Noncommutative Spherical Tight Frames in finitely generated Hilbert C*-modules*

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March 29, 2022

Abstract

Let $A$ be a fixed C*-algebra. In an arbitrary finitely generated projective $A$-module $V \subseteq A^n$, a spherical tight $A$-frame is a set of $k$, $k > n$, elements $f_1, \ldots, f_k$ such that the associated matrix $F = [f_1, \ldots, f_k]$ up-to a constant multiple is a partial isometry of the Hilbert structure on the projective finitely generated $A$-module $V$. The space $\mathcal{F}_{k,n}^A$ of all such $A$-frames form a C*-algebra, generated by a system of partial isometries and the structure of such C*-algebras are well described, especially in the case $A = \mathbb{R}$ or $\mathbb{C}$: The main result of K. Dykema and N. Strawn for these cases are generalized to our general projective finitely generated Hilbert $A$-module case. This generalization gives the possibility to study the universal classifying space.

Keywords: spherical tight frame, $K$-theory

Mathematics Subject Classification 2000: 19K35, 46L80, 46M20

1 Introduction

In the classical (commutative) theory of vector bundles the construction of Stiefel bundles is well-known and gives rise to the classification space for principal bundles. Let us recall ([11], Chapter 7) that for ground field $K = \mathbb{R}$ or $\mathbb{C}$, the work was supported in part by Vietnam National Project of Research in Fundamental Sciences and The Abdus Salam ICTP, UNESCO.

\footnote{The work was supported in part by Vietnam National Project of Research in Fundamental Sciences and The Abdus Salam ICTP, UNESCO.}
Let us recall also that the Grassman manifold $G_k(n)$ is the space of $k$-dimensional subspaces in $K^n$. It is also well-known, see e.g. ([H], Theorem 2.2) that $U_k(k) \hookrightarrow V_k(K^n) \rightarrow G_k(K^n) \cong U_k(n)/U_k(n-k)$ is a principal bundle. Moreover, see e.g. ([H], Theorem 4.1) the natural inclusion $U_k(n) \rightarrow U_k(n+q)$ induces morphism of homotopy groups $\pi_i(U_k(n)) \rightarrow \pi_i(U_k(n+q))$ which are isomorphism for $i \leq c(n+1)-3$ and epimorphism for $i \leq c(n+1)-2$. For $q = \infty$ the hypotheses are satisfied.

From these one deduces, see e.g. ([H], Chapter 7, Theorem 6.1) that the principal bundle $V_k(K^n) \rightarrow G_k(K^n)$ is universal in dimension less than or equal to $c(m+1)-2$ and $V_k(K^{\infty}) \rightarrow G_k(K^{\infty})$ is a universal bundle. This result can be understood, for example in the sense that the space $[X, G_k(K^n)] \cong Vect_k(X)$ of homotopy classes of maps is isomorphic with the space of isomorphic classes of vector bundles $f^*\gamma_k^{k+m}$ on a CW complex $X$ obtained from a universal vector bundle $\gamma_k^{k+m}$ on $G_k(K^n)$ when $n \leq c(m+1)-2$. In other words, all vector bundles on a CW complex $X$ can be regarded as some induced vector bundle, associated with a map from $X$ to the classification space $G_k(K^{k+m})$, for $n \leq c(m+1)-2$.

In the work [DS], K. Dykema and N. Strawn had proved that the space of frames with $k > n$ and with replacement of the orthonormality of the system of vectors $f_i, i \in I \subset \{1, \ldots, k\}$ by the spherical tight condition of frames

$$K^1 \subset K^2 \subset \ldots \subset K^k \subset K^{k+1} \subset \ldots K^\infty := \bigcup_{k=1}^\infty K^k$$

