In Half-Plane Re($s$) ≤ 1, Riemann’s $\zeta(s)$ is Convergent and the Dirichlet Series of $\zeta(s)$ is Divergent, Violating the Law of Non-Contradiction

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Abstract

Propositional logics and Three-Valued Logics (3VL) are useful tools for analyzing the Riemann Hypothesis (RH). Riemann’s "expression" of $\zeta(s)$ claims to be convergent throughout half-plane Re($s$) ≤ 1 (except at $s = 1$). Given that the Dirichlet series "expression" of $\zeta(s)$ is proven to be divergent there, the Law of Non-Contradiction (LNC) holds that $\zeta(s)$ cannot also be convergent there. So $\zeta(s)$ is exclusively divergent throughout half-plane Re($s$) ≤ 1, and has no zeros. So $\zeta(1) \neq 0$, resolving the BSD conjecture in favor of finiteness, and 2D Yang-Mills theory is invalid, because it assumes that Riemann’s $\zeta(s)$ is true.

Consistent with the LNC’s holding that Riemann’s $\zeta(s)$ is false is the holding that the derivation of Riemann’s $\zeta(s)$ is invalid, due to Riemann’s improper use of Cauchy’s integral theorem to equate the branch cut of $f(x) = \log(-x)$ to a Hankel contour. The branch cut is non-holomorphic, and the Hankel contour is open (or closed at infinity, thereby encircling non-holomorphic points). Both the branch cut and Hankel contour contradict prerequisites of Cauchy’s integral theorem.

In set theory, $\zeta(s)$’s zeros form an empty set, so RH ("all zeros are on the critical line") and $\overline{RH}$ ("not all zeros are on the critical line") are both "vacuously true". In classical logics, $\zeta(s)$ has no zeros, so material implication holds that $RH$ and $\overline{RH}$ are both true, and conjunction holds that both are false. So in classical logics and set theory, RH is a paradox, violates LNC and LEM, and causes ECQ.

In contrast, three-valued logics (3VL), e.g. Frege’s, Łukasiewicz’s, Kleene’s reject LEM, and hold RH to be neither true nor false. Priest’s "Logic of Paradox" 3VL further rejects LNC and ECQ, and holds RH to be both true and false. In Frege’s logic, neither is proven to imply both, and vice versa. In Kleene’s and Priest’s 3VLs, the conditional proposition "if RH, then X" is true, if X is true (in Łukasiewicz’s, also if X is both), and does not cause ECQ.

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In formal logic, a contradiction is the signal of defeat, but in the evolution of real knowledge it marks the first step in progress toward a victory.

Alfred North Whitehead

1 Introduction

1.1 Riemann’s Zeta Function and the Logical Laws it Violates

Riemann claims to have derived an "expression" of $\zeta(s)$ that is convergent throughout half-plane $\text{Re}(s) \leq 1$ (except at $s = 1$).\(^1\)\(^2\) However, the Dirichlet series "expression" of $\zeta(s)$ is proven to be divergent throughout this half-plane $\text{Re}(s) \leq 1$.\(^3\)\(^4\) So for all values of $s$ in half-plane $\text{Re}(s) \leq 1$ (except at $s = 1$), Riemann’s definition of $\zeta(s)$ violates the definition of a mathematical function, and all three of Aristotle’s "Laws of Thought":

1. The definition of a mathematical function,\(^5\) due to the one-to-two mapping, from domain to range, of $\zeta(s)$ throughout half-plane $\text{Re}(s) \leq 1$ (except at $s = 1$), if we assume that the Dirichlet series definition of $\zeta(s)$ and Riemann’s definition of $\zeta(s)$ are both true;

2. The Law of Identity (LOI) (a.k.a. the Law of Equivalence, and "Leibniz’s Law")

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1See Riemann [94], p.1: "The function of the complex variable $s$ which is represented by these two expressions [the Euler product and the Dirichlet series], wherever they converge, I denote by $\zeta(s)$. Both expressions converge only when the real part of $s$ is greater than 1; at the same time an expression for the function can easily be found which always remains valid."

2Riemann [94] concedes, in its first phrase of the sentence cited in the preceding footnote, that $\zeta(s)$ is divergent throughout the half-plane $\text{Re}(s) \leq 1$. The second phrase of the sentence contradicts the first phrase, by stating that there is an "expression" of $\zeta(s)$ that is "always valid". So Riemann violates the LNC (and invalidates his entire paper) in this single sentence. Also, note that the Euler product is superfluous in Riemann’s initial definition of $\zeta(s)$ in Riemann [94] p.1.

3See Hardy [41], p.4, Theorem 3: "The series may be convergent for all values of $s$, or for none, or for some only. In the last case there is a number $\sigma_s$ such that the series is convergent for $\sigma > \sigma_s$, and divergent or oscillatory for $\sigma < \sigma_s$. In other words the region of convergence is a half-plane." (Citing Jensen [52]).

4See also Hardy [41], p.5, Example (iii): "The series $\sum n^{-s}$ has $\sigma = 1$ as its line of convergence. It is not convergent at any point of the line of convergence, diverging to $+\infty$ for $s = 1$, and oscillating finitely at all other points of the line." (Citing Bromwich [12]).

5See Stover [112]: "A function is a relation that uniquely associates members of one set with members of another set. More formally, a function from $A$ to $B$ is an object $f$ such that every $a \in A$ is uniquely associated with an object $f(a) \in B$. A function is therefore a many-to-one (or sometimes one-to-one) relation."
due to the two different values of $\zeta(s)$ in half-plane $\text{Re}(s) \leq 1$ (except at $s = 1$), resulting from the two different definitions of $\zeta(s)$; So in said half-plane, divergent values of $\zeta(s)$ are not equivalent to convergent values of $\zeta(s)$, so $\zeta(s)$ is not equivalent to itself;

(3) The Law of the Excluded Middle (LEM) is another of the three Aristotelean "Laws of Thought". It states that every proposition is either true or false, and thus cannot be both (hence the "excluded middle"). However, at any value of $s$ in half-plane $\text{Re}(s) \leq 1$ (except at $s = 1$), the "function" $\zeta(s)$ has two values ("convergent and "divergent"). If we assume that the Dirichlet series definition of $\zeta(s)$ and Riemann’s definition of $\zeta(s)$ are both true, then these two values violate LEM;

(4) Most importantly, Riemann’s $\zeta(s)$ violates the Law of Non-Contradiction (LNC), due to the two contradictory values of $\zeta(s)$ (divergence and convergence) at all values of $s$ in half-plane $\text{Re}(s) \leq 1$ (except at $s = 1$). The Law of Non-Contradiction (LNC) holds that proposition $p$ and its negation $\neg p$ cannot both be true. So $\zeta(s)$ cannot be both convergent and divergent at any value of $s$. For this reason, any "analytic continuation" of $\zeta(s)$ that claims to be convergent at any value of $s$ in half-plane $\text{Re}(s) \leq 1$ violates...
the LNC.  

Moreover, the Dirichlet series definition of $\zeta(s)$ has been proven to be divergent throughout half-plane $\text{Re}(s) \leq 1$, the function $\zeta(s)$ is exclusively divergent in that half-plane, and consequently $\zeta(s)$ has no zeros there.  

(Moreover, $\zeta(s)$ as defined by the Dirichlet series has no zeros at all). So $\zeta(1) \neq 0$, thus resolving the Birch and Swinnerton-Dyer (BSD) conjecture in favor of finiteness.  

Also, in classical logics, "Riemann zeta function regularization" is invalid, and so are all physics theories that assume that it or Riemann's definition of $\zeta(s)$ in half-plane $\text{Re}(s) \leq 1$ are valid (e.g. two-dimensional Yang-Mills theory).  

12See e.g. the "analytic continuations" in Gelbart [35], Abstract: "In this expository article, we describe the two major methods for proving the analytic continuation and functional equations of $L$-functions: the method of integral representations, and the method of Fourier expansions of Eisenstein series." See also Gelbart [35], p.78: "Our point is that the analytic continuation and functional equation for the Eisenstein series furnish an analytic continuation and functional equation for the Riemann $\zeta$-function."  

13See also Gelbart [35], p.78, which states: "To analytically continue $\zeta(s)$, basically 'the constant term' is enough: reading through the spectral proof of the analytic continuation of $\phi(s)$ for $E(z,s)$, one demonstrates that $\xi(s)$ is holomorphic everywhere, save for simple poles at $s = 0$ and 1." However, analytic continuation of $\zeta(s)$ violates the LNC, so the aforementioned cannot be true.  

14Moreover, $L$-functions are generalizations of the Riemann $\zeta(s)$ function (whose "analytic continuation" is invalid). So, analytically-continued $L$-functions are invalid, as are the methods that assume that analytically-continued $L$-functions are true, e.g. those described in Gelbart [35], p.65: "The Dirichlet $L$-functions $L(s,\chi)$ satisfy the properties $E$, $BV$, and $FE$ analogous to those of $\zeta(s)$ (which corresponds to the trivial character)". citing Davenport [19].  

15See also Gelbart [35], pp.60-61, which defines the properties $E$, $BV$, and $FE$ as follows (unfortunately, all are false, because the analytic continuation of $\zeta(s)$ violates the LNC): "We want also to characterize the $\zeta$-function as satisfying the following three classical properties (which are simpler to state in terms of $\xi(s)$, the completed $\zeta$-function). Entirety ($E$): $\xi(s)$ has a meromorphic continuation to the entire complex plane, with simple poles at $s = 0$ and 1. Functional Equation ($FE$): $\xi(s) = \xi(1-s)$. Boundedness in vertical strips ($BV$): $\xi(s) + 1/s + 1/(1-s)$ is bounded in any strip of the form $-\infty < a < \text{Re}(s) < b < \infty$ (i.e. $\xi(s)$ is bounded in vertical strips away from its two poles)."  

16Odlyzko [80] discloses several methods for calculating "the zeros of $\zeta(s)$", using the Euler-Maclaurin formula, Riemann-Siegel formula, Turing’s method, and a method that includes a fast Fourier transform. However, these "zeros" are zeros of those respective formulas. They are not zeros of Riemann’s definition of $\zeta(s)$ (which is invalid), nor of the Dirichlet series definition of $\zeta(s)$.  

17As discussed in Hardy [41], in half-plane $\text{Re}(s) \leq 1$, the Dirichlet series definition of $\zeta(s)$ is divergent, and therefore non-zero. This can be proven by "integration by parts" of the trigonometric version of the Dirichlet series definition of $\zeta(s)$ (which is obtained by substituting Euler’s formula into the Dirichlet series). See e.g., Sharon [104], Appendix E, pp.26-32.  

18In half-plane $\text{Re}(s) > 1$, the Dirichlet series definition of $\zeta(s)$ equals the Euler product of the primes. Neither the Dirichlet series nor the Euler product have any zeros. Each factor of the Euler product is a fraction, with "1" as the numerator. So the Euler product cannot equal zero, because at least one numerator of "0" is necessary for a product to equal zero. Therefore the Dirichlet series cannot equal zero either (in this half-plane).  

19See Clay Mathematics Institute [15]: "[T]his amazing conjecture asserts that if $\zeta(1)$ is equal to 0, then there are an infinite number of rational points (solutions), and conversely, if $\zeta(1)$ is not equal to 0, then there is only a finite number of such points."

20See, e.g. Witten [140]: Eq. 2.20, Eq.2.32, and Eq.3.22.
1.2 The Derivation of Riemann’s Zeta Function is Invalid

Consistent with the LNC’s holding that Riemann’s definition of $\zeta(s)$ is false, is the holding that Riemann’s derivation of his "expression" of $\zeta(s)$ is invalid. This invalidity is due to Riemann’s improper use of Cauchy’s integral theorem to equate the branch cut of $f(x) = \log(-x)$ in the complex plane (i.e. the non-negative real axis) to a Hankel contour. But all of the points on the branch cut are undefined (because $\log(-x)$ is undefined for non-negative real values), and therefore these points are non-holomorphic. Moreover, the Hankel contour is either open or closed (if closed, it is closed at infinity, thereby encircling non-holomorphic points). Regardless of whether the Hankel contour is interpreted as open or closed, both it and the branch cut contradict prerequisites of Cauchy’s integral theorem.

1.3 Riemann’s Zeta Function has No Zeros, So Riemann’s Hypothesis has Vacuous Subjects, and Thus is a Paradox

In regards to the Riemann Hypothesis ("RH"), its traditional phrasing is "all zeros of $\zeta(s)$ are on the critical line". The traditional phrasing of the negation of RH ("$\neg RH$") is "not all zeros of $\zeta(s)$ are on the critical line". However, RH falsely assumes that Riemann’s definition of $\zeta(s)$ is valid and has zeros. In set theory, the zeros of $\zeta(s)$ form an empty set, so both RH and $\neg RH$ are "vacuously true", because both propositions describe attributes of an empty set. So in set theory, RH is a paradox, because it violates LNC and LEM. This results in ECQ, due to RH’s violation of LNC.

The results in classical logics are identical to the results in set theory. Because $\zeta(s)$ has no zeros, both RH and $\neg RH$ are propositions with "vacuous subjects", of the type

21See Edwards [25], pp.10-11.
22See Whittaker [127], pp.85-87, 244-45 and 266.
23See Edwards [25], section 1.9, p.19: "Riemann’s next statement is even more baffling. He states that the number of roots $\rho$ of $\zeta(\rho) = 0$ on the line Res $= 1/2$ is also "about" $T/2\pi \cdot \log T/2\pi - T/2\pi$... He gives no indication of a proof at all, and no one since Riemann has been able to prove (or disprove) this statement... He says he considers it ‘very likely’ that the roots all do lie on Res $= 1/2$, but says that he was not able to prove it". See also Edwards [25], section 7.8, pp.164-166; and chapter 9, pp.182-202.
24Riemann’s statement in [94], p.4, as translated by Wilkins, is: "One now finds indeed approximately this number of real roots of $\xi(t) = 0$ within these limits, and it is very probable that all roots are real. Certainly one would wish for a stricter proof here ...".
25The Dirichlet series "expression" of $\zeta(s)$ has no zeros, the Euler product "expression" of $\zeta(s)$ has no zeros, and the Riemann "expression" of $\zeta(s)$ is invalid.
26See Gardner [34] for many other examples of paradoxes.
27See also Scruton [102], Chapter 27 "Paradox", pp.397-412, and 575.
discussed by Frege, Russell, Strawson, and others. (The most famous example being "The present King of France is bald"). According to Frege, neither RH nor \( \overline{RH} \) have truth-values (a "truth-value gap"). According to Russell, RH and \( \overline{RH} \) are both true and false (a "truth-value glut").

Moreover, the same paradoxical results are obtained by rephrasing RH and \( \overline{RH} \) to expressly state the assumption that \( \zeta(s) \) has zeros. This can be done in two ways: (1) as conditional propositions, or (2) as conjunctions. It turns out that the conditional propositions are negations of the conjunctions (and vice versa).  

If RH is rewritten as a conditional proposition, it becomes RH\(_1\): *if \( \zeta(s) \) has zeros, then all zeros of \( \zeta(s) \) are on the critical line*. Likewise, \( \overline{RH} \) becomes RH\(_2\): *if \( \zeta(s) \) has zeros, then not all zeros of \( \zeta(s) \) are on the critical line*. In classical logics, if \( \zeta(s) \) actually had zeros, then by material implication, one of RH\(_1\) and RH\(_2\) would be true, and the other would be false.

However, \( \zeta(s) \) has no zeros, so material implication holds that the RH\(_1\) and RH\(_2\) are both true, because each has an antecedent portion ("\( \zeta(s) \) has zeros") that is false. Therefore, regardless of the truth or falsity of the consequent portion ("all zeros of \( \zeta(s) \) are on the critical line", or "not all zeros of \( \zeta(s) \) are on the critical line"), the proposition as a whole is true.

If RH is rewritten as a conjunction (RH\(_2\)), it becomes: *\( \zeta(s) \) has zeros, and all zeros of \( \zeta(s) \) are on the critical line*. The negation of RH (\( \overline{RH} \)) becomes RH\(_2\): *\( \zeta(s) \) has zeros, and not all zeros of \( \zeta(s) \) are on the critical line*. In classical logics, if \( \zeta(s) \) actually had zeros, then by conjunction, one of RH\(_2\) and RH\(_2\) would be true, and the other would be false.

However, \( \zeta(s) \) has no zeros, so conjunction holds that the RH\(_2\) and RH\(_2\) are both false, because each has an antecedent portion ("\( \zeta(s) \) has zeros") that is false. Therefore,
regardless of the truth or falsity of the consequent portion ("all zeros of \(\zeta(s)\) are on the critical line", or "not all zeros of \(\zeta(s)\) are on the critical line"), the proposition as a whole is false.

So either RH and its negation \(\neg RH\) are paradoxes, and have either a truth-value glut, or a truth-value gap. These results are impermissible in classical logics, due to the LNC and LEM.

