On axiomatizability and decidability of universal theories of hereditary classes of matroids

A V Il’ev¹, V P Il’ev²
¹Sobolev Institute of Mathematics SB RAS, Omsk, Russia
²Dostoevsky Omsk State University, Omsk, Russia

E-mail: ¹artyom_iljev@mail.ru, ²iljev@mail.ru

Abstract. Two hereditary classes of matroids – the class of matroids of rank doesn’t exceeding a fixed positive integer $k$ and the class of matroids of finite rank – are studied by means of the model theory. The problems of axiomatizability of these two classes of matroids as structures and the problems of algorithmic decidability of universal theories of these classes are considered. It is shown that the first class is finitely axiomatizable whereas the second one is nonaxiomatizable. Decidability of the universal theories of the both classes is proved.

1. Introduction

The problems of axiomatizability of different classes of structures as well as the problems of algorithmic decidability of theories of the classes are the questions of traditional interest in the model theory. The problem of decidability of any theory has a paramount importance for its consideration. Establishing decidability is an important question in studying any theory: decidability of a theory of a class $K$ of structures makes it possible in principle to get an exhaustive list of properties of structures from class $K$.

Studying of universal theories presents special interest, because many combinatorial problems formulated in the first-order logic language can be reduced to the studying of models of universal theories [7]. Moreover, any theory may be transformed into an universal one by extending the language (the skolemization method) [2].

In sixties of the 20th century Lavrov proved the undecidability of the elementary theory of graphs [5]. Therefore there is no algorithm which, given any sentence $\varphi$, decides whether the sentence is true for all graphs. Naturally the problem of decidability of the universal graph theory arises, as well as the problems of decidability of the universal theories of some combinatorial objects closed to graphs. Decidability of the universal graph theory and decidability of the universal theories of some hereditary classes of graphs were proved in [4].

In this paper the problems of axiomatizability of two important hereditary classes of matroids as structures and algorithmic decidability of universal theories of these classes of matroids are studied.

It is shown that the class of matroids of rank doesn’t exceeding $k$ is finitely axiomatizable whereas the class of matroids of finite rank is nonaxiomatizable. Decidability of the universal theories of the both classes is proved.
2. On axiomatizability of hereditary classes of matroids

In this section we will formulate some important notions and results of the matroid theory and the model theory. All necessary definitions can be found in [1,3,6,8]. The problem of axiomatizability of hereditary classes of matroids is studied.

Matroids were introduced by Whitney in paper [8]. At first matroids were defined in finite case.

Let \( U \) be a nonempty finite set. A matroid on \( U \) is a pair \( M = (U, \mathcal{I}) \), where \( \mathcal{I} \) is a nonempty family of subsets of \( U \) (called the independent sets) with the following properties: for all \( I,J \subseteq U \)

\[(1) \text{ if } I \in \mathcal{I} \text{ and } J \subseteq I \text{, then } J \in \mathcal{I} \text{ (hereditary property)};\]

\[(2) \text{ if } I,J \in \mathcal{I} \text{ and } |J| = |I| + 1 \text{, then there exists an element } j \in J \setminus I \text{ such that } I \cup \{j\} \in \mathcal{I} \text{ (exchange property)}.\]

Any maximal independent subset of a set \( A \subseteq U \) is called a base of the set \( A \). It is not difficult to see that in a matroid \( M = (U, \mathcal{I}) \) all bases of any set \( A \subseteq U \) have the same cardinality \( r(A) \); this cardinality is called the rank of the set \( A \). The number \( r(M) = r(U) \) is called the rank of a matroid \( M \).

Later infinite matroids were considered in the literature along with finite matroids. Let \( k \) be a fixed positive integer. In general a matroid of rank doesn’t exceeding \( k \) is defined as a pair \( M = (U, \mathcal{I}) \), where \( U \) is a nonempty set (possibly, infinite) and \( \mathcal{I} \) is a nonempty family of independent subsets of \( U \) with hereditary and exchange properties (11), (12) and the following one:

\[(13) \ |I| \leq k \text{ for all } I \in \mathcal{I}.\]

To define a matroid of finite rank (or a finite-rank matroid) we have to change property (13) by the following condition:

\[(13') \text{ There is a positive integer } n \text{ such that } |I| \leq n \text{ for all } I \in \mathcal{I} \ [1].\]

It is easy to see that the class of matroids of finite rank is the union of all classes of matroids of rank doesn’t exceeding \( k \), where \( k \) runs from 1 to \( \infty \).

