The Polyhedral Geometry of Truthful Auctions

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joint work w/ Michael Joswig, Max Klimm

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Allocation Mechanism

Images: flaticon.com
# Allocation Mechanism

| 4.2 | 3.8 | 4.9 | 3.7 | 6.8 |
|-----|-----|-----|-----|-----|
| 5.0 | 5.0 | 5.0 | 5.0 | 5.0 |
| 4.9 | 7.1 | 3.3 | 4.9 | 2.5 |

Images: flaticon.com
## Allocation Mechanism

| Food     | Green | Red | Blue | Total |
|----------|-------|-----|------|-------|
| Coffee   | 4.2   | 5.0 | 4.9  | 6.8   |
| Beverages| 3.8   | 5.0 | 7.1  |       |
| Burger   | 4.9   |     |      |       |
| Pizza    | 3.7   |     |      |       |
| Cake     | 5.0   |     |      | 5.0   |

Images: flaticon.com
|   |   |   |   |   |
|---|---|---|---|---|
|   | 4.2 | 3.8 | 4.9 | 3.7 | 6.8 |
|   | 4.8 | 4.8 | 4.8 | 4.8 | 4.8 |
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\Omega = \left\{ A \in \{0, 1\}^{n \times m} \mid \sum_{i \in [n]} a_{i,j} = 1 \text{ for all } j \in [m] \right\}
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\((\Theta = \mathbb{R}^{n \times m})\)
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- Agent $i$ will misreport a valuation $\theta'_i$ if it benefits their utility

$$u_i(\theta' | \theta_i) = f_i(\theta') \cdot \theta_i - p_i(\theta')$$

A mechanism $M = (f, p)$ is incentive compatible (IC), if misreporting never benefits the agent.
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  \]

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Example (local mechanism):
One agent, two items.
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$$\max_{a \in \{0,1\}^2} \{ a \cdot (\theta_1, \theta_2) - q_a \}$$

Difference sets: $D_a = \{ \theta \in \Theta \mid u(\theta) \text{ maximized by } a \}$
Example (local mechanism):
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$$f(\theta) = \arg \max_{a \in \{0,1\}^2} \{ a \cdot (\theta_1, \theta_2) - q_a \}$$

$$p(\theta) = q_f(\theta)$$

Difference sets: $D_a = \{ \theta \in \Theta \mid u(\theta) \text{ maximized by } a \}$
Lemma (Nisan et al. - 2007)

$M = (f, p)$ is IC if and only if for all $i \in [n]$ and all $\theta \in \mathbb{R}^{n \times m}$, $p_i$ is given by some function $p_{i,\theta_{-i}} : \{0, 1\}^m \to \mathbb{R}$, and

$$f(\theta) \in \arg\max \left\{ A_i \cdot \theta_i - p_{i,\theta_{-i}}(A_i) \mid A \in \Omega \right\}.$$ 

$A_i$ is the $i$-th row of the matrix $A$. 

$\Rightarrow$ Multi-agent mechanisms are characterized by local one-agent mechanisms.
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\[ \theta_1 + \theta_2 = q_{11} \]

Diagram:

- $D_{01}$
- $D_{00}$
- $D_{10}$
- $D_{11}$

Points:
- $q_{01}$
- $q_{10}$

Right side:
- $\theta_{-i}$
- $\theta'_{-i}$
- $\theta''_{-i}$
- $\theta'''_{-i}$
$\theta_1, \theta_2$

$D_{00}, D_{01}, D_{10}, D_{11}$

$\theta_{-i}, \theta'_{-i}, \theta''_{-i}, \theta'''_{-i}$
Indifference Complex

Definition

The *indifference complex* $\mathcal{I}(f)$ of an allocation function $f$ is the abstract simplicial complex defined as

$$
\mathcal{I}(f) = \left\{ \mathcal{O} \subseteq \Omega \mid \bigcap_{A \in \mathcal{O}} \bar{D}_A \neq \emptyset \right\}.
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$\mathcal{I}$ is an ASC $\iff$ (i) $\mathcal{I} \neq \emptyset$, (ii) $E \subset F, F \in \mathcal{I} \Rightarrow E \in \mathcal{I}$
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Central Question

