LITTLEWOOD–PALEY THEORY FOR TRIANGLE BUILDINGS

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Abstract. For the natural two parameter filtration \((F\lambda : \lambda \in \mathbb{P})\) on the boundary of a triangle building we define a maximal function and a square function and show their boundedness on \(L^p(\Omega_0)\) for \(p \in (1, \infty)\). At the end we consider \(L^p(\Omega_0)\) boundedness of martingale transforms. If the building is of \(GL(3, \mathbb{Q}_p)\) then \(\Omega_0\) can be identified with \(p\)-adic Heisenberg group.

1. Introduction

Let \((\Omega, \mathcal{F}, \pi)\) be a \(\sigma\)-finite measure space. A sequence of \(\sigma\)-algebras \((\mathcal{F}_n : n \in \mathbb{Z})\) is a filtration if \(\mathcal{F}_n \subset \mathcal{F}_{n+1}\). Given \(f\) a locally integrable function on \(\Omega\) by \(E[f|\mathcal{F}_n]\) we denote its conditional expectation value with respect to \(\mathcal{F}_n\). Let \(M^*\) and \(S\) denote the maximal function and the square function defined by

\[
M^*f = \sup_{n \in \mathbb{Z}} |f_n|,
\]

and

\[
Sf = \left( \sum_{n \in \mathbb{Z}} |d_nf|^2 \right)^{1/2},
\]

where \(d_nf = f_n - f_{n-1}\). The Hardy and Littlewood maximal estimate (see [8]) implies that

\[
\pi(\{M^*f > \lambda\}) \leq \lambda^{-1} \int_{M^*f > \lambda} |f| \, d\pi,
\]

from where it is easy to deduce that for \(p \in (1, \infty]\)

\[
\|M^*f\|_{L^p} \leq \frac{p}{p-1} \|f\|_{L^p}.
\]

For the square function, if \(p \in (1, \infty)\) then there is \(C_p > 1\) such that

\[
C_p^{-1} \|f\|_{L^p} \leq \|Sf\|_{L^p} \leq C_p \|f\|_{L^p}.
\]

The inequality (1.2) goes back to Paley [12], and has been reproved in many ways, see for example [2, 3, 4, 7, 10]. Its main application is in proving the \(L^p\)-boundedness of martingale transforms (see [2]), that is, for operators of the form

\[
Tf = \sum_{n \in \mathbb{Z}} a_n d_nf
\]

where \((a_n : n \in \mathbb{Z})\) is a sequence of uniformly bounded functions such that \(a_{n+1}\) is \(\mathcal{F}_n\)-measurable.

In 1975, Cairoli and Walsh (see [5]) have started to generalize the theory of martingales to two parameter case. Let us recall that a sequence of \(\sigma\)-fields \((\mathcal{F}_{n,m} : n, m \in \mathbb{Z})\) is a two parameter filtration if

\[
\mathcal{F}_{n+1,m} \subset \mathcal{F}_{n,m}, \quad \text{and} \quad \mathcal{F}_{n,m+1} \subset \mathcal{F}_{n,m}.
\]

Then \((f_{n,m} : n, m \in \mathbb{Z})\) is a two parameter martingale if

\[
E[f_{n+1,m}|\mathcal{F}_{n,m}] = f_{n,m}, \quad \text{and} \quad E[f_{n,m+1}|\mathcal{F}_{n,m}] = f_{n,m}.
\]

Observe that conditions (1.3) and (1.4) impose a structure only for comparable indices. In that generality, it is hard, if not impossible, to build the Littlewood–Paley theory. This lead to the
introduction of other (smaller) classes of martingales (see [20, 19]). In particular, in [5], Cairoli and Walsh introduced the following condition

\( (F_4) \)

\[
E[f\mid \mathcal{F}_{n,\infty}, \mathcal{F}_{\infty, m}] = E[f\mid \mathcal{F}_{\infty, m}, \mathcal{F}_{n, \infty}] = f_{n,m}
\]

where

\[
\mathcal{F}_{n,\infty} = \sigma\left( \bigcup_{m \in \mathbb{Z}} \mathcal{F}_{n,m} \right), \quad \text{and} \quad \mathcal{F}_{\infty, m} = \sigma\left( \bigcup_{n \in \mathbb{Z}} \mathcal{F}_{n,m} \right).
\]

Under \( (F_4) \), the result obtained by Jensen, Marcinkiewicz and Zygmund in [9] implies that the maximal function

\[
M^* f = \sup_{n, m \in \mathbb{Z}} |f_{n,m}|
\]

is bounded on \( L^p(\Omega) \) for \( p \in (1, \infty) \). In this context the square function is defined by

\[
S f = \left( \sum_{n,m \in \mathbb{Z}} |d_{n,m} f|^2 \right)^{1/2}
\]

where \( d_{n,m} \) denote the double difference operator, i.e.

\[
d_{n,m} f = f_{n-1,m} - f_{n-1,m-1} + f_{n,m-1} - f_{n,m}.
\]

In [11], it was observed by Metraux that the boundedness of \( S \) on \( L^p(\Omega) \) for \( p \in (1, \infty) \) is implied by the one parameter Littlewood–Paley theory. Also the concept of a martingale transform has a natural generalization, that is,

\[
T f = \sum_{n,m \in \mathbb{Z}} a_{n,m} d_{n,m} f
\]

where \( (a_{n,m} : n, m \in \mathbb{Z}) \) is a sequence of uniformly bounded functions such that \( a_{n+1,m+1} \) is \( \mathcal{F}_{n,m} \)-measurable.

In this article we are interested in a case when the condition \( (F_4) \) is not satisfied. The simplest example may be obtained by considering the Heisenberg group together with the non-isotropic two parameter dilations

\[
\delta_s (x, y, z) = (sx, ty, stz).
\]

Since in this setup the dyadic cubes do not posses the same properties as the Euclidean cubes, it is more convenient to work on the \( p \)-adic version of the Heisenberg group. We observe that this group can be identified with \( \Omega_0 \), a subset of a boundary of the building of \( \text{GL}(3, \mathbb{Q}_p) \) consisting of the points opposite to a given \( \omega_0 \). The set \( \Omega_0 \) has a natural two parameter filtration \( (\mathcal{F}_{n,m} : n, m \in \mathbb{Z}) \) (see Section 2 for details). The maximal function and the square function are defined by (1.5) and (1.6), respectively. The results we obtain are summarized in the following three theorems.

**Theorem A.** For each \( p \in (1, \infty] \) there is \( C_p > 0 \) such that for all \( f \in L^p(\Omega_0) \)

\[
||M^* f||_{L^p} \leq C_p ||f||_{L^p}.
\]

**Theorem B.** For each \( p \in (1, \infty) \) there is \( C_p > 1 \) such that for all \( f \in L^p(\Omega_0) \)

\[
C_p^{-1} ||f||_{L^p} \leq ||S f||_{L^p} \leq C_p ||f||_{L^p}.
\]

**Theorem C.** If \( (a_{n,m} : n, m \in \mathbb{Z}) \) is a sequence of uniformly bounded functions such that \( a_{n+1,m+1} \) is \( \mathcal{F}_{n,m} \)-measurable, then the martingale transform

\[
T f = \sum_{n,m \in \mathbb{Z}} a_{n,m} d_{n,m} f
\]

is bounded on \( L^p(\Omega_0) \). for all \( p \in (1, \infty) \).

Let us briefly describe methods we use. First, we observe that instead of \( (F_4) \) the stochastic basis satisfies the remarkable identity (2.2). Based on it we show that the following pointwise estimate holds

\[
M^* (|f|) \leq C (L^* R^* L^* R^* (|f|) + R^* L^* R^* L^* (|f|))
\]

proving the maximal theorem. Thanks to the two parameter Khintchine’s inequality, to bound the square function \( S \), it is enough to show Theorem C. To do so, we define a new square function \( S \) which has a nature similar to the square function used in the presence of \( (F_4) \). Then we adapt
the technique developed by Duoandikoetxea and Rubio de Francia in [6] (see Theorem 3). This implies $L^p$-boundedness of $S$. Since $S$ does not preserve the $L^2$ norm, the lower bound requires an extra argument. Namely, we view the square function $S$ as an operator with values in $L^p(l^2)$ and take its dual. As a consequence of Theorem 3 and the identity (4.7) the later is bounded on $L^p$.

Finally, let us comment on the behavior of the maximal function $M^*$ close to $L^1$. Based on the pointwise estimate (1.7), in view of [8], we conclude that $M^*$ is of weak-type for functions in the Orlicz space $L(\log L)^3$. To better understand the maximal function $M^*$ we investigate exact behavior close to $L^1$. This together with weighted estimates is the subject of the forthcoming paper. It is also interesting how to extend theorems A, B and C to higher rank and other types of affine buildings.

1.1. Notation. For two quantities $A > 0$ and $B > 0$ we say that $A \lesssim B$ ($A \gtrsim B$) if there exists an absolute constant $C > 0$ such that $A \leq CB$ ($A \geq CB$).

If $\lambda \in P$ we set $|\lambda| = \max\{|\lambda_1|, |\lambda_2|\}$.

2. Triangle buildings

2.1. Coxeter complex. We recall basic facts about the $A_2$ root system and the $\tilde{A}_2$ Coxeter group. A general reference is [1]. Let $a$ be the hyperplane in $\mathbb{R}^3$ defined as

$$a = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}.$$ 

We denote by $\{e_1, e_2, e_3\}$ the canonical orthonormal basis of $\mathbb{R}^3$ with respect to the standard scalar product $\langle \cdot, \cdot \rangle$. We set $\alpha_1 = e_2 - e_1$, $\alpha_2 = e_3 - e_2$, $\alpha_0 = e_3 - e_1$ and $I = \{0, 1, 2\}$. The $A_2$ root system is defined by

$$\Phi = \{\pm \alpha_0, \pm \alpha_1, \pm \alpha_2\}.$$ 

We choose the base $\{\alpha_1, \alpha_2\}$ of $\Phi$. The corresponding positive roots are $\Phi^+ = \{\alpha_0, \alpha_1, \alpha_2\}$. Denote by $\{\lambda_1, \lambda_2\}$ the basis dual to $\{\alpha_1, \alpha_2\}$; its elements are called the fundamental co-weights. Their integer combinations, form the co-weight lattice $P$. As in Figure 1, we always draw $\lambda_1$ pointing up and to the left and $\lambda_2$ up and to the right. Likewise $\lambda_1 - \lambda_2$ is drawn pointing directly left, while $\lambda_2 - \lambda_1$ points directly right. Because $\langle \lambda_1, \alpha_0 \rangle = \langle \lambda_2, \alpha_0 \rangle = 1$, we see that for any $\lambda \in P$ the expression $\langle \lambda, \alpha_0 \rangle$ represents the vertical level of $\lambda$. For $\lambda = i\lambda_1 + j\lambda_2$, that level is $i + j$.

