Finding zeros of the Riemann zeta function by periodic driving of cold atoms

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The Riemann hypothesis, which states that the non-trivial zeros of the Riemann zeta function all lie on a certain line in the complex plane, is one of the most important unresolved problems in mathematics. Inspired by the Pólya-Hilbert conjecture, we propose a new approach to finding a physical system to study the Riemann zeros, which in contrast to previous examples, is based on applying a time-periodic driving field. This driving allows us to mould the quasienergies of the system (the analogue of the eigenenergies in the absence of driving), so that they are directly governed by the zeta function. We further show by numerical simulations that this allows the Riemann zeros to be measured in currently accessible cold atom experiments.

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Introduction – The Riemann hypothesis states that the non-trivial zeros of the Riemann function, \( \zeta(s) \), have the form \( s_n = 1/2 \pm iE_n \), where the \( E_n \) are all real. A fascinating approach to treating the Riemann problem is to consider the \( E_n \) to be eigenvalues of a self-adjoint operator, since if such an operator could be identified the \( E_n \) would necessarily be real. Attention has therefore focused on finding a system whose Hamiltonian can be related to such an operator, and thus whose energy spectrum is governed by \( E_n \). This idea, known as the Pólya and Hilbert conjecture, is supported by numerous evidence (see [1, 2] for reviews), notably that the zeros appear to closely follow the GUE statistics [3, 4] of random matrix theory. Berry and Keating [5], and Connes [6], proposed that one such model was given by the xp Hamiltonian and a considerable amount of attention has been directed at developing this proposal, and finding physical realizations of it [7–18].

In this work we propose a very different approach, in which a time-periodic driving potential is used to modify the dynamics of a quantum system. As the system is periodically-driven its dynamics is not described by energy eigenvalues, but by a generalization of these quantities termed “quasienergies”. Our central result is the construction of a driving field for which the corresponding quasienergy spectrum is given by the Riemann \( \Xi \) function [19, 20], or by a smoothed version of this function, \( \Xi^* \), introduced by Pólya. This last has the appealing feature that the appropriate driving field can be expressed in a simple closed form. In both cases the zeros of the \( \Xi \) function correspond to degeneracies, or crossings, of the quasienergies. These are of particular physical significance because they correspond to the phenomenon known as “coherent destruction of tunneling” (CDT) in which the dynamics of the system is frozen [21]. As an example we show how the zeros can be seen directly in cold atom experiments by measuring the expansion rate of a condensate held in a driven optical lattice. This technique thus represents a new and powerful way of finding a physical realization of the Riemann function.

Method – We begin by considering a standard two-level system, driven by a time-periodic function \( f(t) = f(t + T) \)

\[
H(t) = -J \sigma_x + f(t)/2 \sigma_z ,
\]

where \( J \) is the tunneling between the two levels. As \( f(t) \) is time-periodic, the natural framework to treat the problem is given by Floquet theory [22]. In this approach one seeks the eigensystem of the Floquet operator

\[
\mathcal{H}(t) = H(t) - i\partial_t ,
\]

where we have set \( \hbar = 1 \). Henceforth we shall also measure all energies (and frequencies) in units of \( J \). The eigenstates of \( \mathcal{H}(t) \) are \( T \)-periodic functions called Floquet states, and their associated eigenvalues, which play an analogous role to energy eigenvalues for the case of a static Hamiltonian, are called quasienergies. The Floquet states provide a complete basis, and expanding the wavefunction in these states provides similar advantages to the normal procedure of expressing a state in energy eigenstates in the undriven case.

