Non-vanishing theorem for generalized log canonical pairs with a polarization

Kenta Hashizume

Accepted: 26 July 2022
© The Author(s), under exclusive licence to Springer Nature Switzerland AG 2022

Abstract
We prove that the non-vanishing conjecture holds for generalized lc pairs with a polarization.

Keywords Generalized abundance · Non-vanishing theorem · Generalized lc pairs

Mathematics Subject Classification 14E30 · 14J17 · 14J40

Contents
1 Introduction ...............................................
2 Preliminaries ...............................................
  2.1 Generalized pairs ..........................................
  2.2 Property of being log abundant ...................................
3 Generalized abundance for generalized lc pairs .............................
4 Proofs of main results ..........................................
  4.1 Non-vanishing theorem .......................................
  4.2 On the Kodaira type vanishing theorem ...............................
5 Appendix: On non-$\mathbb{R}$-Cartier analogue of non-vanishing theorem ...................
References ..................................................

1 Introduction
We will work over an algebraically closed field of characteristic zero.
In this paper, we deal with generalized pairs, an extended notion of log pairs. The notion was introduced by Birkar and Zhang [8] to study the effective Iitaka fibration problem. In [8], Birkar and Zhang proved effectivity and boundedness of several invariants, known as the effective birationality, the ACC for generalized lc
thresholds, and the global ACC. These results played critical roles in Birkar’s work on the boundedness of complements [2] and the boundedness of Fano varieties with a fixed dimension and mild singularity [3], which led us to a significant development in birational geometry. Currently, generalized pairs are also used as powerful tools to prove results of log pairs [5, 11, 33, 36].

The geometry of generalized pairs is also a topic of independent interest. The structure of generalized pairs naturally appears in log pairs with anti-nef log canonical divisors, the base varieties of lc-trivial fibrations or Iitaka fibrations, and the adjunction formulas for higher codimensional lc centers. A lot of results in birational geometry, in particular, the minimal model program, the canonical bundle formula, and Fujita’s log spectrum conjecture, have been extended to the context of generalized pairs [10, 18, 20–24, 34, 35]. For the current status and open problems, see [4] by Birkar.

In this paper, we study the non-vanishing conjecture for generalized lc pairs. In the case of log pairs, the non-vanishing conjecture for lc pairs predicts that the pseudo-effective log canonical divisor of an lc pair is effective up to $\mathbb{R}$-linear equivalence. The conjecture is one of the most important open problems in minimal model theory. Similar to the case of log pairs, we can define generalized lc pairs, generalized klt pairs, and so on, and we can formulate a generalized lc pair analogue of the non-vanishing conjecture. However, the non-vanishing conjecture for generalized lc pairs does not hold even in the case of dimension one. Hence, a weaker assertion of the conjecture, called weak non-vanishing conjecture (cf. [7, Question 3.5]), is a suitable formulation for generalized lc pairs, and the conjecture is expected to hold. Unfortunately, we cannot reduce the weak non-vanishing conjecture for generalized lc pairs to the non-vanishing conjecture for lc pairs because the class of generalized lc pairs is strictly larger than the class of lc pairs. More precisely, there exists a normal projective variety having a structure of generalized lc pair but not having any boundary divisor such that the corresponding log pair is lc ([26, Remark 4.13 (1)]). So the weak non-vanishing conjecture for generalized lc pairs does not immediately follow from the non-vanishing conjecture for lc pairs.

Though we have the above obstructions, the following main result of this paper shows that the non-vanishing conjecture for generalized lc pairs holds true when we consider generalized lc pairs with a polarization.

**Theorem 1.1** Let $(X, B + M)$ be a projective NQC generalized lc pair with data $X' \to X$ and $M'$. Let $A$ be an ample $\mathbb{R}$-divisor on $X$. Suppose that $K_X + B + A + M$ is pseudo-effective. Then the following statements hold true.

1. There exists an effective $\mathbb{R}$-divisor $D$ on $X$ such that

   $$K_X + B + A + M \sim_{\mathbb{R}} D.$$ 

2. Suppose further that $M$ is $\mathbb{R}$-Cartier. Then, for every real number $\alpha \geq 1$, there exists an effective $\mathbb{R}$-divisor $D_\alpha$ on $X$ such that

   $$K_X + B + A + \alpha M \sim_{\mathbb{R}} D_\alpha.$$
For the definition of NQC property, see [20] by Han and Li or Definition 2.2. Note that the generalized pair \((X, B + \alpha M)\) in Theorem 1.1 (2) is not necessarily generalized lc. So Theorem 1.1 (2) is not a direct consequence of Theorem 1.1 (1). In the generalized klt case, we can reduce the theorem to the case of log pairs which follows from [6]. If \(M = 0\), then Theorem 1.1 is nothing but the non-vanishing theorem for lc pairs with a polarization, which was proved in [29]. We would like to remark that Theorem 1.1 does not directly follow from [29] because of gaps between generalized lc pairs and lc pairs.

As stated before, generalized lc pairs appear in the canonical bundle formulas for lc-trivial fibrations or Iitaka fibrations and the higher codimensional adjunction formulas. In those situations, usually we do not know whether the varieties are \(\mathbb{Q}\)-factorial. This is a reason why we do not assume that the variety in Theorem 1.1 is \(\mathbb{Q}\)-factorial. From the viewpoint, it looks unnatural that \(M\) is assumed to be \(\mathbb{R}\)-Cartier in Theorem 1.1 (2). Hence, it is natural to expect the effectivity of \(K_X + B + \alpha M\) up to \(\mathbb{R}\)-linear equivalence without the \(\mathbb{R}\)-Cartier property of \(M\) in Theorem 1.1 (2). See Sect. 5 for the question of this direction and a related topic. It is not clear that the expectation can be realized, but a partial result holds true.

**Theorem 1.2** Let \((X, B + M)\) be a projective NQC generalized lc pair with data \(X' \xrightarrow{f} X\) and \(M'\). Let \(A\) be an ample \(\mathbb{R}\)-divisor on \(X\). Suppose that \(K_X + B + A\) is the pushdown of a pseudo-effective \(\mathbb{R}\)-divisor on \(X'\). Then, for every real number \(\alpha \geq 0\), there exists an effective \(\mathbb{R}\)-divisor \(D_\alpha\) on \(X\) such that

\[
K_X + B + A + \alpha M \sim_\mathbb{R} D_\alpha
\]

as (not necessarily \(\mathbb{R}\)-Cartier) \(\mathbb{R}\)-divisors.

Theorem 1.2 is a variant of “Generalised Nonvanishing Conjecture”, studied by Lazić and Peternell [31]. In [31] and [32], Lazić and Peternell studied generalized klt pairs (without polarizations) under assumption of the pseudo-effectivity of the log canonical divisors. Especially, in [31], they proved connections between conjectures of minimal model theory for klt pairs and those for generalized klt pairs. Generalised nonvanishing conjecture in [31] asserts that the sum of the log canonical divisor of a pseudo-effective klt pair and a nef \(\mathbb{Q}\)-divisor always belongs to the effective cone. Theorem 1.2 is an analogue of Lazić–Peternell’s generalised nonvanishing conjecture in the context of generalized lc pairs.

Although the above theorems are concerned with only the non-vanishing theorem, we can prove a stronger property, called the property of being log abundant. For the definition of the property of being log abundant, see Definition 2.9. The property of being log abundant for \(\mathbb{R}\)-divisors is much stronger than effectivity up to \(\mathbb{R}\)-linear equivalence (see Sect. 2.2),

**Theorem 1.3** Let \((X, B + M)\) be a projective NQC generalized lc pair with data \(X' \rightarrow X\) and \(M'\). Let \(A \geq 0\) be an ample \(\mathbb{R}\)-divisor on \(X\) such that \((X, (B + A) + M)\) is a generalized lc pair whose generalized lc centers are those of \((X, B + M)\). Let \((Y, \Gamma + M_Y)\) be a \(\mathbb{Q}\)-factorial generalized dlt model of \((X, (B + A) + M)\).
Then, for any sequence of steps of a \((K_Y + \Gamma + M_Y)\)-MMP

\[(Y, \Gamma + M_Y) =: (Y_0, \Gamma_0 + M_0) \rightarrow \cdots \rightarrow (Y_i, \Gamma_i + M_i) \rightarrow \cdots ,\]

the divisor \(K_{Y_i} + \Gamma_i + M_i\) is log abundant with respect to \((Y_i, \Gamma_i + M_i)\) for every \(i \geq 0\).

Theorem 1.3 gives a partial affirmative answer to [7, Question 3.5] by Birkar and Hu. Moreover, preservation of the property of being log abundant in an MMP is closely concerned with the termination of the MMP. Indeed, in the case of log pairs, every log MMP for dlt pairs with scaling of an ample divisor terminates if the log MMP preserves the property of being log abundant ([27, Corollary 1.2]). After this paper was announced, the author proved the termination of an MMP in the situation of Theorem 1.3 (see [28, Theorem 3.17]) by techniques similar to [27]. The remaining important problem in the setting of Theorem 1.3 is the abundance conjecture. Compared to the termination of the MMP, the abundance seems more difficult because the Kodaira type vanishing theorem [13, 14] and the finiteness of pluri-canonical representations [16, 19] for generalized lc pairs are still open. The situation of Theorem 1.3 is also closely concerned with the existence of flips for generalized lc pairs. In [17], Hacon and Liu proved the existence of flips for \(\mathbb{Q}\)-factorial generalized lc pairs. We hope that Theorem 1.3 plays an important role for the existence of flips for generalized lc pairs in the full general case.

Lastly, we give a remark on the Kodaira type vanishing theorem for generalized pairs. As a generalization of the Kodaira type vanishing theorem for lc pairs [15, Theorem 5.6.4], for every projective generalized lc pair \((X, B + M)\) and \(\mathbb{Q}\)-Cartier Weil divisor \(D\) on \(X\) such that \(D - (K_X + B + M)\) is ample, we may expect that \(H^i(X, O_X(D))\) vanishes for every \(i > 0\). This problem will be an important step toward the base point free theorem for generalized lc pairs. Since the non-vanishing theorem for generalized lc pairs holds in the situation of Theorem 1.1 (2), it is natural to expect that the Kodaira type vanishing theorem for generalized lc pairs also holds in the situation as in Theorem 1.1 (2) or Theorem 1.2. Unfortunately, the following result shows that the expectation cannot be realized in general.

**Theorem 1.4** (See Theorem 4.3) There is a projective \(\mathbb{Q}\)-factorial generalized klt pair \((X, B + M)\) and a Cartier divisor \(D\) on \(X\) satisfying the following property: There is a rational number \(t > 1\) such that \(D - (K_X + B + tM)\) is ample and \(H^1(X, O_X(D)) \neq 0\).

The projective generalized pair \((X, B + tM)\) in the theorem is not generalized klt and it satisfies \(\dim X \geq 3\). The two conditions must be satisfied when \(H^i(X, O_X(D)) \neq 0\) for some \(i > 0\) and \(D\) such that \(D - (K_X + B + tM)\) is ample. Indeed, if \((X, B + tM)\) is generalized klt or \(\dim X < 3\), we can find a klt pair \((X, B')\) such that \(D - (K_X + B')\) is ample, so we have \(H^i(X, O_X(D)) = 0\) for every \(i > 0\) by Kawamata–Viehweg vanishing theorem.

The content of this paper is as follows: In Sect. 2, we collect definition and basic properties of generalized pairs, and we define the property of being log abundant. In Sect. 3, we study the generalized abundance for generalized lc pairs. In Sect. 4, we prove theorems 1.1–1.4. In Sect. 5, which is an appendix, we give a small remark on a non-\(\mathbb{R}\)-Cartier analogue of Theorem 1.1 (2).
2 Preliminaries

Notation 2.1 Throughout this paper, \( \mathbb{R} \)-divisors on varieties are not assumed to be \( \mathbb{R} \)-Cartier. In particular, big \( \mathbb{R} \)-divisors are not necessarily \( \mathbb{R} \)-Cartier. For any variety \( V \), \( \mathbb{R} \)-divisor \( D \) on \( V \) and birational contraction \( V \rightarrow W \), unless otherwise stated the birational transformation of \( D \) on \( W \) is denoted by \( D_W \).

Let \( X \) be a normal variety. A prime divisor over \( X \) is a prime divisor on a normal variety admitting a projective birational morphism to \( X \). For any prime divisor \( P \) over \( X \), we denote the image of \( P \) on \( X \) by \( c_X(P) \).

2.1 Generalized pairs

In this subsection, we collect definition and properties of generalized pairs.

Singularities of pairs. A pair \( (X, B) \) consists of a normal variety \( X \) and an effective \( \mathbb{R} \)-divisor \( B \) on \( X \) such that \( K_X + B \) is \( \mathbb{R} \)-Cartier.

Let \( (X, B) \) be a pair and \( P \) a prime divisor over \( X \). Then \( a(P, X, B) \) denotes the log discrepancy of \( P \) with respect to \( (X, B) \). A pair \( (X, B) \) is called a Kawamata log terminal (klt, for short) pair if \( a(P, X, B) > 0 \) for all prime divisors \( P \) over \( X \). A pair \( (X, B) \) is called a log canonical (lc, for short) pair if \( a(P, X, B) \geq 0 \) for all prime divisors \( P \) over \( X \). A pair \( (X, B) \) is called a divisorially log terminal (dlt, for short) pair if all the coefficients of \( B \) belong to \([0, 1]\) and there is a log resolution of \((X, B)\) whose exceptional divisors \( E \) satisfy \( a(E, X, B) > 0 \). When \((X, B)\) is an lc pair, an lc center of \((X, B)\) is the image on \( X \) of a prime divisor \( P \) over \( X \) which satisfies \( a(P, X, B) = 0 \).

Generalized pairs. We define generalized pairs and singularities of generalized pairs. In [8, Definition 1.4], the boundary part of generalized pairs are assumed to have coefficients belonging to \([0, 1]\). However, in Definition 2.2 below we only assume that the boundary parts are effective divisors.

Definition 2.2 (Singularities of generalized pairs) A generalized pair \((X, B + M)\) consists of

- a normal variety \( X \) equipped with a projective morphism \( X \rightarrow Z \),
- an \( \mathbb{R} \)-divisor \( B \geq 0 \) on \( X \), and
- an \( \mathbb{R} \)-Cartier divisor \( M' \) on a normal variety \( X' \) with a projective birational morphism \( \phi : X' \rightarrow X \)

such that \( M' \) is nef over \( Z \) and \( K_X + B + M \) is \( \mathbb{R} \)-Cartier, where \( M := \phi_* M' \).

We usually refer to the generalized pair by saying that \((X, B + M)\) is a generalized pair with data \( X' \xrightarrow{\phi} X \rightarrow Z \) and \( M' \). Because we are interested in the divisor \( K_X + B + M \), there is no harm in replacing \( X' \rightarrow X \) and \( M' \) by any projective birational morphism \( Y \xrightarrow{g} X' \rightarrow X \) and \( g^* M' \). When \( Z \) is not relevant, we usually drop it and do not mention it: in this case one can just assume \( X \rightarrow Z \) is the identity. When \( Z \) is a point, we also drop it and we say that \((X, B + M)\) is a projective generalized pair with data \( X' \xrightarrow{\phi} X \). When \( M' = 0 \), the generalized pair \((X, B + M)\) is a pair with a projective morphism.
Let \((X, B + M)\) be a generalized pair with data \(X' \overset{\phi}{\to} X \to Z\) and \(M'\), and let \(P\) be a prime divisor over \(X\). By replacing \(X'\), we may assume that \(\phi\) is a log resolution of \((X, B)\) such that \(P\) is a prime divisor on \(X'\). Then there is an \(\mathbb{R}\)-divisor \(B'\) on \(X'\) such that

\[
K_{X'} + B' + M' = \phi^*(K_X + B + M).
\]

Then the \textit{generalized log discrepancy} \(a(P, X, B + M)\) of \(P\) is defined by \(a(P, X, B + M) = 1 - \text{coeff}_P(B')\). When \(M' = 0\), the generalized log discrepancies coincide with the log discrepancies with respect to the pair \((X, B)\).

