Pairing in nuclear systems with effective
Gogny and $V_{\text{low } k}$ interactions

A. Sedrakian$^a$, T.T.S. Kuo$^b$, H. Müther$^a$ and P. Schuck$^c$

$^a$ Institut für Theoretische Physik, Universität Tübingen, D-72076 Tübingen, Germany
$^b$ Department of Physics and Astronomy, State University of New York, Stony Brook, NY 11794-3800, U. S. A.
$^c$ Institut de Physique Nucléaire, IN2P3-CNRS, Université Paris-Sud, F-91406 Orsay Cedex, France

Abstract

The pairing properties of nuclear systems are a sensitive probe of the effective nucleon-nucleon interactions. We compare the $^1S_0$ pairing gaps in nuclear and neutron matter derived from the phenomenological Gogny interaction and a renormalization group motivated low-momentum $V_{\text{low } k}$ interaction extracted from realistic interactions. We find that the pairing gaps predicted by these interactions are in an excellent agreement in a wide range of sub-nuclear densities. The close agreement between the predictions of the effective forces remains intact in the case where the single particle spectra in neutron and nuclear matter are renormalized by nuclear interactions.

The effective nuclear forces are the key ingredients in the nuclear structure studies of finite nuclei within the Hartree-Fock-Bogoliubov and related mean field theories. Well-known examples are the zero-range Skyrme [1] and finite-range Gogny [2,3] forces that have extensively been used in the large scale numerical calculations of finite nuclei over several decades. The effective forces are commonly adjusted to the bulk properties of nuclear systems after a mean-field variational minimization of the ground state energy of a collection of nuclei within the density dependent Hartree-Fock or Hartree-Fock-Bogoliubov theories.

A distinctive feature of the finite range Gogny forces, which will be discussed below, is that these were adjusted by minimizing the Hartree-Fock-Bogoliubov energy functional that is a function of the pairing fields [2,3]. Thus, the pairing correlations in the system, seen experimentally in the odd-even staggering effects caused by the pairing in the isospin $T = 1$ states, are encapsulated in...
the effective force. Since, in general, the effective forces are subject to simple parameterizations and have the advantage of reducing the numerical cost of the extensive nuclear structure calculations, it remains an important task to scrutinize the reliability of effective forces in different contexts, in particular their relation to the interactions derived from the underlying microscopic theories.

During the recent years much effort went into formulations of nuclear interactions in terms of effective field theories [4,5,6,7,8,9,10,11]. The main goal of these theories is the separation of the long-range component of the nuclear forces, which is dominated by the pion exchange and is well under control, from the intermediate and short range components, which are dominated by correlated pion and heavy meson exchanges that are poorly known.

A line of approach, developed by the Stony-Brook group, applies the renormalization group arguments to the Lipmann-Schwinger (LS) equation to eliminate the high-momentum modes of a phase-shift equivalent potential $V_{NN}$ which serves as a driving term in the LS equation [12,13,14,15,16,17,18]. Note that low momentum nucleon-nucleon (NN) interactions were also derived by the Bochum-Jülich group by applying a method of unitary transformations to full (un-truncated) meson exchange interactions derived from chiral Lagrangians [10,11]. The Stony-Brook low-momentum NN interaction $V_{\text{low } k}$ is obtained by integrating out the high momentum components of $V_{NN}$ beyond a scale $\Lambda$. The following LS equations for the scattering amplitudes with driving terms $V_{NN}$ and $V_{\text{low } k}$ are considered:

$$T(k', k, k^2) = V_{NN}(k', k) + \int_0^{\infty} q^2 dq V_{NN}(k', q) \frac{1}{k^2 - q^2 + i0^+} T(q, k, k^2), \quad (1)$$

$$T_{\text{low } k}(p', p, p^2) = V_{\text{low } k}(p', p) + \int_0^{\Lambda} q^2 dq V_{\text{low } k}(p', q) \frac{1}{p^2 - q^2 + i0^+} T_{\text{low } k}(q, p, p^2). \quad (2)$$

Note that the intermediate state momentum $q$ is integrated from 0 to $\infty$ and 0 to $\Lambda$ in the first and second equation respectively. The equivalence of the $T$-matrices derived from the LS equation above is required: $T(p', p, p^2) = T_{\text{low } k}(p', p, p^2); \quad (p', p) \leq \Lambda$. As described in Ref. [14], the $V_{\text{low } k}$ interactions are then derived by the Andreozzi-Lee-Suzuki method [19].