In general it is difficult to obtain analytical expressions for the Floquet states. In the strong-driving limit, however, when the frequency \( \omega = 2\pi/T \) is the dominant energy scale, it is possible to make an expansion by first solving just for the time-dependent component of \( H(t) \), and applying the static part as perturbation [23]. In this way one obtains a perturbative series in orders of \( J \). Truncating at first-order gives the simple result \( H_{\text{eff}} = -J_{\text{eff}} \sigma_x \), where the effective tunneling \( J_{\text{eff}} \) is given by

\[
J_{\text{eff}}/J = \frac{1}{T} \int_0^T dt \ e^{-iF(t)} ,
\]
and $F(t) = \int_0^t dt' f(t')$. The quasienergies are given simply by the eigenvalues of $H_{\text{eff}}$, namely $\epsilon_k = \pm | J_{\text{eff}} |$.

From Eq.\ref{eq:3} it is thus straightforward to calculate the behavior of the quasienergies (or equivalently, of the effective tunneling) for a given driving potential $f(t)$. For the case $f(t) = K \cos \omega t$, for example this yields the well-known Bessel function renormalization \cite{21} of tunneling $J_{\text{eff}} = J J_0(K/\omega)$. At zeros of the Bessel function $K/\omega = 2.404, 5.520 \ldots$ the effective tunneling vanishes, producing CDT. This effect has been measured experimentally \cite{22} in the expansion dynamics of a driven gas of ultracold atoms.

We need, however, to solve the inverse problem; to find an $f(t)$ that produces a given behavior of the quasienergies. We shall first explain our technique using Pólya’s function, $\Xi^*(E)$, as this gives a convenient closed-form for the solution, and then go on to consider the more complicated case of the true Riemann $\Xi$ function, deferring most of the details to the Supplemental Material.

Pólya’s function is given by

$$\Xi^*(E) = 4\pi^2 \left( K_{a+iE/2}(2\pi) + K_{a-iE/2}(2\pi) \right),$$

where $K_{\beta}(t)$ is the modified $K$-Bessel function and $a = 9/4$. This is a smoothed version of the Riemann $\Xi$ function

$$\Xi(E) = \frac{1}{4} s(s-1) \Gamma \left( \frac{s}{2} \right) \pi^{-s/2} \zeta(s), \quad s = 1/2 + iE$$

whose zeros coincide with the non-trivial zeros of $\zeta(s)$. Pólya’s zeta function shares many properties with $\Xi(E)$, most importantly that it has the same average distribution of zeros \cite{20, 26}. Pólya further proved that the zeros of $\Xi^*(E)$ are real for any value of the constant $a$. The spectrum of the $xp$-type \cite{9} and Dirac Hamiltonian \cite{18} is given, for example, by the zeros of $\Xi$ with $a = 1/2$.

The modified Bessel function can be conveniently expressed as the integral identity

$$K_\beta(x) = \int_0^\infty dt \cosh(\beta t) e^{-x \cosh t},$$

and thus

$$\Xi^*(E) = 8\pi^2 R \left[ K_{a+iE/2}(2\pi) \right] = 8\pi^2 \int_0^\infty dt \cosh(at) e^{-2\pi \cosh t} \cos(Et/2).$$

We are aiming to obtain the result $R[J_{\text{eff}}(E)] \propto \Xi^*(E)$.

Combining Eqs.\ref{eq:3} and \ref{eq:7} reveals that this requires $F(t)$ to obey the relation

$$\int_0^T dt \cos F(t) = \int_0^\infty dt \alpha \cosh(at) e^{-2\pi \cosh t} \cos(Et/2),$$

where $\alpha$ is an arbitrary constant. As the integrand on the right-hand side decays rapidly with $t$, we can replace the upper limit of integration with $T$, as long as we take $T$ to be sufficiently large. In this case we can then write

$$F(t) = \cos^{-1} [ \alpha \cosh(at) e^{-2\pi \cosh t} \cos(Et/2) ].$$

The value of $\alpha$ is now fixed by noting that we require $F(0) = 0$, and thus $\alpha = e^{\pi i}$. The driving field is then given by $f(t) = \partial_t F(t)$, yielding the final result

$$f(t) = \frac{\phi(t) (a \tanh at - 2\pi \sinh t - E \tan Et)}{\sqrt{1 - \phi(t)^2}}$$

where $\phi(t) = \cosh(at) e^{2\pi (1-\cosh t)} \cos(Et)$. \cite{27}.

