New Results on Higher-Order Daehee and Bernoulli Numbers and Polynomials

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Abstract

We derive new matrix representation for higher order Daehee numbers and polynomials, the higher order \( \lambda \)-Daehee numbers and polynomials and the twisted \( \lambda \)-Daehee numbers and polynomials of order \( k \). This helps us to obtain simple and short proofs of many previous results on higher order Daehee numbers and polynomials. Moreover, we obtained recurrence relation, explicit formulas and some new results for these numbers and polynomials. Furthermore, we investigated the relation between these numbers and polynomials and Stirling numbers, Nörlund and Bernoulli numbers of higher order. The results of this article gives a generalization of the results derived very recently by El-Desouky and Mustafa [6].

Keywords: Daehee numbers, Daehee polynomials, Higher order Daehee numbers, Higher order Daehee polynomials, Higher order Bernoulli polynomials, Matrix representation.

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1 Introduction

For \( \alpha \in \mathbb{N} \), the Bernoulli polynomials of order \( \alpha \) are defined by, see [1]-[15]

\[
\left( \frac{t}{e^t - 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!}.
\]

When \( x = 0 \), \( B_n^{(\alpha)} = B_n^{(\alpha)}(0) \) are the Bernoulli numbers of order \( \alpha \), defined by

\[
\left( \frac{t}{e^t - 1} \right)^\alpha = \sum_{n=0}^{\infty} B_n^{(\alpha)} \frac{t^n}{n!}.
\]

The Daehee polynomials are defined by, see [11], [12] and [15].

\[
\left( \frac{\log(1 + t)}{t} \right) (1 + t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}.
\]

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In the special case, \( x = 0 \), \( D_n = D_n(0) \) are called the Daehee numbers, defined by
\[
\frac{\log (1 + t)}{t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!}.
\]
(1.4)

The Stirling numbers of the first and second kind are defined, respectively, by
\[
(x)_n = \prod_{i=0}^{n} (x - i) = \sum_{l=0}^{n} s_1(n, l)x^l,
\]
(1.5)

where \( s_1(n, 0) = \delta_{n,0}, \ s_1(n, k) = 0, \) for \( k > n, \) and
\[
x^n = \sum_{k=0}^{n} s_2(n, k)(x)_k,
\]
(1.6)

where \( s_2(n, 0) = \delta_{n,0}, \ s_2(n, k) = 0, \) for \( k > n, \) and \( \delta_{n,k} \) is the kronecker delta.

The Stirling numbers of the second kind have the generating function, see [1, 2, 4, 5] and [7].
\[
(e^t - 1)^m = m! \sum_{l=m}^{\infty} s_2(l, m) \frac{t^l}{l!}.
\]
(1.7)

2 Higher order Daehee Numbers and Polynomials

In this section, we derive an explicit formulas and recurrence relations for the higher order Daehee numbers and polynomials of the first and second kinds. Also the relation between these numbers and Nörlund numbers are given. Furthermore, we introduce the matrix representation of some results for higher order Daehee numbers and polynomial obtained by Kim et al. [8] in terms of Stirling numbers, Nörlund numbers and Bernoulli numbers of higher order and investigate a simple and short proofs of these results.

Kim et al. [8] defined the Daehee numbers of the first kind of order \( k, \) by the following generating function
\[
\sum_{n=0}^{\infty} D^{(k)}_n \frac{t^n}{n!} = \left( \frac{\log (1 + t)}{t} \right)^k.
\]
(2.1)

Next, an explicit formula for \( D^{(k)}_n \) is given by the following theorem.

**Theorem 2.1.** For \( n \in \mathbb{Z}, \ k \in \mathbb{N}, \) we have
\[
D^{(k)}_n = n! \sum_{l_1+l_2+\cdots+l_k=n+k} \frac{(-1)^n}{l_1!l_2!\cdots l_k!}.
\]
(2.2)

**Proof.** From Eq. (2.1), we have
\[
\sum_{n=0}^{\infty} D^{(k)}_n \frac{t^{n+k}}{n!} = (\log (1 + t))^k = \left( \sum_{l=1}^{\infty} \frac{(-1)^{l-1}t^l}{l} \right)^k.
\]

Using Cauchy rule of product of series, we obtain
\[
\sum_{n=0}^{\infty} D^{(k)}_n \frac{t^{n+k}}{n!} = \sum_{r=k}^{\infty} \sum_{l_1+l_2+\cdots+l_k=r} \frac{(-1)^{r-k}}{l_1!l_2!\cdots l_k!} t^r,
\]
let \( r - k = n \), in the right hand side, we have
\[
\sum_{n=0}^{\infty} D_n^{(k)} \frac{t^{n+k}}{n!} = \sum_{n=0}^{\infty} \sum_{l_1+l_2+\ldots+l_k = n+k} (-1)^n l_1 l_2 \cdots l_k t^{n+k}.
\]
Equating the coefficients of \( t^{n+k} \) on both sides yields (2.2). This completes the proof. \( \square \)

**Remark 2.2.** It is worth noting that setting \( k = 1 \) in (2.2), we get \( [8], \text{ Eq. (2.2)} \) as a special case.

Kim et al. [8, 2014, Theorem 1] proved that, see [16], for \( n \in \mathbb{Z}, k \in \mathbb{N} \), we have
\[
D_n^{(k)} = \frac{s_1(n+k,k)}{n+k}.
\]  
(2.3)
We can represent the Daehee numbers of the first kind of order \( k \), by \( (n+1) \times (k+1) \) matrix , \( 0 \leq k \leq n \), as follows
\[
D^{(k)} = \begin{pmatrix}
D_0^{(0)} & D_1^{(0)} & D_2^{(0)} & \cdots & D_k^{(0)} \\
D_0^{(1)} & D_1^{(1)} & D_2^{(1)} & \cdots & D_k^{(1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
D_0^{(n)} & D_1^{(n)} & D_2^{(n)} & \cdots & D_k^{(n)}
\end{pmatrix}.
\]
For example if \( 0 \leq n \leq 3, \ 0 \leq k \leq n \), we have
\[
D^{(k)} = \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & -1/2 & -1 & -3/2 \\
0 & 2/3 & 11/6 & 7/2 \\
0 & -3/2 & -5 & -45/4
\end{pmatrix}.
\]
Kim et al. [8, Theorem 4], proved the following result. For \( n \in \mathbb{Z}, k \in \mathbb{N} \), we have
\[
B_n^{(k)} = \sum_{m=0}^{n} D_m^{(k)} s_2(n,m).
\]  
(2.4)

**Remark 2.3.** We can write this relation in the matrix form as follows.
\[
B^{(k)} = S_2 D^{(k)},
\]  
(2.5)
where \( D^{(k)} \) is \( (n+1) \times (k+1) \), \( 0 \leq k \leq n \), matrix for the Daehee numbers of the first kind of order \( k \) and \( S_2 \) is \( (n+1) \times (n+1) \) lower triangular matrix for the Stirling numbers of the second kind and \( B^{(k)} \) is \( (n+1) \times (k+1) \), \( 0 \leq k \leq n \), matrix for the Bernoulli numbers of order \( k \). For example, if setting \( 0 \leq n \leq 3, \ 0 \leq k \leq n \), in (2.5), we have
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 3 & 1
\end{pmatrix} \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & -1/2 & -1 & -3/2 \\
0 & 2/3 & 11/6 & 7/2 \\
0 & -3/2 & -5 & -45/4
\end{pmatrix} = \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & -1/2 & -1 & -3/2 \\
0 & 1/6 & 5/6 & 2 \\
0 & 0 & -1/2 & -9/4
\end{pmatrix}.
\]
Kim et al. [8, Theorem 3] introduced the following result. For \( n \in \mathbb{Z}, k \in \mathbb{N} \), we have
\[
D_n^{(k)} = \sum_{m=0}^{n} s_1(n,m) B_m^{(k)}.
\]  
(2.6)
We can write this relation in the matrix form as follows
\[ D^{(k)} = S_1 B^{(k)}, \]  
(2.7)
where \( S_1 \) is \((n + 1) \times (n + 1)\) lower triangular matrix for the Striling numbers of the first kind.

