Nonlinear Dependence of the Phase Screen Structure Function on the Atmospheric Layer Thickness

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(Dated: January 13, 2013)

The phase structure function accumulated by two parallel rays after transmission through a layer of turbulent air is best known by a proportionality to the 5/3rd power of the lateral distance in the aperture, derived from an isotropic Kolmogorov spectrum of the refractive index. For a von-Kármán spectrum of the refractive index, a dependence involving a modified Bessel function of the ratio of the distance over the outer scale is also known.

A further standard proposition is a proportionality to the path length through the atmospheric layer. The manuscript modifies this factor through a refined calculation of an integral representation of the structure function. The correction establishes a sub-linearity as the lateral distance grows in proportion to the layer thickness; it is more important for large than for small outer scales.

PACS numbers: 95.75.Qr, 42.68.Bz

Keywords: phase screen, structure function, turbulence, phase screen, von Karman, outer scale

I. SCOPE

A. Planar Phase Screen Terminology

A phase screen characterizes the distribution of optical path lengths in the aperture of a receiver depending on the inhomogeneity of the refractive index along the path through the atmosphere up to the emitter. The phase \( \varphi(\mathbf{r}) \) in the receiver’s pupil plane is a function of the two-dimensional spatial vector \( \mathbf{r} \). The phase is the line integral of the product of the vacuum wave number \( \bar{k} = 2\pi/\lambda \) by the index of refraction \( n \) along the path through the atmosphere at the wavelength \( \lambda \) of the electromagnetic spectrum.

The seeing quality, figures of merit like the Strehl ratio, and the requirements on adaptive optics systems aiming to straighten these wave fronts refer to these phase screens. The current work is a contribution to an accurate description of the phases, the measured quantities, in terms of parameters that describe the turbulence (structure constant and outer scale length) and in terms of geometric parameters which determine the sampling (optical path length of the rays and distance between these).

In the remainder of this section, the manuscript recalls how a model of isotropic turbulence is set up, then reviews in Section III how this leads through a procedure of integration of the refractive index along two parallel rays to a standard formula of the phase structure function of von-Kármán turbulence. This is redone in Section III on a more accurate basis that keeps the layer thickness parameter consistent throughout the computation, with the main result that a finite layer thickness generates a smaller phase structure function than with the standard formula.

B. Three-Dimensional Turbulence Spectrum

This section summarizes the basic theory which links the structure functions of three-dimensional indices of refractions to phase covariances measured in pupil planes.

The structure function of refractive indices \( n \) is defined as the expectation value of squared differences if measured at two points separated by a 3-dimensional vector \( \Delta \mathbf{r} \),

\[
\mathcal{D}_n = \langle [n(\mathbf{r}) - n(\mathbf{r} + \Delta \mathbf{r})]^2 \rangle. \tag{1}
\]

Assuming that the expectation value of the mean does not depend on \( \mathbf{r} \), binomial expansion rephrases this as a background value minus two times a correlation function,

\[
\mathcal{D}_n(\Delta \mathbf{r}) = 2\langle n^2 \rangle - 2\langle n(\mathbf{r})n(\mathbf{r} + \Delta \mathbf{r}) \rangle. \tag{2}
\]

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Its Fourier Transform as a function of spatial frequencies is
\[ D_n(\Delta r) = 2 \int \Phi_n(f)[1 - \cos(2\pi f \cdot \Delta r)]d^3f. \] (3)

The imaginary part \( \propto i \sin(2\pi f \cdot \Delta r) \) vanishes as we assume that the structure function has even parity: \( D_n(\Delta r) = D_n(-\Delta r) \). Other notations emerge if the factor \( 2\pi \) is absorbed into the spatial frequency or if the normalization of the Fourier Transform is chosen differently.

We focus on the von-Kármán model of an isotropic power spectrum with zero inner scale, a prefactor \( c_n C_n^2 \) —which is split into a constant \( c_n \) and a structure constant \( C_n^2 \)—, an outer scale wave number \( f_0 \) representing the inverse of the outer scale length, and a power spectral index \( \gamma \),
\[ \Phi_n(f) = c_n C_n^2 (f^2 + f_0^2)^{-(\gamma + 3)/2}. \] (4)

The Kolmogorov Model is obtained by setting \( \gamma = 2/3 \) and setting the outer scale to infinity, i.e., \( f_0 = 0 \). This \( \Phi \) is inserted in (3). Polar coordinates are introduced in wave number space with Jacobian determinant \( \gamma \) and a scalar component \( h \).

