Renormalizability of Topologically Massive Gravity

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Abstract

We consider renormalizability of topologically massive gravity in three space-time dimensions. With a usual parametrization of the metric tensor, we establish the statement that topologically massive gravity is in fact renormalizable. In this proof, we make use of not only a recently found, new infrared regularization method of scalar mode but also a covariant ultraviolet regulator with a specific combination of higher derivative terms which is motivated by the new massive gravity in three dimensions.

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1 Introduction

In recent years there has been a revival of interest in three-dimensional quantum gravity mainly owing to the paper by Witten [1] and that by Li, Song and Strominger [2]. Both the papers deal with not only the AdS/CFT correspondence in three dimensions but also the problem of counting the microscopic degrees of freedom associated with BTZ black holes by utilizing certain holographic dual theories at a boundary which are two-dimensional. Even if three-dimensional quantum gravity is considered in these papers, Witten investigates topological AdS gravity whereas Li et al. studies, what is called, "topologically massive gravity (TMG)”. In this article, we shall focus on the problem of renormalizability of the latter three-dimensional gravity.

Topologically massive gravity (TMG) [3, 4] in three space-time dimensions is described by the action consisted of the Einstein-Hilbert action with wrong sign and gravitational Chern-Simons term which is parity-violating and includes three derivatives, and has one dynamical degree of freedom corresponding to massive graviton of +2 or −2 helicity mode depending on the sign of the overall constant in front of gravitational Chern-Simons term. It is remarkable that despite the presence of three derivatives in gravitational Chern-Simons term, there are neither ghosts nor acausalities in TMG.

Moreover, it is expected that the higher derivative term in TMG would dominate the behavior of the theory at high energies, leading to a stabilization of the ultraviolet divergence and consequently to power-counting renormalizability. Actually, the problem of renormalizability of TMG has been discussed before, but we have not yet had a firm grasp of it since we have no gauge-invariant regularization method in such a way to preserve the desirable power-counting behavior. This absence of the useful ultraviolet regulator is of course related to the fact that TMG includes the Levi-Civita tensor density $\varepsilon^{\mu\nu\rho}$ in the classical action so that we cannot make use of dimensional regularization. Furthermore, it turns out that gauge-invariant, higher derivative regulators spoil the argument of formal renormalizability, non-covariant cutoffs cannot be easily analyzed, and non-local regularization method involves some assumption to be proved [5, 6].

Recently, in three space-time dimensions there has been an interesting progress for obtaining a sensible interacting massive gravity theory [7, 8]. This model has been shown to be equivalent to the Pauli-Fierz massive gravity [18] at the linearized approximation level and thus massive modes of helicities $\pm 2$ are physical propagating ones. A key idea in this model is that one adds a specific combination of higher derivative curvature terms to the Einstein-Hilbert action with the wrong sign in such a way that the trace part of the stress-energy tensor associated with those higher derivative terms is proportional to the original higher derivative Lagrangian. With this idea, it turns out that the scalar mode coming from higher derivative Lagrangian is precisely cancelled out [19] and consequently we have a conformal invariance.

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2See the references [9, 10, 11, 12, 13, 14, 15, 16, 17] for alternative massive gravity models.

3Recall that the massive graviton in the Pauli-Fierz theory possesses $\frac{(D+1)(D-2)}{2}$ propagating modes in a general $D$ dimension.
at least in the higher derivative sector (Of course, the Einstein-Hilbert action breaks the conformal invariance). Afterwards, this new massive gravity model in three dimensions has been studied from various viewpoints such as the unitarity and the impossibility of generalization to higher dimensions [20], relation to the Pauli-Fierz mass term [19], black hole solutions [21, 22], the properties of linearized gravitational excitations in asymptotically AdS space-time [23, 24, 25], AdS waves [26], the \(z=4\) Horava-Lifshitz Gravity [27], the no-go theorem [28] and the effects of torsion [29].

More recently, we have presented a proof that the new massive gravity theory in three dimensions [7, 8] is certainly renormalizable [30]. In this proof, we have used a particular BRST-invariant infrared regularization procedure for the scalar mode [30]. This new regularization method is composed of two steps. The first step is to add the conventional Pauli-Fierz mass term in the new massive theory and then to make the Pauli-Fierz mass term BRST-invariant by applying the Stueckelberg formalism. The next step is to take the massless limit after renormalization procedure. Incidentally, the procedure of making the BRST-invariant Pauli-Fierz mass term was previously considered by Hamamoto [31], in which the motivation was to construct a massive tensor theory with a smooth massless limit.

It is then natural to ask ourselves if this infrared regularization method could be also applied to topologically massive gravity (TMG) in three dimensions or not. The purpose of this article is to show that the answer to this question is affirmative. However, in the process we will soon realize that compared with the new massive gravity [7, 8], we encounter a new difficulty which amounts to the impossibility of making use of dimensional regularization in TMG since there is the Levi-Civita tensor density \(\varepsilon^{\mu\nu\rho}\) in gravitational Chern-Simons term. Note that this difficulty is related to the problem what regularization method for the ultraviolet divergence we should adopt. In this article, we shall adopt a diffeomorphism-invariant, higher derivative regularization method. A peculiar feature of this regulator is that the combination of the higher derivative terms shares the same structure as that of the new massive gravity, by which we have no scalar mode in graviton propagator obtained by the regulator terms.

In the next section, we briefly review on topologically massive gravity. In the third section, we explain how to construct the BRST-invariant Pauli-Fierz massive gravity via the Stueckelberg formalism. In the fourth section, we derive the propagator of the gravitational field on the basis of the gauge fixed, BRST-invariant action obtained in the section 3. In the fifth section, we calculate the superficial degree of divergence and present the ultraviolet regulator. The final section is devoted to conclusion and discussions.

2 Brief review of topologically massive gravity

We start with brief review of topologically massive gravity in three space-time dimensions [3, 4]. The action takes the form

\[
S_c \equiv \int d^3 x \mathcal{L}_c
\]
\[
\int d^3x \left[ -\frac{1}{\kappa^2} \sqrt{-g} R + \frac{1}{2\kappa^2 \mu} \varepsilon^{\lambda \mu \nu} \Gamma^\rho_{\lambda \sigma} \left( \partial_{\mu} \Gamma^\sigma_{\rho \nu} + \frac{2}{3} \Gamma^\sigma_{\mu \tau} \Gamma^\tau_{\nu \rho} \right) \right],
\]

where \( \kappa^2 \equiv 16\pi G \) (\( G \) is the 3-dimensional Newton’s constant) and \( \mu \) is a constant of mass dimension. Let us note that \( \kappa \) has dimension of \((\text{mass})^{-\frac{1}{2}}\), so the theory defined by the action (1) might at first sight appear to be unrenormalizable, but it turns out to be an illusion. The space-time indices \( \mu, \nu, \cdots \) run over 0, 1, 2, we take the metric signature \((-+,+,+)\), and follow the notation and conventions of the textbook of MTW [32]. Finally, the Levi-Civita tensor density is defined as \( \varepsilon^{012} = 1 \).

