CONSTRUCTION OF SOLITARY WAVE SOLUTION OF THE NONLINEAR FOCUSING SCHRÖDINGER EQUATION OUTSIDE A STRICTLY CONVEX OBSTACLE FOR THE $L^2$-SUPERCRITICAL CASE

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Abstract. We consider the focusing $L^2$-supercritical Schrödinger equation in the exterior of a smooth, compact, strictly convex obstacle $\Theta \subset \mathbb{R}^3$. We construct a solution behaving asymptotically as a solitary waves on $\mathbb{R}^3$, as large time. When the velocity of the solitary wave is high, the existence of such a solution can be proved by a classical fixed point argument. To construct solutions with arbitrary nonzero velocity, we use a compactness argument similar to the one that was introduced by F. Merle in 1990 to construct solution of NLS blowing up at several blow-up point together with a topological argument using Brouwer’s theorem to control the unstable direction of the linearized operator at soliton. These solutions are arbitrarily close to the scattering threshold given by a previous work of R. Killip, M. Visan and X. Zhang which is the same as the one on whole Euclidean space.

Contents

1. Introduction 2
Acknowledgements 7
2. Construction of the solution assuming uniform estimates 7
  2.1. Properties of the ground state 7
  2.2. Spectral theory of the linearized operator 7
  2.3. Compactness argument 9
3. Proof of the uniform estimate 12
  3.1. Bootstrap and topological arguments 12
  3.2. Estimate on the modulation parameters 21
4. Fixed point theorem 26
Appendix A. Proof of some Technical results 32
Appendix B. Computation of some estimates 44
References 44
1. Introduction

We consider the focusing nonlinear Schrödinger equation in the exterior of a smooth compact strictly convex obstacle $\Theta \subset \mathbb{R}^3$ with Dirichlet boundary conditions:

\begin{equation}
\text{(NLS}_\Omega) \left\{ \begin{array}{ll}
    i \partial_t u + \Delta_\Omega u = -|u|^{p-1} u & \forall (t,x) \in [T_0, +\infty) \times \Omega, \\
    u(T_0, x) = u_0(x) & \forall x \in \Omega, \\
    u(t, x) = 0 & \forall (t,x) \in [T_0, +\infty) \times \partial \Omega.
\end{array} \right.
\end{equation}

Where $\Omega = \mathbb{R}^3 \setminus \Theta$, $\Delta_\Omega$ is the Dirichlet Laplace operator on $\Omega$ which is a self-adjoint operator with form domain $H^1_0(\Omega)$, $\partial_t$ is the derivative with respect to the time variable and $T_0 \in \mathbb{R}$ is the initial time. Here $u$ is a complex-valued function, $u : [T_0, +\infty) \times \Omega \rightarrow \mathbb{C}$, $(t,x) \mapsto u(t,x)$.

We take the initial data $u_0 \in H^1_0(\Omega)$.

The local Cauchy problem for (NLS$_\Omega$) in $H^1_0(\Omega)$ was studied in several articles. For $1 < p < 5$, L. Vega and F. Planchon proved that (NLS) equation in the exterior of a non-trapping domain in $\mathbb{R}^3$ is locally well-posed, see [28]. After that, F. Planchon and O. Ivanovici extended the result to the quintic Schrödinger equation outside a non-trapping domain see [19].

The solutions of the (NLS$_\Omega$) satisfy the mass and energy conservation laws:

\begin{align*}
M(u(t)) := \int_\Omega |u(t,x)|^2 dx &= M(u_0). \\
E(u(t)) := \frac{1}{2} \int_\Omega |\nabla u(t,x)|^2 dx - \frac{1}{p + 1} \int_\Omega |u(t,x)|^{p+1} dx &= E(u_0).
\end{align*}

Furthermore, the (NLS) equation posed on the whole Euclidean space $\mathbb{R}^3$ is invariant by the scaling transformation, that is,

\[ u(t, x) \mapsto \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t), \quad \text{for } \lambda > 0. \]

This scaling identifies the critical Sobolev space $\tilde{H}^{s_c}$, where the critical regularity $s_c$ is given by $s_c := \frac{3}{2} - \frac{2}{p-1}$. The case when $s_c = 0$ is referred to as mass-critical or $L^2$-critical and the case when $s_c = 1$ is called energy-critical or $H^1$-critical.

Throughout this paper, we will take $\frac{7}{3} < p < 5$. Since the presence of the obstacle does not change the intrinsic dimensionality of the problem, so we may regard (NLS$_\Omega$) equation as being $H^1(\Omega)$-subcritical and $L^2(\Omega)$-supercritical.

Consider solitary waves solution of (NLS$_\Omega$), with $\Omega = \mathbb{R}^3$, that is $u(t, x) = e^{i t \omega} Q_\omega(x)$ where $Q_\omega$ is a solution of the nonlinear elliptic equation:

\begin{equation}
\begin{cases}
    -\Delta Q_\omega + \omega Q_\omega = |Q_\omega|^{p-1} Q_\omega, \\
    Q_\omega \in H^1(\mathbb{R}^3).
\end{cases}
\end{equation}

This elliptic equation admits solutions if and only if $\omega > 0$. In this paper, we will denote by $Q_\omega$ the ground state which is the unique radial positive solution of (1.2). We recall that $Q_\omega$
is smooth and exponentially decaying at infinity and characterized as the unique minimizer for the Gagliardo-Nirenberg inequality up to scaling, space translation and phase shift, see [23].

The (NLS) equation posed on the whole Euclidean space $\mathbb{R}^3$, also enjoys Galilean invariance. If $u(t, x)$ is solution, then $u(t, x - vt) e^{i \frac{x - vt}{2} - \frac{|v|^2}{4} t}$ is also a solution, for $v \in \mathbb{R}^3$.

Applying a Galilean transform to the solution $e^{it\omega} Q_\omega(x)$ of the (NLS) on $\mathbb{R}^3$, we obtain a soliton solution, moving on the line $x = tv$ with velocity $v \in \mathbb{R}^3$:

$$u(t, x) = e^{i \left( \frac{x}{2} - \frac{1}{4} |v|^2 t + t\omega \right)} Q_\omega(x - tv).$$

The soliton (1.3) is a global solution of the focusing nonlinear Schrödinger equation (NLS) posed on the whole space, but is not a solution of (NLS). Our goal is to construct solitary waves of the (NLS) satisfying Dirichlet boundary conditions and behaving asymptotically as the preceding solitary waves $e^{i \left( \frac{x}{2} - \frac{1}{4} |v|^2 t + t\omega \right)} Q_\omega(x - tv)$, as $t \to +\infty$. The main result of this paper is the following.

**Theorem 1.1.** Assume $\frac{7}{3} < p < 5$.

Let $\Psi$ be a $C^\infty$ function such that: \[
\begin{aligned}
\Psi &= 0 \quad \text{near } \Theta, \\
\Psi &= 1 \quad \text{if } |x| \gg 1.
\end{aligned}
\]

Let $v \in \mathbb{R}^3 \setminus \{0\}$ be the velocity, $\omega > 0$. Then there exists $\delta > 0$, $T_0 > 0$ and a function $r_\omega$ defined on $[T_0, +\infty) \times \Omega$ satisfying

$$
\|r_\omega(t)\|_{H^s_0(\Omega)} \leq e^{-\delta \sqrt{|\omega|} t} \quad \forall t \in [T_0, +\infty),
$$

such that,

$$u(t, x) = e^{i \left( \frac{x}{2} - \frac{1}{4} |v|^2 t + t\omega \right)} Q_\omega(x - tv) \Psi(x) + r_\omega(t, x), \quad \forall (t, x) \in [T_0, +\infty) \times \Omega,$n

is a solution of (NLS).

**Remark 1.2.** Theorem 1.1 can be generalized for any dimension $d \geq 3$. Moreover, this result can be extended to the subcritical case $1 < p < \frac{7}{3}$ which are easier to prove due to the stability of solitons.

**Remark 1.3.** The restriction to a strictly convex obstacle is purely technical. In section 2, we will need that the (NLS) equation is well posed on $H^s(\Omega)$, for some $s \in [s_p, 1]$, with $s_p = \frac{3}{2} - \frac{3}{p+1}$ (Cf. Lemma 2.10), for that we need to use a Strichartz estimate from [17] (Cf. Theorem A) and some fractional rules given by [20] for strictly convex obstacle (Cf. Proposition B). Because of this, we shall suppose that the obstacle $\Theta$ is strictly convex.

In [8], T. Duyckaerts, J. Holmer and S. Roudenko have studied the behavior (i.e scattering and global existence) of the solutions of the focusing cubic (i.e $p=3$) nonlinear Schrödinger equation on $\mathbb{R}^3$, whenever the initial data satisfies a smallness criterion given by the ground state threshold. The criterion is expressed in terms of the scale-invariant quantities $\|u_0\|_{L^2} \|\nabla u_0\|_{L^2}$ and $M(u)E(u)$. This result was later extended to arbitrary space dimensions and focusing mass-supercritical power nonlinearities by T. Cazenave, J. Xie and D. Fang, see [11] and by C. Guevara in [15].
Theorem A ([11],[8],[15]). Let \( s = \frac{3}{2} - \frac{2}{p} \) and \( \frac{7}{3} < p < 5 \). Let \( u_0 \in H^1(\mathbb{R}^3) \) satisfy

\[
\|u_0\|_{L^p(\mathbb{R}^3)} \|\nabla u_0\|_{L^2(\mathbb{R}^3)}^s \|Q\|_{L^2(\mathbb{R}^3)}^s \|\nabla Q\|_{L^2(\mathbb{R}^3)}^s,
\]

\[
M(u_0)^{1-s}E(u_0)^s < M(Q)^{1-s}E(Q)^s.
\]

Then \( u \) scatters in \( H^1(\mathbb{R}^3) \).

Theorem A remains true for \((\text{NLS}_\Omega)\) in the exterior of a strictly convex obstacle in three dimension. Indeed, R. Killip, M. Visan and X. Zhang had proved in [21] that the threshold for global existence and scattering is the same as for the cubic equation on \( \mathbb{R}^3 \). Moreover, K. Yang extended this result for \( \frac{7}{3} < p < 5 \), see [34].

The solitary waves constructed in the main Theorem 1.1 prove the optimality of the threshold for scattering given in [34, Theorem 1.3]. Indeed, the solution \( u \) of \((\text{NLS}_\Omega)\) is global, does not scatter for positive time direction and we have

\[
E(u) = \frac{|v|^2}{8} \int |Q|^2 + E(Q).
\]

Since, the velocity \( v \) can be taken arbitrary small, we have proved that for all \( \varepsilon > 0 \) there exists a solution \( u_\varepsilon \) of \((\text{NLS}_\Omega)\) which is global and does not scatter for positive time such that

\[
M(u_\varepsilon) = M(Q), \quad \sup_{t \geq t_0} \|\nabla u_\varepsilon(t)\|_{L^2(\Omega)}^s < \|\nabla Q\|_{L^2(\mathbb{R}^3)}^s + \varepsilon
\]

and

\[
E(u_\varepsilon)^s < E(Q)^s + \varepsilon.
\]

The proof of Theorem 1.1 relies on a compactness argument that uses the structure of the linearized operator around the ground state soliton. If the velocity \( v \) is large enough, we can use a simple fixed point theorem to construct a soliton solution of \((\text{NLS}_\Omega)\).

Theorem 1.4. Assume \( 2 \leq p < 5 \).

Let \( \Omega = \mathbb{R}^3 \setminus \Theta \) where \( \Theta \) is any smooth compact obstacle and \( Q_\omega \) be any solution of \((1.2)\).

Let \( \Psi \) be a \( C^\infty \) function such that:

\[
\begin{cases}
\Psi = 0 & \text{near } \Theta, \\
\Psi = 1 & \text{if } |x| \gg 1.
\end{cases}
\]

Let \( \omega, t_0 > 0 \). Then there exists \( V_0 := V_0(\omega) \gg 1 \) with the following property. Let \( v \in \mathbb{R}^3 \) be the velocity such that \( |v| > V_0 \).

Then there exists \( \delta > 0 \) and a functions \( r_\omega \) defined on \([T_0, +\infty) \times \Omega\) satisfying

\[
\forall t \in [T_0, +\infty) \quad \|r_\omega(t)\|_{H^2(\Omega)} \leq C_\omega |v|^3 e^{-\delta \sqrt{\omega}|v|t},
\]

such that \( u(t, x) = e^{i \left( \frac{1}{2} \|x\|^2 - \frac{1}{4} |v|^2 t + t \omega \right)} Q_\omega(x - tv) \Psi(x) + r_\omega(t, x), \quad \forall (t, x) \in [T_0, +\infty) \times \Omega \), is a solution of \((\text{NLS}_\Omega)\).

Unlike in Theorem 1.1, \( Q_\omega \) is any solution of the nonlinear elliptic equation \((1.2)\) (not necessarily the ground state) and \( \Theta \subset \mathbb{R}^3 \) does not have to be convex, which makes Theorem 1.1 more general for high velocity. However, we can see in (1.6) that the choice of high velocity does not allow us to use Theorem 1.4 to show the optimality of the threshold for scattering in [21] and [34]. Let us mention that, this result can be extended for any dimension \( d \geq 3 \). We will give the proof of the Theorem 1.4 for the cubic case \( p = 3 \). The proof for general \( p \in [2, 5) \) is very similar, see Remark 4.1.
Let us mention that apart from the works of R. Killip, M. Visan and K. Yang cited above and in [22], \((\text{NLS}_\Omega)\) outside obstacle was also studied by N. Burq, P. Gerard and N. Tzvetkov in [3] and F. Abou Shakra in [1]. Let us also mention the recent works on dispersive estimates outside one or several strictly convex obstacles of O. Ivanovici and G. Lebeau in [18] and D. Lafontaine in [24], [25].

