Localization and exact current compensation
in the quantum Hall effect

K. Shizuya
Yukawa Institute for Theoretical Physics
Kyoto University, Kyoto 606-01, Japan

A field-theoretic formulation of a planar Hall-electron system with edges is presented and some fundamental aspects of the integer quantum Hall effect are studied with emphasis on clarifying general symmetry-based consequences of localization. It is shown, in particular, that the immobility of localized electron states and current compensation by extended electron states, both crucial for quantization of the Hall conductance, are derived through the operation of magnetic translation of localized electron states alone. They actually are consequences of gauge invariance and hold under general circumstances with both level mixing and electron edge states taken into account.

73.40.Hm,73.20.Dx,11.15-q

I. INTRODUCTION

One of the key elements in the quantum Hall effect (QHE) is localization of electron states via disorder. The current-carrying properties of Hall electrons are substantially modified by localization: The localized electron states cease to support current. Miraculously, the surviving extended electron states carry more current and exactly compensate for the loss due to the localized states. Aoki and Ando[1] and Prange[2] identified this phenomenon of current compensation as a mechanism responsible for the exact quantization of the Hall conductance; it was originally noted in some specific cases (of a strong magnetic field or a single impurity), but it actually holds under more general circumstances with level mixing and electron edge states properly taken into account.

In view of the principal importance of this phenomenon of current compensation it is desirable to fully explore its content and implications. The purpose of the present paper is to elaborate on this point. It was shown earlier[1,2] for an infinite Hall-electron system that the current compensation theorem is derived through magnetic translation of localized electron states. In this paper we present a refined version of our previous consideration adapted to accommodate the presence of sample edges, and point out that the symmetry principle underlying current compensation is electromagnetic gauge invariance. Special care is taken to treat inter-level degeneracy inherent to edge states.

In Sec. II we present a field-theoretic description of a planar Hall system with sharp edges. In Sec. III we construct the Hamiltonians projected to each impurity-broadened Landau subband. In Sec. IV we study consequences of localization and give a general proof of current compensation. In Sec. V we present another proof within a linear-response treatment, which shows an interplay of the localized and extended states explicitly. Section VI is devoted to concluding remarks.

II. FIELD THEORY OF HALL ELECTRONS

In this section we formulate a field theory of Hall electrons in the presence of disorder. Consider electrons confined to an infinitely long strip of width \( L_y \) (or formally, a strip bent into a loop of circumference \( L_x \)), described by the Hamiltonian:

\[
\mathcal{H} = \int dx dy \, \Psi^\dagger(x, y, t) \mathcal{H} \Psi(x, y, t),
\]

\[
\mathcal{H} = H_0 + U(x, y) - eA_0(y),
\]

\[
H_0 = \frac{1}{2} \omega \left\{ \ell^2 p_y^2 + (1/\ell^2)(y - y_0 - eA_x)^2 \right\},
\]

written in terms of \( \omega \equiv eB/m \), the magnetic length \( \ell \equiv 1/\sqrt{eB} \) and \( y_0 \equiv p_x/(eB) \). Here \( U(x, y) \) stands for a random impurity potential and the Landau-gauge vector potential \( (-By, 0) \) has been used to supply a uniform magnetic field \( B \) normal to the plane. We take explicit account of the two edges \( y = \pm L_y/2 \), where the wave function is bound to vanish.

We shall study the Hall effect in a static setting: The scalar potential \( A_0(y) \) supplies a general Hall field \( E_y(y) = -\partial_y A_0(y) \) that can vary across the strip. We use a constant potential \( A_x \) to detect the total Hall current \( J_x = \int dy \, j_x(x, y, t) \), or more precisely, its spatial average \( (1/L_x) \int dx \, J_x \), that flows in response to \( E_y(y) \).

The kinetic term \( H_0 \) describes the cyclotron motion of an electron. The eigenstates of \( H_0 \) in the sample bulk are Landau levels with discrete energy \( \omega(n + \frac{1}{2}) \), labeled by integers \( n = 0, 1, 2, \ldots \), and \( y_0 = \ell^2 p_x \). The eigenfunctions in the presence of sharp edges are still labeled by \( N = (n, y_0) \):

\[
\psi_N(x, y) \equiv \langle x, y | N \rangle = (2\pi\ell^2)^{-1/2} e^{i x p x} \phi_N(y),
\]

with \( \phi_N(y) \) given by the parabolic cylinder functions \( D_n(\pm \sqrt{2}(y - y_0)/\ell) \) for electrons residing near the edges \( y = \mp L_y/2 \), where \( y_0 \equiv y_0 + e\ell^2 A_x \). Setting \( \phi_N = 0 \) at \( y = \pm L_y/2 \) fixes the energy eigenvalues.
as a function of $\tilde{y}_0$ for each $n$. The spectra $\nu_n(\tilde{y}_0)$ are determined numerically. They rise sharply for $|\tilde{y}_0| \gtrsim L_y/2$, and recover the integer values $n$ as $y_0$ moves into the interior a few magnetic lengths away from $y_0 = \pm L_y/2$ for the $n = 0$ level, $\nu_0(\tilde{y}_0)$ decreases from 1 to 0.003 as $y_0$ moves inward from $\pm L_y/2$ by 2.5$\ell$.

The normalized wave functions
\begin{equation}
\phi_N(y) = \phi_n(y - y_0; y) \equiv \phi_n(y; \tilde{y}_0),
\end{equation}
taken to be real, depend on $A_x$ only through $\tilde{y}_0 = y_0 + e\ell^2A_x$. They are highly localized around $y \sim \tilde{y}_0$ with spread $\Delta y \sim O(\ell)$, and a single-particle state $(n, y_0)$ has its center-of-mass at position $y_{cm} = \tilde{y}_0 - \ell\nu^*_n(\tilde{y}_0)$ on the $y$ axis; see Eq. (2.9) below. This $y_{cm}$ stays within the sample width $[-L_y/2, L_y/2]$. In principle, $y_0 = \ell^2p_x$ has an infinite range $-\infty < y_0 < \infty$, but under circumstances of practical interest (where the number of filled levels is limited) it falls in a reduced range $-L_y/2 \lesssim y_0 \lesssim L_y/2$.

For the description of drift motion of an orbiting electron it is advantageous to express the electron field $\Psi$ in terms of the eigenmodes of $H_0$. Note that in the basis $|N\rangle = |n, y_0\rangle$ the coordinate $x$ has the representation
\begin{equation}
\langle N | x | N' \rangle = -\ell^2[\delta_{mn} k + \langle k \rangle_{mn}] \delta(y_0 - y'_0),
\end{equation}
with $k = -i\partial/\partial y_0$; likewise, $\langle N | y | N' \rangle = (y)_mn \delta(y_0 - y'_0)$. [To obtain Eq. (2.7) replace $x$ by a derivative $-\ell^2 k$ acting on $e^{-iy_0x/\ell^2}$ in $\langle N | x | y \rangle$.] Here we have introduced the notation
\begin{equation}
\langle O \rangle_{mn} = \int_{-L_y/2}^{L_y/2} dy \phi_m(y; \tilde{y}_0) O \phi_n(y; \tilde{y}_0)
\end{equation}
with normalization $(1)_{mn} = \delta_{mn}$.

For later convenience let us denote $Y_{mn}(\tilde{y}_0) \equiv \langle y - y_0 \rangle_{mn}/\ell$ and $Q_{mn}(\tilde{y}_0) \equiv -\ell \langle k \rangle_{mn}$; $Y_{mn}$ are symmetric in $(m,n)$ while $Q_{mn}$ are antisymmetric. They are completely known through the spectra $\nu_n(\tilde{y}_0)$; see Appendix A. Here we quote only the following:
\begin{align}
Y_{mn} &= \langle y - y_0 \rangle_{mn}/\ell = -\ell \nu^*_n(\tilde{y}_0), \\
Y_{mn} &= \sigma_{mn} \ell(\nu_m/\nu_n)^{1/2}/[(\nu_m - \nu_n)^2 - 1], \\
Q_{mn} &= i Y_{mn}/(\nu_m - \nu_n) \quad (m \neq n),
\end{align}
where $\nu_n = \nu_n(\tilde{y}_0)$ and $\nu^n_0 \equiv \partial_{y_0} \nu_n(\tilde{y}_0)$; the overall sign $\sigma_{mn} = 1$ for $y_0 \sim +L_y/2$ while $\sigma_{mn} = (-1)^{1+n+m}$ for $y_0 \sim -L_y/2$. The special case $\nu_m = \nu_n = \pm 1$ arises only in the sample interior $|y_0| < L_y/2 - O(\ell)$, where $Y_{mn}$ and $Q_{mn}$ are reduced to constant hermitian matrices given by
\begin{equation}
Y_{mn}^{(bulk)} + i Q_{mn}^{(bulk)} = \sqrt{2\ell} \delta_{m,n-1}.
\end{equation}

