Spiral instabilities: Linear and nonlinear effects

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ABSTRACT
We present a study of the spiral responses in a stable disc galaxy model to co-orbiting perturbing masses that are evenly spaced around rings. The amplitudes of the responses, or wakes, are proportional to the masses of the perturbations, and we find that the response to a low-mass ring disperses when it is removed – behaviour that is predicted by linear theory. Higher mass rings cause nonlinear changes through scattering at the major resonances, provoking instabilities that were absent before the scattering took place. The separate wake patterns from two rings orbiting at differing frequencies, produce a net response that is an apparently shearing spiral. When the rings have low mass, the evolution of the simulation is both qualitatively and quantitatively reproduced by linear superposition of the two separate responses. We argue that apparently shearing transient spirals in simulations result from the superposition of two or more steadily rotating patterns, each of which is best accounted for as a normal mode of the non-smooth disc.

Key words: galaxies: spiral — galaxies: evolution — galaxies: structure — galaxies: kinematics and dynamics — instabilities

1 INTRODUCTION
In earlier papers (Sellwood & Carlberg 2014, 2019), we argued that spiral patterns in galaxies can result from a recurrent cycle of instabilities in the stellar disc. Each instability is excited by a deficiency in the distribution of stars over a narrow range of angular momentum, that was created by resonant scattering by a previous disturbance. The cycle can be initiated by the infall of a mass clump during disc assembly, say, or by a tidal interaction, or in the unlikely event that neither of these triggers occurs, it can bootstrap out of the noise (Sellwood 2012). Resonant scattering by successive waves raises the level of random motion wherever it occurs, making the surviving disc gradually less responsive so that the recurrent cycle is self-limiting in the absence of a dissipative gas component (Sellwood & Carlberg 1984; Carlberg & Freedman 1985; Toomre 1990; Roskar et al. 2008; Aumer et al. 2016).

The vigorous instabilities we invoke are normal modes of the modified disc. They can be calculated by first order perturbation theory (Kalnajs 1971; Sellwood & Kahn 1991) that assumes an infinitesimal perturbation and neglects possible changes to the background state. A linear instability will grow from any arbitrarily small seed perturbation, but the exponential growth must saturate at some finite amplitude due to the breakdown of the assumption that second- and higher-order terms can be neglected. Scattering at resonances is another nonlinear effect that changes the background state of the disc by wave-particle interactions that are strictly second order (Lynden-Bell & Kalnajs 1972; Mark 1974). Thus our proposed mechanism invokes both linear perturbation theory to account for the growth of the spirals, and nonlinear scattering at resonances to account for the seeding of subsequent instabilities.

Swing-amplification (Goldreich & Lynden-Bell 1965; Toomre 1981; Binney & Tremaine 2008), which also causes wakes to form around co-orbiting perturbing masses (Julian & Toomre 1966; Binney 2020), is another important result from linear perturbation theory in stellar discs in which second order changes are also neglected. But it is not an instability, in the sense of a normal mode, and it merely amplifies an input perturbation by a finite, sometimes large, factor. However, scattering of stars caused by the ultimate resonant damping of the disturbance (Toomre 1981) may change the stability properties of the entire disc (Sellwood 2012), even for tiny perturbations.

A number of authors (Wada et al. 2011; Grand et al. 2012a,b; Roca-Fàbrega et al. 2013; Kawata et al. 2014; Dobbs & Baba 2014; Baba 2015; Michikoshi & Kokubo 2018, 2020) argue for an alternative picture, in which spiral patterns are hardly density waves at all, but wind more tightly over time at a rate that is almost as rapid as if they were material features. These authors find evidence from their simulations that swing-amplification plays a prominent role in the development of the spirals and some aspects of the behaviour are attributed to nonlinearities (Baba et al. 2013; Kumamoto & Noguchi 2016).

A third mechanism for the origin of spirals is ar-

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gued by Toomre (1990), Toomre & Kalnajs (1991), and D’Onghia et al. (2013). These authors propose that galaxies do not possess spiral instabilities at all, and that the patterns are no more than the collective responses of the surrounding disc to co-orbiting density inhomogeneities, such as giant molecular clouds perhaps. The linear theory of this behaviour was originally set out by Julian & Toomre (1966), but D’Onghia et al. (2013) show that massive enough clumps in a low-mass disc can trigger nonlinearities in the response that lead to continuing development of patterns after the seed mass has been removed.

Thus all three of these proposed mechanisms for spiral generation invoke nonlinearities. Here we endeavour to clarify the distinction between linear and nonlinear behaviour in disc galaxy simulations. We present experiments that employ a stable disc model and have large enough particle numbers that the evolution of mild perturbations reflects the predictions of linear theory, while larger disturbances behave differently.

Our purpose in presenting these simplified experiments is to shed additional light on the behaviour of self-gravitating disturbances in shearing discs. Our idealized experiments are more easily understood than those bearing a closer resemblance to real galaxies and illustrate clearly how visual impressions can be misleading.

We do not dispute the numerical results of others, but present an alternative interpretation of shearing spirals in §6. We also confirm one result from D’Onghia et al. (2013) in §7, but show that wakes in more massive discs sculpt the distribution function in a manner to excite new instabilities. Thus we find that the only nonlinearity one needs to invoke to explain the appearance of fresh disturbances seeded by a larger perturbation is scattering of particles at the major resonances of the original disturbance.

2 TECHNIQUE

As we noted before (Sellwood & Carlberg 2019), the dynamical behaviour of fully self-consistent simulations can be hard to unravel, and we therefore find it fruitful to run simplified simulations that can capture the phenomena we wish to study without them being obscured by unrelated activity. We therefore first choose a model that has been proven to be globally stable and study its response to controlled perturbations, while restricting disturbance forces to a single sectoral harmonic. We present a different model in §7.

2.1 Mestel disc model

Once again, we adopt the razor-thin Mestel disc used in the studies by Zang (1976) and Toomre (1981), which is characterized by a constant circular speed \( V_0 \) at all radii. The surface density of the full-mass disc is \( \Sigma_0(R) = V_0^2/(2\pi GR) \).

