AMENABILITY AND PARADOXICALITY IN SEMIGROUPS AND C*-ALGEBRAS

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Abstract. We analyze the dichotomy amenable/paradoxical in the context of (discrete, countable, unital) semigroups and corresponding semigroup rings. We consider also Følner type characterizations of amenability and give an example of a semigroup whose semigroup ring is algebraically amenable but has no Følner sequence.

In the context of inverse semigroups $S$ we give a characterization of invariant measures on $S$ (in the sense of Day) in terms of two notions: domain measurability and localization. Given a unital representation of $S$ in terms of partial bijections on some set $X$ we define a natural generalization of the uniform Roe algebra of a group, which we denote by $R_X$. We show that the following notions are then equivalent: (1) $X$ is domain measurable; (2) $X$ is not paradoxical; (3) $X$ satisfies the domain Følner condition; (4) there is an algebraically amenable dense *-subalgebra of $R_X$; (5) $R_X$ has an amenable trace; (6) $R_X$ is not properly infinite and (7) $[0] \neq [1]$ in the $K_0$-group of $R_X$. We also show that any tracial state on $R_X$ is amenable. Moreover, taking into account the localization condition, we give several C*-algebraic characterizations of the amenability of $X$. Finally, we show that for a certain class of inverse semigroups, the quasidiagonality of $C^*_r(X)$ implies the amenability of $X$. The reverse implication (which is a natural generalization of Rosenberg’s conjecture to this context) is false.

Contents

1. Introduction 2
2. Groups and Uniform Roe Algebras 4
3. Semigroups 6
4. Inverse semigroups 9
   4.1. A characterization of invariant measures 10
   4.2. Domain measurable inverse semigroups 11
   4.3. Representations of inverse semigroups 12
   4.4. Amenable inverse semigroups 18
5. Inverse semigroups, C*-algebras and traces 20
   5.1. Domain measures as amenable traces 21
   5.2. Traces and amenable traces 24
   5.3. Traces in amenable inverse semigroups 25
References 27

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1. Introduction

The notion of an amenable group was first introduced by von Neumann in [42] to explain why the paradoxical decomposition of the unit ball in \( \mathbb{R}^n \) (the so-called Banach-Tarski paradox) occurs only for dimensions greater than two (see [55, 38, 50]). Later, Følner provided in [28] a very useful combinatorial characterization of amenability in terms of nets of finite subsets of the group that are almost invariant under left multiplication. This alternative approach was then used to study amenability in the context of algebras over arbitrary fields by Gromov [31, §1.11] and Elek [22] (see also [8, 14, 4] as well as Definition 2.1 (5)), in operator algebras by Connes [16, 17] (see also Definition 2.2 (2)) and in metric spaces by Ceccherini-Silberstein, Grigorchuk and de la Harpe in [13] (see also [4]).

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Definition 2.2 (2)) and in metric spaces by Ceccherini-Silberstein, Grigorchuk and de la Harpe in [14] (see also [13]).

A central aspect of the study of amenability in different mathematical structures is the dynamics inherent to the structure. From this point of view, one can divide the analysis depending on whether the action is injective (e.g., in the case of groups) or singular (e.g., in the case of algebras or operator algebras, where the presence of non-zero divisors is generic). Following this division, it usually happens that notions that are equivalent to amenability in the case of groups are no longer equivalent in a more general case. Thus, semigroups and, in particular, inverse semigroups, provide an interesting frame to reconsider central notions in the theory of amenability with relation to groups, groupoids and C*-algebras (see, e.g., the pioneering works [18, 47, 36] as well as the recent survey by Lawson in [38] and references therein). In this article we address von Neumann’s original dichotomy - amenable versus paradoxical - in the context of semigroups and semigroup rings, and connect this analysis to C*-algebraic structures associated to inverse semigroups. In particular, we define a C*-algebra for an inverse semigroup which generalizes the uniform Roe-algebra of a group, and then study its trace space in relation to the amenability of the original inverse semigroup.

Amenability of semigroups has been studied since Day’s seminal article (see [18] as well as other classical references [6, 34, 21, 41]). However, the category of semigroups is too broad to obtain classical equivalences like that between amenability, existence of Følner sequences, absence of paradoxical decompositions or algebraic amenability of the corresponding semigroup ring. This has led to a variety of approaches that modify classical definitions and introduce new notions, such as strong and weak Følner conditions or fair amenability to mention only a few [20, 29, 59]. Some other recent results exploring (geo)metrical aspects of discrete semigroups are presented in [27, 30]. Furthermore, following the dynamical point of view mentioned above, semigroups are closer to algebras than they are to groups, since the action of an element \( s \in S \) on subsets of \( S \) can be singular. In the case that \( S \) has a zero element, for instance, its action drastically shrinks the size of any subset of \( S \) under multiplication. As an illustration of the singular dynamics involved we show in Theorem 5.3 that if \( S \) has a Følner sequence but does not have a Følner sequence exhausting the semigroup (which we call proper Følner sequence in Definition 2.1 (3)) then \( S \) has a finite principal left ideal. This behavior is characteristic of the dynamics given by multiplication in an algebra (cf., [4, Theorem 3.9]) and is not present in the context of groups, where one can easily modify a Følner sequence of a group to turn it exhausting.

An alternative approach to understand the dichotomy on a given category is to use operator algebra techniques for a canonical C*-algebra associated to the initial structure. In the special case of groups two important C*-algebras are the reduced group C*-algebra, denoted by \( C_r^*(G) \), and the uniform Roe algebra of a group, which we denote by \( \mathcal{R}_G = \ell^\infty(G) \rtimes_r G \), where \( G \) acts on \( \ell^\infty(G) \) by left translation. Among other things, Rørdam and Sierakowski establish in [49] a relation between paradoxical decompositions of \( G \) and properly infinite projections in \( \mathcal{R}_G \). Nevertheless, it is not obvious how to associate a C*-algebra to a general semigroup, since the naive approach would be to define the generators of a possible C*-algebra via \( V_\delta := \delta \) on the Hilbert space \( \ell^2(S) \). This, in general, gives unbounded operators due to the singular dynamics involved. Therefore, when we connect our analysis to C*-algebras we will restrict to the class of inverse semigroups, where the dynamics induced by left multiplication are only locally injective, i.e., injective on the corresponding domains. Some general references for inverse semigroups and, also, in relation to C*-algebras are [7, 27, 32, 37, 40, 43, 44, 52, 56]. In Theorem 3.19 of [35], Kudryavtseva, Lawson, Lenz and Resende...
prove a Tarski’s type alternative where the invariant measure and the paradoxical decomposition restricts to the space of projections $E(S)$ of the inverse semigroup. In the context of groupoids, Bönicek and Li (see [11]) and Rainone and Sims (see [16]) establish a sufficient condition on an étale groupoid that ensures pure infiniteness of the reduced groupoid C*-algebra in terms of paradoxicality of compact open subsets of the unit space. See also [11, 26, 24] for additional references on the relation between inverse semigroups, C*-algebras and groupoids.

Recall that an inverse semigroup $S$ is a semigroup such that for every $s \in S$ there is a unique $s^* \in S$ satisfying $ss^*s = s$ and $s^*ss^* = s^*$. We will assume that our semigroups are unital, discrete and countable. In Proposition 4.3 we will characterize invariant measures in the sense of Day, i.e., finitely additive probability measures satisfying $\mu(s^{-1}A) = \mu(A)$, $s \in S$, $A \subset S$ (where $s^{-1}A$ denotes the preimage of $A$ by $s$), by means of the two following conditions:

(a) **Localization**: $\mu(A) = \mu(A \cap s^*sA)$, for any $s \in S$, $A \subset S$.

(b) **Domain-measurability**: $\mu(s^*sA) = \mu(sA)$, for any $s \in S$, $A \subset S$.

Fixing a representation $\alpha: S \to \mathcal{I}(X)$ of $S$ in terms of partial bijections on some discrete set $X$ one can consider a natural *-representation $V: S \to \mathcal{B}(\ell^2(X))$. Define $\mathcal{R}_{X,alg}$ as the *-algebra generated by the family of partial isometries $\{V_s : s \in S\}$ and $\ell^\infty(X)$. The C*-algebra $\mathcal{R}_X$ is the norm closure of $\mathcal{R}_{X,alg}$. In particular, taking the left regular representation $\iota: S \to \mathcal{I}(S)$ we obtain a Roe algebra $\mathcal{R}_S$, which is a natural generalization of the uniform Roe algebra $\mathcal{R}_G$ of a discrete group. Recall that uniform Roe algebras associated to general discrete metric spaces are an important class of C*-algebras that naturally encode properties of the metric space, such as amenability, property (A) or lower dimensional aspects (see, e.g., [5, Theorem 4.9] or [39, Theorem 2.2]). We will use this strategy to characterize in different ways amenability aspects of the inverse semigroup. In this context one can define notions like $S$-domain Følner condition and $S$-paradoxical decomposition which correspond, in essence, to the usual notions but restricted to the corresponding domains given by the representation $\alpha$. In this way, one of the main results in this article is

**Theorem 1** (cf., Theorem 5.4). Let $S$ be a countable and discrete inverse semigroup with identity $1 \in S$, and let $\alpha: S \to \mathcal{I}(X)$ be a representation of $S$ on $X$. Then the following conditions are equivalent:

1. $X$ is $S$-domain measurable.
2. $X$ is not $S$-paradoxical.
3. $X$ is $S$-domain Følner.
4. $\mathcal{R}_{X,alg}$ is algebraically amenable.
5. $\mathcal{R}_X$ has an amenable trace.
6. $\mathcal{R}_X$ is not properly infinite.
7. $[0] \neq [1]$ in the $K_0$-group of $\mathcal{R}_X$.

Note that this characterization involves the notions corresponding to domain-measurability (see (b) above). We also characterize the full force of amenability of the action in Theorem 5.4 obtaining in particular that it is equivalent to the fact that no projection associated to an idempotent of $S$ is properly infinite in $\mathcal{R}_X$ (compare with Theorem 1.6).

An important step in the proof of the previous theorem is the construction and analysis of a type semigroup $\text{Typ}(\alpha)$ (see Definition 1.2) associated to the representation $\alpha$. Recall that type semigroups have been considered recently in many interesting situations (see, e.g., [2, 44]).

Moreover, we also show in this section that every tracial state on $\mathcal{R}_X$ is amenable.

**Theorem 2** (cf., Theorem 5.9). Let $S$ be a countable and discrete inverse semigroup with identity $1 \in S$, and let $\alpha: S \to \mathcal{I}(X)$ be a representation. A positive linear functional on $\mathcal{R}_X$ is a trace if and only if it is an amenable trace.

Given the representation $V: S \to \mathcal{B}(\ell^2(X))$ introduced above, one can also consider the reduced semigroup C*-algebra, that is, the C*-algebra $C_r^*(X)$ generated by $\{V_s\}_{s \in S}$. In particular we have the following inclusions:

$$C_r^*(X) := C^* (\{V_s : s \in S\}) \subset \mathcal{R}_X := C^* (\{V_s : s \in S\} \cup \ell^\infty(X)) \subset \mathcal{B}(\ell^2(X)).$$
Lastly, using the theorems above we also prove a generalization to a result by Rosenberg (cf. [19]) in the setting of inverse semigroups.

**Theorem 3** (cf., Theorem 5.13). Let $S$ be a countable and discrete inverse semigroup with identity $1 \in S$ and with a minimal projection. Let $\alpha : S \to \mathcal{I}(X)$ be a representation on some set $X$ and suppose $C^*_\alpha(X)$ is quasidiagonal. Then $X$ is $S$-amenable.

The structure of the article is as follows. In Section 2 we recall different results around the notion of amenability in the context of groups and algebras that partly motivate our analysis. In particular, we introduce the notion of uniform Roe algebra $\mathcal{R}_G$ of a group $G$ and mention in Theorem 2.4 a variety of ways in which one may characterize the amenability of $G$ via $\mathcal{R}_G$. In Section 3 we focus first on amenability and Følner sequences for general semigroups and semigroup rings. We give some questions in relation to this problem.

In the final two sections we restrict our analysis to the case of inverse semigroups. In Section 4 we focus on the algebraic (read as non-$C^*$) aspects of amenability in inverse semigroups. In particular we split Day’s invariance condition for measures over amenable inverse semigroups $S$ into the two notions (a) and (b) above, and introduce the type semigroup construction. In Section 5 we present a type semigroup construction. In Section 6 we introduce the notion of uniform Roe algebra of amenability in the context of groups and algebras that partly motivate our analysis. In particular, we introduce the algebra $\mathcal{R}_X$ and prove that all its traces factor through $\ell^\infty(X)$ via a canonical conditional expectation. We finish the article studying the relation between the quasidiagonality of $C^*_\alpha(X)$ and the $S$-amenability of $X$. We also mention some questions in relation to this problem.

**Conventions:** We denote by $A \sqcup B$ the disjoint union of two sets $A$ and $B$. Unless otherwise specified, any measure $\mu$ on a set $X$ will be a finitely additive probability measure, i.e., $\mu : \mathcal{P}(X) \to [0,1]$, where $\mathcal{P}(X)$ is the power set of $X$, satisfies $\mu(X) = 1$ and $\mu(A \sqcup B) = \mu(A) + \mu(B)$, for every $A, B \subset X$. All semigroups $S$ considered will be countable, discrete and with unit $1 \in S$. A representation of an inverse semigroup $S$ on a set $X$ is a unital semigroup homomorphism $\alpha : S \to \mathcal{I}(X)$, where $\mathcal{I}(X)$ denotes the inverse semigroup of partial bijections of $X$. We will denote by $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators on a complex separable Hilbert space $\mathcal{H}$.

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2. Groups and Uniform Roe Algebras

This section aims to give a brief summary to some aspects of amenability needed later. We begin with classical notions in the context of groups and relate these with $C^*$-algebraic concepts using the uniform Roe algebra of a group.

**Definition 2.1.** Let $G$ be a countable group and $\mathcal{A}$ a unital $C^*$-algebra of countable dimension.

1. $G$ is (left) **amenable** if there exists a (left) invariant measure on $G$, i.e., a finitely additive probability measure $\mu : \mathcal{P}(G) \to [0,1]$ such that $\mu(g^{-1}A) = \mu(A)$ for all $g \in G$, $A \subset G$.

2. $G$ satisfies the **Følner condition** if for every $\varepsilon > 0$ and finite $F \subset G$, there is a finite non-empty $F' \subset G$ such that $|gF \cup F| \leq (1 + \varepsilon)|F|$, for every $g \in F$.

3. $G$ satisfies the **proper Følner condition** if, in addition, the finite set $F$ can be taken to contain any other set $A \subset G$, i.e., for every $\varepsilon > 0, F \subset G$ and finite $A \subset G$ there is a finite non-empty $F' \subset G$ as above such that $A \subset F$.

4. $G$ is **paradoxical** if there are sets $A_1, B_j \subset G$ and elements $a_i, b_j \in G$ such that 
   
   $G = a_1A_1 \sqcup \cdots \sqcup a_nA_n = b_1B_1 \sqcup \cdots \sqcup b_mB_m$ 
   
   $\sqcup A_1 \sqcup \cdots \sqcup A_n \sqcup B_1 \sqcup \cdots \sqcup B_m$.

5. $\mathcal{A}$ is **algebraically amenable** if for every $\varepsilon > 0$ and finite $F \subset \mathcal{A}$ there is a non-zero finite dimensional subspace $W \leq \mathcal{A}$ such that $\dim(\mathcal{A}W + W) \leq (1 + \varepsilon)\dim(W)$ for every $A \in F$.

Along the lines of these notions, but in the $C^*$-scenario, we can define when a $C^*$-algebra $\mathcal{A}$ captures some aspects of amenability, the Følner condition or paradoxicality. For additional motivations and results see, e.g., [16 17 9 10 8] and references therein.
Definition 2.2. Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a unital C*-algebra of bounded linear operators on a complex separable Hilbert space $\mathcal{H}$. A state on $\mathcal{A}$ is a positive and linear functional on $\mathcal{A}$ with norm one.

