MYERS’ TYPE THEOREMS AND SOME RELATED OSCILLATION RESULTS

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Abstract. In this paper we study the behavior of solutions of a second order differential equation. The existence of a zero and its localization allow us to get some compactness results. In particular we obtain a Myers’ type theorem even in the presence of an amount of negative curvature. The technique we use also applies to the study of spectral properties of Schrödinger operators on complete manifolds.

1. Introduction and main results

In 1941 S. B. Myers, [11], obtained his well known and celebrated compactness theorem stating that a complete $m$-dimensional Riemannian manifold $M$ is compact provided its Ricci curvature is bounded from below by a positive constant. By the way, Myers proof also permits to get sharp upper diameter estimates. Since then, this result has been widely extended and improved in several directions. For example, G.J. Galloway, [5], proved compactness and a diameter estimate for $M$ perturbing the constant lower bound for the Ricci curvature by the derivative in radial direction of some bounded function. This is relevant e.g. in the (elliptic) Ricci solitons theory, [13].

Theorem 1 (Myers-Galloway). Let $M$ be a complete Riemannian manifold. Given two different points $p, q \in M$, let $\gamma_{p,q}$ be a minimizing geodesic from $p$ to $q$ parameterized by arc length. Suppose that there exist constants $c > 0$ and $F \geq 0$ such that for each pair of points $p, q$ it holds

$$\text{Ric}(\dot{\gamma}_{p,q}, \dot{\gamma}_{p,q})|_{\gamma_{p,q}(t)} \geq c + \frac{d}{dt}(f \circ \gamma_{p,q}),$$

for some $C^1(M)$ function $f$ satisfying $\sup_{M} |f| \leq F$. Then $M$ is compact and

$$\text{diam}(M) \leq \frac{1}{c} \left[ 2F + \sqrt{4F^2 + \pi^2(m-1)c} \right].$$

Myers’ proof (and Galloway’s generalization) is based on the fact that, by the second variation formula for the length functional, given a minimizing geodesic
\( \gamma(s) : [0, a] \rightarrow M \) between two points and a smooth function \( u \) satisfying \( u(0) = u(a) = 0 \), it holds
\[
0 \leq \int_0^a \left[ \left( \frac{du}{ds} \right)^2 - \frac{\text{Ric}(\dot{\gamma}, \dot{\gamma})}{m-1} u^2(s) \right] ds.
\]

Therefore, if (2) is not satisfied for a suitable choice of \( u \) it follows that \( \gamma \) is not minimizing and repeating the argument for each \( \gamma \) gives the desired conclusion. By the way, as first pointed out by G.J. Galloway in [6], the validity of (2) and an integration by parts show that the compactness of \( M \) depends on the behavior, and in particular on the position of the zeros, of the solution of the differential equation along minimizing geodesics
\[
Ju(t) = 0,
\]
where the differential operator \( J \) is defined as
\[
Ju(t) := -u''(t) - \frac{\text{Ric}(\dot{\gamma}, \dot{\gamma})}{m-1} u(t).
\]

See also [3].

Thus we are reduced to find sufficient conditions on the Ricci curvature for which the solutions of the differential equation [9] have a first zero at finite time. At this point, usually one applies oscillation theory to get geometric assumptions to guarantee that \( M \) is compact; we refer to [15] and [6] for a more detailed discussion on oscillation theory and compactness. In particular, using a result by R. Moore, see [9], we get the following theorem. The case \( \lambda = 0 \) was previously obtained by W. Ambrose in [1]. In what follows we denote
\[
K_\gamma = \frac{\text{Ric}(\dot{\gamma}, \dot{\gamma})}{m-1}.
\]

**Theorem 2** (Ambrose-Moore). Suppose there is a point \( q \in M \) such that along each geodesic \( \gamma : [0, +\infty) \rightarrow M \) parameterized by arc length with \( \gamma(0) = q \) the condition
\[
\int_0^{\infty} t^\lambda K_\gamma(t) dt = +\infty
\]
holds for some \( 0 \leq \lambda < 1 \). Then \( M \) is compact.

Under the further assumption \( \text{Ric} \geq 0 \), condition [1] can be improved. The following result applies a Nehari’s oscillation theorem, see [12].

**Theorem 3** (Nehari). Let \( \text{Ric} \geq 0 \). Suppose there is a point \( q \in M \) such that along each geodesic \( \gamma : [0, +\infty) \rightarrow M \) parameterized by arc length with \( \gamma(0) = q \) the condition
\[
\int_{t_0}^{\infty} t^\lambda K_\gamma(t) dt > \frac{(2 - \lambda)^2}{4(1 - \lambda)} \frac{1}{t_0^{1-\lambda}}
\]
holds for some \( t_0 > 0 \) and \( 0 \leq \lambda < 1 \). Then \( M \) is compact.

As a matter of fact, as we observed above, to conclude that \( M \) is compact oscillation theory is not strictly necessary and one could improve Theorem [2] and Theorem [3] by focusing his attention upon the more general problem of the existence of a zero
for solutions of (3). To the best of our knowledge, a few steps have been done in this direction. We point out the paper [3] by E. Calabi where the same conclusion of Theorem 3 is reached under assumptions which seem to be neither weaker nor stronger than those of Nehari’s result.

**Theorem 4 (Calabi).** Let $\text{Ric} \geq 0$. Suppose there is a point $q \in M$ such that along each geodesic $\gamma : [0, +\infty) \to M$ parameterized by arc length with $\gamma(0) = q$ it holds

$$\limsup_{a \to +\infty} \left\{ \int_0^a \sqrt{K_\gamma(t)} dt - \frac{1}{2\sqrt{m-1}} \log a \right\} = +\infty.$$  

Then $M$ is compact.

Adapting the techniques introduced by Calabi, we are able to extend Theorem 2 and Theorem 3 to the case where the Ricci tensor is bounded from below by a negative constant. Namely, we obtain the following result.

