ANOTHER LOOK AT THE HOFER–ZEHNDER CONJECTURE

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Abstract. We give a different and simpler proof of a slightly modified (and weaker) variant of a recent theorem of Shelukhin extending Franks' "two-or-infinitely-many" theorem to Hamiltonian diffeomorphisms in higher dimensions and establishing a sufficiently general case of the Hofer–Zehnder conjecture. A few ingredients of our proof are common with Shelukhin's original argument, the key of which is Seidel's equivariant pair-of-pants product, but the new proof highlights a different aspect of the periodic orbit dynamics of Hamiltonian diffeomorphisms.

Dedicated to Claude Viterbo on the occasion of his 60th birthday

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1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction. In this paper we give a different and simpler proof of a slightly modified and weaker version of a recent theorem of Shelukhin, [Sh19a], extending Franks’ “two-or-infinitely-many” theorem, [Fr92, Fr96], to higher dimensions.

This celebrated theorem of Franks asserts that every area preserving diffeomorphism of $S^2$ has either exactly two or infinitely many periodic points. (Moreover,
in the setting of Franks’ theorem, there are also strong growth rate results; see, e.g., [FH, LeC, Ke]. A generalization of Franks’ theorem conjectured in [HZ, p. 263] is that a Hamiltonian diffeomorphism \( \varphi \) of a closed symplectic manifold has infinitely many periodic points whenever \( \varphi \) has “more than absolutely necessary” fixed points. (Hence, the title of [Sh19a] and of this paper.) The vaguely stated lower bound “more than absolutely necessary” is usually interpreted as a lower bound arising from some version of the Arnold conjecture, e.g., as the sum of the Betti numbers. For \( \mathbb{CP}^n \), the expected threshold is \( n + 1 \) regardless of the non-degeneracy assumption and, in particular, it is 2 for \( S^2 = \mathbb{CP}^1 \) as in Franks’ theorem. A slightly different interpretation of the conjecture, not directly involving the count of fixed points, is that the presence of a fixed or periodic point that is unnecessary from a homological or geometrical perspective is already sufficient to force the existence of infinitely many periodic points. We refer the reader to [GG14, Gü13, Gü14] for some results in this direction.

We note that whenever \( \varphi \) has finitely many periodic points, by passing to an iterate one can assume them to be fixed points. Furthermore, \( \varphi \) has infinitely many periodic points if and only if it has infinitely many simple, i.e., un-iterated, periodic orbits and the results are often stated in these terms. It is also worth keeping in mind that all known Hamiltonian diffeomorphisms \( \varphi \) with finitely many periodic orbits are strongly non-degenerate, i.e., \( \varphi^k \) is non-degenerate for all \( k \in \mathbb{N} \).

Volume preserving diffeomorphisms or flows with finitely many simple periodic orbits play an important role in dynamics; see, e.g., [FK] and references therein. In the Hamiltonian setting they are sometimes referred to as pseudo-rotations. Recently, symplectic topological methods have been employed to study the dynamics of pseudo-rotations and its connections with symplectic topological properties of the underlying manifold in all dimensions; see [AS, Ban, Br15b, Br15a, ÇGG19, ÇGG20, GG18a, LRS, Sh19b, Sh19c].

The original proof of Franks’ theorem utilized methods from low-dimensional dynamics, and the first purely symplectic topological proof was given in [CKRTZ]. However, that proof and also a different approach from [BH] were still strictly low-dimensional, and Shelukhin’s theorem, [Sh19a, Thm. A], is the first sufficiently general higher-dimensional variant of Franks’ theorem. (Strictly speaking, [Sh19a, Thm. A] and our Theorem 1.1 and Corollary 1.2, which are overall slightly weaker, still fall short of completely reproving Franks’ theorem in dimension two; we will discuss and compare these results in Section 1.2.) Similarly to [Sh19a], the key ingredient of our proof is Seidel’s \( \mathbb{Z}_2 \)-equivariant pair-of-pants product, [Se]. (While we use the original version of the product, [Sh19a] relies on its \( \mathbb{Z}_p \)-equivariant version from [ShZa].) Our proof also uses several simple ingredients from persistent homology theory in the form developed in [UZ] (see also [PS]), although to a much lesser degree than [Sh19a].

Finally, it is worth pointing out that Hamiltonian pseudo-rotations are extremely rare and most of the manifolds do not admit such maps. This statement is known as the Conley conjecture. The state of the art result is that the Conley conjecture holds for a manifold \((M, \omega)\) unless there exists \( A \in \pi_2(M) \) such that \( \langle c_1(TM), A \rangle > 0 \) and \( \langle \omega, A \rangle > 0 \); see [Ci, GG17]. For example, the Conley conjecture holds when \( c_1(TM)|_{\pi_2(M)} = 0 \) or when \( M \) is negative monotone. For many manifolds the conjecture is also known to hold \( C^\infty \)-generically (see [GG09]); we refer the reader to [GG15] for a detailed survey and further references.
1.2. Shelukhin’s theorem. Let $\varphi$ be a Hamiltonian diffeomorphism of a closed monotone symplectic manifold $M$. We view $\varphi$ as the time-one map in a time-dependent Hamiltonian flow and denote by $P_k(\varphi)$ the set of its $k$-periodic points, arising from contractible $k$-periodic orbits. The Hamiltonian diffeomorphism $\varphi$ is said to be $k$-perfect if $P_k(\varphi) = P_1(\varphi)$ and perfect if $\varphi$ is $k$-perfect for all $k \in \mathbb{N}$. (We refer the reader to Sections 2 for further notation and definitions used here.) We call $\varphi$ a non-degenerate pseudo-rotation over a field $\mathbb{F}$ if it is non-degenerate, perfect and the differential in the Floer complex of $\varphi$ over $\mathbb{F}$ vanishes. This condition is independent of the choice of an almost complex structure and, by Arnold’s conjecture, equivalent to that the number of 1-periodic orbits $|P_1(\varphi)|$ is equal to the sum of Betti numbers of $M$ over $\mathbb{F}$. Denote by $\beta(\varphi)$ the barcode norm of $\varphi$ over $\mathbb{F}$, i.e., the length of the maximal finite bar in the barcode of $\varphi$; see Section 4.

One of the goals of this paper is to give a simple proof to the following theorem proved in a slightly different form in [Sh19a].

**Theorem 1.1** (Shelukhin’s Theorem, [Sh19a]). Assume that $\varphi$ is strongly non-degenerate and perfect and that $\beta(\psi)$ over $\mathbb{F}_2 := \mathbb{Z}_2$ is bounded from above for all Hamiltonian diffeomorphisms $\psi$ of $M$ or at least for all iterates $\psi = \varphi^{2^k}$ (e.g., $M = \mathbb{C}P^n$). Then $\varphi$ is a pseudo-rotation.

Applying this to the iterates $\varphi^{2^k}$ we obtain

**Corollary 1.2** ([Sh19a]). Assume that $\varphi$ is strongly non-degenerate, $\beta(\varphi^{2^k})$ over $\mathbb{F}_2$ is bounded from above (e.g., $M = \mathbb{C}P^n$), and $|P_1(\varphi)|$ is strictly greater than the sum of Betti numbers of $M$ over $\mathbb{F}_2$. Then $|P_{2^k}(\varphi)| \to \infty$ as $k \to \infty$.

This theorem is proved in Section 3.2 as an easy consequence of Theorem 3.1, a new result in this paper. (However, at least on the conceptual level, our proof of that theorem is also a subset of Shelukhin’s argument, although the inclusion is rather implicit.)

In the rest of this section we discuss the conditions of Theorem 1.1 and also some of the differences between Corollary 1.2 and the original Shelukhin’s theorem, [Sh19a, Thm. A], which is in several ways more general and more precise.

First of all, the coefficient field in [Sh19a, Thm. A] is $\mathbb{Q}$ rather than $\mathbb{F}_2$ and the assertion is that $P_k(\varphi)$ contains a simple periodic orbit for every large prime $p$. As a consequence, one obtains the growth of order at least $O(k/ \log k)$ for the number of simple periodic orbits of period up to $k$. This difference stems from the fact that the main tool used in [Sh19a] is the $\mathbb{Z}_p$-equivariant pair-of-pants product introduced in [ShZa] while we rely on a somewhat simpler $\mathbb{Z}_2$-equivariant pair-of-pants product defined in [Sc]. We touch upon the $p$-iterated analogues of Theorem 1.1 and Corollary 1.2 in Remark 5.5.

Secondly, [Sh19a, Thm. A] allows for some degeneracy of $\varphi$. Namely, in the setting of Corollary 1.2, the number of 1-periodic orbits $|P_1(\varphi)|$ in the condition that $|P_1(\varphi)|$ is strictly greater than the sum of Betti numbers is replaced by

$$\sum_{x \in P_1(\varphi)} \dim_{\mathbb{F}} \text{HF}(x; \mathbb{F}),$$

where $\text{HF}(x; \mathbb{F})$ is the local Floer (co)homology of $x$ with coefficients in a field $\mathbb{F}$ (see, e.g., [GG10]); $\mathbb{F} = \mathbb{Q}$ in [Sh19a]. Note that, as a consequence, Corollary 1.2
still holds without the non-degeneracy assumption, provided that the number of 1-periodic orbits with \( \text{HF}(x; \mathbb{F}) \neq 0 \) is greater than the sum of Betti numbers. In the setting of this paper, one should take \( \mathbb{F} = \mathbb{F}_2 \) and we will further discuss the degenerate case of Theorem 1.1 and Corollary 1.2 in Section 5.2. Overall, the role of the condition that \( \text{HF}(x; \mathbb{F}) \neq 0 \) is unclear to us beyond the case of \( S^2 \). Franks’ theorem has an analogue for a certain class of symplectomorphisms of surfaces and then, interestingly, this condition becomes essential; see [Bat, GG09].

