Intuitive Analyses via Drift Theory

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Abstract
Humans are bad with probabilities, and the analysis of randomized algorithms offers many pitfalls for the human mind. Drift theory is an intuitive tool for reasoning about random processes. It allows turning expected stepwise changes into expected first-hitting times. While drift theory is used extensively by the community studying randomized search heuristics, it has seen hardly any applications outside of this field, in spite of many research questions which can be formulated as first hitting times.

We state the most useful drift theorems and demonstrate their use for various randomized processes, including approximating vertex cover, the coupon collector process, a random sorting algorithm, and the Moran process. Finally, we consider processes without expected stepwise change and give a lemma based on drift theory applicable in such scenarios without drift. We use this tool for the analysis of the Gambler’s Ruin process, for a coloring algorithm, for an algorithm for 2-SAT, and for a version of the Moran process without bias.

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1 Drift Theory

Suppose that you win a million dollars in a lottery and that you start spending your winnings. You observe that you spend an average of $10k per day. How long will your lottery winnings last? Intuitively, you would divide the million you won by $10k and estimate that your winnings would last for 100 days. But isn’t that confusing a random process with a deterministic one? Well, yes, but the good news is: there is a theorem that tells us that 100 days is the mathematically precise answer, even when the process is randomized. Even better: if you gain money on some days (say, by playing in a casino) but still, in expectation, your balance goes down by $10k a day, the conclusion still holds. There can even be dependencies between the earnings/spendings of different days. The theorem showing that this is the case, is called the additive drift theorem (see Theorem 1). The term drift refers to the expected change of the stochastic process and, the term additive refers to the requirement that two successive values of the process differ, in expectation, by an additive constant.

A similar setting to that of the process described above is the well-known coupon collector process. Suppose there are n collectible kinds of coupons, of which you would like a complete set. Each iteration, you receive a uniformly random kind of coupon. How long does it take until you have a complete set? Note that if you still miss $X_t$ coupons from the complete collection after $t$ iterations, you have a chance of $X_t/n$ of getting a new one. Thus, the expected gain is $X_t/n$. This is a multiplicative expected progress and the multiplicative drift theorem (see Theorem 2) gives an upper bound of $O(n \log n)$. Again, this theorem also holds when there is a possibility of losing coupons, and it even gives a concentration bound. This brings us to the following description of drift theory:
Drift theory is a collection of theorems to turn iteration-wise expected gains into expected first-hitting times.

Drift theory is based on an intricate theorem by Hajek. This theorem was picked up by the community of theoreticians analyzing randomized search heuristics, such as evolutionary algorithms (EAs). EAs work by the principle of variation (mutating an individual by random changes) and selection (accepting improvements and rejecting worsenings), where computing the expected improvements per iteration is a straightforward idea. However, Hajek’s theorem is very complex and not straightforward to apply. Thus, He and Yao inferred from Hajek’s original result the additive drift theorem, which is much easier to understand and apply. The multiplicative drift theorem was originally derived from the additive; a proof not relying on Hajek’s result was given by Doerr and Goldberg. Since then, drift theory has been the dominant method for formally analyzing randomized search heuristics. There are drift theorems for analyzing processes whose drift is neither additive nor multiplicative (the variable drift theorems), processes whose drift goes in the wrong direction (the negative drift theorems), as well as drift theorems that show concentration bounds.

In this paper, we discuss drift theorems formally in Section 2 and then show that drift theory can simplify proofs of many classical results in the analysis of randomized algorithms and stochastic processes. Any researcher working in the field can probably make use of drift theorems, since many questions concerning the analysis of stochastic processes can be formulated as first-hitting times of a real-valued process.

In Section 3, we discuss three theorems where we easily uncover a drift and can apply drift theorems in a straightforward way. The first theorem is about a simple randomized algorithm that finds a 2-approximation for the vertex cover problem, a popular example that highlights the basic principles of a randomized algorithm. Using drift theory, the analysis is very intuitive and short. The second example is a formal proof of the coupon collector process mentioned earlier. Finally, our third example is the use of a drift theorem to analyze a sorting algorithm that swaps two randomly chosen elements whenever these elements are not in order.

In Section 4, we consider the Moran process, a stochastic process that arises in biology and models the spread of genetic mutations in populations. This gives an example of a somewhat more elaborate potential function that needs to be uncovered in order to apply a drift theorem. The analysis of the Moran processes is one of the rare cases where drift theory was used previously outside of the analysis of randomized search heuristics.

In Section 5, we consider stochastic processes that do not exhibit drift, such as martingales, where a direct drift analysis is not applicable. We show how such processes can be transformed in order to derive expected times of certain events. We then apply this idea to bound the first-hitting time of the classical gambler’s ruin process. Furthermore, we use this technique to bound the expected run time of two randomized algorithms – one for finding colorings of a graph and one for finding satisfying assignments of a 2-SAT formula – before we conclude our paper in Section 6.

Overall, we consider drift theory very intuitive and widely applicable, which also makes it a great tool to teach to students.

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1 Making a process real-valued typically requires the invention of a clever potential function, which is at the heart of every proof using drift theory.
2 The Classic Drift Theorems

He and Yao [7,8] derived a very simple yet elegant and tight statement about the first-hitting time from Hajek [6]: the additive drift theorem. It is very general in the sense that it requires almost no assumptions except for the bound on the drift.