We end this section by giving sketch of the proofs of the two theorem. 

**Sketch of the proof of Theorem 1.1.**

The structure of the proof is similar to the one for construction of multi-soliton for \((\text{NLS})\) on \(\mathbb{R}^d\) in the subcritical case in [26] with an additional argument coming from [7] which allows to handle the supercritical character of the non-linearity. The compactness argument used in the present paper is similar to the main argument used in [26], [7], and [27].

Note that, even though we use some similar arguments, a large part of the proof of Theorem 1.1 is different. It is because of the presence of the obstacle \(\Theta\) which makes the calculations more complicated.

Recall that the soliton \(Q_\omega(x - tv)e^{i\frac{1}{2}(x \cdot v) - \frac{1}{4}|v|^2t + t\omega}\) is an exact solution of the \((\text{NLS})\) on the whole space \(\mathbb{R}^3\). So, the proof consists in the construction of a smooth correction \(r_\omega(t, x)\) with some uniform estimates, such that \(R(t, x) + r_\omega(t, x)\) is a solution of the equation \((\text{NLS}_\Omega})\) where

\[
R(t, x) = e^{i\frac{1}{2}(x \cdot v) - \frac{1}{4}|v|^2t + t\omega}Q_\omega(x - tv)\Psi(x).
\]

The paper is organised as follows. In §2.1, we give a review of some properties of the ground state \(Q\). In §2.2, we recall some spectral properties of the linearized Schrödinger operator around the soliton \(e^{itQ}\). That is,

In the subcritical case, Cazenave and Lions [5], Weinstein [33] proved that the solitary waves are stable when \(1 < p < \frac{7}{3}\), which means that the nonlinearity has a \(L^2\)-subcritical growth. From [33], there exits \(\lambda > 0\) such that for any real-valued function \(h \in H^1\),

\[
(h, Q_\omega), (h, \nabla Q_\omega) = 0 \implies \int \{|\nabla h|^2 + \omega |h|^2 - pQ_\omega^{p-1}|Q_\omega|^2\} \geq \lambda \|h\|_{H^1}^2.
\]

In [26], the authors use some modulation in the scaling, phase and translation parameters, to control these two direction.

In §2.3, we suppose that there exists a solution \(u_n\) of \((\text{NLS}_\Omega})\) for \(t \in [T_0, T_n]\) that satisfies some uniform estimate with initial data \(u_n(T_n)\) and \(T_n\) is an increasing sequence of times. Then by compactness argument we construct a solution \(u\) of \((\text{NLS}_\Omega})\) for \([T_0, +\infty)\), with initial data \(u(T_0)\) and \(T_0 > 0\), which concludes the proof of Theorem 1.1.

In Section 3, we prove the existence of the solution \(u_n\) and the uniform estimate assumed in the previous section. For this, we use a modulation for large time in the phase and translation
parameters in the decomposition of the solution as above to obtain some orthogonality conditions. Next, we define a maximal time interval on which hold a suitable exponential estimates of the modulation parameters, the uniform estimate used in §2.3 and others terms expressed in function of \( Y^+ \) and \( Y^- \). In order to control these estimates, we use a bootstrap argument with the coercivity property of the linearized operator. Indeed, the linearized operator \((\mathcal{L}, \cdot, \cdot)\) is positive definite up to the four directions \( Q, \partial_x Q \) and \( Y^\pm \), see [9] and [10]. As in the subcritical case, the two directions \( Q_\omega, \nabla Q_\omega \) are still be controlled due to the orthogonality conditions given by the modulation. The direction \( Y^+ \) is stable in some sense that can be controlled but the other one \( Y^- \) is unstable and cannot be controlled by a scaling argument, even if we introduce an extra parameters in the modulation. Thus, we have to use a topological argument to control this unstable direction and to conclude the proof of the uniform estimate on \([T_0,T_\alpha]\).

**Sketch of the proof of Theorem 1.4.**

In section 4, we will give the proof of Theorem 1.4. We construct a contraction mapping of a complete metric space to itself using the Duhamel formula. By fixed point theorem we prove the existence of a smooth correction \( r_\omega(t,x) \) such that \( u(t,x) = R(t,x) + r_\omega(t,x) \) is a solution of \((\text{NLS}_\Omega)\), where \( R(t,x) = e^{i\left(\frac{1}{2}(x,v) - \frac{1}{4}|v|^2 t + t \omega\right)} Q_\omega(x - tv) \Psi(x) \).

We have

\[
(i\partial_t + \Delta) R(t,x) = -\Psi(x) |H(t,x)|^2 H(t,x) + 2\nabla \Psi(x) \nabla H(t,x) + \Delta \Psi(x) H(t,x),
\]

where \( H(t,x) = e^{i\left(\frac{1}{2}(x,v) - \frac{1}{4}|v|^2 t + t \omega\right)} Q_\omega(x - tv) \).

We look for \( r_\omega \in C([T_0, +\infty), H^2(\Omega) \cap H^1_0(\Omega)) \) such that

\[
(i\partial_t + \Delta) r_\omega = -|R + r_\omega|^2 (R + r_\omega) + \Psi |H|^2 H - 2\nabla \Psi \nabla H - \Delta \Psi H,
\]

\[
r_\omega(t) \longrightarrow 0 \quad t \longrightarrow +\infty \quad \text{in} \quad H^2(\Omega) \cap H^1_0(\Omega).
\]

We shall look for solutions of \((\text{NLS}_\Omega)\) in the following space

\[
E = \left\{ r_\omega \in C \left([T_0, +\infty), H^2(\Omega) \cap H^1_0(\Omega)\right), \ |r_\omega|_E < \infty \right\},
\]

\[
|r_\omega|_E = \sup_{t \geq T_0} \left\{ e^{3\sqrt{\omega}|v|} t \left( \frac{1}{|v|^3} (|r_\omega|_{H^2(\Omega)} + |r_\omega|_{L^2(\Omega)}) \right) \right\}.
\]

Let

\[
\Phi : (B_E, d_E) \longrightarrow (B_E, d_E)
\]

\[
r_\omega \longmapsto \Phi (r_\omega) = -i \int_{t}^{+\infty} S(t - \tau) \left( |R + r_\omega|^2 (R + r_\omega) + \Psi |H|^2 H - 2\nabla \Psi \nabla H - \Delta \Psi H \right) d\tau.
\]

Where \( S(t) \) is the unitary group of the linear Schrödinger equation on \( \Omega \) with Dirichlet boundary conditions.

\[
B_E := B_E(0,1) = \{ h \in E : |h|_E \leq 1 \} \quad \text{and} \quad d_E(h, g) = |h - g|_E.
\]

One can check that \((B_E, d_E)\) is a complete metric space.

Our goal is to solve the integral formulation of (1.7) by the contraction mapping principle. Using the high velocity assumption, we prove that \( \Phi \) is stable on \( B_E \) and it is a contraction
mapping. Thus, by the fixed point theorem we conclude that there exists a unique solution \( r_\omega \) of (1.7) on \( E \).

Appendix A contains the proof of the coercivity property of the linearized Schrödinger operator, the local existence of the equation on \( H^s(\Omega) \), for some \( s \), the modulation for time independent function and other technical results.

Appendix B, contains the computation of some estimates used on the proof of Theorem 1.4.

**Notation:**
If \( a \) and \( b \) are two functions of \( t \) and if \( b \) is positive, we write \( a = O(b) \) when there exists a constant \( C > 0 \) independent of \( t \) such that \( |a(t)| \leq C b(t) \) for all \( t \).

For \( h \in \mathbb{C} \), we denote \( h_1 = \text{Re} h \) and \( h_2 = \text{Im} h \).

Throughout this paper, \( C \) denotes a positive constant independent of \( t \), that may change from line to line and may depend on \( \omega \) and \( \Omega \).

Denote by \( \|\cdot\| \) the IR\(^d\)-norm with \( d = 1, 2, 3 \).

For simplicity, we will write \( \Delta := \Delta_\Omega \).

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2. Construction of the solution assuming uniform estimates

2.1. Properties of the ground state. We recall some well-known properties of the ground state and we refer the reader to [31], [23], [30, Appendix B] and [16] for more details.

**Proposition 2.1** (Exponential decay of \( Q \)). Let \( Q \) be a solution of (1.2) with \( \omega = 1 \) then the following properties hold:
1) \( Q \in W^{3,p}(\mathbb{R}^3) \) for every \( 2 \leq p < +\infty \). In particular, \( Q \in C^2 \) and \( |D^\beta Q(x)| \rightarrow 0 \), as \( |x| \rightarrow \infty \), for all \( |\beta| \leq 2 \).
2) there exists \( \delta > 0 \) such that
\[
e^{\delta|x|} \left( |Q(x)| + |\nabla Q(x)| + |\nabla^2 Q(x)| \right) \in L^\infty(\mathbb{R}^3).
\]

**Proof.** See [2] and [4, chapter 8] for the proof. \( \square \)

We can deduce \( Q_\omega(x) \) from \( Q(x) : Q_\omega(x) = \omega^{\frac{1}{p-1}} Q(\sqrt{\omega} x) \).

Then, there exits \( C \) and \( \delta > 0 \) such that
\[
|Q_\omega(x)| + |\nabla Q_\omega(x)| + |\nabla^2 Q_\omega(x)| \leq C e^{-\delta \sqrt{\omega}|x|}.
\]

2.2. Spectral theory of the linearized operator. Consider a solution \( u \) of the nonlinear Schrödinger equations close to the soliton \( e^{it} Q \). Let \( h \in \mathbb{C} \) such that \( h = h_1 + ih_2 \).

We can write \( u(t, x) \) as,
\[
u(t, x) = e^{it} (Q(t, x) + h(t, x)).
\]
Note that $h$ is the solution of the following equation,

$$\partial_t h + \mathcal{L} h = S(h), \quad \mathcal{L} := \begin{pmatrix} O & -L^- \\ L^+ & 0 \end{pmatrix}$$

where $S(h)$ contains the nonlinear terms on $h$ and the self-adjoint operators $L^-$ and $L^+$ are defined by:

$$L^+ h_1 = -\Delta h_1 + h_1 - p Q^{p-1} h_1 \quad \text{and} \quad L^- h_2 = -\Delta h_2 + h_2 - Q^{p-1} h_2.$$ 

In all the sequel, we assume $\frac{7}{3} < p < 5$. The spectral properties of the linearized operator $\mathcal{L}$ around the ground state are well-known and we refer to [32], [13] and [29] for the following Proposition.

**Proposition 2.2.** Let $\sigma(\mathcal{L})$ be the spectrum of the operator $\mathcal{L}$ defined on $L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ and let $\sigma_{ess}(\mathcal{L})$ be its essential spectrum. Then

$$\sigma_{ess}(\mathcal{L}) = \{ i\xi : \xi \in \mathbb{R}, |\xi| \geq 1 \}, \quad \sigma(\mathcal{L}) \cap \mathbb{R} = \{-e_0, 0, e_0\} \quad \text{with} \quad e_0 > 0.$$ 

Moreover, $e_0$ and $-e_0$ are simple eigenvalues of $\mathcal{L}$ with eigenfunctions $\mathcal{Y}^+$ and $\mathcal{Y}^-$,

$$\mathcal{L} \mathcal{Y}^\pm = \pm e_0 \mathcal{Y}^\pm,$$

and $\overline{\mathcal{Y}^+} = \mathcal{Y}^-$. Furthermore $\mathcal{Y}^+, \mathcal{Y}^- \in \mathcal{S}(\mathbb{R}^3)$, in fact there exists $\delta > 0$ and $C > 0$ such that

$$|\mathcal{Y}^\pm| + |\nabla \mathcal{Y}^\pm| \leq C e^{-\delta |x|}.$$ 

**Remark 2.3.** The null-space of $L^+$ is spanned by $\partial_{x_1} Q$, $\partial_{x_2} Q$ and $\partial_{x_3} Q$ and the null-space of $L^-$ is spanned by $Q$.

Moreover, the operators $L^+$ and $L^-$ satisfies the following coercivity property for the mass super-critical case.

**Lemma 2.4 (Coercivity).** There exists $C > 0$ such that for all $h = h_1 + i h_2 \in H^1(\mathbb{R}^3)$, we have

$$\tag{2.2} \|h\|_{H^1}^2 \leq C \left[ (L^+ h_1, h_1) + (L^- h_2, h_2) + \sum_{j=1}^3 \left( \int \partial_{x_j} Q h_1 \right)^2 + \left( \int Q h_2 \right)^2 \right. \left. + \left( \text{Im} \int \mathcal{Y}^+ \bar{h} \right)^2 + \left( \text{Im} \int \mathcal{Y}^- \bar{h} \right)^2 \right].$$

**Proof.** The proof of this result is well known and for the sake of completeness, we will give it in Appendix A. \qed

**Remark 2.5.** The scalar product $(L^+ h_1, h_1)$ and $(L^- h_2, h_2)$ must be understood in the sense of the quadratic form $\int \| h_1 \|^2 + \| h_2 \|^2 + |h|^2 - \int p Q^{p-1} h_1^2 - \int Q^{p-1} h_2^2.$

Moreover, Lemma 2.4 is still valid with $h \in H^1_0(\Omega)$. Indeed, $h$ can be extended to a $H^1(\mathbb{R}^3)$ function by letting $h(x) = 0$ for $x \in \Theta$.

