Gravitomagnetic effects for polar circular geodesic orbits around a central rotating body

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Summary. — In this paper we are interested in the general relativistic motion around a central rotating body of mass $M$. In particular, we wish to elucidate the gravitomagnetic effect on the motion of a test particle following a polar circular geodesic orbit in a plane containing the proper angular momentum $J$ of the central gravitating source. The first general relativistic correction of order $O(c^{-2})$ to the Keplerian period is proportional to $J^2/\sqrt{M^5}$. Such correction, which turns out not to be a mere coordinate effect, is insensitive to the sense of motion of the test particle on its orbit, contrary to the well known case of a circular, equatorial orbit yielding the gravitomagnetic clock effect for a couple of counter–orbiting test particles. For a LAGEOS like satellite in the gravitational field of the Earth it is of order of $10^{-9}$ s. Unfortunately, due to the uncertainty in the Earth’s $GM$, the error in the classical Keplerian period is of the order of $10^{-5}$ s, so that the obtained gravitomagnetic effect is, at present, undetectable.

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1. – Introduction

One of the most intriguing general relativistic features of the space–time structure generated by mass–energy currents is represented by the gravitomagnetic corrections to the orbital motion of a test particle freely falling in the gravitational field of a central body of mass $M$, radius $R$ and angular velocity $\omega$. They have been extensively worked out by a number of authors.

In regard to the so called gravitomagnetic clock effect on the coordinate sidereal periods of a couple of counter–orbiting test particles following identical, circular equatorial orbits we quote the references [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. As it is well known, the time $T$ required for describing a geodesic circular, equatorial full orbit of radius $r$, as viewed from an asymptotically inertial observer, is given by

\[ T^\pm = T^{(0)} \pm 2\pi \frac{J}{Mc^2}, \]

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where $T^{(0)} = 2\pi \sqrt{r^3/\mathcal{G}M}$ is the classical Keplerian period, $G$ is the Newtonian gravitational constant, $J = \frac{2}{3} MR^2 \omega$ is the proper angular momentum of the central mass, supposed to be spherically symmetric, $c$ is the speed of light in vacuum and the signs $+$ and $-$ refer to the two possible senses of motion of the test particle along its orbit. The sign $+$ refers to the counterclockwise sense of rotation as viewed from the tip of $J$ while the sign $-$ refers to the clockwise sense of rotation. The interesting observable is the difference $\Delta T = T^+ - T^- = 4\pi J/Mc^2$ in which the classical Newtonian terms are canceled out and the leading term is just the general relativistic gravitomagnetic correction.

The Lense–Thirring effect on the Keplerian orbital elements of a test body is currently under measurement in the gravitational field of the Earth with the LAGEOS–LAGEOS II experiment by Ciufolini and coworkers \[11\], and various efforts aimed to enforce the experimental accuracy of such measurement have been recently carried out \[12\]. Also the originally proposed LAGEOS–LARES project \[13\] is currently under revision \[14\].

The ambitious GP–B mission \[15\], aimed to the detection, among other things, of the gravitomagnetic precession of four freely falling gyroscopes in the gravitational field of the Earth, is scheduled to fly in June 2003.

All the gravitomagnetic effects considered up to now, both theoretically and experimentally, are post–Newtonian corrections of order $\mathcal{O}(c^{-2})$ and linear in the angular velocity $\omega$ of the central mass. In this paper we wish to investigate if it is possible to work out some other gravitomagnetic effects which account for higher powers of $\omega$. In particular, we will work out the effect of the square of the proper angular momentum of a central, weakly gravitating astrophysical object on the coordinate sidereal period of a test body freely orbiting it. Moreover, we will check if such new feature is detectable, according to our knowledge of the Earth’s space environment.

The paper is organized as follows. In section 2 we will work out the geodesic motion in a plane containing the spin of the central mass, according to the full Kerr metric. The gravitomagnetic correction to the coordinate period is calculated, as well. Section 3 is devoted to the discussions and the conclusions.