Variation of the classical action (1) yields the equations of motion

\[
G^\mu\nu - \frac{1}{\mu} C^\mu\nu = 0,
\]

where \( C^\mu\nu \) is the Cotton conformal tensor density defined by

\[
C^\mu\nu = \frac{1}{2\sqrt{-g}} (\varepsilon^\mu\alpha\beta \nabla_\alpha R^\nu_\beta + \varepsilon^\nu\alpha\beta \nabla_\alpha R^\mu_\beta - \frac{1}{4} \delta^\nu_\beta \delta^\mu_\alpha R).
\]

In deriving the second equality, we have used an identity

\[
-\varepsilon^\mu\alpha\beta \nabla_\alpha R^\nu_\beta + \varepsilon^\nu\alpha\beta \nabla_\alpha R^\mu_\beta = \frac{1}{2} \varepsilon^\mu\alpha \nabla_\alpha R,
\]

which is obtained by using both the Bianchi identity

\[
\nabla_{[\mu} R_{\nu \rho] \sigma} = 0,
\]

and the relation holding only in three dimensions

\[
R^\rho_{\mu \nu \sigma} = 4\delta^\rho_{[\mu} R_{\nu \sigma]} - \delta^\rho_{\nu} \delta^\sigma_{\mu} R,
\]

where the square bracket denotes the antisymmetrization of indices with a numerical weight. Note that the Cotton tensor density has the properties \( C^\mu_{\mu} = 0, C^\mu\nu = C^\nu\mu \), and \( \nabla_\mu C^\mu\nu = 0 \).

It turns out that the linearized equations of motion are those of a massive scalar field

\[
(\Box + \mu^2) \phi = 0,
\]

where

\[
\phi = (\delta_{ij} + \hat{\partial}_i \hat{\partial}_j) h^{ij},
\]

with the definitions of \( \Box = \eta_{\mu \nu} \partial_\mu \partial_\nu, \hat{\partial}_i = \frac{\partial_i}{\sqrt{-\nabla^2}} \) and \( i, j, \cdots = 1, 2 \). The existence of such a massive mode is also confirmed by examining the effective force between two external gravitational sources and finding the Yukawa-type interaction. Here it should be emphasized
that despite the Klein-Gordon equation (7) the graviton carries spin either +2 or −2 (parity-violating because of the presence of the Levi-Civita tensor density) and the spin sign is correlated with that of the coefficient $\mu$ in front of gravitational Chern-Simons term.

One important remaining problem in TMG is to prove that this theory is perturbatively renormalizable. If so, TMG would give us a scarce example of consistent quantum gravity theories in lower dimensions and bring about a better understanding of how to construct a satisfactory quantum gravity in four dimensions in future. Indeed, this problem has been attacked by two groups [5, 6], but still unsolved completely. The reason why the problem is difficult is quite simple. Although there is an improvement in the ultraviolet behavior owing to gravitational Chern-Simons term with three derivatives and TMG consequently seems to be manifestly power-counting renormalizable, there is a nasty problem associated with the conformal (scalar) mode for which the propagator falls off like $\frac{1}{p^2}$. The origin of the scalar mode lies in the Einstein-Hilbert action since gravitational Chern-Simons term is in itself conformal invariant so that this topological term does not affect the behavior of the scalar mode. Thus, the crucial point for proving the renormalizability of TMG is to find a suitable regularization method for the scalar mode.

In a recent article [30], we have solved this problem in the new massive gravity in three dimensions [7, 8]. The key observation in the solution is to add the BRST-invariant Pauli-Fierz mass term in a theory and use it as an infrared regulator of the scalar mode, and finally take the massless limit after renormalization procedure. In this article, we will apply this regularization method to TMG and see that this method is also effective for the proof of renormalizability of TMG. For that, in the next section, we shall explain the new regularization method in detail.

3 BRST-invariant mass term

Now let us consider the BRST transformations for diffeomorphisms. The BRST transformations of the metric tensor, ghost, antighost and Nakanishi-Lautrup auxiliary field are respectively given by

$$
\begin{align*}
\delta_B g_{\mu\nu} &= -\kappa^3(\nabla_\mu c_\nu + \nabla_\nu c_\mu), \\
\delta_B c^\mu &= -\kappa^3 c^\nu \partial_\nu c^\mu, \\
\delta_B \bar{c}_\mu &= ib_\mu, \\
\delta_B b_\mu &= 0,
\end{align*}
$$

(9)

where the covariant derivative is defined as usual by $\nabla_\mu c_\nu = \partial_\mu c_\nu - \Gamma^\lambda_{\mu\nu} c_\lambda$ with the affine connection $\Gamma^\lambda_{\mu\nu}$. It is straightforward to prove that the BRST transformations (9) are off-shell nilpotent.

Next, we expand the metric around a flat Minkowski background $\eta_{\mu\nu} = \text{diag}(-1, 1, 1)$ as usual

$$
g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}.
$$

(10)
With this definition, the gravitational field $h_{\mu\nu}$ has canonical dimension of $(mass)^{\frac{3}{2}}$. As a result, the Einstein-Hilbert action and the graviton-matter interaction terms have canonical dimensions greater than three, which is the origin of unrenormalizability of Einstein’s general relativity without the higher derivative terms. Later, we will see that the gravitational field $h_{\mu\nu}$ has ultraviolet dimension of $(mass)^0$, whose fact leads to renormalizability of topologically massive gravity under consideration. The BRST transformation of the metric in (9) gives rise to that of the gravitational field $h_{\mu\nu}$ at the linearized level

$$\delta_B h_{\mu\nu} = -\kappa^2 (\partial_\mu c_\nu + \partial_\nu c_\mu),$$

which is nilpotent up to the order $O(\kappa^5)$.