Initially, every drift theorem assumed a bounded state space of the process that is being analyzed. Kötzing and Krejca [12] proved that this restriction is not necessary. Hence, we state the drift theorems for unbounded search spaces. However, for the sake of simplicity, we still assume discrete processes and only condition on the previous state although these restrictions are not necessary [12].

\begin{itemize}
\item \textbf{Theorem 1} (Additive drift [7,8]). Let \((X_t)_{t \in \mathbb{N}}\) be random variables over \(S \subseteq \mathbb{R}\), and let \(T = \inf\{t \in \mathbb{N} \mid X_t \leq 0\}\).
\begin{enumerate}
\item Assume that there is a value \(\delta > 0\) such that, for all \(s \in S \setminus (-\infty, 0]\) and all \(t \in \mathbb{N}\),
\[\mathbb{E}[X_t - X_{t+1} \mid X_t = s] \geq \delta \, ,\]
for all \(t \in \mathbb{N}\), we have \(X_t \geq 0\). Then
\[\mathbb{E}[T] \leq \frac{\mathbb{E}[X_0]}{\delta} \, .\]
\item Assume that there is a value \(\delta > 0\) such that, for all \(s \in S \setminus (-\infty, 0]\) and all \(t \in \mathbb{N}\),
\[\mathbb{E}[X_t - X_{t+1} \mid X_t = s] \leq \delta \, ,\]
there is a value \(c \geq 0\) such that, for all \(s \in S \setminus (-\infty, 0]\) and all \(t \in \mathbb{N}\),
\[\mathbb{E}[|X_t - X_{t+1}| \mid X_t = s] \leq c \, .\]
Then
\[\mathbb{E}[T] \geq \frac{\mathbb{E}[X_0]}{\delta} \, .\]
\end{enumerate}
\end{itemize}

Note that, for case 1, we cannot allow the process to assume smaller values than the target 0. This is demonstrated by the process \((X_t)_{t \in \mathbb{N}}\) with \(X_0 = 1\) and, for all \(t\), with probability \(1 - 1/n\), \(X_{t+1} = X_t\) and otherwise \(X_{t+1} = -n + 1\); even the expected time until \(X_t \leq 0\) is not correctly bounded.

For case 2, the expected step size of the process must be bounded, as demonstrated by the process \((X_t)_{t \in \mathbb{N}}\) with \(X_0 = 1\) and, for all \(t\), with probability \(1/2\), \(X_{t+1} = 0\) and otherwise \(X_{t+1} = 2X_t - 2\delta\).

Various processes exhibit a strength of the drift depending on the current value of the process. This motivates the study of drift theorems which take this variable strength into account. A frequent case is that of a drift proportional to the current value of the process, which is covered by the multiplicative drift theorem.

\begin{itemize}
\item \textbf{Theorem 2} (Multiplicative drift [5] with tail bounds [4]). Let \((X_t)_{t \in \mathbb{N}}\) be random variables over \(\{0, 1\} \cup S\), where \(S \subseteq \mathbb{R}_{>1}\), and let \(T = \inf\{t \in \mathbb{N} \mid X_t < 1\}\).
Assume that there is a value \(\delta > 0\) such that, for all \(s \in S \setminus \{0\}\) and all \(t \in \mathbb{N}\),
\[\mathbb{E}[X_t - X_{t+1} \mid X_t = s] \geq \delta s \, .\]
\end{itemize}
Then
\[ E[T] \leq \frac{1 + \ln E[X_0]}{\delta}. \]

Further, for all \( k > 0 \) and \( s \in S \),
\[ \Pr\left[ T > \frac{k + \ln s}{\delta} \middle| X_0 = s \right] \leq e^{-k}. \]

While the multiplicative drift theorem requires a very specific dependence on the current value of the process, the following variable drift theorem allows for any monotone dependency (meaning that a larger distance to the target has to imply a larger drift). Note that this theorem can be used also for additive or multiplicative drift with essentially the same upper bounds on the expected first-hitting time as when the corresponding theorem above was used.

\[ \text{Theorem 3 (Variable drift \[9,16\]).} \]

Let \((X_t)_{t \in \mathbb{N}}\) be random variables over \( \{0, 1\} \cup S \), where \( S \subset \mathbb{R}_{>1} \), and let \( T = \inf\{t \in \mathbb{N} \mid X_t < 1\} \).

If there exists a monotonically increasing function \( h: \mathbb{R}^+ \to \mathbb{R}_{\geq 0} \) such that, for all \( s \in S \setminus \{0\} \) and all \( t \in \mathbb{N} \),
\[ E[X_t - X_{t+1} \mid X_t = s] \geq h(s), \]
then
\[ E[T] \leq \frac{1}{h(1)} + \int_1^{E[X_0]} \frac{1}{h(x)} \, dx. \]

These three drift theorems are the most common ones and already cover a lot of applications. In fact, in this paper, we only need the additive and the multiplicative drift theorem; we stated the variable drift theorem solely as an illustration. Further drift theorem variants exist, for example, for lower bounds on multiplicative drift \[24\] or on variable drift \[13\]. Furthermore, there are concentrations bounds for hitting times under additive drift \[10\] and occupation probabilities under additive drift \[11\]. When the drift goes away from the target, we speak of negative drift. The negative drift theorem \[19,20\] gives an exponential lower bound in this setting.

