Quantum information in quantum cognition: Framework and applications

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Matthew Fisher recently postulated a mechanism by which quantum phenomena could influence cognition: Phosphorous nuclear spins may resist decoherence for long times. The spins would serve as biological qubits. The qubits may resist decoherence longer when in Posner molecules. We imagine that Fisher postulates correctly. How adroitly could biological systems process quantum information (QI)? We establish a framework for answering. Additionally, we construct applications of biological qubits to quantum error correction, quantum communication, and quantum computation. First, we posit how the QI encoded by the spins transforms as Posner molecules form. The transformation points to a natural computational basis for qubits in Posner molecules. From the basis, we construct a quantum code that detects arbitrary single-qubit errors. Each molecule encodes one qubit. Shifting from information storage to computation, we define the model of Posner quantum computation. To illustrate the model’s quantum-communication ability, we show how it can teleport information incoherently: A state’s weights are teleported. Dephasing results from the entangling operation’s simulation of a coarse-grained Bell measurement. Whether Posner quantum computation is universal remains an open question. However, the model’s operations can efficiently prepare a Posner state usable as a resource in universal measurement-based quantum computation. The state results from deforming the Affleck-Lieb-Kennedy-Tasaki (AKLT) state and is a projected entangled-pair state (PEPS). Finally, we show that entanglement can affect molecular-binding rates, boosting a binding probability from 33.6% to 100% in an example. This work opens the door for the QI-theoretic analysis of biological qubits and Posner molecules.

Fisher recently proposed a mechanism by which quantum phenomena might affect cognition [1]. Phosphorus atoms populate biochemistry. A phosphorus nucleus’s spin, he argued, can store quantum information (QI) for long times. The nucleus has a spin quantum number $s = \frac{1}{2}$. Hence the nucleus forms a qubit, a quantum two-level system. The qubit is the standard unit of QI.

Fisher postulated physical processes that might entangle phosphorus nuclei. Six phosphorus atoms might, with other ions, form Posner molecules, Ca$_9$(PO$_4$)$_6$[2,4]. The molecules might protect the spins’ states for long times. Fisher also described how the QI stored in the spins might be read out. This QI, he conjectured, could impact neuron firing. The neurons could participate in quantum cognition.

These conjectures require empirical testing. Fisher has proposed experiments [1], including with Radzihovsky [5]. Some experiments have begun [6].

Suppose that Fisher conjectures correctly. How effectively could the spins process QI? We provide a framework for answering this question, and we begin answering. We translate Fisher’s physics and chemistry into information theory. The language of molecular binding, heat dissipation, etc., is replaced with the formalism of positive operator-valued measures (POVMs), computational bases, etc. Additionally, we identify and quantify QI-storage, -communication, and -computation capacities of the phosphorus nuclear spins and Posners.

This paper is intended for QI scientists, for chemists, and for biophysicists. Some readers may require background about QI theory. They may wish to consult App. [A] and [7,8]. Next, we overview this paper’s contributions.

**Computational bases before and after molecule formation:** Phosphorus nuclear spins originate outside Posners, in Fisher’s narrative. The spins occupy phosphate ions that join together to form Posners. Molecular formation changes how QI is encoded physically.

Outside of molecules, phosphorus nuclear spins couple little to orbital degrees of freedom (DOFs). Spin states form an obvious choice of computational basis [2]. In a Posner molecule, the nuclei are indistinguishable. They occupy a totally antisymmetric state [4,5]: The spins entangle with orbital DOFs. Which physical states form a useful computational basis is not obvious.

We identify such a basis. Molecule formation, we posit further, maps premolecule spin states to antisymmetric molecule states deterministically. The premolecule orbital state determines the map. We formalize the map with a projector-valued measure (PVM). The mapped-to antisymmetric states form the computational basis, in terms of which Posners’ QI processing can be expressed cleanly.

**Quantum error-correcting and -detecting codes:** The basis elements may decohere quickly: Pos-

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1 Ca$_9$(PO$_4$)$_6$ has been called the Posner cluster and Posner molecule. We call it the Posner, for short.

2 In QI, computations are expressed in terms of a computational basis for the system’s Hilbert space [4]. Basis elements are often represented by bit strings, as in \{00...0, 00...01, ..., 11...1\}.
ners’ geometry protects only spins. The basis elements are spin-and-position entangled states. Do the dynamics protect any states against errors?

Hamiltonians’ ground spaces may form quantum error-correcting and -detecting codes (QEC codes) \[3\]. One might hope to relate the Posner Hamiltonian \( H_{\text{Pos}} \) to a QEC code. \( H_{\text{Pos}} \) was characterized shortly after the present paper’s initial release \[3\]. Even without knowing the form of \( H_{\text{Pos}} \), however, one can construct QEC codes that respect charges expected to be conserved.

\( H_{\text{Pos}} \) likely preserves two observables. One, \( G_C \), generates cyclic permutations of the spins. One such permutation shuffles the spins about the molecule’s symmetry axis, through an angle \( 2\pi/3 \). This permutation preserves the Posner’s geometry \[4\]. The other charge, \( S^z_{1-6} \), is the spins’ total \( z \)-component relative to the lab frame.

The dynamics likely preserve eigenstates shared by \( G_C \) and \( S^z_{1-6} \). Yet \( G_C \) shares many eigenbases with \( S^z_{1-6} \): The charges fail to form a complete set of commuting observables (CSGO). We identify a useful operator to break the degeneracy: \( S^z_{123} \otimes S^z_{156} \) equals a product of the spin-squared operators \( S^2 \) of trios of a Posner’s spins. This operator (i) respects the Posner’s geometry and (ii) facilitates the construction of Posner states that can fuel universal quantum computation (discussed below).

From the eigenbasis shared by \( G_C \), \( S^z_{1-6} \), and \( S^z_{123} \otimes S^z_{156} \) we form QEC codes. A state \( \ket{\psi} \) in one charge sector of \( G_C \) and one sector of \( S^z_{1-6} \) likely cannot transform, under the dynamics, into a state \( \ket{\phi} \) in a second sector of \( G_C \) and a second sector of \( S^z_{1-6} \). Hence \( \ket{\psi} \) and \( \ket{\phi} \) suggest themselves as codewords. Charge preservation would prevent one codeword from evolving into another.

We construct two quantum codes, each partially protected by charge preservation. Via one code, each Posner encodes one qubit. The codewords correspond to distinct eigenvalues of \( G_C \). This code detects arbitrary single-qubit errors. Via the second code, each Posner encodes one qubit. This repetition code corrects two bit flips. The codewords correspond to distinct eigenvalues of \( S^z_{1-6} \).

**Model of Posner quantum computation:** Fisher posits physical processes, such as binding, that Posners may undergo \[4\]. We abstract away the physics, formalizing the computations effected by the processes. These effects Fisher operations form the model of Posner quantum computation.

The model includes the preparation of singlets, \( \frac{1}{\sqrt{2}} \left( \ket{01} - \ket{10} \right) \). The logical state evolves trivially, under the identity \( \mathbb{1} \), when Posners form. But Posner creation associates a heptoctet of qubits with a geometry and with an observable \( G_C \).

A Posner’s six qubits may rotate through angles of up to \( \pi \), though typical angles are expected to be much smaller. Also, measurements can be performed: \( G_C \) has eigenvalues \( \tau = 0, \pm 1 \). Whether Posnors \( A \) and \( B \) satisfy \( \tau_A + \tau_B = 0 \) can be measured projectively. If the equation is satisfied, the twelve qubits can rotate jointly.

Finally, hextuples or to \( G_C \)’s, as Posners break down into their constituent ions. Thereafter, qubits can group together into new hextuples. This model enables us to recast Fisher’s narrative \[1\] as a quantum circuit.

**Entanglement generated by, and quantum communication with, molecular binding:** Two Posners, Fisher conjectures, can bind together \[1\]. Quantum-chemistry calculations support the conjecture \[3\]. The binding is expected to entangle the Posners \[1\]. How much entanglement does binding generate, and entanglement of what sort?

We characterize the entanglement in two ways. First, we compare Posner binding to a Bell measurement \[7\]. A Bell measurement yields one of four possible outcomes—two bits of information. Posner binding transforms a subspace as a coarse-grained Bell measurement. A Bell measurement is performed, and one bit is discarded, effectively.

Second, we present a quantum-communication protocol reliant on Posner binding. We define a qudit (three-level) subspace of the Posner Hilbert space. A Posner \( P \) may occupy a state \( \ket{\psi} = \sum_j c_j \ket{j} \) in the subspace. The coefficients \( |c_j|^2 \) form a probability distribution \( Q \). This distribution has a probability \( p \) of being teleported to another Posner, \( P' \). Another distribution, \( Q \), consists of combinations of the \( |c_j|^2 \)'s. \( Q \) has a probability \( 1 - p \) of being teleported. Measuring \( P' \) in the right basis would yield an outcome distributed according to \( Q \) or according to \( Q \). A random variable is teleported, though \( P \) never interacts with \( P' \) directly.

The weights of \( \ket{\psi} \) (or combinations of the weights) are teleported \[10\]. The coherences are not. We therefore dub the protocol incoherent teleportation. The dephasing comes from the binding’s simulation of a coarse-grained Bell measurement. Bell measurements teleport QI coherently.

Incoherent teleportation effects a variant of superdense coding \[11\]. A trit (a classical three-level system) is communicated effectively, while a bit is communicated directly. The trit is encoded superdensely in the bit, with help from Posner binding.

**Posner-molecule state that can serve as a universal resource for measurement-based quantum computation:** Measurement-based quantum computation (MBQC) \[12\] \[13\] is a quantum-computation model alternative to the circuit model \[13\]. MBQC begins with a many-body entangled state \( \ket{\psi} \). Single qubits are measured adaptively.

MBQC can efficiently simulate universal quantum computation if begun with the right \( \ket{\psi} \). Most quantum states cannot serve as universal resources \[13\]. Cluster states \[12\] \[17\] \[18\] on 2D square lattices can \[12\] \[13\] \[19\] \[20\]. So can the Affleck-Kennedy-Lieb-Tasaki (AKLT) state \[21\] \[22\] on a honeycomb lattice, \( \ket{\text{AKLT}_{\text{hon}}} \). Local measurements can transform \( \ket{\text{AKLT}_{\text{hon}}} \) into the universal cluster state. Hence \( \ket{\text{AKLT}_{\text{hon}}} \) can fuel universal MBQC \[20\] \[24\].
We define a variation $|\text{AKLT}_{\text{hon}}\rangle$ on $|\text{AKLT}_{\text{hon}}\rangle$. $|\text{AKLT}_{\text{hon}}\rangle$ can be prepared efficiently with Posner operations. Preparing $|\text{AKLT}_{\text{hon}}\rangle$, one projects onto a spin-$\frac{3}{2}$ subspace. Preparing $|\text{AKLT}_{\text{hon}}\rangle$, one projects onto a larger subspace. Local measurements (supplemented by Posner hydrolyzation, singlet formation, and Posner creation) can transform $|\text{AKLT}_{\text{hon}}\rangle$ into the universal cluster state. Hence $|\text{AKLT}_{\text{hon}}\rangle$ can fuel universal MBQC as $|\text{AKLT}_{\text{hon}}\rangle$ can.

Whether Posner operations can implement the extra local measurements, or the adaptive measurements in MBQC, remains an open question. Yet the universality of a Posner state, efficiently preparable by a (conjectured) biological system, is remarkable. Most states cannot fuel universal MBQC\cite{16}. The universality of $|\text{AKLT}_{\text{hon}}\rangle$ follows from (i) Posners’ geometry and (ii) their ability to share singlets.

Like $|\text{AKLT}_{\text{hon}}\rangle$, $|\text{AKLT}_{\text{hon}}\rangle$ is a projected entangled-pair state (PEPS)\cite{25}. The state is formed from two basic tensors. Each tensor has three physical qubits and three virtual legs. One virtual leg has bond dimension six. Each other virtual leg has bond dimension two. $|\text{AKLT}_{\text{hon}}\rangle$ is the unique ground state of some frustration-free Hamiltonian $H_{\text{AKLT}}$\cite{26,27}. The relationship between $H_{\text{AKLT}}$ and $H_{\text{Pos}}$ remains an open question. So does whether $H_{\text{AKLT}}$ has a constant-size gap.

**Entanglement’s influence on binding probabilities:** Entanglement, Fisher conjectures, can affect Posners’ probability of binding together\cite{1}. He imagined a Posner $A$ entangled with a Posner $A'$ and a $B$ entangled with a $B'$. Suppose that $A$ has bound to $B$. $A'$ more likely binds to $B'$, Fisher argues, than in the absence of entanglement.

We formulate a scheme for quantifying entanglement’s influence on binding probabilities. Two Posners, $A$ and $B$, illustrate the scheme. First, we suppose that the pair contains no singlets. Then, we raise the number of singlets incrementally. In the final case, $A$ and $B$ are maximally entangled. The binding probability rises from 33.6\% to 100\%. Our technique can be scaled up to Fisher’s four-Posner example\cite{1} and to clouds of entangled Posners.

**Comparison with DiVincenzo’s criteria:** DiVincenzo codified the criteria required for realizing quantum computation and communication\cite{25}. We compare the criteria with Fisher’s narrative. At least most criteria are satisfied, if sufficient control is available. Whether the gate set is universal remains an open question.

**Organization of this paper:** Section \[I\] reviews Fisher’s proposal. Section \[II\] details the physical set-up and models Posner creation. How Posner creation changes the physical encoding of QI appears in Sec. \[III\]. QECD codes are presented in Sec. \[IV\].

The model of Posner quantum computation is defined in Sec. \[V\]. Posner binding is analyzed, and applied to incoherent teleportation, in Sec. \[VI\]. Section \[VII\] showcases the universal resource state $|\text{AKLT}_{\text{hon}}\rangle$.

Section \[VII\] quantifies entanglement’s effect on molecular-binding probabilities. Quantum cognition is compared with DiVincenzo’s criteria in Sec. \[IX\]. Opportunities for further study are detailed in Sec. \[X\].

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\[1\] We focus on Fisher’s quantum-cognition proposal\cite{1}. Alternative proposals appear in, e.g.,\cite{29,30}.

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\[I.\] **REVIEW: FISHER’S QUANTUM-COGNITION PROPOSAL**

Biological systems are warm, wet, and large\cite{4}. Such environments quickly diminish quantum coherences. Fisher catalogued the influences that could decohere nuclear spins in biofluids. Examples include electric and magnetic fields generated by other nuclear spins and by electrons.

These sources, Fisher estimated, decohere the phosphorus-31 ($^{31}\text{P}$) nuclear spin slowly. Coherence times might reach $\sim 1$ s, if the phosphorus occupies a free-floating phosphate ion, or $10^6 - 10^9$ s, if the phosphorus occupies a Posner. No other biologically prevalent atom, Fisher conjectures, has such a long-lived nuclear spin.

Phosphorus atoms inhabit many biological ions and molecules. Examples include the phosphate ion, $\text{PO}_4^{3-}$. Three phosphates feature in the molecule *adenosine triphosphate* (ATP). ATP stores energy that powers chemical reactions. Two phosphates can detach from an ATP molecule, forming a *diphosphate* ion.

A diphosphate can break into two phosphates, with help from the enzyme *pyrophosphatase*. The two phosphates’ phosphorus nuclear spins form a *singlet*, Fisher and Radzihovsky (F&R) conjecture\cite{1,5}. A singlet is a maximally entangled state. Entanglement is a correlation, shareable by quantum systems, stronger than any achievable by classical systems\cite{7}.

Many biomolecules contain phosphate ions. Occupying a small molecule, Fisher argues, could shield the phosphorus nuclear spin: Entanglement with other particles could decohere the spin. Dipole-dipole interactions with external protons threaten the spin most. But protons and small molecules tumble around each other in fluids. The potential experienced by the phosphorus spin is expected to average to zero.

Which small biomolecules could a phosphorus inhabit? An important candidate is Ca$_9$(PO$_4$)$_6$. A Posner consists of six phosphate ions ($\text{PO}_4^{3-}$) and nine calcium ions ($\text{Ca}^{2+}$)\cite{2,4}. Posners form in simulated biofluids and might form in vivo\cite{32,54}. A Posner could contain a phosphate that forms a singlet with a phosphate in another Posner. The Posners would share entanglement.

Two Posners can bind together, according to quantum-chemistry calculations\cite{1,9}. The binding projects the Posners onto a possibly entangled state. Moreover, pre-
existing entanglement could affect the probability that Posners bind.

Bound-together Posners move slowly, Fisher argues. Compound molecules must displace many water molecules, which slow down the pair. Relatedly, the Posner pair has a large moment of inertia. Hence the pair rotates more slowly than separated Posners by the conservation of angular momentum.

Hydrogen ions $\text{H}^+$ can attach easily to slow molecules, Fisher expects. $\text{H}^+$ hydrolyzes Posners, breaking the molecules into their constituent ions. Hence entanglement might correlate hydrolyzation of $A$ and $B$ with hydrolyzation of $A'$ and $B'$. Hydrolyzation would release $\text{Ca}^{2+}$ ions into the neurons. Suppose that many entangled Posners hydrolyzed in these two neurons. The neurons' $\text{Ca}^{2+}$ concentrations could rise. The neurons could fire coordinately due to entanglement.

II. PHYSICAL SET-UP AND POSNER-MOLECULE CREATION

This section concerns (i) the physical set-up and (ii) the joining together of phosphates and calcium ions in Posner molecules. Part of the material appears in [1–4] and is reviewed. Part of the material has not, according to our knowledge, appeared elsewhere.

The phosphorus nuclei are associated with spin and spatial Hilbert spaces in Sec. II A. Section II B reviews, and introduces notation for, the Posner’s geometry. Section II C models the creation of a Posner from close-together ions.

II A. Spin and spatial Hilbert spaces

Each phosphorus nucleus has two relevant DOFs: a spin and a position. We will sometimes call the position the orbital or spatial DOF. Let $\mathcal{H}_\text{spin}^\text{nuc}$ and $\mathcal{H}_\text{orb}^\text{nuc}$ denote the associated Hilbert spaces. The nucleus has a spin quantum number $s = \frac{1}{2}$. Hence $\mathcal{H}_\text{spin}^\text{nuc} = \mathbb{C}^2$. The orbital Hilbert space is infinite-dimensional: $\text{dim}(\mathcal{H}_\text{orb}^\text{nuc}) = \infty$. Each phosphorus nucleus’s Hilbert space decomposes as $\mathcal{H}_\text{nuc} = \mathcal{H}_\text{spin}^\text{nuc} \otimes \mathcal{H}_\text{orb}^\text{nuc}$.

The electrons’ states transform trivially under all relevant operations, Fisher and Radzihovsky (F&R) conjecture [5]. We therefore ignore the electronic DOFs. We ignore calcium ions similarly. We focus on the DOFs that might store QI for long times.

II B. Posner-molecule geometry and notation

Quantum-chemistry calculations have shed light on the shapes available to Posners [2–4, 9]. A Posner’s shape depends on the environment. Posners in biofluids have begun to be studied [4]. We follow [1–5, 9], supposing that more-detailed studies will support [4, 9].

The Posner forms a cube (Fig. 1). At each face’s center sits a phosphate. The Posner lacks cubic symmetry, due to the tetrahedral phosphates’ orientations. But (a stable proposed configuration of) the Posner retains $S_6$ symmetry and one $C_3$ symmetry. The $C_3$ symmetry is an invariance under $2\pi/3$ rotations about a cube diagonal.

This cube diagonal serves as the $z$-axis of a reference frame fixed in the molecule. The atoms’ positions remain constant relative to this internal frame. The internal frame can move relative to the lab frame, denoted by the subscript “lab.” The spins’ Bloch vectors are defined with respect to the lab frame.

Imagine gazing down the diagonal, as in Fig 1a. You would see two triangles whose vertices consisted of phosphates. The triangles would occupy parallel planes pierced by the $z_{\text{in}}$-axis. The three black dots in Fig. 1b represent the phosphates closest to you. We label this trio’s $z_{\text{in}}$-coordinate by $h_+$. Farther back, at $z_{\text{in}} = h_-$, sits the trio represented by gray dots. $z_{\text{in}}$ points oppositely the direction in which we imagined gazing, such that $h_+ > h_-$.

$\phi$ labels the triangles’ orientation, as shown in Fig. 11. We denote by $\varphi_j$ the angular orientation of cube face $j$ (the site of a phosphate): Consider a top-triangle face $j$, at $z_{\text{in}} = h_+$. Imagine rotating the $x_{\text{in}}$-axis counterclockwise until it intersects a phosphate. The angle swept out is $\varphi_j$. One $h_+$ phosphate’s $\varphi_j = \phi$, another’s $\varphi_j = \phi + 2\pi/3$, and another’s $\varphi_j = \phi + 4\pi/3$. Now, consider the triangle at $z_{\text{in}} = h_-$. Each phosphate sits at an angle $\varphi_j + \pi/4$, for $\varphi_j = \phi, \phi + 2\pi/3, \phi + 4\pi/3$. We label the site of phosphate $j$ with an angle and a height: $(\varphi_j, h_j)$.

II C. Qualitative model for the creation of a Posner molecule

Posners form from phosphate and calcium ions. We propose a qualitative model for the formation process. We first review how, according to Fisher, phosphorus nuclear spins might come to form singlets. We then envision phosphates falling into a Lennard-Jones potential as a Posner forms. F&R have discussed the indistinguishability of phosphorus nuclei in a Posner [1–4, 9]. We expand upon this discussion, considering how distinguishable ions become indistinguishable.

4 Reference [9], released shortly after this paper’s initial release, supports [4].
V(x)

FIG. 2: Lennard-Jones potential: The Lennard-Jones potential, \( V_{LJ}(x) = \frac{a}{x^6} - \frac{b}{x^12} \), models van der Waals forces between particles. The real parameters \( a, b > 0 \). We approximate qualitatively, with \( V_{LJ}(x) \), the potential experienced by phosphate ions coalescing into a Posner molecule. \( x \) denotes the distance from a phosphate to the system’s center of mass.

\( \hat{z}_{\text{enz}} \) denote the \( z \)-axis of a reference frame fixed in the enzyme. Let \( \hat{S}_{\text{enz}} \) denote the \( \hat{z}_{\text{enz}} \)-component of a phosphorus nucleus’s spin operator. Let \( |\uparrow\rangle \) and \( |\downarrow\rangle \) denote the \( \hat{S}_{\text{enz}} \) eigenstates: \( \hat{S}_{\text{enz}} |\uparrow\rangle = \frac{\hbar}{2} |\uparrow\rangle \), and \( \hat{S}_{\text{enz}} |\downarrow\rangle = -\frac{\hbar}{2} |\uparrow\rangle \). The singlet has the form

\[
|\Psi^-\rangle := \frac{1}{\sqrt{2}} (|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle).
\]

The singlet is one of the four Bell pairs. The Bell pairs are mutually orthogonal, maximally entangled states of pairs of qubits \([7]\). Bell pairs serve as units of entanglement in QI.

Phosphorus nuclei are identical fermions, as F&R emphasize \([15]\). But some of the nuclei’s DOFs might be distinguishable before Posners form. Consider, for example, two ATP molecules on opposite sides of a petri dish. Call the molecules \( A \) and \( B \). A diphosphate could break off from each ATP molecule. Each diphosphate could hydrolyze into two phosphates, \( A_1 \) and \( A_2 \) or \( B_1 \) and \( B_2 \). Consider the phosphorus nuclear spins of one phosphate pair—say, of \( A_1 \) and \( A_2 \). These spins would be indistinguishable: Neither nucleus could be associated with an upward-pointing spin or with a downward-pointing spin.

But the spatial DOF of \( A_1 \) and \( A_2 \) could be distinguished from the spatial DOF of \( B_1 \) and \( B_2 \): We can imagine painting phosphate pair \( A \) red and phosphate pair \( B \) blue. The phosphate pairs could diffuse to the dish’s center. The red pair and the blue pair could be tracked along their trajectories.

Consider six phosphates (and nine Ca\(^{2+}\) ions) approaching each other. We model the ions qualitatively as subject to a Lennard-Jones potential, as in molecular-dynamics simulations \([33]\). The model encapsulates interatomic forces’ key features.

