BRAID GROUP ACTIONS ON THE $n$-ADIC INTEGERS

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ABSTRACT. We construct an infinite tower of covering spaces over the configuration space of $n-1$ distinct non-zero points in the complex plane. This results in an action of the affine braid group $B_{n-1}$ on the $n$-adic integers $\mathbb{Z}_n$ for all natural numbers $n \geq 2$. With similar constructions we obtain several actions of the Artin braid group $B_n$ on $\mathbb{Z}_n$ and we study some of the properties of these actions such as continuity and transitivity. We also use the braid group actions to define descending sequences of normal subgroups in the braid group, which do not stabilize. The construction of the actions involves a new way of associating to any braid $B$ an infinite sequence of braids, whose braid types are invariants of $B$. We speculate on how such sequences can be useful in the context of invariants of braids, conjugacy classes of braids and links. We present computations for the cases of $n = 2$ and $n = 3$ and use these to show that an infinite family of braids close to links of isolated singularities of real polynomials $\mathbb{R}^4 \rightarrow \mathbb{R}^2$.

1. INTRODUCTION

Actions of a group on a set can offer insights both on properties of the group and symmetries of the set. In the case of the Artin braid group on $n$ strands $B_n$, there is an additional motivation, since the elements of the group correspond to topological objects. One might hope that a braid group action contains information not only about the group structure, but also about the topological interpretation of its elements, say about the link type of their closures for example. By Markov’s Theorem elements of the braid group correspond to braids that close to equivalent links if and only if they are related through a sequence of isotopies, conjugation and (de)stabilisation moves. Actions that contain information that does not change under conjugation and (de)stabilisation can therefore be used to construct link invariants.

It should be noted that actions that are actually representations particularly lend themselves to this procedure as they offer several values that are invariant under conjugation, such as the determinant or the trace of the matrices. This principle has been applied many times, such as in the case of the interpretation of the Alexander polynomial as a normalized determinant of the Burau representation [7] or the Jones polynomial as a normalized Markov trace of a representation of the braid group into a Temperley-Lieb Algebra [11].

In this paper we are going to construct braid group actions on the $n$-adic integers. While these do have an interesting algebraic structure, as well as being a prominent object of study in itself in number theory, we do not expect these actions to lead to representations of the braid group. However, they offer interesting connections between different aspects of the study of braid groups, linking group theoretic properties with the topology of certain configuration spaces and subsets of the space of complex polynomials.

Let $n \in \mathbb{N}$ with $n \geq 2$. Let $\mathbb{V}_n$ be the space of monic complex polynomials $f \in \mathbb{C}[z]$ of degree $n$ with $n$ distinct roots or equivalently the configuration space of $n$ distinct unmarked points in the complex plane. Then $\pi_1(\mathbb{V}_n) = B_n$, where $B_n$ denotes the braid group on $n$ strands. This article is to a large part about the consequences of a result by Beardon, Carne
and \( \text{Ng}^{[1]} \), namely about the subset \( V_n \subset \mathbb{V}_n \) of polynomials \( f \in V_n \) that have \( n - 1 \) distinct non-zero critical values and a constant term equal to 0. Then the set of possible sets of critical values of such a polynomial is

\[
W_n := \{ (v_1, v_2, \ldots, v_{n-1}) \in (\mathbb{C} \setminus \{0\})^{n-1} : v_i \neq v_j \text{ if } i \neq j \}/S_{n-1},
\]

where \( S_{n-1} \) is the symmetric group on \( n - 1 \) elements.

**Theorem 1.1.** (cf. \([1]\)) The map \( \theta_n : V_n \to W_n \) that sends a polynomial \( f \in V_n \) to the set of its critical values \( \{v_1, v_2, \ldots, v_{n-1}\} \) is a covering map of degree \( n^{n-1} \).

Note that the fundamental group of \( W_n \) is the affine braid group \( \mathbb{B}_{n-1}^{aff} \), which is defined in Section 2. Therefore, Theorem 1.1 implies an action of \( \mathbb{B}_{n-1}^{aff} \) on the fibre, which consists of \( n^{n-1} \) points, via the monodromy action. In this paper we use properties of the covering map \( \theta_n \) to construct an infinite tower of covering spaces

\[
\cdots \to V_{n+1} \to V_n \to \cdots \to V_2 \to V_1 = V_n \to W_n.
\]

This results in a continuous action of \( \mathbb{B}_{n-1}^{aff} \) on the set of \( n \)-adic integers \( \mathbb{Z}_n \) via \( \phi_n(\cdot, B) : \mathbb{Z}_n \to \mathbb{Z}_n, B \in \mathbb{B}_{n-1}^{aff} \).

With a view towards link invariants we do not want to restrict ourselves to the case of affine braids, which makes it necessary to consider paths in \( W_n \), rather than loops. We obtain a continuous action of \( \mathbb{B}_n \) on \( \mathbb{Z}_n \) by isometries \( \phi_n(\cdot, B) : \mathbb{Z}_n \to \mathbb{Z}_n, B \in \mathbb{B}_n \).

The remainder of this paper is structured as follows. In Section 2 we provide the reader with the necessary background on the connection between polynomial maps and braids as well as the very basics of the theory of \( n \)-adic integers. In Section 3 we construct the actions \( \phi_n \) and \( \phi_{\alpha} \) and prove that they act by isometries on \( \mathbb{Z}_{n\alpha} \cong \mathbb{Z}_n \) and \( \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}_{n\alpha} \cong \mathbb{Z}_n \) respectively. Section 4 also discusses another action \( \psi_n \) of the braid group on \( \mathbb{Z}/n\mathbb{Z} \) that shares some properties with \( \phi_n \), but makes computations significantly simpler. The definition of \( \psi_n \) involves the construction of a sequence of braids

\[
(B, \{B_1, B_{1,1}, \ldots, B_{1,n^2-1}\}, \{B_2, B_{2,1}, \ldots, B_{2,(n^2-1)}\}, \ldots, \{B_j, B_{j,1}, \ldots, B_{j,(n^j\alpha-1)}\}, \ldots),
\]

which is an invariant of the braid \( B \). We outline in Section 4 how sequences like this can be used to improve invariants of braids or conjugacy classes, i.e., make the invariants better at distinguishing different braids (or different conjugacy classes of braids). In Section 5 we compute the effect of \( \psi_n(\cdot, \sigma) \) on the first coordinates of \( \mathbb{Z}_{n\alpha} \) for the generators \( \sigma_i \) of the braid groups on two and three strands. The results can be used to compute the effect of \( \psi_n(\cdot, B) \) on any fixed number of coordinates for any braid \( B \) on two or three strands in a computational complexity that is only linear in the length of \( B \). Since the actions on \( \mathbb{Z}_{n\alpha}, \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}_{n\alpha} \) and \( \mathbb{Z}_{n\alpha} \) are by isometries, they restrict to actions on \( \mathbb{Z}/(n^{j-1})/\mathbb{Z} \), \( \mathbb{Z}/(n \times (n^{j-1})/\mathbb{Z} \), and \( \mathbb{Z}/(n^j)/\mathbb{Z} \) respectively for all \( j \). In Section 6 we show that while none of the constructed actions is transitive on all of \( \mathbb{Z}_{(n^j-1)} \cong \mathbb{Z}_n \) and \( \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}_{n^j-1} \cong \mathbb{Z}_n \), \( \phi_n \) and \( \phi_{\alpha} \) are transitive when restricted to \( \mathbb{Z}/(n^{j-1})/\mathbb{Z} \) and \( \mathbb{Z}/(n \times (n^{j-1})/\mathbb{Z} \). We also take a brief look at the orbit of points under the repeated application of \( \theta_n \) and \( \theta_{n^{-1}} \) for \( n = 2, 3 \). Note that the restrictions of the actions to \( \mathbb{Z}/(n^{j-1})/\mathbb{Z} \), \( \mathbb{Z}/(n \times (n^{j-1})/\mathbb{Z} \), and \( \mathbb{Z}/(n^j)/\mathbb{Z} \) correspond to sequences of homomorphisms from the (affine) braid group to the symmetric group \( S_{(n^{j-1})}, S_{n \times (n^{j-1})}, \) and \( S_{(n^j)} \) respectively. Section 7 studies the sequences of the normal subgroups that are given by the kernels of these homomorphisms and we show that the sequences of normal subgroups do not stabilize. In Section 8 we employ the computations from Section 5 to show that an infinite family of braids close to real algebraic links, i.e., links of isolated singularities of real polynomials \( \mathbb{R}^4 \to \mathbb{R}^2 \).
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2. Background

In this section we summarise the necessary background on polynomials and their relation to braids, as well as Beardon, Carne and Ng’s covering map from [11] and on the $n$-adic integers. Proofs and more detailed descriptions can be found in [1], [3], [4] and [6], while the basics of $n$-adic integers are available in countless number theory texts such as [8] or [16].

2.1. Polynomials, braids and the covering map. Let $\tilde{V}_n$ be the space of monic complex polynomials in one variable of degree $n$ and with $n$ distinct roots. Consider a parametrised family of polynomials $f_t \in \tilde{V}_n$, $t \in [0,1]$. Alternatively, this can be written as a map $f : \mathbb{C} \times [0,1] \to \mathbb{C}$, $(z,t) \mapsto f_t(z)$, which is a polynomial in $z$. This is a Weierstrass polynomial as discussed in the context of braids in [9], [10] and [15]. Since $f_t$ has $n = \deg f_t$ distinct roots for every $t \in [0,1]$, the nodal set $f^{-1}(0)$ forms a braid on $n$ strands in $\mathbb{C} \times [0,1]$. The underlying principle is the fundamental theorem of algebra that allows us to identify a monic polynomial with distinct roots with its (unordered) set of roots. The map that sends a polynomial in $\tilde{V}_n$ to its roots gives a homeomorphism $\tilde{V}_n \cong C_n := \{(z_1,z_2,\ldots,z_n) \in \mathbb{C}^n : z_i \neq z_j \text{ if } i \neq j\}/S_n$, where $S_n$ is the symmetric group on $n$ elements. This means that the fundamental group of the space of monic complex polynomials of a fixed degree $n$ with distinct roots is $\mathbb{B}_n$, with homotopies of loops in $\tilde{V}_n$ corresponding to braid isotopies. For a path in $\tilde{V}_n$ given by $f_t$, $t \in [0,1]$, we will frequently refer to the braid that is formed by the roots of $f_t$, $t \in [0,1]$ as the braid corresponding to the path $f_t$.

This is of course well-known, but what is less often considered is the possibility of replacing the concept of ‘roots’ in the above construction by other sets such as the critical points or critical values of the polynomials. Instead of demanding that each $f_t$ has $n$ distinct roots, we can require that $f_t$ has $n - 1$ distinct critical points $c_1,c_2,\ldots,c_{n-1}$ with $f_t''(c_i) = \frac{\partial f_t}{\partial z}(c_i) = 0$ for all $i$. Just like above we get a homeomorphism between a space of polynomials (monic with fixed degree $n$, distinct critical points and with constant term equal to 0) and $C_{n-1}$, which tells us that the fundamental group of that space of polynomials is the braid group $\mathbb{B}_{n-1}$. Restricting the space of polynomials to those with a constant term equal to zero was necessary to get a map that is 1-to-1. We might argue that this is not really different from the construction above. We merely used the fact that every polynomial has a unique antiderivative once the integration constant is fixed (i.e., the constant term is set equal to 0) and then employed the homeomorphism between $\tilde{V}_{n-1}$ and $C_{n-1}$.

It is therefore easy to construct parametrised families of polynomials $f_t \in \mathbb{C}[z]$, $t \in [0,1]$ whose nodal sets or whose sets of critical points form a given braid. In the first case, we
only have to find a parametrisation

\begin{equation}
\bigcup_{j=1}^{s} (z_j(t), t) \subset \mathbb{C} \times [0, 1]
\end{equation}

of the braid on \(s\) strands and define \(f_t(z) = \prod_{j=1}^{s} (z - z_j(t))\). Here the polynomial degree \(n\) equals the number of strands \(s\) and the roots trace out the desired braid as \(t\) varies from 0 to 1.

In the latter case, we obtain \(f_t\) via

\begin{equation}
f_t(z) = \int_{0}^{t} \prod_{j} (w - z_j(t)) \, dw.
\end{equation}

Note that here each \(f_t\) has degree \(n = s + 1\) and the critical points trace out the desired braid as \(t\) varies from 0 to 1.

Suppose now we are not interested in the topology of the nodal set or that of the critical set of a family of polynomials, but instead in the topology of the set of critical values, i.e., the values \(f_t(c_i)\) of the polynomials at their critical points \(c_i\). Given a parametrisation

\begin{equation}
\bigcup_{j=1}^{s} (v_j(t), t) \subset \mathbb{C} \times [0, 1]
\end{equation}

of a braid \(B\) we want to construct a \(f(z,t) = f_t(z)\) such that the critical values of \(f_t\) form the braid \(B\), i.e., we want the existence of \(c_1(t), c_2(t), \ldots, c_s(t)\) such that \(f(c_j(t)) = v_j(t)\) and \(f'(c_j) = 0\) for all \(j = 1, 2, \ldots, s\).

We find such a family of polynomials \(f_t\) for a given braid parametrisation by solving a system of polynomial equations for every \(t \in [0, 1]\), which is not very practical. The problem becomes easier if we are content with a family of polynomials \(f_t\) whose critical values form a braid that is isotopic to \(B\), rather than realising a specific parametrisation. There is an extra degree of freedom that can be eliminated by setting the constant term of \(f_t\) to 0 for all \(t\). In this case, we can solve the system of polynomial equations for some fixed values of \(t\), say \(t = t_1, t_2, \ldots, t_m\) for some \(m \in \mathbb{N}\), to obtain polynomials \(f_{t_i}\). In contrast to the earlier examples, these solutions are not unique. We return to the question of the number of solutions later. Interpolating functions through the coefficients of the polynomials \(f_{t_i}\) then provide us with the coefficients of \(f(z,t) = f_t(z)\) as functions of \(t\). For a sufficient choice of many data points, i.e., high values of \(m\), the braid that is formed by the critical values of \(f\) is isotopic to \(B\).

This brings us to the question that started this project initially. Suppose the polynomials \(f_t\) whose critical values form the braid \(B\) all lie in \(\tilde{V}_n\) with \(n = s + 1\). Then the roots of \(f_t\) form a braid too, say \(A\). What can be said about the relation between the braids \(A\) and \(B\), one formed by the roots of \(f_t\), the other by its critical values?

Let

\begin{equation}
V_n := \{ f \in \mathbb{C}[z] : f \text{ monic of degree } n \text{ with distinct roots, distinct critical values and constant term equal to 0} \}.
\end{equation}

If \(f_t \in V_n \subset \tilde{V}_n\), the fact that its roots are distinct implies that none of the critical values \(v_1(t), v_2(t), \ldots, v_{n-1}(t)\) is 0. The space of the possible sets of critical values is therefore

\begin{equation}
W_n := \{(v_1, v_2, \ldots, v_{n-1}) \in (\mathbb{C} \setminus \{0\})^{n-1} : v_i \neq v_j \text{ if } i \neq j\}/S_{n-1},
\end{equation}

where \(S_{n-1}\) denotes the symmetric group on \(n-1\) elements.
The fundamental group of $W_n$ is the affine braid group $B_{n-1}^{aff}$, which has the generators $x, \sigma_2, \sigma_3, \ldots, \sigma_{n-1}$ and defining relations
\begin{align}
\sigma_2 x \sigma_2 x &= x \sigma_2 x \sigma_2, \\
\sigma_i \sigma_j &= \sigma_j \sigma_i, \quad \text{if } |i - j| > 1, \\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \text{if } i = 2, 3, \ldots, n-2, \\
\sigma_i x &= x \sigma_i, \quad \text{if } i > 2.
\end{align}

This corresponds to the usual Artin braid group with the only difference that in any braid word $\sigma_i$ can appear only appear with even exponents, so that $x = \sigma_i^2$ becomes one of the generators. Geometrically, this means that the braid can be parametrised such that the first strand does not move at all, i.e., can be taken to be $(0, t) \subset \mathbb{C} \times [0, 1]$. We often refer to this strand as the 0-strand or the flagpole.

