One-dimensional fermionic systems after interaction quenches and their description by bosonic field theories

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Abstract. We study the time evolution of two fermionic one-dimensional models (spinless fermions with nearest-neighbor repulsion and the Hubbard model) exposed to an interaction quench for short and moderate times. The method used to calculate the time dependence is a semi-numerical approach based on the Heisenberg equation of motion. We compare the results of this approach with the results obtained by bosonization implying power law behavior. Indeed, we find that power laws describe our results well, but our results raise the issue of which exponents occur. For spinless fermions, it seems that the Tomonaga–Luttinger parameters work well, which also describe the equilibrium low-energy physics. But for the Hubbard model this is not the case. Instead, we find that exponents from the bosonization around the initial state work well. Finally, we discuss what can be expected for the long-time behavior.
1. Introduction

Lately, the interest in non-equilibrium physics has risen significantly. This is due to the progress on the experimental side where many experimental setups have been developed, which can be used to study many-body systems far from equilibrium. Among these setups are femtosecond photospectroscopy [1] and cold atomic gases trapped in optical lattices [2, 3]. Due to their excellent decoupling of the atoms from any environment, the time evolution of atoms in optical lattices can be viewed as realizations of closed quantum systems. The relevant issues comprise the temporal evolution on short, intermediate and long time scales. In this paper, we focus on short and intermediate time scales.

Since these systems usually are in highly excited states involving many degrees of freedom developing quickly already on short time scales, a theoretical description is very challenging and requires new theoretical approaches. Over the last few years, techniques have been developed to deal with systems out of equilibrium. Among these are DMFT techniques [4–6], time-dependent DMRG [7, 8], light-cone renormalization [9], methods based on CUT techniques [10], on variational approaches including mean-field theories [11–13], on perturbative renormalization of Keldysh Green functions [14] and on QMC techniques [15], for a review see [16].

There are several ways to realize states far from equilibrium. A widely considered scenario is quenches, i.e. sudden changes in the intrinsic system parameters [17]. In this work, we focus on interaction quenches where the interaction $H_{\text{int}}$ is abruptly changed [10, 11, 14, 15, 18–22]. We will focus on quenches where the interaction is switched from zero to a finite value. Thus, the system is prepared initially in the ground state (GS) of a non-interacting Hamiltonian $H_0$. But the time evolution is governed by the interacting Hamiltonian $H = H_0 + H_{\text{int}}$. In this way, the system is in a highly excited state with respect to $H$.

In this work, we study the time evolution of two generic fermionic lattice models in one dimension (1D). Of course, bosonic models are investigated in other studies as well [12, 13, 19, 23–27]. 1D models play a special role in the context of thermalization because in these models an exhaustive number of conserved quantities may exist. Naturally, the existence of
conserved quantities influences the dynamics of the system strongly [28]. The first model to be studied is spinless fermions with nearest-neighbor repulsion. The quench dynamics of this system is contrasted with the quench dynamics of the second model, the 1D Hubbard model, which includes the spin degree of freedom. Both models are integrable.

In equilibrium, 1D models are tackled by many analytical and numerical methods whose review is far beyond the scope of this paper. For the purpose of the present paper, it is sufficient to note that gapless 1D models with linear dispersion at low energies can be efficiently described by bosonic field theories [29–35]. Often, the bosonic fields can be taken to be without interactions at low energies. For instance, this is the case for spinless fermions. For models including spin, sine-Gordon models include the leading bosonic interaction [33, 35]. Naturally, the question arises whether the same or similar bosonic field theories are also able to describe the non-equilibrium dynamics [36–38]. Quenches of the non-interacting bosonic models are fairly well understood by now [18, 21, 39, 40]. A set of results for the sine-Gordon model is also available [14, 23–26, 41].

The quenches considered in this work start from non-interacting Fermi seas (FSs). For these initial states the momentum distribution displays a jump at the Fermi points at ±k_F, where the occupation drops from unity to zero on passing from |k| < k_F to |k| > k_F. Exposed to the quench, the jump in the momentum distribution of the system evolves in time, which provides a sensitive probe for the quench dynamics.

