Quantum Uncertainty in Decision Theory

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Abstract

An approach is presented treating decision theory as a probabilistic theory based on quantum techniques. Accurate definitions are given and thorough analysis is accomplished for the quantum probabilities describing the choice between separate alternatives, sequential alternatives characterizing conditional quantum probabilities, and behavioral quantum probabilities taking into account rational-irrational duality of decision making. The comparison between quantum and classical probabilities is explained. The analysis demonstrates that quantum probabilities serve as an essentially more powerful tool of characterizing various decision-making situations including the influence of psychological behavioral effects.

Keywords: Decision theory, Choice under uncertainty, Quantum probability, Quantum conditional probability, Behavioral effects
1 Introduction

Decision theory is an interdisciplinary topic widely used in various applications, such as economics, statistics, finances, data analysis, psychology, biology, politics, social sciences, philosophy, in computer and artificial intelligence studies. For example, one has to decide choosing between several portfolios of assets. The basis of decision theory is expected utility theory [1] that is a deterministic normative theory prescribing that with probability one the optimal choice is associated with the largest expected utility. There exist as well stochastic decision theories [2,3], which nevertheless are based on a deterministic approach decorated by superimposed randomness of utility.

However the overwhelming majority of empirical studies demonstrates that the choice of decision makers is not deterministic but in principle probabilistic. It is never happens that among a given group of people all without exception would make the identical choice prescribed by the standard deterministic utility theory. There always exist fractions of subjects preferring different alternatives. That is, there always exists a distribution of decisions over the set of the given alternatives. Moreover, as has been recently summarized by Woodford [4], psychological and neurological studies persuasively demonstrate that even a single decision maker in a single choice acts probabilistically because of the noisy functioning of the brain. This tells us that the correct description of decision making has to be probabilistic.

Two types of probabilities are known, classical and quantum. The natural question is: Which of these two types is more appropriate for describing human decision making? To convincely answer this question, it is necessary to thoroughly compare what situations in decision making can be treated by classical and quantum probabilities. That of them having the wider region of applicability would be more general, hence preferable, although the other one also can be employed in its domain of validity.

The basics of classical probability theory are well known [5]. Quantum probability, as applied to physical measurements, also has been described in several books [6]. Here we give the exposition of quantum probability in the language of decision theory, keeping in mind a decision maker performing a choice between several alternatives. We start from the notion of the probability of a single choice, then consider sequential choices among alternatives and compare these with classical probabilities.

It is important to stress that quantum probabilities can be generalized to the form that allows us to take into account behavioral effects and the dual nature of human decision making, comprising its rational cognitive side as well as irrational emotional feelings. This generalization also is described.

2 Probability of separate alternatives

A decision problem is assumed to consist in choosing one of the alternatives from the set \( \{A_n : n = 1, 2, \ldots, N_A\} \) of alternatives. Each alternative is associated with a vector \( |A_n\rangle \) from a Hilbert space \( \mathcal{H}_A \). Here and in what follows, we employ the Dirac bracket notation [7].

It is possible to accept that the set of the vectors \( |A_n\rangle \) forms a basis of \( \mathcal{H}_A \), although, for generality, we can accept any basis whose span composes the space \( \mathcal{H}_A \). The vectors \( |A_n\rangle \)
are orthonormalized, such that
\[ \langle A_m | A_n \rangle = \delta_{mn} \quad (n = 1, 2, \ldots, N_A) . \] (1)

Since we are planning to develop a probabilistic approach, we need a statistical object in order to introduce later quantum probability. The statistics of the problem at time \( t \) is represented by a statistical operator \( \hat{\rho}(t) \), which is a semipositive trace-class operator on \( \mathcal{H}_A \), briefly called the state. The pair \( \{ \mathcal{H}_A, \hat{\rho}(t) \} \) forms a quantum statistical ensemble. The evolution of the state in time, from an initial state \( \rho(0) \), is given by the relation
\[ \hat{\rho}(t) = \hat{U}(t, 0) \hat{\rho}(0) \hat{U}^+(t, 0) \] (2)
through the action of a unitary evolution operator \( \hat{U}(t, 0) \).

The set of alternatives is usually associated with the eigenvectors of operators of local observables acting on the Hilbert space \( \mathcal{H}_A \). The collection of these operators \( \{ \hat{A} \} \) composes a von Neumann algebra \( A(\mathcal{H}_A) \) which is a norm-closed algebra of bounded self-adjoint operators, containing the identity operator, on the Hilbert space \( \mathcal{H}_A \). The triple
\[ \{ \mathcal{H}_A, \hat{\rho}(t), A(\mathcal{H}_A) \} \]
is termed a quantum statistical system.

