Minority Game With Peer Pressure

H. F. Chau, F. K. Chow and K. H. Ho
Department of Physics, University of Hong Kong, Pokfulam Road, Hong Kong
(Dated: March 22, 2022)

To study the interplay between global market choice and local peer pressure, we construct a minority-game-like econophysical model. In this so-called networked minority game model, every selfish player uses both the historical minority choice of the population and the historical choice of one’s neighbors in an unbiased manner to make decision. Results of numerical simulation show that the level of cooperation in the networked minority game differs remarkably from the original minority game as well as the prediction of the crowd-anticrowd theory. We argue that the deviation from the crowd-anticrowd theory is due to the negligence of the effect of a four point correlation function in the effective Hamiltonian of the system.

PACS numbers: 89.65.Gh, 02.50.Le, 05.45.-a, 89.75.Fb

I. INTRODUCTION

Minority Game (MG) — a game proposed by Challet and Zhang under the inspiration of the El Farol bar problem — is a simple model showing how selfish players cooperate with each other in the absence of direct communication. It succinctly captures the self-organizing global cooperative behavior which is ubiquitously found in many social and economic systems.

In MG, \( N \) inductive reasoning players have to choose one out of two choices independently in each turn. Based only on certain commonly available global information, each player has to decide one’s choice by means of his/her current best working strategy or mental model. Those who end up in the minority side (i.e., the choice with the least number of players) win. Although its rules are remarkably simple, MG exhibits very rich self-organized collective behavior. Moreover, the dynamics of MG can be explained by the so-called crowd-anticrowd theory which stated that fluctuations arisen in the MG is resulted from the interaction between crowds of like-minded agents and their anti-correlated partners.

However, in the real world, people usually do not only consider the global information when they make decisions. In fact, they may also consult the opinion of their neighbors before making decisions. For example, it is not uncommon for people to consider both the recommendation of their peers (local information) and the stock price (global information) in deciding which stock to buy from the market. Hence, it is instructive for us to incorporate the local information into the MG model to gain more insights on this kind of social and economic systems.

In the past few years, many researchers have studied a few variations of MG with local information. However, players can only make use of either the global or local information in each turn in these models. In contrast, people often make their decisions according to both the global information and their local information in many social and economic phenomenon. On the other hand, Quan et al. introduced the local information in the evolutionary minority game (EMG). Since we want to focus on how are players affected by their local information in a non-evolutionary game, we shall not consider the EMG with local information here. With the above consideration in mind, we would like to propose a model of MG where players use both the local and global information in an unbiased manner for making decision.

In Section II, we introduce a new model called the Networked Minority Game (NMG). It is a modified MG model in which all players can make use of not only the global information but also the local information from their neighbors that are disseminated through a network. Results of numerical simulations are presented and discussed in Section III. Lastly, we conclude by giving a brief summary of our work in Section IV.

II. OUR NETWORKED MINORITY GAME

In this Section, we would like to show how to construct the Networked Minority Game (NMG) model. In this repeated game, there are \( N \) heterogeneous inductive reasoning players whose aim is to maximize one’s own profit in the game. In every turn, each player has to choose one out of \( N_c \) alternatives with label \( \{0, 1, \ldots, N_c - 1\} \). The global minority choice, denoted as \( \Omega_g(t) \) at time \( t \), is simply the least popular choice amongst all players in that turn. (Note that the global minority choice is the least popular choice that are chosen by a non-zero number of players; and it is chosen randomly amongst the choices with the least non-zero number of players in case of a tie.) The players picked the global minority choice gain one unit of wealth while all the other lose one.
Unlike in MG, not only the global information is delivered to each player as one’s external information in our model. Nevertheless, the global information given to players are the same in these two models. In NMG, we call it the global history \((\Omega_g(t - M_g), \Omega_g(t - M_g - 1), \ldots, \Omega_g(t - 1))\) which is simply the \(N_c\)-ary string of the global minority choice of the last \(M_g\) turns. The global history can only take on \(P_g \equiv N_c^{M_g}\) different states. We label these states by an index \(\mu_g = 1, \ldots, P_g\) and denote the global history \((\Omega_g(t - M_g), \Omega_g(t - M_g - 1), \ldots, \Omega_g(t - 1))\) by the index \(\mu_g(t)\).

