Cut polytope has vertices on a line

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Abstract

The cut polytope $\text{CUT}(n)$ is the convex hull of the cut vectors in a complete graph with vertex set $\{1, \ldots, n\}$. It is well known in the area of combinatorial optimization and recently has also been studied in a direct relation with admissible correlations of symmetric Bernoulli random variables. That probabilistic interpretation is a starting point of this work in conjunction with a natural binary encoding of the $\text{CUT}(n)$. We show that for any $n$, with appropriate scaling, all encoded vertices of the polytope $1-\text{CUT}(n)$ are approximately on the line $y = x - 1/2$.

Keywords: Cut polytope, Bernoulli correlations.

1 Introduction

The cut polytope $\text{CUT}(n)$ is the convex hull of cut vectors in a complete graph with vertex set $\{1, \ldots, n\}$. It is well known in the area of combinatorial optimization as it can be used to formulate the max-cut problem, which has many applications in various fields, like statistical physics, in relation to spin glasses [1]. It has also been studied in relation with correlations of binary

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random variables. A symmetric Bernoulli random variable is such that takes values 0 and 1 with equal probabilities. The space of all \( n \)-variate symmetric Bernoulli r.v. is denoted by \( B_n \) and its correlation space \( R(B_n) \). It is well known that every correlation matrix belongs to \( E_n \), the set of symmetric positive semi-definite matrices with all diagonal elements equal to 1. For Gaussian marginals, the entirety of \( E_n \) can be realized, but surprisingly enough, for other common distributions very little is known [2]. For multivariate symmetric Bernoulli the problem was recently solved in [3] where the polytope \( R(B_n) \) was characterized by identifying its vertices. A relationship with the \( \text{CUT}(n) \) was established explicitly as \( R(B_n) = 1 - 2 \text{CUT}(n) \). A relation between \( \text{CUT}(n) \) and its approximation by \( E_n \) has also been studied in [4]. In this work the established relationship of \( 1-\text{CUT}(n) \) with \( R(B_n) \) is our starting point. We then use a natural binary encoding to study the vertices of \( 1-\text{CUT}(n) \) as integers. It is shown that, with appropriate scaling, for any \( n \), the encoded vertices of this polytope are approximately on the line \( y = x - 1/2 \). Consequently, the encoded vertices of \( \text{CUT}(n) \) are approximately on the line \( y = -(n-1)/2 \).

2 Cut polytopes

Let \( G = (V, E) \) be a graph with vertex set \( V \) and edge set \( E \). For \( S \subseteq V \) a cut of the graph is a partition \((S, S^C)\) of the vertices. The cut-set consists of all edges that connect a node in \( S \) to a node not in \( S \).

Let \( V_n = [n] = \{1, \ldots, n\} \), \( E_n = \{(i, j) : 1 \leq i \neq j \leq n\} \), and \( K_n = (V_n, E_n) \) be a complete graph with the vertex set \([n]\).

**Definition 2.1** For every \( S \subseteq [n] \) a vector \( \delta(S) \in \{0, 1\}^{E_n} \), defined as

\[
\delta(S)_{ij} = \begin{cases} 
1, & \text{if } |S \cap \{i, j\}| = 1 \\
0, & \text{otherwise}
\end{cases}
\]

for \( (1 \leq i < j \leq n) \), is called a cut vector of \( K_n \).

The cut polytope \( \text{CUT}(n) \) is the convex hull of all cut vectors of \( K_n \):

\[
\text{CUT}(n) = \text{conv}\{\delta(S) : S \subseteq [n]\}.
\]

**Remark 2.2** Since every cut vector is a vertex of \( \text{CUT}(n) \), there are \( 2^{n-1} \) vertices of this polytope [5].

