0 : b \to C(g)$ to the linear map $\theta_0 : g \to C(g)$ by setting

$$
\theta_0(x_a) = x_a \otimes 1 + 1 \otimes \sum_{\beta, \gamma \in \Delta_+} c_{a,\beta}^\gamma x_\beta x_\gamma^*.
$$

We already know that the restriction of $\theta_0$ to $n$ is a Lie algebra homomorphism and

$$
[\theta_0(x), 1 \otimes y] = 1 \otimes [x, y]
$$

for $x, y \in n$.

Although $\theta_0$ is not a Lie algebra homomorphism, we have the following.

**Lemma 2.19.** The restriction of $\theta_0$ to $b_\alpha$ is a Lie algebra homomorphism. We have $[\theta_0(x), 1 \otimes y] = 1 \otimes \text{ad}^*(x)(y)$ for $x \in b_\alpha$, $y \in n^*$, where $\text{ad}^*$ denote the coadjoint action and $n^*$ is identified with $(g/b_\alpha)^*$.\[\square\]

Let $C(g)_+$ denote the subalgebra of $C(g)$ generated by $\theta_0(n)$ and $\Lambda(n) \subset Cl$, and let $C(g)_-$ denote the subalgebra generated by $\theta_0(n)$ and $\Lambda(n^*) \subset Cl$.

**Lemma 2.20.** The multiplication map gives a linear isomorphism

$$
C(g)_- \otimes C(g)_+ \to C(g).
$$

**Lemma 2.21.** The subspaces $C(g)_-$ and $C(g)_+$ are subcomplexes of $(C(g), \text{ad} Q)$. Hence $C(g) \cong C(g)_- \otimes C(g)_+$ as complexes.

**Proof.** The fact that $C(g)_+$ is subcomplex is obvious (see Lemma 2.15). The fact that $C(g)_-$ is a subcomplex follows from the following formula.

$$
[Q, \theta_0(x_\alpha)] = \sum_{b \in \Delta_+ \cup I, \alpha \in \Delta_+} c_{a,b}^\gamma \theta_0(x_b)(1 \otimes x_\alpha^*) - 1 \otimes \sum_{\beta, \gamma \in \Delta_+} c_{a,\beta}^\gamma \chi(\tau_\gamma) x_\beta x_\gamma^*.
$$

$$
[Q, 1 \otimes x_\alpha^*] = -1 \otimes \frac{1}{2} \sum_{\beta, \gamma \in \Delta_+} c_{\beta,\gamma}^\alpha x_\beta^* x_\gamma^*.
$$

$(a \in \Delta_- \cup I, \alpha \in \Delta_+)$.\[\square\]

**Proposition 2.22.** $H^\bullet(C(g)_-, \text{ad} Q) \cong H^\bullet(C(g), \text{ad} Q)$.

**Proof.** By Lemma 2.21 and Kunneth’s Theorem,

$$
H^p(C(g), \text{ad} Q) \cong \bigoplus_{i+j=p} H^i(C(g)_-, \text{ad} Q) \otimes H^j(C(g)_+, \text{ad} Q).
$$

On the other hand we have $\text{ad}(Q)(1 \otimes x_\alpha) = \theta_\chi(x_\alpha) = \theta_0(x_\alpha) - \chi(x_\alpha)$ for $\alpha \in \Delta_-$. Hence $C(g)_-$ is isomorphic to the tensor product of complexes of the form $\mathbb{C} \theta_\chi(x_\alpha) \otimes \Lambda(x_\alpha)$ with the differential $\theta_\chi(x_\alpha) \otimes x_\alpha^*$, where $x_\alpha^*$ denotes the odd derivation of the exterior algebra $\Lambda(x_\alpha)$ with one variable $x_\alpha$ such that $x_\alpha^*$. Each of these complexes has one-dimensional zeroth cohomology and zero first cohomology. Therefore $H^1(C(g)_+, \text{ad} Q) = \delta_{i,0} \mathbb{C}$. This completes the proof.\[\square\]
Hence by Proposition 2.22 we may identify \( Z(\mathfrak{g}) = H^0(\mathcal{Z}(\mathfrak{g}), \text{ad } Q) \) with the subalgebra \( H^0(\mathcal{Z}(\mathfrak{g})_-, \text{ad } Q) = \{ c \in C(\mathfrak{g})_- | \text{ad } Q(c) = 0 \} \) of \( C(\mathfrak{g})_- \).

Consider the decomposition
\[
C(\mathfrak{g})_0 = \bigoplus_{j \leq 0} C(\mathfrak{g})_{-j}, \quad C(\mathfrak{g})_{-j} = \{ c \in C(\mathfrak{g})_0 | [\theta_0(h), c] = 2jc \}.
\]

Note that \( C(\mathfrak{g})_{0,-0} \) is generated by \( \theta_0(h) \) and is isomorphic to \( U(\mathfrak{h}) \). The projection
\[
C(\mathfrak{g})^0_0 \to C(\mathfrak{g})_{0,-0} \cong U(\mathfrak{h})
\]
is an algebra homomorphism, and hence, its restriction
\[
\Upsilon : Z(\mathfrak{g}) = H^0(\mathcal{Z}(\mathfrak{g})_-, \text{ad } Q) \to U(\mathfrak{h})
\]
is also an algebra homomorphism.

**Proposition 2.23.** The map \( \Upsilon \) is an embedding.

Let \( K \cdot C(\mathfrak{g})_\pm \) be the filtration of \( C(\mathfrak{g})_\pm \) induced by the Kazhdan filtration of \( C(\mathfrak{g}) \). We have the isomorphism
\[
C(\mathfrak{g}) = \text{gr}_K C(\mathfrak{g}) \cong \text{gr}_K C(\mathfrak{g})_- \otimes \text{gr}_K C(\mathfrak{g})_+
\]
as complexes. Similarly as above, we have \( H^i(\text{gr}_K C(\mathfrak{g})_+, \text{ad } \tilde{Q}) = \delta_{i,0} \mathbb{C} \), and
\[
H^0(\text{gr}_K C(\mathfrak{g})_-, \text{ad } \tilde{Q}) \cong H^0(\text{gr}_K C(\mathfrak{g})_-, \text{ad } Q).
\]

**Proof of Proposition 2.23.** The filtration \( K \cdot U(\mathfrak{h}) \) of \( U(\mathfrak{h}) \cong C(\mathfrak{g})_{0,-0} \) induced by the Kazhdan filtration coincides with the usual PBW filtration. By (2.12) and Theorem 2.12 the induced map
\[
H^0(\text{gr}_K C(\mathfrak{g})_-, \text{ad } Q) \to \text{gr}_K U(\mathfrak{h})
\]
can be identified with the restriction map
\[
(2.13) \quad \Upsilon : \mathbb{C}[S] = \mathbb{C}[f + \mathfrak{h}]^N \to \mathbb{C}[f + \mathfrak{h}].
\]
It is sufficient to show that \( \Upsilon \) is injective.

If \( \varphi \in \mathbb{C}[f + \mathfrak{h}]^N \) is in the kernel, \( \varphi(g,x) = 0 \) for all \( g \in N \) and \( x \in f + \mathfrak{h} \). Hence it is enough to show that the image of the the action map
\[
(2.14) \quad N \times (f + \mathfrak{h}) \to f + \mathfrak{h}, \quad (g,x) \mapsto \text{Ad}(g)x,
\]
is Zariski dense in \( f + \mathfrak{b} \).

The differential of this morphism at \((1, x) \in N \times (f + \mathfrak{h})\) is given by
\[
n \times \mathfrak{h} \to \mathfrak{b}, \quad (y, z) \mapsto [y, x] + z.
\]
This is an isomorphism if \( x \in f + \mathfrak{h}_{\text{reg}} \), where \( \mathfrak{h}_{\text{reg}} = \{ x \in \mathfrak{h} | n^x = 0 \} \). Hence (2.14) is a dominant morphism as required, see e.g. \[\text{[TY]}\] Theorem 16.5.7. \( \square \)

**Remark 2.24.** The fact that \( \Upsilon \) is injective is in fact well-known. Indeed, under the identifications \( \mathbb{C}[S] \cong \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \), \( \mathbb{C}[f + \mathfrak{h}] \cong \mathbb{C}[\mathfrak{h}] \), \( \Upsilon \) is identified with the Chevalley restriction map \( \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{h}]^W \), where \( W = S_n \).

The advantage of the above proof is that it applies to a general finite W-algebra \([\text{Lyn}]\), and also, it generalizes to the affine setting, see \([\text{Lyn}]\).
2.7. Generalization to an arbitrary simple Lie algebra. It is clear that the above argument works if we replace $\mathfrak{gl}_n$ by $\mathfrak{sl}_n$, and $\mathfrak{a}$ by $\mathfrak{a} \cap \mathfrak{sl}_n$.

More generally, let $\mathfrak{g}$ be an arbitrary simple Lie algebra. Let $f$ be a principal (regular) nilpotent element of $\mathfrak{g}$, $\{e, f, h\}$ an associated $\mathfrak{sl}_2$-triple. One may assume that
\[
f = \sum_{i \in I} f_i,
\]
where $f_i$ is a root vector of roots $\alpha_i$ and $\{\alpha_i\}_{i \in I}$ is the set of simple roots of $\mathfrak{g}$. Define the Kostant slice $\mathcal{S}$ by
\[
\mathcal{S} := f + \mathfrak{g}^e \subset \mathfrak{g} = \mathfrak{g}^* ,
\]
where $\mathfrak{g}^e$ is the centralizer of $e$ in $\mathfrak{g}$.

Then all the statements in previous subsections that make sense hold by replacing the set of companion matrices by the Kostant slice ($[\text{Kos}]$).

2.8. Generalization to finite $W$-algebras. In fact, the above argument works in more general setting of Hamiltonian reduction. In particular for Slodowy slices. Namely, for a non-zero nilpotent element $f$ of a finite-dimensional semisimple Lie algebra $\mathfrak{g}$, we can use Jacobson-Morozov’s theorem to embed $f$ into an $\mathfrak{sl}_2$-triple $\{e, f, h\}$. The Slodowy slice at $f$ is defined to be the affine subspace
\[
\mathcal{S}_f = f + \mathfrak{g}^e
\]
of $\mathfrak{g}$.

The Slodowy slice $\mathcal{S}_f$ has the following properties.

- $\mathcal{S}_f$ intersects the $G$-orbits at any point of $\mathcal{S}_f$, where $G$ is the adjoint group of $\mathfrak{g}$.
- $\mathcal{S}_f$ admits a $\mathbb{C}^*$-action which is contracting at $f$.

As in the case of the set of companion matrices $\mathcal{S}_f$ can be realized by Hamiltonian reduction. Let $\mathfrak{g}_j = \{x \in \mathfrak{g} \mid [h, x] = 2jx\}$, so that
\[
\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j.
\]
Then the subspace $\mathfrak{g}_{1/2}$ admits a symplectic form defined by $\langle x | y \rangle = (f| [x, y])$. Choose a Lagrangian subspace $l$ of $\mathfrak{g}_{1/2}$ with respect to this form, and set $\mathfrak{m} = l + \sum_{j \geq 1} \mathfrak{g}_j$. Then $\mathfrak{m}$ is a nilpotent subalgebra of $\mathfrak{g}$ and $\chi : \mathfrak{m} \to \mathbb{C}$, $x \mapsto (f| x)$, defines a character. Let $M$ be the unipotent subgroup of $G$ corresponding to $\mathfrak{m}$, that is, $\text{Lie}M = \mathfrak{m}$. The adjoint action of $M$ on $\mathfrak{g}$ is Hamiltonian, so we can consider the moment map of this action
\[
\mu : \mathfrak{g}^* \longrightarrow \mathfrak{m}^* ,
\]
which is just a restriction map. Then we have the following realization of the Slodowy slice.
\[
\mathcal{S}_f \cong \frac{\mu^{-1}(\chi)}{M}
\]
To obtain the BRST realization of this Hamiltonian reduction we simply replace the Clifford algebra $Cl$ by $Cl_m$, i.e., the Clifford algebra associated to $m \oplus m^*$. Then we can define the operator $ad \tilde{Q}$ similarly and get a differential cochain complex $(\mathbb{C}[g^*] \otimes \overline{Cl}_m, ad \tilde{Q})$. We have

$$\mathbb{C}[S_f] \cong H^0(\mathbb{C}[g^*] \otimes \overline{Cl}_m, ad \tilde{Q})$$

as Poisson algebras.

As above, this construction has a natural quantization and the quantization $U(\mathfrak{g}, \mathfrak{f})$ of $S_f$ thus defined is called the finite $W$-algebra associated to the pair $(\mathfrak{g}, \mathfrak{f})$.

$U(\mathfrak{g}, \mathfrak{f}) := H^0(U(\mathfrak{g}) \otimes Cl_m, ad \tilde{Q}_+) \cong \text{End}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes U(\mathfrak{m}) \mathbb{C}_\chi)^\text{op},$

where $\mathbb{C}_\chi$ is the one-dimensional representation of $\mathfrak{m}$ defined by $\chi$ (cf. [DSK] A2).

3. Arc spaces, Poisson vertex algebras, and associated varieties of vertex algebras.

3.1. Vertex algebras. A vertex algebra is a vector space $V$ equipped with $|0\rangle \in V$ (the vacuum vector), $T \in \text{End} V$ (the translation operator), and a bilinear product

$$V \times V \to V((z)), \quad (a, b) \mapsto a(z)b,$$

where $a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}$, $a(n) \in \text{End} V$, such that

1. $|0\rangle(z) = \text{id}_V$,
2. $a(z)|0\rangle \in V[[z]]$ and $\lim_{z \to 0} a(z)|0\rangle = a$ for all $a \in V$,
3. $(Ta)(z) = \partial_z a(z)$ for all $a \in V$, where $\partial_z = d/dz$,
4. for any $a, b \in V$, $(z - w)^{N_{a,b}} [a(z), b(w)] = 0$ for some $N_{a,b} \in \mathbb{Z}_+ = \{0, 1, 2, \ldots \}$.

The last condition is called the locality, which is equivalent to the fact that

$$[a(z), b(w)] = \sum_{n=0}^{N_{a,b}-1} (a(n)b(w)) \frac{1}{n!} \partial_w^n \delta(z - w),$$

where $\delta(z - w) = \sum_{n \in \mathbb{Z}} w^n z^{-n-1} \in \mathbb{C}[[z, w, z^{-1}, w^{-1}]].$

A consequence of the definition is the following Borcherds identities:

$$[a_{(m)}, b_{(n)}] = \sum_{i \geq 0} \binom{m}{i} (a_{(i)}b)_{(m+n-i)},$$

$$[a_{(m)}b]_{(n)} = \sum_{j \geq 0} (-1)^j \binom{m}{j} (a_{(m-j)}b_{(n+j)} - (-1)^m b_{(m+n-j)}a_{(j)}).$$

We write (3.1) as

$$[a_\lambda b] = \sum_{n \geq 0} \frac{\lambda^n}{n!} a_{(n)}b \in V[\lambda],$$

and call it the $\lambda$-bracket of $a$ and $b$. (We have $a_{(n)}b = 0$ if $(z - w)^n [a(z), b(w)] = 0.$)
Here are some properties of $\lambda$-brackets.

(3.4) \[ [(Ta)_{\lambda}b] = -\lambda[a_{\lambda}b], \quad [a_{\lambda}(Tb)] = (\lambda + T)[a_{\lambda}b], \]

(3.5) \[ [b_{\lambda}a] = -[a_{-\lambda-T}b], \]

(3.6) \[ [a_{\lambda}[b_{\mu}c]] - [b_{\mu}[a_{\lambda}c]] = [[a_{\lambda}b]_{\lambda+\mu}c]. \]

The normally ordered product on $V$ is defined as: $ab := a(z)b(z)$. We also write $ab : (z) := a(z)b(z) :$. We have

\[ :a(z)b(z) := a(z)_+ b(z) + b(w)a(z)_-, \]

where $a(z)_+ = \sum_{n<0} a_{(n)} z^{-n-1}$, $a(z)_- = \sum_{n\geq 0} a_{(n)} z^{-n-1}$. We have the following non-commutative Wick formula.

(3.7) \[ [a_{\lambda} : bc : ] := [a_{\lambda}b]c : + : [a_{\lambda}c]b : + \int_{0}^{\lambda} [[a_{\lambda}b]_{\mu}c]d\mu, \]

(3.8) \[ [: ab : \lambda c ] := (e^{T_{\lambda}a})[b_{\lambda}c] : + : (e^{T_{\lambda}b})[a_{\lambda}c] : + \int_{0}^{\lambda} [b_{\mu}[a_{\lambda-\mu}c]]d\mu. \]

3.2. **Commutative vertex algebras and differential algebras.** A vertex algebra $V$ is called *commutative* if

\[ [a_{\lambda}b] = 0, \quad \forall a, b \in V, \]

or equivalently, $a_{(n)} = 0$ for $n \geq 0$ in $\text{End} V$ for all $a \in V$. This condition is equivalent to that

\[ [a_{(m)}, b_{(n)}] = 0 \quad \forall a, b \in \mathbb{Z}, \ m, n \in \mathbb{Z} \]

by (3.2).

A commutative vertex algebra has the structure of a unital commutative algebra by the product

\[ a \cdot b := ab := a_{(-1)}b, \]

where the unite is given by the vacuum vector $|0\rangle$. The translation operator $T$ of $V$ acts on $V$ as a derivation with respect to this product:

\[ T(a \cdot b) = (Ta) \cdot b + a \cdot (Tb). \]

Therefore a commutative vertex algebra has the structure of a *differential algebra*, that is, a unital commutative algebra equipped with a derivation. Conversely, there is a unique vertex algebra structure on a differential algebra $R$ with a derivation $T$ such that

\[ Y(a, z) = e^{zT}a \]

for $a \in R$. This correspondence gives the following.

**Theorem 3.1** ([Bor]). The category of commutative vertex algebras is the same as that of differential algebras.
3.3. Arc spaces. Define the (formal) disc as
\[ D = \text{Spec}(\mathbb{C}[\![t]\!] ). \]
For a scheme \( X \), a homomorphism \( \alpha : D \to X \) is called an arc of \( X \).

**Theorem 3.2 (BLR, EM, Ish)**. Let \( X \) be a scheme of finite type over \( \mathbb{C} \), \( Sch \) the category of schemes of finite type over \( \mathbb{C} \), \( Set \) the category of sets. The contravariant functor
\[ \text{Sch} \to \text{Set}, \ Y \mapsto \text{Hom}_{\text{Sch}}(Y \hat{\times} D, X), \]
is represented by a scheme \( JX \), that is,
\[ \text{Hom}_{\text{Sch}}(Y, JX) \cong \text{Hom}_{\text{Sch}}(Y \hat{\times} D, X). \]
for any \( Y \in \text{Sch} \). Here \( Y \hat{\times} D \) is the completion of \( Y \times D \) with respect to the sub-scheme \( Y \hat{\times}\{0\} \).

By definition, the \( \mathbb{C} \)-points of \( JX \) are
\[ \text{Hom}_{\text{Sch}}(\text{Spec} \mathbb{C}, JX) = \text{Hom}_{\text{Sch}}(D, X), \]
that is, the set of arcs of \( X \). The reason we need the completion \( Y \hat{\times} D \) in the definition is that \( A \otimes \mathbb{C}[\![t]\!] \not\subseteq A[\![t]\!] = A \otimes \mathbb{C}[\![t]\!] \) in general.

The scheme \( JX \) is called the arc space, or the infinite jet scheme, of \( X \).

It is easy to describe \( JX \) when \( X \) is affine:
First, consider the case \( X = \mathbb{C}^N = \text{Spec} \mathbb{C}[x_1, x_2, \cdots, x_N] \). The \( \mathbb{C} \)-points of \( JX \) are the arcs \( \text{Hom}_{\text{Sch}}(D, JX) \), that is, the ring homomorphisms
\[ \gamma : \mathbb{C}[x_1, x_2, \cdots, x_N] \to \mathbb{C}[\![t]\!]. \]
Such a map is determined by the image
\[ \gamma(x_i) = \sum_{n \geq 0} \gamma_{i,(-n-1)} t^n \]
of each \( x_i \), and conversely, the coefficients \( \{ \gamma_{i,(-n-1)} \} \) determines a \( \mathbb{C} \)-point of \( JX \).

If we choose coordinates \( x_{i,(-n-1)} \) of \( JX \) as \( x_{i,(-n-1)}(\gamma) = \gamma_{i,(-n-1)} \), we have
\[ JC^N = \text{Spec} \mathbb{C}[x_{i,(n)} | i = 1, 2, \cdots, N, n = -1, -2, \cdots]. \]

Next, let \( X = \text{Spec} \mathbb{R} \), with \( R = \mathbb{C}[x_1, x_2, \cdots, x_N]/(f_1, f_2, \cdots, f_r) \). The arcs of \( X \) are
\[ \text{Hom}_{\text{ring}}(\mathbb{C}[x_1, x_2, \cdots, x_n], \mathbb{C}[\![t]\!]) \subset \text{Hom}_{\text{ring}}(\mathbb{C}[x_1, x_2, \cdots, x_n], \mathbb{C}[\![t]\!]). \]
An element \( \gamma \in \text{Hom}_{\text{ring}}(\mathbb{C}[x_1, x_2, \cdots, x_n], \mathbb{C}[\![t]\!]) \) is an element of this subset if and only if \( \gamma(f_i) = 0 \) for \( i = 1, 2, \cdots, r \). By writing
\[ f_i(x_1(t), x_2(t), \ldots, x_N(t)) = \sum_{m \geq 0} \frac{f_{i,m}}{m!} t^m \]
with \( f_{i,m} \in \mathbb{C}[x_{i,(-n-1)}] \), where \( x_i(t) := \sum_{m \geq 0} x_{i,(-m-1)} t^m \), we get that
\[ JX = \text{Spec} \mathbb{C}[x_{i,(n)} | i = 1, 2, \cdots, N; n = -1, -2, \cdots] / (f_{i,m}(x_{i,(n)}), i = 1, 2, \cdots, r; m \geq 0). \]
Lemma 3.3. Define the derivation $T$ of $\mathbb{C}[x_{i(n)}| i = 1, 2, \ldots, N; n = -1, -2, \ldots]$ by

$$Tx_{i(n)} = -nx_{i(n-1)}.$$  

Then $f_{i,m} = T^n f_i$ for $n \geq 0$. Here we identify $x_i$ with $x_{i(-1)}$.