The distance to the reference point of the structure function is split into a horizontal vector component and a scalar component. The phase \( \varphi \) of the electromagnetic wave accumulated after transmission through the turbulent layer is the product of the optical path length by the vacuum wave number \( k \), and the optical path length is the path integral over the product of geometric path length and refractive index \( n \) along the atmospheric height \( h \):
\[ \varphi(b) = \bar{k} \int_0^{K / \sin a} n(b, h)dh \] (9)
The scale height \( K \) is the vertical thickness of the turbulent layer. The air mass is \( 1/\sin a \) as a function of the star elevation angle \( a \) above the horizon. The structure function of the phase is defined as
\[ D_\varphi = \langle |\varphi(0) - \varphi(b)|^2 \rangle. \] (10)
The expectation value of the square is expanded with the binomial theorem,
\[ D_\varphi = \left\langle \left[ \int_0^{K / \sin(a)} \bar{k} n(0, h)dh - \int_0^{K / \sin(a)} \bar{k} n(b, h)dh \right]^2 \right\rangle \] (11)
\[ = 2\bar{k}^2 \left\{ \int_0^{K / \sin(a)} n(0, h)dh \int_0^{K / \sin(a)} n(0, h')dh' - \int_0^{K / \sin(a)} n(0, h)dh \int_0^{K / \sin(a)} n(b, h')dh' \right\} \] (12)

II. LINE-OF-SIGHT PATH INTEGRALS

A. Phase Structure Function

We review the integration of isotropic structure functions \( D_n \) along the line of sight \[ p. 293 \] (6, (C1)) \[ (7a) \]. The distance to the reference point of the structure function is split into a horizontal vector component \( b \) (baseline) and a scalar component \( h \). The phase \( \varphi \) of the electromagnetic wave accumulated after transmission through the turbulent layer is the product of the optical path length by the vacuum wave number \( \bar{k} \), and the optical path length is the path integral over the product of geometric path length and refractive index \( n \) along the atmospheric height \( h \):
We switch to relative and mean coordinates, $\Delta h \equiv h - h'$ and $\bar{h} = (h + h')/2$, with inverse mapping $h = \bar{h} + \Delta h/2$, $h' = \bar{h} - \Delta h/2$,

$$
D_\varphi = 2\bar{k}^2 \left[ \int_0^{K/(2 \sin(a))} d\bar{h} \int_{-2\bar{h}}^{2\bar{h}} d\Delta h n(0, \bar{h} + \Delta h/2)n(0, \bar{h} - \Delta h/2) \\
+ \int_0^{K/(2 \sin(a))} d\bar{h} \int_{-2\bar{h}}^{2\bar{h}} d\Delta h n(0, \bar{h} + \Delta h/2)n(0, \bar{h} - \Delta h/2) \\
- \int_0^{K/(2 \sin(a))} d\bar{h} \int_{-2\bar{h}}^{2\bar{h}} d\Delta h n(0, \bar{h} + \Delta h/2)n(b, \bar{h} - \Delta h/2) \\
- \int_0^{K/(2 \sin(a))} d\bar{h} \int_{-2\bar{h}}^{2\bar{h}} d\Delta h n(0, \bar{h} + \Delta h/2)n(b, \bar{h} - \Delta h/2) \right].
$$

(13)

In the first two terms of the right hand side, the distance between the locations in the arguments of the refractive indices is $\Delta h$, because these are located on the same ray. In the remaining two terms of the right hand side, the distance is between points located on two different rays, horizontally translated by an amount $b$, rays slanted by the altitude angle $a$ towards the horizontal. The location $(0, \bar{h} + \Delta h/2)$ on the first ray translates into Cartesian coordinates $(h + \Delta h/2)(-\cos A \cos a, \sin A \cos a, \sin a)$ in a suitable topocentric system with star azimuth $A$. The location $(b, \bar{h} - \Delta h/2)$ on the second ray has Cartesian coordinates $(h - \Delta h/2)(-\cos A \cos a, \sin A \cos a, \sin a) + b(-\cos A_b, \sin A_b, 0)$ where $A_b$ is the baseline azimuth $[14]$. The square of the distance between these two locations is $b^2 + (\Delta h)^2 - 2b\Delta h \cos \theta$, where $\cos \theta \equiv \cos a \cos(A_b - A)$ measures the angle $\theta$ between baseline and pointing.