To each alternative, one puts into correspondence a projector
\[ \hat{P}(A_n) = | A_n \rangle \langle A_n | . \] (3)
In quantum measurements, there can occur the so-called degenerate states with projectors given by a sum of projectors. However in decision theory the degenerate states need to be specified, otherwise they do not have much sense. Therefore in decision theory we shall deal with the standard projectors (3). The collection of these projectors constitutes a projector-valued measure
\[ \mathcal{P}(A) = \{ \hat{P}(A_n) : n = 1, 2, \ldots, N_A \} . \] (4)
And the triple
\[ \{ \mathcal{H}_A, \hat{\rho}(t), \mathcal{P}(A) \} \] (5)
is a quantum probability space.

The probability of choosing at the moment of time \( t \) an alternative \( A_n \) is
\[ p(A_n, t) = \text{Tr} \hat{\rho}(t) \hat{P}(A_n) , \] (6)
with the trace over \( \mathcal{H}_A \). The probability is normalized, so that
\[ \sum_n p(A_n, t) = 1 , \quad 0 \leq p(A_n, t) \leq 1 . \] (7)
Unfolding the probability (6), we get
\[ p(A_n, t) = \langle A_n | \hat{\rho}(t) | A_n \rangle . \] (8)
Note that this result does not depend on the used basis, if one takes into account that for any basis \{|j\rangle\}, one has
\[
\sum_j |j\rangle\langle j| = \hat{1} .
\]

Similarly, for any problem represented by an operator \(\hat{B} \in \mathcal{A}(\mathcal{H}_A)\), one has a set of vectors \{\(|B_k\rangle\)\} that are orthonormalized,
\[
\langle B_k | B_p \rangle = \delta_{kp} \quad (k = 1, 2, \ldots, N_B)
\]
and define the corresponding projectors
\[
\hat{P}(B_k) = | B_k \rangle\langle B_k | .
\]
The family of these projectors composes the operator valued measure
\[
\mathcal{P}(B) = \{\hat{P}(B_k) : k = 1, 2, \ldots, N_B\} .
\]
The corresponding quantum probability space is
\[
\{\mathcal{H}_A, \hat{\rho}(t), \mathcal{P}(B)\} .
\]
The probability of choosing at time \(t\) an alternative \(B_k\) is
\[
p(B_k, t) = \text{Tr} \hat{\rho}(t) \hat{P}(B_k)
\]
satisfying the normalization condition
\[
\sum_k p(B_k, t) = 1 , \quad 0 \leq p(B_k, t) \leq 1 .
\]

3 Probability of sequential choices

Considering sequential choices, it is necessary to accurately take into account the temporal evolution of probabilities [8]. The probability of an alternative \(A_n\), defined in Eq. (6), is the a priori expected probability of choosing this alternative. Suppose at the moment \(t_0\) a decision has been made and the alternative \(A_n\) has actually been chosen. Hence a priori form (6) is valid only before this time up to the time \(t_0 - 0\), when
\[
p(A_n, t_0 - 0) = \text{Tr} \hat{\rho}(t_0 - 0) \hat{P}(A_n) .
\]
If at the time \(t_0\) the alternative \(A_n\) has certainly been chosen, this means that the a posteriori probability of choosing \(A_n\) becomes
\[
p(A_n, t_0 + 0) = 1 .
\]
One tells that at the moment \(t_0\) the state reduction has happened,
\[
\hat{\rho}(t_0 - 0) \longrightarrow \hat{\rho}(A_n, t_0 + 0) ,
\]
leading to the probability update

\[ p(A_n, t_0 - 0) \rightarrow p(A_n, t_0 + 0) . \]  

(18)

Condition \((16)\) implies

\[ p(A_n, t_0 + 0) = \text{Tr} \hat{\rho}(A_n, t_0 + 0) \hat{P}(A_n) = 1 . \]  

(19)

The solution to this equation, describing the state reduction \((17)\), reads as

\[ \hat{\rho}(A_n, t_0 + 0) = \hat{P}(A_n) \hat{\rho}(t_0 - 0) \hat{P}(A_n) , \]  

(20)

which is the von Neumann-Lüders state \([6, 9]\). This form serves as a new initial condition for the state evolution

\[ \hat{\rho}(A_n, t) = \hat{U}(t, t_0) \hat{\rho}(A_n, t_0 + 0) \hat{U}^+(t, t_0) \]  

(21)

after the time \(t_0\).