The advantage of using this model is that Toomre (1981) reported that the half-mass Mestel disc, with \( Q = 1.5 \) and a suitable central cut out had no linear instabilities whatsoever. Thus an N-body realization having a large number of particles to beat down shot noise, should not evolve. Tests of this prediction (Sellwood 2012) confirmed that it was true for quite a long time, though he also reported that instabilities eventually appeared in very long integrations. Here we do not run our simulations for nearly long enough for this subtle change of behaviour to affect the results we report.

The particle distribution function (DF), tapers and cut-offs are the same as in eqs. (4) – (6) of Sellwood & Carlberg (2019) and we again adopt units such that \( V_0 = R_0 = G = 1 \), where \( R_0 \) is the central radius of the inner cut out. As before we halve the disc mass, and maintain centrifugal equilibrium by keeping the central attraction \( a_R = -V_0^2/R \) at all times. Figure 1 gives the surface density of the active mass (solid line, upper panel); the tapers change the power law profile (dotted curve) to one that is approximately exponential with a central hole. The solid horizontal line in the lower panel is the circular speed; the contribution that the particles would make is drawn by the dashed curve, indicating that our disc is significantly sub-maximal.

The natural unit of mass of the full-mass Mestel disc is \( V_0^2 R_0/G \), which is the mass enclosed within \( R_0 \), and the disc mass would increase linearly with \( R \) in the absence of tapers. The total mass of the particles in our model, after allowing for the various tapers, truncations and halving of the active mass, is \( M_{\text{act}} \approx 5.40 \).

2.2 Numerical method

The particles in our simulations are constrained to move in a plane over a 2D polar mesh at which the self-gravitational attractions are calculated and interpolated to the position of each particle. A full description of our numerical procedures
is given in the on-line manual (Sellwood 2014) and the code itself is available for download. Table 1 gives the values of the numerical parameters for all the simulations presented in §§3–5. Sellwood (2012) reported, and we have reconfirmed in this study, that all our results are insensitive to reasonable changes to grid resolution, time step and zones, and number of particles. Doubling the softening length does reduce the responsiveness of the disc, but does not alter the qualitative behaviour.

Since the gravitational field is a convolution of the mass density with a Green function that is most efficiently computed by Fourier transforms, it is easy to restrict the sectoral harmonics that contribute to the field when using a polar grid. In the simulations we report here, the disturbance forces arising from the particles are confined to just the \( m = 3 \) sectoral harmonic. Imposing 3-fold symmetry implies slightly more vigorous swing amplification (Toomre 1981; Binney & Tremaine 2008, their Figure 6.21) than for \( m = 2 \) because \( X = 4/m \) in this half-mass Mestel disc.

Not only does our choice of a grid code enable us readily to select active sectoral harmonics, but it is very much faster than tree codes. Sellwood (2014) reported performance benchmarks and also demonstrated that the computed forces acting on each particle are very nearly the same in the two codes. Our method is admittedly less versatile than tree codes, but is ideally suited for the evolution of isolated collisionless models. For this application, its performance advantage over tree codes can be likened to that of FFTs over direct FTs – it gets the same answer in a fraction of the cpu time. Furthermore, our code has been validated by verification of the normal modes predicted by linear theory (Sellwood & Athanassoula 1986), and for non-linear evolution (Inagaki et al. 1984), and confirms the predicted stability of the half-mass Mestel disc (Toomre 1981; Sellwood 2012, and this paper).

### Table 1. Numerical parameters

| Parameter                  | Value                                  |
|----------------------------|----------------------------------------|
| Grid size                  | \( 106 \times 128 \)                   |
| Active sectoral harmonic   | 3                                      |
| \( R_0 \)                  | 8 grid units                           |
| Softening length           | \( R_0/8 \)                            |
| Number of particles        | \( 5 \times 10^7 \)                    |
| Basic time-step            | \( R_0/(80V_0) \)                      |
| Time step zones            | 5                                      |
| Guard zones                | 4                                      |

2.3 Other details

The models we present in §4 generally include three co-orbiting masses equally spaced around a ring of radius \( R_p \), and in some cases we employ two such rings at differing radii. The perturbing masses were each Plummer spheres (Binney & Tremaine 2008) with masses \( M_p \), a parameter we vary, and core radius \( a = 0.05R_0 \). The attraction of these perturbing masses on each disc particle is computed directly, and added to the non-axisymmetric forces from the other particles, computed through the grid, plus the fixed central attraction. The ring particles are moved at each step at fixed speed \( V_0 \) around a circle of radius \( R_p \).

Table 2 lists all the simulations reported in §§3–5.

We will use the radial action of a particle, \( J_R \equiv \int \rho dR/(2\pi) \) (Binney & Tremaine 2008), which has dimensions of angular momentum. It is zero for a circular orbit, and increases with the eccentricity of the orbit. In an axisymmetric potential, the angular momentum, \( L_z \), is the other action for orbits confined to a plane.

We also make extensive use of logarithmic spiral transforms of the particle distribution, which for \( N \) particles of equal mass, as here, is defined as

\[
A(m, \tan \gamma, t) = \frac{1}{N} \sum_{j=1}^{N} \exp[i(m\phi_j + \tan \gamma \ln R_j)],
\]

where \((R_j, \phi_j)\) are the cylindrical polar coordinates of the \( j \)-th particle at time \( t \), and \( \gamma \), the complement to the spiral pitch angle, is the angle between the radius vector and the tangent to an \( m \)-arm logarithmic spiral, with positive values for trailing spirals. We usually plot the time evolution of the amplitude \(|A|\), which ignores the complex phase. If the particles were randomly distributed, the expectation value of \(|A^2| \equiv 1/N\), while \(|\langle A \rangle \equiv [\pi/(4N)]^{1/2}\) for any \( m \) and \( \tan \gamma \) (Mardia 1972).

