Asymptotic Prime Divisors Related to Ext, Regularity of Powers of Ideals, and Syzygy Modules

Thesis
Submitted in partial fulfillment of the requirements
of the degree of
Doctor of Philosophy
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2016
Dedicated To
My Parents and Grandparents
Abstract

This dissertation focuses on the following topics: (1) asymptotic prime divisors over complete intersection rings, (2) asymptotic stability of complexities over complete intersection rings, (3) asymptotic linear bounds of Castelnuovo-Mumford regularity in multigraded modules, and (4) characterizations of regular local rings via syzygy modules of the residue field.

Chapter 1 introduces the concerned research problems and provides an overview of the works presented in this dissertation.

Chapter 2 gives a detailed literature review of the subjects studied in this dissertation.

Now we concentrate on our main four topics mentioned in the first paragraph. For each topic, there is a separate chapter as follows.

Chapter 3 deals with the study of asymptotic behaviour of certain sets of associated prime ideals related to Ext-modules. Let $A$ be a local complete intersection ring. Suppose $M$ and $N$ are two finitely generated $A$-modules and $I$ is an ideal of $A$. We prove that

$$\bigcup_{n \geq 1} \bigcup_{i \geq 0} \text{Ass}_A \left( \text{Ext}_A^i(M, N/I^n N) \right)$$

is a finite set. Moreover, we analyze the asymptotic behaviour of the sets

$$\text{Ass}_A \left( \text{Ext}_A^i(M, N/I^n N) \right) \quad \text{if } n \text{ and } i \text{ both tend to } \infty.$$  

We show that there are non-negative integers $n_0$ and $i_0$ such that for all $n \geq n_0$ and $i \geq i_0$ the set $\text{Ass}_A \left( \text{Ext}_A^i(M, N/I^n N) \right)$ depends only on whether $i$ is even or odd. We also prove the analogous results for complete intersection rings which arise in algebraic geometry.

Chapter 4 shows that if $A$ is a local complete intersection ring, then the complexity $\text{cx}_A(M, N/I^n N)$ is independent of $n$ for all sufficiently large $n$. 

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Chapter 5 is devoted to study the Castelnuovo-Mumford regularity of powers of several ideals. Let \( A \) be a Noetherian standard \( \mathbb{N} \)-graded algebra over an Artinian local ring \( A_0 \). Let \( I_1, \ldots, I_t \) be homogeneous ideals of \( A \), and let \( M \) be a finitely generated \( \mathbb{N} \)-graded \( A \)-module. We show that there exist two integers \( k_1 \) and \( k_1' \) such that
\[
\text{reg}(I_1^{n_1} \cdots I_t^{n_t} M) \leq (n_1 + \cdots + n_t)k_1 + k_1' \quad \text{for all } n_1, \ldots, n_t \in \mathbb{N}.
\]
Moreover, we prove that if \( A_0 \) is a field, then there exist two integers \( k_2 \) and \( k_2' \) such that
\[
\text{reg} \left( \overline{I_1^{n_1}} \cdots \overline{I_t^{n_t}} M \right) \leq (n_1 + \cdots + n_t)k_2 + k_2' \quad \text{for all } n_1, \ldots, n_t \in \mathbb{N},
\]
where \( \overline{I} \) denotes the integral closure of an ideal \( I \) of \( A \).

Chapter 6 is allocated for the syzygy modules of the residue field of a Noetherian local ring. Let \( A \) be a Noetherian local ring with residue field \( k \). We show that if a finite direct sum of syzygy modules of \( k \) maps onto ‘a semidualizing module’ or ‘a non-zero maximal Cohen-Macaulay module of finite injective dimension’, then \( A \) is regular. We also prove that \( A \) is regular if and only if some syzygy module of \( k \) has a non-zero direct summand of finite injective dimension.

We conclude this dissertation by presenting a few open questions in Chapter 7.

**Key words and phrases.** Associate primes; Graded rings and modules; Rees rings and modules; Ext; Tor; Complete intersection rings; Eisenbud operators; Support variety; Complexity; Hilbert functions; Local cohomology; Castelnuovo-Mumford regularity; Regular rings; Syzygy and cosyzygy modules; Semidualizing modules; Injective dimension.
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List of Notations

Throughout this dissertation, unless otherwise specified, all rings (graded or not) are assumed to be commutative Noetherian rings with identity.

Let $A$ be a ring, and let $I$ be an ideal of $A$. Suppose $M$ and $N$ are finitely generated $A$-modules. We use the following list of notations in this dissertation.

Sets

| $\mathbb{N}$ | the set of all non-negative integers |
| $\mathbb{Z}$ | the set of all integers |
| $\Lambda$ | a finite collection of non-negative integers |
| $\phi$ | empty set |
| $f$ | a finite sequence $f_1, \ldots, f_c$ |
| $\text{Spec}(A)$ | the set of all prime ideals of $A$, spectrum of $A$ |
| $\text{Min}(A)$ | the set of all minimal prime ideals of $A$ |
| $\text{Supp}(M)$ | $\{p \in \text{Spec}(A) : M_p \neq 0\}$, support of $M$ |
| $\text{Proj}(A)$ | the set of all homogeneous prime ideals of an $\mathbb{N}$-graded ring $A = \bigoplus_{n \geq 0} A_n$ which do not contain the irrelevant ideal $A_+ = \bigoplus_{n \geq 1} A_n$ |
| $\text{Var}(a)$ | $\{p \in \text{Spec}(A) : a \subseteq p\}$, variety of an ideal $a$ of $A$ |
| $\text{Ass}_A(M)$ | the set of all associated prime ideals of an $A$-module $M$ |
| $D(x)$ | $\{p \in \text{Spec}(A) : x \notin p\}$, where $x$ is an element of $A$ |
| $^*D(x)$ | $\{p \in \text{Proj}(A) : x \notin p\}$, where $A$ is an $\mathbb{N}$-graded ring and $x$ is a homogeneous element of $A$ |
| $^*\text{Ass}_A(M)$ | $\text{Ass}_A(M) \cap \text{Proj}(A)$, the set of relevant associated prime ideals of an $\mathbb{N}$-graded module $M$ over an $\mathbb{N}$-graded ring $A$ |
| $\mathcal{V}(M, N)$ | support variety of $M$ and $N$ |
### Graded Objects

Let $R = \bigoplus_{n \in \mathbb{N}} R_n$ be an $\mathbb{N}^t$-graded ring (where $t$ is a fixed positive integer), and let $L = \bigoplus_{n \in \mathbb{N}^t} L_n$ be an $\mathbb{N}^t$-graded $R$-module. Suppose $A$ is an $\mathbb{N}$-graded ring and $M$ is an $\mathbb{N}$-graded $A$-module. Assume $I$ and $J$ are two homogeneous ideals of $A$.

| $n$ | $t$-tuple $(n_1, n_2, \ldots, n_t)$ over $\mathbb{N}$, i.e., element of $\mathbb{N}^t$ |
|-----|----------------------------------------------------------------------------------|
| $|n|$ | sum of the components of $n$, i.e., $(n_1 + n_2 + \cdots + n_t)$ |
| $\epsilon^i$ | $t$-tuple with all components 0 except the $i$th component which is 1 |
| $0$ | $t$-tuple with all components 0 |
| $L_n$ | the $n$th graded component of an $\mathbb{N}^t$-graded module $L$ |
| $\text{end}(M)$ | $\sup \{n : M_n \neq 0\}$, end of $M$ |
| $L(u)$ | same as $L$ but the grading is twisted by $u$, i.e., $L(u)$ is an $\mathbb{N}^t$-graded module with $L(u)_n := L_n(u+\omega)$ for all $n \in \mathbb{N}^t$ |
| $\text{deg}(a)$ | homogeneous degree of an element $a$ |
| $d(J)$ | $\min \{d : J \text{ is generated by homogeneous elements of degree } \leq d\}$ |
| $\rho_M(I)$ | $\min \{d(J) : J \text{ is an } M\text{-reduction of } I\}$ |
| $\varepsilon(M)$ | the smallest degree of the (non-zero) homogeneous elements of $M$ |
| $A_+$ | $\bigoplus_{n \geq 1} A_n$ which is called the irrelevant ideal of $A$ |
| $a_i(M)$ | $\text{end} \left( H^i_{A_+}(M) \right) $ |

### Rings

| $(A, m)$ | local ring $A$ with its unique maximal ideal $m$ |
| $(A, m, k)$ | local ring $(A, m)$ with its residue field $k := A/m$ |
| $\hat{A}$ | $m$-adic completion of $A$, where $(A, m)$ is a local ring |
| $S^{-1}A$ | localization of $A$ by a multiplicatively closed subset $S$ of $A$ |
| $A_p$ | localization of $A$ at a prime ideal $p$ of $A$ |
| $A_{(p)}$ | homogeneous localization (or degree zero localization) of a graded ring $A$ at a homogeneous prime ideal $p$ of $A$ |
| $A[X_1, \ldots, X_d]$ | polynomial ring in $d$ variables $X_1, \ldots, X_d$ over $A$ |
| $\mathcal{R}(I)$ | $\bigoplus_{n \geq 0} I^nX^n$ which is called the Rees ring associated to $I$ |
| $F(I)$ | $\mathcal{R}(I) \otimes_A k$. It is called fiber cone of $I$ (where $(A, m, k)$ is a local ring) |
| $\mathcal{T}$ | polynomial ring $A[t_1, \ldots, t_c]$ over $A$ in the cohomology operators $t_j$ with $\text{deg}(t_j) = 2$ for all $1 \leq j \leq c$ |
| $\mathcal{S}$ | bigraded ring $\mathcal{R}(I)[t_1, \ldots, t_c]$, where we set $\text{deg}(t_j) = (0, 2)$ for all $1 \leq j \leq c$ and $\text{deg}(u) = (s, 0)$ if $\text{deg}(u)$ in $\mathcal{R}(I)$ is $s$ |
### Ideals

| Notation   | Definition |
|------------|------------|
| $\text{Ann}_A(X)$ | $\{ a \in A : ax = 0 \text{ for all } x \in X \}$, annihilator of $X$, where $X \subseteq M$ |
| $\sqrt{I}$ | $\{ a \in A : a^n \in I \text{ for some } n \geq 1 \}$, radical of $I$ |
| $\text{Soc}(A)$ | $\{ a \in A : am = 0 \}$, socle of a local ring $(A, m)$ |

### Modules

| Notation   | Definition |
|------------|------------|
| $\hat{M}$ | $m$-adic completion of $M$, where $M$ is an $(A, m)$-module |
| $M_g$ | localization of $M$ by the multiplicatively closed set $\{ g^i : i \geq 0 \}$ |
| $M^n$ | direct sum of $n$ copies of $M$, where $n$ is a positive integer |
| $\omega$ | canonical module of $A$ |
| $\Omega^A_n(M)$ | the $n$th syzygy module of $M$, where $n$ is a non-negative integer |
| $\Omega^A_{n}(M)$ | the $n$th cosyzygy module of $M$, where $n$ is a non-negative integer |
| $\text{Image}(\Phi)$ | image of a module homomorphism $\Phi$ |
| $(0 :_M I)$ | $\{ x \in M :Ix = 0 \}$, colon submodule of $M$ |
| $\Rees(\mathcal{I}, N)$ | $\bigoplus_{n \geq 0} I^nN$ of $N$ associated to $I$ |
| $N[X]$ | $N \otimes_A A[X]$ (tensor product of the $A$-modules $N$ and $A[X]$) |
| $\text{gr}_I(N)$ | associated graded module $\bigoplus_{n \geq 0}(I^nN/I^{n+1}N)$ of $N$ with respect to $I$ |
| $N'$ | $\bigoplus_{n \geq 0} N_n$, an $N$-graded $\Rees(I)$-module. We set $\mathcal{L} := \bigoplus_{n \geq 0}(N/I^{n+1}N)$ |

### Few Invariants

| Notation   | Definition |
|------------|------------|
| depth$(A)$ | depth of a local ring $A$ |
| dim$(A)$ | $\sup \{ n : \exists \mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_n, \mathfrak{p}_i \in \text{Spec}(A) \}$, Krull dimension of $A$ |
| dim$_A(M)$ | dimension of an $A$-module $M$, i.e., $\dim(A/\text{Ann}_A(M))$ |
| dim$(\mathcal{V})$ | dimension of a variety $\mathcal{V}$ |
| rank$_k(V)$ | dimension of $V$ as a vector space over a field $k$ |
| $\mu_A(M)$ | the minimal number of generators of $M$ as an $A$-module |
| codim$(A)$ | $\mu_A(m) - \dim(A)$, codimension of a local ring $(A, m)$ |
| $\lambda_A(M)$ | length of $M$ as an $A$-module |
| $\beta^A_n(M)$ | the $n$th Betti number of $M$ |
| projdim$_A(M)$ | projective dimension of an $A$-module $M$ |
| injdim$_A(M)$ | injective dimension of an $A$-module $M$ |
| $\text{cx}_A(M)$ | complexity of $M$ |
| $\text{cx}_A(M, N)$ | complexity of a pair of $A$-modules $(M, N)$ |
| reg$(M)$ | Castelnuovo-Mumford regularity of $M$ |
| $\lim \sup_{n \to \infty} a_n$ | limit supremum value of a sequence $\{ a_n \}$ |
**Homological Tools**

| Notation               | Description                                                                 |
|------------------------|-----------------------------------------------------------------------------|
| $\text{Hom}_A(M, N)$   | the collection of all $A$-linear maps from $M$ to $N$                      |
| $\text{Ext}^i_A(-, -)$ | the $i$th *extension functor* (the right derived functors of $\text{Hom}$)   |
| $\text{Tor}^i_A(-, -)$ | the $i$th *torsion functor* (the left derived functors of tensor product)   |
| $H^i_J(M)$              | the $i$th local cohomology module of $M$ with respect to an ideal $J$       |
| $\text{Ext}^*_A(M, N)$ | $\bigoplus_{i \geq 0} \text{Ext}^i_A(M, N)$, total Ext-module            |
| $C(M, N)$              | $\text{Ext}^*_A(M, N) \otimes_A k$ (where $A$ is a local ring with its residue field $k$) |
| $\mathscr{E}(N)$       | $\bigoplus_{n \geq 0} \bigoplus_{i \geq 0} \text{Ext}^i_A(M, N_n)$, where $N = \bigoplus_{n \geq 0} N_n$ |
| $\mathbb{F}$           | free resolution of $M$                                                      |
| $\mathbb{F}(m)$        | same complex as $\mathbb{F}$ but the components are twisted by $m \in \mathbb{Z}$ |
| $\text{Hom}_A(\mathbb{F}, N)$ | cochain complex induced by applying the functor $\text{Hom}_A(-, N)$ on $\mathbb{F}$ |
Chapter 1

Introduction

In this dissertation, we study the following topics: (1) asymptotic prime divisors over complete intersection rings, (2) asymptotic stability of complexities over complete intersection rings, (3) asymptotic linear bounds of Castelnuovo-Mumford regularity in multi-graded modules, and (4) characterizations of regular local rings via syzygy modules of the residue field.

Throughout this dissertation, unless otherwise specified, all rings (graded or not) are assumed to be commutative Noetherian rings with identity.

1.1 Asymptotic Prime Divisors Related to Ext

Let $A$ be a ring, and let $M$ be a finitely generated $A$-module. A prime ideal $p$ of $A$ is called an associated prime ideal of $M$ if $p$ is the annihilator $\text{Ann}_A(x)$ of some element $x$ of $M$. The set of all associated prime ideals of $M$ is denoted by $\text{Ass}_A(M)$. It is well-known that $\text{Ass}_A(M)$ is a finite set.

Throughout this section, let $M$ and $N$ be two finitely generated $A$-modules, and let $I$ be an ideal of $A$. In [Bro79], M. Brodmann proved that the set of associated prime ideals $\text{Ass}_A(M/I^nM)$ is independent of $n$ for all sufficiently large $n$. Thereafter, L. Melkersson and P. Schenzel generalized Brodmann’s result in [MS93 Theorem 1] by showing that

$$\text{Ass}_A(\text{Tor}_1^A(M, A/I^n))$$

\footnote{The new results appear under the topics (1) and (2) are joint work with T. J. Puthenpurakal; while the results shown in topic (4) are joint work with A. Gupta and T. J. Puthenpurakal.}
is independent of $n$ for all large $n$ and for every fixed $i \geq 0$. Later, D. Katz and E. West proved the above result in a more general way: For every fixed $i \geq 0$, the sets
\[ \text{Ass}_A \left( \text{Tor}^A_i(M, N/I^n N) \right) \quad \text{and} \quad \text{Ass}_A \left( \text{Ext}^i_A(M, N/I^n N) \right) \]
are independent of $n$ for all sufficiently large $n$ (see [KW04, Corollary 3.5]). In particular, for every fixed $i \geq 0$, one obtains that the sets
\[ \bigcup_{n \geq 1} \text{Ass}_A \left( \text{Tor}^i_1(M, N/I^n N) \right) \quad \text{and} \quad \bigcup_{n \geq 1} \text{Ass}_A \left( \text{Ext}^i_A(M, N/I^n N) \right) \]
are finite.

The motivation for our main results on associated prime ideals came from the following two questions. They were raised by W. V. Vasconcelos [Vas98, Problem 3.15] and L. Melkersson and P. Schenzel [MS93, page 936] respectively.

(A1) Is the set $\bigcup_{i \geq 0} \text{Ass}_A \left( \text{Ext}^i_A(M, A) \right)$ finite?

(A2) Is the set $\bigcup_{n \geq 1} \bigcup_{i \geq 0} \text{Ass}_A \left( \text{Tor}^i_1(M, A/I^n) \right)$ finite?

Note that if $A$ is a Gorenstein local ring, then the question (A1) has a positive answer (trivially). If we change the question a little, then we may ask the following:

(A3) Is the set $\bigcup_{i \geq 0} \text{Ass}_A \left( \text{Ext}^i_A(M, N) \right)$ finite?

This is not known for Gorenstein local rings. However, if $A = Q/(f)$, where $Q$ is a local ring and $f = f_1, \ldots, f_c$ is a $Q$-regular sequence, and if $\text{projdim}_Q(M)$ is finite, then the question (A3) has an affirmative answer. This can be seen by using the finite generation of $\bigoplus_{i \geq 0} \text{Ext}^i_A(M, N)$ over the polynomial ring $A[t_1, \ldots, t_c]$ in the cohomology operators $t_j$ over $A$.

The result in the same spirit proved by T. J. Puthenpurakal is the following:

**Theorem 1.1.1.** [Put13, Theorem 5.1] Let $A$ be a local complete intersection ring. Suppose $N = \bigoplus_{n \geq 0} N_n$ is a finitely generated graded module over the Rees ring $\mathcal{R}(I)$. Then
\[ \bigcup_{n \geq 0} \bigcup_{i \geq 0} \text{Ass}_A \left( \text{Ext}^i_A(M, N_n) \right) \]
is a finite set.

Furthermore, there exist $n_0, i_0 (\geq 0)$ such that for all $n \geq n_0$ and $i \geq i_0$, we have
\[ \text{Ass}_A \left( \text{Ext}^{2i}_A(M, N_n) \right) = \text{Ass}_A \left( \text{Ext}^{2i_0}_A(M, N_{n_0}) \right), \]
\[ \text{Ass}_A \left( \text{Ext}^{2i+1}_A(M, N_n) \right) = \text{Ass}_A \left( \text{Ext}^{2i_0+1}_A(M, N_{n_0}) \right). \]
1.1 Asymptotic Prime Divisors Related to Ext

In particular, $N$ can be taken as $\bigoplus_{n \geq 0} I^n N$ or $\bigoplus_{n \geq 0} N/I^n N$ in the above theorem. Note that if $N \neq 0$, then $\bigoplus_{n \geq 0} N/I^n N$ is not finitely generated as an $R(I)$-module. So we cannot take $N$ as $\bigoplus_{n \geq 0} N/I^n N$ in Theorem 1.1.1. In this theme, T. J. Puthenpurakal [Put13, page 368] raised the following question:

\[(A4)\quad \text{Is the set } \bigcup_{n \geq 1} \bigcup_{i \geq 0} \text{Ass}_A \left( \text{Ext}^i_A (M, N/I^n N) \right) \text{ finite?}\]

In Chapter 3, we show that the question (A4) has an affirmative answer for local complete intersection rings. We also analyze the asymptotic behaviour of the sets

\[\text{Ass}_A \left( \text{Ext}^i_A (M, N/I^n N) \right) \quad \text{if } n \text{ and } i \text{ both tend to } \infty.\]

If $A$ is a local complete intersection ring, then we prove that there exist non-negative integers $n_0, i_0$ such that for all $n \geq n_0$ and $i \geq i_0$, we have

\[
\text{Ass}_A \left( \text{Ext}^{2i}_A (M, N/I^n N) \right) = \text{Ass}_A \left( \text{Ext}^{2i_0}_A (M, N/I^{n_0} N) \right), \\
\text{Ass}_A \left( \text{Ext}^{2i+1}_A (M, N/I^n N) \right) = \text{Ass}_A \left( \text{Ext}^{2i_0+1}_A (M, N/I^{n_0} N) \right).
\]

We also show the analogous results for complete intersection rings which arise in algebraic geometry.

Recall that a local ring $A$ is said to be a local complete intersection ring if its completion $\hat{A} = Q/(f)$, where $Q$ is a complete regular local ring and $f = f_1, \ldots, f_c$ is a $Q$-regular sequence. To prove our results, we may assume that $A$ is complete because of the following well-known fact: For a finitely generated $A$-module $D$, we have

\[\text{Ass}_A(D) = \{ q \cap A : q \in \text{Ass}_\hat{A} \left( D \otimes_A \hat{A} \right) \}. \]

So, without loss of generality, we may assume that $A$ is a quotient of a regular local ring modulo a regular sequence. Then the desired results follow from more general results appeared below. Let us fix the following hypothesis for the rest of this section.

**Hypothesis 1.1.2.** Let $Q$ be a ring of finite Krull dimension, and let $f = f_1, \ldots, f_c$ be a $Q$-regular sequence. Set $A := Q/(f)$. Let $M$ and $N$ be finitely generated $A$-modules, where $\text{projdim}_Q(M)$ is finite. Let $I$ be an ideal of $A$.

We prove the announced finiteness and stability results in this set-up. One of the main ingredients we use in the proofs of our results is the following theorem due to T. J.
Puthenpurakal. This theorem concerns the finite generation of a family of Ext-modules. Here we need to understand the theory of cohomology operators which gives the bigraded module structure used in the following theorem. For details, we refer to Section 3.1.

**Theorem 1.1.3.** [Put13, Theorem 1.1] With the Hypothesis 1.1.2 let \( \mathcal{N} = \bigoplus_{n \geq 0} N_n \) be a finitely generated graded module over the Rees ring \( \mathcal{R}(I) \). Then

\[
\bigoplus_{n \geq 0} \bigoplus_{i \geq 0} \operatorname{Ext}^i_A(M, N_n)
\]

is a finitely generated bigraded \( S := \mathcal{R}(I)[t_1, \ldots, t_c] \)-module, where

\[
t_j : \operatorname{Ext}^i_A(M, N_n) \to \operatorname{Ext}^{i+2}_A(M, N_n) \quad (j = 1, \ldots, c)
\]

are the cohomology operators over \( A \).

With the Hypothesis 1.1.2 we prove that the set

\[
\bigcup_{n \geq 1} \bigcup_{i \geq 0} \operatorname{Ass}_A (\operatorname{Ext}^i_A(M, N/I^n N))
\]

is finite; see Theorem 3.2.1. After having this finiteness result, we focus on the asymptotic stability of the sets of associated prime ideals which occurs periodically after a certain stage. We show that there exist \( n', i' \geq 0 \) such that for all \( n \geq n' \) and \( i \geq i' \), we have

\[
\operatorname{Ass}_A (\operatorname{Ext}^{2i}_A(M, N/I^n N)) = \operatorname{Ass}_A \left( \operatorname{Ext}^{2i'}_A(M, N/I^{n'} N) \right),
\]

\[
\operatorname{Ass}_A (\operatorname{Ext}^{2i+1}_A(M, N/I^n N)) = \operatorname{Ass}_A \left( \operatorname{Ext}^{2i'+1}_A(M, N/I^{n'} N) \right);
\]

see Theorem 3.3.1. To prove this result, we take advantage of the notion of Hilbert function. We set

\[
V(n,i) := \operatorname{Ext}^i_A(M, N/I^n N) \quad \text{for all } n, i \geq 0.
\]

Since \( \bigcup_{n \geq 0} \bigcup_{i \geq 0} \operatorname{Ass}_A (V(n,i)) \) is finite, it is enough to prove that for each

\[
p \in \bigcup_{n \geq 0} \bigcup_{i \geq 0} \operatorname{Ass}_A (V(n,i)) ,
\]

there exist some \( n_l, i_l \geq 0 \) such that exactly one of the following alternatives must hold:

- either \( p \in \operatorname{Ass}_A (V(n,2i+l)) \) for all \( n \geq n_l \) and \( i \geq i_l \);
- or \( p \notin \operatorname{Ass}_A (V(n,2i+l)) \) for all \( n \geq n_l \) and \( i \geq i_l \),
1.2 Asymptotic Stability of Complexities

Let $A$ be a local ring with residue field $k$. Let $M$ and $N$ be finitely generated $A$-module. The integer
\[\beta_n^A(M) := \text{rank}_k(\text{Ext}_A^n(M, k))\]
is called the $n$th Betti number of $M$. It is equal to the rank of $F_n$ in a minimal free resolution $F$ of $M$. The notion of complexity of a module was introduced by L. L. Avramov in [Avr89, Definition (3.1)]. The complexity of $M$, denoted $cx_A(M)$ is the smallest non-negative integer $b$ such that $\beta_n^A(M) \leq an^{b-1}$ for some real number $a > 0$ and for all sufficiently large $n$. If no such $b$ exists, then set $cx_A(M) := \infty$.

The complexity of a pair of modules $(M, N)$, introduced in [AB00] by L. L. Avramov and R.-O. Buchweitz, is defined to be the number
\[cx_A(M, N) := \inf \left\{ b \in \mathbb{N} \mid \mu_A(\text{Ext}_A^n(M, N)) \leq an^{b-1} \text{ for some real number } a > 0 \text{ and for all } n \gg 0 \right\},\]
where $\mu_A(D)$ denotes the minimal number of generators of an $A$-module $D$. Clearly, the complexity $cx_A(M, N)$ measures ‘the size’ of $\text{Ext}_A^n(M, N)$. This notion encompasses the asymptotic invariant of $M$: its complexity $cx_A(M) = cx_A(M, k)$, measuring the polynomial rate of growth at infinity of its minimal free resolution.