With the above lemma, we conclude that for the affine scheme $X = \text{Spec} R$, $R = \mathbb{C}[x_1, x_2, \ldots, x_n]/\langle f_1, f_2, \ldots, f_r \rangle$, its arc space $JX$ is the affine scheme $\text{Spec}(JR)$, where

$$JR := \frac{\mathbb{C}[x_{i(n)}| i = 1, 2, \ldots, N; n = -1, -2, \ldots]}{(T^n f_i, i = 1, 2, \ldots, r; n \geq 0)}$$

and $T$ is as defined in the lemma.

The derivation $T$ acts on the above quotient ring $JR$. Hence for an affine scheme $X = \text{Spec} R$, the coordinate ring $JR = \mathbb{C}[JX]$ of its arc space $JX$ is a differential algebra, hence is a commutative vertex algebra.

Remark 3.4. The differential algebra $JR$ has the universal property that

$$\text{Hom}_{\text{diff.alg}}(JR, A) \cong \text{Hom}_{\text{ring}}(R, A)$$

for any differential algebra $A$, where $\text{Hom}_{\text{diff.alg}}(JR, A)$ is the set of homomorphisms $JR \to A$ of differential algebras.

For a general scheme $Y$ of finite type with an affine open covering $\{U_i\}_{i \in I}$, its arc space $JY$ is obtained by gluing $JU_i$ (see [EM, Ish]). In particular, the structure sheaf $\mathcal{O}_{JY}$ is a sheaf of commutative vertex algebras.

There is a natural projection $\pi_\infty : JX \to X$ that corresponds to the embedding $R \hookrightarrow JR$, $x_i \mapsto x_{i(-1)}$, in the case $X$ is affine. In terms of arcs, $\pi_\infty(\alpha) = \alpha(0)$ for $\alpha \in \text{Hom}_{\text{Sch}}(D, X)$, where 0 is the unique closed point of the disc $D$.

The map from a scheme to its arc space is functorial, i.e., a scheme homomorphism $f : X \to Y$ induces a scheme homomorphism $Jf : JX \to JY$ that makes the following diagram commutative:

$$\begin{array}{ccc}
JX & \xrightarrow{Jf} & JY \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y.
\end{array}$$

In terms of arcs, $Jf(\alpha) = f \circ \alpha$ for $\alpha \in \text{Hom}_{\text{Sch}}(D, X)$.

We also have

$$(3.10) \quad J(X \times Y) \cong JX \times JY.$$  

Indeed, for any scheme $Z$,

$$\text{Hom}(Z, J(X \times Y)) = \text{Hom}(Z \times \hat{D}, X \times Y)$$

$$\cong \text{Hom}(Z \times \hat{D}, X) \times \text{Hom}(Y \times \hat{D}, Y)$$

$$= \text{Hom}(Z, JX) \times \text{Hom}(Y, JY)$$

$$\cong \text{Hom}(Z, JX \times JY).$$
Lemma 3.5. The natural morphism $X_{\text{red}} \to X$ induces an isomorphism $JX_{\text{red}} \to JX$ of topological spaces, where $X_{\text{red}}$ denotes the reduced scheme of $X$.

Proof. We may assume that $X = \text{Spec } R$. An arc $\alpha$ of $X$ corresponds to a ring homomorphism $\alpha^*: R \to \mathbb{C}[t]$. Since $\mathbb{C}[t]$ is an integral domain it decomposes as $\alpha^*: R \to R/\sqrt{0} \to \mathbb{C}[t]$. Thus, $\alpha$ is an arc of $X_{\text{red}}$. \hfill \Box

If $X$ is a point, then $JX$ is also a point, since $\text{Hom}(\mathcal{O}, X) = \text{Hom}(\mathcal{O}, \mathbb{C}[t])$ consists of only one element. Thus, Lemma 3.5 implies the following.

Corollary 3.6. If $X$ is zero-dimensional then $JX$ is also zero-dimensional.

Theorem 3.7 ([Kol]). $JX$ is irreducible if $X$ is irreducible.

Lemma 3.8. Let $Y$ be irreducible, and let $f: X \to Y$ be a morphism that restricts to a bijection between some open subsets $U \subset X$ and $V \subset Y$. Then $Jf: JX \to JY$ is dominant.

Proof. $Jf$ restricts to the isomorphism $JU \cong JV$, and the open subset $JV$ is dense in $JY$ since $JY$ is irreducible. \hfill \Box

3.4. Arc space of Poisson varieties and Poisson vertex algebras. Let $V$ be a commutative vertex algebra, or equivalently, a differential algebra. $V$ is called a Poisson vertex algebra if it is equipped with a bilinear map

$$V \times V \to V[\lambda], \quad (a, b) \mapsto \{a, b\} = \sum_{n \geq 0} \frac{\lambda^n}{n!} a_{(n)} b, \quad a_{(n)} \in \text{End } V,$$

also called the $\lambda$-bracket, satisfying the following axioms:

\begin{align}
(3.11) \quad &\{ (Ta)_{\lambda} b \} = -\lambda \{ a_{\lambda} b \}, \quad \{ a_{\lambda} (Tb) \} = (\lambda + T) \{ a_{\lambda} b \}, \\
(3.12) \quad &\{ b_{\lambda} a \} = -\{ a_{-\lambda - T} b \}, \\
(3.13) \quad &\{ a_{\lambda} (b_{\mu} c) \} - \{ b_{\mu} (a_{\lambda} c) \} = \{ (a_{\lambda} b)c + \lambda b_{\lambda} c \}, \quad \{ (ab)_{\lambda} c \} = \{ a_{\lambda + T} c \} \to b + \{ b_{\lambda + T} c \} \to a,
\end{align}

where the arrow means that $\lambda + T$ should be moved to the right, that is, $\{ a_{\lambda + T} c \} \to b = \sum_{n \geq 0} (a_{(n)} c) (\lambda + T)^n b$.

The first equation in (3.13) says that $a_{(n)}$, $n \geq 0$, is a derivation of the ring $V$. (Do not confuse $a_{(n)} \in \text{Der}(V)$, $n \geq 0$, with the multiplication $a_{(n)}$ as a vertex algebra, which should be zero for a commutative vertex algebra.)

Note that (3.11), (3.12), (3.13) are the same as (3.4), (3.5), (3.6), and (3.14) is the same with (3.7) and (3.8) without the third terms. In particular, by (3.13), we have

\begin{equation}
[ a_{(m)}, b_{(n)} ] = \sum_{i \geq 0} \binom{m}{i} (a_{(i)} b)_{(m+n-i)}, \quad m, n \in \mathbb{Z}_+.
\end{equation}

Theorem 3.9 ([A4] Proposition 2.3.1]). Let $X$ be an affine Poisson scheme, that is, $X = \text{Spec } R$ for some Poisson algebra $R$. Then there is a unique Poisson vertex algebra structure on $JR = \mathbb{C}[JX]$ such that

$$\{ a_{(n)} b \} = \{ a, b \} \quad \text{for } a, b \in R \subset JR,$$
where \( \{a, b\} \) is the Poisson bracket in \( R \).

**Proof.** The uniqueness is clear by \( \mathbf{3.4} \) since \( JR \) is generated by \( R \) as a differential algebra. We leave it to the reader to check the well-definedness. \( \square \)

**Remark 3.10.** More generally, let \( X \) be a Poisson scheme which is not necessarily affine. Then the structure sheaf \( \mathcal{O}_{JX} \) carries a unique vertex Poisson algebra structure such that \( \{f\lambda g\} = \{f, g\} \) for \( f, g \in \mathcal{O}_X \subset \mathcal{O}_{JX} \), see [AKM, Lemma 2.1.3.1].

**Example 3.11.** Let \( G \) be an affine algebraic group, \( \mathfrak{g} = \text{Lie} \, G \). The arc space \( JG \) is naturally a proalgebraic group. Regarding \( JG \) as the \( \mathbb{C}[[t]] \)-points of \( G \), we have \( JG = G[[t]] \). Similarly, \( J\mathfrak{g} = \mathfrak{g}[[t]] = \text{Lie}(JG) \).

The affine space \( \mathfrak{g}^\ast \) is a Poisson variety by the Kirillov-Kostant Poisson structure, see \( \mathbf{2.1.3} \). If \( \{x_i\} \) is a basis of \( \mathfrak{g} \), then

\[
\mathbb{C}[\mathfrak{g}^*] = \mathbb{C}[x_1, x_2, \ldots, x_n].
\]

Thus

\[
(3.16) \quad J\mathfrak{g}^* = \text{Spec} \, \mathbb{C}[x_{i(-n)}| i = 1, 2, \ldots, t; n \geq 1].
\]

So we may identify \( \mathbb{C}[J\mathfrak{g}^*] \) with the symmetric algebra \( S(\mathfrak{g}[t^{-1}] t^{-1}) \).

Let \( x = x_{(-1)}[0] = (xt^{-1})[0] \), where we denote by \( [0] \) the unite element in \( S(\mathfrak{g}[t^{-1}] t^{-1}) \). Then \( (3.15) \) gives that

\[
(3.17) \quad [x_{(m)}, y_{(n)}] = [x, y]_{(m+n)}, \quad x, y \in \mathfrak{g}, m, n \in \mathbb{Z}_{\geq 0}.
\]

So the Lie algebra \( J\mathfrak{g} = \mathfrak{g}[[t]] \) acts on \( \mathbb{C}[J\mathfrak{g}^*] \). This action coincides with that obtained by differentiating the action of \( JG = G[[t]] \) on \( J\mathfrak{g}^* \) induced by the coadjoint action of \( G \). In other words, the vertex Poisson algebra structure of \( \mathbb{C}[J\mathfrak{g}^*] \) comes from the \( JG \)-action on \( J\mathfrak{g}^* \).

### 3.5. Canonical filtration of vertex algebras.

Haisheng Li [Li2] has shown that every vertex algebra is canonically filtered. For a vertex algebra \( V \), let \( F^pV \) be the subspace of \( V \) spanned by the elements

\[
a_1^{n_1-1}a_2^{n_2-1}\cdots a_r^{n_r-1}|0\rangle
\]

with \( a^1, a^2, \ldots, a^r \in V, \, n_i \geq 0, \, n_1 + n_2 + \cdots + n_r \geq p \). Then

\[
V = F^0V \supset F^1V \supset \ldots.
\]

It is clear that \( TF^pV \subset F^{p+1}V \).

Set \( (F^pV)_n := \text{span}_\mathbb{C} \{a_n b| a \in F^pV, b \in F^qV\} \).

**Lemma 3.12.** We have

\[
F^pV = \sum_{j \geq 0} (F^0V)_{(j-1)}F^{p-j}V.
\]

**Proposition 3.13.** \( (F^pV)_n (F^qV) \subset F^{p+q-n-1}V \). Moreover, if \( n \geq 0 \), we have \( (F^pV)_n (F^qV) \subset F^{p+q-n}V \).
(2) The filtration $F^*V$ is separated, that is, $\bigcap_{p \geq 0} F^p V = \{0\}$, if $V$ is a positive energy representation over itself.

Proof. It is straightforward to check. ($\text{(2)}$ also follows from Lemma 3.23 below.) $\square$

In this note we assume that the filtration $F^*V$ is separated.

Set $gr V = \bigoplus_{p \geq 0} F^p V / F^{p+1} V$.

We denote by $\sigma_p : F^p V \rightarrow F^p V / F^{p+1} V$ for $p \geq 0$, the canonical quotient map.

Proposition 3.13 gives the following.

**Proposition 3.14** ([Li2]). The space $gr V$ is a Poisson vertex algebra by

\[ \sigma_p(a) \cdot \sigma_q(b) := \sigma_{p+q}(a_{(-1)} b), \quad \sigma_p(a)(n) \sigma_q(b) := \sigma_{p+q-n}(a(n)b) \]

for $a \in F^p V$, $b \in F^q V$, $n \geq 0$.

Set

\[ R_V := F^0 V / F^1 V \subset gr V. \]

Note that $F^1 V = \text{span}_C \{ a_{(-2)} b \mid a, b \in V \}$.

**Proposition 3.15** ([Zhu, Li2]). The restriction of the Poisson structure gives $R_V$ a Poisson algebra structure, that is, $R_V$ is a Poisson algebra by

\[ \bar{a} \cdot \bar{b} := a_{(-1)} b, \quad \{ \bar{a}, \bar{b} \} = a_{(0)} b, \]

where $\bar{a} = \sigma_0(a)$.

Proof. It is straightforward from Proposition 3.14. $\square$

In the literature $F^1 V$ is often denoted by $C_2(V)$ and the Poisson algebra $R_V$ is called Zhu’s $C_2$-algebra.

A vertex algebra $V$ is called finitely strongly generated if $R_V$ is finitely generated as a ring. If the images of vectors $a_1, \ldots, a_N \in V$ generate $R_V$, we say that $V$ is strongly generated by $a_1, \ldots, a_N$.

Below we always assume that a vertex algebra $V$ is finitely strongly generated.

Note that if $\phi : V \rightarrow W$ is a homomorphism of vertex algebras, $\phi$ respects the canonical filtration, that is, $\phi(F^p V) \subset F^p W$. Hence it induces the homomorphism $gr V \rightarrow gr W$ of Poisson vertex algebra homomorphism which we denote by $gr \phi$.

### 3.6. Associated variety and singular support of vertex algebras.

**Definition 3.16.** Define the associated scheme $\tilde{X}_V$ and the associated variety $X_V$ of a vertex algebra $V$ as

\[ \tilde{X}_V := \text{Spec } R_V, \quad X_V := \text{Specm } R_V = (\tilde{X}_V)_{\text{red}}. \]

It was shown in [Li2, Lemma 4.2] that $gr V$ is generated by the subring $R_V$ as a differential algebra. Thus, we have a surjection $JR_V \rightarrow gr V$ of differential algebras by Remark 3.4. This is in fact a homomorphism of Poisson vertex algebras:
Theorem 3.17 (L12 Lemma 4.2, A4 Proposition 2.5.1). The identity map $R_V \to R_V$ induces a surjective Poisson vertex algebra homomorphism

$$JR_V = \mathbb{C}[J\tilde{X}_V] \twoheadrightarrow \text{gr} V.$$ 

Let $a^1, \ldots, a^n$ be a set of strong generators of $V$. Since $\text{gr} V \cong V$ as $\mathbb{C}$-vector spaces by the assumption that $F^•V$ is separated, it follows from Theorem 3.17 that $V$ is spanned by elements

$$a^{i_1}_{(-n_1)} \cdots a^{i_r}_{(-n_r)} |0\rangle$$

with $r \geq 0$, $n_i \geq 1$.

Definition 3.18. Define the singular support of a vertex algebra $V$ as

$$SS(V) := \text{Spec}(\text{gr} V) \subset J\tilde{X}_V.$$

Theorem 3.19. We have $\dim SS(V) = 0$ if and only if $\dim X_V = 0$.

Proof. The “only if” part is obvious since $\pi_\infty(SS(V)) = \tilde{X}_V$, where $\pi_\infty : J\tilde{X}_V \to \tilde{X}_V$ is the projection. The “if” part follows from Corollary 3.6. □

Definition 3.20. We call $V$ lisse (or $C_2$-cofinite) if $\dim X_V = 0$.

Remark 3.21. Suppose that $V$ is $\mathbb{Z}_+$-graded, so that $V = \bigoplus_{i \geq 0} V_i$, and that $V_0 = \mathbb{C} |0\rangle$. Then $\text{gr} V$ and $R_V$ are equipped with the induced grading:

$$\text{gr} V = \bigoplus_{i \geq 0} (\text{gr} V)_i, \quad (\text{gr} V)_0 = \mathbb{C},$$

$$R_V = \bigoplus_{i \geq 0} (R_V)_i, \quad (R_V)_0 = \mathbb{C}.$$

So the following conditions are equivalent:

1. $V$ is lisse.
2. $X_V = \{0\}$.
3. The image of any vector $a \in V_i$ for $i \geq 1$ in $\text{gr} V$ is nilpotent.
4. The image of any vector $a \in V_i$ for $i \geq 1$ in $R_V$ is nilpotent.

Thus, lisse vertex algebras can be regarded as a generalization of finite-dimensional algebras.

Remark 3.22. Suppose that the Poisson structure of $R_V$ is trivial. Then the Poisson vertex algebra structure of $JR_V$ is trivial, and so is that of $\text{gr} V$ by Theorem 3.17. This happens if and only if

$$(F^p V)_{(n)}(F^q V) \subset F^{p+q-n+1}V \quad \text{for all } n \geq 0.$$ 

If this is the case, one can give $\text{gr} V$ yet another Poisson vertex algebra structure by setting

$$\sigma_p(a)_{(n)} \sigma_q(b) := \sigma_{p+q-n+1}(a_{(n)} b) \quad \text{for } n \geq 0.$$

(We can repeat this procedure if this Poisson vertex algebra structure is again trivial).
3.7. Comparison with weight-depending filtration. Let $V$ be a vertex algebra that is $\mathbb{Z}$-graded by some Hamiltonian $H$:

$$V = \bigoplus_{\Delta \in \mathbb{Z}} V_\Delta \quad \text{where} \quad V_\Delta := \{ v \in V | H v = \Delta v \}. $$

Then there is another natural filtration of $V$ defined as follows.

For a homogeneous vector $a \in V_\Delta$, $\Delta$ is called the conformal weight of $a$ and is denoted by $\Delta_a$. Let $G_p V$ be the subspace of $V$ spanned by the vectors

$$a_1^{(-n_1-1)} a_2^{(-n_2-1)} \cdots a_r^{(-n_r-1)} |0\rangle$$

with $\Delta_{a_1} + \cdots + \Delta_{a_r} \leq p$. Then $G \bullet V$ defines an increasing filtration of $V$:

$$0 = G_{-1} V \subset G_0 V \subset \cdots G_1 V \subset \cdots, \quad V = \bigcup_p G_p V.$$

Moreover we have

$$T G_p V \subset G_p V,$$

$$(G_p)_V G_q V \subset G_{p+q} V \quad \text{for} \ n \in \mathbb{Z},$$

$$(G_p)_V G_q V \subset G_{p+q-1} V \quad \text{for} \ n \in \mathbb{Z}^+. $$

It follows that $\text{gr}_G V = \bigoplus G_p V / G_{p-1} V$ is naturally a Poisson vertex algebra.

It is not too difficult to see the following.

Lemma 3.23 ([A4, Proposition 2.6.1]). We have

$$F^p V_\Delta = G_{\Delta-p} V_\Delta,$$

where $F^p V_\Delta = V_\Delta \cap F^p V$, $G_p V_\Delta = V_\Delta \cap G_p V$. Therefore

$$\text{gr} V \cong \text{gr}_G V$$

as Poisson vertex algebras.

3.8. Example: universal affine vertex algebras. Let $\mathfrak{a}$ be a Lie algebra with a symmetric invariant bilinear form $\kappa$. Let

$$\hat{\mathfrak{a}} = \mathfrak{a}[t, t^{-1}] \oplus \mathbb{C} 1$$

be the Kac-Moody affinization of $\mathfrak{a}$. It is a Lie algebra with commutation relations

$$[xt^m, y t^n] = [x, y] t^{m+n} + m \delta_{m+n,0} \kappa(x, y) 1, \quad x, y \in \mathfrak{a}, \ m, n \in \mathbb{Z}, \quad [1, \hat{\mathfrak{a}}] = 0. $$

Let

$$V^\kappa(\mathfrak{a}) = U(\hat{\mathfrak{a}}) \otimes_{U(\mathfrak{a}[t] \oplus \mathbb{C} 1)} \mathbb{C},$$

where $\mathbb{C}$ is one-dimensional representation of $\mathfrak{a}[t] \oplus \mathbb{C} 1$ on which $\mathfrak{a}[t]$ acts trivially and $1$ acts as the identity. The space $V^\kappa(\mathfrak{a})$ is naturally graded: $V^\kappa(\mathfrak{a}) = \bigoplus_{\Delta \in \mathbb{Z}_{\geq 0}} V^\kappa(\mathfrak{a})_\Delta$, where the grading is defined by setting $\deg xt^n = -n$, $\deg |0\rangle = 0$. Here $|0\rangle = 1 \otimes 1$. We have $V^\kappa(\mathfrak{a})_0 = \mathbb{C} |0\rangle$. We identify $\mathfrak{a}$ with $V^\kappa(\mathfrak{a})_1$ via the linear isomorphism defined by $x \mapsto xt^{-1}|0\rangle$. 

There is a unique vertex algebra structure on $V^\kappa(a)$ such that $|0\rangle$ is the vacuum vector and

$$Y(x, z) = x(z) := \sum_{n \in \mathbb{Z}} (xt^n) z^{-n-1}, \quad x \in a.$$  

(So $x(n) = xt^n$ for $x \in a = V^\kappa(a)_1, n \in \mathbb{Z}$).

The vertex algebra $V^\kappa(a)$ is called the universal affine vertex algebra associated with $(a, \kappa)$.

We have $F^1V^\kappa(a) = a[t^{-1}]t^{-2}V^\kappa(a)$, and the Poisson algebra isomorphism

$$\mathbb{C}[a^*] \rightarrow R_{V^\kappa(a)} = V^k(a)/g[t^{-1}]t^{-2}V^\kappa(a)$$

$$x_1 \ldots x_r \mapsto (x_1t^{-1})\ldots (x_rt^{-1})|0\rangle \quad (x_i \in a).$$

Thus

$$X_{V^\kappa(a)} = a^*.$$  

We have the isomorphism

$$(3.20) \quad \mathbb{C}[Ja^*] \simeq \text{gr} V^\kappa(a)$$

because the graded dimensions of both sides coincide. Therefore

$$SS(V^\kappa(a)) = Ja^*.$$  

The isomorphism $(3.20)$ follows also from the fact that

$$G_p V^\kappa(a) = U_p(a[t^{-1}]t^{-1})|0\rangle,$$

where $\{U_p(a[t^{-1}]t^{-1})\}$ is the PBW filtration of $U(a[t^{-1}]t^{-1})$.

3.9. Example: simple affine vertex algebras. For a finite-dimensional simple Lie algebra $g$ and $k \in \mathbb{C}$, we denote by $V^k(g)$ the universal affine vertex algebra $V^{k\kappa_0}(g)$, where $\kappa_0$ is the normalized invariant inner product of $g$, that is,

$$\kappa_0(\theta, \theta) = 2,$$

where $\theta$ is the highest root of $g$. Denote by $V_k(g)$ the unique simple graded quotient of $V^k(g)$. As a $\hat{g}$-module, $V_k(g)$ is isomorphic to the irreducible highest weight representation $L(k\Lambda_0)$ of $\hat{g}$ with highest weight $k\Lambda_0$, where $\Lambda_0$ is the weight of the basic representation of $\hat{g}$.

