Improved QCD expansion and the problem of determination of $\alpha_s$ and the condensates from the $\tau$ decays

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The problem of determination of $\alpha_s$ and the condensates from the moments of invariant mass distribution in the semileptonic $\tau$ decays is considered. It is investigated how the extracted values of $\alpha_s(m^2)$ and the condensates are affected by the the renormalization scheme dependence of the next-to-next-to-leading order perturbative corrections to the spectral moments. A simplified approach is used, in which the nonperturbative contributions are approximated by the terms of dimension six in the SVZ expansion, which arise from the four-quark condensates.

Recently there has been considerable interest in the determination of the QCD parameters from the invariant mass distribution in the semileptonic $\tau$ decays. It appears that using specific moments of the invariant mass distribution one may obtain rather tight experimental constraints for $\alpha_s$ and the QCD condensates. The perturbative QCD corrections to the spectral moments — which are known to next-to-next-to-leading order (NNLO) — play central role in such a determination. These corrections are usually evaluated in the $\overline{MS}$ renormalization scheme. However, in NNLO there is a two-parameter freedom in the choice of the renormalization scheme (RS). The numerical values of the perturbative corrections depend on the choice of the scheme. It is therefore an interesting question, how the values of the QCD parameters, extracted from the $\tau$ decay, may be affected, if a different choice of the RS is made, motivated for example by the principle of minimal sensitivity (PMS).

The theoretical framework adopted in the analysis of the $\tau$ decay data involves the $R_{\tau,V/A}^{kl}$ moments, defined by the relation:

$$R_{\tau,V/A}^{kl} = \frac{3}{2} |V_{ud}|^2 S_{EW} R_0^{kl} (1 + \delta_{V/A}^{kl}),$$

where $|V_{ud}| = 0.9752$ and $S_{EW} = 1.0194$. The $R_0^{kl}$ factor denotes the parton model prediction. The $\delta_{V/A}^{kl}$ term is the perturbative contribution, which is evaluated assuming 3 massless quarks — in this approximation it is universal for the V and A channels. The $\delta_{npt}^{kl}$ term denotes the contribution from the nonperturbative QCD effects, which are estimated using the SVZ approach:

$$\delta_{npt}^{kl} = \sum_{D=4,6} \frac{1}{m_D^2} \sum_j c_{D,j}^{kl} < O_{D,j}^{V/A} >,$$

where $< O_{D,j}^{V/A} >$ are the vacuum expectation values of the gauge invariant operators of dimension $D$ and $c_{D,j}^{kl}$ are coefficients specific for the considered spectral moment and the type of the operator.

The authors of $[\mathbb{4},\mathbb{5}]$ used $R_0^{V/A}$ and the $R_{\tau,V/A}^{kl}$ moments with $k = 1$ and $l = 0, 1, 2, 3$, and fitted the $D = 4, 6, 8$ condensates. Since our goal is primarily to study the theoretical uncertainties in the whole procedure, we shall adopt a simplified approach, in which the nonperturbative contribution to $R_{\tau,V/A}^{kl}$ is approximated by the leading $D = 6$ contribution, coming from the four-quark condensates. (This means we neglect the

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$D = 4$ contribution, which is suppressed by an additional power of $\alpha_s$, the higher order perturbative corrections to the coefficients of the $D = 6$ operators, and the $D \geq 8$ contributions.) We also use the $R_{\tau,V}^{12}$ moment, for which a similar approximation may be made, and for which we have the relation $\delta_{(6)V/A}^{12} = -(70/13)\delta_{(6)V/A}^{00}$. This gives us two equations to fit two parameters: $\Lambda_{(3) MS}$ (or, more conventionally, $\alpha_s^{MS}(m_t^2)$) and one parameter characterizing the nonperturbative contribution, which we take simply to be $\delta_{(6)V/A}^{00}$.

The perturbative QCD corrections $\delta_{pt}^{kl}$ are evaluated using a contour integral expression, which relates them to the QCD correction $\delta_{pt}^{kl}$ to the so-called Adler function, i.e., the logarithmic derivative of the transverse part of the vector/axial-vector current correlator:

$$\delta_{pt}^{kl} = \frac{i}{\pi} \int_C \frac{d\sigma}{\sigma} f^{kl}(\sigma/m_t^2) \delta_{pt}(\sigma),$$

(4)

where $f^{kl}(\sigma/m_t^2)$ is a weight function specific to the considered moment and $C$ in our case is assumed to be a circle $\sigma = -m_t^2 \exp(-i\theta)$, $\theta \in [-\pi, \pi]$. The NNLO renormalization group improved perturbative expansion for $\delta_{pt}^{kl}$ may be written in the form:

$$\delta_{pt}^{kl}(\sigma) = a(-\sigma)[1 + r_1a(-\sigma) + r_2a^2(-\sigma)],$$

(5)

where $a = \alpha_s/\pi = g^2/(4\pi^2)$ denotes the running coupling constant that satisfies the NNLO renormalization group equation:

$$\sigma \frac{da}{d\sigma} = -\frac{b}{2}a^2(1 + c_1a + c_2a^2).$$

(6)

By evaluating numerically the integral (4) with the renormalization group improved expression for $\delta_{pt}^{kl}$ under the integral we resum to all orders some of the corrections, which would appear in the “naive” expansion of $\delta_{pt}^{kl}$ in powers of $\alpha_s(m_t^2)$.