A generalized pair \((X, B + M)\) is called a \textit{generalized klt} (resp. \textit{generalized lc}) pair if \(a(P, X, B + M) > 0\) (resp. \(a(P, X, B + M) \geq 0\)) for all prime divisors \(P\) over \(X\).

A \textit{generalized lc center} of \((X, B + M)\) is the image on \(X\) of a prime divisor \(P\) over \(X\) satisfying \(a(P, X, B + M) = 0\).

A generalized pair \((X, B + M)\) is a \textit{generalized dlt pair} if it is generalized lc and for any generic point \(\eta\) of any generalized lc center of \((X, B + M)\), \((X, B)\) is log smooth near \(\eta\) and \(M'\) coincides with the pullback of \(M\) over a neighborhood of \(\eta\).

Following the convention in [20, Subsection 2.6], we say that a generalized pair \((X, B + M)\) with data \(X' \overset{\phi}{\to} X \to Z\) and \(M'\) is an \(NQC\) \textit{generalized pair} if \(M'\) is an \(\mathbb{R}_{> 0}\)-linear combination of nef \(\mathbb{Q}\)-Cartier divisors. We say that a generalized klt (resp. generalized lc, generalized dlt) pair is an \(NQC\) \textit{generalized klt} (resp. \(NQC\) \textit{generalized lc}, \(NQC\) \textit{generalized dlt}) pair if it is an \(NQC\) generalized pair.

For a given \((X, B + M)\), we call \((Y, \Gamma + M_Y)\) in Theorem 2.3 below a \(\mathbb{Q}\)-\textit{factorial generalized dlt model} of \((X, B + M)\).

**Theorem 2.3** (\(\mathbb{Q}\)-factorial generalized dlt model, cf. [8, Lemma 4.5]) Let \((X, B + M)\) be a generalized lc pair with data \(X' \overset{\phi}{\to} X \to id\) and \(M'\) such that \(X\) is quasi-projective. Then, after replacing \(X'\) and \(M'\) there exists a projective birational morphism \(f: Y \to X\) and a \(\mathbb{Q}\)-factorial generalized dlt pair \((Y, \Gamma + M_Y)\) with data \(X' \to Y \to X\) and \(M'\) such that

\[
K_Y + \Gamma + M_Y = f^*(K_X + B + M)
\]

and \(\Gamma\) is the sum of \(f_*^{-1}B\) and the reduced \(f\)-exceptional divisor.

**Remark 2.4** Let \((X, B + M)\) be a generalized lc pair such that \(X\) is quasi-projective. For every prime divisor \(P\) over \(X\) such that \(a(P, X, B + M) = 0\), by [8, Lemma 4.5] there is a \(\mathbb{Q}\)-factorial generalized dlt model \((Y, \Gamma + M_Y) \to (X, B + M)\) on which \(P\) appears as a divisor. In particular, for any generalized lc center \(S\) of \((X, B + M)\), we can construct a \(\mathbb{Q}\)-factorial generalized dlt model \((Y, \Gamma + M_Y)\) such that \(Y \to X\) induces a surjective morphism \(T \to S\) from a component \(T\) of \(\cup \Gamma_{\downarrow}\).

**Lemma 2.5** Let \((X, B + M)\) be a generalized lc pair with data \(X' \overset{\phi}{\to} X \to Z\) and \(M'\) such that \(\phi\) is a log resolution of \((X, B)\). We define \(B' \geq 0\) and \(E' \geq 0\) on \(X'\) by

\[
K_{X'} + B' + M' = \phi^*(K_X + B + M) + E'
\]
such that $B'$ and $E'$ have no common components. Let $g: Y \to X'$ be a projective birational morphism from a smooth variety $Y$. We define $\Gamma \geq 0$ and $F \geq 0$ on $Y$ by

$$K_Y + \Gamma = g^*(K_{X'} + B') + F$$

such that $\Gamma$ and $F$ have no common components.

Then $\Gamma$ and $g^*E' + F$ have no common components. We note that we have

$$K_Y + \Gamma + g^*M' = (\phi \circ g)^*(K_X + B + M) + g^*E' + F$$

by construction.

**Proof** It is sufficient to show that $\Gamma$ and $g^*E'$ have no common components. Since $B'$ and $E'$ have no common components, by log smoothness of $(X', \text{Supp}(B' + E'))$ and computations of log discrepancies as in [30, Lemma 2.29] and [30, Lemma 2.45], for any prime divisor $P$ over $X'$ if $c_{X'}(P)$ is contained in $\text{Supp}E'$ then we have $a(P, X', B') \geq 1$. Since the log discrepancy of every component of $\Gamma$ with respect to $(X', B')$ is less than one, we see that $\Gamma$ and $g^*E'$ have no common components. Therefore, $\Gamma$ and $g^*E' + F$ have no common components. $\square$

**Divisorial adjunction for generalized pairs.** We will freely use the construction of divisorial adjunction for generalized pairs in [8, Definition 4.7].

**Lemma 2.6** Let $(X, B + M)$ be a projective $\mathbb{Q}$-factorial generalized dlt pair with data $X \xrightarrow{\phi} X$ and $M'$ such that $\phi$ is a log resolution of $(X, B)$. Let $S$ be a component of $\text{Supp}(B + M)$. We define $B' \geq 0$ and $E' \geq 0$ on $X'$ by

$$K_{X'} + S' + B' + M' = \phi^*(K_X + B + M) + E',$$

where $S' = \phi^{-1}_*S$, such that $B'$ and $E'$ have no common components. We define $\mathbb{R}$-divisors $B_{S'}, B_S, N'$, and $N$ by

$$B_{S'} = B'|_{S'}, \quad B_S = \phi|_{S'^*}B_{S'},$$

$$N' = M''|_{S'} \text{ for some } M'' \sim_{\mathbb{R}} M', \quad \text{and} \quad N = \phi|_{S'^*}N'.$$

Then $(S, B_S + N)$ and $(S', B_{S'} + N')$ are generalized lc pairs and

$$a(P, S, B_S + N) = a(P, S', B_{S'} + N')$$

for all prime divisors $P$ on $S$.

**Proof** By definition of $B_{S'}$ and $N'$, we have

$$K_{S'} + B_{S'} + N' = \phi|^*_{S'}(K_S + B_S + N) + E'|_{S'}.$$

The first assertion follows from [8, Remark 4.8]. For the second assertion, the proof of [29, Lemma 2.4] works with no changes. More precisely, by the same argument as in [29, Proof of Lemma 2.4], we see that $B_{S'}$ and $E'|_{S'}$ have no common components and $E'|_{S'}$ is exceptional over $S$. Then the second assertion follows from the definition of log discrepancies for generalized pairs. $\square$
MMP for generalized pairs. Let \((X, B + M)\) be a \(\mathbb{Q}\)-factorial generalized lc pair with data \(X' \to X \to Z\) and \(M'\) such that \(Z\) is quasi-projective and \((X, 0)\) is a klt pair. As in [8, Section 4], we can run a \((K_X + B + M)\)-MMP over \(Z\) with scaling of an ample divisor. By the main result of Hacon and Liu [17], currently we may run an MMP even though \((X, 0)\) is not necessarily klt. In this paper, when we discuss an MMP for \((X, B + M)\), the pair \((X, 0)\) is always klt. We often denote the sequence of the MMP by
\[
(X, B + M) =: (X_0, B_{X_0} + M_{X_0}) \to \cdots \to (X_i, B_{X_i} + M_{X_i}) \to \cdots.
\]

Generalized dlt pairs in this paper are generalized dlt pairs in the sense of [20] by Han and Li. Moreover, the NQC generalized dlt pairs in this paper coincide with those in [20] (see [28, Theorem 6.1]). Hence, we may freely use results in [20]. For fundamental results of MMP for generalized pairs, see [20].

2.2 Property of being log abundant

In this subsection, we introduce the notion of being log abundant for generalized pairs.

Definition 2.7 (Invariant Iitaka dimension) Let \(X\) be a normal projective variety, and let \(D\) be an \(\mathbb{R}\)-Cartier divisor on \(X\). We define the invariant Iitaka dimension of \(D\), denoted by \(\kappa_\text{i}(X, D)\), as follows ([9, Definition 2.2.1], see also [15, Definition 2.5.5]):

If there is an \(\mathbb{R}\)-divisor \(E \geq 0\) such that \(D \sim_\mathbb{R} E\), set \(\kappa_\text{i}(X, D) = \kappa(X, E)\). Here, the right hand side is the usual Iitaka dimension of \(E\). Otherwise, we set \(\kappa_\text{i}(X, D) = -\infty\).

Definition 2.8 (Numerical dimension) Let \(X\) be a normal projective variety, and let \(D\) be an \(\mathbb{R}\)-Cartier divisor on \(X\). We define the numerical dimension of \(D\), denoted by \(\kappa_\sigma(X, D)\), as follows ([37, V, 2.5 Definition]): For any Cartier divisor \(A\) on \(X\), we set

\[
\sigma(D; A) = \max\left\{ k \in \mathbb{Z}_{\geq 0} \mid \limsup_{m \to \infty} \frac{\dim H^0(X, O_X((-mD) + A))}{m^k} > 0 \right\}
\]

if \(\dim H^0(X, O_X((-mD) + A)) > 0\) for infinitely many \(m \in \mathbb{Z}_{>0}\), and otherwise we set \(\sigma(D; A) := -\infty\). Then, we define

\[
\kappa_\sigma(X, D) := \max\{ \sigma(D; A) \mid A\ is\ a\ Cartier\ divisor\ on\ X \}.
\]

See [29, Remark 2.8] for basic properties of the invariant Iitaka dimension and the numerical dimension.

In [37], the numerical dimension is defined for pseudo-effective \(\mathbb{R}\)-Cartier divisors. On the other hand, in this paper the numerical dimension is defined for all \(\mathbb{R}\)-Cartier divisors. It is because we will define the property of being log abundant (see Definition 2.9 below) for all \(\mathbb{R}\)-Cartier divisors and we will discuss preservation of the property of being log abundant under MMP.
Definition 2.9 (Abundant divisors and log abundant divisors) Let $X$ be a normal projective variety, and let $D$ be an $\mathbb{R}$-Cartier divisor on $X$. We say that $D$ is abundant if the equality $\kappa_i(X, D) = \kappa_{\sigma}(X, D)$ holds.

Let $X$ and $D$ be as above, and let $(X, B + M)$ be a projective generalized lc pair. We say that $D$ is abundant if $D$ is abundant and for any generalized lc center $S$ of $(X, B + M)$ with the normalization $S^n \to S$, the pullback $D|_S$ is abundant.

We can define the relative version of the invariant Iitaka dimension, the numerical dimension, and the property of being log abundant ([27, Section 5]). But we will not use them in this paper.

3 Generalized abundance for generalized lc pairs

Lemma 3.1 Let $(X, B + M)$ be a projective generalized lc pair with data $Y \xrightarrow{f} X$ and $M_Y$ such that $f$ is a log resolution of $(X, B)$. Let $X \to X'$ be a birational contraction to a normal projective variety $X'$ such that the induced birational map $Y \to X'$ is a morphism. Let $(Y, \Gamma + M_Y)$ be a projective generalized lc pair. Suppose that

- $K_{X'} + B_{X'} + M_{X'}$ is $\mathbb{R}$-Cartier, where $B_{X'}$ and $M_{X'}$ are the birational transforms of $B$ and $M$, respectively,
- $a(P, X, B + M) \leq a(P, X', B_{X'} + M_{X'})$ for all prime divisors $P$ on $Y$, and
- the relation $K_Y + \Gamma + M_Y = f^*(K_X + B + M) + E$ holds for an effective $f$-exceptional $\mathbb{R}$-divisor $E$.

Then, by running a $(K_Y + \Gamma + M_Y)$-MMP over $X'$ with scaling of an ample divisor we get a projective birational morphism $f': Y' \to X'$ such that

$$K_{Y'} + \Gamma_{Y'} + M_{Y'} = f'^*(K_{X'} + B_{X'} + M_{X'}).$$

Proof We denote $Y \to X'$ by $g$. By the hypothesis of relation on log discrepancies of the generalized pairs, there is a $g$-exceptional $\mathbb{R}$-divisor $F \geq 0$ on $Y$ such that

$$K_Y + \Gamma + M_Y = f^*(K_X + B + M) + E = g^*(K_{X'} + B_{X'} + M_{X'}) + E + F.$$

By construction, $E + F$ is effective and $g$-exceptional. By [20, Proposition 3.9] and running a $(K_Y + \Gamma + M_Y)$-MMP over $X'$ with scaling of an ample divisor, we get a birational contraction $Y \to Y'$ over $X'$ which contracts $E + F$. Then $Y' \to X'$ is the desired morphism.

Proposition 3.2 Let $(X, B + M)$ be a generalized lc pair with data $Y \xrightarrow{f} X \to Z$ and $M_Y$ such that $f$ is a log resolution of $(X, B)$ and $Z$ is quasi-projective. Let $\Gamma$ be an $\mathbb{R}$-divisor on $Y$ such that $(Y, \Gamma)$ is a log smooth lc pair and $f_*\Gamma = B$. Suppose that

- $K_Y + \Gamma + M_Y$ is pseudo-effective over $Z$,
- $K_X + B + M \sim_{\mathbb{R}, Z} 0$, and
• $X$ is of Fano type over $\mathbb{Z}$.

Then there is a birational contraction $Y \rightarrow Y'$ over $\mathbb{Z}$ such that

• $K_{Y'} + \Gamma_{Y'} + M_{Y'}$ is semi-ample over $\mathbb{Z}$, and
• $a(P, Y, \Gamma + M_Y) \leq a(P, Y', \Gamma_{Y'} + M_{Y'})$ for all prime divisors $P$ over $Y$.

**Proof** We prove it in several steps.

**Step 1.** By hypothesis, we can write

$$K_Y + \Gamma + M_Y = f^*(K_X + B + M) + E - F \sim_{\mathbb{R}, \mathbb{Z}} E - F,$$

where $E \geq 0$ and $F \geq 0$ are $f$-exceptional $\mathbb{R}$-divisors having no common components.

Let $G$ be the sum of all $f$-exceptional prime divisors which are neither components of $\Gamma$ nor $F$. Since $(X, B + M)$ is generalized lc, all prime divisors $P$ on $Y$ satisfy $\text{coeff}_P(\Gamma + F - E) \leq 1$.

If $P$ is a component of $F$, then

$$\text{coeff}_P(\Gamma + F) = \text{coeff}_P(\Gamma + F - E) \leq 1.$$ 

If $P$ is not a component of $F$, then

$$\text{coeff}_P(\Gamma + F) = \text{coeff}_P(\Gamma) \leq 1$$

by definition of $\Gamma$. Thus, all prime divisors $P$ on $Y$ satisfy

$$\text{coeff}_P(\Gamma + F) \leq 1.$$ 

Since $(Y, \text{Supp}\Gamma \cup \text{Ex}(f))$ is log smooth, $(Y, \Gamma + t_0(G + F))$ is a log smooth lc pair for some $t_0 \in (0, 1)$. Then $(Y, (\Gamma + t_0(G + F)) + M_Y)$ is a generalized lc pair, and we have

$$K_Y + \Gamma + t_0(G + F) + M_Y \sim_{\mathbb{R}, \mathbb{Z}} E + t_0G - (1 - t_0)F.$$ 

By the definitions of $E$, $F$ and $G$, the divisors $E + t_0G$ and $F$ have no common components.

