A numerical study of Poisson equations: steady heat distribution with one-point source

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Abstract. In this paper, Poisson equations are considered. Especially, equations which governed problems involving steady heat distribution with a point source. The heat distribution problems studied are over thin plates. Generally, these problems may not be solved analytically. Hence, a numerical method called Dual Reciprocity Method (DRM) is employed to solve the problems. Using this method, some of the numerical results are presented and discussed to investigate the influence of source positions to heat distribution over thin plates.

1. Introduction

Problems involving heat distribution over thin plates are some of the physical or engineering problems that are modeled mathematically. One of the purposes modeling the problems mathematically is to study the problems efficiently. To study the problems mathematically, a derivation of mathematical models from the problems is needed. Such derivation of heat distribution problems is presented by Haberman [1]. Resulting mathematical models of the heat distribution problems are then solved to obtained required solutions. To solve the model, we first use analytical method. However, this method may not be applied to solve any heat distribution problems. Hence, we need numerical methods to solve problems that may not be solved analytically.

One of the numerical methods that may be used to solve heat distribution problems is Dual Reciprocity Method (DRM), which is part of Boundary Element Methods (BEM). BEM has been used by researchers to solve various problems. Such researchers are Clements and Lobo [2], Solekhudin and Ang [3], and Yun and Ang [4]. These methods are used widely as these methods are flexible. These methods may be employed to solve problems over any domain bounded by simple closed curve. In this research, DRM is used to solve steady heat distribution over thin plates with a point source. Some numerical solutions obtained are presented and discussed to investigate the influence of source position to distribution of heat over the thin plates.

2. Problem formulation and basic equations

In this section, the mathematical model of steady heat distribution problems over thin plates is presented. A brief derivation of DRM for solving the problems is also presented. Steady heat distribution problems over a region $R$ bounded by a simple closed curve $C$ are governed by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + Q(x, y) = 0,$$

(1)
where \( u \) is the temperature, and \( Q \) is the source. For problems with a point source, the governing equation is

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + Q(x, y)\delta(x, y; a, b) = 0,
\]

(2)

where \((a, b)\) is the coordinate of the source, and \(\delta\) is a Dirac delta function with the source at \((a, b)\).

Equation (1) and Equation (2) may be solved numerically using DRM. To solve Equation (1) and Equation (2) using DRM, we first express their solutions in the form of boundary integral equations. The boundary integral equations for Equation (1) and Equation (2) are

\[
\lambda(\alpha, \beta)u(\alpha, \beta) = \int_C \left[ u(x, y) \frac{\partial}{\partial n} [U(x, y; \alpha, \beta)] - U(x, y; \alpha, \beta) \frac{\partial}{\partial n} [u(x, y)] \right] ds - \int_R U(x, y; \alpha, \beta) Q(x, y) dxdy,
\]

(3)

and

\[
\lambda(\alpha, \beta)u(\alpha, \beta) = \int_C \left[ u(x, y) \frac{\partial}{\partial n} [U(x, y; \alpha, \beta)] - U(x, y; \alpha, \beta) \frac{\partial}{\partial n} [u(x, y)] \right] ds - U(a, b; \alpha, \beta) Q(a, b),
\]

(4)

respectively. Here

\[
\lambda(\alpha, \beta) = \begin{cases} 
1, & (\alpha, \beta) \in R \\
\frac{1}{2}, & (\alpha, \beta) \text{ on the smooth part of } C 
\end{cases},
\]

and

\[
U(x, y; \alpha, \beta) = \frac{1}{2\pi} \sqrt{(x - \alpha)^2 + (y - \beta)^2}
\]

is the fundamental solution of two-dimensional Laplace’s equation.

From Integral equations (3) and (4), systems of linear algebraic equations

\[
\lambda^{(n)}u^{(n)} = \sum_{k=1}^{N} \left\{ u^{(k)} \int_{C^{(k)}} \frac{\partial}{\partial n} [U(x, y; x^{(n)}, y^{(n)})] ds(x, y) - u^{(k)} \int_{C^{(k)}} U(x, y; x^{(n)}, y^{(n)}) ds(x, y) \right\} - \sum_{k=1}^{N+L} \mu^{(nk)} Q^{(k)},
\]

\[
 n = 1, 2, \ldots, N + L
\]

(5)

and

\[
\lambda^{(n)}u^{(n)} = \sum_{k=1}^{N} \left\{ u^{(k)} \int_{C^{(k)}} \frac{\partial}{\partial n} [U(x, y; \alpha, \beta)] ds(x, y) - u^{(k)} \int_{C^{(k)}} U(x, y; \alpha, \beta) ds(x, y) \right\} - U(a, b; x^{(n)}, y^{(n)}) Q(a, b),
\]

\[
 n = 1, 2, \ldots, N + L
\]