However, from our perspective, the most important difference lies in the proofs, which highlight different aspects of the dynamics and Floer theory of \( \varphi \). Our proof focuses on the behavior of the shortest bar \( \beta_{\text{min}} \) in the barcode of \( \varphi \) (rather than the longest bar \( \beta \geq \beta_{\text{min}} \)) or, to be more precise, of the shortest Floer arrow under the iteration from \( \varphi \) to \( \varphi^2 \); see Section 3.1. In particular, we show in Theorem 3.1 that when \( \varphi \) is 2-perfect the shortest arrow persists under such an iteration, although it may migrate into the equivariant domain for \( \varphi^2 \), and the length of the arrow doubles. The shortest non-equivariant arrow for \( \varphi^2 \) is at least as long as the equivariant one. Hence \( \beta_{\text{min}}(\varphi^2) \geq 2\beta_{\text{min}}(\varphi) \), and Theorem 1.1 readily follows from Theorem 3.1 applied to a sequence of period doubling iterations; see Section 3.2. The key ingredient in the proof of Theorem 3.1 is the equivariant pair-of-pants product, introduced in [Se], having a very strong non-vanishing property also proved therein (see Proposition 2.3).

Finally, a few words are due on the requirement in Theorem 1.1 and Corollary 1.2 that \( \beta(\psi) \) is bounded from above. First of all, note that while it would be sufficient to only have an upper bound on \( \beta(\psi) \) where \( \psi = \varphi^k \psi_0 \) or, as in [Sh19a, Thm. A], on \( \beta(\psi) \) where \( \psi = \varphi^p \), all relevant results proved to date are more robust and give an upper bound on \( \beta(\psi) \) for all \( \psi \). (This is the curse (and the blessing) of symplectic topological methods in dynamics: they are very robust and general, but not particularly discriminating; they often tell the same thing about all maps. There are, however, exceptions.)

The simplest manifold for which such an \( \text{a priori} \) bound is established is \( \mathbb{C}P^n \) for any coefficient field (suppressed in the notation), and the result essentially goes back to [EP]. The argument is roughly as follows. (We use here the notation and conventions from Section 2.1.) First recall that

\[
\beta(\psi) \leq \gamma(\psi).
\]

Here \( \gamma(\psi) \) is the \( \gamma \)-norm of \( \psi \) defined, using cohomology, as

\[
\gamma(\psi) = -\left( c_1(\psi) + c_1(\psi^{-1}) \right),
\]

where \( c_\alpha(\psi) \) is the spectral invariant associated with a quantum cohomology class \( \alpha \in \text{HQ}(M) \) and \( \mathbb{I} \) is the unit in the ordinary cohomology \( H(M) \) of \( M \). (We suppress the grading in the cohomology notation when it is irrelevant.) The upper bound (1.2) holds for any closed monotone symplectic manifold and its proof is similar to the proof in [Us] of the upper bound for \( \beta \) by the Hofer norm, but with continuation maps replaced by the multiplications by the image of \( \mathbb{I} \) in \( \text{HF}(\psi) \) and \( \text{HF}(\psi^{-1}) \). (We refer the reader to [KS] for some further results along these lines.) Applying the Poincaré duality in Floer cohomology (see [EP]), it is not hard to show that \( c_1(\psi^{-1}) = -c_\varpi(\psi) \) when \( N \geq n + 1 \), where \( \varpi \) is the generator of \( H^{2n}(M) \) and \( N \) is the minimal Chern number of \( M^{2n} \). In particular, this is true for \( M = \mathbb{C}P^n \) since then \( N = n + 1 \). By construction, for any two classes \( \alpha \) and \( \zeta \) in \( \text{HQ}(M) \) the spectral invariants satisfy the Lusternik–Schnirelmann inequality \( c_{\alpha+\zeta}(\psi) \geq c_\alpha(\psi) \).
Thus, from the identity \( \varpi \ast \zeta = q \mathbb{1} \) where \( \zeta \) is the generator of \( HQ^2(\mathbb{C}P^n) \), we conclude that \( c_1(\psi) \leq c_\omega(\psi) \leq c_1(\psi) + \pi \). These inequalities, combined with (1.2), show that

\[
\beta(\psi) \leq \gamma(\psi) \leq \pi
\]

for any Hamiltonian diffeomorphism \( \psi \) of \( \mathbb{C}P^n \).

A similar upper bound on \( \beta \) holds for all closed monotone manifolds \( M \) such that \( HQ^{even}(M; \mathbb{F}) \) for some field \( \mathbb{F} \) is semi-simple, i.e., splits as an algebra into a direct sum of fields. This is [Sh19a, Thm. B] and, interestingly, this result bypasses the upper bound (1.2) in its original form. In fact, \( HQ(S^2 \times S^2; \mathbb{Q}) \) is semi-simple, but \( \gamma \) is not bounded from above for \( S^2 \times S^2 \); see [Sh19a, Rmk. 7] and also [RP, Thm. 6.2.6]. We are not aware of any algebraic criteria for an \( a \ priori \) bound on the \( \gamma \)-norm. Nor do we know how large the class of monotone symplectic manifolds with semi-simple \( HQ^{even}(M; \mathbb{F}) \) is. In addition to \( \mathbb{C}F^n \) (with any \( \mathbb{F} \)), the complex Grassmannians, \( S^2 \times S^2 \), and the one point blow-up of \( \mathbb{C}P^2 \) with standard monotone symplectic structures are in this class when \( char \mathbb{F} = 0 \) (see [EP] and references therein); but \( S^2 \times S^2 \) is not for \( \mathbb{F} = \mathbb{F}_2 \).

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2. Preliminaries

2.1. Conventions and notation. For the reader’s convenience we set here our conventions and notation and briefly recall some basic definitions. The reader may want to consult this section only as needed.

Throughout this paper, the underlying symplectic manifold \((M, \omega)\) is assumed to be closed and strictly monotone, i.e., \( [\omega]|_{\pi_2(M)} = \lambda c_1(TM)|_{\pi_2(M)} \neq 0 \) for some \( \lambda > 0 \). The minimal Chern number of \( M \) is the positive generator \( N \) of the subgroup \( \langle c_1(TM), \pi_2(M) \rangle \subset \mathbb{Z} \) and the rationality constant is the positive generator \( \lambda_0 = 2N\lambda \) of the group \( \langle \omega, \pi_2(M) \rangle \subset \mathbb{R} \).

A Hamiltonian diffeomorphism \( \varphi = \varphi_H = \varphi_H^1 \) is the time-one map of the time-dependent flow \( \varphi^t = \varphi^t_H \) of a 1-periodic in time Hamiltonian \( H : S^1 \times M \to \mathbb{R} \), where \( S^1 = \mathbb{R}/\mathbb{Z} \). The Hamiltonian vector field \( X_H \) of \( H \) is defined by \( i_{X_H} \omega = -dH \). In what follows, it will be convenient to view Hamiltonian diffeomorphisms together with the path \( \varphi^t_H \), \( t \in [0, 1] \), up to homotopy with fixed end points, i.e., as elements of the universal covering of the group of Hamiltonian diffeomorphisms.

Let \( x : S^1 \to M \) be a contractible loop. A capping of \( x \) is an equivalence class of maps \( A : D^2 \to M \) such that \( A|_{S^1} = x \). Two cappings of \( x \) are equivalent if the integral of \( \omega \) (or of \( c_1(TM) \) since \( M \) is strictly monotone) over the sphere obtained by clutching the cappings is equal to zero. A capped closed curve \( \bar{x} \) is, by definition, a closed curve \( x \) equipped with an equivalence class of cappings, and the presence of capping is indicated by a bar.

The action of a Hamiltonian \( H \) on a capped closed curve \( \bar{x} = (x, A) \) is

\[
A(\bar{x}) = -\int_A \omega + \int_{S^1} H_t(x(t)) \, dt.
\]

The space of capped closed curves is a covering space of the space of contractible loops, and the critical points of \( A_H \) on this space are exactly the capped 1-periodic orbits of \( X_H \).
The $k$-periodic points of $\varphi$ are in one-to-one correspondence with the $k$-periodic orbits of $H$, i.e., of the time-dependent flow $\varphi^t$. Recall also that a $k$-periodic orbit of $H$ is called \textit{simple} if it is not iterated. A $k$-periodic orbit $x$ of $H$ is said to be \textit{non-degenerate} if the linearized return map $D\varphi^k: T_{x(0)}M \to T_{x(0)}M$ has no eigenvalues equal to one. A Hamiltonian $H$ is non-degenerate if all its 1-periodic orbits are non-degenerate. We denote the collection of capped $k$-periodic orbits of $H$ by $\mathcal{P}_k(\varphi)$.