For example, if setting \( 0 \leq n \leq 3, \ 0 \leq k \leq n \), in (2.7), we have
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 2 & -3 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & -1/2 & -1 & -3/2 \\
0 & 1/6 & 5/6 & 2 \\
0 & 0 & -1/2 & -9/4
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & -1/2 & -1 & -3/2 \\
0 & 2/3 & 11/6 & 7/2 \\
0 & -3/2 & -5 & -45/4
\end{pmatrix}.
\]

**Remark 2.4.** Using the matrix form (2.7), we easily derive a short proof of Theorem 4 in Kim et al. \[8\].

Multiplying both sides by the Striling numbers of second kind as follows.
\[ S_2 D^{(k)} = S_2 S_1 B^{(k)} = I B^{(k)} = B^{(k)}, \]
where \( I \) is the identity matrix of order \((n + 1)\).

Kim et al. \[8\] defined the Daehee polynomials of order \( k \) by the generating function as follows.
\[
\sum_{n=0}^{\infty} D_n^{(k)}(x) \frac{t^n}{n!} = \left( \frac{\log(1+t)}{t} \right)^k (1+t)^x.
\]
(2.8)

Liu and Srivastava \[14\] define the Nörlund numbers of the second kind \( b_n^{(x)} \) as follows.
\[
\left( \frac{t}{\log(1+t)} \right)^x = \sum_{n=0}^{\infty} b_n^{(x)} t^n.
\]
(2.9)

Next, we find the relation between the Daehee polynomials of order \( k \) and the Nörlund numbers of the second kind \( b_n^{(x)} \) by the following theorem.

**Theorem 2.5.** For \( m \in \mathbb{Z}, \ k \in \mathbb{N} \), we have
\[
D_m^{(k)}(z) = m! \sum_{n=0}^{m} \binom{z}{m-n} b_n^{(-k)}.
\]
(2.10)

**Proof.** From Eq. (2.9), by multiplying both sides by \((1+t)^z\), we have
\[
\left( \frac{t}{\log(1+t)} \right)^x (1+t)^z = \sum_{n=0}^{\infty} b_n^{(x)} t^n (1+t)^z = \sum_{n=0}^{\infty} b_n^{(x)} t^n \sum_{i=0}^{\infty} \binom{z}{i} t^i
\]
\[
= \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} b_n^{(x)} \binom{z}{m-n} t^m = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \binom{z}{m-n} b_n^{(x)} t^m.
\]
(2.11)

Replacing \( x \) by \(-k\) in (2.11), we have
\[
\left( \frac{\log(1+t)}{t} \right)^k (1+t)^z = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \binom{z}{m-n} b_n^{(-k)} t^m
\]
\[
= \sum_{m=0}^{\infty} m! \sum_{n=0}^{m} \binom{z}{m-n} b_n^{(-k)} \frac{t^m}{m!}.
\]
(2.12)

From (2.8) and (2.12), we have (2.10). This completes the proof. \(\square\)
Corollary 2.1. Setting $k = 1$ in (2.10) we have

$$D_m(z) = m! \sum_{n=0}^{m} \binom{z}{m-n} b_n^{(-1)}.$$  \hfill (2.13)

Setting $z = 0$, in (2.10), we have the following relation between Dahee numbers of higher order and Nörlund numbers of the second kind.

Corollary 2.2. For $k \in \mathbb{N}$, by setting $z = 0$ in (2.10) we obtain

$$D_m^{(k)} = m! b_m^{(-k)}.$$  \hfill (2.14)

The relation between the Bernoulli numbers and Bernoulli polynomials of order $k$ are given by Kimura [13], as follows.

$$B_n^{(k)}(x) = \sum_{j=0}^{n} \binom{n}{j} B_j^{(k)} x^{n-j}.$$  \hfill (2.15)

Therefore, we can represent (2.15) in the matrix form

$$B^{(k)}(x) = P(x) B^{(k)},$$  \hfill (2.16)

where $B^{(k)}(x)$ is $(n+1) \times (k+1)$ matrix, $0 \leq k \leq n$ for Bernoulli polynomials of order $k$ as follows

$$B^{(k)}(x) = \begin{pmatrix} B_0^{(k)}(x) & B_1^{(k)}(x) & B_2^{(k)}(x) & \cdots & B_n^{(k)}(x) \\ B_0^{(k)}(x) & B_1^{(k)}(x) & B_2^{(k)}(x) & \cdots & B_n^{(k)}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_0^{(k)}(x) & B_1^{(k)}(x) & B_2^{(k)}(x) & \cdots & B_n^{(k)}(x) \end{pmatrix},$$

where the column $k$ represents the Bernoulli polynomials of order $k$, $B^{(k)}(x)$ is $(n+1) \times (k+1)$ matrix, $0 \leq k \leq n$ for Bernoulli numbers of order $k$ and the matrix $P(x)$, the Pascal matrix, is $(n+1) \times (n+1)$ lower triangular matrix defined by

$$(P(x))_{ij} = \begin{cases} \binom{i}{j} x^{i-j}, & i \geq j, \\ 0, & \text{otherwise} \end{cases}, \quad i, j = 0, 1, \ldots, n.$$  

For example if setting $0 \leq n \leq 3$, $0 \leq k \leq n$ in (2.16), we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x^2 & 2x & 1 & 0 \\ x^3 & 3x^2 & 3x & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1/2 & -1 & -3/2 \\ 0 & 1/6 & 5/6 & 2 \\ 0 & 0 & -1/2 & -9/4 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & x & x - \frac{1}{2} & x - 1 \\ x^2 & x^2 - x + \frac{1}{2} & x^2 - 2x + \frac{5}{2} & x^2 - 3x + 2 \\ x^3 & x^3 - \frac{3}{2}x^2 + \frac{1}{2} & x^3 - 3x^2 + \frac{5}{2}x - \frac{1}{2} & x^3 - \frac{9}{2}x^2 + 6x - \frac{9}{4} \end{pmatrix}.$$  

Kim et al. [8, Theorem 5] introduced the following result. For $n \in \mathbb{Z}$, $k \in \mathbb{N}$,

$$D_n^{(k)}(x) = \sum_{m=0}^{n} s_1(n, m) B_m^{(k)}(x).$$  \hfill (2.17)
We can write this relation in the matrix form as follows

\[
D^{(k)}(x) = S_1 B^{(k)}(x),
\]

where \(D^{(k)}(x)\) is \((n+1) \times (k+1)\) matrix for the Dahee polynomials of the first kind with order \(k\) and \(B^{(k)}(x)\) is \((n+1) \times (k+1)\) matrix for the Bernoulli polynomials of order \(k\).

