Galois Structure of Zariski Cohomology for Weakly Ramified Covers of Curves

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Abstract. We compute equivariant Euler characteristics of locally free sheaves on curves, thereby generalizing several results of Kani and Nakajima. For instance, we extend Kani’s computation of the Galois module structure of the space of global meromorphic differentials which are logarithmic along the ramification locus from the tamely ramified to the weakly ramified case.

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Introduction

Let $X$ be a smooth projective curve defined over an algebraically closed field $k$ of characteristic $p$, and let $G \subseteq \text{Aut}(X/k)$ be a finite subgroup of automorphisms of $X$. The goal of this paper is to compute the Galois module structure of the Zariski cohomology groups of $X$ with values in an equivariant locally free sheaf $\mathcal{E}$ on $X$ such as the ideal sheaf of a $G$-stable finite subset of points on $X$ or the sheaf of differentials on $X$.

Our first result, see Theorem (3.1), is an explicit formula for the equivariant Euler characteristic

$$
\chi(G, X, \mathcal{E}) := [H^0(X, \mathcal{E})] - [H^1(X, \mathcal{E})]
$$

considered as an element of the Grothendieck group $K_0(G, k)$ of all finitely generated modules over the group ring $k[G]$. It describes $\chi(G, X, \mathcal{E})$ in terms of the rank and degree of $\mathcal{E}$, the genus of the quotient curve $Y := X/G$, the order of $G$ and of the higher
ramification groups $G_{P,s}, P \in X, s \geq 0$, and the representations of the decomposition group $G_P$ on the fibre $\mathcal{E}(P)$ and on the cotangent space $\mathfrak{m}_P/\mathfrak{m}_P^2$ for $P \in X$.

In the case the cover $\pi : X \to Y$ is tamely ramified, this formula becomes Theorem 1.1 in [Ko2], see Remark (3.2). In particular, if the order of $G$ is not divisible by $p$, it implies the main result of the paper [EL] by Ellingsrud and Lønsted, which in turn generalizes the classical Chevalley-Weil formula, see [Ko2]. While Theorem (3.1) of the present paper has the advantage of being available in general, i.e. without any assumption on the ramification of $\pi$ or on the group $G$, it has the disadvantage of computing the equivariant Euler characteristic only in the “weak” Grothendieck group $K_0(G,k)$, i.e. it yields only composition factors. In particular, if $p$ divides the order of $G$, we need a further input to describe the actual $k[G]$-isomorphism class of the cohomology groups $H^0(X,\mathcal{E})$ and $H^1(X,\mathcal{E})$, even if one of them vanishes.

Such an input is provided by Theorem (1.1) parts of which may already be found in the literature, see Remark (1.2). It gives a criterion for any fractional ideal in a local Galois extension to have a normal basis element. In particular, it yields the following fact, see the proof of Theorem (2.1)(a): Let $\pi$ be weakly ramified and let $D = \sum_{P \in X} n_P[P]$ be an equivariant divisor on $X$ such that $n_P \equiv -1 \mod e_P$, for all $P \in X$ (where $e_P$ denotes the order of the (first) ramification group $G_{P,1}$); then the direct image $\pi_* (\mathcal{O}_X(D))$ of the associated equivariant invertible sheaf $\mathcal{O}_X(D)$ is a locally free $\mathcal{O}_Y[G]$-module. (As in Erez’ paper [Er1], the notion weakly ramified means that both tame and the simplest kind of wild ramification are allowed, more precisely, that all second ramification groups $G_{P,2}$, $P \in X$, are trivial.) Using a standard argument in geometric Galois module theory (see Chinburg’s paper [Ch] for the version most suitable for our purposes), we obtain from this fact that the equivariant Euler characteristic $\chi(G,X,\mathcal{O}_X(D))$ lies in the image of the (injective) Cartan homomorphism $K_0(k[G]) \to K_0(G,k)$ and, moreover, that $H^0(X,\mathcal{O}_X(D))$ and $H^1(X,\mathcal{O}_X(D))$ are projective $k[G]$-modules, if one of them vanishes, see Theorem (2.1)(a).

This observation applied to the divisor $E := \sum_{P \in X} (e_P - 1)[P]$ together with the above-mentioned “weak” formula for $\chi(G,X,\mathcal{E})$ applied to $\mathcal{E} = \mathcal{O}_X(E)$ allows us to construct a certain canonical projective $k[G]$-module $N$ (depending only on the action of $G$ on $X$) which embodies a global relation between the local data $\mathfrak{m}_P/\mathfrak{m}_P^2$, $P \in X$, where $\mathfrak{m}_P/\mathfrak{m}_P^2$ means the cotangent space of $X$ at $P$ together with the obvious action of the decomposition group $G_P$, see Theorem (4.3). Moreover, using our “weak” formula again, we express $\chi(G,X,\mathcal{O}_X(D))$ as an integral linear combination of classes of explicit projective $k[G]$-modules one of which is $N$, see Theorem (4.5). These theorems generalize results of Kani and Nakajima from the tamely ramified to the weakly ramified case, see Remarks (4.4) and (4.6). Our approach to these theorems (described above) generalizes the one used in [Ko2] (a special case of which may already be found in Borne’s thesis [Bo1]), but it is quite different from the ones used by Kani and Nakajima.

Finally we give the following two applications of these theorems. Firstly, we compute the $k[G]$-module structure of the first cohomology group $H^1(X,\mathcal{I}(S))$ of $X$ with values in the ideal sheaf $\mathcal{I}(S)$ of any $G$-stable finite subset $S$ of $X$ which contains all wildly
ramified points, see Corollary (4.7). We refer the reader to Pink’s paper [Pi] for the significance of this ideal sheaf in his proof of a \(p\)-adic Grothendieck-Ogg-Shafaravic formula. Secondly, if \(S\) contains not only all wildly ramified but all ramified points, we prove that the direct sum of the \(k[G]\)-module \(N\) with the space \(H^0(X, \Omega_X(S))\) of global meromorphic differentials, which are logarithmic along \(S\), is a free \(k[G]\)-module, see Corollary (4.8). This result generalizes Theorem 2 in Kani’s paper [Ka] again from the tamely ramified to the weakly ramified case. In the tamely ramified case, Kani furthermore deduces the \(k[G]\)-isomorphism class of the space \(H^0(X, \Omega_X)\) of all global holomorphic differentials from this result, see Theorem 3 in [Ka]. It would be interesting to know whether this can also be done in the weakly ramified case.

At this point we moreover mention that the conditions \(\pi\) is weakly ramified and \(n_P \equiv -1 \mod e_P\) for all \(P \in X\) are not only sufficient, but also necessary for \(\pi_*(\mathcal{O}_X(D))\) being locally free over \(\mathcal{O}_Y[G]\), see Theorem (1.1), and, if the degree of \(D\) is sufficiently large, also for \(H^0(X, \mathcal{O}_X(D))\) being a projective \(k[G]\)-module, see Theorem (2.1)(b). Without the assumption on the degree of \(D\), it might be true that these conditions are necessary for \(R\Gamma(X, \mathcal{O}_X(D))\) being quasi-isomorphic to a perfect complex of \(k[G]\)-modules (see Question (2.6)).

The reader may also wish to consult the paper [Vi] by Vinatier for the current state of the art in the Galois module theory of weakly ramified extensions of number fields.

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§1 The Normal Basis Theorem for Fractional Ideals in Local Galois Extensions

In this section we explicitly describe those fractional ideals in any finite Galois extension of local fields for which the normal basis theorem holds, i.e., which are free (of rank 1) over the corresponding group ring.

Let \(L/K\) be a finite Galois extension of local fields with Galois group \(G\) and (positive) residue class field characteristic \(p\). The corresponding extensions of discrete valuation rings, of maximal ideals and of (perfect) residue class fields are denoted by \(\mathcal{O}_L/\mathcal{O}_K\), \(m_L/m_K\) and \(\lambda/\kappa\), respectively. For any \(s \geq -1\), let \(G_s\) denote the \(s\)th ramification group of the extension \(L/K\). We recall (see Chapitre IV in [Sc]): The ramification groups form a chain

\[
G = G_{-1} \supseteq G_0 \supseteq G_1 \supseteq G_2 \supseteq \ldots
\]

of normal subgroups of \(G\), \(G_0 = \ker(G \to \Gal(\lambda/\kappa))\) is the inertia subgroup, \(G_0/G_1\) is a cyclic group of order prime to \(p\), \(G_s/G_{s+1}\) is an abelian group of exponent \(p\) for \(s \geq 1\), and \(G_s\) is the trivial group for \(s\) sufficiently big. The extension \(L/K\) is called
weakly ramified (tamely ramified, unramified), iff \( G_s = 0 \) for \( s = 2 \) (\( s = 1, s = 0 \), respectively).

The following theorem generalizes the well-known result of E. Noether that \( L/K \) is tamely ramified, if and only if \( \mathcal{O}_L \) is free (of rank 1) over the group ring \( \mathcal{O}_K[G] \). It implies in particular that \( L/K \) is weakly ramified, if and only if \( m_L \) is free over \( \mathcal{O}_K[G] \). Furthermore it describes all fractional ideals in \( L \) which are \( \mathcal{O}_K[G] \)-free. For example, if \( L/K \) is not weakly ramified, then there does not exist any \( \mathcal{O}_K[G] \)-free fractional ideal in \( L \) at all.