1.4 Three-Valued Logics Allow Riemann’s Hypothesis to be Neither True nor False, and/or Both True and False

In contrast to classical logic, Frege’s, Łukasiewicz’s, and Kleene’s three-valued logics (3VL) reject LEM, as does Intuitionism in a more limited manner (specifically, in regards to proof and lack thereof), thereby allowing propositions (e.g. the RH) to be neither true nor false. Priest’s "Logic of Paradox" (LP) further rejects LNC (by allowing for paradoxes) and ECQ, thereby allowing propositions (e.g. the RH) to be both true and false. 35 Moreover, in Frege’s logic, neither is proven to imply both, and vice versa. 36 37 In their truth tables for material implication, Kleene’s, Łukasiewicz’s, and Priest’s 3VLs, the proposition "if RH, then X" is true, if X is true, and does not cause ECQ despite RH having the third truth-value. In Łukasiewicz’s 3VL, "if RH, then X" is also true if X has the third truth-value "neither/both". 38

1.5 Additional Introductory Info

The LNC is one of Aristotle’s three "Laws of Thought", 39 and is an axiom or theorem in all classical logics 40 (e.g. Principia Mathematica, Boole’s algebraic symbolism for logic 41), and in many non-classical logics (e.g. intuitionism). 42 The LNC in sequent

\[\neg(A \lor \neg A)\] is equivalent to \[A \land \neg A\] (see Wikipedia [136]). The Łukasiewicz L3 has the same tables for AND, OR, and NOT as the Kleene logic given above, but differs in its definition of implication in that "unknown implies unknown" is true. This section follows the presentation from [Malinowski [65]].

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35See Priest [89]: these four possibilities (true, false, both, neither) form the "catuskoti" (or "tetralemma") of early Buddhist logic, which rejects the LNC.

36See Milne’s [75] p.475, citing Heidelberger [45].

37See also Priest [89], p.27, which recites an explanation for this phenomenon that is different from Heidelberger’s [45]: "Notably, assuming De Morgan’s laws, \(\neg(A \lor \neg A)\) is equivalent to \(A \land \neg A\).

38See Wikipedia [136]: "The Łukasiewicz L3 has the same tables for AND, OR, and NOT as the Kleene logic given above, but differs in its definition of implication in that "unknown implies unknown" is true. This section follows the presentation from [Malinowski [65]]."

39See Gottlieb [36]

40See Gabbay, [33], Chapter 2.6.

41See Boole [11], pp. 48-49, Proposition IV. See also Stillwell [111], p.99: "In fact, if \(p + q\) is taken to mean 'p or q but not both,' then the algebraic rules of propositional logic become exactly the same as those of mod 2 arithmetic."

42But see Priest [90]: "dialetheism amounts to the claim that there are true contradictions."
form 43 is: \( \vdash \neg (A \land \neg A) \). Its verbal characterizations include “opposite assertions cannot both be true simultaneously”, and "no unambiguous statement can be both true and false". 44 45

According to LNC, a function \( f(s) \) of variable \( s \) cannot be both convergent and divergent at any value of \( s \). Therefore, Riemann’s definition of \( \zeta(s) \) violates the LNC, because it claims that \( \zeta(s) \) is convergent at all values of \( s \in \mathbb{C} \) in half-plane \( \text{Re}(s) \leq 1 \) (except \( s = 1 \)), while also the Dirichlet series "expression" of \( \zeta(s) \) is proven to be divergent at the same values of \( s \). Therefore, in all logics that have LNC as an axiom or theorem, Riemann’s definition of \( \zeta(s) \) must be false.

Furthermore, in logics that accept the principle of "explosion" (ECQ) (e.g. classical and intuitionistic logics), violation of LNC causes any proposition, no matter how absurd, to be trivially true. In contrast, paraconsistent bivalent logics reject "explosion" (ECQ), by rejecting the axioms (e.g. disjunctive syllogism and/or disjunction introduction) that lead to ECQ. 46 In paraconsistent logics, unrelated propositions are no longer "trivially true", but propositions that are directly related to the contradictory proposition remain false. (In a paraconsistent bivalent logic, any proposition is false if it assumes that Riemann’s "analytic continuation" of \( \zeta(s) \) is true).

Also, because \( \zeta(s) \) has no zeros, the Riemann Hypothesis (RH), which states that "all the zeros of \( \zeta(s) \) are on the critical line \( \text{Re}(s) = 0.5 \)", is a proposition with non-existent subjects ("vacuous subjects"). When RH is rephrased as "if \( \zeta(s) \) has zeros, then all zeros are on the critical line \( \text{Re}(s) = 0.5 \)", the RH is both true and false in classical logics, due to material implication (a.k.a. "the material conditional"). is violates the LNC. 47 The material conditional states that \( p \rightarrow q \) is logically equivalent to \( \neg (p \land \neg q) \). 48 So, counter-intuitively, in classical and intuitionistic logics the material

43 See Horn [47], Gottlieb [36]; Grishin [38]; and Smith [108], section 11.
44 See Perzanowski [82] p.22, para.4: "The Principle of Non-Contradiction occurs in at least four versions: METAPHYSICAL — no object can, at the same time be and not be such-and-such; LOGICAL — no unambiguous statement can be both true and false; PSYCHOLOGICAL — nobody really and seriously has contradictory experiences, i.e., nobody really sees and does not see (hears and does not hear) simultaneously, etc.; ETHICAL — no one in his right mind would simultaneously demand (or perform) A and not-A."
45 An example use of LNC in the context of the RH is found in Edwards [25], chapter 9, p.202, citing Landau [58], which uses the LNC to prove the theorem that "if there are only a finite number of exceptions to the Riemann hypothesis, then \( S(t) \) cannot be bounded below".
46 See Mortansen [77] and Priest [91].
47 This paradoxical result in classical logic of RH being both true and false is inconsistent with Deligne’s proof of the Weil conjecture III (the function field analogue of the Grand Riemann Hypothesis), and is inconsistent with Hasse’s proof of the RH for elliptic curves of genus 1. See Milne [74], pp.3 and 49.
48 In classical logics (but not in intuitionistic logics), by De Morgan’s Laws this is also equivalent to \( \neg (p \lor q) \).
conditional is true when \( p \) is false. 49 50

Russell’s *On Denoting*, which is not a formal logic (but is still relevant to this situation), states that a sentence with a non-existent subject (e.g. the RH) can be interpreted as either a true statement or as a false one. If the RH is interpreted as "there exist zeros of \( \zeta(s) \), and it is not the case that any of them are located off of the critical line \( \text{Re}(s) = 0.5 \)"; then the RH is false, because \( \zeta(s) \) has no zeros. However, the alternative interpretation is "it is not the case that there exist zeros of \( \zeta(s) \) and any of them are located off of the critical line \( \text{Re}(s) = 0.5 \)." This version of RH is true, because it indeed is not the case that there exist zeros of \( \zeta(s) \). So according to Russell, RH’s ambiguity means that it can be interpreted as either true or false (and thus is both). 51

In contrast, some non-classical logics reject LEM (e.g. multi-valued logics, Frege’s *Über Sinn und Bedeutung*, Strawson’s *On Referring*), thereby enabling a third state in addition to "true" and "false". For example, Frege’s *Über Sinn und Bedeutung* (which can be modeled as a three-valued logic) states that propositions with vacuous subjects (e.g. the RH) lack any truth-value, so they are neither true nor false. Strawson’s reasoning in *On Referring* states that questions with "vacuous subjects" (such as the RH) are "absurd" and therefore not asked, thereby inherently creating three truth-values (true, false, absurd), thereby rejecting the LEM. 52

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49 See Tarski [114], pp.25-26: "The logicians ... adopted the same procedure with respect to the phrase "if ..., then ..." as they had done in the caso of the word "or". For this purpose, they extended the usage of this phrase, considering an implication as a meaningful sentence even if no connection whatsoever exists between its two members, and they made the truth or falsity of an implication dependent exclusively upon the truth or falsity of the antecedent and consequent. To characterize this situation briefly, we say that contemporary logic uses IMPLICATIONS IN MATERIAL MEANING, or simply, MATERIAL IMPLICATIONS; this is opposed to the usage of IMPLICATIONS IN FORMAL MEANING, or simply, FORMAL IMPLICATION, in which case the presence of a certain formal connection between antecedent and consequent is an indispensable condition of the meaningfulness and truth of the implication, The concept of formal implication ... is narrower than that of material implication[.]"]

50 See also Tarski [114], p.26: "In order to illustrate the foregoing remarks, let us consider the following four sentences:

\[
\begin{align*}
\text{if } 2 \cdot 2 = 4, & \text{ then New York is a large city;} \\
\text{if } 2 \cdot 2 = 5, & \text{ then New York is a large city;} \\
\text{if } 2 \cdot 2 = 4, & \text{ then New York is a small city;} \\
\text{if } 2 \cdot 2 = 5, & \text{ then New York is a small city.}
\end{align*}
\]

In everyday language, these sentences would hardly be considered as meaningful, and even less true. From the point of view of mathematical logic, on the other hand, they are all meaningful, the third sentence being false, while the remaining three are true."

51 Note that both axiomatic set theory and Russell’s *On Denoting* assume that the LEM is true.

52 Note: the 160 year history of the RH should be sufficient evidence to invalidate this argument.
1.6 Implications of the Invalidity of Riemann’s Zeta Function

The invalidity of Riemann’s definition of $\zeta(s)$ has wide-reaching implications. Some of these implications are discussed in this paper. In regards to real analysis, the divergence of the trigonometric version of the Dirichlet series definition of $\zeta(s)$ throughout the "critical strip" ($0 < \text{Re}(s) \leq 1$) is sufficient to invalidate both Abel’s Uniform Convergence Test $^{53}$ and Dirichlet’s Convergence Test. $^{54}$

In number theory, the invalidity of "analytic continuation" of $\zeta(s)$ applies also to the generalizations of $\zeta(s)$ (e.g. Dirichlet $L$-functions. $^{55}$), Also, because $\zeta(1) \neq 0$, the Birch and Swinnerton-Dyer (BSD) Conjecture is resolved in favor of finiteness. So are the closely-related Tate-Shaferevich group, the Brauer group, and the Tate conjecture. The Hodge conjecture must be resolved in a manner consistent with these results (otherwise it violates the LNC). The invalidity of Riemann’s definition of $\zeta(s)$ also invalidates generalizations of the RH, such as the Grand Riemann Hypothesis and the Grand Lindelöf Hypothesis (GLH).

In physics, the invalidity of Riemann’s definition of $\zeta(s)$ means that "Zeta Function Regularization" is invalid (in all logics that have the LNC). This paper points out several instances where this invalid "regularization" is used in Yang-Mills theory, the Casimir Effect, Quantum Electrodynamics (QED), Chromodynamics (QCD), Supersymmetry (SUSY), Quantum Field Theory (QFT), and Bosonic String Theory.

2 The Axiomatic Method and Aristotle’s "Laws of Thought"

2.1 Axiom Systems and the Axiomatic Method

Carnap [14] defines "axiom systems" as follows: $^{56}$

By an axiom system (abbreviation: AS) we understand the representation of a theory in such a way that certain sentences of this theory (the axioms) are placed at the beginning, and from them further sentences (the theorems) are derived by means of logical deduction.

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$^{53}$See Weisstein [124].
$^{54}$See Weisstein [125].
$^{55}$See Wiles [138], p.2: "A conjecture going back to Hasse ... predicted that $L(C, s)$ should have a holomorphic continuation as a function of $s$ to the whole complex plane. This has now been proved". This result contradicts the LNC!
$^{56}$See Carnap [14], p.171.
There is a traditional view of AS - current in Euclid’s time, and continuing into our own - that requires its axioms to be self-evident, i.e. immediately clear to the intuition and hence in no need of proof. (Even today, common usage tends to attribute this meaning to the word "axiom"). The modern conception of an AS does not include this requirement; arbitrary sentences may be selected as axioms.

2.2 Overview of Aristotle’s "Laws of Thought"

Aristotle’s "Laws of Thought" (as characterized by Boole and Russell) is a minimalistic axiom system. Aristotle’s three "Laws of Thought" are the Law of Identity (LOI), the Law of Non-Contradiction (LNC), and the Law of Excluded Middle (LEM).

All three are inherited in classical logics, either as axioms or as theorems (e.g. the LNC and LEM are theorems in Russell’s Principia Mathematica, and LOI is often referred to as "Material Equivalence"). Some non-classical logics, such as intuitionistic logics (including minimal logic), accept the LOI and LNC, but reject the LEM when a proposition has neither been proved or disproved. The LNC is also the test for simple consistency of a propositional calculus.

The three "Laws of Thought" originate in Aristotle’s Organon. However, Aristotle discusses the LNC elsewhere in his works, and Plato’s Socratic dialogues (all of

57See Boole, [11], which discusses the LOI in Chapter II, pp.34-36, Para.12-13; the LNC in proposition IV, Chapter III, p.49; and the LEM in pp.8 and 99-100, and in proposition II, Chapter III, p.48

58As stated by Russell [97] at Chapter VII: "On Our Knowledge of General Principles", Aristotle’s "Laws of Thought" are:

(1) The law of identity [LOI]: 'Whatever is, is.'
(2) The law of contradiction [LNC]: 'Nothing can both be and not be.'
(3) The law of excluded middle [LEM]: 'Everything must either be or not be.'

59See also Brittanica [24], citing Dorbolo [22]

60According to Priest [87] p.139, both LNC and LEM as defined in Aristotle’s Metaphysics, Book 4, are not logical principles for Aristotle, but metaphysical principles, governing the nature of beings qua beings. By the time one gets to Leibniz, however, the Laws have been absorbed into the logical canon.

61See Lee [60], pp.193-194, 251

62See Carnap [14] p.173: "Consistency is thus an obvious requisite of any non-trivial [Axiomatic System]." See also Tarsky [115], p.28: "If [theorem] T is inconsistent, two sentences Φ and ¬Φ are valid in T". See also Tarski’s [115] example in pp.46-47.

63See "On Interpretation" at Aristotle [6], the second of the six texts in Aristotle’s "Organon".

64See Gottlieb [36], citing Aristotle’s Metaphysics IV (Gamma) 3–6, especially 4; De Interpretatione; and Posterior Analytics I, chapter 11.
which predate Aristotle) have Socrates discussing the LNC.  

Bertrand Russell argues that these three "laws" are either axioms or theorems of logic: 

'[A]nything implied by a true proposition is true' ... is one of a certain number of self-evident logical principles. Some at least of these principles must be granted before any argument or proof becomes possible. When some of them have been granted, others can be proved, though these others, so long as they are simple, are just as obvious as the principles taken for granted. For no very good reason, three of these principles have been singled out by tradition under the name of 'Laws of Thought.'

Lee [60] states that these "are so general and intuitive that their general forms are accepted as laws of logic." Minto [76] adds that "[i]t is even said that all the doctrines of Deductive or Syllogistic Logic may be deduced from them." (Note: In the "classical logic" of Whitehead and Russell's Principia Mathematica ("PM"), Aristotle's three "Laws of Thought" are theorems derived from other axioms.

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65In the Socratic dialogue Republic, Plato characterizes Socrates as advocating for the LNC: "It is obvious that the same thing will not do or suffer opposites in the same respect in relation to the same thing and at the same time." See Priest [87] pp.137-138, citing Hamilton [109], p.436b.

66In the Socratic dialogue Euthyphro, Plato has Socrates using the LNC in an argument. See Smith [107], p.29: "Euthyphro says that that which is pious is beloved of the gods. Socrates then gets Euthyphro that the gods sometimes quarrel, usually in respect to objects of love or hatred. Socrates then argues that things exist which some gods love but others hate (a position Euthyphro again has to agree with). Socrates next contends that if Euthyphro’s definition of piety is right, then there must be objects that are contemporaneously pious and impious, since they are loved and hated by the gods at the same time. Euthyphro realizes the absurdity of the proposition and is forced to review his understanding of what it is to be pious."

67But Priest [87] interprets the Socratic dialogue Parmenides as advocating against the LNC: "Even if all things come to partake of both [the form of like and the form of unlike], and by having a share of both are both like and unlike one another, what is there surprising in that? ... when things have a share in both or are shown to have both characteristics, I see nothing strange in that, Zeno, nor yet in a proof that all things are one by having a share in unity and at the same time many by sharing in plurality. But if anyone can prove that what is simple unity itself is many or that plurality itself is one, then shall I begin to be surprised." See Priest [87] p.138, citing Hamilton [109], p.129b,c.