FIG. 1: Sketch of geometric parameters. The sections of the two parallel rays are the bold lines. $a$ is the angle between the rays and the horizontal. The vertical distance between the top and bottom of the layer is $K$; so the geometric length of each ray is $K/\sin a$.

In Figure 1, $\theta$ is the angle between $b$ and what is called the delay $D$ in astronomical interferometry,

$$
D = b \cos \theta,
$$

(14)

and the third leg in this rectilinear triangle is the projected baseline

$$
P = b \sin \theta.
$$

(15)

Reverse use of

$$
D_n = 2\langle n(0, h) \rangle^2 - 2\langle n(0, h)n(b, h) \rangle
$$

(16)

and insertion of the aforementioned distance $b^2 + (\Delta h)^2 - 2b\Delta h \cos \theta = P^2 + (D - \Delta h)^2$ yields

$$
D_\varphi = 2\bar{k}^2 \left[ \int_0^{K/(2 \sin(a))} d\bar{h} \int_{-2\bar{h}}^{2\bar{h}} d\Delta h n^2(0, \bar{h}) - \bar{k}^2 \int_0^{K/(2 \sin(a))} d\bar{h} \int_{-2\bar{h}}^{2\bar{h}} d\Delta h D_n(\Delta h) \\
+ 2\bar{k}^2 \int_0^{K/(2 \sin(a))} d\bar{h} \int_{-2\bar{h}}^{2\bar{h}} d\Delta h n^2(0, \bar{h}) - \bar{k}^2 \int_0^{K/(2 \sin(a))} d\bar{h} \int_{-2\bar{h}}^{2\bar{h}} d\Delta h D_n(\Delta h) \\
- 2\bar{k}^2 \int_0^{K/(2 \sin(a))} d\bar{h} \int_{-2\bar{h}}^{2\bar{h}} d\Delta h n^2(0, \bar{h}) + \bar{k}^2 \int_0^{K/(2 \sin(a))} d\bar{h} \int_{-2\bar{h}}^{2\bar{h}} d\Delta h D_n(\sqrt{P^2 + (D - \Delta h)^2}) \\
- 2\bar{k}^2 \int_0^{K/(2 \sin(a))} d\bar{h} \int_{-2\bar{h}}^{2\bar{h}} d\Delta h n^2(0, \bar{h}) + \bar{k}^2 \int_0^{K/(2 \sin(a))} d\bar{h} \int_{-2\bar{h}}^{2\bar{h}} d\Delta h D_n(\sqrt{P^2 + (D - \Delta h)^2}) \right].
$$
The four expectation values of the squared mean values cancel,

\[
\mathcal{D}_\varphi = -\tilde{k}^2 \int_0^{K/(2\sin(a))} \frac{d\bar{h}}{2\bar{h}} d\Delta h D_n(\Delta h) \\
-\tilde{k}^2 \int_0^{K/(2\sin(a))} \frac{d\bar{h}}{2\bar{h}} d\Delta h D_n(\Delta h) \\
+\tilde{k}^2 \int_0^{K/(2\sin(a))} \frac{d\bar{h}}{2\bar{h}} d\Delta h D_n(\sqrt{P^2 + (D - \Delta h)^2}) \\
+\tilde{k}^2 \int_0^{K/(2\sin(a))} \frac{d\bar{h}}{2\bar{h}} d\Delta h D_n(\sqrt{P^2 + (D - \Delta h)^2}).
\]

(18)

In the second and fourth term substitute \(\bar{h}\) for \(\zeta = K/\sin(a) - \bar{h}\) to observe that their contributions double the values of the first and third,

\[
\mathcal{D}_\varphi = -2\tilde{k}^2 \int_0^{K/(2\sin(a))} \frac{d\bar{h}}{2\bar{h}} d\Delta h D_n(\Delta h) \\
+2\tilde{k}^2 \int_0^{K/(2\sin(a))} \frac{d\bar{h}}{2\bar{h}} d\Delta h D_n(\sqrt{P^2 + (D - \Delta h)^2}) \\
= 2\tilde{k}^2 \int_0^{K/(2\sin(a))} \frac{d\bar{h}}{2\bar{h}} d\Delta h \left[D_n(\sqrt{P^2 + (D - \Delta h)^2}) - D_n(\Delta h)\right].
\]

