Crossing Number for Graphs with Bounded Pathwidth

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Abstract

The crossing number is the smallest number of pairwise edge crossings when drawing a graph into the plane. There are only very few graph classes for which the exact crossing number is known or for which there at least exist constant approximation ratios. Furthermore, up to now, general crossing number computations have never been successfully tackled using bounded width of graph decompositions, like treewidth or pathwidth.

In this paper, we for the first time show that crossing number is tractable (even in linear time) for maximal graphs of bounded pathwidth 3. The technique also shows that the crossing number and the rectilinear (a.k.a. straight-line) crossing number are identical for this graph class, and that we require only an \( O(n) \times O(n) \)-grid to achieve such a drawing.

Our techniques can further be extended to devise a 2-approximation for general graphs with pathwidth 3, and a \( 4w^3 \)-approximation for maximal graphs of pathwidth \( w \). This is a constant approximation for bounded pathwidth graphs.

1 Introduction

The crossing number \( cr(G) \) is the smallest number of pairwise edge-crossings over all possible drawings of a graph \( G \) into the plane. Despite decades of lively research, see e.g. [26,27], even most seemingly simple questions, such as the crossing number of complete or complete bipartite graphs, are still open, cf. [24]. There are only very few graph classes, e.g., Petersen graphs \( P(3,n) \) or Cartesian products of small graphs with paths or trees, see [4,21,25], for which the crossing number is known or can be efficiently computed.

Considering approximations, we know that computing \( cr(G) \) is APX-hard [5], i.e., there does not exist a PTAS (unless P = NP). The best known approximation ratio for general graphs with bounded maximum degree is \( O(n^{0.9}) \) [10]. We only know constant approximation

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ratios for special graph classes. In fact, all known constant approximation ratios are based on one of three concepts: Topology-based approximations require that $G$ can be embedded without crossings on a surface of some fixed or bounded genus [14–17,18]. Insertion-based approximations assume that there is only a small (i.e., bounded size) subset of graph elements whose removal leaves a planar graph [6–9]. In either case, the ratios are constant only if we further assume bounded maximum degree. Finally, some approximations for the crossing number exist if the graph is dense [13].

While treewidth and pathwidth have been very successful tools in many graph algorithm scenarios, they have only very rarely been applied to crossing number: Since general crossing number seems not to be describable with second order monadic logic, Courcelle’s result [11] regarding treewidth-based tractability can only be applied if $cr$ itself is bounded [15,19].

The related strategy of “planar decompositions” lead to linear crossing number bounds [28].

Contribution. In this paper, we for the first time show that such graph decompositions, in our case pathwidth, can be used for computing crossing number. We show for maximal graphs $G$ of pathwidth 3 (see Section 3):

- We can compute the exact crossing number $cr(G)$ in linear time.
- The topological $cr(G)$ equals the rectilinear crossing number $\overline{cr}(G)$, i.e., the crossing number under the restriction that all edges need to be drawn as straight lines.
- We can compute a drawing realizing $\overline{cr}(G)$ on an $O(n) \times O(n)$-grid.

We then generalize these techniques to show:

- A 2-approximation for $cr(G)$ and $\overline{cr}(G)$ for general graphs of pathwidth 3, see Section 4.
- A $4w^3$-approximation for $cr(G)$ for maximal graphs of pathwidth $w$, see Section 5. This can be achieved by placing vertices and bend points on a $4n \times wn$ grid.

Observe that in contrast to most previous results, these approximation ratios are not dependent on the graph’s maximum degree. As a complementary side note, we show (in the full version of the paper, see [1]) that the weighted (possibly rectilinear) crossing number is weakly NP-hard already for maximal graphs with pathwidth 4.

Focusing on graphs with bounded pathwidth may seem very restrictive, but in some sense these are the most interesting graphs for crossing minimization because Hliněný showed that crossing-number critical graphs have bounded pathwidth [10].

2 Preliminaries

We always consider a simple undirected graph $G$ with $n$ vertices as our input. A drawing of $G$ is a mapping $\varphi$ of vertices and edges to points and simple curves in the plane, respectively. The curve $\varphi(e)$ of an edge $e = (u,v)$ does not pass through any point $\varphi(w)$, $w \in V(G)$, but has its ends at $\varphi(u)$ and $\varphi(v)$. When asking for a crossing minimum drawing of $G$, we can restrict ourselves to good drawings, which means that adjacent edges do not cross, non-adjacent edges cross at most once, and no three edges cross at the same point of the drawing. For other drawings, straightforward redrawing arguments, see e.g. [26], show that the crossing number can never increase when establishing these properties.

A clique is a complete graph and a biclique is a complete bipartite graph. While the exact crossing number is unknown for general cliques and bicliques, there are upper bound constructions, conjectured to attain the optimal value. In particular the old construction due to Zarankiewicz, attaining $\left\lfloor \frac{n_1}{2} \right\rfloor \left\lfloor \frac{n_1-1}{2} \right\rfloor \left\lfloor \frac{n_2}{2} \right\rfloor \left\lfloor \frac{n_2-1}{2} \right\rfloor$ crossings for $K_{n_1,n_2}$, is known to give the optimum for $n_1 \leq 6$ [20].
A prominent variant of the traditional (“topological”) crossing number \(cr(G)\) is the \(rectilinear\) crossing number \(\overline{cr}(G)\), sometimes also known as geometric or straight-line crossing number. Thereby, edges are required to be drawn as straight line segments without any bends. Interestingly, while we know \(\overline{cr}(G) > cr(G)\) in general (e.g., already for complete graphs), Zarankiewicz’s construction is a straight-line drawing, suggesting that maybe \(cr(G) = \overline{cr}(G)\) for bicoliques.

**Alternating path decompositions and clusters.** There are several equivalent definitions of pathwidth; we use here the one based on tree decompositions, see e.g. [22]. A path decomposition \(P\) of a connected graph \(G\) consists of a finite set of bags \(\{X_i \mid 1 \leq i \leq \xi \in \mathbb{N}\}\), where each bag is a subset of the vertices of \(G\), such that for every edge \((v, w)\) at least one bag contains both \(v\) and \(w\), and for every vertex \(v\) of \(G\) the set of bags containing \(v\) forms an interval (i.e., the underlying graph formed by the bags is a path). The indexing of the bags gives a total ordering and we may speak of first, last, preceding, and succeeding bags. The width of a path decomposition is the maximum cardinality of a bag minus one, i.e., \(\max_{1 \leq i \leq \xi} |X_i| - 1\). The pathwidth \(w := w(G)\) of \(G\) is the smallest width that can be achieved by a path decomposition of \(G\). A maximal pathwidth-\(w\) graph is a graph of pathwidth \(w\) for which adding any edge increases its pathwidth. In particular, this implies that the vertices in each bag form a clique. We assume that \(n > w + 1\); otherwise \(G\) is a clique and the crossing number is 0 for \(w = 3\) and easily approximated within a factor of \(O(1)\) for bigger \(w\) (e.g., via the crossing lemma [23]).

Several additional constraints can be imposed on the bags and the path decomposition without affecting the required width. We use a variant of a nice path decomposition that we call an alternating path decomposition (see Fig. 1); one can easily show that such a decomposition exists:

- There are exactly \(\xi = 2n - 2w - 1\) bags.
- \(|X_i| = w + 1\) if \(i\) is odd and \(|X_i| = w\) if \(i\) is even.
- For any even \(1 < i < \xi\), we have \(X_{i-1} \supset X_i \subset X_{i+1}\).

Note that for any odd \(i\) there is exactly one vertex \(v\) that is in \(X_i\) but not in bag \(X_{i+1}\). We say that \(v\) is forgotten by bag \(X_{i+1}\). Similarly, bag \(X_i\) contains exactly one vertex \(v\) that was not in bag \(X_{i-1}\). We say that \(v\) is introduced by bag \(X_i\). We define the age-order \(\{v_1, \ldots, v_\xi\}\) of the vertices of \(G\) as follows: \(v_1\) is forgotten by \(X_2\); \(v_2, \ldots, v_{w+1}\) are the other vertices of bag \(X_1\) in arbitrary order. The order of the remaining vertices corresponds to the order of the bags by which they are introduced. We say that \(v_i\) is older than \(v_j\) if \(i < j\), so the three oldest vertices are \(v_1, v_2, v_3\). Note that we can choose \(v_2, v_3\) arbitrarily among \(X_1 - \{v_1\}\). In particular, if two vertices \(p, q \in X_1\) are specified, then we can ensure that they are among the three oldest; this will be exploited in Section 4.2.