In Fig.\ref{fig:1} we show the form of $f(t)$ as $E$ is varied. Little structure is visible for $t > 1.5$, and accordingly we now set the period of the driving, and thus the cut-off in the integration in Eq.\ref{eq:8} to be $T = \pi/2$. We shall use this value throughout the rest of the paper. In Fig.\ref{fig:2} we show the full periodic driving field, obtained by periodically repeating cycles of $f(t)$. Although this form of $f(t)$ indeed satisfies \cite{8}, the driving can be made more effective by imposing a further set of conditions on it: $i$) to avoid heating in the cold atom model (see below), the average of $f(t)$ over one period should vanish, $ii)$ discontinuities should be avoided, and $iii)$ for the quasienergy crossings to be well-defined, the Floquet states must be from different parity classes. If condition $iii)$ is not fulfilled, the von Neumann-Wigner theorem implies that the quasienergies cannot cross as $E$ is varied, and will instead form a broad avoided crossing. As the system is periodically-driven, the appropriate generalized parity operator involves both inversion and time-translation $P : x \rightarrow -x$, $t \rightarrow t + T/2$. These three conditions can be
FIG. 2: (a) The driving potential, \( f(t) \), is extended to be a periodic function, by periodically repeating its behavior over the interval \( 0 \leq t < T \). This produces, however, a function with rather poor performance. (b) Construction of a more efficient driving potential. Four copies of \( f(t) \) are joined together, following some reflection transformations, to create a single continuous function, resembling two localized pulses of opposite sign. The vertical dashed lines indicate where the segments are joined. This function is continuous, has a time-average of zero, and \( f(t) = -f(t + T_0/2) \) where \( T_0 = 4T \) is the total period of the signal, satisfying the parity requirement given in the text.

satisfied by joining four copies of the fundamental waveform \( \Xi^* (E) \) as shown in Fig.2, to create what amounts to two pulses of opposite sign, with a total period of \( T_0 = 4T \).

Having obtained the solution for \( \Xi^* (E) \), we now turn to the more important case of the Riemann function itself. This can be written

\[
\Xi(E) = \int_0^\infty dt \ \Phi(t) \cos \left( Et/2 \right) ,
\]

(11)

where \( \Phi(t) = 2\pi e^{5t/4} \sum_{n=1}^\infty \left( 2\pi e^t n^2 - 3 \right) n^2 e^{-\pi n^2 e^t} \). We now follow the same procedure as for the smoothed \( \Xi \) function (details given in the Supplementary Material), to obtain the result

\[
F(t) = \cos^{-1} \left[ \Phi(t)/\Phi(0) \cos \left( Et/2 \right) \right] ,
\]

(12)

with the driving field given by \( f(t) = \partial_t F(t) \). As with the case of the smoothed \( \Xi \) function, for \( F(t) \) to be real the argument of the inverse cosine in Eq.12 must lie between \( \pm 1 \), thus requiring \( |\Phi(t)/\Phi(t = 0)| \leq 1 \). As \( \Phi(t) \) is a concave function with a maximum at \( t = 0 \), this condition is indeed guaranteed.

Results – The perturbational approach we have made is only valid in the high-frequency limit. As we use \( T = \pi/2 \), the frequency of the driving potential \( \omega = 2\pi/T_0 = 1 \), which is, however, not high. Accordingly, to ensure that this condition is satisfied, we scale the driving as \( f(t) \rightarrow \Omega f(\Omega t) \), where \( \Omega > 1 \), while keeping \( T_0 = 2\pi \) constant. This has the effect of narrowing the pulses and increasing their amplitude, while keeping their spacing unchanged. As we show in the Supplementary Material, good precision is obtained for \( \Omega > 4 \), and the results presented use \( \Omega = 128 \). Increasing \( \Omega \) beyond this value was found to introduce instabilities in the numerical integration of the system’s time-evolution.