Finally, we extend the Proposition 2.2 to the linearized operator $\mathcal{L}_\omega$ around the stationary soliton $e^{it \omega} Q_\omega$, by a simple scaling argument.

**Corollary 2.6 ([6]).** Let $\omega > 0$ and $h \in \mathcal{C}$ such that $h = h_1 + h_2$. The linearized operator $\mathcal{L}_\omega$ is defined by

$$\mathcal{L}_\omega h = -L^- h_2 + i L^+_\omega h_1,$$

where,

$$L^+_\omega h_1 = -\Delta h_1 + \omega h_1 - p Q^{p-1}_\omega h_1 \quad \text{and} \quad L^-\omega h_2 = -\Delta h_2 + \omega h_2 - Q^{p-1}_\omega h_2.$$
Moreover, the spectrum $\sigma(\mathcal{L}_\omega)$ of $\mathcal{L}$ satisfies

$$\sigma(\mathcal{L}_\omega) \cap \mathbb{R} = \{-e_\omega, 0, e_\omega\}, \text{ where } e_\omega = \omega^\frac{3}{2}e_0 > 0.$$ 

Furthermore, $e_\omega$ and $-e_\omega$ are simple eigenvalues of $\mathcal{L}_\omega$ with eigenfunctions $\mathcal{Y}_\omega^+$ and $\mathcal{Y}_\omega^-$:

$$\mathcal{L}_\omega \mathcal{Y}_\omega^\pm = \pm e_\omega \mathcal{Y}_\omega^\pm,$$

where,

$$\mathcal{Y}_\omega^+(x) = \omega^\frac{1}{2} \mathcal{Y}^+(\sqrt{\omega}x) \quad \text{and} \quad \mathcal{Y}_\omega^+ = \overline{\mathcal{Y}_\omega^-}.$$ 

**Remark 2.7.** The null-space of $\mathcal{L}_\omega^+$ is spanned by $\partial_{x_1}Q_\omega$, $\partial_{x_2}Q_\omega$ and $\partial_{x_3}Q_\omega$ and the null-space of $\mathcal{L}_\omega^-$ is spanned by $Q_\omega$.

### 2.3. Compactness argument

Denote:

$$R(t, x) = Q_\omega(x - tv)\Psi(x)e^{i\phi(t,x)}$$

$$Y_\pm(t, x) = \mathcal{Y}_\omega^\pm(x - tv)\Psi(x)e^{i\phi(t,x)},$$

where, $\phi(t, x) = \frac{1}{2}(x.v) - \frac{1}{4}|v|^2t + t\omega$.

Let $T_n \to \infty$ be an increasing sequence of times.

**Proposition 2.8.** There exists $n_0 \geq 0$, $T_0 > 0$ and $C > 0$ (independent of $n$) such that the following holds. For each $n \geq n_0$ there exists $\lambda_n := (\lambda_n^+) \in \mathbb{R}^2$ such that

$$|\lambda_n| \leq e^{-\delta \sqrt{|v|}T_n},$$

and the solution $u_n$ of

$$\begin{cases}
    i\partial_t u_n + \Delta u_n = -|u_n|^{p-1}u_n, \\
    u_n(T_n) = R(T_n) + i \lambda_n^+ Y_\pm(T_n),
\end{cases}$$

is defined on the interval time $[T_0, T_n]$ and satisfies

$$\forall t \in [T_0, T_n] \quad \|u_n(t) - R(t)\|_{H^1_0(\Omega)} \leq Ce^{-\delta \sqrt{|v|}t}.$$

**Proof.** We will assume this proposition to prove Theorem 1.1 and we postpone the proof of Proposition 2.8 to Section 3.

Now, we will start the proof of the Theorem 1.1 assuming the main Proposition 2.8. The proof is based on the compactness argument and the uniform estimate (2.4). Renumbering the indices, we can take $n_0 = 0$ in Proposition 2.8.

**Proof of Theorem 1.1 assuming Proposition 2.8.** The proof proceeds in several steps.

- **Step 1**: "compactness argument" The Proposition 2.8 implies that there exists a sequence $u_n(t)$ of solution defined on $[T_0, T_n]$ such that

$$\forall n \in \mathbb{N}, \forall t \in [T_0, T_n], \quad \|u_n(t) - R(t)\|_{H^1_0(\Omega)} \leq Ce^{-\delta \sqrt{|v|}t}.$$ 

**Lemma 2.9.**

$$\lim_{M \to +\infty} \sup_{n \in \mathbb{N}} \int_{|x| \geq M} u_n^2(T_0, x) \, dx = 0.$$
Proof. The proof of the lemma is the same as in [26] for the construction of multi-soliton solutions of (NLS) for the subcritical case on $\mathbb{R}^d$. We give it for the sake of complet.

Let $\varepsilon > 0$ and $T_\varepsilon \geq T_0$ such that: $C^2 e^{-2\delta \sqrt{|v|}T_\varepsilon} < \varepsilon$, where $C$ and $\delta$ are the same constant as in the Proposition 2.8.

For $n$ large enough, so that $T_n \geq T_\varepsilon$ and due to (2.4), we have

$$\int_{\Omega} |u_n(T_\varepsilon) - R(T_\varepsilon)|^2 dx \leq C^2 e^{-2\delta \sqrt{|u|}T_\varepsilon} \leq \varepsilon.$$ 

Let $M(\varepsilon) > 0$ such that

$$\int_{|x| \geq M(\varepsilon)} |R(T_\varepsilon)|^2 dx < \varepsilon,$$

by direct computation,

$$\int_{|x| \geq M(\varepsilon)} |u_n(T_\varepsilon)|^2 dx \leq 4\varepsilon.$$

Now consider a $C^1$ cut-off function $f : \mathbb{R} \to [0, 1]$ such that $f \equiv 0$ on $[-\infty, 1]$ ; $0 < f' < 2$ on $(1, 2)$ ; $f \equiv 1$ on $(2, +\infty)$.

For $K_\varepsilon > 0$ to be specified later, we can check that

$$(2.5) \quad \frac{d}{dt} \int_{\Omega} |u_n(t)|^2 f \left( \frac{|x| - M(\varepsilon)}{K_\varepsilon} \right) dx = -\frac{2}{K_\varepsilon} \text{Im} \int_{\Omega} u_n(t) \left( \nabla \bar{u}_n, \frac{x}{|x|} \right) f' \left( \frac{|x| - M(\varepsilon)}{K_\varepsilon} \right) dx.$$ 

From Proposition 2.8, $\exists \alpha > 0$, $\forall n$ and $\forall t \geq T_0$, $\|u_n(t)\|^2_{H^1} \leq \alpha$. Using (2.5) we get

$$\left| \frac{d}{dt} \int_{\Omega} |u_n(t)|^2 f \left( \frac{|x| - M(\varepsilon)}{K_\varepsilon} \right) \right| \leq \frac{4}{K_\varepsilon} \|u_n(t)\|^2_{H^1} \leq \frac{4}{K_\varepsilon} \alpha.$$ 

Now, we choose $K_\varepsilon > 0$ independently of $n$ such that

$$K_\varepsilon \geq \left( \frac{T_\varepsilon - T_0}{\varepsilon} \right) 4\alpha,$$

which yields

$$\left| \frac{d}{dt} \int_{\Omega} |u_n(t)|^2 f \left( \frac{|x| - M(\varepsilon)}{K_\varepsilon} \right) \right| \leq \frac{\varepsilon}{T_\varepsilon - T_0}.$$ 

Integrating on the time interval $[T_0, T_\varepsilon]$, we get

$$\int_{\Omega} |u_n(T_0)|^2 f \left( \frac{|x| - M(\varepsilon)}{K_\varepsilon} \right) dx - \int_{\Omega} |u_n(T_\varepsilon)|^2 f \left( \frac{|x| - M(\varepsilon)}{K_\varepsilon} \right) dx$$

$$\leq \int_{T_0}^{T_\varepsilon} \left| \frac{d}{dt} \int_{\Omega} |u_n(t)|^2 f \left( \frac{|x| - M(\varepsilon)}{K_\varepsilon} \right) dx \right| dt$$

$$\leq \varepsilon.$$ 

Hence,

$$\int_{\Omega} |u_n(T_0)|^2 f \left( \frac{|x| - M(\varepsilon)}{K_\varepsilon} \right) dx \leq \varepsilon + \int_{\Omega} |u_n(T_\varepsilon)|^2 f \left( \frac{|x| - M(\varepsilon)}{K_\varepsilon} \right) dx.$$
Due to the properties of $f$, we have
\[
\int_{|x|>2K_\varepsilon+M(\varepsilon)} |u_n(T_0)|^2 \, dx \leq \int_{\Omega} |u_n(T_0)|^2 f \left( \frac{|x| - M(\varepsilon)}{K_\varepsilon} \right) \, dx \\
\leq \varepsilon + \int_{|x| \geq M(\varepsilon)} |u_n(T_\varepsilon)|^2 \, dx \\
\leq \varepsilon + 4\varepsilon = 5\varepsilon.
\]
This concludes the proof of the lemma. □

Due to the main proposition we have
\[
\|u_n(T_0)\|_{H^1_0(\Omega)} \leq \alpha.
\]
Since $H^1_0$ is a Hilbert space, there exists a subsequence of $(u_n(t))_n$ that we still denote by $(u_n(t))_n$ to simplify notation and $U_0 \in H^1_0(\Omega)$ such that
\[
u_n(T_0) \rightharpoonup U_0 \quad \text{in} \quad H^1_0(\Omega), \quad \text{as} \quad n \to +\infty.
\]
By the compactness of the embedding of $H^1(\{|x| \leq A\})$ into $L^2(\{|x| \leq A\})$, we have
\[
u_n(T_0) \rightarrow U_0 \quad \text{in} \quad L^2_{loc}.
\]
By the Lemma 2.9, we get $u_n(T_0) \rightarrow U_0$ in $L^2(\Omega)$.

Now using the following interpolation inequality
\[
\forall s \in [0,1], \quad \|u_n(t) - U_0\|_{H^s(\Omega)} \leq \|u_n(t) - U_0\|_{L^2(\Omega)}^{1-s} \|u_n(t) - U_0\|_{H^1_0(\Omega)}^s,
\]
we obtain,
\[
u_n(T_0) \rightarrow U_0 \quad \text{in} \quad H^s(\Omega), \quad \forall s \in [0,1).
\]

• Step 2: Construction of the solution.

**Lemma 2.10** (Well posedness in $H^s(\Omega)$, $s \in [s_p,1)$). Denote by $s_p = \frac{3}{2} - \frac{3}{p+1}$ and let $s \in [0,1)$ such that $s_p \leq s$.

Let $u_0 \in H^s(\Omega)$ then there exists $\tau > 0$ which depends only on $\|u_0\|_{H^s(\Omega)}$, such that for any $T \in [0,\tau]$, the nonlinear Schrödinger equation (NLS$_{\Omega}$) admits a unique solution $u \in C([0,T],H^s(\Omega))$.

Furthermore, the solution $u$ can be extended to a maximal existence interval $[0,T_+)$ and the following alternative holds,

Either $T_+ = +\infty$ (the solution is global) or $T_+ < +\infty$ (the solution blows up in finite time) and
\[
\lim_{t \to T_+} \|u(t,\cdot)\|_{H^s} = +\infty.
\]

**Proof.** see Appendix A. □
Due to the Lemma 2.10, the equation (NLS$_{\Omega}$) is well-posed in $H^s(\Omega)$, for $s_p \leq s < 1$. Let $\tilde{u}$ be the maximal solution of

$$
\begin{cases}
  i\partial_t \tilde{u} + \Delta \tilde{u} = -|\tilde{u}|^{p-1}\tilde{u} & \forall (t, x) \in [T_0, T) \times \Omega, \\
  \tilde{u}(T_0, x) = U_0 & \forall x \in \Omega,
\end{cases}
$$

(2.7)

By (2.6) we have

$$
u_n(T_0) \longrightarrow U_0 = \tilde{u}(T_0, x) \text{ in } H^s(\Omega), \forall s \in [s_p, 1).
$$

(2.8)

For $n$ large enough, $u_n(t)$ is defined for all $t \in [T_0, T)$ and by the continuity of the flow we have

$$
u_n(t) \longrightarrow \tilde{u}(t) \text{ in } H^s(\Omega), \forall s \in [s_p, 1).
$$

Due to the main Proposition 2.8, we know that for $n$ large enough $u_n(t)$ is uniformly bounded in $H^1_0$. Then necessarily,

$$
\forall t \in [T_0, T), \quad u_n(t) \rightharpoonup \tilde{u}(t) \text{ in } H^1_0(\Omega).
$$

Using the property of weak convergence and by the main proposition, it follows that

$$
\forall t \in [T_0, T), \quad \|\tilde{u}(t) - R(t)\|_{H^1_0} \leq \liminf \|u_n(t) - R(t)\|_{H^1_0} \leq Ce^{-\delta\sqrt{\|v\|}t}.
$$

In particular we deduce that, $\tilde{u}$ is bounded in $H^1_0(\Omega)$. Due to the blow up alternative we get $T = +\infty$. Finally, we have $\tilde{u} \in C([T_0, +\infty), H^1_0(\Omega))$ and by (2.4) in Proposition 2.8,

$$
\forall t \in [T_0, +\infty), \quad \|\tilde{u}(t) - R(t)\|_{H^1_0} \leq e^{-\delta\sqrt{\|v\|}t},
$$

which concludes the proof of the Theorem 1.1

\square

3. Proof of the uniform estimate

3.1. Bootstrap and topological arguments. In this section, we prove the main Proposition 2.8. We use some modulation in the phase and translation parameters in the decomposition of the solution to obtain the orthogonality conditions. Next, we use a bootstrap argument to control these parameters and some scalar product that are related to the size of the soliton. Finally, to conclude the proof we use a topological argument to control the unstable direction.