Each \( \delta(\cdot) \) is a 0/1-vector (every coordinate value is either 0 or 1). The convex hulls of finite sets of 0/1-vectors are called 0/1-polytopes, out of which
cut polytopes are a sub-class. An excellent lecture on 0/1-polytopes, including CUT, is given in Ziegler [5]. A thorough treatment of cut polytopes can be found in Deza and Laurent [1]. The starting point for us here are results from Huber and Marić [3] where the cut polytopes are given a new probabilistic interpretation.

3 Cut polytopes via agreement probabilities

Definition 3.1 For an n-dimensional 0/1-vector \( x \) we define its concurrence vector as 
\[
\lambda(x) = (\lambda(x)_{12}, \lambda(x)_{13}, \ldots, \lambda(x)_{1n}, \lambda(x)_{23}, \ldots, \lambda(x)_{2n}, \ldots, \lambda(x)_{n-1n})
\]
where \( \lambda(x)_{ij} = \mathbf{1}(x(i) = x(j)) \), for \( 1 \leq i < j \leq n \).

Here \( \mathbf{1} \) denotes the indicator function: \( \mathbf{1}(A) = 1 \) if \( A \) is true and 0 otherwise. Applying the definition to an example \( x = (0, 1, 1, 0) \), \( \lambda(x) = (0, 0, 1, 1, 0, 0) \). Note that if \( x \) has \( n \) coordinates then \( \lambda(x) \) has \( \binom{n}{2} \) coordinates.

Introduction of the concurrence vector has its motivation from the context of symmetric Bernoulli random variables [2]. Let \( B_1, \ldots, B_n \sim \text{Bern}(1/2) \), that is \( P(B_i = 1) = P(B_i = 0) = 1/2 \), for all \( i \). The random vector \( (B_1, \ldots, B_n) \in \mathcal{B}_n \) takes values in \( \{0, 1\}^n \) and correlations among these variables are explicitly related to concurrence probabilities, i.e. probabilities of two variables taking the same value, \( P(B_i = B_j) \), for \( i \neq j \).

Let’s look at elements of \( \mathcal{B}_n \) (the set of all \( n \)-variate symmetric Bernoulli dist.) that are uniformly distributed over two diagonal points of \( \{0, 1\}^n \), where by a diagonal we mean the set \( \{x, 1-x\} \). There are \( 2^{n-1} \) such distributions and they play an important role for both \( \mathcal{B}_n \) and \( R(\mathcal{B}_n) \). Namely it was shown in [3] that the concurrence vectors associated to those diagonal distributions are precisely vertices of the polytope \( 1 \cdot \text{CUT}(n) \) (obtained by replacing all coordinates \( x_i \) by \( 1 - x_i \)).

3.1 Binary encoding

Let us look now at elements of \( \{0, 1\}^n \) encoded as a binary representation of numbers \( \{0, 1, 2, \ldots, 2^n - 1\} \). For instance we will identify \( (0, 1, 1, 1) \) with a binary number 0111, that is decimal number 7. Note that 00111 also represents decimal number 7, so when needed we will specify the number of bits used in representation of the specific number. The notation in that case will be \( x_{[k]} \), where \( k \) is the number of bits. When it is helpful for easier reading to emphasize that the number is represented in binary, we will add \( b \) in superscript, like \( x_{[k]}^b \). More notation: We will write two strings next to each
other with vertical dots in between to denote concatenation: if \( x = 001 \), then \( 0: x = 0001 \). Also \( \bar{x} \) is the complement of \( x \).

What happens with \( \lambda \) in this encoding? It becomes a function from \( \mathbb{N} \) to \( \mathbb{N} \) and in place of vectors we get integers that perhaps follow some interesting law.

