1 Introduction

Let $X$ be a non-singular $n$-dimensional complex manifold (or algebraic variety over an algebraically closed field $k$ of characteristic zero), and let $D \subset X$ be a hypersurface with reduced defining ideal $I_X$. We denote by $\text{Der}(-\log D)$ the sheaf of vector fields $\chi \in \text{Der}_X$ such that $\chi \cdot I_X \subset I_X$, or, equivalently, such that $\chi$ is tangent to $D$ at its regular points. It is clearly an $\mathcal{O}_X$-module.

**Definition 1.1.** The hypersurface $D \subseteq X$ is a free divisor if $\text{Der}(-\log D)$ is a locally free $\mathcal{O}_X$-module.

Free divisors were introduced by K. Saito in [27]. The simplest example is the normal crossing divisor, but the main source of examples, motivating Saito’s definition, has been the deformation theory of singularities, where discriminants and bifurcation sets are frequently free divisors. If $D$ is the discriminant hypersurface in the base $S$ of a versal deformation of an isolated hypersurface singularity, the module $\text{Der}(-\log D)$ is the kernel of the Kodaira-Spencer map from $\text{Der}_S$ onto the relative $T^1$ of the deformation, and from this freeness follows by an easy homological argument, due initially to Teissier. Variants of this argument show the freeness of the discriminant in the base of a versal deformation in a number of cases: isolated complete intersection singularities ([19]), space-curve singularities ([31]), functions on space curves ([13], [21]), Gorenstein surface singularities in 5-space ([5]), Hilbert schemes of a smooth surface ([3]). Damon, in his paper “The legacy of free divisors” ([7]), has shown, by an essentially similar argument, how the bifurcation set in the base space of a versal deformation of a non-linear section of a free divisor is once again a free divisor, provided a natural condition, namely, the existence of “Morse-type singularities”, is met. Another significant source of examples is the theory of hyperplane arrangements, where many examples of free arrangements have been constructed by combinatorial means (see e.g. [21] Chapter 4).

Saito’s original paper [27] contained the following criterion, now known by his name, for a divisor $D$ to be free:

**Proposition 1.2.** (Saito’s Criterion)

The hypersurface $D \subset X$ is a free divisor in the neighbourhood of a point $x$ if and only if there are germs of vector fields $\chi_1, \ldots, \chi_n \in \text{Der}(-\log D)_x$, such that the determinant of the matrix of coefficients $[\chi_1, \ldots, \chi_n]$, with respect to some, or any, $\mathcal{O}_{X,x}$-basis of $\text{Der}_{X,x}$, is a reduced equation for $D$ at $x$. In this case, $\chi_1, \ldots, \chi_n$ form a basis for $\text{Der}(-\log D)_x$. □

Note that it is clear that the determinant of the matrix of coefficients of any $n$-tuple of vector fields in $\text{Der}(-\log D)$ must vanish identically on $D$, since at any regular point $x \in D$ all $n$ vectors lie in the
n – 1-dimensional vector space $T_xD$. Moreover, since $\text{Der}(-\log D)$ coincides with $\text{Der}_X$ outside $D$, the determinant of the matrix of coefficients of any set of generators of $\text{Der}(-\log D)$ must vanish only on $D$.

In practice, one uses often the following concrete algebraic version of this criterion that does not refer to vector fields directly, rather characterizes the Taylor series of the function $f$ defining a free divisor at some point $x \in X$:

**Proposition 1.3.** A formal power series $f \in P = k[[z_1, \ldots, z_n]]$ defines a (formal) free divisor, if it is reduced, that is, squarefree, and there is an $(n \times n)$-matrix $A$ with entries from $P$ such that

$$\det A = f \quad \text{and} \quad (\nabla f)A \equiv (0, \ldots, 0) \mod f,$$

where $\nabla f = \left( \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right)$ is the gradient of $f$, and the last condition just expresses that each entry of the (row) vector $(\nabla f)A$ is divisible by $f$ in $P$. The columns of $A$ can then be viewed as the coefficients of a basis, with respect to the partial derivatives $\partial/\partial z_i$, of the logarithmic vector fields along the divisor $f = 0$. 

The normal crossing divisor $D = \{x_1 \cdots x_n = 0\}$ provides a simple example: Saito’s criterion shows that the vector fields $x_1 \partial/\partial x_1, \ldots, x_n \partial/\partial x_n$ form a basis for $\text{Der}(-\log D)$. This free divisor has the striking property that $\text{Der}(-\log D)$ has a basis consisting of vector fields that are homogeneous of weight zero with respect to the natural grading. Among free hyperplane arrangements it is the only one with this property ([24] Chapter 4). Until recently, the only other free divisor with this property known to either of the authors of this paper was the “bracelet”, the discriminant in the space of binary cubics (see [12], and [22], where it is described in some detail, though not under this name).

**Definition 1.4.** The free divisor $D$ is linear if $\text{Der}(-\log D)$ has a basis consisting of vector fields of weight zero — that is, all of whose coefficients are linear functions of the variables.

Here we show that far from being uncommon, linear free divisor are abundant. We show that the set of degenerate, or non-generic, orbits in the representation space $\text{Rep}(Q, d)$ of a quiver with dimension vector $d$, is a linear free divisor whenever $d$ is a real Schur root (definition in Section 3) of $Q$, and provided that a natural condition on the existence of “codimension 1” degeneracies holds - a condition which is closely related to Damon’s condition on the existence of “Morse-type singularities” mentioned above.

Since we hope that our paper will be read by singularity theorists, we include some background on quiver representations.

## 2 Linear free divisors

Suppose that $D$ is a linear free divisor, and let $\chi_1, \ldots, \chi_n$ be a basis consisting of weight-zero vector fields. Since the weight of the Lie bracket of any two homogeneous vector fields is the sum of their weights, $\chi_1, \ldots, \chi_n$ form the basis of an $n$-dimensional Lie algebra $L_D$ over $k$, as well as a basis of the free $\mathcal{O}$-module $\text{Der}(-\log D)$. Consider the standard action of $\text{GL}_n(k)$ on $k^n$. The vector field $x_i \partial/\partial x_j$ is the infinitesimal generator of this action corresponding to the elementary matrix $E_{ij}$ (1 in the $i$-th row and $j$-th column, zeroes elsewhere). It follows that $L_D$ is the image, under the infinitesimal action, of an $n$-dimensional Lie subalgebra of $\mathfrak{gl}_n(k)$, which we denote $\mathfrak{g}_D$. In the complex case, if the exponential of $\mathfrak{g}_D$ is a closed subgroup $G_D$ of $\text{GL}_n(\mathbb{C})$, then $G_D$ has an open orbit in $\mathbb{C}^n$ and $D$ is its complement. This follows easily from Mather’s lemma on Lie group actions ([24] Lemma 3.1), which gives sufficient conditions for a connected submanifold of a manifold to lie in a single orbit of the action of a Lie group $G$: that

(i) at each point of $X$, $T_xX$ should be contained in the tangent space to the $G$ orbit of $x$, and
(ii) the dimension of this orbit should be constant for $x \in X$. 


Taking $X = \mathbb{C}^n \setminus D$, both conditions evidently hold here.

In all examples known, this indeed applies. To find linear free divisors one may thus look for $n$–dimensional Lie groups acting on $k^n$ with an open orbit. It is precisely these that the representation theory of quivers offers in abundance. Indeed, in that situation, the Lie groups $G_D$ are reductive. Examples of some nonreductive groups that also give rise to linear free divisors will be presented in [4]. Here we mention just one series.

**Example 2.1.** The group $B_n(k)$ of upper triangular $n \times n$ matrices acts on the space $\text{Sym}_n(k)$ of symmetric matrices by

$$B \cdot S = {}^tB S B.$$  

There is an open orbit; the equation of the complement is the product of $n$ nested symmetric determinants, beginning with the top left hand entry (1×1 determinant) in the symmetric matrix $S$ and continuing with the determinant of the top left hand $2 \times 2$ block, the determinant of the top left hand $3 \times 3$ block, etc.

### 3 Representations of Quivers

A *quiver* is a finite directed graph. That is, it consists of a finite set $Q_0$ of nodes (or vertices), and a finite set of arrows $Q_1$ equipped with two maps $h, t : Q_1 \to Q_0$ that assign to each arrow $\varphi \in Q_1$ its head $h\varphi$ and tail $t\varphi$ in $Q_0$. A *representation* $V$ of a quiver $Q$ consists of a choice of vector space $V_x$ for each node $x$, and a $k$–linear map $V(\varphi) : V_{t\varphi} \to V_{h\varphi}$ for each arrow $\varphi \in Q_1$. The representation is *finite dimensional* if each $V_x$ is a finite dimensional vector space.

If $W$ is a second such representation, then a *morphism of representations* $\psi : W \to V$ is a family of $k$–linear maps $\psi_x : W_x \to V_x, x \in Q_0$, such that for each $\varphi \in Q_1$ the square

$$\begin{array}{ccc}
W_{t\varphi} & \xrightarrow{W(\varphi)} & W_{h\varphi} \\
\psi_{t\varphi} \downarrow & & \downarrow \psi_{h\varphi} \\
V_{t\varphi} & \xrightarrow{V(\varphi)} & V_{h\varphi}
\end{array}$$

commutes. The $k$–vector space of all morphisms of representations from $W$ to $V$ is denoted $\text{Hom}_Q(W, V)$. The so-defined category of (finite dimensional) representations of $Q$ is *abelian*. Moreover, it is *hereditary*, which means that the extension groups in this abelian category — denoted $\text{Ext}^i_Q(W, V)$, or $\text{Ext}^i_{kQ}(W, V)$ if we wish to specify the coefficients — vanish whenever $i \geq 2$.

Once we fix the dimensions of the spaces at each node, by assigning to $Q$ a *dimension vector* $d \in \mathbb{N}^{Q_0}$, we can consider the $k$-vector space of representations

$$\text{Rep}(Q, d) = \prod_{\varphi \in Q_1} \text{Hom}_k(V_{t\varphi}, V_{h\varphi}) \simeq \prod_{\varphi \in Q_1} \text{Hom}_k(k^{d(h\varphi)}, k^{d(t\varphi)}).$$

The group $\text{Gl}(Q, d) = \prod_{x \in Q_0} \text{Gl}_{d(x)}(k)$ acts on $\text{Rep}(Q, d)$ by

$$(g_x)_{x \in Q_0} \cdot (V(\varphi))_{\varphi \in Q_1} = (g_{h\varphi} \circ V(\varphi) \circ g_{t\varphi}^{-1})_{\varphi \in Q_1}.$$  

The orbits of this group action are the isomorphism classes of $Q$–representations with the prescribed dimension vector. It will be from this action that we obtain the generators of $\text{Der}(- \log D)$ for the linear free divisors we construct.

Given $V' \in \text{Rep}(Q, d')$ and $V'' \in \text{Rep}(Q, d'')$, the *direct sum* $V' \oplus V'' \in \text{Rep}(Q, d' + d'')$ is the representation with $(V' \oplus V'')(x) = V'_x \oplus V''_x$ for $x \in Q_0$ and

$$(V' \oplus V'')(\varphi) = \begin{pmatrix} V'(\varphi) & 0 \\ 0 & V''(\varphi) \end{pmatrix}.$$
A given representation $V \in \text{Rep}(Q, d)$ is **decomposable** if it is the direct sum of subrepresentations — that is, if there are representations $V' \in \text{Rep}(Q, d')$ and $V'' \in \text{Rep}(Q, d'')$ such that $V = V' \oplus V''$. In this case, of course, $d = d' + d''$.

