EFFECTIVE CODESCENT MORPHISMS IN SOME VARIETIES OF UNIVERSAL ALGEBRAS

DALI ZANGURASHVILI

Abstract. The paper gives a sufficient condition formulated in a syntactical form for all codescent morphisms of a variety of universal algebras satisfying the amalgamation property to be effective. This result is further used in proving that all codescent morphisms of quasigroups are effective.

Key words and phrases: Effective codescent morphism, variety of universal algebras, amalgamation property, term, quasigroup.

2000 Mathematics Subject Classification: 18C20, 18C05, 03C05, 18A32, 20N05.

1. Introduction

The present paper deals with the problem of describing effective codescent morphisms in varieties of universal algebras. This problem has been solved for modules over a commutative ring with unit (see e.g. G. Janelidze and W. Tholen [2]), commutative rings with units (A. Joyal and M. Tierney (unpublished), B. Mesablishvili [6]), Boolean algebras (M. Makkai (unpublished)) and groups [7]. In this work we consider arbitrary varieties satisfying the amalgamation property and give for them a sufficient condition formulated in a syntactical form, for a codescent morphism to be effective. In particular, this criterion implies the following:

Let a variety of universal algebras satisfy the amalgamation property and the following condition:

(*) Let

\[
\begin{array}{ccc}
B & \xrightarrow{p} & E \\
q & \downarrow & \downarrow q' \\
D & \xrightarrow{p'} & D \sqcup_B E
\end{array}
\]
be a pushout with monomorphic \( p, q \) (and \( p', q' \)). Then for any element \( \alpha \) of \( D \sqcup_B E \) and any subalgebra \( C^{(1)} \) of \( D \) containing \( B \) and such that \( \alpha \) lies in \( C \sqcup_B E \) \(^{2)\}, \) there exists a presentation

\[
\alpha = t(c_1, c_2, ..., c_m, e_1, e_2, ..., e_n)
\]

in which \(^3)\) : \( c_1, c_2, ..., c_m \) are in \( C \); \( e_1, e_2, ..., e_n \) are in \( E \); and \( t \) is an \((m + n)\)-ary term such that, for any \( d_1, d_2, ..., d_m \) in \( D \), the equality

\[
t(c_1, c_2, ..., c_m, e_1, e_2, ..., e_n) = t(d_1, d_2, ..., d_m, e_1, e_2, ..., e_n)
\]

in \( D \sqcup_B E \) implies that

\[
d_1, d_2, ..., d_m \in C.
\]

Then every codescent morphism of the variety is effective.

The condition \((*)\) is easily satisfied, for instance, for modules over a commutative ring with unit and for groups. We show that it is satisfied for quasigroups, too. Since the variety of quasigroups satisfies also the amalgamation property, as proved by J. Ježek and T. Kepka in \([3]\), we obtain that

\[\text{every codescent morphism of quasigroups is effective.}\]

As to an internal characterization of such morphisms, we recall that it is given in \([7]\) for any variety with the so-called strong amalgamation property (and the variety of quasigroups is indeed of this kind \([3]\)).

According to this characterization, a monomorphism \( p : B \rightarrow E \) is a codescent morphism if and only if \( R' \cap (B \times B) = R \) for any congruence \( R \) on \( B \) and its closure \( R' \) in \( E \).

2. Preliminaries

We begin with the needed definitions from descent theory \([2]\) formulated, for convenience, in the dual form.

Let \( C \) be a category with pushouts, and let \( p : B \rightarrow E \) be a morphism in \( C \). It is well known that the change-of-cobase functor

\[
p_* : B/C \longrightarrow E/C
\]

\(^{1}\)In fact here, for any \( \alpha \), we can confine the consideration only to subalgebras \( C \) which are generated by \( B \) and elements \( d_1, d_2, ..., d_k \) from \( D \), for some presentation \( \alpha = t(d_1, d_2, ..., d_k, x_1, x_2, ..., x_l) \) with \( x_1, x_2, ..., x_l \) lying in \( E \).

\(^{2}\)One can easily show (see Lemma 2.3) that \( C \sqcup_B E \) is embedded to \( D \sqcup_B E \).