**Theorem 5.** Let $\text{Ric} \geq -(m-1)B^2$, for some constant $B \geq 0$. Suppose there is a point $q \in M$ such that along each geodesic $\gamma : [0, +\infty) \to M$ parameterized by arc length, with $\gamma(0) = q$, it holds either

(5) \[ \int_a^b tK_\gamma(t)dt > B \left\{ b + a \frac{e^{2Ba} + 1}{e^{2Ba} - 1} \right\} + \frac{1}{4} \log \left( \frac{b}{a} \right). \]

or

(6) \[ \int_a^b t^\lambda K_\gamma(t)dt > B \left\{ b^\lambda + a^\lambda \frac{e^{2Ba} + 1}{e^{2Ba} - 1} \right\} + \frac{\lambda^2}{4(1-\lambda)} \left\{ a^{\lambda-1} - b^{\lambda-1} \right\} \]

for some $0 < a < b$ and $\lambda \neq 1$. Then $M$ is compact.

**Remark 6.** In case $B = 0$ the expressions in Theorem 5 have to be intended in a limit sense. Namely (5) and (6) have to be replaced respectively by

(5') \[ \int_a^b tK_\gamma(t)dt > \frac{1}{4} \log \left( \frac{b}{a} \right) \]

and

(6') \[ \int_a^b t^\lambda K_\gamma(t)dt > \frac{(2 - \lambda)^2}{4(1-\lambda)} \frac{1}{a^{1-\lambda}} - \frac{\lambda^2}{4(1-\lambda)} \frac{1}{b^{1-\lambda}} \]

Moreover we note that for $B > 0$ and $\lambda = 0$ assumption (6) has the more compact expression

(6") \[ (1 - e^{-2Ba}) \int_a^b K_\gamma(t)dt > 2B. \]

**Remark 7.** Consider a manifold $M$ and its universal covering $\tilde{M}$. Since the projection $\pi_M : \tilde{M} \to M$ is a local isometry we note that geodesics of $M$ (not necessarily minimizing) lift to geodesics of $\tilde{M}$ and Ricci curvature is preserved. Supposing we are in the assumptions of one of the theorems above, we have that also $\tilde{M}$ satisfies the same set of assumptions and so it is compact. Hence, as observed in [5], we can also conclude that the fundamental group $\pi_1(M)$ is finite.
Theorem 5 will be proved by finding lower and upper bounds for solutions of (3). This in turn permits to localize the zeros, if any. The same technique can be used to study solutions \( z(t) \) of the more general equation

\[
\begin{aligned}
(v(t)z'(t))' + W(t)v(t)z(t) &= 0 \quad \text{on } (0, +\infty) \\
z'(t) &= O(1) \quad \text{as } t \searrow 0^+, \\
z(0^+) &= z_0 > 0.
\end{aligned}
\]

In case \( v(t) \) and \( W(t) \) are nonnegative functions and satisfy very weak regularity and integrability assumptions, equation (7) has been intensively studied by B. Bianchini, L. Mari and M. Rigoli in [2]. In particular they dealt with the problem of the existence of a first zero, they studied conditions which imply oscillation and obtained an estimate on the distance of two subsequent zeros. Here we will study the case where \( W(t) \) is not necessarily nonnegative, but satisfies the request \( Wv^2 \geq -B^2 \), for some constant \( B \). First, we give an integral assumption on \( Wv \) which guarantees the existence of a first zero.

**Theorem 8.** Let \( v(t) \) and \( W(t) \) be \( L^\infty_{loc}((0, +\infty)) \) functions such that

\[
\begin{aligned}
v(t) &\geq 0, \\
v(t)^{-1} &\in L^\infty_{loc}((0, +\infty)), \\
v^{-1} &\notin L^1(0^+), \\
\lim_{t \to 0^+} v(t) &= 0
\end{aligned}
\]

and

\[
W(t) \geq -\frac{B^2}{v(t)^2}
\]

for some real constant \( B \geq 0 \). Let \( z(t) \in \text{Lip}_{loc}([0, +\infty)) \) be a solution of problem (7). If \( z(t) \neq 0 \) for all \( t \in (0, +\infty) \), then, defining \( V(t_1, t_2) := e^{2B \int_{t_1}^{t_2} ds} \) for every \( t_1, t_2 \in [0, +\infty] \), it holds

\[
\int_a^b W(s)v(s)ds \leq \begin{cases} 2B & \text{if } v^{-1} \notin L^1(+\infty) \\ 2B \frac{V(b, +\infty)}{V(0, +\infty)} & \text{if } v^{-1} \in L^1(+\infty) \end{cases}
\]

for every \( 0 \leq a < b \).

Then, iterating the technique of the proof of Theorem 8, we get an asymptotic condition providing the oscillatory behavior of \( z \).

**Theorem 9.** Let \( v, W \) and \( z \) be defined as in Theorem 8. Then \( z \) is oscillatory provided either \( v^{-1} \in L^1(+\infty) \) and

\[
\lim_{t \to \infty} \sup \left\{ \int_R^t W(s)v(s)ds \int_t^\infty \frac{ds}{v(s)} \right\} > 1
\]

for some \( R > 0 \), or \( v^{-1} \notin L^1(+\infty) \) and

\[
\lim_{t \to \infty} \sup \left\{ \frac{1}{\sup_{t \leq q_1 < q_2 \leq \infty} \int_{q_1}^{q_2} W(s)v(s)ds} \right\} > 2B.
\]

**Remark 10.** In [7], E. Hille studied the differential equation

\[
u''(t) + f(t)u(t) = 0
\]
with \( f \) a nonnegative function. In particular he defined the function \( g(x) = x \int_x^\infty f(t) \, dt \) and showed that if \( \int_0^\infty t^{\alpha-1} f(t) \, dt < \infty \) for some \( \alpha > 0 \), then \( \liminf g \leq \frac{1}{4} \) and \( \limsup g \leq 1 \). In case \( v^{-1} \in L^1(\infty) \), R. Moore, [9] adapted the first of these conditions to study equation (7), showing that \( z \) is oscillatory provided

\[
\lim_{t \to \infty} \frac{1}{\sqrt{v(t)}} \int_t^\infty \frac{v(s)}{v(t)} \, ds \geq \frac{1}{4},
\]

for some constant \( c > \frac{1}{4} \), without any sign assumption on \( W \). Up to imposing (9), condition (10) of Theorem 9 is, in a sense, a “\( \lim \sup \) counterpart” of Moore’s result.