3 A SIMULATION WITHOUT FORCING

Although a smooth stellar fluid model of this disc is globally stable (Toomre 1981), the shot noise in any particle realization induces a swing-amplified response. We therefore first report model 0 to calibrate the evolution of this 50M particle disc in the absence of any driving masses when, for the reasons given in §2, disturbance forces were restricted to \( m = 3 \) only.

Figure 2 shows the time evolution of the amplitude of the \(|A(3,2)|\) component of the transform (1). This trailing spiral component has a pitch angle \((90 - \gamma) \approx 27^\circ\), which is close to the peak of the swing-amplified response, but the behaviour of other trailing components in the range \( 1 \leq \tan \gamma \leq 3 \) is very similar. While manifesting large variations with time, \(|A(3,2)|\) is generally many times higher than that expected from a random distribution, shown by the dashed line. This is because the self-gravity of the disc causes swing-amplified collective responses to the density variations seeded by shot noise; in effect, self-gravity polarizes the particle distribution,
as described in detail by Toomre & Kalnajs (1991). The fluctuations in the net spiral amplitude that are evident in Figure 2 result from destructive and constructive interference between different noise-driven spiral segments that orbit at differing angular frequencies at different radii. The mean excess over the random expectation value depends on the responsiveness of the disc, which is determined by the swing-amplification parameters $X, Q$, the dimensionless shear rate $\Gamma = -d \ln \Omega_d / d \ln R$, as well as the softening length.

Because of the large number of particles, the amplified responses are still quite mild, and remain in the linear regime for a long time, during which any secular growth is slow. We do not show the evolution of model 0 beyond $t = 200$, but have continued it to $t = 1000$, finding that the amplitude rises rapidly later due to nonlinear resonant scattering and three-arm spiral patterns become visible. This behaviour is similar to that reported by Sellwood (2012) for $m = 2$ forces in the same model with the same file of initial particles. None of the forced simulations we report here is affected by this late time nonlinear change in the behaviour because they were run to $t = 200$ only.

### 4 EXPERIMENTS WITH A SINGLE RING OF PERTURBING MASSES

A few snapshots from the evolution of model 1H are shown in Figure 3, which report the early growth of the density response to an imposed ring of heavy co-orbiting particles, marked by the three dots in each frame. As already noted, disturbance forces were restricted to the $m = 3$ sectoral harmonic. The orbit period of the individual masses is $6\pi$ in our units. This Figure is drawn for model 1H, used in §4.2 below because the response to this heavier ring stands out more clearly from the noise, although even in this case the peak fractional over-density of the response is less than 1% by $t = 30$.

#### 4.1 A low-mass ring

Figure 4 presents the time evolution of a trailing component amplitude from several simulations. The blue curve is from model 0, and is reproduced from Figure 2. The green line reports the amplitude evolution in model 1L with a single ring of three perturbing masses at $R_p = 3$, each having a mass $M_p = 2 \times 10^{-4}$, which is $3.7 \times 10^{-5}$ of the total disc mass or that of $\sim 1.852$ particles. The response to this trefoil perturbation rises quickly at first, just as in the unperturbed disc (blue line), because the physics of the response to this type of forcing is identical to that of disc polarization but, because the perturbation is substantially heavier than a single particle, the overall amplitude is a little higher, making the interference from the noise-driven fluctuations relatively weaker. Consistent with Figure 3, the growth of the response slows after a couple of orbits of the perturbing ring. Linear theory (Julian & Toomre 1966) predicts that the response to a single perturbing mass asymptotes to a steady value after about 5 epicyclic periods, which would be by $t \sim 65$ in this case, although it is clear that the amplitude in our model continues to rise slowly to the last moment shown.

The ring of three perturbing masses was present throughout the simulation shown by the green line. In a separate experiment, model 1LR, we removed the perturbing ring of particles at $t = 50$, or after 2.65 orbits, and the amplitude evolution from this time is shown by the red line. As linear theory would predict, the response amplitude begins to decrease as soon as the perturbation is removed, and the red line quickly decays back to the level in model 0, indicated by the blue line.

#### 4.2 A heavier ring

The amplitude evolution in model 1H, a run with five times heavier perturbing masses, is shown in Figure 5. At early times, the response amplitude is close to five times that in model 1L (Figure 4), consistent with linear theory expectations. However, the amplitude given by the red line, model 1HR, indicates that the overall response did not decay in this separate experiment in which the heavier perturbing ring was removed, but continued to rise pretty much as in model 1H when the perturbing masses remained present (green line).

The response amplitude to this heavier ring rises at first to an almost steady value, because it stands out more clearly from the noise, but fluctuations in the green (model 1H) and red (model 1HR) lines in this figure again develop after $t \sim 50$, which reflect beats between multiple disturbances of comparable amplitude having differing angular frequencies. The power spectrum from the period after the ring was removed in model 1HR, displayed in Figure 6, reveals three disturbances having distinct pattern speeds. That with $m\Omega_p \approx 1$ has the frequency of the original forcing ring with corotation at $R = 3$, and Lindblad resonances at $R \approx 1.6$ and $R \approx 4.4$. The other two frequencies must have been excited by other means and, since corotation for each of these disturbances is close to the Lindblad resonance of the original wave, it seems likely they are groove modes excited by deficiencies in the DF created by Lindblad resonance scattering. This conjecture is confirmed in Figure 7, which shows changes by $t = 50$ in model 1H to the density of particles in the space of the two actions, $L_z$ and $J_R$. Forcing by the heavier ring has scattered...
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Figure 3. The growth of the relative response density to a single ring of perturbing masses, marked by the large grey dots. Only the positive parts of the response density, on the given logarithmic scale, and over the range $1.2 < R < 6$ are shown, and the figure is drawn for model 1H, the case of the heavier ring because it is less contaminated by noise. The full-drawn circle marks the corotation resonance and the dashed circles mark the Lindblad resonances.