\(^{3}\)We do not exclude the case where either \( m = 0 \) or \( n = 0 \) (i.e., the corresponding variables are absent).
(pushing out along \( p \)) has a right adjoint \( p' \) composing with \( p \) from the right. \( p \) is called a \textit{codescent} (resp. \textit{effective codescent}) \textit{morphism} if \( p_\ast \) is precomonadic (resp. comonadic), i.e., the comparison functor

\[
\Phi_p : B/C \to \text{Codes}(p),
\]

where \( \text{Codes}(p) \) is the Eilenberg-Moore category of the comonad induced by the adjunction

\[
p_\ast \dashv p',
\]

is full and faithful (resp. an equivalence of categories). Recall that objects of \( \text{Codes}(p) \) (called codescent data with respect to \( p \)) are triples \((C, \gamma, \xi)\) with \( C \in \text{Ob}C \) and \( \gamma, \xi \) being morphisms \( E \to C \) and \( C \to C \sqcup_B E \), respectively, such that the following equalities are valid (see Fig. 1 and 2):

\[
\begin{align*}
\xi \gamma &= i_2, \\
(1_C, \gamma) \xi &= 1_C, \\
(i_1 \sqcup_B 1_E) \xi &= (\xi \sqcup_B 1_E) \xi.
\end{align*}
\]

Moreover, a \( \text{Codes}(p) \)-morphism \((C, \gamma, \xi) \to (C', \gamma', \xi')\) is a \( C \)-morphism \( h : C \to C' \) such that the following diagram commutes:

\[
\begin{array}{c}
\begin{array}{ccc}
E & \xrightarrow{\gamma} & C \\
\gamma & \searrow & \downarrow h \\
\xi & \swarrow & \xi'
\end{array}
& \quad & \\
\begin{array}{ccc}
C \sqcup_B E & \xrightarrow{\gamma'} & C' \\
\xi & \swarrow & \xi'
\end{array}
\end{array}
\]
Also recall that the functor $\Phi_p$ maps every $f : B \rightarrow D$ to

$$(D \sqcup_B E, \bar{i}_2, \bar{i}_1 \sqcup_B 1_E),$$

where $\bar{i}_1$ and $\bar{i}_2$ are the pushouts of $p$ and $f$, respectively, along each other.

From now on it will be assumed that, in addition to pushouts, $C$ also has equalizers.

**Theorem 2.1** ([1], [2]). $p$ is a codescent morphism if and only if it is a universal regular monomorphism, i.e., a morphism such that any of its pushouts is a regular monomorphism.

Let $(C, \gamma, \xi)$ be codescent data with respect to a morphism $p$. Consider the equalizer

$$Q \xrightarrow{q} C \xrightarrow{i_1} C \sqcup_B E \xrightarrow{\xi}$$

(2.4)

It gives the equalizer in $B/C$:

$$Q \xrightarrow{q} C \xrightarrow{i_1} C \sqcup_B E$$

(2.5)

**Theorem 2.2.** For a codescent morphism $p$ and codescent data $(C, \gamma, \xi)$ with respect to $p$, the following conditions are equivalent:

(i) $(C, \gamma, \xi) \approx \Phi_p(f)$, for some $f \in \text{Ob}B/C$;

(ii) $(C, \gamma, \xi) \approx \Phi_p(\delta)$;

(iii) the functor $p_*$ preserves the equalizer (2.5);

(iv) the morphism $(q, \gamma) : Q \sqcup_B E \rightarrow C$ is an isomorphism;

(v) $q \sqcup_B 1_E$ is a monomorphism and there exists a morphism $\theta : C \rightarrow Q \sqcup_B E$ (see Fig. 3) with

$$\xi = (q \sqcup_B 1_E)\theta.$$
If $p_*$ maps regular monomorphisms to morphisms (of the coslice category $B/C$) with monomorphic underlying morphisms, then one can omit “$q \sqcup_B 1_E$ is a monomorphism and” from (v).

Proof. The equivalence of the conditions (i)–(iv) is well known (see e.g. G. Janelidze and W. Tholen [2]). (iii)$ \Rightarrow$ (v) follows from (2.3).

(v)$ \Rightarrow$ (iv): We will show that the morphism $\theta$ is the inverse of $(q, \gamma)$. First we observe that
\[ \theta(q, \gamma) i_1 = \theta q = i_1, \quad (2.6) \]
as follows from the equalities
\[ (q \sqcup_B 1_E) \theta q = \xi q = i_1 q = (q \sqcup_B 1_E) i_1. \]
Moreover,
\[ \theta(q, \gamma) i_2 = \theta \gamma = i_2, \quad (2.7) \]
since by (2.1) we have
\[ (q \sqcup_B 1_E) \theta \gamma = \xi \gamma = i_2 = (q \sqcup_B 1_E) i_2. \]
From (2.6) and (2.7) we obtain that $\theta(q, \gamma) = 1_{Q \sqcup_B E}$. The equality $(q, \gamma) \theta = 1_C$ follows from (2.2) and the trivial observation that $(q, \gamma) = (1_C, \gamma)(q \sqcup_B 1_E)$.

