A Problem of W. R. Scott: Classify the Subgroup of Elements with Many Roots

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Abstract. Let $G$ be an infinite group and let $h$ and $g$ be elements. We say that $h$ is a root of $g$ if some integer power of $h$ is equal to $g$. We define $K(G)$ to be the subgroup of all elements of $G$ for which the number of elements which are not roots is of smaller cardinality than the cardinality of the group. That is, each element in $K$ has almost every element in $G$ as a root. This paper discusses the problem: When can $K(G)$ be non-trivial?

§1. Introduction. Nearly 60 years ago in [8], W. R. Scott defined two subgroups of a given infinite group $G$. The first, $K = K(G)$, is defined as follows:

For $g \in G$, let $\eta(g, G)$ be the set of $h \in G$ such that $h^n = g$ has no solution for $n$. (In other words, $\eta(g) = \{h : g \notin h\}$.) Then

$$K(G) = \{k \in G : |\eta(k)| < |G|\}.$$ 

The second group, $D = D(G)$, is defined by

$$D = \cap\{H \leq G : |H| = |G|\}.$$

Scott proved several interesting theorems about $D$ and $K$; the principal ones follow.

**THEOREM A.** If $G$ is abelian, $D = K = E$ unless $G \cong \mathbb{Z}_{p^\infty} \times F$, $F$ finite, in which case, $D = K \cong \mathbb{Z}_{p^\infty}$.

**THEOREM B.** For any $G$, $K \leq D$. If $G$ is not periodic, $K = E$.

**THEOREM C.** For any $G$,

(i) $K \leq Z(G)$.

(ii) $K$ is either cyclic of order $p^n$ or a $p^\infty$-group for some prime $p$.

(iii) $K$ is a $p^\infty$-group if and only if there exists a central $p^\infty$-subgroup $C$ such that $G/C$ is finite. If such a $C$ exists, then $C = K = D$.

We attack two main questions in this paper.

Question 1. Are there locally finite groups with $D > K$?
Question 2. What are the groups with non-trivial $K$?

We answer the first question affirmatively in Example 4.10.

Question 2 seems much more difficult to answer. For countable binary finite groups we have Corollary 4.6.

**COROLLARY 4.6.** If $G$ is countable and either binary finite or a 2-group such that $K(G) \neq E$, then either (i) $G = (P \times F)c$, with $F$ a finite group, $P$ a $p^\infty$-group, and $C \leq Z(F)$ a cyclic group of order a power of $p$, or (ii) $G = \langle x, G_1 \rangle$, where $G_1 = (P \times F)c$ with $F$ a finite group, $P$ a $2^\infty$-group, $C \leq Z(F)$ a cyclic group of order a power of 2: $x^{-1}zx = z^{-1}$ for every element $z$ of $P$, $x^2 \in G_1$ and there exists an $m$ such that $x^{2m} = a$, the unique element of $P$ of order 2. In the first case, $K(G) = P$, and in the second case, $K(G) = \langle a \rangle$.

For uncountable groups, we have Corollary 4.9.

**COROLLARY 4.9.** If $G$ is an uncountable class 2 nilpotent group, then $K(G) = E$.

Clearly in Jonsson groups [9] $G$, $D(G) = G$. Thus, in discussions of $D$ we must put some restrictions on the class of groups we study, and locally finite seems a reasonable restriction since there are no uncountable locally finite Jonsson groups of regular cardinality (see [2, p.73]). However, the groups in Example 4.10 are nilpotent of class 2 and are even FC.

If $G$ has an equipotent abelian subgroup, then by Theorem A, we know just about all there is to know about $D$ and $K$, so we must expect to consider groups without equipotent abelian subgroups.

This paper is arranged as follows. Section 2 provides the notation. Section 3 gives a number of lemmas which are needed later. Section 4 provides the answer to Question 1 and concentrates on reducing Question 2 to seemingly more manageable questions.

§2. Notation. Let $S$ and $T$ be sets. The cardinality (power) of $S$ is denoted by $|S|$. If $S$ and $T$ have the same cardinality, we say that they are equipotent. If $m$ is an infinite cardinal, $m^+$ is the first cardinal greater than $m$; $2^m$ is the cardinality of the set of all subsets of a set of power $m$; $2^{\sum_{n \in m}} = \sum_{n \in m} 2^n$; $m$ is regular if it is not the sum of a smaller number of smaller cardinals.