Let $\bar{x}$ be a non-degenerate capped periodic orbit. The \textit{Conley–Zehnder index} $\mu(\bar{x}) \in \mathbb{Z}$ is defined, up to a sign, as in [Sa, SZ]. In this paper, we normalize $\mu$ so that $\mu(\bar{x}) = n$ when $x$ is a non-degenerate maximum (with trivial capping) of an autonomous Hamiltonian with small Hessian.

Fixing an almost complex structure, which will be suppressed in the notation, we denote by $(\text{CF}(\varphi), d_{FI})$ and $\text{HF}(\varphi)$ the Floer complex and cohomology of $\varphi$ over $\mathbb{F}_2 = \mathbb{Z}_2$; see, e.g., [MS, Sa]. (Throughout this paper, all complexes and cohomology groups are over $\mathbb{F}_2$.) The complex $\text{CF}(\varphi)$ is generated by the capped 1-periodic orbits $\bar{x}$ of $H$, graded by the Conley–Zehnder index, and filtered by the action. The filtration level (or the action) of a chain $\xi \in \text{CF}(\varphi)$ is defined by

$$
A(\xi) = \min\{A(\bar{x}_i)\}, \quad \text{where } \xi = \sum \bar{x}_i. \tag{2.1}
$$

(Note that the filtration depends on $H$, not just on $\varphi$, making of the notation $\text{CF}(\varphi)$ somewhat misleading.) The differential $d_{FI}$ is the upward Floer differential: it increases the action and also the index by one. The Floer complex $\text{CF}(\varphi)$ is also a finite-dimensional free module over the Novikov ring $\Lambda$. There are several choices of $\Lambda$; see, e.g., [MS]. For our purposes, it is convenient to take the field of Laurent series $\mathbb{F}_2((q))$ with $|q| = 2N$ as $\Lambda$. With this choice, $\Lambda$ naturally acts on $\text{CF}(\varphi)$ by recapping, and multiplication by $q$ corresponds to the recapping by $A \in \pi_2(M)$ with $\langle c_1(TM), A \rangle = N$. Furthermore, $\text{CF}(\varphi)$ is a finite-dimensional vector space over $\Lambda$ with a preferred basis formed by 1-periodic orbits with arbitrarily fixed capping.

Notationally, it is convenient to equip $\text{CF}(\varphi)$ with a non-degenerate $\mathbb{F}_2$-valued pairing $\langle, \rangle$ for which $\mathcal{P}_1(\varphi)$ is an orthogonal basis: $\langle \bar{x}, \bar{y} \rangle = \delta_{\bar{x}\bar{y}}$. Then, essentially by definition,

$$
d_{FI}\bar{x} = \sum \langle d_{FI}\bar{x}, \bar{y} \rangle \bar{y}.
$$

There is a canonical, grading-preserving isomorphism $\text{HF}(\varphi) \xrightarrow{\cong} \text{HQ}(M)[-n]$ where $\text{HQ}(M)$ is the quantum cohomology of $M$; see, e.g., [Sa, MS] and references therein. (Depending on the context, this is the PSS-isomorphism or the continuation map or a combination of the two.) The cohomology groups $\text{HQ}(M)$ and $\text{HF}(\varphi)$ are also modules over a Novikov ring $\Lambda$, and $\text{HQ}(M) \cong H(M) \otimes \Lambda \cong H(\varphi)$ (as a module).

The Floer complex carries a pairing

$$
\text{CF}(\varphi) \otimes \text{CF}(\varphi) \to \text{CF}(\varphi^2)[n]
$$
descending, on the level of cohomology, to the so-called \textit{pair-of-pants product}

$$
\text{HF}(\varphi) \otimes \text{HF}(\varphi) \to \text{HF}(\varphi^2)[n],
$$

which we denote by $\ast$. Thus with our conventions $|\alpha \ast \beta| = |\alpha| + |\beta| + n$. In quantum cohomology, this product corresponds to the \textit{quantum product}, also denoted by $\ast$, which makes it into a graded-commutative algebra over $\Lambda$ with unit $1$. This product is a deformation (in $q$) of the cup product: $\alpha \ast \beta = \alpha \cup \beta + O(q)$. 

2.2. Equivariant Floer cohomology and the pair-of-pants product.

2.2.1. Equivariant Floer cohomology: a brief introduction. The equivariant Floer cohomology $HF_w(\varphi^2)$, introduced in [Sc], is the homology of a certain complex $(\text{CF}_w(\varphi^2), d_w)$ called the equivariant Floer complex. As a graded $\mathbb{Z}_2$-vector space or as a $\Lambda$-module,
\[
\text{CF}_w(\varphi^2) = \text{CF}(\varphi^2)[h]
\]
where $|h| = 1$, and the differential $d_w$ has the form
\[
d_w = d_{\varphi} + h d_1 + h^2 d_2 + \ldots = d_{\varphi} + O(h).
\]
This differential is $\Lambda[h]$-linear and non-strictly action-increasing. It is roughly speaking defined as follows, mimicking Borel’s construction of the $\mathbb{Z}_2$-equivariant Morse cohomology.

Fix a family $\tilde{J}$ of 2-periodic in $t$ almost complex structures on $M$ parametrized by the unit infinite-dimensional sphere $S^\infty \subset \mathbb{R}^\infty$. Here $\mathbb{R}^\infty$ is the direct sum of infinitely many copies of $\mathbb{R}$, i.e., its elements $\xi = (\xi_0, \xi_1, \ldots)$ have only finitely many non-zero components, and $S^\infty = \{ ||\xi|| = 1 \}$ with $||\xi||^2 = \sum_k |\xi_k|^2$. The almost complex structure $\tilde{J}$ is required to satisfy the symmetry condition $\tilde{J}_{-\xi} = \tilde{J}_\xi$, where $\tilde{J}_\xi$ is obtained from $\tilde{J}_\xi$ by the time-shift $t \mapsto t + 1$. Consider the self-indexing quadratic form $f(\xi) = \sum_k k|\xi_k|^2$ on $S^\infty$ and an antipodally symmetric metric such that the natural equatorial embedding $S^\infty \to S^\infty$ given by $(\xi_0, \xi_1, \ldots) \mapsto (0, \xi_0, \ldots)$ is an isometry. (Note also that the pull back of $f$ by this embedding is $f + 1$.) The almost complex structure $\tilde{J}$ must furthermore be constant in $\xi$ near the critical points of $f$, invariant under the equatorial embedding, and satisfy a certain regularity requirement. Denote by $w_k^\pm$ the critical points of $f$ of index $k$.

Next, consider the hybrid Morse-Floer complex of $\mathcal{A} + f$ with respect to $\tilde{J}$ and the metric on $S^\infty$. This complex has pairs $(\tilde{x}, w_k^\pm)$ with $\tilde{x} \in \mathcal{P}_2(\varphi)$ as generators and carries a natural $\mathbb{Z}_2$-action, free on the generators, sending $(\tilde{x}, w_k^\pm)$ to $(\tilde{x}', w_k^\pm)$, where $\tilde{x}'$ is the time-shift of $\tilde{x}$. It is easy to see that the homology of this hybrid complex is equal to $HF(\varphi^2)$. By definition, $\text{CF}_w(\varphi^2)$ is the $\mathbb{Z}_2$-invariant part of this hybrid complex, where we write $\tilde{x} h^k$ for $(\tilde{x}, w_k^\pm)$ ($\tilde{x}'$, $w_k^\pm$). The fact that the differential is $h$-linear follows from the requirement that $f$ (up to a constant) and the auxiliary data are invariant under the equatorial embedding. Thus, in self-explanatory notation,
\[
d_k \tilde{x} = \sum \langle d_k \tilde{x}, h^k \tilde{y} \rangle \tilde{y}, \text{ where } \mu(\tilde{y}) = \mu(\tilde{x}) + 1 - k
\]
and $\langle d_k \tilde{x}, h^k \tilde{y} \rangle$ counts mod 2 the total number of continuation Floer trajectories from $\tilde{x}$ to $\tilde{y}$ along gradient lines of $f$ connecting $w_k^+$ to $w_k^-$ and from $\tilde{x}$ to $\tilde{y}'$ along gradient lines of $f$ connecting $w_k^+$ to $w_k^-$. Clearly, the complex (and hence its cohomology) is filtered by the action $\mathcal{A}$ in addition to the filtration by $\mathcal{A} + f$. On the level of (co)chains the filtration is defined similarly to (2.1), but with the powers of $h$ ignored:
\[
\mathcal{A}(\xi) = \min\{ \mathcal{A}(\tilde{x}_i) \}, \text{ where } \xi = \sum h^{m_i} \tilde{x}_i.
\]
The equivariant complex and the cohomology has natural continuation properties; see [Sc].