The coefficients $r_1$, $r_2$ and $c_2$ are RS dependent, but there exists a RS invariant combination:

$$\rho_2 = c_2 + r_2 - c_1r_1 - r_1^2.$$  

(7)

For the $\delta_{pt}$ we have $\rho_2 = 5.23783$. We shall use $r_1$ and $c_2$ to parametrize the freedom of choice of the RS in NNLO.

The dependence of $\delta_{pt}^{00}$ on the scheme parameters $r_1$ and $c_2$ was discussed in detail in [10] and the RS dependence of $\delta_{pt}^{12}$ was investigated in [11]. In both cases it was found that for moderate values of $\Lambda_{(3) MS}$ the NNLO predictions have a saddle point type of behavior as a function of $r_1$ and $c_2$ and that the position of the saddle point is well approximated by $r_1 = 0$ and $c_2 = 1.5\rho_2 = 7.857$. For very large values of $\Lambda_{(3) MS}$ the RS-dependence pattern is more complicated than a simple saddle point, but even then the scheme parameters distinguished above belong to the region of extremely small RS dependence. We shall therefore accept these parameters as the PMS parameters in NNLO.

In order to obtain a quantitative estimate of the RS dependence of the results we shall use the approach outlined in [12], based on the existence of the RS invariant $\rho_2$. In [12] it was proposed to calculate variation of the predictions over the set of schemes for which the expansion coefficients satisfy the condition:

$$\sigma_2(r_1, r_2, c_2) \leq l |\rho_2|,$$

(8)

where

$$\sigma_2(r_1, r_2, c_2) = |c_2| + |r_2| + c_1 |r_1| + r_1^2.$$  

(9)

A motivation for the condition (8) is that it eliminates schemes in which the expansions (5) and (6) involve unnaturally large expansion coefficients, that introduce large cancellations in the expression for the RS invariant $\rho_2$. The constant $l$ in the condition (8) controls the degree of cancellation that we want to allow in the expression for $\rho_2$. In our case we have for the PMS parameters $\sigma_2(\text{PMS}) \approx 2|\rho_2|$, so in order to take into account the schemes, which have the same — or smaller — degree of cancellation as the PMS scheme we take $l = 2$.

Let us begin with the fits in the vector channel. We use the experimental values reported recently by ALEPH [8]: $R_{\tau,V}^{00} = 1.782 \pm 0.018$, $D_{\tau,V}^{12} = 0.0532 \pm 0.0007$. For simplicity we neglect correlations between experimental errors for $R_{\tau,V}^{00}$ and $D_{\tau,V}^{12}$. For comparison with other determinations of $\alpha_s$ we evolve the fitted value of $\alpha_s^{MS}(m_t^2)$ to the scale of $m_t^2$ using the
NNLO renormalization group equation and the NNLO matching formula \[13\] at \(\mu = 2m_c, 2m_b\).

Using the PMS predictions we obtain from the NNLO fit in the vector channel \(\delta_{(6)V}^{00} = 0.0156 \pm 0.0023\) and \(\Lambda_{\text{MS}}^{(3)} = 421 \pm 30\ \text{MeV}\), which corresponds to \(\alpha_s(m_Z^2) = 0.356 \pm 0.017\) and \(\alpha_s(m_Z^2) = 0.1226 \pm 0.0018\). For comparison, using the MS scheme we obtain \(\Lambda_{\text{MS}}^{(3)} = 441 \pm 32\ \text{MeV}\) and \(\delta_{(6)V}^{00} = 0.0147 \pm 0.0025\), which corresponds to \(\alpha_s(m_Z^2) = 0.367 \pm 0.018\) and \(\alpha_s(m_Z^2) = 0.1238 \pm 0.0018\).

In order to make our calculations more generally useful we show in Fig. 1 the results of the NNLO fit of \(\alpha_s(m_{\tau}^2)\) and \(\delta_{(6)V}^{00}\), obtained using the PMS predictions, as a function of the experimental values of \(R_{\tau,V}^{00}\) and \(D_{\tau,V}^{12}\).