(1.1) Theorem. Let \( b \in \mathbb{Z} \). Then the fractional ideal \( m_L^b \) of \( L \) is free over \( \mathcal{O}_K[G] \), if and only if \( L/K \) is weakly ramified and \( b \equiv 1 \mod |G_1| \).

(1.2) Remark.
(a) The only-if-part of Theorem (1.1) follows from Theorem 3 and the corollary of Proposition 2 in Ullom’s paper [Ul3]. The same case, i.e. \( |G_1| = 1 \), and the case \( b = 1 \), \( G = G_1 \) of the if-part of Theorem (1.1) are proved in Theorem 1 and Theorem 2 in his paper. Unfortunately, he does not state the general case of the if-part of Theorem (1.1) which is essential for this paper. Though it certainly can be proved with the methods he has developed in his papers [Ul1], [Ul2] and [Ul3], we here give a coherent and self-contained proof of Theorem (1.1) for the reader’s convenience.
(b) In the geometric case, Pink has given a “global proof” for the fact that \( L/K \) is weakly ramified, if and only if \( m_L \) is \( \mathcal{O}_K[G] \)-free (see Corollary 3.6 in [Pi]); to be precise, the if-direction is proved there only under the additional assumption \( G = G_1 \). Note that, in his terminology, weakly ramified means of type 2.
(c) Let \( \text{ord}(G) \) be odd. Then Theorem (1.1) also implies Erez’ theorem that \( L/K \) is weakly ramified, if and only if the so-called square root of the inverse different is \( \mathcal{O}_K[G] \)-free (see Theorem 1 in [Er1]).

We will use the following propositions and lemma in the proof of Theorem (1.1).

(1.3) Proposition. Let \( I \) be any fractional ideal of \( L \). Then \( I \) is free over \( \mathcal{O}_K[G] \), if and only if it is projective over \( \mathcal{O}_K[G] \).

Proof. This follows from a theorem of Swan (see Corollary 6.4 on p. 567 in [Sw]). □

We recall that a \( \mathbb{Z}[G] \)-module \( M \) is called cohomologically trivial, iff the Tate cohomology groups \( \hat{H}^i(U, M) \), \( i \in \mathbb{Z} \), vanish for all subgroups \( U \) of \( G \).

(1.4) Proposition. Let \( M \) be any \( \mathcal{O}_K[G] \)-module. Then \( M \) is projective over \( \mathcal{O}_K[G] \), if and only if \( M \) is projective over \( \mathcal{O}_K \) and cohomologically trivial.

Proof. If the ring of coefficients \( \mathcal{O}_K \) is replaced by \( \mathbb{Z} \), this is Théorème 7 on p. 151 in [Sc]. The same proof applies, if the ring of coefficients is any Dedekind domain. See also Proposition 4.1(a) on p. 457 in [Ch]. □

As usual, we denote the multiplication with the norm element \( \sum_{\sigma \in G}[\sigma] \in \mathbb{Z}[G] \) by \( \text{Tr}_{L/K} \), the different of \( L/K \) by \( D_{L/K} \), and the ramification index of \( L/K \) by \( e := e_{L/K} \). Furthermore, for any \( r \in \mathbb{R} \), the standard notation \([r]\) means the greatest integer less than or equal to \( r \).

(1.5) Lemma. Let \( b \in \mathbb{Z} \). Then we have:
(a) $(m^b_L)^G = m^a_K$, where $a = 1 + \left[ \frac{b-1}{e} \right]$.  
(b) $\text{Tr}_{L/K}(m^b_L) = m^{a'}_L$, where $a' = \left[ \frac{\text{ord}(D_{L/K}) + b}{e} \right]$.  

Proof. Let $a \in \mathbb{Z}$ and $a' \in \mathbb{Z}$ be defined by $(m^b_L)^G = m^a_K$ and $\text{Tr}_{L/K}(m^b_L) = m^{a'}_K$, respectively. The obvious relation  
$$m^{ae}_L = m^{a}_K \mathcal{O}_L \subseteq m^{b}_L \subset m^{a-1}_K \mathcal{O}_L = m^{(a-1)e}_L$$  
implies $a \geq \frac{b}{e} > a - 1$, hence $a = 1 + \left[ \frac{b-1}{e} \right]$. This proves assertion (a). Furthermore we have: $\mathcal{D}^{-1}_{L/K} = \{x \in L : \text{Tr}_{L/K}(x \cdot \mathcal{O}_L) \subseteq \mathcal{O}_K \}$. Hence:  
$$m^{-ea'+b} = m^{-a'}_K m^{b}_L \subseteq \mathcal{D}^{-1}_{L/K} \subseteq m^{-a'+1}_K m^{b}_L = m^{-e(a'+1)+b}_L.$$  
Thus $-ea' + b \geq -\text{ord}(\mathcal{D}_{L/K}) > -e(a' + 1) + b$, i.e., $a' = \left[ \frac{\text{ord}(D_{L/K}) + b}{e} \right]$, as was to be shown.  

Proof of Theorem (1.1). We first prove the if-part. By Propositions (1.3) and (1.4), it suffices to show that the Tate cohomology groups $\hat{H}^i(U, \mathcal{m}^b_L)$, $i \in \mathbb{Z}$, vanish for any subgroup $U$ of $G$. Since the extension $L/L^U$ is again weakly ramified and since the first ramification group $U_1$ of the extension $L/L^U$ is contained in $G_1$ (see Proposition 2 on p. 70 in [Se]), it suffices to consider the case $U = G$. The usual spectral sequence argument yields the following fact for any normal subgroup $N$ of $G$: If $\hat{H}^i(N, m^b_L) = 0$ and $\hat{H}^i(G/N, (m^b_L)^N) = 0$ for all $i \in \mathbb{Z}$, then also $\hat{H}^i(G, m^b_L) = 0$ for all $i \in \mathbb{Z}$. Applying this fact to the filtration $G \supseteq G_0 \supseteq G_1 \supseteq \{1\}$ of $G$, we are reduced to show that $\hat{H}^i(G_1, m^b_L) = 0$, that $\hat{H}^i(G_0/G_1, I) = 0$ for any fractional ideal $I$ in $L^{G_1}$ and that $\hat{H}^i(G/G_0, J) = 0$ for any fractional ideal $J$ in $L^{G_0}$ (for all $i \in \mathbb{Z}$). Applying this fact again to a filtration of the abelian group $G_1$ (of exponent $p$) whose successive quotients are cyclic groups of order $p$, we see that it finally suffices to show that $\hat{H}^i(G, m^b_L) = 0$ for all $i \in \mathbb{Z}$ and $b \in \mathbb{Z}$ with $b \equiv 1 \mod |G_1|$ in the following three cases (note that, for any subgroup $U$ of $G_1$ of order $p$, we have $(m^b_L)^U = m^b_L U$ with $a \equiv 1 \mod |G_1/U|$ by Lemma (1.5)(a)):  
(i) $G_1$ is cyclic of order $p$ and $G = G_1$, i.e., $L/K$ is totally wildly ramified of order $p$.  
(ii) $G_1 = \{1\}$ and $G = G_0$, i.e., $L/K$ is totally tamely ramified.  
(iii) $G_0 = 1$, i.e., $L/K$ is unramified.  

We first consider the case (i). Since $G$ is cyclic, it suffices to show $\hat{H}^i(G, m^b_L) = 0$ for $i = 0$ and $i = 1$. Using Hilbert’s formula for the order of the different (see Proposition 4 on p. 72 in [Se]) and the congruence $b \equiv 1 \mod p$, we obtain:  
$$\left\lfloor \frac{\text{ord}(D_{L/K}) + b}{p} \right\rfloor = \left\lfloor \frac{2(p-1) + b}{p} \right\rfloor = 1 + \left\lfloor \frac{b-1}{p} \right\rfloor.$$  
Hence, by Lemma (1.5), we have $\text{Tr}_{L/K}(m^b_L) = (m^b_L)^G$, i.e., $\hat{H}^0(G, m^b_L) = 0$. To show $\hat{H}^1(G, m^b_L) = 0$, it suffices to show that the Herbrand difference  
$$h(m^b_L) := \text{length}_{\mathcal{O}_K}(\hat{H}^0(G, m^b_L)) - \text{length}_{\mathcal{O}_K}(\hat{H}^1(G, m^b_L))$$  
vanishes. Since $m^b_L \otimes_{\mathcal{O}_K} K \cong L$ is $K[G]$-free of rank 1, we can find a free $\mathcal{O}_K[G]$-submodule $M$ of $m^b_L$ such that $m^b_L/M$ is of finite length. Now $h(m^b_L) = 0$ follows.
from the well-known (and easy) facts that the Herbrand difference vanishes for any free \(\mathcal{O}_K[G]\)-module and for any \(\mathcal{O}_K[G]\)-module of finite length, and that the Herbrand difference is additive on short exact sequences.

