Nonlinear harmonic forms and an Exotic Bochner Formula

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1 Introduction

Hopf conjectured that

**Conjecture 1.1.** $S^2 \times S^2$ does not admit a metric of strictly positive sectional curvature.

Let $h$ be a harmonic 2-form on a 4-manifold of positive sectional curvature. In a local orthonormal frame, the Bochner formula gives

$$-\frac{1}{2} \Delta |h|^2 = |\nabla h|^2 - R_{ijkl} h_{kj} h_{il} + \text{Ric}_{il} h_{lj} h_{iq}.$$  (1.2)

Positivity of the sectional curvature implies positivity of the Ricci term in (1.2) but does not imply positivity of the remaining curvature term without additional assumptions.

As $H^2(S^2 \times S^2)$ has an indefinite intersection form, the Hopf conjecture follows immediately from the following conjecture.

**Conjecture 1.3.** A compact oriented 4-manifold with positive sectional curvature with nonvanishing second betti number has definite intersection form.

This conjecture leads one to replace (1.2) with more exotic Bochner formulas. For example, the function $|h|^4 - |h \wedge h|^2$ vanishes if $h$ is self dual or anti self dual. At a point where $|h|^4 - |h \wedge h|^2 \neq 0$, we can choose an oriented orthonormal coframe $\{w^i\}_{i=1}^4$ and coframe $\{e_i\}_{i=1}^4$ in which $h$ takes the form $h = aw^1 \wedge w^2 + bw^3 \wedge w^4$. In this frame, we have (see (2.3))

$$-\frac{1}{4} \Delta \ln(|h|^4 - |h \wedge h|^2) \geq R_{1331} + R_{2442} + R_{2332} + R_{1441}$$

$$- \frac{1}{2} \left( c_{ij}^3 \right)^2 + \left( c_{12}^4 \right)^2 + \left( c_{34}^1 \right)^2 + \left( c_{34}^2 \right)^2,$$  (1.4)

where $[e_i, e_j] = c_{ij}^k e_k$.

The only curvature terms which appear in (1.4) are sectional curvatures, making such exotic Bochner formulas appear well suited to studying metrics of positive sectional curvature.

**Definition 1.5.** Let $\{v_j\}_{j=1}^4$ be any orthonormal basis of $T_p M$, $M$ a 4-manifold. $\frac{1}{4}(R(v_1, v_3, v_3, v_1) + R(v_2, v_4, v_4, v_2))$ is called the biorthogonal curvature associated to the plane $\sigma$ spanned by $\{v_1, v_3\}$ and its orthogonal complement $\sigma^\perp$.

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Renato Bettiol [1] has proved that $S^2 \times S^2$ admits metrics with positive bioriented curvature. The curvature terms appearing in (1.4) occur in bioriented pairs, and therefore this formula appears unsuitable for proving Conjectures [1,1] or [1,3]. In particular, on $S^2 \times S^2$, there cannot be a method for eliminating the nonpositive $c^{ij}_{k}$ terms in this expression. The offending nonpositive terms, however, are precisely the obstructions to integrability of the distributions spanned by $\{e_1, e_2\}$ and $\{e_3, e_4\}$ respectively. This observation leads us to introduce a new object to geometric analysis which we call \textit{nonlinear harmonic forms}. These forms are in some ways intermediate between surfaces and harmonic forms and their analysis is similar to that of harmonic maps. We define these forms simply as forms $z$ which

- represent a fixed class in de Rham cohomology, and
- minimize $\|z\|_{L^2}^2$ subject to a natural nonlinear constraint $p(z) = 0$.

The constraints we consider in this paper are polynomial. Our strongest regularity results require that $p$ is diffeomorphism invariant. As $H^2(S^2 \times S^2)$ has a basis with de Rham representatives satisfying $p(z) = z \wedge z$, our above investigations into the Hopf conjecture lead us to study nonlinear harmonic forms satisfying this constraint. It is not difficult to prove minimizers exist. We have the following elementary theorem.

\textbf{Theorem 1.6.} Let $M^n$ be a compact Riemannian manifold. Let $f$ be a $p$–form with $f \wedge f = 0$. Let $h$ be the harmonic representative of $f$. Set

$$Q := \{y \in H_1 : (h + dy) \wedge (h + dy) = 0, \text{ and } d^* y = 0\}.$$ 

Let $E(y) = \|h + dy\|^2$. Let $\nu_0 := \inf\{E(y) : y \in Q\}$. Then there exists $y \in H_1$ such that $E(y) = \nu_0$. Moreover $z := h + dy$ lies in the Morrey space $L^{2,n-2p}$.