Then, we wish to construct a BRST-invariant gravitational field $h^\prime_{\mu\nu}$ at the linearized level through the Steuckelberg formalism. To do that, we introduce the Steuckelberg vector field $A_\mu$ and define $h^\prime_{\mu\nu}$ as

$$h^\prime_{\mu\nu} = h_{\mu\nu} - \frac{1}{m} (\partial_\mu A_\nu + \partial_\nu A_\mu),$$

and assume the BRST transformation of $A_\mu$ to be

$$\delta_B A_\mu = -\kappa^2 m c_\mu.$$  \hspace{1cm} (13)

Then it is obvious that the $h^\prime_{\mu\nu}$ field is BRST-invariant $\delta_B h^\prime_{\mu\nu} = 0$.

Using this BRST-invariant gravitational field $h^\prime_{\mu\nu}$, let us construct a BRST-invariant Lagrangian for the Pauli-Fierz mass term [18] by

$$\mathcal{L}^h = -\frac{m^2}{4} (h^\prime_{\mu\nu} h^\prime_{\mu\nu} - h^2),$$

where $m$ is a constant of mass dimension. Moreover, we have raised indices by the flat Minkowski metric $\eta^{\mu\nu}$ like $h^\mu_{\nu\omega} = \eta^{\mu\rho} \eta^{\nu\sigma} h^\rho_{\sigma\omega}$ and $h^\prime = \eta^\mu_{\nu\rho} h^\prime_{\mu\nu}$. When expanded in terms of the definition (12), this Lagrangian reads

$$\mathcal{L}^h = -\frac{m^2}{4} (h_{\mu\nu} h^\mu_{\nu\omega} - h^2) - \frac{1}{4} F^2_{\mu\nu} - m (\partial^\mu h_{\mu\nu} - \partial_\nu h) A^\mu,$$  \hspace{1cm} (15)

where we have defined $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

At this stage, we encounter a problem, which is the existence of the ghost $A^0$. In order to kill this negative norm mode, we need to have an extra gauge symmetry. To find it, in particular, let us notice that the last term in Eq. (15) is not invariant under the usual gauge transformation $\delta A_\mu = \partial_\mu \lambda$. Therefore, to remedy this term, we appeal to the Steuckelberg formalism again. If we introduce the Steuckelberg scalar field $\varphi$ via

$$A^\prime_\mu = A_\mu - \frac{1}{m} \partial_\mu \varphi,$$  \hspace{1cm} (16)

4\delta^2_B A_\mu = 0 up to the order $O(\kappa^5)$.
the BRST transformations corresponding to new gauge symmetry are determined such that $\delta_B A'_\mu = 0$ by

$$
\begin{align*}
\delta_B A_\mu &= \kappa^2 \partial_\mu c, \\
\delta_B \varphi &= \kappa^2 mc,
\end{align*}
$$

(17)

where $c$ is the new scalar ghost.

As a consequence of the definitions (12) and (16), and the BRST transformations (11), (13) and (17), we are eventually led to defining a new BRST-invariant gravitational field $\bar{h}_{\mu\nu}$ by

$$
\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{m}(\partial_\mu A_\nu + \partial_\nu A_\mu) + \frac{2}{m^2} \partial_\mu \partial_\nu \varphi,
$$

(18)

and the full BRST transformations at the linearized level by

$$
\begin{align*}
\delta_B h_{\mu\nu} &= -\kappa^2 (\partial_\mu c_\nu + \partial_\nu c_\mu), \\
\delta_B A_\mu &= \kappa^2 (-mc_\mu + \partial_\mu c), \\
\delta_B \varphi &= \kappa^2 mc, \\
\delta_B \bar{c} &= i\beta, \quad \delta_B b = \delta_B c = 0,
\end{align*}
$$

(19)

where the BRST transformations of antighost, Nakanishi-Lautrup field and ghost are also added. Hence, the new BRST-invariant mass term is made out of the BRST-invariant $\bar{h}_{\mu\nu}$ field as

$$
\mathcal{L}_m^h = -\frac{m^2}{4}(\bar{h}_{\mu\nu}\bar{h}^{\mu\nu} - \bar{h}^2) = -\frac{m^2}{4}(h_{\mu\nu}h^{\mu\nu} - h^2) - \frac{1}{4} F_{\mu\nu}^2 - (mA_\mu - \partial_\mu \varphi)(\partial^\nu h_{\mu\nu} - \partial_\nu h). 
$$

(20)

Next, we move on to fixing gauge symmetries. For diffeomorphisms and the new scalar gauge symmetry respectively, we set up the de Donder gauge and extended Lorentz gauge conditions

$$
\begin{align*}
\partial_\mu \tilde{g}^{\mu\nu} &\equiv \partial_\mu (\sqrt{-g}g^{\mu\nu}) = 0, \\
\Box (\partial_\mu A_\mu - \frac{m}{2} h) &= 0.
\end{align*}
$$

(21)

The reason why we have selected a higher derivative gauge condition for the new scalar gauge symmetry will be clarified when we discuss the graviton propagator in the next section.

The Lagrangian corresponding to the gauge fixing plus FP ghost terms is given by the following BRST-exact term:

$$
\mathcal{L}_{GF+FP} = i\delta_B \left[ \frac{1}{\kappa^3} \tilde{c}_\nu \partial_\mu \tilde{g}^{\mu\nu} - \frac{\alpha}{2\kappa} \eta^{\mu\nu} b_\mu \right] + \kappa^2 \bar{c} \Box (\partial_\mu A_\mu - \frac{m}{2} h - \frac{\beta}{2\kappa^2} b)
$$

where $c$ is the new scalar ghost.
equality, we have performed path integration over the Nakanisghi-Lautrup fields \( \alpha, \beta \)
where \( \varphi \) not relevant at least to the argument of renormalizability. However, in particular, if someone integrate over this part. Indeed, such the terms which do not contain interaction terms are BRST-invariant action of the classical action (1) where the gauge condition for diffeomorphisms is the de Donder gauge.