We temporarily approximate each phosphate as having a classical position. Let \( x \) denote some phosphate’s distance from the ions’ center of mass. Figure 2 illustrates
the Lennard-Jones potential,
\[ V_{LJ}(x) = \frac{a}{x^{12}} - \frac{b}{x^6}. \] (2)
The real parameters \( a, b > 0. \)

The potential has two limiting behaviors. The behaviors dominate opposite sides of the point which the derivative vanishes: \( \frac{dV_{LJ}(x)}{dx} = 0 \Rightarrow x = \left(\frac{2a}{b}\right)^{1/6} = x_0. \) At large distances \( x \gg x_0, \) the negative term in Eq. (2) dominates. \( V_{LJ}(x) \) attracts; far-apart ions approach each other. At short distances \( x \ll x_0, \) the positive term dominates. \( V_{LJ}(x) \) repels; the ions cannot coincide.

Consider an ion approaching \( x = 0 \) from afar. \( V_{LJ}(x) \) drops precipitously when the concavity changes from negative to positive: \( \frac{d^2V_{LJ}(x)}{dx^2} = 0 \Rightarrow x = \left(\frac{2a}{b}\right)^{1/6}. \) This point forms a “lip” of the potential. The ions have more energy, separated, than they would have in a molecule. The ions slide down the potential well, releasing binding energy as heat. The heat disrupts the environment, which effectively measures the ions’ state.

At the well’s bottom, the ions constitute a Posner molecule. The phosphorus nuclei’s quantum states have position representations (wave functions) that overlap significantly. The nuclei are indistinguishable \( [5] \): No nuclear pair can be identified as red-painted or as blue-painted. The six phosphorus nuclei occupy a totally antisymmetric spin-and-spatial state. We will abbreviate “totally antisymmetric” as “antisymmetric.”

II D. Formalizing the model for Posner-molecule creation

Let us model, with mathematical tools of QI, the environment’s measuring of the ions, the creation of a Posner, and the antisymmetrization process. Let \( t_{\text{Pos}} \) denote the scale of the time over which the ions slide down the Lennard-Jones well from the lip, emit heat, jostle about, and settle into the Posner geometry.

The environment effectively measures the ions with a frequency \( 1/t_{\text{Pos}}. \) We model the measurement with a *projector-valued measure* (PVM) \( [7] \). Consider the Hilbert space \( (\mathcal{H}_{\text{nuc}})^\otimes 6 \) of the Posner’s six phosphorus nuclei. An antisymmetric subspace \( \mathcal{H}_{\text{no-coll}} \) consists of the states available to the indistinguishable nuclei. (The states are detailed in Sec. [III C]) The subscript stands for “no-colliding-nuclei”. No two nuclei can inhabit the same Posner-cube face.

Let \( \Pi_{\text{no-coll}}^{\text{no-coll}} \) denote the projector onto \( \mathcal{H}_{\text{no-coll}}. \) The PVM has the form
\[ \{ \Pi_{\text{no-coll}}^{\text{no-coll}}, \mathbb{I} - \Pi_{\text{no-coll}}^{\text{no-coll}} \}. \] (3)

One might try to model the environment as measuring the ions continuously. This model is unfaithful: The environment would continuously project the ions onto states inaccessible to a Posner. No Posner could form, due to the quantum Zeno effect \( [36] \). The Posner-creation time \( t_{\text{Pos}} \) sets the measurement’s time scale.

F&R suggest that, upon forming, a molecule is entangled with its environment \( [5] \). Our PVM is consistent with F&R’s model, by the principle of deferred measurement \( [7] \). Let \( S \) denote a general quantum system. A measurement of \( S \) consists of two steps: First, \( S \) is entangled with a memory \( M. \) Second, \( M \) is measured. Suppose that (i) the entanglement is maximal and (ii) the measurement is projective. The measurement projects the system’s state. Suppose that \( S \) evolves after the measurement. This entangling, \( M \) measurement, and evolution is equivalent to the entangling, followed by the \( S \) evolution, followed by the \( M \) measurement. The \( M \) measurement can be deferred until after the evolution. Deferral fails to alter the measurement statistics. Let \( S \) denote the nuclei, and let \( M \) denote the environment. The \( M \) measurement is deferred in F&R’s model, not in ours. The models are equivalent, by the deferred-measurement principle.

III. ENCODED STATES AND THEIR CHANGING PHYSICAL REPRESENTATIONS

Phosphorus nuclear spins cleanly encode QI before Posners form. The spins, Fisher conjectures, are découpled from the nuclei’s positions \( [1] \). Posner creation antisymmetrizes the spin-and-orbital state. The spins become entangled with the positions, no longer encoding QI cleanly.
But, we posit, Posner creation maps each pre-Posner spin state to an antisymmetric Posner state deterministically. Posner creation preserves QI but changes how QI is encoded physically. Hence spin configurations can label a computational basis for the Posner Hilbert space, e.g., \( \uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\equiv 000000 \).

This section is organized as follows. Section III A concerns premolecule phosphorus nuclear spins. Section III B reviews the formalism of encodings. A map between (i) physical states of pre-Posner spins and (ii) logical states is formalized. Logical states are mapped to Posner states in Sec. III C.

### III A. Physical encoding of quantum information in the phosphorus nuclei that will form a Posner molecule

Consider six phosphates that approach each other, soon to form (with Ca\(^{2+}\) ions) a Posner. We index the phosphorus nuclei as \( a = 1, 2, \ldots, 6 \). Each nucleus has a spin DOF and an orbital DOF. Nucleus \( a \) occupies some quantum state \( \rho_a \in \mathcal{D}(\mathcal{H}_{\text{nuc}}) \). \( \mathcal{D}(\mathcal{H}) \) denotes the set of density operators (trace-one linear operators) defined on the Hilbert space \( \mathcal{H} \), \( \rho_a \) may be pure (unentangled with any external DOFs) or mixed (entangled with external DOFs, e.g., another phosphorus nucleus’s spin).

Tracing out the orbital DOF yields the reduced spin state: \( \rho_a^{\text{spin}} := \text{Tr}_{\text{orb}}(\rho_a) \in \mathcal{D}(\mathcal{C}^2) \). The magnetic spin quantum number \( m_a = \pm \frac{1}{2} \) quantifies the spin’s z-direction component.

Shifting focus from chemistry to information theory, we adopt QI notation: We usually omit hats from operators, and we often omit factors of \( \hbar \) and \( \frac{1}{2} \). We often replace the spin operator’s \( \alpha \)-component with the Pauli \( \alpha \)-operator, for \( \alpha = x, y, z \): \( S^\alpha \equiv S^\alpha = \frac{\sqrt{2}}{2} \sigma^\alpha = \sigma^\alpha \). The \( \sigma^z \) eigenstates are often labeled as \( |0\rangle := |\uparrow\rangle \) and \( |1\rangle := |\downarrow\rangle \).

Tracing out the spin DOF from \( \rho_a \) yields the reduced orbital state: \( \rho_a^{\text{orb}} := \text{Tr}_{\text{spin}}(\rho_a) \in \mathcal{D}(\mathcal{H}_{\text{nuc}}^{2}) \). We parameterize \( \mathcal{H}_{\text{nuc}}^{2} \) with the eigenstates \( |x\rangle \) of the position operator, \( \mathbf{x} \). The coordinates are defined with respect to the lab frame. \( \{|x\rangle\} \) forms a continuous set.

The spin and/or orbital DOFs can store QI. But water and other molecules buffet the phosphates. An independent phosphate’s position is expected to be mixed. The spin, in contrast, is expected to remain coherent for long times. (See Sec. IX and II.) The spins encode protected QI.

The nuclear spins form six qubits. The qubits correspond to the Hilbert space \( \mathcal{H}_{\text{nuc}}^{\text{spin}} \otimes \mathcal{C}^{12} \), which has dimensionality \( 2^6 = 64 \). A useful basis for \( \mathcal{H}_{\text{nuc}}^{\text{spin}} \) consists of tensor products of \( \sigma^z \) eigenstates: \( \mathcal{B}_{\text{comp}} := \{ |0, 0, \ldots, 0\rangle, |0, 0, \ldots, 1\rangle, \ldots, |1, 1, \ldots, 1\rangle \} \). The notation \( |A, B, \ldots, K\rangle \equiv |A\rangle \otimes |B\rangle \otimes \cdots \otimes |K\rangle \). The set \( \mathcal{B}_{\text{comp}} \) constitutes the computational basis for the physical states.

Consider \( N \) hextuples of phosphates (\( N \) sets of six phosphates). The phosphorus nuclei correspond to a spin space \( \mathcal{C}^{6N} \). We suppose, without loss of generality, that the \( 6N \) spins occupy a pure joint state \( |\psi\rangle \). Each hextuple could contain three singlets, for example. Or a spin in some hextuple \( A \) could form a singlet with a spin in some hextuple \( B \).

### III B. Notation and quick review: Encodings

Imagine an agent Alice who wishes to send another agent, Bob, a message. A quantum message is a quantum state \( |\psi_L\rangle \in \mathcal{H}_L \), called the logical state. Let \( \mathcal{B}_\text{comp}^L \) denote a preferred basis for the Hilbert space \( \mathcal{H}_L \). Operations are expressed in terms of this computational basis for the logical space.

Alice must encode \( |\psi_L\rangle \) in the state of a physical system. The agents would choose a code, a dictionary between the computational basis \( \mathcal{B}_\text{comp}^L \) for the logical space and the computational basis \( \mathcal{B}_\text{comp}^L \) for the physical space. Alice would decompose \( |\psi_L\rangle \) in terms of \( \mathcal{B}_\text{comp}^L \) elements \( |j_L\rangle \); replace each \( |j_L\rangle \) with a \( \mathcal{B}_\text{comp}^L \) element \( |j\rangle \); and prepare the resultant physical state: \( |\psi\rangle = \sum_j c_j |\psi_L\rangle = \sum_j c_j |j_L\rangle = |\psi\rangle \).

\( \mathcal{H}_L \) cannot be arbitrarily large, if the encoding is faithful. A faithful encoding can be reversed, yielding the exact form of \( |\psi_L\rangle \). The six-qubit state \( |\psi\rangle \) can faithfully encode a \( |\psi_L\rangle \) of \( \leq 6 \) qubits, called logical qubits. The phosphorus nuclear spins—the physical DOFs that encode the logical qubits—are called physical qubits.

Suppose that \( |\psi_L\rangle \) is a state of six logical qubits. We label the logical space’s computational basis as \( \mathcal{B}_\text{comp}^L \equiv \{ |00\ldots0\rangle, |00\ldots01\rangle, \ldots, |11\ldots1\rangle \} \). A simple code from \( \mathcal{B}_\text{comp}^L \) to \( \mathcal{B}_\text{comp}^L \) has the form

\[
|m_1, \ldots, m_6\rangle \equiv |m_1 \ldots m_6\rangle,
\]

for \( m_1, \ldots, m_6 = 0, 1 \). For example, all six physical qubits’ pointing upward is equivalent to all six logical qubits’ pointing upward: \( |0, \ldots, 0\rangle = |0 \ldots 0\rangle \).

### III C. Transformation of the encoding during Posner-molecule creation

Consider six phosphates that join together, forming a Posner. The phosphorus nuclei might begin with distinguishable DOFs (Sec. II C). The spins entangle with each other and with orbital DOFs I I I I I. The QI \( |\psi_L\rangle \) stored in the spins “spills” into the orbital DOFs.

But, we posit, Posner creation maps each pre-Posner spin state to an antisymmetric Posner state deterministically. The physical qubits change from spins to spin- and orbital DOFs. The physical state’s form changes from \( |\psi\rangle \in \mathcal{C}^{12} \) to some \( |\psi'\rangle \in \mathcal{H}_{\text{no-coll.}} \). The Posner state \( |\psi'\rangle \) encodes \( |\psi_L\rangle \) faithfully.

Reparameterizing position will prove useful. We labeled by \( x \) a pre-Posner phosphorus nucleus’s position. A Posner’s phosphorus nuclei occupy the centers of cube
of spin quantum numbers equals the in-Posner nuclei’s set. But Posner creation prevents any particular m from corresponding, anymore, to any particular nucleus. The nuclei delocalize across the cube-face centers.

Let us mathematize this physics. The one-nucleus states (5) combine into the antisymmetric six-nucleus states

\[
\frac{1}{\sqrt{6!}} \sum_{\alpha=1}^{6!} 6! \prod_{\beta=1}^{6} (-1)^{\pi_\alpha} |m_{\pi_\alpha(j)}, r_{\pi_\alpha(j)}\rangle
\]

(7)

Each term contains a tensor product of six one-nucleus kets. Each ket is labeled by one tuple \((m_{\pi_\alpha(j)}, r_{\pi_\alpha(j)})\). No tuple equals any other tuple in the same term, by Pauli’s exclusion principle. Permuting one term’s six tuples yields another term, to within a minus sign.

\(\pi_\alpha\) denotes the \(\alpha^{th}\) term’s permutation. The permutation’s sign, \((-1)^{\pi_\alpha}\) parity of permutation, equals the term’s sign. \(\alpha\) \(=\) \((-1)^{\pi_\alpha}\) equals the term’s sign. The semicolon in Eq. (7) separates the \(h_+\) spins from the \(h_-\) spins. \(\alpha\) is equivalent to a Slater determinant.\[37\]

If not for the Posner’s geometry, two tuples could contain the same position variables. \(r_1\) could equal \(r_3\), for example, if \(m_1\) did not equal \(m_3\). But each cube face can house only one phosphate. The phosphorus nuclei’s state occupies the \(\text{no-colliding-nuclei subspace} \mathcal{H}_{\text{no-coll}}\). The antisymmetric subspace.

Posner creation, we posit, projects the nuclei’s state onto \(\mathcal{H}_{\text{no-coll}}\). The projector has the form

\[
\Pi_{\text{no-coll}}^{-} := \sum' \left| (m_1, r_1)(m_2, r_2)(m_3, r_3) ; (m_4, r_4)(m_5, r_5)(m_6, r_6) \right\rangle \left\langle (m_1, r_1)(m_2, r_2)(m_3, r_3) ; (m_4, r_4)(m_5, r_5)(m_6, r_6) \right|.
\]

(8)

The left-hand side (LHS) represents an element of the computational basis \(\mathcal{B}_{\text{comp}}\) for the space \(\mathcal{H}_{\text{no-coll}}\) of the pre-Posner physical qubits. The right-hand side (RHS) represents an element of the computational basis \(\mathcal{B}_{\text{comp}}^{\text{no-pos}}\) for the space \(\mathcal{H}_{\text{no-coll}}\) of the in-Posner physical qubits.

Each pre-Posner state consists of a unique assignment of \(m\)-values to nuclei, a unique distribution of six fixed \(m\)-values across six kets. Similarly, each Posner state consists of a unique assignment of \(m\)-values to positions, a unique distribution of six fixed \(m\)-values across six \(r\)-values. Sixty-four pre-Posner \(\mathcal{B}_{\text{comp}}\) states exist. Hence 64 \(\mathcal{B}_{\text{comp}}^{\text{no-pos}}\) basis elements must exist. A counting argument in App. [C] confirms this conclusion.

Let us combine the map [9] with the simple code [4]. The result is another simple code. This code maps between (i) elements of the computational basis \(\mathcal{B}_{\text{comp}}^{\text{no-pos}}\) for the Posner space and (ii) elements of the computational states (5) combine into the antisymmetric six-nucleus states

\[
\frac{1}{\sqrt{6!}} \sum_{\alpha=1}^{6!} 6! \prod_{\beta=1}^{6} (-1)^{\pi_\alpha} |m_{\pi_\alpha(j)}, r_{\pi_\alpha(j)}\rangle
\]

(7)

Each term contains a tensor product of six one-nucleus kets. Each ket is labeled by one tuple \((m_{\pi_\alpha(j)}, r_{\pi_\alpha(j)})\). No tuple equals any other tuple in the same term, by Pauli’s exclusion principle. Permuting one term’s six tuples yields another term, to within a minus sign.

\(\pi_\alpha\) denotes the \(\alpha^{th}\) term’s permutation. The permutation’s sign, \((-1)^{\pi_\alpha}\) parity of permutation, equals the term’s sign. \(\alpha\) \(=\) \((-1)^{\pi_\alpha}\) equals the term’s sign. The semicolon in Eq. (7) separates the \(h_+\) spins from the \(h_-\) spins. \(\alpha\) is equivalent to a Slater determinant.\[37\]

If not for the Posner’s geometry, two tuples could contain the same position variables. \(r_1\) could equal \(r_3\), for example, if \(m_1\) did not equal \(m_3\). But each cube face can house only one phosphate. The phosphorus nuclei’s state occupies the \(\text{no-colliding-nuclei subspace} \mathcal{H}_{\text{no-coll}}\). The antisymmetric subspace.

Posner creation, we posit, projects the nuclei’s state onto \(\mathcal{H}_{\text{no-coll}}\). The projector has the form

\[
\Pi_{\text{no-coll}}^{-} := \sum' \left| (m_1, r_1)(m_2, r_2)(m_3, r_3) ; (m_4, r_4)(m_5, r_5)(m_6, r_6) \right\rangle \left\langle (m_1, r_1)(m_2, r_2)(m_3, r_3) ; (m_4, r_4)(m_5, r_5)(m_6, r_6) \right|.
\]

(8)

The left-hand side (LHS) represents an element of the computational basis \(\mathcal{B}_{\text{comp}}\) for the space \(\mathcal{H}_{\text{no-coll}}\) of the pre-Posner physical qubits. The right-hand side (RHS) represents an element of the computational basis \(\mathcal{B}_{\text{comp}}^{\text{no-pos}}\) for the space \(\mathcal{H}_{\text{no-coll}}\) of the in-Posner physical qubits.

Each pre-Posner state consists of a unique assignment of \(m\)-values to nuclei, a unique distribution of six fixed \(m\)-values across six kets. Similarly, each Posner state consists of a unique assignment of \(m\)-values to positions, a unique distribution of six fixed \(m\)-values across six \(r\)-values. Sixty-four pre-Posner \(\mathcal{B}_{\text{comp}}\) states exist. Hence 64 \(\mathcal{B}_{\text{comp}}^{\text{no-pos}}\) basis elements must exist. A counting argument in App. [C] confirms this conclusion.

Let us combine the map [9] with the simple code [4]. The result is another simple code. This code maps between (i) elements of the computational basis \(\mathcal{B}_{\text{comp}}^{\text{no-pos}}\) for the Posner space and (ii) elements of the computational
basis $B^L_{\text{comp}}$ for the logical space:
\[
| (m_1, r_1) (m_2, r_2) (m_3, r_3) ; (m_4, r_4) (m_5, r_5) (m_6, r_6) \rangle \\
= | m_1 m_2 \ldots m_6 \rangle. \tag{10}
\]

Equation (10) shows how the QI, initially stored in pre(Posner spin states, is encoded faithfully in spin-and-orbital states. We will often replace the physical state’s label (the LHS) with the logical state’s label (the RHS), to streamline notation.

IV. CHARGE-PROTECTED ENCODINGS FOR QUANTUM INFORMATION STORED IN POSNER MOLECULES

The computational-basis elements (10) are states of spin-and-orbital DOFs. The Posner’s dynamics conserve the spins’ states for long times, Fisher hypothesizes (1). The dynamics might not conserve the orbital DOFs’ states. Hence the dynamics might not conserve the orbital spin-and-orbital DOFs. The Posner’s dynamics conserve certain charges: (i) the generator $G_C$ of a permutation operator $C$ (Sec. IV A) and (ii) the total spin operator’s $z_{\text{lab}}$-component, $S^z_{1\ldots6}$ (Sec. IV B). Eigenstates shared by these charges (Sec. IV C) may be conserved.

The spins (represented loosely by the $a$’s) undergo a pseudospin transformation. Let the operator $C$ represent this pseudospin transformation. $C$ cyclically permutes the $h_+\text{s}$ and $h_-\text{s}$ while identically permuting the $h_0\text{s}$ (the final three $m$-values). $C$ transforms the $B^L_{\text{comp}}$ elements (10) as
\[
C : | m_1 m_2 m_3 m_4 m_5 m_6 \rangle \\
\equiv | (m_1, r_1) (m_2, r_2) (m_3, r_3) ; (m_4, r_4) (m_5, r_5) (m_6, r_6) \rangle \\
\mapsto | (m_3, r_3) (m_1, r_1) (m_2, r_2) ; (m_6, r_6) (m_4, r_4) (m_5, r_5) \rangle \\
\equiv | m_3 m_1 m_2 m_6 m_4 m_5 \rangle. \tag{11}
\]

The $B^L_{\text{comp}}$ elements (10) are not $C$ eigenstates.

But $C$ eigenstates can be constructed. We adopt F&R’s notation for the eigenvalues,
\[
\omega^7, \quad \text{wherein } \omega = e^{i2\pi/3} \text{ and } \tau = 0, 1, 2 \text{ or, equivalently, } \tau = 0, \pm 1.
\]

F&R call $\tau$ a three-level “pseudospin.” We call $\tau$ the eigenvalue of the observable $G_C$ that generates $C$ (10). The general form of a $C$ eigenstate appears in (5). F&R use second quantization.

We translate into QI. We also extend (5) by characterizing the eigenspaces of $C$ and by identifying a useful basis for each eigenspace (Sec. IV C). The $\tau = 0$ eigenspace has degeneracy 24; the $\tau = 1$ eigenspace, degeneracy 20; and the $\tau = -1$ eigenspace, degeneracy 20. The $\tau = 0$ eigenspace will play an important role in Posner resource states for universal quantum computation (Sec. VII).

IV B. Conserved charge 2: The total-spin operator $S^z_{1\ldots6}$

Fisher conjectures that Posners’ phosphorus nuclear spins have long coherence times (1). We infer that the Posner Hamiltonian $H_{\text{pos}}$ conserves $S^z_{1\ldots6} = \sum_{j=1}^6 1^{(j-1)} \otimes S^z_{j} \otimes 1^{(6-j)}$. The total magnetic spin number, $m_{1\ldots6} = \sum_{j=1}^6 m_j$, remains constant. Appendix D supports this argument with conjectured interactions between a Posner’s phosphorus nuclear spins. Interactions with the environment are expected to conserve $S^z_{1\ldots6}$ approximately (for long times).

We decompose $H_{\text{no-coll.}}$ into composite-spin subspaces in App. E. That appendix also reviews the addition of quantum angular momentum.

---

10 A pseudospin is a physical DOF that transforms according to a certain rule. $\tau$ is, rather, the eigenvalue of an observable. Suppose that $\tau$ were a three-level quantum pseudospin. $\tau$ would occupy a quantum state in some three-dimensional effective Hilbert space $H_{\text{pseudo}}$. No such space can be associated uniquely with a Posner, to our knowledge. Rather, the Posner Hilbert space $H_{\text{no-coll.}}$ has dimensionality 64. $H_{\text{no-coll.}}$ equals a direct sum of the three $G_C$ eigenspaces: $H_{\text{no-coll.}} = H_{\tau=0} \oplus H_{\tau=1} \oplus H_{\tau=2}$. Each subspace is degenerate. Hence no subspace can serve as an element in a basis for any $H_{\text{pseudo}}$. One could conjure up a $H_{\text{pseudo}}$ by choosing one state $|\tau=0\rangle \in H_{\tau=0}$, one $|\tau=1\rangle \in H_{\tau=1}$, and one $|\tau=2\rangle \in H_{\tau=2}$; then constructing $H_{\text{pseudo}} = \langle \{\tau=0\}, |\tau=1\rangle, |\tau=2\rangle \rangle$. We do so in Sections IV D and VI. But the choice of $|\tau=0\rangle$ is nonunique, as is the choice of $|\tau=1\rangle$, as is the choice of $|\tau=2\rangle$. Hence no unique three-level Hilbert space corresponds to a Posner, to our knowledge. Hence $\tau$ appears not to label a unique quantum pseudospin.

11 A related characterization, and alternative bases, appeared in (5), shortly after the present paper’s initial release.
IV C. Eigenbasis shared by the conserved charges

We introduced the computational basis $B_{\text{comp}}$ for $\mathcal{H}_{\text{no-coll.}}$ in Eq. (7). Most $B_{\text{comp}}$ elements transform non-trivially under $C$ [Eq. (11)]. The Posner dynamics conserve $C$. So, too, would the dynamics ideally conserve quantum codewords. We therefore seek a useful $C$ eigenbasis from which to construct QEC codes.

The $C$ eigenspaces have degeneracies. Which basis should we choose for each eigenspace? A basis shared with $S^{2\text{lab}}_{1\ldots 6}$, the other conserved charge.