Figure 4. The time evolution of the $|A(3, 2)|$ component of the transform (eq. 1) in three simulations. The green line, from model 1L, is when the low-mass perturbing ring at $R_p = 3$ is retained throughout, the red (model 1LR) is when the ring is removed at $t = 50$, and the blue line (model 0) and dashed line are reproduced from Fig. 2.

Figure 5. The time evolution of the $|A(3, 2)|$ component of the transform (eq. 1). The green line is from model 1H in which the high-mass perturbing ring at $R_p = 3$ was retained throughout, the red is from model 1HR in which the ring was removed at $t = 50$, the cyan line is from model 1HRS in which the perturbing ring has been removed and the particle coordinates scrambled in azimuth to erase the existing density response, while the blue and dashed lines are reproduced from Fig. 2.

Figure 7 also reveals that some particles have crossed corotation, which is reminiscent of radial migration (Sellwood & Binney 2002). The deficiency for small $J_R$ at $L_z < 3$ and excess at $L_z > 3$ indicates there were greater numbers of outward movers over inward, which is consistent with the negative gradient $\partial f/\partial L_z |_{R_0}$ near $L_z = 3$ (see Sellwood 2012, Fig. 1). However, the parallel with the process discussed by Sellwood & Binney (2002) is not exact because here the disturbance was the forced response to a co-orbiting ring of perturbers. The disc response to constant forcing grows quickly at first (green line in Figure 5), but becomes almost steady well before $t = 50$; changes to home...
Figure 6. The grey scale shows the logarithm of power in the $m = 3$ sectoral harmonic of the disturbance density over the time interval $51 \leq t \leq 200$ in model 1HR after the heavier forcing ring was removed. The solid curve indicates the circular angular frequency $m \Omega_c$, and the dashed curves $m \Omega_c \pm \kappa$. Data interior to the dotted line were excluded. Each horizontal streak indicates a disturbance of fixed pattern speed.

Figure 7. The difference between density of particles in action space at $t = 50$ and $t = 0$ in model 1H. The lines record the loci of the major resonances of the driving ring: full-drawn for corotation and dashed lines for the Lindblad resonances.

Radii at one end of a horseshoe orbit are exactly undone at the other end when the disturbance amplitude remains constant; in effect the particle is trapped in the resonance. Thus the net changes near corotation can have occurred only as the particles became trapped.

We have used our mode fitting software (Sellwood & Athanassoula 1986) to estimate the frequencies of the three disturbances in model 1HR visible in Figure 6, finding $\omega_1 \approx 1.054 - 0.012i$, $\omega_2 \approx 1.66 + 0.001i$, and $\omega_3 \approx 0.772 + 0.015i$. The real parts of these frequencies are the angular rotation rates, which are in good agreement with the frequencies indicated in Figure 6. However, the imaginary parts, which correspond to the growth rates, are of doubtful significance; that for disturbance 2 is probably an underestimate, because the instability had almost saturated over the time interval of the fit. Note also that although we have fitted a “mode” to disturbance 1, it should not be thought of as a mode, since it is the decaying response to a driving perturbation that was suddenly removed. An imaginary part to $\omega_1$ was required by the functional form adopted for the fit, and while the negative value indicates decreasing amplitude, in reality the decay is not exponential.

Figures 7 and 8 compare the changes in action space density in model 1H, the heavier ring, with that in model 1L having the lighter ring. It is important to notice the five-fold difference in the colour scale between the two figures. The lighter ring indeed causes some very mild resonance scattering, but it is evidently too weak to excite new modes, and the density response simply decays away (Fig. 4). This is a clear illustration of the difference between linear and nonlinear behaviour.

The cyan line in Figure 5 reports the amplitude evolution in model 1HRS in which the forcing ring was not only removed, but the particle azimuths were rearranged at random in order to disperse the wake response to the imposed ring. The power spectrum from model 1HRS, shown in Figure 9, reveals at least three coherent waves; not just the two groove modes excited at the Lindblad resonances of the original forced ring but also, unexpectedly, a disturbance with $m \Omega_p \geq 1$, which was close to the original forcing frequency. Figure 7 manifested changes in the phase space density across corotation, and it seemed possible that the deficiency at slightly lower $L_z$ than the resonance had also seeded an instability.

In order to check this possibility, we spliced the distribution of particles near corotation into a pristine undisturbed DF, as we did in Sellwood & Carlberg (2019), in order to suppress the instabilities associated with the grooves created by Lindblad resonance scattering. A simulation with this modified
DF, model 1HSp manifested just a single instability standing out from the noise, as shown in Figure 10. The fact that the pattern speed of this instability is very similar to that of the middle disturbance in Figure 9, and that \( m\Omega_p > 1 \) in both cases confirms that the deficiency just inside corotation in Figure 7 also excites a groove mode.

4.3 Discussion

The results reported in this section have clarified that a very low-mass perturbation produces a linear response, that leads to no lasting change. The response decays when the driving masses are removed (Fig. 4), and the changes to the DF caused by the mild forcing by the low-mass ring (Fig. 8) were not sufficiently strong or coherent to provoke new instabilities. We stress that this result could be obtained only if the disc has no instabilities. An externally imposed disturbance in a disc that possesses even mild instabilities must always seed a growing response that would not disperse were the driving perturbation to be removed.

The situation is quite different when the forcing masses were five times heavier, though each was still only 0.017% of the disc mass. The disc response was strong enough to cause coherent scattering at the major resonances (Fig. 7), which in turn excited instabilities where there had been none in the absence of the perturbation. The nature of the nonlinear response to this still quite mild perturbation is now evident: scattering at the major resonances created deficiencies in the DF at low-\( J_R \), thereby seeding new linear instabilities.

5 EXPERIMENTS WITH TWO RINGS

We next report experiments in which we introduced two rings at radii \( R_{p,1} = 2.5 \) and \( R_{p,2} = 4 \). In our first such case, model 2L, each ring has three forcing particles of mass \( M_p = 0.0001 \), i.e. half those in the lighter ring used in §4.1, and we ran the simulation to \( t = 200 \) with this forcing continuously applied. We also present a second simulation, model 2H, having ten times heavier forcing masses.

