Antiproton low-energy collisions with Ps-atoms and true muonium atoms ($\mu^+\mu^-$) 
$\Pi$ and $\Pi_\mu$ three-body formation reactions

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Abstract Three-charge-particle collisions with participation of ultra-slow antiprotons ($\bar{\Pi}$) is the subject of this work. Specifically we compute the total cross sections and corresponding thermal rates of the following three-body reactions: $\bar{\Pi} + (e^+e^-) \rightarrow \bar{\Pi} + e^-$ and $\bar{\Pi} + (\mu^+\mu^-) \rightarrow \bar{\Pi}_\mu + \mu^-$, where $e^-(\mu^-)$ is an electron (muon) and $e^+(\mu^+)$ is a positron (antimuon) respectively, $\Pi = (\bar{\Pi}e^+)$ is an antihydrogen atom and $\Pi_\mu = (\bar{\Pi}\mu^+)$ is a muonic antihydrogen atom, i.e. a bound state of $\bar{\Pi}$ and $\mu^+$. A set of two-coupled few-body Faddeev-Hahn-type (FH-type) equations is numerically solved in the framework of a modified close-coupling expansion approach.

Keywords Ultra-slow antiproton · Antihydrogen · Muonic antihydrogen · Few-body systems

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1 Introduction

Recently created ultra-low energy antiprotons are of great scientific interest because of the possible formation of slow antihydrogen atoms [1,2]. The main motivation of the antihydrogen and antimatter physics research is to check and confirm (or not confirm) certain fundamental laws and theories of modern physics. For example, one of the most important subjects in the field is to check the charge conjugation, parity, and time reversal (CPT) symmetry of quantum electrodynamics. In other words: a charged particle and its antiparticle should have equal/opposite charges, equal masses, lifetimes, and gyromagnetic ratios.
In quantum field theory the CPT theorem plays a very important and fundamental role. It states that any canonical (local, Lorentz-covariant) quantum field theory is invariant under the CPT operation. The CPT symmetry also predicts that hydrogen and antihydrogen atoms should have identical spectra. In order to test these fundamental laws of physics new experiments are in progress. It is planed to test whether \( H \) and \( \overline{H} \) have such properties. In such sensitive experiments, it would be important to have a certain quantity of \( \overline{H} \) atoms at low kinetic energies, ideally at rest: \( T \sim 0 \) K. Next, in this context it would be useful to mention muonic physics problems and related muonic-atomic few-body systems too. In this field a scientific breakthrough has recently been achieved: it was found that the size of a proton in the muonic hydrogen atom \( (p\mu^+) \) differs by \( \sim 4\% \) from the size of a proton in normal hydrogen atom \([3]\). It is clear, that it would be extremely interesting to undertake the same comparisons between antihydrogen and muonic antihydrogen atoms too. The author of \([4]\) pointed out that the muonic antihydrogen atom, \( \overline{H}_\mu \), which is a bound state of \( \overline{H} \) and an antimuon, \( \mu^+ \), could be an even better choice to check the CPT law than the usual antihydrogen atom. This is because the size of this atom is \( \sim 207 \) times smaller than the size of a normal \( \overline{H} \) atom. Therefore, as mentioned in \([4]\): the short range CPT violating interaction with an extremely heavy boson can be easily detected within the system. Further, in a relatively old work \([5]\) the authors considered the reaction of \( \mu^- \)-capture by hydrogen nuclei, i.e. the following process:

\[
\mu^- + p \to n + \nu. \quad (1)
\]

The study of the capture of negative muons by atomic nuclei can provide valuable information about the weak interaction \([6]\) and the reaction \((1)\) can be used to determine the weak interaction constants \([5]\). One of the most important characteristics of the process is the rate of the nuclear \( \mu^- \)-capture. The rate of \((1)\) significantly depends on the mutual muon and nuclear spin orientation. In \([5]\) this process was considered within the muonic molecular ion \( (p\mu^-)^+ \). Now, in our opinion, it would also be very interesting to transfer this consideration to antimatter physics, i.e. consider the antimuon \( \mu^- \)-capture by antiprotons. For example, in slow collisions between \( \overline{H} \) and \( \overline{H}_\mu \) one can form an antimuonic molecular ion \( (\overline{p}\overline{p}\mu^+)^- \):

\[
\overline{H} + \overline{H}_\mu \to (\overline{p}\overline{p}\mu^+)^- + e^+. \quad (2)
\]