**Theorem 3.24.** The vertex algebra $V_k(g)$ is lisse if and only if $V_k(g)$ is integrable as a $\hat{g}$-module, which is true if and only if $k \in \mathbb{Z}_+.$

**Lemma 3.25.** Let $(R, \partial)$ be a differential algebra over $\mathbb{Q}$, $I$ a differential ideal of $R$, i.e., $I$ is an ideal of $R$ such that $\partial I \subset I$. Then $\partial \sqrt{I} \subset \sqrt{I}$.

**Proof.** Let $a \in \sqrt{I}$, so that $a^m \in I$ for some $m \in \mathbb{N} = \{1, 2, \ldots\}$. Since $I$ is $\partial$-invariant, we have $\partial^m a^m \in I$. But

$$\partial^m a^m = \sum_{0 \leq i \leq m} \binom{m}{i} a^{m-i}(\partial a)^i \equiv ml(\partial a)^m \pmod{\sqrt{I}}.$$  

Hence $(\partial a)^m \in \sqrt{I}$, and therefore, $\partial a \in \sqrt{I}$.  

\[\square\]
Proof of the “if” part of Theorem 3.24. Suppose that $V_k(\mathfrak{g})$ is integrable. This condition is equivalent to that $k \in \mathbb{Z}^+$ and the maximal submodule $N_k$ of $V^k(\mathfrak{g})$ is generated by the singular vector $(e_\theta t^{-1})^{k+1}|0\rangle$ ([Kac2]). The exact sequence $0 \to N_k \to V^k(\mathfrak{g}) \to V_k(\mathfrak{g}) \to 0$ induces the exact sequence

$$0 \to I_k \to R_V^k(\mathfrak{g}) \to R_{V_k(\mathfrak{g})} \to 0,$$

where $I_k$ is the image of $N_k$ in $R_V^k(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]/I_k$. The image of the singular vector in $I_k$ is given by $e_k \theta$. Therefore, $e_\theta \in \sqrt{I}$. On the other hand, by Lemma 3.25, $\sqrt{I}$ is preserved by the adjoint action of $\mathfrak{g}$. Since $\mathfrak{g}$ is simple, $\mathfrak{g} \subset \sqrt{I}$. This proves that $X_{V_k(\mathfrak{g})} = \{0\}$ as required. □

The proof of “only if” part follows from [DM]. We will give a different proof using $W$-algebras in Remark 6.21.

In view of Theorem 3.24, one may regard the lisse condition as a generalization of the integrability condition to an arbitrary vertex algebra.

4. Zhu’s algebra

In this section we will introduce and study the Zhu’s algebra of a vertex algebra, which plays an important role in the representation theory. See [Kac3] in this volume for the definition of modules over vertex algebras.

4.1. Zhu’s $C_2$-algebra and Zhu’s algebra of a vertex algebra. Let $V$ be a $\mathbb{Z}$-graded vertex algebra. Zhu’s algebra $Zhu(V)$ is defined as

$$Zhu(V) := V/V \circ V$$

where $V \circ V := \text{span}\{a \circ b|a, b \in V\}$ and

$$a \circ b := \sum_{i \geq 0} \left(\frac{\Delta a}{i}\right)_{(i-2)} b$$

for homogeneous elements $a, b$ and extended linearly. It is an associative algebra with multiplication defined as

$$a * b := \sum_{i \geq 0} \left(\frac{\Delta a}{i}\right)_{(i-1)} b$$

for homogeneous elements $a, b \in V$.

For a simple positive energy representation $M = \bigoplus_{n \in \mathbb{Z}_+} M_{\lambda+n}$, $M_{\lambda} \neq 0$, of $V$, let $M_{\text{top}}$ be the top degree component $M_{\lambda}$ of $M$. Also, for a homogeneous vector $a \in V$, let $o(a) = a_{(\Delta_{\lambda}-1)}$, so that $o(a)$ preserves the homogeneous components of any graded representation of $V$.

The importance of Zhu’s algebra in vertex algebra theory is the following fact that was established by Yonchang Zhu.

**Theorem 4.1 (Zhu).** For any positive energy representation $M$ of $V$, $\pi \mapsto o(\pi)$ defines a well-defined representation of $Zhu(V)$ on $M_{\text{top}}$. Moreover, the correspondence $M \mapsto M_{\text{top}}$ gives a bijection between the set of isomorphism classes of irreducible positive energy representations of $V$ and that of simple $Zhu(V)$-modules.
A vertex algebra $V$ is called a *chiralization* of an algebra $A$ if $\text{Zhu}(V) \cong A$.

Now we define an increasing filtration of Zhu’s algebra. For this, we assume that $V$ is $\mathbb{Z}_+$-graded: $V = \bigoplus_{\Delta \geq 0} V_\Delta$. Then $V_{\leq p} = \bigoplus_{\Delta = 0}^p V_\Delta$ gives an increasing filtration of $V$. Define

$$\text{Zhu}_p(V) := \text{im}(V_{\leq p} \rightarrow \text{Zhu}(V)).$$

Obviously, we have

$$0 = \text{Zhu}_{-1}(V) \subset \text{Zhu}_0(V) \subset \text{Zhu}_1(V) \subset \cdots \text{, and } \text{Zhu}(V) = \bigcup_{p \geq -1} \text{Zhu}_p(V).$$

Also, since $a_{(i)} b \in V_{\Delta_a + \Delta_b - 1}$ for $a \in V_{\Delta_a}, b \in V_{\Delta_b}$, we have

\[ (4.1) \quad \text{Zhu}_p(V) * \text{Zhu}_q(V) \subset \text{Zhu}_{p+q}(V). \]

The following assertion follows from the skew symmetry.

**Lemma 4.2.** We have

$$b * a \equiv \sum_{i \geq 0} \binom{\Delta_a - 1}{i} a_{(i-1)} b \quad (\text{mod } V \circ V),$$

and hence,

$$a * b - b * a \equiv \sum_{i \geq 0} \binom{\Delta_a - 1}{i} a_{(i)} b \quad (\text{mod } V \circ V).$$

By Lemma 4.2, we have

\[ (4.2) \quad [\text{Zhu}_p(V), \text{Zhu}_q(V)] \subset \text{Zhu}_{p+q-1}(V). \]

By (4.1) and (4.2), the associated graded $\text{gr } \text{Zhu}(V) = \bigoplus_p \text{Zhu}_p(V)/\text{Zhu}_{p-1}(V)$ is naturally a graded Poisson algebra.

Note that $a \circ b \equiv a_{(-2)} b \pmod{\bigoplus_{\Delta < \Delta_a \Delta_b} V_\Delta}$ for homogeneous elements $a, b \in V$.

**Lemma 4.3** (Zhu, see [DSK, Proposition 2.17(c)], [ALY1, Proposition 3.3]). The following map defines a well-defined surjective homomorphism of Poisson algebras.

\[ \eta_V : R_V \rightarrow \text{gr } \text{Zhu}(V) \]

$$\bar{a} \mapsto a \pmod{V \circ V + \bigoplus_{\Delta < \Delta_a} V_\Delta}.$$  

**Remark 4.4.** The map $\eta_V$ is not an isomorphism in general. For an example, let $g$ be the simple Lie algebra of type $E_8$ and $V = V_1(g)$. Then $\dim R_V > \dim \text{Zhu} V = 1$.

**Corollary 4.5.** If $V$ is lisse then $\text{Zhu} V$ is finite dimensional. Hence the number of isomorphic classes of simple positive energy representations of $V$ is finite.

In fact the following stronger facts are known.

**Theorem 4.6** ([ABD]). Let $V$ be lisse. Then any simple $V$-module is a positive energy representation. Therefore the number of isomorphic classes of simple $V$-modules is finite.
Theorem 4.7 ([DLM] [MNT]). Let $V$ be lisse. Then the abelian category of $V$-modules is equivalent to the module category of a finite-dimensional associative algebra.

4.2. Computation of Zhu’s algebras. We say that a vertex algebra $V$ admits a PBW basis if $R_V$ is a polynomial algebra and the map $\mathbb{C}[JX_V] \to \text{gr } V$ is an isomorphism.

Theorem 4.8. If $V$ admits a PBW basis, then $\eta_V : R_V \to \text{gr Zhu } V$ is an isomorphism.

Proof. We have $\text{gr Zhu}(V) = V/\text{gr}(V \circ V)$, where $\text{gr}(V \circ V)$ is the associated graded space of $V \circ V$ with respect to the filtration induced by the filtration $V_{\leq p}$. We wish to show that $\text{gr}(V \circ V) = F^1 V$. Since $a \circ b \equiv a_{(-2)} b \pmod{F_{\leq \Delta_n + \Delta_b} V}$, it is sufficient to show that $a \circ b \neq 0$ implies that $a_{(-2)} b \neq 0$.

Suppose that $a_{(-2)} = (Ta)_{(-1)} b = 0$. Since $V$ admits a PBW basis, $\text{gr } V$ has no zero divisors. That fact that $V$ admits a PBW basis also shows that $T a = 0$ implies that $a = c(0)$ for some constant $c \in \mathbb{C}$. Thus, $a$ is a constant multiple of $|0\rangle$, in which case $a \circ b = 0$.

Example 4.9 (Universal affine vertex algebras). The universal affine vertex algebra $V^\kappa(a)$ (see [DLM]) admits a PBW basis. Therefore

$$
\eta_{V^\kappa(a)} : R_{V^\kappa(a)} = \mathbb{C}[a^*] \to \text{gr Zhu } V^\kappa(a).
$$

On the other hand, from Lemma 4.2 one finds that

$$
(4.3) \quad U(a) \to \text{Zhu } (V^\kappa(a))
$$

$$
\quad a \ni x \mapsto \bar{x} = x_{(-1)} |0\rangle
$$

gives a well-defined algebra homomorphism. This map respects the filtration on both sides, where the filtration in the left-hand-side is the PBW filtration. Hence it induces a map between their associated graded algebras, which is identical to $\eta_{V^\kappa(a)}$. Therefore (4.3) is an isomorphism, that is to say, $V^\kappa(a)$ is a chiralization of $U(a)$.

Exercise 1. Extend Theorem 4.8 to the case that $a$ is a Lie superalgebra.

Example 4.10 (Free fermions). Let $n$ be a finite-dimensional vector space. The Clifford affinization $\hat{\mathcal{C}}I$ of $n$ is the Clifford algebra associated with $n[t, t^{-1}] \oplus n^* [t, t^{-1}]$ and its symmetric bilinear form defined by

$$(xtm|ftn) = \delta_{m+n,0} f(x), \quad (xtm|gt^n) = 0 = (ftm|gt^n)$$

for $x, y \in n$, $f, g \in n^*$, $m, n \in \mathbb{Z}$.

Let $\{x_{\alpha}\}_{\alpha \in \Delta_+}$ be a basis of $n$, $\{x_{\beta}^*\}$ its dual basis. We write $\psi_{\alpha, m}$ for $x_{\alpha} t^m \in \hat{\mathcal{C}}I$ and $\psi_{\alpha, m}^*$ for $x_{\alpha}^* t^m \in \hat{\mathcal{C}}I$, so that $\hat{\mathcal{C}}I$ is the associative superalgebra with

- odd generators: $\psi_{\alpha, m}, \psi_{\alpha, m}^*, m \in \mathbb{Z}, \alpha \in \Delta_+$,
- relations: $[\psi_{\alpha, m}, \psi_{\beta, n}] = [\psi_{\alpha, m}^*, \psi_{\beta, n}^*] = 0, [\psi_{\alpha, m}, \psi_{\beta, n}] = \delta_{\alpha, \beta} \delta_{m+n,0}$. 

Define the charged fermion Fock space associated with $\mathfrak{n}$ as

$$F_n := \hat{C}l/(\sum_{m \geq 0} \hat{C}l\psi_{\alpha,m} + \sum_{k \geq 1} \hat{C}l\psi_{\beta,k}).$$

It is an irreducible $\hat{C}l$-module, and as $\mathbb{C}$-vector spaces we have

$$F_n \cong \Lambda(n^*[t^{-1}]) \otimes \Lambda(n[t^{-1}]t^{-1}).$$

There is a unique vertex (super)algebra structure on $F_n$ such that the image of 1 is the vacuum $|0\rangle$ and

$$Y(\psi_{\alpha,-1}|0\rangle, z) = \psi_{\alpha}(z) := \sum_{n \in \mathbb{Z}} \psi_{\alpha,n} z^{-n-1},$$

$$Y(\psi^{*}_{\alpha,0}|0\rangle, z) = \psi^{*}_{\alpha}(z) := \sum_{n \in \mathbb{Z}} \psi^{*}_{\alpha,n} z^{-n}.$$ 

We have $F^1F_n = n^*[t^{-1}]t^{-1}F_n + n[t^{-1}]t^{-2}F_n$, and it follows that there is an isomorphism

$$\overline{C}l \rightarrow R_{F_n},$$

$$x_{\alpha} \mapsto \psi_{\alpha,-1}|0\rangle,$$

$$x^{*}_{\alpha} \mapsto \psi^{*}_{\alpha,0}|0\rangle$$

as Poisson superalgebras. Thus,

$$X_{F_n} = T^*\Pi n,$$

where $\Pi n$ is the space $\mathfrak{n}$ considered as a purely odd affine space. The arc space $JT^*\Pi n$ is also regarded as a purely odd affine space, such that $\mathbb{C}[JT^*\Pi n] = \Lambda(n^*[t^{-1}]) \otimes \Lambda(n[t^{-1}]t^{-1})$. The map $\mathbb{C}[JX_{F_n}] \rightarrow \text{gr}F_n$ is an isomorphism and $F_n$ admits a PBW basis. Therefore we have the isomorphism

$$\eta_{F_n} : R_{F_n} = \overline{C}l \rightarrow \text{Zhu}(F_n)$$

by Exercise[11] On the other hand the map

$$C l \rightarrow \text{Zhu}(F_n)$$

$$x_{\alpha} \mapsto \psi_{\alpha,-1}|0\rangle,$$

$$x^{*}_{\alpha} \mapsto \psi^{*}_{\alpha,0}|0\rangle$$

gives an algebra homomorphism that respects the filtration. Hence we have

$$\text{Zhu}(F_n) \cong C l.$$ 

That is, $F_n$ is a chiralization of $C l$.

5. $W$-algebras

We are now in a position to define $W$-algebras. We will construct a differential graded vertex algebra, so that its cohomology algebra is a vertex algebra and that will be our main object to study.

For simplicity, we let $\mathfrak{g} = \mathfrak{gl}_n$ and we only consider the principal nilpotent case. However the definition works for any simple Lie algebra. The general definition for
an arbitrary nilpotent element will be similar but one does need a new idea (see [KRW] for the most general definition).

5.1. The BRST complex. Let $\mathfrak{g}, \mathfrak{n}$ be as in §2.1. Denote by $\kappa_\mathfrak{g}$ the Killing form on $\mathfrak{g}$ and $\kappa_0 = \frac{1}{2n} \kappa_\mathfrak{g}$, so that $\kappa_0(\theta, \theta) = 2$.

Choose any symmetric invariant bilinear form $\kappa$ on $\mathfrak{g}$ and let $V^\kappa(\mathfrak{g})$ be the universal affine vertex algebra associated with $(\mathfrak{g}, \kappa)$ (see §3.8) and let $\mathcal{F} = \mathcal{F}_n$ be the fermion Fock space as in Example 4.10.

We have the following commutative diagrams:

\[
\begin{array}{c}
\mathbb{C}[J\mathfrak{g}^\ast] \xrightarrow{\text{gr}(\theta)} V^\kappa(\mathfrak{g}) \xrightarrow{\text{Zhu}(\theta)} \mathbb{C}[\mathfrak{g}^\ast] \\
\downarrow \quad \quad \downarrow \\
\mathbb{C}[\mathfrak{g}^\ast] \xrightarrow{\text{gr}(\theta)} U(\mathfrak{g}),
\end{array}
\]

\[
\begin{array}{c}
\mathbb{C}[JT^\ast \Pi n] \xrightarrow{\text{gr}(\theta)} \mathcal{F} \xrightarrow{\text{Zhu}(\theta)} \mathbb{C} \xrightarrow{\text{gr}(\theta)} Cl \\
\downarrow \quad \quad \downarrow \\
\mathbb{C}[JT^\ast \Pi n] \xrightarrow{\text{gr}(\theta)} Cl
\end{array}
\]

Define

\[C^\kappa(\mathfrak{g}) := V^\kappa(\mathfrak{g}) \otimes \mathcal{F}.\]

Since it is a tensor product of two vertex algebras, $C^\kappa(\mathfrak{g})$ is a vertex algebra. We have

\[R_{C^\kappa(\mathfrak{g})} = R_{V^\kappa(\mathfrak{g})} \otimes R_{\mathcal{F}} = \mathbb{C}[\mathfrak{g}^\ast] \otimes Cl = \mathcal{C}(\mathfrak{g}),\]

and

\[\text{Zhu}C^\kappa(\mathfrak{g}) = \text{Zhu}V^\kappa(\mathfrak{g}) \otimes \text{Zhu}\mathcal{F} = U(\mathfrak{g}) \otimes Cl = C(\mathfrak{g}).\]

Thus, $C^\kappa(\mathfrak{g})$ is a chiralization of $C(\mathfrak{g})$ considered in §2.5. Further we have

\[\text{gr}C^\kappa(\mathfrak{g}) = \text{gr}V^\kappa(\mathfrak{g}) \otimes \text{gr}\mathcal{F} = \mathbb{C}[J\mathfrak{g}^\ast] \otimes \mathbb{C}[JT^\ast \Pi n].\]

So we have the following commutative diagram:

\[
\begin{array}{c}
\mathbb{C}[J\mathfrak{g}^\ast] \xrightarrow{\text{gr}(\theta)} V^\kappa(\mathfrak{g}) \xrightarrow{\text{Zhu}(\theta)} \mathbb{C}[\mathfrak{g}^\ast] \\
\downarrow \quad \quad \downarrow \\
\mathbb{C}[\mathfrak{g}^\ast] \xrightarrow{\text{gr}(\theta)} U(\mathfrak{g})
\end{array}
\]

Define a gradation

\[(5.1) \quad \mathcal{F} = \bigoplus_{p \in \mathbb{Z}} \mathcal{F}^p\]

by setting $\text{deg} \psi_{\alpha,m} = -1, \text{deg} \psi_{\alpha,k}^* = 1, \forall i, j \in I, m, k \in \mathbb{Z}, \text{deg} [0] = 0$. This induces a $\mathbb{Z}$-grading (that is different from the conformal grading) on $C^\kappa(\mathfrak{g})$:

\[(5.2) \quad C^\kappa(\mathfrak{g}) = V^\kappa(\mathfrak{g}) \otimes \mathcal{F} = \bigoplus_{p \in \mathbb{Z}} C^{\kappa, p}(\mathfrak{g}), \quad \text{where} \quad C^{\kappa, p}(\mathfrak{g}) := V^\kappa(\mathfrak{g}) \otimes \mathcal{F}^p.\]

Let $V(\mathfrak{n})$ be the universal affine vertex algebra associated with $\mathfrak{n}$ and the zero bilinear form, which is identified with the vertex subalgebra of $V^\kappa(\mathfrak{g})$ generated by $x_\alpha(z)$ with $\alpha \in \Delta_+$. 

\[\text{Zhu}\mathcal{F}(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^\ast] \otimes \mathbb{C}[\mathfrak{g}^\ast] \otimes \mathbb{C}[\mathfrak{g}^\ast] \otimes \mathbb{C}[\mathfrak{g}^\ast] \otimes \mathbb{C}[\mathfrak{g}^\ast].\]
Lemma 5.1. The following defines a vertex algebra homomorphism.
\[ \hat{\rho} : V(n) \rightarrow F \]
\[ x_\alpha(z) \mapsto \sum_{\beta, \gamma \in \Delta_+} c_{\alpha, \beta}^\gamma \psi_\beta^*(z) \psi_\gamma(z). \]

Remark 5.2. In the above formula the normally ordered product is not needed because \( n \) is nilpotent.

The map \( \hat{\rho} \) induces an algebra homomorphism
\[ \text{Zhu} V(n) = U(n) \rightarrow \text{Zhu} F = Cl \]
and a Poisson algebra homomorphism
\[ R V(n) = \mathbb{C}[n^\ast] \rightarrow R F = \overline{\mathbb{C}l} \]
that are identical to \( \rho \) and \( \bar{\rho} \) (see Lemma 2.8 and 2.9), respectively.

Recall the character \( \chi : n \rightarrow \mathbb{C}, x \mapsto (f|x) \).

Lemma 5.3. The following defines a vertex algebra homomorphism.
\[ \hat{\theta}_\chi : V(n) \rightarrow C^\ast(g) \]
\[ x_\alpha(z) \mapsto (x_\alpha(z) + \chi(x_\alpha)) \otimes \text{id} + \text{id} \otimes \hat{\rho}(x_\alpha(z)). \]

The map \( \hat{\theta}_\chi \) induces an algebra homomorphism
\[ \text{Zhu} V(n) = U(n) \rightarrow \text{Zhu} C^\ast(g) = C(g) \]
and a Poisson algebra homomorphism
\[ R V(n) = \mathbb{C}[n^\ast] \rightarrow R F = \overline{C(g)} \]
that are identical to \( \theta_\chi \) and \( \bar{\theta} \), respectively (see Lemmas 2.10 and 2.14).

The proof of the following assertion is similar to that of Lemma 2.11.

Proposition 5.4. There exists a unique element \( \hat{Q} \in C^{k,1}(g) \) such that
\[ [\hat{Q}_\lambda(1 \otimes \psi_\alpha)] = \hat{\theta}_\chi(x_\alpha), \quad \forall \alpha \in \Delta_. \]
We have \( [\hat{Q}_\lambda \hat{Q}] = 0 \).