A quiver $Q$ is a **Dynkin quiver** if the underlying undirected graph $\overline{Q}$ is a disjoint union of Dynkin diagrams of type $A_n$, $D_n$, $E_6$, $E_7$, or $E_8$. Dynkin quivers are ubiquitous in the theory of representations of quivers, and central in this paper.

**Example 3.1.** Let $Q$ be the Dynkin quiver of type $A_3$

$$
\begin{array}{c}
\bullet \\
A \\
\bullet \\
B \\
\bullet 
\end{array}
$$

(i) With dimension vector $(1, 1, 1)$ any representation in which each of the morphisms is non-zero is indecomposable.

(ii) Indeed, these are the only indecomposable representations whose dimension vector is **sincere**, meaning that it is nonzero at each node. For example, if $d = (1, 2, 1)$ there is no indecomposable representation. Representations in which $BA \neq 0$ decompose as the direct sum

$$
k \xrightarrow{A} \text{im } A \cong k \xrightarrow{B/\text{im } A} k \oplus 0 \rightarrow \ker B \cong k \rightarrow 0
$$

Representations in which $BA = 0$ and $A \neq 0$ decompose as

$$
k \xrightarrow{A} \text{im } A \cong k \rightarrow 0 \oplus 0 \rightarrow k^2/\text{im } A \cong k \xrightarrow{B} k
$$

where the middle term in the second summand can be viewed as a complement to $\text{im } A$. Representations in which $A = 0$ decompose as

$$
k \rightarrow 0 \rightarrow 0 \oplus 0 \rightarrow k^2 \xrightarrow{B} k.
$$

Similarly, any representation with $d = (l, m, n)$ for $l, m, n \geq 0$ decomposes as:

$$
(0 \rightarrow k \rightarrow 0)_{(a)}\oplus (0 \rightarrow k \rightarrow 0)_{(b)}\oplus (0 \rightarrow k)_{(c)}\oplus (k \rightarrow k)_{(d)}\oplus (0 \rightarrow k \rightarrow 1)_{(e)}\oplus (k \rightarrow k \rightarrow 1)_{(f)},
$$

where

- $a = \dim \ker A$ ,
- $b = \dim \ker B/(\text{im } A \cap \ker B)$ ,
- $c = \dim \cok B$ ,
- $d = \dim \ker BA/\ker A$ ,
- $e = \dim \text{im } B/\text{im } BA$ ,
- $f = \dim \text{im } BA = l - a - d = m - b - d - e = n - c - e$.

**Definition 3.2.** The dimension vector $d$ is a root of $Q$ if $\text{Rep}(Q, d)$ contains an indecomposable representation. The root is real if $\text{Rep}(Q, d)$ contains exactly one orbit of, necessarily isomorphic, indecomposable representations. It is imaginary if there is a family of non-isomorphic indecomposable representations. If a general representation in $\text{Rep}(Q, d)$ is indecomposable, then $d$ is a Schur root.\(^1\)

The frequent use of the term “root” in these definitions is no coincidence, as we will see below.

A crucial role in the representation theory of quivers is played by the **Euler form**, a bilinear form on the space $\mathbb{R}^Q$ of dimension vectors. It is defined by

$$
\langle e, d \rangle = \sum_{x \in Q_0} e_x d_x - \sum_{\varphi \in Q_1} e_{t\varphi} d_{h\varphi} = \dim \prod_{x \in Q_0} \text{Hom}(W_x, V_x) - \dim \prod_{\varphi \in Q_1} \text{Hom}_k(W_{t\varphi}, V_{h\varphi})
$$

\(^1\)Some authors define a Schur root as a root $d$ for which $\text{Rep}(Q, d)$ contains a ‘brick’ — a representation $V$ for which $\text{End}_Q(V) = k$. If $d$ is a Schur root in this sense, then by the upper semicontinuity of $\text{dim End}_Q(V)$ with respect to $V$, the general representation also has endomorphism ring $k$, and so is indecomposable. Conversely, 2.7 of [18] shows that if the general representation is indecomposable then it is a brick. So the two versions of the definition are equivalent.
for any \( W \in R(Q, e) \) and \( V \in R(Q, d) \), and accordingly we sometimes denote \( \langle e, d \rangle \) by \( \langle W, V \rangle \).

The Tits form on the space of dimension vectors is the associated quadratic form, \( q(d) = \langle d, d \rangle \).

Observe that the Tits form does not depend on the orientation of the arrows. Indeed, it is used to calculate the members of the root system of the Kac–Moody Lie algebra attached to the underlying graph \( \overline{Q} \), and those roots with nonnegative components are precisely the roots for \( Q \), regardless of the orientation of the arrows, see \[16\]. For example, if \( \overline{Q} \) is a Dynkin diagram then \( d \in \mathbb{Q}^{\mid Q \mid} \) is a root of the corresponding semi-simple Lie algebra, in the classical sense, if and only if \( q(d) = 1 \). In particular, all roots are real in this case.

Choosing an ordering of the nodes in \( Q_0 \), we may write \( \langle e, d \rangle = e E d^T \), where \( e, d \) are thought of as row vectors, and \( E \) is the corresponding Euler matrix. Its entries are \( E_{x,y} = \delta^y_x - \# \{ \varphi \in Q_1 \mid t\varphi = x, h\varphi = y \} \), with \( \delta^y_x \) denoting the Kronecker delta. Put differently, \( E = I_{\mid Q_0 \mid} - A \), where \( I_{\mid Q_0 \mid} \) is the identity matrix of the indicated size and the matrix entry \( A_{x,y} \) records the number of arrows from \( x \) to \( y \) in \( Q_1 \). The matrix associated to the Tits form is then \( C = E + E^T \), the Cartan matrix of \( Q \), which coincides with the usual Cartan matrix of the associated Dynkin diagram \( \overline{Q} \), in case \( Q \) is a Dynkin quiver\(^2\).

The following simple result is useful for the actual calculation of the linear free divisors below.

**Lemma 3.3.** If \( Q \) is a finite quiver without oriented cycles, then its Euler matrix is invertible. The inverse is given by \( E^{-1} = I_{\mid Q_0 \mid} + A' \), where \( A'_{x,y} \) equals the number of directed paths from \( x \) to \( y \). \( \square \)

Now we recall the trichotomy of the representation theory of quivers:

**Definition 3.4.** A quiver \( Q \) is of finite representation type if \( Q \) has only finitely many indecomposable representations, up to isomorphism. The quiver is wild if its representation theory is at least as complicated as that of the quiver

\[
\begin{array}{c}
\text{C} \\
\text{C}
\end{array}
\]

The quiver is tame if it is neither of finite representation type, nor wild\(^3\).

Gabriel (\[10\], \[11\]) showed

**Theorem 3.5.** A connected quiver \( Q \) is of finite representation type if and only if it is a Dynkin quiver. Assigning to an isomorphism class of indecomposable representations of \( Q \) its dimension vector induces then a bijection between these classes and the positive roots of the underlying Dynkin diagram. \( \square \)

The last part of this result can be restated thus: if \( d \) is a positive root of the underlying Dynkin diagram (as listed, for example, in the appendix to \[3\]) then \( d \) is also a root of any associated Dynkin quiver \( Q \), in the sense of Definition \[\text{P}\] Moreover, in this case each root is a real Schur root: there is a (unique) open orbit in \( \text{Rep}(Q, d) \) whose points correspond to indecomposable representations. A good account of all this can be found in \[2\].

The class of tame quivers has a similar characterisation:

**Theorem 3.6.** (\[9\], \[23\]) A connected quiver is tame if and only if the underlying undirected graph is an extended Dynkin diagram. \( \square \)

Finally, in what follows we will need a result of V.Kac (\[16\]):

**Proposition 3.7.** Let \( Q \) be a connected quiver whose proper subquivers are all either of finite or tame type. Then a dimension vector \( d \) is a real root if and only if \( q(d) = 1 \), and it is an imaginary root if and only if \( q(d) \leq 0 \). \( \square \)

\(^2\) More generally, \( C \) is the Cartan matrix of the Kac–Moody Lie algebra associated to \( \overline{Q} \), for an arbitrary finite quiver \( Q \) without oriented cycles, see \[16\] again.

\(^3\) The reader should be aware that the definition often is “tame” = “not wild”, thus, different from our usage here.
4 The fundamental exact sequence

Let $V$ and $W$ be representations of the quiver $Q$. In [25], Ringel introduced the following exact sequence $\mathcal{E}_V^W$ of vector spaces:

$$0 \to \text{Hom}_Q(W,V) \to \prod_{x \in Q_0} \text{Hom}_k(W_x, V_x) \xrightarrow{d_V^W} \prod_{\varphi \in Q_1} \text{Hom}_k(W(t(\varphi)), V(h(\varphi))) \xrightarrow{e_V^W} \text{Ext}^1_Q(W,V) \to 0. \quad (1)$$

The morphism $d_V^W$ is defined by

$$d_V^W((\psi_x)_{x \in Q_0}) = (\psi_{h(\varphi)} \circ W(\varphi) - V(\varphi) \circ \psi_{t(\varphi)})_{\varphi \in Q_1};$$

the component of $d_V^W((\psi_x))$ corresponding to $\varphi \in Q_1$ measures non-commutativity of the diagram

\[
\begin{array}{ccc}
W_{t\varphi} & \xrightarrow{W(\varphi)} & W_{h\varphi} \\
\psi_{t\varphi} \downarrow & & \downarrow \psi_{h\varphi} \\
V_{t\varphi} & \xrightarrow{V(\varphi)} & V_{h\varphi}
\end{array}
\]

whence it is clear that $\ker d_V^W$ is indeed equal to $\text{Hom}_Q(W,V)$.

To define $e_V^W$, from $\theta = (\theta_{\varphi})_{\varphi \in Q_1}$ we construct a new representation $Z$ of $Q$ and an exact sequence,

$$e_V^W(\theta) \equiv 0 \to V \xrightarrow{i} Z \xrightarrow{j} W \to 0,$$

by the following recipe: $Z_x = V_x \oplus W_x$ for each $x \in Q_0$, $i_x : V_x \to V_x \oplus W_x$ and $j_x : V_x \oplus W_x \to W_x$ are the standard inclusion and projection, and for each $\varphi \in Q_1$, $Z(\varphi) : V_{t\varphi} \oplus W_{t\varphi} \to V_{h\varphi} \oplus W_{h\varphi}$ has matrix

$$\begin{pmatrix}
W(\varphi) & \theta_{\varphi} \\
0 & V(\varphi)
\end{pmatrix}.$$

It is straightforward to check that the short exact sequence $e_V^W(\theta)$ of representations of $Q$ is split if and only if $\theta = d_V^W(\psi)$ for some $\psi \in \prod_{x \in Q_0} \text{Hom}(W_x, V_x)$, and that $e_V^W$ is onto.

Exactness of the sequence $\mathcal{E}_V^W$ implies that

$$\langle e, d \rangle = \dim_k \text{Hom}_Q(W,V) - \dim_k \text{Ext}^1_Q(W,V)$$

for any $W \in R(Q, e)$ and $V \in R(Q, d)$, so that the expression on the right hand side depends only on the dimension vectors and not on the choice of representations, although evidently the dimensions of $\text{Ext}^1_Q(W,V)$ and $\text{Hom}_Q(W,V)$ do depend on the choice of $V \in R(Q, d)$ and $W \in R(Q, e)$.

The fundamental sequence $\mathcal{E}_V^W$ plays two roles in what follows. In the next section we show how to reinterpret it in terms of the deformation theory of representations, where it may become more familiar to singularity-theorists. From this we will see how free divisors appear naturally in this context.