Let $\mathcal{M}$ be a class of $\textbf{C}$-morphisms. Recall (see e.g. E. W. Kiss, L. Márki, P. Pröhle and W. Tholen [1]) that $\textbf{C}$ is said to satisfy the amalgamation property with respect to $\mathcal{M}$ if, for any pushout
\[
\begin{array}{ccc}
C & \xrightarrow{\nu} & D \\
\mu \downarrow & & \downarrow \mu' \\
C' & \xrightarrow{\nu'} & D'
\end{array}
\]
with $\mu, \nu \in \mathcal{M}$, we have $\mu', \nu' \in \mathcal{M}$, too. If, in addition, any such pushout is also a pullback square (or, equivalently, in the case of varieties of universal algebras, $\nu'(C') \cap \mu'(D) = \nu' \mu(C)$), then $\textbf{C}$ is said to satisfy the strong amalgamation property. If the class $\mathcal{M}$ is not indicated, it is meant to be the class of all monomorphisms.

**Lemma 2.3.** Let $\textbf{C}$ satisfy the amalgamation property with respect to a morphism class $\mathcal{M}$ containing all regular monomorphisms. If $p$ is a codescent morphism, then the functor $p_*$ preserves $\mathcal{M}$-morphisms (in the obvious sense).
Proof. It is sufficient to observe that if the back square and the upper inclined one in the diagram

\[
\begin{array}{c}
B \quad \xrightarrow{p} \quad E \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
C' \quad \xrightarrow{\mu} \quad C' \sqcup_B E \\
C \quad \xrightarrow{\mu} \quad C \sqcup_B E
\end{array}
\]

are pushouts, then so is the lower inclined square. \qed

Before continuing our discussion, let us recall the following recent result.

**Theorem 2.4** (8). Let \((\mathcal{E}, \mathcal{M})\) be a factorization system on \(\mathcal{C}\) with \(\mathcal{E} \subset \mathcal{E} \text{pi}\mathcal{C}\). A codescent morphism \(p\) is effective if and only if, for any morphism \(p'\) lying in \(\mathcal{M}\) and being the pushout of \(p\) along an \(\mathcal{E}\)-morphism and any codescent data \((C', \gamma', \xi')\) with respect to \(p'\) such that \(\gamma' p' \in \mathcal{M}\), there exists \(f'\) from the corresponding coslice category, such that \((C', \gamma', \xi')\) is isomorphic to \(\Phi_{p'}(f')\). If \(\mathcal{C}\) satisfies the amalgamation property with respect to \(\mathcal{M}\), then the statement remains valid provided that “\(\gamma' p' \in \mathcal{M}\)” is replaced by “\(\gamma' \in \mathcal{M}\)”.

Similarly to (2.4), for codescent data \((C', \gamma', \xi')\) with respect to a morphism \(p'\), let a monomorphism \(q' : Q' \rightarrow C'\) be the equalizer of the pair \((i'_1, \xi')\) with obvious \(i'_1\). Applying Theorem 2.2, Lemma 2.3 and Theorem 2.4, we obtain

**Proposition 2.5.** Let \((\mathcal{E}, \mathcal{M})\) be a proper factorization system on \(\mathcal{C}\) (i.e. a factorization system with \(\mathcal{E} \subset \mathcal{E} \text{pi}\mathcal{C}\) and \(\mathcal{M} \subset \text{Mono}\mathcal{C}\)), and let \(\mathcal{C}\) satisfy the amalgamation property with respect to \(\mathcal{M}\). A codescent morphism \(p\) is effective if and only if, for any morphism \(p' : B' \rightarrow E'\) lying in \(\mathcal{M}\) and being the pushout of \(p\) along an \(\mathcal{E}\)-morphism and any codescent data \((C', \gamma', \xi')\) with respect to \(p'\) such that \(\gamma' \in \mathcal{M}\), there exists a morphism \(\theta'\) with \(\xi' = (q' \sqcup_B 1_{E'})\theta'\).\(^4\)

For the proof we only observe that the inclusion \(\mathcal{E} \subset \mathcal{E} \text{pi}\mathcal{C}\) obviously implies the inclusion \(\text{Reg}\text{Mono}\mathcal{C} \subset \mathcal{M}\). \(\square\)

\(^4\) In the same manner one can derive criteria for the effectiveness of \(p\) by using also the conditions (ii) and (iii) of Theorem 2.2, but for our further purposes we need only the present criterion.
3. Effective Codescent Morphisms in Varieties with the Amalgamation Property

Throughout this section, if it is not specified otherwise, we assume that $C$ is a variety of universal algebras (of any type $\mathcal{F}$) satisfying the amalgamation property.