**Step 2.** We run a $(K_Y + \Gamma + M_Y)$-MMP over $X$ with scaling of an ample divisor. Since $E \geq 0$ is $f$-exceptional, [20, Proposition 3.9] implies that after finitely many steps we get a morphism $f_0: Y_0 \rightarrow X$ such that $E$ is contracted by the map $Y \rightarrow Y_0$ of the MMP.

Let $\Gamma_0$, $F_0$ and $G_0$ be the birational transforms of $\Gamma$, $F$ and $G$ on $Y_0$, respectively. Then the relation $K_{Y_0} + \Gamma_0 + M_{Y_0} \sim_{\mathbb{R}, \mathbb{Z}} -F_0$ holds and the inequality

$$a(P, Y, \Gamma + M_Y) \leq a(P, Y_0, \Gamma_0 + M_{Y_0})$$

holds for all prime divisors $P$ over $Y$. By the first condition of Proposition 3.2, we see that $K_{Y_0} + \Gamma_0 + M_{Y_0}$ is pseudo-effective over $\mathbb{Z}$, so $F_0$ is vertical over $\mathbb{Z}$. In particular, there is $H_0 \geq 0$ on $Y_0$ such that $K_{Y_0} + \Gamma_0 + M_{Y_0} \sim_{\mathbb{R}, \mathbb{Z}} H_0$. 

For any $t \in (0, t_0]$, we have

$$
K_{Y_0} + \Gamma_0 + t(G_0 + F_0) + M_{Y_0} \sim_{\mathbb{R},Z} tG_0 - (1 - t)F_0 \\
\sim_{\mathbb{R},Z} t(G_0 + F_0) + H_0.
$$

(♠)

We take $t_0 > 0$ in Step 1 sufficiently small. Then the birational map $Y \dashrightarrow Y_0$ is a sequence of steps of a $(K_Y + \Gamma + t_0(G + F) + M_Y)$-MMP. Then the generalized pair $(Y_0, (\Gamma_0 + t_0(G_0 + F_0)) + M_{Y_0})$ is a $\mathbb{Q}$-factorial generalized dlt pair.

**Step 3.** Pick a strictly decreasing sequence of positive real numbers $\{t_i\}_{i \geq 1}$ such that $0 < t_i < t_0$ for every $i$ and $\lim_{i \to \infty} t_i = 0$. In this step, we prove that for each $i \geq 1$ there exists a sequence of steps of a $(K_{Y_0} + \Gamma_0 + t_i(G_0 + F_0) + M_{Y_0})$-MMP over $Z$

$$
(Y_0, (\Gamma_0 + t_i(G_0 + F_0)) + M_{Y_0}) \dashrightarrow (Y_i, (\Gamma_i + t_i(G_i + F_i)) + M_{Y_i})
$$

such that $G_0$ is contracted by the map $Y_0 \dashrightarrow Y_i$ (i.e., $G_i = 0$) and $K_{Y_i} + \Gamma_i + t_i(G_i + F_i) + M_{Y_i}$ is semi-ample over $Z$. From now to the end of this step, we fix $i$.

We run a $(K_{Y_0} + \Gamma_0 + t_i(G_0 + F_0) + M_{Y_0})$-MMP over $X$ with scaling of an ample divisor. By (♠) in Step 2, this MMP is an MMP for $t_iG_0 - (1 - t_i)F_0$ over $X$. By [20, Proposition 3.9], after finitely many steps we get a morphism $f'_0 : Y'_0 \to X$ such that $G_0$ is contracted by the birational map $Y_0 \dashrightarrow Y'_0$ of the MMP. In this way, we get a sequence of steps of a $(K_{Y_0} + \Gamma_0 + t_i(G_0 + F_0) + M_{Y_0})$-MMP over $Z$

$$
\tau : (Y_0, (\Gamma_0 + t_i(G_0 + F_0)) + M_{Y_0}) \dashrightarrow (Y'_0, (\Gamma'_0 + t_iF'_0) + M_{Y'_0}).
$$

Here, the divisors $\Gamma'_0$ and $F'_0$ are the birational transforms of $\Gamma_0$ and $F_0$, respectively.

By construction, $\Gamma + G + F$ contains all $f$-exceptional prime divisors. Therefore, $\Gamma'_0 + t_iF'_0$ contains all $f'_0$-exceptional prime divisors in its support. Since $X$ is of Fano type over $Z$, there is a klt pair $(X, \Delta)$ such that $\Delta$ is big over $Z$ and $K_X + \Delta \sim_{\mathbb{R},Z} 0$.

Let $\Delta'_0$ be an $\mathbb{R}$-divisor on $Y'_0$ defined by the equation

$$
K_{Y'_0} + \Delta'_0 = f'_0^*(K_X + \Delta).
$$

Since $\Delta$ is big over $Z$ and $\Gamma'_0 + t_iF'_0$ contains all $f'_0$-exceptional prime divisors in its support, there is $u > 0$ such that

$$
\Psi'_0 := \frac{1}{1 + u}(\Gamma'_0 + t_iF'_0) + \frac{u}{1 + u}\Delta'_0
$$

is effective and big over $Z$. Since $(Y'_0, (\Gamma'_0 + t_iF'_0) + M_{Y'_0})$ is generalized lc, $(Y'_0, \Psi'_0 + \frac{1}{1 + u}M_{Y'_0})$ is generalized klt. By perturbing the coefficients of $\Psi'_0 + \frac{1}{1 + u}M_{Y'_0}$, we may find an $\mathbb{R}$-divisor $\Theta'_0 \geq 0$ on $Y'_0$ which is big over $Z$ such that $\Theta'_0 \sim_{\mathbb{R},Z} \Psi'_0 + \frac{1}{1 + u}M_{Y'_0}$ and the pair $(Y'_0, \Theta'_0)$ is klt.
By [6], there is a sequence of steps of a \((K_{Y_0} + \Theta_0)\)-MMP over \(Z\) terminating with a good minimal model. By construction, we have

\[
K_{Y_0} + \Theta_0 \sim_{\mathbb{R}, \mathbb{Z}} K_{Y_0} + \Psi_0 + \frac{1}{1 + u} M_{Y_0}
\]

\[
= K_{Y_0} + \frac{1}{1 + u} (\Gamma_0' + t_i F_0') + \frac{u}{1 + u} \Delta_0 + \frac{1}{1 + u} M_{Y_0}'
\]

\[
= \frac{1}{1 + u} \left( K_{Y_0}' + \Gamma_0' + t_i F_0' + M_{Y_0}' \right) + \frac{u}{1 + u} \left( K_{Y_0}' + \Delta_0' \right)
\]

\[
\sim_{\mathbb{R}, \mathbb{Z}} K_{Y_0} + \Gamma_0' + t_i F_0' + M_{Y_0}'.
\]

The final relation follows from \(K_{Y_0}' + \Delta_0' = f_0^*(K_X + \Delta) \sim_{\mathbb{R}, \mathbb{Z}} 0\). Thus, there is a sequence of steps of a \((K_{Y_0}' + \Gamma_0' + t_i F_0' + M_{Y_0}')\)-MMP over \(Z\)

\[
\tau_i : \left( Y_0', (\Gamma_0' + t_i F_0') + M_{Y_0}' \right) \rightarrow (Y_i, (\Gamma_i + t_i F_i) + M_{Y_i})
\]

such that \(K_{Y_i} + \Gamma_i + t_i F_i + M_{Y_i}\) is semi-ample over \(Z\).

By the above discussions, the composition

\[
\tau_i \circ \tau : \left( Y_0, (\Gamma_0 + t_i (G_0 + F_0)) + M_{Y_0} \right) \rightarrow (Y_i, (\Gamma_i + t_i F_i) + M_{Y_i})
\]

is the desired \((K_{Y_0} + \Gamma_0 + t_i (G_0 + F_0) + M_{Y_0})\)-MMP over \(Z\). We finish this step.

**Step 4.** With this step we complete the proof.

For each \(i \geq 1\), let

\[
(Y_0, (\Gamma_0 + t_i (G_0 + F_0)) + M_{Y_0}) \rightarrow (Y_i, (\Gamma_i + t_i F_i) + M_{Y_i})
\]

be the \((K_{Y_0} + \Gamma_0 + t_i (G_0 + F_0) + M_{Y_0})\)-MMP over \(Z\) constructed in Step 3. We recall that \(G_0\) is contracted by \(Y_0 \rightarrow Y_i\) and \(K_{Y_i} + \Gamma_i + t_i F_i + M_{Y_i}\) is semi-ample over \(Z\). By (♠) in Step 2, we see that prime divisors contracted by \(Y_0 \rightarrow Y_i\) are the components of \(G_0 + F_0 + H_0\), which are independent of \(t_i\). Therefore, by replacing \{\(t_i\)\}_{i \geq 1} with a subsequence, we may assume that all maps \(Y_0 \rightarrow Y_i\) contract the same divisors, so all \(Y_i\) are isomorphic in codimension one.

We prove that the birational contraction \(Y \rightarrow Y_1\) satisfies the two conditions of Proposition 3.2. We first check that \(K_{Y_1} + \Gamma_1 + M_{Y_1}\) is semi-ample over \(Z\). Because \(G\) is contracted by \(Y \rightarrow Y_1\), with (♠) in Step 2, we have

\[
K_{Y_1} + \Gamma_1 + t_1 F_1 + M_{Y_1} \sim_{\mathbb{R}, \mathbb{Z}} -(1 - t_1) F_1.
\]

From this relation and the fact that \(K_{Y_1} + \Gamma_1 + t_1 F_1 + M_{Y_1}\) is semi-ample over \(Z\), we see that

\[
K_{Y_1} + \Gamma_1 + M_{Y_1} \sim_{\mathbb{R}, \mathbb{Z}} -F_1 \sim_{\mathbb{R}, \mathbb{Z}} \frac{1}{1 - t_1} (K_{Y_1} + \Gamma_1 + t_1 F_1 + M_{Y_1})
\]
is semi-ample over $Z$. Therefore, the first condition of Proposition 3.2 holds true.

Next, we prove $a(P, Y, \Gamma + M_Y) \leq a(P, Y_1, \Gamma_1 + M_{Y_1})$ for all prime divisors $P$ over $Y$. Since $K_{Y_1} + \Gamma_1 + M_{Y_1}$ is semi-ample over $Z$, we see that

$$K_{Y_1} + \Gamma_1 + t_i F_1 + M_{Y_1} \sim_{\mathbb{R}, Z} (1 - t_i)(K_{Y_1} + \Gamma_1 + M_{Y_1})$$

is semi-ample over $Z$ for any $i$. Recall that the induced birational map $Y_1 \dashrightarrow Y_i$ are small. By taking a common resolution of $Y_1 \dashrightarrow Y_i$ and the negativity lemma, we obtain

$$a(P, Y_i, (\Gamma_i + t_i F_i) + M_{Y_i}) = a(P, Y_1, (\Gamma_1 + t_i F_1) + M_{Y_1})$$

for every $i > 1$. Combining this with construction of the map $Y_0 \dashrightarrow Y_i$ in Step 3, we obtain

$$a(P, Y_0, (\Gamma_0 + t_i (G_0 + F_0)) + M_{Y_0}) \leq a(P, Y_1, (\Gamma_1 + t_i F_1) + M_{Y_1})$$

for every $i$. By taking the limit $i \to \infty$, we have

$$a(P, Y_0, \Gamma_0 + M_{Y_0}) \leq a(P, Y_1, \Gamma_1 + M_{Y_1}).$$

We combine this inequality with $(\ast)$ in Step 2. Then we have

$$a(P, Y, \Gamma + M_Y) \leq a(P, Y_0, \Gamma_0 + M_{Y_0}) \leq a(P, Y_1, \Gamma_1 + M_{Y_1}).$$

In this way, we see that $Y \dashrightarrow Y_1$ is the desired birational contraction.

We complete the proof.

**Remark 3.3** By a different argument, Birkar independently obtained a similar result in the case of $\mathbb{Q}$-divisors (see [5, Lemma 4.10]).

**Proposition 3.4** Let $\pi : X \to Z$ be a surjective morphism of normal projective varieties, and let $(X, B + M)$ be a projective NQC generalized klt pair with data $Y \to X$ and $M_Y$ such that there exists an open subset $U \subset Z$ over which we have

$$(K_X + B + M)|_{\pi^{-1}(U)} \sim_{\mathbb{R}, U} 0.$$ 

Let $A_Z$ be an ample $\mathbb{R}$-divisor on $Z$. Then, $K_X + B + M + \pi^* A_Z$ is abundant. Furthermore, if $K_X + B + M + \pi^* A_Z$ is pseudo-effective, then the divisor birationally has the Nakayama–Zariski decomposition with semi-ample positive part.

**Proof** The argument is very similar to [25, Proof of Lemma 3.1].

**Step 1.** By taking the Stein factorization of $\pi$, we may assume that $\pi$ is a contraction. We may assume that $f$ is a log resolution of $(X, B)$. We can find $\mathbb{R}$-divisors $\Gamma \geq 0$ and $E \geq 0$ on $Y$ such that $(Y, \Gamma)$ is a log smooth klt pair, $E$ is $f$-exceptional, and

$$K_Y + \Gamma + M_Y = f^*(K_X + B + M) + E.$$
Note that $\Gamma$ and $E$ may have common components. We apply the weak semistable reduction ([1, Proof of Theorem 2.1], [1, Proposition 4.4] and [1, Remark 4.5]) to the morphism $(Y, \Gamma) \to Z$. Then we have the following diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\psi} & Y' \\
\downarrow{\pi \circ f} & & \downarrow{\pi'} \\
Z & \xleftarrow{g} & Z'
\end{array}
\]

with a normal projective variety $Y'$ and a smooth projective variety $Z'$ such that

- $\psi : Y' \to Y$ and $g : Z' \to Z$ are birational morphisms,
- $\pi' : Y' \to Z'$ is a contraction and all fibers of $\pi'$ have the same dimensions, and
- $(Y', 0)$ is a $\mathbb{Q}$-factorial klt pair and $(Y', \mathrm{Supp}\psi^{-1}_*\Gamma \cup \mathrm{Ex}(\psi))$ is an lc pair.

For details, see [25, Step 1 in the proof of Lemma 3.1].

By the third condition, there is an $\mathbb{R}$-divisor $\Gamma'$ on $Y'$ such that $(Y', \Gamma')$ is a $\mathbb{Q}$-factorial klt pair and $K_{Y'} + \Gamma' - \psi^*(K_Y + \Gamma)$ is effective and $\psi$-exceptional. Set $f' = f \circ \psi : Y' \to X$ and put $M' = \psi^*M_Y$. Then $(Y', \Gamma' + M')$ is a projective NQC generalized klt pair, and we may write

\[K_{Y'} + \Gamma' + M' = f'^*(K_X + B + M) + E'
\]

for some $f'$-exceptional $\mathbb{R}$-divisor $E' \geq 0$.