In our algorithms and proofs, we will work with special subsets of bags called clusters. Let \(G\) be a connected graph of pathwidth 3 with an alternating path decomposition \(P = \{X_i\}_{1 \leq i \leq \xi}\). Consider a set of three vertices \(Y\) that constitute at least one bag (this bag has an even index). There can be several such bags with exactly those vertices, but all bags containing \(Y\) are consecutive. For any such \(Y\), we define a cluster \(C\) as the maximal consecutive set of bags that all contain \(Y\). We say that \(T(C) := Y\) is the anchor-triplet of \(C\). Any cluster has at least 3 bags. They alternate between size 4 and 3, starting and ending with size-4 bags. Two consecutive clusters overlap in exactly one bag (which consequently has size 4). The order of the bags induces a unique order of the clusters \(\{C_1, \ldots, C_\xi\} := C\).

Note that a cluster \(C\) can be described as a set of bags, or by its anchor-triplet. Denote the vertices that appear in the union of bags of \(C\) by \(V(C)\), and let \(n(C) := |V(C)|\). The
We define the emerging vertex of \( C_i \), denoted by \( x^+_i \), as the vertex introduced by the last bag of \( C_i \). Note that \( x^+_i \) belongs to the anchor-triplet of the next cluster \( C_{i+1} \) if \( i < \kappa \). We define the lost vertex of \( C_i \), denoted by \( x^-_i \), as the vertex that was forgotten by the second bag of \( C_i \). Note that \( x^-_i \) belongs to the anchor-triplet of the previous cluster \( C_{i-1} \) if \( i > 1 \), but not to the anchor-triplet of \( C_i \). Observe that \( x^-_1 = v_1 \), \( x^+_1 = v_0 \), \( x^-_{i+1} \neq x^+_i \) and \( T(C_i) = T(C_{i-1}) \cup \{ x^+_{i-1}, x^-_{i+1} \} \) for all \( 2 \leq i \leq \kappa \). For notational simplicity, we define \( x^+_0 := v_2 \). Any vertex \( x \) that belongs to \( C_i \) but is not in \( T(C_i) \cup \{ x^+_1, x^-_1 \} \) is called a singleton of \( C_i \). Vertex \( x \) belongs to a “middle” bag of \( C_i \) and only appears in this bag; it belongs to no cluster other than \( C_i \). See Fig. 1 for an example.

### 3 Exact Algorithm for Maximal Pathwidth-3 Graphs

Let \( G \) be a maximal pathwidth-3 graph and fix an alternating path decomposition of width 3. By maximality, all bags form cliques, and in particular, each anchor-triplet induces a triangle in the graph, called anchor triangle consisting of anchor edges.

The general idea to draw \( G \) is to iterate through the clusters \( C_1, \ldots, C_\kappa \). When considering cluster \( C_i \), its first bag will already be drawn and the anchor triangle will form the outer face of the current drawing. About half of the vertices introduced by \( C_i \) will be drawn inside the anchor triangle while the other half will be drawn outside, mimicking Zarankiewicz’ construction locally. The number of crossings that these vertices add will be exactly the minimum number of crossings needed to draw the biclique \( K_{3,n(C_i)-3} \) of cluster \( C_i \), hence leading to an optimal drawing.

We start with drawing bag \( X_1 = \{ v_1, v_2, v_3, v_4 \} \) as a planar drawing of \( K_4 \) with the vertices \( T(C_1) = X_2 = \{ v_2, v_3, v_4 \} \) on the outer face. Now we iterate over all clusters \( C_i \), \( 1 \leq i \leq \kappa \), drawing their bags with the following invariants:

- The drawing is good and straight-line.
- Before drawing \( C_i \), the outer face contains the three vertices \( T(C_i) \).
- For any \( j \leq i \), the anchor edges of \( C_j \) are drawn without crossings.

Let \( \ell \) be the number of singleton vertices in \( C_i \) (possibly \( \ell = 0 \)). We need to place the \( \ell \) singletons and the emerging vertex \( x^+_i \). We will add \( \ell_i := \lceil (\ell + 1)/2 \rceil \leq \ell \) vertices into an
inner face of the current drawing and \( \ell_2 = \lceil (\ell + 1)/2 \rceil \geq 1 \) vertices on the outside. Note that \( \ell_1 + \ell_2 = \ell + 1 \).

**Placement on the inside.** By the invariant the outer face consists of the edges connecting \( T(C_i) = \{x_{i-1}^+, p, q\} \) for some \( p, q \). \( \text{W.l.o.g.} \) assume that \( x_{i-1}^+, p, \) and \( q \) occur in clockwise order walking along the outer face. By maximality, and because \( x_{i-1}^+ \) has just been introduced, \( x_{i+1}^- \) has degree 3 in the current graph, and its neighbors are \( p, q, x_{i-1}^- \).

Let \( R \) be the open region obtained by the intersection of three open “wedges” \( W_1, W_2, W_3 \) defined as follows: Wedge \( W_1 \) emanates from \( x_{i-1}^+ \) between edges \((x_{i-2}^+, p)\) and \((x_{i-1}^+, x_{i-1}^-)\) in the interior of the triangle induced by \( T(C_i) \). Wedge \( W_2 \) \((W_3)\) emanates at \( p \) \((q)\) inside of \( T(C_i) \) and runs along edge \((p, x_{i-1}^-)\) \((q, x_{i-1}^-)\) respectively with a sufficiently small angle such that it crosses only edges incident to \( x_{i-1}^- \). Any point inside \( R \) can be connected to all of \( p, q, x_{i-1}^- \), with straight lines and a single crossing (with edge \((x_{i-1}^+, x_{i}^-)\)).

Consider a straight line \( s \) through \( R \) but not through any of \( p, q, x_{i-1}^- \). Place \( \ell_1 \) vertices (for \( \ell_1 \) singletons of \( C_i \)) along \( s \) within \( R \), and connect each of them to all of \( p, q, x_{i-1}^- \). All generated crossings are with edge \((x_{i-1}^+, x_{i}^-)\) or among the added edges. The drawing is straight-line and good (no three edges cross in a point), and the number of added crossings is \( \ell_1 + \lceil \ell_2/2 \rceil = \frac{3}{2} \ell_1 (\ell_1 + 1) \).

**Placement on the outside.** The outer face of the drawing is still formed by the edges connecting \( T(C_i) \), since all vertices from the paragraph above were added inside \( R \) and thus in the interior of \( T(C_i) \). We know that the vertex \( x_{i+1}^- \) in \( T(C_i) \) will be lost in the next cluster \( C_{i+1} \) (if there is any); it will play a prominent role now. Since we may or may not have \( x_{i+1}^- = x_{i-1}^+ \), we label the vertices of \( T(C_i) \) as fresh as \( \{x_{i+1}^+, p', q'\} \).

Define an open wedge \( W \) in the exterior of \( T(C_i) \) emanating from \( x_{i+1}^- \) between the extensions of the edges \((p', x_{i+1}^-)\) and \((q', x_{i+1}^-)\) beyond \( x_{i+1}^- \). Any point inside \( W \) can be connected via straight lines to all of \( p', q' \) without any crossings. Consider a straight line \( s' \) through \( W \), not through any of \( x_{i+1}^+, p', q' \), and crossing \((p', q')\). Now place \( \ell_2 \) vertices along \( s' \) within \( W \), and connect all of them to all of \( x_{i+1}, p', q' \) via straight lines. All generated crossings are among the added edges. The drawing is still straight-line and good, and the number of added crossings is \( \ell_2/2 \). The outer face of the resulting drawing is again a triangle with two corners being \( p' \) and \( q' \) and the third corner being a vertex that was added on \( s' \). We assign this latter vertex the role of the emerging vertex \( x_{i+1}^- \); the other inserted vertices are the necessary singletons. With this, the invariant holds since \( T(C_{i+1}) = T(C_i) \cup \{x_{i+1}^+\} \setminus \{x_{i+1}^-\} \).

This finishes the description of the drawing algorithm. We claim that the final drawing has the minimum possible number of crossings: We first give an upper bound on the number
of crossings that we achieve, and then show that any drawing requires this number.

\textbf{Lemma 2.} The above algorithm produces at most \( \sum_{i=1}^{n} \frac{1}{2} (n(C_i) - 3) \) \( \frac{1}{2} (n(C_i) - 4) \) crossings.

\textbf{Proof.} The algorithm started with a planar drawing of \( K_4 \). We argued above that the \( i \)-th iteration (drawing \( C_i \), which contains \( \ell \) singletons) added

\[ \frac{1}{2} \ell_1 (\ell_1 + 1) + \frac{1}{2} \ell_2 (\ell_2 - 1) = \frac{1}{2} (\ell + 1) \frac{1}{2} (\ell + 2) \]

crossings, where \( \ell_1 = \lceil (\ell + 1)/2 \rceil \) and \( \ell_2 = \lfloor (\ell + 1)/2 \rfloor \). Finally, observe that \( \ell = n(C_i) - 5 \) since all vertices of \( C_i \) except \( T(C_i) \cup \{ x^+_i, x^-_i \} \) are singletons.