Let $G$ be a group, $H$ be a subgroup, and $S$ be a subset of $G$. The centralizer of $S$ in $G$ is denoted by $C_G(S)$. We denote the center of $G$ by $Z(G) = C_G(G)$. If $H = \{1\}$, the trivial subgroup, we often denote $H$ by $E$. If $x, y \in G$, the commutator of $x$ and $y$ is
\[ [x, y] = x^{-1}y^{-1}xy \]. The subgroup generated by \( S \) is denoted by \( <S> \). The derived (commutator) group of \( G \) is \( G' = \langle \{[x, y] : x, y \in G \} \rangle \). The index of \( H \) in \( G \) is denoted by \( [G : H] \). The conjugate class of \( S \) in \( G \) is \( Cl(S) = Cl_G(S) = \{g^{-1}Sg : g \in G \} \). An element \( g \) of \( G \) is a \( p' \)-element if its order, \( |g| \), is relatively prime to the prime \( p \). The exponent of \( G \) is the smallest integer \( n \) (if one exists) such that \( g^n = 1 \) for all \( g \in G \); it is denoted \( \exp(G) \). \( G \) is nilpotent of class \( n \) if \( n \) is the length of the upper central series. A section of \( G \) is a factor group of a subgroup. \( G \) is FC if every element \( y \) has finitely many distinct conjugates \( g^{-1}yg \) for \( g \in G \). If \( G \) is a \( p \)-group, \( \Omega_1(G) \) is the group generated by the set of all elements of order \( p \). A Jonsson group is an infinite group which has no proper equipotent subgroups. \( G \) is locally (binary) finite if every finite (two-element) subset generates a finite subgroup. The cartesian product of \( H \) and \( C \leq G \) with amalgamated subgroup \( A \) is denoted by \( (H \times C)_A \). Additional terminology can be found in [6] and [7].

We also have occasion to use the following construction. Let \( V \) and \( W \) be vector spaces over a field \( K \) and \( \rho : V \times V \to W \) be a nonzero alternating bilinear function. If \( \gamma : V \times V \to W \) is any bilinear function such that \( \rho(x, y) = \gamma(x, y) - \gamma(y, x) \), then \( V \times W \) can be given the structure of a nilpotent group of class 2, denoted by \( G = V\gamma W \), by defining \( (x, a)(y, b) = (x + y, a + b + \gamma(x, y)) \). Note that \( [(x, a), (y, b)] = (0, \rho(x, y)) \).

§3. Preliminary Lemmas. Throughout this section \( G \) is a group and \( H \) is a subgroup. If \( H < G \) and \( S \leq G \), we denote the set \( SH / H \) by \( \bar{S} \).

**Lemma 3.1.** [8, p.189]

(i) \( \eta(1) = \emptyset \).

(ii) \( \eta(x^{-1}) = \eta(x) \).

(iii) \( \eta(x_1x_2) \leq \eta(x_1) \cup \eta(x_2) \).

(iv) \( \eta(\sigma(x), \sigma(G)) \leq \sigma(\eta(x, G)) \) for any homomorphism \( \sigma \) of \( G \).

(v) \( \eta(h, H) = H \cap \eta(h, G) \) if \( h \in H \leq G \).

(vi) \( \eta(x) \geq G - C_G(x) \).

**Lemma 3.2.**

(i) \( x_1 \notin \eta(x_2) \) and \( x_2 \notin \eta(x_3) \) imply that \( x_1 \notin \eta(x_3) \).

(ii) \( g \notin \eta(y) \) and \( g \in \eta(a) \) imply that \( y \in \eta(a) \).
PROOF.
(i) If \( x_1 \not\in \eta(x_2) \), there exists an \( n \) such that \( x_2 = x_1^n \). If \( x_2 \not\in \eta(x_3) \), there exists an \( m \) such that \( x_3 = x_2^m \). Thus \( x_3 = x_2^m \), so \( x_1 \not\in \eta(x_3) \).

Of course, (ii) is a contrapositive of (i).

**Lemma 3.3.** Let \( K = < a >, \eta(a) = \{ g : a \not\in < g > \} \). If \( \alpha \in \text{Aut}(G) \), then 
\[ \alpha(\eta(a)) = \eta(a) \].

**PROOF.** We know by [8, p.193] that \( K \) is a characteristic subgroup. Suppose \( g \in \eta(a) \). If \( [\alpha(g)]^n = a \), then \( \alpha(g^n) = a \), so \( g^n = \alpha^{-1}(a) \in K \). Since \( \alpha(K) = K \), \( \alpha^{-1}(a) \) is a generator of \( K \). Thus there exists \( m \) such that \( (g^n)^m = (\alpha^{-1}(a))^m = a \), so \( g \not\in \eta(a) \). This contradiction shows that \( \alpha(g) \in \eta(a) \). Since \( \alpha^{-1} \) is also an automorphism, \( \alpha^{-1}(\eta(a)) \subseteq \eta(a) \).

**Lemma 3.4.** If \( H \triangleleft G \), \( |G/H| = |G| \) and \( a \in K \), then \( \alpha \in K(G/H) \).

**PROOF.** This follows from Lemma 3.1.(iv).

Next we generalize [8; Corollary 3].