Remark 3.1. In this section, to simplify notations we will write $r$ instead of $r_\omega$ and we will drop the index $n$ for most variables. Hence, we will write $u$ for $u_n$, $\lambda^\pm$ for $\lambda^\pm_n$ etc. Except the sequence of times that will be written with the index. As Proposition 2.8 is proved for given $n$, this should not be a source of confusion. We possibly drop the first terms of the sequence $T_n$, so that, for all $n$, $T_n$ is large enough for our purposes.

3.1.1. Modulated final data.

Lemma 3.2 (modulation for time independent function). There exists $C, \epsilon > 0$ such that the following holds.

Given $x \in \mathbb{R}^3$ and $\theta \in \mathbb{R}$. If $u(x) \in L^2$ is such that

$$
\|u - R\|_{L^2} \leq \epsilon.
$$

Then there exists modulation parameters $y = (y_i)_i \in \mathbb{R}^3$ and $\mu \in \mathbb{R}$, such that setting

$$
r(x) = u(x) - \tilde{R}(x),
$$

\begin{align*}
&\text{with } \|r\|_{L^2} \leq \epsilon, \\
&\text{and } \|y - \mu \omega\|_\infty \leq \epsilon, \quad \|\mu - \lambda^\pm\|_{L^2(\mathbb{R})} \leq \epsilon.
\end{align*}
the following holds
\[ \| r \|_{L^2} + |y| + |\mu| \leq C \| u - R \|_{L^2}, \]
and
\[ \text{Re} \int r(x) \partial_x \tilde{Q}_\omega(x) \Psi(x) e^{-i \frac{1}{2} (x,v) + \theta} e^{-i \mu} dx = \text{Im} \int r(x) \overline{R}(x) dx = 0, \quad j = 1, 2, 3, \]
where,
\[ R(x) = Q_\omega(x - \alpha(x)) \Psi(x) e^{i \frac{1}{2} (x,v) + \theta}, \]
\[ \tilde{Q}_\omega(x) = Q_\omega(x - \alpha - y), \]
\[ \tilde{R}(x) = \tilde{Q}_\omega(x) \Psi(x) e^{i \frac{1}{2} (x,v) + \theta} e^{i \mu}. \]
Furthermore, \( u \mapsto (r, y, \mu) \) is a smooth \( C^1 \)-diffeomorphism.

Proof. see Appendix A. \( \square \)

Note that the previous lemma applies to time independent functions. A consequence of this modulation in the decomposition of fixed \( u \) is the the following result on a solution \( u(t) \) of (2.3).

**Corollary 3.3.** There exists \( C, \epsilon > 0 \) such that the following holds for all \( t \in [T,T_n] \), for \( T > T_0 \), if \( u(t, \cdot) \in L^2_t \) satisfies
\[ \| u(t) - R(t) \|_{L^2} \leq \epsilon. \]
Then there exits a \( C^1 \)-functions \( y : [T,T_n] \to \mathbb{R}^3 \) and \( \mu : [T,T_n] \to \mathbb{R} \) such that if we set
\[ r(t, x) = u(t, x) - \tilde{R}(t, x), \]
the following holds
\[ \| r(t) \|_{L^2} + |y(t)| + |\mu(t)| \leq C \| u(t) - R(t) \|_{L^2}, \]
and
\[ \text{Re}\int r(t, x) \partial_x \tilde{Q}_\omega(t, x) \Psi(x) e^{-i \frac{1}{2} (x,v) + \theta(t)} e^{-i \mu(t)} dx = 0 \quad j = 1, 2, 3, \]
\[ \text{Im}\int r(t, x) \overline{R}(t, x) dx = 0, \]
where,
\[ R(t, x) = Q_\omega(x - \alpha(t)) \Psi(x) e^{i \frac{1}{2} (x,v) + \theta(t)}, \text{ with } \alpha(t) := tv \text{ and } \theta(t) := -\frac{1}{4} |v|^2 t + t \omega. \]
\[ \tilde{Q}_\omega(t, x) = Q_\omega(x - \alpha(t) - y(t)). \]
\[ \tilde{R}(t, x) = \tilde{Q}_\omega(t, x) \Psi(x) e^{i \frac{1}{2} (x,v) + \theta(t)} e^{i \mu(t)}. \]

Proof. For small \( \lambda \), the solution \( u(t) \) is closed to the soliton \( R(t) \) for \( t \) close to \( T_n \). Assume that \( u(t) \) satisfies (3.3) on \( [T,T_n] \). Applying Lemma 3.2 to \( u(t) \) for any \( t \in [T,T_n] \) and since the map \( t \mapsto u(t) \) is continuous in \( H^1_0 \), we obtain the existence of continuous functions \( y : [T,T_n] \to \mathbb{R}^3 \) and \( \mu : [T,T_n] \to \mathbb{R} \) such that (3.1) and (3.2) holds. \( \square \)
Notation: $u(t)$ is defined and modulable around $R(t)$ for $t$ close to $T_n$, in the sense of the previous Corollary.

$R(t, x) = Q_\omega(x - tv)\Psi(x)e^{i\varphi(t, x)}$, where $\varphi(t, x) = \frac{1}{2}x.v - \frac{1}{4}|v|^2t + t\omega$.

$\bar{Q}_\omega(t, x) = Q_\omega(x - tv - y(t))$.

$\bar{R}(t, x) = \bar{Q}_\omega(t, x)\Psi(x)e^{i\bar{\varphi}(t, x)}$, where $\bar{\varphi}(t, x) = \frac{1}{2}x.v - \frac{1}{4}|v|^2t + t\omega + \mu(t)$.

$\bar{\mathcal{Y}}^\pm_\omega(t, x) = \mathcal{Y}^\pm_\omega(x - tv - y(t))$.

$\bar{\mathcal{Y}}_\omega(t, x) = e^{i\bar{\varphi}(t, x)}$ and $\quad \alpha^\pm(t) = \text{Im} \int \bar{\mathcal{Y}}^\pm_\omega(t, x)\bar{\eta}(t, x)dx.$

$\bar{L}_\omega h_1 = -\Delta h_1 + \omega h_1 - p\bar{Q}_\omega^{p-1} h_1$ and $\quad \bar{L}_\omega h_2 = -\Delta h_2 + \omega h_2 - \bar{Q}_\omega^{p-1} h_2$.

**Lemma 3.4** (Modulated final data). There exists $C > 0$ (independent of $n$) such that for all $\alpha^+ \in B_{\mathcal{H}}(e^{-\delta\sqrt{\omega}|v|}T_n)$ there exists a unique $\lambda$ such that

$$|\lambda| \leq C |\alpha^+|,$$

and the modulation parameters $(r(T_n), y(T_n), \mu(T_n))$ of $u(T_n)$ satisfies

$$\begin{cases}
\alpha^+(T_n) = \alpha^+, \\
\alpha^-(T_n) = 0.
\end{cases}$$

Proof. See Appendix A.

Let $T_0$ to be specified later, independent of $n$. Let $\alpha^+$ to be chosen, $\lambda$ be given by Lemma 3.4 and let $u$ be the corresponding solution of (2.3). We now define the maximal time interval $[T(\alpha^+), T_n]$, on which suitable exponential estimates hold.

**Definition 3.5.** Let $T(\alpha^+)$ be the infimum of $T \geq T_0$ such that the following properties hold for all $t \in [T, T_n]$:

Closeness to $R(t)$:

$$\|u(t) - R(t)\|_{\mathcal{H}^1} \leq \varepsilon.$$  

In particular, this ensures that $u(t)$ is modulable around $R(t)$ in the sense of Lemma 3.2. Estimates on the modulation parameters: There exists $M > 0$ and $M' > 0$ to be specified later,

$$\begin{align*}
\|r(t)\|_{\mathcal{H}^1} &\leq Me^{-\delta\sqrt{\omega}|v|t} \\
|y(t)| &\leq M'e^{-\delta\sqrt{\omega}|v|t} \\
|\mu(t)| &\leq M'e^{-\delta\sqrt{\omega}|v|t} \\
|\alpha^\pm(t)| &\leq e^{-\delta\sqrt{\omega}|v|t}.
\end{align*}$$

Note that, if for all $n$ we can find $\alpha^+$ such that $T(\alpha^+) = T_0$ then the Proposition 2.8 is proved. It remains to prove the existence of such value of $\alpha^+$.

Denote $h(t, x) = e^{-i\bar{\varphi}(t, x)}r(t, x)$. Recall that,

$$u(t, x) = \bar{R}(t, x) + r(t, x) = e^{i\bar{\varphi}(t, x)}(\bar{Q}_\omega(t, x)\Psi(x) + h(t, x)).$$
Lemma 3.6. Let $t \in [T(\alpha^+), T_n]$ and let $C, \delta > 0$. We have

\begin{equation}
 i \partial_t h + \Delta h - \omega h + (\frac{p+1}{2}) \tilde{Q}_\omega^{p-1} \Psi^{p-1} h + (\frac{p-1}{2}) \tilde{Q}_\omega^{p-1} \Psi^{p-1} h + i v \nabla h - \frac{d\mu(t)}{dt} h \\
+ \tilde{Q}_\omega^p \Psi (\Psi^{p-1} - 1) + 2 \nabla \tilde{Q}_\omega \nabla \Psi + \tilde{Q}_\omega \Delta \Psi + i v \tilde{Q}_\omega \nabla \Psi - i \frac{dy(t)}{dt} \nabla \tilde{Q}_\omega \Psi - \frac{d\mu(t)}{dt} \tilde{Q}_\omega \Psi + \beta(t, x) = 0,
\end{equation}

where $\beta(t, x)$ is a remainder terms on $h$.

\begin{equation}
\left| \frac{d\mu(t)}{dt} \right| + \left| \frac{dy(t)}{dt} \right| \leq C \| h(t) \|_{H^1_0}^2 + C e^{-2\delta \sqrt{\omega} |v| t}.
\end{equation}

\begin{equation}
\left| \frac{d\alpha^\pm(t)}{dt} \right| \leq C \| h(t) \|_{H^3_0}^3 + C e^{-2\delta \sqrt{\omega} |v| t}.
\end{equation}

Proof. For the equation (3.8) of $h$ it suffices to plug the above expression of $u(t, x)$ on the nonlinear Schrödinger equation: $i \partial_t u + \Delta u = -|u|^{p-1} u$. Using, the elliptic equation (1.2) of $Q_\omega$ and the Taylor expansion for the nonlinear term, we get (3.8), with $\| \beta(t) \|_{L^2} \leq C \| h(t) \|_{H^1_0}^2$.

For the proof of (3.9) and (3.10), we claim the following estimates.

Claim 3.7.

\begin{align*}
\Im \int \partial_t \tilde{h}(t, x) \tilde{Q}_\omega(t, x) \Psi(x) dx &= \sum_{k=1}^{3} \Im \int \tilde{h}(t, x)(v_k + \frac{dy_k}{dt}(t)) \partial_{x_k} \tilde{Q}_\omega(t, x) \Psi(x) dx. \\
\Re \int \partial_t \tilde{h}(t, x) \partial_{x_j} \tilde{Q}_\omega(t, x) \Psi(x) dx &= \sum_{k=1}^{3} \Re \int \tilde{h}(t, x)(v_k + \frac{dy_k}{dt}(t)) \partial_{x_k} \partial_{x_j} \tilde{Q}_\omega(t, x) \Psi(x) dx, j = 1, 2, 3.
\end{align*}

Proof. It is just a consequence of the orthogonality conditions in Lemma 3.2. So, we have

\begin{equation}
\Re \int h(t, x) \partial_{x_j} \tilde{Q}_\omega(t, x) \Psi(x) dx = \Im \int h(t, x) \tilde{Q}_\omega(t, x) \Psi(x) dx = 0, j = 1, 2, 3.
\end{equation}

Differentiating each equality with respect to the time variable $t$, the Claim 3.7 follows. \(\square\)

Now let us estimate $\frac{dy(t)}{dt}$ and $\frac{d\omega(t)}{dt}$ in (3.9). Multiply by $\partial_{x_j} \tilde{Q}_\omega \Psi$ and take the imaginary part of the equation (3.8). Using the Claim 3.7 and the fact that $Q_\omega$ is radial, so that

\begin{equation}
\begin{cases}
Q_\omega(x_1, x_2, x_3) = Q_\omega(-x_1, x_2, x_3), \\
\partial_{x_1} Q_\omega(x_1, x_2, x_3) = -\partial_{x_1} Q_\omega(-x_1, x_2, x_3),
\end{cases}
\end{equation}

which yields

\begin{equation}
\int \partial_{x_1} Q_\omega(x_1, x_2, x_3) Q_\omega(x_1, x_2, x_3) dx = - \int \partial_{x_1} Q_\omega(x_1, x_2, x_3) Q_\omega(x_1, x_2, x_3) dx.
\end{equation}

Hence

\begin{equation}
\int \partial_{x_j} Q_\omega(t, x) Q_\omega(t, x) dx = 0, \quad \text{for } j = 1, 2, 3.
\end{equation}
We obtain the following equality on $\frac{dy(t)}{dt}$.