Let \( M = (U, \mathcal{I}) \) be a matroid and \( A \subseteq U \). Then the set \( A \) with the family \( \mathcal{I}_A = \{ I \in \mathcal{I} | I \subseteq A \} \) is called the submatroid of \( M \) induced by the set \( A \). A submatroid \( M_A = (A, \mathcal{I}_A) \) is also being a matroid on \( A \) [1].

A matroid of fixed rank \( k \in \mathbb{Z}_+ \) can be defined in terms of bases as a pair \( M = (U, B) \), where \( U \) is a nonempty set, \( B \) is a nonempty family of its bases with the following properties:

\[(B1) \text{ if } B_1, B_2 \subseteq B \text{ and } B_1 \neq B_2 \text{, then } B_1 \not\subseteq B_2 \text{ and } B_2 \not\subseteq B_1;\]

\[(B2) \text{ if } B_1, B_2 \subseteq B \text{, then for all } b_1 \in B_1 \text{ there exists } b_2 \in B_2 \text{ such that } (B_1 \setminus b_1) \cup b_2 \subseteq B;\]

\[(B3) \ |B| = k \text{ for all } B \subseteq B.\]

Let \( L \) be a language. An \( L \)-structure \( M_1 = (U, L) \) is called a substructure of \( L \)-structure \( M_2 = (V, L) \) if

1) \( U \subseteq V \);

2) functions and predicates in \( M_1 \) are the contractions on \( U \) of respective functions and predicates in \( M_2 \);

3) the set \( U \) is closed under functions.

Two \( L \)-structures \( M_1 \) and \( M_2 \) are called elementary equivalent if for every \( L \)-sentence \( \varphi \) the sentence \( \varphi \) is true in \( M_1 \) if and only if \( \varphi \) is true in \( M_2 \).

Let \( I \) be a nonempty set. A filter on \( I \) is a nonempty family \( F \) of subsets of \( I \) satisfying the following conditions:

1) \( \emptyset \not\in F \);

2) if \( X \in F \), \( X \subseteq Y \subseteq I \), then \( Y \in F \);
3) if $X,Y \in F$, then $X \cap Y \in F$.

A maximal filter is called an ultrafilter.

**Lemma 1** [3]. For every filter $F$ on $I$ there exists an ultrafilter on $I$ containing $F$.

**Lemma 2** [3]. A filter $F$ on $I$ is an ultrafilter if and only if either $X \in F$ or $I \setminus X \in F$ for all $X \subseteq I$.

Let $F$ be an ultrafilter on $I$ and $M_i$ be an $L$-structure for each $i \in I$. Then the reduced product of the structures $M_i$ using $F$ is called the ultrapower of $M_i$ using $F$.

**Theorem 1** [2,6]. An $L$-sentence $\varphi$ is true in the ultrapower of $L$-structures $M_i$, using $F$ if and only if the set of numbers of the factors $M_i$ in which $\varphi$ is true belongs to the ultrafilter $F$.

An abstract class of $L$-structures is a family of $L$-structures which is closed under isomorphism. We will consider only abstract classes. A class of $L$-structures is called hereditary if it is closed under substructures.

A class $K$ of $L$-structures is called axiomatizable if there is a set $Z$ of $L$-sentences such that $K$ consists of all models of $Z$, i.e., of all $L$-structures that satisfy $Z$. The set $Z$ is said to be a set of axioms for class $K$. If the set $Z$ is finite, then $K$ is called finitely axiomatizable. A class $K$ is called recursive axiomatizable if it can be defined by a recursive set of axioms, i.e., there is an algorithm which correctly decides whether any $L$-sentence belongs to the set $Z$.

**Theorem 2** [3]. A class of structures is axiomatizable if and only if it is closed under elementary equivalence and ultraproducts.

Let $K$ be a class of $L$-structures. The (elementary) theory of a class $K$ is the set $\text{Th}(K)$ of all $L$-sentences each of which is true in all structures from $K$.