For example, if setting \(0 \leq n \leq 3,\ 0 \leq k \leq n\), in (2.18), we have

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 2 & -3 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 \\
x & x - \frac{1}{2} & x - 1 & x - \frac{3}{2} \\
x^2 - x & x^2 - 2x + \frac{3}{2} & x^2 - 3x + \frac{11}{6} & x^2 - 4x + \frac{7}{2} \\
x^3 - 3x^2 + 2x & x^3 - \frac{9}{2}x^2 + \frac{11}{2}x - \frac{3}{2} & x^3 - 6x^2 + \frac{21}{2}x - 5 & x^3 - \frac{15}{2}x^2 + 17x - \frac{45}{4}
\end{bmatrix}
= \begin{bmatrix}
1 \\
x \\
x^2 - x \\
x^3 - 3x^2 + 2x
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 \\
x & x - \frac{1}{2} & x - 1 & x - \frac{3}{2} \\
x^2 - x & x^2 - 2x + \frac{3}{2} & x^2 - 3x + \frac{11}{6} & x^2 - 4x + \frac{7}{2} \\
x^3 - 3x^2 + 2x & x^3 - \frac{9}{2}x^2 + \frac{11}{2}x - \frac{3}{2} & x^3 - 6x^2 + \frac{21}{2}x - 5 & x^3 - \frac{15}{2}x^2 + 17x - \frac{45}{4}
\end{bmatrix}.
\]

Kim et al. [8, Theorem 7] introduced the following result. For \(n \in \mathbb{Z},\ k \in \mathbb{N}\),

\[
B^{(k)}_n(x) = \sum_{m=0}^{n} D^{(k)}_m(x) S_2(n, m).
\]

We can write Eq. (2.19) in the matrix form as follows

\[
B^{(k)}(x) = S_2 D^{(k)}(x).
\]

For example, if setting \(0 \leq n \leq 3,\ 0 \leq k \leq n\), in (2.20), we have

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 3 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
x \\
x^2 - x \\
x^3 - 3x^2 + 2x
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 \\
x - \frac{1}{2} & x - 1 & x - \frac{3}{2} \\
x^2 - 2x + \frac{3}{2} & x^2 - 3x + \frac{11}{6} & x^2 - 4x + \frac{7}{2} \\
x^3 - 6x^2 + \frac{21}{2}x - 5 & x^3 - \frac{15}{2}x^2 + 17x - \frac{45}{4} \\
x^3 - \frac{9}{2}x^2 + \frac{11}{2}x - \frac{3}{2} & x^3 - 6x^2 + \frac{21}{2}x - 5 & x^3 - \frac{15}{2}x^2 + 17x - \frac{45}{4} \\
x^3 - 3x^2 + 2x & x^3 - \frac{9}{2}x^2 + \frac{11}{2}x - \frac{3}{2} & x^3 - 6x^2 + \frac{21}{2}x - 5 & x^3 - \frac{15}{2}x^2 + 17x - \frac{45}{4}
\end{bmatrix}
= \begin{bmatrix}
1 \\
x \\
x^2 - x \\
x^3 - 3x^2 + 2x
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 \\
x & x - \frac{1}{2} & x - 1 & x - \frac{3}{2} \\
x^2 - x & x^2 - 2x + \frac{3}{2} & x^2 - 3x + \frac{11}{6} & x^2 - 4x + \frac{7}{2} \\
x^3 - 3x^2 + 2x & x^3 - \frac{9}{2}x^2 + \frac{11}{2}x - \frac{3}{2} & x^3 - 6x^2 + \frac{21}{2}x - 5 & x^3 - \frac{15}{2}x^2 + 17x - \frac{45}{4}
\end{bmatrix}.
\]

Remark 2.6. We can prove Theorem 7 in Kim et al. [8] by using the matrix form (2.18) as follows. Multiplying both sides of (2.18) by the Striling number of second kind, we have

\[
S_2 D^{(k)}(x) = S_2 S_1 B^{(k)}(x) = I B^{(k)}(x) = B^{(k)}(x).
\]

Kim et al. [8] defined the Dahee numbers of the second kind of order \(k\) by the generating function as follows.

\[
\sum_{n=0}^{\infty} \hat{D}_n^{(k)}(t) \frac{t^n}{n!} = \left( \frac{(1 - t) \log(1 - t)}{-t} \right)^k.
\]

Kim et al. [8, Theorem 8] introduced the following result. For \(n \in \mathbb{Z},\ k \in \mathbb{N}\),

\[
\hat{D}_n^{(k)} = \sum_{l=0}^{n} \binom{n}{l} B_2^{(k)}.
\]
where \[ \binom{n}{t} = (-1)^{n-t}s_1(n, l) = |s_1(n, k)| = \mathfrak{s}(n, k), \] where \( \mathfrak{s}(n, k) \) is the signless Stirling numbers of the first kind, see \cite{2} and \cite{4, 5}.

We can write this theorem in the matrix form as follows

\[
\hat{D}^{(k)} = \mathfrak{S} \mathcal{B}^{(k)},
\] (2.23)

where \( \hat{D}^{(k)} \) is \((n + 1) \times (k + 1)\) matrix of Dahee numbers of the second kind with order \( k \) and \( \mathfrak{S} \) is \((n + 1) \times (n + 1)\) lower triangular matrix for the signless Stirling numbers of first kind.

For example, if setting \( 0 \leq n \leq 3, \ 0 \leq k \leq n \) in (2.23), we have

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & -1/2 & -1 & -3/2 \\
0 & 1/6 & 5/6 & 2 \\
0 & 0 & -1/2 & -9/4
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & -1/2 & -1 & -3/2 \\
0 & -1/3 & -1/6 & 1/2 \\
0 & -1/2 & 0 & 3/4
\end{pmatrix}.
\]

Kim et al. \cite{8, Theorem 9} introduced the following result. For \( n \in \mathbb{Z}, \ k \in \mathbb{N} \), we have

\[
B^{(k)}_n = \sum_{m=0}^{n} (-1)^{n-m}s_2(n, m)\hat{D}^k_m([k]).
\] (2.24)

We can write Eq. (2.24) in the matrix form as follows

\[
\mathcal{B}^{(k)} = \tilde{S}_2 \hat{D}^{(k)}.
\] (2.25)

where \( \tilde{S}_2 \) is \((n + 1) \times (n + 1)\) lower triangular matrix for signed Stirling numbers of the second kind defined by

\[
(\tilde{S}_2)_{ij} = \begin{cases} (-1)^{i-j}s_2(i, j), & i \geq j, \\ 0, & \text{otherwise}. \end{cases}, \ i, j = 0, 1, \ldots, n.
\]

For example, if setting \( 0 \leq n \leq 3, \ 0 \leq k \leq n \) in (2.25), we have

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & -1/2 & -1 & -3/2 \\
0 & -1/3 & -1/6 & 1/2 \\
0 & -1/2 & 0 & 3/4
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & -1/2 & -1 & -3/2 \\
0 & 1/6 & 5/6 & 2 \\
0 & 0 & -1/2 & -9/4
\end{pmatrix}.
\]

**Remark 2.7.** We can prove Theorem 9 in Kim et al. \cite{8} by using the matrix form (2.23) as follows. Multiplying both sides of (2.23) by the matrix of sign Stirling numbers of second kind \( \tilde{S}_2 \) we have

\[
\tilde{S}_2 \hat{D}^{(k)} = \tilde{S}_2 \mathfrak{S} \mathcal{B}^{(k)} = \mathcal{I} \mathcal{B}^{(k)} = \mathcal{B}^{(k)},
\]

we obtain Eq. (2.25), where we used the identity, \( \tilde{S}_2 \mathfrak{S} = \mathcal{I} \).