Second, following Schofield [29], we use it to generate semi-invariants of the representation space $R(Q, d) = \prod_{\varphi \in Q_1} \text{Hom}_k(k^{d(t\varphi)}, k^{d(h\varphi)})$, and thereby find explicit equations for the free divisors, in Sections 8 and 10.
5 Deformations of representations

Recall that the group $\text{Gl}(Q, d)$ acts on $\text{Rep}(Q, d)$ by

$$
(g_x)_{x \in Q_0} \cdot (V(\varphi))_{\varphi \in Q_1} = (g_{h \varphi} \circ V(\varphi) \circ g_{t \varphi}^{-1})_{\varphi \in Q_1}.
$$

The orbit of $V$ in $\text{Rep}(Q, d)$ is open if and only if the associated map

$$
\alpha_V : \text{Gl}(Q, d) \to \text{Rep}(Q, d)
$$

sending $g$ to $g : V$ is a submersion, and for this it is enough that it be a submersion at the identity. The Lie algebra $\mathfrak{gl}(Q, d)$ of $\text{Gl}(Q, d)$ is

$$
\prod_{x \in Q_0} \text{End}(k^d(x)) = \prod_{x \in Q_0} \text{Hom}(k^d(x), k^d(x)),
$$

and the tangent space to $\text{Rep}(Q, d)$ at $V$ is $\text{Rep}(Q, d)$ itself, i.e. $\prod_{x \in Q_0} \text{Hom}_k(k^d(\varphi x), k^d(\varphi))$. The derivative of $\alpha_V$ at the identity in $\text{Gl}(Q, d)$ is precisely the map $d\alpha^V_V$ of the exact sequence $E_V$. In fact we may canonically identify $\text{Ext}^1_{\text{Rep}}(V, V)$ with $T^1(V)$ for the associated deformation theory, though we will not make any formal use of this identification.

A deformation, in the analytic category, of a representation $V$ is, by definition, the germ of an analytic map $(B, 0) \to (\text{Rep}(Q, d), V)$. If $(B, 0)$ is smooth, a deformation $\mathcal{V} : (B, 0) \to (\text{Rep}(Q, d), V)$ is versal if and only if it is complete, that is, if every other deformation $\mathcal{V}' : (B', 0) \to (\text{Rep}(Q, d), V)$ is equivalent to one induced from it by base-change $\eta : (B', 0) \to (B, 0)$. The equivalence here is the existence of a map-germ $g : (B, 0) \to (\text{Gl}(Q, d), 1)$ such that

$$
\mathcal{V}'(b') = g(b') \cdot \mathcal{V}(\eta(b')).
$$

Thus it is evident that $\text{Rep}(Q, d)$ itself, or more precisely the identity map $(\text{Rep}(Q, d), V) \to (\text{Rep}(Q, d), V)$, is a versal deformation; for any other deformation $\mathcal{V}'$, the base change map $\eta$ is simply $\mathcal{V}'$ itself, and $g$ is the constant map taking the value 1. The slice theorem from the theory of smooth group actions is now enough to establish the versality of any deformation obtained from this one by restricting its domain to any smooth space-germ transverse to the orbit of $V$, or indeed by pulling it back by any map-germ $(B, 0) \to (\text{Rep}(Q, d), V)$ transverse to the orbit of $V$. These considerations imply the Artin–Voigts Lemma: that the dimension of $\text{Ext}^1_{\text{Rep}}(V, V) \cong T^1(V)$ equals the codimension of the orbit of $V$ in $\text{Rep}(Q, d)$. In particular, if there is an open orbit, then the representations therein have no self-extensions: they are rigid as representations.

Now we consider the relative $T^1$, obtained by regarding the coefficients of the morphisms $V(\varphi)$ as variables. This can be done in the analytic, formal or algebraic category, and amounts to no more than tensoring the exact sequence $E^V$ with the appropriate ring, or sheaf, of functions — $\mathcal{O}_{\text{Rep}(Q, d)}$, $k[\text{Rep}(Q, d)^*]$ or $k[[\text{Rep}(Q, d)^*]]$. We refer to these indistinctly as $R$. The module (sheaf) of vector fields on $\text{Rep}(Q, d)$ is $\theta_R = \text{Der}_k(R) \cong \text{Rep}(Q, d) \otimes_k R$, and the $k$-linear map $\mathfrak{gl}(Q, d) \to \text{Rep}(Q, d)$ extends to a morphism of $R$-modules $\mathfrak{gl}(Q, d) \otimes_k R \to \theta_R$ whose cokernel can be viewed both as $\text{Ext}^1_{\text{Rep}}(M, M)$ for the universal representation $M$ of the quiver $Q$ with coefficients in $R$, and as the relative $T^1$ of the versal deformation $i : \text{Rep}(Q, d) \to \text{Rep}(Q, d)$, denoted $T^1(i/\text{Rep}(Q, d))$. The surjection $\theta_R \to T^1(i/\text{Rep}(Q, d))$ is the Kodaira–Spencer map of the versal deformation $i$.

The kernel of this projection is the space of simultaneous endomorphisms of the representations $V \in \text{Rep}(Q, d)$, or, in other words, the endomorphism ring of the universal representation $M$. Provided the general representation in $\text{Rep}(Q, d)$ is indecomposable, this ring is isomorphic to $R$. Let us understand why this is so. It is clear that if $V \in \text{Rep}(Q, d)$ is any representation then for each $\lambda \in k^*$ we have $(\lambda I_{d_x})_{x \in Q_0} \in \text{Aut}_k(V)$, and similarly $(\lambda I_{d_x})_{x \in Q_0} \in \text{End}_Q(V)$ for $\lambda \in k$. If $V$ is stably indecomposable (that
is, if there is a neighbourhood of $V$ in $\text{Rep}(Q, d)$ consisting of indecomposable representations) then this copy of $k$ accounts for all of $\text{End}_Q(V)$ (see e.g. [5, 2.7]). Now if the general representation in $\text{Rep}(Q, d)$ is indecomposable — which means that $d$ is a Schur root — then at each of these representations, any endomorphism of the universal representation $M$ must be a scalar. By density, the same must be true everywhere, and so $\text{End}_Q(M)$ can be identified with $R$.

The cokernel of the inclusion of Lie algebras $0 \to k \to \mathfrak{gl}(Q, d)$ is, by definition, $\mathfrak{pgl}(Q, d)$, and we can identify the cokernel of the inclusion of free $R$-modules $0 \to R \to \mathfrak{gl}(Q, d) \otimes R$ with $\mathfrak{pgl}(Q, d) \otimes R$. Thus, provided the generic representation in $\text{Rep}(Q, d)$ is indecomposable, we have a short exact sequence

$$0 \to \mathfrak{pgl}(Q, d) \otimes_k R \overset{\tilde{d}_M}{\longrightarrow} \theta_R \to \text{Ext}^1_{RQ}(M, M) \to 0. \quad (2)$$

Even without generic indecomposability, we still have an exact sequence

$$\mathfrak{pgl}(Q, d) \otimes_k R \overset{\tilde{d}_M}{\longrightarrow} \theta_R \to \text{Ext}^1_{RQ}(M, M) \to 0. \quad (3)$$

Let $D$ be the support of $\text{Ext}^1_{RQ}(M, M) = T^1(\text{Rep}(Q, d))$, with (possibly non-reduced) coordinate ring $R[D] = R/\mathcal{F}_0(\text{Ext}^1_{RQ}(M, M))$, where $\mathcal{F}_0$ means zero'th Fitting ideal.

**Proposition 5.1.** (i) $D$ is the set of non-rigid representations. Its open complement is the set of rigid representations$^4$.

If $q(d) = 1$ and the general representation in $\text{Rep}(Q, d)$ is indecomposable, thus, $d$ is a Schur root, then

(ii) $D$ is a divisor in $\text{Rep}(Q, d)$.

(iii) $\text{Ext}^1_{RQ}(M, M)$ is a maximal Cohen-Macaulay $R[D]$-module.

(iv) The image of $\tilde{d}_M : \mathfrak{pgl}(Q, d) \otimes_k R \to \theta_R$ is contained in $\text{Der}(- \log D)$.

**Proof**

(i) Let $m_V$ be the maximal ideal of $R$ corresponding to $V \in \text{Rep}(Q, d)$. By right-exactness of tensor product, tensoring the sequence (2) with $R/m_V$ gives the exact sequence

$$\mathfrak{pgl}(Q, d) \otimes_k R \overset{d'_V}{\longrightarrow} \text{Rep}(Q, d) \to \text{Ext}^1_Q(V, V) = T^1(V) \to 0.$$

This establishes (i).

(ii) Since now $\mathfrak{pgl}(Q, d) \otimes_k R$ and $\theta_R$ are free $R$-modules of the same rank, $\mathcal{F}_0(\text{Ext}^1_{RQ}(M, M))$ is generated by $\det(\tilde{d}_M)$, and so $D = \text{supp}(\text{Ext}^1_{RQ}(M, M)) = V(\mathcal{F}_0(\text{Ext}^1_{RQ}(M, M))) = V(\det(\tilde{d}_M))$.

(iii) Exactness of the sequence (2) implies, by the Auslander-Buchsbaum formula, that

$$\text{depth}_R(\text{Ext}^1_{RQ}(M, M)) = \dim R - 1 = \dim \text{Ext}^1_{RQ}(M, M),$$

where “dim” here refers to Krull dimension.

Hence $\text{Ext}^1_{RQ}(M, M)$ is a Cohen-Macaulay $R$-module. It is annihilated by $\mathcal{F}_0(\text{Ext}^1_{RQ}(M, M))$, so is an $R[D]$-module, and as such, a maximal Cohen-Macaulay module.

(iv) The vector fields in $\tilde{d}_M(\mathfrak{pgl}(Q, d) \otimes_k R)$ are infinitesimal generators of the action of $\text{Gl}(Q, d)$ on $\text{Rep}(Q, d)$, and are thus tangent to all its orbits. So they are tangent to $D$, which is a union of orbits.

---

$^4$Singularity theorists might prefer ‘stable’ to ‘rigid’; however, in representation theory the term ‘stable’ often refers to its meaning in geometric invariant theory, so here we use ‘rigid’.
Note that by \(3.5\) if \(Q\) is a Dynkin quiver then in order for (ii)-(iv) to hold we need only require that \(q(d) = 1\).

If the conditions of \(5.1\)(ii)-(iv) hold, and moreover the vector fields in \(\tilde{d}_M^d(pgl(Q, d) \otimes_k R)\) generate \(\text{Der}(-\log D)\) then \(D\) is a linear free divisor, since by exactness of \(2\), \(\text{Der}(-\log D)\) is free over \(R\). Saito’s criterion (1.2 above) shows that in order that they do generate, it is enough that \(\det(\tilde{d}_M^d)\) be reduced.

Thus, we obtain the following result:

**Corollary 5.2.** With the conditions of 5.1(ii)-(iv), suppose in addition that \(D\) is reduced. Then it is a linear free divisor. \(\square\)

From now on we will refer to the divisor \(D\) of non-rigid representations in \(\text{Rep}(Q, d)\) as the discriminant and call \(\Delta = \det(\tilde{d}_M^d)\) its canonical equation.

Suppose that \(D\) is reduced at \(V\). Then by Saito’s criterion, the vector fields in \(\tilde{d}_M^d(pgl(Q, d) \otimes_k R)\) generate the stalk at \(V\) of the sheaf \(\text{Der}(-\log D)\). If \(V\) is a regular point of \(D\), then the tangent space \(T_V D\) is equal to \(d_V^d(pgl(Q, d)) \subseteq \text{Rep}(Q, d)\). It follows that the deformation of \(V\) obtained by following any smooth curve transverse to \(D\) is versal. For the same reason, any deformation in a direction tangent to \(D\) is infinitesimally trivial at \(V\). Since the same holds at any nearby point, any deformation of \(V\) in the smooth part of \(D\) is globally trivial. In terms of the group action, this means that then each irreducible component of \(D\) contains a dense open orbit, and for each representation \(V\) in such an orbit, \(T^1(V)\) will be one-dimensional. We now investigate further the relation between the dimension of \(T^1(V)\) for a generic representation on such a component and the multiplicity with which that component occurs in the discriminant.