A formula $\varphi$ is called an universal formula or $\forall$-formula if $\varphi = \forall x_1 \ldots x_n \psi$, where $\psi$ is a quantifier free formula. The set of all $\forall$-formulas of $\text{Th}(K)$ is called the universal theory or $\forall$-theory of a class $K$.

It is easy to see that both the class of matroids of rank doesn’t exceeding a fixed positive integer $k$ and the class of matroids of finite rank are hereditary. Here we will prove that only one of these classes is axiomatizable.

**Theorem 3.** Let $k$ be a fixed positive integer. The class of matroids of rank doesn’t exceeding $k$ is finitely axiomatizable.

**Proof.** A matroid of rank doesn’t exceeding $k$ can be defined as the structure $M = (U, I_k)$, where $U$ is a nonempty set and the language $L_k = (I_0, I_1, \ldots, I_k, \neg)$ consists of the equality predicate and $k + 1$ predicates of independence, arities of which are equal to their numbers. The predicates of independence satisfy the following axioms:

1. $\forall x_1 \ldots x_n [I_n(x_1, \ldots, x_n) \rightarrow \bigwedge_{\pi} I_n(\pi(x_1), \ldots, \pi(x_n))]$, where $\pi$ runs through all permutations of variables $x_1, \ldots, x_n$, $n \in \{1, \ldots, k\}$;
2. $\forall x_1 \ldots x_n [I_n(x_1, \ldots, x_n) \rightarrow \bigwedge_{x \neq x_i} (x \neq x_i)]$, $n \in \{1, \ldots, k\}$;
3. $\forall x_1 \ldots x_n [I_n(x_1, \ldots, x_n) \rightarrow I_{n-1}(x_2, \ldots, x_n) \land \ldots \land I_{n-1}(x_2, \ldots, x_n)] \land I_0]$, $n \in \{2, \ldots, k\}$;
4. $\forall x_1 \ldots x_n \forall y_1 \ldots \forall y_{n+1} [I_n(x_1, \ldots, x_n) \land \bigwedge_{y \neq [1, \ldots, n+1]} I_{n+1}(y_1, \ldots, y_{n+1})] \rightarrow \bigvee_{y \in [1, \ldots, n+1]} I_{n+1}(x_1, \ldots, x_n, y)]$, $n \in \{1, \ldots, k - 1\}$.

To similarly define a matroid with a fixed rank $k$, add the next axiom to the above definition:

5. $\exists x_1 \ldots \exists x_k I_k(x_1, \ldots, x_k)$.

Thus, the class of matroids of rank doesn’t exceeding $k$ is finitely axiomatizable.
The theorem is proved.

**Theorem 4.** The class of matroids of finite rank is nonaxiomatizable.

**Proof.** We will prove that the class of matroids of finite rank is not closed under ultraproducts.

Consider the Frechet filter on the set $N$ of all positive integers consisting of all sets whose complements are finite. By lemma 1, there exists an ultrafilter $F$ on $N$ containing the Frechet filter. By lemma 2, $F$ contains no finite set.

Consider a countable set of matroids $M_i$ of finite rank, where the rank of each $M_i$ equals its number $i \in N$. Consider the sentence $\varphi_i = \exists x_1...\exists x_i I_i(x_1,....,x_i)$ that means the existence of an independent set of the cardinality $i$. Obviously, $\varphi_i$ is true in all matroids $M_j$, where $j \geq i$. Thus, for an arbitrary large $i$ there is the countable set of matroids $M_j$ in which the sentence $\varphi_i$ is true. Then by theorem 1, all $\varphi_i$ is true in the ultraprodct $M$ of $M_i$ using $F$. Since $M$ contains independent sets of an arbitrary large cardinality, hence $M$ is not a matroid of finite rank.

Thus, the class of matroids of finite rank is not closed under ultraproducts. By theorem 2, this class is nonaxiomatizable.

The theorem is proved.

3. Decidability of universal theories of matroids

In this section we will show that both the universal theory of matroids of finite rank and the universal theory of matroids of finite rank are decidable.