We calculate the time evolution of the jump in the momentum distribution by a semi-numerical approach based on the Heisenberg equations of motion. The method is used to investigate the behavior of the two models at short and intermediate times after the quench. Here we focus on the non-oscillatory, smooth decay of the jump; thus we study not too strong quenches that remain in the metallic regime. The oscillatory behavior occurring for strong quenches into insulating phases or close to them was studied separately [42] for the sake of clarity. For the accessible times, the dynamics of the jump can be described by the non-interacting bosonic field theories. But we find evidence that the underlying parameters differ from their values at equilibrium. We discuss various scenarios for the behavior at longer times.

The paper is organized as follows. In the next section, the models under study are presented, the semi-numerical method used to calculate the time dependence of the jump is explained and the expectations from bosonic field theories are recalled. In section 3, results for the spinless fermion model and Hubbard model are shown and compared with the field theoretic expectations. Section 4 discusses the long-time behavior. In the last section, the results are summarized.

2. Models and methods

2.1. Models

The first model under study is spinless fermions

\[ H_{\text{NN}} = -J \sum_{\langle i,j \rangle} (\hat{c}_i^\dagger \hat{c}_j + \text{h.c.}) + U(t) \sum_i \hat{n}_i \hat{n}_{i+1}, \]  

(1)

Note that version 1 of preprint [44] dealt with both subjects in a very short and not yet complete way so that we have decided to split and to extend the discussion into the present article and [42].
with nearest-neighbor repulsion (NN). The operator $\hat{c}_j^\dagger (\hat{c}_j)$ creates (annihilates) a particle at site $j$ and $\hat{n}_j = \hat{c}_j^\dagger \hat{c}_j$. By a Jordan–Wigner transformation it can be mapped to an anisotropic spin $S = 1/2$ XXZ chain [44]. It is integrable by Bethe ansatz [45, 46].

The second model is the Hubbard model [47–49] comprising spin $\sigma = \uparrow, \downarrow$

$$H_{\text{Hu}} = -J \sum_{\langle i,j \rangle; \sigma} (\hat{c}_{i,\sigma}^\dagger \hat{c}_{j,\sigma} + \text{h.c.}) + U(t) \sum_i \hat{n}_{i,\uparrow} \hat{n}_{i,\downarrow},$$

(2)

which is governed by the hopping $J$ and the local repulsion $U$. This model as well is exactly solvable by Bethe ansatz [50, 51].

We focus on quenches where the interaction is switched on $U(t) = \Theta(t)U \geq 0$ abruptly and consider FSs as initial states, i.e. the GS for $U = 0$. Throughout, the band width $W = 4J$ is used as natural energy scale and consequently time is measured in the inverse band width $1/W$ since we set $\hbar$ to unity.

2.2. Methods

Below some quantities with spin $\sigma$ are denoted for the Hubbard model. In the corresponding quantities without spin, i.e. for the spinless fermion model, the subscript $\sigma$ is to be omitted.

For $U = 0$, the momentum distribution shows a jump at the Fermi momentum $k = k_F$. Under the influence of the quench the jump

$$\Delta_n(t) = \lim_{k \to k_F^+} n_{k,\sigma}(t) - \lim_{k \to k_F^-} n_{k,\sigma}(t)$$

(3)

is reduced.

The momentum distribution of the system can be calculated by the Fourier transformation of the one-particle correlation function

$$G_{\sigma}(\vec{r}, t) = \langle 0 | \hat{c}_{\sigma}(\vec{r}, t) \hat{c}_{\sigma}(0, t)^\dagger | 0 \rangle,$$

(4)

where the expectation value is taken with respect to the non-interacting FS $|0\rangle$. Thus, the time dependence of the operators $\hat{c}_{\sigma}$ and $\hat{c}_{\sigma}^\dagger$ is needed.

To capture the time evolution of these operators, we use the the following ansatz [21]:

$$\hat{c}_{\sigma}^\dagger(\vec{r}, t) = \hat{T}_{\vec{r}}^\dagger + \hat{T}_{\vec{r}} \left( \hat{T}^\dagger_{\vec{r}} \hat{L}^\dagger_{\vec{r}} \right) + \cdots$$

(5)

with $\hat{T}^\dagger (\hat{L}^\dagger)$ denoting a general superposition of particle (hole) creation operators. For instance, $\hat{T}_{\vec{r}}^\dagger$ is given by

$$\hat{T}_{\vec{r}}^\dagger = \sum_{|\delta| \lesssim v_{\text{max}}t} \sum_{\sigma} h_0(\delta, t) \hat{c}_{\vec{r}+\delta,\sigma}^\dagger,$$