Besides global information, local information is also distributed via a network to all players in NMG such that individual player receive one’s local information from a different source. In this game, we arrange all the players with label \{1, \ldots, \(N\)\} on a ring (see fig. 1) where the local information of a player is based on the choices of this player and his/her nearest neighbors on the ring. Specifically, the local information given to the \(i\)th player is the so-called local history \((\Omega_l(t - M_l), \Omega_l(t - M_l - 1), \ldots, \Omega_l(t - 1))\) which is simply the \(N_c\)-ary string of the local minority choice of this player of the last \(M_l\) \(\neq 0\) turns. (Note that we do not consider the case of \(M_l = 0\) since it is equivalent to MG.) Here, the local minority choice of the \(i\)th player at time step \(t\), \(\Omega_l^i(t)\), refers to the least popular choice amongst this player and his/her nearest neighbor on the ring at this time step. (In other words, \(\Omega_l^i(t)\) is the least popular choice amongst \((i - 1)\)th, \(i\)th and \((i + 1)\)th player.) However, unlike the global minority choice, the local minority choice could be an alternative that nobody chooses. In case of a tie, \(\Omega_l^i(t)\) is chosen randomly amongst the choices with the least number of players. The local history can only take on \(P_l \equiv N_c^{M_l}\) different states. We again label these states by an index \(\mu_l = 1, \ldots, P_l\) and denote the local history of the \(i\)th player \((\Omega_l^i(t - M_l), \Omega_l^i(t - M_l - 1), \ldots, \Omega_l^i(t - 1))\) by the index \(\mu_l^i(t)\). By the way, it is easy to extend NMG to the case where players are connected on a different topology. For example, we can arrange players on a dynamically random chain in which each player is connected to two randomly chosen players and all the connections between players change at every time step.

In brief, each player is given a \(N_c\)-ary string of length \(M = M_l + M_g\) storing both the global and local history to decide his/her choice. Players can only interact indirectly with each other through the global history \(\mu_g(t)\) and their local history \(\mu_l^i(t)\). However, how does each player make use of such external information to decide one’s choice in the NMG? He/She does so by employing strategies to predict the next minority choice according to both the global and local information where a strategy is a map sending individual combination of global and local history \((\mu_g, \mu_l)\) to the choice \{0, 1, \ldots, \(N_c - 1\)\}. A strategy \(s\) can be represented by a vector \(\vec{s} \equiv (\chi^{(1,1)}, \chi^{(1,2)}, \ldots, \chi^{(P_g, P_l)})\) where \(\chi^{(\mu_g, \mu_l)}\) is the minority choice predicted by the strategy \(s\) for the input \((\mu_g, \mu_l)\).

In our model, each player picks \(S\) randomly drawn strategies from the strategy space before the game commences. (We will discuss about the strategy space in depth later on.) Just like in MG, strategies in NMG are not evolving, i.e., players are not allowed to revise their own \(S\) strategies during the game. At each time step, each player uses his/her own best working strategy to guess the next global minority choice. How does a player decide which strategy is the best? Players use the virtual score, which is simply the hypothetical profit for using a single strategy throughout the game, to evaluate the performance of a strategy. The strategy with the highest virtual score is considered as the best one.

