NLSE for quantum plasmas with the radiation damping

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We consider contribution of the radiation damping in the quantum hydrodynamic equations for spinless particles. We discuss possibility of obtaining of corresponding non-linear Schrodinger equation (NLSE) for the macroscopic wave function. We compare contribution of the radiation damping with weakly (or semi-) relativistic effects appearing in the second order by v/c. The radiation damping appears in the third order by v/c. So it might be smaller than weakly relativistic effects, but it gives damping of the Langmuir waves which can be considerable.

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I. INTRODUCTION

Some relativistic effects have been considered in classic and quantum plasmas [1]-[10]. In many cases one-particle Schrodinger, Pauli, Klein-Gordon, and Dirac equations have been used to derive set of quantum hydrodynamic equations [11], [12], [13], [14]. Some collective effects in classic relativistic plasmas with the radiation damping were considered in Ref. [14]. A method of rigorous derivation of non-relativistic and semi-relativistic hydrodynamic equations from many-particle Schrodinger equation was suggested in Ref. [12] and developed in Refs. [8], [9], [16], [17]. Semi-relativistic effects appears in the second order by the parameter v/c showing ratio of the particle velocity v to the speed of light c. In this paper we are interested in contribution of the radiation damping in the evolution of quantum plasmas working in terms of quantum hydrodynamics (QHD). This effect appears in the third order by v/c, when electromagnetic radiation of particles arises in theory. So system can not be described in terms of Hamiltonian of particles, the electromagnetic field have to be explicitly accounted. However the method of classic hydrodynamic derivation suggested in Refs. [18], [19] does not apply Hamiltonian description using the Newton equations. Recent applications and discussions of this method can be found in Refs. [4] and [20]. Hence this method gives possibility to derive hydrodynamics with the radiation damping. Comparing final equations with similar QHD equation we make generalization of obtained equation on quantum plasmas.

Hydrodynamic equations appears as natural representation of classic and quantum dynamics of many-particle systems. In some cases the set of continuity and Euler equations including the quantum Bohm potential can be represented as non-linear Schrodinger equation (NLSE) for macroscopic wave function defined in terms of the particle concentration and the potential of velocity field [13], [21]. Famous examples of NLSE are the Gross-Pitaevskii equation for Bose-Einstein condensates of neutral atoms and Ginsburg-Landau equation for superconductors. Different methods for dealing with NLSEs have been developed, hence they can be used for weakly relativistic quantum plasmas with the radiation damping as well.

We gave an old example of studying of the radiation damping in plasmas [14]. However it is a topic under discussion in recent papers as well. For instance laser-plasma interactions in ultra-relativistic regime were considered in Ref. [22] to calculate the nonlinear dielectric permittivity, ponderomotive and dissipative forces acting in plasmas. The motion of a cold electron fluid accounting for the radiation reaction force in the Lorentz-Abraham-Dirac form was used (see formula 3). Some radiative and quantum electrodynamics effects were numerically modeled for ultra-relativistic laser-plasma interactions as well [23]. The scalar and spinor quantum electrodynamics in the presence of strong laser fields in plasmas were considered in Ref. [24].

II. CONSTRUCTION OF MACROSCOPIC EQUATIONS

Microscopic classic motion of each particle in plasmas obeys the Newton equation

\[ m_i \ddot{v}_i = F_i, \]

where \( i \) is the number of particle and force acting on the particle includes the radiation damping \( f_i \) along with the Lorentz force

\[ F_i = q_i E_i + \frac{q_i}{c} [v_i, B_i] + f_i, \]

where \( E_i \) and \( B_i \) are electric and magnetic fields acting on \( i \)th particle and creating by other particles of the system, \( m_i \) are masses of particles, \( q_i \) are charges of particles, \( v_i \) is the velocity of particles, and \( c \) is the speed of light.

Non-relativistic force for the radiation damping appears as

\[ f_i \approx \frac{2q_i^2}{3c^3} \dot{v}_i \approx \frac{2q_i^3}{3m_i c^3} E_i + \frac{2q_i^4}{3m_i^2 c^3} [E_i, B_i] \]

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(see Ref. 25 section 9, paragraph 75, formula 75.8). The second identity in formula 3 has been made taking derivative of the acceleration with respect to time and neglecting \( \dot{\mathbf{f}} \). In the second term of the Lorentz force we have used \( \mathbf{v} = e\mathbf{E} \), and we have not considered time derivative of the magnetic field. Doing these approximations we keep considering terms in the third order on \( v/c \).