\textbf{Lemma 3.} Any good drawing of \( G \) requires at least \( \sum_{i=1}^{n} [\frac{1}{2} (n(C_i) - 3)] [\frac{1}{2} (n(C_i) - 4)] \) crossings.

\textbf{Proof.} From Observation 1 we know that each cluster \( C_i \) contains a biclique \( B(C_i) := K_{3, n(C_i) - 3} \). By Zarankiewicz’s formula, \( K_{3, m} \) needs \( \lceil m/2 \rceil \lceil (m - 1)/2 \rceil \) crossings in any drawing. Thus, within each cluster we only introduce the optimal number of crossings.

However, we must argue that it is impossible for one crossing to belong to two or more clusters in an optimal drawing. This holds because nearly all of \( V(C_i) \) does not belong to other clusters. More precisely, assume \( C_j \) shares vertices with \( C_i \); we may assume \( j < i \). Then all common vertices must appear in the first bag \( X = T(C_i) \cup \{ x^-_i \} \). However, only three edges of those induced by \( X \) are in \( B(C_i) \), and all three of them are incident to \( x^-_i \). Since adjacent edges do not cross in a good drawing, no crossing can be shared between \( B(C_i) \) and \( B(C_j) \).

\textbf{Theorem 4.} There is a linear time algorithm to compute the exact crossing number \( cr(G) \) of any maximal pathwidth-3 graph \( G \). Furthermore, \( cr(G) = \pi(G) \), and the algorithm gives rise to a straight-line drawing where the anchor edges are not crossed.

\textbf{Proof.} Optimality follows from Lemmas 2 and 3. The second part of the claim follows from the first and third invariant in the above algorithmic description. It remains to argue linear running time. Computing a path decomposition of width 3 (if it exists) can be done in linear time. This path decomposition can be turned into an alternating path decomposition in linear time as well. On it we compute \( cr(G) \) as the sum in Lemma 2 in linear time.

Assume we are interested in the drawing achieving this solution. The drawing algorithm uses \( O(n) \) operations, but this does not immediately imply linear time, since coordinates may become very small. We also cannot list all crossings, as there can be \( \Theta(n^2) \) many. If, however, we are careful about how to place anchor-triplets, then singletons can be inserted while keeping all vertices at grid-points of an \( O(n) \times O(n) \)-grid, and thus we require only linear time to compute and output the drawing. Details are given in the full version of the paper [1, Appendix B]. We summarize:

\textbf{Theorem 5.} Every maximal pathwidth-3 graph on \( n \) vertices has a crossing-minimum drawing that is good, straight-line, and lies on a \( 28n \times 29n \)-grid. It can be found in \( O(n) \) time.

\section{4 Approximation Algorithm for Pathwidth-3 Graphs}

We now give an algorithm that draws graphs of pathwidth 3 (not necessarily maximal) such that the number of crossings is within a factor of 2 of the optimum. Roughly speaking, if the
graph is 3-connected (technically, we will define a slightly weaker assumption 3-traceable),
then the algorithm for maximal pathwidth-3 graphs is applied, and the number of crossings
is within a factor of 2. If the graph is not 3-traceable, then it can be split and the arising
subdrawings can be “glued” together without increasing the approximation ratio.

4.1 3-traceable graphs

We first analyze graphs that satisfy a condition that is weaker than 3-connectivity. Define a
non-anchor vertex to be a vertex that occurs in exactly one bag. Those are exactly $v_1$, $v_n$,
and all the singletons defined earlier.

Definition 6 (3-traceable graph). A graph $G$ with an alternating path decomposition $\mathcal{P}$ of
width 3 is 3-traceable if every non-anchor vertex has degree at least 3, and for all $1 \leq i \leq \kappa$,
edge $(x^+_{i-1}, x^-_{i})$ exists.

Assume we are given a 3-traceable graph $G$ with an alternating path decomposition $\mathcal{P}$ of
width 3. We can first maximize $G$ (obtaining $G'$) by adding all edges that have both ends in
one bag, but are not in $G'$ yet. We then apply the algorithm described in Section 3 to $G'$,
and finally delete the temporarily added edges again. We will show:

Lemma 7. Let $G$ be a 3-traceable graph. Then the algorithm of Theorem \[ gives a drawing of $G$
with at most $2cr(G)$ crossings.

We first give a sketch of the proof. The main challenge is that a cluster $C$ now does
not necessarily contain a biclique $K_{3,n(C)-3}$. However, we can argue that $G$ contains a
subdivision of $K_{3,n(C)-3}$ that uses mostly vertices of $C$, but “borrows” a non-anchor vertex
each (to play the role of $x_i^+$ and $x_i^-$) from the nearest preceding and succeeding cluster
that has such vertices. This subdivided $K_{3,n(C)-3}$ requires $cr(K_{3,n(C)-3})$ crossings. The
main work is then in arguing that these subdivided bicliques cannot overlap much, or more
precisely, that any crossing can belong to at most 2 of them. Lemma 7 then follows by
applying the upper bound given in Lemma 3.

As before, let $C_1, \ldots, C_\kappa$ be the clusters of $G$ with anchor-triplets $T(C_1), \ldots, T(C_\kappa)$, and
recall that we have an age-order $\{v_1, \ldots, v_n\}$.

There are three types of edges in $G$. Type I are edges that are incident to non-anchor
vertices. Type II are edges that have the form $(x^+_{i-1}, x^-_{i})$ for some $2 \leq i \leq \kappa$. Finally, Type
III are the remaining edges (they connect vertices of some anchor-triplet $T(C_i)$, $1 \leq i \leq \kappa$).

Observation 8. Consider a 3-traceable graph. For any $1 \leq i < j \leq \kappa$, there are three
vertex-disjoint paths $\Pi_{i,j}$ from $T(C_i)$ to $T(C_j)$ that are either single vertices or consist exactly
of the Type II edges $(x^+_{i-1}, x^-_{i})$ for $i < k \leq j$. Every non-anchor vertex attaches to the three
different paths $\Pi := \Pi_{1,n}$.

Proof. For any $1 \leq i < \kappa$, we have $T(C_{i+1}) = T(C_i) \cup \{x^+_{i+1}\} \setminus \{x^-_{i+1}\}$. By 3-traceability of $G$,
edge $(x^+_{i+1}, x^-_{i+1})$ exists and $\Pi_{i,i+1}$ consists of two paths of length 0 (the common vertices of
the triplets) and the third path being this edge. We obtain arbitrary $\Pi_{i,j}$ by extending $\Pi_{i,i+1}$
via $\Pi_{i+1,j}$. Since $G$ is 3-traceable, the non-anchor vertices have degree 3 and are adjacent to
the vertices of the anchor-triplet of their unique cluster; those lie on distinct paths of $\Pi$.

This shows that $G$ has $K_{3,n'}$ as a minor, where $n'$ is the number of non-anchor vertices.
Unfortunately this is not sufficient for crossing number arguments as contracting edges may
increase the crossing number. Instead, we will use the above structure to extract a subdivision
of $K_{3,n(C)-3}$ for each cluster $C$ in such a way that these bicliques do not overlap “much.”
Definition 9. Let $C_i$, $1 \leq i \leq \kappa$, be a cluster with at least one singleton. The cluster biclique of $C_i$, denoted $B(C_i)$, is a subdivision of $K_{3,n(C_i)−3}$ obtained as follows, cf. Fig. 3:

(a) The 3-side is formed by the three vertices of $T(C_i)$.
(b) Every singleton $w$ that belongs to $C_i$ (there are $n(C_i)−5$ of them) is one of the vertices on the side that will have $n(C_i)−3$ vertices. We know that $\deg(w) = 3$ by 3-traceability, and it is adjacent to all of $T(C_i)$ as required for the biclique.
(c) Let $i− < i$ ($i+ > i$) be maximal (minimal) such that cluster $C_{i−}$ ($C_{i+}$, respectively) has a non-anchor vertex; among its non-anchor vertices, let $w_− (w_+)$ be the youngest (oldest, respectively). If $i = 1$, we simply set $w_− := v_1$; if $i = \kappa$, we set $w_+ := v_n$. By Observation 8, we can establish three disjoint paths from $w_−$ and $w_+$ to $T(C_i)$. Hence, add $w_−$ and $w_+$ to the “big” side of $B(C_i)$. Observe that in either case, $w_−$ and $w_+$ are distinct from the singletons of $C_i$ and their paths to $T(C_i)$.

Lemma 10. Let $e_1, e_2$ be two edges of $G$ without common endpoint. There are at most two cluster bicliques that contain both $e_1$ and $e_2$.

