O(N) Sigma Model as a Three Dimensional Conformal Field Theory

‡S. Guruswamy, ‡S. G. Rajeev and ‡‡P. Vitale

‡Department of Physics, University of Rochester
Rochester, NY 14627 USA
e-mail: guruswamy, rajeev@urhep.pas.rochester.edu
and

‡‡Dipartimento di Scienze Fisiche, Università di Napoli
and I.N.F.N. Sez. di Napoli,
Mostra d’Oltremare Pad. 19, 80125 Napoli ITALY.
e-mail: vitale@axpna1.na.infn.it

Abstract

We study a three dimensional conformal field theory in terms of its partition function on arbitrary curved spaces. The large $N$ limit of the nonlinear sigma model at the non-trivial fixed point is shown to be an example of a conformal field theory, using zeta–function regularization. We compute the critical properties of this model in various spaces of constant curvature ($R^2 \times S^1$, $S^1 \times S^1 \times R$, $S^2 \times R$, $H^2 \times R$, $S^1 \times S^1 \times S^1$ and $S^2 \times S^1$) and we argue that what distinguishes the different cases is not the Riemann curvature but the conformal class of the metric. In the case $H^2 \times R$ (constant negative curvature), the $O(N)$ symmetry is spontaneously broken at the critical point. In the case $S^2 \times R$ (constant positive curvature) we find that the free energy vanishes, consistent with conformal equivalence of this manifold to $R^3$, although the correlation length is finite. In the zero curvature cases, the correlation length is finite due to finite size effects. These results describe two dimensional quantum phase transitions or three dimensional classical ones.
1. Introduction

The correlation functions of statistical mechanical systems exhibit scale invariance at a phase transition of second or higher order. It is possible to describe such a phase transition in terms of a scale invariant euclidean quantum field theory, a typical classical statistical system corresponding to a regularized euclidean quantum field theory. The correlation length (inverse mass of the particles) of this field theory will diverge as the coupling constants approach a ‘fixed point’. The theory defined by the fixed point in the space of coupling constants will be scale invariant and corresponds to a phase transition point of the statistical mechanical system. The immediate neighborhood of the fixed point describes quantum field theories with masses of the particles small compared to the cut–off.

At present, the only general way to construct quantum field theories is as such limiting cases of statistical mechanical systems. It is of great interest to construct scale invariant quantum field theories directly. This should be possible since they are ‘finite’ (i.e., correspond to a fixed point of the renormalization group); moreover it could eventually lead to a definition of a general quantum field theory as a perturbation to a scale invariant quantum field theory without the usual cumbersome procedure of renormalization. Any progress in that direction is of importance to particle physics, where the standard model should eventually be constructed by such an intrinsic procedure.

Scale invariant quantum field theories are also interesting for phenomenological reasons: they describe experimentally accessible phase transition phenomena. The critical exponents of such transitions have been calculated with great accuracy in many realistic cases. Also, in the case of two dimensional systems, many systems have been solved exactly. That is, their partition and correlation functions have been obtained in terms of the special functions of classical mathematics.

The key to this success in two dimensions is that the theories often have a symmetry much larger than scale invariance: conformal invariance. In two dimensions, infinitesimal conformal transformations (position dependent scale transformations) are the same as complex analytic co–ordinate transformations. This large symmetry puts strong constraints on the correlation functions; in the case of ‘minimal models’ these constraints are strong enough to determine them completely. This is similar in spirit to the solution of classical potential problems in two dimensions by conformal (Schwarz–Christoffel) transformations. For example, the general boundary value problem of the two dimensional Laplace equation can be solved by mapping the boundary to one of a small class of standard boundaries.
Most interesting classical phase transitions occur in three dimensional systems. One should not expect a complete generalization of conformal techniques to three dimensions (as an analogy, it is not possible to solve the general boundary value problem of the Laplace equation in three dimension by conformal transformation to a standard boundary). One should expect that conformal invariance is a powerful constraint even in this case, although not strong enough to completely determine the correlation functions. In this paper we will begin a study of three dimensional conformal field theory. (We study the \( O(N) \) non-linear sigma model). It is seen that in three dimensions also, the partition and correlation functions at a second or higher order transition are invariant under conformal transformations of the metric tensor. We will also compute the partition function in some special cases to confirm this general picture.

In three dimensions, it is necessary to study field theories in a curved space to fully understand conformal invariance. In two dimensions all manifolds are conformally flat, so the only geometric information about the manifold that can affect critical systems is a finite number of (Teichmuller) moduli parameters. This is replaced in the three dimensional case by an infinite dimensional space of conformal structures (see appendix A for a precise definition). The partition function of the system should be viewed as a functional of the metric density \( \hat{g}_{ij} = g^{-\frac{1}{2}}g_{ij} \), where \( g = \det g_{ij} \); it is the generating functional of the correlation functions of the stress tensor. The point is that conformal invariance alone will not determine this functional, even for ‘minimal’ models. The correlation functions of the stress tensor are physically measurable quantities; therefore the partition function on curved spaces is of physical interest even when the true metric of space is flat.

There is another reason to study critical phenomena in curved spaces. If we subject a system to external stresses, we can deform the underlying microscopic structure (such as a lattice) so that the effective distance between points is no longer the usual one. At a critical point, universality suggests that the details of the microscopic structures do not matter; but the system is still sensitive to the deformation through the effective metric tensor density.

In three dimensions there is no conformal anomaly; hence a conformal field theory can be defined as one whose partition function is a function on the space of conformal structures (if there were an anomaly, it would have been a section of a real line bundle). That is, its partition function should satisfy

\[
\mathcal{Z}[e^{2f}g] = \mathcal{Z}[g] \tag{1}
\]
\( g \) being the metric tensor. It is not obvious that interacting theories of this kind exist; we will show that the large \( N \) limit of the \( O(N) \) sigma model \([1], [2], [3], [4], [5]\) is such a conformal field theory, at its non-trivial fixed point. Moreover a primary field is one whose correlation functions transform homogeneously under a conformal transformation:

\[
<\phi(x_1)\cdots\phi(x_n)>_{e^{2fg}} = e^{\Delta(f(x_1)+\cdots+f(x_n))} <\phi(x_1)\cdots\phi(x_n)>_g
\] (2)

Again, the field \( \phi_i \) of the \( O(N) \) sigma model is an example of such a primary field. These definitions are the appropriate generalizations of the point of view of Ref. \([6]\). For previous work on conformal field theory in dimensions greater than two, see Refs. \([7], [8], [9], [10]\).

Although there is no conformal anomaly in three dimensions, there could be a parity anomaly in general. The the particular example we are studying, the \( O(N) \) sigma model does not have such an anomaly. We plan to return to this issue in the context of fermionic and Grassmannian sigma models \([11]\).

The following is the lay-out of this paper. Sect. 2 is a discussion of the large \( N \) limit of the \( O(N) \) sigma model on arbitrary three-dimensional manifolds. Sect. 3 is a brief review of the results on \( R^3 \). Sect. 4 discusses the divergence structure of the theory. Sect. 5 describes the zeta function regularization and an outline of the proof of the conformal invariance of the free energy density is given. Sect. 6 and Sect. 7 contain the study of the theory on special manifolds. Appendix A is a pedagogic discussion of conformal geometry in three dimensions. This appendix contains material for the reader who is interested in the geometric context for the definition of a conformal field theory in three dimensions. In particular the differences between two and three dimensional conformal geometry and the special nature of conformal curvature in three dimensions are stressed. Appendix B is a derivation of the Poisson sum formula on \( S^2 \). A table summarizing the results is given at the end of the last section.

2. \( O(N) \) non-linear sigma model in three dimensions

In this section we study the \( O(N) \) non-linear sigma model [1-5] on a Riemannian manifold \((M, g)\). Of particular interest to us is the case \( M = \Sigma \times R \) with \( \Sigma \) being an arbitrary 2-dimensional curved space: this describes a quantum phase transition at zero temperature, the temperature being the inverse radius of \( R \). We will study the theory in detail at its non-trivial fixed point for manifolds \( \Sigma \) with zero, constant positive and constant negative Riemannian curvature.
The euclidean partition function of the $O(N)$ non-linear sigma model in 3-dimensions in the presence of a background metric $g^{\mu\nu}(x)$ is given by,

$$
Z[g] = \int \mathcal{D}[\phi] e^{-\frac{1}{2\lambda} \int d^3x \sqrt{g(x)} [\phi^i(x) g^{\mu\nu}(x) \partial_\mu \phi_i \partial_\nu \phi_i]}
$$

where $i = 1, 2, \cdots, N$ and the $\phi^i(x)$ satisfy the constraint $\phi^i(x) \phi_i(x) = 1$. $\lambda$ is a coupling constant. As it stands, the action is not invariant under the conformal transformation of the metric, $g^{\mu\nu}(x) \rightarrow \Omega^2(x) g^{\mu\nu}(x)$. The action becomes conformally invariant \cite{12} when the laplacian $-\frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g(x)} g^{\mu\nu}(x) \partial_\nu)$ is modified to the “conformal laplacian”, $-\Box_g = -\frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g(x)} g^{\mu\nu}(x) \partial_\nu) + \xi R$, with $\phi$ transforming as $\phi(x) \rightarrow \Omega^{1-\frac{d}{2}}(x) \phi(x)$. $R$, in the conformal laplacian, denotes the Ricci scalar and $\xi$ is a numerical constant. $\xi = \frac{d-2}{4(d-1)}$ and is equal to 1/8 for dimensions $d = 3$. The generating functional then reads,

$$
Z[g] = \int \mathcal{D}[\phi] e^{-\frac{1}{2\lambda} \int d^3x \sqrt{g(x)} [-\phi^i \phi_i + \sigma(\phi^i)^2 - \phi^2 \frac{\lambda}{\Lambda}]}.
$$

However, the constraint $\phi^i(x) \phi_i(x) = 1$ still violates conformal invariance. The constraint on the $\phi$ fields can be implemented by a Lagrange multiplier, in the form of an auxiliary field $\sigma(x)$ with no dynamics, as follows:

$$
Z[g] = \int \mathcal{D}[\phi] \mathcal{D}[\sigma] e^{-\frac{1}{4\lambda} \int d^3x \sqrt{g(x)} [-\phi^i \phi_i + \sigma(\phi^i)^2 - 1]}.
$$

Although, classically this is not conformally invariant, we will show that there is a non-trivial fixed point for the quantum theory at which it is conformally invariant. Following Wilson’s approach, we regularize the generating functional $Z$ in the ultraviolet by introducing a cut-off, $\Lambda$, in the momentum space. Before doing that, let us note the canonical dimensions of the fields and the couplings in our action. Rescaling $\phi^i$ to $\sqrt{\lambda} \phi^i$, we have,

$$
Z[g] = \int \mathcal{D}[\phi] \mathcal{D}[\sigma] e^{-\int d^3x \sqrt{g(x)} [-\frac{1}{\lambda} \phi^i \Box_g \phi_i + \frac{1}{2} \sigma(\phi^i)^2 - \frac{\sigma}{\lambda} \phi^2 \frac{\lambda}{\Lambda}]}.
$$

The canonical dimensions in mass units can be read off from the action:

$$
[\phi] = 1/2 \quad , \quad [\sigma] = 2 \quad , \quad [\frac{1}{\lambda}] = 1.
$$

Let us redefine our coupling constant $\frac{1}{\lambda(\Lambda)}$ to $\frac{1}{\lambda(\Lambda)}$ so that the coupling constant is now dimensionless. The regularized partition function can now be formally written as,

$$
Z[g, \Lambda, \lambda(\Lambda)] = \int \mathcal{D}_\Lambda[\phi] \mathcal{D}_\Lambda[\sigma] e^{-\int d^3x \sqrt{g(x)} [-\frac{1}{2} \phi^i (-\Box_g + \sigma) \phi_i - \frac{\sigma}{\lambda(\Lambda)} \phi^2 \frac{\lambda}{\Lambda}]}.
$$
where \( \mathcal{D}_\Lambda[\phi] = \prod_{|k| < \Lambda} d\phi(k) \) and similarly \( \mathcal{D}_\Lambda[\sigma] \).