**Lemma 3.5.** If \( H \trianglelefteq K(G) \) with \( H \) finite, then \( \overline{K} = K(\overline{G}) \).

**PROOF.** \( \overline{K} \trianglelefteq K(\overline{G}) \) by Lemma 3.4. Suppose \( \overline{a} \in K(\overline{G}) \). Then \( |\eta(\overline{a})| < |\overline{G}| \). Now
\[ \eta(\overline{x}) = \{ \overline{g} : \overline{x} \not\in < \overline{g} > \} = \{ \overline{g} : x \not\in < g > \} \).

Let \( \eta = \{ g : \overline{g} \in \eta(\overline{x}) \} = \{ g : x \not\in < g > \} \). Then \( |\eta| = |H| |\eta(\overline{x})| < |G| \). If \( g \not\in \eta \cup ( \cup_{h \in H} \eta(h)) \), then \( g^n = hx \) for some \( n \) and some \( h \in H \) and also \( g^m = h^{-1} \) for some \( m \). Hence \( g^{m+n} = x \), so \( g \not\in \eta(x) \). Thus
\[ |\eta(x)| \leq |\eta| + \sum_{h \in H} |\eta(h)| < |G| \],

so \( x \in K \).

**Lemma 3.6.** Suppose \( G = C(\eta(a)) \), \( 1 \neq a \in K \), \( |a| = p^n \). Then the set \( H \) of \( p^\prime \)-elements is an abelian subgroup and \( H \trianglelefteq \eta(a) \).
PROOF. Clearly if \( x \) is a \( p' \)-element, \( a \notin < x > \). Since \( \eta(a) \) is abelian, the \( p' \)-elements form an abelian group.

**Lemma 3.7.** Suppose \( G = C(\eta(a)) \), \( 1 \neq a \in K \), \( |a| = p^n \). Then \( \bar{a} \in K(\bar{G}) \) and \( \bar{G} = G/H \) is a \( p \)-group, where \( H \) is the set of \( p' \)-elements.

**Proof.** This follows from Lemma 3.1.(iv) and Lemma 3.6 and its proof.

**Lemma 3.8.** If \([h,a] = 1\), \( h \in \eta(a) \) and \( |a| = p^n \), then \( ah \notin \eta(a) \) if and only if \( (p,|< h >/(< h > \cap < a >)|) = 1 \).

**Proof.** Note first that \( h \in \eta(a) \) implies that \( < h > \cap < a > \leq < a^p > \). Let \( m = |(< h > \cap < a >)| \). Then \( h^m = (a^p)^s \) for some \( s \). If \( 1 = p^nk + ml \), then \( (ah)^{ml} = a^m h^ml = a^{l-p^k} a^p t = a^{psl+1} \). Since \( (p, psl+1) = 1 \), there exists a \( t \) such that \( (ah)^{mst} = (a^{psl+1})t = a \), so \( ah \notin \eta(a) \). In the other direction, if \( (ah)^q = a \), then \( a^{q-1} = h^{-q} \in < h > \cap < a > \leq < a^p > \). This gives both \( p | q - 1 \) and \( m | q \), from which we conclude \( (p, m) = 1 \).

**Lemma 3.9.** Suppose \( g \in \eta(a) \) and \( x^p = a \). Then if \( g \notin \eta(y) \), \( y \in \eta(x) \).

**Proof.** By Lemma 3.2 (ii), \( y \in \eta(a) \). Thus \( x^p = a \notin < y > \), so \( x \notin < y > \).

**Lemma 3.10.** [8; p.191]. If \( H \) is an equipotent subgroup of \( G \), then \( K(G) \leq K(H) \).

**Definition.** Suppose \( G \) has a normal \( p^\infty \)-subgroup \( C \) of finite index. We say

(i) \( G \) is of type \( T_1 \) for the prime \( p \) if \( C \) is central;

(ii) \( G \) is of type \( T_2 \) if \( p = 2 \), \( C \) is not central, and every element \( x \) not in \( C \) satisfies \( x^{-1}cx = c^{-1} \) for all \( c \in C \) and there exists an \( m \) such that \( x^{2m} = a \), the unique element in \( C \) of order 2.