To close this section, we should make a comment on the massless limit \( m \rightarrow 0 \) of the action (23). Note that the action (23) has a well-defined massless limit and reduces to the form

\[
S_{m=0} = \int d^3x \left[ -\frac{1}{\kappa^2} \sqrt{-g} R + \frac{1}{2\kappa^2 \mu^2} \varepsilon^{\lambda \mu \nu} \Gamma^\varphi_\lambda \sigma (\partial_\mu \Gamma^\varphi_\rho \sigma + \frac{2}{3} \Gamma^\varphi_\rho \sigma \Gamma^\rho_\varphi_\mu \nu) - \frac{1}{2\alpha \kappa^2} (\partial_\mu \tilde{g}^{\mu \nu})^2 + i \partial_\mu \tilde{c}_\nu D^{\mu \nu} c^\rho - \frac{1}{4} F_{\mu \nu}^2 - \frac{1}{4} \kappa^4 (\partial_\mu A^\mu - \frac{m}{2} h) \Box (\partial_\nu A^\nu - \frac{m}{2} h) \right] + \partial_\mu \varphi (\partial^\mu h_{\mu \nu} - \partial_\nu h) - \frac{1}{4} F_{\mu \nu}^2 - \frac{1}{4} \kappa^4 \partial_\mu A^\mu \Box \partial_\nu A^\nu - i \kappa^4 \Box^2 c].
\]

This reduced action (24) consists of two parts. The first part is nothing but the gauge fixed, BRST-invariant action of the classical action (1) where the gauge condition for diffeomorphisms is the de Donder gauge.

The second one is a free action made out of only quadratic terms, so that we can simply integrate over this part. Indeed, such the terms which do not contain interaction terms are not relevant at least to the argument of renormalizability. However, in particular, if someone worries the presence of the mixing term between \( \varphi \) and \( h_{\mu \nu} \) in (24), it is easy to show that this term can be absorbed into the redefinition of the Nakanishi-Lautrup auxiliary field \( b_\mu \) at the linearized level if we take slightly modified gauge conditions \( \partial_\mu (gg^{\mu \nu}) = 0 \), which also turns out to lead to the desired property of the graviton propagator so that this gauge choice is also admissible. Actually, at the linearized order, one has

\[
\partial_\mu (gg^{\mu \nu}) = \kappa (\partial_\mu h^{\mu \nu} - \partial^\nu h) + \cdots,
\]

where \( \alpha, \beta \) are gauge parameters and \( D^{\mu \nu}_\rho = \tilde{g}^{\mu \nu} \delta^\rho_\sigma \partial_\sigma + \tilde{g}^{\mu \nu} \delta^\rho_\sigma \partial_\sigma - \tilde{g}^{\mu \nu} \partial_\rho - (\partial_\rho \tilde{g}^{\mu \nu}) \). In the last equality, we have performed path integration over the Nakanishi-Lautrup fields \( b_\mu \) and \( b \).

In this way, we arrive at the gauge fixed, BRST-invariant action

\[
S \equiv \int d^3x L
\equiv \int d^3x (L_c + L_m + L_{GF+FP})
= \int d^3x \left[ -\frac{1}{\kappa^2} \sqrt{-g} R + \frac{1}{2\kappa^2 \mu^2} \varepsilon^{\lambda \mu \nu} \Gamma^\varphi_\lambda \sigma (\partial_\mu \Gamma^\varphi_\rho \sigma + \frac{2}{3} \Gamma^\varphi_\rho \sigma \Gamma^\rho_\varphi_\mu \nu) - \frac{m^2}{4} (h_{\mu \nu} h^{\mu \nu} - h^2) - \frac{1}{4} F_{\mu \nu}^2 - (m A^\mu - \partial^\mu \varphi)(\partial^\nu h_{\mu \nu} - \partial_\nu h) - \frac{1}{2\alpha \kappa^2} (\partial_\mu \tilde{g}^{\mu \nu})^2 + i \partial_\mu \tilde{c}_\nu D^{\mu \nu} c^\rho - \frac{1}{4} \kappa^4 (\partial_\mu A^\mu - \frac{m}{2} h) \Box (\partial_\nu A^\nu - \frac{m}{2} h) \right].
\]
where dots imply the higher-order terms in \( \kappa \). With the Landau gauge \( \alpha = 0 \), the gauge fixing and FP ghost Lagrangian for only the gauge condition \( \partial_\mu (gg^{\mu \nu}) = 0 \) takes the form at the lowest level

\[
\mathcal{L}_{GF+FP}' = i\delta_B \left[ \frac{1}{\kappa^3} \bar{c}_\nu \partial_\mu (gg^{\mu \nu}) \right] = -\frac{1}{\kappa^2} b^\mu (\partial^\nu h_{\mu \nu} - \partial_\mu h) + i\bar{c}_\mu (\Box c^\mu - \partial^\mu \partial_\nu c^\nu).
\]

(26)

Thus, at the massless limit, up to irrelevant terms for the present argument the Lagrangian reads

\[
\mathcal{L}_{m=0}' = -\frac{1}{\kappa^2} (b^\mu - \kappa^2 \partial^\mu \varphi) (\partial^\nu h_{\mu \nu} - \partial_\mu h) + i\bar{c}_\mu (\Box c^\mu - \partial^\mu \partial_\nu c^\nu) \\
\rightarrow -\frac{1}{\kappa^2} b^\mu (\partial^\nu h_{\mu \nu} - \partial_\mu h) + i\bar{c}_\mu (\Box c^\mu - \partial^\mu \partial_\nu c^\nu),
\]

(27)

where we have redefined \( b^\mu - \kappa^2 \partial^\mu \varphi \rightarrow b^\mu \) at the linearized level. In this way, we can nullify the mixing term between \( \varphi \) and \( h_{\mu \nu} \). Let us recall that the theory at hand is independent of the choice of the gauge condition as well as the gauge parameter, so the mixing term can be ignored safely in the present argument.

Finally, of course, this massless limit must be taken after the whole renormalization procedure is completed. In this sense, the physical content in the present formalism is the same as that of TMG in the massless limit although the BRST transformation of the mass term is nilpotent only approximately.

