Homoclinic points of symplectic partially hyperbolic systems with 2D centre

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ABSTRACT

We consider a generic symplectic partially hyperbolic diffeomorphism close to direct/skew products of symplectic Anosov diffeomorphisms with area-preserving diffeomorphisms and prove that every hyperbolic periodic point has transverse homoclinic points.

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1. Introduction

Let \( f : M \to M \) be a diffeomorphism on a closed manifold \( M \), \( p \) be a hyperbolic periodic point of \( f \), and \( W^s, u(p) \) be the stable and unstable manifolds of \( p \), respectively. A point \( x \in W^s(p) \cap W^u(p) \setminus \{ p \} \) is called a homoclinic point of \( p \), and the intersection \( W^s(p) \cap W^u(p) \) at a homoclinic point \( x \) is said to be transverse if \( T_x W^s(p) + T_x W^u(p) = T_x M \). The fundamental importance of transverse homoclinic points was first uncovered by Poincaré in the study of the restricted three-body problem [28,29]. It is further revealed by Birkhoff [3] that there exist infinitely many hyperbolic periodic points accumulating on any transverse homoclinic point. Smale introduced in [38] a geometric model, now called Smale horseshoe, for the dynamics around a transverse homoclinic point, and started a systematic study of general hyperbolic sets.

Xia and Zhang proved an interesting result in [43] that periodic points are dense for a \( C^r \)-generic symplectic partially hyperbolic diffeomorphism close to a direct product of a symplectic Anosov diffeomorphism with an area-preserve diffeomorphism. In this paper we obtain the existence of homoclinic points of hyperbolic periodic points of such systems. Let \((N, \omega)\) be a closed symplectic manifold, \( S \) be a closed surface with an area-form \( \mu \). Then \( \Omega = \omega \oplus \mu \) is a symplectic form on the product manifold \( M = N \times S \). Let \( f : N \to N \) be a symplectic Anosov diffeomorphism, \( g : S \to S \) be an area-preserving diffeomorphism such that the direct product \( f \times g \) is partially hyperbolic with centre bundle \( E^c_{(x,s)} = \{ 0 \} \times T_s S \). Replacing \( f \) by \( f^n \) for a larger \( n \) if necessary, we may assume \( f \times g \) is 4-normally hyperbolic
on $M$ (see Section 2.1 for more details). Then there exists a $C^1$ open neighbourhood $\mathcal{U}$ of $f \times g$ such that each map $\Phi \in \mathcal{U}$ is partially hyperbolic, 4-normally hyperbolic, dynamically coherent and plaque expansive (see Section 2.2). Moreover, the centre foliation $\mathcal{F}_\Phi^c$ is leaf conjugate to the trivial foliation $\mathcal{F}_{f \times g}^c = \{ \{ x \} \times S : x \in N \}$. Therefore, the centre leaf $\mathcal{F}_\Phi^c(p)$ is diffeomorphic to the surface $S$ for each $p \in M$. Our first result is

**Theorem 1.1:** Suppose $r \geq 1$, $f : N \to N$ be a $C^r$ symplectic Anosov diffeomorphism, $g : S \to S$ an area-preserving diffeomorphism such that $f \times g$ is partially hyperbolic and 4-normally hyperbolic. Then there is a $C^1$-open neighbourhood $\mathcal{U} \subset \text{Diff}_\Omega^r(M)$ of $f \times g$ such that for a $C^r$-generic map $\Phi \in \mathcal{U}$, every hyperbolic periodic point of $\Phi$ admits transverse homoclinic points.

Recall that a property (P) holds $C^r$-generically in a given $C^1$-open subset $\mathcal{U} \subset \text{Diff}_\Omega^r(M)$ if there is a $C^r$-residual subset $\mathcal{R} \subset \mathcal{U}$ such that every $f \in \mathcal{R}$ satisfies the property (P).

More generally, let $f : N \to N$ be a symplectic Anosov diffeomorphism, and $g : N \to \text{Diff}_\Omega^r(S)$ be a $C^r$ smooth cocycle. This induces a skew product on the manifold $M = N \times S$ by $(f, g) : M \to M$, $(x, s) \mapsto (f(x), g(x)(s))$. Replacing $f$ by $f^n$ for large enough $n$ if necessary, we may assume $(f, g)$ is partially hyperbolic and 4-normally hyperbolic. Our main result is

**Theorem 1.2:** Suppose $r \geq 1$, $f : N \to N$ be a $C^r$ symplectic Anosov diffeomorphism, $g : N \to \text{Diff}_\Omega^r(S)$ be a $C^r$ smooth cocycle such that the skew-product $(f, g)$ is partially hyperbolic and 4-normally hyperbolic. Then there is a $C^1$-open neighbourhood $\mathcal{U} \subset \text{Diff}_\Omega^r(M)$ of $(f, g)$ such that for a $C^r$-generic map $\Phi \in \mathcal{U}$, every hyperbolic periodic point of $\Phi$ admits transverse homoclinic points.

Our results are related to the long-standing conjecture of Poincaré [29] (Chapter 3 in Vol. 1 and Chapter 33 in Vol. 3).

**Conjecture 1.3:** Let $(M, \omega)$ be a closed symplectic manifold and $\text{Diff}_\omega^r(M)$ be the set of $C^r$ symplectic diffeomorphisms on $M$. Then the following hold for a generic $f \in \text{Diff}_\omega^r(M)$:

- (P1) The set of periodic points of $f$ is dense in the space $M$.
- (P2) There are transverse homoclinic points for every hyperbolic periodic point of $f$.

The above conjecture is closely related to the Closing Lemma and Connecting Lemma. See [30,32] for the proof of $C^1$ Closing Lemma, and [12] for the proof of $C^1$ Connecting Lemma. Among $C^1$ diffeomorphisms, (P1) was proved by Pugh [31], (P2) was proved by Takens [39], and a stronger version of (P2) was proved by Xia [40]. There are a few results for diffeomorphisms of higher regularity, most of which are on surfaces. More precisely, (P1) has been proved in [7,8], (P2) has been proved on $S^2$ by Pixton [27], on $\mathbb{T}^2$ by Oliveira [24], and on surfaces of higher genus by Le Calvez and Sambarino [16]. See [1,9,10,15,17,34,41] for results related to (P1), [25,36,42] for results related to (P2) and [6,14,44,45,47] for related results on dynamical systems of geometric origin.

**Organization of the paper:** In Section 2, we introduce definitions and preliminary results. In Section 3, we construct a perturbation to change the centre-leaf twist coefficient for a
nonhyperbolic periodic point. The proof of Theorem 1.2 is given in Section 4. It is clear that Theorem 1.1 is a special case of Theorem 1.2.

2. Preliminaries
We introduce necessary definitions and preliminary results that will be needed later.

2.1. Partial hyperbolicity
Let $M$ be a closed manifold and Diff$^r(M)$ be the set of $C^r$ diffeomorphisms on $M$. There are many different versions of definitions for a map $f \in$ Diff$^r(M)$ being partially hyperbolic [13]. Our definition follows [5]. See [11] for results about dominated splittings.

Definition 2.1: A diffeomorphism $f : M \rightarrow M$ is said to be partially hyperbolic if there exist a $Df$-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ and a Riemannian metric on $M$ for which we can choose four continuous positive functions $\nu$, $\hat{\nu}$, $\gamma$ and $\hat{\gamma}$ on $M$ with $\nu, \hat{\nu} < 1$ and $\nu < \gamma < \gamma^{-1} < \hat{\gamma}^{-1}$, such that for any $x \in M$, for any unit vector $v \in T_xM$,

\[ \|D_xf(v)\| < \nu(x) \quad \text{if} \quad v \in E^s, \]
\[ \gamma(x) < \|D_xf(v)\| < \hat{\gamma}(x)^{-1} \quad \text{if} \quad v \in E^c, \]
\[ \hat{\nu}(x)^{-1} < \|D_xf(v)\| < \nu(x)^{-1} \quad \text{if} \quad v \in E^u. \]

The three subbundles $E^s$, $E^c$ and $E^u$ are called the stable, the centre and the unstable bundles, respectively. Moreover, $f$ is said to be Anosov (or equivalently, uniformly hyperbolic) if $E^c = \{0\}$.