(19)

Insertion of the Wiener spectrum \[4\] and substitutions \(\Delta h = Py\) and \(\bar{h} = Pz/2\) of the integration variables transform this into

\[
\mathcal{D}_\varphi = 16\pi c_n C_n^2 \tilde{k}^2 P \int_0^\infty \frac{f^2 df}{(f^2 + f_0^2)^{3/2}} \int_0^{K/(2\sin(a))} \frac{d\bar{h}}{2\bar{h}} \int_{-\bar{h}}^{\bar{h}} dy \left[j_0(2\pi f Py) - j_0(2\pi f P \sqrt{1 + (b \cos \theta/P - y)^2})\right] \\
= 8\pi c_n C_n^2 \tilde{k}^2 P^2 \int_0^\infty \frac{f^2 df}{(f^2 + f_0^2)^{3/2}} \int_0^{K/(2\sin(a))} \frac{d\bar{h}}{2\bar{h}} \int_{-\bar{h}}^{\bar{h}} dy \left[j_0(2\pi f Py) - j_0(2\pi f P \sqrt{1 + (D/P - y)^2})\right].
\]

(20)

(21)

B. Review of the Standard Theory

The standard proceeding assumes that the contributions of the two spherical Bessel functions in \[20\] cancel well for large \(y\). In consequence, the limits of the innermost integral are extended to \(\bar{h}/P \to \infty\), emitting a Bessel Function of order zero \[5, 12, 21\]:

\[
\int_{-\infty}^{\infty} dy j_0(2\pi f Py) - \int_{-\infty}^{\infty} dy j_0(2\pi f P \sqrt{1 + (b \cos \theta/P - y)^2}) = \frac{1}{2fP} - \frac{1}{2fP} J_0(2\pi f P).
\]

(22)

This reduces the integration over \(\bar{h}\) to a simple factor \[3, 8\]

\[
\mathcal{D}_\varphi = 4\pi c_n C_n^2 \tilde{k}^2 \frac{K}{\sin a} \int_0^\infty \frac{f df}{(f^2 + f_0^2)^{3/2}} \left[1 - J_0(2\pi f P)\right].
\]

(23)

The remaining frequency integrals are known in the literature \[10, 6.565.4\] the frequency-to-real-space matching factor \[8\] is also inserted, and the structure function for the von Kármán spectrum is condensed into the well-known \[2, 4\]

\[
\mathcal{D}_\varphi(P) = -\tilde{k}^2 \frac{K}{\sin a} C_n^2 P^{1+\gamma} \frac{\sqrt{\pi}}{\Gamma(\gamma/2)(\pi P f_0)^{1+\gamma}} \left[\Gamma\left(\frac{1+\gamma}{2}\right) - 2(\pi P f_0)^{(1+\gamma)/2} K_{(1+\gamma)/2}(2\pi P f_0)\right],
\]

(24)

where \(K(.)\) are modified Bessel Functions. \[1, 9.6\] The factors that depend on the projected baseline (phase) measured in units of the outer scale,

\[
P_0 = 2\pi f_0 P,
\]

(25)
are the topic further below. The Kolmogorov limit of infinite outer scale is

\[ \lim_{f_0 \to 0} \mathcal{D}_\varphi(P) = \bar{k}^2 \frac{K}{\sin a} C_n^2 P^{1+\gamma} g(\gamma) \]  

(26)

where

\[ g(\gamma) = \sqrt{\frac{\pi}{\Gamma(-1/2 - \gamma/2) \Gamma(-\gamma/2)}} \approx 2.9143808 \]  

(27)

at \( \gamma = 2/3 \). In overview, observations away from the zenith multiply the scale height with the air mass, and employ an effective distance between the two beams (which scales \( \propto 5/3 \)) equal to the projected baseline.

III. CORRECTION FOR FINITE LAYER THICKNESS

A. Aim

Quantitative understanding of effects of the finite thickness of turbulent layers of air is of interest where (i) the thickness is smaller than the entire atmosphere, cases such as subdivisions in models for multi-conjugated adaptive optics or jet-streams, or (ii) where the lateral beam separation is comparatively large, and the scaling down to small separations is be done with a single, consistent structure constant, for example comparing interferometric observations with atmospheric site monitor data [13].