68But Brownstein [13], pp.49-50, interprets the same section of Parmenides as not contradicting the LNC. Brownstein assumes that \(a\) is a red circle, \(b\) is a red square, and \(c\) is a green circle. "Thus \(a\) and \(b\) are qualitatively similar to one another [in color] ... but dissimilar to \(c\). ... Thus \(a\) and \(c\) are similar to each other [in shape] while both are not similar to \(b\). We might describe this situation as one in which objects \(a\), \(b\), and \(c\) are both alike and unlike ... Plato makes it clear that he does not regard the kind of situation I have described as an absurdity at all." However, we cannot describe this situation as one in which objects \(a\), \(b\), and \(c\) are both circles and non-circles, or both red and non-red.

69See Russell [97], Chapter VII: "On Our Knowledge of General Principles".

70See Lee [60], Chapter VII: "On Our Knowledge of General Principles".

71See also Minto [76], p.29
Moreover, in the "classical logic" of PM, Aristotle’s three laws are interchangeable, due to the inter-definition of the connectives of the three laws.)

2.3 The Law of Identity (LOI)

"Leibniz’s Law" is: "\(x = y\) if, and only if, \(x\) has every property which \(y\) has, and \(y\) has every property which \(x\) has". 74 This law is also referred to as the "Identity of Indiscernibles". 75 By substituting "\(x\)" for "\(y\)" we obtain the law that "\(x = x\) if, and only if, \(x\) has every property which \(x\) has, and \(x\) has every property which \(x\) has". 76 This can be simplified to "\(x = x\) if, and only if, \(x\) has every property which \(x\) has". 77 78

This last variation of Leibniz’s Law is equivalent to the Law of Identity (LOI), which is the first of Aristotle’s three "Laws of Thought", 79 and is an axiom of classical and intuitionistic propositional logics. The Law of Identity (LOI) states that a proposition \((P)\) "is the same with itself and different from another", which can be written as \(P \equiv P\). In the notation of Whitehead and Russell’s Principia Mathematica 80, the corresponding propositional logic sequent is: \(\vdash .p \equiv p\).

Another definition of the LOI, in the context of propositions, is that the definition of a proposition must be consistent throughout a logical discourse. Changing the definition of a proposition in the course of an argument is "equivocation". Aristotle states that "[t]he identity of subject and of predicate must not be 'equivocal'". 81

Therefore, at every value of domain \(s\), \(\zeta(s)\) cannot have two different definitions, if they produce two or more different values of \(\zeta(s)\), because this would be "equivocation". It would mean that the same proposition (in this case, \(\zeta(s)\)) is different from itself \((x \neq x, \text{ or alternatively, } p \neq p)\), thereby violating the LOI.

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72 73 Moreover, in the "classical logic" of PM, Aristotle’s three laws are interchangeable, due to the inter-definition of the connectives of the three laws.)

74See Langer [59], p.305, which states that the LNC is proved in Th. 3.24 of Principia Mathematica. Whitehead and Russell, where the two famous authors state that "in spite of its fame, we have found few occasions for its use."

73See also Andrews [3], p.54, which states that all three of the Laws of Thoughts are theorems in Principia Mathematica: The LNC in Th. 3.24, and the LEM in Th. 2.11. and the Principle of Identity in Th. 2.08. (Russell states that the ‘Law of Identity’ is inferred later in PM from the Principle of Identity). See also Principia Mathematica to *56, Cambridge, 1967, pp. 99, 101, 111.

74See Tarski, [114], p.55.

75See Forrest, [27], which formulates it as: "[I]f, for every property \(F\), object \(x\) has \(F\) if and only if object \(y\) has \(F\), then \(x\) is identical to \(y\). Or in the notation of symbolic logic: \(\forall(Fx \leftrightarrow Fy) \rightarrow x = y\)"

76See Tarski, [114], p.56

77See Tarski, [114], p.56

78See also Sruton [102], pp.144-146: "But what is identity? Philosophers agree on the following four characteristics: ... (ii) Identity is reflexive: everything is identical with itself: \((x)(x = x)\).

79See, e.g. Russell [97], at Chapter VII: "On Our Knowledge of General Principles".

80See Langer [59], p.307, sequent (*4.2).

81See part 6 of "On Interpretation" at Aristotle [6].
2.4 The Law of the Excluded Middle (LEM)

The Law of the Excluded Middle (LEM) is another of the three Aristotelean "Laws of Thought". It states that every proposition is either true or false, and thus cannot be both (hence the "excluded middle"). According to Aristotle: \(^{82}\)

In the case of that which is or which has taken place, propositions, whether positive or negative, must be true or false. Again, in the case of a pair of contradictories, either when the subject is universal and the propositions are of a universal character, or when it is individual, as has been said, one of the two must be true and the other false[.]

The sequent of the LEM is written as: \(\forall P \vdash (P \lor \neg P)\). Counter-intuitively, the truth table of the logical disjunction "\(\lor\)" is that of the Boolean "Inclusive OR", not that of the Boolean "Exclusive OR (XOR)". \(^{83}\) (So only in logics that have both the LEM and the LNC is the middle indeed excluded).

In the context of non-classical logics, the LEM is a frequently rejected "law". The LEM is rejected by intuitionistic logics, \(^{84}\) by multi-valued logics, and by other closely related logics/models (e.g. Frege’s Über Sinn und Bedeutung, Strawson’s On Referring, and Russell’s On Denoting).

2.5 The Law of Non-Contradiction (LNC)

The Law of the Non-Contradiction (LNC) is the third axiom of Aristotle’s "Laws of Thought". One expression of this law \(^{85}\) is the sequent: \(\forall P \vdash \neg(P \land \neg P)\). The LNC states that a proposition \((P)\) and its negation \((\neg P)\) cannot both be true simultaneously \((\neg(A \land \neg A))\). Another expression of LNC is that "no unambiguous statement can be both true and false." \(^{86}\) Yet another version is that either a proposition \((P)\), or its negation \((\neg P)\), is true. \(^{87}\) (This last version is true only in logics that accept the LEM). \(^{88}\)

\(^{82}\)See part 9 of "On Interpretation" at Aristotle [6].

\(^{83}\)See Aloni [2]; and Horn’s [47] section 2: "LEM and LNC".

\(^{84}\)See Moschovakis [78].

\(^{85}\)See Horn [47], Gottlieb [36]; Grishin [38]; and Smith [108], section 11.

\(^{86}\)See Perzanowski [82] p.22, para.4: "The Principle of Non-Contradiction occurs in at least four versions: METAPHYSICAL — no object can, at the same time be and not be such-and-such; LOGICAL — no unambiguous statement can be both true and false; PSYCHOLOGICAL — nobody really and seriously has contradictory experiences, i.e., nobody really sees and does not see (hears and does not hear) simultaneously, etc.; ETHICAL — no one in his right mind would simultaneously demand (or perform) A and not-A."

\(^{87}\)In the alternative: "Any proposition is either true or false". See Langer [59], pp.262-283, and 300.

\(^{88}\)This is a major issue for the intuitionists.
According to Aristotle:

A simple proposition is a statement, with meaning, as to the presence of something in a subject or its absence, in the present, past, or future, according to the divisions of time.\(^{89}\)

An affirmation is a positive assertion of something about something, a denial a negative assertion ... Those positive and negative propositions are said to be contradictory which have the same subject and predicate.\(^{90}\)

We see that in a pair of this sort both propositions cannot be true.\(^{91}\)

When discussing the LNC, it is important to note that the terms "convergent series" and "divergent series" (which are used extensively in this paper) are mutually exclusive, and therefore contradictory: \(^{92}\)

\[
[A] \text{ series } \\
\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \ldots \tag{2.1}
\]

is said to be *convergent*, to the sum \(s\), if the 'partial sum'

\[
s_n = a_0 + a_1 + \ldots + a_n \tag{2.2}
\]

tends to a finite limit \(s\) when \(n \to \infty\); and a series which is not convergent is said to be *divergent*.

Therefore, by definition, an infinite series is *either* convergent or divergent. Such a series either converges to a value, or it does not. For example, an oscillating series such as \(1 - 1 + 1 - 1 + \ldots\) does not converge to any value, and therefore by definition is divergent.\(^{93}\) Also by definition, a given series cannot be *both* divergent and convergent.

So if proposition \((P)\) states that a given series is convergent, then the negation of that proposition (proposition \((\neg P)\)) states that the given series is divergent. The converse is also true: if the proposition \((P)\) states that the series is divergent, then the negation of that proposition (proposition \((\neg P)\)) states that the series is convergent.

According to the LNC, \((P)\) and \((\neg P)\) cannot both be true simultaneously \((\neg (A \land \neg A))\). So the infinite series \(\zeta (s)\) cannot be both convergent and divergent at *any* value of \(s\).

\(^{89}\)See part 5 of "On Interpretation" at Aristotle [6].
\(^{90}\)See part 6 of "On Interpretation" at Aristotle [6].
\(^{91}\)See part 7 of "On Interpretation" at Aristotle [6].
\(^{92}\)See Hardy [40], p.1.
\(^{93}\)See Hardy [40], p.1.
2.6 LNC is the Test for Consistency of an Axiom System

Langer [59] further defines the "axiomatic method" as follows (emphasis added in bold font): 94

All we ask of a postulate [axiom] is (1) that it shall belong to the system, i.e. be expressible entirely in the language of the system ["coherence"]; (2) that it shall imply further propositions of the system ["contributiveness"]; (3) that it shall not contradict any other accepted postulate, or any proposition implied by such another postulate ["consistency"]; and (4) that it itself shall not be implied by other accepted postulates, jointly or singly taken ["independence"].

Langer [59] also states that "Contradictory theorems cannot follow from consistent postulates." 95

Therefore, the LNC is the test for consistency of a axiom system. According to Carnap [14], the LNC is a "sentential formula" that is a tautology. 96 97 Further according to Carnap (emphasis added in bold font)[14]: 98

An [Axiomatic System] AS is said to be inconsistent provided that among its theorems is one of the form \(\mathcal{E}_i\) and another of the form \(\sim \mathcal{E}_i\). An AS is said to be consistent provided that is not inconsistent. In view of T6-15, 99 any sentence of the language is derivable from \(\mathcal{E}_i\) and \(\sim \mathcal{E}_i\) together; the theorems of an inconsistent AS therefore include all the sentences of the language \(L'\), and the AS in consequence is trivial and useless for practical purposes. Consistency is thus an obvious requisite of any non-trivial AS.

Tarski [114] affirms these comments (emphasis added in bold font): 100

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94See Langer [59], pp.185-186.
95See Langer [59], p.202.
96See Carnap [14] p.26: "T8-1. The following formulas are tautologies and hence L-true", followed by two alternate expression of the LEM, and the LNC: (a) \(p\lor \sim p\), (b) \(p \lor p\), and (c) \(p \sim p\).
97See also Carnap [14] p.42: "For suppose it is not raining here now ... E.g. the modal sentence "it is impossible that it is raining and it is not raining" is true, whereas the sentence "it is impossible that it is raining" (produced therefrom by the indicated replacement) is false - for while it is not the case that it is raining here now, this case is nevertheless logically possible. Thus symbolic languages with modality symbols are generally not extensional.
98See Carnap [14] p.173.
99See Carnap [14] p.23: "The class comprising the sentential formulas \(\mathcal{E}_i\) and \(\sim \mathcal{E}_i\) L-implies every sentential formula; and likewise the conjunction \(\mathcal{E}_i \sim \mathcal{E}_i\) L-implies every sentential formula." This corresponds to "explosion" / ex contradictione (sequitur) quodlibet (ECQ).
100See Tarski [114] p.135.
A deductive theory is called CONSISTENT or NON-CONTRADICTORY if no two asserted statements of this theory contradict each other, or, in other words, if of any two contradictory statements (cf. Section 7) at least one cannot be proved. A theory is called COMPLETE, on the other hand, if of any two contradictory sentences formulated exclusively in the terms of the theory under consideration (and the theories preceding it) at least one sentence can be proved in this theory. Of a sentence which has the property that its negation can be proved in a given theory, it is usually said that it can be DISPROVED in that theory. In this terminology we can say that a deductive theory is consistent if no sentence can be both proved and disproved in it[.]

Also Langer [59] reaffirms these comments (emphasis added in bold font. Italic font is in the original): ¹⁰¹

The same theorem may follow from more than one possible selection of premises ... But contradictory theorems can never follow from consistent postulates. No matter how widely developed the system, how far removed a theorem may be from the original assumptions, they and they only are its ultimate premises; if two theorems in a system are incompatible, and there has been no error in the process of deduction, then the postulates, no matter how obvious and simple they appear, are inconsistent[.]

However, Langer [59] is wrong. Occasionally, contradictory theorems do arise from consistent postulates. These are called "paradoxes". They arise, for example, from propositions that have "vacuous subjects", and from contradictory self-referential statements (e.g. the Liar Paradox).

Furthermore, the above quotations stress the importance of the LNC. Therefore they are derived from a logic that assumes bivalence (most likely classical logic), and thus ignore multi-valued logics (which tolerate contradictions). This is discussed in greater detail later in this paper.

### 3 Classical (Propositional) Logics

"Classical logics" are logics that accept the following as axioms or theorems: ¹⁰²

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¹⁰¹See Langer [59] p.135.

¹⁰²See Wikipedia [128], citing Gabbay, [33], Chapter 2.6. See also Lee [60], p.251.
| Name | Synonym | Sequent | Description |
|------|---------|---------|-------------|
| Law of the Excluded Middle (LEM) | | $p \lor \neg p$ | |
| Double Negation (DN) | Double Negative Elimination | $p \equiv \neg \neg p$ | |
| Law of Non-Contradiction (LNC) | | $\neg (p \land \neg p)$ | |
| LNC and Principle of Explosion | Ex Contradictione Quodlibet (ECQ) | $\forall p, \forall q: (p \land \neg p) \vdash q$ | |
| Monotonicity of Entailment | Weakening | $p \vdash q$ | Adding presumption $a$ results in $p, a \vdash q$ |
| Idempotency of Entailment | Contraction | $p, p, a \vdash q$ | Deleting one of presumptions $p$ results in $p, a \vdash q$ |
| Commutativity (Com) of Conjunction | | $(p \land q) \equiv (q \land p)$ | |
| De Morgan’s Duality (DeM) | | $\neg (p \land q) \equiv (\neg p \lor \neg q)$ | Every logical operator is dual to another |
| DeM continued | | $\neg (p \lor q) \equiv (\neg p \land \neg q)$ | |

Also, most semantics of classical logic are *bivalent*, meaning all of the possible denotations of propositions can be categorised as either true or false. Any higher-order logic that is based on a "classical logic" inherits all of these properties, *in addition to* the three "Laws of Thought".

The most famous of the classical logics is Whitehead and Russell’s *Principia Mathe-
matica. It is often referred to as "the" classical logic. 107 108 Other examples of classical logic include George Boole’s algebraic reformulation of Aristotellean logic, 109 and the second-order logic found in Gottlob Frege’s Begriffsschrift (when applied to "judgable content"). 110 111

In classical propositional logics, the three "Laws of Thought" can be either axioms or theorems. For example, Kleene lists all three of the "Laws of Thought" as axioms of a classical propositional calculus. 112 113 In contrast, in Principia Mathematica, the three "Laws of Thought" are theorems. 114 115

107See F. E. Andrews [3], p.54, footnote 3: "In this century the logic of Principia Mathematica [henceforth PM] has so succeeded that it is now called "Classical logic"."

108See also Priest [88], p.xvii, "Around the turn of the twentieth century, a major revolution occurred in logic. Mathematical techniques of a quite novel kind were applied to the subject, and a new theory of what is logically correct was developed by Gottlob Frege, Bertrand Russell and others. This theory has now come to be called the 'classical logic'. The name is rather inappropriate, since the logic has only a somewhat tenuous connection with logic as it was taught and understood in Ancient Greece or the Roman Empire. But it is classical in another sense of that term, namely standard."

109See Boole [11], especially Propositions III and IV on pp. 48-49, that correspond to the LEM and LNC, respectively.

110See Lotter [63], Section 3a: "Frege’s early semantics is based on the notion of a conceptual content, that is, it is based on that part of meaning that is relevant for logical inferences. The class of conceptual contents in turn is divided up into judgable and non-judgable ones, whereby the former are logically composed of and can be decomposed into the latter. What Frege may have had in mind – although he does not put it exactly this way – with his distinction between judgable and non-judgable contents is the following consideration: a judgable content is such that we can reasonably either affirm or deny it."

111Note: Riemann [94] was published in 1859. Riemann died in 1866. Sigwart’s work was published in 1873, Frege [29] in 1892, Russell [96] in 1905, and Russell’s Principia Mathematica in 1910-1913. So Riemann had no knowledge of any of these before his death. In contrast, Boole [11] discusses the LOI in Chapter II, pp.34-36, Para.12-13; the LNC in proposition IV, Chapter III, p.49; and the LEM in pp.8 and 99-100, and in proposition II, Chapter III, p.48. Boole [11] was published in 1854, a few years before Riemann’s 1859 paper, but Riemann does not appear to have been aware of it or its implications.