This description of $\pi_1(W_n)$ as $B_{n-1}^{aff}$ with the above generators and relations assumes that we have chosen a basepoint in $W_n$ that is a $n-1$-tuple of complex numbers, whose real parts are all positive. If we consider different basepoints, loops correspond to conjugates (in the braid group) of affine braids with the generators and relations above, but setting $x = \sigma_i^2$. In general, any loop in $W_n$ corresponds to a braid, whose permutation of the n-1 strands has (at least) one fixed point. Sometimes we will be somewhat inexact and call such a braid affine as well. It should be clear from the context if we refer to a braid, whose permutation fixes the first strand or the $i$th strand for some $i = 1, 2, \ldots n$, i.e., which of the strands is the 0-strand.

The fact that loops in $W_n$ are affine braids is a first hint that the braid word of $B$, which is formed by the critical values, does not carry all relevant information about the parametrisation $(v_1(t), v_2(t), \ldots, v_{n-1}(t))$. The way in which the strands twist around the flagpole $(0, t) \subset \mathbb{C} \times [0, 1]$ is important. We should focus on the braid that is formed by $(0, v_1(t), v_2(t), \ldots, v_{n-1}(t))$ instead of $B$ itself.

The question above is related to the problem of constructing polynomial fibrations, which can be stated as follows. For which fibred links $L$ can we explicitly construct a polynomial $p : \mathbb{R}^2 \to \mathbb{R}^2$ such that $p^{-1}(0) \cap S^3 = L$ and $\arg f|_{S^3} \neq \arg f|_{[0, 1]}$ is a fibration? This question is clearly motivated by results by Milnor on isolated singularities of complex plane curves [13].

Suppose the given parametrisation $(v_1(t), v_2(t), \ldots, v_{n-1}(t))$ is such that $\frac{\partial \arg v_i}{\partial t}(h) \neq 0$ for all $i = 1, 2, \ldots, n-1$ and all $h \in [0, 1]$. Then any polynomial $f(z, t) = f(t)$ with the critical values equal to $v_j(t)$ gives a fibration of the braid complement via $\arg f|_{\mathbb{C} \times [0, 1]} f^{-1}(0)$. In [3, 4] we show that in this case we can construct a polynomial $p$ as above. In fact, $p$ can be written as polynomial in complex variables $u, v$ and $v$, a property that we call semiholomorphic.

**Definition 2.1.** A braid $B$ on $s$ strands is called homogeneous if it can be written as a word $w$ such that for every $i = 1, 2, \ldots, s-1$ the generator $\sigma_i$ appears in $w$ if and only if $\sigma_i^{-1}$ does not appear.

In [4, 5] we show that homogeneous braids can be parametrised in such a way that the critical values have no turning points, i.e., $\frac{\partial \arg v_j}{\partial t}(h) \neq 0$ for all $j = 1, 2, \ldots, n-1$ and all $h \in [0, 1]$, and the fibrations of the complements of their closures can therefore be explicitly constructed as arguments of semiholomorphic polynomials. The proof uses the first partial result concerning the question about the relation between the braid formed by the roots and the braid formed by the critical values and the strand $(0, t)$. It can be already found in some way in [19]. The most detailed account at the moment is probably in [6].
Proposition 2.2. Let

\[ Y_i = \sigma_1^{-1} \sigma_2^{-1} \ldots \sigma_{i-1}^{-1} \sigma_i^2 \sigma_i^3 \ldots \sigma_i \quad \text{if } i \geq 2 \]

and \( Y_1 = \sigma_1^2 \) be braids on \( s \) strands. Furthermore, let

\[ X_i = \begin{cases} \frac{Y_{i+1}}{\sigma_i} & \text{if } i \text{ is odd,} \\ \frac{Y_i}{\sigma_i^{i+1}} & \text{if } i \text{ is even.} \end{cases} \]

Then for every parametrisation \((0, v_1(t), v_2(t), \ldots, v_s(t))\) of any braid of the form \( \prod_{j=1}^k X_{ij}^{\epsilon_j} \), there is a parametrised family of polynomials \( f_t \in \mathbb{V}_{s+1} \) such that the roots of \( f(z, t) = f_t(z) \) form a braid that is conjugate to \( \prod_{j=1}^k \sigma_{ij}^{\epsilon_j} \) and the critical values of \( f_t \) are \( v_1(t), v_2(t), \ldots, v_s(t) \).

While this result is clearly related to the question above we would like to emphasize that it only states that the \( f_t \) are in \( \mathbb{V}_{s+1} \) and not necessarily in \( \mathbb{V}_{s+1} \) itself, i.e., the constant term is not necessarily 0 for all of them.

The construction of polynomial fibrations for closures of homogeneous braids then follows from the fact that affine braids of the form \( \prod_{j=1}^k X_{ij}^{\epsilon_j} \) can be parametrised such that the non-zero strands have no turning points. In this case, the corresponding braid that is formed by the roots is a homogeneous braid \( \prod_{j=1}^k \sigma_{ij}^{\epsilon_j} \).

The remarkable thing about Proposition 2.2 is that the braid word of the braid that is formed by the roots, \( \prod_{j=1}^k \sigma_{ij}^{\epsilon_j} \), does not depend on the explicit parametrisation of the critical values \((v_1(t), v_2(t), \ldots, v_s(t))\), but only on the isotopy type of the affine braid \((0, v_1(t), v_2(t), \ldots, v_s(t)) \subset \mathbb{C} \times [0, 1]\). This can be explained with a result by Beardon, Carne and Ng. They showed that the map that sends a polynomial in \( \mathbb{V}_n \) to its set of critical values is a covering map of degree \( n^{n-1} \) (cf. Theorem 1.1 in Section 1). This tells us not only that for every set of critical values in \( \mathbb{W}_n \) there are \( n^{n-1} \) polynomials in \( \mathbb{V}_d \) that have the given critical values (answering the question about the number of solutions to the relevant system of polynomial equations mentioned earlier), but more importantly allows us to use the homotopy lifting property. Let \( \gamma \) be a loop in \( \mathbb{W}_n \). Since a loop in the base space \( \mathbb{W}_n \) corresponds to an affine braid and homotopies of a loop to isotopies of the affine braid, the homotopy types of the \( n^{n-1} \) lifts of \( \gamma \) only depend on the isotopy class of the affine braid corresponding to \( \gamma \). Since \( \mathbb{V}_n \subset \mathbb{V}_n \), every path in \( \mathbb{V}_n \) can be interpreted as a braid on \( n \) strands by considering the \( n \) distinct roots of each polynomial and every homotopy between paths in \( \mathbb{V}_n \) corresponds to an isotopy between the corresponding braids (but not vice versa). In particular, the braid types of the \( n^{n-1} \) lifts of a loop \( \gamma \subset \mathbb{W}_n \) only depend on the affine braid type of the braid corresponding to \( \gamma \), not on its particular parametrisation.

Note that all polynomials in \( \mathbb{V}_n \) have 0 as one of their roots, since their constant term is by definition equal to 0. Therefore the braid \( A \) that is formed by the roots of a parametrised family of polynomials \( f_t \in \mathbb{V}_n \), \( t \in [0, 1] \) and the 0-strand \((0, t) \subset \mathbb{C} \times [0, 1]\) form a braid too. We can therefore not only consider relations between the braids formed by the roots and the critical values of such a family of polynomials, but also their relation to the braid that is formed by the critical points.
2.2. The \( n \)-adic integers. We only need the very basics of the theory of \( n \)-adic integers \( \mathbb{Z}_n \). They are defined as the inverse limit \( \lim_{\leftarrow k} \mathbb{Z}/n^k\mathbb{Z} \) for \( k \geq 1 \). Therefore it is the set of infinite sequences

\[
(12) \quad a = (a_1, a_2, a_3, \ldots)
\]

such that \( a_i \in \mathbb{Z}/n^i\mathbb{Z} \) and \( a_{i+1} = a_i \mod n^i \) for all \( i \geq 1 \).

The integers \( \mathbb{Z} \) embed into \( \mathbb{Z}_n \) by sending \( m \in \mathbb{Z} \) to the sequence \( (m \mod n, m \mod n^2, \ldots) \).

The \( n \)-adic integers form an abelian group. They are in fact rings (possibly with zero divisors), but we are not really going to need these facts. There are several equivalent definitions of \( \mathbb{Z}_n \). We could for example define it as the set of infinite sequences of elements in \( \mathbb{Z}/n\mathbb{Z} \). This is because in Equation (12) there are \( n \) choices for \( a_1 \) and while in principle there are \( n^i \) choices for \( a_i \), only \( n \) of them satisfy the condition of \( a_{i+1} = a_i \mod n^i \). The way to move between these two definitions is to interpret a sequence of elements \( b_i \) in \( \mathbb{Z}/n\mathbb{Z} \) as the coefficients of an infinite formal expansion in powers of \( n \),

\[
(13) \quad b_1 + b_2 n + b_3 n^2 \ldots,
\]

and see the corresponding sequence of elements \( a_i \) in \( \mathbb{Z}/n^i\mathbb{Z} \) as the \( \mod n^i \) reduction of that expansion

\[
(14) \quad a_i = \sum_{j=1}^{\infty} b_j n^{j-1} \mod n^i.
\]

If \( n = \prod_{i=1}^{m} p_i^{k_i} \) with \( p_i \) prime, \( p_i \neq p_j \) for all \( i \neq j \) and \( k_i \geq 1 \), then

\[
(15) \quad \mathbb{Z}_n \cong \mathbb{Z}_{p_1^{k_1}} \times \mathbb{Z}_{p_2^{k_2}} \times \ldots \times \mathbb{Z}_{p_m^{k_m}}
\]

as abelian groups. This (and the existence of zero-divisors) is one explanation why in most cases we are only interested in the case where \( n \) is a prime. In this paper however, we are less concerned with the arithmetic and algebraic properties of \( \mathbb{Z}_n \), but instead focus on the definition as the limit of an inverse system. Furthermore, in our case the number \( n \) is going to be related to the degree of a polynomial or the number of strands of a braid, so from a geometric viewpoint it is not advisable to factor \( n \) into primes.

We can put an \( n \)-adic valuation (and with it a metric and a topology) on \( \mathbb{Z}_n \) by defining

\[
(16) \quad \text{ord}_n(a) = \min\{k \geq 1 : a_i = 0 \text{ for all } i < k\}.
\]

The order of the 0-sequence \( \text{ord}_n(0) \) is defined as \( \infty \). The \( n \)-adic value of \( a = (a_1, a_2, a_3, \ldots) \) is

\[
(17) \quad |a|_n = n^{-\text{ord}_n(a)}.
\]

This means an \( n \)-adic integer is considered small if its sequence starts with a lot of zeros. Consequentially, two \( n \)-adic integers are considered close to each other if their sequences start with many identical terms. A possible basis of the open sets on \( \mathbb{Z}_n \) is given by

\[
(18) \quad U_k(m) = \{a \in \mathbb{Z}_n : |a - m|_n \leq n^{-k}\}, \quad k \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z} \cap [1, n^k].
\]

Note that with this topology the group isomorphisms in Equation (15) are isomorphisms of topological groups.
3. Braid group actions

In this section we describe actions of the affine braid group and the Artin braid group on the \( n \)-adic integers and show that they are continuous. In fact, they are isometries on spaces that are homeomorphic to the \( n \)-adic integers. By Theorem 2.4 \( V_n \) is a covering space of \( W_n \) of degree \( n^{n-1} \). Therefore, the fundamental group of \( W_n \), which is the affine braid group \( \mathbb{B}^{\text{aff}}_{n-1} \), acts on the fibre consisting of \( n^{n-1} \) points. Furthermore, recall that the polynomials in \( V_n \) have a constant term equal to 0 and therefore one of their roots is equal to 0. Hence \( V_n \) can be embedded in \( W_n \) by sending a polynomial \( f \in V_n \) to its \( n-1 \) distinct non-zero roots.

Let \( V_n^1 := V_n \) and define \( V_n^{j+1} \) to be the space of monic polynomials \( f \in V_n \) with (by definition distinct) critical values \( (v_1, v_2, \ldots, v_{n-1}) \) such that the polynomial \( z \prod_{i=1}^{n-1} (z - v_i) \) is in \( V_n^j \). We obtain an infinite tower of covering spaces

\[
\ldots \to V_n^{j+1} \to V_n^j \to \ldots \to V_n^2 \to V_n^1 = V_n \to W_n,
\]

where each covering map is \( \theta_n \), the map that sends a polynomial to its set of critical values, restricted to the relevant \( V_n^j \) and composed with the map that sends a set of critical values \( (v_1, v_2, \ldots, v_{n-1}) \) to the polynomial \( z \prod_{i=1}^{n-1} (z - v_i) \). We often ignore the second map and move freely between interpreting a point in \( V_n^j \) as a \((n-1)\)-tuple of complex numbers and as the corresponding polynomial and simply write \( \theta_n \) for the covering map. It follows that \( V_n^j \) is a covering space of \( W_n \) of degree \((n^{n-1})^j\).

Suppose for example \( v = (v_1, v_2, \ldots, v_{n-1}) \in W_n \) and let \( f_j \in V_n \), \( j = 1, 2, \ldots, n^{n-1} \) be the \( n^{n-1} \) preimages of \( v \) under the covering map, i.e., the polynomials that have the set \( \{v_1, v_2, \ldots, v_{n-1}\} \) as their set of critical values. Let \( \{z_1^j, z_2^j, \ldots, z_{n-1}^j\} \) denote the \( n-1 \) non-zero roots of \( f_j \). Since \( (z_1^j, z_2^j, \ldots, z_{n-1}^j) \in W_n \), there are exactly \( n^{n-1} \) polynomials in \( V_n \) that have \( (z_1^j, z_2^j, \ldots, z_{n-1}^j) \) as their sets of critical values. In summary, for a given \( v = (v_1, v_2, \ldots, v_{n-1}) \in W_n \) there are \((n^{n-1})^2\) polynomials \( g_k \) in \( V_n \) that have the property that their critical values \( (z_1, z_2, \ldots, z_{n-1}) \) define a polynomial \( z \prod_{i=1}^{n-1} (z - z_i) \) that has 0 and the critical values of \( g_k \) as its roots and \( v \) as its set of critical values.

The affine braid group acts not only on the fibre in \( V_n \), but on the fibre in each covering space in Equation \((19)\). We thus have an action of \( \mathbb{B}^{\text{aff}}_{n-1} \) on the set of \((n^{n-1})^j\) points or equivalently on the set \(\mathbb{Z}/(n^{n-1})^j\mathbb{Z}\) for every \( j \in \mathbb{Z}_{\geq 1} \). Furthermore, the different actions are compatible with each other in the sense that \( \theta_n(x, \gamma) = \theta_n(x) \cdot \gamma \) for all \( x \) in the fibre in \( V_n^j \) and all \( \gamma \in \mathbb{B}^{\text{aff}}_{n-1} \).