Let PH$^r(M)$ be the set of $C^r$ partially hyperbolic diffeomorphisms on $M$. Note that the stable bundle $E^s$ is uniquely integrable. Let $\mathcal{F}^s$ be the stable foliation of $f$, whose leaves $\mathcal{F}^s(x)$ are $C^r$ immersed submanifolds. The same holds for the unstable bundle $E^u$. Denote by $\mathcal{F}^u$ the unstable foliation. However, the centre bundle $E^c$ may be non-integrable.

Definition 2.2: Let $f \in$ PH$^r(M)$, and $\nu, \gamma, \hat{\nu}$ and $\hat{\gamma}$ be the functions given in Definition 2.1. Then $f$ is said to be $k$-normally hyperbolic if $\nu < \gamma^k$ and $\hat{\nu} < \hat{\gamma}^k$.

It follows from Definition 2.1 that every partially hyperbolic diffeomorphism is $k$-normally hyperbolic for some $k \geq 1$.

2.2. Dynamical coherence and plaque expansiveness
A diffeomorphism $f \in$ PH$^r(M)$ is said to be dynamically coherent if the subbundles $E^c$, $E^c \oplus E^s$ and $E^c \oplus E^u$ integrate to invariant foliations $\mathcal{F}^c$, $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ respectively, $\mathcal{F}^c$ and $\mathcal{F}^s$ subfoliate $\mathcal{F}^{cs}$, $\mathcal{F}^c$ and $\mathcal{F}^u$ subfoliate $\mathcal{F}^{cu}$. Note that there are different versions of definitions of dynamical coherence in the literature. See [4] for more details.

Proposition 2.3 ([13,26]): Let $1 \leq k \leq r$, $f \in$ PH$^r(M)$ be dynamically coherent and $k$-normally hyperbolic. Then $\mathcal{F}^c(p)$ is a $C^k$ smooth submanifold for each $p \in M$. 

Hirsh, Pugh and Shub [13, §7] introduced the property *plaque expansiveness* for dynamically coherent partially hyperbolic maps. Here we follow the definition given in [33]. More precisely, given a $c$-dimensional foliation $\mathcal{F}$ on a closed $d$-dimensional manifold $M$, one can pick finitely many foliation boxes for $\mathcal{F}$, say $\phi_i : D^c \times D^{d-c} \rightarrow M$, $1 \leq i \leq I$, such that the corresponding half size foliation boxes $\phi_i(D^c \times \frac{1}{2}D^{d-c})$, $1 \leq i \leq I$, cover $M$. Each piece $\phi_i(D^c \times \{y\})$ is called a plaque of $\mathcal{F}$, and together they form a plaquation $\mathcal{P}$ of the foliation $\mathcal{F}$ which cover the leaves of $\mathcal{F}$ in a uniform fashion.

Let $f \in \text{PH}^r(M)$ be dynamically coherent. A sequence $(x_n)_{n \in \mathbb{Z}}$ is said to be an $\epsilon$-pseudo orbit if $d(f(x_n), x_{n+1}) < \epsilon$ for each $n \in \mathbb{Z}$, and is said to respect the plaquation $\mathcal{P}$ if for each $n \in \mathbb{Z}$ there exists a plaque $P_n \in \mathcal{P}$ containing both $f(x_n)$ and $x_{n+1}$. Then $f$ is said to be *plaque expansive* if there exist $\epsilon > 0$ and a plaquation $\mathcal{P}$ of the centre foliation $\mathcal{F}_c^f$ such that for any two $\epsilon$-pseudo orbits $(p_n)_{n \in \mathbb{Z}}$ and $(q_n)_{n \in \mathbb{Z}}$ that respect $\mathcal{P}$, if $d(p_n, q_n) < \epsilon$ for all $n \in \mathbb{Z}$, then for each $n \in \mathbb{Z}$ there exists a plaque $P_n \in \mathcal{P}$ containing both $p_n$ and $q_n$. Plaque expansiveness can be viewed as a generalization of the expansiveness from hyperbolic systems to partially hyperbolic ones.

**Proposition 2.4 ([13, Theorem 7.1]):** Suppose $f \in \text{PH}^1(M)$ is dynamically coherent and plaque expansive. Then there is a $C^1$-neighbourhood $\mathcal{U} \subset \text{PH}^1(M)$ of $f$ such that every map $g \in \mathcal{U}$ is dynamically coherent and plaque expansive, and $(g, \mathcal{F}_c^g)$ is canonically leaf conjugate to $(f, \mathcal{F}_c^f)$.

See also Theorem 1 and Theorem A in [33]. Recall that $(g, \mathcal{F}_c^g)$ is canonically leaf conjugate to $(f, \mathcal{F}_c^f)$ if there is a homeomorphism $h : M \rightarrow M$ such that $h(\mathcal{F}_c^g(x)) = \mathcal{F}_c^f(h(x))$ and $h(\mathcal{F}_c^g(gx)) = \mathcal{F}_c^f(h(fx))$ for every $x \in M$.

Plaque expansiveness is a desirable property but is not easy to be verified directly. There are some simple and sufficient conditions for plaque expansiveness. See [33] for more details.

**Proposition 2.5 ([13, Theorem 7.2]):** Suppose $f \in \text{PH}^1(M)$ admits an invaraint centre foliation $\mathcal{F}_c^f$. If $\mathcal{F}_c^f$ is a $C^1$ foliation, then $f$ is plaque expansive.

### 2.3. Symplectic diffeomorphisms

A 2-form $\omega$ on an even-dimensional manifold $M$ is said to be *symplectic* if it is nondegenerate and closed. A symplectic manifold is a pair $(M, \omega)$ where $M$ is a smooth manifold and $\omega$ is a symplectic form on $M$. A map $f \in \text{Diff}^r(M)$ is said to be symplectic if $f^*\omega = \omega$. Let $\text{Diff}^r_{\omega}(M) \subset \text{Diff}^r(M)$ be the set of symplectic diffeomorphisms on $M$. For convenience, let $d_{C^r}(f, g)$ be the $C^r$-distance between two diffeomorphisms $f, g \in \text{Diff}^r_{\omega}(M)$ and $B_{C^r}(f, \epsilon)$ be the $\epsilon$-ball of diffeomorphisms $g \in \text{Diff}^r_{\omega}(M)$ with $d_{C^r}(f, g) < \epsilon$. The following result is proved by Zehnder [46, Theorem 1].

**Proposition 2.6:** Let $(M, \omega)$ be a symplectic manifold. Then the set $\text{Diff}^\infty_{\omega}(M)$ is dense in $\text{Diff}^r_{\omega}(M)$ for each $r \geq 1$.

Let $E \subset TM$ be a continuous subbundle with $\dim E = i$, that is, $\dim(E_x) = i$ for any $x \in M$. The *symplectic complement* of $E$, denoted by $E^\omega$, is a subbundle of $TM$ with fibre $E^\omega_x = \{X \in TM : \omega(X, X) = 0\}$. The symplectic complement $E^\omega$ is a subbundle of $TM$. For any $\epsilon > 0$, let $\mathcal{F}_c^\omega$ be the plaque of the symplectic complement $E^\omega$. A plaque of $\mathcal{F}_c^\omega$ is called a *plaque of $\mathcal{F}_c^\omega$*. The plaque of $\mathcal{F}_c^\omega$ is then a plaque of $\mathcal{F}$, and together they form a plaquation $\mathcal{P}$ of the foliation $\mathcal{F}$ which cover the leaves of $\mathcal{F}$ in a uniform fashion.
\[ \{ v \in T_x M : \omega(v, w) = 0 \text{ for any } w \in E_x \}. \] Clearly \( \dim E^\omega = \dim M - \dim E \). A subbundle \( E \subset TM \) is said to be isotropic if \( E \subset E^\omega \), is said to be coisotropic if \( E \supset E^\omega \), is said to be symplectic if \( E \cap E^\omega = 0 \), and is said to be Lagrangian if \( E = E^\omega \). A submanifold \( S \subset M \) is said to be a symplectic submanifold if \( T_x S \) is a symplectic subspace of \( T_x M \) for every \( x \in S \). In this case, the restriction \( \omega|_S \) serves as the symplectic form on \( S \).