$$
\frac{dy_j(t)}{dt} \|\partial_{x_j} \tilde{Q}_t \psi\|^2_{L^2} = \int_{I^1_h} h_1(t, x) \frac{dy(t)}{dt} \cdot \nabla(\partial_{x_j} \tilde{Q}_t(t, x)) \psi(x) dx - \int_{I^1_h} h_2(t, x) \partial_{x_j} \tilde{Q}_t(t, x) \psi(x) dx
$$

$$
- \int_{I^2_h} \tilde{L}_1 h_1(t, x) \tilde{Q}_t(t, x) \psi(x) dx + \int_{I^2_h} p \tilde{Q}_t^{-1}(t, x) h_1(t, x)(\Psi^{-1}(x) - 1) dx
$$

$$
+ \int_{I^2_h} h_1(t, x) \partial_{x_j} \tilde{Q}_t(t, x) v.\nabla \Psi(x) dx + O(\|h(t)\|^2_{H^1_h}).
$$

Taking the scalar product with $\tilde{Q}_t \psi$ and the equation (3.8) on $h$. Using the same argument as above, we get the following equality on $\frac{du(t)}{dt}$.

$$
\frac{du(t)}{dt} \|\tilde{Q}_t \psi\|^2_{L^2} = \int_{I^1_h} h_2(t, x) \frac{dy(t)}{dt} \cdot \tilde{Q}_t(t, x) \psi(x) dx - \int_{I^1_h} \frac{du(t)}{dt} h_1(t, x) \tilde{Q}_t(t, x) \psi(x) dx
$$

$$
- \int_{I^2_h} \tilde{L}_1 h_1(t, x) \tilde{Q}_t(t, x) \psi(x) dx + \int_{I^2_h} p \tilde{Q}_t^{-1}(t, x) h_1(t, x)(\Psi^{-1}(x) - 1) dx
$$

$$
- \int_{I^2_h} h_2(t, x) \tilde{Q}_t(t, x) v.\nabla \psi(x) dx + \int_{I^2_h} \tilde{Q}_t^{-1}(t, x) \psi^2(x)(\Psi^{-1}(x) - 1) dx
$$

$$
+ \int_{I^2_h} \tilde{Q}_t^2(t, x) \Delta \psi(x) \psi(x) dx + O \left(\|h(t)\|^2_{H^1_h}\right).
$$

Summing the absolute values of the two equalities above and using the fact that

$$
\|\tilde{Q}_t \psi\|^2_{L^2} = \|Q_\omega\|^2_{L^2} + O(e^{-2\delta \sqrt{\alpha}|v|t}) \quad \text{and} \quad \|\nabla \tilde{Q}_t \psi\|^2_{L^2} = \|\nabla Q_\omega\|^2_{L^2} + O(e^{-2\delta \sqrt{\alpha}|v|t}),
$$

We obtain the left hand side on the estimate (3.9) Next, we have to estimate the right hand side in both equalities.

$$
|V^1_1| := \int_{I^1_h} h_1(t, x) \frac{dy(t)}{dt} \cdot \nabla(\partial_{x_j} \tilde{Q}_t(x)) \psi(x) dx \leq C_1 \frac{dy(t)}{dt} \|h(t)\|_{L^2}
$$

$$
\leq C_1 \frac{dy(t)}{dt} \|h(t)\|_{C^0(T_0)}
$$

$$
\leq \frac{1}{10} \frac{dy(t)}{dt} \|\partial_{x_j} Q_\omega\|^2_{L^2}.
$$

Provided

$$
M e^{-\delta \sqrt{\alpha}|v|T_0} \leq \frac{1}{10 C_1} \|\partial_{x_j} Q_\omega\|^2_{L^2}, \ j = 1, 2, 3.
$$
\[ |I^\mu_h| := \left| \frac{d\mu(t)}{dt} \int h_2(t,x)\partial_{x_j} \tilde{Q}_\omega(x)\Psi(x)dx \right| \leq C \left| \frac{d\mu(t)}{dt} \right| \| h(t) \|_{L^2} \]
\[ \leq C_2 \left| \frac{d\mu(t)}{dt} \right| M e^{-\delta \sqrt{\omega}|v|T_0} \]
\[ \leq \frac{1}{10} \left| \frac{d\mu(t)}{dt} \right| \| Q_\omega \|_{L^2}^2 , \]

If the following condition is satisfied,

\[ (3.13) \quad M e^{-\delta \sqrt{\omega}|v|T_0} \leq \frac{1}{10}C_2 \| Q_\omega \|_{L^2}^2 . \]

\[ |J^\gamma_h| := \left| \int h_2(t,x) \frac{dy(t)}{dt} \nabla \tilde{Q}_\omega(x)\Psi(x)dx \right| \leq C \left| \frac{dy(t)}{dt} \right| \| h(t) \|_{L^2} \]
\[ \leq C \left| \frac{dy(t)}{dt} \right| M e^{-\delta \sqrt{\omega}|v|T_0} \]
\[ \leq \frac{1}{10} \left| \frac{dy(t)}{dt} \right| \| \partial_{x_j} Q_\omega \|_{L^2}^2 , \]

If the condition (3.12) holds.

\[ |J^\mu_h| := \left| \frac{d\mu(t)}{dt} \int h_1(t,x)\tilde{Q}_\omega(x)\Psi(x)dx \right| \leq C \left| \frac{d\mu(t)}{dt} \right| \| h(t) \|_{L^2} \]
\[ \leq C \left| \frac{d\mu(t)}{dt} \right| M e^{-\delta \sqrt{\omega}|v|T_0} \]
\[ \leq \frac{1}{10} \left| \frac{d\mu(t)}{dt} \right| \| \partial_{x_j} Q_\omega \|_{L^2}^2 , \]

If the condition (3.13) is verified.

We next treat the terms \( I_h := I^\mu_h + I^\gamma_h \) and \( J_h := J^\mu_h + J^\gamma_h \) that depends on \( h \). We will estimate the main integral for both terms, where appears the self-adjoint operator \( \tilde{L}_\omega^- \) and \( \tilde{L}_\omega^- \).

\[ \left| \int \tilde{L}_\omega^- h_2(t,x)\partial_{x_j} \tilde{Q}_\omega(x)\Psi(x)dx \right| = \left| \int h_2(t,x)\tilde{L}_\omega^- \left( \partial_{x_j} \tilde{Q}_\omega(x)\Psi(x) \right)dx \right| \leq C \| h(t) \|_{H^1_0}. \]

Similarly, we can estimate the integral on \( \tilde{L}_\omega^- \). We obtain

\[ |I_h| + |J_h| \leq C \| h \|_{L^2} . \]

Finally, we have to estimate \( J_1 \) and \( J_2 \). Using the exponential decay of \( Q \) and the fact that \( \Delta \Psi \) and \( (\Psi^{p-1} - 1) \) have a compact support, we get

\[ |J_1 + J_2| := \left| \int \tilde{Q}_\omega^{p+1}(x)\Psi(x)^2(\Psi^{-1} - 1) + \int \tilde{Q}_\omega^2(x)\Delta \Psi \Psi dx \right| \leq C e^{-2\delta \sqrt{\omega}|v|t}. \]
We have proved the estimate (3.9), if conditions (3.12) and (3.13) on $M$ hold. For $T_0$ large enough,

\begin{equation}
Me^{-\delta \sqrt{\omega}|t|} \leq \frac{1}{10C'} \min \left( \|\partial_x Q_\omega\|_{L^2}^2, \|Q_\omega\|_{L^2}^2 \right),
\end{equation}

where $C' = \max(C_1, C_2)$.

Next, we have to prove the last estimate (3.10). Let us recall that

$$\alpha^\pm(t) = \text{Im} \int \tau(t, x) \tilde{Y}_\pm(t, x) dx = \text{Im} \int \overline{\mathcal{H}}(t, x) \tilde{Y}_\omega(t, x) \Psi(x) dx.$$ 

$$\frac{d}{dt} \alpha^\pm(t) = -\text{Im} \int \overline{\mathcal{H}}(t, x) \frac{dy(t)}{dt} . \nabla \tilde{Y}_\omega(t, x) \Psi(x) dx + \text{Im} \int \partial_t \overline{\mathcal{H}}(t, x) \tilde{Y}_\omega(t, x) \Psi(x) dx.$$

Due to (3.9) and the exponential decay properties of the eigenfunctions of the linearized operator. We get

$$|I_1| = \left| \text{Im} \int \overline{\mathcal{H}}(t, x) \frac{dy(t)}{dt} . \nabla \tilde{Y}_\omega(t, x) \Psi(x) dx \right| \leq C \left| \frac{dy(t)}{dt} \right| \|h\|_{L^2} \leq C \|h(t)\|^2_{H^1_0} + Ce^{-\delta \sqrt{\omega} |t|}.$$

One can check that the second integral $I_2$ will be simplified with a term from $I_3$.

Now, let us estimate $I_3$. For this we have to use the equation (3.8) of $h$. One can see that the main terms is the following

$$\partial_t \overline{\mathcal{H}} = -i \Delta \overline{\mathcal{H}} + i \omega \overline{\mathcal{H}} - i \left( \frac{p + 1}{2} \right) \overline{Q}_\omega^{p-1} \Psi^{p-1} \overline{\mathcal{H}} - i \left( \frac{p - 1}{2} \right) \overline{Q}_\omega^{p-1} \Psi^{p-1} h + f$$

Where $f$ contains all others terms of the equation (3.8). Let $h = h_1 + ih_2$.

$$-i \Delta \overline{\mathcal{H}} + i \omega \overline{\mathcal{H}} - i \left( \frac{p + 1}{2} \right) \overline{Q}_\omega^{p-1} \Psi^{p-1} \overline{\mathcal{H}} - i \left( \frac{p - 1}{2} \right) \overline{Q}_\omega^{p-1} \Psi^{p-1} h = i \overline{\cal L}_\omega^+ h_1 + \overline{\cal L}_\omega^- h_2 + \overline{Q}_\omega^{p-1} h_2 (1 - \Psi^{p-1}) + i p \overline{Q}_\omega^{p-1} h_1 (1 - \Psi^{p-1}).$$

Multiply (3.8) by $\tilde{Y}_\omega^\mp(t, x) \Psi(x)$ and take the imaginary part, we obtain $I_3$ on the left hand side. The terms containing the linearized operator will be treated later. To estimate the other terms, we use the fact that $Q_\omega$ and $Y_\omega^\pm$ are radial, exponentially decaying at infinity and the compact support of $\nabla \Psi$ and $(1 - \Psi^{p-1})$. Also, we have to use the estimate (3.9) to obtain the right hand side of the estimate (3.10).

To complete the proof we have to compute the terms of the linearized operator. Let $y_1^\mp(t, x) = \text{Re}(\tilde{Y}_\omega^\mp(t, x))$ and $y_2^\mp(t, x) = \text{Im}(\tilde{Y}_\omega^\mp(t, x))$. Thus,

$$\begin{align*}
\overline{\cal L}_\omega^+ y_1^\mp &= \mp e_\omega y_2^\mp, \\
\overline{\cal L}_\omega^- y_2^\mp &= \mp e_\omega y_1^\mp.
\end{align*}$$

\begin{equation}
(3.15)
\end{equation}
Recall that $L^\pm$ are self-adjoint operator.

\[
\text{Im} \int (i \tilde{L}_\omega^+ h_1 + \tilde{L}_\omega^- h_2)(y_1^\mp + iy_2^\mp) \Psi \, dx = \text{Im} \int i (\tilde{L}_\omega^+ h_1) y_1^\mp \Psi + i (\tilde{L}_\omega^- h_2) y_2^\mp \Psi \, dx \\
= \text{Im} \int i h_1 (\tilde{L}_\omega^+ y_1^\mp \Psi) + i h_2 (\tilde{L}_\omega^- y_2^\mp \Psi) \, dx \\
= \text{Im} \int i h_1 (\mp e_\omega y_2^\mp \Psi) + i h_2 (\pm e_\omega y_1^\mp \Psi) \, dx + O(e^{-2\delta\sqrt{\omega}|t|}) \\
= \mp e_\omega \text{Im} \int \bar{h} \bar{y}^\mp \Psi \, dx + O(e^{-2\delta\sqrt{\omega}|t|}) \\
= \mp e_\omega \alpha^\pm (t, x) + O(e^{-2\delta\sqrt{\omega}|t|}).
\]

This concludes the proof of the Lemma 3.6

\[\square\]

3.1.2. Control of the modulation parameters. We claim the following estimate of $v(t), \mu$ and $y$ on $[T(\alpha^+), T_n]$.

**Lemma 3.8** (Control of $r, y$ and $\mu$). For $T_0$ large enough independent of $n$ and $\forall \alpha^+$ such that

\[|\alpha^+| \leq e^{-\delta\sqrt{\omega}|v|T_n}.
\]

the following holds

(3.16) \[\forall t \in [T(\alpha^+), T_n], \quad \|u(t) - R(t)\|_{H^2} \leq Ce^{-\delta\sqrt{\omega}|v|t} \leq \frac{\epsilon}{2}
\]

(3.17) \[\|r(t)\|_{H^2} \leq \frac{M}{2} e^{-\delta\sqrt{\omega}|v|t}
\]

(3.18) \[|\mu(t)| + |y(t)| \leq \frac{M'}{2} e^{-\delta\sqrt{\omega}|v|t}.
\]

We postpone the proof of Lemma 3.8 to the end of this section.