In this note, we only establish the most elementary properties of nonlinear harmonic forms. A careful analysis of their singularities requires further study.

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2 Bochner formulas adapted to studying duality of harmonic forms

Let $h$ be a strongly harmonic 2 form on a riemannian 4–manifold $M$. In a neighborhood of a point where $h$ is neither self dual nor antiself dual, we may choose an oriented local orthonormal frame $\{e_i\}_{i=1}^4$ and dual frame $\{w^i\}_{i=1}^4$ so that

$$h = aw^1 \wedge w^2 + bw^3 \wedge w^4.$$ 

Write

$$\nabla_{e_i} e_j = \gamma^k_{ij} e_k, \text{ and } c^k_{ij} := \gamma^{k}_{ij} - \gamma^{k}_{ji},$$

In this notation, $dh = 0$ gives

$$0 = a_3 - a(\gamma^3_{11} + \gamma^3_{22}) + bc^4_{12},$$

$$0 = a_4 - a(\gamma^4_{11} + \gamma^4_{22}) - bc^3_{12},$$

$$0 = b_1 + ac^3_{34} - b(\gamma^1_{33} + \gamma^1_{44}),$$

$$0 = b_2 - ac^4_{34} - b(\gamma^2_{33} + \gamma^2_{44}).$$
and $d^* h = 0$ gives

\[ 0 = a_1 + bc_{34}^2 - a(\gamma_{33}^1 + \gamma_{44}^1), \]
\[ 0 = a_2 - bc_{34}^1 - a(\gamma_{33}^2 + \gamma_{44}^2), \]
\[ 0 = b_3 - b(\gamma_{11}^3 + \gamma_{22}^3) + ac_{12}^4, \]
\[ 0 = b_4 - b(\gamma_{44}^4 + \gamma_{22}^4) - ac_{12}^3. \]

Hence

\[ 0 = aa_1 - bb_1 - (a^2 - b^2)(\gamma_{33}^1 + \gamma_{44}^1), \]
\[ 0 = aa_2 - bb_2 - (a^2 - b^2)(\gamma_{33}^2 + \gamma_{44}^2), \]
\[ 0 = aa_3 - bb_3 - (a^2 - b^2)(\gamma_{11}^3 + \gamma_{22}^3), \]
\[ 0 = aa_4 - bb_4 - (a^2 - b^2)(\gamma_{44}^4 + \gamma_{22}^4). \]

Let $f = \ln(\sqrt{a^2 - b^2})$. Let

\[ H_a = (\gamma_{11}^3 + \gamma_{22}^3)e_3 + (\gamma_{44}^3 + \gamma_{22}^4)e_4, \]

and

\[ H_b = (\gamma_{33}^1 + \gamma_{44}^1)e_1 + (\gamma_{33}^2 + \gamma_{44}^4)e_2. \]

Then

\[ \nabla f = H_a + H_b. \]

Formally we may think of $H_a$ and $H_b$ as the mean curvatures of the distributions spanned by $\{e_1, e_2\}$ and $\{e_3, e_4\}$ respectively and the flow induced by $\nabla f$ as a mean curvature flow. Taking second derivatives, we have

\[
\begin{align*}
  f_{11} &= \gamma_{33,1}^1 + \gamma_{44,1}^1, \\
  f_{22} &= \gamma_{33,2}^2 + \gamma_{44,2}^2, \\
  f_{33} &= \gamma_{11,3}^3 + \gamma_{22,3}^3, \\
  f_{44} &= \gamma_{11,4}^4 + \gamma_{22,4}^4.
\end{align*}
\]

Hence

\[ \sum_j f_{jj} = \gamma_{33,1}^1 + \gamma_{44,1}^1 + \gamma_{33,2}^2 + \gamma_{44,2}^2 + \gamma_{11,3}^3 + \gamma_{22,3}^3 + \gamma_{11,4}^4 + \gamma_{22,4}^4. \quad (2.1) \]

In this frame, the sectional curvatures are given by

\[ R_{ijji} = \gamma_{j,j,i}^i + \gamma_{m,n,j}^i \gamma_{j,i}^m - \gamma_{j,j}^i \gamma_{i,j}^m - (\gamma_{i,j}^m - \gamma_{j,i}^m) \gamma_{i,j}^m. \quad (2.2) \]

For example,

\[ R_{1331} + \gamma_{11}^3 \gamma_{33}^m - \gamma_{31}^m \gamma_{13}^m + (\gamma_{13}^m - \gamma_{31}^m) \gamma_{m3}^1 = \gamma_{33,1}^1 + \gamma_{33,1}^3. \]