Let $A$ be a local complete intersection ring, and let $I$ be an ideal of $A$. Puthenpurakal (in [Put13, Theorem 7.1]) proved that $cx_A(M, I^nN)$ is constant for all sufficiently large $n$. In Chapter 4 of this dissertation, we show that...
(C1) \( cx_A(M, N/I^nN) \) is independent of \( n \) for all sufficiently large \( n \).

To prove this result, we use the notion of support variety which was introduced by Avramov and Buchweitz in the same article [AB00, 2.1].

### 1.3 Castelnuovo-Mumford Regularity of Powers of Several Ideals

Let \( A \) be a standard \( \mathbb{N} \)-graded algebra, where \textit{standard} means \( A \) is generated by elements of \( A_1 \). Let \( A_+ \) be the ideal of \( A \) generated by the homogeneous elements of positive degree. Let \( M \) be a finitely generated \( \mathbb{N} \)-graded \( A \)-module. For every integer \( i \geq 0 \), we denote the \( i \)th local cohomology module of \( M \) with respect to \( A_+ \) by \( H^i_{A_+}(M) \). The \textit{Castelnuovo-Mumford regularity} of \( M \) is the invariant

\[
\text{reg}(M) := \max \left\{ \text{end} \left( H^i_{A_+}(M) \right) + i : i \geq 0 \right\},
\]

where \( \text{end}(D) \) denotes the maximal non-vanishing degree of an \( \mathbb{N} \)-graded \( A \)-module \( D \) with the convention \( \text{end}(D) = -\infty \) if \( D = 0 \). It is a natural extension of the usual definition of the Castelnuovo-Mumford regularity in the case \( A \) is a polynomial ring over a field.

Let \( S = k[X_1, \ldots, X_d] \) be a polynomial ring over a field \( k \) with its usual grading, i.e., each \( X_i \) has degree 1. By the Hilbert’s Syzygy Theorem, every finitely generated \( \mathbb{N} \)-graded \( S \)-module \( N \) has a finite graded minimal free resolution:

\[
0 \rightarrow F_p \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow N \rightarrow 0,
\]

where \( F_i = \bigoplus_{j=1}^{n_i} S(-b_{ij}) \) for some integers \( b_{ij} \). Then

\[
\text{reg}(N) = \max_{i,j} \{b_{ij} - i\}.
\]

One can write it in terms of the maximal non-vanishing degree of Tor-modules:

\[
\text{reg}(N) = \max \left\{ \text{end} \left( \text{Tor}^S_i(N, k) \right) - i : i \geq 0 \right\}.
\]

There has been a surge of interest on the behaviour of the function \( \text{reg}(I^n) \), where \( I \) is a homogeneous ideal in a polynomial ring over a field. A detailed survey of this topic can
be found in Section 2.2 of Chapter 2. In [Swa97, Theorem 3.6], I. Swanson proved that
\( \text{reg}(I^n) \leq kn \) for all \( n \geq 1 \), where \( k \) is some integer. Later, S. D. Cutkosky, J. Herzog and N. V. Trung [CHT99, Theorem 1.1] and V. Kodiyalam [Kod00, Theorem 5] independently showed that \( \text{reg}(I^n) \) can be expressed as a linear function of \( n \) for all sufficiently large \( n \).

Recently, N. V. Trung and H.-J. Wang [TW05, Theorem 3.2] proved that if \( A \) is a standard \( \mathbb{N} \)-graded ring, \( I \) is a homogeneous ideal of \( A \), and \( M \) is a finitely generated \( \mathbb{N} \)-graded \( A \)-module, then \( \text{reg}(I^n M) \) is asymptotically a linear function of \( n \).

In this context, a natural question arises: What happens when we consider several ideals instead of just considering one ideal? More precisely, if \( I_1, \ldots, I_t \) are homogeneous ideals of \( A \), and \( M \) is a finitely generated \( \mathbb{N} \)-graded \( A \)-module, then what will be the behaviour of \( \text{reg}(I_1^{n_1} \cdots I_t^{n_t} M) \) as a function of \( (n_1, \ldots, n_t) \)?

Let \( A = A_0[x_1, \ldots, x_d] \) be a standard \( \mathbb{N} \)-graded algebra over an Artinian local ring \( (A_0, m) \). In particular, \( A \) can be the coordinate ring of a projective variety over a field with usual grading. Let \( I_1, \ldots, I_t \) be homogeneous ideals of \( A \), and let \( M \) be a finitely generated \( \mathbb{N} \)-graded \( A \)-module. One of the main results of this dissertation is the following: There exist two integers \( k_1 \) and \( k'_1 \) such that

\[
(R1) \quad \text{reg}(I_1^{n_1} \cdots I_t^{n_t} M) \leq (n_1 + \cdots + n_t)k_1 + k'_1 \quad \text{for all } n_1, \ldots, n_t \in \mathbb{N}.
\]

In Chapter 5, we prove this result in a quite general set-up. As a consequence, we also obtain the following: If \( A_0 \) is a field, then there exist two integers \( k_2 \) and \( k'_2 \) such that

\[
(R2) \quad \text{reg}(\overline{I}_1^{n_1} \cdots \overline{I}_t^{n_t} M) \leq (n_1 + \cdots + n_t)k_2 + k'_2 \quad \text{for all } n_1, \ldots, n_t \in \mathbb{N},
\]

where \( \overline{I} \) denotes the integral closure of an ideal \( I \) of \( A \).

The basic technique of the proof of our results is the use of \( \mathbb{N}^{t+1} \)-grading structures on

\[
\bigoplus_{\underline{n} \in \mathbb{N}^t} (I_1^{n_1} \cdots I_t^{n_t} M) \quad \text{and} \quad \bigoplus_{\underline{n} \in \mathbb{N}^t} (\overline{I}_1^{n_1} \cdots \overline{I}_t^{n_t} M).
\]

Let \( R = A[I_1T_1, \ldots, I_tT_t] \) be the Rees algebra of \( I_1, \ldots, I_t \) over the graded ring \( A \), and let \( L = M[I_1T_1, \ldots, I_tT_t] \) be the Rees module of \( M \) with respect to the ideals \( I_1, \ldots, I_t \). We give \( \mathbb{N}^{t+1} \)-grading structures on \( R \) and \( L \) by setting \((\underline{n}, i)\)th graded components of \( R \) and \( L \) as the \( i \)th graded components of the \( \mathbb{N} \)-graded \( A \)-modules \( I_1^{n_1} \cdots I_t^{n_t} A \) and \( I_1^{n_1} \cdots I_t^{n_t} M \).
respectively. Then clearly, $R$ is an $\mathbb{N}^{t+1}$-graded ring and $L$ is a finitely generated $\mathbb{N}^{t+1}$-graded $R$-module. Note that $R$ is not necessarily standard as an $\mathbb{N}^{t+1}$-graded ring. Also note that for every $\underline{n} \in \mathbb{N}^t$, we have

$$R_{(\underline{n},\ast)} := \bigoplus_{i \in \mathbb{N}} R_{(\underline{n},i)} = I_1^{n_1} \cdots I_t^{n_t} A$$

and

$$L_{(\underline{n},\ast)} := \bigoplus_{i \in \mathbb{N}} L_{(\underline{n},i)} = I_1^{n_1} \cdots I_t^{n_t} M.$$ 

Since $R = A[I_1 T_1, \ldots, I_t T_t] = R_{(0,\ast)}(R_{(1,\ast)}, \ldots, R_{(t,\ast)})$, we may consider $R = \bigoplus_{\underline{n} \in \mathbb{N}^t} R_{(\underline{n},\ast)}$ as a standard $\mathbb{N}^t$-graded ring.

In a similar way as above, we can give an $\mathbb{N}^{t+1}$-graded $R$-module structure on $L := \bigoplus_{\underline{n} \in \mathbb{N}^t} (I_1^{n_1} \cdots I_t^{n_t} M)$.

In this case also, $L$ is a finitely generated $\mathbb{N}^{t+1}$-graded $R$-module provided $A_0$ is a field; see Example 5.2.10. Keeping these two examples in mind, let us fix the following hypothesis for the rest of this section.

**Hypothesis 1.3.1.** Let

$$R = \bigoplus_{(\underline{n},i) \in \mathbb{N}^{t+1}} R_{(\underline{n},i)}$$

be an $\mathbb{N}^{t+1}$-graded ring, which need not be standard. Let

$$L = \bigoplus_{(\underline{n},i) \in \mathbb{N}^{t+1}} L_{(\underline{n},i)}$$

be a finitely generated $\mathbb{N}^{t+1}$-graded $R$-module. For each $\underline{n} \in \mathbb{N}^t$, we set

$$R_{(\underline{n},\ast)} := \bigoplus_{i \in \mathbb{N}} R_{(\underline{n},i)} \quad \text{and} \quad L_{(\underline{n},\ast)} := \bigoplus_{i \in \mathbb{N}} L_{(\underline{n},i)}.$$ 

Also set $A := R_{(\underline{0},\ast)}$. Suppose $R = \bigoplus_{\underline{n} \in \mathbb{N}^t} R_{(\underline{n},\ast)}$ and $A = R_{(\underline{0},\ast)}$ are standard as $\mathbb{N}^t$-graded ring and $\mathbb{N}$-graded ring respectively. Assume that $A_0 = R_{(\underline{0},0)}$ is an Artinian local ring. Suppose $A = A_0[x_1, \ldots, x_d]$ for some $x_1, \ldots, x_d \in A_1$. Set $A_+ := \langle x_1, \ldots, x_d \rangle$.

With the Hypothesis 1.3.1, we now give $\mathbb{N}^t$-grading structures on

$$R = \bigoplus_{\underline{n} \in \mathbb{N}^t} R_{(\underline{n},\ast)} \quad \text{and} \quad L = \bigoplus_{\underline{n} \in \mathbb{N}^t} L_{(\underline{n},\ast)}$$

in the obvious way, i.e., by setting $R_{(\underline{n},\ast)}$ and $L_{(\underline{n},\ast)}$ as the $n$th graded components of $R$ and $L$ respectively. Then clearly, $R = \bigoplus_{\underline{n} \in \mathbb{N}^t} R_{(\underline{n},\ast)}$ is a standard $\mathbb{N}^t$-graded ring and
1.3 Castelnuovo-Mumford Regularity of Powers of Several Ideals

$L = \bigoplus_{n \in \mathbb{N}^t} L(n, \ast)$ is a finitely generated $\mathbb{N}^t$-graded $R$-module. From now onwards, by $R$ and $L$, we mean $\mathbb{N}^t$-graded ring $\bigoplus_{n \in \mathbb{N}^t} R(n, \ast)$ and $\mathbb{N}^t$-graded $R$-module $\bigoplus_{n \in \mathbb{N}^t} L(n, \ast)$ (satisfying the Hypothesis 1.3.1) respectively. To prove our main results on regularity, it is now enough to show that there exist two integers $k$ and $k'$ such that

\begin{equation}
\text{reg}(L(n, \ast)) \leq (n_1 + \cdots + n_t)k + k' \quad \text{for all } n \in \mathbb{N}^t.
\end{equation}

We prove (R3) in several steps by using induction on an invariant which we introduce here. We call this invariant the saturated dimension of a multigraded module. Suppose $M = \bigoplus_{n \in \mathbb{N}^t} M_n$ is a finitely generated $\mathbb{N}^t$-graded module over a standard $\mathbb{N}^t$-graded ring $R = \bigoplus_{n \in \mathbb{N}^t} R_n$. Then we prove that there exists $\underline{v} \in \mathbb{N}^t$ such that

\begin{equation}
\text{Ann}_R(M_n) = \text{Ann}_R(M_{\underline{v}}) \quad \text{for all } n \geq \underline{v},
\end{equation}

and hence $\dim_R(M_n) = \dim_R(M_{\underline{v}})$ for all $n \geq \underline{v}$; see Lemma 5.2.2. We call such a point $\underline{v} \in \mathbb{N}^t$ an annihilator stable point of $M$. Then the saturated dimension of $M$ is defined to be $s(M) := \dim_{R_0}(M_{\underline{v}})$. We use induction on the saturated dimension $s(L)$ of $L = \bigoplus_{n \in \mathbb{N}^t} L(n, \ast)$ in order to prove (R3).

In the base case, i.e., in the case when $s(L) = 0$, we have $\dim_A(L(n, \ast)) = 0$ for all $n \geq \underline{v}$, where $\underline{v}$ is an annihilator stable point of $L$. Then, in view of Grothendieck’s Vanishing Theorem, we obtain that

\begin{equation}
H^i_A(L(n, \ast)) = 0 \quad \text{for all } i > 0 \text{ and } n \geq \underline{v},
\end{equation}

which gives

\begin{equation}
\text{reg}(L(n, \ast)) = \max \{ \mu : H^0_{A_+}(L(n, \ast))_\mu \neq 0 \} \quad \text{for all } n \geq \underline{v}.
\end{equation}

In this situation, we show the linear boundedness result (in Theorem 5.3.1) by using the following fact: There exists a positive integer $k$ such that

\begin{equation}
(A_+)^k L(n, \ast) \cap H^0_{A_+}(L(n, \ast)) = 0 \quad \text{for all } n \in \mathbb{N}^t;
\end{equation}

see Lemma 5.2.1.

The inductive step is shown in Theorem 5.3.2 by using the following well-known result on regularity: For a finitely generated $\mathbb{N}$-graded $A$-module $N$, we have

\begin{equation}
\text{reg}(N) \leq \max \{ \text{reg}(0 :_N x), \text{reg}(N/xN) - l + 1 \},
\end{equation}

where $x$ is a homogeneous element of $A$ with degree $l \geq 1$. In this step, we prove that there exist $\underline{u} \in \mathbb{N}^t$ and an integer $k$ such that

\begin{equation}
\text{reg}(L(n, \ast)) < (n_1 + \cdots + n_t)k + k \quad \text{for all } n \geq \underline{u}.
\end{equation}
In particular, this shows that if \( t = 1 \), then there exist two integers \( k \) and \( k' \) such that
\[
\text{reg}(L_{(n, s)}) \leq nk + k' \quad \text{for all } n \in \mathbb{N}.
\]

Finally, in Theorem \ref{5.3.3}, we prove (R3) by using induction on \( t \).

### 1.4 Characterizations of Regular Rings via Syzygy Modules

In the present section, \( A \) always denotes a local ring with maximal ideal \( \mathfrak{m} \) and residue field \( k \). For every non-negative integer \( n \), we denote the \( n \)th syzygy module of \( k \) by \( \Omega^A_n(k) \).

One of the most influential results in commutative algebra is the result of Auslander, Buchsbaum and Serre: The local ring \( A \) is regular if and only if \( \text{projdim}_A(k) \) is finite. Note that \( \text{projdim}_A(k) \) is finite if and only if some syzygy module of \( k \) is a free \( A \)-module. There are a number of characterizations of regular local rings in terms of syzygy modules of the residue field. In \cite[Dut89, Corollary 1.3]{Dutta1989}, S. P. Dutta gave the following characterization of regular local rings.

**Theorem 1.4.1** (Dutta). \( A \) is regular if and only if \( \Omega^A_n(k) \) has a non-zero free direct summand for some integer \( n \geq 0 \).

Later, R. Takahashi generalized Dutta’s result by giving a characterization of regular local rings via the existence of a semidualizing direct summand of some syzygy module of the residue field. Let us recall the definition of a semidualizing module.

**Definition 1.4.2** (\cite{Gol84}). A finitely generated \( A \)-module \( M \) is said to be a **semidualizing module** if the following hold:

(i) The natural homomorphism \( A \rightarrow \text{Hom}_A(M, M) \) is an isomorphism.

(ii) \( \text{Ext}^i_A(M, M) = 0 \) for all \( i \geq 1 \).

Note that \( A \) itself is a semidualizing \( A \)-module. So the following theorem generalizes the above result of Dutta.

**Theorem 1.4.3**. \cite[Tak06, Theorem 4.3]{Takahashi2006} \( A \) is regular if and only if \( \Omega^A_n(k) \) has a semidualizing direct summand for some integer \( n \geq 0 \).
If \( A \) is a Cohen-Macaulay local ring with canonical module \( \omega \), then \( \omega \) is a semidualizing \( A \)-module. Therefore, as a corollary of Theorem 1.4.3, Takahashi obtained the following:

**Corollary 1.4.4.** [Tak06, Corollary 4.4] Let \( A \) be a Cohen-Macaulay local ring with canonical module \( \omega \). Then \( A \) is regular if and only if \( \Omega^n_A(k) \) has a direct summand isomorphic to \( \omega \) for some integer \( n \geq 0 \).

Now recall that the canonical module (if exists) over a Cohen-Macaulay local ring has finite injective dimension. Also it is well-known that \( A \) is regular if and only if \( k \) has finite injective dimension. So, in this theme, a natural question arises that “if \( \Omega^n_A(k) \) has a non-zero direct summand of finite injective dimension for some integer \( n \geq 0 \), then is the ring \( A \) regular?” In the present study, we show that this question has an affirmative answer (see Theorem 6.2.7).

Kaplansky conjectured that if some power of the maximal ideal of \( A \) is non-zero and of finite projective dimension, then \( A \) is regular. In [LV68, Theorem 1.1], G. Levin and W. V. Vasconcelos proved this conjecture. In fact, their result is even stronger:

**Theorem 1.4.5** (Levin and Vasconcelos). If \( M \) is a finitely generated \( A \)-module such that \( mM \) is non-zero and of finite projective dimension (or of finite injective dimension), then \( A \) is regular.

In [Mar96, Proposition 7], A. Martsinkovsky generalized Dutta’s result in the following direction. He also showed that the above result of Levin and Vasconcelos is a special case of the following theorem:

**Theorem 1.4.6** (Martsinkovsky). If a finite direct sum of syzygy modules of \( k \) maps onto a non-zero \( A \)-module of finite projective dimension, then \( A \) is regular.

In this direction, we prove the following result which considerably strengthens Theorem 1.4.3. The proof presented here is very simple and elementary.

**Theorem 1.4.7** (See Corollary 6.2.2). If a finite direct sum of syzygy modules of \( k \) maps onto a semidualizing \( A \)-module, then \( A \) is regular.

Furthermore, we raise the following question:

**Question 1.4.8.** If a finite direct sum of syzygy modules of \( k \) maps onto a non-zero \( A \)-module of finite injective dimension, then is the ring \( A \) regular?
In Chapter 6 we give a partial answer to this question as follows:

**Theorem 1.4.9** (See Corollary 6.2.4). If a finite direct sum of syzygy modules of \( k \) maps onto a non-zero maximal Cohen-Macaulay \( A \)-module \( L \) of finite injective dimension, then \( A \) is regular.

If \( A \) is a Cohen-Macaulay local ring with canonical module \( \omega \), then one can take \( L = \omega \) in the above theorem.

Theorems 1.4.7 and 1.4.9 are deduced as consequences of a more general result appeared below. Let \( \mathcal{P} \) be a property of modules over local rings. We say that \( \mathcal{P} \) is a \((\ast)\)-property if \( \mathcal{P} \) satisfies the following:

(i) An \( A \)-module \( M \) satisfies \( \mathcal{P} \) implies that the \( A/(x) \)-module \( M/xM \) satisfies \( \mathcal{P} \), where \( x \in A \) is an \( A \)-regular element.

(ii) An \( A \)-module \( M \) satisfies \( \mathcal{P} \) and \( \text{depth}(A) = 0 \) together imply that \( \text{Ann}_A(M) = 0 \).

It can be noticed that the properties ‘semidualizing modules’ and ‘non-zero maximal Cohen-Macaulay modules of finite injective dimension’ are two examples of \((\ast)\)-properties; see Examples 6.1.4 and 6.1.5. Therefore we obtain Theorems 1.4.7 and 1.4.9 as corollaries of the following general result. Here we denote a finite collection of non-negative integers by \( \Lambda \).

**Theorem 1.4.10** (See Theorem 6.2.1). Assume that \( \mathcal{P} \) is a \((\ast)\)-property. Let

\[
f : \bigoplus_{n \in \Lambda} (\Omega_n^A(k))^{j_n} \longrightarrow L \quad (j_n \geq 1 \text{ for each } n \in \Lambda)
\]

be a surjective \( A \)-module homomorphism, where \( L (\neq 0) \) satisfies \( \mathcal{P} \). Then \( A \) is regular.

We establish a relationship between the socle of the ring and the annihilator of the syzygy modules: If \( A \neq k \) (i.e., if \( m \neq 0 \)), then

\[
\text{Soc}(A) \subseteq \text{Ann}_A(\Omega_n^A(k)) \quad \text{for all } n \geq 0;
\]
see Lemma 6.1.1. We use this fact in order to prove Theorem 1.4.10 when \( \text{depth}(A) = 0 \). Considering this as the base case, the general case follows from an inductive argument; see the proof of Theorem 6.2.1.
1.4 Characterizations of Regular Rings via Syzygy Modules

We obtain one new characterization of regular local rings. It follows from Dutta’s result (Theorem 1.4.1) that $A$ is regular if and only if some syzygy module of $k$ has a non-zero direct summand of finite projective dimension. We prove the following counterpart for injective dimension.

**Theorem 1.4.11** (See Theorem 6.2.7). $A$ is regular if and only if some syzygy module of $k$ has a non-zero direct summand of finite injective dimension.

Moreover, this result has a dual companion which says that $A$ is regular if and only if some cosyzygy module of $k$ has a non-zero finitely generated direct summand of finite projective dimension; see Corollary 6.2.8.

Till now we have considered surjective homomorphisms from a finite direct sum of syzygy modules of $k$ onto a ‘special module’. One may ask “what happens if we consider injective homomorphisms from a ‘special module’ to a finite direct sum of syzygy modules of $k$?” More precisely, if

$$f : L \longrightarrow \bigoplus_{n \in A} \left( \Omega^A_n(k) \right)^{j_n}$$

is an injective $A$-module homomorphism, where $L$ is non-zero and of finite projective dimension (or of finite injective dimension), then is the ring $A$ regular? We give an example which shows that $A$ is not necessarily a regular local ring in this situation; see Example 6.2.9.
Chapter 2

Review of Literature

The literature survey in this chapter concentrates on the following subjects: (1) asymptotic prime divisors related to derived functors Ext and Tor, (2) Castelnuovo-Mumford regularity of powers of ideals, and (3) characterizations of regular local rings via syzygy modules of the residue field.

2.1 Asymptotic Prime Divisors Related to Derived Functors

Throughout this section, let \( A \) be a ring. Let \( I \) be an ideal of \( A \) and \( M \) be a finitely generated \( A \)-module. M. Brodmann [Bro79] proved that the set of associated prime ideals \( \text{Ass}_A(M/I^nM) \) is independent of \( n \) for all sufficiently large \( n \). He deduced this result by proving that \( \text{Ass}_A(I^nM/I^n+1M) \) is independent of \( n \) for all large \( n \).

Thereafter, in [MS93, Theorem 1], L. Melkersson and P. Schenzel generalized Brodmann’s result by showing that for every fixed \( i \geq 0 \), the sets

\[
\text{Ass}_A \left( \text{Tor}_i^A(M, I^n/I^{n+1}) \right) \quad \text{and} \quad \text{Ass}_A \left( \text{Tor}_i^A(M, A/I^n) \right)
\]

are independent of \( n \) for all sufficiently large \( n \). By a similar argument, one obtains that for a given \( i \geq 0 \), the set

\[
\text{Ass}_A \left( \text{Ext}_i^A(M, I^n/I^{n+1}) \right)
\]

is independent of \( n \) for all large \( n \). Later, D. Katz and E. West proved the above results in a more general way [KW04, Corollary 3.5]; if \( N \) is a finitely generated \( A \)-module, then
for every fixed $i \geq 0$, the sets
\[ \Ass_A(\Tor^i_A(M, N/I^n N)) \quad \text{and} \quad \Ass_A(\Ext^i_A(M, N/I^n N)) \]
are stable for all sufficiently large $n$. So, in particular, for every fixed $i \geq 0$, the sets
\[ \bigcup_{n \geq 1} \Ass_A(\Tor^i_A(M, N/I^n N)) \quad \text{and} \quad \bigcup_{n \geq 1} \Ass_A(\Ext^i_A(M, N/I^n N)) \]
are finite. However, for a given $i \geq 0$, the set
\[ \bigcup_{n \geq 1} \Ass_A(\Ext^i_A(A/I^n, M)) \]
need not be finite, which follows from the fact that the set of associated prime ideals of the $i$th local cohomology module
\[ H^i_I(M) \cong \lim_{n \to \infty} \Ext^i_A(A/I^n, M) \]
need not be finite, due to an example of A. K. Singh [Sin00, Section 4].

Recently, T. J. Puthenpurakal [Put13, Theorem 5.1] proved that if $A$ is a local complete intersection ring and $\mathcal{N} = \bigoplus_{n \geq 0} N_n$ is a finitely generated graded module over the Rees ring $\mathcal{R}(I)$, then
\[ \bigcup_{n \geq 0} \bigcup_{i \geq 0} \Ass_A(\Ext^i_A(M, N_n)) \]
is a finite set. Moreover, he proved that there exist $n_0, i_0 \geq 0$ such that
\[ \Ass_A(\Ext^{2i}_A(M, N_n)) = \Ass_A(\Ext^{2i_0}_A(M, N_{n_0})), \]
\[ \Ass_A(\Ext^{2i+1}_A(M, N_n)) = \Ass_A(\Ext^{2i+1}_A(M, N_{n_0})) \]
for all $n \geq n_0$ and $i \geq i_0$. In particular, $\mathcal{N}$ can be taken as
\[ \bigoplus_{n \geq 0} (I^n N) \quad \text{or} \quad \bigoplus_{n \geq 0} (I^n N/I^{n+1} N). \]
He showed these results by proving the finite generation of a family of Ext-modules: If $A = Q/f$, where $Q$ is a local ring and $f = f_1, \ldots, f_c$ is a $Q$-regular sequence, and if $\text{projdim}_Q(M)$ is finite, then
\[ \mathcal{E}(\mathcal{N}) := \bigoplus_{n \geq 0} \bigoplus_{i \geq 0} \Ext^i_A(M, N_n) \]
is a finitely generated bigraded $\mathcal{J} = \mathcal{R}(I)[t_1, \ldots, t_c]$-module, where $t_j$ are the cohomology operators over $A$. 
In the present study, we prove that if $A$ is a local complete intersection ring, then the set of associate primes
\[ \bigcup_{n \geq 1} \bigcup_{i \geq 0} \text{Ass}_A \left( \text{Ext}^i_A(M, N/I^nN) \right) \]
is finite. Moreover, there are non-negative integers $n_0$ and $i_0$ such that for all $n \geq n_0$ and $i \geq i_0$, the set $\text{Ass}_A \left( \text{Ext}^i_A(M, N/I^nN) \right)$ depends only on whether $i$ is even or odd; see Chapter 3.

