QUANTUM CONVOLUTION INEQUALITIES ON FROBENIUS VON NEUMANN ALGEBRAS

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Abstract. In this paper, we introduce Frobenius von Neumann algebras and study quantum convolution inequalities. In this framework, we unify quantum Young’s inequality on quantum symmetries such as subfactors, and fusion bi-algebras studied in quantum Fourier analysis. Moreover, we prove quantum entropic convolution inequalities and characterize the extremizers in the subfactor case. We also prove quantum smooth entropic convolution inequalities. We obtain the positivity of comultiplications of subfactor planar algebras, which is stronger than the quantum Schur product theorem. All these inequalities provide analytic obstructions of unitary categorification of fusion rings stronger than Schur product criterion.

Key words. Frobenius von Neumann algebra, quantum Young’s inequality, quantum entropic convolution inequality, unitary categorification

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1. Introduction

In [2], Beckner remarkably proved the sharp Young’s inequality [29]. Recently, quantum Young’s inequality for convolution has been established on quantum symmetries, such as subfactors [11, 7], fusion bi-algebras [21], Kac algebras [22], locally compact quantum groups [12], etc., see further discussions in the framework of quantum Fourier analysis [10]. Moreover, the extremal pairs of quantum Young’s inequality are characterized on subfactor planar algebras [13] and kac algebras [22].

In 2021, A. Wigderson and Y. Wigderson [27] unified various classical (smooth) uncertainty principles related to discrete Fourier transforms. In [9], the authors unified various quantum (smooth) uncertainty principles on quantum symmetries related to quantum Fourier transforms.

One goal of this paper is to unify inequalities on quantum symmetries related to the convolution. The study of the comultiplication turns out to be more fundamental than the study of convolution.

Theorem 1.1 (See Theorem 2.4). Suppose $\mathcal{P}_\ast$ is a subfactor planar algebra. Then the comultiplication $\Delta: \mathcal{P}_{2,\pm} \rightarrow \mathcal{P}_{2,\pm} \otimes \mathcal{P}_{2,\pm}$ is positive.

Theorem 1.1 implies the positivity of convolution (called quantum Schur product theorem, see Theorem 4.1 in [18]), which led to an analytic criterion of unitary categorification of fusion rings (called Schur product criterion, see Proposition 8.3 in [21]). Theorem 1.1 leads to a stronger criterion of unitary categorification.
Theorem 1.2 (Positivity of Comultiplication Criterion). For a fusion ring $\mathcal{A}$, let $N_{i,j}^k$ be the fusion coefficients and $M_k = (N_{i,j}^k)_{i,j}$ be the fusion matrices, $1 \leq i,j,k \leq n$. If $\mathcal{A}$ admits a unitary categorification, then for any $v \in \mathbb{C}^n$,

$$\sum_{k=1}^{n} v^* M_k v \geq 0.$$ (1.1)

To study convolution inequalities on quantum symmetries captured by a tracial von Neumann algebra $(\mathcal{M}, \tau)$, we introduce a comultiplication $\Delta: \mathcal{M} \to \mathcal{M} \otimes \mathcal{M}$ and a convolution $\ast: L^1(\mathcal{M}) \times L^1(\mathcal{M}) \to L^1(\mathcal{M})$, which are dual to each other. We call $\Delta$ a good $k$-comultiplication if $\Delta \geq 0$ and $\Delta(I) = kI$, $k > 0$. It induces a good $k$-convolution (See Proposition 2.2), namely

1. Positivity:

$$x \ast y \geq 0, \quad \forall x, y \geq 0, \ x, y \in \mathcal{M};$$

2. Primary Young’s inequality:

$$\|x \ast y\|_1 \leq k \|x\|_1 \|y\|_1, \quad \forall x, y \in \mathcal{M};$$

3. Haar measure:

$$\tau(x \ast y) = k \tau(x) \tau(y), \quad \forall x, y \geq 0, \ x, y \in \mathcal{M}.$$}

Furthermore, a trace-preserving anti $\ast$-isomorphism $\rho$ on $\mathcal{M}$ is called an antipode if the Frobenius Reciprocity holds:

$$\tau((x \ast y)z) = \tau((\rho(z) \ast x)\rho(y)), \quad \forall x, y, z \in \mathcal{M}.$$

We call the quadruple $(\mathcal{M}, \tau, \ast, \rho)$ a Frobenius von Neumann $k$-algebra (FN $k$-algebra). This notion is highly inspired by subfactor theory and Hopf algebras. Specific examples of FN $k$-algebras come from subfactor planar algebras, fusion bi-algebras, Kac algebras, etc. In Section 3 we prove the quantum Young’s inequality on FN $k$-algebras, which unifies the quantum Young’s inequality on subfactor planar algebras (See Theorem 4.13 in [11]) and fusion bi-algebras (See Theorem 5.11 in [21]).

Theorem 1.3 (See Theorem 3.7). Let $(\mathcal{M}, \tau, \ast, \rho)$ be a FN $k$-algebra. Then for any $x, y \in \mathcal{M}$, $1 \leq p, q, r \leq \infty$ with $1 + 1/r = 1/p + 1/q$, we have

$$\|x \ast y\|_r \leq k \|x\|_p \|y\|_q.$$ (1.2)

In Section 4 we prove quantum entropic convolution inequalities (qECI) on FN $k$-algebras. Moreover, we characterize the extremizers of qECI when the FN $k$-algebras are from subfactors. We introduce the notion of smooth convolution entropy.
**Definition 1.1.** Let \((\mathcal{M}, \tau)\) be a tracial von Neumann algebra with a good \(k\)-convolution \(\ast\). For any positive operators \(x, y \in \mathcal{M}\), \(\epsilon, \eta \in [0, 1]\) and \(p,q \in [1, \infty]\), the smooth convolution entropy is defined by

\[
H_{\epsilon,\eta}^{p,q}(x \ast y) := \inf \{ H(z \ast w) : z, w \in \mathcal{M}, z, w \geq 0, \|x - z\|_p \leq \epsilon, \|y - w\|_q \leq \eta \}. \tag{1.4}
\]

For any \(x \in \mathcal{M}\), \(x \geq 0\), the smooth entropy of \(x\) with \(1 \leq p \leq \infty, 0 \leq \epsilon \leq 1\) is defined as

\[
H_{\epsilon}^p(x) := \sup \{ H(y) : y \in \mathcal{M}, y \geq 0, \|y - x\|_p \leq \epsilon \}.
\]

For a finite dimensional FN \(k\)-algebra \((\mathcal{M}, \tau, \ast, \rho)\), we call \(\tau(I)\) the Frobenius-Perron dimension of \(\mathcal{M}\), denoted by \(d\). We set \(\lambda = \min \{\tau(e) : e\) is a projection in \(\mathcal{M}\}\). In Section 4, we prove the quantum smooth entropic convolution inequality.

**Theorem 1.5** (See Theorem 4.5). Let \((\mathcal{M}, \tau, \ast, \rho)\) be a finite dimensional FN \(k\)-algebra. Let \(p,q \in [1, \infty]\), \(\epsilon, \eta \in [0, 1]\), \(\epsilon + \eta \leq 1\). For any positive operators \(x, y \in \mathcal{M}\) with \(\|x\|_1 = \|y\|_1 = k^{-1}, 0 \leq \theta \leq 1\), we have

\[
H_{\epsilon,\eta}^{p,q}(x \ast y) \geq \theta H_{\epsilon}^p(x) + (1 - \theta) H_{\eta}^q(y) - O_{d,\lambda,k}(|\epsilon \log \epsilon|) - O_{d,\lambda,k}(|\eta \log \eta|). \tag{1.5}
\]

The paper is organized as follows: In Section 2, we prove the positivity of comultiplications on subfactors, which provides analytic obstructions of unitary categorifications of fusion rings stronger than the one from the positivity of convolution \([21]\). Section 3 is devoted to the proof of the quantum Young’s inequalities on FN \(k\)-algebras, which unifies the quantum Young’s inequalities on subfactor planar algebras and fusion bi-algebras. In the last section, we show the quantum (smooth) entropic convolution inequalities on FN \(k\)-algebras. Moreover, the extremizers of qECI are characterized when the FN \(k\)-algebras are from subfactors.

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## 2. Positivity of Comultiplications and Unitary categorification Criterion

In this section, we will introduce the notion of comultiplications on von Neumann algebras. We prove the positivity of the comultiplications on subfactor planar algebras, which leads to an analytic criterion of unitary categorifications of fusion rings stronger than Schur product criterion (See Proposition 9.3 in \([21]\)).

### 2.1. Preliminaries.

We first recall some basic definitions of von Neumann algebras and non-commutative \(L^p\) spaces. Let \(\mathcal{M}\) be a von Neumann algebra acting on a Hilbert space \(\mathcal{H}\) with a faithful normal tracial positive linear functional \(\tau\), see e.g. \([16]\). We simply call \(\tau\) a trace and the pair \((\mathcal{M}, \tau)\) as a tracial von Neumann algebra in the rest of the paper. We refer the readers to e.g. \([25]\) and \([26]\) for details about non-commutative \(L^p\) spaces.
A closed densely defined operator $x$ affiliated with $\mathcal{M}$ is call $\tau$-measurable if for all $\epsilon > 0$ there exists a projection $P \in \mathcal{M}$ such that $P\mathcal{H} \subseteq \mathcal{D}(x)$, and $\tau(I - P) \leq \epsilon$, where $\mathcal{D}(x)$ is the domain of $x$. We denote the set of all $\tau$-measurable closed densely defined operators by $\widetilde{\mathcal{M}}$. Then $\widetilde{\mathcal{M}}$ is a $*$-algebra with respect to a strong sum, strong product, and adjoint operation. If $x$ is a positive self-adjoint $\tau$-measurable operator, then $x^\alpha \log x$ is also $\tau$-measurable for any $\alpha \in \mathbb{C}$ with positive real part.