At a fixed point, we may choose our frame so that $\gamma_{13}^4 = 0 = \gamma_{11}^2$. In such a frame adapted to a given point, we have

\[ R_{1331} + (\gamma_{11}^3)^2 + \left(\frac{\gamma_{12}^3 + \gamma_{21}^3}{2}\right)^2 - \left(\frac{\gamma_{12}^3}{2}\right)^2 + (\gamma_{33}^1)^2 + \left(\frac{\gamma_{34}^1 + \gamma_{43}^1}{2}\right)^2 - \left(\frac{\gamma_{34}^1}{2}\right)^2 = \gamma_{33,1}^1 + \gamma_{11,3}^3. \]
Inserting this and related equations for other sectional curvatures into (2.1) yields

\[
\sum_j f_{jj} = R_{1331} + R_{2332} + R_{1441} + R_{2442} \\
+ (\gamma_{33}^1)^2 + (\gamma_{11}^1)^2 + (\gamma_{22}^2)^2 + (\gamma_{33}^2)^2 + (\gamma_{11}^2)^2 + (\gamma_{44}^1)^2 + (\gamma_{44}^2)^2 + (\gamma_{44}^3)^2 \\
+ (\frac{\gamma_{32}^2 + \gamma_{23}^2}{\sqrt{2}})^2 + (\frac{\gamma_{44}^1 + \gamma_{44}^3}{\sqrt{2}})^2 + (\frac{\gamma_{44}^2 + \gamma_{44}^3}{\sqrt{2}})^2 + (\frac{\gamma_{12} + \gamma_{21}}{\sqrt{2}})^2 \\
- \left(\frac{c_{12}^3}{\sqrt{2}}\right)^2 - \left(\frac{c_{12}^4}{\sqrt{2}}\right)^2 - \left(\frac{c_{34}^3}{\sqrt{2}}\right)^2.
\]  

(2.3)

If the \(c_{ij}^k\) terms in the above expressions vanished, Conjecture 1.3 would follow immediately from (2.3). Let \(\Sigma(a)\) and \(\Sigma(b)\) denote the distributions spanned by \(\{e_1, e_2\} \) and \(\{e_3, e_4\}\) respectively. These distributions are only defined where \(a \neq b\). They are integrable if and only if \(c_{12}^1, c_{12}^2, c_{34}^1, \) and \(c_{34}^2\) vanish. Hence the possible nonintegrability of \(\Sigma(a)\) and \(\Sigma(b)\) at a maximum of \(f\) is an obstruction to utilizing (2.3) using only sectional curvature data.

Set

\[
\tau := \frac{1}{4} \ln\left(\frac{(a + b)^2}{(a - b)^2}\right) = \ln\left(\frac{|h_+|^2}{|h_-|^2}\right),
\]  

(2.4)

where \(h = h_+ + h_-\) is the decomposition of \(h\) into self dual and anti self dual forms respectively. Then

\[
\nabla \tau = -c_{34} e_1 + c_{34} e_2 - c_{12} e_3 + c_{12} e_4,
\]  

(2.5)

and the only curvature term appearing in \(-\Delta T\) is \(-2R_{1234}\). Thus the gradient of the ratio of the norms of the self dual and anti self dual components of \(h\) measures the nonintegrability of \(\Sigma(a)\) and \(\Sigma(b)\).

As we noted in the introduction, Renato Bettiol’s theorem \([1]\) that \(S^2 \times S^2\) admits metrics of positive biorthogonal curvature suggests that we should be unable to eliminate the \(-(c_{ij}^k)^2\) terms in (2.3). One obvious way to attempt to circumvent this difficulty is to use the fact that at a maximum, the hessian of \(\ln(f)\) is negative semidefinite, not just its trace. At a maximum of \(h\), rotate coordinates so that \(c_{12}^1 = 0 = c_{34}^2\). Then we have the following curvature expressions.