**Lemma 5.3.** Let \(D_j\) be an irreducible component of \(D\), and \(h_j\) its reduced equation, \(m_j\) the multiplicity of \(h_j\) in \(\det(\tilde{d}_M^d)\), and \(V_j\) a generic representation on \(D_j\). One has then \(m_j \geq \dim_k T^1(V_j)\) and equality holds if and only if \(h_j\) annihilates \(\text{Ext}^1_{RQ}(M, M)\). In particular, \(m_j = 1\) forces \(\dim T^1(V_j)\) to be one-dimensional and the orbit generated by \(V_j\) to be dense in \(D_j\).

**Proof** Let \(p\) be the ideal \((h_j)\). Then the localisation \(R_p\) is a discrete valuation ring. We must have

\[
\text{Ext}^1_{RQ}(M, M) \otimes_R R_p \cong \bigoplus_{\alpha} R_p/(pR_p)^{\alpha}
\]

for some positive integers \(\alpha\); it follows that the matrix \(\tilde{d}_M^d\) is equivalent, over \(R_p\), to a matrix of the form \(\text{diag}(h_j^{\alpha_1}, \ldots, h_j^{\alpha_{\ell}}) \oplus \text{I}_{r-\ell}\), a block matrix formed of the indicated diagonal matrix and the identity matrix \(I_{r-\ell}\), where \(r = \dim_k pgl(Q, d)\). Evidently \(\det(\tilde{d}_M^d) = (h_j)^{\sum_{\alpha=1}^\ell \alpha} \in R_p\), and so \(\sum_{\alpha=1}^\ell \alpha = m_j\). Moreover, by \(4\), \(\sum_{\alpha=1}^\ell \alpha\) is also the rank of \(\text{Ext}^1_{RQ}(M, M)\) at a generic point \(V_j\) of \(D_j\). Dividing by the maximal ideal \(m_{V_j}\), we see that then \(\ell\) is equal to \(\dim_k \text{Ext}^1_Q(V_j, V_j)\). Therefore, \(m_j = \sum_{\alpha=1}^\ell \alpha = \ell = \dim_k \text{Ext}^1_Q(V_j, V_j) = \dim T^1(V_j)\). Clearly, \(m_j = \ell\) if and only if each \(\alpha = 1\) if and only if \(h_j\) annihilates \(\text{Ext}^1_{RQ}(M, M)\). \(\square\)

In the case of Dynkin quivers, it follows that the discriminant is indeed reduced, as we show next.

**Proposition 5.4.** Let \(d\) be a real Schur root of a Dynkin quiver \(Q\) and assume that \(V \in \text{Rep}(Q, d)\) satisfies \(\dim T^1(V) = 1\). If \(D' \subseteq D\) denotes the irreducible component of the discriminant that contains \(V\) and \(h' = 0\) is its reduced equation, then \(h'\) divides \(\Delta = \det(\tilde{d}_M^d)\) with multiplicity one.

**Proof** We begin by clarifying in general what it means that \(D'\) appears with multiplicity one, if we know already that the generic representation on it has one-dimensional \(T^1\): As \(T^1(V) = \text{Ext}^1_Q(V, V)\) is one-dimensional, the semi-universal deformation of \(V\) as a representation of \(Q\) has a one-dimensional base. Because \(V\) deforms into a rigid representation generically, its reduced discriminant consists just of the origin. By Openness of Versality, it suffices to prove that the discriminant in that semi-universal
deformation is indeed reduced. If $\mathfrak{G}$ is the universal module over $k[t]$, the (formal) base ring of the semiuniversal deformation, it suffices to show that $\text{Ext}^1_{k[t][Q]}(\mathfrak{G}, \mathfrak{G})$ is a one-dimensional vector space. Now $\text{Ext}^1_{k[t][Q]}(\mathfrak{G}, \mathfrak{G})$ is concentrated on the discriminant, thus a finite dimensional vector space. Moreover, $\text{Ext}^1_{k[t][Q]}(\mathfrak{G}, \mathfrak{G}) \otimes_{k[t]} k \cong \text{Ext}^1_{Q}(V, V) \cong k$, whence as $k[t]$–module $\text{Ext}^1_{k[t][Q]}(\mathfrak{G}, \mathfrak{G}) \cong k[t]/(t^m)$ for some $m$. We need to show that $m = 1$, and this can be achieved by establishing that the following natural projection, in its various guises:

$$
\begin{align*}
\text{Ext}^1_{k[t][Q]}(\mathfrak{G}, \mathfrak{G}) \otimes_{k[t]} k[t]/(t^2) & \cong \text{Ext}^1_{k[t][Q]}(\mathfrak{G}, \mathfrak{G}) \otimes_{k[t]} k[t] \cong \text{Ext}^1_{Q}(V, V) \\
\text{Ext}^1_{k[t][Q]}(\mathfrak{G}, \mathfrak{G}) \otimes_{k[t]} k & \cong \text{Ext}^1_{k[t][Q]}(\mathfrak{G}, \mathfrak{G}) \otimes_{k[t]} k[t] \cong \text{Ext}^1_{Q}(V, V) \cong k
\end{align*}
$$

is an isomorphism. To this end, let

$$
0 \to V \xrightarrow{i} W \xrightarrow{p} V \to 0
$$

represent a nontrivial element in the one-dimensional vector space $\text{Ext}^1_{Q}(V, V)$. Define an action of $t$ on $W$ through $t(w) = ip(w)$. Clearly, $t^2 = ipip = 0$ on $W$, whence the $Q$–representation $W$ becomes as well a $k[t]/(t^2)$–module. Infinitesimal deformation theory says that indeed $W \cong \mathfrak{G}/t^2 \mathfrak{G}$, and that the extension above can be viewed as an extension of $k[t]$–modules,

$$
0 \to V \cong \mathfrak{G}/t^2 \mathfrak{G} \xrightarrow{i \circ t \circ x} W \cong \mathfrak{G}/t^3 \mathfrak{G} \xrightarrow{p \circ \delta \circ t \circ q} V \cong \mathfrak{G}/t \mathfrak{G} \to 0.
$$

Now apply $\text{Hom}_{k[t][Q]}(\mathfrak{G}, -)$ to this exact sequence to obtain the following long exact sequence of $k[t]$–modules, with $\delta$ denoting the connecting homomorphism:

$$
\begin{array}{ccccccccc}
0 & \to & \text{Hom}_{k[t][Q]}(\mathfrak{G}, V) & \to & \text{Hom}_{k[t][Q]}(\mathfrak{G}, W) & \to & \text{Hom}_{k[t][Q]}(\mathfrak{G}, V) & \delta \\
& & \text{Ext}^1_{k[t][Q]}(\mathfrak{G}, V) & \to & \text{Ext}^1_{k[t][Q]}(\mathfrak{G}, W) & \to & \text{Ext}^1_{k[t][Q]}(\mathfrak{G}, V) & 0
\end{array}
$$

The map $\pi = \text{Ext}^1_{k[t][Q]}(\mathfrak{G}, p)$ is the same as the projection alluded to above, which we wish to show is an isomorphism. Using the various identifications, we may rewrite this long exact sequence as

$$
\begin{array}{ccccccccc}
0 & \to & \text{End}_{Q}(V) & \to & \text{End}_{(k[t]/(t^2))Q}(W) & \to & \text{End}_{Q}(V) & \delta \\
& & \text{Ext}^1_{Q}(V, V) & \to & \text{Ext}^1_{Q}(\mathfrak{G}, W) & \pi & \text{Ext}^1_{Q}(V, V) & 0
\end{array}
$$

As $\mathbf{d}$ is a Schur root, and $\dim T^1(V) = \dim \text{Ext}^1_{Q}(V, V) = 1$, we see that $\pi$ is an isomorphism if and only if $\delta \neq 0$ if and only if there exists a $Q$–endomorphism of $V$ that cannot be lifted to a $k[t]$–linear $Q$–endomorphism of $W$. While these considerations apply to any quiver, we now show that $\delta \neq 0$, thereby establishing that $\pi$ is indeed an isomorphism, for any Schur root of a Dynkin quiver.

By assumption, $q(V) = 1$ and $\dim \text{Ext}^1_{Q}(V, V) = 1$, whence $V$ is decomposable, say, $V = V' \oplus V''$ for nonzero $Q$–representations $V', V''$. It follows from $\dim \text{End}_{Q}(V) = 2$ that $\text{End}_{Q}(V) \cong \text{End}_{Q}(V') \oplus \text{End}_{Q}(V'')$, and that the endomorphism rings of $V', V''$ are one-dimensional, in particular these representations are indecomposable. This means that their dimension vectors are real Schur roots as well, and so the representations are rigid. From $\text{Ext}^1_{Q}(V, V) \cong \text{Ext}^1_{Q}(V' \oplus V'', V' \oplus V'')$, it then follows that exactly one of the groups $\text{Ext}^1_{Q}(V', V'')$ or $\text{Ext}^1_{Q}(V'', V')$ is nonzero — and then one-dimensional. Assume $\text{Ext}^1_{Q}(V', V'') \neq 0$. The associated nontrivial extension

$$
0 \to V'' \xrightarrow{j} W' \xrightarrow{p} V' \to 0
$$

(6)
gives rise to the following nonzero extension class in $\text{Ext}_Q^1(V,V)$:

$$
\begin{array}{cccc}
V' & \to & V' \\
\oplus & & \oplus \\
0 & \to & V'' & \xrightarrow{i} & W' & \xrightarrow{p} & V' & \to & 0 \\
\oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
V'' & \to & V''
\end{array}
$$

Note that $W'$ has dimension vector $d$, as that is the sum of the dimension vectors of $V''$ and $V'$, equal to the dimension vector of $V$. It is now a general fact that $V = V' \oplus V''$ deforms into the middle term $W'$, for any extension. As the sequence does not split, $W' \not\cong V$, and, as $V$ has a one-dimensional semi-universal deformation, $W'$ must be the indecomposable representation of dimension vector $d$. Using the observation following [5], the $k[t]$-module structure on the middle term $W = V' \oplus W' \oplus V''$ is as follows:

$$t(V') = 0 \quad , \quad t|_{W'} = p \quad , \quad t|_{V''} = i.$$  

With $W'$ an indecomposable $Q$-representation and the action of $t$ as described, it follows easily that $W = V'' \oplus W' \oplus V'$ is indecomposable as a $Q$-representation over $k[t]$. Accordingly, its endomorphism ring $\text{End}_{(k[t]/(t^2))Q}(W)$ contains only the trivial idempotents, thus none of the idempotents in $\text{End}_Q(V)$ that corresponds to the projections onto the indecomposable factors of $V$ can be lifted, and the natural ring homomorphism $\text{End}_{(k[t]/(t^2))Q}(W) \to \text{End}_Q(V)$ is not surjective. This yields the claim. 

**Corollary 5.5.** If $Q$ is a Dynkin quiver and $d$ is a real root of $Q$ then the discriminant in $\text{Rep}(Q,d)$ is a linear free divisor.

**Proof**  
By Gabriel’s theorem $Q$ is of finite representation type. Therefore at a generic point $V$ on each irreducible component of $D$, any deformation of $V$ inside $D$ is trivial. Thus $T^1(V)$ is 1-dimensional. □

Everything we have said so far only depends on the support of the dimension vector $d$, that is, the full subquiver whose nodes are those $x \in Q_0$ with $d(x) \neq 0$. A dimension vector is sincere if its support is all of $Q_0$.