Let $p : B \to E$ be a codescent morphism in $C$, and let $R$ be a congruence on $B$. It is well known that the pushout of $p$ along the projection $B \to B' = B/R$ is the obvious morphism $p' : B' \to E' = E/R'$, where $R'$ is the closure of $R$ in $E$. It is obvious that $p'$ is a codescent morphism as well.

Let $C'$ be an extension of $B'$. Below we will deal with the free product of the algebras $C'$ and $E'$ with the amalgamated subalgebra $B'$ (see Fig. 4).

For convenience, when no confusion might arise, we identify $C'$ and $E'$ with their images under $i'_1$ and $i'_2$, respectively, and hence consider them as subalgebras of $C' \sqcup_{B'} E'$ (whose intersection is, in general, wider than $B'$). We adopt the similar convention for the free product of several algebras with an amalgamated subalgebra.

All the terms considered below are meant to be of type $\mathcal{F}$. When using the notation $t(c_1, c_2, \ldots, c_m, e_1, e_2, \ldots, e_n)$ for a term we do not exclude the case where either $m = 0$ or $n = 0$ (i.e., the corresponding variables are absent).

**Proposition 3.1.** The following conditions are equivalent:

(i) $p$ is effective;

(ii) for any congruence $R$ on $B$ and any codescent data $(C', \gamma', \xi')$ (with respect to $p'$) with monomorphic $\gamma'$, there exists a homomorphism $\theta' : C' \to Q' \sqcup_{B'} E'$ with

$$\xi' = (q' \sqcup_{B'} 1_{E'})\theta';$$

(iii) for any congruence $R$ on $B$ and any codescent data $(C', \gamma', \xi')$ (with respect to $p'$) with monomorphic $\gamma'$, one has

$$\xi'(C') \subset (q' \sqcup_{B'} 1_{E'})(Q' \sqcup_{B'} E');$$

(iv) for any congruence $R$ on $B$, any codescent data $(C', \gamma', \xi')$ (with respect to $p'$) with monomorphic $\gamma'$ and any $c \in C'$, there exists a
presentation of \( \xi'(c) \) as a term \( t \) over the set \( C' \cup E' \) of variables, such that, for any variable \( c' \) from \( C" \setminus E' \) involved in \( t \), one has

\[ \xi'(c') = c'. \]

Proof. The equivalence (i)\( \Leftrightarrow \) (ii) is precisely the contents of Proposition 2.5 for \( \mathcal{M} = \text{MonoC} \) (and \( \mathcal{E} = \text{Reg\pi C} \)). (ii)\( \Leftrightarrow \) (iii) follows from Lemma 2.3, while (iii)\( \Leftrightarrow \) (iv) is obvious. \( \square \)

**Proposition 3.2.** For the following conditions, one has (i)\( \Rightarrow \) (ii)\( \Rightarrow \) (iii).

(i) For any congruence \( R \) on \( B \), any extensions \( B' \rightarrow C' \rightarrow D' \) and any element \( \alpha \) of \( C' \sqcup_{B'} E' \), there exists a presentation

\[ \alpha = t(c_1, c_2, \ldots, c_m, e_1, e_2, \ldots, e_n) \]  

(3.1)

in which: \( c_1, c_2, \ldots, c_m \) are in \( C' \); \( e_1, e_2, \ldots, e_n \) are in \( E' \); and \( t \) is an \((m + n)\)-ary term such that, for any \( d_1, d_2, \ldots, d_m \) in \( D' \), the equality

\[ t(c_1, c_2, \ldots, c_m, e_1, e_2, \ldots, e_n) = 
= t(d_1, d_2, \ldots, d_m, e_1, e_2, \ldots, e_n) \]  

(3.2)

in \( D' \sqcup_{B'} E' \) implies that

\[ d_1, d_2, \ldots, d_m \in C'. \]  

(3.3)

(ii) For any congruence \( R \) on \( B \), any extension \( C' \) of \( B' \) and any element \( \alpha \) of \( C' \sqcup_{B'} E' \), there exists a presentation (3.1) in which: \( c_1, c_2, \ldots, c_m \) are in \( C' \); \( e_1, e_2, \ldots, e_n \) are in \( E' \); and \( t \) is an \((m + n)\)-ary term such that for any elements \( d_1, d_2, \ldots, d_m \) in \( C' \sqcup_{B'} E' \), the equality

\[ t(c_1, c_2, \ldots, c_m, e_1, e_2, \ldots, e_n) \]

(3.2)

in \( C' \sqcup_{B'} E' \sqcup_{B'} E' \), where the variables \( e_1, e_2, \ldots, e_n \) (in the case of their existence) in all their occurrences on both sides of (3.2) are considered as representatives of the third cofactor of \( C' \sqcup_{B'} E' \sqcup_{B'} E' \) implies (3.3);

(iii) \( p \) is an effective codescent morphism.