2. – Polar circular geodesics

Let us consider, for the sake of concreteness, a central body of mass $M$, and proper angular momentum $J$ directed along the $z$ axis of an asymptotically inertial frame $K \{x, y, z\}$ whose origin is located at the center of mass of the body. By adopting, as usual, the Boyer-Lindquist coordinates

\[
x^0 = ct, \quad x^1 = r, \quad x^2 = \theta, \quad x^3 = \phi,
\]

the components of the Kerr space–time metric tensor are

\[
go_{00} = 1 - \frac{Rr}{\vartheta^2}, \quad g_{11} = -\frac{\vartheta^2}{\Delta}.
\]
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\[ g_{22} = -\dot{g}^2, \]
\[ g_{33} = -\sin^2 \theta \left[ r^2 + a_g^2 + \frac{R_s r}{\dot{g}^2} a_g^2 \sin^2 \theta \right], \]
\[ g_{03} = \frac{R_s r}{\dot{g}^2} a_g \sin^2 \theta, \]

with
\[ \dot{g}^2 = r^2 + a_g^2 \cos^2 \theta, \]
\[ \Delta = r^2 - R_s r + a_g^2, \]
\[ R_s = \frac{2GM}{c^2}, \]
\[ a_g = \frac{J}{Mc}. \]

In the following subsections we will consider a circular geodesic orbit in an arbitrary approximately fixed \( [16] \) azimuthal plane\(^{(1)} \) passing through the spin of the central body, i.e. we will work at fixed \( r \) and \( \phi \).

2.1. The asymptotically inertial period. – We will first calculate the time \( T \) required for describing a full polar orbit passing from \( \theta = 0 \) to \( \theta = 2\pi \) as viewed from an asymptotically inertial observer. To this aim, we will start by considering the radial geodesic equation. As it is well known, the geodesic equations of motions are

\[ \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0, \quad \mu = 0, 1, 2, 3 \]

in which \( \tau \) is the proper time: the Christoffel symbols are given by

\[ \Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\beta} \left( \frac{\partial g_{\beta\nu}}{\partial x^\rho} + \frac{\partial g_{\beta\rho}}{\partial x^\nu} - \frac{\partial g_{\nu\rho}}{\partial x^\beta} \right), \quad \mu, \nu, \rho = 0, 1, 2, 3. \]

Since we are working at fixed \( r \) and \( \phi \), we need only \( \Gamma^1_{00}, \Gamma^1_{22} \) and \( \Gamma^1_{20} \). It can be easily seen that, since \( g^{11} = 1/g_{11}, \)

\[ \Gamma^1_{00} = -\frac{1}{2} g^{11} \frac{\partial g_{00}}{\partial x^1} = \frac{1}{2} \frac{\Delta}{\dot{g}^2} \frac{R_s (r^2 - a_g^2 \cos^2 \theta)}{\Delta}, \]
\[ \Gamma^1_{22} = \frac{1}{2} g^{11} \frac{\partial g_{22}}{\partial x^1} = r \frac{\Delta}{\dot{g}^2}, \]
\[ \Gamma^1_{20} = 0. \]

\( ^{(1)} \) In fact, the orbital plane is not fixed in the asymptotically inertial space because it undergoes the Lense–Thirring precession of the longitude of the ascending node \( \Omega \) which, in this case, coincides to \( \phi \) \[16\]. However, it can be neglected over the temporal scale fixed by a typical orbital revolution of a near–Earth artificial satellite. Indeed, for, e.g., the LAGEOS satellite the period of the Lense–Thirring nodal precession amounts to \( 1.3 \times 10^{15} \) s while its Keplerian orbital period is \( 1.3 \times 10^4 \) s.
Then, the radial equation yields

$$R_s(r^2 - a_g^2 \cos^2 \theta) \left(\dot{x}^0\right)^2 - r \left(\dot{\theta}\right)^2 = 0,$$

where the overdot stands for the derivative with respect to the proper time $\tau$. By defining the adimensional quantity $\alpha = a_g/r$, from eq. (20) it can be obtained

$$dt = \pm \frac{1}{n} \frac{1 + \alpha^2 \cos^2 \theta}{\sqrt{1 - \alpha^2 \cos^2 \theta}} d\theta,$$

where $n = \sqrt{GMr^{-3}}$ is the Keplerian mean motion. The signs + and - refer to opposite rotations. By neglecting terms of order $O(\alpha^k)$, $k \geq 4$, which is well adequate, e.g., for an artificial satellite orbiting the Earth since $a_g \approx 3.3 \text{ m}$, eq. (21) becomes

$$dt \simeq \pm \frac{1}{n} \left(1 + \frac{3}{2} \alpha^2 \cos^2 \theta\right) d\theta.$$