The field \( \hat{Q}(z) \) is given explicitly as
\[ \hat{Q}(z) = \sum_{\alpha \in \Delta_+} (x_\alpha + \chi(x_\alpha)) \otimes \psi_\alpha^*(z) - \text{id} \otimes \frac{1}{2} \sum_{\alpha, \beta, \gamma \in \Delta_+} c_{\alpha, \beta}^\gamma \psi_\beta^*(z) \psi_\gamma(z). \]

Since \( \hat{Q} \) is odd and \( [\hat{Q}_\lambda \hat{Q}] = 0 \), we have
\[ \hat{Q}_{(0)}^2 = 0. \]
(Recall that we write \( \hat{Q}(z) = \sum_{n \in \mathbb{Z}} \hat{Q}_{(n)} z^{-n-1} \)). So \( (C^\ast(g), \hat{Q}_{(0)}) \) is a cochain complex.

Lemma 5.5. If it is nonzero, the cohomology \( H^\ast(C^\ast(g), \hat{Q}_{(0)}) \) inherits the vertex algebra structure from \( C^\ast(g) \).
Theorem 5.7. We have morphism 
\( (\text{gr} J\mu, \hat{\mathcal{Q}}(0)) \) as Poisson vertex algebras, where 

\[ \text{Proof.} \] Set \( Z := \{ v \in C^\infty(\g) \mid \hat{\mathcal{Q}}(0)v = 0 \} \), \( B = \hat{\mathcal{Q}}(0)C^\infty(\g) \subset Z \), so that \( H^\bullet(C^\infty(\g), \hat{\mathcal{Q}}(0)) = Z/B \). From the commutator formula (3.2), we know that 

\[ [\hat{\mathcal{Q}}(0), a(m)] = (\hat{\mathcal{Q}}(0)a)(m) \quad \forall a \in C^\infty(\g), m \in \mathbb{Z}. \]

Thus, if \( a, b \in Z \), then \( \hat{\mathcal{Q}}(0)(a(m)b) = 0 \), that is, \( a(m)b \in Z \). It follows that \( Z \) a vertex subalgebra of \( C^\infty(\g) \). Further, if \( a \in Z \) and \( b = \hat{\mathcal{Q}}(0)b' \in B \), then \( a(m)b = a(m)\hat{\mathcal{Q}}(0)b' = \hat{\mathcal{Q}}(0)(a(m)b) \in B \). Hence \( B \) is an ideal of \( Z \). This completes the proof. \( \Box \)

Definition 5.6. The \( W \)-algebra \( W^\infty(\g) = W^\infty(\g, f) \) associated to \( (\g, f, \kappa) \) is defined to be the zero-th cohomology of the cochain complex \( (C^\infty(\g), \hat{\mathcal{Q}}(0)) \), that is, 

\[ W^\infty(\g) := H^0(\hat{\mathcal{Q}}(0)). \]

This definition of \( W^\infty(\g) \) is due to Feigin and Frenkel [FF1]. In [FF1], we show that the above \( W^\infty(\g) \) is identical to the original \( W \)-algebra defined by Fateev and Lukyanov [FL].

5.2. Cohomology of associated graded. We have \( \hat{\mathcal{Q}}(0)F^pC^\infty(\g) \subset F^pC^\infty(\g) \), so \( (\text{gr}^F C^\infty(\g), \hat{\mathcal{Q}}(0)) \) is also a cochain complex. The cohomology \( H^\bullet(\text{gr}^F C^\infty(\g), \hat{\mathcal{Q}}(0)) \) inherits a Poisson vertex algebra structure from \( \text{gr}^F C^\infty(\g) \).

Theorem 5.7. We have \( H^i(\text{gr}^F C^\infty(\g), \hat{\mathcal{Q}}(0)) = 0 \) for \( i \neq 0 \) and 

\[ H^0(\text{gr}^F C^\infty(\g), \hat{\mathcal{Q}}(0)) \cong \mathbb{C}[JS] \]

as Poisson vertex algebras, where \( S \) is the slice defined in 2.12.

\[ \text{Proof.} \] The proof is an arc space analogue of that of Theorem 2.12.

The moment map \( \mu : \g^* \rightarrow n^* \) for the \( N \)-action on \( \g \) induces a \( JN \)-equivariant morphism 

\[ J\mu : J\g^* \rightarrow Jn^*. \]

The pullback \( (J\mu)^* : \mathbb{C}[Jn^*] \rightarrow \mathbb{C}[J\g^*] \) is an embedding of vertex Poisson algebras.

The point \( \chi = J\chi \) of \( Jn^* \) corresponds to the arc \( \alpha \in \text{Hom}(D, n^*) = \text{Hom}(\mathbb{C}[n^*], \mathbb{C}[t]) \) such that \( \alpha(f) = \chi(x) \) for \( x \in n \subset \mathbb{C}[n^*] \).

We have 

\[ (J\mu)^{-1}(\chi) = J(\mu^{-1}(\chi)) = \chi + Jb \subset J\g^*, \]

and the adjoint action gives the isomorphism 

\[ JN \times JS \cong J\mu^{-1}(\chi) \]

by Theorem 2.13 and 3.10.

Now put 

\[ C := \text{gr} C^\infty(\g) = \mathbb{C}[J\g^*] \otimes \Lambda(\mathbb{N}[t^{-1}]) \otimes \Lambda(\mathbb{N}[t^{-1}]) \]

and define a bigrading on \( C \) by 

\[ C = \bigoplus_{i \leq 0, j \geq 0} C^{i, j}, \quad \text{where} \quad C^{i, j} = \mathbb{C}[J\g^*] \otimes \Lambda^{-i}(\mathbb{N}[t^{-1}]) \otimes \Lambda^j(\mathbb{N}[t^{-1}]). \]
As before, we can decompose the operator $\hat{Q}(0)$ as the sum of two suboperators such that each of them preserves one grading but increase the other grading by 1. Namely, we have

$$\hat{Q}(0) = \hat{d}_+ + \hat{d}_-,$$

$$\hat{d}_- : C^{i,j} \longrightarrow C^{i,j+1}, \quad \hat{d}_+ : C^{i,j} \longrightarrow C^{i+1,j}.$$ 

This shows that

$$(\hat{d}_+)^2 = (\hat{d}_-)^2 = [\hat{d}_+, \hat{d}_-] = 0.$$ 

Thus we can get a spectral sequence $E_r \Longrightarrow H^*(C, \hat{Q}(0))$ such that

$$E_1 = H^*(C, \hat{d}_-), \quad E_2 = H^*(H^*(C, \hat{d}_-), \hat{d}_+) .$$

This is a converging spectral sequence since $C$ is a direct sum of subcomplexes $F^p C^\infty(g)/F^{p+1} C^\infty(g)$, and the associated filtration is regular on each subcomplex.

The complex $(C, \hat{d}_-)$ is the Koszul complex with respect to the sequence

$$x_1 t^{-1} - \chi(x_1), \ldots, x_N t^{-1} - \chi(x_N), x_1 t^{-2}, x_2 t^{-2}, \ldots, x_N t^{-2}, x_1 t^{-3}, x_2 t^{-3}, \ldots$$

where $N = \dim g.$ Hence we have

$$(5.6) \quad H^i(C, \hat{d}_-) = \delta_{i,0} \mathbb{C}[J\mu^{-1}(\chi)] \otimes \Lambda(n^*[t^{-1}]).$$

Next, by (5.6), the complex $(H^0(C, \hat{d}_-), \hat{d}_+)$ is identical to the Chevalley complex for the Lie algebra cohomology $H^*(Jn, \mathbb{C}[J\mu^{-1}(\chi)]) = H^*(n[[t]], \mathbb{C}[J\mu^{-1}(\chi)])$. By (5.3),

$$H^i(Jn, \mathbb{C}[J\mu^{-1}(\chi)]) = H^i(Jn, \mathbb{C}[JN] \otimes \mathbb{C}[JS]) = H^i(Jn, \mathbb{C}[JN]) \otimes \mathbb{C}[JS] = \delta_{i,0} \mathbb{C}[JS].$$

We conclude that

$$H^i(H^j(C, \hat{d}_-), \hat{d}_+) = \delta_{i,0} \delta_{j,0} \mathbb{C}[S].$$

Thus, the spectral sequence $E_r$ collapses at $E_2 = E_{\infty}$, and we get the desired isomorphisms.

**Theorem 5.8 ([FF1][FBZ]).** We have $H^0(C^\infty(g), \hat{Q}(0)) = 0$ for $i \neq 0$ and

$$\gr W^k(g) = \gr H^0(C^\infty(g), \hat{Q}(0)) \cong H^0(\gr C^\infty(g), \hat{Q}(0)) = \mathbb{C}[JS].$$

In particular, $R_{W^k(g)} \cong \mathbb{C}[S] \cong \mathbb{C}[g]^G$, so $\hat{X}_{W^k(g)} = S$, $SS(W^k(g)) = JS$.

The proof of Theorem 5.8 will be given in (5.6). Note that there is a spectral sequence for $H^*(C^\infty(g), \hat{Q}(0))$ such that $E_1^* = H^0(\gr C^\infty(g), \hat{Q}(0))$. Hence Theorem 5.8 would immediately follow from Theorem 5.7 if this spectral sequence converges. However, this is not clear at this point because our algebra is not Noetherian.
**Remark 5.9.** The complex \((C^\kappa(\mathfrak{g}), \hat{Q}_0)\) is identical to Feigin’s standard complex for the semi-infinite \(n[t, t^{-1}]\)-cohomology \(H^{\Delta +} n[t, t^{-1}], V^\kappa(\mathfrak{g}) \otimes C_\chi\) with coefficient in the \(\mathfrak{g}[t, t^{-1}]\)-module \(V^\kappa(\mathfrak{g}) \otimes C_\chi\) (\([\text{Pei}]\)), where \(C_\chi\) is the one-dimensional representation of \(n[t, t^{-1}]\) defined by the character \(\chi: n[t, t^{-1}] \to \mathbb{C}, x t^n \mapsto \delta_{n,-1} \chi(x)\):

\[
H^\bullet(C^\kappa(\mathfrak{g}), \hat{Q}_0) \cong H^{\Delta +} n[t, t^{-1}], V^\kappa(\mathfrak{g}) \otimes C_\chi.
\]

**5.3. W-algebra associated with \(\mathfrak{sl}_n\).** It is straightforward to generalize the above definition to an arbitrary simple Lie algebra \(\mathfrak{g}\). In particular, by replacing \(V^\kappa(\mathfrak{gl}_n)\) with \(V^k(\mathfrak{sl}_n), k \in \mathbb{C}\), we define the \(\mathcal{W}\)-algebra

\[
\mathcal{W}^k(\mathfrak{sl}_n) := H^0(C^k(\mathfrak{sl}_n), \hat{Q}_0)
\]

associated with \((\mathfrak{sl}_n, f)\) at level \(k\).

We have \(V^\kappa(\mathfrak{gl}_n) = \pi\kappa \otimes V^\kappa(\mathfrak{sl}_n)\), where \(\kappa|_{\mathfrak{sl}_n \times \mathfrak{sl}_n} = k \kappa_0\) and \(\pi\kappa\) is the rank 1 Heisenberg vertex algebra generated by \(I(z) = \sum_{i=1}^n c_i(z)\) with \(\lambda\)-bracket \([I, I] = \kappa(I, I)\lambda\). It follows that \(C^k(\mathfrak{gl}_n) = \pi\kappa \otimes C^k(\mathfrak{sl}_n)\). As easily seen, \(\hat{Q}_0 I = 0\). Hence \(H^\bullet(C^\kappa(\mathfrak{gl}_n)) = \pi\kappa \otimes H^\bullet(C^k(\mathfrak{sl}_n))\), so that

\[
\mathcal{W}^\kappa(\mathfrak{gl}_n) = \mathcal{W}^k(\mathfrak{sl}_n) \otimes \pi\kappa.
\]

In particular if we choose the form \(\kappa\) to be \(k \kappa_0\), we find that \(\pi\kappa\) belongs to the center of \(\mathcal{W}^\kappa(\mathfrak{gl}_n)\) as \(\pi\kappa\) belongs to the center of \(C^\kappa(\mathfrak{gl}_n)\). Thus, \(\mathcal{W}^k(\mathfrak{sl}_n)\) is isomorphic to the quotient of \(\mathcal{W}^{k\kappa_0}(\mathfrak{gl}_n)\) by the ideal generated by \(I(-1)0\).

**5.4. The grading of \(\mathcal{W}^\kappa(\mathfrak{g})\).** The standard conformal grading of \(C^\kappa(\mathfrak{g})\) is given by the Hamiltonian \(H\) defined by

\[
H(0) = 0, \quad [H, x_{(n)}] = -n x_{(n)} \quad (x \in \mathfrak{g}), \quad [H, \psi_{\alpha, n}] = -n \psi_{\alpha, n}, \quad [H, \psi^*_{\alpha, n}] = -n \psi^*_{\alpha, n}.
\]

However \(H\) is not well-defined in \(\mathcal{W}^\kappa(\mathfrak{g})\) since \(H\) does not commute with the action of

\[
\hat{Q}_0 = \sum_{\alpha \in \Delta_+} \sum_{k \in \mathbb{Z}} (x_{\alpha})_{(k)} \psi_{\alpha, k} + \sum_{\alpha \in \Delta_+} \chi(x_{\alpha}) \psi_{\alpha, 1} - \sum_{\alpha, \beta \in \Delta_+} \sum_{k+l+m=0} c_{\alpha, \beta}^\gamma \psi_{\alpha, k} \psi_{\beta, l} \psi_{\gamma, m}.
\]

Here and below we omit the tensor product sign.

To remedy this, define the linear operator \(H_W\) by

\[
H_W(0) = 0, \quad [H_W, (x_i)_{(n)}] = -n (x_i)_{(n)} \quad (i \in I), \quad [H_W, (x_{\alpha})_{(n)}] = (\alpha(\rho^\vee) - n) (x_{\alpha})_{(n)} \quad (\alpha \in \Delta), \quad [H_W, \psi_{\alpha, n}] = (\alpha(\rho^\vee) - n) \psi_{\alpha, n}, \quad [H_W, \psi^*_{\alpha, n}] = -\alpha(\rho^\vee) - n) \psi^*_{\alpha, n}, \quad (\alpha \in \Delta_+).
\]

Here \(\rho^\vee = 1/2h\), where \(h\) is defined in \([\text{2.4}]\). Set \(C^\kappa(\mathfrak{g})_{\Delta, \text{new}} = \{v \in C^\kappa(\mathfrak{g}) \mid H_Wc = \Delta c\}\). Then

\[
C^\kappa(\mathfrak{g}) = \bigoplus_{\Delta \in \mathbb{Z}} C^\kappa(\mathfrak{g})_{\Delta, \text{new}}.
\]

Since \([\hat{Q}, H_W] = 0\), \(C^\kappa(\mathfrak{g})_{\Delta, \text{new}}\) is a subcomplex of \(C^\kappa(\mathfrak{g})\). We have

\[
H^\bullet(C^\kappa(\mathfrak{g}), \hat{Q}_0) = \bigoplus_{\Delta \in \mathbb{Z}} H^\bullet(C^\kappa(\mathfrak{g}), \hat{Q}_0)_{\Delta}, \quad H^\bullet(C^\kappa(\mathfrak{g}), \hat{Q}_0)_{\Delta} = H^\bullet(C^\kappa(\mathfrak{g})_{\Delta, \text{new}}, \hat{Q}_0).
In particular \( W^k(\mathfrak{g}) = \bigoplus_{\Delta \in \mathbb{Z}} W^k(\mathfrak{g})_{\Delta} \). Note that the grading \([5.3]\) is not bounded from below.

If \( k \neq -n \) then the action of \( H_W \) on the vertex subalgebra \( W^k(\mathfrak{sl}_n) \) of \( W^k(\mathfrak{g}) \) is inner: Set

\[
L(z) = L_{\text{sug}}(z) + \rho^\vee(z) + L_F(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-1},
\]

where \( L_{\text{sug}}(z) \) is the Sugawara field of \( V^k(\mathfrak{sl}_n) \):

\[
L_{\text{sug}}(z) = \frac{1}{2(k + n)} \sum_a : x_a(z)x^a(z) :,
\]

and

\[
L_F(z) = \sum_{\alpha \in \Delta_+} (\text{ht}(\alpha) : \partial_z \psi_\alpha(z)\psi^*_\alpha(z) : + (1 - \text{ht}(\alpha)) : \partial_z \psi^*_\alpha(z)\psi_\alpha(z) :).
\]

Here \( \{x_a\} \) is a basis of \( \mathfrak{sl}_n \) and \( \{x^a\} \) is the dual basis of \( \{x_a\} \) with respect to \( (\ | \ ) \).

Then \( \tilde{Q}(0)L = 0 \), and so \( L \) defines an element of \( W^k(\mathfrak{sl}_n) \). It is a conformal vector of \( W^k(\mathfrak{sl}_n) \), that is to say, \( L_0 = H_W \) and \( L_{-1} = T \) and

\[
[L_m, L_n] = (m-n)L_{m+n} + \frac{m^2 - m}{12} \delta_{m,n}c,
\]

where \( c \in \mathbb{C} \) is the central charge of \( L \), which is in this case given by

\[
(n - 1)(1 - n(n + 1)(n + k - 1)^2)/(n + k).
\]

5.5. Decomposition of BRST complex. We extend the map in \([2.6]\) to the linear map \( \tilde{\theta}_0 : \mathfrak{g}[t, t^{-1}] \to C^\omega(\mathfrak{g}) \) by setting

\[
\tilde{\theta}_0(x_a(z)) = x_a(z) + \sum_{\beta, \gamma \in \Delta_+} c_{a,\beta,\gamma} : \psi_\alpha(z)\psi^*_\beta(z) :.
\]

Proposition 5.10.  (1) The correspondence

\[
x_a(z) \mapsto J_a(z) := \tilde{\theta}_0(x_a(z)) \quad (x_a \in \mathfrak{b}_-)
\]

defines a vertex algebra embedding \( V^\kappa(\mathfrak{b}) \hookrightarrow C^\omega(\mathfrak{g}) \), where \( \kappa_\mathfrak{b} \) is the bilinear form on \( \mathfrak{b} \) defined by \( \kappa_\mathfrak{b}(x, y) = \kappa(x, y) + \frac{1}{2} \kappa_\mathfrak{g}(x, y) \). We have

\[
[J_a \lambda \psi^*_\alpha] = \sum_{\beta \in \Delta_+} c_{a,\beta,\gamma}^\alpha : \psi_\alpha(z)\psi^*_\beta(z) :.
\]

(2) The correspondence

\[
x_\alpha(z) \mapsto J_\alpha(z) := \tilde{\theta}_0(x_\alpha) \quad (x_\alpha \in \mathfrak{n})
\]

defines a vertex algebra embedding \( V(\mathfrak{n}) \hookrightarrow C^\omega(\mathfrak{g}) \). We have

\[
[J_\alpha \lambda \psi^*_\beta] = \sum_{\gamma \in \Delta_+} c_{\alpha,\beta,\gamma}^\alpha : \psi_\alpha(z)\psi^*_\beta(z) :.
\]
Let \( C^\kappa(g)_+ \) denote the subalgebra of \( C^\kappa(g) \) generated by \( J_\alpha(z) \) and \( \psi_\alpha(z) \) with \( \alpha \in \Delta_+ \), and let \( C^\kappa(g)_- \) denote the subalgebra generated by \( J_\alpha(z) \) and \( \psi^*_\alpha(z) \) with \( \alpha \in \Delta_- \cup I, \alpha \in \Delta_+ \).

The proof of the following assertions are parallel to that of Lemma 2.20 and Proposition 2.22.

**Lemma 5.11.** The multiplication map gives a linear isomorphism
\[
C^\kappa(g)_- \otimes C^\kappa(g)_+ \to C^\kappa(g).
\]

**Lemma 5.12.** The subspaces \( C^\kappa(g)_- \) and \( C^\kappa(g)_+ \) are subcomplexes of \( (C^\kappa(g), \hat{Q}(0)) \). Hence \( C^\kappa(g) \cong C^\kappa(g)_- \otimes C^\kappa(g)_+ \) as complexes.

**Theorem 5.13** (GT2, FZ). We have \( H^i(C^\kappa(g)_-, \hat{Q}(0)) = \delta_{i,0} \mathbb{C} \). Hence \( H^\bullet(C^\kappa(g)_-, \hat{Q}(0)) = H^\bullet(C^\kappa(g)_-, \hat{Q}(0)) \). In particular \( \mathcal{W}^\kappa(g) = H^0(C^\kappa(g)_-, \hat{Q}(0)) \).

Since the complex \( C^\kappa(g)_- \) has no positive cohomological degree, its zeroth cohomology \( \mathcal{W}^\kappa(g) = H^0(C^\kappa(g)_-, \hat{Q}(0)) \) is a vertex subalgebra of \( C^\kappa(g)_- \). Observe also that \( C^\kappa(g)_- \) has no negative degree with respect to the Hamiltonian \( H_\mathcal{W} \), and each homogeneous space is finite-dimensional:
\[
C^\kappa(g)_- = \bigoplus_{\Delta \in \mathbb{Z}_-} C^\kappa(g)_{-\Delta, new}, \quad \dim C^\kappa(g)_{-\Delta, new} < \infty.
\]

Here \( C^\kappa(g)_{-\Delta, new} = C^\kappa(g)_- \cap C^\kappa(g)_{\Delta, new} \).

**5.6. Proof of Theorem 5.8** As \( \hat{Q}(0) F^p C^\kappa(g)_- \subset F^p C^\kappa(g)_- \), one can consider a spectral sequence for \( H^\bullet(C^\kappa(g)_-, \hat{Q}(0)) \) such that the \( E_1 \)-term is \( H^\bullet(\text{gr } C^\kappa(g)_-, \hat{Q}(0)) \). This spectral sequence clearly converges, since \( C^\kappa(g)_- \) is a direct sum of finite-dimensional subcomplexes \( C^\kappa(g)_{-\Delta, new} \).

