Susceptibility to disorder of the optimal resetting rate in the Larkin model of directed polymers

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Abstract

We consider the Larkin model of a directed polymer with Gaussian-distributed random forces, with the addition of a resetting process whereby the transverse position of the end-point of the polymer is reset to zero with constant rate $r$. We express the average over disorder of the mean time to absorption by an absorbing target at a fixed value of the transverse position. Thanks to the independence properties of the distribution of the random forces, this expression is analogous to the mean time to absorption for a diffusive particle under resetting, which possesses a single minimum at an optimal value $r^*$ of the resetting rate. Moreover, the mean time to absorption can be expanded as a power series of the amplitude of the disorder, around the value $r^*$ of the resetting rate. We obtain the susceptibility of the optimal resetting rate to disorder in closed form, and find it to be positive.

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1 Introduction

In a one-dimensional search problem, one may cut off long excursions into the wrong direction by returning to the starting point after a duration of search considered excessive. In a model of such a strategy, the searcher was assimilated in [1] to a diffusive random walker on a line with an absorbing target, resetting its position to its starting point at stochastic times, distributed according to a Poisson law of fixed rate $r$. The corresponding non-equilibrium stationary state was worked out. Moreover, the mean first passage time at the target is made finite by the resetting process, and can be minimised by setting the resetting rate to a value proportional to the diffusion constant divided by the square of the distance between the initial position and the target, up to a numerical constant given in terms of a transcendental equation [2].

The study of systems under resetting has since become a source of developments in out-of-equilibrium statistical physics [3–6], with applications including RNA polymerisation processes [7,8], active matter [9–11], randomised searching problems [12] and lifting of entropy barriers [13]. The corresponding renewal arguments [1, 2, 10] have been applied to models of active matter [10–14], predator-prey dynamics [15,16], population dynamics [17,18], and stochastic processes [19–22] (see [23] for a review, and references therein for more applications). For experimental realisations, see [24].

It is natural to ask how the one-dimensional picture of optimal resetting is deformed by the presence of a weak disorder in the environment. In the presence of random forces, a diffusive random walker is described as the transverse coordinate of the end-point of a directed polymer, described by the random-force model (or Larkin model). In this model of a directed polymer, random forces are transverse. Their amplitude depends only on the longitudinal direction, moreover their amplitudes are independent and identically distributed. This model emerged as a model for pinning in superconductivity [25–27], and its free-energy has been studied for fixed and free boundary conditions in [28,29]. The linear structure of the contribution of the disorder to the energy of the polymer makes the model the most elementary modification of the diffusion by disorder. The model enjoys exact-solvability properties because it gives rise to Gaussian path integrals (for more general developments on path integrals for systems under resetting, see [30]).

In Section 2, we will review the model in the absence of resetting, following the derivation of [29] (Gaussian integrals are exposed in Appendix A). We will work out the probability density of the transverse position of the end-point of the polymer. In Section 3 we will express the mean first-passage time at a fixed absorbing target, following the general arguments of [31], which are valid thanks to the independence of the configurations of random forces in distinct intervals between resetting times. The optimal resetting rate is defined as the rate that minimises this mean time (which is known exactly in the absence of disorder). The susceptibility of the optimal rate of disorder will be worked out as the quotient of two derivatives of the mean time. These two derivatives are obtained in closed form in Appendix B.
2 Model and quantities of interest

2.1 The random-force model

Let us consider a directed polymer in 1 + 1 dimension. An elastic line grows randomly on a plane. The plane is endowed with Cartesian coordinates, and we think of $x$, the directing coordinate, as time. The configuration of the polymer is therefore described by the transverse displacement field

$$\phi : x \in [0, L] \mapsto \phi(x) \in \mathbb{R}. \quad (1)$$

Let us assume the polymer start as a point-like object at the origin:

$$\phi(0) = 0. \quad (2)$$

Consider a disordered environment described by a random potential $V$ depending on the coordinate $x$ and the transverse displacement field. Moreover, let us assume that this potential corresponds to a random force which depends only on the coordinate $x$ and not on the current value of the transverse displacement, and which we denote by $f(x)$. The corresponding random potential is therefore expressed as $f(x)\phi(x)$. Let $c$ denote the elasticity constant of the polymer. The energy of a configuration of the polymer at a fixed value $L$ of the directing coordinate is the sum of the elastic energy and the potential energy:

$$H[\phi, L, f] := \int_0^L dx \left[ \frac{c}{2} \left( \frac{d\phi}{dx}(x) \right)^2 + f(x)\phi(x) \right]. \quad (3)$$

The distribution of the random force field $f$ is assumed to be a centered Gaussian, with variance denoted by $u$. Denoting averages over disorder by bars, we write

$$\overline{f(x)} = 0, \quad \overline{f(x)f(x')} = u\delta(x - x'). \quad (4)$$

This choice of disordered potential is referred to as the random-force model (or Larkin model [25–27]). Setting the parameter $u$ we recover the free case, in which the end-point of the polymer is a diffusive Brownian random walker (the diffusion constant will be expressed in terms of the temperature and diffusion constant in Eq. (19)).

2.2 Partition function of a sector with fixed ends, for a given realisation of the random forces

Consider a given realisation of the force field $f$. For a value $L$ of the directing coordinate, consider the sector consisting of all the transverse displacement fields such that the end-point of the polymer is at a fixed value $y$, meaning $\phi(L) = y$. If thermal agitation is described by the parameter $\beta$, this sector corresponds to the following Boltzmann weight:

$$Z[L, y; f] := \int_{\phi(0) = 0}^{\phi(L) = y} [D\phi(x)] \exp \left( -\beta H[\phi, f] \right). \quad (5)$$
This functional integral is Gaussian and can therefore be explicitly evaluated. Let us follow the derivation of [29] and impose the boundary conditions by changing function from the displacement field \( \phi \) to the field \( \varphi \) defined by the linear shift:

\[
\phi(x) = \frac{x}{L}y + \varphi(x), \quad \text{with} \quad \varphi(0) = \varphi(L) = 0.
\]

\[
Z[L, y; f] = \exp \left( -\beta \frac{y^2}{2L} - \beta f(x) \int_0^L dx \frac{x}{L}y \right) \times \int_{\varphi(0) = 0}^{\varphi(L) = 0} [D\varphi(x)] \exp \left( -\beta \left[ \int_0^L dx \left( \frac{c}{2} \varphi'(x))^2 + cy\varphi'(x) + f(x)\frac{x}{L}y + f(x)\varphi(x) \right) \right).
\]

Let us denote by \( \varphi_q \) the field that makes the energy stationary, and by \( \delta \varphi \) the fluctuations around this value:

\[
\varphi(x) = \varphi_q(x) + \delta \varphi(x),
\]

where \( \varphi_q \) satisfies the equation

\[
c\varphi_q''(x) = f(x), \quad x \in [0, L].
\]

The fluctuations decouple in the following sense:

\[
Z[L, y; f] = \exp \left( -\beta \frac{y^2}{2L} - \beta y \int_0^L dx f(x) \right) \exp \left( -\beta H[\varphi_q, \delta \varphi] \right) \times \int_{\delta \varphi(0) = 0}^{\delta \varphi(L) = 0} [D\delta \varphi(x)] \exp \left( -\beta \int_0^L dx \frac{c}{2} (\delta \varphi')^2 \right) \times \exp \left( +\frac{y}{L} \int_0^L dx (\varphi_q'(x) + (\delta \varphi)')(x) + \int_0^L dx (c\varphi_q'(x)(\delta \varphi)'(x) + f(x)\delta \varphi(x)) \right).
\]