and also $U_k(\infty) = \bigcup_{k=1}^\infty U_k(k)$. The Stiefel variety, by definition is the subspace $V_k(K^n)$ of $K^{kn}$ consisting of orthonormal frames (k-tuples $[v_1, \ldots, v_k]$ of orthonormal vectors) in $K^k$, for $k \leq n = 1, 2, \ldots, \infty$. One defines the action of $U_k(k)$ on $V_k(K^n)$ by the map $\eta : U_k(k) \rightarrow V_k(K^n)$ via the formula $\eta(u) = (u(e_1), \ldots, u(e_k))$, where $e_1, \ldots, e_k$ is the standard orthonormal basis in $K^n$. With this action of $U_k(k)$ the space $V_k(K^n)$ becomes a principal bundle with structural group $U_k(k)$. It is well-known that $V_k(K^n) \cong U_k(n)/U_k(n-k)$, in particular it is a homogeneous space, called Stiefel variety and for example for $k = n$, $V_n(K^n) = U_k(n)$, for $k = 1$ $V_1(K^n)$ is the sphere $S^m$, where $m = n-1$ in the case $F = R$, $m = 2n-1$ in the case $F = C$ and $m = 4n-1$ in the case $F = H$, for $k = n-1$, $V_{n-1}(K^n) \cong U_k(n)/U_k(1) \cong SU_k(n)$.

R, C or H, denote $U_k(K)$ the orthogonal group $O_k(k)$ for $K = R$, the unitary group $U(k)$ for $K = C$ and the symplectic groups $Sp(k)$ for $K = H$. Denote

$K^1 \subset K^2 \subset \ldots \subset K^k \subset K^{k+1} \subset \ldots K^\infty := \bigcup_{k=1}^\infty K^k$
admits also a manifold structure. The aim of this paper is to use some non-
commutative analog of this manifold structure to construct noncommutative
universal classifying spaces. We introduce, in Section 2 a noncommutative
setting necessary for C*-algebras Hilbert projective A-modules of finite type.
In Section 3, we proved that the space of all noncommutative spherical tight
A-frames indeed admits a structure of a C*-algebra, generated by a system
of partial isometries and which can be decomposed into an orthogonal sum
of simple ones. In Section 4 we apply this result to construct the universal
classifying spaces in general noncommutative situation.

2 Equivalence classes of noncommutative spherical tight A-frames

We are interested in a class of noncommutative Serre Fibrations (NCSF) \([\text{\cite{D1}}]\)
of type
\[
f : A \longrightarrow B,
\]
i.e. a morphism from A to B in the category of C*-algebras with NCCW
structure. In that case, B is endowed with a structure of an \(A\)-module:
\[
a \in A, b \in B \mapsto a.b := f(a)b.
\]
A \textit{section} of the noncommutative Serre fibration \(f : A \longrightarrow B\), is defined
as a morphism \(s : B \rightarrow A\) such that \(f \circ s = Id_B\).

\textbf{Definition 2.1} A \textit{noncommutative spherical tight A-frame} is a collection
\(F = [f_i]_{i \in I}\) of vectors \(f_i\) in a finitely generated Hilbert \(A\)-module satisfying
the condition
\[
b ||v||_A^2 = \sum_{i \in I} ||\langle v, f_i \rangle||_A^2, \quad \forall v \in V,
\]
where \(b\) is some positive constant and \(||.||_A\) is the norm on the C*-algebra \(A\).

\textbf{Lemma 2.2} If \(F\) is a spherical tight A-frame in a submodule \(V\) of the standard Hilbert \(A\)-module \(\ell^2_A\), then there is an orthonormal basis of \(V\), in which
\(b^{-1/2}F = W_{k,n}U\), with \(U\) is a unitary operator in the \(A\)-module.
Proof. Indeed let us denote $e_1, \ldots, e_n$ the standard basis of the free $A$-module $A^n$. Then

$$\langle b^{-1/2}Fe_i, b^{-1/2}Fe_j \rangle = \langle e_i, e_j \rangle = \delta_{i,j}.$$ 

Let us denote $V = \langle b^{-1/2}Fe_1, \ldots, b^{-1/2}Fe_n \rangle$ the submodule of the Hilbert $A$-module $B$ and by $V^\perp$ the orthogonal complement

$$V^\perp = \{ v \in V; \langle v, e_i \rangle = 0, \forall i = 1, n \}.$$

Choose orthonormal basis in Hilbert $A$-modules $V$ as follows.

$$f_i = b^{-1/2}Fe_i, \forall i = 1, n,$$

in $V$ then we have matrix form of the partial isometry

$$b^{-1/2}F = \begin{bmatrix} b^{-1/2}F|_V & 0 \\ 0 & b^{-1/2}F|_{V^\perp} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} = [I_n|0_{n,k-n}].$$

□

Theorem 2.3 Any spherical tight $A$-frame in a finitely generated projective $A$-module $V$ can be realized as a matrix of type

$$F = b^{1/2}W_{k,n}U, \quad U \in \mathcal{O}^A(n),$$

where $\mathcal{O}^A(n)$ is denotes the orthogonal $A$-automorphism group of $A$-module $V$ and $W_{k,n} = [I_n|0_{n,k-n}]$.

