The Geroch group in the Ashtekar formulation

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Abstract

We study the Geroch group in the framework of the Ashtekar formulation. In the case of the one-Killing-vector reduction, it turns out that the third column of the Ashtekar connection is essentially the gradient of the Ernst potential, which implies that the both quantities are based on the “same” complexification. In the two-Killing-vector reduction, we demonstrate Ehlers’ and Matzner-Misner’s SL(2,R) symmetries, respectively, by constructing two sets of canonical variables that realize either of the symmetries canonically, in terms of the Ashtekar variables. The conserved charges associated with these symmetries are explicitly obtained. We show that the gl(2,R) loop algebra constructed previously in the loop representation is not the Lie algebra of the Geroch group itself. We also point out that the recent argument on the equivalence to a chiral model is based on a gauge-choice which cannot be achieved generically.

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I. INTRODUCTION

The Geroch group has been known, since his discovery in 1971, as a symmetry group acting on the solutions of Einstein’s equation [1]. Developing the early day’s result on the (actually two types of) SL(2,R) moduli in the presence of one or two Killing vector fields [2,3], he showed that the symmetry of the solutions is enlarged to infinite-dimensional in consequence of the interplay between two different SL(2,R) symmetries. This result was further developed as solution-generating techniques in terms of general relativity, and several approaches to generate solutions have been subsequently evolved [4]. Among them was shown the integrability of the Ernst equation [5], which determines the solutions of Einstein’s equation for stationary axisymmetric spacetime, by explicitly constructing the Lax pair for this equation [6-7].

Some years later, particle physicists also became interested in the Geroch group, based on the recognition that the emergence of the extra symmetry can be realized in the same context as that of hidden symmetries [8] in the Kaluza-Klein reduction of supergravity theories. This was motivated by the work of B.Julia [9], who showed in 1980 that the Lie algebra of the Geroch group is in fact the $\hat{sl}(2,R)$ affine Kac-Moody algebra. Moreover, he pointed out the existence of the non-zero central term of this algebra even at the classical level, which acts as a constant rescaling on the conformal factor of the zweibein in the resulting two-dimensional field theory. The group-theoretical structure was further elaborated [10] in connection with the non-linear sigma model, and the generalization to the Einstein-Maxwell theory was discussed [11]. The reduction of supergravity theory was also explored as an integrable system [12]. An evidence of a symmetry of a hyperbolic algebra was revealed in the reduction to one dimension [13].

In canonical gravity, on the other hand, an important breakthrough was brought about by A. Ashtekar in 1986 [14]. He found a new set of canonical variables, in terms of which a drastic simplification occured in the canonical constraints. Making use of the new variables allows us to write them in a polynomial form, and what is more, a large class of solutions
for the quantum constraints can be found by introducing the self-dual and the loop representation [14] - [16], although the interpretation of these solutions still remains to be a difficult problem. This formalism has been subsequently applied to spacetime with one or more Killing vector fields ("mini-superspace") [17] - [22] ([23] is an exhaustive reference list for the literature related to the Ashtekar variable.).

One of the notable features of the model with Killing-field isometries is the existence of physical observables in the sense of Dirac. Associated with the hidden symmetry arising through the reduction to lower dimensions, one may always have the symmetry charges, which by definition weakly commute with all the constraints. This would be helpful for better understanding of both classical and quantum gravity, since, no such functional is known in the ordinary four-dimensional gravity theory, except for the constraints themselves. The cosmological model with a closed space manifold which allows two commuting Killing vector fields has been known as the Gowdy model [24]. The Ashtekar formulation was applied to the three-torus topology model, and a set of operators which forms a GL(2,R) loop algebra are constructed in the loop representation [14]. It was conjectured that this loop algebra \( \hat{\mathfrak{gl}}(2, R) \) would be related to the Geroch group. One of the aims of this paper is to clarify this point.

We first consider the one-Killing-vector reduction, and show that the complex Ernst potential is a "natural" variable in the Ashtekar formulation. One of the distinguished properties of the Ashtekar connection is its being a complex canonical variable. This requires the reality condition in order to recover the ordinary general relativity theory. On the other hand, the complex nature of the Ernst potential is originated from the complex structure of the target space of the coset non-linear sigma model (the upper-half plane in our case), which arises through the dimensional reduction of Einstein’s action. Rather unexpectedly, it turns out that \( A_3 \) and \( iA_{\alpha} \) (\( \alpha = 1, 2 \)) are nothing but its gradient. Moreover, the upper-indexed elements \( A_\mu^\alpha \) and \( A_\mu^3 \) are real (or pure imaginary) function (not a "real-valued" function, of course) of the Ernst potential. In this sense one may say that these two kinds of complex variables are based on the "same" complexification. Another reason why this
relation is non-trivial is that, on the course of reduction, the Ernst potential is defined only after the duality relation is invoked; the Ashtekar connections are so constructed that this step may be already included in its definition. As a consequence, $A_{za} \ (a = 1, 2, 3)$ transform in a simple manner under Ehlers’ SL(2,R) transformation, known as one of two SL(2,R) subgroups whose Lie algebras generate the whole $\hat{\mathfrak{sl}}(2, R)$ through the Serre relation.

We then go further to the case of two-Killing-vector reduction, and examine how Matzner-Misner’s SL(2,R), which is the other SL(2,R), is seen in this scheme. We will see that the GL(2,R) charges in ref. [19] act as a product of this SL(2,R) and the center of the Geroch group. We will also show, however, that the GL(2,R) loop algebra constructed in the loop representation does not contain Ehlers’ SL(2,R). Therefore it does not coincide with the Lie algebra of the Geroch group itself, but is something else. To realize Ehlers’ symmetry canonically, we are forced to work with canonical variables obtained through a non-local canonical transformation from the original ones. This is in some sense expected, since a similar difficulty has been known for a long time in a chiral model, which is a much simpler system than the present one, when one realizes canonically the non-local Kac-Moody symmetry [25].

The plan of this paper is as follows. In Sec.II we review the two different SL(2,R) symmetries of the reduced system which allows the presence of two commuting Killing vectors in spacetime. In Sec.III we consider the reduction from four dimensions to three. First we describe the general settings for the U(1) symmetric spacetime in Subsec.III.A, and comment on the integrality condition considered in ref. [26]. The relation between the Ashtekar connection and the Ernst potential is revealed in Subsec.III.B. We prove Ehlers’ SL(2,R) symmetry in the framework of the Ashtekar formulation in Subsec.III.C, and derive the conserved charges in Subsec.III.D. Sec.IV is devoted to the study of the reduced model from four dimensions to two. Matzner-Misner’s SL(2,R) symmetry is demonstrated and the associated conserved charges are obtained in Subsec.IV.A and IV.B, respectively. In Subsec.IV.C we show that the GL(2,R) charges in ref. [19] act as a product group of Matzner-Misner’s SL(2,R) and the center of the Geroch group. In Subsec.IV.D we examine whether
or not the loop algebra of ref. [13] includes Ehlers’ SL(2,R), and see that it does not. Finally in Sec.V, we conclude our result, and comment on the recent argument on the equivalence to a chiral model [27]. We point out that the resulting linear system for the flat-space SL(2,R) chiral model is a consequence of a gauge-choice which can not be achieved generically.

In this paper we have to group the spacetime and the Lorentz indices in varieties of ways. Throughout the paper we use the following notations. The capital $M, N, \ldots$ stand for the four-dimensional spacetime indices $\{t, x, y, z\}$, and $m, n, \ldots$ for space indices $\{x, y, z\}$. In the one-Killing reduction, $m', n', \ldots$ represent the reduced three-dimensional spacetime indices $\{t, x, y\}$ and $\mu, \nu, \ldots$ do the two-dimensional space indices $\{x, y\}$, while $z$ is taken as the direction along the Killing vector. In the two-Killing reduction, $\bar{m}, \bar{n}, \ldots$ range over the reduced two-dimensional spacetime indices $\{t, x\}$, while $\tilde{m}, \tilde{n}, \ldots$ are used for the “compactified” coordinates $\{y, z\}$. Correspondingly, the internal Lorentz indices $A, B, \ldots, a, b, \ldots, a', b' \ldots, \alpha, \beta, \ldots, \dot{a}, \dot{b}, \ldots$ run over $\{0, 1, 2, 3\}, \{1, 2, 3\}, \{0, 1, 2\}, \{1, 2\}, \{0, 1\}$ and $\{2, 3\}$, respectively. We will sometimes repeat this definition of those indices if needed in the subsequent sections. We take the signature of the metric as $(- + + +)$. The Levi-Civita anti-symmetric tensors $\epsilon_{abc}$ and $\epsilon_{a'b'c'}$ are so defined that $\epsilon_{123} = +1$ and $\epsilon_{012} = +1$, respectively. We restrict ourselves only to the case in which all the Killing vector fields are space-like in this paper.

II. EHLERS’ AND MATZNER-MISNER’S SL(2,R)

In this section we review the Killing-vector reduction in the Lagrangian formulation, and explain the basic two SL(2,R) symmetries arising as a result of the reduction. Let us first consider the reduction from four dimensions to three by introducing a Killing vector field along the $z$ axis. As usual in the Kaluza-Klein theory [28], we start from the following four-dimensional metric

$$g_{MN} = \begin{pmatrix} \Delta^{-1}g_{m'n'}^{(3)} + \Delta B_{m'}B_{n'} & \Delta B_{m'} \\ \Delta B_{n'} & \Delta \end{pmatrix},$$

(1)
where all the components are assumed to be independent of the \( z \)-coordinate. This metric can be achieved by taking the vierbein as

\[
E^A_M = \begin{pmatrix}
\Delta^{-\frac{1}{2}} f_{m'}^a \ & \Delta^{\frac{1}{2}} B_{m'} \\
0 & \Delta^{\frac{1}{2}}
\end{pmatrix},
\]

(2)

where \( g^{(3)}_{m'n'} = f_{m'}^a \eta_{a'b'} f^{b'}_{n'} \). The Lagrangian is reduced to up to a total derivative

\[
L = \sqrt{-g} R
\]

\[
= \sqrt{-g^{(3)}} \left[ R^{(3)} - \frac{1}{4} \Delta^2 F_{m'n'} F^{m'n'} - \frac{1}{2} g^{(3)m'n'} \Delta^{-2} \partial_{m'} \Delta \partial_{n'} \Delta \right],
\]

(3)

where \( F_{m'n'} = \partial_{m'} B_{n'} - \partial_{n'} B_{m'} \) and \( F^{m'n'} = g^{(3)m'k'} g^{(3)n'l'} F_{k'l'} \). We would like to treat \( F^{m'n'} \) as an independent field. To this end we add the following term to the Lagrangian

\[
L' = L - \frac{1}{2} B \cdot \sqrt{-g^{(3)}} \epsilon^{m'n'k'} \partial_{k'} F_{m'n'}. \]

(4)

Here \( B \) is the Lagrange multiplier, and \( \frac{1}{2} \) is inserted for convenience. This term guarantees that \( F^{m'n'} \) is locally a rotation. The equation of motion of \( F^{m'n'} \) then becomes

\[
\Delta^2 F^{m'n'} = \epsilon^{m'n'k'} \partial_{k'} B.
\]

(5)

Substituting (5) into (4), we obtain the SL(2,R)/U(1) coset non-linear sigma model Lagrangian

\[
L' = \sqrt{-g^{(3)}} \left[ R^{(3)} - \frac{1}{2} g^{(3)m'n'} \Delta^{-2} (\partial_{m'} B \partial_{n'} B + \partial_{m'} \Delta \partial_{n'} \Delta) \right].
\]

(6)

Setting \( Z^{(E)} = B + i \Delta \), this action is invariant under

\[
Z^{(E)} \rightarrow a Z^{(E)} + b \quad \text{for any real numbers } a, b, c, d.
\]

(7)

for any real numbers \( a, b, c, d \). Since the simultaneous scaling on them obviously results in the same transformation, we may impose \( ad - bc = 1 \). Naming after the seminal work of J. Ehlers [2], we will call this symmetry “Ehlers’ SL(2,R)”. \( Z^{(E)} \) is related to the so called Ernst potential \( E \) [3] by \( E = iZ^{(E)} \).
It turns out that another SL(2,R) symmetry arises if we further reduce the spacetime dimension from three to two by introducing an additional Killing vector field along the $y$ axis. Roughly speaking, this is a symmetry of rotating in the $yz$-plane. $f^{m'}_{m}$ is now assumed to be in the form

$$f^{m'}_{m} = \begin{pmatrix} f^{\tilde{a}}_{\tilde{m}} \rho A_{m'} & \rho \lambda \delta^{\tilde{a}}_{\tilde{m}} \rho A_{\tilde{m}} \\ 0 & 0 \end{pmatrix}. \quad (8)$$

Here we have taken the conformal gauge for the zweibein of the reduced two-dimensional field theory. It was shown by R. Geroch that, if one would like to have infinite-dimensional symmetry, one must assume some two constants to vanish \[1\]. The easiest way to satisfy this requirement is to take \[9\]

$$A_{\tilde{m}} = B_{\tilde{m}} = 0. \quad (9)$$

This means that the four-dimensional metric is assumed to be in a block diagonal form consisting of $g_{\tilde{m}\tilde{n}}$ and $g_{\tilde{m}\tilde{n}}$. Evidently, it is essential that the metric can be recasted in this form only by such a diffeomorphism that keeps $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ being commuting Killing vectors.