Figure 11 presents the time evolution of the logarithmic spiral response in model 2L (red) and in model 2H (green). Since the circular frequency at each ring is \( \Omega_{c,1} = 0.4 \) and \( \Omega_{c,2} = 0.25 \), the beat frequency between the two disturbances

\[ m(\Omega_{c,1} - \Omega_{c,2}) = 0.45 \]

and the beat period is \( \approx 14.0 \), which is consistent with the amplitude variations of the red and cyan curves in the Figure, and of the green curve to \( t \approx 50 \).

5.1 Lighter ring pair

We have fitted two separate waves to the disturbance density in model 2L over the time interval 50 < \( t < 200 \), finding \( m\Omega_{p,1} = 1.212 \) and \( m\Omega_{p,2} = 0.755 \), in good agreement with the two driving frequencies. The fit allowed for exponential growth of these waves and returned the imaginary part of the frequency \( \Im(x) \approx 0.0055 \) in both cases. Since these disturbances are forced responses and not modes, we do not attach much significance to this growth rate that is < 1% of the real part of either frequency.

The positive parts of the fitted waves in model 2L are drawn in Figure 12, and in order emphasize that the two responses are almost perfect scaled versions of each other, we changed the spatial scale between the two panels so that the corotation circle (full-drawn) has the same size in each. This gratifying result stems from the self-similarity of the Mestel disc, but can hold only where the tapers to the infinite disc have little effect; recall that the tapers were centered on \( L_x = 1 \) and \( L_x = 11.5 \), and will not much affect the density of low-eccentricity orbits, that would dominate the response, near either forcing ring.

The cyan line in Figure 11 is the superposition of the logarithmic spiral components fitted to these two separate waves, and its time evolution is in reasonable qualitative and quantitative agreement with the red curve from model 2L. We present a movie in Figure 13 in which, in the top right panel, we reconstruct the time evolution of the disturbance density as the linear superposition of the two fitted waves, which is to be compared with the density variations at each moment in the simulation in the top left panel. Notice in this movie how the superposed spiral arms appear to wind up for a while before each breaks briefly into two separate spiral segments that quickly rejoin to make a more open pattern that winds again. Each of the perturbing rings has a low enough mass to ensure the response remains linear, as we showed above for a single ring. The close agreement in the movie between the time evolution of model 2L and its reconstruction is consistent with linear theory.

\[ m(\Omega_{c,1} - \Omega_{c,2}) = 0.45 \]
Figure 12. The separate steady responses to the two low-mass rings of forcing particles fitted to the disturbance density in model 2L over the time range $50 \leq t \leq 200$. The full-drawn circles mark corotation at $R \approx 2.5$ (upper panel) and at $R \approx 4$ (lower panel), while the dashed circles mark the Lindblad resonances for each disturbance. Note that the spatial scale, in units of $R_0$, differs between the two panels in order that the corotation circle is the same size in each.

The lower two panels compare the logarithmic spiral spectrum measured from the simulation, on the left, with the superposition of the fitted transforms of each of the two separate waves on the right. Here one can see a repeating pattern of a peak moving to the right, from leading to trailing as the amplitude rises to a maximum for an open trailing spiral near $\tan \gamma = +2$, followed by a decrease as the spiral continues to wind up. This behaviour is reminiscent of swing-amplification and the “dust to ashes” figure from Toomre (1981), even though we know that the bottom right panel was constructed by superposition of two steady waves.

A figure illustrating similar time evolution of the logarithmic spiral transform in an unforced simulation was first presented by Sellwood & Carlberg (1984), who interpreted it at the time as evidence that the spiral pattern was caused by swing-amplification, as have many subsequent authors. However, Sellwood (1989) later presented power spectra of the same simulation, which showed that the apparently shearing and swing-amplified pattern in fact resulted from the superposition of a few steadily rotating waves.

The appearance of a disc having multiple spiral patterns, perhaps not all having the same $m$-fold symmetry, would be more complicated. But there can be no doubt that the resulting superposed spiral arms would also appear to wind up, since all that is required is that those patterns closer to the disc centre should have higher pattern speed, which is always true. We emphasize this point in §6.

### 5.2 Heavier ring pair

We next report an experiment with the masses of the forcing rings increased ten-fold so that the forcing particles have the same masses as in the heavier case with a single ring. The amplitude evolution of this coefficient from model 2H, green curve in Figure 11 closely resembles that of the red line from model 2L over the early part of the evolution, except that it is ten times higher, but this ceases to be true after $t \sim 60$.

The power spectrum from the time interval $50 < t < 200$ of model 2H is presented in Figure 14, where the driven responses to the two forcing rings have frequencies $m_1 \Omega_p = 1.2$ and $m_2 \Omega_p = 0.75$, but a third disturbance has appeared having $m_3 \Omega_p \lesssim 2.0$. This is almost certainly a groove mode that has been excited by resonance scattering, since corotation for this third pattern is close to the ILR of the higher forcing frequency. It is reasonable that this should be the first additional disturbance to appear, since the dynamical clock runs faster at smaller radii.

The presence of this new instability causes the net amplitude of the logarithmic spiral coefficient in model 2H, green curve in Figure 11, to depart from a scaled up version of that from model 2L, red curve, at later times. In fact, the recognizably regular beats that characterize the red curve have become something that looks almost chaotic when there are three dominant disturbances.

The coloured lines in Figure 15 present the radial velocity dispersion at $t = 200$ in both models 2L (red line) and 2H (green line). Scattering at the ILR in model 2H has caused significant heating in the inner disc; although the heated particles have a narrow range of $L_z$ (cf. Fig. 7), they have large $J_R$ and therefore significant radial velocities that dominate in the very center where the unaffected disc is tapered away. However, there was no change in $\sigma_R$ from the start (dotted line) at any radius in model 2L, which is further evidence that driving with the lower mass rings remained in the linear regime.

