Quantum circuit for accurate simulation of qudit channels

Dong-Sheng Wang\textsuperscript{1} and Barry C. Sanders\textsuperscript{1,2}

\textsuperscript{1}Institute for Quantum Science and Technology, University of Calgary, Alberta T2N 1N4, Canada
\textsuperscript{2}Program in Quantum Information Science, Canadian Institute for Advanced Research, Toronto, Ontario M5G 1Z8, Canada

We construct a classical algorithm that designs quantum circuits for algorithmic quantum simulation of arbitrary qudit channels within a pre-specified error tolerance with respect to diamond-norm distance. The classical algorithm is constructed by decomposing a quantum channel into a convex combination of extreme channels by optimization of a set of nonlinear coupled algebraic equations. The resultant circuit is a randomly chosen extreme-channel circuit whose run-time is logarithmic with respect to the error tolerance and quadratic with respect to Hilbert space dimension, which requires only a single ancillary qudit plus one classical bit.

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Algorithmic quantum simulation\textsuperscript{1,3} is important for simulating many-body dynamics\textsuperscript{4,5}, quantum-state generation and dissipative quantum-state engineering\textsuperscript{6,7}, quantum thermodynamics\textsuperscript{8,9}, nonequilibrium quantum phase transitions\textsuperscript{10,11}, testing element distinctness\textsuperscript{12}, and solving linear equations\textsuperscript{13} and differential equations\textsuperscript{14}. Experimental quantum simulation\textsuperscript{15} has been demonstrated in quantum computing implementations such as ion traps\textsuperscript{16,17}, atoms in optical lattice\textsuperscript{18,19}, and superconducting circuits\textsuperscript{21}. Whereas unitary evolution generated by a self-adjoint Hamiltonian has so far been the major research focus, algorithmic quantum simulation of quantum channels (i.e., completely-positive trace-preserving mappings)\textsuperscript{22-24} and open-system dynamics\textsuperscript{25} is a nascent and exciting research area both theoretically\textsuperscript{26,27} and experimentally\textsuperscript{11,28}.

Our aim is to develop a classical algorithm that designs quantum circuits which simulate accurately an arbitrary qudit (corresponding to states described by positive semi-definite trace-class operators $\rho$ acting on the $d$-dimensional Hilbert space $H_d \cong C^d$) channel $E$ within a specified error tolerance $\epsilon$ with respect to the diamond norm$\textsuperscript{23}$. This circuit needs to be executed efficiently with respect to $\epsilon$, i.e., with space and time resources scaling as $\text{polylog}\frac{1}{\epsilon}$. Previously two different cases have been considered: Markovian qudit channel simulation but with the Lindvillian rather than the channel as input$\textsuperscript{24}$ and for a general single-qubit channel as input with an efficient simulation circuit as output$\textsuperscript{27}$ employing qubit extreme channel theory$\textsuperscript{23,30}$.

Generalizing from simulating qubit channels to qudit channels is not straightforward because decomposing an arbitrary qudit channel into a convex combination of extreme channels is an open problem in quantum information$\textsuperscript{31}$. We circumvent this obstacle by decomposing approximately, rather than exactly, into a convex sum of extreme channels, and we devise circuits for simulating extreme channels so that the entire qudit channel can be simulated by random concatenations of extreme-channel simulators. The circuits devised by our algorithm show how to realize experimentally algorithmic quantum-channel simulation.

We approach the problem of constructing a qudit channel circuit simulator by constructing an algorithm whose inputs are the description of $E$, the tolerance $\epsilon$, and the dimension $d$ of the qudit Hilbert space and delivering an output comprising a description of a simulation circuit and the actual error $\tilde{\epsilon}$. In contrast to the case for unitary channels, which can be constructed as a concatenation of other unitary channels, the non-unitary channel is not such a simple sequence of channels$\textsuperscript{32}$, thereby resulting in a more complicated approach to quantum channel simulation.