A very different approach at understanding random processes via their stepwise changes is given by Wormald \[25\], modeling the processes as solutions of a system of differential equations.

### 3 Vertex Cover, Coupon Collector, and Random Inversion Sort

Our first example is a randomized algorithm for finding, in expectation, a 2-approximation of the classical vertex cover problem. For an undirected graph \((V,E)\), a subset \( C \subseteq V \) such that, for all \( \{u,v\} \in E \), \( u \) or \( v \) is in \( C \) is called a vertex cover. By using the additive drift theorem (Thm. \[1\]), we can easily bound the expected size of the vertex cover that the algorithm constructs.

\[ \text{Theorem 4.} \]

Given an undirected graph, iteratively choose an uncovered edge and add randomly an endpoint to the cover. Then, in expectation, the resulting cover is a 2-approximation of an optimal vertex cover of the given graph.
Proof. Let a graph \( G \) be given. Furthermore, fix a minimum vertex cover \( C \). For all \( t \), let \( D_t \) be the set of vertices chosen by the algorithm after \( t \) iterations. Let \( X_t \) be 0 if \( D_t \) is a vertex cover, and otherwise let \( X_t \) be the number of vertices of \( C \) that are not in \( D_t \). Clearly, the algorithm terminates exactly when \( X_t = 0 \). Furthermore, in each step, the algorithm selects a vertex from \( C \) with probability at least \( 1/2 \), since, for every edge of \( G \), at least one of the endpoints is in \( C \). Let \( s \) denote an outcome of \( X_t \) before we have a vertex cover. We get \( \mathbb{E}[X_t - X_{t+1} | X_t = s] \geq \frac{1}{2} \). Hence, using the additive drift theorem (Thm. 1), we get that the algorithm terminates in expectation after choosing \( 2 |C| \) vertices.

Next, we formalize the example of the coupon collector process given in the introduction.

\textbf{Theorem 5.} Suppose we want to collect at least one of each kind of \( n \) coupons. Each round, we are given one coupon chosen uniformly at random from the \( n \) kinds. Then, in expectation, we have to collect for at most \( n(1 + \ln n) \) iterations. Furthermore, overshooting this time by \( kn \) has a probability of \( e^{-(k+1)} \).

Proof. Let \( X_t \) be the number of coupons missing after \( t \) iterations and \( s \) an outcome of \( X_t \) before we collected all coupons. The probability of making progress (of 1) with coupon \( t + 1 \) is \( X_t/n \). Thus, \( \mathbb{E}[X_t - X_{t+1} | X_t = s] = s/n \). An application of the multiplicative drift theorem (Thm. 2) gives the desired result. \( \blacksquare \)

One can easily derive a lower bound on the expected first-hitting time with the same asymptotic growth with a lower-bounding multiplicative drift theorem [24]. Using the analogous proof, one can directly analyze a generalized version of the coupon collector process as follows.

\textbf{Theorem 6.} Suppose we want to collect at least one of each kind of \( n \) coupons. For each kind of coupon and each round, we get this kind of coupon with probability at least \( p \). Then, in expectation, we have to collect for at most \( (1 + \ln n)/p \) iterations. Furthermore, overshooting this time by \( k/p \) has a probability of \( e^{-(k+1)} \).

Proof. Let \( X_t \) be the number of coupons missing after \( t \) iterations and \( s \) an outcome of \( X_t \) before we collected all coupons. The expected progress is \( \mathbb{E}[X_t - X_{t+1} | X_t = s] \geq ps \), since the expected number of missing coupons that we get in the next iteration is \( ps \). An application of the multiplicative drift theorem (Thm. 2) gives the desired result. \( \blacksquare \)

Note that this generalized version does not make any assumptions on how many coupons we get per iteration, or whether these indicator random variables are in any way correlated.

Our next example is a simple randomized sorting algorithm. This and similar sorting algorithms were considered in [22]. The analysis via the multiplicative drift theorem is short, easy, and intuitive.

\textbf{Theorem 7.} Consider the sorting algorithm which, given an input array \( A \) of length \( n \), iteratively chooses two different positions of the array uniformly at random and swaps them if they are out order. Then the algorithm obtains a sorted array after \( \Theta(n^2 \log n) \) iterations in expectation.

Proof. Let an array \( A \) of length \( n \) be given. An ordered pair \((i, j)\) with \( 0 \leq i < j < n \) is called an inversion if and only if \( A[i] > A[j] \). Note that the maximum number of inversions is \( \binom{n}{2} \). Let \( X_t \) be the number of inversions after \( t \) iterations. If the algorithms chooses a pair which is not an inversion, nothing changes. If the algorithm chooses an inversion \((i, j)\), then this inversion is removed; for any other inversion, only indices \( k \) with \( i < k < j \) are
Drift: Intuitive Analyses via Drift Theory

relevant. If \( A[k] < A[j] \) (\( < A[i] \)), then \((i, k)\) is an inversion before and after the swap, while \((k, j)\) is neither an inversion before nor after the swap; similarly for \( A[k] > A[i] \) (\( > A[j] \)). Finally, if \( A[j] < A[k] < A[i] \), then \((i, k)\) and \((k, j)\) are inversions before the swap but are not afterwards. Overall, this shows that the number of inversions goes down by at least 1 whenever the algorithm chooses an inversion for swapping. Let \( s \) denote an outcome of \( X_t \) before \( A \) is sorted. Since the probability of the algorithm choosing an inversion is \( X_t/\binom{n}{2} \), we get \( E[X_t - X_{t+1} | X_t = s] \geq s/\binom{n}{2} \). An application of the multiplicative drift theorem (Thm. 2) gives the desired upper bound.