Since inductive/reasoning players do not know whether a strategy is good or not before the game commences, we cannot restrict players to have “good strategies” only. As a result, all strategies in our model must be unbiased to the properties and dynamics of the game. How can we construct the reduced strategy space for NMG? To answer this question, let us look at the reduced strategy space of MG first. For MG, the reduced strategy space consists of mutually uncorrelated and anti-correlated strategies only. Challet and Zhang showed that the maximal reduced strategy space of the original 2-choice MG, denoted as \(V_2(M)\) for a 2-choice game with length of global history \(M\), is composed of \(2^M\) pairs of mutually anti-correlated strategies where any two strategies from different anti-correlated strategy pairs are uncorrelated with each other \(2^4\). Therefore, there is a total of \(2^M\) different strategies in \(V_2(M)\). (Note that there are other smaller reduced strategy spaces consisting of less number of strategies for MG \(\underline{12} \underline{16} \underline{17}\).) It can be shown that, in general, the maximal reduced strategy space \(V_{N_c}(M)\) for \(N_c\)-choice MG is composed of \(N_c^M\) ensembles of mutually anti-correlated strategies where any two strategies from different anti-correlated strategy ensembles are uncorrelated with each other \(\underline{12} \underline{16} \underline{17}\). Moreover, \(V_{N_c}(M)\) consists of \(N_c^{M+1}\) distinct strategies.

For NMG, it seems reasonable for us to define the reduced strategy space \(R_{N_c}(M)\) to be the maximal reduced strategy space of MG. However, we find that the cooperative behavior of the players in NMG using \(V_{N_c}(M)\) and the full strategy space are greatly different from each other. Indeed, such a discrepancy is due to the bias of the strategies.
of $V_{N_c}(M)$ to some of the input (the global and local history) in our model. Hence, we should not use $V_{N_c}(M)$ to substitute the full strategy space in NMG. In fact, we must define the reduced strategy space $R_{N_c}(M)$ in a different way.

For NMG, a strategy $\hat{s}$ is unbiased to the input if it satisfies the following conditions:

1) Simply looking at the predictions of this strategy for a given global history does not give us any information about its predictions for any other global history.

2) Simply looking at the predictions of this strategy for a given local history does not give us any information about its predictions for any other local history.

Accordingly, we define the reduced strategy space of NMG as follows:

$$R_{N_c}(M) = \begin{cases} \{(s_1, s_2, \ldots, s_{P_l}) : s_1, s_2, \ldots, s_{P_l} \in V_{N_c}(M_g)\} & \text{if } M_l \leq M_g, \\ \{(s_1, s_2, \ldots, s_{P_g}) : s_1, s_2, \ldots, s_{P_g} \in V_{N_c}(M_l)\} & \text{otherwise,} \end{cases}$$

(1)

where the $i$th player uses the so-called segment strategy $s_i$ to predict the global minority choice for the global history $\mu_g(t)$ if $j = \mu_l(t)$ in case of $M_l \leq M_g$ and respectively for the local history $\mu_l(t)$ if $j = \mu_g(t)$ in case of $M_l > M_g$. Table II illustrates how a strategy in $R_{N_c}(M)$ gives the prediction of the next minority choice. It is easy to show that the number of distinct strategies in $R_{N_c}(M)$ is either $N_c^{(M_g+1)P_l}$ or $N_c^{(M_l+1)P_g}$ depending on the ratio of $M_g/M_l$. Note that we should not define $R_{N_c}(M)$ to be given by just one of the above expressions over all range of $M_g/M_l$. Otherwise, there is a large redundancy of strategies in $R_{N_c}(M)$ since many strategies are very similar for such case. (However, it does not matter if we define $R_{N_c}(M)$ by the second expression when $M_l = M_g$.) Indeed, we have verified that the cooperative behavior of the players in NMG using $R_{N_c}(M)$ and the full strategy space agree well with each other; i.e., $R_{N_c}(M)$ can successfully characterize the diversity of strategy in the full strategy space.

| global history $\mu_g$ | local history $\mu_l$ | prediction $\chi_s^{(\mu_g, \mu_l)}$ |
|------------------------|------------------------|-------------------------|
| (0,0)                  | (0)                    | 1                       |
| (0,1)                  | (0)                    | 1                       |
| (1,0)                  | (0)                    | 1                       |
| (1,1)                  | (0)                    | 1                       |
| (0,0)                  | (1)                    | 0                       |
| (0,1)                  | (1)                    | 0                       |
| (1,0)                  | (1)                    | 1                       |
| (1,1)                  | (1)                    | 1                       |