1. A state $\tau$ on $\mathcal{A}$ is called an amenable trace if there is a state $\phi$ on $\mathcal{B}(\mathcal{H})$ extending $\tau$, i.e., $\phi|_{\mathcal{A}} = \tau$, and satisfying
   $$\phi(AT) = \phi(TA), \quad T \in \mathcal{B}(\mathcal{H}), \quad A \in \mathcal{A}.$$ 

The state $\phi$ is called a hypertrace for $\mathcal{A}$. The concrete C*-algebra $\mathcal{A}$ is a called a Følner $C^*$-algebra if it has an amenable trace.

2. $\mathcal{A}$ satisfies the Følner condition if for every $\varepsilon > 0$ and every finite $\mathcal{F} \subset \mathcal{A}$ there is a non-zero finite rank orthogonal projection $P \in \mathcal{B}(\mathcal{H})$ such that $||PA - AP||_2 \leq \varepsilon ||P||_2$ for every $A \in \mathcal{F}$, where $|| \cdot ||_2$ denotes the Hilbert-Schmidt norm.

3. A projection $P \in \mathcal{A}$ is properly infinite if there are $V, W \in \mathcal{A}$ such that $P = V^*V = W^*W \geq VV^* + WW^*$. Note that, in this case, the range projections $VV^*$ and $WW^*$ are orthogonal. The algebra $\mathcal{A}$ is called properly infinite when $1 \in \mathcal{A}$ is properly infinite.

Remark 2.3. The class of Følner C*-algebras is also known in the literature as weakly hypertracial C*-algebras (cf., [9]). In [3, Section 4] the first and second authors gave an abstract (i.e., representation independent) characterization of this class of algebras in terms of a net of unital completely positive (u.c.p.) maps into matrices which are asymptotically multiplicative in a weaker norm than the operator norm (see also [5, Theorem 3.8]). It can be shown that an abstract C*-algebra possesses an amenable trace (cf., [3, Theorem 4.14]). However, if $\mathcal{A}$ is a unital nuclear C*-algebra, then it is a Følner C*-algebra if and only if $\mathcal{A}$ admits a tracial state (see [12, Proposition 6.3.4]). Note that this fact implies that every stably finite unital nuclear C*-algebra is in the Følner class.

A classical construction relating C*-algebras and groups is given via the so-called left regular representation: the unitary representation $\lambda: G \to \mathcal{B}(l^2(G))$ defined by $(\lambda_g(f))(h) := f(g^{-1}h)$. The uniform Roe algebra $\mathcal{R}_G$ of $G$ is the C*-algebra generated by $\{\lambda_g\}_{g \in G}$ and $l^\infty(G)$ viewed as multiplication operators in $l^2(G)$, that is,

$$\mathcal{R}_G := C^* \left\{ \lambda_g \mid g \in G \right\} \cup l^\infty(G) \subset \mathcal{B} \left( l^2(G) \right).$$

The following result shows how one can characterize amenability and paradoxicality of the group in terms of C*-properties of the algebra $\mathcal{R}_G$.

Theorem 2.4. Let $G$ be a countable and discrete group. The following are equivalent:

1. $G$ is amenable.
2. $G$ is not paradoxical.
3. $G$ has a Følner sequence.
4. $\mathcal{C}G$ is algebraically amenable.
5. $\mathcal{R}_G$ has an amenable trace (and hence is a Følner C*-algebra).
6. $\mathcal{R}_G$ is not properly infinite.
7. $[0] \neq [1]$ in the $K_0$-group of $\mathcal{R}_G$.

Proof. The equivalences $(1) \iff (2) \iff (3)$ are classical (see, e.g., [14, 33]). Their equivalence to $(4)$ is due to Bartholdi [8]. To show the equivalences $(1) \iff (5) \iff (6)$ recall that the uniform Roe algebra of the group $G$ can be also seen as a reduced crossed product, i.e., $\mathcal{R}_G = l^\infty(G) \rtimes_r G$, where the action of $G$ on $l^\infty(G)$ is given by left translation of the argument (see, e.g., [12, Proposition 5.1.3]). Rørdam and Sierakowski show in [49, Proposition 5.5] a direct equivalence between paradoxicality of $G$ and proper infiniteness for this class of crossed products. In fact, they show that $E \subset G$ is paradoxical if and only if the characteristic function $\mathcal{F}_E$ is properly infinite in $l^\infty(G) \rtimes_r G$ (see also [5, Theorem 4.9]). Finally, using the reasoning in [5, Theorem 4.6] one can also prove $(1) \iff (7)$. □
Remark 2.5. For a general study of the relation between Følner $C^*$-algebras and crossed products see also [9] [10]. Moreover, note that the $C^*$-algebra $R_G$ contains the reduced group $C^*$-algebra $C^*_r(G) = C^*(\{\lambda_g | g \in G\})$.

However the characterization in terms of proper infiniteness in the preceding theorem would not be true if we replaced the former by the latter. In fact, it is well known (see [15]) that the reduced $C^*$-algebra of the free group on two generators $\mathbb{F}_2$ has no non-trivial projection and hence is vacuously not properly infinite. However $\mathbb{F}_2$ is indeed paradoxical. Thus observe that a $C^*$-algebraic characterization of amenability via proper infiniteness requires the existence of non-trivial projections in the $C^*$-algebra, and that in $R_G$ the existence of nontrivial projections is guaranteed by the characteristic functions in $\ell^\infty(G)$.

3. Semigroups

We begin next our analysis of amenability in the context of semigroups. We will see that, from a dynamical point of view, semigroups are closer to algebras than to groups. At this level of generality it is not possible to define a natural $C^*$-algebra which provides the variety of characterizations given in Theorem 2.4.

Recall that a semigroup is a non-empty set $S$ equipped with an associative binary operation $(s, t) \mapsto st$. The notions treated in Section 2 do have an analogue in the semigroup scenario, which relies on the preimage of a set. Given $s \in S$ and $A \subset S$, the preimage of $A$ by $s$ is defined by

$$s^{-1}A := \{t \in S | st \in A\}.$$

The following definition for semigroups is due to Day [18] (see also [28, 41]). Recall that by Definition 3.1 and we will omit the prefix left.

**Definition 3.1.** Let $S$ be a semigroup.

1. $S$ is (left) amenable if there exists an (left) invariant measure on $S$, i.e., a probability measure $\mu: P(S) \to [0, 1]$ such that $\mu(s^{-1}A) = \mu(A)$ for every $s \in S$, $A \subset S$.

2. $S$ satisfies the (left) Følner condition if for all $\varepsilon > 0$ and finite $F \subset S$ there is a finite non-empty $F \subset S$ such that $|sF \cup F| \leq (1 + \varepsilon)|F|$ for every $s \in F$.

3. $S$ satisfies the proper Følner condition if, in addition, the Følner set $F$ can be taken to contain any other set $A \subset S$, i.e., for every $\varepsilon > 0$, finite $F \subset S$ and finite $A \subset S$ there is a finite non-empty $F \subset S$ as in (2) that, in addition, satisfies $A \subset F$.

**Remark 2.2.**

1. For the rest of the text we will just consider left amenability as defined in Definition 3.1 and we will omit the prefix left.

2. We mention the Følner condition given in (2) is equivalent to the existence of a net (a sequence if $S$ is countable) $\{F_i\}_{i \in I}$ of finite non-empty subsets of $S$ such that $|F_i \setminus F_i| / |F_i| \to 0$ for all $s \in S$. These conditions will be used indistinctly throughout the text.

We introduce next a stronger notion than amenability in the context of semigroups.

**Definition 3.3.** A semigroup $S$ is called measurable if there is a probability measure $\mu$ on $S$ such that $\mu(sA) = \mu(A)$, $s \in S$, $A \subset S$.

It is a standard result that any measurable semigroup is amenable as well. The reverse implication is false in general, although it holds in some classes of semigroups, e.g., for left cancellative ones (see Sorenson’s Ph.D. thesis [51] as well as Klawe [34]).

The following proposition justifies why we can assume a semigroup $S$ to be countable and unital, as we will normally do in the following sections. In general, given a possibly non-unital semigroup $S$ we can always consider its unitization $S' := S \cup \{1\}$ and define a multiplication in $S'$ extending that of $S$ so that 1 behaves as a unit. Moreover, as in the case of groups and algebras, the property of amenability is in essence a countable one, at least for a large class of semigroups (including the inverse). Recall from [29] that a semigroup $S$ satisfies the Klawe condition whenever $sx = sy$ for $s, x, y \in S$ implies there is some $t \in S$ such that $xt = yt$. As mentioned in [29], the Klawe condition is very general and, in particular, left cancellative as well as inverse semigroups satisfy it.
Proposition 3.4. Let $S$ be a semigroup and denote by $S'$ its unitization. Then

(i) $S$ is amenable if and only if $S'$ is amenable.

(ii) If any countable subset in $S$ is contained in an amenable countable subsemigroup of $S$, then $S$ is amenable. If, in addition, $S$ satisfies the Klawe condition, then the reverse implication is also true.

Proof. (i) The proof directly follows from the definition. Indeed, an invariant measure on $S$ can be extended to an invariant measure on $S'$ defining $\mu(\{1\}) = 0$. Conversely, $\{1\}$ is null for any invariant measure on $S'$, so any invariant measure on $S'$ is also an invariant measure on $S$.

(ii) For the first part, let $\mathcal{A}(S)$ denote the set of countable and amenable subsemigroups of $S$. Furthermore, for $T \in \mathcal{A}(S)$ denote by $\mu_T$ an invariant measure on $T \subset S$. We may, without loss of generality, extend it to $S$ by defining $\mu_T(S \setminus T) := 0$. Observe that then $\mu_T(t^{-1}A) = \mu_T(A)$ for every $t \in T$, $A \subset T$. Consider the measure:

$$\mu: \mathcal{P}(S) \to [0, 1], \quad A \mapsto \mu(A) := \lim_{T \in \mathcal{A}(S)} \mu_T(A) = \lim_{T \in \mathcal{A}(S)} \mu_T(A \cap T),$$

where the limit is taken along a free ultrafilter of $\mathcal{A}(S)$. It follows from a straightforward computation that $\mu$ is an invariant measure on $S$.

For the second part we follow a similar route to that of [41 Proposition 3.4]. Recall from [29] that a semigroup satisfying the Klawe condition is amenable if and only if for every $\epsilon > 0$ and finite $\mathcal{F} \subset S$ there is a $(\epsilon, \mathcal{F})$-Følner set $F \subset S$ such that $|F| = |sF|$ for every $s \in \mathcal{F}$ (see Theorem 2.6 in [29] in relation with the notion of strong Følner condition).

Let $C = \{c_n\}_{n \in \mathbb{N}} \subset S$ be a countable subset. In order to construct an amenable semigroup $T \supset C$ we define an increasing sequence $\{T_n\}_{n \in \mathbb{N}}$ of countable subsemigroups of $S$ by:

- $T_0$ is the subsemigroup generated by $C$.

- Suppose $T_k = \{t_j\}_{j \in \mathbb{N}}$ has been defined. By [29] and the previous paragraph, for every $k \in \mathbb{N}$ we may find an $(1/k, \{t_1, \ldots, t_k\})$-Følner set $F_k \subset S$ such that $|t_jF_k| = |F_k|$ for every $j = 1, \ldots, k$. We thus define the semigroup $T_{k+1}$ to be the semigroup generated by $T_k \cup F_1 \cup F_2 \cup \ldots$.

Finally, consider the semigroup $T = \cup_{i \in \mathbb{N}} T_i$. It is straightforward to prove that then $T$ is amenable, countable and contains $C$. \qed

Remark 3.5. As in the case of metric spaces or algebras (see, e.g., [5 Section 2.1] and [3 Section 4]), an amenable semigroup can have non-amenable sub-semigroups. For instance, take $S := \mathbb{F}_2 \cup \{0\}$, where $0 \omega = \omega 0 = 0$ for every $\omega \in \mathbb{F}_2$. This semigroup $S$ is amenable, since it has a $0$ element, but has a non-amenable sub-semigroup. A more striking fact is that amenable groups may contain non-amenable semigroups. For instance, the group $G_2$ of isometries of $\mathbb{R}^2$ is a solvable group containing a non-commutative free semigroup (see [53 Theorem 1.8, Theorem 14.30]).

For the purpose of this article the main difference between a semigroup and a group is the lack of injectivity under left multiplication. This fact, among other things, makes it impossible to define a canonical regular representation in the general semigroup case. We recall next some well-known facts.

Theorem 3.6. Let $S$ be a countable discrete semigroup. Consider the assertions:

1. $S$ is amenable.

2. $S$ has a Følner sequence.

3. $\mathbb{C}S$ is algebraically amenable.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof. Følner proved the implication (1) $\Rightarrow$ (2) in the case of groups and, later, Frey and Namioka extended the proof for semigroups (cf., [28, 111]). To show (2) $\Rightarrow$ (3) choose a Følner sequence $\{F_n\}_{n \in \mathbb{N}}$ for $S$. Then the linear span of these subsets $W_n := \text{span} \{f \mid f \in F_n\}$ defines a Følner sequence for $\mathbb{C}S$. In fact, note that $\dim(W_n) = |F_n|$ and for any $s \in S$ we have

$$\frac{\dim(sW_n + W_n)}{\dim(W_n)} \leq \frac{|sF_n \cup F_n|}{|F_n|} \xrightarrow{n \to \infty} 1,$$
which concludes the proof.

We remark that none of the reverse implications in Theorem 3.6 hold in general. It is well known that a finite semigroup may be non-amenable and any such semigroup is a counterexample to the implication $(2) \Rightarrow (1)$, because if $S$ is finite it has a trivial (constant) Følner sequence $F_n = S$. A concrete example was first given by Day in [18]: let $S = \{a, b\}$, where $ab = aa = a$ and $ba = bb = b$. Note that in this case any invariant measure $\mu$ must satisfy $\mu(b^{-1}\{a\}) = \mu(a^{-1}\{b\}) = \mu(\emptyset) = 0$. Therefore, no probability measure on $S$ can be invariant and, hence, $S$ not amenable.

The following example is a counterexample to the implication $(3) \Rightarrow (2)$ in Theorem 3.6.

**Example 3.7.** Consider the additive semigroup of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and the free semigroup on two generators $\mathbb{F}_2^+ = \{a, b, ab, \ldots\}$, where we assume that the semigroup $\mathbb{F}_2^+$ has no identity. Denote by $\alpha$ the action of $\mathbb{F}_2^+ \cap \mathbb{N}$ given by $\alpha : \mathbb{F}_2^+ \to \text{End}(\mathbb{N})$, $a, b \mapsto \alpha_a(n) = \alpha_b(n) = n - 1$ when $n \geq 1$ and $\alpha_a(0) = \alpha_b(0) = 0$.

We claim the semigroup $S := \mathbb{N} \rtimes \mathbb{F}_2^+$ does not satisfy the Følner condition, while its complex group algebra is algebraically amenable. Note that the element $s = (0, a) - (1, a) \in \mathbb{C}S$ clearly satisfies $(n, \omega)s = 0$ for every $(n, \omega) \in S$. Therefore $W := \mathbb{C}s$ is trivially a Følner subspace for $\mathbb{C}S$, since it is a one-dimensional left ideal. This proves that $\mathbb{C}S$ is algebraically amenable.