Regarding the lower bound, consider the array \( A \) which is almost sorted but the first and second element are swapped, the third and fourth, and so on. Then the algorithm effectively performs a coupon collector process on \( n/2 \) coupons, where each has a probability of \( 1/\binom{n}{2} \) to be collected. This takes an expected time of \( \Omega(n^2 \log n) \).

4 The Moran Process

The Moran process is a stochastic process introduced in biology to model the spread of genetic mutations in populations. In the Moran process, as introduced in [18], we are given a population of \( n \) individuals that can be of two types: mutants and non-mutants. The process has a parameter \( r \), which is the fitness of mutants. All non-mutants have fitness 1. The fitness of the population is the sum of the fitness of the individuals. At each time step, an individual \( x \) is chosen for reproduction with probability proportional to its fitness with respect to the fitness of the population. The chosen individual then replaces another individual \( y \) of the population, chosen uniformly at random, with a new individual of the same type as \( x \). Thus, if \( x \) is a mutant, then it will replace \( y \) with a mutant, and if \( x \) is a non-mutant, it will replace \( y \) with a non-mutant. When running this process indefinitely with \( n-1 \) non-mutants and a single mutant as a starting state, it will either reach a state where the population will consist exclusively of mutants, which we call fixation, or reach a state where the population will consist exclusively of non-mutants, which we call extinction.

One of the core purposes of the Moran process is to compute the fixation probability (or, equivalently, the extinction probability) of a population.

Lieberman, Hauert, and Nowak [14] extended the original Moran process by introducing structured populations in the form of directed graphs. Earlier work on the time of absorption (that is, either fixation or extinction) of the Moran process is about simple graphs, such as complete graphs, stars, and regular undirected graphs (see [1][23]). A drift theorem was used by Díaz et al. [2] to show that, on any undirected graph, the expected time of absorption of the Moran process is \( O(n^6) \) when \( r > 1 \), \( O(n^5) \) when \( r = 1 \), and \( O(n^3) \) when \( r < 1 \), and they also show concentration for these bounds. Díaz et al. [3] show that there are families of directed graphs that have, in expectation, exponential time of absorption.

We will show how to use drift to obtain an upper bound on the time to absorption of the original Moran process. In the proof, we will see how a clever potential can lead to very good bounds.

Theorem 8. The expected time to absorption of the original Moran process on \( n \) individuals is, for \( r > 1 \), \( O\left(\frac{n^6}{r} n \log n\right) \) and, for \( r < 1 \), \( O\left(\frac{n^5}{r} n \log n\right) \).

Proof. First, assume that \( r > 1 \). For all \( t \), let \( Y_t \) be the number of non-mutants in the population after \( t \) iterations of the process. Thus, \((Y_t)_{t \in \mathbb{N}}\) is a random process on \( \{0, \ldots, n\} \) with starting value \( n-1 \) and absorbing states 0 and \( n \). For any \( k \) with \( 0 < k < n \), we have

\[
\Pr\{Y_{t+1} = Y_t + 1 | Y_t = k\} = \frac{k}{k + r(n-k)} \cdot \frac{n-k}{n}.
\]
We use this function to define a potential of each non-mutant for reproduction. The second term is because there are \( n - k \) mutants among \( n \) individuals that can be replaced from the reproduction of a non-mutant. We abbreviate, for all \( k \) with \( 0 < k < n \),

\[
p(k) = \frac{k}{k + r(n - k)} \cdot \frac{n - k}{n}
\]
as the probability that the number of non-mutants increases when we currently have \( k \) mutants. From similar reasoning, we get

\[
\Pr[Y_{t+1} = Y_t - 1 | Y_t = k] = \frac{r(n - k)}{k + r(n - k)} \cdot \frac{k}{n} = r \cdot p(k).
\]

In particular, for all \( k \) with \( 0 < k < n \), we have \( \mathbb{E}[Y_{t+1} - Y_t | Y_t = k] = (r - 1) \cdot p(k) \).

In order to apply the additive drift theorem, we need to lower-bound this term by a value independent of \( k \), which is currently not the case, as we have a factor of \( p(k) \). However, since \( p(k) = \frac{k(n - k)}{(r n)^2} \geq (1/2n) \), we get \( \mathbb{E}[Y_{t+1} - Y_t | Y_t = k] \geq \frac{r n}{2n} \) and can use the additive drift theorem (Thm. 1) to get an upper bound of \( O\left(\frac{r n}{2n}\right) \).

Note that applying the variable drift theorem (Thm. 3) is not possible, since \( p(k) \) is not monotone in \( k \). Thus, to get an easy lower bound, we could use the same bound as above and get the same hitting time.