TABLE I: The prediction of the minority choice $\chi_s^{(\mu_g, \mu_l)}$ by the strategy $s = (s_1, s_2)$ where $\hat{s}_1 \equiv (1,1,1,1)$ and $\hat{s}_2 \equiv (0,0,1,1)$ for $M_g = 2$ and $M_l = 1$.

To investigate how well players cooperate with each other in our game, a quantity of interest is the attendance of an alternative $A_j(t)$ which is the number of players choosing the alternative $j$ at time $t$. For players gaining the maximum profit, the expectation value of the attendance of any alternative should be equal to $N/N_{nc}$. Accordingly, the variance of the attendance $\sigma_j^2 = \langle(A_j(t))^2 \rangle - \langle A_j(t) \rangle^2$ represents the loss of players in the game. (Here, $\langle \rangle$ denotes the average over time.) Hence, we would like to study $\langle A_j(t) \rangle$ and $\sigma_j^2$ as a function of the complexity of the system for our model. Which parameter can be used as a measure of the complexity of the system? It is the so-called control parameter $\alpha \equiv \frac{\beta}{\gamma}$ which is the ratio of the strategy space size to the number of strategies at play. For NMG, the control parameter $\alpha$ is equal to either $N_c^{(M_g+1)P_l}/NS$ for $M_l \leq M_g$ or $N_c^{(M_l+1)P_g}/NS$ for $M_l > M_g$.

On the other hand, we also want to compare the variance of the attendance with the prediction by the crowd-anticrowd theory [2, 4, 5] in order to investigate the crowding effect in our model. Since the strategies used in NMG (which are picked from $R_{N_c}(M)$) are neither anti-correlated nor uncorrelated, we cannot only consider the interactions of the anti-correlated strategies in the crowd-anticrowd calculation of the variance just like in MG [8, 9] and the multichoice MG [10]. In fact, each strategy used in NMG is a set of segment strategies from either $V_{N_c}(M_l)$ or $V_{N_c}(M_g)$ whereas different segment strategies are used for different $\mu_l(t)$ or $\mu_g(t)$ (see Eq. [1]). Thus we can apply the crowd-anticrowd theory to estimate the variance simply by counting the crowd-anticrowd cancellation of the anti-correlated segment strategies. Accordingly, the crowd-anticrowd prediction of the variance in NMG is given
as follows:

\[
\sigma^2 = \begin{cases} 
  \left\langle \frac{1}{N_c} \sum_{S_g \in V_{N_c}(M_g)} \sum_{a \in S_g} \left\{ \frac{1}{N_g^2} \left[ \sum_{b \in S_g \setminus \{a\}} (N_a - N_b) \right]^2 \right\} \right\rangle & \text{if } M_l \leq M_g, \\
  \left\langle \frac{1}{N_c} \sum_{S_l \in V_{N_c}(M_l)} \sum_{a \in S_l} \left\{ \frac{1}{N_l^2} \left[ \sum_{b \in S_l \setminus \{a\}} (N_a - N_b) \right]^2 \right\} \right\rangle & \text{otherwise},
\end{cases}
\]

where \(S_g\) and \(S_l\) denotes the mutually anti-correlated strategy ensembles in \(V_{N_c}(M_g)\) and \(V_{N_c}(M_l)\) respectively, and \(N_a\) is the number of players making decision according to the segment strategy \(a\). We should aware that the variance of attendance for different alternatives must equal when averaged over time and initial choice of strategies since there is no bias for any alternative in our game.