In Fig.3, we compare the quasienergies, obtained by the direct integration of the equation of motion (1) under the driving potential given in Eq.11 with the exact behaviour of \( \Xi^* (E) \). The agreement is seen to be excellent. Similarly in Fig.3b we compare the Riemann \( \Xi \) function with the quasienergies resulting from driving the system with the potential derived from Eq.12 and again see essentially perfect agreement. As the \( \Xi \) functions decay roughly exponentially with \( E \), we show in Fig.3c the same data plotted logarithmically. The cusps visible in this plot correspond to zeros of the \( \Xi \) functions, and thus to crossings of the quasienergies at which \( J_{\text{eff}} \) vanishes. We see that the quasienergies accurately reproduce the behavior of the \( \Xi \) functions over at least six orders of magnitude, although eventually precision effects do lead to deviations at large values of \( E \).

A way of directly measuring \( J_{\text{eff}} \) in experiment is to observe the expansion of a gas of cold atoms \( \cite{21, 22} \). If the atoms are held in an optical lattice potential they can be described well by a tight-binding model

\[
H_0 = -J \sum_{\langle i,j \rangle} (a_i^\dagger a_j + \text{H.c.}) + \sum_j V(r_j) n_j ,
\]

(13)

where \( i \) labels the lattice site, \( \langle i,j \rangle \) are nearest-neighbors, and \( V(r) \) is a trap potential. By “shaking” the optical lattice \( \cite{28} \) it is possible to introduce a time-periodic driving potential \( H(t) = H_0 + f(t) \sum_j r_j n_j \) which generalizes Eq.10 from a two-level model describing two sites to the case of \( N \) lattice sites. For a parabolic trap potential, the initial state of the system will be Gaussian. If the trap potential is then released, this Gaussian wavepacket will undergo free expansion at a rate governed by \( |J_{\text{eff}}| \) \( \cite{24} \).

In Fig.4a we show the spread of the wavepacket with time, \( \sigma(t) = \sqrt{(\langle x^2 \rangle - \langle x \rangle^2)} \), under the periodic driving corresponding to \( \Xi^* (E) \). For \( E = 0 \) the wavepacket expands rapidly, and soon enters the ballistic regime in which \( \sigma \propto t \). For \( E = 4 \) the expansion is slower since \( J_{\text{eff}} \) is smaller, and for \( E = 9 \) the wavepacket barely expands at all, indicating that this value of \( E \) is close to a zero of \( \Xi^* (E) \).

These estimations can be made quantitative. In Fig.4b we show the values of \( |J_{\text{eff}}| \) obtained by measuring the spread of the wavepacket after an expansion time of \( 50T_0 \) and fitting it to the result \( |J_{\text{eff}}| \propto \sqrt{\sigma(t)^2 - \sigma_0^2} \), where \( \sigma_0 \) is the initial width of the condensate. Both of the driving potentials show excellent agreement with the exact
level system driven by the potential $f(t)$ given by Eq. (11), such that $J_{\text{eff}} \propto \Xi(E)$. For $E = 0$ the wavepacket spreads quickly, but for $E = 4$ the wavepacket spreads less rapidly, indicating that $J_{\text{eff}}$ is smaller. For $E = 9$ the wavepacket hardly expands at all, indicating that CDT is occurring. (b) The solid black curve shows $|\Xi(E)|$, and the black circles the values of $|J_{\text{eff}}|$ measured from the expansion curves. The agreement between the two is excellent. Similarly the dotted red line shows $|\Xi(E)|$, and the red squares the measured values of $|J_{\text{eff}}|$ for a system driven by the potential defined by (12). Again the agreement is perfect.