The $V_{\text{low } k}$ so derived reproduces the empirical deuteron binding energy, NN scattering phase shifts up to $E_{\text{lab}} = 2\hbar^2 \Lambda^2 / M$. Experiments give us information about phase shifts only up to $E_{\text{lab}} \sim 350$ MeV. Thus an appropriate choice for $\Lambda$ is $\sim 2$ fm$^{-1}$. Beyond this momentum, $V_{NN}$ is model dependent and lacks physical ground. An interesting feature of the resulting effective interaction $V_{\text{low } k}$ is that it is largely independent of the underlying microscopic force that was used as a driving term in the LS equation [18]. This property clearly re-
fects the fact that the low-momentum part of the microscopic interactions is well constrained by the experimental data, i.e., they equally well reproduce the binding energy of the deuteron and are phase-shift equivalent [14,18]. The initial applications of the $V_{\text{low } k}$ interaction are in the shell model calculations [14,16,17] and in various treatments of the equation of state and the pairing in nuclear systems [15,20]. Clearly, the renormalization group based decimation procedure, adopted to derive the effective $V_{\text{low } k}$ interaction, bridges the gap between the effective and microscopic interactions in a controlled manner, thus it provides a good starting point for mean-field calculations.

The purpose of this Letter is to compare the predictions of the Gogny and $V_{\text{low } k}$ interactions for the pairing in nuclear systems in the $^1S_0$ interaction channel. We will be concerned with infinite, zero-temperature matter, parameterized in terms of the Fermi-momentum $k_F$ and will mainly focus on effects of the force on pairing. We shall briefly comment on the renormalization of the single particle spectrum in the mean-field approximation, but leave aside the issues of the vertex and propagator renormalizations beyond the mean field. Some aspects of the pairing, which are complementary to this study, are explored in Refs. [15,20] and [21,22] using the $V_{\text{low } k}$ and Gogny interactions, respectively. Our work, in part, is motivated by the observation that the Gogny interactions predict pairing properties that are surprisingly close to those derived from the bare realistic (phase shift equivalent) nuclear interactions [21].

The effective Gogny interactions are of the generic form

\begin{equation}
V(r_1 - r_2) = t_0 (1 + x_0 P_\sigma) \delta(r_1 - r_2) \rho^d
+ \sum_{m=1}^{2} [W_m + B_m P_\sigma - H_m P_\tau - M_m P_\sigma P_\tau] \exp \left( -\frac{|r_1 - r_2|^2}{\mu^2_m} \right),
\end{equation}

i.e. they contain two Gaussian terms which reflect the finite-range of the interaction, and a contact density-dependent term responsible for short range correlations. Here $\rho$ is the density and $P_\sigma, P_\tau$ are the spin and isospin exchange operators. For completeness we reproduce the values of the parameters according to the D1S parameterization in Table 1 [3]. (Compared to the original D1 parameterization [2] the D1S force gives a lower surface tension and, at the same time, a smaller even-odd staggering which is closer to the experimental values.)

The $V_{\text{low } k}$ interaction we shall employ is based on the Nijmegen 93 potential with a momentum cut-off $\Lambda = 2.5$ fm$^{-1}$, which corresponds to a distance scale $\Lambda^{-1} \sim 0.4$ fm. The choice of the underlying realistic potential is not important, as the effective interactions derived from various realistic interactions are practically identical (c.f. [18]).
The gap in the quasiparticle spectrum of infinite nuclear systems is governed by the Bardeen-Cooper-Schrieffer (BCS) integral equation for the gap function

$$\Delta(k) = -\frac{1}{2} \int_0^\Lambda dk' k'^2 V(k, k') \frac{\Delta(k')}{\sqrt{(\varepsilon_{k'} - \varepsilon_F)^2 + \Delta(k')^2}}, \quad (4)$$

where $V(k, k')$ is the momentum space effective pairing interaction, $\varepsilon_k$ is the quasiparticle spectrum, $\varepsilon_F$ is the Fermi-energy, and $\Lambda$ is momentum space cut-off. The high-momentum behaviour of the pairing interaction, for the case of Gogny force, is controlled by the finite range of the Gaussians; hence we can take safely the limit $\Lambda \to \infty$ in Eq. (4). The $V_{\text{low } k}$ interactions have a sharp cut-off at high-momenta, and it is natural to identify $\Lambda$ with the cut-off in the interaction. Note that when transformed in the momentum space the Gogny interaction depends on the momentum transfer in the process, and hence on the angle between the relative incoming and outgoing momenta of the particles. It is then suitable to average the matrix elements over the angle between the vectors $k$ and $k'$. The angle averaged $^1S_0$-wave pairing interaction can be written as [21]