As the result of manipulations described above we have an approximate equations for classic motion of electrons with the radiation damping. We can use it as framework to derive classic hydrodynamic equations describing collective evolution of considering system. To this end we use method suggested in Ref. 18 and briefly reviewed in Ref. 20. This method gives the following continuity and Euler equations

\[
\partial_t n + \nabla (n \mathbf{v}) = 0 \tag{4}
\]

and

\[
m_n (\partial_t + \mathbf{v} \nabla) \mathbf{v} + \nabla p = q n \mathbf{E}
+ \frac{q}{c} n \left[ \nabla \mathbf{v} + \frac{2q^3}{3mc^3} \mathbf{E} + \frac{2q^4}{3m^2c^4} \mathbf{E}, \mathbf{B} \right] \tag{5}
\]

written in the self-consistent field approximation allowing to truncate the chain of HD equations at using of an equation of state for the thermal pressure. In equations 4 and 5 we have next physical quantities

\[ n(\mathbf{r}, t), \mathbf{v}(\mathbf{r}, t), p(\mathbf{r}, t), \mathbf{E}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t) \]

and has following form

\[
\nabla \mathbf{E}(\mathbf{r}, t) = 4\pi \sum_a q_a n_a(\mathbf{r}, t), \tag{6}
\]

\[
\nabla \times \mathbf{E}(\mathbf{r}, t) = 0, \tag{7}
\]

\[
\nabla \mathbf{B}(\mathbf{r}, t) = 0, \tag{8}
\]

and

\[
\nabla \times \mathbf{B}(\mathbf{r}, t) = \frac{4\pi}{c} \sum_a q_a n_a(\mathbf{r}, t) \mathbf{v}_a(\mathbf{r}, t). \tag{9}
\]

We do not have time derivatives of field here, since only trace of radiation in the approximation under consideration is the radiation damping.

We see a field force caused by the radiation damping. Comparing classic and quantum hydrodynamic equations we can get contribution of the radiation damping in the set of QHD equations. The quantum Euler equation differs from the classic one by the quantum Bohm potential and has following form

\[
m_n (\partial_t + \mathbf{v} \nabla) \mathbf{v} + \nabla p
- \frac{\hbar^2}{2m} \nabla \left( \frac{\Delta n}{n} - \frac{(\nabla n)^2}{2n^2} \right) = q n \mathbf{E}
+ \frac{q}{c} n \left[ \nabla \mathbf{v} + \frac{2q^3}{3mc^3} \mathbf{E} + \frac{2q^4}{3m^2c^4} \mathbf{E}, \mathbf{B} \right]. \tag{10}
\]

Considering quantum plasmas of charged Fermi particles we should consider the Fermi pressure instead of thermal pressure.

Having equations 4 and 10 we can derive a non-linear Schrodinger equation for the macroscopic wave function 17 neglecting by the magnetic field and assuming that the electric field is potential \( \mathbf{E} = -\nabla \phi \)

\[
\Phi = \sqrt{n} \exp \left( \frac{i}{\hbar} m \phi \right), \tag{11}
\]

where \( \phi \) is the potential of the velocity field \( \mathbf{v} = \nabla \phi \).

Considering the potential electric field we can introduce an effective potential electric field including contribution of the radiation damping

\[
\tilde{\mathbf{E}} = -\nabla \varphi = -\left( 1 + \frac{2e^2}{3mc^3} \partial_t \right) \nabla \varphi. \tag{12}
\]

Differentiating the macroscopic wave function 11 with respect to time we obtain a non-linear Schrodinger equation

\[
i\hbar \partial_t \Phi(\mathbf{r}, t) = \left( -\frac{\hbar^2 \nabla^2}{2m} + 3\pi^2 \right) \frac{\hbar^2}{2m} n \hat{\mathbf{e}}(\mathbf{r}, t) + q \varphi \right) \Phi(\mathbf{r}, t) \tag{13}
\]

with \( n = |\Phi|^2 \).