(6)

are respectively derived. Here \( N \) is the number of segments or elements on boundary \( C \), \( C^{(1)}, C^{(2)}, \ldots, C^{(N)} \) are the segments satisfy \( C \cong C^{(1)} \cup C^{(2)} \cup \ldots \cup C^{(N)} \). Number \( L \) is the number of interior collocation points. Points \((x^{(1)}, y^{(1)}), (x^{(2)}, y^{(2)}), \ldots, (x^{(N)}, y^{(N)})\) are the midpoints of segments \( C^{(1)}, C^{(2)}, \ldots, C^{(N)} \), respectively. Points \((x^{(N+1)}, y^{(N+1)}), (x^{(N+2)}, y^{(N+2)}), \ldots, (x^{(N+L)}, y^{(N+L)})\), are the interior collocation points.
\[ \begin{align*}
\chi^{(n)} &= \chi(x^{(n)}, y^{(n)}), \\
u^{(n)} &= u(x^{(n)}, y^{(n)}), \\
u_{n}^{(n)} &= \frac{\partial u}{\partial n}(x,y) = (x^{(n)}, y^{(n)}), \\
Q^{(n)} &= Q(x^{(n)}, y^{(n)}),
\end{align*} \]

and

\[\mu^{(nk)} = \sum_{m=1}^{N+L} \Psi^{(nm)} \omega^{(mk)}.\]

Notation \( \Psi^{(nm)} \) and \( \omega^{(mk)} \) are defined as

\[\Psi^{(nm)} = \chi(a^{(n)}, b^{(n)}) \chi(a^{(n)}, b^{(n)}; a^{(m)}, b^{(m)}) + \int_{C} [U(x, y; a^{(n)}, b^{(n)}) \frac{\partial}{\partial n} [\chi(x, y; a^{(m)}, b^{(m)})] - \chi(x, y; a^{(m)}, b^{(m)}) \frac{\partial}{\partial n} [U(x, y; a^{(n)}, b^{(n)})] ] ds, \]

and

\[\sum_{k=1}^{N+L} \omega^{(mk)} \rho^{(kl)} = \begin{cases} 1, & \text{if } k = m \\ 0, & \text{if } k \neq m \end{cases}, \]

Where

\[\rho^{(kl)} = 1 + r^{2} (x^{(k)}, y^{(k)}; x^{(l)}, y^{(l)}) + r^{3} (x^{(k)}, y^{(k)}; x^{(l)}, y^{(l)}), \]

And

\[\chi(x, y; a^{(m)}, b^{(m)}) = \frac{1}{4} r^{2} (x, y; a^{(m)}, b^{(m)}) + \frac{1}{16} r^{4} (x, y; a^{(m)}, b^{(m)}) + \frac{1}{25} r^{5} (x, y; a^{(m)}, b^{(m)}). \]

Function \( r \) is defined as

\[r(x, y; a, b) = \sqrt{(x-a)^{2} + (y-b)^{2}}. \]

Systems of linear algebraic equations (5) and (6) may be solved to obtain numerical solutions at collocation points. Using these solutions, numerical solution at \((\eta, \xi) \in \mathbb{R} \cup \mathbb{C}\) may be obtained.

3. Results and discussion

In this section, the method presented in the preceding section is employed to solve problems involving Poisson equations. The first problem is a problem with the analytic solution. The purpose of presenting this problem is to investigate the accuracy of the method. The second problem is a problem without analytic solution. This problem is a problem involving steady heat distribution over thin plates with a point source.

3.1. Problem with the analytic solution

We consider a problem involving Poisson equation

\[\frac{\partial^{2} u}{\partial x^{2}} + \frac{\partial^{2} u}{\partial y^{2}} - 2(x^{2} + y^{2}) = 0, \quad (7)\]

defined over a square region with boundary conditions presented in Figure 1.
Figure 1. Region and boundary conditions of Equation (7).