The sets

$$U(\epsilon, \eta) = \{ x \in \widetilde{\mathcal{M}} : \exists \text{ a projection } P \in \mathcal{M} \text{ satsfying } P\mathcal{H} \subseteq \mathcal{D}(x), \|xP\| \leq \epsilon, \tau(I - P) \leq \eta \},$$

where $\epsilon, \eta > 0$, form a neighborhood basis of 0 that makes $\widetilde{\mathcal{M}}$ into a topological vector space. Now $\widetilde{\mathcal{M}}$ is a complete Hausdorff topological $*$-algebra and $\mathcal{M}$ is a dense subset of $\widetilde{\mathcal{M}}$.

For any positive self-adjoint operator $x$ affiliated with $\mathcal{M}$, we set

$$\tau(x) = \sup_{n \in \mathbb{N}} \tau(\int_0^n tde_t),$$

where $x = \int_0^\infty tde_t$ is the spectral decomposition of $x$. Then for $p \in [1, \infty)$, the non-commutative $L^p$ space $L^p(\mathcal{M})$ with respect to $\tau_\mathcal{M}$ is defined as

$$L^p(\mathcal{M}) := \{ x \text{ closed, densely defined, affiliated with } \mathcal{M} : \tau(|x|^p) < \infty \}.$$

The $p$-norm of $x$ is given by $\|x\|_p = \tau(|x|^p)^{1/p}$, where $|x| = (x^*x)^{1/2}$. In particular, $\|x\|_\infty = \|x\|$, the operator norm. We have that $L^p(\mathcal{M}) \subseteq \widetilde{\mathcal{M}}$. The following non-commutative Hölder’s inequality will be used frequently in the whole paper.

**Proposition 2.1 (Hölder’s inequality).** For any $x, y, z \in \mathcal{M}$, we have

1. $|\tau(xy)| \leq \|x\|_p \|y\|_q$, where $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$;
2. $|\tau(xyz)| \leq \|x\|_p \|y\|_q \|z\|_r$, where $1 \leq p, q, r \leq \infty$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$;
3. $\|xy\|_r \leq \|x\|_p \|y\|_q$, where $0 < p, q, r \leq \infty$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

**Proof.** See e.g. Theorems 5.2.2 and 5.2.4 in [28].

### 2.2. Comultiplication and Convolution

The comultiplication and convolution appear as a pair of dual operations on von Neumann algebras. The study of the comultiplication turns out to be more fundamental than the study of the convolution while establishing convolution inequalities on quantum symmetries.

**Definition 2.1.** Let $\mathcal{M}$ be a tracial von Neumann algebra. A $k$-comultiplication $\Delta$ is a linear normal map from $\mathcal{M}$ into the spatial tensor product $\mathcal{M} \overline{\otimes} \mathcal{M}$ with operator norm $\|\Delta\| = \sup_{x \in \mathcal{M}} \frac{\|\Delta(x)\|}{\|x\|} = k$.

We shall note that a $k$-comultiplication may not be a homomorphism, which is different from the case of Kac algebras. A $k$-co-multiplication $\Delta$ induces a $k$-convolution $*$ defined as follows:

$$\langle x \ast y, z \rangle = \langle x \otimes y, \Delta(z) \rangle, \quad \forall x, y, z \in \mathcal{M}.$$  \hfill (2.1)
By Hölder’s inequality, we have
\[ |\langle x * y, z \rangle| = |\langle x \otimes y, \Delta(z) \rangle| \leq \|x \otimes y\|_1 \|\Delta(z)\| \leq k \|x\|_1 \|y\|_1 \|z\|.\]
Therefore,
\[ \|x * y\|_1 = \sup_{\|z\|=1} \{|\langle x * y, z \rangle|\} \leq k \|x\|_1 \|y\|_1, \]
which implies that \(x * y \in L^1(M)\). Consequently, a \(k\)-convolution satisfies the primary Young’s inequality:
\[ \|x * y\|_1 \leq k \|x\|_1 \|y\|_1, \quad \forall x, y \in M. \tag{2.2} \]
The properties of convolutions inherit the properties of comultiplications naturally.

**Proposition 2.2.** Let \(\Delta\) be a \(k\)-comultiplication and \(*\) be the induced \(k\)-convolution. Then for any \(x, y, z \in M\), the following statements holds:

1. \(\Delta \geq 0\) implies \(x * y \geq 0, \ x, y \geq 0\);
2. \(\Delta(I) = kI\) implies \(\tau(x * y) = k\tau(x)\tau(y)\);
3. \((\Delta \otimes I)\Delta = (I \otimes \Delta)\Delta\) implies \((x * y) * z = x * (y * z)\).

**Proof.**

(1) Let \(x, y, z \geq 0, \ x, y, z \in M\). We have
\[ \langle x * y, z \rangle = \langle x \otimes y, \Delta(z) \rangle \geq 0. \]

Therefore, \(x * y \geq 0\).

(2) We have
\[ \tau(x * y) = \langle x * y, I \rangle = \langle x \otimes y, \Delta(I) \rangle = \langle x \otimes y, kI \rangle = k\tau(x)\tau(y). \]

(3) For any \(x, y, z, w \in M\), we have
\[ \langle (x * y) * z, w \rangle = \langle (x * y) \otimes z, \Delta(w) \rangle = \langle x \otimes y \otimes z, (\Delta \otimes I)\Delta(w) \rangle = \langle x \otimes (y * z), \Delta(w) \rangle = \langle x * (y * z), w \rangle, \]
which implies \((x * y) * z = x * (y * z)\). \(\square\)

Note that every linear map \(\tilde{\Delta}: M \to M \overline{\otimes} M\) satisfying \(\tilde{\Delta} \geq 0\) and \(\tilde{\Delta}(I) = kI\) has operator norm \(k\) due to Russo-Dye Theorem \([24]\): every unital positive linear map between two \(C^*\)-algebras is contractive.

**Definition 2.2.** A \(k\)-comultiplication \(\Delta: M \to M \overline{\otimes} M\) is called a **good \(k\)-comultiplication** if \(\Delta \geq 0\) and \(\Delta(I) = kI\). We simply call \(\Delta\) a **good comultiplication** when \(k = 1\), i.e., \(\Delta\) is a unital positive linear map.

**Remark 2.1.** A good \(k\)-comultiplication \(\Delta\) induces a **good \(k\)-convolution** \(*\) satisfying the following properties from Proposition 2.2 and inequality (2.2):
(1) **Positivity:**
\[ x \ast y \geq 0, \quad \forall \; x, y \geq 0, \; x, y \in \mathcal{M}; \]

(2) **Primary Young’s inequality:**
\[ \|x \ast y\|_1 \leq k \|x\|_1 \|y\|_1, \quad \forall \; x, y \in \mathcal{M}; \]

(3) **Haar measure:**
\[ \tau(x \ast y) = k \tau(x) \tau(y), \quad \forall \; x, y \in \mathcal{M}. \]

When \( k = 1 \), we also simply call \( \ast \) a **good convolution**.

**Proposition 2.3.** The Haar measure implies Positivity.

**Proof.** For any \( x, y \geq 0, \; x, y \in \mathcal{M} \), we have
\[ \|x\|_1 \|y\|_1 = \tau(x) \tau(y) = k^{-1} \tau(x \ast y) \leq k^{-1} \|x \ast y\|_1 \leq \|x\|_1 \|y\|_1. \]

Hence \( \tau(x \ast y) = \|x \ast y\|_1 \). This implies \( x \ast y \geq 0 \).

The positivity plays a fundamental role in subfactor theory such as in the proof of the remarkable Jones index theorem \cite{14}, and it is formulated as reflection positivity in subfactor planar algebras \cite{15}.

### 2.3. Positivity of Comultiplication and Categorification Criterion.

The positivity of the convolution is called quantum Schur product theorem in subfactor planar algebras (See Theorem 4.1 in \cite{18}), which provides an analytic tool in classification of subfactor planar algebras \cite{18, 6, 20}. The Schur product property on the dual of fusion rings gives an analytic criterion of unitary categorifications of fusion rings (called Schur product criterion, see Proposition 8.3 in \cite{21}). In this section, we will show the positivity of the comultiplication on subfactor planar algebras, which leads to a stronger criterion for unitary categorification.

Suppose \( \mathcal{P}_i \) is a subfactor planar algebra with finite index \( \delta \), \( \delta > 0 \). Recall that the convolution on the 2-box spaces is defined as follows:

\[
\begin{array}{c}
\begin{array}{ccc}
& & \\
& x & \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
& & \\
& y & \\
\end{array}
\end{array}
\]

The convolution is also called coproduct (See e.g. page 61 in \cite{4} and page 11 in \cite{11}). Recall that a biprojection is a projection with respect to both multiplication and convolution (up to a scalar). We refer to \cite{3, 4, 5} for more details. The biprojection has been applied to characterize the extremizers of uncertain principles \cite{11} and quantum Young’s inequality \cite{13} and will appear in Theorems 3.9 and 4.3 in this paper.
We consider the comultiplication $\Delta: \mathcal{P}_{2,\pm} \to \mathcal{P}_{2,\pm} \otimes \mathcal{P}_{2,\pm}$ as the adjoint operator of $*$ with respect to the unnormalized faithful Morkov trace $\text{Tr}$:

$$\langle \Delta(z), x \otimes y \rangle = \langle z, x^* y \rangle$$

(2.3)

for any $x, y, z \in \mathcal{P}_{2,\pm}$. Switching the input disc and output discs of the convolution tangle, we obtain a surface tangle for comultiplication, see [19] for the theory of surface tangles.