\[
R_{1331} + (\gamma_{11}^1)^2 + (\frac{\gamma_{12}^2 + \gamma_{21}^2}{2})^2 - (\frac{\gamma_{32}^1}{2})^2 + (\frac{\gamma_{33}^1 + \gamma_{43}^1}{2})^2 - (\frac{c_{12}^3}{2})^2 = \gamma_{33,1}^1 + \gamma_{11,1}^3,
\]

\[
R_{1441} + (\gamma_{11}^4)^2 + (\frac{\gamma_{12}^2 + \gamma_{21}^2}{2})^2 + (\frac{\gamma_{11}^4 + \gamma_{14}^1}{2})^2 - (\frac{c_{12}^3}{2})^2 = \gamma_{44,1}^1 + \gamma_{11,4},
\]

\[
R_{2332} + (\gamma_{22}^3)^2 + (\frac{\gamma_{12}^2 + \gamma_{21}^2}{2})^2 - (\frac{c_{12}^3}{2})^2 + (\frac{\gamma_{33}^2}{2})^2 + (\frac{\gamma_{34}^2 + \gamma_{43}^2}{2})^2 = \gamma_{33,2}^2 + \gamma_{22,3},
\]

\[
R_{2442} + (\gamma_{44}^2)^2 + (\frac{\gamma_{42}^2 + \gamma_{24}^2}{2})^2 + (\frac{\gamma_{44}^2}{2})^2 + (\frac{\gamma_{34}^2 + \gamma_{43}^2}{2})^2 = \gamma_{44,2}^2 + \gamma_{22,4}.
\]

We see that no offending \(c_{ij}^k\) terms remain in the expression for \(R_{2442}\). The expression for this curvature involves both \(\gamma_{22,4}^4\) and \(\gamma_{22,4}^2\). In order to express these in terms of the hessian of \(f\), we also need \(\gamma_{33,2}^2\) and \(\gamma_{11,4}\). These two terms appear in the expressions for \(R_{1441}\) and \(R_{2332}\). Continuing in this way, we see that using the preceding equations, an expression relating hessian terms to curvature terms eventually requires all the above curvature terms and all the hessian terms, simply returning us to (2.3). We now introduce nonlinear harmonic forms for an (ultimately unsuccessful) attempt to improve this argument.
3 Heuristic Properties of nonlinear harmonic forms

In this section, in order to motivate the study of nonlinear harmonic forms, we formally consider minimizers of $\|z\|_2^2$ in a fixed degree 2 cohomology class on a compact 4-manifold, subject to the constraint that $z \wedge z = 0$. We study the consequences of Euler Lagrange equations. Again, we emphasize that the discussion in this section is heuristic only, we will not even specify the function spaces needed to give exact meaning to the formal equalities until the next section.

Let $f$ be a closed 2-form satisfying $f \wedge f = 0$. Suppose we consider closed 2-forms $z$ cohomologous to $f$ such that $\|z\|_2^2$ is minimal subject to the constraint $z \wedge z = 0$. The linearization of the constraint is $2z \wedge \dot{z} = 0$. Hence formally $\langle d^* z, b \rangle_{L^2} = 0$ for all $b$ satisfying $z \wedge db = 0$. Therefore $d^* z$ is perpendicular to the kernel of $e(f) d$, where $e(\phi)$ denotes exterior multiplication on the left by $\phi$. Thus $d^* z$ is in the image of $d^* e^*(z)$. That is

$$d(*z - \mu z) = 0,$$

for some function $\mu$. Let $\{e_i\}_{i=1}^4$ and $\{w^j\}_{j=1}^4$ be an oriented orthonormal frame and dual frame so that $z = aw^1 \wedge w^2$, with $a \geq 0$. $dz = 0$ implies

$$0 = a_3 - a(\gamma_{11}^3 + \gamma_{22}^3),$$
$$0 = a_4 - a(\gamma_{11}^4 + \gamma_{22}^4),$$
$$0 = a c_{34}^2,$$
$$0 = a c_{44}^2.$$

So, $\{e_3, e_4\}$ span an integrable distribution where $a \neq 0$.

The constraint on $d^* z$ gives

$$0 = a(c_{12}^3 - \mu_4),$$
$$0 = a(c_{12}^4 + \mu_3),$$
$$0 = a_1 - a(\gamma_{33}^1 + \gamma_{44}^1),$$
$$0 = a_2 - a(\gamma_{33}^2 + \gamma_{44}^2).$$

The $\mu$ relations imply

$$aT \mu = 0,$$

where

$$T := c_{12}^3 e_3 + c_{12}^4 e_4.$$

We may view $T$ as the Hamiltonian vector field for $\mu$ restricted to the leaves of the foliation associated to the integrable distribution spanned by $\{e_3, e_4\}$, equipped with the symplectic form $w^3 \wedge w^4$ pulled back to the leaves. Let $\phi_s$ denote the one parameter family of diffeomorphisms generated by $aT$ on the set where $a > 0$. We have seen that $\phi_s^* \mu = \mu$. We also have $\phi_s^* z = z$ since $i_T z = 0$ and $z$ is closed; here $i_X$ denotes interior product with $X$. The volume form is also invariant under $\phi_s$ since

$$di_T dvol = d(c_{12}^3 a w^1 \wedge w^2 \wedge w^4 - c_{12}^4 a w^1 \wedge w^2 \wedge w^3) = d(d \mu \wedge z) = 0.$$