Let $H$ be the family of affine hyperplanes, called walls,

$$H_{j,k} = \{x \in a : \langle x, \alpha_j \rangle = k\}.$$ 

Figure 1. A_2 root system
where \( j \in I, \ k \in \mathbb{Z} \). To each wall \( H_{j,k} \) we associate \( r_{j,k} \) the orthogonal reflection in \( a \), i.e.

\[
r_{j,k}(x) = x - (\langle x, \alpha_j \rangle - k)\alpha_j.
\]

Set \( r_1 = r_{1,0}, r_2 = r_{2,0} \) and \( r_0 = r_{0,1} \). The finite Weyl group \( W_0 \) is the subgroup of \( \text{GL}(a) \) generated by \( r_1 \) and \( r_2 \). The affine Weyl group \( W \) is the subgroup of \( \text{Aff}(a) \) generated by \( r_0, r_1 \) and \( r_2 \).

Let \( C \) be the family of open connected components of \( a \setminus \bigcup_{H \in H} H \). The elements of \( C \) are called chambers. By \( C_0 \) we denote the fundamental chamber, i.e.

\[
C_0 = \{ x \in a : \langle x, \alpha_1 \rangle > 0, \langle x, \alpha_2 \rangle > 0, \langle x, \alpha_0 \rangle < 1 \}.
\]

The group \( W \) acts simply transitively on \( C \). Moreover, \( C_0 \) is a fundamental domain for the action of \( W \) on \( a \) (see e.g. [1, VI, §1-3]). The vertices of \( C_0 \) are \( \{0, \lambda_1, \lambda_2\} \). The set of all vertices of all \( C \in C \) is denoted by \( V(\Sigma) \). Under the action of \( W \), \( V(\Sigma) \) is made up of three orbits, \( W(0) \), \( W(\lambda_1) \), and \( W(\lambda_2) \). Vertices in the same orbit are said to have the same type. Any chamber \( C \in C \) has one vertex in each orbit or in other words one vertex of each of the three types.

The family \( C \) may be regarded as a simplicial complex \( \Sigma \) by taking as the simplices all non-empty subsets of vertices of \( C \), for all \( C \in C \). Two chambers \( C \) and \( C' \) are \( i \)-adjacent for \( i \in I \) if \( C = C' \) or if there is \( w \in W \) such that \( C = wC_0 \) and \( C' = wC_0 \). Since \( r_i^2 = 1 \) this defines an equivalence relation.

The fundamental sector is defined by

\[
S_0 = \{ x \in a : \langle x, \alpha_1 \rangle > 0, \langle x, \alpha_2 \rangle > 0 \}.
\]

Given \( \lambda \in P \) and \( w \in W_0 \) the set \( \lambda + wS_0 \) is called a sector in \( \Sigma \) with base vertex \( \lambda \). The angle spanned by a sector at its base vertex is \( \pi/3 \).

2.2. The definition of triangle buildings. For the theory of affine buildings we refer the reader to [13]. See also the first author’s expository paper [14], for an elementary introduction to the \( p \)-adics, and to precisely the sort of the buildings which this paper deals with.

A simplicial complex \( \mathcal{X} \) is an \( \mathbb{A}_2 \) building, or as we like to call it, a triangle building, if each of its vertices is assigned one of the three types, and if it contains a family of subcomplexes called apartments such that

\begin{itemize}
  \item[(i)] each apartment is type-isomorphic to \( \Sigma \),
  \item[(ii)] any two simplexes of \( \mathcal{X} \) lie in a common apartment,
  \item[(iii)] for any two apartments \( \mathcal{A} \) and \( \mathcal{A}' \) having a chamber in common there is a type-preserving isomorphism \( \psi : \mathcal{A} \to \mathcal{A}' \) fixing \( \mathcal{A} \cap \mathcal{A}' \) pointwise.
\end{itemize}

We assume also that the system of apartments is complete, meaning that any subcomplex of \( \mathcal{X} \) type-isomorphic to \( \Sigma \) is an apartment. A simplex \( C \) is a chamber in \( \mathcal{X} \) if it is a chamber for some apartment. Two chambers of \( \mathcal{X} \) are \( i \)-adjacent if they are \( i \)-adjacent in some apartment. For \( i \in I \) and for a chamber \( C \) of \( \mathcal{X} \) let \( q_i(C) \) be equal to

\[
q_i(C) = |\{ C' \in \mathcal{X} : C' \sim_i C \}| - 1.
\]

It may be proved that \( q_i(C) \) is independent of \( C \) and of \( i \). Denote the common value by \( q \), and assume local finiteness: \( q < \infty \). Any edge of \( \mathcal{X} \), i.e., any 1-simplex, is contained in precisely \( q + 1 \) chambers.

It follows from the axioms that the ball of radius one about any vertex \( x \) of \( \mathcal{X} \) is made up of \( x \) itself, which is of one type, \( q^2 + q + 1 \) vertices of a second type, and a further \( q^2 + q + 1 \) vertices of the third type. Moreover, adjacency between vertices of the second and third types makes them into, respectively, the points and the lines of a finite projective plane.

A subcomplex \( \mathcal{X} \) is called a sector of \( \mathcal{X} \) if it is a sector in some apartment. Two sectors are called equivalent if they contain a common subsector. Let \( \Omega \) denote the set of equivalence classes of sectors. If \( x \) is a vertex of \( \mathcal{X} \) and \( \omega, \omega' \in \Omega \), there is a unique sector denoted \( \langle x, \omega \rangle \) which has base vertex \( x \) and represents \( \omega \).

Given any two points \( \omega \) and \( \omega' \in \Omega \), one can find two sectors representing them which lie in a common apartment. If that apartment is unique, we say that \( \omega \) and \( \omega' \) are opposite, and denote the unique apartment by \( \langle \omega, \omega' \rangle \). In fact \( \omega \) and \( \omega' \) are opposite precisely when the two sectors in the common apartment point in opposite directions in the Euclidean sense.
2.3. Filtrations. We fix once and for all an origin vertex $O \in \mathcal{X}$ and a point $\omega_0 \in \Omega$. Choose $O$ so that it has the same type as the origin of $\Sigma$. Let $\mathcal{X}_0 = [O, \omega_0]$ be the sector representing $\omega_0$ with base vertex $O$. By $\mathcal{X}_0$ we denote the subset of $\Omega$ consisting of $\omega$’s opposite to $\omega_0$. For purposes of motivation only, we recall that if $\mathcal{X}$ is the building of $\text{GL}(3, \mathbb{Q}_p)$, then $\Omega_0$ can be identified with the $p$-adic Heisenberg group (see Appendix A for details).

Let $\omega_0$ be any apartment containing $\mathcal{X}_0$. By $\psi$ we denote the type-preserving isomorphism between $\omega_0$ and $\Sigma$ such that $\psi(\mathcal{X}_0) = -S_0$. We set $\rho = \psi \circ \rho_0$ where $\rho_0$ is the retraction from $\mathcal{X}$ to $\omega_0$. With these definitions, $\rho : \mathcal{X} \to \Sigma$ is a type-preserving simplicial map, and for any $\omega \in \Omega_0$ the apartment $[\omega, \omega_0]$ maps bijectively to $\Sigma$ with $\omega_0$ mapping to the bottom (of Figure 1) and $\omega$ mapping to the top.

For any vertex $x \in \mathcal{X}$ define the subset $E_x \subset \Omega_0$ to consist of all $\omega$’s such that $x$ belongs to $[\omega, \omega_0]$; an equivalent condition is that $[x, \omega_0] \subset [\omega, \omega_0]$. Fix $\lambda \in P$. By $\mathcal{F}_\lambda$ we denote the $\sigma$-field generated by sets $E_x$ for $x \in \mathcal{X}$ with $\rho(x) = \lambda$. There are countably many such $x$, and the corresponding sets $E_x$ are mutually disjoint, hence $\mathcal{F}_\lambda$ is a countably generated atomic $\sigma$-field.

Let $\preceq$ denote the partial order on $P$ where $\lambda \preceq \mu$ if and only if $(\lambda - \mu, \alpha_1) \leq 0$ and $(\lambda - \mu, \alpha_2) \leq 0$. If we draw and orient $\Sigma$ as in Figure 1, then $\lambda \preceq \mu$ exactly when $\mu$ lies in the sector pointing upwards from $\lambda$.

**Proposition 2.1.** If $\lambda \preceq \mu$ then $\mathcal{F}_\lambda \subset \mathcal{F}_\mu$.

**Proof.** Choose any vertex $x$ so that $\rho(x) = \mu$. Because $\lambda \preceq \mu$, there is a unique vertex $y$ in the sector $[x, \omega_0]$ so that $\rho(y) = \lambda$. For any $\omega \in E_x$, the apartment $[\omega, \omega_0]$ contains $x$, hence it contains $[x, \omega_0]$, hence it contains $y$. This establishes that $E_x \subseteq E_y$. In other words, each atom of $\mathcal{F}_\mu$ is a subset of some atom of $\mathcal{F}_\lambda$. Hence each atom of $\mathcal{F}_\lambda$ is a disjoint union of atoms of $\mathcal{F}_\mu$. $\square$

In fact, Proposition 2.1 says that $(\mathcal{F}_\lambda : \lambda \in P) = (\mathcal{F}_{\lambda_{i,j}(2,1)} : i, j \in \mathbb{Z})$ is a two parameter filtration. Let

$$\mathcal{F} = \sigma \left( \bigcup_{\lambda \in P} \mathcal{F}_\lambda \right).$$

Let $\pi$ denote the unique $\sigma$-additive measure on $(\Omega_0, \mathcal{F})$ such that for $E_x \in \mathcal{F}_\lambda$ $\pi(E_x) = q^{-2(\lambda, \omega_0)}$.

All $\sigma$-fields in this paper should be extended so as to include $\pi$-null sets.