Yet $C$ and $S^{2\text{lab}}_{1\ldots 6}$ do not form a complete set of commuting observables (CSCO) [38]. Many eigenbases of $C$ are eigenbases of $S^{2\text{lab}}_{1\ldots 6}$. Another operator is needed to break the degeneracy, to complete the CSCO. We choose the spin-squared sum $S^2_z$.

$$S^2_{123} + S^2_{456} \equiv (S_1 + S_2 + S_3)^2 + (S_4 + S_5 + S_6)^2 \quad (13)$$

$$\equiv \sum_{a=1}^{3} \mathbb{1} \otimes (a-1) \otimes S_a \otimes \mathbb{1} \otimes (3-a)$$

$$+ \sum_{a=4}^{6} \mathbb{1} \otimes (a-4) \otimes S_a \otimes \mathbb{1} \otimes (6-a) . \quad (14)$$

The CSCO consists of $S^2_{123}$; $S^2_{456}$; the $C$ analog that permutes just qubits 1, 2, and 3; the $C$ analog that permutes just qubits 4, 5, and 6; $S^{2\text{lab}}_{123}$; and $S^{2\text{lab}}_{456}$.

Geometry and measurement-based quantum computation (Sec. VII B) motivate the choice of $S^2_{123} + S^2_{456}$: A Posner contains two triangles of spins (Fig. 1A). The positions in the $h_+$ triangle are labeled $r_1$, $r_2$, and $r_3$. Hence the first three tuples in Eq. (10) correspond to the $h_+$ triangle. Hence the magnetic spin quantum numbers $m_1$, $m_2$, and $m_3$ may be viewed as occupying the $h_+$ triangle. These spins’ joint state is equivalent to a three-qubit logical state, $|m_1 m_2 m_3\rangle$. An analogous argument concerns $h_-$. Hence the antisymmetric state $|10\rangle$ is equivalent to a product of two three-logical-qubit states $|13\rangle$.

$$\begin{align*}
|10\rangle |10\rangle |10\rangle & \equiv (m_1 m_2 m_3) |m_4 m_5 m_6\rangle . \quad (21)
\end{align*}$$

12 Swift et al. choose the total-spin operator $S^2_{1\ldots 6}$ and the Hamiltonian. Their choice can be used to define an alternative computational basis. Our choice clarifies the preparation of the universal quantum-computation resource state in Sec. VII.

13 $|10\rangle |10\rangle |10\rangle$ is equivalent to Eq. (10). Equation (17) is equivalent to Eq. (16). Equation (18) follows from the tensor product’s associativity. Equation (19) consists of a rewriting with new notation. Equation (20) is analogous to Eq. (4).

The trios function logically as independent units. Such trios can be used to prepare universal quantum-computation resource states (Sec. VII B). Hence the spin-operator trios in Eq. (13).

Each qubit trio corresponds to a Hilbert space $\mathbb{C}^6$. Let us focus on qubits 1-3. For concreteness, $C$, $S^{2\text{lab}}_{123}$, and $S^{2\text{lab}}_{456}$ share the basis in Table IV. Each basis element is symmetric with respect to cyclic permutations of the three logical qubits.

Tensoring together two one-triangle states yields a state of a Posner’s phosphorus nuclear spins: $|000\rangle, |000\rangle |W\rangle, \ldots, |\omega^2\rangle |\omega^2\rangle$. Sixty-four such states exist. We classify them with quantum numbers in App. F.

We have pinpointed an eigenbasis shared by the conserved charges. The Posner dynamics are expected not to map states in one charge sector to states in another. Hence different-sector states suggest themselves as quantum codewords. We present partially charge-protected QEC codes next.

IV D. Quantum error-detecting and -correcting codes accessible to Posner molecules

We exhibit two codes formed from states accessible to Posners. Each codeword is an eigenstate of a conserved charge, $C$ or $S^{2\text{lab}}_{1\ldots 6}$. Each code’s codewords correspond to distinct eigenvalues of the charge. Hence the Posner dynamics likely do not map any codeword into any other.

Section IV D 1 introduces a quantum error-detecting code. One Posner, we show, can encode one logical qutrit. The code detects one arbitrary physical-qubit error. Section IV D 2 shows how to implement a repetition code with Posner states. The code corrects two bit flips.

More Posner codes, we expect, await discovery. Opportunities are detailed in Sec. X.

We have already discussed an encoding of logical states in physical systems (Sec. III B). Earlier, the logical Hilbert space $\mathcal{H}_L$ shared the physical Hilbert space’s dimensionality, 64. Section III B concerned a bijective, injective map between the spaces. QEC encodes a small logical space in a larger physical space. Notation will reflect the distinction between Sec. III B and QECED: Script subscripts $\mathcal{L}$ (as in $\mathcal{H}_L$) will replace the Roman L (as in $\mathcal{H}_L$). QEC is reviewed in App. A.

IV D 1. Qutrit error-detecting code formed from Posner-molecule states

One Posner, we show, can encode one logical qutrit. The code detects arbitrary single-physical-qubit errors. The physical qubits are the spin-and-orbital DOFs of Sec. III C.

The code has the form

$$\mathcal{H}^{\text{qutrit}}_L = \text{span} \{ |0_\mathcal{L}\rangle, |1_\mathcal{L}\rangle, |2_\mathcal{L}\rangle \} , \quad (22)$$
TABLE I: Symmetric basis for a trio of qubits: A Posner molecule consists of two triangles of spins (Fig. 1). In accordance with Eq. (21), each triangle functions as a trio of logical qubits. The three physical qubits correspond to an eight-dimensional Hilbert space, $C^8$. A useful basis is an eigenbasis shared by the conserved charges, $C$ (a permutation operator) and $S_{123}^z$ (the $z$-component, relative to the lab’s $z_{lab}$-axis, of the total spin). These operators share many basis. The eigenbasis shared also by $S_{123}^z$ proves useful in the preparation of universal quantum-computation resource states (Sec. VII). $\tau$ describes, here, how a triangle transforms under the permutation represented by $C$.

| State         | Decomposition | $S_{123}$ | $n_{123}$ | $\tau$ |
|---------------|---------------|-----------|-----------|--------|
| $|000\rangle$  | $|000\rangle$ | 3/2       | 3/2       | 0      |
| $|W\rangle$   | $\frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$ | 3/2 | 1/2 | 0 |
| $|\bar{W}\rangle$ | $\frac{1}{\sqrt{3}}(|011\rangle + |101\rangle + |110\rangle)$ | 3/2 | -1/2 | 0 |
| $|111\rangle$  | $|111\rangle$ | 3/2 | -3/2 | 0 |
| $|\omega\rangle$ | $\frac{1}{\sqrt{8}}(|100\rangle + \omega^2|010\rangle + \omega|001\rangle)$ | 1/2 | 1/2 | 1 |
| $|\bar{\omega}\rangle$ | $\frac{1}{\sqrt{8}}(|011\rangle + \omega^2|101\rangle + \omega|110\rangle)$ | 1/2 | -1/2 | 1 |
| $|\omega^2\rangle$ | $\frac{1}{\sqrt{8}}(|100\rangle + \omega|010\rangle + \omega^2|001\rangle)$ | 1/2 | 1/2 | 2 |
| $|\bar{\omega}^2\rangle$ | $\frac{1}{\sqrt{8}}(|011\rangle + \omega|101\rangle + \omega^2|110\rangle)$ | 1/2 | -1/2 | 2 |

Each logical state $|j_C\rangle$ occupies the $\tau = j$ subspace. The codewords satisfy the two quantum error-detection criteria [8, 39–43]. First, the states are locally indistinguishable:

$$\langle j_C|\sigma^x|j_C\rangle = \langle j_C|\sigma^y|j_C\rangle = 0,$$

(26)

$$\langle j_C|\sigma^z|j_C\rangle = \frac{1}{2}$$

(27)

for all $j$. That is, the codewords satisfy the diagonal criterion. (See App. A 3 for background.) Second, the codewords satisfy the off-diagonal criterion,

$$\langle j_C|\sigma^\alpha \sigma^\beta|k_C\rangle = 0 \quad \forall \alpha = x, y, z, \quad \forall j \neq k$$

(28)

by direct calculation.

IV D 2. Repetition code formed from Posner-molecule states

The repetition code originated in classical error correction [44]. Each logical bit is cloned until $n$ copies exist: $0 \rightarrow 00\ldots0$, and $1 \rightarrow 11\ldots1$. Suppose that errors flip under half the bits. For example, 000000 may transform into 011000. One decodes the bit string by counting the zeroes, counting the ones, and following majority rule. More physical bits end as 0s than as 1s in our example. A logical zero, the receiver infers, was likely sent.

The repetition code can be translated into quantum states. For example, let $H_C^{rep} = \{|0_C\rangle, |1_C\rangle\}$, wherein $|0_C\rangle = |000000\rangle$ and $|1_C\rangle = |111111\rangle$. This code corrects two $\sigma^x$ errors. But each codeword is unentangled [15]. Hence $H_C^{rep}$ fails to satisfy the off-diagonal error-detection criterion,

$$\langle j_C|\sigma^\alpha \sigma^\beta|k_C\rangle = 0 \quad \forall j \neq k$$

(29)

whenever $\alpha \neq x$ and/or $\beta \neq x$.

V. THE MODEL OF POSNER QUANTUM COMPUTATION

Fisher has conjectured that several physical processes occur in biofluids [1]. We reverse-engineer two more. We abstract away the physics, identifying the computations that the processes effect. We call the computations Posner operations [15]. The operations form a model of quantum computation, Posner quantum computation.

The model’s operations, we will show, can be used (i) to teleport QI incoherently and (ii) to prepare, efficiently, universal resource states for measurement-based quantum computation (Sections VII-VII). Whether the model’s operations can realize universal quantum computation remains an open question (Sec. IX).

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14 More precisely, the element $|000000\rangle$ of the computational basis for the logical space (Sec. III C) is unentangled. The spin-and-orbital state represented by $|000000\rangle$ [by Eq. (10)] is entangled.

15 We occasionally call the physical processes, too, “Posner operations.”
Posner quantum computation is defined in Sec. V A. The model is analyzed in Sec. V B. We discuss the model’s ability to entangle qubits and the control required to perform QI-processing tasks. Fisher’s narrative [1] is also cast as a quantum circuit.

V A. Definition of Posner quantum computation

Terminological notes are in order. When discussing physical processes, we discuss phosphorus nuclear spins, spin-and-orbital DOFs, and Posners. When discussing logical DOFs, we discuss qubits. A circuit-diagram element represents each operation (Figures 3-7):

1. Singlet-state preparation (Fig. 3): Arbitrarily many singlets |ψ−⟩ can be prepared. Singlets are prepared when an enzyme hydrolyzes diphosphates into entangled phosphate pairs (Sec. II D).

These singlets are prepared differently than in conventional quantum circuits. Conventionally, one prepares two qubits in the state |0⟩⊗2; performs a Hadamard† on the first qubit; and performs a CNOT‡ controlling on the first qubit: CNOT(H ⊗ 1)|00⟩ = |ψ−⟩.

In contrast, Fisher posits that enzymes prepare singlets by projective measurements [1]. We formalize Fisher’s statement as follows. A diphosphate’s phosphorus nuclear spins occupy some state |ρdiphos⟩. The diphosphate enters a pyrophosphatase enzyme. The enzyme measures the PVM |Ψ⟩−⟨Ψ−|1. The gate has a geometric interpretation expressed in terms of the Bloch sphere: The state evolves through 180° about the axis \(\sqrt{2}(z + \bar{z})\).

Suppose, instead, that the diphosphate leaves the enzyme uncleaved. The second possible measurement outcome has obtained. The diphosphate cannot form a Posner molecule with other ions. Hence the diphosphate cannot participate in quantum cognition. Hence the diphosphate plays no role in Posner quantum computation. Hence the PVM’s \(\mathbb{1} - |Ψ−⟩⟨Ψ−|\) outcome plays no role. Any diphosphate that remains uncleaved “is discarded,” in QI language. The |Ψ−⟩⟨Ψ−| outcome is classically postselected on. Classical postselection provides no superquantum computational power; see footnote 20.

In summary, Posner operations include the preparation of |ψ−⟩. Quantum-cognition systems prepare |ψ−⟩ by measuring a PVM nondestructively, then postselecting classically on the “yes” outcome. “No”-outcome ions do not participate in later chemical events of interest.

2. Hextuple formation (Fig. 4): Qubits can group together in hextuples (groups of six). Hextuple formation evolves the logical qubits trivially, under the operator \(\mathbb{I}\). But hextuple formation associates the qubits with a geometry and with an observable \(\mathcal{G}_C\).

Logical qubits form hextuples as ions bind together, forming Posners. A logical qubit can occupy, at most, one hextuple. Section III C explains why hextuple formation fails to change logical qubits’ states: The spins’ state changes, suggesting that the logical qubits’ state changes. But the logical information’s physical encoding changes, too.

Hextuple creation impacts the logical system in three ways: (i) Each hextuple has an observable \(\mathcal{G}_C\). (ii) Hextuple creation induces a geometry that influences operation \(\mathcal{H}_G\). (iii) The six logical qubits’ Hilbert space transforms from \(\mathbb{C}^1\) to the isomorphic \(\mathbb{C}^6\). Let us detail these three effects.

First, creating a hextuple creates an observable \(\mathcal{G}_C\) (Sec. IV A). \(\mathcal{G}_C\) has eigenvalues \(\tau = 0, 1, 2\) (equivalently, \(\tau = 0, \pm 1\)). \(\tau\) impacts operation 5.

Second, hextuple creation induces a geometry. Each
logical qubit is assigned to a cube face, in accordance with Sec. III C. The six qubits can be distributed across the six faces in any of $6!$ ways. Physically, different assignments follow from different pre-Posner orbital states (App. B). The six qubits form two triangles, called trios below, in accordance with Fig. 1. This geometry limits the single-qubit unitaries that can evolve the six qubits (operation 3b). The geometry also influences our construction of universal quantum-computation resource states (Sec. VII).

Third, hextuple creation changes the system’s Hilbert space from $(\mathbb{C}^2)^{\otimes 6}$ to $H_{\text{no-coll}}$. $(\text{Sec. III C})$. $H_{\text{no-coll}}$ is isomorphic to $(\mathbb{C}^2)^{\otimes 6}$, as the map (10) is injective and bijective. Hence we will keep referring to the logical space as $(\mathbb{C}^2)^{\otimes 6}$.

### 3. Hextuple unitaries: Any hextuple can undergo any sequence of any instances of the following operations:

(a) **Hextuple permutation** (Fig. 5): The qubits can undergo the permutation $(231)(564)$ \textcolor{Blue}{[Eq. (11)]}. The qubits are permuted as the Posner rotates about its symmetry axis, $\hat{z}_{\text{in}}$, through an angle $2\pi/3$. Posners rotate while tumbling in the fluid.

(b) **Hextuple-coordinated single-qubit rotations** (Fig. 6): The qubits in a hextuple can undergo identical single-qubit rotations simultaneously. The qubits rotate due to the magnetic field generated as neurons spike.

Fisher’s narrative allows for, though does not include, this operation. We approximate the magnetic field as roughly constant over the Posner’s length scale, $\sim 1$ nm \textcolor{Red}{[13]}. The formalism can easily be generalized to accommodate short-distance field fluctuations, however. The unitary $[U_{\hat{n}}(\theta)]^{\otimes 6}$ rotates the qubits through an angle $\theta$ about an axis $\hat{n}$ relative to the lab frame. Angles of up to $\theta \approx \pi$ might be reached. Smaller angles are expected to be typical, however. Section IX and App. K detail the rotation mechanism and scales.

### 4. Hextuple break-up (Fig. 7): One hextuple can break down into its constituents: The qubits can cease to correspond to geometries or to observables $G_C$.

Different regions of the body have different pH’s and different magnesium-ion ($Mg^{2+}$) concentrations. A Posner can migrate to a region packed with $H^+$ and/or with $Mg^{2+}$. These ions can bind to $PO_3^{4-}$, as $Ca^{2+}$ can. The higher the $H^+$ and $Mg^{2+}$ concentrations, the more $H^+$ and $Mg^{2+}$ ions dislodge Posners’ $Ca^{2+}$ ions \textcolor{Red}{[6]}. The dislodging hydrolyzes the molecules.

Fisher’s narrative allows for, though does not include, hextuple break-up. Operation 4 can be used to prepare Posners, efficiently, in resource states that can power universal measurement-based quantum computation (Sec. VII).

### 5. Posner-binding measurement (Figures 8 and 9):

Let $A$ and $B$ denote two hextuples formed via operation \textcolor{Blue}{3b}. Whether the hextuples’ $G_C$ eigenvalues sum to zero can be measured nondestructively: $\tau_A + \tau_B = 0$.

First, we discuss the measurement’s physical manifestation. Then, we mathematize the operation with a PVM.

The measurement manifests in the binding, or failure to bind, of two Posners. Fisher conjectures as follows \textcolor{Red}{[1]}, supported by quantum-chemistry calculations \textcolor{Red}{[9]}: Two Posners, $A$ and $B$, can bind together. They bind upon approaching each other such that their directed symmetry axes (Fig. 1a) lie side-by-side and point oppositely. One Posner forms a mirror image of the other.

Such Posners bind, Fisher conjectures \textcolor{Red}{[1]}, when and only when $\tau_A + \tau_B = 0$. F&R support the conjecture...
with a Berry-phase argument. They formalize the conjecture as a “quantum dynamical selection rule” \[5\]. Hence if \( A \) and \( B \) approach with the right orientation, whether they bind depends entirely on whether \( \tau_A + \tau_B = 0 \). The molecules’ bound-together-or-not status serves as a classical measurement record. So does the environment, as in Posner creation (Sec. \( \Pi \)): Posner binding releases about 1 eV of heat \[1\], \[9\].

Let us formalize the measurement, using the mathematics of QI. We define a projector on \( (H_{\text{no-coll.}}) \otimes_2 \):

\[
\Pi_{AB} := (\Pi_{\tau_A=0} \otimes \Pi_{\tau_B=0}) + (\Pi_{\tau_A=\pm 1} \otimes \Pi_{\tau_B=\mp 1}).
\]

The PVM

\[
\{\Pi_{AB}, \mathbb{1} - \Pi_{AB}\}
\]

(31)

\[\rho \mapsto \Pi_{AB} \rho \Pi_{AB} \quad \text{Tr} (\Pi_{AB} \rho \Pi_{AB}) \].

(32)

The twelve qubits form a \( \text{dodectuple} \). Suppose, instead, that the second outcome obtains (that the Posners fail to bind). The joint state updates as

\[
\rho \mapsto \frac{\rho - \{\Pi_{AB}, \rho\} + \Pi_{AB} \rho \Pi_{AB}}{1 - \text{Tr} (\Pi_{AB} \rho)}.
\]

(33)

The anticommutator of operators \( O \) and \( O' \) is denoted by \( \{O, O'\} \).

6. Dodectuple operations: Suppose that hextuples \( A \) and \( B \) have been measured with the PVM \[31\]. Suppose that outcome \( \Pi_{AB} \) has obtained. The twelve logical qubits can undergo operation \( \text{6a} \) followed by \( \text{6b} \) or \( \text{6c} \).

(a) Dodectuple-coordinated single-qubit unitaries (Fig. \(10\)): The two hextuple’s qubits can undergo approximately identical single-qubit rotations: \( [U_h(\theta)] \otimes_{12} \). The qubits rotate as neuron firings generate magnetic fields. See the comments about operation \( \text{3b} \).

(b) Dodectuple \( \rightarrow 2 \) hextuples (Fig. \(11\)): A dodectuple can separate into independent hextuples, the hextuples that joined together. The Posners can drift apart. They can return to undergoing operations \( \text{3b} \) and \( \text{4} \).
V B. Analysis of Posner quantum computation

We have dissected Fisher’s narrative into physical processes, then abstracted out the computations that the processes effect. Fisher’s narrative \[1\] can now be cast as a quantum circuit. The circuit appears in Fig. 13.

Four features of Posner operations merit analysis. Two operations entangle logical qubits. The entanglement generated is discussed in Sec. V B 1. Section V B 2 concerns control: To perform the QI-processing tasks introduced in Sections VII-VIII one might need fine control over Posners. Biofluids might not exert such control. But assuming control facilitates first-step QI analyses. One operation merits its own section: The measurement \(31\) is compared with a Bell measurement, and applied in QI-processing tasks, in Sec. VI.

(c) Dodectuple break-up (Fig. 12): The hextuples can break down into their constituents: The qubits can cease to correspond to meaningful geometries or to observables \(G_C\). The qubits thereafter behave independently. They can, again, undergo operation 2.

The hextuples break down as the Posners hydrolyze. Fisher conjectures that bound-together Posners hydrolyze more often than separated Posners \[1\], as reviewed in this paper’s Sec. I.

V B 1. Entanglement generation

Entanglement enables quantum computers to solve certain problems quickly\[19\]. Two Posner operations create entanglement: Bell-pair creation (operation 1) and the Posner-binding measurement (operation 5).

Bell pairs serve as units of entanglement in QI \[7\]. We present two implications of Bell-pair creation for Posners. First, Bell-pair creation (operation 1), with the Posners’ geometry, can efficiently prepare a state that fuels universal measurement-based quantum computation (Sec. VII). Second, distributing Bell pairs across Posners can affect their binding probabilities (Sec. VIII).

The role played by Bell pairs in QI processing is well-known. Less obvious is how much, and which kinds of, entanglement \(\Pi_{AB}\) creates and destroys. We characterize this entanglement in two ways (Sec. VI A). The PVM (31), we show, transforms a subspace as a coarse-grained Bell measurement. Bell measurements facilitate quantum teleportation \[10\]. The PVM (31) facilitates incoherent teleportation: A state’s weights are teleported; the coherences are not.

One might expect Posner binding to render Posner quantum computation universal: Conventional wisdom says, nearly any entangling gate, plus all single-qubit unitaries, form a universal gate set \([48, 52]\). Posner operations include entangle qubits (via operation 5) and rotate qubits (via operations 3b and 6a). (Arbitrary rotations through angles of up to \(\pi\) might be realized. Typical angles are expected to be smaller. See App. K.)

But the conventional wisdom appears inapplicable to Posner operations, for three reasons. First, conventional-wisdom gates evolve the system unitarily. The Posner-binding measurement (31) does not. (Hence our shift to measurement-based quantum computation in Sec. VII.) Second, many universality proofs decompose a desired

\[19\] More precisely, contextuality does \([46, 47]\).
FIG. 13: Circuit representation of Fisher’s quantum-cognition narrative: Fisher conjectures that certain chemical processes occur, in a certain sequence, in the body [1]. The sequence is reviewed in this paper’s introduction. We abstracted out the computations effected by the chemical processes, in Sec. V A. The abstraction enables us to recast Fisher’s narrative as a quantum circuit. Time progresses from left to right in the figure (from the bottom to the top of the page). The sets of six qubits are labeled $a, b, a', \text{ and } b'$, as in [1 p. 5, Fig. 3]. The circuit elements are defined in Sec. V A.
entangling measurement seems unlikely to decompose. Third, conventional-wisdom entangling gates are defined in terms of qubits’ states. The Posner-binding measurement is defined in terms of \( \tau \). \( \tau \) is an eigenvalue of an observable \( G_C \) of a hextuple of qubits. One must deduce how the measurement transforms any given qubit. Does this indirect entangler of qubit states, with single-qubit rotations (operations \([33]\) and \([34]\)), form a universal set? The answer merits further study.

Finally, one might wonder how the permutations \([33]\) alter entanglement. Each permutation decomposes into two-party swaps, as \( (231) = (321)(132) \). Swaps shift entanglement amongst subsystems, rather than creating entanglement.

\[ V \text{ B 2. Control required to perform quantum-information-processing tasks with Posner molecules} \]

In Sections \([VI]\)\([VII]\), we concatenate Posner operations to form QI-processing protocols. Implementing the protocols may require fine control over the chemical processes that effect the computations. The body might seem unlikely to realize fine control. We illustrate with two examples. Then, we justify the assumption of fine control.

Consider, as a first example, running an arbitrary quantum circuit. Arbitrary qubits must rotate through arbitrary angles \( \theta \), about arbitrary axes \( \hat{n} \), arbitrarily precisely. In the quantum-cognition setting, logical qubits rotate as Posners experience magnetic fields generated by neural currents (via operations \([33]\) and \([34]\)). The field experienced depends on the Posner’s location, which depends on the Posner’s collisions with other particles. Fluid particles collide randomly. Random collisions appear unlikely to facilitate the precise rotations required for a given circuit.

The \( \tau_A + \tau_B = 0 \) measurement (operation \([5]\)) provides a second example. Consider Posners \( A \) and \( B \) that approach each other. One might wish to infer, from the Posners’ binding or lack thereof, whether \( \tau_A + \tau_B = 0 \). But the inference is justified only if \( A \) and \( B \) were oriented such that whether they would bind depended only on whether \( \tau_A + \tau_B \) vanished.