Hence, the nonlinear harmonic form comes equipped with a foliation with a symplectic structure, and a distinguished Hamiltonian whose Hamiltonian vector field preserves the nonlinear harmonic form and which after a rescaling preserves the volume.
Returning to Bochner formulas, let now $f = \ln(a)$. At a critical point $p$ of $f$, the mean curvature of the leaf through $p$ is zero. The Bochner formula is again (2.4) and (2.3)

$$-\Delta f = \gamma^1_{33,1} + \gamma^1_{34,1} + \gamma^2_{33,2} + \gamma^2_{44,2} + \gamma^3_{11,3} + \gamma^3_{22,3} + \gamma^4_{11,4} + \gamma^4_{22,4}$$

$$= R_{1331} + R_{2332} + R_{1441} + R_{2442} + \frac{(\gamma^1_{11})^2 + (\gamma^2_{22})^2 + (\gamma^3_{33})^2 + (\gamma^4_{44})^2 - (c^2_{12})^2}{\sqrt{2}}$$

$$\gamma^1_{33,1} + \gamma^1_{34,1} + \gamma^2_{33,2} + \gamma^2_{44,2} + \gamma^3_{11,3} + \gamma^3_{22,3} + \gamma^4_{11,4} + \gamma^4_{22,4}$$

$$= R_{1331} + R_{2332} + R_{1441} + R_{2442} + \frac{(\gamma^1_{11})^2 + (\gamma^2_{22})^2 + (\gamma^3_{33})^2 + (\gamma^4_{44})^2 - (c^2_{12})^2}{\sqrt{2}}$$

$$\gamma^1_{33,1} + \gamma^1_{34,1} + \gamma^2_{33,2} + \gamma^2_{44,2} + \gamma^3_{11,3} + \gamma^3_{22,3} + \gamma^4_{11,4} + \gamma^4_{22,4}$$

Once again the curvatures appear in biorthogonal pairs. Let $f$ have a maximum at $p \in M$. Considering the positivity of the hessian rather than simply its trace, we still have insufficient data to use the maximum principle to force $a = 0$ when the sectional curvatures are positive, as we now illustrate.

Suppose $T$ is nonvanishing near $p$; otherwise the maximum principle leads to a contradiction to the existence of $z$. Rotate $\{e_3, e_4\}$ so that $c^3_{12} = 0$ in a neighborhood of $p$. If $\gamma^1_{33}e_1 + \gamma^2_{33}e_2$ is nonvanishing at $p$, rotate $\{e_1, e_2\}$ so that $\gamma^3_{33}$ vanishes near $p$. Then we have

$$R_{1331} + (\gamma^1_{11})^2 + \frac{(\gamma^2_{12} + \gamma^3_{21})^2}{2} + \frac{(\gamma^3_{13} + \gamma^4_{23})^2}{2} = \gamma^2_{11,1,3},$$

$$R_{1441} + (\gamma^1_{11})^2 + \frac{(\gamma^4_{12} + \gamma^2_{21})^2}{2} + \frac{(\gamma^3_{13} + \gamma^4_{13})^2}{2} = \gamma^2_{11,1,4},$$

$$R_{2332} + (\gamma^2_{22})^2 + \frac{(\gamma^3_{12} + \gamma^2_{31})^2}{2} + \frac{(\gamma^3_{23} + \gamma^4_{23})^2}{2} = \gamma^2_{33,2} + \gamma^2_{22,3},$$

$$R_{2442} + (\gamma^4_{22})^2 + \frac{(\gamma^4_{12} + \gamma^2_{21})^2}{2} + \frac{(\gamma^3_{13} + \gamma^4_{13})^2}{2} = \gamma^2_{44,2} + \gamma^2_{22,4}.$$

At the maximum of $f$, we have $\gamma^4_{14,1} \leq 0$. The curvature relations imply $\gamma^4_{11,1,4} > 0$. The hessian then implies $\gamma^2_{22,4} < 0$. The curvature relations imply $\gamma^2_{44,2} > 0$. The hessian implies $\gamma^2_{33,2} < 0$, and we run out of useful sign data at this step.