In order to get a better result than \( O\left(\frac{r n}{2n}\right) \), we define the following function for all \( k \) with \( 1 \leq k \leq n - 2 \):

\[
g(n) = g(n - 1) = \frac{1}{p(n - 1)} \quad \text{and} \quad g(k) = \frac{r - 1}{r \cdot p(k)} + \frac{g(k + 1)}{r}.
\]

We use this function to define a potential \( \phi \) of \( Y_t \) in the following way: \( \phi(Y_t) = \sum_{i=1}^{Y_t} g(i) \). We want to apply the additive drift theorem (Thm. 1) to \( \phi(Y_t) \).

First, we consider the drift of \( \phi(Y_t) \) with \( Y_t = n - 1 \):

\[
\mathbb{E}[\phi(Y_t) - \phi(Y_{t+1}) | Y_t = n - 1] = r \cdot p(n - 1) \cdot \frac{1}{p(n - 1)} - p(n - 1) \cdot \frac{1}{p(n - 1)} = r - 1.
\]

Now, we consider the drift of \( \phi(Y_t) \) with \( Y_t = k \) for \( 1 \leq k \leq n - 2 \):

\[
\mathbb{E}[\phi(Y_t) - \phi(Y_{t+1}) | Y_t = k] = r \cdot p(k) g(k) - p(k) g(k + 1) = r - 1 + p(k) g(k + 1) - p(k) g(k + 1) = r - 1.
\]

As of now, the drift for \( Y_t = n \) is not positive, since it is an absorbing state. To fix this, we transform \( Y_t \) into the process \( Y'_t \) that behaves exactly like \( Y_t \) but goes deterministically into \( 0 \) if \( Y'_t = n \). This way, the drift for \( Y'_t \) with \( 1 \leq Y'_t \leq n - 1 \) is the same as the one of \( Y_t \), but the drift for \( Y'_t = n \) is \( \phi(n) \), which can be easily lower bounded by \( r - 1 \).

Overall, we get the expected time to absorption \( T \), noting that \( Y'_n = n - 1 \):

\[
\mathbb{E}[T] \leq \sum_{k=1}^{n-1} \frac{g(k)}{r - 1} \leq \frac{1}{r - 1} \sum_{k=1}^{n-1} \frac{r - 1}{r \cdot p(k)} \cdot \frac{1}{r} \leq \frac{1}{r - 1} \sum_{k=1}^{n-1} \frac{r}{r \cdot p(k)} \cdot \frac{n}{r - 1} = \frac{1}{r - 1} \sum_{k=1}^{n-1} \frac{2r n}{k} \cdot \frac{n}{n - k} \leq \frac{2}{r - 1} \sum_{k=1}^{n-1} \frac{2r n}{k} = O\left(\frac{r}{r - 1} n \log n\right).
\]
Drift:8  Intuitive Analyses via Drift Theory

For the case of \( r < 1 \), we consider \( n - Y_t \) and observing that we have an analogous process where the transition probability to go left is \( 1/r \) times larger than the probability to go right. Further, we start at 1, not at \( n - 1 \). Thus, using \( r < 1 \), we get for the time to absorption \( T \) in this case, noting that \( Y'_0 = 1 \),

\[
E[T] \leq \frac{g(1)}{r - 1} = \frac{r \cdot g(1)}{1 - r} = \frac{r}{1 - r} \sum_{k=1}^{n-1} \frac{1 - r}{p(k)} \leq \frac{r}{1 - r} \sum_{k=1}^{n-1} \frac{1}{p(k)} \cdot \frac{1 - r}{r} = O\left(\frac{n \log n}{1 - r}\right),
\]
as we have already estimated this sum above. This concludes the proof.

5 Drift without Drift

In order to apply any drift theorem, it is necessary to define a potential that moves, in expectation, closer to a desired goal. When considering martingales\(^2\) this is, by definition, not the case. It could even be that the process at hand represents a single number and, thus, never changes over time. In this case, a positive drift will never exist. We exclude such martingales by only considering those that have the possibility to move, that is, they have a positive variance. Additionally, we need to specify targets for our process to hit. Since moving martingales do so unbiasedly into either direction, we look at two targets: one to the left and one to the right of the martingale’s starting position.

Since we cannot use drift theory directly on the process, we transform it such that the expected difference is the variance and we are able to use drift theory in order to bound the hitting time. This is formalized in the following lemma.

\[\text{ Lemma 9.}\] Let \((X_t)_{t \in \mathbb{N}}\) be a martingale over \( S \subset \mathbb{R} \), let \([\alpha, \beta] \subseteq S \) be an interval, and let \( T = \inf\{t \mid X_t \notin (\alpha, \beta)\} \).

1. Assume that there is a value \( \delta > 0 \) such that, for all \( s \in (\alpha, \beta) \) and all \( t \in \mathbb{N} \),

\[
\text{Var}[X_t - X_{t+1} \mid X_t = s] \geq \delta , \quad \text{and that,}
\]

for all \( t \in \mathbb{N} \), we have \( X_t \in [\alpha, \beta] \). Then

\[
E[T] \leq \frac{(E[X_0] - \alpha)(\beta - E[X_0])}{\delta}.
\]

2. Assume that there is a value \( \delta > 0 \) such that, for all \( s \in (\alpha, \beta) \) and all \( t \in \mathbb{N} \),

\[
\text{Var}[X_t - X_{t+1} \mid X_t = s] \leq \delta.
\]