Aiming at making a choice among the set of alternatives \(\{B_k\}\), we deal with the quantum probability space

\[ \{ \mathcal{H}_A, \hat{\rho}(A_n, t), \mathcal{P}(B) \} . \]  

(22)

The related probability plays the role of the \textit{quantum conditional probability}

\[ p(B_k, t|A_n, t_0) = \text{Tr} \hat{\rho}(A_n, t) \hat{P}(B_k) \]  

(23)

of choosing an alternative \(B_k\) at a time \(t\), after the alternative \(A_n\) has certainly been chosen at the time \(t_0\).

Introducing the notation

\[ p(B_k, t, A_n, t_0) \equiv \text{Tr} \hat{U}(t, t_0) \hat{P}(A_n) \hat{\rho}(t_0 - 0) \hat{P}(A_n) \hat{U}^+(t, t_0) \hat{P}(B_k) \]  

(24)

allows us to represent the conditional probability \((23)\) in the form

\[ p(B_k, t|A_n, t_0) = \frac{p(B_k, t, A_n, t_0)}{p(A_n, t_0 - 0)} . \]  

(25)

This relation is similar to the classical relation between conditional and joint probabilities, which justifies the admissibility of naming the probability \((24)\) \textit{quantum joint probability} of choosing an alternative \(A_n\) at the time \(t_0\) and an alternative \(B_k\) at the time \(t > t_0\). The difference with the classical probabilities is that now the choices are made at different times, but not simultaneously.

An important particular case is when the second choice is made immediately after the first one \([10]\). Then the evolution operator reduces to the identity,

\[ \hat{U}(t_0 + 0, t_0) = \hat{1} . \]  

(26)

This simplifies the conditional probability

\[ p(B_k, t_0 + 0|A_n, t_0) = \text{Tr} \hat{\rho}(A_n, t_0 + 0) \hat{P}(B_k) \]  

(27)
and the joint probability

\[ p(B_k, t_0 + 0, A_n, t_0) = \text{Tr} \hat{P}(A_n) \hat{\rho}(t_0 - 0) \hat{P}(A_n) \hat{P}(B_k). \]  

(28)

The latter is analogous to the Wigner probability [11]. The relation (25) now reads as

\[ p(B_k, t_0 + 0|A_n, t_0) = \frac{p(B_k, t_0 + 0, A_n, t_0)}{p(A_n, t_0 - 0)}. \]

(29)

Accomplishing the trace operation in the joint probability (28) yields

\[ p(B_k, t_0 + 0, A_n, t_0) = |\langle B_k | A_n \rangle|^2 p(A_n, t_0 - 0). \]

(30)

Hence the conditional probability (29) becomes

\[ p(B_k, t_0 + 0|A_n, t_0) = |\langle B_k | A_n \rangle|^2. \]

(31)

4 Symmetry properties of probabilities

It is important to study the symmetry properties of the quantum probabilities when the choice order reverses, that is, first one chooses an alternative \( B_k \) at the time \( t_0 \) and then one considers the probability of choosing an alternative \( A_n \) at the time \( t \). The symmetry properties should be compared with those of classical probabilities. Not to confuse the latter with the quantum probabilities, denoted by the letter \( p \), we shall denote the classical probabilities by the letter \( f \). Thus the classical conditional probability of two events, \( A_n \) and \( B_k \), is

\[ f(B_k|A_n) = \frac{f(B_kA_n)}{f(A_n)}, \]

(32)

where \( f(B_kA_n) \) is the classical joint probability, which is symmetric:

\[ f(A_nB_k) = f(B_kA_n), \]

(33)

while the conditional probability is not,

\[ f(A_n|B_k) \neq f(B_k|A_n). \]

(34)

For the quantum probabilities with the reversed order, acting as in the previous section, we obtain the conditional probability

\[ p(A_n, t|B_k, t_0) = \text{Tr} \hat{\rho}(B_k, t) \hat{P}(A_n), \]

(35)

with the state

\[ \hat{\rho}(B_k, t) = \hat{U}(t, t_0) \hat{\rho}(B_k, t_0 + 0) \hat{U}^+(t, t_0), \]