3.1.3. Control of the stable direction.

**Lemma 3.9.** For $T_0$ large enough, independent of $n$ and $\forall \alpha^+$ such that $|\alpha^+| \leq e^{-\delta\sqrt{\omega}|v|T_n}$. The following holds

\[\forall t \in [T(\alpha^+), T_n], \quad |\alpha^-(t)| \leq \frac{1}{2} e^{-\delta\sqrt{\omega}|v|t}.
\]

Proof.

\[
\frac{d}{dt} (\alpha^-(t)e^{-\omega t}) = (\frac{d}{dt} \alpha^-(t) - e_\omega \alpha^-(t))e^{-\omega t}.
\]

Due to (3.10) and (3.17), we have

\[
\left| \frac{d}{dt} (\alpha^-(t)e^{-\omega t}) \right| \leq \left( C_3 \frac{M^3}{8} e^{-2\delta\sqrt{\omega}|v|t} + C_4 e^{-\delta\sqrt{\omega}|v|t} \right) e^{-\delta\sqrt{\omega}|v|t} e^{-\omega t}.
\]

Then, we obtain by integration on $[t, T_n]$ and using that $\alpha^-(T_n) = 0$, we get

\[
|\alpha^-(t)| \leq \left( C_3 \frac{M^3}{8} e^{-2\delta\sqrt{\omega}|v|t} + C_4 e^{-\delta\sqrt{\omega}|v|t} \right) e^{-\delta\sqrt{\omega}|v|t}.
\]

Hence,

\[\forall t \in [T(\alpha^+), T_n], \quad |\alpha^-(t)| \leq \frac{1}{2} e^{-\delta\sqrt{\omega}|v|t}.
\]
If the following conditions are satisfied

\begin{align}
C_3 \frac{M^3}{8} e^{-2\delta \sqrt{\omega} |v| T_0} &\leq \frac{1}{4}, \\
C_4 e^{-\delta \sqrt{\omega} |v| T_0} &\leq \frac{1}{4}.
\end{align}

3.1.4. Control of the unstable direction by a topological argument. Finally, we have to control \( \alpha^+(t) \). For this, we will provide the existence of a suitable value of \( \alpha^+ \).

**Lemma 3.10.** For \( \delta > 0 \) small enough and \( T_0 \) large enough, there exists \( \alpha^+ \) such that

\[ |\alpha^+| \leq e^{-\delta \sqrt{\omega} |v| t} \quad \text{and} \quad T(\alpha^+) = T_0. \]

**Proof.** We argue by contradiction. Assume that, \( \forall \alpha^+ \) such that \( |\alpha^+| \leq e^{-\delta \sqrt{\omega} |v| t} \), one has \( T(\alpha^+) > T_0 \).

From Lemma 3.8 and 3.9 we have

\[ \left\| u(T(\alpha^+)) - R(T(\alpha^+)) \right\|_{H_0^1} \leq \frac{\epsilon}{2}, \]

\[ \left\| r(T(\alpha^+)) \right\|_{H_0^1} \leq \frac{M}{2} e^{-\delta \sqrt{\omega} |v| T(\alpha^+)} \]

\[ |y(T(\alpha^+))| + |\mu(T(\alpha^+))| \leq \frac{M'}{2} e^{-\delta \sqrt{\omega} |v| T(\alpha^+)} \]

\[ |\alpha^-(T(\alpha^+))| \leq \frac{1}{2} e^{-\delta \sqrt{\omega} |v| T(\alpha^+)} \]

By the definition of \( T(\alpha^+) \) and the continuity of the flow, one must have

\[ |\alpha^+(T(\alpha^+))| = e^{-\delta \sqrt{\omega} |v| T(\alpha^+)} \]

Let \( T < T(\alpha^+) \) be close enough to \( T(\alpha^+) \) so that the solution \( u(t) \) and its modulation are well-defined on \([T, T_n]\).

For \( t \in [T, T_n] \), let \( \mathcal{N}(\alpha^+(t)) = \mathcal{N}(t) = \left| e^{\delta \sqrt{\omega} |v| t} \alpha^+(t) \right|^2 \).

\begin{equation}
\frac{d}{dt} \mathcal{N}(t) = e^{2\delta \sqrt{\omega} |v| t} \left[ 2\delta \sqrt{\omega} |v| \alpha^+(t) + 2 \frac{d}{dt} \alpha^+(t) \right] \alpha^+(t)
\end{equation}

Multiply by \( 2|\alpha^+(t)| \) the estimate (3.10), we obtain

\[ \left| 2\alpha^+(t) \frac{d}{dt} \alpha^+(t) + 2e_\omega \alpha^+(t)^2 \right| \leq C \left| \alpha^+(t) \right| \left( \left\| h(t) \right\|_{H_0^1}^3 + e^{-2\delta \sqrt{\omega} |v| t} \right) \]

which yields

\[ \frac{d}{dt} |\alpha(t)|^2 + 2e_\omega |\alpha(t)|^2 \leq C \left| \alpha^+(t) \right| \left( \left\| h(t) \right\|_{H_0^1}^3 + e^{-2\delta \sqrt{\omega} |v| t} \right) \]

By (3.21), it follows that

\[ \frac{d}{dt} \mathcal{N}(t) = e^{2\delta \sqrt{\omega} |v| t} \left[ 2\delta \sqrt{\omega} |v| - 2e_\omega \right] |\alpha^+(t)|^2 + O \left( e^{2\delta \sqrt{\omega} |v| t} \left| \alpha^+(t) \right| \left( \left\| h(t) \right\|_{H_0^1}^3 + e^{-2\delta \sqrt{\omega} |v| t} \right) \right) \]

Due to (3.17) we have

\[ e^{2\delta \sqrt{\omega} |v| t} |\alpha^+(t)| \left( \left\| h(t) \right\|_{H_0^1}^3 + e^{-2\delta \sqrt{\omega} |v| t} \right) \leq C \sqrt{\mathcal{N}(t)} \left( \frac{M^3}{8} e^{-2\delta \sqrt{\omega} |v| t} + e^{-\delta \sqrt{\omega} |v| t} \right) \].
Let $\delta > 0$ such that $2e_\omega - 2\delta \sqrt{\omega} |v| \geq e_\omega$, so that
\[
\frac{d}{dt} N(t) \leq -e_\omega N(t) + \left( C_5 \frac{M^3}{8} e^{-2\delta \sqrt{\omega}|v|t} + C_6 e^{-\delta \sqrt{\omega}|v|t} \right) \sqrt{N(t)}.
\]

We consider the above estimate at $t = T(\alpha^+) \geq T_0$, so large such that
\[
C_5 \frac{M^3}{8} e^{-2\delta \sqrt{\omega}|v|T_0} \leq \frac{1}{4} e_\omega,
\]
\[
C_6 e^{-\delta \sqrt{\omega}|v|T_0} \leq \frac{1}{4} e_\omega.
\]

Using that $N(T(\alpha^+)) = 1$, we get
\[
\forall \alpha^+ \in B(e^{-\delta \sqrt{\omega}|v|T_n}), \quad \frac{d}{dt} N(T(\alpha^+)) \leq -\frac{1}{2} e_\omega.
\]

From (3.24), a standard argument says that the map: $\alpha^+ \mapsto T(\alpha^+)$ is continuous.

Indeed, by (3.24), $\forall \varepsilon > 0, \exists \eta > 0$ such that
\[
N(T(\alpha^+) - \varepsilon) > 1 + \eta,
\]
and
\[
N(t) < 1 - \eta, \quad \forall t \in [T(\alpha^+) + \varepsilon, T_n] \quad \text{(possibly empty)}.
\]

By continuity of the flow of the (NLS) equation, it follows that $\exists \theta > 0$ such that, for all $||\tilde{\alpha}^+ - \alpha^+|| \leq \theta$, the corresponding $\tilde{\alpha}^+(t)$ satisfies
\[
|N(\tilde{\alpha}^+(t)) - N(\alpha^+(t))| \leq \frac{\eta}{2} \quad \forall t \in [T(\alpha^+) - \varepsilon, T_n].
\]

In particular, $T(\alpha^+) - \varepsilon < T(\tilde{\alpha}^+) < T(\alpha^+) + \varepsilon$.

Now we consider the continuous map
\[
P : B_{\mathbb{R}}(e^{-\delta \sqrt{\omega}|v|T_n}) \longrightarrow S_{\mathbb{R}}(e^{-\delta \sqrt{\omega}|v|T_n})
\]
\[
\alpha^+ \mapsto e^{-\delta \sqrt{\omega}|v|(T_n - T(\alpha^+))} \alpha^+(T(\alpha^+))
\]

Let $\alpha^+ \in S_{\mathbb{R}}(e^{-\delta \sqrt{\omega}|v|T_n})$, from (3.24) it follows that $T(\alpha^+) = T_n$ and $P(\alpha^+) = \alpha^+$, which means that $P|S_{\mathbb{R}}(e^{-\delta \sqrt{\omega}|v|T_n}) = Id$. But this contradicts Brouwer’s fixed point theorem.

So, $\exists \alpha^+ \in B_{\mathbb{R}}(e^{-\delta \sqrt{\omega}|v|T_n})$ such that $T(\alpha^+) = T_0$. 

### 3.2. Estimate on the modulation parameters.

**Proof.** This section is devoted to the proof of the Lemma 3.8. For that, we claim the following results which will be proved at the end of the proof.

Let us recall that $\tilde{R}(t, x) = e^{i\tilde{\varphi}(t, x)} \tilde{Q}_\omega(t, x) \Psi(x)$.

**Claim 3.11.**
\[
(3.25) \quad \left| \frac{d}{dt} \left( E(\tilde{R}(t)) + \left( \frac{\omega}{2} + \frac{|v|^2}{8} \right) M(\tilde{R}(t)) - \frac{v}{2} P(\tilde{R}(t)) \right) \right| \leq Ce^{-2\delta \sqrt{\omega}|v|t} + M^2 e^{-3\delta \sqrt{\omega}|v|t}.
\]
Claim 3.12.

\[
\begin{align*}
(E(u(t)) + \left(\frac{\omega}{2} + \frac{|v|^2}{8}\right)M(u(t)) - \frac{v}{2}P(u(t)) - E(\tilde{R}(t)) + \left(\frac{\omega}{2} + \frac{|v|^2}{8}\right)M(\tilde{R}(t)) - \frac{v}{2}P(\tilde{R}(t))
\end{align*}
\]

From Lemma 3.2 and Lemma 3.4 we have

Claim 3.13. There exists \( C > 0 \) such that,

\[
\|h(t)\|^2_{H_0^1} \leq C \left[ \left( \tilde{L}_+ h_1(t), h_1(t) \right) + \left( \tilde{L}_- h_2(t), h_2(t) \right) \right]
\]

Now, we start the proof of Lemma 3.8. Let \( t \in [T(\alpha^\pm), T_n] \), integrating (3.25) on \([t, T_n]\) we get

\[
\left| \left( E(\tilde{R}(T_n)) + \left(\frac{\omega}{2} + \frac{|v|^2}{8}\right)M(\tilde{R}(T_n)) - \frac{v}{2}P(\tilde{R}(T_n)) \right) - \left( E(\tilde{R}(t)) + \left(\frac{\omega}{2} + \frac{|v|^2}{8}\right)M(\tilde{R}(t)) - \frac{v}{2}P(\tilde{R}(t)) \right) \right| 
\]

From the above estimate and (3.26), we have

\[
\left| \left( \tilde{L}_+ h_1(T_n), h_1(T_n) \right) + \left( \tilde{L}_- h_2(T_n), h_2(T_n) \right) \right| \leq C \|h(T_n)\|^2_{H_0^1} \leq C |\lambda|^2 \leq C e^{-2\delta \sqrt{|v|} t}.
\]

We deduce from (3.28), (3.29) and the Claim 3.13 that

\[
\|h(t)\|^2_{H_0^1} \leq C (\tilde{L}_+ h_1(t), h_1(t)) + C (\tilde{L}_- h_2(t), h_2(t)) + C (\alpha^\pm(t))^2 + C M e^{-4\delta \sqrt{|v|} t} \]

If \( T_0 \) satisfies

\[
C_7 e^{-2\delta \sqrt{|v|} T_0} \leq \frac{1}{4},
\]

\[
C_8 M e^{-3\delta \sqrt{|v|} T_0} \leq \frac{1}{4}.
\]

Then, we have provide

\[
\|h(t)\|^2_{H_0^1} \leq \frac{M}{2} e^{-\delta \sqrt{|v|} t}.
\]

If conditions (3.14), (3.19), (3.20), (3.23), (3.22), (3.30) and (3.31) on \( M \) and \( T_0 \) hold. However it is easy to find \( T_0 \) and \( M \) satisfying these conditions. We take \( T_0 \) large enough such that

\[
\max(C_4, C_6, C_7) e^{-\delta \sqrt{|v|} T_0} \leq \frac{1}{4} \min(1, e_\omega),
\]
and we take $M$ such that
\begin{equation}
Me^{-\delta\sqrt{\omega_v}|t|T_0} \leq \frac{1}{10C} \min \left( \|\partial_x Q\Psi\|_{L^2}^2, \|Q\Psi\|_{L^2}^2 \right),
\end{equation}
\begin{equation}
\max(C_3, C_5) \frac{M^3}{8} e^{-2\delta\sqrt{\omega_v}|t|T_0} \leq \frac{1}{4} \min(1, e_\omega).
\end{equation}
\begin{equation}
C_M e^{-\delta\sqrt{\omega_v}|t|T_0} \leq \frac{1}{4}
\end{equation}

