SOME GENERALIZED NONLINEAR GAMIDOV TYPE
INTEGRAL INEQUALITIES WITH MAXIMA IN TWO
VARIABLES AND THEIR WEAKLY SINGULAR ANALOGUES

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Abstract. In this paper, some new nonlinear Gronwall-Bellman-Gamidov type integral inequalities with maxima in two variables and their weakly singular analogues are discussed. By using analysis techniques, such as change of variable, amplification method, differential and integration, inverse function, we estimated the upper bounds of the unknown functions. For illustrating the validity of the inequalities established, some examples are given to study the boundedness and uniqueness of solutions of a certain Gamidov type weakly singular integral equations.

1. Introduction

With the development of the theory of differential equations, The Gronwall-Bellman inequality [1,2] and Bihari inequality [3] are widely used in the qualitative and quantitative analysis of differential equations, as it can provide explicit bound for an unknown function lying in the inequality. The study of inequality has always been a hot topic, see the literature [4-39]. During the past few years, many researchers have established various generalizations of the Gronwall-Bellman inequality. For example, in [7-8], the authors discussed some Gronwall-Like type inequalities; in [9-14], some inequalities with weakly singular kernel were studied; in [4-6, 15-17], the authors researched some Gamidov type inequalities; in [18-20], generalized Volterra-Fredholm type inequalities were investigated.

In 1992, Banov and Simeonov [5] established the following useful integral inequality:

\[ u(t) \leq c + \int_{\alpha}^{t} f(s)u(s)ds + \int_{\alpha}^{\beta} g(s)u(s)ds, \quad t \in [\alpha, \beta]. \] (1.1)

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In 2007, Wu-Sheng Wang [8] discussed a generalized retard Gronwall-Like inequality in two variables:

\[ u^p(x,y) \leq a(x,y) + \sum_{i=1}^{n} \int_{b_i(x_0)}^{c_i(y)} f_i(x,y,s,t) \varphi_i(u(s,t)) \, dt \, ds. \tag{1.2} \]

In 2014, Kelong Cheng et al. [15] researched a generalized nonlinear Gronwall-Bellman-Gamidov type integral inequality:

\[ u^m(t) \leq a(t) + b(t) \int_0^t f(s) u^n(s) \, ds + c(t) \int_0^T g(s) u^r(s) \, ds, \quad t \in [0,T], \tag{1.3} \]

and its weakly singular analogue:

\[ u^m(t) \leq a(t) + b(t) \int_0^t \int_0^t (t^{\alpha_1} - s^{\alpha_1})^{\beta_1-1} s^{\gamma_1-1} f(s) u^n(s) \, ds \]

\[ + c(t) \int_0^T \int_0^T (T^{\alpha_2} - s^{\alpha_2})^{\beta_2-1} s^{\gamma_2-1} g(s) u^r(s) \, ds, \quad t \in [0,T], \tag{1.4} \]

where \( m \geq n \geq r \geq 0 \).

Along with the development of automatic control theory and its applications to computational mathematics and modeling, many Gronwall-Bellman integral inequalities with the maxima of the unknown function are established, see [21-24].

In 2013, Yong Yan [23] studied a generalized nonlinear Gronwall-Bellman inequalities with maxima in two variables:

\[ u(x,y) \leq a(x,y) + \sum_{i=1}^{m} \int_{a_i(x_0)}^{b_i(y_0)} \beta_i(s,t) h_i(u(s,t)) \, ds \, dt \]

\[ + \sum_{j=m+1}^{m+n} \int_{a_j(x_0)}^{b_j(y_0)} \beta_j(s,t) h_j \left( \max_{\xi \in [s-h,\infty]} u(\xi, t) \right) \, ds \, dt, \quad (x,y) \in \Delta, \tag{1.5} \]

\[ u(x,y) \leq \psi(x,y), \quad (x,y) \in \Psi. \tag{1.6} \]

In 2015, Yong Yan [24] investigated a generalized nonlinear weakly singular Volterra integral inequalities with maxima:

\[ \varphi(u(t)) \leq a(t) + \sum_{i=1}^{m} \int_{b_i(t_0)}^{c_i(y_0)} (t^{\alpha_i} - s^{\alpha_i})^{k_i(\beta_i-1)} s^{\gamma_i(\gamma_i-1)} g_i(t,s) \omega_i(u(s)) \, ds \]

\[ + \sum_{j=m+1}^{m+n} \int_{b_j(t_0)}^{c_j(y_0)} (t^{\alpha_j} - s^{\alpha_j})^{k_j(\beta_j-1)} s^{\gamma_j(\gamma_j-1)} g_j(t,s) \omega_j \]

\[ \left( \max_{\xi \in [c_j(s) - h, c_j(s)]} f(u(\xi)) \right) \, ds, \quad t \in [t_0,t_1), \tag{1.7} \]

\[ u(t) \leq \psi(t), \quad t \in [b^*(t_0) - h,t_0], \tag{1.8} \]
where \( b^*(t_0) = \min \left\{ \min_{1 \leq i \leq m} b_i(t_0), \min_{m+1 \leq j \leq m+n} c_j(b_j(t_0)) \right\} \).

However, we notice that there are few literature to investigate the Gamidov type integral inequalities with the unknown function that composed with the given function in the integrals. In this paper, based on the work of Wang [8], Cheng [15] and Yan [24], we deal with some classes of nonlinear Gamidov type integral inequalities with maxima in two variables and their weakly singular analogues. The other importance of the paper is the applications that show the boundedness and uniqueness of solutions for weakly singular Gamidov type integral equations with maxima.

2. Preliminary knowledge

In what follows, \( R \) denotes the set of real numbers, \( R_+ = [0, +\infty), R_0 = (0, +\infty) \). \( C^1(U, V) \) denotes the class of all continuously differentiable functions defined on set \( U \) with range in the set \( V \), \( C(U, V) \) denotes the class of all continuous functions defined on set \( U \) with range in the set \( V \).

Consider the sets \( \Delta, \Psi, \Lambda \) defined by:
\[
\Delta = \left\{ (x, y) \in R^2 : x \in [x_0, M], y \in [y_0, N] \right\};
\]
\[
\Psi = \left\{ (x, y) \in R^2 : x \in [\beta b_*(x_0), x_0], y \in [y_0, N] \right\};
\]
\[
\Lambda = \left\{ (x, y) \in R^2 : x \in [\beta b_*(x_0), M], y \in [y_0, N] \right\} = \Delta \cup \Psi;
\]
where \( b_*(x_0) = \min_{1 \leq i \leq m} b_i(x_0) \), \( 0 < \beta < 1 \).

For convenience, we cite some useful lemmas in the discussion of our proof as follows.

**Lemma 2.1.** (See [25]). Assume that \( a \geq 0, m \geq n \geq 0, \) and \( m \neq 0 \). then for any \( K > 0 \),
\[
a^{\frac{m}{m}} \leq \frac{n}{m} K^{\frac{m-n}{m}} a + \frac{m-n}{m} K^{\frac{m}{m}}.
\]

**Lemma 2.2.** (See [9]). Let \( \alpha, \beta, \gamma \) and \( p \) be positive constants. Then
\[
\int_0^t (t^\alpha - s^\alpha)^p(\beta - 1)s^p(\gamma - 1) ds = \frac{t^\theta}{\alpha} B\left[p(\gamma-1)+1, p(\beta-1)+1\right], \quad t \in R_+,
\]
where \( B[\xi, \eta] = \int_0^1 s^{\xi-1}(1-s)^{\eta-1} ds \) (Re \( \xi > 0; \) Re \( \eta > 0 \)) is the well-known \( B \)-function and \( \theta = p(\alpha(\beta - 1) + \gamma - 1) + 1 \geq 0 \).

**Lemma 2.3.** (Discrete Jensen Inequality). Let \( A_1, A_2, \ldots, A_n \) be nonnegative real numbers and \( r > 1 \) is a real number. Then
\[
(A_1 + A_2 + \cdots + A_n)^r \leq n^{r-1}(A_1^r + A_2^r + \cdots + A_n^r).
\]
3. Main results

THEOREM 3.1. Let the following conditions be fulfilled:

(i) The functions $b_i \in C^1([x_0, M], R_+)$ and $c_i \in C^1([y_0, N], [y_0, N]) \ (i = 1, 2, 3, 4)$ are nondecreasing with $b_i(x) \leq x$ on $[x_0, M]$, $c_i(y) \leq y$ on $[y_0, N]$ and $c_i(y_0) = y_0$;

(ii) The functions $a(x, y) \in C(\Delta, [1, \infty))$, $f_i(x, y) \in C(\Delta, R_+) \ (i = 1, 2, 3, 4)$ are nondecreasing in each of the variables, $h_i(x, y) \in C(\Delta, R_+)$ ($i = 1, 2, 3, 4$);

(iii) The function $\phi(x, y) \in C(\Psi, R_+)$ satisfies $\max_{t \in [\tau, \xi]} \phi(t, \tau) \leq a(x_0, y_0)$;

(iv) The functions $\omega_i \in C(R_+, C_+)$ are nondecreasing with $\omega_i(u) > 0$ for $u > 0$ ($i = 1, 2$), such that $\omega_1 \propto \omega_2$, and $\omega_i$ are submultiplicative, that is $\omega_i(tx) \geq t\omega_i(x)$ for $0 \leq t \leq 1$;

(v) The function $u \in C(\Lambda, R_+)$ satisfies:

$$u(x, y) \leq na(x, y) + f_1(x, y) \int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} h_1(s, t)\omega_1(u(s, t))dsdt$$

$$+ f_2(x, y) \int_{b_2(x_0)}^{b_2(x)} \int_{c_2(y_0)}^{c_2(y)} h_2(s, t)\omega_2\left(\max_{\xi \in [s, t]} u(\xi, t)\right)dsdt$$

$$+ f_3(x, y) \int_{b_3(x_0)}^{b_3(M)} \int_{c_3(y_0)}^{c_3(N)} h_3(s, t)u(s, t)dsdt$$

$$+ f_4(x, y) \int_{b_4(x_0)}^{b_4(M)} \int_{c_4(y_0)}^{c_4(N)} h_4(s, t)\max_{\xi \in [s, t]} u(\xi, t)dsdt, \quad (x, y) \in \Delta, \quad (3.1)$$