**Lemma 3.11.** The group \( G \) has an infinite abelian subgroup of finite index, and \( K(G) \neq E \) if and only if one of the following holds:

(i) \( G \) is of type \( T_1 \), in this case \( K(G) = C \);

(ii) \( G \) is of type \( T_2 \), in this case \( K(G) = < a > \).
PROOF. If $G$ has an infinite abelian subgroup of finite index, then $G$ has an infinite normal abelian subgroup $A$ of finite index. Now $K(G) \leq K(A) = E$ unless $A$ has a characteristic $p^n$-subgroup $C$ of finite index. Thus $C \triangleleft G$, and $G$ is a finite extension of $C$. Suppose $C$ is not central in $G$. Then $K = K(G) \leq C$ is central and finite and contains all the elements of $C$ which have order $p$. Let $x$ be any element in $G$ which fails to centralize $C$. Then conjugation by $x$, $\alpha_x$, is a non-trivial automorphism of $C$, which fixes every element of order $p$. By a theorem of Baer (see [6; Lemma 3.28]), $\alpha_x$ has infinite order unless $p = 2$. Thus we have a contradiction unless $p = 2$. For $p = 2$, we have the same contradiction if $\alpha_x$ fixes every element of order 4, so $\alpha_x^2 = 1$, $x^2 \in C_G(C)$ and $\alpha_x(c) = c^{-1}$ for every $c \in C$. Now $(xc)^2 = x^2(x^{-1}cx)c = x^2$ for each $c \in C$. Since $\eta(a)$ is finite where $a$ is the element of order 2 in $C$, $a \in < xc >$ for all but finitely many $c$. Suppose that there are infinitely many $c \in C$ such that $a = (xc)^m$ for $m_e$ odd. Then $(xc)^m = x^m c$, and since $x$ has finite order, there are infinitely many $m_e$ which are equal to $m$, some fixed integer. This gives infinitely many $c \in C$ such that $x^m c = a$ for a fixed integer $m$, clearly a contradiction. Thus for all but finitely many $c$, $a = (xc)^m$, with $m_e$ even. For these $c$, $a = (xc)^m = x^m$, so, in fact, there exists an even integer $2m$ such that $a = x^{2m}$. This shows that $G$ is type $T_2$. Clearly $K(G) = < a >$ since no other non-trivial element of $C$ is central.

Conversely, if $G$ is of type $T_1$, $K(G) = C$ by Theorem C(iii). If $G$ is of type $T_2$, let us compute $\eta(a)$. Since each element $x$ not in $C_G(C)$ satisfies $x^2 = a$ for some $m$, these $x$ are not in $\eta(a)$. On the other hand, $C_G(C)$ is a finite central extension of a $2^m$-group, so $\eta(a)$ contains only finitely many elements in $C_G(C)$. This shows that $K(G) \geq < a >$. In addition, $K(G) \leq C$, and $a$ is the only non-trivial central element in $C$. Thus $K(G) = < a >$.

§4. Main Results. First we characterize the groups of type $T_1$ and type $T_2$ (see Lemma 3.11).

THEOREM 4.1.

(i) $G$ is of type $T_1$ for the prime $p$ if and only if $G = (H \times C)_A$, where $H$ is a finite group, $C$ is a $p^n$-group, and $A \leq Z(H)$ is a cyclic group of order $p^n$;

(ii) $G$ is of type $T_2$ if and only if $G = < x, G_1 >$, where $G_1 = (H \times C)_A$ is of type $T_1$ for the prime 2, $x^{-1}cx = c^{-1}$ for every element $c$ of $C$, $[G : G_1] = 2$ and there exists an $m$ such that $x^{2m} = a$, the unique element of $C$ of order 2.

PROOF.
(i) If $G$ is of type $T_1$ then $G = \bigcup_{i \in \mathbb{N}} x_iC$. Let $H = \langle x_i : i \leq n \rangle$. Since $G$ is locally finite, $H$ is finite. Clearly $G = HC$, $H \cap C \leq Z(G)$, and is cyclic.

(ii) If $G$ is of type $T_2$, then $G_1 = C_{G_1}(C)$ is of type $T_1$ for the prime 2. Suppose $x$ and $y \notin C_{G_1}(C)$. Then $x^{-1}cx = c^{-1} = y^{-1}cy$ for every $c \in C$, so $yx^{-1} \in C_{G_1}(C)$. Thus $|G : C_{G_1}(C)| = 2$ and $G = \langle x, G_1(C) \rangle$ for any $x \notin C_{G_1}(C)$. In the other direction, note that for every $g \in G$ either $g^{-1}cg = c^{-1}$ for every $c \in C$ or $g \in C_{G_1}(C)$, depending upon the parity of the number of times that $x$ appears in the word which generates $g$. Again, we deduce $|G : C_{G_1}(C)| = 2$, so $G_1 = C_{G_1}(C)$.

EXAMPLE 4.2.

(i) Let $H$ be a non-abelian group of order $p^3$ and exponent $p$. Let $A$ be the center of $H$. Then $G = (H \times C)_A$ is not isomorphic to $C \times F$ for any finite group $F$ since $|G/C| = p^2$ and if, $|F| = p^2$, then $C \times F$ would be abelian.