### The large \( N \) limit

Since it is not possible to solve this theory exactly, we have to use some approximation methods. At this point there are a few different approximation methods one could use in studying this problem. These are the \( 2 + \epsilon \) (or \( 4 - \epsilon \)) expansion methods, or the large \( N \) expansion. We choose to study the problem in the large \( N \) limit. In this paper we do all our analysis to the leading order in the large \( N \) expansion.

In the large \( N \) limit, by which we mean \( N \to \infty \) keeping \( N \lambda (\Lambda) \) fixed, the generating functional can be calculated using the saddle point approximation. Also, we will assume that the number \( n \) of components of the field \( \phi_i \) that have non–zero values in the ground state remains finite in this limit. We will therefore redefine \( (N - n) \lambda (\Lambda) \) in the action as \( \lambda (\Lambda) \) which will remain fixed as \( N \to \infty \). We also rescale the fields \( \phi_i \) to \( \sqrt{N - n} \phi_i \) for \( i = 1, 2 \cdots n \). We can now integrate out the first \( N - n \) components of the \( \phi \) field and rewrite \( Z \) as,

\[
Z[g, \Lambda, \lambda(\Lambda)] = \int \mathcal{D}_\Lambda[\phi] \mathcal{D}_\Lambda[\sigma] e^{-\frac{(N-n)}{2} \left[ \text{Tr} \log_A \left( -\Box + \sigma \right) + \int d^3x \sqrt{g} \sum_{i=1}^{n} \phi^i (-\Box + \sigma) \phi_i - \frac{1}{\lambda (\Lambda)} \sigma(x) \right]}
\]

(6)

We use the \( O(N) \) symmetry to rotate the unintegrated components to the subspace indexed by \( i = 1 \cdots n \) and denote these components by \( b_i(x) \).

We first discuss the general case of the \( O(N) \) sigma model on a manifold with an arbitrary metric. This is given by the following set of equations ("gap equations") obtained by extremizing the action with respect to \( b_i(x) \) keeping \( \sigma(x) \) fixed and vice–versa.

\[
(-\Box_g + \sigma)b_i = 0,
\]

(7)

\[
\sum_{i=1}^{n} b_i^2 = \frac{\Lambda}{\lambda (\Lambda)} - G_A(x, x; \sigma, g),
\]

(8)

where \( G_A(x, x; \sigma, g) = \langle x \mid (-\Box_g + \sigma)^{-1} \mid x \rangle \). \( G(x', x; \sigma, g) \) is the two point correlation function of the \( \phi \) fields.

Once we find the saddle point solutions of the action, we can compute the free energy
The density, $W$, of the system at the saddle point, to the leading order in the $\frac{1}{N}$ expansion.

$$W[g, \Lambda, \lambda(\Lambda)] = \frac{N}{2} [\text{Tr} \log (-\Box_g + \sigma) - \int d^3x \sqrt{g} \sigma \frac{\Lambda}{\lambda(\Lambda)}].$$ \hspace{1cm} (9)

The gap equations are the equations of motion of a classical field theory: the large $N$ limit of the $O(N)$ sigma model. The ground state will be given by the solution with the least free energy. Other solutions representing solitons or instantons of the $\frac{1}{N}$ expansion are also of interest. If the background metric is homogenous, we expect the ground state solution of the gap equations to be constant. Then we can assume (using $O(N)$ invariance) that just one component of $b_i$ (say $b_1 = b$) is non-zero: this has the meaning of spontaneous magnetization. If $\sigma = m^2$ is positive, $m$ can be viewed as the mass of the scalar field fluctuations around this background magnetization.

However, this interpretation need not always make sense: stability of the ground state only demands that $-\Box_g + \sigma$ be positive, which does not always imply that $\sigma$ itself be positive. The correlation function for fluctuations around the background $b_i$ is the inverse of the operator $-\Box + \sigma$. The condition of stability thus coincides with the condition that these correlations not increase with distance. The correlation length of the system should be defined as $\frac{1}{\sqrt{\lambda}}$, $\lambda$ being the smallest eigenvalue of the operator $-\Box_g + \sigma$. In flat space this is the same as $m^{-1}$, but in curved spaces, in general this is not the case.

3. Flat space ($R^3$)

We will see in the next section that the short distance divergences of the Green’s function, $G_\Lambda(x,x;m^2,g)$, are independent of the curvature of the space. It is therefore natural to first study our theory on $R^3$. Though this is a very well studied case, [2], [4], let us briefly recapitulate the study of the fixed points in this theory. On $R^3$, the Ricci scalar is zero. Therefore the saddle point solutions are given by the gap equations,

$$m^2 b = 0$$ \hspace{1cm} (10)

$$b^2 = \frac{\Lambda}{\lambda(\Lambda)} - G_\Lambda(x,x;m^2,g).$$ \hspace{1cm} (11)

In flat space, $-\Box_{R^3} = -(\partial_x^2 + \partial_y^2 + \partial_z^2)$. The spectrum of the conformal laplacian is $\text{Sp}(-\Box_{R^3}) = k^2$ where $\vec{k}$ is a vector in $R^3$. Therefore,

$$G_\Lambda(x,x;m^2,g) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{(k^2 + m^2)}$$
The gap equation (10) admits 3 possible solutions with $m^2 = 0, b \neq 0$ or $m^2 \neq 0, b = 0$ or both $m^2$ and $b$ vanish.

(1) When $m^2 = 0$ and $b \neq 0$,

$$b^2 = \frac{\Lambda}{\lambda(\Lambda)} - \frac{1}{(2\pi)^3} \int d^3k \frac{k^2}{k^2}$$

At the critical coupling $\lambda_c(\Lambda)$ given by

$$\frac{\Lambda}{\lambda_c(\Lambda)} = \frac{1}{(2\pi)^3} \int d^3k \frac{k^2}{k^2},$$

$$b^2 = \frac{\Lambda}{\lambda(\Lambda)} - \frac{\Lambda}{\lambda_c(\Lambda)}$$

vanishes.

(2) When $m^2 \neq 0$ and $b = 0$,

$$\frac{\Lambda}{\lambda_c(\Lambda)} - \frac{\Lambda}{\lambda(\Lambda)} = \frac{m^2}{(2\pi)^3} \int d^3k \frac{k^2}{k^2(k^2 + m^2)}$$

$$= \frac{m}{4\pi}$$

from which it is seen that $m > 0$ only when $\lambda > \lambda_c$. This corresponds to the unbroken phase of the $O(N)$ symmetry. When $\lambda < \lambda_c$, $m < 0$ which is unphysical. If we do a more careful analysis as in [2], we will be able to see that $m = 0$ when $\lambda < \lambda_c$. This is the broken phase where the $O(N)$ symmetry is spontaneously broken and has given rise to $N - 1$ Goldstone bosons with $m = 0$. At $\lambda = \lambda_c$, which separates the two phases, $m$ goes to zero. This is the non–trivial fixed point of the theory, the trivial fixed point being $\lambda_c = 0$. We therefore see that at $m = 0$ and $b = 0$, we have the critical theory at the non–trivial fixed point. This fixed point is UV stable.

4. Short distance divergence structure of the Green’s function

We would next like to study our theory on curved 3–dimensional manifolds $M$. Before we proceed, we would like to establish that the only divergences that arise in the Green’s function, in the gap equation in curved space, are those which arise on $R^3$.

$$G_\Lambda(x, x; \sigma, g) = \langle x | (-\Box_g + \sigma)^{-1} | x \rangle_\Lambda = \int_0^\infty dt \langle x | e^{-(-\Box_g + \sigma)t} | x \rangle$$

(12)
Let $\lambda_n$ and $\psi_n(x)$ be the eigenvalues and eigenvectors of $-\Box_g + \sigma$. The Green’s function can be written as follows:

$$G_\Lambda(x, x; \sigma, g) = \int_0^\infty dt \ h(t; x, x)$$

where $h(t; x, x)$ is the heat kernel of the operator $-\Box_g + \sigma$ and has the formal expansion

$$h(t; x, x) = \sum_n e^{-\lambda_n t} \psi_n^*(x) \psi_n(x).$$

The divergence in the Green’s function comes from short distance and hence from small $t$ region. Let us therefore isolate the divergent part, from the finite part in the Green’s function. This can be done by invoking the asymptotic expansion of the heat kernel, [13] which, in $d$ dimensions, is given by

$$h(t; x, y) \sim \frac{e^{-l^2(x, y)/(4t)}}{(4\pi t)^{d/2}} \sum_{n=0}^\infty a_n(x, y) t^n,$$  \hfill (13)

where $l(x, y)$ is the Riemannian distance between points $x$ and $y$ on the manifold $M$. The short distance behaviour of the Green’s function is determined by the short time behaviour of the heat kernel. In particular, the divergence in $G_\Lambda(x, x, \sigma, g)$ is determined by the leading terms in the short time expansion of the heat kernel. If we formally substitute the asymptotic expansion of the heat kernel for $d = 3$, we find that the leading term, with $a_0(x) = 1$, is the only term that contributes to the divergence in the Green’s function:

$$G_\Lambda(x, x; \sigma, g) = \frac{\Lambda}{4(\pi)^{d/2}} + \text{finite part}.$$  

We see that the divergent part of the Green’s function is independent of the metric $g_{\mu\nu}(x)$. Thus the critical value of the coupling constant is independent of the metric. This argument is not quite mathematically rigorous, since the expansion we are using is only asymptotic. However, this method has been used successfully in many renormalization calculations and we believe it can be made rigorous using bounds on the short distance behaviour of the heat kernel.