Example 2.1. Assume that $\varphi$ is 2-perfect and $\varphi^2$ admits a regular 1-periodic almost complex structure $\tilde{J}$, i.e., for every pair $\tilde{x}$ and $\tilde{y}$ of 2-periodic orbits the space of
Floer trajectories connecting \( \bar{x} \) to \( \bar{y} \) has dimension \( \mu(\bar{y}) - \mu(\bar{x}) \). In particular, this space is empty when \( \mu(\bar{y}) \leq \mu(\bar{x}) \), except when \( \bar{y} = \bar{x} \) and the space comprises one constant trajectory. Set \( \hat{J} = J \) to be a constant (i.e., independent of \( \xi \)) almost complex structure. Then \( \hat{J} \) is also regular and \( d_j = 0 \) for \( j \geq 1 \) since continuation trajectories for a constant homotopy are just Floer trajectories. Thus, in this case, \( HF_{eq}(\varphi^2) = HF(\varphi)[h] \) for any interval of action. These conditions are met, for instance, when \( \varphi = \varphi_H \) is generated by a \( C^2 \)-small autonomous Hamiltonian \( H \). As a consequence, for any \( \varphi \) the global cohomology \( HF_{eq}(\varphi^2) \) is not a particularly interesting object: it is simply isomorphic to \( HQ(M)[h] \) via the equivariant continuation (or the PSS map); see [Wi18a, Wi18b] for further details.

**Remark 2.2 (Polynomials vs. Formal Power Series).** One difference between our definition of \( CF_{eq}(\varphi^2) \) and the one in [Se] is that there \( CF_{eq}(\varphi^2) = CF(\varphi^2)[|h|] \); for in that setting the expansion \( d_{\varphi, \bar{x}} = \sum h^k d_k \bar{x} \) may have infinitely many non-vanishing terms. However, as already pointed out in [Se, Sect. 7], when \( M \) is strictly monotone this expansion is necessarily finite. Indeed, otherwise it would involve capped orbits \( \bar{y} \in \mathcal{P}_2(\varphi) \) with arbitrarily small index \( \mu(\bar{y}) \). However, due to monotonicity and since \( \mathcal{P}_2(\varphi) \) is finite, such orbits would eventually have action strictly smaller than that of \( \bar{x} \), which is impossible. This difference is essential for our proof as at some point in the argument we evaluate the elements of \( CF_{eq}(\varphi^2) \) at \( h = 1 \).

### 2.2.2. Equivariant pair-of-pants product.
For our purposes, the most important feature of the equivariant Floer complex is that it is the target space of the equivariant pair-of-pants product, also defined in [Se]. On the level of complexes this product is a chain map

\[
\varphi: C(Z_2; CF(\varphi) \otimes CF(\varphi)) \to CF_{eq}(\varphi^2).
\]

The domain of \( \varphi \) is the group cochain complex

\[
C(Z_2; CF(\varphi) \otimes CF(\varphi)) := CF(\varphi) \otimes CF(\varphi)[h]
\]

with the differential

\[
d_{Z_2} = d_{\varphi, \bar{x}} + h(id + \tau).
\]

Here \( \tau \) is the involution \( \tau(\bar{x} \otimes \bar{y}) = \bar{y} \otimes \bar{x} \) and the first term is induced by the Floer differential on \( CF(\varphi) \otimes CF(\varphi) \). Note also that in these formulas and throughout the paper, all tensor products are over \( \mathbb{F}_2 \) unless specified otherwise. Furthermore, we distinguish between \( \mathbb{F}_2 \) and \( Z_2 \): the former is a field and the latter is a group.

The equivariant pair-of-pants product is bilinear over \( \Lambda[h] \) and respects the action filtration. In particular, it can also be defined for a fixed action interval \( [a, b] \) in the domain and \( [2a, 2b] \) in the target, but here we will not need the filtered version of this construction. The map \( \varphi \) is a perturbation of the ordinary pair-of-pants product:

\[
\varphi(\bar{x} \otimes \bar{y}) = \bar{x} \ast \bar{y} + O(h), \tag{2.2}
\]

and the \( O(h) \) part is again polynomial in \( h \) involving only finitely many terms (depending on \( \bar{x} \) and \( \bar{y} \)).

The cohomology of the domain of \( \varphi \) is the group cohomology \( H(Z_2; CF(\varphi) \otimes CF(\varphi)) \) of \( Z_2 \) with coefficients in \( CF(\varphi) \otimes CF(\varphi) \). Thus, on the level of cohomology, the equivariant pair-of-pants product turns into a homomorphism

\[
H(Z_2; CF(\varphi) \otimes CF(\varphi)) \cong H(Z_2; HF(\varphi) \otimes HF(\varphi)) \to HF_{eq}(\varphi^2). \tag{2.3}
\]
(The first isomorphism is a consequence of the fact that $\text{CF}(\varphi) \otimes \text{CF}(\varphi)$ and $\text{HF}(\varphi) \otimes \text{HF}(\varphi)$ are equivariantly quasi-isomorphic.) The map (2.3) obviously kills the $h$-torsion in the domain; it is a deformation in $h$ of the standard pair-of-pants product due to (2.2) and is closely related to a quantum deformation of the Steenrod squares; see [Se, Wi18a, Wi18b] and also [CGG20] for a short introduction. The map (2.3) is a monomorphism modulo $h$-torsion; [Sh19a]. For symplectically aspherical manifolds, but not in the strictly monotone case, (2.3) is also onto and hence an isomorphism modulo $h$-torsion i.e., the kernel and the cokernel are torsion modules; see [Se].

On the level of complexes $\varphi$ has the following extremely important feature:

**Proposition 2.3** (Seidel’s non-vanishing theorem; [Se], Prop. 6.7). For every $\bar{x} \in \mathcal{P}_1(\varphi)$, we have

$$\varphi(\bar{x} \otimes \bar{x}) = h^m \bar{x}^2 + \ldots,$$

where $\bar{x}^2 \in \mathcal{P}_2(\varphi)$ is the second iterate of $\bar{x}$ and $m = 2\mu(\bar{x}) - \mu(\bar{x}^2) + n$ and the dots stand for a sum of capped orbits with action strictly greater than $2A(\bar{x})$.

This non-vanishing property points to a stark difference between the equivariant and non-equivariant pair-of-pants products: $\bar{x} \ast \bar{x} = \bar{x}^2 + \ldots$ only when $\mu(\bar{x}^2) = 2\mu(\bar{x}) + n$, i.e., $m = 0$ in (2.4); cf. [CGG19].

**Remark 2.4.** A generalization of the equivariant pair-of-pants product to the $p$-th iterates $\varphi^p$, where $p$ is a prime, replacing $\mathbb{Z}_2$ by $\mathbb{Z}_p$ and $F_2$ by $F_p$ is constructed in [ShZa]. This construction and the analogue of Seidel’s non-vanishing theorem for the $p$-th iterate plays a crucial role in the original proof of Shelukhin’s theorem in [Sh19a]; cf. Remark 5.5.

### 3. Floer graphs

**3.1. Main result.** The key to the statement of our main result is the following admittedly naive and obvious construction which has been used, at least on an informal level, for quite some time.

Let $\varphi$ be a non-degenerate Hamiltonian diffeomorphism of a closed monotone symplectic manifold $M$. Consider the directed graph $\Gamma(\varphi)$ whose vertices are capped fixed points of $\varphi$, and two vertices $\bar{x}$ and $\bar{y}$ are connected by an arrow (from $\bar{x}$ to $\bar{y}$) if and only if $\mu(\bar{y}) = \mu(\bar{x}) + 1$ and there is an odd number of Floer trajectories from $\bar{x}$ to $\bar{y}$, i.e., $(d_{\text{F}}, \bar{x}, \bar{y}) = 1$. The length of an arrow is the difference of actions of $\bar{y}$ and $\bar{x}$. We call $\Gamma(\varphi)$ the *Floer graph* of $\varphi$.

When $M$ is strictly monotone as is always assumed in this paper, the group $\mathbb{Z}$ acts freely on $\Gamma(\varphi)$ by simultaneous recapping, preserving the arrow length. Sometimes it is convenient to consider the *reduced Floer graph* $\tilde{\Gamma}(\varphi) := \Gamma(\varphi)/\mathbb{Z}$. The length of an arrow in $\tilde{\Gamma}(\varphi)$ is still well-defined. Note that, unless $M$ is symplectically aspherical, both $\Gamma(\varphi)$ and $\tilde{\Gamma}(\varphi)$ are infinite, but the latter has finitely many arrows. In particular, if $d_{\text{F}} \neq 0$, there exists a shortest arrow. Such an arrow might not be unique, although it is unique for a generic $\varphi$, but obviously all shortest arrows have the same length.

The *equivariant Floer graph* $\Gamma_{eq}(\varphi^2)$ of $\varphi^2$ is defined in a similar fashion. (We are assuming that $\varphi^2$ is non-degenerate, and hence $\varphi$ is also non-degenerate.) Its vertices are capped two-periodic orbits of $\varphi$. The vertices $\bar{x}$ and $\bar{y}$ are connected by an arrow if and only if $\bar{y}$ enters $d_{\text{eq}}(\bar{x})$ with non-zero coefficient. In other words,
now we do not require the index difference to be 1, and $\bar{x}$ and $\bar{y}$ are connected by an arrow if and only if $\bar{x}$ and $h^n \bar{y}$, where $m = \mu(\bar{x}) - \mu(\bar{y}) + 1$, are connected by an odd number of equivariant Floer trajectories. The length of an arrow is again the difference of actions. As in the non-equivariant case, the reduced equivariant Floer graph $\bar{\Gamma}_\varphi(\varphi^2) := \bar{\Gamma}_\varphi(\varphi^2)/\mathbb{Z}$ has only finitely many arrows, and hence the shortest arrows exist.