### 2.2 Castelnuovo-Mumford Regularity of Powers of Ideals

Let $I$ be a homogeneous ideal of a polynomial ring $S = K[X_1, \ldots, X_d]$ over a field $K$ with usual grading. In [BEL91, Proposition 1], A. Bertram, L. Ein and R. Lazarsfeld have initiated the study of the Castelnuovo-Mumford regularity of $I^n$ as a function of $n$ by proving that if $I$ is the defining ideal of a smooth complex projective variety, then $\text{reg}(I^n)$ is bounded by a linear function of $n$.

Thereafter, A. V. Geramita, A. Gimigliano and Y. Pitteloud [GGP95, Theorem 1.1] and K. A. Chandler [Cha97, Theorem 1] proved that

\[
\text{if } \dim(S/I) \leq 1, \text{ then } \text{reg}(I^n) \leq n \cdot \text{reg}(I) \text{ for all } n \geq 1.
\]

This result does not hold true for higher dimension. A first counter example was given by Terai in characteristic different from 2. Later, B. Sturmfels [Stu00, Section 1] exhibited a monomial ideal $J$ with 8 generators for which

\[ \text{reg}(J^2) > 2 \cdot \text{reg}(J) \]

in any characteristic. However, in arbitrary dimension, I. Swanson ([Swa97, Theorem 3.6]) first proved that for a homogeneous ideal $I$, there exists an integer $k$ such that

\[ \text{reg}(I^n) \leq kn \text{ for all } n \geq 1. \]

Later, S. D. Cutkosky, J. Herzog and N. V. Trung [CHT99, Theorem 1.1] and V. Kodiyalam [Kod00, Theorem 5] independently proved that $\text{reg}(I^n)$ can be expressed as a linear function of $n$ for all sufficiently large $n$. Recently, in [TW05, Theorem 3.2], N. V. Trung and H.-J. Wang proved this result in a more general way. Let $A$ be a standard
graded ring, and let $M$ be a finitely generated graded $A$-module. Then they showed that for a homogeneous ideal $I$ of $A$, there exists an integer $e \geq \varepsilon(M)$ such that

$$\text{reg}(I^nM) = \rho_M(I) \cdot n + e \quad \text{for all } n \gg 1,$$

where the invariants $\rho_M(I)$ and $\varepsilon(M)$ are defined as follows. The number $\varepsilon(M)$ denotes the smallest degree of the (non-zero) homogeneous elements of $M$. A homogeneous ideal $J \subseteq I$ is said to be an $M$-reduction of $I$ if $I^{n+1}M = JI^nM$ for some $n \geq 0$. Define

$$d(J) := \min\{d : J \text{ can be generated by homogeneous elements of degree } \leq d\}.$$ 

The invariant $\rho_M(I)$ is defined to be the number

$$\rho_M(I) := \min\{d(J) : J \text{ is an } M\text{-reduction of } I\}.$$ 

In the present study, we deal with several ideals instead of just considering one ideal. Let $A = A_0[x_1, \ldots, x_d]$ be a standard graded algebra over an Artinian local ring $A_0$. Let $I_1, \ldots, I_t$ be homogeneous ideals of $A$, and $M$ be a finitely generated graded $A$-module. In Chapter 5 we prove that there exist two integers $k$ and $k'$ such that

$$\text{reg}(I_1^{n_1} \cdots I_t^{n_t}M) \leq (n_1 + \cdots + n_t)k + k' \quad \text{for all } n_1, \ldots, n_t \in \mathbb{N}.$$ 

### 2.3 Characterizations of Regular Rings via Syzygy Modules

Throughout this section, let $A$ denote a local ring with maximal ideal $m$ and residue field $k$. Let $\Omega^n_A(k)$ be the $n$th syzygy module of $k$. In [Dut89, Corollary 1.3], S. P. Dutta gave the following characterization of regular local rings.

**Theorem 2.3.1** (Dutta). $A$ is regular if and only if $\Omega^n_A(k)$ has a non-zero free direct summand for some integer $n \geq 0$.

Later, R. Takahashi generalized Dutta’s result by giving a characterization of regular local rings in terms of the existence of a semidualizing direct summand of some syzygy module of the residue field. Note that $A$ itself is a semidualizing $A$-module; see Definition 1.4.2. So the following theorem generalizes the above result of Dutta.
2.3 Characterizations of Regular Rings via Syzygy Modules

**Theorem 2.3.2.** [Tak06, Theorem 4.3] A is regular if and only if $\Omega^n_A(k)$ has a semidualizing direct summand for some integer $n \geq 0$.

If $A$ is a Cohen-Macaulay local ring with canonical module $\omega$, then $\omega$ is a semidualizing $A$-module. Therefore, as an application of the above theorem, Takahashi obtained the following:

**Corollary 2.3.3.** [Tak06, Corollary 4.4] Let $A$ be a Cohen-Macaulay local ring with canonical module $\omega$. Then $A$ is regular if and only if $\Omega^n_A(k)$ has a direct summand isomorphic to $\omega$ for some integer $n \geq 0$.

Kaplansky conjectured that if some power of the maximal ideal of $A$ is non-zero and of finite projective dimension, then $A$ is regular. In [LV68, Theorem 1.1], G. Levin and W. V. Vasconcelos proved this conjecture. In fact, their result is even stronger:

**Theorem 2.3.4 (Levin and Vasconcelos).** If $M$ is a finitely generated $A$-module such that $mM$ is non-zero and of finite projective dimension (or of finite injective dimension), then $A$ is regular.

Later, A. Martsinkovsky generalized Dutta’s result in the following direction. He also showed that the above result of Levin and Vasconcelos is a special case of the following theorem. We denote a finite collection of non-negative integers by $\Lambda$.

**Theorem 2.3.5.** [Mar96, Proposition 7] Let

$$f : \bigoplus_{n \in \Lambda} (\Omega^n_A(k))^{j_n} \longrightarrow L \quad (j_n \geq 1 \text{ for each } n \in \Lambda)$$

be a surjective $A$-module homomorphism, where $L$ is non-zero and of finite projective dimension. Then $A$ is regular.

Thereafter, L. L. Avramov proved a much more stronger result than the above one.

**Theorem 2.3.6.** [Avr96, Corollary 9] Each non-zero homomorphic image $L$ of a finite direct sum of syzygy modules of $k$ has maximal complexity, i.e., $\text{cx}_A(L) = \text{cx}_A(k)$.

Recall that the complexity of a finitely generated $A$-module $M$ is defined to be the number

$$\text{cx}_A(M) := \inf \left\{ b \in \mathbb{N} \mid \limsup_{n \to \infty} \left( \frac{\text{rank}_k \left( \text{Ext}^n_A(M, k) \right)}{n^{b-1}} \right) < \infty \right\}.$$
It can be noted that $\text{cx}_A(M) \geq 0$ with equality if and only if $\text{projdim}_A(M)$ is finite. Therefore one obtains Theorem 2.3.5 as a consequence of Theorem 2.3.6.

In Chapter 6, we prove a few variations of the above results. Suppose $L$ is a non-zero homomorphic image of a finite direct sum of syzygy modules of $k$. We prove that $A$ is regular if $L$ is either ‘a semidualizing module’ or ‘a maximal Cohen-Macaulay module of finite injective dimension’. We also obtain one new characterization of regular local rings. We show that $A$ is regular if and only if some syzygy module of $k$ has a non-zero direct summand of finite injective dimension.
Chapter 3

Asymptotic Prime Divisors over Complete Intersection Rings

Assume $A$ is either a local complete intersection ring or a geometric locally complete intersection ring. Throughout this chapter, let $M$ and $N$ be finitely generated $A$-modules, and let $I$ be an ideal of $A$. The main goal of this chapter is to show that the set

$$\bigcup_{n \geq 1} \bigcup_{i \geq 0} \text{Ass}_A \left( \text{Ext}^i_A(M, N/I^n N) \right)$$

is finite.

Moreover, we analyze the asymptotic behaviour of the sets

$$\text{Ass}_A \left( \text{Ext}^i_A(M, N/I^n N) \right) \quad \text{if } n \text{ and } i \text{ both tend to } \infty.$$

We prove that there are non-negative integers $n_0$ and $i_0$ such that for all $n \geq n_0$ and $i \geq i_0$, we have that

$$\text{Ass}_A \left( \text{Ext}^{2i}_A(M, N/I^n N) \right) = \text{Ass}_A \left( \text{Ext}^{2i_0}_A(M, N/I^{n_0} N) \right),$$

$$\text{Ass}_A \left( \text{Ext}^{2i+1}_A(M, N/I^n N) \right) = \text{Ass}_A \left( \text{Ext}^{2i_0+1}_A(M, N/I^{n_0} N) \right).$$

Here we describe in brief the contents of this chapter. In Section 3.1, we give some graded module structures which we use in order to prove our main results of this chapter. The finiteness results on asymptotic prime divisors are proved in Section 3.2, while the stability results are shown in Section 3.3. Finally, in Section 3.4, we prove the analogous results on associate primes for complete intersection rings which arise in algebraic geometry.
We refer the reader to [Mat86, §6] for all the basic results on associate primes which we use in this chapter.

## 3.1 Module Structures on Ext

In this section, we give the graded module structures which we are going to use in order to prove our main results.

Let $Q$ be a ring, and let $f = f_1, \ldots, f_c$ be a $Q$-regular sequence. Set $A := Q/(f)$. Let $M$ and $D$ be finitely generated $A$-modules.

### 3.1.1 (Eisenbud Operators and Total Ext-module)

Let\[ F : \cdots \rightarrow F_n \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow 0 \]
be a projective resolution of $M$ by finitely generated free $A$-modules. Let\[ t'_j : F(+2) \rightarrow F, \quad 1 \leq j \leq c \]
be the *Eisenbud operators* defined by $f = f_1, \ldots, f_c$ (see [Eis80, Section 1]). In view of [Eis80, Corollary 1.4], the chain maps $t'_j$ are determined uniquely up to homotopy. In particular, they induce well-defined maps\[ t_j : \Ext_A^i(M, D) \rightarrow \Ext_A^{i+2}(M, D), \]
(for all $i$ and $1 \leq j \leq c$), on the cohomology of $\Hom_A(F, D)$. In [Eis80, Corollary 1.5], it is shown that the chain maps $t'_j$ ($j = 1, \ldots, c$) commute up to homotopy. Thus\[ \Ext_A^*(M, D) := \bigoplus_{i \geq 0} \Ext_A^i(M, D) \]
turns into a graded $\mathcal{T} := A[t_1, \ldots, t_c]$-module, where $\mathcal{T}$ is the graded polynomial ring over $A$ in the *cohomology operators* $t_j$ defined by $f$ with $\deg(t_j) = 2$ for all $1 \leq j \leq c$. We call $\Ext_A^*(M, D)$ the *total Ext-module* of $M$ and $D$. These structures depend only on $f$, are natural in both module arguments and commute with the connecting maps induced by short exact sequences.

### 3.1.2 (Gulliksen’s Finiteness Theorem)

T. H. Gulliksen [Gul74, Theorem 3.1] proved that if either $\text{projdim}_Q(M)$ is finite or $\text{injdim}_Q(D)$ is finite, then $\Ext_A^*(M, D)$ is a finitely generated graded $\mathcal{T} = A[t_1, \ldots, t_c]$-module; see also [Avr89, Theorem (2.1)].
3.1.3 (Bigraded Module Structure on Ext). Let $I$ be an ideal of $A$. Let $R(I)$ be the Rees ring $\bigoplus_{n \geq 0} I^n X^n$ associated to $I$. We consider $R(I)$ as a subring of the polynomial ring $A[X]$. Let $N = \bigoplus_{n \geq 0} N_n$ be a graded $R(I)$-module. Let $u \in R(I)$ be a homogeneous element of degree $s$. Consider the $A$-linear maps given by multiplication with $u$:

$$N_n \xrightarrow{u} N_{n+s} \quad \text{for all } n.$$ 

By applying $\text{Hom}_A(F, -)$ on the above maps and using the naturality of the Eisenbud operators $t'_j$, we have the following commutative diagram of cochain complexes:

$$\text{Hom}_A(F, N_n) \xrightarrow{t'_j} \text{Hom}_A(F(2), N_n)$$

$$\downarrow u \quad \downarrow u$$

$$\text{Hom}_A(F, N_{n+s}) \xrightarrow{t'_j} \text{Hom}_A(F(2), N_{n+s}).$$

Now, taking cohomology, we obtain the following commutative diagram of $A$-modules:

$$\text{Ext}_A^i(M, N_n) \xrightarrow{t_j} \text{Ext}_A^{i+2}(M, N_n)$$

$$\downarrow u \quad \downarrow u$$

$$\text{Ext}_A^i(M, N_{n+s}) \xrightarrow{t_j} \text{Ext}_A^{i+2}(M, N_{n+s})$$

for all $n, i$ and $1 \leq j \leq c$. Thus

$$\mathcal{E}(N) := \bigoplus_{n \geq 0} \bigoplus_{i \geq 0} \text{Ext}_A^i(M, N_n)$$

turns into a bigraded $\mathcal{S} := R(I)[t_1, \ldots, t_c]$-module, where we set $\text{deg}(t_j) = (0, 2)$ for all $1 \leq j \leq c$ and $\text{deg}(uX^s) = (s, 0)$ for all $u \in I^s$, $s \geq 0$.

3.1.4 (Bigraded Module Structure on Ext). Suppose $N$ is a finitely generated $A$-module. Set $L := \bigoplus_{n \geq 0} (N/I^{n+1}N)$. Note that $R(I, N) = \bigoplus_{n \geq 0} I^n N$ and $N[X] = N \otimes_A A[X]$ are graded modules over $R(I)$ and $A[X]$ respectively. Since $R(I)$ is a graded subring of $A[X]$, we set that $N[X]$ is a graded $R(I)$-module. Therefore $L$ is a graded $R(I)$-module, where the graded structure is induced by the sequence

$$0 \longrightarrow R(I, N) \longrightarrow N[X] \longrightarrow L(-1) \longrightarrow 0.$$ 

Therefore, by the observations made in Section 3.1.3, we have that

$$\mathcal{E}(L) = \bigoplus_{n \geq 0} \bigoplus_{i \geq 0} \text{Ext}_A^i(M, N/I^{n+1}N)$$

is a bigraded module over $\mathcal{S} = R(I)[t_1, \ldots, t_c]$.
Chapter 3, Section 3.2

One of the main ingredients we use in this chapter is the following finiteness result, due to T. J. Puthenpurakal [Put13, Theorem 1.1].

**Theorem 3.1.5** (Puthenpurakal). Let $Q$ be a ring of finite Krull dimension, and let $f = f_1, \ldots, f_c$ be a $Q$-regular sequence. Set $A := Q/(f)$. Let $M$ be a finitely generated $A$-module, where $\text{projdim}_Q(M)$ is finite. Let $I$ be an ideal of $A$, and let $N = \bigoplus_{n \geq 0} N_n$ be a finitely generated graded $R(I)$-module. Then

$$E(N) := \bigoplus_{n \geq 0} \bigoplus_{i \geq 0} \text{Ext}_A^i(M, N_n)$$

is a finitely generated bigraded $\mathcal{R} = R(I)[t_1, \ldots, t_c]$-module.

We first prove the main results of this chapter for a ring $A$ which is of the form $Q/(f)$, where $Q$ is a regular local ring and $f = f_1, \ldots, f_c$ is a $Q$-regular sequence. Then we deduce the main results for a local complete intersection ring with the help of the following well-known lemma:

**Lemma 3.1.6.** Let $(A, m)$ be a local ring, and let $\widehat{A}$ be the $m$-adic completion of $A$. Let $D$ be a finitely generated $A$-module. Then

$$\text{Ass}_A(D) = \left\{ q \cap A : q \in \text{Ass}_{\widehat{A}} \left( D \otimes_A \widehat{A} \right) \right\}.$$

3.2 Asymptotic Associate Primes: Finiteness

In this section, we prove the announced finiteness result for the set of associated prime ideals of the family of Ext-modules $\text{Ext}_A^i(M, N/I^n N)$, $(n, i \geq 0)$, where $M$ and $N$ are finitely generated modules over a local complete intersection ring $A$, and $I \subseteq A$ is an ideal; see Corollary [3.2.2]

**Theorem 3.2.1.** Let $Q$ be a ring of finite Krull dimension, and let $f = f_1, \ldots, f_c$ be a $Q$-regular sequence. Set $A := Q/(f)$. Let $M$ and $N$ be finitely generated $A$-modules, where $\text{projdim}_Q(M)$ is finite, and let $I$ be an ideal of $A$. Then the set

$$\bigcup_{n \geq 1} \bigcup_{i \geq 0} \text{Ass}_A \left( \text{Ext}_A^i(M, N/I^n N) \right)$$

is finite.

**Proof.** For every fixed $n \geq 0$, we consider the short exact sequence of $A$-modules:

$$0 \to I^n N/I^{n+1} N \to N/I^{n+1} N \to N/I^n N \to 0.$$
Taking direct sum over \( n \geq 0 \) and setting

\[
\mathcal{L} := \bigoplus_{n \geq 0} (N/I^{n+1}N),
\]

we obtain the following short exact sequence of graded \( \mathcal{R}(I) \)-modules:

\[
0 \rightarrow \text{gr}_I(N) \rightarrow \mathcal{L} \rightarrow \mathcal{L}(-1) \rightarrow 0,
\]

which induces an exact sequence of graded \( \mathcal{R}(I) \)-modules for each \( i \geq 0 \):

\[
\text{Ext}^i_A(M, \text{gr}_I(N)) \rightarrow \text{Ext}^i_A(M, \mathcal{L}) \rightarrow \text{Ext}^i_A(M, \mathcal{L}(-1)).
\]

Taking direct sum over \( i \geq 0 \) and using the naturality of the cohomology operators \( t_j \), we get the following exact sequence of bigraded \( \mathcal{S} = \mathcal{R}(I)[t_1, \ldots, t_c] \)-modules:

\[
\bigoplus_{n,i \geq 0} \text{Ext}^i_A \left( M, \frac{I^nN}{I^{n+1}N} \right) \xrightarrow{\Phi} \bigoplus_{n,i \geq 0} V_{(n,i)} \xrightarrow{\Psi} \bigoplus_{n,i \geq 0} V_{(n-1,i)},
\]

where

\[
V_{(n,i)} := \text{Ext}^i_A(M, N/I^{n+1}N)
\]

for each \( n \geq -1 \) and \( i \geq 0 \). Now we set

\[
U = \bigoplus_{n,i \geq 0} U_{(n,i)} := \text{Image}(\Phi).
\]

Then, for each \( n, i \geq 0 \), considering the exact sequence of \( A \)-modules:

\[
0 \rightarrow U_{(n,i)} \rightarrow V_{(n,i)} \rightarrow V_{(n-1,i)},
\]

we have

\[
\text{Ass}_A \left( V_{(n,i)} \right) \subseteq \text{Ass}_A \left( U_{(n,i)} \right) \cup \text{Ass}_A \left( V_{(n-1,i)} \right) \subseteq \text{Ass}_A \left( U_{(n,i)} \right) \cup \text{Ass}_A \left( U_{(n-1,i)} \right) \cup \text{Ass}_A \left( V_{(n-2,i)} \right) \subseteq \cup_{0 \leq j \leq n} \text{Ass}_A \left( U_{(j,i)} \right) [\text{as } \text{Ass}_A \left( V_{(-1,i)} \right) = \phi \text{ for each } i \geq 0].
\]

Taking union over \( n, i \geq 0 \), we obtain that

\[
\bigcup_{n,i \geq 0} \text{Ass}_A \left( V_{(n,i)} \right) \subseteq \bigcup_{n,i \geq 0} \text{Ass}_A \left( U_{(n,i)} \right). \tag{3.2.1}
\]

Since \( \text{gr}_I(N) \) is a finitely generated graded \( \mathcal{R}(I) \)-module, by Theorem 3.1.5,

\[
\bigoplus_{n,i \geq 0} \text{Ext}^i_A \left( M, \frac{I^nN}{I^{n+1}N} \right)
\]
is a finitely generated bigraded $\mathcal{S}$-module, and hence $U$ is a finitely generated bigraded $\mathcal{S}$-module. Therefore, in view of [Wes04, Lemma 3.2], we obtain that

$$\bigcup_{n,i \geq 0} \text{Ass}_{\mathcal{A}} (U_{(n,i)})$$

is a finite set. (3.2.2)

Now the result follows from (3.2.1) and (3.2.2).

As an immediate corollary, we obtain the following desired result.

**Corollary 3.2.2.** Let $(A,\mathfrak{m})$ be a local complete intersection ring. Let $M$ and $N$ be finitely generated $A$-modules, and let $I$ be an ideal of $A$. Then the set

$$\bigcup_{n \geq 1} \bigcup_{i \geq 0} \text{Ass}_{\mathcal{A}} \left( \text{Ext}^i_A(M, N/I^n N) \right)$$

is finite.

**Proof.** Since $A$ is a local complete intersection ring, $\hat{A} = Q/(f)$, where $Q$ is a regular local ring and $f = f_1, \ldots, f_c$ is a $Q$-regular sequence. Since $Q$ is a regular local ring, $\text{projdim}_Q(M)$ is finite. Then, by applying Theorem 3.2.1 for the ring $\hat{A}$, we have that

$$\bigcup_{n,i \geq 0} \text{Ass}_{\hat{A}} \left( \text{Ext}^i_{\hat{A}}(M, N/\hat{I}^n \hat{N}) \otimes_{\hat{A}} \hat{A} \right) = \bigcup_{n,i \geq 0} \text{Ass}_{\hat{A}} \left( \text{Ext}^i_{\hat{A}}\left( \hat{M}, \hat{N}/(\hat{I}\hat{A})^n \hat{N} \right) \right)$$

is a finite set, and hence the result follows from Lemma 3.1.6.

3.3 Asymptotic Associate Primes: Stability

In the present section, we analyze the asymptotic behaviour of the sets of associated prime ideals of Ext-modules $\text{Ext}^i_A(M, N/I^n N)$, $(n, i \geq 0)$, where $M$ and $N$ are finitely generated modules over a local complete intersection ring $A$, and $I \subseteq A$ is an ideal (see Corollary 3.3.4). We first prove the following theorem:

**Theorem 3.3.1.** Let $Q$ be a ring of finite Krull dimension, and let $f = f_1, \ldots, f_c$ be a $Q$-regular sequence. Set $A := Q/(f)$. Let $M$ and $N$ be finitely generated $A$-modules, where $\text{projdim}_Q(M)$ is finite, and let $I$ be an ideal of $A$. Then there exist two non-negative integers $n_0, i_0$ such that for all $n \geq n_0$ and $i \geq i_0$, we have that

$$\text{Ass}_A \left( \text{Ext}^i_A(M, N/I^n N) \right) = \text{Ass}_A \left( \text{Ext}^{2i}_A(M, N/I^{n_0} N) \right),$$

$$\text{Ass}_A \left( \text{Ext}^{2i+1}_A(M, N/I^n N) \right) = \text{Ass}_A \left( \text{Ext}^{2i+1}_A(M, N/I^{n_0} N) \right).$$
We give the following example which shows that two sets of stable values of associate primes can occur.

**Example 3.3.2.** Let \( Q = k[[u, x]] \) be a ring of formal power series in two indeterminates over a field \( k \). We set \( A := Q/(ux) \), \( M = N := Q/(u) \) and \( I = 0 \). Clearly, \( A \) is a local complete intersection ring, and \( M, N \) are \( A \)-modules. Then, for each \( i \geq 1 \), we have that

\[
\text{Ext}^{2i-1}_A(M, N) = 0 \quad \text{and} \quad \text{Ext}^{2i}_A(M, N) \cong k;
\]

see [AB00, Example 4.3]. So, in this example, we see that

\[
\text{Ass}_A \left( \text{Ext}^{2i-1}_A(M, N/I^i N) \right) = \emptyset \quad \text{and} \quad \text{Ass}_A \left( \text{Ext}^{2i}_A(M, N/I^i N) \right) = \text{Ass}_A(k) \quad (\neq \emptyset)
\]

for all positive integers \( n \) and \( i \).

Now we prove Theorem 3.3.1. To prove this, we assume the following lemma which we prove at the end of this section.

**Lemma 3.3.3.** Let \( (Q, n) \) be a local ring with residue field \( k \), and let \( f = f_1, \ldots, f_c \) be a \( Q \)-regular sequence. Set \( A := Q/(f) \). Let \( M \) and \( N \) be finitely generated \( A \)-modules, where \( \text{projdim}_Q(M) \) is finite, and let \( I \) be an ideal of \( A \). Then, for every fixed \( l = 0, 1, \ldots \)

\[
\text{either} \quad \text{Hom}_A(k, \text{Ext}^{2i+1}_A(M, N/I^n N)) \neq 0 \quad \text{for all} \quad n, i \gg 0;
\]

\[
\text{or} \quad \text{Hom}_A(k, \text{Ext}^{2i+1}_A(M, N/I^n N)) = 0 \quad \text{for all} \quad n, i \gg 0.
\]

**Proof of Theorem 3.3.1** By virtue of Theorem 3.2.1 we may assume that

\[
\bigcup_{n \geq 1} \bigcup_{i \geq 0} \text{Ass}_A \left( \text{Ext}^i_A(M, N/I^n N) \right) = \{p_1, p_2, \ldots, p_l\}.
\]

Set \( V_{(n, i)} := \text{Ext}^i_A(M, N/I^n N) \) for each \( n, i \geq 0 \), and \( V := \bigoplus_{n, i \geq 0} V_{(n, i)} \).

We first prove that there exist some \( n', i' \geq 0 \) such that

\[
\text{Ass}_A \left( V_{(n', i')} \right) = \text{Ass}_A \left( V_{(n, i')} \right) \quad \text{for all} \quad n \geq n' \quad \text{and} \quad i \geq i'. \tag{3.3.1}
\]

To prove the claim (3.3.1), it is enough to prove that for each \( p_j \), where \( 1 \leq j \leq l \), there exist some \( n_{j_0}, i_{j_0} \geq 0 \) such that exactly one of the following alternatives must hold:

\[
\text{either} \quad p_j \in \text{Ass}_A \left( V_{(n, i)} \right) \quad \text{for all} \quad n \geq n_{j_0} \quad \text{and} \quad i \geq i_{j_0};
\]

\[
\text{or} \quad p_j \notin \text{Ass}_A \left( V_{(n, i)} \right) \quad \text{for all} \quad n \geq n_{j_0} \quad \text{and} \quad i \geq i_{j_0}.
\]
Localizing at $p_j$, and replacing $A_{p_j}$ by $A$ and $p_jA_{p_j}$ by $m$, it is now enough to prove that there exist some $n', i' \geq 0$ such that

either $m \in \text{Ass}_A (V_{(n, 2i)})$ for all $n \geq n'$ and $i \geq i'$; \hspace{1cm} (3.3.2)

or $m \notin \text{Ass}_A (V_{(n, 2i)})$ for all $n \geq n'$ and $i \geq i'$. \hspace{1cm} (3.3.3)

But, in view of Lemma 3.3.3, we get that there exist some $n', i' \geq 0$ such that

either $\text{Hom}_A (k, V_{(n, 2i)}) \neq 0$ for all $n \geq n'$ and $i \geq i'$;

or $\text{Hom}_A (k, V_{(n, 2i)}) = 0$ for all $n \geq n'$ and $i \geq i'$,

which is equivalent to that either (3.3.2) is true, or (3.3.3) is true.