A direct procedure for quantum simulation of a quantum channel is to employ Stinespring dilation$\textsuperscript{22}$, which replaces the qudit channel $E$ with a up to $d^3$-dimensional unitary channel $U$ followed by a partial trace over the environment to recover the description of the channel $E$. The circuit for simulating the channel by a dilated unitary channel acting on a Hilbert space of dimension $d^3$ generically requires $O(d^3)$ single-qudit and two-qudit gates$\textsuperscript{33,34}$, obtained from a small universal instruction set using a Solovay-Kitaev gate decomposition approach$\textsuperscript{35,37}$. Therefore, the time cost is $O(d^3)$ and the space cost is three qudits.

In the interest of bringing algorithmic quantum-channel simulation to its lowest possible cost for experimental expediency, we employ the procedure of approximately decomposing the channel into a convex combination of extreme channels. The simulation circuit has a time cost of $O(d^2 \log \frac{1}{\epsilon})$, which is the same as the time cost for simulating a unitary qudit channel, hence a lower bound$\textsuperscript{33,38}$. Furthermore this procedure requires a spatial resource of just two qudits plus one random bit (a $d$-dimensional digit, $d = 2$ for bit), hence reduces the quantum space cost by a third.

The first step in developing the circuit-design procedure is to decompose the channel into a convex sum of extreme channels. A quantum channel $E \in S_d$, the set of all channels for qudits of dimension $d$, can be represented as $E(\rho) = \sum_{i=0}^{m} K_i \rho K_i^\dagger$ for all states $\rho$ with Kraus oper-
Any channel \( \mathcal{E} \in \mathcal{J}_d \) can be decomposed
\[
\mathcal{E} = \sum_{i=1}^{d} p_i \mathcal{E}_i^e, \quad \sum_{i=1}^{d} p_i = 1, \quad 0 \leq p_i \leq 1, \quad (1)
\]
for extreme channels \( \{ \mathcal{E}_i^e \} \subset \mathcal{J}_d \) of Kraus rank \( d \).

**Proof.** In the channel representation given by the Choi-Jamiolkowski state \( \mathcal{C} \equiv \mathcal{E} \otimes \mathbb{I} (|\eta\rangle \langle \eta|) \), \( |\eta\rangle = \sum_{i=1}^{d-1} |i\rangle |i\rangle \in \mathcal{H}_d \otimes \mathcal{H}_d \),
\[
\mathcal{C} = \sum_{i=1}^{d} p_i \mathcal{C}_i^e = \sum_{i=1}^{m'} p_i \mathcal{C}_i^e
\]
and \( \{ |i\rangle \} \) the computational basis of \( \mathcal{H}_d \), Eq. (1) is equivalent to proving \( \mathcal{C} = \sum_{i=1}^{m'} p_i \mathcal{C}_i^e \) with each extreme state \( \mathcal{C}_i^e \) corresponding to an extreme channel \( \mathcal{E}_i^e \). Convexity of \( \mathcal{J}_d \) is equivalent to the convexity of the set of \( \mathcal{C} \). The property of extreme points of convex set and positivity of \( \mathcal{C} \) ensures the existence of a non-unique decomposition \( \mathcal{C} = \sum_{i=1}^{m'} p_i \mathcal{C}_i^e \), where \( m' \) can be bounded above by \( d^2 \), since the maximum rank (the same as Kraus rank) of a general state \( C \) is \( d^2 \), and the minimum rank of \( C^e \) is one. As the maximum rank of \( C^e \) is \( d \), then rank \( \left( \sum_{i=1}^{m'} p_i \mathcal{C}_i^e \right) \leq d^2 \), which means any channel can be written as a convex sum of up to \( d \) extreme channels each with rank up to \( d \).

Following Proposition 1 any channel can be decomposed non-uniquely in the form (1). Next we propose a Kraus operator-sum representation and quantum circuit for an arbitrary \( d \)-extreme channel. First, we construct the sum representation using the Heisenberg-Weyl basis \( \{ X_j, Z_j; i,j \in \mathbb{Z}_d \} \) for \( X_i = \sum_{\ell=0}^{d-1} |\ell\rangle \langle \ell+i| \) and \( Z_j = \sum_{\ell=0}^{d-1} e^{2 \pi i j \ell/d} |\ell\rangle \langle \ell| \).