Proof. For (i)\( \Rightarrow \) (ii) it is sufficient to take \( D' = C' \sqcup_{B'} E' \).

(ii)\( \Rightarrow \) (iii): We will verify the validity of the condition (iv) of Proposition 3.1. To this end, consider any codescent data \((C', \gamma', \xi')\) with respect to \( p' \) such that \( \gamma' \) is a monomorphism, and any \( c \in C' \). If \( \xi'(c) \in C' \), then, according to (2.2), \( \xi'(c) = c \). If \( \xi'(c) = e \in E' \), then the term \( e \) is obviously the desired one. Suppose that \( \xi'(c) \) does not lie in \( C' \sqcup E' \). Consider the presentation (3.1) of \( \alpha = \xi'(c) \). From (2.3) we have the equality (3.2), where \( d_1 = \xi'(c_1), d_2 = \xi'(c_2), \ldots, d_m = \xi'(c_m) \) and the variables \( e_1, e_2, \ldots, e_n \) are considered as representatives of the second \( E' \) in \( C' \sqcup_{B'} E' \sqcup_{B'} E' \). Therefore, for all \( i \) \((1 \leq i \leq m)\), we have \( \xi'(c_i) \in C' \) and hence, by (3.2) we have \( \xi'(c_i) = c_i \). \( \square \)

Proposition 3.2 immediately implies
**Theorem 3.3.** Let $C$ be a variety of universal algebras that satisfies the amalgamation property and the following condition:

(*) Let

\[
\begin{array}{c}
B @>{p}>> E \\
\downarrow^{q} & & \downarrow^{q'} \\
D @>{p'}>> D \sqcup_B E
\end{array}
\]

be a pushout in $C$ with monomorphic $p$, $q$ (and $p'$, $q'$). Then for any element $\alpha$ of $D \sqcup_B E$ and any subalgebra $C$ of $D$ containing $B$ and such that $\alpha$ lies in $C \sqcup_B E$ (contained in $D \sqcup_B E$), there exists a presentation (3.1) of $\alpha$ in which: $c_1, c_2, ..., c_m$ are in $C$; $e_1, e_2, ..., e_n$ are in $E$; and $t$ is an $(m+n)$-ary term such that, for any $d_1, d_2, ..., d_m$ in $D$, the equality (3.2) in $D \sqcup_B E$ implies that

\[d_1, d_2, ... d_m \in C.\]

Then all codescent morphisms of $C$ are effective.

**Remark 3.4.** In (*), for any $\alpha$ from $D \sqcup_B E$, we can confine the consideration only to subalgebras $C$ which are generated by $B$ and elements $d_1, d_2, ..., d_k$ from $D$, for some presentation $\alpha = t(d_1, d_2, ..., d_k, x_1, x_2, ..., x_l)$ with $x_1, x_2, ..., x_l$ lying in $E$.

**Example 3.5.** (i) It is well-known that all codescent morphisms of the variety of Abelian groups (more generally, modules over a commutative ring with unit) are effective. The condition (*) is obviously satisfied in this case.

(ii) In [7] we have shown that every codescent morphism of the variety of groups is effective. Let us show that this variety satisfies the condition (*). To this end, let us first recall the well-known fact related to the free product $G$ of groups $G_1$ and $G_2$ with an amalgamated subgroup $B$ (we assume that $G_1 \cap G_2 = B$)[5].

For any right coset of $G_1$ and of $G_2$ by $B$, except for $B$ itself, we choose a representative. We denote the set of all chosen representatives by $A$. Then every element of $G$ can be uniquely written as a product

\[b a_1 a_2 ... a_n, \tag{3.4}\]

where $n \geq 0$, $b \in B$, all $a_j$ lie in $A$ and no two $a_j, a_{j+1}$ belong to one and the same $G_j$. Form (3.4) is called $A$-canonical. The procedure how an element

\[a'_1 a'_2 ... a'_n, \tag{3.5}\]
of $G$ (taken in un cancellable form) can be reduced to the canonical form is described in [5] (see also [7]). Roughly speaking, we, beginning from the right, pick out the left $B$-coefficient from a current factor in (3.5), and then multiply it to the left neighbor.