Let \( v \in W_n \) and consider the set \( X \) of infinite sequences \((a_1, a_2, a_3, \ldots)\) with \( a_j \in V_n^j \), \( \theta_n(a_1) = v \) and \( \theta_n(a_{j+1}) = a_j \) for all \( j \geq 1 \). Since there are exactly \((n^{n-1})^j\) choices for the \( j \)th term \( a_j \), of which only \( n^{n-1} \) satisfy the compatibility condition, this set can be identified with the \( n^{n-1} \)-adic integers \( \mathbb{Z}_{n^{n-1}} \). In other words, there is a bijection between the inverse system defining \( X \) and the inverse system defining \( \mathbb{Z}_{n^{n-1}} \), as in Section 2.2.

The group \( \pi_1(W_n) = \mathbb{B}^{\text{aff}}_{n-1} \) acts on \( X \) and therefore on \( \mathbb{Z}_{n^{n-1}} \), since the action of \( \mathbb{B}^{\text{aff}}_{n-1} \) on the fibres in the different \( V_n^j \) satisfies \( \theta_n(a_j, \gamma) = \theta_n(a_j) \cdot \gamma \). Since \( \mathbb{Z}_{n^{n-1}} \cong \mathbb{Z}_n \), this shows that the affine braid group on \( n-1 \) strands is acting on the \( n \)-adic integers, which we denote by \( \varphi_n : \mathbb{Z}_n \times \mathbb{B}^{\text{aff}}_{n-1} \to \mathbb{Z}_n \).

**Proposition 3.1.** The affine braid group \( \mathbb{B}^{\text{aff}}_{n-1} \) acts on \( \mathbb{Z}_{n^{n-1}} \) by isometries. Therefore, the action \( \varphi_n (\cdot, B) : \mathbb{Z}_n \to \mathbb{Z}_n \) is continuous.
Proof. The topology on $\mathbb{Z}_{n^{-1}}$ is induced by the metric, which itself is derived from the $n^{-1}$-adic valuation. Let $x = (x_1, x_2, x_3, \ldots) \in \mathbb{Z}_{n^{-1}}$ and $y = (y_1, y_2, y_3, \ldots) \in \mathbb{Z}_{n^{-1}}$ such that $|y - \varepsilon| = n^{-m-1}$. This is equivalent to $x$ and $y$ agreeing on the first $m$ terms of the sequence, i.e., $x_i = y_i$ for all $i = 1, 2, \ldots, m$, but $x_{m+1} \neq y_{m+1}$.

Therefore, the first $m$ terms of the sequences $x, y$ and $x, y$ also agree with each other for all $y \in \mathbb{B}_{n^{-1}}$, which means the distance between $x, y$ and $y, y$ is at most $(n/n - 1) - m - 1$. On the other hand, applying $y^{-1}$ to $x, y$ and $y, y$ shows that their distance is at least $(n/n - 1) - m - 1$. Hence $|y - x, y| = n^{-m-1}$ and the action on $\mathbb{Z}_{n^{-1}}$ is by isometries. The $\varepsilon - \delta$ criterion then tells us that the action is continuous on $\mathbb{Z}_{n^{-1}}$ and by Section 2.2 continuous on $\mathbb{Z}_n$.

We would like to obtain a similar action for the whole braid group, not only for the affine braids. One straightforward way to do this is to use that the braid group on $B$ strands is a subgroup of $W$, however be interpreted as a set of paths in $W$. However other ways to obtain braid group actions and they seem to lead to more interesting results.

If a braid $B$ on $n$ strands is not affine, it does not correspond to a loop in $W_n$. It can however be interpreted as a set of paths in $W_n$ as follows. Let $x \in W_n$ be such that all its $n - 1$ coordinates have distinct, non-zero real parts and such that the real parts of $i - 1$ of the coordinates are negative and the real parts of $n - i + 1$ coordinates are positive. Then we say $x$ has a $0$ in $i$th position. Consider for example the $n$ points $x_1, x_2, \ldots, x_n$ in $W_n$ with $x_1 = (1, 2, \ldots, n-1), x_2 = ((n-1), 2, 3, \ldots, n-1), x_3 = ((n-1), (n-2), 3, 4, \ldots, n-1), \ldots, x_n = ((n-1), (n-2), \ldots, 2, 1)$. Then $x_i$ has a $0$ in $i$th position.

Choose one of the $x_i$ and parametrise it such that the $i$th strand is constant $0$, the parametrisation of the remaining $n - 1$ strands starts at $x_i$ and ends at one of the $x_j$. It is thus a path $\gamma_B$ in $W_n$, from $x_i$ to some $x_j$. The permutation representation $\pi_B$ of the braid group sends a generator $\sigma_i$ to the transposition $(i + 1)$. Then the end point of $\gamma_B$ is $x_{\pi_B(i)}$. Note that braid isotopies now directly correspond to homotopies of $\gamma_B$ with fixed start and end points. Therefore, the specific parametrisation does not matter. Both points, $x_i$ and $x_{\pi_B(i)}$, have $n^{-1}$ preimage points (under $\theta_n$) in $V_n$ and $B$ induces a bijection between the preimage points of $x_i$ and the preimage points of $x_{\pi_B(i)}$. In order to get a nice algebraic structure it is not enough to consider one arbitrary choice of $x_i$. We have to consider all $n$ of them. We obtain $n$ paths in $W_n$, say $\gamma_B, i = 1, 2, \ldots, n$, which give us first of all a permutation of the points $\{x_1, x_2, \ldots, x_n\}$ in $W_n$ by sending $x_i$ to $x_{\pi_B(i)}$. This is by definition simply the permutation representation of $B$. Furthermore, we obtain an action on the $n \times n^{-1} = n^{1-n}$ preimage points of the $x_i$ in $V_n$, which satisfies that for all $i$ every preimage point of $x_i$ must be send to a preimage point of $x_{\pi_B(i)}$. The action is again given by mapping a point $x \in \theta_n^{-1}(\{x_1, x_2, \ldots, x_n\})$ to the endpoint of the unique lifted path of any of the $\gamma_{B, i}$ that starts at $x$.

In exactly the same manner we have an action of the braid group $\mathbb{B}_n$ on the $n \times (n^{-1})^2$ preimage points of $x_1, x_2, \ldots, x_n$ in $\theta_n^{-1}(\{x_1, x_2, \ldots, x_n\}) \subset V_n^2$ that is compatible with the action on $\theta_n^{-1}(\{x_1, x_2, \ldots, x_n\})$. More generally, we have an action of $\mathbb{B}_n$ on the fibre of $\{x_1, x_2, \ldots, x_n\} \in V_n^k$ for all $k$ and we have $\theta_n(x, B) = \theta_n(x, B) B$ for all $x \in V_n^k$ and $B \in \mathbb{B}_n$. We thus have defined an action of $\mathbb{B}_n$ on the set $X$ of infinite sequences $(a_1, a_2, a_3, \ldots)$ such that $a_i \in \mathbb{Z}/(n \times (n^{-1})^{i-1} \mathbb{Z})$ and $a_i = a_i \mod n \times (n^{-1})^{i-2}$. This definition of $X$ is very
similar to the definition of the \( n \)-adic integers in Section 2.2 and it is not hard to see that there is a bijection between \( X \) and \( \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}_{n^1} \), which is isomorphic to \( \mathbb{Z}_n \). We have thus constructed an action of \( \mathbb{B}_n \) on \( \mathbb{Z}_n \), which we denote by \( \phi_n : \mathbb{Z}_n \times \mathbb{B}_n \to \mathbb{Z}_n \). Exactly the same arguments as in the proof of Proposition 3.1 imply that it is continuous and by isometries on \( \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}_{n^1} \). We have thus shown the following result.

**Proposition 3.2.** The braid group \( \mathbb{B}_n \) acts on \( \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}_{n^1} \) by isometries. Therefore, the action \( \phi_n(\cdot, B) : \mathbb{Z}_n \to \mathbb{Z}_n \) is continuous.

It is worth noting that the action \( \phi_n \) of the affine braid group can be found inside \( \phi_n \). If a braid \( B \) is affine, then \( \pi(B)(1) = 1 \) and hence the parametrisation of \( B \) starting at \( x_1 \) must also end at \( x_1 \) and is therefore a loop. Thus the fibre of \( x_1 \) in the tower of covering spaces is an invariant subspace of \( \phi_n(\cdot, \mathbb{B}_{n^1}) \) and the permutation \( \phi_n(\cdot, B) \) of the points in the fibre is by definition given by \( \phi_n(\cdot, B) \).

Explicit calculations of \( \phi_n \) are rather elaborate. In order to compute the action of the generators \( \sigma_i \) of \( \mathbb{B}_n \) on \( \mathbb{Z}_n \), we need to find \( n \) parametrisations of each \( \sigma_i \), each of which starts and ends at one of the \( x_i \)'s and lift these paths to the covering spaces \( V_i \). Every such lifting procedure corresponds to solving a system of \( n \) polynomial equations for sufficiently many data points \( \{t_i\}_{i=1,2,\ldots,n} \subset [0,1] \). Therefore, in order to compute the action of \( \sigma_i \) on the first \( k+1 \) coordinates of \( \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}_{n^1} \), \( k \geq 1 \), we need to solve \( mn(n^{k-1}) \) systems of \( 2(n-1) \) polynomial equations, each of which has \( n^{k-1} \) solutions.

Once we have found the lifts, we can read off the permutations on the preimage points of the \( x_i \)'s in the different \( V_i \), \( j = 1,2,\ldots,k \), that are induced by \( \sigma_i \). From these we can build the effect of any \( B \in \mathbb{B}_n \) on the first \( k+1 \) coordinates by composing the permutations for the individual generators (and their inverses). The action on the first coordinates on \( \mathbb{Z}_n \) can then be determined via the isomorphism between \( \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}_{n^1} \) and \( \mathbb{Z}_n \). The composition of permutations can be done in a number of steps that grows linearly with the length of the braid word of \( B \). What makes this calculation impractical is the huge number of systems of polynomial equations that we have to solve.

Since we want to avoid such unnecessarily expensive computations, we define a different action \( \psi_n \) of \( \mathbb{B}_n \) on \( \mathbb{Z}_{n^1} \), which is very similar to \( \phi_n \), but requires solving only \( mn \) systems of \( 2(n-1) \) polynomials, no matter how many coordinates of \( \mathbb{Z}_n \) we are interested in. As an illustration of the concept we solve the corresponding systems of equations for the cases of \( n = 2 \) and \( n = 3 \) in Section 5 and illustrate how to use the solutions to compute the action of any braid \( B \in \mathbb{B}_n \) on any given number of coordinates of \( \mathbb{Z}_n \).

We now define the action \( \psi_n \). Let \( B \in \mathbb{B}_n \). Like in the definition of \( \phi_n \) we think of \( B \) as the collection of \( n \) paths in \( W_n \), each of which starts at a different one of the \( x_i \) and ends at \( x_i(\pi(B)) \). Lifting these paths to \( V_1 \) results in \( n \times n^{n-1} \) paths in \( V_n \), which give us a permutation of the \( n \times n^{n-1} = n^n \) preimage points \( \partial_n^{-1}\{x_1, x_2, \ldots, x_n\} \) that is compatible with \( \pi(B) \). The action of \( B \) on the first coordinate of \( \mathbb{Z}_{n^1} \) via \( \psi_n \) is therefore the same as its action on the second coordinate of \( \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}_{n^1} \) via \( \phi_n(B,B) \). We are going to refer to this map more often, so we give it a name: \( \sigma : \mathbb{B}_n \to S_{n^1} \).

Now comes the part that distinguishes the new action from \( \phi_n \). Each of the \( n \times n^{n-1} \) paths in \( V_n \) corresponds to a family of polynomials \( f_{jt} \subset V_n \), \( j = 1,2,\ldots,n^1 \) such that the union of the critical values of \( f_{j,t} \) and the zero strand \( (0,t) \subset \mathbb{C} \times [0,1] \) form the braid \( B \) as \( t \) varies from 0 to 1. Since \( f_{j,t} \in V_n \), the roots of \( f_{j,t} \) also form a braid as \( t \) varies from 0 to 1, say \( B_j \). Instead of lifting the \( n \times n^{n-1} \) paths in \( V_n \) to \( V_n^2 \) as before (i.e., using the particular parametrisations of \( B_j \) that we obtained through the lifting procedure), we parametrize each \( B_j \) as \( n \) paths in \( W_n \) starting and ending at the different \( x_i \)'s just like we did
for the original braid $B$. We can lift these $n^\rho$ paths in $W_n$ to $V_n$ and obtain $n^\rho$ preimages of the $n^\rho$ preimage points $\theta_n^{-1}(\{x_1, x_2, \ldots, x_n\})$. By definition these permutations are simply $\sigma(B_j)$.

We can write these $n^\rho$ permutations $\sigma(B_j)$ of $n^\rho$ points as one permutation of $(n^\rho)^2$ points that is compatible with the permutation of $n^\rho$ points, i.e., compatible with the action on the first coordinate of $\mathbb{Z}_{n^\rho}$, as follows. Label the points with 0 through $n^\rho - 1$, so that every number can be expressed uniquely as $n^\rho k + i$ with $k, i = 0, 1, 2, \ldots, n^\rho - 1$. Then the permutation of $(n^\rho)^2$ points is

$$n^\rho k + i \mapsto n^\rho \sigma(B_j)(k) + \sigma(B)(i).$$

(20)

Note that the residue class mod $n^\rho$ of the image is given by $\sigma(B)$, which means that this permutation on $(n^\rho)^2$ points is compatible with $\sigma(B)$ on $n^\rho$ points. In each residue class are $n^\rho$ elements that $n^\rho k + i$ could be mapped to by a permutation that satisfies this compatibility condition. The permutation $\sigma(B_j)$ specifies which one is the actual image point. From this description it is (hopefully) clear that we can associate to every braid $B$ a permutation of $(n^\rho)^2$ points that is compatible with $\sigma(B)$ on $n^\rho$ points. It might not be entirely obvious that we have actually constructed an action on the set of $(n^\rho)^2$ points.

Since $\theta_n$ is a covering map, the homotopy types of the lifted paths only depend on the homotopy types of the $n$ original paths in $W_n$. Equivalently, the braid types $B_j$ of the braids formed by the roots of the lifts $f_{ij}$ are invariants of the original braid $B$. In particular, neither the choice of parametrisations of $B$, nor the choice of parametrisations of the lifted braids $B_j$ changes the resulting permutation.