Then

\[
E[T] \geq \frac{(E[X_0] - \alpha)(\beta - E[X_0])}{\delta}.
\]

\[\text{ Proof.}\] Let \( Y_t = (X_t - \alpha)(\beta - X_t) \). Note that \( Y_t \leq 0 \) if and only if \( X_t \notin (\alpha, \beta) \), and it is positive otherwise. We now determine the drift of \( Y \) with respect to \( X \). Thm. \[\] does not allow to

\[\text{ We define a martingale } X \text{ to be a random process } (X_t)_{t \in \mathbb{N}} \text{ such that } E[X_{t+1} \mid X_t = s] = s. \text{ That is, we only condition on the last point in time and not on an arbitrary filtration.}\]
do so, but the more general theorems by Kötzing and Krejca [12] do without adding any restrictions. Hence, for any \( s \in (\alpha, \beta) \), we get

\[
E[Y_t - Y_{t+1} | X_t = s] = (s - \alpha)(\beta - s) - E((X_{t+1} - \alpha)(\beta - X_{t+1}) | X_t = s) \\
= -s^2 + (\alpha + \beta)s - \alpha\beta - E[-X_{t+1}^2 + (\alpha + \beta)X_{t+1} - \alpha\beta | X_t = s] \\
= -s^2 + E[X_{t+1}^2 | X_t = s] + (\alpha + \beta)s - (\alpha + \beta)E[X_{t+1} | X_t = s] - \alpha\beta + \alpha\beta \\
= E[X_{t+1} | X_t = s] - s^2 \\
= E[X_{t+1} | X_t = s]^2 \\
= \text{Var}[X_{t+1} | X_t = s].
\]

Since we can bound the variance by assumption, we get the desired result by applying the additive drift theorem (Thm. 1). Note that, for case 2 we still need to check that \( E[(Y_t - Y_{t+1}) | X_t = s] \) is bounded, which is implied by the bound on the variance of \( X \).

We note that Lemma 9 is also applicable to martingales that have access to values further in the past, that is, to martingales with respect to any filtration. This follows from a more general version of the additive drift theorem by Kötzing and Krejca [12], which is applicable to any filtration.

### 5.1 Random Walks on the Line

Many random processes can be described as a random walk on an interval of integers, where either both ends or one end is absorbing, and for the states in between there is no expected change. If we are interested in the time until an absorbing state is hit, we can use Lemma 9 to fully exhibit the argument in generality for these cases, we give the following definitions of two- and one-barrier processes; we will use these tools in Section 5.2.

Consider a random walk on the line with \( P_n = v_0 \ldots v_n \) as its underlying graph, and with the following transition probabilities: for each \( i \in \{1, \ldots, n-1\} \), when the walk is at vertex \( v_i \), then, with probability \( p \), the walk will transition to \( v_{i+1} \), with probability \( p \), the walk will transition to \( v_{i-1} \), and, with probability \( 1 - 2p \), the walk remains in its current state, where \( 0 \leq p \leq 1/2 \). Let \( v_0 \) and \( v_n \) be the absorbing states, that is, when the walk reaches \( v_0 \) or \( v_n \), it will remain in that state forever. We can model this walk as the following stochastic process.

**Definition 10.** The two-barrier process with parameter \( p \) is the stochastic process \( X_t \in \{0, \ldots, n\} \) with transition probabilities, where \( s \in \{1, \ldots, n-1\} \) and \( s' \in \{0, n\} \),

\[
\begin{align*}
\Pr[X_{t+1} = X_t + 1 \mid X_t = s] &= p, & \Pr[X_{t+1} = X_t - 1 \mid X_t = s] &= p, \\
\Pr[X_{t+1} = X_t \mid X_t = s] &= 1 - 2p, & \Pr[X_{t+1} = X_t \mid X_t = s'] &= 1,
\end{align*}
\]

and hitting time \( T = \inf\{t \in \mathbb{N} \mid X_t \in \{0, n\}\} \).

Thus, if the random walk is at state \( i \) at time \( t \), for the above process, we have \( X_t = i \). Using Lemma 9 we can easily calculate the expected hitting time of the two-barrier process.

**Theorem 11.** Let \( X_t \) be the two-barrier process with parameter \( p \). \( E[T] = \frac{E[X_0][n-E[X_0]]}{2p} \).

**Proof.** Since, for any \( s \in \{0, \ldots, n\} \), we have \( E[X_{t+1} | X_t = s] = s \), \( X \) is a martingale. Calculating the variance of \( X \), we have

\[
\text{Var}[X_{t+1} | X_t = s] = E[X_{t+1}^2 | X_t = s] - E[X_{t+1} | X_t = s]^2 \\
= p(s + 1)^2 + p(s - 1)^2 + (1 - 2p)s^2 - s^2 = 2p.
\]
Applying Lemma 9 concludes the proof.

We now turn to the other frequent case: a walk on a set of integers where only one boundary is absorbing.

> **Definition 12.** The one-barrier process with parameter $p$ is the stochastic process $X_t \in \{0, \ldots, n\}$ with transition probabilities, where $s \in \{1, \ldots, n-1\}$,

\[
\begin{align*}
\Pr[X_{t+1} = X_t + 1 \mid X_t = s] &= p, \\
\Pr[X_{t+1} = X_t - 1 \mid X_t = s] &= p, \\
\Pr[X_{t+1} = X_t \mid X_t = s] &= 1 - 2p, \\
\Pr[X_{t+1} = 1 \mid X_t = 0] &= 2p, \\
\Pr[X_{t+1} = 0 \mid X_t = 1] &= 1 - 2p,
\end{align*}
\]

and hitting time $T = \inf\{t \in \mathbb{N} \mid X_t = n\}$.

