WEAK WIDTH OF SUBGROUPS

RITA GITIK

Abstract. We say that the weak width of an infinite subgroup $H$ of $G$ in $G$ is $n$ if there exists a collection of $n$ strongly essentially distinct conjugates 
\[ \{H, g_1^{-1}Hg_1, \ldots, g_{n-1}^{-1}Hg_{n-1}\} \] of $H$ in $G$ such that the intersection $H \cap g_i^{-1}Hg_i$ is infinite for all $1 \leq i \leq n-1$ and $n$ is maximal possible. We prove that a quasiconvex subgroup of a negatively curved group has finite weak width in the ambient group. We also give examples demonstrating that height, width, and weak width are different invariants of a subgroup.

1. Introduction

A subgroup $H$ of $G$ is malnormal in $G$ if for any $g \in G$ such that $g \notin H$ the intersection $H \cap g^{-1}Hg$ is trivial. Most subgroups are neither normal nor malnormal, so the study of the intersection pattern of conjugates of a subgroup is an interesting problem. It is closely connected to the study of the behavior of different lifts of subspaces of topological spaces in covering spaces.

Malnormality of a subgroup has been generalized in different ways. One of them, namely the height, introduced in [3], has been used by Agol in [1] and [2] in his proof of Thurston’s conjecture that 3-manifolds are virtual bundles. In this paper we introduce yet another generalization of malnormality. It is a new invariant of the conjugacy class of a subgroup $H$ of $G$, which we call the weak width of a subgroup. Like malnormality, the weak width measures only the cardinality of the intersections of $H$ with its conjugates in $G$. In section 4 we prove that quasiconvex subgroups of negatively curved groups have finite weak width, which might simplify Agol’s proof. In section 2 we review the definitions and the basic properties of the width and the height of a subgroup. In section 3 we give examples showing that height, width, and weak width are different invariants of a subgroup.

Remark 1. Note that if $g_i \in Hg_jH$, hence $g_i = h_1g_jh_2$ with $h_1$ and $h_2$ in $H$, then $H \cap g_i^{-1}Hg_i = H \cap (h_1g_jh_2)^{-1}H(h_1g_jh_2) = H \cap (h_2^{-1}g_j^{-1}Hg_jh_2) = h_2^{-1}(H \cap g_0^{-1}Hg_0)h_2$. So the cardinality of the set $H \cap g_i^{-1}Hg_i$ is equal to the cardinality of the set $H \cap g_j^{-1}Hg_j$.

Remark [1] motivates the following definitions.

Definition 1. Let $H$ be a subgroup of a group $G$. We say that the elements \{g_i|1 \leq i \leq n\} of $G$ are strongly $H$-essentially distinct if $Hg_iH \neq Hg_jH$ for $i \neq j$. Conjugates $g_i^{-1}Hg_i$ of $H$ by strongly $H$-essentially distinct elements are called strongly essentially distinct conjugates.

Date: November 14, 2017.

2010 Mathematics Subject Classification. Primary: 20E45; Secondary: 20F67.
Definition 2. We say that the weak width of an infinite subgroup $H$ of $G$ in $G$, denoted $\text{WeakWidth}(H, G)$, is $n$ if there exists a collection of $n$ strongly essentially distinct conjugates \{\(H, g_1^{-1}Hg_1, \cdots, g_n^{-1}Hg_n\}\} of $H$ in $G$ such that the intersection $H \cap g_i^{-1}Hg_i$ is infinite for all $1 \leq i \leq n-1$ and $n$ is maximal possible. We define the weak width of a finite subgroup of $G$ to be 0.

Note that if $\text{WeakWidth}(H, G) = n$, then in any set of $n+1$ strongly essentially distinct conjugates \{\(H, g_1^{-1}Hg_1, \cdots, g_n^{-1}Hg_n\)\} of $H$ in $G$ there exists an element $g_i^{-1}Hg_i$ which has finite intersection with $H$.

2. Height and width.

The following definitions were introduced in [3] and [4].

Definition 3. Let $H$ be a subgroup of a group $G$. We say that the elements \{\(g_i|1 \leq i \leq n\)\} of $G$ are $H$-essentially distinct if $Hg_i \neq Hg_j$ for $i \neq j$. Conjugates $g_i^{-1}Hg_i$ of $H$ by $H$-essentially distinct elements are called essentially distinct conjugates.

