Exact solutions of Bianchi-type I and V spacetimes in the $f(R)$ theory of gravity

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Abstract
In this paper, the crucial phenomenon of the expansion of the universe has been discussed. For this purpose, we study the vacuum solutions of Bianchi-type I and V spacetimes in the framework of $f(R)$ gravity. In particular, we find two exact solutions in each case using the variation law of Hubble parameter. These solutions correspond to two models of the universe. The first solution gives a singular model, while the second solution provides a non-singular model. The physical behavior of these models is discussed. Moreover, the function of the Ricci scalar is evaluated for both models in each case.

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1. Introduction

The accelerating expansion of the universe has attracted much attention in recent years. Why is the universe expanding at an increasing rate and spreading its contents over greater dimensions of space? This is the most interesting question in modern day cosmology. There is another issue of dark energy and dark matter which has been discussed widely. Einstein initiated the concept of dark energy by introducing the cosmological constant. Later, he remarked that the introduction of the cosmological term was the biggest blunder of his life.

Carmeli predicted for the first time in 1996 that the universe expansion was accelerating [1]. Later, the supernova experiments [2] became known and it was found that the major part of the universe was filled with dark matter and dark energy. The results from these experiments provided evidence for the accelerating expansion of the universe. The cosmological constant is considered as one of the candidates responsible for dark energy. Modified theories of gravity provide an alternative approach to studying the universe.

There are some useful aspects of modified theories of gravity [3]. Modified gravity gives an easy unification of early-time inflation and late-time acceleration. It provides a natural gravitational alternative to dark energy. The explanation of cosmic acceleration is obtained just by introducing the term $1/R$ which is essential at small curvatures. Modified
gravity also describes the transition phase of the universe from deceleration to acceleration. It can be used as an explanation of the hierarchy problem in high energy physics. The $f(R)$ theory of gravity is one of the modified theories which is considered most suitable due to cosmologically important $f(R)$ models. These models consist of higher order curvature invariants as functions of the Ricci scalar. Viable $f(R)$ gravity models [4] have been proposed which show the unification of early-time inflation and late-time acceleration. The problem of dark matter can also be addressed by using viable $f(R)$ gravity models.

Singularity has been an important issue in general relativity (GR). The occurrence of spacetime singularity is a general feature of any cosmological model under some reasonable conditions. It may be possible to avoid these undesired singularities in the context of modified theories. Kanti et al showed that there did not exist any cosmological singularity by considering higher order curvature terms [5]. Nojiri and Odintsov proposed some realistic singularity-free models in modified $f(R)$ gravity [6]. Bamba et al proved that a non-minimal gravitational coupling can remove the finite-time future singularity in modified gravity [7]. Thus, there is a strong reason to study solutions of the field equations in modified theories to address problems such as dark energy and singularity.

Weyl [8] and Eddington [9] studied $f(R)$ actions in 1919 and 1922, respectively. Buchdahl [10] explored these actions in the context of non-singular oscillating cosmologies. Cognola et al [11] investigated $f(R)$ gravity at the one-loop level in the de Sitter universe. It was found that one-loop effective action can be useful for the study of constant curvature black hole nucleation rate. Spherical symmetry is the closest approach to nature because one can compare the results from solar system observations. Thus, the most commonly explored exact solutions in $f(R)$ gravity are the spherically symmetric solutions. Multamäki and Vilja [12] studied spherically symmetric vacuum solutions for the first time in this theory. The same authors [13] also investigated perfect fluid solutions and showed that pressure and density did not uniquely determine $f(R)$. Capozziello et al [14] explored spherically symmetric solutions of $f(R)$ theories of gravity via the Noether symmetry approach. Hollenstein and Lobo [15] analyzed the exact solutions of static spherically symmetric spacetimes in $f(R)$ gravity coupled to nonlinear electrodynamics.