### 6 APPARENTLY SHEARING PATTERNS

Many authors (see §1 for references) argue that spirals are not density waves having fixed pattern speeds in the traditional picture, but are instead shearing disturbances that
Figure 13. An animation (which will be included in the paper, but must be viewed separately for now) showing the evolution of the disturbance density in the model 2L (top left) and its reconstruction from superposition of the two fitted waves from Fig. 12. The bottom row shows the evolution of the logarithmic spiral spectrum measured from the simulation (left) and its reconstruction from combining the fitted waves (right). The comparison is very successful.

Figure 14. As for fig. 6 but for model 2H with the two heavier rings. This is computed over the time interval $50 < t < 200$.

Figure 15. The radial variation of the rms radial velocity at in the cases with two rings. The black dotted line is at $t = 0$ for both cases, while the coloured curves are at $t = 200$; red for model 2L and green for model 2H.

wind up over time at a rate close to that expected for material features. We have argued elsewhere, and continue to do so here, that the apparently shearing spirals are merely the superposition of two or more rigidly rotating density waves.

In order to make this point more forcefully, we present Figure 16, which is drawn to resemble similar figures presented by Baba et al. (2013), and we have followed their procedure to create it. They divide the disc into a set of rings and, at frequent time intervals, they expand the relative disturbance density in sectoral harmonics:

$$\frac{\Sigma(R, \phi, t)}{\Sigma_0(R, t)} = \sum_{m=0}^{\infty} A_m(R, t) \cos [m \phi - \phi_m(R, t)], \quad (2)$$

where $A_m(R, t)$ is the amplitude and $\phi_m(R, t)$ the phase of the $m$-th harmonic. In our case, we have restricted disturbance forces to the $m = 3$ terms only, while they focused on $m = 4$. Following their procedure, we compute
a rough estimate of $m$ times the “pattern speed” from $\phi_3(R, t_2) - \phi_3(R, t_1)/(t_2 - t_1)$ at two closely spaced times $t_1$ and $t_2$ at each radius throughout a selected period of the evolution. We then histogram the values so obtained, weighting each by the amplitude $A_3(R, t_2)$, and present the results in Figure 16.

The appearance of our Figure 16 resembles that in figures 3 and 4 of Baba et al. (2013), in the sense that the dominant values in this 2D histogram are strung out along a locus of decreasing frequency with increasing radius that lies interior to the circular orbit frequency, marked by the full-drawn curve. The rough first-order estimate of the pattern speed used in these plots inevitably leads to some scatter in the plotted values, but outliers are down weighted when the amplitude is low. In our case the values at smaller radii become roughly constant at the pattern speed of the inner ring $m\Omega_c = 1.2$, and there are hints of flattening again at larger radii to values near $m\Omega_c = 0.75$ for the outer ring. But the key point is that what we know to be the superposition of two driven disturbances having distinct and radially constant frequencies, appears in this plot, and in the movie shown in Figure 13, as a shearing pattern.

The relative amplitudes of the two waves (Fig. 12) change continuously over the radial range of the decline. Since the faster pattern dominates for $R \lesssim 2$ while the slower dominates at $R \gtrsim 3$, the estimated frequency changes continuously in between. We have verified that doubling the radius, and therefore halving the frequency, of the outer ring shifts the region of overlap of the two waves to larger radii; in this case the declining ridge starts outside corotation of the faster wave and drops more steeply to inside corotation for the slower. Thus the locations of ridges in figures like 16 is purely a consequence of the radial range over which the relative amplitudes of the two disturbances changes.

Thus there can be no doubt that the superposition of two waves, can appear as a shearing spiral. In order to confirm the converse, that an apparently shearing spiral results from the superposition of separate patterns of fixed frequency, one must compute power spectra, such as that shown in Figure 14, which is computed from a longer period in the same simulation (model 2H). A power spectrum is the temporal transform of the coefficients from the surface density decomposition (eq. 2); the longer the time period used the higher the frequency resolution becomes, enabling the combined density evolution to be separated into distinct peaks. We have always found that this analysis decomposes the evolving disturbance density variations into a small number of separate waves having frequencies that are independent of radius.

Baba et al. (2013) further claim nonlinear behaviour that caused some particles in their simulations to gain random energy. We agree that wave-particle interactions that create random motion are manifestations of nonlinear scattering that is expected to happen at Lindblad resonances (Lynden-Bell & Kalnajs 1972). Baba et al. (2013) argue that because the spiral in their simulation is shearing, the heating must be some new kind of nonlinear behaviour, but we suggest that the shearing spiral they studied was in fact the superposition of two or more density waves, and the heating they observed took place at the Lindblad resonances of those steadily rotating patterns.

Kumamoto & Noguchi (2016) report experiments that were forced by co-orbiting rings of particles in a similar manner to those here, although they generally employed six particles per ring, as was more appropriate for their very low mass discs. The “wakelets” in their terminology were the forced responses created swing amplification (Julian & Toomre 1966; Binney 2020), as we have also described above. Kumamoto & Noguchi (2016) argued that the greater amplitude when their wakelets from two rings align was a nonlinear effect, but we have shown that the net amplitude fluctuates at the beat frequency between the two disturbances, which peaks when the ridge lines of the two separate waves align – behaviour that can be reconstructed by linear superposition (Figure 13). They further report that the spirals created by the superposed wakelets are shearing disturbances, but we have also shown that the apparent evolution of the pitch angle of the combined response is reproduced by the superposition of the two steady driven disturbances.

In summary, we are unconvinced by the alternative picture that these, and many other, authors have embraced. Here we have constructed apparent shearing transients from the forced responses to two rings of driving masses and shown (Figures 13 and 16) that they have most of the properties that these authors describe. We have observed shearing transient spirals in most of our simulations, and indeed were the first to report the phenomenon (Sellwood & Carlberg 1984), but power spectra of those same simulations (Sellwood 1989) and every subsequent case have revealed that shearing transients result from the superposition of a few steadily rotating waves.