### 2.4. Symplectic partially hyperbolic systems

Let \((M, \omega)\) be a symplectic manifold, \( \text{PH}_\omega^r(M) := \text{Diff}_\omega^r(M) \cap \text{PH}_\omega^r(M) \) be the set of symplectic partially hyperbolic diffeomorphisms on \( M \). Note that there might be different ways of formulating the partially hyperbolic splitting of a map \( f \in \text{PH}_\omega^r(M) \), see [4, Section 3] for some interesting examples. As we will see below, the centre bundle \( E^c \) can always be chosen to be a symplectic subbundle of \( TM \).

**Proposition 2.7 ([37]):** Suppose \( f \in \text{Diff}_\omega^r(M) \) admits a dominated splitting \( TM = E \oplus F \) with \( \dim E \leq \dim F \). Then \( f \) is partially hyperbolic with \( E^s = E, E^c = E^\omega \cap F \) and \( E^u = (E^s)^\omega \cap F \). Moreover, \( E^s \) and \( E^u \) are isotropic, \( E^s \oplus E^u \) and \( E^c \) are symplectic and are symplectic-complement to each other.

From now on, the centre bundle \( E^c \) of a map \( f \in \text{PH}_\omega^r(M) \) is *always* assumed to be symplectic.

**Remark 2.8:** It is proved in [37] that symplectic partially hyperbolic maps are symmetric. That is, one can take \( \tilde{\nu} = \nu \) and \( \tilde{\gamma} = \gamma \) in Definition 2.1. Then the normal hyperbolicity condition in Definition 2.2 for general partially hyperbolic maps admits a simpler form in the symplectic case. That is, a map \( f \in \text{PH}_\omega^r(M) \) is \( k \)-normally hyperbolic if the functions \( \nu \) and \( \gamma \) in Definition 2.1 satisfy \( \nu < \gamma^k \).

**Proposition 2.9 ([43]):** Suppose \( f \in \text{PH}_\omega^r(M) \) is dynamically coherent. Then each centre leaf \( \mathcal{F}^c(x) \) is a symplectic submanifold of \( M \) with respect to the restricted symplectic form \( \omega|_{\mathcal{F}^c(x)} \). Moreover, the restriction \( \tilde{f} : \mathcal{F}^c_f(x) \to \mathcal{F}^c_f(fx) \) is a symplectic diffeomorphism for every \( x \in M \).

A centre leaf \( \mathcal{F}^c(x) \) is said to be periodic if \( f^k \mathcal{F}^c(x) = \mathcal{F}^c(x) \) for some \( k \geq 1 \). In [23] Niţică and Török proved the following.

**Proposition 2.10:** Suppose \( f \in \text{PH}_\omega^r(M) \) is dynamically coherent and plaque expansive. Then the periodic centre leaves of \( \mathcal{F}^c \) are dense in \( M \).

### 2.5. Birkhoff normal form and nonlinear stability

Let \( S \) be a closed surface, \( \mu \) be an area form on \( S \), \( f : S \to S \) be a \( C^4 \) symplectic map, and \( p \) be an elliptic fixed point of \( f \), that is, \( |\lambda_p| = 1 \) and \( \lambda_p \neq \pm 1 \), where \( \lambda_p \) is an eigenvalue of the linear map \( D_pf : T_pS \to T_pS \). Recall that an elliptic fixed point \( p \) is said to be nonresonant if \( \lambda_p^j \neq 1 \) for each \( 1 \leq j \leq 4 \). Birkhoff [2] showed that there exist a unique real
number \( \tau_1 \) and a symplectic embedding \( h : U \to S \) on a neighbourhood \( U \) of \( 0 \in \mathbb{C} \) with \( h(0) = p \in S \) such that
\[
h^{-1} \circ f \circ h(z) = \lambda_p \cdot z \cdot e^{i \tau_1 |z|^2} + O(|z|^4).
\] (1)

See also [22, Theorem 2.12]. The number \( \tau_1 = \tau_1(f, p) \) is called the first twist coefficient of \( f \) around the fixed point \( p \), the map \( h \) is called the first-order Birkhoff normalization, and the map of the form (1) is called the first-order Birkhoff Normal Form of \( f \) at \( p \).

**Definition 2.11:** An elliptic fixed point \( p \) of a surface map \( f : S \to S \) is said to be **nonlinearly stable**, if there is a sequence of nesting neighbourhoods \( \{D_n : n \geq 1\} \) of \( p \) such that for each \( n \geq 1 \), \( f(D_n) = D_n \) and the restriction of \( f \) on \( \partial D_n \simeq S^1 \) is a transitive circle map.

Note that nonlinearly stable periodic points are isolated from the dynamics in the sense that it cannot be reached from any invariant curve whose starting point lies outside some \( D_n \). The following is Moser’s **Twisting Mapping Theorem** [21]. See also [22, Theorem 2.13].

**Theorem 2.12:** Let \( r \geq 4 \), \( f \in \text{Diff}^r_\mu(S) \) and \( p \) be a nonresonant elliptic fixed point of \( f \). If the first twist coefficient of \( f \) at \( p \) is nonzero, then \( p \) is nonlinearly stable.

### 2.6. Homoclinic intersections for surface diffeomorphisms

Let \( S \) be a closed surface of genus \( g_S \), \( \mu \) be an area form on \( S \), and \( \mathcal{G}_\mu^r(S) \subset \text{Diff}^r_\mu(S) \) be the set of \( C^r \) symplectic diffeomorphisms \( f : S \to S \) satisfying the following conditions:

- **(G1)** Every periodic point of \( f \) is either elliptic or hyperbolic. Moreover, if \( p \) is an elliptic periodic point of period \( n \), then the eigenvalues of \( Df^n(p) \) are not roots of unity.
- **(G2)** Stable and unstable branches of hyperbolic points that intersect must also intersect transversally.
- **(G3)** Every elliptic periodic point of \( f \) is nonlinearly stable.

The following alternative condition for (G3) is used in [16]:

- **(G3)’** Every elliptic periodic point of \( f \) admits a nesting sequence of topological disks whose boundaries consist of finitely many pieces of stable and unstable manifolds of some hyperbolic periodic points.

Both (G3) and (G3)’ are sufficient when applying Mather’s prime-end theory [18,19]. The main difference between (G3) and (G3)’ is that (G3)’ is a generic condition for \( C^r \), \( r \geq 1 \), while (G3) is a generic condition only for \( r \geq 4 \). Since the existence of transverse homoclinic points is a \( C^1 \)-open property for one periodic point and a \( G_5 \)-property for diffeomorphisms, we only need to prove the \( C^r \)-denseness of diffeomorphisms with transverse homoclinic points for every hyperbolic periodic points. Combining with Proposition 2.6, we have that the \( C^r \)-generic existence of transverse homoclinic points implies the \( C^k \)-generic existence for any \( r \geq k \). Therefore, one can replace (G3)’ in [16] by (G3) when defining the set \( \mathcal{G}_\mu^r(S) \).
Proposition 2.13: Let $S = S^2$ or $T^2$, $f \in \mathcal{G}_\mu'(S)$. Then there are transverse homoclinic points for any hyperbolic periodic point of $f$.

The case $S = S^2$ is proved by Pixton [27], and the case $T^2$ is proved by Oliveira [24]. See also [16, Theorem 1.5].

Now we consider a closed surface $S$ of genus $g_S \geq 2$ and a map $f \in \mathcal{G}_\mu'(S)$. Let $P(f) \subset S$ be the set of periodic points of $f$, and $P_h(f) \subset P(f)$ be the set of hyperbolic ones. Le Calvez and Sambarino [16] obtained several important characterizations for such maps. Here we only list a few that are needed in this paper. See [16, Proposition 1.4, Theorem 1.5 and 1.6] for more details.

Proposition 2.14: Let $S$ be a closed surface of genus $g_S \geq 2$, $f \in \mathcal{G}_\mu'(S)$. Then $|P_h(f)| \geq 2g_S - 2$. Moreover, the following dichotomy holds:

1. $|P_h(f)| > 2g_S - 2$: every hyperbolic periodic point of $f$ has transverse homoclinic points;
2. $|P_h(f)| = 2g_S - 2$: every periodic point of $f$ is hyperbolic, and each stable (resp. unstable) branch of every hyperbolic periodic point is dense on $S$.