In this way, we have a diagram

\[
\begin{array}{ccc}
X & \xleftarrow{f'} & Y' \\
\downarrow{\pi} & & \downarrow{\pi'} \\
Z & \xleftarrow{g} & Z'
\end{array}
\]

and a projective $\mathbb{Q}$-factorial NQC generalized klt pair $(Y', \Gamma' + M')$ such that

1. $f' : Y' \to X$ and $g : Z' \to Z$ are birational morphisms and $Z'$ is smooth,
2. $\pi' : Y' \to Z'$ is a contraction and all fibers of $\pi'$ have the same dimensions, and
3. $K_{Y'} + \Gamma' + M' = f'^*(K_X + B + M) + E'$ for some $f'$-exceptional $\mathbb{R}$-divisor $E' \geq 0$.

By (3), it follows that $K_X + B + M + \pi^*A_Z$ is abundant and birationally has the Nakayama–Zariski decomposition (see [37]) with semi-ample positive part if and only if the same assertions hold for $K_{Y'} + \Gamma' + M' + \pi'^*g^*A_Z$.

**Step 2.** By (3) in Step 1 and that $(K_X + B + M)|_{\pi^{-1}(U)} \sim_{\mathbb{R}, U} 0$ for some open $U \subset Z$, there exists an $\mathbb{R}$-divisor $F' \geq 0$ on $Y'$ such that

\[K_{Y'} + \Gamma' + M' \sim_{\mathbb{R}, Z'} F'.\]
We can write $F' = F_h + F_v$ such that all components of $F_h$ dominates $Z'$ and all components of $F_v$ are vertical over $Z'$. Since all the fibers of $\pi'$ have the same dimensions (see (2) in Step 1), the image of any component of $F_v$ on $Z'$ is a divisor.

Since $Z'$ is smooth, we can consider

$$v_P := \sup \{ v \in \mathbb{R}_{\geq 0} \mid F_v - v \pi'^* P \geq 0 \}$$

for any prime divisor $P$ on $Z'$. Then it is easy to see that $v_P > 0$ for only finitely many prime divisors $P$. By (2) in Step 1, we see that $F_v - \sum p v_P \pi'^* P$ is an effective $\mathbb{R}$-divisor which is very exceptional over $Z'$. By replacing $F_v$ with $F_v - \sum p v_P \pi'^* P$, we may assume that $F_v$ is very exceptional over $Z'$.

We run a $(K_{Y'} + \Gamma' + M')$-MMP over $Z'$ with scaling of an ample divisor

$$(Y', \Gamma' + M') =: (Y_0, \Gamma_0 + M_{Y_0}) \rightarrow \cdots \rightarrow (Y_i, \Gamma_i + M_{Y_i}) \rightarrow \cdots.$$ 

We prove that after finitely many steps we get $(Y_m, \Gamma_m + M_{Y_m})$ such that $K_{Y_m} + \Gamma_m + M_{Y_m} \sim_{\mathbb{R}, Z'} 0$. By shrinking $U$, we may assume $g : Z' \rightarrow Z$ is an isomorphism over $U$, so $U$ can be thought of an open subset of $Z'$. For each $i$, let $V_i$ be the inverse image of $U$ to $Y_i$. Then

$$(V_0, \Gamma_0 | V_0 + M_{Y_0} | V_0) \rightarrow \cdots \rightarrow (V_i, \Gamma_i | V_i + M_{Y_i} | V_i) \rightarrow \cdots$$

is a sequence of steps of a $(K_{V_0} + \Gamma_0 | V_0 + M_{Y_0} | V_0)$-MMP over $U$, and the generalized klt pair $(V_0, \Gamma_0 | V_0 + M_{Y_0} | V_0)$ has a relatively trivial minimal model over $U$ ((3) in Step 1 and $(K_X + B + M)_{|\pi^{-1}(U)} \sim_{\mathbb{R}, U} 0$). By [20, Theorem 4.1], for any $i \gg 0$ we obtain

$$K_{V_i} + \Gamma_i | V_i + M_{Y_i} | V_i \sim_{\mathbb{R}, U} 0.$$ 

Because we have

$$K_{Y'} + \Gamma' + M' \sim_{\mathbb{R}, Z'} F_h + F_v$$

and all the components of $F_h$ dominate over $Z'$, it follows that $F_h$ is contracted by $Y' \rightarrow Y_i$ for every $i \gg 0$. We pick $m > 0$ such that $F_h$ is contracted by $Y' \rightarrow Y_m$. Since the $(K_{Y'} + \Gamma' + M')$-MMP over $Z'$ occurs only in Supp($F_h + F_v$), the birational transform of $F_v$ on $Y_m$ is very exceptional over $Z'$. Applying [20, Proposition 3.9] to $(Y_m, \Gamma_m + M_{Y_m})$ with the morphism $Y_m \rightarrow Z'$ and replacing $m$, we may assume that $F_v$ is also contracted by $Y \rightarrow Y_m$. From this discussion, there exists $m > 0$ such that

$$K_{Y_m} + \Gamma_{Y_m} + M_{Y_m} \sim_{\mathbb{R}, Z'} 0.$$
Step 3. We denote the natural morphism $Y_m \to Z'$ by $\pi'_m$. Now we have the following diagram.

$$
\begin{array}{ccc}
X & \xrightarrow{f'} & Y' \\
\downarrow{\pi} & & \downarrow{\pi'_m} \\
Z & \xleftarrow{g} & Z'
\end{array}
$$

Since $Y' \to Y_m$ is a sequence of steps of a $(K_{Y'} + \Gamma' + M' + \pi'^{\ast}g^{\ast}A_Z)$-MMP, $K_{Y'} + \Gamma' + M' + \pi'^{\ast}g^{\ast}A_Z$ is abundant and birationally has the Nakayama–Zariski decomposition with semi-ample positive part if and only if the same assertions hold for $K_{Y_m} + \Gamma_{Y_m} + M_{Y_m} + \pi'^{\ast}m^{\ast}g^{\ast}A_Z$.

We apply the generalized canonical bundle formula [24, Theorem 1.2] to $(Y_m, \Gamma_{Y_m} + M_{Y_m}) \to Z'$, and we get a projective generalized klt pair $(Z', B_{Z'} + N)$ such that

$$K_{Y_m} + \Gamma_{Y_m} + M_{Y_m} \sim_{\mathbb{R}} \pi'^{\ast}_m(K_{Z'} + B_{Z'} + N).$$

Since $g^{\ast}A_Z$ is nef and big, we can find a big $\mathbb{R}$-divisor $\Delta_{Z'}$ on $Z'$ such that $(Z', \Delta_{Z'})$ is klt and

$$K_{Z'} + B_{Z'} + N + g^{\ast}A_Z \sim_{\mathbb{R}} K_{Z'} + \Delta_{Z'}.$$

By [6, Theorem 1.2], $(Z', \Delta_{Z'})$ has a good minimal model or a Mori fiber space. In particular, $K_{Z'} + \Delta_{Z'}$ is abundant, and the divisor birationally has the Nakayama–Zariski decomposition with semi-ample positive part when it is pseudo-effective. Because

$$K_{Y_m} + \Gamma_{Y_m} + M_{Y_m} + \pi'^{\ast}_m g^{\ast}A_Z \sim_{\mathbb{R}} \pi'^{\ast}_m(K_{Z'} + \Delta_{Z'}),$$

the left hand side is also abundant. If the divisor is pseudo-effective, then it birationally has the Nakayama–Zariski decomposition with semi-ample positive part when it is pseudo-effective. We finish the proof. $\square$

**Theorem 3.5** Let $\pi : X \to Z$ be a morphism of normal projective varieties, and let $(X, B + M)$ be a projective NQC generalized lc pair with data $Y \xrightarrow{f} X$ and $M_Y$ such that there is an effective $\mathbb{R}$-Cartier divisor $C$ on $X$ satisfying the following properties.

(I) $(X, (B + tC) + M)$ is generalized lc for some $t > 0$, and

(II) $K_X + B + C + M \sim_{\mathbb{R}, Z} 0$.

Let $A_Z$ be an ample $\mathbb{R}$-divisor on $Z$, and pick $0 \leq A \sim_{\mathbb{R}} \pi^{\ast}A_Z$ such that $(X, (B + A) + M)$ is a generalized lc pair. Then, $K_X + B + A + M$ is abundant.

Before proving the theorem, we prove the following lemma.
**Lemma 3.6** Assume Theorem 3.5 for all projective NQC generalized lc pairs of dimension at most $n - 1$. Let $\pi : X \to Z$ be a morphism, $(X, B + M)$ a projective NQC generalized lc pair with data $Y \xrightarrow{f} X$ and $M_Y$, and let $C, A_Z$ and $A$ be $\mathbb{R}$-Cartier divisors as in Theorem 3.5 such that $\dim X \leq n - 1$ and $f : Y \to X$ is a log resolution of $(X, B)$. Let $\Gamma \geq 0$ be an $\mathbb{R}$-divisor on $Y$ such that $(Y, \Gamma + f^*A)$ is an lc pair and the effective part of the divisor $K_Y + \Gamma + M_Y - f^*(K_X + B + M)$ is $f$-exceptional. Then, $K_Y + \Gamma + M_Y + f^*A$ is abundant.

**Proof** We put $A_Y = f^*A$. We can write

$$K_Y + \Gamma + M_Y - f^*(K_X + B + M) + E_+ - E_-$$

such that $E_+ \geq 0$ and $E_- \geq 0$ have no common components and $E_+$ is $f$-exceptional. We run a $(K_Y + \Gamma + M_Y + A_Y)$-MMP over $X$ with scaling of an ample divisor. By applying [20, Proposition 3.9] to $(Y, (\Gamma + A_Y) + M_Y)$ over $X$, after finitely many steps we get a projective generalized lc pair $(Y', (\Gamma' + A') + M_{Y'})$ with a birational morphism $f' : Y' \to X$ such that

$$K_{Y'} + \Gamma' + A' + M_{Y'} = f'^*(K_X + B + A + M) - E'_-,$$

where $E'_-$ is the birational transform of $E_-$ on $Y'$. Then $A' = f'^*A$, and we have

$$K_{Y'} + \Gamma' + (uf'^*C + E'_-) + M_{Y'} = f'^*(K_X + B + uC + M)$$

for all real numbers $u \geq 0$. By (I) in Theorem 3.5, we see that the generalized pair $(Y', (\Gamma' + tf'^*C + E'_-) + M_{Y'})$ is generalized lc for some $t > 0$. We put $t' = \min\{1, t\}$. Then $(Y', (\Gamma' + t'(f'^*C + E'_-)) + M_{Y'})$ is a generalized lc pair. Furthermore, by (II) in Theorem 3.5, we have

$$K_{Y'} + \Gamma' + (f'^*C + E'_-) + M_{Y'} = f'^*(K_X + B + C + M) \sim_{\mathbb{R}, Z} 0.$$

By construction, $(Y', (\Gamma' + A') + M_{Y'})$ is a generalized lc pair. From these facts, we can apply Theorem 3.5 to $\pi \circ f' : Y' \to Z$, $(Y'', \Gamma' + M_{Y''})$, $f'^*C + E'_-, A_Z$ and $A'$. We see that $K_{Y'} + \Gamma' + A' + M_{Y'}$ is abundant. Then $K_Y + \Gamma + M_Y + A_Y$ is also abundant. \(\square\)

**Proof of Theorem 3.5** The argument is very similar to [29, Proof of Theorem 5.4]. We prove Theorem 3.5 by induction on $\dim X$.

If $\dim X = 1$, then Theorem 3.5 is clear. Assume Theorem 3.5 for all projective generalized lc pairs of dimension $\leq n - 1$. Let $\pi : X \to Z$ be a projective morphism, $(X, B + M)$ a projective NQC generalized lc pair with data $Y \xrightarrow{f} X$ and $M_Y$, and let $C, A_Z$ and $A$ be $\mathbb{R}$-Cartier divisors as in Theorem 3.5 such that $\dim X = n$. We may assume that $K_X + B + A + M$ is pseudo-effective, and we may assume that $\pi$ is a contraction by taking the Stein factorization of $\pi$. By replacing $A$ with a general one, we may assume that $B$ and $A$ have no common components and all generalized lc centers of $(X, (B + A) + M)$ are those of $(X, B + M)$. By replacing $(X, B + M)$...
with a generalized $\mathbb{Q}$-factorial dlt model and replacing $C$ and $A$ with the pullbacks, we may assume that $(X, B + M)$ is a $\mathbb{Q}$-factorial NQC generalized dlt pair.

From now on, we divide the proof into several steps.

**Step 1.** In this step, we prove Theorem 3.5 in the case where

$$K_X + B - \epsilon L B \downarrow + A + M$$

is pseudo-effective for some real number $\epsilon > 0$. This case includes the generalized klt case.

Since $K_X + B + C + M \sim_{\mathbb{R}, Z} 0$, by restricting $K_X + B - \epsilon L B \downarrow + A + M$ to a general fiber of $\pi$, we see that $L B \downarrow$ and $C$ are vertical over $Z$. Then there is an open subset $U \subset Z$ such that

$$(K_X + B - \epsilon L B \downarrow + M)|_{\pi^{-1}(U)} \sim_{\mathbb{R}, U} 0.$$ 

By Proposition 3.4, $K_X + B - \epsilon L B \downarrow + A + M$ is abundant, so

$$K_X + B - \epsilon L B \downarrow + A + M \sim_{\mathbb{R}} G$$

for some $\mathbb{R}$-divisor $G \geq 0$. Similarly, we see that $K_X + B - \frac{\epsilon}{2} L B \downarrow + A + M$ is abundant. Now we have

$$K_X + B + A + M \sim_{\mathbb{R}} G + \frac{\epsilon}{2} L B \downarrow$$

and

$$K_X + B + A + M \sim_{\mathbb{R}} G + \epsilon L B \downarrow.$$ 

By [29, Remark 2.8], we have

$$\kappa_0(X, K_X + B + A + M) = \kappa_0(X, K_X + B - \frac{\epsilon}{2} L B \downarrow + A + M)$$

$$= \kappa_1(X, K_X + B - \frac{\epsilon}{2} L B \downarrow + A + M)$$

$$= \kappa_1(X, K_X + B + A + M).$$

In this way, we see that $K_X + B + A + M$ is abundant.

**Step 2.** By Step 1, we may assume that $K_X + B - \epsilon L B \downarrow + A + M$ is not pseudo-effective for every real number $\epsilon > 0$. Then there exists a component $S$ of $L B \downarrow$ such that

$$K_X + B - \epsilon S + A + M$$

is not pseudo-effective for every real number $\epsilon > 0$. In this step, we will apply [23, Proof of Lemma 4.3] (see also [5, Lemma 2.10]).

For any $\epsilon' \in (0, 1]$, running a $(K_X + B - \epsilon' S + A + M)$-MMP with scaling of an ample divisor we get a birational contraction $X \dasharrow X'$ and a projective generalized lc pair $(X', (B' - \epsilon' S' + A') + M_{X'})$ with the structure of a Mori fiber space $X' \to Z'$. In other words, we have
• $\dim Z' < \dim X'$,
• $-(K_{X'} + B' - \epsilon'S' + A' + M_{X'})$ is ample over $Z'$, and
• the relative Picard number is one.

With the ACC for generalized lc thresholds ([8, Theorem 1.5]), we can find $\epsilon_0 \in (0, 1]$ such that if $\epsilon' \in (0, \epsilon_0)$, then $(X', (B' + A') + M_{X'})$ is a projective generalized lc pair.