Suppose that one Posner’s \( \hat{z}_m \)-axis stood tip-to-tail with the other Posner’s \( \hat{z}_m \)-axis, rather than side-by-side. The Posners would fail to bind. But one could not infer that \( \tau_A + \tau_B \neq 0 \). Only finely tuned two-Posner encounters reflect whether \( \tau_A + \tau_B = 0 \). Only finely tuned encounters constitute measurements \([20]\).

But assuming perfect control can facilitate QI-theoretic analyses. Many QI protocols are phrased in the language of “agents.” One imagines intelligent agents, Alice and Bob, who wish to process QI. One specifies and analyzes protocols in terms of the agents’ intents and actions. Alice and Bob are often assumed to perform certain operations with perfect control. Examples of such “allowed operations” include local operations and classical communications \([54]\).

Control partitions (i) what can be achieved in principle from (ii) what can be achieved easily with today’s knowledge and techniques. Item (ii) shifts with our understanding and technology. Item (i) is permanent and is the focus of much QI theory.

A few decades ago, for example, experimentalists had trouble performing CNOT gates. Many groups have mastered the gate by now. These groups implement protocols devised before CNOTs appeared practical.

Similarly, precise phosphate rotations appear impractical. But some precise-rotation mechanism could be discovered. Also, by assuming perfect control, we derive a limit on what Posners can achieve without perfect control. We ascertain what QI processing is possible in principle.

\[ VI. \text{ THE POSNER-BINDING MEASUREMENT AND APPLICATIONS THEREOF TO QUANTUM INFORMATION PROCESSING} \]

The measurement \([51]\) entangles two Posners’ states. Yet the measurement projectors, \( \Pi_{AB} \) and \( 1 - \Pi_{AB} \), entangle states in different ways. How much either projector entangles is not obvious. Neither is the PVM’s potential for processing QI.

This section sheds light on these unknowns. We compare the PVM to a \textit{Bell measurement}, a standard QI operation (Sec. \([VI \text{ A}]\)). The next two sections detail applications of the PVM: The PVM facilitates incoherent teleportation (Sec. \([VI \text{ B}]\)). Also, the PVM can be used to project Posners onto their \( \tau = 0 \) eigenspaces (Sec. \([VI \text{ C}]\)).

---

20 If two Posners bind, then \( \tau_A + \tau_B = 0 \); the inference is justified. But binding does not constitute merely a measurement. Binding constitutes a measurement followed by classical postprocessing. By \textit{classical postprocessing}, we mean the following. Consider performing some protocol in each of several trials. Let the protocol involve a measurement. Consider the data collected throughout trials. Consider discarding some of the data, keeping only the data collected during the trials in which the measurement yielded some outcome \( x \). One has classically postselected on \( x \). If two Posners bind, then (i) whether \( \tau_A + \tau_B = 0 \) is measured and (ii) the “yes” outcome is classically postselected on. If two Posners bind, step (i) alone is not implemented; a measurement alone is not performed.

Classical postprocessing differs from the postselection in, e.g., \([23]\). The latter postselection affords computational power unlikely to grace quantum systems. In contrast, classical postprocessing happens in today’s laboratories.
VI A. Comparison of the Posner-binding measurement with a Bell measurement

First, we review Bell states and measurements [7]. A Bell measurement prepares an entangled state of two qubits. Four maximally entangled states span the two-qubit Hilbert space, $\mathbb{C}^4$. The orthonormal Bell basis is

\[
\{|\Phi^+\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad |\Phi^-\rangle := \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \quad |\Psi^+\rangle := \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \quad |\Psi^-\rangle := \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\}.
\]

A Bell measurement is represented by the PVM

\[
\{|\Phi^+\rangle\langle\Phi^+|, |\Phi^-\rangle\langle\Phi^-|, |\Psi^+\rangle\langle\Psi^+|, |\Psi^-\rangle\langle\Psi^-|\}.
\]

Many QI protocols involve Bell measurements. Examples include quantum teleportation [10], superdense coding (the effective transmission of two bits via the direct transmission of just one bit, with help from entanglement) [11], and teleportation-based quantum computation [59–58].

Posner binding simulates a coarse-grained Bell measurement. The Bell-state projectors (34) are defined on $\mathbb{C}^2$. In contrast, the Posner Hilbert space $\mathcal{H}_{\text{no-coll}}$ is isomorphic to $\mathbb{C}^6$. We therefore define an effective qubit subspace. Let $|1\_\tau\rangle$ denote an arbitrary $\tau = 1$ eigenstate of $C$; and $|2\_\tau\rangle$, an arbitrary $\tau = 2$ eigenstate. $|1\_\tau\rangle$ and $|2\_\tau\rangle$ serve analogously to $|0\rangle$ and $|1\rangle$ in span $\{|1\_\tau\rangle, |2\_\tau\rangle\}$.

**Proposition 1.** Let $A$ and $B$ denote two Posners. The measurement [51] transforms the effective two-qubit space

\[
\text{span} \{|1\_\tau, 1\_\tau\rangle, |1\_\tau, 2\_\tau\rangle, |2\_\tau, 1\_\tau\rangle, |2\_\tau, 2\_\tau\rangle\}
\]

identically to the coarse-grained Bell measurement

\[
\{|\Phi^+\rangle\langle\Phi^+| + |\Phi^-\rangle\langle\Phi^-|, |\Psi^+\rangle\langle\Psi^+| + |\Psi^-\rangle\langle\Psi^-|\}.
\]

**Proof.** The projector (30) transforms the two-qubit space as

\[
\Pi_{AB} = |1\_\tau, 2\_\tau\rangle\langle 1\_\tau, 2\_\tau| + |2\_\tau, 1\_\tau\rangle\langle 2\_\tau, 1\_\tau|.
\]

Let us relabel $1\_\tau$ as 0 and $2\_\tau$ as 1. The projector becomes

\[
\Pi_{AB} = |\Psi^+\rangle\langle\Psi^+| + |\Psi^-\rangle\langle\Psi^-|.
\]

Direct substitution into the RHS yields the LHS.

Consider the complementary projector in the measurement (31). $1 - \Pi_{AB}$ transforms the effective two-qubit space as

\[
1 - \Pi_{AB} = |1\_\tau, 1\_\tau\rangle\langle 1\_\tau, 1\_\tau| + |2\_\tau, 2\_\tau\rangle\langle 2\_\tau, 2\_\tau|.
\]

Relabeling and direct substitution show that

\[
1 - \Pi_{AB} = |\Phi^+\rangle\langle\Phi^+| + |\Phi^-\rangle\langle\Phi^-|.
\]
VI B 1. Set-up and notation

Let \(|j_\tau\rangle\) denote an arbitrary \(\tau = j\) eigenstate of \(C\), for \(j = 0, 1, 2\). The \(|j_\tau\rangle\)'s form the computational basis for the qutrit space span\(|0_\tau\rangle, |1_\tau\rangle, |2_\tau\rangle\). This basis serves, in incoherent teleportation, similarly to the \(\sigma_z\) eigenbasis in conventional teleportation.

Consider restricting the projector \((30)\) to the space of two qutrits:

\[
\Pi'_{AB} := |0_\tau, 0_\tau\rangle\langle 0_\tau, 0_\tau| + |1_\tau, 2_\tau\rangle\langle 1_\tau, 2_\tau| + |2_\tau, 1_\tau\rangle\langle 2_\tau, 1_\tau|.
\]

Let \(|+_\tau\rangle := \frac{1}{\sqrt{3}} (|0_\tau\rangle + |1_\tau\rangle + |2_\tau\rangle)\).

VI B 2. Incoherent-teleportation protocol

Let \(A\), \(B\), and \(C\) denote three Posners. Suppose that \(B\) and \(C\) begin in \(|+_\tau, +_\tau\rangle\), then bind together\(^{21}\). The joint state becomes

\[
\Pi_{BC} |+_\tau, +_\tau\rangle = \frac{1}{\sqrt{3}} (|0_\tau, 0_\tau\rangle + |1_\tau, 2_\tau\rangle + |2_\tau, 1_\tau\rangle).
\]

In the first term's absence, \((46)\) would be a triplet. A triplet is a Bell pair, a maximally entangled state that can fuel quantum teleportation. \((46)\), we will show, fuels incoherent teleportation.

Suppose that, after \((46)\) is prepared, Posners \(B\) and \(C\) drift apart. (In quantum-computation language, Alice and Bob share a Bell pair.) Let \(B\) approach \(A\). Let \(A\) occupy an arbitrary state

\[
|\psi\rangle = c_0|0_\tau\rangle + c_1|1_\tau\rangle + c_2|2_\tau\rangle.
\]

The complex coefficients satisfy the normalization condition \(\sum_{j=0}^{2} |c_j|^2 = 1\). (In quantum-computation language, \(|\psi\rangle\) is the unknown state that contains information that Alice will teleport to Bob.) The three Posners occupy the joint state

\[
|\chi\rangle := |\psi\rangle (\Pi_{BC} |+_\tau, +_\tau\rangle).
\]

Suppose that Posners \(A\) and \(B\) bind together. (During the analogous quantum-teleportation step, Alice performs a Bell measurement of her qubits.) The three-Posner state becomes

\[
\Pi_{AB} |\chi\rangle / \langle \chi | \Pi_{AB} |\chi\rangle = c_0|0_\tau, 0_\tau, 0_\tau\rangle + c_1|1_\tau, 2_\tau, 1_\tau\rangle + c_2|2_\tau, 1_\tau, 2_\tau\rangle
\]

\[
=: |\chi'\rangle.
\]

Posner \(C\) occupies (Bob holds) the reduced state

\[
\rho_C := \text{Tr}_{AB} (|\chi'\rangle \langle \chi'|) \]

\[
= |c_0|^2|0_\tau\rangle\langle 0_\tau| + |c_1|^2|1_\tau\rangle\langle 1_\tau| + |c_2|^2|2_\tau\rangle\langle 2_\tau|.
\]

Posner \(C\)'s state encodes information about \(|\psi\rangle\), the square moduli of the coefficients in Eq. \((47)\). Yet \(C\) has never interacted with \(A\) directly. Information has teleported from \(A\) to \(C\), with help from \(|+_\tau, +_\tau\rangle\) and from Posner binding.

Posners \(A\) and \(B\) had a probability

\[
p_{|\chi\rangle} = \text{Tr} (\Pi_{AB} \Pi_C (|\chi\rangle \langle \chi|))
\]

of binding together. (An analogous probability can be introduced into quantum teleportation: Let Alice have a nonzero probability of failing to perform her Bell measurement.)

Suppose, instead, that \(A\) and \(B\) fail to bind together. The projector

\[
1 - \Pi_{AB} = (\Pi_{\tau_A=0} \otimes \Pi_{\tau_B=1}) + (\Pi_{\tau_A=0} \otimes \Pi_{\tau_B=2})
\]

\[
+ (\Pi_{\tau_A=1} \otimes \Pi_{\tau_B=0}) + (\Pi_{\tau_A=1} \otimes \Pi_{\tau_B=1})
\]

\[
+ (\Pi_{\tau_A=2} \otimes \Pi_{\tau_B=0}) + (\Pi_{\tau_A=2} \otimes \Pi_{\tau_B=2})
\]

projects the state of \(AB\). The three-Posner state \(|\chi\rangle\) [Eq. \((48)\)] updates to

\[
[(1 - \Pi_{AB}) \otimes 1] |\chi\rangle = \frac{1}{2} [c_0(|0_\tau, 1_\tau, 2_\tau\rangle + |0_\tau, 2_\tau, 1_\tau\rangle)
\]

\[
+ c_1(|1_\tau, 0_\tau, 0_\tau\rangle + |1_\tau, 1_\tau, 2_\tau\rangle)
\]

\[
+ c_2(|2_\tau, 0_\tau, 0_\tau\rangle + |2_\tau, 2_\tau, 1_\tau\rangle)]
\]

\[
=: |\chi''\rangle.
\]

Posner \(C\) occupies (Bob holds) the reduced state

\[
\text{Tr}_{AB} (|\chi''\rangle \langle \chi''|) = \frac{1}{2} [(|c_0|^2 + |c_2|^2)|0_\tau\rangle\langle 0_\tau|)
\]

\[
+ (|c_2|^2 + |c_0|^2)|1_\tau\rangle\langle 1_\tau| + (|c_0|^2 + |c_1|^2)|2_\tau\rangle\langle 2_\tau|).
\]

Again, \(C\) contains information about \(|\psi\rangle\), despite never having interacted directly with \(A\).

Suppose that Bob measures \(G_C\), the observable that generates the unitary \(C\). Bob samples from a random variable whose values 0, 1, and 2 are distributed according to \((p'_0 = |c_1|^2 + |c_2|^2, p'_1 = |c_2|^2 + |c_0|^2, p'_2 = |c_0|^2 + |c_1|^2)\).
VI B 3. Analysis of incoherent teleportation

Five points merit analysis. First, we quantify the classical information teleported. Second, we characterize the QI not teleported. Third, we compare the resources required for incoherent teleportation to the resources required for quantum teleportation. Incoherent teleportation, we show fourth, implements superdense coding—the effective sending of much classical information via the direct sending of little classical information, with help from entanglement. Fifth, we explain how to prepare $|+\rangle$ and $|\psi\rangle$ with Posner operations.

a. Quantification of the information teleported: Posners $A$ and $B$ teleport a trit to $C$. A trit is classical random variable that can assume one of three possible values. Imagine preparing a Posner in the state $|\psi\rangle$ [Eq. (47)] and measuring $G_C$. The measurement has a probability $p_0 = |c_0|^2$ of yielding 0, a probability $p_1 = |c_1|^2$ of yielding 1, and a probability $p_2 = |c_2|^2$ of yielding 2. So does a $G_C$ measurement of $C$, if $A$ binds to $B$ [Eq. (53)]. The distribution has been teleported from $A$ to $C$.

Suppose that $A$ fails to bind to $B$. Measuring Posner $C$ has a probability $p_0' = \frac{1}{2}(|c_1|^2 + |c_2|^2)$ of yielding 0, a probability $p_1' = \frac{1}{2}(|c_1|^2 - |c_2|^2)$ of yielding 1, and a probability $p_2' = \frac{1}{2}(|c_1|^2 + |c_2|^2)$ of yielding 2 [Eq. (57)]. The measurement of Posner $C$ is equivalent to an encoded generalized measurement of $|\psi\rangle$.

A positive-operator-valued measure (POVM) $\{M_1, M_2, \ldots, M_L\}$ represents a generalized quantum measurement $\mathbf{1}$. The measurement elements are positive operators $M_k > 0$. They satisfy the completeness condition $\sum_k M_k = I$. The $M_k$’s need not be projectors, unlike POVM elements. Consider the POVM

$$\begin{align*}
\{ |0\rangle \rangle |0\rangle &= \frac{1}{\sqrt{2}}(|1\rangle \rangle |1\rangle + |2\rangle \rangle |2\rangle |, \\
|1\rangle \rangle |1\rangle &= \frac{1}{\sqrt{2}}(|2\rangle \rangle |2\rangle + |0\rangle \rangle |0\rangle |, \\
|2\rangle \rangle |2\rangle &= \frac{1}{\sqrt{2}}(|0\rangle \rangle |0\rangle + |1\rangle \rangle |1\rangle |. 
\end{align*}$$

(58, 59, 60)

Measuring this POVM is equivalent to measuring the encoded observable $G_C := \sum_j M_j |\omega_j\rangle \langle \omega_j|$. Measuring the $G_C$ of $|\psi\rangle$ has a probability $p_j'$ of yielding the encoded outcome $j'$.

Suppose that Posners $A$ and $B$ fail to bind. A measurement of the $G_C$ of $C$ simulates an encoded measurement of the $G_C$ of $|\psi\rangle$.

b. Classicality of the teleported information: Only the square moduli $|c_j|^2$ are teleported. The coefficients’ phases are not. Hence incoherent teleportation achieves less than quantum teleportation does.

Section VI A clarifies why: Quantum teleportation involves Bell measurements. Incoherent teleportation involves measurements of whether $\tau_A + \tau_B = 0$. The $\tau_A + \tau_B = 0$ measurement simulates a coarse-grained Bell measurement.

c. Comparison of resources required for incoherent teleportation with resources required for quantum teleportation: In quantum teleportation, qubit $C$ undergoes a local unitary conditioned on the Bell measurement’s outcome. Our Posner $C$ needs no such conditional correcting.

Yet part of our story depends on the Posner-binding measurement’s outcome: the interpretation of the outcome of a $G_C$ measurement of Posner $C$. Suppose that $A$ binds to $B$. A $G_C$ measurement of Posner $C$ simulates a measurement of the $G_C$ of $|\psi\rangle$. Suppose, instead, that $A$ fails to bind to $B$. A $G_C$ measurement of Posner $C$ simulates a measurement of the $G_C$ of $|\psi\rangle$.

d. Incoherent teleportation as superdense coding: Incoherent teleportation offers less power than classical communication. Suppose that Alice has incoherently teleported $|\psi\rangle$. Bob may wish to know which probability distribution he holds, $\{p_0, p_1, p_2\}$ or $\{p_0', p_1', p_2'\}$. Alice should send Bob a bit directly: a zero if $A$ bound to $B$ and a one otherwise.

Alice would directly send Bob a bit, while effectively sending a trit, with help from entanglement and Posner binding. A trit is equivalent to $\log_2(3) > 1$ bits. Hence Alice packs much classical information (a trit) into a small classical system (a bit).

Much classical information packs into a small classical system, with help from a Bell pair and a Bell measurement, in superdense coding [11]. Conventional superdense coding packs two bits into one. Our protocol packs information less densely.

e. Preparing $|\psi\rangle$ and $|+\rangle$ and $|\tau, +\rangle$: Incoherent teleportation involves two coherent quantum states, $|\psi\rangle$ and $|+\rangle$ and $|\tau, +\rangle$. Instances of these states can be prepared with Posner operations. We illustrate with an example in App. H To construct each state, one arranges singlets in each Posner. One then rotates one spin per Posner through an angle $\frac{\pi}{4}$ about the $y_{ab}$-axis. Alternative preparation protocols might exist.

\[\text{22} \] How could such classical communication manifest in biological systems? In ordinary QI protocols, classical communication manifests as telephone calls. Today’s phones do not fit in human brains. But one can envision classical channels in a biofluid. For example, if $A$ and $B$ bind, they shove water molecules away together. If $A$ and $B$ fail to bind, water propagates away from them differently. The patterns in the fluid’s motion may be distinguished. The fluid-motion pattern would encode the bit.
VI C. Application 2 of binding Posner molecules: Projecting Posner molecules onto their $\tau = 0$ subspaces

The AKLT state can be prepared via projections onto subspaces associated with the spin quantum number $s = \frac{3}{2}$. Posners can come to occupy a variation AKLT’ on the AKLT state. The Posners must be projected onto their $\tau = 0$ subspaces (Sec. VII). Posner-binding measurements can effect these projections.

**Proposition 2.** Let $A, B, C, \ldots, M$ label $m$ Posner molecules. The following sequence of events projects each Posner’s state onto the $\tau = 0$ eigenspace:

1. $A$ and $B$ bind together, then drift apart.
2. $B$ and $C$ bind together, then drift apart.
3. $C$ and $A$ bind together, then drift apart. $A$, $B$, and $C$ have been projected onto their $\tau = 0$ subspaces.
4. Each remaining Posner ($D, \ldots, M$) binds to a projected Posner, then drifts away.

**Proof.** First, we prove that steps 1–3 project $A$, $B$, and $C$ onto their $\tau = 0$ subspaces. Then, we address step 4.

A product $\Pi_{123}$ of projectors represents the sequence 1–3 of bindings:

$$\Pi_{123} = \left[ \left( \Pi_{\tau_A=0} \otimes \Pi_{\tau_B=0} \otimes 1^{(m-2)} \right) + \left( \Pi_{\tau_A=\pm 1} \otimes \Pi_{\tau_B=\mp 1} \otimes 1^{(m-2)} \right) \right] \times \left[ \left( 1 \otimes \Pi_{\tau_C=0} \otimes 1^{(m-3)} \right) + \left( 1 \otimes \Pi_{\tau_B=\pm 1} \otimes \Pi_{\tau_C=0} \otimes 1^{(m-3)} \right) \right] \times \left[ \left( \Pi_{\tau_A=0} \otimes 1 \otimes \Pi_{\tau_C=0} \otimes 1^{(m-3)} \right) + \left( \Pi_{\tau_A=\pm 1} \otimes 1 \otimes \Pi_{\tau_C=0} \otimes 1^{(m-3)} \right) \right]$$

$$= \Pi_{\tau_A=0} \otimes \Pi_{\tau_B=0} \otimes \Pi_{\tau_C=0} \otimes 1^{(m-3)} \times \Pi_{\tau_A=0} \otimes \Pi_{\tau_D=0} \times \Pi_{\tau_B=\pm 1} \otimes \Pi_{\tau_D=\mp 1}.$$  (61)

Equation (61) can be understood in terms of a frustrated lattice, as explained in App. I.

Step 4 of Proposition 2 is proved as follows. Suppose that Posners $A$ and $D$ bind together, then drift apart. The joint state of $AD$ is acted on by

$$\left( \Pi_{\tau_A=0} \otimes \Pi_{\tau_D=0} + \Pi_{\tau_A=\pm 1} \otimes \Pi_{\tau_D=\mp 1} \right).$$  (63)

The state of $A$ was projected onto the $\tau_A = 0$ subspace during steps 1–3. Hence the final term in Eq. (63) annihilates the $AD$ state. Hence $\Pi_{\tau_D=0}$ projects the state of $D$. \hfill $\square$

**FIG. 14:** AKLT’ state: Spins on a honeycomb lattice can occupy the Affleck-Lieb-Kennedy-Tasaki (AKLT) state $|\text{AKLT}_{\text{hon}}\rangle$. $|\text{AKLT}_{\text{hon}}\rangle$ serves as a universal resource in measurement-based quantum computation (MBQC). So does the similar state $|\text{AKLT}_{\text{hon}}\rangle$, which Posner operations can prepare efficiently. Each dashed oval encloses the spins in a Posner molecule. Each molecule consists of two trios of phosphorous nuclear spins. Each large black dot represents a trio. Each small white dot represents a spin. Each thin black line connects the two spins in a singlet. This figure resembles Fig. 3a of [24], as $|\text{AKLT}_{\text{hon}}\rangle$ resembles $|\text{AKLT}_{\text{hon}}\rangle$. This figure does not illustrate the spatial arrangement of Posners in $|\text{AKLT}_{\text{hon}}\rangle$. Rather, the figure illustrates the entanglement in $|\text{AKLT}_{\text{hon}}\rangle$.

VII. EFFICIENT PREPARATION OF POSNER MOLECULES IN UNIVERSAL QUANTUM-COMPUTATION RESOURCE STATES

How complicated an entangled state can Posner operations (Sec. V) prepare efficiently? Many measures quantify multipartite entanglement. We study computational resourcefulness. Posner operations, we show, can efficiently prepare a state that fuels universal MBQC: By operating on the state locally, one can efficiently simulate a universal quantum computer.

The Posner state is a variation on an Affleck-Lieb-Kennedy-Tasaki (AKLT) state. AKLT first studied a one-dimensional (1D) chain of spin-1/2 particles. They constructed a nearest-neighbor antiferromagnetic Hamiltonian [21,23]. The ground state, $|\text{AKLT}_{1D}\rangle$, has a known form. A constant gap, independent of the system size, separates the lowest two energies.

$|\text{AKLT}_{1D}\rangle$ has many applications in quantum computation [20,24,59,68]. For example, $|\text{AKLT}_{1D}\rangle$ was the first state recognized as a matrix product state
Measurements are defined. Spin-$\frac{3}{2}$ particles on a honeycomb lattice can occupy the state $|{\text{AKLT}}_{\text{hon}}\rangle$ \cite{62,63}. Local operations on $|{\text{AKLT}}_{\text{hon}}\rangle$ can efficiently simulate universal quantum computation \cite{20,24}. We will draw on the proof by Wei et al. \cite{20,24}. Reference \cite{20} contains results related to the preparation similarly to \cite{21}. The graph $G$ is random, depending on the set $A$ of measurement outcomes. Also the encoding depends on $A$. The overline in $|{\text{AKLT}}_{\text{hon}}\rangle$ represents the encoding. Wei et al. prescribe local measurements of a few qubits. The measurements convert $|G(A)\rangle$ into a cluster state on a 2D square lattice (if $A$ is a typical set) \cite{22}. Such cluster states serve as resources in universal MBQC \cite{12,13,19,20}. By measuring single qubits adaptively, one can efficiently simulate a universal quantum computer.