In the argument of the last exponential factor, the first integral equals zero because of the boundary conditions, and the second integral equals zero because \( \varphi_q \) satisfies the stationarity condition of Eq. (9). Let us extend the transverse displacement \( \varphi_q \) and force field \( f \) to the interval \([-L, L]\), as odd functions. An odd function \( g \) defined on \([-L, L]\) can be expanded as a Fourier series:

\[
g_m := \int_{-L}^{L} dx g(x) \sin(k_m x), \quad \text{where} \quad k_m = \frac{m\pi}{L}, \quad m \in \mathbb{N},
\]

\[
g(x) = \frac{1}{L} \sum_{m=1}^{\infty} g_m \sin(k_m x), \quad x \in [-L, L].
\]

The integrals needed in the expression of the energy are expressed as

\[
\int_0^L dx f(x) = \frac{1}{L} \sum_{m=1}^{\infty} f_m \sin(k_m x) = \frac{1}{L} \sum_{m=1}^{\infty} f_m \int_0^L x \sin(k_m x) dx = -\frac{1}{L} \sum_{m=1}^{\infty} (-1)^m \frac{f_m}{k_m}.
\]
\[ H[\varphi, f\varphi] = \frac{c}{2} \int_{-L}^{L} dx \left[ (\varphi'(x))^2 - \varphi''(x)\varphi(x) \right] \]
\[ = -\frac{c}{4} \int_{-L}^{L} dx (\varphi'_q(x))^2 \]
\[ = -\frac{c}{4} \sum_{m \geq 1} \frac{k^2_m \varphi^2_{qm}}{L^2} \int_{-L}^{L} \cos^2(k_m x) dx \]
\[ = -\frac{1}{4cL} \sum_{m \geq 1} \frac{f^2_m}{k^2_m} . \]  

where we used integration by parts and \(-ck^2_m (\varphi_q)_m = f_m\), which is the \(m\)-th Fourier coefficient of Eq. (3). The partition function of the sector with transverse displacement \(y\) at time \(L\) therefore reads for a fixed realisation of the random force field:
\[ Z[L, y; f] = Z_0 \exp \left( -\beta c \frac{y^2}{2L} + \frac{1}{2} \sum_{m \geq 1} (-1)^m \frac{f_m}{k_m} + \frac{\beta}{4cL} \sum_{m \geq 1} \frac{f^2_m}{k^2_m} \right) \]
with \( Z_0 := \int_{\varphi(0) = 0} [D\varphi(x)] \exp \left( -\beta \int_0^L dx \frac{c}{2} (\delta \varphi)'(x)^2 \right) \), (15)

The partition function of fluctuations with both ends fixed, denoted by \(Z_0\), does not depend on the position \(y\) of the end-point of the polymer. In [29], this result was used to study the free energy of the random-force model with fixed boundary condition. We are going to use it to study the random transverse position of the end-point of the polymer instead.

### 2.3 Probability law of the transverse position of the end-point

The last factor in Eq. (15) is the partition function of fluctuations with both ends fixed, which does not depend on the position \(y\) of the end-point of the polymer. We can therefore obtain the probability law of the random transverse transverse position \(y\) of the end-point of the polymer by normalising, which amounts to computing a Gaussian integral. The normalisation factor is the integral of the expression obtained in Eq. (15) w.r.t. the coordinate \(y\):
\[ \int_{-\infty}^{+\infty} Z[L, y; f] dy = Z_0 \sqrt{\frac{2\pi L}{\beta c}} \exp \left( +\frac{L}{2\beta c L^2} \left( \sum_{m=1}^{\infty} \frac{(-1)^m f_m}{k_m} \right)^2 \right) \exp \left( \sum_{m=1}^{\infty} \frac{\beta f^2_m}{4cLk^2_m} \right) . \]  

Let us denote by \(P(y, t|y_0, t_0, [f])\) the probability density of the transverse position of the end-point at coordinate \(y\) at time \(t\), conditional on the coordinate \(y_0\) at time \(t_0 < t\), and of a particular realisation of forces \(f\) (as the values of the force \(f\) at different times are independent, conditioning on the realisation \(f\) is equivalent to conditioning on its restriction to the interval \([t_0, t]\)).
For a fixed realisation of forces, these notations yield
\[
P(y, L | 0, 0, [f]) := \frac{Z[L, y; f]}{\int_{-\infty}^{\infty} Z[L, y; f] dy} = \sqrt{\frac{\beta c}{2\pi L}} \exp \left(-\frac{\beta c y^2}{2L}\right) \exp \left(\frac{\beta}{L} \sum_{m=1}^{\infty} \frac{(-1)^m f_m}{k_m} - \frac{\beta}{2Lc} \left(\sum_{m=1}^{\infty} \frac{(-1)^m f_m}{k_m}\right)^2\right),
\]
which is Gaussian in the linear combination of Fourier modes of the force.

The free evolution of the elastic line is the special case were forces are identically zero (or equivalently the amplitude \(u\) is zero in Eq. (14)). The probability law of the transverse position of the end-point becomes that of the position of an ordinary diffusive random walker in one dimension. Let us use the symbol \(t\) for the current value of the directing coordinate, and write Eq. (17) in the case of zero disorder:
\[
P(y, t | 0, 0, [f = 0]) = \frac{1}{\sqrt{4\pi D t}} \exp \left(-\frac{y^2}{4Dt}\right),
\]
where we read off the diffusion constant \(D\) in terms of the tempeature and elastic constant of the problem as
\[
D := \frac{1}{2\beta c}.
\]

Turning on a small amplitude of noise \(u > 0\) allows non-zero configurations of the forces gives rise to deformed Gaussian probability laws of the transverse coordinate \(y\), for each realisation of the forces (Eq. (17)). Averaging over the disorder by taking into account the Gaussian distribution of forces (Eq. (4)) should induce one-parameter deformations of all the properties of the diffusive random walk, including the optimal resetting rate.

Let us denote by \(\pi_L\) the probability density of the Fourier components of the forces (at a fixed amplitude \(u > 0\) of the disorder). We are interested in the average over disorder of the probability density of the position of the end-point of the polymer over disorder (conditional on the position \(y_0\) at time \(t_0 < t\), expressed as the following integral:
\[
\mathcal{P}(y, t | y_0, t_0) := \left(\prod_{n=1}^{\infty} \int_{-\infty}^{+\infty} df_n \pi_L(f_n)\right) P(y, t | y_0, t, [f]).
\]

The density \(\pi_L\) is Gaussian, it is induced by the Gaussian distribution of the forces (Eq. (14)) and the induced Gaussian integral over the vector of Fourier components is worked out in Appendix A. We obtain the following probability density of the transverse position of the end-point of the directed polymer at time \(t\)
\[
\mathcal{P}(y, t | 0, 0) = \frac{1}{\sqrt{4\pi D t (1 + \epsilon t^2)}} \exp \left(-\frac{y^2}{4Dt (1 + \epsilon t^2)}\right),
\]
with corrections w.r.t. the free diffusive case encoded by the parameter
\[
\epsilon := \frac{\beta}{3c} u.
\]
Moreover, starting the process at time $t_0 < t$ with the condition $\phi(t_0) = y_0$, we can repeat the above derivation using the fact that the forces $f_{[t_0,t]}$ have the same distribution as $f_{[0,t-t_0]}$, and obtain

$$P(y, t|y_0, t_0) = P(y - y_0, t - t_0|0,0).$$  \hspace{1cm} (23)

This result concludes the review of the Larkin model in the absence of resetting for our purposes.