Proof. We are done if at least one of $e_1$ and $e_2$ is of Type III, because then it belongs to no cluster biclique at all. Assume that one of $e_1$ and $e_2$ is of Type II, say $e_1 = (x_{i−1}, x_−)$ for some $2 \leq i \leq \kappa$. Edge $e_1$ may be used only for the cluster bicliques $B(C_{i−})$ and $B(C_{j+})$ where $j− < i$ ($j+ ≥ i$) is the maximal (minimal) index such that cluster $C_{j−}$ ($C_{j+}$, respectively) has singletons. The fact that $e_1$ belongs to at most two cluster bicliques proves the claim.

Finally, assume that both $e_1$ and $e_2$ are of Type I, i.e., incident to distinct non-anchor vertices, say $y_1 \in C_{i}$ and $y_2 \in C_{i'}$. Let $C' \subseteq C$ be the ordered subsequence of clusters that have at least one non-anchor vertex. A non-anchor vertex $x$ can belong to at most three cluster bicliques, refer to Definition 9 the one of its “own” cluster $C \in C'$, and those of the directly preceding and succeeding cluster in $C'$. Assume that $y_1$ and $y_2$ are in three cluster bicliques. If $i = i'$, $y_1$ and $y_2$ are singletons of different age in $C_i$, and the two clusters directly preceding and succeeding $C_i$ would have chosen distinct singletons of $C_i$, a contradiction. If $i \neq i'$, any overlap of three-element subsequences of $C'$ with distinct middle clusters has size at most 2, a contradiction.

Proof of Lemma 7 We know from Lemma 2 that the algorithm of Theorem 4 gives a drawing with at most $\sum_{C \in C} \frac{1}{2} (n(C)−3)\left\lfloor \frac{1}{2}(n(C)−4) \right\rfloor$ crossings. We need to consider only clusters $C$ that have at least one singleton; for any other cluster we have $n(C) = 5$.
and therefore its summand is 0. For any cluster $C$ that has a singleton, we have $B(C)$, a subdivision of $K_{3,3(n(C)-3)}$, which requires at least $\frac{1}{4}(n(C) - 3)\left\lfloor \frac{1}{2}(n(C) - 4) \right\rfloor$ crossings in any good drawing $\mathcal{D}$ of $G$. Any crossing in $\mathcal{D}$ is created by two edges without common endpoints, and by Lemma 10, any such pair belongs to at most two cluster bicliques. Hence any drawing of $G$ has at least $\frac{1}{2} \sum_{C \in \mathcal{C}} \left\lfloor \frac{1}{2}(n(C) - 3)\right\rfloor \left\lfloor \frac{1}{2}(n(C) - 4) \right\rfloor$ crossings, yielding the $2$-approximation.

### 4.2 General pathwidth-$3$ graphs

A pair of vertices $\{u, v\}$ of a $2$-connected graph $G$ is called a separation pair if $G - \{u, v\}$ is not connected. Assume that the pathwidth-$3$ graph $G$ is $2$-connected but not $3$-traceable. We will show that we can split the graph at separation pairs within anchor-triplets, draw the cut-components recursively, and merge them without introducing additional crossings. We start with a more general auxiliary statement whose proof is in [1, Appendix C].

- **Lemma 11.** Let $G$ be a $2$-connected graph with a separation pair $\{u, v\}$. Consider a partition of $G$ into two edge-disjoint connected subgraphs $H_1, H_2$ with $H_1 \cap H_2 = \{u, v\}$. Define $H_i^+ = H_i \cup \{(u, v)\}$ for $i = 1, 2$. Then $cr(H_1^+) + cr(H_2^+) \leq cr(G)$.

We will draw cut-components inside triangles bounded by their three oldest vertices.

- **Lemma 12.** Let $G$ be a $2$-connected graph with an alternating path decomposition $\mathcal{P}$ of width $3$. Then there exists an algorithm to create a straight-line drawing of $G$ with at most $2cr(G)$ crossings. All anchor-edges are drawn without crossings, and the three oldest vertices $\{v_1, v_2, v_3\}$ form the corners of the triangular convex hull of the drawing.

**Proof.** We prove the result by induction on the structure and size of the graph.

**Base case:** $G$ is $3$-traceable or a $K_4$. If $G = K_4$, the claim is obvious. Otherwise, we apply Lemma 2. However, the algorithm of Theorem 3 used therein grows the drawing “outwards”, while we would now like the oldest vertices to form the outer triangle. Thus we apply the algorithm for the reverse path decomposition; this makes (by suitably placing the last vertex) $T(C_1) = \{v_1, v_2, v_3\}$ the outer face and draws it as a triangle.

**Induction Step:** $G$ is neither $3$-traceable nor a $K_4$. For every non-anchor vertex $w \neq v_1$ of degree $2$, let $p_w, q_w$ be its adjacent anchor vertices. We can temporarily remove $w$ from $G$, ensure that the reduced graph contains edge $(p_w, q_w)$, draw the reduced graph, and—since $(p_w, w), (w, q_w)$ crossing-free close to the drawing of $(p_w, q_w)$. Similarly, we can remove $v_1$ if it has degree $2$: We can choose an age-order of the reduced graph $G'$ such that the neighbors of $v_1$ are among the three oldest vertices of $G'$ and hence draw $G'$ such that the neighbors of $v_1$ are on the outer-triangle; then $v_1$ can be reinserted on the outside to form the desired outer triangle. If the graph became $3$-traceable by these operations, we are done (base case). Otherwise, we can now assume that all non-anchor vertices have degree $3$.

Since $G$ is not $3$-traceable, $(x_{i-1}^+, x_i^-) \not\subseteq G$ for some $2 \leq i \leq \kappa$. There exists a unique bag $X_j$, the common bag of $C_{i-1}$ and $C_i$, that contains both $x_{i-1}^+$ and $x_i^-$. Let $p, q$ be the two other vertices in this bag, and observe that $T(C_{i-1}) = \{p, q, x_{i-1}^-\}$ while $T(C_i) = \{p, q, x_i^+\}$. Let $G_{\ell}$ be the graph induced by all vertices that appear in bags $\mathcal{P}_{\ell} := [X_1, X_{j-2}]$, and let $G_r$ be the graph induced by all vertices that appear in bags $\mathcal{P}_r := [X_{j+2}, X_\kappa]$. Any edge of $G$ appears in $G_{\ell}$ or $G_r$, since $\{x_{i-1}^-, x_{i-1}^+, x_i^-\}$ is the only vertex-pair that existed in bags of $\mathcal{P}$, but neither of $\mathcal{P}_{\ell}$ nor $\mathcal{P}_r$. Clearly, $\{p, q\}$ is a separation pair with $G_{\ell} \cap G_r = \{p, q\}$.

Define $G_{\ell}^- = G_{\ell} \cup \{(p, q)\}$ and $G_r^+ = G_r \cup \{(p, q)\}$. By the addition of edge $(p, q)$ (if it did not already exist), both graphs are $2$-connected. Apply induction to $G_{\ell}^+$ (with path
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decomposition $\mathcal{P}_v$ and $G_v^+$ (with the path decomposition $\mathcal{P}_v$). Since $p,q$ belong to the first bag of $\mathcal{P}_v$, we can ensure that they are among the three oldest vertices of $G_v^+$. We obtain two drawings $D_v^+, D_v^-$ in both of which $(p,q)$ is not crossed. We can insert (affinely transformed) $D_v^+, D_v^-$, which has $(p,q)$ on its bounding triangle, along $(p,q)$ in $D_v^+$ without additional crossings. Finally, we remove edge $(p,q)$ from the resulting drawing if $(p,q) \notin E(G)$.

By induction hypothesis, $cr(D_v^+) \leq 2cr(G_v^+)$ and $cr(D_v^-) \leq 2cr(G_v^-)$. By Lemma 11, $cr(G_v^+) + cr(G_v^-) \leq cr(G)$ and since the gluing gave no new crossings, the claim follows. ▷

We are now ready to establish the theorem for general pathwidth-3 graphs.

**Theorem 13.** Let $G$ be any pathwidth-3 graph. We have $\overline{cr}(G) \leq 2cr(G)$, and a linear time algorithm to create a good straight-line drawing of $G$ with at most $2cr(G)$ crossings.

**Proof.** (Sketch) If $G$ is 2-connected, then the result holds by Lemma 12. It is well known that $cr(G)$ is additive over the 2-connected components of $G$. When gluing at cut-vertices, the cut-vertex must be on the outer face of the drawing to be inserted into the other. We can achieve this while maintaining a straight-line drawing by choosing appropriate path decompositions; see [1] Appendix D. The running time follows as in Theorem 4. ▷

5 Approximation Algorithm for Graphs of Higher Pathwidth

We now study the crossing number of graphs that have pathwidth $w \geq 4$, and are maximal within this class. We give an algorithm to draw such graphs, and show that the number of crossings in the resulting drawing is within a factor of $4w^3$ of the crossing number. As opposed to Section 3, the drawings we create here are not straight-line drawings.