**Theorem 5.** The universal theory of matroids of finite rank is decidable.

**Proof.** The following algorithm decides whether an arbitrary universal $L_I$-sentence $\varphi$ belongs to the universal theory of matroids of finite rank.

**Algorithm A**

*Step 0.* The algorithm formulates the sentence $\neg \varphi = \exists x_1...\exists x_p \psi$, where $\psi$ is a quantifier free formula, $p \in N$. Then $\neg \varphi$ is transformed to the equivalent sentence $\neg \psi_0 = \exists x_1...\exists x_p (\psi_1 \lor ... \lor \psi_m)$ in the prenex disjunctive form, where every $\psi_i$ is a conjunct, $i = 1,...,m$. After that the algorithm looks sequentially all conjuncts $\psi_i$ and performs steps 1–8 for each conjunct.

*Step 1.* The algorithm looks sequentially all factors of the conjunct $\psi_i$. If it finds a factor $x = y$, then the variable $y$ is replaced by $x$ in the conjunct $\psi_i$, and the factor $x = x$ is deleted from $\psi_i$.

*Step 2.* For every factor $I_n(x_1,...,x_n)$ of the conjunct $\psi_i$, the algorithm seeks in $\psi_i$ and deletes all factors $I_n(\pi(x_1),...,\pi(x_n))$, where $\pi$ is an arbitrary permutation of variables $x_1,...,x_n$.

*Step 3.* If the conjunct $\psi_i$ contains a factor $x \neq x$, then the algorithm deletes this conjunct as identically false and goes to considering the next conjunct.

*Step 4.* For every factor $I_n(x_1,...,x_n)$ of the conjunct $\psi_i$, the algorithm seeks in $\psi_i$ the factor $\neg I_n(\pi(x_1),...,\pi(x_n))$, where $\pi$ is an arbitrary permutation of variables $x_1,...,x_n$. If such factor is found, then the algorithm deletes the conjunct $\psi_i$ and goes to considering the next conjunct.

*Step 5.* If the conjunct $\psi_i$ contains a factor $I_n(x_1,...,x_s,...,x_n)$, where $x_i = x_i$ and $s \neq t$, then the algorithm deletes this conjunct and goes to considering the next conjunct.

*Step 6.* For every factor $I_n(x_1,...,x_n)$ of the conjunct $\psi_i$ the algorithm seeks in $\psi_i$, the factor $\neg I_n-1(y_1,...,y_{n-1})$, where the set of variables $\{y_1,...,y_{n-1}\}$ is an $(n - 1)$-subset of $\{x_1,...,x_n\}$. If such factor is found, then the algorithm deletes the conjunct $\psi_i$ and goes to considering the next conjunct.
Step 7. If the conjunct $\psi_i$ contains the factor $-I_0$, then the algorithm deletes this conjunct and goes to considering the next conjunct.

Step 8. For every pair of factors $I_a(x_1,\ldots,x_n)$ and $I_{a+1}(y_1,\ldots,y_{a+1})$ of the conjunct $\psi_i$ the algorithm seeks in $\psi_i$ the multiplication of factors $\bigwedge_{q \in \{0,\ldots,a+1\}} -I_{a+1}(\pi(x_1),\ldots,\pi(x_n),\pi(y_q))$, where $\pi$ is an arbitrary permutation of variables $x_1,\ldots,x_n,y_q$. If such multiplication is found, then the algorithm deletes the conjunct $\psi_i$ and goes to considering the next conjunct.

If all conjuncts of the sentence $-\varphi_0$ are deleted after performing steps 1–8, then the algorithm yields YES and finishes.

If some conjuncts are not deleted after performing steps 1–8, then the algorithm goes to step 9.

Step 9. The algorithm looks sequentially all remaining conjuncts and for each conjunct $\psi_i$ tries to find a finite matroid with finite rank in which this conjunct $\psi_i$ is true. This construction is realized in the following stages.

1) Let $X_i = \{x_1,\ldots,x_n\}$ be the set of variables of the conjunct $\psi_i$, and $k$ be the largest arity of its factors $I_k(y_1,\ldots,y_k)$. All $k$-subsets of the set $X_i$ are enumerated in lexicographic order and the set of these numbers is denoted as $K_i$. Let $L_i$ be the set of all numbers from $K_i$ that enter to the factors $I_k(y_1,\ldots,y_k)$, and $M_i$ be the set of all numbers from $K_i$ that enter to the factors $-I_k(y_1,\ldots,y_k)$ in $\psi_i$.