(6)

where the creation operators acting on site $\vec{r} + \delta$ are summed weighted with the prefactor $h_0(\delta, t)$. The shifts $\delta$ are roughly bounded by the distance over which quasi-particles can move in the time $t$, which is given by $v_{\text{max}}t$. This is the light-cone effect [52], first derived by Lieb and Robinson [53]. The effect of correlations beyond this bound is exponentially small. The more complex terms $\hat{T}_{\vec{r}}^\dagger (\hat{T}^\dagger \hat{L}^\dagger)$ are given by the superpositions of two creation and one annihilation operator. Another particle–hole pair $T^\dagger L^\dagger$ adds another creation and annihilation operator and so on.
The prefactors \( h(\vec{\delta}, t) \) contain the whole time dependence of the operators \( \hat{T}_r^\dagger \) and \( \hat{L}_r^\dagger \) and thus they determine the time dependence of \( \hat{c}^\dagger(\vec{r}, t) \). To calculate the time dependence of the prefactors, we use the equation of motion

\[
\partial_t \hat{A}(\vec{r}, t) = i \left[ \hat{H}, \hat{A}(\vec{r}, t) \right]
\]

for the time derivative of any operator \( \hat{A} \). The advantage of dealing with the time dependence for the operators instead of the one for the quantum states is the dependence on the size of the system. For dealing with time-dependent states, we would be obliged to treat states in an infinite quantum system, which is very difficult. In contrast, the appearance of the commutators in the time dependence of the operators implies a linked-cluster property. In other words, up to a certain order \( m \) in the time \( t \), one has to deal only with a finite number of operators while dealing with the infinite system in the thermodynamic limit. We stress that all results presented here refer to this limit. The caveat of the approach is that the number of operators grows exponentially with the order \( m \).

When calculating the commutator \( [\hat{H}, \hat{c}^\dagger(\vec{r}, t)] \), we encounter two cases. (i) The commutation with the non-interacting part of the Hamiltonian \( \hat{H}_0 \) leads to a shift of single-fermionic operators. (ii) The commutation with the interaction term \( \hat{H}_{\text{int}} \) creates or annihilates additional particle–hole pairs \( (\hat{T}^\dagger \hat{L}) \). Iterating the commutation leads to the ansatz (5) and extends it step by step. Each commutation creates more and more terms with higher and higher number of particles and holes involved.

Comparing the coefficients of the left-hand side and the right-hand side of the Heisenberg equation (7) yields differential equations for the prefactors \( h \). Then, this set of differential equations is solved numerically. The initial conditions of the prefactors are \( h_0(0, 0) = 1 \) and \( h_i(\vec{r}, t) = 0 \) \( \forall i \neq 0 \).

Since each commutation adds the coefficients necessary to describe another order in time \( t \), the results become more and more accurate on increasing number of commutations (loops). In this way a calculation with \( n \) commutations provides results for \( \hat{c}^\dagger(t) \), which are exact up to order \( t^n \). To quantify the convergence of the results, calculations with different numbers of commutations are performed and compared. One may introduce a time \( t_{\text{runaway}} \) up to which the deviations between the results do not exceed a certain threshold, for instance \( 10^{-2} \). Thus, the precise definition of \( t_{\text{runaway}} \) depends on the threshold, but for a given reasonable value of the threshold one can clearly see that the results become more and more accurate for increasing number of loops. This has been performed for results for the Hubbard model in appendix A of [42] and results for the spinless fermions can be found in the appendix below. The results in appendix A of [42] and those in the appendix show that \( t_{\text{runaway}} \) increases roughly quadratically with the number of loops \( m \), i.e. \( t_{\text{runaway}} \propto m^2 \).