Cylindrical symmetry is close to spherical symmetry, and may be used to study the exact solutions of the field equations in $f(R)$ gravity. Azadi et al [16] studied cylindrically symmetric vacuum solutions in the metric $f(R)$ theory of gravity. Momeni and Gholizadi [17] extended this work to the general cylindrically symmetric solution. Recently, we have explored static plane symmetric vacuum solutions in the $f(R)$ theory of gravity [18]. The field equations are solved using the assumption of constant scalar curvature which may be zero or non-zero. However, no attempt has been made so far for solutions with non-constant scalar curvature.

The accelerating expansion of the universe can be studied using Bianchi-type I and V spacetimes which are the generalization of FRW spacetimes. Due to their spatially homogeneous and isotropic nature, many authors have studied these spacetimes in different contexts [19–24]. Berman [25] introduced a different method to solve the field equations using the variation law of Hubble’s parameter. The main feature of the variation law is that it gives constant value of deceleration parameter. Using this law, Singh et al explored perfect fluid solutions of the Bianchi-type V spacetime [26]. Recently, Kumar and Singh studied solutions of the field equations in the presence of a perfect fluid using the Bianchi-type I spacetime in GR [27]. The same authors investigated perfect fluid solutions using the Bianchi-type I spacetime in scalar–tensor theory [28].

In this paper, we focus our attention on the vacuum solutions of Bianchi-type I and V spacetimes in $f(R)$ theories of gravity using the metric approach. The paper is organized as
follows. In section 2, we give a brief introduction about the field equations in the context of f(R) gravity. Sections 3 and 4 are used to find exact vacuum solutions and the singularity analysis of these solutions. In the last section, we summarize and conclude the results.

2. f(R) gravity formalism

The f(R) theory of gravity is the generalization of GR. The action for this theory is given by

$$ S = \int \sqrt{-g} \left( \frac{1}{16\pi G} f(R) + L_m \right) d^4x. $$

Here f(R) is a general function of the Ricci scalar and L_m is the matter Lagrangian. It is noted that this action is obtained just by replacing R by f(R) in the standard Einstein–Hilbert action. The corresponding field equations are found by varying the action with respect to the metric g_{\mu\nu},

$$ F(R)R_{\mu\nu} - \frac{1}{2} f(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu F(R) + g_{\mu\nu} \Box F(R) = \kappa T_{\mu\nu}, $$

where

$$ F(R) \equiv \frac{df(R)}{dR}, \quad \Box \equiv \nabla^\mu \nabla_\mu, $$

\nabla_\mu is the covariant derivative and T_{\mu\nu} is the standard matter energy–momentum tensor derived from the Lagrangian L_m. These are the fourth-order partial differential equations in the metric tensor. The fourth order is due to the last two terms on the left-hand side of the equation. If we take f(R) = R, these equations reduce to the field equations of GR.

Now contracting the field equations, it follows that

$$ F(R)R - 2f(R) + 3 \Box F(R) = \kappa T $$

and in vacuum, we have

$$ F(R)R - 2f(R) + 3 \Box F(R) = 0. $$

This gives an important relationship between f(R) and F(R) which will be used to simplify the field equations and to evaluate f(R).

3. Exact Bianchi-type I solutions

Here we shall find exact solutions of the Bianchi I spacetime in f(R) gravity. For the sake of simplicity, we take the vacuum field equations.

3.1. Bianchi-type I spacetime

The line element of the Bianchi-type I spacetime is given by

$$ ds^2 = dt^2 - A^2(t) dx^2 - B^2(t) dy^2 - C^2(t) dz^2, $$

where A, B and C are cosmic scale factors. The corresponding Ricci scalar becomes

$$ R = -2 \left[ \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{A}B}{AB} + \frac{\dot{B}C}{BC} + \frac{\dot{C}A}{CA} \right], $$

where a dot represents derivative with respect to t.