Then, we have the following explicit estimation

$$u(x, y) \leq a(x, y)W_1^{-1}\left\{W_2^{-1}\left\{W_1\left(1 + G_1^{-1}(B_1(M, N))\right) + A_1(x, y)\right\} + B_1(x, y)\right\}, \quad (3.3)$$

for all $(x, y) \in \Delta$, where $W_i^{-1}$ is the inverse function of $W_i$:

$$W_1(z) = \int_c^z \frac{ds}{\omega_1(s)}, \quad W_2(z) = \int_c^z \frac{\omega_1(W_1^{-1}(s))}{\omega_2(W_1^{-1}(s))}ds, \quad (3.4)$$

$c > 0$ is a given constant.

$$A_1(x, y) = f_1(x, y) \int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} h_1(s, t)dsdt, \quad (3.5)$$

$$B_1(x, y) = f_2(x, y) \int_{b_2(x_0)}^{b_2(x)} \int_{c_2(y_0)}^{c_2(y)} h_2(s, t)dsdt, \quad (3.6)$$

$$D_1(M, N) = f_3(M, N) \int_{b_3(x_0)}^{b_3(M)} \int_{c_3(y_0)}^{c_3(N)} h_3(s, t)\frac{\bar{a}(s, t)}{a(x_0, y_0)}dsdt$$

$$+ f_4(M, N) \int_{b_4(x_0)}^{b_4(M)} \int_{c_4(N)}^{c_4(N)} h_4(s, t)\frac{\bar{a}(s, t)}{a(x_0, y_0)}dsdt > 0, \quad (3.7)$$
where \( G_1(u) \) is increasing in \( R^+ \).

**Proof.** Define the nondecreasing function \( \tilde{a}(x,y) \in C(\Lambda, [1, +\infty)) \) by

\[
\tilde{a}(x,y) = \begin{cases} 
  a(x,y), & (x,y) \in \Delta, \\
  a(x_0,y_0), & (x,y) \in \Psi.
\end{cases}
\]

From inequalities (3.1), (3.2), conditions (iii), (iv) and \( \frac{1}{a(x,y)} \leq 1 \), we obtain

\[
\frac{u(x,y)}{a(x,y)} \leq 1 + f_1(x,y) \int_{b_1(x_0)}^{c_1(y)} \int_{c_1(y_0)}^{b_1(x_0)} h_1(s,t) \omega_1 \left( \frac{u(s,t)}{a(s,t)} \right) dsdt \\
+ f_2(x,y) \int_{b_2(x_0)}^{c_2(y)} \int_{c_2(y_0)}^{b_2(x_0)} h_2(s,t) \omega_2 \left( \frac{\max_{\xi \in [b_2(s),s]} u(\xi,t)}{a(s,t)} \right) dsdt \\
+ f_3(M,N) \int_{b_3(x_0)}^{c_3(N)} \int_{c_3(y_0)}^{b_3(x_0)} h_3(s,t) \frac{a(s,t)}{a(x_0,y_0)} \cdot \frac{u(s,t)}{a(s,t)} dsdt \\
+ f_4(M,N) \int_{b_4(x_0)}^{c_4(y_0)} \int_{c_4(y_0)}^{b_4(x_0)} h_4(s,t) \frac{\tilde{a}(s,t)}{a(x_0,y_0)} \cdot \max_{\xi \in [b_4(s),s]} u(\xi,t) \frac{u(\xi,t)}{\tilde{a}(s,t)} dsdt,
\]

\((x,y) \in \Delta, \quad (x,y) \in \Psi. \tag{3.10}\)

For \( s \in [b_+(x_0), b_+(M)], \ t \in [y_0, N], \) we have

\[
\frac{\max_{\xi \in [b_2(s),s]} u(\xi,t)}{\tilde{a}(s,t)} = \frac{u(\xi_1,t)}{a(s,t)} \leq \frac{u(\xi_1,t)}{\tilde{a}(\xi_1,t)} \leq \frac{\max_{\xi \in [b_2(s),s]} u(\xi,t)}{\tilde{a}(\xi,t)}. \tag{3.12}\]

Let

\[
z(x,y) = \frac{u(x,y)}{a(x,y)}. \tag{3.13}\]
From (3.12), it follows that the inequalities (3.10), (3.11) may be written in the form

\[ z(x,y) \leq 1 + f_1(x,y) \int_{b_1(x)}^{b_1(x_0)} \int_{c_1(y_0)}^{c_1(y)} h_1(s,t) \omega_1(z(s,t)) \, ds \, dt \\
+ f_2(x,y) \int_{b_2(x_0)}^{b_2(x)} \int_{c_2(y_0)}^{c_2(y)} h_2(s,t) \omega_2 \left( \max_{\xi \in [\beta x, s]} z(\xi, t) \right) \, ds \, dt + C(M,N), \]

\[(x,y) \in \Delta, \quad (x,y) \in \Psi, \tag{3.14} \]

where

\[ C(M,N) = f_3(M,N) \int_{b_3(x_0)}^{b_3(M)} \int_{c_3(y_0)}^{c_3(N)} h_3(s,t) \frac{\tilde{a}(s,t)}{a(x_0,y_0)} z(s,t) \, ds \, dt \\
+ f_4(M,N) \int_{b_4(x_0)}^{b_4(M)} \int_{c_4(y_0)}^{c_4(N)} h_4(s,t) \frac{\tilde{a}(s,t)}{a(x_0,y_0)} \max_{\xi \in [\beta s, x]} z(\xi, t) \, ds \, dt. \tag{3.16} \]

\[ \forall X \in [x_0,M], \quad Y \in [y_0,N], \text{ for all } (x,y) \in [x_0,X] \times [y_0,Y] \triangleq \Delta_1, \text{ we have} \]

\[ z(x,y) \leq 1 + f_1(X,Y) \int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} h_1(s,t) \omega_1(z(s,t)) \, ds \, dt \\
+ f_2(X,Y) \int_{b_2(x_0)}^{b_2(x)} \int_{c_2(y_0)}^{c_2(y)} h_2(s,t) \omega_2 \left( \max_{\xi \in [\beta s, x]} z(\xi, t) \right) \, ds \, dt + C(M,N), \tag{3.17} \]

for \((x,y) \in [\beta b_+(x_0), x_0] \times [y_0,Y] \triangleq \Psi_1\), we have

\[ z(x,y) \leq 1. \tag{3.18} \]

Define the function \( z_1(x,y) \in C(\Lambda_1, R_+) \) (\( \Lambda_1 = \Delta_1 \cup \Psi_1 \)) by:

\[ z_1(x,y) = \begin{cases} 
1 + f_1(X,Y) \int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} h_1(s,t) \omega_1(z(s,t)) \, ds \, dt \\
+ f_2(X,Y) \int_{b_2(x_0)}^{b_2(x)} \int_{c_2(y_0)}^{c_2(y)} h_2(s,t) \omega_2 \left( \max_{\xi \in [\beta s, x]} z(\xi, t) \right) \, ds \, dt \\
+ C(M,N), \quad (x,y) \in \Delta_1, \\
1 + C(M,N), \quad (x,y) \in \Psi_1. 
\end{cases} \]

Which is positive and nondecreasing in each of the variables, and

\[ z_1(x_0,y) = 1 + C(M,N). \tag{3.19} \]

From (3.17), (3.18) and the definition of the \( z_1(x,y) \), we have

\[ z(x,y) \leq z_1(x,y), \quad (x,y) \in \Lambda_1, \tag{3.20} \]
\[
\max_{\xi \in [\beta, x]} z(\xi, y) \leq \max_{\xi \in [\beta, x]} z_1(\xi, y) \\
\leq z_1(x, y), \quad (x, y) \in [b_*(x_0), b_*(X)] \times [y_0, Y]. \tag{3.21}
\]

Differentiating \( z_1(x, y) \) on \( \Delta_1 \) with respect to \( x \), and from (3.20), (3.21), we have
\[
\frac{\partial z_1(x, y)}{\partial x} = f_1(X, Y) b'_1(x) \int_{c_1(y_0)}^{c_1(y)} h_1(b_1(x), t) \omega_1(z(b_1(x), t)) dt \\
\quad + f_2(X, Y) b'_2(x) \int_{c_2(y_0)}^{c_2(y)} h_2(b_2(x), t) \omega_2 \left( \max_{\xi \in [\beta, b_2(x), b_2(x)]} z(\xi, t) \right) dt \\
\leq f_1(X, Y) b'_1(x) \int_{c_1(y_0)}^{c_1(y)} h_1(b_1(x), t) \omega_1(z_1(b_1(x), t)) dt \\
\quad + f_2(X, Y) b'_2(x) \int_{c_2(y_0)}^{c_2(y)} h_2(b_2(x), t) \omega_2(z_1(b_2(x), t)) dt, \tag{3.22}
\]

by the monotonicity of \( \omega_i \), \( z_1 \) and the property of \( b_i, c_i \) (\( i = 1, 2 \)), we get
\[
\frac{(\partial / \partial x) z_1(x, y)}{\omega_1(z_1(x, y))} \leq f_1(X, Y) b'_1(x) \int_{c_1(y_0)}^{c_1(y)} h_1(b_1(x), t) dt \\
\quad + f_2(X, Y) b'_2(x) \int_{c_2(y_0)}^{c_2(y)} h_2(b_2(x), t) \frac{\omega_2(z_1(b_2(x), t))}{\omega_1(z_1(b_2(x), t))} dt. \tag{3.23}
\]

