The topology of multi-coupling deformations
of CFT

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Abstract: We discuss the topological properties of the manifold of cou-
pling constants for multi-coupling deformations of conformal field theo-
ries. We calculate the Euler and Betti numbers and briefly discuss physical appli-
cations of these results.

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1 Introduction

The study of the space of quantum field theories is an interesting important topic. In higher space-time dimensions not much is known about this space, (although the recent results of Seiberg and collaborators may be changing this [1]). In 1 + 1 dimensions the situation is much better, though, and all our considerations will concern $D = 2$.

Recently geometrical methods have been applied to understanding the renormalization group (RG) flow [2]. In these articles the RG flow was studied within the framework of local differential geometry. Questions concerning the global topological structure was not discussed. We are going to consider the global topological structure of the RG flow from an ultraviolet to an infrared fixed point and the relation between the topological invariants of the RG flow and the local geometrical structure. Similar ideas were previously pursued by Vafa [3], but whereas he considers the space of all 2D quantum field theories (QFTs), our considerations are restricted to the space of coupling constants relevant for the description of the flow of a QFT from its ultraviolet (UV) to its infrared (IR) fixed point.

The fixed points of the RG flow are conformal field theories (CFTs). Let us consider a two-dimensional QFT with one coupling constant. It can be viewed as a deformation of a CFT by a spinless operator $\Phi$:

$$S = S_{CFT} + \lambda \int d^2 x \Phi(x).$$

(1)

This CFT describes the UV limit of the theory if the scaling dimension of $\Phi$ is $d < 2$, so that the perturbation is by a relevant operator, and $\lambda$ has mass dimension $2 - d$. If $d = 2$ the QFT is asymptotically free and $\Phi$ is a marginal operator. The UV CFT describes the short distance behavior of an off-critical QFT. The long distance behaviour is described by an IR CFT. When the coupling constant is dimensionful, (1) can be considered effectively as a perturbation away from an IR CFT by an irrelevant operator.

The situation described above is the general one for 1+1 off-critical QFTs. Particular deformations may result in integrable systems. In that case one can solve the theory non-perturbatively within a bootstrap program [4].

The really interesting problem is multi-coupling deformations of a CFT. However, integrability is lost, e.g., already in a two-coupling deformation of the Ising model (or the tricritical Ising model) [4]. This may be traced back
to the different null-vector conditions satisfied by the perturbing relevant fields. It is an important task to find tools to study nonperturbative effects in such models. The aim of the present letter is to provide a step in this direction.

The content of the letter is as follows: In section 2 we outline the geometric interpretation of manifold of coupling constants. In section 3 we show that Zamolodchikov’s C-function can be considered as a Morse function for this manifold. We also calculate the Euler and Betti numbers. The topology of a manifold corresponding to deformations with even number of coupling constants differs from that corresponding to an odd number of deformations. In section 4 we briefly discuss some physical applications of our knowledge of the topological properties of the manifold of coupling constants. The concluding remarks are summarized in section 5.

2 The manifold of coupling constants

We study the deformation of a CFT by some finite number of relevant operators. Near the fixed point we have,

$$S = S_{CFT} + \sum_{i=1}^{N} \lambda_i \int d^2 x \Phi_i(x).$$

(2)

The operators $\Phi_i$ have dimensions $d_i$ with respect to the UV CFT. The dimensionful coupling constants can be represented as

$$\lambda_i = g_i \mu^{2-d_i},$$

(3)

where $\mu$ is a mass parameter and the $g_i$’s are dimensionless coupling constants. We will consider the action (2) to describe the RG flow from an UV CFT to an IR CFT for a QFT. Basic examples of this situation are provided by the deformations of the unitary minimal models with central charges

$$c = 1 - \frac{6}{m(m+1)}, \quad m > 1.$$  

(4)

The number of relevant operators for these models is $2(m - 2)$. 

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The appropriate objects in a renormalization group setting are in fact the running coupling constants $g_i(\mu)$ which satisfy the equations

$$\frac{\mu}{d\mu}g_i(\mu) = \beta_i(g) \quad (5)$$

$$g_i(\mu_*) = g_{i0}$$

where $g_{i0}$ are defined at some renormalization scale $\mu_*$, and $\beta_i(g)$ are the Gell-Mann-Low $\beta$ functions. We define the manifold of coupling constants $\mathcal{M}$ to be the manifold coordinatized by $\{g_i\}$. The geometry of this manifold will be defined by the renormalization of the QFT. The RG is one-parameter group of motions on $\mathcal{M}$ defined through $\beta_i(g)$.