Applying a similar procedure as in the even case, we obtain that there exist some $n'', i'' \geq 0$ such that

$$\text{Ass}_A (V_{(n, 2i+1)}) = \text{Ass}_A (V_{(n'', 2i''+1)})$$

for all $n \geq n''$ and $i \geq i''$. Now $(n_0, i_0) := \max\{(n', i'), (n'', i'')\}$ satisfies the required result of the theorem. \hfill \square

An immediate corollary of the Theorem 3.3.1 is the following:

**Corollary 3.3.4.** Let $(A, m)$ be a local complete intersection ring. Let $M$ and $N$ be finitely generated $A$-modules, and let $I$ be an ideal of $A$. Then there exist two non-negative integers $n_0$ and $i_0$ such that for all $n \geq n_0$ and $i \geq i_0$, we have that

$$\text{Ass}_A (\text{Ext}_A^{2i} (M, N/I^nN)) = \text{Ass}_A (\text{Ext}_A^{2i_0} (M, N/I^{n_0}N)),$$

$$\text{Ass}_A (\text{Ext}_A^{2i+1} (M, N/I^nN)) = \text{Ass}_A (\text{Ext}_A^{2i_0+1} (M, N/I^{n_0}N)).$$

**Proof.** Assume $\hat{A} = Q/(f)$, where $Q$ is a regular local ring and $f = f_1, \ldots, f_c$ is a $Q$-regular sequence. Then, by applying Theorem 3.3.1 for the ring $\hat{A}$, we see that there exist $n_0, i_0 \geq 0$ such that for all $n \geq n_0$ and $i \geq i_0$, we have that

$$\text{Ass}_{\hat{A}} (\text{Ext}_A^{2i} (M, N/I^nN) \otimes_A \hat{A}) = \text{Ass}_{\hat{A}} (\text{Ext}_A^{2i_0} (M, N/I^{n_0}N) \otimes_A \hat{A}),$$

$$\text{Ass}_{\hat{A}} (\text{Ext}_A^{2i+1} (M, N/I^nN) \otimes_A \hat{A}) = \text{Ass}_{\hat{A}} (\text{Ext}_A^{2i_0+1} (M, N/I^{n_0}N) \otimes_A \hat{A}).$$

The result now follows from Lemma 3.1.6. \hfill \square

We need the following result to prove Lemma 3.3.3.
Lemma 3.3.5. Let \((Q, n)\) be a local ring with residue field \(k\), and let \(f = f_1, \ldots, f_c\) be a \(Q\)-regular sequence. Set \(A := Q/(f)\). Let \(M\) and \(N\) be finitely generated \(A\)-modules, where \(\text{projdim}_Q(M)\) is finite, and let \(I\) be an ideal of \(A\). Then

\[
\lambda_A \left( \text{Hom}_A (k, \text{Ext}_A^{2i}(M, N/I^m N)) \right) \quad \text{and} \quad \lambda_A \left( \text{Hom}_A (k, \text{Ext}_A^{2i+1}(M, N/I^m N)) \right)
\]

are given by polynomials in \(n, i\) with rational coefficients for all sufficiently large \((n, i)\).

Here we need to use the Hilbert-Serre Theorem for standard bigraded rings. Let us recall the definition of standard bigraded rings.

Definition 3.3.6. A bigraded ring \(R = \bigoplus_{(n, i) \in \mathbb{N}^2} R_{(n, i)}\) is said to be a standard bigraded ring if there exist \(a_1, \ldots, a_r \in R_{(1, 0)}\) and \(b_1, \ldots, b_s \in R_{(0, 1)}\) such that

\[
R = R_{(0, 0)}[a_1, \ldots, a_r, b_1, \ldots, b_s].
\]

Let us now recall the Hilbert-Serre Theorem for standard bigraded rings.

Theorem 3.3.7 (Hilbert-Serre). [Rob98, Theorem 2.1.7] Let \(R = \bigoplus_{(n, i) \in \mathbb{N}^2} R_{(n, i)}\) be a standard bigraded ring, where \(R_{(0, 0)}\) is an Artinian ring. Let \(L = \bigoplus_{(n, i) \in \mathbb{N}^2} L_{(n, i)}\) be a finitely generated bigraded \(R\)-module. Then there is a polynomial \(P(z, w) \in \mathbb{Q}[z, w]\) and an element \((n_0, i_0) \in \mathbb{N}^2\) such that

\[
\lambda_{R_{(0, 0)}} (L_{(n, i)}) = P(n, i) \quad \text{for all} \quad (n, i) \geq (n_0, i_0).
\]

As a corollary of this theorem, one obtains the following:

Corollary 3.3.8 (Hilbert-Serre). Let \(R = \bigoplus_{(n, i) \in \mathbb{N}^2} R_{(n, i)}\) be a standard bigraded ring. Let \(L = \bigoplus_{(n, i) \in \mathbb{N}^2} L_{(n, i)}\) be a finitely generated bigraded \(R\)-module, where \(\lambda_{R_{(0, 0)}} (L_{(n, i)})\) is finite for every \((n, i) \in \mathbb{N}^2\). Then there is a polynomial \(P(z, w) \in \mathbb{Q}[z, w]\) and an element \((n_0, i_0) \in \mathbb{N}^2\) such that

\[
\lambda_{R_{(0, 0)}} (L_{(n, i)}) = P(n, i) \quad \text{for all} \quad (n, i) \geq (n_0, i_0).
\]

Proof. We set \(S := R/\text{Ann}_R(L)\). Since \(R\) is a standard bigraded ring, it can be observed that \(S\) is also a standard bigraded ring. Note that \(\lambda_{R_{(0, 0)}} (L_{(n, i)}) = \lambda_{S_{(0, 0)}} (L_{(n, i)})\) for all \((n, i) \in \mathbb{N}^2\). Therefore, by virtue of Theorem 3.3.7, it is enough to show that \(S_{(0, 0)}\) is Artinian.
Assume that $L$ is generated (as an $R$-module) by a collection $\{m_1, \ldots, m_l\}$ of homogeneous elements of $L$. Let $\text{deg}(m_j) = (a_j, b_j)$ for all $1 \leq j \leq l$. We now consider the following map:

$$
\varphi : R \rightarrow \bigoplus_{j=1}^{l} L(a_j, b_j)
$$

$$
r \mapsto (r m_1, \ldots, r m_l) \quad \text{for all } r \in R.
$$

Clearly, $\varphi$ is a homogeneous $R$-module homomorphism, where $\text{Ker}(\varphi) = \text{Ann}_R(L)$. So we have an embedding of graded $R$-modules:

$$
S \hookrightarrow \bigoplus_{j=1}^{l} L(a_j, b_j),
$$

which yields an embedding of $R_{(0,0)}$-modules:

$$
S_{(0,0)} \hookrightarrow \bigoplus_{j=1}^{l} L(a_j, b_j).
$$

Therefore $S_{(0,0)}$ has finite length, and hence it is Artinian, which completes the proof of the corollary.

The following result is well-known. But we give a proof for the reader’s convenience.

**Theorem 3.3.9.** Let $A$ be a ring and $I$ an ideal of $A$. Suppose that $\mathcal{R}(I)$ is the Rees ring of $I$. Let $\mathcal{I} = \mathcal{R}(I)[t_1, \ldots, t_c]$ with $\text{deg}(t_j) = (0, 2)$ for all $1 \leq j \leq c$ and $\text{deg}(I^s) = (s, 0)$ for all $s \geq 0$. Let $L = \bigoplus_{(n,i) \in \mathbb{N}^2} L(n,i)$ be a finitely generated bigraded $\mathcal{I}$-module, where $\lambda_A(L(n,i))$ is finite for every $(n, i) \in \mathbb{N}^2$. Set

$$
f_1(n, i) := \lambda_A(L(n,2i)) \quad \text{for all } (n, i) \in \mathbb{N}^2,
$$

$$
f_2(n, i) := \lambda_A(L(n,2i+1)) \quad \text{for all } (n, i) \in \mathbb{N}^2.
$$

Then there are polynomials $P_1(z, w), P_2(z, w) \in \mathbb{Q}[z, w]$ such that

$$
f_1(n, i) = P_1(n, i) \quad \text{for all } n, i \gg 0,
$$

$$
f_2(n, i) = P_2(n, i) \quad \text{for all } n, i \gg 0.
$$

**Proof.** We set

$$
L^{\text{even}} := \bigoplus_{(n,i) \in \mathbb{N}^2} L(n,2i) \quad \text{and} \quad L^{\text{odd}} := \bigoplus_{(n,i) \in \mathbb{N}^2} L(n,2i+1).
$$
In view of \( \mathcal{S} = \mathcal{R}(I)[t_1, \ldots, t_c] \), we build a new standard bigraded ring

\[
\mathcal{S}' := \mathcal{R}(I)[u_1, \ldots, u_c],
\]

where \( \deg(u_j) := (0, 1) \) for all \( 1 \leq j \leq c \) and \( \deg(I^s) := (s, 0) \) for all \( s \geq 0 \). We give \( \mathbb{N}^2 \)-graded structures on \( L^{\text{even}} \) and \( L^{\text{odd}} \) by setting the \( (n,i) \)th components of \( L^{\text{even}} \) and \( L^{\text{odd}} \) as \( L_{(n,2i)} \) and \( L_{(n,2i+1)} \) respectively. Now we define the action of \( \mathcal{S}' := \mathcal{R}(I)[u_1, \ldots, u_c] \) on \( L^{\text{even}} \) and \( L^{\text{odd}} \) as follows: Elements of \( \mathcal{R}(I) \) act on \( L^{\text{even}} \) and \( L^{\text{odd}} \) as before; while the action of \( u_j \) \( (1 \leq j \leq c) \) is defined by

\[
u_j \cdot m := t_j \cdot m \quad \text{for all } m \in L^{\text{even}} \quad \text{(resp. } L^{\text{odd}}).\]

It can be noticed that for every \( 1 \leq j \leq c \), we have

\[
u_j \left( L_{(n,2i)} \right) \subseteq L_{(n,2(i+1))}, \quad \text{i.e., } u_j \left( L^{\text{even}}_{(n,i)} \right) \subseteq L^{\text{even}}_{(n,i+1)} \quad \text{for all } (n,i) \in \mathbb{N}^2;
\]

\[
u_j \left( L_{(n,2i+1)} \right) \subseteq L_{(n,2(i+1)+1)}, \quad \text{i.e., } u_j \left( L^{\text{odd}}_{(n,i)} \right) \subseteq L^{\text{odd}}_{(n,i+1)} \quad \text{for all } (n,i) \in \mathbb{N}^2.
\]

In this way, we obtain that \( L^{\text{even}} \) and \( L^{\text{odd}} \) are bigraded modules over the standard bigraded ring \( \mathcal{S}' := \mathcal{R}(I)[u_1, \ldots, u_c] \). Note that

\[
\lambda_A \left( L^{\text{even}}_{(n,i)} \right) = \lambda_A \left( L_{(n,2i)} \right) < \infty \quad \text{for all } (n,i) \in \mathbb{N}^2;
\]

\[
\lambda_A \left( L^{\text{odd}}_{(n,i)} \right) = \lambda_A \left( L_{(n,2i+1)} \right) < \infty \quad \text{for all } (n,i) \in \mathbb{N}^2.
\]

So, in view of Corollary 3.3.8, it is now enough to show that \( L^{\text{even}} \) and \( L^{\text{odd}} \) are finitely generated as \( \mathcal{S}' := \mathcal{R}(I)[u_1, \ldots, u_c] \)-modules. We only show that \( L^{\text{even}} \) is finitely generated as \( \mathcal{S}' \)-module. \( L^{\text{odd}} \) can be shown to be finitely generated \( \mathcal{S}' \)-module in a similar manner.

Let \( L \) be generated (as an \( \mathcal{S} \)-module) by a collection \( \{m_1, m_2, \ldots, m_r, m'_1, m'_2, \ldots, m'_s\} \) of homogeneous elements of \( L \), where

\[
m_j \in L_{(n_j,2i_j)} \quad \text{for some } (n_j, i_j) \in \mathbb{N}^2, 1 \leq j \leq r;
\]

\[
m'_l \in L_{(n'_l,2i'_l+1)} \quad \text{for some } (n'_l, i'_l) \in \mathbb{N}^2, 1 \leq l \leq s.
\]

We claim that \( L^{\text{even}} \) is generated by \( \{m_1, m_2, \ldots, m_r\} \) as an \( \mathcal{S}' \)-module. To prove this claim, we consider an arbitrary homogeneous element \( m \) of \( L^{\text{even}} \). Then \( m \in L_{(n,2i)} \) for some \( (n,i) \in \mathbb{N}^2 \). Since \( L \) is generated by \( \{m_1, \ldots, m_r, m'_1, \ldots, m'_s\} \), the element \( m \) can be written as an \( \mathcal{S} \)-linear combination as follows:

\[
m = \alpha_1 m_1 + \cdots + \alpha_r m_r + \beta_1 m'_1 + \cdots + \beta_s m'_s \quad \text{(3.3.4)}
\]
for some homogeneous elements \( \alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s \) of \( \mathcal{S} = R(I)[t_1, \ldots, t_c] \), where \( \alpha_j m_j \) and \( \beta_i m'_i \) are in \( L_{(n,2i)} \). Note that any homogeneous element \( \alpha \) of \( \mathcal{S} \) can be written as a finite sum of homogeneous elements (with same degree) of the following type:

\[
a \cdot t_1^{l_1} t_2^{l_2} \cdots t_c^{l_c} \quad \text{for some } a \in \mathbb{F} (p \geq 0) \text{ and for some } l_1, l_2, \ldots, l_c \geq 0.
\]

Observe that

\[
\alpha L_{(n',i')} \subseteq L_{(n'+p,i'+2q)} \quad \text{for some } p,q > 0.
\]

Therefore, for every \( 1 \leq l \leq s \), since \( m'_i \in L_{(n'_i,2i'_i+1)} \), we get that

\[
\beta_i m'_i \in L_{(n'_i+p_i,i'_i+q_i+1)} \quad \text{for some } p_i, q_i > 0.
\]

But, in view of (3.3.4), we have that \( \beta_i m'_i \in L_{(n,2i)} \). Therefore \( \beta_i m'_i \) must be zero for all \( 1 \leq l \leq s \), and hence \( m = \alpha_1 m_1 + \cdots + \alpha_r m_r \). Replacing \( t_j \) by \( u_j \) in \( \alpha_1, \ldots, \alpha_r \), we obtain that \( m \) can be written as an \( \mathcal{S}' := R(I)[u_1, \ldots, u_c] \)-linear combination of \( m_1, \ldots, m_r \). Thus \( L^{\text{even}} \) is generated by \( \{m_1, m_2, \ldots, m_r\} \) as an \( \mathcal{S}' \)-module, which completes the proof of the theorem.

In a similar way as above, one can prove the following well-known result for single graded case:

**Theorem 3.3.10.** Let \( A \) be a ring. Let \( \mathcal{S} = A[t_1, \ldots, t_c] \) with \( \deg(t_j) = 2 \) for all \( 1 \leq j \leq c \). Let \( L = \bigoplus_{i \in \mathbb{N}} L_i \) be a finitely generated graded \( \mathcal{S} \)-module, where \( \lambda_A(L_i) \) is finite for every \( i \in \mathbb{N} \). Then there are polynomials \( P_1(z), P_2(z) \in \mathbb{Q}[z] \) such that

\[
\lambda_A(L_{2i}) = P_1(i) \quad \text{for all } i \gg 0,
\]

\[
\lambda_A(L_{2i+1}) = P_2(i) \quad \text{for all } i \gg 0.
\]

Here we give:

**Proof of Lemma 3.3.5.** We set \( V_{(n,i)} := \text{Ext}_A^n(M, N/I^n N) \) for every \( n,i \geq 0 \), and \( V := \bigoplus_{n,i \geq 0} V_{(n,i)} \). We only prove that the length \( \lambda_A \left( \text{Hom}_A \left( k, V_{(n,2i)} \right) \right) \) is given by a polynomial in \( n,i \) with rational coefficients for all sufficiently large \( (n,i) \). For \( \lambda_A \left( \text{Hom}_A \left( k, V_{(n,2i+1)} \right) \right) \), the proof goes through exactly the same way.
For every fixed $n \geq 0$, we consider the short exact sequence of $A$-modules:

$$0 \to I^n N \to N \to N/I^n N \to 0,$$

which induces an exact sequence of $A$-modules for each $n, i$:

$$\text{Ext}^i_A(M, I^n N) \to \text{Ext}^i_A(M, N) \to \text{Ext}^i_A(M, N/I^n N) \to \text{Ext}^{i+1}_A(M, I^n N).$$

Taking direct sum over $n, i$ and using the naturality of the cohomology operators $t_j$, we obtain an exact sequence of bigraded $S = \mathcal{R}(I)[t_1, \ldots, t_c]$-modules:

$$U \to T \to V \to U(0, 1),$$

where

$$U = \bigoplus_{n,i \geq 0} U_{n,i} := \bigoplus_{n,i \geq 0} \text{Ext}^i_A(M, I^n N),$$

$$T = \bigoplus_{n,i \geq 0} T_{n,i} := \bigoplus_{n,i \geq 0} \text{Ext}^i_A(M, N),$$

$$V = \bigoplus_{n,i \geq 0} V_{n,i} := \bigoplus_{n,i \geq 0} \text{Ext}^i_A(M, N/I^n N),$$

and

$U(0, 1)$ is same as $U$ but the grading is twisted by $(0, 1)$. Setting

$$X := \text{Image}(U \to T), \ Y := \text{Image}(T \to V) \quad \text{and} \quad Z := \text{Image}(V \to U(0, 1)),$$

we have the following commutative diagram of exact sequences of bigraded $\mathcal{I}$-modules:

$$\begin{array}{cccccc}
U & \to & T & \to & V & \to U(0, 1) \\
0 & \to & X & \to & Y & \to Z & \to 0,
\end{array}$$

which gives the following short exact sequences of bigraded $\mathcal{I}$-modules:

$$0 \to X \to T \to Y \to 0 \quad \text{and} \quad 0 \to Y \to V \to Z \to 0.$$
where

\[ C := \text{Image } (\text{Hom}_A(k, Y) \to \text{Ext}_A^1(k, X)) \quad \text{and} \]
\[ D := \text{Image } (\text{Hom}_A(k, V) \to \text{Hom}_A(k, Z)). \]

By virtue of Theorem 3.1.5, we have that
\[ U = \bigoplus_{n \geq 0} \bigoplus_{i \geq 0} \text{Ext}_A^i(M, I^n N) \]
is a finitely generated bigraded \(\mathcal{S}\)-module, and hence \(X = \text{Image}(U \to T)\) is so. Therefore \(\text{Hom}_A(k, X)\) and \(\text{Ext}_A^1(k, X)\) are finitely generated bigraded \(\mathcal{S}\)-modules. Being an \(\mathcal{S}\)-submodule of \(\text{Ext}_A^1(k, X)\), the bigraded \(\mathcal{S}\)-module \(C\) is also finitely generated. Since \(\text{Hom}_A(k, X)\) and \(\text{Ext}_A^1(k, X)\) are annihilated by the maximal ideal of \(A\), \(\text{Hom}_A(k, X_{(n,i)})\) and \(C_{(n,i)}\) both are finitely generated \(k\)-modules, and hence they have finite length as \(A\)-modules for each \(n, i \geq 0\). Therefore, by applying Theorem 3.3.9 to the bigraded \(\mathcal{S} = \mathcal{R}([t_1, \ldots, t_c]; \text{Hom}_A(k, X) \text{ and } C \text{ (where deg}(t_j) = (0, 2) \text{ for all } j = 1, \ldots, c, \text{ and deg}(I^s) = (s, 0) \text{ for all } s \geq 0)\), we obtain that

\[ (n, i) \mapsto \lambda_A (\text{Hom}_A(k, X_{(n,2i)})) \quad \text{(3.3.7)} \]
\[ (n, i) \mapsto \lambda_A (C_{(n,2i)}) \quad \text{(3.3.8)} \]

are given by polynomials in \(n, i\) with rational coefficients for all sufficiently large \((n, i)\).

For every fixed \(n \geq 0\), we have that \(\bigoplus_{i \geq 0} T_{(n,i)} = \bigoplus_{i \geq 0} \text{Ext}_A^i(M, N)\) is a finitely generated graded \(A[t_1, \ldots, t_c]\)-module (where \(\text{deg}(t_j) = 2\) for all \(j = 1, \ldots, c\)), and hence \(\text{Hom}_A(k, \bigoplus_{i \geq 0} T_{(n,i)})\) is also so. By a similar argument as before, we have that \(\lambda_A (\text{Hom}_A(k, T_{(n,i)}))\) is finite for every \(i \geq 0\). Therefore, in view of Theorem 3.3.10 for each \(n \geq 0\), we obtain that

\[ i \mapsto \lambda_A (\text{Hom}_A(k, T_{(n,2i)})) \]

is given by a polynomial in \(i\) for all sufficiently large \(i\). Note that these polynomials are the same polynomial for all \(n \geq 0\). Therefore

\[ (n, i) \mapsto \lambda_A (\text{Hom}_A(k, T_{(n,2i)})) \quad \text{(3.3.9)} \]
is asymptotically given by a polynomial in \(n, i\) (which is independent of \(n\)).

Now considering 3.3.9, we have an exact sequence of \(A\)-modules:
\[ 0 \to \text{Hom}_A(k, X_{(n,i)}) \to \text{Hom}_A(k, T_{(n,i)}) \to \text{Hom}_A(k, Y_{(n,i)}) \to C_{(n,i)} \to 0 \]
for each $n, i \geq 0$. So, the additivity of the length function gives
\[ \lambda_A \left( \text{Hom}_A \left( k, X_{(n,2i)} \right) \right) - \lambda_A \left( \text{Hom}_A \left( k, T_{(n,2i)} \right) \right) + \lambda_A \left( \text{Hom}_A \left( k, Y_{(n,2i)} \right) \right) - \lambda_A \left( C_{(n,2i)} \right) = 0 \]
for each $n, i \geq 0$. So, in view of (3.3.7), (3.3.8) and (3.3.9), we obtain that
\[ (n, i) \mapsto \lambda_A \left( \text{Hom}_A \left( k, Y_{(n,2i)} \right) \right) \] (3.3.10)
is asymptotically given by a polynomial in $n, i$.

Recall that by Theorem 3.1.5 $U$ is a finitely generated bigraded $\mathcal{S}$-module. Now observe that $Z$ is a bigraded submodule of $U(0,1)$. Therefore $Z$ is a finitely generated bigraded $\mathcal{S} = \mathcal{R}(I)[t_1, \ldots, t_c]$-module, hence $\text{Hom}_A(k, Z)$ is so, and hence
\[ D = \text{Image} \left( \text{Hom}_A(k, V) \to \text{Hom}_A(k, Z) \right) \]
is also so. Observe that $\lambda_A \left( D_{(n,i)} \right)$ is finite for each $n, i \geq 0$. Therefore, once again by applying Theorem 3.3.9 we have that
\[ (n, i) \mapsto \lambda_A \left( D_{(n,2i)} \right) \] (3.3.11)
is asymptotically given by a polynomial in $n, i$.

Now considering (3.3.6), we have an exact sequence of $A$-modules:
\[ 0 \to \text{Hom}_A \left( k, Y_{(n,i)} \right) \to \text{Hom}_A \left( k, V_{(n,i)} \right) \to D_{(n,i)} \to 0 \]
for each $n, i \geq 0$, which gives
\[ \lambda_A \left( \text{Hom}_A \left( k, Y_{(n,2i)} \right) \right) - \lambda_A \left( \text{Hom}_A \left( k, V_{(n,2i)} \right) \right) + \lambda_A \left( D_{(n,2i)} \right) = 0. \]
Therefore, in view of (3.3.10) and (3.3.11), we obtain that
\[ (n, i) \mapsto \lambda_A \left( \text{Hom}_A \left( k, V_{(n,2i)} \right) \right) \]
is asymptotically given by a polynomial in $n, i$ with rational coefficients, which completes the proof of the lemma.

To prove Lemma 3.3.3 we also need the following result:

**Lemma 3.3.11.** Let $A$ be a ring and $I$ an ideal of $A$. Let $\mathcal{R}(I)$ be the Rees ring of $I$. Let $\mathcal{S} = \mathcal{R}(I)[t_1, \ldots, t_c]$, where $\deg(t_j) = (0, 2)$ for all $j = 1, \ldots, c$ and $\deg(I^s) = (s, 0)$ for all $s \geq 0$. Let $L = \bigoplus_{(n,i) \in \mathbb{N}^2} L_{(n,i)}$ be a finitely generated bigraded $\mathcal{S}$-module. Then, for every fixed $l = 0, 1$, we have that

- either $L_{(n,2i+l)} \neq 0$ for all $n, i \gg 0$;
- or $L_{(n,2i+l)} = 0$ for all $n, i \gg 0$. 

Proof. In view of [Wes04, Proposition 5.1], we get that there is \((n_0, i_0) \in \mathbb{N}^2\) such that
\[
\text{Ass}_A \left( L_{(n, 2i)} \right) = \text{Ass}_A \left( L_{(n_0, 2i_0)} \right) \quad \text{for all } (n, i) \geq (n_0, i_0);
\]
\[
\text{Ass}_A \left( L_{(n, 2i+1)} \right) = \text{Ass}_A \left( L_{(n_0, 2i_0+1)} \right) \quad \text{for all } (n, i) \geq (n_0, i_0).
\]

It is well-known that an \(A\)-module \(M\) is zero if and only if \(\text{Ass}_A(M) = \emptyset\). Therefore the lemma follows from the above stability of the sets of associated prime ideals.