(ii) Let $G_1 = (H \times C)_A$ be a group of type $T_1$. Let $y$ be any element in $G_1 - C$ such that $y^n = a$. Suppose there exists an automorphism $\alpha$ of $G_1$ which fixes $y$, $\alpha(c) = c^{-1}$ for each $c \in C$ and $\alpha^2(g_1) = y^{-1}g_1y$ for every $g_1 \in G$. Then there exists a group $G$ such that $G = \langle x, G_1 \rangle$ with $G/G_1 \cong Z_2 \times Z_2$, $x^2 = y$, $x^{-1}g_1x = \alpha(g_1)$ for every $g_1 \in G$ (see [7, 9.7.1]). If $G_1$ is abelian and $m = 1$, then such an $\alpha$ exists. One can easily construct examples with $m \neq 1$. For example, let $G_1 = \langle z \rangle \times C$, $z^2 = 1$. Then if $y = (z, a_2)$ with $a_2$ an element of order 4, we can define $\alpha$ by $\alpha(zc) = za_2^2c^{-1}$ for each $c \in C$.

Next we consider countable groups with non-trivial finite $K$.

THEOREM 4.3. If $G$ is a countable group with $K(G) \cong Z_{p^m}$, then $G$ has an infinite section $H$ of finite index with $K(H) = \langle a \rangle \cong Z_{p^n}$, where $H$ is a $p$-group and $\eta(a, H) = E$.

PROOF. First let us show that every group $G$ which is of type $T_2$ has a section $S$ of finite index which is infinite generalized quaternion, that is, $S = \langle x, C \rangle$ with $C \equiv Z_{2^m}$, $x^{-1}cx = c^{-1}$ for every element $c$ of $C$, and $x^2 = a$, the unique element of $C$ of order 2. Note that such a group $S$ has $K(S) = \langle a \rangle$ and $\eta(a) = E$. We shall induct on the integer $m$ in the definition of type $T_2$. Let $G = \langle x, G_1 \rangle$ be of type $T_2$ as in Theorem 4.1 (ii) with $G_1 = (H \times C)_A$, etc. Let $a_2$ be an element of $C$ of order 4. Let $z = x^ma_2$. We have $z^{-1}cz = x^{-m}cx^m = c^{(-1)^m}$ for every $c \in C$ and $z^2 = (x^ma_2)^2 = x^ma_2x^{-m}a_2 = a_2^{(-1)^m}a_2a$.
If $m$ is odd, $< z, C >$ is the desired section. If $m$ is even, say $m = 2k$, then $S_1 \leq x, C > / N$ with $N = \{1, z, za, a\}$ has the desired generalized quaternion section by induction. To see this, note that we have shown that $(za)^2 = z^2 = 1$; $[z, c] = [x^m, c] = 1$ for every $c \in C$, $x^{-1}zx = xx^m a_x a_z = x^m a_z^{-1} = za$ so that $N \leq S_1$; $a_2 N$ is the unique element in $CN / N$ of order 2 since $a_2 \notin N$; and finally, $x^{2k} = x^m = za z a^{-1} = za a_z$, so $x^{2k} N = a_2 N$.

Let $K(G) = \langle a \rangle \cong Z_{p^n}$. We may assume that $G$ does not have a subgroup $H_1$ of finite index which is type $T_2$, otherwise $H_1$ has a subgroup $H$ which satisfies the conditions of the theorem by the argument just given.

Since by Lemma 3.3, $\alpha(\eta(a)) = \eta(a)$ for every $\alpha \in Aut(G)$, and since $\eta(a)$ is finite, each element in $\eta(a)$ has finitely many conjugates. Thus $H = C(\eta(a))$ has finite index in $G$. If $C = K(H)$ were a $p^{\infty}$-group, then since $C$ has finite index in $H$ by Theorem C(iii), $C$ would have finite index in $G$. Since $K(G)$ is finite, $C$ could not be central. Thus by Lemma 3.11, we would have $K(G) = E$; a contradiction. Thus $K(H)$ is finite. Since by Lemma 3.10, $K(G) \leq K(H)$, $K(H)$ is non-trivial, and we may suppose $G = C(\eta(a))$. The set $N$ of $p'$-elements forms an abelian subgroup and $N \leq \eta(a)$ by Lemma 3.6. Now $\bar{a} \in K(G / N)$ by Lemma 3.7. If $P / N = K(G / N) \cong Z_{p^n}$, then $[G / N : P / N] < \infty$. Now by [7,3.2.10], $P$ is abelian, hence $P = Q \times N$, where $Q$ is a $p^{\infty}$-group and $N$ is a finite $p'$-group. Thus $[G : Q] < \infty$, so $K(G) \cong Z_{p^n}$; a contradiction. Thus we may assume that $G$ is a $p$-group.