If $g_i$ and $g_j$ are not $H$-essentially distinct, then $g_i^{-1}Hg_i = g_j^{-1}Hg_j$, hence it is interesting to investigate the intersections of the conjugates of $H$ only if they are $H$-essentially distinct. However, essentially distinct conjugates need not be distinct.

For example, let $G = \langle a_1, a_2 | a_1a_2a_1 = a_2a_1 \rangle$ be a free abelian group of rank 2 and let $H = \langle a_1 \rangle$ be a subgroup of $G$. The conjugates $a_2^{-1}Ha_2$ and $H$ are essentially distinct, but $a_2^{-1}Ha_2 = H$.

Definition 4. We say that the height of an infinite subgroup $H$ of $G$ in $G$, denoted by $\text{Height}(H, G)$, is $n$ if there exists a collection of $n$ essentially distinct conjugates of $H$ in $G$ such that the intersection of all the elements of the collection is infinite and $n$ is maximal possible. We define the height of a finite subgroup of $G$ to be 0.

Note that if $\text{Height}(H, G) = n$ then the intersection of any set of $n+1$ essentially distinct conjugates of $H$ in $G$ is finite. It was shown in [3] that subgroups of negatively curved groups have finite height in the ambient group.

Definition 5. We say that the width of an infinite subgroup $H$ of $G$ in $G$, denoted by $\text{Width}(H, G)$, is $n$ if there exists a collection of $n$ essentially distinct conjugates of $H$ in $G$ such that the intersection of any two elements of the collection is infinite and $n$ is maximal possible. We define the width of a finite subgroup of $G$ to be 0.

Note that if $\text{Width}(H, G) = n$ then in any set of $n+1$ essentially distinct conjugates of $H$ in $G$ there exist two elements with finite intersection. It was shown in [4] and, later, in [5] that quasiconvex subgroups of negatively curved groups have finite width in the ambient group.

It follows from the above definitions that $\text{Width}(H, G)$ and $\text{Height}(H, G)$ are invariants of the conjugacy class of $H$ in $G$.

Note also that $\text{Height}(H, G) \leq \text{Width}(H, G)$, however, it is not clear if there is any relationship between $\text{WeakWidth}(H, G)$ and $\text{Width}(H, G)$.

Infinite normal subgroups of infinite index have infinite height, width, and weak width in the ambient group. More generally, if an infinite subgroup has infinite index in its normalizer, then the subgroup has infinite height, width, and weak width in the ambient group.

If $G$ is torsion-free and $H$ is infinite, then $H$ is malnormal in $G$ if and only if $\text{Height}(H, G) = \text{Width}(H, G) = \text{WeakWidth}(H, G) = 1$. 

3. Examples

The following examples demonstrate that \( \text{WeakWidth}(H,G), \text{Width}(H,G), \) and \( \text{Height}(H,G) \) are distinct invariants of the conjugacy class of \( H \) in \( G \).

Let \( X \) be a set and let \( X^* = \{ x, x^{-1} \mid x \in X \} \), where for \( x \in X \) we define \((x^{-1})^{-1} = x\). Denote the equality of two words in \( X^* \) by “\( \equiv \)”.

**Example 1.** Let \( F \) be a free group of rank 4 generated by the elements \( x_1, x_2, x_3, x_4 \), let \( G = \langle t, t^i = 1, t^{-1}x_it = x_{(i+1) \mod 4} \mid 1 \leq i \leq 4 \rangle \), and let \( H_1 = \langle x_1, x_2 \rangle \). We claim that \( \text{WeakWidth}(H_1, G) = 3 \), but \( \text{Height}(H_1, G) = \text{Width}(H_1, G) = 2 \).

In order to prove the claim we will list all essentially distinct and all strongly essentially distinct conjugates of \( H_1 \) in \( G \) which have non-trivial intersection with \( H_1 \).

Let \( H_1 = \langle x_i, x_{(i+1) \mod 4} \mid 1 \leq i \leq 4 \rangle \). \( t^{i} \) be conjugates of \( H_1 \) in \( G \). As \( t^{i} \notin F \) for \( i \neq 0 \) (mod 4), the conjugates \( \{ H_i \mid 1 \leq i \leq 4 \} \) are essentially distinct.

Let \( H_1 \) be conjugates of \( H_1 \) in \( G \). Let \( H \) be a non-trivial reduced word such that \( g^{-1}v_g = t^{-k}H_1t^{k} = H_{1+k} \), and the intersection pattern of the subgroups \( \{ H_i \mid 1 \leq i \leq 4 \} \) is described above.