In the cases (ii) and (iii), we have to show that \(\hat{H}^i(G, I) = 0\) (for all \(i \in \mathbb{Z}\)) for any arbitrary fractional \(I\) of \(L\). Let \(\pi_K\) be a prime element in \(\mathcal{O}_K\). Then we have \(\hat{H}^i(G, I/\pi_K I) = 0\) for all \(i\). Indeed, in the case (ii) this is a consequence of Corollaire 1 on p. 138 in [Se], since \(I/\pi_K I\) is annihilated by a power of \(p\) and the order of \(G\) is prime to \(p\); in the case (iii) it is a consequence of the classical normal basis theorem and of Proposition (1.4), since \(I/\pi_K I\) is isomorphic to \(\lambda\) as a \(\kappa[G]\)-module. The long exact sequence associated with the short exact sequence of \(G\)-modules

\[
0 \rightarrow I \rightarrow I/\pi_K I \rightarrow 0
\]

now shows that \(\pi_K \hat{H}^i(G, I) = \hat{H}^i(G, I)\) for all \(i \in \mathbb{Z}\). Finally, Nakayama’s Lemma implies that \(\hat{H}^i(G, I) = 0\) for all \(i \in \mathbb{Z}\). This completes the proof of the if-direction in Theorem (1.1).

We now prove the only-if-direction. So, let \(b \in \mathbb{Z}\) such that \(m^b_L\) is \(\mathcal{O}_K[G]\)-free. We first show that \(L/K\) is weakly ramified. Let \(s\) be the greatest integer such that \(G_s \neq \{1\}\). We may assume that \(s \geq 1\). Then there is a cyclic subgroup \(U\) of \(G_s\) of order \(p\). By Proposition (1.4) we have \(\hat{H}^0(U, m^b_L) = 0\), i.e., \(\text{Tr} \, L/L^{U}(m^b_L) = (m^b_L)^U\). Furthermore, we have \(\text{ord}(\mathcal{D}_{L/L^U}) = (s+1)(p-1)\) by Proposition 2 on p. 70 and Proposition 4 on p. 72 in [Se]. Now Lemma (1.5) applied to the extension \(L/L^U\) implies that

\[
s + 1 + \left[ \frac{b - 1 - s}{p} \right] = \left[ \frac{(s+1)(p-1) + b}{p} \right] = 1 + \left[ \frac{b - 1}{p} \right].
\]

Hence \(s\) must be equal to 1, i.e., \(L/K\) is weakly ramified. It remains to show that \(b \equiv 1 \mod |G_1|\). By Proposition (1.4) we have \(\hat{H}^0(G_1, m^b_L) = 0\), i.e., \(\text{Tr} \, L/L^{G_1}(m^b_L) = (m^b_L)^{G_1}\). Furthermore, we have \(\text{ord}(\mathcal{D}_{L/L^{G_1}}) = 2 \cdot (|G_1| - 1)\) by Proposition 4 on p. 72 in [Se]. Now, Lemma (1.5) applied to the extension \(L/L^{G_1}\) implies:

\[
2 + \left[ \frac{b - 2}{|G_1|} \right] = 1 + \left[ \frac{b - 1}{|G_1|} \right].
\]

Hence \(b \equiv 1 \mod |G_1|\), as desired. Thus, the proof of Theorem (1.1) is complete. \(\square\)

§2 Projectivity of Zariski Cohomology

Let \(k\) be an algebraically closed field of characteristic \(p > 0\), \(X\) a connected smooth projective curve over \(k\) and \(G\) a finite subgroup of the automorphism group \(\text{Aut}(X/k)\) of order \(n\).

In this section we give sufficient resp. necessary conditions under which the Zariski cohomology groups of \(X\) with values in an equivariant invertible \(\mathcal{O}_X\)-module are projective over the group ring \(k[G]\).
Let $\pi : X \to Y := X/G$ denote the canonical projection, and let $g_X$ resp. $g_Y$ denote the genus of $X$ resp. $Y$. Furthermore, for any $P \in X$, the decomposition group $\{ \sigma \in G : \sigma(P) = P \}$ is denoted by $G_P$, the ramification index of $\pi$ at the place $P$ by $e_P$, the higher ramification groups (see Chapitre IV in [Se]) by $G_{P,s}$, $s \geq 0$, the wild part of the ramification index, i.e. $|G_{P,1}|$, by $e_P^w$ and the tame part of the ramification index, i.e. $|G_P/G_{P,1}|$, by $e_P^t$. We say that $\pi$ is weakly ramified, iff $G_{P,s}$ is trivial for $s \geq 2$ and all $P \in X$.

We denote the Grothendieck group of all finitely generated $k[G]$-modules (resp., of all finitely generated projective $k[G]$-modules) by $K_0(G,k)$ (resp., by $K_0(k[G])$). We recall from classical representation theory that the set of isomorphism classes of irreducible $k[G]$-modules (resp., of indecomposable projective $k[G]$-modules) forms a basis of $K_0(G,k)$ (resp., of $K_0(k[G])$) and that the Cartan homomorphism $K_0(k[G]) \to K_0(G,k)$ is injective.

We recall that a locally free $G$-module on $X$ is a locally free $\mathcal{O}_X$-module $\mathcal{E}$ together with $\mathcal{O}_X$-isomorphisms $g^*(\mathcal{E}) \to \mathcal{E}$, $g \in G$, which satisfy the usual composition rules. For instance, if $D = \sum_{P \in X} n_P [P]$ is an equivariant divisor on $X$ (i.e., $n_{\sigma(P)} = n_P$ for all $\sigma \in G$ and $P \in X$), then the $\mathcal{O}_X$-module $\mathcal{O}_X(D)$ is a locally free $G$-module on $X$ of rank 1. The Zariski cohomology groups $H^i(X, \mathcal{E})$, $i \geq 0$, are then $k$-representations of $G$ in the obvious way. Let

$$\chi(G, X, \mathcal{E}) := [H^0(X, \mathcal{E})] - [H^1(X, \mathcal{E})] \in K_0(G,k)$$

denote the equivariant Euler characteristic of $X$ with values in $\mathcal{E}$.

(2.1) Theorem. Let $D = \sum_{P \in X} n_P [P]$ be an equivariant divisor on $X$.

(a) Let $\pi$ be weakly ramified and $n_P \equiv -1 \mod e_P^w$ for all $P \in X$; then there exists a bounded complex $L^*$ of finitely generated projective $k[G]$-modules such that the $k[G]$-module $H^i(X, \mathcal{O}_X(D))$ is isomorphic to the $i$th cohomology module $H^i(L^*)$ for all $i \in \mathbb{Z}$; in particular we have:

(i) The equivariant Euler characteristic $\chi(G, X, \mathcal{O}_X(D)) \in K_0(G,k)$ lies in the image of the Cartan homomorphism $K_0(k[G]) \to K_0(G,k)$.

(ii) If one of the two cohomology groups $H^i(X, \mathcal{O}_X(D))$, $i = 0,1$, vanishes, then the other one is a projective $k[G]$-module.

(b) Let $\deg(D) > 2g_X - 2$. If the $k[G]$-module $H^0(X, \mathcal{O}_X(D))$ is projective, then $\pi$ is weakly ramified and $n_P \equiv -1 \mod e_P^w$ for all $P \in X$.

Proof.

(a) Theorem (1.1) implies that, for any $P \in X$, the $\mathcal{O}_{Y,\pi(P)}[G_P]$-module $\mathcal{O}_X(D)_P = m_P^{-pP}$ is free after completion. From Corollary (76.9) on p. 533 in [CR] we obtain that this is even true without completion. Hence the direct image $\pi_* (\mathcal{O}_X(D))$ is a locally free $\mathcal{O}_Y[G]$-module. Furthermore we have $H^i(X, \mathcal{O}_X(D)) = H^i(Y, \pi_*(\mathcal{O}_X(D)))$ for all $i \in \mathbb{Z}$. Now Theorem 1.1 on p. 447 and Proposition 4.1(a) on p. 457 in Chinburg’s paper [CR] imply the first assertion of Theorem (2.1)(a). Statement (i) is an immediate consequence of this assertion, and statement (ii) can be derived from it as in the proof of Theorem 2 in Nakajima’s paper [Na1].