With the expansion of the electron field $\Psi(x, y, t) = \sum_N \psi_N(x, y) a_N(t)$, the Hamiltonian (2.1) is rewritten as
\begin{equation}
\mathcal{H} = \int dy_0 \sum_{m,n=0}^{\infty} a^\dagger_m(y_0, t) H_{mn} a_n(y_0, t),
\end{equation}
where we have set $a_N(t) \rightarrow a_n(y_0, t)$. The matrix Hamiltonian $H_{mn}$ is given by
\begin{equation}
H_{mn} = \delta_{mn} \omega \{\nu_n(\tilde{y}_0) + 1/2\} + [U(\tilde{y}_0, k)]_{mn} - e[A_0(\tilde{y}_0 + \ell Y)]_{mn},
\end{equation}
with $k = -i\partial/\partial y_0$ stands for the impurity potential in the $N$ representation obtained from $U(x, y)$ through substitution $x \rightarrow -\ell^2k + \ell Q$ and $y \rightarrow \tilde{y}_0 + \ell Y$. [From now on we shall frequently suppress Landau-level indices and employ matrix notation; notation $[\cdot \dot{\cdot}]_{mn}$ refers to some specific components.]

In the Hamiltonian (2.13) the motion of a Hall electron is decomposed into relative cyclotron motion described by matrix dynamics and c.m. motion described by a one-dimensional field theory with coordinate $y_0$ and its conjugate $k$. Physically $x_0 = -\ell^2k$ and $\tilde{y}_0$ stand for the center coordinates of an orbiting electron. They are the generators of magnetic translation and obey $[x_0, \tilde{y}_0] = i\ell^2$. Note that the relative coordinates $Y$ and $Q$ obey a nontrivial commutation relation
\begin{equation}
[Y, Q] = i (1 + \ell \partial_{y_0} Y),
\end{equation}
which follows from the trivial relation $[x, y] = 0$.

The presence of sample edges has effectively generated strong potentials $\omega(\nu_n(\tilde{y}_0))$ that confine Hall electrons to finite width $-L_y/2 \lesssim y_0 \lesssim L_y/2$ in $y_0$ space. These confining potentials drive orbiting electrons and thus make electrons residing near the sample edges characteristically different from those in the sample bulk. Indeed, as seen clearly in the impurity-free ($U \rightarrow 0$) case where an electron state $(n, y_0)$ acquires the group velocity
\begin{equation}
v_x = \omega^2 \nu^*_n(\tilde{y}_0) + E_y(\tilde{y}_0)/B + \ldots,
\end{equation}
with $n \approx \omega^0 \nu^*_n(\tilde{y}_0) \sim \pm \ell^0$ much larger than the field-induced drift velocity $E_y/B$. [Numerically $v_x^{disp} \sim \omega \ell \sim 10^7$ cm/s for typical values $\omega \sim 10$ meV and $\ell \sim 100$ $\mathrm{Å}$ while $v_x^{bulk} \sim E_y/B \sim 10^3$ cm/s for $E_y = -1$ V/cm and $B = 5$ T.] Classically these edge states are visualized as electrons hopping along the sample edge. They travel in opposite directions at opposite edges, and the currents they carry at the two edges combine to cancel in equilibrium. For distinction we refer to electron states in the sample bulk as bulk states.

III. SUBBAND HAMILTONIANS

In this section we derive Hamiltonians projected to each impurity-broadened Landau level. To start with let us note that a concise expression for the $x-$averaged current of our interest, $J_x = (1/L_x) \int dx \int dy J_x$, is
\[
J_x = -(1/L_x) \int dx dy \Psi^\dagger \partial_A H \Psi, \tag{3.1}
\]

where \(\partial_A H \equiv \partial H/\partial A_x\). The use of this representation lies in the fact that the relevant current is calculated from the \(A_x\) dependence of the eigenvalues of the Hamiltonian \(H\).

Our discussion below makes extensive use of unitary transformations. It will therefore be useful to consider how a unitary transformation \(G\), which may in general depend on \(A_x\), affects the current operator \(\hat{J}_x\). With the transformation \(\Psi \rightarrow \Psi^G = G \Psi\) and \(H^G = G^\dagger H G^{-1}\), \(J_x\) reads

\[
J_x = -(1/L_x) \int dx dy (\Psi^G)^\dagger \{\partial_A H^G - [\partial_A G G^{-1}, H^G]\} \Psi^G, \tag{3.2}
\]

Note that the commutator term has a vanishing expectation value \(\langle \alpha | [\partial_A G G^{-1}, H^G] | \alpha \rangle = 0\) for each eigenstate \(| \alpha \rangle\) of \(H^G\) as long as the matrix element \(\langle \alpha | \partial_A G G^{-1} | \alpha \rangle\) exists. Thus, when \(G\) itself is a well-defined operator, no explicit account of the commutator term is needed in calculating the physical expectation value of the current \(J_x\), which is determined solely from the \(\partial_A H^G\) part.

As an example, the wave functions \((x, y) \psi(N)\) play the role of a unitary transformation \(G_{N \in \mathbb{N}} = (N|x)\) that connects the \(x = (x, y)\) and \(N = (n, y)\) representations. The \(A_x\) dependence of this \(G\) is characterized by \(Q\):

\[
\langle N | [\partial_A G G^{-1}, N^\dagger] | N^\prime \rangle = i e \ell \delta(y_0 - y_0^\prime) Q_{nn^\prime}(y). \tag{3.3}
\]

The current \(J_x\), written as \(J_x \propto \int dy_0 a^\dagger_m Y_{nn^\prime} a_n \) in the \(N\) representation, therefore differs from an equivalent representation \(J_x \propto \int dy_0 a^\dagger_m (\partial_A H_{nn^\prime}) a_n\) by a term \(\propto \langle Q, H \rangle\).

Let us take local disorder into account. Impurities scatter electrons and turn each Landau level into a broadened subband. When the disorder \(U(x, y)\) is weak compared with the level gap \(\sim \omega\), one can diagonalize the Hamiltonian \((2.14)\) with respect to Landau-level labels by a unitary transformation of the form \(W(y_0, k)\):

\[
H_{sb}^{\text{ub}}(y_0, k) = W \{\omega \nu(y_0) + \omega/2 + \hat{U} - eA_0\} W^{-1}, \tag{3.4}
\]

where \(\nu(y_0) = \text{diag} [\nu_n(y_0)]\). In particular, when the edge states are ignored, one may adjust the operator-valued matrix \([W(y_0, k)]_{nm}\) successively to each power of \((\hat{U} - eA_0)/\omega\) so that off-diagonal terms disappear from \(H_{sb}^{\text{ub}}\), obtaining a subband Hamiltonian \([H_{sb}^{\text{ub}}]_{nn}\) of the form

\[
\omega (n + 1/2) + U^A_{nn}(y_0, k) + \frac{1}{\omega} \sum_{m \neq n} \frac{1}{n - m} U^A_{mm}(y_0, k) U^A_{nn}(y_0, k) + \cdots, \tag{3.5}
\]

where \(U^A_{mn} = [\hat{U}(y_0, k) - eA_0(y_0 + \ell Y)]_{mn}\).

Edge states are affected by inter-level degeneracy; i.e., the edge states of a given level get degenerate in energy with some electron states of the lower levels. Still it is possible to carry out diagonalization by a unitary transformation of the form \(W(y_0, k)\). See Appendix B for details. Actually, relevant to our discussion below is only the fact that \(W\) and \(H_{sb}^{\text{ub}}\) consist of \(\bar{y}_0\) and \(k\).