## 4 Existence of quadratic nonlinear harmonic forms

In this section, using standard techniques, we prove the existence of nonlinear harmonic forms in $L_2$ subject to quadratic constraints. Let $H_s$ denote the Sobolev space of differential forms $\phi$ with $(1 + \Delta)^s/2\phi \in L_2$ and norm $||\phi||_{H_s} = ||(1 + \Delta)^s/2\phi||_{L_2}$.

**Theorem 4.1.** Let $M^n$ be a compact Riemannian manifold. Let $f := (f_1, \ldots, f_d)$ be a $d$-tuple of closed differential forms with

$$p^k(f) := \sum_{i,j} p^k_{ij} f_i \wedge f_j = 0, \; 1 \leq k \leq D.$$ 

for some smooth differential forms $p^k_{ij}$. Let $h_i$ denote the harmonic representative of $f_i$. Set

$$Q := \{y = (y_1, \ldots, y_d) \in H_1 : p^k(h + dy) = 0, \; 1 \leq k \leq D \text{ and } d^*y_i = 0\}.$$ 

Let $E(y) = ||h + dy||^2 = \sum_i ||h_i + dy_i||^2$. Let $\nu_0 := \inf\{E(y) : y \in Q\}$. Then there exists $y \in H_1$ such that $E(y) = \nu_0$. 

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Proof. Let \( \{y^n\}_{n=1}^{\infty} \subset Q \) be a minimizing sequence for \( E \). Then the sequence is bounded in \( H_1 \). By passing to a subsequence, we may assume that \( y^n \stackrel{H_1}{\rightharpoonup} y \) and \( y^n \stackrel{L^2_2}{\rightharpoonup} y \), as \( n \to \infty \), for some \( y \in H_1 \). Then for all smooth forms \( \phi \) and \( \forall k \),

\[
0 = \lim_{n \to \infty} \int \phi \wedge p^k(h + dy^n) = \int \phi \wedge p^k(h + dy) + \lim_{n \to \infty} \int \phi \wedge p^k(dy^n - dy)
\]

\[
= \int \phi \wedge p^k(h + dy) + \lim_{n \to \infty} \int d(\phi p^k_{ij}) \wedge (y^n_i - y_i) \wedge (dy_j^n - dy_j) = \int \phi \wedge p^k(h + dy),
\]

where \( \tilde{p}^k_{ij} = \pm p^k_{ij} \), depending on implied degrees. Hence \( y \in Q \). Next we observe that

\[
v_0 = \lim_{n \to \infty} \|h + dy^n\|^2 = \|h + dy\|^2 + \lim_{n \to \infty} \|dy^n - dy\|^2
\]

implies \( E(y) = v_0 \), and \( \lim_{n \to \infty} \|dy^n - dy\|^2 = 0 \). In particular, \( y \stackrel{H_1}{\rightharpoonup} y \).

We call

\[
z := h + dy
\]

a nonlinear harmonic representative of \( f \). Observe \( d^*z \in H_{-1} \).

4.1 Price Monotonicity

Recall the Morrey norm defined by

\[
\|u\|_{L^{p,\mu}} := \left( \sup_{x \in M, 0 < r < \text{diam}(M)} r^{-\mu} \int_{B_r(x)} |u|^p dv \right)^{\frac{1}{p}}.
\]

Then the Morrey space \( L^{p,\mu} \) is defined to be the subset of \( L^p \) with finite \( L^{p,\mu} \) norm.

**Theorem 4.4.** If the nonlinear harmonic form \( z \) defined in \( \ref{4.4} \) is homogeneous of degree \( p \), and if the coefficients \( p^k_{ij} \) are constants, then \( z \in L^{2,n-2p} \).

**Proof.** Let \( Y \) be a smooth vectorfield. Let \( \phi_i \) be the one parameter family of diffeomorphisms generated by \( Y \). Then \( \phi_i \) preserves \( H_1 \) and since the \( p^k_{ij} \) are constant, \( p^k(f^*\zeta) = f^*p^k(\zeta) = 0 \) if \( p^k(\zeta) = 0 \). Hence \( \|\phi_i z\|^2_{L_2} \geq \|z\|^2_{L_2} \). In particular, (see \cite{3}, p. 142)

\[
0 = \int \langle z, (-\frac{1}{2} \text{div}(Y) + Y^j_i c(w^j) i_{e_i}) z \rangle.
\]

For smooth \( z \), this is equivalent to

\[
\langle d^*z, i_Y z \rangle = 0.
\]

Equation \( \ref{4.5} \) allows us to prove monotonicity formulas in the usual fashion. We include the details for the convenience of the reader.