We have \( \text{gr } C^\kappa(g)_- \cong S(b_-[t^{-1}]^{-1}) \otimes \Lambda(n[t^{-1}]^{-1}) \cong \mathbb{C}[J_{\mu^{-1}(\chi)}] \otimes \Lambda(n[t^{-1}]^{-1}) \), and the complex \( (\text{gr } C^\kappa(g)_-, \hat{Q}(0)) \) is identical to the Chevalley complex for the Lie algebra cohomology \( H^\bullet(n[t], \mathbb{C}[J_{\mu^{-1}(\chi)}]) \). Therefore
\[
H^i(\text{gr } C^\kappa(g)_-, \hat{Q}(0)) \cong \delta_{i,0} \mathbb{C}[JS].
\]

Thus the spectral sequence collapses at \( E_1 = E_\infty \), and we get
\[
\text{gr}\ C^\kappa(g)_-(\hat{Q}(0)) \cong H^i(\text{gr } C^\kappa(g)_-, \hat{Q}(0)) \cong \delta_{i,0} \mathbb{C}[JS].
\]

Here \( \text{gr}\ C^\kappa(g)_-(\hat{Q}(0)) \) is the associated graded space with respect to the filtration \( G^\bullet H^0(C^\kappa(g)_-, \hat{Q}(0)) \) induced by the filtration \( F^\bullet C^\kappa(g)_- \), that is,
\[
G^p H^0(C^\kappa(g)_-, \hat{Q}(0)) = \text{im}(H^i(F^p C^\kappa(g)_-, \hat{Q}(0)) \to H^i(C^\kappa(g)_-, \hat{Q}(0))).
\]

We claim that the filtration \( G^\bullet H^0(C^\kappa(g)_-, \hat{Q}(0)) \) coincides with the canonical filtration of \( H^0(C^\kappa(g)_-, \hat{Q}(0)) = \mathcal{W}^\kappa(g) \). Indeed, from the definition of the canonical filtration we have \( F^p \mathcal{W}^k(g) \subset G^p \mathcal{W}^k(g) \) for all \( p \), and hence, there is a Poisson vertex algebra homomorphism
\[
\text{gr } \mathcal{W}^\kappa(g, f) \to \text{gr} G \mathcal{W}^\kappa(g, f) \cong \mathbb{C}[JS].
\]
that restricts to a surjective homomorphism

\[ W^\kappa(g)/F^1W^\kappa(g) \to W^\kappa(g)/G^1W^\kappa(g) \cong \mathbb{C}[S]. \]

Since \( \mathbb{C}[JS] \) is generated by \( \mathbb{C}[S] \) as differential algebras it follows that \( \text{(1.12)} \) is surjective. On the other hand the cohomology vanishing and the Euler-Poincaré principle imply that the graded character of \( W^\kappa(g) \) and \( \mathbb{C}[JS] \) are the same. Therefore \( \text{(5.12)} \) is an isomorphism, and thus, \( G^pW^\kappa(g) = F^pW^\kappa(g) \) for all \( p \).

Finally the embedding \( \text{gr} \, C^\kappa(g) \to \text{gr} \, C^\kappa(g) \) induces an isomorphism

\[ H^0(\text{gr} \, C^\kappa(g), \hat{Q}(0)) \cong H^0(\text{gr} \, C^\kappa(g), \hat{Q}(0)) \]

by Theorem 5.7 and 5.11. This completes the proof. \( \square \)

5.7. Zhu’s algebra of \( W \)-algebra. Let \( \text{Zhu}_{\text{new}}(C^\kappa(g)) \) be Zhu’s algebra of \( C^\kappa(g) \) with respect to the Hamiltonian \( H_W \), \( \text{Zhu}_{\text{old}}(C^\kappa(g)) \) Zhu’s algebra of \( C^\kappa(g) \) with respect to the standard Hamiltonian \( H \). We have

\[ \text{Zhu}_{\text{new}}(C^\kappa(g)) \cong \text{Zhu}_{\text{old}}(C^\kappa(g)) \cong C(g), \]

see [A7, Proposition 5.1] for the details. Then it is legitimate to write Zhu\( (C^\kappa(g)) \) for \( \text{Zhu}_{\text{new}}(C^\kappa(g)) \) or \( \text{Zhu}_{\text{old}}(C^\kappa(g)) \).

By the commutation formula, we have

\[ \hat{Q}(0)(C^\kappa(g) \circ C^\kappa(g)) \subset C^\kappa(g) \circ C^\kappa(g). \]

Here the circle \( \circ \) is defined as in the definition of the Zhu algebra (with respect to the grading \( H_W \)). So \( (\text{Zhu}_{\text{new}}C^\kappa(g), \hat{Q}(0)) \) is a differential, graded algebra, which is identical to \( (C(g), \text{ad} \, Q) \).

Theorem 5.14 (A2). We have

\[ \text{Zhu}W^\kappa(g) \cong H^0(\text{Zhu}_{\text{new}}C^\kappa(g), \hat{Q}(0)) \cong Z(g). \]

Proof. By Theorem 5.8 it follows that \( W^\kappa(g) \) admits a PBW basis. Hence \( \eta_{W^\kappa(g)} : \text{gr} \, \text{Zhu}W^\kappa(g) \to R_{W^\kappa(g)} \) is an isomorphism by Theorem 4.8. On the other hand we have a natural algebra homomorphism \( \text{Zhu}W^\kappa(g) \to H^0(\text{Zhu}C^\kappa(g), \hat{Q}(0)) \) which makes the following diagram commute.

\[ \begin{array}{ccc}
\text{gr} \, \text{Zhu}W^\kappa(g) & \xrightarrow{\eta_{W^\kappa(g), \text{Zhu}}} & R_{W^\kappa(g)} \\
\downarrow & & \downarrow \cong \text{Theorem 4.8} \\
\text{gr} \, Z(g) & \cong & \mathbb{C}[S].
\end{array} \]

Note that we have the isomorphisms \( H^0(R_{C^\kappa(g)}, \hat{Q}(0)) \cong H^0(C^\kappa(g), \text{ad} \, \hat{Q}(0)) \cong \mathbb{C}[S] \) and \( \text{gr} \, H^0(\text{Zhu}_{\text{new}}C^\kappa(g), \hat{Q}(0)) \cong \text{gr} \, Z(g) \) in the diagram. Now the other three isomorphisms will give the desired isomorphism. \( \square \)
We conclude that we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathbb{C}[JS] & \overset{\text{gr}(2)}{\rightarrow} & \mathcal{W}^\kappa(\mathfrak{g}) \\
\text{Zhu}(?) & \downarrow R_\tau & \downarrow \text{Zhu}(?) \\
\mathbb{C}[S] & \overset{\text{gr}(1)}{\rightarrow} & \mathcal{Z}(\mathfrak{g}).
\end{array}
\]

**Remark 5.15.** The same proof applies for an arbitrary simple Lie algebra \( \mathfrak{g} \). In particular, we have \( \text{Zhu}(\mathcal{W}^k(\mathfrak{sl}_n)) \cong \mathcal{Z}(\mathfrak{sl}_n) \). In fact the same proof applies for the \( \mathcal{W} \)-algebra associated with a simple Lie algebra \( \mathfrak{g} \) and an arbitrary nilpotent element \( f \) of \( \mathfrak{g} \) to show its Zhu’s algebra is isomorphic to the finite \( \mathcal{W} \)-algebra \( U(\mathfrak{g}, f) \) ([DSK]).

5.8. **Explicit generators.** It is possible to write down the explicit generators of \( \mathcal{W}^\kappa(\mathfrak{g}) \subset C^\kappa(\mathfrak{g}) \).

Recall that the *column-determinant* of a matrix \( A = (a_{ij}) \) over an associative algebra is defined by

\[
\text{cdet} A = \sum_{\sigma \in \mathfrak{S}_n} \text{sgn} \sigma \cdot a_{\sigma(1)}a_{\sigma(2)} \cdots a_{\sigma(n)n}.
\]

Introduce an extended Lie algebra \( \mathfrak{b}[t^{-1}]t^{-1} \oplus \mathbb{C}\tau \), where the element \( \tau \) commutes with \( 1 \), and

\[
[\tau, x(-n)] = nx(-n) \quad \text{for} \quad x \in \mathfrak{b}, n \in \mathfrak{n},
\]

where \( x(-n) = xt^{-n} \). This induces an associative algebra structure on the tensor product space \( U(\mathfrak{b}[t^{-1}]t^{-1}) \oplus \mathbb{C}[\tau] \).

Consider the matrix

\[
B = \begin{bmatrix}
\alpha \tau + (e_{11})(-1) & -1 & 0 & \ldots & 0 \\
(e_{21})(-1) & \alpha \tau + (e_{22})(-1) & -1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
(e_{n-1})(-1) & (e_{n-2})(-1) & \ldots & \alpha \tau + (e_{n-1})(-1) & -1 \\
(e_{n1})(-1) & (e_{n2})(-1) & \ldots & \ldots & \alpha \tau + (e_{nn})(-1)
\end{bmatrix}
\]

with entries in \( U(\mathfrak{b}[t^{-1}]t^{-1}) \otimes \mathbb{C}[\tau] \otimes \mathbb{C}[^\alpha] \), where \( \alpha \) is a parameter.

For its column-determinant\(^1\) we can write

\[
\text{cdet } B = \tau^n + W_\alpha^{(1)}\tau^{n-1} + \ldots + W_\alpha^{(n)}
\]

for certain coefficients \( W_\alpha^{(r)} \) which are elements of \( U(\mathfrak{b}[t^{-1}]t^{-1}) \otimes \mathbb{C}[\alpha] \). Set

\[
W^{(i)} = W_\alpha^{(i)}|_{\alpha=k+n-1}.
\]

This is an element of \( U(\mathfrak{b}[t^{-1}]t^{-1}) \), which we identify with \( V^{\kappa_\ast}(\mathfrak{b}) \subset C^\kappa(\mathfrak{g}) \).

**Theorem 5.16** ([AMol]). \( \mathcal{W}^k(\mathfrak{g}) \) is strongly generated by \( W^{(1)}, \ldots, W^{(n)} \).

\(^1\)It is easy to verify that \( \text{cdet } B \) coincides with the *row-determinant* of \( B \) defined in a similar way.
5.9. **Miura Map.** The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ acts on $C^\kappa(\mathfrak{g})_+$ by $x_i \mapsto (J_i)_{(\alpha_i)}$, $i \in I$, see Proposition 5.10. Let $C^\kappa(\mathfrak{g})^\alpha_+$ be the weight space of weight $\lambda \in \mathfrak{h}^*$ with respect to this action. Then

$$C^\kappa(\mathfrak{g})_+ = \bigoplus_{\lambda \leq 0} C^\kappa(\mathfrak{g})^\alpha_+, \quad C^\kappa(\mathfrak{g})^0_+ = V^\kappa(\mathfrak{h}) \subset V^\kappa(\mathfrak{b}).$$

The vertex algebra $V^\kappa(\mathfrak{h})$ is the *Heisenberg vertex algebra* associated with $\mathfrak{h}$ and the bilinear form $\kappa_{\mathfrak{h}} := \kappa_{\mathfrak{b}\mid \mathfrak{h}}$.

The projection $C^\kappa(\mathfrak{g})_+ \to C^\kappa(\mathfrak{g})^0_+ = V^\kappa(\mathfrak{h})$ with respect to this decomposition is a vertex algebra homomorphism. Therefore its restriction

$$\hat{\Upsilon} : W^\kappa(\mathfrak{g}) \to V^\kappa(\mathfrak{h})$$

is also a vertex algebra homomorphism that is called the **Miura map**.

**Theorem 5.17.** The Miura map is injective for all $k \in \mathbb{C}$.

**Proof.** The induced Poisson vertex algebra homomorphism

$$\text{gr} \hat{\Upsilon} : \text{gr} W^\kappa(\mathfrak{g}) = \mathbb{C}[J\mathcal{S}] \to \text{gr} V^\kappa(\mathfrak{h}) = \mathbb{C}[J\mathfrak{h}^*] = \mathbb{C}[J(f + \mathfrak{b})]$$

is just a restriction map and coincides with $J\hat{\Upsilon}$, where $\hat{\Upsilon}$ is defined in (2.13). Clearly, it is sufficient to show that $J\hat{\Upsilon}$ is injective.

Recall that the action map gives an isomorphism

$$N \times (f + \mathfrak{b}_{\text{reg}}) \cong U \subset f + \mathfrak{b},$$

where $U$ is some open subset of $f + \mathfrak{b}$, see the proof of Proposition 2.20. Therefore, by Lemma 3.8, the action map $JN \times J(f + \mathfrak{b}) \to J(f + \mathfrak{b})$ is dominant. Thus, the induced map $\mathbb{C}[J(f + \mathfrak{b})] \to \mathbb{C}[JN \times J(f + \mathfrak{b})]$ is injective, and so is $J\hat{\Upsilon} : C'[J(f + \mathfrak{b})]^{JN} \to \mathbb{C}[JN \times J(f + \mathfrak{b})]^{JN} = \mathbb{C}[J(f + \mathfrak{b})]$. \(\square\)

**Remark 5.18.** It is straightforward to generalize Theorem 5.17 for the $W$-algebra $W^k(\mathfrak{g})$ associated with a general simple Lie algebra $\mathfrak{g}$.

**Theorem 5.19.** Let $x_i = E_{ii} \in \mathfrak{h} \subset \mathfrak{g} = \mathfrak{sl}_n$, and $J_i(z)$ the corresponding field of $V^\kappa(\mathfrak{h})$. The image $\Upsilon(W^{(i)}(z))$ of $W^{(i)}(z)$ by the Miura map is described by

$$\sum_{i=0}^n \Upsilon(W^{(i)}(z))(\partial z)^{n-i} =: (\alpha \partial z + J_1(z))(\alpha \partial z + J_2(z)) \cdots (\alpha \partial z + J_N(z)),$$

where $\alpha = k + n - 1$, $W^{(0)}(z) = 1$, $[\partial z, J_i(z)] = \frac{d}{dz} J_i(z)$.

**Proof.** It is straightforward from Theorem 5.10. \(\square\)

Note that if we choose $\kappa$ to be $k\kappa_0$ and set $\sum_{i=1}^N J_i(z) = 0$, we obtain the image of the generators of $W^k(\mathfrak{sl}_n)$ by the Miura map $\hat{\Upsilon}$. For $k + n \neq 0$, this expression can be written in a more symmetric manner: Set $b_i(z) = \frac{J_i(z)}{\sqrt{k+n}}$, so that $\sum_{i=1}^n b_i(z) = 0$, and

$$[(b_i)_\lambda b_j] = \begin{cases} (1 - \frac{i}{n})\lambda & \text{if } i = j, \\ -\frac{j}{n}\lambda & \text{if } i \neq j. \end{cases}$$
Then we obtain the following original description of the $W^k(sl_n)$ due to Fateev and Lukyanov \[FL\].

**Corollary 5.20.** Suppose that $k+n \neq 0$. Then the image of $W^k(sl_n)$ by the Miura map is the vertex subalgebra generated by fields $\tilde{W}_2(z) \ldots , \tilde{W}_n(z)$ defined by

$$
\sum_{i=0}^{n} \tilde{W}_i(z)(\alpha_0 \partial_z)^{n-i} =: (\alpha_0 \partial_z + b_1(z))(\alpha_0 \partial_z + b_2(z)) \ldots (\alpha_0 \partial_z + b_n(z)) :,
$$

where $\alpha_0 = \alpha_+ + \alpha_-, \alpha_+ = \sqrt{k+n}, \alpha_- = -1/\sqrt{k+n}, \tilde{W}_0(z) = 1, \tilde{W}_1(z) = 0.$

**Corollary 5.21.** Suppose that $k + n \neq 0$. We have

$$W^k(sl_n) \simeq W^{\ell k}(sl_n),$$

where $\ell k$ is defined by $(k + n)(\ell k + n) = 1.$

**Example 5.22.** Let $g = sl_2$, $k \neq -2$. Set $b(z) = \sqrt{2}\partial_1(z) = -\sqrt{2}\partial_2(z)$, so that $[\partial_1, \partial_2] = \lambda$. Then the right-hand-side of the formula in Corollary 5.22 becomes

$$(\alpha_0 \partial_z + \frac{1}{\sqrt{2}} b(z))(\alpha_0 \partial_z - \frac{1}{\sqrt{2}} b(z)) : = \alpha_0^2 \partial_z^2 - L(z),$$

where

$$L(z) = \frac{1}{2} : b(z)^2 : + \frac{\alpha_0}{\sqrt{2}} \partial_z b(z).$$

It is well-known and is straightforward to check that the field generates the Virasoro algebra of central charge $1 - 6(k+1)^2/(k+2)$. Thus $W^k(sl_2), k \neq -2,$ is isomorphic to the universal Virasoro vertex algebra of central charge $1 - 6(k+1)^2/(k+2)$.

In the case that $\kappa = \kappa_c := -\frac{1}{2} \kappa g$, then it follows from Theorem 5.17 that $W^{\kappa_c}(gl_n)$ is commutative since $V^{(\kappa_c)}(g^*)$ is commutative. In fact the following fact is known: Let $Z(V^{\kappa}(g)) = \{ z \in V^{\kappa}(g) | [z(n), a(n)] = 0 \}$, the center of $V^{\kappa}(g)$.

**Theorem 5.23 (FF2).** We have the isomorphism

$$Z(V^{\kappa}(g)) \simeq W^{\kappa_c}(g), \quad z \mapsto [z \otimes 1].$$

This is a chiralization of Kostant’s Theorem 2.17 in the sense that we recover Theorem 2.17 from Theorem 5.23 by considering the induced map between Zhu’s algebras of both sides. The statement of Theorem 5.23 holds for any simple Lie algebra $g$ (FF2).

**Remark 5.24.** For a general simple Lie algebra $g$, the image of the Miura map for a generic $k$ is described in terms of screening operators, see [FBZ] 15.4. Theorem 5.19 for $g = gl_n$ also follows from this description (the proof reduces to the case $g = sl_2$). An important application of this realization is the Feigin-Frenkel duality which states

$$W^k(g) \simeq W^{\ell k}(L g),$$

where $L g$ is the Langlands dual Lie algebra of $g$, $r^\vee (k + h^\vee)(\ell k + L h^\vee) = 1$. Here $r^\vee$ is the maximal number of the edges of the Dynking diagram of $g$ and $L h^\vee$ is the
dual Coxeter number of $Lg$. In [FFZ] this isomorphism was stated only for a generic $k$, but it is not too difficult to see the isomorphism remains valid for an arbitrary $k$ using the injectivity of the Miura map.

The Miura map is defined [KRW] for the $W$-algebra $W^k(g, f)$ associated with an arbitrary $f$, which is injective as well since the proof of Theorem 5.17 applies. Recently Naoki Genra [Gen] has obtained the description of the image by the Miura map in terms of screening operators for the $W$-algebra $W^k(g, f)$ associated with an arbitrary nilpotent element $f$.

5.10. Classical $W$-algebras. Since the Poisson structure of $C[S]$ is trivial, we can give $gr W^k(g)$ a Poisson vertex algebra structure by the formula (3.18). The Poisson structure of $R_{V^k+(h)} = C[h]$ is also trivial, hence $gr V^k+(h) = C[h^+]$ is equipped with the Poisson vertex algebra structure by the formula (3.18) as well. Then the map $gr \hat{Y} : gr W^k(g) \to gr V^k+(h)$ is a homomorphism of Poisson vertex algebras with respect to these structures. Set $\kappa = k\kappa_0$, $k \in C$, and consider its restriction $gr \hat{Y} : gr W^k(sl_n) \to gr V^k+(h')$, where $h'$ is the Cartan subalgebra of $sl_n$.

In $gr V^k+(h')$ we have

$$\{h \lambda h'\} = \kappa h(h, h') = (k + n)\kappa_0(h, h'),$$

and this uniquely determines the $\lambda$-bracket of $gr V^k+(h')$. Hence it is independent of $k$ provided that $k \neq -n$. Since the image of $gr W^k(sl_n)$ is strongly generated by elements of $C[(h')]^W$, it follows that the Poisson vertex algebra structure of $gr W^k(sl_n), k \neq -n$, is independent of $k$. We denote this Poisson vertex algebra by $W^{cl}(sl_n)$.

The Poisson vertex algebra $W^{cl}(sl_n)$ is called the classical $W$-algebra associated with $sl_n$, which appeared in the works of Adler [Adl], Gelfand-Dickey [GD78] and Drinfeld-Sokolov [DS]. Thus, the $W$-algebra $W^k(sl_n), k \neq -n$, is a deformation of $W^{cl}(sl_n)$.

On the other hand the $W$-algebra $W^{-n}(sl_n)$ at the critical level can be identified with the space of the $sl_n$-opers [BD2] on the disk $D$. We refer to [FBZ, Fre07] for more on this subject.

6. Representations of $W$-algebras

From now on we set $g = sl_n$ and study the representations of $W^k(g)$ (see 5.8).

6.1. Poisson modules. Let $R$ be a Poisson algebra. Recall that a Poisson $R$-module is a $R$-module $M$ in the usual associative sense equipped with a bilinear map

$$R \times M \to M, \quad (r, m) \mapsto ad r(m) = \{r, m\},$$

which makes $M$ a Lie algebra module over $R$ satisfying

$$\{r_1, r_2\} m = \{r_1, r_2\} m + \{r_2, r_1\} m, \quad \{r_1 r_2, m\} = r_1 \{r_2, m\} + r_2 \{r_1, m\}$$

for $r_1, r_2 \in R$, $m \in M$. Let $R$-PMod be the category of Poisson modules over $R$. 
Lemma 6.1. A Poisson module over $\mathbb{C}[g^*]$ is the same as a $\mathbb{C}[g^*]$-module $M$ in the usual associative sense equipped with a Lie algebra module structure $g \to \text{End} M$, $x \mapsto \text{ad}(x)$, such that

$$\text{ad}(x)(fm) = \{x, f\}m + f \cdot \text{ad}(x)(m)$$

for $x \in g$, $f \in \mathbb{C}[g^*]$, $m \in M$.

6.2. Poisson vertex modules.

Definition 6.2. A Poisson vertex module over a Poisson vertex algebra $V$ is a $V$-module $M$ as a vertex algebra equipped with a linear map

$$V \mapsto (\text{End} M)[[z^{-1}]]z^{-1}, \quad a \mapsto Y^M_a(z, \bar{z}) = \sum_{n \geq 0} a^{(n)}_M z^{-n-1},$$

satisfying

\begin{align*}
(6.1) & \quad a^{(n)}_M m = 0 \quad \text{for } n \gg 0, \\
(6.2) & \quad (Ta)^M_{(n)} = -na^{M}_{(n-1)}, \\
(6.3) & \quad a^{(n)}_M (b\cdot v) = (a^{(n)}_M b)\cdot v + b\cdot (a^{(n)}_M v), \\
(6.4) & \quad [a^{(m)}_M, b^{(n)}_M] = \sum_{i=0}^{\infty} \binom{m}{i} (a^{(i)}_M b^{(m+n-i)}_M), \\
(6.5) & \quad (ab)^M_{(n)} = \sum_{i=0}^{\infty} (a^{(i-1)}_M b^{(m+n-i)}_M + b^{(i-1)}_M a^{(m+n+i)}_M)
\end{align*}

for all $a, b \in V$, $m, n \geq 0$, $v \in M$.