## 3 The random-force model under resetting

### 3.1 First-passage time at a fixed target, averaged over disorder

Let us subject the directed polymer to stochastic resetting to a point-like configuration. At random times distributed according to a Poisson process with rate $r$, the position of the random walker (i.e. the transverse displacement of the polymer) is reset to 0. Moreover, there is an absorbing wall at a fixed value $Y$ of the transverse coordinate: when the transverse coordinate of the end-point of the directed polymer reaches the value $Y$, the process stops. Let us denote by $\langle T(Y, r, u) \rangle$ the average over disorder of the first passage time of the end-point of the polymer at the transverse position $Y$. In the case where $u = 0$, the problem reduces to ordinary one-dimensional Brownian motion (with diffusion constant $D$). The mean first passage time at $\langle T(Y, r, u = 0) \rangle$ has been calculated, and found to exhibit a unique minimum for value of the resetting rate \cite{1,2} denoted by $r^*$. This optimal value of the resetting rate depends only on the position $Y$ of the absorbing target, and on the diffusion constant (the derivation is reviewed at the beginning of Appendix B).

For a non-zero value $u$ of the amplitude of the disorder, the mean first passage time can again be expressed in integral form, as the arguments leading to it are quite general \cite{31}. Let us denote by $S_{r,u}(Y, t|[f])$ the survival probability of the process until time $t$, for a fixed realisation of forces on the interval $[0,t]$ (and an initial tranverse position 0 at time 0):

$$S_{r,u}(Y, t|[f]) = \text{Prob} \left( y < Y, \forall t' \in [0,t] | [f_{[0,t]}] \right).$$  \hspace{1cm} (24)

Its average over disorder is denoted by $\overline{S_{r,u}}(Y, t)$.

The mean first passage time at $Y$ is the integral of time against the decreasing rate of the survival probability $\overline{S_{r,u}}$ by integrating by parts:

$$\langle T(Y, r, u) \rangle = \int_0^{\infty} dt \ t \left( - \frac{\partial \overline{S_{r,u}}(Y, t)}{\partial t} \right) = \int_0^{\infty} \overline{S_{r,u}}(Y, t) dt = \widetilde{\overline{S_{r,u}}}(Y, 0),$$  \hspace{1cm} (25)

where we denoted with a tilde the Laplace transform in the time variable:

$$\tilde{g}(s) := \int_0^{\infty} dt \ e^{-st} g(t).$$  \hspace{1cm} (26)
3.2 Renewal equation

On the other hand, the Laplace transform of the survival probability under resetting \( \overline{S}_{r,u} \) can be expressed in terms of the Laplace transform of the survival probability in the ordinary process (with zero resetting rate and a disorder of amplitude \( u \)), taken at the resetting rate. Consider a fixed realisation of the random forces on the interval \([0, t] \). Conditioning on the last resetting event in the interval \([0, t] \) induces a renewal equation:

\[
S_{r,u}(Y, t|f) = e^{-rt}S_{0,u}(Y, t|f) + r \int_0^t d\tau e^{-r\tau}S_{r,u}(Y, t - \tau|[f_{[0,t-\tau]}])S_{0,u}(Y, \tau|[f_{[t-\tau,t]}]),
\]

where the first term corresponds to no resetting in the interval \([0, t] \) (which is the case with probability \( e^{-rt} \)) and the second term to at least one resetting event in this interval, the last of which occurs at \( t - \tau \). Each factor in the integrand of the second term is conditioned on the forces, but in the first factor the condition depends only on the forces on the interval \([0, t - \tau] \[, and in the second factor the condition depends only on the forces on the interval \([t - \tau, t] \].

To take the average of this renewal equation over disorder, we use the independence of the two random configurations of forces in the intervals \([0, t - \tau] \) and \([t - \tau, t] \) to write the average of the integrand as a product of averages. Moreover, the restriction \( f_{[t-\tau,t]} \) has the same distribution as \( f_{[0,\tau]} \), hence

\[
\overline{S}_{r,u}(Y, t) = e^{-rt}\overline{S}_{0,u}(Y, t) + r \int_0^t d\tau e^{-r\tau}\overline{S}_{0,u}(Y, t - \tau)\overline{S}_{0,u}(Y, \tau),
\]

(28)

The Laplace transform w.r.t. \( t \) yields

\[
\overline{S}_{r,u}(Y, s) = \overline{S}_{0,u}(Y, s + r) + r\overline{S}_{0,u}(Y, s)\overline{S}_{r,u}(Y, s),
\]

(29)

hence the averages over disorder of the survival probabilities satisfy the equation

\[
\overline{S}_{r,u}(Y, s) = \frac{\overline{S}_{0,u}(Y, s + r)}{1 - r\overline{S}_{0,u}(Y, s + r)},
\]

(30)

which reduces when \( u = 0 \) to the known equation satisfied by the survival probability without disorder. From Eqs \( \underline{25} \) and \( \underline{30} \) we see that it is enough to evaluate the Laplace transform of the survival probability \( \overline{S}_{0,u} \) in the process without resetting, to express the time \( \langle T(Y, r, u) \rangle \).

For the directed polymer in a fixed realisation of the forces, and with no resetting, let us denote by \( \phi_0(Y, t|f) \) the probability density of reaching the transverse displacement \( Y \) for the first time at \( t \) (after having started at transverse position 0 at time 0). This quantity describes the leaking of survival probability through the absorbing target, hence

\[
\phi_0(Y, t|f) = - \frac{\partial}{\partial t}S_{0,u}(Y, t|f),
\]

(31)
which upon average over the disorder and Laplace transform yields the quantity we need in Eq. (30):

$$\tilde{\phi}_0(Y, s) = 1 - s\tilde{S}_0(u)(Y, s).$$ (32)

On the other hand, the density $\phi_0$ is related to the probability density of the position of the end-point of the polymer by conditioning on the time $T$ at which the end-point of the polymer reaches the transverse displacement $Y$ for the first time. we write

$$P(Y, t|0, 0, [f]) = \int_0^t dT \phi_0(Y, T|0) P(Y, t|t-T, [f]).$$ (33)

where the second factor in the integrand describes the return of the end-point to the transverse position $Y$ at $t$. The two factors in the integrand depend on the realisation of forces $f$ though their restrictions to the intervals $[0, T]$ and $[T, t]$ respectively. As the forces at distinct times are independent and identically distributed, the average over disorder of Eq. (33) reads

$$\mathcal{P}(Y, t|0, 0) = \int_0^t dT \tilde{\phi}_0(Y, T) \mathcal{P}(Y, t|t-T)$$

$$= \int_0^t dT \tilde{\phi}_0(Y, T) \mathcal{P}(Y, t-T|0)$$

$$= \int_0^t dT \tilde{\phi}_0(Y, T) \mathcal{P}(0, t-T|0, 0),$$ (34)

where in the last step we used the fact the probability of return to the initial position in a fixed time is independent of the value of this initial position. Let us make the dependence on the amplitude of the disorder explicit by introducing the following notation for the relevant Laplace transforms:

$$\mathcal{L}(y, r, u) := \int_0^{\infty} dt e^{-rt} \mathcal{P}(y, t|0, 0).$$ (35)

The Laplace transform of Eq. (34) w.r.t. the variable $t$ (taken at the value $r$) reads

$$\mathcal{L}(Y, r, u) = \tilde{\phi}_0(Y, r) \mathcal{L}(0, r, u).$$ (36)