It is also clear that the Killing vectors are hyper-surface orthogonal. By this choice the three-dimensional Lagrangian \[4\] becomes

$$\mathcal{L}' = -\frac{1}{2} \rho \eta^{\tilde{m}\tilde{n}} \left[ -4 \partial_{\tilde{m}} \log \rho \partial_{\tilde{n}} \log \lambda + \frac{1}{\Delta^2} \left\{ \partial_{\tilde{m}} B \partial_{\tilde{n}} B + \partial_{\tilde{m}} \Delta \partial_{\tilde{n}} \Delta \right\} \right]. \quad (10)$$

We may, on the other hand, perform the dimensional reduction from four to two directly. Using \[2\], \[8\] and \[9\], the Lagrangian \[3\] is simplified to

$$\mathcal{L} = -\frac{1}{2} \rho \eta^{\tilde{m}\tilde{n}} \left[ -4 \partial_{\tilde{m}} \log \rho \partial_{\tilde{n}} \log(\lambda \Delta^{-\frac{1}{2}} \rho^\frac{1}{2}) + \frac{\Delta^2}{\rho^2} \left\{ \partial_{\tilde{m}} B y \partial_{\tilde{n}} B y + \partial_{\tilde{m}} \left( \frac{\rho}{\Delta} \right) \partial_{\tilde{n}} \left( \frac{\rho}{\Delta} \right) \right\} \right]. \quad (11)$$

In terms of the variable $Z^{(MM)} = B_y + i \frac{\rho}{\Delta}$, the SL(2,R) transformation is expressed in this case

$$Z^{(MM)} \to \frac{a Z^{(MM)} + b}{c Z^{(MM)} + d}, \quad (12)$$
under which (11) is manifestly invariant. Following ref. [9], we call this “Matzner-Misner’s SL(2,R)”. Clearly the two Lagrangians (10) and (11) are made completely identical by the transformation

\[ B \leftrightarrow B_y, \quad \Delta \leftrightarrow \frac{\rho}{\Delta}, \quad \lambda \leftrightarrow \lambda \Delta^{-\frac{1}{2}} \rho^{\frac{1}{4}}, \quad \rho \leftrightarrow \rho, \]  

(13)

which was found by D. Kramer and G. Neugebauer [29].

R. Geroch noticed that the infinitesimal transformations of these two SL(2,R) are not commutative on the solution of Einstein’s equation, but generate infinitely many different solutions by their successive applications [1]. In fact, this is isomorphic to the affine \( \hat{\mathfrak{sl}}(2,R) \) algebra [9] (See also Subsec.IV.C for further explanations.). We will study in the subsequent sections the structure of the realization of these groups in the Ashtekar formulation.

III. KALUZA-KLEIN REDUCTION TO THREE DIMENSIONS

A. U(1) symmetric spacetime

As seen in the previous section, the first symmetry, Ehlers’ SL(2,R), already shows up at the stage of the reduction from four dimensions to three. Let us examine in this section how this is seen in the Ashtekar formulation. Although we would like to discuss its local transformation property in this paper (since even this does not seem to have been studied in detail before), we begin with describing a slightly more general setting for the topology of our spacetime that admits one Killing vector field. This allows us to fix our notations and to comment on Moncrief’s integrality condition [26].

Our starting point is that we assume the spacetime to be a direct product of a total space of a U(1) principal bundle \( \Sigma \) and time R. The base manifold \( \tilde{\Sigma} \sim \Sigma/U(1) \) is assumed to be a compact, connected and orientable two-dimensional manifold. The fiber of the bundle is topologically \( S^1 \), and the Killing vector field assumed to exist is tangent to the fiber. Geroch’s fundamental requirement for the reduction to three dimensions is thus satisfied by this U(1) gauge symmetry (of the bundle). When we discuss the reduction to two dimensions
in Sec. IV, we will consider another Killing vector field on the base manifold $\tilde{\Sigma}$ in addition to the one above.

We next introduce the U(1)-adapted coordinate $(t, x, y, z)$. Let $t$ represent time and $(x, y)$ be a local coordinate system of $\tilde{\Sigma}$ on each local patch. $z$ is a coordinate of the fiber so normalized that the Killing vector field is written as $\frac{\partial}{\partial z}$, and $0 \leq z \leq 2\pi$. This means that all derivatives with respect to $z$ are zero for any field that appears in the present model.

As usual in the ADM formalism [30], we take a vector normal to the Cauchy surface $\Sigma$ at each point with respect to the given metric $g_{MN}$, $M, N = t, x, y, z$. This induces a three-dimensional metric $h_{mn}$, $m, n = x, y, z$ on $\Sigma$. $h_{mn}$ can be further decomposed orthogonally with respect to $\frac{\partial}{\partial z}$, which induces a two-dimensional metric $h'_{\mu\nu}$, $\mu, \nu = x, y$ on $\tilde{\Sigma}$. The spacetime metric is then written as

$$ds^2 = g_{MN}dx^M dx^N = -N^2 dt^2 + h_{mn}(dx^m + N^m_{(0)}dt)(dx^n + N^n_{(0)}dt)$$

$$= -N^2 dt^2 + \left(h_{\mu\nu} - \frac{h_{\mu z}h_{\nu z}}{h_{zz}}\right)(dx^\mu + N^\mu_{(0)}dt)(dx^\nu + N^\nu_{(0)}dt)$$

$$+ h_{zz} \left\{ dz + N^z_{(0)} dt + \frac{h_{\mu z}}{h_{zz}}(dx^\mu + N^\mu_{(0)} dt) \right\}^2.$$  \hspace{1cm} (14)

We set

$$h_{zz} = \Delta,$$

$$\frac{h_{\mu z}}{h_{zz}} = B_\mu,$$

$$N^z_{(0)} + \frac{h_{\mu z}}{h_{zz}}N^\mu_{(0)} = N'^\alpha,$$

$$(N)^2 = \Delta^{-1}(N')^2,$$

$$h_{\mu\nu} - \frac{h_{\mu z}h_{\nu z}}{h_{zz}} = \Delta^{-1}h'_{\mu\nu},$$ \hspace{1cm} (15)

and

$$f^\alpha_{\mu \nu} N^\mu_{(0)} = N'^\alpha,$$ \hspace{1cm} (16)

where $h'_{\mu\nu} = f^\alpha_{\mu} \delta^\nu_\alpha f^\beta_\nu$. We can then read off the corresponding vierbein
\[
E_M^A = \begin{pmatrix}
N' & N^{\alpha} & N'^3 \\
0 & f_\mu^\alpha & B_\mu \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\Delta^{-\frac{1}{2}} \\
0 & \Delta^{-\frac{1}{2}} \\
\Delta^{\frac{1}{2}}
\end{pmatrix}
\equiv \begin{pmatrix}
N & N^a \\
0 & e_m^a
\end{pmatrix}.
\]

Here the Lorentz frame indices \( A, B, \ldots, a, b, \ldots \) and \( \alpha, \beta, \ldots \) take \{0, 1, 2, 3\}, \{1, 2, 3\} and \{1, 2\}, respectively. \( N'^3 \) is sometimes denoted by \( B_3 \) in this paper, when it is more appropriate to be regarded as a part of component of the Kaluza-Klein vector rather than as an element of the shift. We further define

\[
f_{m'}^{a'} = \begin{pmatrix}
N' & N'^{\alpha} \\
0 & f_\mu^\alpha
\end{pmatrix},
\]

where we use \( m' = t, x, y \) and \( a' = 0, 1, 2 \) as the reduced three-dimensional spacetime and Lorentz frame indices. Also we write the inverse as

\[
E_A^M = \begin{pmatrix}
\Delta^{\frac{1}{2}} \\
0 & \Delta^{\frac{1}{2}} \\
\Delta^{-\frac{1}{2}}
\end{pmatrix}
\begin{pmatrix}
N'^{-1} - N'^{-1} N^\mu & N'^{-1} (B_\alpha N'^{\alpha} - N'^3) \\
0 & f_\mu^\alpha \\
0 & 0 & -B_\alpha
\end{pmatrix}
\equiv \begin{pmatrix}
N^{-1} & -N^{-1} N^m \\
0 & e_m^a
\end{pmatrix},
\]

where \( B_\alpha = f_\mu^\alpha B_\mu \). The upper-left submatrix of the second factor is \( f_{m'}^{a'} \). For convenience for the calculation we write \( e_m^a \) and \( e_a^m \) explicitly

\[
e_m^a = \begin{pmatrix}
e_\mu^\alpha & e_\mu^3 \\
e_\alpha^\alpha & e_\alpha^3
\end{pmatrix}
= \begin{pmatrix}
f_\mu^\alpha B_\mu \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\Delta^{-\frac{1}{2}} \\
\Delta^{\frac{1}{2}}
\end{pmatrix},
\]

\[
e_a^m = \begin{pmatrix}
e_\alpha^\mu & e_\alpha^z \\
e_3^\mu & e_3^z
\end{pmatrix}
= \begin{pmatrix}
\Delta^{\frac{1}{2}} \\
\Delta^{-\frac{1}{2}}
\end{pmatrix}
\begin{pmatrix}
f_\alpha^\mu & -B_\alpha \\
0 & 1
\end{pmatrix}.
\]

10
\[
\begin{align*}
&= \begin{pmatrix}
\Delta^{\frac{1}{2}} f_\alpha^\mu - \Delta^{\frac{1}{2}} B_\alpha \\
0 & \Delta^{-\frac{1}{2}}
\end{pmatrix}.
\end{align*}
\tag{21}
\]

Due to the assumption that we consider the U(1) bundle, the 1-form

\[ \eta \equiv dz + B_m dx^m(= dz + B_\mu dx^\mu + N^\beta dt) \tag{22} \]

is a section of the U(1) bundle defined locally on each coordinate patch. For illustrative purposes let \( \bar{\Sigma} \) be \( S^2 \) and let \( H^+(+) \) and \( H^(-)(-) \) be its local patches covering each hemisphere with the intersection \( H^+(+) \cap H^(-)(-) \sim S^1 \) at the equator in common. We parameterize this \( S^1 \) by \( \theta \in [0, 2\pi] \). Then \( \eta \)'s defined on each patch differ by the U(1) gauge transformation

\[ \eta^+(+) = \eta^(-)(-) + d\varphi, \tag{23} \]

for some \( \varphi \) at the equator, satisfying \( \varphi(\theta = 2\pi) - \varphi(\theta = 0) = 2\pi n, n \in \mathbb{Z} \). This integer, referred to as first Chern class, characterizes the U(1) bundle under consideration. A closed two form \( \Phi \) obtained by pulling \( d\eta \) back to the two-dimensional base manifold \( \bar{\Sigma} \) must satisfy the integrality condition

\[ \int_{\bar{\Sigma}} \Phi = 2\pi n, \tag{24} \]

or, using the component fields,

\[ \int_{\bar{\Sigma}} (\partial_x B_y - \partial_y B_x) = 2\pi n. \tag{25} \]

It may be easily noticed that this is nothing but the quantization condition of Kaluza-Klein monopole [31]. (Here this is nothing but the Dirac monopole; see, e.g. [32]). Indeed, the integrand of (25) is just the Kaluza-Klein “magnetic” field, whose total flux is determined by the cohomology class of the transition function (23) characterizing the bundle. This is the simplest example of the known fact that the solitonic solution of the Kaluza-Klein field is classified by \( \pi^1 \) (first fundamental group) of the isometry group generated by the Killing vector fields [33]. The number of “monopole charge” controls the topology of spacetime; for
example, in the case of $\Sigma \sim S^2$, spacetime is a direct product $S^2 \times S^1$ if monopole charge is zero, and $S^3$ if monopole charge is one.