Remark 11. When \( W \geq 0 \) (i.e. for \( B = 0 \)) and \( v^{-1} \in L^1(\infty) \), in [2] the authors defined a critical function

\[
\chi(t) := \left[ \left( \frac{1}{2} \log \int_t^\infty \frac{ds}{v(s)} \right) \right]^2
\]

and prove that \( z \) is oscillatory provided

\[
\limsup_{t \to +\infty} \int_t^T \left( \sqrt{W(s)} - \chi(s) \right) \, ds = +\infty
\]

for some constant \( T > 0 \). An easy computation shows that condition (13) is equivalent to

\[
\limsup_{t \to +\infty} \left\{ e^{2 \int_t^\infty \sqrt{W(s)} \, ds} \int_t^\infty \frac{ds}{v(s)} \right\} = +\infty.
\]

The relation between (14) and (10) is not so clear. Apparently condition (14) does not completely contain assumption (10).

Remark 12. If \( v^{-1} \notin L^1(\infty) \) we can deduce that \( z \) is oscillatory provided \( \int_0^\infty \sqrt{Wv} = +\infty \). This was obtained by W. Leighton without any sign assumption on \( W \); see [8].

Now, consider the Schrödinger operator \( L_w = \Delta + w(x) \), where \( w \in C^0(M) \) and \( \Delta \) is the Laplace-Beltrami operator on a complete non-compact Riemannian manifold \( M \). Denote with \( B_t \) the geodesic ball centered at some origin \( o \in M \), define \( v(t) = \text{Vol}(\partial B_t) \) and let \( W(t) \) be the spherical mean of the potential \( w \), that is

\[
W(t) = (\text{Vol}(\partial B_t))^{-1} \int_{\partial B_t} w(x) \, ds
\]

integrated in the \((m - 1)\)-dimensional Hausdorff measure \( ds \). By Rayleigh characterization the bottom of the spectrum on the bounded domain \( \Omega \subset M \) is defined as

\[
\lambda^L_w(\Omega) := \inf_{\varphi \in L^2(\Omega)} \frac{\int_{\Omega} |\nabla \varphi|^2 - \int_{\Omega} w \varphi^2}{\int_{\Omega} \varphi^2}
\]
Suppose there exists a solution \( z \in \text{Lip}(\Omega) \) of the problem (7) with \( z(0) = z(t_2) = 0 \) for some \( t_2 > 0 \). As in [2], we consider the function \( \varphi_z : M \to \mathbb{R} \) defined as

\[
\varphi_z(x) := \begin{cases} 
  z(r(x)) & r(x) \leq t_2 \\
  0 & r(x) > t_2,
\end{cases}
\]

where \( r(x) \) is the distance function from \( o \in M \). Then, integrating by parts,

\[
\lambda_{L^w}(B_{t_3}) \leq \frac{\int_{B_{t_3}} |\nabla \varphi_z|^2 - \int_{B_{t_3}} w\varphi_z^2}{\int_{B_{t_3}} \varphi_z^2} = \frac{\int_{0}^{t_3} v(z')^2 - \int_{0}^{t_3} Wvz^2}{\int_{0}^{t_3} vz^2} = 0,
\]

for every \( t_3 > t_2 \). By the domain monotonicity of eigenvalues we get \( \lambda_{L^w}(M) < 0 \). Moreover, with analogous computations, one can prove that the oscillation of solutions imply

\[
\lambda_{L^w}(M \setminus B_R) < 0, \quad \text{for all } R \geq 0.
\]

Recall that, given a bounded domain \( \Omega \subset M \), the index of \( L_w \) is defined as the number of negative eigenvalues of \( -L_w \). Hence conditions (15) together with a result by D. Fisher-Colbrie, [4], gives that \( L_w \) has infinite index, that is

\[
\text{ind}_{L_w}(M) := \sup_{\Omega \subset M \text{ bounded}} \{ \text{ind}_{L_w}(\Omega) \} = +\infty.
\]

This is the content of the next

**Theorem 13.** Let \( w(x) \in C^0(M) \) be defined on a complete non-compact Riemannian manifold \( M \). Let \( B \geq 0 \) be a constant and set \( v := \text{Vol}(\partial B_r) \). Suppose that the spherical mean \( W(r) \) of \( w(x) \) satisfies

\[
W(r)v^2(r) \geq -B^2
\]

for all \( r > 0 \).

i) Defining \( V(t_1, t_2) \) as in Theorem 8 then \( \lambda_{L^w}(M) < 0 \) provided there exist \( 0 < a < b \) such that

\[
\int_{B_b \setminus B_a} w(x) dx > \begin{cases} 
  2B \frac{V(b, +\infty)}{V(a, +\infty)} & \text{if } v^{-1} \notin L^1(1, +\infty) \\
  2B & \text{if } v^{-1} \in L^1(1, +\infty).
\end{cases}
\]

ii) \( L_w \) is unstable at infinity, i.e. \( \lambda_{L^w}(M \setminus B_R) < 0 \) for every \( R > 0 \), provided either \( v \in L^1(1, +\infty) \) and

\[
\lim_{t \to \infty} \left\{ \int_{B_t \setminus B_R} w(x) dx \int_{t}^{\infty} ds \frac{1}{v(s)} \right\} > 1
\]

for some \( R > 0 \), or \( v \notin L^1(1, +\infty) \) and

\[
\lim_{t \to \infty} \left\{ \sup_{t \leq r_1 < r_2 \leq t} \int_{B_{r_2} \setminus B_{r_1}} w(x) dx \right\} > 2B.
\]
In particular, in the assumption ii) \( L_w \) has infinite index.