Which indifference complexes arise from IC mechanisms?
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Theorem (Joswig, Klimm, S.; cf. Frongillo, Kash - 21)

An indifference complex $\mathcal{I}$ for $m$ items and one agent arises from a local IC mechanism if and only if it corresponds to a regular subdivision of the $m$-cube.
Definition

Let $S \subset \mathbb{R}^n$ be finite.
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$$P(S, \lambda) = \text{conv} \left\{ (x, \lambda(x)) \in \mathbb{R}^{n+1} \mid x \in S \right\}.$$

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Regular Subdivisions

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Not all subdivisions are regular, e.g.:

$$\lambda = (6, 5, 7, 8, 5)$$
Theorem (Joswig, Klimm, S.; cf. Frongillo, Kash - 21)

An indifference complex $\mathcal{I}$ for $m$ items and one agent arises from a local IC mechanism if and only if it corresponds to a regular subdivision of the $m$-cube.

Number of triangulations of the $m$-cube:

$m$ | Total Triangulations
--- | ---
3 | 74,743,743
4 | 92,487,256
87,959,448
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- A mechanism is nondegenerate, if the associated regular subdivision is a triangulation.
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| $m$ | all     | regular |
|-----|---------|---------|
| 2   | 2       | 2       |
| 3   | 74      | 74      |
| 4   | 92,487,256 | 87,959,448 |
• $S_m$ acts by permuting the coordinates of the cube.
  $\rightarrow$ corresponds to permutation of items
Symmetries of the Cube

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| $m$ | all     | regular | $S_m$-orbits | $\Gamma_m$-orbits |
|-----|---------|---------|--------------|-------------------|
| 2   | 2       | 2       | 2            | 1                 |
| 3   | 74      | 74      | 23*          | 6                 |
| 4   | 92,487,256 | 87,959,448 | 3,706,261*  | 235,277          |

*Computations made using MPTOPCOM
$\Gamma_3$-Orbits

Type A
(4 reg, 2 $S_3$)

Type B
(8 reg, 4 $S_3$)

Type C
(24 reg, 6 $S_3$)

Type D
(24 reg, 6 $S_3$)

Type E
(12 reg, 3 $S_3$)

Type F
(2 reg, 2 $S_3$)

Type A – E have been found by Vidali (2009).
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Sensitivity of Mechanisms

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- Hamming distance: \( d_h(a, b) = |a - b|_1 \rightarrow \mu_h(f) \)
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- Hamming distance: $d_h(a, b) = |a - b|_1 \rightarrow \mu_h(f)$
- What is $M_c(m) = \min_{f \in \Phi_m} \mu_c(f)$? (Resp. $M_h(m)$?)
  \[\Phi_m = \text{set of local allocation functions for } m \text{ items}\]
Proposition (Joswig, Klimm, S.)

The minimal cardinality sensitivity of an IC single agent mechanism for \( m \) items is \( M_c(m) = 1 \).

Proposition (Joswig, Klimm, S.)

The minimal Hamming sensitivity of an IC single agent mechanism for \( m \geq 3 \) items is bounded by \( 2 \leq M_h(m) \leq m - 1 \).
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Proof. Cut the cube with the hyperplanes

\[
H_k = \left\{ x \in \mathbb{R}^m \mid \sum_{i \in [m]} x_i = k \right\}.
\]

The resulting subdivision proves the claim. It can be obtained with the prices \( q_a = |a|^2_1 \).
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Proof. Upper bound, \( m \) odd: Cut off all corners with even number of ones \( \Rightarrow \) no antipodal vertices in the same cell.