Proof. Indeed, Every finitely generated projective $A$-module $V$ can be included in some free $A$-module $A^n$, as a direct summand and the latter can be included in the standard Hilbert $A$-module $\ell^2_A$. From the definition we see that

$$b||v||^2_A = \sum_{i \in I} ||\langle v, f_i \rangle||^2_A.$$ 

This means that $b^{-1/2}F$ is a partial isometry in the Hilbert module $A^n$. Following the previous Lemma, we can conclude that in a standard basis we have $b^{-1/2}F = W_{k,n}U$, where $W_{k,n} = [I_n|0_{n,k-n}]$ and $U$ is an element in the unitary group $U(A)$ with entries from $A$. □
3 The structure of the C*-algebra of NC spherical tight A-frames

Let us denote by $\mathcal{F}^A_{k,n}$ the set of all spherical tight A-frames in the finitely generated free A-module $A^n$.

**Proposition 3.1** Let $F = [f_1, \ldots, f_k] \in \mathcal{F}^A_{k,n}$ and let $I \subset \{1, \ldots, k\}$ be a subset. Let $e_1, \ldots, e_k, e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ be the standard basis of $A^k$, $Q_I : A^k \to A^k$ be the projection onto the subspace $\langle e_1, \ldots, e_k \rangle_A = \text{span}_A \{e_i| i \in I\}$. Then $F^*F$ is a projector and commutes with the projector $Q_I$ if and only if there is a submodule $V \subseteq A^n$ such that $[f_i]_{i \in I}$ is a tight A-frame in sub-A-module $V$, and $[f_i]_{i \in I^c}$ is a spherical tight A-frame in the sub-A-module $V^\perp$. Moreover, in that case (where $F^*F$ commutes with $Q_I$), the cardinality of $I$ is a multiple of $k/d$, where $d = \gcd(k, n)$.

**Proof.** Without loss of generality, up-to a change of order of $f_i$, which is equal to multiplication by $F$ on the right by a permutation matrix, we may assume that $F = [F_1|F_2]$, i.e. we may assume that $I = \{1, \ldots, p\}$ for some $p \in \{1, \ldots, k\}$. Then $F^* = \begin{bmatrix} F_1^* \\ F_2^* \end{bmatrix}$ and therefore

$$F^*F = \begin{bmatrix} F_1^*F_1 & F_1^*F_2 \\ F_2^*F_1 & F_2^*F_2 \end{bmatrix}.$$ 

In the case $F^*F$ commutes with $Q_I$, we have $F_1^*F_2 = (F_2^*F_1)^* = 0$ and the proof is achieved.

**Definition 3.2** In the situation of the previous Proposition 3.1 if $I$ is a proper nonempty subset of $\{1, \ldots, k\}$, we say that the spherical tight A-frame $F = [f_1, \ldots, f_k]$ is ortho-decomposable, $[f_i]_{i \in I}$ is a spherical tight A-frame in the sub-A-module $V$ and $[f_i]_{i \in I^c}$ is a spherical tight A-frame in the sub-A-module $V^\perp$. In the opposite case the A-frame is called ortho-indecomposable.
Theorem 3.3 Let $A$ be a C*-algebra, $\sigma$ a partition of the set $\{1, \ldots, k\}$, denote $\hat{M}_{k,n}^A$ the set of all $A$-frames $F \in \mathcal{F}_{k,n}^A$ which are otho-indecomposable, $\hat{M}_{k,n}^A(\sigma)$ the set of all $A$-frames $F \in \mathcal{F}_{k,n}^A$ such that $\rho_F = \sigma$.