(b) We first prove this in the case that $G$ is cyclic of order $p$. We fix a point $P \in X$.
Let $N$ denote the greatest integer such that $G_{P,N}$ is not trivial. We may assume that $N \geq 1$. By definition of the higher ramification groups we have $N + 1 = \text{ord}_P(\sigma(x) - x)$ where $\sigma \in G \setminus \{1\}$ and $x$ is any prime element of the local ring $\mathcal{O}_{X,P}$. From Theorem 1 on p. 86 in [Na3] we obtain that the non-negative integers

$$m_j := \frac{N}{p} + \left\langle \frac{n_p - jN}{p} \right\rangle - \left\langle \frac{n_p - (j-1)N}{p} \right\rangle, \quad j = 1, \ldots, p - 1,$$

are zero; here, for a rational number $a$, $\langle a \rangle$ denotes the fractional part of $a$, i.e., $0 \leq \langle a \rangle < 1$ and $a - \langle a \rangle$ is an integer. Since $N$ is not divisible by $p$ (see Lemma 1 on p. 87 in [Na3]), there is a solution $j_0 \in \{0, \ldots, p - 1\}$ of the congruence $n_p \equiv (j_0 - 1)N \mod p$. If $j_0 \neq 0$, then one of the integers $m_j$, $j = 1, \ldots, p - 1$, would be positive, namely $m_{j_0} = \frac{N}{p} + \left\langle \frac{n_p - j_0N}{p} \right\rangle$. Hence $j_0 = 0$, i.e., $N \equiv -n_p \mod p$; therefore

$$m_j = \frac{N}{p} + \left\langle \frac{-jN}{p} - \frac{N}{p} \right\rangle - \left\langle \frac{-jN}{p} \right\rangle \quad \text{for all } j = 1, \ldots, p - 1.$$ 

Let now $j_1 \in \{1, \ldots, p - 1\}$ be a solution of the congruence $-j_1N \equiv 1 \mod p$. If $N \not\equiv 1 \mod p$, then

$$m_{j_1} = \frac{N}{p} + \left\langle \frac{1 - \frac{N}{j_1}}{p} - 1 \right\rangle = \frac{N}{p} + 1 - \left\langle \frac{N}{p} \right\rangle$$

would be positive. Hence $N \equiv 1 \mod p$; therefore

$$m_j = \frac{N}{p} + \left\langle \frac{-j - 1}{p} \right\rangle - \left\langle \frac{-j}{p} \right\rangle = \frac{N}{p} + \frac{p - j - 1}{p} - \frac{p - j}{p} = \frac{N - 1}{p}$$

for all $j = 1, \ldots, p - 1$. Hence we have $N = 1$ and $n_P \equiv -1 \mod p$, as was to be shown.

We now consider the general case. We fix a point $P \in X$ with $e_P^w \neq 1$. (If such a point does not exist, we are done.) Let $N$ denote the greatest integer such that $G_{P,N}$ is not trivial. We choose a cyclic subgroup $H$ of $G_{P,N}$ of order $p$. Let $\eta : X \to Z := X/H$ denote the corresponding cover. Since $H^0(X, \mathcal{O}_X(D))$ is also projective as a $k[H]$-module and since, for any $s \geq 0$, the intersection $G_{P,s} \cap H$ is the $s$th ramification group of the cover $\eta$ at $P$ (see Proposition 2 on p. 70 in [Se]), we obtain from the case considered above that $N = 1$ and that $n_{P'} \equiv -1 \mod f_{P'}$ for all $P' \in X$ (where $f_{P'}$ denotes the ramification index of $\eta$ at $P'$). It remains to show that $n_P \equiv -1 \mod e_P^w$. We prove this by induction on $n := \text{ord}(G)$. The case $n = 1$ is trivial. Let $g_Z$ denote the genus of $Z$, and let the divisor $E$ on $Z$ be defined by the equality $\mathcal{O}_Z(E) = \eta^*(\mathcal{O}_X(D))$ of subsheaves of the constant sheaf $K(Z)$, the function field of $Z$. Then by Lemma (1.5)(a), the multiplicity of $E$ at any point $Q \in Y$ is $- \left(1 + \left\lceil \frac{n_q - 1}{f_Q} \right\rceil \right) = -1 + \frac{n_Q + 1}{f_Q}$ where $\tilde{Q} \in Y$ is any point in the fibre $\eta^{-1}(Q)$. Thus we have:

$$p \cdot \text{deg}(E) = p \cdot \sum_{Q \in Z} \left( -1 + \frac{n_{\tilde{Q}} + 1}{f_{\tilde{Q}}} \right) =$$

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\[\sum_{P' \in X} (-f_{P'} + n_{P'} + 1) = \deg(D) - \sum_{P' \in X} (f_{P'} - 1) \geq 2(g_X - 1) - \sum_{P' \in X} 2(f_{P'} - 1) = p \cdot (2g_Z - 2)\]

by Hurwitz' formula (see Corollary 2 on p. 301 in [Ha]) and Hilbert's formula (see Proposition 4 on p. 72 in [Se]). Furthermore \(H^0(Z, \mathcal{O}_Z(E)) = H^0(X, \mathcal{O}_X(D))^H\) is projective as a \(k[G_{P,1}/H]\)-module. Now the induction hypothesis applied to the Galois cover \(Z \to X/G_{P,1}\) (with Galois group \(G_{P,1}/H\)) and to the divisor \(E\) on \(Z\) implies that \(\sum_{P_e \in \Sigma} e^w_{P_e} \equiv -1 \mod e^w_P\), hence \(n_P \equiv -1 \mod e^w_P\), as was to be shown. This completes the proof of Theorem (2.1).

\[(2.2)\text{ Corollary.}\] The following three assertions are equivalent:
(a) The cover \(\pi\) is tamely ramified.
(b) For every equivariant divisor \(D\) on \(X\) satisfying \(H^1(X, \mathcal{O}_X(D)) = 0\), the \(k[G]\)-module \(H^0(X, \mathcal{O}_X(D))\) is projective.
(c) There exists a divisor \(D\) on \(X\) of the form \(D = \pi^*(E)\), \(E\) any divisor on \(Y\), with \(\deg(D) > 2g_X - 2\) such that the \(k[G]\)-module \(H^0(X, \mathcal{O}_X(D))\) is projective.

\[\text{Proof.}\] This immediately follows from Theorem (2.1), since, for any divisor \(D = \sum_{P \in X} n_P[P]\) on \(X\) of the form \(D = \pi^*(E)\), we have \(n_P \equiv 0 \mod e_P\) for all \(P \in X\).

\[(2.3)\text{ Corollary.}\] Let \(S\) be any non-empty \(G\)-stable set of closed points on \(X\) containing all ramified points, and let \(\Omega_X(S) := \Omega_X \otimes \mathcal{O}_X(\sum_{P \in S}[P])\) denote the sheaf of meromorphic differentials on \(X\) which are logarithmic along \(S\). Then, \(\pi\) is weakly ramified, if and only if the \(k[G]\)-module \(H^0(X, \Omega_X(S))\) is projective.

\[\text{Proof.}\] By Théorème 2.34 on p. 44 in [Bo1], there is an equivariant divisor \(K_X = \sum_{P \in X} n'_P[P]\) on \(X\) such that \(\mathcal{O}_X(K_X) \cong \Omega_X\). Then the divisor \(D = K_X + \sum_{P \in S}[P]\) satisfies \(\deg(D) > 2g_X - 2\) (by Example 1.3.3 on p. 296 in [Ha]). Hence, Theorem (2.1)(b) implies the if-part of Corollary (2.3). We now prove the only-if-part. We have a natural short exact sequence

\[0 \to \pi^*(\Omega_Y) \to \Omega_X \to \Omega_{X/Y} \to 0\]

of coherent \(G\)-modules on \(X\). Let the divisor \(K_Y = \sum_{Q \in Y} r_Q[Q]\) on \(Y\) be defined by the equality \(\pi^*(\Omega_Y) = \mathcal{O}_X(\pi^*(K_Y))\) of subsheaves of \(\mathcal{O}_X(K_X)\). If \(\pi\) is weakly ramified, Hilbert’s formula (see Proposition 4 on p. 72 in [Se]) then implies that \(n'_P - e_P r_{\pi(P)} = (e_P - 1) + (e^w_P - 1)\) for all \(P \in X\). In particular, the divisor \(D\) satisfies the condition of Theorem (2.1)(a), and statement (ii) of Theorem (2.1)(a) finally implies the only-if-part of Corollary (2.3).

\[(2.4)\text{ Remark.}\]
(a) Corollary (2.2) is Theorem 2 in [Na3].
(b) Let \(S\) and \(\Omega_X(S)\) be as in Corollary (2.3). Kani states in Theorem 1 of [Ka] that \(H^0(X, \Omega_X(S))\) is a projective \(k[G]\)-module, if and only if \(\pi\) is tamely ramified.
Since there exist weakly ramified covers which are not tamely ramified (see the case 
\( r = 1 \) in Example (2.5) below), the only-if-direction of Corollary (2.3) contradicts 
the only-if-direction of Kani’s result which seems to be wrong. (More precisely, the 
final displayed equation and the computation of \( m_y \) for \( y \in Y_{\text{ram}} \) in the proof of the 
only-if-part of that theorem seem to be wrong.)

The following example (taken from Hasse’s paper \([\text{Has}]\)) shows that every theoretically 
possible “type” of ramification for Galois covers of smooth projective curves can occur 
even under the rather restrictive conditions that the base curve is the projective line, 
that the Galois group is cyclic of order \( p \) and that only one point is ramified.

\((2.5)\) Example. Let \( r \in \mathbb{N} \) such that \( p \) does not divide \( r \). Let \( k(x, y) \) be the cyclic field 
extension of the rational function field \( k(x) \) of degree \( p \) given by the Artin-Schreier 
equation \( y^p - y = x^r \). Let \( \pi : C \to \mathbb{P}^1_k \) denote the corresponding cover of nonsingular 
curves. Then \( \pi \) is unramified precisely over \( \mathbb{A}^1_k \subset \mathbb{P}^1_k \), and, at the unique point \( P \in C \) 
lying over \( \infty \in \mathbb{P}^1_k \), the greatest integer \( N \) such that \( G_{P, N} \) is not trivial is equal to \( r \). 
Furthermore, the genus of \( C \) is equal to \( \frac{(r-1)(p-1)}{2} \).