The importance of the geometry of $\mathcal{M}$ is clear from the fact that, introducing a renormalization scale $\mu$, one can treat the correlation functions as functions on $\mathcal{M}$. Namely, consider a correlation function

$$D(p_1, p_2, ..., p_n, g_1, g_2, ..., g_N) = <O_1(p_1)O_2(p_2)O_n(p_n)>, \quad (6)$$

where $p_1, p_2, ..., p_n$ are the momenta and $O_1, O_2, ..., O_n$ some local operators. As a consequence of the RG equations this function can be rewritten as

$$D(p_1, p_2, ..., p_n, g_1, g_2, ..., g_N) = \mu^d F\left(\frac{p_1}{\mu}, \frac{p_2}{\mu}, ..., \frac{p_n}{\mu}, g_1, g_2, ..., g_N\right), \quad (7)$$

where $d$ is the full mass dimension of the correlation function. This is a consequence of the RG equations. If we fix $p_1, p_2, ..., p_n$, $F$ becomes a function on $\mathcal{M}$. This is equivalent to introducing the normalization scale $\mu$ in the correlation function

$$<O_1 O_2 ... O_n > |_{p_1=\mu, p_2=\mu, ..., p_n=\mu} \quad (8)$$

In this manner we may consider the correlation functions as functions on $\mathcal{M}$.

### 3 The C-theorem and the topology of $\mathcal{M}$

Two dimensional QFT offers a unique opportunity to study the global topology of $\mathcal{M}$ using Zamolodchikov’s C-theorem [5]. This theorem about the RG
flow for 1+1 QFT establishes that there exists a function $C(g_1, g_2, ..., g_N)$ defined on $\mathcal{M}$ which is non-increasing along the RG trajectories and is stationary at the fixed points only. There it coincides with the central charge of the corresponding CFT. The proof of the C-theorem assumes renormalizability, rotational and translational invariance, reflection positivity and conservation of the stress-energy tensor.

The first observation on the relevance of the C-theorem to the topology of 2D QFTs was made by Vafa [3] in discussing the space of all 2D QFTs. We will treat an arbitrary QFT with CFT fixed points and consider the C-function as a Morse function on the relevant $\mathcal{M}$, Note that this restricts us to the RG flow between the UV and the IR fixed points. What happens after that (branching et c) concerns other QFTs. In Morse theory knowledge of the critical points of a function on a manifold and the behavior of the function near these critical point carries information about the global topological propeties of manifold. The C-function on $\mathcal{M}$ corresponding to the QFT with action (2) has two critical points, (the UV and IR points), and at these points it satisfies the constraint

$$\frac{\partial C}{\partial g_i} = 0, \quad i = 1, 2, ..., N.$$  \hfill (9)

In a neighbourhood of the UV critical point the C-function can be represented as

$$C(g_1, g_2, ..., g_N) = c - \sum_{i=1}^{N} 3(2 - d_i)g_i^2 + O(g^3)$$ \hfill (10)

where $c$ is the central charge of the UV CFT. The Hessian for the C-function is thus

$$det\left(\frac{\partial^2 C}{\partial g_i \partial g_j}\right) = (-1)^N \prod_{i=1}^{N} 6(2 - d_i) + O(g).$$ \hfill (11)

The symmetric bilinear form in neighbourhood of critical point is

$$dC^2 = \sum_{i=1}^{N} -6(2 - d_i)dg_i^2.$$ \hfill (12)

From (11) and (12) we see that the UV critical point is nondegenerate and that the index of this point is equal to $N$.

\footnote{The index of the critical point is number of negative squares in form $d^2 C$.}
The IR critical point has no relevant directions (cf the comment below (1)). This means that the index of the IR point is 0. The nondegeneracy of the IR point is not obvious. We believe that the theory near the IR point can be effectively described through a perturbation of the IR CFT by irrelevant operators. The C-function may then be represented also in neighbourhood of the IR point as in (10) and we may conclude that this point too is non-degenerate.

Let us make some mathematical comments. For our considerations below to be valid, the manifold \( M \) and the function \( C \) must satisfy certain restrictions. The manifold \( M \) must be smooth and separable. This is true in the present case, since \( M \) has a Riemannian structure given by Zamolodchikov's metric. The function \( C \) has to be bounded from below, (in our case \( C > 0 \)), have a finite number of (nondegenerate) critical points and near every critical point there should exist an \( \epsilon > 0 \) such that the set \( \{ x \in M : c - \epsilon \leq C(x) \leq c + \epsilon \} \) is compact. Here \( c \) is the value of \( C \) at the critical point \( [8] \). This last requirement can be realized on \( M \) by a suitable coordinatization near the critical points.

We have shown that the C-function is a Morse function with two nondegenerate critical points. Let us now calculate the Euler number for \( M \),

\[
\chi(M) = (-1)^N + (-1)^0 = \begin{cases} 0, & N \text{ odd} \\ 2, & N \text{ even.} \end{cases}
\]

A relevant object for further discussing the topology of \( M \) is the following polynomial corresponding to the C-function \( K \)

\[
K(y) = y^N + 1.
\]

With every manifold one also associates a Poincaré polynomial defined by

\[
P(y) = \sum_{k=0}^{N} \dim H^k(M) y^k, \tag{15}
\]

where \( H^k(M) \) are the homology groups. Morse theory tells us that in general these two polynomials connected the following way

\[
K(y) - P(y) = (1 + y)T(y), \tag{16}
\]

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where $T(y)$ is polynomial with positive and integer coefficients. For $M$ we obtain the dimensions of the homology groups $(\dim H^k(M))$, called the Betti numbers

\[
dim H^0(M) = 1, \quad \dim H^1(M) = 0, \quad \ldots, \quad \dim H^{N-1}(M) = 0, \quad \dim H^N(M) = 1.
\]

(17)

We have found both the Euler number and the Betti numbers for $M$ from the C-function. For two dimensional QFT we thus have the exceptional situation that one can obtain information about the global RG topology from general principles.