If \( v \) is a reduced word in a free group \( F \), then exist decompositions \( w \equiv \prod w_1 \cdot w_2 \) (where \( \equiv \) denotes equality of words) with \( w^{-1}v_w = (w_1^{-1}w_2^{-1})(w_1w_2) = w_1^{-1}w_0^{-1}w_2^{-1}w_2 \) is a reduced word in \( H_{1-k} \mod 4 \). Then \( v_0 \in H_{1-k} \mod 4 \) and \( w_2 \in H_{1-k} \mod 4 \). As \( v \in H_1 \), it follows that \( w_1 \in H_1 \) and \( v_0 \in H_1 \). However, as \( g = wt^k = w_1w_2t^k \) is shortest in the coset \( H_1g \), \( w_1 \) should be trivial. Hence \( w = w_2 \in H_{1-k} \mod 4 \). As a non-trivial word \( v_0 \) belongs to \( H_1 \cap H_{1-k} \mod 4 \), it follows that \( (1-k) (mod 4) \) is equal to either 1, 2 or 4. Hence if \( (1-k) (mod 4) \equiv 3 \), so \( k = 2 \), then for any \( r \in F \) the intersection \( \langle rt^2 \rangle^{-1}H_1(\langle rt^2 \rangle) \cap H_1 \) is trivial.

If \( (1-k) (mod 4) \equiv 1 \) then \( w = w_2 \in H_1 \), contradicting again the fact that \( g \) is shortest in the coset \( H_1g \). Hence either \( (1-k) (mod 4) \equiv 2 \) and \( k = 3 \), or \( (1-k) (mod 4) \equiv 4 \) and \( k = 1 \).

If \( k = 3 \), then \( g = wt^3 \) with \( w \in H_2 \). Note that the essentially distinct elements of the infinite collection of the conjugates \( \{ (w^3)^{-1}H_1(w^3) \} \) intersect each other trivially. Indeed, consider \( w_0 \in H_2 \) and \( w \in H_2 \) such that the intersection \( \langle t^{-3}w^{-1} \rangle H_1(w^3) \cap \langle t^{-3}w_0^{-1} \rangle H_1(w_0^3) \) is non-trivial. Then the intersection \( H_1 \cap \langle w^3 \rangle \langle t^{-3}w^{-1} \rangle H_1(w^3) \langle t^{-3}w_0^{-1} \rangle H_1(w_0^3) \) is non-trivial. As \( H_1 \) is malnormal in \( F \), it follows that \( w_0w_0^{-1} = (w_0^3)(t^{-3}w^{-1}) \in H_1 \), so the conjugates \( \langle t^{-3}w^{-1} \rangle H_1(w^3) \) and \( \langle t^{-3}w_0^{-1} \rangle H_1(w_0^3) \) are not essentially distinct. Therefore the family of the conjugates \( \{ (w^3)^{-1}H_1(w^3) \} \) does not contribute to \( \text{Width}(H_1, G) \).
Similarly, if \( k = 1 \), hence \( g = u t \) with \( u \in \mathcal{H}_4 \), the essentially distinct elements of the infinite collections of the conjugates \( \{(u t)^{-1} H_1(u t) | u \in \mathcal{H}_4\} \) intersect each other trivially.

Also for \( w \in \mathcal{H}_2 \) and \( u \in \mathcal{H}_4 \) the intersection \( (t^{-3} w^{-1}) H_1(w t^3) \cap (t^{-1} u^{-1}) H_1(u t) \) is trivial. Indeed, the cardinality of that intersection is equal to the cardinality of the intersection \( (u t)(t^{-3} w^{-1}) H_1(t^{-1} u^{-1}) \cap H_1 \). However, \( (w t^3)(t^{-1} u^{-1}) = w t^2 u^{-1} = (w t^2 u^{-1} t^{-2})t^2 = rt^2 \) with \( r \in F \), and we have mentioned above that for all \( r \in F \) the intersection \( (r t^2)^{-1} H_1(r t^2) \cap H_1 \) is trivial. So the infinite family of conjugates \( \{(u t)^{-1} H_1(u t) | u \in \mathcal{H}_4\} \) does not contribute to \( \text{Width}(H_1, G) \), therefore \( \text{Height}(H_1, G) = \text{Width}(H_1, G) = 2 \).