1. $\hat{M}_{k,n}^A$ is a C*-algebra, generated by a otho-indecomposable system of partial isometries.

2. Let $d = \gcd(k, n)$ and let $k' = k/d$, $n' = n/d$. Let $\mathcal{P}(k, k')$ the set of all partitions of the set $\{1, \ldots, k\}$ into the subsets whose cardinalities are multiple of $k'$. Then $\mathcal{F}_{k,n}^A$ is a direct sum of the algebras $\hat{M}(\sigma)$, for $\sigma \in \mathcal{P}(k, k')$,

$$\mathcal{F}_{k,n}^A = \bigoplus_{\sigma \in \mathcal{P}(k, k')} \hat{M}_{k,n}^A(\sigma).$$

3. If $\sigma = \{A_1, \ldots, A_\ell\} \in \mathcal{P}(k, k')$ with $|A_i| = m_i k'$, then it is a tensor product of algebras

$$\hat{M}_{k,n}^A(\sigma) \cong \bigotimes_{i=1}^\ell \hat{M}_{m_i k', m_i n'}^A.$$

**Proof.** The theorem is proven in the same way as it was done in the commutative case, see (loc. cit., §4) for a more detailed analysis. The first assertion is exactly deduced from the definition of C*-algebra generated by a system of partial isometries, as the $A$-span of those partial isometries. \[\square\]

4 Application to noncommutative classification spaces

It is easy to see that $A = \text{Mat}_1(A)$ can be included in $\text{Mat}_n(A)$ and can be regarded as some noncommutative Serre fibration (NCSF), if $A$ admits some NCCW complex structure. The algebra $\text{Mat}_\infty(A)$ can then be regarded as some universal NCSF, in the sense that any finite rank NCSF can be obtained as induced one from a universal NCSF

$$\omega_W: \text{Mat}_\infty^n(A) = \lim_{\substack{\rightarrow \kappa}} \text{Mat}_\kappa^n(A) \to W.$$
**Definition 4.1** Let us consider the space of all infinite matrix of
\[ \text{Mat}_\infty^n(A) = \lim_{k \to \infty} \text{Mat}_k^n(A). \]
Suppose that \( \text{Mat}_\infty^n(A) \to W \) is a \( \text{Mat}_\infty^n(A) \)-module, then the pushout diagram
\[
\begin{array}{ccc}
A \ast M\infty^n(A) & \xleftarrow{\sim} & W \\
\uparrow & & \uparrow \\
A & \xleftarrow{\sim} & M\infty^n(A)
\end{array}
\]
of the two morphisms \( M\infty^n(A) \to A \), defined by \( m = [m_{ij}] \mapsto m_{11} \) and the fibration \( M\infty^n(A) \to W \) gives an \( A \)-module \( A \ast M\infty^n(A) \ast W \), which is called \textit{NC induced bundle}, what is indeed an induced \( A \)-module.

**Theorem 4.2** Any rank \( n \) projective \( A \)-module \( V \) can be obtained as some induced bundle \( A \otimes M\infty^n(A) \ast W \) from a finitely generated universal projective \( \text{Mat}_\infty(A) \)-module \( \omega_W \).

**Proof.** Indeed there is one-to-one correspondence between finitely generated projective \( A \)-modules and idempotents \( e^2 = e^* = e \) in \( \text{Mat}_\infty(A) \), in one hand side and one-to-one correspondence between finitely generated projective \( M\infty^n(A) \)-modules and idempotents \( E^2 = E^* = E \) in \( \text{Mat}_\infty(M\infty^n(A)) \cong M\infty^n(A) \otimes \text{Mat}_\infty(C) \cong \text{Mat}_\infty(A) \). Let us denote \( A \ast B \) the free product of two algebras \( A \) and \( B \).

**Theorem 4.3 (The Milnor universal bundle)** We have a natural noncommutative bundle
\[ \omega_A : \text{Mat}_\infty(A) \ast \text{Mat}_\infty(A) \ast \ldots \to \text{Mat}_\infty(A). \]

**Proof.** It is clear that \( \text{Mat}_\infty(A) \) and \( \text{Mat}_\infty(A) \ast \text{Mat}_\infty(A) \ast \ldots \) are NCCW. The maps from \( \text{Mat}_\infty(A) \ast \text{Mat}_\infty(A) \ast \ldots \to \text{Mat}_\infty(A) \) is defined as the natural product of factors from the free product to the algebra. Up-to homotopy equivalence in the category of NCCW every map is a NCSF.