We introduce a variation on $|\text{AKLT}_{\text{hon}}\rangle$. We call the variation the AKLT stage and denote the state by $|\text{AKLT}_{\text{hon}}\rangle$. Figure 14 illustrates the state. $|\text{AKLT}_{\text{hon}}\rangle$ is prepared similarly to $|\text{AKLT}_{\text{hon}}\rangle$, resembles $|\text{AKLT}_{\text{hon}}\rangle$ locally, and fuels universal MBQC similarly.

This section is organized as follows. Section VII A reviews the set-up and the state construction of Wei et al. $|\text{AKLT}^\prime_{\text{hon}}\rangle$ is defined in Sec. VII B. How to construct $|\text{AKLT}_{\text{hon}}\rangle$ efficiently from phosphorus nuclear spins, using Posner operations, is detailed. Section VII C describes the reduction of $|\text{AKLT}^\prime_{\text{hon}}\rangle$ to a 2D cluster state, known to fuel universal MBQC. The protocol is analyzed in Sec. VII D.

$|\text{AKLT}^\prime_{\text{hon}}\rangle$ holds interest not only as a computational resource, but also in its own right. The state is analyzed in Sec. VII E. For instance, AKLT$^\prime$ is shown to be a PEPS.

VII A. Set-up by Wei et al.

Wei et al. consider a 2D honeycomb lattice, illustrated in Fig. 3a of \cite{23}. (Figure 14 has nearly the same form.) A black dot represents each site. At each site sit three spin-$\frac{1}{2}$ DOFs. White dots represent these DOFs, called virtual spins.

Let $s_{123}$ and $m_{123}$ denote a site’s total spin quantum number and total magnetic spin quantum number. These numbers can assume the values $(s_{123}, m_{123}) = (\frac{1}{2}, \pm \frac{1}{2})$, $(\frac{1}{2}, \pm \frac{1}{2})$, and $(\frac{1}{2}, \pm \frac{1}{2})$. The qubit trio can behave as one physical spin of $s_{123} = \frac{1}{2}$ or $\frac{3}{2}$. $|\text{AKLT}_{\text{hon}}\rangle$ may be prepared as follows \cite{21}:

1. Consider two nearest-neighbor sites. Choose a virtual spin in each site. Form a singlet $|\Psi^-\rangle$ between these spins. Perform this process on every pair of nearest-neighbor sites.

2. Project each physical spin (each site) onto its $s_{123} = \frac{3}{2}$ subspace. $|\text{AKLT}_{\text{hon}}\rangle$ is trivalent: Each site links, via singlets, to three other sites.

VII B. Preparing Posner molecules in $|\text{AKLT}^\prime_{\text{hon}}\rangle$.

Posner operations (Sec. V) can nearly prepare $|\text{AKLT}_{\text{hon}}\rangle$. Whether Posner operations can project trios onto their $s_{123} = \frac{3}{2}$ subspaces remains unknown. But Posner operations can project onto a molecule’s $\tau = 0$ subspace.

The $\tau = 0$ subspace decomposes into a direct sum of tensor products of two three-qubit subspaces. The first three-qubit subspace is labeled by $s_{123}$, the total spin quantum number of the qubit triangle at $z_{\text{in}} = h_{\text{in}}$. The second three-qubit subspace is labeled by $s_{456}$. The $\tau = 0$ subspace has the form $|\Psi^{-}\rangle = \frac{1}{\sqrt{2}} \left( |\frac{3}{2} \otimes \frac{3}{2} \rangle \right)$. (See Appendices A and B for a derivation. See \cite{69} for background and notation.) The first term represents the space $s_{123} \otimes s_{456} = \frac{3}{2} \otimes \frac{3}{2}$ of two spin-$\frac{3}{2}$ particles. Wei et al. project onto this space, in step 2.

Projecting onto the larger $\tau = 0$ space yields $|\text{AKLT}^\prime_{\text{hon}}\rangle$. We now detail how Posners can come to occupy $|\text{AKLT}^\prime_{\text{hon}}\rangle$. The steps are explained in physical terms (of molecules, binding, etc.). Figure 15 recasts the protocol in computational terms, as a quantum circuit:

1. Pyrophosphatase enzymes cleave some number $N$ of diphosphates. $N$ singlets $|\Psi^-\rangle$ are prepared (via operation 1).

2. The phosphates group together in trios. Singlets connect the trios as thin black lines connect the white dots in Fig. 14.

3. Each trio, with a nearest-neighbor trio, forms a Posner molecule (via operation 2).

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23 Measurements are adaptive if earlier measurements’ outcomes dictate which measurements are performed later.

24 A graph state is defined in terms of a graph $G$. Each vertex corresponds to a spin. Consider the Hamiltonian

$$H_G = \sum_{i \in G} \left( \sigma_i^x \bigotimes_{k \in \text{NB}(i)} \sigma_k^z \right).$$

$i$ indexes the vertices in $G$. The nearest neighbors of $i$ are indexed by $k \in \text{NB}(i)$. $H_G$ has a unique ground state, called a graph state \cite{12,15}.

25 A cluster state is a graph state associated with a regular lattice $G$ \cite{12,17,15}.
4. Posners approach each other with the prebinding orientation (see operation 5), then drift apart, as described in Proposition 2. For simplicity, we focus on the case in which each Posner $P$ approaches only Posners $P'$ that are nearest neighbors of $P$ in the hexagonal lattice (Fig. 14). But this assumption is unnecessary.

Suppose that each approach leads to binding. Proposition 2 is realized. Every Posner’s state is projected onto the $\tau = 0$ subspace. But two approaching Posners might fail to bind. The success probability $\frac{26}{128} \approx 0.203$. Suppose that Posners $P$ and $P'$ fail to bind. Suppose that $P$ has already been projected onto its $\tau_P = 0$ subspace. $P'$ can be “refreshed”: Let $P'$ drift into a region of high pH and/or high $\text{Mg}^{2+}$ concentration. $P'$ likely hydrolyzes (undergoes operation 4). Two of the phosphorus nuclear spins used to form a singlet internal to the Posner. These spins form a singlet no longer, due to the binding failure. These two spins can drift away; a fresh singlet can replace them. Four other phosphorus nuclear spins remain. They continue to form singlets with spins in other Posners. The two new, and four old, phosphates can form a Posner $\tilde{P}'$, via operation 2. $\tilde{P}'$ occupies the state that $P$ occupied before the binding failure. $\tilde{P}'$ can approach $P$ with the prebinding orientation. If the binding fails, $\tilde{P}'$ can be refreshed again.

VII C. Reduction of $|\text{AKLT}_{\text{hon}}\rangle$ to a cluster state known to fuel universal MBQC

Local operations can reduce $|\text{AKLT}_{\text{hon}}\rangle$ to a cluster state on a 2D square lattice. Such cluster states serve as universal resources in MBQC. Section VII C 1 reviews the reduction in Wei et al. Section VII C 2 explains the need to deviate from this reduction. Section VII C 3 details the deviation.

VII C 1. Model: Reduction of $|\text{AKLT}_{\text{hon}}\rangle$ to a cluster state

Wei et al. prescribe two steps. First, each site is measured with a POVM. The measurements yield a state equivalent, under local unitaries, to a random graph

This probability is calculated as follows. The $N$ Posners occupy some pure state $|\psi\rangle$. Consider the two Posners’ joint reduced state, $\rho_{PP'}$. The Posners share one singlet. The Posner pair contains ten other phosphorus nuclear spins. Let $a$ denote an arbitrary one of these other spins. $a$ forms a singlet with a spin in some other Posner, $P''$. $P''$ is traced out from $|\psi\rangle$ in this calculation of $\rho_{PP'}$. Hence $\rho_{PP'}$ equals a tensor product of $\frac{1}{2}$ and $|\Psi^\perp\rangle$: $\rho_{PP'} = \left(\frac{1}{2}\right)^{\frac{13}{2}} \otimes |\Psi^\perp\rangle$. Posners $P$ and $P'$ have a probability $\approx \text{Tr}(\mathcal{P}_{PP'}\rho_{PP'}) = \frac{11}{12} \approx 0.336$ of binding. $\mathcal{P}_{PP'}$ is defined as in Eq. (30). This approximation does not account for all correlations amongst sites but is expected to capture the greatest contribution to the probability. Exact calculations are left as an opportunity for future study.

This claim can be checked via direct calculation. Computational resources limited our calculation to the reduced state of 13 spins. Whether longer-range correlations affect the results is left for future study. See footnote 26.
The POVMs reduce each site's Hilbert space to a qubit spanned by the analogous $S_{123}^{\alpha}$ eigenstates, for $\alpha = x, y, z$.

Each subspace has dimensionality two. Hence the POVM projects onto the subspace spanned by the analogous $S_{123}^{\alpha}$ eigenstates, for $\alpha = x, y, z$.

The $F_{v,z}$ projects onto the subspace spanned by the $S_{123}^{\alpha}$ eigenstates associated with the magnetic spin quantum numbers $m_{123} = \pm \frac{\hbar}{2}$. $F_{v,\alpha}$ projects onto the subspace spanned by the analogous $S_{123}^{\alpha}$ eigenstates, for $\alpha = x, y, z$.

The POVM reduces each site's Hilbert space to a qubit space. The $\frac{\sqrt{2}}{2}$ leads to the completeness relation $\sum_{\alpha=x,y,z} F_{v,\alpha} = 1$.

$\mathcal{A}$ denotes the set of POVM outcomes. The POVMs yield a state $|G(\mathcal{A})\rangle$. $G(\mathcal{A})$ denotes a random graph whose form depends on $\mathcal{A}$. $|G(\mathcal{A})\rangle$ denotes the graph state associated with $G(\mathcal{A})$. The overline denotes an encoding dependent on $\mathcal{A}$. The system occupies a state equivalent, via the encoding, to $|G(\mathcal{A})\rangle$.

A random graph $G(\mathcal{A})$ defines $|G(\mathcal{A})\rangle$. In contrast, a regular graph defines a cluster state. The cluster state on a 2D square lattice serves as a universal resource in MBQC. This cluster state can be distilled from $|G(\mathcal{A})\rangle$, if $\mathcal{A}$ is typical. The distillation consists of a few single-qubit Pauli measurements [23].

### VII C 2. **Towards a reduction of $|\text{AKLT}_{\text{hon}}\rangle$ to a cluster state**

The Wei et al. system differs from the Posner system in two ways. First, Posner operations cannot necessarily simulate (i) the POVM [65] or (ii) the Pauli measurements in Sec. VII C 1. Whether Posner operations can remain an open question. Second, Wei et al. invoke $|\text{AKLT}_{\text{hon}}\rangle$. Posner operations can prepare $|\text{AKLT}_{\text{hon}}\rangle$.

Posners therefore require a step absent from [24]. To facilitate the explanation, we invoke the agent framework of QI (Sec. VII B 2). Different experimentalists can perform different operations easily. An agent Alice might run a biochemistry lab. She might be able to effect Posner operations. An agent Bob might be able to perform local POVMs but not to create and arrange singlets.

Together, Alice and Bob could produce cluster states. Alice would create $|\text{AKLT}_{\text{hon}}\rangle$ and pass the state to Bob. Bob would perform local POVMs. (He might ask Alice to refresh a few Posners.) Together, the agents would form cluster states that fuel universal MBQC.

### VII C 3. **Reduction of $|\text{AKLT}_{\text{hon}}\rangle$ to a cluster state**

Bob will perform the protocol in Sec. VII C 1. But first, he measures each Posner’s $S_{123}^2 \otimes S_{456}^z$. Suppose that Posner $P$ yields the outcome labeled by $\frac{\sqrt{2}}{2}$. The measurement has succeeded.

Now, suppose that Posner $P$ yields the outcome labeled by $\frac{\sqrt{2}}{2}$. The measurement has failed. Bob returns $P$ to Alice. Alice hydrolyzes $P$ via operation [1]. She refreshes the internal singlet, as in step [1] in Sec. VII B. Let $P$ denote the refreshed Posner. Bob measures the $S_{123}^2 \otimes S_{456}^z$ of $P$. He and Alice “repeat until success” (until obtaining the $\frac{3}{2}$ outcome).

Bob holds an $|\text{AKLT}_{\text{hon}}\rangle$ state. He now follows the prescription of Wei et al. (Sec. VII C 1).

### VII D. **Analysis of $|\text{AKLT}_{\text{hon}}\rangle$ preparation**

Posner operations, we have shown, can prepare $|\text{AKLT}_{\text{hon}}\rangle$ efficiently. A few local measurements reduce $|\text{AKLT}_{\text{hon}}\rangle$ to a cluster state on a 2D square lattice. This cluster state can be used directly in universal MBQC. Posners’ universality is remarkable: Most quantum states cannot power universal MBQC [1]. The Posners’ singlets and their geometry (the decomposition of Posners into triangles, and the triangles’ trivalence), underlie the state’s universality [27].

Opportunities for enhancing and simplifying our protocol exist:

1. A precise structure—a honeycomb lattice—underlies $|\text{AKLT}_{\text{hon}}\rangle$. In contrast, biomolecules drift randomly. Biological singlets might not form honeycombs of their own accord. Random graphs might likely arise. Such graphs underlie states that might power MBQC.

Such graphs might have two or three dimensions. Three-dimensional (3D) graph states offer particular opportunities.

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28 $P$ contained four spins apart from the internal singlet. Each of these spins remains in a singlet with a spin in another Posner. See the calculational comments in footnote 27.

29 $|\text{AKLT}_{\text{hon}}\rangle$ is not the simplest universal Posner resource state. Singlets on a trivalent lattice would suffice. The $\tau = 0$ projections are unnecessary, by the second equality in Eq. (31) of [24]. Yet $|\text{AKLT}_{\text{hon}}\rangle$ merits defining, for three reasons. First, preparing $|\text{AKLT}_{\text{hon}}\rangle$ is natural: In $|\text{AKLT}_{\text{hon}}\rangle$, pairs of sites are projected onto their $s_{123} \otimes s_{456} = \frac{1}{2} \otimes \frac{1}{2}$ subspaces. In $|\text{AKLT}_{\text{hon}}\rangle$, pairs of sites are projected onto slightly larger subspaces. Hence $|\text{AKLT}_{\text{hon}}\rangle$ resembles $|\text{AKLT}_{\text{hon}}\rangle$ locally. Second, suppose that Alice did not project the Posners onto their $\tau = 0$ subspaces. Bob would obtain more “error” outcomes, labeled by $\frac{3}{2}$, in Sec. VII C 3. Third, $|\text{AKLT}_{\text{hon}}\rangle$ holds interest in its own right (Sec. VII E).

30 Onuma and Ito offer hope that Posners form regular lattices, however. Hydroxyapatite is the mineral $\text{Ca}_9(\text{PO}_4)_6(\text{OH})_2$. A variation on hydroxyapatite forms much of bone. Onuma and Ito proposed that hydroxyapatite crystals grow as Posners form hexagonal lattices [55].
2. Bob might avoid returning Posners to Alice. The Posners’ triangles (Fig. 1) form the sites in a hexagonal lattice (Fig. 14). Consider coarse-graining two sites into one. Triangle pairs are coarse-grained into Posners’ triangles (Fig. 1) form the sites in a hexagonal lattice. The coarse-grained lattice is more analogous properties, of AKLT states approximately. The 1D AKLT state resembles the AKLT state |AKLT\rangle on a honeycomb lattice. AKLT states have remarkable properties. We discuss analogous properties, and opportunities to seek more analogous properties, of |AKLT\rangle.

First, classical resources can compactly represent AKLT states approximately. The 1D AKLT state |AKLT_{1D}\rangle is an MPS [62, 63]. |AKLT\rangle is a PEPS [64, 65, 76]. |AKLT\rangle is a PEPS, illustrated in Fig. 17 and detailed in App. 4.

Hence |AKLT\rangle is the ground state of some local, frustration-free Hamiltonian \(H_{\text{AKLT}}\). The ground state is unique [27]. The relationship between \(H_{\text{AKLT}}\) and the Posner Hamiltonian \(H_{\text{Pos}}\) merits study. So does whether \(H_{\text{AKLT}}\) has a constant-size gap [21, 22]. If \(H_{\text{AKLT}}\) has, |AKLT\rangle can be prepared efficiently via cooling.

Third, |AKLT\rangle results from deforming |AKLT\rangle. AKLT states have been deformed via another strategy [73, 77, 79]: Let \(H_{\text{AKLT}}\) denote the Hamiltonian.

Bob’s site-deletion probability exceeds the threshold: \(p \approx \frac{2}{3} > 0.59 \approx p_*\). A large, richly connected component spans Bob’s graph. Such components underlie universality [24].

Bob returns to regarding triangles, rather than Posners, as vertices. The lattice looks hexagonal but contains holes. A large connected component spans also this graph. Hence the Wei et al. prescription (Sec. VII C 1), or a related prescription, appears likely to transform the state into a universal cluster state.

To check, one might refer to [71–73]. The authors consider faulty lattices: Sites might be deleted, as by measurement.

3. Universal quantum computation is unnecessary for achieving quantum supremacy [73]. Suppose that |AKLT\rangle has been converted into a cluster state on a 2D square lattice. Consider measuring single qubits nonadaptively. A random distribution \(P\) is sampled. Classical computers are expected not to be able to sample from \(P\) efficiently [75].

Aside from these opportunities, the calculational technique in footnote 26 may be rendered more precise.

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FIG. 16: Coarse-graining the hexagonal lattice into a square lattice: The black lines form a hexagonal lattice (see Fig. 14). Three qubits (small white dots) occupy each site (large black dot). Two neighboring sites form a Posner molecule (encircled with a dashed hoop). The lattice can be coarse-grained: Each Posner’s two sites can be lumped together (into a red dot). The coarse-grained lattice is square (as shown by the long, red lines). This coarse-graining might facilitate a universality protocol simpler than the one in Sec. VII C 3.

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31 This probability is estimated via the technique described in footnote 26.

32 Site percolation is a topic in graph theory and statistical mechanics. Let \(G\) denote a graph of \(N\) sites. Consider deleting each site \(v\) with probability \(1 - p\). If \(v\) is deleted, so are the edges that terminate on \(v\). Let \(G'\) denote the remaining graph. Does a path of edges traverse \(G'\) from top to bottom? If so, \(G'\) percolates. \(p_*\) denotes the percolation threshold. If \(p \geq p_*\), \(G'\) percolates in the limit as \(N \to \infty\). \(G'\) does not if \(p < p_*\). A phase transition occurs at \(p = p_*\).
whose ground state is the AKLT state of interest. $H_{\text{AKLT}}$ is transformed with a deformation operator $D(a)$ \cite{73}. The parameter $a$ is tuned, changing the ground state.

$|\text{AKLT}'_{\text{hon}}\rangle$ follows from a different deformation. We start not from a Hamiltonian, but from the Hilbert space $(\mathbb{C}^3)^{\otimes 2}$. Singlets are arranged; then the state is projected onto the $\tau = 0$ eigenspace $H_{\tau=0}$. $H_{\tau=0}$ contains the $3^2 \otimes 3^2$ subspace. Projecting onto the latter subspace would yield an AKLT state. Enlarging the projector deforms the state.

Wei et al. study an AKLT state’s computational power as a function of $a$ \cite{73}. Our state’s computational power might be studied as a function of the projected-onto space.

VIII. ENTANGLEMENT’S EFFECT ON MOLECULAR-BINDING RATES

Consider two Posners approaching each other with the prebinding orientation described below operation $5$. The Posners might bind together. They could form subsystems in a many-body entangled system. Entanglement affects the binding probability, Fisher proposes \cite{1}.

Fisher illustrates with an example \cite{II} p. 5, Fig. 3. Let $a$, $a'$, $b$, and $b'$ denote Posners. Let $a$ be entangled with $a'$, and let $b$ be entangled with $b'$. Suppose that $a$ has bound to $b$. Suppose that $a'$ approaches $b'$ with the prebinding orientation. $a'$ and $b'$ have a higher probability of binding, Fisher argues, than in the absence of entanglement. We recast this narrative as a quantum circuit in Fig. 13.

Fisher supports his proposal by analyzing position-and-spin states. We use, instead, the Posner-binding PVM \cite{31}. Checking Fisher’s example lies beyond our (classical) computational power. But we check the principle behind his example quantitatively: Entanglement, we show, can affect the probability that two Posners bind (Fig. 18 and Sec. VIII A). We identify a percent increase of 198%. This technique can be scaled up to analyze arbitrarily many Posners. Random rotations, we find in Sec. VIII B, can eliminate entanglement’s effect on average binding rates.

VIII A. Illustration: Entanglement’s effect on binding rates

Let $A$ and $B$ denote two Posners. We illustrate with four cases.

1. Base case: no singlets: Suppose that the $AB$ system contains no singlets (Fig. 18a). Every phosphorus nuclear spin forms a singlet with an external molecule. The reduced state of $AB$ equals a product of maximally mixed states: $\rho_{AB} = \frac{1}{64} \otimes \frac{1}{64}$. The identity operator defined on $\mathbb{C}^k$ is denoted by $\mathbb{1}_k$.

Suppose that the Posners approach each other with the prebinding orientation. The Posners have a probability

$$p_{AB} = \text{Tr} (\Pi_{AB} \rho_{AB}) = \frac{43}{128} \approx 0.336 \quad (66)$$

of binding.

2. 2 singlets shared: Consider any two qubits in the same triangle of $A$, e.g., qubits 4 and 5 in Fig. 18b. Let each of these qubits share a singlet with the corresponding qubit in either triangle of $B$. For example, let 4 share a singlet with 7, and let 5 share a singlet with 8. The Posners occupy the state

$$\rho'_{AB} = \left(\frac{1}{2}\right)^{\otimes 4} \otimes 4,7|\Psi^+\rangle\langle\Psi^-|4,7$$

$$\otimes 5,8|\Psi^+\rangle\langle\Psi^-|5,8 \otimes \left(\frac{1}{2}\right)^{\otimes 4}. \quad (67)$$

Suppose that the Posners approach each other with the prebinding orientation. They have a probability

$$p'_{AB} = \text{Tr}(\Pi_{AB} \rho'_{AB}) \quad (68)$$

$$= 0.34375 \approx 0.344 \quad (69)$$
of binding. Entanglement raises the binding probability by a fraction $p_{AB}^{\prime\prime} - 1 \approx 0.238$, or by $\approx 2.38\%$.

3. **3 singlets shared**: Let one qubit trio in $A$ be maximally entangled with a qubit trio in $B$. Let one singlet link each $A$ qubit to the geometrically analogous qubit in the $B$ trio. For example, qubits 4, 5, and 6 can form singlets with qubits 7, 8, and 9, respectively (Fig. 18c). The Posners would occupy the state

$$\rho_{AB}^{\prime\prime} = \left( \frac{1}{2} \right)^{\otimes 3} \otimes 4.7|\Psi^{-}\rangle \langle \Psi^{-}| + 6.9|\Psi^{-}\rangle \langle \Psi^{-}| + \left( \frac{1}{2} \right)^{\otimes 3}. \; (70)$$

Suppose that the Posners approached each other with the prebinding orientation. They would have a probability

$$\rho_{AB}^{\prime\prime} = \text{Tr}(\Pi_{AB} \rho_{AB}^{\prime\prime}) = 0.375 \; (71)$$

of binding. The probability has risen by $\frac{\rho_{AB}^{\prime\prime}}{\rho_{AB}^{\prime}} - 1 \approx 0.116$, or by $11.6\%$, from the base case.

4. **Maximal entanglement: 6 singlets shared**: Suppose that each $A$ qubit forms a singlet with the corresponding $B$ qubit (Fig. 18d). The qubits occupy the state

$$\rho_{AB}^{\prime\prime\prime} = \left( |\Psi^{-}\rangle \langle \Psi^{-}| \right)^{\otimes 4}. \; (72)$$

The binding probability rises to

$$\rho_{AB}^{\prime\prime\prime} = \text{Tr}(\Pi_{AB} \rho_{AB}^{\prime\prime\prime}) = 1, \; (73)$$

in an approximately $198\%$ increase over the base case. Such maximal entanglement ensures that the Posners always bind.

Linking the Posners with just one singlet appears not to raising the binding probability above the baseline $p_{AB}$. The technique illustrated here can be scaled up. With more classical computational power, one can store larger quantum states. The four-Posner conjecture illustrated in Fig. 13 can be checked. So can entanglement’s effect on the binding probabilities of swarms of Posners.