With level mixing now properly taken care of, each subband becomes independent:

\[
\mathcal{H} = \int dy_0 \sum_{n=0}^{\infty} b_n^\dagger (y_0, t) [H_{sb}^{\text{ub}}(y_0, k)]_{nn} b_n(y_0, t), \tag{3.6}
\]

where \(b_n(y_0, t) = \sum_m [W(y_0, k)]_{nm} a_m(y_0, t)\). Correspondingly we shall henceforth concentrate on a single subband, say, the \(n\)th one, and write

\[
\tilde{H}(\bar{y}_0, k) \equiv [H_{sb}^{\text{ub}}(\bar{y}_0, k)]_{nn}, \tag{3.7}
\]

for short. Let \(\{ \langle \alpha \rangle \} \) denote the eigenstates of \(\tilde{H}(\bar{y}_0, k)\), which are taken to form an orthonormal set. The current \(J_x\) the \(n\)th subband supports is calculated from the \(A_x\) dependence of the spectrum:

\[
J_x^{(n)} = -\frac{e^2}{L_x} \sum_\alpha \langle \langle \alpha | \partial_{y_0} \tilde{H}(\bar{y}_0, k) | \alpha \rangle, \tag{3.8}
\]

where the sum \(\sum_\alpha\) runs over all the occupied states \(\alpha\) within the \(n\)th subband.

The Hamiltonian \(\tilde{H}(\bar{y}_0, k)\) as an operator formally acts on the \(n\)th subband spanned by the \(y_0\)-diagonal basis \(\{ |y_0\rangle; -\infty < y_0 < \infty\}\), which forms a complete set of the eigenstates of \(\tilde{H}(\bar{y}_0, k)\) with \(k \to 0\). (For brevity suffix \(n\) for \(y_0\) is suppressed.) Here associated with the basis vectors \(\langle y_0 \rangle\) are the coordinate-space wave functions \(\langle x, y | W^{-1} | n, y_0 \rangle\), which, owing to dominant admixture of the plane-wave mode \(\langle x, y | n, y_0 \rangle\), are extended in \(x\).

**IV. LOCALIZATION AND ITS CONSEQUENCES**

In the absence of impurities all electron states \(|n, y_0\rangle\) are plane waves extended in \(x\), though localized in \(y\) with spread \(\Delta y \sim O(\ell)\). Such spatial characteristics of electron states are modified in the presence of disorder: Electrons, scattered repeatedly by impurities, tend to be confined in finite domains of space, and in two-dimensional Hall samples the majority of electron states are considered to be localized. In this section we study consequences of localization by use of magnetic translation.

**A. Localized states v.s. extended states**

As preliminaries, note first that the subband Hamiltonian \(\tilde{H}(\bar{y}_0, k)\) depends on \(A_x\) only through \(\bar{y}_0 = y_0 + e\ell^2 A_x\), and try to translate each electron mode by \(-e\ell^2 A_x\) in the \(y_0\) direction:

\[
b_n^{\text{ub}}(y_0, t) = e^{-i\ell^2 A_x k} b_n(y_0, t), \tag{4.1}
\]
where \( k = -i\partial/\partial y_0 \). This makes the transformed Hamiltonian \( e^{-i\epsilon^2 A_z k}\hat{H}(y_0, k) e^{i\epsilon^2 A_z k} = \hat{H}(y_0, k) \) independent of \( A_z \). Thus this transformation, if allowed to perform, would imply that there is no current. A resolution to this apparent puzzle turns out revealing, as explained below.

In connection with \( J_x \) in Eq. (3.2) we have noted that a commutator of the form \([\mathcal{O}, \hat{H}^{(0)}]\) has a vanishing expectation value for each eigenstate \( |\alpha\rangle \) of \( \hat{H}^{(0)} \). Such expectation values need not vanish in case \( \langle\alpha|\mathcal{O}|\alpha\rangle \) are ill-defined or singular. For example, consider the impurity-free Hamiltonian \( h_0 = \omega \{y(y_0) + 1/2\} \) and its eigenstates \( |N\rangle \). While the commutator \( i[k, h_0] = \omega^2 \mu'_{\alpha}(y_0) \) itself is a well-defined operator, its expectation values \( \langle N|ik, h_0||N\rangle = \omega^2 \mu'_{\alpha}(y_0) \) fail to vanish because \( \langle N|k||N\rangle \sim O(L_2/\ell^2) \) are singular. Physically this singular behavior derives from the fact that the expectation value for each eigenstate \( |\alpha\rangle \) is singular. For example, consider the impurity-

The solution to the puzzle is now clear: The magnetic \n
B. Immobility of localized states

Since the transformation makes sense for localized states, one may still choose to translate them and ask what would happen. An immediate result is that the localized states cease to support current, as shown below.

For the proof we shall examine the \( A_x \) dependence of the exact spectrum of \( \hat{H}(y_0, k) \) by starting with the zeroth-order Hamiltonian \( \hat{H}^{(0)}(y_0, k) = \hat{H}(y_0, k) \). Let \( \{\langle\alpha|\rangle \} \) denote the eigenstates of \( \hat{H}^{(0)} \), which are obtained from the eigenstates \( \{\langle\alpha\rangle\} \) of \( \hat{H} \) by letting \( A_x \to 0 \). We divide them into localized states \( \{\langle\alpha|\rangle \} \) and extended states \( \{\sigma\} \), i.e., \( \langle\alpha| \hat{H}^{(0)}(y_0, k)|\alpha\rangle = \epsilon_\alpha(y_0)|\alpha\rangle \) for \( \alpha = \lambda, \sigma; \) obviously both eigenvalues \( \epsilon_\alpha \) and eigenfunctions \( \{\langle\alpha|\rangle\} \) are independent of \( A_x \). We suppose that \( A_x \) is very weak so that both \{\alpha\} and the paired state \{\alpha\} share the same extended or localized character.

With the expansion of the field operators \( b_\alpha(y_0, t) = \sum_\alpha \langle\alpha|\alpha\rangle b_\alpha(t) \) in terms of the eigenfunctions of \( \hat{H}^{(0)} \), the Hamiltonian \( \hat{H}^{(0)} \) for the nth subband is rewritten as \( \hat{H}_n = \sum_{\alpha, \beta} b_\alpha \hat{H}_{\alpha\beta} b_\beta \), where \( \hat{H}_{\alpha\beta} = \langle\alpha|\hat{H}(y_0, k)|\beta\rangle \). We now translate only the localized modes \( b_\lambda \) so that

\[
\epsilon_\lambda = T_{\lambda\lambda} b_\lambda \quad \text{with} \quad T = e^{-i\epsilon^2 A_z k},
\]

where \( T_{\lambda\lambda} \equiv \langle\lambda|T|\lambda\rangle \). Then the Hamiltonian reads

\[
\hat{H}_n = c_{\lambda}^T T_{\lambda\lambda} \hat{H}_\lambda \hat{H}_\lambda^\dagger \epsilon_{\lambda} + b_\lambda^\dagger \hat{H}_{\lambda\alpha} b_\alpha + c_{\lambda}^4 T_{\lambda\lambda} \hat{H}_\lambda \epsilon_{\lambda} T_{\lambda\lambda} b_\lambda^\dagger \hat{H}_{\lambda\alpha} (T^{-1})_{\alpha\lambda} \epsilon_{\lambda} T_{\lambda\lambda}^\dagger,
\]

where \( \hat{H}_{\lambda\lambda} \equiv \langle\lambda|\hat{H}(y_0, k)|\lambda\rangle \), etc. Summations over repeated labels \( \lambda, \sigma, ... \) are understood from now on. The \( T_{\lambda\lambda} \) is a unitary transformation within the space of localized states \( \{\lambda\} \).

Let us next rewrite the \( \hat{T} \hat{H} T^{-1} \) term in Eq. (4.3) in terms of the \( y_0 \)-diagonal basis \( \{|y_0\rangle\} \) in Eq. (1.10).

\[
\hat{H}_n = c_{\lambda}^T T_{\lambda\lambda} \hat{H}_\lambda \hat{H}_\lambda^\dagger \epsilon_{\lambda} + b_\lambda^\dagger \hat{H}_{\lambda\alpha} b_\alpha + c_{\lambda}^4 T_{\lambda\lambda} \hat{H}_\lambda \epsilon_{\lambda} T_{\lambda\lambda} b_\lambda^\dagger \hat{H}_{\lambda\alpha} (T^{-1})_{\alpha\lambda} \epsilon_{\lambda} T_{\lambda\lambda}^\dagger,
\]

where \( T_{\lambda\lambda} \equiv \langle\lambda|T|\lambda\rangle \). Actually, in our discussion below it suffices to fix \( T \) only to \( O(A_x) \) and thus to take \( T_{\lambda\lambda} = \delta_{\lambda\lambda} - i\epsilon^2 A_x (\lambda|\lambda\rangle + \cdots). \)