### 4 Graviton propagator

On the basis of the gauge fixed, BRST-invariant action (23), we wish to derive the propagator of the gravitational field \( h_{\mu \nu} \) and check that the propagator really has desired features. To this end, it is useful to take account of the spin projection operators in 3 space-time dimensions [33, 20]. A set of the spin operators \( P^{(2)} \), \( P^{(1)} \), \( P^{(0,s)} \), \( P^{(0,w)} \), \( P^{(0,sw)} \) and \( P^{(0,ws)} \) form a complete set in the space of second rank symmetric tensors and are defined as

\[
P^{(2)}_{\mu \nu, \rho \sigma} = \frac{1}{2} (\theta_{\mu \rho} \theta_{\nu \sigma} + \theta_{\mu \sigma} \theta_{\nu \rho}) - \frac{1}{2} \theta_{\mu \nu} \theta_{\rho \sigma},
\]

\[
P^{(1)}_{\mu \nu, \rho \sigma} = \frac{1}{2} (\theta_{\mu \rho} \omega_{\nu \sigma} + \theta_{\mu \sigma} \omega_{\nu \rho} + \theta_{\nu \rho} \omega_{\mu \sigma} + \theta_{\nu \sigma} \omega_{\mu \rho}),
\]

\[
P^{(0,s)}_{\mu \nu, \rho \sigma} = \frac{1}{2} \theta_{\mu \nu} \theta_{\rho \sigma}.
\]

\[\text{It is true that there are also off-diagonal propagators between gravitons and the other fields. However, since there are no interaction terms, it is easy to see that they do not make any contribution to the effective action, so we can safely neglect them in the argument of renormalizability. Thus we here consider only the graviton propagator.}\]
\[ P^{(0,w)}_{\mu\nu,\rho\sigma} = \omega_{\mu\nu} \omega_{\rho\sigma}, \]
\[ P^{(0,sw)}_{\mu\nu,\rho\sigma} = \frac{1}{\sqrt{2}} \theta_{\mu\nu} \omega_{\rho\sigma}, \]
\[ P^{(0,ws)}_{\mu\nu,\rho\sigma} = \frac{1}{\sqrt{2}} \omega_{\mu\nu} \theta_{\rho\sigma}. \] (28)

Here the transverse operator \( \theta_{\mu\nu} \) and the longitudinal operator \( \omega_{\mu\nu} \) are defined as

\[ \theta_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{\Box} \partial_{\mu} \partial_{\nu} = \eta_{\mu\nu} - \omega_{\mu\nu}, \]
\[ \omega_{\mu\nu} = \frac{1}{\Box} \partial_{\mu} \partial_{\nu}. \] (29)

It is straightforward to show that the spin projection operators satisfy the orthogonality relations

\[ P^{(i,a)}_{\mu\nu,\rho\sigma} P^{(j,b)}_{\rho\sigma,\lambda\tau} = \delta^{ij} \delta^{ab} P^{(i,a)}_{\mu\nu,\lambda\tau}, \]
\[ P^{(i,ab)}_{\mu\nu,\rho\sigma} P^{(j,c)}_{\rho\sigma,\lambda\tau} = \delta^{ij} \delta^{bc} P^{(i,ac)}_{\mu\nu,\lambda\tau}, \]
\[ P^{(i,a)}_{\mu\nu,\rho\sigma} P^{(j,ac)}_{\rho\sigma,\lambda\tau} = \delta^{ij} \delta^{bc} P^{(i,ac)}_{\mu\nu,\lambda\tau}, \]
\[ P^{(i,ab)}_{\mu\nu,\rho\sigma} P^{(j,c)}_{\rho\sigma,\lambda\tau} = \delta^{ij} \delta^{bc} P^{(i,ac)}_{\mu\nu,\lambda\tau}. \] (30)

with \( i, j = 0, 1, 2 \) and \( a, b, c, d = s, w \) and the tensorial relation

\[ [P^{(2)} + P^{(1)} + P^{(0,s)} + P^{(0,w)}]_{\mu\nu,\rho\sigma} = \frac{1}{2} (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho}). \] (31)

In order to accommodate with gravitational Chern-Simons term, one has to add two operators to the whole spin projection operators \([34, 19]\)

\[ S_{1\mu\nu,\rho\sigma} = \frac{1}{4} \Box (\epsilon_{\mu\rho\lambda} \partial_{\sigma} \omega^{\lambda}_{\nu} + \epsilon_{\mu\sigma\lambda} \partial_{\rho} \omega^{\lambda}_{\nu} + \epsilon_{\nu\rho\lambda} \partial_{\sigma} \omega^{\lambda}_{\mu} + \epsilon_{\nu\sigma\lambda} \partial_{\rho} \omega^{\lambda}_{\mu}), \]
\[ S_{2\mu\nu,\rho\sigma} = -\frac{1}{4} \Box (\epsilon_{\mu\rho\lambda} \eta_{\sigma\nu} + \epsilon_{\mu\sigma\lambda} \eta_{\rho\nu} + \epsilon_{\nu\rho\lambda} \eta_{\sigma\mu} + \epsilon_{\nu\sigma\lambda} \eta_{\rho\mu}) \partial^{\lambda}. \] (32)

These operators together with the spin projection operators satisfy the following relations

\[ S_{1} S_{1} = \frac{1}{4} \Box^{3} P^{(1)}, \]
\[ S_{1} S_{2} = S_{2} S_{1} = -\frac{1}{4} \Box^{3} P^{(1)}, \]
\[ S_{2} S_{2} = \Box^{3} (P^{(2)} + \frac{1}{4} P^{(1)}), \]
\[ P^{(1)} S_{1} = S_{1} P^{(1)} = S_{1}, \]
\[ P^{(1)} S_{2} = S_{2} P^{(1)} = -S_{1}, \]
\[ P^{(2)} S_{2} = S_{2} P^{(2)} = S_{1} + S_{2}. \] (33)
where the matrix indices on operators are to be understood.

It is then straightforward to extract the quadratic fluctuations in \( h_{\mu\nu} \) from each term in the action (23) and express them in terms of the spin projection operators and \( S \) operators:

\[
\mathcal{L}_{EH} \equiv -\frac{1}{\kappa^2} \sqrt{-g} R = -\frac{1}{4} h_{\mu\nu} \left[ P^{(2)} - P^{(0,s)} \right]_{\mu\nu,\rho\sigma} \Box h^{\rho\sigma},
\]

\[
\mathcal{L}_{GCS} \equiv \frac{1}{2\kappa^2 \mu} \varepsilon^{\lambda\mu\nu} \Gamma^\rho_\lambda \left( \partial_\mu \Gamma^\sigma_\nu + \frac{2}{3} \Gamma^\sigma_\nu \Gamma^\tau_\mu \right) = \frac{1}{4\mu} h^{\mu\nu} (S_1 + S_2)_{\mu\nu,\rho\sigma} h^{\rho\sigma},
\]

\[
\mathcal{L}_{PF} \equiv -\frac{m^2}{4}(h_{\mu\nu}h^{\mu\nu} - h^2) = -\frac{m^2}{4} h_{\mu\nu} \left[ P^{(2)} + P^{(1)} - P^{(0,s)} - \sqrt{2}(P^{(0,sw)} + P^{(0,ws)}) \right]_{\mu\nu,\rho\sigma} \Box h^{\rho\sigma},
\]