A function $f(\omega)$ on $\Omega_0$ is $\mathcal{F}_\lambda$-measurable if it depends only on that part of the apartment $[\omega, \omega_0]$ which retracts under $\rho$ to the sector pointing downwards from $\lambda$. For $i, j \in \mathbb{Z}$ set

$$\mathcal{F}_{i,\infty} = \sigma \left( \bigcup_{j' \in \mathbb{Z}} \mathcal{F}_{\lambda_{i,j'j} + j'\lambda_2} \right), \quad \mathcal{F}_{\infty,j} = \sigma \left( \bigcup_{i' \in \mathbb{Z}} \mathcal{F}_{\lambda_{i'j} + i'\lambda_2} \right).$$

A function $f(\omega)$ on $\Omega_0$ is $\mathcal{F}_{i,\infty}$-measurable (respectively $\mathcal{F}_{\infty,j}$-measurable) if it depends only on that part of the apartment which retracts to a certain “lower” half-plane with boundary parallel to $\lambda_2$ (respectively $\lambda_1$).

If $\mathcal{F}'$ is $\sigma$-subfield of $\mathcal{F}$, we denote by $\mathbb{E}[f|\mathcal{F}']$ the Radon–Nikodym derivative with respect to $\mathcal{F}'$. If $\mathcal{F}''$ is another $\sigma$-subfield of $\mathcal{F}$ we write

$$\mathbb{E}[f|\mathcal{F}'|\mathcal{F}''] = \mathbb{E}[\mathbb{E}[f|\mathcal{F}']|\mathcal{F}''].$$

The $\sigma$-field generated by $\mathcal{F}' \cup \mathcal{F}''$ is denoted by $\mathcal{F}' \vee \mathcal{F}''$. We write $f_\lambda = \mathbb{E}_{\lambda} f = \mathbb{E}[f|\mathcal{F}_\lambda]$ for $\lambda \in P$. If $\lambda \preceq \mu$, then it follows from Proposition 2.1 that $\mathbb{E}_{\mu} \mathbb{E}_{\lambda} = \mathbb{E}_{\lambda} \mathbb{E}_{\mu} = \mathbb{E}_{\lambda}$.

We note that the Cairoli–Walsh condition $(F_4)$ introduced in [5] is not satisfied, i.e.

$$\mathbb{E}_{\lambda + \lambda_1} \mathbb{E}_{\lambda + \lambda_2} \neq \mathbb{E}_{\lambda}.$$

Instead of $(F_4)$ we have

**Lemma 2.2.** For a locally integrable function $f$ on $\Omega_0$

\begin{align}
(2.1) \quad & \mathbb{E}[f_{\lambda + \lambda_1}|\mathcal{F}_{\lambda + \lambda_2}|\mathcal{F}_{\lambda + \lambda_3}] = q^{-1} f_{\lambda + \lambda_1} - q^{-1} \mathbb{E}[f_{\lambda + \lambda_1}|\mathcal{F}_{\lambda + \lambda_2} \vee \mathcal{F}_{\lambda}] + f_{\lambda}, \\
(2.2) \quad & (\mathbb{E}_{\lambda + \lambda_2} \mathbb{E}_{\lambda + \lambda_3})^2 = q^{-1} \mathbb{E}_{\lambda + \lambda_2} \mathbb{E}_{\lambda + \lambda_3} + (1 - q^{-1}) \mathbb{E}_{\lambda},
\end{align}

and likewise if we exchange $\lambda_1$ and $\lambda_2$. 

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Proof. For the proof of (2.1) it is enough to consider \( f = 1_{E_{p_1}} \) where \( p_1 \) is a vertex in \( \mathcal{X} \) such that \( \rho(p_1) = \lambda + \lambda_1 \). Let \( \mathcal{F} \) be the sector \([p_1, \omega_0]\) and let \( x \) be the unique vertex of \( \mathcal{F} \) with \( \rho(x) = \lambda \). The ball in \( \mathcal{X} \) of radius 1 around \( x \) has the structure of a finite projective plane. In Figure 2

![Figure 2. Residue of x](image)

the spot marked \( x \) is for vertices of \( \mathcal{X} \) which retract via \( \rho \) to \( \lambda \). Recall that \( E_x \) is an atom of the \( \sigma \)-field \( \mathcal{F}_\lambda \). The spot marked \( p_1 \) is for vertices retracting to \( \lambda + \lambda_1 \); the spot marked \( l \) is for vertices retracting to \( \lambda + \lambda_2 \); the spot marked \( l_1 \) is for vertices retracting to \( \lambda + \lambda_1 - \lambda_2 \); etc. In the ball of radius 1 around \( x \), only \( x \) itself retracts to the spot marked \( x \). The line type vertex known as \( l_0 \) is the only vertex in the ball retracting to its spot; \( q \) line type vertices retract to the same spot as \( l_1 \); the remaining \( q^2 \) line type vertices retract to the spot marked \( l \). Likewise, \( p_0 \) is the unique point type vertex of the ball retracting to its spot; \( q \) point type vertices retract to the spot marked \( p \); \( q^2 \) retract to the same spot as \( p_1 \). It follows that

\[
E[1_{E_{p_1}} | \mathcal{F}_\lambda] = q^{-2}1_{E_x} = q^{-2} \sum_{\lambda' \neq l_0} 1_{E_{\lambda'}} = q^{-2} \sum_{\lambda' \neq p_0} 1_{E_{\lambda'}}
\]

and

\[
E[1_{E_{p_1}} | \mathcal{F}_{\lambda+\lambda_1} \lor \mathcal{F}_\lambda] = q^{-1}1_{E_{x \cap E_{\lambda}}} = q^{-1} \sum_{\lambda' \neq l_1} 1_{E_{\lambda'}}
\]

where \( \lambda' \) runs through the point type vertices of the ball, \( l \) runs through the line type vertices of the ball, and \( \sim \) stands for the incidence relation. We have

\[
(2.3) \quad E[1_{E_{p_1}} | \mathcal{F}_{\lambda+\lambda_2}] = q^{-1} \sum_{\lambda' \neq p_0} 1_{E_{\lambda'}}.
\]

Therefore, we obtain

\[
E[1_{E_{p_1}} | \mathcal{F}_{\lambda+\lambda_3} | \mathcal{F}_{\lambda+\lambda_1}] = q^{-2} \sum_{\lambda' \neq l_1} 1_{E_{\lambda'}} = q^{-1}1_{E_{p_1}} + q^{-2} \sum_{\lambda' \neq l_0} 1_{E_{\lambda'}}
\]

\[
(2.4) \quad = q^{-1}1_{E_{p_1}} + q^{-2} \sum_{\lambda' \neq l_0} 1_{E_{\lambda'}} - q^{-2} \sum_{\lambda' \neq l_1} 1_{E_{\lambda'}},
\]

which finishes the proof of (2.1). Applying one more average to the next to the last expression of (2.4) we get

\[
E[1_{E_{p_1}} | \mathcal{F}_{\lambda+\lambda_2} | \mathcal{F}_{\lambda+\lambda_1} | \mathcal{F}_{\lambda+\lambda_3}] = q^{-2} \sum_{\lambda' \neq l_1} 1_{E_{\lambda'}} + q^{-3} \sum_{\lambda' \neq l_0} 1_{E_{\lambda'}}.
\]

For any line \( l \neq p_0 \) there are \( q \) points \( p' \) such that \( p' \sim l \) and \( p' \neq l_0 \) and among them there is exactly one incident to \( l_1 \). Hence in the last sum each line \( l \neq p_0 \) appears \( q - 1 \) times. Thus, we
can write
\[ q^{-3} \sum_{p' \neq p_0 \text{ and } l' \neq l, l' \neq l_1} = q^{-3}(q - 1) \sum_{l \neq p_0} 1_{E_l} = (1 - q^{-1})E[1_{E_{p_0}} | F_\lambda] \]
proving (2.2).

The following lemma describes the composition of projections on the same level.

**Lemma 2.3.** If \( k, j \in \mathbb{Z} \) are such that \( k \geq j \geq 0 \) or \( k \leq j \leq 0 \) then
\[
E_{\lambda + k(\lambda_2 - \lambda_1)}E_\lambda = E_{\lambda + k(\lambda_2 - \lambda_1)}E_{\lambda + j(\lambda_2 - \lambda_1)}E_\lambda.
\]

**Proof.** We do the proof for \( k \geq j \geq 0 \). For any \( \omega \in \Omega_0 \), there is a connected chain of vertices \((x_i : 0 \leq i \leq k) \subseteq [\omega, \omega_0]\) with \( \rho(x_i) = \lambda + k(\lambda_2 - \lambda_1) \). Suppose, conversely, that \((x_i : 0 \leq i \leq k)\) is a connected chain of vertices and that \( \rho(x_i) = \lambda + k(\lambda_2 - \lambda_1) \). Construct a subcomplex \( B \subseteq B' \) by putting together \([x_i, \omega_0] : 0 \leq i \leq k\), the edges between the \( x_i \)'s and the triangles pointing downwards from those edges to \( \omega_0 \). Referring to Figure 3, the extra triangle pointing downward from the first edge has vertices \( x_0, x_1 \), and \( y_0 \). Note that \([x_0, \omega_0] \cap [x_1, \omega_0] = [y_0, \omega_0]\). Proceeding one step at a time, one may verify that the restriction of \( \rho \) to \( B \) is an injection and that \( B \) and \( \rho(B) \) are isomorphic complexes.

**Figure 3. The complex \( B \)**

By basic properties of affine buildings, one knows it is possible to extend \( B \) to an apartment. Any such apartment will retract bijectively to \( \Sigma \), and will be of the form \([\omega, \omega_0]\) where \( \omega \) is the equivalence class represented by the upward pointing sectors of the apartment. Moreover, using the definition of \( \pi \) one may calculate that
\[
\pi([\{\omega \in \Omega_0 : B \subseteq [\omega, \omega_0]\}) = q^{-2(\lambda,\alpha_0)} - k.
\]
The important point is that the measure of the set depends only on the level of \( \lambda \) and the length of the chain.

Basic properties of affine buildings imply that any apartment containing \( x_0 \) and \( x_k \) contains the entire chain. Hence
\[
\pi(E_{x_0} \cap E_{x_k}) = \pi([\{\omega \in \Omega_0 : B \subseteq [\omega, \omega_0]\}) = q^{-2(\lambda,\alpha_0)} - k.
\]

Fix \( x_0 \). Proceeding one step at a time, one sees there are \( q^k \) connected chains \((x_i : 0 \leq i \leq k)\) with \( \rho(x_i) = \lambda + k(\lambda_2 - \lambda_1) \). Consequently
\[
E_{\lambda + k(\lambda_2 - \lambda_1)}1_{x_0} = q^{-k} \sum_{0 \leq i \leq k} 1_{x_i}.
\]
Likewise
\[
E_{\lambda + k(\lambda_2 - \lambda_1)}E_{\lambda + j(\lambda_2 - \lambda_1)}1_{x_0} = q^{-j}E_{\lambda + k(\lambda_2 - \lambda_1)} \sum_{0 \leq i \leq j} 1_{x_j}.
\]
which is the same thing.