> **Theorem 13.** Let $X_t$ be the one-barrier process with parameter $p$ and with $X_0 = 0$. Then $E[T] = \frac{n^2}{2p}$.

**Proof.** Consider the extension $Y_t$ of $X_t$ by mirroring the process around 0. That is, $Y_t \in \{-n, \ldots, n\}$ is the process with transition probabilities, where $s \in \{-n+1, \ldots, n-1\}$ and $s' \in \{-n, n\}$,

\[
\begin{align*}
\Pr[Y_{t+1} = Y_t + 1 \mid Y_t = s] &= p, \\
\Pr[Y_{t+1} = Y_t - 1 \mid Y_t = s] &= p, \\
\Pr[Y_{t+1} = Y_t \mid Y_t = s] &= 1 - 2p, \\
\Pr[Y_{t+1} = Y_t \mid Y_t = s'] &= 1,
\end{align*}
\]

and hitting time $T_Y = \inf\{t \in \mathbb{N} \mid X_t \in \{-n, n\}\}$. Since $X_t$ is the same process as $|Y_t|$, we have $E[T] = E[T_Y]$.

Consider the process $X'_t = Y_t + n$. It is easy to see that $X'_t \in \{0, \ldots, 2n\}$ is a two-barrier process as in Definition 10 with $X'_0 = n$ and hitting time $T' = \inf\{t \in \mathbb{N} \mid X'_t \in \{0, 2n\}\}$. Since $E[T'] = E[T_Y] = E[T]$, the theorem follows from Theorem 11.

### 5.2 Applications

We now turn to some processes from the analysis of randomized algorithms which can easily be analyzed using the tools from Section 5.1.

#### Gambler’s Ruin

The most classic example of an unbiased random walk is the gambler’s ruin example. In fact, the two-barrier process emerged as a generalization of the gambler’s ruin process.

> **Corollary 14.** Suppose you have $n$ coins and bet 1 coin each round. With probability 1/2 you lose that coin, with probability 1/2 you gain another coin on top. Then the expected time until you have either 0 or 2$n$ coins is $n^2$.

**Proof.** Let $X_i$ be the number of coins after $i$ iterations. We see that $X_i$ is the two-barrier process $X_t$, with $p = 1/2$, $N = 2n$ and $X_0 = n$. The corollary follows from Theorem 11.

#### The RECOLOUR Algorithm

McDiarmid [15] studies the following simple randomized algorithm for coloring an undirected graph, called RECOLOUR: on input $G$, RECOLOUR starts with an arbitrary 2-coloring of $G$. At every step, it checks whether the current coloring has a monochromatic triangle. If so, RECOLOUR changes the color of one of the vertices of this triangle uniformly at
random. Otherwise, the 2-coloring has no monochromatic triangles and it is the output of RECOLOUR.

McDiarmid shows that when RECOLOUR is applied to a 3-colorable graph \( G \), it will return a 2-coloring of \( G \) such that no triangle is monochromatic in expected time \( O(n^4) \). His analysis shows that the expected run time of the algorithm can be bounded above by the expected hitting time of a random walk on the line with two absorbing states. This analysis in turn relies on previous results on one-dimensional random walks, which usually require lengthy calculations.

We present a simple and self-contained proof of the \( O(n^4) \) expected run time of the RECOLOUR algorithm for finding a 2-coloring with no monochromatic triangles on 3-colorable graphs. Our proof follows the proof of McDiarmid [15] to reduce the problem to the two-barrier process and then uses Theorem [11]. A similar analysis can be used to derive an upper bound on the run time of RECOLOUR on hypergraph colorings.

\[ \textbf{Theorem 15 (McDiarmid '93).} \quad \text{The expected run time of RECOLOUR on a 3-colorable graph is } O(n^4). \]

**Proof.** Let \( G = (V, E) \) be a 3-colorable graph and let \( \chi : V \to \{1, 2, 3\} \) be a 3-coloring of \( G \). Let \( U = \{ v \in V \mid \chi(v) \in \{1, 2\} \} \) be the set of all vertices which are colored with colors 1 and 2. It is easy to see that any 2-coloring of \( G \) which agrees with \( \chi \) on the vertices from \( U \) is a 2-coloring of \( G \) with no monochromatic triangles. Thus, the run time of RECOLOUR is bounded by the expected time that RECOLOUR takes to find such a coloring.

Let \( \chi_t \) be the 2-coloring found by RECOLOUR at time \( t \). Let \( Y_t \) be the number of vertices \( u \in U \) such that \( \chi(u) = \chi_t(u) \). The algorithm terminates when \( Y_t \in \{0, |U|\} \), since agreeing on all vertices of \( U \) is a coloring without monochromatic triangles, but disagreeing on all vertices from \( U \) is also such a valid coloring, since the use of the colors is symmetric.