By construction, we may find a non-negative real number $t_{\epsilon'} < \epsilon'$ such that $K_{X'} + B' - t_{\epsilon'} S' + A' + M_{X'}$ is numerically trivial over $Z'$. Let $F'$ be a general fiber of $X' \to Z'$. Then

$$K_{F'} + (B' - t_{\epsilon'} S' + A' + M_{X'})|_{F'} \equiv 0.$$  

By applying the global ACC [8, Theorem 1.6] to the set

$$\{(F', (B' - t_{\epsilon'} S' + A')|_{F'} + M_{X'}|_{F'}) \mid \epsilon' \in (0, \epsilon_0)\},$$

we can find $\epsilon' \in (0, \epsilon_0)$ such that $t_{\epsilon'} = 0$.

By the above discussions, choosing $\epsilon' \in (0, 1]$ appropriately, we get a birational contraction $X \to X'$ and a contraction $X' \to Z'$ such that $(X', 0)$ is a $\mathbb{Q}$-factorial klt pair, $(X', (B' + A') + M_{X'})$ is a projective generalized lc pair, and $K_{X'} + B' + A' + M_{X'} \sim_{\mathbb{R}, Z'} 0$.

**Step 3.** The goal of this step is to construct a diagram

$$\begin{array}{ccc}
Y_0 & \dasharrow & Y_0' \\
\downarrow & & \downarrow \\
X & & Z_0
\end{array}$$

having good properties, where $Y_0 \to X$ is a log resolution of $(X, B)$ factoring through $f : Y \to X$, the map $Y_0 \dasharrow Y_0'$ is a sequence of steps of an MMP over $Z'$ of an $\mathbb{R}$-divisor $K_{Y_0} + B_0 + A_0 + M_0$ such that the property of being abundant for $K_X + B + A + M$ follows from that for $K_{Y_0} + B_0 + A_0 + M_0$, and $Y_0' \to Z_0$ is a contraction over $Z'$ induced by the birational transform of $K_{Y_0} + B_0 + A_0 + M_0$ on $Y_0'$.

Since $(X', (B' + A') + M_{X'})$ is generalized lc, the property of being generalized lc for $(X', (B' + A') + M_{X'})$ is preserved after replacing $A$ by an other general member of $|A|_{\mathbb{R}}$. Thus, we may freely replace $A$ by an other general member of $|A|_{\mathbb{R}}$ keeping that $(X', (B' + A') + M_{X'})$ is generalized lc and $K_{X'} + B' + A' + M_{X'} \sim_{\mathbb{R}, Z'} 0$.

Replacing $f : Y \to X$, we may assume that the induced birational map $f' : Y \dasharrow X'$ is a morphism and the morphisms $f$ and $f'$ form a common log resolution of $(X, B) \dasharrow (X', B')$. By [29, Lemma 2.2] and replacing $A$ if necessary, we may assume that $f$ (resp. $f'$) is a log resolution of $(X, B + A)$ (resp. $(X', B' + A')$) and that $\text{Supp} f^* A$ and $\text{Supp} f'^{-1} B \cup \text{Ex}(f) \cup \text{Ex}(f')$ have no common divisorial components. Then $f^* A = f'^{-1} A$. Putting $A_Y = f^* A$, we can write

$$K_Y + B_Y + A_Y + M_Y = f^*(K_X + B + A + M) + E_Y$$  

(1)
with \( B_Y \geq 0 \) and \( E_Y \geq 0 \) which have no common components. Then \( B_Y + A_Y \) and \( E_Y \) have no common components.

We see that the generalized lc pair \((X', (B' + A') + MX')\) with data \( Y \to X' \to Z'\) and \( M_Y \) and the generalized lc pair \((Y, (B_Y + A_Y) + M_Y)\) satisfy the conditions of Proposition 3.2. Indeed, the first and the second conditions of Proposition 3.2 follow from construction. The third condition of Proposition 3.2 follows from the facts that \((X', 0)\) is \( \mathbb{Q} \)-factorial klt and \( X' \to Z' \) is a \((K_{X'} + B' - \epsilon'S' + A' + M_{X'})\)-Mori fiber space. Thus, we can apply Proposition 3.2, and there is a birational contraction \( Y \to Y' \) over \( Z' \) such that

(i) \( K_{Y'} + B_{Y'} + A_{Y'} + M_{Y'} \) is semi-ample over \( Z' \), and

(ii) \( a(P, Y, (B_Y + A_Y) + M_Y) \leq a(P, Y', (B_{Y'} + A_{Y'}) + M_{Y'}) \) for any prime divisor \( P \) over \( Y \).

Now we have the following diagram.

\[
\begin{array}{ccc}
Y & \to & Y' \\
X & \to & X' \\
\pi & \downarrow & \downarrow \\
Z & \to & Z'
\end{array}
\]

Take a log resolution \( g : Y_0 \to Y \) of \((Y, B_Y)\) such that the induced birational map \( Y_0 \to Y' \) is a morphism. Putting \( A_0 = g^*A_Y \), we can write

\[
K_{Y_0} + B_0 + A_0 = g^*(K_Y + B_Y + A_Y) + E_0
\]

with \( B_0 \geq 0 \) and \( E_0 \geq 0 \) having no common components. Replacing \( A \), we may assume \( A_0 = g_0^{-1}A_Y \) and that \((Y_0, B_0 + A_0)\) is a log smooth lc pair. Put \( M_0 = g^*M_Y \).

By the above conditions (i) and (ii), we may apply Lemma 3.1 to \((Y, (B_Y + A_Y) + M_Y), Y \to Y', g : Y_0 \to Y\), and \((Y_0, (B_0 + A_0) + M_0)\). By applying Lemma 3.1, we get a birational contraction \( Y_0 \to Y'_0 \) over \( Y' \), which is a sequence of steps of a \((K_{Y_0} + B_0 + A_0 + M_0)\)-MMP, and a projective \( \mathbb{Q} \)-factorial generalized dlt pair \((Y'_0, (B'_0 + A'_0) + M'_0)\) such that the divisor

\[
K_{Y'_0} + B'_0 + A'_0 + M'_0
\]

is the pullback of \( K_{Y'} + B_{Y'} + A_{Y'} + M_{Y'} \) over \( Z' \) by (i), hence the divisor defines a contraction \( Y'_0 \to Z'_0 \) over \( Z' \). The induced morphism \( Z_0 \to Z' \) is birational because the restriction of \( K_{Y'_0} + B'_0 + A'_0 + M'_0 \) to a general fiber of \( Y'_0 \to Z' \) is numerically trivial.
Step 4. We put $f_0 = f \circ g : Y_0 \to X$. We have constructed the diagram

$$
\begin{array}{ccc}
Y_0 & \to & Y_0' \\
\downarrow f_0 & & \downarrow \\
X & \to & Z_0 \\
\downarrow \pi & & \downarrow \\
Z & & \\
\end{array}
$$

and projective $\mathbb{Q}$-factorial generalized dlt pairs $(Y_0, (B_0 + A_0) + M_0)$ and $(Y_0', (B_0' + A_0') + M_0')$ such that

- $(Y_0, B_0 + A_0)$ is a log smooth lc pair, and
- $K_{Y_0'} + B_0' + A_0' + M_0' \sim_{\mathbb{R}} Z_0$.

By the relations (1) and (2) in Step 3 and Lemma 2.5, we can write

$$
K_{Y_0} + B_0 + A_0 + M_0 = f_0^* (K_X + B + A + M) + g^* E_Y + E_0 \quad (3)
$$

and $B_0 + A_0$ and $g^* E_Y + E_0$ have no common components. By (3) and that $Y_0 \to Y_0'$ is a sequence of steps of a $(K_{Y_0} + B_0 + A_0 + M_0)$-MMP, to prove Theorem 3.5, it is sufficient to prove that

$$
K_{Y_0'} + B_0' + A_0' + M_0'
$$

is abundant.

We recall that $S$ is a component of $B$ and $S'$ is ample over $Z'$. Let $S_0$ (resp. $S_0'$) be the birational transform of $S$ on $Y_0$ (resp. $Y_0'$). Then $S_0'$ dominates $Z_0$ because $Z_0 \to Z'$ is birational. Then the natural morphism $S_0' \to Z_0$ is surjective, and therefore it is sufficient to prove that $(K_{Y_0'} + B_0' + A_0' + M_0')|_{S_0'}$ is abundant ([29, Remark 2.8 (2)]).

With divisorial adjunction for generalized pairs, we construct the following generalized lc pairs:

$$
(S, B_S + M_S), \quad (S_0, B_{S_0} + M_{S_0}), \quad \text{and} \quad (S_0', B_{S_0'} + M_{S_0'}).
$$

By taking a suitable resolution of the normalization of the graph of $Y_0 \to Y_0'$, we can find a common resolution $W \to Y_0$ and $W \to Y_0'$ and a subvariety $T \subset W$ which is birational to $S_0$ and $S_0'$ such that the induced morphisms $\tau : T \to S_0$ and $\tau' : T \to S_0'$ form a common log resolution of $(S_0, B_{S_0}) \to (S_0', B_{S_0'})$. By taking $W$ appropriately, we may assume that $(S, B_S + M_S)$ (resp. $(S_0, B_{S_0} + M_{S_0})$, $(S_0', B_{S_0'} + M_{S_0'})$) is an NQC generalized pair with data $T \to S$ (resp. $T \to S_0$, $T \to S_0'$) and a nef $\mathbb{R}$-divisor $M_T$ on $T$.

We set

$$
A_S = A|_S, \quad A_{S_0} = A_0|_{S_0}, \quad \text{and} \quad A_{S_0'} = A_0'|_{S_0'}.
$$
By construction of divisorial adjunction for generalized pairs \([8, \text{Definition 4.7}]\), the generalized pair \((S, (B_S + A_S) + M_S)\) coincides with the generalized pair constructed with the divisorial adjunction for \((X, (B + A) + M)\). Therefore, \((S, (B_S + A_S) + M_S)\) is generalized lc. By the same reason, \((S_0, (B_{S_0} + A_{S_0}) + M_{S_0})\) and \((S'_0, (B_{S'_0} + A_{S'_0}) + M_{S'_0})\) are generalized lc.

Since

\[ K'_{S'_0} + B'_{S'_0} + A'_{S'_0} + M'_{S'_0} \sim_R (K'_{Y'_0} + B'_0 + A'_0 + M'_0)|_{S'_0}, \]

to prove Theorem 3.5, it is sufficient to prove that \(K'_{S'_0} + B'_{S'_0} + A'_{S'_0} + M'_{S'_0}\) is abundant. Since \(A_{S_0} = f_0^* A|_S\), by replacing \(A\) with the aid of \([29, \text{Lemma 2.1}]\) and \([29, \text{Lemma 2.2}]\), we may assume that

- \(A_{S_0} \geq 0\) and \(A_{S'_0} \geq 0\),
- \(\tau^* A_{S_0} \leq \tau'^{-1} A_{S'_0}\), and
- \(\tau': T \to S'_0\) is a log resolution of \((S'_0, B_{S'_0} + A_{S'_0})\).

**Step 5.** Let \(\pi_S: S \to Z\) be the restriction of \(\pi: X \to Z\) to \(S\), and let \(f_{S_0}: S_0 \to S\) be the restriction of \(f_0: Y_0 \to X\) to \(S_0\). We have the following diagram

\[
\begin{array}{ccc}
T & \xrightarrow{\tau} & S_0 \\
\downarrow f_{S_0} & & \downarrow f \\
S & \xrightarrow{\pi_S} & Z \\
\downarrow & & \downarrow \\
Z_0 & & \end{array}
\]

and generalized lc pairs \((S, (B_S + A_S) + M_S), (S_0, (B_{S_0} + A_{S_0}) + M_{S_0}),\) and \((S'_0, (B_{S'_0} + A_{S'_0}) + M_{S'_0})\). In this step, we will study the morphism \((S, (B_S + A_S) + M_S) \to Z\), and we will construct a generalized lc pair on \(T\) by using \((S'_0, (B_{S'_0} + A_{S'_0}) + M_{S'_0})\).

We check that \(\pi_S: S \to Z, (S, B_S + M_S), C|_S,\) and \(A_S \sim_{\mathbb{R}} \pi_S^* A_Z\) satisfy the conditions of Theorem 3.5. By construction, \((S, B_S + M_S)\) is a NQC generalized lc pair. Since \((X, (B + tC) + M)\) is generalized lc for some \(t > 0\), the divisor \(C|_S\) is a well-defined effective \(\mathbb{R}\)-Cartier divisor on \(S\). By \([8, \text{Definition 4.7}]\), \((S, (B_S + tC|_S) + M_S)\) coincides with the generalized pair constructed with the divisorial adjunction for \((X, (B + tC) + M)\), hence \((S, (B_S + tC|_S) + M_S)\) is generalized lc. From this fact, (I) of Theorem 3.5 is satisfied. By construction, we have

\[ K_S + B_S + C|_S + M_S \sim_{\mathbb{R}} (K_X + B + C + M)|_S \sim_{\mathbb{R},Z} 0. \]

Thus, (II) of Theorem 3.5 is satisfied. From these discussions, it follows that \(\pi_S: S \to Z, (S, B_S + M_S), C|_S,\) and \(A_S \sim_{\mathbb{R}} \pi_S^* A_Z\) satisfy the conditions of Theorem 3.5. By the induction hypothesis of Theorem 3.5, the divisor \(K_S + B_S + A_S + M_S\) is abundant.
Put $AT = \tau^* A_{S_0}$. By the relation $\tau^* A_{S_0} \leq \tau'^{-1} A_{S'_0}$, we can write

$$KT + \Psi_T + AT + MT = \tau'^*(K_{S'_0} + B_{S'_0} + A_{S'_0} + M_{S'_0}) + E_T,$$  \hspace{1cm} (4)

where $\Psi_T + AT \geq 0$ and $ET \geq 0$ have no common components. By construction, $(T, \Psi_T + AT)$ is a log smooth lc pair. By (4), it is sufficient to prove that

$$KT + \Psi_T + AT + MT$$

is abundant.

**Step 6.** In this step, we prove that $KT + \Psi_T + AT + MT$ is abundant.

We may write

$$KT + \Psi_T + AT + MT = (f_{S_0} \circ \tau)^*(K_S + B_S + A_S + M_S) + \Xi_+ - \Xi_-,$$

where $\Xi_+ \geq 0$ and $\Xi_- \geq 0$ have no common components. To apply Lemma 3.6, we show that $\Xi_+$ is exceptional over $S$.

Suppose by contradiction that there is a component $Q$ of $\Xi_+$ which is not exceptional over $S$. Then

$$a(Q, T, (\Psi_T + AT) + MT) < a(Q, S, (B_S + A_S) + M_S) \leq 1.$$  

By (3) in Step 4 and Lemma 2.6, we have

$$a(Q, S, (B_S + A_S) + M_S) = a(Q, S_0, (B_{S_0} + A_{S_0}) + M_{S_0}).$$

Since $Y_0 \to Y'_0$ is constructed with a $(K_{Y_0} + B_0 + A_0 + M_0)$-MMP, by taking a suitable common resolution of $Y_0 \to Y'_0$ and the negativity lemma (that is the generalized pair analogue of [12, Lemma 4.2.10]), we have

$$a(Q, S_0, (B_{S_0} + A_{S_0}) + M_{S_0}) \leq a(Q, S'_0, (B_{S'_0} + A_{S'_0}) + M_{S'_0}).$$

Finally, since $\Psi_T + AT$ and $ET$ have no common components (see (4) in Step 5), we can write

$$a(Q, T, (\Psi_T + AT) + MT) = \min\{a(Q, S'_0, (B_{S'_0} + A_{S'_0}) + M_{S'_0}), 1\}.$$  

By combining these relations, we obtain

$$a(Q, S, (B_S + A_S) + M_S) = \min\{a(Q, S, (B_S + A_S) + M_S), 1\}$$

$$= \min\{a(Q, S_0, (B_{S_0} + A_{S_0}) + M_{S_0}), 1\}$$

$$\leq \min\{a(Q, S'_0, (B_{S'_0} + A_{S'_0}) + M_{S'_0}), 1\}$$

$$= a(Q, T, (\Psi_T + AT) + MT)$$

$$< a(Q, S, (B_S + A_S) + M_S),$$
which is a contradiction. Thus, \( \Xi_+ \) is exceptional over \( S \).