A Poisson vertex algebra $R$ is naturally a Poisson vertex module over itself.

Example 6.3. Let $M$ be a Poisson vertex module over $\mathbb{C}[Jg^*]$. Then by (6.4), the assignment

$$x^m \mapsto x^{(m)}_M \quad x \in g \subset \mathbb{C}[g^*] \subset \mathbb{C}[Jg^*], \quad n \geq 0,$$

defines a $Jg = g[[t]]$-module structure on $M$. In fact, a Poisson vertex module over $\mathbb{C}[Jg^*]$ is the same as a $\mathbb{C}[Jg^*]$-module $M$ in the usual associative sense equipped with an action of the Lie algebra $Jg$ such that $(xt^n)m = 0$ for $n \gg 0$, $x \in g$, $m \in M$, and

$$(xt^n) \cdot (am) = (x^{(m)}_M a)m + a(x^n) \cdot m$$

for $x \in g$, $n \geq 0$, $a \in \mathbb{C}[Jg^*]$, $m \in M$.

Below we often write $a^{(m)}_M$ for $a^{(m)}_M$. The proofs of the following assertions are straightforward.

Lemma 6.4. Let $R$ be a Poisson algebra, $E$ a Poisson module over $R$. There is a unique Poisson vertex $JR$-module structure on $JR \otimes_R E$ such that

$$a^{(n)}_R (b \otimes m) = (a^{(n)}_R b) \otimes m + \delta_{n,0} b \otimes \{a, m\}$$

for $n \geq 0$, $a \in R \subset JR$, $b \in JR$, $m \in E$ (Recall that $JR = \mathbb{C}[J \text{Spec } R]$.)
Lemma 6.5. Let \( R \) be a Poisson algebra, \( M \) a Poisson vertex module over \( JR \). Suppose that there exists a \( R \)-submodule \( E \) of \( M \) (in the usual commutative sense) such that \( a_{(n)}E = 0 \) for \( n > 0 \), \( a \in R \), and \( M \) is generated by \( E \) (in the usual commutative sense). Then there exists a surjective homomorphism

\[
\begin{array}{c}
JR \otimes_R E \twoheadrightarrow M
\end{array}
\]

of Poisson vertex modules.

6.3. Canonical filtration of modules over vertex algebras. Let \( V \) be a vertex algebra graded by a Hamiltonian \( H \). A compatible filtration of a \( V \)-module \( M \) is a decreasing filtration

\[
M = \Gamma^0 M \supset \Gamma^1 M \supset \cdots
\]

such that

\[
a_{(n)}\Gamma^q M \subset \Gamma^{p+q-n-1} M \quad \text{for} \quad a \in F^p V, \quad \forall n \in \mathbb{Z},
\]

\[
a_{(n)}\Gamma^q M \subset \Gamma^{p+q-n} M \quad \text{for} \quad a \in F^p V, \quad n \geq 0,
\]

\[
H \cdot \Gamma^p M \subset \Gamma^p M \quad \text{for} \quad a \in F^p V, \quad n \geq 0,
\]

\[
\bigcap_p \Gamma^p M = 0.
\]

For a compatible filtration \( \Gamma^M \) the associated graded space

\[
gr^M M = \bigoplus_{p \geq 0} \Gamma^p M / \Gamma^{p+1} M
\]

is naturally a graded vertex Poisson module over the graded vertex Poisson algebra \( gr^F V \), and hence, it is a graded vertex Poisson module over \( JR_V = \mathbb{C}[\tilde{X}_V] \) by Theorem 3.17.

The vertex Poisson \( JR_V \)-module structure of \( gr^M M \) restricts to the Poisson \( R_V \)-module structure of \( M / \Gamma^1 M = \Gamma^0 M / \Gamma^1 M \), and \( a_{(n)}(M / \Gamma^1 M) = 0 \) for \( a \in R_V \subset JR_V \), \( n > 0 \). It follows that there is a homomorphism

\[
JR_V \otimes_{R_V} (M / \Gamma^1 M) \rightarrow gr^M M, \quad a \otimes \bar{m} \mapsto a \bar{m},
\]

of vertex Poisson modules by Lemma 6.5.

Suppose that \( V \) is positively graded and so is a \( V \)-module \( M \). We denote by \( F^* M \) the Li filtration \([12]\) of \( M \), which is defined by

\[
F^p M = \text{span}_C \{ a^{(r_{n_1} - 1)} \cdots a^{r_{(n_r - 1)}} m \mid a^i \in V, \ m \in M, \ n_1 + \cdots + n_r \geq p \}.
\]

It is a compatible filtration of \( M \), and in fact is the finest compatible filtration of \( M \), that is, \( F^p M \subset \Gamma^p M \) for all \( p \) for any compatible filtration \( \Gamma^* M \) of \( M \). The subspace \( F^1 M \) is spanned by the vectors \( a^{(r_1 - 2)} m \) with \( a \in V \), \( m \in M \), which is often denoted by \( C_2(M) \) in the literature. Set

\[
\tilde{M} = M / F^1 M (= M / C_2(M)),
\]

which is a Poisson module over \( R_V = \tilde{V} \). By \([12]\) Proposition 4.12, the vertex Poisson module homomorphism

\[
JR_V \otimes_{R_V} \tilde{M} \rightarrow gr^F M
\]
is surjective.

Let \( \{ a^i; i \in I \} \) be elements of \( V \) such that their images generate \( R_V \) in usual commutative sense, and let \( U \) be a subspace of \( M \) such that \( M = U + F^1M \). The surjectivity of the above map is equivalent to that

\[
(6.7) \quad \text{span}_C \{ a^{i_1}_{(-n_1-1)} \cdots a^{i_r}_{(-n_r-1)}m \mid m \in U, n_i \geq 0, n_1 + \cdots + n_r \geq p; i_1, \ldots, i_r \in I \}.
\]

**Lemma 6.6.** Let \( V \) be a vertex algebra, \( M \) a \( V \)-module. The Poisson vertex algebra module structure of \( gF^pM \) restricts to the Poisson module structure of \( M := M/F^1M \) over \( R_V \), that is, \( M \) is a Poisson \( R_V \)-module by

\[
\hat{a} \cdot \hat{m} = \hat{a} \cdot \hat{m}, \quad \text{ad}(\hat{a})(\hat{m}) = \hat{a}(\hat{m}).
\]

A \( V \)-module \( M \) is called **finitely strongly generated** if \( M \) is finitely generated as a \( R_V \)-module in the usual associative sense.

### 6.4. Associated varieties of modules over affine vertex algebras.

A \( \hat{g} \)-module \( M \) of level \( k \) is called smooth if \( x(z) \) is a field on \( M \) for \( x \in \mathfrak{g} \), that is, \( xt^n m = 0 \) for \( n \gg 0 \), \( x \in \mathfrak{g} \), \( m \in M \). Any \( V^k(\mathfrak{g}) \)-module \( M \) is naturally a smooth \( \hat{\mathfrak{g}} \)-module of level \( k \). Conversely, any smooth \( \hat{\mathfrak{g}} \)-module of level \( k \) can be regarded as a \( V^k(\mathfrak{g}) \)-module. It follows that a \( V^k(\mathfrak{g}) \)-module is the same as a smooth \( \hat{\mathfrak{g}} \)-module of level \( k \).

For a \( V = V^k(\mathfrak{g}) \)-module \( M \), or equivalently, a smooth \( \hat{\mathfrak{g}} \)-module of level \( k \), we have

\[
\hat{M} = M/\mathfrak{g}[t^{-1}]t^{-2}M,
\]

and the Poisson \( \mathbb{C}[\mathfrak{g}^*] \)-module structure is given by

\[
x \cdot \hat{m} = xt^{-1}m, \quad \text{ad}(x)(\hat{m}) = x\hat{m}.
\]

For a \( \mathfrak{g} \)-module \( E \) let

\[
V^k_E := U(\hat{\mathfrak{g}}) \otimes U(\mathfrak{g}[t]) \otimes \mathbb{C}K E,
\]

where \( E \) is considered as a \( \mathfrak{g}[t] \oplus \mathbb{C}K \)-module on which \( \mathfrak{g}[t] \) acts via the projection \( \mathfrak{g}[t] \to \mathfrak{g} \) and \( K \) acts as multiplication by \( k \). Then

\[
\overline{V^k_E} \cong \mathbb{C}[\mathfrak{g}^*] \otimes E,
\]

where the Poisson \( \mathbb{C}[\mathfrak{g}^*] \)-module structure is given by

\[
f \cdot g \otimes v = (fg) \otimes v, \quad \text{ad}(x)(f \otimes v) = \{ x, f \} \otimes v + f \otimes xv,
\]

for \( f, g \in \mathbb{C}[\mathfrak{g}^*], \, v \in V \).

Let \( \mathcal{O}_k \) be the category \( \mathcal{O} \) of \( \hat{\mathfrak{g}} \) of level \( k \) \((\text{Kac1})\), \( \mathbf{KL}_k \) the full subcategory of \( \mathcal{O}_k \) consisting of modules \( M \) which are integrable over \( \mathfrak{g} \). Note that \( \overline{V^k_E} \) is a object of \( \mathbf{KL}_k \) for a finite-dimensional representation \( E \) of \( \mathfrak{g} \). Thus, \( V^k(\mathfrak{g}) = V^k_\mathbf{E} \) and its simple quotient \( \overline{V^k_\mathbf{E}}(\mathfrak{g}) \) are also objects of \( \mathbf{KL}_k \).

Both \( \mathcal{O}_k \) and \( \mathbf{KL}_k \) can be regarded as full subcategories of the category of \( V^k(\mathfrak{g}) \)-modules.

**Lemma 6.7.** For \( M \in \mathbf{KL}_k \) the following conditions are equivalent.
(1) \( M \) is finitely strongly generated as a \( V^k(\mathfrak{g}) \)-module,
(2) \( M \) is finitely generated as a \( \mathfrak{g}^{[t^{-1}]} \)-module,
(3) \( M \) is finitely generated as a \( \hat{\mathfrak{g}} \)-module.

For a finitely strongly generated \( V^k(\mathfrak{g}) \)-module \( M \) define its associated variety \( X_M \) by
\[
X_M = \text{supp}_{R^V}(\bar{M}) \subset X_V,
\]
equipped with a reduced scheme structure.

**Example 6.8.** \( X_{V^k} = \mathfrak{g}^* \) for a finite-dimensional representation \( E \) of \( \mathfrak{g} \).

**6.5. Ginzburg’s Correspondence.** Let \( \mathcal{HC} \) be the full subcategory of the category of Poisson \( \mathbb{C}[\mathfrak{g}^*] \)-modules on which the Lie algebra \( \mathfrak{g} \)-action (see Lemma 6.1) is integrable.

**Lemma 6.9.** For \( M \in \mathbb{KL}_k \), the Poisson \( \mathbb{C}[\mathfrak{g}^*] \)-module \( \bar{M} \) belongs to \( \mathcal{HC} \).

By Lemma 6.9 we have a right exact functor
\[
\mathbb{KL}_k \to \mathcal{HC}, \quad M \mapsto \bar{M}.
\]

For \( M \in \mathcal{HC}, \) \( M \otimes \mathbb{C}l \) is naturally a Poisson module over \( \bar{C}(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*] \otimes \mathbb{C}l \). (The notation of Poisson modules natural extends to the Poisson supralgebras.) Thus, \( (M \otimes \mathbb{C}l, \text{ad} \bar{Q}) \) is a differential graded Poisson module over the differential graded Poisson module \( (\bar{C}(\mathfrak{g}), \text{ad} \bar{Q}) \). In particular its cohomology \( H^*(M \otimes \mathbb{C}l, \text{ad} \bar{Q}) \) is a Poisson module over \( H^*(\bar{C}(\mathfrak{g}), \text{ad} \bar{Q}) = \mathbb{C}[S] \). So we get a functor
\[
\mathcal{HC} \to \mathbb{C}[S] \text{-Mod}, \quad M \mapsto H^0(M) := H^0(M \otimes \mathbb{C}l, \text{ad} \bar{Q}).
\]

The following assertion is a restatement of a result of Ginzburg \cite{Gin} (see \cite[Theorem 2.3]{A7}).

**Theorem 6.10.** Let \( M \in \mathcal{HC} \). Then \( H^i(M) = 0 \) for \( i \neq 0 \), and we have an isomorphism
\[
H^0(M) \cong (M/ \sum_i \mathbb{C}[\mathfrak{g}^*](x_i - \chi(x_i))M)^N.
\]
In particular if \( M \) is finitely generated \( H^0(M) \) is finitely generated over \( \mathbb{C}[S] \) and
\[
\text{supp}_{\mathbb{C}[S]} H^0(M) = (\text{supp}_{\mathbb{C}[\mathfrak{g}^*]} M) \cap S.
\]

**Corollary 6.11.** The functor \( \mathcal{HC} \to \mathbb{C}[S] \text{-Mod}, \) \( M \mapsto H^0(M) \), is exact.

Denote by \( \mathcal{N} \) the set of nilpotent elements of \( \mathfrak{g} \), which equals to the zero locus of the augmentation ideal \( \mathbb{C}[\mathfrak{g}^*]_+ \subset \mathbb{C}[\mathfrak{g}^*] \) under the identification \( \mathfrak{g} = \mathfrak{g}^* \) via \( (|) \).

Since the element \( f \) (defined in \( (2.3) \) is regular (or principal), the orbit
\[
\mathcal{O}_{\text{prin}} := G.f \subset \mathfrak{g} = \mathfrak{g}^*
\]
is dense in \( \mathcal{N} \):
\[
\mathcal{N} = \overline{\mathcal{O}_{\text{prin}}}.
\]
The transversality of $S$ implies that

$$S \cap N = \{ f \}.$$ 

**Theorem 6.12 (Gins).** Let $M$ be a finitely generated object in $\mathcal{HC}$.

1. $H^0(M) \neq 0$ if and only if $N \subset \text{supp}_{C[[g^*]]} M$.
2. $H^0(M)$ is nonzero and finite-dimensional if $\text{supp}_{C[[g^*]]} M = N$.

**Proof.** (1) Note that $\text{supp}_{C[[g^*]]} H^0(M)$ is invariant under the $C^*$-action \(2.7\) on $S$, which contracts the point \(\{ f \}\). Hence $\text{supp}_{C[[g^*]]} H^0(M) = (\text{supp}_{C[[g^*]]} M) \cap S$ is nonempty if and only if $f \in \text{supp}_{C[[g^*]]} H^0(M)$. The assertion follows since $\text{supp}_{C[[g^*]]} H^0(M)$ is $G$-invariant and closed. (2) Obvious since the assumption implies that $\text{supp}_{C[[g^*]]} H^0(M) = \{ f \}$. \(\square\)

6.6. **Losev’s correspondence.** Let $\mathcal{HC}$ be the category of Harish-Chandra bimodules, that is, the full subcategory of the category of $U(g)$-bimodules on which the adjoint action of $g$ is integrable.

**Lemma 6.13.** Every finitely generated object $M$ of $\mathcal{HC}$ admits a good filtration, that is, an increasing filtration $0 = F_0M \subset F_1M \subset \ldots$ such that $M = \bigcup F_pM$,

$$U_p(g) \cdot F_qM : U_r(g) \subset F_{p+q+r}M, \quad [U_p(g), F_pM] \subset F_{p+q-1}M,$$

and $gr^F M = \bigoplus F_pM/F_p-1M$ is finitely generated over $C[g^*]$.

If $M \in \mathcal{HC}$ and $F_\bullet M$ is a good filtration, then $gr^F M$ is naturally a Poisson module over $C[g^*]$. Therefore, it is an object of $\mathcal{HC}$.

Let $M$ be a finitely generated object in $\mathcal{HC}$. It is known since Bernstein that

$$\text{Var}(M) := \text{supp}_{C[[g^*]]} (gr^F M) \subset g^*$$

in independent of the choice of a good filtration $F_\bullet M$ of $M$.

For $M \in \mathcal{HC}$, $M \otimes Cl$ is naturally a bimodule over $C(g) = U(g) \otimes Cl$. Thus, $(M \otimes Cl, \text{ad} Q)$ is a differential graded bimodule over $C(g)$, and its cohomology

$$H^\bullet(M) := H^\bullet(M \otimes Cl, \text{ad} Q)$$

is naturally a module over $H^0(C(g), \text{ad} Q)$ that is identified with $Z(g)$ by Theorem 2.17. Thus, we have a functor

\[(6.8) \quad \mathcal{HC} \to Z(g) - \text{Mod}, \quad M \mapsto H^0(M).\]

Let $M \in \mathcal{HC}$ be finitely generated, $F_\bullet M$ a good filtration. Then $F_p(M \otimes Cl) := \sum_{i+j=p} F_iM \otimes Cl_j$ defines a good filtration of $M \otimes Cl$, and the associated graded space $gr^F(M \otimes Cl) = \sum_0 F_p(M \otimes Cl)/F_{p-1}(M \otimes Cl)$ is a Poisson module over $gr C(g) = \bar{C}(g)$.

The filtration $F_\bullet (M \otimes Cl)$ induces a filtration $F_\bullet H^\bullet(M)$ on $H^\bullet(M)$, and $gr^F H^\bullet(M) = \bigoplus F_p H^\bullet(M)/F_{p-1} H^\bullet(M)$ is a module over $gr Z(g) = C[S]$.

For a finitely generated $Z(g)$-module $M$, set $\text{Var}(M) = \text{supp}_{C[[S]]}(gr M)$, $gr M$ is the associated graded $M$ with respect to a good filtration of $M$.

The following assertion follows from Theorems 6.10 and 6.12.
Theorem 6.14 ([Gin, Los]). (1) We have \( H^i(M) = 0 \) for all \( i \neq 0 \), \( M \in \mathcal{HC} \).

Therefore the functor \( \mathbb{B} \) is exact.

(2) Let \( M \) be a finitely generated object of \( \mathcal{HC} \), \( F_\bullet M \) a good filtration. Then \( \text{gr} F^0 H^0(M) \cong H^0(\text{gr} F_\bullet M) \). In particular \( H^0(M) \) is finitely generated, \( F_\bullet H^0(M) \) is a good filtration of \( H^0(M) \).

(3) For a finitely generated object \( M \) of \( \mathcal{HC} \), \( \text{Var}(H^0(M)) = \text{Var}(M) \cap \mathcal{S} \).

6.7. Frenkel-Zhu’s bimodules. Recall that for a graded vertex algebra \( V \), Zhu’s algebra \( \text{Zhu}(V) = V/V \circ V \) is defined. There is a similar construction for modules due to Frenkel and Zhu [FZ]. For a \( V \)-module \( M \) set \( \text{Zhu}(M) = M/V \circ M \), where \( V \circ M \) is the subspace of \( M \) spanned by the vectors \( a \circ m = \sum_{i \geq 0} \binom{\Delta a}{i} a_{(i-2)} m \) for \( a \in V_{\Delta a}, \Delta a \in \mathbb{Z}, \) and \( m \in M \).

Proposition 6.15 ([FZ]). \( \text{Zhu}(M) \) is a bimodule over \( \text{Zhu}(V) \) by the multiplications \( a \ast m = \sum_{i \geq 0} \binom{\Delta a}{i} a_{(i-1)} b, \quad m \ast a = \sum_{i \geq 0} \binom{\Delta a - 1}{i} a_{(i-1)} m \) for \( a \in V_{\Delta a}, \Delta a \in \mathbb{Z}, \) and \( m \in M \).

Thus, we have a right exact functor \( V \)-Mod \( \rightarrow \text{Zhu}(V) \)-biMod, \( M \mapsto \text{Zhu}(M) \).

Lemma 6.16. Let \( M = \bigoplus_{d \in \mathbb{Z}_+} M_d \) be a positive energy representation of a \( \mathbb{Z}_+ \)-graded vertex algebra \( V \). Define an increasing filtration \( \{ \text{Zhu}_p(M) \} \) on \( \text{Zhu}(V) \) by \( \text{Zhu}_p(M) = \text{im}(\bigoplus_{d=h}^{h+p} M_p \rightarrow \text{Zhu}(M)) \).

(1) We have \( \text{Zhu}_p(V) \cdot \text{Zhu}_q(M) \cdot \text{Zhu}_r(V) \subset \text{Zhu}_{p+q+r}(M), \) \( [\text{Zhu}_p(V), \text{Zhu}_q(M)] \subset \text{Zhu}_{p+q-1}(M) \).

Therefore \( \text{gr} \text{Zhu}(M) = \bigoplus_p \text{Zhu}_p(M)/\text{Zhu}_{p-1}(M) \) is a Poisson \( \text{gr} \text{Zhu}(V) \)-module, and hence is a Poisson \( R_V \)-module through the homomorphism \( \eta_V : R_V \twoheadrightarrow \text{gr} \text{Zhu}(V) \).

(2) There is a natural surjective homomorphism \( \eta_M : \bar{M} (= M/F^1 M) \rightarrow \text{gr} \text{Zhu}(M) \)

of Poisson \( R_V \)-modules. This is an isomorphism if \( V \) admits a PBW basis and \( \text{gr} M \) is free over \( \text{gr} V \).
Example 6.17. Let $M = V^k_E$. Since $\text{gr } V^k_E$ is free over $\mathbb{C}[J^*]$, we have the isomorphism

$$\eta_{V^k_E} : \overline{V^k_E} = E \otimes \mathbb{C}[g^*] \cong \text{gr } V^k_E.$$ 

On the other hand, there is a $U(g)$-bimodule homomorphism

$$E \otimes U(g) \to \text{Zhu}(V^k_E),$$

(6.9)

$$v \otimes x_1 \ldots x_r \mapsto (1 \otimes v) \ast (x_1 t^{-1}) \ast (x_1 t^{-1}) + V^k(g) \circ V^k_E$$

which respects the filtration. Here the $U(g)$-bimodule structure of $U(g) \otimes E$ is given by

$$x(v \otimes u) = (xv) \otimes u + v \otimes xu, \quad (v \otimes u) x = v \otimes (ux),$$

and the filtration of $U(g) \otimes E$ is given by $\{U_i(g) \otimes E\}$. Since the induced homomorphism between associated graded spaces (6.9) coincides with $\eta_{V^k_E}$, (6.9) is an isomorphism.