In this reaction a part of the excess energy after the formation of \( (\overline{p}\overline{p}\mu^+)^- \) is taken by free \( e^+ \). In conclusion, the low energy formation of \( \overline{H}_\mu \) can be achieved with the use of the true muonium atom \( (\mu^+\mu^-) \): the smallest pure QED atom with a Bohr radius only \( \sim 512 \) fm.

These ideas appear be interesting, and therefore it would be useful to compute the formation cross sections and rates of \( \overline{H}_\mu \) at low energy collisions, for example, from \( \sim 1 \) eV down to \( \sim 10^{-5} \) eV. Thus, in this work we consider the following three-body reactions of antihydrogen \( \overline{H} \) and muonic antihydrogen \( \overline{H}_\mu \) formation:

\[
\overline{p} + (e^+e^-)_{1s} \to \overline{H} + e^-, \quad (3)
\]
\[ \overline{\text{p}} + (\mu^+ \mu^-)_{1s} \rightarrow \overline{\text{H}}_{\mu} + \mu^- . \]  

(4)

At such low energies the quantum-mechanical Coulomb few-body dynamics becomes important, especially in the case of heavy charge transfer, i.e. \( \mu^+ \). Also, it would be quite appropriate to mention that exotic atomic and antiatomic systems like a true muonium atom, \( (\mu^+ \mu^-) \), or a simple muonic hydrogen atom, \( \text{H}_{\mu} = (p^+ \mu^-) \), are always of great interest in nuclear, atomic and few-body physics. There is also another three-charge-particle reaction of interest in \( \text{H}_{\mu} \) formation:

\[ \overline{\text{p}} + \text{Mu} \rightarrow \text{H}_{\mu} + e^- . \]  

(5)

Here, Mu is the muonium atom, i.e. a bound state of a positive muon (antimuon) \( \mu^+ \) and an electron: \( \text{Mu} = (\mu^+ e^-) \). This is a very interesting and challenging computation example of a heavy charge, \( \mu^+ \), transfer reaction.

Using this perspective we develop a quantum-mechanical approach which should be reliable at low and very low collision energies, i.e. when the quantum-mechanical few-body dynamics of three Coulomb particles becomes important. The method is formulated for arbitrary masses of the particles, that is when the dynamics of lighter and heavier particles are not separated from each other. In the current work we apply a few-body approach based on a set of coupled two-component FH-type equation formalism [7,8].

### 2 Basic equations and results

Let us define the system of units to be \( e = \hbar = m_3 = 1 \) and denote antiproton \( \overline{\text{p}} \) by 1, a negative muon \( \mu^- \) by 2, and a positive muon \( \mu^+ \) by 3. Before the three-body breakup threshold two cluster asymptotic configurations are possible in the three-body system, i.e. (23)−1 and (13)−2 being determined by their own Jacobi coordinates \( \{ r_{j3}, \rho_k \} \):

\[
\begin{align*}
    r_{j3} &= r_3 - r_j, \\
    \rho_k &= (r_3 + m_jr_j)/(1 + m_j) - r_k, \quad (j \neq k = 1, 2).
\end{align*}
\]

(6)

Here \( r_\xi, \ m_\xi \) are the coordinates and the masses of the particles \( \xi = 1, 2, 3 \) respectively. This suggests a Faddeev formulation which uses only two components. In this approach the three-body wave function is represented as follows:

\[ \Psi = \psi_1(r_{23}, \rho_1) + \psi_2(r_{13}, \rho_2), \]