In the rest of this paper, we do not consider this global applicability \(^1\) of Geroch’s transformation, but restrict ourselves to focusing on only local properties of the Ashtekar connection under the transformation.

**B. The Ashtekar connection and the Ernst potential**

In order to clarify Geroch’s symmetry in the Ashtekar formulation, let us first express the Ashtekar connection using the parameterization (17). We follow the notation \([36]\) for the Ashtekar formulation in this paper. The Ashtekar connection is given by

$$A_{ma} = -\frac{1}{2} \epsilon_{abc} \omega_{mbc} \pm 2i\hat{\omega}_{ma}$$

$$= -\frac{1}{4} \epsilon_{abc} (2\Omega_{dbc} - \Omega_{bcd}) e^d_m \pm ie_{mb} \Omega_{0(ab)},$$

where $\Omega_{ABC} = 2E^M_A E^N_B \partial_M E_{NC}$. The symmetrization and the anti-symmetrization are the ones of strength one. $\Omega_{ABC}$ are explicitly written as

$$\Omega_{\alpha\beta\gamma} = 2\Delta f_{[\alpha}^\mu f_{\beta]}^\nu \partial_\mu (\Delta^{-\frac{1}{2}} f_{\gamma}),$$

$$\Omega_{\alpha\beta3} = 2\Delta^\frac{3}{2} f_{[\alpha}^\mu f_{\beta]}^\nu \partial_\mu B_\nu,$$

$$\Omega_{3\beta\gamma} = 0,$$

$$\Omega_{3\beta3} = -f_{\beta}^\mu \partial_\mu (\Delta^\frac{3}{2}),$$

$$\Omega_{ab0} = 0 \quad (a, b = \alpha (= 1, 2, 3)),$$

$$\Omega_{0\beta0} = -\Delta f_{[\alpha}^\mu \partial_\mu (\Delta^{-\frac{3}{2}} N'),$$

$$\Omega_{030} = 0,$$

\(^1\)The possibility of transition between the two U(1) bundles with distinct monopole charges is examined in ref. \([34]\). The appearance of unphysical singularities after Geroch’s transformation is discussed in refs. \([35, 36]\).
\[ \Omega_{033} = f^{m'}_0 \partial_{m'}(\Delta^{\frac{1}{2}}), \]
\[ \Omega_{03\gamma} = 0, \]
\[ \Omega_{0;33} = \Delta^{\frac{1}{2}} f^\nu_\beta f^{m'}_0 (\partial_{m'} B_\nu - \partial_\nu B_{m'}) \]
\[ \Omega_{0;3\gamma} = \Delta^{\frac{1}{2}} f^{m'}_0 f^\nu_\beta (\partial_{m'} f_{\nu\gamma} - \partial_\nu f_{m'\gamma}) - \delta_{\beta\gamma} f^{m'}_0 \partial_{m'}(\Delta^{\frac{1}{2}}). \] (27)

Plugging these expressions into eq. (26), we obtain
\[ A_{z3} = \pm i \frac{1}{2} f^{m'}_0 \partial_{m'} \Delta + \frac{1}{2} \Delta^2 \epsilon_{3\alpha\beta} f^\mu_\sigma f^\nu_\beta \partial_\mu B_\nu, \] (28)
\[ A_{z\alpha} = \frac{1}{2} \epsilon_{3\alpha\beta} f^\mu_\beta \partial_\mu \Delta \pm \frac{1}{2} \Delta^2 f^\nu_\alpha f^{m'}_0 (\partial_{m'} B_\nu - \partial_\nu B_{m'}). \] (29)

To express these formulas in terms of the Ernst potential \( \mathcal{E}_\pm \equiv \Delta \pm iB \), we now invoke the duality relation
\[ \Delta^2 (\partial_{m'} B_{n'} - \partial_{n'} B_{m'}) = f^{a'}_m f^{b'}_n f^{c'}_p \epsilon^{a'b'c'} \partial^{\mu'} B. \] (5)

Use of this equation leads to the following simple result
\[ A_{z3} = \pm i \frac{1}{2} f^{m'}_0 \partial_{m'} \mathcal{E}_\pm, \] (30)
\[ A_{z\alpha} = \frac{1}{2} \epsilon_{3\alpha\beta} f^\mu_\beta \partial_\mu \mathcal{E}_\pm, \] (31)

where \( \pm \) depends on the choice of the sign in (26). The third column of the Ashtekar connection is therefore nothing but (essentially) the gradient of the Ernst potential. What may be a remarkable thing is that the Ashtekar connection is readily complexified and partially dualized (in the sense that the second term of the Ashtekar connection has a factor \( \epsilon_{abc} \)) to give directly (the gradient of) the complex Ernst potential. This is the first observation that shows the close relationship between the Ashtekar connection and the Ernst potential. Consequently \( A_{z3} \) and \( A_{z\alpha} \) transform as
\[ \pm i \mathcal{E}_\pm \rightarrow \frac{\pm ia \mathcal{E}_\pm + b}{\pm ic \mathcal{E}_\pm + d}, \]
\[ A_{z3} \rightarrow \frac{1}{(\pm ic \mathcal{E}_\pm + d)^2} A_{z3}, \]
\[ A_{z\alpha} \rightarrow \frac{1}{(\pm ic \mathcal{E}_\pm + d)^2} A_{z\alpha}. \] (32)
where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.

Remaining components of the Ashtekar connection $A_{\mu \alpha}$ and $A_{\mu 3}$ are linear combinations of $A_{\mu \alpha}$ and $A_{\mu 3}$, which are calculated to be

$$A_{\mu \alpha} = \pm i \frac{1}{2} f_{\alpha}^m f_{\alpha}^{m'} \partial_{m'} \Delta - \frac{1}{2} \Delta^2 \epsilon_{\gamma \beta} f_{\alpha}^{\mu} f_{\gamma}^{\nu} \partial_{\lambda} B_{\nu} \pm i \Delta \omega_{(f)\mu}^{\alpha 0},$$

$$A_{\mu 3} = \frac{1}{2} \epsilon_{\gamma \beta} f_{\alpha}^{\mu} f_{\beta}^{\nu} \partial_{\nu} \Delta \pm i \frac{1}{2} \Delta^2 f_{\alpha}^{\mu} f_{\alpha}^{\nu} f_{\alpha}^{m'} (\partial_{m'} B_{\nu} - \partial_{\nu} B_{m'}) - \Delta \omega_{(f)\mu}^{12},$$

where $\omega_{(f)m'_{b'c'}}$ is the spin-connection with respect to $f_{a'}^m$

$$\omega_{(f)m'_{b'c'}} = \frac{1}{2} (\Omega_{(f)}^{a'b'c'} - \Omega_{b'c'a'}^{(f)} + \Omega_{c'a'b'}^{(f)}) f_{m'}^{m'},$$

$$\Omega_{(f)}^{a'b'c'} = 2 f_{m'}^{m'} f_{n'}^{n'} \partial_{m'} f_{n'}^{n'},$$

and $B^\mu = f_{\alpha}^{\mu} B_{\alpha}$. The index $\mu$ of the Ashtekar connection is raised by

$$g^{mn} = e_{m}^{a} e_{n}^{b} \delta_{ab}$$

$$= \begin{pmatrix} \Delta f_{\alpha}^{\mu} f_{a}^{\alpha} & - \Delta B^\mu \\ - \Delta B^\nu & \Delta B_{\alpha} B^{\alpha} + \Delta^{-1} \end{pmatrix},$$

which is the inverse of the submatrix of the original metric

$$g_{mn} = e_{m}^{a} e_{n}^{b} \delta_{ab}$$

$$= \begin{pmatrix} \Delta^{-1} f_{\alpha}^{\mu} f_{a}^{\alpha} + \Delta B_{\mu} B^{\alpha} \Delta B_{\mu} \\ \Delta B_{\nu} \Delta \end{pmatrix}.$$

Making use of eq.(5) in (33) and (34), we find

$$A_{\mu \alpha} = \pm i \left[ - \frac{1}{2} f_{\alpha}^{m} f_{\alpha}^{m'} \partial_{m'} \mathcal{E}_{\pm} + \Delta \omega_{(f)\mu}^{\alpha 0} \right],$$

$$A_{\mu 3} = \frac{1}{2} \epsilon_{\gamma \beta} \left[ f_{\alpha}^{\mu} f_{\beta}^{\nu} \partial_{\nu} \mathcal{E}_{\pm} - \Delta \omega_{(f)\mu}^{\alpha \beta} \right].$$

Thus $A_{\mu \alpha}$ and $A_{\mu 3}$ can be also expressed in compact forms in terms of the Ernst potential.

$A_{\mu \alpha}$ and $A_{\mu 3}$ are, on the other hand, related to them by

$$A_{\lambda 3} = \Delta^{-1} f_{\lambda}^{\beta} f_{\mu}^{\alpha} A_{\mu 3} + B_{\lambda} A_{z 3},$$

$$A_{\lambda \alpha} = \Delta^{-1} f_{\lambda}^{\beta} f_{\mu}^{\alpha} A_{\mu \alpha} + B_{\lambda} A_{z \alpha}.$$
Their explicit dependence on $B_\mu$ shows that they transform non-locally under (32). We will see, however, that such $B_\mu$ dependence will be drop in all the first class constraints.

Comparing (29) (28) and (33) (34), we find that the following relations hold without using the duality relation

$$A^\mu_\alpha = -f^\mu_\alpha A_{z3} \pm i\Delta \omega^{(f)\mu}_\alpha,$$

$$A^\mu_3 = f^\mu_\alpha A_{z\alpha} - \Delta \omega^{(f)\mu}_1.$$ (42)

These relations will turn out to be useful in a moment.

Before concluding this subsection, a comment is in order. We have succeeded to relate the Ashtekar connections to the Ernst potential with the help of the duality relation (5), which is originally a solution of the field equation of $B_{m'}$ in the Lagrangian formalism. Therefore, in the framework of Hamiltonian formalism, we have to make the origin of this equation clear. As shown in ref. [26], a part of the relations can be obtained by solving the diffeomorphism constraint of $z$-coordinate, while $r$ defined by

$$r \equiv f f^\mu_\alpha f^\nu_\beta \epsilon^{\alpha\beta\mu} \partial_\nu B_\rho,$$ (43)

$$f \equiv \det f^\alpha_\mu$$ (44)

is treated as an independent canonical variable, whose transformation rule should be imposed on itself. We will elaborate further on this point in the next subsection.

C. Ehlers’ SL(2, R) symmetry in the Ashtekar formulation

We will now show Ehlers’ SL(2,R) symmetry of this model within the framework of Hamiltonian formalism. In the Hamiltonian formalism in general, we have to clarify the following two points to prove a symmetry of the system. First, we must show, of course, the invariance of the Hamiltonian (which is zero in the context of gravity) and the constraints under the transformation. Second, we have to make sure that the new variables resulting from the action on the canonical variables again satisfy the canonical Poisson algebra. We
do not directly take these steps, but rather we will do it alternatively in the following way:
we first look for canonical pairs which transform in a simple way under the transformation,
show the invariance of their Liouville form, and then we rewrite the constraints in terms of
these new canonical variables and see the invariance of the constraints. The advantage of
this approach is that the second point we mentioned above is automatically guaranteed by
the invariance of the Liouville form, and also, of course, that the invariance of the constraints
is expected to be seen more neatly in terms of these special canonical variables than the
original ones, which transform in a non-trivial manner. Ehlers’ SL(2,R) invariance is already
shown in ref. [26] in this way in the ADM formalism. In this paper, however, we will use
the Ashtekar formulation here, in which the Ernst potential and its complex conjugate will
naturally arise as such canonical variables on which Ehlers’ SL(2,R) acts as a linear fractional
transformation.