Combining Eqs (25, 30, 32, 36) yields

$$\langle T(Y, r, u) \rangle = \frac{1}{r} \left(1 - \tilde{\phi}_0(Y, r)\right) \frac{1}{\tilde{\phi}_0(Y, r)} = \frac{1}{r} \left(\frac{\mathcal{L}(0, r, u)}{\mathcal{L}(Y, r, u)} - 1\right),$$ (37)

which is formally identical to the expression of the mean time to absorption in the absence of disorder in terms of the Laplace transform of the propagator of the process without resetting.
3.3 Susceptibility to disorder of the optimal resetting rate

We would like to calculate the response rate of the optimal resetting rate to a disorder of small amplitude, which we denote by $\frac{\delta r^*}{\delta u}$. Let us denote by $\rho(u)$ the optimal resetting rate for a small value of $u$ (at fixed values of the diffusion constant $D$, and fixed position $Y$ of the target), so that $\rho(0) = r^*$, and the the desired susceptibility is

$$\frac{\delta r^*}{\delta u} = \rho'(0).$$

The optimality condition defining $\rho(u)$ takes the form

$$\frac{\partial}{\partial r} \langle T(Y, r = \rho(u), u) \rangle = 0. \quad (39)$$

The derivative $\rho'(0)$ can be calculated from the first-order term in the expansion of this optimality condition in powers of the amplitude of the disorder:

$$\frac{\partial}{\partial u} \left( \frac{\partial}{\partial r} \langle T(Y, r = \rho(u), u) \rangle \right) |_{u=0} = 0. \quad (40)$$

The susceptibility of the optimal rate to the disorder is therefore expressed as

$$\frac{\delta r^*}{\delta u} = - \left( \frac{\partial^2}{\partial r^2} \langle T(Y, r^*, 0) \rangle \right)^{-1} \frac{\partial^2}{\partial r \partial u} \langle T(Y, r^*, 0) \rangle. \quad (41)$$

Thanks to the expression of the mean first passage time in Eq. (37), the two second derivatives we are instructed to compute can be evaluated from a second-order Taylor expansion of the Laplace transform of the probability density of the end-point of the polymer without resetting (Eq. (23)), around the values $r = r^*$ and $u = 0$. By dominated convergence, the terms we need can be obtained by expanding the integrands in the Laplace transforms in powers of $r - r^*$ (up to order two) and $u$ (up to order one). The derivation can be found in Appendix B. Thanks to the optimality condition satisfied by $r^*$ in the absence of disorder, the susceptibility can be expressed in closed form in terms of values of modified Bessel functions at the point $\gamma^*$ (Eq. (61)). Numerically we find

$$\frac{\delta r^*}{\delta u} \simeq 2.6394 \frac{\beta}{c r^*} \simeq 0.5197 \beta^2 Y^2. \quad (42)$$

The product $\beta^2 Y^2$ is the only monomial in the dimensionful parameters $\beta, c, Y$ with the correct unit, because $u$ has dimension of the square of of a force divided by time. The numerical factor is expressed in terms of the constant $\gamma^*$ only (see Eq. (100)). It is therefore beneficial on average to increase the resetting rate in presence of disorder with a small amplitude.

4 Conclusion

In this paper we have expressed the expectation value of the mean first passage time to a fixed absorbing target (averaged over disorder) in the Larkin model of a directed polymer in random
forces, subjected to stochastic resetting to a point-like configuration. The end-point of the polymer becomes an ordinary Brownian random walker when the amplitude of the disorder goes to zero. The probability density of the transverse position of the end-point of the polymer can be averaged over disorder thanks to the Gaussian nature of the involved integrals.

Thanks to the equivalence \( P(y, t) \sim \exp(-y^2/(4D\epsilon t^3)) \) at large \( t \), the Larkin model is of the class addressed in [3]. In particular, it a non-equilibrium steady state on a space-dependent time scale. In this paper we have been concerned in the regime of time in which the effect of disorder is still small, and the behaviour of the end-point of the polymer is still close to a Brownian motion. Technically, the involved expansions in powers of the amplitude of the disorder are valid by dominated convergence, because they occur in the integrand of a Laplace transform. They yield the susceptibility of the optimal resetting rate in one dimension to disorder. This susceptibility is proportional to the square of the position of the target and to the inverse of the square of the temperature (for dimensional reasons), up to a positive coefficient expressed in closed form.

Studying more complex disordered systems would typically involve the Laplace transform of the probability distribution of the propagator of the system in the absence of resetting, taken at the resetting rate. In the case of the Matheron-de Marsily model of a layered flow (with random velocities of the layers), special values of the Laplace transform of the propagator have been worked out in the absence of resetting by path-integral methods [32]. Extension of the calculation of the Laplace transform to the resetting rate is usually a formidable task [33]. However, in the case of the Matheron-de Marsily model, the transverse position has been shown in [34] to be a fractional Brownian motion, which thanks to the large-deviation arguments of [3] induces the relaxation dynamics to a non-equilibrium steady state under resetting.

**Appendix A**

From Eq. (4) we work out the mean and variance of the Fourier components of the force field \( f_{[-L,L]} \), the odd function obtained from \( f_{0,L} \):

\[
\overline{f}_m = 0, \quad \overline{f}_m^2 = \int_{-L}^{L} dx \int_{-L}^{L} dy f(x) f(y) \sin(k_m x) \sin(k_m y) = 2uL. \tag{43}
\]

Moreover, all the moments of higher order of \( f_m \) are zero, just as the moments of higher order of the forces. Hence the probability density of \( f_m \) (denoted by \( \pi_L \), as it depends on the interval of time on which random forces are studied) is the following centered Gaussian:

\[
\pi_L(f_m) = \frac{1}{\sqrt{4\pi uL}} \exp \left( -\frac{f_m^2}{4uL} \right). \tag{44}
\]

To average the probability density of the position of the end-point of the polymer, let us rewrite Eq. (17) as

\[
P(y, L|0, 0, [f]) = \sqrt{\frac{\beta c}{2\pi L}} \exp \left( -\frac{\beta cy^2}{2L} \right) \exp \left( \lambda y \vec{f} \cdot \vec{a} - \mu (\vec{f} \cdot \vec{a})^2 \right), \tag{45}
\]
with the vector notations (using \( k_m = m\pi/L \), Eq. (11)): 

\[
\vec{f} := \sum_{n \geq 1} f_n \vec{e}_n, \quad \vec{a} = \frac{\sqrt{6}}{\pi} \sum_{n \geq 1} \frac{(-1)^n}{n} \vec{e}_n, \tag{46}
\]

and the coefficients

\[
\lambda := \frac{\beta}{\sqrt{6}}, \quad \mu := \frac{\beta L}{12c}. \tag{47}
\]

The vector \( \vec{a} \) is normalised.