Note that for any \( w \in \mathcal{H}_2 \), \( wt^3 = t^3 (t^{-3} wt^3) \in t^3 H_1 \subseteq t^i H_1 \), hence all the elements \( \{(wt^3) | w \in \mathcal{H}_2\} \) are strongly \( H_1 \)-equivalent to \( t^3 \), so the conjugates of \( H_1 \) by those elements do not contribute to the weak width of \( H_1 \). Similarly, all the elements \( \{(ut)^{-1} | u \in \mathcal{H}_4\} \) are strongly \( H_1 \)-equivalent to \( t \), so the conjugates of \( H_1 \) by those elements do not contribute to the weak width of \( H_1 \) either. Therefore, \( \text{WeakWidth}(H, G) = 3 \).

\[
\Box
\]

Example 2. Let \( G \) be as in Example 1 and let \( L_1 = \langle x_1, x_2, x_3 \rangle \). We claim that \( \text{WeakWidth}(L_1, G) = \text{Width}(L_1, G) = 4 \), but \( \text{Height}(L_1, G) = 3 \).

Let \( L_i = \langle x_i, x_{i+1} \rangle \) for \( i \leq 4 \). Then there are only three essentially distinct conjugates of \( L_1 \) by these elements intersect \( L_1 \) non-trivially. They are \( \{(wt^3) | w \in \mathcal{L}_2\}, \{(ut) | u \in \mathcal{L}_4\} \), and \( \{st^2 | s \in \mathcal{L}_3\} \). Just as in Example 1, the malnormality of \( L_1 \) in \( G \) implies that the essentially distinct conjugates in each family intersect each other trivially, hence \( \text{Width}(L_1, G) = 4 \). Also as in Example 1 these elements are strongly \( L_1 \)-essentially equivalent to \( t^3, t \), and \( t^2 \), respectively, so \( \text{WeakWidth}(L_1, G) = 4 \).

Since \( \text{Height}(H, G) \geq 4 \). Then there are 3 essentially distinct conjugates \( M_2, M_3, \) and \( M_4 \) of \( L_1 \) such that the intersection \( \bigcap_{i=1}^{4} L_i \) is infinite. The preceding paragraph implies that the \( M_i \)’s must come one from each of the families of conjugates of \( L_1 \) described above, i.e. \( M_2, M_3, \) and \( M_4 \) are conjugates of \( L_1 \) by \( wt^3, ut, \) and \( st^2 \) respectively, with \( w \in \mathcal{L}_2, u \in \mathcal{L}_4, \) and \( s \in \mathcal{L}_3 \).

Let \( h_1, h_2, h_3, \) and \( h_4 \) in \( L_1 \) be such that \( h_4 = t^{-3} w^{-1} h_1 w t^3 = t^{-2} s^{-1} h_2 s t^2 = t^{-1} u^{-1} h_3 u t \). Note that \( t^{-3} w \in \mathcal{L}_1, t^{-3} h_1 t^3 \in \mathcal{L}_4, t^{-2} s t^2 \in \mathcal{L}_1, t^{-2} h_2 s t^2 \in \mathcal{L}_3, t^{-1} u t \in \mathcal{L}_1, \) and \( t^{-1} h_3 t \in \mathcal{L}_2 \). Then \( t^{-3} w^{-1} h_1 w t^3 = r_1^{-1} q_1 r_1 \), with \( r_1 \in L_1 \) and \( q_1 \in L_4 \), \( t^{-2} s^{-1} h_2 s t^2 = r_2^{-1} q_2 r_2 \), with \( r_2 \in L_1 \) and \( q_2 \in L_3 \), and \( t^{-1} u^{-1} h_3 u t = r_3^{-1} q_3 r_3 \), with \( r_3 \in L_1 \) and \( q_3 \in L_2 \). As \( r_1^{-1} q_1 r_1 = r_2^{-1} q_2 r_2 = r_3^{-1} q_3 r_3 \), it follows that \( q_2 = l_1^{-1} q_1 l_1 = l_2^{-1} q_2 l_2 = l_3^{-1} q_3 l_3 \) and \( k_1, k_2, \) and \( k_3 \) are reduced. Then, as in Example 1 there exist decompositions \( l_1 = p_1 p_2 \) and
q₁ ≡ p₁q'₁p₁⁻¹ such that q₂ = (p₁p₂)⁻¹(p₁q'₁p₁⁻¹)(p₁p₂) = p₂⁻¹q'₂p₂, and p₂⁻¹q'₂p₂ is a reduced word in F.