We have shown in Lemma 3.5 that $K(G / \langle a^n \rangle) = \langle \bar{a} \rangle$, so we may assume that $K(G) \cong Z_p$. Consider $Z(G)$. If it is infinite, then by Theorem A, $K(Z(G)) = E$ unless $Z(G) = Q \times F$, where $Q$ is a $p^{\infty}$-group and $F$ is finite. If $Z(G)$ is finite, $Z(G) = Q \times F$, where $Q$ is cyclic and $F$ is finite. Thus $Z(G) = Q \times F$, where $Q$ is cyclic or quasicyclic and $a \in Q$. Now $\eta(a) = \{qf : q \in Q, f \in F, |q| \leq |f|\}$.

Let us compute in $\bar{G} = G / F$. Suppose $|\eta(\bar{x})| < \infty$, with $x \notin Z(G)$. Then $|G - C(x)|$ is infinite and so is $|G - C(x) / F|$. Since $C(xf) = C(x)$ for all $f \in F$, $G - C(x) = G - C(xf) \leq \eta(xf)$. If $y \in G - C(x)$, but $y \notin \eta(\bar{x})$, then $\bar{x} = \bar{y}^n$ or $y^n = xf$ for some $n$ and $f \in F$, contradicting $y \in \eta(xf)$. This shows that $|\eta(\bar{x})|$ is finite only if $x \in Z(G)$.

Let us show that $\eta(\bar{a}) = \{\bar{1}\}$. Suppose $\bar{z}^p = \bar{1}$, $\bar{z} \neq \bar{1}$, If $z \in \eta(a)$, then $z = qf$, with $q \in Q, f \in F, |q| \leq |f|$. Since $q^p f^p = z^p \in F, q^p = 1$, so $q \in \langle a \rangle$. Thus $< \bar{z}^p > = < \bar{a} >$. If $z \notin \eta(a)$, then since $z^p \in F \leq \eta(a), z^p = 1$. Since $z \notin F$, $< z > = < a >$ and again
$\langle \overline{z} \rangle = \langle \overline{a} \rangle$. Thus $\langle \overline{a} \rangle$ is the only subgroup of $G$ of order $p$. It follows that $\eta(\overline{a}) = \{1\}$.

If $\eta(\overline{x})$ is finite, then $x \in Z(G)$, so $x = qf$ where $q \in Q$. We shall show that $\eta(q) = \langle q^p \rangle$. We have shown that $\eta(\overline{a}) = \{1\}$. Suppose that we have established the equality $\eta(q) = \langle q^p \rangle$ for all $q \in Q$ with $\langle q^p \rangle < a > < p^m$, and there is a $q \in Q$, with $q^m = a$. Suppose $\overline{y} \in \eta(q^p)$, then $\overline{y} \notin \eta(q^p)$, so $\overline{y}^n = q^p$ for some $n$. Since $\overline{y} \notin \eta(q^p)$, we may assume that $p \mid n$, say, $n = pl$. Thus $\overline{y}^n = (\overline{y}^l)^p = q^p$, where $\overline{z} = \overline{y}^l \in \eta(q) \subset q^p$. We have $\overline{y}^n = (\overline{y}^l)^p = 1$. Since $\overline{y} \in \eta(q)$, $\overline{z} \neq q^p$. Since $\overline{z} \in \eta(q) \subset < q^p >$, we have $\overline{z} \neq q^p$. Thus $\overline{z} \neq q^p$. We have shown that $K(G) \equiv Q$ and that $\eta(q) = \langle q^p \rangle$ for every non-trivial $q \in Q$. If $Q$ is a $p^\infty$-group, then $[G : K(G)] < \infty$, so $[G : Q] < \infty$, which gives $K(G) \equiv Z_{p^\infty}$, which is a contradiction. Thus there is a group $G$ with $K(G) = \langle a >$ and $\eta(x) = \langle x^p \rangle$ for each non-trivial $x \in < a >$. Now in $\overline{G} = G/\langle a >$, $K(\overline{G}) = \langle \overline{a} > \equiv Z_p$ by Lemma 3.5 and $\eta(\overline{a}) \leq \overline{a} = \langle \overline{a}^p \rangle = \{1\}$. This proves the theorem.

We need the following lemma which is stated in [10]. The main ideas for the proof can also be found in [6; p. 70, part 1] and [2; I.G.4 and I.G.6].

**Lemma 4.4.** Let $G$ be infinite and either binary finite or a 2-group. If $G$ has a finite maximal elementary abelian subgroup, then it has an infinite abelian subgroup of finite index.

**Theorem 4.5.** If $G$ is a countable group such that $K(G) \neq E$, then one of the following holds:

(i) $G$ is of type $T_1$: $G = (P \times F)_C$, with $F$ a finite group; $P$ a $p^\infty$-group; and $C \leq Z(F)$, a cyclic group of order a power of $p$. In this case $K(G) = P$.