2.7. Symplectic partially hyperbolic systems with 2D centre

Let $\mathcal{P}_{\phi}(M, 2)$ be the set of symplectic partially hyperbolic diffeomorphisms with 2D centre bundles. As we have mentioned right after Proposition 2.7, the centre bundles will be assumed to be symplectic. Given a map $f \in \mathcal{P}_{\phi}(M, 2)$ and a periodic point $p$ of minimal period $n$, the splitting $T_pM = E_p^s \oplus E_p^c \oplus E_p^u$ at $p$ is $D_p f^n$-invariant. It follows from Definition 2.1 that the eigenvalues of $D_p f^n$ along the subspace $E_p^s$ (resp. $E_p^u$) have modulus smaller (resp. larger) than 1. Moreover, it follows from Proposition 2.9 that the two eigenvalues of $D_p f^n$ along the 2D symplectic subspace $E_p^c$ are of the from $\lambda_c(p, fn)$ and $\lambda_c(p, fn)^{-1}$. Therefore, we have the following dichotomy:

1. either $|\lambda_c(p, fn)| \neq 1$: then $p$ is a hyperbolic periodic point of $f$;
2. or $|\lambda_c(p, fn)| = 1$: then $p$ is nonhyperbolic with a 2D neutral subspace.

The stable manifold $W^s(p)$ of a periodic point $p$ of period $n$ (not necessarily hyperbolic) is defined to be the $f^n$-invariant submanifold tangent to the generalized eigenspace of eigenvalues $\lambda$ of $D_p f^n$ with $|\lambda| < 1$. It coincides with the stable leaf $\mathcal{F}^s(p)$ when $p$ is nonhyperbolic and strictly contains the stable leaf $\mathcal{F}^s(p)$ when $p$ is a hyperbolic periodic point. Note that $W^s(p)$ may be thin along the centre direction. Given a positive number $\delta > 0$, we can define the stable disk $W^s(p, \delta)$ centred at $p$ of radius $\delta$ with respect to the induced submanifold metric on $W^s(p)$. Similarly one can define the unstable manifold $W^u(p)$ and the unstable disk $W^u(p, \delta)$.

2.8. Kupka–Smale property

Robinson [35] extended the Kupka–Smale property to symplectic diffeomorphisms. For convenience, we will restrict to $\mathcal{P}_{\phi}(M, 2)$, which is an open subset of $\text{Diff}^r_{\phi}(M)$. Let
Let \( f \in \mathrm{PH}^r_\omega(M, 2) \) and \( p \) be a nonhyperbolic periodic point of minimal period \( n \), that is, \(|\lambda_c(p, f^n)| = 1\), see § 2.7. Then \( p \) is said to be \emph{centre-elliptic} if \( \lambda_c(p, f^n) \neq \pm 1 \), and is \emph{centre-nonresonant} if \( \lambda_c(p, f^n)^k \neq 1 \) for each \( 1 \leq k \leq 4 \). This generalizes of the definition of (nonresonant) elliptic periodic points given in Section 2.5. For each \( n \geq 1 \), let \( P_n(f) \) be the set of points fixed by \( f^n \). Clearly \( P_n(f) \) is a closed set. Robinson proved in [35] the following

**Proposition 2.15:** There exists a \( C^1 \)-open and \( C^r \)-dense subset \( U'_n \subset \mathrm{PH}^r_\omega(M, 2) \) such that for each \( f \in U'_n \),

1. \( P_n(f) \) is finite and varies continuously;
2. each periodic point in \( P_n(f) \) is either hyperbolic or centre-nonresonant;
3. \( W^u_f(p, n) \cap W^s_f(q, n) \) (possibly empty) for any \( p, q \in P_n(f) \).

Let \( R_{KS}(2) = \bigcap_{n \geq 1} U'_n \), which is a \( C^r \)-residual subset of \( \mathrm{PH}^r_\omega(M, 2) \). It follows that

**Corollary 2.16:** Let \( f \in R_{KS}(2) \). Then the following hold:

1. the set of periodic points \( P(f) \) is countable;
2. each periodic point of \( f \) is either hyperbolic or centre-elliptic. Moreover, if \( p \) is a centre-elliptic periodic point of period \( n \), then the centre-eigenvalues of \( Df^n(p) \) are not roots of unity;
3. \( W^u_f(p) \cap W^s_f(q) \) (possibly empty) for any periodic points \( p \) and \( q \) of \( f \).

**Remark 2.17:** The last item of the above Kupka–Smale property says that, when \( W^s_f(p) \) and \( W^u_f(q) \) have a nontrivial intersection, the intersection is actually transverse. However, it does not address the question whether \( W^s_f(p) \) and \( W^u_f(q) \) can have any nontrivial intersection. Theorem 1.2 states that there are homoclinic intersections for every hyperbolic periodic point generically.

### 3. Perturbations of the centre-leaf twist coefficients

Let \((M, \omega)\) be a closed symplectic manifold, \( 1 \leq k \leq r \), and \( \mathcal{N}_k^r(2) \subset \mathrm{PH}^r_\omega(M, 2) \) be the set of symplectic partially hyperbolic diffeomorphisms on \( M \) that are \( k \)-normally hyperbolic, dynamically coherent and plaque expansive. It follows from Definition 2.2 and Proposition 2.4 that \( \mathcal{N}_k^r(2) \) is a \( C^1 \)-open subset of \( \mathrm{PH}^r_\omega(M, 2) \). Moreover, it follows from Propositions 2.3, 2.7 and 2.9 that for each \( f \in \mathcal{N}_k^r(2) \), the centre leaf \( \mathcal{F}^c_f(p) \) is a \( C^k \) symplectic submanifold with symplectic form \( \omega|_{\mathcal{F}^c_f(p)} \), and the restriction \( f : (\mathcal{F}^c_f(p), \omega|_{\mathcal{F}^c_f(p)}) \to (\mathcal{F}^c_f(fp), \omega|_{\mathcal{F}^c_f(fp)}) \) is a \( C^k \) symplectic diffeomorphism for each \( p \in M \). In order to apply Propositions 2.13 and 2.14 to these centre-leaf mappings, we need to show that \( C^r \)-generically, the conditions (G1)–(G3) hold for all periodic centre leaves. Note that (G1) and (G2) have been established in Proposition 2.15, see also Corollary 2.16. For (G3), it follows from Theorem 2.12 that we only need to check the centre-leaf twist coefficient \( \tau_1 \neq 0 \) for nonhyperbolic periodic orbits.
**Proposition 3.1:** Suppose \( r \geq 4 \). Then there exists a \( C^1 \)-open and \( C^r \)-dense subset \( \mathcal{V}_n \subset \mathcal{N}_s'(2) \) such that for each \( f \in \mathcal{V}_n \) and each periodic point \( p \in P_n(f) \), either \( p \) is hyperbolic, or the centre-leaf twist coefficient \( \tau_1(p, F^c_f(p)) \neq 0 \), where \( k \) is the minimal period of the point \( p \).

**Proof:** Let \( \mathcal{U}_n^r(2) = \mathcal{N}_s'(2) \cap \mathcal{U}_n^r \), where \( \mathcal{U}_n^r \) is the \( C^1 \)-open and \( C^r \)-dense subset of \( \text{PH}_s^r(M, 2) \) given in Proposition 2.15. Let \( f \in \mathcal{U}_n^r(2) \) and \( p \in P_n(f) \) be a nonhyperbolic periodic point, and \( k \) be the minimal period of \( p \). Then \( k | n \).

Since \( f \in \mathcal{U}_n^r \), the periodic point \( p \) is centre-nonresonant. Let \( h_c : U_c \to \mathcal{F}_f^c(p) \) be the symplectic embedding of an open set \( U_c \subset \mathbb{C} \) given in Section 2.5 such that

\[
h_c^{-1} \circ f^k | \mathcal{F}_f^c(p) \circ h_c(z) = \lambda_p ze^{\tau_1(z^2)} + O(|z|^4),
\]

where \( \tau_1 = \tau_1(p, f^k, \mathcal{F}_f^c(p)) \) be the first twist coefficient of the centre-leaf map \( f^k | \mathcal{F}_f^c(p) \) at \( p \).