### 2.3. Bosonization results

We briefly recall what is to be expected for the jump in the momentum distribution in Tomonaga–Luttinger models of non-interacting bosons [18, 21, 39, 40]. For the spinless case one finds [21]

\[
\Delta n(t) = \left[ \frac{r^2}{r^2 + (2vt)^2} \right]^{2y(1+y)},
\]

(8)
where $v$ is the dressed velocity in the system and $r$ the characteristic length scale of the interaction. The leading power law in time $t$ without the length scale $r$ in the denominator was first derived by Cazalilla [18]. In the above formula, the interaction is assumed to range over all momenta, i.e. from $-\infty$ to $\infty$. If a finite range in momentum space is assumed, oscillations occur. The occurrence of oscillations stemming from the high-energy cutoff is indeed generic [23, 40, 52]. Note that such a finite range is natural in microscopic models because the finite extension of the Brillouin zone limits the range of the interaction in momentum space.

The exponent in (8) is determined by $\gamma$ which is related to the standard bosonization parameter in the usual way

$$\gamma = (K + K^{-1} - 2)/4.$$  \hspace{1cm} (9)

Note that the expression (8) is governed by an exponent related to the one occurring in the equilibrium [31] except that $\gamma$ in the equilibrium has been replaced by $2\gamma(1 + \gamma)$ after the quench [21]. Thus, for small values, one finds a factor of 2 similar to the observation after quenches of other systems [54]. We stress that the replacement $\gamma \rightarrow 2\gamma(1 + \gamma)$ is an inherent property of the Tomonaga–Luttinger model. It is not related to any underlying microscopic model.

In the presence of spin, the above formulae are modified. The Hamiltonian is given by the non-interacting sum of the spin part and charge part. Consequently, the single-particle correlation (4) and hence the jump are given by the product of the responses in the spin and charge channels [21]

$$\Delta n(t) = \left[\frac{r^2_\rho}{r^2_\rho + (2v_\rho t)^2}\right]^{\gamma_\rho(1+\gamma_\rho)} \left[\frac{r^2_\sigma}{r^2_\sigma + (2v_\sigma t)^2}\right]^{\gamma_\sigma(1+\gamma_\sigma)},$$  \hspace{1cm} (10)

where $\nu \in \{\rho, \sigma\}$ stands for charge ($\rho$) or spin ($\sigma$) channel. Again, the parameters $r_\nu$ are the characteristic length scales of the interaction, $v_\nu$ the velocities and $\gamma_\nu$ the equilibrium exponents. The exponents can be expressed through the anomalous dimensions $K_\nu$ in the usual way

$$\gamma_\nu = (K_\nu + K_\nu^{-1} - 2)/4.$$  \hspace{1cm} (11)

Note that the exponents in each channel separately take only half the value of their counterpart in the spinless case. This is again the same as in equilibrium [31, 33]. But also in the case with spin, the non-equilibrium exponents are obtained from the equilibrium ones by the replacement $\gamma_\nu \rightarrow 2\gamma_\nu(1 + \gamma_\nu)$.

3. Results

3.1. Spinless fermions

In this section, we present the results obtained by the equation of motion approach for the behavior of the spinless fermion model after the quench. We focus on short and intermediate times after the quench and show results for the jump $\Delta n(t)$. We explain the data from the equation of motion approach by comparing them with the bosonization results.

In figure 2 the jump $\Delta n(t)$ is shown for the spinless fermion model with nearest-neighbor repulsion. The data agree very well with the data by Karrasch et al [36] in figure 1, obtained by time-dependent infinite-size DMRG, for exemplary comparisons see figure 1.
Figure 1. Upper panel: jump $\Delta n(t)$ in the half-filled spinless fermions model for various loop numbers. Lower panel: $\Delta n(t)$ for increasing $U$ (from top to bottom at small $t$) in 11 loops. The dotted lines display exemplary DMRG data from Karrasch et al [36] illustrating the quantitative agreement in the range where our data are converged.

Figure 2. Solid lines: $\Delta n(t)$ in the half-filled model of spinless fermions for increasing $U$ (from top to bottom) in 11 loops. Dashed (black) lines: $\Delta n(t)$ as in equation (8) with the GS exponents from equation (12). Dashed–dotted (red) lines: $\Delta n(t)$ with the FS exponents from equation (14).
Figure 3. Results for the anomalous dimensions \( K_\nu, \nu \in \{\rho, \sigma\} \) obtained by bosonization. Dashed lines: results obtained by bosonization around the GS based on Bethe ansatz. Solid lines: \( K_\nu \) values obtained by bosonization around the FS. Left panel: results for the spinless fermion model with nearest-neighbor repulsion. Right panel: results for the Hubbard chain at quarter filling; the Bethe ansatz is difficult to evaluate so that slight inaccuracies imply some minor wiggling of the dashed curve for \( K_\rho \).