Consider \( E_\lambda E_\mu \). If \( \lambda \preceq \mu \) then the product is equal to \( E_\lambda \); similarly if \( \mu \preceq \lambda \). If \( \lambda \) and \( \mu \) are incomparable, the following lemma allows us to reduce to the case where \( \lambda \) and \( \mu \) are on the same level.
Lemma 2.4. Suppose $\lambda \in P$ and
\[ \lambda' = \lambda - i\lambda_1, \quad \mu = \lambda' + k(\lambda_2 - \lambda_1), \quad \tilde{\mu} = \mu + (\lambda_2 - \lambda_1) \]
for $i, k \in \mathbb{N}$. Then for any locally integrable function $f$ on $\Omega_0$
\[
\begin{align*}
(2.6) & \quad \mathbb{E}[f|F_\lambda|F_\mu] = \mathbb{E}[f|F_\lambda|F_{\mu'}], \\
(2.7) & \quad \mathbb{E}[f|F_\mu|F_\lambda] = \mathbb{E}[f|F_\mu|F_{\lambda'}], \\
(2.8) & \quad \mathbb{E}[f|F_\mu \lor F_\lambda] = \mathbb{E}[f|F_\mu|F_{\lambda'}] \\
(2.9) & \quad \mathbb{E}[f|F_\mu \lor F_\lambda|F_{\lambda'}] = \mathbb{E}[f|F_\mu|F_{\lambda'}]
\end{align*}
\]
and likewise if we exchange $\lambda_1$ and $\lambda_2$.

Proof. We first prove (2.6) for $i = 1$ and $k = 1$. Because $\mathbb{E}[f|F_\lambda] = \mathbb{E}[f|F_\lambda|F_{\lambda'}]$, it is sufficient to consider $f = \mathbb{1}_{E_{p_1}}$, where $\rho(p_1) = \lambda$. Use Figure 2 to fix the notation, and note that if $p_1$ retracts to $\lambda$, then $x$ retracts to $\lambda'$ and $p$ to $\mu$. One calculates:
\[
\mathbb{E}[\mathbb{1}_{E_{p_1}}|F_\lambda|F_{\mu}] = \mathbb{E}[\mathbb{1}_{E_{p_1}}|F_{\mu}] = q^{-3} \sum_{p \neq p_1} \mathbb{1}_{E_{p}} = q^{-2} \mathbb{E}[\mathbb{1}_{E_{p_1}}|F_{\mu}] = \mathbb{E}[\mathbb{1}_{E_{p_1}}|F_{\lambda'}|F_{\mu}].
\]
Next consider the case $i = 1$, $k > 1$. Set $\mu' = \mu + \lambda_1$, $\nu = \mu + \lambda_1 - \lambda_2$ and $\nu' = \nu + \lambda_1$ (see Figure 4). Since $F_{\mu}$ is a subfield of $F_{\mu'}$, we have
\[
\mathbb{E}[f|F_\lambda|F_{\mu'}] = \mathbb{E}[f|F_\lambda|F_{\nu'}|F_{\mu'}].
\]
Thus, applying Lemma 2.3 we obtain
\[
\mathbb{E}[f|F_\lambda|F_\mu] = \mathbb{E}[f|F_\lambda|F_{\mu'}|F_\mu] = \mathbb{E}[f|F_\lambda|F_{\nu'}|F_{\mu'}] = \mathbb{E}[f|F_\lambda|F_{\nu'}|F_\mu] = \mathbb{E}[f|F_\lambda|F_\nu|F_\mu].
\]
where in the last step we have used the case $k = 1$. Now apply induction on $k$ and Lemma 2.3 again to get
\[
\mathbb{E}[f|F_\lambda|F_{\mu'}] = \mathbb{E}[f|F_\lambda|F_{\nu'}|F_{\mu'}] = \mathbb{E}[f|F_\lambda|F_{\nu'}|F_{\mu}] = \mathbb{E}[f|F_{\lambda'}|F_{\mu'}].
\]
To extend to the case $i > 1$, use induction on $i$ and observe that
\[
\mathbb{E}[f|F_\lambda|F_{\nu'}] = \mathbb{E}[f|F_\lambda|F_{\nu'}|F_{\mu'}] = \mathbb{E}[f|F_{\lambda'}|F_{\nu'}|F_{\mu'}] = \mathbb{E}[f|F_{\lambda'}|F_{\nu'}|F_{\mu}].
\]
The proof of (2.8) is analogous, starting with the case $i = 1$, $k = 0$. Identity (2.6) can be read as $\mathbb{E}_x \mathbb{E}_\lambda = \mathbb{E}_\mu \mathbb{E}_\lambda$. The expectation operators are orthogonal projections with respect to the usual inner product, and taking adjoints gives $\mathbb{E}_\lambda \mathbb{E}_\mu = \mathbb{E}_\lambda \mathbb{E}_{\mu'}$, which is (2.7). To be more precise, one takes the inner product of either side of (2.7) with some nice test function, applies self-adjointness, and reduces to (2.6). Likewise, (2.9) follows from (2.8).

Lemma 2.5. Suppose $i\lambda_1 + j\lambda_2$, $\mu = \lambda + k(\lambda_1 - \lambda_2)$. Then for any locally integrable function $f$ on $\Omega_0$
\[
\mathbb{E}[f|F_\mu|F_\lambda] = \begin{cases} 
\mathbb{E}[f|F_\mu|F_{i,\infty}] & \text{if } k \geq 0, \\
\mathbb{E}[f|F_\mu|F_{\infty,j}] & \text{if } k \leq 0.
\end{cases}
\]
Proof. Suppose \( k \geq 0 \). By Lemma 2.4 for any \( j' \geq 0 \) we have
\[
E_{\mu}E_{\lambda+j'\lambda_0} = E_{\mu}E_{\lambda}.
\]
So if \( g \) is \( F_{\lambda+j'\lambda_2} \)-measurable and compactly supported, then
\[
\langle g, E_{i,\infty}E_{\mu}, f \rangle = \langle E_{\mu}E_{i,\infty}g, f \rangle = \langle E_{\mu}g, f \rangle = \langle E_{\mu}E_{\lambda+j'\lambda_0}g, f \rangle = \langle E_{\mu}E_{\lambda}g, f \rangle = \langle g, E_{\lambda}E_{\mu}f \rangle.
\]
The test functions \( g \) which we use are sufficient to distinguish between one \( F_{i,\infty} \)-measurable function and another. Since \( E_{i,\infty}E_{\mu}f \) and \( E_{\lambda}E_{\mu}f \) are both \( F_{i,\infty} \)-measurable the proof is done. \( \square \)

3. Littlewood-Paley theory

3.1. Maximal functions. The natural maximal function \( M^* \) for a locally integrable function \( f \) on \( \Omega_0 \) is defined by
\[
M^*f = \max_{\lambda \in P} |f_{\lambda}|.
\]
Additionally, we define two auxiliary single parameter maximal functions
\[
L^*f = \max_{\mu \in \Omega} E[|f|]_{F_{i,\infty}}, \quad R^*f = \max_{\mu \in \Omega} E[|f|]_{F_{\infty,j}}.
\]

Lemma 3.1. Let \( \lambda \in P \) and \( k \in \mathbb{N} \). For any non-negative locally integrable function \( f \) on \( \Omega_0 \)
\[
(E_{\lambda+k\lambda_2}E_{\lambda+k\lambda_1})^2 f \geq (1 - q^{-1})E_{\lambda}f.
\]

Proof. We may assume \( \lambda = 0 \). Let us define (see Figure 5)
\[
\mu = k\lambda_1, \quad \mu' = \lambda_1 + (k-1)\lambda_2, \quad \mu'' = k\lambda_2, \quad \nu = (k-1)\lambda_1, \quad \nu' = \lambda_1 + (k-2)\lambda_2, \quad \nu'' = (k-1)\lambda_2.
\]
We show
\[
E_{\mu'}E_{\mu}E_{\mu''}E_{\mu} = q^{-1}E_{\mu'}E_{\mu}E_{\mu''}E_{\mu} = E_{\nu'}E_{\nu}E_{\nu''}E_{\nu} = q^{-1}E_{\nu'}E_{\nu}E_{\nu''}E_{\nu}.
\]
Let \( g = E[f]_{\mathcal{F}_{\mu}} \). By two applications of Lemma 2.3 we can write
\[
E[g]_{\mathcal{F}_{\mu''}}E_{\mu} = E[g]_{\mathcal{F}_{\mu'}}E_{\mu} + E[g]_{\mathcal{F}_{\mu''}} - q^{-1}E[g]_{\mathcal{F}_{\mu'}}E_{\mu}.
\]
and by Lemma 2.2
\[
E[g]_{\mathcal{F}_{\mu'}}E_{\mu''}E_{\mu} = q^{-1}E[g]_{\mathcal{F}_{\mu'}} + E[g]_{\mathcal{F}_{\mu'}} - q^{-1}E[g]_{\mathcal{F}_{\mu'}}E_{\mu}.
\]
Hence,
\[
E[g]_{\mathcal{F}_{\mu''}}E_{\mu} - q^{-1}E[g]_{\mathcal{F}_{\mu'}}E_{\mu} = E[g]_{\mathcal{F}_{\mu''}}E_{\mu} - q^{-1}E[g]_{\mathcal{F}_{\mu'}}E_{\mu}.
\]
By repeated application of Lemma 2.4 we have
\[
E[g]_{\mathcal{F}_{\mu''}}E_{\mu} = E[f]_{\mathcal{F}_{\mu}}E_{\mu''}E_{\mu} = E[f]_{\mathcal{F}_{\mu}}E_{\mu''}E_{\mu}.
\]
and
\[
E[g]_{\mathcal{F}_{\mu'}}E_{\mu''}E_{\mu} = E[f]_{\mathcal{F}_{\mu}}E_{\mu''}E_{\mu} = E[f]_{\mathcal{F}_{\mu}}E_{\mu''}E_{\mu}.
\]
which finishes the proof of (3.1). By iteration of (3.1) we obtain
\[
E_{\mu'}E_{\mu}E_{\mu''}E_{\mu} = q^{-1}E_{\mu'}E_{\mu}E_{\mu''}E_{\mu} = E_{\lambda_2}E_{\lambda_2}E_{\lambda_2}E_{\lambda_1} - q^{-1}E_{\lambda_2}E_{\lambda_1}E_{\lambda_1}
\]
which together with Lemma 2.2 implies
\[ \mathbb{E}_{\nu'}\mathbb{E}_{\nu''}\mathbb{E}_{\mu} = q^{-1}\mathbb{E}_{\nu'}\mathbb{E}_{\nu''}\mathbb{E}_{\mu} + (1 - q^{-1})\mathbb{E}_{0}. \]

\[ \text{Theorem 1.} \quad \text{For each } p \in (1, \infty) \text{ there is } C_p > 0 \text{ such that} 
\begin{align*}
(3.2) \quad \|L^*f\|_{L^p} & \leq C_p\|f\|_{L^p}, \\
(3.3) \quad \|M^* f\|_{L^p} & \leq C_p\|f\|_{L^p}.
\end{align*}
\]