2) Each family of $k$-subsets of $X_i$ can be written as the subset of numbers from $K_i$, disposed in ascending order. These families also are numerated in ascending lexicographic order. Let $B_s$ be the family with number $s$. Now all families of $k$-subsets consisting of variables of conjunct $\psi_i$ can be sequentially considered.

3) The algorithm looks sequentially all $B_s$ in ascending order of numbers. If in the current $B_s$ both conditions $L_i \subseteq B_s$ and $M_i \cap B_s = \emptyset$ hold, then it checks this family for satisfying all other conditions of the conjunct $\psi_i$. To do this, the algorithm firstly looks all its factors $I_l(y_1,\ldots,y_l)$, where $l < k$, and for each of them seeks in $B_s$ an element containing the set $\{y_1,\ldots,y_l\}$ as a subset. If the corresponding element in $B_s$ is not found for some factor, then the family $B_s$ does not satisfy the conditions of conjunct $\psi_i$ and the algorithm goes to considering the next family $B_{s+1}$. Otherwise, algorithm looks all factors $-I_l(y_1,\ldots,y_l)$ of the conjunct $\psi_i$, where $l < k$, and for each of them also seeks an element of $B_s$ containing $\{y_1,\ldots,y_l\}$ as a subset. If the corresponding element in $B_s$ is found for some factor, then the family $B_s$ does not satisfy the conditions of the conjunct $\psi_i$ and the algorithm proceeds to the next family $B_{s+1}$. Otherwise, it is go to stage 4.

4) By using a special procedure (see below) the algorithm checks whether the current family $B_s$ is the family of bases of some matroid with rank $k$ over the set $X_i$. If this procedure gives the affirmative answer, then the matroid $M = (X_i,B_s)$ of rank $k$, in which the conjunct $\psi_i$ is true, is found. Then the algorithm gives the answer NO and finishes. Otherwise, the algorithm goes to the stage 3 and considers the family $B_{s+1}$.
If for any family $B_s$ matroid with rank $k$, in which the conjunct $\psi_i$ is true, is not found, then there is no such matroid. In this case the algorithm deletes the conjunct $\psi_i$ and goes to considering the next conjunct.

If all conjuncts of the sentence $\neg\varphi_0$ are deleted, then the algorithm finishes with answer YES.

**Procedure checking whether the family $B_s$ is the family of bases of some matroid with rank $k$**

Property (B1) does not need to check, because it follows from step 5. Therefore, go to property (B2). All pairs of subsets of variables from $X_i$ whose numbers enter in $B_s$, are sequentially looked in ascending lexicographic order. For each pair the procedure seeks in $B_s$ subsets satisfying (B2). If such subsets are found, then the next pair is considered, otherwise the procedure finishes. If all pairs of subsets of variables from $X_i$ whose numbers enter in $B_s$ are successfully considered, then $B_s$ is the family of bases of the matroid $M = (X, B_s)$ with rank $k$. Otherwise, there is no matroid with the family of bases $B_s$.

**Proof of Theorem 5.** Prove that algorithm A yields YES if the $L_j$-sentence $\varphi$ belongs to the universal theory $T$ of matroids of finite rank, and NO otherwise.

Algorithm A takes as an input an arbitrary universal $L_j$-sentence $\varphi$. The negation $\neg\varphi$ is transformed to the equivalent sentence $\neg\varphi_0$ in the prenex disjunctive form $\neg\varphi_0 = \exists x_1 \ldots \exists x_p (\psi_1 \lor \ldots \lor \psi_m)$, where $\psi_i$ are conjuncts, $i = 1, \ldots, m$.

The sentence $\varphi$ doesn’t belong to the theory $T$ if and only if the sentence $\neg\varphi$, and hence the sentence $\neg\varphi_0$, is true in some matroid $M$ of finite rank. The last holds if and only if the value of some conjunct $\psi_i$ is true in $M$.