The core of the following is to substitute the approximate calculation of Section 2.2 by transforming (21) into (45). The nature of the approximation reviewed above is sampling the difference of the two structure functions (19) beyond the precise limits of the integral; as the structure function is an increasing function of its argument, the net effect of the approximation is an exaggeration of \( \mathcal{D}_\varphi \), at least in the case \( D = 0 \) where the argument of the square root remains larger than \( \Delta h \).

We first constrain the turbulence model to the limit of infinite outer scale, because the mathematics reduces to simpler special functions then (Section III B). The generic case is phrased in terms of integrals over MacDonald Functions in the main part (Section III C).

B. In the Kolmogorov Limit

Taking the limit \( f_0 \to 0 \), (27) is inserted into (19),

\[ \mathcal{D}_\varphi = 2C_n^2 \bar{k}^2 \int_0^{K/(2\sin a)} d\bar{h} \int_{-2\bar{h}}^{2\bar{h}} d\Delta h \left\{ \left[ P^2 + (D - \Delta h)^2 \right]^{\gamma/2} - |\Delta h|^\gamma \right\} \]

\[ = C_n^2 \bar{k}^2 P^{1+\gamma} \frac{K}{\sin a} \frac{P \sin a}{K} \int_0^{K/(P \sin a))} dz \int_{-z}^z dy \left\{ \left[ 1 + \frac{(D - P - y)^2}{P^2} \right]^{\gamma/2} - |y|^\gamma \right\} . \]  

(28)

Binomial expansion of the integrand in hypergeometric series of \( y \) transforms \( \mathcal{D}_\varphi \) into

\[ \mathcal{D}_\varphi = C_n^2 \bar{k}^2 P^{1+\gamma} \frac{K}{\sin a} g(\gamma, P_K) \]  

(29)

where the dimensionless

\[ P_K \equiv \frac{P \sin a}{K} \]  

(30)

is the beam separation in units of the path length, and where

\[ g(\gamma, P_K) = \frac{P_K}{2} \left[ F(1) - 2F(0) + F(-1) \right] - \frac{1}{(1 + \gamma/2)(1 + \gamma)} P_K^{1+\gamma} \]  

(31)

with

\[ F(1) = \binom{D}{P + j/P_K}^2 3F_2 \left( \begin{array}{c} -\gamma/2, 1/2, 1 \end{array} | -\left( \frac{D + j}{P_K} \right)^2 \right) \]  

(32)
generalizes (27) in terms of three generalized hypergeometric functions. Aspects of the implementation are discussed in Appendix A.

Figure 2 illustrates the magnitude of the effect. Plotting \( g \) as a function of \( \frac{P}{K} \) focuses on the relative deviation of the full computation in comparison with the standard formula (26); non-horizontal lines indicate a nonlinearity between \( D_\varphi \) and \( K/\sin a \) in (29). In general, the phase fluctuations are smaller than the factor 2.9 predicted in the approximate (27), which had been derived in the limit \( K/P \rightarrow \infty \) of infinite layer thickness. At optical and infrared wavelengths, the thickness \( K \) multiplied by the air mass is limited to the order of 10 km representing the entire atmosphere (troposphere), and \( P \) is of the order of 2 m for a single telescope up to 100 m for an interferometer. Whence \( \frac{P}{K} \) is in the range \( 2 \times 10^{-4} \) to 0.01, and Figure 2 demonstrates that this finite layer thickness places \( g \) in the range 2.3 to 2.7. Long baseline interferometry “drops” power of \( D_\varphi \) through this geometric sampling effect.

An increase of \( D_\varphi \) is induced if \( \frac{D}{P} = \cot \theta \) turns nonzero, i.e., if the pointing direction is not perpendicular to the baseline. This tendency is qualitatively understood from Figure 1, since the contribution at \( D = 0 \) integrates correlations between the two beams at shortest distances \( P \), and nonzero \( D \) essentially requires parallel sampling of the refractive index structure function correlated over an effective distance \( b > P \).