Let us now rewrite the Hamiltonian \( \hat{H}_n \) in terms of the \( y_0 \)-diagonal basis \( \{|y_0\rangle\} \) in Eq. (1.10) in the \( y_0 \)-diagonal basis \( \{y_0\} \) and \( \{\lambda\} \) are well-defined. For ease of notation we shall henceforth reserve \( \lambda, \lambda', \cdots \) for localized states, \( \sigma, \sigma', \cdots \) for extended states, and \( \alpha, \beta \) for general states.\n
\[
\hat{H}_n = c_{\lambda}^T T_{\lambda\lambda} \hat{H}_\lambda \hat{H}_\lambda^\dagger \epsilon_{\lambda} + b_\lambda^\dagger \hat{H}_{\lambda\alpha} b_\alpha + c_{\lambda}^4 T_{\lambda\lambda} \hat{H}_\lambda \epsilon_{\lambda} T_{\lambda\lambda} b_\lambda^\dagger \hat{H}_{\lambda\alpha} (T^{-1})_{\alpha\lambda} \epsilon_{\lambda} T_{\lambda\lambda}^\dagger,
\]

where \( T_{\lambda\lambda} \equiv \langle\lambda|T|\lambda\rangle \) and \( \hat{H}_{\lambda\lambda} \equiv \langle\lambda|\hat{H}(y_0, k)|\lambda\rangle \), etc. Here, through the introduction of the transformation

\[
T_{\lambda\alpha} = \langle\lambda|T|y_0\rangle = \langle\lambda|y_0\rangle - i\epsilon^2 A_x (\lambda|y_0\rangle + \cdots),
\]

we have let \( T \) act on the extended states \( \{\sigma\} \) as well. Note that the transformations involved in Eq. (4.4) such as \( T_{\lambda\sigma} = -i\epsilon^2 A_x (\lambda|\sigma\rangle + \cdots \) are all well-defined to \( O(A_x) \); they do not involve any matrix elements of the kind \( \langle\sigma|\sigma'\rangle \) connecting extended states.

As desired, the \( T_{\lambda\alpha} \hat{H}_{y_0y_0}(T^{-1})_{y_0y_0} \) term in Eq. (4.4) becomes independent of \( A_x \):

\[
\langle\lambda|T \hat{H}(y_0, k) T^{-1} |\nu\rangle = \langle\lambda|\hat{H}(y_0, k)|\nu\rangle,
\]

which equals \( \epsilon_\lambda \delta_{\lambda\nu} \). By virtue of \( T_{\lambda\lambda} = O(A_x) \) and \( \hat{H}_{\sigma\sigma} = O(A_x) \), the remaining term \( C_{\lambda\lambda} \) is only of \( O(A_x^2) \)
and no other terms in $\mathcal{H}_n$ of Eq. (4.3) give rise to an $O(A_z)$ energy shift for each localized state $\lambda$. This proves that the localized states support no current, i.e., 
\[
\langle \lambda| \partial_{\mathbf{y}_0} \hat{H}(\mathbf{y}_0, k)|\lambda \rangle = 0.
\]

C. Edge states v.s. bulk states

Nontrivial consequences of disorder all derive from the $x$ dependence of the impurity potential $U(x, y)$ or the $k$ dependence of $\hat{U}(\mathbf{y}_0, k)$. Indeed, if one sets $k \to 0$ in $\hat{U}$, the Hamiltonian $\hat{H}(\mathbf{y}_0, k)$ becomes local in $\mathbf{y}_0$, and all the eigenstates are plane waves extended in the $x$ direction, leading to no localization.

Physically this $k$ dependence is tied to the impurity-induced drift of each electron in the $y$ direction with velocity $v_y \sim \ell^2 \partial_x U$. Electrons in the sample bulk, drifting slowly with velocity $v_x^{\text{bulk}} \sim E_y/B$, are readily scattered by impurities. In contrast, the electrons at the sample edges, traveling faster with velocity $v_x^{\text{edge}} \sim \omega \ell$, are much less sensitive to such impurity-induced drift. Accordingly the spread $\Delta y_0$ of the wave functions $(\mathbf{y}_0|\alpha)$ in $y_0$ space provides a natural measure to distinguish between the edge and bulk states: The bulk states acquire typical spread $\Delta y_0 \gtrsim \ell$ whereas the edge states have smaller spread $\ell^2/L_x \ll \Delta y_0 < \ell$. (Remember here that a plane-wave state $|\mathbf{y}_0\rangle$ has tiny spread $\Delta y_0 = 2\pi\ell^2/L_x$ in $y_0$ space while it has spread $\Delta y \sim O(\ell$) in real space.) The spread $\Delta y_0$ gets even smaller as an edge state travels faster, i.e., as it gets closer to the edge; this is readily verified by a numerical simulation.

The edge states are thus close to plane-wave states, though not identical, and this feature provides a way to label them. Suppose that, as we let $k \to 0$ in $\hat{H}(\mathbf{y}_0, k)$, an edge state $|\alpha\rangle$ is reduced to a plane-wave state $|\mathbf{y}_0\rangle$ of $\hat{H}(\mathbf{y}_0, 0)$. We can use $\mathbf{y}_0$ to label the edge states $|\alpha\rangle$, i.e., set $\alpha \to \mathbf{y}_0$ and $\sum_\alpha \propto \int d\mathbf{y}_0$. Then the edge-state spectrum $\epsilon_\alpha = \epsilon(\mathbf{y}_0)$ depends on $A_x$ only through $\mathbf{y}_0 = \mathbf{y}_0 + \ell^2 A_x$. [Note here that physical quantities depend on $p_x$ only through the combination of a covariant derivative $\gamma_0 = \ell^2(p_x + eA_x)$; this $\gamma_0$ dependence of the edge-state spectrum is also directly verified by calculating $\epsilon_\alpha$ perturbatively from the spectrum of $\hat{H}(\mathbf{y}_0, 0)$.] We suppose that the edge-state spectrum $\epsilon(\mathbf{y}_0)$ continuously rises (or falls) with $\mathbf{y}_0$, as in the impurity-free case. For extremely fast electrons the influence of disorder can be neglected so that $\epsilon(\mathbf{y}_0) \to \hat{H}(\mathbf{y}_0, 0)$ for $\mathbf{y}_0 \to \pm\infty$.

In general there is such one-to-one correspondence between the eigenstates of $\hat{H}(\mathbf{y}_0, k)$ and those of $\hat{H}(\mathbf{y}_0, 0)$ (because the electron number is conserved). Unlike the edge-state spectrum, however, the bulk-state spectrum is not a smooth function of $\mathbf{y}_0$.

D. Quantum Hall effect

We are now ready to study the quantum Hall effect. Suppose we fill a subband with electrons by varying $B$ or electron population gradually. Since the confining potential is higher near the sample edges, the electrons are accommodated in the sample bulk first. No net current flows while only localized states arise, and a lower Hall plateau develops. The current starts to flow as soon as extended states emerge. When the extended states in the bulk-state spectrum are filled up, the edge states begin to emerge. Figure 1 schematically shows the spectrum of one such well-filled subband when a Hall field is turned on. [The spectrum gets tilted in the presence of a potential $A_0(y)$] There each electron state is labeled in terms of the related eigenstate of $\hat{H}(\mathbf{y}_0, 0)$; the $\mathbf{y}_0$ axis therefore refers to $\mathbf{y}_0$. The bulk-state spectrum, being discontinuous in $\mathbf{y}_0^\pm$, is depicted as a broadened spectrum lying in the interval $\eta^- \leq y_0 \leq \eta^+$. The edge-state spectrum $\epsilon(\mathbf{y}_0)$ is drawn with solid curves, and is filled up to the energy $\epsilon^\pm \equiv \epsilon(\mathbf{y}_0^\pm)$ at the two sample edges.

Suppose now that the extended states in the bulk-state spectrum are filled up in Fig. 1. Then this subband supports a fixed amount of current determined by the Hall voltage alone, as explained below. It is clear that one can regard the localized states as all occupied in calculating the current (since they carry no current). Our task therefore is to calculate the current supported by the states filling the interval $y^-_0 \leq y_0 \leq y^+_0$. To this end let us first examine the case where the spectrum of filled states extends over a wider range $y^-_0 \leq y_0 \leq y^+_0$ and take $\epsilon^- \equiv \epsilon(y^-_0) \gg \epsilon^-$ and $\epsilon^+ \equiv \epsilon(y^+_0) \gg \epsilon^+$ so that the edge states $y_0 \ll y^-_0$ and $y_0 \gg y^+_0$, traveling extremely fast, are well described by the plane-wave states of $\hat{H}(\mathbf{y}_0, 0)$. The vacant edge states with $y_0 < y^-_0$ and $y_0 > y^+_0$ are naturally orthogonal to the occupied states.