**Theorem 2.4** (Positivity of Comultiplication). Suppose $\mathcal{P}_\bullet$ is a subfactor planar algebra with finite index $\delta^2$, $\delta > 0$. Then the comultiplication $\Delta: \mathcal{P}_{2,\pm} \to \mathcal{P}_{2,\pm} \otimes \mathcal{P}_{2,\pm}$ is positive. Moreover, if $\mathcal{P}_\bullet$ is irreducible then $\Delta$ is a good $\delta^{-1}$-comultiplication.

**Proof.** Let $E$ be the conditional expectation from $\mathcal{P}_{4,\pm}$ onto $\mathcal{P}_{2,\pm} \otimes \mathcal{P}_{2,\pm}$. From equality (2.3), we have

$$\Delta(z) = E$$

which is a composition (up to a scalar) of a $*$-homomorphism and a conditional expectation. Thus, $\Delta$ is positive. Now suppose $\mathcal{P}_\bullet$ is irreducible. Note that

$$\langle x \otimes y, \Delta(I) \rangle = \langle x^* y, I \rangle = \text{Tr}(x^* y) = \frac{1}{\delta} \text{Tr}(x) \text{Tr}(y).$$

Since $\text{Tr}$ is faithful, so $\Delta(I) = \delta^{-1}I$. Therefore, $\Delta$ is a good $\delta^{-1}$-comultiplication. \hfill \Box

**Remark 2.2.** From Proposition 2.2(1), Theorem 2.4 implies Theorem 4.1 in [18].

**Corollary 2.5.** Suppose $\mathcal{P}_\bullet$ is an irreducible subfactor planar algebra with finite index $\delta^2$, $\delta > 0$. For any $x, y \in \mathcal{P}_{2,\pm}$, we have

$$\|x^* y\|_1 \leq \frac{\|x\|_1 \|y\|_1}{\delta}.$$ 

**Proof.** It follows from Theorem 2.4 and Remark 2.1. \hfill \Box

Let $\mathfrak{A}$ be a fusion ring with basis $\{x_1 = I, x_2, \ldots, x_n\}$ satisfying the following fusion rules:

$$x_k x_j = \sum_{i=1}^{n} N_{k,j}^i x_i, \quad N_{k,j}^i \in \mathbb{N}.$$

Let $M_k = (N_{k,j}^i)_{i,j}$ be the fusion matrix of $x_k$.

**Definition 2.3** (See page 18 in [21]). We define a linear map $\Delta_1: \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{A}$ such that

$$\Delta_1(x_k) = \frac{1}{d(x_k)} x_k \otimes x_k, \quad \Delta_1(x_k^*) = \Delta_1(x_k)^*.$$

Then $\Delta_1$ is a $*$-preserving linear map.
Proposition 2.6. If a fusion ring $\mathcal{A}$ admits a unitary categorification, then the linear map $\Delta_1$ is positive, i.e., $\Delta_1(x) \geq 0$, for any $x \geq 0$, $x \in \mathcal{A}$.

Proof. It follows from Theorem 2.4 and Proposition 2.25 in [21]. \hfill \square

Proposition 2.7. The linear map $\Delta_1$ is positive if and only if for any $v \in \mathbb{C}^n$

$$T = \sum_{k=1}^{n} \frac{v^* M_k v}{\| M_k \|_\infty} M_k \otimes M_k \geq 0.$$ (2.4)

Proof. Since every positive operator in $\mathcal{A}$ has the form $\left( \sum_{i=1}^{n} a_i x_i \right) \left( \sum_{j=1}^{n} a_j x_j \right)^*$, $a_i \in \mathbb{C}$, so the linear map $\Delta_1$ is positive if and only if $\Delta_1(\sum_{i=1}^{n} a_i x_i)(\sum_{j=1}^{n} a_j x_j)^*$ is positive. Therefore,

$$\Delta_1\left( \sum_{i=1}^{n} a_i x_i \right) \left( \sum_{j=1}^{n} a_j x_j \right)^* = \Delta_1\left( \sum_{i,j=1}^{n} a_i \alpha_{ij} x_i x_j^* \right) = \Delta_1\left( \sum_{i,j=1}^{n} a_i \alpha_{ij} \sum_{k=1}^{N_{i,j}^k} x_k \right) = \sum_{k=1}^{n} \frac{1}{d(x_k)} \left( \sum_{i,j=1}^{n} a_i \alpha_{ij} N_{i,j}^k \right) x_k \otimes x_k \text{ Frobenius reciprocity}$$

$$= \sum_{k=1}^{n} \frac{v^* M_k v}{d(x_k)} x_k \otimes x_k \geq 0.$$ (2.5)

Now we consider the left regular representation of $\mathcal{A} \otimes \mathcal{A}$ on $L^2(\mathcal{A} \otimes \mathcal{A})$. Then inequality (2.5) is equivalent to inequality (2.6). \hfill \square

Theorem 2.8 (Positivity of Comultiplication Criterion). For a fusion ring $\mathcal{A}$, let $N_{i,j}^k$ be the fusion coefficients and $M_k = (N_{i,j}^k)_{i,j}$ be the fusion matrices, $1 \leq i,j,k \leq n$. If $\mathcal{A}$ admits a unitary categorification, then for any $v \in \mathbb{C}^n$,

$$T = \sum_{k=1}^{n} \frac{v^* M_k v}{\| M_k \|_\infty} M_k \otimes M_k \geq 0.$$ (2.6)

Proof. It follows from Propositions 2.6 and 2.7. \hfill \square

Remark 2.3. For any $v_s \in \mathbb{C}^n$, $s = 1, 2, 3$, inequality (2.6) implies

$$\langle T(v_2 \otimes v_3), v_2 \otimes v_3 \rangle \geq 0.$$
Therefore, we obtain the non-commutative Schur product criterion (see Proposition 7.3 in [21]):

\[
\sum_{k=1}^{n} \frac{1}{\|M_k\|} \prod_{s=1}^{3} v^*_s M_k v_s \geq 0.
\]  

(2.7)

Note that a matrix \(M\) acting on \(\mathbb{C}^n \otimes \mathbb{C}^n\) such that \(\langle M v \otimes w, v \otimes w \rangle \geq 0\) for any \(v, w \in \mathbb{C}^n\) does not assure that \(M\) is positive-semidefinite. For example, let

\[
M = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Then \(M\) is not positive-semidefinite since the determinant of \(M\) is negative. However, for \(v = (v_1, v_2)^t, w = (w_1, w_2)^t \in \mathbb{C}^2\), we have

\[
\langle M v \otimes w, v \otimes w \rangle = |v_1|^2 |w_1|^2 + 2 \text{Re}(v_1 w_2 \overline{v_2 w_1}) + |v_2|^2 |w_2|^2 = |\langle v, w \rangle|^2 \geq 0.
\]

This example supports the idea that the positivity of comultiplication criterion is stronger than the non-commutative Schur product criterion.

Now let \((\mathcal{A}, \mathcal{B}, d, \tau, F)\) be a fusion bi-algebra (see Definition 2.13 in [21]), where \(\mathcal{A}\) and \(\mathcal{B}\) are finite dimensional \(C^\ast\)-algebras, \(\mathcal{A}\) is commutative, and \(F\) is a unitary from \(\mathcal{A}\) onto \(\mathcal{B}\) preserving 2-norm. Let \(\{I = x_1, x_2, \ldots, x_n\}\) be the unique \(\mathbb{R}_{\geq 0}\)-basis of \(\mathcal{B}\) such that \(F^{-1}(x_i)\) is a multiple of minimal projection in \(\mathcal{A}\) and satisfying the fusion rules:

\[
x_k x_j = \sum_{i=1}^{n} N_{k,j}^i x_i, \quad N_{k,j}^i \geq 0.
\]

The unitary \(F\) induces a convolution \(*\) of \(\mathcal{A}\):

\[
\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}, \quad x \otimes y \mapsto x * y = F^{-1}(F(x)F(y)).
\]

We define a linear map \(\Delta_2: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}\) as the adjoint operator of \(*\) with respect to the trace \(d\):

\[
\langle \Delta_2(z), x \otimes y \rangle = \langle z, x * y \rangle
\]

(2.8)

for any \(x, y, z \in \mathcal{A}\).

**Proposition 2.9.** Let \((\mathcal{A}, \mathcal{B}, d, \tau, F)\) be a fusion bi-algebra. The linear map \(\Delta_2: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}\) defined in equality (2.8) is a good comultiplication.