We start from the parameterization of the inverse vierbein (19). The densitized (inverse)
dreibein reads
\[ \tilde{e}_a^m = f \begin{pmatrix} f_\alpha^\mu - B_\alpha & 0 \\ 0 & \Delta^{-1} \end{pmatrix}. \] (45)

In the Ashtekar formulation the first class constraints are given by [14]
\[ \mathcal{H} \equiv \epsilon^{abc} \tilde{e}_a^m \tilde{e}_b^n F_{mnc}, \]
\[ \mathcal{H}_n \equiv \tilde{e}_a^m F_{mn}^a, \]
\[ \mathcal{G}_a \equiv D_m \tilde{e}_a^m. \] (46)

\( \mathcal{G}_a \) is the Lorentz constraint, while \( \mathcal{H} \) and \( \mathcal{H}_n \) are the Hamiltonian and diffeomorphism
constraint modulo the Lorentz constraint. The diffeomorphism is actually generated by the
following linear combination
\[ C_n = A_{na} \mathcal{G}_a + \mathcal{H}_n. \] (47)

Since we have taken \( \tilde{e}_3^\mu = 0 \), three of the two Lorentz constraints
\[ G_\alpha = \equiv D_m \tilde{e}_\alpha^m = 0 \]  

(48)

become second class, while other constraints still remain first class. Solving (48), \( A_{\mu 3} \) are written in terms of other canonical variables

\[ \tilde{e}_\alpha^\mu A_{\mu 3} = -\epsilon_{3\alpha\beta} \partial_\nu \tilde{e}_\beta^\nu + A_{3\alpha} \tilde{e}_3^\nu - A_{33} \tilde{e}_3^\alpha. \]  

(49)

We may use this equality in the strong sense to eliminate \( A_{\mu 3} \) in what follows.

We next consider \( C_z \). In our case it is reduced to

\[ C_z = \tilde{e}_\alpha^\mu \partial_\mu A_{2\alpha} - \epsilon_{\alpha\beta} A_{23} (\tilde{e}_\alpha^\mu A_{\mu 3} + \tilde{e}_3^\alpha A_{33}) \]

\[ = \partial_\mu (\tilde{e}_\alpha^\mu A_{2\alpha}), \]  

(50)

where we used (49) and the assumption that any field does not depend on \( z \)-coordinate.

Now let us assume here that we have imposed any gauge-fixing condition on the \( z \)-coordinate diffeomorphism degrees of freedom, so that the constraint \( C_z \) is already of second class and can be imposed strongly. For example, one may take

\[ N^{\prime z} = 0 \]  

(51)

or equivalently,

\[ N^\alpha \tilde{e}_3^\alpha + N^3 \tilde{e}_3^z = 0 \]  

(52)

as a gauge-fixing condition. Alternatively, one may choose

\[ \tilde{e}_1^z = 0 \]  

(53)

as was done in ref. [19]. In any case, one may easily verify that the constraint \( C_z \) together with either of these conditions actually become second class. We may then solve the equation

\[ \tilde{e}_1^y = 0 \]  

(54)

Note that \( \tilde{e}_1^y \) was also set to zero to fix \( C_y \) in ref. [19] since they considered a two-Killing reduced model. We also take this gauge-choice in the next section.
\[ C_z = 0 \quad (54) \]

explicitly. Due to Stokes' theorem, (54) with (50) means that there exists some \( \phi \) such that \( \tilde{e}^\mu_\alpha A_{z\alpha} \) can be written locally

\[ \tilde{e}^\mu_\alpha A_{z\alpha} = f f^\mu_\alpha f^\nu_\beta \epsilon_{3\alpha\beta} \partial_\nu \phi. \quad (55) \]

In fact, \( \phi \) is nothing but (locally) a half of the Ernst potential: \( \phi = \mathcal{E}_\pm / 2 \) (\( \pm \) depends on the choice of the sign in the definition of the Ashtekar connection (26)), as can be checked by substituting (29) into (55).

In the ADM formalism, the diffeomorphism constraint of \( z \)-coordinate amounts to a requirement that the \( B_\mu \) field should be divergence free [26]. This can be similarly solved to ensure that the \( B_\mu \) can be written as a Hodge dual of the gradient of \( B^\mu_0 \), which is an imaginary part of the Ernst potential. In our case, we have used complex the Ashtekar connection to write down \( C_z \) to find \( \tilde{e}^\mu_\alpha A_{z\alpha} \) be divergence free, being led directly to the complex Ernst potential. Hence, in that sense, both the Ashtekar connection and the Ernst potential are complexified "in the same way".

The equation (53) reproduces only a part of the duality relations (3). To see this, let us write \( \tilde{e}^\mu_\alpha A_{z\alpha} \), using (29), explicitly in terms of the components of dreibein

\[ \tilde{e}^\mu_\alpha A_{z\alpha} = f f^\mu_\alpha \left( \frac{1}{2} \epsilon_{3\alpha\beta} f_\beta^\mu \partial_\mu \Delta \pm i \frac{1}{2} \Delta^2 f^\nu f^{m'}_{\beta} \left( \partial_{m'} B_\nu - \partial_\nu B_{m'} \right) \right). \quad (56) \]

The first term is already in the form of a rotation. Hence (53) forces the second term to be \( f f^\mu_\alpha f^\beta_\beta \epsilon \partial_\nu B \) for some \( B \). This may be obtained by setting \( a' = 0 \) in the equation below equivalent to (3)

\[ \Delta^2 f^{m'}_{\alpha'} \left( \partial_{m'} B_\mu - \partial_\mu B_{m'} \right) = f^{\beta}_{\mu} f^{\nu}_{\rho} \epsilon_{\alpha' \beta' \rho} \partial_{\nu'} B. \quad (57) \]

On the other hand, the equation for the \( xy \)-space rotation of \( B_\mu \) can not be derived by (55), but rather such a field is contained in the degrees of freedom of canonical momenta, whose transformation rule should be independently imposed.

\[^3 B \] is denoted by \( \bar{\psi} \) in ref. [14], and \( \omega \) in ref. [26].
Let us now look for “good” canonical pairs for Ehlers’ SL(2,R). Since we would like to take \( \phi(= E_\pm/2) \) as a canonical variable, we write the Liouville form in terms of \( \tilde{e}_\alpha^\mu A_{z\alpha} \) as a first step.

\[
A_\mu \tilde{e}_\alpha^\mu + A_{z\alpha} \tilde{e}_\alpha^z + \dot{A}_{z3} \tilde{e}_3^z \quad = -(\tilde{e}_\alpha^\mu A_{z\alpha}) B_\mu + \dot{\tilde{e}}_\alpha^\mu (A_{z\alpha} B_\mu - A_\mu \alpha) + \dot{A}_{z3} \tilde{e}_3^z \\
= -\dot{\phi} r + (A_\mu \alpha - A_{z\alpha} B_\mu) \tilde{e}_\alpha^\mu + \dot{A}_{z3} \tilde{e}_3^z, \tag{58}
\]

where the equalities hold up to total derivatives. \( r \) is defined by (43) in the previous section.

Using the formulas (41)(42), this is further rewritten in the following form

\[
= -\dot{\phi} r + (f \dot{A}_{z3}) f^{-1} \tilde{e}_3^z \pm i \omega_\mu^{(f)} \tilde{e}_\alpha^\mu, \tag{59}
\]

with \( \pm \) depending on the sign in the definition of the Ashtekar connection. Finally the relation

\[
A_{z3} - \Delta^2 f^{-1} r = -\overline{A_{z3}}, \tag{60}
\]

allows us to rewrite (59) as

\[
= f \Delta^{-2} (\overline{\phi A_{z3}} - \overline{\dot{\phi} A_{z3}}) \pm i \omega_\mu^{(f)} \tilde{e}_\alpha^\mu. \tag{61}
\]

The relation (60) can be immediately shown from (28). Since the first two terms of (61) give their imaginary part, the total Liouville form is purely imaginary. The new canonical variables are thus \((\overline{\phi}, f \Delta^{-2} A_{z3})\), \((\phi, -f \Delta^{-2} \overline{A_{z3}})\) and \((\pm i \omega_\mu^{(f)}, \tilde{e}_\alpha^\mu)\). The Liouville form (61) is invariant if these canonical variables transform as follows

\[
(\overline{\phi}, f \Delta^{-2} A_{z3}) \rightarrow \left( \frac{1}{i} \cdot \frac{ia\phi + b}{ic\phi + d}, (ic\phi + d)^2 \cdot f \Delta^{-2} A_{z3} \right), \\
(\phi, -f \Delta^{-2} \overline{A_{z3}}) \rightarrow \left( -\frac{1}{i} \cdot \frac{-ia\phi + b}{-ic\phi + d}, (-ic\phi + d)^2 \cdot (-f \Delta^{-2} \overline{A_{z3}}) \right), \\
(\pm i \omega_\mu^{(f)}, \tilde{e}_\alpha^\mu) \rightarrow (\pm i \omega_\mu^{(f)}, \tilde{e}_\alpha^\mu). \tag{62}
\]

This is Ehlers’ SL(2,R) transformation in the Hamiltonian formalism. What we have seen here is that we could take the Ernst potential and its complex conjugate as canonical variables (although we have used its half \( \phi \) actually to avoid the appearance of the factor 2
everywhere), and then their canonical conjugate turned out to be proportional to the $A_{z3}$ and the $\overline{A_{z3}}$. The complex conjugate field has naturally appeared to give manifestly pure imaginary Liouville form and, as we see below, the constraints. These phenomena also reflect a nice structure of Ehlers’ symmetry in the Ashtekar formulation. This transformation is generically non-local with respect to the original Ashtekar variables.

We will now write the first class constraints in manifestly invariant forms in terms of the canonical variables above. The Hamiltonian constraint is reduced to

$$\mathcal{H} = \Delta^{-1} f^2 \left[ 2\epsilon_{3\alpha\beta} \epsilon_\gamma^\mu \epsilon_\alpha^\mu F_{z\mu\beta} + 2\epsilon_{3\alpha\beta} \epsilon_\gamma^\mu \epsilon_\beta^\mu F_{z\mu3} + \epsilon_{3\alpha\beta} \epsilon_\alpha^\mu \epsilon_\beta^\mu F_{\mu\nu3} \right], \quad (63)$$

where

$$F_{z\mu\beta} = -\partial_\mu A_{z\beta} + \epsilon_{\beta\gamma3} (A_{z\gamma} A_{\mu3} - A_{z3} A_{\mu\gamma}),$$

$$F_{z\mu3} = -\partial_\mu A_{z3} + \epsilon_{3\alpha\beta} A_{z\alpha} A_{\mu\beta},$$

$$F_{z\mu3} = \partial_\mu A_{\nu3} - \partial_\nu A_{\mu3} + \epsilon_{3\alpha\beta} A_{\mu\alpha} A_{\nu\beta}. \quad (64)$$

There appears no $F_{\mu\nu\alpha}$. Substituting (41) (42) into (63) (64), we find that all the explicit dependence on $B_\mu$ cancel, obtaining

$$\mathcal{H} = -f^2 \Delta^{-2} \left[ -A_{z3} \overline{A_{z3} - A_{z3} \overline{A_{z3}}} + \epsilon_{3\alpha\beta} f_\alpha^\mu f_\beta^\mu \left( \partial_\mu \omega^{(f)}_{\nu12} - \partial_\nu \omega^{(f)}_{\mu12} \right) + \epsilon_{3\alpha\beta} \epsilon_{3\gamma\delta} \omega^{(f)}_{\alpha\gamma0} \omega^{(f)}_{\beta\delta0} \right]$$

$$= \frac{1}{2} (\phi + \overline{\phi})^{-2} \tilde{e}_\alpha^\mu \tilde{e}_\alpha^\nu \partial_\mu \phi \partial_\nu \overline{\phi} + \frac{1}{2} (\phi + \overline{\phi})^2 (f \Delta^{-2} A_{z3})(f \Delta^{-2} \overline{A_{z3}})$$

$$- 2\epsilon_{3\alpha\beta} \tilde{e}_\alpha^\mu \tilde{e}_\alpha^\nu \partial_\mu \omega^{(f)}_{\nu12} - \epsilon_{3\alpha\beta} \epsilon_{3\gamma\delta} \tilde{e}_\alpha^\mu \omega^{(f)}_{\mu\gamma0} \tilde{e}_\beta^\nu \omega^{(f)}_{\nu\delta0}, \quad (65)$$

where in the first line we have used

$$A_{z3} - \epsilon_{3\alpha\beta} f_\beta^\mu \partial_\mu \Delta = -\overline{A_{z3}}. \quad (66)$$

The last two terms of (65) are trivially invariant (Note that $\omega^{(f)}_{\nu12}$ is a function of $\tilde{e}_\alpha^\mu$). The first two are also invariant since

$$\phi + \overline{\phi} \rightarrow \frac{\phi + \overline{\phi}}{|c\phi + i\delta|},$$

$$\partial_\mu \overline{\phi} \rightarrow \frac{\partial_\mu \overline{\phi}}{(ic\overline{\phi} - d)^2},$$

$$\partial_\mu \phi \rightarrow \frac{\partial_\mu \phi}{(ic\phi + d)^2}. \quad (67)$$
Thus the expression \((65)\) is invariant under Ehlers’ SL(2,R) transformation \((62)\). \((65)\) is independent of the sign choice of the Ashtekar connection.