Going back to the set \( \{0, 1, \ldots, 2^n - 1\} \), label the upper half of the points by \( x_1 = 2^{n-1}, x_2 = 2^{n-1} + 1, \ldots, x_{2^n-1} = 2^n - 1 \) (all binary). Take as an example \( n = 4 \), then \( x_1 = 1000, x_2 = 1001, \ldots, x_8 = 1111 \). Using the definition we can easily calculate the concurrence vectors associated to the numbers: \( \lambda(1000) = 000111, \lambda(1001) = 001100, \ldots, \lambda(1111) = 111111 \). As mentioned previously, these points (i.e. the associated 0/1 vectors) are vertices of the polytope \( 1\text{-CUT}(n) \). Obviously, out of \( 2^n \) vertices, \( v_1, \ldots, v_{2^n-1} \) we can directly identify two vertices of the polytope: \( v_1 = \lambda(x_1) = 2^{n-1} - 1 \) and \( v_{2^n-1} = \lambda(x_{2^n-1}) = 2^n - 1 \). Actually, for every \( k \) evaluating \( \lambda(x_k) \) is straightforward but it is interesting to see if there is a more general law between these integers.

**Remark 3.2** Note that \( \lambda(0:x) = \bar{x}: \lambda(x) \) and \( \lambda(1:x) = x: \lambda(x) \).

For example \( \lambda(0111) = 000111 = \overline{1111:1} \).

**Proposition 3.3** For \( x, y \) written using same number of bits and with 1 as a leading digit, if \( x < y \) then \( \lambda(x) < \lambda(y) \).

**Proof.** Suppose \( x \) and \( x + 1 \) can be written using same number of bits and have 1 as a leading digit. Then they can be written as \( x = 2^{n-1} + z \) and \( x + 1 = 2^{n-1} + z + 1 \). Then, \( \lambda(x + 1) = \lambda(1:z + 1) = z + 1: \lambda(z + 1) > z:11\ldots1 \geq z: \lambda(z) = \lambda(x) \). The strict inequality here is due to the fact that 1 at any position to the left from the \( : \) has more weight than all ones at the right side. \( \square \)

This proposition tells us that the sequence \( v_1, \ldots, v_{2^n-1} \) is increasing. Moreover, with appropriate scaling they are approximately on the line \( y = x - 1/2 \). This is the statement of the next theorem.

**Theorem 3.4** For \( k = 1, \ldots, 2^{n-1} \) and \( v_k = \lambda(x_k) \)

\[
\left| \frac{v_k}{2^{(n-1)} - (k - 1/2)} \right| < 1/2.
\]

That is, \( v_k / 2^{(n-1)} \) are approximately on the line \( y = k - 1/2 \) with residuals being at most 1/2.
Proof.

\[ \lambda(x_k) = \lambda((k - 1)^b_{[n-1]}) = (k - 1)^b_{[n-1]} \lambda((k - 1)^b_{[n-1]}) = (k - 1)2^{(n-1)/2} + \lambda((k - 1)^b_{[n-1]}) \]

Note that \( \lambda((k - 1)^b_{[n-1]}) \) has \( \binom{n-1}{2} \) bits and therefore \( \lambda((k - 1)^b_{[n-1]}) < 2^{(n-1)/2} \). Then

\[ \frac{\lambda(x_k)}{2^{(n-1)/2}} - (k - 1/2) = -1/2 + \lambda((k - 1)^b_{[n-1]})/2^{(n-1)/2} \]

which finishes the proof. \( \square \)

In the following figures \( v_k/2^{(n-1)/2} \) is plotted versus \( k = 1, \ldots, 2^{n-1} \), for \( n = 8 \) and \( n = 12 \). The fitted line \( y = x - 0.5 \) is obtained using linear regression (MATLAB). For \( n = 12 \) the residuals plot is also shown, and it can be clearly seen that residuals, in absolute value, are bounded by 1/2 as stated in the above theorem.
Note that the above analysis refers to the vertices of a polytope $\mathbf{1} \cdot \text{CUT}(n)$. If we denote by $c_k$ the appropriate encoded vertices of $\text{CUT}(n)$, then $c_k = 2^{\binom{n}{2}} - 1 - v_k$. A corollary of the Theorem 3.4 then says that $c_k/2^{\binom{n-1}{2}}$ are approximately on the line $y = -k + 2^{n-1} + 1/2$.

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