Fix geodesic coordinates \( \{x^i\} \) centered at some \( p \in M \), and choose

\[
Y := y \left( \frac{r^2}{2s} \right) x^j \frac{\partial}{\partial x^j}.
\]

Then

\[
\text{div}(Y) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} (\sqrt{g} \left( \frac{r^2}{2s} \right) x^j) = ny \left( \frac{r^2}{2s} \right) + \frac{r^2}{s} y \left( \frac{r^2}{2s} \right) + \frac{r^2}{2s} \frac{r^2}{2s} \frac{\partial \ln(g)}{\partial r}
\]

\[
= ny \left( \frac{r^2}{2s} \right) - 2s \frac{d}{ds} y \left( \frac{r^2}{2s} \right) + \frac{r^2}{2s} \frac{r^2}{2s} \frac{\partial \ln(g)}{\partial r}.
\]
and
\[ Y^i_j e(v^j)_{i,\gamma} = y(\frac{r^2}{2s}) e(dv^j) i_{\frac{\partial}{\partial s}} + y(\frac{r^2}{2s}) e(dr) i_{\frac{\partial}{\partial r}} + y(\frac{\frac{r^2}{2s}}{2s}) x^m \Gamma^i_j m e(dv^j) i_{\frac{\partial}{\partial v^i}} \]
\[ = y(\frac{r^2}{2s}) e(dv^j) i_{\frac{\partial}{\partial s}} - 2s \frac{d}{ds} y(\frac{r^2}{2s}) e(dr) i_{\frac{\partial}{\partial r}} + y(\frac{\frac{r^2}{2s}}{2s}) x^m \Gamma^i_j m e(dv^j) i_{\frac{\partial}{\partial v^i}} \]

Hence
\[ s^{p-1-\frac{n}{4}} (-\frac{1}{2} dv(Y) + Y^i_j e(v^j)_{i,\gamma}) z = (\frac{d}{ds} (s^{p-\frac{n}{4}} y) - 2s^{p-\frac{n}{4}} \frac{dy}{ds} e(dr) i_{\frac{\partial}{\partial r}} + s^{p-1-\frac{n}{4}} yw) z, \]
where \( w := -\frac{\partial \ln(z)}{\partial r} + x^m \Gamma^i_m e(dv^j) i_{\frac{\partial}{\partial v^i}} \). Inserting this expression into (4.5) and multiplying by \( e^{\lambda s} \) gives
\[ \frac{d}{ds} (e^{\lambda s} s^{p-\frac{n}{4}} \int y |z|^2 dv) = e^{\lambda s} s^{p-\frac{n}{4}} \int (2 \frac{d}{ds} |(\frac{\partial}{\partial r}) z|^2 + y(z, (\lambda - \frac{w}{s})) dv. \] (4.7)
Integrate this equality in \( s \) from \( \sigma^2 \) to \( \tau^2 \) to get
\[ e^{\lambda \tau^2 \tau^{2p-n}} \int y(\frac{r^2}{2\tau^2}) |z|^2 dv - e^{\lambda \sigma^2 \sigma^{2p-n}} \int y(\frac{r^2}{2\sigma^2}) |z|^2 dv \]
\[ = \int_{\sigma^2}^{\tau^2} e^{\lambda s} s^{p-\frac{n}{4}} \int (2 \frac{d}{ds} |(\frac{\partial}{\partial r}) z|^2 + y(z, (\lambda - \frac{w}{s})) dv) ds. \] (4.8)
Replace \( y \) with a sequence \( \{y_n\} \) of \( C^1 \) monotone decreasing functions, supported in \([0, 1]\) and converging to the characteristic function of \([0, \frac{1}{2}]\). Using the support of \( y_n \) and the fact that \( |w| = O(r^2) \), we see that we can pick \( \lambda > 0 \) depending on geometric data so that \( \lambda > \frac{2}{r} \) on the support of \( y \). Hence, the right hand side of (4.8) is nonnegative, yielding
\[ e^{\lambda \tau^2 \tau^{2p-n}} \int_{B_{\tau}} |z|^2 dv \geq e^{\lambda \sigma^2 \sigma^{2p-n}} \int_{B_{\sigma}} |z|^2 dv. \] (4.9)
This monotonicity relation immediately implies \( z \in L^{2, n-2p} \).