First, we compare the data with the power law behavior (8). This still leaves the question which parameters are to be used. Certainly, a first trial is the parameters that describe the models in equilibrium. Since we are dealing with the systems at zero temperature, ‘equilibrium’ refers to the GS and its immediate vicinity, i.e. the elementary excitations. Then, the anomalous dimension \( K \) and the velocity can be determined by Bethe ansatz [29, 45]. They read

\[
K_{\text{GS}} = \frac{\pi}{[2(\pi - \arccos(2U/W))]},
\]

\[
\nu_{\text{GS}} = \frac{\pi \sin(\arccos(2U/W))}{[2 \arccos(2U/W)]},
\]

where we use the subscript GS for ‘ground state’ to emphasize that these parameters pertain to the behavior of the model at the lowest energies in the vicinity of the GS. The cutoff length \( r \) of the curves is fitted and evolves from 0.2 to 0.6 on increasing \( U \), assuming that the lattice constant is set to unity. The formulae (12) are reasonable only up to \( U = W/2 \), where the system enters a gapped phase. The resulting anomalous dimension \( K \) is depicted in the left panel of figure 3 as a dashed curve.

Inspecting figure 2 we see that the microscopic model displays oscillations in \( \Delta n(t) \), which are absent in equation (8). These oscillations are to be ascribed to the momentum cutoff of the interaction in microscopic models [23, 40, 52]. Otherwise, the power law (8) nicely describes the accessible dynamics, at least for not too large values of the interaction. This observation agrees with the one by Karrasch et al [36].

We point out, however, that the agreement deteriorates for larger values of the interaction, namely for \( U = 0.45W \) and for \( 0.5W \). Thus, we pose the question whether the equilibrium exponent \( \gamma \) is still the relevant one for non-equilibrium situations. We stress that the answer to
Figure 4. Excitation energies $\Delta E$ per site defined in (13) in dependence on the interaction strength $U$ for the spinless fermions model at half filling (upper panel) and for the quarter-filled Hubbard model (lower panel); the behavior at low values of $U$ is quadratic even though the difficult evaluation of the Bethe ansatz equations hindered us to reach very high precision. The results are very accurate at large $U$.

In order to assess how far a quench puts the microscopic model away from equilibrium, i.e. from the GS, we define the quench energy $\Delta E$

$$\Delta E := \langle \text{FS} | H(t > 0) | \text{FS} \rangle - \langle \text{GS} | H(t > 0) | \text{GS} \rangle,$$

where $| \text{FS} \rangle$ stands for the initial state, which in our case is a FS, while $| \text{GS} \rangle$ stands for the ground state of the Hamiltonian after the quench. Thus, $\Delta E$ as defined above measures the total excitation energy above the GS induced by the quench. In [26], this quantity is called the heat. It is conserved since the energy in a closed constant quantum system is a conserved quantity. We stress that quenches in imaginary time [36], which obey $| t \rangle = \exp(-H\tau)| \text{FS} \rangle$, do not conserve $\Delta E$ and thus behave differently.

In Figure 4, it is plotted for the two models under study as a function of the quenched interaction. For the spinless fermion model, we note the quenched energy remains fairly small.
\[ \lesssim 3 \times 10^{-3} W \] for quenches below the phase transition at \( U = W/2 \) [45, 46]. Thus, one may not be surprised that the GS parameters (12) yield good agreement. In [36], it was concluded that the dynamics of interaction quenches is universally governed by the equilibrium exponents.

An alternative argument to reach the parameters for the relevant field theoretic model after a quench is the following. The dynamics after the quench starts from the initial state, here |FS\rangle. The short time dynamics implies the iterated gradual excitation of particle–hole pairs [21], so that up to moderate times only a limited number of particle–hole pairs needs to be described. Thus, bosonization of the density fluctuations around the initial states |FS\rangle is expected to yield a valid description of the dynamics at short and intermediate times.