We define the average scale factor a as

$$ a = \sqrt{ABC} $$

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and the volume scale factor as
\[ V = a^3 = ABC. \]  (9)

We also define the generalized mean Hubble parameter \( H \) in the form
\[ H = \frac{1}{3} (H_1 + H_2 + H_3), \]  (10)
where \( H_1 = \frac{\dot{A}}{\dot{A}}, H_2 = \frac{\dot{B}}{\dot{B}}, H_3 = \frac{\dot{C}}{\dot{C}} \) are the directional Hubble parameters in the directions of the x-, y- and z-axes, respectively. Using equations (8)–(10), we obtain
\[ H = \frac{1}{3} \frac{\dot{V}}{V} = \frac{1}{3} (H_1 + H_2 + H_3) = \frac{\dot{a}}{a}. \]  (11)

It follows from equation (5) that
\[ f(R) = \frac{3}{2} \Box F(R) + \frac{F(R)R}{4}. \]  (12)

Inserting this value of \( f(R) \) in the vacuum field equations, we have
\[ F(R) R_{\mu\nu} - \nabla_\mu \nabla_\nu F(R) = \frac{F(R)R - \Box F(R)}{4}. \]  (13)

Since the metric (6) depends only on \( t \), one can view equation (13) as the set of differential equations for \( F(t), A, B \) and \( C \). It follows from equation (13) that the combination
\[ A_\mu = \frac{F(R) R_{\mu\nu} - \nabla_\mu \nabla_\nu F(R)}{g_{\mu\nu}} \]  (14)

is independent of the index \( \mu \) and hence \( A_\mu - A_\nu = 0 \) for all \( \mu \) and \( \nu \). Consequently, \( A_0 - A_1 = 0 \) gives
\[ \frac{\dot{B}}{B} \frac{\dot{C}}{C} + \frac{\dot{A}}{A} + \frac{\dot{C}}{A} + \frac{\dot{A}}{F} - \frac{\dot{F}}{F} = 0. \]  (15)

Also, \( A_0 - A_2 = 0 \) and \( A_0 - A_3 = 0 \) yield respectively
\[ \frac{\dot{A}}{A} - \frac{\dot{C}}{C} + \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{\dot{F}}{F} - \frac{\dot{F}}{F} = 0, \]  (16)
\[ \frac{\dot{A}}{A} - \frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{\dot{A}}{A} + \frac{\dot{C}}{C} + \frac{\dot{F}}{F} - \frac{\dot{F}}{F} = 0. \]  (17)

Thus we get three nonlinear differential equations with four unknowns, namely \( A, B, C \) and \( F \). The solution of these equations can be found by following the approach of Saha [29].

3.2. Solution of the field equations

Subtracting equation (16) from (15), equation (17) from (16) and equation (17) from (14), we get respectively
\[ \frac{\dot{A}}{A} - \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) + \frac{\dot{F}}{F} \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) = 0, \]  (18)
\[ \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \left( \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right) + \frac{\dot{F}}{F} \left( \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right) = 0, \]  (19)
\[ \frac{\dot{A}}{A} - \frac{\dot{C}}{C} \left( \frac{\dot{A}}{A} - \frac{\dot{C}}{C} \right) + \frac{\dot{F}}{F} \left( \frac{\dot{A}}{A} - \frac{\dot{C}}{C} \right) = 0. \]  (20)
These equations imply that
\[
\frac{B}{A} = d_1 \exp \left[ \frac{c_1}{a^3} \right],
\]
\[
\frac{C}{B} = d_2 \exp \left[ \frac{c_2}{a^3} \right],
\]
\[
\frac{A}{C} = d_3 \exp \left[ \frac{c_3}{a^3} \right],
\]
where \( c_1, c_2, c_3 \) and \( d_1, d_2, d_3 \) are constants of integration which satisfy the relation
\[
c_1 + c_2 + c_3 = 0, \quad d_1 d_2 d_3 = 1.
\]
Using equations (21)–(23), we can write the metric functions explicitly as
\[
A = a p_1 \exp \left[ \frac{q_1}{a^3} \right],
\]
\[
B = a p_2 \exp \left[ \frac{q_2}{a^3} \right],
\]
\[
C = a p_3 \exp \left[ \frac{q_3}{a^3} \right],
\]
where
\[
p_1 = (d_1^{-2} d_2^{-1})^{\frac{1}{3}}, \quad p_2 = (d_1 d_2^{-1})^{\frac{1}{3}}, \quad p_3 = (d_1 d_2^{2})^{\frac{1}{3}}
\]
and
\[
q_1 = -\frac{2c_1 + c_2}{3}, \quad q_2 = \frac{c_1 - c_2}{3}, \quad q_3 = \frac{c_1 + 2c_2}{3}.
\]
Note that \( p_1, p_2, p_3 \) and \( q_1, q_2, q_3 \) also satisfy the relation
\[
p_1 p_2 p_3 = 1, \quad q_1 + q_2 + q_3 = 0.
\]
Now we use the power-law assumption to solve the integral part in the above equations. The power-law relation between scale factor and scalar field has already been used by Johri and Desikan [30] in the context of Robertson–Walker Brans–Dicke models. However, in a recent paper [31], Kotub Uddin et al have established a result in the context of \( f(R) \) gravity which shows that
\[
F \propto a^m,
\]
where \( m \) is an arbitrary constant. Thus using power-law relation between \( F \) and \( a \), we have
\[
F = k a^m,
\]
where \( k \) is the constant of proportionality and \( m \) is any integer. The deceleration parameter \( q \) in cosmology is the measure of the cosmic acceleration of the universe expansion and is defined as
\[
q = -\frac{\ddot{a}a}{\dot{a}^2}.
\]
It is mentioned here that the negative sign and the name ‘deceleration parameter’ are historical. Initially, \( q \) was supposed to be positive but recent observations from supernova experiments suggest that it is negative. Thus, the behavior models of the universe depend upon the sign of \( q \). A positive deceleration parameter corresponds to a decelerating model while a negative
value indicates inflation. We also use a well-known relation [32] between the average Hubble parameter $H$ and average scale factor $a$ given as