We will proceed to study the $O(N)$ sigma model on some curved manifolds at the critical point $\lambda_c(\Lambda)$. We find that at this critical point the free energy is finite, confirming the above argument.
5. Zeta function regularization

Although regularizations such as the Pauli-Villars regularization are easier to understand physically, the zeta function regularization is more tractable when we are in curved space. We will thus be using this regularization in our work. Since the critical value of the coupling constant at which the theory becomes finite is independent of the background metric, we compute this critical coupling on $\mathbb{R}^3$. The Green’s function, $G(x, x; m^2, g)$, is regularized as,

$$G_s(x, x; m^2, g) = \langle x | (-\Box_g + m^2)^{-s} | x \rangle = \zeta_g(s, x).$$

$\zeta_g(s, x)$ is the ‘local zeta function’,

$$\zeta_g(s, x) = \sum_{\lambda_n \neq 0} |\lambda_n|^{-s} |\psi_n(x)|^2$$

where $\{\psi_n(x)\}$ is an orthonormal basis in the $\lambda_n$-eigenspace; $\lambda_n$’s being the eigenvalues of $(-\Box_g + m^2)$ and the sum includes degeneracies. If the eigenvalues are continuous the sum gets replaced by an integral. Then,

$$G(x, x; m^2, g) = \lim_{s \to 1} \zeta_g(s, x). \quad (14)$$

(On homogeneous spaces such as the ones we will be considering in this paper, $\zeta(s, x)$ turns out to be independent of $x$.) The gap equation on $\mathbb{R}^3$ in this regularization is

$$\lim_{s \to 1} \left[ \frac{1}{\lambda(s)} \right] = b^2 + \int \frac{d^3k}{(2\pi)^3(k^2 + m^2)^s} = b^2 + \frac{1}{2\pi^2} \int_0^\infty dt \frac{t^{s-1}}{\Gamma(s)} \int k^2 dk e^{-(k^2 + m^2)t},$$

where the regularized coupling $\frac{\Lambda}{\lambda(\Lambda)}$ in the Pauli-Villars regularization has been replaced by $\frac{1}{\lambda(s)}$ in the zeta function regularization. Here we have used the Mellin transform to analytically continue the zeta function. Note that

$$\zeta(s, x) = \int \frac{d^3k}{(2\pi)^3(k^2 + m^2)^s}$$

has no pole at $s = 1$. (The local zeta function is seen to be independent of $x$ due to the homogeneity of the space $\mathbb{R}^3$). It is now easy to verify that,

$$\lim_{s \to 1} \frac{1}{\lambda(s)} = b^2 - \frac{m^{3-2s}}{(4\pi)^{\frac{3}{2}}} \frac{\Gamma(s - \frac{3}{2})}{\Gamma(s)}.$$
Recall, by analytic continuation of the gamma function, we can see that \( \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi} \) and thus,
\[
\lim_{s \to 1} \frac{1}{\lambda(s)} = b^2 + \frac{m}{4\pi}.
\]
We have seen though from the analysis in the case of \( M = R^3 \) that at the critical point, \( m = 0 \) and \( b = 0 \). \( m \) and \( b \) are physical quantities and are regularization independent. Therefore in the zeta function regularization, \( \lim_{s \to 1} \frac{1}{\lambda_c(s)} = 0 \). We will be using this value of the critical coupling in all our future calculations.

The critical coupling, while independent of the background metric, does depend on the regularization scheme. In the \( \zeta \)-function regularization, it is defined through some analytic continuations and \( \frac{1}{\lambda_c} \) has the value zero. This means that one of the phases of the theory would correspond to negative values of the coupling constant. There is no contradiction here, this is just a peculiarity of this regularization method. Physical quantities such as free energy still make sense.

In the zeta function regularization, the free energy density of the \( O(N) \) sigma model in the large \( N \) limit has a rather simple form:
\[
W(g) = \frac{N}{2} \text{TrLog}(-\Box_g + m^2).
\]
Let us now consider the conformal invariance of the free energy density in the case of a general curved space. Temporarily we consider the situation in arbitrary dimension \( d \), which will make clear the special situation in an odd dimension such as three (i.e., there is no conformal anomaly).

If \( g_{ij} \to e^{2f} g_{ij} \) and \( \phi \to e^{(1-\frac{d}{2})f} \phi \), the action \(-\int \sqrt{g} d^d x \phi \Box_g \phi \) is invariant. If we assign the transformation \( \sigma \to e^{-2f} \sigma \), the action \( \int \phi [-\Box_g + \sigma] \phi \sqrt{g} d^d x \) is still conformally invariant. We can define a determinant for this operator by
\[
\det[-\Box_g + \sigma] = \begin{cases} 
0, & \text{if } \dim(\ker[-\Box_g + \sigma]) \neq 0 \\
(-1)^v e^{-\zeta'(0)}, & \text{if } \dim(\ker[-\Box_g + \sigma]) = 0
\end{cases}
\]
where \( v \) is the number of negative eigenvalues (assumed to be finite in number) of \(-\Box_g + \sigma\). Also, the zeta function, \( \zeta(s) = \sum_{\lambda_n \neq 0} |\lambda_n|^{-s} \), \( \lambda_n \) being the eigenvalues of \(-\Box_g + \sigma\).

**Theorem.** Given a \( d \) dimensional compact manifold \( M \), \( \det[-\Box_g + \sigma] \) is conformally invariant under the transformation of the metric \( g_{\mu\nu} \to e^{2f(x)} g_{\mu\nu} \) and \( \sigma \to e^{-2f} \sigma \), if \( M \) is odd dimensional. If \( M \) is even dimensional,
\[
\delta_f \det[-\Box_g + \sigma] = -(4\pi)^{\frac{d}{2}} \det[-\Box_g + \sigma] \int_M 2f(x) \text{tr}_g \frac{d}{2}(x).
\]
$a_d(x)$ is one of the coefficients in the asymptotic expansion of the heat kernel $h(t; x, y)$ which is a solution of the diffusion equation,

$$(\partial_t - \Box_g + \sigma)h(t; x, y) = 0, \quad t > 0; \quad h(0; x, y) = \delta(x, y).$$

The asymptotic expansion of $h(t; x, y)$, as we have seen before, is,

$$h(t; x, y) \sim e^{-l^2(x,y)/4t} \sum_{k=0}^{\infty} a_k(x,y) t^k$$

where $l(x,y)$ is the distance between the points $x$ and $y$ on $M$.

A proof of this theorem in the case $\sigma = 0$, in the zeta function formulation is given in [14]. This proof can be extended without much difficulty to our case where the operator is $-\Box_g + \sigma(x)$. The explicit expression for $a_k(x)$ will change, but this is not of interest in the odd dimensional case.

Rather than grind through the proof with the minimal changes required to Ref. [14] we will give a simple (but not rigorous) argument which we believe captures the essence of the conformal anomaly. Under an infinitesimal conformal transformation, the operator $-\Box_g + \sigma$ transforms like $\sigma$ itself: a density of weight $-2$. The variation of the $\zeta$-function is,

$$\delta \text{Tr}[-\Box_g + \sigma]^{-s} = -s \text{Tr}[-\Box_g + \sigma]^{-s-1} \delta[-\Box_g + \sigma] = (2s) \text{Tr}[-\Box_g + \sigma]^{-s} \delta f.$$ \hspace{1cm} (16)

Differentiating the two sides with respect to $s$ and setting $s = 0$, we get

$$\delta \log \det[-\Box_g + \sigma] = -\delta \zeta'(0) = -2 \lim_{s \to 0} \text{Tr}[-\Box_g + \sigma]^{-s} \delta f.$$ \hspace{1cm} (17)

Although this statement is in essence true, to justify the limits, some more work is necessary.

Formally the r.h.s. is the trace of the identity operator weighted by $\delta f$. This can be evaluated by putting in the asymptotic form of the heat kernel and performing a Mellin transform. The only term which will contribute (in the limit) is $a_d$, as can be verified easily. These steps are common in the computation of anomalies in the physics literature. The proofs in [14] are somewhat more precise.

To summarize, the large $N$ limit of the $O(N)$ sigma model at the fixed point can be viewed as a classical field theory with action

$$\Gamma[b_i, \sigma] = \frac{1}{2} \int b_i[-\Box_g + \sigma]b_i \sqrt{g} d^3 x + \frac{1}{2} \log \det[-\Box_g + \sigma].$$ \hspace{1cm} (18)
The gap equations are the Euler–Lagrange equations of this non–local action. This action is conformally invariant under the above transformations. In this sense the large $N$ limit of the $O(N)$ sigma model is a conformal field theory. The field $b_i$ is a primary field in the sense that its correlation functions are conformally covariant.

We are now ready to study specific examples. The next two special cases we investigate are the $O(N)$ sigma model on the manifolds, $R^2 \times S^1$ and $S^1 \times S^1 \times R$, both of zero Riemann curvature. They are however not conformally equivalent to $R^3$. We will see that in these cases $m$ is non-zero at the critical point. In these cases, $m$ is the inverse correlation length and therefore the correlation length is finite at criticality. This is due to the finite size of the manifold in some directions. More generally, whenever the manifold is not conformally equivalent to $R^3$, we should expect the value of $m$ at criticality to be non-zero. Our computations confirm this conjecture.

6. Finite size effects on $R^2 \times S^1_\beta$  

This case has been well studied in the papers [3], [4] in the Pauli-Villars regularization. We will study it in the zeta function regularization [15] to make later comparisons easier. Also, the calculation is technically simpler in the zeta function regularization.

On $R^2 \times S^1$ we choose the metric tensor to be $g_{\mu\nu} = (1, 1, \beta^2)$. The radius of the circle, $\beta$, can be thought of as inverse temperature for a system on the 2–dimensional flat space $R^2$. The conformal laplacian on this space is

$$-\Box_g = -(\partial_x^2 + \partial_y^2 + \frac{1}{\beta^2} \partial_\theta^2),$$

where $(x, y)$ are co-ordinates in $R^2$ and $\theta$ is the local co-ordinate on $S^1$. The spectrum of $-\Box_g$ is,

$$Sp(-\Box_g) = k^2 + \frac{4\pi^2 n^2}{\beta^2},$$

$n = 0, \pm 1, \pm 2, \cdots$. The gap equations are

$$m^2 b = 0$$  \hspace{1cm} (19)

$$\lim_{s \to 1} |b^2| = \frac{1}{\lambda(s)} - \frac{1}{\beta} \sum_n \int \frac{d^2 k}{(2\pi)^2(k^2 + \frac{4\pi^2 n^2}{\beta^2} + m^2)^s}. \hspace{1cm} (20)$$
At the critical point \( \lim_{s \to 1} \frac{1}{\lambda(s)} = 0 \). Taking the Mellin transform of the above equation, we get,

\[
- \lim_{s \to 1} \frac{1}{\beta} \int_0^\infty dt \frac{t^{s-1}}{\Gamma(s)} \sum_n \int_0^\infty dk \frac{k}{2\pi} e^{-(\frac{4n^2}{\beta^2} + m^2)t} = b^2.
\]

We integrate over \( k \) and use the Poisson sum formula to separate out the divergent part in the small \( t \) region of the sum so that we are now able to interchange the sum and the integral. The Poisson sum formula in this case is,

\[
\sum_n e^{-\left(\frac{4n^2}{\beta^2}\right)t} = \frac{\beta}{(4\pi t)^{\frac{1}{2}}} \sum_n e^{-\frac{n^2}{4t}}.
\]  

(21)