Let $z_0$ through $z_{n^\rho - 1}$ denote the points in $\theta_n^{-1}(\{x_1, x_2, \ldots, x_n\})$. Let $\mathbb{B}_n^n$ be the direct product of $n^\rho$ copies of $\mathbb{B}_n$. The symmetric group $S_{n^\rho}$ acts on $\mathbb{B}_n^n$ by permutation of the components. Algebraically, the lifting procedure gives us a homomorphism $h$ from $\mathbb{B}_n$ to the semidirect product, or wreath product, $\mathbb{B}_n^n \rtimes S_{n^\rho}$, where $h$ sends a braid $B$ to the list of lifted braids $B_j$ starting at $z_j$ and the permutation $\sigma(B)$.

$$h : B \mapsto (B_0, B_1, B_2, \ldots, B_{n^\rho - 1}, \sigma(B)).$$

(21)

It is a homomorphism because addition in $\mathbb{B}_n^n \rtimes S_{n^\rho}$ corresponds to concatenation of paths (i.e., braids) with matching start and end points. Composing $h$ with $n^\rho$ copies of $\sigma$ we obtain a homomorphism $\mathbb{B}_n \to S_{n^\rho} \times S_{n^\rho}$. We can now check that the map that sends an element $(\sigma(B_1), \sigma(B_2), \ldots, \sigma(B_{n^\rho}), \sigma(B))$ to the permutation in Equation (20) is a homomorphism $S_{n^\rho} \times S_{n^\rho} \to S_{(n^\rho)^2}$, which altogether results in a homomorphism $\mathbb{B}_n \to S_{(n^\rho)^2}$. We have

$$n^\rho \sigma(B_{\sigma(A)(i)})(\sigma(A_j)(k)) + \sigma(B)(\sigma(A)(i)) = n^\rho \sigma(A_{B_{\sigma(A)(i)}})(k) + \sigma(AB)(i)$$

(22)

We have therefore constructed an action $\rho_{n,2}$ of $\mathbb{B}_n$ on $(n^\rho)^2$ points that is compatible with the action $\rho_{n,1} = \sigma$ on $n^\rho$ points. We iterate this process to actions on $(n^\rho)^j$ points that are compatible with the action on $(n^\rho)^{j-1}$ points.

Applying the homomorphism $h$ to each of the $B_j$ results in $n^\rho$ elements of $\mathbb{B}_n \rtimes S_{n^\rho}$. Previously, we denoted the image of a braid $B$ under $h$ by $\rho_{n,2}(B)$. In order to avoid excessive use of subscripts we write $B_{2j, B_2, n^\rho + i}, B_{2j, (2 \times n^\rho) + i}, \ldots, B_{2j, (n^\rho - 1) \times n^\rho + i}$ for the braids that correspond to the lifts of $B_i$ instead of $B_0, B_1, \ldots, B_{n^\rho - 1}$. Note in particular that all braids that are lifts of $B_i$ have a second index that is in the residue class $i$ mod $n^\rho$. We have chosen this labelling such that the map $B \mapsto \rho_{n,2}(B)$ becomes a homomorphism from $\mathbb{B}_n$ to $\mathbb{B}_n^{(n^\rho)^2} \rtimes S_{(n^\rho)^2}$. Exactly as in the previous case we
define the action of \( B \) on \((n^n)^3\) points by
\[
(n^n)^2 k+i \mapsto (n^n)^2 \sigma(B_{j,i})(k)+\rho_{n,j}(B)(i),
\]
where \( k = 0,1,2,\ldots,n^n-1 \) and \( j = 0,1,2,\ldots,(n^n)^2-1 \).

It is important to note that not only the braid types \( B_{1,j} = B_j \) of the lifted paths of the parametrisations of \( B \) are invariants of \( B \). As such the braid types of the lifts of the parametrisations of \( B_{1,j} \) are also invariants of \( B \), say \( B_{2,kn^j+k}, k = 0,1,2,\ldots,n^n-1 \). Continuing like this we obtain a sequence of braids
\[
\begin{align*}
&B_1 = \{B_{1,0},B_{1,1},B_{1,2},\ldots,B_{1,n^n-1}\}, \{B_{2,0},B_{2,1},B_{2,2},\ldots,B_{2,(n^n)^2-1}\}, \ldots,
&B_2 = \{B_{j,0},B_{j,1},B_{j,2},\ldots,B_{j,(n^n)^2-1}\}, \ldots
\end{align*}
\]
that is an invariant of \( B \). This should be understood as follows. If two braids, \( A \) and \( B \), with sequences
\[
\begin{align*}
&A_1 = \{A_{1,0},A_{1,1},A_{1,2},\ldots,A_{1,n^n-1}\}, \{A_{2,0},A_{2,1},A_{2,2},\ldots,A_{2,(n^n)^2-1}\}, \ldots,
&A_2 = \{A_{j,0},A_{j,1},A_{j,2},\ldots,A_{j,(n^n)^2-1}\}, \ldots
\end{align*}
\]
and
\[
\begin{align*}
&B_1 = \{B_{1,0},B_{1,1},B_{1,2},\ldots,B_{1,n^n-1}\}, \{B_{2,0},B_{2,1},B_{2,2},\ldots,B_{2,(n^n)^2-1}\}, \ldots,
&B_2 = \{B_{j,0},B_{j,1},B_{j,2},\ldots,B_{j,(n^n)^2-1}\}, \ldots
\end{align*}
\]
are isotopic, then for every \( j,i \) the braid \( A_{j,i} \) is isotopic to \( B_{j,i} \).

We can define an action \( \rho_{n,j+1} \) on \((n^n)^{j+1}\) points inductively via
\[
(n^n)^j k+i \mapsto (n^n)^j \sigma(B_{j,i})(k)+\rho_{n,j}(B)(i),
\]
where \( \rho_{n,j} \) is the action on \((n^n)^j\) points and \( k = 0,1,2,\ldots,n^n-1 \) and \( i = 0,1,2,\ldots,(n^n)^j-1 \). Note that the residue classes mod \((n^n)^j\) are permuted according to \( \rho_{n,j} \), i.e., the action on \((n^n)^{j+1}\) points is compatible with the action on \((n^n)^j\) points. We thus obtain a braid group action \( \psi_n \) on \( \mathbb{Z}_{n^n} \cong \mathbb{Z}_n \). Since \( \psi_n \) is constructed from permutations in every coordinate of \( \mathbb{Z}_{n^n} \) it acts by isometries and is therefore continuous, both by exactly the same arguments as in the case of \( \phi_n \) and \( \phi_n \) (cf. Proposition 3.1 and Proposition 3.2).

It should be noted that this new action \( \psi_n(\cdot,B) \) carries less (or equal) information about \( B \) than \( \phi_n(\cdot,B) \). This is because \( V_n \) is a proper subset of \( V_n \). In particular, there are non-homotopic paths \( \gamma_1, \gamma_2 \) in \( V_n \) corresponding to families of polynomials \( f_1 \) and \( g_2 \) such that the braids that are formed by the roots of \( f_1 \) and \( g_2 \) are isotopic. On the other hand, \( \psi_n(\cdot,B) \) is completely determined by \( h(B) \in \mathbb{B}_n^{e^n} \times S_m \).

The construction of all actions in this section is possible because \( V_n \) can be embedded into \( W_n \) by sending a polynomial to the set of its non-zero roots. There is another way of embedding \( V_n \) into \( W_n \), which sends a polynomial to the set of its critical points. We again obtain a tower of covering spaces \( \tilde{V}_n \) and can define analogues of the actions \( \phi_n, \phi_n \) and \( \psi_n \). In fact, all results from this section remain true for these actions.

Note that for \( j > 1 \) the spaces \( \tilde{V}_n^j \) are different from \( V_n^j \) as they are the spaces of monic polynomials \( f \) such that the set of critical values of \( f \) is the set of critical points of a polynomial in \( \tilde{V}_n^{j-1} \). We will see in Section 5 that the analogue of \( \psi_3 \) is different from \( \psi_3 \).

4. Sequences of Braids

This section aims at using sequences of braids to make braid invariants stronger. Say we have a way of associating to any braid \( B \in \mathbb{B}_n \) a sequence of braids, whose braid types are invariants of \( B \). We take a braid invariant \( I_n : \mathbb{B}_n \rightarrow K_n \) with values in some set \( K_n \) that
is not very good at distinguishing braids, but easy to compute, such as the exponent sum or the permutation representation, which both grow in complexity linearly with the length of the braid word. We can evaluate $I_n$ on the first $k$ terms of the sequence associated to $B$ and obtain a sequence of braid invariants that is presumably much stronger than the original invariant $I_n$. If two braids $A$ and $B$ are isotopic, then not only are their exponent sums equal, but also every braid in the sequence associated to $A$ is isotopic to its counterpart in the sequence associated to $B$ and hence these exponent sums must be equal too. Furthermore, if it is not hard to compute the sequence corresponding to a braid $B$, then the whole sequence of invariants is relatively easy to compute too.

In the previous section (cf. Eq. (24)) we have encountered such a sequence of braids. There are some other candidates that are just as suitable for our purposes. We could for example consider the collection of lifts in $V^j_n$ of $n$ paths in $W_n$ that correspond to the braid $B$. As discussed in the previous section, we can interpret each braid as the collection of $n$ paths in $W_n$ starting and ending at points $x_i \in W_n$ that have 0 in the $i$th position. The lifts of these paths are paths in $V^j_n$ and since $V^j_n$ embeds into $W_n$, we can interpret each of these paths as a braid. We obtain again a sequence of braids $\{ B_{j,i} \}_i=0,1,2,...,n^n-1; \{ \hat{B}_{j,i} \}_i=0,1,2,...,n \times (n^n-1)^2-1,\ldots\}$ similar to Equation (24), which are invariants of $B$. This time however, the sequence is not determined by the entries $B'_{1,0}, B'_{1,1}, \ldots, B'_{1,n^n-1}$ and the permutation in $S_{n^2}$. This is essentially because $V^j_n$ is a proper subset of $W_n$. In particular, there are paths in $V^j_n$ with the same start and end points that correspond to the same braid (i.e., are homotopic in $W_n$), but are not homotopic in $V^j_n$.

There are two different embeddings of $V^j_n$ into $W_n$, the one that sends a polynomial to its non-zero roots and the one that sends a polynomial to its critical points. Both of these give rise to an infinite tower of covering spaces $V^j_n$ and $\tilde{V}^j_n$ and the procedure discussed above works for both of them. Hence we have in fact three sequences of braids, $B_{j,i}$, $B'_{j,i}$, and $\hat{B}_{j,i}$, say, that are candidates for improving braid invariants.

The sequences $B_{j,i}$, $B'_{j,i}$ and $\hat{B}_{j,i}$ are infinite sequences of braid invariants of $B$. In order to understand if they are useful for the improvement of braid invariants, we need to know how hard it is to compute them. Once we have computed the first $k$ terms of the corresponding sequences and permutations for each generator of $B_n$, we can calculate the first $k$ terms of the sequences $B'_{j,i}$ and $\hat{B}_{j,i}$ in a number of steps that grows linearly with the length of the braid word of $B$ as it is simply addition in semi-direct products of groups. In the case of $B'_{j,i}$ and $\hat{B}_{j,i}$ the first $k$ terms of the sequences that correspond to a generator can be found by solving $k$ systems of polynomial equations. If $k$ is too large, this becomes impractical. Therefore, even though we have in theory infinite sequences of invariants at our disposal, in practice we will only calculate the first $k$ entries for some relatively low number $k$. In the case of $B_{j,i}$ we can only have to solve one system of polynomial equations to find the first term of the sequence $\{ B_{j,i} \}_i=0,1,2,...,n^n-1$ corresponding to a generator $B = \sigma_m$. From this first term we can calculate $B_{j,i}$ for any braid $B$ and any $j, i$ in a number of steps that grows linearly with the length of $B$. The sequence $B_{j,i}$ is therefore significantly easier to compute than $B'_{j,i}$ or $\hat{B}_{j,i}$.

The braids $B'_{j,i}$ and $\hat{B}_{j,i}$ in the sequences are open braids in the sense that in general the corresponding paths in $V_n$ and $\tilde{V}_n$ respectively are not necessarily loops. We can associate braids to the paths nonetheless, since isotopies of the braids formed by the roots of the polynomials that make up the paths are equivalent to homotopies of paths that keep the start and end points fixed. However, there is also a way to associate to a braid $B$ a sequence
of braids $B_{j,C}$ that correspond to loops in $V_n^I$. These can be used to obtain a sequence of invariants of conjugacy classes of braids.

Denote by $\phi_{n,j}(B)$ the restriction of the action $\phi_{n}(\cdot, B)$ to an action on $n \times (n^n - 1)^J$ points and let $C_j$ the set of cycles of $\phi_{n,j}(B)$. Then every cycle

$$C = (i \phi_{n,j}(B)(i) \phi_{n,j}(B)^2(i) \ldots \phi_{n,j}(B)^{|C|(i)}) \in C_j,$$

where $|C|$ is the length of the cycle $C$, can be associated with the braid corresponding to the loop $\gamma_C$ in $V_n^I$ that is the concatenation of the lifts starting at $z_{j,i}, z_{j,\phi_{n,j}(B)(i)}, z_{j,\phi_{n,j}(B)^2(i)}$ and so on, where we denote the points in the fibres in $V_n^I$ over the $x_k \in W_n$ by $z_{j,i}$.

We thus obtain a sequence of braids

$$\{B_i \{B_{C_{i,j}}\}_{i=1,2,...,|C_j|}, \{B_{C_{i,j}}\}_{i=1,2,...,|C_j|,\ldots}\}$$

where we have labelled the cycles in $C_j$ by $C_{i,j}$ and $|C_j|$ denotes the number of cycles of $\phi_{n,j}(B)$.

The above procedure again really gives rise to several sequences of braids, depending on how we choose to associate a braid to a path in $V_n^I$, the braid that is formed by its roots $B'_{j,i}$ or by its critical points $\bar{B}_{j,i}$.

Now every braid in the sequence (29) (apart from $B$) comes from a loop in $V_n^I$, which was obtained from a lifting procedure. This means that the sequence (29) is an invariant of the conjugacy class of $B$ in the sense that if $A$ is conjugate to $B$, then for every $j$ there is a bijection $g_j$ between the sets of cycles of $\phi_{n,j}(B)$ and $\phi_{n,j}(A)$ that maps each cycle to a cycle of the same length. Furthermore, $A_{\psi_j(C_{j,i})}$ is conjugate to $B_{C_{j,i}}$. We can now apply an invariant $J_n$ of braid conjugacy classes in $B_n$ to the sequence in Eq. (29) and obtain a sequence of invariants that is (presumably) stronger than the original invariant $J_n$.

At this moment it is not clear how such a sequence of braids changes under stabilization and destabilization, i.e., how the sequence corresponding to $B_{\sigma_{n+1}}$ is related to the sequence corresponding to a $n$-strand braid $B$. The hope is that there are some properties that stay invariant, which can be used to define a sequence of link invariants analogously to the sequence of invariants of braids and braid conjugacy classes in the previous paragraphs.

## 5. Computations for $n = 2$ and $n = 3$

In this section we compute how the generators of the braid groups $B_n$ act on $\mathbb{Z}_n^{\otimes n} \cong \mathbb{Z}_n$, for $n = 2$ and $n = 3$ via $\psi_n$. The effect of more complicated braid words on two or three strands can then be obtained by composing the permutations coming from the relevant generators.

### 5.1. The case of $n = 2$

The case of $n = 2$ is probably the only case that is simple enough to be done by hand. We have to pick two points in $W_2$, say $x_1 = x$ and $x_2 = y$ with $\text{Re}(x) > 0$ and $\text{Re}(y) < 0$. Take for example $x = e^{ix}$ and $y = -e^{ix}$ with a small $\varepsilon > 0$. The only generator $\sigma_1$ of $B_2 \cong \mathbb{Z}$ forms an odd braid, so any parametrisation that starts at $x_1$ and ends at one of the $x_j$ must end at $x_2$. Similarly, a parametrisation that starts at $x_2$ ends at $x_1$.

The generator can be parametrised as a path that starts at $x_1$ and ends at $x_2$ via $x(t) = e^{i(t\varepsilon)}$, where $t$ is going from $0$ to $\pi$. Similarly, a parametrisation of $\sigma_1$ that starts at $x_2$ and ends at $x_1$ is $y(t) = e^{i(t\varepsilon + \pi)}$, where $t$ is going from $0$ to $\pi$.