**Proposition 2.** A rank- \( d \) extreme channel \( \mathcal{E}^e \in \mathcal{J}_d \) can be represented by
\[
\mathcal{E}^e(\rho) = \sum_{i=0}^{d-1} K_i \rho K_i^e \quad (3)
\]
for any Kraus operators satisfying
\[
K_i := W F_i V, \quad F_i := X_i E_i, \quad E_i := \sum_{j=0}^{d-1} a_{ij} Z_j, \quad i \in \mathbb{Z}_d, \quad (4)
\]
for any unitary operators \( V, W \in SU(d) \), provided that \( \{ a_{ij} \subset \mathbb{C} \} \) is chosen such that the set \( \{ F_i^\dagger F_j \} \) is linearly independent and \( \sum_{i=0}^{d-1} F_i^\dagger F_i = \mathbb{I} \) is satisfied.

**Proof.** Per definition, Eq. (3) holds for any rank- \( d \) extreme channel with \( \{ K_i K_j \} \) being linearly independent. Thus, the proof focuses on showing that the construction (4) yields linearly independent operators \( \{ K_i K_j \} \). Subsequent to the establishment of linearity independence, the proof showing extremality of the channel (3) is complete.

Linear independence of \( \{ K_i K_j \} \) requires that
\[
\Xi := \sum_{i,j=0}^{d-1} \gamma_{ij} K_i^e K_j = 0 \iff \gamma_{ij} = 0 \forall i,j. \quad (5)
\]
From Eq. (4), \( \Xi = V\left( \sum_{i,j} \gamma_{ij} F_i^\dagger F_j \right) V \). This is a unitary conjugation of the sum in parentheses so we ignore \( V \) in the proof. Therefore, we need to require linear independence of \( \{ F_i^\dagger F_j \} \).

For \( \{ F_i^\dagger F_j \} \) to be a linearly independent set, each subset must be linearly independent. For each subset, \( \Xi_{i,j} := \sum_{\mu=0}^{d-1} \gamma_{i,j} F_i^\dagger F_j \). Then \( \Xi_{i,j} = 0 \) implies \( \Xi_{i,j} = 0 \forall i,j \).

Now we establish linear independence of \( \{ F_i^\dagger F_j \} \) by constraining each subset. First we map each matrix \( F_i^\dagger F_j \) to a vector \( b_{ij} := (b_{ij}) \). Then linear independence of \( \{ F_i^\dagger F_j \} \) can be ensured by the condition that the determinant of each matrix \( B_{ij} := (b_{ij}) \) is nonzero; i.e., \( \det B_{ij} \neq 0 \forall \mu \) (except for a subset of values of \( a_{ij} \) of measure zero). Then \( \Xi_{i,j} = 0 \) implies \( \gamma_{i,j} = 0 \forall \mu \), which establishes linear independence of \( \{ F_i^\dagger F_j \} \) hence also \( \{ K_i K_j \} \).

**Corollary 3.** The set of Kraus operators \( \{ F_i \} \) of an extreme channel is a rank- \( d \) positive semi-definite matrix. There are at most \( d^2 \) real parameters in \( C^e \). Constrained by normalization, there are at most \( d^2 - d \) independent parameters in \( \{ F_i \} \).

Now that we have the sum representation of the extreme channel \( \mathcal{E}^e \) with respect to \( \{ F_i \} \) (4), we construct
Given any channel by employing Stinespring dilation \[^{[22]}\]. The Kraus operators \(\{F_i\}\) can be realized by a channel

\[
U(\bullet) := U \bullet U^\dagger, \quad U \in SU(d^2)
\]

with \(U\) acting on the system (s) qudit and an ancillary \(d\)-dimensional ancilla (a) qudit such that \(F_i = _s \langle i | U | 0 \rangle_a\). Such Kraus operators \(\{F_i\}\) trivially satisfy \(\{F_i^\dagger F_i\}\) being linearly independent and \(\sum_i F_i^\dagger F_i = 1\).

In principle the quantum circuit for an extreme channel could be constructed in three stages: solve the Kraus decomposition in Proposition\[^{[2]}\] then use the Kraus operators \(\{F_i\}\) to construct the unitary operator based on Stinespring dilation, and finally decompose it into a quantum circuit comprising gates from a finite universal instruction set. However, this method is stymied by the intractability of the nonlinear algebraic equations that arise from the Kraus decomposition so this approach is not viable.