\[
\mathcal{L}_\alpha \equiv -\frac{1}{2\alpha\kappa^2} (\partial_\mu \phi^{\mu\nu})^2 = \frac{1}{2\alpha} h_{\mu\nu} \left[ \frac{1}{2} P^{(1)} + \frac{1}{2} P^{(0,s)} + \frac{1}{4} P^{(0,w)} - \frac{1}{2\sqrt{2}} (P^{(0,sw)} + P^{(0,ws)}) \right]_{\mu\nu,\rho\sigma} \Box h^{\rho\sigma},
\]

\[
\mathcal{L}_\beta \equiv -\frac{1}{2\beta \kappa^4} \frac{m^2}{4} h \Box h = -\frac{1}{2\beta} \kappa^4 \frac{m^2}{4} h_{\mu\nu} \left[ 2P^{(0,s)} + P^{(0,w)} + \sqrt{2}(P^{(0,sw)} + P^{(0,ws)}) \right]_{\mu\nu,\rho\sigma} \Box h^{\rho\sigma}.
\]

Using the relations (34), the quadratic part in \( h_{\mu\nu} \) of the action (23) is expressed in terms of the spin projection operators and \( S \) operators

\[
S = \int d^3 x \frac{1}{2} h^{\mu\nu} \mathcal{P}_{\mu\nu,\rho\sigma} h^{\rho\sigma},
\]

where \( \mathcal{P}_{\mu\nu,\rho\sigma} \) is defined as

\[
\mathcal{P}_{\mu\nu,\rho\sigma} = \left[ -\frac{1}{2}(\Box + m^2) P^{(2)} + \frac{1}{2} \left( \frac{1}{\alpha} - m^2 \right) P^{(1)} + \frac{1}{2} \left\{ (1 + \frac{1}{\alpha} - \frac{1}{\beta} \kappa^4 m^2) \Box + m^2 \right\} P^{(0,s)} + \frac{1}{4} \left( \frac{1}{\alpha} - \frac{1}{\beta} \kappa^4 m^2 \right) \Box P^{(0,w)} - \frac{1}{2\sqrt{2}} \left\{ (1 + \frac{1}{\beta} \kappa^4 m^2) \Box - 2m^2 \right\} (P^{(0,sw)} + P^{(0,ws)}) + \frac{1}{2\mu} (S_1 + S_2) \right]_{\mu\nu,\rho\sigma}.
\]

Then, the propagator of \( h_{\mu\nu} \) is defined in a standard way by

\[
< 0| T(h_{\mu\nu}(x)h_{\rho\sigma}(y)) |0> = i \mathcal{P}_{\mu\nu,\rho\sigma}^{-1}\delta^{(3)}(x-y).
\]
With the help of the relations (30), (31) and (33), the inverse of the operator $P$ is easily calculated to be

$$P^{-1}_{\mu\nu,\rho\sigma} = [x_1 P^{(2)} + x_2 P^{(1)} + x_3 P^{(0,s)} + x_4 P^{(0,w)} + x_5 (P^{(0,sw)} + P^{(0,ws)}) + x_6 (S_1 + S_2)]_{\mu\nu,\rho\sigma},$$

where $x_i (i = 1, 2, \cdots, 6)$ are given by

\begin{align*}
x_1 &= \frac{2\mu(\Box + m^2)}{\frac{1}{\alpha} \Box^3 - \mu(\Box + m^2)^2}, \\
x_2 &= \frac{2}{\frac{1}{\alpha} \Box - m^2}, \\
x_3 &= \frac{2(\frac{1}{\alpha} - \frac{1}{\beta} \kappa^4 m^2) \Box}{I}, \\
x_4 &= \frac{4[(1 + \frac{1}{\alpha} - \frac{1}{\beta} \kappa^4 m^2) \Box + m^2]}{I}, \\
x_5 &= \frac{2\sqrt{2}[(\frac{1}{\alpha} + \frac{1}{\beta} \kappa^4 m^2) \Box - 2m^2]}{I}, \\
x_6 &= \frac{2}{\frac{1}{\alpha} \Box^3 - \mu(\Box + m^2)^2},
\end{align*}

with $I$ being defined by

$$I = \left(\frac{1}{\alpha} - \frac{1}{\beta}(1 + \frac{4}{\alpha}) \kappa^4 m^2\right) \Box^2 + \left(\frac{5}{\alpha} + \frac{3}{\beta} \kappa^4 m^2\right) m^2 \Box - 4m^4.$$

These expressions provide us how to choose the gauge parameters $\alpha, \beta$ in order to have the desired graviton propagator. Before doing so, let us recall where the difficult point is in the proof of renormalizability of TMG [5]. Since gravitational Chern-Simons term includes three derivatives, we expect that the graviton propagator would fall off like $\frac{1}{p^3}$ for large momenta. In fact, this $\frac{1}{p^3}$ behavior turns out to be required for power counting renormalizability of TMG. However, in particular, the propagator of the scalar excitation ($x_3$) behaves as $\frac{1}{p^2}$ and thus decreases more slowly in the large momentum limit. The reason why the scalar mode behaves like so is simple. This part of the propagator comes from the conformal mode in the Einstein-Hilbert action and is not affected by the higher derivative gravitational Chern-Simons term since the gravitational Chern-Simons term is conformally invariant.

This thorny problem is resolved in the present formalism as follows: First, let us notice that the coefficients $x_2, x_3, x_4$ and $x_5$ depend on the gauge parameters $\alpha, \beta$ whereas $x_1$ and $x_6$ are independent of them. Second, the troublesome spin 0 component $x_3$ projected by $P^{(0,s)}$ can be vanished by selecting the condition $\beta = \alpha \kappa^4 m^2$. Third, the spin 1 component $x_2$ projected by $P^{(1)}$ vanishes when we take the limit $\alpha \to 0$. In other words, we choose the gauge parameters to be

$$\beta = \alpha \kappa^4 m^2 \to 0.$$
Finally, it is easy to see that with this condition (41) on the gauge parameters, the other gauge-variant terms \( x_4 \) and \( x_5 \) become zero as well.