(7)

where each Faddeev-type component is determined by its own Jacobi coordinates. Moreover, \( \psi_1(r_{23}, \rho_1) \) is quadratically integrable over the variable \( r_{23} \), and \( \psi_2(r_{13}, \rho_2) \) over the variable \( r_{13} \). To define \( \psi \), \( (l = 1, 2) \) a set of two coupled FH-type equations can be written:

\[
\begin{align*}
    \left( E - \tilde{H}_0 - V_{23}(r_{23}) \right) \psi_1(r_{23}, \rho_1) &= \left( V_{23}(r_{23}) + V_{12}(r_{12}) \right) \psi_2(r_{13}, \rho_2), \\
    \left( E - \tilde{H}_0 - V_{13}(r_{13}) \right) \psi_2(r_{13}, \rho_2) &= \left( V_{13}(r_{13}) + V_{12}(r_{12}) \right) \psi_1(r_{23}, \rho_1).
\end{align*}
\]

(8) (9)

Here, \( \tilde{H}_0 \) is the kinetic energy operator of the three-particle system, \( V_{ij}(r_{ij}) \) are paired interaction potentials \( (i \neq j = 1, 2, 3) \), \( E \) is the total energy.
The constructed equations satisfy the Schrödinger equation exactly. For the energies below the three-body break-up threshold these equations exhibit the same advantages as the Faddeev equations, because they are formulated for energies below the three-body break-up threshold these equations exhibit the

\begin{align*}
\Psi_{\text{tot}}(r_1, r_2, r_3) &\approx \sum_{i} \int f_{n_i}^{(1)}(\rho_1) \varphi_{n_i}^{(1)}(r_1) \varphi_{n_i}^{(2)}(r_2) \varphi_{n_i}^{(3)}(r_3) \tag{10}
\end{align*}

It provides us with a set of one-dimensional integral-differential equations after the partial-wave projection. A further advantage of the Faddeev-type method is the fact that the Faddeev-components are smoother functions of the coordinates than the total wave function. Also, the Faddeev decomposition avoids overcompleteness problems, because two-body subsystems are treated in an equivalent way, and the correct asymptotes are guaranteed. After a proper angular momentum expansion one can obtain an infinite set of coupled integral-differential equations for the unknown functions \(f_{\alpha}^{(1)}(\rho_1)\) and \(f_{\alpha'}^{(2)}(\rho_2)\) [7,8]:

\begin{align*}
\left( k_n^2 + \frac{\partial^2}{\partial \rho_1^2} - \frac{\lambda(\lambda + 1)}{\rho_1^2} \right) f_{\alpha}^{(1)}(\rho_1) &= g \sum_{\alpha'} \sqrt{\frac{(2\lambda + 1)(2\lambda + 1)}{(2L + 1)}} \\
\int_0^\infty d\rho_2 f_{\alpha}^{(2)}(\rho_2) \int_0^{\pi} d\omega \sin \omega R_{\alpha\alpha'}^{(i)}(r_1, 3) (V_{\alpha'}^{(i)}(r_1, 3) + V_{\alpha'}^{(ii)}(r_1, 3)) \rho_1 \rho_2 \\
R_{\alpha\alpha'}^{(i)}(r_1, 3) \sum_{\lambda m} D_{\lambda m}^L(0, \omega, 0) C_{\lambda \alpha m}^{Lm'} Y_{\lambda m}(\nu_1, \pi) Y_{\nu m'}^{*}(\nu_2, \pi). \tag{11}
\end{align*}