As before we assume that we have an alternating path decomposition $\mathcal{P} = \{X_i\}_{1 \leq i \leq \ell}$ of width $w$. We again use the age-order $\{v_1, \ldots, v_n\}$ of the vertices of $G$. Define $G_i$ to be the graph induced by vertices $v_1, \ldots, v_i$, and use $\deg_{G_i}(v)$ to denote the number of neighbors that $v$ has within graph $G_i$. For any $1 \leq i \leq n$, let the predecessors of vertex $v_i$ be those neighbors that are older. We will only use this concept for $i \geq w + 1$, which implies that $v_i$ has exactly $w$ predecessors by maximality of $G$. We enumerate them as $\{p^i_1, \ldots, p^i_w\}$ in age-order, with $p^i_1$ the oldest.

**Drawing algorithm.** We create a drawing of $G$ by starting with $G_{w+1}$ (the graph induced by $v_1, \ldots, v_{w+1}$) and then iteratively adding vertex $v_i$. We maintain the following invariants for the drawing of $G_i$ (see also Figure 4):

- Vertex $v_i$ is drawn at $(j, 0)$ for all $1 \leq j \leq i$.
- The drawing is contained in the half-space $\{(x,y): x \leq i\}$.
- All vertices $w$ in the bag introducing $v_i$ are bottom-visible, i.e., the vertical ray downward from $w$ does not intersect any edge.

We start by placing $v_1, \ldots, v_{w+1}$ at their specified coordinates, and draw the edges between them as half-circles above the $x$-axis. This satisfies the above invariants and gives rise to $\binom{w+1}{4}$ crossings since crossings are in 1-to-1-correspondence with subsets of 4 vertices.

Assume $G_{i-1}$ is drawn and consider $v_i$, for $i \geq w + 2$. Place $v_i$ as specified, i.e., to the right of all previous vertices and edges. Let $p^i_1, \ldots, p^i_w$ be the predecessors of $v_i$, all of which are bottom-visible by the invariant. We draw the edges to them using two different methods (and then redraw previous edges as a third step for each $i$). See also Figure 4:

- The edge to $p^i_1$ (the oldest predecessor) is routed counterclockwise around the drawing of $G_{i-1}$ until it is below but slightly to the left of $p^i_1$, from where it connects to $p^i_1$. We need no crossings, and all predecessors remain bottom-visible.
Figure 4 The construction for higher pathwidth: edge routings when adding vertex $v_i$.

- All other $w - 1$ edges incident to $v_i$ are routed together as a bundle from $v_i$ leftward below the drawing of $G_{i-1}$. This allows $v_i$ to be bottom-visible. Whenever the bundle is slightly to the right of some $p_k^i$, $w \geq k \geq 2$, one of the bundle’s lines (the lowest one) connects to $p_k^i$. The remaining bundle lines go counterclockwise around $p_k^i$, in its direct vicinity, until they are to the left of $p_k^i$ and below $G_{i-1}$. The bundle hence crosses every edge incident to $p_k^i$ in $G_{i-1}$, but no other edges, and $p_k^i$ remains bottom-visible. This drawing scheme continues until the last bundle line connects to $p_2^i$.

- Finally, we redraw the edges $(p_{k-1}^i, p_k^i)$ for $3 \leq k \leq w$: they exist by maximality. Both ends of any such edge are bottom-visible, so we can redraw it without crossing below the entire drawing, including the newly drawn edges from $v_i$. We remove the previous drawings of these edges and retain bottom-visibility of the vertices in the current bag.

In the full paper \cite{Biedl2017} Appendix E] we analyze the number of crossings and obtain:

\textbf{Theorem 14.} Let \( G \) be a maximal graph of pathwidth $w \geq 4$. The described algorithm runs in linear time and finds a drawing of $G$ with at most $2(w - 1)(w - 2)(2w - 4)\text{cr}(G) \leq 4w^3\text{cr}(G)$ crossings. In particular, for any constant pathwidth $w$, we have an $O(1)$-approximation of the crossing number. The drawing is poly-line on a $4n \times wn$ grid.

### 6 Conclusions and Open Questions

We have shown that the path decomposition of a graph can be used to efficiently compute or bound the crossing number of a graph. This is the first successful use of such graph decomposition for crossing numbers (besides the use of a tree decomposition in the special case that $\text{cr}(G)$ is bounded by a constant \cite{Biedl2015,Bodlaender1996}). Several interesting questions remain:

- Can we attain stronger approximation results for general pathwidth-3 graphs? The proven ratio of 2 may simply be due to a too weak lower bound, and we, in fact, do currently not know an instance where the algorithm does not obtain the optimum.

- Can we approximate $\text{cr}(G)$ for arbitrary (not maximal) pathwidth-$w$-graphs?

- In \cite{Biedl2017} we only showed weak NP-completeness for the weighted crossing number version on pathwidth-restricted graphs. Can this be strengthened to unweighted graphs? Finally, there is of course the question whether we can use the stronger tool of tree decompositions, instead of path decompositions, to achieve crossing number results.

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The weighted rectilinear crossing number problem asks: Given a graph $G = (V, E)$, edge weights $w: E \rightarrow \mathbb{N}_0^+$, and a threshold $K$, is there a straight-line drawing $D$ of $G$ such that

$$wcr(D) := \sum_{e_1, e_2 \in E, e_1 \text{ and } e_2 \text{ cross in } D} w(e_1) \cdot w(e_2) \leq K ?$$

In this section, we prove the following:

**Theorem 15.** The weighted and weighted rectilinear crossing number problems are weakly NP-hard already for (maximal) pathwidth-4 graphs that have non-weighted crossing number 1.

Our reduction is from Partition, defined as follows. Given $n$ positive integers $a_1, \ldots, a_n$ with $\sum_{i=1}^n a_i = 2S$, does there exist a $J \subset \{1, \ldots, n\}$ such that $\sum_{i \in J} a_i = S$. Given a Partition instance $\mathcal{I}$, define graph $G$ as described in the proof sketch as a $2n+2$-cycle $Q$ and $n$ chords $e_i = (x_i, y_i)$ with weight $a_i$ for $i = 1, \ldots, n$. We must show that $\mathcal{I}$ is a yes-instance if and only if $G$ has a straight-line drawing $D$ with $wcr(D) \leq S^2 - c$, where $c = \frac{1}{2} \sum_{i=1}^n a_i^2$ depends only on $\mathcal{I}$.

Assume first that there exists some $J \subset \{1, \ldots, n\}$ with $\sum_{i \in J} a_i = S$. Figure 5 shows how to create a straight-line drawing of $G$: Place vertices $x_1, \ldots, x_n$ on the left legs of an X-shape, and vertices $y_1, \ldots, y_n$ on the right legs of the X, using the upper/lower leg depending on whether $i \in J$. With the help of $x_0$ and $y_0$, the cycle can then be completed without crossing.

Consider a pair $i, j$ with $i \in J$ and $j \not\in J$. Then $e_i$ is drawn between the two upper legs of the X (hence inside $Q$) and while $e_j$ is drawn between the two lower legs of the X (hence outside $Q$), which means that they cannot cross. Also no edge of $Q$ has a crossing. In consequence, the number of crossings is at most

$$\sum_{i, j \in J} a_i \cdot a_j + \sum_{i, j \not\in J} a_i \cdot a_j = \frac{1}{2} \left( \sum_{i \in J} a_i^2 - \left( \sum_{i \not\in J} a_i^2 \right) \right) + \frac{1}{2} \left( \sum_{i \not\in J} a_i^2 - \left( \sum_{i \in J} a_i^2 \right) \right)$$

$$= \frac{1}{2} \left( S^2 - \left( \sum_{i \not\in J} a_i^2 \right) \right) + \frac{1}{2} \left( S^2 - \left( \sum_{i \in J} a_i^2 \right) \right) = S^2 - c$$

as desired.