\( m \) even: Consider \( m \)-cube as prism over \((m-1)\)-cube. Cut off corners as before. Cells of \( m \)-cube are prisms over cells of \((m-1)\)-cube.
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Summary

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- Indifference complexes arise from local IC mechanisms if and only if they correspond to a regular subdivision of the cube.
- The sensitivity measures how drastically an outcome may change by only small perturbations of the valuations.
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Indifference complexes arise from local IC mechanisms if and only if they correspond to a regular subdivision of the cube.

The sensitivity measures how drastically an outcome may change by only small perturbations of the valuations.

Thank You for Your attention!
Affine Maximizers

Allocation space for $n$ agents and $m$ items:

$$\Omega = \left\{ A \in \{0, 1\}^{n \times m} \middle| \sum_{i \in [n]} A_{i,j} = 1 \text{ for all } j \in [m] \right\}$$

$f$ is an affine maximizer $\iff$ There exist $w_1, \ldots, w_n \in \mathbb{R}$ and $c_A \in \mathbb{R}$ for all $A \in \Omega$, such that

$$f(\theta) \in \arg \max \left\{ c_A + \sum_{i \in [n]} w_i \theta_i \cdot A_i \middle| A \in \Omega \right\}.$$
Affine maximizer:

\[ f(\theta) \in \arg \max \left\{ c_A + \sum_{i \in [n]} w_i \theta_i \cdot A_i \ \bigg| \ A \in \Omega \right\} . \]

**Theorem (Joswig, Klimm, S.)**

An indifference complex \( \mathcal{I} \) for \( n \) agents and \( m \) items arises from an affine maximizer if and only if it corresponds to a regular subdivision of \( \Delta_{n-1}^m \).
Symmetries of $\Delta_{n-1}^m$

$$\Omega = \left\{ A \in \{0, 1\}^{n \times m} \left| \sum_{i \in [n]} A_{i,j} = 1 \text{ for all } j \in [m] \right. \right\}$$

- Regular subdivisions of $\Delta_{n-1}^2$ have been studied before.

- Denote by $S_n \times S_n$ the automorphism group which permutes the vertices of each simplex separately.

- Denote by $S_m \times S_n$ the automorphism group which permutes the rows and columns of allocations $A \in \{0, 1\}^{n \times m}$. 
Symmetries of $\Delta_{n-1}^m$

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Results for $m = 2$:

| $n$ | regular | $[S_2 \times S_n]$-orbits | $[S_n \times S_n]$-orbits |
|-----|---------|---------------------------|---------------------------|
| 3   | 108     | 21                        | 5                         |
| 4   | 4,494,288 | 96,722                   | 7,869                     |

Computations made using MPTOPCOM
Triangulations of $\Delta_2^2$

Type A
- 6 regular
- $3 \times S_3 \times S_3$

Type B
- 12 regular
- $4 \times S_3 \times S_3$

Type C
- 36 regular
- $5 \times S_3 \times S_3$

Type D
- 36 regular
- $5 \times S_3 \times S_3$

Type E
- 18 regular
- $4 \times S_3 \times S_3$
Cardinality distance: \( d_c(a, b) = |a|_1 - |b|_1 \). The cardinality sensitivity of an affine maximizer \( f \) is

\[
\mu_c(f) = \max \{ d_c(A_i, B_i) \mid A, B \in F \text{ for some } F \in \mathcal{I}(f) \text{ and } i \in [n] \}
\]

**Proposition (Joswig, Klimm, S.)**

The minimal cardinality sensitivity of affine maximizers for \( n \geq 3 \) agents and \( m \) items is bounded by \( \mu_c(f) \leq \left\lceil \frac{m}{2} \right\rceil \).

This sensitivity can be achieved by the allocation biases

\[
c_A = -\max_{i \in [n]} \left( \sum_{j \in [m]} a_{i,j} \right)^2
\]