From Lemma 3.6 we have
\[
\left| \frac{d\mu(t)}{dt} \right| + \left| \frac{dy(t)}{dt} \right| \leq C \|h(t)\|_{\dot{H}_0^1}^2 + C e^{-2\delta\sqrt{\omega_v}|t|} + Ce^{-2\delta\sqrt{\omega_v}|t|}.
\]

We integrate the above estimate on some time interval $[t, T_n]$, for $t \in [T(\alpha^+), T_n]$.
\[
|\mu(t)| + |y(t)| \leq |\mu(T_n)| + |y(T_n)| + C M^2 e^{-2\delta\sqrt{\omega_v}|t|} + C e^{-2\delta\sqrt{\omega_v}|t|}.
\]

Furthermore, due to the definition of $T(\alpha^+)$ we get
\[
|\mu(t)| + |y(t)| \leq C_1 e^{-2\delta\sqrt{\omega_v}|t|} + C_2 M^2 e^{-2\delta\sqrt{\omega_v}|t|}.
\]

Then, we can deduce that
\[
|\mu(t)| + |y(t)| \leq M' e^{-\delta\sqrt{\omega_v}|t|}.
\]

Provided, for $T_0$ large enough
\begin{equation}
C_1 e^{-\delta\sqrt{\omega_v}|T_0|} \leq \frac{M'}{4},
\end{equation}
and we take $M'$ such that
\begin{equation}
C_2 M^2 e^{-2\delta\sqrt{\omega_v}|T_0|} \leq \frac{M'}{4}.
\end{equation}

Finally, we obtain
\[
\|u(t) - R(t)\|_{H_0^1} \leq \left\| R(t) - \tilde{R}(t) \right\|_{H_0^3} + \|h(t)\|_{H_0^1}^3 \leq C |y(t)| + \|h(t)\|_{H_0^1}^3 \leq C e^{-\delta\sqrt{\omega_v}|t|} \leq \frac{\varepsilon}{2},
\]
which concludes the proof of Lemma 3.8, by taking $T_0$ large enough. \qed

**Proof of Claim 3.11.** Recall that \(R(t, x) = e^{i\tilde{\phi}(t, x)} \tilde{Q}_\omega(t, x) \Psi(x)\).

\[
\nabla \tilde{R}(t, x) = \left[ i \frac{\tilde{\omega}}{2} \tilde{Q}_\omega \Psi + \nabla (\tilde{Q}_\omega \Psi) \right] e^{i\tilde{\phi}(t, x)}, \quad |\nabla \tilde{R}(t, x)|^2 = \frac{|v|^2}{4} \tilde{Q}_\omega^2 \Psi^2 + |\nabla (\tilde{Q}_\omega \Psi)|^2.
\]
\[ E(\bar{R}(t)) = \frac{1}{2} \int |\nabla \bar{R}(t)|^2 \, dx - \frac{1}{p+1} \int \bar{Q}_\omega^{p+1} \Psi^{p+1} \, dx, \]

\[ \frac{d}{dt} E(\bar{R}(t)) = \frac{1}{2} \frac{d}{dt} \left[ \int |v|^2 \bar{Q}_\omega \Psi^2 + |\nabla (\bar{Q}_\omega \Psi)|^2 \, dx - \frac{1}{p+1} \int \bar{Q}_\omega^{p+1} \Psi^{p+1} \, dx \right] \]

\[ = \frac{1}{2} \int \frac{|v|^2}{4} 2(-v - \frac{dy(t)}{dt}) \nabla \bar{Q}_\omega \bar{Q}_\omega \Psi^2 + 2(-v - \frac{dy(t)}{dt}) \nabla (\nabla (\bar{Q}_\omega \Psi)) \nabla (\bar{Q}_\omega \Psi) \, dx \]

\[ - \frac{1}{p+1} \int (p+1)(-v - \frac{dy(t)}{dt}) \nabla \bar{Q}_\omega \bar{Q}_\omega \Psi^{p+1} \, dx \]

\[ = (-v - \frac{dy(t)}{dt}) \left[ \int \frac{|v|^2}{4} \nabla \bar{Q}_\omega \bar{Q}_\omega \Psi^2 + \nabla (\nabla \bar{Q}_\omega \Psi) \nabla (\bar{Q}_\omega \Psi) \, dx - \int \nabla \bar{Q}_\omega \bar{Q}_\omega \Psi^{p+1} \right], \]

where, \((-v - \frac{dy(t)}{dt}) \nabla ((\nabla \bar{Q}_\omega \Psi)) \nabla (\bar{Q}_\omega \Psi) = \sum_{k=1}^{3} \sum_{j=1}^{3} (-v_k - \frac{dy_k(t)}{dt}) \partial_{x_k} \partial_{x_j} (\bar{Q}_\omega \Psi) \partial_{x_j} (\bar{Q}_\omega \Psi). \]

\[ M(\bar{R}(t)) = \int |\bar{R}(t)|^2 \, dx, \]

\[ \frac{d}{dt} M(\bar{R}(t)) = \frac{d}{dt} \int \bar{Q}_\omega^2 \Psi^2 \, dx = 2(-v - \frac{dy(t)}{dt}) \int \nabla \bar{Q}_\omega \bar{Q}_\omega \Psi^2 \, dx. \]

\[ P(\bar{R}(t)) = \text{Im} \int \nabla \bar{R}(t) \bar{R}(t) \, dx, \]

\[ \frac{d}{dt} P(\bar{R}(t)) = \frac{d}{dt} \left( \frac{v}{2} \int \bar{Q}_\omega^2 \Psi^2 \, dx \right) = v \int (-v - \frac{dy(t)}{dt}) \nabla \bar{Q}_\omega \bar{Q}_\omega \Psi^2 \, dx. \]

Hence, we have

\[ \frac{d}{dt} \left[ E(\bar{R}(t)) + \left( \frac{\omega}{2} + \frac{|v|^2}{8} \right) M(\bar{R}(t)) \right] - \frac{v}{2} P(\bar{R}(t)) \]

\[ = \frac{\omega}{2} (-v - \frac{dy(t)}{dt}) \int \nabla \bar{Q}_\omega \bar{Q}_\omega \Psi^2 \]

\[ + (-v - \frac{dy(t)}{dt}) \left[ \int \nabla (\nabla \bar{Q}_\omega \Psi) \nabla (\bar{Q}_\omega \Psi) \, dx - \int \nabla \bar{Q}_\omega \bar{Q}_\omega \Psi^{p+1} \right] . \]

For the first integral, we have

\[ \left\| \nabla \bar{Q}_\omega \bar{Q}_\omega \Psi^2 \right\| = \left\| \frac{1}{2} \int \nabla \bar{Q}_\omega^2 \Psi^2 \right\| = \left\| \int \bar{Q}_\omega^2 \nabla \Psi \, dx \right\| \leq C e^{-2\delta \sqrt{|v|^2} t}. \]

Using (3.9) and the fact that the support of the derivatives of \( \Psi \) is compact, furthermore, in the second integral, we have some terms with \( \Psi \) which doesn’t have a compact support. For this terms, we have to use the fact that \( Q_\omega \) is a radial function, concluding the proof of the Claim 3.11.

\[ \square \]

**Proof of Claim 3.12.** Recall that,

\[ u(t, x) = e^{i\varphi(t,x)} \left( \bar{Q}_\omega(t, x) \Psi(x) + h(t, x) \right), \]

\[ E(u(t)) = E(e^{i\varphi} \left[ \bar{Q}_\omega \Psi + h \right]) \]

\[ = \frac{1}{2} \int_{\Omega} \left| \nabla (e^{i\varphi} (\bar{Q}_\omega \Psi + h)) \right|^2 \, dx - \frac{1}{p+1} \int_{\Omega} \left| \bar{Q}_\omega \Psi + h \right|^{p+1} \, dx. \]
Using Taylor expansion,
\[
\left| \tilde{Q}_\omega \Psi + h \right|^{p+1} = \tilde{Q}_\omega^{p+1} \Psi^{p+1} + \left( \frac{p + 1}{2} \right) \tilde{Q}_\omega^p \Psi^p (h + \tilde{h}) \\
+ \frac{1}{2} \left( \frac{p + 1}{2} \right) \left( \frac{p - 1}{2} \right) \tilde{Q}_\omega^{p-1} \Psi^{p-1} \left( h^2 + \tilde{h}^2 \right) + \left( \frac{p + 1}{2} \right)^2 \tilde{Q}_\omega^{p-1} \Psi^{p-1} h \tilde{h} + \beta(t, x).
\]
and
\[
\left| \nabla \left( e^{i\tilde{\varphi}} (\tilde{Q}_\omega \Psi + h) \right) \right|^2 = \left| e^{i\tilde{\varphi}} \left( i \tilde{\varphi} \tilde{Q}_\omega \Psi + h + (\nabla (\tilde{Q}_\omega \Psi) + \nabla h) \right) \right|^2 \\
= \left| v^2 \tilde{Q}_\omega \Psi + h \right|^2 - v \nabla (\tilde{Q}_\omega \Psi) h_2 + v \tilde{Q}_\omega \Psi \nabla h_2 + v (h_1 \nabla h_2 - h_2 \nabla h_1) \\
+ \left| \nabla (\tilde{Q}_\omega \Psi) \right|^2 + 2 \nabla (\tilde{Q}_\omega \Psi) \nabla h_1 + |\nabla h|^2.
\]
Here and until the end the proof: \( \int \) denote the integral over \( \Omega \).

We have
\[
E(u(t)) - E(\tilde{R}(t)) = \left| \frac{v}{2} \right|^2 \int \tilde{Q}_\omega \Psi h_1 + \left| \frac{v}{2} \right|^2 \int |h|^2 + \frac{1}{2} \int |\nabla h|^2 + \int \nabla (\tilde{Q}_\omega \Psi) \nabla h_1 - \int \tilde{Q}_\omega^p \Psi^p h_1 \\
- \int v \nabla (\tilde{Q}_\omega \Psi) h_2 + \int \frac{v}{2} (h_1 \nabla h_2 - h_2 \nabla h_1) - \frac{p}{2} \int \tilde{Q}_\omega^{p-1} \Psi^{p-1} h_2 \\
+ \frac{1}{2} \int \tilde{Q}_\omega^{p-1} \Psi^{p-1} h_2^2 + \beta(t, x).
\]

Then we have,
\[
\left[ E(u(t)) + \left( \frac{\omega}{2} + \frac{|v|^2}{8} \right) M(u(t)) - \frac{v}{2} P(u(t)) \right] - \left[ E(\tilde{R}(t)) + \left( \frac{\omega}{2} + \frac{|v|^2}{8} \right) M(\tilde{R}(t)) - \frac{v}{2} P(\tilde{R}(t)) \right] \\
= \frac{1}{2} \left[ (\tilde{L}_\omega^+ h_1, h_1) + (\tilde{L}_\omega^- h_2, h_2) \right] - \frac{p}{2} \int \tilde{Q}_\omega^{p-1} h_1^2 (\Psi^{p-1} - 1) - \frac{1}{2} \int \tilde{Q}_\omega^{p-1} h_2^2 (\Psi^{p-1} - 1) \\
+ \int -\Delta (\tilde{Q}_\omega \Psi) h_1 dx - \int \tilde{Q}_\omega^p \Psi^p h_1 + \int \omega \tilde{Q}_\omega \Psi h_1 + \beta(t, x) \\
= \frac{1}{2} \left[ (\tilde{L}_\omega^+ h_1, h_1) + (\tilde{L}_\omega^- h_2, h_2) \right] - \frac{p}{2} \int \tilde{Q}_\omega^{p-1} h_1^2 (\Psi^{p-1} - 1) - \frac{1}{2} \int \tilde{Q}_\omega^{p-1} h_2^2 (\Psi^{p-1} - 1) \\
+ \int (\Delta \tilde{Q}_\omega + \omega \tilde{Q}_\omega - Q_\omega^p) \Psi h_1 + \int \tilde{Q}_\omega \nabla h_1 - \int \tilde{Q}_\omega \Delta h_1 - \int \tilde{Q}_\omega \Psi (\Psi^{p-1} - 1) h_1 + \beta(t, x).
\]

Using the fact that \( \nabla \Psi, \Delta \Psi \) and \( (\Psi^{p-1} - 1) \) has a compact support, to conclude the proof of Claim 3.12. \( \square \)
Proof of Claim 3.13. The proof of (3.27) is a standard consequence of Lemma 2.4 and the following orthogonality conditions, \( \text{Re} \int \partial_x \tilde{Q}_\omega \Psi \overline{h} \, dx = 0, \text{Im} \int \tilde{Q}_\omega \Psi \overline{h} \, dx = 0. \)

Due to (2.2), there exits \( C > 0 \) such that
\[
\|h(t)\|^2_{H^1_0} \leq C \left[ \left( \tilde{L}_\omega^+ h_1(t, x), h_1(t, x) \right) + \left( \tilde{L}_\omega^- h_2(t, x), h_2(t, x) \right) + \sum_{j=1}^3 \left( \int \partial_x \tilde{Q}_\omega(t, x) h_1(t, x) \, dx \right)^2 \\
+ \left( \int \tilde{Q}_\omega(t, x) h_2(t, x) \, dx \right)^2 + \left( \text{Im} \int \tilde{Y}^\pm(t, x) \overline{h}(t, x) \, dx \right)^2 \right].
\]

Using the orthogonality conditions, we get
\[
\int \partial_x \tilde{Q}_\omega h_1 = \int \partial_x \tilde{Q}_\omega (1 - \Psi) h_1 \quad \text{and} \quad \int \tilde{Q}_\omega h_2 = \int \tilde{Q} (1 - \Psi) h_2.
\]

Due to the exponential decay of \( Q \) and the compact support of \( (1 - \Psi) \), we have
\[
\left| \int \partial_x \tilde{Q}_\omega(t) h_1(t) \right| \leq C M e^{-2 \delta \sqrt{|v|} t} \quad \text{and} \quad \left| \int \tilde{Q}_\omega(t) h_2(t) \right| \leq C M e^{-2 \delta \sqrt{|v|} t}
\]
\[
\text{Im} \int \tilde{Y}^\pm(t, x) \overline{h}(t, x) = \alpha^\pm(t) + \text{Im} \int \tilde{Y}^\pm(t, x) \overline{h}(t, x)(1 - \Psi(x)) \, dx
\]
\[
= \alpha^\pm(t) + O \left( M e^{-2 \delta \sqrt{|v|} t} \right)
\]
This concludes the proof of the Claim 3.13.

\[ \square \]

4. Fixed point theorem

Proof. This section is devoted to the proof of Theorem 1.4.