(36)

where

\[ \hat{\rho}(B_k, t_0 + 0) = \frac{\hat{P}(B_k)\hat{\rho}(t_0 - 0)\hat{P}(B_k)}{\text{Tr}\hat{\rho}(t_0 - 0)\hat{P}(B_k)}. \]

(37)
Introducing the notation of the joint probability
\[ p(A_n, t, B_k, t_0) \equiv \text{Tr} \, \hat{U}(t, t_0) \, \hat{P}(B_k) \hat{\rho}(t_0 - 0) \, \hat{P}(B_k) \, \hat{U}^+(t, t_0) \, \hat{P}(A_n) , \quad (38) \]
results in the relation
\[ p(A_n, t|B_k, t_0) = \frac{p(A_n, t, B_k, t_0)}{p(B_k, t_0 - 0)} . \quad (39) \]

For different times \( t \) and \( t_0 \), neither conditional nor joint quantum probabilities are symmetric:
\[ p(A_n, t|B_k, t_0) \neq p(B_k, t|A_n, t_0) , \]
\[ p(A_n, t, B_k, t_0) \neq p(B_k, t, A_n, t_0) \quad (t > t_0) . \quad (40) \]

The quantum and classical probabilities satisfy the same normalization conditions, such as for the conditional probability
\[ \sum_k p(B_k, t|A_n, t_0) = \sum_n p(A_n, t|B_k, t_0) = 1 . \quad (41) \]
and for the joint probability
\[ \sum_k p(B_k, t, A_n, t_0) = p(A_n, t_0 - 0) , \]
\[ \sum_n p(A_n, t, B_k, t_0) = p(B_k, t_0 - 0) . \quad (42) \]

Then the normalization condition follows:
\[ \sum_{nk} p(B_k, t, A_n, t_0) = \sum_{nk} p(A_n, t, B_k, t_0) = 1 , \quad (43) \]
from which
\[ \sum_{nk} [ p(B_k, t, A_n, t_0) - p(A_n, t, B_k, t_0) ] = 0 . \quad (44) \]

In the case when in the second choice at the time \( t_0 + 0 \) one estimates the probability of an alternative \( A_n \) immediately after the first choice at the time \( t_0 \) has resulted in an alternative \( B_k \), similarly to the previous section, we find the joint probability
\[ p(A_n, t_0 + 0, B_k, t_0) = |\langle A_n | B_k \rangle|^2 \, p(B_k, t_0 - 0) \quad (45) \]
and the conditional probability
\[ p(A_n, t_0 + 0|B_k, t_0) = |\langle A_n | B_k \rangle|^2 . \quad (46) \]

Therefore the conditional probability is symmetric, while the joint probability is not:
\[ p(A_n, t_0 + 0|B_k, t_0) = p(B_k, t_0 + 0|A_n, t_0) , \]
\[ p(A_n, t_0 + 0, B_k, t_0) \neq p(B_k, t_0 + 0, A_n, t_0) \quad (t = t_0 + 0) , \quad (47) \]
which is contrary to the classical case \((33)\) and \((34)\).

If the projectors \(\hat{P}(A_n)\) and \(\hat{P}(B_k)\) commute, then the joint probability \((28)\) becomes symmetric:

\[
p(B_k, t_0 + 0, A_n, t_0) = \text{Tr} \hat{\rho}(t_0 - 0) \hat{P}(B_k) \hat{P}(A_n) = p(A_n, t_0 + 0, B_k, t_0),
\]

provided that \([\hat{P}(A_n), \hat{P}(B_k)] = 0\). Taking into account the form of the joint probability \((45)\), we get the equality

\[
p(B_k, t_0 - 0) = p(A_n, t_0 - 0) \quad \left( [\hat{P}(A_n), \hat{P}(B_k)] = 0 \right).
\]

Thus in that case both the joint and conditional probabilities are symmetric, which contradicts the asymmetry of the classical conditional probability.

If the repeated choice is made among the same set of alternatives, say \(\{A_n\}\), that is when \(B_k = A_k\), we obtain

\[
p(A_k, t_0 + 0|A_n, t_0) = \delta_{nk}.
\]

This equation represents the principle of the choice reproducibility, according to which, when the choice, among the same set of alternatives, is made twice, immediately one after another, the second choice reproduces the first one. This sounds reasonable for decision making. Really, when a decision maker accomplishes a choice immediately after another one, there is no time for deliberation, hence this decision maker just should repeat the previous choice \([12]\).