Instead we adopt a different tack, which is to find the quantum circuit by optimization. In this approach, we construct the circuit for extreme channel as a sequence of instruction-set gates, and optimize over the set of circuits such that the diamond-norm distance \[^{[29]}\] (rather than the induced Schatten one-norm \[^{[22]}\]) between the input channel \(\mathcal{E}\) and the approximate channel \(\tilde{\mathcal{E}}\) satisfies

\[
\|\mathcal{E} - \tilde{\mathcal{E}}\|_d := \|\mathcal{E} \otimes 1 - \tilde{\mathcal{E}} \otimes 1\|_{1\to 1} \leq \epsilon.
\]

The diamond-norm distance is preferred as it gives worst-case gate error, and has the operational meaning that the probability of distinguishing between the two channels from their outputs is \(\frac{\epsilon}{2}\).

Next we present the single- and two-qudit gate set for this circuit construction. Three types of single-qudit gates are specified by the generalized Pauli gate \(X_{jk} = |j\rangle_k \langle k + |k\rangle \langle j|\), by the Givens rotation \(G_{jk}(\theta) = \cos \theta (|j\rangle_k \langle k + |k\rangle \langle j| + \sin \theta (|k\rangle_k \langle j - |j\rangle \langle k|\) \[^{[25]}\] and by the gate \(X_i\) from the Heisenberg-Weyl basis \((i, j, k \in \mathbb{Z}_d)\). Although the Givens rotation is parameterized by continuous \(\theta\), it can be decomposed efficiently into a sequence of single-qudit gates according to Solovay-Kitaev algorithms \[^{[35,37]}\].

We augment these gates by their two-qudit controlled counterparts \(CX_{jk} := |j\rangle_s \langle j| \otimes X_{jk}\) and \(CG_{jk}(\theta) := |j\rangle_s \langle j| \otimes G_{jk}(\theta)\) with the system as control, and \(CX_i := X_i \otimes |i\rangle_s \langle i|\) with the ancilla as control. We introduce a qudit multiplexer, which generalizes the qubit case \[^{[41]}\], as a sequence of two controlled Givens rotations \(M_{jk}(\alpha, \beta) := CG_{jk}(\alpha)CG_{kj}(-\beta)\) \(\text{(Fig.\ 1)}\), with the proof of the circuit equivalence following straightforwardly from the qubit case \[^{[39]}\].

**Proposition 4.** Given any \(\mathcal{V}(\bullet) := V \bullet V^\dagger\) and \(\mathcal{W}(\bullet) := W \bullet W^\dagger\) with \(V, W \in SU(d)\), any channel \(\mathcal{W}(tr_{d})\) \(\mathcal{V}\) is extreme provided that

\[
U := \prod_{i=d-1}^{1} CX_i \prod_{j=d-1}^{0} M_{jk}(\alpha_{jk}, \beta_{jk}),
\]

for all but a zero-measure subset of the rotation-angle sets \(\{\alpha_{jk}\}\) and \(\{\beta_{jk}\}\) with at most \((d^2 - d)/2\) elements per set.

**Proof.** We prove the theorem by showing that the partial trace of \(U(10)\) yields Kraus operators \(F_i = _a \langle i | U | 0 \rangle_a\) that satisfy the normalization and linear independence conditions of Prop.\[^{[2]}\]. To this end we define \(U'\) as a product of controlled-Givens rotations such that \(U = \prod_{i=d-1}^{1} CX_i U'\).

We define \(u_{d(i)} \in \mathbb{R}; i, \ell \in \mathbb{Z}_d\) such that \(U'(|0\rangle_a |\ell\rangle_s = \sum_{i=0}^{d-1} u_{d(i)} |\ell\rangle_a |i\rangle_s\). The unitary operator \(U'\) corresponds to a channel with diagonal Kraus operators \(\{E_i\}\) such that \(E_i |\ell\rangle_s = u_{d(i)} |\ell\rangle_s\) as

\[
U'(|0\rangle_a |\ell\rangle_s = \sum_{i=0}^{d-1} |i\rangle_a \langle i| U'| |0\rangle_a |\ell\rangle_s = \sum_{i=0}^{d-1} |i\rangle_a E_i |\ell\rangle_s.
\]

We can identify \(E_i\) in Eq. \((11)\) with \(E_i\) in Eq. \((4)\) by setting \(u_{d(i)} \equiv \bar{a}_{d(i)} e^{2\pi i \ell / d}\).