**VIII B. Random rotations can eliminate entanglement’s effect on average molecular-binding rates.**

Consider twelve phosphorus nuclear spins in a joint state $\rho_{AB}^{\prime}$. Suppose that independent currents randomly rotate the nuclei. Each qubit $a$ evolves under some unitary $U_{a}(\theta_{a})$. The rotation axis $\hat{n}_{a}$ and the rotation angle $\theta_{a}$ may be distributed uniformly.

Suppose that six nuclei form Posner $A$, while the other nuclei form Posner $B$. The molecules occupy the joint state

$$\rho_{AB}^{\prime\prime} = \left( \prod_{a=1}^{12} U_{\hat{n}_{a}}(\theta_{a}) \right) \rho_{AB}^{\prime} \left( \prod_{a=1}^{12} U_{\hat{n}_{a}}(\theta_{a}) \right)^{\dagger}. \; (74)$$

FIG. 18: Illustration of entanglement’s effect on molecular-binding rates: The large black circles represent Posner molecules $A$ and $B$. Each molecule contains six phosphorus nuclear spins, represented by small black circles. The circles group together in trios (see Fig. 1). A dashed line represents the separation between a Posner’s trios. Two small circles connected by a green, wavy line form a singlet $|\Psi^{-}\rangle$ [Eq. (1)]. Some spins form singlets with spins in external molecules. Hence some green, wavy lines have ends that do not terminate on spins in the diagram. Figure 18(a) shows the base case, a Posner pair that contains no singlets. These Posners have a probability $p_{AB}^{\prime\prime} - 1 \approx 0.238$, or by $\approx 2.38\%$.

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**VIII B. Random rotations can eliminate entanglement’s effect on average molecular-binding rates.**

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Suppose that six nuclei form Posner $A$, while the other nuclei form Posner $B$. The molecules occupy the joint state

$$\rho_{AB}^{\prime\prime} = \left( \prod_{a=1}^{12} U_{\hat{n}_{a}}(\theta_{a}) \right) \rho_{AB}^{\prime} \left( \prod_{a=1}^{12} U_{\hat{n}_{a}}(\theta_{a}) \right)^{\dagger}. \; (74)$$
Suppose that $A$ and $B$ approach each other with the pre-binding orientation. The Posners have a probability

$$\rho''_{AB} \left\{ \hat{n}_a \right\} = \text{Tr} \left( \Pi_{AB} \rho''_{AB} \left\{ \hat{n}_a \right\} \right) \quad (75)$$

of binding.

On average over rotations, the Posners have a probability

$$\overline{p''}_{AB} = \frac{1}{m} \sum_{\{\hat{n}_a\}} \text{Tr} \left( \Pi_{AB} \rho''_{AB} \left\{ \hat{n}_a \right\} \right) \quad (76)$$

$$= \text{Tr} \left( \Pi_{AB} \left[ \left( \frac{1}{2} \right)^{\otimes 3} \otimes \int \left\{ \hat{n}_a \right\}_{a=1}^{10} \right] \left| \hat{\Psi}_- \right\rangle \left\langle \hat{\Psi}_- \right| \left( \bigotimes_{a=3}^{10} U_{\hat{n}_a}^{\dagger} (\theta_a) \right) \left( \frac{1}{2} \right)^{\otimes 3} \right) \quad (77)$$

$$= \text{Tr} \left( \Pi_{AB} \left( \frac{1}{2} \right)^{\otimes 12} \right) \quad (78)$$

$$= p_{AB} \quad (79)$$

of binding. Uniformly random rotations effectively decohere the internal singlets, on average. The binding probability reduces to its non-singlet-enhanced value.

### IX. MEASURING QUANTUM COGNITION AGAINST DIVINCENZO’S CRITERIA FOR QUANTUM COMPUTATION AND COMMUNICATION

Consider attempting to realize universal quantum computation and quantum communication with any physical platform. Which requirements must the physical components and processes satisfy? DiVincenzo catalogued these requirements [22]. Five of diVincenzo’s criteria underpin quantum computation. Quantum communication requires another two criteria. Quantum cognition, we find, satisfies DiVincenzo’s criteria, except perhaps universality.

We continue assuming that Posner operations can be performed with fine control. This assumption is discussed in Sec. V B 2. The assumption may appear questionable here: Practicabilities partially concerned DiVincenzo. Yet the fine-control assumption facilitates a first-step analysis of what the model can achieve in principle. Incorporating randomness forms an opportunity for future research.

1. “A scalable physical system with well-characterized qubits”: Different physical DOFs encode QI at different stages of a quantum-cognition computation (Sec. III). Initially, phosphorus-31 ($^{31}$P) atoms’ nuclear spins serve as qubits. These atoms occupy free-floating phosphate ions.

Six phosphates can join together, forming a Posner molecule. Each state of six independent spins transforms into one antisymmetrized state [7]. Hence anti-symmetrized spin-and-position states store QI.

Third, each Posner has an observable $G_C$ (Sec. IV A). The eigenvalue $\tau$ assumes one of three possible values: $\tau = 0, \pm 1$. The eigenspaces $\mathcal{H}_\tau$ have 24-, 20-, and 20-fold degeneracies. A state $|\tau=j\rangle$ can be chosen from each $\mathcal{H}_\tau$ eigenspace. span $\{|\tau=0\rangle, |\tau=1\rangle, |\tau=2\rangle\}$ forms an effective qutrit space (Sections IV D 1 and VI). But Posners are not associated uniquely with effective qutrits, to our knowledge (footnote 10).

QI-storing Posner systems can be scaled up spatially. $^{31}$P nuclear spins can form singlets distributed across Posners. Entangled Posners can form lattices that can power universal MBQC (Sec. VII).

2. “The ability to initialize the state of the qubits to a simple fiducial state, such as $|000\ldots\rangle$”: Phosphorus nuclear spins can be prepared in singlets $|\Psi^\prime\rangle$ (operation 1). A singlet forms as the enzyme pyrophosphatase cleaves a diphosphate ion. The resultant two phosphates are projected onto $|\Psi^\prime\rangle$.

3. “Long relevant decoherence times, much longer than the gate-operation time”: Fisher has catalogued sources of decoherence and has estimated coherence times [1, 6, 9, 80]. Nearby spins and electrons generate electric and magnetic fields. Protons H$^+$ threaten $^{31}$P spins the most. The $^{31}$P spins can entangle with external DOFs via magnetic dipole-dipole coupling.

But phosphates, Posners, and other small particles tumble in solution. As the particles move around each other, the fields experienced by a $^{31}$P changes. The fields are expected to vanish on average over tumbles.

$^{31}$P nuclear spins in free phosphates, Fisher estimates, remain coherent for about a second. $^{31}$P nuclear spins in Posners remain coherent for $\sim 10^5 - 10^6$ s.

Fisher writes also that $\tau$ labels a pseudospin “very isolated from the environment, with potentially extremely long (days, weeks, months, …) decoherence times” [80]. As explained in Sec. IV A, to our knowledge, $\tau$ does not label any unique quantum three-level pseudospin. We translate Fisher’s statement as follows: Consider preparing a $G_C$ eigenstate associated with the eigenvalue $\tau = j$. Consider waiting, then measuring $G_C$. How long would you have to wait to have a high probability of obtaining an outcome $\tau = k \neq j$? At least days.

These coherence times, we expect, exceed entangling-gate times. Posner binding (operation 5) consists of electronic dynamics. Electrons have much shorter time scales than nuclei, in general. Posner binding releases about 1 eV of energy to the environment [1, 9]. A rough estimate for the binding’s time scale is $t_{\text{bind}} \sim \frac{\hbar}{k_B T} \sim 10^{-15}$ s $\ll 10^5$ s. As many as $10^{20}$ entangling gates might be performed before the spins decohere.

Could many single-qubit rotations be performed? We estimate that a qubit can rotate through an angle $\sim \pi$ in a time $t_{\text{rot}} \sim 10$ s $\ll 10^5$ s, in the best case (App. K). Hence the best-case single-qubit-rotation time scale is much less than the out-of-Posner decoherence time.

4. “A ‘universal’ set of quantum gates”: The biological qubits can undergo Posner operations, introduced in Sec. IV A. The operations include single-qubit
rotations and entangling operations. Analyses appear in Sections V.B and VII.

We detail the rotation mechanism in App. K. Neuron firing generates magnetic fields $B$. We model these fields as external and classical. (These fields contrast with the influence, mentioned earlier, of $H^+$ spins. $H^+$ spins can entangle with $^{31}\text{P}$ nuclear spins, decohering the qubits. External classical fields cannot, to a good approximation.) Let $\mu$ denote a $^{31}\text{P}$ nucleus’s spin magnetic moment. Neuronal-current fields could rotate the spins via the Hamiltonian $H_{\text{mag}} = -\mu \cdot B$. In the best case, spins could rotate through angles of up to $\pi$. We expect typical rotations to be through much smaller angles, though.

Qubit gates induce gates on the effective qutrits. The induced gates depend on how the qutrits are defined.

Whether the gates form a universal set—in transforming the qubits or the qutrits—remains an open question. Posner operations can efficiently prepare a state $|\psi\rangle$ (Section V.B and VII), so the question of whether two Posners have bound remains unknown.

5. “A qubit-specific measurement capability”: Posner binding (operation $\Pi_{\text{no-coll}}^\tau$) measures whether $\tau_A + \tau_B = 0$. Whether two molecules have bound is a classical property. Each Posner’s center of mass is a classical DOF: Water and other molecules bounce off the Posner frequently. The bounces measure the Posner’s position, precluding coherence. Hence two Posners’ closeness is a classical DOF. Closeness serves as a proxy for binding. Hence whether two Posners have bound is a classical DOF. This classical memory records whether $\tau_A + \tau_B = 0$.

Fisher proposed that the readout might be amplified further. Posner binding could impact later Posner binding, then the hydrolyzation of Posners, then neurons’ Ca$^{2+}$ concentrations, and then neuron firing. This process is overviewed in this paper’s Sec. I.

6. “The ability to interconvert stationary and flying qubits”: Computing is often easiest with unmov- ing hardware. Stationary qubits remain approximately fixed. They undergo computation. Then, their state can be transferred to flying qubits. Flying qubits move easily. They can bring states together for joint processing.

Consider, for example, a quantum algorithm that contains subroutines. Different labs’ quantum dots could implement different subroutines. The quantum dots’ states could be converted into photonic states. The photons could travel down optical fibers to a central lab. There, the algorithm’s final steps could be implemented.

Phosphorus nuclear spins could serve as stationary qubits and as flying qubits. Phosphorus atoms occupy lone phosphates and Posners. In each setting, the nuclear spins undergo computations (Sec. V.A). But Posners protect the spins from decoherence better than lone phosphates do. Hence Posners form better flying qubits.

The projector $\Pi_{\text{no-coll}}^\tau$ transforms phosphate states into a Posner state. The projection forms a one-to-one map. Hence “stationary” phosphates’ states are converted into a “flying” Posner’s state faithfully.

7. “The ability faithfully to transmit flying qubits between specified locations”: Posners diffuse through intracellular and extracellular fluid. A protein could transport Posners into neurons I: the vesicular glutamate transporter (VGLUT) [81–84], alias the brain-specific (B) sodium-dependent (Na$^+$) inorganic-phosphate (Pi) cotransporter (BNPI) [55]. VGLUT sits in cell membranes, through which the protein could ferry Posners. Posners protect in-transit $^{31}\text{P}$ nuclear spins for $\sim 10^{-9} - 10^{-6}$ s I, as discussed above.

For how long do Posners diffuse between neurons? We estimate by dimensional analysis. The diffusion constant $D$ has dimensions of distance$^2$/time:

$$D \sim \frac{\ell^2}{t_{\text{diff}}}. \quad (80)$$

The time scale over which a Posner diffuses between neurons is denoted by $t_{\text{diff}}$. A typical synapse has an area of $10^{-2} \text{m}^2$ [80, Fig. 2].

We estimate $D$ via the Einstein-Stokes relation,

$$D = \frac{k_B T}{6\pi \eta r}. \quad (81)$$

Equation (81) describes a radius-$r$ sphere in a viscosity-$\eta$ fluid. Water has a viscosity $\eta \sim 10^{-3} \text{N} \cdot \text{s}/\text{m}^2$. A Posner molecule has a radius $r \sim 10 \text{ Å}$ I.

We substitute these numbers, with $k_B \sim 10^{-23} \text{ J/K}$, $T \sim 10^2 \text{ K}$, and $6\pi \times 10$, into Eq. (81): $D \sim 10^{-10} \text{ m}^2/\text{s}$. We substitute into Eq. (80), upon solving for $t_{\text{diff}}$:

$$t_{\text{diff}} \sim \frac{\ell^2}{D} \sim \frac{10^{-2}(10^{-6} \text{ m})^2}{10^{-10} \text{ m}^2/\text{s}} \quad (82)$$

$$\approx 0.1 \text{ ms} \ll 10^5 \text{ s}. \quad (83)$$

Hence Posners are expected to be able to traverse a synapse before their phosphorus nuclear spins decohere.

X. OUTLOOK

This paper establishes a framework for the QI-theoretic analysis of Posner chemistry. The paper also presents applications of Posners to QI processing: to QI storage and protection, to quantum communication, and to quantum computation. Many QI applications of Posners await discovery, we expect. In turn, QI motivates quantum-chemistry questions. Opportunities are discussed below.

Quantum error-correcting and -detecting codes: We presented one quantum error-detecting code and one error-correcting code accessible to Posners. Other accessible codes might protect more information against more errors.

Furthermore, one conserved charge “protects” each of our codes. In the error-detecting code, for example, the codewords $|j_+\rangle$ correspond to distinct eigenvalues of $G_C$. The natural dynamics protect $G_C$. Hence the dynamics should not map any codeword $|j_+\rangle$ into any other $|k_{-}\rangle$. 
But the dynamics could map \( |j_j \rangle \) to another state \( |j'_j \rangle \) in the \( \tau = j \) eigenspace.

Imagine a more robust code: A complete set of quantum numbers (e.g., \( \{ \tau, m_{1,6,\ldots} \} \)) would label each codeword. The dynamics could not map any codeword \( |\tau, m_{1,6,\ldots} \rangle \) into any other codeword \( |\tau', m'_{1,6,\ldots} \rangle \). Such a code would enjoy considerable protection by charge preservation.

Relatedly, quantum codes have been cast as the ground spaces of Hamiltonians. Every code’s states, \( |\psi \rangle \), occupy a Hilbert space \( \mathcal{H} \). Suppose that \( \mathcal{H} \) is the ground space of a Hamiltonian \( H \). Suppose that the system is in thermal equilibrium at a low temperature \( T = \frac{1}{k_B} \). The system has a high probability of remaining in \( \mathcal{H} \). Entropy suppresses errors. Equivalently, the code detects errors. The Posner Hamiltonian \( H_{\text{Pos}} \) was characterized shortly after this paper’s initial release [9]. The ground space might point to an entropically preserved a code.

Quantum algorithms: Posners might perform quantum algorithms of two types: (i) Known algorithms [87] might decompose into Posner operations. (ii) Posner operations could inspire hitherto-unknown quantum algorithms.

Reverse-engineering: QI processing could guide conjectures about quantum chemistry. Fisher reverse-engineered physical mechanisms by which entanglement could impact cognition [1]. Similarly, one might reverse-engineer physical mechanisms by which Posners could process QI. This paper motivates reverse-engineering opportunities:

1. Section [VII B] details how Posner operations can efficiently prepare states that can fuel universal MBQC. To use the states, one performs the operations in Sec. [VII C]. Example operations include (i) measurements of the POVM \( \{F_x, F_y, F_z \} \) [Eq. (34)] and (ii) adaptive single-qubit measurements. Could biological systems implement these operations?

2. Reverse-engineer a measurement of the generator \( G_C \) of the permutation operator \( C \). If \( G_C \) can be measured, incoherently teleported random variables can be used easily (Sec. [VI B 2]).

Quantum computational complexity and universality: Posner operations (Sec. [V]) constitute a model of quantum computation. Which set of problems can this model solve efficiently? Let PosQP denote the class of computational problems solvable efficiently with Posner quantum computation.

Whether Posner quantum computation is universal remains an open question. (See Sec. [V B 1] for an elaboration.) Suppose that the model were universal. PosQP would equal BQP (the class of problems that a quantum computer can solve in polynomial time [2]). But perhaps PosQP \( \subset \) BQP. PosQP merits characterization.

AKLT’ state and MBQC protocol: Posner operations can efficiently prepare a state \( |\text{AKLThom} \rangle \) that fuels universal MBQC (Sec. [VII]). The state preparation may be simplified. Opportunities are detailed in Sec. [VII D]. Also, \( |\text{AKLThom} \rangle \) holds interest outside of MBQC. Properties to explore are discussed in Sec. [VII E].

Entanglement’s effect on binding rates and biological Bell tests: Entanglement between Posners affects binding rates. So Fisher conjectured in [1]. The conjecture grew from analyses of spin-and-orbital states. We supported the conjecture with a two-Posner example, using a PVM (Sec. [VIII]). The example illustrates how to check Fisher’s conjecture with the formalism of QI. Larger-scale calculations could test (i) Fisher’s four-Posner conjecture and (ii) entanglement’s effects on the binding probabilities of swarms of Posners.

Moreover, the QI formalism could lead to a framework for biological Bell tests. Such tests might be cast as nonlocal games [88]. The Clauser-Holt-Shimony-Hauser (CHSH) game, which illustrates Bell’s theorem [89, 90], can serve as a model.

Quantum chemistry: Physical conjectures populate Sections [I C, I D] and [III C]. These conjectures merit testing and refinement. First, Posner creation was modeled with a Lennard-Jones potential. Second, pre-Posner spin states were assumed to transform deterministically into antisymmetric Posner states. The pre-Posner orbital state was assumed to determine the map. Third, Posner creation was assumed to preserve each spin’s \( S_{123}^\text{lab} \) essentially. Fourth, Posner dynamics were assumed to preserve \( C \) and \( S_{1,6}^\text{lab} \).

Randomness: Our QI-processing protocols involve perfect executions of Posner operations. But Posners suffer magnetic fields somewhat randomly. Randomness could hinder some, and improve some, QI processing.

For example, Sec. [VII] features a honeycomb lattice. Singlets might not form a honeycomb in solution. (See footnote [50] for a reason why regular graphs might form.) They might have a greater probability of forming a random graph. Randomness could improve the state’s connectivity. Improved connectivity might lower the bar for fueling universal MBQC (Sec. [VII D]). What randomness helps, and what randomness hinders, merits investigation.

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Appendix A  BACKGROUND: QUANTUM INFORMATION THEORY

Quantum systems can process information more efficiently, transmit information more compactly, and secure information more reliably than classical systems can. Consider a system of $N$ qubits, e.g., $N$ phosphorus nuclear spins. The system corresponds to a Hilbert space $\mathcal{H}$ of dimensionality $2^N$. Let $\{|\phi_j\rangle\}$ denote an orthonormal basis for $\mathcal{H}$. The system can occupy a quantum state $|\psi\rangle = \sum_j c_j |\phi_j\rangle \in \mathcal{H}$. The $2^N$ coefficients $c_j \in \mathbb{C}$ satisfy the normalization condition $\sum_j |c_j|^2 = 1$. Consider specifying one of the $2^N$ basis elements $|\phi_j\rangle$. One must use $2^N$ bits (two-level units of classical information). The specification requires only $N$ qubits. One can leverage this discrepancy to process information quickly, using quantum systems. The state $|\psi\rangle$ constitutes QI.

QI can be processed with help from entanglement [7, 8]. Entanglement manifests in correlations stronger than any shareable by classical systems. Entanglement facilitates quantum computation, communication, and cryptography. We briefly review efficiency, quantum computational models and universality, and quantum error correction. Readers seeking more background are referred to [7, 8].

A 1  Efficiency

Quantum computers can efficiently solve certain problems that, according to widespread belief, classical computers cannot. Efficiently loosely means the following. Consider a family $F$ of computational problems. For example, consider receiving a number $N$ whose prime factors you must identify. An instance of $F$ consists of, e.g., the number $N$ to be factored. Let $n$ quantify the resources required to specify an instance of $F$. For example, $n$ might equal the number of bits needed to represent $N$. Let $t$ denote the time required to solve the instance. Suppose that the time grows, at most, polynomially in the amount of resources: $t \sim (\text{const.})n^k$, for some $k \geq 0$. The problems in $F$ can be solved efficiently.

Quantum computers can factor arbitrary numbers more quickly than classical computers can [91]. Imagine using a quantum computer to solve a problem more quickly than any classical computer. One would achieve quantum speedup, or quantum supremacy [74, 83]

A 2  Quantum-computation models and universality

A general quantum process consists of state preparations, evolutions, and measurements. Which operations can be implemented easily (which states $|\psi\rangle$ can be prepared easily, etc.) varies from platform to platform. Consider, for example, a nuclear-magnetic resonance (NMR) experiment. Let $N$ denote the number of nuclear spins. Preparing the pure state $|0\rangle^\otimes N$ is difficult. Preparing a maximally mixed state $\frac{1}{2^N-1} \otimes (N-1)$ spins, tensored with one pure $|0\rangle$, is easier [92]. A set of quantum resources—of performable quantum operations—forms a model for quantum computation. DiVincenzo catalogued the ingredients needed to realize a quantum-computation model physically [28].

Certain computational models are universal [93]. A universal quantum computer can perform every conceivable quantum computation. Every universal model can simulate every other universal model efficiently.

Many quantum-computation models exist. Two prove most pertinent to this paper: the circuit model [15] and measurement-based quantum computation (MBQC) [12, 14]. Other models include the quantum Turing machine [93], the one-clean-qubit model [92], adiabatic quantum computation [94], anyonic quantum computation [95], teleportation-based quantum computation [55, 58], quantum walks on graphs [96], and permutational quantum computation [97].

The circuit model is used most widely [15]. One solves a problem by running a quantum circuit, illustrated by a circuit diagram (e.g., Fig. 13). Wires represents the qubits, which are often prepared in pure states $|0\rangle$. Rectangles represent unitary operations $U$. The $U$’s evolve the qubits, implementing gates. A rectangle inscribed with a dial easy, but it would be especially presumptuous to believe that our limited classical minds can divine the future course of quantum information science.” Posners suggest that we have better chances than Preskill expected.
represents a measurement. Single qubits can be measured with respect to some orthonormal basis, e.g., \{\ket{0}, \ket{1}\}, wherein \(\langle 0|1 \rangle = 0\).

*Depth* quantifies a circuit’s length, or complexity. Consider grouping together the operations that can be performed simultaneously. For example, qubit 1 can interact with qubit 2 while qubit 3 interacts with qubit 4. Each group of gates occurs during one time slice. The number of time slices in a circuit equals the circuit’s depth. Suppose that the depth does not depend on the number of qubits. Such a circuit has constant depth.

Primitive unitaries can be implemented directly. Composing primitives simulates more-complicated operations. One universal primitive set \([7, 98]\) is natural to compare with \(^{31}\)P dynamics: (i) Each qubit’s state can rotate through a fixed angle \(\theta\) about a fixed axis \(\hat{n}\) of the Bloch sphere; \(\theta\) must be an irrational multiple of \(2\pi\). (ii) Each qubit can rotate through a fixed angle \(\theta'\) about a fixed axis \(\hat{n}'\not= \hat{n}\). (iii) Any two qubits can be entangled via some fixed unitary.

No unitary is known to entangle Posners’ phosphorus nuclear spins. Hence we turn from the circuit model to MBQC \([12,14]\). To implement MBQC, one prepares a many-qubit entangled state \(\ket{\psi}\). One measures single qubits adaptively. Measurements are *adaptive* if earlier measurements’ outcomes dictate later measurements’ forms.

Certain states \(\ket{\psi}\) enable one to simulate efficiently, via MBQC, a universal quantum computer. Example states include the Affleck-Kennedy-Lieb-Tasaki (AKLT) state on a honeycomb lattice, \(\ket{\text{AKLT}_{\text{hon}}}\) \([12,13,19,20]\). Posners can occupy a similar state, \(\ket{\text{AKLT}_{\text{hon}}}\). \(\ket{\text{AKLT}_{\text{hon}}}\) can fuel universal MBQC (Sec. VII).

### A 3 Quantum error correction

Two sources of error threaten quantum computers. First, the operations performed might differ from the target operations. Consider, for example, trying to rotate a qubit through an angle \(\frac{\pi}{2}\) about the z-axis. One might overshoot or undershoot. The qubit would rotate through an angle \(\frac{\pi}{2} + \epsilon\), for some \(\epsilon \not= 0\).

Second, a quantum computer might entangle with its environment. The environment decoheres the computer’s state. QI leaks from the computer into the environment.