Let us now take $G_1 = D$, $G_2 = E$ and choose a set $A$ of representatives of right cosets. Then take $G_1 = C$, $G_2 = E$ and choose a set $A'$ of representatives such that $A' \subset A$. In other words, for cosets of both $C$ and $E$ by $B$ we take the already chosen representatives. Let us consider any element $\alpha$ from $C \sqcup_B E$ and take its $A'$-canonical form

$$bc_1 \varepsilon_1 c_2 \varepsilon_2 \ldots c_{m-1} \varepsilon_m,$$  \hspace{1cm} (3.6)

with $c_1, c_2, \ldots, c_m \in C$ and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m-1} \in E$. We show that (3.6) is the desired representation of $\alpha$, where $b$ is considered as an element of $E$ (the proof of the statement in the case where (3.6) is ended by an element of $E$ is similar). To this end, let us consider $d_1, d_2, \ldots, d_m \in D$ with

$$bc_1 \varepsilon_1 c_2 \varepsilon_2 \ldots c_{m-1} \varepsilon_m = bd_1 \varepsilon_1 d_2 \varepsilon_2 \ldots c_{m-1} \varepsilon_m d_m.$$ \hspace{1cm} (3.7)

Let us reduce the right hand part of (3.7) to the $A$-canonical form. For that we consider the presentation

$$d_m = b_m d'_m,$$

with $d'_m \in A$ and $b_m \in B$. From the uniqueness of the canonical form, we conclude that $d'_m = c_m$ and hence $d_m \in C$. Let $1 \leq k < m$. We have

$$d_k \varepsilon_k' = b_k d'_{k+1},$$

for some $d'_{k+1} \in A$ and $b_k, \varepsilon_k' \in B$. Again, from the uniqueness of the canonical form, we have $d'_{k+1} = c_{k+1}$, which implies that $d_k \in C$.

(iii) Let $\mathcal{F}$ contain only nullary and unary operations. Then $D \sqcup_B E$, as a set, is obviously isomorphic to the corresponding pushout $(D \sqcup_B E)_{\text{Set}}$ in the category of sets. Therefore, the condition (*) holds in that case, too.

(iv) Let $\mathbf{C}$ be the variety of all algebras of type $\mathcal{F}$. Then, as is well known, each element of $D \sqcup_B E$ can be uniquely presented as a term over $(D \sqcup_B E)_{\text{Set}}$ such that the variables of none of its subterm lie in one and the same cofactor of $D \sqcup_B E$. This implies that the condition (*) of Theorem 3.3 holds.

We conclude that all codescent morphisms are effective in both cases (iii) and (iv).
4. Effective Codescent Morphisms of Quasigroups

Let us now pass to the case where $C$ is the variety of quasigroups. Take its usual presentation $(\mathcal{F}, \Sigma)$. Recall that here $\mathcal{F}$ consists of three binary operations $\circ, /, \backslash$, while $\Sigma$ is the set of the identities

\[
\backslash (x_1, \circ (x_1, x_2)) = x_2, \quad (4.1)
\]
\[
/ (\circ (x_1, x_2), x_2) = x_1, \quad (4.2)
\]
\[
\circ (\backslash (x_1, x_2), x_2) = x_2, \quad (4.3)
\]
\[
\circ (/ (x_1, x_2), x_2) = x_1. \quad (4.4)
\]

Let $B$ and $A_i$ $(1 \leq i \leq k, k \in \mathbb{N})$ be quasigroups such that, for any $i$, $x_1, x_2 \notin A_i$. For simplicity, it is assumed that $A_i \cap A_j = B$, for any distinct $i, j$.

To distinguish between terms over $X = \{x_1, x_2\}$ and those over $\bigcup_{i=1}^k A_i$, we will use different notations for them: the former will be denoted by the capital letter $T$ (perhaps with (co)indices), while the latter by the small-case letter $t$ (perhaps with (co)indices). Unless specified otherwise, we will use the word “term” to mean a term over the set $\bigcup_{i=1}^k A_i$.

In the set of terms we introduce transformations of the following forms, which below will be called reduction transformations or, for short, reductions:

(i) if a term $t$ contains a subterm

\[
f(a_1, a_2)
\]

with $f$ one of $\circ, /, \backslash$ and $a_1, a_2$ variables from one and the same $A_i$, then we replace (4.5) in $t$ by the corresponding element of $A_i$; in that case the term (4.5) is called the replaced term of the reduction, while the value of (4.5) in $A_i$ is called the replacing term of the reduction;

(ii) if, for some identity

\[
T = x_i
\]

from (4.1)-(4.4), a term $t$ contains a subterm $t'$ obtained from $T$ by replacing the variables $x_1$ and $x_2$ by some terms $t_1$ and $t_2$, then we replace $t'$ in $t$ by $t_i$. The subterm $t'$ is called the replaced term of the reduction and $t_i$ is called the replacing term of the reduction.

A reduction transformation is said to be performed on an occurrence $o$ of an operation in $t$ if $o$ is the first (from the left) among all occurrences of operations in the replaced term of the reduction.