**Claim.** Let \( \mathcal{U}_f \subset \mathcal{U}_n^r \) be a \( C^4 \)-open neighbourhood of \( f \) such that \( P_n(\cdot) \) is a finite subset of the same cardinalty and varies continuously on \( \mathcal{U}_f \). If \( \tau_1(p, f^k, \mathcal{F}_f^c(p)) \neq 0 \), then there exists a \( C^4 \)-open neighbourhood \( \mathcal{U}(f, p) \subset \mathcal{U}_f \) of \( f \) such that \( \tau_1(p_g, g^k, \mathcal{F}_g^c(p_g)) \neq 0 \) for all \( g \in \mathcal{U}(f, p) \).

**Proof of Claim:** Note that the periodic point \( p \) is nondegenerate. Let \( p_g \) be the continuation of \( p \) for a map \( g \) that is close to \( f \). Moreover, the partially hyperbolic splitting of the map \( g \) varies continuously, and \( g \) admits a \( g \)-invariant centre foliation \( \mathcal{F}_g^c \). Therefore, the map \( g \mapsto (g^k, \mathcal{F}_g^c(p_g)) \) varies continuously, so is the first twist coefficient \( g \mapsto \tau_1(p_g, g^k, \mathcal{F}_g^c(p_g)) \). This completes the proof of Claim.

In the following we consider the case that \( \tau_1(p, f^k, \mathcal{F}_f^c(p)) = 0 \). We will add a small positive twist to the Birkhoff normal form on a small neighbourhood of the centre leaf at \( p \). More precisely, let \( \epsilon \) and \( \delta \) be two small positive numbers (to be specified later), \( b : [0, \infty) \to [0, 1] \) be a smooth bump function with \( b(t) = 1 \) for \( t \leq 1/3 \) and \( b(t) = 0 \) for \( t \geq 2/3 \), and \( \hat{g}_c \) an integrable twist map on an open ball \( B_c(0, \epsilon) \subset U_c \) given by

\[
\hat{g}_c(z) = ze^{ibb(|z|/\epsilon)|z|^2}.
\]

Note that \( \hat{g}_c(0) = 0 \), \( \hat{g}_c(z) = z \) when \( |z| \geq 2\epsilon/3 \), and the \( C^r \)-norm of \( \hat{g}_c - 1 \text{Id} \) can be made arbitrarily small by reducing the parameter \( \delta \). Then consider the map \( g_c : U_c \to U_c \) defined by \( g_c = h_c \circ \hat{g}_c \circ h_c^{-1} \). Note that \( g_c \) is symplectic since both \( h_c \) and \( \hat{g}_c \) are symplectic. Then it is easy to see that the Birkhoff coefficient \( \tau_1(p; f^k \circ g_c, \mathcal{F}_f^c(p)) = \delta b(0) > 0 \). Note that \( k \) is the period of \( p \), not necessarily the period of the centre leaf \( \mathcal{F}_f^c(p) \). In particular, it is possible that \( f^j \mathcal{F}_f^c(p) = \mathcal{F}_f^c(p) \) for some \( j | k \). In this case, the intersection \( \mathcal{O}(p, f) \cap \mathcal{F}_f^c(p) \) is a finite set, and the support of \( g_c \) can be made small enough such that it does not interfere with the intermediate returns of \( p \) to \( \mathcal{F}_f^c(p) \). Note that the map \( g_c \) has yet to be defined on \( M \setminus \mathcal{F}_f^c(p) \).

Next we will extend \( g_c \) to the whole manifold \( M \). By Darboux’s theorem [20], one can extend the local coordinate system \((x_1, y_1)\) on \( U_c \subset \mathcal{F}_f^c(p) \) to a local neighbourhood \( U \subset \mathcal{F}_f^c(p) \).
M containing \( U_c \), say \((x_i, y_i)_{1 \leq i \leq d} \), such that \( p = (0, 0, \ldots, 0) \) and \( \omega = \sum_i dx_i \wedge dy_i \), where \( 1 \leq i \leq d \). Suppose \( g_c(x_1, y_1) = (X_1(x_1, y_1), Y_1(x_1, y_1)) \), \((x_1, y_1) \in U_c \). It follows from the definition (3) that the support of the map \( g_c \) is contained in the ball \( B_c(0, \epsilon) \subset U_c \). Note that both \( h_c \) and \( \hat{g}_c \) are close to identity, so is \( g_c \). It follows from \([20, \text{Lemma 9.2.1}]\) that there exists a \( C^{r+1} \)-small function \( V_c(X_1, y_1) \) supported on \( B_c(0, \epsilon) \subset U_c \) such that \( g_c(x_1, y_1) = (X_1, Y_1) \) if and only if

\[
X_1 - x_1 = \frac{\partial V_c}{\partial y_1}(X_1, y_1), \quad Y_1 - y_1 = -\frac{\partial V_c}{\partial X_1}(X_1, y_1). \tag{4}
\]

Then we extend the above function \( V_c \) to a \( C^{r+1} \)-small function \( V \) supported on a small ball \( B(0, \epsilon') \subset U \) with \( V|_{U_c} = V_c \) (reducing \( \epsilon \) and \( \delta \) if necessary). Let \( g \) be the symplectic diffeomorphism on \( U \) generated by the function \( V \) using the vector form of Equation (4): \( g(x, y) = (X, Y) \) if and only if

\[
X_i - x_i = \frac{\partial V_c}{\partial y_i}(X, y), \quad Y_i - y_i = -\frac{\partial V_c}{\partial X_i}(X, y), \quad 1 \leq i \leq d. \tag{5}
\]

Note that \( g \) is supported on \( B(0, \epsilon') \subset U \). So we can extend \( g \) to the whole manifold \( M \) by setting \( g = Id \) on \( M \setminus U \). It follows that \( g \) is \( C^r \)-close to identity, and \( g = g_c \) on a small neighbourhood of \( p \) in \( F_c^r(p) \). Let \( \hat{f} = f \circ g \). Then we have \( \hat{f}^i(p) = f^i \circ g(p) = f^i(p) \) for each \( 1 \leq i \leq k \), \( \hat{f}^k(F_c^r(p)) = F_c^r(p) \) and \( \tau_1(p, \hat{f}^k, F_c^r(p)) = \tau_1(p, f^k \circ h_c, F_c^r(p)) > 0 \). Note that an invariant normally hyperbolic manifold is isolated and persists under perturbations. The fact \( F_c^r(p) \) is a normally hyperbolic manifold of \( \hat{f}^k \) implies that \( F_c^r(p) = F_c^r(p) \).

Therefore, we can rewrite the above conclusion as \( \tau_1(p, \hat{f}^k, F_c^r(p)) > 0 \).

As we have shown in the Claim, there is a \( C^4 \)-open neighbourhood \( U(p, \hat{f}) \subset U \) of \( \hat{f} \) such that for any \( h \in U(p, \hat{f}) \), the continuation \( p_h \) satisfies \( \tau_1(p_h, h^k, F_c^r_h(p_h)) \neq 0 \). Let \( \ell = |P_n(\hat{f})| \), which is constant on \( U \). Then by induction, we can find a \( C^4 \)-open subset \( U_f^\ell(\ell) \subset U(p, \hat{f}) \) arbitrarily \( C^r \)-close \( f \), such that for each \( h \in U_f^\ell(\ell) \) and each periodic point \( p \in P_n(h) \), either it is hyperbolic or the centre-leaf Birkhoff coefficient \( \tau_1(p, h^k, F_c^r_h) \neq 0 \), where \( k \) is the minimal period of the point \( p \).

Note that the map \( f \) is chosen arbitrarily in \( U_f^\ell(\ell) \), and \( U_f^\ell(\ell) \) is a \( C^4 \)-open set with elements \( C^r \)-close to \( f \). Putting these sets \( U_f^\ell(\ell) \) together, we get a \( C^4 \)-open and \( C^r \)-dense subset in \( U_f^\ell(\ell) \), say \( \mathcal{V}_n \), such that for each \( f \in \mathcal{V}_n \) and each periodic point \( p \in P_n(f) \), either \( p \) is hyperbolic, or \( p \) is centre-nonresonant and the centre-leaf Birkhoff coefficient \( \tau_1(p, f^k, F_c^r) \neq 0 \), where \( k \) is the minimal period of \( p \). Then it follows that \( \mathcal{V}_n \) is a \( C^4 \)-open and \( C^r \)-dense subset of \( \mathcal{N}_4^r(2) \).