First of all, we need to find the preimage points of $x_1$ and $x_2$ under $\theta_2$, i.e., we have to find monic quadratic polynomials with one (simple) root equal to 0 and the critical value
equal to $x = e^{i\varepsilon}$ (and $y = -e^{i\varepsilon}$, respectively). We thus have to solve
\[ c(c - z) = x = e^{i\varepsilon} \]
(30)
\[ 2c - z = 0 \]
as well as
\[ c(c - z) = y = -e^{i\varepsilon} \]
(31)
for the non-zero root $z$. In the first case, we obtain $z = \pm 2e^{i(\varepsilon + \pi)/2}$, which are the $n^{n-1} = 2$ preimage points of $x_1$, corresponding to the polynomials $u + 2e^{i(\varepsilon + \pi)/2} \in V_2$. In the latter case, we have $z = \pm 2e^{i\varepsilon/2}$. We label these points as follows: $z_1 = 2e^{i\varepsilon/2}$, $z_2 = 2e^{i(\varepsilon + \pi)/2}$, $z_3 = -2e^{i\varepsilon/2}$ and $z_0 = -2e^{i(\varepsilon + \pi)/2}$. The reason for our choice of $x_1$ and $x_2$ was that the real parts of their preimages are non-zero, which allows us to read off the braid words corresponding to lifted paths. Now we calculate how the two paths $x(t)$ and $y(t)$ in $W_2$ permute these four preimage points. We already know that the preimage points of $x_1$ get mapped to preimage points of $x_2$ and vice versa.

We solve
\[ c(t)(c(t) - z(t)) = x(t) = e^{i(r+\varepsilon)} \]
(32)
\[ 2c(t) - z(t) = 0 \]
as well as
\[ c(t)(c(t) - z(t)) = y(t) = e^{i(t+\varepsilon+\pi)} \]
(33)
for the non-zero root $z(t)$. In the first case, we obtain $z(t) = 2e^{i(t+\varepsilon+\pi)/2}$ and in the latter $z(t) = \pm 2e^{i(t+\varepsilon)/2}$. The lifted path that starts at $z_1 = 2e^{i\varepsilon}$ is $2e^{i(t+\varepsilon)/2}$ and it ends at $t = \pi$ at $2e^{i(\varepsilon + \pi)/2}$, which is $z_2$. The lifted path that starts at $z_2 = 2e^{i(\varepsilon + \pi)/2}$ is $2e^{i(t+\varepsilon+\pi)/2}$ and it ends at $t = \pi$ at $2e^{i(\varepsilon/2 + \pi)} = -2e^{i\varepsilon/2} = z_3$. Since we know that all lifted paths that start at preimage points of $x_1$ end at preimage points of $x_2$ and vice versa, this is enough to conclude that the permutation sends $z_i$ to $z_{i+1}$, where the index is taken mod 4.

We could interpret each of the four paths as paths in $W_2$ and lift them too, which would tell us the action of $\sigma_1$ on the third coordinate of $\mathbb{Z}_2$ in Proposition 3.2, i.e., the restriction of $\phi_2$ to an action on eight points. However, this would require a calculation of the eight preimage points of $\{z_1, z_2, z_3, z_0\}$, so after a number of iterations this procedure becomes unnecessarily long. Instead we are going to focus on the braids that each lifted path corresponds to, as in the construction of $\psi_2$. Recall that each of the lifted paths corresponds to a family of polynomials, one of whose roots is 0 and the other given by the different $z_i$. The points $x_1$ and $x_2$ were chosen such that their preimage points are not purely imaginary, which makes it possible to read off the braid word from a parametric plot of the roots corresponding to a lifted path, namely the 2-strand braid that is formed out of the 0-strand and $z(t)$.

For the paths that start at $z_1$ and $z_3$ we find that the corresponding braid is $\sigma_1$ and for the paths that start at $z_2$ and $z_0$ we obtain the trivial braid (cf. Figure 1). In the notation of the previous section, $B_{1,1} = B_{1,3} = \sigma_1$ and $B_{1,2} = B_{1,0} = e$. Each of these braids can be parametrised as paths in $W_2$ with start and end points at $\{x_1, x_2\}$. This information is enough to compute the action of any 2-strand braid $B$ on $\mathbb{Z}_2$. We would like to give a bit more insight into the calculations and also compute the action of $\sigma_1$ on the second coordinate of $\mathbb{Z}_4$, i.e., on 16 points. For each of the four braids $B_{1,j}$ we obtain again an
The braids that are formed by the roots of the polynomials corresponding to the lifts of $\sigma_1$ in $V_2$ with $\varepsilon = 0.8$. a) The lift that starts at $z_1$ has the corresponding braid $B_{1,1} = \sigma_1$. b) The lift that starts at $z_2$ corresponds to the trivial braid, $B_{1,2} = e$. c) The lift that starts at $z_3$ has the corresponding braid $B_{1,3} = \sigma_1$. d) The lift that starts at $z_0$ corresponds to the trivial braid, $B_{1,0} = e$.

We arrange these four permutations of four points into one permutation of 16 points, labelled 0 through 15, as indicated in Section 3. Points that have a label $j \mod 4$ must be mapped to a point with a label $j + 1 \mod 4$ in order to be consistent with the action that $\sigma_1$ induces on 4 points. Therefore, the point with the label 1 can go to four possible values, 2, 6, 10 and 14. The label 1 corresponds to the braid $B_{1,1} = \sigma_1$, which induces the cyclic permutation on four points. We thus send the point 1, the first of the points 1 mod 4, to the second possible point that it can go to, in this case 6. Similarly, the point 5, the second of points with a label 1 mod 4, is mapped to 10, which is the third possible value that it can take. In exactly the same way 9 is mapped to 14, 13 to 2, 3 (the first point 3 mod 4, which is the other residue class $j \mod 4$ whose corresponding braid is $B_{1,j} = \sigma_1$) to 4 (the second point 0 mod 4), 7 to 8, 11 to 12 and 15 to 16. The points that have even labels are start points of the lifts $B_{1,2}$ and $B_{1,0}$, both of which are trivial and therefore induce trivial permutations. Therefore, the point 2, which is the first in the list of points with label 2 mod 4 should be mapped to the first point of the list of points 3 mod 4, i.e., 3, and so on.
In other words, the permutation on 4\(^4\) points dictates how the residue classes mod 4\(^4\) are permuted. Therefore, every point \(x \in \{0, 1, 2, \ldots, 4^{i+1} - 1\}\) has 4 possible image values. Which one of these it is mapped to is determined by the permutation on 4 points that is induced by the braid \(B_{j,x \mod 4^i}\).

Therefore the permutation of 16 points induced by the generator \(\sigma_1\) is given by

\[
\begin{align*}
4k + 1 &\mapsto 4(k + 1) + 2 & \mod 16, \\
4k + 2 &\mapsto 4k + 3 & \mod 16, \\
4k + 3 &\mapsto 4(k + 1) & \mod 16, \\
4k &\mapsto 4k + 1 & \mod 16,
\end{align*}
\]

(34)

where \(k = 0, 1, 2, 3\). In cycle notation this is

\[
(1 \ 6 \ 7 \ 8 \ 9 \ 14 \ 15 \ 0)(2 \ 3 \ 4 \ 5 \ 10 \ 11 \ 12 \ 13).
\]

For a general 2-strand braid \(B = \sigma_1^k, k \in \mathbb{Z}\) the action on \(\mathbb{Z}_2\) can now be described as follows. For the first coordinate of \(\mathbb{Z}_2\) there are two points, 0 and 1, and \(B\) permutes non-trivially if and only if \(k\) is odd. The action of \(B\) on the four possible choices of the second coordinate is given by \(\tau^k\), where \(\tau\) is the cyclic permutation of four elements, \((1 \mapsto 2 \mapsto 3 \mapsto 0 \mapsto 1)\). Note that for a given parity of \(k\) there are again only two possible permutations and the action is trivial if and only if \(k\) is divisible by four. The braid words that are formed by the four lifted paths are each \(\sigma_1^{k/2}\) if \(k\) is even and twice \(\sigma_1^{[k/2]}\) and twice \(\sigma_1^{[k/2]}\) if \(k\) is odd. Using the notation of Section 3 we have

\[
(36) \quad B_{1,1} = B_{1,2} = B_{1,3} = B_{1,0} = \sigma_1^{k/2}
\]

if \(k\) is even and

\[
(37) \quad B_{1,1} = B_{1,3} = \sigma_1^{[k/2]}, \\
B_{1,2} = B_{1,0} = \sigma_1^{[k/2]},
\]

if \(k\) is odd.

If we take for example \(B = \sigma_1^{12}\), then the action on the four preimage points of \(\{x_1, x_2\}\) is given by the twelfth power of the cyclic permutation, i.e., the identity. The four lifted paths each has the braid \(\sigma_1^{12}\). Each of these paths gives us a permutation of the four preimage points of \(\{x_1, x_2\}\). Since all four braids are identical, we obtain the same permutation every time, the 6th power of the cyclic permutation, i.e., \((1 \mapsto 3 \mapsto 1, 2 \mapsto 4 \mapsto 2)\). These can be arranged into a permutation of 16 elements that is compatible with \(\tau_1(B) = \text{id}\), namely

\[
(38) \quad \rho_{2,2}(\sigma_1^{12}) = (1 \ 9)(5 \ 13)(2 \ 10)(6 \ 14)(3 \ 11)(7 \ 15)(4 \ 12)(8 \ 0).
\]

This permutation was obtained as follows. Because \(\tau_1(B) = \text{id}\) every number must be mapped to a number which lies in the same residue class mod 4. Since we are constructing a permutation on 16 elements, there are four possible images of each number, for example 1 could go to 1, 5, 9 or 13. The permutation \((1 \mapsto 3 \mapsto 1, 2 \mapsto 0 \mapsto 2)\) now tells us how these four numbers are permuted, namely the first and the third number in this residue class (1 and 9) are swapped and the second and the fourth number in this residue class (5 and 13) are swapped. Analogous computations apply to the other residue classes.

The braid words of the 16 lifted paths are each \(\sigma_1^3\), which we can use in a similar fashion to construct the action on 64 elements as

\[
(39) \quad 16k + i \mapsto 16(k + 3) + \rho_{2,2}(\sigma_1^{12})(i),
\]
since $\sigma(\sigma_1^3) = (1 2 3 0)^3$. The braids corresponding to the lifts of $\sigma_1^3$ are either $\sigma_2^2$ or $\sigma_1$ depending on the start point. We can use these to compute the action on 256 points and then 4 powers by repeating the process.

**Proposition 5.1.** The action $\psi_2$ of $B_2$ on $Z_2$ is faithful.

*Proof.* Consider a braid $B = \sigma_1^k$ with $k \in Z$ and $k \neq 0$, i.e., $k = 2^m q$ for some $m \in N_0$ and some odd $q \in Z$. As we have seen above, the induced permutation is non-trivial if $m = 0$.

If $m > 0$, the computations from the previous paragraphs tell us that after the $m-1$th iteration of the lifting procedure results in $4^{m-1}$ paths in $V_2$ that each form the braid $\sigma_1^{2^m}$.

In the notation of the previous section, we have $B_{m,j} = \sigma_1^{2^m}$ for all $j = 1, 2, \ldots, 4^m$.

This means that the permutation on the $m$th coordinate of $Z_{4^m}$ is non-trivial. For example, the elements $1, 4, 2 \times 4^{m-1} + 1$ and $3 \times 4^{m-1} + 1$ are permuted in a non-trivial way, namely

\[(1 \mapsto 4^{m-1} + 1 \mapsto 2 \times 4^{m-1} + 1 \mapsto 3 \times 4^{m-1} + 1)^{2q \text{ mod } 4},\]

where $q$ is odd. Therefore, the trivial braid (with $k = 0$) is the only braid that leads to a trivial permutation of the $n$-adic integers. In other words, the action on $Z_4$ is faithful and hence $\psi_2$ is faithful on $Z_2$. $\square$

We can see from the calculations that the action is not transitive. Recall from Equation [35] the permutation of 16 points that is induced by the generator $\sigma_1$, i.e., the action of $\sigma_1$ on the second coordinate of $Z_4$. Since there are two distinct cycles, no power of $\sigma_1$ can map a 4-adic integer whose second coordinate is 1 to a 4-adic integer whose second coordinate is 2.

Note that the fact that the critical points of a polynomial in $V_n$ are branch points with deficiency 2 are reflected in the halving of the exponent of $\sigma_1$ with each lifting procedure.

We mentioned before that we can use the other embedding of $V_n$ into $W_n$, the one that sends a polynomial to its set of critical points rather than its non-zero roots, to define actions on $Z_n$ completely analogously to the constructions in Section 3. We can perform similar calculations for the analogue of $\psi_n$ as above. In principle, we could obtain different permutations and lifted braids (see the case of $n = 3$). However, in the case of $n = 2$, the critical point of a monic polynomial with constant term equal to 0 is precisely half of the non-zero root. Hence, we obtain the same permutations and braids.

5.2. The case of $n = 3$. For the case of $n = 3$ we have to consider the two generators $\sigma_1$ and $\sigma_2$. In order to perform our computations we first have to fix three points in $W_3$, namely $x_i$, $i = 1, 2, 3$, with 0 in $i$th position. As we have seen in the example of $n = 2$ we need something more. In order to be able to read off braid words from the lifted paths, we require that the preimage points of \{x_1, x_2, x_3\} under $\Theta_3$ also are tuples with distinct non-zero real parts.

One such choice of points in $W_3$ is $x_1 = (3/2e^{i\pi/100} - 1/2, 2)$, $x_2 = (-2, 2)$ and $x_3 = (-2, -3/2e^{i\pi/100} + 1/2)$. We have chosen these points such that $x_1 = -x_3$ and $x_2 = -x_2$.

This means that we can take the negative of a parametrisation of $\sigma_1$ that is starting at $x_i$ as a parametrisation of $\sigma_2$ that is starting at $x_i mod 3$. 
The preimage set $\theta_3^{-1}(\{x_1, x_2, x_3\})$ consists of $3^3 - 1 = 27$ points in $V_3$, i.e., tuples of complex numbers, which can be identified by solving the corresponding system of polynomial equations, $(f(c_1), f(c_2)) = x_j, f'(c_i) = 0$, as

\begin{align*}
  z_0 &= (-1.8676 - i0.3641, -2.9070 + i0.7149), \\
  z_1 &= (1.8676 + i0.3641, 2.9070 - i0.71491), \\
  z_2 &= (1.7321, 3.4641), \\
  z_3 &= (2.9070 - i0.7149, 1.0394 - i1.0790), \\
  z_4 &= (1.4541 + i0.3606, 2.0726 + i2.1601), \\
  z_5 &= (0.8660 + i1.5000, 1.7321 + i3.0000), \\
  z_6 &= (2.0726 + i2.1601, 0.6185 + i1.7995), \\
  z_7 &= (0.4147 + i1.4396, 1.2491 - i1.4354), \\
  z_8 &= (0.8660 - i1.5000, 1.7321 - i3.0000), \\
  z_9 &= (1.2491 - i1.4353, 0.8344 - i2.8750), \\
  z_{10} &= (-0.4147 - i1.4396, 0.8344 - i2.8750), \\
  z_{11} &= (-0.8660 + i1.5000, 0.8660 - i1.5000), \\
  z_{12} &= (1.4541 + i0.3606, -0.6185 - i1.7995), \\
  z_{13} &= (-1.8676 - i0.3641, 1.0394 - i1.0790), \\
  z_{14} &= (-1.7321, 1.7321), \\
  z_{15} &= (1.8676 + i0.3641, -1.0394 + i1.0790), \\
  z_{16} &= (-1.4541 - i0.3606, 0.6185 + i1.7995), \\
  z_{17} &= (-0.8660 - i1.5000, 0.8660 + i1.5000), \\
  z_{18} &= (0.4147 + i1.4396, -0.8344 + i2.8750), \\
  z_{19} &= (-1.2491 + i1.4354, -0.8344 + i2.8750), \\
  z_{20} &= (-1.7321 + i3.0000, -0.8660 + i1.5000), \\
  z_{21} &= (-0.4147 - i1.4396, -1.2491 + i1.4354), \\
  z_{22} &= (-2.0726 - i2.1601, -0.6185 - i1.7995), \\
  z_{23} &= (-1.7321 - i3.0000, -0.8660 - i1.5000), \\
  z_{24} &= (-1.4541 - i0.3606, -2.0726 - i2.1601), \\
  z_{25} &= (-2.9070 + i0.7149, -1.0394 + i1.0790), \\
  z_{26} &= (-3.4641, -1.7321).
\end{align*}

(41)

Numbers are rounded to four decimal points. The labels are such that the preimage points of $x_i$ have a label from the residue class $i$ mod 3.