A term is called irreducible if it admits no reduction transformation.
Lemma 4.1. For any identity
\[ T = x_i \]
from (4.1) – (4.4), performing any sequence of reductions over the term \( t \) obtained from \( T \) by replacing the variables \( x_1 \) and \( x_2 \) by any irreducible terms \( t_1 \) and \( t_2 \), we arrive either at a reducible term or at the term \( t_i \).

Proof. Let us assume that a term \( t \) has, for instance, the form
\[ \backslash(t_1, o(t_1, t_2)) \]  
with irreducible \( t_1 \) and \( t_2 \). If \( o(t_1, t_2) \), too, is irreducible, then the only possible reduction is the one on the depicted in (4.6) occurrence of the operation \( \backslash \) and obviously giving the term \( t_2 \). Let now \( o(t_1, t_2) \) be reducible. Then it admits a reduction \( r \) on the first occurrence of the operation \( o \). If \( r \) is of the form (i), then both \( t_1 \) and \( t_2 \) are variables from some \( A_i \) and the element of \( A_i \) corresponding to \( t \) is \( t_2 \). If \( r \) is of the form (ii), then \( o(t_1, t_2) \) is equal either to \( o(t_1 \backslash(t_1, t_3)) \) or to \( o(/(t_4, t_2), t_2) \). Let us consider the first case. Then, performing the reduction \( r \), from \( t \) we obtain the term \( \backslash(t_1, t_3) \) which is irreducible and equal to \( t_2 \). In a likewise manner we obtain \( t_2 \) in the second case too. The proof of the assertion in the case, where \( t \) is obtained from the left part of anyone of (4.2)–(4.4) by replacing the variables \( x_1 \) and \( x_2 \) by some irreducible terms, is analogous. \( \square \)

Lemma 4.1 immediately implies

Lemma 4.2. Any two reductions performed on one and the same occurrence \( o \) of an operation in a term give one and the same result provided that \( o \) has irreducible arguments.

Lemma 4.3. Applying only reduction transformations, each term can be reduced to a unique irreducible form.

Proof. The existence of the needed term is obvious. In the particular case, where a given term \( t \) has the form described in Lemma 4.1, the uniqueness immediately follows from this lemma. For the general case we apply the principle of mathematical induction on the length \( l \) of \( t \).

If \( l = 1 \), then the validity of the statement is clear. Assume that \( l > 1 \) and that the statement is valid for all terms whose length is less than \( l \). Let \( t \) have the form
\[ f(t_1, t_2), \]  
where \( f \) is one of the operations \( o, /, \backslash \), while \( t_1 \) and \( t_2 \) are some terms. Let us introduce the following sequence \( S \) of reduction transformations: first both \( t_1 \) and \( t_2 \) are reduced (in a certain manner) to irreducible
forms and then the reduction is performed, if possible, on the occurrence of \( f \) depicted in (4.7).

Let \( S' \) be any sequence of reductions applicable to (4.7) and yielding an irreducible term. If the first transformation from \( S' \) is the reduction of some \( t_i \), then, by the assumption of induction, the transformations \( S \) and \( S' \) give one and the same result.

Let the first member of \( S' \) be the reduction on the first occurrence of \( f \) in \( t \). If this transformation is of the form (i), then both \( t_1 \) and \( t_2 \) are merely variables from one and the same \( A_i \) and thus the results of \( S \) and \( S' \) coincide. Now assume that the first member of \( S' \) has the form (ii). Then \( t \) is obtained from the term \( T \) by replacing the variables \( x_1 \) and \( x_2 \) by some terms which without loss of generality can be assumed to be irreducible. But Lemma 4.1 implies that both sequences \( S \) and \( S' \) give one and the same term.

\( \square \)

**Lemma 4.4.** Given two terms, if there are two reduction transformations such that by applying one of them to one of these terms and the other transformation to the second term we obtain one and the same term, then there exists a term that can be reduced by sequences of reduction transformations to both original terms.

*Proof.* Let reductions \( r_1 \) and \( r_2 \) transform terms \( t_1 \) and \( t_2 \) to a term \( t \), and let the replaced (resp. replacing) term of \( r_i \) be \( t_{1i} \) (resp. \( t_{2i} \)), \( i = 1, 2 \). Assume that \( t_{12} \) and \( t_{22} \) are not subterms of each other in \( t \). Then the desired term \( t' \) is the term obtained from \( t \) by replacing \( t_{12} \) by \( t_{11} \) and \( t_{22} \) by \( t_{21} \).