\textit{Proof (extracted from \([\text{Has}]\)).} The Galois group \( G = \text{Gal}(k(x, y)/k(x)) \) is generated 
by the automorphism \( \sigma \) given by \( \sigma(y) = y + 1 \). Thus, clearly, \( \pi \) is unramified over \( \mathbb{A}^1_k \). 
Let \( P \in C \) be any point with \( \pi(P) = \infty \). Since \( x^{-1} \) is a prime element at \( \infty \in \mathbb{P}^1_k \), 
the equality \( y^p - y = x^r \) implies that \( \text{ord}_P(y) < 0 \) and, more precisely, that 
\[
\text{ord}_P(y) = \frac{\text{ord}_P(y^p)}{p} = \frac{\text{ord}_P(y^p - y)}{p} = \frac{\text{ord}_P(x^r)}{p} = -r \cdot \text{ord}_P(x^{-1}).
\]

Hence, the cover \( \pi \) is ramified at \( P \) and we have \( \text{ord}_P(y) = -r \). We choose \( u, v \in \mathbb{N} \) 
such that \( -uv + vp = 1 \). Then \( y^ux^{-v} \) is obviously a prime element at \( P \) and we have:
\[
N + 1 = \text{ord}_P(\sigma(y^ux^{-v}) - y^ux^{-v}) = \text{ord}_P(((y + 1)^u - y^u)x^{-v}) = \text{ord}_P(uy^{u-1}x^{-v}) = r + 1,
\]
as was to be shown. Finally, the genus \( g_C \) of \( C \) is determined by the formula \( 2(g_C-1) = -2p + (N + 1)(p-1) \) which is a consequence of Hurwitz’ formula (see Corollary 2.4 
on p. 301 in \([\text{Hs}]\)) and Hilbert’s formula (see Proposition 4 on p. 72 in \([\text{Sc}]\)). Hence 
\( g_C = \frac{(r-1)(p-1)}{2} \), as was to be shown.

\((2.6)\) Question. Recall that a bounded complex of finitely generated projective \( k[G] \)-
modules is called a perfect complex. The proof of Theorem (2.1)(a) actually yields 
the following slightly stronger result: If \( \pi \) is weakly ramified and \( n \equiv -1 \mod e_P^w \) 
for all \( P \in X \), then \( R\Gamma(X, \mathcal{O}_X(D)) \) is quasi-isomorphic to a perfect complex. If 
\( \deg(D) > 2g_X - 2 \), then Theorem (2.1)(b) shows that also the converse statement is 
true. Is the converse statement true in general?

\section*{§3 Computing Equivariant Euler Characteristics}

In the paper \([\k\delta]\) we have given a formula for the Euler characteristic of equivariant 
Zariski sheaves on curves in the tamely ramified case. In this section we generalize
that formula to the general case.

We keep the assumptions and notations introduced in Section 2. We do not assume any condition on the ramification of the cover $\pi : X \to Y := X/G$ in this section. The main focus in this paper is on the case $p = \text{char}(k) > 0$, but everything in this and the following section is true and interesting (though classical) also in the case $p = 0$ (if the condition prime to $p$ is regarded as the empty condition and the term $p$-group means trivial group).

Let $E$ be a locally free $G$-module on $X$ of rank $r$. For any $P \in X$, we view the fibre $E(P) := E_P / m_P E_P$ as a $k$-representation of the decomposition group $G_P$. Furthermore, the obvious representation of $G_P$ on the cotangent space $m_P / m_P^2$ (or the corresponding character $G_P \to k^\times$) is denoted by $\chi_P$.

The following theorem computes the equivariant Euler characteristic $\chi(G, X, E)$ of $X$ with values in $E$.

(3.1) Theorem. We have in $K_0(G, k)_Q$:

$$\chi(G, X, E) = \left( (1 - g_Y)r + \frac{1}{n} \deg(E) - \frac{r}{2n} \sum_{P \in X} \left( (e^w_P - 1)(e^i_P + 1) + \sum_{s \geq 2} (|G_{P,s}| - 1) \right) \right) [k[G]]$$

$$- \frac{1}{n} \sum_{P \in X} e^w_P \sum_{d=1}^{e^i_P - 1} d \cdot \left[ \text{Ind}_{G_P}^G (E(P) \otimes \chi^d_P) \right].$$

(3.2) Remark.
(a) If $\pi$ is tamely ramified, then the upper sum over $P \in X$ obviously vanishes. In particular, Theorem (3.1) generalizes Theorem 1.1 in [Kô2]. In the next section, we will apply Theorem (3.1) in the weakly ramified case, i.e., when $\sum_{s \geq 2} (|G_{P,s}| - 1)$ vanishes for all $P \in X$.
(b) The reader may wish to compare Theorem (3.1) with Théorème 3.18 in Borne’s paper [Bo2] which gives an alternative expression of the left hand side.

Proof of Theorem (3.1). By classical representation theory (see §82 in [CR]) it suffices to show that both sides coincide after restricting to any cyclic subgroup $H$ of $G$ of order prime to $p$. Therefore it suffices to show that, by restricting to $H$, we obtain the corresponding formula for the action of $H$ on $X$ and that the character values (of the original formula) coincide at any element of $G$ whose order is prime to $p$.

We first show that character values at $\sigma = 1$, i.e. the $k$-dimensions, coincide. The $k$-dimension of the right hand side is

$$n(1 - g_Y)r + \deg(E) - \frac{r}{2} \sum_{P \in X} \left( (e^w_P - 1)(e^i_P + 1) + \sum_{s \geq 2} (|G_{P,s}| - 1) \right)$$

$$- \frac{1}{n} \sum_{P \in X} e^w_P \sum_{d=1}^{e^i_P - 1} d \cdot r \cdot \frac{n}{e^i_P}.$$
By Hurwitz’ formula (see Corollary 2.4 on p. 301 in [Ha]) and Hilbert’s formula for the order of the different (see Proposition 4 on p. 72 in [Se]), this is equal to

\[
(1 - g_X)^r + \frac{r}{2} \sum_{P \in X} \sum_{s \geq 0} (|G_{P,s}| - 1) + \deg(E)
\]

\[
- \frac{r}{2} \sum_{P \in X} \left( e_P e_P^t - e_P^t + e_P^w - 1 + \sum_{s \geq 2} (|G_{P,s}| - 1) \right)
\]

\[
- \sum_{P \in X} r \cdot \frac{e_P^t (e_P^t - 1)}{2}
\]

\[
= (1 - g_X)^r + \deg(E) + \frac{r}{2} \sum_{P \in X} (e_P^t - 1)
\]

\[
- \frac{r}{2} \sum_{P \in X} (e_P^t - 1)
\]

\[
= (1 - g_X)^r + \deg(E).
\]

This is equal to the \(k\)-dimension of the left hand side by the theorem of Riemann-Roch (see §1 in Chapter IV of [Ha] and Exercise 6.11 on p. 149 in [Ha]).

We now fix an element \(\sigma \in G \setminus \{1\}\) whose order is prime to \(p\) and show that the character values of both sides coincide at \(\sigma\). Using \(\text{Trace}(\sigma|k[G]) = 0\) and, more generally, the well-known formula for an induced character (see formula (38.3) on p. 266 in [CR]), we obtain for the character value of the right hand side at \(\sigma\):

\[
- \frac{1}{n} \sum_{P \in X} \sum_{d=1}^{e_P^t - 1} d \cdot \text{Trace}(\sigma|\text{Ind}_{G_P}^G (E(P) \otimes \chi_P^d)) =
\]

\[
= - \frac{1}{n} \sum_{P \in X} \sum_{d=1}^{e_P^t - 1} e_P^w \cdot d \cdot \sum_{e_P} \text{Trace}(\tau^{-1} \sigma \tau | E(P)) \cdot \chi_P^d (\tau^{-1} \sigma \tau)
\]

\[
= - \frac{1}{n} \sum_{P \in X} \sum_{d=1}^{e_P^t - 1} d \cdot \sum_{\tau \in G, \sigma \in G_{\tau}(P)} \text{Trace}(\sigma | E(\tau(P))) \cdot \chi_P^d (\tau(P)) (\sigma)
\]

\[
= - \sum_{P \in X^\sigma} \text{Trace}(\sigma | E(P)) \cdot \frac{1}{e_P^t} \sum_{d=1}^{e_P^t - 1} d \cdot \chi_P^d (\sigma)
\]

where \(X^\sigma := \{P \in X : \sigma(P) = P\}\). Since \(G_{P,1}\) is a \(p\)-group (see Corollaire 3 on p. 75 in [Se]), the character \(\chi_P : G_P \to k^\times\) factors modulo \(G_{P,1}\), and the induced character \(\bar{\chi}_P : G_P/G_{P,1} \to k^\times\) is injective by Corollaire 1 on p. 75 in [Se] (for all \(P \in X\)). Hence \(\chi_P(\sigma) \in k\) is a non-trivial \(e_P^t\)th root of unity and we obtain

\[
- \frac{1}{e_P^t} \sum_{d=1}^{e_P^t - 1} d \cdot \chi_P^d (\sigma) = (1 - \chi_P(\sigma))^{-1}
\]
by Lemma (3.3) below. Thus the character value of the right hand side at \( \sigma \) is equal to
\[
\sum_{P \in X^\sigma} \frac{\text{Trace}(\sigma | \mathcal{E}(P))}{1 - \chi_P(\sigma)}
\]
which in turn is equal to the character value of the left hand side at \( \sigma \) by the Lefschetz fixed point formula (see Chapter VI, §9 in [FT] or Example 3 in [Ko1]).