III. RESULTS OF THE GAME

In all the simulations reported in this paper, each set of data was recorded from 1000 independent runs. In each run, we run 10000 steps starting from initialization before making any measurements. We have checked that it is already enough for the system to attain equilibrium in all the cases reported here. Then we took the average values on 15000 steps after the equilibration. To be computational feasible, we choose the case where \(N_c = 2, S = 2\) and \(M = M_l + M_g = 5\).

A. Comparison of NMG with MG

In this section, we compare the performance of players in NMG with that in MG. Since the system exhibits peculiar properties in NMG with \(M_g = 0\), we delay the discussion about this case to Section III C.

Let us begin by investigating the properties of the mean attendance as a function of the control parameter \(\alpha\) in NMG. Obviously, the mean attendance in NMG is similar to MG for all values of \(M_l/M_g\). That is to say, it always fluctuates around \([N/2]\) such that players can maximize their global profit.

Next we evaluate the performance of the players in NMG by studying the variance of the attendance per player \(\sigma^2/N\) versus the control parameter \(\alpha\) as shown in fig. 2 (Note that the variance of the two choices must be the same by symmetry for a 2-choice game; and all the variance mentioned in this section are divided by the number of player \(N\) for objective comparison.) We find that the variance in NMG is always much smaller than that in MG for small \(\alpha\) no matter what is the value of \(M_g/M_l\). In other words, players perform much better in NMG than in MG through the introduction of their own local information when \(\alpha\) is small. When the reduced strategy space size is relatively small comparing with the number of strategies at play, the fluctuation of the attendance is large in MG due to the overcrowding effect of player’s strategies. Under the overcrowding effect, players tend to choose the same alternative for the same global history since they use some very similar strategies. However, in NMG, players can choose different alternatives even they use the same strategy since they consider both their local information and the global information in deciding their choice. Such phenomenon will dilute the overcrowding effects of the strategies in NMG and so players can perform much better than in MG when \(\alpha\) is small.

When the control parameter \(\alpha\) increases, the overcrowding effect will be suppressed in both NMG and MG. In MG, the maximal cooperation amongst the players can be achieved subsequently when the number of strategies at play is approximately equal to the reduced strategy space size. However, in NMG, players need to cooperate with each other through both the global information and their local information. These mixed types of information obtained from different sources makes the cooperation amongst all players becomes much more difficult for all value of \(M_g/M_l\). So the variance in NMG is larger than in MG around the critical point \(\alpha_c\) and it tends to the coin-toss value (which is the variance resulting from players making random choices throughout the game) when \(\alpha\) increases further.

B. NMG with global information (i.e. \(M_g \neq 0\))

Fig. 2 shows the variance of attendance per player \(\sigma^2/N\) as a function of the control parameter \(\alpha\) in NMG for different ratio of \(M_g\) to \(M_l\). When \(M_l < M_g\), the variance becomes larger as \(M_l\) increases regardless of the value of the control parameter \(\alpha\). Moreover, the variance tends to the coin-toss value for \(M_l \approx M_g\) over all range of \(\alpha\).
Furthermore, when \( M_t > M_g \), the performance of players becomes better if more local information is available no matter what is the value of \( \alpha \). To account for this phenomenon, we should consider the structure of strategies in the reduced strategy space \( R_{N_c}(M) \). When \( M_t < M_g \), each strategy is composed of more segment strategies belonging to a smaller strategy space \( V_{N_c}(M_g) \) as \( M_t \) increases. Therefore, players are more likely to use similar strategies and the overcrowding effect of strategies will dominate which results in large fluctuations of the attendance. Similarly, when \( M_t > M_g \), the segment strategies are drawn from \( V_{N_c}(M_t) \) and thus the variance becomes larger as \( M_g \) increase due to the overcrowding effect of strategies. In particular, when \( M_t \approx M_g \), each strategy is composed of the largest number of segment strategies drawn from the smallest strategy space \( (V_{N_c}(M_t) \text{ or } V_{N_c}(M_g)) \) and so the variance is the largest for such case.