We now give:

**Proof of Lemma 3.3.3.** We prove the lemma for \(l = 0\) only. For \(l = 1\), the proof goes through exactly the same way. We set
\[
f(n, i) := \lambda_A \left( \text{Hom}_A \left( k, \text{Ext}_A^2(M, N/I^nN) \right) \right) \quad \text{for all } n, i \geq 0. \tag{3.3.12}
\]

By virtue of Lemma 3.3.5, we get that \(f(n, i)\) is given by a polynomial in \(n, i\) with rational coefficients for all sufficiently large \((n, i)\). If \(f(n, i) = 0\) for all \(n, i \gg 0\), then we have nothing to prove. Suppose this is not the case. Then we claim that
\[
\text{Hom}_A \left( k, \text{Ext}_A^2(M, N/I^nN) \right) \neq 0 \quad \text{for all } n, i \gg 0. \tag{3.3.13}
\]

For every fixed \(n \geq 0\), we consider the following short exact sequence of \(A\)-modules:
\[
0 \longrightarrow I^nN/I^{n+1}N \longrightarrow N/I^{n+1}N \longrightarrow N/I^nN \longrightarrow 0,
\]
which yields an exact sequence of \(A\)-modules for each \(n, i\):
\[
\text{Ext}_A^i(M, I^nN/I^{n+1}N) \longrightarrow \text{Ext}_A^i(M, N/I^{n+1}N) \longrightarrow \text{Ext}_A^{i+1}(M, I^nN/I^{n+1}N).
\]
Taking direct sum over \(n, i\), and using the naturality of the cohomology operators \(t_j\), we obtain an exact sequence of bigraded \(\mathcal{S} = \mathcal{R}(I)[t_1, \ldots, t_c]\)-modules:
\[
U \longrightarrow V(1, 0) \longrightarrow V \longrightarrow U(0, 1),
\]
where
\[
U = \bigoplus_{n, i \geq 0} U_{(n, i)} := \bigoplus_{n, i \geq 0} \text{Ext}_A^i(M, I^nN/I^{n+1}N) \quad \text{and}
\]
\[
V = \bigoplus_{n, i \geq 0} V_{(n, i)} := \bigoplus_{n, i \geq 0} \text{Ext}_A^i(M, N/I^nN).
\]
3.3 Asymptotic Associate Primes: Stability

Setting

\[ X := \text{Image} \left( U \to V(1, 0) \right), \quad Y := \text{Image} \left( V(1, 0) \to V \right) \quad \text{and} \quad Z := \text{Image} \left( V \to U(0, 1) \right), \]

we have the following short exact sequences of bigraded \( \mathcal{S} \)-modules:

\[
0 \to X \to V(1, 0) \to Y \to 0 \quad \text{and} \quad 0 \to Y \to V \to Z \to 0.
\]

Now applying \( \text{Hom}_A(k, -) \) to these short exact sequences, we obtain the following exact sequences of bigraded \( \mathcal{S} \)-modules:

\[
0 \to \text{Hom}_A(k, X) \to \text{Hom}_A(k, V(1, 0)) \to \text{Hom}_A(k, Y) \to C \to 0, \quad (3.3.14)
\]
\[
0 \to \text{Hom}_A(k, Y) \to \text{Hom}_A(k, V) \to D \to 0, \quad (3.3.15)
\]

where

\[
C := \text{Image} \left( \text{Hom}_A(k, Y) \to \text{Ext}^1_A(k, X) \right) \quad \text{and} \quad \quad D := \text{Image} \left( \text{Hom}_A(k, V) \to \text{Hom}_A(k, Z) \right).
\]

In view of Theorem 3.1.5, we have that

\[
U = \bigoplus_{n \geq 0} \bigoplus_{i \geq 0} \text{Ext}^i_A \left( M, \frac{I^n N}{I^{n+1} N} \right)
\]
is a finitely generated bigraded \( \mathcal{S} \)-module, and hence

\[
X = \text{Image} \left( U \to V(1, 0) \right) \quad \text{and} \quad Z = \text{Image} \left( V \to U(0, 1) \right)
\]
are so. Therefore \( \text{Hom}_A(k, X) \), \( \text{Ext}^i_A(k, X) \) and \( \text{Hom}_A(k, Z) \) are finitely generated bigraded \( \mathcal{S} \)-modules. Hence \( C \) and \( D \) are finitely generated bigraded \( \mathcal{S} = \mathcal{R}(I)[t_1, \ldots, t_c] \)-modules. So, in view of Lemma 3.3.11, we obtain that

\[
\begin{align*}
\text{either} & \quad \text{Hom}_A \left( k, X_{(n,2i)} \right) \neq 0 \text{ for all } n, i \gg 0, \\
\text{or} & \quad \text{Hom}_A \left( k, X_{(n,2i)} \right) = 0 \text{ for all } n, i \gg 0; \\
\text{either} & \quad C_{(n,2i)} \neq 0 \text{ for all } n, i \gg 0, \\
\text{or} & \quad C_{(n,2i)} = 0 \text{ for all } n, i \gg 0; \\
\text{either} & \quad D_{(n,2i)} \neq 0 \text{ for all } n, i \gg 0, \\
\text{or} & \quad D_{(n,2i)} = 0 \text{ for all } n, i \gg 0.
\end{align*}
\]
For every \(n, i \geq 0\), the \((n, 2i)\)th components of (3.3.14) and (3.3.15) give the following exact sequences of \(A\)-modules:

\[
0 \rightarrow \text{Hom}_A(k, X_{(n, 2i)}) \rightarrow \text{Hom}_A(k, V_{(n+1, 2i)}) \rightarrow \text{Hom}_A(k, Y_{(n, 2i)}) \rightarrow C_{(n, 2i)} \rightarrow 0,
\]

(3.3.19)

\[
0 \rightarrow \text{Hom}_A(k, Y_{(n, 2i)}) \rightarrow \text{Hom}_A(k, V_{(n, 2i)}) \rightarrow D_{(n, 2i)} \rightarrow 0.
\]

(3.3.20)

We now prove the claim (3.3.13) (i.e., \(\text{Hom}_A(k, V_{(n, 2i)}) \neq 0\) for all \(n, i \gg 0\)) by considering the following four cases:

**Case 1.** Assume that \(\text{Hom}_A(k, X_{(n, 2i)}) \neq 0\) for all \(n, i \gg 0\). Then, in view of (3.3.19), we have that \(\text{Hom}_A(k, V_{(n, 2i)}) \neq 0\) for all \(n, i \gg 0\). So, in this case, we are done.

**Case 2.** Assume that \(C_{(n, 2i)} \neq 0\) for all \(n, i \gg 0\). So again, in view of (3.3.19), we get that \(\text{Hom}_A(k, Y_{(n, 2i)}) \neq 0\) for all \(n, i \gg 0\). Hence (3.3.20) yields that \(\text{Hom}_A(k, V_{(n, 2i)}) \neq 0\) for all \(n, i \gg 0\). So, in this case also, we are done.

**Case 3.** Assume that \(D_{(n, 2i)} \neq 0\) for all \(n, i \gg 0\). In this case, by considering (3.3.20), we obtain that \(\text{Hom}_A(k, V_{(n, 2i)}) \neq 0\) for all \(n, i \gg 0\), and we are done.

Now note that if none of the above three cases holds, then by virtue of (3.3.16), (3.3.17) and (3.3.18), we have the following case:

**Case 4.** Assume that \(\text{Hom}_A(k, X_{(n, 2i)}) = 0\) for all \(n, i \gg 0\), \(C_{(n, 2i)} = 0\) for all \(n, i \gg 0\), and \(D_{(n, 2i)} = 0\) for all \(n, i \gg 0\). Then the exact sequences (3.3.19) and (3.3.20) yield the following isomorphisms:

\[
\text{Hom}_A(k, V_{(n+1, 2i)}) \cong \text{Hom}_A(k, Y_{(n, 2i)}) \cong \text{Hom}_A(k, V_{(n, 2i)}) \quad \text{for all } n, i \gg 0. \tag{3.3.21}
\]

These isomorphisms (with the setting (3.3.12)) give the following equalities:

\[
f(n+1, i) = f(n, i) \quad \text{for all } n, i \gg 0. \tag{3.3.22}
\]

We write the polynomial expression of \(f(n, i)\) in the following way:

\[
f(n, i) = h_0(i)n^a + h_1(i)n^{a-1} + \cdots + h_{a-1}(i)n + h_a(i) \quad \text{for all } n, i \gg 0, \tag{3.3.23}
\]

where \(h_j(i) (j = 0, 1, \ldots, a)\) are polynomials in \(i\) over \(\mathbb{Q}\). Without loss of generality, we may assume that \(h_0\) is a non-zero polynomial. So \(h_0\) may have only finitely many roots. Let \(i' \geq 0\) be such that \(h_0(i) \neq 0\) for all \(i \geq i'\). In view of (3.3.22) and (3.3.23), there
exist some \( n_0 (\geq 0) \) and \( i_0 (\geq i', \text{ say}) \) such that for all \( n \geq n_0 \) and \( i \geq i_0 \), we have

\[
\begin{align*}
  f(n+1,i) &= f(n,i) \quad \text{and} \\
  f(n,i) &= h_0(i)n^a + h_1(i)n^{a-1} + \cdots + h_{a-1}(i)n + h_a(i).
\end{align*}
\]

Therefore \( a \) must be equal to 0, and hence \( f(n,i) = h_0(i) \) for all \( n \geq n_0 \) and \( i \geq i_0 \). Thus we obtain that \( f(n,i) \neq 0 \) for all \( n \geq n_0 \) and \( i \geq i_0 \), i.e.,

\[
\text{Hom}_A(k, V_{(n,2i)}) \neq 0 \quad \text{for all } n \geq n_0 \text{ and } i \geq i_0,
\]

which completes the proof of the lemma.

\[\square\]

### 3.4 Asymptotic Associate Primes: The Geometric Case

Let \( V \) be an affine or projective variety over an algebraically closed field \( K \). Let \( A \) be the coordinate ring of \( V \). Then \( V \) is said to be a \textit{locally complete intersection variety} if all its local rings are complete intersection local rings. Thus:

(i) in the affine case, \( A_p \) is a local complete intersection ring for every \( p \in \text{Spec}(A) \);

(ii) in the projective case, \( A(p) \) is a local complete intersection ring for every \( p \in \text{Proj}(A) \).

Recall that \( A(p) \) is called the \textit{homogeneous localization} (or \textit{degree zero localization}) of \( A \) at the homogeneous prime ideal \( p \) which is defined to be the degree zero part of the graded ring \( S^{-1}A \), where \( S \) is the collection of all homogeneous elements in \( A \setminus p \).

In this section, we prove the results analogous to Theorems 3.2.1 and 3.3.1 for the coordinate rings of locally complete intersection varieties. In the affine case, we prove the following general results.

**Theorem 3.4.1.** Let \( A = Q/\mathfrak{a} \), where \( Q \) is a regular ring of finite Krull dimension and \( \mathfrak{a} \subseteq Q \) is an ideal so that \( \mathfrak{a}_q \subseteq Q_q \) is generated by a \( Q_q \)-regular sequence for each \( q \in \text{Var}(\mathfrak{a}) \). Let \( M \) and \( N \) be finitely generated \( A \)-modules, and let \( I \) be an ideal of \( A \). Then the set

\[
\bigcup_{n \geq 0} \bigcup_{i \geq 0} \text{Ass}_A \left( \text{Ext}_A^i(M, N/I^nN) \right)
\]
Moreover, there exist some non-negative integers $n_0$ and $i_0$ such that for all $n \geq n_0$ and $i \geq i_0$, we have that
\[
\operatorname{Ass}_A(\operatorname{Ext}_A^{2i}(M, N/I^nN)) = \operatorname{Ass}_A(\operatorname{Ext}_A^{2i_0}(M, N/I^{n_0}N)),
\]
\[
\operatorname{Ass}_A(\operatorname{Ext}_A^{2i+1}(M, N/I^nN)) = \operatorname{Ass}_A(\operatorname{Ext}_A^{2i_0+1}(M, N/I^{n_0}N)).
\]

Proof. For each $x \in A$, we set
\[
D(x) := \{p \in \text{Spec}(A) : x \not\in p\}.
\]
As in [Put13, page 384, Proof of Theorem 6.1], we have
\[
\text{Spec}(A) = D(g_1) \cup \cdots \cup D(g_m)
\]
for some $g_1, \ldots, g_m \in A$ such that the localization $A_{g_j}$ by $\{g_j^l : l \in \mathbb{N}\}$ has the form $Q_j/a_j$ for some regular ring $Q_j$ of finite Krull dimension and some ideal $a_j$ of $Q_j$ generated by a $Q_j$-regular sequence.

It can be easily observed from (3.4.1) that for a finitely generated $A$-module $E$, we have
\[
\operatorname{Ass}_A(E) = \bigcup \left\{q \cap A : q \in \operatorname{Ass}_{A_{g_j}}(E_{g_j}) \text{ for some } j = 1, \ldots, m \right\}.
\]
Since localization $A_{g_j}$ is flat over $A$, we have
\[
\left(\operatorname{Ext}_A^i(M, N/I^nN)\right)_{g_j} = \operatorname{Ext}_{A_{g_j}}^i(M_{g_j}, N_{g_j}/(IA_{g_j})^nN_{g_j})
\]
for all $n, i \geq 0$ and $j = 1, \ldots, m$. Therefore, in view of (3.4.2) and (3.4.3), it is enough to prove the result for the ring $A_{g_j} = Q_j/a_j$ for each $j$. Note that $Q_j$ is a regular ring of finite Krull dimension, and hence $\text{projdim}_{Q_j}(M_{g_j})$ is finite. Therefore the result now follows by applying the Theorems 3.2.1 and 3.3.1 to each $A_{g_j} = Q_j/a_j$. \qed

Now we prove the analogous results to Theorem 3.4.1 in the projective case. Let us fix the following hypothesis:

Hypothesis 3.4.2. Let $K$ be a field not necessarily algebraically closed, and let $Q = K[X_0, X_1, \ldots, X_r]$ be a polynomial ring over $K$, where $\deg(X_i) = 1$ for all $i = 0, 1, \ldots, r$. Let $a$ be a homogeneous ideal of $Q$. Set $A := Q/a$. Suppose $A_{(p)}$ is a complete intersection ring for every $p \in \text{Proj}(A)$. 

Let $\mathfrak{m}$ be the unique maximal homogeneous ideal of $A$. Let $E$ be a finitely generated graded $A$-module. Note that all the associate primes of $E$ are homogeneous prime ideals. Define the set of relevant associated prime ideals of $E$ as

$$\ast \text{Ass}_A(E) := \text{Ass}_A(E) \cap \text{Proj}(A) = \text{Ass}_A(E) \setminus \{\mathfrak{m}\}.$$ 

In the projective case, we prove the following:

**Theorem 3.4.3.** With the Hypothesis 3.4.2, let $M$ and $N$ be finitely generated graded $A$-modules, and let $I$ be a homogeneous ideal of $A$. Then the set

$$\bigcup_{n \geq 0} \bigcup_{i \geq 0} \ast \text{Ass}_A \left( \text{Ext}^i_A(M, N/I^nN) \right)$$

is finite. Moreover, there exist some non-negative integers $n_0$ and $i_0$ such that for all $n \geq n_0$ and $i \geq i_0$, we have that

$$\ast \text{Ass}_A \left( \text{Ext}^{2i}_A(M, N/I^nN) \right) = \ast \text{Ass}_A \left( \text{Ext}^{2i}_A(M, N/I^{n_0}N) \right),$$

$$\ast \text{Ass}_A \left( \text{Ext}^{2i+1}_A(M, N/I^nN) \right) = \ast \text{Ass}_A \left( \text{Ext}^{2i+1}_A(M, N/I^{n_0}N) \right).$$

**Proof.** For each homogeneous element $x \in A$, we set

$$\ast D(x) = \{p \in \text{Proj}(A) : x \notin p\}.$$ 

As in [Put13, page 386, Proof of Theorem 6.3], we have

$$\text{Proj}(A) = \ast D(g_1) \cup \cdots \cup \ast D(g_m)$$

for some homogeneous $g_1, \ldots, g_m \in A$ such that $A_{g_j} = Q_j/a_j$ for some regular ring $Q_j$ of finite Krull dimension and some ideal $a_j$ of $Q_j$ generated by a $Q_j$-regular sequence. Clearly, for any graded $A$-module $E$, we obtain

$$\ast \text{Ass}_A(E) = \bigcup \left\{q \cap A : q \in \text{Ass}_{A_{g_j}}(E_{g_j}) \text{ for some } j = 1, \ldots, m \right\}.$$ 

Similarly, as in the proof of Theorem 3.4.1, the result now follows by applying Theorems 3.2.1 and 3.3.1 to each $A_{g_j} = Q_j/a_j$. 

$\square$
Chapter 4

Asymptotic Stability of Complexities over Complete Intersection Rings

Throughout this chapter, let \((A, \mathfrak{m}, k)\) be a local complete intersection ring of codimension \(c\). Let \(M\) and \(N\) be finitely generated \(A\)-modules. Recall that the complexity of the pair \((M, N)\) is defined to be the number

\[
\text{cx}_A(M, N) = \inf \left\{ b \in \mathbb{N} \middle| \limsup_{n \to \infty} \frac{\mu(\Ext^n_A(M, N))}{n^{b-1}} < \infty \right\},
\]

where \(\mu(D)\) denotes the minimal number of generators of an \(A\)-module \(D\). Let \(I\) be an ideal of \(A\). In this chapter, we show that

\[(C1) \quad \text{cx}_A(M, N/I^jN) \text{ is independent of } j \text{ for all sufficiently large } j;\]

see Theorem 4.2.1. To prove this result, we take advantage of the notion of support variety which was introduced by L. L. Avramov and R.-O. Buchweitz in [AB00, 2.1].

The organization of this chapter is as follows. In Section 4.1, we recall how the notions of complexity and support variety are related. We also give some preliminaries which we need to prove our result on complexity. Finally, in Section 4.2, we prove Theorem 4.2.1.

4.1 Support Varieties

To prove Theorem 4.2.1, we first reduce to the case when our local complete intersection ring \(A\) is complete and its residue field \(k\) is algebraically closed.
4.1.1 (Complexity through Flat Local Extension). Suppose \((A', m')\) is a flat local extension of \((A, m)\) such that \(m' = mA'\). It is shown in [Avr98, Theorem 7.4.3] that \(A'\) is also a local complete intersection ring. For an \(A\)-module \(E\), we set \(E' = E \otimes_A A'\). Note that \(I' \cong IA'\). So we may consider \(I'\) as an ideal of \(A'\). It can be easily checked that
\[
\operatorname{cx}_{A'}(M', N'/(I')^jN') = \operatorname{cx}_A(M, N/I^jN) \quad \text{for all } j \geq 0.
\]

4.1.2 (Reduction to the Case when \(A\) is Complete with Algebraically Closed Residue Field). By [Bou83, Chapitre 9, appendice, corollaire du théorème 1, p. IX.41], there exists a flat local extension \(A \subseteq \tilde{A}\) such that \(\tilde{m} = m\tilde{A}\) is the maximal ideal of \(\tilde{A}\) and the residue field \(\tilde{k}\) of \(\tilde{A}\) is an algebraically closed extension of \(k\). Therefore, by the observations made in Section 4.1.1, we may assume \(k\) to be algebraically closed field. We now consider the completion \(\hat{A}\) of \(A\). Since \(\hat{A}\) is a flat local extension of \(A\) such that \(m\hat{A}\) is the maximal ideal of \(\hat{A}\), we may as well assume that our local complete intersection ring \(A\):

(i) is complete, and hence \(A = Q/(f)\), where \((Q, n)\) is a regular local ring and \(f = f_1, \ldots, f_c \in n^2\) is a \(Q\)-regular sequence;

(ii) has an algebraically closed residue field \(k\).

4.1.3 (Total Ext-module). Let \(U\) and \(V\) be two finitely generated \(A\)-modules. Recall from Section 3.1.1 that
\[
\operatorname{Ext}^*_A(U, V) := \bigoplus_{i \geq 0} \operatorname{Ext}^i_A(U, V)
\]
is the total Ext-module of \(U\) and \(V\) over the graded ring \(A[t_1, \ldots, t_c]\) of cohomology operators \(t_j\) defined by \(f\), where \(\deg(t_j) = 2\) for all \(j = 1, \ldots, c\). We set
\[
\mathcal{C}(U, V) := \operatorname{Ext}^*_A(U, V) \otimes_A k.
\]
Since \(\operatorname{Ext}^*_A(U, V)\) is a finitely generated graded \(A[t_1, \ldots, t_c]\)-module, \(\mathcal{C}(U, V)\) is a finitely generated graded module over
\[
\overline{T} := A[t_1, \ldots, t_c] \otimes_A k = k[t_1, \ldots, t_c].
\]

4.1.4 (Support Variety and Complexity). Define the support variety \(\mathcal{V}(U, V)\) of \(U, V\) as the zero set in \(k^c\) of the annihilator of \(\mathcal{C}(U, V)\) in \(\overline{T}\), that is
\[
\mathcal{V}(U, V) := \{(b_1, \ldots, b_c) \in k^c : P(b_1, \ldots, b_c) = 0 \text{ for all } P \in \operatorname{Ann}_{\overline{T}}(\mathcal{C}(U, V))\} \cup \{0\}.
\]
It is shown in [AB00, Proposition 2.4(2)] that
4.1 Support Varieties

\[(\dagger) \quad \text{cx}_A(U, V) = \dim(\mathcal{V}(U, V)) = \dim_{\mathcal{T}}(\mathcal{C}(U, V)).\]

The following lemma is well-known. We give its proof also for the reader’s convenience.

**Lemma 4.1.5.** Let \( R = \bigoplus_{n \geq 0} R_n \) be a standard \( \mathbb{N} \)-graded ring, and let \( M = \bigoplus_{n \geq 0} M_n \) be a finitely generated \( \mathbb{N} \)-graded \( R \)-module. Then there exists some non-negative integer \( j_0 \) such that

\[
\dim_{R_0}(M_j) = \dim_{R_0}(M_{j_0}) \quad \text{for all } j \geq j_0.
\]

**Proof.** Since \( M \) is a finitely generated \( \mathbb{N} \)-graded module over a standard \( \mathbb{N} \)-graded ring \( R \), there exists some \( j' \geq 0 \) such that

\[
M_j = (R_1)^{j-j'}M_{j'} \quad \text{for all } j \geq j',
\]

which gives the following ascending chain of ideals:

\[
\text{Ann}_{R_0}(M_{j'}) \subseteq \text{Ann}_{R_0}(M_{j'+1}) \subseteq \text{Ann}_{R_0}(M_{j'+2}) \subseteq \cdots.
\]

Since \( R_0 \) is Noetherian, there exists some \( j_0 \) \((\geq j')\) such that \( \text{Ann}_{R_0}(M_j) = \text{Ann}_{R_0}(M_{j_0}) \), and hence \( \dim_{R_0}(M_j) = \dim_{R_0}(M_{j_0}) \) for all \( j \geq j_0 \).

**4.1.6 (Asymptotic Stability of Complexities)\text{ cx}_A(M, I^jN/I^{j+1}N).** Since

\[
\text{gr}_I(N) = \bigoplus_{j \geq 0} I^jN/I^{j+1}N
\]

is a finitely generated graded \( \mathcal{R}(I) \)-module, in view of Theorem 3.1.5, we have that

\[
\bigoplus_{j \geq 0} \text{Ext}_A^i(M, I^jN/I^{j+1}N) = \bigoplus_{j \geq 0} \bigoplus_{i \geq 0} \text{Ext}_A^i(M, I^jN/I^{j+1}N)
\]

is a finitely generated graded \( \mathcal{R}(I)[t_1, \ldots, t_c] \)-module, and hence

\[
\bigoplus_{j \geq 0} \text{Ext}_A^*(M, I^jN/I^{j+1}N) \otimes_A k
\]

is a finitely generated graded module over

\[
\mathcal{R}(I)[t_1, \ldots, t_c] \otimes_A k = F(I)[t_1, \ldots, t_c],
\]

where \( F(I) \) is the fiber cone of \( I \) which is a finitely generated \( k \)-algebra. Writing

\[
F(I)[t_1, \ldots, t_c] = k[x_1, \ldots, x_m][t_1, \ldots, t_c] = \mathcal{T}[x_1, \ldots, x_m],
\]
we can say that
\[ \bigoplus_{j \geq 0} \Ext^*_A(M, I^j N/I^{j+1} N) \otimes_A k \]
is a finitely generated graded \( \mathbb{T}[x_1, \ldots, x_m] \)-module. So, by using Lemma 4.1.5, we have
\[ \dim \mathbb{T}(\Ext^*_A(M, I^j N/I^{j+1} N) \otimes_A k) \]
is constant for all \( j \gg 0 \).
Therefore, in view of (†) in Section 4.1.4 we obtain that
\[ (C2) \quad \text{cx}_A(M, P^j N/I^{j+1} N) \quad \text{is constant for all } j \gg 0. \]

4.2 Asymptotic Stability of Complexities

Now we are in a position to prove the main result of this chapter.

**Theorem 4.2.1.** Let \((A, m, k)\) be a local complete intersection ring. Let \( M \) and \( N \) be two finitely generated \( A \)-modules, and let \( I \) be an ideal of \( A \). Then
\[ \text{cx}_A(M, N/I^j N) \quad \text{is constant for all } j \gg 0. \]

**Proof.** By the observations made in Section 4.1.2 we may assume that \( A \) is complete and its residue field \( k \) is algebraically closed.