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\(^1\)We thank S. Palcoux for providing this example.
Proof. For any \(x_i, x_j, x_k\), we have
\[
\langle F^{-1}(x_i) \otimes F^{-1}(x_j), \Delta_2(F^{-1}(x_k)) \rangle = \langle F^{-1}(x_i) * F^{-1}(x_j), F^{-1}(x_k) \rangle
\]
\[
= \langle F^{-1}(\sum_{s=1}^{n} N_{i,j}^s x_s), F^{-1}(x_k) \rangle
\]
\[
= \sum_{s=1}^{n} N_{i,j}^s d(x_s x_k^*)
\]
\[
= N_{i,j}^k \geq 0.
\]
Since \(A\) is commutative and
\[
\sum_{i=1}^{n} d(F^{-1}(x_i)) F^{-1}(x_i) = I,
\]
thus \(\Delta_2 \geq 0\). By Proposition 2.5 in [21], we further have
\[
\langle F^{-1}(x_i) \otimes F^{-1}(x_j), \Delta_2(I) \rangle = \langle F^{-1}(x_i) * F^{-1}(x_j), I \rangle
\]
\[
= d(F^{-1}(x_i) * F^{-1}(x_j)) = d(F^{-1}(x_i))d(F^{-1}(x_j)),
\]
which implies \(\Delta_2\) preserves identity. Therefore, \(\Delta_2\) is a good comultiplication. \(\square\)

Quantum inequalities were studied and applied in [18, 6, 20, 21]. In Sections 3 and 4, we will unify various quantum inequalities including quantum Young’s inequality, quantum (smooth) entropic convolution inequality on Frobenius von Neumann \(k\)-algebras. These inequalities can be applied as criterions of unitary categorification as well.

3. Quantum Young’s inequality on Frobenius von Neumann \(k\)-algebras

In this section, we will prove the quantum Young’s inequality on Frobenius von Neumann \(k\)-algebras, which unifies the quantum Young’s inequality on subfactor planar algebras [11] and fusion bi-algebras [21].

3.1. Frobenius von Neumann \(k\)-algebras. To unify quantum inequalities on quantum symmetries, we introduce the notion of Frobenius von Neumann \(k\)-algebras.

Definition 3.1. Let \((\mathcal{M}, \tau)\) be a tracial von Neumann algebra with a \(k\)-convolution \(*\), \(k > 0\). A trace-preserving anti *-isomorphism \(\rho\) on \(\mathcal{M}\) is called an antipode if the Frobenius Reciprocity holds:
\[
\tau((x \ast y)z) = \tau((\rho(z) \ast x)\rho(y)), \quad \forall \, x, y, z \in \mathcal{M}.
\] (3.1)

Proposition 3.1. For any \(x \in \mathcal{M}\), \(0 < p \leq \infty\), the antipode \(\rho\) satisfies \(\|\rho(x)\|_p = \|x\|_p\).

Proof. For any positive rational number \(r = \frac{m}{n}\), \(m, n \in \mathbb{N}_+\), we have
\[
(|\rho(x)|^r)^n = |\rho(x)|^m = \rho(|x|)^m = \rho(|x|^m), \quad \rho(|x|^r)^n = \rho(|x|^m) = \rho(|x|^m).
\]
Since $|\rho(x)|^r$ and $\rho(|x^*|^r)$ are positive operators, we have $|\rho(x)|^r = \rho(|x^*|^r)$. Therefore,

$$\|\rho(x)\|_r = \tau(\|\rho(x)|^r)^{1/r} = \tau(\rho(|x^*|^r))^{1/r} = \tau(|x^*|^r)^{1/r} = \|x^*\|_r = \|x\|_r.$$ 

Since positive rational numbers are dense in positive real numbers and $\tau$ is continuous with respect to operator norm, we have $\|\rho(x)\|_p = \|x\|_p$ for any $0 < p < \infty$. Note that the operator norm $\|x\| = \lim_{n \to \infty} \tau(|x|^n)^{1/n}$, we have $\|\rho(x)\| = \|x\|$. Therefore $\|\rho(x)\|_p = \|x\|_p$ for any $0 < p \leq \infty$. \hfill \Box

**Definition 3.2.** Let $(\mathcal{M}, \tau)$ be a tracial von Neumann algebra with a good $k$-convolution $\ast$ and an antipode $\rho$. Then we call the quadruple $(\mathcal{M}, \tau, \ast, \rho)$ a *Frobenius von Neumann $k$-algebra* (FN $k$-algebra). We simply call $(\mathcal{M}, \tau, \ast, \rho)$ a *Frobenius von Neumann algebra* (FN algebra) when $k = 1$.

**Remark 3.1.** Let $(\mathcal{M}, \tau, \ast, \rho)$ be a FN $k$-algebra. For any $\lambda_1, \lambda_2 > 0$, define

$$\tau_{\lambda_1}(x) = \lambda_1^{-1}\tau(x), \quad x \ast_{\lambda_2} y = \lambda_2^{-1}x \ast y, \quad \forall x, y \in \mathcal{M}.$$ 

Then $(\mathcal{M}, \tau_{\lambda_1}, \ast_{\lambda_2}, \rho)$ is a FN $\lambda_1 k/\lambda_2$-algebra. In particular, $(\mathcal{M}, \tau_{\lambda_1}, \ast_{\lambda_2}, \rho)$ is a FN algebra when $\lambda_1/\lambda_2 = 1/k$.

**Example 3.1.** Let $\mathcal{P}_{2,\pm}$ be the 2-box space of an irreducible subfactor planar algebra with finite index $\delta^2$, $\delta > 0$. For any $x \in \mathcal{P}_{2,\pm}$, let $\rho(x)$ be the contragredient of $x$. Then $\rho$ is an antipode on $\mathcal{P}_{2,\pm}$ by Lemma 3.4 in [11]. The quadruple $(\mathcal{P}_{2,\pm}, \text{Tr}, \ast, \rho)$ is a FN $\delta^{-1}$-algebra.

**Example 3.2.** Let $(\mathcal{A}, \mathcal{B}, d, \tau, \mathcal{F})$ be a fusion bi-algebra. For any $x \in \mathcal{A}$, let $\rho(x) = J(x)^\ast$, where $J = \mathcal{F}^{-1}(\mathcal{F}(x)^\ast)$ is an anti-linear, $\ast$-isomorphism on $\mathcal{A}$ (See Definition 2.12 in [21]). Then $\rho$ is an antipode on $\mathcal{A}$ by Proposition 2.21 in [21]. The quadruple $(\mathcal{A}, d, \ast, \rho)$ is a FN algebra.

### 3.2. Quantum Young’s Inequality

In this section, we will prove the quantum Young’s inequality on Frobenius von Neumann $k$-algebras. The interpolation theorem for bounded linear maps between two tracial von Neumann algebras would be helpful in the proof.

**Proposition 3.2** (Interpolation Theorem, see e.g. [17]). Let $\mathcal{M}$ and $\mathcal{N}$ be two von Neumann algebras with traces $\tau_1$ and $\tau_2$ respectively. Suppose $T : \mathcal{M} \to \mathcal{N}$ is a linear map. If

$$\|Tx\|_{p_1, \tau_1} \leq K_1\|x\|_{q_1, \tau_1}, \quad \|Tx\|_{p_2, \tau_2} \leq K_2\|x\|_{q_2, \tau_2},$$

then

$$\|Tx\|_{p_0, \tau_0} \leq K_1^{1-\theta}K_2^\theta\|x\|_{q_0},$$

where $\frac{1}{p_0} = \frac{1}{p_1} + \frac{\theta}{p_2}$, $\frac{1}{q_0} = \frac{1}{q_1} + \frac{\theta}{q_2}$, $1 \leq p_1, q_1, p_2, q_2 \leq \infty$, $0 \leq \theta \leq 1$.

For a FN $k$-algebra $(\mathcal{M}, \tau, \ast, \rho)$, we shall note from the definition of good $k$-convolution (see Remark [21]) that the primary Young’s inequality holds:

$$\|x \ast y\|_1 \leq k\|x\|_1\|y\|_1$$

(3.2)

for any $x, y \in \mathcal{M}$.

**Lemma 3.3.** For any $x, y \in \mathcal{M}$, we have

$$\|x \ast y\|_\infty \leq k\|x\|_1\|y\|_\infty.$$
Proof. We have
\[ \|x \ast y\|_\infty = \sup_{\|z\|_1 = 1} |\tau((x \ast y)z)| = \sup_{\|z\|_1 = 1} |\tau((\rho(z) \ast x)\rho(y))| \]
\[ \leq \sup_{\|z\|_1 = 1} \|\rho(z) \ast x\|_1 \|\rho(y)\|_\infty \leq k \|x\|_1 \|y\|_\infty. \]
The first inequality is Hölder’s inequality and the second uses primary Young’s inequality and Lemma 3.1.

Lemma 3.4. For any \( x, y \in M \), we have
\[ \|x \ast y\|_\infty \leq k \|x\|_\infty \|y\|_1. \]

Proof. We have
\[ \|x \ast y\|_\infty = \sup_{\|z\|_1 = 1} |\tau((x \ast y)z)| = \sup_{\|z\|_1 = 1} |\tau((\rho(z) \ast x)\rho(y))| \]
\[ \leq \sup_{\|z\|_1 = 1} \|\rho(z) \ast x\|_1 \|\rho(y)\|_1 \leq k \|x\|_1 \|y\|_1. \]
The first inequality is Hölder’s inequality and the second uses Lemmas 3.3 and 3.1.

Lemma 3.5. For any \( x, y \in M \), \( 1 \leq p \leq \infty \), we have
\[ \|x \ast y\|_p \leq k \|x\|_1 \|y\|_p, \quad \|x \ast y\|_p \leq k \|x\|_p \|y\|_1. \]

Proof. It follows from primary Young’s inequality, Lemmas 3.3 and 3.4 and Proposition 3.2.