Analytic solution of Equation (7) subject to boundary conditions in Figure 1 is

\[ u = x^2 y^2. \]

To solve the problem using the DRM, a number of elements and interior points are needed. We consider two sets of elements and interior points, namely Set A and Set B. In Set A, the number of elements is 80 and the number of interior collocation points is 81. In Set B, the numbers of elements and interior collocation points are 200 and 196, respectively. Some of the numerical results are presented in Table 1.

| Point      | Analytic | Set A       | Set B       | Absolute Error Set A | Absolute Error Set B |
|------------|----------|-------------|-------------|-----------------------|-----------------------|
| (0.2,0.2)  | 0.001600 | 0.001528    | 0.001585    | 0.000072              | 0.000015              |
| (0.2,0.8)  | 0.025600 | 0.025419    | 0.025552    | 0.000181              | 0.000048              |
| (0.4,0.4)  | 0.025600 | 0.025324    | 0.025540    | 0.000276              | 0.000060              |
| (0.6,0.6)  | 0.129600 | 0.128926    | 0.129459    | 0.000674              | 0.000141              |
| (0.8,0.2)  | 0.025600 | 0.025419    | 0.025552    | 0.000181              | 0.000048              |
| (0.8,0.8)  | 0.409600 | 0.408046    | 0.409288    | 0.001554              | 0.000312              |

Table 1 shows numerical solutions at selected points obtained using the DRM with Set A and Set B. The corresponding analytic solutions are also presented in the table. It can be seen that numerical solutions obtained using the DRM are in good accuracy. It seems that the absolute errors of the numerical solutions obtained using Set A and Set B are less than 0.002 and 0.0004 respectively. It can also be seen that Set B results in better accuracy than Set A. Hence for the problems without analytic solutions, the number of elements and interior points is about the same as that in Set B.

3.2. Steady heat distribution with a point source

In this part, we apply the DRM to solve a steady heat distribution problem from a point source. Since this study is a qualitative study, dimensions are omitted. We are given a thin plate of the form of a square. The temperature of its sides is maintained at a temperature of 0. A source with temperature of 100 is placed at a point on the plate. Let \((a, b)\) be the point of source. The problem described is governed by

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 100\delta(x, y; a, b) = 0,
\]  

(8)
subject to boundary conditions described in Figure 2.

![Figure 2](image1.png)

**Figure 2.** Region and boundary conditions of steady heat distribution problem with a point source.

In this part, we consider two cases. In the first case, the source is placed at point \((0.01,0.5)\). In the second case, point \((0.5,0.5)\) is the point of source. The DRM is implemented with 200 elements and 225 interior collocation points. Some of the results are presented in Figure 3 and Figure 4, and Table 2.

![Figure 3](image2.png)  ![Figure 4](image3.png)

**Figure 3.** Distribution of temperature on the plate for source at point \((0.01,0.5)\).

**Figure 4.** Distribution of temperature on the plate for source at point \((0.5,0.5)\).

Figure 3 and Figure 4 show the distribution of temperature on the plate. As can be seen in the figures, the maximum temperature when the source placed at \((0.01,0.5)\) is less than 18. For the source located at \((0.5,0.5)\), the maximum temperature is more than 60. These results imply that source placed in the middle of the plate produces higher maximum temperature on the plate than that placed near the boundary. The total amount of temperature on the plate is given in Table 2.
Table 2. The total temperature on the plates.

| Source at (0.01,0.5) | Source at (0.5,0.5) |
|----------------------|---------------------|
| Total temperature    | 34.157              | 736.923            |

The total amount of temperature resulted by placing the source at point (0.01,0.5) and (0.5,0.5) is presented in Table 2. These values are computed numerically using numerical solutions at 99 × 99 equally spaced points on the plate. The total amount of temperature is computed using formula

\[
(0.01)^2 \left( \sum_{i=1}^{99} \sum_{j=1}^{99} u(i,j) + \sum_{i=1}^{99} u(i,1) + \sum_{j=1}^{99} u(1,j) + u(99,99) \right),
\]

where \( u(i,j) \) is the numerical value of \( u \) at point \((0.01i, 0.01j)\).

From the results presented in Table 2, the total amount of temperature on the plate with the source at (0.01,0.5) is about 34.157. This amount is relatively small compared to that on the plate with the source at (0.5,0.5), which is about 736.923. These results imply that the source placed at the middle of the plate results in a higher amount of temperature on the plate than that placed at other locations, especially at location near the boundary. For the cases considered in this paper, the total amount of temperature of plate with the source at (0.5,0.5) is about 21 times higher than that of the plate with the source at (0.01,0.5).

4. Concluding remarks
Problems involving the Poisson equation have been solved numerically using a DRM. Numerical solutions obtained using the DRM are accurate with the corresponding analytic solutions. The problem involving steady heat distribution with a point source is also solved numerically using the DRM. Source located in the middle of the plate results in higher maximum temperature and total amount of temperature on the plate.

5. References
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