![Figure 1](image1.png)

Figure 1. Plot of the fitted values of \(\alpha_s(m_{\tau}^2)\) and \(\delta_{(6)V}^{00}\) in the vector channel as a function of \(R_{\tau,V}^{00}\) and \(D_{\tau,V}^{12}\), obtained using the NNLO PMS predictions. The dashed lines indicate the change in the plot when the MS NNLO predictions are used instead.

In Fig. 2 we show how the value of \(\alpha_s(m_{\tau}^2)\) resulting from the fit depends on the parameters \(r_1\) and \(c_2\) specifying the renormalization scheme in NNLO. By varying the RS parameters in the region satisfying the condition \(8\) with \(l = 2\) we obtain the variation \(0.347 < \alpha_s(m_{\tau}^2) < 0.367\) \((403\ \text{MeV} < \Lambda_{\text{MS}}^{(3)} < 440\ \text{MeV}, 0.1217 < \alpha_s(m_Z^2) < 0.1238)\). Analyzing in a similar way the RS dependence of \(\delta_{(6)V}^{00}\) we obtain \(0.0145 < \delta_{(6)V}^{00} < 0.0182\).

![Figure 2](image2.png)

Figure 2. The contour plot of the fitted value of \(\alpha_s(m_{\tau}^2)\) in the vector channel as a function of the RS parameters \(r_1\) and \(c_2\). The region of the scheme parameters satisfying the condition \(8\) with \(l = 2\) has been also indicated.

It is of some interest to perform the same fits using instead the NLO predictions. (The PMS parameters in NLO are \(r_1 = -0.76\) for \(\delta_{pt}^{00}\) and \(r_1 = -0.64\) for \(\delta_{pt}^{12}\)). Using the NLO predictions in the PMS scheme we obtain \(\delta_{(6)V}^{00} = 0.0150\) and \(\Lambda_{\text{MS}}^{(3)} = 465\ \text{MeV}\). Using the NLO predictions in the \(\overline{\text{MS}}\) scheme, we obtain \(\delta_{(6)V}^{00} = 0.0148\) and...
$\Lambda^{(3)}_{\overline{MS}} = 527 \text{ MeV.}$

Similar fits may be performed in the axial-vector channel. We use the experimental values reported recently by ALEPH [4]:

$$R^{00}_{\tau,A} = 1.711 \pm 0.019, \quad D^{12}_{\tau,A} = 0.0639 \pm 0.0005.$$  

Using the PMS scheme we obtain in NNLO

$$\delta^{00}_{(6)A} = -0.0165 \pm 0.0018 \quad \text{and} \quad \Lambda^{(3)}_{\overline{MS}} = 380 \pm 34 \text{ MeV},$$

which corresponds to $\alpha_s(m_Z^2) = 0.335 \pm 0.018$ and $\alpha_s(m_\tau^2) = 0.1203 \pm 0.0021$. For comparison, using the $\overline{MS}$ scheme we obtain

$$\delta^{00}_{(6)A} = -0.0168 \pm 0.0021 \quad \text{and} \quad \Lambda^{(3)}_{\overline{MS}} = 398 \pm 37 \text{ MeV},$$

which corresponds to $\alpha_s(m_Z^2) = 0.344 \pm 0.019$ and $\alpha_s(m_\tau^2) = 0.1213 \pm 0.0021$.

Performing the variation of the RS parameters in the region satisfying the condition $[8]$ with $l = 2$ we obtain for the axial-vector channel

$$0.326 < \alpha_s(m_Z^2) < 0.343 \quad \text{and} \quad 397 \text{ MeV} < \Lambda^{(3)}_{\overline{MS}} < 397 \text{ MeV},$$

$$0.1193 < \alpha_s(m_\tau^2) < 0.1212 \quad \text{and} \quad 0.015 < -\delta^{00}_{(6)A} < 0.017.$$

Summarizing, we find that in NNLO the change from the $\overline{MS}$ scheme to the PMS scheme results in

the reduction of the fitted value of $\alpha_s(m_\tau^2)$ by approximately 0.01. ($\alpha_s(m_Z^2)$ is reduced by 0.001.) Also, the difference between the NLO and NNLO results is much smaller in the PMS scheme than in the $\overline{MS}$ scheme. The extracted values of the QCD parameters appear to be relatively stable with respect to change of the RS. However, varying the scheme parameters $r_1$ and $c_2$ in the region satisfying the condition $[8]$ with $l = 2$ we obtain an uncertainty in the extracted value of $\alpha_s(m_\tau^2)$ of approximately 0.02 (uncertainty in $\alpha_s(m_Z^2)$ is 0.002).

More details about the calculations and the results reported above may be found in [4].

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