$$H = la^{-n}, \quad (33)$$

where $l > 0$ and $n \geq 0$. This is an important relation because it gives a constant value of the deceleration parameter.

Using equations (11) and (33), it follows that

$$\dot{a} = la^{1-n} \quad (34)$$

and consequently the deceleration parameter turns out to be

$$q = n - 1 \quad (35)$$

which is obviously a constant. Integrating equation (34), it follows that

$$a = (nlt + k_1)^{\frac{1}{n}}, \quad n \neq 0 \quad (36)$$

and

$$a = k_2 \exp(qlt), \quad n = 0, \quad (37)$$

where $k_1$ and $k_2$ are constants of integration. Thus, we obtain two values of the average scale factor that correspond to two different models of the universe.

3.3. Singularity analysis

The Riemann tensor is useful to determine whether a singularity is essential or coordinate. If the curvature becomes infinite at a certain point, then the singularity is essential. By constructing scalars from the Riemann tensor, it can be checked whether they become infinite somewhere or not. It is obvious that infinitely many scalars can be constructed from the Riemann tensor. However, symmetry considerations can be used to show that there are only a finite number of independent scalars. All others can be expressed in terms of these. In a four-dimensional Riemann spacetime, there are only 14 independent curvature invariants. Some of these are

$$R_1 = R = g^{ab} R_{ab}, \quad R_2 = R_{ab} R^{ab}, \quad R_3 = R_{abc} R^{ab}, \quad R_4 = R_{abcd} R^{abcd}. \quad (31)$$

Here we give analysis for the first invariant commonly known as the Ricci scalar. For the Bianchi-type I spacetime, it is given by equation (7). For the special case when $m = -2$, it follows from equation (31) that

$$F = ka^{-2}. \quad (38)$$

After some manipulations, we can write

$$R_1 = -2 \left[ \frac{3k^2 (a \ddot{a} + \dot{a}^2) - (q_1 q_2 + q_2 q_3 + q_3 q_1)}{k^2 a^2} \right] \quad (39)$$

which shows that singularity occurs at $a = 0$.