As observed in [2], if \( s(x) \) denotes the scalar curvature of the \( m \)-dimensional manifold \((M, \langle \cdot, \cdot \rangle)\) and \( c_m = 4(m - 1)/(m - 2) \), then, setting \( w(x) = -c_m^{-1}s(x) \), the negativity of \( \lambda^L_w \) can be used to prove the existence of positive solutions \( u \) of the Yamabe equation

\[
c_m \Delta u + s(x)u - k(x)u^{\frac{m+2}{m-2}} = 0.
\]

Here \( k(x) \) is the prescribed scalar curvature of the conformally deformed metric \( \tilde{\langle \cdot, \cdot \rangle} = u^{\frac{4}{m-2}} \langle \cdot, \cdot \rangle \). Hence we obtain the following

**Theorem 14.** Suppose that the dimension of \( M \) is \( m \geq 3 \) and that the spherical mean \( S(r) \) of \( s(x) \) satisfies

\[
S(r) = c_m \frac{B^2}{\text{Vol}(\partial B_r)} , \quad r > 0,
\]

for some positive constant \( B \). Let \( k(x) \in C^\infty(M) \) be non-positive on \( M \) and strictly negative outside a compact set. Set \( \mathcal{K}_0 = k^{-1}(0) \) and

\[
\lambda^L_w(\mathcal{K}_0) = \sup_D \lambda^L_w(D),
\]

where

\[
L_w = \Delta - \frac{1}{c_m} s(x),
\]

and \( D \) varies among all open sets with smooth boundary containing \( \mathcal{K}_0 \). Suppose \( \lambda^L_w(\mathcal{K}_0) > 0 \).

Defining \( V(t_1,t_2) \) as in Theorem 8, then the background metric can be conformally deformed to a new metric of scalar curvature \( k(x) \) provided there exist \( 0 < a < b \) such that

\[
\int_{B_b \setminus B_a} (-s(x))dx > \begin{cases} 2c_m B^2 & \text{if } v^{-1} \notin L^1(+\infty) \\ 2c_m B^2 V(b,+) & \text{if } v^{-1} \in L^1(+\infty). \end{cases}
\]

2. Compactness results

Since it will be used in the sequel, observe that the existence of a solution of the Cauchy problems involved in our study is guaranteed by minor changes to Proposition A.1 in [2]. In fact both problem (7) with assumptions on the functions as in Theorem [8] and problem (3) with initial condition \( u(0) = 0 \) admit a locally Lipschitz solution globally defined in \([0, +\infty)\).

To start with, we recall the following well known lemma. For a proof avoiding the use of the second variation formula for arc-length see [14].

**Lemma 15.** Let \((M, \langle \cdot, \cdot \rangle)\) be a complete Riemannian manifold. Fix \( o \in M \) and let \( r(x) = \text{dist}(x,o) \). For any point \( q \in M \), let \( \gamma_q : [0, r(q)] \to M \) be a minimizing geodesic from \( o \) to \( q \) such that \( |\dot{\gamma}_q| = 1 \). If \( g \in \text{Lip}_{loc}(\mathbb{R}) \) is such that \( g(0) = g(r(q)) = 0 \), then for every \( q \in M \), it holds

\[
0 \leq \int_0^{r(q)} (g')^2 ds - \int_0^{r(q)} g^2K_{\gamma_q}(t)ds.
\]
Myers’ theorem shows how a positive lower bound on the Ricci curvature of $M$ suffices to conclude that $M$ is compact. Nevertheless Lemma [15] can be used to find weaker conditions for compactness. This is the content of the next theorem, due to G. J. Galloway [6].

**Theorem 16** (Galloway). Let $M$ be an $m$-dimensional complete Riemannian manifold. Suppose there exists a point $q \in M$ such that for all geodesic $\gamma : [0, +\infty) \to M$, parameterized by arc length, with $\gamma(0) = q$, the differential equation

$$J u(t) = -u''(t) - K_\gamma(t) u(t) = 0$$

has a non trivial weak solution $\tilde{u}$ with $\tilde{u}(t_1) = \tilde{u}(t_2) = 0$ for some $0 \leq t_1 < t_2$ depending on $\gamma$. Then $M$ is compact and

$$\text{diam } M \leq 2 \max_{\gamma; \gamma(0) = q} t_2.$$ 

For the sake of completeness we provide a somewhat direct proof.

**Proof.** First, we fix $\gamma$ and show that $\gamma$ stops minimizing beyond $t_2$. Without loss of generality we can suppose $\gamma$ minimizes distances on $[0,t_2]$. Moreover we can assume $t_2$ is the first zero of $\tilde{u}$ greater than $t_1$. This is well defined since $\tilde{u}(t) > 0$ on $[t_1, t_1 + \eta]$ for some $\eta$ small enough. Indeed $\tilde{u}$ is an eigenfunction of $J$ on $[t_1, t_2]$ corresponding to the eigenvalue 0. If, by contradiction $\tilde{u}(t) = 0$ on a sequence $\{t_1 + \eta_n\}_1^\infty$ for some $\eta_n \searrow 0$, it would be $\tilde{u} \equiv 0$ on $[t_1, t_1 + \eta]$ by the unique continuation principle of eigenfunctions. Hence, up to change sign, we take $\tilde{u} > 0$ on $(t_1, t_2)$. Denote the bottom of the spectrum of the operator $J$ restricted to the interval $[t_1, t_2]$ by

$$\lambda_{1,\gamma}^{-J}([t_1, t_2]) = \inf_{u \in H^2([t_1, t_2]), u(t_1) = u(t_2) = 0} \frac{\int_{t_1}^{t_2} u Ju}{\int_{t_1}^{t_2} u^2}.$$ 