## 5 Duality in decision making

Human decision making is known to be of dual nature, including the rational (slow, cognitive, conscious, objective) evaluation of alternatives and their irrational (fast, emotional, subconscious, subjective) appreciation \([13–16]\). This feature of decision making that can be called rational-irrational duality, or cognition-emotion duality, or objective-subjective duality, can be effectively described in the language of quantum theory that also possesses a dual nature comprising the so-called particle-wave duality. To take into account the dual nature of decision making, the quantum decision theory has been advanced \([17–22]\). In the frame of this theory, quantum probability, taking account of emotional behavioral effects, becomes behavioral probability \([23]\). Below, we briefly delineate quantum decision theory following the recent papers \([8, 12]\).

The space of alternatives \(\mathcal{H}_A\) is composed of the state vectors characterizing the rational representation of these alternatives whose probabilities can be rationally and objectively evaluated. Since there also exist subjective emotional feelings, for taking them into account, the space of the state vectors has to be extended by including the subject space

\[
\mathcal{H}_S = \text{span} \{ |\alpha\rangle \}
\]

formed by the vector representations \(|\alpha\rangle\) of all admissible elementary feelings. These vectors \(|\alpha\rangle\) form an orthonormal basis,

\[
\langle \alpha | \beta \rangle = \delta_{\alpha\beta}.
\]
Thus, the total decision space is the tensor product
\[ \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_S. \] (52)

The statistical state \( \hat{\rho}(t) \) now acts on the decision space (52) where it evolves as
\[ \hat{\rho}(t) = \hat{U}(t, 0) \hat{\rho}(0) \hat{U}^+(t, 0). \] (53)

Respectively, the quantum statistical ensemble is
\( \{ \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_S, \hat{\rho}(t) \}. \) (54)

Each alternative \( A_n \) is accompanied by a related set of emotions \( x_n \) that is represented in the subject space by an emotion vector \( | x_n \rangle \in \mathcal{H}_S \), which can be written as an expansion
\[ | x_n \rangle = \sum_{\alpha} b_{n\alpha} | \alpha \rangle. \] (55)

Strictly speaking, emotions are contextual and are subject to variations, which means that the coefficients \( b_{n\alpha} \) can vary and, generally, fluctuate with time depending on the state of a decision maker and the corresponding surrounding.

The emotion vectors can be normalized,
\[ \langle x_n | x_n \rangle = \sum_{\alpha} | b_{n\alpha} |^2 = 1, \] (56)

but they are not necessarily orthogonal, so that
\[ \langle x_m | x_n \rangle = \sum_{\alpha} b_{m\alpha}^* b_{n\alpha}. \] (57)

is not compulsorily a Kronecker delta.

An emotion operator
\[ \hat{P}(x_n) = | x_n \rangle \langle x_n | \] (58)

is idempotent,
\[ [ \hat{P}(x_n) ]^2 = \hat{P}(x_n), \] (59)

but different operators are not orthogonal, since
\[ \hat{P}(x_m) \hat{P}(x_n) = \langle x_m | x_n \rangle x_m \langle x_n |. \] (60)

The emotion operators of elementary feelings \( | \alpha \rangle \), forming a complete orthonormal basis in the space (51), sum to one
\[ \sum_{\alpha} \hat{P}(\alpha) = \sum_{\alpha} | \alpha \rangle \langle \alpha | = \hat{1}, \]

but the emotion operators (58) do not necessarily sum to one, giving
\[ \sum_n \hat{P}(x_n) = \sum_n \sum_{\alpha\beta} b_{n\alpha} b_{n\beta}^* | \alpha \rangle \langle \beta |. \]
from where
\[ \langle \alpha | \sum_n \hat{P}(x_n) | \beta \rangle = \sum_n b_{n\alpha} b_{n\beta}^*. \]

The emotion vectors \(x_n\) do not form a basis, hence the emotion operators (58) do not have to sum to one. The projector (58) projects onto the subspace of feelings associated with the alternative \(A_n\).