In order to prove that $S$ does not satisfy the Følner condition, we shall prove that for any non-empty finite subset $F \subset S$ either $|(0, a) F \setminus F| \geq |F|/50$ or $|(0, b) F \setminus F| \geq |F|/50$. First observe that $|(0, a) F| \geq |F|/2$, and that equality holds if and only if

\begin{equation}
F = \{(0, w_1), (1, w_1), \ldots, (0, w_k) \} \text{ for some } w_i \in \mathbb{F}_2^+, i = 1, \ldots, k.\end{equation}

Indeed, if $F$ is of this form then clearly $|(0, a) F| = |F|/2$. And, conversely, given $(n, u) \neq (m, v)$ one has that $(0, a) (n, u) = (0, a) (m, v)$ only when $u = v$ and $n = 0, m = 1$ or $n = 1, m = 0$.

Now suppose $F$ is of the form given in Eq. (3.1) and satisfies $|(0, a) F \setminus F| \leq |F|/5$. Note that, by the observation in the previous paragraph, $|(0, a) F \setminus F| \geq |F|/2 - N_a$, where $N_a$ is the number of words $w_i$ of $F$ that begin with $a$. Since a word cannot begin with $a$ and with $b$, it follows that the number $N_b$ of words that begin with $b$ satisfies $N_b \leq |F|/5$. Therefore, again, we conclude that $|(0, b) F \setminus F| \geq |F|/2 - |F|/5 \geq |F|/5$, as desired. This proves that no set of the form (3.1) can be Følner.

Finally, given an arbitrary $F \subset S$ we may decompose it into $F = F_a \cup F_b$, where $F_a$ is of the form (3.1) and $F_b$ does not contain pairs of elements of the form $(0, \omega), (1, \omega)$, with $\omega \in \mathbb{F}_2^+$. We have

\[|(0, a) F \setminus (0, b) F | \geq |(0, a) F_a| + |(0, a) F_b| + |(0, b) F_a| - |(0, b) F_a| \cap F_a| = |F_b|.
\]

Note that the last equality follows from the fact that $|(0, a) F_a| = |F_a| = |(0, b) F_b|$. Therefore, if $F$ is relatively large when compared to $F_a$, then $F$ itself will not be Følner. Suppose hence that $|F_a| \leq |F|/25$ and $|(0, a) F \setminus F| \leq |F|/25$. Then we have $|F_a| \geq (24/25) |F|$. Now observe that

\[(0, a) F_a \setminus F_a = [(0, a) F_a \setminus F] \cup [(0, a) F_a \cap F_a] \subset [(0, a) F \setminus F] \cup F_b,
\]

and so

\[|(0, a) F_a \setminus F_a| \leq \frac{|F|}{25} + \frac{|F|}{25} \leq \frac{2 \cdot 25 \cdot |F_a|}{25} = \frac{|F_a|}{5}.
\]

Since $F_a$ is of the form (3.1), it follows that $|(0, b) F_a \setminus F_a| \geq |F_a|/5$. Hence

\[|(0, b) F_a \setminus F_a| \geq \frac{|F_a|}{5} \geq \frac{24|F|}{5 \cdot 25}.
\]

Finally,

\[|(0, b) F \setminus F| \geq |(0, b) F_a \setminus F_a| = |(0, b) F_a \setminus F_a| - |(0, b) F_a \cap F_a| \geq \frac{24|F|}{5 \cdot 25} = \frac{|F|}{25}.
\]
It remains to consider what happens when $|F'| \geq |F|/25$. In this case, by the above computation, we get

$$2 \max \{|0,a)F \setminus F|, |0,b)F \setminus F| \} \geq |\{(0,a)F \cup (0,b)F) \setminus F| \geq |F'| \geq \frac{|F|}{25},$$

and we deduce that either $|(0,a)F \setminus F|$ or $|(0,b)F \setminus F|$ is greater or equal than $|F|/50$.

We conclude that no non-empty finite subset $F \subset S$ can be $(\varepsilon,\{a,b\})$-invariant for $\varepsilon < 1/50$, which proves that $S$ itself does not satisfy the Følner condition.

In the following result we establish the difference between the Følner condition and the proper Følner condition. This result is analogous to \[5, Proposition 2.15\] (see also \[5, Theorem 3.9\]). We will use this statement in Proposition \[4.3.1\]. Its proof is inspired by the corresponding result in the algebra setting \[5, Theorem 3.9\].

**Theorem 3.8.** Let $S$ be a semigroup. Suppose that $S$ satisfies the Følner condition but not the proper Følner condition. Then there is an element $a \in S$ such that $|Sa| < \infty$.

**Proof.** Given $\varepsilon > 0$ and a non-empty finite subset $F \subset S$ define

$$\text{Føl}(\varepsilon,F) := \left\{ F \subset S \mid 0 < |F| < \infty \text{ and } \max_{\varepsilon \in F} \frac{|sF \setminus F|}{|F|} \leq \varepsilon \right\},$$

$$M_{\varepsilon,F} := \sup_{F \in \text{Føl}(\varepsilon,F)} |F| \in \mathbb{N} \cup \{\infty\}.$$ 

Since $S$ is not properly Følner there is a pair $(\varepsilon_0,F_0)$ with finite $M_{\varepsilon_0,F_0}$. Note that the pairs $(\varepsilon,F)$ are partially ordered by $(\varepsilon_1,F_1) \leq (\varepsilon_2,F_2)$ if and only if $F_1 \subset F_2$ and $\varepsilon_2 \leq \varepsilon_1$. This partial order induces a partial order on $M_{\varepsilon,F}$ and thus we may suppose that $\varepsilon_0 |M_{\varepsilon_0,F_0} < 1$. Indeed, simply substitute $\varepsilon_0$ with some $\varepsilon_0 < \min \{\varepsilon_0,1/M_{\varepsilon_0,F_0}\}$.

We first claim that for any $\varepsilon \in (0,\varepsilon_0]$ and $F \supset F_0$ we have $\text{Føl}(0,F) = \text{Føl}(\varepsilon,F)$. Indeed, the inclusion $\subset$ is obvious. Moreover, for $F \in \text{Føl}(\varepsilon,F)$ and $s \in F$ we have

$$|sF \setminus F| \leq \varepsilon |F| \leq \varepsilon M_{\varepsilon,F} \leq \varepsilon_0 |M_{\varepsilon_0,F_0} < 1$$

and hence $|sF \setminus F| = 0$. Therefore $F \in \text{Føl}(0,F)$. Thus it makes sense to consider the largest Følner sets with $\varepsilon = 0$:

$$\text{Føl}_{\text{max}}(0,F) := \left\{ F \in \text{Føl}(0,F) \mid |F| \geq |F'| \text{ for all } F' \in \text{Føl}(0,F) \right\}.$$ 

Next we claim that if $F \subset F'$ and $F'_m \in \text{Føl}_{\text{max}}(0,F)$, $F'_m \in \text{Føl}_{\text{max}}(0,F')$, then $F'_m \subset F_m$. Indeed, suppose the contrary. Then $\tilde{F} := F_m \cup F'_m$ would be in $\text{Føl}_{\text{max}}(0,F)$ and strictly larger than $F_m$, contradicting the maximality condition in the definition of $\text{Føl}_{\text{max}}(0,F)$. In particular, this means that $\text{Føl}_{\text{max}}(0,F)$ has only one element, for if $F_1,F_2 \in \text{Føl}_{\text{max}}(0,F)$ then $F_1 \subset F_2 \subset F_1$.

Finally, denote by $F_F$ the element of $\text{Føl}_{\text{max}}(0,F)$ and consider the net $\{\{|F|\} \}_{F \supset F_0}$, where $F := \{ F \subset S \mid |F| < \infty \text{ and } F_0 \subset F \}$. This net is decreasing and contained in $[1,|F_0|] \cap \mathbb{N}$ and, thus, has a limit, which is attained by some $F_1$. This means that $sF_{F_1} \subset F_{F_1}$ for all $s \in S$. Therefore any $a \in F_{F_1}$ will meet the requirements of the theorem. \[\square\]

4. **Inverse semigroups**

In the rest of the article we will incorporate into the analysis notions of paradoxical decompositions and the relation to C*-algebras in the category of inverse semigroups, i.e., where one only has a locally injective action. While the rest of the text will be devoted to inverse semigroups, this section focuses only on the algebraic (meaning non-C*) properties and Section \[5\] will focus on how these properties of $S$ translate into properties of a C*-algebra defined as a generalization of the uniform Roe algebra of a group.

First we recall the definition of inverse semigroup as well as some important structures and examples.

**Definition 4.1.** An inverse semigroup is a semigroup $S$ such that for every $s \in S$ there is a unique $s^* \in S$ satisfying $ss^*s = s$ and $s^*ss^* = s^*$. 


Example 4.2. The most important example of an inverse semigroup is that of the set of partial bijections on a given set $X$, denoted by $I(X)$. Elements $(s, A, B) \in I(X)$ are bijections $s: A \to B$, where $A, B \subset X$. The operation of the semigroup is just the composition of maps where it can be defined. This semigroup contains both a zero element, namely $(0, 0, 0)$, and a unit, namely $(id, X, X)$. Just as the elements of a group $G$ can be thought of as bijections of $G$ on itself by left multiplication, every inverse semigroup $S$ can be thought as contained in $I(S)$ via the Wagner-Preston representation (see, e.g., [56][44][7]).

Remark 4.3. Given an inverse semigroup $S$, the set $E(S) = \{ s^*s \mid s \in S \}$ is the set of all idempotents (or projections) of $S$, i.e., elements satisfying $e = e^2 \in S$. Observe that in an inverse semigroup all idempotents commute and satisfy $e^* = e$ (see [56] or [37], Theorem 3). Moreover, $E(S)$ has the structure of a meet semi-lattice with respect to the order $e \leq f \iff ef = e$, and $S$ is a group if and only if $E(S)$ only has a single element (the identity in the group). If one considers $S$ as contained in $I(S)$, then an idempotent $e \in S$ will be identified with the identity function $id_e: eS \to eS$.

Note also that $S$ is unital if and only if $E(S)$ has a greatest element. We shall assume that all our inverse semigroups are unital with unit denoted by $1$.

We will show in Theorems 4.27 and 5.4 that all the different amenability notions are again related in the inverse semigroup case, but not quite as elegantly intertwined as in groups (see Section 2). Such a conclusion might seem surprising, since it is known that the amenability of an inverse semigroup is closely related to the amenability of its group homomorphic image $G(S)$, as the following result of Duncan and Namioka in [21] shows. In the literature, this fact has led to the opinion that amenability of the inverse semigroup case can be traced back to the group case. Our results later will refine this line of thought.

Theorem 4.4. A countable discrete inverse semigroup $S$ is amenable if and only if the group $G(S)$ is amenable, where $G(S) = S/\sim$ and $s \sim t$ if and only if $es = et$ for some projection $e \in E(S)$.

4.1. A characterization of invariant measures. Recall from Definition 3.1 that a semigroup $S$ is called amenable if there is an invariant probability measure $\mu: \mathcal{P}(S) \to [0, 1]$. One handicap to this definition of amenability is that one loses contact with the notion of paradoxical decomposition, which was, from the beginning, close to amenability. In addition, a non-amenable semigroup does not naturally provide elements $s_1, t_2$ whose regular representation induce properly infinite projections in the $C^*$-algebra $\mathcal{R}_G$, as in the group case. Avoiding this drawback will be a critical step in the proof of Theorem 5.4. We will present an alternative approach taking into account the domain of the action of the semigroup. In this way we can directly relate the non-amenability of $S$ with the proper infiniteness of the identity of the associated $C^*$-algebra. However, before developing the new approach, we present some basic results for inverse semigroups. In particular, the following lemma, whose proof is elementary, will be very useful in the rest of the text.

Lemma 4.5. Let $S$ be an inverse semigroup. For any $s \in S$ and $A, B \subset S$ the following relations hold:

(i) $s (s^{-1}A \cap s^*ss^{-1}A) = A \cap ss^*A = ss^{-1}A \subset A \subset s^{-1}sA$.

(ii) $ss^* (A \cap ss^*B) = A \cap ss^*B$.

(iii) $s^{-1} (A \setminus ss^*A) = \emptyset$.

Proof. The inclusions $ss^{-1}A \subset A \subset s^{-1}sA$ follow directly from the definition and (ii) is straightforward to check. To show $s (s^{-1}A \cap s^*ss^{-1}A) = A \cap ss^*A$ choose $t \in s^{-1}A \cap s^*ss^{-1}A$. Then $st \in A$ and $t = s^*sq$ for some $q \in S$ with $sq \in A$, hence $st = ss^*sq \in ss^*A$. To show the reverse inclusion consider $t \in A \cap ss^*A$, i.e., $A \ni t = ss^*a$ for some $a \in A$. Then $s^*a \in s^{-1}A$ and $t = ss^*s(s^*a) \in s (s^{-1}A \cap s^*ss^{-1}A)$. The remaining equalities are proved in a similar vein.

The following result gives a useful characterization of invariant measures that avoids the use of preimages.

Proposition 4.6. Let $S$ be a countable and discrete inverse semigroup with identity $1 \in S$ and $\mu$ be a probability measure on it. Then the following conditions are equivalent:
(1) $\mu$ is invariant, i.e., $\mu(A) = \mu(s^{-1}A)$ for all $s \in S$, $A \subset S$. 
(2) $\mu$ satisfies the following conditions for all $s \in S$, $A \subset S$:

(2.a) $\mu(A) = \mu(A \cap ss^*A)$.
(2.b) $\mu(ss^*A) = \mu(A)$. 

Proof. For notational simplicity we show conditions (2.a) and (2.b) interchanging the roles of $s$ and $s^*$. The implication (1) $\Rightarrow$ (2) follows from two simple observations. First, note that 

$$\mu(A \setminus ss^*A) = \mu(s^{-1}(A \setminus ss^*A)) = \mu(\emptyset) = 0.$$ 

Therefore $\mu(A) = \mu(A \cap ss^*A) + \mu(A \setminus ss^*A) = \mu(A \cap ss^*A)$, as required. Secondly, observe that $ss^*A \subset ss^*A$. Thus 

$$\mu(ss^*A) = \mu(s^{-1}ss^*A) = \mu(s^{-1}ss^*A \setminus s^*A) + \mu(s^*A)$$

$$= \mu(s^{-1}(s^{-1}ss^*A \setminus s^*A)) + \mu(s^*A) = \mu(s^*A).$$

The reverse implication (2) $\Rightarrow$ (1) follows from (2.a), (2.b) and Lemma 4.5(i). In fact, 

$$\mu(s^{-1}A) = \mu(s^{-1}A \cap ss^*A) = \mu(s(s^{-1}A \cap ss^*A)) = \mu(A \cap ss^*A) = \mu(A),$$

which proves (1). \qed 

Remark 4.7. Observe that this characterization indeed restricts to the usual one in the case of groups since then $s^* = s^{-1}$ and $A \cap ss^*A = A$. Therefore condition (2.a) is empty in the group case. 

In the following corollary we combine conditions (2.a) and (2.b) into a single one. 

Corollary 4.8. Let $S$ be a countable and discrete inverse semigroup and $\mu$ be a probability measure on it. Then $\mu$ is invariant if and only if $\mu(A) = \mu(s(A \cap ss^*A))$ for all $s \in S$, $A \subset S$. 

Proof. Assume that $\mu$ is invariant, hence satisfies conditions (2.a) and (2.b). Since $A \cap ss^*A \subset ss^*A$ we have 

$$\mu(A) = \mu(A \cap ss^*A) = \mu(s(A \cap ss^*A)) = \mu(A \cap ss^*A).$$

To show the reverse implication we prove first condition (2.a) which follows from 

$$\mu(A) = \mu(s(A \cap ss^*A)) = \mu(s(s(A \cap ss^*A) \cap ss^*s(A \cap ss^*A)))$$

$$= \mu(s^*s(A \cap ss^*A) \cap ss^*s(A \cap ss^*A)) = \mu(A \cap ss^*A),$$

where for the last equation we used Lemma 4.5(ii). The condition (2.b) follows directly from $\mu(A) = \mu(s(A \cap ss^*A))$ just by replacing the set $A$ by $ss^*A$. \qed 

4.2. Domain measurable inverse semigroups. The characterization of an invariant measure $\mu$ given in Proposition 4.3 means that $\mu$ is measurable (see Definition 3.3) via $s$ but only when the action of $s$ is restricted to its domain (namely $\mu(s^*sA) = \mu(sA)$). In addition, the measure of any set $A$ is localized within the domain of every $s \in S$ (namely $\mu(A) = \mu(A \cap ss^*A)$).