**Theorem 4.4 (Comparison with the free product construction)** There is a natural map from the Milnor universal bundle \( \omega_A \) to the universal bundle \( \omega_W \), following the commutative diagram.
\[
\begin{array}{ccc}
\text{Mat}_\infty(A) & \longrightarrow & W \\
\omega_A & \uparrow & \\
\text{Mat}_\infty(A) \ast \text{Mat}_\infty(A) \ast \ldots & \longrightarrow & \text{Mat}_\infty^n(A)
\end{array}
\]

where \( W = \mathcal{F}_n(A^\infty) \) is the universal totualogical \( M^n_\infty(A) \) module.

**Proof.** The horizontal arrows are defined following the push-out diagram. They provide the isomorphism between two fibrations in vertical columns. \( \square \)

**Remark 4.5** The above theorem explains the construction of J. Cuntz and D. Quillen for classifying NC space as \( qA \). We construct some NC analog of the Stiefel construction of classifying spaces.

**Theorem 4.6** The K-theory \( K_*(A) \) of \( A \) and the \( KK^*(A, C) \) are isomorphic.

**Proof.** The KK functor in Cuntz setting is

\[ KK(A, B) = [[B, q \text{Mat}_\infty(A)]] , \]

where

\[ q \text{Mat}_\infty = \ker \{ \text{Mat}_\infty(A) \ast \text{Mat}_\infty(A) \ast \ldots \rightarrow \text{Mat}_\infty(A) \} \]

and \( [[A, B]] \) is the set of all homotopy classes of quasi-isomorphisms from \( A \) to \( B \). From the other side, we have \( M^n_\infty(A) \) is the classification space of rank \( n \) projective \( A \)-modules,

\[ K_*(A) = [C, \text{Mat}_\infty(A)] . \]

We have a natural map see [MT] from \([B, \text{Mat}_\infty(A)]\) into \([[B, q \text{Mat}_\infty(A)]]\), induced from the natural morphism

\[ q \text{Mat}_\infty(A) = \ker \{ \text{Mat}_\infty(A) \ast \text{Mat}_\infty(A) \ast \ldots \rightarrow \text{Mat}_\infty(A) \} \rightarrow \text{Mat}_\infty(A) . \]

And if this morphism becomes isomorphism up to a homotopy, then \( K_*(A) \cong KK^*(A, C) \).

In one hand we have the KK-theory of the algebras \( A \) and \( B = C \). In the other hand we have K-theory of \( A \) in the sense of the group of stable classes of projective finitely generated \( A \)-modules over \( M^n_\infty(A) \), i.e. the ordinary \( K \)-homology \( K_*(A) \). The theorem is proved. \( \square \)
Definition 4.7 A spherical tight $A$-frame in the universal Hilbert $A$-module $\ell^2(A)$ is a set $F = [f_1, \ldots, f_k]$ of $k$ vectors in $A^n \subset \ell^2(A)$, such that $k > n$.

Corollary 4.8 Any spherical tight $A$-frame can be obtained from some spherical tight $A$-frame on the universal Hilbert $A$-module $\ell^2(A)$.

Proof. In $A$-module $A^n F = [f_1, \ldots, f_k]$ is of form $F = b^{1/2} W_{k,n} U$, with $U \in \mathcal{O}^n(A)$. The latter group $\mathcal{O}^A(n)$ is a subgroup in the $\mathcal{O}^A(\infty)$ and therefore is also a spherical tight $A$-frame in a Mat$_\infty(A)$-module $\ell^2(A)$. □

Acknowledgments

The author would like to thank Abdus Salam ICTP for an excellent scientific stay and especially to Professor Le Dung Trang for invitation to participate the Commutative and Noncommutative Geometry Year Program in ICTP. The author expresses his deep thanks to Professor Claude Schochet for pointing out the work [DS].

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