The pair of an alternative \(A_n\) and the set of the related emotions \(x_n\) composes a prospect \(A_n x_n\) whose representation in the decision space (52) is given by the vector
\[ | A_n x_n \rangle = A_n \rangle \otimes x_n \rangle = \sum_\alpha b_{n\alpha} | A_n \alpha \rangle . \quad (61) \]

These vectors are orthonormalized,
\[ \langle x_m A_m | A_n x_n \rangle = \delta_{mn} . \]

The prospect projector
\[ \hat{P}(A_n x_n) = A_n x_n \rangle \langle x_n A_n | = \hat{P}(A_n) \otimes \hat{P}(x_n) \quad (62) \]
is idempotent,
\[ [ \hat{P}(A_n x_n) ]^2 = \hat{P}(A_n x_n) . \quad (63) \]

The projectors of different prospects are orthogonal,
\[ \hat{P}(A_m x_m) \hat{P}(A_n x_n) = \delta_{mn} \hat{P}(A_n x_n) \quad (64) \]
and commute with each other,
\[ [ \hat{P}(A_m x_m), \hat{P}(A_n x_n) ] = 0 . \]

The vectors \( | A_n \alpha \rangle \) generate a complete basis in the decision space (52), because of which
\[ \sum_{n\alpha} \hat{P}(A_n \alpha) = \sum_{n\alpha} | A_n \alpha \rangle \langle \alpha A_n | = \hat{1} . \quad (65) \]

But the prospect projectors (62) on subspaces do not necessarily sum to one,
\[ \sum_n \hat{P}(A_n x_n) = \sum_n \sum_{\alpha\beta} b_{n\alpha} b_{n\beta}^* | A_n \alpha \rangle \langle \alpha A_n | , \]
as far as
\[ \langle \alpha A_m | \sum_n \hat{P}(A_n x_n) | A_n \beta \rangle = \delta_{mn} b_{n\alpha}^* b_{n\beta} . \]

However, it is admissible to require that the prospect projectors would sum to one on average, so that
\[ \text{Tr} \hat{\rho}(t) \sum_n \hat{P}(A_n x_n) = 1 , \quad (66) \]
which is equivalent to the condition
\[
\sum_n \sum_{\alpha\beta} b_{\alpha n}^* b_{n\beta} \langle \alpha A_n | \hat{\rho}(t) | A_n \beta \rangle = 1 .
\] (67)

The trace operation in Eq. (66) and below is over the total decision space (52).

The projection-valued measure on the space (52) is
\[
P(Ax) = \{ \hat{P}(A_n x_n) : n = 1, 2, \ldots, N_A \},
\] (68)
so that the quantum probability space is
\[
\{ \mathcal{H}, \hat{\rho}(t), P(Ax) \}. 
\] (69)

The prospect probability reads as
\[
p(A_n x_n, t) = \text{Tr} \hat{\rho}(t) \hat{P}(A_n x_n)
\] (70)
and satisfies the normalization conditions
\[
\sum_n p(A_n x_n, t) = 1, \quad 0 \leq p(A_n x_n, t) \leq 1.
\] (71)

In expression (70), it is possible to separate the diagonal part
\[
f(A_n x_n, t) \equiv \sum_\alpha |b_{n\alpha}|^2 \langle \alpha A_n | \hat{\rho}(t) | A_n \alpha \rangle
\] (72)
and the nondiagonal part
\[
q(A_n x_n, t) \equiv \sum_{\alpha \neq \beta} b_{\alpha n}^* b_{n\beta} \langle \alpha A_n | \hat{\rho}(t) | A_n \beta \rangle .
\] (73)

The diagonal part has the meaning of the rational fraction of the total probability (70),
because of which it is called the \textit{rational fraction} and is assumed to satisfy the normalization condition
\[
\sum_n f(A_n x_n, t) = 1, \quad 0 \leq f(A_n x_n, t) \leq 1 .
\] (74)

The rational fraction satisfies the standard properties of classical probabilities.

The nondiagonal part is caused by the quantum interference of emotions and fulfills the conditions
\[
\sum_n q(A_n x_n, t) = 0, \quad -1 \leq q(A_n x_n, t) \leq 1.
\] (75)

As far as emotions describe the quality of alternatives, the quantum term (75) can be called the \textit{quality factor}. Being due to quantum interference, it also can be named the \textit{quantum factor}. And since the quality of alternatives characterizes their attractiveness, the term (75) can be called the \textit{attraction factor}.