112See Kleene [55], p.8: "Now we make one further assumption about the atoms, which is characteristic of classical logical logic. We assume that each atom (or the proposition it expresses) is either true or false but not both."

113See also Kleene [55], p.16, formulas *1, *50, and *51.

114See Langer [59], p.305, which states that the LNC is proved in Th. 3.24 of Principia Mathematica. Whitehead and Russell: "In spite of its fame, we have found few occasions for its use."

115See also Andrews [3], p.54, which states that all three of the Laws of Thoughts are theorems in Principia Mathematica: The LNC in Th. 3.24, and the LEM in Th. 2.11. and the Principle of Identity in Th. 2.08. (Russell states that the 'Law of Identity' is inferred later in PM from the Principle of Identity). See Principia Mathematica to *56, Cambridge, 1967, pp. 99, 101, 111.
4 Non-Classical (Propositional) Logics

4.1 The Variety of Non-Classical Logics

Non-classical logics, by definition, do not accept all of the properties of the classical logics.\(^{116}\)

Intuitionistic logics reject the Law of the Excluded Middle (LEM), Double Negation (DN), and part of De Morgan’s laws;\(^{117}\) Many-valued logic rejects bivalence, allowing for truth values in addition to "true" and "false". The most popular forms are three-valued logic, as initially developed by Jan Łukasiewicz, and infinitely-valued logics such as fuzzy logic, which permits any real number between 0 and 1 as a truth value.\(^{118}\)

Paraconsistent logic (e.g., relevance logic) rejects the Principle of Explosion (ECQ), and has a close relation to Dialetheism;\(^{119}\) Relevance logic, linear logic, and non-monotonic logic reject monotonicity of entailment;\(^{120}\) Non-reflexive logic (also known as "Schrödinger logics") rejects or restricts the law of identity.\(^{121}\)

The non-classical logics that reject the LEM (to at least some extent) include Intuitionistic logics, multi-valued logics, and arguably Frege’s *Begriffsschrift* when applied to "non-judgable content".\(^{122}\) Higher-order logics that are based on such non-classical logics will also reject the LEM to at least some extent.\(^{123}\)\(^{124}\)\(^{125}\)

4.2 Intuitionistic Logics

4.2.1 Intuitionistic Logics Reject the LEM (in Certain Circumstances)

Classical logics accept both the LEM and the LEC, so both of these "laws" must be true in these logics. As a result, only one of \( (P) \) and \( (\neg P) \) can be true, as in Boolean "exclusive OR (XOR)".\(^{126}\)

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\(^{116}\)See Priest [88]. See also Sadegh-Zadeh [98], p.1030: "Consequently, a large number of such non-classical logics have developed. ... Each of them effectively dismantles the classical logic in a particular way."

\(^{117}\)See Wikipedia [128], citing Gabbay, [33], Chapter 2.6.

\(^{118}\)Id.

\(^{119}\)Id.

\(^{120}\)Id.

\(^{121}\)Id., citing da Costa [18].

\(^{122}\)See Lotter [63], Section 3a

\(^{123}\)See Sakharov [99]: "The set of axiom schemata of first-order predicate calculus is comprised of the axiom schemata of propositional calculus together with the two following axiom schemata".

\(^{124}\)See also Andrews [4] p.201: "So far we have been concerned with first-order logic, and its subsystem propositional calculus, which we might regard as zeroth-order logic."

\(^{125}\)See also Kleene [55], p.74: "The predicate calculus includes the propositional calculus".

\(^{126}\)See Plisko [83]; and Stanford [5].
This use of LEM together with LNC enables the technique of proving that \((P)\) is true, by instead proving that \((\neg P)\) is false. This technique is called "proof by contradiction". According to classical logic, which accepts the LEM, proof by contradiction is a valid form of proof. However, in logics that reject LEM (e.g. intuitionistic logics and multi-valued logics), proof by contradiction is not a valid form of proof.

Some non-classical logics reject the LEM, and thus reject proof by contradiction. The intuitionistic school of logic, founded by L.E.J. Brouwer, is one of the non-classical schools of logic that reject the LEM in certain instances. According to Moschovakis [78]:

Intuitionistic logic can be succinctly described as classical logic without the Aristotelian law of excluded middle (LEM) \((A \lor \neg A)\) or the classical law of double negation elimination \((\neg \neg A \rightarrow A)\), but with the law of contradiction \((A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)\) and ex falso quodlibet \((\neg A \rightarrow (A \rightarrow B))\)

Kleene [56] agrees, proving that "either of \(\neg \neg A \supset A\) [Principle of Double Negation] or \(A \lor \neg A\) [LEM] can be chosen as the one non-intuitionistic postulate of the classical system." 127 (Kleene [56] also states that the LNC is valid in intuitionistic logic). 128

However, the Brouwer–Heyting–Kolmogorov (BHK) interpretation of intuitionistic logic does accept the LEM, in the following two circumstances: either (1) proposition \((P)\) has been proven to be true, or (2) an impossibility proof exists for \((P)\). 129 According to Iemlhoff [48]:

The BHK-interpretation is not a formal definition because the notion of construction is not defined and therefore open to different interpretations. Nevertheless, already on this informal level one is forced to reject one of the logical principles ever-present in classical logic: the principle of the excluded middle \((A \lor \neg A)\). According to the BHK-interpretation[,] this statement holds intuitionistically if the creating subject knows a proof of \(A[.]\) or a proof that \(A\) cannot be proved. In the case that neither for \(A\) nor for its negation a proof is known, the statement \((A \lor \neg A)\) does not hold.

Further according to Iemlhoff [48]:

Indeed, there are propositions, such as the Riemann hypothesis, for which there exists currently neither a proof of the statement nor of its...
negation. Since knowing the negation of a statement in intuitionism means that one can prove that the statement is not true, this implies that both $A$ and $\neg A$ do not hold intuitionistically, at least not at this moment.

As for the relationship between the LEM and the Riemann hypothesis, the situation is more interesting than as described in the quote above. Chapter 6.1 of this paper discusses the relationship between the LEM and the Riemann hypothesis in greater detail.

### 4.2.2 Minimal Logic rejects LEM and ECQ

One variant of intuitionistic logic is minimal logic. Minimal logic rejects not only LEM, but also ECQ ($\bot \vdash B$). However, minimal logic does derive a special case of ECQ ($\bot \vdash \neg B$). Adding ECQ to minimal logic results in intuitionistic logic, and adding Double Negation (DN) to intuitionistic logic results in classical logic.  

### 4.3 Multi-Valued Logics

#### 4.3.1 Multi-Valued Logics Inherently Reject the LEM

Multi-valued logics inherently reject the LEM, because they are not "bi-valent". They have at least one value in addition to the two values "true" and "false". For example, Frege’s *Über Sinn und Bedeutung* ("On Sense and Denotation") "claimed that an utterance of a sentence containing a non-referring singular is neither true nor false." More specifically, Frege states the following:

The sentence ‘Scylla has six heads’ is not true, but the sentence ‘Scylla does not have six heads’ is not true either; for it to be true the proper name ‘Scylla’ would have to designate something.

Therefore, according to Frege’s logic, a proposition can have no truth-value, which means that a proposition has three possible states: true, false, or neither. As Marques [68] states (emphasis added):  

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130 See Wikipedia [134], citing Johansson [53] and Troelstra [121], p.37.  
131 See Marques, [68] p.70, and Frege [29].  
132 See Marques, [68] p.71, citing Frege [28], p.127. "Scylla" refers to the creature from Greek mythology.  
133 See Marques [68], p.71
This gives expression to two natural ideas: i) a sentence such as ‘Scylla
does not have six heads’ is the negation of ‘Scylla has six heads’; and ii)
‘Scylla has six heads’ is false if and only if its negation is true (that is, if
‘Scylla does not have six heads’ is true). When a sentence has no truth-
value, the result of embedding the sentence, for instance under the scope of
negation, also can have no truth-value.

See also Milne [75], who states: "Frege holds that any sentence containing a bearerless
name in a direct/non-oblique context is neither true nor false." 134

Frege holds that any sentence containing a bearerless name in a direct/non-
oblique context is neither true nor false. ... He terms the thought expressed
by such a sentence 'fictitious' and a 'mock thought' ('Logic', p.130); they
are such exactly and only in that they fail to be about actually existing
objects. In particular he says 'Scylla has six heads' is not true, and 'Scylla
does not have six heads' is not true. Lack of a bearer for a singular term
spreads lack of truth-value pervasively to logically complex sentences.

4.3.2 LNC and the Principle of Explosion (ECQ)

One of the theorems in both classical and most non-classical logics (but not multi-
valued logics!) is the "Principle of Explosion". In Latin: Ex Contradictione (Sequitur)
Quodlibet (ECQ): "from contradiction, anything (follows)".

According to this theorem, the result of a contradiction (a violation of LNC) is that
any statement whatsoever can be proven. In other words, "a false proposition implies
any proposition". 135 So a single contradiction in a theorem results in an "explosion"
of false theorems that incorrectly assume the original contraction to be true.

4.4 Paraconsistent Logics

4.4.1 Dialetheism Rejects the LNC

"Aristotle introduced (what was later to be called) the LNC as “the most certain of all
principles” — firmissimum omnium principiorum, as the Medieval theologians said."

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134See Milne [75], p.473, citing Frege’s "Logic". Milne’s [75] p.474 reproduces Smiley’s truth tables
for Frege’s three-valued logic (Citing Smiley [106], pp.125-35.). The "third value" in these truth tables
is an absence of any truth-value (a.k.a "a truth-value gap"). Milne’s [75] p.474: “Beware! The bar
[symbol] is not a third truth-value, it signifies the absence of a truth-value. Where both A and B have
truth-values, the connectives behave classically."

135See Langer [59], p.284.
"The LNC has been an (often unstated) assumption, felt to be so fundamental to rationality that some claim it cannot be defended."  

Yet "[f]rom the very dawn of Greek thought ... these principles [of LNC] have been contested, first by some rhetoricians and sophists, later on by certain metaphysicists, and recently even by several logicians and mathematicians."  

Dialetheism is this view that rejects the LNC, by holding that there exist propositions that are simultaneously true and false (i.e. paradoxes / "antinomies").  

"As a challenge to the LNC, therefore, dialetheism assails what most philosophers take to be unassailable common sense, calling into question the rules for what can be called into question".  

One of the first logicians to question the status of the logical version of the LNC was Jan Łukasiewicz, father of the Polish school of logic. In his book On the Principle of Consistency in Aristotle (1910), "Łukasiewicz endorsed only the ethical version of the principle of non-contradiction".  

Moreover, contrary to Beziau’s argument, dialetheism does not reject the LNC for all propositions, thereby arguing that all propositions are both true and false. It merely does so for propositions that have a truth-value gap or a truth-value glut. Beziau’s "trivial dialetheism" reduces dialetheism (and by extension, 3VLs) to absurd trivialities.

4.4.2 The Dialectic and the LNC

The Socratic dialectic is not a logic, but rather a method of analysis. According to Smith [107], p.29 (citing the Socratic dialogue Euthyphro as an example):

The [Socratic] dialectic approach requires a subject, idea or argument to be exposed to a series of probing questions. After a process of intense cross-examination, the answers reached increase the knowledge and wisdom of those involved. It forces participants to hone the exact terms of reference of any given philosophical position and to test how durable it is. Where an idea falls down, it must either be discarded or revised. As such, Socrates’s approach may be regarded as the basis for modern scientific investigation.
Therefore, the Socratic dialectic most closely resembles Jaśkowski’s paraconsistent logic, as described above.

In contrast, Hegel claims that his dialectic is a non-classical logic. In broad terms, Hegel’s dialectic logic is that a "thesis" gives rise to an "antithesis" that contradicts or negates the original thesis, and the conflict between the two is resolved by their "synthesis". "Whereas Plato’s 'opposing sides' were people (Socrates and his interlocutors), however, what the 'opposing sides' are in Hegel’s work depends on the subject matter he discusses. In [Hegel’s] work on logic, for instance, the 'opposing sides' are different definitions of logical concepts that are opposed to one another." 144

There is a debate over whether Hegel’s dialectic is "logical". 145 146 147

[This] may be fueled in part by discomfort with his particular brand of logic, which, unlike today’s symbolic logic, is not only syntactic, but also semantic. While some of the moves from stage to stage are driven by syntactic necessity, other moves are driven by the meanings of the concepts in play. Indeed, Hegel rejected what he regarded as the overly formalistic logics that dominated the field during his day. 148

However, it must be noted that Popper [84] argues that Hegel’s dialectic does not actually reject the LNC, 149 and Kazumi [54] characterizes Jaśkowski’s [51] paraconsistent logic as describing the dialectic within the framework of classical formal logic. 150 This would leave dialetheism as the only logic that expressly rejects the LNC. 151

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144See Maybee [69].
145See Maybee [69], Section 3 "Is Hegel’s dialectical method logical?".
146See also Wójcicki, [145], p.18, regarding the early post-WWII era in Poland: "During the early postwar period it was customary to avoid the term "logic" as either the name or part of the name given to sections of the mathematics departments in which logicians were grouped. The preferred term was "foundations of mathematics". This terminology was a form of camouflage. Even though the ideologists of the communist party considered no branch of science to be ideologically neutral, logic (viewed as part of philosophy and thus part of a discipline supposed to be the chief tool of [the] "ideological offensive") was the subject of their special attention. At different times and in different countries, it was demanded of logicians (with varying degrees of vigor) to suppress formal logic and supplant it with the so-called "dialectical logic" implicit in the writings of Hegel and the "classics" of Marxism."
147See also the Wikipedia entry on Dialectical logic [129]: "During the Sino-Soviet split, dialectical logic was used in China as a symbol of Marxism–Leninism against the Soviet rehabilitation of formal logic", citing Meissner [71] at pp.171-172, in turn citing the 1946 edition of Mao [66].
148See Maybee [69], Section 3 "Is Hegel’s dialectical method logical?", citing Hegel [44], Remark to §162.
149See also Popper [84]: "The only 'force' which propels the dialectic development is, therefore, our determination not to accept or to put up with the contradiction between the thesis and the antithesis."
150See also Kazumi [54], abstract: "A dialectical contradiction can be appropriately described within the framework of classical formal logic", citing Jaśkowski [51]
151See Priest [90], para.2: "dialetheism amounts to the claim that there are true contradictions. As such, dialetheism opposes—contradicts—the Law of Non-Contradiction (LNC)".
4.4.3 Paraconsistent Logics Accept LNC, But Reject ECQ

A paraconsistent logic is one that rejects "explosion" (ECQ). Paraconsistency must be distinguished from dialetheism. 

"In the literature, especially in the part of it that contains objections to paraconsistent logic, there has been some tendency to confuse paraconsistency with dialetheism". 

Paraconsistent logic (the rejection of ECQ) does not entail dialetheism (rejection of LNC). "Paraconsistency is a property of a consequence relation, whereas dialetheism is a view about truth ... The fact that one can construct a model where a contradiction holds but not every sentence of the language holds (or where this is the case at some world) does not mean that the contradiction is true per se. "

Here is a specific example: "Suppose I have proved that the Russell set is and is not a member of itself. Why should it follow from this that there is a donkey braying loudly in my bedroom?" "The question of relevance (just what has a donkey to do with set theory?) is one that has plagued classical logic for a long time, and is one that makes classical logic a hard pill to swallow to first-time students of logic, who are often told that 'this is the way it is' in logic. Fortunately for those students, paraconsistency provides an alternative."

Stanisław Jaśkowski (Łukasiewicz’s pupil) produced a paraconsistent logic which "accommodates" inconsistencies, and allows for their investigation, without "explosion" (ECQ). "Jaśkowski’s point of departure was a discourse, the situation of a discussion. When one asks: Is it the case that A?, and does not know the answer, one often considers both possibilities at once. Likewise, when defending A, one respects, at least during a honest discussion, an opponent who claims not-A. Which logic applies here?"

"Firstly, [Jaśkowski] created a discursive calculus D2, which fulfilled all the formal criteria we tend to impose on interesting paraconsistent logics. Secondly, his construction in its deep structure enables us to consider inconsistencies occurring in a theory T as contingent statements in a related modal theory M(T) playing the role of its metatheory. Thirdly, it often allows for the consistent examination of a given inconsistency. Sometimes even for the understanding of its mechanism and sources"

A famous physics problem that needs such a logic (without the ECQ) is the "para-

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152 But see Priest [91], citing Asmus [7].
153 Priest [91]
154 Priest [91]
155 See McKubre-Jordens [70].
156 See McKubre-Jordens [70].
157 See Perzawoski [82], p.23, para.8.
158 See Perzawoski [82], p.23, para.10.
159 See Perzawoski [82], p.24, para.13:
dox” of Schrödinger’s Cat. By enabling the consideration of both possibilities (the cat being alive, and the cat being dead) simultaneously, without ECQ, there is no worry that the entire theory is reduced to triviality. According to paraconsistent logics (because they accept the LNC), Riemann’s definition of \( \zeta(s) \) is invalid. Moreover, because paraconsistent logics reject ECQ, the RH is not automatically "trivially true".