In many interesting cases of multi-coupling deformation of CFT the manifold $M$ can be consider to be a compact, closed and connected. This is possible, e.g., when theory has an IR fixed point at finite values of the coupling constants. Also an IR point at infinite values of the coupling constants is included in considering $M$, because we can think of this manifold as compact. It is a well-known fact that if there is a smooth function $C$ with only two critical points (perhaps degenerate) then this manifold is homeomorphic to the sphere $S^n$.

4 Physical applications

Global topological invariants may be expressible in terms of local geometric quantities. A well known example is the Gauss-Bonnet theorem for two dimensional compact manifolds. We will briefly discuss an application of this theorem. Let us consider a two-coupling deformation of an UV CFT

\[
S = S_{CFT} + g_1 \mu^{2-d_1} \int d^2 x \Phi_1(x) + g_2 \mu^{2-d_2} \int d^2 x \Phi_2(x).
\]

(18)

We introduce the function (cf. (8)),

\[
G_{ij}(g_1, g_2) = x^4 < \Phi_i(x) \Phi_j(0) > |_{x^2 = x_0^2}
\]

(19)

where $x_0$ is the renormalization scale. The symmetric matrix $G_{ij}(g_1, g_2)$ is positive definite and may be thought of as a metric on $M$. For this metric

\footnote{In analogy to viewing $R^2 \times \{\infty\}$ as homeomorphic to the sphere $S^2$}
we define the curvature form $R(g_1, g_2)$. The Gauss-Bonnet theorem gives us a relation between the Euler number and the integral of the curvature form

$$\chi(\mathcal{M}) = \frac{1}{2\pi} \int_{\mathcal{M}} R = \frac{1}{2\pi} \int_{\mathcal{M}} KdS.$$  \hspace{1cm} (20)

Here $K$ is the Gaussian (extrinsic) curvature and $dS$ is the area element. For the case of a two-coupling deformation we know the Euler number and metric (19). One may then write the curvature form in terms of the metric and use the Gauss-Bonnet theorem to obtain

$$4\pi = \int_{\mathcal{M}} S(x^4 < \Phi_i(x)\Phi_j(0) > |_{x^2=\beta_0^2})dg_1 \wedge dg_2,$$  \hspace{1cm} (21)

where $S$ is function defined by

$$S(G_{ij})dg_1 \wedge dg_2 = R = KdS.$$  \hspace{1cm} (22)

We thus obtain the non-perturbative integral equality (21) for the correlation function $< \Phi_i(x)\Phi_j(0) >$. We believe that this equality can be used to study non-perturbative effects in multicoupling deformations of CFTs. The relation (21) bears a similarity to Cardy’s sum rule [7]

$$c_{UV} - c_{IR} = \frac{3}{4\pi} \int_{|x|<\epsilon} d^2xx^2 < \Theta(x)\Theta(0) >,$$  \hspace{1cm} (23)

where $c_{UV}$ and $c_{IR}$ are the central charges of the UV and IR conformal theories and $\Theta$ is the trace of the stress-energy tensor, which may be expressed in terms of the relevant operators $\Phi_i(x)$ as

$$\Theta = 2\pi \beta_i(g)\Phi_i(x).$$  \hspace{1cm} (24)

(We hope to return to the exact relation in the future).

The above discussion is immediately generalised to an arbitrary even number of coupling constants.

We introduce the Euler class $e(\mathcal{M})$ which is an element of the cohomology group $H_N(\mathcal{M}, Z)$. The Euler class is expressible in terms of the curvature form. Again, we can start from the metric $x^4 < \Phi_i(x)\Phi_j(0) > |_{x^2=\beta_0^2}$ and construct different geometric structures. For example, the integral of the Euler class over $\mathcal{M}$ gives

$$\chi(\mathcal{M}) = \int_{\mathcal{M}} e(\mathcal{M}).$$  \hspace{1cm} (25)

where the Euler number is two. We thus again find a nontrivial equality for the correlation function.
5 Conclusions

We have shown that in 1 + 1 QFT the C-function has the properties of a Morse function on the manifold coupling constants $\mathcal{M}$. If the normalization point is introduced in the correlation functions, we can treat them as functions defined on $\mathcal{M}$. The Zamolodchikov metric on $\mathcal{M}$ gives it a Riemannian structure. The Gauss-Bonnet-Chern-Avez theory relates the global topological invariant (the Euler number) to the differential-geometric structure (the Euler class) on an even dimensional manifold. Using this theorem we obtained a nonperturbative equality for correlation functions for multicoupling deformations of a CFT.

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