We will show in Theorem 5.3 that a necessary and sufficient condition for the C*-algebra $\mathcal{R}_S$ to have an amenable trace is the measurability condition on domains given by $\mu(s^*sA) = \mu(sA)$. This fact justifies the next definition. 

Definition 4.9. Let $S$ be a countable and discrete inverse semigroup and $A \subset S$ a subset. Then $A$ is domain measurable if there is a measure $\mu : \mathcal{P}(S) \to [0, \infty]$ such that the following conditions hold:

(1) $\mu(A) = 1$.
(2) $\mu(s^*sB) = \mu(sB)$ for all $s \in S$ and $B \subset S$. 

We say that $S$ is domain measurable when the latter holds for $A = S$ and call the corresponding measures domain measures. 

Domain measurable semigroups can be understood as a possible generalization of the amenable groups. To further specify this idea see Theorem 6.4 and compare it to Theorem 2.4.
Remark 4.10. Recall from [20] that a semigroup is called fairly amenable if it has a probability measure μ such that

\[(4.1) \mu(A) = \mu(sA) \text{ if } s \text{ acts injectively on } A \subseteq S.\]

Observe that if μ satisfies Eq. (4.1), then it satisfies condition (2.b) in Proposition 4.8 as well, since s acts injectively on s^*sA. Therefore, if S is fairly amenable then it is also domain measurable. However, a domain measure satisfying the condition of domain measurability need not satisfy (4.1). In fact, consider an inverse semigroup S with a 0 element and some other element s ∈ S, s ≠ 0. Then S is domain measurable since it is amenable with an invariant measure μ satisfying μ(\{0\}) = 1. This measure, however, cannot implement fair amenability.

Example 4.11. We build a class of non-amenable, domain measurable semigroups. Let A, N be disjoint inverse semigroups, with A amenable and N non-amenable. Consider then the semigroup S = A ∪ N, where an := n =: na for every a ∈ A, n ∈ N. It is routine to show S is an inverse semigroup. Furthermore, we claim that it is non-amenable and domain measurable. Indeed, suppose it is amenable and let μ be an invariant measure on it. For any n ∈ N, μ(A) = μ(n^{-1}A) = μ(∅) = 0 and hence μ(N) = 1. Therefore μ would restrict to an invariant mean on N, contradicting the hypothesis.

To prove now that S is domain measurable, just choose an invariant measure ν on A (that exists since A is amenable) and extend it to S as \(\hat{\nu}(A' \cup N') = ν(A')\), for any A' ⊆ A and N' ⊆ N. This measure will satisfy \(\hat{\nu}(s^*sB) = \nu(sB)\) for every s ∈ S and B ⊆ S.

4.3. Representations of inverse semigroups. Following [37], we define a representation of a unital inverse semigroup S on a (discrete) set X as a unital semigroup homomorphism \(α : S \to \mathcal{I}(X)\). One can check that any action θ of S on X gives a representation α of S on X by the rule

\[α_s = (θ_s|_{θ^s(X)} : θ_{s^*s}(X) \to θ_{ss^*}(X)).\]

Indeed the domain for θ_s|_{θ^s(X)} ∩ θ_s^*s(X) is

\[θ_t(θ_{lt^*}(X) \cap θ_{s^*s}(X)) = θ_t(X) \cap θ_{t^*s}(X) = θ_{t^*s}s(X) = θ_{(st)^*s}(X).\]

If α is a representation, we denote by \(D_{ss^*}\) the domain of α_s. Note that α_s is a bijection from \(D_{ss^*}\) onto \(D_{ss^*}\), with inverse α_s^*.

Definition 4.12. Let \(α : S \to \mathcal{I}(X)\) be a representation of the inverse semigroup S on a set X. We define the type semigroup Typ(α) as the commutative monoid generated by symbols [A] with \(A ∈ \mathcal{P}(X)\) and relations

1. \([∅] = 0\).
2. \([A] = [α_s(A)]\) if \(A \subseteq D_{ss^*}\).
3. \([A ∪ B] = [A] + [B]\) if \(A ∩ B = ∅\).

This definition is very natural and allows to easily check if a map from Typ(α) to another semigroup is a homomorphism. We show next that Typ(α) is indeed isomorphic to a type semigroup which is constructed based on Tarski’s original ideas.

Definition 4.13. Let \(α : S \to \mathcal{I}(X)\) be a representation of the inverse semigroup S on a set X. We say A, B ⊆ X are equidecomposable, and write A ∼ B, if there are sets \(A_i ⊆ X\) and elements \(s_i ∈ S\), \(i = 1, …, n\), such that \(A_i ⊆ D_{s_is^*i}\) for \(i = 1, …, n\), and

\[A = A_1 ∪ … ∪ A_n \text{ and } α_{s_1}(A_1) ∪ … ∪ α_{s_n}(A_n) = B.\]

It is routine to show that ∼ is an equivalence relation. Indeed, note that since 1 ∈ S we have \(A ∼ A\). Furthermore the relation ∼ is clearly symmetric by choosing \(B_i := α_{s_i}(A_i)\) and the dynamics \(t_i := s_i^*\). Finally, if \(A ∼ B ∼ C\) then there are \(A_i, B_j ⊆ X\) and \(s_i, t_j ∈ S\) such that \(A_i ⊆ D_{s_is^*i}, B_j ⊆ D_{tjt_j}\), and

\[A = A_1 ∪ … ∪ A_n \text{ and } α_{s_1}(A_1) ∪ … ∪ α_{s_n}(A_n) = B,\]

\[B = B_1 ∪ … ∪ B_m \text{ and } α_{t_1}(B_1) ∪ … ∪ α_{t_m}(B_m) = C.\]
In this case the sets $A_{ij} = \alpha_{s_i}^j (\alpha_{s_i} (A_i) \cap B_j)$ and the elements $r_{ij} = t_j s_i$ implement the relation $A \sim C$.

Given a representation $\alpha : S \to \mathcal{I}(X)$, consider the following extensions:

- The semigroup $S \times \text{Perm} (\mathbb{N})$, where $\text{Perm} (\mathbb{N})$ is the finite permutation group of $\mathbb{N}$, that is the group of permutations moving only a finite number of elements.
- A set $A \subseteq X \times \mathbb{N}$ is called bounded if $A \subseteq X \times F$, with $F \subseteq \mathbb{N}$ finite. These sets are sometimes expressed as $A_1 \times \{i_1\} \cup \cdots \cup A_k \times \{i_k\}$ and, if so, each $A_j$ is called a level.

Then there is an obvious representation of $S \times \text{Perm}$ on $X \times \mathbb{N}$ given coordinate-wise, which will be also denoted by $\alpha$. Hence, it makes sense to ask when two bounded sets $A, B \subseteq X \times \mathbb{N}$ are equidecomposable. Define

$$\hat{X} := \{ A \subseteq X \times \mathbb{N} \mid A \text{ is bounded}\} / \sim .$$

This set has the natural structure of a commutative monoid with $0 = \emptyset$ and sum defined as follows. Given two bounded sets $A, B \subseteq X \times \mathbb{N}$, let $k \in \mathbb{N}$ be such that $A \cap B^c = \emptyset$, where $B^c := \{(b, n + k) \mid (b, n) \in B\}$. Then define $[A] + [B] := [A \cup B^c]$. One can verify that $+$ is well-defined, associative and commutative. This construction was first done by Tarski in [53], and has been used since then extensively (see, e.g., [55, 48, 45, 43]). We only need the notation $\hat{X}$ temporarily and after the proof of the following result we will only use the symbol $\text{Typ}(\alpha)$.

**Proposition 4.14.** Let $\alpha$ be a representation of the unital inverse semigroup $S$ on $X$. Then the map

$$\gamma : \text{Typ}(\alpha) \to \hat{X}, \quad \gamma([A]) = [A \times \{1\}]$$

is a monoid isomorphism.

**Proof.** Since $[A \times \{1\}] = [A \times \{i\}]$ in $\hat{X}$, we easily see that this map is well-defined and surjective. To show it is injective, assume that $\gamma(\sum_{i=1}^n [A_i]) = \gamma(\sum_{j=1}^m [B_j])$ for subsets $A_i, B_j$ of $X$. Then

$$A := \bigsqcup_{i=1}^n A_i \times \{i\} \sim \bigsqcup_{j=1}^m B_j \times \{j\} =: B,$$

and so by definition there are subsets $W_1, \ldots, W_l$ of $X$, and numbers $n_1, \ldots, n_l, m_1, \ldots, m_l \in \mathbb{N}$ and elements $s_1, \ldots, s_l \in S$ such that $W_k \subseteq D_{s_k}^{s_k}$ for $k = 1, \ldots, l$ and

$$A = \bigsqcup_{k=1}^l W_k \times \{n_k\}, \quad B = \bigsqcup_{k=1}^l \alpha_{s_k} (W_k) \times \{m_k\}.$$

It follows that there is a partition $\{1, \ldots, l\} = \bigsqcup_{i=1}^n I_i$ such that for each $i \in \{1, \ldots, n\}$ we have $A_i = \bigsqcup_{j \in I_i} W_j$. We thus get in $\text{Typ}(\alpha)$

$$\sum_{i=1}^n [A_i] = \sum_{i=1}^n \sum_{j \in I_i} [W_j] = \sum_{i=1}^n \sum_{j \in I_i} [\alpha_{s_j} (W_j)] = \sum_{j=1}^m [B_j],$$

showing injectivity. \qed

For simplicity we will often denote $\alpha_s(x) \in X$ by $sx$ and $sA$ will stand for $\alpha_s(A)$ for any $s \in S$, $x \in X$ and $A \subseteq X$. Recall that $sx$ is defined only if $x \in D_{s^*} s$. We extend next Definition 4.9 above to representations.

**Definition 4.15.** Let $\alpha : S \to \mathcal{I}(X)$ be a representation of $S$ and let $A \subseteq X$ be a subset.

1. The set $A$ is $S$-domain measurable if there is a measure $\mu : \mathcal{P}(X) \to [0, \infty]$ satisfying the following conditions:
   - (a) $\mu(A) = 1$.
   - (b) $\mu(B) = \mu(sB)$ for all $s \in S$ and $B \subseteq D_{s^*} s$.

   We say that $X$ is $S$-domain measurable when the latter holds for $A = X$.  

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**AMENABILITY AND PARADOXICALITY IN SEMIGROUPS AND C*-ALGEBRAS 13**
(2) The set $A$ is $S$-domain Folner if there is a sequence $\{F_n\}_{n \in \mathbb{N}}$ of finite, non-empty subsets of $A$ such that
\[
\lim_{n \to \infty} \frac{|s(F_n \cap D^*_s s) \setminus F_n|}{|F_n|} = 0
\]
for all $s \in S$.

(3) The set $A$ is $S$-paradoxical if there are $A_i, B_j \subset X$ and $s_i, t_j \in S$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, such that $A_i \subseteq D^*_s s_i$, $B_j \subseteq D^*_t t_j$ and
\[
A = s_1 A_1 \cup \ldots \cup s_n A_n = t_1 B_1 \cup \ldots \cup t_m B_m
\]
\[
\supseteq A_1 \cup \ldots \cup A_n \cup B_1 \cup \ldots \cup B_m.
\]

Remark 4.16. (i) Note that $S$ is domain measurable in the sense of Definition 4.9 precisely when $S$ is $S$-domain measurable with respect to the canonical representation $\alpha: S \to \mathcal{I}(S)$.

(ii) Note also that $A \subset X$ being paradoxical is the same as saying that $2[A] \leq [A]$ in $\text{Typ}(\alpha)$.

Recall that, in a commutative semigroup $S$, we denote by $n \cdot \beta$ the sum $\beta + \cdots + \beta$ of $n$ terms. Also, the only (pre-)order that we use on $S$ is the so-called algebraic pre-order, defined by $x \leq y$ if and only if $x + z = y$ for some $z \in S$.

Lemma 4.17. Let $\alpha: S \to \mathcal{I}(X)$ be a representation of $S$, and consider the type semigroup $\text{Typ}(\alpha)$ constructed above. Then the following hold:

1. For any bounded sets $A, B \subset X \times X$ if $A \sim B$, then there exists a bijection $\phi: A \to B$ such that for any $C \subset A$ and $D \subset B$ one has $C \sim \phi(C)$ and $D \sim \phi^{-1}(D)$.
2. For any $[A], [B] \in \text{Typ}(\alpha)$, if $[A] \leq [B]$ and $[B] \leq [A]$, then $[A] = [B]$.
3. A subset $A$ of $X$ is $S$-paradoxical if and only if $[A] = 2 \cdot [A]$.
4. For any $[A], [B] \in \text{Typ}(\alpha)$ and $n \in \mathbb{N}$, if $n \cdot [A] = n \cdot [B]$, then $[A] = [B]$.
5. If $[A] \in \text{Typ}(\alpha)$ and $(n + 1) \cdot [A] \leq n \cdot [A]$ for some $n \in \mathbb{N}$, then $[A] = 2 \cdot [A]$.

Proof. The proof of this lemma is virtually the same as in the group case (see, e.g. [50 p. 10]). For convenience of the reader we include a sketch of the proofs.

To construct the bijection $\phi: A \to B$ in (1) just define it by multiplication by $s_i$ in each of the subsets $A_i$, where $A = \bigcup_i A_i$ and $B = \bigcup_i s_i A_i$.

(2) There are $[A_0], [B_0] \in \text{Typ}(\alpha)$ such that $[A] + [A_0] = [B]$ and $[B] + [B_0] = [A]$. In this case without loss of generality we can suppose that $A \cap A_0 = \emptyset = B \cap B_0$. Choose $\phi: A \cup A_0 \to B$ and $\psi: B \cup B_0 \to A$ as in (1) and consider
\[
C_0 := A_0, \ C_{n+1} := \psi(\phi(C_n)) \text{ and } C := \cup_{n=0}^\infty C_n.
\]
It then follows that $(B \cup B_0) \setminus \phi(C) = \psi^{-1}(A \setminus C) = \psi^{-1}(A \cup A_0 \setminus C)$ and hence
\[
A \cup A_0 = (A \setminus C) \cup C \sim \psi^{-1}(A \setminus C) \cup \phi(C) = (B \cup B_0) \setminus \phi(C) \cup \phi(C) = B \cup B_0.
\]
Therefore $B \sim A \cup A_0 \sim B \cup B_0 \sim A$.

Now (3) follows from the definitions and (2).

Claim (1) uses graph theory and follows from König’s Theorem (see [43 Theorem 0.2.4]). If $n \cdot [A] = n \cdot [B]$ then there are sets $A_i, B_j$ with the following properties:

(a) $A_1, \ldots, A_n$ are pairwise disjoint, just as $B_1, \ldots, B_n$.
(b) $n \cdot [A] = [A_1] + \cdots + [A_n] = [B_1] + \cdots + [B_n] = n \cdot [B]$.
(c) For every $i = 1, \ldots, n$ we have $A_i \sim A$ and $B_i \sim B$.