Thus the prospect probability (70) reads as the sum of the rational fraction and the quality factor:
\[
p(A_n x_n, t) = f(A_n x_n, t) + q(A_n x_n, t).
\] (76)
6 Conditional behavioral probability

Sequential choices in quantum decision theory can be treated by analogy with Sec. 3. The a priori probability at any time $t < t_0$ is defined in Eq. (70), provided no explicit choice has been done before the time $t_0$. Just until this time, the a priori probability of an alternative $A_n$ is

$$p(A_n x_n, t_0 - 0) = \text{Tr} \hat{\rho}(t_0 - 0) \hat{P}(A_n x_n).$$

(77)

If at the moment of time $t_0$ a choice has been made and an alternative $A_n$ is certainly chosen, then the a posteriori probability becomes

$$p(A_n x_n, t_0 + 0) = 1.$$

(78)

This implies the reduction of the probability

$$p(A_n x_n, t_0 - 0) \rightarrow p(A_n x_n, t_0 + 0)$$

and the related state reduction

$$\hat{\rho}(t_0 - 0) \rightarrow \hat{\rho}(A_n x_n, t_0 + 0).$$

(80)

Equation (78), asserting that

$$\text{Tr} \hat{\rho}(A_n x_n, t_0 + 0) \hat{P}(A_n x_n) = 1,$$

(81)

possesses the solution

$$\hat{\rho}(A_n x_n, t_0 + 0) = \frac{\hat{P}(A_n x_n) \hat{\rho}(t_0 - 0) \hat{P}(A_n x_n)}{\text{Tr} \hat{\rho}(t_0 - 0) \hat{P}(A_n x_n)}.$$

(82)

The state (82) serves as an initial condition for the new dynamics prescribed by the equation

$$\hat{\rho}(A_n x_n, t) = \hat{U}(t, t_0) \hat{\rho}(A_n x_n, t_0 + 0) \hat{U}^+(t, t_0).$$

(83)

The a priori probability of choosing an alternative $B_k$ at any time $t > t_0$, after the alternative $A_n$ has certainly been chosen, is the conditional probability

$$p(B_k x_k, t|A_n x_n, t_0) = \text{Tr} \hat{\rho}(A_n x_n, t) \hat{P}(B_k x_k).$$

(84)

Introducing the joint behavioral probability

$$p(B_k x_k, t, A_n x_n, t_0) \equiv \text{Tr} \hat{U}(t, t_0) \hat{P}(A_n x_n) \hat{\rho}(t_0 - 0) \hat{P}(A_n x_n) \hat{U}^+(t, t_0) \hat{P}(B_k x_k),$$

(85)

allows us to represent the conditional probability in the form

$$p(B_k x_k, t|A_n x_n, t_0) = \frac{p(B_k x_k, t, A_n x_n, t_0)}{p(A_n x_n, t_0 - 0)}.$$

(86)

If the probability of choosing an alternative $B_k$ is evaluated immediately after $t_0$, then we need to consider the conditional probability

$$p(B_k x_k, t_0 + 0|A_n x_n, t_0) = \text{Tr} \hat{\rho}(A_n x_n, t_0 + 0) \hat{P}(B_k x_k)$$

(87)
and the joint probability
\[ p(B_k x_k, t_0 + 0, A_n x_n, t_0) = \text{Tr} \hat{P}(A_n x_n) \hat{\rho}(t_0 - 0) \hat{P}(A_n x_n) \hat{P}(B_k x_k). \] (88)

As a result, the conditional probability \((87)\) becomes
\[ p(B_k x_k, t_0 + 0 | A_n x_n, t_0) = \frac{p(B_k x_k, t_0 + 0, A_n x_n, t_0)}{p(A_n x_n, t_0 - 0)}. \] (89)

For the joint probability, we obtain
\[ p(B_k x_k, t_0 + 0, A_n x_n, t_0) = |\langle x_k B_k | A_n x_n \rangle|^2 p(A_n x_n, t_0 - 0). \] (90)

Therefore the conditional behavioral probability is
\[ p(B_k x_k, t_0 + 0 | A_n x_n, t_0) = |\langle x_k B_k | A_n x_n \rangle|^2. \] (91)

## 7 Symmetry of behavioral probabilities

Considering the symmetry properties of behavioral probabilities, it is useful to remember that, strictly speaking, emotions are contextual and can vary in time. In an approximate picture, it is possible to assume that emotions are mainly associated with the corresponding alternatives and are approximately the same at all times. Then the symmetry properties of the probabilities can be studied with respect to the interchange of the order of the prospects \(A_n x_n\) and \(B_k x_k\). Keeping in mind this kind of the order interchange, we can conclude that the symmetry properties of behavioral probabilities are similar to the order symmetry of the quantum probabilities examined in Sec. 4.