The probability density of the position of the end-point of the polymer can be averaged over disorder by evaluating a Gaussian integral w.r.t. each of the Fourier components of the random force:

\[
\overline{P}(y, L|0, 0) = \left( \prod_{n=1}^{\infty} \int_{-\infty}^{+\infty} df_n \pi_L(f_n) \right) P(y, L|0, 0, [f])
\]

\[
= \sqrt{\frac{\beta c}{2\pi L}} \exp \left( -\frac{\beta cy^2}{2L} \right) \left( \prod_{n=1}^{\infty} \int_{-\infty}^{+\infty} \frac{df_n}{\sqrt{4\pi uL}} \right) \exp \left( \lambda y \vec{f} \cdot \vec{a} - \mu \left( \vec{f} \cdot \vec{a} \right)^2 - \frac{1}{4uL} \vec{f} \cdot \vec{f} \right). \tag{48}
\]

The quadratic terms in the Gaussian kernel are the sum of a multiple of the identity and a multiple of the projector onto the normalised vector \( \vec{a} \):

\[
\mu \left( \vec{f} \cdot \vec{a} \right)^2 + \frac{1}{4uL} \vec{f} \cdot \vec{f} =: \vec{f} \cdot M \vec{f}, \tag{49}
\]

where

\[
M_{ij} = \frac{1}{4uL} \delta_{ij} + \mu a_i a_j. \tag{50}
\]

We can invert \( M \) as follows (where \( I \) denotes the identity and \( \pi_a \) denotes the projector onto the vector \( \vec{a} \)):

\[
M^{-1} = 4uL (I + 4uL\mu \pi_a)^{-1} = 4uL \left( I - \frac{4uL\mu}{1 + 4uL\mu} \pi_a \right). \tag{51}
\]

By Gram–Schmidt orthonormalisation, the determinant of \( M \) reads

\[
\det M = (1 + 4uL\mu) \det \left( \frac{I}{4uL} \right) = (1 + 4uL\mu) \prod_{n=1}^{\infty} \frac{1}{4uL}. \tag{52}
\]

The last factor in the above determinant compensates the infinite product in the denominator in
Eq. (48). The average over disorder therefore reads

\[
\mathcal{F}(y, L|0, 0) = \sqrt{\frac{\beta c}{2\pi L}} \exp\left(-\frac{\beta cy^2}{2L}\right) \frac{1}{\sqrt{1 + 4uL\mu}} \exp\left(\frac{\lambda^2 y^2 L}{1 + \frac{4uL\mu}{1 + 4uL\mu}}\right)
\]

which is the expression reported in Eq. (23).

Appendix B

Bessel-function identities

We will repeatedly use the following identity (see Section 4.5 of [35]):

\[
\int_0^\infty dt \, t^{\nu-1} \exp\left(-\frac{\alpha}{t} - \chi t\right) = 2 \left(\frac{\alpha}{\chi}\right)^{\nu/2} K_\nu(2\sqrt{\alpha\chi}),
\]

where \(K_\nu\) denotes the modified Bessel function of the second kind of order \(\nu\). The case \(\nu = 1/2\) appears when calculating the Laplace transform of the probability density of an ordinary random walk:

\[
K_{1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x}.
\]

This identity holds by continuity at \(\alpha = 0\) for \(\nu > 0\), because

\[
K_\nu(z) \sim 2^{\nu-1}\Gamma(\nu)z^{-\nu}.
\]

As we will focus on the neighborhood of the optimal resetting rate in the absence of disorder, all the values of modified Bessel functions that we will need will be at the special point \(\gamma^*\) in terms of which the optimal resetting rate of the diffusive random walker can be expressed (see the next subsection for a review of the derivation, and Eq. (61) for the definition of \(\gamma^*\)). We will also use the special values \(\Gamma(1/2) = \sqrt{\pi}, \Gamma(3/2) = \sqrt{\pi}/2, \Gamma(5/2) = 3\sqrt{\pi}/4, \Gamma(7/2) = 15\sqrt{\pi}/8\).

Review of the optimal resetting rate in the absence of disorder

At \(u = 0\) we are in the situation of the optimisation of resetting for diffusion in the absence of random forces. The optimal rate minimises:

\[
\langle T(\gamma, r, 0) \rangle = \frac{1}{r} \left(\frac{\mathcal{L}(0, r, 0)}{\mathcal{L}(\gamma, r, 0)} - 1\right).
\]
The quantities $\mathcal{L}(Y, r, 0)$ and $\mathcal{L}(0, r, 0)$ are just Laplace transforms in time diffusive propagator of Eq. (18). 

\[
\mathcal{L}(y, r, 0) = \int_0^\infty P(y, t|0, 0, [f = 0])e^{-rt}dt \\
= \int_0^\infty \frac{1}{\sqrt{4\pi Dt}} \exp \left(-\frac{y^2}{4Dt} - rt \right) dt \\
= \frac{1}{\sqrt{4\pi D}} \times 2 \left(\frac{y^2}{4Dr}\right)^{1/4} K_{1/2} \left(r \sqrt{\frac{y}{D}}\right) \\
= \frac{1}{\sqrt{4D}} \exp \left(-y \sqrt{\frac{r}{D}}\right),
\]

where we used Eqs (54,55).

The optimal rate $r^*$ therefore minimises:

\[
\langle T(Y, r, u = 0) \rangle = \frac{1}{r} \left(\exp \left(\frac{Y}{\sqrt{D}} \sqrt{r}\right) - 1\right)
\]

for constant position $Y$ of the absorbing target, which yields the condition

\[
\frac{1}{2} \sqrt{\frac{r^*}{D}} Y = 1 - \exp \left(-\sqrt{\frac{r^*}{D}} Y\right).
\]

The optimal rate is therefore expressed in terms of the position of the absorbing target, the diffusion constant, and the solution $\gamma^*$ to a transcendental equation (11, see also Section 3 of 23), so that:

\[
r^* = \frac{D(\gamma^*)^2}{Y^2}, \quad \text{with} \quad \frac{\gamma^*}{2} = 1 - \exp(-\gamma^*), \quad \gamma^* \approx 1.5936.
\]

In the Taylor expansions around the optimal rate, the order-zero terms will involve the values of the Laplace transforms worked out in Eq. (58) for $r = r^*$:

\[
\mathcal{L}(Y, r^*, 0) = \frac{1}{2\sqrt{Dr^*}} \exp \left(-Y \sqrt{\frac{r^*}{D}}\right) = \frac{Y}{2D\gamma^*} \exp(-\gamma^*),
\]

\[
\mathcal{L}(0, r^*, 0) = \frac{1}{2\sqrt{Dr^*}} = \frac{Y}{2D\gamma^*}.
\]

**Taylor expansions**

To evaluate the expressions in Eqs (37,11), we need a Taylor expansion of the Laplace transform $\mathcal{L}(y, r^*, u)$ around $(r = r^*, u = 0)$, at order two in the rate and order one in the amplitude of the
disorder (and we will substitute the values $Y$ and 0 to $y$). With the notations for the diffusion constant $D$ and noise parameter $\epsilon$ introduced in Eqs 19 and 22, we write

$$L(y, r^* + h, u = 3c\beta^{-1}\epsilon) = \int_0^\infty dt \frac{1}{\sqrt{4\pi Dt}} \exp \left(-\frac{y^2}{4Dt} \left(1 + \epsilon t^2\right)\right),$$

(64)

We will specialise the result to the values 0 and $Y$ of the transverse coordinate $y$.