Fix \( j \geq 0 \). Consider the short exact sequence of \( A \)-modules
\[ 0 \longrightarrow I^j N/I^{j+1} N \longrightarrow N/I^{j+1} N \longrightarrow N/I^j N \longrightarrow 0, \]
which induces the following exact sequence of \( A \)-modules for each \( i \):
\[ \Ext^i_A(M, N/I^j N) \longrightarrow \Ext^i_A(M, N/I^{j+1} N) \longrightarrow \Ext^{i+1}_A(M, N/I^j N). \]
Taking direct sum over \( i \) and setting
\[ U_j := \bigoplus_{i \geq 0} \Ext^i_A(M, P^j N/I^{j+1} N) \quad \text{and} \quad V_j := \bigoplus_{i \geq 0} \Ext^i_A(M, N/P^j N), \]
we obtain an exact sequence of \( A[t_1, \ldots, t_c] \)-modules:
\[ V_j(-1) \overset{\varphi_1}{\longrightarrow} U_j \overset{\varphi_2}{\longrightarrow} V_{j+1} \overset{\varphi_3}{\longrightarrow} V_j \overset{\varphi_4}{\longrightarrow} U_j(1). \]
Now we set
\[ Z_j := \text{Image}(\varphi_1), \quad X_j := \text{Image}(\varphi_2) \quad \text{and} \quad Y_j := \text{Image}(\varphi_3). \]

Thus we have the following commutative diagram of exact sequences:
\[
\begin{array}{cccccccc}
V_j(-1) & \longrightarrow & U_j & \longrightarrow & V_{j+1} & \longrightarrow & V_j & \longrightarrow & U_j(1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Z_j & \longrightarrow & X_j & \longrightarrow & Y_j & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & Z_j(1) & \longrightarrow & 0.
\end{array}
\]

Consider the following two short exact sequences of \( A[t_1, \ldots, t_c] \)-modules:
\[
0 \longrightarrow X_j \longrightarrow V_{j+1} \longrightarrow Y_j \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow Y_j \longrightarrow V_j \longrightarrow Z_j(1) \longrightarrow 0.
\]

Tensoring these sequences with \( k \) over \( A \), we get the following exact sequences of \( T = k[t_1, \ldots, t_c] \)-modules:
\[
X_j \otimes_A k \xrightarrow{\Phi_j} V_{j+1} \otimes_A k \longrightarrow Y_j \otimes_A k \longrightarrow 0,
\]
\[
Y_j \otimes_A k \xrightarrow{\Psi_j} V_j \otimes_A k \longrightarrow Z_j(1) \otimes_A k \longrightarrow 0.
\]

Now for each \( j \geq 0 \), we set
\[ X_j' := \text{Image}(\Phi_j) \quad \text{and} \quad Y_j' := \text{Image}(\Psi_j) \]

to get the following short exact sequences of \( T \)-modules:
\[
0 \longrightarrow X_j' \longrightarrow V_{j+1} \otimes_A k \longrightarrow Y_j \otimes_A k \longrightarrow 0, \tag{4.2.1}
\]
\[
0 \longrightarrow Y_j' \longrightarrow V_j \otimes_A k \longrightarrow Z_j(1) \otimes_A k \longrightarrow 0. \tag{4.2.2}
\]

By the observations made in Section 4.1.6, we have that \( \bigoplus_{j \geq 0} U_j \) is a finitely generated graded \( R[I][t_1, \ldots, t_c] \)-module, and hence its submodule \( \bigoplus_{j \geq 0} Z_j \) is also so. Therefore \( \bigoplus_{j \geq 0} (Z_j \otimes_A k) \) is a finitely generated graded module over
\[
R(I) \otimes_A k = F(I)[t_1, \ldots, t_c] = T[x_1, \ldots, x_m].
\]

Therefore, by Lemma 4.1.5 \( \dim_T(Z_j \otimes_A k) = z \) for all sufficiently large \( j \), where \( z \) is some constant. Now considering the short exact sequences \( (4.2.1) \) and \( (4.2.2) \), we obtain that
\[
\dim_T(V_{j+1} \otimes_A k) = \max\{\dim_T(X_j'), \dim_T(Y_j \otimes_A k)\}, \tag{4.2.3}
\]
\[
\dim_T(V_j \otimes_A k) = \max\{\dim_T(Y_j'), z\} \geq z. \tag{4.2.4}
\]
for all sufficiently large \( j \), say \( j \geq j_0 \).

Note that \( \dim_T(V_j \otimes_A k) = \text{cx}_A(M, N/P^j N) \) for all \( j \geq 0 \); see (†) in Section 4.1.4. Therefore it is enough to prove that the stability of \( \dim_T(V_j \otimes_A k) \) holds for all sufficiently large \( j \).

If \( \dim_T(V_j \otimes_A k) = z \) for all \( j \geq j_0 \), then we are done. Otherwise there exists some \( j \geq j_0 \) such that \( \dim_T(V_j \otimes_A k) > z \), and hence for this \( j \), we have

\[
\dim_T(V_j \otimes_A k) = \dim_T(Y'_j) \leq \dim_T(Y'_j \otimes_A k) \leq \dim_T(V_{j+1} \otimes_A k).
\]

First equality above occurs from (4.2.4), second inequality occurs because \( Y'_j \) is a quotient module of \( Y_j \otimes_A k \), and the last inequality occurs from (4.2.3). Note that \( \dim_T(V_{j+1} \otimes_A k) > z \). So, by applying a similar procedure, we have that

\[
\dim_T(V_{j+1} \otimes_A k) \leq \dim_T(V_{j+2} \otimes_A k).
\]

In this way, we obtain a bounded non-decreasing sequence of non-negative integers:

\[
\dim_T(V_j \otimes_A k) \leq \dim_T(V_{j+1} \otimes_A k) \leq \dim_T(V_{j+2} \otimes_A k) \leq \cdots \leq \dim(T) < \infty,
\]

which eventually stabilizes somewhere, and hence the required stability holds. \( \square \)
Chapter 5

Asymptotic Linear Bounds of Castelnuovo-Mumford Regularity

Suppose $A$ is a standard $\mathbb{N}$-graded algebra over an Artinian local ring $A_0$. Let $I_1, \ldots, I_t$ be homogeneous ideals of $A$, and let $M$ be a finitely generated $\mathbb{N}$-graded $A$-module. Our main goal in this chapter is to show that there exist two integers $k_1$ and $k_1'$ such that

$$\text{reg}(I_1^{n_1} \cdots I_t^{n_t} M) \leq (n_1 + \cdots + n_t)k_1 + k_1'$$

for all $n_1, \ldots, n_t \in \mathbb{N}$.

We prove this result in a quite general set-up; see Hypothesis 5.2.7 and Theorem 5.3.3. As a consequence, we also obtain the following: If $A_0$ is a field, then there exist two integers $k_2$ and $k_2'$ such that

$$\text{reg}(\overline{I_1^{n_1}} \cdots \overline{I_t^{n_t}} M) \leq (n_1 + \cdots + n_t)k_2 + k_2'$$

for all $n_1, \ldots, n_t \in \mathbb{N}$, where $\overline{I}$ denotes the integral closure of an ideal $I$ of $A$; see Definition 5.2.9.

We use the following notations throughout this chapter.

**Notations 5.1.** Throughout, $\mathbb{N}$ denotes the set of all non-negative integers and $t$ is any fixed positive integer. We use small letters with underline (e.g., $\underline{n}$) to denote elements of $\mathbb{N}^t$, and we use subscripts mainly to denote the coordinates of such an element, e.g., $\underline{n} = (n_1, n_2, \ldots, n_t)$. In particular, for every $1 \leq i \leq t$, $e^i$ denotes the $i$th standard basis element of $\mathbb{N}^t$. We denote $\underline{0}$ the element of $\mathbb{N}^t$ with all components $0$. Throughout, we use the partial order on $\mathbb{N}^t$ defined by $\underline{n} \geq \underline{m}$ if and only if $n_i \geq m_i$ for all $1 \leq i \leq t$. For every $\underline{n} \in \mathbb{N}^t$, we set $|\underline{n}| := n_1 + \cdots + n_t$. If $R$ is an $\mathbb{N}$-graded ring and $L$ is an $\mathbb{N}$-graded $R$-module, then by $L_{\underline{n}}$, we always mean the $\underline{n}$th graded component of $L$. 49
By a standard multigraded ring, we mean a multigraded ring which is generated in total degree one, i.e., $R$ is a standard $\mathbb{N}^l$-graded ring if $R = R_0[R_{e_1}, \ldots, R_{e_l}]$.

The rest of this chapter is organized as follows. We start by recalling some of the well-known basic results on Castelnuovo-Mumford regularity in Section 5.1 while in Section 5.2 we give some preliminaries on multigraded modules which we use in order to prove our main results on regularity. The announced linear boundedness results of regularity are proved in Section 5.3 through several steps. Finally, in Section 5.4, we discuss about asymptotic linearity of regularity for powers of several ideals.

5.1 Castelnuovo-Mumford Regularity

Let $A = A_0[x_1, \ldots, x_d]$ be a standard $\mathbb{N}$-graded ring. Let $A_+$ be the irrelevant ideal $\langle x_1, \ldots, x_d \rangle$ of $A$ generated by the homogeneous elements of positive degree. Let $M$ be a finitely generated $\mathbb{N}$-graded $A$-module. For every integer $i \geq 0$, we denote the $i$th local cohomology module of $M$ with respect to $A_+$ by $H_{A_+}^i(M)$. For every integer $i \geq 0$, we set

$$a_i(M) := \begin{cases} \max \{ \mu : H_{A_+}^i(M)_\mu \neq 0 \} & \text{if } H_{A_+}^i(M) \neq 0 \\ -\infty & \text{if } H_{A_+}^i(M) = 0. \end{cases}$$

Recall that the Castelnuovo-Mumford regularity of $M$ is defined by

$$\text{reg}(M) := \max \{ a_i(M) + i : i \geq 0 \}.$$

For a given short exact sequence of graded modules, by considering the corresponding long exact sequence of local cohomology modules, we can prove the following well-known result:

**Lemma 5.1.1.** Let $A$ be a standard $\mathbb{N}$-graded ring. Let $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ be a short exact sequence of finitely generated $\mathbb{N}$-graded $A$-modules. Then we have the following inequalities:

(i) $\text{reg}(M_1) \leq \max\{\text{reg}(M_2), \text{reg}(M_3) + 1\}$.

(ii) $\text{reg}(M_2) \leq \max\{\text{reg}(M_1), \text{reg}(M_3)\}$.

(iii) $\text{reg}(M_3) \leq \max\{\text{reg}(M_1) - 1, \text{reg}(M_2)\}$. 

The following lemma is well-known, but we cannot locate a reference for it. So we give a proof here for the completeness.

**Lemma 5.1.2.** Let $A$ be a standard $\mathbb{N}$-graded ring, and let $M$ be a finitely generated $\mathbb{N}$-graded $A$-module. Let $x$ be a homogeneous element in $A$ of positive degree $l$. Then we have the following inequality:

$$\text{reg}(M) \leq \max\{\text{reg}(0 :_M x), \text{reg}(M/xM) - l + 1\}. $$

Over polynomial rings over fields, if $x$ is such that $\dim(0 :_M x) \leq 1$, then the inequality could be replaced by equality.

**Proof.** We set $N := (0 :_M x)$. To prove the first part of the lemma, it is enough to prove that at least one of the following is true:

$$\text{reg}(M) \leq \text{reg}(N) \quad \text{or} \quad \text{reg}(M) \leq \text{reg}(M/xM) - l + 1. \tag{5.1.1}$$

We prove (5.1.1) by contradiction. If (5.1.1) is not true, then we have the following:

$$\text{reg}(M) > \text{reg}(N) \quad \text{and} \quad \text{reg}(M) > \text{reg}(M/xM) - l + 1. \tag{5.1.2}$$

Now consider the following short exact sequences of graded $A$-modules:

$$0 \longrightarrow N(-l) \longrightarrow M(-l) \xrightarrow{x} xM \longrightarrow 0, \tag{5.1.3}$$

$$0 \longrightarrow xM \longrightarrow M \longrightarrow M/xM \longrightarrow 0, \tag{5.1.4}$$

where $M(-l)$ is same as $M$ but the grading is twisted by $-l$, and the second map of each of the sequences is the inclusion map. It directly follows from the definition of regularity that

$$\text{reg}(M(-l)) = \text{reg}(M) + l.$$

Then, from the short exact sequences (5.1.3) and (5.1.4), by using Lemma 5.1.1, we have

$$\text{reg}(M) + l \leq \max\{\text{reg}(N) + l, \text{reg}(xM)\}$$

$$= \text{reg}(xM) \quad [\text{as } \text{reg}(M) > \text{reg}(N)], \tag{5.1.5}$$

$$\text{reg}(xM) \leq \max\{\text{reg}(M), \text{reg}(M/xM) + 1\}$$

$$= \text{reg}(M/xM) + 1 \quad [\text{as } \text{reg}(xM) > \text{reg}(M) \text{ by (5.1.5)}]$$

$$< \text{reg}(M) + l \quad [\text{by (5.1.2)}]. \tag{5.1.6}$$

Clearly (5.1.5) and (5.1.6) give a contradiction.

For the second part of the lemma, we refer the reader to [Cha07, Remark 1.4.1].
5.2 Preliminaries on Multigraded Modules

Here we give some preliminaries on $\mathbb{N}^t$-graded modules which we use in the next section.

We start with the following lemma:

**Lemma 5.2.1.** Let $R = \bigoplus_{n \in \mathbb{N}^t} R_n$ be an $\mathbb{N}^t$-graded ring, and let $L = \bigoplus_{n \in \mathbb{N}^t} L_n$ be a finitely generated $\mathbb{N}^t$-graded $R$-module. Set $A := R_0$. Let $J$ be an ideal of $A$. Then there exists a positive integer $k$ such that

\[ J^m L_n \cap H^0_J(L_n) = 0 \quad \text{for all } n \in \mathbb{N}^t \text{ and } m \geq k. \]

**Proof.** Let $I = JR$ be the ideal of $R$ generated by $J$. Since $R$ is Noetherian and $L$ is a finitely generated $R$-module, then by the Artin-Rees Lemma, there exists a positive integer $c$ such that

\[ (I^m L) \cap H^0_I(L) = I^{m-c}((I^c L) \cap H^0_I(L)) \quad \text{for all } m \geq c \]

\[ \subseteq I^{m-c}H^0_I(L) \quad \text{for all } m \geq c. \tag{5.2.1} \]

Now consider the ascending chain of submodules of $L$:

\[ (0 :_L I) \subseteq (0 :_L I^2) \subseteq (0 :_L I^3) \subseteq \cdots. \]

Since $L$ is a Noetherian $R$-module, there exists some $l$ such that

\[ (0 :_L I^l) = (0 :_L I^{l+1}) = (0 :_L I^{l+2}) = \cdots = H^0_I(L). \tag{5.2.2} \]

Set $k := c + l$. Then, in view of (5.2.1) and (5.2.2), we have

\[ (I^m L) \cap H^0_I(L) \subseteq I^{m-c}(0 :_L I^{m-c}) = 0 \quad \text{for all } m \geq k, \]

which gives

\[ J^m L_n \cap H^0_J(L_n) = 0 \quad \text{for all } n \in \mathbb{N}^t \text{ and } m \geq k. \]

Now we are aiming to obtain some invariant of multigraded module with the help of the following lemma:

**Lemma 5.2.2.** Let $R$ be a standard $\mathbb{N}^t$-graded ring and $L$ an $\mathbb{N}^t$-graded $R$-module finitely generated in degrees $\leq \underline{n}$. Set $A := R_0$. Then we have the following:
5.2 Preliminaries on Multigraded Modules

(i) For every $v \geq u$, $\text{Ann}_A(L_n) \subseteq \text{Ann}_A(L_m)$ for all $n \geq v$, and hence $\dim_A(L_n) \geq \dim_A(L_m)$ for all $n \geq v$.

(ii) There exists $v \in \mathbb{N}^t$ such that $\text{Ann}_A(L_n) = \text{Ann}_A(L_v)$ for all $n \geq v$, and hence $\dim_A(L_n) = \dim_A(L_v)$ for all $n \geq v$.

Proof. (i) Let $v \geq u$. Since $R$ is standard and $L$ is an $\mathbb{N}^t$-graded $R$-module finitely generated in degrees $\leq u$, for every $n \geq v$ ($\geq u$), we have

$$L_n = R^{n_1-v_1}e_1 R^{n_2-v_2}e_2 \cdots R^{n_t-v_t}e_t L_v,$$

which gives $\text{Ann}_A(L_w) \subseteq \text{Ann}_A(L_n)$, and hence $\dim_A(L_n) \geq \dim_A(L_m)$ for all $n \geq v$.

(ii) Consider $C := \{\text{Ann}_A(L_n) : n \geq u\}$, a collection of ideals of $A$. Since $A$ is Noetherian, $C$ has a maximal element $\text{Ann}_A(L_w)$, say. Then, by part (i), it follows that $\text{Ann}_A(L_n) = \text{Ann}_A(L_w)$ for all $n \geq v$, and hence $\dim_A(L_n) = \dim_A(L_v)$ for all $n \geq v$. \qed

Let us introduce the following invariant of multigraded module on which we apply induction to prove our main result.

Definition 5.2.3. Let $R$ be a standard $\mathbb{N}^t$-graded ring and $L$ a finitely generated $\mathbb{N}^t$-graded $R$-module. We call $v \in \mathbb{N}^t$ an annihilator stable point of $L$ if

$$\text{Ann}_{R_w}(L_n) = \text{Ann}_{R_v}(L_n) \quad \text{for all} \quad n \geq v.$$

In this case, we call $s := \dim_{R_v}(L_v)$ the saturated dimension of $L$.

Remark 5.2.4. Existence of an annihilator stable point of $L$ (with the hypothesis given in the Definition 5.2.3) follows from Lemma 5.2.2(ii). Let $v, w \in \mathbb{N}^t$ be two annihilator stable points of $L$, i.e.,

$$\text{Ann}_{R_w}(L_n) = \text{Ann}_{R_v}(L_n) \quad \text{for all} \quad n \geq v$$

and

$$\text{Ann}_{R_w}(L_n) = \text{Ann}_{R_w}(L_n) \quad \text{for all} \quad n \geq w.$$

If we denote $\dim_{R_v}(L_v)$ and $\dim_{R_w}(L_w)$ by $s(v)$ and $s(w)$ respectively, then observe that $s(v) = s(w)$. Thus the saturated dimension of $L$ is well-defined.

Let us recall the following result from [Wes04, Lemma 3.3].
Lemma 5.2.5. Let $R$ be a standard $\mathbb{N}^t$-graded ring, and let $L$ be a finitely generated $\mathbb{N}^t$-graded $R$-module. For any fixed integers $1 \leq i \leq t$ and $\lambda \in \mathbb{N}$, set

$$S_i := \bigoplus_{\{\underline{n} \in \mathbb{N}^t : n_i = 0\}} R_{\underline{n}} \quad \text{and} \quad M_{i\lambda} := \bigoplus_{\{\underline{n} \in \mathbb{N}^t : n_i = \lambda\}} L_{\underline{n}}.$$ 

Then $S_i$ is a Noetherian standard $\mathbb{N}^{t-1}$-graded ring and $M_{i\lambda}$ is a finitely generated $\mathbb{N}^{t-1}$-graded $S_i$-module.

Discussion 5.2.6. Let

$$R = \bigoplus_{(\underline{n},i) \in \mathbb{N}^{t+1}} R_{(\underline{n},i)} \quad \text{and} \quad L = \bigoplus_{(\underline{n},i) \in \mathbb{N}^{t+1}} L_{(\underline{n},i)}$$

be an $\mathbb{N}^{t+1}$-graded ring and a finitely generated $\mathbb{N}^{t+1}$-graded $R$-module respectively. For every $\underline{n} \in \mathbb{N}^t$, we set

$$R_{(\underline{n},\star)} := \bigoplus_{i \in \mathbb{N}} R_{(\underline{n},i)} \quad \text{and} \quad L_{(\underline{n},\star)} := \bigoplus_{i \in \mathbb{N}} L_{(\underline{n},i)}.$$ 

We give $\mathbb{N}^t$-grading structures on

$$R = \bigoplus_{\underline{n} \in \mathbb{N}^t} R_{(\underline{n},\star)} \quad \text{and} \quad L = \bigoplus_{\underline{n} \in \mathbb{N}^t} L_{(\underline{n},\star)}$$

in the obvious way, i.e., by setting $R_{(\underline{n},\star)}$ and $L_{(\underline{n},\star)}$ as the $\underline{n}$th graded components of $R$ and $L$ respectively. Then clearly, for any $\underline{m}, \underline{n} \in \mathbb{N}^t$, we have

$$R_{(\underline{m},\star)} \cdot R_{(\underline{n},\star)} \subseteq R_{(\underline{m}+\underline{n},\star)} \quad \text{and} \quad R_{(\underline{m},\star)} \cdot L_{(\underline{n},\star)} \subseteq L_{(\underline{m}+\underline{n},\star)}.$$ 

Thus $R$ is an $\mathbb{N}^t$-graded ring and $L$ is an $\mathbb{N}^t$-graded $R$-module. Since we are changing only the grading, $R$ is anyway Noetherian. Since $L$ is finitely generated $\mathbb{N}^{t+1}$-graded $R$-module, it is just an observation that $L = \bigoplus_{\underline{n} \in \mathbb{N}^t} L_{(\underline{n},\star)}$ is finitely generated as $\mathbb{N}^t$-graded $R$-module. Now we set $A := R_{(\underline{0},\star)}$. Note that $A$ is a Noetherian $\mathbb{N}$-graded ring, and for every $\underline{n} \in \mathbb{N}^t$, $R_{(\underline{n},\star)}$ and $L_{(\underline{n},\star)}$ are finitely generated $\mathbb{N}$-graded $A$-modules.

We are going to refer the following hypothesis repeatedly in the rest of the present chapter.

Hypothesis 5.2.7. Let

$$R = \bigoplus_{(\underline{n},i) \in \mathbb{N}^{t+1}} R_{(\underline{n},i)}$$
be an \( \mathbb{N}^{t+1} \)-graded ring, which need not be standard. Let
\[
L = \bigoplus_{(\underline{n}, i) \in \mathbb{N}^{t+1}} L_{(\underline{n}, i)}
\]
be a finitely generated \( \mathbb{N}^{t+1} \)-graded \( R \)-module. For every \( \underline{n} \in \mathbb{N}^t \), we set
\[
R_{(\underline{n}, \star)} := \bigoplus_{i \in \mathbb{N}} R_{(\underline{n}, i)} \quad \text{and} \quad L_{(\underline{n}, \star)} := \bigoplus_{i \in \mathbb{N}} L_{(\underline{n}, i)}.
\]
Also set \( A := R_{(\underline{0}, \star)} \). Suppose \( R = \bigoplus_{n \in \mathbb{N}^t} R_{(n, \star)} \) and \( A = R_{(\underline{0}, \star)} \) are standard as \( \mathbb{N}^t \)-graded ring and \( \mathbb{N} \)-graded ring respectively, i.e.,
\[
R = R_{(\underline{0}, \star)}[R_{(n_1^1, \star)}, R_{(n_2^2, \star)}, \ldots, R_{(n_t^t, \star)}] \quad \text{and} \quad R_{(\underline{0}, \star)} = R_{(\underline{0}, 0)}[R_{(\underline{0}, 1)}].
\]
Assume \( A_0 = R_{(\underline{0}, 0)} \) is Artinian local with the maximal ideal \( \mathfrak{m} \). Since \( A \) is a Noetherian standard \( \mathbb{N} \)-graded ring, we assume that \( A = A_0[x_1, \ldots, x_d] \) for some \( x_1, \ldots, x_d \in A_1 \). Let \( A_+ = (x_1, \ldots, x_d) \).

With the Hypothesis 5.2.7 in view of Discussion 5.2.6, we have the following:

1. \( R = \bigoplus_{(\underline{n}, i) \in \mathbb{N}^{t+1}} R_{(\underline{n}, i)} \) is not necessarily standard as an \( \mathbb{N}^{t+1} \)-graded ring.
2. \( R = \bigoplus_{\underline{n} \in \mathbb{N}^t} R_{(\underline{n}, \star)} \) is a standard \( \mathbb{N}^t \)-graded ring.
3. \( A = R_{(\underline{0}, \star)} \) is a standard \( \mathbb{N} \)-graded ring.
4. \( L = \bigoplus_{\underline{n} \in \mathbb{N}^t} L_{(\underline{n}, \star)} \) is a finitely generated \( \mathbb{N}^t \)-graded \( R \)-module.
5. For every \( \underline{n} \in \mathbb{N}^t \), \( R_{(\underline{n}, \star)} \) and \( L_{(\underline{n}, \star)} \) are finitely generated \( \mathbb{N} \)-graded \( A \)-modules.

We now give two examples which satisfy the Hypothesis 5.2.7.

**Example 5.2.8.** Let \( A \) be a standard \( \mathbb{N} \)-graded algebra over an Artinian local ring \( A_0 \). Let \( I_1, \ldots, I_t \) be homogeneous ideals of \( A \), and let \( M \) be a finitely generated \( \mathbb{N} \)-graded \( A \)-module. Let \( R = A[I_1T_1, \ldots, I_tT_t] \) be the Rees algebra of \( I_1, \ldots, I_t \) over the graded ring \( A \), and let \( L = M[I_1T_1, \ldots, I_tT_t] \) be the Rees module of \( M \) with respect to the ideals \( I_1, \ldots, I_t \). We give \( \mathbb{N}^{t+1} \)-grading structures on \( R \) and \( L \) by setting \( (\underline{n}, i) \)th graded components of \( R \) and \( L \) as the \( i \)th graded components of the \( \mathbb{N} \)-graded \( A \)-modules \( I_1^{n_1} \cdots I_t^{n_t} A \) and \( I_1^{n_1} \cdots I_t^{n_t} M \) respectively. Then clearly, \( R \) is an \( \mathbb{N}^{t+1} \)-graded ring, and \( L \) is a finitely
generated $\mathbb{N}^{t+1}$-graded $R$-module. Note that $R$ is not necessarily standard as an $\mathbb{N}^{t+1}$-graded ring. Also note that for every $n \in \mathbb{N}^{t}$, we have
\begin{align*}
R(e_i, \bullet) &= I_i \quad \text{for all } 1 \leq i \leq t.
\end{align*}
Therefore $R = A[I_1 T_1, \ldots, I_t T_t] \cong R(0, \bullet) \oplus R(e_1, \bullet) \oplus \cdots \oplus R(e_t, \bullet)$, and hence $R$ is standard as an $\mathbb{N}^{t}$-graded ring. Thus $R$ and $L$ are satisfying the Hypothesis 5.2.7.

Let us recall the definition of the integral closure of an ideal.

**Definition 5.2.9.** Let $I$ be an ideal of a ring $A$. An element $r \in A$ is said to be integral over $I$ if there exist an integer $n$ and elements $a_i \in I_i$, $i = 1, \ldots, n$ such that
\begin{align*}
r^n + a_1 r^{n-1} + a_2 r^{n-2} + \cdots + a_{n-1} r + a_n = 0.
\end{align*}
The set of all elements that are integral over $I$ is called the integral closure of $I$, and is denoted $\overline{I}$.

Here is another example satisfying the Hypothesis 5.2.7.

**Example 5.2.10.** Let $A, I_1, \ldots, I_t$ and $M$ be as in Example 5.2.8. Let $A_0$ be a field. Set $R := A[I_1 T_1, \ldots, I_t T_t]$ as above. Here we set
\begin{align*}
L := \bigoplus_{n \in \mathbb{N}^t} \left( \overline{I_1^n} \cdots \overline{I_t^n} M \right) T_1^{m_1} \cdots T_t^{m_t}.
\end{align*}
We give $\mathbb{N}^{t+1}$-grading structure on $L$ by setting the $n$th graded component of $L$ as the $n$th graded component of the $\mathbb{N}$-graded $A$-module $\overline{I_1^n} \cdots \overline{I_t^n} M$. For a homogeneous ideal $I$ of $A$, since
\begin{align*}
I^m \overline{I^n} \subseteq \overline{I^m} \overline{I^n} \subseteq \overline{I^{m+n}}
\end{align*}
for all $m, n \in \mathbb{N}$, $L$ is an $\mathbb{N}^{t+1}$-graded $R$-module. Again for a homogeneous ideal $I$ of $A$, there exists an integer $n_0$ such that for all $n \geq n_0$, we have $\overline{I^n} = I^{n-n_0} \overline{I^{n_0}}$ (see [SH06, 5.3.4(2)]). Therefore $L$ is a finitely generated $\mathbb{N}^{t+1}$-graded $R$-module. Hence in this case also $R$ and $L$ are satisfying the Hypothesis 5.2.7.