## 6  A Criterion for $D$ to be a Linear Free Divisor

The group $\text{Gl}(Q,d)$ acts on the ring $R$ of polynomial functions on $\text{Rep}(Q,d)$ by the contragredient action, as described earlier in Section[5] A polynomial $f \in R$ is a semi-invariant of weight $\chi$, where $\chi$ is a character of $\text{Gl}(Q,d)$, if for all $g \in \text{Gl}(Q,d)$ we have $g \cdot f = \chi(g)f$. As the characters of $\text{Gl}_n(k)$ are just integral powers of det, the characters of $\text{Gl}(Q,d)$ are in bijection with elements of $Z^{Q_0}$. The weight $w(f)$ of a semi-invariant $f$ is usually identified with the image in $Z^{Q_0}$ of its associated character.

**Theorem 6.1.**  (Sato-Kimura [28]) Let the connected algebraic group $G$ act on the vector space $V$. If there is an open orbit then the ring $SI(G,V)$ spanned by the semi-invariants is a polynomial ring:

$$SI(G,V) = k[f_1, \ldots, f_s]$$

for some collection of algebraically independent and irreducible semi-invariants $f_1, \ldots, f_s$. Moreover if $f_i \in SI(G,V)_{\chi_i}$ then the $\chi_i$ are linearly independent in the space of characters of $G$. □

**Corollary 6.2.** Under the assumptions of the theorem, the set of characters $\chi$ such that $SI(G,V)_{\chi} \neq 0$ forms a free abelian semigroup, isomorphic to $\mathbb{N}^s$. In particular, if $f$ is any semi-invariant, of weight $\chi$, then $f = uf_1^{a_1} \cdots f_s^{a_s}$, where $u$ is a unit in $k$ and the $a_i \geq 0$ are the unique integers such that $\chi = \sum_{i=1}^s a_i \chi_i$ in the space of characters of $G$. □
Suppose that $d$ is a real Schur root of $Q$, and let $D$ be the discriminant in $\text{Rep}(Q,d)$. As $D$ is preserved under the action of $\text{Gl}(Q,d)$, its canonical equation $\Delta$ is a semi-invariant. If $V \notin D$, and $f$ is a non-zero semi-invariant, then $f(V)$ cannot vanish; if it did, then it would vanish everywhere on the orbit of $V$, which is dense. In other words, the zero locus of any semi-invariant must be contained in the discriminant. In particular, with the $f_i$ as in [6.1], $f_1 \cdots f_s$ is necessarily a reduced equation for $D$, and so $\Delta = uf_1^{a_1} \cdots f_s^{a_s}$, with $u$ a unit in $k$, and uniquely determined integers $a_i > 0$.

Moreover, Kac has shown in [17, p.153] that the discriminant for a real Schur root $d$ contains precisely $n - 1$ irreducible components, where $n$ is the number of nodes in the support of $d$, thus, there are $s = n - 1$ fundamental semi-invariants $f_i$ in $SI(\text{Gl}(Q,d), \text{Rep}(Q,d))$. This gives us a first combinatorial criterion for $D$ to be a linear free divisor.

**Proposition 6.3.** Suppose that $d$ is a real Schur root of $Q$, supported on $n$ nodes. Assume further that $g_1, \ldots, g_{n-1}$ are semi-invariants on $\text{Rep}(Q,d)$ with linearly independent weights $w_i = w(g_i)$. If the weight of the discriminant $D$ satisfies $w(D) = \sum_{i=1}^{n-1} a_i w_i$, for integers $a_i \geq 1$, then $\Delta = w g_1^{a_1} \cdots g_{n-1}^{a_{n-1}}$ for some unit $u \in k$. If we know further that the weights $w_i$ generate the semigroup of all weights occurring in $SI(\text{Gl}(Q,d), \text{Rep}(Q,d))$, then the $g_i$ constitute the reduced equations of the components of $D$, and $D$ is a linear free divisor if and only if each $a_i = 1$.

Derksen and Weyman in [8] describe in general the semigroup of weights occurring in $SI(\text{Gl}(Q,d), \text{Rep}(Q,d))$ through a single equation\(^5\) and integral inequalities that depend upon the dimension vectors of generic sub-representations, whence the criterion can be applied, at least in principle. We may as well turn the criterion around to determine all semi-invariants if we already know that $D$ is a linear free divisor, such as for real roots whose support is a Dynkin quiver:

**Corollary 6.4.** Assume the discriminant $D$ in $\text{Rep}(Q,d)$, for $d$ a real Schur root, is a free divisor and its canonical equation factors as $\Delta = g_1 \cdots g_{n-1}$ for semi-invariant polynomials $g_i$ with linearly independent weights. If $n$ is the number of nodes in the support of $d$, then the factors $g_i$ are algebraically independent and irreducible polynomials that generate the ring of semi-invariants $SI(\text{Gl}(Q,d), \text{Rep}(Q,d))$.

\(^5\)Namely that the ordinary scalar product of the weight of a semi-invariant with the dimension vector $d$ has to vanish, that is, $w \cdot d = 0$.
Lemma 6.6. Let \( \mathbf{d}, \mathbf{e} \) be dimension vectors for the quiver \( Q \) with \( \langle \mathbf{e}, \mathbf{d} \rangle = 0 \). The weight of the \( \text{Gl}(\mathbf{e}) \times \text{Gl}(\mathbf{d}) \) semi-invariant polynomial \( c(W, V) \) in the character group \( \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \) is

\[
w(c(W, V)) = (\mathbf{d} - \text{out}_\mathbf{d}, -\mathbf{e} + \text{in}_\mathbf{e}) = (\mathbf{d}^T, -\mathbf{e}) ,
\]

while that of the \( \text{Gl}(
\mathbf{d}) \) semi-invariant \( c^W \) in \( \mathbb{Z}^{Q_0} \) is

\[
w(c^W) = -\mathbf{e} + \text{in}_\mathbf{e} = -\mathbf{e} \]

and the weight of the discriminant in \( \text{Rep}(Q, \mathbf{d}) \) equals

\[
w(\Delta) = \text{in}_\mathbf{d} - \text{out}_\mathbf{d} = \mathbf{d}(E^T - E) .
\]

Proof. Let \( V, W \) be two representations with dimension vectors \( \mathbf{d}, \mathbf{e} \) such that \( \langle \mathbf{e}, \mathbf{d} \rangle = 0 \). The map \( d^W_\mathbf{V} \) can be viewed as a linear map

\[
d^W_\mathbf{V} : \bigoplus_{x \in Q_0} V_x \otimes W_x^* \rightarrow \bigoplus_{\varphi \in Q_1} V_{h\varphi} \otimes W_{t\varphi}^* ,
\]

where \((-)^*\) denotes the \( k \)-dual. Denoting by \( \Lambda(-) \) the highest exterior power of a vector space, and observing that

\[
\Lambda(U^*) \cong \Lambda(U)^*, \quad \Lambda(U \oplus U') \cong \Lambda(U) \otimes \Lambda(U') , \quad \Lambda(U \otimes U') \cong \Lambda(U)^{\dim U'} \otimes \Lambda(U')^{\dim U} ,
\]

for vector spaces \( U, U' \), the determinant of \( d^W_\mathbf{V} \) can be represented as

\[
\det d^W_\mathbf{V} \cong \Lambda(d^W_\mathbf{V}) : \bigotimes_{x \in Q_0} \Lambda(V_x)^{\mathbf{d}(x)} \otimes \Lambda(W_x^*)^{\mathbf{e}(x)} \rightarrow \bigotimes_{\varphi \in Q_1} \Lambda(V_{h\varphi})^{\mathbf{e}(\varphi)} \otimes \Lambda(W_{t\varphi}^*)^{\mathbf{d}(\varphi)} .
\]

One reads off that as a semi-invariant for \( \text{Gl}(\mathbf{e}) \times \text{Gl}(\mathbf{d}) \) the determinant of \( d^W_\mathbf{V} \) transforms according to

\[
\left( \prod_{\varphi \in Q_1} \det \left( \text{Gl}(\mathbf{d}(\varphi)) \right) \right)^{\mathbf{e}(\varphi)} \det \left( \text{Gl}(\mathbf{e}(\varphi)) \right)^{-\mathbf{d}(\varphi)} \left( \prod_{x \in Q_0} \det \left( \text{Gl}(\mathbf{d}(x)) \right) \right)^{-\mathbf{e}(x)} \det \left( \text{Gl}(\mathbf{e}(x)) \right)^{\mathbf{d}(x)}
\]

thus, its weight, in the character group \( \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \) of \( \text{Gl}(\mathbf{e}) \times \text{Gl}(\mathbf{d}) \), is given on a pair of nodes \( (y, x) \) by

\[
w(\det d^W_\mathbf{V})(y, x) = \mathbf{d}(y) - \sum_{t\varphi = y} \mathbf{d}(h\varphi) - \mathbf{e}(x) + \sum_{h\varphi = x} \mathbf{e}(t\varphi)
\]

thus,

\[
w(\det d^W_\mathbf{V}) = (\mathbf{d} - \text{out}_\mathbf{d}, -\mathbf{e} + \text{in}_\mathbf{e}) = (\mathbf{d}^T, -\mathbf{e}) \in \mathbb{Z}^{Q_0 \times Q_0} .
\]

For \( V = W \), the diagonal summand \( k \subseteq \oplus_{x \in Q_0} \text{Hom}(V_x, V_x) \) does not contribute to the weight of the determinant, and restricting \( w(\det d^V_\mathbf{V}) \) to the diagonal \( y = x \) yields the claimed formula for the discriminant.

Now we are ready to study some examples.
7 Examples

To illustrate the results and to exhibit explicit linear free divisors arising from Dynkin quivers, we concentrate mainly on the most complicated ones, those corresponding to the highest root of a Dynkin diagram viewed as the dimension vector of some Dynkin quiver. Recall that the connected Dynkin diagrams are in natural bijection with the binary polyhedral groups, the congruence classes of finite subgroups of $\text{SL}(2, \mathbb{C})$. One has the following simple relation between the dimension of the representation variety associated to the highest root and the order of the corresponding finite group.

**Lemma 7.1.** Let $Q$ be a connected Dynkin quiver, $d$ the highest root of the underlying Dynkin diagram, and $\Gamma$ the associated binary polyhedral group. The dimension of $\text{Rep}(Q, d)$, equal to the degree of the discriminant $D$, is then $\text{dim Rep}(Q, d) = |\Gamma| - 2$.

**Proof.** By the McKay correspondence, the components $d(x)$ of the highest root are in bijection with the dimensions of the isomorphism classes of irreducible and nontrivial representations of $\Gamma$. Accordingly,

$$|\Gamma| = 1 + \sum_{x \in Q_0} d(x)^2 = 2 + \text{dim pgl}(d) = 2 + \text{dim Rep}(Q, d).$$

\[\square\]

**Example 7.2.** Let $Q$ be a Dynkin quiver of type $A_n$ with any orientation, and let $d$ be its highest root, the dimension vector assigning 1 at each vertex. Then $\text{Rep}(Q, d)$ can be identified with $k|Q_1| = k^n - 1$ by associating to each morphism its $1 \times 1$ matrix. Each of the coordinates is a semi-invariant, and $D$ is the normal crossing divisor in $n - 1$ variables. Notice that $D$ is independent of the orientation of the arrows.

**Example 7.3.** Consider the two Dynkin quivers $Q^{(1)}$ and $Q^{(2)}$ of type $E_6$ with the highest root as dimension vector as shown. Each space $\text{Rep}(Q^{(i)}, d)$ has dimension $22 = 24 - 2$, as the corresponding binary tetrahedral group has order 24.