Lemma 3.6. For any \( x, y \in M \), \( 1 \leq p \leq \infty \), \( \frac{1}{p} + \frac{1}{q} = 1 \), we have
\[ \|x \ast y\|_p \leq k \|x\|_1 \|y\|_q. \]

Proof. We have
\[ \|x \ast y\|_\infty = \sup_{\|z\|_1 = 1} |\tau((x \ast y)z)| = \sup_{\|z\|_1 = 1} |\tau((\rho(z) \ast x)\rho(y))| \]
\[ \leq \sup_{\|z\|_1 = 1} \|\rho(z) \ast x\|_p \|\rho(y)\|_q \leq k \|x\|_p \|y\|_q. \]
The second inequality uses Lemma 3.5.

Theorem 3.7 (Quantum Young’s inequality). Let \((M, \tau, \ast, \rho)\) be a FN \( k \)-algebra. For any \( x, y \in M \), \( 1 \leq p, q, r \leq \infty \) with \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} + 1 \), we have
\[ \|x \ast y\|_r \leq k \|x\|_p \|y\|_q. \] (3.3)

Proof. It follows from Lemma 3.5, Lemma 3.6 and Proposition 3.2.

Remark 3.2. We have the following statements:
(1) Theorem 3.7 implies Theorem 4.13 in [11].
(2) Theorem 3.7 implies Theorem 5.11 in [21].
Theorem 3.8. Let \((M, \tau_M, *, \rho)\) be a FN \(k\)-algebra. For fixed \(x_i, y_i \in M, i = 1, 2, \ldots, n, 1 \leq p, q, r \leq \infty\) with \(1 + 1/r = 1/p + 1/q\), there exists \(t_0 \in [0, 1]\) such that

\[
\left\| \sum_{i=1}^{n} x_i \ast y_i \right\|_r \leq k \left\| \sum_{j=1}^{n} e^{2\pi i j t} x_j \right\|_p \left\| \sum_{k=1}^{n} e^{-2\pi i k t} y_k \right\|_q.
\]

Proof. Note that

\[
\int_0^1 e^{2\pi i (j-k)t} dt = \delta_{j,k}, \quad j, k \in \mathbb{N}.
\]

We have

\[
\sum_{i=1}^{n} x_i \ast y_i = \int_0^1 \sum_{1 \leq j, k \leq n} e^{2\pi i (j-k)t} x_j \ast y_k dt
\]

\[
= \int_0^1 \left( \sum_{j=1}^{n} e^{2\pi i j t} x_j \right) \ast \left( \sum_{k=1}^{n} e^{-2\pi i k t} y_k \right) dt.
\]

By Minkowski inequality,

\[
\left\| \sum_{i=1}^{n} x_i \ast y_i \right\|_r = \left\| \int_0^1 \left( \sum_{j=1}^{n} e^{2\pi i j t} x_j \right) \ast \left( \sum_{k=1}^{n} e^{-2\pi i k t} y_k \right) dt \right\|_r
\]

\[
\leq \int_0^1 \left\| \left( \sum_{j=1}^{n} e^{2\pi i j t} x_j \right) \ast \left( \sum_{k=1}^{n} e^{-2\pi i k t} y_k \right) \right\|_r dt
\]

\[
\leq k \int_0^1 \left\| \sum_{j=1}^{n} e^{2\pi i j t} x_j \right\|_p \left\| \sum_{k=1}^{n} e^{-2\pi i k t} y_k \right\|_q dt \quad \text{quantum Young's inequality}
\]

\[
\leq \sup_{t \in [0,1]} k \left\| \sum_{j=1}^{n} e^{2\pi i j t} x_j \right\|_p \left\| \sum_{k=1}^{n} e^{-2\pi i k t} y_k \right\|_q.
\]

Since \([0, 1]\) is compact, there exists some \(t_0\) to obtain the supremum. \(\Box\)

3.3. Quantum reverse Young’s inequality. In the end of this section, we will give two versions of quantum reverse Young’s inequality on subfactor planar algebras.

Theorem 3.9 (Quantum reverse Young’s inequality (1)). Let \(\mathcal{P}\) be an irreducible planar algebra with index \(\delta^2, \delta > 0\). For any positive operators \(x, y \in \mathcal{P}_{2, \pm}, 0 < r, s, t \leq 1\) with \(1 + 1/r = 1/s + 1/t\), we have

\[
\|x \ast y\|_r \geq \delta^{1-2/r} \|x\|_s \|y\|_t.
\]  

(3.4)

Proof. Let \(p = id \otimes e_1\) be the biprojection in \(\mathcal{P}_{4, \pm}\). Define a map: \(\mathcal{P}_{4, \pm} \to p\mathcal{P}_{4, \pm}p, x \mapsto pxp\), which is a unital positive linear map when considering \(p\) as the identity of the \(C^*\)-algebra
Note that \( f(x) = x^r \), \( 0 < r \leq 1 \), is an operator-concave function, we have (See e.g. Theorem 2.1 in [8])

\[
px^r p \leq (pxp)^r, \quad x \geq 0.
\]

We have

\[
\|x \ast y\|_r = \|(x \ast y)^r\|_1^{1/r} \\
= \delta\|(px \otimes yp)^r\|_1^{1/r} \\
\geq \delta\|px^r \otimes y^r p\|_1^{1/r} \quad \text{inequality (3.5)} \\
= \delta^{1-1/r}\|x^r \ast y^r\|_1^{1/r} \\
= \delta^{1-2/r}\|x\|_s\|y\|_t. \\
\]

Thus, we obtain the inequality (3.4).

**Remark 3.3.** Suppose \( \|x \ast y\|_r = \delta^{1-2/r}\|x\|_s\|y\|_t \) for some \( 0 < r < 1 \). Then \( \|x\|_r = \|x\|_s \) and \( \|y\|_r = \|x\|_s \) from the proof, which implies \( x, y \) are both trace-one projections. Note that (See Proposition 4.7) \( S(x^r \ast y^r) \leq S(x)S(y) = 1 \), where \( S(x) = \text{Tr}(R(x)) \), \( \text{Tr} \) is the unnormalized Markov trace, \( R(x) \) is the range projection of \( x \). Thus, \( S(x^r \ast y^r) = 1 \) and \( \delta x \ast y \) is a trace-one projection. We have \( \delta^{-1} = \|x \ast y\|_r > \delta^{1-2/r} = \delta^{1-2/r}\|x\|_s\|y\|_t \). So inequality (3.4) is not sharp.

**Conjecture 3.1.** Let \( \mathcal{P}_\ast \) be an irreducible planar algebra with index \( \delta^2, \delta > 0 \). For any positive operators \( x, y \in \mathcal{P}_{2, \pm} \), \( 0 < r, s, t \leq 1 \) with \( 1 + 1/r = 1/s + 1/t \), the following sharp inequality holds:

\[
\|x \ast y\|_r \geq \delta^{-1}\|x\|_s\|y\|_t. \quad (3.6)
\]

**Remark 3.4.** Suppose inequality (3.6) holds. Take \( t = 1 \), then \( s = r \). We have

\[
\lim_{r \to 0^+}\|x \ast y\|_r \geq \lim_{r \to 0^+} \delta^{-r}\|x\|_r\|y\|_1^r.
\]

It implies the following sum set estimate:

\[
S(x \ast y) \geq S(x). \quad (3.7)
\]

See also Theorem 4.1 in [13].

To study the sharp inequality, we introduce another version of quantum reverse Young’s inequality. Let \( (\mathcal{M}, \tau, \ast, \rho) \) be a finite dimensional FN \( k \)-algebra. We set

\[
\lambda = \min\{\tau(e) : \ e \text{ is a projection in } \mathcal{M}\}.
\]

**Theorem 3.10.** Let \( (\mathcal{M}, \tau, \ast, \rho) \) be a finite dimensional FN \( k \)-algebra. For any positive operators \( x, y \in \mathcal{M} \), \( 0 < r, s, t \leq 1 \) with \( 1 + 1/r = 1/s + 1/t \), we have

\[
\|x^r \ast y^r\|_r \geq \lambda^{1/r-r}k\|x\|_s^r\|y\|_t^r. \quad (3.8)
\]
Proof. By Remark 3.13 \((M, \tau, *_{\lambda}, \rho)\) is also a FN \(k\)-algebra. Define
\[
\|x\|_{\lambda,p} = \tau_{\lambda}(|x|^p)^{1/p}, \quad 0 < p \leq \infty.
\]
We have
\[
\|x\|_{\lambda,\infty} \leq \|x\|_{\lambda,p}, \quad x \in M, \quad 1 \leq p \leq \infty.
\]
By Positivity of \("*_{\lambda}"\),
\[
x^t *_{\lambda} y^s \leq (\|x\|_{\lambda,\infty}^{t-r} x^r) *_{\lambda} (\|y\|_{\lambda,\infty}^{s-r} y^r) \leq \|x\|_{\lambda,t}^{t-r} \|y\|_{\lambda,s}^{s-r} x^r *_{\lambda} y^r.
\]
By Haar measure property,
\[
\|x\|_{\lambda,t}^{r} \|y\|_{\lambda,s}^{r} = k^{-1} \|x\|_{\lambda,t}^{r-t} \|y\|_{\lambda,s}^{r-s} \tau_{\lambda}(x^r *_{\lambda} y^r).
\]
Therefore,
\[
\|x\|_{\lambda,t}^{r} \|y\|_{\lambda,s}^{r} \leq k^{-1} \tau_{\lambda}(x^r *_{\lambda} y^r)
\]
\[
= k^{-1} \tau_{\lambda}((x^r *_{\lambda} y^r)^r (x^r *_{\lambda} y^r)^{1-r})
\]
\[
\leq k^{-1} \| (x^r *_{\lambda} y^r)^r \|_{\lambda,1} \| (x^r *_{\lambda} y^r)^{1-r} \|_{\lambda,\infty} \quad \text{Hölder’s inequality}
\]
\[
= k^{-1} \|x^r *_{\lambda} y^r\|_{\lambda,r}^{r} \|x^r *_{\lambda} y^r\|_{\lambda,\infty}^{1-r}
\]
\[
\leq k^{-1} \|x^r *_{\lambda} y^r\|_{\lambda,r}^{r} \|x^r *_{\lambda} y^r\|_{\lambda,1/r}^{1-r}
\]
\[
\leq k^{-r} \|x^r *_{\lambda} y^r\|_{\lambda,r}^{r} \|x^r\|_{\lambda,t/r}^{1-r} \|y^r\|_{\lambda,s/r}^{1-r} \quad \text{quantum Young’s inequality}
\]
\[
= k^{-r} \|x^r *_{\lambda} y^r\|_{\lambda,r}^{r} \|x^r\|_{\lambda,t/r}^{r(1-r)} \|y^r\|_{\lambda,s}^{r (1-r)}.
\]
Thus,
\[
\|x^r *_{\lambda} y^r\|_{\lambda,r} \geq k \|x\|_{\lambda,t}^{r} \|y\|_{\lambda,s}^{r}.
\]
It is equivalent to
\[
\|x^r \|_{\lambda,r} \geq \lambda^{1/r} k \|x\|_{\lambda,t}^{r} \|y\|_{\lambda,s}^{r}.
\]
\(\Box\)