\text{Proof.} Inequalities (3.2) are two instances of Doob’s well-known maximal inequality for single parameter martingales (see e.g. [15]). To show (3.3) consider a non-negative \( f \in L^p(\Omega_0, \mathcal{F}_\mu) \). Fix \( \lambda \in \mathbb{P} \). Since \( f \in L^p(\Omega_0, \mathcal{F}_\mu) \) for any \( \mu' \geq \mu \) we may assume \( \mu \geq \lambda \). Let
\[ \nu = \lambda + (\mu - \lambda, \alpha_0)\lambda_1, \quad \nu'' = \lambda + (\mu - \lambda, \alpha_0)\lambda_2. \]

By Lemma 3.1
\[ (1 - q^{-1})\mathbb{E}_{\nu} f \leq \mathbb{E}_{\nu'}\mathbb{E}_{\nu''}\mathbb{E}_{\mu} f. \]

If \( \lambda = i\lambda_1 + j\lambda_2 \), then repeated application of Lemma 2.5 gives
\[ \mathbb{E}_{\nu'}\mathbb{E}_{\nu''}\mathbb{E}_\nu f = \mathbb{E}_{\nu'}\mathbb{E}_{\nu''}\mathbb{E}_\mu f = \mathbb{E}[f|\mathcal{F}_{i,\infty}]\mathbb{E}[f|\mathcal{F}_{i,\infty}]\mathbb{E}[f|\mathcal{F}_{i,\infty}] \leq L^* R^* L^* R^* f. \]

By taking the supremum over \( \lambda \in \mathbb{P} \) we get
\[ (1 - q^{-1})M^* f \leq L^* R^* L^* R^* f. \]

Hence, by (3.2) we obtain (3.3) for \( f \in L^p(\Omega_0, \mathcal{F}_\mu) \). Finally, a standard Fatou’s lemma argument establishes the theorem for arbitrary \( f \in L^p(\Omega_0)). \)

\[ \text{□} \]

3.2. Square function. Let \( f \) be a locally integrable function on \( \Omega_0 \). Given \( i, j \in \mathbb{Z} \) we define projections
\[ L_i f = \mathbb{E}[f|\mathcal{F}_{i,\infty}] - \mathbb{E}[f|\mathcal{F}_{i-1,\infty}], \quad R_j f = \mathbb{E}[f|\mathcal{F}_{\infty,j}] - \mathbb{E}[f|\mathcal{F}_{\infty,j-1}]. \]

Note that \( L_i \) (respectively \( R_j \)) is the martingale difference operator for the filtration \( (\mathcal{F}_{i,\infty}: i \in \mathbb{Z}) \) (respectively \( (\mathcal{F}_{\infty,j}: j \in \mathbb{Z}) \)). For \( \lambda = i\lambda_1 + j\lambda_2 \) we set
\[ D_\lambda f = L_i R_j f, \quad D_\lambda^* f = R_j L_i f. \]

The following development is inspired by that of Stein and Street in [17]. We start by defining the corresponding square function.
\[ S f = \left( \sum_{\lambda \in \mathbb{P}} |D_\lambda f|^2 \right)^{1/2}. \]

We will also need its dual counterpart
\[ S^* f = \left( \sum_{\lambda \in \mathbb{P}} |D_\lambda^* f|^2 \right)^{1/2}. \]

\[ \text{Theorem 2.} \quad \text{For every } p \in (1, \infty) \text{ there is } C_p > 1 \text{ such that} 
\begin{align*}
C_p^{-1}\|f\|_{L^p} & \leq \|S f\|_{L^p} \leq C_p\|f\|_{L^p}, \\
C_p^{-1}\|f\|_{L^p} & \leq \|S^* f\|_{L^p} \leq C_p\|f\|_{L^p}.
\end{align*}
\]

Moreover, on \( L^2(\Omega_0) \) square functions \( S \) and \( S^* \) preserve the norm.

\text{Proof.} Since
\[ S_L(f) = \left( \sum_{i \in \mathbb{Z}} |L_i f|^2 \right)^{1/2} \] and \[ S_R(f) = \left( \sum_{j \in \mathbb{Z}} |R_j f|^2 \right)^{1/2} \]
preserve the norm on \( L^2(\Omega_0) \) we have
\[ \int \sum_{i,j \in \mathbb{Z}} |L_i R_j f|^2 \, d\mu = \sum_{j \in \mathbb{Z}} \int \sum_{i \in \mathbb{Z}} |L_i R_j f|^2 \, d\mu = \sum_{j \in \mathbb{Z}} \int |R_j f|^2 \, d\mu = \int |f|^2 \, d\mu. \]

Hence, \( S \) preserves the norm.
For $p \neq 2$ we use the two parameter Khintchine inequality (see [12]) and bounds on single parameter martingale transforms (see [2, 15, 18]). Let $(\epsilon_i : i \in \mathbb{Z})$ and $(\epsilon_j : j \in \mathbb{Z})$ be sequences of real numbers, with absolute values bounded above by 1. For $N \in \mathbb{N}$ we consider the operator

$$T_N = \sum_{|i|, |j| \leq N} \epsilon_i \epsilon_j D_{i\lambda_1 + j\lambda_2}$$

which may be written as a composition $\mathcal{L}_N \mathcal{R}_N$ where

$$\mathcal{L}_N = \sum_{|i| \leq N} \epsilon_i L_i, \quad \mathcal{R}_N = \sum_{|j| \leq N} \epsilon_j R_j.$$ 

Since by Burkholder’s inequality (see [2, 15]) the operators $\mathcal{R}_N$ and $\mathcal{L}_N$ are bounded on $L^p(\Omega_0)$ with bounds uniform in $N$ we have

$$\|T_N f\|_{L^p} \lesssim \|f\|_{L^p}.$$ 

Setting $r_k$ to be the Rademacher function, by Khintchine’s inequality we get

$$\int \left( \sum_{|i|, |j| \leq N} |D_{i\lambda_1 + j\lambda_2}|^2 \right)^{p/2} \, d\pi \lesssim \int \int_0^1 \int_0^1 \left| \sum_{|i|, |j| \leq N} r_i(s) r_j(t) D_{i\lambda_1 + j\lambda_2} f \right|^p \, ds \, dt \, d\pi,$$

which is bounded by $\|f\|_{L^p}^p$. Finally, let $N$ approach infinity and use the monotone convergence theorem to get

$$\|Sf\|_{L^p} \lesssim \|f\|_{L^p}.$$ 

For the opposite inequality, we take $f \in L^p(\Omega_0) \cap L^2(\Omega_0)$ and $g \in L^{p'}(\Omega_0) \cap L^2(\Omega_0)$ where $1/p' + 1/p = 1$. By polarization of (3.4) and the Cauchy–Schwarz and Hölder inequalities we obtain

$$(f, g) = \int \sum_{\lambda \in P} D_{\lambda} f D_{\lambda}^* g \, d\pi \leq \|Sf\|_{L^p} \|Sg\|_{L^{p'}} \lesssim \|Sf\|_{L^p} \|g\|_{L^{p'}}.$$

Given a set $\{v_\lambda : \lambda \in P\}$ of vectors in a Banach space, we say that $\sum_{\lambda \in P} v_\lambda$ converges unconditionally if, whenever we choose a bijection $\phi : \mathbb{N} \to P$,

$$\sum_{n=1}^{\infty} v_{\phi(n)}$$

exists, and is independent of $\phi$.

Equivalently, we may ask that for any increasing, exhaustive sequence $(F_N : N \in \mathbb{N})$ of finite subsets of $P$, the limit

$$\lim_{N \to \infty} \sum_{\lambda \in F_N} v_\lambda$$

exists.

The following proposition provides a Calderón reproducing formula.

**Proposition 3.2.** For each $p \in (1, \infty)$ and any $f \in L^p(\Omega_0)$,

$$f = \sum_{\lambda \in P} D_{\lambda} D_{\lambda}^* f$$

where the sum converges in $L^p(\Omega_0)$ unconditionally.

**Proof.** Fix an increasing and exhaustive sequence $(F_N : N \in \mathbb{N})$ of finite subsets of $P$. Let

$$I_N(f) = \sum_{\lambda \in F_N} D_{\lambda} D_{\lambda}^* f.$$ 

For $f \in L^p(\Omega_0)$ and $g \in L^{p'}(\Omega_0)$, where $1/p + 1/p' = 1$, we have

$$|\langle I_N(f) - I_M(f), g \rangle| = \left| \sum_{\lambda \in F_N \setminus F_M} \langle D_{\lambda}^* f, D_{\lambda}^* g \rangle \right|$$

(3.5)

$$\leq \left\| \left( \sum_{\lambda \in F_N \setminus F_M} (D_{\lambda}^* f)^2 \right)^{1/2} \right\|_{L^p} \|S^* (g)\|_{L^{p'}}.$$ 

In particular,

$$|\langle I_N(f), g \rangle| \leq \|S^* (f)\|_{L^p} \|S^* (g)\|_{L^{p'}}.$$
whence $\|I_N(f)\|_{L^p} \lesssim \|f\|_{L^p}$ uniformly in $N$. Consequently, it is enough to prove convergence for $f \in L^p(\Omega_0) \cap L^2(\Omega)$. From (3.5) and the bounded convergence theorem it follows that for any positive $\varepsilon$, $\|I_N(f) - I_M(f)\|_{L^p} \leq \varepsilon$ whenever $M$ and $N$ are large enough. This shows that the limit exists. Finally, for $g \in L^p(\Omega_0) \cap L^2(\Omega)$, the polarized version of (3.4) gives
\[
\lim_{N \to \infty} \langle I_N(f), g \rangle = \lim_{N \to \infty} \sum_{\lambda \in F_N} \langle D^*_\lambda f, D^*_\lambda g \rangle = \langle f, g \rangle. \tag*{□}
\]

**Theorem 3.** Let $(T_\lambda : \lambda \in P)$ be a family of operators such that for some $\delta > 0$ and $p_0 \in (1, 2)$
\[
\tag{3.6}
\|T_\lambda\|_{L^1 \to L^1} \lesssim 1,
\]
\[
\tag{3.7}
\|T_\lambda^* T_\mu\|_{L^2 \to L^2} \lesssim q^{-\delta|\mu - \lambda|} \quad \text{and} \quad \|T^*_\mu T_\lambda\|_{L^2 \to L^2} \lesssim q^{-\delta|\mu - \lambda|},
\]
\[
\tag{3.8}
\|D_\lambda T_\mu D_\lambda^*\|_{L^2 \to L^2} \lesssim q^{-\delta(\lambda - \mu)} q^{-\delta|\lambda'|},
\]
\[
\tag{3.9}
\|\sup_{\lambda \in P} \|T_\lambda f\|_{L^p} \| \| \sup_{\lambda \in P} \|f\|_{L^p}.
\]

Then for any $p \in (p_0, 2]$ the sum $\sum_{\lambda \in P} T_\lambda$ converges unconditionally in the strong operator topology for operators on $L^p(\Omega_0)$.