One sees easily that codimension 1 degeneracies are given, for $Q^{(1)}$, by the vanishing of any of

$\det[EB], \det[EC], \det[B|CD], \det[BA|C], \det[EBA|ECD].$

The third of these measures the independence of the images of $B$ and $CD$ in the 3-dimensional space attached to the central node; the fourth and fifth are to be understood similarly. The degrees of the corresponding equations, equal to 4, 4, 4, and 6, add to 22, and their weights are easily seen to be linearly independent. Thus these form a complete list of the factors, and the linear free divisor $D$ is the union of these five, necessarily irreducible components.

For $Q^{(2)}$, four codimension 1 degeneracies are defined by the vanishing of

$\det[EB], \det[CB], \det \left[ \begin{array}{c} E \\ D \\ C \end{array} \right], DCBA.$

One further degeneracy is easier to describe verbally than by an equation (however, see Section 8 and in particular Example 8.1 below): it is the failure of general position, in the 3-dimensional space at the central node, of the three lines $\text{im}(BA), \ker(E), \ker(C)$.

\[\text{We indicate by } X|Y \text{ the concatenation of two matrices } X, Y \text{ with the same number of rows.}\]
In both cases, each equation of degree 4 has 12 monomials and the equation of degree 6 has 48. Moreover, the complements of the discriminants \( D^{(1)} \subset \text{Rep}(Q^{(1)}, \mathbf{d}) \) and \( D^{(2)} \subset \text{Rep}(Q^{(2)}, \mathbf{d}) \) are isomorphic to one another, being orbits, with trivial isotropy, of the groups \( \mathbb{F}\text{Gl}(Q^{(i)}, \mathbf{d}) \), which are themselves isomorphic to one another. However, the two discriminants are not isomorphic. Essentially, this is because the equations involve different numbers of variables. In the first case, the five equations involve, respectively, 12, 12, 14, 14, and 22 variables, while in the second the five equations involve 12, 12, 14, 16, and 20 variables. A Macaulay calculation confirms that the spaces of vector fields with constant coefficients tangent to the germ at \( 0 \in \text{Rep}(Q^{(i)}, \mathbf{d}) \) of the five components have dimensions 10, 10, 8, 8, and 0 in the first case, and 10, 10, 8, 6, and 2 in the second. Any isomorphism \( D^{(1)} \cong D^{(2)} \) must map 0 to 0, since because of the presence of the Euler field, in each case 0 is the only point where all of the vector fields in \( \text{Der}(-\log D^{(i)}) \) vanish. It follows that these dimensions are geometrical invariants: the dimension corresponding to the irreducible component \( D^{(i)}_{j} \) of \( D^{(i)} \) is the maximum dimension of a non-singular factor in a product decomposition \( (D^{(i)}_{j}, 0) \cong (E^{(i)}_{j}, 0) \times (F^{(i)}_{j}, 0) \).

**Proposition 7.4.** Let \( Q \) be the quiver whose nodes consist of \( n + 1 \) sources surrounding one sink, with an arrow going from each source to the sink. The discriminant with respect to the dimension vector that assigns 1 to each of the sources and \( n \) to the sink is a linear free divisor. It is of the form \( \Delta_{1} \cdots \Delta_{n+1} \), where the \( \Delta_{i} \) are the maximal minors of a generic \( n \times (n+1) \)-matrix.

**Proof** We can identify \( \text{Rep}(Q, \mathbf{d}) \) with the space of \( n \times (n+1) \)-matrices, with the matrix of each of the arrows determining a column. The degree of the discriminant \( D \) equals \( n(n+1) \). The generic representation describes \( n+1 \) distinct lines in a vector space of dimension \( n \), with no \( n \) of them lying in a hyperplane. Such a representation is indecomposable and lies in an open orbit, with the group \( \text{Gl}(n) \) acting transitively on the set of such line configurations in general position. Accordingly, the dimension vector is a real Schur root. There are \( n+1 \) codimension 1 degeneracies, each one determined by the vanishing of an \( n \times n \) minor of the \( n \times (n+1) \) matrix. The product of these minors has degree \( n(n+1) \), equal to the degree of \( D \), and the weights, assigning \(-1\) to each source contributing to the minor, \(0\) to the remaining source, and \(1\) to the sink, are clearly linearly independent. Thus each is present in \( \det d^{M}_{M} \) with multiplicity 1. \( \square \)

Note that from Theorem 5.1 we recover the classical result that these maximal minors are algebraically independent.

**Example 7.5.** Consider the four quivers shown below, in which the underlying undirected graph is the extended Dynkin diagram of type \( \tilde{D}_{4} \). Assign to each the dimension vector with 1 at each outer node and 3 at the central node. According to Kac’s result quoted as Proposition 6.7 above, the dimension vector shown is a real root. In (i)–(iii), it is also a Schur root, but in case (iv), it is not. In case (i), the discriminant is a linear free divisor, according to Proposition 7.4 above, but in cases (ii) and (iii) this fails. In case (iv), the discriminant is the whole space, and there is no rigid representation.

(\begin{figure}[h]
\begin{center}
\includegraphics{quiver.png}
\end{center}
\end{figure})

In case (ii), there is a modulus attached to the codimension 1 degeneracy in which the images of \( B, C \) and \( D \) lie in a plane \( P \); these three lines, together with the fourth line \( P \cap \ker A \), determine a cross-ratio. Any representation \( V \) of this type therefore has \( T^{1}_{V} \) of dimension (at least) 2, and so the multiplicity of the corresponding component in \( D \) is also at least 2. In fact it is exactly 2: the remaining three components
of $D$ are $\det AB, \det AC, \det AD$, each of degree 2. Together with twice the degree of $\det [B|C|D]$ these add up to 12, the degree of the (non-reduced) equation $\det M_D$ of $D$. As the four components described have linearly independent weights, the multiplicity of the non-reduced component is exactly 2.

Case (iii), obtained by reversing all of the arrows, is dual to (ii): here the non-reduced component of $D$ is where the kernels of the three outgoing arrows $B, C, D$ meet along a line $L$. Together with the plane $L + \text{im } A$, these make four planes in the pencil of planes containing $L$, and thus once again determine a cross ratio.

In the fourth quiver, the given dimension vector is not a Schur root. For there is no open orbit. In a general representation $V$, im $A$ and im $C$ span a plane $P$. The intersections with $\ker B$ and $\ker D$ determine two further lines in $P$, and thus a cross-ratio. Since thus $\dim \text{Ext}^1_Q(V,V) \geq 1$, it follows that $\dim \text{Hom}_Q(V,V) \geq 2$, and $V$ must be decomposable. Indeed, it is easily verified that the intersection $\ker B \cap \ker D$, concentrated on the central node, splits off. By Kac’s theorem, there is exactly one orbit of indecomposable representations. We invite the reader to find it.

**Proposition 7.6.** Suppose that $d$ is a real Schur root of the connected quiver $Q$, and let $Q^{\text{opp}}$ be obtained from $Q$ by reversing all of the arrows. If the discriminant in $\text{Rep}(Q,d)$ is a linear free divisor then the same holds in $\text{Rep}(Q^{\text{opp}},d)$.

**Proof** This is essentially projective duality. Transposition determines an isomorphism of representation spaces $\text{Rep}(Q,d) \rightarrow \text{Rep}(Q^{\text{opp}},d)$ which maps orbits to orbits. \qed

**Example 7.7.** Suppose $Q$ is a quiver and $x \in Q_0$ is a node. Construct a new quiver $Q_x$ by replacing the node $x$ by a pair of nodes $x', x''$ connected by an arrow $F$ from $x'$ to $x''$, and attaching the arrows previously attached to $x$ either to $x'$ or to $x''$. Two possible outcomes of this process are shown in the figure below. If $d$ is any dimension vector for $Q$, we define a dimension vector $d_x$ for $Q_x$ by setting $d_x(y) = d(y)$ if $y \neq x', x''$, $d_x(x') = d_x(x'') = d(x)$. Then $\langle d_x, d_x \rangle = \langle d, d \rangle$. If the generic representation in $\text{Rep}(Q,d)$ is indecomposable, then the same is true in $\text{Rep}(Q_x,d_x)$, since generically $V(F)$ is an isomorphism. So it is reasonable to hope that if $\langle d, d \rangle = 1$ and $D \subset \text{Rep}(Q,d)$ is a linear free divisor, then the discriminant in $\text{Rep}(Q_x,d_x)$ is also a linear free divisor. The following examples show that this is sometimes but not always the case.

The quivers $Q_2$ and $Q_3$ shown below are obtained from $Q_1$ by the operation just described. Assign to $Q_1$ the dimension vector $d$ with 1’s at all the sources and 4 at the central sink, and define $d_x$ accordingly. By Table 7A the discriminant $D_1 \subset \text{Rep}(Q_1,d)$ is a linear free divisor with components given by the vanishing of

$$\det[A|B|C|D], \det[A|B|C|E], \det[A|B|D|E], \det[A|C|D|E], \det[B|C|D|E].$$

In $\text{Rep}(Q_2,d_x)$, these become

$$\det[F|A|F|B|F|C|D], \det[F|A|F|B|F|C|E], \det[F|A|F|B|D|E], \det[F|A|F|C|D|E], \det[F|B|F|C|D|E], \det[F].$$

In $\text{Rep}(Q_3,d_x)$, they become

$$\det[A|B|C|F|D], \det[A|B|C|F|E], \det[A|B|F|D|E], \det[A|C|F|D|E], \det[B|C|F|D|E], \det[F].$$

The degrees of the (reduced) discriminants $D_2 \subset \text{Rep}(Q_2,d_x)$ and $D_3 \subset \text{Rep}(Q_3,d_x)$ are thus 36 and 32 respectively. So $D_2$ is a linear free divisor, whereas $D_3$ is not. The exponent of $\det F$ in the canonical equation $\Delta_3$ is 2.
One can easily show, by the same technique of counting degrees, that if one performs this operation on the central node in the quiver of Proposition 7.4, then one obtains a linear free divisor if and only if just two of the arrows coming from the outer nodes are attached to \( x'' \), and the rest are attached to \( x' \).

By applying the same construction to Dynkin quivers and their roots, one can obtain further examples of linear free divisors. In particular, one easily deals with the case \( D_n \) in this way:

**Proposition 7.8.** Let \( Q \) be the Dynkin quiver of type \( D_n \) with the following orientation

\[
\begin{array}{cccccccccc}
1 & \searrow & A & \searrow & C_1 & \cdots & C_{n-4} & D \\
& & \downarrow & & \downarrow & & \uparrow & & \\
& & 2 & & 2 & & 2 & & 1 \\
& & \nearrow & & \nearrow & & \nearrow & & \\
1 & & B & & C_1 & & \cdots & & C_{n-4} & \\
\end{array}
\]

The indicated dimension vector \( d \) is the highest root of \( D_n \). The discriminant in \( \text{Rep}(Q, d) \) is a linear free divisor of degree \( 4n - 10 \) with \( n - 1 \) factors

\[
\det[A|B], \det C_1, \ldots, \det C_{n-4}, DC_{n-4} \cdots C_1 A, DC_{n-4} \cdots C_1 B,
\]

where the first \( n - 3 \) factors are of degree 2, the last two of degree \( n - 2 \).

Changing the orientation of arrows in \( Q \) results in an isomorphic linear free divisor.

**Proof.** The criterion 6.3 shows immediately that the factors are correct, as they represent semi-invariants with linearly independent weights. For the last assertion, note that changing the direction of the arrow underlying the matrix \( C_i \), say, results in the same linear free divisor as the one already established, provided one replaces \( C_i \) by its adjoint matrix. Similarly, changing, say, the direction of the arrow underlying \( A \), amounts to replacing \( A = (a_1, a_2) \) by \( A' = (a_2, -a_1) \) in the above factors, and the situation for \( B, D \) is analogous.