3.4. Model of the universe when $n \neq 0$

Now we discuss model of the universe when $n \neq 0$, i.e., $a = (nl t + k_1)^{\frac{1}{n}}$. For this model, $F$ becomes

$$F = k(nlt + k_1)^{-\frac{1}{n}}. \quad (40)$$
Using this value of $F$ in equations (25)–(27), the metric coefficients $A$, $B$ and $C$ turn out to be

$$A = p_1(nlt + k_1)^{\frac{1}{n}} \exp \left[ \frac{q_1(nlt + k_1)^{\frac{1}{n}}}{kl(n - 1)} \right], \quad n \neq 1$$

(41)

$$B = p_2(nlt + k_1)^{\frac{1}{n}} \exp \left[ \frac{q_2(nlt + k_1)^{\frac{1}{n}}}{kl(n - 1)} \right], \quad n \neq 1$$

(42)

$$C = p_3(nlt + k_1)^{\frac{1}{n}} \exp \left[ \frac{q_3(nlt + k_1)^{\frac{1}{n}}}{kl(n - 1)} \right], \quad n \neq 1.$$  

(43)

The directional Hubble parameters $H_i (i = 1, 2, 3)$ take the form

$$H_i = \frac{l}{nlt + k_1} + \frac{q_i}{k(nlt + k_1)^{\frac{1}{n}}}.$$  

(44)

The mean generalized Hubble parameter becomes

$$H = \frac{l}{nlt + k_1}$$

(45)

while the volume scale factor turns out to be

$$V = (nlt + k_1)^{\frac{2}{n}}.$$  

(46)

Moreover, the function of Ricci scalar, $f(R)$, can be found using equation (12)

$$f(R) = \frac{k}{2}(nlt + k_1)^{-\frac{2}{n}} R + 3kl^2(nlt + k_1)^{\frac{2n-3}{n}}.$$  

(47)

It follows from equation (39) that

$$R \equiv R_1 = -2 \left[ 3l^2(2 - n)(nlt + k_1)^{-2} - \frac{(q_1q_2 + q_2q_3 + q_3q_1)}{k^2}(nlt + k_1)^{\frac{2}{n}} \right],$$

(48)

which clearly indicates that $f(R)$ cannot be explicitly written in terms of $R$. However, by inserting this value of $R$, $f(R)$ can be written as a function of $t$, which is true as $R$ depends upon $t$. For a special case when $n = \frac{1}{2}$, $f(R)$ turns out to be

$$f(R) = \frac{k}{2} \left[ \frac{9l^2 \pm \sqrt{81l^4 + \frac{8(q_1q_2 + q_2q_3 + q_3q_1)}{k^2}R}}{2R} \right]^{-2} R$$

$$= \frac{3kl^2}{2} \left[ \frac{9l^2 \pm \sqrt{81l^4 + \frac{8(q_1q_2 + q_2q_3 + q_3q_1)}{k^2}R}}{2R} \right]^{-3}. $$

(49)

This gives $f(R)$ only as a function of $R$.

### 3.5. Model of the universe when $n = 0$

The average scale factor for this model of the universe is $a = k^2 \exp(lt)$ and hence $F$ takes the form

$$F = \frac{k}{k^2} \exp(-2lt).$$

(50)

Inserting this value of $F$ in equations (25)–(27), the metric coefficients $A$, $B$ and $C$ become

$$A = p_1k^2 \exp(lt) \exp \left[ -\frac{q_1 \exp(-lt)}{klk^2} \right].$$

(51)
\[ B = p_1 k_2 \exp(\ell t) \exp\left[ -\frac{q_1 \exp(-\ell t)}{k \ell k_2} \right], \] (52)

\[ C = p_2 k_2 \exp(\ell t) \exp\left[ -\frac{q_2 \exp(-\ell t)}{k \ell k_2} \right]. \] (53)

The directional Hubble parameters \( H_i \) and the mean generalized Hubble parameter will become

\[ H_i = l + \frac{q_i k}{k k_2} \exp(-\ell t), \quad H = l. \] (54)

The volume scale factor turns out to be

\[ V = k_2 \exp(3lt), \] (55)

while \( f(R) \) takes the form

\[ f(R) = \frac{k}{2k_2} \exp(-2lt)(R - 6l^2). \] (56)