The diffeomorphism constraints \(\mathcal{C}_\mu\) can be also made manifestly symmetric

\[
\mathcal{C}_\mu = \pm i \left\{ \partial_\nu (\overline{\omega}^{\nu}_{\mu \alpha 0}) - \overline{\omega}^{\nu}_{\mu \alpha 0} \partial_\nu \phi \right\} - (\partial_\mu \overline{\omega} \cdot f \Delta^{-2} A_{z3} - \partial_\mu \phi \cdot f \Delta^{-2} \overline{A_{z3}}). \tag{68}
\]

Finally the Lorentz constraint \(\mathcal{G}_3\) is simply written as

\[
\mathcal{G}_3 = \pm i \epsilon_{\alpha \beta \gamma} \overline{\omega}^{\mu}_{\beta \gamma} \omega^{(f)}_{\mu \alpha 0}. \tag{69}
\]

This completes the proof of Ehlers’ SL(2,R) symmetry in the Ashtekar formulation.

**D. Conserved charges**

It is now easy to calculate the conserved charges associated with the invariance under \((62)\). Corresponding to the Chevalley generators of sl(2,R)

\[
h \equiv \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad e \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{70}
\]

we will write the infinitesimal Ehlers’ SL(2,R) action on the canonical variables as \(h^{(E)}, e^{(E)}, f^{(E)}\), which we may read off from \((62)\). We have

\[
h^{(E)}(\overline{\phi}) = -2\overline{\phi}, \quad e^{(E)}(\overline{\phi}) = \frac{1}{i}, \quad f^{(E)}(\overline{\phi}) = \frac{1}{i} \overline{\phi}^2, \quad f^{(E)}(\phi) = -\frac{1}{i}, \quad f^{(E)}(\phi) = -\frac{1}{i} \phi^2, \tag{71}
\]

while

\[
h^{(E)}(\pm i \omega^{(f)}_{\mu \alpha 0}) = e^{(E)}(\pm i \omega^{(f)}_{\mu \alpha 0}) = f^{(E)}(\pm i \omega^{(f)}_{\mu \alpha 0}) = 0. \tag{72}
\]

Setting \(f \Delta^{-2} A_{z3} \equiv p_{\overline{\phi}}, -f \Delta^{-2} \overline{A_{z3}} \equiv p_{\phi}\), the conserved charges are given by

\[
K_{h^{(E)}} \equiv \int -2 \left( p_{\overline{\phi}} \overline{\phi} + p_{\phi} \phi \right), \\
K_{e^{(E)}} \equiv \int \frac{1}{i} \left( p_{\overline{\phi}} - p_{\phi} \right), \\
K_{f^{(E)}} \equiv \int \frac{1}{i} \left( p_{\overline{\phi}}^2 - p_{\phi} \phi^2 \right). \tag{73}
\]
It is an easy exercise to check that their Poisson bracket algebra satisfy the commutation relation of \( sl(2,\mathbb{R}) \). We may also verify that these conserved charges coincide the ones obtained in ref. \[26\] up to trivial constant factors. We will see in the next section that the conserved charges associated with Matzner-Misner’s \( SL(2,\mathbb{R}) \) can be expressed precisely in the same form as above.

IV. REDUCTION FROM FOUR TO TWO DIMENSIONS

A. Invariance under Matzner-Misner’s \( SL(2,\mathbb{R}) \)

To reduce the spacetime dimensions to two, we introduce another Killing vector field on \( \tilde{\Sigma} \). Since we do not discuss the global applicability of the Geroch group, we simply assume that we may take a global coordinate \( y \) such that \( \frac{\partial}{\partial y} \) is a Killing vector field. \( \tilde{\Sigma} \) is not necessarily be compact. We adopt the same gauge-fixing condition as was set in ref. \[19\] to fix the diffeomorphism degrees of freedom of \( y \)- and \( z \)-coordinates

\[
\tilde{e}_y^1 = \tilde{e}_z^1 = 0. 
\]

These conditions allow us to solve \( C_y = C_z = 0 \), which are now of second class. The solution is found to be \[19\]

\[
A_{y1} = A_{z1} = 0. 
\]

They simply determine the Lagrange parameters \( N^y \) and \( N^z \) so that \( \Omega_{10\epsilon} = 0 \). This means that \( \partial_x N^y = \partial_x N^z = 0 \). In ref. \[19\] the gauge-fixing condition for the Lorentz constraints \( \mathcal{G}_2, \mathcal{G}_3 \) are also set by requiring

\[
\tilde{e}_2^x = \tilde{e}_3^x = 0. 
\]

We have, however, already set \( \tilde{e}_2^y = \tilde{e}_3^y = 0 \) to distinguish the coordinate of the \( U(1) \) fiber. Consequently there remain no degrees of freedom of Lorentz rotations. The solution for the Lorentz constrains \( \mathcal{G}_2 = \mathcal{G}_3 = 0 \) is \[19\]
\[ A_{x2} = A_{x3} = 0. \quad (77) \]

It turns out that these equations arise no further restriction on the Lagrange parameters than the ones arising from \([75]\). To summarize, the non-vanishing components of the densitized inverse dreibein are

\[
\tilde{e}_a^m = \begin{pmatrix}
\tilde{e}_1^x \\
\tilde{e}_2^y \\
\tilde{e}_2^z \\
\tilde{e}_3^z
\end{pmatrix} = \begin{pmatrix}
\rho \\
f_1^1 - f_1^1 \psi \\
\rho \\
f_1^1 \Delta
\end{pmatrix}, \quad (78)
\]

where we also wrote their parameterization suitable for the description of Matzner-Misner's SL(2,R) symmetry. The vanishing Ashtekar connections are

\[ A_{x2} = A_{x3} = A_{y1} = A_{z1} = 0. \quad (79) \]

The independent canonical pairs are thus \((\tilde{e}_1^x, A_{x1}), (\tilde{e}_2^y, A_{y2}), (\tilde{e}_2^z, A_{x1})\) and \((\tilde{e}_3^z, A_{z3})\). Finally, \(A_{y3}\) is non-vanishing, but is not independent, obeying

\[ \tilde{e}_2^y A_{y3} = \partial_x \tilde{e}_1^x + \tilde{e}_3^z A_{x2} - \tilde{e}_2^z A_{z3}. \quad (80) \]

In the similar way we have done in the previous section, let us first write the Ashtekar connections in terms of the parameterization \([78]\). The coefficients of anholonomy are in this case

\[
\Omega_{1\bar{b}\bar{c}} = e_1^\bar{r} e_b^\bar{n} \partial_x \tilde{e}_{\bar{r} \bar{n}}, \\
\Omega_{0\bar{b}\bar{c}} = E_{\bar{m}}^\bar{n} e_b^\bar{n} \partial_{\bar{m}} \tilde{e}_{\bar{r} \bar{n}}, \\
\Omega_{10\bar{c}} = e_1^\bar{r} (E_{\bar{m}}^\bar{n} \partial_x E_{\bar{m}} - E_{\bar{m}}^\bar{n} \partial_{\bar{m}} \tilde{e}_{\bar{r} \bar{n}}) = 0, \\
\Omega_{101} = e_1^\bar{r} E_{\bar{m}}^\bar{n} (\partial_x E_{\bar{n}1} - \partial_{\bar{n}} E_{\bar{x}1}), \\
\Omega_{100} = -e_1^\bar{r} N^{-1} \partial_x N,
\]

otherwise = 0, \quad (81)

where the dreibein is now \(^4\)

\(^4\)We use the notation \(\tilde{e}^\bar{a}_\bar{m}\) for the lower right two by two block of the dreibein, following ref. [10].

This is, of course, no complex conjugate of anything.
\[ e^a_m = \begin{pmatrix} e^1_x \\ e^\bar{a}_m \end{pmatrix} = \begin{pmatrix} f_x^1 \\ \rho \ \psi \end{pmatrix} \begin{pmatrix} \Delta^{-\frac{1}{2}} \\ \Delta^{-\frac{1}{2}} \end{pmatrix}. \] (82)

We have set \( f^2_y \equiv \rho, B_y \equiv \psi \) for simpler notations (but will leave \( f^1_x \) as it is, to remember that it belongs to the “uncompactified” sector, on which the center of the Geroch group acts as a scale transformation, as we shall see below). The Ashtekar connections are calculated to be

\[
\begin{align*}
A_{z3} &= \frac{\alpha}{2} f_0^m \partial_m \Delta + \frac{1}{2} \Delta^2 \rho^{-1} f_x^1 \partial_x \psi, \\
A_{z2} &= -\frac{1}{2} f_x^1 \partial_x \Delta + \frac{\alpha}{2} \Delta^2 \rho^{-1} f_0^m \partial_m \psi, \\
A_{y3} &= \Delta^{-1} \rho A_{z2} + \psi A_{z3} + f_1^1 \partial_x \rho, \\
A_{y2} &= -\Delta^{-1} \rho A_{z3} + \psi A_{z2} + \alpha f_0^m \partial_m \rho, \\
A_{x1} &= -\frac{1}{2} \rho \partial_x \psi + \alpha \sqrt{\Delta} N' \left( \partial_t \left( \frac{f_x^1}{\sqrt{\Delta}} \right) - \partial_x \left( \frac{N'}{\sqrt{\Delta}} \right) \right),
\end{align*}
\] (83)

where, from the definition (78),

\[
\begin{align*}
\rho &= \tilde{e}_1^m, \\
f_x^1 &= \tilde{e}_2^y, \\
\psi &= -\tilde{e}_2^z, \\
\Delta &= \frac{\tilde{e}_1^x \tilde{e}_2^y}{\tilde{e}_3^z},
\end{align*}
\] (84)

and \( \alpha \) stands for \( \pm i \) in the definition of the Ashtekar connection (26). The expression (83) is not very illuminating. However, let us consider the following combinations

\[
\begin{align*}
(\pm \pm) &\equiv \tilde{e}_\pm^m A_{\pm m} \\
&= \pm i f_x^1 \rho (\alpha f_0^m \pm if_1^m) \frac{\Delta}{\rho} \partial_m (\psi \pm i \rho \Delta), \\
(\mp \pm) &\equiv \tilde{e}_\mp^m A_{\pm m} \\
&= f_x^1 (\alpha f_0^m \pm if_1^m) \partial_m \rho. \quad (85)
\end{align*}
\]
Recalling (13), we see that they are good variables to describe Matzner-Misner’s symmetry.