One notable subset of paraconsistent logics is that of relevance logics. They are para-consistent because the existence of a contradiction will not cause "explosion" (ECQ). This follows from the fact that in relevance logics, "a conditional with a contradictory antecedent that does not share any propositional or predicate letters with the consequent cannot be true (or derivable)."

Therefore, according to relevance logics (because they accept the LNC), Riemann’s definition of \( \zeta(s) \) is invalid. Moreover, because relevance logics use "predicate letters", the RH is cannot be true.

5 Riemann’s Zeta Function, the LOI, and the LNC

5.1 Aristotle’s Laws of Thought, Applied to the Zeta Function

When applied to the Zeta function \( \zeta(s) \), the LOI holds that \( \zeta(s) \) cannot have two different values at any value of \( s \), because this would mean that if \( \zeta(s) \) is the proposition \( P \), then \( P \) is not equal to itself \( (P \neq P) \). The LNC is more specific. It states that a proposition \( P \) and its contradiction cannot both be true. Using the function \( \zeta(s) \) as an example, \( \zeta(s) \) cannot be both convergent and divergent at the same value of \( s \), because this would mean that a proposition \( P \) and its negation \( \neg P \) are both true.

So given that the Dirichlet series definition of \( \zeta(s) \) is proven to be divergent in half-plane \( \text{Re}(s) \leq 1 \), according to the LOI and the LNC, Riemann’s definition of \( \zeta(s) \) cannot be valid, nor can any other "analytic continuation" of \( \zeta(s) \) to half-plane \( \text{Re}(s) \leq 1 \).
So, according to all classical and intuitionistic propositional logics that have LOI and LNC as axioms or theorems, $\zeta(s)$ is defined exclusively by the Dirichlet series (which has no zeros). This means that the "trivial zeros" and the "non-trivial zeros" of the Riemann Hypothesis (RH) do not exist. The non-existent zeros are "vacuous subjects" of a proposition, like "the present King of France" in Bertrand Russell’s famous proposition: "The present King of France is bald." 166

5.2 Two Conflicting Versions of the Zeta Function (Dirichlet’s vs. Riemann’s)

The Dirichlet series version of $\zeta(s)$ was proven by Euler to be divergent along the half-line $s \leq 1$, for $s \in \mathbb{R}$. This is easily confirmed by use of the Integral test for convergence, 167 or the so-called "p-series" test for convergence. 168

Moreover, Riemann [94] refers to the divergence of $\zeta(s)$ in the half-plane $\text{Re}(s) \leq 1$ as a given fact in his seminal article about $\zeta(s)$ that was presented in 1859. It is easily proven that the Dirichlet series version of $\zeta(s)$ is divergent in the half-plane $\text{Re}(s) \leq 1$ of the complex plane $s \in \mathbb{C}$, by rewriting the Dirichlet series into its trigonometric form, and then performing integration by parts. 169

In contrast, Riemann’s paper [94] claims that there also exists "an expression for the function ... which always remains valid", except at the point $s = 1$. 170 Riemann’s version of $\zeta(s)$ is: 171

$$\zeta(s) = \frac{\Pi(-s)}{2\pi i} \int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \cdot \frac{dx}{x} \quad (5.1)$$

Therefore, Riemann’s version of $\zeta(s)$ claims to be convergent throughout the half-plane $\text{Re}(s) \leq 1$ of the complex plane $s \in \mathbb{C}$, except at the point $s = 1$.

tory) in case its range is the null range, i.e. it is false for every value-assignment. Every L-false sentence is evidently false; moreover, its falsity resides entirely in the sense of the sentence and is independent of the facts."

166See Russell [96], pp.483-485 and 490.
167See, e.g. Guichard [39].
168See, e.g. Department of Mathematics Website, Oregon State University [20], and Birdsong [10].
169See, e.g., Sharon [104], Appendix E, pp.26-32.
170See Edwards, [25], p.11; and Riemann [94], p.2.
171See Edwards [25], pp.10-11.
5.3 LNC and the Two Contradictory Zeta Functions in Half-Plane $\text{Re}(s) \leq 1$

Riemann’s version of $\zeta(s)$ violates the LNC.

In logic, the law of identity is the first of the three classical laws of thought. It states that "each thing is the same with itself and different from another". ... In logical discourse, violations of the Law of Identity (LOI) result in the informal logical fallacy known as equivocation.

If Riemann’s claim is true, that the alternative version of $\zeta(s)$ is convergent for all $s \in \mathbb{C}$, $s \neq 1$, then $\zeta(s)$ is both convergent and divergent throughout half-plane $\text{Re}(s) \leq 1$, where Riemann’s definition of $\zeta(s)$ and the Dirichlet series definition of $\zeta(s)$ disagree. The sole exception is the pole at $s = 1$, where both the Dirichlet series definition of $\zeta(s)$ and Riemann’s definition of $\zeta(s)$ agree on divergence.

In other words, Riemann’s claim is that at half-plane $\text{Re}(s) \leq 1$, for all $s \in \mathbb{C}$, $s \neq 1$, both a proposition ($P$) and its negation ($\neg P$) are simultaneously true. So this claim contradicts the LNC, which states that a proposition ($P$) and its negation ($\neg P$) cannot both be true simultaneously ($\neg (A \land \neg A)$). Thus the LNC and Riemann’s definition of $\zeta(s)$ cannot both be true.

The Law of Non-Contradiction (LNC) is "derivable in classical as well as in intuitionistic constructive propositional calculus", 172 so Riemann’s definition of $\zeta(s)$ is invalid according to both the classical and the intuitionistic schools of propositional logic. So according to both of these logics, the LNC by itself is sufficient to invalidate Riemann’s version of $\zeta(s)$. 173

6 The Riemann Hypothesis

6.1 The Vacuous Subjects of the Riemann Hypothesis

The Riemann Hypothesis ($RH$) states that "all non-trivial zeros of $\zeta(s)$ are on the critical line $\sigma = 0.5$". The $RH$ has been described as "[e]legant, crisp, falsifiable, and far-reaching" and "the epitome of what a conjecture should be". 174 It may indeed be

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172 See Grishin[38].
173 In contrast, Weierstrass’s "unit disk" analytic continuation of the Taylor series expansion of $f(s) = 1/(1 - s)$. This method does not claim that $f(s) = 1/(1 - s)$ is both convergent and divergent at $s = 1$. Also, if each disk is a distinct proposition, then the method contains no directly contradictory propositions. Also, unlike the derivation of Riemann’s definition of $\zeta(s)$, the prerequisites of Cauchy’s integral theorem are strictly observed (e.g. no non-holomorphic points, such as the pole at $s = 1$, permitted in any disk). See, e.g., Weyl [126], pp.1-4, and Coleman [16], pp.1-2.
174 See Sarnak [100], first page; and Iwaniec [49], p.712.
"elegant", but as to being "falsifiable": that is a different matter entirely.

According to the LNC, any "analytic continuation" of $\zeta(s)$ to the half-plane $\text{Re}(s) \leq 1$ is invalid. So the function $\zeta(s)$ is defined exclusively by the Dirichlet series definition of $\zeta(s)$, which has no zeros. Therefore, none of the zeros assumed by the RH exist. There are no "trivial" zeros, and there are no "non-trivial" zeros either. These non-existent zeros of $\zeta(s)$ constitute vacuous subjects of a proposition, just like Russell’s famous example of "the present King of France" in the proposition "the present King of France is bald", or the proposition "Socrates is ill" when stated either before his birth, or after his death.

So given that the RH is a proposition with vacuous subjects, is the RH true? False? Both? Neither? The answer: it can be any of these, depending on the system of logic that is applied. Factors such as whether or not the system of logic accepts or rejects the LEM affect whether a proposition with a vacuous subject (e.g. the RH) is both true and false, or neither true nor false.

6.2 Truth-Value Glut: RH is Both True and False

6.2.1 Axiomatic Set Theory

In set theory, the material conditional produces a result of "true" when $\mathcal{p}$ is an empty set. When applied to RH, the RH must be true, again because the RH states:

If $\zeta(s) = 0$, then $\text{Re}(s) = 0.5$.

So if $\zeta(s)$ has no zeros, the RH is "vacuously true", because the set of zeros of $\zeta(s)$ is an empty set, and also because the RH states that all the members of this empty set have a given property (they are on the critical line $\text{Re}(s) = 0.5$).

The following description of "vacuous truth" in axiomatic set theory is taken from O'Searcoid:

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175Also, Riemann’s functional equation of $\zeta(s)$ is invalid when $\zeta(s)$ is defined exclusively by the Dirichlet series definition of $\zeta(s)$, because $\zeta(s)$ is divergent if $s \leq 1$, and $\zeta(1-s)$ is divergent if $s > 1$. Therefore, contrary to Gelbart [35], p.60, it is incorrect to argue that we "know that $\zeta(s)$ extends meromorphically to the entire complex plane and satisfies the functional equation".

176But see Gelbart [35], p.60: "The Riemann Hypothesis: $\zeta(s) \neq 0$ for $\text{Re}(s) > 1/2." This formulation of RH avoids the 'vacuous subjects', and thereby avoids all of the issues discussed below. This formulation of RH is unambiguously "true" given that the derivation of Riemann’s definition of $\zeta(s)$ is not valid. However, this version of RH would also be true if $\exists \{ \zeta(s) = 0 \}$ on the line $\text{Re}(s) = 1/2$, so this formulation assumes that $\zeta(s)$ has zeros, based on Gelbart’s statement (p.60): "Our role here is not so much to focus on the zeroes of $\zeta(s)$, but in some sense rather on its poles."

177O'Searcoid [81], p.248.

178Note, however, the flaws specific to O'Searcoid’s example: what constitutes "irrefutable proof" that there are no inhabitants of Mars? If no such "irrefutable proof" exists, then a basic assumption is
Some statements are true because they assert nothing: for this reason, they may seem less convincing than other statements of truth. Consider a statement $P(x)$ that refers to a variable $x$ that takes values in some set $A$. Suppose we discover that the set $A$ is empty. We then say without more ado that the statement $P(x)$ is \textit{vacuously true} for all $x$ in $A$. The statement is true - no less true than any other true statement - simply because it claims nothing. For example, if there is irrefutable evidence that there are no inhabitants of Mars, then we can confidently say that the statement 'All blue Martians wear silk hats' is true; a lawyer might read other things into it, but pure logic dictates that it is true simply because there are no blue Martians. The reader who is not convinced might like to try refuting the statement by finding a counterexample; that is, by producing a blue Martian who does not wear a silk hat. The abstract logical position is this: the statement 'for all $x$ in $A$, $P(x)$ holds' is the negation of the statement 'there exists some $x$ in $A$ such that $P(x)$ does not hold', and when $A$ is empty, the latter statement is certainly false, so that the former, its negation, is true.

Therefore, the Riemann Hypothesis can be phrased as the following statement $P(x)$: All "non-trivial" zeros of the function $\zeta(s)$ are on the critical line $\text{Re}(s) = 0.5$.

So when we 'consider a statement $P(x)$ that refers to a variable $x$ that takes values in some set $A$', we consider the RH to be a statement $P(x)$, that refers to the "variable" $\zeta(s)$. Set $A$ is the set of values of $s$ at which $\zeta(s) = 0$. Because set $A$ is empty, we say that the statement $P(x)$ (the RH) is \textit{vacuously true} for all values of $s$ at which $\zeta(s) = 0$.

Moreover, note the following. The negation of RH is: "There exists a 'non-trivial' zero of $\zeta(s)$ that is not on the critical line $\text{Re}(s) = 1/2". Because also this set of values of $s$ at which $\zeta(s) = 0$ ("set $A$") is empty, we know that the negation of RH is a false statement. Axiomatic set theory accepts the principle of double negation, so this double negation produces the same as the original RH. According to the LEM, the original statement must therefore be true (or as we have mentioned, \textit{vacuously true}).

Here we again see the reasons for Brouwer's objections to the LEM and to the principle of double negation (and also to axiomatic set theory, which accepts both "laws"). The negation of the RH is clearly false, but is it reasonable to conclude that therefore the RH is true? Or even "vacuously true"? "No", according to Brouwer. This is why he developed Intuitionist logic, which (in one version) does not include the Principle of Double Negation.

\textbf{not valid. It is important to avoid falling for the "black swan fallacy": The logical error of discounting the possibility of "X" because "x" has not yet been observed. Also, the definition of "wearing" and "hat" have not been defined. What about a microbe that slides under a rock in order to shade itself from sunlight?}
6.2.2 Classical Logic: Russell’s "On Denoting"

Russell’s *On Denoting* [96] (which like axiomatic set theory, accepts the LEM) holds that a proposition with a vacuous subject (e.g. the Riemann hypothesis) is ambiguous, because it can be interpreted in two ways. Therefore, depending on how such a statement (e.g. the RH) is interpreted, it can be either true or false. (In its ambiguous state, it has both meanings).

In contrast, the negation ("the present King of France is not bald") can be interpreted as the conjunction of the following three propositions: 179

i. There is at least one King of France. \( \exists x (Kx) \)

ii. There is at most one King of France. \((x)(y)(Kx \land Ky \rightarrow x = y)\)

iii. Whatever is King of France is not bald. \((x)(Kx \rightarrow \neg Bx)\)

When these three propositions are conjoined, we get: "There is one and only one present King of France and he is not bald." In standard logical notation, this first sentence is: 180

\[ \exists x \left( Kx \land (\forall y) \left( (Ky \rightarrow x = y) \land \neg Bx \right) \right) \]

This sentence is *false*, because it quantifies over a non-existent entity. ("There is one and only one present King of France" is false).

A second interpretation of the sentence is: "It is not the case that that there exists a present King of France and he is bald". The second interpretation is *true*, because it is indeed not the case that that there exists a present King of France. In standard logical notation, this second sentence is: 181

\[ \neg \exists x \left( Kx \land (\forall y) \left( (Ky \rightarrow x = y) \land Bx \right) \right) \]

If the RH is interpreted according to Russell’s first interpretation, as "there exist zeros of \( \zeta(s) \) and they are not located off of the critical line \( \text{Re}(s) = 0.5 \)", then the RH is *false*, because it quantifies over non-existent entities (the non-existent zeros of \( \zeta(s) \)).

However, if the RH is interpreted according to Russell’s second interpretation, as "it is not the case that there exist zeros of \( \zeta(s) \) and they are located off of the critical line \( \text{Re}(s) = 0.5 \)", then it is *true*, because indeed it is *not* the case that there exist zeros of \( \zeta(s) \). (Note: Both axiomatic set theory and Russell’s *On Denoting* assume that the LEM is true).

Moreover, if we apply Russell’s first interpretation to RH and its negation \( \overline{\text{RH}} \) ("not

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179 See Batty [8], "1. Russell Recap".
180 See Russell [96], p.490, and Jacquette [50], pp.5-6.
181 See Russell [96], p.490, and Jacquette [50], pp.5-6.
all zeros of $\zeta(s)$ are on the critical line"), then paradoxically both are false.

### 6.2.3 Classical Logic: Material Conditional

The material conditional states that $p \rightarrow q$ is logically equivalent to $\neg(p \land \neg q)$. In classical logics (but not in intuitionistic logics), by De Morgan’s Laws this is also equivalent to $(\neg p \lor q)$. So, counter-intuitively, in classical and intuitionistic logics the material conditional is true when $p$ is false. Tarski describes the material conditional as follows:

The logicians ... adopted the same procedure with respect to the phrase "if ..., then ..." as they had done in the caso of the word "or". For this purpose, they extended the usage of this phrase, considering an implication as a meaningful sentence even if no connection whatsoever exists between its two members, and they made the truth or falsity of an implication dependent exclusively upon the truth or falsity of the antecedent and consequent.

To characterize this situation briefly, we say that contemporary logic uses IMPLICATIONS IN MATERIAL MEANING, or simply, MATERIAL IMPLICATIONS; this is opposed to the usage of IMPLICATIONS IN FORMAL MEANING, or simply, FORMAL IMPLICATION, in which case the presence of a certain formal connection between antecedent and consequent is an indispensable condition of the meaningfulness and truth of the implication. The concept of formal implication ... is narrower than that of material implication[.]

Tarski also illustrates the material conditional, as follows:

In order to illustrate the foregoing remarks, let us consider the following four sentences:

- if $2 \cdot 2 = 4$, then New York is a large city;
- if $2 \cdot 2 = 5$, then New York is a large city;
- if $2 \cdot 2 = 4$, then New York is a small city;
- if $2 \cdot 2 = 5$, then New York is a small city.