Furthermore, we have reported (Sellwood & Carlberg 2014, 2019), that the amplitudes of spirals in our simulations do not change as the number of particles is increased by several orders of magnitude, although it takes longer to reach the common amplitude as $N$ is increased. The most natural explanation of this fact is that the spirals are linearly unstable modes that exponentiate out of the noise, which is reduced as $N$ is increased, and they saturate at a common amplitude due to nonlinear effects. The authors of papers who suggest that shearing transients are the fundamental phenomenon point to swing amplification for their origin but, because that mechanism merely amplifies an input signal by a fixed amount, it would predict that the amplitudes of
resulting spirals should decrease as $N^{-1/2}$, which is not what we have reported.

Thus, no convincing mechanism has been presented in this alternative view of the origin of spirals in simulations that can account for the similar amplitudes of the spiral patterns as the number of disc particles employed is varied by orders of magnitude. Since we find that power spectra always reveal a few discrete, steadily rotating waves, we argue that spiral activity in simulations results from the superposition of unstable modes (Sellwood & Carlberg 2014, 2019).

7 A SINGLE PERTURBING MASS

In this section we present models having a single perturbing mass and no imposed rotational symmetry.

7.1 A low mass disc

We first match one of the experiments reported by D’Onghia et al. (2013, hereafter DVH), who employed an exponential disc having a scalelength $R_d = 3.13$ kpc and mass of $M_d = 1.9 \times 10^{10} \, M_\odot$, embedded in a Hernquist halo of mass 50$M_d$ and radial parameter 10$R_d$. The rotation curve is shown in the upper panel of Figure 17; the disc particle velocities were chosen to be in rotational balance and to have the $Q$-profile shown in the lower panel, with a minimum $Q \approx 1.2$. As usual, we prefer units in which $G = M_d = R_d = 1$; our unit of velocity is therefore 161.1 km s$^{-1}$ and our unit of time is 19 Myr.

Since the disc of this model best supports smaller-scale structures having rotational symmetries in the range $6 \lesssim m \lesssim 12$ (DVH), we refined our 2D polar grid to 340 $\times$ 512, included sectoral harmonics in the range $0 \leq m \leq 16$, adopted a shorter softening length $\epsilon = 0.025R_d$ and doubled the number of particles to match the number employed by DVH.

Following these authors, we introduced a single perturbing mass on a circular orbit at $R = 2R_d$ having a mass $M_d/2000$. The orbit period of this particle is $\sim 12.5$, and the wake that dressed the perturber after one orbit closely resembled that shown in the upper left panel of Fig. 10 of DVH.

For a quantitative measure of the total non-axisymmetric amplitude, which is not confined to a single sectoral harmonic, we compute

$$D = \left\langle (\mu(i,j) - \bar{\mu}_j)^2 \right\rangle^{1/2} / M_d, \tag{3}$$

where $\mu(i,j)$ is the mass of particles assigned to the grid point $(i,j)$ and $\bar{\mu}_j$ is the average of the values on the $j$ th ring. The quantity $D$ is therefore the rms relative mass variations averaged over the entire grid. The green curve in Figure 18 shows that non-axisymmetric mass variations increase steadily in response to the perturbing mass. The red curve shows that non-axisymmetric variations also increase, but more slowly, without a co-orbiting perturber, reflecting the onset of polarization of the disc and possible mild instabilities, since this model is not known to be stable.

Figure 18. The time evolution of the rms relative mass variations on the entire grid in four separate simulations of models resembling those of D’Onghia et al. (2013).

Figure 19. The phase space changes caused by a single perturbing mass in our reconstruction of the simulation having a single perturbing mass by D’Onghia et al. (2013, top left panel of their Fig. 10). The perturbing mass had $L_z = 2.06$ in our units.

Figure 17. Above: the rotation curve (full drawn), the disc (dotted), and halo (dot-dashed) contributions and below the $Q$ profile of our model intended to match that of D’Onghia et al. (2013)
The blue curve shows how $D$ evolves when the evolution is continued after removing the perturber, when the disc manifested very similar on-going activity of the kind reported by DVH. However, the cyan line reveals that activity grew scarcely more rapidly after scrambling this disc than it did in the separate model without any perturbations (red curve) and no clear frequencies stood out in the power spectra of its evolution.

This null result implies that lasting changes wrought by the perturber did not seed any vigorous new instabilities, and the continuing activity shown by the blue line probably is caused by something like the nonlinear mechanism proposed by DVH. In order to understand why the modified disc has no new instabilities, we present Figure 19, which shows the phase space changes caused by the single orbit of this perturbing mass. Since Lindblad resonances lie ever closer to corotation as $m$ increases, the dominant higher sectoral harmonics caused the wake to have both a smaller radial extent and the resonances to blur together. Thus we see that resonant scattering took place over almost the entire region affected by the wake. Since a groove of a fixed depth excites a instability only when sufficiently narrow (Sellwood & Kahn 1991), the deficiency of nearly circular orbits over a broad range of angular momentum (Figure 19) is not destabilizing.

### Table 3. Dominant instabilities

| $m$ | frequency | corotation |
|-----|-----------|------------|
| 1   | 2.125 ± 0.025 + (0.638 ± 0.005)i | 1.60 ± 0.03 |
| 2   | 0.716 ± 0.003 + (0.015 ± 0.010)i | 2.79 ± 0.01 |
| 3   | 0.434 ± 0.005 + (0.016 ± 0.005)i | 4.61 ± 0.05 |
| 4   | 1.110 ± 0.003 + (0.025 ± 0.005)i | 2.70 ± 0.01 |

Continuing the evolution after removing the perturber, and scrambling the azimuthal coordinates to disperse the wake, we found that the model now possessed several unstable modes. The dominant ones are listed in Table 3; three are bisymmetric and one has three-fold symmetry. The last column lists the implied corotation radius, which in each case is close to a deficiency of near-circular orbits visible in Figure 20.