We now fix an arbitrary subgroup \( H \) of \( G \) and show that, by restricting to \( H \), we obtain the corresponding formula for the action of \( H \) on \( X \). This is obviously true for the left hand sides:
\[
\text{Res}^G_H (\chi(G, X, \mathcal{E})) = \chi(H, X, \mathcal{E}) \quad \text{in} \quad K_0(H, k).
\]

In order to verify the corresponding equality for the right hand sides, we first show the following formula for any \( Q \in Y \) and \( d \in \mathbb{Z} \):
\[
(1) \quad \text{Res}^G_H \left( \text{Ind}^G_{\tilde{\sigma} Q} \left( \mathcal{E}(\tilde{Q}) \otimes \chi^d_Q \right) \right) \cong \bigoplus_{R \in X/H, R \to Q} \text{Ind}^H_{\tilde{R}} \left( \mathcal{E}(\tilde{R}) \otimes \chi^d_R \right);
\]
here, \( \tilde{Q} \in X \) is a preimage of \( Q \) under \( \pi \), \( \tilde{R} \in X \) is a preimage of \( R \) under the canonical projection \( X \to X/H \), \( H_P \) denotes the decomposition group of the action of \( H \) on \( X \) at the place \( P \), and \( \mathcal{E}(P) \otimes \chi^d_P \) denotes the obvious representation of both \( G_P \) and \( H_P \) (for \( P \in X \)). Note that neither the left hand side nor the right hand side of the formula (1) depend on the chosen points in the fibre of \( \pi \) or of \( X \to X/H \). By Mackey’s double coset theorem (see Theorem (44.2) on p. 324 in [CR]), the left hand side of (1) is equal to
\[
\bigoplus_{\sigma \in H \setminus G / \tilde{\sigma} Q} \text{Ind}^H_{H \cap \sigma G \tilde{\sigma} Q^{-1}} \left( (\sigma^{-1})^* \left( \mathcal{E}(\tilde{Q}) \otimes \chi^d_Q \right) \right);
\]
Furthermore, the association \( \sigma \mapsto \overline{\sigma(\tilde{Q})} \) yields a bijection between \( H \setminus G / \tilde{\sigma} Q \) and \( \{ R \in X/H : R \to Q \} \), and we have: \( H \cap \sigma G \tilde{\sigma} Q^{-1} = H \cap G_{\sigma(\tilde{Q})} = H_{\sigma(\tilde{Q})} \) and \( (\sigma^{-1})^* \left( \mathcal{E}(\tilde{Q}) \otimes \chi^d_Q \right) \cong \mathcal{E}(\sigma(\tilde{Q})) \otimes \chi^d_{\sigma(\tilde{Q})} \). This proves formula (1). We now verify the above-mentioned equality for the right hand sides. Since we already know that the \( k \)-dimensions agree, it suffices to verify this equality modulo \( Q[k[H]] \). Using formula (1), we obtain for the restriction of the right hand side of the formula in Theorem (3.1) to \( H \) modulo \( Q[k[H]] \):
\[
-\text{Res}^G_H \left( \frac{1}{n} \sum_{P \in X} e_P \sum_{d=0}^{e_P-1} d \cdot \left[ \text{Ind}^G_{\tilde{\sigma} P} \left( \mathcal{E}(P) \otimes \chi^d_P \right) \right] \right) =
\]
\[
= -\text{Res}^G_H \left( \sum_{Q \in Y} \frac{1}{e_Q} \sum_{d=0}^{e_Q-1} d \cdot \left[ \text{Ind}^G_{\tilde{\sigma} Q} \left( \mathcal{E}(\tilde{Q}) \otimes \chi^d_Q \right) \right] \right)
\]
\[
= -\sum_{Q \in Y} \frac{1}{e_Q} \sum_{d=0}^{e_Q-1} d \sum_{R \in X/H, R \to Q} \left[ \text{Ind}^H_{\tilde{R}} \left( \mathcal{E}(\tilde{R}) \otimes \chi^d_R \right) \right]
\]
\[
\begin{align*}
= - \sum_{R \in X/H} \frac{1}{e^t_R} \sum_{d=0}^{e^t_R-1} d \cdot \left[ \text{Ind}^H_{H_R} \left( \mathcal{E}(\tilde{R}) \otimes \chi^d_{\tilde{R}} \right) \right].
\end{align*}
\]

Let now \(e^t_R(H), e^w_R(H)\) denote the ramification indices of the cover \(X \to X/H\) at the place \(\tilde{R}\) which are defined analogously to those of the cover \(\pi\). Furthermore we put \(f^t_R := e^t_R/e^t_R(H) \in \mathbb{N}\). Then we can write the latter term as follows:

\[
\begin{align*}
&= - \sum_{R \in X/H} \frac{1}{e^t_R} \sum_{d=0}^{e^t_R-1} \sum_{a=0}^{f^t_R} (d + a \cdot e^t_R(H)) \cdot \left[ \text{Ind}^H_{H_R} \left( \mathcal{E}(\tilde{R}) \otimes \chi^{d+a e^t_R(H)}_{\tilde{R}} \right) \right] = \\
&= - \sum_{R \in X/H} \frac{1}{e^t_R} \sum_{d=0}^{e^t_R-1} \sum_{a=0}^{f^t_R} a \cdot e^t_R(H) \cdot \left[ \text{Ind}^H_{H_R} \left( \mathcal{E}(\tilde{R}) \otimes \chi^{d+e^t_R(H)}_{\tilde{R}} \right) \right].
\end{align*}
\]

Since the first ramification group \(H_{\tilde{R},1}\) is a \(p\)-group (see Corollaire 3 on p. 75 in [Se]) and since \(H_{\tilde{R}}\) is the semi-direct product of \(H_{\tilde{R},1}\) and the cyclic group \(H_{\tilde{R}/H_{\tilde{R},1}}\) (see Corollaire 4 on p. 75 in [Se]), we obtain

\[
\sum_{d=0}^{e^t_R(H)-1} \left[ \chi^d_{\tilde{R}} \right] = \left[ k[H_{\tilde{R}}/H_{\tilde{R},1}] \right] = e^w_R(H)^{-1} \cdot \left[ k[H_{\tilde{R}}] \right]
\]

by Lemma (3.4) below. Hence the latter term is, modulo \(\mathbb{Q}[k[H]]\), congruent to:

\[
\begin{align*}
&= - \sum_{R \in X/H} \frac{1}{e^t_R(H)} \sum_{d=0}^{e^t_R(H)-1} d \cdot \left[ \text{Ind}^H_{H_R} \left( \mathcal{E}(\tilde{R}) \otimes \chi^d_{\tilde{R}} \right) \right] = \\
&= - \frac{1}{\left| H \right|} \sum_{P \in X} e^w_P(H) \sum_{d=0}^{e^t_P(H)-1} d \cdot \left[ \text{Ind}^H_{H_P} \left( \mathcal{E}(P) \otimes \chi^d_P \right) \right],
\end{align*}
\]

as was to be shown. This completes the proof of Theorem (3.1).

The following easy lemma is also a crucial step in an alternative approach to Theorem 5.2 of [Ch], see section 2a of [Er2].

\textbf{(3.3) Lemma.} Let \(m \in \mathbb{N}\) and \(\zeta \neq 1\) an \(m\)th root of unity in \(k\). Then we have:

\[
m(\zeta - 1)^{-1} = \sum_{d=1}^{m-1} d\zeta^d.
\]

\textit{Proof.} \((\sum_{d=1}^{m-1} d\zeta^d)(\zeta - 1) = \sum_{d=1}^{m-1} d\zeta^{d+1} - \sum_{d=1}^{m-1} d\zeta^d = (m - 1)\zeta^m - \sum_{d=1}^{m-1} \zeta^d = m.\)
Lemma. Let $H$ be the semi-direct product of a finite $p$-group $P$ and an (arbitrary) group $C$ which acts on $P$. Then we have in $K_0(H, k)$:

$$[k[H]] = |P| \cdot [k[C]].$$

Proof. Let $I$ denote the augmentation ideal of the group ring $k[P]$. The group $C$ acts on $k[P]$ in the obvious way, and the ideals $I^r$, $r \geq 0$, of $k[P]$ are clearly $C$-stable. Since $([\sigma] - [1])_{|P|} = [\sigma^{P}] - [1] = 0$ for all $\sigma \in P$, we have $I^N = 0$ for $N$ sufficiently big. Furthermore, the group $P$ acts trivially on the successive quotients $I^r/I^{r+1}$, $r \geq 0$. Thus we have a finite filtration

$$k[H] = k[P] \ast C = I^0 \ast C \supseteq I^1 \ast C \supseteq I^2 \ast C \supseteq \ldots \supseteq I^N \ast C = 0$$

of the regular representation $k[H]$ by $k[H]$-submodules such that the successive quotients split into a direct sum of $k[H]$-modules of the form $k[C] = k[H/P]$. (Here, $I^r \ast C$ denotes the ideal $\oplus_{\sigma \in C} I^r[\sigma]$ in the twisted group ring $\oplus_{\sigma \in C} k[P][\sigma] = k[P] \ast C = k[H]$.) This proves Lemma (3.4).