**Proposition 3.2:** Let \( \mathcal{V}_n \) be the \( C^4 \)-open and \( C^r \)-dense subset of \( \mathcal{N}_4^r(2) \) given in Proposition 3.1, and \( \mathcal{R} = \bigcap_n \mathcal{V}_n \). Then \( \mathcal{R} \) contains a \( C^r \)-residual subset of \( \mathcal{N}_4^r(2) \) such that for each \( f \in \mathcal{R} \),

1. \( P_n(f) \) is finite, and each periodic point is either hyperbolic or centre-nonresonant;
2. \( W^s(p) \cap W^u(q) \) for any two hyperbolic periodic points \( p, q \).
(3) the centre Birkhoff coefficient \( \tau_1(p, f^k, F^c(p)) \neq 0 \) for each centre-nonresonant periodic point \( p \).

4. Proof of the main theorem

Let \((N, \omega)\) be a closed symplectic manifold, \( f : N \to N \) be a symplectic Anosov diffeomorphism, \( S \) be a closed surface with an area form \( \mu \), and \( g : N \to \text{Diff}^c_\mu(S) \) be a smooth cocycle such that the skew-product \((f, g) \in PH^c_\mu(M)\) is 4-normally hyperbolic, where \( M = N \times S \) and \( \Omega = \omega \oplus \mu \). Moreover, the centre leaf \( F^c_{\Omega}(x) = \{ x \} \times S \) for every \((x, s) \in M\). It follows from Proposition 2.5 that \((f, g)\) is dynamically coherent and plaque expansive.

Let \( U_n \subset PH^c_\mu(M) \) be the \( C^1 \)-neighbourhood of \((f, g)\) given by Proposition 2.4 such that every \( \Phi \in U_n \) is 4-normally hyperbolic, dynamically coherent and plaque expansive, and \((\Phi, F^c_\Phi)\) is canonically leaf conjugate to \(((f, g), F^c_{(f, g)})\). In particular, \( U_n \subset N^r_4(2) \), and the leaves of the centre foliation \( F^c_\Phi \) are diffeomorphic to \( S \). Assume \( r \geq 4 \) for the moment. Then it follows from Propositions 2.3 and 2.9 that the centre leaf \( F^c_\Phi(x) \) is a \( C^4 \) symplectic submanifold diffeomorphic to \( S \), and the restriction \( \Phi : F^c_\Phi(x) \to F^c_\Phi(\Phi x) \) is a \( C^4 \) symplectic diffeomorphism for every \( x \in M \).

We will divide the remaining of the proof of Theorem 1.2 into two cases depending on the genus of the surface \( S \). The proof when \( S = S^2 \) or \( \mathbb{T}^2 \) is easier mainly due to Proposition 2.13. We give a proof of Theorem 1.2 in these two special cases first and deal with the general cases later.

Proof of Theorem 1.2. Part 1: Let \( U_n^* \) be the subset given in Proposition 2.15, \( \mathcal{V}_n \subset N^r_4(2) \cap U_n^* \) be the subset given in Proposition 3.1 and \( \mathcal{R} = \bigcap_n \mathcal{V}_n \). Then \( \mathcal{R}_* := \mathcal{R} \cap U_* \) is a residual subset of \( U_n \).

Let \( \Phi \in \mathcal{R}_* \). Then for any hyperbolic periodic point \( p \) of \( \Phi \) with minimal period \( n \), the centre leaf \( F^c_\Phi(p) \) is diffeomorphic to the surface \( S \) and is invariant under \( \Phi^n \), where \( n \) is the minimal period of \( p \). It follows from Theorem 2.12 and Proposition 3.2 that every elliptic periodic point of the centre-leaf map \( \Phi^n : F^c_\Phi(p) \to F^c_\Phi(p) \) is nonlinearly stable. Combining with Proposition 2.15, we have that the map \( \Phi^n|_{F^c_\Phi(p)} \) satisfies all three conditions \((G1)-(G3)\) given in Section 2.6. That is, \( \Phi^n|_{F^c_\Phi(p)} \in G^4_{\Omega|\mathcal{F}_\Phi^c(p)}(F^c_\Phi(p)) \). Then it follows from Proposition 2.13 that the hyperbolic periodic point \( p \) admit transverse homoclinic points with respect to the surface map \( \Phi^n|_{F^c_\Phi(p)} \). These points are also transverse homoclinic points of \( p \) for \( \Phi \) on the ambient manifold \( M \). This holds for any hyperbolic periodic point \( p \) and for any map \( \Phi \in \mathcal{R}_* \). So Theorem 1.2 holds for every \( r \geq 4 \) when \( S = S^2 \) or \( \mathbb{T}^2 \). The \( C^r \)-generic existence of transverse homoclinic points with \( 1 \leq r \leq 3 \) follows directly from the \( C^4 \)-generic existence since it is a \( G_\delta \) property.

In the case \( S = S^2 \) or \( \mathbb{T}^2 \), no secondary perturbation is needed during the proof of Theorem 1.2. Next we deal with the remaining case that \( S \) is a closed surface of genus \( g_S \geq 2 \).

Proof of Theorem 1.2. Part 2: Suppose \( g_S \geq 2 \). Let \( U_n^* \) be the subset given in Proposition 2.15, \( \mathcal{V}_n \) be the subset given in Proposition 3.1, \( \mathcal{R} = \bigcap_n \mathcal{V}_n \), and \( \mathcal{R}_* = \mathcal{R} \cap U_* \) be the same residual subset of \( U_n \) as in the first part of the proof. Applying Proposition 2.14
to the periodic centre leaves of a map $\Phi \in \mathcal{R}_s$, one has the dichotomy that either every hyperbolic periodic point in that leaf admits (leafwise) transverse homoclinic points, or the stable and unstable manifolds of every hyperbolic periodic points are dense on the periodic centre leaf. To prove the theorem, we need the following:

**Claim:** Let $n \geq 1$, $\mathcal{W}_n$ be the set of maps $\Phi \in \mathcal{V}_n \cap \mathcal{U}_s$ such that every hyperbolic periodic point $p \in P_n(\Phi)$ admits transverse homoclinic points. Then $\mathcal{W}_n$ is a $C^1$-open and $C^\gamma$-dense subset of $\mathcal{U}_s$.

**Proof of Claim:** Since $\mathcal{V}_n \subset \mathcal{U}_s^\gamma$, it follows directly from Proposition 2.15 the set $P_n(\Phi)$ is finite and varies continuously on $\mathcal{V}_n \cap \mathcal{U}_s$. Moreover, the existence of transverse homoclinic points for all hyperbolic periodic points in $P_n(\Phi)$ is a $C^1$-open condition in $\mathcal{V}_n \cap \mathcal{U}_s$. Therefore, the subset $\mathcal{W}_n$ is $C^1$-open by its definition. Next, we will prove that $\mathcal{W}_n$ is also $C^\gamma$-dense in $\mathcal{U}_s$.

Let $\Phi \in \mathcal{U}_s$ be fixed. Pick a $C^\gamma$-small neighbourhood $\mathcal{X}_n \subset \mathcal{V}_n$ of $\Phi$ on which the function $\Psi \in \mathcal{X}_n \mapsto |P_n(\Psi)|$ is constant. Name the hyperbolic ones in $P_n(\Psi)$ by $p_i(\Psi)$, $1 \leq i \leq I_n$, for some $I_n \geq 0$. Recall that $B_{C^\gamma}(\Phi, \delta)$ is the $\delta$-ball of maps $\Psi \in \text{Diff}_{C^\gamma}(M)$ with $C^\gamma$-distance $d_{C^\gamma}(\Psi, \Phi) < \delta$. Pick $\epsilon > 0$ such that $B_{C^\gamma}(\Phi, (1 + I_n)\epsilon) \subset \mathcal{X}_n$ and a map $\Psi_0 \in B_{C^\gamma}(\Phi, \epsilon) \cap \mathcal{R}$. Then the restriction $\Psi_0^{|F_{\Psi_0}^0(p_i)} : F_{\Psi_0}^0(p_i) \to F_{\Psi_0}^0(p_i)$ satisfies $\psi_0^{|F_{\Psi_0}^0(p_i)} \in C_\omega^4(F_{\Psi_0}^0(p_i))$ for each $1 \leq i \leq I_n$.