Lemma 6.18. For $M \in \text{KL}_k$ we have $\text{Zhu}(M) \in \mathcal{H}C$. If $M$ is finitely generated, then so is $\text{Zhu}(M)$.

6.8. Zhu’s two functors commute with BRST reduction. For a smooth $\mathfrak{g}$-module $M$ over level $k$, $C(M) := M \otimes \mathcal{F}$ is naturally a module over $C^k(g) = V^k(g) \otimes \mathcal{F}$. Thus, $(C(M), Q_{(0)})$ is a cochain complex, and its cohomology $H^*(M) := H^*(C(M), Q_{(0)})$ is a module over $W^k(g) = H^*(C^k(g), Q_{(0)})$. Thus we have a functor

$$V^k(g) \otimes \text{Mod} \to W^k(g) \otimes \text{Mod}, \quad M \mapsto H^0(M).$$

Here $V \otimes \text{Mod}$ denotes the category of modules over a vertex algebra $V$.

Theorem 6.19. (1) ([FG10, A6]) We have $H^i(M) = 0$ for $i \neq 0$, $M \in \text{KL}_k$.

In particular the functor

$$\text{KL}_k \to W^k(g) \otimes \text{Mod}, \quad M \mapsto H^0(M),$$

is exact.

(2) (A6) For a finitely generated object $M$ of $\text{KL}$,

$$\overline{H^0(M)} \cong H^0(M)$$

as Poisson modules over $R_{W^k(g)} = \mathbb{C}[S]$. In particular $H^0(M)$ is finitely strongly generated and

$$X_{H^0(M)} = X_M \cap S.$$

(3) ([A7]) For a finitely generated object $M$ of $\text{KL}$,

$$\text{Zhu}(H^0(M)) \cong H^0(\text{Zhu}(M))$$

as bimodules over $\text{Zhu}(W^k(g)) = Z(g)$.

Let $W_k(g)$ denote the unique simple graded quotient of $W^k(g)$. Then $X_{W_k(g)}$ is a $\mathbb{C}$*-invariant subvariety of $S$. Therefore $X_{W_k(g)}$ is lisse if and only if $X_{W_k(g)} = \{ f \}$ since the $\mathbb{C}$*-action on $S$ contracts to the point $f$. 
Corollary 6.20.  
(1) $H^0(V_k(\mathfrak{g}))$ is a quotient of $W^k(\mathfrak{g}) = H^0(V_k(\mathfrak{g}))$. In particular $W_k(\mathfrak{g})$ is a quotient of $H^0(V_k(\mathfrak{g}))$ if $H^0(V_k(\mathfrak{g}))$ is nonzero.
(2) $H^0(V_k(\mathfrak{g}))$ is nonzero if and only if $X_{V_k(\mathfrak{g})} \supset \overline{G.f} = N$.
(3) The simple $W$-algebra $W_k(\mathfrak{g})$ is lisse if $X_{V_k(\mathfrak{g})} = \overline{G.f} = N$.

Proof. (1) follows from the exactness statement of Theorem 6.19.  
(2) $H^0(V_k(\mathfrak{g}))$ is nonzero if and only if $X_{V_k(\mathfrak{g})} \supset \overline{G.f} = N$.  
(3) If $X_{V_k(\mathfrak{g})} = \overline{G.f}$, $X_{H^0(V_k(\mathfrak{g}))} = X_M \cap S = \{ f \}$, and thus, $H^0(V_k(\mathfrak{g}))$ is lisse, and thus, so its quotient $W_k(\mathfrak{g})$.

Remark 6.21.  
(1) The above results hold for $W$-algebras associated with any $\mathfrak{g}$ and any $f \in N$ without any restriction on the level $k$ ([A6, A7]).  
In particular we have the vanishing result
\begin{equation}
H^j_i(M) = 0 \quad \text{for } i \neq 0, \quad M \in \mathbf{KL}_k,
\end{equation}
for the BRST cohomology $H^j_i(M)$ of the quantized Drinfeld-Sokolov reduction functor associated with $f$ in the coefficient in an object $M$ of $\mathbf{KL}_k$.  
Thus the functor
$$
\mathbf{KL}_k \rightarrow W^k(\mathfrak{g}, f)\text{-Mod}, \quad M \mapsto H^j_i(M),
$$
is exact, and moreover,
\begin{equation}
X_{H^0_i(V_k(\mathfrak{g}))} = X_{V_k(\mathfrak{g})} \cap S_f.
\end{equation}
where $S_f$ is the Slodowy slice at $f$ (see §2.3).  
In particular
\begin{equation}
H^0_i(V_k(\mathfrak{g})) \neq 0 \iff X_{V_k(\mathfrak{g})} \supset \overline{G.f}.
\end{equation}

(2) In the case that $f = f_\theta$, a minimal nilpotent element of $\mathfrak{g}$, then we also have the following result [A1]:
\begin{equation}
H_{f_\theta}(V_k(\mathfrak{g})) = \begin{cases} 
W_k(\mathfrak{g}, f_\theta) & \text{if } k \not\in \mathbb{Z}_+, \\
0 & \text{if } k \in \mathbb{Z}_+.
\end{cases}
\end{equation}

Here $W_k(\mathfrak{g}, f_\theta)$ is the simple quotient of $W^k(\mathfrak{g}, f_\theta)$.  
Together with (6.11), this proves the “only if” part of Theorem 3.24.  
Indeed, if $V_k(\mathfrak{g})$ is lisse, then $H_{f_\theta}(V_k(\mathfrak{g})) = 0$ by (6.11), and hence, $k \in \mathbb{Z}_+$.

7. Irreducible Representations of $W$-Algebras

In this section we quickly review results obtained in [A2].

Since $Z(W^k(\mathfrak{g})) \cong Z(\mathfrak{g})$, by Zhu’s theorem irreducible positive energy representations of $W^k(\mathfrak{g})$ are parametrized by central characters of $Z(\mathfrak{g})$.  
For a central character $\gamma : Z(\mathfrak{g}) \rightarrow \mathbb{C}$, let $L(\gamma)$ be the corresponding irreducible positive energy representations of $W^k(\mathfrak{g})$.  
This is a simple quotient of the Verma module $M(\gamma)$ of $W^k(\mathfrak{g})$ with highest weight $\gamma$, which has the character
\begin{align*}
\text{ch} M(\gamma) := \text{tr}_{M(\gamma)} q^L &= \frac{q^{\frac{\gamma(0)}{2}}}{\prod_{j \geq 1} (1-q^j)^{v_k \theta}}.
\end{align*}
in the case that $k$ is non-critical, where $\Omega$ is the Casimir element of $U(g)$.

In Theorem 6.19 we showed that the functor $\text{KL}_k \to W^k(g)\text{-Mod}, M \mapsto H^0(M)$, is exact. However in order to obtain all the irreducible positive energy representation we need to extend this functor to the whole category $O_k$. However the functor $O_k \to W^k(g)\text{-Mod}, M \mapsto H^0(M)$, is not exact in general except for the case $g = sl_2$. Nevertheless, we can [FKW] modify the functor to obtain the following result.

**Theorem 7.1 ([A2]).** There exists an exact functor $O_k \to W^k(g)\text{-Mod}, M \mapsto H^0(\mathcal{L}(\lambda))$ (called the “$-$”-reduction functor in [FKW]), which enjoys the following properties.

1. $H^0(\mathcal{L}(\lambda)) \cong \mathcal{M}(\gamma_{\bar{\lambda}})$, where $\mathcal{L}(\lambda)$ is the Verma module of $\hat{g}$ with highest weight $\lambda$, and $\gamma_{\bar{\lambda}}$ is the evaluation of $Z(g)$ at the Verma module $M_{\hat{g}}(\bar{\lambda})$ of $\hat{g}$ with highest weight $\bar{\lambda}$.

2. $H^0(\mathcal{L}(\lambda)) \cong \begin{cases} \mathcal{L}(\gamma_{\bar{\lambda}}) & \text{if } \bar{\lambda} \text{ is anti-dominant (that is, } M_{\hat{g}}(\bar{\lambda}) \text{ is simple),} \\ 0 & \text{otherwise.} \end{cases}$

**Corollary 7.2.** Write $\text{ch} \mathcal{L}(\lambda) = \sum_{\mu} c_{\lambda,\mu} \text{ch} M(\mu)$ with $c_{\lambda,\mu} \in \mathbb{Z}$. If $\bar{\lambda}$ is anti-dominant, we have

$$\text{ch} \mathcal{L}(\gamma_{\bar{\lambda}}) = \sum_{\mu} c_{\lambda,\mu} \text{ch} M(\gamma_{\bar{\mu}}).$$

In the case that $k$ is non-critical, then it is known by Kashiwara and Tanisaki [KT2] that the coefficient $c_{\lambda,\mu}$ is expressed in terms of Kazhdan-Lusztig polynomials. Since any central character of $Z(g)$ can be written as $\gamma_{\bar{\lambda}}$ with anti-dominant $\bar{\lambda}$, Corollary 7.2 determines the character of all the irreducible positive energy representations of $W^k(g)$ for all non-critical $k$.

On the other hand, in the case that $k$ is critical, all $\mathcal{L}(\gamma_{\bar{\lambda}})$ are one-dimensional since $W^{-n}(g)$ is commutative. This fact with Theorem 7.1 can be used in the study of the critical level representations of $\hat{g}$, see [AF].

The results in this section hold for arbitrary simple Lie algebra $g$.

**Remark 7.3.** The condition $\bar{\lambda} \in \mathfrak{h}^*$ is anti-dominant does not imply that $\lambda \in \mathfrak{h}^*$ is anti-dominant. In fact this condition is satisfied by all non-degenerate admissible weights $\lambda$ (see below) which are regular dominant.

**Remark 7.4.** Theorem 7.1 has been generalized in [A3]. In particular the character of all the simple ordinary representations (=simple positive energy representations with finite-dimensional homogeneous spaces) has been determined for $W$-algebras associated with all nilpotent elements $f$ in type $A$.

### 8. KAC-WAKIMOTO ADMISSIBLE REPRESENTATIONS AND FRENKEL-KAC-WAKIMOTO CONJECTURE

We continue to assume that $g = sl_n$, but the results in this section holds for arbitrary simple Lie algebra $g$ as well with appropriate modification unless otherwise stated.
8.1. **Admissible affine vertex algebras.** Let \( \hat{\mathfrak{g}} \) be the Cartan subalgebra \( \mathfrak{h} \oplus CK \) of \( \hat{\mathfrak{g}} \), \( \hat{\mathfrak{h}} = \mathfrak{h} \oplus CK \oplus \mathbb{C}D \) the extended Cartan subalgebra, \( \hat{\Delta} \) the set of roots of \( \hat{\mathfrak{g}} \) in \( \hat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \Lambda_0 \oplus \mathbb{C} \delta \), where \( \Lambda_0(K) = 1 = \delta(d) \), \( \Lambda_0(\mathfrak{h} + \mathbb{C}D) = \delta(\mathfrak{h} \oplus CK) = 0 \), \( \hat{\Delta}_+ \) the set of positive roots. \( \Delta^{re} \subset \hat{\Delta} \) the set of real roots, \( \Delta^{re}_+ = \Delta^{re} \cap \hat{\Delta}_+ \). Let \( \hat{W} \) be the affine Weyl group of \( \hat{\mathfrak{g}} \).

**Definition 8.1 (KW2).** A weight \( \lambda \in \hat{\mathfrak{h}}^* \) is called *admissible* if

1. \( \lambda \) is regular dominant, that is,
   \[
   \langle \lambda + \rho, \alpha' \rangle \not\in \mathbb{Z}_+ \quad \text{for all} \quad \alpha \in \hat{\Delta}^{re},
   \]
2. \( \mathbb{Q}\hat{\Delta}(\lambda) = \mathbb{Q}\hat{\Delta}^{re} \), where \( \hat{\Delta}(\lambda) = \{ \alpha \in \hat{\Delta}^{re} \mid \langle \lambda + \rho, \alpha' \rangle \in \mathbb{Z} \} \).

The irreducible highest weight representation \( L(\lambda) \) of \( \hat{\mathfrak{g}} \) with highest weight \( \lambda \in \hat{\mathfrak{h}}^* \) is called *admissible* if \( \lambda \) is admissible. Note that an irreducible integrable representations of \( \hat{\mathfrak{g}} \) is admissible.

Clearly, integrable representations of \( \hat{\mathfrak{g}} \) are admissible.

For an admissible representation \( L(\lambda) \) we have \( \text{KW1} \)

\[
\text{ch } L(\lambda) = \sum_{w \in \hat{W}(\lambda)} (-1)^{f_\lambda(w)} \text{ch } M(w \circ \lambda)
\]

since \( \lambda \) is regular dominant, where \( \hat{W}(\lambda) \) is the *integral Weyl group* \( \text{KT1 MP} \) of \( \lambda \), that is, the subgroup of \( \hat{W} \) generated by the reflections \( s_\alpha \) associated with \( \alpha \in \hat{\Delta} \) and \( w \circ \lambda = w(\lambda + \rho) - \rho \). Further the condition (2) implies that \( \text{ch } L(\lambda) \) is written in terms of certain theta functions. Kac and Wakimoto \( \text{KW2} \) showed that admissible representations are *modular invariant*, that is, the characters of admissible representations form an \( SL_2(\mathbb{Z}) \) invariant subspace.

Let \( \lambda, \mu \) be distinct admissible weights. Then the condition (1) implies that

\[
\text{Ext}_L^2(L(\lambda), L(\mu)) = 0.
\]

Further, the following fact is known by Gorelik and Kac \( \text{GK} \).

**Theorem 8.2 (GK).** Let \( \lambda \) be admissible. Then \( \text{Ext}_L^2(L(\lambda), L(\lambda)) = 0 \).

Therefore admissible representations form a semisimple fullsubcategory of the category of \( \hat{\mathfrak{g}} \)-modules.

Recall that the simple affine vertex algebra \( V_k(\mathfrak{g}) \) is isomorphic to \( L(k\Lambda_0) \) as an \( \hat{\mathfrak{g}} \)-module.

**Lemma 8.3.** The following conditions are equivalent.

1. \( k\Lambda_0 \) is admissible.
2. \( k\Lambda_0 \) is regular dominant and \( k \in \mathbb{Q} \).
3. \( k + h^\vee = p/q, p, q \in \mathbb{N}, (p, q) = 1, p \geq h^\vee = n \).

If this is the case, the level \( k \) is called admissible for \( \hat{\mathfrak{g}} \), and \( V_k(\mathfrak{g}) \) is called an admissible affine vertex algebra.

For an admissible number \( k \) let \( Pr_k \) be the set of admissible weights of \( \hat{\mathfrak{g}} \) of level \( k \). (For \( \mathfrak{g} = \mathfrak{sl}_n \), \( Pr_k \) is the same as the set of *principal admissible weights* of level \( k \).)
8.2. Feigin-Frenkel Conjecture and Adamović-Milas Conjecture. The following fact was conjectured by Feigin and Frenkel and proved for the case that $\mathfrak{g} = \mathfrak{sl}_2$ by Feigin and Malikov $[FM]$.

**Theorem 8.4 ([A6]).** The associated variety $X_{\hat{V}_k(g)}$ is contained in $N$ if $k$ is admissible.

In fact the following holds.

**Theorem 8.5 ([A6]).** Let $k$ be admissible, and let $q \in N$ be the denominator of $k$, that is, $k + h^\vee = p/q$, $p \in N$, $(p, q) = 1$. Then

$$X_{\hat{V}_k(g)} = \{ x \in g \mid (\text{ad} \, x)^{2q} = 0 \} = \overline{\mathcal{O}_q},$$

where $\mathcal{O}_q$ is the nilpotent orbit corresponding to the partition

$$\begin{cases} (n) & \text{if } q \geq n, \\ (q, q, \ldots, q, s) & (0 \leq s \leq n - 1) \text{ if } q < n. \end{cases}$$

The following fact was conjectured by Adamović and Milas $[AdM]$.

**Theorem 8.6 ([AS]).** Let $k$ be admissible. Then an irreducible highest weight representation $L(\lambda)$ is a $V_k(g)$-module if and only if $k \in P_{R_k}$. Hence if $M$ is a finitely generated $V_k(g)$-module on which $\hat{\mathfrak{n}}_+$ acts locally nilpotently and $\hat{\mathfrak{h}}$ acts locally finitely then $M$ is a direct sum of $L(\lambda)$ with $\lambda \in P_{R_k}$.

8.3. Outline of proofs of Theorems 8.4, 8.5 and 8.6. The idea of the proofs of Theorem 8.4 and Theorem 8.6 is to reduce to the $\mathfrak{sl}_2$-cases.

Let $\mathfrak{sl}_{2,i} \subset \mathfrak{g}$ be the copy of $\mathfrak{sl}_2$ spanned by $e_i := e_{i,i+1}$, $h_i := e_i - e_{i+1,i+1}$, $f_i := e_{i+1,i}$, and let $p_i = \mathfrak{sl}_{2,i} + \mathfrak{h} \subset \mathfrak{g}$, the associated minimal parabolic subalgebra. Then

$$p_i = l_i \oplus m_i,$$

where $l_i$ is the Levi subalgebra $\mathfrak{sl}_{2,i} + \mathfrak{h}$, and $m_i$ is the nilradical $\bigoplus_{1 \leq p < q \leq n \atop (p, q) \neq (i, i+1)} \mathbb{C}e_{p,q}$.

Consider the semi-infinite cohomology $H^{\leq 0}(\mathfrak{sl}_i[t, t^{-1}], M)$. It is defined as a cohomology of Feigin’s complex $C(m_i[t, t^{-1}], M, d)$ ([Fe]). There is a natural vertex algebra homomorphism

$$V^{k_i}(\mathfrak{sl}_2) \to H^{\leq 0}(\mathfrak{sl}_i[t, t^{-1}], M),$$

where $k_i = k + n - 2$, see, e.g. [HT]. Note that if $k$ is an admissible number for $\hat{\mathfrak{g}}$ then $k_i$ is an admissible number for $\hat{\mathfrak{sl}}_2$.

**Theorem 8.7 ([A5]).** Let $k$ be an admissible number. The map (8.2) factors through the vertex algebra embedding

$$V_k(\mathfrak{sl}_2) \to H^{\leq 0}(\mathfrak{sl}_i[t, t^{-1}], V_k(g)).$$

**Outline of proof of Theorem 8.4.** First, consider the case that $\mathfrak{g} = \mathfrak{sl}_2$. Let $N_k$ be the maximal submodule of $V^k(g)$, and let $I_k$ be the image of $N_k$ in $R_{V^k(g)} = \mathbb{C}[g^*]$, so that $R_{V_k(g)} = \mathbb{C}[\mathfrak{g}^*]/I_k$. It is known by Kac and Wakimoto $[KW1]$ that $N_k$ is
generated by a singular vector, say \( v_k \). The projection formula \([\text{MFF}]\) implies that the image \([v_k] \) of \( v_k \) in \( I_k \) is nonzero. Since \([v_k] \) is a singular vector of \( \mathbb{C}[\mathfrak{g}] \) with respect to the adjoint action of \( \mathfrak{g} \), Kostant’s Separation Theorem implies that
\[
[v_k] = e^m \Omega^n
\]
for some \( m, n \in \mathbb{N} \) up to constant multiplication, where \( \Omega = ef + fe + \frac{1}{2} h^2 \). Now suppose that \( X_{V_k(\mathfrak{g})} \not\subset \mathcal{N} \) and let \( \lambda \in X_{V_k(\mathfrak{g})} \setminus \mathcal{N} \), so that \( \Omega(\lambda) \neq 0 \). Then \( e(\lambda) = 0 \). Since \( X_{V_k(\mathfrak{g})} \) is \( G \)-invariant this implies that \( x(\lambda) = 0 \) for any nilpotent element \( x \) of \( \mathfrak{g} \). Because any element of \( \mathfrak{g} \) can be written as a sum of nilpotent elements we get that \( \lambda = 0 \). Contradiction.

Next, consider the case that \( \mathfrak{g} \) is general. Note that since \( X_{V_k(\mathfrak{g})} \) is \( G \)-invariant and closed, the condition \( X_{V_k(\mathfrak{g})} \subset \mathcal{N} \) is equivalent to that \( X_{V_k(\mathfrak{g})} \cap \mathfrak{h}^* = \{0\} \).

Now the complex structure of \( C(m_i[t, t^{-1}], V_k(\mathfrak{g})) \) induces the complex structure on Zhu’s \( C_2 \)-algebra \( R_C(m_i[t, t^{-1}], V_k(\mathfrak{g})) \). The embedding in Theorem \([6.7]\) induces a homomorphism
\[
R_{V_k(\mathfrak{s}\mathfrak{l}_2)} \to H^0(R_C(m_i[t, t^{-1}], V_k(\mathfrak{g})), d)
\]
of Poisson algebra. Since \( \Omega \) is nilpotent in \( R_{V_k(\mathfrak{s}\mathfrak{l}_2)} \), so is its image \( \Omega_i = c_i f_i + f_i c_i + \frac{1}{2} h_i^2 \) in \( H^0(R_C(m_i[t, t^{-1}], V_k(\mathfrak{g})), d) \). It follows that \( h_i^n \equiv 0 \pmod{n_i R_{V_k(\mathfrak{g})} + n_i R_{V_k(\mathfrak{g})}} \) in \( R_{V_k(\mathfrak{g})} \) for all \( i = 1, \ldots, n - q \), and we get that \( X_{V_k(\mathfrak{g})} \cap \mathfrak{h}^* = \{0\} \) as required.

**Outline of proof of Theorem \([8.6]\)** The proof is done by determining the variety \( X_{V_k(\mathfrak{g})} \).

By Theorem \([5.3]\) \( X_{V_k(\mathfrak{g})} \) is a finite union of nilpotent orbits. Thus it is enough to know which nilpotent element orbits are contained in \( X_{V_k(\mathfrak{g})} \). On the other hand, \([6.1]\) says \( X_{V_k(\mathfrak{g})} \supset \mathcal{G} \mathfrak{f} \) if and only \( H^0(V_k(\mathfrak{g})) \neq 0 \). Thus, it is sufficient to compute the character of \( H^0(V_k(\mathfrak{g})) \). This is in fact possible since we know the explicit formula \([6.1]\) of the character of \( V_k(\mathfrak{g}) \), and thanks of the vanishing theorem \([6.10]\) and the Euler-Poincaré principle.