By applying Lemma 3.6 to \((S, B_S + M_S) \rightarrow Z\) with data \( T \rightarrow S \) and \( M_T \),
\( A_S \sim_\mathbb{R} \pi_S^* A_Z \) and \( \Psi_T \), we see that the divisor
\[
K_T + \Psi_T + A_T + M_T
\]
is abundant. We finish this step.

By (4) in Step 5, the divisor \( K_{S_0} + B_{S_0} + A_{S_0} + M_{S_0} \) is abundant. By the arguments in Step 4, the divisor \( K_{Y_0'} + B_0' + A_0' + M_0' \) is abundant. From this, we see that \( K_X + B + A + M \) is abundant. We are done. \( \square \)

4 Proofs of main results

In this section, we prove the main results of this paper.

4.1 Non-vanishing theorem

In this subsection, we firstly prove Theorem 1.3, then we prove Theorem 1.1, and finally we prove Theorem 1.2.

**Theorem 4.1** Let \( \pi : X \rightarrow Z \) be a morphism of normal projective varieties, and let \((X, B + M)\) be a projective \( \mathbb{Q} \)-factorial NQC generalized dlt pair with data \( Y \rightarrow X \) and \( M_Y \) such that there is an effective \( \mathbb{R} \)-Cartier divisor \( C \) on \( X \) satisfying the following properties.

(I) \((X, (B + tC) + M)\) is generalized lc for some \( t > 0 \), and
(II) \( K_X + B + C + M \sim_\mathbb{R} Z \).

Let \( A_Z \) be an ample \( \mathbb{R} \)-divisor on \( Z \), and pick \( 0 \leq A \sim_\mathbb{R} \pi^* A_Z \) such that \((X, (B + A) + M)\) is a generalized lc pair whose generalized lc centers are those of \((X, B + M)\).

We put \( \Delta = B + A \).

Then, for any sequence of steps of a \((K_X + \Delta + M)\)-MMP
\[
(X, \Delta + M) =: (X_0, \Delta_0 + M_0) \rightarrow \cdots \rightarrow (X_i, \Delta_i + M_i) \rightarrow \cdots,
\]
the divisor
\[
K_{X_i} + \Delta_i + M_i
\]
is log abundant with respect to \((X_i, \Delta_i + M_i)\) for every \( i \). Furthermore, if \((X, B + M)\) is generalized klt and \( K_X + \Delta + M \) is pseudo-effective, then \( K_X + \Delta + M \) birationally has the Nakayama–Zariski decomposition with semi-ample positive part.

**Proof** By taking the Stein factorization of \( \pi \), we may assume that \( \pi \) is a contraction. When \((X, B + M)\) is generalized klt and \( K_X + \Delta + M \) is pseudo-effective, \( C \) is vertical over \( Z \). Thus, we can find an open subset \( U \subset Z \) such that \((K_X + B + M)|_{\pi^{-1}(U)} \sim_\mathbb{R} U \).
0. Then the second assertion directly follows from Proposition 3.4. So it is enough to show the property of being log abundant.

The strategy is very similar to [29, Proof of Theorem 5.5]. Let $\pi : X \to Z$, $(X, B + M)$, and $A$ be as in Theorem 4.1. Put $\Delta = B + A$, and let

$$(X, \Delta + M) =: (X_0, \Delta_0 + M_0) \dashrightarrow \cdots \dashrightarrow (X_i, \Delta_i + M_i) \dashrightarrow \cdots$$

be a sequence of steps of a $(K_X + \Delta + M)$-MMP. Fix an index $i$. By replacing $Y \to X$, we may assume that the induced birational map $Y \dashrightarrow X_i$ is a morphism.

By Theorem 3.5, the divisor $K_X + \Delta + M$ is abundant, so

$$K_{X_i} + \Delta_i + M_i$$

is also abundant. Therefore, it is sufficient to prove that

$$(K_{X_i} + \Delta_i + M_i)|_{S_i}$$

is abundant for every generalized lc center $S_i$ of $(X_i, \Delta_i + M_i)$. We fix a generalized lc center $S_i$ of $(X_i, \Delta_i + M_i)$. Then there is a generalized lc center $S$ of $(X, \Delta + M)$ such that the map $X \dashrightarrow X_i$ induces a birational map $S \dashrightarrow S_i$. Let $f : (X', \Delta' + M') \to (X, \Delta + M)$ be a $\mathbb{Q}$-factorial generalized dlt model such that there is a component $S'$ of $\Delta'$ from which $f$ induces a surjective morphism $S' \to S$. Such a model always exists by Remark 2.4. Taking a lift of the $(K_X + \Delta + M)$-MMP (see, for example, [20, Subsection 3.5]), we get a sequence of steps of a $(K_{X'} + \Delta' + M')$-MMP

$$(X', \Delta' + M') =: (X'_0, \Delta'_0 + M'_0) \dashrightarrow \cdots \dashrightarrow (X'_{k_i}, \Delta'_{k_i} + M'_{k_i})$$

and a $\mathbb{Q}$-factorial generalized dlt model

$$f_i : (X'_{k_i}, \Delta'_{k_i} + M'_{k_i}) \to (X_i, \Delta_i + M_i)$$

such that the birational transform $S'_{k_i}$ of $S'$ on $X'_{k_i}$ is a component of $\Delta'_{k_i}$ and $f_i : X'_{k_i} \to X_i$ induces a surjective morphism $S'_{k_i} \to S_i$. By construction, we have

$$K_{X'_{k_i}} + \Delta'_{k_i} + M'_{k_i} = f_i^*(K_{X_i} + \Delta_i + M_i),$$

thus $(K_{X_i} + \Delta_i + M_i)|_{S_i}$ is abundant if and only if

$$(K_{X'_{k_i}} + \Delta'_{k_i} + M'_{k_i})|_{S'_{k_i}}$$

is abundant. Furthermore, it is easy to check that (I) and (II) in Theorem 4.1 hold after replacing $(X, \Delta + M)$ by $(X', \Delta' + M')$.

From the above arguments, by replacing the $(K_X + \Delta + M)$-MMP and $S_i$, we may assume that $S_i$ is a component of $\Delta_i$. Then $S$ is a component of $\Delta B$. Let $B_i$ and $A_i$
be the birational transforms of $B$ and $A$ on $X_i$, respectively. Then $S_i$ is a component of $\cup B_{i,j}$.

Replacing $Y \to X$ by a suitable resolution of the normalization of the graph of $X \dasharrow X_i$, we may assume that the birational morphisms $Y \to X$ and $Y \to X_i$ form a common resolution of $X \dasharrow X_i$ and there is a subvariety $T \subset Y$ which is birational to $S$ and $S_i$ such that the induced morphisms $\tau : T \to S$ and $\tau' : T \to S_i$ form a common resolution of $S \dasharrow S_i$. We pick $M_T \sim_{\mathbb{R}} M_Y|_T$. Applying divisorial adjunction to $(X, B + M)$ and $(X_i, B_i + M_i)$, we construct NQC generalized pairs

$$\begin{align*}
(S, B_S + M_S) \text{ with data } T \to S \text{ and } M_T, \quad \text{and} \\
(S_i, B_{S_i} + M_{S_i}) \text{ with data } T \to S_i \text{ and } M_T.
\end{align*}$$

Replacing $Y$ if necessary and replacing $T$ and $M_T$ accordingly, we may assume that $\tau : T \to S$ and $\tau' : T \to S_i$ form a common log resolution of $(S, B_S) \dasharrow (S_i, B_{S_i})$ and $\tau_{\ast}^{-1}B_{S_i} \cup \operatorname{Ex}(\tau) \cup \operatorname{Ex}(\tau')$ is a simple normal crossing divisor.

We set

$$A_S = A|_S \quad \text{and} \quad A_{S_i} = A_i|_{S_i}. $$

By construction of divisorial adjunction for generalized pairs [8, Definition 4.7], the generalized pair $(S, (B_S + A_S) + M_S)$ coincides with the generalized pair constructed by applying the divisorial adjunction to $(X, \Delta + M)$. Therefore, $(S, (B_S + A_S) + M_S)$ is generalized lc. By the same reason, $(S_i, (B_{S_i} + A_{S_i}) + M_{S_i})$ is also generalized lc. By [29, Lemma 2.1] and [29, Lemma 2.2] and replacing $A$ with a general one, we may assume that

- $A_S \geq 0$ and $A_{S_i} \geq 0$,
- $\tau^*A_S \leq \tau'^{-1}_{\ast}A_{S_i}$, and
- $\tau' : T \to S_i$ is a log resolution of $(S_i, B_{S_i} + A_{S_i})$.

Since

$$K_{S_i} + B_{S_i} + A_{S_i} + M_{S_i} \sim_{\mathbb{R}} (K_{X_i} + \Delta_i + M_i)|_{S_i},$$

it is sufficient to prove that $K_{S_i} + B_{S_i} + A_{S_i} + M_{S_i}$ is abundant.

Since $\tau^*A_S \leq \tau'^{-1}_{\ast}A_{S_i}$, we can write

$$K_T + \Psi_T + \tau^*A_S + M_T = \tau'^* (K_{S_i} + B_{S_i} + A_{S_i} + M_{S_i}) + E_T$$

with $\Psi_T \geq 0$ and $E_T \geq 0$ on $T$ such that $\Psi_T + \tau^*A_S$ and $E_T$ have no common components. Then $(T, \Psi_T + \tau^*A_S)$ is a log smooth lc pair. Now we can write

$$K_T + \Psi_T + \tau^*A_S + M_T = \tau^* (K_S + B_S + A_S + M_S) + \Xi_+ - \Xi_-$$

such that $\Xi_+ \geq 0$ and $\Xi_- \geq 0$ have no common components.
Pick a component $Q$ of $\Xi_+$. By the negativity lemma, we have
\[ a(Q, S, (B_S + A_S) + M_S) \leq a(Q, S_i, (B_{S_i} + A_{S_i}) + M_{S_i}). \]

By construction of $\Psi_T$ and $E_T$, we can write
\[ a(Q, T, (\Psi_T + \tau^* A_S) + M_T) = \min\{a(Q, S_i, (B_{S_i} + A_{S_i}) + M_{S_i}), 1\}. \]

Thus, we obtain
\[
\begin{align*}
a(Q, S, (B_S + A_S) + M_S) &> a(Q, T, (\Psi_T + \tau^* A_S) + M_T) \\
&= \min\{a(Q, S_i, (B_{S_i} + A_{S_i}) + M_{S_i}), 1\} \\
&\geq \min\{a(Q, S, (B_S + A_S) + M_S), 1\}.
\end{align*}
\]

Thus, it follows that $a(Q, S, (B_S + A_S) + M_S) > 1$ which shows that $Q$ is $\tau$-exceptional. Therefore, $\Xi_+$ is $\tau$-exceptional.

Now we can apply Lemma 3.6 to $(S, B_S + M_S)$ with data $T \to S$ and $M_T, S \to Z, C|S, A_S \sim_{\mathbb{R}} \pi|_S^* A_Z$, and $\Psi_T$. Then we see that
\[ K_T + \Psi_T + \tau^* A_S + M_T \]
is abundant. By (♣), the divisor
\[ K_{S_i} + B_{S_i} + A_{S_i} + M_{S_i} \sim_{\mathbb{R}} (K_{X_i} + \Delta_i + M_i)|_{S_i} \]
is abundant. Since $i$ is an arbitrary index, we see that $K_{X_i} + \Delta_i + M_i$ is log abundant with respect to $(X_i, \Delta_i + M_i)$ for every $i$. So we are done. $\square$

**Proof of Theorem 1.3** The theorem is the case of Theorem 4.1 where $\pi$ is the morphism of a $\mathbb{Q}$-factorial dlt model and $C = 0$. $\square$

**Proof of Theorem 1.1** The first statement immediately follows from Theorem 3.5. Thus, it is enough to prove the second statement for every real number $\alpha > 1$.

Suppose that $M$ is $\mathbb{R}$-Cartier. Fix a real number $\alpha > 1$. We pick $\epsilon \in (0, \alpha - 1)$ such that $A + \epsilon M$ is ample, and we pick a general member $A_\epsilon \in |A + \epsilon M|_{\mathbb{R}}$. Then \(\frac{\alpha - \epsilon}{1 - \epsilon} > 1\), the divisor $K_X + B + A_\epsilon + (1 - \epsilon)M$ is pseudo-effective, and we have
\[ K_X + B + A + \alpha M \sim_{\mathbb{R}} K_X + B + A_\epsilon + \frac{\alpha - \epsilon}{1 - \epsilon} (1 - \epsilon)M. \]

By replacing $M', A$ and $\alpha$ with $(1 - \epsilon)M', A_\epsilon$ and $\frac{\alpha - \epsilon}{1 - \epsilon}$ respectively, we may assume that the generalized pair $(X, B + (1 + \epsilon')M)$ is generalized lc for some real number $\epsilon' > 0$.

By replacing $f : X' \to X$ if necessary, we may assume that $f$ is a log resolution of $(X, B)$. We may write
\[ K_{X'} + B' + M' = f^*(K_X + B + M) + E' \]
with $B' \geq 0$ and $E' \geq 0$ having no common components. Furthermore, since $M$ is $\mathbb{R}$-Cartier, we may write

$$M' + F' = f^* M$$

for some $F' \geq 0$ which is exceptional over $X$. Since $(X, B + (1 + \epsilon')M)$ is generalized lc for some $\epsilon' > 0$, we can find a real number $\delta > 0$ such that $(X', B' + \delta F')$ is a log smooth lc pair. By construction, we have

$$K_{X'} + B' + f^* A + \alpha M' = f^*(K_X + B + A + \alpha M) + E' - (\alpha - 1)F'$$

and $K_{X'} + B' + f^* A + \alpha M'$ is pseudo-effective.

We run a $(K_{X'} + B' + \alpha M')$-MMP over $X$ with scaling of an ample divisor. By the negativity lemma, after finitely many steps we obtain a projective birational morphism $g: X'' \to X$ and a projective $\mathbb{Q}$-factorial generalized dlt pair $(X'', B'' + \alpha M'')$ such that the effective part of $E' - (\alpha - 1)F'$ is contracted by $X' \to X''$. By replacing $\delta$ with a smaller one, we may assume that $X'/\to X''$ is a sequence of steps of a $(K_{X'} + B' + \delta F' + \alpha M')$-MMP.

Let $E''$ and $F''$ be the birational transforms of $E'$ and $F'$ on $X''$, respectively. By construction, we see that

- $(X'', (B'' + \delta F'') + \alpha M'')$ is generalized lc, and
- defining $C''$ to be $(\alpha - 1)F'' - E''$, then $C'' \geq 0$.