Reincorporating the gates \(CX_i\) yields

\[
U(|0\rangle_a |\ell\rangle_s = \sum_{i=0}^{d-1} |i\rangle_a X_i E_i |\ell\rangle_s = \sum_{i=0}^{d-1} |i\rangle_a F_i |\ell\rangle_s.
\]

A projection \(|i\rangle_a \langle i|\) on the ancilla corresponds to the action of \(F_i\) on the system. The angles \(\alpha_{jk}\) and \(\beta_{jk}\) can be chosen \(\text{(e.g., randomly)}\) to satisfy the linear independence of \(\{F_i, F_j\}\). This means the circuit \(U\) realizes the Kraus operators \(\{F_i\}\) for an extreme channel. As there are \(\left(\frac{3}{2}\right)^2 = (d^2 - d)/2\) multiplexers, the total number of independent parameters is consistent with Corollary\[^{[3]}\] \(\square\).

Now we describe the algorithm for the simulation of a general qudit channel. The algorithm accepts the dimension \(d\) of the Hilbert space, the description of a channel \(\mathcal{E}\) and an error tolerance \(\epsilon\) as input. The output is a quantum circuit and a bound \(\tilde{\epsilon}\) on the resultant circuit with respect to the actual channel \(\mathcal{E}\) being simulated.

Our algorithmic procedure is as follows. Prop.\[^{[1]}\] shows that any channel can be decomposed into a \(d\)-fold sum of
extreme channels, and we know from Prop. 4 a description of the circuit for any extreme channel, with rank $< d$ extreme channels being realized when some of the Kraus operators are zero. Thus, Prop. 4 and Prop. 3 together inform us that a quantum circuit for the qudit channel can be realized by choosing extreme-channel circuits randomly with each $i$th circuit chosen with probability $p_i$.

Our algorithm initially chooses a set of $d$ extreme channels randomly and tests whether the resultant guessed channel $\tilde{E}$ is within distance $\epsilon$ of the correct channel $E$. Typically the guessed channel fails to be within the error tolerance so we employ an optimization algorithm to pick a new set of parameters for $E$ and try again. This procedure is repeated until a satisfactory circuit is found. The procedure is aborted if the optimization routine fails to find a good circuit within a pre-specified number of trials.

We now determine the number of parameters in $E$ over which optimization takes place. From Prop. 1 we see that there are $d - 1$ parameters of $\{p_i\}$. The set of rotation angles $\{\alpha_{jk}, \beta_{jk}\}$ has $d^2 - d$ independent real parameters. The unitary matrices $V$ and $W$ in Prop. 2 could be constructed as products $V = \prod_i V_i$ and $W = \prod_j W_j$ with as many unitary operators in the two products as needed to provide enough parameters for the optimization to work. Let us refer to the total number of elements in the two products as $\kappa$.

As there are $d$ extreme channels and $d^2 - 1$ free parameters in SU($d$), we have $\kappa(d^2 - 1)$ free parameters associated with $V$ and $W$. We add this number of parameters to the number of parameters for $d$ extreme channels, namely $d(d^2 - d)$ with $d^2 - d$ the number of free rotation angles $\{\alpha_{jk}, \beta_{jk}\}$, and then add these to the number of free probabilities $\{p_i\}$. The total number of parameters for the approximate channel should satisfy the inequality $\kappa(d^2 - 1) + d(d^2 - 1) + (d - 1) \geq d^4 - d^2$ with the right-hand side corresponding to the number of parameters that specify the qudit channel. For the most efficient simulation, we minimize $\kappa$ so

$$\kappa = \left\lceil \frac{(d - 1)(d^2 + d + 1)}{d(d + 1)} \right\rceil.$$  \hspace{2cm} (13)

As an example, a qutrit channel has 72 parameters, but our optimization is not that case. In that case $d = 2$ so the channel $E$ has 12 parameters whereas the optimization of $E$ is over 17 parameters. We note that the optimization precludes an efficient circuit-design algorithm even in the qubit case, contrary to the earlier claim [27].