For considerations above, it is $\lambda_{1,\gamma}^{-J}([t_1, t_2]) \leq 0$. On the other hand, by Lemma [15] and integrating by parts, we have that

$$\int_{t_1}^{t_2} u(t) Ju(t) dt = -\int_{t_1}^{t_2} u^2(t) K_\gamma(t) dt + \int_{t_1}^{t_2} u''(t) dt \geq 0$$

for all $0 \leq u \in \text{Lip}_{loc}(\mathbb{R})$ such that $u(t_1) = u(t_2) = 0$. In particular, replacing $\tilde{u}$ to $u$ in (17) gives that $\lambda_{1,\gamma}^{-J}([t_1, t_2]) \geq 0$. Thus $\lambda_{1,\gamma}^{-J}([t_1, t_2]) = 0$. Now, fix $\varepsilon > 0$ and define a new function $\tilde{u}_\varepsilon$ on $[t_1, t_2 + \varepsilon]$ as

$$\tilde{u}_\varepsilon(t) := \begin{cases} \tilde{u}(t) & t \in [t_1, t_2] \\ 0 & t \in [t_2, t_2 + \varepsilon]. \end{cases}$$

We have that $\tilde{u}_\varepsilon \in H^2([t_1, t_2 + \varepsilon])$ since it is $H^2$ on both $[t_1, t_2]$ and $[t_2, t_2 + \varepsilon]$ and it is $\text{Lip}_{loc}([t_1, t_2 + \varepsilon])$. This gives

$$\lambda_{1,\gamma}^{-J}([t_1, t_2 + \varepsilon]) = \inf_{u \in H^2([t_1, t_2 + \varepsilon]), u(t_1) = u(t_2 + \varepsilon) = 0} \frac{\int_{t_1}^{t_2+\varepsilon} u Ju}{\int_{t_1}^{t_2+\varepsilon} u^2} \leq \frac{\int_{t_1}^{t_2} u Ju}{\int_{t_1}^{t_2} u^2} = 0.$$
We show that the inequality is strict. By contradiction, let $\lambda^{-J}_{t}(\{t_{1}, t_{2} + \varepsilon\}) = 0$. Since $\tilde{u}$ realizes the minimum in (13), it would be an eigenfunction. Then it would be $\tilde{u}_{\varepsilon} \equiv 0$ by unique continuation. Contradiction.

Thus there exists an eigenfunction $v$ on $[t_{1}, t_{2} + \varepsilon]$ such that $v(t_{1}) = v(t_{2} + \varepsilon) = 0$, $v \geq 0$ and $Jv = \lambda^{-J}_{t}(\{t_{1}, t_{2} + \varepsilon\})v$ is non-positive and not identically 0. Applying Lemma 15 we obtain that $\gamma$ cannot minimize distances on $[t_{1}, t_{2} + \varepsilon]$, hence it stops minimizing at $t_{2}$ as claimed.

Now, fix a point $q \in M$ and let $\Gamma$ be the set of geodesics $\gamma_{q}$ parameterized by arc length such that $\gamma_{q}(0) = q$, define

$$\text{conj}(q, \gamma_{q}) := \inf_{\gamma_{q} \in \Gamma} \{ t : \gamma_{q} \text{ does not minimize on } [0, t] \}.$$ 

Set $\text{conj}(q) = \cup_{\gamma_{q}(0) = q} \text{conj}(q, \gamma)$. Since $M$ is complete, $M$ is compact provided $\text{conj}(q)$ is bounded (see [1]). This is trivial since the function $\text{conj}(q, \gamma)$ is continuous with respect to the outgoing geodesic vector $\gamma(0) \in \mathbb{S}^{m}$ by a result of Morse (Lemma 13.1 in [10]).

Finally let $p_{1}, p_{2} \in M$ and consider the geodesics $\gamma_{1}$ and $\gamma_{2}$ joining respectively $p_{1}$ and $p_{2}$ to $q$. Both $\gamma_{1}$ and $\gamma_{2}$ are shorter than $\max_{\gamma_{1}, \gamma_{2}(0) = q} t_{2}$. Hence (10) is proved because of the arbitrariness of $p_{1}$ and $p_{2}$.

In the following proofs we will use a comparison result for Riccati equations, which is a generalization of Corollary 2.2 in [14].

**Lemma 17 (Riccati Comparison).** Let $G$ and $0 < v$ be $C^{0}([0, +\infty))$ functions and let $q_{i} \in AC((\bar{t}, T_{i}))$, $i = 1, 2$, be solutions of the Riccati differential inequalities

$$q_{i}(t) - \frac{q_{i}^{2}(t)}{v(t)} - G(t) \geq 0, \quad q_{i}'(t) - \frac{q_{i}^{2}(t)}{v(t)} - G(t) \leq 0,$$

a.e. in $(\bar{t}, T_{i})$ satisfying $q_{1}(\bar{t}) = q_{2}(\bar{t})$ for some $\bar{t} > 0$. Then $T_{1} \leq T_{2}$ and $q_{1}(t) \geq q_{2}(t)$ in $[\bar{t}, T_{1})$. Conversely, if $q_{i} \in AC((T_{i}, \bar{t}))$, $i = 1, 2$, are solutions of (19) a.e. in $(T_{i}, \bar{t})$ satisfying $q_{1}(\bar{t}) = q_{2}(\bar{t})$, then $T_{1} \geq T_{2}$ and $q_{1}(t) \leq q_{2}(t)$ in $(T_{1}, \bar{t})$.

This lemma is proven with minor changes to the proof of Corollary 2.2 in [14] and we refer to this for more details.