For this model, \( R \) becomes

\[ R = -2 \left[ 6l^2 - \frac{(q_1 q_2 + q_2 q_3 + q_3 q_1)}{k^2 k_2^2 \exp(2lt)} \right]. \] (57)

Here we can get the general function \( f(R) \) in terms of \( R \)

\[ f(R) = \frac{k^3}{2(q_1 q_2 + q_2 q_3 + q_3 q_1)} \left[R^2 + 6l^2 R - 72l^4\right] \] (58)

which corresponds to the general function \( f(R) \) [33],

\[ f(R) = \sum a_n R^n, \] (59)

where \( n \) may take the values from negative or positive.

### 4. Exact Bianchi-type V solutions

Here we shall find exact solutions of the Bianchi-type V spacetime in \( f(R) \) gravity for the vacuum field equations.

#### 4.1. Bianchi-type V spacetime

The metric for the Bianchi-type V spacetime is

\[ ds^2 = dt^2 - A^2(t) \, dx^2 - e^{2m} \left[B^2(t) \, dy^2 + C^2(t) \, dz^2 \right], \] (60)

where \( A, B \) and \( C \) are cosmic scale factors and \( m \) is an arbitrary constant. The corresponding Ricci scalar is

\[ R = -2 \left[ \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} - \frac{3m^2}{A^2} + \frac{AB}{AB} + \frac{BC}{BC} + \frac{CA}{CA} \right]. \] (61)

With the help of equation (14), we can write \( A_0 - A_1 = 0, A_0 - A_2 = 0 \) and \( A_0 - A_3 = 0 \) respectively as

\[ -\frac{\dot{B}}{B} - \frac{\dot{C}}{C} - \frac{2m^2}{A^2} + \frac{AB}{AB} + \frac{CA}{CA} + \frac{AF}{AF} - \frac{F}{F} = 0, \] (62)

\[ -\frac{\dot{A}}{A} - \frac{\dot{C}}{C} - \frac{2m^2}{A^2} + \frac{AB}{AB} + \frac{BC}{BC} + \frac{BF}{BF} - \frac{F}{F} = 0. \] (63)
\[-\frac{A}{A} - \frac{B}{B} - \frac{2m^2}{A^2} + \frac{BC}{BC} + \frac{CA}{CA} + \frac{CF}{CF} - \frac{F}{F} = 0. \quad (64)\]

The 01-component can be written by using equation (2) in the following form:

\[\frac{2A}{A} - \frac{B}{B} - \frac{C}{C} = 0. \quad (65)\]

We solve these equations using the same procedure as for the Bianchi-type I solutions.

### 4.2. Solution of the field equations

Here we get the same equations (18)–(20) as obtained previously with the difference of the constraint equations (using equation (65))

\[p_1 = 1, \quad p_2 = p_3^{-1} = P, \quad q_1 = 0, \quad q_2 = -q_3 = Q. \quad (66)\]

Consequently, the metric functions become

\[A = a, \quad B = aP \exp \left[ Q \int \frac{dt}{a^3F} \right], \quad C = aP^{-1} \exp \left[ -Q \int \frac{dt}{a^3F} \right]. \quad (67)\]

Moreover, \(R_1\) turns out to be

\[R_1 = -2 \left[ \frac{3k^2(\dot{a}a + \dot{a}^2) + Q^2 - 3m^2k^2}{k^2a^2} \right] \quad (68)\]

which also yields a singularity at \(a = 0\).