### C. NMG with no global information (i.e. \( M_g = 0 \))

When \( M_g = 0 \), only local information is available to each player. From numerical simulation, the variance in NMG with \( M_g = 0 \) is found to be smaller than in MG and NMG with \( M_g \neq 0 \) when the control parameter \( \alpha \) is small and approaches zero. It implies that the cooperation amongst players in NMG with \( M_g = 0 \) is much better than in MG and NMG with \( M_g \neq 0 \) for small \( \alpha \). In fact, player’s cooperation in NMG with no global information is resulted from their local interaction. To investigate the local interaction, we calculate the correlation function for the local minority choices of two neighboring players as follows:

\[
\rho_{i-1,i} = \frac{\langle \Omega_i^{-1}(t) \Omega_i^{-1}(t) \rangle - \langle \Omega_i^{-1}(t) \rangle \langle \Omega_i^{-1}(t) \rangle}{\langle \Omega_i^{-1}(t) \rangle^2 - \langle \Omega_i^{-1}(t) \rangle^2},
\]

where \( \Omega_i^{-1}(t) \) is the local minority choice of the \( i \)th player at time \( t \). We reveal that the correlation function \( \rho_{i-1,i} \) is equal to infinity for many neighboring players when \( M_g = 0 \). That is to say, many players always choose the same alternative as their nearest neighbors throughout the game. In such case, we found that the local histories of these players remain the same such that their local minority choices are completely frozen. In fact, the decision of a frozen player is dominated by the components of segment strategies corresponding to \( \mu_g = (0, \ldots, 0) \) or \( \mu_g = (1, \ldots, 1) \) whenever \( M_g = 0 \). For instance, suppose the components of segment strategies corresponding to \( \mu_g = (0, \ldots, 0) \) equal 1 for all \( S \) strategies of two nearest neighbors. Once their local histories are \((0, \ldots, 0)\), their own local minority will then be 0 in this turn and their local histories will be \((0, \ldots, 0)\) again at the next time step. It is a fixed point of the dynamics leading to the dimension reduction of the reduced strategy space. If similar configurations arise along the ring, the choices of some players can be determined before each turn because their actions are frozen. The freezing effect of players will minimize the fluctuations of attendance since it is only contributed by few non-frozen players. Indeed, it is the negligence of global information that allows the strong local interactions amongst players who are connected on a ring. On the contrary, players can only weakly interact with each other locally if they are placed on a dynamically random chain in which each player is connected to two randomly chosen players with all the connections between players change at every time step. Hence, whenever \( M_g = 0 \), the freezing effect of players disappears and the variance tends to the coin-toss value for the case of the dynamically random chain as shown in fig. 3.

On the other hand, numerical results show that the probability for the freezing of players decreases exponentially when \( M_g \) increases for \( M_t > M_g \). It is because each strategy is composed of more number of segment strategies and the disturbance from the global information becomes more significant when more global information is available. So the variance becomes larger as \( M_t \) decreases for \( M_t > M_g \).

### D. Comparison of the numerical results of NMG with the predictions by the crowd-anticrowd theory

In order to investigate the crowding effect in NMG, we compare the numerical results of the variance with the predictions by the crowd-anticrowd theory. We find a large discrepancy between them whenever \( M_g \gg M_t \) or \( M_t \gg M_g \) although their trends are consistent with each other for all value of \( M_t/M_g \). According to the crowd-anticrowd theory, the variance of attendance for a given realization of the quenched disorder \( \Omega \) (i.e. the initial configuration of the system) is given by

\[
\sigma^2_{\Omega} = \sum_R \langle a_R^{\mu(t)}(n_R(t))^2 \rangle + \sum_R \langle a_R^{\mu(t)} a_{R'}^{\mu(t)} n_R(t) n_{R'}(t) \rangle + \sum_{R \neq R' \neq R} \langle a_R^{\mu(t)} a_{R'}^{\mu(t)} n_R(t) n_{R'}(t) \rangle,
\]