**Proof.** Let $q_{i} \in AC((\bar{t}, T_{i}))$, $i = 1, 2$, be solutions of (19) a.e. in $(\bar{t}, T_{i})$, with $q_{1}(\bar{t}) = q_{2}(\bar{t})$. Setting $y_{i} = -q_{i}$ we obtain that

$$y_{1}'(t) + \frac{y_{1}^{2}(t)}{v(t)} + G(t) \leq 0, \quad y_{2}'(t) + \frac{y_{2}^{2}(t)}{v(t)} + G(t) \geq 0.$$ 

Let $\phi_{i} \in C^{1}([\bar{t}, T_{i}])$ be the positive function on $[\bar{t}, T_{i})$ defined by

$$\phi_{i} = \exp \left\{ \int_{\bar{t}}^{t} \left( \frac{y_{i}(s)}{v(s)} \right) ds \right\}.$$
Then $\phi_i(t) = 1$, $\phi_i > 0$ on $(\bar{t}, T_i)$, $\phi_i' \in AC(\bar{t}, T_i)$ and a straightforward computation shows that
\[
\phi_i'(t) = \frac{y_i}{v} \phi_i(t),
\]
\[
\phi_i'(\bar{t}) = \frac{y_i(\bar{t})}{v(\bar{t})} \phi_1(\bar{t}) = \frac{y_2(\bar{t})}{v(\bar{t})} \phi_2(\bar{t}) = \phi_2'(\bar{t})
\]
and
\[
(v\phi_i')' + G\phi_i \leq 0 \text{ a.e. in } (\bar{t}, T_1), \quad (v\phi_2')' + G\phi_2 \leq 0 \text{ a.e. in } (\bar{t}, T_2).
\]
Adapting the Sturm comparison result of Theorem 2.1 in [14] to the differential inequalities (22) we have that if $\phi_i \in C^1([\bar{t}, T_i])$ are solutions of (22) with the properties obtained above then
\[
\frac{\phi_1'}{\phi_1} \leq \frac{\phi_2'}{\phi_2}, \quad T_1 \leq T_2 \quad \text{and} \quad \phi_1 \leq \phi_2 \text{ on } [\bar{t}, T_1).
\]
This shows that $-q_1 = y_1 = \frac{\phi_1'}{\phi_1} v \leq \frac{\phi_2'}{\phi_2} v = y_2 = -q_2$ on $(\bar{t}, T_1)$, as required. The second part of the lemma can be proven similarly making a change of variable from $t$ to $-t$. □

We are now in the position to prove Theorem 3.

Proof. (of Theorem 3). First consider the case $B > 0$. Suppose $M$ is non compact. By Theorem 16 for each $q \in M$ there exists a geodesic $\gamma$ parameterized by arc length with $\gamma(0) = q$ such that each non trivial $\text{Lip}_{\text{loc}}$ solution $u$ of the problem
\[
\begin{cases}
u'''(t) + K_\gamma(t)u(t) = 0 \\
u(0) = 0,
\end{cases}
\]
which exists by the considerations at the beginning of this section, should satisfy $u(t) \neq 0$ for all $t > 0$. Hence the function $h(t) := -\frac{u(t)}{u'(t)}$ is well defined and continuous in $(0, +\infty)$. Moreover, since $u'' = -K_\gamma u \in L^\infty_{\text{loc}}([0, +\infty))$ implies $u'$ is locally Lipschitz, we have that $h$ satisfies the differential equation
\[
h'(t) = h^2(t) + K_\gamma(t).
\]
We want to prove that
\[
-\frac{e^{2Bt} + 1}{e^{2Bt} - 1} \leq \frac{h(t)}{B} \leq 1,
\]
for all $t > 0$. To this purpose consider the functions
\[
\hat{h}_C(t) = B\frac{C + e^{2Bt}}{C - e^{2Bt}}, \quad C \geq 1,
\]
which are solutions of the equation
\[
\hat{h}'(t) = \hat{h}^2(t) - B^2.
\]
and note that for all \( t > 0 \) the lower bound on Ricci yields \( h'(t) \geq \hat{h}_c'(t) \) each time \( h(t) = \hat{h}_c(t) \). Moreover \( h'(t), \hat{h}_c'(t) \geq 0 \) where \( |h(t)| \geq B \) and
\[
\hat{h}_c(t) \to +\infty, \quad \text{as} \ t \to (\log C/(2B))^-, \quad C > 1, \\
\hat{h}_c(t) \to -\infty, \quad \text{as} \ t \to (\log C/(2B))^+, \quad C \geq 1.
\]
By contradiction, suppose there is a value \( t_1 \) for which \( h(t_1) = H_1 > B \). Then we have that
\[
\hat{h}_c(t_1) = H_1 = h(t_1), \quad \text{for} \ C_1 = \frac{H_1 + B}{H_1 - B} e^{2Bt_1} > 1.
\]
Applying the first part of Lemma 17 with \( q_1 = h, \ q_2 = \hat{h}_c, \ G \equiv -B^2, \ v \equiv 1 \) and \( \bar{t} = t_1 \), we can conclude that \( h(t) \to +\infty \) as \( t \to t_0 \) for some \( 0 < t_0 < \frac{\log C_1}{2B} \). Thus \( h \) is not globally defined. Contradiction. Similarly, suppose there is a value \( t_2 \) for which
\[
h(t_2) = H_2 < -B e^{2Bt_2} + 1.
\]
Then we have that
\[
\hat{h}_c(t_2) = H_2 = h(t_2), \quad \text{for} \ C_2 = \frac{H_2 + B}{H_2 - B} e^{2Bt_2} > 1.
\]
As above, we achieve a contradiction by applying the second part of Lemma 17 with \( q_1 = h, \ q_2 = \hat{h}_c, \ G \equiv -B^2, \ v \equiv 1 \) and \( \bar{t} = t_2 \).