For the other direction, assume that we have a straight-line drawing $D$ of $G$ with $wcr(D) \leq S^2 - c$. Since $c > 0$, no edge of $Q$ can have a crossing. Define $J$ to be the indices
of all those edges \( e_i \) that are drawn inside \( Q \). Any two such edges must cross each other, since the order of their endpoints is interleaved on \( Q \). Likewise, any two edges \( e_i, e_j \) with \( i, j \notin Q \) must cross each other. In consequence, we have

\[
\text{wcr}(D) \geq \sum_{i,j \notin J} a_i \cdot a_j + \sum_{i,j \notin J} a_i \cdot a_j = \frac{1}{2} \left( \sum_{i \in J} a_i \right)^2 + \frac{1}{2} \left( \sum_{i \notin J} a_i \right)^2 - c
\]

Define \( d = \sum_{i \in J} a_i - S = S - \sum_{i \notin J} a_i \) (note that \( d \) could be positive or negative). Then

\[
\text{wcr}(D) \geq \frac{1}{2} (S - d)^2 + \frac{1}{2} (S + d)^2 - c = S^2 + d^2 - c.
\]

But we assumed \( \text{wcr}(D) \leq S^2 - c \), which implies \( d^2 = 0 = d \) and hence \( \sum_{i \notin J} a_i = S \) as desired.

### B Proof of Theorem \[5\]

We explain how to place points for the algorithm in Section 3 so that the resulting drawing has linear coordinates. This involves a paradigm-shift in explaining how the drawing is created. In Section 3, we added vertices from the point of view of adding cluster \( C_i \). This added half of the singletons near \( (x_i^-, x_i^+ - 1) \), and the other half near \( (x_i^+, x_i^+ + 1) \). We now change this around, and describe the algorithm in terms of all those singletons (coming from both \( C_i \) and \( C_{i-1} \)) that need to be added near one edge \( (x_i^-, x_i^+ - 1) \). Let there be \( s_i \) such singletons (in terms of the notation of Section 3, we have \( s_i = \ell_2(C_{i-1}) - 1 + \ell_1(C_i) \)).

We first explain how to place all anchor vertices and \( v_1, v_n \). We first split the vertices except \( v_1 \) into three groups. We put \( v_2 \) in group \( G_T \) (“top”), \( v_3 \) in \( G_L \) (“lower left”), and \( v_4 \) in \( G_R \) (“lower right”). For any edge \( (x_i^-, x_i^+ - 1) \), \( i \geq 2 \), its incident vertices are in the same group. We now place the considered vertices as follows (see also Fig. 3[4]).

- \( v_1 \) is placed at the origin.
- \( v_2 \) is placed at \((0, 10n)\), i.e., on the vertical upward ray from \( v_1 \).
- \( v_3 \) is placed at \((-10n, -10n)\), i.e., on the diagonal downward-left ray from \( v_1 \).
- \( v_4 \) is placed at \((10n, -10n)\), i.e., on the diagonal downward-right ray from \( v_1 \).

Now, iteratively for \( i = 2, \ldots, \kappa \), consider edge \((x_i^-, x_i^+ - 1)\). Vertex \( x_i^- \) has already been placed, while we do not have a placement for \( x_i^+ - 1 \) yet.

- If \( x_i^- \in G_T \), then place \( x_i^+ - 1 \) on the vertical ray upward from \( v_1 \), and \( s_i + 5 \) units higher than \( x_i^- \).
- If \( x_i^- \in G_L \), then place \( x_i^+ - 1 \) on the diagonal downward-left ray from \( v_1 \), and \( s_i + 4 \) units farther left of and \( s_i + 4 \) units farther down from \( x_i^- \).
- If \( x_i^- \in G_R \), then place \( x_i^+ - 1 \) on the diagonal downward-right ray from \( v_1 \), and \( s_i + 4 \) units farther right of and \( s_i + 4 \) units farther down from \( x_i^- \).

One immediately verifies that this placement gives a planar drawing of the graph induced by the so-far considered vertices: Any edge either lies on a ray or connects two different rays, and as we go along in age-order, the current anchor triangle always forms the outer-face and the next vertex is placed outside of it. We briefly analyze the size of this drawing:

- **Claim 16.** The drawing uses only points in the range \((-14n, 14n) \times (-14n, 15n)\).

---

1 The coordinates are chosen to be easy to define and analyze; the constant factor could likely be improved by making more careful choices.
**Figure 6** The overall layout (to scale).

**Proof.** Consider the topmost vertex above $v_1$. In the worst case, all the vertices are placed above $v_1$. Thus, the largest $y$-coordinate is at most $10n + (\kappa - 1)5 + (n - \kappa) < 15n$, where $n - \kappa$ is the upper bound on the number of all singletons. Similarly, any vertex on the other two rays has horizontal and vertical distance less than $10n + 4n$ from $v_1$, and the claim follows.

**Claim 17.** Any edge $(u, v)$ from the left-down ray to the right-down ray has slope in $(-\frac{1}{5}, \frac{1}{5})$.

**Proof.** We know that $x(u) = y(u) = -10n - k$ for some $0 \leq k < 4n$, and $y(v) = -x(v) = -10n - \ell$ for some $0 \leq \ell < 4n$. Assume $\ell \leq k$, i.e., the slope is non-negative (the other case is symmetric). The slope of the edge is hence

$$\frac{-10n - \ell - (-10n - k)}{10n + \ell - (-10n - k)} = \frac{k - \ell}{20n + \ell + k} < \frac{4n}{20n} = \frac{1}{5}.$$

**Claim 18.** Any edge $(u, v)$ from the left-down ray to the vertical-up ray has slope in $(\frac{1}{5}, 2.9)$.

**Proof.** We know that $x(u) = y(u) = -10n - k$ for some $0 \leq k < 4n$, $x(v) = 0$, and $y(v) = 10n + \ell$ for some $0 \leq \ell < 5n$. The slope of the edge is hence

$$\frac{10n + \ell - (-10n - k)}{-(-10n - k)} = \frac{20n + k + \ell}{10n + k}.$$
and we observe

\[
\frac{10}{7} = \frac{20n}{14n} < \frac{20n + \ell}{10n + k} < \frac{29n}{10n} = 2.9.
\]

We must now add the points for singletons. Observe that any such vertex is placed “near” an edge \((x_i^-, x_{i-1}^-)\) for some index \(i \geq 2\), and is then connected either to all of \(T(C_{i-1})\), or to all of \(T(C_i)\). We must hence argue that near any edge \((x_i^-, x_{i-1}^-)\), we can find \(s_i\) grid points, each of which allows straight lines to all of \(T(C_i)\) and \(T(C_{i-1})\) while intersecting only edge \((x_i^-, x_{i-1}^-)\). We distinguish cases depending on to which of the groups \(G_T, G_L, G_R\) the two vertices \(x_i^-, x_{i-1}^-\) belong.

**Case 1:** \(x_i^-, x_{i-1}^- \in G_T\): Let \(p, q\) be the two vertices in \(T(C_i) \cap T(C_{i-1})\). They are not in \(G_T\), and on different rays, say \(p \in G_L\) and \(q \in G_R\). From a point \(x\), we can see the four vertices \(\{p, q, x_{i-1}^-, x_i^-\}\) in the required way if \(x\) is within the triangle \(\{p, x_{i-1}^-, x_i^-\}\) and above the extension of the edge \((q, x_i^-)\) into that triangle.

Let \(P\) be the set of points that are one unit left of the drawing of \((x_i^-, x_{i-1}^-)\), ends included. We have \(|P| = s_i + 6\) by our construction. Edge \((p, x_{i-1}^-)\) has slope less than 2.9, so at most 3 points of \(P\) are above \((p, x_{i-1}^-)\). Edge \((p, x_i^-)\) has positive slope, so all points of \(P\) are above \((p, x_i^-)\). Edge \((q, x_i^-)\) has slope more than 2.9 (by a symmetric argument), so at most 3 points of \(P\) are below the extension of \((q, x_i^-)\). This leaves at least \(s_i\) points. We use the top points for the singletons of \(C_i\) (i.e., connecting to \(x_{i-1}^-\)) and the bottom points for the singletons of \(C_{i-1}\) (i.e., connecting to \(x_i^-\)). The total number of crossings created matches the number achieved in Section 3.

**Case 2:** \(x_i^-, x_{i-1}^- \in G_L\) (the case \(x_i^-, x_{i-1}^- \in G_R\) is symmetric): Edge \((x_i^-, x_{i-1}^-)\) is drawn with slope 1. Let \(p, q\) be the two vertices in \(T(C_i) \cap T(C_{i-1})\), say \(p \in G_T\), and \(q \in G_R\).