We note that $\Gamma(\varphi^2)$ and $\Gamma_\varphi(\varphi^2)$ (and their reduced counterparts) have the same vertices. Furthermore, since $d_{x_0} = d_{y_0} + O(h)$, every arrow in $\Gamma(\varphi^2)$ is also an arrow in $\Gamma_\varphi(\varphi^2)$, i.e., the equivariant Floer graph is obtained from its non-equivariant counterpart by adding arrows. Note that in the process the shortest arrow length can only get shorter or remain the same. Also, observe that there is a natural one-to-one map from the vertices of $\bar{\Gamma}(\varphi)$ to the vertices of $\bar{\Gamma}(\varphi^2)$ sending $\bar{x}$ to $\bar{x^2}$; likewise for un-reduced graphs. However, even when $\varphi$ is 2-perfect, this map is not onto unless $M$ is symplectically aspherical.

The main new result of the paper is the following theorem which relates the Floer graphs for $\varphi$ and its second iterate $\varphi^2$.

**Theorem 3.1.** Assume that $\varphi$ is 2-perfect and $\varphi^2$ is non-degenerate. Then $\bar{x}$ and $\bar{y}$ are connected by one of the shortest arrows in $\Gamma(\varphi)$ if and only $\bar{x^2}$ and $\bar{y^2}$ are connected by one of the shortest arrows in $\Gamma_\varphi(\varphi^2)$.

This theorem is proved in Section 5.1 after we recall in Section 4 a few relevant facts about barcodes.

**Remark 3.2 (The role of an almost complex structure).** The Floer graph of $\varphi$ depends on the choice of an almost complex structure $J$, and hence should rather be denoted by $\Gamma(\varphi, J)$. Likewise, the equivariant Floer graph depends on the parametrized almost complex structure. However, in both cases, the collection of shortest arrows is independent of this choice. This fact implicitly follows from Theorem 3.1 or can be proved directly by a continuation argument.

Note also that Floer graphs are stable under small perturbations of $\varphi$ and $J$. To be more precise, $\Gamma(\varphi, J) = \Gamma(\bar{\varphi}, J)$ whenever $\bar{\varphi}$ is sufficiently close to $\varphi$ and $\bar{J}$ is close to $J$. The same is true in the equivariant setting.

**3.2. Implications and the proof of Theorem 1.1.** Theorem 3.1 shows that when $\varphi$ is perfect, the shortest arrow (or, to be more precise, every shortest arrow) persists from $\varphi$ to $\varphi^2$, although in the process it might move to the equivariant domain. This happens exactly when the difference of indices changes: $\mu(\bar{y}) - \mu(\bar{x}) = 1$ but $\mu(\bar{y}^2) - \mu(\bar{x}^2) \neq 1$. Moreover, in this case, we necessarily have $\mu(\bar{y}^2) - \mu(\bar{x}^2) < 1$. On the other hand, if the difference of indices remains equal to one, the orbits continue to be connected by one of the shortest non-equivariant arrows.

Denote by $\beta_{\text{min}}(\varphi) = \mathcal{A}(\bar{y}) - \mathcal{A}(\bar{x})$ the length of a shortest arrow. As follows from Proposition 4.3, $\beta_{\text{min}}(\varphi)$ is exactly equal to the shortest bar in the barcode of $\varphi$. Since every non-equivariant arrow for $\varphi^2$ is also an equivariant arrow, the shortest equivariant arrow length $\beta_{\text{min}}^e(\varphi^2)$ for $\varphi^2$ does not exceed $\beta_{\text{min}}(\varphi^2)$, i.e.,

$$\beta_{\text{min}}^e(\varphi^2) \leq \beta_{\text{min}}(\varphi^2).$$

In the setting of Theorem 3.1,

$$\beta_{\text{min}}^e(\varphi^2) = \mathcal{A}(\bar{y}^2) - \mathcal{A}(\bar{x}^2) = 2\beta_{\text{min}}(\varphi).$$
We conclude that
\[ 2 \beta_{\text{min}}(\varphi^{2^k}) \leq \beta_{\text{min}}(\varphi^{2^{k+1}}) \]
as long as the iterates of \( \varphi \) remain perfect and non-degenerate, and hence
\[ 2^k \beta_{\text{min}}(\varphi) \leq \beta_{\text{min}}(\varphi^{2^k}). \]
In particular, when \( \varphi \) is perfect, the longest finite bar \( \beta(\varphi) \) (and even the shortest bar) in the barcode cannot be bounded from above for the iterates of \( \varphi \). This proves Theorem 1.1.

Remark 3.3. An interesting question that arises from Theorem 3.1 is if a shortest arrow could persist in the non-equivariant domain for all iterates \( \varphi^{2^k} \), assuming that \( \varphi \) is perfect. As discussed above, this would be the case if and only if \( \mu(\bar{y}^{2^k}) - \mu(\bar{x}^{2^k}) = 1 \) for all \( k \in \mathbb{N} \). Using a slightly simplified version of the index divisibility theorem from [GG18b] one can show that this is impossible when \( \varphi \) is replaced by a suitable iterate \( \varphi^m \). (This is non-obvious.) Passing to an iterate is apparently essential because there exist pairs of strongly non-degenerate elements \( A \) and \( B \) in \( \tilde{\text{Sp}}(2n) \) such that \( \mu(A^{2^k}) - \mu(B^{2^k}) = 1 \) for all \( k = 0, 1, 2, \ldots \).

4. A few words about the shortest bar

In this section we recall a few facts, well-known to experts, about persistent homology in the context of Hamiltonian Floer theory. All results discussed here are contained in, e.g., [UZ], although in some instances implicitly and usually in a much more general setting. A reader sufficiently familiar with the material can easily skip this section. There are, however, two points the reader might want to keep in mind. Namely, our emphasis here is on the shortest bar rather than the longest finite bar (aka the boundary depth) which is more frequently used in applications to dynamics. Secondly, our sign conventions are different from those in [UZ] due to the fact that we are working with Floer cohomology.

Consider the Floer complex \( C := CF(\varphi) \) of a non-degenerate Hamiltonian diffeomorphism \( \varphi \) of a strictly monotone symplectic manifold, equipped with the standard action filtration. Clearly, \( C \) is a finite-dimensional vector space over \( \Lambda \) and the collection of 1-periodic orbits of \( \varphi \) with fixed capping forms a basis of \( C \).

A finite set of vectors \( \xi_i \in C \) is said to be orthogonal if for any collection of coefficients \( \lambda_i \in \Lambda \) we have
\[ A(\sum \lambda_i \xi_i) = \min A(\lambda_i \xi_i). \]
(Recall that with our conventions,
\[ A(\xi) := \min A(\bar{x}_i) \text{ when } \xi = \sum \bar{x}_i; \]
see (2.1).) It is not hard to show that an orthogonal set is necessarily linearly independent over \( \Lambda \).

Example 4.1. Assume that all capped 1-periodic orbits of \( \varphi \) have distinct actions. Write \( \xi_i = \bar{x}_i + \ldots \), where the dots stand for the orbits with action strictly greater than \( \bar{x}_i \). Then it is easy to see that the set \( \xi_i \) is orthogonal if and only if the capped orbits \( \bar{x}_i \) are distinct.

Definition 4.2. A basis \( \mathcal{B} = \{\alpha_i, \eta_j, \gamma_j\} \) of \( C \) over \( \Lambda \) is said to be a singular decomposition if
It is shown in [UZ, Sections 2 and 3] that $C$ admits a singular decomposition. For the sake of brevity we omit the proof of this fact. In what follows we will order the pairs $(\eta_j, \gamma_j)$ so that

$$A(\gamma_1) - A(\eta_1) \leq A(\eta_2) - A(\gamma_2) \leq \ldots.$$  

This increasing sequence is usually referred to as the barcode of $\varphi$ (or to be more precise the collection of finite bars). The maximal entry in the sequence is called the barcode norm $\beta(\varphi)$ or the boundary depth, [Us]. The barcode is independent of the choice of a singular decomposition (see, e.g., [UZ]), but here we do not use this fact. Instead, we need the following characterization of the shortest bar $\beta_{\min} = \beta_{\min}(\varphi)$:

**Proposition 4.3 ([UZ]).** Set

$$\beta_{\min} := A(\gamma_1) - A(\eta_1).$$

Then

$$\beta_{\min} = \inf \{ A(\bar{y}) - A(\bar{x}) \mid \langle d_{F^1}\bar{x}, \bar{y} \rangle = 1 \} = \inf \{ A(d_{F^1}\xi) - A(\xi) \mid \xi \in C, \xi \neq 0 \}.$$  

Here, in the first equality, the infimum is taken over all capped 1-periodic orbits $\bar{x}$ and $\bar{y}$ such that $\bar{y}$ enters $d_{F^1}\bar{x}$ with non-zero coefficient and, in the second, over all non-zero $\xi \in C$. In particular, $\beta_{\min}(\varphi)$ is the shortest arrow length in $\Gamma(\varphi)$.

Note that the infimums in (4.2) and (4.3) are actually attained and thus can be replaced by minima, and that the proposition can be thought of as an analogue for $C$ of the Courant-Fischer minimax theorem giving a variational interpretation of the eigenvalues of a quadratic form. For the sake of completeness we include a proof of Proposition 4.3.