Quantum error correction preserves QI. Imagine wishing to process a state \(\ket{\psi}\) of \(k\) qubits. One chooses an error-correcting code. The code maps \(\ket{\psi}\) to a state \(\ket{\bar{\psi}}\) of \(n > k\) qubits. \(\ket{\bar{\psi}}\) undergoes physical processes that effect *logical operations* on the encoded state. The logical operations constitute a computation.

Throughout the computation, certain observables \(O\) are measured. Which \(O\)'s depends on the code. The measurements’ outcomes imply whether an error has occurred and, if so, which sort of error. The code dictates how to counteract the error. The state is typically corrected with some unitary \(U\). After the computation and correction terminate, the state is decoded. The computational problem’s answer is read out.

A code can detect more errors than it can correct. Suppose that, according to the \(O\) measurements, many errors have corrupted \(\ket{\psi}\). Suppose that the code cannot correct all those errors. The state must be scrapped; and the computation, reinitiated. We present a quantum error-detecting code and an error-correcting code formed from states accessible to Posners (Sec. IV D).

Let us review the mathematics of quantum error correction and detection (QECD). Consider encoding \(k < n\) logical qubits in \(n\) physical qubits. The physical Hilbert space \(\mathbb{C}^{2n}\) has dimensionality \(2^n\). A QECD code is a subspace \(\mathcal{H}_c \subset \mathbb{C}^{2n}\) of dimensionality \(2^k < 2^n\). Let \(\mathcal{L}_c^{\text{emp}} = \{|L\rangle\}\) denote the code’s computational basis. (See Sec. III B for an introduction to computational bases.)

Each quantum error-correcting/-detecting code corresponds to a set \(\{E_\alpha\}\) of correctable/detectable errors. For example, a code of \(n = 9\) physical qubits has been constructed \([29]\). This code corrects the set of single-qubit Pauli errors, \(\{\sigma_1^x, \sigma_2^x, \ldots, \sigma_9^x, \sigma_1^z, \ldots, \sigma_9^z\}\). The shorthand \(\sigma_j^a \equiv 1^\otimes(j-1) \otimes \sigma_j^a \otimes 1^\otimes(n-j)\). The ability to correct \(\sigma^y\) errors follows from the ability to correct \(\sigma^x\) and \(\sigma^z\).

Under what conditions can a code \(\mathcal{H}_c\) detect a set \(\{E_\alpha\}\) of errors? The code and set must satisfy the *quantum error-detection criteria*,

\[
\langle j|E_\alpha|k\rangle = C_\alpha \delta_{jk} \quad \forall j,k,\alpha . \tag{A1}
\]

The Kronecker delta is denoted by \(\delta_{jk}\). \(C_\alpha\) denotes a constant dependent only on the error \(E_\alpha\), not on the codeword labels \(j\) and \(k\). Equation (A1) decomposes into two subcriteria: the off-diagonal criterion, in which \(j \not= k\), and the diagonal criterion, in which \(j = k\).

---

\(^{34}\) The *Bloch sphere* represents pure qubit states geometrically \([7]\). A general pure qubit state has the form \(\ket{\psi} = \cos \frac{\theta}{2} \ket{0} + e^{i\varphi} \sin \frac{\theta}{2} \ket{1}\), wherein \(\theta, \varphi \in [0,2\pi]\). The state is equivalent to the *Bloch vector* \((\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)\). The Bloch vector lies on the unit sphere, or Bloch sphere. Points inside the sphere represent mixed states \(\rho \not= \ket{\psi}\rangle\langle\psi|\).
The off-diagonal error-detecting criterion has the form
\[ \langle j_\ell | E_\alpha | k_\ell \rangle = 0 \quad \forall j \neq k. \] \hfill (A2)

No \( E_\alpha \) maps any codeword \( |k_\ell\rangle \) into any other codeword \( |j_\ell\rangle \). The logical states retain their integrity under detectable errors.

The diagonal criterion has the form
\[ \langle j_\ell | E_\alpha | j_\ell \rangle = C_\alpha \quad \forall j, \alpha. \] \hfill (A3)

Suppose that \( |j_\ell\rangle \) is prepared. The environment might effectively measure \( \langle E_\alpha \rangle \). The environment gains no information about the state, according to Eq. \( (A3) \). Every codeword’s expectation value equals every other codeword’s. Typical detectable errors \( E_\alpha \) operate nontrivially on just a few close-together qubits. The codewords are locally indistinguishable with respect to \( \{E_\alpha\} \).

Local indistinguishability protects QI: Suppose that the environment had “learned” about \( |j_\ell\rangle \). Information would have leaked out of the system. Highly entangled states are locally indistinguishable: Entanglement distributes information throughout the system. Local operations cannot extract the distributed information.

We have reviewed the error-detection criteria. Under what conditions can a code \( H_\ell \) correct \( \{E_\alpha\} \)? The code must satisfy the quantum error-correction criteria \[ \{8, 39–43\} \].

\[ \langle j_\ell | E_\alpha^\dagger E_\beta | k_\ell \rangle = C_{\alpha\beta} \delta_{jk} \quad \forall j, k, \alpha, \beta. \] \hfill (A4)

Equation \( (A4) \) is interpreted similarly to Eq. \( (A1) \). A code that corrects \( (d - 1)/2 \) errors detects \( (d - 1) \) errors. We refer readers to \[ \{8\} \] for more background.

Appendix B  MULTIPLICITY OF (NO-COLLIDING-NUCLEI ANTISYMMETRIC) SUBSPACES ACCESSIBLE TO A POSNER MOLECULE

A subtlety about \( H_{\text{no-coll.}}^{\text{pos-n}} \) was glossed over in Sec. II C. Consider Eq. \( (7) \). In every term, the spin quantum number \( m_{\pi_\alpha(j)} \) appears alongside the position \( r_{\pi_\alpha(j)} \). The tuple \( (m_{\pi_\alpha(j)}, r_{\pi_\alpha(j)}) \) occupies different kets in different terms. But \( m_{\pi_\alpha(j)} \) remains hitched to the same position \( r_{\pi_\alpha(j)} \) throughout the terms. How are the \( m_{\pi_\alpha(j)} \)'s assigned to positions?

This question has a two-part answer. The choice of coordinate system partially determines the assignments. So do initial conditions, the pre-Posner phosphates’ positions and momenta.

The choice of coordinate system determines the \( \phi \)-value associated with a given \( m \)-value. For example, suppose that \( m_1 = 0 \). Should this spin variable be assigned to \( r_1 = (\phi, h) \), to \( r_1 = (\phi + 2\pi/3, h) \), or to \( r_1 = (\phi + 4\pi/3, h) \)? (Whether \( h = h_+ \) or \( h = h_- \) is irrelevant.) This assignment is a convention, because the orientation of \( x_\text{in} \) is a convention.

We illustrate the answer’s second part with an example. Suppose that three singlets, \[ |\Psi^-\rangle^\otimes 3 = \frac{1}{\sqrt{2}} (|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle) \otimes \frac{1}{\sqrt{2}} (|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle) \otimes \frac{1}{\sqrt{2}} (|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle), \] \hfill (B1)

join together to form a Posner. Molecule creation is assumed to preserve the entanglement within each pair of spins (Sec. II D). Posner creation maps each six-spin term in \( (B1) \) to a sum \( (7) \). Suppose we choose an intra-Posner coordinate system such that \( r_1 = (0, h) \), for \( h = h_+ \) or \( h = h_- \). Given that coordinate system, which value should \( r_1 \) assume? Should the spin at \( (0, h_{\pm}) \) form a singlet with the spin at \( (0, h_{\pm}) \), with the spin at \( (2\pi/3, h_{\pm}) \), etc.? Different answers generate qualitatively different Posner states: The states transform differently under \( C \).

The correct answer, we posit, is determined by the positions and momenta that the phosphates had at the lip of the Lennard-Jones potential (Sec. II C). Different projections of the same initial state, we posit, would release different amounts of heat to the environment.

The PVM model in Sec. II D can now be refined: Posner creation projects the phosphates’ state with the projector \( \Pi_{\text{no-coll.}}^{-} \) onto some no-colliding-nuclei subspace \( H_{\text{no-coll.}}^{-} \) of the antisymmetric subspace. \( 6! \) such subspaces exist; \( 6! \) possible forms are available to \( \Pi_{\text{no-coll.}}^{-} \). One subspace and projector correspond to entanglement between the \( (0, h_{\pm}) \) spin and the \( (0, h_{\pm}) \) spin; one subspace and projector correspond to entanglement between the \( (0, h_{\pm}) \) spin and \( (2\pi/3, h_{\pm}) \) spin; etc. Hence pre-Posner positions and momenta, with a choice of coordinate system, determine to which position each spin variable (e.g., \( m_1 \)) is assigned during Posner creation.
When a Posner forms, we posit, $\Pi_{\text{no-coll}}^-$ projects the phosphorus nuclei’s joint state [Eq. (8)]. $\Pi_{\text{no-coll}}^-$ defines a map that preserves the dimensionality of the space available for storing QI, 64. A counting argument shows why.

Imagine that the phosphorus nuclei were classical and distinguishable. A tuple $(m_j, r_j)$ would label the $j^{\text{th}}$ nucleus’s state. The spin variable $m_j$ could assume one of two possible values. The position $r_j$ could assume one of six possible values, $\uparrow$ or $\downarrow$. The tuple could therefore assume one of twelve possible values:

\[
\begin{align*}
(\uparrow, (\phi, h_+) ), & \quad (\uparrow, (\phi, h_- ) ), \\
(\uparrow, (\phi + 2\pi/3, h_+) ), & \quad (\uparrow, (\phi + 2\pi/3, h_- ) ), \\
(\uparrow, (\phi + 4\pi/3, h_+) ), & \quad (\uparrow, (\phi + 4\pi/3, h_- ) ), \\
(\downarrow, (\phi, h_+) ), & \quad (\downarrow, (\phi, h_- ) ), \\
(\downarrow, (\phi + 2\pi/3, h_+) ), & \quad (\downarrow, (\phi + 2\pi/3, h_- ) ), \\
(\downarrow, (\phi + 4\pi/3, h_+) ), & \quad (\downarrow, (\phi + 4\pi/3, h_- ) ).
\end{align*}
\]

The nuclei would be “dodequits”: $\dim(H_{\text{nuc}})$ would equal $2 \times 6 = 12$.

Let us return to reality: The phosphorus nuclei are indistinguishable fermions. A hextuple of nuclei can occupy the antisymmetric basis state \( \Phi \text{pos} \). This state is labeled by a set of six tuples. Each tuple must differ from each other tuple, for the state to be antisymmetric. To label a joint state, we choose six of the twelve possible tuples.

Hence we pair up the twelve possible tuples. Each pair’s constituent tuples have the same positions and different spin states:

1. \((\uparrow, (\phi, h_+)), (\downarrow, (\phi, h_-))\)
2. \((\uparrow, (\phi, h_-)), (\downarrow, (\phi, h_+))\)
3. \((\uparrow, (\phi + 2\pi/3, h_+)), (\downarrow, (\phi + 2\pi/3, h_-))\)
4. \((\uparrow, (\phi + 2\pi/3, h_-)), (\downarrow, (\phi + 2\pi/3, h_+))\)
5. \((\uparrow, (\phi + 4\pi/3, h_+)), (\downarrow, (\phi + 4\pi/3, h_-))\)
6. \((\uparrow, (\phi + 4\pi/3, h_-)), (\downarrow, (\phi + 4\pi/3, h_+))\)

We have formed six pairs of tuples. We choose one tuple from each pair, to label an antisymmetric joint basis state.

Let us count the ways in which we can choose the six tuples. We can choose one tuple from each pair in two ways. We choose from each of six pairs. Hence we have $2^6 = 64$ choices of labels for an antisymmetric joint state.

**Appendix D WHY THE POSNER’S HAMILTONIAN IS EXPECTED TO CONSERVE $S_{1\ldots6}^{z_{\text{nuc}}}$**

A Posner’s phosphorus nuclear spins resist decoherence for long times, according to Fisher. We infer that the Posner Hamiltonian $H_{\text{Pos}}$ preserves $S_{1\ldots6}^{z_{\text{nuc}}}$ (the $z$-component, relative to the Posner’s lab frame, of the six phosphorus nuclei’s total spin). We support this interpretation by identifying candidate interactions that preserve the Posner’s $C_3$ symmetry. These interactions, we show, preserve $S_{1\ldots6}^{z_{\text{nuc}}}$. Swift et al. studied intra-Posner interactions more rigorously after the present paper’s initial release [9, Eq. (3)]. Their findings—including their form for $H_{\text{Pos}}$—are consistent with ours.

The nuclei within a molecule can interact, in general. Intramolecule interactions include the Coulomb exchange, kinetic exchange, and superexchange. These interactions have the Heisenberg form

\[
S_j \cdot S_k = S_j^+ S_k^- + S_j^- S_k^+ .
\]

The $j^{\text{th}}$ single-nucleus spin operator is denoted by $S_j$. Raising and lowering operators are denoted by $S_j^\pm := \frac{1}{2}(S_j^x \pm iS_j^y)$.

Suppose that arbitrary phosphorus nuclear spins in a Posner interact via Eq. (D1):

\[
H_{\text{int}} = \sum_{j=1}^{6} \sum_{k<j} J_{jk} s_j \cdot s_k .
\]

The pair-dependent interaction strength is denoted by $J_{jk}$. This $H_{\text{int}}$ remains invariant under permutations of the spins via $C$. $C$ represents the rotation that preserves the Posner’s geometry. Hence the Posner’s intrinsic Hamiltonian might contain $H_{\text{int}}$.

The first term in Eq. (D1) conserves each spin’s $S_j^z$, relative to an arbitrary reference frame. The second term does not. But suppose that any spin flips upward via $S_j^+$. Another spin flips downward via $S_k^-$. The compensation preserves the total spin’s $z$-component.
Appendix E  DECOMPOSITION OF THE POSNER-MOLECULE HILBERT SPACE $\mathcal{H}_{\text{no-coll}}$ IN TERMS OF COMPOSITE SPIN OPERATORS

A Posner encodes logical qubits (Sec. [III]). Three qubits correspond to the $h_\perp$ triangle in Fig. 1a via Eq. (10) [35]. We label these qubits 1, 2, and 3. The $h_\perp$ triangle corresponds to logical qubits 4, 5, and 6. The 64-dimensional logical space decomposes into a direct sum of subspaces. Different subspaces transform in different ways under quantum-mechanics textbooks, such as [69], for background.

Let us focus on one triangle (one trio of qubits) first. Each trio corresponds to a Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. Each factor is replaced with the corresponding subsystem’s spin quantum number, in useful conventional notation: $s_1 \otimes s_2 \otimes s_3 = \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2}$. This tensor product can be rewritten as a direct sum.

To derive the direct sum, we follow rules for adding angular-momentum quantum numbers. Two spin quantum numbers, $s_1$ and $s_2$, sum as

$$s_{\text{tot}} = |s_1 - s_2|, |s_1 - s_2| + 1, \ldots s_1 + s_2 - 1, s_1 + s_2. \tag{E1}$$

Two magnetic spin quantum numbers, $m_1$ and $m_2$, sum as

$$m_{\text{tot}} = m_1 + m_2. \tag{E2}$$

We need not use Eq. (E2) here, however.

Since tensor products distribute across direct sums,

$$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = (0 \otimes 1) \otimes \frac{1}{2} = \frac{1}{2} \oplus \left( \frac{1}{2} \oplus \frac{3}{2} \right). \tag{E3}$$

We can check Eq. (E4): A space that transforms with spin quantum number $s$ has dimensionality $2s + 1$. That is, $s$ corresponds to $2s + 1$ possible magnetic spin quantum numbers $m$. According to the LHS of Eq. (E3), therefore, each triangle corresponds to a space of dimensionality $(2 \times \frac{1}{2} + 1)^3 = 2^3 = 8$. Equation (E4) implies the same dimensionality: $2 + 2 + 4 = 8$.

Each Posner consists of two triangles. A triangle pair corresponds to the Hilbert space $\left( \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2} \right)^2$. Distributing the tensor product across the direct sums yields

$$\left( \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2} \right)^2 = \left( \frac{1}{2} \oplus \frac{1}{2} \right)^4 \oplus \left( \frac{1}{2} \oplus \frac{3}{2} \right)^4 \oplus \left( \frac{3}{2} \oplus \frac{3}{2} \right)^4 \tag{E5}$$

$$= (0 \oplus 1)^4 \oplus (1 \oplus 2)^4 \oplus (0 \oplus 1 \oplus 2 \oplus 3)^4 \tag{E6}$$

$$= 0^4 \oplus 1^4 \oplus 2^4 \oplus 3^4.$$ \tag{E7}

Let us check Eq. (E7). According to the LHS of Eq. (E5), a Posner corresponds to a space of dimensionality $(2 + 2 + 4)^2 = 8^2 = 64$. Equation (E7) implies the same dimensionality: $5 + (3 \times 9) + (5 \times 5) + 7 = 64$.

Appendix F  PREFERRED EIGENBASIS OF THE PERMUTATION OPERATOR C

The permutation operator $C$ was introduced in Sec. [IV A]. The Posner dynamics is assumed to conserve $C$, as well as $S_{1\ldots6}^{z\text{lab}}$. An eigenbasis shared by $C$ and $S_{1\ldots6}^{z\text{lab}}$ can facilitate the construction of natural quantum error-correcting codes (Sec. [IV D]).

Several eigenbases of $C$ are eigenbases of $S_{1\ldots6}^{z\text{lab}}$. The operator $S_{123}^z \otimes S_{456}^z$ breaks the degeneracy satisfactorily, as discussed in Sections [IV C] and [VII C], $S_{1\ldots6}^{z\text{lab}}$, and $S_{123}^z \otimes S_{456}^z$ share the eigenbasis in Tables [II], [III], and [IV]. Each table corresponds to one value of $\tau = 0, \pm 1$ (equivalently, $\tau = 0, 1, 2$).

---

35 No particular nucleus can be associated with any particular pure spin-and-position state, by Pauli’s principle. But a spin can be associated with a position. Loosely speaking, some nucleus $A$ occupies the spin state $|m_j\rangle$ if and only if $A$ occupies the position state $|r_j\rangle$. 
TABLE II: Preferred eigenbasis for the $\tau = 0$ eigenspace of the permutation operator $C$: Twenty-four states span the eigenspace. Each basis element equals a product of two three-qubit states. The final column displays the product, explained in Sec. IV C. The state’s first factor represents a state of the qubits (labeled $j = 1, 2, 3$) in the top triangle in Fig. 1a. The second factor represents a state of the qubits (labeled $j = 4, 5, 6$) in the bottom triangle. Each factor is an eigenstate shared by the total-spin operators $S_{123}$ and $S_{123}^\dagger$, or by $S_{456}$ and $S_{456}^\dagger$. The operators are defined in Sec. IV C. Table 1 displays the three-qubit eigenstates. The spin quantum number $m_{123}$ denotes the eigenvalue of $S_{123}^z$. The magnetic spin quantum number $m_{123}$ denotes the eigenvalue of $S_{123}^z$. $s_{123}$ and $m_{123}$ are defined analogously. The total magnetic spin quantum number $m_{123} = m_{123} + m_{456}$. The notation in column two follows from [69]: Eigenspaces of $S_{123}^z$ and $S_{456}^z$ bear the label $s_{123} \otimes s_{456}$. Column six is notated similarly. $\tau_{123}$ denotes the eigenvalue of the permutation operator that cyclically permutes qubits 1, 2, and 3. $\tau_{456}$ is defined analogously. The permutation eigenvalues multiply to $\tau_{123} \times \tau_{456} = \tau$.

| State | $s_{123} \otimes s_{456}$ | $m_{123}$ | $m_{456}$ | $m_{123}$ | $\tau_{123} \otimes \tau_{456}$ | Decomposition |
|-------|-----------------------------|----------|----------|----------|-----------------------------|---------------|
| $|c_{123}^0\rangle$ | $\frac{1}{2} \otimes \frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 3 | $1 \otimes 1$ | (000)(000) |
| $|c_{123}^1\rangle$ | $\frac{1}{2} \otimes \frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 2 | $1 \otimes 1$ | (000)|$|W\rangle$ |
| $|c_{123}^2\rangle$ | $\frac{1}{2} \otimes \frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $1 \otimes 1$ | (000)|(W)|$000\rangle$ |
| $|c_{123}^3\rangle$ | $\frac{1}{2} \otimes \frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $1 \otimes 1$ | (W)|$W\rangle$ |
| $|c_{123}^4\rangle$ | $\frac{1}{2} \otimes \frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | -1 | $1 \otimes 1$ | (W)|$W\rangle$ |
| $|c_{123}^5\rangle$ | $\frac{1}{2} \otimes \frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $1 \otimes 1$ | (W)|$W\rangle$ |
| $|c_{123}^6\rangle$ | $\frac{1}{2} \otimes \frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | -2 | $1 \otimes 1$ | (W)|$W\rangle$ |
| $|c_{123}^7\rangle$ | $\frac{1}{2} \otimes \frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $1 \otimes 1$ | (111)(000) |
| $|c_{123}^8\rangle$ | $\frac{1}{2} \otimes \frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $\omega \otimes \omega^2$ | (|$\omega\rangle$|$(\omega^2)$ |
| $|c_{123}^9\rangle$ | $\frac{1}{2} \otimes \frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $\omega^2 \otimes \omega$ | (|$\omega\rangle$|$(\omega^2)$ |
| $|c_{123}^{10}\rangle$ | $\frac{1}{2} \otimes \frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\omega \otimes \omega^2$ | (|$\omega\rangle$|$(\omega^2)$ |
| $|c_{123}^{11}\rangle$ | $\frac{1}{2} \otimes \frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | -1 | $\omega \otimes \omega^2$ | (|$\omega\rangle$|$(\omega^2)$ |
| $|c_{123}^{12}\rangle$ | $\frac{1}{2} \otimes \frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | $\omega \otimes \omega^2$ | (|$\omega\rangle$|$(\omega^2)$ |
| $|c_{123}^{13}\rangle$ | $\frac{1}{2} \otimes \frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | -2 | $\omega \otimes \omega^2$ | (|$\omega\rangle$|$(\omega^2)$ |
| $|c_{123}^{14}\rangle$ | $\frac{1}{2} \otimes \frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | $\omega \otimes \omega^2$ | (|$\omega\rangle$|$(\omega^2)$ |

Appendix G QUANTIFICATION OF THE INFORMATION ENCODED IN THE OUTCOME OF A POSNER-BINDING MEASUREMENT: ANALYSIS 2

The Posner-binding measurement is analyzed in Sec. VI. The measurement yields an outcome that encodes classical information. This information is quantified in Sec. VI A. The quantification is explained alternatively here.

Imagine wishing to measure the $\tau_A$ and $\tau_B$ of Posners $A$ and $B$. Each measurement would yield one of three possible outcomes (0, 1, or 2). The pair of measurements would yield one of nine possible outcomes. The pair of outcomes could be recorded in $\lceil \log_2(9) \rceil = 4$ bits.

Whether two Posners bind is equivalent to a measurement of whether $\tau_A + \tau_B = 0$. The yes-or-no answer constitutes one bit. You forfeit three of the bits you wanted, measuring just whether the Posners bind. Three is the number of bits you would need to specify the value of $(\tau_A, \tau_B)$, given that $\tau_A + \tau_B \neq 0$. Why? Suppose that $\tau_A + \tau_B \neq 0$. 

Imagine learning, instead, that \( \tau_A + \tau_B = 0 \). Given this information, would you need three bits to specify the value of \( (\tau_A, \tau_B) \)? No: \( (\tau_A, \tau_B) \) can assume one of three possible values. Hence you would need \( \lceil \log_2(3) \rceil = 2 \) bits. But you could encode the tuple’s value in three bits.