Let \(P\) be the set of grid points that are one unit left of the drawing of \((x_i^-, x_{i-1}^-)\) excluding the lowest such grid point. We have \(|P| = s_i + 4\) by construction. Edge \((x_{i-1}^-, p)\) has slope more than \(\frac{4}{3}\), while the line from \(x_{i-1}^-\) to the fourth point from the left of \(P\) has slope \(\frac{4}{3} < \frac{10}{7}\), so at most 3 points of \(P\) are left of edge \((x_{i-1}^-, p)\). Edge \((x_i^-, p)\) has slope > 1, so all

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**Figure 7** Adding singletons near edge \((x_{i-1}^+, x_i^-)\) if it is (left) vertical with \(s_i = 7\), or (right) on the lower-left diagonal with \(s_i = 4\). For clarity, not all singletons are shown.
points of $P$ are left of $(p, x_i^-)$. Edge $(q, x_i^-)$ has slope less than $\frac{1}{2}$, while the line from $x_i^-$ to the second point from the right of $P$ has slope $\frac{1}{2}$, so only 1 point of $P$ is above the extension of $(q, x_i^-)$. This leaves at least $s_i$ points in $P$ that are inside the face and can see $q$ while only crossing $(x_i^-, x_i^+)$). We use the bottom points for single-cluster vertices of $C_i$ (i.e., connecting to $x_i^-$) and the top points for single-cluster vertices of $C_{i-1}$ (i.e., connecting to $x_i^-$). The total number of crossings created again matches the number achieved in Section 3.

All singletons are placed in an inner face of the drawing. The size of the drawing is thus determined by the coordinates of the vertices placed on the rays in the first step of the algorithm. This proves Theorem 13.

C Proof of Lemma 11

Let $D$ be a drawing achieving $cr(G)$, and let $D_i$ be the subdrawing of $D$ corresponding to $H_i$. Each of the latter gives rise to a planarly embedded graph $L_i$ of $H_i$, where crossings in $D_i$ are substituted by degree-4 vertices. We call edges in $L_i$ subedges. We call a $u$-$v$-path in $L_i$ an $i$-path, and for each $i = 1, 2$, we choose an $i$-path $P_i$. Let $D_i^+ \supset D_i$ be a drawing of $H_i^+$ where $(u, v)$ is drawn into $D_i$ (without $(u, v)$ if it already existed) following the route of $P_{3-i}$; we have $cr(H_i^+) \leq cr(D_i^+)$. Clearly, any crossing in any $D_i^+$ has a counterpart in $D$. Inversely, any crossing in $D$ can show up in at most one of $D_i^+$, except for crossings between edges of $P_1$ and $P_2$—so-called path-crossings. We show that for each crossing that we count in both $D_i^+$ and $D_j^+$, there is at least one other crossing in $D$ that is in neither $D_i^+$, $D_j^+$.

We can assume that any choice of $P_1, P_2$ gives path-crossings, as otherwise we would be done. Furthermore, for $i = 1, 2$, we can assume there are no two subedge-disjoint $i$-paths; otherwise, we can pick the one with fewer paths-crossings with $P_{3-i}$ as $P_i$ and be done. Similarly, we can account for crossings on a subpath $P'_i = (w \rightarrow w') \subseteq P_i$ if there is another subedge-disjoint subpath connecting $w$ to $w'$ in $L_i$. Let $F_i \subset P_i$, $i = 1, 2$, be the subedges that are in every $i$-path. We only have to account for crossings between $F_1$ and $F_2$.

Assume there is a path-crossing $(e_1, e_2)$, $e_i \in F_i$, even though we choose a crossing minimal insertion route for $P_1$ in $L_2$. Therefore the ends of $e_1$ lie in different faces of the planar graph $L_2$. In consequence $e_2$ lies on a cycle $Q \in L_2$ separating the ends of $e_1$ from each other. Let $P'_2$ and $P''_2$ be the subpaths after deleting $e_2$ from $P_2$. The subgraph $S = P'_2 \cup (Q \setminus \{e_2\}) \cup P''_2$ connects $u$ to $v$ in $L_2$ even though $e_2 \notin S$—a contradiction to $e_2 \in F_2$.

D Details for Theorem 13

It remains to argue how 2-connected components can be merged while maintaining straight-line drawings. For this, we show that one vertex can be forced to appear at the outer-face.

Lemma 19. Let $G$ be a graph with a path decomposition $\mathcal{P}$ of width 3, and let $p$ be a vertex in bag $X_1$. Then there exists a straight-line drawing of $G$ with at most $2cr(G)$ crossings that has $p$ on the convex hull.

Proof. Convert $\mathcal{P}$ into an alternating path decomposition; this can be done while keeping $p$ in the first bag. We prove the claim by induction on the number of 2-connected components; in the base case (no cut-vertex) the claim holds by Lemma 12. If $G$ has a cut-vertex $v$, then let $G_1, \ldots, G_k$ be the cut-components of $v$, named such that $G_1$ contains $p$. Recursively obtain a drawing $D_1$ of $G_1$ that has $p$ on the convex hull, using the induced path decomposition.
Consider $i \geq 2$ and the path decomposition $\mathcal{P}_i$ of $G_1$ induced by $\mathcal{P}$. If $v$ happens to be in the first bag of $\mathcal{P}_i$, then draw $G_1$ recursively with $v$ on the convex hull, and merge (after an affine transformation) the result in the vicinity of the drawing of $v$ in $\mathcal{D}_1$.

If $\mathcal{P}_i$ does not contain $v$ in its first bag, then we modify it. Let $X_j$ be the first bag of $\mathcal{P}$ that does contain $v$, and let $X_h$ be any bag with $h < j$ that contains vertices of $G_i$. Within $G_1$ there exists a path $P$ from $p$ to $v$, hence from $X_1$ to $X_j$, hence $X_h$ contains at least one vertex of $P$. Since $v \notin X_h$, $X_h$ must contain at least one vertex of $G_1 - \{v\}$, i.e., not in $G_i$. Hence in $\mathcal{P}_i$ we have $|X_h| \leq 3$ and can add $v$ to this bag. Doing this for all $X_h$, we obtain a path decomposition of $G_i$ that has $v$ in its first bag and that is still alternating. ◀

### E Approximation Algorithm for Graphs of Higher Pathwidth

In this section, we provide the analysis of the number of crossings achieved by the algorithm presented in Section 5.

#### E.1 Upper-bounding the number of crossings

With the routing as described, some edges cross twice for $w \geq 5$ (e.g., edge $(p^2_3, v_i)$ crosses edge $(p^3_3, p^2_5)$ both near $p^2_3$ and near $p^5_1$). We can avoid such crossings by local re-drawings, which can only improve the overall number of crossings. But in our counting of crossings we will not take advantage of this.

We want to bound the number of crossings incurred when drawing vertex $v_i$, $i \geq w + 2$. No new crossings occur in the vicinity of $p^3_1$ or $p^4_2$. Consider the routing of edge $(p^j_3, v_i)$ in the vicinity of $p^i_k$ for some $3 \leq j < k \leq w$. This edge crosses any edge incident to $p^i_k$ with two exceptions: It does not cross $(p^k_{i-1}, v_j)$, since we ordered edges within the bundle appropriately. And it does not cross the edge $(p^i_{k-1}, p^k_{i-1})$, since we re-routed that edge to be without crossings after the introduction of $v_i$. Therefore edge $(p^j_3, v_i)$ crosses at most $\deg_{G_i}(p^i_k) - 2$ other edges in the vicinity of $p^i_k$. Summing up over all $k$ and over the $w - 1$ edges added within the bundle of $v_i$ gives:

**Observation 20.** Drawing vertex $v_i$ gives at most $\sum_{j=3}^{w}(j - 2)(\deg_{G_i}(p^j_3) - 2)$ new crossings.

To simplify this bound, we upper-bound the degrees.

**Observation 21.** For all $k \geq 4$, $\deg_{G_i}(p^i_k) \geq \deg_{G_i}(p^i_3)$. Thus, drawing vertex $v_i$ adds at most $\frac{(w-1)(w-2)}{2}(\deg_{G_i}(p^i_3) - 2)$ new crossings.

**Proof.** Vertices $p^i_k$ and $p^i_3$ are adjacent. Besides this, any predecessor $u$ of $p^i_k$ is a predecessor of $p^i_3$, or it was introduced after $p^i_3$. In both cases, $u$ is adjacent to $p^i_3$ as well. Since we are looking at $G_i$ (and not full $G$), any vertex so far introduced after $p^i_k$ is adjacent to both $p^i_k$ and $p^i_3$. This proves the first part of the claim and the second follows from Observation 20. ◀

Define again (and compatible to before) an anchor-triplet $T$ to be three vertices that are the oldest vertices of some bag $X \neq X_1$. Note that, again, $T$ forms a triangle by maximality. Also, $T$ again defines a cluster consisting of all bags that contain all of $T$. Clearly, the bags of a cluster are again consecutive. However, in contrast to before, clusters may overlap in more than one bag. Figure 5 gives an example.

We say a vertex $u$ is introduced by cluster $C$ if $u$ appears in $C$, but not in $G_{w+1}$ or in any cluster that ends at an earlier bag. (This is quite similar to the concept of singletons used earlier, except that a vertex that belongs to only one bag may now belong to multiple clusters, and is considered to be introduced only by the cluster that ends earliest.) Let $i(C)$ be the number of vertices introduced by a cluster $C$. 