In everyday language, these sentences would hardly be considered as meaningful, and even less true. From the point of view of mathematical logic, on the other hand, they are all meaningful, the third sentence being false, while the remaining three are true.

When applied to RH, the RH must be true, because the RH states:

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182 See Tarski [114], pp.25-26
183 See also Tarski [114], p.26.
If \( \zeta(s) = 0 \), then \( \text{Re}[\zeta(s)] = 0.5 \).

So if \( \zeta(s) \) has no zeros, then RH is true.

### 6.3 Truth-Value Gap: RH is Neither True Nor False

#### 6.3.1 Intuitionistic Logic

#### 6.3.2 Frege

In the alternative, some logics that reject the LEM hold the Riemann hypothesis to be neither true nor false, because in these logics, some propositions are not assigned a (classical) truth-value.

In favor of such a "truth-value gap", Russell argues the following regarding the statement "the present King of France is bald": 184

> By the law of the excluded middle [LEM], either 'A is B' or 'A is not B' must be true. Hence either "the present King of France is bald" or "the present King of France is not bald" must be true. Yet if we enumerated the things that are bald, and then the things that are not bald, we should not find the present King of France in either list.

In those systems that embrace truth-value gaps (Strawson, Frege) or non-classically-valued systems (Łukasiewicz, Bochvar, Kleene), some sentences or statements are not assigned a (classical) truth-value.

However, in the specific case of Strawson’s *On Referring*, its reasoning is inapplicable to the Riemann hypothesis, for reasons that will be discussed below.

#### 6.3.3 Strawson’s "On Referring"

Accordingly, Aristotle’s and Russell’s logics (which accept the LEM) hold the RH to be false, but Frege’s and Strawson’s logics hold that the RH cannot be used to make a true or false assertion (thereby rejecting the LEM). 185 More specifically, according to Horn:

> In those systems that do embrace truth-value gaps (Strawson, arguably Frege) or non-classically-valued systems (Łukasiewicz, Bochvar, Kleene), some sentences or statements are not assigned a (classical) truth-value; in Strawson’s famous dictum, the question of the truth-value of “The king of

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184 See Russell [96] p.485.
185 See Horn [47], "4. Gaps and Gluts: LNC and Its Discontents".
France is wise”, in a world in which France is a republic, simply fails to arise. The negative form of such vacuous statements, e.g. “The king of France is not wise”, is similarly neither true nor false. This amounts to a rejection of LEM, as noted by Russell [in "On Denoting"].

In contrast to Russell’s *On Denoting* [96], Strawson’s *On Referring* [113] states that a statement with a vacuous subject (a subject term that has no referent, e.g. "the present King of France") is *not* false. Instead, it is "absurd" and therefore not asked. So, it is neither true nor false (and thus belongs in a third category, whose existence is a rejection of LEM). Strawson provides the following example:

A literal-minded and childless man asked whether all his children are asleep will certainly not answer "Yes" on the ground that he has none; but nor will he answer "No" on this ground. Since he has no children, the question does not arise.

However, Strawson assumes that the potential questioner knows that the question has a vacuous subject. The 160 year history of the RH shows that this is not always the case. Speranza’s [110] quotation of Christoph Sigwart reinforces this point: 186

For Strawson, as for his intellectual predecessor Frege [1892], the notion of presupposition has semantic status as a necessary condition on true or false assertion ... In fact, the earliest pragmatic treatments of the failure of existential presupposition predate Frege’s analysis by two decades. Here is Christoph Sigwart [1873] on the problem of vacuous subjects:

"As a rule, the judgement A is not B presupposes the existence of A in all cases when it would be presupposed in the judgement A is B ... 'Socrates is not ill' presupposes in the first place the existence of Socrates, because only on the presupposition [Vorausestuzung] of his existence can there be any question of his being ill." (Sigwart [1873/1895: 122], ...)

In the context of the Riemann Hypothesis, Strawson’s argument is clearly wrong. Over the course of the past 160 years, many mathematicians have asked if all of the zeros of $\zeta(s)$ indeed fall on the critical line $\text{Re}(s) = 0.5$. The question has arisen, because in contrast to Strawson’s examples ("the present King of France", the children of a man well-known to be childless), it has not been common knowledge that Riemann’s definition of $\zeta(s)$ is invalid (and that $\zeta(s)$ thus has no zeros). Instead, Riemann’s definition of $\zeta(s)$ was widely assumed to indeed have zeros. So an axiom of Strawson’s logic (common knowledge that subject of the question is vacuous) is clearly false in the context of the Riemann Hypothesis.

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186See Speranza [110], p.148.
6.4 Comparison of Truth-Value Gluts to Truth-Value Gaps

6.4.1 Comparison of Truth Tables

For the sake of brevity, the Three-Valued Logic Truth Tables are referred to as "Kleene" Truth Tables, and the Truth-Value Gap Truth Tables are referred to as "Frege" Truth Tables.

Remember that the "third value" in the Frege truth tables is the absence of any truth-value (a.k.a. "a truth-value gap"). "Beware! The bar [ - ] is not a third truth-value, it signifies the absence of a truth-value. Where both [variables] have truth-values, the connectives behave classically." 187

The "Kleene" tables are "essentially those of Kleene’s and Łukasiewicz’s three valued logics". 188 The values are: ● = True (only), ○ = False (only), and ☒ = Both (True and False). The conditional →, follows Kleene’s three valued logic, 189 190 and material equivalence ↔, is defined as "means the same as".

187 See Milne [75], p.473, citing Frege’s "Logic". Milne’s [75] p.474 reproduces Smiley’s truth tables for Frege’s three-valued logic (Citing Smiley [106], pp.125-35.).
188 See Priest [91], section 3.6.
189 See Priest [91], section 3.6.
190 See also Wikipedia [136]: "The Łukasiewicz L3 has the same tables for AND, OR, and NOT as the Kleene logic given above, but differs in its definition of implication in that 'unknown implies unknown' is true", citing Malinowski [65]
6.4.2 Every Truth-Value Gap Implies a Glut

The following natural deduction rules in classical logic fail in Frege’s Truth-value gap logic: v-introduction, →-introduction (conditional proof), reductio ad absurdum, ex falso quodlibet (ECQ), the law of the excluded middle (LEM).\footnote{See Milne’s [75] p.474.}

However, enough of classical logic remains valid to prove the following:\footnote{See Milne’s [75] p.475, citing Heidelberger [45].}

It’s not true that \( P \) and it’s not false that \( P \) only if it’s both true that \( P \) and false that \( P \).

So in Frege’s logic, whenever there is a truth-value gap, there is also a truth-value glut (and vice versa).\footnote{See also Priest [89], p.27, which recites an explanation for this phenomenon that is different from Heidelberger’s [45]: "Notably, assuming De Morgan’s laws, \( \neg(A \lor \neg A) \) is equivalent to \( A \land \neg A \)."}

But according to another interpretation, this is a not a paradox, because the difference lies in the definition of tautologies.\footnote{See Heis [46]: "Frege, of course, would resolve this paradox by prescribing that a logically perfected language have no bearerless names. Milne [75] advocates instead adopting a semantic (as opposed to Frege’s functional) theory of negation. He rejects Frege’s solution because it precludes a plausible semantics for ordinary language, and because the set-theoretic paradoxes show that even a scientific language such as Frege’s own needs to allow for the possibility of singular terms (like "the extension of \( x \notin x \)" that are nevertheless bearerless."}

However, if we assume that this is indeed a paradox, then we should always apply logic based on truth-value gluts (e.g. Kleene’s three-valued logic) instead of logic based on truth-values.
on truth-value gaps (e.g. Frege’s logic), because the former is "truth preserving".  

Perhaps the most interesting result in the "Kleene" three-valued truth tables is that of material implication, \( A \to B \). In classical logic, the material implication \( A \to B \) is equivalent to \( \neg A \lor B \) (this can be seen in the "Frege" truth tables). So it is true if \( A \) is false, regardless of whether \( B \) is true or false.

In a three-valued logic, the material implication \( A \to B \) remains equivalent to \((\neg A \lor B)\). So if \( A \) is "both true and false", then the material implication is not false, regardless of the value of \( B \). This can be seen in the "Kleene" (But Not "Frege") truth tables.

So if the \( RH \) is "both true and false", then all theorems that assume \( RH \) is true cannot be proven true in classical logic. But in three valued logic, they are proven true by use of the material conditional.

6.4.3 Three-Valued Logic is Paraconsistent

7 The Derivation of Riemann’s Zeta Function is Invalid

7.1 The Derivation of Riemann’s Zeta Function

Riemann’s version of the Zeta function \( \zeta(s) \), that he claims "remains valid for all \( s \)”, is derived as follows:  

First, Riemann substitutes \( nx \) for \( x \) in Euler’s integral expression for \( \prod(s-1) \), which results in:

\[
\int_0^\infty e^{-nx}x^{s-1} \, dx = \frac{\prod(s-1)}{n^s}
\]

wherein \( (s > 0, n = 1, 2, 3, \ldots) \).

Then, Riemann sums the \( n^{-s} \) term over \( n = 1 \to \infty \) on the right side of the equation, and uses \( \sum_{n=1}^\infty r^{-n} = (r - 1)^{-1} \) to replace \( e^{-nx} \) in the integral on the left side of the equation.

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\[200\] See Priest [91], section 3.6: "Let \( t \) [true] and \( b \) [both] be the designated values. These are the values that are preserved in valid inferences. If we define a consequence relation in terms of preservation of these designated values, then we have the paraconsistent logic \( LP \). In \( LP \). ECQ is invalid", citing Priest [85].

\[201\] See also MacFarlane [64], p.14: "The idea is to keep the classical idea that validity is truth preservation, but give up the classical assumption that the same sentence cannot be both true and false."

\[202\] See Riemann [94] pp.1-2; Edwards [25], pp.9-11; and Whittaker [127], pp.265-266.
equation with $1/(e^x - 1)$, thereby obtaining

$$
\int_0^\infty \frac{x^{s-1}}{e^x - 1} \, dx = \prod (s - 1) \cdot \sum_{n=1}^\infty \frac{1}{n^s} \tag{7.2}
$$

Then, since $\zeta(s) = \sum n^{-s}$, the above equation can be written as

$$
\int_0^\infty \frac{x^{s-1}}{e^x - 1} \, dx = \prod (s - 1) \cdot \zeta(s) \tag{7.3}
$$

Next, Riemann considers the following integral

$$
\int_+^\infty \frac{(x)^s}{(e^x - 1)} \cdot \frac{dx}{x} \cdot \frac{dx}{x} \tag{7.4}
$$

Edwards [25] states: 203

The limits of integration are intended to indicate a path of integration which begins at $+\infty$, moves to the left down the positive real axis, circles the origin once in the positive (counterclockwise) direction, and returns up the positive real axis to $+\infty$. The definition of $(-x)^s$ is $(-x)^s = \exp[s \cdot \log(-x)]$, where the definition of $\log(-x)$ conforms to the usual definition of $\log(z)$ for $z$ not on the negative real axis as the branch which is real for positive real $z$; thus $(-x)^s$ is not defined on the positive real axis and, strictly speaking, the path of integration must be taken to be slightly above the real axis as it descends from $+\infty$ to 0 and slightly below the real axis as it goes from 0 back to $+\infty$.

This is the Hankel contour. 204 When written in three terms, with the first term a slight distance above the real axis as it descends from $+\infty$ to $\delta$, the middle term representing the circle with radius $\delta$ around the origin, and the third term a slight distance below the real axis as it goes from $\delta$ back to $+\infty$, it is:

$$
\int_+^\delta \frac{(-x)^s}{(e^x - 1)} \cdot \frac{dx}{x} + \int_{|z|=\delta} \frac{(-x)^s}{(e^x - 1)} \cdot \frac{dx}{x} + \int_+^\infty \frac{(-x)^s}{(e^x - 1)} \cdot \frac{dx}{x} \tag{7.5}
$$

Edwards [25] further states: 205

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203 See Edwards [25], p.10.
204 See Whittaker [127], pp.244-45 and 266.
205 See Edwards [25], p.10.
The middle term is $2\pi i$ times the average value of $(-x)^s \cdot (e^x - 1)^{-1}$ on the circle $|x| = \delta$ [because on this circle $i \cdot d\theta = (dx/x)$]. Thus the middle term approaches zero as $\delta \to 0$ provided $s > 1$ [because $x(e^x - 1)^{-1}$ is nonsingular near $x = 0$]. The other two terms can then be combined to give

$$
\int_{+\infty}^{+\infty} \frac{(-x)^s \cdot dx}{e^x - 1} = \lim_{\delta \to 0} \left[ \int_{+\infty}^{\delta} \frac{\exp[s(\log x - i\pi)] \cdot dx}{(e^x - 1)} + \int_{\delta}^{+\infty} \frac{\exp[s(\log x + i\pi)] \cdot dx}{(e^x - 1)} \right]
$$

resulting in

$$
\int_{+\infty}^{+\infty} \frac{(-x)^s \cdot dx}{e^x - 1} = (e^{is\pi} - e^{-is\pi}) \cdot \int_{0}^{\infty} \frac{x^{s-1} \cdot dx}{e^x - 1}
$$

(7.6)

Since $(e^{is\pi} - e^{-is\pi}) = 2i \sin(\pi s)$, this can be rewritten as

$$
\int_{+\infty}^{+\infty} \frac{(-x)^s \cdot dx}{e^x - 1} = 2i \sin(\pi s) \cdot \int_{0}^{\infty} \frac{x^{s-1} \cdot dx}{e^x - 1}
$$

(7.7)

Riemann then substitutes the right side of Equation 7.3 into Equation 7.8, resulting in

$$
\int_{+\infty}^{+\infty} \frac{(-x)^s \cdot dx}{e^x - 1} = 2i \sin(\pi s) \cdot \prod(s - 1) \cdot \zeta(s)
$$

(7.8)

Then, both sides of the equation are multiplied by $\prod(-s) \cdot s/2\pi is$, resulting in

$$
\prod(-s) \cdot s \cdot \int_{+\infty}^{+\infty} \frac{(-x)^s \cdot dx}{e^x - 1} = \prod(-s) \cdot s \cdot 2i \sin(\pi s) \cdot \prod(s - 1) \cdot \zeta(s)
$$

(7.9)

Which when simplified and rearranged, is

$$
\prod(-s) \cdot s \cdot \int_{+\infty}^{+\infty} \frac{(-x)^s \cdot dx}{e^x - 1} = \prod(-s) \cdot \prod(s - 1) \cdot s \cdot \frac{\sin(\pi s)}{\pi s} \cdot \zeta(s)
$$

(7.10)

Next, the identity $\prod(s) = s \cdot \prod(s - 1)$ is substituted into 7.11, resulting in

$$
\prod(-s) \cdot s \cdot \int_{+\infty}^{+\infty} \frac{(-x)^s \cdot dx}{e^x - 1} = \prod(-s) \cdot \prod(s) \cdot \frac{\sin(\pi s)}{\pi s} \cdot \zeta(s)
$$

(7.12)

Finally, the identity $\sin(\pi s) = \pi s \cdot \left[ \prod(-s) \prod(s) \right]^{-1}$ is substituted into the right side of 7.12, resulting in

$$
\zeta(s) = \frac{\prod(-s) \cdot \int_{+\infty}^{+\infty} \frac{(-x)^s \cdot dx}{e^x - 1}}{2\pi i}
$$

(7.13)
This is the Riemann Zeta Function.

7.2 The Hankel Contour

In regards to the Hankel contour of Equation 7.5: \(206\)

\[
\int_{+\infty}^{+\infty} \frac{(-x)^s}{(e^x - 1)} \cdot \frac{dx}{x} = \int_{+\infty}^{\delta} \frac{(-x)^s}{(e^x - 1)} \cdot \frac{dx}{x} + 
\int_{|z|=\delta} \frac{(-x)^s}{(e^x - 1)} \cdot \frac{dx}{x} + 
\int_{\delta}^{+\infty} \frac{(-x)^s}{(e^x - 1)} \cdot \frac{dx}{x} \tag{7.14}
\]

Edwards [25] states: \(207\)

[T]hus \((-x)^s\) is not defined on the positive real axis and, strictly speaking, the path of integration must be taken to be slightly above the real axis as it descends from \(+\infty\) to 0 and slightly below the real axis as it goes from 0 back to \(+\infty\).

This is the Hankel contour. Riemann copied this solution directly from Hankel’s derivation of the Gamma function \(\Gamma(s)\). \(208\)

However, Edwards [25] provides no rational basis for equating the first contour (the branch cut), to the second contour (the Hankel contour that is "slightly above the real axis as it descends from \(+\infty\) to 0 and slightly below the real axis as it goes from 0 back to \(+\infty\")). On what basis can we equate the two contours, given that the logarithm function is undefined at all points on the branch cut?