Replace \( x \) to \( s \), and integrating it from \( x_0 \) to \( x \), we obtain
\[
W_1(z_1(x, y)) \leq W_1(z_1(x_0, y)) + f_1(X, Y) \int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} h_1(s, t) ds dt \\
\quad + f_2(X, Y) \int_{b_2(x_0)}^{b_2(x)} \int_{c_2(y_0)}^{c_2(y)} h_2(s, t) \frac{\omega_2(z_1(s, t))}{\omega_1(z_1(s, t))} ds dt \\
\leq W_1(z_1(x_0, y)) + A_1(X, Y) \\
\quad + f_2(X, Y) \int_{b_2(x_0)}^{b_2(x)} \int_{c_2(y_0)}^{c_2(y)} h_2(s, t) \frac{\omega_2(z_1(s, t))}{\omega_1(z_1(s, t))} ds dt, \tag{3.24}
\]

where \( A_1(X, Y) \) is defined in (3.5). Let \( z_2(x, y) \) denote the function of the right-hand side of (3.24), which is positive and nondecreasing in each of the variables, and
\[
z_2(x_0, y) = W_1(z_1(x_0, y)) + A_1(X, Y), \tag{3.25}
\]
\[
z_1(x, y) \leq W_1^{-1}(z_2(x, y)). \tag{3.26}
\]

Differentiating \( z_2(x, y) \) on \( \Delta_1 \) with respect to \( x \), and from (3.26), we have
\[
\frac{\partial z_2(x, y)}{\partial x} = f_2(X, Y) b'_2(x) \int_{c_2(y_0)}^{c_2(y)} h_2(b_2(x), t) \frac{\omega_2(z_1(b_2(x), t))}{\omega_1(z_1(b_2(x), t))} dt \\
\leq f_2(X, Y) b'_2(x) \int_{c_2(y_0)}^{c_2(y)} h_2(b_2(x), t) \frac{\omega_2(W_1^{-1}(z_2(b_2(x), t)))}{\omega_1(W_1^{-1}(z_2(b_2(x), t)))} dt. \tag{3.27}
\]
From the condition $\omega_1 \propto \omega_2$, we obtain that $\frac{\omega_2}{\omega_1}$ is nondecreasing, by the monotonicity of $z_2$ and the property of $b_2$, $c_2$, we get

$$\frac{\omega_1(W_1^{-1}(z_2(x,y)))(\partial / \partial x)z_2(x,y)}{\omega_2(W_1^{-1}(z_2(x,y)))} \leq f_2(X,Y)b'_2(x)\int_{c_2(y_0)}^{c_2(y)} h_2(b_2(x),t)dt. \quad (3.28)$$

Replace $x$ to $s$, and integrating it from $x_0$ to $x$, we obtain

$$W_2(z_2(x,y)) \leq W_2(z_2(x_0,y)) + B_1(x,y,X,Y), \quad (3.29)$$

where

$$B_1(x,y,X,Y) = f_2(X,Y)\int_{b_2(x_0)}^{b_2(x)} \int_{c_2(y_0)}^{c_2(y)} h_2(s,t)dsdt, \quad (3.30)$$

obviously, $B_1(x,y,x,y) = B_1(x,y)$, which is defined in (3.6). From (3.19), (3.20), (3.25), (3.26) and (3.29), we get

$$z(x,y) \leq z_1(x,y) \leq W_1^{-1}(z_2(x,y)) \leq W_1^{-1}\left\{ W_2^{-1}\left[ W_2\left( W_1(1 + C(M,N)) + A_1(X,Y) \right) + B_1(x,y,X,Y) \right] \right\}. \quad (3.31)$$

Since $X$, $Y$ are chosen arbitrarily, we have

$$z(x,y) \leq z_1(x,y) \leq W_1^{-1}\left\{ W_2^{-1}\left[ W_2\left( W_1(1 + C(M,N)) + A_1(x,y) \right) + B_1(x,y) \right] \right\}, \quad (x,y) \in \Delta. \quad (3.32)$$

By (3.20), (3.21) and the definition of $C(M,N)$, we have

$$C(M,N) \leq z_1(M,N)D_1(M,N) \leq W_1^{-1}\left\{ W_2^{-1}\left[ W_2\left( W_1(1 + C(M,N)) + A_1(M,N) \right) + B_1(M,N) \right] \right\}D_1(M,N),$$

i.e.

$$W_2\left[ W_1\left( \frac{C(M,N)}{D_1(M,N)} \right) \right] - W_2\left[ W_1\left( 1 + C(M,N) \right) + A_1(M,N) \right] \leq B_1(M,N), \quad (3.33)$$

where $D_1(M,N)$ is defined in (3.7). By (3.8), we have

$$C(M,N) \leq G_1^{-1}(B_1(M,N)). \quad (3.34)$$

Combining (3.13), (3.32) and (3.34), we get the desired result (3.3). \qed

If $\omega_1(u) = \omega_2(u) = u$ in Theorem 3.1, we get an interesting result as follows.
COROLLARY 3.2. Assume that the conditions (i)-(iii) of Theorem 3.1 are satisfied, \( u(x,y) \in C(\Lambda, R_+) \) satisfies

\[
\begin{align*}
    u(x,y) &\leq a(x,y) + f_1(x,y) \int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} h_1(s,t) u(s,t) ds dt \\
    &+ f_2(x,y) \int_{b_2(x_0)}^{b_2(x)} \int_{c_2(y_0)}^{c_2(y)} h_2(s,t) \max_{\xi \in [\beta_2,s]} u(\xi,t) ds dt \\
    &+ f_3(x,y) \int_{b_3(x_0)}^{b_3(M)} \int_{c_3(y_0)}^{c_3(N)} h_3(s,t) u(s,t) ds dt \\
    &+ f_4(x,y) \int_{b_4(x_0)}^{b_4(M)} \int_{c_4(y_0)}^{c_4(N)} h_4(s,t) \max_{\xi \in [\beta_3,s]} u(\xi,t) ds dt, \quad (x,y) \in \Delta, \\
\end{align*}
\]

\( u(x,y) \leq \phi(x,y), \quad (x,y) \in \Psi. \) (3.36)

Then

\[
    u(x,y) \leq \frac{a(x,y)}{1 - D_2(M,N)} \exp(A_1(x,y) + B_1(x,y)), \quad (x,y) \in \Delta, \quad (3.37)
\]

where

\[
D_2(x,y) = f_3(M,N) \int_{b_3(x_0)}^{b_3(M)} \int_{c_3(y_0)}^{c_3(N)} \frac{\tilde{a}(s,t)}{a(x_0,y_0)} h_3(s,t) \exp(A_1(s,t) + B_1(s,t)) ds dt \\
+ f_4(M,N) \int_{b_4(x_0)}^{b_4(M)} \int_{c_4(y_0)}^{c_4(N)} \frac{\tilde{a}(s,t)}{a(x_0,y_0)} h_4(s,t) \exp(A_1(s,t) + B_1(s,t)) ds dt < 1. \quad (3.38)
\]

Proof. Applying Theorem 3.1 to (3.35), (3.36), then

\[
W_1(z) = W_2(z) = \int_{c}^{z} \frac{ds}{s} = \ln z - \ln c, \quad G_1(u) = \frac{u}{D_2(M,N)} - u,
\]

obviously, \( G_1(u) \) is a strictly increasing function, we get the desired result. \( \square \)

THEOREM 3.3. Let the following conditions be fulfilled:
(i) The conditions (i), (ii) of Theorem 3.1 are satisfied;
(ii) The functions \( \omega_i \in C(R_+, R_+) \) are nondecreasing with \( \omega_i(u) > 0 \) for \( u > 0 \) \( (i = 1, 2) \), such that \( \omega_1 < \omega_2 \), and \( \omega_i \) are subadditive and submultiplicative, that is
\( \omega_i(x+y) \leq \omega_i(x) + \omega_i(y), \quad \omega_i(tx) \geq t \omega_i(x) \) for \( 0 \leq t \leq 1; \)
(iii) The function \( \phi \in C(\Psi, R_+) \) satisfies \( \max_{1,(i) \in \Psi} \phi(1, \tau) \leq a^\tau(x_0,y_0); \)
(iv) The function \( u \in C(\Lambda, R_+) \) satisfies:

\[
\begin{align*}
    u^1(x,y) &\leq a(x,y) + f_1(x,y) \int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} h_1(s,t) \omega_1(u(s,t)) ds dt \\
    &+ f_2(x,y) \int_{b_2(x_0)}^{b_2(x)} \int_{c_2(y_0)}^{c_2(y)} h_2(s,t) \omega_2(\max_{\xi \in [\beta_2,s]} u(\xi,t)) ds dt
\end{align*}
\]
Then, we have the following explicit estimation

$$
\begin{align*}
+ f_3(x,y) & \int_{b_3(x_0)}^{b_3(M)} \int_{c_3(y_0)}^{c_3(N)} h_3(s,t) u''(s,t) dsdt \\
+ f_4(x,y) & \int_{b_4(x_0)}^{b_4(M)} \int_{c_4(y_0)}^{c_4(N)} h_4(s,t) \max_{\xi \in [x,s]} u'(\xi, t) dsdt,
\end{align*}
$$

where \( l, m, r \) are constants and satisfy \( l \geq m \geq 0, l \geq r \geq 0, l \geq 1 \). Then, we have the following explicit estimation

$$
\begin{align*}
u(x,y) \leq & \left\{ a(x,y) + E_1(x,y) W_1^{-1} \left\{ W_2^{-1} \left( W_1 \left( 1 + G_2^{-1}(B_1(M,N)) \right) \right) \right. \\
+ & A_1(x,y) \right\} \right\}^{-\frac{1}{2}}, \quad (x,y) \in \Delta,
\end{align*}
$$

where

$$
E_1(x,y) = 1 + f_1(x,y) \int_{b_1(x_0)}^{b_1(M)} \int_{c_1(y_0)}^{c_1(N)} h_1(s,t) \omega_1 \left( \frac{1}{l} K^{\frac{1-l}{l}} \tilde{a}(s,t) + \frac{l-1}{l} K^{\frac{l}{l}} \right) dsdt \\
+ f_2(x,y) \int_{b_2(x_0)}^{b_2(M)} \int_{c_2(y_0)}^{c_2(N)} h_2(s,t) \omega_2 \left( \frac{1}{l} K^{\frac{1-l}{l}} \tilde{a}(s,t) + \frac{l-1}{l} K^{\frac{l}{l}} \right) dsdt \\
+ f_3(x,y) \int_{b_3(x_0)}^{b_3(M)} \int_{c_3(y_0)}^{c_3(N)} h_3(s,t) \left( \frac{m}{l} K^{\frac{m-l}{l}} \tilde{a}(s,t) + \frac{l-m}{l} K^{\frac{m}{l}} \right) dsdt \\
+ f_4(x,y) \int_{b_4(x_0)}^{b_4(M)} \int_{c_4(y_0)}^{c_4(N)} h_4(s,t) \left( \frac{r}{l} K^{\frac{r-l}{l}} \tilde{a}(s,t) + \frac{l-r}{l} K^{\frac{r}{l}} \right) dsdt,
\end{align*}
$$

for any constant \( K \geq 1 \).