We now look for canonical pairs which transform under Matzner-Misner’s SL(2,R) in a simple way. $A_{z3}$ and $A_{z2}$ are written in terms of the variables (85) as

$$A_{z3} = \frac{1}{4\tilde{e}_3^3} \{(++) + (--) + (++) + (--)\},$$

$$A_{z2} = \frac{1}{4i\tilde{e}_3^3} \{(++) + (--) - (++) - (--)\}.$$  \hspace{1cm} (86)

$A_{y2}$ can also be written as

$$A_{y2} = -\frac{1}{4} \left[ \frac{\tilde{e}_2}{\tilde{e}_3^3} \{(++) - (--)\} - \frac{\tilde{e}_2^y}{\tilde{e}_3^3} \{(++) - (--)\} \right].$$  \hspace{1cm} (87)

Hence up to a total derivative we may then write

$$\dot{A}_{y2}\tilde{e}_2^y + \dot{A}_{z2}\tilde{e}_2^z + \dot{A}_{z3}\tilde{e}_3^z$$

$$= -A_{y2}\dot{\tilde{e}}_2^y - A_{z2}\dot{\tilde{e}}_2^z - A_{z3}\dot{\tilde{e}}_3^z$$

$$= -\frac{\tilde{e}_2^y}{4\tilde{e}_3^3} \left( \frac{\tilde{e}_2}{\tilde{e}_3^3} \right) \cdot (++) - \frac{\tilde{e}_2^z}{4\tilde{e}_3^3} \left( \frac{\tilde{e}_3}{\tilde{e}_3^3} \right) \cdot (--) - \frac{(\tilde{e}_2^y\tilde{e}_3^3)}{4\tilde{e}_3^3\tilde{e}_3^3} \cdot \{(++) + (--)\}$$

$$+ \frac{1}{2} \partial_x \tilde{e}_1^1 \cdot \frac{\tilde{e}_2^y}{\tilde{e}_3^3} \left( \frac{\tilde{e}_3}{\tilde{e}_3^3} \right),$$  \hspace{1cm} (88)

where we have used

$$(++) - (--) = 2i\partial_x \tilde{e}_1^1,$$  \hspace{1cm} (89)

which can be shown by the relation (81). Although the first three terms of (88) are good canonical pairs obeying a simple transformation rule, the last term is not because it contains $\left( \frac{\tilde{e}_3}{\tilde{e}_3^3} \right) = \psi$. If one looks at the expression of Ashtekar’s connection (83), one may guess what the good canonical variables are. It turns out that the combinations

$$[++ i \tilde{e}_2^y/\tilde{e}_3^3 \partial_x \left( \tilde{e}_2^y/\tilde{e}_3^3 \right) \text{ and } (--) + i \tilde{e}_2^y/\tilde{e}_3^3 \partial_x \left( \tilde{e}_2^y/\tilde{e}_3^3 \right),$$  \hspace{1cm} (90)

rather than $(++)$ and $(--)$ themselves, are such good variables (if multiplied by $-\frac{\tilde{e}_2^y}{4\tilde{e}_3^3}$). By this replacement the first two terms of (88) read
(the first two terms of (88))

\[
- \frac{e^y}{4e^3} \left[ \left( \frac{\dot{e}^z}{e^2} \right) \cdot \left\{ (++\) - ie_1 \frac{e^y}{e^3} \partial_x \left( i \frac{\dot{e}^z}{e^3} \right) \right\} \right] + \left( \frac{\dot{e}^z}{e^2} \right) \cdot \left\{ (--) + ie_1 \frac{e^y}{e^3} \partial_x \left( i \frac{\dot{e}^z}{e^3} \right) \right\}
\]

\[
+ \frac{1}{2} \frac{e^y}{e^3} \left( \frac{e^y}{e^2} \right)^2 \left[ \left( \frac{\dot{e}^z}{e^3} \right) \cdot \partial_x \left( \frac{e^z}{e^2} \right) - \left( \frac{\dot{e}^z}{e^3} \right) \cdot \partial_x \left( \frac{e^z}{e^2} \right) \right].
\]

(91)

The sum of the last term of (88) and the second term of (91) simply gives \( \frac{1}{2} \frac{e^y}{e^3} \cdot \frac{e^y}{e^2} \partial_x \left( \frac{e^z}{e^2} \right) \) up to a total derivative term. Thus we have succeeded to rewrite the Liouville form as follows:

\[
A_{x1} \dot{e}^x_1 + A_{y2} \dot{e}^y_2 + A_{z3} \dot{e}^z_3 + A_{x3} \dot{e}^z_3
\]

\[
= \left( \frac{\dot{e}^x_1}{e^2} \right) p_+ + \left( \frac{\dot{e}^z_2}{e^2} \right) p_- + (\frac{e^y_2 e^z_3}{e^2}) p_{23} + \dot{e}^z_1 p_1,
\]

(92)

where

\[
p_+ = - \frac{e^y}{4e^3} \left\{ (--) + ie_1 \frac{e^y}{e^3} \partial_x \left( i \frac{\dot{e}^z}{e^3} \right) \right\},
\]

\[
p_- = - \frac{e^y}{4e^3} \left\{ (++\) - ie_1 \frac{e^y}{e^3} \partial_x \left( i \frac{\dot{e}^z}{e^3} \right) \right\},
\]

\[
p_{23} = - \frac{1}{4e^2 e^3} \{(-+ \) \} ,
\]

\[
p_1 = - A_{x1} + \frac{e^y_2 e^z_3}{e^2} \partial_x \left( \frac{e^z_2}{e^2} \right).
\]

(93)

Matzner-Misner’s SL(2,R) acts on these canonical variables in the following simple way:

\[
\left( \frac{\dot{e}^z_1}{e^2}, \ p_+ \right) \rightarrow \left( \frac{1}{i} \cdot \frac{i a \dot{e}^z_1}{e^2} + b \right) \left( \frac{\dot{e}^z_1}{e^2} + d \right)^2 p_+,
\]

\[
\left( \frac{\dot{e}^z_2}{e^2}, \ p_- \right) \rightarrow \left( -\frac{1}{i} \cdot \frac{-i a \dot{e}^z_2}{e^2} + b \right) \left( -\frac{\dot{e}^z_2}{e^2} + d \right)^2 p_-,
\]

\[
\left( \frac{e^y_2 e^z_3}{e^2}, \ p_{23} \right) \rightarrow \left( \frac{e^y_2 e^z_3}{e^2}, \ p_{23} \right),
\]

\[
\left( \frac{\dot{e}^x_1}{e^2}, \ p_1 \right) \rightarrow \left( \dot{e}^x_1, \ p_1 \right).
\]

(94)

Obviously \( \frac{\dot{e}^z_1}{e^2} / \frac{\dot{e}^z_2}{e^2} \) correspond to the Ernst potential and its complex conjugate in the case of Ehlers’ SL(2,R). The Liouville form (92) is manifestly invariant under the transformation (94).
Let us rewrite again the constraints into manifestly invariant forms by using these canonical variables. The Hamiltonian constraint in this reduced model is given by

\[ H = -2\epsilon_1 \epsilon_2 D_x A_{m\bar{n}} \cdot \bar{e}_x^m \bar{e}_1^x + 2(A_{y2} A_{z3} - A_{y3} A_{z2}) \bar{e}_2^y \bar{e}_3^z, \]  

(95)

where \( D_x \) stands for the covariant derivative with respect to the Ashtekar connection. The first term of (95) may be written as

\[
i\bar{e}_1^x \left[ \bar{e}_m^m \partial_x A_{m+} - \bar{e}_m^m \partial_x A_{m-} - iA_{x1} \cdot \{(+-) + (+-)\} \right].
\]

(96)

Making use of the relations (89) and (80), one may further rewrite this as

\[
e = \bar{e}_1^x \frac{\bar{e}_2^y}{2\bar{e}_3^z} \left\{ \partial_x \left( \frac{\bar{e}_3^z}{\bar{e}_2^y} \right) (++) - \partial_x \left( \frac{\bar{e}_2^y}{\bar{e}_3^z} \right) (--) \right\} - p_1 \{(+-) + (+-)
\]

\[
+ 2\partial_x \partial_x \bar{e}_1^x - \partial_x \bar{e}_1^x \cdot \frac{\partial_x (\bar{e}_2^y \bar{e}_3^z)}{\bar{e}_2^y \bar{e}_3^z} \right].
\]

(97)

It is easy to see that the second term of (95) is equal to

\[-\frac{1}{2} (++) (--) + \frac{1}{2} (--) (+-).\]

(98)

Summing up (97) and (98), we end up with the following manifestly invariant expression of \( H \):

\[
H = -\frac{1}{2} \left[ \left( \frac{4\bar{e}_3^z}{\bar{e}_2^y} \right)^2 p_+ p_- - \left( \frac{\bar{e}_3^z}{\bar{e}_2^y} \right)^2 \partial_x \left( \frac{\bar{e}_3^z}{\bar{e}_2^y} \right) \partial_x \left( \frac{\bar{e}_3^z}{\bar{e}_2^y} \right) \right] + 4\bar{e}_1^x \bar{e}_2^y \bar{e}_3^z p_{23} p_1
\]

\[
+ \bar{e}_1^x \left( 2\partial_x \partial_x \bar{e}_1^x - \partial_x \bar{e}_1^x \cdot \frac{\partial_x (\bar{e}_2^y \bar{e}_3^z)}{\bar{e}_2^y \bar{e}_3^z} \right) + 2(\bar{e}_2^y \bar{e}_3^z)^2 (p_{23})^2 + \frac{1}{2} (\partial_x \bar{e}_1^x)^2.
\]

(99)

For the diffeomorphism constraints, the only remaining first class one is

\[
C_x = \partial \bar{e}_1^x \cdot A_{x1} - \bar{e}_a^m \partial_x A_{\bar{m}\bar{a}}.
\]

(100)

After some similar rearrangement of formulas we find

\[
C_x = \left[ \partial_x \bar{e}_1^x \cdot p_1 + \partial_x \left( \frac{\bar{e}_1^x}{\bar{e}_2^y} \right) \cdot p_+ + \partial_x \left( \frac{\bar{e}_1^x}{\bar{e}_2^y} \right) \cdot p_- + \partial_x (\bar{e}_2^y \bar{e}_3^z) \cdot p_{23} \right],
\]

(101)

which is clearly invariant under Matzner-Misner’s SL(2,R) (94).
B. conserved charges

It is also straightforward to calculate the conserved charges for Matzner-Misner’s SL(2,R). Corresponding to the generators of sl(2,R), we may in this case write down the infinitesimal Matzner-Misner’s SL(2,R) action on the canonical variables as \( h^{(MM)}, e^{(MM)}, f^{(MM)} \). For \( \tilde{e}_1^\pm / \tilde{e}_2^y \equiv q_\pm \) we have

\[
\begin{align*}
h^{(MM)}(q_\pm) & = -2q_\pm, \\
e^{(MM)}(q_\pm) & = \pm \frac{1}{i}, \\
f^{(MM)}(q_\pm) & = \pm \frac{1}{i}(q_\pm)^2,
\end{align*}
\]

while

\[
\begin{align*}
h^{(MM)}(\tilde{e}_1^x) & = e^{(MM)}(\tilde{e}_1^x) = f^{(MM)}(\tilde{e}_1^x) = h^{(MM)}(\tilde{e}_2^y \tilde{e}_3^z) = e^{(MM)}(\tilde{e}_2^y \tilde{e}_3^z) = f^{(MM)}(\tilde{e}_2^y \tilde{e}_3^z) = 0.
\end{align*}
\]

The conserved charges read

\[
\begin{align*}
K_{h^{(MM)}} & \equiv \int -2(p_+q_+ + p_-q_-), \\
K_{e^{(MM)}} & \equiv \int \frac{1}{i}(p_+ - p_-), \\
K_{f^{(MM)}} & \equiv \int \frac{1}{i}(p_+(q_+)^2 - p_-(q_-)^2).
\end{align*}
\]

These expressions are completely the same as the ones in the previous section if one replaces \((\phi, p_\phi)\) and \((\phi, p_\phi)\) by \((q_+, p_+)\) and \((q_-, p_-)\), respectively. This is a consequence of Kramer-Neugebauer’s transformation which relates the two SL(2,R) symmetry in the two-Killing reduced model [29].