\[ D_\varphi = 8\pi c_\nu K_\varphi^2 P^{2+\gamma} \int_0^\infty \frac{f^2 df}{[f^2 + (f_0P)^2]^{(\gamma+3)/2}} \int_0^{1/K_\varphi} dz \int_{-z}^z dy \left[ j_0(2\pi f y) - j_0(2\pi f \sqrt{1 + (D/P - y)^2}) \right] \\
= 8\pi c_\nu K_\varphi^2 P^{2+\gamma} \int_0^\infty \frac{f^2 df}{[f^2 + (f_0P)^2]^{(\gamma+3)/2}} \int_0^{1/P_\varphi} dz \int_0^z dy \left[ 2j_0(2\pi f y) - j_0(2\pi f \sqrt{1 + (D/P + y)^2}) \right] \\
= 8\pi c_\nu K_\varphi^2 P^{1+\gamma} K \sin a \int_0^\infty \frac{f^2 df}{[f^2 + (f_0P)^2]^{(\gamma+3)/2}} \int_0^{1/P_\varphi} dy [1 - P_\varphi y] \left[ 2j_0(2\pi f y) - j_0(2\pi f \sqrt{1 + (D/P + y)^2}) \right] (33) \]

with the convention that \( \pm \) or \( \mp \) indicate summation over both signs in the corresponding term.

1. First Ray

The next step is to evaluate the previous equation. It contains a simpler term with a spherical Bessel function with argument \( y \), which will be called the contribution of the first ray, and another term depending on \( D/P \), contribution of the second ray. Each splits into two terms, one with and one without a factor \( yP_\varphi \). Evaluating the \( f \)-integrals...
first, one term is [10, 3.771.5]  

\[
\int_0^\infty \frac{f^2 df}{[f^2 + (f_0 P)^2]^{(\gamma+3)/2}} J_0(2\pi f y) = (2\pi y)^\gamma \int_0^\infty \frac{t dt}{[t^2 + (2\pi y f_0 P)^2]^{(\gamma+3)/2}} \sin t = \frac{\sqrt{\pi}}{2^{\gamma+2}(f_0 P)^\gamma \Gamma(3/2 + \gamma/2)} (2\pi f_0 P y)^{\gamma/2} K_{\gamma/2}(2\pi f_0 P).
\]  

(34)

Its \( y \)-integral is

\[
\int_0^{1/P_K} (P_0 y)^{\gamma/2} K_{\gamma/2}(P_0 y) dy = \frac{1}{P_0} V_\nu(P_0/P_K),
\]

(35)

at

\[
\nu \equiv \gamma/2,
\]

(36)

where we define [10, 6.561.4]

\[
V_\nu(x) \equiv \int_0^x t^\nu K_\nu(t) dt = 2^{\nu-1} \sqrt{\pi} \Gamma(1/2 + \nu) x [K_\nu(x) L_{\nu-1}(x) + K_{\nu-1}(x) L_\nu(x)],
\]

(37)

with \( L \) representing the modified Struve functions [1, 12.2]. The other contribution of the first ray is [11, 11.3.27]

\[
\int_0^{1/P_K} y(P_0 y)^{\gamma/2} K_{\gamma/2}(P_0 y) dy = -\frac{1}{P_0^2} [(P_0/P_K)^{1+\nu} K_{1+\nu}(P_0/P_K) - \Gamma(1 + \nu) 2^\nu].
\]

(38)

2. Second Ray

The companion of (34) from the second ray is

\[
\int_0^\infty \frac{f^2 df}{[f^2 + (f_0 P)^2]^{(\gamma+3)/2}} J_0(2\pi f \sqrt{1 + |y + D/P|^2}) = \frac{\sqrt{\pi}}{2^{\gamma+2}(f_0 P)^\gamma \Gamma(3/2 + \gamma/2)} [P_0 \sqrt{1 + |y + D/P|^2}]^\nu K_\nu(P_0 \sqrt{1 + |y + D/P|^2}).
\]

(39)

Its \( y \)-integral is

\[
\int_0^{K/(P \sin \alpha)} (1 - P_K y)(P_0 \sqrt{1 + |y + D/P|^2})^\nu K_\nu(P_0 \sqrt{1 + |y + D/P|^2}) dy = \int_{\frac{D}{P}}^{1/P_K + \frac{D}{P}} (1 - P_K y \pm \frac{P_K D}{P}) [P_0 \sqrt{1 + y^2}]^\nu K_\nu(P_0 \sqrt{1 + y^2}) dy.
\]

(40)