Kim et al. \cite{8} defined the Dahee polynomials of the second kind of order \( k \) by the generating function as follows.

\[
\sum_{n=0}^{\infty} \hat{D}^k_n([k])x^n \frac{t^n}{n!} = \left( \frac{1 - t}{1 - t^x} \right) \frac{k}{-t} (1 - t)^x.
\] (2.26)

Kim et al. \cite{8, Eq. (31)} introduced the following result. For \( n \in \mathbb{Z}, \ k \in \mathbb{N} \),

\[
\hat{D}^k_n([k])x = \sum_{m=0}^{n} (-1)^{n-m}s_1(n, m)B_m^{(k)}(-x).
\] (2.27)
Eq. (2.27) is equivalent to
\[ \hat{D}_n^k[(k)](x) = \sum_{m=0}^{n} s(n, m) B_m^{(k)}(-x). \] (2.28)

We can write Eq. (2.28) in the matrix form as follows
\[ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -x & -x - \frac{1}{2} & -x - 1 & -x - \frac{3}{2} \\ x^2 & x^2 + x + \frac{1}{6} & x^2 + 2x + \frac{5}{6} & x^2 + 3x + 2 \\ -x^3 - \frac{3}{2}x^2 - \frac{1}{2}x & -x^3 - 3x^2 - \frac{9}{2}x - \frac{1}{2} & -x^3 - \frac{15}{2}x^2 - 6x - \frac{9}{4} \end{bmatrix} = \begin{bmatrix} 1 \\ -x \\ -x - \frac{1}{2} \\ x^2 - x \end{bmatrix} \begin{bmatrix} 1 \\ -x - \frac{1}{2} \\ -x - 1 \\ -x - \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 \\ x^2 - \frac{1}{2} \\ x^2 + x - \frac{1}{6} \\ x^2 + 2x + \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ -x - \frac{1}{2} \\ -x - 1 \\ -x - \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 \\ x^2 + x + \frac{1}{6} \\ x^2 + 2x + \frac{5}{6} \\ x^2 + 3x + 2 \end{bmatrix} \begin{bmatrix} 1 \\ -x^3 - \frac{3}{2}x^2 - \frac{1}{2} \\ -x^3 - 3x^2 - \frac{9}{2}x - \frac{1}{2} \\ -x^3 - \frac{15}{2}x^2 - 6x - \frac{9}{4} \end{bmatrix}. \]

Kim et al. [8, Theorem 11] introduced the following result. For \( n \in \mathbb{Z}, \ k \in \mathbb{N}, \)
\[ B_n^{(k)}(-x) = \sum_{m=0}^{n} (-1)^{n-m} s_2(n, m) \hat{D}_m^k[(k)](x). \] (2.30)

We can write Eq. (2.30) in the matrix form as follows
\[ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -x & -x - \frac{1}{2} & -x - 1 & -x - \frac{3}{2} \\ x^2 - x & x^2 - \frac{1}{2} & x^2 + x - \frac{1}{6} & x^2 + 2x + \frac{1}{2} \\ 3x^2 - x^3 - 2x & 3x^2 - x^3 - \frac{9}{2}x - \frac{1}{2} & \frac{3x^2}{2} - x^3 & x - \frac{3x^2}{2} - x^3 + \frac{3}{4} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -x & -x - \frac{1}{2} & -x - 1 & -x - \frac{3}{2} \\ x^2 - \frac{1}{2} & x^2 + x + \frac{1}{6} & x^2 + 2x + \frac{5}{6} & x^2 + 3x + 2 \\ -x^3 - \frac{3}{2}x^2 - \frac{1}{2} & -x^3 - 3x^2 - \frac{9}{2}x - \frac{1}{2} & -x^3 - \frac{15}{2}x^2 - 6x - \frac{9}{4} \end{bmatrix}. \]

Remark 2.8. We can prove Eq. (2.27), [8, Theorem 11], directly by using the matrix form (2.29) as follows. Multiplying both sides of (2.29) by \( \tilde{S}_2 \) as follows.
\[ \tilde{S}_2 \hat{D}_n^k(x) = \tilde{S}_2 \otimes B_n^{(k)}(-x) = I B_n^{(k)}(-x) = B_n^{(k)}(-x), \]
thus, we have Eq. (2.37).
3 The $\lambda$-Daehee Numbers and Polynomials of Higher Order

In this section we introduce the matrix representation for the $\lambda$-Daehee numbers and polynomials of higher order given by Kim et al. [9]. Hence, we can derive these results in matrix representation and prove these results simply by using the given matrix forms.

The $\lambda$-Daehee polynomials of the first kind with order $k$ can be defined by the generating function

$$
\left( \frac{\lambda \log (1 + t)}{(1 + t)^\lambda - 1} \right)^k (1 + t)^x = \sum_{n=0}^{\infty} D_{n,\lambda}^{(k)}(x) \frac{t^n}{n!}.
$$

(3.1)

When $x = 0$, $D_{n,\lambda}^{(k)}(0)$ are called the $\lambda$-Daehee numbers of order $k$.

$$
\left( \frac{\lambda \log (1 + t)}{(1 + t)^\lambda - 1} \right)^k = \sum_{n=0}^{\infty} D_{n,\lambda}^{(k)} \frac{t^n}{n!}.
$$

(3.2)

It is easy to see that $D_n^{(k)}(x) = D_{n,1}^{(k)}(x)$ and $D_{n,\lambda}(x) = D_{n,\lambda}^{(1)}(x)$. Kim et al. [9, Theorem 3] obtained the following results. For $n \geq 0$, $k \in \mathbb{N}$,

$$
D_{n,\lambda}^{(k)}(x) = \sum_{m=0}^{n} s_1(n,m) \lambda^m B_m^{(k)} \left( \frac{x}{\lambda} \right),
$$

(3.3)

and

$$
\lambda^n B_n^{(k)} \left( \frac{x}{\lambda} \right) = \sum_{m=0}^{n} s_2(n,m) D_{m,\lambda}^{(k)}(x),
$$

(3.4)

we can write these results in the following matrix forms

$$
D_{\lambda}^{(k)}(x) = S_1 \Lambda B^{(k)} \left( \frac{x}{\lambda} \right),
$$

(3.5)

and

$$
\Lambda B^{(k)} \left( \frac{x}{\lambda} \right) = S_2 D_{\lambda}^{(k)}(x),
$$

(3.6)

where, $D_{\lambda}^{(k)}(x)$ is $(n + 1) \times (k + 1)$ matrix for the $\lambda$-Daehee polynomials of the first kind with order $k$; $B^{(k)}(x/\lambda)$ is $(n + 1) \times (k + 1)$ matrix for the Bernoulli polynomials of order $k$, when $x \to x/\lambda$ and $\Lambda$ is $(n + 1) \times (n + 1)$ diagonal matrix with elements, $(\Lambda)_{ii} = \lambda^i$, $i = j = 0, 1, \cdots, n$.

For example, if setting $0 \leq n \leq 3$, $0 \leq k \leq n$, in (3.3), we have

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 2 & -3 & 1
\end{pmatrix}
\times
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 \\
0 & 0 & \lambda^2 & 0 \\
0 & 0 & 0 & \lambda^3
\end{pmatrix}
= \begin{pmatrix}
x & x^2 - \frac{x^2}{2} + \frac{x}{2} & \frac{x^2}{2} - \frac{x}{2} & \frac{x^2}{2} - \frac{x}{2} \\
\frac{x^2}{2} & \frac{x^2}{2} - \frac{x^2}{2} + \frac{x}{2} & \frac{x^2}{2} - \frac{x}{2} + \frac{5}{6} & \frac{x^2}{2} - \frac{x}{2} - \frac{1}{2} \\
\frac{x^2}{2} & \frac{x^2}{2} - \frac{x^2}{2} + \frac{x}{2} & \frac{x^2}{2} - \frac{x}{2} + \frac{5}{6} & \frac{x^2}{2} - \frac{x}{2} - \frac{1}{2} \\
\frac{x^2}{2} & \frac{x^2}{2} - \frac{x^2}{2} + \frac{x}{2} & \frac{x^2}{2} - \frac{x}{2} + \frac{5}{6} & \frac{x^2}{2} - \frac{x}{2} - \frac{1}{2}
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
x & \frac{x^2}{2} - \frac{x^2}{2} + \frac{x}{2} & \frac{x^2}{2} - \frac{x}{2} + \frac{5}{6} & \frac{x^2}{2} - \frac{x}{2} - \frac{1}{2} \\
x(x - 1) & \frac{x^2}{6} - \frac{x^2}{2} + \frac{x}{2} + x^2 - x & \frac{5x^2}{6} - 2\lambda x + \lambda + x^2 - x & 2\lambda^2 - 3\lambda x + \frac{5}{3} + x^2 - x \\
D_{n,\lambda}^{(0)}(x) & D_{n,\lambda}^{(1)}(x) & D_{n,\lambda}^{(2)}(x) & D_{n,\lambda}^{(3)}(x)
\end{pmatrix}
\]
Using Cauchy rule of product of series, we obtain

\[ D_{3,\lambda}^{(0)}(x) = x(x-1)(x-2), \quad D_{3,\lambda}^{(1)}(x) = -\frac{1}{2}(\lambda - 2x + 2)(\lambda - 2x + x^2 - \lambda x), \]
\[ D_{3,\lambda}^{(2)}(x) = -\frac{1}{2}(\lambda - x + 1)(\lambda^2 - 4\lambda x + 4\lambda + 2x^2 - 4x), \]
\[ D_{3,\lambda}^{(3)}(x) = -\frac{1}{4}(3\lambda - 2x + 2)(3\lambda^2 - 6\lambda x + 6\lambda + 2x^2 - 4x). \]