The first-order terms in the noise parameter $\epsilon$ are extracted from

$$\frac{1}{\sqrt{(1 + \epsilon t^2)}} = 1 - \frac{t^2}{2} \epsilon + o(\epsilon),$$

(65)

$$\exp \left(-\frac{y^2}{4Dt (1 + \epsilon t^2)}\right) = \exp \left(-\frac{y^2}{4Dt} (1 - \epsilon t^2 + o(\epsilon))\right) \quad \text{or} \quad \exp \left(-\frac{y^2}{4Dt} \left(1 + \frac{y^2t}{4D} \epsilon + o(\epsilon)\right)\right),$$

(66)

The terms of order one and two in $h$ come only from the Taylor expansion of the exponential function in the factor $e^{-ht}$, present in both integrands:

$$e^{-(r^*+h)t} = e^{-r^*t} \left(1 - th + \frac{t^2}{2} h^2 + o(h^2)\right).$$

(67)

$$L(y, r^* + h, u = 3c\beta^{-1}\epsilon) = \int_0^\infty dt \frac{1}{\sqrt{4\pi Dt}} \exp \left(-r^*t - \frac{y^2}{4Dt} \left(1 - th + \frac{t^2}{2} h^2 + o(h^2)\right)\right)$$

$$\times \left(1 + \frac{y^2t}{4D} \epsilon + o(\epsilon)\right) \times \left(1 - \frac{t^2}{2} \epsilon + o(\epsilon)\right)$$

$$= \int_0^\infty dt \frac{1}{\sqrt{4\pi Dt}} \exp \left(-r^*t - \frac{y^2}{4Dt} \left(1 - th + \frac{t^2}{2} h^2 + o(h^2)\right)\right)$$

$$\times \left(1 + \frac{y^2t}{4D} \epsilon - \frac{t^2}{2} \epsilon + o(\epsilon)\right)$$

$$= :L(y, r^*, 0) + \kappa_{10}(y) h + \kappa_{01}(y) \epsilon + \kappa_{11}(y) h \epsilon + \kappa_{20} h^2 + \ldots,$$

where we can read off the expression of the coefficients in the Taylor expansions in integral form, and apply the identity of Eq. 54 to each term.

$$\kappa_{10}(y) = -\frac{1}{\sqrt{4\pi D}} \int_0^\infty dt t^{1/2} \exp \left(-r^*t - \frac{y^2}{4Dt}\right)$$

$$= -\frac{1}{\sqrt{4\pi D}} \times 2 \left(\frac{y^2}{4Dr^*}\right)^{3/4} K_{3/2} \left(y\sqrt{\frac{r^*}{D}}\right).$$

(69)
\[ \kappa_{01}(y) = \frac{y^2}{4D \sqrt{4\pi D}} \int_0^\infty dt t^{3/2} e^{-\frac{y^2}{4Dt}} - \frac{1}{2\sqrt{4\pi D}} \int_0^\infty dt t^{3/2} e^{-\frac{y^2}{4Dt}} \]

\[ = \frac{y^2}{4D \sqrt{4\pi D}} \times 2 \left( \frac{y^2}{4Dr^*} \right)^{3/4} K_{3/2} \left( y \sqrt{\frac{r^*}{D}} \right) - \frac{1}{2\sqrt{4\pi D}} \times 2 \left( \frac{y^2}{4Dr^*} \right)^{5/4} K_{5/2} \left( y \sqrt{\frac{r^*}{D}} \right), \]

\[ \kappa_{11}(y) = -\frac{y^2}{4D \sqrt{4\pi D}} \int_0^\infty dt t^{3/2} e^{-\frac{y^2}{4Dt}} + \frac{1}{2\sqrt{4\pi D}} \int_0^\infty dt t^{5/2} e^{-\frac{y^2}{4Dt}} \]

\[ = -\frac{y^2}{4D \sqrt{4\pi D}} \times 2 \left( \frac{y^2}{4Dr^*} \right)^{5/4} K_{5/2} \left( y \sqrt{\frac{r^*}{D}} \right) + \frac{1}{2\sqrt{4\pi D}} \times 2 \left( \frac{y^2}{4Dr^*} \right)^{7/4} K_{7/2} \left( y \sqrt{\frac{r^*}{D}} \right). \]

\[ \kappa_{20}(y) = \frac{1}{2\sqrt{4\pi D}} \int_0^\infty dt t^{3/2} e^{-\frac{y^2}{4Dt}} \]

\[ = \frac{1}{2\sqrt{4\pi D}} \times 2 \left( \frac{y^2}{4Dr^*} \right)^{5/4} K_{5/2} \left( y \sqrt{\frac{r^*}{D}} \right). \]

The argument of each of the modified Bessel functions is \( y \sqrt{r^* / D} \) in the above expressions, which for \( y = Y \) reduces to the constant \( \gamma^* \), defined in Eq. (61). Moreover, this definition can be used to eliminate the symbol \( r^* \) from the above expression if \( y = Y \). This yields the following polynomial expressions in the position \( Y \) of the absorbing target:

\[ \frac{Y^2}{4Dr^*} = \frac{Y^4}{4\gamma^*^2 D^2}. \]  

The values we will need are therefore obtained by substituting \( Y \) and 0 to the variable \( y \).

\[ \kappa_{10}(Y) = -\frac{1}{\sqrt{4\pi D}} \times 2 \left( \frac{Y^4}{4D^2 \gamma^*} \right)^{3/4} K_{3/2} \left( \gamma^* \right), \]

\[ \kappa_{01}(Y) = \frac{Y^2}{4D \sqrt{4\pi D}} \times 2 \left( \frac{Y^4}{4D^2 \gamma^*} \right)^{3/4} K_{3/2} \left( \gamma^* \right) - \frac{1}{\sqrt{4\pi D}} \left( \frac{Y^4}{4D^2 \gamma^*} \right)^{5/4} K_{5/2} \left( \gamma^* \right), \]

\[ \kappa_{11}(Y) = -\frac{Y^2}{4D \sqrt{4\pi D}} \times 2 \left( \frac{Y^4}{4D^2 \gamma^*} \right)^{5/4} K_{5/2} \left( \gamma^* \right) + \frac{1}{\sqrt{4\pi D}} \times 2 \left( \frac{Y^4}{4D^2 \gamma^*} \right)^{7/4} K_{7/2} \left( \gamma^* \right), \]

\[ \kappa_{20}(Y) = \frac{1}{2\sqrt{4\pi D}} \times 2 \left( \frac{Y^4}{4D^2 \gamma^*} \right)^{5/4} K_{5/2} \left( \gamma^* \right). \]

To obtain the needed values at \( y = 0 \), let us use the equivalents of modified Bessel functions in Eq. (60). Again let us use the definition of the optimal resetting rate (Eq. (61)) to make the dependence on \( Y \) explicit. We are going to encounter limits of the form

\[ \lim_{y \to 0} \left( 2 \left( \frac{y^2}{4Dr^*} \right)^{\nu/2} K_{\nu} \left( y \sqrt{\frac{r^*}{D}} \right) \right) = \frac{\Gamma(\nu/2)}{D^{\nu} (\gamma^*)^{2\nu}} Y^{2\nu}. \]
The needed coefficients at $y = 0$ therefore read:

$$
\kappa_{10}(0) = \lim_{y \to 0} \left( -\frac{1}{\sqrt{4\pi D}} \times 2 \left( \frac{y^2}{4D r^*} \right)^{3/4} K_{3/2} \left( y\sqrt{\frac{r^*}{D}} \right) \right) = -\frac{1}{\sqrt{4\pi D}} \frac{\Gamma(3/2)}{D^{3/2}(\gamma^*)^3} Y^3, \tag{79}
$$

$$
\kappa_{01}(0) = \lim_{y \to 0} \left( \frac{y^2}{4D\sqrt{4\pi D}} \times 2 \left( \frac{y^2}{4D r^*} \right)^{3/4} K_{3/2} \left( y\sqrt{\frac{r^*}{D}} \right) - \frac{1}{\sqrt{4\pi D}} \left( \frac{y^2}{4Dr^*} \right)^{5/4} K_{5/2} \left( y\sqrt{\frac{r^*}{D}} \right) \right) = -\frac{1}{2\sqrt{4\pi D} D^{5/2}(\gamma^*)^5} Y^5, \tag{80}
$$

$$
\kappa_{11}(0) = \frac{1}{2\sqrt{4\pi D} D^{7/2}(\gamma^*)^7} Y^7, \tag{81}
$$

$$
\kappa_{20}(0) = \frac{1}{2\sqrt{4\pi D} D^{9/2}(\gamma^*)^9} Y^9. \tag{82}
$$