Theorem (3.1) implies the following global relation in $K_0(G, k)$ between the representations $\text{Ind}_{G_P}^{G}(\chi_P^d)$, $P \in X$, $d = 0, \ldots, e_P^t - 1$.

(3.5) Corollary. Let $p$ be odd. For $P \in X$ and $d \in \{0, \ldots, e_P^t - 1\}$ we put

$$n_{P,d} := e_P^w \left( d + \frac{(e_P^w - 1)(e_P^t + 1) + \sum_{s \geq 2}(|G_P| - 1)}{2} \right) \in \mathbb{N}.$$ 

Then the element

$$\sum_{P \in X} \sum_{d=0}^{e_P^t-1} n_{P,d} \cdot [\text{Ind}_{G_P}^{G}(\chi_P^d)] \in K_0(G, k)$$

is divisible by $n = |G|$ in $K_0(G, k)$.

Proof. Apply Theorem (3.1) to $\mathcal{E} = \mathcal{O}_X$ and use the equality

$$[k[G]] = e_P^w \sum_{d=0}^{e_P^t-1} [\text{Ind}_{G_P}^{G}(\chi_P^d)]$$

which follows from Lemma (3.4).

§4 Galois Structure in the Weakly Ramified Case

In this section we will generalize several results of Kani and Nakajima on the Galois module structure of Zariski cohomology groups of curves from the tamely ramified to the weakly ramified case.
We keep the assumptions and notations introduced in §2 and §3. In addition we assume in this section that the cover \( \pi : X \to Y \) is weakly ramified.

We begin with recalling the following crucial properties of weakly ramified covers.

**Lemma.** For any \( P \in X \), the first ramification group \( G_{P,1} \) is an abelian group of exponent \( p \), the factor group \( G_{P}/G_{P,1} \) is cyclic of order prime to \( p \) and the natural action of \( G_{P}/G_{P,1} \) on \( G_{P,1}\setminus\{1\} \) is free. In particular, \( G_{P} \) is the semidirect product of \( G_{P,1} \) and \( G_{P}/G_{P,1} \) and we have: \( e_{P}^{w} \equiv 1 \mod e_{P}^{w} \).

**Proof.** This is proved on the pages 74-77 in Serre’s book [Se], see in particular Proposition 9 and the corollaries of Proposition 7. \( \square \)

**Lemma.** Let \( H \) be the semi-direct product of a finite \( p \)-group \( P \) with a finite group \( C \) which acts on \( P \). We assume that the action of \( C \) on \( P\setminus\{1\} \) is free, and we put \( a := (|P|−1)/|C| \in \mathbb{N} \). Furthermore let \( V \) be a \( k \)-representation of \( C \) (of finite dimension) which we view also as a \( k \)-representation of \( H \) via the canonical projection \( H \to C \). Then we have:

(a) The induced representation \( \text{Ind}^{H}_{C}(V) \) is the \( k[H] \)-projective cover of \( V \).

(b) \( \text{Ind}^{H}_{C}(V) = [V] + a \cdot \dim_{k}(V) \cdot [k[C]] \) in \( K_{0}(H, k) \).

**Proof.** The order of \( C \) is prime to \( p \) by assumption. Hence \( V \) is a projective \( k[C] \)-module and \( \text{Ind}^{H}_{C}(V) \) is a projective \( k[H] \)-module. Furthermore we have an obvious \( k[H] \)-epimorphism \( \text{Ind}^{H}_{C}(V) \to V \). Since \( V \cong \text{Ind}^{H}_{C}(V)/\text{rad}(k[H])\text{Ind}^{H}_{C}(V) \), the \( k[H] \)-module \( \text{Ind}^{H}_{C}(V) \) is minimal with these properties. Thus we have proved part (a). We now prove part (b). For all \( x \in P \) and \( \xi, \eta \in C \) we obviously have:

\[
(x, \xi) \cdot (1, \eta) \cdot (x, \xi)^{-1} = (x \cdot (\xi \eta \xi^{-1})(x^{-1}), \xi \eta \xi^{-1}) \quad \text{in} \quad P \rtimes C = H.
\]

Hence, by assumption on the action of \( C \) on \( P \), the intersection \( C \cap \sigma C \sigma^{-1} \) is trivial for all \( \sigma \in H \setminus C \). Using Mackey’s double coset theorem (see Theorem (4.4.2) on p. 324 in [CR]), we thus obtain:

\[
\text{Res}^{H}_{C}(\text{Ind}^{H}_{C}(V)) \cong \bigoplus_{\sigma \in C \setminus G/C} \text{Ind}^{G}_{C \rtimes \sigma G \sigma^{-1}}(\text{Res}^{G}_{C \rtimes \sigma G \sigma^{-1}}((\sigma^{-1})^{*}(V)))
\]

\[= V \oplus \left( a \cdot \dim_{k}(V) \oplus k[C] \right). \]

Furthermore the restriction homomorphism \( \text{Res}^{H}_{C} : K_{0}(H, k) \to K_{0}(C, k) \) is bijective since the irreducible \( k \)-representations of \( C \) considered as \( k \)-representations of \( H \) are the only irreducible \( k \)-representations of \( H \) (because \( k[H]/\text{rad}(k[H]) \cong k[C] \)). This proves part (b). \( \square \)

In the sequel, the \( k[G_{P}] \)-projective cover of the \( k[G_{P}] \)-module \( \chi_{P}^{d} \) is denoted by \( \text{Cov}(\chi_{P}^{d}) \) (for all \( P \in X \) and \( d \in \mathbb{Z} \)). The following theorem gives a **global** relation between the projective \( k[G] \)-modules \( \text{Ind}^{G}_{G_{P}}(\text{Cov}(\chi_{P}^{d})) \), \( P \in X \), \( d = 1, \ldots, e_{P}^{w}−1 \).

**Theorem.** There is a (unique) projective \( k[G] \)-module \( N \) such that

\[
\bigoplus_{P \in X} N \cong \bigoplus_{d=1}^{e_{P}^{w}−1} e_{P}^{w} \cdot d \oplus \bigoplus_{P \in X} \text{Ind}^{G}_{G_{P}}(\text{Cov}(\chi_{P}^{d})).
\]
(4.4) Remark. If \( \pi \) is assumed to be not only weakly but tamely ramified, then the projective \( k[G] \)-module on the right hand side has the following somewhat simpler shape:

\[
\bigoplus_{P \in X} e_P^{p-1} d \oplus \bigoplus_{d=1} \text{Ind}_{G_P}^G (\chi_P^d).
\]

In particular, Theorem (4.3) generalizes the first part of Theorem 2 in Kani’s paper [Ka] and Theorem 2(i) in Nakajima’s paper [Na2].

Proof of Theorem (4.3). Let \( E \) denote the equivariant divisor \( E := \sum_{P \in X} (e^w_P - 1)[P] \) on \( X \). Then we have in \( K_0(G, k) \):

\[
\chi(G, X, \mathcal{O}_X(E)) = (1 - g_Y)[k[G]]
\]

\[
= \frac{1}{n} \sum_{P \in X} \left( (e^w_P - 1) - \frac{(e^w_P - 1)(e^t_P + 1)}{2} \right) [k[G]]
\]

\[
- \frac{1}{n} \sum_{P \in X} e^w_P \sum_{d=1}^{e_P^p - 1} d \cdot \left[ \text{Ind}_{G_P}^G (\chi_P^{d+1-e_P^t}) \right] \quad \text{(by Theorem (3.1))}
\]

\[
= (1 - g_Y)[k[G]]
\]

\[
- \frac{1}{n} \sum_{P \in X} \frac{(e^t_P - 1)(e^w_P - 1)}{2} [k[G]]
\]

\[
- \frac{1}{n} \sum_{P \in X} e^w_P \sum_{d=0}^{e_P^p - 1} d \cdot \left[ \text{Ind}_{G_P}^G (\chi_P^d) \right] \quad \text{(since } e^w_P \equiv 1 \text{ mod } e^t_P \text{ by Lemma (4.1))}
\]

\[
= (1 - g_Y)[k[G]]
\]

\[
- \frac{1}{n} \sum_{P \in X} \frac{(e^t_P - 1)e^t_P}{2} \cdot \frac{e^w_P - 1}{e^t_P} [k[G]]
\]

\[
- \frac{1}{n} \sum_{P \in X} e^w_P \sum_{d=0}^{e_P^p - 1} d \cdot \left( \text{Ind}_{G_P}^G (\text{Cov}(\chi_P^d)) - \frac{e^w_P - 1}{e^t_P} \left[ \text{Ind}_{G_P}^G (k[G_P/G_P,1]) \right] \right) \quad \text{(by Lemma (4.1) and Lemma (4.2))}
\]

\[
= (1 - g_Y)[k[G]]
\]

\[
- \frac{1}{n} \sum_{P \in X} e^w_P \sum_{d=0}^{e_P^p - 1} d \cdot \left[ \text{Ind}_{G_P}^G (\text{Cov}(\chi_P^d)) \right] \quad \text{(by Lemma (4.1) and Lemma (3.4))}.
\]