From now onwards, by $R$ and $L$, we mean $\mathbb{N}^{t}$-graded ring $\bigoplus_{n \in \mathbb{N}^t} R(\omega, \bullet)$ and $\mathbb{N}^{t}$-graded $R$-module $\bigoplus_{n \in \mathbb{N}^t} L(\omega, \bullet)$ (satisfying the Hypothesis 5.2.7) respectively.
5.3 Linear Bounds of Regularity

In this section, we are aiming to prove that the regularity of $L_{(\underline{n},*)}$ as an $\mathbb{N}$-graded $A$-module is bounded by a linear function of $\underline{n}$ by using induction on the saturated dimension of the $\mathbb{N}^t$-graded $R$-module $L$. Here is the base case.

**Theorem 5.3.1.** With the Hypothesis 5.2.7 let $L = \bigoplus_{\underline{n} \in \mathbb{N}} L_{(\underline{n},*)}$ be generated in degrees $\leq \underline{u}$. If $\dim_A(L_{(\underline{v},*)}) = 0$ for some $\underline{v} \geq \underline{u}$, then there exists an integer $k$ such that

$$\text{reg}(L_{(\underline{n},*)}) < |\underline{n} - \underline{u}|k + k \quad \text{for all} \quad \underline{n} \geq \underline{v}.$$  

**Proof.** Let $\dim_A(L_{(\underline{v},*)}) = 0$ for some $\underline{v} \geq \underline{u}$. Then, by virtue of Lemma 5.2.2(i), we obtain $\dim_A(L_{(\underline{n},*)}) = 0$ for all $\underline{n} \geq \underline{v}$.

In view of Grothendieck’s Vanishing Theorem ([BS13, 6.1.2]), we get

$$H^i_{A_+}(L_{(\underline{n},*)}) = 0 \quad \text{for all} \quad i > 0 \text{ and } \underline{n} \geq \underline{v}.$$  

Therefore in this case, we have

$$\text{reg}(L_{(\underline{n},*)}) = \max \{ \mu : H^0_{A_+}(L_{(\underline{n},*)})_{\mu} \neq 0 \} \quad \text{for all} \quad \underline{n} \geq \underline{v}. \quad (5.3.1)$$

Now consider the finite collection

$$\mathcal{D} := \{ R_{(e^1,*)}, R_{(e^2,*)}, \ldots, R_{(e^t,*)}, L_{(\underline{u},*)} \}.$$  

Since every member of $\mathcal{D}$ is a finitely generated $\mathbb{N}$-graded $A = A_0[x_1, \ldots, x_d]$-module, we may assume that every member of $\mathcal{D}$ is generated in degrees $\leq k_1$ for some $k_1 \in \mathbb{N}$. Since $L$ is a finitely generated $\mathbb{N}^t$-graded $R$-module and $A_+$ is an ideal of $A (= R_{(\underline{u},*)})$, in view of Lemma 5.2.1, there exists a positive integer $k_2$ such that

$$(A_+)^{k_2}L_{(\underline{n},*)} \cap H^0_{A_+}(L_{(\underline{n},*)}) = 0 \quad \text{for all} \quad \underline{n} \in \mathbb{N}^t. \quad (5.3.2)$$

Now set $k := k_1 + k_2$. We claim that

$$H^0_{A_+}(L_{(\underline{n},*)})_{\mu} = 0 \quad \text{for all} \quad \underline{n} \geq \underline{v} \quad \text{and} \quad \mu \geq |\underline{n} - \underline{u}|k + k. \quad (5.3.3)$$

To show (5.3.3), fix $\underline{n} \geq \underline{v}$ and $\mu \geq |\underline{n} - \underline{u}|k + k$. Assume $X \in H^0_{A_+}(L_{(\underline{n},*)})_{\mu}$. Note that the homogeneous (with respect to $\mathbb{N}$-grading over $A$) element $X$ of

$$L_{(\underline{n},*)} = R_{(e^1,*)}^{n_1-n_1}R_{(e^2,*)}^{n_2-n_2} \cdots R_{(e^t,*)}^{n_t-n_t}L_{(\underline{u},*)}$$

can be written as a finite sum of elements of the following type:

\((r_{11} r_{12} \cdots r_{1 n_1-u_1})(r_{21} r_{22} \cdots r_{2 n_2-u_2}) \cdots (r_{t1} r_{t2} \cdots r_{t n_t-u_t}) Y\)

for some homogeneous (with respect to \(\mathbb{N}\)-grading over \(A\)) elements

\(r_{i1}, r_{i2}, \ldots, r_{i n_i-u_i} \in R_{(g^i,*)}\) for all \(1 \leq i \leq t\), and \(Y \in L_{(\underline{w},*)}\).

Considering the homogeneous degree with respect to \(\mathbb{N}\)-grading over \(A\), we have

\[\text{deg}(Y) + \sum_{i=1}^{t} \{\text{deg}(r_{i1}) + \text{deg}(r_{i2}) + \cdots + \text{deg}(r_{i n_i-u_i})\} = \mu \geq |\underline{n} - \underline{u}| k + k,\]

which gives at least one of the elements

\(r_{11}, r_{12}, \ldots, r_{1 n_1-u_1}; \ldots; r_{t1}, r_{t2}, \ldots, r_{t n_t-u_t}\) and \(Y\)

is of degree \(\geq k\). In first case, we consider \(\text{deg}(r_{ij}) \geq k\) for some \(i, j\). Since \(R_{(g^i,*)}\) is an \(\mathbb{N}\)-graded \(A\)-module generated in degrees \(\leq k_1\), we have

\[r_{ij} \in (R_{(g^i,*)})_{\text{deg}(r_{ij})} = (A_1)^{\text{deg}(r_{ij})-k_1} (R_{(g^i,*)})_{k_1} \subseteq (A_+)^{k_2} L_{(\underline{w},*)} \quad [\text{as } \text{deg}(r_{ij}) - k_1 \geq k - k_1 = k_2].\]

In another case, we consider \(\text{deg}(Y) \geq k\). In this case also, since \(L_{(\underline{w},*)}\) is an \(\mathbb{N}\)-graded \(A\)-module generated in degrees \(\leq k_1\), we have

\[Y \in (L_{(\underline{w},*)})_{\text{deg}(Y)} = (A_1)^{\text{deg}(Y)-k_1} (L_{(\underline{w},*)})_{k_1} \subseteq (A_+)^{k_2} L_{(\underline{w},*)} \quad [\text{as } \text{deg}(Y) - k_1 \geq k - k_1 = k_2].\]

In both cases, the typical element \((r_{11} r_{12} \cdots r_{1 n_1-u_1}) \cdots (r_{t1} r_{t2} \cdots r_{t n_t-u_t}) Y\) is in

\[\begin{align*}
(A_+)^{k_2} R_{g_1}^{n_1-u_1} R_{g_2}^{n_2-u_2} \cdots R_{g_t}^{n_t-u_t} L_{(\underline{w},*)} = (A_+)^{k_2} L_{(\underline{w},*)},
\end{align*}\]

and hence \(X \in (A_+)^{k_2} L_{(\underline{w},*)}\). Therefore

\[X \in (A_+)^{k_2} L_{(\underline{w},*)} \cap H^0_{A_+} (L_{(\underline{w},*)}),\]

which gives \(X = 0\) by [5.3.2]. Thus we have

\[H^0_{A_+} (L_{(\underline{w},*)})_\mu = 0 \quad \text{for all } \underline{n} \geq \underline{v} \quad \text{and} \quad \mu \geq |\underline{n} - \underline{u}| k + k,\]

and hence the theorem follows from [5.3.1].
5.3 Linear Bounds of Regularity

Now we give the inductive step to prove the following linear boundedness result.

**Theorem 5.3.2.** With the Hypothesis 5.2.7 there exist \( u \in \mathbb{N}^t \) and an integer \( k \) such that

\[
\operatorname{reg}(L_{(u, \star)}) < |u|k + k \quad \text{for all} \quad n \geq u.
\]

In particular, if \( t = 1 \), then there exist two integers \( k \) and \( k' \) such that

\[
\operatorname{reg}(L_{(n, \star)}) \leq nk + k' \quad \text{for all} \quad n \in \mathbb{N}.
\]

**Proof.** Let \( v \in \mathbb{N}^t \) be an annihilator stable point of \( L \) and \( s \) the saturated dimension of \( L \). Without loss of generality, we may assume that \( L \) is finitely generated as \( R \)-module in degrees \( \leq v \). We prove the theorem by induction on \( s \). If \( s = 0 \), then the theorem follows from Theorem 5.3.1 by taking \( u := v \). Therefore we may as well assume that \( s \geq 1 \) and the theorem holds true for all such finitely generated \( \mathbb{N}^t \)-graded \( R \)-modules of saturated dimension \( \leq s - 1 \).

Let \( n := m \oplus A_+ \) be the maximal homogeneous ideal of \( A \). We claim that

\[
n \notin \operatorname{Min} \left( \frac{A}{\operatorname{Ann}_A(L_{(v, \star)})} \right).
\]

Since the collection of all minimal prime ideals of \( A \) containing \( \operatorname{Ann}_A(L_{(v, \star)}) \) are associated prime ideals of \( \frac{A}{\operatorname{Ann}_A(L_{(v, \star)})} \), they are homogeneous, and hence they must be contained in \( n \). Thus if the above claim is not true, then we have

\[
\operatorname{Min} \left( \frac{A}{\operatorname{Ann}_A(L_{(v, \star)})} \right) = \{ n \},
\]

and hence \( s = \dim_A(L_{(v, \star)}) = 0 \), which is a contradiction. Therefore the above claim is true, and hence by Prime Avoidance Lemma and using the fact that \( n \) is the only homogeneous prime ideal of \( A \) containing \( A_+ \) (as \( (A_0, m) \) is Artinian local), we have

\[
A_+ \notin \bigcup \left\{ P : P \in \operatorname{Min} \left( \frac{A}{\operatorname{Ann}_A(L_{(v, \star)})} \right) \right\}.
\]

Then, by the Graded Version of Prime Avoidance Lemma, we may choose a homogeneous element \( x \) in \( A \) of positive degree such that

\[
x \notin \bigcup \left\{ P : P \in \operatorname{Min} \left( \frac{A}{\operatorname{Ann}_A(L_{(v, \star)})} \right) \right\}.
\]

Since \( v \) is an annihilator stable point of \( L \) and \( s \) is the saturated dimension of \( L \) (Definition 5.2.3), we have that \( \operatorname{Ann}_A(L_{(v, \star)}) = \operatorname{Ann}_A(L_{(u, \star)}) \) for all \( n \geq v \), and \( \dim_A(L_{(v, \star)}) = s \).
Therefore, for all \( n \geq v \), we obtain that
\[
\dim_A \left( L_{(n, \star)} / xL_{(n, \star)} \right), \quad \dim_A \left( 0 : L_{(n, \star)} x \right) \leq \dim_A( L_{(n, \star)} ) - 1 = s - 1
\]
as \( \text{Ann}_A \left( L_{(n, \star)} / xL_{(n, \star)} \right), \quad \text{Ann}_A \left( 0 : L_{(n, \star)} x \right) \supseteq \langle \text{Ann}_A( L_{(n, \star)} ), x \rangle.
\]

Now observe that \( L/xL \) and \( (0 : L x) \) are finitely generated \( \mathbb{N}^t \)-graded \( R \)-modules with saturated dimensions \( \leq s - 1 \). Therefore, by induction hypothesis, there exist \( w \) and \( w' \) in \( \mathbb{N}^t \) and two integers \( k_1, k_2 \) such that
\[
\text{reg} \left( L_{(n, \star)} / xL_{(n, \star)} \right) < |n| k_1 + k_1 \quad \text{for all} \quad n \geq w,
\]
and \( \text{reg} \left( 0 : L_{(n, \star)} x \right) < |n| k_2 + k_2 \quad \text{for all} \quad n \geq w'\).

Set \( k := \max \{ k_1, k_2 \} \) and \( u := \max \{ w, w' \} \) (i.e., \( u_i := \max \{ w_i, w'_i \} \) for all \( 1 \leq i \leq t \), and \( u := (u_1, \ldots, u_t) \)). Then, by Lemma 5.1.2, we have
\[
\text{reg}( L_{(n, \star)} ) \leq \max \left\{ \text{reg} \left( L_{(n, \star)} / xL_{(n, \star)} \right), \text{reg} \left( 0 : L_{(n, \star)} x \right) \right\}
\]
\[
< |u| k + k \quad \text{for all} \quad n \geq u.
\]
This completes the proof of the first part of the theorem.

To prove the second part, assume \( t = 1 \). Then, from the first part, there exist \( u \in \mathbb{N} \) and an integer \( k \) such that
\[
\text{reg}( L_{(n, \star)} ) < nk + k \quad \text{for all} \quad n \geq u.
\]
Set \( k' := \max \{ k, \text{reg}( L_{(n, \star)} ) : 0 \leq n \leq u - 1 \} \). Then clearly, we have
\[
\text{reg}( L_{(n, \star)} ) \leq nk + k' \quad \text{for all} \quad n \in \mathbb{N},
\]
which completes the proof of the theorem.

Above theorem gives the result that \( \text{reg}( L_{(n, \star)} ) \) has linear bound for all \( n \geq u \), for some \( u \in \mathbb{N}^t \). Now we prove the result for all \( n \in \mathbb{N}^t \).

**Theorem 5.3.3.** With the Hypothesis 5.2.7, there exist two integers \( k \) and \( k' \) such that
\[
\text{reg} \left( L_{(n, \star)} \right) \leq (n_1 + \cdots + n_t) k + k' \quad \text{for all} \quad n \in \mathbb{N}^t.
\]

**Proof.** We prove the theorem by induction on \( t \). If \( t = 1 \), then the theorem follows from the second part of the Theorem 5.3.2. Therefore we may as well assume that \( t \geq 2 \) and the theorem holds true for \( t - 1 \).
By Theorem 5.3.2 there exist \( u \in \mathbb{N}' \) and an integer \( k_1 \) such that
\[
\operatorname{reg} (L(\underline{u}, \cdot)) < |\underline{u}|k_1 + k_1 \quad \text{for all } \underline{u} \geq \underline{u}.
\]
(5.3.4)

Now for each \( 1 \leq i \leq t \) and \( 0 \leq \lambda < u_i \), we set
\[
S_i := \bigoplus_{\underline{n} \in \mathbb{N}'_t, n_i = 0} R(\underline{n}, \cdot) \quad \text{and} \quad M_{i\lambda} := \bigoplus_{\underline{n} \in \mathbb{N}'_t, n_i = \lambda} L(\underline{n}, \cdot).
\]

Then, by Lemma 5.2.2, \( S_i \) is a standard \( \mathbb{N}^{t-1} \)-graded ring and \( M_{i\lambda} \) is a finitely generated \( \mathbb{N}^{t-1} \)-graded \( S_i \)-module. Therefore, by induction hypothesis, for each \( 1 \leq i \leq t \) and \( 0 \leq \lambda < u_i \), there exist two integers \( k_{i\lambda} \) and \( k'_{i\lambda} \) such that
\[
\operatorname{reg} (L(n_1, \ldots, n_{i-1}, \lambda, n_{i+1}, \ldots, n_t, \cdot)) \leq (n_1 + \cdots + n_{i-1} + n_{i+1} + \cdots + n_t)k_{i\lambda} + k'_{i\lambda}
\]
\[
= (n_1 + \cdots + n_{i-1} + \lambda + n_{i+1} + \cdots + n_t)k_{i\lambda} + k''_{i\lambda} \quad \text{for all } n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_t \in \mathbb{N},
\]
(5.3.5)

where \( k''_{i\lambda} = k'_{i\lambda} - \lambda k_{i\lambda} \). Now set
\[
k := \max \{k_1, k_{i\lambda} : 1 \leq i \leq t, 0 \leq \lambda < u_i\} \quad \text{and} \quad k' := \max \{k_1, k''_{i\lambda} : 1 \leq i \leq t, 0 \leq \lambda < u_i\}.
\]

We claim that
\[
\operatorname{reg} (L(\underline{u}, \cdot)) \leq |\underline{u}|k + k' \quad \text{for all } \underline{u} \in \mathbb{N}'^t.
\]
(5.3.6)

To prove (5.3.6), consider an arbitrary \( \underline{u} \in \mathbb{N}'^t \). If \( \underline{u} \geq \underline{u} \), then (5.3.6) follows from (5.3.4). Otherwise if \( \underline{u} \not\geq \underline{u} \), then we have \( n_i < u_i \) for at least one \( i \in \{1, \ldots, t\} \), and hence in this case, (5.3.6) holds true by (5.3.5). \( \square \)

Now we have arrived at the main goal of this chapter.

**Corollary 5.3.4.** Let \( A \) be a standard \( \mathbb{N} \)-graded algebra over an Artinian local ring \( A_0 \). Let \( I_1, \ldots, I_t \) be homogeneous ideals of \( A \), and let \( M \) be a finitely generated \( \mathbb{N} \)-graded \( A \)-module. Then there exist two integers \( k \) and \( k' \) such that
\[
\operatorname{reg}(I_1^n \cdots I_t^n M) \leq (n_1 + \cdots + n_t)k + k' \quad \text{for all } n_1, \ldots, n_t \in \mathbb{N}.
\]

**Proof.** Let \( R = A[I_1T_1, \ldots, I_tT_t] \) be the Rees algebra of \( I_1, \ldots, I_t \) over the graded ring \( A \) and let \( L = M[I_1T_1, \ldots, I_tT_t] \) be the Rees module of \( M \) with respect to the ideals \( I_1, \ldots, I_t \). We give \( \mathbb{N}^{t+1} \)-grading structures on \( R \) and \( L \) by setting \((n, i)\)th graded components of \( R \) and \( L \) as the \( i \)th graded components of the \( \mathbb{N} \)-graded \( A \)-modules \( I_1^n \cdots I_t^n A \)
and \( I_1^{n_1} \cdots I_t^{n_t} M \) respectively. From Example 5.2.8 note that \( R \) and \( L \) are satisfying the Hypothesis 5.2.7, and in this case

\[
L_{(\underline{n}^*)} = I_1^{n_1} \cdots I_t^{n_t} M \quad \text{for all} \quad \underline{n} \in \mathbb{N}^t.
\]
Therefore the corollary follows from Theorem 5.3.3. \( \square \)

Here is another corollary of the Theorem 5.3.3

**Corollary 5.3.5.** Let \( A \) be a standard \( \mathbb{N} \)-graded algebra over a field \( A_0 \). Let \( I_1, \ldots, I_t \) be homogeneous ideals of \( A \), and let \( M \) be a finitely generated \( \mathbb{N} \)-graded \( A \)-module. Then there exist two integers \( k \) and \( k' \) such that

\[
\text{reg} \left( I_1^{n_1} \cdots I_t^{n_t} M \right) \leq (n_1 + \cdots + n_t)k + k' \quad \text{for all} \quad n_1, \ldots, n_t \in \mathbb{N}.
\]

**Proof.** We set \( R \) and \( L \) as in Example 5.2.10. From Example 5.2.10 note that \( R \) and \( L \) are satisfying the Hypothesis 5.2.7, and in this case

\[
L_{(\underline{n}^*)} = I_1^{n_1} \cdots I_t^{n_t} M \quad \text{for all} \quad \underline{n} \in \mathbb{N}^t.
\]
Hence the corollary follows from Theorem 5.3.3. \( \square \)

### 5.4 About Linearity of Regularity

In [CHT99] page 252, S. D. Cutkosky, J. Herzog and N. V. Trung remarked that over a polynomial ring \( S = k[X_1, \ldots, X_d] \) over a field \( k \), the asymptotic linearity of regularity holds for a collection of ideals, i.e.,

\[
\text{reg}(I_1^{n_1} \cdots I_t^{n_t}) = a_1n_1 + \cdots + a_t n_t + b \quad \text{for all} \quad n_1, \ldots, n_t \gg 0
\]

and for some constants \( a_1, \ldots, a_t, b \). However, their proof has a gap which we now describe.

**5.4.1.** In the proof of Theorem 3.4 in [CHT99], it is used repeatedly that “the maximum of finitely many linear functions is asymptotically linear”. But this is not true in general for functions of more than one variable (see Example 5.4.2). Therefore we cannot conclude that for every \( i \geq 0 \), \( \text{reg}_i(I_1^{n_1} \cdots I_t^{n_t}) \) is linear in \( (n_1, \ldots, n_t) \) for all sufficiently large \( n_1, \ldots, n_t \), where for a finitely generated \( \mathbb{N} \)-graded \( S \)-module \( N \), \( \text{reg}_i(N) \) is defined to be

\[
\text{reg}_i(N) := \sup\{n : \text{Tor}_i^S(N, k)_n \neq 0\} - i.
\]
5.4 About Linearity of Regularity

Even if \( \text{reg}_i(I_1^{n_1} \cdots I_t^{n_t}) \) is asymptotically linear for all \( i \geq 0 \), then also it is not clear whether

\[
\text{reg}(I_1^{n_1} \cdots I_t^{n_t}) = \max\{\text{reg}_i(I_1^{n_1} \cdots I_t^{n_t}) : i \geq 0\}
\]

is linear in \((n_1, \ldots, n_t)\) for all sufficiently large \( n_1, \ldots, n_t \).

Here is an example which shows that the maximum of finitely many linear functions need not be asymptotically linear.

**Example 5.4.2.** Set \( h(m, n) := \max\{2m + n, m + 3n\} \) for all \( m, n \in \mathbb{N} \). We claim that \( h(m, n) \) is not asymptotically a linear function in \((m, n)\).

If possible, assume that \( h(m, n) = am + bn + c \) for all \( m, n \gg 0 \), say for all \((m, n) \geq (u, v)\), where \( a, b \) and \( c \) are some constants.

Note that for every fixed \( n \in \mathbb{N} \), \( h(m, n) = 2m + n \) for all \( m \gg 0 \). Therefore for every fixed \( n \gg v \), we have \( am + bn + c = 2m + n \) for all \( m \gg 0 \), which implies that \( a = 2 \), and hence \( bn + c = n \). Thus for all \( n \gg v \), we have \( bn + c = n \), which gives \( b = 1 \) and \( c = 0 \). In this way, we have \( a = 2, b = 1 \) and \( c = 0 \).

Again for every fixed \( m \in \mathbb{N} \), \( h(m, n) = m + 3n \) for all \( n \gg 0 \). Therefore for every fixed \( m \gg u \), we have \( am + bn + c = m + 3n \) for all \( n \gg 0 \), which implies that \( b = 3 \), and hence \( am + c = m \). Thus for all \( m \gg u \), we have \( am + c = m \), which gives \( a = 1 \) and \( c = 0 \). In this way, we have \( a = 1, b = 3 \) and \( c = 0 \), which gives a contradiction. Therefore \( h(m, n) \) is not asymptotically a linear function in \((m, n)\).
Chapter 6

Characterizations of Regular Local Rings via Syzygy Modules

This chapter shows some instances where properties of a local ring are closely connected with the homological properties of a single module.

Let $A$ be a local ring with residue field $k$. The aim of this chapter is to prove that if a finite direct sum of syzygy modules of $k$ maps onto ‘a semidualizing module’ or ‘a non-zero maximal Cohen-Macaulay module of finite injective dimension’, then $A$ is regular (see Corollaries 6.2.2 and 6.2.4). We also obtain one new characterization of regular local rings, namely, the local ring $A$ is regular if and only if some syzygy module of $k$ has a non-zero direct summand of finite injective dimension; see Theorem 6.2.7. Moreover, this result has a dual companion.

We use the following notations throughout this chapter.

Notations 6.1. Throughout the present chapter, $A$ always denotes a local ring with maximal ideal $m$ and residue field $k$. Let $M$ be a finitely generated $A$-module. For a non-negative integer $n$, we denote $\Omega_n^A(M)$ (resp. $\Omega_{-n}^A(M)$) the $n$th syzygy (resp. cosyzygy) module of $M$. We denote a finite collection of non-negative integers by $\Lambda$.

The structure of this chapter is as follows. In Section 6.1 we provide some preliminaries on syzygy modules which we use in the next section. Finally, we prove our main results of this chapter in Section 6.2.
Chapter 6, Section 6.1

6.1 Preliminaries on Syzygy Modules

In the present section, we give some preliminaries which we use in order to prove our main results of this chapter. We start with the following lemma which gives a relation between the socle of the ring and the annihilator of the syzygy modules.

Lemma 6.1.1. With the Notation 6.1, for every positive integer \( n \), we have

\[
\operatorname{Soc}(A) \subseteq \operatorname{Ann}_A \left( \Omega^n_A(M) \right).
\]

In particular, if \( A \neq k \) (i.e., if \( \mathfrak{m} \neq 0 \)), then

\[
\operatorname{Soc}(A) \subseteq \operatorname{Ann}_A \left( \Omega^n_A(k) \right) \quad \text{for all } n \geq 0.
\]

Proof. Fix \( n \geq 1 \). If \( \Omega^n_A(M) = 0 \), then we are done. So we may assume \( \Omega^n_A(M) \neq 0 \). Consider the following commutative diagram in a minimal free resolution of \( M \):

\[
\cdots \rightarrow A^{b_n} \rightarrow A^{b_{n-1}} \rightarrow \cdots.
\]

Let \( a \in \operatorname{Soc}(A) \), i.e., \( am = 0 \). Suppose \( x \in \Omega^n_A(M) \). Since \( f \) is surjective, there exists \( y \in A^{b_n} \) such that \( f(y) = x \). Note that \( \delta(ay) = a\delta(y) = 0 \) as \( \delta(A^{b_n}) \subseteq \mathfrak{m}A^{b_{n-1}} \) and \( am = 0 \). Therefore \( g(ax) = g(f(ay)) = \delta(ay) = 0 \), which gives \( ax = 0 \) as \( g \) is injective. Thus \( \operatorname{Soc}(A) \subseteq \operatorname{Ann}_A \left( \Omega^n_A(M) \right) \).

For the last part, note that \( \operatorname{Soc}(A) \subseteq \mathfrak{m} = \operatorname{Ann}_A \left( \Omega^0_A(k) \right) \) if \( \mathfrak{m} \neq 0 \).

Let us recall the following well-known result initially obtained by Nagata.

Proposition 6.1.2. [Tak06, Corollary 5.3] Let \( x \in \mathfrak{m} \setminus \mathfrak{m}^2 \) be an \( A \)-regular element. Set \( (-) := (-) \otimes_A A/(x) \). Then we have

\[
\Omega^n_A(k) \cong \Omega^n_A(k) \oplus \Omega^{n-1}_A(k) \quad \text{for every integer } n \geq 1.
\]

We notice two properties satisfied by semidualizing modules (see Definition 1.4.2) and maximal Cohen-Macaulay modules of finite injective dimension.