**Case 1.** $|P_n(\Psi_0^{|F_{\Psi_0}^0(p_i)})| > 2g_\delta - 2$ for each $1 \leq i \leq I_n$. It follows from Proposition 2.14 that the hyperbolic periodic point $p_i$ has (leafwise) transverse homoclinic points, $1 \leq i \leq I_n$. Therefore, $\Psi_0 \in \mathcal{W}_n$.

**Case 2.** $|P_n(\Psi_0^{|F_{\Psi_0}^0(p_i)})| = 2g_\delta - 2$ for some (or all) $1 \leq i \leq I_n$. Relabeling these points if necessary, we assume this equality holds for each $1 \leq j \leq I_n$ for some $1 \leq I_n \leq I_n$. In the following, we will construct a map $\Psi_1 \in B_{C^\gamma}(\Psi_0, \epsilon)$ for which the point $p_{i}$ admits transverse homoclinic points. Moreover, the perturbation $\Psi_1$ is $C^\gamma$-close enough to $\Psi_0$ such that the existing transverse homoclinic points for hyperbolic periodic points in $P_n(\Psi_0)$ persist under the perturbation $\Psi_1$. Since $\mathcal{R}_s$ is $C^\gamma$-dense in $\mathcal{U}_s$ and the existence of transverse homoclinic points is $C^1$-open, we can assume $\Psi_1 \in \mathcal{R}_s$, too. Then by induction, for each $2 \leq j \leq I_n$, we obtain a map $\Psi_j \in B_{C^\gamma}(\Psi_{j-1}, \epsilon)$ close enough to $\Psi_{j-1}$ for which the point $p_j$ admits transverse homoclinic points and the existing transverse homoclinic points for hyperbolic periodic points in $P_n(\Psi_{j-1})$ persist under the perturbation $\Psi_j$. It follows that every hyperbolic periodic point in $P_n(\Psi_{j-1})$ admits transverse homoclinic points. That is, $\Psi_{I_n} \in \mathcal{W}_n \cap B_{C^\gamma}(\Phi, (1 + I_n)\epsilon) \subset \mathcal{W}_n \cap \mathcal{X}_n$. Since $\mathcal{X}_n$ is an arbitrarily chosen neighbourhood of an arbitrarily chosen map $\Phi \in \mathcal{U}_s$, it follows that $\mathcal{W}_n$ is $C^\gamma$-dense. This will complete the proof of the claim.

The construction of the perturbation $\Psi_1$ follows the approach in [43, §4], combining Proposition 2.14 on generic surface diffeomorphisms.

For $\epsilon > 0$ given as above, pick $\delta > 0$ (much smaller than $\epsilon$) such that for any points $x, y \in M$ with $d(x, y) < \delta$, $F^s_{\Psi_0}(x, 3\epsilon)$ and $F^u_{\Psi_0}(y, 3\epsilon)$ intersect at a unique point, and $F^s_{\Psi_0}(x, 3\epsilon)$ and $F^u_{\Psi_0}(y, 3\epsilon)$ intersect at a unique point. Applying Proposition 2.10, we can pick another periodic centre leaf, say $F_{\Psi_0}^0(\hat{p})$ for some point $\hat{p} \in B(p_1, \delta)$, such that $d(F_{\Psi_0}^0(\hat{p}), F_{\Psi_0}^0(p_1)) < \delta$. Let $\hat{n}$ be the period of the centre leaf $F_{\Psi_0}^0(\hat{p})$. By the choice of $\Psi_0$, we have $\Psi_0^{|F_{\Psi_0}^0(\hat{p})} \in C_\omega^4(F_{\Psi_0}^0(\hat{p}))$. 
The initial choice of the point \( \hat{p} \) in the periodic centre leaf \( F_{\psi_0}^c(\hat{p}) \) might be nonperiodic. Since \( g_S \geq 2 \), it follows from Proposition 2.14 that \( |P_h(\Psi_n^h|_{F_{\psi_0}^c(\hat{p})})| \geq 2g_S - 2 \geq 2 \).

In particular, there do exist hyperbolic periodic points on \( F_{\psi_0}^c(\hat{p}) \). Let \( q \) be such a hyperbolic periodic point on \( F_{\psi_0}^c(\hat{p}) \) and \( m \) be the minimal period of \( q \). Note that \( m \) can be much larger than the number \( n \). In the following we will use the point \( q \) instead of \( \hat{p} \) as the marked point on the centre leaf \( F_{\psi_0}^c(\hat{p}) = F_{\psi_0}^c(q) \).

Pick a point \( \hat{q} \in F_{\psi_0}^c(p_1) \) with \( d(\hat{q}, \hat{q}) < \delta \). Then \( F_{\psi_0}^s(q, \epsilon) \) and \( F_{\psi_0}^{cu}(\hat{q}, \epsilon) \) intersect at a unique point, say \( v \). That is, \( v \in F_{\psi_0}^s(q, \epsilon) \cap F_{\psi_0}^{cu}(x, \epsilon) \) for some \( x \in F_{\psi_0}^c(\hat{q}, \epsilon) \subset F_{\psi_0}^c(p_1) \). Similarly, \( F_{\psi_0}^{su}(q, \epsilon) \) and \( F_{\psi_0}^{cu}(\hat{q}, \epsilon) \) intersect at a unique point, say \( w \). That is, \( w \in F_{\psi_0}^{su}(q, \epsilon) \cap F_{\psi_0}^c(y, \epsilon) \) for some \( y \in F_{\psi_0}^c(\hat{q}, \epsilon) \subset F_{\psi_0}^c(p_1) \). Since \( |P_h(\Psi_n^h|_{F_{\psi_0}^c(p_1)})| = 2g_S - 2 \), it follows from Proposition 2.14 that the leafwise stable and unstable manifolds \( W^{s,u}(p_1, \Psi_0^n|_{F_{\psi_0}^c(p_1)}) \) of \( p_1 \) are dense on the whole leaf \( F_{\psi_0}^c(p_1) \). Therefore, we can pick

1. a sequence of points \( x_j \in W^u(p_1, \Psi_0^n|_{F_{\psi_0}^c(p_1)}) \) that converge to \( x \)
2. a sequence of points \( y_j \in W^s(p_1, \Psi_0^n|_{F_{\psi_0}^c(p_1)}) \) that converge to \( y \).

Note that \( \Psi_0^{-kn}(v) \) and \( \Psi_0^{kn}(w) \) converge to the centre leaf \( F_{\psi_0}^c(p_1) \) as \( k \to +\infty \) and \( \Psi_0^{km}(v) \) and \( \Psi_0^{-km}(w) \) converge to the centre leaf \( F_{\psi_0}^c(q) \) as \( k \to +\infty \). These two points being non-recurrence makes the \( C^r \)-perturbations around these two points straightforward. This is similar to setting as in [43, §4]. Following the same argument, one can find a \( C^r \)-small perturbation \( \Psi_1 \) of \( \Psi_0 \) supported on two disjoint small neighbourhoods of \( v \) and \( w \), respectively, such that

1. \( v \in F_{\psi_1}^s(q, \epsilon) \cap F_{\psi_1}^{su}(x_j, \epsilon) \) for some \( x_j \) sufficiently close to \( x \)
2. \( w \in F_{\psi_1}^{su}(q, \epsilon) \cap F_{\psi_1}^c(y_j, \epsilon) \) for some \( y_j \) sufficiently close to \( y \).

Note that \( \Psi_1 = \Psi_0 \) on both centre leaves \( F_{\psi_0}^c(p_1) \) and \( F_{\psi_0}^c(q) \). It follows that \( v \in W_{\psi_1}^s(q) \cap W_{\psi_1}^{cu}(p_1) \) and \( w \in W_{\psi_1}^{cu}(q) \cap W_{\psi_1}^s(p_1) \). That is, there is a heteroclinic cycle between the two hyperbolic periodic points \( p_1 \) and \( q \) for the perturbed map \( \Psi_1 \). Making a further perturbation if necessary, we may assume that the heteroclinic intersections at both \( v \) and \( w \) are transverse. Then it follows from the Lambda Lemma that there are transverse homoclinic points for the hyperbolic periodic point \( p_1 \). This completes the construction of the perturbation \( \Psi_1 \) and the proof of the claim.