$$
D_3(M,N) = f_3(M,N) \int_{b_3(x_0)}^{b_3(M)} \int_{c_3(y_0)}^{c_3(N)} m \tilde{E}_1(s,t) dsdt \\
+ f_4(M,N) \int_{b_4(x_0)}^{b_4(M)} \int_{c_4(y_0)}^{c_4(N)} r \tilde{E}_1(s,t) dsdt,
$$

$$
\tilde{E}_1(x,y) = \begin{cases} 
E_1(x,y), & (x,y) \in \Delta, \\
E_1(x_0,y_0), & (x,y) \in \Psi.
\end{cases}
$$

$$
G_2(u) = W_2 \left( W_1 \left( \frac{u}{D_3(M,N)} \right) \right) - W_2 \left( W_1(1+u) + A_1(M,N) \right)
$$

where \( G_2(u) \) is increasing in \( R^+ \). \( W_1, W_2 \) are defined as in (3.4).
Proof. Define the functions \( z(x,y) \in C(\Lambda, R_+) \) by

\[
z(x,y) = \begin{cases} 
  f_1(x,y) \int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} h_1(s,t) \omega_1(u(s,t)) ds dt \\
  + f_2(x,y) \int_{b_2(x_0)}^{b_2(x)} \int_{c_2(y_0)}^{c_2(y)} h_2(s,t) \omega_2(\max_{\xi \in [\beta_s,x]} u(\xi,t)) ds dt \\
  + f_3(x,y) \int_{b_3(M)}^{b_3(x_0)} \int_{c_3(y_0)}^{c_3(N)} h_3(s,t) u^m(s,t) ds dt \\
  + f_4(x,y) \int_{b_4(x_0)}^{b_4(M)} \int_{c_4(y_0)}^{c_4(N)} h_4(s,t) \max_{\xi \in [\beta_s,x]} u'(\xi,t) ds dt,
\end{cases}
\]

\( \Delta \), \( (x,y) \in \Delta \),

\[
f_3(x_0,y_0) \int_{b_3(x_0)}^{b_3(M)} \int_{c_3(y_0)}^{c_3(N)} h_3(s,t) u^m(s,t) ds dt \\
  + f_4(x_0,y_0) \int_{b_4(x_0)}^{b_4(M)} \int_{c_4(y_0)}^{c_4(N)} h_4(s,t) \max_{\xi \in [\beta_s,x]} u'(\xi,t) ds dt,
\]

\( \Psi \).

Obviously, \( z(x,y) \) is nonnegative and nondecreasing in \( x \) and \( y \). From inequalities (3.39), (3.40), and by the Lemma 2.1, we have

\[
u(x,y) \leq \left( a(x,y) + z(x,y) \right) \frac{1}{t} \leq \left( \tilde{a}(x,y) + z(x,y) \right) \frac{1}{t} \leq \frac{1}{l} K_\frac{1}{l-t} \left( \tilde{a}(x,y) + z(x,y) \right) + \frac{l}{l} K_\frac{1}{t} \frac{1}{l}, \quad (x,y) \in \Delta,
\]

(3.46)

\[
u(x,y) \leq \phi(x,y) \leq a_\frac{1}{t} (x_0,y_0) \leq \left( \tilde{a}(x,y) + z(x,y) \right) \frac{1}{t} \leq \frac{1}{l} K_\frac{1}{l-t} \left( \tilde{a}(x,y) + z(x,y) \right) + \frac{l}{l} K_\frac{1}{t} \frac{1}{l}, \quad (x,y) \in \Psi,
\]

(3.47)

where \( \tilde{a}(x,y) \) is defined in (3.9). Moreover, from above inequalities, for \( (x,y) \in [b_*(x_0), b_*(M)] \times [y_0,N] \), we have

\[
\max_{\xi \in [\beta_s,x]} u(\xi,y) \leq \left( \tilde{a}(x,y) + z(x,y) \right) \frac{1}{t} \leq \frac{1}{l} K_\frac{1}{l-t} \left( \tilde{a}(x,y) + z(x,y) \right) + \frac{l}{l} K_\frac{1}{t},
\]

\[
u^m(x,y) \leq \left( \tilde{a}(x,y) + z(x,y) \right) \frac{1}{t} \leq \frac{m}{l} K_\frac{1}{l-t} \left( \tilde{a}(x,y) + z(x,y) \right) + \frac{l}{l} K_\frac{1}{t},
\]

\[
\max_{\xi \in [\beta_s,x]} u'(\xi,y) \leq \left( \tilde{a}(x,y) + z(x,y) \right) \frac{1}{t} \leq \frac{r}{l} K_\frac{1}{l-t} \left( \tilde{a}(x,y) + z(x,y) \right) + \frac{l}{l} K_\frac{1}{t},
\]

(3.48)

for any constant \( K \geq 1 \). From (3.48) and property of \( \omega_i \) \( (i = 1,2) \), the function \( z(x,y) \) may be written in the form:

\[
z(x,y) \leq f_1(x,y) \int_{b_1(x_0)}^{b_1(x)} \int_{c_1(y_0)}^{c_1(y)} h_1(s,t) \omega_1 \left[ \frac{1}{l} K_\frac{1}{l-t} \left( \tilde{a}(s,t) + z(s,t) \right) + \frac{l}{l} K_\frac{1}{t} \right] ds dt
\]

\[
  + f_2(x,y) \int_{b_2(x_0)}^{b_2(x)} \int_{c_2(y_0)}^{c_2(y)} h_2(s,t) \omega_2 \left[ \frac{1}{l} K_\frac{1}{l-t} \left( \tilde{a}(s,t) + z(s,t) \right) + \frac{l}{l} K_\frac{1}{t} \right] ds dt
\]
where \( E_1(x,y) \) is defined in (3.42), which is nonnegative and nondecreasing in each of the variables. Inequalities (3.49), (3.50) satisfies the conditions of Theorem 3.1, applying the result of Theorem 3.1, we get

\[
z(x,y) \leq E_1(x,y)W_1^{-1} \left\{ W_2^{-1} \left\{ W_2 \left[ W_1 \left( 1 + G_2^{-1}(B_1(M,N)) \right) + A_1(x,y) \right] + B_1(x,y) \right\} \right\},
\]

where \( A_1(x,y) \), \( B_1(x,y) \) and \( G_2(u) \) are defined in (3.5), (3.6) and (3.45), respectively. Combining (3.46) and (3.51), we get the desired result. \( \square \)

Take \( l = 2 \), \( m = r = 1 \), \( \omega_1(u) = \omega_2(u) = u \) in Theorem 3.3, a new Gamidov-Ou-Lang type inequalities is obtained as follows.

**Corollary 3.4.** Suppose that the conditions (i) of Theorem 3.3 are satisfied, and the function \( \phi \in C(\Psi, R_+) \) satisfies \( \max_{(t,\tau)\in\Psi} \phi(t, \tau) \leq a^2(x_0, y_0) \). If \( u \in C(\Lambda, R_+) \) satisfies

\[
u^2(x,y) \leq a(x,y) + f_1(x,y) \int_{b_1(x,y)}^{c_1(y)} h_1(s,t)u(s,t)dsdt
+ f_2(x,y) \int_{b_2(x,y)}^{c_2(y)} h_2(s,t) \max_{\xi \in [\beta s, s]} u(\xi, t)dsdt
\]
Then, we have the following explicit estimation

$$u(x,y) \leq \left[ a(x,y) + \frac{E_2(x,y)}{1 - D_4(M,N)} \exp(A_1(x,y) + B_1(x,y)) \right]^{\frac{1}{4}}, \quad (x,y) \in \Delta, \quad (3.54)$$

where

$$E_2(x,y) = 1 + f_1(x,y) \int_{b_1(x_0)}^{b_1(y_0)} \int_{c_1(x_0)}^{c_1(y_0)} h_1(s,t) \left( \frac{1}{2} K^{-\frac{1}{2}} \tilde{a}(s,t) + \frac{1}{2} K \right) dsdt$$

$$+ f_2(x,y) \int_{b_2(x_0)}^{b_2(y_0)} \int_{c_2(x_0)}^{c_2(y_0)} h_2(s,t) \left( \frac{1}{2} K^{-\frac{1}{2}} \tilde{a}(s,t) + \frac{1}{2} K \right) dsdt$$

$$+ f_3(x,y) \int_{b_3(x_0)}^{b_3(M)} \int_{c_3(x_0)}^{c_3(y_0)} h_3(s,t) \left( \frac{1}{2} K^{-\frac{1}{2}} \tilde{a}(s,t) + \frac{1}{2} K \right) dsdt$$

$$+ f_4(x,y) \int_{b_4(M)}^{b_4(y_0)} \int_{c_4(x_0)}^{c_4(y_0)} h_4(s,t) \left( \frac{1}{2} K^{-\frac{1}{2}} \tilde{a}(s,t) + \frac{1}{2} K \right) dsdt, \quad (3.55)$$

$$D_4(M,N) = f_3(M,N) \int_{b_3(x_0)}^{b_3(M)} \int_{c_3(x_0)}^{c_3(y_0)} \tilde{E}_2(s,t) \left( \frac{2E_2(x_0,y_0)}{2E_2(x_0,y_0)} \right) h_3(s,t) K^{-\frac{1}{2}} \exp \left( A_1(s,t) + B_1(s,t) \right) dsdt$$

$$+ f_4(M,N) \int_{b_4(M)}^{b_4(y_0)} \int_{c_4(x_0)}^{c_4(y_0)} \tilde{E}_2(s,t) \left( \frac{2E_2(x_0,y_0)}{2E_2(x_0,y_0)} \right) h_4(s,t) K^{-\frac{1}{2}} \exp \left( A_1(s,t) + B_1(s,t) \right) dsdt < 1. \quad (3.56)$$

$$\tilde{E}_2(x,y) = \begin{cases} 
E_2(x,y), & (x,y) \in \Delta, \\
E_2(x_0,y_0), & (x,y) \in \Psi.
\end{cases} \quad (3.57)$$