$$V(k, k') = \frac{1}{\sqrt{\pi} kk'} \sum_{m=1}^{2} C_m \exp \left[ -\frac{\mu_m^2}{4} (k^2 + k'^2) \right] \sinh \left( \frac{\mu_m^2 kk'}{2} \right), \quad (5)$$

where $C_m \equiv \mu_m (W_m - B_m - H_m + M_m)$. Note that only the density independent part of the Gogny interaction contributes to the pairing in the $^1S_0$-channel. The angle averaged pairing interaction thus contains two Gaussian describing long range attraction and a short range repulsion (the terms $\propto C_1$ and $C_2$, respectively).

Next we need to specify the single particle spectrum, $\varepsilon_k$, in the gap equation (4). We shall employ two approximations. First, the single particle spectrum in a non-interacting limit will be used to understand the correlations between the pairing gap and the pairing force. Second, the single particle spectrum will be renormalized within the mean-field approximation. We shall use the Hartree-Fock single particle spectra for the Gogny interaction, which are derived below. For the $V_{\text{low } k}$ interaction we shall employ the Brueckner-Hartree-Fock scheme (see Ref. [20] and references therein.) It should be kept in mind that when the

| $m$ | $\mu_m$ | $W_m$  | $B_m$   | $H_m$   | $M_m$   |
|-----|---------|--------|---------|---------|---------|
| 1   | 0.7     | -1720.3| 1300.0  | -1813.53| 1397.6  |
| 2   | 1.2     | 103.639| -163.483| 162.812 | -223.934|

Table 1

The parameters of the D1S Gogny interaction [3]. The ranges of the interaction $\mu_m$ are in fm while the remainder coefficients are in MeV. The values of the parameters of the contact term are $t_0 = 1390$ MeV fm$^4$, $x_0 = 1$, and $d = 1/3$. 

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effective interactions depend on energy, e.g. when one is dealing with time-retarded interactions, the wave-function renormalization differs from unity and tends to counter-act the reduction of the mass caused by the momentum dependent self-energies. Thus, the net effect of medium renormalization of particle mass could be an overestimate.

To define the single-particle spectrum, it is useful to start with the expression for the ground state energy (at zero temperature) in the mean-field approximation

\[
E = \sum_i \frac{\hbar^2 k_i^2}{2m} n_i + \frac{1}{2} \sum_{ij} \langle ij | V | ij - jj \rangle n_i n_j , \quad (6)
\]

where the first term is the kinetic energy, the second term is the potential energy of the mean-field interaction; the indices refer to the nucleonic states, \( n_i \) are their occupation probabilities. In the case where the interaction is density dependent the single particle potential is given by the functional derivative of the second term \( E_{\text{int}} \) in Eq. (6), i.e. \( U_i \equiv \delta E_{\text{int}} / \delta n_i \), or explicitly

\[
U_\tau i = \sum_j \langle ij | V | ij - jj \rangle n_j + \frac{1}{2\Omega} \sum_{jl} \langle jl | \frac{\partial V}{\partial \rho_\tau} | jl - lj \rangle n_j n_l , \quad (7)
\]

where \( \tau (= n, p) \) is the isospin index, \( \Omega \) is the volume. Equation (7) should be solved self-consistently with the normalization condition for the total density \( \sum_i n_i = \rho \).

The matrix elements defining the single-particle potential are evaluated in the mean-field approximation using plane waves for the nucleon states. We find

\[
U_\tau(k) = \frac{t_0}{4} (2 + x_0) (2 + d) \rho^{d+1} - \frac{t_0}{4} (1 + 2x_0) \left[ 2x_\tau + d(x_p^2 + x_n^2) \right] \rho^{d+1} \\
+ \frac{1}{2} \sum_{m=1}^2 \rho F_m(0) \left[ (2W_m + B_m) - (2H_m + M_m) x_\tau \right] \\
+ \sum_{m=1}^2 \sum_{k'} F_m(k - k') \left[ (H_m + 2M_m) [n_p(k') + n_n(k')] - (W_m + 2B_m) n_\tau(k') \right] , \quad (8)
\]