**Proof.** First, recall that the Cotlar–Stein Lemma (see e.g. [16]) states that (3.7) implies the unconditional convergence of $\sum_{\lambda \in P} T_\lambda$ in the strong operator topology on $L^2(\Omega_0)$. Let $(F_N : N \in \mathbb{N})$ be an arbitrary increasing and exhaustive sequence of finite subsets of $P$. For $N > 0$ we set
\[
V_N = \sum_{p \in F_N} T_\mu, \quad I_N = \sum_{\lambda \in F_N} D_\lambda D_\lambda^*.
\]

By (3.6), (3.7) and interpolation, each $T_\mu$ is bounded on $L^p$ for $p \in [1, 2]$ and the same holds for the finite sum $V_N$. We consider $f \in L^p(\Omega_0)$ for $p \in (p_0, 2)$. By Proposition 3.2 and Theorem 2, we have
\[
\|V_M I_N(f)\|_{L^p} \lesssim \|S(V_M I_N(f))\|_{L^p} = \left\| \left( \sum_{\mu \in F_M} \sum_{\lambda \in F_N} D_\lambda T_\mu D_\lambda^* f : \lambda \in P \right) \right\|_{L^p(P)}
\]
\[
\leq \left\| \left( \sum_{\gamma, \gamma' \in P} 1_{F_N}(\lambda + \gamma + \gamma') 1_{F_M}(\lambda + \gamma) D_\lambda T_\lambda D_\lambda^* D_\lambda^* f : \lambda \in P \right) \right\|_{L^p(P)}
\]
\[
\leq \sum_{\gamma, \gamma' \in P} \left\| \left( 1_{F_N}(\lambda + \gamma + \gamma') 1_{F_M}(\lambda + \gamma) D_\lambda T_\lambda D_\lambda^* D_\lambda^* f : \lambda \in P \right) \right\|_{L^p(P)}.
\]

Finally, by change of variables we get
\[
\|V_M I_N(f)\|_{L^p} \lesssim \sum_{\gamma, \gamma' \in P} \| (D_{\lambda + \gamma + \gamma'} T_{\lambda + \gamma} D_\lambda f : \lambda \in F_N) \|_{L^p(P)}.
\]

Assuming there is $\delta_0 > 0$ such that
\[
\| (D_{\lambda + \gamma + \gamma'} T_{\lambda + \gamma} D_\lambda f : \lambda \in P)\|_{L^p(P)} \lesssim q^{-\delta_0(\gamma + |\gamma'|)} \| (f : \lambda \in P)\|_{L^p(P)} \tag{3.10}
\]
we can estimate
\[
\|V_M I_N(f)\|_{L^p} \lesssim \sum_{\gamma, \gamma' \in P} q^{-\delta_0(\gamma + |\gamma'|)} \| (D_\lambda^* f : \lambda \in F_N)\|_{L^p(P)} \tag{3.11}
\]
\[
\lesssim \left\| \left( \sum_{\lambda \in F_N} (D_\lambda^* f) \right)^{1/2} \right\|_{L^p}.
\]

Theorem 2, Proposition 3.2 and (3.11) imply that the $V_M$ are uniformly bounded on $L^p$.

For the proof of (3.10), we consider an operator $T$ defined for $\tilde{f} \in L^p(\pi, \ell^2(P))$ by
\[
T\tilde{f} = (D_{\lambda + \gamma + \gamma'} T_{\lambda + \gamma} f_{\lambda} : \lambda \in P).
\]

Since $\|D_\lambda\|_{L^1 \to L^1} \lesssim 1$ and $\|T_\mu\|_{L^1 \to L^1} \lesssim 1$ we have
\[
\|T\tilde{f}\|_{L^1(\ell^2)} \lesssim \|\tilde{f}\|_{L^1(\ell^2)},
\]
Also, by (3.8), we can estimate
\[ \|T f\|_{L^2(\mathbb{F})}^2 = \sum_{\lambda \in P} \|D_{\lambda+\gamma}T_{\lambda+\gamma}D_\lambda f\|_{L^2}^2 \lesssim q^{-\delta(|\gamma| + |\gamma'|)} \sum_{\lambda \in P} \|f_\lambda\|_{L^2}^2. \]

Therefore, using interpolation between $L^1(\pi, \ell^1(\mathbb{P}))$ and $L^2(\pi, \ell^2(\mathbb{P}))$ we obtain that there is $\delta' > 0$ such that
\[ \|T f\|_{L^{p_0}(\mathbb{P})} \lesssim q^{-\delta'(|\gamma| + |\gamma'|)} \|f\|_{L^{p_0}(\mathbb{P})}. \]

Because $|D_\lambda g| \lesssim L^* R^*(g)$, and because Theorem 1 says that $L^*$ and $R^*$ are bounded on $L^{p_0}$, we know that $(D_\lambda : \lambda \in P)$ is bounded on $L^{p_0}(\pi, \ell^\infty(\mathbb{P}))$. Of course the same holds for $(D_{\lambda+\gamma+\gamma'} : \lambda \in P)$. Hence, by (3.9) we get
\[ \|T f\|_{L^{p_0}(\mathbb{P})} \lesssim \|f\|_{L^{p_0}(\mathbb{P})}. \]

Next, interpolating between $L^{p_0}(\pi, \ell^{p_0}(\mathbb{P}))$ and $L^{p_0}(\pi, \ell^{\infty}(\mathbb{P}))$ gives a $\delta'' > 0$ such that
\[ \|T f\|_{L^{p_0}(\mathbb{P})} \lesssim q^{-\delta''(|\gamma| + |\gamma'|)} \|f\|_{L^{p_0}(\mathbb{P})}. \]

Finally, interpolating between $L^{p_0}(\pi, \ell^{p_0}(\mathbb{P}))$ and $L^{2}(\pi, \ell^{2}(\mathbb{P}))$ we obtain (3.10).

To finish the proof, we are going to show that $(V_N f : N \in \mathbb{N})$ is a Cauchy sequence in $L^p(\Omega_0)$. Let us consider $g \in L^p(\Omega_0) \cap L^2(\Omega_0)$. Setting
\[ a = \frac{2(p - p_0)}{4 - p - p_0}, \quad \text{and} \quad \hat{p} = \frac{p + p_0}{2}, \]
and using the log-convexity of the $L^q$-norms we get
\[ \|V_M g - V_N g\|_{L^p}^p \leq \|V_M g - V_N g\|_{L^2}^a \|V_M g - V_N g\|_{L^p}^{1-a}. \]

Since $(V_N g : N \in \mathbb{N})$ converges in $L^2(\Omega_0)$ and is uniformly bounded on $L^p(\Omega_0)$ it is a Cauchy sequence in $L^p(\Omega_0)$. For an arbitrary $f \in L^p(\Omega_0)$ use the density of $g$’s as above. We have
\[ \|V_M f - V_N f\|_{L^p} \lesssim \|f - g\|_{L^p} + \|V_N g - M g\|_{L^p}. \]

Thus $(V_N f : N \in \mathbb{N})$ also converges, and this finishes the proof of the theorem. \hfill \Box

4. DOUBLE DIFFERENCES

The martingale transforms are expressed in terms of double differences defined for a martingale $f = (f_\lambda : \lambda \in P)$ as
\[ d_\lambda f = f_\lambda - f_{\lambda - \lambda_1} - f_{\lambda - \lambda_2} + f_{\lambda - \lambda_1 - \lambda_2}. \]

4.1. Martingale transforms. The following proposition is our key tool.

Proposition 4.1. Let $f \in L^2(\Omega_0)$ and $\lambda \in P$. If $f_{\lambda - \lambda_1} = 0$ for $j \in \mathbb{N}$ then for each $k \geq j$
\[ \|E[f_\lambda | \mathcal{F}_{\lambda - k(\lambda_1 - \lambda_2)}]\|_{L^2} \leq 2q^{-(k-j+1)/2} \|f_\lambda\|_{L^2}. \]

Analogously, for $\lambda_1$ and $\lambda_2$ exchanged.

Proof. Suppose $j = 1$. We are going to show that if $f_{\lambda - \lambda_1} = 0$ then for all $k \geq 1$
\[ \|E[f_\lambda | \mathcal{F}_{\lambda - k(\lambda_1 - \lambda_2)}]\|_{L^2} \leq q^{-k/2} \|f_\lambda\|_{L^2}. \]

Indeed, if $k = 1$ then by (2.1) of Lemma 2.2
\[ \|E[f_\lambda | \mathcal{F}_{\lambda - \lambda_1 + \lambda_2}]\|_{L^2}^2 = \|E[f_\lambda | \mathcal{F}_{\lambda - \lambda_1 + \lambda_2} | \mathcal{F}_{\lambda_1}]f_\lambda\|_{L^2}^2 = q^{-1} \|f_\lambda\|_{L^2}^2 - q^{-1} \|E[f_\lambda | \mathcal{F}_{\lambda - \lambda_1} \lor \mathcal{F}_{\lambda - \lambda_2}]\|_{L^2}^2. \]

If $k > 1$, we use Lemma 2.3 to write
\[ \|E[f_\lambda | \mathcal{F}_{\lambda - k(\lambda_1 - \lambda_2)}\| \leq \|E[f_\lambda | \mathcal{F}_{\lambda - (\lambda_1 - \lambda_2)} | \mathcal{F}_{\lambda - k(\lambda_1 - \lambda_2)}]\|_{L^2}. \]