### 4.3. Model of the universe when \(n \neq 0\)

Using the value of \(F\) from equation (40) in (67), the metric coefficients \(A, B\) and \(C\) become

\[A = (nlt + k_1)^\frac{1}{3}, \quad (69)\]

\[B = P(nlt + k_1)^\frac{1}{3} \exp \left[ \frac{Q(nlt + k_1)^\frac{2}{3}}{k(n - 1)} \right], \quad n \neq 1 \quad (70)\]

\[C = P^{-1}(nlt + k_1)^\frac{1}{3} \exp \left[ -\frac{Q(nlt + k_1)^\frac{2}{3}}{k(n - 1)} \right], \quad n \neq 1. \quad (71)\]

The directional Hubble parameters \(H_1, H_2\) and \(H_3\) take the following form:

\[H_1 = \frac{l}{nlt + k_1}, \quad (72)\]

\[H_2 = \frac{l}{nlt + k_1} + \frac{Q}{k(nlt + k_1)^\frac{1}{2}}, \quad (73)\]

\[H_3 = \frac{l}{nlt + k_1} - \frac{Q}{k(nlt + k_1)^\frac{1}{2}}. \quad (74)\]

Note that the mean generalized Hubble parameter \(H\), the volume scale factor \(V\) and \(f(R)\) turn out to be the same as for the Bianchi-type I spacetime.
4.4. Model of the universe when $n = 0$

For this model, the metric coefficients $A$, $B$ and $C$ turn out to be

$$A = k_2 \exp(lt),$$  \hspace{1cm} (75)$$

$$B = Pk_2 \exp(lt) \exp\left[-\frac{Q \exp(-lt)}{klk_2}\right],$$  \hspace{1cm} (76)$$

$$C = P^{-1}k_2 \exp(lt) \exp\left[-\frac{Q \exp(-lt)}{klk_2}\right].$$  \hspace{1cm} (77)$$

The directional Hubble parameters $H_1$, $H_2$ and $H_3$ take the form

$$H_1 = l, \hspace{1cm} H_2 = l + \frac{Q \exp(-lt)}{kk_2}, \hspace{1cm} H_3 = l - \frac{Q \exp(-lt)}{kk_2}.$$  \hspace{1cm} (78)$$

Here we also have the same mean generalized Hubble parameter $H$, volume scale factor $V$ and $f(R)$ as given in equations (54)–(56).

5. Summary and conclusion

The main purpose of this paper is to discuss the well-known phenomenon of the universe expansion in the context of $f(R)$ gravity. For this purpose, we have investigated exact solutions of the Bianchi-type I and V spacetimes using the vacuum field equations. We have found two exact solutions for both spacetimes using the variation law of the Hubble parameter. This yields a constant value of the deceleration parameter. These solutions correspond to two models of the universe. The first solution gives a singular model with power-law expansion and positive deceleration parameter, while the second solution provides a non-singular model with exponential expansion and negative deceleration parameter. It is mentioned here that the solutions for both spacetimes correspond to perfect fluid solutions [26, 27] in GR. We have also evaluated function of the Ricci scalar, $f(R)$, for both models in each case. In particular, the general function $f(R)$ includes the squared power of the Ricci scalar for the non-singular model. The physical behavior of these models is given below.

First we discuss the singular model of the universe. This model corresponds to $n \neq 0$ with average scale factor $a = (nlt + k_1)^\frac{1}{n}$. It has a point singularity at $t = t_\text{s} = -\frac{k_1}{nl}$. The physical parameters $H_1$, $H_2$, $H_3$ and $H$ are all infinite at this point but the volume scale factor vanishes. The function of the Ricci scalar, $f(R)$, is also infinite while the metric functions $A$, $B$ and $C$ vanish at this point of the singularity. Thus, we can conclude from these observations that the model starts its expansion with zero volume at $t = t_\text{s}$ and it continues to expand for $0 < n < 1$.

The non-singular model of the universe corresponds to $n = 0$ with average scale factor $a = k_2 \exp(lt)$. It is non-singular because an exponential function is never zero and hence there does not exist any physical singularity for this model. The physical parameters $H_1$, $H_2$, $H_3$ are all finite for all finite values of $t$. The mean generalized Hubble parameter $H$ is constant while $f(R)$ is also finite here. The metric functions $A$, $B$ and $C$ do not vanish for this model. The volume scale factor increases exponentially with time which indicates that the universe starts its expansion with zero volume from an infinite past.

It would be worthwhile to study other Bianchi-type spacetimes, especially by removing the vacuum condition. It would also be interesting to explore Bianchi-type I and V solutions with a perfect fluid. These are under progress.
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