Note that this spectrum of occupied states derives from the plane-wave states $|\mathbf{y}_0\rangle$ of $\hat{H}(\mathbf{y}_0, 0)$ over the same interval $y^-_0 \leq y_0 \leq y^+_0$ and, in view of the completeness of these states, rewrite the sum $\sum_\alpha$ in Eq. (3.2) as a sum over the plane-wave states $|\mathbf{y}_0\rangle$, with the result

\[
J_x^{\text{wide}} = -\frac{e^2}{L_x} \int_{y^-_0}^{y^+_0} dy_0 \partial_{y_0} \langle \mathbf{y}_0| \hat{H}(\mathbf{y}_0, k)|\mathbf{y}_0\rangle
\]

\[
= -\frac{e}{2\pi}(\epsilon^+ - \epsilon^-),
\]

where the last line is reached by noting that $\hat{H}(\mathbf{y}_0, k)$ is reduced to the plane-wave spectrum $\hat{H}(\mathbf{y}_0, 0)$ for $y_0 \sim y^-_0$ and $y_0 \sim y^+_0$; we have set $A_x = 0$ and used the normalization $\langle \mathbf{y}_0| \mathbf{y}_0 \rangle = \delta(\ell^2 p_x = 0) = L_x/(2\pi\ell^2)$.

The edge states occupying the intervals $y^-_0 \leq y_0 \leq y^+_0$ and $y^-_0 \leq y_0 \leq y^+_0$ are not necessarily plane-wave states, but the current they carry $\propto \int dy_0 \partial_{y_0} \epsilon(\mathbf{y}_0)$ is exactly calculated from the spectrum:

\[
\Delta J_x^{\text{wide}} = -\frac{e}{2\pi}(\epsilon^+ - \epsilon^- + \epsilon^- - \epsilon^-).
\]
Consequently the current carried by the states filling the range $y_1 \leq y \leq y_2$ of our concern turns out to be determined by the Hall voltage $V_H = (e^+ - e^-)/e$ alone:

$$J_x^{(n)} = - (e^2/2\pi) V_H.$$  \hspace{1cm} (4.11)

This shows that each filled subband gives rise to a single unit $-e^2/2\pi \hbar = -e^2/h$ of the Hall conductance; the emerging Hall plateaus become visible when a significant portion of electrons gets localized in the sample interior.

Equation (4.11), when combined with the immobility of the localized states, offers a general but rather indirect proof of the current compensation theorem. In the next section we present a more direct proof, based on a linear-response treatment, which demonstrates an interplay of the localized states and extended states explicitly.

V. LINEAR RESPONSE

In this section we examine what would happen to the subband described by $[H^{sb}(y_0, k)]_{nn}$ when the Hall potential $A_0(y)$ is varied by a small amount $\delta A_0(y)$. The Hamiltonian we consider now is

$$H^{new}(y_0, k) = H^{sb}(y_0, k) + V(y_0, k), \hspace{1cm} (5.1)$$

$$V(y_0, k) = -eW(y_0, k) \delta A_0(y) + \nabla V W(y_0, k)^{-1}. \hspace{1cm} (5.2)$$

The $H^{sb}$ is diagonal in $(m, n)$. The potential variation $[V(y_0, k)]_{nn} = \delta_{nn} \delta A_0(y_0) + \cdots$ in general has off-diagonal pieces which cause transitions to different subbands. Fortunately, it turns out unnecessary to take explicit account of such level mixing, as argued below.

The current flowing in response to $\delta A_0$ is calculated from the energy shift to $O(V)$ of each eigenstate $|\alpha\rangle$ of $[H^{sb}]_{nn}$. The $[H^{sb}]_{nn}$ in general has degenerate eigenvalues as well as nondegenerate ones. For a state $|\alpha\rangle$ with a nondegenerate eigenvalue the $O(V)$ energy shift is given by $dH/\langle\alpha|\alpha\rangle$. For a degenerate eigenvalue one has to treat all the eigenstates belonging to it on an equal footing. Let $D = \{|\alpha\rangle\}$ denote one such set of degenerate states. The states in $D$ get mixed by the perturbation $V$, and one can always form, by solving a secular equation, a new basis $D = \{|\tilde{\alpha}\rangle\}$ such that $V$ becomes diagonal in it. This change of bases $D = \{|\alpha\rangle\} \rightarrow D = \{|\tilde{\alpha}\rangle\}$ is a unitary transformation and, when $D$ represents inter-level degeneracy, causes inter-level mixing. As is familiar from degenerate perturbation theory, such $|\tilde{\alpha}\rangle$ are a correct zeroth-order choice of basis vectors, and the $O(V)$ energy shifts of the states $|\tilde{\alpha}\rangle$ are given by $\langle\tilde{\alpha}|V(y_0, k)|\tilde{\alpha}\rangle$.

It is practically impossible to isolate the current component carried by each of the modes $\tilde{\alpha}$ in $D$ that are still almost degenerate in energy. It only makes sense to treat each degeneracy set $D$ as a whole. The effect of the mixing $\{|\alpha\rangle\} \rightarrow \{|\tilde{\alpha}\rangle\}$ then apparently disappears from the sum of the energy shifts:

$$\sum_{\alpha \in D} \langle\tilde{\alpha}|V(y_0, k)|\tilde{\alpha}\rangle = \sum_{\alpha \in D} \langle\alpha|V(y_0, k)|\alpha\rangle. \hspace{1cm} (5.3)$$

This implies that no explicit account of level mixing is needed in calculating the Hall current; $V$ does cause inter-subband mixing but the total Hall current, or the Hall conductance, is insensitive to it.

In view of this effective independence of each subband, we shall henceforth concentrate on a single subband, the $n$th one, and write $H^{new}(y_0, k) = [H^{new}(y_0, k)]_{nn}$ and $V(y_0, k) = [V(y_0, k)]_{nn}$ for short so that

$$H^{new}(y_0, k) = H(y_0, k) + \tilde{V}(y_0, k). \hspace{1cm} (5.4)$$

Let us choose anew the eigenstates $|\alpha\rangle$ (with energy $\epsilon_\alpha$) of $H$ so that they evolve into the eigenstates $|\alpha\rangle$ of $H^{new}$ smoothly as $\tilde{V}$ is turned on. The current $J_x^{(n)} \equiv J_x^{(n)}[A_0 + \delta A_0]$ supported by the subband in the presence of the potential $A_0 + \delta A_0$ is given by Eq. (3.8) with $H \rightarrow H^{new}$ and $|\alpha\rangle \rightarrow |\alpha\rangle$. Expressing $|\alpha\rangle$ in terms of $|\alpha\rangle$ gives rise to the linear-response expression:

$$J_x^{(n)} = -\frac{e\ell^2}{L_x} \sum_{\alpha} \left[ \langle\alpha|\partial_{y_0} H^{new}|\alpha\rangle + \sum_{\beta} \rho_{\alpha\beta} \left( \epsilon_\alpha - \epsilon_\beta \right)^2 \right]. \hspace{1cm} (5.5)$$

$$\rho_{\alpha\beta} = (\epsilon_\alpha - \epsilon_\beta) \left\{ \langle\alpha|\tilde{V}|\beta\rangle \langle\beta|\partial_{y_0} H\rangle + (\alpha \leftrightarrow \beta) \right\} \hspace{1cm} (5.6)$$

where the sum $\sum_{\alpha}$ runs over all the occupied states $|\alpha\rangle$ and $\sum_{\beta}$ over all possible intermediate states $|\beta\rangle$ not degenerate in energy with $|\alpha\rangle$ within the same subband. The $\rho_{\alpha\beta}$, which derive from electron scattering by the potential $\delta A_0$, are antisymmetric in $(\alpha, \beta)$.

Equation (5.3) is a generalization of the Kubo formula so as to include electron edge states and a general potential $A_0 + \delta A_0$. The original formula is recovered for electron bulk states in a strong magnetic field ($W \rightarrow 1$) and a uniform field $E_y$; in this case, $[\tilde{H}, \tilde{V}] \rightarrow i\ell^2 E_y \partial_{x}, [\tilde{U}]_{nn}, \partial_{y_0} \tilde{H} \rightarrow \partial_{y} [\tilde{U}]_{nn},$ and $\partial_{y_0} \tilde{H}^{new} \rightarrow eE_y$.