(ii) $G$ is of type $T_2$: $G = < x, G_1 >$, where $G_1 = (P \times F)_C$ with $F$ a finite group; $P$ a $2^\infty$-group; $C \leq Z(F)$, a cyclic group of order a power of 2, $x^{-1}zx = z^{-1}$ for every element $z$ of $P$. $[G : G_1] = 2$ and there exists an $m$ such that $x^{2m} = a$, the unique element of $C$ of order 2. In this case $K(G) = < a >$.

(iii) $G$ has an infinite section $H$, which is a 2-generated $p$-group for an odd prime $p$ and $K(H) \equiv Z_p$. 

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PROOF. We know by Theorems C(iii) and 4.1 that \( K(G) \cong \mathbb{Z}_p \) if and only if (i) holds. On the other hand, if \( K(G) \cong \mathbb{Z}_p \) then by Theorem 4.3 there is an infinite section \( H \) with \( K(H) = \langle a \rangle \cong \mathbb{Z}_p \), with \( H \) a \( p \)-group and \( \eta(a) = \{1\} \). Clearly \( \langle a \rangle \) is a maximal elementary abelian subgroup. If \( H \) is binary finite or a 2-group, then by Lemma 4.4, \( H \) has an infinite abelian subgroup of finite index. By Lemma 3.11, this is only possible if (ii) holds. Otherwise \( p \neq 2 \) and \( H \) has an infinite subgroup \( L \) which is 2-generated with \( a \in L \) and \( K(L) \) finite. By Lemma 3.5, \( L \) has a factor group which satisfies (iii).

COROLLARY 4.6. If \( G \) is countable and either binary finite or a 2-group such that \( K(G) \neq E \), then either \( G \) is of type \( T_1 \) and \( K(G) \) is \( p^n \)-group or \( G \) is of type \( T_2 \) and \( K(G) \cong \mathbb{Z}_2 \).

REMARKS. We have not been able to improve Theorem 4.5 to include in (iii) the statement \( \langle a \rangle \cong \mathbb{Z}_p \) with \( \eta(a) = \{1\} \). Of course, the likely place to look for groups of the type described in 4.5 (iii) is a central extension of \( \mathbb{Z}_p \) by a group \( B \) which is either a Novikov-Adjan group (see [4]) or a Tarski-Monster constructed by Ol’shanskii [5]. This necessitates the study of maps \( W : B \times B \to \mathbb{Z}_p \) which satisfy

\[
W(xy, z) + W(x, y) = W(x, yz) + W(y, z),
\]

which seems quite difficult.

Now we examine some uncountable groups.

THEOREM 4.7. If \( G \) is an uncountable class 2 nilpotent \( p \)-group for \( p \) odd, then \( |\Omega_1(G)| = |G| \). For each infinite cardinal \( m \) there is a class 2 nilpotent 2-group \( G \) of power \( 2^m \) such that \( |\Omega_1(G)| = 2^m \).

PROOF. Consider the tree \( T = 2^{<m} \) of functions from ordinals less than \( m \) into 2, ordered by function extension. \( T \) has \( 2^{<m} \) nodes and \( 2^m \) paths (a path corresponds to a function from \( m \) into 2.) Let \( V \) be the \( 2^m \)-dimensional vector space over \( \mathbb{Z}_2 \) with basis \( \Sigma \), the set of paths of \( T \). Let \( W \) be the \( 2^{<m} \)-dimensional vector space over \( \mathbb{Z}_2 \) with basis \( N \), the set of nodes of \( T \). We define for each \( f \) and \( g \) in \( \Sigma \), \( \rho(f, g) \) to be the largest element of \( N \) common to both \( f \) and \( g \) if \( f \neq g \) and \( \rho(f, f) = 0 \). Extend \( \rho \) by bilinearity to all of \( V \). Let \( \gamma \) be any bilinear function from \( V \times V \) to \( W \) such that \( \gamma(v, v) \neq 0 \) for \( v \in \Sigma \) and \( \rho(u, v) = \gamma(u, v) + \gamma(v, u) \). Consider the class 2 nilpotent 2-group \( G = V\gamma W \) (see [1]). If \( (v, a)^2 = 1 \) with \( 0 \neq v \in V \) and \( a \in W \), then \( \gamma(v, v) = 0 \). Suppose \( v = \sum_{i=1}^{n} f_i \) with
If $G$ is an uncountable class 2 nilpotent $p$-group for $p$ odd with $|\Omega_1(G)| < |G|$, then since the derived group $G'$ is abelian, $|G'| < |G|$ (see [8; p. 184, Corollary 1]). Assume that $G$ is an example with smallest cardinality. It is clear that $|G|$ is a successor cardinal. Since $G / G'$ is abelian, it has an elementary abelian subgroup $H / G'$ with $|H / G'| = |G / G'| = |G|$. If $x, y \in H$, then $[x, y] \in Z(G)$ and $x^p \in G' \leq Z(G)$, so $[x, y]^p = [x^p, y] = 1$, and hence $\exp H' = p$. Let $\{x^\alpha H'\}$ be a basis for $H / H'$. Since $x_\alpha^p \in H'$, there is a set $T$ of power $|G|$ and an $a \in H'$ such that $x_\alpha^p = a$ for every $\alpha \in T$.