We are therefore instructed to extract the relevant corrections from:

$$
\langle T(Y, r^* + h, \epsilon) \rangle = \frac{1}{r^* + h} \left( \frac{\mathcal{L}(0, r^*, 0) + \kappa_{10}(0)h + \kappa_{01}(0)\epsilon + \kappa_{11}(0)h\epsilon + \kappa_{20}(0)h^2 + \ldots}{\mathcal{L}(Y, r^*, 0) + \kappa_{10}(Y)h + \kappa_{01}(Y)\epsilon + \kappa_{11}(Y)h\epsilon + \kappa_{20}(Y)h^2 + \ldots} - 1 \right), \tag{83}
$$

Factorising the dominant terms yields:

$$
\langle T(Y, r^* + h, \epsilon) \rangle = \frac{1}{r^*} \left( 1 - \frac{1}{r^*} h + \frac{1}{(r^*)^2} h^2 + o(h^2) \right) \times \left( \frac{\mathcal{L}(0, r^*, 0) + \kappa_{10}(0)h + \kappa_{01}(0)\epsilon + \kappa_{11}(0)h\epsilon + \kappa_{20}(0)h^2 + \ldots}{\mathcal{L}(Y, r^*, 0) + \kappa_{10}(Y)h + \kappa_{01}(Y)\epsilon + \kappa_{11}(Y)h\epsilon + \kappa_{20}(Y)h^2 + \ldots} - 1 \right) = \frac{1}{r^*} \left( 1 - \frac{1}{r^*} h + \frac{1}{(r^*)^2} h^2 + o(h^2) \right) \times \left( \frac{\mathcal{L}(0, r^*, 0)}{\mathcal{L}(Y, r^*, 0)} \times \frac{1 + L_{10}(0)h + L_{01}(0)\epsilon + L_{11}(0)h\epsilon + L_{20}(0)h^2 + \ldots}{1 + L_{10}(Y)h + L_{01}(Y)\epsilon + L_{11}(Y)h\epsilon + L_{20}(Y)h^2 + \ldots} - 1 \right), \tag{84}
$$

with the notations

$$
L_\delta(y) = \frac{K_\delta(y)}{\mathcal{L}(y, r^*, 0)}, \tag{85}
$$

for every pair of indices $\delta$. 

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The explicit values, using Eqs (62,63) are as follows:

\[ L_{10}(Y) = -\frac{1}{\sqrt{4\pi D}} \times 2 \left( \frac{Y^4}{4D^2\gamma^*} \right)^{3/4} K_{3/2}(\gamma^*) \frac{2D\gamma^* e^{\gamma^*}}{\sqrt{2\pi\gamma^*}} = -\frac{K_{3/2}(\gamma^*) e^{\gamma^*} Y^2}{2\sqrt{2\pi\gamma^*} D}, \]

\[ L_{01}(Y) = \left( \frac{K_{3/2}(\gamma^*) e^{\gamma^*}}{4\sqrt{2\pi\gamma^*}} - \frac{1}{4\sqrt{2\pi}} \frac{K_{5/2}(\gamma^*) e^{\gamma^*}}{(\gamma^*)^{3/2}} \right) \frac{Y^4}{D^2}, \]

\[ L_{11}(Y) = -\frac{Y^2}{2D} \frac{K_{5/2}(\gamma^*) e^{\gamma^*}}{4\sqrt{2\pi(\gamma^*)^{3/2}}} \frac{Y^4}{D^2} + \frac{1}{2\sqrt{4\pi D}} \times 2 \left( \frac{Y^4}{4D^2\gamma^*} \right)^{7/4} K_{7/2}(\gamma^*) \frac{2D\gamma^* e^{\gamma^*}}{Y} \]

\[ L_{20}(Y) = \frac{K_{5/2}(\gamma^*) e^{\gamma^*}}{4\sqrt{2\pi(\gamma^*)^{3/2}}} \frac{Y^4}{D^2}. \]

\[ L_{10}(0) = -\frac{\Gamma(3/2)}{\sqrt{\pi(\gamma^*)^2}} \frac{Y^2}{D}, \quad L_{01}(0) = -L_{20}(0) = -\frac{\Gamma(5/2)}{2\sqrt{\pi(\gamma^*)^4}} \frac{Y^4}{D^2}, \quad L_{11}(0) = \frac{\Gamma(7/2)}{2\sqrt{\pi(\gamma^*)^6}} \frac{Y^6}{D^3}, \]

\[ \frac{1 + L_{10}(0)h + L_{01}(0)\epsilon + L_{11}(0)h\epsilon + L_{20}(0)h^2 + \ldots}{1 + L_{10}(0)h + L_{01}(0)\epsilon + L_{11}(0)h\epsilon + L_{20}(0)h^2 + \ldots} = 1 + (L_{10}(0) - L_{10}(Y))h \]

\[ + (L_{01}(0) - L_{01}(Y))\epsilon \]

\[ + (L_{11}(0) - L_{11}(Y) - L_{01}(0)L_{10}(Y) - L_{10}(0)L_{01}(Y))h\epsilon \]

\[ + (L_{20}(0) - L_{20}(Y) - L_{10}(0)L_{11}(Y) + L_{11}(0)Y^2)h^2 + \ldots \]

\[ =: 1 + M_{10}h + M_{01}\epsilon + M_{11}h\epsilon + M_{20}h^2 + \ldots \]

\[ \langle T(Y, r^*, h, \epsilon) \rangle = \frac{1}{r^*} \frac{\mathcal{L}(0, r^*, 0)}{\mathcal{L}(Y, r^*, 0)} \left( 1 - \frac{1}{r^*} h + \frac{1}{(r^*)^2} h^2 + o(h^2) \right) \]

\[ \times \left( 1 - \frac{\mathcal{L}(Y, r^*, 0)}{\mathcal{L}(0, r^*, 0)} + M_{10}h + M_{01}\epsilon + M_{11}h\epsilon + M_{20}h^2 + \ldots \right), \]

\[ =: \frac{1}{r^*} \frac{\mathcal{L}(0, r^*, 0)}{\mathcal{L}(Y, r^*, 0)} \left( 1 - \frac{\mathcal{L}(Y, r^*, 0)}{\mathcal{L}(0, r^*, 0)} + \tau_{01}h + \tau_{11}h\epsilon + \tau_{20}h^2 + \ldots \right), \]

with \( \tau_{01} = M_{01} \)

\[ \tau_{10} = M_{10} - \frac{1}{r^*} \left( 1 - \frac{\mathcal{L}(Y, r^*, 0)}{\mathcal{L}(0, r^*, 0)} \right), \]

\[ 18 \]
\[ \tau_{11} = -\frac{1}{r^*} M_{01} + M_{11} \]  
\[ \tau_{20} = \frac{1}{(r^*)^2} \left( 1 - \frac{\mathcal{L}(Y, r^*, 0)}{\mathcal{L}(0, r^*, 0)} \right) + M_{20} - \frac{1}{r^*} M_{10}. \]

The stationarity condition satisfied by the rate \( r^* \) implies \( \tau_{10} = 0 \):

\[ M_{10} = L_{10}(0) - L_{10}(Y) = \frac{1}{r^*} \left( 1 - \frac{\mathcal{L}(Y, r^*, 0)}{\mathcal{L}(0, r^*, 0)} \right). \]