A substitution \( P_0 \sqrt{1 + y^2} = t \) calls to split the interval into the regions \( y > 0 \) and \( y < 0 \), which is qualified with the step-function, \( \Theta(x) \equiv 1 \) if \( x > 0 \) and \( \Theta(x) \equiv 0 \) if \( x < 0 \). The contribution from positive \( y \) to the integral is

\[
\frac{\Theta(1/P_K + D/P)}{P_0} \left[ (1 \mp P_K \frac{D}{P}) U_\nu(t, P_0) + \frac{P_K}{P_0} t^{\nu+1} K_{\nu+1}(t) \right]_{t = P_0 \sqrt{1 + \max^2(0, \mp D/P)}}.
\]

(41)

where

\[
U_\nu(z, u) \equiv \int_u^z \frac{t^{1+\nu}}{\sqrt{t^2 - u^2}} K_\nu(t) dt
\]

(42)

is an integral described in Appendix [B]. The contribution from negative \( y \) is

\[
\frac{\Theta(\mp D/P)}{P_0} \left[ (1 \mp P_K \frac{D}{P}) U_\nu(t, P_0) - \frac{P_K}{P_0} t^{\nu+1} K_{\nu+1}(t) \right]_{t = P_0 \sqrt{1 + \min^2(0, 1)/P_K + D/P}}.
\]

(43)
Collecting (35), (38), (41) and (43) puts (33) into the format

$$D\phi = C^2 g^2 P^{1+\gamma} \frac{K}{\sin a} g(\gamma, P_0, P_K),$$

with

$$g(\gamma, P_0, P_K) = \frac{-2^{1+\nu}}{\Gamma(-\nu)P_0^{1+\gamma}} \left( 2V_\nu(P_0/P_K) + 2\frac{P_K}{P_0} (\nu+1)K_{1+\nu}(P_0/P_K) - \Gamma(1+\nu)2^\nu \right)$$

$$-\Theta(1/P_K \mp D/P) \left( (1 \mp P_K \frac{D}{P})U_\nu(t, P_0) + \frac{P_K}{P_0} t^{\nu+1} K_{\nu+1}(t) \right)$$

$$-\Theta(\pm D/P) \left( (1 \mp P_K \frac{D}{P})U_\nu(t, P_0) - \frac{P_K}{P_0} t^{\nu+1} K_{\nu+1}(t) \right)$$

(45)

at $\nu = \gamma/2$. A summation over the upper and lower sign is still implied for the terms with the two $\Theta$-function factors. [Note that there are massive cancellations between the group of terms containing $V_\nu$ and $U_\nu$, and also between the other terms depending on $K_{1+\nu}$, which can be traced back to the differences of the spherical Bessel functions $j_0$.]

This is the main result. Equation (24) represents the concerted limit $P_K \to 0$, $D/P \to 0$, equation (27) the limit $P_0 \to 0$, $P_K \to 0$, $D/P \to 0$, equation (31) the limit $P_0 \to 0$, and equation (A2) the limit $P_0 \to 0$, $D/P \to 0$.

Figure 3 shows again $g^{(0)}$ to point out at which lateral distances the full theory starts to deviate from the straight lines predicted by the standard theory. The upper index indicates that this is the contribution from the zeroth order of an expansion in powers of $(D/P)$, as in App. A.

A numerical example: The curve $P_0 = 0.1$ is equivalent to a separation $P = 40$ cm if the outer scale is $1/f_0 = 25$ m. The standard formula (24) assumes a value of $g = 1.83$ at the left edge of the plot; the refined calculation reduces this by 25% to $g = 1.37$ if $P\sin a/K = 0.137$, i.e., if the path length through the layer is $K/\sin a \approx 2.9$ m. As layers are thicker in practice, the more advanced calculation has little impact here. This seems to contradict the importance claimed in Section III B, but the following mechanism conciliates both views: As $P_0$ increases, i.e., as the outer scale shrinks, the curves in Figure 3 remain horizontal over an increasingly wide initial range on the $P_K$ axis. The standard formula remains accurate down to thin turbulent layers, because the clamping of $D_n$ introduced by the outer scale reduces the importance of correlations over large (including large vertical) scales. The additional strict cut within the refined calculation targets smaller values of $D_n$, so the relative effect becomes less important.
Inclusion of nonzero “shear” ratios $D/P$ leads to Figure 4, which shows no other features than those expected by combining the shapes of Figures 2 and 3: a hump appears in $g$ if the delay $D$ occupies a major fraction of the path length, and $g$ falls off with an approximate $-5/6$th power at large $P_0$.

\[ g(\gamma = 2/3) = \frac{P \sin(a)}{K} \]

FIG. 4: The factor $(45)$ as a function of the dimensionless $P/K$ for two different dimensionless measures $P_0$ of the outer scale and three different ratios $D/P$.