**Remark 3.1.** In fact, we can prove Eq. (3.6), simply by multiplying Eq. (3.5) by \( S_2 \) as follows.

\[ S_2 D^{(k)}_{\lambda}(x) = S_2 S_1 \Lambda B^{(k)}(\frac{x}{\lambda}) = \Lambda AB^{(k)}(\frac{x}{\lambda}) = \Lambda B^{(k)}(\frac{x}{\lambda}). \]

The following Theorem gives the relation between the Daheepolynomials of higher order and \( \lambda \)-Dahee polynomials of higher order.

**Theorem 3.2.** For \( m \geq 0 \), we have

\[ D^{(k)}_{m,\lambda}(\lambda x) = m! \sum_{n=0}^{m} \sum_{i_1+i_2+\cdots+i_n=m} \frac{D^{(k)}_{n}(x)}{n!} \left( \frac{\lambda}{i_1} \right) \left( \frac{\lambda}{i_2} \right) \cdots \left( \frac{\lambda}{i_n} \right). \]  

(3.7)

**Proof.** From (2.8), replacing \((1+t)\) by \((1+t)^{\lambda}\), we have

\[ \left( \frac{\lambda \log (1+t)}{(1+t)^{\lambda} - 1} \right)^k (1+t)^{\lambda x} = \sum_{n=0}^{\infty} D^{(k)}_{n}(x) \frac{(1+t)^{\lambda} - 1}{n!}, \]

thus from (3.1), we get

\[ \sum_{m=0}^{\infty} D^{(k)}_{m,\lambda}(\lambda x) \frac{t^m}{m!} = \sum_{n=0}^{\infty} \frac{D^{(k)}_{n}(x)}{n!} \left( \sum_{i=0}^{\lambda} \left( \frac{\lambda}{i} \right) t^i - 1 \right)^n = \sum_{n=0}^{\infty} \frac{D^{(k)}_{n}(x)}{n!} \left( \sum_{i=1}^{\lambda} \left( \frac{\lambda}{i} \right) t^i \right)^n. \]

Using Cauchy rule of product of series, we obtain

\[ \sum_{m=0}^{\infty} D^{(k)}_{m,\lambda}(\lambda x) \frac{t^m}{m!} = \sum_{n=0}^{\infty} \frac{D^{(k)}_{n}(x)}{n!} \sum_{m=0}^{\infty} \sum_{i_1+i_2+\cdots+i_n=m} \left( \frac{\lambda}{i_1} \right) \cdots \left( \frac{\lambda}{i_n} \right) t^m \]
\[ = \sum_{m=0}^{\infty} m! \sum_{n=0}^{m} \sum_{i_1+i_2+\cdots+i_n=m} \frac{D^{(k)}_{n}(x)}{n!} \left( \frac{\lambda}{i_1} \right) \cdots \left( \frac{\lambda}{i_n} \right) t^m. \]

Equating the coefficients of \( t^m \) on both sides yields (3.7). This completes the proof.

Setting \( x = 0 \), in (3.7), we have the following corollary as a special case.

**Corollary 3.1.** For \( m \geq 0 \), we have

\[ D^{(k)}_{m,\lambda} = m! \sum_{n=0}^{m} \sum_{i_1+i_2+\cdots+i_n=m} \frac{D^{(k)}_{n}(x)}{n!} \left( \frac{\lambda}{i_1} \right) \left( \frac{\lambda}{i_2} \right) \cdots \left( \frac{\lambda}{i_n} \right). \]  

(3.8)
Remark 3.3. We can write (3.10) and (3.11), respectively, in the following matrix forms as follows.

\[
\begin{pmatrix}
 \lambda \log (1 + t) \\
 (1 + t)^{\lambda} - 1
\end{pmatrix}
\left(1 + t\right)^{\lambda k + \lambda x} = \sum_{n=0}^{\infty} \hat{D}_{n,\lambda}^{(k)}(x) \frac{t^n}{n!},
\]

(3.9)

Kim et al. [9, Theorem 5] proved that

\[
\hat{D}_{m,\lambda}^{(k)}(x) = \sum_{l=0}^{m} s_1(m, l) \lambda^l B_l^{(k)} \left(k + \frac{x}{\lambda}\right),
\]

(3.10)

and

\[
\lambda^m B_m^{(k)} \left(k + \frac{x}{\lambda}\right) = \sum_{n=0}^{m} s_2(m, n) \hat{D}_{n,\lambda}^{(k)}(x).
\]

(3.11)

Also, Kim et al. [9, Eq. (35)] introduced the following result

\[
B_n^{(k)}(k - x) = (-1)^n B_n^{(k)}(x).
\]

(3.12)

**Remark 3.3.** We can write (3.10) and (3.11), respectively, in the following matrix forms

\[
\hat{D}_{\lambda}^{(k)}(x) = S_1 \Lambda_1 B^{(k)} \left(-\frac{x}{\lambda}\right),
\]

(3.13)

and

\[
\Lambda_1 B^{(k)} \left(-\frac{x}{\lambda}\right) = S_2 \hat{D}_{\lambda}^{(k)}(x),
\]

(3.14)

where \(\hat{D}_{\lambda}(x)\) is \((n + 1) \times (n + 1)\) matrix for the \(\lambda\)-Dahee polynomials of the second kind of order \(k\) and \(\Lambda_1\) is \((n + 1) \times (n + 1)\) diagonal matrix with elements \((\Lambda_1)_{i,i} = (-\lambda)^i\), for \(i = j = 0, 1, \cdots, n\).

For example, if setting \(0 \leq n \leq 3\), \(0 \leq k \leq n\), in (3.13), we have

\[
\begin{pmatrix}
 1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1
\end{pmatrix}
\left(
\begin{pmatrix}
 1 & 0 & 0 & 0 \\
 0 & -\lambda & 0 & 0 \\
 0 & 0 & \lambda^2 & 0 \\
 0 & 0 & 0 & -\lambda^3
\end{pmatrix}
\right) =
\begin{pmatrix}
 1 & \frac{1}{\lambda} & \frac{1}{\lambda^2} & \frac{1}{\lambda^3} \\
 \frac{x}{\lambda} & \frac{x^2}{\lambda} & \frac{x^3}{\lambda} & \frac{x^4}{\lambda} \\
 \frac{1}{\lambda^2} & \frac{1}{\lambda^3} & \frac{1}{\lambda^4} & \frac{1}{\lambda^5} \\
 \frac{1}{\lambda^3} & \frac{1}{\lambda^4} & \frac{1}{\lambda^5} & \frac{1}{\lambda^6}
\end{pmatrix}
\]