The bosonization in the vicinity of the FS corresponds to the bosonization in leading order in \( U \) because no feedback effects due to the interaction need to be included. This leads us to the expressions [29, 30, 32–34]

\[
K_{FS} = \sqrt{\frac{\pi v_F - U}{\pi v_F + 3U}}, \quad (14a)
\]

\[
v_{FS} = \left(\frac{1}{\pi}\right)\sqrt{\left(\pi v_F + U\right)^2 - 4U^2}. \quad (14b)
\]

The anomalous dimensions \( K \) are shown in the left panel of figure 3. The implied power laws for the jump \( \Delta n \) are included in figure 2 as dashed–dotted curves. The fitted cutoff length \( r \) varies from 0.2 to 0.1 on raising \( U \).

Obviously, the difference between the curves from the GS and FS parameters is small, cf figure 2. This can be understood easily in view of the small excitation energies \( \Delta E \). Nevertheless, we point out that for the largest interactions \( U \) the FS curves fit better to the numerical data than the GS curves. This underlines the relevance of the question which field theoretic model is the appropriate one off equilibrium?

We note that the bosonization in the vicinity of the FS leaves out certain effects that will become important at even stronger quenches, for instance the curvature of the dispersion or the effect of higher order umklapp scattering. (At leading order in \( U \), no umklapp scattering occurs for spinless fermions.) This restricts the validity of the FS formulae (14) to values of \( U \) below \( \pi v_F \approx 1.57W \), where \( K_{FS} \) and \( v_{FS} \) vanish.

### 3.2. One-dimensional Hubbard model

Quenches in a model with spin are by themselves of great interest because such models are much closer to what is realized in strongly correlated systems, such as Mott insulators. Moreover, they display dynamical transitions at half filling if quenched strongly in infinite dimensions [6, 11] and in 1D [42].

In this paper, however, we focus on weaker quenches for fillings off half filling. The reason is that we want to focus on models that are quenched within the metallic phase. Thus, we focus on quarter filling which also suppresses umklapp scattering, at least to leading order, so that a quench into a Mott insulating phase is avoided. Furthermore, the Hubbard model at quarter filling displays two further interesting features.

The first is shown in the right panel of figure 3 where the anomalous dimensions in the charge and spin channel are depicted. The GS parameters are deduced from the GS properties as obtained from the Bethe ansatz [51, 55]. Note that \( K_{s,GS} = 1 \) by spin rotational symmetry [35]. The FS parameters are again those of the bosonization around the FS amounting to the bosonization [32, 33] in leading order in the quenched interaction \( U \):

\[
K_{\rho,FS} = \sqrt{2\pi v_F/(2\pi v_F + 2U)}, \quad (15a)
\]
Figure 5. Jump $\Delta n(t)$ for the Hubbard model at quarter-filling at $U = 0.8W$. Lines as in figure 2. For the GS exponents $r_\rho \approx 0.01$ is fitted; $r_\sigma$ does not occur. For the FS exponents $r_\rho \approx 1$ and $r_\sigma \approx 0.6$ are fitted.

\begin{align}
K_{\sigma,FS} &= \sqrt{2\pi v_F/(2\pi v_F - 2U)}, \\
v_{\rho,FS} &= v_F\sqrt{1 + U/(\pi v_F)}, \\
v_{\sigma,FS} &= v_F\sqrt{1 - U/(\pi v_F)}.
\end{align}

The right panel of figure 3 shows that the charge dimensions agree surprisingly well in both calculations $K_{\rho,GS} \approx K_{\rho,FS}$. In contrast, the spin dimensions $K_{\sigma,GS}$ and $K_{\sigma,FS}$ differ significantly. It is well known from the analysis of the renormalization of the underlying sine-Gordon model that $K_{\sigma,GS}$ converges to its final value only for exponentially small energy scales $[35]$.

The second interesting aspect is the fact that in the Hubbard model, a moderate interaction quench implies much larger excitation energies, see lower panel of figure 4. For instance, a quench to $U = 0.8W$ yields an excitation energy of about 0.036W. Thus, in the Hubbard model we can study more easily the effects of a larger distance from the equilibrium situation.

The behavior of the system after strong quenches can be found in $[42]$. Figure 5 shows the data for a quench to $U = 0.8W$ in the quarter-filled Hubbard model. The oscillations are again to be attributed to the finite moment cutoff of the interaction in a lattice model.

Inspecting the power laws, it is obvious that the GS and the FS power laws differ significantly as we expected from the significant differences in the anomalous dimensions and the significant excitation energy of the quench. We observe that the power law with the FS exponents fits much better than the power law with the GS exponents. As can be seen in the inset of figure 5, the GS exponents can only be used to describe the curve over a small range and the fitted range $r_\sigma$ is unreasonably small.