Corollary 3.11 (Quantum reverse Young’s inequality (2)). Let \(P\) be an irreducible planar algebra with index \(\delta^2, \delta > 0\). For any positive operators \(x, y \in P_{2,\pm}\), \(0 < r, s, t \leq 1\) with 
\(1 + 1/r = 1/s + 1/t\), we have
\[
\|x^r * y^s\|_{\lambda} \geq \delta^{-1} \|x\|_{\lambda,t}^{r} \|y\|_{\lambda,s}^{s}.
\]  
(3.9)

Remark 3.5. Take \(x = y = e_1\), the Jones projection. Then \(\|x^r * y^s\|_{\lambda} = \delta^{-1} \|x\|_{\lambda,t}^{r} \|y\|_{\lambda,s}^{s}\). So inequality (3.9) is sharp.

Theorem 3.12 (Sum set estimate). Let \((M, \tau, *, \rho)\) be a FN \(k\)-algebra. For any \(x, y \in M\), we have
\[
S(\mathcal{R}(x) * \mathcal{R}(y)) \geq \max\{S(x), S(y)\}.
\]
Proof. We have
\[ S(x)S(y) = \|R(x)\|_1\|R(y)\|_1 \]
\[ = k^{-1}\|R(x) * R(y)\|_1 \quad \text{Haar measure} \]
\[ \leq k^{-1}\|R(R(x) * R(y))\|_2\|R(x) * R(y)\|_2 \quad \text{Hölder’s inequality} \]
\[ \leq S(R(x) * R(y))^{1/2}\|R(x)\|_1\|R(y)\|_2 \quad \text{quantum Young’s inequality} \]
\[ = S(R(x) * R(y))^{1/2}S(x)S(y)^{1/2}. \]
Therefore, \( S(R(x) * R(y)) \geq S(y) \). Similarly, \( S(R(x) * R(y)) \geq S(x) \). \qed

Remark 3.6. We have that
(1) Theorem 3.12 implies Theorem 4.1 in [13];
(2) Theorem 3.12 implies Theorem 5.19 in [21].

4. Quantum Entropic Convolution Inequality

In this section, we will prove the quantum entropic convolution inequality on Frobenius von Neumann \( k \)-algebras and characterize the extremiziers of the qECI on subfactor planar algebras. Moreover, we introduce the notion of (smooth) convolution entropy and establish the quantum smooth entropic convolution inequality.

4.1. Quantum Entropic Convolution Inequality.

Theorem 4.1 (Quantum entropic convolution inequality). Let \((\mathcal{M}, \tau, *, \rho)\) be a FN \( k \)-algebra. For any positive operators \( x, y \in \mathcal{M} \) with \( \|x\|_1 = \|y\|_1 = k^{-1}, 0 \leq \theta \leq 1 \), we have
\[ H(x * y) \geq \theta H(x) + (1 - \theta)H(y), \tag{4.1} \]
where \( H(x) = \tau(-x \log x) \) is the von Neumann entropy.

Proof. Let \( 0 \leq \theta \leq 1 \). For any \( r \geq 1 \), define
\[ p = \frac{r}{1 - \theta + \theta r}, \quad q = \frac{r}{(1 - \theta)r + \theta}. \]
Then \( 1 + 1/r = 1/p + 1/q \). Define
\[ f(r) = \|x * y\|_r - k\|x\|_p\|y\|_q. \]
Then \( f(1) = \|x * y\|_1 - k\|x\|_1\|y\|_1 = 0 \) by Haar measure property and \( f(r) \leq 0 \) by quantum Young’s inequality (3.3). So \( f'(1) \leq 0 \). Since
\[ \frac{d}{dr}\tau(x^r) = \tau(x^r \log x), \]
we have
\[ \frac{d}{dr}\|x\|_r \bigg|_{r=1} = \frac{d}{dr}\exp\left(\frac{1}{r}\log\tau(x^r)\right) \bigg|_{r=1} = -H(x) - \tau(x) \log \tau(x). \]
Therefore,
\[
\frac{d}{dr} \| x \ast y \|_r \left|_{r=1} - k \frac{d}{dr} \| x \|_p \right|_{r=1} \| y \|_1 - k \| x \|_1 \frac{d}{dr} \| y \|_q \right|_{r=1} \leq 0,
\]
which implies
\[
H(x \ast y) \geq (1 - \theta)H(x) + \theta H(y).
\]
Take \( \theta = 0, 1 \) respectively. We could obtain the inequality (4.1). □

Note that a good \( k \)-convolution \( \ast \) may not satisfy associative law.

**Corollary 4.2.** For any positive operators \( x_i \in \mathcal{M} \) with \( \| x_i \|_1 = k^{-1}, 0 \leq \lambda_i \leq 1, \ i = 1, 2, 3 \), we have
\[
H((x_1 \ast x_2) \ast x_3) \geq \sum_{i=1}^{3} \lambda_i H(x_i)
\]
and
\[
H(x_1 \ast (x_2 \ast x_3)) \geq \sum_{i=1}^{3} \lambda_i H(x_i)
\]
For any \( n \geq 4 \), we also have the same argument.

We characterize the extremizers of qECI on subfactor planar algebras.

**Theorem 4.3.** Suppose \( \mathcal{P}_* \) is an irreducible subfactor planar algebra with finite index \( \delta^2, \delta > 0 \). For any positive operators \( x, y \in \mathcal{P}_{2, \pm} \) with \( \| x \|_1 = \| y \|_1 = \delta, 0 \leq \theta \leq 1 \), we have
\[
H(x \ast y) \geq (1 - \theta)H(x) + \theta H(y).
\]
Moreover, the equality holds for some \( 0 < \theta < 1 \) if and only if \( x, y \) are multiples of right shifts of biprojections such that \( \mathcal{R}(\mathcal{F}^{-1}(x)^\ast) = \mathcal{R}(\mathcal{F}^{-1}(y)) \).

**Proof.** Applying Theorem 4.1 to the FN \( \delta^{-1} \)-algebra \( (\mathcal{P}_{2, \pm}, \text{Tr}, \ast, \rho) \), we could obtain the inequality. We now assume that
\[
H(x \ast y) = (1 - \theta)H(x) + \theta H(y)
\]
for some \( 0 < \theta < 1 \). Let
\[
g(z) = \text{Tr} \left[ \left( x^{(1-\theta)z+\theta} y^{\theta z + 1-\theta} \right) (x \ast y)^{1-z} \right], \quad 0 \leq \text{Re}(z) \leq 1.
\]
Then \( g(1) = \delta \). By Hölder’s inequality and quantum Young’s inequality, we have that
\[
|g(z)| \leq \left\| x^{(1-\theta)z+\theta} y^{\theta z + 1-\theta} \right\|_{\text{Re}(z)}^{-1} \left\| (x \ast y)^{1-z} \right\|_{1-\text{Re}(z)}^{-1}
\]
\[
\leq \frac{1}{\delta} \left\| x^{(1-\theta)z+\theta} \right\|_{(1-\theta)\text{Re}(z)+\theta} \left\| y^{\theta z + 1-\theta} \right\|_{\theta \text{Re}(z)+1-\theta} \delta^{1-\text{Re}(z)}
\]
\[
= \delta.
\]
Differentiating $g(z)$ with respect to $z$, we obtain that
\[ g'(z) = (1 - \theta)\text{Tr} \left[ (x^{(1-\theta)z+\theta} \log x \ast y^{\theta z+1-\theta}) (x \ast y)^{1-z} \right] \\
+ \theta \text{Tr} \left[ (x^{(1-\theta)z+\theta} \ast y^{\theta z+1-\theta} \log y) (x \ast y)^{1-z} \right] \\
- \text{Tr} \left[ (x^{(1-\theta)z+\theta} \ast y^{\theta z+1-\theta}) (x \ast y)^{1-z} \log (x \ast y) \right], \]
i.e.
\[ g'(1) = -(1 - \theta)H(x) - \theta H(y) + H(x \ast y) = 0. \]

By Proposition 6.3 in [11], we have that $g(z) \equiv \delta$ for $0 \leq \text{Re}(z) \leq 1$. Hence
\[ \|x^{(1-\theta)z+\theta} \ast y^{\theta z+1-\theta}\|_{H(z)} \leq \frac{1}{\delta} \|x^{(1-\theta)z+\theta}\|_{(1-\theta)H(z)+\theta} \|y^{\theta z+1-\theta}\|_{\theta H(z)+1-\theta}. \]

By Theorem 1.3 in [13] and $x, y \geq 0$, we have that $x, y$ are multiples of right shifts of biprojections such that
\[ \mathcal{R}(\mathcal{F}^{-1}(x)) = \mathcal{R}(\mathcal{F}(y)). \]

Conversely, assume $x, y$ are multiples of right shifts of biprojections such that $\mathcal{R}(\mathcal{F}^{-1}(x)) = \mathcal{R}(\mathcal{F}^{-1}(y))$. Then $x \ast y$ is a multiple of a projection. By Theorem 1.3 in [13], we have that inequality (4.4) becomes equality. Hence $g(z)$ is constant and $g'(1) = 0$ and $H(x \ast y) = (1 - \theta)H(x) + \theta H(y)$. \qed

We have partial characterization for $x, y$ when $H(x \ast y) = H(x)$.