The preimage set of one fixed $x_i$ has some symmetries. If $y = (y_1, y_2) \in V_3$ maps to $x_i$, so does $e^{2\pi i/3} y = (e^{2\pi i/3} y_1, e^{2\pi i/3} y_2)$. This, or rather its analogue with $e^{2\pi i/n}$, is in fact easy to check for polynomials of any degree $n$, not just $n = 3$. The other symmetry is a bit more mysterious (at least to the author). It seems like if $y = (y_1, y_2) \in V_3$ maps to $x_i$, there is another point $y' = (y_1', y_2') \in V_3$, which maps to $x_j$ and has $y_1'$ equal to $-y_1$. Furthermore, the point $(-y_2, -y_2')$ is also in $V_3$ and also maps to $x_j$. The few examples that we have explicitly studied seem to suggest that with these symmetries all preimage points of $x_i$ can be calculated starting from one preimage point $y \in \theta_3^{-1}(x_i)$. 

The generator $\sigma_1$ can be parametrised as

\begin{align}
(42) & \quad (v_1(t), v_2(t), 0) = \left( \frac{3}{2} e^{i \frac{3\pi}{10} t} - \frac{1}{2}, 2, 0 \right), \quad t \in \left[ 0, \frac{65}{100} \right], \\
(43) & \quad (v_1(t), v_2(t), 0) = \left( -\frac{3}{2} e^{i \pi t} - \frac{1}{2}, 2, 0 \right), \quad t \in \left[ 0, \frac{135}{100} \right],
\end{align}

\begin{align}
(v_1(t), v_2(t), 0) &= \left( \frac{\sqrt{5.0951}}{2} \cos(t\pi + 2.5079) + \frac{1}{2} \left( -2 - \frac{3}{2} \cos(\pi \frac{35}{100}) + 1/2 \right) \\
& \quad + i \left( \frac{\sqrt{5.0951}}{2} \sin(t\pi + 2.5079) \right) - \frac{3}{4} \sin\left( \pi \frac{35}{100} \right), \\
& \quad \sqrt{5.0951} \cos((t + 1)\pi + 2.5079) + \frac{1}{2} \left( -2 - \frac{3}{2} \cos(\pi \frac{35}{100}) + 1/2 \right) \\
& \quad + i \left( \frac{\sqrt{5.0951}}{2} \sin((t + 1)\pi + 2.5079) \right) - \frac{3}{4} \sin\left( \pi \frac{35}{100} \right), 0 \right), \quad t \in [0, 1].
\end{align}

The parametrisation in Equation (42) starts at $x_1$ and ends at $x_2$, while the parametrisation in Equation (43) starts at $x_2$ and ends at $x_1$. The start and end point of the parametrisation in Equation (44) are both $x_3$. We can lift these paths in $W_3$ to paths in $V_3$ with start and end points in $\theta_3^{-1}(\{x_1, x_2, x_3\})$. We do this by solving a system of polynomial equations,

$$f_k(c_k(t)) = c_k(t) \prod_{l=1}^{2} (c_k(t) - u_l(t)) = v_k(t), \quad k = 1, 2,$$

$$\frac{\partial f_k}{\partial u}(c_k(t)) = 0, \quad k = 1, 2,$$

for $u_l$ for values $t = j/100, j = 0, 1, \ldots, 100$ in Equation (44) and for a similar set of values for $t$ in Equation (42) and Equation (43). With a set of 101 data points for each strand, it is easy to identify the lifted paths in $V_3$ and the corresponding braids that are formed by the roots of the polynomials that make up the paths.

For example, the lift of the path in Equation (42) that starts at $z_1$ ends at $z_2$. A braid that interpolates the 101 data points for the set of roots of the polynomials that form this lift is shown in Figure 2a) and it is obviously the trivial braid. Performing this lifting procedure for every $z_i$ gives us the permutation of the 27 points in $\theta_3^{-1}(\{x_1, x_2, x_3\})$, namely

\begin{align}
\rho_{3,1}(\sigma_1) = (1 2)(3 12 9)(4 5 16 17)(6 18 15)(7 11 10 8)(13 14 25 26) \\
&(19 20)(21 0 24)(22 23).
\end{align}

Note that because $\pi(\sigma_1) = (1 2)$, every lift that starts at a point with index 1 mod 3 must end at a point with index 2 mod 3 and vice versa. Therefore the permutation above can be split into those cycles that only contain numbers 0 mod 3 and those that permute numbers that are not 0 mod 3.

From the lifted paths we find the braid words of the corresponding braids

\begin{align}
(A_{1,0}, A_{1,1}, A_{1,2}, \ldots, A_{1,26}) = (\sigma_1^{-1}, e, \sigma_2, \sigma_1^{-1}, e, \sigma_1, \sigma_1, e, e, \sigma_1, e, \sigma_1, e, \sigma_2, \sigma_1^{-1}, e, \sigma_1, e, \sigma_1, e, e, \sigma_1, e, \sigma_2),
\end{align}

where $e$ denotes the trivial braid and $A_{1,i}$ is the braid corresponding to the lift that starts at $z_i$. 
Since $x_1 = -x_3$ and $x_2 = -x_2$, the negatives of the parametrisations of $\sigma_1$ in Equations (42)–(44) can be taken as parametrisations of $\sigma_2$. For example, $z_0$ is the negative of $z_1$. The lift of the parametrisation in Equation (42) that starts at $z_1$ is the trivial braid and ends at $z_2$. Therefore, the lift of the negative of Equation (42) that starts at $z_0$ ends at the negative of $z_2$, which is $z_{20}$, and the corresponding braid is the trivial braid.

We obtain

$$\rho_{3,1}(\sigma_2) = (1 4 7)(2 1 5 14 3)(5 6)(8 9)(10 13 22)(11 18 20 21)
(12 17 24 23)(16 19 25)(26 0)$$

Note that because $\pi(\sigma_2) = (2 3)$, every lift that starts at a point with index 2 mod 3 must end at a point with index 0 mod 3 and vice versa. Therefore the permutation above can be split into those cycles that only contain numbers 1 mod 3 and those that permute numbers that are not 1 mod 3.

The corresponding braids are

$$(B_{1,0}, B_{1,1}, B_{1,2}, ..., B_{1,26}) = (e, \sigma_2^{-1}, \sigma_1, e, \sigma_2, \sigma_2, e, e, e, \sigma_2, e, e, \sigma_2, e, e, \sigma_2^{-1}, \sigma_1, e, \sigma_2, \sigma_2, \sigma_2, e, e, \sigma_2, \sigma_2, e, \sigma_2^{-1}, \sigma_1, e)$$

Example: We consider the braid $\beta = \sigma_1^{12} \sigma_2^{12} \sigma_1^{-12} \sigma_2^{-12}$. Then

$$\rho_{3,1}(\beta) = \rho_{3,1}(\sigma_1)^{12} \rho_{3,1}(\sigma_2)^{12} \rho_{3,1}(\sigma_1)^{-12} \rho_{3,1}(\sigma_2)^{-12} = \text{id.}$$

Some of the $\beta_{1,i}$s are trivial braids. For example,

$$\beta_{1,1} = \prod_{k=0}^{11} A_{1,1}(\rho_{3,1}(\sigma_1))^4(1) \prod_{k=0}^{11} B_{1,1}(\rho_{3,1}(\sigma_2))^4(1) \prod_{k=1}^{12} A_{1,1}^{-1}(\rho_{3,1}(\sigma_1))^{-4}(1) \prod_{k=1}^{12} B_{1,1}^{-1}(\rho_{3,1}(\sigma_2))^{-4}(1)$$

$$= (A_{1,1} A_{1,2})^6(B_{1,1} B_{1,2} B_{1,7})^4(A_{1,1}^{-1} A_{1,2})^6(B_{1,1}^{-1} B_{1,2}^{-1} B_{1,7})^4$$

$$= \sigma_2^6(\sigma_2^{-1} \sigma_2)^4 \sigma_2^{-6}(\sigma_2^{-1} \sigma_2)^4$$

$$= e.$$
We have
\begin{equation}
(\beta_{1,0}, \beta_{1,1}, \beta_{1,2}, \ldots, \beta_{1,26}) = (e, e, S, e, e, T, e, e, T, e, e, e, e, T, e, e, T, e, e, T, e, e, T, e, e, S, e, e, T, e, e, T, e, e, T, e, e, S),
\end{equation}
where \( S = \sigma_2^6 \sigma_1^6 \sigma_6^{-6} \sigma_1^{-6} \) and \( T = \sigma_2^6 \sigma_1^6 \sigma_1^{-6} \).

Hence, like in the case of \( n = 2 \), we witness a certain halving of even exponents. Note that both \( \rho_{2,1}(S) \) and \( \rho_{2,1}(T) \) are non-trivial, so that the action of \( \beta \) on \( 27^2 = 729 \) points is non-trivial. For example \( 27 \times 2 + 2 \) gets sent to \( 27 \times 26 + 2 \).

Recall from Section 5 that \( h(\beta) = (B_{1,0}, B_{1,1}, B_{1,2}, \ldots, B_{1,n^6-1}, \sigma(B)) \). Knowing \( h(\sigma_1) \) and \( h(\sigma_2) \) we can compute \( h(\beta) \) for any braid word \( B \) in a number of steps that grows linearly with the length of the braid word. Thus we can also compute \( h(B_{j,i}) \) in a linear time, which define \( \rho_{3,j+1}(B) \). Note that the length of \( B_{j,i} \) is at most the length of \( B \) for all \( j, i \). Therefore we can compute each \( \rho_{3,j}(B) \), i.e., the action \( \psi_3(B) \) on the \( j \)th coordinate of \( \mathbb{Z}_{n^6} \) in linear time with respect to the length of \( B \). Note however that the number of lifted braids \( B_{j,i} \) for a given \( j \) is \( (n^6)^j \) and thus grows exponentially with \( j \). Therefore in particular actions of braids with a large number of strands \( n \) are still quite expensive to compute.

The braids listed in Equations (46) and (48) are formed by the roots of paths in the space of polynomials corresponding to the lifts of \( \hat{A} = \sigma_1 \) and \( B = \sigma_2 \). Note that \( A_{1,j} = A'_{1,j} \) and \( B_{1,j} = B'_{1,j} \) as introduced in Section 3. We can perform the same computations for the other embedding of \( V_n \) into \( W_n \), i.e., the braid associated to a path in \( V_n \) is given by the union of the 0-strand and the critical points of the corresponding polynomials. In the case of \( \hat{A} = \sigma_1 \) and \( B = \sigma_2 \) these are:
\begin{equation}
(\hat{A}_{1,0}, \hat{A}_{1,1}, \hat{A}_{1,2}, \ldots, \hat{A}_{1,26}) = (\sigma_1^{-1}, e, e, \sigma_2 \sigma_1, e, e, \sigma_1, e, e, \sigma_1, e, e, e, \sigma_1^{-1}, e, e, \sigma_1, e, e, e, \sigma_1, e, e, e, \sigma_2),
\end{equation}
and
\begin{equation}
(\hat{B}_{1,0}, \hat{B}_{1,1}, \hat{B}_{1,2}, \ldots, \hat{B}_{1,26}) = (e, e, \sigma_2^{-1}, e, e, \sigma_1, e, e, e, \sigma_2, e, e, e, \sigma_1, e, e, e, \sigma_2, e, e, e, \sigma_1, e, e, e, \sigma_2). \end{equation}

We find that the braids that are formed by the critical points are often, but not always the same as those formed by the roots. One difference is that in the case of the roots, the lifted braids \( A_{1,j} \) and \( B_{1,j} \) had at most the length of the original braid \( \hat{A} \) or \( B \), while \( \hat{A}_{1,4} \) for example has length 2.

We can use the embedding of \( V_n \) into \( W_n \) where a polynomial is mapped to its set of critical points to construct an action of \( \mathbb{B}_n \) on \( \mathbb{Z}_n \) completely analogous to the construction of \( \psi_n \). By definition the action on \( n^6 \) points of this action is identical with that defined from \( \psi_n \). For bigger values of \( j \) however, the actions on \( (n^6)^j \) points are different, since the braids in Equations (52) and (53) are different from those in (46) and (48).

6. Dynamics

The previous sections give rise to two types of maps, whose dynamics might be worth studying. One is produced by the braid group actions on \( \mathbb{Z}_n \). The problem of describing this system includes for example questions about the orbits of an \( n \)-adic integer under the actions defined in Section 3. No action of the braid group on the \( n \)-adic integers can be transitive, since \( \mathbb{B}_n \) is countable, while \( \mathbb{Z}_n \) is uncountable, but we can prove that the corresponding actions on \( \mathbb{Z}/(n^6-1)/\mathbb{Z} \) and \( \mathbb{Z}/(n \times (n^6-1))/\mathbb{Z} \) are transitive for all \( j \).
The other problem that seems somewhat related to dynamical systems is concerning the image of a point \( v \in W_n \) under repeated application of \( \theta_n^{-1} \) and the image of a point \( v \in W_n \) under repeated application of \( \theta_n \). In this section we give a brief overview of the kind of questions that arise here.

6.1. Transitivity.

**Proposition 6.1.** \( V^j_n \) is path-connected for every \( n \) and every \( j \).

**Proof.** Let \( x, y \in V^j_n \). Then \( x \) and \( y \) are unordered \((n - 1)\)-tuples of non-zero complex numbers \((x_1, x_2, \ldots, x_{n-1})\) and \((y_1, y_2, \ldots, y_{n-1})\) in \( C^{n-1}/S_{n-1} \). Consider a corresponding ordered \((n - 1)\)-tuple, i.e., \( \tilde{x} = (x_1, x_2, \ldots, x_{n-1}) \) and \( \tilde{y} = (y_1, y_2, \ldots, y_{n-1}) \) as points in \( C^{n-1} \). Take a complex affine line \( L \subset C^{n-1} \) containing both \( \tilde{x} \) and \( \tilde{y} \).

A point \( z = (z_1, z_2, \ldots, z_{n-1}) \) in \( C^{n-1} \) corresponds to a point in \( W_n \) if and only if \( z_i \neq z_j \), for all \( i, j \) and \( z_i \neq 0 \) for all \( i \). The space of points in \( C^{n-1} \) that correspond to points in \( W_n \) is therefore given by the complement of a finite number of algebraic varieties in \( C^{n-1} \). Since the polynomials that determine these varieties \( (z_i - z_j) \) are holomorphic, they have only a countable number of zeros on \( L \). So the complement of the algebraic varieties in \( L \) is still path-connected and therefore the set of points in \( C^{n-1} \) corresponding to points in \( W_n \) is path-connected. The quotient by \( S_{n-1} \) preserves path-connectedness, so \( V^j_n \) is path-connected. There are a lot easier ways to see this, but it illustrates how to proceed in the proof of the path-connectedness of \( V^j_n \).