Consider then the bijections $\phi_j: A_1 \to A_j$, $\psi_j: B_i \to B_j$ and $\chi: n \cdot [A] \to n \cdot [B]$ induced by $\sim$, as in (1). For $a \in A_1$ denote by $\overline{a}$ the set $\{\phi_1(a), \ldots, \phi_n(a)\}$ (and analogously for $b \in B_1$). Consider now the bipartite graph defined by:

- Its sets of vertices are $X = \{\overline{a} \mid a \in A_1\}$ and $Y = \{\overline{b} \mid b \in B_1\}$.
- The vertices $\overline{a}$ and $\overline{b}$ are joined by an edge if $\chi(\phi_j(a)) \in \overline{b}$ for some $j = 1, \ldots, n$. 
Then this graph is \( n \)-regular and, by König’s Theorem, it has a perfect matching \( F \). In this case it can be checked that the sets
\[
C_{j,k} := \{ a \in A_1 \mid \exists b \in B_1 \text{ such that } (\pi, b) \in F \text{ and } \chi (\phi_j (a)) = \psi_k (b) \},
\]
\[
D_{j,k} := \{ b \in B_1 \mid \exists a \in A_1 \text{ such that } (\pi, b) \in F \text{ and } \chi (\phi_j (a)) = \psi_k (b) \},
\]
are respectively a partition of \( A_1 \) and \( B_1 \). Furthermore \( \psi_k^{-1} \circ \chi \circ \phi_j \) is a bijection from \( C_{j,k} \) to \( D_{j,k} \) implementing the relations \( C_{j,k} \sim D_{j,k} \). These, in turn, implement \( A_1 \sim A_1 \sim B_1 \sim B_1 \).

Finally, \( \square \) follows from (2) and (4). Indeed, from the hypothesis
\[
2 \cdot [A] + n \cdot [A] = (n+1) \cdot [A] + [A] \leq n \cdot [A] + [A] = (n+1) \cdot [A] \leq n \cdot [A].
\]
Iterating this argument we get \( 2n \cdot [A] \leq n \cdot [A] \) and, since the other inequality trivially holds, \( n \cdot [A] = 2n \cdot [A] \). Applying (4) we conclude that \([A] = 2 \cdot [A] \). \( \square \)

Finally, we next recall one of Tarski’s fundamental results \( 53 \) (see also \( 50 \) Theorem 0.2.10]).

**Theorem 4.18.** Let \( (S, +) \) be a commutative semigroup with neutral element 0 and let \( \epsilon \in S \). The following are then equivalent:

(i) \((n+1) \cdot \epsilon \leq n \cdot \epsilon \) for all \( n \in \mathbb{N} \).

(ii) There is a semigroup homomorphism \( \nu : (S, +) \to ([0, \infty], +) \) such that \( \nu (\epsilon) = 1 \).

In order to prove the main result of the section (see Theorem 4.23) we need to introduce several actions of the inverse semigroup \( S \) on canonical spaces associated with \( X \). In particular, we will consider the behavior of domain measures as functionals on \( \ell^\infty (X) \). Given a representation \( \alpha : S \to I(X) \) we define the action of \( S \) on \( \ell^\infty (X) \) by
\[
(s f)(x) := \begin{cases} f(s^* x) & \text{if } x \in D_{s^* s}, \\ 0 & \text{if } x \notin D_{s^* s}. \end{cases}
\]
The next result establishes an invariance condition in the context of states on \( \ell^\infty (X) \).

**Proposition 4.19.** Let \( \alpha : S \to I(X) \) be a representation of \( S \). If \( X \) is domain measurable, with domain measure \( \mu \) (cf., Definition 4.17), then there is a state \( m : \ell^\infty (X) \to \mathbb{C} \) such that
\[
m (s f) = m (f P_{s^* s}), \quad f \in \ell^\infty (X), \quad s \in S,
\]
where \( P_{s^* s} \) denotes the characteristic function of \( D_{s^* s} \subset X \).

**Proof.** For a set \( B \subset S \) define \( m (P_B) := \mu (B) \), where \( P_B \) denotes the characteristic function on \( B \), and extend the definition by linearity to simple functions and by continuity to all \( \ell^\infty (X) \). Then \( m \) satisfies Eq. (4.3) if and only if it does for any characteristic function \( P_B \), and this is a consequence of the domain measurability of \( \mu \). Indeed, observe that
\[
s P_B = P_{s(B \cap D_{s^* s})}
\]
and hence, by domain measurability, we obtain
\[
m (s P_B) = m (P_{s(B \cap D_{s^* s})}) = \mu (s (B \cap D_{s^* s})) = \mu (B \cap D_{s^* s}) = m (P_{B \cap D_{s^* s}}) = m (P_B P_{s^* s}),
\]
as claimed. \( \square \)

We next observe that the functional \( m \) in the latter proposition can be approximated by functionals of finite support.

**Lemma 4.20.** Let \( \alpha : S \to I(X) \) be a representation of \( S \). If \( X \) is domain measurable then for every \( \epsilon > 0 \) and finite \( F \subset S \) there is a positive \( h \in \ell^1 (X) \) of norm one such that \( ||s^* h - s^* s h||_1 < \epsilon \) for every \( s \in F \). Furthermore, the support of \( h \) is finite.

**Proof.** We denote by \( \Omega \) the set of positive \( h \in \ell^1 (X) \) of norm one and finite support. By Proposition 4.19 there is a functional \( m : \ell^\infty (X) \to \mathbb{C} \) such that \( m (s f) = m (s^* s f) \) for every \( s \in S, f \in \ell^\infty (X) \). Since the normal states are weak-* dense in \( \ell^\infty (X)^* \) there is a net \( \{ h_\lambda \}_{\lambda \in \Lambda} \) in \( \Omega \) such that
\[
|\phi_{h_\lambda} (s f) - \phi_{h_\lambda} (s^* s f)| = |\phi_{s^* s h_\lambda} (f) - \phi_{s^* s h_\lambda} (f)| \to 0,
\]
for every \( s \in S \).
where \( \phi_h(f) = \sum_{x \in X} h(x) f(x) \) and \( h \in \Omega, f \in \ell^\infty(X) \). In order to transform the latter weak convergence to norm convergence, we shall use a variation of a standard technique (see [18, 11, 16]). Consider the space \( E = (\ell^1(X))^S \), which, when equipped with the product topology, is a locally convex linear topological space. Consider the map

\[
T: \ell^1(X) \to E, \quad h \mapsto T(h) = (s^* h - s^* s h)_s \in S.
\]

Since the weak topology coincides with the product of weak topologies on \( E \), it follows from Eq. (4.4) that 0 belongs to the weak closure of \( T(\Omega) \). Furthermore, since \( E \) is locally convex and \( T(\Omega) \) is convex, its closure in the weak topology and in the product of the norm topologies are the same. Thus we may suppose that the net \( \{h_\lambda\}_{\lambda \in \Lambda} \) actually satisfies that \( \|s^* h_\lambda - s^* s h_\lambda\|_1 \to 0 \) for all \( s \in S \), which completes the proof.

The following lemma is straightforward to check, but we mention it for convenience of the reader.

**Lemma 4.21.** Let \( X \) be a set. Any \( h \in \ell^1(X)_+ \) of norm 1 and finite support can be written as

\[
h = (\beta_1/|A_1|) P_{A_1} + \cdots + (\beta_N/|A_N|) P_{A_N}
\]

for some finite \( A_1 \supset A_2 \supset \cdots \supset A_N \), where \( \beta_i \geq 0 \) and \( \sum_{i=1}^N \beta_i = 1 \).

**Proof.** Let \( 0 = a_0 < a_1 < \cdots < a_N \) be the distinct values of the function \( h \). Then, defining \( A_i := \{x \in X \mid a_i \leq h(x)\} \) we have that \( A_1 \supset A_2 \supset \cdots \supset A_N \). Furthermore \( h = \sum_{i=1}^N \gamma_i P_{A_i} \), where \( \gamma_i = a_i - a_{i-1} \) for \( i \geq 1 \). To conclude the proof put \( \beta_i := \gamma_i/|A_i|, i = 1, \ldots, N \), and note that \( \|h\|_1 = 1 \) implies \( \sum_{i=1}^N \beta_i = 1 \).

**Lemma 4.22.** Let \( \alpha: S \to \mathcal{I}(X) \) be a representation of \( S \) and consider \( s \in S, A \subset X \). Then \( (s^* P_A - s^* s P_A)(x) < 0 \) if and only if \( x \in A \cap D_{s^* s} \setminus s^*(A \cap D_{s^* s}) \).

**Proof.** By definition of the action given in Eq. (4.2) we compute

\[
(s^* P_A - s^* s P_A)(x) = \begin{cases} 
1 & \text{if } x \in D_{s^* s} \setminus A \text{ and } s x \in A \\
-1 & \text{if } x \in A \setminus D_{s^* s} \text{ and } s x \not\in A \\
0 & \text{otherwise.}
\end{cases}
\]

Thus, if \( (s^* P_A - s^* s P_A)(x) < 0 \) then \( x \in A \cap D_{s^* s} \setminus s^*(A \cap D_{s^* s}) \). The other implication is clear.

We can finally establish the main theorem of the section, characterizing the domain measurable representations of an inverse semigroup.

**Theorem 4.23.** Let \( S \) be a countable and discrete inverse semigroup with identity \( 1 \in S \) and \( \alpha: S \to \mathcal{I}(X) \) be a representation of \( S \) on \( X \). The following are then equivalent:

1. \( X \) is \( S \)-domain measurable.
2. \( X \) is not \( S \)-paradoxical.
3. \( X \) is \( S \)-domain Følner.

**Proof.** \( \Box \Rightarrow \blacksquare \). Suppose \( X \) is \( S \)-paradoxical. Then, choosing a domain measure \( \mu \) and an \( S \)-paradoxical decomposition of \( X \) we would have

\[
1 = \mu(X) \geq \mu(A_1) + \cdots + \mu(A_n) + \mu(B_1) + \cdots + \mu(B_m)
\]

\[
= \mu(s_1 A_1 \sqcup \cdots \sqcup s_n A_n) + \mu(t_1 B_1 \sqcup \cdots \sqcup t_m B_m) = \mu(X) + \mu(X) = 2,
\]

which gives a contradiction.

\( \blacksquare \Rightarrow \Box \). Consider the type semigroup \( \text{Typ}(\alpha) \) of the action. As \( X \) is not \( S \)-paradoxical we know that \( [X] \) and \( 2 \cdot [X] \) are not equal in \( \text{Typ}(\alpha) \). It follows from Lemma 4.17(5) that \( (n + 1) \cdot [X] \not\leq n \cdot [X] \) and hence, by Tarski’s Theorem [11, 18], there exists a semigroup homomorphism \( \nu: \text{Typ}(\alpha) \to [0, \infty] \) such that \( \nu([X]) = 1 \). Then we define \( \mu(B) := \nu([B]) \) which satisfies \( \mu(X) = 1 \) and \( \mu(B) = \mu(s B) \) for every \( B \subset D_{s^* s} \), proving that \( X \) is \( S \)-domain measurable.

\( \Box \Rightarrow \blacksquare \). Let \( \{F_n\}_{n \in \mathbb{N}} \) be a sequence witnessing the \( S \)-domain Følner property of \( X \) and let \( \omega \) be a free ultrafilter on \( \mathbb{N} \). Consider the measure \( \mu \) defined by

\[
\mu(B) := \lim_{n \to \omega} |B \cap F_n| / |F_n|.
\]
It follows from $\omega$ being an ultrafilter that $\mu$ is a finitely additive measure. Thus it remains to prove that $\mu(B) = \mu(sB)$ for any $B \subset D_{ss^*}$. Observe first that $s$ acts injectively on $B$. Therefore we have

$$|B \cap F_n| = |s(B \cap F_n)| \leq |sB \cap (F_n \cap D_{ss^*}) \cap F_n| + |(sB \cap s(F_n \cap D_{ss^*})) \setminus F_n|$$

and hence, normalizing by $|F_n|$ and taking ultralimits on both sides, we obtain $\mu(B) \leq \mu(sB)$. The other inequality follows from a similar argument, noting that

$$|s^*(sB \cap F_n)| = |sB \cap F_n|$$

since $s^*$ acts injectively on $sB$.

By Lemma 4.22, the function $h = \sum_{i=1}^{N} \beta_i P_{A_i}$ where $A_1 \supset A_2 \supset \cdots \supset A_N$ and $\sum_{i=1}^{N} \beta_i = 1$. Consider now the set $B_s := \bigcup_{i=1}^{N} (A_i \cap D_{ss^*}) \setminus s^*(A_i \cap D_{ss^*})$, By Lemma 4.22 the function $s^*h - s^*sh$ is non-negative on $X \setminus B_s$ and hence

$$\varepsilon / |F| \geq ||s^*h - s^*sh||_1 \geq \sum_{x \in X \setminus B_s} s^*h(x) - s^*sh(x) = \sum_{i=1}^{N} \beta_i \left( \sum_{x \in X \setminus B_s} \left( s^*P_{A_i} (x) - s^*sP_{A_i} (x) \right) \right)$$

$$= \sum_{i=1}^{N} \beta_i (s^*(A_i \cap D_{ss^*}) \setminus A_i \cap D_{ss^*}) \setminus A_i |A_i|$$

(4.5)

Observe that the last inequality follows from the fact that the sets $A_i$ are nested. Indeed, writing $Z_i = A_i \cap D_{ss^*}$ and $T_i = s^*(A_i \cap D_{ss^*})$ and denoting by $Y^c$ the complement in $X$ of a subset $Y$, we need to show that

$$Z_i^c \cap T_i \subset B_s^c = \bigcap_{j=1}^{N} (Z_j^c \cup T_j).$$

Now, if $i \geq j$ then $A_i \subset A_j$ and so $T_i \subset T_j$ which implies that $Z_i^c \cap T_i \subset Z_j^c \cup T_j$. If $i < j$ then $A_j \subset A_i$ and so $Z_j \subset Z_i$ which implies $Z_i^c \subset Z_j^c$ and so $Z_i^c \cap T_i \subset Z_j^c \cup T_j$. This shows (4.6). The rest of the proof is similar to [11]. Denote by $I = \{1, \ldots, N\}$ and consider the measure on $I$ given by $\mu(J) = \sum_{j \in J} \beta_j$ for every $J \subset I$ and put $\mu(\emptyset) := 0$. For $s \in F$ consider the set

$$K_s := \{ i \in I \mid |s(A_i \cap D_{ss^*}) \setminus A_i | < \varepsilon |A_i| \}.$$

From Eq. (4.5) it follows that

$$\varepsilon / |F| \geq \sum_{i=1}^{N} \beta_i \left( s^*(A_i \cap D_{ss^*}) \setminus A_i |A_i| \right) \geq \varepsilon \sum_{i \in I \setminus K_s} \beta_i = \varepsilon \mu(I \setminus K_s)$$

and, thus, $\mu(I \setminus K_s) < 1 / |F|$. From this and since $F = F^*$ we obtain

$$1 - \mu(\cap_{s \in F} K_s) = \mu(I \setminus \cap_{s \in F} K_s) = \mu(\cup_{s \in F} I \setminus K_s) \leq \sum_{s \in F} \mu(I \setminus K_s) < 1.$$
Remark 4.24. We mention here that the theory of type semigroups for representations of inverse semigroups includes the corresponding theory for partial actions of groups. Given a (discrete) group \(G\) and a non-empty set \(X\), Exel defines the notion of a partial action of \(G\) on \(X\) (see, e.g., [25]). In this context one can associate in a natural way the type semigroup \(\text{Typ}(X,G)\) to the given partial action (see e.g. [2, Section 7]). Moreover, in [24] Exel associates to each group \(G\) an inverse semigroup \(S(G)\) such that the partial actions of \(G\) on \(X\) are in bijective correspondence with the representations \(\alpha: S(G) \to \mathcal{I}(X)\). (Note that representations of inverse semigroups are called actions in [24].) In this context, it can be shown, using the abstract definitions of these semigroups, that the type semigroup \(\text{Typ}(\alpha)\) introduced in Definition 4.12 is naturally isomorphic to the type semigroup \(\text{Typ}(X,G)\) of the corresponding partial action of \(G\) on \(X\).