**Proof.** Applying Theorem 3.3 to (3.52), (3.53), then

$$W_1(z) = W_2(z) = \int_{c}^{\xi} \frac{ds}{s} = \ln z - \ln c, \quad G_2(u) = \frac{u}{D_4(M,N)} - u,$$

obviously, $G_2(u)$ is a strictly increasing function, we get the desired result. □

**Theorem 3.5.** Let the following conditions be fulfilled:

(i) The conditions (i)-(ii) of Theorem 3.3 are satisfied;

(ii) The function $\phi \in C(\Psi, R_+)$ satisfies $0 < \max_{(t,\tau) \in \Psi} \phi(t,\tau) \leq S \frac{q-1}{q} a^\tau(x_0,y_0)$;

(iii) $\alpha_i \in (0,1]$, $\beta_i \in (0,1)$ and $p(\gamma_i - 1) + 1 > 0$, $p(\beta_i - 1) + 1 > 0$ such that $\frac{1}{p}$ +
The function \( u \in C(\Lambda, R_+^\epsilon) \) satisfies:

\[
\begin{align*}
\alpha_k (\beta_i - 1) + \gamma_k - 1 & \geq 0 \quad (p > 1; \ k = 1, 2; \ i = 1, 2, 3, 4); \\
(iv) \text{The function } u & \in C(\Lambda, R_+^\epsilon) \text{ satisfies:} \\
u^I(x,y) \leq a(x,y) \\
+ f_1(x,y) \int_{b_1(x)}^{c_1(x)} (x^{\alpha_{11} - \alpha_{11}} \beta_{11} - 1 \gamma_{11} - 1 (y^{\alpha_{21} - \alpha_{21}} - t^{\alpha_{21}}) \beta_{21} - 1 \gamma_{21} - 1 h_1(s,t) \\
\times \omega_1(u(s,t)) ds dt \\
+ f_2(x,y) \int_{b_2(x)}^{c_2(x)} (x^{\alpha_{12} - \alpha_{12}} \beta_{12} - 1 \gamma_{12} - 1 (y^{\alpha_{22} - \alpha_{22}} - t^{\alpha_{22}}) \beta_{22} - 1 \gamma_{22} - 1 h_2(s,t) \\
\times \omega_2\left( \max_{\xi \in [\beta, \epsilon]} \frac{u(\xi, t)}{s} \right) ds dt \\
+ f_3(x,y) \int_{b_3(x)}^{c_3(x)} (M^{\alpha_{13} - \alpha_{13}} \beta_{13} - 1 \gamma_{13} - 1 (N^{\alpha_{23} - \alpha_{23}} - t^{\alpha_{23}}) \beta_{23} - 1 \gamma_{23} - 1 h_3(s,t) \\
\times h_3(s,t) \max_{\xi \in [\beta, \epsilon]} u^I(\xi, t) ds dt, \quad (x,y) \in \Lambda, \\
(3.58)
\end{align*}
\]

\[
\begin{align*}
u(x,y) & \leq f(x,y), \quad (x,y) \in \Psi. \\
(3.59)
\end{align*}
\]

Then, we have the following explicit estimation

\[
\begin{align*}
u(x,y) & \leq \left\{ 5^{q-1} a^q (x,y) + E_3(x,y) \right\} \\
+ A_2(x,y) \\
+ B_2(x,y) \right\} \right\} \right\} \right\} \right\}, \quad (x,y) \in \Lambda, \\
(3.60)
\end{align*}
\]

where

\[
E_3(x,y) = \begin{align*}
+ F_1(x,y) & \int_{b_1(x)}^{c_1(x)} H_1(s,t) 2^{q-1} \omega_1^q \left( \frac{1}{ql} K^{\frac{1+ql}{ql}} a_1(s,t) + \frac{ql - 1}{ql} K^{\frac{1}{ql}} \right) ds dt \\
+ F_2(x,y) & \int_{b_2(x)}^{c_2(x)} H_2(s,t) 2^{q-1} \omega_2^q \left( \frac{1}{ql} K^{\frac{1+ql}{ql}} a_1(s,t) + \frac{ql - 1}{ql} K^{\frac{1}{ql}} \right) ds dt \\
+ F_3(x,y) & \int_{b_3(x)}^{c_3(x)} H_3(s,t) \left( \frac{m}{l} K^{-\frac{m}{l}} a_1(s,t) + \frac{l - m}{l} K^{\frac{m}{l}} \right) ds dt \\
+ F_4(x,y) & \int_{b_4(x)}^{c_4(x)} H_4(s,t) \left( \frac{r}{l} K^{-\frac{r}{l}} a_1(s,t) + \frac{l - r}{l} K^{\frac{r}{l}} \right) ds dt, \\
(3.61)
\end{align*}
\]

\[
F_i(x,y) = 5^{q-1} f_i^q (x,y) e_i^q (x,y), \quad (i = 1, 2), \\
F_j(x,y) = 5^{q-1} f_j^q (x,y) e_j^q (M,N), \quad (j = 3, 4),
\]

with
\[ H_i(x,y) = h_i^q(x,y), \quad (i = 1, 2, 3, 4), \quad (3.62) \]

\[ e_i(x,y) = \left( \frac{\theta_1(x,y)}{\alpha_1 \alpha_2} \right)^{\frac{1}{p}} \]

\[ \times \left[ B \left( \frac{p(\gamma_i - 1)}{\alpha_i} + \frac{p(\beta_i - 1)}{1} \right) B \left( \frac{p(\gamma_i - 1)}{\alpha_i} + \frac{p(\beta_i - 1)}{1} \right) \right]^{\frac{1}{p}}, \]

\[ \theta_{ki} = p[\alpha_{ki}(\beta_i - 1) + \gamma_{ki} - 1] + 1 \geq 0, \quad (k = 1, 2; \quad i = 1, 2, 3, 4), \quad (3.63) \]

\[ W_1(z) = \int_c^z ds/\omega_1^q(s), \quad W_2(z) = \int_c^z \omega_1^q(W_1^{-1}(s)) ds, \quad (3.64) \]

\[ A_2(x,y) = F_1(x,y) \int_{b_1(x,y)}^{c_1(x,y)} H_1(x,t)2^{q-1} ds dt, \quad (3.65) \]

\[ B_2(x,y) = F_1(x,y) \int_{b_1(x,y)}^{c_1(x,y)} H_2(x,t)2^{q-1} ds dt, \quad (3.66) \]

\[ D_5(M,N) = F_3(M,N) \int_{b_3(x,y)}^{c_3(x,y)} H_3(x,t)2^{q-1} ds dt \]

\[ + F_4(M,N) \int_{b_4(x,y)}^{c_4(x,y)} H_4(x,t)2^{q-1} ds dt, \quad (3.67) \]

\[ G_3(u) = W_2 \left( \frac{u}{D_5(M,N)} \right) - W_2 \left( W_1(1 + u) + A_2(M,N) \right), \quad (3.68) \]

where \( G_3(u) \) is increasing on \( R_+ \).

**Proof.** Let \( \frac{1}{p} + \frac{1}{q} = 1, \quad p > 1 \), then \( q > 0 \). Since \( p(\beta_i - 1) + 1 > 0, \quad p(\gamma_{ki} - 1) + 1 > 0, \quad \frac{1}{p} + \alpha_{ki}(\beta_i - 1) + \gamma_{ki} - 1 \geq 0 \) for \( k = 1, 2; \quad i = 1, 2, 3, 4 \). Using the Hölder inequality to (3.58), and applying Lemma 2.2, we get

\[ u^q(x,y) \leq a(x,y) \]

\[ + f_1(x,y) \]

\[ \times \left[ \int_{b_1(x,y)}^{c_1(x,y)} \left( x^{\alpha_{11} - s^{\alpha_{11}}} + s^{\alpha_{21} - r^{\alpha_{21}}} + \gamma_{21} - 1 \right) ds dt \right]^{\frac{1}{p}} \]

\[ \times \left[ \int_{b_1(x,y)}^{c_1(x,y)} h_i^q(s,t)2^{q-1} ds dt \right]^{\frac{1}{q}} \]

\[ + f_2(x,y) \]

\[ \times \left[ \int_{b_2(x,y)}^{c_2(x,y)} \left( x^{\alpha_{12} - s^{\alpha_{12}}} + s^{\alpha_{22} - r^{\alpha_{22}}} + \gamma_{22} - 1 \right) ds dt \right]^{\frac{1}{p}} \]

\[ \times \left[ \int_{b_2(x,y)}^{c_2(x,y)} h_i^q(s,t)2^{q-1} ds dt \right]^{\frac{1}{q}} \]
\[ f_3(x, y) + f_4(x, y) \times \left[ \int_{b_3(y_0)}^{c_3(N)} (M^{\alpha_{13}} - s_{\alpha_{13}}) p(\beta_{3-1}) s_{p(\gamma_{13}-1)} (N^{\alpha_{23}} - t_{\alpha_{23}}) p(\beta_{3-1}) t_{p(\gamma_{23}-1)} dsdt \right]^{1/\rho} \]

\[ \times \left[ \int_{b_3(y_0)}^{c_3(N)} h_2^q(s, t) u_{qm}(s, t) dsdt \right]^{1/\eta} \]

\[ \leq a(x, y) + f_1(x, y) e_1(x, y) \left[ \int_{b_1(x_0)}^{c_1(y_0)} h_1^q(s, t) \omega_1^q(u(s, t)) dsdt \right]^{1/\eta} \]

\[ + f_2(x, y) e_2(x, y) \left[ \int_{b_2(x_0)}^{c_2(y_0)} h_2^q(s, t) \omega_2^q(\max_{\xi \in [b, s]} u(\xi, t)) dsdt \right]^{1/\eta} \]

\[ + f_3(x, y) e_3(M, N) \left[ \int_{b_3(x_0)}^{c_3(N)} h_3^q(s, t) u_{qm}(s, t) dsdt \right]^{1/\eta} \]

\[ + f_4(x, y) e_4(M, N) \left[ \int_{b_4(x_0)}^{c_4(N)} h_4^q(s, t) \max_{\xi \in [b, s]} u_{qr}(\xi, t) dsdt \right]^{1/\eta}, \quad (x, y) \in \Delta, \quad (x, y) \in \Psi \]