### 8 Equations for \( D \)

To find equations for \( D \) in general, one can use the following recipe due to Schofield [29] that is based on his result 6.5 above. We quote it in the slightly simplified form that is all that we require here. Assume that \( Q \) is a finite connected quiver without oriented cycles and fix the sincere real Schur root \( d \) and a generic representation \( V \in \text{Rep}(Q, d) \).

To apply 6.5 one looks for roots \( e \) of \( Q \) such that \( \langle e, d \rangle = 0 \), and computes, for generic \( W \) in \( \text{Rep}(Q, e) \), the polynomial \( c^W \). If \( \text{Hom}_Q(W, V) \neq 0 \), then the square matrix underlying \( c^W \) has a nontrivial kernel and \( c^W \) vanishes on the open orbit, thus, identically. In view of this, one needs only to consider representations \( W \) that lie in the left orthogonal category \( \perp V \), the full subcategory of all those finite dimensional representations \( W \) of \( Q \) that satisfy

\[
\text{Hom}_Q(W, V) = \text{Ext}^1_Q(W, V) = 0.
\]

Schofield shows that this left orthogonal category is equivalent to the category of finite dimensional representations of some new quiver \( Q' \) that has \( n - 1 \) nodes and contains no oriented cycles. In [3] (Lemma 1) it is pointed out that a short exact sequence

\[
0 \to W' \to W \to W'' \to 0
\]

\(^7\)One may as well work throughout with the right orthogonal category \( V^\perp \), the treatment is symmetric.
of representations of \( Q \) leads either to the factorisation
\[
c^W = c^{W''} c^{W'}
\]
if \( \langle W', V \rangle = \langle W'', V \rangle = 0 \), or to the conclusion that \( c^W = 0 \) if \( \langle W', V \rangle < 0 \). So, if the generic representation in \( \text{Rep}(Q, e) \) is not simple in \( ^\perp V \), the semi-invariant we obtain will either be zero or a non-trivial product of others. Accordingly, one needs to consider only the \( n - 1 \) simple objects \( W \) in \( ^\perp V \), and those must provide the factors of the discriminant via the associated determinants \( c^W \). Indeed, the dimension vectors \( e_i \) of the simple objects \( W_i \), for \( i = 1, ..., n - 1 \) form the unique basis of the free abelian semigroup of dimension vectors \( N_0 \) for \( ^\perp V \), and their associated characters \( \langle e_i, ? \rangle = w(e_i) = -e_i + \text{in}_e = -e_i E \), see \( 6.6 \), form the unique basis of the free abelian semigroup of weights for the semi-invariants of \( \text{Rep}(Q, d) \).

Conversely, knowing the weights \( w_i \) of the generating semi-invariants, one may calculate the dimension vectors \( e_i \) through
\[
e_i = -w_i(E - 1),
\]
with \( E - 1 \) as exhibited in \( 3.3 \).

The map \( N_0 \to N_0 \) that maps the \( i \)th basis vector to \( e_i \) is an isometry with respect to the Euler forms on \( Q' \) and \( Q \), and as the simple representations for \( Q' \) have real Schur roots as their dimension vectors, the same must hold true for the dimension vectors \( e_i \). Thus, in case of a Dynkin quiver \( Q \), we simply need to go through the list of positive roots that are perpendicular to \( d \) and find among them the uniquely determined basis for the semigroup \( N_0 \).

More generally, if \( d \) is the dimension vector of a preprojective or pre-injective representation, as is the case for any Schur root of a Dynkin quiver, (see, e.g. \( 11 \) VIII.1 for the definitions and result), then one can read off the roots \( e_i \) from the Auslander–Reiten quiver of \( Q \), as explained in \( 15 \) Proof of Proposition 2.1. In that case, the quiver \( Q' \) is obtained from \( Q \) by deletion of a node along with its incident arrows and possibly some changes in the orientation of the remaining arrows. It is noteworthy that conversely for any quiver, any dimension vector of a preprojective or pre-injective representation is a real Schur root, thus providing a huge reservoir for potentially linear free divisors. Given that in this situation one can easily determine the simple objects of the orthogonal category from the Auslander–Reiten quiver, it seems reasonable to expect that one should be able to decide in general which of these roots give rise to linear free divisors.

We now turn to the two most complex Dynkin quivers, those of type \( E_7 \) and \( E_8 \), and demonstrate how the algorithm described here works in practice.

**Example 8.1.** Consider the Dynkin quiver of type \( E_7 \) with Schur root as shown - the highest root of \( E_7 \).

```
A - B - C - D - E
1  2  3  4  3  2

F
```

The representation space has dimension \( 46 = 48 - 2 \) as the associated binary polyhedral group, the binary octahedral group, is a double cover of the symmetric group on four letters.

By \( 17 \) p.153 the discriminant \( D \) has 6 irreducible components. Of these, five may be found by inspection: they are the four described by the equations
\[
\det[CBA|D], \quad \det[CB|DE], \quad \det[F|DE], \quad \det[CB|F],
\]
and the component corresponding to the degeneracy \( \text{im} C \cap \text{im} D \cap \text{im} F \neq 0 \), for which an equation is less obvious. One further component remains to be found. We obtain all of them using Schofield’s recipe.
Consider first

where solid arrows indicate maps within a quiver, and dotted arrows indicate maps from the quiver $W$ to the quiver $V$ (a convention we adhere to from now on). Note that the dimension vector $e$ of $W$ is a root with support a Dynkin diagram of type $A_5$, the “type” of $e$, that satisfies $(e,d) = 0$. We have

$$d^W_1(S_1,\ldots,S_3) = (AS_1 - S_2a, BS_2 - TS_3b, CS_3 - S_4c, DS_5 - S_4d).$$

Thus $d^W_1$ has matrix

$$
\begin{array}{ccccc}
A & -aI_2 & 0 & 0 & 0 \\
0 & B & -bI_3 & 0 & 0 \\
0 & 0 & C & -cI_4 & 0 \\
0 & 0 & 0 & -dI_4 & D \\
\end{array}
$$

where the five columns refer to the five maps $S_1,\ldots,S_5$ and the four rows to the four maps $T_1,\ldots,T_4$. Here for each $p,q$ we have ordered the natural basis vectors $E_{ij}, 1 \leq i \leq q, 1 \leq j \leq p$, of Hom$(k^p,k^q)$ lexicographically. Assuming $abc \neq 0$, row operations transform this successively to

$$
\begin{array}{ccccc}
& & & & \\
A & -aI_2 & 0 & 0 & 0 \\
\frac{1}{a}BA & 0 & -bI_3 & 0 & 0 \\
0 & 0 & C & -cI_4 & 0 \\
0 & 0 & 0 & -dI_4 & D \\
\end{array}
$$

so that $C(V,W) = \pm d \det [CBA|D]$, and fixing $d \neq 0$ we obtain the first of the degeneracies listed above. Note that the indicated root $e$ underlying $W$ predicts, by 6.3, the following weight of the semi-invariant $c^W$:

$$w(c^W) = -e + \text{in}_e : \begin{pmatrix} 0 \\ 1 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}$$

which is indeed the weight of $\det [CBA|D]$. The reader will have no difficulty checking that the next three semi-invariants listed above can be obtained, by the same procedure, from the first three roots in the diagram
The last root gives rise to the matrix

\[
\begin{bmatrix}
C & -cI_4 & 0 & 0 \\
0 & -dI_4 & D & 0 \\
0 & -fI_4 & 0 & F
\end{bmatrix}
\]

and assuming \( c \neq 0 \), column and row operations transform this into

\[
\begin{bmatrix}
0 & -cI_4 & 0 & 0 \\
-\frac{2}{3}C & 0 & D & 0 \\
-\frac{2}{3}C & 0 & 0 & F
\end{bmatrix}
\]

If also \( df \neq 0 \), then this determinant vanishes if and only if that of

\[
\begin{bmatrix}
-C & D & 0 \\
-C & 0 & F
\end{bmatrix}
\]

vanishes, which is the case when \( \text{im } C \cap \text{im } D \cap \text{im } F \neq 0 \); this can be seen by noting that if \( Cu = Dv = Fw \) then the vector \((u, v, w)^t\) lies in its kernel, and vice versa.

The sixth and last component of \( D \) is given by the vanishing of the semi-invariant arising from the root represented by \( W \) in the diagram

![Diagram](image)

The resulting determinant is

\[
\begin{bmatrix}
A & -aI_2 & 0 & 0 & 0 & 0 & 0 \\
0 & B & -b_{11}I_3 & -b_{21}I_3 & 0 & 0 & 0 \\
0 & 0 & C & 0 & -c_{11}I_4 & -c_{21}I_4 & 0 & 0 \\
0 & 0 & 0 & C & -c_{12}I_4 & -c_{22}I_4 & 0 & 0 \\
0 & 0 & 0 & 0 & -d_{11}I_4 & -d_{21}I_4 & D & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -eI_3 & E & 0 \\
0 & 0 & 0 & 0 & -f_{11}I_4 & -f_{21}I_4 & 0 & 0 & F
\end{bmatrix}
\]

where the columns and rows refer, in this order, to the maps \( S_1, \ldots, S_7 \) and \( T_1, \ldots, T_6 \) respectively. Row and column operations, and the deletion of rows and columns containing only an invertible matrix, transform this to the matrix

\[
\begin{bmatrix}
\frac{1}{e}CBA & 0 & (c_{12}b_{21} - c_{11}b_{11})I_4 & (c_{22}b_{21} - c_{21}b_{11})I_4 & 0 & 0 \\
0 & C & -c_{12}I_4 & -c_{22}I_4 & 0 & 0 \\
0 & 0 & -d_{11}I_4 & -d_{21}I_4 & \frac{1}{e}DE & 0 \\
0 & 0 & -f_{11}I_4 & -f_{21}I_4 & 0 & F
\end{bmatrix}
\]
and now permuting columns brings it to the form

\[
\begin{array}{cccccc}
CBA & 0 & 0 & 0 & \lambda_1 I_4 & \mu_1 I_4 \\
0 & C & 0 & 0 & \lambda_2 I_4 & \mu_2 I_4 \\
0 & 0 & DE & 0 & \lambda_3 I_4 & \mu_3 I_4 \\
0 & 0 & 0 & F & \lambda_4 I_4 & \mu_4 I_4
\end{array}
\]

where the \( \lambda_i \) and \( \mu_j \) are polynomials in the coefficients \( a, b, \ldots \) of the representation \( W \), and we have multiplied some rows and columns by other such polynomials to simplify the expression (since we choose a generic \( W \) in \( \text{Rep}(Q, e) \) to obtain the polynomial \( C^W \), this multiplication has the effect only of multiplying \( C^W \) by a scalar).

The geometrical significance of the vanishing of the determinant is that the three lines \( \text{im} \ DE \cap \text{im} \ C \), \( \text{im} \ F \cap \text{im} \ C \) and \( \text{im} \ CBA \) fail to span \( \text{im} \ C \). The reader will note the similarity in the geometric description of the last two semi-invariant factors. This can be understood by looking at their weights. They are given by

\[
\begin{pmatrix}
0 & 0 & -1 & 2 & -1 & 0 \\
-1 & 0 & -1 & 0 & 2 & 0 & -1
\end{pmatrix}
\]

According to Derksen and Weyman \[8, \text{p.477, Step 2}\], if the weight of \( W \) is not sincere, as in these cases, one may simplify the calculation by removing successively nodes not in the support, adding instead one arrow for each pair of ingoing and outgoing arrows. In the first case at hand, this produces a weight with support a Dynkin quiver of type \( D_4 \), in the second a weight of type \( D_5 \). For the first four orthogonal roots listed, the type of the weight equals \( A_3 \), explaining the similarity in the description of the corresponding semi-invariants. Once one has modified the quiver in this fashion, one can then simply calculate the corresponding semi-invariant on the new quiver, where one drops from \( d \) as well the nodes not in the support of the weight, and substituting at the end the actual composition of the maps along each pair of ingoing and outgoing arrow into the resulting semi-invariant. Revisiting, for example, the first orthogonal root considered above and its corresponding weight of type \( A_3 \); see e.g. the table below; it becomes thus transparent that the semi-invariant obtained, \( \det[CBA|D] \), has indeed to be a polynomial in the entries of \( CBA \) and \( D \).