Hence, by Theorem (2.1)(a)(i), the class of the projective \( k[G] \)-module

\[
\bigoplus_{P \in X} e_P^{p-1} d e^w_P \oplus \bigoplus_{d=1} \text{Ind}_{G_P}^G (\text{Cov}(\chi_P^d))
\]

in \( K_0(k[G]) \) is divisible by \( n \) in \( K_0(k[G]) \). Writing the elements of \( K_0(k[G]) \) as integral linear combinations of a basis of \( K_0(k[G]) \) consisting of indecomposable projective
**Theorem (4.5).** Let \( D = \sum_{P \in X} n_P[P] \) be an equivariant divisor on \( X \) with \( n_P \equiv -1 \) mod \( e_P^w \) for all \( P \in X \). For any \( P \in X \), we write

\[
n_P = (e_P^w - 1) + (l_P + m_P e_P^t) e_P^w
\]

with \( l_P \in \{0, \ldots, e_P^t - 1\} \) and \( m_P \in \mathbb{Z} \). Furthermore, for any \( Q \in Y \), we choose a preimage \( Q \in X \) of \( Q \) under \( \pi \). Then we have in \( K_0(k[G]) \):

\[
\chi(G, X, \mathcal{O}_X(D)) = -[N] + \sum_{Q \in Y} \sum_{d=1}^{l_Q} \left[ \text{Ind}_{G_Q}^G \left( \text{Cov}(\chi_Q^{-d}) \right) \right] + \left( 1 - g_Y + \sum_{Q \in Y} m_Q \right) [k[G]].
\]

**Remark (4.6).** Note that \( l_P = 0 = m_P \) for all but finitely many \( P \in X \). If \( \pi \) is tamely ramified at \( P \), then \( l_P \) is obviously the unique number in \( \{0, \ldots, e_P^t - 1\} \) such that \( \mathcal{O}_X(D)(P) \cong \chi_P^{-l_P} \) (as \( k[G_P] \)-modules). If \( \pi \) is tamely ramified everywhere, then Theorem (4.5) implies the congruence

\[
\chi(G, X, \mathcal{O}_X(D)) \equiv -[N] + \sum_{Q \in Y} \sum_{d=1}^{l_Q} \left[ \text{Ind}_{G_Q}^G (\chi_Q^{-d}) \right] \mod \mathbb{Z}[k[G]]
\]

for an arbitrary equivariant divisor \( D \) on \( X \). This congruence is a reformulation of Theorem 2(ii) in [Na2] applied to \( E := \mathcal{O}_X(D) \).

**Proof of Theorem (4.5).** Let \( E \) denote the divisor \( \sum_{P \in X} (e_P^w - 1)[P] \) as in the proof of Theorem (4.3). We first compute the difference \( \chi(G, X, \mathcal{O}_X(D)) - \chi(G, X, \mathcal{O}_X(E)) \) in \( K_0(G, k)_Q \).

\[
\chi(G, X, \mathcal{O}_X(D)) - \chi(G, X, \mathcal{O}_X(E)) = \frac{1}{n} \sum_{P \in X} n_P[k[G]] - \frac{1}{n} \sum_{P \in X} (e_P^w - 1)[k[G]]
\]

\[
- \frac{1}{n} \sum_{P \in X} e_P^w \sum_{d=1}^{e_P^t - 1} d \left[ \text{Ind}_{G_P}^G (\chi_P^{-d n_P}) \right] + \frac{1}{n} \sum_{P \in X} e_P^w \sum_{d=1}^{e_P^t - 1} d \left[ \text{Ind}_{G_P}^G (\chi_P^{d+1-e_P^t}) \right]
\]

(by Theorem (3.1))

\[
= \frac{1}{n} \sum_{P \in X} (l_P + m_P e_P^t) e_P^w [k[G]]
\]

\[
- \frac{1}{n} \sum_{P \in X} e_P^w \left( \sum_{d=0}^{e_P^t - 1} d \left[ \text{Ind}_{G_P}^G (\chi_P^{d-l_P}) \right] - \sum_{d=0}^{e_P^t - 1} d \left[ \text{Ind}_{G_P}^G (\chi_P^d) \right] \right)
\]
This result for the difference \( \chi_m \)ula \( (4.5) \).

□

The ideal sheaf of the reduced effective divisor on \( \text{rem} \) now obviously implies the desired formula in Theo-

rem. The following corollary computes the Galois module structure of its first cohomology group.

(4.7) Corollary. Let \( S \) be a \( G \)-stable non-empty finite set of closed points on \( X \) which contains all wildly ramified points, i.e. all points \( P \in X \) with \( e_P^w \neq 1 \). Let \( \mathcal{I}(S) \) denote the ideal sheaf of \( S \). Then the \( k[G] \)-module \( H^1(X, \mathcal{I}(S)) \) is stably isomorphic to \( N \oplus \left( \bigoplus_{Q \in \pi(S)} \text{Ind}_{G_Q}^G \left( \text{Cov}(\chi_Q^0) \right) \right) \). More precisely we have:

\[
H^1(X, \mathcal{I}(S)) \oplus k[G] \cong N \oplus \left( \bigoplus_{Q \in \pi(S)} \text{Ind}_{G_Q}^G \left( \text{Cov}(\chi_Q^0) \right) \right) \oplus \left( g_Y \oplus k[G] \right).
\]

Proof. Since \( S \neq \emptyset \), we have \( H^0(X, \mathcal{I}(S)) = 0 \). Hence we obtain the following equality in \( K_0(G, k) \):

\[
[H^1(X, \mathcal{I}(S))] = -\chi \left( G, X, \mathcal{O}_X \left( \sum_{P \in S} -[P] \right) \right)
\]

\[
= [N] - \sum_{Q \in \pi(S)} \sum_{d=1}^{e_Q^w - 1} \left[ \text{Ind}_{G_Q}^G \left( \text{Cov}(\chi_Q^{-d}) \right) \right] + \sum_{Q \in \pi(S)} [k[G]] + (g_Y - 1)[k[G]].
\]
\[ [N] + \sum_{Q \in \pi(S)} \text{Ind}_{G_Q}^G \left( \text{Cov}(\chi_Q^0) \right) + (g_Y - 1)[k[G]] \]

(by Theorem (4.5))

Furthermore, \( H^1(X, \mathcal{I}(S)) \) is a projective \( k[G] \)-module by Theorem (2.1)(a)(ii). Since two projective \( k[G] \)-modules, whose classes are equal in \( K_0(G, k) \), are already isomorphic, Corollary (4.7) is now proved. \( \square \)

The following corollary computes the Galois module structure of the space of global meromorphic differentials on \( X \) which are logarithmic along all ramified points. It generalizes the second part of Theorem 2 in Kani’s paper [Ka] from the tamely ramified to the weakly ramified case.

(4.8) Corollary. Let \( S \) be a \( G \)-stable non-empty finite set of closed points on \( X \) which contains all ramified points. Let \( \Omega_X(S) \) denote the sheaf of meromorphic differentials on \( X \) which are logarithmic along \( S \). Then the \( k[G] \)-module \( H^0(X, \Omega_X(S)) \oplus N \) is free of rank \( |S/G| + g_Y - 1 \).

Proof. We use the notations introduced in the proof of Corollary (2.3). Then the divisor \( D = K_X + \sum_{P \in S} [P] \) satisfies the condition of Theorem (4.5) and the corresponding integers \( l_P, m_P, P \in X \), are given as follows: \( l_P = 0 \) for all \( P \in X \), \( m_P = r_{\pi(P)} + 1 \) for \( P \in S \) and \( m_P = r_{\pi(P)} \) for \( P \in X \setminus S \). Thus we obtain:

\[ [H^0(X, \Omega_X(S)) \oplus N] = [H^0(X, O_X(D))] + [N] = (1 - g_Y + \sum_{Q \in Y} r_Q + |S/G|)[k[G]] \quad \text{(by Theorem (4.5))} \]

\[ = (g_Y - 1 + |S/G|)[k[G]] \quad \text{in } K_0(k[G]) \quad \text{(since } deg(\Omega_Y) = 2g_Y - 2) \]

As in Corollary (4.7), this implies Corollary (4.8). \( \square \)

(4.9) Remark. Alternatively, Corollary (4.8) can be derived from Corollary (4.7) using the Serre duality isomorphism \( H^0(X, \Omega_X(S)) \cong H^1(X, \mathcal{I}(S))^* \) and the isomorphism

\[ N^* \oplus \left( \oplus_{Q \in \pi(S)} \text{Ind}_{G_Q}^G \left( \text{Cov}(\chi_Q^0) \right)^* \right) \oplus N \cong \frac{|S/G|}{k[G]} \]

which may easily be checked.

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