We recall that even at equilibrium $K_\sigma$ reaches its final (GS) value only for exponentially small energy scales $\epsilon$. This implies that one must know the equal-time Green function at two points in space with exponentially large distance $\vec{r}$ according to $\epsilon \approx v_F/|\vec{r}|$ with the Fermi velocity $v_F$.

After the quench, the correlations due to the switched interaction will develop gradually and spread out in time according to $|\vec{r}| \approx v_{\text{max}}t$, i.e. $1/t$ acts as a low-energy cutoff. Thus, effects at exponentially low energies are expected to show up only at exponentially large
Here we state that the behavior at small and intermediate times is described better by the FS parameters (15). To support this finding, figure 6 shows the jump $\Delta n(t)$ for the Hubbard model at quarter filling for various interaction strengths $U$. As can be seen in the upper panel, for small values $U \lesssim 0.3W$ the differences between the GS and the FS power laws are small so that they are indistinguishable on the accessible time scales.

For larger values of $U$, the time evolution of the jump is described very well by the FS power law, while the GS exponents do not fit the slope of the decay of the jump toward larger times. Only the oscillations due to the high-energy cutoff [23, 40, 52] are missed by the power law (10).

For larger $U \gtrsim W$ (lower panel) the agreement between microscopic data and the FS power laws deteriorates. We attribute this deterioration to the breakdown of the Tomonaga–Luttinger liquid description in terms of bosonic modes without interaction. Backscattering and the curvature of the single-particle dispersion are neglected. Hence for large excitation energies $\Delta E$, which take the system even further way from its equilibrium, these effects need to be included. For short and intermediate times, however, they are of reduced relevance. For example, the leading correction due to backscattering is a term $\propto \int \cos(\sqrt{8}\Phi(x)) \, dx$. But the expectation value of this and of any of its higher derivatives vanish with respect to the FS because in the latter the fluctuations $\langle \Phi(x)^2 \rangle$ diverge, which smears out the cosine term completely to zero, see also the discussion in [25].

### 4. Behavior for longer times

The next intriguing question is what happens for longer times $t$? To date this question cannot be answered definitely. Numerically, no tools can treat long times far off equilibrium. Analytically,
no general renormalization group theory for non-equilibrium physics exists. A study of the sine-Gordon model on the basis of Keldysh Green functions has been carried out [14, 41]. The non-interacting bosonic system is quenched at $t = 0$, but the sine term is switched on slowly afterwards. At present, it is unclear to what extent this calculation captures the scenario we are studying in this paper where the interaction is quenched abruptly at $t = 0$ in fermionic models.

To decide whether the results of this paper are relevant also for longer times, we have to consider various energy cutoffs. The first was already mentioned above, namely the inverse time $\propto 1/t$. Given the accessible times, see figures 1, 2 and 6, we find that this cutoff is of order of about $0.1W$, the precise value depends on the value of the interaction, the filling and the model under study. This is fairly large so that one may expect that the observed power laws will still gradually change for larger times.

But we come back to the distance of a quenched system to the true GS, which is measured by the excitation energy $\Delta E$ per site in (13), see figure 4. We anticipate that this distance prevents the parameters of the effective low-energy model reaching their fixed-point values, which would define the relevant low-energy model just above the GS. At present, it is unclear whether $\Delta E/L$ itself sets the cutoff scale or whether a fictitious temperature $T_{\text{quench}}$, which induces the same excitation energy by thermal fluctuations, or yet another quantity sets the relevant scale. The fictitious temperature $T_{\text{quench}}$ would be the most promising candidate if the system relaxed toward a thermal state on moderate time scales. But so far all numerical and analytical evidence in 1D systems points against such a relaxation. In particular, there is a growing evidence that integrable models, such as the sine-Gordon model, do not thermalize [23–26]. Their steady state may be characterized by a generalized Gibbs ensemble [28].

Mitra and Giamarchi [14, 41] argue that in interacting systems the high-energy modes act as a thermal bath to the low-energy modes so that the latter behave on long time scales as if they were thermally excited. A (small) dissipation rate is computed which sets the inverse time scale for the relaxation toward this apparent thermal behavior. Although this scenario appears intuitively plausible, the studied peculiar quench in the integrable sine-Gordon model leaves the question still open, how generic it is?