Let \( t_{12} \) be a subterm in \( t \) of \( t_{22} \). If \( r_2 \) is a reduction transformation of the form (ii), then \( t_{21} \) contains both \( t_{22} \) and \( t_{12} \) as subterms. Hence, in that case the desired term is obtained from \( t_2 \) by replacing \( t_{12} \) in \( t_{21} \) by \( t_{11} \).

Let \( r_2 \) be of the form (i). Then \( t_{12} = t_{22} \). If \( r_1 \) is of the form (ii), then the existence of the term we want to find follows from the foregoing arguments. Otherwise it is obtained from the term \( t_1 \) by replacing \( t_{11} \) by

\[
/(\circ(t_{11}, \setminus(t_{12}, \circ(t_{21}, t_{12}))), t_{12}).
\]

\( \square \)

Let us now consider the free product of \( A_i \) (1 \( \leq \) \( i \) \( \leq \) \( k \)) with the amalgamated subquasigroup \( B \). It is well known that it is isomorphic to the quotient of the \( \mathcal{F} \)-algebra of terms over the set \( \bigcup_{i=1}^{k} A_i \) of variables with respect to the congruence \( R \), where a term \( t \) is \( \tilde{R} \)-equivalent to
a term $t'$ if and only if either $t = t'$ or $t'$ can be obtained from $t$ by a sequence of transformations being reductions or their inverses.

**Lemma 4.5.** For any two $R$-equivalent terms there exists a term which by applying only reduction transformations can be reduced to both ones.

The proof easily follows from Lemma 4.4.

Lemma 4.3 and Lemma 4.5 immediately give rise to

**Lemma 4.6.** Any element of the free product of quasigroups $A_i$ ($1 \leq i \leq k$) with the amalgamated subquasigroup $B$ can be uniquely presented as an irreducible term over the set $\bigcup_{i=1}^{k} A_i$ of variables.

Recall that the variety of quasigroups satisfies the amalgamation property (J. Ježek and T. Kepka [3]). Therefore, from Lemma 4.6 and Theorem 3.3 we obtain

**Theorem 4.7.** Every codescent morphism of quasigroups is effective.

**Proof.** Let us verify the validity of the condition (*) of Theorem 3.3. For any element $\alpha$ of $C \sqcup_B E$ consider its irreducible presentation $t$. Let the term $t'$ obtained from $t$ by replacing all available variables $c_1, c_2, \ldots, c_m$ from $C$ by $d_1, d_2, \ldots, d_m$ from $D$ present $\alpha$ in $D \sqcup_B E$. If $t$ is irreducible, then, according to Lemma 4.6, $c_i = d_i$, for all $i$. Assume now that $t'$ is not irreducible. Then reducing $t'$, we obtain an irreducible term $t''$, $R$-equivalent to $t$ and such that its length is less than that of $t$. This contradicts Lemma 4.6.

**Acknowledgement**

The author gratefully acknowledges valuable discussions with George Janelidze on the subject of this paper. The work was partially supported by Volkswagen Foundation Ref.: I/84 328 and Georgian National Science Foundation Ref.: ST06/3-004.

**References**

[1] G. Janelidze and W. Tholen, How algebraic is the change-of-base functor? *Lecture Notes in Math.* 1488 (Springer, Berlin, 1991), 174–186.

[2] G. Janelidze and W. Tholen, Facets of descent. I. *Appl. Categ. Structures* 2 (1994), 1–37.

[3] J. Ježek and T. Kepka, Varieties of quasigroups determined by short strictly balanced identities. *Czechoslovak Math. J.* 29 (104) (1979), 84–96.
[4] E. W. Kiss, L. Márki, P. Pröhle and W. Tholen, Categorical algebraic properties. A compendium on amalgamation, congruence extension, epimorphisms, residual smallness and injectivity. *Studia Sci. Math. Hungarica* **18**(1983), 79–141.

[5] A. G. Kurosh, Theory of groups. ”Nauka” Moscow, 1967 (in Russian).

[6] B. Mesablishvili, Pure morphisms of commutative rings are effective descent morphisms for modules – a new proof. *Theory Appl. Categ.* **7**(2000), no. 3, 38–42.

[7] D. Zangurashvili, The strong amalgamation property and (effective) codescent morphisms. *Theory Appl. Categ.* **11**(2003), no. 20, 438–449.

[8] D. Zangurashvili, Effective codescent morphisms, amalgamations and factorization systems. *J. Pure Appl. Algebra* **209**(2007), no. 1, 255–267.

Authors address:

Andrea Razmadze Mathematical Institute,
Tbilisi Centre for Mathematical Sciences
1 Alexidze Str., 0193, Tbilisi, Georgia
E-mail: dalizan@rmi.acnet.ge