In contrast, Whittaker [127] expressly states \(209\) that the Cauchy integral theorem’s path equivalence corollary is the basis for equating the Hankel contour to the branch cut (as domains of the contour integral). \(210\) However, both the Hankel contour and the branch cut contradict the prerequisites of the Cauchy integral theorem, \(211\)

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\(^{206}\)As in Edwards [25], pp.10-11. See also Whittaker [127], p.244.

\(^{207}\)See Edwards [25], p.10.

\(^{208}\)See Whittaker [127], pp.244-245 and 266.

\(^{209}\)See Whittaker [127], p.244

\(^{210}\)See Whittaker [127], p.244, which states that "by §5.2 corollary 1, the path of integration may be deformed (without affecting the value of the integral) into the path of integration which starts at \(\rho\), proceeds along the real axis to \(\lambda\), describes a circle of radius \(\lambda\) counter-clockwise round the origin and returns to \(\rho\) along the real axis". The cited corollary appears at the top of Whittaker’s p.87, and is a corollary of Cauchy’s integral theorem, discussion of which begins on Whittaker’s p.85.

\(^{211}\)See Whittaker [127], p.85. The integrated function must be analytic at all points on and inside the contour of integration.
and therefore of the corollary as well. \(^{212}\) This is the logical contradiction that invalidates Riemann’s version of \(\zeta(s)\)

### 7.3 Cauchy’s Integral Theorem

More specifically, Cauchy’s integral theorem states that if \(f(z)\) is a function of complex variable \(z\), if \(f(z)\) is holomorphic at all points on a simple closed curve ("contour") \(C\), and if \(f(z)\) is holomorphic at all points inside the contour, then the contour integral of \(f(z)\) is equal to zero: \(^{213}\)

\[
\int_{(C)} f(z) \cdot dz = 0 \tag{7.15}
\]

The path equivalence corollary of Cauchy’s integral theorem \(^{214}\) states that:

1. If there exist two points \(z_0\) and \(Z\) in the complex domain, connected by two distinct paths \(z_0AZ\) and \(z_0BZ\), and

2. If \(f(z)\) is a function of complex variable \(z\) that is holomorphic at all points on these two paths, and holomorphic at all points enclosed by these two paths, then

3. Any line integral connecting the two points \(z_0\) and \(Z\) in this region has the same value, regardless of whether the path of integration is \(z_0AZ\), or \(z_0BZ\), or any other path disposed between \(z_0AZ\) and \(z_0BZ\).

### 7.4 Prerequisites of Cauchy Integral Theorem are Contradicted

However, the contour representing Riemann’s definition of \(\zeta(s)\) falls directly on the branch cut of the function \(f(z) = \ln(-z)\). Therefore \(f(z)\) is undefined at all of the non-positive values of \(z\) on the branch cut. It is not possible to calculate a derivative at a point on a curve where the curve itself is undefined. What is the value of \(f(z) = \ln(-z)\) if \(z\) is a non-negative real number? What is the first derivative, \(f'(z)\), if \(z\) is a non-negative real number? Both of these are undefined.

So no point of the contour integral on the branch cut is holomorphic. Therefore the path equivalence corollary of Cauchy’s integral theorem cannot be applied to the branch cut, and thus cannot be used to equate the branch cut to any other path.

Moreover, the Hankel contour that surrounds the branch cut, is open (or closed at infinity, thereby encircling non-holomorphic points). Therefore, if the Hankel contour is open, it is inapplicable for the Cauchy integral theorem (which only applies to closed contours).

\(^{212}\)See Whittaker [127], top of p.87.  
\(^{213}\)See Whittaker [127], p.85.  
\(^{214}\)See Whittaker [127], p.87, Corollary 1.
Even if we assume, for the sake of argument, that the Hankel contour is indeed closed at \(+\infty\) on the branch cut, \(^{215}\) then it would still be inapplicable for the Cauchy integral theorem. This is because the closed Hankel contour would enclose the entire branch cut (which consists entirely of non-holomorphic points). There would also be a non-holomorphic point on the Hankel contour itself, at the point where it intersects the branch cut at \((+\infty, 0)\). Both of these reasons disqualify the use of the Cauchy integral theorem.

For these reasons it is improper to use the Cauchy integral theorem’s path equivalence corollary to equate the branch cut to the Hankel open contour as domains of the contour integral. So Riemann’s \(\zeta(s)\) is not valid. Hankel’s derivation of the Gamma function \(\Gamma(s)\) is invalid for the same reasons.

7.5 Strictly Speaking, the Hankel Contour is Undefined

Further in regards to the Hankel contour of Equation 7.5: \(^{216}\)

\[
\int_{+\infty}^{+\infty} \frac{(-x)^s}{(e^x - 1)} \cdot \frac{dx}{x} = \int_{+\infty}^{\delta} \frac{(-x)^s}{(e^x - 1)} \cdot \frac{dx}{x} + \int_{|z|=\delta} \frac{(-x)^s}{(e^x - 1)} \cdot \frac{dx}{x} + \int_{\delta}^{+\infty} \frac{(-x)^s}{(e^x - 1)} \cdot \frac{dx}{x}
\]

(7.16)

Edwards \(^{217}\) states:

The definition of \((-x)^s\) is \((-x)^s = \exp[s \cdot \log(-x)]\), where the definition of \(\log(-x)\) conforms to the usual definition of \(\log(z)\) for \(z\) not on the negative real axis as the branch which is real for positive real \(z\); thus \((-x)^s\) is not defined on the positive real axis[.]

and also:

\[\text{[T]he middle term is } 2\pi i \text{ times the average value of } (-x)^s \cdot (e^x - 1)^{-1} \text{ on the circle } |x| = \delta [\text{because on this circle } i \cdot d\theta = (dx/x)]. \text{ Thus the middle term approaches zero as } \delta \to 0 \text{ provided } s > 1 [\text{because } x(e^x - 1)^{-1} \text{ is nonsingular near } x = 0].\]

\(^{215}\)As described in Whittaker \[^{127}\], p.245.

\(^{216}\)See also Whittaker \[^{127}\], p.266.

\(^{217}\)See Edwards \[^{25}\], p.10.
so as \( \delta \to 0 \) (provided \( s > 1 \)), the two "non-middle" terms are written as follows:

\[
\int_{+\infty}^{+\infty} \frac{(-x)^s}{(e^x - 1)} \cdot \frac{dx}{x} = \int_{+\infty}^{0} \frac{(-x)^s}{(e^x - 1)} \cdot \frac{dx}{x} + \int_{+0}^{\infty} \frac{(-x)^s}{(e^x - 1)} \cdot \frac{dx}{x}
\] (7.17)

Strictly speaking, based on the notation used, both of these two terms are located directly on the "branch cut" of \( \log(-x) \), where they are undefined. (There is no indication in the math notation to indicate otherwise). Any assignment of a value to this expression contradicts the definition of the logarithm \( \log(-x) \), which by definition is undefined for all non-negative Real values of \( x \).

8 Some Implications in Number Theory

8.1 The Birch and Swinnerton-Dyer Conjecture

8.1.1 Hasse–Weil Zeta Function

In regards to the Birch and Swinnerton-Dyer Conjecture, the Hasse–Weil zeta function \(^{218}\) holds that \( Z_{E,\mathbb{Q}}(s) \) of elliptic curve \( E \) over rational number field \( \mathbb{Q} \) of conductor \( N \) is

\[
Z_{E,\mathbb{Q}}(s) = \frac{\zeta(s) \cdot \zeta(s-1)}{L(E, s)}
\] (8.1)

Given that Riemann’s definition of \( \zeta(s) \) is not valid, \( \zeta(s) \) is defined exclusively by the Dirichlet series definition of \( \zeta(s) \), which has no zeros. So neither \( \zeta(s) \) nor \( \zeta(s-1) \) can equal zero, and therefore \( Z_{E,\mathbb{Q}}(s) \neq 0 \) for all \( s \in \mathbb{C} \). Moreover, rearranging the terms of the Hasse–Weil zeta function results in

\[
L(E, s) = \frac{\zeta(s) \cdot \zeta(s-1)}{Z_{E,\mathbb{Q}}(s)}
\] (8.2)

Since neither \( \zeta(s) \) nor \( \zeta(s-1) \) can equal zero, \( L(E, s) \neq 0 \) for all \( s \in \mathbb{C} \). So at \( s = 1 \), the function \( L(E, 1) \neq 0 \). Also, \( L(E, s) \) cannot be divergent at any \( s \in \mathbb{C} \), because \( Z_{E,\mathbb{Q}}(s) \neq 0 \). Given that \( L \)-functions are generalizations of \( \zeta(s) \), this is consistent with the result that Riemann’s "analytic continuation" of \( \zeta(s) \) is invalid. This result also directly contradicts the proof of the "analytical continuation" of \( L(C, s) \) cited by Wiles \(^{138}\), which must now be re-evaluated. \(^{219}\) Given that at \( s = 1 \), the function

\(^{218}\)See Wikipedia \(^{132}\), citing Silverman’s \(^{105}\) Section C.16, and Serre \(^{103}\).

\(^{219}\)See Wiles \(^{138}\), p.2: "A conjecture going back to Hasse ... predicted that \( L(C, s) \) should have a holomorphic continuation as a function of \( s \) to the whole complex plane. This has now been proved".

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\( L(E, 1) \neq 0 \), all modular elliptic curves \( E \) have rank 0, and thus are finite.  

8.1.2 Zeta Function is Divergent at \( s=1 \)

The Birch and Swinnerton-Dyer Conjecture is also resolved by confirming that \( \zeta(s) \neq 0 \) at \( s = 1 \). According to the Clay Mathematics Institute’s [15] description of the Birch and Swinnerton-Dyer (BSD) conjecture:

\[
\text{[T]his amazing conjecture asserts that if } \zeta(1) \text{ is equal to 0, then there are an infinite number of rational points (solutions), and conversely, if } \zeta(1) \text{ is not equal to 0, then there is only a finite number of such points.}
\]

Riemann’s definition of \( \zeta(s) \) is not valid, nor is any other "analytic continuation" of the Dirichlet series definition of \( \zeta(s) \) to half-plane \( \text{Re}(s) \leq 1 \). So, according to the Dirichlet series, \( \zeta(1) \neq 0 \). In fact, at \( s = 1 \) the Dirichlet series definition of \( \zeta(s) \) is the famous harmonic series, which is well known to be divergent.  

(Moreover, both the Dirichlet series definition of \( \zeta(s) \) and Riemann’s definition of \( \zeta(s) \) agree on divergence at \( s = 1 \), so \( \zeta(1) \) would be divergent (and thus \( \zeta(1) \neq 0 \)) even if Riemann’s definition of \( \zeta(s) \) were valid).

8.2 Finiteness of the Tate–Shafarevich Group

Wiles’s [138] official Clay Foundation description of the BSD conjecture states the following:

There is an analogous conjecture for elliptic curves over function fields. It has been proved in this case by Artin and Tate [116] that the \( L \)-series has a zero of order at least \( r \), but the conjecture itself remains unproved. In the function field case it is now known to be equivalent to the finiteness of the Tate–Shafarevich group.  

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220 See Wiles [138] (citing Kolyvagin [57]): "Kolyvagin showed in 1990 that for modular elliptic curves, if \( L(C, 1) \neq 0 \) then \( r = 0 \) and if \( L(C, 1) = 0 \) but \( L'(C, 1) \neq 0 \) then \( r = 1 \)."

221 See Wikipedia [131]: "The fact that the harmonic series diverges was first proven in the 14th century by Nicole Oresme, but this achievement fell into obscurity. Proofs were given in the 17th century by Pietro Mengoli, Johann Bernoulli, and Jacob Bernoulli." The Wikipedia entry includes citations to the original Latin-language publications.

222 Wiles's [138] p.2, citing Tate [116], and citing Milne's [73] Corollary 9.7.
So, the result of $L(E, s) \neq 0$ for all $s \in \mathbb{C}$ for the BSD conjecture implies a similar result for its analogous conjecture for elliptic curves over function fields, and the finiteness of the Tate–Shafarevich group.

Totaro [119] [120] states that the "finiteness" result for the Birch and Swinnerton-Dyer Conjecture implies finiteness for other conjectures in number theory, such as the finiteness of the Tate–Shafarevich group over function fields. More specifically, Totaro [120] states:

To spell out the relations between the Tate conjecture and finiteness problems, let $X$ be a smooth projective surface over a finite field $k$, and let $f$ be a morphism with connected fibers from $X$ onto a smooth projective curve $C$. Assume that the generic fiber $F$ of $f$, which is a curve over the function field $k(C)$, is smooth over $k(C)$. Let $J$ be the Jacobian of $F$; thus $J$ is an abelian variety over the global field $k(C)$. Then the following are equivalent:

- the Tate conjecture holds for divisors on $X$;
- the Brauer group of $X$ is finite;
- the Tate–Shafarevich group of $J$ is finite;
- the Birch–Swinnerton-Dyer conjecture holds for $J$.

### 8.3 The Brauer Group, the Tate Conjecture, and the Hodge Conjecture

Totaro [119][120] further states that the "finiteness" result for the Birch and Swinnerton-Dyer Conjecture implies finiteness for the Brauer group over finite fields, and the Tate conjecture for divisors.

However, regarding the Hodge conjecture, Totaro [120] states: "Although there are no obvious implications between the Tate and Hodge conjectures, there have been implications for particular classes of varieties". So the Hodge conjecture must be resolved in a manner consistent with these other results, in order to avoid violating the LNC.

### 8.4 Other Related Number Theory Conjectures

Moreover, based on discussions in Iwaniec [49] and Sarnak [100], the invalidity of Riemann’s definition of $\zeta(s)$ implies that the Grand Riemann Hypothesis (GRH) and the Grand Lindelöf Hypothesis (GLH) are invalid.

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223 Citing Ulmer [123], Proposition 5.1.2 and Theorem 6.3.1.
9 Some Implications in Physics

9.1 Riemann Zeta Function Regularization

Hawking [42] describes the use of Riemann Zeta function regularization as:

... a technique for obtaining finite values to path integrals for fields (including the gravitational field) on a curved spacetime background or, equivalently, for evaluating the determinants of differential operators such as the four-dimensional Laplacian or D’Alembertian.

According to Dittrich [21]:

[In] many local relativistic quantum field theory models of elementary particles, ... Riemann’s results are of utmost importance for handling infinities with the aid of his zeta-function regularization.

Moreover, according to Aguilera-Damia [1]:

Using $\zeta$-function regularization, we study the one-loop effective action of fundamental strings in $AdS_5 \times S^5$ dual to the latitude $\frac{1}{4}$-BPS Wilson loop in $N = 4$ super-Yang-Mills theory. To avoid certain ambiguities inherent to string theory on curved backgrounds we subtract the effective action of the holographic $\frac{1}{2}$-BPS Wilson loop. We find agreement with the expected field theory result at first order in the small latitude angle expansion but discrepancies at higher order.

Unfortunately, "Riemann Zeta function regularization" is merely the substitution of the invalid Riemann $\zeta(s)$ in place of the Dirichlet series of $\zeta(s)$, and therefore this "regularization" is invalid. The proposition that a divergent series is equal to a convergent series is clear violation of the LOI and LNC.

So Riemann Zeta function regularization is invalid. As a direct consequence, all physics models that include Riemann Zeta function regularization are invalid. The models that the author is aware of are briefly described below.

\footnote{See Dittrich [21], p.3.}
\footnote{See Aguilera-Damia [1], abstract.}
9.2 Two-Dimensional Yang-Mills Theory

Witten [140] describes two-dimensional quantum Yang-Mills Theory (YM T) from three different "points of view":

1. By standard physical methods,

2. By relating YMT to the large $k$ limit of three-dimensional Chern-Simons theory, and two-dimensional conformal field theory, and

3. By relating the weak coupling limit of YMT to the theory of Reidmeister-Ray-Singer Torsion.

The abstract of Witten [140] states that the results obtained from these three points of view are in agreement, and "give formulas for the volumes of the moduli spaces of representations of fundamental groups of two dimensional surfaces." However, each of these three points of view use Riemann's version of $\zeta(s)$, which is in half-plane $\text{Re}(s) \leq 1$. So all three "points of view" of YMT are invalid.