5 Euler Lagrange Equations

Write
\[ P(z) = (p^1(z), \ldots, p^D(z)), \]
with \( p^k(z) \) defined as in Theorem 4.1 still assumed to be quadratic. Here Let \( V = P^{-1}(0) \).
Let \( z(t) = h + db(t) \) be a curve in \( V \), with \( b(t) \) a \( C^1 \) curve in \( H_1 \). In components, write \( z_i(t) = h_i + db_i(t) \), with \( h_i \) harmonic and \( d^* b_i = 0 \). Then
\[ 0 = \frac{d}{dt} P(z(t)) = DP_z db = 0, \]
where \( DP_z \) denotes the derivative of \( P \) at \( z \). Let \( K_z \) denote the kernel of \( DP_z d \) in \( H_1 \).

Definition 5.1. We call \( K_z \) the formal tangent space to \( V \) at \( z \). We say \( V \) is unobstructed at \( z \) if for each \( v \in K_z \), there is a differentiable curve \( c(t) \) in \( V \) so that \( c(0) = z \) and \( \dot{c}(0) = v \).
For simplicity, we now assume that dim $M = 4$ and that the $p_{ij}^k$ are closed. Then since $H_1 \subset L^4$ and $z \in L_2$, we may view $DP_z$ as a bounded linear map from $H_1 \to H_{-1}$. In the quadratic case under consideration, this map is an algebraic operator given by the adjoint of left multiplication by the product of smooth forms and components of $z$. For example, when $p(z) = z \wedge z$, $DP_z = 2e^*(z)$. Hence $dDP_z$ defines a bounded linear map from $H_1 \to H_{-2}$. Here we have used $dp^k_{ij} = 0$ to substitute $dDP_z$ for $DP_z d$. We let $d^*$ and $DP_z^*$ denote the formal $L_2$ adjoint of these two operators, and $(dDP_z)^* = (1 + \Delta)^{-1} DP_z^* d^*(1 + \Delta)^{-2}$ denote the Hilbert space adjoint of $dDP_z : H_1 \to H_{-2}$. Then
\[
K_z = (\text{Im} (dDP_z)^*)^{\perp_{H_1}} = ((1 + \Delta)^{-1} DP_z^* d^* H_2)^{\perp_{H_1}}.
\] (5.2)

Similarly, if $E(h + db) := \|h + db\|^2$, then
\[
\frac{d}{dt} E(b(t)) = \sum_i \langle d^* z_i, \dot{b}_i \rangle_{L_2} = \sum_i \langle (1 + \Delta)^{-1} d^* z_i, \dot{b}_i \rangle_{H_1} := DE_b \dot{b}.
\]

If $z = h + db$ is an $E$ minimizer, then
\[
DE_b v = 0, \; \forall v \in K_z,
\]
such that there exists a differentiable curve $\beta(t)$ in $V$ with $b = \beta(0)$ and $v = \dot{\beta}(0)$.

Thus, if $z = h + db$ minimizes $E$ and $K_z$ is unobstructed, then the Euler Lagrange equation becomes
\[
d^* z = \lim_{j \to \infty} DP_z^* d^* \mu_j, \; \text{in} \; H_{-1},
\]
for some sequence $\mu_j$ in $H_2$. When the image is closed, we can, of course, replace the limit with $d^* z = DP_z^* d^* \mu$, some $\mu \in H_2$.

6 Similarities to Harmonic Maps

To see the similarities between the analysis of nonlinear harmonic forms and harmonic maps, it is useful to consider the gradient flows associated to the energy $E$. The natural $H_1$ gradient flow for the energy $E$ is
\[
0 = \dot{b} + p_{z(t), H_1} (1 + \Delta)^{-1} \Delta b(t),
\]
and the natural $L_2$ gradient flow is
\[
0 = \dot{b} + p_{z(t), L_2} \Delta b(t),
\]
(6.1)
where $p_{z, H_1}$ denotes the $H_1$ projection onto $K_z$ and $p_{z(t), L_2}$ denotes the $L_2$ projection.

On the other hand, consider the harmonic map flow for
\[
f : M \times [0, T] \to N \subset \mathbb{R}^l : \quad 0 = \dot{f} + d_{f, TN}^* df = \dot{f} + p_N \Delta f,
\]
(6.2)
where $d_{f, TN}^*$ and $d_{f, TR}^*$ are the adjoints of the exterior derivatives associated to the vector bundle $f^*TN$ and the trivial bundle $f^*\mathbb{R}^l$ respectively. Here $p_N$ denotes orthogonal projection $f^*\mathbb{R}^l \to f^*TN$. Hence we see that (6.1) and (6.2) have the same structure. They are distinguished primarily by the properties of their projections. The projection $p_N$ is local with finite dimensional kernel and image. The projection $p_{z, L_2}$ is nonlocal with infinite dimensional kernel and image.
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