**Proof.** Let us denote the right-hand sides in (4.2) and (4.2) by $\beta_{\min}'$ and, respectively, $\beta_{\min}''$. We claim that $\beta_{\min}' = \beta_{\min}''$. Indeed, setting $\xi = \bar{x}$, in (4.3), it is easy to see that $\beta_{\min}' \leq \beta_{\min}''$. On the other hand, writing $\xi = \bar{x}_1 + \bar{x}_2 + \ldots$ in the order of increasing action and $d_{F^1}\xi = \sum d_{F^1}\bar{x}_i = \bar{y} + \ldots$, we observe that $\langle \bar{y}, d_{F^1}\bar{x}_i \rangle = 1$ for some $i$. Then

$$A(d_{F^1}\xi) - A(\xi) = A(\bar{y}) - A(\bar{x}_1) \geq A(\bar{y}) - A(\bar{x}_i) \geq \beta_{\min}'',$$

and thus $\beta_{\min}'' \geq \beta_{\min}'$.

Next, clearly, $\beta_{\min} \geq \beta_{\min}''$. Therefore, it remains to show that $\beta_{\min} \leq \beta_{\min}''$. To this end, let us decompose $\xi$ in the basis $\mathcal{B}$ over $\Lambda$:

$$\xi = \sum \lambda_j \eta_j + \sum \lambda_j' \gamma_j + \sum \lambda_j'' \alpha_k.$$

Then

$$d_{F^1}\xi = \sum \lambda_j \gamma_j.$$

By orthogonality,

$$A(d_{F^1}\xi) = \min \{ A(\lambda_j \gamma_j) = A(\lambda_k \gamma_k)$$

such that

$$d_{F^1}\alpha_i = 0,$$

$$d_{F^1}\eta_j = \gamma_j,$$

and $\mathcal{B}$ is orthogonal.
for some $k$, and, again by orthogonality,
\[ A(\xi) \leq \min A(\lambda_j \eta_j) \leq A(\lambda_k \eta_k). \]
Therefore,
\[ A(d_F \xi) - A(\xi) = A(\lambda_k \gamma_k) - A(\lambda_k \eta_k) \]
\[ \geq A(\gamma_k) - A(\eta_k) \]
\[ \geq A(\gamma_1) - A(\eta_1) = \beta_{\min}. \]
As a consequence, $\beta_{\min} \leq \beta''_{\min}$, which finishes the proof of the proposition. \[ \square \]

Remark 4.4. In conclusion, we point out that all results in this section are purely algebraic and extend in a straightforward way to any un-graded finite-dimensional complex over $\Lambda$ with an “action filtration” having expected properties; see [UZ].

5. Proof of theorem 3.1 and further remarks

5.1. Proof of theorem 3.1. We begin by proving the theorem under the additional background assumption that all actions and action differences for $\varphi$ and $\varphi^2$ are distinct modulo the rationality constant $\lambda_0$. Then, in the last step of the proof, we will show how to remove this extra assumption. Note that in particular this assumption guarantees that the shortest arrow is unique for $\Gamma(\varphi)$ and $\Gamma_{eq}(\varphi^2)$.

Remark 5.1. It is worth pointing out that while this background assumption is satisfied $C^\infty$-generically, it is not quite innocuous in the context of pseudo-rotations or perfect Hamiltonian diffeomorphisms. Indeed, in this case one can expect certain “resonance relations” between actions or actions and mean indices to hold; see [GK, GG09].

The proof is carried out in three steps.

Step 1: The shortest arrow for $\varphi$. In this step we simply apply the machinery from Section 4 to $CF(\varphi)$. Let $B = \{ \alpha_i, \eta_j, \gamma_j \}$ be a singular decomposition for $CF(\varphi)$ over $\Lambda$; see Definition 4.2. Due to the background assumption, the inequalities in (4.1) are strict:
\[ A(\gamma_1) - A(\eta_1) < A(\gamma_2) - A(\eta_2) < \ldots. \] (5.1)

Let us write
\[ \gamma_1 = \bar{y}_s + \ldots \quad \text{and} \quad \eta_1 = \bar{x}_s + \ldots, \]
where dots stand for higher action terms, and $\bar{x}_s$ and $\bar{y}_s$ are unique by the background assumption. Then, by definition,
\[ A(\gamma_1) = A(\bar{y}_s) \quad \text{and} \quad A(\eta_1) = A(\bar{x}_s), \]
and hence
\[ \beta_{\min} := A(\gamma_1) - A(\eta_1) = A(\bar{y}_s) - A(\bar{x}_s). \]
We claim that
\[ \langle d_F \bar{x}_s, \bar{y}_s \rangle = 1. \] (5.2)
Indeed, $\langle d_F \bar{x}, \bar{y}_s \rangle = 1$ for some $\bar{x}$ entering $\eta_1$. Then
\[ \beta_{\min} = A(\bar{y}_s) - A(\bar{x}_s) \geq A(\bar{y}_s) - A(\bar{x}) \geq \beta_{\min}. \]
It follows that the first inequality is in fact an equality and $\bar{x} = \bar{x}_s$ due to the background assumption.
Therefore, by Proposition 4.3 and (5.2), $\bar{x}_* \text{ and } \bar{y}_*$ are connected by the shortest arrow in $\Gamma(\varphi)$.

**Step 2:** The shortest arrow for $\varphi^2$. In the previous step we have shown that $\bar{x}_*$ and $\bar{y}_*$ are connected by the shortest arrow in $\text{CF}(\varphi)$. Our goal now is to prove the following key fact.

**Lemma 5.2.** The iterated orbits $\bar{x}_2^*$ and $\bar{y}_2^*$ are connected by the shortest arrow in $\Gamma_{eq}(\varphi^2)$.

Since under the background assumption the shortest arrows in $\tilde{\Gamma}(\varphi)$ and $\Gamma_{eq}(\varphi^2)$ are unique, this will establish the theorem.

**Proof of Lemma 5.2.** In the notation from Section 2.2, set

$$\hat{\alpha}_i = \wp(\alpha_i \otimes \alpha_i),$$

$$\hat{\eta}_j = h\wp(\eta_j \otimes \eta_j) + \wp(\eta_j \otimes \gamma_j),$$

$$\hat{\gamma}_j = \wp(\gamma_j \otimes \gamma_j).$$

Then, by Seidel’s non-vanishing theorem (Proposition 2.3),

$$\hat{\eta}_1 = h^m \bar{x}_2^* + \ldots$$

for some $m \geq 0$, and

$$\hat{\gamma}_1 = h^{m'} \bar{y}_2^* + \ldots$$

for some $m' \geq 0$, where the dots again stand for higher action terms.

Since $\wp$ is a chain map, i.e., $\wp \circ d_{\varphi} = d_{\varphi} \circ \wp$, we have

$$d_{\varphi} \hat{\alpha}_i = 0$$

and

$$d_{\varphi} \hat{\eta}_j = h\wp(\gamma_j \otimes \eta_j) + h\wp(\eta_j \otimes \gamma_j)$$

$$+ \wp(h\eta_j \otimes \gamma_j + h\gamma_j \otimes \eta_j)$$

$$= \hat{\gamma}_j.$$

This indicates that the collection $\mathcal{B} := \{\hat{\alpha}_i, \hat{\eta}_j, \hat{\gamma}_j\}$ can be thought of as a singular decomposition of $\text{CF}_{eq}(\varphi^2)$ with the minimal bar given by

$$A(\hat{\gamma}_1) - A(\hat{\eta}_1) = A(\bar{y}_2^*) - A(\bar{x}_2^*),$$

and, arguing similarly to Step 1, we should be able to show that $\bar{x}_2^*$ and $\bar{y}_2^*$ are connected by the shortest arrow. A minor technical difficulty that arises at this stage is that $\text{CF}_{eq}(\varphi^2)$ does not fit in with the algebraic framework of Section 4 or [UZ]. Namely, $\text{CF}_{eq}(\varphi^2)$ is not finite-dimensional over $\Lambda$; it is finite-dimensional over $\Lambda[h]$, but the latter is not a field. We circumvent this difficulty by a trick which essentially amounts to setting $h = 1$. (This is the point where our choice of working with polynomials in $h$ rather than formal power series as in [Se] is essential; cf. Remark 2.2.)

Consider the ungraded complex $\tilde{C}$ defined as follows: $\tilde{C} := \text{CF}(\varphi^2) \subset \text{CF}_{eq}(\varphi^2)$ as a vector space over $\Lambda$ with the differential $d_{eq} := d_{eq} \alpha|_{h=1}$ for $\alpha \in \tilde{C}$. Since $d_{eq}$ is $h$-linear, we have $d^2 = 0$. More formally, $\tilde{C}$ is the quotient complex in the short exact sequence of ungraded complexes

$$0 \to \text{CF}_{eq}(\varphi^2) \xrightarrow{1+h} \text{CF}_{eq}(\varphi^2) \xrightarrow{\pi} \tilde{C} \to 0$$
over $\Lambda$, where $\pi$ is the $h = 1$ evaluation map.