Consequently, with the condition (41) we have the propagator of the graviton whose essential part is controlled by

\[
\mathcal{P}_{\mu\nu,\rho\sigma}^{-1} = \left[ \frac{2\mu(\Box + m^2)}{\mu^3 - \mu(\Box + m^2)^2} \mathcal{P}^{(2)}(S_1 + S_2) \right]_{\mu\nu,\rho\sigma},
\]

(42)

An interesting feature is that this propagator is described entirely by the transverse operator \( \theta_{\mu\nu} \) so it vanishes when multiplied by the momenta \( p^\mu \).

We will soon realize that this propagator damps cubically like \( \frac{1}{P^3} \) for large momenta, instead of the quadratic one \( \frac{1}{P^2} \) as was obtained by the Einstein-Hilbert action, since the \( S \) operators involve the momentum factor \( p^2 p^\mu \). Thus, the dimension of the gravitational field \( h_{\mu\nu} \), concerning its ultraviolet behavior, should be assigned to be \( (mass)^0 \) in place of the original canonical one \( (mass)^{\frac{1}{2}} \). With this assignment of the dimension, interaction terms existing in the Einstein-Hilbert action have the dimension \( (mass)^2 \) solely coming from two derivatives and the index \( \delta_{EH} = 2 - 3 = -1 \), thereby implying that they are super-renormalizable. Similarly, interaction terms in gravitational Chern-Simons term have the dimension \( (mass)^3 \) and hence the index \( \delta_{GCS} = 3 - 3 = 0 \), so they are of marginally renormalizable type.

As a final remark, it is worthwhile to ask ourselves what has become of one dynamical degree of freedom associated with the scalar mode since we know that the number of dynamical degrees of freedom remains unchanged in perturbation theory. The answer lies in the fact that there appears the propagator for the Steuckelberg field \( A_\mu \), which has one dynamical degree of freedom owing to gauge invariance. In other words, because of the BRST-invariant regulator the massless pole \( \frac{1}{P^2} \) of the scalar mode is changed to that of the Steuckelberg field \( A_\mu \).

5 Superficial degree of divergence and ultraviolet regulator

We now turn our attention to the analysis of structure of the divergences. To this aim, let us first add sources \( K_{\mu\nu} \) (anti-commuting, ghost number = \(-1\), dimension = \( \frac{3}{2} \)), \( L_\mu \) (commuting, ghost number = \(-2\), dimension = 1), \( M_\mu \) (anti-commuting, ghost number = \(-1\), dimension = 1) and \( N \) (anti-commuting, ghost number = \(-1\), dimension = 1) for the BRST transformations of \( \tilde{g}^{\mu\nu} \), \( c_\mu \), \( A_\mu \) and \( \varphi \) to the action (23), respectively \(^6\):

\[
\tilde{S} \equiv \int d^3x \tilde{\Sigma}
\]

\(^6\)Here for simplicity we have regarded \( \tilde{g}^{\mu\nu} \) as a basic gravitational field.
\[
\begin{align*}
&= \int d^3x [\mathcal{L} + \frac{1}{\kappa^3} K_{\mu\nu}\delta_B g^{\mu\nu} + \frac{1}{\kappa^3} L_\mu \delta_B c^\mu + \frac{1}{\kappa^3} M_\mu \delta_B A^\mu + \frac{1}{\kappa^3} N \delta_B \varphi] \\
&= \int d^3x [\frac{1}{\kappa^2} \sqrt{-g} R + \frac{1}{2\kappa^2} \varepsilon^{\lambda\mu\nu} \Gamma^\rho_{\lambda\sigma} (\partial_\mu \Gamma^\sigma_{\rho\nu} + \frac{2}{3} \Gamma^\sigma_{\mu\nu} \Gamma^\rho_{\nu\rho}) \\
&\quad - \frac{m^2}{4} (h_{\mu\nu} h^{\mu\nu} - h^2) - \frac{1}{4} F^2_{\mu\nu} - (mA^\mu - \partial^\mu \varphi)(\partial^\nu h_{\mu\nu} - \partial_\mu h) \\
&\quad - \frac{1}{2\alpha \kappa^2} (\partial_\mu g^{\mu\nu})^2 - \frac{1}{2\beta} \kappa^4 (\partial_\mu A^\mu - \frac{m}{2} h) \Box (\partial_\nu A^\nu - \frac{m}{2} h) + (K_{\mu\nu} + i \partial_\mu \bar{c}_\nu) D^\mu_{\rho} c^\rho \\
&\quad - i \kappa^4 \Box^2 c - L_\mu c^\nu \partial_\nu c^\mu + \frac{1}{\kappa} M_\mu (-mc^\mu + \partial^\mu c) + \frac{m}{\kappa} N c].
\end{align*}
\]

Next, based on this action, let us consider the superficial degree of divergence for 1PI (one particle irreducible) Feynman diagrams. Then, it is convenient to introduce the following notation:

\begin{align*}
&n_R = \text{number of graviton vertices with two derivatives}, \\
&n_G = \text{number of ghost vertices}, \\
&n_K = \text{number of K-graviton-ghost vertices}, \\
&n_L = \text{number of L-ghost-ghost vertices}, \\
&I_G = \text{number of internal ghost propagators and} \\
&I_E = \text{number of internal graviton propagators}.
\end{align*}

Note that the fields $A_\mu, \varphi, c$ and $\bar{c}$ are free so we can exclude such the fields from the counting of the superficial degree of divergence. Using this fact and the above notation, the superficial degree of divergence for an arbitrary Feynman diagram $\gamma$ can be easily calculated to be

\[
\omega(\gamma) = \sum n_i d_i + (3 - 3) I_E + (3 - 2) I_G - 3(\sum n_i - 1) = 3 - n_R - n_G - 2n_K - 2n_L + I_G,
\]

where $d_i$ denotes the number of derivatives in the interaction terms. Here we have made use of the fact that the graviton propagator behaves like $p^{-3}$ for large momenta as mentioned in the previous section.

Furthermore, using the relation

\[
2n_G + 2n_L + n_K = 2I_G + E_c + E_{\bar{c}},
\]

with the notation that $E_c = \text{number of external ghosts} c^\mu$ and $E_{\bar{c}} = \text{number of external antighosts} \bar{c}_\mu$, Eq. (44) is cast to the form

\[
\omega(\gamma) = 3 - n_R - \frac{3}{2} n_K - n_L - \frac{1}{2} E_c - \frac{1}{2} E_{\bar{c}}.
\]

Accordingly, we reach the conclusion

\[
\omega(\gamma) \leq 3.
\]

which indicates that the bound on the superficial degree of divergence for 1PI diagrams $\gamma$ is cubic to all orders, and the theory is power-counting renormalizable.