4.4. Amenable inverse semigroups. The goal of this section is to prove the analogue of Theorem 4.23 but considering amenable representations instead of the weaker notion of domain measurable ones. Therefore we will have to refine the reasoning of the previous section including the localization condition. In fact, let us first recall that by Proposition 4.6 the classical definition of invariant measure given by Day can be characterized by domain measurability and the condition

\[\mu \left( A \cap s^* sA \right) = \mu \left( A \right), \quad s \in S, \quad A \subset S.\]

Noting that, since \(A \cap s^* sA = A \cap s^* sS\), we can call this property localization, for the measure \(\mu\) is concentrated in the domain of the projection \(s^* s \in E(S)\). We now extend this definition to the context of representations.

Definition 4.25. Let \(\alpha: S \to \mathcal{I}(X)\) be a representation of the inverse semigroup \(S\) and let \(A \subset X\) be a subset. Then \(A\) is \(S\)-amenable when there is a measure \(\mu: \mathcal{P}(X) \to [0, \infty]\) such that:

1. \(\mu(A) = 1\).
2. \(\mu(B) = \mu(\alpha_s(B))\) for all \(s \in S\) and \(B \subset D^*_s s\).
3. \(\mu(B) = \mu(B \cap D^*_t t)\) for all \(t \in S\) and \(B \subset X\).

We say that \(X\) is \(S\)-amenable when the latter holds for \(A = X\).

The following Lemma is just a simple observation, but it will be useful for later use.

Lemma 4.26. Every countable and inverse semigroup \(S\) has a decreasing sequence of projections \(\{e_n\}_{n \in \mathbb{N}}\) that is eventually below every other projection, that is, \(e_n \geq e_{n+1}\) and for every \(f \in E(S)\) there is some \(n_0 \in \mathbb{N}\) such that \(f \geq e_{n_0}\).

Proof. Since \(S\) is countable we can enumerate the set of projections \(E(S) = \{f_1, f_2, \ldots\}\). The Lemma follows by letting \(e_n := f_1 \ldots f_n\). \(\square\)

The localization property of the measure can be included in the reasoning leading to Theorem 4.23 and this yields the following theorem.

Theorem 4.27. Let \(S\) be a countable and discrete inverse semigroup with identity \(1 \in S\) and \(\alpha: S \to \mathcal{I}(X)\) be a representation of \(S\) on \(X\). Then the following conditions are equivalent:

1. \(X\) is \(S\)-amenable.
2. \(D_e\) is not \(S\)-paradoxical for any \(e \in E(S)\).
3. For every \(\varepsilon > 0\) and finite \(F \subset X\) there is a finite non-empty \(F \subset X\) such that \(F \subset D^*_s s\) and \(|sF \setminus F| < \varepsilon |F|\) for all \(s \in F\).

Proof. Observe the equivalence between (1) and (2) follows from Theorem 4.23 and Lemma 4.26. Indeed, one can check that the proof of (1) \(\Leftrightarrow\) (2) in Theorem 4.23 works for any subset \(A \subset X\), in particular if \(A = D_e\). Thus, if \(D_e\) is not paradoxical for any projection \(e\) then there are measures \(\mu_e\) on \(X\) such that \(\mu_e(D_e) = 1\). Now, by Lemma 4.26 let \(\{e_n\}_{n \in \mathbb{N}}\) be a decreasing sequence of projections that is eventually below every other projection. A measure in \(X\) can be given by:

\[\mu(B) := \lim_{n \to \omega} \mu_{e_n}(B \cap D_{e_n}), \quad B \subset X,\]

where \(\omega\) is a free ultrafilter of \(\mathbb{N}\). It is routine to show that \(\mu\) is then a probability measure on \(X\) satisfying the domain measurability and localization conditions mentioned above, i.e., \(\mu(A) = \mu(sA)\) when \(A \subset D^*_s s\) and \(\mu(B) = \mu(B \cap D^*_s s)\) for every \(s \in S\), \(B \subset X\).
In order to prove \((\mathfrak{I}) \Rightarrow (\mathfrak{II})\) observe that the condition \((\mathfrak{I})\) ensures the existence of a domain Følner sequence \(\{F_n\}_{n \in \mathbb{N}}\) such that for every \(s \in S\) there is a number \(N \in \mathbb{N}\) with \(F_n \subset D_{s^*s}\) for all \(n \geq N\). Consider then a free ultrafilter \(\omega\) on \(\mathbb{N}\) and the measure
\[
\mu(B) := \lim_{n \to \omega} |B \cap F_n| / |F_n|.
\]
It follows from Theorem 4.23 that \(\mu\) is a domain measure, which, in addition, satisfies that
\[
\mu(D_{s^*s}) = \lim_{n \to \omega} |D_{s^*s} \cap F_n| / |F_n| = \lim_{n \to \omega} |F_n| / |F_n| = 1,
\]
for all \(s \in S\), i.e., the measure is localized.

We will only sketch the proof \((\mathfrak{II}) \Rightarrow (\mathfrak{I})\) since it is just a refinement of the same reasoning as in Theorem 4.23. Let \(\mu\) be an invariant measure on \(X\). The corresponding mean \(m: \ell^\infty(X) \to C\) (see Proposition 4.19) satisfies \(m(P_{s^*s}) = 1\) for all \(s \in S\). Then, any net \(h_\lambda\) converging to \(m\) in norm must also satisfy \(\|h_\lambda(1 - P_{s^*s})\|_1 \to 0\) for all \(s \in S\). In particular, this must also be the case for the approximation \(h\) appearing in Lemma 4.20. To get the desired Følner set, we have to cut \(h\) so that its whole support is within \(D_{s^*s}\) for all \(s \in F\). For this, consider \(F = \{s_1, \ldots, s_k\}\) and define the function
\[
g := \frac{h P_{s_1^*s_1 \ldots s_k^*s_k}}{\|h P_{s_1^*s_1 \ldots s_k^*s_k}\|_1}.
\]
The function \(g\) has norm 1, is positive and has finite support, which is contained in \(D_{s^*s}\) for all \(s \in F\). Furthermore \(\|h - g\|_1 \leq \varepsilon\). Thus, by substituting \(h\) by \(g\) in the proof of Theorem 4.23 and following the same construction, we obtain a Følner set \(F\) within the support of \(g\), that is, a Følner set within the requirements of the theorem.

**Remark 4.28.** Following results in [29, Theorem 3.1] one has that for inverse semigroups amenability is equivalent to the Følner condition and a local injectivity condition on the Følner sets. The preceding theorem is an improvement of Gray and Kambites’ result applied to inverse semigroups. Amenability gives, in fact, that Følner sets can be taken within the corresponding domains and, thus, the local injectivity condition is guaranteed by the localization property of the measures.

Note that Theorem 4.27 is not constructive in the sense that if \(X\) is not amenable, then one knows some \(D_{e_0}\) is paradoxical, but Theorem 4.27 does not tell which element \(e_0\) satisfies this condition. In the case \(S\) has a minimal projection \(e_0\) then one can improve the preceding Theorem. To do this we first remark the following simple and useful lemma.

**Lemma 4.29.** Let \(S\) be a discrete and countable inverse semigroup with a minimal projection \(e_0 \in E(S)\). Then \(e_0\) commutes with every \(s \in S\).

**Proof.** Note that \(e_0 = e_0 se_0s^* = s^*e_0se_0\) by the minimality of \(e_0\). Thus, we obtain
\[
e_0 s = e_0 se_0s^* s = e_0 e_0 s = s s^* e_0 e_0 = s e_0,
\]
where, for the first equality we have multiplied the identity \(e_0 = e_0 se_0s^*\) from the right by \(s\), and for the last equality we have multiplied the identity \(e_0 = s^*e_0se_0\) from the left by \(e_0\). This proves the claim. \(\square\)

**Proposition 4.30.** Let \(S\) be a discrete, countable inverse semigroup with a minimal projection \(e_0 \in E(S)\) and let \(\alpha: S \to \mathcal{T}(X)\) be a representation. Then the following conditions are equivalent:

1. \(X\) is \(S\)-amenable.
2. \(D_{e_0}\) is not \(S\)-paradoxical.
3. \(D_{e_0}\) is \(S\)-domain Følner.

**Proof.** The implication \((\mathfrak{II}) \Rightarrow (\mathfrak{I})\) is a particular case of Theorem 4.27. For \((\mathfrak{I}) \Rightarrow (\mathfrak{II})\) note that it follows from Lemma 4.29 that \(D_{e_0} \subset X\) is an invariant subset for the action. Indeed, for any \(s \in S\)
\[
s(D_{e_0} \cap D_{s^*s}) = sD_{e_0} \subset D_{e_0}.
\]
The implication \((\mathfrak{II}) \Rightarrow (\mathfrak{III})\) therefore follows from Theorem 4.23 by considering, if necessary, the induced action of \(S\) on \(D_{e_0}\). Finally, \((\mathfrak{III}) \Rightarrow (\mathfrak{II})\) can be proven in a similar fashion to that of \((\mathfrak{III}) \Rightarrow (\mathfrak{I})\) in Theorem 4.27. \(\square\)
As an example of an inverse semigroup $S$ with a minimal projection we consider the case where $S$ satisfies the Følner condition but not the proper Følner condition (see Definition 3.1).

**Proposition 4.31.** Let $S$ be a countable and discrete inverse semigroup. Suppose $S$ satisfies the Følner condition but not the proper Følner condition. Then $S$ has a minimal projection.

**Proof.** Following Theorem 3.8 there is an element $a \in S$ such that $|Sa| < \infty$. Suppose $Sa = \{s_1 a, \ldots, s_k a\}$. Then we claim $e := s_1^* s_1 \ldots s_k^* s_k a a^* \in S$ is a minimal projection. Indeed, for any other projection $f \in E(S)$ there is an $i$ such that $s_i a = f a$. In this case we have

$$a^* f a = (f a)^* f a = (s_i a)^* s_i a = a^* s_i^* s_i a.$$  

Thus, multiplying by $a$ from the left, $aa^* f a = f a = aa^* s_i^* s_i a = s_i^* s_i a$. Therefore

$$f \geq f a a^* = s_i^* s_i a a^* \geq e,$$

proving that $e$ is indeed minimal. 

Note that, in order to produce an example of an inverse semigroup $S$ that is Følner but not proper Følner, as in the hypothesis of Proposition 4.31, $S$ must have a minimal projection $e_0 \in E(S)$. Moreover, by Proposition 4.30 the domain $E_0$ must be domain-Følner. It can thus be shown that $S := \mathbb{F}_2 \cup \{0\}$ satisfies the Følner condition but not the proper one.

5. **Inverse semigroups, C*-algebras and traces**

In this final part of the article we connect the analysis of the previous section with properties of a C*-algebra $R_X$ generalizing the uniform Roe algebra $R_G$ of a group $G$ presented in Section 2. In particular, we will show that domain measurability completely characterizes the existence of traces in $R_X$.

Let $S$ be a discrete inverse semigroup with identity and consider a representation $\alpha : S \to \mathcal{I}(X)$ on a set $X$. As before, we will denote $\alpha_s(x)$ simply by $sx$ for any $s \in S, x \in D_{s^* s}$. To construct the C*-algebra $R_X$ consider first the representation $V : S \to B(\ell^2(X))$ given by

$$V_s \delta_x := \begin{cases} \delta_{sx} & \text{if } x \in D_{s^* s} \\ 0 & \text{if } x \notin D_{s^* s} \end{cases},$$

where $\{\delta_x\}_{x \in X} \subset \ell^2(X)$ is the canonical orthonormal basis. $V$ is a *-representation of $S$ in terms of partial isometries of $\ell^2(X)$. Define the unital *-subalgebra $R_X, \text{alg}$ in $B(\ell^2(X))$ generated by $\{V_s \mid s \in S\}$ and $\ell^\infty(X)$. The C*-algebra $R_X$ is defined as the norm closure of $R_X, \text{alg}$, i.e.,

$$R_X := R_{X, \text{alg}} = \text{C}^* \left( \{V_s \mid s \in S\} \cup \ell^\infty(X) \right) \subset B(\ell^2(X)).$$

Note that conjugation by $V_s$ implements the action of $s \in S$ on subsets $A \subset X$. It is straightforward to show the following intertwining equation for the generators of $R_X$:

$$P_A V_s = V_s P_{s^* (A \cap D_{s^* s})}, \quad s \in S, \ A \subset X,$$

where, as before, $P_A \in B(\ell^2(X))$ is the orthogonal projection onto the closure of span $\{\delta_a \mid a \in A\}$. Note also that for any $s \in S, f \in \ell^\infty(X)$ (which we interpret as multiplication operators on $\ell^2(X)$) we have the following commutation relations between the generators of $R_X$

$$V_s f = (sf) V_s \quad \text{or, equivalently,} \quad f V_s = V_s (s^* f).$$

From this commutation relations, it follows that the algebra $R_X$ is actually the closure in operator norm of the linear span of operators of the form $V_s P_A$, where $s \in S, A \subset D_{s^* s}$. This fact will be used throughout the section.
5.1. Domain measures as amenable traces. Before proving the next theorem we first need to introduce some notation and some preparing lemmas. The first result defines a canonical conditional expectation from \(B(ℓ^2(X)) \) onto \(ℓ^∞(X)\). The proof is virtually the same as in the case of groups (see, e.g., [12]).

**Lemma 5.1.** The linear map \(E: B(ℓ^2(X)) \to ℓ^∞(X)\) given by

\[
E(T) = \sum_{x \in X} P_x T P_x,
\]

is a conditional expectation, where the sum is taken in the strong operator topology.

To analyze the dynamics on Følner sets of inverse semigroups it is convenient to introduce the following equivalence relation in \(X\). Let \(α : S \to Γ(X)\) be a representation of \(S\) on \(X\). Given a pair \(u,v \in X\), we write \(u \approx v\) if there is some \(s \in S\) such that \(u \in D_{s^*s}\) and \(sv = v\). The relation \(\approx\) is an equivalence relation: in fact, since \(S\) is unital, \(u \approx u\), and if \(u \approx v\) then \(v \approx u\) by considering \(v = su \in D_{ss^*}\). For transitivity, if \(u \approx v \approx w\) then \(u \in D_{ss'v'v}\), where \(s,t\) witness \(u \approx v\) and \(v \approx w\) respectively. We will see in Lemma [5.3] that if a set \(F \subset X\) has only one \(\approx\)-class, then the corner \(P_F R_X P_F\) has dimension \(|F|^2\) as a vector space.

The next lemma guarantees the existence of transitive domain Følner sets, i.e., of domain Følner sets \(F\) such that \(F/\approx\) is a singleton.

**Lemma 5.2.** Let \(α : S \to Γ(X)\) be a representation of \(S\) on \(X\). If \(A \subset X\) is domain Følner then for every \(ε > 0\) and finite \(F \subset S\), there is an \((ε, F)\)-domain Følner \(F_0 \subset A\) with exactly one \(\approx\)-equivalence class.