Now we want to use (23) and (24) to contradict (26). Then, for \( \lambda \neq 1 \),
\[
\int_a^b t^\lambda K_\gamma(t) dt = \int_a^b (t^\lambda h'(t) - t^\lambda h^2(t)) dt \\
= \int_a^b \left[ (t^\lambda h(t))' - t^\lambda \left( h(t) + \frac{\lambda}{2t^\lambda} \right)^2 + \frac{\lambda^2}{4} t^{-\lambda-2} \right] dt \\
\leq b^\lambda h(b) - a^\lambda h(a) + \frac{\lambda^2}{4(\lambda - 1)} \left[ b^{\lambda-1} - a^{\lambda-1} \right] \\
\leq B \left( b^\lambda + a^\lambda e^{2Ba} + 1 \right) + \frac{\lambda^2}{4(1 - \lambda)} \left[ a^{\lambda-1} - b^{\lambda-1} \right]
\]
for all \( b > a > 0 \). The case \( \lambda = 1 \) can be treated similarly. Finally observe that the computations above work even if we intend all the expressions in a limit sense as \( B \to 0 \). This concludes the proof. \( \square \)

**Remark 18.** Reasoning as in the proof of Theorem 5, we can even find diameter estimates as follows. Suppose \( \text{diam} \ M > D \). Hence by Theorem 16 there exists a geodesic \( \gamma \), with \( \gamma(0) = q \), such that \( \gamma \) is minimizing at least on \((0, D/2)\). With notations as above, we have that \( h \) has to be defined and continuous at least on \((0, D/2)\). In analogy with (24), this fact and Riccati comparison force \( h \) to satisfy
\[
-B e^{2Bt} + 1 \leq h(t) \leq B e^{2B(\frac{t}{B} - t)} + 1.
\]
This estimate, together with the fact that \( K_\gamma = h' - h^2 \), leads to obtain integral conditions on \( K_\gamma \), in the spirit of (25). For instance one can prove that\( \text{diam } M \leq D \) provided that \( 2 \int_0^{D/4} t^2 K_\gamma(t) dt > D \).

3. Oscillatory behavior and spectral applications

In this final section we give the proofs of the results concerning the behavior of solutions of problem (7) and their geometrical applications; for further details on the proof of these latter see [2].

**Proof. (of Theorem 5).** By assumption, \( z(t) \in \text{Lip}_{\text{loc}}([0, +\infty)) \) is a solution of problem (7) such that \( z(t) \neq 0 \) for all \( t \in (0, +\infty) \). Defining the function \( y(t) := -v(t)^2/z(t) \), we have that \( y \) is well defined in \( (0, +\infty) \), is locally Lipschitz by considerations as in the proof of Theorem 5 and it satisfy the differential equation

\[
\begin{cases}
y'(t) = \frac{y^2(t)}{v(t)} + W(t)v(t) \\
y(0) = 0.
\end{cases}
\]

First of all assume that \( v^{-1} \notin L^1(+\infty) \). Proceeding as in the proof of Theorem 5 we want to prove that

\[
-1 \leq \frac{y(t)}{B} \leq 1,
\]

for all \( t > 0 \). To this purpose consider the one-parameter family of functions

\[
\tilde{y}_C(t) = B \frac{C + V(1,t)}{C - V(1,t)}, \quad C > 0,
\]

which are solutions of the equation

\[
y'(t) = \frac{\tilde{y}^2(t) - B^2}{v(t)}
\]

and note that for all \( t > 0 \) the lower bound on \( W(t) \) yields \( y'(t) \geq \tilde{y}_C(t) \) each time \( y(t) = \tilde{y}_C(t) \). Moreover \( y'(t), \tilde{y}_C(t) \geq 0 \) where \( |y(t)| \geq B \) and

\[
i \lim_{t \to 0^+} \tilde{y}_C(t) = B^+; \quad ii \lim_{t \to +\infty} \tilde{y}_C(t) = -B^-; \quad iii \lim_{t \to \tilde{t}_C} \tilde{y}_C(t) = +\infty,
\]

where \( t_C \) is such that \( \int_0^{t_C} v^{-1}(s) ds = \log C/(2B) \). By contradiction, suppose there are values \( t_i, i = 1, 2 \), for which \( y(t_i) = Y_i \) with a) \( Y_1 > B \) or b) \( Y_2 < -B \). Then we have that

\[
\tilde{y}_{C_i}(t_i) = Y_i = y(t_i), \quad \text{for } C_i = \frac{Y_i + B}{Y_i - B} V(1,t_i).
\]

Choose \( q_1 = y, q_2 = \tilde{h}_{C_1} \), \( G = -B^2/v \) and \( \bar{t} = t_i \). Applying the first part of Lemma 17 for \( i = 1 \) and the second part for \( i = 2 \), we can conclude that a) yields \( y(t) \to +\infty \) as \( t \to t' \) for some \( t_1 < t' < t_{C_1} \) while b) leads to conclude that
$y(t) \to -\infty$ as $t \to t''^+$ for some $0 < t_{C,2} < t'' < t_2$. Thus $y$ is not globally defined. This contradiction implies the validity of (28), which gives

$$\int_a^b W(s)v(s)ds \leq \int_a^b y'(s)ds = y(b) - y(a) \leq 2B. \tag{31}$$

Now, let $v^{-1} \in L^1(+\infty)$. In this case the limit (30.iii) holds only for $C < V(1, +\infty)$, since otherwise $\tilde{y}_C$ is well defined all over $(0, +\infty)$. Note that also (30.ii) is satisfied with a different limit, but this has no importance to our purpose. Hence the estimate (28) gets modified in

$$-1 \leq \frac{y(t)}{B} \leq \frac{V(t, +\infty) + 1}{V(t, +\infty) - 1}, \tag{32}$$

which in turn implies

$$\int_a^b W(s)v(s)ds \leq \int_a^b y'(s)ds = y(b) - y(a) \leq \frac{2BV(b, +\infty)}{V(b, +\infty) - 1}.$$ 

Proof. (of Theorem 7). First, we assume $v^{-1} \notin L^1(+\infty)$ and consider the functions $\tilde{y}_C$ defined as in (29). By contradiction, suppose $z$ is not oscillatory. Hence there exists $T > 0$ such that $z$ has no zeros in $(T, +\infty)$, which in turn implies that the function $y(t) = -\frac{v(t)z'(t)}{z(t)}$ is globally defined in this interval. As shown in the proof of Theorem 8, this forces $y(t) \leq B$ for all $t > T$. In fact we can prove

$$-B \frac{V(T, t) + 1}{V(T, t) - 1} \leq y(t) \leq B, \quad t > T. \tag{33}$$

Indeed the RHS of (33) is exactly the function $\tilde{y}_C$ for $\tilde{C} = V(1, T)$. By (30), we get that, for $C > \tilde{C}$, $\tilde{y}_C$ is a monotone non decreasing function with a vertical asymptote in some $t_C > T$. If there exists a point $t_1 > T$ such that (33) is not verified in $t_1$, we contradict the global definition of $y$ in $(T, +\infty)$ by applying Lemma 17 as in the previous proofs.