**Outline of proof of Theorem \([8.7]\)** Let \( L(\lambda) \) be a \( V_k(\mathfrak{g}) \)-module. Then, the space \( H^{\pm i}(m_i[t, t^{-1}], L(\lambda)) \), \( i \in \mathbb{Z} \), is naturally a \( H^{\pm i}(m_i[t, t^{-1}], V_k(\mathfrak{g})) \)-module. By Theorem \([8.7]\) this means that \( H^{\pm i}(m_i[t, t^{-1}], L(\lambda)) \) is in particular a module over the admissible affine vertex algebra \( V_k(\mathfrak{s}\mathfrak{l}_2) \). Therefore Theorem \([8.6]\) for \( \mathfrak{g} = \mathfrak{s}\mathfrak{l}_2 \) that was established by Adamović and Milas \([\text{AdM}]\) implies that \( H^{\pm i}(m_i[t, t^{-1}], L(\lambda)) \) must be a direct sum of admissible representations of \( \mathfrak{s}\mathfrak{l}_2 \). This information is sufficient to conclude that \( L(\lambda) \) is admissible.

Conversely, suppose that \( L(\lambda) \) is an admissible representation of level \( k \). If \( L(\lambda) \) is integrable over \( \mathfrak{g} \), then it has been already proved by Frenkel and Malikov \([\text{FM}]\) that \( L(\lambda) \) is a \( V_k(\mathfrak{g}) \)-module. But then an affine analogue of Duflo-Joseph Lemma \([\text{A8} \text{ Lemma 2.6}]\) implies that this is true for a general admissible representation as well.

### 8.4. Lisse property of \( W \)-algebras

An admissible number \( k \) is called **non-degenerate** if \( X_{V_k(\mathfrak{g})} = \mathcal{N} \). By Theorem \([8.5]\) this condition is equivalent to that
\[
k + n = \frac{p}{q}, \quad p, q \in \mathbb{N}, \quad (p, q) = 1, \quad p \geq n, \quad q \geq n.
\]
The following assertion follows immediately from Corollary \[6.20\]

**Theorem 8.8 (\[A6\]).** Let \( k \) be a non-degenerate admissible number. Then the \( W \)-algebra \( W_k(g) \) is lisse.

### 8.5. Minimal models of \( W \)-algebras.

A vertex algebra \( V \) is called rational if any \( V \)-module is completely reducible. To a lisse and rational conformal vertex algebra \( V \) one can associate rational 2d conformal field theory, and in particular, the category \( V \mathrm{-Mod} \) of \( V \)-modules forms a modular tensor category [BK], as in the case of the category of integrable representation of \( g \) at a positive level and the category of minimal series representations [BPZ] of the Virasoro algebra.

An admissible weight \( \lambda \) is called non-degenerate if \( \lambda \) is anti-dominant. Let \( \mathcal{P} \) be the set of non-degenerate admissible weights of level \( k \) of \( g \). It is known [FKW] that \( \mathcal{P} \) is non-empty if and only if \( k \) is non-degenerate.

By Theorem 7.1, for \( \lambda \in \mathcal{P} \), \( \mathcal{H}^0(L(\lambda)) \) is a (non-zero) simple \( \mathcal{W}^k(g) \)-module if and only of \( \lambda \in \mathcal{P} \). Theorem 8.8 (\[A6\]) of \( \mathcal{W}^k(g) \)-algebras.

Let \( \mathcal{P} \) be a non-degenerate admissible number. Then the \( \mathcal{P} \) of \( \mathcal{W}^k(g) \)-algebra.

The following assertion was conjectured by Frenkel, Kac and Wakimoto [FKW].

**Theorem 8.9 (\[A7\]).** Let \( k \) be a non-degenerate admissible number. Then the simple \( W \)-algebra \( W_k(g) \) is rational, and \( \{L(\gamma) = H^0(L(\lambda)) \mid \lambda \in \mathcal{P} \} \) forms the complete set of isomorphism classes of simple \( W_k(g) \)-modules.

In the case that \( g = sl_2 \), Theorems 8.8 and 8.9 have been proved in [BFM, Wan], and the above representations are exactly the minimal series representations of the Virasoro algebra.

The representations  
\[ \{L(\gamma) \mid \lambda \in \mathcal{P} \} \]

are called the minimal series representations of \( W^k(g) \), and if \( k + n = p/q, p, q \in \mathbb{N}, (p, q) = 1 \), then the rational \( W \)-algebra \( W_k(g) \) is called the \( (p, q) \)-minimal model of \( W^k(g) \). Note that the \( (p, q) \)-minimal model and the \( (q, p) \)-minimal model are isomorphic due to the duality, see Corollary \[5.24\].

**Outline of the proof of Theorem 8.9.** Let \( k \) be a non-degenerate admissible number. We have

\[ H^0(V_k(g)) \cong W_k(g) \]

by [A2]. Hence by Theorem \[6.19\] (3)

\[ \text{Zhu}(W_k(g)) = \text{Zhu}(H^0(V_k(g))) = H^0(\text{Zhu}(V_k(g))). \]
From this together with Theorem 8.6, it is not too difficult to obtain the classification is the simple $\mathcal{W}_k(\mathfrak{g})$-modules as stated in Theorem 8.9. One sees that the extensions between simple modules are trivial using the linkage principle that follows from Theorem 7.1. □

Remark 8.10.

(1) We have $\mathcal{W}_k(\mathfrak{g}) = L(\gamma_{-(k+n)\rho})$ for a non-degenerate admissible number $k$. (Note that $k\Lambda_0 \notin Pr_{\text{non-deg.}}$)

(2) Let $\lambda \in Pr_k$. From Corollary 7.2 and (8.1), we get

$$\text{ch}L(\gamma_{\bar{\lambda}}) = \sum_{w \in \hat{W}(\lambda)} \epsilon(w) \text{ch}M(\gamma_{w^\circ \lambda}).$$

(8.3)

This was conjectured by [FKW].

(3) When it is trivial (that is, equals to $\mathbb{C}$), $\mathcal{W}_k(\mathfrak{g})$ is obviously lisse and rational. This happens if and only if $\mathcal{W}_k(\mathfrak{g})$ is the $(n, n+1)$-minimal model (=the $(n+1, n)$-minimal model). In this case the character formula (8.3) for $\mathcal{W}_k(\mathfrak{g}) = L(\gamma_{\bar{\lambda}})$, $\lambda = -(k+n)\rho + k\Lambda_0$, gives the following denominator formula:

$$\sum_{w \in \hat{W}(\lambda)} \epsilon(w)q^{\frac{\text{wt}(w, \lambda)}{k+1}} = \prod_{j=1}^{n-1} (1 - q^j)^{n-1}.$$ 

In the case that $\mathfrak{g} = \mathfrak{sl}_2$, we get the denominator formula for the Virasoro algebra, which is identical to Euler’s pentagonal identity.

(4) As a generalization of the GKO construction [GKO] it has been conjectured [KW3] that the $(p, q)$-minimal model of $\mathcal{W}_k(\mathfrak{g})$, with $p > q$, is isomorphic to the commutant of $V_{l+1}(\mathfrak{g})$ inside $V_l(\mathfrak{g}) \otimes V_1(\mathfrak{g})$, where $l + n = q/(p - q)$. (Note that $V_l(\mathfrak{g})$ and $V_{l+1}(\mathfrak{g})$ are admissible.) This conjecture has been proved in [ALY2] for the special case that $(p, q) = (n+1, n)$.

A similar conjecture exists in the case that $\mathfrak{g}$ is simply laced.

(5) The existence of rational and lisse $W$-algebras has been conjectured for general $W$-algebras $\mathcal{W}^k(\mathfrak{g}, f)$ by Kac and Wakimoto [KW1]. This has been proved in [A9] in part including all the cases in type $A$. See [Kaw, AMor] for a recent development in the classification problem of rational and lisse $W$-algebras.

REFERENCES

[A2] Tomoyuki Arakawa. Representation theory of $W$-algebras. Invent. Math., 169(2):219–320, 2007.

[A2] Tomoyuki Arakawa. Representation theory of superconformal algebras and the Kac-Roan-Wakimoto conjecture. Duke Math. J., 130(3):435–478, 2005.

[Adl] Mark Adler. On a trace functional for formal pseudo differential operators and the symplectic structure of the Korteweg-de Vries type equations. Invent. Math., 50(3):219–248, 1978/79.

[AdM] Dražen Adamović and Antun Milas. Vertex operator algebras associated to modular invariant representations for $A_1^{(1)}$. Math. Res. Lett., 2(5):563–575, 1995.

[ABD] Toshiyuki Abe, Geoffrey Buhl, and Chongying Dong. Rationality, regularity, and $C_2$-cofiniteness. Trans. Amer. Math. Soc., 356(8):3391–3402 (electronic), 2004.
[A3] Tomoyuki Arakawa. Representation theory of W-algebras, II. In Exploring new structures and natural constructions in mathematical physics, volume 61 of Adv. Stud. Pure Math., pages 51–90. Math. Soc. Japan, Tokyo, 2011.

[A4] Tomoyuki Arakawa. A remark on the $C_2$ cofiniteness condition on vertex algebras. Math. Z., 270(1-2):559–575, 2012.

[A5] Tomoyuki Arakawa. Two-sided BGG resolution of admissible representations. Represent. Theory, 19(3):183–222, 2014.

[A6] Tomoyuki Arakawa. Associated varieties of modules over Kac-Moody algebras and $C_2$-cofiniteness of W-algebras. Int. Math. Res. Not., 2015:11605–11666, 2015.

[A7] Tomoyuki Arakawa. Rationality of W-algebras: principal nilpotent cases. Ann. Math., 182(2):565–694, 2015.

[A8] Tomoyuki Arakawa. Rationality of admissible affine vertex algebras in the category $O$. Duke Math. J., 165(1):67–93, 2016.

[A9] Tomoyuki Arakawa. Rationality of W-algebras II. in preparation.

[AF] Tomoyuki Arakawa and Peter Fiebig. On the restricted Verma modules at the critical level. Trans. Amer. Math. Soc., 364(9):4683–4712, 2012.

[AKM] Tomoyuki Arakawa, Toshiro Kuwabara, and Fyodor Malikov. Localization of Affine W-Algebras. Comm. Math. Phys., 335(1):143–182, 2015.

[ALY1] Tomoyuki Arakawa, Ching Hung Lam, and Hiromichi Yamada. Zhu’s algebra, $C_2$-algebra and $C_2$-cofiniteness of parafermion vertex operator algebras. Adv. Math., 264:261–295, 2014.

[ALY2] Tomoyuki Arakawa, Ching Hung Lam, and Hiromichi Yamada. Parafermion vertex operator algebras and W-algebras, arXiv:1701.06229 [math.RT].

[AMol] Tomoyuki Arakawa and Alexander Molev. Explicit generators in rectangular affine W-algebras of type $A$. Lett. Math. Phys. 107(1), 47-59, 2017.

[AMor] Tomoyuki Arakawa and Anne Moreau. Joseph ideals and lisse minimal W-algebras. J. Inst. Math. Jussieu, published online.

[BD1] Alexander A. Beilinson and Vladimir G. Drinfeld. Quantization of Hitchin’s fibration and Langlands’ program. In Algebraic and geometric methods in mathematical physics (Kaciveli, 1993), volume 19 of Math. Phys. Stud., pages 3–7. Kluwer Acad. Publ., Dordrecht, 1996.

[BD2] Alexander Beilinson and Vladimir Drinfeld. Opers. preprint, 2005. arXiv:math/0501398v1 [math.AG].

[BFM] Alexander Beilinson, Boris. Feigin, and Barry Mazur. Introduction to algebraic field theory on curves. preprint.

[BK] Bojko Bakalov and Alexander Kirillov, Jr. Lectures on tensor categories and modular functors, volume 21 of University Lecture Series. American Mathematical Society, Providence, RI, 2001.

[BLR] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. Néron models, volume 21 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1990.

[Bor] Richard E. Borcherds. Vertex algebras, Kac-Moody algebras, and the Monster. Proc. Nat. Acad. Sci. U.S.A., 83(10):3068–3071, 1986.

[BPZ] Aleksander A. Belavin, Alexander. M. Polyakov, and Aleksandr. B. Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. Nuclear Phys. B, 241(2):333–380, 1984.

[dBT1] Jan de Boer and Tjark Tjin. Quantization and representation theory of finite W algebras. Comm. Math. Phys., 158(3):485–516, 1993.

[dBT2] Jan de Boer and Tjark Tjin. The relation between quantum W algebras and Lie algebras. Comm. Math. Phys., 160(2):317–332, 1994.

[BS] Peter Bouwknegt and Kareljan Schoutens, editors. W-symmetry, volume 22 of Advanced Series in Mathematical Physics. World Scientific Publishing Co. Inc., River Edge, NJ, 1995.
[DLM] Chongying Dong, Haisheng Li, and Geoffrey Mason. Vertex operator algebras and associative algebras. *J. Algebra*, 206(1):67–96, 1998.

[DM] Chongying Dong and Geoffrey Mason. Integrability of $C_2$-cofinite vertex operator algebras. *Int. Math. Res. Not.*, pages Art. ID 80468, 15, 2006.

[DSK] Alberto De Sole and Victor G. Kac. Finite vs affine W-algebras. *Japan. J. Math.*, 1(1):137–261, 2006.

[DS] Vladimir G. Drinfel’d and Vladimir V. Sokolov. Lie algebras and equations of Korteweg-de Vries type. In *Current problems in mathematics, Vol. 24*, Itogi Nauki i Tekhniki, pages 81–180. Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984.

[EM] Lawrence Ein and Mircea Mustaţă. Jet schemes and singularities. In *Algebraic geometry—Seattle 2005. Part 2*, volume 80 of *Proc. Sympos. Pure Math.*, pages 505–546. Amer. Math. Soc., Providence, RI, 2009.

[FL] Vladimir A. Fateev and Sergei L. Lykyanov. The models of two-dimensional conformal quantum field theory with $Z_n$ symmetry. *Internat. J. Modern Phys. A*, 3(2):507–520, 1988.

[Fo] Boris L. Feigin. Semi-infinite homology of Lie, Kac-Moody and Virasoro algebras. *Uspekhi Mat. Nauk*, 39(2(236)):195–196, 1984.

[FF1] Boris Feigin and Edward Frenkel. Quantization of the Drinfel’d-Sokolov reduction. *Phys. Lett. B*, 246(1-2):75–81, 1990.

[FF2] Boris Feigin and Edward Frenkel. Affine Kac-Moody algebras at the critical level and Gel’fand-Dikii algebras. In *Infinite analysis, Part A, B (Kyoto, 1991)*, volume 16 of *Adv. Ser. Math. Phys.*, pages 197–215. World Sci. Publ., River Edge, NJ, 1992.

[FM] Boris Feigin and Fyodor Malikov. Modular functor and representation theory of $\hat{\mathfrak{sl}}_2$ at a rational level. In *Operads: Proceedings of Renaissance Conferences (Hartford, CT/Luminy, 1995)*, volume 202 of *Contemp. Math.*, pages 357–405, Providence, RI, 1997. Amer. Math. Soc.

[Fre07] Edward Frenkel. *Langlands correspondence for loop groups*, volume 103 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2007.

[FBZ] Edward Frenkel and David Ben-Zvi. *Vertex algebras and algebraic curves*, volume 88 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2004.

[FKW] Edward Frenkel, Victor Kac, and Minoru Wakimoto. Characters and fusion rules for W-algebras via quantized Drinfel’d-Sokolov reduction. *Comm. Math. Phys.*, 147(2):295–328, 1992.

[FM] Igor Frenkel and Fyodor Malikov. Kazhdan-Lusztig tensoring and Harish-Chandra categories. *preprint*, 1997. [arXiv:q-alg/9703010]

[FG10] Edward Frenkel and Dennis Gaitsgory. Weyl modules and opers without monodromy. In *Arithmetic and geometry around quantization*, volume 279 of *Progr. Math.*, pages 101–121. Birkhäuser Boston Inc., Boston, MA, 2010.

[FZ] Igor B. Frenkel and Yongchang Zhu. Vertex operator algebras associated to representations of affine and Virasoro algebras. *Duke Math. J.*, 66(1):123–168, 1992.

[GD78] Izrail’ M. Gel’fand and Leonid A. Dikii. A family of Hamiltonian structures connected with integrable nonlinear differential equations. *Akad. Nauk SSSR Inst. Prikl. Mat. Preprint*, (136):41, 1978.

[Gen] Naoki Genra. Screening operators for W-algebras. [arXiv:1606.00060 [math.RT]], to appear in *Sel. Math. New Ser.*

[Gin] Victor Ginzburg. Harish-Chandra bimodules for quantized Slodowy slices. *Represent. Theory*, 13:236–271, 2009.

[GK] Maria Gorelik and Victor Kac. On complete reducibility for infinite-dimensional Lie algebras. *Adv. Math.*, 226(2):1911–1972, 2011.

[GKO] Peter Goddard, Adrian Kent, and David Olive. Virasoro algebras and coset space models. *Phys. Lett. B*, 152(1-2):88–92, 1985.
Shinobu Hosono and Akihiro Tsuchiya. Lie algebra cohomology and $N = 2$ SCFT based on the GKO construction. *Comm. Math. Phys.*, 136(3):451–486, 1991.

Yi-Zhi Huang. Rigidity and modularity of vertex tensor categories. *Commun. Contemp. Math.*, 10(suppl. 1):871–911, 2008.

Shihoko Ishii. Jet schemes, arc spaces and the Nash problem. *C. R. Math. Acad. Sci. Soc. R. Can.*, 29(1):1–21, 2007.

Victor G. Kac. Infinite-dimensional Lie algebras, and the Dedekind $\eta$-function. *Funkcional. Anal. i Prilozhen.*, 8(1):77–78, 1974.

Victor G. Kac. *Infinite-dimensional Lie algebras*. Cambridge University Press, Cambridge, third edition, 1990.

Victor G. Kac. Introduction to vertex algebras, Poisson vertex algebras, and integrable Hamiltonian PDE.

Victor Kac, Shi-Shyr Roan, and Minoru Wakimoto. Quantum reduction for affine superalgebras. *Comm. Math. Phys.*, 241(2-3):307–342, 2003.

Victor G. Kac and Minoru Wakimoto. Modular invariant representations of infinite-dimensional Lie algebras and superalgebras. *Proc. Nat. Acad. Sci. U.S.A.*, 85(14):4956–4960, 1988.

Victor G. Kac and Minoru Wakimoto. Classification of modular invariant representations of affine algebras. In *Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988)*, volume 7 of *Adv. Ser. Math. Phys.*, pages 138–177. World Sci. Publ., Teaneck, NJ, 1989.

Victor G. Kac and Minoru Wakimoto. Branching functions for winding subalgebras and tensor products. *Acta Appl. Math.*, 21(1-2):3–39, 1990.

Victor G. Kac and Minoru Wakimoto. On rationality of $W$-algebras. *Transform. Groups*, 13(3-4):671–713, 2008.

Masaki Kashiwara and Toshiyuki Tanisaki. Kazhdan-Lusztig conjecture for symmetrizable Kac-Moody Lie algebras. III. Positive rational case. *Asian J. Math.*, 2(4):779–832, 1998.

Masaki Kashiwara and Toshiyuki Tanisaki. Characters of irreducible modules with non-critical highest weights over affine Lie algebras. In *Representations and quantizations (Shanghai, 1998)*, pages 275–296. China High. Educ. Press, Beijing, 2000.

Kazuya Kawasetsu. $W$-algebras with non-admissible levels and the Deligne exceptional series. *Int. Math. Res. Notices*, published online.

Ellis R. Kolchin. *Differential algebra and algebraic groups*. Academic Press, New York-London, 1973. Pure and Applied Mathematics, Vol. 54.

Bertram Kostant. On Whittaker vectors and representation theory. *Invent. Math.*, 48(2):101–184, 1978.

Bertram Kostant and Shlomo Sternberg. Symplectic reduction, BRS cohomology, and infinite-dimensional Clifford algebras. *Ann. Physics*, 176(1):49–113, 1987.

Haisheng Li. Vertex algebras and vertex Poisson algebras. *Commun. Contemp. Math.*, 6(1):61–110, 2004.

Haisheng Li. Abelianizing vertex algebras. *Comm. Math. Phys.*, 259(2):391–411, 2005.

Ivan Losev. Finite-dimensional representations of $W$-algebras. *Duke Math. J.*, 159(1):99–143, 2011.

Thomas E. Lynch. *Generalized Whittaker vectors and representation theory*. PhD thesis, M.I.T., 1979.

Sergei L. Luk'yanov and Vladimir A. Fateev. Exactly soluble models of conformal quantum field theory associated with the simple Lie algebra $D_4$. *Yadernaya Fiz.*, 49(5):1491–1504, 1989.

Fedor G. Malikov, Boris L. Feigin, and Dmitry B. Fuks. Singular vectors in Verma modules over Kac-Moody algebras. *Funktional. Anal. i Prilozhen.*, 20(2):25–37, 96, 1986.

Atsushi Matsuo, Kiyokazu Nagatomo, and Akihiro Tsuchiya. Quasi-finite algebras graded by Hamiltonian and vertex operator algebras. In *Moonshine: the first quarter century and
beyond, volume 372 of *London Math. Soc. Lecture Note Ser.*, pages 282–329. Cambridge Univ. Press, Cambridge, 2010.

[MP] Robert V. Moody and Arturo Pianzola. *Lie algebras with triangular decompositions*. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons Inc., New York, 1995. A Wiley-Interscience Publication.

[Pre] Alexander Premet. Special transverse slices and their enveloping algebras. *Adv. Math.*, 170(1):1–55, 2002. With an appendix by Serge Skryabin.

[TY] Patrice Tauvel and Rupert W. T. Yu. *Lie algebras and algebraic groups*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2005.

[Wan] Weiqiang Wang. Rationality of Virasoro vertex operator algebras. *Internat. Math. Res. Notices*, (7):197–211, 1993.

[Zam] Alexander B. Zamolodchikov. Infinite extra symmetries in two-dimensional conformal quantum field theory. *Teoret. Mat. Fiz.*, 65(3):347–359, 1985.

[Zhu] Yongchang Zhu. Modular invariance of characters of vertex operator algebras. *J. Amer. Math. Soc.*, 9(1):237–302, 1996.

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502 JAPAN

*E-mail address: arakawa@kurims.kyoto-u.ac.jp*