**Proposition 4.4.** Suppose $\mathcal{P}_\ast$ is an irreducible subfactor planar algebra with finite index $\delta^2$, $\delta > 0$. Let $x, y \in \mathcal{P}_{2, \pm}, x, y \geq 0$, with $\|x\|_1 = \|y\|_1 = \delta$, $0 \leq \theta \leq 1$. Suppose $x$ is a multiple of a projection, then $H(x \ast y) = H(x)$ if and only if $y$ and $x \ast y$ are multiple of projections.

**Proof.** Let $y = \sum_{i=1}^{n} \alpha_i \frac{\delta P_i}{\text{Tr}(P_i)}$ be the spectral decomposition of $y$, where $P_i$ are projections, $0 \leq \alpha_i \leq 1$. Then $\sum_{i=1}^{n} \alpha_i = 1$. From the concavity of entropy, we have
\[ H(x \ast y) = H(x \ast \sum_{i=1}^{n} \alpha_i \frac{\delta P_i}{\text{Tr}(P_i)}) \\
= H(\sum_{i=1}^{n} \alpha_i x \ast \frac{\delta P_i}{\text{Tr}(P_i)}) \geq \sum_{i=1}^{n} \alpha_i H(x \ast \frac{\delta P_i}{\text{Tr}(P_i)}) \\
\geq \sum_{i=1}^{n} \alpha_i H(x) \hspace{1cm} \text{Theorem 4.3} \\
= H(x). \]

From $H(x \ast y) = H(x)$, we have $y = \frac{\delta P_{i_0}}{\text{Tr}(P_{i_0})}$ for some $i_0$. Let
\[ g(z) = \text{Tr} \left[ (x^z \ast y) (x \ast y)^{1-z} \right], \hspace{1cm} 0 \leq \text{Re}(z) \leq 1. \tag{4.5} \]
Then $g(1) = \delta$. By Hölder’s inequality and quantum Young’s inequality, we have that
\[
|g(z)| \leq \left\| x^z \right\| \left\| y \right\| \frac{1}{\text{Re}(z)} \left\| (x \ast y)^{1-z} \right\| \frac{1}{1-\text{Re}(z)}
\]
\[
\leq \frac{1}{\delta} \left\| x^z \right\| \left\| y \right\| \frac{1}{\text{Re}(z)} z^{1-\text{Re}(z)} = \delta.
\]
(4.6)

Differentiating $g(z)$ with respect to $z$, we obtain that
\[
g'(1) = -H(x) + H(x \ast y) = 0.
\]
By Proposition 6.3 in [11], we have that $g(z) \equiv \delta$ for $0 \leq \text{Re}(z) \leq 1$. Thus inequality (4.6) becomes equality. We have
\[
\left\| x^z \right\| \left\| y \right\| \frac{1}{\text{Re}(z)} z^{1-\text{Re}(z)} = \delta.
\]
(4.7)

Note that $x$ is a multiple of some projection $P$. Thus
\[
\| Q \ast P_{10} \| \frac{1}{\text{Re}(\theta)} = \frac{1}{\delta} \| Q \| \frac{1}{\text{Re}(\theta)} \| P_{10} \| .
\]
By Theorem 1.5 in [13], we have $Q \ast P_{10}$ is a multiple of projection. Therefore, $x \ast y$ is a multiple of projection.

Conversely, assume that $y$ and $x \ast y$ are multiple of projections. Then equality (4.7) holds, which implies inequality (4.6) becomes equality. Thus $g(z) \equiv \delta$ and $g(1)' = 0$. We have that $H(x \ast y) = H(x)$.

Next, we will see a class of good convolutions such that the quantum entropic convolution inequality holds while the quantum Young’s inequality does not hold.

**Example 4.1.** Let $\mathcal{M} = M_n(\mathbb{C})$, $\tau = \text{Tr}$, the unnormalized trace of matrices. Let $U$ be a unitary matrix in $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$, define
\[
\Delta : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) : x \mapsto U^*(x \otimes I_1)U.
\]
It is clear that $\Delta$ is a unital positive linear map, thus is a good co-multiplication. By computation, we have that the reduced good convolution $\ast$ is given by
\[
M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}) : x \otimes y \mapsto x \ast y = \text{Tr}_2(Ux \otimes yU^*),
\]
where $\text{Tr}_2$ is the partial trace at the second position. In particular, let
\[
U_\theta = \sqrt{\theta}I + i\sqrt{\theta(1-\theta)}S, \quad 0 \leq \theta \leq 1,
\]
where $S(A \otimes B) = B \otimes A$ is the swap operator. Let $\Delta_\theta$ be the corresponding co-multiplication. Then the reduced good convolution $\ast_\theta$ is given by
\[
x \ast_\theta y = \text{Tr}_2(U_\theta x \otimes yU_\theta^*) = \theta x + (1-\theta)y + i\sqrt{\theta(1-\theta)}[x, y]
\]
for any density matrices \( x, y \). The quantum entropic convolution inequality for this convolution was proved in [1]:

\[
H(x \ast_\theta y) \geq \theta H(x) + (1 - \theta) H(y).
\]

However, quantum Young’s inequality does not hold for this good convolution.

4.2. Quantum smooth entropic convolution inequality. In this section, we will prove the H"older continuity of von Neumann entropy and convolution entropy. Using the continuity of entropy, we establish the quantum smooth entropic convolution inequality. For a finite dimensional FN \( k \)-algebra \( (\mathcal{M}, \tau, \ast, \rho) \), we call \( \tau(I) \) the Frobenius-Perron dimension of \( \mathcal{M} \), denoted by \( d \). Recall that \( \lambda = \min\{\tau(e) : e \text{ is a projection in } \mathcal{M}\} \). We have

\[
\|x\| \leq \lambda^{-1/p} \|x\|_p, \quad \forall x \in \mathcal{M}.
\] (4.8)

**Theorem 4.5** (Quantum smooth entropic convolution inequality). Let \((\mathcal{M}, \tau, \ast, \rho)\) be a finite dimensional FN \( k \)-algebra. Let \( p, q \in [1, \infty] \), \( \epsilon, \eta, \in [0, 1] \), \( \epsilon + \eta \leq \frac{1}{(d + 1)(1 + k(d + 1))} \). For any positive operators \( x, y \in \mathcal{M} \) with \( \|x\|_1 = \|y\|_1 = k^{-1} \), \( 0 \leq \theta \leq 1 \), we have

\[
H^{p,q}_{\epsilon, \eta}(x \ast y) \geq \theta H^p_{\epsilon}(x) + (1 - \theta) H^q_{\eta}(y) - O_{d, \lambda, k}(|\epsilon \log \epsilon|) - O_{d, \lambda, k}(|\eta \log \eta|).
\] (4.9)

Before proving Theorem 4.5, we need some preparations.

4.2.1. Smooth convolution entropy. The convolution entropy of two positive operator \( x, y \in \mathcal{M} \) is defined as follows.

**Definition 4.1.** Let \((\mathcal{M}, \tau)\) be a tracial von Neumann algebra with a good \( k \)-convolution \( \ast \). For any positive operators \( x, y \in \mathcal{M} \), the convolution entropy is defined by

\[
H(x \ast y) = \tau(-x \ast y \log x \ast y).
\] (4.10)

Let \((\mathcal{P}_2, \pm, \text{Tr}, \ast, \rho)\) be the FN \( \delta^{-1} \)-algebra from subfactor planar algebras. We have that the convolution entropy \( H(e_1 \ast x) = H(x \ast e_1) = H(x) \), the von Neumann entropy of \( x \), where \( e_1 \) is the Jones projection.

Let \((\mathcal{M}, \tau, \ast, \rho)\) be a FN \( k \)-algebra. From the quantum entropic convolution inequality (See Theorem 4.1), we know that the convolution entropy of two positive operators with traces \( k^{-1} \) is larger than or equal to the entropy of either of them.

To study the continuity of convolution entropy, we shall introduce the notion of smooth convolution entropy.