Since, by Lemma 2.4,
\[ \|E[f_\lambda | \mathcal{F}_{\lambda - (\lambda_1 - \lambda_2)} | \mathcal{F}_{\lambda - \lambda_1 - \lambda_2}] = E[f_\lambda | \mathcal{F}_{\lambda - \lambda_1} | \mathcal{F}_{\lambda - \lambda_1 - \lambda_2}] = 0 \]
we can use induction to obtain
\[ \|E[f_\lambda | \mathcal{F}_{\lambda - k(\lambda_1 - \lambda_2)} | \mathcal{F}_{\lambda - k(\lambda_1 - \lambda_2)}]\|_{L^2} \leq q^{-(k-1)/2} \|E[f_\lambda | \mathcal{F}_{\lambda - (k-1)(\lambda_1 - \lambda_2)}]\|_{L^2} \leq q^{-k/2} \|f_\lambda\|_{L^2}. \]
Let us consider \( j > 1 \). For each \( i = 0, 1, \ldots, j - 1 \) we set
\[
g_i = f_{\lambda - i\lambda_1} - f_{\lambda - (i+1)\lambda_1}.
\]
By Lemma 2.4 and (4.1)
\[
\|E[g_i | \mathcal{F}_{\lambda-k(\lambda_1-\lambda_2)}]\|_{L^2} = \|E[g_i | \mathcal{F}_{\lambda-(k\lambda_1-k\lambda_2)}]\|_{L^2} \leq q^{-(k-i)/2} \|g_i\|_{L^2} \leq q^{-(k-i)/2} \|f_\lambda\|_{L^2}.
\]
Hence,
\[
\|E[f_\lambda | \mathcal{F}_{\lambda-k(\lambda_1-\lambda_2)}]\|_{L^2} \leq \sum_{i=0}^{j-1} \|E[g_i | \mathcal{F}_{\lambda-k(\lambda_1-\lambda_2)}]\|_{L^2} \leq \sum_{i=0}^{j-1} q^{-(k-i)/2} \|f_\lambda\|_{L^2}
\]
which finishes the proof since
\[
\sum_{i=0}^{j-1} q^{i/2} \leq 2q^{(j-1)/2}.
\]
□

We have the following

**Proposition 4.2.** For any \( \lambda, \lambda', \mu \in P \) and \( m \geq 1 \)
\[
\|D_\lambda \delta^m_{\lambda'} D_\lambda\|_{L^2 \to L^2} \lesssim q^{-|\mu-\lambda|/4} q^{-|\mu-\lambda'|/4},
\]
\[
\|d^m_{\lambda'} \delta^m_{\mu}\|_{L^2 \to L^2} \lesssim q^{-|\lambda-\mu|/2}.
\]

**Proof.** We observe that for \( f \in L^2(\Omega_0) \), \( d_\mu f \in L^2(\pi, \mathcal{F}_\mu) \) and
\[
\mathbb{E}[d_\mu f | \mathcal{F}_\nu] = 0
\]
whenever \( \langle \nu, \alpha_0 \rangle \leq \langle \mu, \alpha_0 \rangle - 2 \). For the proof it is enough to analyze the case \( \nu = \mu - 2\lambda_2 \). By Lemma 2.4, we can write
\[
\mathbb{E}[f_{\mu-\lambda_1} | \mathcal{F}_{\mu-2\lambda_2}] = \mathbb{E}[f_{\mu-\lambda_1} | \mathcal{F}_{\mu-\lambda_1-\lambda_2} | \mathcal{F}_{\mu-2\lambda_2}] = \mathbb{E}[f_{\mu-\lambda_1-\lambda_2} | \mathcal{F}_{\mu-2\lambda_2}].
\]
Suppose \( \lambda = i\lambda_1 + j\lambda_2 \). Let us consider \( R_j d_\mu \). If \( j \geq \langle \mu, \alpha_2 \rangle + 1 \) then \( R_j d_\mu f = 0 \). For \( j \leq \langle \mu, \alpha_2 \rangle - 2 \), in view of (4.2) we can use Proposition 4.1 to estimate
\[
\|R_j d_\mu f\|_{L^2} \lesssim q^{-(\mu-\lambda, \alpha_2)/2} \|d_\mu f\|_{L^2}.
\]
Next, if \( \langle \lambda, \alpha_0 \rangle \geq \langle \mu, \alpha_0 \rangle + 2 \) then \( D_\lambda d_\mu f = 0 \), because \( d_\mu f \) is \( \mathcal{F}_\mu \)-measurable. For \( \langle \lambda, \alpha_0 \rangle \leq \langle \mu, \alpha_0 \rangle - 4 \) and \( \langle \lambda, \alpha_2 \rangle \leq \langle \mu, \alpha_2 \rangle \), by Lemma 2.5 we can write \( D_\lambda d_\mu f = L_i g \) where
\[
g = \mathbb{E}[R_j d_\mu f | \mathcal{F}_\nu]
\]
and \( \nu = \langle (\mu, \alpha_0) - j \rangle \lambda_1 + j \lambda_2 \). By Lemma 2.5, we have
\[
R_j d_\mu f = \mathbb{E}[d_\mu f | \mathcal{F}_\nu] - \mathbb{E}[d_\mu f | \mathcal{F}_{\nu+\lambda_1-\lambda_2}].
\]
We notice that by Lemma 2.4 and (4.2)
\[
\mathbb{E}[d_\mu f | \mathcal{F}_\nu] | \mathcal{F}_{\nu+2\lambda_1}] = \mathbb{E}[d_\mu f | \mathcal{F}_{\mu-2\lambda_2}] | \mathcal{F}_{\nu+2\lambda_1}] = 0.
\]
Similarly, one can show
\[
\mathbb{E}[d_\mu f | \mathcal{F}_{\nu+\lambda_1-\lambda_2}] | \mathcal{F}_{\nu+2\lambda_1}] = 0.
\]
Therefore, \( \mathbb{E}[g | \mathcal{F}_{\nu+2\lambda_1}] = 0 \). Now, by Proposition 4.1, we obtain
\[
\|L_i g\|_{L^2} \lesssim q^{-(\nu-\lambda, \alpha_0)/2} \|R_j d_\mu f\|_{L^2}.
\]
Combining (4.4) with (4.3) we get
\[
\|D_\lambda d_\mu f\|_{L^2} \lesssim q^{-(\mu-\lambda, \alpha_0)/2} q^{-(\mu-\lambda, \alpha_2)/2} \|d_\mu f\|_{L^2}
\]
since \( \langle \nu, \alpha_0 \rangle = \langle \mu, \alpha_0 \rangle \). By analogous reasoning one can show the corresponding norm estimates for \( D_\lambda^2 d_\mu \). Hence, taking adjoint
\[
\|d_\mu D_\lambda f\|_{L^2} \lesssim q^{-(\mu-\lambda', \alpha_0)/2} q^{-(\mu-\lambda', \alpha_2)/2} \|f\|_{L^2}.
\]
Finally, (4.5) and (4.6) allow us to conclude the proof of the first inequality.
For the second, we may assume $0 \leq \langle \mu - \lambda, \alpha_0 \rangle \leq 1$. Suppose $\langle \mu - \lambda, \alpha_0 \rangle = 0$ and $\langle \mu - \lambda, \alpha_2 \rangle \geq 2$. Since $d_\mu f \in L^2(\pi, F_\mu)$, by (4.2) and Proposition 4.1

$$\|E[d_\mu f|\mathcal{F}_\lambda]\|_{L^2} \lesssim q^{-(\mu - \lambda, \alpha_2)/2}\|d_\mu f\|_{L^2}. $$

Similarly, we deal with the case $\langle \mu - \lambda, \alpha_0 \rangle = 1$. We can assume $\langle \mu - \lambda, \alpha_2 \rangle \geq 1$. By Lemma 2.4, we have

$$E[d_\mu f|\mathcal{F}_\lambda] = E[d_\mu f|\mathcal{F}_{\mu - \lambda_2}] = E[\lambda_{\lambda_1 - \lambda_2} - f_{\lambda_1}|\mathcal{F}_\lambda].$$

Hence, by Proposition 4.1

$$\|E[d_\mu f|\mathcal{F}_\lambda]\|_{L^2} \lesssim q^{-(\mu - \lambda, \alpha_2)/2}\|f\|_{L^2}. $$

Let $(a_\lambda : \lambda \in P)$ be an uniformly bounded predictable family of functions, i.e. each function $a_\lambda$ is measurable with respect to $\mathcal{F}_{\lambda_1 - \lambda_2}$ and

$$\sup_{\omega \in \Omega_0} |a_\lambda(\omega)| \leq M.$$ Predictability is the condition needed to ensure that $d_\lambda(a_\lambda f) = a_\lambda d_\lambda f$. By Theorem 3, Theorem 1, Proposition 4.2 and duality when $p > 2$, we get

**Theorem 4.** For each $p \in (1, \infty)$ and $m \in \mathbb{N}$ the series

$$\sum_{\lambda \in P} a_\lambda d^m_\lambda$$

converges unconditionally in the strong operator topology for the operators on $L^p(\Omega_0)$, and defines the operator with norm bounded by a constant multiply of

$$\sup_{\lambda \in P} \sup_{\omega \in \Omega_0} |a_\lambda(\omega)|.$$  

4.2. Martingale square function. For a martingale $f = (f_\lambda : \lambda \in P)$ there is the natural square function defined by

$$Sf = \left( \sum_{\lambda \in P} (d_\lambda f)^2 \right)^{1/2}.$$ 

Although $S$ does not preserve $L^2$ norm we have

**Theorem 5.** For every $p \in (1, \infty)$ there is $C_p > 0$ such that

$$C_p^{-1}\|f\|_{L^p} \leq \|Sf\|_{L^p} \leq C_p\|f\|_{L^p}. $$

**Proof.** We start from proving the identity

$$d^2_\lambda - d^2_\lambda - q^{-1}d^2_\lambda + q^{-1}d_\lambda = 0. $$

Let us notice that

$$d_\lambda E_\lambda = d_\lambda,$$

$$d_\lambda E_{\lambda - \lambda_2} = -E_{\lambda - \lambda_1}E_{\lambda_2} + E_{\lambda - \lambda_1 - \lambda_2},$$

$$d_\lambda E_{\lambda - \lambda_1} = -2E_{\lambda - \lambda_1 - \lambda_2}.$$ 

Therefore, consecutively we have

$$d^2_\lambda = d_\lambda + E_{\lambda - \lambda_1}E_{\lambda - \lambda_2} + E_{\lambda - \lambda_1}E_{\lambda - \lambda_1} - 2E_{\lambda - \lambda_1 - \lambda_2},$$

$$d^3_\lambda = d^2_\lambda - E_{\lambda - \lambda_1}E_{\lambda - \lambda_2}E_{\lambda - \lambda_1} - E_{\lambda - \lambda_1}E_{\lambda - \lambda_1}E_{\lambda - \lambda_2} + 2E_{\lambda - \lambda_1},$$

$$d^4_\lambda = d^3_\lambda + (E_{\lambda - \lambda_1}E_{\lambda - \lambda_2})^2 + (E_{\lambda - \lambda_1}E_{\lambda - \lambda_1})^2 - 2E_{\lambda - \lambda_1}. $$

Hence, by Lemma 2.2,

$$d_1^4 = d_1^4 + q^{-1}E_{\lambda - \lambda_1}E_{\lambda - \lambda_2} + q^{-1}E_{\lambda - \lambda_2}E_{\lambda - \lambda_1} - 2q^{-1}E_{\lambda - \lambda_1}. $$

which together with (4.8) implies (4.7).