FIG. 4: (a) Expansion of a Gaussian wavepacket in an optical lattice with $N = 200$ sites under the periodic driving potential $f(t)$ [10], such that $J_{\text{eff}} \propto \Xi(E)$. For $E = 0$ the wavepacket spreads quickly, but for $E = 4$ the wavepacket spreads less rapidly, indicating that $J_{\text{eff}}$ is smaller. For $E = 9$ the wavepacket hardly expands at all, indicating that CDT is occurring. (b) The solid black curve shows $|\Xi(E)|$, and the black circles the values of $|J_{\text{eff}}|$ measured from the expansion curves. The agreement between the two is excellent. Similarly the dotted red line shows $|\Xi(E)|$, and the red squares the measured values of $|J_{\text{eff}}|$ for a system driven by the potential defined by (12). Again the agreement is perfect.

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Conclusions – Other physical implementations of the Riemann functions have been suggested previously, such as measuring the escape of particles from a leaky billiard [29], or by evaluating the Fourier transform of a suitable initial wave using either opto-mechanical means [30] or far-field diffraction [31]. We take a rather different approach of using a time-dependent driving potential to modify the dynamics of a quantum system, so that its quasienergy spectrum mimics the desired Riemann function. We provide a systematic scheme for calculating the appropriate potential, and suggest a physical realization using driven cold atoms. Although the observation of high Riemann zeros is hindered by the rapid decay of the $\Xi$ function, this could be compensated in experiment by increasing the tunneling strength (which depends exponentially on the optical lattice depth), or by using a form of the zeta function which decays more slowly, such as $\xi(s)/s$ [30, 31]. An intriguing possibility for future study would be to apply this technique to more general trigonometric integrals of the function $\Phi(t)$, which could thereby lead to the measurement of a new bound on the de Bruijn-Newman constant, whose value is related to the Riemann hypothesis [32].

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Supplementary Material

DERIVATION OF THE DRIVING POTENTIAL FOR Ξ(E)

We begin with the expression for the Riemann Ξ function given in Eq. (11)
\[
Ξ(E) = \int_0^\infty dt \, \cos(E t/2) ,
\]
where the function Φ(t), related to the Fourier transform of Ξ(E), is defined as (19)
\[
Φ(t) = 2\pi e^{5t/4} \sum_{n=1}^\infty (2\pi e^2 n^2 - 3) n^2 e^{-\pi n^2 t}, \quad 0 \leq t < \infty.
\]
(15)
Following the same procedure as for the smoothed Riemann function, we seek a driving potential such that \( \Re [J_{eff}(E)] \propto Ξ(E) \). This requires finding the solution of the equation
\[
\int_0^T dt \, \cos F(t) = \alpha \int_0^\infty dt \, \cos(E t/2) ,
\]
(16)
where \( \alpha \) is a constant. We show the behavior of Φ(t) in Fig. 5. Since \( Φ(t) \) decreases rapidly with t, being fitted reasonably well by a simple Gaussian function, we can substitute the upper limit of integration on the right-hand side of Eq. (16) to be T, as long as T is sufficiently large for the integral to converge. We can then write
\[
F(t) = \cos^{-1} [\alpha Φ(t) \cos(E t/2)] .
\]
(17)
Imposing the boundary condition \( F(0) = 0 \) requires \( \alpha \) to take the value \( \alpha = 1/Φ(0) \). As Φ(0) is the global maximum of the function, this guarantees that \( |α Φ(t)| \leq 1 \) for all values of \( t \), and thus \( F(t) \) is real and well-defined.

![FIG. 5: Behaviour of Φ(t) and its derivative. Solid black line: Φ(t), defined in Eq. (19). It has a global maximum at \( t = 0 \), for which \( Φ(0) \approx 0.89339 \), and decays rapidly as \( t \) increases. Dashed red line: the derivative, \( Ψ(t) = \partial_\Phi(t) \), evaluated from Eq. (19). The series expansions for both Φ(t) and Ψ(t) were truncated at 30 terms.](image-url)
small differences. These minor differences lead to the two contour plots), there are nonetheless significant differences. We show here the driving functions for \( E = 4 \) for the two Riemann functions we consider.