\(x(x - 1)\)

\[
\hat{D}_{\lambda}^{(0)}(x) = \hat{D}_{\lambda}^{(1)}(x) = \hat{D}_{\lambda}^{(2)}(x) = \hat{D}_{\lambda}^{(3)}(x) =
\]

where

\[
\hat{D}_{\lambda}^{(0)}(x) = x(x - 1)(x - 2),
\]

\[
\hat{D}_{\lambda}^{(1)}(x) = \frac{1}{2}(\lambda + x - 1)(\lambda^2 + 4\lambda x - 4\lambda + 2x^2 - 4x),
\]

\[
\hat{D}_{\lambda}^{(2)}(x) = \frac{1}{4}(3\lambda + 2x - 2)(3\lambda^2 + 6\lambda x - 6\lambda + 2x^2 - 4x)),
\]

and

\[
\hat{D}_{\lambda}^{(3)}(x) = \frac{1}{4}(3\lambda + 2x - 2)(3\lambda^2 + 6\lambda x - 6\lambda + 2x^2 - 4x)).
\]

**Remark 3.4.** We can prove Eq. (3.11) easily by using the matrix form, multiplying Eq. (3.13) by \(S_2\) as follows.

\[
S_2 \hat{D}_{\lambda}^{(k)}(x) = S_2 S_1 \Lambda_1 B^{(k)} \left(-\frac{x}{\lambda}\right) = I \Lambda_1 B^{(k)} \left(-\frac{x}{\lambda}\right) = \Lambda_1 B^{(k)} \left(-\frac{x}{\lambda}\right).
\]
4 The Twisted $\lambda$-Daehee Numbers and Polynomials of Higher Order

Kim et al. [10] defined the twisted $\lambda$-Daehee polynomials of the first kind of order $k$ by the generating function

$$
\left( \frac{\lambda \log (1 + \xi t)}{(1 + \xi t)^\lambda - 1} \right)^k (1 + \xi t)^x = \sum_{n=0}^{\infty} D_{n,\xi,\lambda}^{(k)}(x|\lambda) \frac{t^n}{n!}.
$$

(4.1)

In the special case, $x = 0$, $D_{n,\xi,\lambda}^{(k)} = D_{n,\xi}^{(k)}(0|\lambda)$ are called the twisted $\lambda$-Daehee numbers of the first kind of order $k$.

$$
\left( \frac{\lambda \log (1 + \xi t)}{(1 + \xi t)^\lambda - 1} \right)^k = \sum_{n=0}^{\infty} D_{n,\xi,\lambda}^{(k)} \frac{t^n}{n!}.
$$

(4.2)

The twisted Bernoulli polynomials of order $r \in \mathbb{N}$ are defined by the generating function, see [3]

$$
\left( \frac{t}{\xi e^t - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} B_{n,\xi}^{(r)}(x) t^n, \quad (4.3)
$$

The relation between the twisted $\lambda$-Daehee polynomials and $\lambda$-Daehee polynomials of order $k$, can be obtained by the following corollary.

Corollary 4.1. For $n \geq 0$, $k \in \mathbb{N}$, we have

$$
D_{n,\xi,\lambda}^{(k)}(x|\lambda) = \xi^n D_{n,\lambda}^{(k)}(x).
$$

(4.4)

Proof. Replacing $t$ with $\xi t$ in (4.1), we have

$$
\left( \frac{\lambda \log (1 + \xi t)}{(1 + \xi t)^\lambda - 1} \right)^k (1 + \xi t)^x = \sum_{n=0}^{\infty} D_{n,\lambda}^{(k)}(x) \frac{(\xi t)^n}{n!} = \sum_{n=0}^{\infty} \xi^n D_{n,\lambda}^{(k)}(x) \frac{t^n}{n!}, \quad (4.5)
$$

hence by virtue of (4.1) and (4.5), we get (4.4). This completes the proof.

If we put in (4.4), $x = 0$ and $\lambda = 1$, respectively, we have

$$
D_{n,\xi}^{(k)}(x) = \xi^n D_{n,\lambda}^{(k)}(x).
$$

and

$$
D_{n,\xi,\lambda}^{(k)} = \xi^n D_{n,\lambda}^{(k)}.
$$

Kim et al. [10, Theorem 1] proved the following relation. For $m \in \mathbb{Z}$, $k \in \mathbb{N}$,

$$
D_m^{(k)}(x|\lambda) = \xi^m \sum_{l=0}^{m} S_1(m, l) \lambda^l B_l^{(k)} \left( \frac{x}{\lambda} \right),
$$

(4.6)

and

$$
\lambda^m B_m^{(k)} \left( \frac{x}{\lambda} \right) = \sum_{n=0}^{m} D_n^{(k)}(x|\lambda) \xi^{-n-x} s_2(m, n).
$$

(4.7)

where $B_m^{(k)} (\frac{x}{\lambda})$ is defined by Kim et al. [10, Eq. 15], as follows

$$
\left( \frac{\lambda t}{\xi^m e^{\lambda t} - 1} \right)^k (\xi e^t)^x = \xi^x \sum_{m=0}^{\infty} \lambda^m B_m^{(k)} \left( \frac{x}{\lambda} \right) \frac{t^m}{m!}.
$$

(4.8)
Remark 4.1. We can write (4.6) in the following matrix form
\[ D^{(k)}_{\xi}(x|\lambda) = \Xi S_1 \Lambda B^{(k)} \left( \frac{x}{\lambda^3} \right), \]
where \( D^{(k)}_{\xi}(x|\lambda) \) is \((n+1) \times (k+1)\) matrix for the twisted Dahee numbers of the first kind of the order \( k \) and \( \Xi \) is \((n+1) \times (n+1)\) diagonal matrix with elements \((\Xi)_{ii} = \xi^i\) for \( i = j = 0, 1, \cdots, n \).

For example, if setting \( 0 \leq n \leq 3, \ 0 \leq k \leq n \), in (4.9), we have

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \xi & 0 & 0 \\
0 & 0 & \xi^2 & 0 \\
0 & 0 & 0 & \xi^3
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 \\
0 & 0 & \lambda^2 & 0 \\
0 & 0 & 0 & \lambda^3
\end{pmatrix}
\times
\begin{pmatrix}
\frac{1}{x^2} & \frac{1}{x} & -\frac{1}{2} & 0 \\
\frac{1}{x^2} & \frac{1}{x} & 0 & -\frac{1}{2} \\
\frac{5}{x^3} & \frac{1}{x} & 0 & 0 \\
\frac{5}{x^3} & \frac{1}{x} & 0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 \\
0 & 0 & \lambda^2 & 0 \\
0 & 0 & 0 & \lambda^3
\end{pmatrix}
\begin{pmatrix}
\xi x \\
\xi^2 x(x-1) \\
D^{(0)}_{3,\xi}(x|\lambda) \\
D^{(1)}_{3,\xi}(x|\lambda)
\end{pmatrix}
\begin{pmatrix}
\xi(\frac{5}{6} \lambda^2) \\
\xi(\frac{5}{6} \lambda^2 - 2\lambda x) \\
\xi(\frac{5}{6} \lambda^2 + 2\lambda x) \\
\xi(\frac{5}{6} \lambda^2 - \lambda x)
\end{pmatrix}
\begin{pmatrix}
1 \\
-\frac{\lambda - 2x}{\lambda x} \\
D^{(2)}_{3,\xi}(x|\lambda) \\
\lambda x - 2x - x)
\end{pmatrix}
\begin{pmatrix}
-\frac{\lambda - 2x}{\lambda x} \\
D^{(3)}_{3,\xi}(x|\lambda)
\end{pmatrix}.
\]

where

\[
D^{(0)}_{3,\xi}(x|\lambda) = \xi^3 x(x-1)(x-2) \quad D^{(1)}_{3,\xi}(x|\lambda) = -\frac{\xi^3}{2} (\lambda - 2x + 2)(\lambda - 2x + 2 - \lambda x),
\]
\[
D^{(2)}_{3,\xi}(x|\lambda) = -\frac{\xi^3}{2} (\lambda - x + 1)(\lambda^2 - 4\lambda x + 4\lambda + 2\lambda x - 4x),
\]
\[
D^{(3)}_{3,\xi}(x|\lambda) = -\frac{\xi^3}{2} (3\lambda - 2x + 2)(3\lambda^2 - 6\lambda x + 6\lambda + 2\lambda x - 4x).
\]

Remark 4.2. In fact, it seems that the statement in (4.7) is not correct, the second equation of, Kim et al. [10, Theorem 1]. From (4.9), multiplying both sides by \( \Xi^{-1} \), we have,
\[
\Xi^{-1} D^{(k)}_{\xi}(x|\lambda) = \Xi^{-1} \Xi S_1 \Lambda B^{(k)} \left( \frac{x}{\lambda^3} \right) = S_1 \Lambda B^{(k)} \left( \frac{x}{\lambda^3} \right),
\]
then multiplying both sides by \( S_2 \), we have
\[
S_2 \Xi^{-1} D^{(k)}_{\xi}(x|\lambda) = S_2 S_1 \Lambda B^{(k)} \left( \frac{x}{\lambda^3} \right) = \Lambda B^{(k)} \left( \frac{x}{\lambda^3} \right),
\]
From (4.7) and (4.10), it is clear that there is a contradiction.