Let us again explore consequences drawn by translating the localized states, with the Hamiltonian $H^{new}(y_0, k) = H^{(0)}(y_0, k) + \tilde{F}$ this time, where

$$\tilde{F}(y_0, k) \equiv \tilde{V}(y_0, k) + e\ell^2 A_x \partial_{y_0} H^{(0)}(y_0, k) + \cdots. \hspace{1cm} (5.7)$$

As before, some care is needed to handle degeneracy inherent to $H^{(0)} = H(y_0, k)$. We choose its eigenstates $|\alpha\rangle$ so that $|\alpha\rangle F|\beta\rangle$ becomes a diagonal matrix within each set of degenerate states belonging to the same eigenvalue. Each state $|\alpha\rangle$ thereby evolves into the corresponding eigenstate $|\alpha\rangle$ of $H^{new}$ as $A_x$ and $\delta A_0$ are turned on. We suppose that during the evolution $|\alpha\rangle \rightarrow |\alpha\rangle \rightarrow |\alpha\rangle$ each mode $\alpha$ remains either localized or extended.

We go through the procedure of Sec. IV again: Expand the field operators $b_\alpha(y_0, t)$ in terms of the eigenfunctions...
of $\hat{H}^{(0)}$, and translate only the localized modes $b_\lambda$, as in Eq. (4.2). Equations (4.3) through (4.7) remain intact, except for obvious replacement $\hat{H} \rightarrow \hat{H}^{\text{new}}$. This time, however, $C_{\lambda\lambda'}$ in Eq. (4.5) yields an $O(A_x)$ energy shift of the form for state $\lambda$:

$$C_{\lambda\lambda} = i e^2 A_x \sum_{\sigma} \left[ k_{\lambda\sigma} V_{\sigma\lambda} - \hat{V}_{\lambda\sigma} k_{\sigma\lambda} \right] + O(A_x^2), \quad (5.8)$$

where $k_{\lambda\sigma} \equiv \langle \lambda | k | \sigma \rangle$ and $\hat{V}_{\lambda\sigma} \equiv \langle \lambda | \hat{V} | \sigma \rangle$. The sum $\sum'_{\sigma}$ is taken over all possible extended states $|\sigma\rangle$ not degenerate with $|\lambda\rangle$; this is because $\langle \hat{H}^{\text{new}} \rangle_{\lambda\sigma} = 0$ for states $|\sigma\rangle$ and $|\lambda\rangle$ belonging to the same eigenvalue.

At the same time, the last two terms in $\mathcal{H}_n$ of Eq. (4.3) combine to cause the $\lambda \rightarrow \sigma \rightarrow \lambda$ virtual transitions that give rise to an $O(A_x)$ energy shift of the form:

$$\Delta \epsilon_\lambda \equiv \sum_{\sigma} \left( \langle \hat{H}^{\text{new}} \rangle_{\lambda\sigma} \langle \hat{H}^{\text{new}} \rangle_{\sigma\lambda} \right) / (\epsilon_\lambda - \epsilon_\sigma) = -i e^2 A_x \sum_{\sigma} \left[ k_{\lambda\sigma} V_{\sigma\lambda} - \hat{V}_{\lambda\sigma} k_{\sigma\lambda} \right] + \cdots, \quad (5.9)$$

where use has been made of the relation $\langle \hat{H}^{\text{new}} \rangle_{\lambda\sigma} = V_{\sigma\lambda} + i e^2 A_x k_{\sigma\lambda} (\epsilon_\lambda - \epsilon_\sigma) + O(A_x^2)$. This $\Delta \epsilon_\lambda$ cancels the $O(A_x)$ terms in $C_{\lambda\lambda}$ of Eq. (5.8). This proves again that the localized states support no current.

For an extended state $\sigma$ the $b_\sigma^\dagger \langle \hat{H}^{\text{new}} \rangle_{\lambda\sigma} b_\sigma$, term in $\mathcal{H}_n$ of Eq. (4.3) gives rise to an $O(A_x)$ energy shift of the form $\langle \partial_\lambda \hat{H}^{\text{new}} \rangle_{\sigma\sigma}$, which is to be combined with another $O(A_x)$ energy shift coming from the $\lambda \rightarrow \sigma \rightarrow \sigma$ virtual transitions. Consequently the current $\mathcal{J}^{(n)}_x$ is written as

$$\mathcal{J}^{(n)}_x = \frac{e^2}{L_x} \sum_{\sigma} \left[ \langle \sigma | \partial_{y_0} \hat{H}^{\text{new}} | \sigma \rangle + \sum_{\beta} \rho_{\sigma\beta} \left( \epsilon_\sigma - \epsilon_\beta \right)^2 \right], \quad (5.10)$$

where $\rho_{\sigma\beta}$ is given by the same expression as Eq. (5.6) with $A_x \rightarrow 0$. [One is free to set $A_x \rightarrow 0$ in Eq. (5.6), in which case $\hat{H} \rightarrow \hat{H}^{(0)}$ and $|\alpha\rangle \rightarrow |\alpha\rangle$.] Here the sum $\sum_{\sigma}$ runs over all the occupied extended states $\sigma$.

The two formulas (5.3) and (5.10) are essentially the same, except that the immobility of localized states is already built in the latter. Actually, with the characterization of localized states now at hand, we can verify their equivalence directly: Simply substitute the relation

$$\langle \beta | \partial_{y_0} \hat{H} | \lambda \rangle = i (\epsilon_\lambda - \epsilon_\beta) \langle \beta | k | \lambda \rangle, \quad (5.11)$$

valid for a localized state $\lambda$, into Eq. (5.3) and rewrite the contribution of the $\lambda \rightarrow \{ \beta \} \rightarrow \lambda$ transitions as

$$\sum_{\beta} \rho_{\beta\beta} \left( \epsilon_\sigma - \epsilon_\beta \right)^2 = -i \langle \lambda | \partial_{y_0} \hat{H}^{\text{new}} | \lambda \rangle, \quad (5.12)$$

which indeed cancels the $\langle \lambda | \partial_{y_0} \hat{H}^{\text{new}} | \lambda \rangle = i \epsilon_\lambda$ term in Eq. (5.3). A remark regarding this proof is in order here: The immobility of the localized state $|\lambda\rangle$ can be concluded from $i \langle \lambda | [k, \hat{H}] | \lambda \rangle = 0$ or Eq. (5.11) directly. In our approach, however, the vanishing of such expectation values is derived from the response of the localized states to magnetic translation. This situation is quite similar to a statement of Noether’s theorem that the existence of conserved quantities is directly read from the invariance of the Lagrangian without explicit use of the equations of motion. A symmetry principle underlying our approach will be clarified in Sec. VI.

In view of the antisymmetry $\rho_{\lambda\sigma} = -\rho_{\sigma\lambda}$, the $\sigma \rightarrow \lambda \rightarrow \sigma$ and $\lambda \rightarrow \sigma \rightarrow \lambda$ virtual transitions contribute to $\mathcal{J}^{(n)}_x$ in equal magnitude but in opposite sign; physically this is a manifestation of Fermi statistics. To reveal their interplay let us again consider the current carried by the states filling the interval $y_0^- \leq y_0 \leq y_0^+$. One may now take the sums $\sum_{\sigma}$ and $\sum_{\beta}$ in Eq. (5.10) simply over these states (since the vacant states are plane-wave states orthogonal to them). Let us write $\sum_{\beta} = \sum_{\sigma'} + \sum_{\lambda}$.

The effects of $\sigma \rightarrow \sigma'$ $\rightarrow \sigma$ transitions combine to vanish. On the other hand, the $\sigma \rightarrow \{ \lambda \} \rightarrow \sigma$ transitions, when summed over all possible extended states $\sigma$, yield

$$\sum_{\sigma} \left( \sum_{\lambda} \frac{\rho_{\sigma\lambda}}{(\epsilon_\sigma - \epsilon_\beta)^2} \right) = \sum_{\lambda} \langle \lambda | \partial_{y_0} \hat{H}^{\text{new}} | \lambda \rangle. \quad (5.13)$$

which demonstrates how the extended states combine to recover the loss due to the localized states. The current $\mathcal{J}^{(n)}_x$ of our concern is now given by the drift term $\langle \partial_\lambda \hat{H}^{\text{new}} | \lambda \rangle$, thus leading to essentially the same conclusion as Eq. (5.13). Note finally that Eq. (5.13) can also be derived from Eq. (5.12); thus the immobility of localized states and current compensation are correlated.

For completeness we remark that Eq. (5.11) can also be derived through a current operator directly; see Appendix C for details.