On using this formula, the gap equation simplifies to,

\[
- \lim_{s \to 1} \left[ \sum_{n=1}^\infty \frac{2}{(4\pi)^{\frac{1}{2}}} \int_0^\infty dt \frac{t^{s-\frac{1}{2}}}{\Gamma(s)} e^{-(m^2 t + \frac{n^2}{4t})^2} + \frac{1}{(4\pi)^{\frac{1}{2}}} \int_0^\infty dt \frac{t^{s-\frac{1}{2}}}{\Gamma(s)} e^{-m^2 t} \right] = b^2.
\]

We use the standard integral,

\[
\int_0^\infty dt \ t^{-\nu} e^{-(\frac{\sigma}{\gamma} + \gamma t)} = 2 \left( \frac{\sigma}{\gamma} \right)^{\frac{\nu}{2}} K_\nu(2\sqrt{\sigma\gamma}); \quad \text{Re}\sigma > 0, \text{Re}\gamma > 0,
\]

(22)

where \( K_\nu \) is the MacDonald’s function. The gap equation is then:

\[
- \lim_{s \to 1} \left[ \sum_{n=1}^\infty \frac{4}{(4\pi)^{\frac{1}{2}}} \left( \frac{n^2}{4m^2} \right)^{\frac{s}{2}} K_{s-\frac{1}{2}}(m\beta n) + \frac{m^{3-2s}}{(4\pi)^{\frac{1}{2}}} \Gamma(s - \frac{3}{2}) \right] = b^2
\]

Noting that, \( K_{-\frac{1}{2}}(m\beta n) = K_{\frac{1}{2}}(m\beta n) = \left( \frac{\pi}{2m\beta n} \right)^{\frac{1}{2}} e^{-m\beta n} \), in the limit \( s \to 1 \) the gap equation reduces to a very simple equation:

\[
\frac{1}{2\pi_\beta} \log 2 \sinh \left( \frac{m\beta}{2} \right) = b^2.
\]  

(23)

We know from equation (19) that either \( m = 0 \) or \( b = 0 \) or both. The equation above is not satisfied for \( m = 0 \) since \( b^2 \) is always positive. Hence we require \( b = 0 \) to satisfy equation (19). This gives us \( m = 2T \log \tau \), as in [3], where \( \tau \) is the Golden mean \( \frac{1+\sqrt{5}}{2} \) and \( T = \frac{1}{\beta} \) and can be interpreted as the temperature of the system.
Free energy density on $R^2 \times S^1_eta$

The regularized free energy density, at the critical point, on $R^2 \times S^1$ is,

$$\frac{W_c(\beta)}{N} = \frac{1}{2} \text{TrLog}(-\Box + m^2) = -\frac{1}{2} \zeta'_{R^2 \times S^1}(0).$$

More explicitly,

$$\frac{W(\beta)}{N} = -\lim_{s \to 0} \frac{d}{ds} \frac{1}{2 \beta} \sum_n \int \frac{d^2 k}{(2\pi)^2 (k^2 + \frac{4\pi^2 n^2}{\beta^2} + m^2)^s}. \quad (24)$$

We Mellin transform the r.h.s and simplify the above expression as before to,

$$\frac{W(\beta)}{N} = -\frac{2}{(4\pi)^{3/2}} \left[ -\frac{\sqrt{\pi}}{3} m^3 - \left( \frac{2m}{\beta} \right)^{3/2} \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} K_{3/2}(m\beta n) \right]. \quad (25)$$

Recalling, $K_{\frac{3}{2}}(x) = -2\frac{d}{dx} K_{\frac{1}{2}}(x) - K_{\frac{1}{2}}(x)$, we obtain,

$$\frac{W(\beta)}{N} = -\frac{1}{(4\pi)^{3/2}} \left[ \frac{m^3}{3} + \left( \frac{2m}{\beta} \right) \sum_{n=1}^{\infty} \frac{e^{-(m\beta n)}}{n^2} + \left( \frac{2}{\beta^3} \right) \sum_{n=1}^{\infty} \frac{e^{-(m\beta n)}}{n^3} \right]. \quad (25)$$

We know though that at the critical point, $m\beta = \log \tau^2 = -\log(2 - \tau)$. We also recall the power series representation of the polylogarithm, $\sum_{n=1}^{\infty} \frac{x^n}{n^p} = \text{Li}_p(x)$. Putting all these together, the free energy density, in agreement with [3], is

$$\frac{W(\beta)}{N} = \frac{1}{(2\pi \beta^3)^3} \left[ \frac{1}{6} \log^3(2 - \tau) + \log(2 - \tau) \text{Li}_2(2 - \tau) - \text{Li}_3(2 - \tau) \right] = -\frac{2}{5\pi \beta^3} \text{Li}_3(1) \quad (26)$$

The last equality can be arrived at by using polylogarithm identities and is derived in Ref. [3]. Using the expression for the regularized free energy density obtained from hyperscaling for a d–dimensional slab geometry [16],

$$\frac{W(\beta)}{N} = -\frac{\Gamma(d/2)\zeta(d)}{\pi^{d/2} \beta^3} \hat{c},$$

it is seen that, $\hat{c} = \frac{4}{5}$ (a rational number). (Note, here $\zeta(d)$ is the Riemann zeta function.) It is therefore of some mathematical interest to study the free energy of the large $N$–limit of the $O(N)$ $\sigma$ model. There is a possibility that this is a three dimensional analogue of rational conformal field theory. We hope that the explicit calculations described here will be useful to test this conjecture.
7. Study of the $O(N)$ sigma model on manifolds of the type $\Sigma \times R$

In the next three subsections we will be studying the $O(N)$ sigma model on a manifold $M = \Sigma \times R$. We therefore discuss the general form of the gap equations on such manifolds before we consider specific cases of $\Sigma$.

The Green’s function we need can be written in terms of the geometry of the 2-manifold $\Sigma$:

$$G_s(x,x;m^2,g) = \langle x|(-\bar{\Box}_g + m^2)^{-s}|x\rangle$$

$$= \int_0^\infty \frac{dt}{\Gamma(s)} t^{s-1} \langle x|e^{-[-\nabla^2_\Sigma + \xi R - \partial^2_t + m^2]t}|x\rangle$$

$$= \frac{1}{\sqrt{4\pi}} \int_0^\infty \frac{dt}{\Gamma(s)} t^{s-\frac{3}{2}} \langle x|e^{-[-\nabla^2_\Sigma + \xi R + m^2]t}|x\rangle$$

$$= \frac{1}{\sqrt{4\pi}} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \zeta_\Sigma(s - \frac{1}{2}, x, m^2)$$

(27)

Here,

$$\zeta_\Sigma(s, x, m^2) = \langle x|[-\nabla^2_\Sigma + \xi R + m^2]^{-s}|x\rangle.$$

At the critical point, the gap equations will then be of the form,

$$(-\bar{\Box}_g + m^2(x))b_i(x) = 0$$

(28)

$$\sum_i b_i(x)^2 = -\frac{1}{2}\zeta_\Sigma(\frac{1}{2}, x, m^2).$$

(29)

The zeta function on the r.h.s. is analytic at $s = \frac{1}{2}$ so that this equation is finite (the singularities of the zeta function of an even dimensional manifold occur at negative integer values of $s$).

To evaluate $\zeta_\Sigma(s - \frac{1}{2}, x, m^2)$, we have to find the spectrum of the conformal laplacian, $-\nabla^2_\Sigma + \xi R + m^2$, on the space $\Sigma$. The specific examples we are going to consider are the ones with $\Sigma$ having zero, constant positive and constant negative curvature. In this case we can look for solutions of the gap equations with $m^2$ and $b_i$ constant; then $b_i$ can be chosen to have only one non–zero component.
7.1. $S^1 \times S^1 \times R$

For simplicity let us study the case where the radii of the two circles are the same and are denoted by $\rho$. This is a space of zero curvature. The Ricci scalar $R = 0$ and the conformal laplacian is just the ordinary laplacian and is given by

$$-\Box_g = -\frac{1}{\rho^2}(\partial_\theta^2 + \partial_\phi^2) + \partial_z^2,$$

where $\theta$ and $\phi$ are local co–ordinates on the two circles and $z$ the co–ordinate on $R$. The spectrum of the conformal laplacian is,

$$\text{Sp}(-\Box_g) = \frac{4\pi^2}{\rho^2}(p^2 + q^2) + k^2$$

where, $p, q = 0, \pm 1, \pm 2, \cdots$ and $k$ takes values on the real line. The gap equations on $S^1 \times S^1 \times R$ are,

$$m^2 b = 0 \quad (30)$$

$$\lim_{s \to 1} \left[ \frac{1}{\lambda(s)} - \frac{1}{2} \zeta_{S^1 \times S^1}(s - \frac{1}{2}, x) = b^2 \right] \quad (31)$$

At the critical point, $\lim_{s \to 1} \frac{1}{\lambda_c(s)} = 0$. We use the integral representation of the zeta function,

$$\zeta_{S^1 \times S^1}(s - \frac{1}{2}, x) = \frac{1}{\rho^2} \int_0^\infty dt \frac{t^{s-\frac{3}{2}}}{\Gamma(s - \frac{1}{2})} \sum_{p,q} e^{-\frac{4\pi^2}{\rho^2}(p^2 + q^2)t}m^2 t$$

and after performing calculations similar to that in the example of $R^2 \times S^1$, at the critical point, (31) simplifies to:

$$-\frac{1}{4} - \frac{1}{m\rho} \log(1 - e^{-m\rho}) + \sum_{p,q=1}^{\infty} \frac{e^{-\sqrt{(p^2 + q^2)m\rho}}}{m\rho \sqrt{p^2 + q^2}} = -\frac{\pi b^2}{m} \quad (32)$$

This equation is difficult to solve as the double sum we are left with is not an obvious one. We can see that $m \neq 0$ without actually solving the equation. In essence, we put in an ansatz $m \to 0$ and we will show that this is inconsistent with the gap equation and hence $m \neq 0$ in this case at the critical point. If $m$ were small, we could approximate the double sum by a double integral and the equation (32) can be written as,

$$-\frac{1}{4} - \frac{1}{m\rho} \log(1 - e^{-m\rho}) + \int_0^\infty dp \int_0^\infty dq \frac{e^{-\sqrt{(p^2 + q^2)m\rho}}}{m\rho \sqrt{(p^2 + q^2)}} = -\frac{\pi b^2}{m}$$

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The integral in the above expression can be easily performed using polar co-ordinates and is evaluated to be \( \frac{\pi}{2(\pi \rho)^2} \). Also, when \( m \to 0 \), we can approximate \( \log(1 - e^{-m \rho}) \) to \( \log m \rho \) and the above equation reduces to a transcendental equation for \( m \rho \),

\[
\frac{(m \rho)^2}{4} + (m \rho) \log m \rho - \frac{\pi}{2} = \pi b^2 m \rho^2.
\]  

(33)

It is immediately apparent from the above equation that \( m = 0 \) cannot be one of its solutions. This implies that \( b \) has to be zero at the critical point in order to satisfy the gap equation (30). Hence the solutions to the gap equations in the case of \( M = S^1 \times S^1 \times R \) are \( m \neq 0 \) and \( b = 0 \) as we expected and \( m \) is the solution to (32). Since \( m \neq 0 \) here, the smallest eigenvalue of \(-\Box_\rho + m^2\) is non-zero and therefore the correlation function will decay exponentially.