where \( \sum_R \) denotes the sum over the strategies in the corresponding reduced strategy space, \( a_R^{\mu(t)} \) is the action of strategy \( R \) to the history \( \mu \) at time \( t \) (\( a_R^{\mu(t)} = \pm 1 \) corresponds to the two alternatives) and \( n_R(t) \) is the number
of players using strategy $R$ at time $t$. In MG, $\sum_{R \neq R'} \langle a_{R}^{\mu(t)} a_{R'}^{\mu(t)} n_{R}(t) n_{R'}(t) \rangle$ should be equal to zero and thus is dropped. (For MG, the variance in fact plays the role of the effective Hamiltonian of the spin-glass-like system while the strategies can be interpreted as the quenched disorder [18, 19]... Thus, we can state that the four-point correlation function is dropped in the effective Hamiltonian of the system for MG.) Such expectation is based on the fact that the number of players using an uncorrelated strategy pair are, on average of the history $\mu$, independent from the response of the strategy pair to the history in MG. However, from numerical simulation, we find that $\sum_{R \neq R'} \langle a_{R}^{\mu(t)} a_{R'}^{\mu(t)} n_{R}(t) n_{R'}(t) \rangle$ is equal to a large negative number for NMG with $M_{g} \gg M_{l}$. In other words, the uncorrelated segment strategy pairs are no longer independent with each other. Through the local interaction, players tend to use the uncorrelated segment strategy pairs whenever the pairs choose the opposite choice. It is a favorable solution of the dynamics because the uncorrelated segment strategies can now contribute to the crowd-anticrowd cancellation which will in turn lead to the further reduction of the fluctuation of attendance. In fact, the crowd-anticrowd cancellation by the uncorrelated strategies is only possible in NMG since players have the freedom to choose which uncorrelated segment strategies to be used on the basis of the local information. We can correct the semi-analytical results of the crowd-anticrowd theory by counting the contribution of both the anti-correlated and uncorrelated segment strategies to the crowd-anticrowd cancellation. The corrected semi-analytical results is found to match with the numerical one as shown in fig. [4] So we conclude that the discrepancy is mainly due to the crowd-anticrowd cancellation by the uncorrelated strategies for $M_{g} \gg M_{l}$.

When $M_{l} \gg M_{g}$, the discrepancy of the predictions of the crowd-anticrowd theory from the numerical results is due to the freezing of players. The ergodicity of the local histories of frozen players is broken down under the freezing effects. In other words, some possible states of the local histories will never be visited for frozen players since the effective dimension of the strategy space is reduced. Of course, this will violate the ergodicity assumption in the crowd-anticrowd theory. So the results predicted by the crowd-anticrowd theory show a large discrepancy from the numerical results when $M_{l} \gg M_{g}$.

IV. CONCLUSION

In summary, we find that NMG exhibits a remarkably different behavior from MG. In NMG with non-zero global information, selfish players on the ring can cooperate with each other to reduce their loss; moreover, their cooperation is much better than in MG by using the local information that are disseminated through the ring except when the number of strategies at play is approximately equal to the strategy space size. Such phenomenon is believed to be due to the dilution of the crowding effect of players which is resulted from their local interactions on the ring.

In NMG with no global information, many players on the ring are found to be frozen where their action and also their local minority choice remain the same throughout the game. We reveal that the freezing occurs when their decisions are dominated by the components of segment strategies corresponding to $\mu_{2} = (0, \ldots, 0)$ or $\mu_{2} = (1, \ldots, 1)$. Such domination arises because of the strong local interaction on the ring due to the absence of the global information.

On the other hand, we find that the predictions of the crowd-anticrowd theory deviate very much from the numerical results for NMG. Such discrepancy is found to be due to the crowd-anticrowd cancellation contributed by the uncorrelated strategies in NMG whereas it is impossible for the uncorrelated strategies to contribute to the crowd-anticrowd cancellation in MG.