Let \( s \) denote an outcome of \( Y_t \) before the algorithm terminates. We then have that \( \Pr[Y_{t+1} = Y_t + 1 | Y_t = s] = 1/3 \), as, for every monochromatic triangle, there is exactly one vertex in \( u \in U \) with \( \chi(u) \neq \chi_t(u) \). Similarly, \( \Pr[Y_{t+1} = Y_t - 1 | Y_t = s] = 1/3 \). Thus, \( Y_t \) is the two-barrier process with \( p = 1/3 \) and first-hitting time \( T = \inf\{ t \mid X_t \in \{0, |V|\} \} \). From Theorem [11] we get

\[
E[T] = \frac{3E[Y_0]|(U| - E[Y_0])}{2} \leq \frac{3n^2}{8}.
\]

At each step, the algorithm requires \( O(n^2) \) time to transition to the next coloring (since there are only \( O(n^2) \) triangles to consider, as otherwise the graph would not be 3-colorable), which concludes the proof.

The analysis of the RECOLOUR algorithm for finding 2-colorings with no monochromatic triangles appears as an exercise in [17, Exercise 7.10].

**Random 2-SAT**

Papadimitriou [21] studies the following simple randomized algorithm that will return a satisfying assignment of a satisfiable 2-SAT formula \( \phi \) with \( n \) variables within \( O(n^3) \) time in expectation: the algorithm starts with a random assignment of the variables of \( \phi \). At every step, the algorithm checks whether there is an unsatisfied clause for this assignment. If so,
the algorithm changes the assignment of one of the vertices of this assignment uniformly at random. Otherwise, the assignment is satisfying and it is the output of the algorithm.

The analysis given is similar to the RECOLOUR algorithm, and it also relies on the previous results on one-dimensional random walks. An extensive analysis of this algorithm appears in [17, Section 7.1.1]. Here, we present a simpler proof that uses Theorem 13.

**Theorem 16.** The randomized 2-SAT algorithm, when run on a satisfiable 2-SAT formula, will terminate in $O(n^2)$ time.

**Proof.** Let $\phi$ be a satisfiable 2-SAT formula and let $a$ be a satisfying assignment. At each time step $t$, the randomized 2-SAT algorithm finds a (not necessarily satisfying) assignment $a_t$. Let $X_t$ be the random variable denoting the number of variables that have the same truth assignment in both $a$ and $a_t$. Let $T$ be the first time the algorithm will reach a satisfying assignment for $\phi$. Assume that a clause $x \lor y$ is not satisfied by $a_t$. Since $a$ is a satisfying assignment, $a$ and $a_t$ will differ in the assignment of at least one of the variables in this clause. Let $s$ denote an outcome of $X_t$ before a satisfying assignment has been found. Thus, $\Pr[X_{t+1} = X_t + 1 \mid X_t = s] \geq 1/2$ and $\Pr[X_{t+1} = X_t - 1 \mid X_t = s] \leq 1/2$. When $a_t = a$, the algorithm will terminate. Therefore, $X_t$ is dominated by the one-barrier process $Y_t$ with parameter $p = 1/2$, and first-hitting time $T_{Y_t} = \inf \{ t \in \mathbb{N} \mid X_t = n \}$. Hence, we get from Theorem 13 that $\mathbb{E}[T] \leq \mathbb{E}[T_{Y_t}] \leq n^2$. To transition from $a_t$ to $a_{t+1}$, the algorithm requires $O(n^2)$ time (since a 2-SAT formula has $O(n^2)$ distinct clauses), so the theorem follows. 

The Moran Process Revisited

In Theorem 8, we considered the Moran process where $r = 1$ and observed a drift. If we now consider the case of $r = 1$, that is, mutants and non-mutants have the same fitness, we do not have a drift anymore. However, we can apply Lemma 9 and arrive at the following theorem.

**Theorem 17.** The expected time to absorption of the original Moran process on $n$ individuals is $O(n^2)$ when $r = 1$.

**Proof.** We let $Y_t$ be the number of non-mutants in the population after $t$ iterations of the process. Thus, $(Y_t)_{t \in \mathbb{N}}$ is a martingale and we have, for all $k \in \{1, \ldots, n-1\}$,

$$\Pr[Y_{t+1} = Y_t + 1 \mid Y_t = k] = \Pr[Y_{t+1} = Y_t - 1 \mid Y_t = k] = \frac{n-k}{n} \cdot \frac{k}{n} = p(k).$$

Since the process is a walk on $\{0, \ldots, n\}$ starting at $n-1$, we can apply Lemma 9. For this, we need a lower bound on the variance of the process while none of the absorbing states is found, which we compute as

$$\text{Var}[Y_{t+1} \mid Y_t = k] = 2p(Y_t) \geq 2 \frac{n-1}{n} \cdot \frac{1}{n}.$$ 

Thus, Lemma 9 gives an expected first-hitting time of the absorbing states of at most $n^2/2$. 

6 Conclusions

The beauty in working with drift theory is that it is very intuitive – expected gains translate into expected times to gain a certain amount. Typically, the process at hand does not need to be fitted too much to suit the needs of a drift theorem, since different drift theorems are
available for different needs. In the theory of randomized search heuristics, there are several instances where drift theory has been used to significantly simplify previous analyses.

We encourage our readers to apply drift theory themselves the next time they are facing a randomized process and the research question can be formulated as a hitting time. The biggest challenge then will most likely be to come up with potentials that are well-suited for the process of interest. In order to construct highly non-trivial potentials that end up yielding the desired results, a lot of trial and error is necessary -- as always. We hope that we sparked the reader’s interest in drift theory and wish them much success in applying it.

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