But then $(x_\beta^{-1} x_\alpha)^p = [x_\alpha, x_\beta^{-1}]^p [x_\beta^{-1}]^p = 1$, so $|\Omega_1(G)| = |G|$.

**COROLLARY 4.8.** For each infinite cardinal $m$ there is a class 2 nilpotent 2-group $G$ of power $2^m$ such that every abelian subgroup $A$ satisfies $|A| \leq 2^m$.

**PROOF.** This is clear since if $|A| > 2^m$, then $|\Omega_1(G)| \geq |\Omega_1(A)| = |A| > 2^m$; a contradiction.

**REMARK.** The uncountable group constructed in [3] is similar.

**COROLLARY 4.9.** If $G$ is an uncountable class 2 nilpotent group, then $K(G) = E$.

**PROOF.** If $G$ is a counter-example of smallest cardinality and $1 \neq a \in K(G)$, then $|\eta(a)| < |G|$, so we may assume $|\eta(a)| \leq m$ and $|G| = m^+$. Thus, by arguments like those used in the proof of the theorem, we need only prove the corollary for groups $G$ such that $G / G'$ and $G'$ have exponent $p$. If $p$ is odd, then Theorem 4.7 gives a contradiction, so we suppose that $p = 2$. We have $a^2 = 1$, and if $x \in G - \eta(a) - \{a\}$, then $x^2 = a$. Let $\{x_\alpha\}$ be such that $\{x_\alpha G\}$ is a basis for $G / G'$. Let $x$ be any fixed element of $T = \{x_\alpha\} - \eta(a)$. If $x_\alpha \in T$, then $(xx_\alpha)^2 = x^2 x_\alpha^2 [x, x_\alpha] = [x, x_\alpha]$. Thus there must be $S_0 \leq T$ such that $|S_0| = |T|$ and $[x, x_\alpha] = a$ for all $x_\alpha \in S_0$. Fix $y \neq x \in S_0$. Again there is $S_1 \leq S_0$ such that $|S_1| = |S_0|$ and $[y, x_\alpha] = a$ for all $x_\alpha \in S_1$. Fix $z \neq x, y \in S_1$ and find $S_2 \leq S_1$ such that $|S_2| = |S_1|$ and $[z, x_\alpha] = a$ for all $x_\alpha \in S_2$. Thus for each $x_\alpha \in S_2$, $(xyzx_\alpha)^2 = (xy)^2 (zx_\alpha)^2 [x, y][x, z][y, z][x, x_\alpha][y, x_\alpha] = a^6 = 1$. But $xyzx_\alpha$ are distinct members of $\eta(a)$ for $x_\alpha \in S_2$, which is a contradiction.
REMARK. Although it seems quite likely that Corollary 4.9 holds for all uncountable locally nilpotent groups, we have not even been able to extend it to class 3 nilpotent groups.

EXAMPLE 4.10. There are class 2 nilpotent $p$-groups with $D > K = E$. To see this, consider one of the groups $G$ constructed (using G.C.H.) in [1]. $G$ has the properties ($p$ is a prime):
(i) $[G] = \lambda^+$, where $\lambda$ is an infinite cardinal;
(ii) $G' = Z(G) \cong Z_p$;
(iii) if $A$ is an abelian subgroup of $G$, $|A| \leq \lambda$;
(iv) $\exp(G/G') = p$.

Thus if $G_a \leq G$, where $|G_a| < |G|$, $G_a' = G'$. It follows that $D = G'$. On the other hand, we know that $K \leq G' = Z(G)$. Let $1 \neq a \in G'$. Since $\eta(a) = \{g \in G : g^p = 1\} - \langle \langle a \rangle \rangle$, if $|\eta(a)| \leq \lambda$, take $x_0 \in G - \eta(a)$, $x_0^p = a$. Then $|C(x_0) - \eta(a)| = \lambda^+$, and for every $x \in C(x_0) - \eta(a)$, $x^p = a$. Then $(x^{-1}x_0)^p = x^{-p}x_0^p = 1$, so $x^{-1}x_0 \in \eta(a)$. Thus $|\eta(a)| = \lambda^+$, and so $K = E$.

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