For the sake of simplicity \(a \equiv (\pi \lambda^2)\) are quantum numbers of a three-body state [7] and \(L\) is the total angular momentum of the three-body system, \(g = 4\pi M_i/\gamma^3\), \(k_n^2 = \sqrt{2M_i(E - E_n^i)}\), where \(E_n^i\) is the binding energy of the

| \(E, \text{eV}\) | \(\sigma_{\text{tot}}\), \(\text{cm}^2\) | \(\sigma_{\text{tot}}^v, \text{cm}^2\) | \(\sigma_{\text{tot}}^{v, \text{cm}}, \text{cm}^3/\text{s}\) | \(\sigma_{\text{tot}}^v, \text{cm}^2\) | \(\sigma_{\text{tot}}^{v, \text{cm}}, \text{cm}^3/\text{s}\) |
|---|---|---|---|---|---|
| 0.5E-10 | 0.16E-12 | 0.67E-08 | 0.67E-08 | 0.18E-16 | 0.60E-12 |
| 1.0E-05 | 0.50E-13 | 0.67E-08 | 0.67E-08 | 0.58E-17 | 0.59E-12 |
| 1.0E-01 | 0.42E-15 | 0.56E-08 | 0.56E-08 | 0.82E-17 | 0.59E-12 |
| 1.0E-02 | 0.15E-14 | 0.63E-08 | 0.55E-08 | 0.58E-18 | 0.59E-12 |
| 1.0E-03 | 0.50E-14 | 0.66E-08 | 0.58E-08 | 0.59E-12 | 0.59E-12 |
| 1.0E-04 | 0.16E-13 | 0.67E-08 | 0.58E-08 | 0.18E-16 | 0.60E-12 |
| 1.0E-05 | 0.50E-13 | 0.67E-08 | 0.59E-08 | 0.59E-12 | 0.59E-12 |
| 1.0E-06 | 0.16E-12 | 0.67E-08 | 0.60E-08 | 0.18E-16 | 0.60E-12 |
| 1.0E-07 | 0.23E-18 | 0.62E-12 | 0.59E-12 | 0.60E-12 | 0.60E-12 |

Table 1 The total cross sections \(\sigma_{\text{tot}}\) and \(\sigma_{\text{tot}}^v\) for the reactions (3) and (4) respectively. The product of these cross sections and the corresponding center-of-mass velocities \(v_{\text{cm}}\) between \(\mu\) and \(\text{Ps}=(e^+e^-)\), i.e. \(\sigma_{\text{tot}}^v, \text{cm}^3/\text{s}\), and between \(\mu\) and the true muonium atom \(\text{Ps}_{\mu} = (\mu^+\mu^-)\), i.e. \(\sigma_{\text{tot}}^v, \text{cm}^3/\text{s}\), are presented.
Fig. 1 Low energy cross sections of $H$ (upper plot) and $\mu H$ (middle plot) formation reactions are shown. On the lower plot the cross section for the $\mu^-$ transfer from a proton to a deuteron is also included as a test. This result provides the muon transfer thermal rate close to the experimental data at low temperatures [7]. The cross sections are computed within the two-level $2 \times 1s$, four-level $2 \times (1s+2s)$, and six-level $2 \times (1s+2s+2p)$ approximations.