It follows from Claim that \( \mathcal{W}_n \) is \( C^1 \)-open and \( C^r \)-dense for each \( n \geq 1 \). Then the set \( \mathcal{R}_* := \mathcal{R}_* \cap (\cap_{n \geq 1} \mathcal{W}_n) \) is a residual subset of \( \mathcal{U}_* \). Moreover, for each \( \Phi \in \mathcal{R}_* \), every hyperbolic periodic point of \( \Phi \) admits transverse homoclinic points. This completes the proof of Theorem 1.2.

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References

[1] Asaoka M. and Irie K., A $C^\infty$-closing lemma for Hamiltonian diffeomorphisms of closed surfaces, Geom. Funct. Anal. 26(5) (2016), pp. 1245–1254.
[2] Birkhoff G., Surface transformations and their dynamical applications, Acta. Math. 43 (1922), pp. 1–119.
[3] Birkhoff G., Nouvelles recherches sur les systemes dynamiques, Memoriae Pont Acad. Sci. Novi Lyncaei (1935), pp. 85–216.
[4] Burns K. and Wilkinson A., Dynamical coherence and center bunching, Discrete Contin. Dyn. Syst. 22 (1/2) (2008), pp. 89–100.
[5] Burns K. and Wilkinson A., On the ergodicity of partially hyperbolic systems, Ann. Math. 171(1) (2010), pp. 451–489.
[6] Contreras G. and Oliveira F., Homoclinics for geodesic flows of surfaces, arXiv:2205.14848.
[7] Cristofaro-Gardiner D., Prasad R., and Zhang B., Periodic Floer homology and the smooth closing lemma for area-preserving surface diffeomorphisms, arXiv:2110.02925.
[8] Edtmair O. and Hutchings M., PFH spectral invariants and $C^\infty$ closing lemmas, arXiv:2110.02463.
[9] Franks J. and Le Calvez P., Regions of instability for non-twist maps, Ergod. Th. Dynam. Syst. 23(1) (2003), pp. 111–141.
[10] Gan S. and Shi Y., $C^r$-Closing lemma for partially hyperbolic diffeomorphisms with 1D-center bundle, arXiv:2004.06855.
[11] Gourmelon N., Adapted metrics for dominated splittings, Ergod. Th. Dynam. Syst. 27(6) (2007), pp. 1839–1849.
[12] Hayashi S., Connecting invariant manifolds and the solution of the $C^1$-stability and stability conjectures for flows, Ann Math. 145 (1) (1997), pp. 81–137.
[13] Hirsch M., Pugh C., and Shub M., Invariant Manifolds, Vol. 583, Lect. Notes Math, Springer-Verlag, Berlin, 1977.
[14] Irie K., Dense existence of periodic reeb orbits and ECH spectral invariants, J. Mod. Dyn. 9(1) (2015), pp. 357–363.
[15] Koropecki A., Le Calvez P., and Nassiri M., Prime ends rotation numbers and periodic points, Duke Math. J. 164(3) (2015), pp. 403–472.
[16] Le Calvez P. and Sambarino M., Homoclinic orbits for area preserving diffeomorphisms of surfaces, Ergod. Th. Dynam. Syst. 42(3) (2022), pp. 1122–1165.
[17] Marin K., $C^r$-density of (non-uniform) hyperbolicity in partially hyperbolic symplectic diffeomorphisms, Comment. Math. Helv. 91(2) (2016), pp. 357–396.
[18] Mather J., Invariant subsets of area-preserving homeomorphisms of surfaces, Adv. Math. Suppl. Stud.7B (1981), pp. 531–62.
[19] Mather J., Topological proofs of some purely topological consequences of caratheodory’s theory of prime ends, in Selected Studies, T. M. Rassias, G. M. Rassias, eds., North-Holland Publishing Co., Amsterdam, 1982, pp. 225–255.
[20] McDuff D. and Salamon D., Introduction to Symplectic Topology, 3rd ed., Oxford Grad. Texts Math., Oxford University Press, Oxford, 2017.
[21] Moser J., On invariant curves of area-preserving mappings of an annulus, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II 1962 (1962), pp. 1–20.
[22] Moser J., Stable and Random Motions in Dynamical Systems. Annals Math. Studies, Vol. 77, Princeton University Press, Princeton, NJ, 1973.
[23] Niţică V. and Török A., An open dense set of stably ergodic diffeomorphisms in a neighborhood of a non-ergodic one, Topology 40(2) (2001), pp. 259–278.
[24] Oliveira F., On the generic existence of homoclinic points, Ergod. Th. Dynam. Syst. 7(4) (1987), pp. 567–595.
[25] Oliveira F., On the $C^\infty$ genericity of homoclinic orbits, Nonlinearity 13(3) (2000), pp. 653–662.
[26] Pesin Y., Lectures on partial hyperbolicity and stable ergodicity, Zürich Lectures in Advanced Mathematics, EMS, Zürich, 2004.
[27] Pixton D., Planar homoclinic points, J. Differ. Equ. 44(3) (1982), pp. 365–382.
[28] Poincaré H., Sur le problème des trios corps et les équations de la dynamique, Acta Math. 13 (1890), pp. 211–270. (French) [On the Three-Body Problem and the Equations of Dynamics.]
[29] Poincaré H., Les méthodes nouvelles de la mécanique céleste (French). [New Methods of Celestial Mechanics]. Gauthier-Villars, Paris, vol. 1 in 1892; vol. 2 in 1893; vol. 3 in 1899.
[30] Pugh C., The closing lemma, Amer. J. Math. 89(4) (1967), pp. 956–1009.
[31] Pugh C., An improved closing lemma and a general density theorem, Amer. J. Math. 89(4) (1967), pp. 1010–1021.
[32] Pugh C. and Robinson C., The $C^1$ closing lemma including Hamiltonians, Ergod. Theor. Dyn. Sys. 3(2) (1983), pp. 261–313.
[33] Pugh C., Shub M., and Wilkinson A., Hölder foliations, revisited, J. Mod. Dyn 6(1) (2012), pp. 79–120.
[34] Qu H. and Xia Z., A $C^\infty$ closing lemma on torus, arXiv:2106.08844.
[35] Robinson C., Generic properties of conservative systems, Amer. J. Math. 92(3) (1970), pp. 562–603.
[36] Robinson C., Closing stable and unstable manifolds in the two-sphere, Proc. Am. Math. Soc. 41(1) (1973), pp. 299–299.
[37] Saghin R. and Xia Z., Partial hyperbolicity or dense elliptic periodic points for $C^1$-generic symplectic diffeomorphisms, Trans. Amer. Math. Soc. 358(11) (2006), pp. 5119–5138.
[38] Smale S., Diffeomorphisms with Many Periodic Points, Differential and Combinatorial Topology, Princeton Univ. Press, Princeton, NJ, 1965, pp. 63–80.
[39] Takens F., Homoclinic points in conservative systems, Invent. Math. 18(3–4) (1972), pp. 267–292.
[40] Xia Z., Homoclinic points in symplectic and volume-preserving diffeomorphisms, Comm. Math. Phys. 177(2) (1996), pp. 435–449.
[41] Xia Z., Area-preserving surface diffeomorphisms, Comm. Math. Phys. 263(3) (2006), pp. 723–735.
[42] Xia Z., Homoclinic points for area-preserving surface diffeomorphisms, arxiv:math/0606291.
[43] Xia Z. and Zhang H., A $C^\alpha$ closing lemma for a class of symplectic diffeomorphisms, Nonlinearity 19(2) (2006), pp. 511–516.
[44] Xia Z. and Zhang P., Homoclinic points for convex billiards, Nonlinearity 27(6) (2014), pp. 1181–1192.
[45] Xia Z. and Zhang P., Homoclinic intersections for geodesic flows on convex spheres, Contemp. Math. 698 (2017), pp. 221–238.
[46] Zehnder E., Note on Smoothing Symplectic and Volume-Preserving Diffeomorphisms, Geometry and Topology, Lecture Notes in Math., Vol. 597, Springer, Berlin, 1977, pp. 828–854.
[47] Zhang P., Convex billiards on convex spheres, Ann. Inst. H. Poincaré Anal. Non Linéaire 34(4) (2017), pp. 793–816.