We have $\text{Supp} C'' \subset \text{Supp} F''$, and we have

$$K_{X''} + B'' + \alpha M'' + C'' = g^*(K_X + B + A + \alpha M).$$

The NQC generalized dlt pair $(X'', B'' + \alpha M'')$ and the morphism $g: X'' \to X$ satisfy the conditions of Theorem 3.5. Indeed, the divisor $C''$ satisfies (II) of Theorem 3.5 because of the above relation, and $C''$ satisfies (I) of Theorem 3.5 because $(X'', (B'' + \delta F'') + \alpha M'')$ is a generalized lc pair and $\text{Supp} C'' \subset \text{Supp} F''$ which show the existence of a real number $t > 0$ such that $(X'', (B'' + tC'') + \alpha M'')$ is a generalized lc pair. By applying Theorem 3.5 to $X'' \to X$, $(X'', B'' + \alpha M'')$, and $A$, we see that the divisor $K_{X''} + B'' + \alpha M'' + g^* A$ is abundant. From this fact, there is an $\mathbb{R}$-divisor $D''_\alpha \geq 0$ on $X''$ such that

$$D''_\alpha \sim \mathbb{R} K_{X''} + B'' + \alpha M'' + g^* A.$$ 

By putting $D_\alpha = g_* D''_\alpha$, we obtain

$$D_\alpha \sim \mathbb{R} K_X + B + A + \alpha M.$$

Thus, the second statement of Theorem 1.1 holds. We are done. $\square$
Proof of Theorem 1.2} Fix $\alpha \in \mathbb{R}_{\geq 0}$. By [26, Proposition 4.12], with notations as in [26], the pair $(X, B)$ of $X$ and $B$ is pseudo-lc in the sense of [26, Definition 4.2]. By [26, Theorem 1.2], there exists a small birational morphism

$$h : Y \to X$$

from a normal projective variety $Y$ such that if we put $B_Y = h_*^{-1}B$, then $K_Y + B_Y$ is an $h$-ample $\mathbb{R}$-Cartier divisor on $Y$. Put $M_Y = h_*^{-1}M$. Then $M_Y$ is $\mathbb{R}$-Cartier.

We can find a real number $\epsilon > 0$ such that

$$\epsilon(K_Y + B_Y + h^*A)$$

is ample. Putting $A_Y = \frac{1}{1-\epsilon}(\epsilon(K_Y + B_Y + h^*A))$, we have

$$K_Y + B_Y + h^*A = (1 - \epsilon)(K_Y + B_Y + A_Y).$$

Since $K_X + B + A$ is the pushdown of a pseudo-effective $\mathbb{R}$-divisor, it follows that $K_Y + B_Y + A_Y$ is pseudo-effective. Therefore the relation

$$K_Y + B_Y + h^*A + \alpha M_Y = (1 - \epsilon) \left( K_Y + B_Y + A_Y + \frac{\alpha}{1 - \epsilon} M_Y \right)$$

holds and $K_Y + B_Y + A_Y + \frac{\alpha}{1 - \epsilon} M_Y$ is pseudo-effective. By replacing $X$ (resp. $B$, $A$, $\alpha$) with $Y$ (resp. $B_Y$, $A_Y$, $\frac{\alpha}{1 - \epsilon} M_Y$), we may assume that $M$ is $\mathbb{R}$-Cartier.

If $\alpha = 0$, then Theorem 1.2 follows from [29, Theorem 1.5]. Therefore, we may assume $\alpha > 0$. Put $t' = \min\{1, \alpha\}$. By applying Theorem 1.1 (2) to $(X, B + t'M)$ and $K_X + B + A + \alpha M$, we can find an effective $\mathbb{R}$-divisor $D_\alpha$ on $X$ such that $K_X + B + A + \alpha M \sim_\mathbb{R} D_\alpha$.

The following result is out of interests, and we write down for possibility of the future use.

**Theorem 4.2** Let $(X, B + M)$ be a projective NQC generalized lc pair with data $X' \xrightarrow{f} X$ and $M'$. Let $A$ be an ample $\mathbb{R}$-divisor on $X$. Then the following statements hold.

1. If $K_X + B + A$ is the pushdown of a pseudo-effective $\mathbb{R}$-divisor on $X'$, then there exists an effective $\mathbb{R}$-divisor $D$ on $X$ such that

   $$K_X + B + A \sim_\mathbb{R} D.$$  

2. Let $G$ be a $\mathbb{Q}$-divisor on $X$ such that $G \sim_\mathbb{R} K_X + B + A$. Then the graded ring

   $$\mathcal{R}(X, G) = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} H^0(X, \mathcal{O}_X(\lfloor mG \rfloor))$$

   is finitely generated.
Proof The first assertion is the case of $\alpha = 0$ of Theorem 1.2. We will prove the second assertion. By [26, Proposition 4.12], notations as in [26], the pair $(X, B)$ of $X$ and $B$ is pseudo-lc in the sense of [26, Definition 4.2]. By [26, Theorem 1.2], there exists a small birational morphism $h: Y \to X$ from a normal projective variety $Y$ such that if we put $B_Y = h_*^{-1}B$, then $K_Y + B_Y$ is an $h$-ample $R$-Cartier divisor on $Y$. Then $t(K_Y + B_Y) + h^*A$ is an ample $R$-divisor on $Y$ for some $t \in (0, 1)$. Taking a general member $A_Y \sim_{IR} 1 - t(K_Y + B_Y + A_Y)$, we have

$$K_Y + B_Y + h^*A \sim_{IR} (1 - t)(K_Y + B_Y + A_Y).$$

Since $h$ is small, for each $m$ we have

$$H^0(X, O_X(\ell mG)) \cong H^0(Y, O_Y(\ell m h_*^{-1}G)).$$

Therefore, the graded ring $\mathcal{R}(X, G)$ is finitely generated if and only if

$$\mathcal{R}(Y, h_*^{-1}G) = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} H^0(Y, O_Y(\ell m h_*^{-1}G))$$

is finitely generated. Furthermore, we have

$$h_*^{-1}G \sim_{IR} (1 - t)(K_Y + B_Y + A_Y).$$

By [29, Theorem 1.5] and [29, Theorem 1.7], $\mathcal{R}(Y, h_*^{-1}G)$ is finitely generated, thus $\mathcal{R}(X, G)$ is finitely generated. We finish the proof. \hfill $\square$

4.2 On the Kodaira type vanishing theorem

We give an example which shows that the Kodaira type vanishing theorem does not always hold in the situation of Theorem 1.1 (2) or Theorem 1.2. The following theorem is a stronger statement than Theorem 1.4.

Theorem 4.3 Let $d$ be a positive integer. Then, there is a projective $\mathbb{Q}$-factorial generalized klt pair $(X, B + M)$, an ample $\mathbb{Q}$-divisor $A$ on $X$, and a Cartier divisor $D$ on $X$ such that

- $\dim X = 3$ and $K_X + B + A$ is pseudo-effective,
- $D \sim_{\mathbb{Q}} K_X + B + A + tM$ for some $t > 1$, and
- $\dim H^1(X, O_X(D)) \geq d$.

Proof We prove the theorem in several steps.

Step 1. We construct some varieties and divisors.

We put $V = \mathbb{P}^1$, and fix a very ample Cartier divisor $H_V$ such that $O_V(H_V) = O_V(1)$. Let $W$ be an elliptic curve, and fix a very ample Cartier divisor $H_W$ on $W$. We set

$$p_V: V \times W \to V \quad \text{and} \quad p_W: V \times W \to W$$
as projections. The divisor $p^*_V H_V + p^*_W H_W$ is a very ample Cartier divisor on $V \times W$. We construct a $\mathbb{P}^1$-bundle

$$Y := \mathbb{P}_{V \times W}(O_{V \times W} \oplus O_{V \times W}(-p^*_V H_V - p^*_W H_W)) \xrightarrow{f} V \times W.$$ 

Let $T$ be the unique section corresponding to $O_Y(1)$. It is obvious that $(Y, T)$ is a log smooth lc pair. We define a Cartier divisor $L$ on $Y$ by

$$L = T + f^*(p^*_V H_V + p^*_W H_W).$$

By construction, we have

$$K_{V \times W} = p^*_V K_V + p^*_W K_W \sim -2p^*_V H_V.$$ 

From this, we see that $L$ is base point free and we have the relation

$$K_Y + T + L \sim -2f^*p^*_V H_V.$$ 

Let $Y \to Z$ be the contraction induced by $L$. Since $L + f^*p^*_W H_W$ is base point free, we get a contraction $\pi : Y \to X$ induced by $L + f^*p^*_W H_W$. By construction, these morphisms are isomorphisms outside $T$. By calculations of intersection numbers of curves contracted by $\pi$, we see that all curves contracted by $\pi$ are also contracted by $Y \to Z$. Thus, the induced birational map $X \dashrightarrow Z$ is a morphism. We denote $Y \to Z$ by $\phi$ and $X \to Z$ by $\psi$.

![Diagram]

Then

$$L = \phi^*\phi_*L = \pi^*\psi^*\phi_*L.$$ 

Therefore, $\pi_*L$ is a nef and big $\mathbb{Q}$-Cartier divisor on $X$ and $L = \pi^*\pi_*L$. Since we have

$$L + f^*p^*_W H_W = \pi^*\pi_*(L + f^*p^*_W H_W),$$

the divisor $\pi_*f^*p^*_W H_W$ is $\mathbb{Q}$-Cartier and

$$f^*p^*_W H_W = \pi^*\pi_*f^*p^*_W H_W.$$
Since $H_W$ is very ample, $\pi_* f^* p_W^* H_W$ is semi-ample and $\pi_* f^* p_W^* H_W$ induces a contraction $g : X \to W$ such that $g \circ \pi = p_W \circ f$. We put $H_X = \pi_* f^* p_V^* H_V$. By the definition of $L$, we see that $H_X$ is $\mathbb{Q}$-Cartier and

$$\pi^* H_X = \pi^* (\pi_* L - \pi_* f^* p_W^* H_W) = L - f^* p_W^* H_W = T + f^* p_V^* H_V.$$ 

By these discussion, we obtain the following diagram

\[ \begin{array}{ccc} & Y & \\
V \times W & \pi & X \\
V & & W \\
p_V & & p_W \\
\end{array} \]

and the divisors

- $H_V$, $H_W$, $T$, $L = T + f^* (p_V^* H_V + p_W^* H_W)$, and
- $H_X = \pi_* f^* p_V^* H_V$

such that

1. $(Y, T)$ is a log smooth lc pair,
2. $H_V$ and $H_W$ are very ample and $K_{V \times W} \sim -2 p_V^* H_V$,
3. $K_Y + T + L + 2 f^* p_V^* H_V \sim 0$,
4. $\pi_* L$ is a nef and big $\mathbb{Q}$-Cartier divisor on $X$ such that $\pi^* \pi_* L = L$, and
5. $H_X$ is $\mathbb{Q}$-Cartier and $\pi^* H_X = T + f^* p_V^* H_V$.

**Step 2.** In this step, we study properties of $X$ constructed in Step 1. More precisely, we show that $X$ is $\mathbb{Q}$-factorial and $\pi^* K_X = K_Y - T$.

Pick any Weil divisor $D$ on $X$. Then $\pi_*^{-1} D$ is linearly equivalent to the sum of a multiple of $T$ and the pullback of a Cartier divisor $G$ on $V \times W$. Since $V = \mathbb{P}^1$, we can find an integer $\alpha$ and a Cartier divisor $G_W$ on $W$ such that $G \sim \alpha p_V^* H_V + p_W^* G_W$. Thus, we have

$$D \sim \pi_* f^* G \sim \alpha \pi_* f^* p_V^* H_V + \pi_* f^* p_W^* G_W = \alpha H_X + g^* G_W.$$ 

Since $H_X$ is $\mathbb{Q}$-Cartier, $D$ is also $\mathbb{Q}$-Cartier. Therefore, $X$ is $\mathbb{Q}$-factorial.

By (3) in Step 1, we have $K_X + \pi_* L + 2 H_X \sim 0$. The negativity lemma shows

$$K_Y + T + L + 2 f^* p_V^* H_V = \pi^* (K_X + \pi_* L + 2 H_X).$$

By using (4) and (5) in Step 1, we obtain $\pi^* K_X = K_Y - T$.

**Step 3.** In this step, we prove $H^2(Y, \mathcal{O}_Y (-m K_Y + (m - 1) T)) = 0$ for all positive integers $m$. 
We fix a positive integer $m$. Put $h = p_W \circ f : Y \to W$. Applying the Leray spectral sequence to $\mathcal{O}_Y(-mK_Y + (m - 1)T)$ and $h : Y \to W$, we see that it is sufficient to prove the following facts.

1. $H^2(W, h_*\mathcal{O}_Y(-mK_Y + (m - 1)T)) = 0$,
2. $H^1(W, R^1h_*\mathcal{O}_Y(-mK_Y + (m - 1)T)) = 0$, and
3. $H^0(W, R^2h_*\mathcal{O}_Y(-mK_Y + (m - 1)T)) = 0$.

The first fact is clear because dim$W = 1$. We also see that

$$-mK_Y + (m - 1)T \sim K_Y + (m + 1)(T + L + 2f^*p_V^*H_V) + (m - 1)T$$

$$= K_Y + 2mT + (m + 1)L + (2m + 2)f^*p_V^*H_V$$

$$= K_Y + (3m + 1)L + 2f^*p_V^*H_V - 2mh^*H_W,$$

where the final relation follows from $L = T + f^*(p_V^*H_V + p_W^*H_W)$. Since $H_V$ is very ample and $L$ is nef and big, the relative Kawamata–Viehweg vanishing theorem implies

$$R^1h_*\mathcal{O}_Y(-mK_Y + (m - 1)T) = R^2h_*\mathcal{O}_Y(-mK_Y + (m - 1)T) = 0.$$

From this, the second and the third facts stated above follow. In this way, we have

$$H^2(Y, \mathcal{O}_Y(-mK_Y + (m - 1)T)) = 0,$$

which is what we wanted to prove in this step.

**Step 4.** In this step, we prove

$$\dim H^1(X, \mathcal{O}_X(-lK_X)) \geq \dim H^0(W, \mathcal{O}_W(2lH_W))$$

for every positive integer $l$ such that $lK_X$ is Cartier.

Since $X$ is $\mathbb{Q}$-factorial (see Step 2) and $T$ is the unique $\pi$-exceptional divisor, $-T$ is ample over $X$. Since $\pi^*K_X = K_Y - T$ (Step 2), it follows that $Y$ is Fano over $X$. Thus, we have $R^i\pi_*\mathcal{O}_Y = 0$ for every $i > 0$. By the Leray spectral sequence, it follows that

$$\dim H^1(X, \mathcal{O}_X(-lK_X)) = \dim H^1(Y, \mathcal{O}_Y(-l\pi^*K_X))$$

$$= \dim H^1(Y, \mathcal{O}_Y(-lK_Y + lT)).$$

We consider the exact sequence

$$0 \to \mathcal{O}_Y(-lK_Y + (l - 1)T) \to \mathcal{O}_Y(-lK_Y + lT) \to \mathcal{O}_T((-lK_Y + lT)|_T) \to 0.$$

Since we have $H^2(Y, \mathcal{O}_Y(-lK_Y + (l - 1)T)) = 0$ by Step 3, taking the cohomology long exact sequence, we obtain the exact sequence

$$H^1(Y, \mathcal{O}_Y(-lK_Y + lT)) \to H^1(T, \mathcal{O}_T((-lK_Y + lT)|_T)) \to 0.$$
Since \((T + f^*(p_Y^*H_Y + p_W^*H_W))|_T \sim 0\), we have
\[\left((-lK_T + lT)|_T \sim -lK_T - 2lf^*(p_Y^*H_Y + p_W^*H_W)|_T.\]

By identifying \(T\) with \(V \times W\) and using \(K_{V \times W} \sim -2p_Y^*H_Y\), we obtain
\[\dim H^1(T, \mathcal{O}_T((-lK_T + lT)|_T)) = \dim H^1(V \times W, \mathcal{O}_{V \times W}(-2lp_W^*H_W)).\]

Since \(V = \mathbb{P}^1\), using the Leray spectral sequence we have
\[\dim H^1(V \times W, \mathcal{O}_{V \times W}(-2lp_W^*H_W)) = \dim H^1(W, \mathcal{O}_W(-2lH_W)).\]

Since \(W\) is an elliptic curve, by Serre duality, we have
\[\dim H^1(W, \mathcal{O}_W(-2lH_W)) = \dim H^0(W, \mathcal{O}_W(2lH_W)).\]

Combining these relations, we obtain
\[\dim H^1(X, \mathcal{O}_X(-lK_X)) = \dim H^1(Y, \mathcal{O}_Y(-lK_Y + lT)) \geq \dim H^1(T, \mathcal{O}_T((-lK_T + lT)|_T)) = \dim H^1(V \times W, \mathcal{O}_{V \times W}(-2lp_W^*H_W)) = \dim H^1(W, \mathcal{O}_W(-2lH_W)) = \dim H^0(W, \mathcal{O}_W(2lH_W)).\]

We finish this step.