 subsystem ($i'3$), $M_1 = m_1(m_2 + m_3)/(m_1 + m_2 + m_3)$ and $M_2 = m_2(m_1 + m_3)/(m_1 + m_2 + m_3)$ are the reduced masses, $\gamma = 1 - m_i m_{i'}/((m_i + 1)(m_{i'} + 1))$, $D_{mn}\gamma(0, \omega, 0)$ the Wigner functions, $C^{Lm}_{\lambda\lambda'}$, the Clebsh-Gordon coefficients, $Y_{lm}$ are the spherical functions, $\omega$ is the angle between the Jacobi coordinates $\rho_i$ and $\rho_{i'}$, $\nu_i$ is the angle between the Jacobi coordinates $r_{ij}$ and $p_i$, $\nu_{i'}$ is the angle between $r_{i'j}$ and $p_{i'}$. We numerically solve the set of coupled integral-differential equations [11] together with specific boundary conditions which are appropriate
for the three-body rearrangement scattering problems (3) and (4). Also, additionally we compute the \( \mu^- \) transfer reaction from one hydrogen isotope to another heavier hydrogen isotope. Now, below we briefly discuss our computational results. All these different three-body Coulomb systems have been computed in the framework of a unique quantum-mechanical method, i.e. the FH-type equation formalism (8)-(9) and (10). The details of the method have been presented in our recent preprint [7]. The goal of these works is to carry out a quantum-mechanical calculation of the formation cross sections and corresponding thermal rates of the \( \overline{\text{H}} \) and \( \overline{\text{H}}_{\mu} \) atoms at very low collision energies, i.e. the reactions (3) and (4). The coupled integral-differential Eqs. (11) have been solved numerically for the case of the total angular momentum \( L = 0 \) within the two-level \( 2 \times (1s) \), four-level \( 2 \times (1s+2s) \), and six-level \( 2 \times (1s+2s+2p) \) close coupling approximations in Eqs. (10). The sign "\( 2 \times \)" indicates that two different sets of expansion functions are applied. To compute the charge transfer cross sections a \( K \)-matrix formalism has been used [7,8]. Table I shows our results for the cross sections \( \sigma_{tr} \) and \( \sigma_{tr_{\mu}} \) for the reactions (3) and (4) respectively. Also, Table I represents our data for the products \( \sigma_{tr_{\text{v}_{\text{cm}}}} \) and \( \sigma_{tr_{\mu_{\text{v}_{\text{cm}}}}} \), where \( \text{v}_{\text{cm}} \) is relative velocities between colliding particles. One can see, that at very low energies they take almost constant values. These results are in good agreement with the general rule of the quantum-mechanical rearrangement scattering theory: \( \sigma_{tr} \sim 1/v_0 \). where \( \sigma_{tr} \) is the transfer cross-section and \( v_0 \) is the velocity in the input channel. Now these quantities can be used for actual computation of \( \overline{\text{H}} \) and \( \overline{\text{H}}_{\mu} \) production rates [7]. Finally, Fig. 1 shows our cross sections for the antihydrogen processes (3) and (4) and one muon transfer reaction, i.e. \( \text{d} + (\text{p}_{\mu})_{1s} \to (\text{d}_{\mu})_{1s} + \text{p} \). These results are shown within different close-coupling expansion approximations in Eqs. (10). One can see that contribution of the polarization interaction \( (p\text{-wave}) \) becomes significant while the collision energy decreases. Among the considered systems in this work the biggest \( p \text{-wave} \) contribution at low energies is identified in the interesting \( \text{p} + (\mu^+\mu^-) \) collision. There may be different physical reasons for this phenomenon as, for example, the mass symmetry, inertia, and the charge asymmetry in the target. Presumably, the vacuum polarization effects(Casimir) and Casimir-Polder-type forces may also be significant in this exotic system.

References
1. Gabrielse, G., et al., (ATRAP Collaboration): Adiabatic cooling of antiprotons. Phys. Rev. Lett. 106, 073002 (2011).
2. Andresen, G.B. et al., (ALPHA Collaboration): Evaporative cooling of antiprotons to cryogenic temperatures. Phys. Rev. Lett. 105, 013003 (2010).
3. Pohl, R., et al.: The size of the proton. Nature 466, 213 (2010).
4. Nagamine, K.: Introductory muon science, Cambridge University Press, Cambridge (2003).
5. Bakalov, D.D., Faifman, M.P., Ponomarev, L.I., and Vinitsky, S.I.: \( \mu \)-Capture and ortho-para transitions in the muonic molecule \( (\text{pp}_{\mu}) \). Nucl. Phys. A 384, 302 (1982).
6. Primakoff, H.: Theory of muon capture. Rev. Mod. Phys. 31, 802 (1959).
7. Sultanov, R.A. and Guster, D.: Antihydrogen (\( \overline{\text{H}} \)) and muonic antihydrogen (\( \overline{\text{H}}_{\mu} \)) formation in low energy three-charge-particle collisions. arXiv: 1304.2434v2.
8. Sultanov, R.A. and Guster, D.: Muonic antihydrogen (\( \overline{\text{H}}_{\mu} \)) formation in low-energy three-body reactions. Few-Body Syst. 54, 1157 (2013).