Once we solve for \( m \), the free energy density can be computed at the critical point with this value of \( m \).

7.2. \( S^2_\rho \times R \) - example of a space of constant positive curvature

Since the space \( S^2 \) with radius \( \rho \) has finite volume, we might tend to expect \( m \) to be non-zero at criticality. But we will see that this is not true; the manifold \( S^2 \times R \) is conformal to \( R^3 - \{0\} \) and it turns out that in fact \( m = 0 \). Thus we see that what matters, for \( m \) to be zero or otherwise, is the conformal class of the metric and not the ‘size’ of the system. \( m = 0 \) does not however mean an infinite correlation length on \( S^2 \times R \); the correlation length at criticality is in fact finite.

In order to find the conformal laplacian on \( S^2 \times R \), we need to calculate the Ricci scalar on \( S^2 \). This is a standard calculation and will in the end yield \( R = \frac{2}{\rho^2} \) and therefore \( \xi R = \frac{1}{4\rho^2} \). Since conformal curvature plays an important role in our study of the critical theory, it is worthwhile to demonstrate this equivalence of \( S^2 \times R \) to \( R^3 - \{0\} \).

Let the metric on \( S^2_\rho \times R \) be denoted by \( g_1 \) and that on \( R^3 - \{0\} \) by \( g \). The line element on \( S^2_\rho \times R \) is,

\[
\text{d}s^2_{S^2 \times R} = \rho^2(du^2 + d\Omega^2)
\]

where, \( u \) is the co-ordinate on \( R \) and \( d\Omega \) is the solid angle. The line element on \( R^3 - \{0\} \) in spherical polar co-ordinates is,

\[
\text{d}s^2_{R^3 - \{0\}} = dr^2 + r^2d\Omega^2.
\]
On writing $r$ as $r = \rho e^u$, the line element on $R^3 - \{0\}$ becomes,
\[
   ds^2_{R^3-\{0\}} = \rho^2 e^{2u} (du^2 + d\Omega^2).
\]

We immediately see that the metrics $g$ and $g_1$ are related by a conformal transformation, $g_1 = e^{2f} g$ with $f(x) = -u$. Thus the manifolds $S^2 \times R$ and $R^3 - \{0\}$ are conformally equivalent. We can now use this fact to fix $\xi R$ on $S^2 \times R$. For a conformal transformation, $g \to g_1 = e^{2f} g$, the scalar curvature term $\xi R$ transforms as follows [14]:
\[
   \xi R_1 = e^{-(d+2)f} \Box_g e^{(d-2)f}.
\]

Note, $\Box_g = \nabla^2_g$ since the Ricci scalar is zero for $R^3 - \{0\}$. Also, in our case, $d = 3$ and $f(x)$ is equal to $-u$. Hence,
\[
   \xi R_1 = e^{\frac{5u}{2}} \nabla^2_g e^{-\frac{u}{2}}
\]
where,
\[
   \nabla^2_g = \rho^{-2} e^{-2u} (\partial_u^2 + \partial_\theta^2 + \frac{1}{\sin^2 \theta} \partial_\phi^2).
\]
This gives $\xi R_1 = \frac{1}{4\rho^2}$. The conformal laplacian on $S^2 \times R$ is therefore given by
\[
   -\Box_{S^2 \times R} = -\nabla^2_{S^2 \times R} + \frac{1}{4\rho^2}
\]
with the spectrum as
\[
   \text{Sp}(-\Box_{S^2 \times R}) = \left[ \frac{(l + \frac{1}{2})^2}{\rho^2} + k^2 \right],
\]
where $l = 0, 1, 2, \cdots$ and $k \in R$ with degeneracy $(2l + 1)$. Notice that the conformal laplacian on $S^2 \times R$ has no zero modes.

The gap equations on $S^2 \times R$ are
\[
   \left( \frac{1}{4\rho^2} + m^2 \right) b = 0,
\]
\[
   \lim_{s \to 1} \left[ \frac{1}{\lambda(s)} - \frac{1}{2} \zeta_{S^2_{\rho}} (s - \frac{1}{2}, x) = b^2 \right].
\]

From equation (35) we see that $b = 0$ since $m^2 + \frac{1}{4\rho^2}$ cannot be equal to zero because both $m$ and $\rho$ are positive.
\[
   \zeta_{S^2_{\rho}} (s - \frac{1}{2}, x) = \frac{1}{\rho^2} \sum_{l=\frac{1}{2}}^{\infty} \frac{2l}{(l^2 + m^2)^{s-\frac{1}{2}}}.
\]
On using the Mellin transform of the above zeta function, at the critical point, (36) gives,

\[
\lim_{s \to 1} \int_0^\infty dt \frac{t^{s-\frac{3}{2}}}{\Gamma(s)} e^{-m^2 t} \sum_{l=\frac{1}{2}}^{\infty} 2l e^{-\frac{l^2}{2^s} t} = 0. \tag{38}
\]

As before we see that the sum is divergent in the small \( t \) region and we therefore have to separate out the divergent piece in the sum before we can interchange the sum and the integral over \( t \). To do this we need an analog of the Poisson sum formula for the case of the sum over half-integers \( l \) which turns out to be the following:

\[
\frac{1}{2\pi} \sum_{l=\frac{1}{2}}^{\infty} 2l e^{-\frac{l^2}{n^s} t} = \frac{\rho^2}{4\pi t} + \frac{\rho^2}{(4\pi t)\frac{3}{2}} P \int_{-\infty}^{\infty} dx \left( \frac{x}{2\rho} \csc \frac{x}{2\rho} - 1 \right) e^{-\frac{x^2}{4\pi}} \tag{39}
\]

where, by \( P \int_{-\infty}^{\infty} dx \left( \frac{x}{2\rho} \csc \frac{x}{2\rho} - 1 \right) e^{-\frac{x^2}{4\pi}} \) we mean the principal value of the integral; \((\frac{x}{2\rho} \csc \frac{x}{2\rho} - 1)\) has simple poles at all non-zero integral multiples of \( 2\pi \rho \). We give a brief derivation of this formula in appendix B. On using the Poisson sum formula (39), in the limit \( s \to 1 \), the gap equation reduces to,

\[
\frac{\rho}{\sqrt{\pi \rho}} \int_{-\infty}^{\infty} dx \left( x \csc x - 1 \right) \int_0^\infty dt \left( t^{-2} e^{-\frac{x^2}{2^s} - m^2 t} + m \Gamma\left( -\frac{1}{2} \right) \right) = 0. \tag{40}
\]

The integral over \( t \) can be easily performed using (22) and will give a Macdonald’s function. But this way we find it hard to extract the solution for \( m \) from the expression we get. We expect \( m = 0 \) to be the solution, though, since we showed that \( S^2 \times R \) is conformally equivalent to \( R^3 - \{0\} \). Let us therefore try putting the ansatz \( m = 0 \) as the solution in the l.h.s of the above gap equation and see if this is a consistent solution. On putting \( m = 0 \) in the l.h.s of (40) and performing the integral over \( t \), we get,

\[
\text{L.H.S} = \frac{1}{\sqrt{\pi \rho}} P \int_{-\infty}^{\infty} dx \frac{1}{x} \left( \csc x - \frac{1}{x} \right)
\]

We can easily check that the above integral is zero, thus giving us the l.h.s of the gap equation to be zero, consistent with the r.h.s. Hence \( m = 0 \) is indeed the correct solution to the gap equation in this case. We therefore find that at the critical point, \( m = 0 \) and \( b = 0 \) on \( S^2 \times R \).

Although \( m = 0 \) at criticality, the correlation length of this system is not infinite. If we consider the correlation function \( \langle \phi_i(x,u)\phi_j(y,0) \rangle \) as a function of \( x, y \in S^2 \) and \( u \in R \), it will decay like an exponential in \( u \) as \( u \to \infty \). (In the \( x, y \) directions the space has finite radius.) This is because the operator \( -\nabla^2_{\Sigma} + \xi R + m^2 \) has no zero modes.
Free energy density on $S^2 \times R$

The regularized free energy density on $S^2 \times R$ is given by,

$$ W(\rho) = -\lim_{s \to 0} \frac{d}{ds} \frac{1}{4\pi \rho^2} \int_{-\infty}^{\infty} dk \sum_{l=1}^{\infty} \frac{2l}{(k^2 + \frac{l^2}{\rho^2} + m^2) s} - \lim_{s \to 1} \int d^3 x \sqrt{g} \frac{m^2}{\lambda(s)}. $$

Using the analytic continuation of the zeta function, the Poisson sum formula (39) and performing the calculations which are fairly straightforward, we get, at the critical point,

$$ W_c(\rho) N = \frac{1}{16\pi} \left[ P \int_{-\infty}^{\infty} dx \left( \frac{x}{2\rho} \csc \frac{x}{2\rho} - 1 \right) \int_0^\infty dt \frac{\log t}{\Gamma(s)} e^{-\frac{x^2}{4t}} - P \int_{-\infty}^{\infty} dx \left( \frac{4}{x^2} \right)^2 \left( \frac{x}{2\rho} \csc \frac{x}{2\rho} - 1 \right) \right]. $$

The first term vanishes since in the limit $s \to 0$, $\Gamma(s) \to \frac{1}{s}$. We can again perform the integral over $x$ in the second term and it can be verified that it is zero. We see therefore that the regularized free energy density,

$$ \frac{W_c(\rho)}{N} = 0 \quad (41) $$

on $S^2 \times R$. This just means that the free energy density on $S^2 \times R$ is the same as that on $R^3$ at the critical point which is what we should expect from general considerations of conformal equivalence of the spaces $S^2 \times R$ and $R^3 - \{0\}$.

7.3. $H^2 \times R$ - example of a space of constant negative curvature

Let us consider as an example of constant negative curvature $M = H^2 \times R$, where $H^2$ is a two-dimensional hyperboloid. Let the co-ordinate on $R$ be denoted by $u$ and the hyperboloid be parameterized as

$$ H^2 = \{ z = (x, y) : x \in R, 0 < y < \infty \} $$

with line element and laplacian given by

$$ ds^2 = \frac{\rho^2}{y^2} (dx^2 + dy^2) \quad \text{and} \quad \nabla^2_{H^2} = \frac{y^2}{\rho^2} (\partial_x^2 + \partial_y^2) $$

respectively. The Ricci scalar on this space is $R = -\frac{2}{\rho^2}$. Therefore $\xi R = -\frac{1}{4\rho^2}$. The gap equations are,

$$ (-\frac{1}{4\rho^2} + m^2) b = 0 \quad (42) $$

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\[
\lim_{s \to 1} \left[ \frac{1}{\lambda(s)} - \frac{1}{2} \zeta_{H^2}(s - \frac{1}{2}, x) \right] = b^2. \tag{43}
\]

We have to now calculate \(\zeta_{H^2}(s - \frac{1}{2}, x)\). At the expense of wearying the reader, we give more calculational details in this example as it is qualitatively different from those we considered earlier.