(3.69)

(3.70)

where \( e_i(x, y) \) are defined in (3.63). It is easy to see that \( e_i(x, y) \) is nondecreasing in each of the variables. Applying Lemma 2.3 to (3.69), we get

\[ u_ql(x, y) \leq a_1(x, y) + F_1(x, y) \int_{b_1(x_0)}^{c_1(y_0)} H_1(s, t) \omega_1^q(u(s, t)) dsdt \]

\[ + F_2(x, y) \int_{b_2(x_0)}^{c_2(y_0)} H_2(s, t) \omega_2^q(\max_{\xi \in [b, s]} u(\xi, t)) dsdt \]

\[ + F_3(x, y) \int_{b_3(x_0)}^{c_3(N)} H_3(s, t) u_{qm}(s, t) dsdt \]

\[ + F_4(x, y) \int_{b_4(x_0)}^{c_4(N)} H_4(s, t) \max_{\xi \in [b, s]} u_{qr}(\xi, t) dsdt, \quad (x, y) \in \Delta, \quad (x, y) \in \Psi \]

(3.71)

where \( F_i(x, y) \), \( H_i(x, y) \) \((i = 1, 2, 3, 4)\) are defined in (3.62), \( a_1(x, y) = 5^{q-1} a^q(x, y) \), and \( F_i(x, y) \), \( a_1(x, y) \) are nondecreasing in each of the variables. Let

\[ z(x, y) = u^q(x, y). \]

(3.72)
From (3.70) and (3.71), we have

\[
\begin{align*}
  z^1(x, y) &\leq a_1(x, y) + F_1(x, y) \int_{b_1(x_0)}^{c_1(y)} H_1(s, t) \omega_1^{\frac{1}{q}}(z^1(s, t)) ds dt \\
  &\quad + F_2(x, y) \int_{b_2(x_0)}^{c_2(y)} H_2(s, t) \omega_2^{\frac{1}{q}} \left( \max_{\xi \in [\beta s, s]} z^1(\xi, t) \right) ds dt \\
  &\quad + F_3(x, y) \int_{b_3(x_0)}^{c_3(N)} H_3(s, t) z^m(s, t) ds dt \\
  &\quad + F_4(x, y) \int_{b_4(x_0)}^{c_4(N)} H_4(s, t) \max_{\xi \in [\beta s, s]} z^r(\xi, t) ds dt, \quad (x, y) \in \Delta, \\
  z(x, y) &\leq S^{\frac{q-1}{q}} a_1^{\frac{q}{q}}(x_0, y_0) = a_1^{\frac{q}{q}}(x_0, y_0), \quad (x, y) \in \Psi.
\end{align*}
\]  

Inequalities (3.73), (3.74) are similar to (3.39), (3.40), separately. Let

\[
v(x, y) = \begin{cases} 
  F_1(x, y) \int_{b_1(x_0)}^{c_1(y)} H_1(s, t) \omega_1^{\frac{1}{q}}(z^1(s, t)) ds dt \\
  + F_2(x, y) \int_{b_2(x_0)}^{c_2(y)} H_2(s, t) \omega_2^{\frac{1}{q}} \left( \max_{\xi \in [\beta s, s]} z^1(\xi, t) \right) ds dt \\
  + F_3(x, y) \int_{b_3(x_0)}^{c_3(N)} H_3(s, t) z^m(s, t) ds dt \\
  + F_4(x, y) \int_{b_4(x_0)}^{c_4(N)} H_4(s, t) \max_{\xi \in [\beta s, s]} z^r(\xi, t) ds dt, \\
  (x, y) \in \Delta,
\end{cases}
\]

Obviously, \(v(x, y)\) is a positive and nondecreasing function. And from (3.73), (3.74), we have

\[
\begin{align*}
  z(x, y) &\leq (a_1(x, y) + v(x, y))^\frac{1}{q} \leq (\tilde{a}_1(x, y) + v(x, y))^\frac{1}{q}, \quad (x, y) \in \Delta, \\
  z(x, y) &\leq a_1^{\frac{q}{q}}(x_0, y_0) \leq (\tilde{a}_1(x, y) + v(x, y))^\frac{1}{q}, \quad (x, y) \in \Psi,
\end{align*}
\]

where

\[
\tilde{a}_1(x, y) = \begin{cases} 
  a_1(x, y), & (x, y) \in \Delta, \\
  a_1(x_0, y_0), & (x, y) \in \Psi,
\end{cases}
\]

which is nondecreasing in each of the variables. Moreover, from (3.75) and (3.76), for \((x, y) \in \Lambda\), we have

\[
\begin{align*}
  z^1(x, y) &\leq (\tilde{a}_1(x, y) + v(x, y))^\frac{1}{q} \leq \frac{1}{ql} K^{\frac{1-q}{ql}} (\tilde{a}_1(x, y) + v(x, y)) + \frac{ql - 1}{ql} K^{\frac{1}{ql}}, \\
  \max_{\xi \in [\beta s, x]} z^1(\xi, y) &\leq \max_{\xi \in [\beta s, x]} \left( \tilde{a}_1(\xi, y) + v(\xi, y) \right)^\frac{1}{q} \leq \left( \max_{\xi \in [\beta s, x]} \tilde{a}_1(\xi, y) + \max_{\xi \in [\beta s, x]} v(\xi, y) \right)^\frac{1}{q}.
\end{align*}
\]
\[
\begin{align*}
\frac{1}{ql} K_{\frac{1-ql}{ql}} \left( \frac{q-1}{ql} K_{\frac{1}{ql}} \right) + \frac{ql}{ql} K_{\frac{1-ql}{ql}},
\end{align*}
\]
\[
\begin{align*}
z_1^{m}(x,y) \leq \left( \tilde{a}_1(x,y) + v(x,y) \right) \frac{m}{l} K_{\frac{m}{m+l}} \left( \tilde{a}_1(x,y) + v(x,y) \right) + \frac{l-m}{l} K_{\frac{l}{m+l}},
\end{align*}
\]
\[
\begin{align*}
\max_{\xi \in [\beta_x, x]} \hat{z}^{q}(\xi, y) & \leq \max_{\xi \in [\beta_x, x]} \left( \tilde{a}_1(\xi, y) + v(\xi, y) \right) \hat{\omega} \leq \frac{l}{l} K_{\frac{l}{l}} \left( \tilde{a}_1(x,y) + v(x,y) \right) + \frac{l-r}{l} K_{\frac{l}{l}},
\end{align*}
\]
\[
(3.78)
\]
for any constant \( K \geq 1 \). By the subadditive of \( \omega_i \) \((i = 1, 2)\) and the Lemma 2.3, we have
\[
\begin{align*}
\omega_1^q \left( z^{\frac{1}{q}}(x,y) \right) & \leq \omega_1^q \left[ \frac{1}{ql} K_{\frac{1-ql}{ql}} \left( \tilde{a}_1(x,y) + v(x,y) \right) + \frac{ql-1}{ql} K_{\frac{1}{ql}} \right] \tag{3.79}
\end{align*}
\]
\[
\begin{align*}
\omega_2^q \left( \max_{\xi \in [\beta_x, x]} \hat{z}^{q}(\xi, y) \right) & \leq 2^{l-1} \omega_2^q \left( \frac{1}{ql} K_{\frac{1-ql}{ql}} \tilde{a}_1(x,y) + \frac{ql-1}{ql} K_{\frac{1}{ql}} \right) + 2^{l-1} \omega_2^q(v(x,y)),
\end{align*}
\]
Substituting (3.78) and (3.79) into the definition of \( v(x,y) \), and applying the method of proof of Theorem 3.3, we get
\[
\begin{align*}
z(x,y) \leq \left\{ a_1(x,y) + E_3(x,y) \overline{W}_1^{-1} \left\{ \overline{W}_2^{-1} \left\{ \overline{W}_1 \left( 1 + G_3^{-1}(B_2(M,N)) \right) \right\} + A_2(x,y) \right\} \right\}^{\frac{1}{q}}, \quad (x,y) \in \Delta,
\end{align*}
\]
where \( E_3(x,y) \), \( \overline{W}_1(z) \), \( A_2(x,y) \), \( B_2(x,y) \) and \( G_3(u) \) are defined in (3.61), (3.64), (3.65), (3.66) and (3.68), separately. Combining (3.72) and (3.80), we can easily get (3.60).

If \( \alpha_i = \alpha \gamma_i = \gamma_i = 1 \), \( 0 < 0 < 1 \) \((i = 1, 2, 3, 4)\) in (3.58), we get the following result.