It is not a coincidence that both of the conserved charges (73) and (104) consist of a part of realization of the classical \( w_\infty \) algebra [37] in terms of canonical pairs. It is known

\[\text{(Note that the conserved charges for Ehlers’ SL(2,R) are still given by (73) if the integration is performed in the one-dimensional coset space, since the further reduction from three to two dimensions affect only the invariant sector for Ehlers’ SL(2,R).)\]
that, in general, if one has a canonical pair \((q,p)\) with a Poisson bracket \(\{q, p\} = 1\), one can realize the classical \(w_\infty\) algebra by assigning

\[
W_n^{(l)} = p^{l-1} q^{n+l-1} \quad (l \geq 1, \ n \geq -l + 1).
\]

Their Poisson bracket satisfies the commutation relation of the \(w_\infty\) algebra

\[
\{W_n^{(l)}, W_m^{(k)}\} = ((k - 1)n - (l - 1)m)W_{n+m}^{(l+k-2)},
\]

known as an algebra of the area-preserving diffeomorphism. Indeed, the canonical transformation is by definition a transformation that preserves the area of \((q, p)\) phase space.

\(W_n^{(l)}\) is then the generating function of the canonical transformation. This algebra contains “half” of the Virasoro (Witt) subalgebra \((L_n \geq -1)\) generated by \(L_n = W_n^{(2)}\), and this Virasoro algebra further contains therein the \(sl(2,\mathbb{R})\) generated by \(\{L_{-1} = p, L_0 = pq, L_1 = pq^2\}\). Both conserved charges (73) and (104) are diagonal sums of \(sl(2,\mathbb{R})\) generators made out of two sets of canonical pairs. Both for Ehlers’ and Matzner-Misner’s case, we have succeeded to take two canonical pairs, only on which the \(sl(2,\mathbb{R})\) in question act as canonical transformations. Therefore, the algebra generated by the symmetry charges is necessarily a subalgebra of the canonical transformation on these two canonical pairs, and this subalgebra contains the diagonal \(w_\infty\) algebra as a special case. What we have observed here is that the diagonal \(sl(2,\mathbb{R})\) in this diagonal \(w_\infty\) realizes the symmetry algebra.

\[\text{C. GL}(2,\mathbb{R}) \text{ in [19] and the central extension in the Geroch group}\]

In ref. [19] a set of conserved quantities was found in the two-Killing reduced Einstein gravity. They are given by in our notation

\[
K_{\bar{m}\bar{n}} = \int \bar{e}_{\bar{a}}^\bar{m} A_{\bar{n}\bar{a}}
\]

\((\bar{m}, \bar{n} = y, z, \text{and } \bar{a} \text{ runs over } \{2, 3\})\). The Poisson brackets between two of these quantities form the \(gl(2,\mathbb{R}) \sim sl(2,\mathbb{R}) \oplus \mathbb{R}\). Let us examine to which symmetry this \(gl(2,\mathbb{R})\) corresponds
in the Geroch group. In fact, these conserved charges in the sl(2,R) sector are precisely the ones associated with Matzner-Misner’s symmetry $^6$. Indeed, the three generators

$$\{K^z_z - K^y_y, K^y_y, K^z_y\}$$

constitute the sl(2,R), corresponding to $\{h, e, f\}$, respectively, while the trace $K^z_z + K^y_y$ commutes with any of these elements. Using (80), (87), (86) and (89) and integrating by parts, it is straightforward to check that the conserved quantities (108) precisely reproduce the conserved charges (104).

The rather complicated look of the conserved charges (104) is in fact an artifact of the gauge-fixing (78). Indeed, Matzner-Misner’s symmetry can be seen as a symmetry that mixes the $y$ and $z$ indices. If we do not fix $\bar{\epsilon}_3^a = 0$ but restore the full $\bar{\epsilon}_a^m (\bar{m} = y, z; \bar{a} = 2, 3)$, such a variation can be written as

$$\delta(\bar{e}_a^m) = \bar{e}_a^m X_{\bar{n}}^m$$

for some $X_{\bar{n}}^m \in$ sl(2,R). If we take $X_{\bar{n}}^m = -h, e, f$ defined in (70) (The minus sign for $h$ is because the variation is a right action on $\bar{e}_a^m$.), the conserved charges associated with the invariance under these variations, which can be verified in the expressions of the constraints in ref. [19] $^7$ are nothing but the conserved quantities (107) obtained there.

What role does the trace of the gl(2,R) play in the Geroch group, then? To answer this question, let us first recall how the two sl(2,R) Lie algebras are combined to give the affine Kac-Moody algebra $\hat{\text{sl}}(2, \mathbb{R})$. The affine Kac-Moody algebra $\hat{\text{sl}}(2, \mathbb{R})$ is defined by the following commutation relations:

$$[H_n, E_m] = 2E_{n+m},$$

$^6$This fact has been already pointed out by H.Nicolai $^7$.

$^7$From the fact that all the indices $\bar{n}$ of $\bar{e}_a^m$ (corresponding to the index “$\alpha$” in ref. [19]’s notation) are contracted by those of $A_{\bar{n}\bar{b}}$. 30
\[
[H_n, F_m] = -2F_{n+m},
\]
\[
[E_n, F_m] = nk\delta_{n+m,0} + H_{n+m},
\]
\[
[H_n, H_m] = [E_n, E_m] = [F_n, F_m] = 0,
\]

where \( k \in \mathbb{R} \) belongs to the center of this algebra, and \( n, m \in \mathbb{Z} \). This contains following two \( \text{sl}(2,\mathbb{R}) \) subalgebras:

\[
[H_0, E_0] = 2E_0, \quad [H_0, F_0] = -2F_0, \quad [E_0, F_0] = H_0,
\]
\[
[k - H_0, F_1] = 2F_1, \quad [k - H_0, E_{-1}] = -2E_{-1}, \quad [F_1, E_{-1}] = k - H_0.
\]

Conversely, let us assume that the two sets of \( \text{sl}(2,\mathbb{R}) \) generators \( \{h_i, e_i, f_i\} \) \((i = 0, 1)\) satisfy

\[
[h_i, h_j] = 0, \quad [h_i, e_j] = A_{ij} e_j, \quad [h_i, f_j] = -A_{ij} f_j, \quad [e_i, f_j] = \delta_{ij} h_j \tag{112}
\]

and

\[
(\text{ad} e_i)^{1-A_{ij}}(e_j) = 0, \quad (\text{ad} f_i)^{1-A_{ij}}(f_j) = 0 \tag{113}
\]

for \( i \neq j \), where the Cartan matrix \( A_{ij} \) reads in this case

\[
A_{ij} = \begin{pmatrix}
2 & -2 \\
-2 & 2
\end{pmatrix}.
\]

One may define the \( \hat{\text{sl}}(2,\mathbb{R}) \) algebra as a Lie algebra generated by any successive multiplication of commutators satisfying (112)/(113). The set of (112)/(113) is called the Serre relation.

It has been shown that (the Lie algebra of) the Geroch group can be obtained by making use of two \( \text{sl}(2,\mathbb{R}) \) algebras, one of which is Ehlers’ and the other of which is Matzner-Misner’s, as the ones required in the Serre relation. One of the interesting features of the Geroch group is that it realizes the central-extended \( \hat{\text{sl}}(2,\mathbb{R}) \) algebra even at the classical level already. Usually, a central term such as in (111) arises as a consequence of anomaly in quantum field theory. Here the situation is different; identifying the two \( \text{sl}(2,\mathbb{R}) \)’s as the ones

31
corresponding to the two simple roots of the $\hat{\mathfrak{sl}}(2, \mathbb{R})$, the central element $k = H_0 + (k - H_0) = h_0 + h_1$ then acts non-trivially on the fields, although we are still considering classical gravity theory.

Let us now see how this central element acts on the Ashtekar variables. We may identify $h_0 = h^{(E)}$ and $h_1 = h^{(MM)}$. For $h^{(E)}$, we have seen in sect.3 that it acts on $\tilde{\phi}, \phi, \tilde{e}_\alpha^\mu$ as,

$$h^{(E)}(\tilde{\phi}) = -2\tilde{\phi}, \quad h^{(E)}(\phi) = -2\phi, \quad h^{(E)}(\tilde{e}_\alpha^\mu) = 0.$$  \hfill (115)

This means that

$$h^{(E)}(\Delta) = -2\Delta, \quad h^{(E)}(B) = -2B.$$  \hfill (116)

Hence, in the present two-Killing reduced model, it acts on the parameters in (78) as

$$h^{(E)}(\Delta) = -2\Delta, \quad h^{(E)}(\psi) = +2\psi, \quad h^{(E)}(f_1^x) = h^{(E)}(\rho) = 0,$$  \hfill (117)

where we have used the relation (57). We also know, on the other hand, the $h^{(MM)}$ action (102) on the parameters (78), which reads

$$h^{(MM)}(\Delta) = +2\Delta, \quad h^{(MM)}(\psi) = -2\psi, \quad h^{(MM)}(f_1^x) = +f_1^x, \quad h^{(MM)}(\rho) = 0.$$  \hfill (118)

The action of the central element $k$ is thus given by

$$k(f_1^x) = +f_1^x, \quad k(\Delta) = k(\psi) = k(\rho) = 0.$$  \hfill (119)

Therefore it causes a scale transformation only on $f_1^x$ without doing anything on the other parameters. This confirms the known fact that the central element acts as a rescaling on the

$^8$The Virasoro algebra generated by the Sugawara form of this $\hat{\mathfrak{sl}}(2, \mathbb{R})$ should not be confused with the Virasoro subalgebra of the $w_\infty$ in the last subsection. Obviously they are different things. In particular, the latter has only the half of generators $L_n$ for $n \geq -1$ and hence the “central extension” has no meaning.

32
conformal factor of the zweibein for the “uncompactified” sector. On the Ashtekar variables we have

\[ k(\tilde{e}_a^m) = \tilde{e}_a^m, \quad k(A_{\bar{m}\bar{a}}) = -A_{\bar{m}\bar{a}}, \quad k(\tilde{e}_1^x) = k(A_{x1}) = 0, \]

(120)

where \((\bar{m}, \bar{a}) \neq (y, 3)\) in our gauge. We thus find that the central element \(k\) acts as a scale transformation on the “compactified” sector \((\tilde{e}_a^m, A_{\bar{m}\bar{a}})\), while keeping the “uncompactified” sector \((\tilde{e}_1^x, A_{x1})\) invariant.

Let us go back to the question of the role of the center in the \(GL(2,\mathbb{R})\). From the analysis in the previous paragraphs, it is now obvious that the action of the trace of \(gl(2,\mathbb{R})\) \((107)\) is precisely the same as that of the central element of the Geroch group. Indeed, the variation on \(\tilde{e}_a^m\)’s from the action of the trace of \(gl(2,\mathbb{R})\) can be achieved by taking \(X^m_{\bar{n}}\) in \((109)\) to be the identity matrix.

We can show that \(K_y^y + K_z^z\) acts as a rescaling on \(\tilde{e}_a^m\) directly also as follows. Writing

\[ \tilde{e}_a^y A_{ya} = \frac{1}{2(a + b)} [-b(++) + a(-+) + b(+-) - a(--)], \]
\[ \tilde{e}_a^z A_{z\bar{a}} = \frac{1}{2(a + b)} [b(++) + a(-+) + b(+-) + a(--)], \]

(121)

where \(a = \tilde{e}_+^z\) and \(b = \tilde{e}_-^z\), we find

\[ K_y^y + K_z^z = \frac{1}{2} \int [(-+) + (++)] \]
\[ = -2 \int p_{23} q_{23}, \]

(122)

where we set \(q_{23} \equiv \sqrt{q_{23}}\). On the other hand, all \(\tilde{e}_2^y, \tilde{e}_2^z\) and \(\tilde{e}_3^z\) are functions of \(q_{\pm}\) and \(q_{23}\), and in particular they are proportional to \(\sqrt{q_{23}}\). Hence, the fact that \(K_y^y + K_z^z\) scales \(\tilde{e}_2^y, \tilde{e}_2^z\) and \(\tilde{e}_3^z\) while leaves \(\tilde{e}_1^x\) unchanged immediately follows from the following equations:

\[ \{\sqrt{q_{23}}, \quad -2 \int p_{23} q_{23}\} = \sqrt{q_{23}}, \]
\[ \{\tilde{e}_1^x, \quad -2 \int p_{23} q_{23}\} = 0. \]

(123)

To summarize what we have shown in this subsection, the (finite, non-affine) \(GL(2,\mathbb{R})\) symmetry found in ref. [19] is indeed a part of the Geroch group, where the \(SL(2,\mathbb{R})\) and
the trace sector correspond to the Matzner-Misner’s SL(2,R) and the central element of the Geroch group, respectively.