where \( x_\tau = \rho_\tau / \rho \) are the relative concentrations of the neutrons and protons, \( n_\tau(k) = \theta(k_F - k) \) are their occupation probabilities, and we defined a momentum dependent form-factor as

\[
F_m(k - k') = \pi^{3/2} \mu_m^3 e^{-\mu_m^2 |k-k'|^2 / 4} . \quad (9)
\]
The first two terms in Eq. (8) correspond to the contributions of the contact interaction, where the terms proportional to $d$ originate from the rearrangement interaction. The third and the fourth terms are the direct and the exchange contributions of the finite range part of interaction. In the zero-temperature limit of interest the phase space integrals over the Fock term in Eq. (8) can be done analytically,

$$
U_{\tau}^{\text{Fock}}(k) = -\frac{1}{2} \sum_{m=1}^{2} \left[ W_m + 2(B_m - M_m) - H_m \right] \left\{ \frac{2}{\sqrt{\pi} \mu_m k} \left[ \exp \left( -\frac{\mu_m^2}{4}(k + k_{Fq})^2 \right) - \exp \left( -\frac{\mu_m^2}{4}(k - k_{Fq})^2 \right) \right] \right. \\
+ \text{Erf} \left[ \frac{\mu_m}{2}(k + k_{Fq}) \right] - \text{Erf} \left[ \frac{\mu_m}{2}(k - k_{Fq}) \right] \right\},
$$

and the net single particle energy $U_{\tau} = U_{\tau}^{\text{Hartree}} + U_{\tau}^{\text{Fock}}$, where $U_{\tau}^{\text{Hartree}}$ corresponds to the first three terms in Eq. (8), becomes an analytical function of the Fermi-momenta of neutrons ($k_{Fn}$) and protons ($k_{Fp}$). The single particle spectrum is then defined as $\epsilon_{\tau}(k) = k^2/2m + U_{\tau}(k)$. Although the equations above are specified for arbitrary isospin asymmetry, we shall further concentrate on the two special cases $x_p = x_n$ (symmetric nuclear matter) and $x_p = 0$ (pure neutron matter).

To understand the correlations between the effective pairing interactions and the pairing gap it is useful to fix the argument of the gap function on the left-hand-side of Eq. (4) at the Fermi-momentum, i.e.,

$$
\Delta(k) = -\frac{1}{2} \int_0^\Lambda dk' k'^2 V(k, k') \frac{\Delta(k')}{\sqrt{(\epsilon_{k'} - \epsilon_F)^2 + \Delta(k')^2}}.
$$

The kernel of the gap equation is a product of the momentum space matrix element and $V(k, k)$ which, as we shall see below, is a smooth function of momentum $k$ and the anomalous propagator (the remainder multiplier in the kernel) which is bell-shaped with the maximum $1/2$ for $\epsilon_k = \epsilon_F$. Since the main contribution to the integral comes from the vicinity of the Fermi-surface the differences in the pairing gaps reflect the differences in the effective forces in the vicinity of the Fermi-surface. This observation motivates an approximation where we replace $V(k, k)$ by $V(k, k_F)$. Such an approximation permits to solve Eq. (11) analytically and the result is the well know BCS weak-coupling formula. Table 2 lists the values of the gaps calculated numerically (without approximations) using free single-particle spectrum along with the diagonal elements of the effective interactions. A clear correlation is seen between the ratios of the gaps derived from Gogny and $V_{\text{low } k}$ interactions and correspond-
Table 2
The pairing gaps for different Fermi-momenta $k_F$ computed with the Gogny (second column) and the $V_{\text{low } k}$ interactions (third column). The fourth column shows the ratio, $R_{\Delta}$, of the gap values given in the second and third columns to make the variations with the effective interaction explicit. The fifth and sixth columns show the diagonal elements of the interactions $V_{\text{Gogny}}(k_F, k_F)$ and $V_{\text{low } k}(k_F, k_F)$ and the last column - their ratio, $R_{V}$.