Finally, as in (31) we get

$$\int_a^b W(s)v(s)ds \leq \frac{2BV(T, a)}{V(T, a) - 1}, \tag{34}$$

for all $b > a > T$. Hence the existence of such a $T$ contradicts (11), since RHS of (34) tends to $2B$ as $a \to \infty$.

Now, let $v^{-1} \in L^1(+\infty)$. As above, suppose $z$ has no zeros in $(T, +\infty)$ for some $T > R$. As in (32) we get

$$y(t) \leq B \frac{V(t, +\infty) + 1}{V(t, +\infty) - 1},$$

since otherwise $y(t)$ is forced to have a vertical asymptote at some finite $t_0 > t$. Moreover, reasoning exactly as in the case $v^{-1} \notin L^1(+\infty)$, we get the lower estimate

$$y(t) \geq -B \frac{V(T, t) + 1}{V(T, t) - 1},$$
for \( t > T \). This estimates in turn give

\[
\int_a^b W(s)v(s)ds \leq B \left\{ \frac{V(b, +\infty) + 1}{V(b, +\infty) - 1} + \frac{V(T, a) + 1}{V(T, a) - 1} \right\},
\]

for all \( b > a > T \). By assumption (10) there exists a \( \delta > 0 \) and a sequence \( \{t_n\}_{n=1}^\infty \) such that

\[
\int_{t_n}^{t_n+1} W(s)v(s)ds \int_{t_n}^{t_n+1} \frac{ds}{v(s)} > 1 + 3\delta
\]

for all \( n \geq 1 \). Since \( v^{-1} \in L^1(+\infty) \), there exists \( N_1 \in \mathbb{N} \) such that

\[
\int_{t_n}^{t_n+1} W(s)v(s)ds \int_{t_n}^{t_n+1} \frac{ds}{v(s)} < \delta
\]

for all \( n > N_1 \). This latter combined with (36) gives

\[
\int_{t_n}^{t_n+1} W(s)v(s)ds \int_{t_n}^{t_n+1} \frac{ds}{v(s)} > 1 + 2\delta
\]

for all \( n > N_1 \). We note that

\[
\varepsilon \sim e^\varepsilon - 1 \sim 2e^{\varepsilon} - 1 \sim \frac{e^\varepsilon}{e^\varepsilon + 1}, \quad \text{as} \ \varepsilon \to 0^+.
\]

Since \( \int_{t_n}^{t_n+1} v^{-1} \to 0 \) as \( n \to \infty \), there exists \( N_2 \in \mathbb{N} \) such that

\[
2\frac{V(t_n, +\infty) - 1}{V(t_n, +\infty) + 1} > \frac{\delta + 1}{2\delta + 1} B \int_{t_n}^{t_n+1} \frac{ds}{v(s)}
\]

for all \( n > N_2 \). Then (37) and (38) imply

\[
\frac{V(t_n, +\infty)}{V(t_n, +\infty) + 1} \int_{T+1}^{t_n} \frac{W(s)v(s)}{B}ds \geq \frac{\delta + 1}{2\delta + 1} \int_{t_n}^{t_n+1} \frac{ds}{v(s)} \int_{T+1}^{t_n} W(s)v(s)ds \to 1 + \delta
\]

for all \( n > \max \{N_1, N_2\} \). Moreover, since \( v^{-1} \in L^1(+\infty) \), (35) gives

\[
\int_{T+1}^{t_n} W(s)v(s)ds = \left( \int_{R}^{t_n} W(s)v(s)ds - \int_{R}^{T+1} W(s)v(s)ds \right) \to +\infty
\]

as \( n \to \infty \), which in turn implies there exists \( N_3 \in \mathbb{N} \) such that

\[
\frac{V(R, T+1) - 1}{V(R, T+1) + 1} \int_{T+1}^{t_n} W(s)v(s)ds > \frac{(1 + \delta)(2 + \delta)}{\delta}
\]
for all \( n > N_3 \). Choose \( a = T + 1 \) and \( b = t_n \). Combining (35), (39) and (40) we get

\[
1 \geq \int_{T+1}^{t_n} \frac{W(s)v(s)}{B} \left\{ \frac{V(t_n, +\infty) + 1}{V(t_n, +\infty) - 1} + \frac{V(R, T+1) + 1}{V(R, T+1) - 1} \right\}^{-1} ds \]

\[
= \left\{ \frac{V(t_n, +\infty) + 1}{V(t_n, +\infty) - T} \left( \int_{T+1}^{t_n} \frac{W(s)v(s)}{B} ds \right)^{-1} + \frac{V(R, T+1) + 1}{V(R, T+1) - T} \left( \int_{T+1}^{t_n} \frac{W(s)v(s)}{B} ds \right)^{-1} \right\}^{-1} \cdot \left\{ \frac{1}{1 + \delta} + \frac{\delta}{(1 + \delta)(2 + \delta)} \right\}^{-1} \]

\[
= 1 + \frac{\delta}{2} > 1
\]

for all \( n > \max\{N_1, N_2, N_3\} \). Contradiction. \(\square\)

**Proof.** (of Theorem 13). Proposition 1.2 and the considerations at the beginning of Section 2 yield assumptions (8) are satisfied and there exists a locally Lipschitz solution of (3). Then Theorem 13 is implied by Theorems 8 and 9 as in the proof of Theorem 1.4 in [2]. \(\square\)

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