In view of the above arguments, one may expect three temporal regimes in 1D quenched systems: (i) short time dynamics governed by power laws with the FS exponents, (ii) intermediate time dynamics governed by power laws with slowly varying exponents which approach the GS values, but do not reach them because their evolution is stopped by the distance of the system to its equilibrium GS and (iii) if the systems are not integrable, the dynamics of low-energy models displays relaxation to thermal states at very long times.

Since at present, no scenario of the fascinating long time dynamics of 1D systems, let alone of two- or three-dimensional systems, is firmly established, further analytical and numerical work is called for. In particular, the investigation of the potential crossover regimes deserves further attention as well as the regime of very long times.

5. Conclusions

In this work, we investigated the time evolution of the jump in the momentum distribution of two 1D fermionic models after interaction quenches. The method used was the integration of the multiply iterated Heisenberg equations of motion for the operators (up to about $10^7$ equations).
This approach circumvents the paramount difficulty in treating quantum states of infinite systems, because it focuses on the observables and their dynamics. In this dynamics, the commutation with the local Hamilton operator implies a linked-cluster property \([21]\) that keeps the problem finite and tractable for each finite order of powers in the time \(t\).

The first model studied comprises spinless fermions with nearest-neighbor interaction at half filling. For this model, a power law as found for the Tomonaga–Luttinger model provides a good description for the time evolution of the jump for short to intermediate times. The exponents can be taken from the GS properties known from Bethe ansatz or from first-order bosonization around the FS. They do not differ much as long as the quenched systems are still metallic. The excitation energy induced by the quench turns out to be remarkably small. The FS exponents fit slightly better for the quenches close to the phase transition at \(U = W/2\).

The second model studied is the Hubbard model at quarter filling, for which much larger excitation energies can be induced for \(U\) values of the order of the band width \(W\). Also, the anomalous dimension \(\kappa_\sigma\) in the spin channel differs significantly depending on the way it is determined, either around the FS or around the GS. On the accessible time scales, we found that the FS parameters describe the temporal evolution better than the equilibrium parameters.

We discussed possible scenarios for longer times on the basis of energy cutoffs such as the inverse time \(1/t\), the excitation energy \(\Delta E/L\) or a fictitious temperature \(T_{\text{quench}}\) leading thermally to the same excitation energy. The likely scenario is the occurrence of power laws with gradually changing exponents which start from the FS values and approach the equilibrium values without reaching them due to the distance of the quenched system to its equilibrium. For very long times, the low-energy modes may even show thermal relaxation if a macroscopic number of conserved quantities does not prevent it. But many further studies are called for to clarify whether this likely scenario is really the true one.

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**Appendix. Convergence in the number of loops**

In order to quantify the convergence properties of the approach, we study the difference of the results for the jump in momentum distribution \(\Delta n_m(t)\) obtained in \(m\) loops. Taking the result with the highest number of loops (11) as reference, figure A.1 depicts how the deviations from the 11-loop curve increase with time. Setting a certain threshold for the deviation, here 0.01, we determine up to which time \(t_{\text{runaway}}\) the deviation remains below the threshold. Figure A.1 shows data for the half-filled spinless fermion case. Analogous data for the Hubbard model can be found in \([42]\).

The choice of the threshold value is to some extent arbitrary, but it helps to illustrate the main point: the results converge for increasing number of loops \(m \to \infty\). Finally, we plot the resulting inverse runaway times in the double logarithmic plot in figure A.2 as a function.
Figure A.1. Absolute difference of the jump $\Delta n_m(t)$ at various numbers of loops $m$ relative to the 11-loop result $\Delta n_{11}(t)$ for the half-filled spinless fermion model quenched by nearest-neighbor interaction. Horizontal black line: threshold for the determination of the runaway time.

Figure A.2. Double logarithmic plot of the inverse runaway time versus the inverse number of loops of the corresponding calculation. Dashed line: power law fit of the data with an exponent of about 2.02.

of $1/m$. Clearly, the times up to which the results are reliable quickly increase on increasing loop number $m$. The exponent is found to be of the order of 2. Again, this is in agreement with the previous findings for the Hubbard model [42].

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