Again let \( z = (z_1, z_2, \ldots, z_{n-1}) \in C^{n-1} \). The map that sends \( z \) to the coefficients of the polynomial \( p(u) = \int_0^u n \prod_{i=1}^{n-1} (w - z_i) dw \) is holomorphic in \( z_i \). Note that \( p \) is a monic polynomial that has \( z \) as its set of critical points and whose constant term is equal to 0. Therefore its critical values \( p(z_i), i = 1, 2, \ldots, n - 1 \), are holomorphic functions of \( z \). If \( j = 1 \), i.e., \( x, y \in V^j_n \), we denote by \( z_x \) and \( z_y \) the sets of critical points of the polynomials \( u \prod_{i=1}^{n-1} (u - x_i) \) and \( u \prod_{i=1}^{n-1} (u - y_i) \) respectively. Consider a path \( \gamma \) in \( C^{n-1} \) from \( z_x \) to \( z_y \). Note that we can choose \( \gamma \) such that the coordinates of \( \gamma \) are always pairwise distinct. In other words, if we write \( \gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_{n-1}(t)) \), \( t \in [0, 1] \), then \( \gamma_1(t) \neq \gamma_2(t) \) for all \( z \in [0, 1] \), \( i \neq k \). Furthermore, we can deform \( \gamma \) to a path that consists of finitely many straight lines \( \ell_i, \ i = 1, 2, \ldots, m \) in \( C^{n-1} \), so it is piecewise linear. We can now pick complex lines \( L_i, i = 1, 2, \ldots, m \) such that \( \ell_i \subset L_i \). We want to show that there is a path in \( C^{n-1} \) from \( z_x \) to \( z_y \) that lifts to a path from \( x \) to \( y \) in \( V^j_n \).

We proceed similarly as in the case of \( W_n \). The set of paths in \( C^{n-1} \) that lift to paths in \( V_n \) are precisely those whose coordinate are pairwise distinct and that avoid points \( z = (z_1, z_2, \ldots, z_{n-1}) \in C^{n-1} \) whose corresponding polynomial \( p(u) = \int_0^u n \prod_{i=1}^{n-1} (w - z_i) dw \) has two non-distinct critical values or a critical value that is equal to 0. Again these points correspond to zeros of a finite number of holomorphic functions on the \( L_i \). These functions are not constant 0 on \( L_1 \) because it contains \( z_x \) and not constant zero on \( L_{d+1} \), because they have a countable number of zeros on \( L_d \cap L_{d+1} \). Thus on all of \( \bigcup L_i \) there are only a countable number of points that a path has to avoid. Furthermore, since the piecewise-linear path \( \gamma \) has pairwise distinct coordinates, we can pick neighbourhoods \( U_i \subset L_i \) of the line segments, in which this property is always satisfied. It follows that the complement of the points we want to avoid in \( \bigcup U_i \) is path-connected. In other words, there is a path from \( z_x \) to \( z_y \) that lifts to a path from \( x \) to \( y \) in \( V^j_n \) and thus \( V^j_n \) is path-connected.

The general case of \( V^j_n \) is again only a small variation of this technique. We use induction on \( j \), so we assume that \( V^{j-1}_n \) is path connected, as shown for \( j = 2 \) above. We start with a given path in \( V^{j-1}_n \) and show that in its neighbourhood there is a path that is actually
in $V_n$ using the fact that the points that are in $V^{j-1}_n$, but not in $V^j_n$ are the zeros of a finite number of holomorphic functions.

Since $x$ and $y$ are in $V^j_n$, they are also in $V^{j-1}_n$. Since $V^{j-1}_n$ is by induction-hypothesis path-connected, there is a path from $x$ to $y$ in $V^{j-1}_n$. Its image under $j-1$ applications of $\theta_n$ is a path in $W_n$. Let this path be given by $(\tilde{z}_1(t), \tilde{z}_2(t), \ldots, \tilde{z}_{n-1}(t)) \subset C^{n-1}$. The critical points of the polynomial $u\prod_{i=1}^{n-1} (u - \tilde{z}_i)$ also form a path in $C^{n-1}$ that we can deform to have pairwise distinct coordinates and to consist of finitely many line segments. The same arguments as before apply. We pick complex lines $L_i$ containing the line segments. The points we need to avoid are zeros of finitely many non-zero, holomorphic functions and therefore there are only countably many of them. Therefore the union of neighbourhoods of the line segments in $L_i$ is path-connected. A path from the set of critical points of $u\prod_{i=1}^{n-1} (u - \tilde{z}_i(0))$ to the set of critical points of $u\prod_{i=1}^{n-1} (u - \tilde{z}_i(1))$ lifts all the way to a path from $x$ to $y$ in $V^j_n$, which finishes the proof.

$\square$

It follows from the definition of $\varphi_n$ and $\phi_n$ that they reduce to actions on $(n^{a-1})^j$ points and $n \times (n^{a-1})^j$ points respectively. These are given by the actions on the points in the fibres in $V^j_n$.  

**Corollary 6.2.** The actions of $B_{n-1}^{a\text{ff}}$ and $B_n$ on $\mathbb{Z}/(n^{a-1})^j \mathbb{Z}$ and $\mathbb{Z}/(n \times (n^{a-1})^j) \mathbb{Z}$ that are reduced from $\varphi_n$ and $\phi_n$ respectively are transitive for all $n$ and $j$.

**Proof.** It is well-known that the monodromy action on a fibre is transitive if the covering space is path-connected, which proves the statement for $\varphi_n$. Strictly speaking the action $\phi_n$ is not a monodromy action, but it is obviously very closely related as it also maps a point in the covering space to the end point of a lifted path. The only difference is that the paths in the base space are not necessarily loops, but paths starting and ending at a prescribed set of points. Thus the action permutes points on a fixed set of fibres. The standard proof that path-connectedness of the covering space implies transitivity is completely unchanged by this difference.

$\square$

Note that the transitivity on each level does not contradict the lack of transitivity on their inverse limit $\mathbb{Z}_{n-1} \cong \mathbb{Z}_0$ and $\mathbb{Z}/(n \mathbb{Z} \times \mathbb{Z}_{n-1}) \cong \mathbb{Z}_0$, since in general the orbit map does not commute with the inverse limit. Some conditions when they do commute are given in [20], but they are not satisfied here.

With exactly the same methods as in Proposition 6.1 and Corollary 6.2 we can prove the analogous statements for the spaces $V_n$, defined from the embedding of $V_n$ into $W_n$ that sends to a polynomial to its critical points.

### 6.2. Preimage and image sets.

We study the image of a given point in $W_n$ under repeated application of $\theta_n^{-1}$. This is a sequence of points, indexed by the $n$-adic integers, in $W_n$ and so the points are unordered $n-1$-tuples of non-zero complex numbers. We have already mentioned a certain symmetry among the preimage points under $\theta_n$, namely if $(z_1, z_2, \ldots, z_{n-1}) \in V_n$ maps to $x \in W_n$, then $\xi^k (z_1, z_2, \ldots, z_{n-1})$ also maps to $x$ for all $k = 0, 1, 2, \ldots, n-2$, where $\xi = e^{2\pi i/n}$ is an $n$th root of unity and multiplication is defined componentwise. It follows by induction on $j$ that the set of complex numbers that are components of the preimage set of $j$ applications of $\theta_n$ of a point $x \in W_n$ is symmetric under rotation by $\frac{2\pi}{m}$. Let $\mathcal{Z}_x^j$ be the set of the complex numbers $z$ such that there exist distinct non-zero complex numbers $z_i \neq z$, $i = 1, 2, \ldots, n-2$ such that $j$ repeated applications of $\theta_n$ to $(z_1, z_2, \ldots, z_{n-2}, z)$ result in $x \in W_n$. Let $\mathcal{Z}_x = \bigcup_j \mathcal{Z}_x^j$. Then it follows that the arguments of numbers $z \in \mathcal{Z}_x$ is dense in $S^1$ for all $x \in W_n$. 

unordered (is still well-defined as the map that sends a polynomial $u$ seem to lie in a finite interval. We do not have to stop here though. The map $\theta$ of $x$ where at least two of them are equal. We do not have to stop here though. The map $\theta$ no matter how many times we apply one maximum with positive real value and one minimum with negative real value. Hence, $\mathbb{W}$ in Therefore, if $x$ is chosen arbitrarily. We can see that the moduli of points in $\bigcup_{j=1}^{k} \mathcal{C}_{j}^{f}$ for $x = 1$ and $k = 4, 7, 10$.

For the case of $n = 3$ we plot $\bigcup_{j=1}^{k} \mathcal{C}_{j}^{f}$ for $k = 3, 4$ in Figure 4. The base point $(\sqrt{3}/2, -1)$ is chosen arbitrarily. We can see that the moduli of points in $\bigcup_{j=1}^{k} \mathcal{C}_{j}^{f}$ for $n = 3$ seem to lie in a finite interval.

The sets $\mathcal{C}_{j}$ offer insight in the complex numbers that arise under repeated application of $\theta_n^{-1}$ to $x$. We could just as well repeatedly apply $\theta_n$ to $x$. There are different cases to consider. If $x \in V_n^j$, then the first $j$ applications of $\theta_n$ will result in points that are in $W_n$. Therefore, if $x \in \bigcap_j V_n^j$, then no matter how many times we apply $\theta_n$, we always end up in $W_n$. If we take for example any tuple that consists of one positive real number $x_1$ and one negative real number $x_2$. Then the corresponding polynomial $u \prod_{i=1}^{n} (u - x_i)$ has again one maximum with positive real value and one minimum with negative real value. Hence, no matter how many times we apply $\theta_n$, we always obtain a set of critical values that are non-zero and disjoint and thus in $\mathbb{W}$.

In contrast, if $x \in V_n^j \setminus V_n^{j+1}$, then the $j + 1$th application of $\theta_n$ results in a point $y$ that is not in $W_n$. This means it gives an unordered $(n - 1)$-tuple of non-zero complex numbers, where at least two of them are equal. We do not have to stop here though. The map $\theta_n$ is still well-defined as the map that sends a polynomial $u \prod_{i=1}^{n-1} (u - z_i)$ (or equivalently an unordered $(n - 1)$-tuple $(z_1, z_2, \ldots, z_{n-1})$) to its set of critical values. It is just not a covering
map anymore. Since at least two components of \( y \) are equal, the corresponding polynomial has a double root. Hence \( \theta_n(y) \) is an unordered \((n - 1)\)-tuple of complex number with at least one of them equal to 0. The polynomial corresponding to this tuple now has a double root at 0, so its image under \( \theta_n \) also contains 0. Therefore, from now on every additional application of \( \theta_n \) will result in a tuple that contains 0.

We can study the sequence \( a_j \in \mathbb{N} \) of number of zeros in \( \theta_n^j(x) \). Note that it is monotone increasing and obviously bounded above by \( n - 1 \). Therefore, it must be constant after a while and there is a well-defined limit value \( a = \lim_{j \to \infty} a_j \). In fact, the polynomial \( p \equiv 0 \) is the only polynomial with \((0, 0, \ldots, 0)\) as its set of critical values. Therefore, the sequence is actually bounded by \( n - 2 \).

For points like in the example above \((n = 3, \) one positive, one negative entry\), the sequence \( a_j \) is constant 0, but for all points that are not in \( \bigcap_j V_n^j \) the limit is at least 1. In the case of \( n = 3 \) this implies that the limit is 0 for \( \bigcap_j V_3^j \) and 1 in all other cases.

Again, the discussion in this section is focused on the relation between the roots and the critical values of polynomials. We could easily proceed analogously to study the relation between critical points and critical values. We take a sequence \( c = (c_1, c_2, \ldots, c_{n-1}) \) and identify it with the polynomial \( f(w) = \int_1^a \prod_{i=1}^{n-1} (w - c_i)dw \). We then send \( c \) to the set of critical values of \( f \). If \( f \) is in \( V_n \), then the image of this map is in \( W_n \). In any case we obtain a new \((n - 1)\)-tuple, which we can use to iterate the process. In this case, it is not the number of zeros that is increasing, but the number of pairs of identical entries.

7. Normal subgroups

The three actions \( \phi_n, \phi_n, \) and \( \psi_n \) on \( \mathbb{Z}_n \) each provide us with a sequence of homomorphisms \( \mathbb{B}_{n-1}^{\text{aff}} \to S_{(n^2 - 1)/j} \), \( \mathbb{B}_n \to S_{n \times (n^2 - 1)/j} \), and \( \mathbb{B}_n \to S_{(n^2)} \), respectively. These correspond to the restricted actions on the fibre in \( V_n^j \) and its analogues for \( \phi_n \) and \( \psi_n \). Therefore, we have a descending series of normal subgroups \( N_j \) of \( \mathbb{B}_{n-1}^{\text{aff}} \) and two descending series of normal subgroups \( M_j, H_j \) of \( \mathbb{B}_n \) corresponding to the kernels of these homomorphisms. The fact that they are descending sequences, i.e., \( N_{j+1} \subseteq N_j, M_{j+1} \subseteq M_j \) and \( H_{j+1} \subseteq H_j \) for all \( j \in \mathbb{N} \), follows from the compatibility of the homomorphisms for different \( j \), which follows from the definition of the actions.

In the case of \( \phi_n \) and \( \psi_n \), the action on \( n^n \) points (i.e., \( j = 1 \)) is compatible with the permutation representation \( \pi : \mathbb{B}_n \to S_n \). This means that the normal subgroups \( M_j \) and \( H_j \) are in fact normal subgroups of the pure braid group.

The existence of these descending series of normal subgroups raises the question whether they stabilize. In other words, is there a \( j \) such that \( N_j = N_k \) for all \( k \geq j \) and similarly for \( M_j \) and \( H_j \)? We say a group \( G \) is Artinian with respect to normal subgroups if every descending series of normal subgroups \( G = G_0 \geq G_1 \geq G_2 \geq \ldots \) stabilizes.

In this section we show that the constructed descending sequences of normal subgroups do not stabilize. The braid group \( \mathbb{B}_n \) is therefore not Artinian with respect to normal subgroups.

With the exception of \( \psi_2 \) (cf. Section 5) it is not known if the presented actions \( \phi_n, \phi_n, \) and \( \psi_n \) are faithful. In the case of \( \psi_2 \), the calculations in Section 5 show us not only that \( \bigcap_{j=1}^\infty H_j = \{ e \} \), but also that \( \sigma_2^{2j+1} \in H_j \). Thus for \( n = 2 \) the descending sequence \( H_j \) does not stabilize.

**Lemma 7.1.** The descending sequences \( N_j \) of \( \mathbb{B}_{n-1}^{\text{aff}} \) and \( M_j \) of \( \mathbb{B}_n \) do not stabilize for any \( n \).
Proof. Since $V_n^j$ is path-connected for all $n$ and $j$ (cf. Proposition 6.1), the image of $\mathbb{B}_n$ in $S_{n \times (n^n-1)^j}$ consists of at least $n \times (n^n-1)^j$ permutations, which is a lower bound for the index of $M_j$. Since every $M_j$ has finite index, this shows that the sequence cannot stabilize. The proof for $N_j$ is completely analogous. □

Corollary 7.2. For every $n$ the braid group $\mathbb{B}_n$ and the affine braid group $\mathbb{B}_{n-1}^{aff}$ are not Artinian with respect to normal subgroups.