In fact, all the above arguments should still be valid when players are connected on a network with different topology in NMG provided that each player always obtains the local information from the same neighbors; and thus we believe that the cooperative behavior of players are similar in all these cases.

Finally, we would like to point out that the order parameter is also worth to be studied for NMG. From this study, we may learn whether there is a phase transition from a symmetric phase to an asymmetric phase as the complexity of the system increases just like that in MG.

[1] D. Challet and Y. C. Zhang, Physica A 246, 407 (1997).
[2] Y. C. Zhang, Europhys. News 29, 51 (1998).
[3] W. B. Arthur, Amer. Econ. Assoc. Papers and Proc. 84, 406 (1994).
[4] D. Challet and Y. C. Zhang, Physica A 256, 514 (1998).
[5] R. Savit, R. Manuca and R. Riolo, Phys. Rev. Lett. 82, 2203 (1999).
[6] N. F. Johnson, S. Jarvis, R. Jonson, P. Cheung, Y. R. Kwong and P. M. Hui, Physica A 258, 230 (1998).
[7] M. Hart, P. Jefferies, N. F. Johnson and P. M. Hui, Physica A 298, 537 (2001).
[8] M. Hart, P. Jefferies, N. F. Johnson and P. M. Hui, Eur. Phys. J. B 20, 547 (2001).
[9] M. L. Hart and N. F. Johnson, cond-mat/0212088.
[10] M. Paczuski, K. E. Bassler and A. Corral, Phys. Rev. Lett. 84, 3185 (2000).
[11] T. Kalinowski, H.-J. Schulz and M. Briese, Physica A 277, 502 (2000).
[12] S. Moelbert and P. De Los Rios, Physica A 303, 217 (2002).
[13] E. Burgos, H. Ceva and R. P. J. Perazzo, cond-mat/0212635.
[14] H.-J. Quan, B.-H. Wang, P. M. Hui and X.-S. Luo, Physica A 321, 300 (2003).
[15] F. K. Chow and H. F. Chau, Physica A 319, 601 (2003).
[16] H. F. Chau and F. K. Chow, Physica A 312, 277 (2002).
[17] F. K. Chow and H. F. Chau, cond-mat/0210608.
[18] D. Challet and M. Marsili, Phys. Rev. E 60, R6271 (1999).
[19] D. Challet, M. Marsili and R. Zecchina, Phys. Rev. Lett. 84, 1824 (2000).
FIG. 1: The network of $N$ players on a ring for NMG.
FIG. 2: The variance of the attendance per player $\sigma^2/N$ (opaque circles) versus the control parameter $\alpha$ for NMG with $N_c = 2$, $S = 2$ and $M = M_g + M_l = 5$ where players are connected on a ring. The crosses are the predictions of the crowd-anticrowd theory whereas the dashed lines indicate the coin-toss value, i.e., the corresponding value in the random choice game. For comparison purpose, the solid lines indicate the corresponding numerical results in the original MG.
FIG. 3: The variance of the attendance per player $\sigma^2/N$ (opaque circles) versus the control parameter $\alpha$ for NMG with $N_c = 2$, $S = 2$ and $M = M_g + M_l = 5$ where players are connected on a dynamically random chain. The crosses are the predictions of the crowd-anticrowd theory whereas the dashed lines indicate the coin-toss value.

\[ \alpha = 2^{(M_g+1)2^M/NS} \]

FIG. 4: The variance of the attendance per player $\sigma^2/N$ (opaque circles) versus the control parameter $\alpha$ for NMG with $N_c = 2$, $S = 2$ and $M = M_g + M_l = 5$ where players are connected on a ring. The crosses are the corrected semi-analytical results predicted by the crowd-anticrowd theory (by counting the contribution of both the anti-correlated and uncorrelated segment strategies to the crowd-anticrowd cancellation) whereas the dashed lines indicate the coin-toss value.

\[ \alpha = 2^{(M_g+1)2^M/NS} \]