VI. CONCLUDING REMARKS

In this paper we have examined some general consequences of localization in the QHE. In particular, with a simple characterization of localized electron states as those with a definite notion of c.m. position, we have shown that (i) the immobility of the localized states and (ii) exact current compensation by the surviving extended states follow from the response of the Hall system to magnetic translation of the localized states alone. Our consideration relies heavily on a field-theoretic description of Hall electrons in $(n, y_0)$ space rather than in real $(x, y)$ space. We have noted that the electron edge states and bulk states are best distinguished in $y_0$ space, especially in terms of the spread of the wave functions in $y_0$ space.

It will be important to clarify the symmetry principle underlying our use of magnetic translation $T = e^{-i e^2 A_x k}$ in Secs. IV and V. A clue is to note that a special gauge transformation $\Psi_{\Omega} = \Omega \Psi$ with $\Omega = e^{i e A_x x}$ acts like $T$. 

7
Indeed, $\Omega$ formally removes $A_x$ from the Hamiltonian $H$ in Eq. (2.2); $H^\Omega = \Omega H \Omega^{-1}$ equals $H$ with $A_x \to 0$. Like $T$, this gauge transformation affects extended electron states substantially [unless $A_x = (2\pi/L_x) \times \text{integers}$], and it is clear by now that one cannot conclude from the $A_x$ independence of $H^\Omega$ alone that there is no current. To reveal the relation between $T$ and $\Omega$ let us consider how this $\Omega$ acts on the electron operators $a_n(y_0,t)$, which are projections of $\Psi$ to each Landau subband. Recall that $b = Wa = WG\Psi$ in matrix form, where the unitary transformation $G_{Nx} = (N \times k)$ takes $\Psi(x,y,t)$ to $a_n(y_0,t)$ and $W(y_0,k)$ projects $a_n(y_0,t)$ to each exact subband. Analogously, $\Psi^\Omega$ is projected to each subband so that $b^\Omega = W_0 G_0 \Psi^\Omega$, where $W_0 G_0$ stands for $W G$ with $A_x \to 0$. Therefore the desired gauge transformation law is

$$b^\Omega = W_0 G_0 \Omega G^{-1} W^{-1} b,$$

$$= W_0 G_0 G^{-1} e^{ieA_x(-\ell k + Q)} W^{-1} b. \quad (6.1)$$

Noting that $G_0 G^{-1} = 1 - ie\ell^2 A_x Q + O(A_x^3)$, which follows from Eq. (3.3), and that $e^{-ie\ell^2 A_x k} W^{-1} e^{ie\ell^2 A_x k} = W_0^{-1}$, one finds

$$b^\Omega = e^{-ie\ell^2 A_x k} b + O(A_x^2). \quad (6.2)$$

(The $a_n(y_0,t)$ also obey the same transformation law.)

This shows that the magnetic translation $T = e^{-ie\ell^2 A_x k}$ is nothing but the gauge transformation $\Omega = e^{ieA_x x}$ projected to each Landau subband. This connection holds to $O(A_x)$, i.e., to the order relevant to our consideration. Consequently the immobility of localized states and current compensation can be formulated upon the assumption that localized electron states remain essentially unchanged under the gauge transformation $\Omega$; they are thus consequences of gauge invariance. Actually the magnetic translation $T$ is regarded as a transformation of a still larger $(W_x)$ gauge symmetry.

Laughlin gave a simple and general explanation for the QHE on the basis of localization and gauge invariance. Aoki and Ando, and Prange also noted the importance of localization and identified current compensation as a key mechanism for the QHE. In a sense our approach unites these two approaches and promotes current compensation to a general phenomenon, resting on the principle of gauge invariance and valid in the presence of level mixing and edge states.

The Hall potential $A_0(y)$ has so far been treated as an external potential, not necessarily weak. However, it could equally be a potential induced internally as a result of charge redistribution caused by an injected current. In reality the current distribution and potential distribution across a sample are determined self-consistently and $A_0(y)$ can be regarded as such a self-consistent potential. In this sense, while no explicit account of the Coulomb interaction has so far been made, our consideration actually takes it into account within the Hartree approximation. [Remember that our consideration refers to no explicit form of $A_0$, which could even be a general potential depending on both $x$ and $y$.]
APPENDIX A: MATRIX ELEMENTS

In this appendix we derive some formulas relating the matrix elements $Y_{mn} = \langle y - y_0 \rangle_{mn} / \ell$, $Q_{mn} = -\ell \langle k \rangle_{mn}$, etc., to the spectra $\nu_n(y_0)$. Let us recall that $\phi_n(y; y_0)$ are the eigenfunctions of the Hamiltonian $H_0 = (\omega/2)[\ell^2 p_y^2 + (1/\ell^2)(y - y_0)^2]$ with eigenvalues $\omega (\nu_n + 1/2)$ and the boundary condition $\phi_n(y = \pm L_y/2) = 0$. Consider first the commutator $[k, H_0] = i(\omega/\ell^2)(y - y_0)$ and evaluate its matrix elements, with the result

$$Y_{mn} = -\ell \nu'_n(y_0), \quad Q_{mn} = i Y_{mn}/(\nu_m - \nu_n) \quad (m \neq n).$$

(A1) (A2)

Similarly, evaluating the matrix elements of the commutator $[p_y, H_0] = -i(\omega/\ell^2)(y - y_0)$ with particular attention to integrations by parts yields

$$i(\nu_m - \nu_n)\langle p_y \rangle_{mn} + (1/\ell) Y_{mn} = G_{-m}^n - G_{m}^n,$$

(A3)

where $G_{m}^n$ refer to the gradients of the wave functions $\phi_n(y; y_0)$ at the boundary $y = \pm L_y/2$:

$$G_{m}^n(y_0) \equiv (\ell^2/2)[(\partial_y \phi_m)(\partial_y \phi_n)]_{y=\pm L_y/2}.$$  

(A4)

In view of Eqs. (A1) and (A3), these gradients are related to $\nu'_n$ through $G_{m}^n \geq 0$:

$$\nu'_n(y_0) = G_{m}^n(y_0) - G_{-m}^n(y_0),$$

(A5)

which shows that $\nu'_n > 0$ for $y_0 \sim +L_y/2$ while $\nu'_n < 0$ for $y_0 \sim -L_y/2$.

Finally, the commutator $[y - y_0, H_0]$ leads to

$$\langle p_y \rangle_{mn} = (i/\ell)(\nu_m - \nu_n)Y_{mn},$$

(A6)

which is combined with Eq. (A3) to give the expression

$$Y_{mn} = \sigma_{mn} \ell (\nu'_m \nu'_n)^{1/2}/[(\nu_m - \nu_n)^2 - 1]$$

(A7)

for $y_0 \sim \pm L_y/2$ (where $\nu_m - \nu_n \neq \pm 1$ in general). Now $Q_{mn}$ and $\langle \nu_y \rangle_{mn}$ also are known explicitly. Here the over-all sign $\sigma_{mn} \equiv \text{sgn}(\partial_y \phi_m) \text{sgn}(\partial_y \phi_n) \text{sgn}(y_0)$ refers to the signs of $\partial_y \phi_m$ at the relevant boundary $y = \pm L_y/2$ and $y_0 \sim \pm L_y/2$. The standard choice of the bulk-state wave functions leading to Eq. (2.13) gives $\sigma_{mn} = 1$ for $y_0 \sim +L_y/2$ and $\sigma_{mn} = (-1)^{1+n+m}$ for $y_0 \sim -L_y/2$. Actually $\sigma_{mn}$ maintains these values because it is a topological invariant in each separate edge region $y_0 \sim \pm L_y/2$, where $\partial_y \phi_n(y = \pm L_y/2)$ are nonvanishing, as seen from Eq. (A5).

APPENDIX B: SUBBAND HAMILTONIANS

In this appendix we outline the derivation of the subband Hamiltonian $H^{sb}(y_0, k)$ in Eq. (3.4). The Hamiltonian $H(y_0, k) = \omega (\nu + 1/2) + U - eA_0$ in Eq. (2.14) acts on a whole set of Landau levels spanned by the bases $\{|m, y_0); m = 0, 1, 2, \ldots, -\infty < y_0 < \infty\}$, and is brought to a diagonal form by a suitable unitary transformation acting on this whole space. Inter-level mixing among degenerate states is thereby properly taken care of.