The correlations between the momentum dependent pairing gaps $\Delta(k)$ calculated numerically (without approximations) with the momentum space matrix elements $V(k_F, k)$, which would correspond to a pairing gap derived from Eq. (11), can be observed in Fig. 1. Quite generally, the momentum dependence of the pairing gap reflects the momentum dependence of the pairing potential. This relation becomes explicit if one approximates the pairing force by a
Fig. 1. The pairing gaps in the $^1S_0$ channel and the corresponding pairing potentials $V(k_F, k)$ as functions of the momentum $k$ for several fixed Fermi-momenta $k_F$. The black and grey (green) lines refer to the $V_{\text{low } k}$ and the Gogny interaction respectively. The solid, dashed, dashed-dotted and dotted lines correspond to the values of $k_F$ equal 0.2, 0.6, 1.0 and 1.4 fm$^{-1}$.

separable potential. Writing, schematically, $V(k, k') = g(k)g(k')$ and inserting this form in the gap equation, one finds that the solutions are of the form $\Delta(k) = Cg(k)$ where $C$ is a constant.

The $V_{\text{low } k}$ interaction is attractive below the cut-off scale $\Lambda$ for the Fermi-momenta of interest and vanishes above it; therefore there is always an associated non-zero solution to the gap equation which likewise vanishes above
the cut-off scale. The high momentum tail of the Gogny interaction \((k \geq \Lambda)\) is slightly repulsive and the pairing gap changes its sign above this scale and vanishes at much higher momenta; clearly, these high-momentum components do not affect the values of the gaps at their Fermi-surface, since the kernel of the gap equation is sharply peaked at the Fermi-momentum. As seen in Fig. 1 the differences between the predictions of the effective forces for the gap functions are closely correlated with their differences in the vicinity of respective Fermi-momenta.

Fig. 2. The pairing gaps in the \(^1S_0\) channel as functions of Fermi-momentum (density) of the matter for two effective interactions. The heavy and light lines refer to the \(V_{\text{low } k}\) and the Gogny interaction respectively. The solid lines correspond to the non-interacting single-particle spectrum, the dashed and dashed-dotted lines - to renormalized single particle spectrum in symmetric nuclear matter and neutron matter, respectively. The single particle spectra are computed in the Hartree-Fock theory for the Gogny interaction and the Brueckner-Hartree-Fock theory for the \(V_{\text{low } k}\) interaction.

Fig. 2 shows the density dependences of the pairing gaps in neutron and nuclear matter \((k_F\) refers to the Fermi-momentum of nucleons and neutrons respectively) and the effects of single particle renormalization. The single par-
ticle spectra for the Gogny interactions were computed in the Hartree-Fock theory, as described above. For the $V_{\text{low } k}$ interactions we used the Brueckner-Hartree-Fock theory with continuous choice of the single-particle spectrum. If one adopts free single particle spectra for nucleons, the Gogny interaction predicts pairing gaps that are systematically larger than those predicted by the $V_{\text{low } k}$ interaction for momenta larger than about 1 fm$^{-1}$ and slightly smaller for momenta below this value; this picture is consistent with the behaviour of the potentials (see Fig. 1). The renormalization of the single particle spectra reduces the density of states at the Fermi-surface and, hence, the magnitude of the pairing gap. Independent of the chosen interaction this reduction is larger in the symmetrical nuclear matter than in the neutron matter, since in the former case the single particle spectra are steeper at the Fermi-surface mainly due to the tensor channel $^{3}S_{1} - ^{3}D_{1}$ interaction. For neutron matter, where the renormalization of single particle spectrum is mild, there is a close agreement between the gaps computed with the two different interactions for $k_F \leq 1$ fm$^{-1}$ and the deviations at larger densities are not dramatic. For the symmetrical nuclear matter the deviations are larger, indicating that the differences in the single particle spectra are more important for the evaluation of the gap than the differences in the residual pairing interaction.

Extrapolations of the results above to finite nuclei depend on the extent the high-density region contributes to the average pairing gap. If such a contribution is important, the pairing gaps predicted by the $V_{\text{low } k}$ interaction must be systematically smaller than those predicted by the Gogny interaction. A reduction of the pairing gaps by a factor of two compared to the experimental values was observed with realistic Argonne interaction in Sn isotopes [23], which suggests that this might also be the case for the $V_{\text{low } k}$ interaction.

To conclude, we observed a close agreement in the predictions of the pairing properties of nuclear systems by two effective interactions which have largely different origins - the Gogny phenomenological interaction, with parameters fitted to reproduce nuclear properties in Hartree-Fock-Bogoliubov calculations, and the $V_{\text{low } k}$ interaction, which is a renormalization group motivated low-momentum reduction of the realistic interactions. We made explicit the correlations between the values of the pairing gaps and the diagonal and “half on the Fermi-surface” matrix elements of the effective interactions.

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