**Corollary 3.6.** Let the conditions (i)-(ii) of Theorem 3.5 be satisfied, suppose that \( u \in C(\Lambda, R_+) \) satisfies:
\[
\begin{align*}
u^i(x,y) \leq & a(x,y) + f_1(x,y) \int_{b_1(x_0)}^{b_1(y)} \int_{c_1(y_0)}^{c_1(y)} (x-s)^{\beta_1(t)} \beta_1^{-1}(y-t)^{\beta_1(s)} \omega_1(u(s,t))dsdt \\
& + f_2(x,y) \int_{b_2(x_0)}^{b_2(y)} \int_{c_2(y_0)}^{c_2(y)} (x-s)^{\beta_2(t)} \beta_2^{-1}(y-t)^{\beta_2(s)} \omega_2 \left( \max_{\xi \in [\beta_x, x]} u(\xi, t) \right)dsdt
\end{align*}
\]
Then, we have the following explicit estimation
\[
uxy \leq \left\{ 5^{q-1}d^q(x, y) + E_4(x, y)W^{-1}_1 \left\{ W_2^{-1} \left\{ W_1 \left( 1 + G_4^{-1}(B_3(M, N)) \right) \right\} \right\} \right\}^{1/\varphi}, \quad (x, y) \in \Delta,
\]
(3.83)
\[ + \tilde{F}_4(M,N) \int_{b_4(x_0)}^{c_4(y_0)} \int_{b_4(x_0)}^{c_4(y_0)} \tilde{E}_4(s,t) H_4(s,t) K^{\gamma_1} dsdt, \quad (3.89) \]

\[ G_4(u) = W_2 \left( \frac{u}{D_6(M,N)} \right) - W_2 \left( W_1(1 + u) + A_3(M,N) \right), \quad (3.90) \]

where \( G_4(u) \) is increasing on \( R_+ \).

4. Applications

In this section, we present some examples to show applications in the boundedness and uniqueness of a certain Gamidov type weakly singular integral equation with maxima. Consider the following weakly singular integral equations:

\[ u^I(x,y) = a(x,y) + \int_{x_0}^{x} \int_{y_0}^{y} (x-s)^{\beta_1-1}s^{\gamma_1-1}(y-t)^{\beta_1-1}t^{\gamma_1-1} P_1(s,t,x,y,u(s,t)) dsdt \]

\[ + \int_{x_0}^{x} \int_{y_0}^{y} (x-s)^{\beta_2-1}s^{\gamma_2-1}(y-t)^{\beta_2-1}t^{\gamma_2-1} P_2(s,t,x,y, \max_{\xi \in [\beta s,t]} u(\xi,t)) dsdt \]

\[ + \int_{x_0}^{x} \int_{y_0}^{y} (M-s)^{\beta_3-1}s^{\gamma_3-1}(N-t)^{\beta_3-1}t^{\gamma_3-1} P_3(s,t,x,y,u(s,t)) dsdt \]

\[ + \int_{x_0}^{x} \int_{y_0}^{y} (M-s)^{\beta_4-1}s^{\gamma_4-1}(N-t)^{\beta_4-1}t^{\gamma_4-1} P_4(s,t,x,y, \max_{\xi \in [\beta s,t]} u(\xi,t)) dsdt, \quad (4.1) \]

\[ u(x,y) = \phi(x,y), \quad (x,y) \in [\beta x_0,x_0] \times [y_0,N] \triangleq \Psi, \quad (4.2) \]

where \( u(x,y) \in C(\Delta, R) \), \( a(x,y) \in C(\Delta, R) \), \( P_i \in C(\Delta^2 \times R, R) \) \( (i = 1, 2, 3, 4) \), \( \phi(x,y) \in C(\Psi, R) \), and \( 0 < \beta < 1 \), \( I \geq 1 \) are constants.

First, we give the estimate for the solution of problem (4.1) (4.2). Suppose that the following conditions are satisfied:

\[ |P_1(s,t,x,y,u(s,t))| \leq f_1(x,y)h_1(s,t) \omega_1(|u(s,t)|), \]

\[ |P_2(s,t,x,y, \max_{\xi \in [\beta s,t]} u(\xi,t))| \leq f_2(x,y)h_2(s,t) \omega_2( \max_{\xi \in [\beta s,t]} |u(\xi,t)|), \]

\[ |P_3(s,t,x,y,u(s,t))| \leq f_3(x,y)h_3(s,t)|u(s,t)|^m, \]

\[ |P_4(s,t,x,y, \max_{\xi \in [\beta s,t]} u(\xi,t))| \leq f_4(x,y)h_4(s,t) \max_{\xi \in [\beta s,t]} |u(\xi,t)|^r. \quad (4.3) \]

**THEOREM 4.1.** Let the following conditions be fulfilled:

(i) The functions \( f_i(x,y), h_i(x,y) \in C(\Delta, R_+) \) \( (i = 1, 2, 3, 4) \), and \( f_i(x,y) \) are nondecreasing in each of the variables;

(ii) The functions \( \omega_i(u) \in C(R_+, R_+) \) are nondecreasing on \( R_+ \) and positive on \( (0, +\infty) \) such that \( \omega_1 \propto \omega_2 \), and \( \omega_3 \) are subadditive and submultiplicative, that is \( \omega_i(x+y) \leq \omega_i(x) + \omega_i(y), \omega_i(tx) \geq t \omega_i(x) \) for \( 0 \leq t \leq 1 \);
Then every solution $u(x,y)$ of equations (4.1), (4.2) has the estimate

$$ |u(x,y)| \leq \left\{ 5^{q-1}\overline{\sigma}_l(x,y) + E_5(x,y)\overline{W}_1^{-1}\left\{ \overline{W}_2^{-1}\left\{ \overline{W}_1 \left( 1 + G_5^{-1}(B_4(M,N)) \right) \right\} + A_4(x,y)B_4(x,y) \right\} \right\}^{\frac{1}{q}}, \quad (x,y) \in \Delta, \quad (4.4) $$

where

$$ \overline{\sigma}(x,y) = \max_{(t,\tau) \in [x_0, x] \times [y_0, y]} |a(t, \tau)|, \quad (4.5) $$

$$ E_5(x,y) = 1 + \overline{F}_1(x,y) \int_{x_0}^{x} \int_{y_0}^{y} H_1(s,t)2^{q-1}\omega_1\left( \frac{1}{ql}K^{-\frac{1}{ql}}\overline{a}_2(s,t) + \frac{q-1}{ql}K^{-\frac{1}{ql}} \right) dsdt \quad \text{and} \quad \overline{\theta}_{ki} = p[\beta_i + \gamma_{ki} - 2] + 1 \geq 0, \quad (k = 1, 2; \quad i = 1, 2, 3, 4). \quad (4.8) $$

$$ \overline{a}_2(x,y) = \begin{cases} 5^{q-1}\overline{\sigma}_l(x,y), & (x,y) \in \Delta, \\ 5^{q-1}\overline{\sigma}_l(x_0,y_0), & (x,y) \in \Psi, \end{cases} \quad \overline{E}_5(x,y) = \begin{cases} E_5(x,y), & (x,y) \in \Delta, \\ E_5(x_0,y_0), & (x,y) \in \Psi. \end{cases} \quad (4.12) $$
\[ G_5(u) = W_2 \left( \frac{u}{W_1(D_2(M,N))} \right) - W_2 \left( W_1(1 + u) + A_4(M,N) \right) \tag{4.13} \]

where \( G_5(u) \) is increasing on \( R_+ \).

**Proof.** By (4.5), we obtain \( \alpha(x,y) \in C(\Delta, R_+) \) is nondecreasing in each of the variables. From equation (4.1) and the condition (4.3), we get

\[ |u(x,y)|^t \leq \alpha(x,y) \]

Applying Theorem 3.5 to (4.14), (4.15) with \( |u(x,y)| \)

From equation (4.2) and the condition (iv), we obtain

\[ |u(x,y)| \leq \phi(x,y) \leq 5^{\frac{g-1}{g'}} |a(x_0,y_0)|^t, \quad (x,y) \in \Psi. \tag{4.15} \]

Applying Theorem 3.5 to (4.14), (4.15) with \( a(x,y) = \alpha(x,y), \ b_i(x) = x, \ c_i(y) = y, \ \alpha_{ki} = 1, \ (k = 1, 2; \ i = 1, 2, 3, 4) \), we can obtain the estimation (4.4).

Next, we give the conditions of the uniqueness of solutions for problem (4.1), (4.2). Suppose that the following conditions are satisfied:

\[ |P_i(s,t,x,y,u) - P_i(s,t,x,y,v)| \leq f_i(x,y)h_i(s,t)\omega_i(|u - v|), \quad i = 1, 2, \]

\[ |P_j(s,t,x,y,u) - P_j(s,t,x,y,v)| \leq f_j(x,y)h_j(s,t)|u - v|, \quad j = 3, 4. \tag{4.16} \]

**Theorem 4.2.** Assume that the conditions (i-iii) of Theorem 4.1 are satisfied, and \( l = 1 \). Then the problem (4.1), (4.2) has at most one unique solution.

**Proof.** Assume that there exist two different solutions \( u(x,y) \) and \( v(x,y) \) of the problem (4.1), (4.2) defined in \( \Lambda \), then the functions \( u(x,y) \) and \( v(x,y) \) satisfy (4.1), (4.2) and

\[
v'(x,y) = a(x,y) + \int_{x_0}^{x} \int_{y_0}^{y} (x-s)^{\beta_1-1}s^{\gamma_1-1}(y-t)^{\beta_1-1}t^{\gamma_2-1}P_1(s,t,x,y,v(s,t))dsdt \\
+ \int_{x_0}^{x} \int_{y_0}^{y} (x-s)^{\beta_2-1}s^{\gamma_2-1}(y-t)^{\beta_2-1}t^{\gamma_3-1}P_2(s,t,x,y,\max_{\xi \in \beta_{0,1}} v(\xi,t))dsdt
\]
respectively. Then the norm of difference of the solutions $u(x,y)$ and $v(x,y)$ satisfies the inequalities