The derivative of \( \Phi(t) \) can be evaluated by differentiating Eq.13 term by term, to give

\[
\Phi'(t) = \frac{\pi e^{5t/4}}{2} \sum_{n=1}^{\infty} \left( 30e^t n^2 \pi - 8e^{2t} n^4 \pi^2 - 15 \right) n^2 e^{-\pi n^2 e^t},
\]

which we also show in Fig.3.

We show the full behavior of \( f(t) \) as a function of time and the parameter \( E \) in Fig.4. Despite its complicated form, this plot shows a striking similarity to Fig.1 of the main paper, in which the \( f(t) \) giving rise to \( \Xi^*(E) \) is shown. This should not be unexpected as \( \Xi^*(E) \) is simply a smoothed version of the Riemann function, and consequently the main features of the driving functions must be the same.

Just as for the case of the smoothed Riemann function, a more effective driving potential is obtained by joining four copies of \( f(t) \), to produce a continuous function with a definite parity and zero time average (see Fig.2 of the main paper). In Fig.7 we show the form of the driving function we obtain in this way, and compare it with the driving function that produces \( \Xi^*(E) \). Examining the two curves in detail, reveals that although the functions share the same general form (as seen in the similarity of the two contour plots), there are nonetheless significant small differences. These minor differences lead to the different location of zeros for \( \Xi(E) \) and \( \Xi^*(E) \), demonstrating the well-known analytical sensitivity of the zeta function.

**HIGH FREQUENCY LIMIT**

The expression for the effective tunneling, \( J_{\text{eff}} \), that we use in this work comes from a first-order perturbation theory calculation of the driven system. In this perturbation theory, the “small parameter” is \( J/\omega \), and so this result is only valid in the high-frequency limit \( \omega \gg J \). This would require the driving period, \( T = 2\pi/\omega \), to be small; however this contradicts the requirement that \( T \) be as large as possible so that the integral (16) is well-converged. To be able to satisfy both these requirements, we therefore scale the driving as \( f(t) \to \Omega f(\Omega t) \), where \( \Omega > 1 \). Note that this is not a trivial rescaling of time, as its effect is to make the pulses, such as those shown in Fig.7, narrower and taller, while keeping their spacing constant at \( T \).

We show the effect of increasing \( \Omega \) on the pulse-shape in Fig.8. It can be clearly seen that the pulses become progressively more localized and of higher amplitude. As the pulses become shorter and more intense, the quasienergy spectrum of the system approaches that of the high-frequency limit, as we show in Fig.8. In particular we can see that the locations of the quasienergy crossings converge toward the zeros of the Riemann function. This evolution of the quasienergy spectrum as the driving frequency increases is a general feature of periodically-driven systems, including for example, sinusoidal, squarewave, and triangular driving.

In principle \( \Omega \) should be made as large as possible, to ensure that the system is well within the high-frequency regime. Our numerical results for the time-evolution of the driven systems were all obtained using a fourth-order
Runge-Kutta scheme with an adaptive timestep. With this algorithm we found an effective limit of $\Omega \sim 150$ above which numerical instabilities set in. Accordingly we made use of $\Omega = 128$ throughout our work.

FIG. 8: (a) Change of the driving function as $\Omega$ is increased. The pulses become progressively narrower and taller, while their spacing remains the same. (b) As $\Omega$ increases, the quasienergy spectrum asymptotically approaches that of the infinite frequency limit, and the quasienergy crossings move towards the zeros of the Riemann function. Here we show the quasienergies for a system driven to reproduce the smoothed Riemann function, $\Xi^*(E)$. Note how the first crossing asymptotically approaches the first zero of $\Xi^*(E)$ at $E = 8.993$ as $\Omega$ increases.