Next, we consider an operator $\mathcal{T}$ defined for a function $f \in L^p(\Omega_0)$ by

$$\mathcal{T}f = (d_\lambda f : \lambda \in P).$$

We also need an operator $\mathcal{T'}$ acting on $g \in L^{p'}(\Omega_0)$ as

$$\mathcal{T'}g = (-qd_\lambda g + qd_\lambda g + d_\lambda g : \lambda \in P).$$
We observe that by two parameter Khinchine’s inequality and Theorem 4 we have
\[ \|\mathcal{T} f\|_{L^p(\mathbb{F})} \lesssim \|f\|_{L^p}, \quad \text{and} \quad \|\mathcal{T}^* g\|_{L^{p'}(\mathbb{F})} \lesssim \|g\|_{L^{p'}}. \]

The dual operator \( \mathcal{T}^* : L^{p'}(\pi, \ell^2(\mathbb{Z}^2)) \to L^p(\Omega_0) \) is given by
\[ \mathcal{T}^* g = \sum_{\lambda \in \mathcal{P}} d_\lambda g_\lambda. \]
Since \( \mathcal{T} g \in L^p(\pi, \ell^2(\mathbb{Z}^2)) \), by (4.7) and Theorem 4,
\[ \mathcal{T}^* \mathcal{T} g = \sum_{\lambda \in \mathcal{P}} d_\lambda g = g \]
Therefore, by Cauchy–Schwarz and Hölder inequalities
\[ \langle f, g \rangle = \langle f, \mathcal{T}^* \mathcal{T} g \rangle \leq \|\mathcal{T} f\|_{L^p(\mathbb{F})} \|\mathcal{T}^* g\|_{L^{p'}(\mathbb{F})} \lesssim \|\mathcal{T} f\|_{L^p(\mathbb{F})} \|g\|_{L^{p'}} \]
and since \( \|\mathcal{T} f\|_{L^p(\mathbb{F})} = \|\mathcal{T} f\|_{L^p} \) the proof is finished. \( \square \)

Finally, the method of the proof of Theorem 3, together with Theorem 4 and Theorem 5 shows the following

**Theorem 6.** Let \( (T_\lambda : \lambda \in \mathcal{P}) \) be a family of operators such that for some \( \delta > 0 \) and \( p_0 \in (1, 2) \)
\[ \|T_\lambda\|_{L^1 \to L^1} \lesssim 1, \]
\[ \|T_\mu T_\lambda\|_{L^2 \to L^2} \lesssim q^{-\delta|\mu-\lambda|} \]
and
\[ \|T_\mu T_\lambda\|_{L^2 \to L^2} \lesssim q^{-\delta|\mu-\lambda|}. \]

Then for any \( p \in (p_0, 2) \) the sum \( \sum_{\lambda \in \mathcal{P}} T_\lambda \) converges unconditionally in the strong operator topology for the operators on \( L^p(\Omega_0) \).

**Appendix A. About \( \Omega_0 \) and Heisenberg Group**

In some cases \( \Omega_0 \) can be identified with a Heisenberg group over a nonarchimedean local field. Let us recall, that \( F \) is a nonarchimedean local field if it is a topological field \(^1\) that is locally compact, second countable, non-discrete and totally disconnected. Since \( F \) together with the additive structure is a locally compact topological group it has a Haar measure \( \mu \) that is unique up to multiplicative constant. Observe that for each \( x \in F \), the measure \( \mu_x(B) = \mu(xB) \) is also a Haar measure. We set
\[ |x| = \frac{\mu_x(B)}{\mu(B)}, \]
where \( B \) is any measurable set with finite and positive measure. By \( \mathcal{O} = \{ x \in F : |x| \leq 1 \} \), we denote the ring of integers in \( F \). We fix \( \pi \in \mathfrak{p} - \mathfrak{p}^\circ \), where
\[ \mathfrak{p} = \{ x \in F : |x| < 1 \}. \]

We are going to sketch the construction of a building associated to \( \text{GL}(3, F) \). For more details we refer to [14]. A lattice is a subset \( L \subset F^3 \) of the form
\[ L = \mathcal{O}v_1 + \mathcal{O}v_2 + \mathcal{O}v_3, \]
where \( \{v_1, v_2, v_3\} \) is a basis of \( F^3 \). We say that two lattices \( L_1 \) and \( L_2 \) are equivalent if and only if \( L_1 = a L_2 \) for some nonzero \( a \in F \). Then \( \mathcal{X} \), the building of \( \text{GL}(3, F) \), is the set of equivalence classes of lattices in \( F^3 \). For \( x, y \in \mathcal{X} \) there are a basis \( \{v_1, v_2, v_3\} \) of \( F^3 \) and integers \( j_1 \leq j_2 \leq j_3 \) such that (see [14, Proposition 3.1])
\[ x = \mathcal{O}v_1 + \mathcal{O}v_2 + \mathcal{O}v_3, \quad \text{and} \quad y = \pi^{j_1} \mathcal{O}v_1 + \pi^{j_2} \mathcal{O}v_2 + \pi^{j_3} \mathcal{O}v_3. \]

We say that \( x \) and \( y \) are joined by an edge if and only if \( 0 = j_1 \leq j_2 \leq j_3 = 1 \). The subset
\[ \mathcal{A} = \{ \pi^{j_1} \mathcal{O}v_1 + \pi^{j_2} \mathcal{O}v_2 + \pi^{j_3} \mathcal{O}v_3 : j_1, j_2, j_3 \in \mathbb{Z} \} \]

\(^1\)A topological field is an algebraic field with a topology making addition, multiplication and multiplicative inverse a continuous mappings.
is called an apartment. A sector in $\mathcal{A}$ is a subset of the form
\[ S = \{ x + \pi^1 v_1 + \pi^2 v_2 + \pi^3 v_3 : j_{\sigma(1)} \leq j_{\sigma(2)} \leq j_{\sigma(3)}, j_1, j_2, j_3 \in \mathbb{Z} \}, \]
where $\sigma$ is a permutation of $\{1, 2, 3\}$ and $x \in \mathcal{A}$. Thus, a subsector of $S$ is
\[ \{ x + \pi^1 v_1 + \pi^2 v_2 + \pi^3 v_3 : j_{\sigma(1)} \leq j_{\sigma(2)} \leq j_{\sigma(3)}, j_1, j_2, j_3 \in \mathbb{Z} \}, \]
for some $0 \leq k_{\sigma(1)} \leq k_{\sigma(2)} \leq k_{\sigma(3)}$. Finally, two sectors
\[ S = \{ x + \pi^1 v_1 + \pi^2 v_2 + \pi^3 v_3 : j_{\sigma(1)} \leq j_{\sigma(2)} \leq j_{\sigma(3)}, j_1, j_2, j_3 \in \mathbb{Z} \}, \]
and
\[ S' = \{ x' + \pi^1 v_1 + \pi^2 v_2 + \pi^3 v_3 : j_{\sigma(1)} \leq j_{\sigma(2)} \leq j_{\sigma(3)}, j_1, j_2, j_3 \in \mathbb{Z} \}, \]
are equivalent if and only if their intersection contains a sector. By $\Omega$ we denote the equivalence classes of sectors in $\mathcal{X}$. Let $\omega_0$ and $\omega'_0$ be the equivalence class of
\[ \mathcal{X}_0 = \{ \pi^1 v_1 + \pi^2 v_2 + \pi^3 v_3 : j_1 \leq j_2 \leq j_3, j_1, j_2, j_3 \in \mathbb{Z} \}, \]
and
\[ \mathcal{X}'_0 = \{ \pi^1 v_1 + \pi^2 v_2 + \pi^3 v_3 : j_1 \leq j_2 \leq j_3, j_1, j_2, j_3 \in \mathbb{Z} \}, \]
respectively. Two sectors $\mathcal{X}$ and $\mathcal{X}'$ are opposite in $\mathcal{X}$ if there are subsectors of $\mathcal{X}$ and $\mathcal{X}'$ opposite in a common apartment. By $\Omega_0$ we denote the equivalence classes of sectors opposite to $\mathcal{X}_0$.

Suppose that $\omega' \in \Omega_0$. Let $\{v_1, v_2, v_3\}$ be a basis of $F^3$, and $k_1 \leq k_2 \leq k_3$ and $k'_1 \geq k'_2 \geq k'_3$ be integers such that
\begin{equation}
\{ \pi^{j_1+k_1} v_1 + \pi^{j_2+k_2} v_2 + \pi^{j_3+k_3} v_3 : j_1 \leq j_2 \leq j_3, j_1, j_2, j_3 \in \mathbb{Z} \},
\end{equation}
and
\begin{equation}
\{ \pi^{j_1+k'_1} v_1 + \pi^{j_2+k'_2} v_2 + \pi^{j_3+k'_3} v_3 : j_1 \geq j_2 \geq j_3, j_1, j_2, j_3 \in \mathbb{Z} \},
\end{equation}
belong to $\omega_0$ and $\omega'$, respectively. Since the sector (A.1) belongs to $\omega_0$, we have
\[ v_1 = b_{11} e_1, \quad v_2 = b_{21} e_1 + b_{22} e_2, \quad v_3 = b_{31} e_1 + b_{32} e_2 + b_{33} e_3, \]
for some $b_{ij} \in F$ such that $b_{11}, b_{22}, b_{33} \neq 0$. Hence, the matrix
\[ g = \begin{pmatrix} b_{11} & b_{21} & b_{31} \\ b_{21} & b_{22} & b_{32} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \]
fulfills $g e_j = v_j$. In particular, $g \omega'_0 = \omega'$. Therefore, the group of upper triangular matrices acts transitively on $\Omega_0$. Observe also that the stabilizer of $\omega'_0$ in $\text{GL}(3, F)$ is the group of lower triangular matrices. Thus the group
\[ \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in F \right\} \]
acts simply transitively on $\Omega_0$.

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