**Proof.** Since \(A\) is domain Følner, for any \(ε > 0\) and finite \(F \subset S\) there is a finite \(F \subset A\) such that

\[
|s(F \cap D_{s^*s}) \setminus F| < \frac{ε}{|F|} |F|, \text{ for all } s \in F.
\]

Decomposing \(F\) into its \(\approx\)-classes we get \(F = F_1 \sqcup \cdots \sqcup F_L\), where \(u \approx v\) if and only if \(u,v \in F_i\) for some \(i\). To prove the claim it is enough to prove that some \(F_j\) must be \((ε, F)\)-domain Følner. Indeed, suppose for all \(j = 1, \ldots, L\) there is an \(s_j \in F\) such that

\[
|s_j(F_j \cap D_{s_j^*s_j}) \setminus F_j| \geq ε |F_j|.
\]

Observe that the choice of \(s_j\) is not unique, but we can consider a particular fixed choice. Arrange then the indices according to the following: for \(s \in F\), consider \(Λ_s := \{j \in \{1, \ldots, L\} \mid s_j = s\}\). Note that some \(Λ_s\) might be empty. Define \(F_s := \sqcup_{i \in Λ_s} F_i\) and observe

\[
|s(F \cap D_{s^*s}) \setminus F_s| = \sum_{j \in Λ_s} |s_j(F_j \cap D_{s_j^*s_j}) \setminus F_j| \geq ε \sum_{j \in Λ_s} |F_j| = ε |F_s|.
\]

Taking the sum over all \(s \in F\) we get

\[
ε |F| > \sum_{s \in F} |s(F \cap D_{s^*s}) \setminus F| = \sum_{s \in F} |s(F_1 \cap D_{s^*s}) \setminus F_1| \geq ε \sum_{s \in F} |F_s| = ε |F|.
\]

This is a contradiction and, thus, some \(F_{j_0}\) must witness the domain Følner condition. \(\square\)

The next lemma computes the dimension of a certain corner of the algebra \(R_X\).

**Lemma 5.3.** Let \(α : S \to Γ(X)\) be a representation of \(S\) on \(X\). Let \(F_1, F_2 \subset X\) be finite sets such that \(F_1 \cup F_2\) has only one \(\approx\)-class. Then \(W := P_{F_2} R_X P_{F_1}\) has linear dimension \(|F_1| |F_2|\).

**Proof.** To prove the claim it suffices to show that for every \(u_i \in F_i, i = 1, 2\) the matrix unit

\[
M_{u_2,u_1}δ_x = \left\{ \begin{array}{ll} δ_{u_2} & \text{if } x = u_1 \\ 0 & \text{otherwise} \end{array} \right.
\]

AMENABILITY AND PARADOXICALITY IN SEMIGROUPS AND C*-ALGEBRAS 21
is contained in \( W \). Since \( u_1 \approx u_2 \) there must be an element \( s \in S \) such that \( u_1 \in D_{s^{*}s} \) and \( su_1 = u_2 \). It is straightforward to prove that in this case

\[
M_{u_2,u_1} = P_{F_2}P_{u_2}V_sP_{u_1}P_{F_1} \in P_{F_2}R_XP_{F_1} = W,
\]
hence \( \dim W = |F_1||F_2| \).

We are now in a position to show the main theorem of the section, which characterizes the domain measurability of the action in terms of amenable traces of the algebra \( R_X \).

**Theorem 5.4.** Let \( S \) be a countable and discrete inverse semigroup with identity \( 1 \in S \), and let \( \alpha : S \to T(X) \) be a representation of \( S \) on \( X \). Then the following are equivalent:

1. \( X \) is \( S \)-domain measurable.
2. \( X \) is not \( S \)-paradoxical.
3. \( X \) is \( S \)-domain Følner.
4. \( R_{X,alg} \) is algebraically amenable.
5. \( R_X \) has an amenable trace (and hence is a Følner \( C^* \)-algebra).
6. \( R_X \) is not properly infinite.
7. \( [0] \neq [1] \) in the \( K_0 \)-group of \( R_X \).

**Proof.** The equivalences \( (1) \leftrightarrow (2) \leftrightarrow (3) \) follow from Theorem 1.23.

\( (1) \Rightarrow (5) \). Consider the conditional expectation \( \mathcal{E} : \mathcal{B}(\ell^2(X)) \to \ell^\infty(X) \) introduced in Lemma 5.1. Since \( X \) is \( S \)-domain measurable, by Proposition 1.23 there is a mean \( m : \ell^\infty(X) \to \mathbb{C} \) satisfying \( m(sf) = m(s^*sf) \). We claim that then \( \phi(T) := m(\mathcal{E}(T)) \) is a hypertrace on \( R_X \). Indeed, observe that linearity, positivity and normalization follow from those of \( m \) and \( \mathcal{E} \). Hence we only have to prove the hypertrace property for the generators of \( R_X \). Note that since \( \mathcal{E} \) is a conditional expectation we have \( \phi(fT) = \phi(Tf) \) for any \( f \in \ell^\infty(X) \), \( T \in \mathcal{B}(\ell^2(X)) \). To show the same relation for the generator \( V_s \) note first that for any \( s,t \in S \) the following relation holds:

\[
\mathcal{E}(V_sT)(x) = \begin{cases} T_{s^{*}x,x} & \text{if } x \in D_{s^{*}s} \\ 0 & \text{if } x \notin D_{s^{*}s} \end{cases}
\]

\[
\mathcal{E}(TV_s)(y) = \begin{cases} T_{y,sy} & \text{if } y \in D_{s^{*}s} \\ 0 & \text{if } y \notin D_{s^{*}s} \end{cases}
\]

It follows from the action introduced in Eq. 1.2 that \( s \mathcal{E}(TV_s) = \mathcal{E}(V_sT) \) and \( \mathcal{E}(TV_s) = \mathcal{E}(TV_s)P_{s^{*}s} \) and thus

\[
\phi(V_sT) = m(\mathcal{E}(V_sT)) = m(s \cdot \mathcal{E}(TV_s)) = m(\mathcal{E}(TV_s)P_{s^{*}s}) = m(\mathcal{E}(TV_s)) = \phi(TV_s),
\]

where we used the invariance of the mean in the third equality.

\( (5) \Rightarrow (6) \). Suppose that \( R_X \) is properly infinite and has a hypertrace \( \phi \). Then we obtain a contradiction from

\[
1 = \phi(1) \geq \phi(W_1W_1^{*}) + \phi(W_2W_2^{*}) = \phi(W_1^{*}W_1) + \phi(W_2^{*}W_2) = \phi(1) + \phi(1) = 2,
\]

where \( W_1, W_2 \) are the isometries witnessing the proper infiniteness of \( 1 \in R_X \).

\( (6) \Rightarrow (2) \). Suppose that \( s_i, t_j, A_i, B_j, i = 1, \ldots, n, j = 1, \ldots, m \), implement the \( S \)-paradoxicality of \( X \), that is, \( A_i \subseteq D_{s^{*}s_i}, B_j \subseteq D_{t^{*}t_j} \), and

\[
X = s_1A_1 \sqcup \ldots \sqcup s_nA_n = t_1B_1 \sqcup \ldots \sqcup t_mB_m
\]

\( \sqcup A_1 \sqcup \ldots \sqcup A_n \sqcup B_1 \sqcup \ldots \sqcup B_m \).

Consider now the operators

\[
W_1 := V_{s_1}P_{s_1A_1} + \cdots + V_{s_n}P_{s_nA_n} \quad \text{and} \quad W_2 := V_{t_1}P_{t_1B_1} + \cdots + V_{t_m}P_{t_mB_m}.
\]

These are both partial isometries, since \( V_{s_i}P_{s_iA_i} \) and \( V_{t_j}P_{t_jB_j} \) are partial isometries with pairwise orthogonal domain and range projections. Furthermore, \( W_1^{*}W_1 \) is the projection onto the union of the domains of \( V_{s_i}P_{s_iA_i} \), which is the whole space \( \ell^2(X) \). The same argument proves that \( W_2^{*}W_2 = 1 \). Therefore, to prove the claim we just have to show that \( W_1W_1^{*} \) and \( W_2W_2^{*} \) are orthogonal projections.
But these correspond to projections onto \( \bigcup_{i=1}^n A_i \) and \( \bigcup_{j=1}^m B_j \), respectively, which are disjoint sets in \( X \).

(3) \( \Rightarrow \) (4). By Lemma 5.2 given \( \varepsilon > 0 \) and finite \( F \subset S \), there is a finite non-empty \( F \subset X \) with exactly one \( \approx \)-class and such that

\[
|s (F \cap D_{s^*}) \setminus F| < \varepsilon |F|, \text{ for all } s \in F.
\]

Consider the space \( W := P_F \mathcal{R}_X P_F = P_F \mathcal{R}_X \text{alg} P_F \) and observe

\[
V_s W = \{ V_s P_F T P_F \mid T \in \mathcal{R}_X \text{alg} \} = \{ P_s(F \cap D_{s^*}) V_s T P_F \mid T \in \mathcal{R}_X \text{alg} \} \subset \{ P_s(F \cap D_{s^*}) F \} V_s T P_F \mid T \in \mathcal{R}_X \text{alg} \} + \{ P_F P_s(F \cap D_{s^*}) V_s T P_F \mid T \in \mathcal{R}_X \text{alg} \}.
\]

Therefore

\[
V_s W + W \subset P_s(F \cap D_{s^*}) \mathcal{F} \mathcal{R}_X \text{alg} P_F + P_F \mathcal{R}_X \text{alg} P_F,
\]

and, by Lemma 5.3

\[
\dim (W + V_s W) \leq |F|^2 + |F| |s (F \cap D_{s^*}) \setminus F| \leq (1 + \varepsilon) \dim (W),
\]

which proves the algebraic amenability of \( \mathcal{R}_X \text{alg} \).

(4) \( \Rightarrow \) (5). This follows from one of the main results of [5], which states that if a pre-C*-algebra is algebraically amenable then its closure has an amenable trace (see [5, Theorem 3.17]) and hence is a Forrester C*-algebra (cf., Definition 2.2 (1)).

(5) \( \Rightarrow \) (6). Any trace \( \phi \) on \( \mathcal{R}_X \), by the universal property of the K0 group, induces a group homomorphism \( \phi_0 : K_0(\mathcal{R}_X) \to \mathbb{R} \) such that \( \phi_0 ([P]) = \phi (P) \) for any projection \( P \in \mathcal{R}_X \). In this case \( \phi_0 ([1]) = \phi (1) = 1 \) while \( 0 = \phi (0) = \phi_0 ([0]) \). In particular, \([1] \neq [0]\) in the K0 group of \( \mathcal{R}_X \).

(6) \( \Rightarrow \) (7). If \( X \) is S-paradoxical, then it follows from Lemma 1.17(3) and the same argument as in the implication (3) \( \Rightarrow \) (2) that \([1] = [1] + [1]\) in \( K_0(\mathcal{R}_X) \). Therefore \([1] = [0]\) in \( K_0(\mathcal{R}_X) \).

Theorem 5.4 can be generalized, along the lines of [49], to hold for any set \( A \subset X \).

**Corollary 5.5.** Let \( S \) be a countable and discrete inverse semigroup with identity \( 1 \in S \), and let \( \alpha : S \to \mathcal{I}(X) \) be a representation of \( S \) on \( X \). Let \( A \subset X \), then the following conditions are equivalent:

1. \( A \) is S-domain measurable.
2. \( A \) is not S-paradoxical.
3. There is a tracial weight \( \psi : \mathcal{R}_X^+ \to [0, \infty] \) such that \( \psi (P_A) = 1 \).
4. \( P_A \in \mathcal{R}_X \) is not a properly infinite projection.

**Proof.** Most of the proof is similar as in the reasoning leading to Theorem 5.4. The only difference regards condition (3), and the replacement of the trace with a weight. To prove that (3) \( \Rightarrow \) (1), consider the measure \( \mu (B) = \psi (P_B) \). This \( \mu \) will then be a measure on \( X \) such that \( \mu (A) = 1 \). Furthermore, invariance follows from \( \psi \) being tracial:

\[
\mu (B) = \psi (V_{s^*} P_B V_s) = \psi (V_s P_B V_{s^*}) = \psi (P_B) = \mu (\alpha_s (B))
\]

for every \( B \subset D_{s^*} \).

To prove (1) \( \Rightarrow \) (3) we adapt the ideas in [49, Proposition 5.5]. Given a domain measure \( \mu \) normalized at \( A \), denote by \( \mathcal{P}_{\text{fin}}(X) \) the (upwards directed) set of \( K \subset X \) with finite measure, i.e., \( \mu (K) < \infty \). Given \( K \in \mathcal{P}_{\text{fin}}(X) \) consider the finite measure \( \mu_K (B) = \mu (K \cap B) \) and extend it, as in Proposition 4.19, to a functional \( m_K \). Given a non-negative \( f \in \ell^\infty (X) \) define

\[
m (f) := \sup_{K \in \mathcal{P}_{\text{fin}}(X)} m_K (f).
\]

Then \( m \) is \( \mathbb{R}_+ \)-linear, lower-semicontinuous, normalized at \( P_A \), and satisfies that \( m (sf) = m (s^* sf) \), for every \( s \in S, f \in \ell^\infty (X) \). Finally, the weight given by \( \psi := m \circ \mathcal{E} : \mathcal{R}_X^+ \to [0, \infty] \) is a tracial weight. \( \square \)
In our last result of the section we point out that one can translate the Følner sequences of $X$ onto Følner sequences of projections in $\mathcal{R}_X$ (cf., Definition 2.2(2)). To prove it we first recall the following known result, which states that Følner sequences are preserved under $C^*$-closure (see, e.g., [10]).

**Lemma 5.5.** Let $T \subset B(\mathcal{H})$ be a set of operators on a separable Hilbert space $\mathcal{H}$. Suppose $T$ has a Følner sequence $\{P_n\}_{n=1}^{\infty}$. Then $\{P_n\}_{n=1}^{\infty}$ is a Følner sequence for the $C^*$-algebra generated by $T$.

**Proposition 5.6.** Let $S$ be a countable and discrete inverse semigroup and $\alpha: S \to \mathcal{I}(X)$ be a representation. If $X$ is domain measurable, then there is a sequence of projections $\{P_n\}_{n=1}^{\infty} \subset \mathcal{R}_X$ which is a Følner sequence of projections of every operator $T \in \mathcal{R}_X$.

**Proof.** Choose a $S$-domain Følner sequence $\{F_n\}_{n=1}^{\infty}$ of $X$, that exists since $X$ is domain measurable (see Proposition 6.3.4). Consider the orthogonal projection $P_n$ onto span $\{\delta_f \mid f \in F_n\} \subset \ell^2(X)$. Clearly, $P_n$ lies within $\mathcal{R}_X$, so, by Lemma 5.5 it is enough to show that it is a Følner sequence for all generating elements $V_sP_A$, $s \in S$, $A \subset X$. For this we compute

$$(V_sP_AP_n) \delta_x = \begin{cases} \delta_{sx}, & \text{if } x \in D_{ss^*} \cap F_n \cap A \\ 0, & \text{otherwise} \end{cases}$$

Thus we have the following estimates in the Hilbert-Schmidt norm:

$$\|V_sP_AP_n - P_nV_sP_A\|_2^2 \leq |\{x \in F_n \cap D_{ss^*} \cap F_n \mid sx \notin F_n\}| + |s^*(F_n \cap D_{ss^*}) \setminus F_n|$$

Noting that $\|P_n\|_2^2 = |F_n|$ the result follows from normalizing by $|F_n|$ and taking limits on both sides of the inequality.

5.2. **Traces and amenable traces.** The aim of this section is to prove that either all traces in $\mathcal{R}_X$ are amenable or this algebra has no traces at all. Similar results hold for nuclear $C^*$-algebras (see [12] Proposition 6.3.4) and for uniform Roe algebras over metric spaces (see [10] Corollary 4.15). We begin by proving that traces of $\mathcal{R}_X$ factor through $\ell^\infty(X)$ via the conditional expectation $\mathcal{E}$ (see Lemma 5.1).

**Lemma 5.7.** Let $\alpha: S \to \mathcal{I}(X)$ be a representation and let $\phi: \mathcal{R}_X \to \mathbb{C}$ be a trace. Then $\phi(T) = \phi(\mathcal{E}(T))$ for every $T \in \mathcal{R}_X$, where $\mathcal{E}$ denotes the canonical conditional expectation onto $\ell^\infty(X)$.

**Proof.** Since the closure of the linear span of the elements of the form $V_sP_A$, $s \in S$, $A \subset D_{ss^*}$, is dense in $\mathcal{R}_X$ it is enough to show the claim for these elements.