**Step 5.** With this step we will finish the proof of Theorem 4.3. In other words, for any positive integer \(d\), we find a structure of generalized klt pair \((X, B + M)\) on \(X\), an ample \(\mathbb{Q}\)-divisor \(A\), and a Cartier divisor \(D\) satisfying all the conditions of Theorem 4.3.

We fix a positive integer \(d\). We pick a positive integer \(l\) such that \(lK_X\) is Cartier, \(K_X + (l + 1)\pi_*L\) is big, and \(\dim H^0(W, \mathcal{O}_W(2lH_W)) \geq d\). We can find such \(l\) since \(H_W\) is very ample and \(\pi_*L\) is big. By Step 2, \((X, 0)\) is \(\mathbb{Q}\)-factorial klt. Since \(\pi_*L\) is nef and big ((4) in Step 1), we can find a \(\mathbb{Q}\)-divisor \(B \geq 0\) and an ample \(\mathbb{Q}\)-divisor \(A\) on \(X\) such that \((X, B + A)\) is klt and \((l + 1)\pi_*L \sim_{\mathbb{Q}} A + B\). We define
\[N_Y := 2f^*p_Y^*H_Y\quad\text{and}\quad N := \pi_*N_Y.\]

Since \((X, B + A)\) is \(\mathbb{Q}\)-factorial klt, we can find \(u \in (0, 1] \cap \mathbb{Q}\) such that the generalized pair
\[(X, B + uN)\quad\text{with data} \ Y \to X \quad\text{and} \quad uN_Y\]
is generalized klt. We put
\[M_Y = uN_Y, \quad M = uN, \quad\text{and}\quad D = -lK_X.\]
We show that $(X, B+M)$, $A$ and $D$ satisfy all the conditions of Theorem 4.3. Clearly $\dim X = 3$, and $X$ is $\mathbb{Q}$-factorial by Step 2. Since $K_X + A + B \sim_{\mathbb{Q}} K_X + (l + 1)\pi_\ast L$, the divisor $K_X + B + A$ is big. Therefore, the first condition of Theorem 4.3 holds true. By (3) in Step 1, we have $K_X + \pi_\ast L + N \sim 0$. Since $M = uN$ and $D = -lK_X$, we have

$$D = K_X - (l + 1)K_X \sim_{\mathbb{Q}} K_X + B + A + \frac{l+1}{u}M.$$ 

Since $u \in (0, 1]$, we have $\frac{l+1}{u} > 1$. Thus, the second condition of Theorem 4.3 holds. Finally, by Step 4 and our choice of $l$, we have

$$\dim H^1(X, \mathcal{O}_X(D)) \geq \dim H^0(W, \mathcal{O}_W(2lH_W)) \geq d.$$ 

Therefore, the third condition of Theorem 4.3 holds.

We complete the proof of Theorem 4.3. \qed

**Proof of Theorem 1.4** It is clear from Theorem 4.3. \qed

**Acknowledgements** The author was partially supported by JSPS KAKENHI Grant Number JP16J05875 and JP19J00046. Part of the work was done while the author was visiting University of Cambridge in October–November 2018 and January 2020. The author thanks staffs of the university, Professor Caucher Birkar, Doctor Roberto Svaldi, and Doctor Yanning Xu for their hospitality. He is grateful to Professor Caucher Birkar, Professor Yoshinori Gongyo, and Doctor Yanning Xu for discussions and giving him advice. He thanks Professors Caucher Birkar, Osamu Fujino, and Yoshinori Gongyo for comments. He thanks Doctor Sho Ejiri for answering questions. He is grateful to the referee for useful comments which improved the paper considerably.

**5 Appendix: On non-$\mathbb{R}$-Cartier analogue of non-vanishing theorem**

In this appendix, we give a small remark on the following non-$\mathbb{R}$-Cartier analogue of Theorem 1.1 (2).

**Question 5.1** (cf. Theorem 1.1) Let $(X, B + M)$ be a projective generalized lc pair with data $X' \to X$ and $M'$ such that $B$ is a $\mathbb{Q}$-divisor and $M'$ is a $\mathbb{Q}$-Cartier divisor. Let $A$ be an ample $\mathbb{Q}$-divisor on $X$. Suppose that $K_X + B + A + M$ is pseudo-effective. Then, is there an effective $\mathbb{Q}$-divisor $D_\alpha$ on $X$ such that

$$K_X + B + A + \alpha M \sim_{\mathbb{Q}} D_\alpha$$

for every rational number $\alpha > 1$?

Theorem 1.2 shows that the statement holds in the case where $K_X + B + A$ is the pushdown of a pseudo-effective $\mathbb{R}$-divisor on $X'$. In particular, the statement holds when $M' \equiv 0$.

If we can solve Question 5.1 affirmatively, there is an application to anti-nef canonical divisors of klt varieties. The author learned the following question in a discussion with Gongyo.
Question 5.2 (Non-vanishing for nef anti-canonical divisors) Let $(X, 0)$ be a projective klt pair. Suppose that $-K_X$ is nef. In this situation, is there an effective $\mathbb{Q}$-divisor $D$ on $X$ such that $-K_X \sim \mathbb{Q}D$?

Theorem 5.3 Assume that the statement of Question 5.1 holds for all generalized lc pairs and ample $\mathbb{Q}$-divisors. Then, Question 5.2 can be solved affirmatively.

Proof Assume that the statement of Question 5.1 holds for all generalized lc pairs and ample $\mathbb{Q}$-divisors. Let $(X, 0)$ be a projective klt pair such that $-K_X$ is nef. We may assume that $-K_X$ is not big because otherwise the non-vanishing for $-K_X$ is obvious.

Fix a very ample Cartier divisor $H$ on $X$, and consider the $\mathbb{P}^1$-bundle

$$Y := \mathbb{P}_X(\mathcal{O}_X \oplus \mathcal{O}_X(-H)) \xrightarrow{f} X.$$ 

We put $T$ as the unique section of $\mathcal{O}_Y(1)$. By construction, $(Y, T)$ is plt, $T + f^*H$ is base point free, and we have

$$K_Y + T + (T + f^*H) \sim f^*K_X.$$ 

Let $\pi : Y \to Z$ be the contraction induced by $T + f^*H$. Then, $\pi$ is birational and $T$ is the unique $\pi$-exceptional divisor. We pick an ample $\mathbb{Q}$-divisor $AZ$ on $Z$ such that $T + f^*H \sim \mathbb{Q}\pi^*AZ$.

We put $M = -\pi_*f^*K_X$. Since $K_Y + T + \pi^*AZ - f^*K_X \sim \mathbb{Q}0$, we see that $(Z, M)$ is a projective generalized lc pair with data $Y \to Z$ and $-f^*K_X$ such that $K_Z + M + AZ \sim \mathbb{Q}0$.

By applying the statement of Question 5.1 to $(Z, M)$ and $AZ$, we can find an effective $\mathbb{Q}$-divisor $D_Z$ on $Z$ such that

$$-\pi_*f^*K_X = M \sim \mathbb{Q}K_Z + AZ + 2M \sim \mathbb{Q}D_Z.$$ 

Pick an integer $m > 0$ so that $-m\pi_*f^*K_X \sim mD_Z$. Then we can find a positive integer $n$ such that

$$H^0(Y, \mathcal{O}_Y(-mf^*K_X + nT)) \neq 0.$$ 

For each integer $n' \leq n$, we define $L_{n'} := -mf^*K_X + n'T$, and we consider the exact sequence

$$0 \to H^0(Y, \mathcal{O}_Y(L_{n'-1})) \to H^0(Y, \mathcal{O}_Y(L_{n'})) \to H^0(T, \mathcal{O}_T(L_{n'}|T))$$

which is induced by the exact sequence

$$0 \to \mathcal{O}_Y(L_{n'-1}) \to \mathcal{O}_Y(L_{n'}) \to \mathcal{O}_T(L_{n'}|T) \to 0.$$ 

By using $(T + f^*H)|_T \sim 0$ and $T \simeq X$, we have

$$H^0(T, \mathcal{O}_T(L_{n'}|T)) \simeq H^0(X, \mathcal{O}_X(-mK_X - n'H)).$$
If $n' > 0$, then $H^0(X, \mathcal{O}_X(-mK_X - n'H)) = 0$ because otherwise $-K_X$ is big, which is a contradiction. Therefore, for every $0 < n' \leq n$ we have

$$H^0(Y, \mathcal{O}_Y(L_{n'-1})) \simeq H^0(Y, \mathcal{O}_Y(L_{n'})).$$

Since $H^0(Y, \mathcal{O}_Y(L_n)) \neq 0$, by a descending induction on $n'$, we obtain

$$H^0(Y, -mf^*K_X) \neq 0.$$

Thus there is an effective $\mathbb{Q}$-divisor $D$ on $X$ such that $D \sim_{\mathbb{Q}} -K_X$. We finish the proof of Theorem 5.3. □

References

1. Abramovich, D., Karu, K.: Weak semistable reduction in characteristic 0. Invent. math. 139(2), 241–273 (2000)
2. Birkar, C.: Anti-pluricanonical systems on Fano varieties. Ann. of Math. 19(2), 345–463 (2019)
3. Birkar, C.: Singularities of linear systems and boundedness of Fano varieties. Ann. of Math 193(2), 347–405 (2021)
4. Birkar, C.: Generalised pairs in birational geometry. EMS Surv. Math. Sci. 8, 5–24 (2021)
5. Birkar, C.: On connectedness of non-klt loci of singularities of pairs, to appear in J. Differential Geom
6. Birkar, C., Cascini, P., Hacon, C.D., McKernan, J.: Existence of minimal models for varieties of log general type. J. Am. Math. Soc. 23(2), 405–468 (2010)
7. Birkar, C., Hu, Z.: Polarized pairs, log minimal models, and Zariski decompositions. Nagoya Math. J. 215, 203–224 (2014)
8. Birkar, C., Zhang, D.Q.: Effectivity of Iitaka fibrations and pluricanonical systems of polarized pairs. Publ. Math. Inst. Hautes Études Sci. 123(1), 283–331 (2016)
9. Choi, S.R.: The geography of log models and its applications, PhD Thesis, Johns Hopkins University (2008)
10. Filipazzi, S.: On a generalized canonical bundle formula and generalized adjunction. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) XXI, 1187–1221 (2020)
11. Filipazzi, S., Svaldi, R.: On the connectedness principle and dual complexes for generalized pairs (2020). preprint arXiv:2010.08018
12. Fujino, O.: Special termination and reduction to pl flips. In: Flips for 3-folds and 4-folds, Oxford University Press (2007)
13. Fujino, O.: Non-vanishing theorem for log canonical pairs. J. Algebr. Geom. 20(4), 771–783 (2011)
14. Fujino, O.: Fundamental theorems for the log minimal model program. Publ. Res. Inst. Math. Sci. 47(3), 727–789 (2011)
15. Fujino, O.: Foundations of the minimal model program, vol. 35. Mathematical Society in Japan, Tokyo (2017)
16. Fujino, O., Gongyo, Y.: Log pluricanonical representations and abundance conjecture. Compos. Math. 150(4), 593–620 (2014)
17. Hacon, C.D., Liu, J.: Existence of flips for generalized lc pairs (2021). preprint arXiv:2105.13590
18. Hacon, C.D., Moraga, J.: On weak Zariski decompositions and termination of flips. Math. Res. Lett. 27(5), 1393–1421 (2020)
19. Hacon, C.D., Xu, C.: On finiteness of $B$-representation and semi-log canonical abundance. Minimal Models and extremal rays–Kyoto, 2011. Adv. Stud. Pure Math. 70, 361–378 (2016)
20. Han, J., Li, Z.: Weak Zariski decompositions and log terminal models for generalized polarized pairs, Math. Z. (2022)
21. Han, J., Li, Z.: On accumulation points of pseudo-effective thresholds. Manuscripta Math. 165(3), 537–558 (2021)
22. Han, J., Li, Z.: On Fujita’s conjecture for pseudo-effective thresholds. Math. Res. Lett. 27(2), 377–396 (2020)
23. Han, J., Liu, W.: On numerical nonvanishing for generalized log canonical pairs. Doc. Math. 25, 93–123 (2020)
24. Han, J., Liu, W.: On generalized canonical bundle formula for generically finite morphisms. Ann. Inst. Fourier (Grenoble) 71(5), 2047–2077 (2021)
25. Hashizume, K.: Remarks on special kinds of the relative log minimal model program. Manuscripta Math. 160(3), 285–314 (2019)
26. Hashizume, K.: A class of singularity of arbitrary pairs and log canonicalizations. Asian J. Math. 24(2), 207–238 (2020)
27. Hashizume, K.: Finiteness of log abundant log canonical pairs in log minimal model program with scaling (2020). preprint arXiv:2005.12253
28. Hashizume, K.: Iitaka fibrations for dlt pairs polarized by a nef and log big divisor (2022). preprint arXiv:2203.05467
29. Hashizume, K., Hu, Z.: On minimal model theory for log abundant lc pairs. J. Reine Angew. Math. 767, 109–159 (2020)
30. Kollár, J., Mori, S.: Birational geometry of algebraic varieties. With the collaboration of C. H. Clemens and A. Corti. Translated from the 1998 Japanese original. Cambridge Tracts in Mathematics, 134. Cambridge University Press, Cambridge (1998)
31. Lazić, V., Peternell, T.: On generalised abundance I. Publ. Res. Inst. Math. Sci. 56(2), 353–389 (2020)
32. Lazić, V., Peternell, T.: On generalised abundance II. Peking Math. J. 3(1), 1–46 (2020)
33. Lazić, V., Tsakanikas, N.: On the existence of minimal models for log canonical pairs. Publ. Res. Inst. Math. Sci. 58(2), 311–339 (2022)
34. Li, Z.: Fujita's conjecture on iterated accumulation points of pseudo-effective thresholds. Selecta Math. 27(9), 13 (2021)
35. Liu, J.: Sarkisov program for generalized pairs. Osaka J. Math. 58(4), 899–920 (2021)
36. Moraga, J.: Termination of pseudo-effective 4-fold flips (2018). preprint arXiv:1802.10202
37. Nakayama, N.: Zariski-decomposition and abundance, vol. 14. Mathematical Society of Japan, Tokyo (2004)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.