By integrating eq. (22) from 0 to $2\pi$ for the clockwise direction and from $2\pi$ to 0 for the counterclockwise direction we obtain, for both senses of motion

$$T \simeq \frac{2\pi}{n} + \frac{3\pi a_g^2}{2nr^2} = T^{(0)} + \frac{3\pi J^2}{2c^2 \sqrt{GMr}} = T^{(0)} + \frac{6\pi R^4 \omega^2}{25c^2 \sqrt{GMr}}.$$

At this point it may be instructive a comparison with the equatorial case for $r$ and $\theta$ constant. Since for $\theta = \frac{\pi}{2}$

$$\Gamma^1_{00} = \frac{1}{2} \frac{\Delta}{g^2} \frac{R_s}{r^2},$$

$$\Gamma^1_{33} = \frac{1}{2} \frac{\Delta}{g^2} \left(-2r + \frac{R_s a_g^2}{r^2}\right),$$

$$\Gamma^1_{30} = -\frac{1}{2} \frac{\Delta}{g^2} \frac{R_s a_g}{r^2},$$

the geodesic radial equation can be written as

$$\left(\frac{dt}{d\phi}\right)^2 - 2 \frac{a_g}{c} \left(\frac{dt}{d\phi}\right) + \left(\frac{a_g^2}{c^2} - \frac{2r^3}{R_s c^2}\right) = 0.$$ 

From eq. (24), which has been written without any approximation, it can be straightforwardly obtained in an exact way

$$\frac{dt}{d\phi} = \frac{a_g}{c} \pm \frac{1}{n},$$

from which the well known result of eq. (11) follows. Notice that the terms in $a_g^2$ of the diagonal part of the Kerr metric entering eq. (27) cancel out in obtaining eq. (28).

So, in the equatorial case, there is only one general relativistic correction. It is of order $O(c^{-2})$ and is linear in $J$. 


3. – Discussion and conclusions

In this paper we have investigated the general relativistic motion of a test particle along a geodesic, circular orbit lying in a plane containing the proper angular momentum $J$ of a central, weakly gravitating body. By assuming that the motion occurs in an almost fixed plane for $r$ and $\phi$ constant in the full Kerr metric, we have found that there is a small relativistic correction to the coordinate time $T$ required to describe a full orbit. It is given by eq. (23). Its main features are the following.

- It is independent of the sense of motion of the test particle on its orbit, contrary to the well known gravitomagnetic correction due to the off–diagonal component of the metric. This implies, e.g., that for a pair of counter–rotating test particles on polar orbits there is no gravitomagnetic coordinate time shift $\Delta T = T^+ - T^- = 0$

- It is an effect of order $O(c^{-2})$, as it happens in the off–diagonal case. However, while eq. (23) is an approximate result at lowest order in $a_2^2$, eq. (1) is exact

- It depends on the square of the angular velocity of the central body, while the off–diagonal correction is linear in it.

- It depends on the Newtonian gravitational constant $G$, the mass $M$ of the central body and the radius $r$ of the orbit via $1/\sqrt{GMr}$, contrary to the off–diagonal correction which is independent of them. The dependence on $M$ is an important point because if it was dependent purely on $a_2^2$ some doubts could arise on its physical meaning. Indeed, every effect dependent only on $a_2^2$ could be thought as derived from the Kerr metric in the limit $M \to 0$. But the diagonal part of the Kerr metric for $M \to 0$ is simply the flat Minkowskian space–time written in the spheroidal coordinates

\[
\begin{align*}
  x &= \sqrt{r^2 + a_2^2} \sin \theta \cos \phi, \\
  y &= \sqrt{r^2 + a_2^2} \sin \theta \sin \phi, \\
  z &= r \cos \theta
\end{align*}
\]

- In the limit $a_2^2 \to 0$ it reduces to the Keplerian period $T^{(0)} = 2\pi/n$ for the asymptotically inertial observer

- Eq. (23) yields for the gravitomagnetic correction in the field of the Earth a value of $(2.5 \times 10^{-5} \text{ cm s}^{-1})/\sqrt{r}$: for LAGEOS it amounts to $1.2 \times 10^{-9}$ s. Such a correction is two orders of magnitude smaller than the equatorial gravitomagnetic clock effect $\Delta T$. It is, at present, undetectable, mainly because of the error in the knowledge of the Keplerian period, of the order of $10^{-5}$ s, caused by the uncertainty in $GM_\oplus$.

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