Observation 22. Let $C$ be a cluster with $T(C) = \{p_1, p_2, p_3\}$ in age-order. Then the first bag of $C$ introduces $p_3$, $i(C) \leq n(C) - (w + 1)$, and for any vertex $v_i$ introduced by $C$ we have $\deg_{G_i}(p_3) \leq n(C) - 1$.

Proof. Vertex $p_3$ is adjacent to $\{p_1, p_2\}$ and so the bag $X$ introducing $p_3$ contains $T(C)$. But no earlier bag contains $p_3$, so $X$ is the first bag of $C$. Any vertex in $X$ appears in some earlier cluster (or in $G_{w+1}$) and so was not introduced by $C$. Finally, $G_i$ considers only bags of $C$ or earlier clusters, and so any neighbour of $p_3$ in $G_i$ belongs to $C$.

We can now restate the number of crossings achieved as follows:

Lemma 23. The above drawing algorithm for a maximal graph of pathwidth $w \geq 4$ produces at most the following number of crossings:

$$\left(\frac{w+1}{4}\right) + \sum_{C \in \mathcal{C}} 2(w - 1)(w - 2) \left\lfloor \frac{n(C) - 3}{2} \right\rfloor \left\lfloor \frac{n(C) - 4}{2} \right\rfloor.$$  

Proof. Graph $G_{w+1}$ contributes $\left(\frac{w+1}{4}\right)$ crossings. Each vertex $v_i$ introduced by some cluster $C$ adds at most $\frac{(w-1)(w-2)}{2}(\deg_{G_i}(p_3) - 2)$ crossings from Observation 21. Observe that $p_3$ is the youngest vertex $p_i \in T(C)$. Applying Observation 22 and summing over the $i(C) \leq n(C) - 5$ vertices introduced by $C$ (Observation 22 and $w \geq 4$), the number of crossings added by $C$ is at most

$$\frac{(w - 1)(w - 2)}{2}(n(C) - 5)(n(C) - 3) \leq 2(w - 1)(w - 2) \left\lfloor \frac{n(C) - 3}{2} \right\rfloor \left\lfloor \frac{n(C) - 4}{2} \right\rfloor.$$  

E.2 Lower-bounding the crossing number

We know that our initial graph $G_{w+1} = K_{w+1}$ requires at least $\Theta(w^4)$ crossings, see [12] for the currently best bounds. For us, the rather trivial $cr(G_{w+1}) \geq \frac{1}{4}\left(\frac{w+1}{4}\right)$ will suffice.

Every cluster $C$ contains $B(C) := K_{3,n(C)-3}$, its cluster biclique with $T(C)$ as one partition set, and thus needs at least $\left\lfloor \frac{n(C)-3}{2} \right\rfloor \left\lfloor \frac{n(C)-4}{2} \right\rfloor$ crossings in any drawing by Zarankiewicz’ formula. However, any one crossing may belong to multiple cluster bicliques, and so may be counted repeatedly.

Lemma 24. Consider a good drawing of a maximal graph $G$ of pathwidth $w \geq 4$. Any crossing belongs to at most $\mu = 2w - 5$ cluster-bicliques.

Proof. We want to show that in any good drawing of a maximum pathwidth-$w$-graph any crossing belongs to at most $\mu := 2w - 5$ cluster bicliques. Let $\chi := \{x_1, x_2, x_3, x_4\}$ in age order be the four distinct endpoints of edges involved in a specific crossing. For any cluster $C$ whose biclique $K(C)$ may contain this crossing, we have $\chi \subseteq V(C)$ and $|T(C) \cap \chi| = 2$, since $K(C)$ is bipartite. Let $X_i$ be the bag where $x_4$ (the youngest of $\chi$) is introduced. Let $X_k$ be the first size-$w$ bag where one of $\chi$ (say $x'$) has been forgotten. We have two cases:

Case 1: $k < i$, i.e., vertex $x'$ is forgotten before $x_4$ is introduced. All bags containing $x'$ are $X_{i-2}$ or before, and all bags containing $x_4$ are $X_i$ or after. Any cluster $C$ that uses $\chi$ must hence contain $X_{i-1}$, a $w$-sized bag. Observe that any bag $X$ belongs to at most $|X| - 2$ clusters since, starting with the oldest three vertices of $X$ as anchor-triplet, each next cluster containing $X$ forgets one of the anchor vertices and adds one other vertex of $X$ to obtain its anchor-triplet. Hence there are at most $|X_i| - 2 = w - 2 \leq 2w - 5$ clusters containing $\chi$. 


Case 2: \( i \leq k \). All bags between \( X_k \) and \( X_{k-1} \) contain all vertices \( \chi \). Consider a cluster \( C \) that uses the crossing, and let \( X_h \) be the oldest bag of \( C \). Since \( x_4 \) must belong to \( C \), we have \( h \geq i \). We have two subcases:

- Assume first that \( h \geq k \). Then the size-\( w \) bag \( X_h \) belongs to \( C \). As argued above bag \( X_k \) belongs to at most \( |X_h| - 2 \) clusters, so there are at most \( |X_h| - 2 = w - 2 \) clusters using the crossing with \( h \geq k \).

- Now assume that \( h < k \), which by \( h \geq i \) means that \( X_h \) contains all of \( \chi \). Recall that the anchor-triangle \( T(C) \) is defined to be the three oldest vertices in \( X_h \). Since \( \chi \subseteq X_h \) and \( |T(C) \cap \chi| = 2 \), it follows that neither \( x_3 \) nor \( x_4 \) can be in \( T(C) \). Therefore at least one anchor-vertex of \( C \) is older than \( x_4 \), which means that \( C \) starts to the left of \( X_i \). Also, the anchor-triangle of cluster \( C \) uses one of the \( w - 1 \) vertices in \( X_i - \{x_3, x_4\} \). We hence have at most \( w - 3 \) clusters \( C \) that fall into this case.

Putting the two bounds together, we have at most \( 2w - 5 \) cliques that use a crossing. ▶

We can show that this bound is tight. Figure 8 shows an example of a path decomposition of width \( w = 5 \) for which the vertex set \( \chi = \{4, 5, 8, 9\} \) belongs to \( 5 = 2w - 5 \) clusters, all of which have exactly two vertices of \( \chi \) in their anchor triangle.

![Figure 8](image_url)

Figure 8 A path decomposition of width 5 with clusters.

Corollary 25. Any good drawing of \( G \) has at least the following number of crossings:

\[
\frac{1}{\mu + 1} \left( \frac{1}{5} \left( \frac{w + 1}{4} \right) + \sum_{C \in C} \left\lfloor \frac{n(C) - 3}{2} \right\rfloor \left\lfloor \frac{n(C) - 4}{2} \right\rfloor \right)
\]

Proof. Graph \( G_{w+1} \) needs at least \( \frac{1}{5} \left( \frac{w + 1}{4} \right) \) crossings. Any cluster biclique \( B(C) \) needs at least \( \left\lfloor \frac{n(C) - 3}{2} \right\rfloor \left\lfloor \frac{n(C) - 4}{2} \right\rfloor \) crossings. In any drawing of \( G \), crossings are counted in at most \( \mu \) bicliques, and also in \( G_{w+1} \). ▶
Combining the upper and lower bound immediately gives the main result:

**Proof of Theorem 14.** The approximation ratio comes from combining the upper bound of Lemma 23 with the lower bound of Corollary 25, and the observation that \(5 < 2(w-1)(w-2)\) for \(w \geq 4\). The runtime for the decomposition has already been argued in Theorem 4, all the remaining algorithmic steps can be done in linear time as well.

It remains to argue the complexity of the grid. For each vertex, we add one extra (vertex-free) column just before and one just after it. Whenever we need to route “around” some vertex \(p_i^j\), we use its three columns to place all necessary bends (cf. Fig. 4). Furthermore, we use one additional column for each edge from \(v_i\) to its oldest predecessor \(p_i^1\). Therefore, we need no more than \(4n\) columns for all vertices and bends.

Now subdivide each edge with a dummy-node whenever it crosses a column without having a bend- or endpoint there. What results is a so-called hierarchical drawing (turned sideways). We can rearrange this easily, column by column, so that the height of the drawing is dominated by the column with the maximum number of vertices, bends, or dummy-nodes. Any of the columns used for routings to first predecessors is crossed by at most \(n\) edges, each edge crossing twice or having two bends. Thus these columns require a height of at most \(2n\). Any of the other columns could be crossed by almost all edges, but all edges are routed \(x\)-monotonically within there, and hence cross any column at most once. A graph of pathwidth \(w\) has at most \(wn\) edges, and so the bound follows. ◁