As r₃⁻¹q₁r₁ = r₂⁻¹q₂r₂ = r₁⁻¹q₃r₃ = h₄ ∈ L₁, it follows that q₁ ∈ L₁ ∩ L₄ = < x₁, x₂ >, q₂ ∈ L₁ ∩ L₃ = < x₁, x₃ >, and q₃ ∈ L₁ ∩ L₂ = < x₂, x₃ >. As q₁ ∈ < x₁, x₂ > and q₂ ∈ < x₁, x₃ >, it follows that q₁³ = x¹ₙ for n ∈ N.

Similarly, there exist decompositions l₂ ≡ c₁c₂ and q₃ ≡ c₁q'₃c₁⁻¹ such that q₂ = (c₁c₂)⁻¹(c₁q'₃c₁⁻¹)(c₁c₂) = c₂⁻¹q'₄c₂, and c₂⁻¹q'₄c₂ is a reduced word in F. As q₃ ∈ < x₂, x₃ > and q₂ ∈ < x₁, x₃ >, it follows that q₃³ = x″ₙ for m ∈ N. Then a conjugate of q'₁ = x¹₁ is equal to a conjugate of q₃³ = x″ₙ in a free group F. This can happen only if q'₁ and q₃ are trivial, hence q₂ is trivial. Therefore, the intersection of L₁ with all three families of conjugates is trivial, so Height(L₁, G) = 3.

4. Quasiconvex subgroups of negatively curved groups have finite weak width

We will use the following notation.

Let G be a group generated by the set X*. As usual, we identify the word in X* with the corresponding element in G. Let Cayley(G) be the Cayley graph of G with respect to the generating set X*. The set of vertices of Cayley(G) is G, the set of edges of Cayley(G) is G × X*, and the edge (g, x) joins the vertex g to gx.

Definition 6. The label of the path p = (g, x₁)(gx₁, x₂) · · · (gx₁x₂ · · · xₙ₋₁, xₙ) in Cayley(G) is the word Lab(p) ≡ x₁ · · · xₙ. The length of the path p, denoted by |p|, is the number of edges forming it. The inverse of a path p is denoted by p⁻¹.

Remark 2. Let H be a K-quasiconvex subgroup of G, let η be a geodesic in Cayley(G) with Lab(η) ∈ H and let η'η'' be any decomposition of η. There exists a path c with |c| ≤ K which begins at the terminal vertex of η' such that Lab(η'c) ∈ H and Lab(cη'') ∈ H. Indeed, if η begins (and, hence, ends) at an element of H such c exists by the definition of K-quasiconvexity. In the general case, we can find such c using translation in Cayley(G).

The following result was essentially proven in [4]. We include a streamlined version of the proof.

Theorem 1. If H is a quasiconvex subgroup of a negatively curved group G, then WeakWidth(H, G) is finite.

Proof. As G is finitely generated, there exists a finite number N of elements in G of length not greater than 2K + 2δ, hence there exist at most N strongly H-essentially distinct elements {gᵢ ∈ G} such that the shortest representative of the double coset HgᵢH is not longer than 2K + 2δ. Then Lemma 1 implies that the only strongly H-essentially distinct conjugates of H which might have infinite intersection with H are the conjugates of H by the elements in the set {gᵢ|1 ≤ i ≤ N}. Therefore WeakWidth(H, G) ≤ N.

Lemma 1. Let H be a K-quasiconvex subgroup of a δ-negatively curved group G and g be an element in G. If every element of the double coset HgH is longer than 2K + 2δ, then the intersection H ∩ g⁻¹Hg is finite.

Proof. Remark 1 implies that it is sufficient to prove Lemma 1 for a shortest representative g₀ of the double coset HgH.
We will show that all the elements in the intersection $H \cap g_0^{-1}Hg_0$ are shorter than $2K + 8\delta + 2$, so that the intersection is finite, as required.