Thus for any time \(t > t_0\), both the conditional and the joint behavioral probabilities are not order symmetric with respect to the prospect interchange,
\[ p(A_n x_n, t | B_k x_k, t_0) \neq p(B_k x_k, t | A_n x_n, t_0) \quad (t > t_0), \] (92)
and
\[ p(A_n x_n, t, B_k x_k, t_0) \neq p(B_k x_k, t, A_n x_n, t_0) \quad (t > t_0), \] (93)
which is analogous to Eq. (40).

When the second decision is being made at the time \(t = t_0 + 0\) immediately after the first choice has been accomplished at the time \(t_0\), the conditional behavioral probability is order symmetric,
\[ p(A_n x_n, t_0 + 0 | B_k x_k, t_0) = p(B_k x_k, t_0 + 0 | A_n x_n, t_0), \] (94)
but the joint probability, generally, is not order symmetric,
\[ p(A_n x_n, t_0 + 0, B_k x_k, t_0) \neq p(B_k x_k, t_0 + 0, A_n x_n, t_0), \] (95)
which is similar to property \((47)\).

If one makes the immediate sequential choices, and in addition the prospect projectors commute with each other, so that
\[ [\hat{P}(A_n x_n), \hat{P}(B_k x_k)] = 0, \]
then the joint behavioral probability becomes order symmetric,

\[ p(A_n x_n, t_0 + 0, B_k x_k, t_0) = p(B_k x_k, t_0 + 0, A_n x_n, t_0) . \] (96)

This property is in agreement with Eq. (48).

Recall that the conditional probability (93), because of its form (91), is symmetric in any case, whether the prospect operators commute or not.

As is seen, the symmetry properties of the quantum probabilities are in variance with the properties (33) and (34) of the classical probabilities, according to which the joint classical probability is order symmetric, while the conditional classical probability is not order symmetric.

The absence of the order symmetry in classical conditional probability is evident from the definition (32). Empirical investigations [24, 25] also show that the conditional probability is not order symmetric. However, as is seen from equality (94), the quantum conditional probability is order symmetric. Does this mean that the quantum conditional probability cannot be applied to the realistic human behavior?

To answer this question, it is necessary to concretize the realistic process of making decisions. In reality, any decision is not a momentary action, but it takes some finite time. The modern point of view accepted in neurobiology and psychology is that the cognition process, through which decisions are generated, involves three stages: the process of stimulus encoding through which the internal representation is generated, followed by the evaluation of the stimulus signal and then by decoding of the internal representation to draw a conclusion about the stimulus that can be consciously reported [4, 26]. It has been experimentally demonstrated that awareness of a sensory event does not appear until the delay time up to 0.5 s after the initial response of the sensory cortex to the arrival of the fastest projection to the cerebral cortex [26, 27]. About the same time is necessary for the process of the internal representation decoding. So, the delay time of about 1 s is the minimal time for the simplest physiological processes involved in decision making. Sometimes the evaluation of the stimulus signal constitutes the total response time, necessary for formulating a decision, of about 10 s [28]. In any case, the delay time of order 1 s seems to be the minimal period of time required for formulating a decision. This assumes that in order to consider a sequential choice as following immediately after the first one, as is necessary for the quantum conditional probability (89) or (91), the second decision has to follow in about 1 s after the first choice. However, to formulate the second task needs time, as well as some time is required for the understanding the second choice problem. This process demands several minutes.

In this way, the typical situation in the sequential choices is when the temporal interval between the decisions is of the order of minutes, which is much longer than the time of 1 s necessary for taking a decision. Therefore the second choice cannot be treated as following immediately after the first one, hence the form of the conditional probability (91) is not applicable to such a situation. For that case, one has to use expression (86) which is not order symmetric, in agreement with the inequality (92) and empirical observations.

Thus the decisions can be considered as following immediately one after the other provided the temporal interval between them is of the order of 1 s. Such a short interval between subsequent measurements could be realized in quantum experiments, but it is not realizable in human decision making, where the interval between subsequent decisions is usually much longer than 1 s. Hence the form of the conditional probability (91), that one often calls the
Lüders probability, is not applicable to human problems, but expression (86), valid for a finite time interval between decisions, has to be employed. The latter is not order symmetric similarly to the classical conditional probability.

Concluding, quantum probabilities, whose definition takes into account dynamical processes of taking decisions, are more general than simple classical probabilities (32), hence can be applied to a larger class of realistic human decision problems.

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