IV. SUMMARY

We developed a formula of the phase structure function depending on the spectral index, on the lateral separation, on the air mass and layer thickness, and on the outer scale, which rigorously samples the von-Kármán statistics of isotropic homogeneous turbulence along two parallel rays.

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Appendix A: Kolmogorov Limit

The generalized Hypergeometric Function \( _3F_2 \left( \frac{1/2, 1, -\gamma/2}{3/2, 2} \mid z \right) \) reduces effectively to Gaussian Hypergeometric Functions \([9, 17]\):

\[
3F_2 \left( \frac{1/2, 1, -\gamma/2}{3/2, 2} \mid z \right) = 2 \cdot 2F_1 \left( \frac{-\gamma/2, 1/2}{2} \mid z \right) - 2F_1 \left( \frac{-\gamma/2, 1}{2} \mid z \right).
\]

A useful Taylor expansion of (31) in powers of \( D/P \) is

\[
g(\gamma, P_K) = g(0) + g(2) + g(4) + \cdots \quad (A1)
\]

with constant order

\[
g(0)(\gamma, P_K) = -\frac{1}{1 + \gamma/2} \left[ P_K \left\{ 2F_1 \left( \frac{-1/2, -\gamma/2 - 1}{1/2} \mid -\frac{1}{P_K^2} \right) - 1 \right\} + \frac{1}{(1 + \gamma)P_1^{1+\gamma}} \right], \quad (A2)
\]

quadratic order

\[
g(2) = P_K(\psi^{\gamma/2} - 1)(D/P)^2, \quad (A3)
\]

and biquadratic order

\[
g(4) = P_K^2 \psi^{\gamma/2}(2 + \gamma\psi - \gamma - \psi) - \psi^2(D/P)^4, \quad (A4)
\]

where

\[
\psi \equiv 1 + 1/P_K^2. \quad (A5)
\]

Appendix B: Integral \( U \)

If the upper limit \( z \) of the integral (42) is large, it is numerically advantageous to use the complementary \([10, 6.592.12]\)

\[
U_\nu(z, u) = \sqrt{\pi/2}u^{\nu+1/2}K_{\nu+1/2} - \int_z^\infty \frac{t^{\nu+1}}{\sqrt{t^2 - u^2}}K_\nu(t)dt. \quad (B1)
\]

Further binomial expansion in a power series of \( u/z \),

\[
\int_z^\infty \frac{t^{\nu+1}}{\sqrt{t^2 - u^2}}K_\nu(t)dt = \sum_{l=0}^{\infty} \left( -\frac{1}{2} \right)_l (-u^2)^l \int_z^\infty t^{\nu-2l}K_\nu(t)dt \quad (B2)
\]

leads to a set of integrals where the parameter \( l \) is recursively reduced by partial integration if \( l > 0 \) \([16]\),

\[
\int_z^\infty t^{\nu-2l}K_\nu(t)dt = \frac{z^{\nu-2l+1}}{2l - 1}K_\nu(z) + \frac{1}{1 - 2l} \int_z^\infty t^{-2(l-1)}t^{\nu-1}K_{\nu-1}(t)dt \quad (B3)
\]

until the case \( l = 0 \) is covered by \([37]\) via \([10, 6.561.16]\)

\[
\int_z^\infty t^{\nu}K_\nu(t)dt = \sqrt{\pi}2^{\nu-1}\Gamma(\nu + 1/2) - V_\nu(z). \quad (B4)
\]

On the other hand, if \( z \) is not larger than approximately 1.8u, the repeated partial integration represents a converging series

\[
U_\nu(z, u) = \sum_{k=0}^{\infty} \frac{1}{1 \cdot 3 \cdot 5 \cdots (2k + 1)(2k + 3)}(z^2 - u^2)^{k+1/2}z^{\nu-k}K_{\nu-k}(z). \quad (B5)
\]