This result establishes that scattering by isolated, co-orbiting perturbing masses can seed groove instabilities in heavier discs.

### 7.2 Mestel disc again

Responses to perturbations in the previous low-mass disc model are most vigorous for sectoral harmonics $6 \lesssim m \lesssim 12$. In this circumstance, the strongest Lindblad resonance scattering from a co-orbiting point mass takes place close to the radius of the perturber and the resonances are sufficiently close together that the separate scattering peaks from each resonance are blurred together, as shown in Figure 19. In order to create scattering features that are not blurred together, we must employ a disc model that responds more vigorously to lower sectoral harmonics, for which the Lindblad resonances will lie farther from the perturber and be more widely spaced – i.e. a more massive disc.

We therefore returned to using the half-mass Mestel disc, in which we inserted a single co-orbiting mass. Since the swing-amplification parameter $X = 4/m$ in the half-mass Mestel disc, it is sufficient to include sectoral harmonics $1 \leq m \leq 5$ only in order to capture almost the entire self-consistent response (Toomre 1981; Binney & Tremaine 2008). We inserted a perturbing mass $M_p = 0.0025$ at $R_p = 5$, which has an orbit period of 10$t$.

The mild density response to this low-mass perturbation was barely visible. However, by $t = 100$, or $\sim 3$ full orbits, the resulting phase space changes, Figure 20, reveal strong scattering at the $m = 2$ ILR, weaker scattering at the $m = 3$ ILR, some changes near CR that result from trapping as the response to the perturber has grown (see §4.2 for a fuller explanation), but little scattering at the OLRs.

### 8 CONCLUSIONS

In this paper, we have presented a study of spiral responses in simulations that employed a simplified thin disc model of the stellar component of a galaxy having an exactly flat rotation curve. We selected this model because a normal mode analysis of the smooth stellar fluid by Toomre (1981) had predicted it to be stable. Although the behaviour in simulations is much richer because of swing amplification of the unavoidable shot noise from particles, we were able effectively to suppress particle noise by employing large numbers of particles. We then studied the consequences of external forcing that was both linear, for mild forcing, and nonlinear for slightly heavier perturbing masses. Here, nonlinearities are significant changes to the phase space density of the particles caused by resonance scattering.

At first, we forced our models with sets of three co-orbiting heavy particles arranged symmetrically around rings and also restricted self-gravity forces from the disc particles to the $m = 3$ sectoral harmonic. The forcing masses quickly induced spiral responses in the manner predicted by Julian & Toomre (1966, see also Binney 2020) that reached nearly full amplitude after one full orbit, but continued to grow more slowly thereafter. When each perturbing mass was just $\sim 2000$ times more massive than the individual disc particles, we observed the mild response to disperse with no lasting effects when the forcing masses were removed after almost three orbits. This behaviour is possible only in a disc that is stable; forcing masses in a disc having even mild instabilities would seed...
their growth, which would continue after the forcing masses were removed.

Heavier perturbing masses, in both single rings and in two separate rings, produced lasting changes to the disc that later caused additional self-excited patterns to develop. These new instabilities, which were not present in the unperturbed disc, resulted from nonlinear scattering of particles at the principal resonances of the original driving perturbations and they persisted after we removed the perturbations.

Our experiments with two rings of perturbing masses were particularly illuminating, as they manifested the kind of shearing transient spirals that have been reported in many recent papers, cited in the introduction. We were able to demonstrate in the movie (Figure 13) that the two separate steady responses shown in Figure 12 could be superposed to create the evolving disturbance density that appeared to wind up for a while and then reform at the beat period. The responses with heavier forcing masses were proportionately stronger at first, and we analyzed this period by the method employed by Baba et al. (2013) (Figure 16), which revealed that the net disturbance density behaved as all these authors argue even though it resulted from the superposition of two responses driven at different frequencies. Only power spectra (e.g. Figure 6) are able to resolve the evolving disturbance density into separate coherent, rigidly rotating patterns – behaviour that we have always found in our work.

We also exposed the connection between our work and that of D’Onghia et al. (2013). We reproduced their simulation with a single co-orbiting mass, in the response whose low mass disc was dominated by sectoral harmonics in the range \(6 \leq m \leq 12\). Since Lindblad resonances lie ever closer to corotation as \(m\) increases, the wake both had a small radial extent and resonant scattering took place over almost the entire region affected by the wake, creating a deficiency of circular orbits over broad range of \(L_z\). We found no new instabilities were present after we removed the perturber, consistent with the prediction (Sellwood & Kahn 1991) that broader grooves are not expected to be destabilizing. Galaxy discs generally prefer 2- and 3-arm spirals (Davis et al. 2012; Hart et al. 2016; Yu et al. 2018), which is indicative of heavier discs. We therefore also presented experiments with the half-mass Mestel disc in which swing amplification favors \(m = 2\) and \(3\), and showed that a single perturbing mass caused narrow scattering features at the ILRs of these sectoral harmonics that did excite new linear instabilities.

In this, and previous papers (Sellwood & Carlberg 2014, 2019), we have presented compelling evidence that spirals develop as a result of linearly unstable modes in a non-smooth disc, and that nonlinear scattering at resonances is responsible for subsequent instabilities. The unstable modes are disturbances having a fixed frequency over a broad range of radii from the inner to (generally) the outer Lindblad resonance. Here, we have shown that the popular alternative shearing spiral description is no more than the consequence of the superposition of two or more steadily rotating patterns. We have confirmed that something like the mechanism proposed by D’Onghia et al. (2013) does in fact operate in unrealistically low-mass discs only, but new instabilities are again provoked by resonance scattering caused by a single co-orbiting mass in realistically heavy discs. Thus, we believe we now fully understand the origin of spiral patterns in simulations of isolated disc galaxies. Whether it is also the origin of spirals in nature will be harder to establish.

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DATA AVAILABILITY

The data from the simulations reported here can be made available on request.

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