\footnote{See Witten [140], p.154, description of Eq. 1.2: "... and $\Sigma$ [is] a Riemann surface of genus $g$, one finds $\text{Vol}(M) = 2 \cdot (2\pi^2)^{1-g} \cdot \zeta(2g - 2)$, where $\zeta(s)$ is the Riemann zeta function". Riemann's $\zeta(s)$ is invalid, so $\text{Vol}(M)$ is divergent at $g=0$ and $g=1$.}

\footnote{Witten's [140], p.154, description of Eq. 1.2 also refers (in regards to Eq. 3.18) to the Hurwitz zeta function and Dirichlet $L$-functions. These generalizations of Riemann's $\zeta(s)$ inherit Riemann's $\zeta(s)$ invalidity in half-plane $\text{Re}(s) \leq 1$.}

\footnote{Witten's [140], p.174 (last para.) states: "We will formulate this in a way that exhibits the relation to IRF models - which also appear, after a much more difficult analysis, in computing Wilson line expectation values in three dimensional Chern-Simons theory [25]. For convenience, we will consider first the case that $\Sigma$ has genus zero." However, Eq.1.2 with $g=0$ produces a divergent $\text{Vol}(M)$.}

\footnote{Witten's [140], p.159, description of Eq. 2.20: "With an explicit choice (such as zeta function regularization) for defining the determinants that appear in evaluating the left and right-hand sides of (2.20), an \textit{a priori} computation of $\Delta v$ can be given." The so-called "Zeta function regularization" replaces the Dirichlet series of $\zeta(s)$ with the invalid Riemann $\zeta(s)$, for values of $s$ in half-plane $\text{Re}(s) \leq 1$.}

\footnote{See also Witten's [140], p.161, description of Eq. 2.28: "We will ensure this by using the zeta function definition of determinants [3]". But the Dirichlet series of $\zeta(s)$ is divergent at $s=0$.}

\footnote{See also Witten's [140], p.178, Eq. 3.8, which includes Riemann's $\zeta(s)$. Eq. 3.8 is divergent at $g = 0$ and $g = 1$.}

\footnote{See also Witten's [140], p.180, Eq.3.22: "The Hurwitz zeta function ... is then continued holomorphically throughout the complex $z$ plane, except for a pole at $z = 1$". This is false.}

\footnote{See also Witten's [140], p.201, Eq. 4.95: "with $\zeta(s)$ the Riemann zeta function". This is divergent if $\text{Re}(s) \leq 1$.}
9.3 Casimir Effect, QED, and QCD

Dittrich [21] states that "Riemann Zeta Function Regularization" is used to derive the Casimir effect. Tong [117] confirms this for "Casimir Energy". Dittrich [21] also states that: "The same procedure finds application in QED and QCD." Therefore the Casimir effect, Quantum Electrodynamics (QED), and Quantum Chromodynamics (QCD) are all invalid.

9.4 Supersymmetry (SUSY) and Quantum Field Theory (QFT)

According to Elizalde [26], Supersymmetry (SUSY) incorporates Riemann Zeta Function Regularization:

Regularization and renormalization procedures are essential issues in contemporary physics — without which it would simply not exist, at least in the form known today (2000). They are also essential in supersymmetry calculations. Among the different methods, zeta-function regularization — which is obtained by analytic continuation in the complex plane of the zeta-function of the relevant physical operator in each case — might well be the most beautiful of all. Use of this method yields, for instance, the vacuum energy corresponding to a quantum physical system (with constraints of any kind, in principle).

Therefore, Supersymmetry (SUSY) is invalid too. Also according to Elizalde [26], Riemann Zeta Function Regularization is used in Quantum Field Theory (QFT):

These mathematically simple-looking relations involve very deep physical concepts (no wonder that understanding them took several decades in the recent history of quantum field theory, QFT). The zeta-function method is unchallenged at the one-loop level, where it is rigorously defined and where many calculations of QFT reduce basically (from a mathematical point of view) to the computation of determinants of elliptic pseudo-differential operators ...

Therefore, also Quantum Field Theory (QFT) is invalid.

\[^{234}\text{See Dittrich [21], pp.30-34.}\]
\[^{235}\text{See Tong [117], p.13.}\]
\[^{236}\text{See Dittrich [21], p.34.}\]
9.5 Bosonic String Theory

There are several examples in Bosonic string theory of the use of Riemann’s definition of \( \zeta(s) \), and its functional equation (that describes a relationship between \( \zeta(s) \) and \( \zeta(1-s) \)).

The He [43] reference links Riemann’s definition of \( \zeta(s) \) to expressions of the Veneziano amplitude that describe the scattering of four bosonic open strings with tachyonic masses. This is done based on work pertaining to the p-adic string, by Freund and Witten [31]. Turco [122] argues that Riemann’s functional equation of \( \zeta(s) \) is linked to the Veneziano amplitude.

Toppan [118] provides a description of the heat-kernel method and of the (invalid) generalized Riemann’s zeta-functions associated to elliptic operators. Toppan [118] then defines their role in defining one-loop partition functions for Euclidean Field Theories. Toppan [118] then applies these (invalid) results to the Polyakov functional quantization of the closed bosonic string, to derive its (invalid) critical dimensionality of \( D = 26 \). Nunez [79] confirms the use of (invalid) Zeta function regularization in obtaining the (invalid) \( D = 26 \) dimensionality. 237

In Bosonic string theory, the mass of states in lightcone gauge is

\[
M^2 = \frac{4}{\alpha'} \left[ \sum_{n=1}^{\infty} \alpha_{-n} \alpha_n + \frac{D-2}{2} \left( \sum_{n=1}^{\infty} n^{-1} \right) \right] \tag{9.1}
\]

The Dirichlet series definition of \( \zeta(s) \) is divergent at \( s = -1 \), but Riemann’s definition of \( \zeta(s) \) at \( s = -1 \) is

\[
\zeta(-1) = -1/12 \tag{9.2}
\]

If the value of Riemann’s definition of \( \zeta(s) \) at \( s = -1 \) is substituted for the divergent Dirichlet series definition of \( \zeta(s) \) at \( s = -1 \) (thereby violating the LOI and LNC), the mass of states is

\[
M^2 = \frac{4}{\alpha'} \left( N - \frac{(D-2)}{24} \right) \tag{9.3}
\]

At the ground state \( N = 0 \), the formula simplifies to

\[
M^2 = \frac{-(D-2)}{6 \cdot \alpha'} \tag{9.4}
\]

which corresponds to a particle with an imaginary mass, known as a tachyon. Moreover, at the first excited state (\( N = 1 \)), the Equation 9.1 is massless (\( M^2 = 0 \)) at \( D = 26 \).

Needless to say, these results are invalid if Riemann’s definition of \( \zeta(s) \) is invalid. If

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237See Nunez, bottom of p.17.
\( \zeta(s) \) is defined exclusively by the Dirichlet series, the mass of states in lightcone gauge (Equation 9.1) becomes as follows: At \( D = 2 \),

\[
M^2 = \frac{4 \cdot N}{\alpha'}
\]  
(9.5)

At all other values of \( D \), the value of \( M^2 \) is divergent. Moreover, Equation 9.1 is massless \((M^2 = 0)\) only if both \( D = 2 \) and \( N = 0 \), or if \( D = 2 \) and \( \alpha' \) is infinitesimal. At \( D \neq 2 \) and \( \alpha' \) is infinitesimal, the value of \( M^2 \) is indeterminate.

### 9.6 Riemann’s Zeta Function and the Failure of LOI in Quantum Physics

As stated at the beginning of the present paper, the convergence of Riemann’s \( \zeta(s) \) in half-plane \( \text{Re}(s) \leq 1 \) (except at \( s = 1 \)), where the Dirichlet series of \( \zeta(s) \) is divergent, is a violation of the Law of Identity in regards to the function \( \zeta(s) \). Given that quantum physics theories extensively use Riemann’s definition of \( \zeta(s) \) (as seen in the immediately preceding sections of the present paper), it comes as no surprise to see published articles that state that LOI fails in quantum physics: \(^{238}\)

However, it has also been argued that quantum physics is in fact compatible with a metaphysics of individual objects, but that such objects are indistinguishable in a sense which leads to the violation of Leibniz’s famous Principle of the Identity of Indiscernibles. This last claim has recently been contested in a way that has reinvigorated the debate over the impact of the theory.

The question, then, is whether LOI is preserved in quantum physics after all theories that assume the validity of Riemann’s \( \zeta(s) \) are expunged. An additional question is whether the remaining theories are sufficient to explain experimental results.

\(^{238}\)See French, [30].
10 Some Implications in Logic

Russell argued that symbolic logic is "practically identical" to mathematics. 239 240 After Gödel’s blow to Russell and Frege’s "logicism" project, 241 242 and to Hilbert’s "formalism" project", 243 the relationship between logic and mathematics remains unclear.

However, Russell’s concept of "logic" was limited to classical logic. Today there is a wide variety of non-classical logics in addition to classical logic. 244 So if mathematics is "merely logic in another guise", 245 then which logic is math in another guise? (Or perhaps: which logic is equivalent to which portion of mathematics?) Russell assumed that classical logic would be the logic equivalent to all math, but perhaps the entire body of mathematics is equivalent to some other logic (e.g. a 3VL), or to the entire (inconsistent) body of logic instead?

In contrast to Russell, Boole’s algebra 246 indicates that Aristotle’s "Laws of Thought" correspond to a specific subset of mathematics (Boolean algebra). By extension, each non-classical logic should have its own corresponding algebra. Therefore, the entire body of logic should correspond to a subset of mathematics.

In contrast to both Russell and Boole (and perhaps most relevant to this paper), Wittgenstein defined philosophy (which since Aristotle is defined as including logic) as "all those primitive propositions which are assumed as true without proof by the various sciences". 247 These propositions are assumed as true by the sciences, and therefore are investigated by philosophy (which includes logic). Which leads to the question: by which logic(s) are these propositions to be evaluated?

Today, there are a wide variety of logics (classical and non-classical) that contradict one another by least one axiom. 248 And certain "primitive propositions" are imper-

239 See Russell’s [95] definition of mathematics in p.157, para.106: "This definition brought Mathematics into very close relation to Logic, and made it practically identical with Symbolic Logic."
240 See also Scruton [102], p.77: "... as Russell believed, that mathematics is, in the last analysis, merely logic in another guise."
241 See Scruton, [102], p.395: "The final blow to the logicist programme was struck by Gödel, in his famous meta-mathematical proof that there can be no proof of the completeness of arithmetic which permits a proof of its consistency, and vice versa."
242 See also Wikipedia [133] (citing Crossley [17], pp. 52–53): "Roughly speaking, in proving the first incompleteness theorem, Gödel used a modified version of the liar paradox, replacing 'this sentence is false' with 'this sentence is not provable', called the 'Gödel sentence G'."
243 See Scruton [102], p.395: "It follows too that we cannot treat mathematics as Hilbert wished, merely as strings of provable formulae: the theory of 'formalism' is false."
244 See e.g., Priest’s [88] thick book on non-classical logics.
245 Scruton’s phrasing of Russell’s argument.
246 See Boole [11].
247 See Wikipedia [137], citing Wittgenstein [142] p.332, citing Wittgenstein [141] p.89.
248 Again, see Priest [88].
missible in certain logics (e.g. paradoxes in classical logics, due to LNC). So which "primitive propositions" are included, for example, in the foundations of mathematics? Is LNC included in the foundations of mathematics?

And more specifically, regarding the Riemann Hypothesis (RH) problem, which logic is assumed to be in its foundation? This paper shows that classical and intuitionistic logics are sufficient for proving the invalidity of the Riemann Zeta Function $\zeta(s)$, but are unable to resolving the Riemann Hypothesis (RH) problem. Instead, the three-valued logic (3VL) called "Logic of Paradox" ($LP$)\footnote{See Priest’s articles \cite{85} and \cite{86} on "Logic of Paradox".} is well suited to resolve the RH. $LP$ has a third truth-value for paradoxes, which avoids ECQ.

This result is in agreement with Wittgenstein’s statement, because by selecting foundation logics for the derivation of Riemann’s $\zeta(s)$, and for the RH, we are selecting propositions that are assumed (without proof) to be true.

Using $LP$ as the underlying logic of the RH takes the paradox from "a triviality unworthy of serious consideration"\footnote{See Priest, \cite{85}, p.219} to the central consideration of the logical model.

### 10.1 Gödel’s First Incompleteness Theorem

This use of $LP$ as the underlying logic also provides new interpretations of Gödel’s first incompleteness theorem, and the subsequent abandonment of the "logicist" project. Gödel’s first incompleteness theorem is:

**First Incompleteness Theorem:** "Any consistent formal system $F$ within which a certain amount of elementary arithmetic can be carried out is incomplete; i.e., there are statements of the language of $F$ which can neither be proved nor disproved in $F." \text{\footnote{Wikipedia \cite{130}, citing Raatikainen \cite{92}.}}\text{\footnote{Gödel’s first incompleteness theorem has been called a "restatement of the Liar paradox", \text{\footnote{Crossley \cite{17}, pp. 52–53}: "Roughly speaking, in proving the first incompleteness theorem, Gödel used a modified version of the liar paradox".}} and of course the Liar Paradox can neither be proven nor disproven. This is problematic in classical logic. In contrast, in $LP$, a third truth-value is assigned to the Liar Paradox (and to all other paradoxes).

So one interpretation of Gödel’s incompleteness theorem is that it is merely a tautology: that paradoxes exist, and that classical logic does not know what do with paradoxes (due to the LNC and LEM, and the resulting lack of a third truth-value).
In LP, Gödel’s first incompleteness theorem can be interpreted as another tautology: that there exist propositions that have a third truth-value (neither true nor false).

10.2 Gödel’s Second Incompleteness Theorem

The use of LP as the underlying logic also provides a new interpretation of Gödel’s second incompleteness theorem. Gödel’s second incompleteness theorem is:

**Second Incompleteness Theorem**: "Assume F is a consistent formalized system which contains elementary arithmetic. Then $F \not\vdash \text{Cons}(F)$." \(^{253}\)

In LP, Gödel’s second incompleteness theorem can be interpreted as a tautology: the canonical consistency statement $\text{Cons}(LP)$ is not provable in LP, because LP rejects the LNC, and tolerates inconsistency (i.e. statements with the third truth-value).

Moreover, if LP is indeed the foundational logic underlying the RH problem, does applying LP’s axioms to solve the RH correspond to "adding new rules from 'outside' of number theory in order to solve RH"? No, because the axioms of LP are inherited into the axioms of the RH problem.

11 Conclusion

Riemann’s definition of $\zeta(s)$ violates the LNC, because it contradicts the proven divergence of the Dirichlet series definition of $\zeta(s)$ in the half-plane $\text{Re}(s) \leq 1$. According to the LOI and LNC, Riemann’s definition of $\zeta(s)$ is invalid.

Moreover, in classical and intuitionistic logics, both of which include ECQ, the invalidity of Riemann’s definition of $\zeta(s)$ trivializes all theorems that assume the validity of Riemann’s definition of $\zeta(s)$ or that of any other "analytic continuation" of the Dirichlet series definition of $\zeta(s)$ to the half-plane $\text{Re}(s) \leq 1$. Therefore, in classical and intuitionistic logics, every theorem that assumes the validity of Riemann’s definition of $\zeta(s)$ is "trivially true".

Moreover, the Riemann Hypothesis (RH) is both true and false, due to its "vacuous subjects" - the non-existent zeros of $\zeta(s)$. This paradoxical result of being both true and false violates the LNC. So theorems that assume RH to be true cannot be salvaged in classical logic, but can be true according to the material conditional in a Three-Valued Logic (3VL).

\(^{253}\)Wikipedia [130], citing Raatikainen [92].
Moreover, the holding that \( RH \) is a third value in a 3VL is inconsistent with proofs that analogies of the \( RH \) in other areas of mathematics are exclusively true (e.g. Deligne’s proof of the Weil conjecture III, and Hasse’s proof of the \( RH \) for elliptic curves of genus 1). \(^{254}\) Further research is needed to determine why those versions of the \( RH \) are proven to be (exclusively) true. One obvious possibility is that they are "trivially true" in classical logic, due to the conjunction of: (1) a false assumption that "analytic continuation" is valid, (2) LNC, and ECQ.

Finally, as stated elsewhere in this paper, Langer’s \(^{59}\) statement that "[c]ontradictory theorems cannot follow from consistent postulates" \(^{255}\) is wrong. Contradictory theorems do follow from consistent postulates, if the theorems are directed to "vacuous subjects", or if a postulate is self-referential and contradictory. Therefore, MacFarlane’s \(^{64}\) and Meyer’s \(^{72}\) quotes on the topic are correct (emphasis added in bold):

> There’s no good reason to assume that mathematics must be consistent. If math is about a supersensible realm of objects, why should we assume they’re like ordinary empirical objects with respect to consistency? But if math is a free human creation, why can’t it be inconsistent? \(^{256}\)

> ... for certain purposes an inconsistent system might be more useful, more beautiful, and even—at the furthest metaphysical limits—as the case may be, more accurate. \(^{257}\)

**Classical logic forces math to be consistent or trivial.** \(^{258}\)

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\(^{254}\)See Milne \(^{74}\), pp.3 and 49.

\(^{255}\)See Langer \(^{59}\), p.202.

\(^{256}\)See MacFarlane \(^{64}\), p.1.

\(^{257}\)See MacFarlane \(^{64}\), p.1 (citing Meyer \(^{72}\), p.814).

\(^{258}\)See MacFarlane \(^{64}\), p.1.
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