To determine \(\zeta_{H^2}(s - \frac{1}{2}, x)\), we need to find the spectrum of the conformal laplacian, \(-\nabla^2_{H^2} + \xi R\) on the hyperbolic space. This spectrum is continuous and therefore we have to find the density of states \(\mu(\lambda)\). The object that is of interest to us is,

\[
\zeta(s - \frac{1}{2}, x) = \langle x | -\nabla^2_{H^2} - \frac{1}{4\rho^2} + m^2 \rangle^{-(s-\frac{3}{4})} | x \rangle = \int d\lambda \ \mu(\lambda)[\lambda - \frac{1}{4\rho^2} + m^2]^{-(s-\frac{3}{4})}
\]

(for simplicity, we assume that \(\rho = 1\)). If we define \(\nu\) by \(\lambda = \nu(1 - \nu)\), the resolvent,

\[
R(\nu) = [-\nabla^2_{H^2} - \nu(1 - \nu)]^{-1}
\tag{44}
\]

is a single valued function of \(\nu\). This means that the resolvent has a branch cut on the complex \(\lambda\) plane with \(\lambda = \frac{1}{4}\) as the branch point. The explicit form of the resolvent is given in [17]:

\[
R(\nu)f(z) = \int_{y > 0} \phi_{\nu}(u(z, z')) f(z') \frac{dx' dy'}{y'^2}
\]

where, \(z = (x, y), u(z, z') = \frac{|z - z'|^2}{4yy'}\) and,

\[
\phi_{\nu}(u) = \frac{1}{4\pi} \int_{0}^{1} dt \ [t(1-t)]^{\nu-1}(t + u)^{-\nu}. \tag{45}
\]

If we consider the matrix elements,

\[
\langle z | R(\nu) | z' \rangle = \phi_{\nu}(u(z, z'))
\]

we can get the density of states \(\mu(\lambda)\) which is given by the discontinuity of the resolvent of the Laplace operator across the branch cut,

\[
\mu(\lambda) = \frac{1}{2\pi i} [\langle z | R(\lambda + i\epsilon) | z' \rangle - \langle z | R(\lambda - i\epsilon) | z' \rangle]_{z = z'} \tag{46}
\]

to be,

\[
\mu(\lambda) = \frac{1}{8\pi} \Theta(-\frac{1}{4} + \lambda) \tanh \pi \sqrt{-\frac{1}{4} + \lambda}. \tag{47}
\]

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We can now compute $\zeta_{H^2}(s - \frac{1}{2}, x)$.

$$\zeta_{H^2}(s - \frac{1}{2}, x) = \int_0^\infty dt \frac{t^{s-\frac{3}{2}}}{\Gamma(s - \frac{1}{2})} \int_\frac{1}{4}^\infty d\lambda \tanh \pi \sqrt{-\frac{1}{4} + \lambda e^{-(\lambda - \frac{1}{4} + m^2)t}}$$

We change variables, $k^2 = -\frac{1}{4} + \lambda$ and perform the integral over $t$. On substituting the resultant expression for $\zeta_{H^2}(s - \frac{1}{2}, x)$ in the gap equation (43), we obtain, at the critical point,

$$[m + \int_0^\infty dk \frac{k}{\sqrt{k^2 + m^2}}(1 - \tanh \pi k)] = b^2. \quad (48)$$

We see that each term in the l.h.s is manifestly positive. Therefore, $b = 0$ cannot be a solution when $m \neq 0$. If $m = 0$, again it is easy to check that the l.h.s is non-zero (it is equal to $\frac{\ln 2}{\pi}$) and hence $b = 0$ cannot satisfy the above equation. Therefore, in order to satisfy the gap equation (42), we require,

$$m = \frac{1}{2 \rho}$$

(since we set $\rho = 1$, $m = \frac{1}{2}$) at the critical point. We see that negative curvature has induced the symmetry to be spontaneously broken at the critical point and there is a non-zero spontaneous magnetization given by the order parameter $b$.

The value $m = \frac{1}{2 \rho}$ is special in that $[-\nabla^2_{\Sigma} + \xi R + m^2]b = 0$ has a constant solution for that value. The correlation functions decay like a power law in the variable $u(z, z')$, but an exponential in the variable $d = \rho \text{arcosh} (1 + 2u)$ (which has the meaning of geodesic distance).

We can now evaluate the order parameter $b$ at criticality from (48). We expand $\frac{1}{\sqrt{k^2 + m^2}}$ binomially and simplify (48) to the following:

$$b^2 = [m + 2 \int_0^\infty dk \sum_{r=0}^\infty \left(\frac{1}{2}\right)^r \frac{k^{2r+1}}{(2r+1)!} e^{2\pi k}].$$

The integral over $k$ can be performed and and after restoring the appropriate factors of $\rho$, we can express $b$ at criticality as,

$$b^2 \rho = \sqrt{\frac{\pi}{2}} [1 + \sum_{r=0}^\infty \left(\frac{1}{2}\right)^r \frac{2}{\pi^{2r+2}}(2r + 1)!(1 - 2^{-2r-1})\zeta(2r + 2)]. \quad (49)$$
(this sum can also be written as a similar sum over Bernoulli numbers).

**Free energy density on** $H^2 \times R$

The free energy density on $H^2 \times R$ is given by,

$$W(\rho) = \frac{N}{2} \left[ \frac{1}{2\pi} \int dp \int d\lambda \; \mu(\lambda) \; \log(p^2 + \frac{\lambda}{\rho^2} + \frac{1}{4\rho^2} + m^2) - \lim_{s \to 1} \int \frac{d^3x}{\sqrt{g}} \frac{m^2}{\lambda(s)} \right]$$  \hspace{1cm} (50)

At the critical point, the regularized free energy density is given by (we again set $\rho = 1$),

$$\frac{W_c(\rho)}{N} = -\lim_{s \to 0} \frac{d}{ds} \left[ \frac{1}{4\pi} \int dp \int dt \frac{t^{s-1}}{\Gamma(s)} \int d\lambda \; \tanh \pi \sqrt{\frac{1}{4} + \lambda e^{-(\rho^2 + \frac{1}{4} + m^2)t}} \right]$$

where $m = \frac{1}{4\rho}$.

We first perform the integral over $p$ and change variables, $k^2 = -\frac{1}{4} + \lambda$ and then integrating over $t$ gives us the following expression for the free energy density:

$$\frac{W_c(\rho)}{N} = -\lim_{s \to 0} \frac{d}{ds} \left[ \frac{1}{4\sqrt{\pi}} \Gamma(s - \frac{3}{2}) \Gamma(s - \frac{1}{2}) \int_0^{\infty} dk \; k(m^2 + k^2)^{\frac{s-3}{2}} (\tanh \pi k - 1) \right].$$  \hspace{1cm} (51)

We once again use the binomial expansion of $(m^2 + k^2)^{\frac{1}{2} - s}$ and perform the integral over $k$ and obtain,

$$\frac{W_c(\rho)}{N} = -\lim_{s \to 0} \frac{d}{ds} \frac{1}{4\sqrt{\pi}} \Gamma(s - \frac{3}{2}) \Gamma(s - \frac{1}{2}) \int_0^{\infty} dk \; k(m^2 + k^2)^{\frac{1}{2} - s} \frac{m^{2r-2s+1}}{(2\pi)^{2r+2}} (2r + 1)!(1 - 2^{-2r-1})\zeta(2r + 2) \right].$$

After restoring the appropriate factors of $\rho$, this simplifies to the following expression:

$$\frac{W_c(\rho)}{N} = -\frac{1}{24\rho^3} - \frac{1}{4\rho^3} \sum_{r=0}^{\infty} \left( \frac{1}{2} \right) \frac{1}{\pi^{2r+2}} (2r + 1)!(1 - 2^{-2r-1})\zeta(2r + 2).$$  \hspace{1cm} (52)

(In the above equation and in (49), $\zeta(2r + 2)$ is the Riemann zeta function.)
O(N) sigma model at finite temperature

The examples considered above in this section described quantum phase transitions at zero temperature. The extensions to finite temperatures can be easily worked out. We will just state the results here, for the cases $S^1 \times S^1 \times S^1$ (a torus, which has zero curvature) and $S^2 \times S^1$ (constant positive curvature). In the example $S^1_\rho \times S^1_\rho \times S^1_\rho$, we find that $b = 0$ and the critical value of $m$ is a solution to the equation,

$$\int_0^\infty dt \ t^{-\frac{3}{2}} e^{-m^2 t/4} \left[ \theta_3^3(0, e^{-\frac{4m^2}{\pi^2}}) - 1 \right] + m \Gamma(-\frac{1}{2}) = 0 \quad (53)$$

where $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$ and the Theta function, $\theta_3(0, q)$, has the power series representation, $\theta_3(0, q) = \sum_n q^n$, $|q| < 1$. We can compute the free energy using this critical value of $m$.

In the case where the manifold is $M = S^2 \times S^1$, at the critical point, we find that $b = 0$. The critical value of $m$ is a solution to the following (finite) equation and the free energy can be evaluated with this value of $m$.

$$\sum_{n=-\infty}^\infty \ P \int_{-\infty}^\infty dx \frac{x \ cosec \frac{x}{2\rho} - 1}{x \ cosec \frac{x}{2\rho} - 1} \left( \frac{K_1(\sqrt{m(x^2 + n^2\beta^2)})}{\sqrt{x^2 + n^2\beta^2}} \right) - \frac{4\pi}{m\beta} \ Log2 \ sinh\left( \frac{m\beta}{2} \right) = 0 \quad (54)$$

where $K_1(x)$ is the MacDonald function.

**Discussion**  The following table is a summary of results obtained on various manifolds at the critical point. $m$ denotes the mass, $b$ the spontaneous magnetization and $W$ is the regularized free energy density.

| Manifold          | $m$       | $\langle \phi(x)\phi(y) \rangle$ | $b$   | $W$   |
|-------------------|-----------|----------------------------------|-------|-------|
| $\mathbb{R}^3$    | zero      | power law                        | zero  | zero  |
| $\mathbb{R}^2 \times S^1_\beta$ | $2\log \tau / \beta$ | exponential decay               | zero  | $\neq 0$ |
| $S^1 \times S^1 \times R$ | $\neq 0$ | exponential decay               | zero  | $\neq 0$ |
| $S^2 \times R$    | zero      | exponential decay               | zero  | zero  |
| $H^2_\rho \times R$ | $1/2\rho$ | power law in $u$ (exponential in $d$) | $\neq 0$ | $\neq 0$ |

These results, obtained by explicit calculations, show some surprises. The first line in the table is well-known; it is just the statement that the $O(N)$ $\sigma$-model has a second order phase transition. The spontaneous magnetization vanishes and the correlation length diverges at the transition point. In fact the ‘mass’ (vacuum expectation value of the $\sigma$ field) is in that case the inverse of the correlation length. But in more general curved
geometries, this relation need not hold. For example in the case $S^2 \times R$, the correlation length is actually finite although $m = 0$. It is finite in the $S^2$ direction because of the finite size of $S^2$. In the $R$ direction, the correlation decays exponentially. This case is conformally equivalent to $R^3$ which explains why $m, b$ and the free energy density vanish at criticality. On the other hand for the manifold $H^2 \times R$, the parameter $m \neq 0$; in fact the value of $m$ is such that correlation decays like an exponential (in the geodesic distance) in all three directions. The cases $R^2 \times S^1$ and $S^1 \times S^1 \times R$ have zero curvature and seem to fit with naive expectations. In the case $H^2 \times R$ we also find that at the critical point the spontaneous magnetization is non–zero. This could be relevant to the phase transition in two dimensional anti–ferromagnets (such as copper oxides [18]) when the lattice is subject to an external stress that bends it into a hyperboloid.