**Remark 5.3.** This exact sequence, for any action interval, gives rise to the exact triangle in Floer cohomology relating $H(\tilde{\mathcal{C}})$ and $HF_{eq}(\varphi^2)$ via multiplication by $1+h$. As any map of the form $id + O(h)$, this multiplication map in Floer cohomology is one-to-one, and thus

$$H(\tilde{\mathcal{C}}) \cong HF_{eq}(\varphi^2)/(1+h)HF_{eq}(\varphi^2),$$

and hence $\dim_{\mathbb{F}_2} H(\tilde{\mathcal{C}}) = \text{rk}_{\mathbb{F}_2[1]} HF_{eq}(\varphi^2)$, for any action interval. For global cohomology, $H(\tilde{\mathcal{C}}) \cong HF(\varphi^2)$ as ungraded $\Lambda$-modules by the continuation argument and Example 2.1.

Since, by construction, $\tilde{\mathcal{C}}$ is a finite-dimensional vector space over $\Lambda$, now the machinery from [UZ] applies literally; see Remark 4.4. In self-explanatory notation,

$$\langle d_{\varphi} \bar{e}, h^m \bar{z}' \rangle \neq 0 \text{ where } m = \mu(\bar{e}) - \mu(\bar{z}') + 1 \iff \langle d\bar{e}, \bar{z}' \rangle \neq 0$$

for $\bar{e}$ and $\bar{z}'$ in $\bar{P}_2(\varphi)$. Furthermore, we can also form the Floer graph for $\tilde{\mathcal{C}}$ and this graph is identical to the equivariant Floer graph $\Gamma_{eq}(\varphi^2)$.

**Claim 5.4.** The subset $\tilde{\mathcal{B}} := \pi(\tilde{\mathcal{B}})$ in $\tilde{\mathcal{C}}$ formed by $\tilde{\alpha}_i := \pi(\alpha_i)$ and $\tilde{\eta}_j := \pi(\eta_j)$ and $\tilde{\gamma}_j := \pi(\gamma_j)$ is a singular decomposition for $\tilde{\mathcal{C}}$.

Putting aside the proof of the claim, let us first show how Lemma 5.2 follows from it. Observe that

$$A(\tilde{\gamma}_j) - A(\tilde{\eta}_j) = 2(A(\gamma_j) - A(\eta_j)).$$

Indeed, set

$$\eta_j = \bar{x}_j + \ldots,$$
$$\gamma_j = \bar{y}_j + \ldots,$$

where as usual the dots stand for strictly higher action terms. (Thus $\bar{x}_* = \bar{x}_1$ and $\bar{y}_* = \bar{y}_1$.) By Seidel’s non-vanishing theorem (Proposition 2.3), we have

$$\tilde{\eta}_j = h^m \bar{x}_j^2 + \ldots,$$
$$\tilde{\gamma}_j = h^{m'} \bar{y}_j^2 + \ldots$$

for some $m_j \geq 0$ and $m'_j \geq 0$, and hence

$$\eta_j = \bar{x}_j^2 + \ldots,$$
$$\gamma_j = \bar{y}_j^2 + \ldots.$$ 

Therefore,

$$A(\tilde{\gamma}_j) - A(\tilde{\eta}_j) = A(\bar{y}_j^2) - A(\bar{x}_j^2) = 2(A(\bar{y}_j) - A(\bar{x}_j)) = 2(A(\gamma_j) - A(\eta_j)),$$

which proves (5.3).

In particular, similarly to (5.1), we have

$$A(\tilde{\gamma}_1) - A(\tilde{\eta}_1) < A(\tilde{\gamma}_2) - A(\tilde{\eta}_2) < \ldots.$$ 

Therefore,

$$\beta_{\min}(\tilde{\mathcal{C}}) := A(\tilde{\gamma}_1) - A(\tilde{\eta}_1) = A(\bar{y}_*^2) - A(\bar{x}_*^2)$$

is the shortest bar for $\tilde{\mathcal{C}}$. As in Step 1, we infer that

$$\langle d\bar{x}_*^2, \bar{y}_*^2 \rangle = 1.$$
Hence there is an arrow connecting these two orbits in the Floer graph for \( \tilde{\mathcal{C}} \) and this is the shortest arrow. The Floer graph for \( \tilde{\mathcal{C}} \) is defined similarly and in fact identical to the equivariant Floer graph \( \Gamma_\text{eq}(\varphi^2) \). Therefore, this arrow is also the shortest arrow in \( \Gamma_\text{eq}(\varphi^2) \), completing the proof of Lemma 5.2 modulo Claim 5.4.

**Proof of Claim 5.4.** Since \( \pi \) is a homomorphism of complexes, we have \( \tilde{d}\tilde{\alpha}_i = 0 \) and \( \tilde{d}\tilde{\gamma}_j = \tilde{\gamma}_j \). Therefore, we only need to show that \( \tilde{B} \) is an orthogonal basis. For this we do not need to distinguish between different types of elements of \( B \). Write \( B = \{ \xi_i \} \), where \( \xi_i = \tilde{z}_i + \ldots \) with the dots denoting the entries of strictly higher action. Then, by the definition of \( \tilde{B} \) and Seidel's non-vanishing theorem, \( \tilde{B} = \{ \tilde{\xi}_i \} \) comprises the elements

\[
\tilde{\xi}_i := \pi(\tilde{\xi}_i) = \tilde{z}_i^2 + \ldots
\]

Now, as in Example 4.1, the orthogonality for \( B \) is equivalent to that the orbits \( \tilde{z}_i \) are distinct. Similarly, the orthogonality for \( \tilde{B} \) is equivalent to that the orbits \( \tilde{z}_i^2 \) are again distinct. It follows that \( \tilde{B} \) is orthogonal if (in fact, iff) \( B \) is orthogonal which is a part of its definition. As a consequence, \( \tilde{B} \) is linearly independent over \( \Lambda \).

Finally, since \( \tilde{\mathcal{C}} = \text{CF}(\varphi^2) \) as \( \Lambda \)-modules and \( \varphi \) is 2-perfect, we have

\[
\dim_\Lambda \tilde{\mathcal{C}} = \dim_\Lambda \text{CF}(\varphi^2) = \dim_\Lambda \text{CF}(\varphi) = |B| = |\tilde{B}|,
\]

and \( \tilde{B} \) is a basis.

This concludes the proof of Lemma 5.2.

**Step 3: Removing the background assumption.** Recall that the Floer graphs \( \Gamma(\varphi) \) and \( \Gamma_{\text{eq}}(\varphi^2) \) are stable under small perturbations of \( \varphi \). With this in mind, we can replace \( \varphi \) by a \( C^\infty \)-small perturbation \( \tilde{\varphi} \) meeting the background assumption, since the latter is a \( C^\infty \)-generic condition. More precisely, one can change the action of a single orbit by a small amount (positive or negative) using a localized \( C^\infty \)-small perturbation \( \tilde{\varphi} \). Hence, given any arrow in the Floer graphs \( \tilde{\Gamma}(\varphi) \) and \( \tilde{\Gamma}_{\text{eq}}(\varphi^2) \), pick some small \( \epsilon > 0 \). Then one can apply local perturbations at the two ends to shorten its length by \( 2\epsilon \) while not changing the lengths of the remaining arrows more than \( \epsilon \). It follows that every shortest arrow in the Floer graphs \( \tilde{\Gamma}(\varphi) \) and \( \tilde{\Gamma}_{\text{eq}}(\varphi^2) \) can be perturbed into the unique shortest arrow. Now, Theorem 3.1 for \( \varphi \) follows from that theorem for \( \tilde{\varphi} \).

**Remark 5.5 (The \( \mathbb{Z}_p \)-equivariant analogue).** This argument extends with only very minor changes to the \( p \)-th iterates \( \varphi^p \), where \( p \) is a prime, proving the analogue of Theorem 3.1 for \( \mathbb{Z}_p \)-equivariant cohomology of \( \varphi^p \) over \( \mathbb{F}_p \) and relying on the results from [ShZa]; cf. Remark 2.4. As a consequence, as in the proof of Theorem 1.1, if \( \varphi \) is strongly non-degenerate, \( \beta \) is a priori bounded from above and \( |\mathcal{P}(\varphi)| \) is greater than the sum of Betti numbers of \( M \) over \( \mathbb{Q} \), then there exists a simple \( p \)-periodic orbit for every sufficiently large prime \( p \) as is shown in [Sh19a].

5.2. **Degenerate case.** Perhaps, the simplest way to extend our arguments and, in particular, Theorem 1.1 and Corollary 1.2 to include some degenerate Hamiltonian diffeomorphisms as in [Sh19a] is by bypassing Theorem 3.1 and using a somewhat less precise argument. Below we outline the key steps of this generalization, some of which again overlap with [Sh19a]. The account is deliberately brief. The main new
point here is the construction of the (equivariant) Floer graph in the degenerate case.

Assume that $\varphi$ is 2-perfect and that the second iteration is admissible: $-1$ is not an eigenvalue of $D\varphi_x$ for any $x \in \mathcal{P}_1(\varphi)$. (The latter requirement is satisfied once $\varphi$ is replaced by its sufficiently high iterate $\varphi^{2k}$.) Then, as shown in [GG10], for every $\bar{x} \in \bar{\mathcal{P}}_1(\varphi)$ we have a canonical isomorphism in local Floer cohomology:

$$HF(\bar{x}) \overset{\cong}{\to} HF(\bar{x}^2) \quad (5.4)$$

up to a shift of grading. By the Smith inequality in local Floer cohomology, which can be proved by exactly the same argument as in [Sc] (see also [CG, Sh19a]), we have $HF_{eq}(\bar{x}^2) \cong HF(\bar{x}^2)[h]$, where, strictly speaking, on the left we have the graded module associated with the $h$-adic filtration of $HF_{eq}(\bar{x}^2)$. (We expect that in this situation $d_{eq} = d_{eq}$, and hence $HF_{eq}(\bar{x}^2) \cong HF(\bar{x}^2)[h]$ literally, without passing to graded modules, but we have not been able to prove this.)