The final step for the algorithm is to construct the fitness function for the optimization problem. The fitness is inversely related to the distance between the correct channel, which is an input to the algorithm, and the computed circuit. Mathematically we represent the correct channel by the Choi-Jamiolkowski state $C$ [2], the approximate circuit by the state $C' = \sum_i p_i C_i'$, and their trace distance by $D_t(C, C')$, which bounds the $\cdot$-norm distance $\|E - E'\|_\sigma \leq \epsilon$. \hspace{2cm} (14)

The fitness of the optimization is $F := 1 - \frac{1}{4D_t(C, C')}$, whose value is 0 in the worst case and unity for $C' = C$. Optimization is achieved by finding the circuit with maximum fitness

$$F_{\text{max}} := 1 - \min_{\{p_i, C_i\}} \frac{D_t(C, C')}{d} \geq 1 - \frac{\epsilon}{2d}.$$  \hspace{2cm} (15)

which can be solved numerically to yield $\{p_i, C_i\}$. We have tested our numerical optimization algorithm for systems up to four dimensions successfully using the simple nonlinear programming method [33] on channels generated by partial trace of Haar-random-generated unitary operators on the dilated space [44].

Now that we have the classical circuit-design algorithm, we consider the time and space cost for the quantum circuit to simulate an extreme qudit channel $\tilde{E}^\circ$ on a quantum computer based on qubits and single- and two-qubit unitary gates. The extreme qudit channel is dilated to a unitary operator $U$ on two qudits, then represented by qubits and decomposed into a sequence of a small universal qubit instruction set such as the Hadamard, $T$ gate ($\pi/8$ gate) and controlled-NOT gates 22. This decomposition is achieved with the Solovay-Kitaev theorem and algorithms [32, 37] thereby realizing an approximate unitary operator $U$.

The circuit corresponding to $\tilde{U}$ yields an approximation $\tilde{E}^\circ$ to the desired extreme channel $E^\circ$. From the relations $\|E^\circ - \tilde{E}^\circ\|_{1 \rightarrow 1} \leq 2\|U - \tilde{U}\|$ 27 and $\|U - \tilde{U}\| = \|U \otimes I - \tilde{U} \otimes I\|$, we obtain

$$\|U - \tilde{U}\| \leq \frac{\epsilon}{2} \implies \|E^\circ - \tilde{E}^\circ\|_\sigma \leq \epsilon.$$  \hspace{2cm} (16)

From strong convexity and the chain property of trace distance, relations (14) and (16) above together ensure the desired simulation accuracy [9].

The extreme channel circuit operation $U$ can be decomposed into a product of $O(d^2)$ controlled-NOT gates and continuously-parameterized single-qubit gates 33 [34]. From the error tolerance $\epsilon$, which is an algorithmic input for circuit design, any single-qubit gate can be approximated by an $O\left(\log \frac{D^2}{\epsilon}\right)$ sequence of $T$ gates and Hadamard gates 35, 37. As a result, the number of elementary gates, hence computational time cost, of the extreme-channel circuit is $O\left(d^2 \log \frac{d^2}{\epsilon}\right)$, and the space cost is two qudits.

Finally simulating an arbitrary channel is implemented by probabilistically implementing different extreme channels according to the distribution $\{p_i\}$ [11]. The space and time costs of a single-shot implementation of the channel are one dit and two qudits for space and the classical time.
cost for generating the random dit plus $O\left(d^2 \log \frac{d}{\varepsilon}\right)$ quantum gates. In other words, the quantum computational cost for simulating a random qudit channel is the same as for simulating the extreme channel, and the additional cost is only classical: one dit plus running a random-number generator. This cost can be explained by recognizing that the qudit channel simulator is simply a randomized extreme channel simulator. On the other hand, estimating qudit observables accurately could require many shots, with the number of shots depending on the particular observable.

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