Writing explicit form of the potential of effective electric field we represent the NLSE as follows

\[
i\hbar \partial_t \Phi(\mathbf{r}, t) = \left( -\frac{\hbar^2 \nabla^2}{2m} + 3\pi^2 \right) \frac{\hbar^2}{2m} n \hat{\mathbf{e}}(\mathbf{r}, t) + q \varphi \frac{2q^3}{3mc^3} \partial_t \varphi \right) \Phi(\mathbf{r}, t), \tag{14}
\]

where the potential of electric field depends on the particle concentration via the Maxwell equations 8, 9, and consequently it depends on the macroscopic wave function. Hence we have closed set of equations consisting of the NLSE and the Maxwell equations.
The dispersion dependence of semi-relativistic Langmuir waves in absence of the radiation damping was
obtained in Ref. [8]. It has the following form

$$\omega_{SR}^2(k) = \omega_{Le}^2 \left(1 - \frac{\hbar^2 k^2}{4m^2 c^2} \right)$$

$$+ \frac{\hbar^2 k^4}{8m^4 c^2} + \frac{\gamma T_0}{m} k^2,$$

where we included shifts of the Langmuir frequency by both the semi-relativistic part of kinetic energy and the
Darwin interaction presented by term proportional to $\varsigma$. Simultaneous account of both effects gives $\varsigma = 0$, for
details see Ref. [8]. The last part of the first term appears as consequence of the thermal-semi-relativistic force field.
Other terms have following meaning: the quantum Bohm potential, the semi-relativistic part of the quantum Bohm
potential, and the contribution of the thermal pressure.

In linear approximation on small perturbations of hydrodynamic variables $\delta n = N \exp(-i\omega t + ikr)$ and
$\delta v = U \exp(-i\omega t + ikr)$ we can get a dispersion equation giving $\omega(k)$

$$\omega^2 + \lambda \omega - \Omega^2 = 0. \quad (16)$$

Dispersion dependence $\omega(k)$ arises as

$$\omega = \frac{1}{2} \left( -i \lambda \pm 2 \Omega \sqrt{1 - \frac{\lambda^2}{4 \Omega^2}} \right), \quad (17)$$

including weakly-relativistic nature of the radiation damping we can give approximate formula for the disper-
sion dependence

$$\omega \approx \Omega \left[ 1 - \frac{\lambda^2}{8 \Omega^2} - i \frac{\lambda}{2 \Omega} \right], \quad (18)$$

containing

$$\lambda = \frac{8 \pi e^4 n_0}{3 m^2 c^3} = \frac{2 \omega_{Le}^2 r_e}{3 c}, \quad (19)$$

where $r_e \equiv e^2/(mc^2)$ is the classic radius of electron, and $\omega_{Le}$ is the Langmuir frequency, and

$$\Omega^2 = \frac{1}{m} \frac{\partial^2}{\partial n^2} k^2 + \frac{\hbar^2 k^4}{4m^2} + \frac{\gamma T_0}{m}, \quad (20)$$

with $\frac{\partial n}{\partial t} = \gamma T_0$, and $\gamma = 3$.

In formula (18) we can see imaginary part of the fre-
quency, which is negative and gives to damping of the
semi-relativistic quantum Langmuir waves.

Presence of the semi-relativistic effects given by for-

mula (15) changes $\Omega$, but it does not affect structure of
solution (17).

Proper consideration based on full spinless semi-relativistic
theory [8] gives $\omega_{SR}^2(k)$ (15) instead of $\Omega$. Fi-

nally we can rewrite formula (18) as follows

$$Re \omega = \omega_{SR}(k) - \frac{1}{18} \frac{r_e^2}{c} \sqrt{\frac{\omega_{Le}^4}{2} + \frac{3T_0 k^2}{m} + \frac{\hbar^2 k^4}{4m^2}}, \quad (21)$$

and

$$Im \omega = -\frac{1}{3} \frac{\omega_{Le}^2 r_e}{c}. \quad (22)$$

Formula shows that the damping of Langmuir waves
caused by the radiation damping do not contain contribu-
tion of quantum effects.

III. CONCLUSION

We have considered weakly relativistic evolution of
quantum plasmas including radiation damping. We have
derived corresponding equations of quantum hydro-
dynamics. Solving these equations we obtained spectrum
of the Langmuir waves. We have found that the radia-
tion damping gives a shift of the real part of frequency of
the Langmuir waves, but it also leads to damping of
the waves. We should mention that this damping is not
affected by the quantum effects in considered approxima-
tion.

We have shown that the QHD equations with the ra-
diation damping can be represented in form of a NLSE
for the wave function in medium. This equation may
get its place in list of the Ginsburg-Landau and Gross-
Pitaevskii equations. Each of them has been very useful
in their own fields. Therefore the NLSE derived in
this paper opens similar possibilities for quantum plasmas.

Obtained in this paper equations allows to study dif-
derent effects in the weakly relativistic quantum plasmas
with the radiation damping.

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