Consider $h \in H \cap g_0^{-1}Hg_0$. Let $h_0 \in H$ be such that $h = g_0^{-1}h_0g_0$. Let $p_1,p_{h_0},p_2$ and $p_h$ be geodesics in Cayley$(G)$ such that $p_1p_{h_0}p_2p_h$ is a closed path, $p_1$ (hence also $p_h$) begins at 1, $\text{Lab}(p_1) = g_0^{-1}, \text{Lab}(p_{h_0}) = h_0, \text{Lab}(p_2) = g_0,$ and $\text{Lab}(p_h) = h = g_0^{-1}h_0g_0$.

Let $v$ be a middle vertex of $p_h$ and let $q$ be the initial subpath of $p_h$ ending at $v$. As $\text{Lab}(p_h) = h \in H$, Remark [2] implies that there exists a path $s$ with $|s| \leq K$ which begins at $v$ such that $\text{Lab}(qs) \in H$. Let $t$ be a shortest path which begins at $v$ and ends at some vertex $w$ of $p_{h_0}$ and let $q'$ be the initial subpath of $p_{h_0}$ terminating at $w$. As $\text{Lab}(p_{h_0}) = h_0 \in H$, Remark [2] implies that there exists a path $s'$ with $|s'| \leq K$ which begins at $w$ such that $\text{Lab}(q's') \in H$. Then $\text{Lab}(\tilde{p}_1) = g_0 = \text{Lab}(q's')\text{Lab}(\tilde{s}'s)\text{Lab}(\tilde{s}q)$. As $\text{Lab}(q's') \in H$ and $\text{Lab}(\tilde{s}q) = \text{Lab}^{-1}(qs) \in H$, it follows that $\text{Lab}(s'ts) \in Hg_0H$.

As $Hg_0H = Hg_0H$, the assumption of Lemma [1] that any element of the double coset $Hg_0H$ is longer than $2K + 2\delta$ implies that $|s't's| > 2K + 2\delta$. But then $|t| > 2K + 2\delta - |s| - |s'| > 2\delta$, hence the distance from $v$ to $p_{h_0}$ is greater than $2\delta$.

As $G$ is $\delta$-negatively curved, a side $p_h$ of the geodesic 4-gon $p_1p_{h_0}p_2p_h$ belongs to the $2\delta$-neighborhood of the union of the other three sides, so the above discussion implies that $v$ belongs to the $2\delta$-neighborhood of $p_1 \cup p_2$. Assume that there exists a path $y$ of length less than $2\delta$ which begins at a vertex $u$ of $p_1$ and ends at $v$. Consider the decomposition $p_1 = p'_1p''_1$, where $p'_1$ ends at $u$. As $g_0 = \text{Lab}(\tilde{p}_1) = \text{Lab}(\tilde{p}'_1ys)\text{Lab}(\tilde{s}q)$ and $\text{Lab}(\tilde{s}q) \in H$, it follows that $\text{Lab}(\tilde{p}'_1ys) \in g_0H$. As $g_0$ is a shortest representative of $g_0H$, it follows that $|g_0| = |p'_1| + |p''_1| \leq |p''_1ys| = |s| + |y| + |p'_1|$. Hence $|p'_1| \leq |s| + |y| \leq K + 2\delta$, so $|q| \leq |p'_1| + |y| \leq K + 4\delta$. However, as $v$ is a middle point of $p_h$, $|h| = |p_h| \leq 2|q| + 1 < 2K + 8\delta + 2$.

Similarly, if $v$ belongs to the $2\delta$-neighborhood of $p_2$, it follows that $|h| < 2K + 8\delta + 2$, proving Lemma [1].

5. Question

Is there a simple relation between the width and the weak width?

Acknowledgment

The author would like to thank Shmuel Weinberger for his support.

References

[1] I. Agol, The virtual Haken conjecture. With an appendix by Agol, Daniel Groves, and Jason Manning. Doc. Math. 18 (2013), 1045–1087.
[2] I. Agol, D. Groves, and J. Manning, Residual finiteness, QCERF and fillings of hyperbolic groups, Geom. Topol. 13 (2009), 1043–1073.
[3] R. Gitik and E. Rips, Heights of Subgroups, MSRI preprint 027-95.
[4] R. Gitik, M. Mitra, E. Rips, and M. Sageev, Widths of Subgroups, Trans. AMS 350 (1998), 321–329.
[5] G.C. Hruska and D.T. Wise, Packing subgroups in relatively hyperbolic groups, Geom. Topol. 13 (2009), 1945–1988.

E-mail address: ritagtk@umich.edu

Department of Mathematics, University of Michigan, Ann Arbor, MI, 48109