---

**TABLE III: Preferred eigenbasis for the \( \tau = 1 \) eigenspace of the rotation symmetry operator \( C \):** The notation is defined below Table I.

\[
\begin{array}{cccccccc}
\text{State} & s_1 123 \otimes s_4 256 & m_{123} & m_{256} & m_{1...6} & \tau_1 123 \otimes \tau_4 566 & \text{Decomposition} \\
|c_{\tau = 11}^1\rangle & \frac{1}{4} \otimes \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & 2 & 1 \otimes \omega & |000\rangle |\omega\rangle \\
|c_{\tau = 11}^2\rangle & \frac{1}{4} \otimes \frac{1}{4} & \frac{1}{2} & -\frac{1}{2} & 1 & 1 \otimes \omega & |000\rangle |\omega\rangle \\
|c_{\tau = 12}^2\rangle & \frac{1}{4} \otimes \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & 1 & 1 \otimes \omega & \langle W\rangle |\omega\rangle \\
|c_{\tau = 11}^3\rangle & \frac{1}{4} \otimes \frac{1}{4} & \frac{1}{2} & -\frac{1}{2} & 0 & 1 \otimes \omega & \langle W\rangle |\omega\rangle \\
|c_{\tau = 11}^4\rangle & \frac{1}{4} \otimes \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & 0 & 1 \otimes \omega & \langle W\rangle |\omega\rangle \\
|c_{\tau = 12}^4\rangle & \frac{1}{4} \otimes \frac{1}{4} & \frac{1}{2} & -\frac{1}{2} & -1 & 1 \otimes \omega & \langle W\rangle |\omega\rangle \\
|c_{\tau = 11}^5\rangle & \frac{1}{4} \otimes \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & 1 & 1 \otimes \omega & |\omega\rangle |000\rangle \\
|c_{\tau = 12}^5\rangle & \frac{1}{4} \otimes \frac{1}{4} & \frac{1}{2} & -\frac{1}{2} & -1 & 1 \otimes \omega & \langle W\rangle |\omega\rangle \\
|c_{\tau = 11}^6\rangle & \frac{1}{4} \otimes \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & 0 & 1 \otimes \omega & \langle W\rangle |\omega\rangle \\
|c_{\tau = 12}^6\rangle & \frac{1}{4} \otimes \frac{1}{4} & \frac{1}{2} & -\frac{1}{2} & -1 & 1 \otimes \omega & \langle W\rangle |\omega\rangle \\
|c_{\tau = 11}^7\rangle & \frac{1}{4} \otimes \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & 1 & 1 \otimes \omega & \langle W\rangle |\omega\rangle \\
|c_{\tau = 12}^7\rangle & \frac{1}{4} \otimes \frac{1}{4} & \frac{1}{2} & -\frac{1}{2} & -2 & 1 \otimes \omega & \langle W\rangle |\omega\rangle \\
|c_{\tau = 11}^8\rangle & \frac{1}{4} \otimes \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & 0 & 1 \otimes \omega & \langle W\rangle |\omega\rangle \\
|c_{\tau = 12}^8\rangle & \frac{1}{4} \otimes \frac{1}{4} & \frac{1}{2} & -\frac{1}{2} & -2 & 1 \otimes \omega & \langle W\rangle |\omega\rangle \\
|c_{\tau = 11}^9\rangle & \frac{1}{4} \otimes \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & 1 & 1 \otimes \omega & \langle W\rangle |\omega\rangle \\
|c_{\tau = 12}^9\rangle & \frac{1}{4} \otimes \frac{1}{4} & \frac{1}{2} & -\frac{1}{2} & -1 & 1 \otimes \omega & \langle W\rangle |\omega\rangle \\
|c_{\tau = 11}^{10}\rangle & \frac{1}{4} \otimes \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & 1 & 1 \otimes \omega & \langle W\rangle |\omega\rangle \\
|c_{\tau = 12}^{10}\rangle & \frac{1}{4} \otimes \frac{1}{4} & \frac{1}{2} & -\frac{1}{2} & -1 & 1 \otimes \omega & \langle W\rangle |\omega\rangle \\
\end{array}
\]

\( (\tau_A, \tau_B) \) can equal one of six possible values, \((0, 1), (0, 2), (1, 0), (1, 1), (2, 0), \) or \((2, 2) \). Specifying one of six possible values requires \( \lceil \log_2(6) \rceil = 3 \) bits.\(^{36}\) Hence measuring Posner binding is equivalent to each of two QI processes:

1. Measuring \((\tau_A, \tau_B) \) and coarse-graining away three bits (all information except whether \( \tau_A + \tau_B = 0 \)).

2. Measuring the Bell basis and coarse-graining away one bit (whether \( a + \) outcome or \( a - \) outcome occurred).

**Appendix H HOW TO PREPARE, WITH POSNER OPERATIONS, STATES USED IN INCOHERENT TELEPORTATION**

Section VI B details how Posners can teleport QI incoherently. The protocol involves states \( |\pm\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \) and \( |\psi\rangle = c_0 |0\rangle + c_1 |1\rangle + c_2 |2\rangle \). Each \( |j\rangle \) denotes an arbitrary state in the \( \tau = j \) subspace. How can Posner operations (Sec. V A) prepare a \(|\pm\rangle \) and a \(|\psi\rangle \)? One protocol is described below. Other protocols may await discovery.

Each state is of one Posner and is pure. Hence the Posner contains three singlets. Consider preparing three singlets via operation \( I \). Consider rotating one spin about the \( y \)-axis, through an angle \( \theta \).

Consider forming a Posner from the spins, via operation \( I \). Let the singlets be arranged as in Fig. 19. Recall that a Posner contains two triangles of phosphorus nuclear spins (Sec. \( H \)). One triangle sits at \( z_{mn} = h_{+} \); and the other

\(^{36}\) Imagine learning, instead, that \( \tau_A + \tau_B = 0 \). Given this information, would you need three bits to specify the value of \( (\tau_A, \tau_B) \)? No: \( (\tau_A, \tau_B) \) can assume one of three possible values. Hence you would need \( \lceil \log_2(3) \rceil = 2 \) bits. But you could encode the tuple’s value in three bits.
TABLE IV: Preferred eigenbasis for the $\tau = -1$ eigenspace (equivalently, the $\tau = 2$ eigenspace) of the rotation symmetry operator $C$: The notation is defined below Table II.

triangle, at $z_{tn} = h_{\tau}$. Each triangle contains one singlet (illustrated with a green, wavy line). One singlet extends from the $h_{\tau}$ triangle to the $h_{\tau}$ triangle. (How a singlet corresponds to positions in a Posner is discussed in Sec. III C and App. B.) The red hoop encircles the rotated spin. The rotated spin is entangled with a spin in the same triangle.

Let $|\phi(\theta)\rangle$ denote the Posner’s state. $|\phi(\theta)\rangle$ can have weight on each $\tau = j$ eigenspace:

$$|\phi(\theta)\rangle = \sum_{j=0}^{2} \sum_{\lambda_j=1}^{d_j} C_{j,\lambda_j}(\theta) |\tau_{\tau=\lambda_j}\rangle .$$

(H1)

The $\tau = j$ eigenspace has degeneracy $d_j$. The degeneracy parameter is denoted by $\lambda_j$. The coefficients $C_{j,\lambda_j}(\theta)$ satisfy the normalization condition $\sum_{j} \sum_{\lambda_j} |C_{j,\lambda_j}(\theta)|^2 = 1$.

The dependence on $\theta$ can be calculated analytically: The state has an amount

$$\sum_{\lambda_0=1}^{24} |C_{0,\lambda_0}(\theta)|^2 = \frac{1}{6} |\cos(2\theta) + 2|$$

(H2)

of weight on the $\tau = 0$ eigenspace, an amount

$$\sum_{\lambda_1=1}^{20} |C_{1,\lambda_1}(\theta)|^2 = \frac{1}{12} |4 - \cos(2\theta)|$$

(H3)
FIG. 19: Posner-molecule state usable in incoherent teleportation: Each black dot represents a phosphorus nuclear spin. The internal z-axis $\hat{z}_{\text{in}}$ remains fixed with respect to the atoms' positions. Three spins sit at $z_{\text{in}} = h_+$; and three spins, at $z_{\text{in}} = h_-$. The spins occupy a pure state of three singlets. Each green, wavy line represents one singlet. The red hoop encircles a spin that has been rotated through an angle $\theta$. The rotation is about the $y_{\text{lab}}$-axis, which remains fixed relative to the lab that contains the Posner. The angle labels the Posner’s state, $|\phi(\theta)\rangle$. Instances of $|\phi(\theta)\rangle$ can serve as the $|+\tau\rangle$ and the $|\psi\rangle$ in incoherent teleportation (Sec. VI B 2).

on the $\tau = 1$ eigenspace, and an amount

$$\sum_{\lambda_j=1}^{20} |C_{2,\lambda_j}(\theta)|^2 = \frac{1}{12} [4 - \cos(2\theta)]$$

(H4)

on the $\tau = 2$ eigenspace.

At which $\theta$-value does the weight on each eigenspace equal the weight on every other? Let us equate [H2], [H3], and [H4]. Solving for the angle yields $\theta = \frac{\pi}{4}$. The corresponding state can serve as the equal-weight superposition $|+\tau\rangle$:

$$|+\tau\rangle = |\phi(\pi/4)\rangle.$$  

(H5)

The basis vectors $|j_\tau\rangle$ inherit the definition

$$|j_\tau\rangle = \sum_{\lambda_j=1} C_{j,\lambda_j} (\pi/4) |c_{j}^{\lambda_j}\rangle.$$  

(H6)

Now, let $\theta$ assume an arbitrary value. Information about $|\phi(\theta)\rangle$ can be teleported incoherently:

$$|\psi\rangle = |\phi(\theta)\rangle.$$  

(H7)

Granted, $|\phi(\theta)\rangle$ might not decompose as $\sum_{j=0}^{2} c_j |j_\tau\rangle$, in terms of the $|j_\tau\rangle$'s defined in Eq. [H6]. Yet the incoherent-teleportation protocol continues to work: Equation [H7] defines new basis elements $|j_\tau(\theta)\rangle$:

$$|j_\tau(\theta)\rangle = \sum_{\lambda_j=1} C_{j,\lambda_j(\theta)} |c_{j}^{\lambda_j}\rangle.$$  

(H8)

States $|j_\tau\rangle$ of Posner $A$ appear in Eqs. [47] and [50]. Each such $|j_\tau\rangle$ must be replaced with a $|j_\tau(\theta)\rangle$. The projector $\Pi_{AB}$ transforms the $|j_\tau(\theta)\rangle$’s as it would transform the $|j_\tau\rangle$’s.

Appendix I FRUSTRATED-LATTICE INTUITION ABOUT PROJECTING ONTO THE $\tau = 0$ SUBSPACE

We can understand Eq. [62] in terms of a frustrated lattice. Consider a triangular lattice of three sites, $A$, $B$, and $C$. Let a spin-1 DOF occupy each site. The site-$K$ magnetic spin quantum number $m_K = 0, \pm 1$ stands in place of $\tau_K$. 
Let us regress to Eq. (61). We ignore the final $m-3$ identity operators in each term. How does $\Pi_{123}$ transform the lattice’s state? Consider multiplying out the terms in the RHS. We label as a cross-term each term that contains at least one $\Pi_{\tau_k}=0$ and one $\Pi_{\tau_k}=\pm 1$, for some $K=A,B,C$. These projects annihilate each other; the cross-terms vanish. Each surviving term in $\Pi_{123}$ contains only $\tau_k=0$ projectors or only $\tau_k=\pm 1$ projectors.

Each $\tau_k=\pm 1$ projector represents an antiferromagnetic interaction between two lattice sites. The $\tau_k=\pm 1$ projectors form a term that represents a frustrated lattice. No set $(\tau_A,\tau_B,\tau_C)$ satisfies all the constraints encoded in the frustration term. Hence the lattice must occupy its $\tau_A=\tau_B=\tau_C=0$ subspace.

**Appendix J PEPS REPRESENTATION OF $|\text{AKLT}_{\text{mon}}\rangle$**

The AKLT’ PEPS is a repeating pattern of two tensors, $T^+$ and $T^-$ (Fig. 17). We will focus primarily on $T^+$. The tensor has six indices. Three ($v_1^+,$ $v_2^+$, and $v_3^+$) are virtual. Three more indices ($a_1^+$, $a_2^+$, and $a_3^+$) are physical.

Each small, black dot represents a virtual spin. Each short leg, extending upward from the plane occupied by the large circle, represents a physical qubit. We denote the physical qubits’ computational-basis states by $|a_1^+ a_2^+ a_3^+\rangle$. For each $j = 1, 2, 3$, the physical index $a_j^+ = 0, 1$.

Each long leg, extended across the plane occupied by the large circle, represents a virtual index. The $v_1^+$ and $v_2^+$ lines represent singlets. Consider, as an illustration, the physical qubit associated with $a_1^+$. This qubit forms a singlet with some physical qubit in another tensor. Suppose that $a_2^+ = 0$. The $T^+$ physical qubit points upward. Hence the partner physical qubit must point downward: The partner qubit’s $a$ must equal one. This necessity is conveyed to the second tensor by the virtual index $v_1^+$; If $a_2^+ = 0$, $T_{a_1^+, a_2^+, a_3^+, v_1^+, v_2^+, v_3^+}$ $\neq 0$ only if $v_1^+ = 0$.

The virtual index $v_3^+$ differs from $v_1^+$ and $v_2^+$. The tensor lacks isotropy. $v_3^+$ connects two tensors associated with the same Posner, $T^+$ and $T^-$. The two tensors, together, determine which $C$ eigenspace the Posner occupies. Hence $v_3^+$ carries not only “singlet” information about one physical qubit. $v_3^+$ conveys also how the $T^+$ qubit trio transforms under $C_3$ (the final column in Table II). This $C$ information dictates how the $T^-$ physical qubits must transform, such that the Posner occupies the $\tau = 0$ eigenspace.

We ascribe to $v_3^+$ a tuple ($v_3^+, \tau^+$). The first entry conveys information about the $a_3^+$ physical qubit. Only if $v_3^+ = a_3^+$ can the tensor have a nonzero value. The second entry, $\tau^+$, equals 0, 1, or 2. Hence $v_3^+$ assumes one of six possible values:

$$v_3^+ = (v_3^+, \tau^+) = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)\}$$

where $\tau^+$ is determined by the physical state $\tau = 0, 1, 2$.

The virtual index $v_3^+$ has a bond dimension of six.

Having overviewed the tensor’s six indices, we consider the whole tensor, $T_{a_1^+, a_2^+, a_3^+, v_1^+, v_2^+, v_3^+}$. This tensor equals the coefficient that multiplies the computational-basis state $|a_1^+, a_2^+, a_3^+\rangle$ when the virtual indices have the values $v_1^+, v_2^+$, and $v_3^+$. Suppose, for simplicity, that the $T^+$ triangle lacked connections to any other triangles. The triangle would occupy the physical state

$$|\text{const.}| \sum_{a_1^+, a_2^+, a_3^+, v_1^+, v_2^+, v_3^+} T_{a_1^+, a_2^+, a_3^+, v_1^+, v_2^+, v_3^+}^+ |a_1^+, a_2^+, a_3^+\rangle$$

The $v$’s do not label the ket, because they are virtual.

The tensor can be evaluated, with help from Table I, after a normalization convention is chosen. We illustrate with three examples.

First, let us evaluate $T_{000000}^+$. Since $v_3^+ = 0$, Eq. (J2) implies that $T_{000000}^+ = 0$ only if $a_3^+ = 0$. Indeed, $a_3^+ = 0$. In fact, every $a$ vanishes. This tensor equals the coefficient of the one-triangle state $|a_1^+, a_2^+, a_3^+\rangle = |000\rangle$. This state occupies the $\tau = 0$ eigenspace, according to Table II. We choose the following normalization condition: $|000\rangle$ appears once, with a unit coefficient, in the table’s second column. Hence we choose for $T_{000000}^+$ to equal one.

The second example consists of $T_{a_1^+, a_2^+, a_3^+, 001}^+$ wherein the $a$’s have arbitrary values. According to the final three indices [and Eq. (J2)], the tensor can be nonzero only if $a_j^+ = 0$ for all $j = 1, 2, 3$. That is, $T_{a_1^+, a_2^+, a_3^+, 001}^+$ is 0 except, perhaps, if the coefficient of $|a_1^+ a_2^+ a_3^+\rangle = |000\rangle$. The tensor’s final index implies that $v_3^+ = 1$. Hence, by Eq. (J2), the qubit trio transforms under $C$ with $\tau^+ = 1$. No qubit-trio state (ii) transforms with $\tau^+$ and (ii) equals a linear combination of computational-basis states including $|000\rangle$, by Table II. Hence $T_{000000}^+ = 0$.

The final example consists of $T_{100100}^-$. The physical indices “agree with” the virtual indices: $a_1^+ = v_1^+,$ $a_2^+ = v_2^+$, and $a_3^+ = v_3^+$. Hence the tensor does not necessarily vanish. This tensor multiplies the physical one-triangle
ket $|a\uparrow a\downarrow a\uparrow\rangle = |100\rangle$. This ket appears three times in the second column of Table I. Only one of those appearances is relevant: Since $v_3^2 = 0$, Eq. (12) implies that $T_+ = 0$. Hence the physical qubit trio occupies the first ket in the $|W\rangle$ decomposition (in the third row of Table I). This ket multiplies a $\frac{1}{\sqrt{3}}$ in the table’s second column. We might wish to ascribe the value $\frac{1}{\sqrt{3}}$ to $T_{100100}^+$. But the physical qubits’ state is constructed from singlets. Singlets carry minus signs. We must incorporate these minus signs into our convention. We choose for the tensor to carry a factor of $(-1)^{\alpha_j}$ for each $j = 1, 2, 3$. Hence $T_{100100}^+ = -\frac{1}{\sqrt{3}}$.

Appendix K HOW LOGICAL QUBITS COULD BE ROTATED

Firing neurons, we propose, generate a magnetic field that could rotate Posners’ phosphorus nuclear spins significantly. We review the interaction Hamiltonian. Then, we quantify the magnetic field generated by firing neurons. We form the rotation unitary $U(t)$, then infer the time $t_{\text{rot}}$ for which the spin must rotate. $t_{\text{rot}}$, we expect, is much less than the time $t_{\text{firing}}$ for which a neuron fires. But a spin could rotate significantly over several firings. The spin would not decohere significantly during this time, if in a Posner.

These estimates are order-of-magnitude. We often focus on the best possible case.

**Hamiltonian:** Consider a spin of magnetic moment $\mu$. A magnetic field $B$ can evolve the spin under the Hamiltonian $H_{\text{mag}} = -\mu \cdot B$. The $^{31}$P nuclear spin has a magnetic moment of magnitude $\mu = 1.13\mu_N$ [100, 101]. The Bohr magneton is denoted by $\mu_N = \frac{e\hbar}{2m_p}$, and the proton mass, by $m_p$.

**Magnetic-field strength:** Firing causes a current to run down a neuron. The current generates a magnetic field $B$, by the Biot-Savart law. Luo et al. model the in-brain $B$ generated by neural tissue [45]. Table 3 on their p. 15 suggests that the field can reach tens of nano-Tesla (nT). Hence we approximate $B \approx 10^{-8}$ T.

Subtleties merit bearing in mind. First, the in-tissue field has a mean of $10^{-1}$ nT and a standard deviation of $10^{-2} - 10^{-1}$ nT [45]. Our focus on the best case justifies the use of a greater $B$.

Second, the study of the in-brain $B$ has fluctuated over the past decade (e.g., [45, 102–104]). Relatedly, magnetoencephalography (MEG) has guided studies of the in-brain field. But MEG measures the field outside the skull. “Many current sources in the cortex are expected to cancel,” Blagoev et al. write [103], “leading to a small magnetic field outside the skull. Hence, using the MEG-measured magnetic field strength to calculate the magnitude of the field within the cortex might lead to an underestimation.” If $B$ is stronger than believed, single-qubit unitaries can be implemented more quickly than expected.

**Angle of rotation:** Let $\hat{n}$ denote the axis of rotation. Let $\theta$ denote the angle through which a spin rotates. How large must $\theta$ be for the global quantum state to change significantly? One might naively guess $\pi$.

But quantum-cognition spins are prepared in singlets $|\Psi^\text{−}\rangle$ (via operation $\hat{\Pi}$). $|\Psi^\text{−}\rangle$ remains invariant under arbitrary identical rotations of both qubits, $U_\alpha := U_\alpha(\frac{\pi}{2}) \otimes U_\alpha(\frac{\pi}{2})$. $U_\alpha$ transforms $|\Psi^\text{−}\rangle$ as does the identity operation, $I$: $|\Psi^\text{−}\rangle = U_\alpha|\Psi^\text{−}\rangle$. Hence the qubits need not rotate physically to undergo $U_\alpha$ effectively.

Consider rotating the qubits oppositely, physically, with $U_\beta := U_\beta(\frac{\pi}{2}) \otimes U_\beta(\frac{\pi}{2})$. The two unitaries combine group-theoretically: $|\Psi^\text{−}\rangle = U_\alpha|\Psi^\text{−}\rangle \Rightarrow U_\beta U_\alpha|\Psi^\text{−}\rangle = [U_\alpha(\theta) \otimes I]|\Psi^\text{−}\rangle$. Hence rotating qubit 1 through $\theta$ counterclockwise is equivalent to (i) rotating qubit 1 through $\frac{\pi}{2}$ counterclockwise while (ii) rotating qubit 2 through $\frac{\pi}{2}$ clockwise.

Hence the time $t_{\text{rot}}$ required to rotate a qubit effectively through an angle $\pi$ equals the time required to rotate a qubit physically through an angle $\theta = \frac{\pi}{2}$. The unitary

$$\exp \left(-i \frac{\theta}{2} \hat{n} \cdot \sigma \right) \tag{K1}$$

rotates a qubit through an angle $\theta$. Since $\theta = \frac{\pi}{2}$, $\frac{\theta}{2} = \frac{\pi}{4}$. The order-of-magnitude estimate will eliminate the $\frac{1}{2}$, but the half is worth being aware of.

**Equation of unitaries and solution for $t_{\text{rot}}$:** The Hamiltonian generates the unitary

$$\exp \left(-i \frac{\hbar}{\hbar} H_{\text{mag}} t \right) \approx \exp \left(1.13e \frac{Bt}{2m_p} \right). \tag{K2}$$

The exponentials $\frac{eB}{m_p}$ and $\frac{1.13eB}{2m_p}$ are equal each when $t = t_{\text{rot}}$. We equate the exponentials’ arguments and neglect order-one factors: $\frac{eB}{m_p} t_{\text{rot}} \approx 1$. Solving for the time scale’s order of magnitude yields

$$t_{\text{rot}} \approx \frac{m_p}{eB} \approx \frac{10^{-27} \text{ kg}}{(10^{-19} \text{ C})(10^{-8} \text{ T})} \approx 1 \text{ s.} \tag{K3}$$
Let us compare this required rotation time to the duration $t_{\text{fire}}$ of one neuron firing.

**Duration of neuron firing:** Xue et al. attribute 5 – 10 ms to a firing \(^{[102]}\), citing \(105^{[105]}107^{[107]}\). We therefore approximate $t_{\text{fire}} \approx 10 \text{ ms} = 10^{-2} \text{ s}$. One firing does not last long enough to rotate a qubit through an angle $\frac{\pi}{2}$: $t_{\text{fire}} \approx 10^{-2} \text{ s} \ll 1 \text{ s} \approx t_{\text{rot}}$. But 10² firings could rotate the neuron enough.

**Frequency of neuron firing:** Blagoev et al. write that “a ‘typical’ neuron spikes 0.1 to 10 times a second” \(^{[103]}\). We focus on the best case of ten firings per second. One hundred firings would consume about ten seconds. Hence rotating a qubit through an angle $\sim \frac{\pi}{2}$ would take $t_{\text{rot}}' \approx 10 \text{ s}$. Let us compare this rotation time to two time scales that characterize the qubit.

**Comparison with spin lifetime:** Consider a phosphorus nuclear spin in a lone phosphate. The spin is expected to have a lifetime of $\sim 1 \text{ s} \ll t'_{\text{rot}}[1]$. The spin will decohere before rotating appreciably. But a spin in a Posner is expected to have a lifetime of $\sim 10^{5} - 10^{6} \text{ s} [1]$. In-Posner qubits could undergo $\sim 10^{4} - 10^{5}$ single-qubit gates before decohering.

**Comparison with diffusion time:** A Posner could diffuse between the neuron firings. Let us estimate the distance diffused. We estimated the Posner’s diffusion constant in Sec. IX: $D \sim 10^{-10} \text{ m}^{2}/\text{s}$. Solving $D \sim \frac{\ell^{2}}{t}$ for distance yields $\ell \sim \sqrt{D}t \sim \sqrt{(10^{-10} \text{ m}^{2}/\text{s}) (10 \text{ s})} = 1 \text{ mm}$.

One millimeter equals approximately another relevant length: The in-tissue magnetic field appears as a function of two-dimensional position in \(^{[45]}\) Fig. 3. The tallest spikes represent field strengths $B \approx 10 \text{ nT}$. About a millimeter separates neighboring peaks. Hence a Posner could diffuse from peak to peak, rotating maximally during each firing.

We do not expect a Posner to hit peak after peak typically. But we have presented the best possible case. At best, a qubit could effectively rotate through angles up to $\pi$.

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