Next, we have to take into consideration the ultraviolet regularization in order to deal with divergent integrals. Here we meet another difficulty. Because TMG includes the Levi-Civita tensor density explicitly, we cannot use dimensional regularization beyond one-loop.
Furthermore, it turned out that covariant, higher derivative regularization spoils the argument of renormalizability when the unusual parametrization of the metric tensor is used [5].

On the other hand, with the usual parametrization of the metric (10), this problem can be solved by using a covariant, higher derivative regularization as will be explained shortly. Note that there are two criteria of selecting a suitable regulator. One criterion is that the regulator should push all divergences to only one-loop where we can make use of powerful dimensional regularization. Another criterion is that regularization should not generate the graviton propagator, in particular, the propagator of the scalar mode, which falls off like $\frac{1}{p^6}$ for large momenta. It is of interest that the simplest candidate is provided from the new massive gravity [7, 8]. As mentioned explicitly in Refs. [19, 30], at the quadratic order in $\kappa^2$, a salient feature of the specific combination of $R_{\mu\nu} R_{\mu\nu}$ terms in the new massive gravity is the disappearance of the spin projection operator corresponding to the spin 0 scalar mode:

$$\sqrt{-g}(R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2) \approx \frac{\kappa^2}{4} h^{\mu\nu} P_{\mu\nu,\rho\sigma} \Box^2 h^{\rho\sigma},$$

(48)

thereby the higher derivative sector becoming conformally invariant [28].

Using this observation, we shall propose the following covariant, higher derivative regulator satisfying the above criteria

$$S_{\Lambda} = \frac{1}{\kappa^2 \Lambda^6} \int d^3 x \sqrt{-g}(R_{\mu\nu} \Box^2_g R^{\mu\nu} - \frac{3}{8} R \Box^2_g R) \approx \frac{1}{4 \Lambda^6} \int d^3 x h^{\mu\nu} P_{\mu\nu,\rho\sigma} \Box^4 h^{\rho\sigma},$$

(49)

where $\Box^2_g \equiv g^{\mu\nu} g^{\rho\sigma} \nabla_\mu \nabla_\nu \nabla_\rho \nabla_\sigma$. Note that $S_{\Lambda}$ has the graviton propagator damping like $\frac{1}{p^6}$ for large momenta and vertices with eight derivatives. Indeed, with the condition (41), $P_{\mu\nu,\rho\sigma}^{-1}$ is now replaced with

$$P_{\mu\nu,\rho\sigma}^{-1} = \left[ \frac{2\mu(-\frac{1}{\Lambda^6} \Box^4 + \Box + m^2)}{\mu \Box^3 - \mu(-\frac{1}{\Lambda^6} \Box^4 + \Box + m^2)^2} P^{(2)} + \frac{2}{\mu \Box^3 - \mu(-\frac{1}{\Lambda^6} \Box^4 + \Box + m^2)^2} (S_1 + S_2) \right]_{\mu\nu,\rho\sigma},$$

(50)

which certainly falls off like $\frac{1}{p^6}$ for large momenta.

Repeating the calculation of the superficial degree of divergence with this regulator, we now find that

$$\omega(\gamma) = 3 - n_R - 5(I_E - n_\Lambda) - \frac{3}{2} n_K - n_L - \frac{1}{2} E_{c} - \frac{1}{2} E_{\bar{c}},$$

(51)

where $n_\Lambda$ is number of graviton vertices with eight derivatives. This $\omega(\gamma)$ also satisfies the inequality (47) since $I_E \geq n_\Lambda$ for 1PI diagrams. The topological relation among numbers of loops, internal lines and vertices yields the equation

$$L = I_E + I_G - n_R - n_{GCS} - n_\Lambda - n_G - n_K - n_L + 1,$$

(52)
with $n_{GCS}$ being number of graviton vertices with three derivatives, so we can rewrite the superficial degree of divergence in terms of loop number $L$

$$\omega(\gamma) = 8 - 5L - 6n_R - 4n_K - n_L - 5n_{GCS} - 3E_c - 3E_{\bar{c}}. \quad (53)$$

Hence, since $\omega(\gamma)$ becomes negative for $L \geq 2$ while it has a possibility of becoming positive for $L = 1$, we have divergent integrals only in the one-loop amplitudes where we can use dimensional regularization to evaluate such divergent ones.

Although we do not calculate divergent terms in this article explicitly, it is sufficient to prove renormalizability of TMG by using the analysis argued so far. Since the divergent part is local, BRST-invariant (that is, diffeomorphism-invariant), and of dimension 3 at most, the possible form of the divergent counter-terms reads

$$S_{\text{counter}} = \int d^3x[a_1\sqrt{-g} + a_2\sqrt{-g}R + a_3\epsilon^{\lambda\mu\nu}\Gamma^\rho_{\lambda\sigma}(\partial_\mu \Gamma^\sigma_{\rho\nu} + \frac{2}{3}\Gamma^\sigma_{\mu\tau}\Gamma^\tau_{\nu\rho})], \quad (54)$$

where $a_i (i = 1, 2, 3)$ are divergent coefficients. These divergences can be absorbed by renormalizations of the Newton’s constant and the parameter $\mu$ in front of gravitational Chern-Simons term, and by adding the cosmological counter-term. We have thus completed the proof of renormalizability of topologically massive gravity in three dimensions.

6 Discussions

In this article, we have presented a proof of renormalizability of topologically massive gravity (TMG) in three dimensions [3, 4]. This problem has been already studied in Refs. [5, 6], but their proofs were incomplete, unfortunately. One of the differences between their approach and ours is that they have used the unconventional parametrization of the metric tensor whereas we have done the usual parameterization. The two (closely related) problems, those are, the one associated with the scalar propagator and the other being the ultraviolet regulator, are resolved in our approach. In particular, we should notice that our proof relies on the existence of both the BRST symmetry and the BRST-invariant infrared regulator.

Furthermore, we have proposed the ultraviolet regulator whose form is motivated by the new massive gravity in three dimensions [7, 8]. Here it is worthwhile to point out that TMG and the new massive gravity share one common feature, that is, conformal invariance in the higher derivative sector. We think that this common feature might be an essential ingredient of renormalizability of quantum gravity in three dimensions.

Since it has been already shown that TMG is an interactive and unitary theory for the massive graviton of spin +2 or −2, our proof of renormalizability supports that this theory is a satisfactory example of quantum gravity though it is formulated only in three dimensions.
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