**D. GL(2,R) loop algebra?**

Finally, let us discuss the relation between the Geroch group and the GL(2,R) loop algebra constructed in ref. [19], where the following operators are considered in the loop representation:

$$L[\bar{\phi}] \equiv \int \phi_{\bar{n}} \bar{f}^m \frac{\delta}{\delta \bar{f}^n}. \quad (124)$$

The $x$-coordinate is compactified into $S^1$, $0 \leq x \leq 2\pi$. $\bar{f}^m$ are some two densities and $\phi_{\bar{n}}^m$ are any functions on this $S^1$. $L[\delta_{\bar{n}} \delta^m]$ reduces to the generators of GL(2,R) in the last subsection, where $\delta_{\bar{n}} \delta^m$ denotes a $2 \times 2$ matrix in which only the ($\bar{n}, \bar{m}$) component is 1 and otherwise 0.

It was speculated [19] that the loop algebra generated by (124) might be related to the Geroch group. As we have shown, the GL(2,R) algebra generated by $K_{\bar{n}}^m$ consists of Matzner-Misner’s SL(2,R) and the central element of Geroch group. Hence the GL(2,R) loop algebra indeed includes Matzner-Misner’s SL(2,R). Does this GL(2,R) loop algebra also includes Ehlers’ SL(2,R)?

In fact, this is not the case. This can be seen as follows. First, let us examine the Serre generators among (124). The basis of the $\hat{gl}(2, R)$ are obtained by expanding the function $\phi_{\bar{n}}^m$ in (124) in terms of the Fourier modes

$$L[e^{ikx} \delta_{\bar{n}} \delta^m], \ k \in Z. \quad (125)$$

The Serre generators of its $\hat{sl}(2, R)$ sector are

$$\{L[\delta_2 \delta^2] - L[\delta_y \delta^y], L[\delta_y \delta^y], L[\delta_2 \delta^2]) \quad \text{and} \quad \{-L[\delta_2 \delta^2] + L[\delta_y \delta^y], L[e^{ix} \delta_2 \delta^2], L[e^{-ix} \delta_y \delta^y]$.}

(126)
The first set is Matzner-Misner’s SL(2,R) as we saw in the last subsection, while the second one does not correspond to the conserved charges for Ehlers’ SL(2,R) (73) which acts non-locally on the Ashtekar variables.

Another reason to believe the absence of Ehlers’ SL(2,R) is that the \( \hat{\mathfrak{sl}}(2,\mathbb{R}) \) defined by the traceless generators of (125) has vanishing central charge. Basically, the affine Kac-Moody algebra constructed as a loop algebra has no central term until a cocycle term is introduced \([39]\). Here, as we have seen in the last subsection, the central element of the Geroch group is already included in the trace \( L[\delta z \delta z] + L[\delta y \delta y] \), but such a term never results from any commutators between the two generators of (123). We are thus led to the conclusion that the GL(2,R) loop algebra generated by the operators (124) is not the same as the Lie algebra of the Geroch group itself, but contains only Matzner-Misner’s SL(2,R) and the central element of the Geroch group as its GL(2,R) subalgebra.

V. CONCLUSION AND COMMENT

We have studied the realization of the Geroch group in the Ashtekar formulation in this paper. Our first observation was the relation between the Ashtekar connection and the Ernst potential. In the history of the gravity theory, the discovery of the former has brought us a chance to construct a consistent quantum theory, where its complexification is unavoidable in order to achieve the simplification of the constrains. On the other hand, the latter, introduced long before the discovery of the former, plays a central pole in the integrable Ernst equation, where its complex nature is the one inherited from the complex structure of the target space of the sigma model. There seem no reasons that they are necessarily related, and that makes this coincidence interesting.

We have constructed for each case of Ehlers’ and Matzner-Misner’s SL(2,R) a set of canonical variables that realize canonically either of them, but could not find the one that realizes both at the same time. To realize the first one, we are forced to use canonical variables which are non-local with respect to the original ones. Difficulties to realize canonically
the full Geroch group may be guessed from a known example on the canonical realization of
the non-local symmetry in a chiral model [25], which is much simpler than the present one.

We have shown that the action of the GL(2, R) charges constructed in ref. [19] corresponds
to a subgroup of the Geroch group, i.e. the product of Matzner-Misner’s SL(2, R) and the
center of the Geroch group. We have further examined whether or not their GL(2, R) loop
algebra contains Ehlers’ SL(2, R), but it does not. This can be seen either by comparing the
Serre generators with the Ehlers’ SL(2, R) symmetry charges, or by noticing the absence of
the central term in their loop algebra. Therefore their loop algebra is not the Lie algebra of
the Geroch group itself, but something else.

Finally we would like to comment on the recent argument that the two-Killing reduced
model has the same linear system as the one for a flat-space chiral model [27]. However, the
derivation is based on an unusual gauge-choice, which in fact can not be achieved generically.
The author of [27] starts from a block-diagonal dreibein, which is written in our notation

\[
e^a_m = \begin{pmatrix} e_1^a \\ \tilde{e}_m^a \end{pmatrix},
\]

or equivalently

\[
\tilde{e}_a^m = \begin{pmatrix} \tilde{e} \\ e_1^a \tilde{e} e_\tilde{m}^a \end{pmatrix}
\]

(\(m = y, z; \tilde{a} = 2, 3; \tilde{e} = \text{det} \tilde{e}_m^a\)). \(A_{x\tilde{a}}\) and \(A_{m1}\) are also set to zero similarly to Sec.IV. At this
stage we still have three gauge degrees of freedom generated by the first class constraints \(\mathcal{H},
\mathcal{C}_x\) and \(\mathcal{G}^1\). By adopting the gauge-fixing conditions

\[
A_{x1} = 0,
\]

\[
J \equiv \epsilon^{\tilde{a}\tilde{b}} A_{\tilde{m}a} \tilde{e}_m^\tilde{b} = 0,
\]

\[
K \equiv A_{\tilde{m}\tilde{a}} \tilde{e}_a^\tilde{m} = \text{const.},
\]

which means in consequence
\( \tilde{e}_1 = \tilde{e} = \text{const.}, \)
\( N = \text{const.}, \)
\( N^x = \text{const.}, \)

(130)

the author of [27] was led to a linear system of the flat-space SL(2,R) chiral model. However, in reality, the gauge-choice (130) can not be achieved generically. Indeed, the Lorentz rotation generated by \( G^1 \) can not affect the determinant \( \tilde{e} \), the lapse \( N \) nor the shift \( N^x \).

Hence we may employ only the other two gauge degrees of freedom to fix three independent elements of the vierbein, which is not be achievable in general unless the spacetime is flat from the beginning. Therefore the argument of ref. [27] does not show that the two-Killing reduced model is equivalent to a flat space chiral model.

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REFERENCES

[1] R. Geroch, J. Math. Phys. 12, 918 (1971); 13, 394 (1972).

[2] J. Ehlers, Dissertation, Univ. Hamburg (1957).

[3] R. Matzner and C. Misner, Phys. Rev. 154, 1229 (1967).

[4] “Exact solutions of Einstein’s Equations: Techniques and Results”, Lecture notes in Physics 205, eds. C. Hoenselaers and W. Dietz, Springer Verlag, Berlin (1984).

[5] F. J. Ernst, Phys. Rev. 167, 1175 (1968).

[6] V. A. Belinskii and V. E. Sakharov, Zh. Eksp. Teor. Fiz. 75, 1955 (1978); 77, 3 (1979).

[7] D. Maison, Phys. Rev. Lett. 41, 521 (1978).

[8] E. Cremmer and B Julia, Nucl. Phys. 159B, 141 (1979).

[9] B. Julia, in: “Proceedings of the John Hopkins Workshop on Particle Theory”, Baltimore (1981).

[10] P. Breitenlohner and D. Maison, Ann. Inst. Henri Poincaré, 46, 215 (1987).

[11] P. Breitenlohner, D. Maison and G. W. Gibbons, Commun. Math. Phys. 120, 295 (1988).

[12] H. Nicolai, in: “Recent Aspects of Quantum Fields”, Proceedings, Schladming 1991, eds. H. Mitter and H. Gausterer, Springer Verlag, Berlin (1991).

[13] H. Nicolai, Phys. Lett. B276, 333 (1992).

[14] A. Ashtekar, Phys. Rev. Lett. 57 (1986) 2244; Phys. Rev. D36, 1587 (1987); “Non Perturbative Canonical Gravity”, World Scientific, Singapore (1991).

[15] C. Rovelli and L. Smolin, Phys. Rev. Lett. 61, 1155 (1988); Nucl. Phys. B331, 80 (1990).
[16] T. Jacobsen and L. Smolin, Nucl. Phys. B299, 295 (1988).

[17] I. Bengtsson, Class. Quantum Grav. 5, L139 (1988); 7, 27 (1989).
S. Koshtii and N. Dadhich, Class. Quantum Grav. 6, L223 (1989).
L. Bombelli and R. J. Torrence, Class. Quantum Grav. 7, 1747 (1990).
H. Kastrup and T. Thiemann, Nucl. Phys. B399, 211 (1993).

[18] H. Kodama, Prog. Theor. Phys. 80, 1024 (1988); Phys. Rev. D42, 2548 (1990).

[19] V. Husain and L. Smolin, Nucl. Phys. B327, 205 (1989).

[20] A. Ashtekar and J. Pullin, Ann. Israel Phys. Soc. 9, 65 (1990).
Ashtekar, R. Tate and C. Uggla, Intern. J. Mod. Phys. D2, 15 (1993).

[21] V. Husain and J. Pullin, Mod. Phys. Lett. A5, 733 (1990).

[22] N. Manojlović and G. A. Mena Marugán, Phys. Rev. D48, 3704 (1993).
N. Manojlović and A Miković, Class. Quantum Grav. 10, 559 (1993).

[23] “Bibliography of Publications related to Classical and Quantum Gravity in terms of the
Ashtekar Variables”, updated by T. A. Schilling, gr-qc/9409031, (1994).

[24] R. Gowdy, Ann. Phys. 83, 203 (1974); Phys. Rev. Lett. 27, 826 (1971); 1102 (erratum).

[25] M. Lüscher and K. Pohlmeyer, Nucl. Phys. B137, 46 (1978).
M. C. Davies, P. J. Houston, J. M. Leinaas and A. J. Macfarlane, Phys. Lett. 119B, 187 (1982).

[26] V. Moncrief, Ann. Phys. 167, 118 (1986).

[27] V. Husain, “Observables for space-times with two Killing field symmetries”, Alberta U.
preprint, ALBERTA-THY-55-93 (1994).

[28] T. Kaluza, Sitzungsberichte der preussischen Akademie der Wissenschaften, 996 (1921)
O. Klein, Z. Phys. 37, 895 (1926).
[29] G. Neugebauer and D. Kramer, Commun. Math. Phys. 10, 132 (1968).

[30] R. Arnowitt, S. Deser and C. Misner, in: “Gravitation: An Introduction to Current Research”, ed. L. Witten, Wiley, New York (1962).

[31] D. J. Gross and M. J. Perry, Nucl. Phys. B226, 29 (1983).

[32] T. Eguchi, P. B. Gilkey and A. J. Hanson, Phys. Rep. 66, 213 (1980).

[33] M. J. Perry, in: “Monopole ’83”, Ann Arbor, 29 (1983).

[34] V. Moncrief, Class. Quantum Grav. 4, 1555 (1987).

[35] A. P. Veselov, Theor. Math. Phys. 54, 155 (1983).

G. Neugebauer and D. Kramer, in [3].

D. Korotkin, Class. Quantum Grav. 10, 2587 (1993).

[36] H. Nicolai and H. -J. Matschull, Journ. Geom. Phys. 11, 15 (1993).

[37] I. Bakas, Phys. Lett. B228, 57 (1989).

C. N. Pope, X. Sen and L. J. Romans, Nucl. Phys. B339, 191 (1990).

[38] V. G. Kac, “Infinite Dimensional Lie Algebras”, Cambridge, Cambridge Univ. Press (1990).

[39] A. Pressley and G. Segal, “Loop Groups”, Oxford, Clarendon Press (1986).