In the following theorem we obtained the corrected relation as follows.

Theorem 4.3. For \( m \in \mathbb{Z}, \ k \in \mathbb{N} \), we have
\[
\lambda^m B^{(k)}_m \left( \frac{x}{\lambda^3} \right) = \sum_{n=0}^{m} D^{(k)}_{n,\xi}(x|\lambda) \xi^{-n} s_2(m, n).
\]
Proof. From Eq. (4.11), replacing \( t \) by \( (e^t - 1)/\xi \), we have
\[
\left( \frac{\lambda \log \left( 1 + \frac{\xi(e^t-1)}{\xi} \right)}{1 + \frac{\xi(e^t-1)}{\xi}} - 1 \right)^k \left( 1 + \frac{\xi(e^t-1)}{\xi} \right)^x = \sum_{n=0}^{\infty} D_{n,\xi}(x|\lambda) \frac{(e^t-1)^n}{n!\xi^n}
\]
\[
\left( \frac{\lambda t}{e^\lambda - 1} \right)^k e^{\lambda t} = \sum_{n=0}^{\infty} D_{n,\xi}(x|\lambda) \frac{(e^t-1)^n}{n!\xi^n}.
\] (4.12)

Substituting from (1.7) into (4.12), we have
\[
\left( \frac{\lambda t}{e^\lambda - 1} \right)^k e^{\lambda t(\xi)} = \sum_{n=0}^{\infty} D_{n,\xi}(x|\lambda) \xi^{-n} \sum_{m=n}^{\infty} s_2(m, n) \frac{t^m}{m!}
\]
\[
= \sum_{m=0}^{\infty} \sum_{n=0}^{m} D_{n,\xi}(x|\lambda) \xi^{-n} s_2(m, n) \frac{t^m}{m!}.
\] (4.13)

From (1.1) and (4.13), we have
\[
\sum_{m=0}^{\infty} \lambda^m D_{m}(x|\lambda) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \sum_{n=0}^{m} D_{n,\xi}(x|\lambda) \xi^{-n} s_2(m, n) \frac{t^m}{m!}.
\] (4.14)

Equating the coefficients of \( t^m \) on both sides gives (4.11). This completes the proof.

Moreover, we can represent Equation (4.11), in the following matrix form as (4.10).
\[
B^{(k)} \left( \frac{x}{\lambda} \right) = \Lambda^{-1} S_2 \Xi^{-1} D_{\xi}(x|\lambda).
\] (4.15)

For example, if setting \( 0 \leq n \leq 3, \ 0 \leq k \leq n \), in (4.15), we have
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & \frac{1}{3!} & 0 \\
0 & 0 & 0 & \frac{1}{4!}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & \frac{1}{3!} & 0 \\
0 & 0 & 0 & \frac{1}{4!}
\end{pmatrix}
\begin{pmatrix}
1 & \frac{1}{2} & 0 & 0 \\
\xi \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\xi^2(\xi + \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\xi^3 \frac{3}{2} & \frac{3}{2} & 0 & 0 \\
\xi^4 \frac{6}{2} & \frac{6}{2} & 0 & 0 \\
\xi^5 \frac{10}{2} & \frac{10}{2} & 0 & 0 \\
\xi^6 \frac{15}{2} & \frac{15}{2} & 0 & 0 \\
\xi^7 \frac{21}{2} & \frac{21}{2} & 0 & 0 \\
\xi^8 \frac{28}{2} & \frac{28}{2} & 0 & 0
\end{pmatrix}
\]

where
\[
D_{3,\xi}(x|\lambda) = \xi^3 x(x-1)(x-2), \quad D_{3,\xi}(1)(x|\lambda) = -\frac{\xi^3}{2}(\lambda - 2x + 2)(\lambda - 2x + x^2 - \lambda x),
\]
\[
D_{3,\xi}(2)(x|\lambda) = -\frac{\xi^3}{2}((\lambda - x + 1)(\lambda^2 - 4\lambda x + 4\lambda x + 2x^2 - 4x),
\]
\[
D_{3,\xi}(3)(x|\lambda) = -\frac{\xi^3}{4}(3\lambda - 2x + 2)(3\lambda^2 - 6\lambda x + 6\lambda + 2x^2 - 4x).
\]
Kim et al. [10] introduced the twisted $\lambda$-Dahee polynomials of the second kind of order $k$ as follows:

$$
\left( \frac{\lambda \log (1 + \xi t)(1 + \xi t)^{\lambda}}{(1 + \xi t)^{\lambda} - 1} \right)^{k} (1 + \xi t)^{x} = \sum_{n=0}^{\infty} D_{n,\xi}(x|\lambda) \frac{t^{n}}{n!}.
$$

(4.16)

Setting $x = 0$, $\hat{D}_{n,\xi,\lambda}^{(k)} = \hat{D}_{n,\xi}^{(k)}(0|\lambda)$, we have the twisted Dahee numbers of second kind of order $k$.

$$
\left( \frac{\lambda \log (1 + \xi t)(1 + \xi t)^{\lambda}}{(1 + \xi t)^{\lambda} - 1} \right)^{k} = \sum_{n=0}^{\infty} \hat{D}_{n,\xi,\lambda}^{(k)} \frac{t^{n}}{n!}.
$$

(4.17)

Kim et al. [10, Theorem 2], proved that. For $m \in \mathbb{Z}$, $k \in \mathbb{N}$, we have

$$
\xi^{-m}\hat{D}_{n,\xi}(x|\lambda) = \sum_{l=0}^{m} s_{l}(m, l) \lambda^{l} B_{l}^{(k)} \left( k + \frac{x}{\lambda} \right),
$$

(4.18)

and

$$
\lambda^{m} D_{m,\xi,\lambda}^{(k)} \left( k + \frac{x}{\lambda} \right) = \sum_{n=0}^{\infty} \hat{D}_{n,\xi}(x|\lambda) s_{2}(m, n) \xi^{-n-\lambda k-x}.
$$

(4.19)

Using Eq. (3.12), we can write (4.18) in the following matrix form.

$$
\hat{D}_{\xi}^{(k)}(x|\lambda) = \Xi S_{1} A_{1} B^{(k)} \left( -\frac{x}{\lambda} \right),
$$

(4.20)

where $\hat{D}_{\xi}^{(k)}(x|\lambda)$ is $(n+1) \times (k+1)$ matrix for the twisted Dahee numbers of the second kind of the order $k$.