First suppose that $A$ has no fixed points under $s$, i.e., $\{a \in A \mid sa = a\} = \emptyset$. Consider the graph whose vertices are the elements of $A$ and such that two vertices $a, b$ are joined by an edge if and only if $b = sa$. Since the action of $s$ is injective on $A$ it is clear that every vertex has at most degree 2 and no loops. Therefore it can be colored by 3 colors and, thus, there is a partition $A = B_1 \cup B_2 \cup B_3$ such that if $a \in B_i$ then $sa \not\in B_i$. This allows us to decompose $V_sP_A$ as

$$V_sP_A = V_sP_{B_1} + V_sP_{B_2} + V_sP_{B_3}.$$ 

Taking traces on each side of the equality gives

$$\phi(V_sP_A) = \phi(P_{B_1}V_sP_{B_1}) + \phi(P_{B_2}V_sP_{B_2}) + \phi(P_{B_3}V_sP_{B_3}) = 0.$$ 

But in this case we also have $\mathcal{E}(V_sP_A) = 0$ and the equality $\phi = m \circ \mathcal{E}$ follows.

Second, for arbitrary $s \in S$ and $A \subset D_{ss^*}$ one can decompose $A = B \cup C$, where $B := \{a \in A \mid sa = a\}$ is the set of fixed points and $C := \{a \in A \mid sa \neq a\}$. In this case

$$V_sP_A = V_sP_B + V_sP_C.$$ 

By the above paragraph it follows that $\phi(V_sP_C) = 0$, while $V_sP_B$ is a projection in $\ell^\infty(S)$. Thus

$$\phi(V_sP_A) = \phi(V_sP_B) + \phi(V_sP_C) = \phi(\mathcal{E}(V_sP_B)) + \phi(\mathcal{E}(V_sP_A)),$$

which concludes the proof.

□
The following theorem is a consequence of Theorem 5.4 and the latter lemma.

**Theorem 5.9.** Let $S$ be a countable and discrete inverse semigroup with identity $1 \in S$, and let $\alpha: S \to \mathcal{I}(X)$ be a representation. Consider a positive linear functional $\phi$ on $\mathcal{R}_X$. Then the following conditions are equivalent:

1. $\phi$ is an amenable trace on $\mathcal{R}_X$.
2. $\phi$ is a trace on $\mathcal{R}_X$.
3. $\phi = \phi|_{\mathcal{L}^\infty(X)} \circ \mathcal{E}$ and the measure $\mu(A) := \phi(P_A)$ satisfies domain measurability, i.e., $\mu(A) = \mu(sA)$ for all $s \in S, A \subset D_{s^*s}$.

**Proof.** The fact that (1) $\Rightarrow$ (2) is obvious. For (2) $\Rightarrow$ (3) note that by Lemma 5.8 it remains only to prove that $\mu(A) = \mu(sA)$ for every $s \in S, A \subset D_{s^*s}$. This follows from $\phi$ being a trace and Eq. (5.1). Finally, (3) $\Rightarrow$ (1) is proved as the implication (1) $\Rightarrow$ (5) in Theorem 5.4. $\square$

We summarize next some important consequences of the previous theorems.

**Corollary 5.10.** Let $S$ be a countable and discrete inverse semigroup with identity $1 \in S$, and let $\alpha: S \to \mathcal{I}(X)$ be a representation. Then:

1. $X$ is $S$-domain measurable if and only if there is a trace on $\mathcal{R}_X$, in which case every trace on $\mathcal{R}_X$ is amenable.
2. There is a canonical bijection between the space of measures on $X$ such that $\mu(A) = \mu(sA)$ when $s \in S, A \subset D_{s^*s}$ and the space of traces of $\mathcal{R}_X$.

### 5.3. Traces in amenable inverse semigroups

The goal of this last section is to state the analogue of Theorem 5.4 but considering amenable semigroups instead of the weaker notion of domain measurable ones. That is, we will give additional C*-algebraic characterizations of amenable inverse semigroups. We also prove that the reverse implication is false (the so called Rosenberg conjecture in the case of groups, see [12, Corollary 7.1.17]). The proof of the following result is straightforward (see the proof of Theorem 5.4).

**Theorem 5.11.** Let $S$ be a countable and discrete inverse semigroup with identity $1 \in S$ and let $\alpha: S \to \mathcal{I}(X)$ be a representation. Then the following conditions are equivalent:

1. $X$ is $S$-amenable.
2. $D_e$ is not $S$-paradoxical for any $e \in E(S)$.
3. $\mathcal{R}_X$ has a trace $\phi$ such that $\phi(V_e) = 1$ for all $e \in E(S)$.
4. No projection $V_e \in \mathcal{R}_X$ is properly infinite.

In the appendix to [19], Rosenberg showed that any countable discrete group $G$ with a quasidiagonal left regular representation is amenable (see [12]). This result implies that if $C^*_r(G)$ is quasidiagonal then $G$ is amenable (cf., [12, Corollary 7.1.17]). That the reverse implication also holds was recently shown in [54, Corollary C]. Quasidiagonality of a C*-algebra can be defined in terms of a net of unital completely positive (u.c.p.) maps (see, for example, [12, Definition 7.1.1] and [57, 58]):

**Definition 5.12.** A unital separable C*-algebra $\mathcal{A}$ is called quasidiagonal if there exists a sequence of u.c.p. maps $\varphi_n: \mathcal{A} \to M_k(\mathbb{C})$ which is both asymptotically multiplicative, i.e., $\|\varphi_n(AB) - \varphi_n(A)\varphi_n(B)\| \to 0, A, B \in \mathcal{A}$, and asymptotically isometric, i.e., $\|A\| = \lim_{n \to \infty} \|\varphi_n(A)\|, A \in \mathcal{A}$.

We conclude this article by showing that Rosenberg’s implication still holds for some special class of inverse semigroups. We also prove that the reverse implication is false (the so called Rosenberg conjecture in the case of groups, see [12, 54]). Recall that the reduced C*-algebra of an inverse semigroup is the C*-algebra generated by the left regular representation:

$$C^*_r(X) := C^* \{[V_s]_{s \in S} \} \subset \mathcal{R}_X.$$  

Note that the following theorem can only be true for the reduced C*-algebras (either in this context or for groups), since the uniform Roe algebras $\mathcal{R}_X$ (and $\mathcal{R}_G$) are almost never finite, and thus almost never quasi-diagonal (see [12, Proposition 7.1.15]).
Theorem 5.13. Let $S$ be a countable and discrete inverse semigroup with identity $1 \in S$, and let $\alpha : S \to \mathcal{I}(X)$ be a representation. Then:

1. If $C_r^*(X)$ is quasidiagonal then $X$ is $S$-domain measurable.
2. If $C_r^*(X)$ is quasidiagonal and $S$ has a minimal projection then $X$ is $S$-amenable.
3. There are amenable inverse semigroups with and without minimal projections with non-quasidiagonal reduced $C^*$-algebras.

Proof. As is customary in the literature, we will denote by $\text{tr}_{k_n}$ the normalized trace in $M_{k_n}(\mathbb{C})$, while $\text{Tr}$ will stand for the usual trace, i.e., $\text{tr}_{k_n}(\cdot) = \text{Tr}(\cdot)/k_n$.

The proof of (1) is a particular case of Proposition 7.1.6 in [12]. Indeed, let $\varphi_n : C_r^*(X) \to M_{k_n}(\mathbb{C})$ be a sequence of u.c.p. maps that are asymptotically multiplicative and isometric. Then any cluster point $\tau$ of $\{\text{tr}_{k_n} \circ \varphi_n\}_{n \in \mathbb{N}}$ is an amenable trace of $C_r^*(X)$. Thus let $\Phi$ be any hypertrace extending $\tau$, and consider the measure $\mu (A) := \Phi (P_A)$. It follows from Eq. (5.1) that $\mu$ is a domain measure.

Let $e_0 \in E(S)$ be the minimal projection and consider $\varphi_n : C_r^*(X) \to M_{k_n}(\mathbb{C})$ a sequence of unital completely positive (u.c.p.) maps that are asymptotically multiplicative and asymptotically isometric (see Definition 5.12). To prove (2) we will construct a new sequence of u.c.p. maps $\phi_n$ that are asymptotically multiplicative, asymptotically isometric and with asymptotically normalized trace in $V_{e_0}$. For this, recall that, by Lemma [12.29] $e_0$ commutes with every $s \in S$.

Observe that $\varphi_n (V_{e_0})$ is a positive matrix, whose norm is greater than $1 - \varepsilon_n$ and such that $||\varphi_n (V_{e_0}) - \varphi_n (V_{e_0}) \varphi_n (V_{e_0})|| < \varepsilon_n$ for some $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ when $n \to \infty$. We will assume that $\varepsilon_n < 1/4$ for all $n$. A routine exercise shows that the spectrum of $\varphi_n (V_{e_0})$ is contained in $[0, \delta_n) \cup (1 - \delta_n, 1]$ where $\delta_n = \frac{1}{2} (1 - \sqrt{1 - 4 \varepsilon_n})$. Let $r_n$ be the number of eigenvalues of $\varphi_n (V_{e_0})$ in $(1 - \delta_n, 1]$, and note that $r_n \geq 1$ for large $n \in \mathbb{N}$, since $||\varphi_n (V_{e_0})|| \geq 1 - \varepsilon_n$. Let $W_n \subset C^{k_n}$ be the subspace generated by the eigenvectors of $\varphi_n (V_{e_0})$ of eigenvalues close to 1. Finally, let $Q_n : C^{k_n} \to C^n_r$ be a linear map such that $Q_n |w_n$ is an isometry onto $C^n_r$ and $Q_n |w_n = 0$, i.e.,

\[ Q_n : C^{k_n} \to C^n_r, \quad Q_n Q_n^* = 1_{r_n} \quad \text{and} \quad Q_n^* Q_n = P_{W_n}. \]

For each $n \in \mathbb{N}$ let $m_n \in \mathbb{N}$ be large enough such that

\[ \frac{m_n r_n}{k_n + m_n r_n} (1 - \delta_n) \geq 1 - 2 \delta_n. \]

Consider the maps:

\[ \phi_n : C_r^*(X) \to M_{k_n + m_n r_n}(\mathbb{C}), \quad A \mapsto \varphi_n (A) \oplus (Q_n \varphi_n (A) Q_n^* \otimes 1_{m_n}). \]

By construction it is clear that the maps $\phi_n$ are unital, completely positive and asymptotically isometric. They are also asymptotically multiplicative. For this first observe that, using the minimality of $e_0$ and Lemma [12.29] $e_0$ commutes with every $A \in C_r^*(X)$. Thus, by summing and substracting $\varphi_n (V_{e_0}) \varphi_n (A), \varphi_n (A) \varphi_n (V_{e_0})$ and $\varphi_n (V_{e_0}A)$, we have

\[ \||Q_n^* Q_n \varphi_n (A) - \varphi_n (A) Q_n^* Q_n|| \leq 2 ||\varphi_n (A)|| ||\varphi_n (V_{e_0}) - Q_n^* Q_n|| + ||\varphi_n (V_{e_0}) \varphi_n (A) - \varphi_n (V_{e_0}A)|| + ||\varphi_n (A) \varphi_n (V_{e_0}) - \varphi_n (AV_{e_0})|| \]

\[ \leq 2 ||A|| ||\varphi_n (V_{e_0}) - Q_n^* Q_n|| + ||\varphi_n (V_{e_0}) \varphi_n (A) - \varphi_n (V_{e_0}A)|| + ||\varphi_n (A) \varphi_n (V_{e_0}) - \varphi_n (AV_{e_0})|| \xrightarrow{n \to \infty} 0. \]

This asymptotic commutation gives the asymptotic multiplicativity of $\phi_n$ by a straightforward computation since, for any $A, B \in C_r^*(X)$

\[ ||\phi_n (AB) - \phi_n (A) \phi_n (B)|| \leq ||\varphi_n (AB) - \varphi_n (A) \varphi_n (B)|| + ||Q_n \varphi_n (AB) Q_n^* - Q_n \varphi_n (A) Q_n^* Q_n \varphi_n (B) Q_n^*|| \]

\[ \leq 2 ||\varphi_n (AB) - \varphi_n (A) \varphi_n (B)|| + ||B|| ||Q_n^* Q_n \varphi_n (A) - \varphi_n (A) Q_n^* Q_n|| \xrightarrow{n \to \infty} 0. \]
Furthermore, the maps $\phi_n$ have the desired asymptotically normalized trace property at $V_e$:

$$1 \geq \text{tr} (\phi_n (V_{e_0})) = \text{tr} (\varphi_n (V_{e_0})) + m_n \text{tr} (Q_n \varphi_n (V_{e_0}) Q_n^*) \geq \frac{m_n}{k_n + mnr_n} \text{Tr} (Q_n \varphi_n (V_{e_0}) Q_n^*)$$

$$= \frac{m_n}{k_n + mnr_n} \text{Tr} (Q_n^* \varphi_n (V_{e_0})) \geq \frac{mnr_n}{k_n + mnr_n} (1 - \delta_n) \geq 1 - 2 \delta_n \xrightarrow{n \to \infty} 1,$$

where the last inequality follows from the choice of $m_n$, see Eq. (5.2). By the discussion in (11) and Theorem 5.13 any cluster point $\Phi$ of $\{\text{tr}_{k_n+mnr_n} \circ \phi_n\}_{n \in \mathbb{N}}$ will give an amenable trace of $C_r^*(X)$ normalized at $V_{e_0}$. Indeed, observe that

$$\mu (D_{e_0}) = \Phi (V_{e_0}) = \lim_{n \to \infty} \text{tr}_{k_n+mnr_n} (\phi_n (V_{e_0})) = 1.$$

We conclude that $\mu$ is a domain-measure localized at $D_{e_0}$ and, by Proposition 4.30 and Theorem 5.11, $X$ is then $S$-amenable.

Finally, for (3), consider the bicyclic inverse monoid $S = \langle a, a^* \mid a^* a = 1 \rangle$. It is routine to show that $E(S) = \{1, aa^*, a^2(a^*)^2, \ldots \}$ and, thus, $S$ has no minimal projection. Moreover, it is amenable (see [21, pp. 311]) and $C_r^*(S)$ is not quasidiagonal (since it is not even finite, see [12, Proposition 7.1.15]).

For the case when $S$ does have a minimal projection, consider the semigroup $T = \mathbb{F}_2 \cup \{0\}$, where $0$ is a zero element. Since $T$ has a zero element it follows that $T$ is amenable (take $\mu$ atomic with total mass at 0) and has a minimal projection (namely 0). Furthermore, note that $C_r^*(S)$ is not quasidiagonal, since it contains the reduced group C*-algebra of $\mathbb{F}_2$, which is not quasidiagonal (see [12, Proposition 7.1.10] and [54, Corollary C]).

**Remark 5.14.** The condition of the existence of a minimal projection $e_0$ in Theorem 5.13 is assumed to guarantee that it commutes with every element $s \in S$. In particular, the proof of (2) in Theorem 5.13 can also be applied to the more general setting of a unital and separable C*-algebra $A$ and a projection $P \in A$, with $P$ commuting with every $A \in A$. Indeed, suppose that $A$ is also quasidiagonal. Then, by the same argument, the construction of Eq. (5.3) gives a quasidiagonal approximation of $A$ with asymptotic normalized trace at $P$. In both cases, however, the condition that $AP = PA$ for every $A \in A$ is essential.

We conclude with some natural questions. Suppose $\alpha : S \to \mathcal{I}(X)$ is a representation of an inverse semigroup $S$ on a set $X$.

**Q1:** Suppose that $C_r^*(X)$ is quasidiagonal. Is $X$ then $S$-amenable?

**Q2:** Is there a stronger notion than the $S$-ameness of $X$ that guarantees the quasidiagonality of $C_r^*(X)$?

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