We can summarize the information gathered so far for the discriminant in the representation variety of the highest root of \( E_7 \) in the given orientation through the following table, where we list the opposite of the weights to display fewer minus signs:

| Polynomial      | Deg | Root\text{-d} | -Weight | Type (Root, Weight) |
|-----------------|-----|--------------|---------|---------------------|
| \( P_1 = \det[CBA|D] \) | 6   | 0            | 0       | \((A_5, A_3)\)     |
| \( P_2 = \det[C|DE] \)      | 8   | 0            | 0       | \((A_5, A_3)\)     |
| \( P_3 = \det[F|DE] \)      | 6   | 1            | 1       | \((A_4, A_3)\)     |
| \( P_4 = \det[C|F] \)       | 6   | 1            | 1       | \((A_4, A_3)\)     |
| \( P_5 = \det \begin{pmatrix} -C & D \\ -C & 0 \end{pmatrix} \) | 8   | 1            | 1       | \((D_4, D_4)\)     |
| \( P_6 \)                  | 12  | 1            | 1       | \((E_7, D_5)\)     |
| \( \Delta = (\text{unit})P_1 \cdots P_6 \) | 46  | 1            | 1       |                     |

\[ \begin{pmatrix}
1 & 1 & 2 & 2 & 1 & 1 & 1 & 0 & 1 & -2 & 2 & 0 & 1 \\
4 & 2 & 2 & -8 & 2 & 3
\end{pmatrix} \]
The following interlude will allow us to find the equations for semi-invariants such as $P_5$ or $P_6$ above in a more direct form, using some commutative algebra.

9 An Interlude from Commutative Algebra

Let $0 \to M \xrightarrow{j} R^{m+a} \xrightarrow{\varphi} R^a \xrightarrow{p} T \to 0$ be an exact sequence of modules over a commutative normal (and noetherian) domain $R$, with integers $m, a > 0$, and $T$ a torsion $R$-module. Assume given moreover an $R$-linear map $\psi : R^{m+a} \to R^m$. The module $M$ has a (constant) rank, equal to $m$, and its determinant is by definition the reflexive $R$-module $\det M = (\Lambda_R^m M)^{\vee \vee}$, where $(-)^{\vee}$ denotes the $R$-dual module. In words, $\det M$ is the reflexive hull of the $m^{th}$ exterior power of $M$ over $R$. It is isomorphic to $R$, and the composition $\psi j$ induces an $R$-linear map $\det(\psi j) : R \cong \det M \to \det R^m \cong R$. At issue now is to find a closed form for that determinant.

**Lemma 9.1.** The determinant of $\psi j$ satisfies $\det(\psi j) = \det(\psi, \varphi)$.

**Proof.** Consider the following commutative diagram whose rows are exact

\[
\begin{array}{ccccccccc}
0 & \to & M & \xrightarrow{j} & R^{m+a} & \xrightarrow{\varphi} & R^a & \xrightarrow{p} & T & \to 0 \\
& & \downarrow{\psi j} & & \downarrow{(\psi, \varphi)} & & \downarrow{\varphi} & & \downarrow{p} & \\
0 & \to & R^m & \to & R^m \oplus R^a & \xrightarrow{pr_2} & R^a & \to & 0
\end{array}
\]

The multiplicativity of the determinant shows first that $\det M \cong R$ and then yields $\det(\psi j) = \det(\psi, \varphi)$. \hfill \Box

**Example 9.2.** We use this result to find a closed form for the semi-invariant $P_6$ for the highest weight of $E_7$ described in the last section. Namely, with the same notations as there, that invariant measures whether the three lines $\im DE \cap \im C$, $\im F \cap \im C$, and $\im CBA$ span $\im C$. To translate this into multilinear algebra, note that it is equivalent to say that the fibre product $X$ of $DE$ with $C$ over their common target, the fibre product $Y$ of $F$ with $C$ over the common target, and the image $Z$ of $BA$ do not span the domain of $C$. Each of $X, Y, Z$ is a rank one submodule of the domain of $C$, which is a free module of rank 3 over $R$, the ring of the representation variety. We thus expect the corresponding invariant to be $\det[X|Y|Z]$, and the preceding lemma lets us make this precise: In the following diagram, the top row is a direct sum of three short exact sequences of graded $R$–modules

\[
\begin{array}{ccccccccc}
X & \oplus & (i_1, j_1) & R^3 \oplus R^2 & (C, -F) & R(1)^4 \\
0 & \to & Y & \xrightarrow{(i_2, j_2)} & R^3 \oplus R(-1)^2(C, -DE) & \to & R(1)^4 & \to & 0 \\
& \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus \\
R(-2) & & \id_R & & R(-2) & & 0 & & 0 \\
& \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus \\
(i_1, i_2, BA) & \downarrow & & & & & & & \\
0 & \to & R^3 & \oplus & (i_1, j_1) & \oplus & R(1)^4 & \oplus & R(1)^4 \\
& \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus \\
& & & & & & & & \\
M & \downarrow & & & & & & & \\
R^3 & \oplus & R(1)^4 & \oplus & R(1)^4 & \oplus & R(1)^4 & \oplus & R(1)^4 \\
& \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus \\
& & & & & & & & \\
0 & \to & R^3 & \oplus & R(1)^4 & \oplus & R(1)^4 & \oplus & R(1)^4 \\
& \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus & \oplus \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
\]
where the maps \( i_1, i_2, j_1, j_2, n_1 \) are the natural inclusions, \( pr_{23} \) the projection onto the sum of second and third factor, and the matrix \( M \) is of the form:

\[
\begin{array}{cccccc}
R^3 & R^2 & R^3 & R(-1)^2 & R(-2) \\
R^3 & I & 0 & I & 0 & BA \\
R(1)^4 & C & -F & 0 & 0 & 0 \\
R(1)^4 & 0 & 0 & C & -DE & 0 \\
\end{array}
\]

The desired semi-invariant is now \( \det(i_1, i_2, BA) \), which equals the determinant of \( M \) in view of the lemma above. Subtracting (a multiple of) the first column from the third and fifth results in the following simpler form

\[
\begin{array}{cccccc}
R^3 & R^2 & R^3 & R(-1)^2 & R(-2) \\
R^3 & I & 0 & 0 & 0 & 0 \\
R(1)^4 & C & -F & -C & 0 & -CBA \\
R(1)^4 & 0 & 0 & C & -DE & 0 \\
\end{array}
\]

whence the desired semi-invariant is seen to be the determinant of an \( 8 \times 8 \) matrix,

\[
P_6 = \det \begin{bmatrix} F & C & 0 & CBA \\ 0 & C & -DE & 0 \end{bmatrix}
\]

whose degree can be read off to be 12 as stated in the table above.

10 The case of \( E_8 \) with the centre as only sink

As our final example, we determine the discriminant in the representation variety that belongs to the highest root of the Dynkin quiver of type \( E_8 \) with all arrows oriented towards the central trivalent vertex:

The capital letters \( A, ..., G \) stand for the corresponding matrices of independent indeterminates, and the coordinate ring of \( \text{Rep}(E_8, d) \) is \( R = K[A, B, C, D, E, F, G] \), a polynomial ring in \( 118 = 120 - 2 \) variables, where 120 is the order of the binary icosahedral group.

We will also need below three additional auxiliary vertices, denoted by \( \circ \), and corresponding maps \( X, Y, Z \), as indicated by the dashed arrows here:

The map \( X \) is the natural one from the fibre product of \( B \) and \( C \) to the central node. The fibre product itself is an \( R \)-module of rank 1. The map \( Y \) indicated above is the natural one from the fibre product of \( D \) and \( C \) to the central node. This fibre product has rank 2. Finally, the map \( Z \) is the natural one from the fibre product of \( C \) and \( DE \) to the central node. Again, the fibre product has rank 1.

The discriminant \( D \) in question is of degree 118 and has 7 irreducible components, thus, its canonical equation \( \Delta \) is a product of 7 irreducible polynomials \( P_i \) in the entries of the 7 matrices \( A \) through \( G \). Moreover, we obtain from 6.6 that it is a semi-invariant belonging to the weight 

\[
-4 \ -4 \ 12 \ -2 \ -2 \ -3
\]

\[
-6
\]
We can spot immediately three semi-invariants:

\[ P_1 = \det[BA|DE], \quad P_2 = \det[C|DEF], \quad P_3 = \det[B|DEFG], \]

each of degree 12 and belonging to weights of type \( A_3 \). The remaining four can be described thus

- The failure of \( \text{im} \ X = \text{im} \ B \cap \text{im} \ C \) and \( \text{im} \ D \) to generate the vector space at the central node. According to [9.1], the corresponding polynomial is \( P_4 = \det[X|D] \), the determinant of

\[
\begin{array}{ccc}
R(-1)^3 & R(-1)^5 & R(-1)^9 \\
R^6 & B & 0 & D \\
R^6 & B & -C & 0
\end{array}
\]

It is of degree 12.

- The failure of \( \text{im} \ BA, \text{im} \ X, \text{im} \ DEF \) to generate the vector space at the central node. Again using [9.1], the corresponding polynomial is \( P_5 = \det[BA|X|DEF] \), the determinant of

\[
\begin{array}{cccc}
R(-2)^2 & R(-1)^3 & R(-1)^5 & R(-3)^5 \\
R^6 & BA & B & 0 & DEF \\
R^6 & 0 & B & -C & 0
\end{array}
\]

It is of degree 20.

- The failure of \( \text{im} \ BA, \text{im} \ Y, \text{im} \ DEFG \) to generate the vector space at the central node. The corresponding polynomial is \( P_6 = \det[BA|Y|DEFG] \), the determinant of

\[
\begin{array}{cccc}
R(-2)^2 & R(-1)^3 & R(-1)^5 & R(-4)^2 \\
R^6 & BA & C & 0 & DEFG \\
R^6 & 0 & C & -D & 0
\end{array}
\]

It is also of degree 20. The three semi-invariants \( P_4 \) through \( P_6 \) belong to weights of type \( D \), as can easily be seen from the geometric description. Now we turn to the last and biggest one:

- The rank of \( BA \) and \( DEFG \) is 2, that of \( X, Z \) is 1. Their images in the central vector space are thus expected to generate. The failure will be measured by the polynomial \( P_7 = \det[BA|X|Z|DEFG] \), which is the determinant of

\[
\begin{array}{cccc}
R(-2)^2 & R(-1)^3 & R(-1)^5 & R(-2)^4 \\
R^6 & BA & C & 0 & DEFG \\
R^6 & 0 & C & -D & 0 \\
R^6 & 0 & 0 & 0 & C & -DE & 0
\end{array}
\]

It is of degree 30 and its weight is of type \( E_6 \).

We summarize the results again in a table:
Theorem 10.1. The above table is correct.

Proof. Inspection shows that each of the polynomials $P_i$ is a semi-invariant and that its weight is as listed. The indicated weights are easily seen to be linearly independent, and add up to the weight of $\Delta$. Thus, their product must describe the discriminant up to a unit. \qed

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