Corollary 7.2 is already known from another example of a non-stabilizing descending series of normal subgroups, namely the lower central series of the pure braid groups [12].

In order to prove that the normal subgroups $H_j$ do not stabilize either, we need several lemmas.

Lemma 7.3. Let $x \in V_n$ with 0 in $i$th position and $y \in V_n$ with 0 in $j$th position. Let $B \in \mathbb{B}_n$ be a braid with $\pi_B(i) = j$. Then there is a path from $x$ to $y$ in $V_n$ that corresponds to a parametrisation of $B$.

Proof. This proof is a variation of the proof of Proposition 6.1. Let $x = (x_1, x_2, \ldots, x_{n-1})$, $y = (y_1, y_2, \ldots, y_{n-1}) \in V_n$. If $\pi_B(i) = j$, there is a path $\gamma'(t) = (\gamma'_1(t), \gamma'_2(t), \ldots, \gamma'_{n-1}(t))$, $t \in [0, 1]$ from $x$ to $y$ in $\mathbb{C}^{n-1}/S_{n-1}$ that is a parametrisation of $B$. Recall that in the proof of Proposition 6.1 we showed the path-connectedness of $V_n$ by starting with a path $\gamma$ in $\mathbb{C}^{n-1}$ from $z_1$ to $z_2$, the (arbitrarily ordered) sets of critical points of the polynomials $u\prod_{i=1}^{n-1}(u-x_i)$ and $u\prod_{i=1}^{n-1}(u-y_i)$. We can choose this path $\gamma$ to be the sets of critical points of the polynomials $u\prod_{i=1}^{n-1}(u-\gamma'_i(t))$. The proof of Proposition 6.1 then proceeds by showing that $\gamma$ can be deformed to a path that lifts to a path from $x$ to $y$ in $V_n$. Since this deformation also lifts to a homotopy of the original path $\gamma'$, the resulting path is still a parametrisation of $B$. □

Lemma 7.4. Let $n > 2$. There is a basepoint $z_i \in W_n$ that has 0 in the $\lfloor \frac{n+1}{2} \rfloor$th position and that has two preimage points $x, y \in V_n$ that have 0 in the $\lfloor \frac{n+1}{2} \rfloor$th position.

Proof. Consider an $n-1$-tuple $x$ of real numbers with 0 in $\lfloor \frac{n+1}{2} \rfloor$th position. Note that $z = \theta_n(x)$ is also an $n-1$-tuple of real numbers with 0 in $\lfloor \frac{n+1}{2} \rfloor$th position. Recall that the preimage set of a point in $W_n$ under $\theta_n$ is invariant under multiplication by $e^{2\pi i/n}$. Multiplying $x$ by $e^{2\pi i/n}$ thus results in another preimage point of $z$, say $y$. If $n \neq 2, 4$, then $y$ also has 0 in $\lfloor \frac{n+1}{2} \rfloor$th position and $y$ is different from $x$ because it does not consist of real numbers. For $n = 4$ we can apply the same argument to an $n-1$-tuples $x$ of complex numbers that have an argument of $\pm \epsilon$, for some small $\epsilon > 0$, and 0 in $\lfloor \frac{n+1}{2} \rfloor$th position. □

Proposition 7.5. The sequence $H_j$ does not stabilize.

Proof. Consider the points $x, y \in V_n$ given by Lemma 7.4. Since $V_n$ is path-connected, there is a path from $x$ to $y$ and a path from $x$ to $y$. Applying $\theta_n$ to these paths results in two loops in $W_n$ with basepoint $z = \theta_n(x)$. The two loops correspond to two braids, say $B_1$ and $B_2$, and $\pi_B(\lfloor \frac{n+1}{2} \rfloor) = \lfloor \frac{n+1}{2} \rfloor$ for $j = 1, 2$. Note that $B_1$ and $B_2$ induce different permutations on $\mathbb{Z}/n\mathbb{Z}$. One of them fixes $x$, while the other maps $x$ to $y$.

Since $x$ and $y$ are in $V_n$, they are also both in $W_n$ and since they both have 0 in $\lfloor \frac{n+1}{2} \rfloor$th position, there is a path in $W_n$ from $x$ to $y$ that corresponds to a parametrisation of $B_j$, $j = 1, 2$. The same is true for the existence of a path from $x$ to $y$ corresponding to $B_j$, $j = 1, 2$. Thus in total we have four paths in $W_n$, two corresponding to $B_1$ and two to $B_2$. By Lemma 7.3 there are four paths in $V_n$, two from $x$ to $x$ and two from $x$ to $y$, corresponding to parametrisations of $B_1$ and $B_2$. Applying $\theta_n$ to these four paths gives four loops in $W_n$. □
based at \( z \), corresponding to four different braids, whose permutation representation fixes \( \frac{2j + 1}{2} \). Note that by construction all of these four braids induce different permutations on \( \mathbb{Z}/(n^n)^2 \mathbb{Z} \). This process can be iterated an arbitrary number of times and we obtain \( 2^j \) different braids that induce different permutations on \( \mathbb{Z}/(n^n)^2 \mathbb{Z} \). In other words, the index of \( H_j \) is bounded below by \( 2^j \). Exactly like in the proof of Lemma 7.1 this implies that the sequence \( H_j \) does not stabilize.

Note that the study of the normal subgroups \( N_j, M_j \) and \( H_j \) is closely related to properties of the corresponding actions. In particular, \( \phi_n \) is faithful if and only if the intersection of all subgroups in the sequence is the trivial braid, \( \bigcap_{j=1}^{\infty} N_j = \{ e \} \). Analogous statements for \( \phi_n \) and \( \psi_n \) hold.

Again, the results from this section remain true, when one considers the alternative embedding of \( V_n \) into \( W_n \), given by the map that sends a polynomial to its critical points.

8. CONSTRUCTIONS OF REAL ALGEBRAIC LINKS

We can use the computations from Section 5 to construct real algebraic links in \( S^3 \). We use this term in the sense of Perron [17] as the real analogues of Milnor’s algebraic links, links of isolated critical points of polynomials \( f : \mathbb{R}^4 \rightarrow \mathbb{R}^2 \). This should not be confused with knotted algebraic varieties in \( \mathbb{R}P^3 \) as they were introduced by Viro [21], which are also called real algebraic links.

**Definition 8.1.** A link \( L \) is real algebraic if there exists a polynomial \( p : \mathbb{R}^4 \rightarrow \mathbb{R}^2 \) such that

- \( p \) has an isolated singularity at the origin, i.e., \( p(0) = 0, \nabla p(0) = 0 \) and there is a neighbourhood \( U \) of 0 such that 0 is the only point in \( U \) where the rank of \( \nabla p \) is not full,
- \( p^{-1}(0) \cap S^3_0 = L \) for all small enough radii \( \rho \).

The number 0 in Definition 8.1 refers to the origin in \( \mathbb{R}^4 \) and \( \mathbb{R}^2 \) and the zero matrix of size 4-by-2, respectively.

Milnor showed that all real algebraic links are fibred [14], but in contrast to the algebraic links the real algebraic links are not classified yet. Benedetti and Shiota conjectured that all fibred links are real algebraic links [2]. So far however, the set of links that are known to be real algebraic is still comparatively small. There are of course the algebraic links, but also the connected sum \( K \# \bar{K} \) of any fibred knot \( K \) with itself [13]. Perron [17] and Rudolph [18] constructed polynomials for the figure-eight knot. We showed in [5] that all closures of squares of homogeneous braids are real algebraic (cf. Definition 2.2).

The proof in [5] can be summarised as follows. Recall from Section 2 and in particular Proposition 2.2 that for every homogeneous braid \( B \) there is a loop in the space of monic polynomials of degree \( n \) with distinct roots \( f_i \in \mathbb{V}_n \) such that the roots of \( f(\cdot,t) = f_i(\cdot) \) form the braid \( B \) and \( \arg f|_{C \times [0,2\pi]} \) \( B \rightarrow S^1 \) is a fibration. We can approximate the strands of \( B \), given by the roots of \( f_i \) as parametric curves, by trigonometric polynomials \( F_C, G_C : [0,2\pi] \rightarrow \mathbb{R} \) for every component \( C \) of the closure of \( B \) such that

\[
\bigcup_{C} \bigcup_{j=1}^{n_C} \left( F_C \left( \frac{t + 2\pi j}{n_C} \right), G_C \left( \frac{t + 2\pi j}{n_C} \right), t \right), \quad t \in [0,2\pi]
\]

is a parametrisation of \( B \), where \( n_C \) denotes the number of strands in the component \( C \) of the closure of \( B \). Furthermore, since trigonometric polynomials are dense in the set of
periodic functions with respect to the $C^1$-norm, we can do this approximation such that $\arg g_\lambda$ is a fibration for all $\lambda > 0$, where
\begin{equation}
(55)
g_\lambda : \mathbb{C} \times [0, 2\pi] \to \mathbb{C}, \quad g_\lambda(u, t) = \prod_{c}^{n_c} \prod_{j=1}^{n_c} \left( u - \lambda F_C \left( \frac{t + 2\pi j}{s_c} \right) \right) \left( 1 + \frac{t + 2\pi j}{s_c} \right).
\end{equation}

Note that the roots of $g_\lambda$ are precisely the parametrisation of $B$ given by Equation (54) scaled by the factor $\lambda$.

We then define $p_{\lambda, k} : \mathbb{C}^2 \to \mathbb{C}$
\begin{align}
(56) & \quad \bar{p}_{\lambda, k}(u, re^{it}) = r^{2nk} g_\lambda \left( \frac{u}{\sqrt{2}}, t \right), \\
(57) & \quad \bar{p}_{\lambda, k}(u, 0) = u^n
\end{align}
where $n$ is the number of strands of $B$ and $k$ a sufficiently large integer. Then $\bar{p}_{\lambda, k}$ is a polynomial in $u, v, \varphi$ and $\sqrt{\varphi}$. Changing the variable from $t$ to $2t$ results in
\begin{align}
(58) & \quad p_{\lambda, k}(u, re^{it}) = r^{2nk} g_\lambda \left( \frac{u}{\sqrt{2}}, 2t \right), \\
(59) & \quad p_{\lambda, k}(u, 0) = u^n,
\end{align}
which can be written as a complex polynomial in $u, v$ and $\varphi$ and is thus a polynomial map $\mathbb{R}^4 \to \mathbb{R}^2$. The polynomial has an isolated singularity at the origin and the closure of $B^2$ is the link of that singularity if $\lambda$ is chosen sufficiently small (cf. (5)). The fact that $\arg g_\lambda$ is a fibration ensures that the singularity is isolated, while the 2-periodicity of the braid makes $p$ into a polynomial.

If we want to generalise this construction, it suffices to find other braids $B$ that can be parametrised as in Equation (54) such that the resulting polynomial $g_\lambda$ in Equation (55) has no argument-critical points, i.e., $\arg g_\lambda$ is a fibration. Then the closure of $B^2$ is again real algebraic. This can be summarized in the context of the covering map $\theta_n : V_n \to W_n$ as follows.

**Proposition 8.2.** Let $\gamma = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_{n-1}(t)), \ t \in [0, 1]$, be a loop in $W_n$ such that $\frac{\partial \gamma_i}{\partial t}(h) \neq 0$ for all $i = 1, 2, \ldots, n - 1$ and all $h \in [0, 1]$ and such that one of the lifts of $\gamma$ is a loop $\tilde{\gamma}$ in $V_n$. We denote the braid that is traced out by the roots of $\tilde{\gamma}$ by $B$. Then the closure of $B^2$ is again real algebraic.

In the case of homogeneous braids we found that this is possible by finding a family of braids, $\prod_{j=1}^{k} \gamma_{ij}$, that can be parametrised by $(0, v_1(t), v_2(t), \ldots, v_{n-1}(t))$ such that $\frac{\partial \arg \gamma_i}{\partial t}$ never vanishes and identifying $\prod_{j=1}^{k} \sigma_{ij}$ as one of its lifts in $V_n$ (5). We now focus on the case of $n = 3$.

**Corollary 8.3.** Let $B$ be a 3-strand braid of the form $\prod_{j=1}^{k} \sigma_{i_1}^{2k} \sigma_{i_2}^{-1} \sigma_{i_3} \sigma_{i_3}$, where all $k_j, l_j$ are non-zero integers with $\text{sign}(k_i) = \text{sign}(k_j)$ and $\text{sign}(l_i) = \text{sign}(l_j)$ for all $i, j = 1, 2, \ldots, k$. Then the closure of $B^2$ is real algebraic.

**Proof.** The computations of Section 5 tell us all lifts of any 3-strand braid. Some of these are only paths in $V_3$ and not loops and are therefore of no use for us. However, we find that the lift of $\sigma_{i_3}$ that starts at $z_{10}$ is a loop and forms the braid $(\sigma_{i_3}^2)^{1,10} = \sigma_{i_3}^2$. Furthermore, the lift of $Y_3 = \sigma_{i_3}^{-1} \tau_{i_2}^2 \sigma_{i_1}^2$ that starts at $z_{10}$ is also a loop and forms the braid $\sigma_{i_3}^{-1} \sigma_{i_2} \sigma_{i_1}$. Thus the lift of $\prod_{j=1}^{k} \sigma_{i_3}^{2k} \sigma_{i_2}^{-1} \sigma_{i_3} \sigma_{i_2}$ that starts at $z_{10}$ is a loop in $V_3$ and the corresponding braid is $B = \prod_{j=1}^{k} \sigma_{i_3}^{2k} \sigma_{i_2}^{-1} \sigma_{i_3} \sigma_{i_2}$. 


Figure 5. If all $k_j$ and all $l_j$ have matching signs, then the braid $B$ can be parametrised as $(0, v_1(t), v_2(t))$, $t \in [0, 2\pi]$, where the $v_j$ are moving on ellipses around the origin and never change direction (e.g., from clockwise to anti-clockwise). a) The movement of $(0, v_1(t), v_2(t))$ in the complex plane. b) The strands of the braid $\sigma_1^{-2} \sigma_2^{-1} \sigma_1^2 \sigma_2$ can be parametrised by $(0, v_1(t), v_2(t))$ such that the projection gives the curves in a).

Note that $\prod_{j=1}^k \sigma_1^{4k_j} Y_j$ can be parametrised as desired by $(0, v_1(t), v_2(t))$ such that $\frac{\partial \arg v_j}{\partial t}$ never vanishes as long as all $k_j$ have the same sign and all $l_j$ have the same sign. It can for example be arranged that each $v_j$ moves on an ellipse as in Figure 5 where the direction of movement (clockwise or anti-clockwise) is determined by the sign of $k_j$ (for $v_1$) and the sign of $l_j$ (for $v_2$). Thus the arguments from [5] apply and hence the closure of $B^2$ is real algebraic.

Just like in [5] we approximate the strands of the braid corresponding to the roots of the lift in $V_3$ by trigonometric polynomials and define the polynomial $g_\lambda$ as in Equation (55) such that $\arg g_\lambda$ is a fibration. Then $p_{\lambda, k}$ as in Equation (58) is the desired real polynomial with an isolated singularity at the origin and the closure of $B^2$ is the link of that singularity. □

In a follow up paper we study more systematically which affine braids have a parametrisation without turning points and that have a loop as one of their lifts, whose corresponding braid is not already covered by Corollary 8.3 or [5].

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