**Definition 4.2.** Let \((\mathcal{M}, \tau)\) be a tracial von Neumann algebra with a good \( k \)-convolution \( \ast \). For any positive operators \( x, y \in \mathcal{M} \), \( \epsilon, \eta \in [0, 1] \) and \( p, q \in [1, \infty] \), the smooth convolution entropy is defined by

\[
H^{p,q}_{\epsilon, \eta}(x \ast y) := \inf\{H(z \ast w) : z, w \in \mathcal{M}, z, w \geq 0, \|x - z\|_p \leq \epsilon, \|y - w\|_q \leq \eta\}.
\] (4.11)

The smooth entropy is defined as follows.
**Definition 4.3.** Let \((M, \tau)\) be a tracial von Neumann algebra. For any positive operator \(x \in M\), \(p \in [1, \infty]\), \(\epsilon \in [0, 1]\), the \((p, \epsilon)\) smooth entropy of \(x\) is defined by
\[
H^p_\epsilon(x) := \sup \{H(y) : y \in M, y \geq 0, \|y - x\|_p \leq \epsilon\}.
\]

**Remark 4.1.** We refer to another smooth entropy studied in quantum smooth uncertainty principles in \([9]\) and smooth Renyi entropy studied by R. Renner and S. Wolf in quantum information in \([23]\).

### 4.2.2. Continuity of convolution entropy

The following lemma would be useful to prove the H"older continuity of von Neumann entropy.

**Lemma 4.6.** For any \(0 \leq s \leq t \leq r\), \(t - s \leq r/2\), we have
\[
|t \log t - s \log s| \leq -(t - s) \log(t - s) + 2\log r|(t - s)|.
\]

**Proof.** We first assume that \(r = 1\). Let \(\gamma = t - s\), \(f(x) = (x + \gamma) \log(x + \gamma) - x \log(x)\). Since \(f'(x) = \log(x + \gamma) - \log(x) \geq 0\), we have
\[
|t \log t - s \log s| = |f(s)| \leq \max\{|f(0)|, |f(1 - \gamma)|\} = -(t - s) \log(t - s).
\]

Applying the above process to \(s/r\) and \(t/r\), we could obtain the conclusion. \(\square\)

The von Neumann entropy is H"older continuous.

**Proposition 4.7.** Suppose \((M, \tau)\) is a finite dimensional von Neumann algebra. Let \(p \in [1, \infty]\), \(\epsilon \in [0, 1]\), \(h > 0\). Let \(x, y \in M\), \(x, y \geq 0\), \(\|x\|_1 \leq h\), \(\|y\|_1 \leq h\). If \(\|x - y\|_p \leq \epsilon\), then
\[
|H(x) - H(y)| \leq d^{1/p}\epsilon \log \epsilon + O_{d, \lambda, h}(\epsilon).
\]  

**Proof.** Since every finite dimensional von Neumann algebra is a direct sum of matrix algebras, we may assume that
\[
M = \sum_{i=1}^{m} \oplus M_{n_i}(\mathbb{C}), \quad n_i \in \mathbb{N}^*, \quad \delta_i > 0.
\]

Let \(\text{Tr}_i\) be the unnormalized trace on \(M_{n_i}(\mathbb{C})\), we have
\[
\tau = \sum_{i=1}^{m} \delta_i \text{Tr}_i, \quad d = \sum_{i=1}^{m} \delta_i n_i.
\]

Let
\[
x = \sum_{i=1}^{m} x_i, \quad y = \sum_{i=1}^{m} y_i, \quad x_i, y_i \in M_{n_i}(\mathbb{C}).
\]

Let \(\lambda_j(x_i)\) and \(\lambda_j(y_i)\) be the \(j\)-th largest eigenvalues of \(x_i\) and \(y_i\) respectively. Take
\[
r = 2\lambda^{-1}h \geq 2\lambda^{-1} \max\{\|x\|_1, \|y\|_1\} \geq 2 \max\{\|x\|, \|y\|\}.
\]

The second inequality is due to inequality \((4.8)\). Thus
\[
|\lambda_j(x_i) - \lambda_j(y_i)| \leq r/2.
\]
By Lemma 3.26 in [9], we have
\[ \|\lambda(x) - \lambda(y)\|_1 = \sum_{i=1}^{m} \delta_i \sum_{j=1}^{n_i} |\lambda_j(x_i) - \lambda_j(y_i)| \leq \|x - y\|_1. \]  
(4.13)

Let \( f(t) = -t \log t \). We have
\[ |H(x) - H(y)| = \left| \sum_{i=1}^{m} \delta_i \text{Tr}_i(x_i \log x_i - y_i \log y_i) \right| \]
\[ \leq \sum_{i=1}^{m} \delta_i \sum_{j=1}^{n_i} \left| f(\lambda_j(x_i)) - f(\lambda_j(y_i)) \right| \]
\[ \leq \sum_{i=1}^{m} \delta_i \sum_{j=1}^{n_i} \left( f(|\lambda_j(x_i) - \lambda_j(y_i)|) + 2|\log r||\lambda_j(x_i) - \lambda_j(y_i)| \right) \quad \text{Lemma 4.6} \]
\[ \leq f(\|\lambda(x) - \lambda(y)\|_1) + (\log d + 2|\log r|)\|\lambda(x) - \lambda(y)\|_1 \quad \text{Jensen’s inequality}. \]

By Hölder’s inequality, we have
\[ \|\lambda(x) - \lambda(y)\|_1 \leq \|x - y\|_1 \leq d^{1-1/p}\|x - y\|_p \leq d^{1-1/p}\epsilon. \]  
(4.14)

Thus,
\[ f(\|\lambda(x) - \lambda(y)\|_1) \leq |d^{1-1/p}\epsilon \log d^{1-1/p}\epsilon| + d^{1-1/p}\epsilon \]
\[ \leq d^{1-1/p}\epsilon |\log \epsilon| + d^{1-1/p}\epsilon(1 + (1 - 1/p)|\log d|). \]

Therefore
\[ |H(x) - H(y)| \leq d^{1-1/p}\epsilon |\log \epsilon| + d^{1-1/p}\epsilon(1 + (1 - 1/p)|\log d| + |\log d| + 2|\log r|). \]

Note that \( d^{1-1/p} \leq d + 1 \), we could obtain the inequality (4.12). \qed

From Proposition 4.7 and primary Young’s inequality for good \( k \)-convolutions, we obtain the Hölder continuity of convolution entropy.

**Proposition 4.8.** Suppose \((\mathcal{M}, \tau)\) is a finite dimensional von Neumann algebra with a good \( k \)-convolution \(*\). Let \( p, q \in [1, \infty], \epsilon, \eta \in [0, 1], h > 0, \epsilon + \eta \leq \frac{1}{kh(d + 1)} \). Let \( x, y, z, w \in \mathcal{M}, \)
\( x, y, z, w \geq 0, \|x\|_1, \|y\|_1, \|z\|_1, \|w\|_1 \leq h. \) If \( \|x - z\|_p \leq \epsilon, \|y - w\|_q \leq \eta, \) then
\[ |H(x * y) - H(z * w)| \leq O_{d, \lambda, h, k}(|\epsilon \log \epsilon|) + O_{d, \lambda, h, k}(|\eta \log \eta|). \]  
(4.15)
Proof. By primary Young’s inequality and Hölder’s inequality, one could obtain
\[ \| x * y - z * w \|_1 \leq \| (x - z) * y \|_1 + \| z * (y - w) \|_1 \]
\[ \leq kh \| x - z \|_1 + kh \| y - w \|_1 \]
\[ \leq kh (d^{1-1/p} \epsilon + d^{1-1/q} \eta) \]
\[ \leq kd (d + 1) (\epsilon + \eta) \]
\[ \leq 1. \]

Note that \( \| x * y \|_1, \| z * w \|_1 \leq kh^2 \). Applying Proposition 4.7 to \( x * y \) and \( z * w \), we have
\[ | H(x * y) - H(z * w) | \leq kd (d + 1) (\epsilon + \eta) \log kd (d + 1) (\epsilon + \eta) + O_{d,\lambda,k,h} (\epsilon + \eta). \] (4.16)

It is a technical computation from inequality (4.16) to inequality (4.15). □

Now we are ready to prove the quantum smooth entropic convolution inequality.

Proof of Theorem 4.5. Take \( h = k^{-1} + d + 1 \). Then \( \| x - z \|_p \leq \epsilon \) implies
\[ \| z \|_1 \leq \| x \|_1 + \| x - z \|_1 \leq k^{-1} + d^{1-1/p} \| x - z \|_p \leq h. \]
Similarly, \( \| y - w \|_q \leq \eta \) implies \( \| w \|_1 \leq h. \) From Proposition 4.7 we have
\[ H^p_\epsilon (x) = \sup \{ H(z) : z \in \mathcal{M}, \| x - z \|_p \leq \epsilon \} \leq H(x) + d^{1-1/p} \epsilon \log \epsilon + O_{d,\lambda,k,h} (\epsilon), \] (4.17)
and
\[ H^q_\eta (y) = \sup \{ H(w) : w \in \mathcal{M}, \| y - w \|_q \leq \eta \} \leq H(y) + d^{1-1/q} \eta \log \eta + O_{d,\lambda,k,h} (\eta). \] (4.18)

Note that \( \epsilon + \eta \leq 1/kh (d + 1) \). From Proposition 4.8, we have
\[ H^p_{\epsilon,\eta} (x * y) \geq H(x * y) - O_{d,\lambda,k,h} (| \epsilon \log \epsilon |) - O_{d,\lambda,k,h} (| \eta \log \eta |). \] (4.19)

Combining inequalities (4.17), (4.18), (4.19) with Theorem 4.1 we obtain the inequality (4.9) □
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