$$\begin{aligned}
&|u(x,y) - v(x,y)| \\
&\leq f_1(x,y) \int_{x_0}^x \int_{y_0}^y (x-s)^{\beta_1-1}s^{\gamma_1-1}(y-t)^{\beta_1-1}t^{\tau_1-1}h_1(s,t)\omega_1(|u(s,t) - v(s,t)|)dsdt \\
&+ f_2(x,y) \int_{x_0}^x \int_{y_0}^y (x-s)^{\beta_2-1}s^{\gamma_2-1}(y-t)^{\beta_2-1}t^{\tau_2-1}h_2(s,t) \\
&\omega_2\left(\left|\max_{\xi \in [\beta_1,s]} u(\xi,t) - \max_{\xi \in [\beta_1,s]} v(\xi,t)\right|\right)dsdt \\
&+ f_3(x,y) \int_{x_0}^x \int_{y_0}^y (x-s)^{\beta_3-1}s^{\gamma_3-1}(y-t)^{\beta_3-1}t^{\tau_3-1}h_3(s,t)|u(s,t) - v(s,t)|dsdt \\
&+ f_4(x,y) \int_{x_0}^x \int_{y_0}^y (x-s)^{\beta_4-1}s^{\gamma_4-1}(y-t)^{\beta_4-1}t^{\tau_4-1}h_4(s,t) \\
&\leq f_1(x,y) \int_{x_0}^x \int_{y_0}^y (x-s)^{\beta_1-1}s^{\gamma_1-1}(y-t)^{\beta_1-1}t^{\tau_1-1}h_1(s,t)\omega_1(|u(s,t) - v(s,t)|)dsdt \\
&+ f_2(x,y) \int_{x_0}^x \int_{y_0}^y (x-s)^{\beta_2-1}s^{\gamma_2-1}(y-t)^{\beta_2-1}t^{\tau_2-1}h_2(s,t) \\
&\omega_2\left(\left|\max_{\xi \in [\beta_1,s]} u(\xi,t) - \max_{\xi \in [\beta_1,s]} v(\xi,t)\right|\right)dsdt \\
&+ f_3(x,y) \int_{x_0}^x \int_{y_0}^y (x-s)^{\beta_3-1}s^{\gamma_3-1}(y-t)^{\beta_3-1}t^{\tau_3-1}h_3(s,t)|u(s,t) - v(s,t)|dsdt \\
&+ f_4(x,y) \int_{x_0}^x \int_{y_0}^y (x-s)^{\beta_4-1}s^{\gamma_4-1}(y-t)^{\beta_4-1}t^{\tau_4-1}h_4(s,t) \\
&\max_{\xi \in [\beta_1,s]} |u(\xi,t) - v(\xi,t)|dsdt, \quad (x,y) \in \Delta, \\
&|u(x,y) - v(x,y)| \leq 0, \quad (x,y) \in \Psi. 
\end{aligned}$$

Set $z(x,y) = |u(x,y) - v(x,y)|$ for $(x,y) \in \Delta$. Applying Theorem 3.5 to (4.19), (4.20) with $a(x,y) = 0$, $\phi(x,y) = 0$, $b_i(x) = x$, $c_i(y) = y$, $\alpha_{ki} = 1$ $(k = 1, 2; i = 1, 2, 3, 4)$. We obtain that $z(x,y) \leq 0$ for $(x,y) \in \Delta$, which implies the inequality $u(x,y) = v(x,y)$ for $(x,y) \in \Delta.$  \(\square\)
REFERENCES

[1] T.H. Gronwall, Note on the derivatives with respect to a parameter of the solutions of a system of differential equations, Annals of Mathematics, vol. 20, no. 4, pp. 292–296, 1919.

[2] R. Bellman, The stability of solutions of linear differential equations, Duke Mathematical Journal, vol. 10, pp. 643–647, 1943.

[3] I. Bihari, A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations, Acta Mathematica Academia Scientiarum Hungar, vol. 7, pp. 81–94, 1956.

[4] S.G. Gamidov, Certain integral inequalities for boundary value problems of differential equations, Differential Equations, Vol. 5, pp. 463–472, 1969.

[5] D. Banov and P. Simeonov, Integral Inequalities and Applications, vol. 57 of Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1992.

[6] B.G. Pachpatte, Inequalities for differential and integral equations, vol. 197 of Mathematics in Science and Engineering, Academic Press, San Diego, Calif, USA, 1998.

[7] R. P. Agarwal, S. Deng and W. Zhang, Generalization of a retarded Gronwall-like inequality and its applications, Applied Mathematics and Computation, vol. 165, no. 3, pp. 599–612, 2005.

[8] W. S. Wang, A generalized retarded Gronwall-like inequality in two variables and applications to BVP, Applied Mathematics and Computation, vol. 191, no. 1, pp.144–154, 2007.

[9] Q. H. Ma and E. H. Yang, Estimates on solutions of some weakly singular Volterra integral inequalities, Acta Mathematica Sinica, vol. 25, no. 3, pp. 505–515, 2002.

[10] Q. H. Ma and J. Pečarić, Some new explicit bounds for weakly singular integral inequalities with applications to fractional differential and integral equations, Journal of Mathematical Analysis and Applications, vol. 341, pp. 894–905, 2008.

[11] W. S. Cheung, Q. H. Ma and S. Tseng, Some new nonlinear weakly singular integral inequalities of Wendroff type with applications, Journal of Inequalities and Applications, vol. 2008, Article ID 909156, 13 pages, 2008.

[12] Run Xu, Fanwei Meng, Some New Weakly Singular Integral Inequalities and Their Applications to Fractional Differential Equations, Journal of Inequalities and Applications, (2016) 2016:78, 1–16.

[13] F. Lakhal, A new nonlinear integral inequality of Wendroff type with continuous and weakly singular kernel and its application, Journal of Mathematical Inequalities, vol. 6, no. 3, pp. 367–379, 2012.

[14] K. L. Zheng, Bounds on some new weakly singular Wendroff-type integral inequalities and applications, Journal of Inequalities and Applications, vol. 2013, no. 1, 11 pages, 2013.

[15] K. L. Cheng, C. X. Guo and M. Tang, Some nonlinear Gronwall-Bellman-Gamidov integral inequalities and their weakly singular analogues with applications, Abstract and Applied Analysis, vol. 2014, Article ID 562691, 9 pages, 2014.

[16] K. L. Cheng and C. X. Guo, New explicit bounds on Gamidov type integral inequalities for functions in two variables and their applications, Abstract and Applied Analysis, vol. 2014, Article ID 539701, 9 pages, 2014.

[17] B. Zheng, Explicit bounds derived by some new inequalities and applications in fractional integral equations, Journal of Inequalities and Applications, vol. 4, no. 1, 12 pages, 2014.

[18] Q. H. Ma and J. Pečarić, Estimates on solutions of some new nonlinear retarded Volterra-Fredholm type integral inequalities, Nonlinear Analysis, vol. 69, pp. 393–407, 2008.

[19] L. M. Zhao, S. H. Wu and W. S. Wang, A generalized nonlinear Volterra-Fredholm type integral inequality and its application, Journal of Applied Mathematics, vol. 2014, Article ID 865136, 13 pages, 2014.

[20] Y. G. Huang, W. S. Wang and Y. Huang, A class of Volterra-Fredholm type weakly singular difference inequalities with power functions and their applications, Journal of Applied Mathematics, vol. 2014, Article ID 826173, 9 pages, 2014.

[21] A. Golov, S. Hristova and A. Rahnev, An algorithm for approximate solving of differential equations with maxima, Computational Mathematic and Application, vol. 60, pp. 2771–2778, 2010.

[22] S. G. Hristova and K. V. Stefanova, Some integral inequalities with maximum of the unknown functions, Advances in Dynamical Systems and Applications, vol. 6, no. 1, pp. 57–69, 2011.

[23] Y. Yan, A generalized nonlinear Gronwall-Bellman inequality with maxima in two variables, Journal of Applied Mathematics, vol. 2013, Article ID 853476, 10 pages, 2013.

[24] Y. Yan, On some new weakly singular Volterra integral inequalities with maxima and their applications, Journal of Inequalities and Applications, vol. 369, 16 pages, 2015.
[25] Fangcui Jiang and Fanwei Meng, Explicit bounds on some new nonlinear integral inequalities with delay, Journal of Computational and Applied Mathematics, vol. 205, no. 1, pp. 479–486, 2007.

[26] Fanwei Meng, JING SHAO, Some new Volterra-Fredholm type dynamic integral inequalities on time scales, Applied Mathematics and Computation, 2013, 223(3): 444–451.

[27] Run Xu, Fanwei Meng, Cuihua Song, On Some Integral Inequalities on Time Scales and Their Applications, Journal of Inequalities and Applications, Volume 2010, Article ID 464976, 1–13.

[28] Tonglin Wang, Run Xu, Some integral inequalities in two independent variables on time scales, Journal of Mathematical Inequalities, 2012, 6(1), 107–118.

[29] Tonglin Wang, Run Xu, Bounds for Some New Integral Inequalities With Delay on Time Scales, Journal of Mathematical Inequalities, 2012, 6(3), 355–366. 2012.09.

[30] Liwei Du, Run Xu, Some New Pachpatte Type Inequalities on Time Scales and their applications, Journal of Mathematical Inequalities, 2012, 6(2), 229–240. 2012.06.

[31] Run Xu, Ying Zhang, Generalized Gronwall fractional summation inequalities and their applications, Journal of Inequalities and Applications, (2015) 2015:242.

[32] Lingling Wan, Run Xu, Some generalized integral inequalities and there applications, Journal of Mathematical Inequalities, Vol 7, Number 3 (2013).

[33] Run Xu and Xiangting Ma, Some new retarded nonlinear Volterra-Fredholm type integral inequalities with maxima in two variables and their applications, Journal of Inequalities and Applications, (2017) 2017:187.

[34] Run Xu, Some New Nonlinear Weakly Singular Integral Inequalities and Their Applications, Journal of Mathematical Inequalities, Volume 11, Number 4 (2017), 1007–1018.

[35] Haidong Liu, Fanwei Meng, Some new generalized Volterra-Fredholm type discrete fractional sum inequalities and their applications, Journal of Inequalities and Applications (2016) 2016:213.

[36] Haidong Liu, Fanwei Meng, Some new nonlinear integral inequalities with weakly singular kernel and their applications to FDEs, Journal of Inequalities and Applications, 2015, (2015): 209.

[37] Haidong Liu, Some new integral inequalities with mixed nonlinearities for discontinuous functions, Advances in Difference Equations (2018), 2018: 22.

[38] Haidong Liu, A class of retarded Volterra-Fredholm type integral inequalities on time scales and their applications, Journal of Inequalities and Applications (2017) 2017:293.

[39] Haidong Liu, Fanwei Meng, Nonlinear retarded integral inequalities on time scales and their applications, Journal of Mathematical Inequalities. 12 (1) (2018) 219–234.

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