It gives rise to the following simplifications:

\[ \tau_{20} = M_{20}. \]

\[ M_{20} = L_{20}(0) - L_{20}(Y) + L_{10}(Y)(L_{10}(Y) - L_{10}(0)) \]
\[ = L_{20}(0) - L_{20}(Y) - \frac{L_{10}(Y)}{r^*} \left( 1 - \frac{\mathcal{L}(Y, r^*, 0)}{\mathcal{L}(0, r^*, 0)} \right) \]
\[ = L_{20}(0) - L_{20}(Y) - \frac{L_{10}(Y)}{r^*} (1 - e^{-\gamma}) \]
\[ = L_{20}(0) - L_{20}(Y) - \frac{\gamma^* L_{10}(Y)}{2} \]
\[ = L_{20}(0) - L_{20}(Y) - L_{10}(Y) \frac{Y^2}{2D \gamma^*} \]
\[ = \left( \frac{\Gamma(5/2)}{2\sqrt{\pi}(\gamma^*)^4} - \frac{K_{5/2}(\gamma^*)e^{\gamma^*}}{4\sqrt{2\pi}(\gamma^*)^{3/2}} + \frac{K_{3/2}(\gamma^*)e^{\gamma^*}}{2\sqrt{2\pi}(\gamma^*)^{3/2}} \right) \frac{Y^4}{D^2} \]
\[ = \left( \frac{\Gamma(5/2)}{2\gamma^*} - \frac{K_{5/2}(\gamma^*)e^{\gamma^*}}{4\sqrt{2\pi}(\gamma^*)^{3/2}} + \frac{K_{3/2}(\gamma^*)e^{\gamma^*}}{2\sqrt{2\pi}(\gamma^*)^{3/2}} \right) \frac{Y^4}{\sqrt{\pi}D^2} \]

The desired susceptibility is therefore obtained as

\[ \frac{\delta r^*}{\delta u} = -\frac{\beta}{3c} \frac{\tau_{11}}{2\tau_{20}} \]
\[ = +\frac{\beta}{6cr^*} \frac{M_{01} - r^* M_{11}}{M_{20}}. \]

Moreover, the quantities \( M_{01} \) and \( r^* M_{11} \) also carry a dimensionful factor of \( Y^4/D^2 \):

\[ M_{01} = L_{01}(0) - L_{01}(Y) \]
\[ = \left( -\frac{\Gamma(5/2)}{2\sqrt{\pi}(\gamma^*)^4} - \frac{K_{3/2}(\gamma^*)e^{\gamma^*}}{4\sqrt{2\pi}\gamma^*} + \frac{1}{4\sqrt{2\pi}} \frac{K_{5/2}(\gamma^*)e^{\gamma^*}}{(\gamma^*)^{3/2}} \right) \frac{Y^4}{D^2} \]
\[ = \left( -\frac{3\sqrt{\pi}}{8(\gamma^*)^4} - \frac{K_{3/2}(\gamma^*)e^{\gamma^*}}{4\sqrt{2\gamma^*}} + \frac{K_{5/2}(\gamma^*)e^{\gamma^*}}{4\sqrt{2(\gamma^*)^{3/2}}} \right) \frac{Y^4}{\sqrt{\pi}D^2} \]
\[ r^*M_{11} = r^*[L_{11}(0) - L_{11}(Y) - L_{01}(0)L_{10}(Y) - L_{10}(0)L_{01}(Y)] \]
\[ = \frac{\gamma^2 D}{Y^2} \left[ \frac{\Gamma(7/2)}{2\sqrt{\pi}(\gamma^*)^6} Y^6 - \left( \frac{K_{3/2}(\gamma^*)e^{\gamma^*}}{8\sqrt{2}\pi(\gamma^*)^{3/2}} + \frac{K_{7/2}(\gamma^*)e^{\gamma^*}}{8\sqrt{2}\pi(\gamma^*)^{5/2}} \right) \frac{Y^6}{D^3} \right. \]
\[ - \frac{\Gamma(5/2)}{2\sqrt{\pi}(\gamma^*)^4} Y^4 \times \frac{K_{3/2}(\gamma^*)e^{\gamma^*}}{\sqrt{2}\pi\gamma^*} \frac{Y^2}{D} + \frac{\Gamma(3/2)}{\sqrt{\pi}(\gamma^*)^2} \frac{Y^2}{D} \times \left( \frac{K_{3/2}(\gamma^*)e^{\gamma^*}}{4\sqrt{2}\pi\gamma^*} - \frac{1}{4\sqrt{2}\pi(\gamma^*)^{3/2}} \right) \frac{Y^4}{D^2} \]
\[ = \frac{\gamma^2 Y^4}{\sqrt{\pi} D^2} \left[ \frac{\Gamma(7/2)}{2(\gamma^*)^6} + \frac{K_{5/2}(\gamma^*)e^{\gamma^*}}{8\sqrt{2}(\gamma^*)^{3/2}} - \frac{K_{7/2}(\gamma^*)e^{\gamma^*}}{8\sqrt{2}(\gamma^*)^{5/2}} \right] \]
\[ = \frac{\gamma^2 Y^4}{\sqrt{\pi} D^2} \left[ \frac{15\sqrt{\pi}}{16(\gamma^*)^4} + \frac{K_{5/2}(\gamma^*)e^{\gamma^*}}{8\sqrt{2}(\gamma^*)^{3/2}} - \frac{K_{7/2}(\gamma^*)e^{\gamma^*}}{8\sqrt{2}(\gamma^*)^{5/2}} \right] \]
\[ = \frac{Y^4}{\sqrt{\pi} D^2} \left[ \frac{15\sqrt{\pi}}{16(\gamma^*)^4} + \frac{K_{5/2}(\gamma^*)e^{\gamma^*}}{8\sqrt{2}(\gamma^*)^{3/2}} - \frac{K_{7/2}(\gamma^*)e^{\gamma^*}}{8\sqrt{2}(\gamma^*)^{5/2}} \right] \]
\[ - \frac{3\sqrt{\pi}}{8(\gamma^*)^2} \times \frac{K_{3/2}(\gamma^*)e^{\gamma^*}}{\sqrt{2}\pi\gamma^*} + \frac{\sqrt{\pi}}{2} \times \left( \frac{K_{3/2}(\gamma^*)e^{\gamma^*}}{4\sqrt{2}\pi\gamma^*} - \frac{K_{5/2}(\gamma^*)e^{\gamma^*}}{4\sqrt{2}\pi(\gamma^*)^{3/2}} \right) \]
\[ = \frac{Y^4}{\sqrt{\pi} D^2} \left[ \frac{15\sqrt{\pi}}{16(\gamma^*)^4} + \frac{K_{5/2}(\gamma^*)e^{\gamma^*}}{8\sqrt{2}(\gamma^*)^{3/2}} - \frac{K_{7/2}(\gamma^*)e^{\gamma^*}}{8\sqrt{2}(\gamma^*)^{5/2}} \right] \]
\[ - \frac{3K_{3/2}(\gamma^*)e^{\gamma^*}}{8\sqrt{2}(\gamma^*)^{5/2}} + \frac{1}{8\sqrt{2}} \times \left( \frac{K_{3/2}(\gamma^*)e^{\gamma^*}}{\sqrt{\pi\gamma^*}} - \frac{K_{5/2}(\gamma^*)e^{\gamma^*}}{(\gamma^*)^{3/2}} \right) \]
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