For every $\bar{x} \in \bar{\mathcal{P}}_1(\varphi)$, fix a basis $\xi_{i,\bar{x}}$ in $HF(\bar{x})$ so that this system of bases is recapping-invariant. Applying (5.4) to this system, we obtain bases $\xi_{i,\bar{x}}^{eq}$ in $HF(\bar{x}^2)$ with $\bar{x} \in \bar{P}_1(\varphi)$, and this system extends to a recapping-invariant system over the entire $P_2(\varphi)$.

We also have a recapping-invariant system of bases in $HF_{eq}(\bar{x}^2)$ arising from $\phi(\xi_{i,\bar{x}} \otimes \xi_{i,\bar{x}}) \in HF_{eq}(\bar{x}^2)$. To be more precise, it is convenient to replace the equivariant cohomology (local or global) by the homology of the ungraded complex $\mathcal{C}$ obtained by setting $h = 1$ as in the proof of Theorem 3.1. For the sake of brevity, we keep the notation $HF_{eq}$ for this cohomology suppressing the projection $\pi$ in the notation. Set $\xi_{eq}^{eq} := \phi(\xi_{i,\bar{x}} \otimes \xi_{i,\bar{x}})$. We claim that this is a basis in $HF_{eq}(\bar{x}^2)$ which is now just a vector space over $\mathbb{F}_2$. Then, extending, we get a recapping invariant family of bases over $P_2(\varphi)$.

To show that $\{\xi_{eq}^{eq}\}$ is indeed a basis, we first recall that, without changing $D\varphi_x$ and the local cohomology, $\varphi$ can be deformed near $x$ to the direct product of degenerate and totally non-degenerate maps; see [GG10, Sect. 4.5]. This essentially reduces the question to the case, which for the sake of brevity we will focus on, where $x$ is totally degenerate, i.e., all eigenvalues of $D\varphi_x$ are equal to 1 and in particular $\varphi$ can be made $C^1$-close to the identity. Furthermore, recall that $HF(\bar{x}) \cong HF(\varphi_f) \cong HM(f)$ by [Gi, Sect. 3.3 and 6], where $HM$ stands for the local Morse cohomology, $f$ is the generating function of $\varphi$ and $\varphi_f$ is the germ of the Hamiltonian diffeomorphism generated by $f$. These isomorphisms come from continuation maps and there are similar isomorphisms (equivariant and non-equivariant) for $\bar{x}^2$ and $\varphi_{2f} = \varphi^2$, where we can replace the generating function for $\varphi^2$ by $2f$; see [GG10, Sect. 4.3]. Now, as in Example 2.1 and Remark 5.3, we arrive at the continuation map identifications

$$HF_{eq}(\bar{x}^2) \cong HF(\bar{x}^2) \cong HF(\bar{x}) \cong H(Y_f), \quad (5.5)$$

where $Y_f$ is a certain topological space (the Conley index) associated with the critical point $x$ of $f$. Furthermore, the map $\alpha \mapsto \phi(\alpha \otimes \alpha)$ turns into the Steenrod square $Sq$ on $H(Y_f)$; see [Wi18a]. Thus, with these identifications in mind, $\xi_{x,i} = \xi_{\bar{x},i}$ and $\xi_{eq}^{eq} = Sq(\xi_{\bar{x},i}) = \xi_{\bar{x},i} + \ldots$, \quad (5.6)
where the dots stand for the terms of *higher degree* in $\text{H}(Y_f)$. It follows that the vectors $\xi_{x,i}^\varphi$ are linearly independent and, since $\dim_{\mathbb{F}_2} \text{HF}_\varphi(\bar{x}^2) = \dim_{\mathbb{F}_2} \text{HF}(\bar{x})$ by (5.5), this system is a basis.

The action filtration spectral sequence in Floer cohomology has $E_1 = \bigoplus_x \text{HF}(\bar{x})$ and converges to $\text{HF}(\varphi)$. With bases fixed, we can canonically collapse this spectral sequence into one complex with the same features as the ordinary Floer complex including the action filtration and cohomology equal to $\text{HF}(\varphi)$; cf. [GG19, Sect. 2.1.3 and 2.5]. This data is sufficient to define the Floer graph $\Gamma(\varphi)$ of $\varphi$ with vertices $\xi_{x,i}$. (Note that the orbits with $\text{HF}(\bar{x}) = 0$ do not contribute to $\Gamma(\varphi)$ and the graph depends on the choice of the bases $\{\xi_{x,i}\}$.) It is also worth keeping in mind that even in the non-degenerate case this graph and the complex might differ from the Floer graph as defined in Section 3 and from the Floer complex. However, they have the same formal properties as $\text{CF}(\varphi)$ and the original graph, and the resulting homology is isomorphic to the Floer cohomology $\text{HF}(\varphi)$; cf. [GG19].

A similar construction applies to $\varphi^2$ in the ordinary and equivariant settings and $\xi_{x,i}^\varphi \leftrightarrow \xi_{x,i}^{\varphi^2}$ gives rise to an action-preserving one-to-one correspondence between the vertices of $\Gamma(\varphi^2)$ and $\Gamma_{\varphi^2}(\varphi^2)$. The condition that the sum (1.1) with $\mathcal{F} = \mathbb{F}_2$ is strictly greater than the sum of Betti numbers guarantees that the graph $\Gamma(\varphi)$, and hence $\Gamma(\varphi^2)$ and $\Gamma_{\varphi^2}(\varphi^2)$, have at least one arrow.

Denote by $\beta_{\min}$ the length of the shortest arrows in a Floer graph. Our goal is to show that $\varphi$ cannot be $2^k$-perfect, where $k$ is sufficiently large, assuming an *a priori* upper bound on $\beta_{\min}(\varphi^2)$ as in Theorem 1.1. (Note that in contrast with the non-degenerate case the Floer graphs are now sensitive to small perturbations of $\varphi$ and we usually cannot make the shortest arrow unique without changing the graph unless $\dim_{\mathbb{F}_2} \text{HF}(x) = 1$ for all $x \in \mathcal{P}_1(\varphi)$.)

The equivariant pair-of-pants product $\varphi$ extends to the complexes we have constructed, and Seidel’s non-vanishing theorem takes the form

$$\varphi(\xi_{x,i} \otimes \xi_{x,i}) = \xi_{x,i}^\varphi + \ldots,$$

where now the dots stand for terms with action greater than or equal to the action of $\xi_{x,i}^\varphi$, but with the proviso that the first term enters the whole sum with non-zero coefficient. (This is a consequence of (5.6) and Seidel’s non-vanishing theorem applied to the non-degenerate part in the splitting of $\varphi$ at $x$.)

Pick one of the shortest arrows, say $\nu$, in $\Gamma_{\varphi^2}(\varphi^2)$. After recapitulating, we can ensure that the beginning of $\nu$ has the form $\xi_{x,i}^\varphi$. Using (5.7) and the facts that $\varphi$ is a chain map and $\nu$ is a shortest arrow, it is not hard to see that $\xi_{x,i}$ is the beginning of an arrow in $\Gamma(\varphi)$ whose length is at most $\beta_{\min}(\varphi)/2$. Hence,

$$2\beta_{\min}(\varphi) \leq \beta_{\min}(\varphi^2).$$

(This proves a somewhat weaker version of Theorem 3.1: every shortest equivariant arrow comes from an arrow for $\varphi$.)

On the other hand,

$$\beta_{\min}^e(\varphi^2) \leq \beta_{\min}(\varphi^2).$$

Indeed, $\dim_{\mathbb{F}_2} \text{HF}^I(\varphi^2) \geq \text{rk}_{\mathbb{F}_2[h]} \text{HF}^I_{\varphi^2}(\varphi^2)$ for any action interval $I$, as is easy to see from the $h$-adic filtration spectral sequence. Applying this to an interval tightly enclosing one of the shortest arrows in $\Gamma_{\varphi^2}(\varphi^2)$ we obtain (5.9). In fact, we expect that, as in the non-degenerate case, $\Gamma_{\varphi^2}(\varphi^2)$ incorporates all arrows of $\Gamma(\varphi^2)$ (and,
perhaps, more). This is a stronger statement than (5.9), but (5.9) is sufficient for our purposes.

Combining (5.8) and (5.9), we see that $2\beta_{\min}(\varphi) \leq \beta_{\min}(\varphi^2)$. As a consequence, $\beta_{\min}(\varphi^2) \geq 2^k \beta_{\min}(\varphi)$ as long as $\varphi$ is $2^k$-perfect. When $\beta_{\min}(\varphi^2)$ is bounded from above, this is impossible for large $k$.

We note in conclusion that in the non-degenerate case this proof reduces to an argument which does not rely on persistence homology and is ultimately simpler and more direct, although arguably less structured, than our proof of Theorem 1.1 via Theorem 3.1.

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