Any factor of $\psi_i$ is an equality predicate $x = y$, or its negation, or a predicate of independence, or its negation. It is easy to see that if conjunct $\psi_i$ contains an identically false factor $x \neq x$ or factors containing independence predicates that contradict to axioms (1)–(4), then this conjunct can be deleted from the sentence $\neg\varphi_0$ without loss of satisfiability of $\neg\varphi_0$ in the class of matroids of finite rank. Algorithm A deletes such false conjuncts at steps 3–8.

If all conjuncts of the sentence $\neg\varphi_0$ are deleted after performing steps 1–8, then there exist no matroids of finite rank in which the sentence $\neg\varphi$ is true. In this case the initial sentence $\varphi$ belongs to the universal theory $T$, and algorithm A yields YES.

Otherwise, if a conjunct $\psi_i$ under consideration is not deleted after steps 1–8, then at step 9 algorithm A tries to construct a matroid of finite rank in which $\psi_i$ is true. To this purpose the algorithm looks over all the families of $k$-subsets of the set $\{x_1, \ldots, x_n\}$ of conjunct variables and check whether anyone of them satisfies the conditions of $\psi_i$ and is the family of bases of some matroid with rank $k$.

Since the set of these families is finite, then in the worst case looking over all families, algorithm A either finds a finite matroid $M$ with rank $k$ in which $\psi_i$ is true as well as the sentence $\neg\varphi$, or it informs that there is no such a matroid. In the first case, the sentence $\varphi$ doesn’t belong to the theory $T$, and algorithm A finishes with answer NO. In the second case, there is no matroid of finite rank in which $\psi_i$ is true, and the algorithm deletes the conjunct $\psi_i$ and goes to considering the next conjunct of the sentence $\neg\varphi_0$.
The latter is correct, since if such matroid (finite or infinite) exists, then its submatroid induced by the set \( \{ x_1, \ldots, x_n \} \) is also the matroid of finite rank in which \( \psi_i \) is true. Moreover, it is easy to verify that for any matroid on the set \( \{ x_1, \ldots, x_n \} \) with rank greater than \( k \) satisfying the condition of conjunct \( \psi_i \), its independent sets of cardinality less or equal \( k \) form a matroid with rank \( k \) satisfying the condition of this conjunct.

If all conjuncts of the sentence \( \neg \varphi \) are deleted, then there is no matroid of finite rank in which the sentence \( \neg \varphi \) is true. In this case the initial sentence \( \varphi \) belongs to the universal theory of matroids of finite rank and algorithm A yields YES. The theorem is proved. The following theorem is proved similarly.

**Theorem 6.** The universal theory of matroids whose rank doesn’t exceeding a fixed positive integer \( k \) is decidable.

### 4. Conclusion

The problems of axiomatizability and finite axiomatizability of different classes of structures as well as the problems of algorithmic decidability of theories and universal theories of the classes are of traditional interest. In this paper, the problems of axiomatizability of two important hereditary classes of matroids and the problems of decidability of universal theories of these classes are studied. It is shown that the class of matroids of rank doesn’t exceeding a fixed positive integer \( k \) is finitely axiomatizable whereas the class of matroids of finite rank is nonaxiomatizable. Decidability of the universal theories of the both classes is proved.

The author A. V. Il’ev was supported by Russian Foundation of Basic Research (project 18-31-00330).

### References

1. Aigner M 1979 *Combinatorial theory* (New York: Springer-Verlag)
2. Ershov Yu L 1980 *Problems of decidability and constructive models* (Moscow: Nauka)
3. Ershov Yu L and Palyutin E A 1987 *Mathematical logic* (Moscow: Nauka)
4. Il’ev A V 2016 Decidability of universal theories and axiomatizability of hereditary classes of graphs *Trudy Instituta Matematiki i Mekhaniki UrO RAN* 22 pp 100-111
5. Lavrov I A 1963 Effective inseparability of the sets of identically true and finitely refutable formulas for certain elementary theories *Algebra i Logika* 2 pp 5-18
6. Marker D 2002 *Model theory: an introduction* (New York: Springer)
7. Razborov A A 2007 Flag algebras *Journal of Symbolic Logic* 72 pp 1239-1282
8. Whitney H 1935 On the abstract properties of linear dependence *American Journal of Mathematics* 57 pp 509-533