Suppose we have determined all the eigenstates of $H(y_0, k)$ via such diagonalization. Let $\{|\alpha\rangle\}$ denote the whole set of eigenstates forming the nth subband. The associated multi-component eigenfunctions $\langle m, y_0|\alpha\rangle$ (with $m = 0, 1, 2, \ldots$) are functions of $y_0 = y_0 + e\ell^2 A_x$. We assume that they form a complete set in $y_0$ space (spanned by a complete $y_0$-diagonal basis $\{|y_0); -\infty < y_0 < \infty\}$); that is, as in the impurity-free case, each subband is taken to be complete with respect to c.m. motion of an electron.

From these (known) eigenfunctions one can construct $H^{ab}(y_0, k)$ in the following way: In the $\{|\alpha\rangle\}$ basis the coordinate operator $y_0$ is represented by a matrix $(\alpha|y_0\alpha')$, which may be diagonalized to give a complete set of wave functions $\langle y_0|\alpha \rangle \equiv f_\alpha(y_0)$ in the $\{|\alpha\rangle\}$ basis. In this $\{|\alpha\rangle\}$ basis the Hamiltonian for the $n$th subband is expressed, in terms of eigenvalues $\epsilon_\alpha$, as a matrix:

$$\langle y_0|H^{sb}(y_0)|y_0\rangle = \sum_\alpha \epsilon_\alpha f_\alpha^\dagger(y_0)f_\alpha(y_0'),$$

(B1)

which is a function of $\bar{y}_0 = y_0' + e\ell^2 A_x$ and $\bar{y}_0' = y_0'' + e\ell^2 A_x$. This matrix is readily cast in an operator form. Indeed, with an expansion of the form

$$\sum_\alpha \epsilon_\alpha f_\alpha^\dagger(y_0)f_\alpha(y_0') = \sum_{i,j} c_{ij} \phi_i(y_0) \phi_j(y_0')$$

in terms of the eigenfunctions $\phi_i(y_0)$ ($i = 0, 1, 2, \ldots$) of the harmonic oscillator formed of $d = (\bar{y}_0/\ell + i\ell k)/\sqrt{2}$ and $d^\dagger$ with $[d, d^\dagger] = 1$, one obtains the expression

$$[H^{sb}(y_0, k)]_{nn} = \sum_{i,j=0}^\infty \langle \bar{y}_j|\sqrt{i\ell k}^\dagger|\bar{y}_i\rangle : (d^\dagger)^i d^j e^{-d^\dagger d} : ,$$

(B2)

where normal ordering such that $: dd^\dagger := d^\dagger d$ is understood. (For a check it is enlightening to verify that the choice $c_{ij} = i \delta_{ij}$ recovers the number operator $d^\dagger d$.) Likewise one can derive the operator expression for $W(y_0, k)$ from the matrix elements:

$$\langle m, y_0|W^{-1}|n, y_0\rangle = \sum_\alpha \langle m, y_0|\alpha\rangle f_\alpha^\dagger(y_0).$$

(B3)

APPENDIX C: CURRENT OPERATORS

We have passed from $a_n(y_0, t)$ to $b_{\alpha}(y_0, t)$ via a unitary transformation $W(y_0, k)]_{nm}$, which being finite for the potential $U = eA_0$, is a well-defined operator. One can therefore adopt

$$J_x' = -(1/\ell^2) b_{\alpha}^\dagger (\partial_\alpha \hat{H})_{\alpha\beta} b_\beta$$

(C1)

as a current operator physically equivalent to $J_x$ in Eq. (3.1): $(\partial_\alpha \hat{H})_{\alpha\beta} = (\alpha|\partial_\alpha \hat{H}|\beta)$. 

9
Let us arrange \( \hat{H}_{\lambda\lambda'}, \hat{H}_{\lambda\sigma'}, \hat{H}_{\sigma\lambda'}, \hat{H}_{\sigma\sigma'} \) in the form of a \( 2 \times 2 \) matrix \( \hat{H}^{\text{mat}} \) acting on \( \langle b_{\lambda'}, b_{\sigma'} \rangle \), and also \( k_{\lambda\lambda'} \) in an analogous matrix form \( K \). Then the commutator \([K, \hat{H}^{\text{mat}}]\), written in the form

\[
\Delta H_n = b^\dagger_{\lambda}[k_{\lambda\lambda'} \hat{H}_{\lambda\lambda'} - \hat{H}_{\lambda\lambda'} k_{\lambda\lambda'}] b_{\lambda'} + b^\dagger_{\lambda} k_{\lambda\lambda'} \hat{H}_{\lambda\lambda'} b_{\lambda} - b^\dagger_{\lambda} \hat{H}_{\lambda\lambda'} k_{\lambda\lambda'} b_{\lambda'},
\]

(C2)

has a vanishing expectation value for each eigenstate of \( \hat{H} \). One may thus add it to the current \( J_x' \) without modifying the physical content. Actually the translation \( \langle 4.2 \rangle \) takes us to the redefined current \( J_x^{\text{red}} = J_x' + (etL_x)\Delta H_n \). It is now a simple exercise to verify the current compensation theorem directly with \( J_x^{\text{red}} \). [Set \( \hat{H} \rightarrow \hat{H}^{\text{new}} \) in \( J_x^{\text{red}} \) to get to Eq. (5.10).]

---

1 J. Zak, Phys. Rev. A 71, 1879 (1983); C. Wexler and D. J. Thouless, Phys. Rev. Lett. 33, 1209 (1979); P. F. Fontein, J. M. Kikkert, and M. C. Youns, Solid State Commun. 73, 1973 (1997).

2 For a review see, The Quantum Hall Effect, edited by R.E. Prange and S.M. Girvin (Springer-Verlag, Berlin, 1987).

3 H. Aoki and T. Ando, Solid State Commun. 38, 1079 (1981). See also, T. Ando, Y. Matsumoto, and Y. Uemura, J. Phys. Soc. Jpn. 39, 279 (1975).

4 R. E. Prange, Phys. Rev. B 23, 4802 (1981).

5 R. B. Laughlin, Phys. Rev. B 23, 5632 (1981).

6 B. I. Halperin, Phys. Rev. B 25, 2185 (1982).

7 D. J. Thouless, M. Kohmoto, M. N. Nightingale, and M. den Nijs, Phys. Rev. Lett. 49, 405 (1982); Q. Niu, D. J. Thouless, and Y.-S. Wu, Phys. Rev. B 31, 3372 (1985); Q. Niu and D. J. Thouless, Phys. Rev. B 35, 2188 (1987).

8 H. Levine, S. B. Libby, and A. M. M. Pruisken, Nucl. Phys. B 240 [FS12], 49 (1984).

9 A. H. MacDonald, and P. Streda, Phys. Rev. B 29, 1616 (1984).

10 P. Streda, J. Kucera and A. H. MacDonald, Phys. Rev. Lett. 59, 1973 (1987); J. K. Jain and S. A. Kivelson, Phys. Rev. B 37, 4276 (1988); M. Büttiker, Phys. Rev. B 38, 9375 (1988).

11 X.-G. Wen, Phys. Rev. B 43, 11025 (1991); M. Stone, Ann. Phys. (N.Y.) 207, 38 (1991).

12 K. Shizuya, Phys. Rev. B 45, 11143 (1992); Phys. Rev. B 52, 2747 (1995).

13 K. Shizuya, Phys. Rev. Lett. 73, 2907 (1994).

14 E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, (Cambridge University Press, London, 1927).

15 This \( \bar{U}(\eta, k) \) is equivalent to an alternative expression \( \bar{U}_{mn} \) used earlier in Ref. 3.

16 J. Zak, Phys. Rev. 134, A1602 (1964).

17 R. E. Peierls, Surprises in Theoretical Physics, (Princeton University Press, Princeton, 1979).

18 Note \( \langle \eta_0 | \hat{H}(\eta_0, k) | \eta'_0 \rangle = (\partial_{\eta_0} + \partial_{\eta'_0}) \langle \eta_0 | \hat{H}(\eta_0, k) | \eta'_0 \rangle \).

19 B. Sakita, Phys. Lett. B 315, 124 (1993).

20 A. H. MacDonald, T. M. Rice, and W. F. Brinkman, Phys. Rev. B 28, 3648 (1983).

21 O. Heinonen and P. L. Taylor, Phys. Rev. B 32, 633 (1985).

22 D. B. Chklovskii, B. I. Shklovskii, and L. I. Glazman, Phys. Rev. B 46, 4026 (1992).

23 D. J. Thouless, Phys. Rev. Lett. 71, 1879 (1993); C. Wexler and D. J. Thouless, Phys. Rev. B 49, 4815 (1994).

24 P. F. Fontein, et al., Phys. Rev. B 43, 12090 (1991).