Appendix A: Conformal Geometry in Three Dimensions

Let $(M, g)$ be an oriented Riemannian manifold of dimension $d$. We are mostly interested in metrics of positive (Euclidean) signature. Two Riemann metrics $g, \tilde{g}$ are said to be conformally equivalent if there is a smooth function $f : M \to R$ such that $\tilde{g} = e^{2f} g$. Under such a transformation of the metric, $g \to e^{2f} g$, the angles between two vectors will be unchanged, but the lengths of vectors will change. A metric is conformally flat if it is conformally equivalent to a flat metric. The set of conformal transformations form an abelian group, the group of smooth functions $C^\infty(M)$ under addition.

We can define a ‘conformal structure’ on an oriented manifold $M$ to be a non–degenerate symmetric tensor density $\hat{g}_{ij}$ of weight $−2$ and with $\det \hat{g} = 1$. (We find it convenient to use the convention that the volume is a density of weight $d$). The determinant of $\hat{g}_{ij}$ is a scalar, so the condition that it be equal to one is diffeomorphism invariant. Given a Riemannian metric, a conformal structure is determined by $\hat{g}_{ij} = g^{-\frac{d}{2}} g_{ij}$, where $g$ is the determinant of $g_{ij}$. Clearly $\hat{g}_{ij}$ is invariant if we change $g_{ij} \to e^{2f} g_{ij}$, $\hat{g}_{ij}$ determines the equivalence class of the metric tensor under conformal transformations. It is reasonable to call $\hat{g}_{ij}$ the metric tensor density, since it determines the angle $\theta(u, v)$ between two vectors:

$$\cos \theta(u, v) = \frac{\hat{g}_{ij} u^i v^j}{\sqrt{\hat{g}_{kl} u^k u^l \hat{g}_{mn} v^m v^n}}. \quad (55)$$

This is a scalar although the ‘length’ of a vector $\sqrt{[\hat{g}_{kl} u^k u^l]}$ is a scalar density of weight $−2$. 

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If $\hat{\Gamma}$ is the space of conformal structures on $M$, the group $\text{Diff } M$ acts on it by pull-back:

$$\phi \in \text{Diff } M, \phi : \hat{\Gamma} \to \hat{\Gamma}, \hat{g} \mapsto \phi^* \hat{g}. \quad (56)$$

Explicitly in terms of co-ordinates,

$$\tilde{g}_{ij}(x) = [\det \partial \phi]^{-\frac{2}{d}} \hat{g}_{kl}(\phi(x)) \frac{\partial \phi^k}{\partial x^i} \frac{\partial \phi^l}{\partial x^j}. \quad (57)$$

Two metric densities that differ only by such a diffeomorphism are to be regarded as equivalent, as they only differ by ‘change of co-ordinates’. We will thus be interested in objects of conformal geometry that are covariant under this action of the group of diffeomorphisms $\text{Diff } M$ of $M$. These objects will be defined on the space $\hat{\Gamma} / \text{Diff } M$ of equivalence classes of conformal structures under the diffeomorphism group. This space however, is not in general a manifold, since the action of the group can have fixed points.

It is useful to study first the change of $\hat{g}_{ij}$ under the action of an infinitesimal diffeomorphism. This is the Lie derivative with respect to a vector field,

$$[\mathcal{L}_v \hat{g}]_{ij} = v^k \partial_k \hat{g}_{ij} + \partial_i v^k \hat{g}_{kj} + \partial_j v^k \hat{g}_{ik} - \frac{2}{d} \partial_k v^k \hat{g}_{ij}. \quad (58)$$

The last term arises from the fact that $\hat{g}_{ij}$ is a tensor density, of weight $-2$.

By the way, this might also be thought of as a covariant derivative that maps the covariant vector density $\hat{v}_i = \hat{g}_{ij} v^j$ to a symmetric traceless tensor density. After some calculation,

$$[\mathcal{L}_v \hat{g}]_{ij} = (D\hat{v})_{ij} = \partial_i \hat{v}_j + \partial_j \hat{v}_i - \frac{2}{d} \partial_k \hat{v}_i \hat{g}^{kl} \hat{g}_{ij} - 2 \hat{\Gamma}^k_{ij} \hat{v}_k. \quad (59)$$

Here,

$$\hat{\Gamma}^k_{ij} = \frac{1}{2} \hat{g}^{kl} [\partial_i \hat{g}_{lj} + \partial_j \hat{g}_{li} - \frac{2}{d} \hat{g}^{mn} \hat{g}_{ij} \partial_m \hat{g}_{nl}]. \quad (60)$$

Our construction shows that $D$ is a covariant derivative that maps covariant vector densities of weight $-2$ to symmetric traceless covariant tensor densities of weight $-2$. $\hat{\Gamma}^k_{ij}$ are the conformal geometric analogues of the Christoffel symbols of Riemannian geometry. It would be interesting to understand conformal curvature as a ‘commutator’ of such conformally covariant derivatives.

If $[\mathcal{L}_v \hat{g}]_{ij} = 0$, $v^i$ is a conformal Killing vector; then this infinitesimal diffeomorphism has $\hat{g}_{ij}$ as a fixed point. The compact manifold with the maximum number of such conformal Killing vectors is $S^n$; they form the Lie algebra $O(n + 1, 1)$. This space can also be
thought of as \( R^n \); the conformal killing vectors correspond then to translations, rotations, dilatations and some ‘special conformal transformations’:

\[
v_i = a_i + \theta_{ij} x^j + \lambda x_i + x^2 b_i - 2x_i b.x. \tag{61}
\]

The quotient space \( \hat{\Gamma} / \text{Diff } M \) will not then be a Hausdorff topological space and hence not a manifold. This difficulty can be avoided by restricting to the open dense subset of \( \hat{\Gamma} \) which does not have any conformal Killing vectors. Alternatively, we can restrict to the subgroup \( \text{Diff} \ p M \) which agrees with the identity map up to two derivatives, at one point \( p \). Even in the case with largest number of conformal Killing vectors, this condition will remove all of them. This will remove all conformal Killing vectors when \( M \) is compact.

There could still be fixed points due to finite conformal isometries; but they only lead to ‘orbifold’ type singularities in the quotient which can be removed by passing to a covering space. All objects of interest in conformal geometry will be sections of vector bundles over the space \( Q = \hat{\Gamma} / \text{Diff} \ p M \). A similar point of view was found to be very useful in studying Yang–Mills theories \[19\].

There is another point of view on conformal structures in low dimensions that is useful. Recall that on any oriented manifold the Levi–Civita symbols \( \epsilon_{i_1 \cdots i_d} \) and \( \epsilon^{i_1 \cdots i_d} \) are natural anti–symmetric tensor densities of weight \(-d\) and \(d\) respectively. If \( d = 2 \), this allows us to describe a conformal structure also by a tensor \( J^b_a = \hat{g}_{ac} \epsilon^{cb} \). We can see now that \( J^b_a J^b_c = \det \hat{g}_a c^{eb} \). This shows that a conformal structure is the same as a complex structure in dimension two (the integrability condition on \( J^b_a \) is trivial in two dimensions). The analogue when \( d = 3 \) is a tensor density \( c^{jk}_i = \hat{g}_{il} \epsilon^{ijk} \) of weight 1 satisfying

\[
c^{jk}_i c_{ml} + \text{cyclic}[jkl] = 0, \quad \det[c^{jk}_i c_{jl}] = 1. \tag{62}
\]

This will define a ‘cross product’ among covariant vector densities of weight \(-1\). The conformal metric can be recovered from the contraction \( \hat{g}^{ij} = \frac{1}{2} c^{ki}_i c^{lj}_l \). Thus in three dimensions a conformal structure is the same as a Lie algebra structure, isomorphic to \( SU(2) \), (or \( SU(1,1) \)) on vector densities of weight \(-1\). This point of view needs to be investigated further.

Returning to the study of \( Q \), let us first get an idea of how big \( Q \) is. We have a principal bundle \( \text{Diff} \ p M \to \hat{\Gamma} \to Q \). Let us ask what conditions a one–form in \( \hat{\Gamma} \) must satisfy in order that it be the pullback of a one–form in \( Q \) by the natural projection \( \pi : \hat{\Gamma} \to Q \). This will give an idea of the size of the cotangent space of \( Q \). A tangent vector
to $\hat{\Gamma}$ is a traceless symmetric covariant tensor density of weight $-2$ in $M$. A cotangent vector (one-form) in $\hat{\Gamma}$ is a traceless symmetric contravariant tensor density of weight $d+2$. The contraction of a vector $h$ in $\hat{\Gamma}$ with a one–form $t$ is then,

$$i_h t = \int_M t^{ij} h_{ij} d^d x.$$  \hfill (63)

Given a vector field in $M$, there is a vertical vector field in $\hat{\Gamma}$, given by the infinitesimal action:

$$h_{ij} = [D \hat{v}]_{ij}$$  \hfill (64)

where $D$ is the covariant derivative of $\hat{v}_i = \hat{g}_{ij} v^j$ defined earlier. If a one–form $t$ in $\hat{\Gamma}$ is the pull–back of a one–form in $Q$ it must annihilate all vertical vector fields:

$$\int_M t^{ij} [Dv]_{ij} d^d x = 0$$  \hfill (65)

This implies that the covariant divergence of $t^{ij}$ should be zero:

$$[D^* t]^i = 0$$  \hfill (66)

(Strictly speaking, $[D^* t]^i$ must be zero when contracted with vector fields that vanish up two derivatives at $p$. This means that $[D^* t]^i$ is a combination of derivatives of the delta function concentrated at $p$. If the manifold does not admit conformal Killing vectors, this subtlety can be ignored.)

This covariant divergence $[D^* t]^i$ can be defined in terms of $\hat{g}_{ij}$ alone:

$$[D^* t]^i = \partial_i t^{ij} - \hat{\Gamma}_{ik}^j t^{ik}$$  \hfill (67)

$\hat{\Gamma}_{ik}^j$ being the conformal analogues of the Christoffel symbols defined earlier. The covariant divergence of such a tensor density has a meaning within conformal geometry, without any reference to Riemann metric. This divergence is in fact a vector density of weight $d+2$.