For example, if setting $0 \leq n \leq 3$, $0 \leq k \leq n$, in (4.20), we have

$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \xi & 0 & 0 \\
0 & 0 & \xi^{2} & 0 \\
0 & 0 & 0 & \xi^{3}
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & -\lambda & 0 & 0 \\
0 & 2 & -3 & 1
\end{bmatrix}
\times
\begin{bmatrix}
1 & 1 & 1 \\
-\frac{x}{\lambda} & -\frac{x}{\lambda} - \frac{1}{2} \\
-\frac{x}{\lambda} - \frac{1}{2} \\
-\frac{x}{\lambda} - \frac{1}{2} - \frac{x}{\lambda} - \frac{1}{2}
\end{bmatrix}
= \begin{bmatrix}
1 & 1 & 1 & 1 \\
-\frac{x}{\lambda} & -\frac{x}{\lambda} - \frac{1}{2} & -\frac{x}{\lambda} - \frac{1}{2} - \frac{x}{\lambda} - \frac{1}{2} \\
-\frac{x}{\lambda} - \frac{1}{2} & -\frac{x}{\lambda} - \frac{1}{2} - \frac{x}{\lambda} - \frac{1}{2} & -\frac{x}{\lambda} - \frac{1}{2} - \frac{x}{\lambda} - \frac{1}{2} & -\frac{x}{\lambda} - \frac{1}{2} - \frac{x}{\lambda} - \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
\xi x \\
\xi^{2} x(x-1) \\
\xi^{2}(\frac{\lambda}{2} + \lambda x - \frac{x}{2} + x^{2} - x) \\
\xi^{2}(\frac{\lambda^{2}}{2} + 2\lambda x - \lambda + x^{2} - x)
\end{bmatrix}
\begin{bmatrix}
\xi(\lambda + x) \\
\xi(\lambda + x) \\
\xi^{2}(2\lambda^{2} + 3\lambda x - \frac{3}{2}\lambda + x^{2} - x) \\
\xi^{2}(3\xi + 2x)
\end{bmatrix}
\begin{bmatrix}
\hat{D}_{\xi}^{(0)}(x|\lambda) \\
\hat{D}_{\xi}^{(1)}(x|\lambda) \\
\hat{D}_{\xi}^{(2)}(x|\lambda) \\
\hat{D}_{\xi}^{(3)}(x|\lambda)
\end{bmatrix},
$$

where

$$
\hat{D}_{\xi}^{(0)}(x|\lambda) = \xi^{3} x(x-1)(x-2), \quad \hat{D}_{\xi}^{(1)}(x|\lambda) = -\frac{\xi^{3}}{2}(\lambda + 2x - 2)(\lambda + 2x - x^{2} - \lambda x),
$$

$$
\hat{D}_{\xi}^{(2)}(x|\lambda) = \frac{\xi^{3}}{2}(\lambda + x - 1)(\lambda^{2} + 4\lambda x - 4\lambda + 2x^{2} - 4x),
$$

$$
\hat{D}_{\xi}^{(3)}(x|\lambda) = \frac{\xi^{3}}{4}(3\lambda + 2x - 2)(3\lambda^{2} + 6\lambda x - 6\lambda + 2x^{2} - 4x).
$$
Remark 4.4. In fact, it seems that there is something not correct in (4.19), the second equation of Kim et al. [10, Theorem 2].

From (4.20), multiplying both sides by $\Xi^{-1}$, we have,
\[
\Xi^{-1} \hat{D}_\xi^{(k)}(x|\lambda) = \Xi^{-1} \Xi S_1 A_1 B^{(k)} \left( -\frac{x}{\lambda} \right) = S_1 A_1 B^{(k)} \left( -\frac{x}{\lambda} \right)
\]

multiplying both sides by $S_2$, we have
\[
S_2 \Xi^{-1} \hat{D}_\xi^{(k)}(x|\lambda) = S_2 S_1 A_1 B^{(k)} \left( -\frac{x}{\lambda} \right) = I A_1 B^{(k)} \left( -\frac{x}{\lambda} \right) = A_1 B^{(k)} \left( -\frac{x}{\lambda} \right).
\]

From (4.19) and (4.21) there is a contradiction.

We obtained the corrected relation in the following theorem as follows.

Theorem 4.5. For $m \in \mathbb{Z}$, $k \in \mathbb{N}$, we have
\[
\lambda^m B_m^{(k)} \left( k + \frac{x}{\lambda} \right) = \sum_{n=0}^{m} \hat{D}_{n,\xi}^{(k)}(x|\lambda) \xi^{-n} s_2(m, n).
\]

Proof. From Eq. (4.16), replacing $t$ by $(e^t - 1)/\xi$, we have
\[
\left( \frac{\lambda \log \left( 1 + \frac{\xi(e^t-1)}{\xi} \right) (1 + \frac{\xi(e^t-1)}{\xi})^\lambda - 1}{1 + \frac{\xi(e^t-1)}{\xi}} \right)^k \left( 1 + \frac{\xi(e^t-1)}{\xi} \right)^x = \sum_{n=0}^{\infty} \hat{D}_{n,\xi}^{(k)}(x|\lambda) \frac{(e^t - 1)^n}{n! \xi^n},
\]

\[
\left( \frac{\lambda t}{e^{\lambda t} - 1} \right)^k e^{(k+\xi)t} = \sum_{n=0}^{\infty} \hat{D}_{n,\xi}^{(k)}(x|\lambda) \frac{(e^t - 1)^n}{n! \xi^n},
\]

\[
\left( \frac{\lambda t}{e^{\lambda t} - 1} \right)^k e^{\lambda t(k+\xi)} = \sum_{n=0}^{\infty} \hat{D}_{n,\xi}^{(k)}(x|\lambda) \xi^{-n} \frac{(e^t - 1)^n}{n!}.
\]

Substituting from Eq. (1.7) into (4.23), we have
\[
\left( \frac{\lambda t}{e^{\lambda t} - 1} \right)^k e^{\lambda t(k+\xi)} = \sum_{n=0}^{\infty} \hat{D}_{n,\xi}^{(k)}(x|\lambda) \xi^{-n} \sum_{m=n}^{\infty} s_2(m, n) \frac{t^m}{m!}
\]

\[
= \sum_{m=0}^{\infty} \sum_{n=0}^{m} \hat{D}_{n,\xi}^{(k)}(x|\lambda) \xi^{-n} s_2(m, n) \frac{t^m}{m!}.
\]

From Eq. (1.1) and (4.24), we have
\[
\sum_{m=0}^{\infty} \lambda^m B_m^{(k)} \left( k + \frac{x}{\lambda} \right) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \hat{D}_{n,\xi}^{(k)}(x|\lambda) \xi^{-n} s_2(m, n) \frac{t^m}{m!}.
\]

Equating the coefficients of $t^m$ on both sides gives (4.22). This completes the proof. \qed
Moreover, by using Eq. (3.12), we can represent Equation (4.22), in the following matrix form.

\[
B^{(k)} \left( -\frac{x}{\lambda} \right) = \Lambda^{-1}_1 S_2 \Xi^{-1} \hat{D}^{(k)}_\xi (x|\lambda),
\]

(4.26)

where \( \Lambda B^{(k)} (k + \frac{x}{\xi}) = \Lambda_1 B^{(k)} (-\frac{x}{\xi}) \).

For example, if setting \( 0 \leq n \leq 3, \ 0 \leq k \leq n \), in (4.26), we have

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -\frac{1}{\lambda} & 0 & 0 \\
0 & 0 & \frac{1}{\lambda} & 0 \\
0 & 0 & 0 & -\frac{1}{\lambda^2}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{1}{\xi} & 0 & 0 \\
0 & 0 & \frac{1}{\xi} & 0 \\
0 & 0 & 0 & \frac{1}{\xi^2}
\end{pmatrix}
\]

where

\[
\hat{D}^{(0)}_{3, \xi} (x|\lambda) = \xi x(x-1), \quad \hat{D}^{(1)}_{3, \xi} (x|\lambda) = \frac{\xi^3}{2} - \frac{\xi}{2} (\lambda + x - 1)(\lambda + 2x - x^2),
\]

\[
\hat{D}^{(2)}_{3, \xi} (x|\lambda) = \frac{\xi^3}{4} (3\lambda + 2x - 2)(3\lambda^2 + 6\lambda x - 6\lambda + 2x^2 - 4x).
\]

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