Definition 6.1.3. Let \( \mathcal{P} \) be a property of modules over local rings. We say that \( \mathcal{P} \) is a \((\ast)\)-property if \( \mathcal{P} \) satisfies the following:
(i) An $A$-module $M$ satisfies $\mathcal{P}$ implies that the $A/(x)$-module $M/xM$ satisfies $\mathcal{P}$, where $x \in A$ is an $A$-regular element.

(ii) An $A$-module $M$ satisfies $\mathcal{P}$ and $\text{depth}(A) = 0$ together imply that $\text{Ann}_A(M) = 0$.

Now we give a few examples of ($\ast$)-properties.

**Example 6.1.4.** The property $\mathcal{P}_1 := \text{‘semidualizing modules over local rings’}$ is a ($\ast$)-property.

**Proof.** Let $C$ be a semidualizing $A$-module. It is shown in [Gol84 page 68] that $C/xC$ is a semidualizing $A/(x)$-module, where $x \in \mathfrak{m}$ is an $A$-regular element. Since $\text{Hom}_A(C, C) \cong A$, we have $\text{Ann}_A(C) = 0$ (without any restriction on $\text{depth}(A)$).

Here is another example of ($\ast$)-property.

**Example 6.1.5.** The property $\mathcal{P}_2 := \text{‘non-zero maximal Cohen-Macaulay modules of finite injective dimension over Cohen-Macaulay local rings’}$ is a ($\ast$)-property.

**Proof.** Let $A$ be a Cohen-Macaulay local ring, and let $L$ be a non-zero maximal Cohen-Macaulay $A$-module of finite injective dimension. Suppose $x \in A$ is an $A$-regular element. Since $L$ is a maximal Cohen-Macaulay $A$-module, $x$ is $L$-regular as well. Therefore $L/xL$ is a non-zero maximal Cohen-Macaulay module of finite injective dimension over the Cohen-Macaulay local ring $A/(x)$ (see [BH98 Corollary 3.1.15]).

Now further assume that $\text{depth}(A) = 0$. Then $A$ is an Artinian local ring, and $\text{injdim}_A(L) = \text{depth}(A) = 0$. Therefore, by [BH98 Theorem 3.2.8], we have that $L \cong E^r$, where $E$ is the injective hull of $k$, and $r = \text{rank}_k(\text{Hom}_A(k, L))$. It is well-known that $\text{Hom}_A(E, E) \cong A$ as $A$ is an Artinian local ring. Hence $\text{Ann}_A(L) = \text{Ann}_A(E) = 0$.

### 6.2 Characterizations of Regular Local Rings

Now we can achieve the aim of this chapter. First of all, we prove that if a finite direct sum of syzygy modules of the residue field maps onto a non-zero module satisfying a ($\ast$)-property, then the ring is regular.
Theorem 6.2.1. Assume that \( \mathcal{P} \) is a \((\ast)\)-property (see Definition 6.1.3). Let
\[
f : \bigoplus_{n \in \Lambda} (\Omega^A_n(k))^{j_n} \longrightarrow L \quad (j_n \geq 1 \text{ for each } n \in \Lambda)
\]
be a surjective \( A \)-module homomorphism, where \( L \neq 0 \) satisfies \( \mathcal{P} \). Then \( A \) is regular.

Proof. We prove the theorem by using induction on \( t := \text{depth}(A) \). Let us first assume that \( t = 0 \). If possible, let \( A \neq k \), i.e., \( m \neq 0 \). Since \( \text{depth}(A) = 0 \), we have \( \text{Soc}(A) \neq 0 \). But, by virtue of Lemma 6.1.1, we obtain
\[
\text{Soc}(A) \subseteq \bigcap_{n \in \Lambda} \text{Ann}_A (\Omega^A_n(k))^{j_n} \subseteq \text{Ann}_A(L) \quad [\text{as } f : \bigoplus_{n \in \Lambda} (\Omega^A_n(k))^{j_n} \longrightarrow L \text{ is surjective}]
\]
\[
= 0 \quad [\text{as } L \text{ satisfies } \mathcal{P} \text{ which is a } (\ast)\text{-property}].
\]
This gives a contradiction. Therefore \( A (= k) \) is a regular local ring.

Now we assume that \( t \geq 1 \). Suppose the theorem holds true for all such rings of depth smaller than \( t \). Since \( \text{depth}(A) \geq 1 \), there exists an element \( x \in m \setminus m^2 \) which is \( A \)-regular. We set \( \overline{(-)} := (-) \otimes_A A/(x) \). Clearly,
\[
\overline{f} : \bigoplus_{n \in \Lambda} (\Omega^A_n(k))^{j_n} \longrightarrow \overline{L}
\]
is a surjective \( \overline{A} \)-module homomorphism, where the \( \overline{A} \)-module \( \overline{L} \neq 0 \) satisfies \( \mathcal{P} \) as \( \mathcal{P} \) is a \((\ast)\)-property. Since \( x \in m \setminus m^2 \) is an \( A \)-regular element, by Proposition 6.1.2, we have
\[
\bigoplus_{n \in \Lambda} (\Omega^A_n(k))^{j_n} \cong \bigoplus_{n \in \Lambda} \left( \Omega^A_n(k) \oplus \Omega^A_{n-1}(k) \right)^{j_n} \quad [\text{by setting } \Omega^A_{-1}(k) := 0].
\]
Since \( \text{depth}(\overline{A}) = \text{depth}(A) - 1 \), by the induction hypothesis, \( \overline{A} \) is a regular local ring, and hence \( A \) is a regular local ring as \( x \in m \setminus m^2 \) is an \( A \)-regular element. \( \square \)

As a few applications of the above theorem, we obtain the following necessary and sufficient conditions for a local ring to be regular.

Corollary 6.2.2. Let
\[
f : \bigoplus_{n \in \Lambda} (\Omega^A_n(k))^{j_n} \longrightarrow L
\]
be a surjective \( A \)-module homomorphism, where \( L \) is a semidualizing \( A \)-module. Then \( A \) is regular.
Proof. The corollary follows from Theorem 6.2.1 and Example 6.1.4.

Remark 6.2.3. We can recover Theorem 1.4.3 (in particular, Theorem 1.4.1 because \( A \) itself is a semidualizing \( A \)-module) as a consequence of Corollary 6.2.2. In fact the above result is even stronger than Theorem 1.4.3.

Now we give a partial answer to Question 1.4.8.

Corollary 6.2.4. Let
\[
f : \bigoplus_{n \in \Lambda} (\Omega^A_n(k))^i \rightarrow L
\]
be a surjective \( A \)-module homomorphism, where \( L \neq 0 \) is a maximal Cohen-Macaulay \( A \)-module of finite injective dimension. Then \( A \) is regular.

Proof. Existence of a non-zero finitely generated \( A \)-module of finite injective dimension ensures that \( A \) is Cohen-Macaulay (see [BH98, Corollary 9.6.2 and Remark 9.6.4(a)(ii)] and [Rob87]). Therefore we may assume that \( A \) is a Cohen-Macaulay local ring. Then the corollary follows from Theorem 6.2.1 and Example 6.1.5.

Remark 6.2.5. It is clear from the above corollary that Question 1.4.8 has an affirmative answer for Artinian local rings.

Let \( A \) be a Cohen-Macaulay local ring. Recall that a maximal Cohen-Macaulay \( A \)-module \( \omega \) of type 1 and of finite injective dimension is called the canonical module of \( A \). It is well-known that the canonical module \( \omega \) of \( A \) is a semidualizing \( A \)-module; see, e.g., [BH98, Theorem 3.3.10]. So, both Corollary 6.2.2 and Corollary 6.2.4 yield the following result (independently) which strengthens Corollary 1.4.4.

Corollary 6.2.6. Let \((A, m, k)\) be a Cohen-Macaulay local ring with canonical module \( \omega \), and let
\[
f : \bigoplus_{n \in \Lambda} (\Omega^A_n(k))^i \rightarrow \omega
\]
be a surjective \( A \)-module homomorphism. Then \( A \) is regular.

Here we obtain one new characterization of regular local rings. The following characterization is based on the existence of a non-zero direct summand with finite injective dimension of some syzygy module of the residue field.

Theorem 6.2.7. The following statements are equivalent:
(1) $A$ is a regular local ring.

(2) $\Omega^A_n(k)$ has a non-zero direct summand of finite injective dimension for some non-negative integer $n$.

Proof. (1) $\Rightarrow$ (2). If $A$ is regular, then $\Omega^A_0(k) = k$ itself is a non-zero $A$-module of finite injective dimension. Hence the implication follows.

(2) $\Rightarrow$ (1). Without loss of generality, we may assume that $A$ is complete. Existence of a non-zero finitely generated $A$-module of finite injective dimension ensures that $A$ is Cohen-Macaulay. Therefore we may as well assume that $A$ is a Cohen-Macaulay complete local ring.

Suppose that $L$ is a non-zero direct summand of $\Omega^A_n(k)$ with $\text{injdim}_A(L)$ finite for some integer $n \geq 0$. We prove the implication by induction on $d := \dim(A)$. If $d = 0$, then the implication follows from Corollary 6.2.4.

Now we assume that $d \geq 1$. Suppose the implication holds true for all such rings of dimension smaller than $d$. Since $A$ is Cohen-Macaulay and $\dim(A) \geq 1$, there exists $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ which is $A$-regular. We set

$$(-) := (-) \otimes_A A/(x).$$

If $n = 0$, then the direct summand $L$ of $\Omega^A_0(k) (= k)$ must be equal to $k$, and hence $\text{injdim}_A(k)$ is finite, which gives $A$ is regular. Therefore we may assume that $n \geq 1$. Hence $x$ is $\Omega^A_n(k)$-regular. Since $L$ is a direct summand of $\Omega^A_n(k)$, $x$ is $L$-regular as well. This gives $\text{injdim}_A(L)$ is finite. Now we fix an indecomposable direct summand $L'$ of $L$. Then $\text{injdim}_A(L')$ is also finite. Note that the $\overline{A}$-module $\overline{L}$ is a direct summand of $\Omega^A_n(k)$. Hence $L'$ is an indecomposable direct summand of $\Omega^A_n(k)$. Since $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ is an $A$-regular element, by Proposition 6.1.2, we have

$$\overline{\Omega^A_n}(k) \cong \Omega^\overline{A}_n(k) \oplus \Omega^\overline{A}_{n-1}(k).$$

The uniqueness of Krull-Schmidt decomposition ([Lam01, Theorem (21.35)]) yields that $L'$ is isomorphic to a direct summand of $\overline{\Omega^A_n}(k)$ or $\Omega^\overline{A}_{n-1}(k)$. Since $\dim(\overline{A}) = \dim(A) - 1$, by the induction hypothesis, $\overline{A}$ is a regular local ring, and hence $A$ is a regular local ring as $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ is an $A$-regular element. \qed
Let $M$ be a finitely generated $A$-module. Consider the augmented minimal injective resolution of $M$:

$$0 \rightarrow M \xrightarrow{d^{-1}} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \rightarrow \cdots \rightarrow I^{n-1} \xrightarrow{d^{n-1}} I^n \rightarrow \cdots.$$ 

Recall that the $n$th cosyzgy module of $M$ is defined by

$$\Omega^A_{-n}(M) := \text{Image}(d^{n-1}) \quad \text{for all } n \geqslant 0.$$

The following result is dual to Theorem 6.2.7 which gives another characterization of regular local rings via cosyzgy modules of the residue field.

**Corollary 6.2.8.** The following statements are equivalent:

1. $A$ is a regular local ring.
2. Cosyzgy module $\Omega^A_{-n}(k)$ has a non-zero finitely generated direct summand of finite projective dimension for some integer $n \geqslant 0$.

**Proof.** (1) $\Rightarrow$ (2). If $A$ is regular, then $\Omega^A_0(k) (= k)$ has finite projective dimension. Hence the implication follows.

(2) $\Rightarrow$ (1). Without loss of generality, we may assume that $A$ is complete. Suppose

$$\Omega^A_{-n}(k) \cong P \oplus Q \quad \text{for some integer } n \geqslant 0,$$

where $P$ is a non-zero finitely generated $A$-module of finite projective dimension. Consider the following part of the minimal injective resolution of $k$:

$$0 \rightarrow k \rightarrow E \rightarrow E^{\mu_1} \rightarrow \cdots \rightarrow E^{\mu_{n-1}} \rightarrow \Omega^A_{-n}(k) \cong P \oplus Q \rightarrow 0,$$

where $E$ is the injective hull of $k$. Dualizing (6.2.1) with respect to $E$, and using

$$\text{Hom}_A(k, E) \cong k \quad \text{and} \quad \text{Hom}_A(E, E) \cong A$$

(cf. [BH98, Proposition 3.2.12(a) and Theorem 3.2.13(a)]), we get the following part of a minimal free resolution of $k$:

$$0 \rightarrow \text{Hom}_A(P, E) \oplus \text{Hom}_A(Q, E) \cong \Omega^A_{n}(k) \rightarrow A^{\mu_{n-1}} \rightarrow \cdots \rightarrow A^{\mu_1} \rightarrow A \rightarrow k \rightarrow 0.$$

Clearly, $\text{Hom}_A(P, E)$ is non-zero and of finite injective dimension as $P$ is non-zero and of finite projective dimension. Therefore the implication follows from Theorem 6.2.7. 

\[\square\]
Now we give an example to ensure that the existence of an injective homomorphism from a ‘special module’ to a finite direct sum of syzygy modules of the residue field does not necessarily imply that the ring is regular.

**Example 6.2.9.** Let \((A, \mathfrak{m}, k)\) be a Gorenstein local domain of dimension \(d\). Then \(\Omega^A_d(k)\) is a maximal Cohen-Macaulay \(A\)-module; see, e.g., [BH98, Exercise 1.3.7]. Since \(A\) is Gorenstein, \(\Omega^A_d(k)\) is a reflexive \(A\)-module (cf. [BH98, Theorem 3.3.10]), and hence it is torsion-free. Then, by mapping 1 to a non-zero element of \(\Omega^A_d(k)\), we get an injective \(A\)-module homomorphism

\[ f : A \rightarrow \Omega^A_d(k). \]

Note that \(\text{injdim}_A(A)\) is finite. But a Gorenstein local domain need not be a regular local ring.
Chapter 7

Summary and Conclusions

In this chapter, we give a brief summary of the work discussed in this dissertation. We also present a few open questions for possible further future work.

7.1 Asymptotic Prime Divisors Related to Derived Functors

Let $A$ be a ring, and let $I$ be an ideal of $A$. Let $M$ and $N$ be finitely generated $A$-modules. The main problem we study in Chapter 3 is the following:

**Question 7.I.** Is the set $\bigcup_{i \geq 0} \bigcup_{n \geq 1} \text{Ass}_A \left( \text{Ext}^i_A(M, N/I^nN) \right)$ finite?

We prove that Question 7.I has an affirmative answer for local complete intersection rings. Moreover, we show that if $A$ is a local complete intersection ring, then there are non-negative integers $n_0$ and $i_0$ such that for all $n \geq n_0$ and $i \geq i_0$, the set

$$\text{Ass}_A \left( \text{Ext}^i_A(M, N/I^nN) \right)$$

depends only on whether $i$ is even or odd.

Next rings to consider after complete intersection rings are Gorenstein rings. So one may ask Question 7.II for Gorenstein local rings.

In this dissertation, we prove the finiteness of the set of associate primes of a certain family of Ext-modules. It seems natural to ask the following counterpart for Tor-modules:

**Question 7.II.** Is the set $\bigcup_{i \geq 0} \bigcup_{n \geq 1} \text{Ass}_A \left( \text{Tor}^i_A(M, N/I^nN) \right)$ finite?
More particularly, one can ask the following:

**Question 7.III.** Is the set \( \bigcup_{i \geq 0} \Ass_A \left( \Tor^A_i(M, N) \right) \) finite?

If \( A \) is a regular local ring, then

\[
\Tor^A_i(M, N/I^nN) = 0 \quad \text{for all} \quad i > \dim(A) \quad \text{and for all} \quad n \geq 1,
\]

since \( \text{projdim}_A(M) \leq \dim(A) \). Therefore it follows from the result of D. Katz and E. West ([KW04, Corollary 3.5]) that Questions 7.II and 7.III have positive answers for regular local rings. But these two questions are open for local complete intersection rings.

### 7.2 Castelnuovo-Mumford Regularity of Powers of Several Ideals

In Chapter 5, we study the Castelnuovo-Mumford regularity of powers of several ideals. Let \( A = A_0[x_1, \ldots, x_d] \) be a standard \( \mathbb{N} \)-graded algebra over an Artinian local ring \((A_0, \mathfrak{m})\).

Let \( I_1, \ldots, I_t \) be homogeneous ideals of \( A \), and let \( M \) be a finitely generated \( \mathbb{N} \)-graded \( A \)-module. We show that there exist two integers \( k_1 \) and \( k'_1 \) such that

\[
\text{reg}(I_1^{n_1} \cdots I_t^{n_t} M) \leq (n_1 + \cdots + n_t)k_1 + k'_1 \quad \text{for all} \quad n_1, \ldots, n_t \in \mathbb{N}.
\]

Moreover, we prove that if \( A_0 \) is a field, then there exist two integers \( k_2 \) and \( k'_2 \) such that

\[
\text{reg}\left( \mathcal{T}_1^{n_1} \cdots \mathcal{T}_t^{n_t} M \right) \leq (n_1 + \cdots + n_t)k_2 + k'_2 \quad \text{for all} \quad n_1, \ldots, n_t \in \mathbb{N},
\]

where \( \mathcal{T} \) denotes the integral closure of an ideal \( I \) of \( A \).

Keeping the single ideal case (i.e., the result [TW05, Theorem 3.2] of N. V. Trung and H.-J. Wang) in mind, we are now interested in the following question:

**Question 7.IV.** Are there some integers \( a_1, \ldots, a_t \) and \( b \) such that

\[
\text{reg}(I_1^{n_1} \cdots I_t^{n_t} M) = a_1n_1 + \cdots + a_tn_t + b \quad \text{for all} \quad n_1, \ldots, n_t \gg 0?
\]

We observe that the method of Trung and Wang does not generalize to the several ideals case. So we need to build some new techniques to solve the above question.

In [TW05, Corollary 3.4], Trung and Wang showed that if \( A \) is a standard \( \mathbb{N} \)-graded algebra over a field \( A_0 \), and if \( I \) is a homogeneous ideal of \( A \), then \( \text{reg}(\mathcal{T}^n) \) is asymptotically a linear function of \( n \). So one may ask now the following natural question for several ideals.
Question 7.V. Are there some integers $a_1, \ldots, a_t$ and $b$ such that

$$\text{reg} \left( \prod_{i=1}^{t} n_i I_i^m M \right) = a_1 n_1 + \cdots + a_t n_t + b \quad \text{for all } n_1, \ldots, n_t \gg 0?$$

Another question which seems to be interesting is the following:

Question 7.VI. What is the behaviour of $\text{reg} \left( \prod_{i=1}^{t} n_i I_i^m M \right)$ as a function of $(n_1, \ldots, n_t)$?

7.3 Characterizations of Regular Rings via Syzygy Modules

In Chapter 6, we obtain a few necessary and sufficient conditions for a local ring to be regular in terms of syzygy modules of the residue field. Let $A$ be a local ring with residue field $k$. We show that if a finite direct sum of syzygy modules of $k$ maps onto a semidualizing $A$-module, then $A$ is regular.

In [Mar96, Proposition 7], A. Martsinkovsky proved that if a finite direct sum of syzygy modules of $k$ maps onto a non-zero $A$-module of finite projective dimension, then $A$ is regular. So it seems natural to ask the following counterpart for injective dimension:

Question 7.VII. If a finite direct sum of syzygy modules of $k$ maps onto a non-zero $A$-module of finite injective dimension, then is the ring $A$ regular?

This is not known in general. However, we give a partial answer to this question by proving that if a finite direct sum of syzygy modules of $k$ maps onto a non-zero maximal Cohen-Macaulay $A$-module of finite injective dimension, then $A$ is regular. We also obtain one new characterization of regular local rings, namely, the local ring $A$ is regular if and only if some syzygy module of $k$ has a non-zero direct summand of finite injective dimension.
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List of Publications

Published Papers in Journals

(1) Dipankar Ghosh, *Asymptotic linear bounds of Castelnuovo-Mumford regularity in multigraded modules*, J. Algebra **445** (2016), 103–114.

(2) Dipankar Ghosh and Tony J. Puthenpurakal, *Asymptotic prime divisors over complete intersection rings*, Math. Proc. Cambridge Philos. Soc. **160** (2016), 423–436.

Accepted Paper in Journal

(3) Dipankar Ghosh, Anjan Gupta and Tony J. Puthenpurakal, *Characterizations of regular local rings via syzygy modules of the residue field*, To appear in Journal of Commutative Algebra.
Acknowledgements

It is a great pleasure to express my sincere gratitude to all those who contributed in many ways to the successful completion of this dissertation.

First and foremost, I would like to express my heartfelt gratitude to my supervisor Prof. Tony J. Puthenpurakal for his constant support, motivation and encouragement which helped me to grow as a researcher. Under Prof. Puthenpurakal’s guidance, the journey towards completion of this dissertation has become a fulfilling, enjoyable and unforgettable experience.

Besides my supervisor, it is a great pleasure for me to thank the rest of my research progress committee members, Prof. Jugal K. Verma and Prof. Ananthnarayan Hariharan, for their insightful comments, suggestions and encouragement during the entire course of my Ph.D journey. Their advice on both research as well as on my career have been priceless. My Ph.D. career has been blessed with beneficial courses taught by Prof. Ameer Athavale, Prof. Shripad M. Garge, Prof. Sudhir R. Ghorpade, Prof. Ananthnarayan Hariharan, Prof. Manoj K. Keshari, Prof. Ravi Raghunathan, Prof. Gopala K. Srinivasan, Prof. Tony J. Puthenpurakal and Prof. Jugal K. Verma.

I would like to express my gratefulness to the anonymous reviewers for their careful reading of this dissertation, and for giving many insightful comments and suggestions which helped me to improve the overall quality of the dissertation.

I express my sincere thanks and gratitude to all my teachers during my master’s degree program at IIT Guwahati, especially, Prof. Anupam Saikia, Prof. Srirupam Bandopadhay, Prof. Guru P. Prasad and Prof. Anjan K. Chakrabarty who motivated me for better prospects of my career. It was my great pleasure to have Prof. Saikia not only as my master’s project guide but also as a good friend. My deep sense of gratitude is extended to my teacher, Prof. Deepankar Das, who taught me mathematics during my bachelor’s degree
program and encouraged me to pursue my career at IIT. All my teachers’ contributions in my academic career are much appreciated.

I would like to express my gratefulness to Vivek Sadhu and Husney Parvez Sarwar for initiating the weekly students’ algebra seminar in our department which we have continued later on. I have learned a lot from these seminars. I express my sincere thanks to all the members of this seminar series, especially, Rakesh Reddy, Prasant, Rajiv, Parangama, Jai Laxmi and Sudeshna for making it successful.

I owe my deep sense of gratitude and thanks to some special individuals, specifically, to Debu da for always being there for me in my difficult time; to Vivek da for his motivation and generous guidance throughout my doctoral study; to Anjan da for his encouragement on mathematical discussions and for his practical suggestions all the time; to Priyo da for his support and unconditional help all the time in many purposes; to Manna da and Dipjyoti da for having given us their valuable time and place where we used to have our weekly get together and enjoyed delicious food. My heartfelt thanks to all who were there in those weekly get together, and made it fun-filled and memorable.

I would like to acknowledge my sincere thanks to all of my seniors, especially, Soham, Amiya, Shubhankar, Ramesh M., Swapnil, Abhay, Parvez, Rakesh R., Sayan, Ishapathik, Ali Zinna, Satya Prakash, Sajith, Harsh, Ashok, Rakesh K., Ramesh K., Nidhin, Mahendra, Triveni, Aditya, Zafar and Asha for their encouragement, continuous support and help in many ways during the last five years.

I express my heartfelt thanks to all my batchmates: Arunava, Bhimsen, Ganesh, Gouranga, Harsh, Mrinmoy, Ojas, Parangama, Prasant, Radhika, Rajiv, Ram Murti, Saranya, Shuchita, Sunil, Tushar for their companionship, support and many good times which we shared during my stay at IIT Bombay. I am also thankful to my other fellow research scholars: Jai Laxmi, Provanjan, Arindam, Samriddho, Avijit, Gobinda, Praphulla, Sudeshna, Sudeep, Kuldeep, Rakhi, Souvik, Kriti, Venkitesh, Dibyendu, Hiranya, Mukesh, Akansha, Pramod and Vivek for their love and moral support.

Many thanks to all of my hostel mates Oayes Midda, Bapi Bera, Nirmalya Chatterjee, Siddhartha Sarkar, Ashok Shaw, Saurav Mitra, Tanmoy Chakrabarty, Pritam Biswas, Arif Iqbal Mallick, Anjan Roy, Deb Datta Mandal, Tanmoy Sarkar, Rakhahari Saha, Mahitosh Biswas, Ananta Sarkar, Anal Sarkar, Sandip Mandal, Sudip Das, Arup Chakraborty and many more for the great time we had in our hostel, especially, over lunch and dinner shall
always be cherished.

Many people have walked in and out of my life. But very few have left their footprints in my heart. My heartiest thanks to Sunil for being one of them, and for being there with me always, especially, when times were hard. Many thanks to Gouranga for being a part of this journey which has started from IIT Guwahati. My heartfelt thanks to all of my post graduate friends, more specially, Sahadeb, Abhishek and Debarchana, for their constant support, love and encouragement. My academic journey with Sahadeb which has started from Midnapore College was cheerful and memorable. I thank my childhood friend Pintu for making my village life fun-filled and enjoyable.

A lot of thanks to IIT Bombay for providing an excellent atmosphere and all the facilities which are suitable not only for good research but also to live an enjoyable life. I express my gratitude to all the staff members of Mathematics Department for their timely help and support.

Many thanks to National Board for Higher Mathematics, Dept. of Atomic Energy, Govt. of India for providing financial support for this study. Without this support, it would have been difficult for me to continue my research.

And finally, needless to say, I am deeply and forever indebted to my parents, grandparents for their unconditional love, patience, continuous support and encouragement throughout my entire life. This dissertation is dedicated to my parents and grandparents. I am very much fortunate to have two lovable sisters, Papiya and Priyanka, who have always been my source of inspiration. I cannot express how much my parents and sisters mean to me. I thank them for just being who they are. It is my pleasure to thank my cousins Kinkar, Shubhankar, Debashri, Bidhan, Raz, my sister-in-law Mina and my nephew Shyam for their love and affection. The appreciations as well as high expectations of my relatives about my career always motivate me to do something better. My heartfelt thanks to all my well wishers.

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2016