So far we have considered $t_{ij}$ as a one–form at a point $\hat{g}_{ij}$ of $\hat{\Gamma}$. If it is the pull–back of a one–form in $Q$, $t_{ij}$ must change along the vertical direction in a way that is determined by the action of $\text{Diff}_p M$:

$$[\mathcal{L}_v t]^{ij} + \int \frac{\delta t^{ij}}{\delta \hat{g}_{kl}}(y) [D\hat{v}]_{kl}(y) d^d y = 0.$$  \hfill (68)

Conversely, any one–form in $\hat{\Gamma}$ satisfying the above conditions is the pullback of a one–form in $Q$.  

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The size of $Q$ is given by the number of independent solutions to the equation, \([D^* t]^i = 0\), among traceless symmetric tensor densities. A traceless tensor density has \(\frac{(d+2)(d-1)}{2}\) independent components at each point of $M$. The condition of having zero divergence is given by $d$ equations at each point of $M$. Hence the number of independent components in a one–form of $Q$ is \(\frac{(d-2)(d+1)}{2}\). A way to formalize this statement is that $Q$ is a manifold modelled over a vector space \([C^\infty(M)]^{(d-2)(d+1)}\).

The conditions of being traceless and divergence free are familiar properties of stress tensor of a conformal field theory. Indeed the stress tensor of a conformal field theory is a one-form on $Q$, when there is no ‘conformal anomaly’ (for example in three dimensions). Moreover, in general, the stress tensor will be a closed 1–form; it is not always exact. If the parity anomaly vanishes it is exact. Similar considerations arise in the canonical formalism of general relativity [20].

In particular, if $d = 2$, there are no local degrees of freedom in such a $t^{ij}$: the equations $[D^* t]^i = 0$ form an elliptic system. This does not mean that $Q$ is trivial, just that it is finite dimensional. The dimension of $Q$ (the number of independent solutions of $[D^* t]^i = 0$) is $6h - 6$ for a compact manifold of genus $h \geq 2$. The fact that $Q$ has no local degrees of freedom does imply that all 2–manifolds are locally conformally equivalent and hence conformally flat.

In the case of most interest to us, $d = 3$, there are two independent degrees of freedom per point of $M$. Thus there must be a conformal curvature tensor, which measures the local deviation from conformal flatness. We will discuss this tensor soon.

**Conformal Geometry from Riemannian Geometry**

We can view the space $\hat{\Gamma}$ as the quotient of the space of Riemannian metrics $\Gamma$ by the group $C^\infty(M)$. The principal bundle $C^\infty(M) \to \hat{\Gamma} \to \Gamma$ is defined by the group action

\[ g_{ij} \to e^{2f} g_{ij}. \]  

(69)

Thus it is also possible to study conformal geometry by looking at structures in Riemannian geometry invariant under the semi–direct product \(\text{Diff} \, M \otimes C^\infty(M)\).

The curvature (or Riemann) tensor transforms as follows under a conformal transformation [21]:

\[
R_{ijkl} \rightarrow \tilde{R}_{ijkl} = e^{2f}[R_{ijkl} + g_{il} f_{jk} + g_{jk} f_{il} - g_{ik} f_{jl} - g_{jl} f_{ik} + (g_{il} g_{jk} - g_{ik} g_{jl})|df|^2].
\]
Here,
\[ f_{ij} = \nabla_i \partial_j f - \partial_i f \partial_j f, \quad |df|^2 = g^{mn} f_{,m} f_{,n}. \] (70)

(For explicit computations there is still nothing better than the classical coordinate notation of tensor calculus.) From this, it follows that the Ricci tensor and Ricci scalar transform as follows:
\[ R_{ij} \rightarrow \tilde{R}_{ij} = R_{ij} + (d - 2) f_{ij} + g_{ij} \Delta f + (d - 2) |df|^2. \] (71)

\[ R \rightarrow \tilde{R} = e^{2f} [R + 2(d - 1) \Delta f + (d - 1)(d - 2) |df|^2] \] (72)

Here, \( \Delta f = g^{ij} f_{,ij} \) is the Laplacian. The traceless part of the Riemann tensor,
\[ C^h_{ijk} = R^h_{ijk} - \frac{1}{d - 2} \left[ \delta^h_i R_{jk} - \delta^h_k R_{ij} + g_{ik} R^h_j - g_{ij} R^h_k \right] + \frac{1}{(d - 1)(d - 2)} \left[ \delta^h_k g_{ij} - \delta^h_j g_{ik} \right] R \] (73)

is called the Weyl tensor. It is identically zero unless \( d \geq 4 \). The Weyl tensor is invariant under conformal transformations:
\[ C^h_{ijk} \rightarrow C^h_{ijk}. \] (74)

If \( d \geq 4 \), a manifold is conformally flat if and only if the Weyl tensor vanishes.

If \( d = 1 \) every manifold is flat, so in particular conformally flat. If \( d = 2 \), every manifold is locally conformally equivalent to flat space. This was already seen in the previous discussion on \( Q \).

When \( d = 3 \), the situation is more subtle: the Weyl tensor vanishes, but not every 3–manifold is even locally conformally flat. This is already clear from the previous discussion of the number of degrees of freedom of \( Q \). There is a tensor density, special to three dimensions that measures conformal curvature [21]:
\[ \Omega^{ij} = g^{1/4} \epsilon^{ikl} \nabla_k [R^j_i - \frac{1}{4} \delta^j_i R] \] (75)

Here \( \epsilon^{ij} \) is the Levi–Civita tensor (which depends on the choice of orientation on \( M \)) and \( g = \det g_{ij} \). This is a tensor density of weight 5 that is invariant under conformal transformations.

Thus \( \Omega^{ij} \) is a geometric object on the space \( \hat{\Gamma} = \Gamma / C^\infty(M) \): it is in fact a one–form. Moreover, a 3–manifold is locally conformally flat iff \( \Omega^{ij} = 0 \). In fact it is now possible to verify that it satisfies the conditions
\[ \Omega^{ij} = \Omega^{ji}, \quad \hat{g}_{ij} \Omega^{ij} = 0, \quad [D^* \Omega]^i = 0. \] (76)
Moreover, it satisfies the condition that it is a geometrical tensor, depending only on $g_{ij}$,
\[ \mathcal{L}_v \Omega^{ij} + \int \frac{\delta \Omega^{ij}}{\delta g_{kl}(y)} \mathcal{L}_v g_{kl}(y) d^3y = 0. \] (77)

These conditions have a simple interpretation: $\Omega^{ij}$ defines a 1–form on $\mathcal{Q}$. In fact it is a closed form. This can be verified by explicit computation of its exterior derivative. A better strategy is to use the fact [22] that the conformal curvature is the derivative of the Chern–Simons term.

Thus there is something special about conformal geometry in three dimensions: the conformal curvature is given by a tensor density peculiar to three dimensions. Also, the space of conformal structures carries a 1–form that measures the deviation of each point from local conformal flatness. There are two independent components in $\Omega^{ij}$ per point of $M$.

$\Omega^{ij}$ being zero only implies local conformal flatness. Typically, for compact $M$, there is a finite dimensional manifold of inequivalent conformal structures with $\Omega^{ij} = 0$ on some manifold $M$. This finite dimensional space is analogous to Teichmuller space. The equations $\Omega^{ij} = 0$ are the conformal analogues of the field equations of Chern–Simons theory; the solutions are parameterized by conjugacy classes of homomorphisms of the fundamental group of $M$ to the conformal isometry group of $R^3$, which is $O(4,1)$ (recall that the Teichmuller space of a Riemann surface surface is the set of conjugacy classes of homomorphisms $\pi_1(\Sigma) \to O(2,1)$) However, the study of the theory on such conformally flat manifolds does not seem to give sufficiently detailed information on phase transition. The major difference between two and three dimensional conformal field theory is that in three dimension, the effect of conformal curvature needs to be taken into account. This is why we believe that generalizations of conformal field theory to three (or higher) dimensions based on invariance under $O(n + 1, 1)$ describe only part of the story.

Appendix B: Poisson sum formula on $S^2$

The Poisson sum formula (21) is quite standard. We give here a brief derivation of the Poisson sum formula (39): We start with the general Poisson sum formula,
\[ \frac{1}{\beta} \sum_n e^{-4\pi^2 \frac{1}{\beta^2} (n+\sigma)^2} e^{2\pi i \frac{1}{\beta} (n+\sigma)} = \frac{1}{\sqrt{4\pi t}} \sum_n e^{-\frac{(x+n\beta)^2}{4t}} e^{-2\pi i n \sigma} \] (78)

If we choose $\sigma = \frac{1}{2}$ and $\beta = 2\pi$, and differentiate both sides of the above equation w.r.t $x$,
we obtain,
\[
\frac{i}{2\pi} \sum_n \left( n + \frac{1}{2} \right) e^{-t(n+\frac{1}{2})^2} e^{i(x(n+\frac{1}{2}))} = -\frac{1}{2t\sqrt{4\pi}\gamma} \sum_n \left( x + 2\pi n \right) e^{-\frac{(x+2\pi n)^2}{4t}} e^{-\pi in} \tag{79}
\]

Recall that,
\[
\sum_{l=\frac{1}{2}\text{ integers}} \text{sgn}(l) e^{ilx} = \sum_{l=\frac{1}{2}} \left( e^{ilx} - e^{-ilx} \right) = i \text{cosec} \frac{x}{2}
\]

Therefore,
\[
\text{sgn}(l) = \frac{i}{2\pi} \int_0^{2\pi} dx \ \text{cosec} \frac{x}{2} e^{-ilx} \tag{80}
\]

Use equation (80) to write (79) as,
\[
\frac{1}{2\pi} \sum_{l=\frac{1}{2}\text{ integers}} l \text{sgn}(-l) e^{-l^2t} = -\frac{1}{(4\pi t)^{\frac{3}{2}}} \sum_{n=-\infty}^{\infty} (-1)^n \int_0^{2\pi} dx \ (x + 2\pi n) e^{-\frac{(x+2\pi n)^2}{4t}}
\]

On letting \( x + 2\pi n \to x \) and on further simplification, we get,
\[
\frac{1}{2\pi} \sum_{l=\frac{1}{2}} l e^{-l^2t} = \frac{1}{(4\pi t)^{\frac{3}{2}}} \sum_{n=-\infty}^{2\pi n+2\pi} \int_{2\pi n}^{2\pi n} dx \ \frac{x}{2} \ \text{cosec} \frac{x}{2} e^{-\frac{x^2}{4t}} \tag{81}
\]

Rescaling \( t \), \( t \to \frac{t}{\rho^2} \), and extracting completely the small \( t \) divergence of the integral,
\[
\frac{1}{2\pi} \sum_{l=\frac{1}{2}} l e^{-\frac{t^2}{\rho^2}} = \frac{\rho^2}{(4\pi t)^{\frac{3}{2}}} \int_{-\infty}^{\infty} dx \ \left( \frac{x}{2\rho} \text{cosec} \frac{x}{2\rho} - 1 \right) e^{-\frac{x^2}{4\pi t}} + \frac{\rho^2}{4\pi t}
\]

which is the required Poisson sum formula (39).

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