Recall that, if \( \Theta = \emptyset \) then \( H(t, x) = e^{i \varphi(t, x)} Q_\omega(x - tv) \), where \( \varphi(t, x) = \frac{1}{2} (x, v) - \frac{1}{2} |v|^2 t + t \omega \), is an exact soliton solution of (NLS).

Let \( R(t, x) = e^{i \varphi(t, x)} Q_\omega(x - tv) \Psi(x) \). Write
\[
(i \partial_t + \Delta) R = -\Psi |H|^2 H + 2 \nabla \Psi \nabla H + \Delta \Psi H.
\]

We look for \( r_\omega \in C([T_0, +\infty), H^2(\Omega) \cap H^1_0(\Omega)) \) such that
\[
(i \partial_t + \Delta) r_\omega = -|R + r_\omega|^2 (R + r_\omega) + \Psi |H|^2 H - 2 \nabla \Psi \nabla H - \Delta \Psi H,
\]
\[
r_\omega(t) \to 0 \quad \text{as} \quad t \to +\infty \quad \text{in} \quad H^2(\Omega) \cap H^1_0(\Omega).
\]

Set
\[
A_0(t, x) = \Psi(x)(1 - \Psi^2(x)) |H(t, x)|^2 H(t, x) - 2 \nabla \Psi(x) \nabla H(t, x) - \Delta \Psi(x) H(t, x),
\]
\[
A_1(r_\omega(t, x)) = -R(t, x)^2 r_\omega(t, x) - 2 R(t, x)^2 r_\omega(t, x),
\]
\[
A_2(r_\omega(t, x)) = -\overline{R}(t, x) r_\omega^2(t, x) - 2 R(t, x) |r_\omega(t, x)|^2,
\]
\[
A_3(r_\omega(t, x)) = -|r_\omega(t, x)|^2 r_\omega(t, x).
\]

We shall look for solutions of (4.1) in this space:
\[
E = \{ r_\omega \in C([T_0, +\infty), H^2(\Omega) \cap H^1_0(\Omega)), \|r_\omega\|_E < \infty \},
\]
such that
\[
\| r_\omega \|_E = \sup_{t \geq T_0} \left\{ e^{\delta \sqrt{\| v \|} t} \left( \frac{1}{|v|^3} \| r_\omega \|_{H^2(\Omega)} + \| r_\omega \|_{L^2(\Omega)} \right) \right\}.
\]

Let
\[
\Phi : (B_E, d_E) \longrightarrow (B_E, d_E)
\]
\[
r_\omega \mapsto \Phi(r_\omega) = -i \int_0^{+\infty} S(t - \tau) A_0(\tau) d\tau - i \sum_{k=1}^{3} \int_t^{+\infty} S(t - \tau) A_k(r_\omega(\tau)) d\tau.
\]

Where \( B_E = B_E(0, 1) = \{ h \in E, \| h \|_E \leq 1 \} \) and \( d_E(h, g) = \| h - g \|_E \).

One can check that \((B_E, d_E)\) is a complete metric space.

Here \( S(t) \) is the unitary group of the linear Schrödinger equation with Dirichlet boundary conditions.

Denote,
\[
J_0(t) = \int_t^{+\infty} S(t - \tau) A_0(\tau) d\tau,
\]
\[
J_k(r_\omega(t)) = \int_t^{+\infty} S(t - \tau) A_k(r_\omega(\tau)) d\tau, \quad k = 1, 2, 3.
\]

Remark 4.1. For \( 2 \leq p < 5 \), the proof is also based on a fixed point theorem as the cubic case. Indeed, we have to use Taylor expansion for the non-linearity
\[
\Phi
\]
that is a contraction mapping on the complete metric space \( \Omega \) and in the second step we will prove that \( B_E \) is stable by \( \Phi \).

In step 1, we will prove that the ball \( B_E \) is stable by \( \Phi \) and in the second step we will prove that \( \Phi \) is a contraction mapping on the complete metric space \((B_E, d)\). Finally, in step 3 we will conclude by fixed point theorem the existence of the solution of the \((\text{NLS}_\Omega)\).

- Step 1 : Stability of \( B_E \) by \( \Phi \).

Lemma 4.2. There exists \( C_\omega > 0 \) and \( \delta > 0 \) such that,
\[
\| J_0 \|_E \leq \frac{C_\omega}{|v|}
\]
\[
\| J_1(r_\omega) \|_E \leq \frac{C_\omega}{|v|} \| r_\omega \|_E
\]
\[
\| J_2(r_\omega) \|_E \leq C_\omega |v|^4 e^{-\delta \sqrt{|v|} |T_0|} \| r_\omega \|_E^2
\]
\[
\| J_3(r_\omega) \|_E \leq C_\omega |v|^5 e^{-2\delta \sqrt{|v|} |T_0|} \| r_\omega \|_E^3
\]
\[
\forall r_\omega \in B_E, \quad \| \Phi(r_\omega) \|_E \leq 1
\]

Proof. (1) Estimate For \( J_0 \).

Recall that \( A_0(t, x) = \Psi(x)(1 - \Psi^2(x)) |H(t, x)|^2 H(t, x) - 2\nabla \Psi(x) \nabla H(t, x) - \Delta \Psi(x) H(t, x) \),
where \( H(t, x) = Q_\omega(x - tv)e^{it\frac{1}{2}(x, v) - \frac{|v|^2}{4} t + t\omega} \).

Let us prove that there exists \( C_\omega > 0 \) such that,
\[
\| A_0(t) \|_{H^2} \leq C_\omega |v|^3 e^{-\delta \sqrt{|v|} |t|}, \quad \forall t \in [T_0, +\infty).
\]
It suffices to estimate the $L^2$ norm of $A_0$ and $\nabla^2 A_0$, due to the following elementary interpolation inequality, if $f \in H^2$,

$$
\|\nabla f\|_{L^2} \leq \|\nabla^2 f\|_{L^2}^{\frac{1}{2}} \|f\|_{L^2}^{\frac{1}{2}}.
$$

(4.8)

We will use the fact that $\Psi(1 - \Psi^2)$, $\nabla \Psi$ and $\Delta \Psi$ have a compact support. We will suppose that their support is include in $\{|x| < M\}$, for some $M > 0$.

Let $x \in \text{supp} \{\Psi(1 - \Psi^2)\} \subset \{|x| < M\}$ then $\{ t \mid v - M \leq |x - tv| \}$.

By (2.1), we have

$$
\begin{cases}
|Q_\omega(x - tv)| \leq C_\omega e^{\delta \sqrt{\omega} M} e^{-\delta \sqrt{\omega}|v|t}, \\
|\nabla Q_\omega(x - tv)| \leq C_\omega e^{\delta \sqrt{\omega} M} e^{-\delta \sqrt{\omega}|v|t}.
\end{cases}
$$

(4.9)

Then,

$$
\|A_0\|_{L^2} \leq C_\omega |v| e^{-\delta \sqrt{\omega}|v|t}.
$$

Now, let us estimate $\nabla^2 A_0$.

Recall that $A_0 = \Psi(1 - \Psi^2) |H|^2 H - 2 \sum_{k=1}^{3} \partial_{x_k} \Psi \partial_{x_k} H - \Delta \Psi H$.

$$
\partial_{x_j} \partial_{x_i} A_0(t, x) = \partial_{x_j} \partial_{x_i} \left[ \Psi(1 - \Psi^2) |H|^2 H + \partial_{x_i} \left[ \Psi(1 - \Psi^2) \right] \partial_{x_j} \left[ |H|^2 H \right] \right]
+ \partial_{x_j} \left[ \Psi(1 - \Psi^2) \right] \partial_{x_i} \left[ |H|^2 H \right] + \left[ \Psi(1 - \Psi^2) \right] \partial_{x_i} \partial_{x_j} \left[ |H|^2 H \right]
- 2 \sum_{k=1}^{3} \partial_{x_j} \partial_{x_i} \left[ \partial_{x_k} \Psi \partial_{x_k} H \right] + \partial_{x_i} \left[ \partial_{x_k} \Psi \partial_{x_j} \partial_{x_k} H \right]
- 2 \sum_{k=1}^{3} \partial_{x_j} \partial_{x_k} \left[ \partial_{x_k} \Psi \partial_{x_j} \partial_{x_i} \partial_{x_k} H \right] + \partial_{x_k} \Psi \partial_{x_j} \partial_{x_i} \partial_{x_k} H - \partial_{x_j} \partial_{x_i} \left[ \Delta \Psi \right] H - \partial_{x_j} \left[ \Delta \Psi \right] \partial_{x_i} H - \partial_{x_j} \left[ \Delta \Psi \right] \partial_{x_i} H - \partial_{x_j} \partial_{x_i} \left[ \Delta \Psi \right] H

Claim 4.3.

$$
\|\nabla^{4-k} \Psi(x) \nabla^k H(t, x)\| \leq C_\omega |v|^k e^{-\delta \sqrt{\omega}|v|t}, \text{ where } k = 1, 2, 3.
$$

$$
\|\nabla^{2-k} \left( \Psi(x)(1 - \Psi^2(x)) \right) \nabla^k(|H(t, x)|^2 H(t, x))\| \leq C_\omega |v|^k e^{-\delta \sqrt{\omega}|v|t}, \text{ where } k = 1, 2.
$$

Proof. We postpone the proof of Claim 4.3 to Appendix B.

By the Claim 4.3, we have

$$
\|\nabla^2 A_0(t)\|_{L^2} \leq C_\omega |v|^3 e^{-\delta \sqrt{\omega}|v|t}.
$$

This concludes the proof of (4.7).
Thus, we obtain
\[ \|J_0(t)\|_{H^2} \leq \int_t^{+\infty} \|A_0(\tau)\|_{H^2} \, d\tau \leq C_\omega |v|^2 e^{-\delta \sqrt{|v|} |t|}, \]
\[ \|J_0\|_E \leq \frac{C_\omega}{|v|}. \]

(2) **Estimate for** \( J_1 \).
Recall that \( A_1(r_\omega(t,x)) = -R(t,x)\bar{r}_\omega(t,x) - 2|R(t,x)|^2 r_\omega(t,x) \).

Using the elementary interpolation inequality (4.8), we have
\[ \|J_1(r_\omega(t))\|_{H^2} \leq \int_t^{+\infty} \|A_1(r_\omega(\tau))\|_{H^2} \, d\tau \]
\[ \leq C \int_t^{+\infty} \|A_1(r_\omega(\tau))\|_{L^2} \, d\tau + C \int_t^{+\infty} \|\nabla^2 A_1(r_\omega(\tau))\|_{L^2} \, d\tau. \]

Let us prove that there exists \( C_\omega > 0 \) such that
\[ \int_t^{+\infty} \|A_1(r_\omega(\tau))\|_{L^2} \, d\tau \leq \frac{C_\omega}{|v|} e^{-\delta \sqrt{|v|} |t|} \|r_\omega\|_E, \]
\[ \int_t^{+\infty} \|\nabla^2 A_1(r_\omega(\tau))\|_{L^2} \, d\tau \leq C_\omega |v|^2 e^{-\delta \sqrt{|v|} |t|} \|r_\omega\|_E. \]

This prove the first estimate. Now, let us look to the second estimate
\[ \int_t^{+\infty} \|\nabla^2 A_1(r_\omega(\tau))\|_{L^2} \, d\tau \leq C \int_t^{+\infty} \|\nabla^2 R^2(\tau)\|_{L^\infty} \|r_\omega(\tau)\|_{L^2} \, d\tau \]
\[ + C \int_t^{+\infty} \|\nabla R^2(\tau)\|_{L^\infty} \|\nabla r_\omega(\tau)\|_{L^2} \, d\tau \]
\[ + C \int_t^{+\infty} \|R^2(\tau)\|_{L^\infty} \|\nabla^2 r_\omega(\tau)\|_{L^2} \, d\tau. \]

It is easy to see that
\[ |I_1| \leq C_\omega |v| e^{-\delta \sqrt{|v|} |t|} \|r_\omega\|_E, \]
\[ |I_3| \leq C_\omega |v|^2 e^{-\delta \sqrt{|v|} |t|} \|r_\omega\|_E. \]

For \( I_2 \) we use the elementary interpolation inequality (4.8),
\[ \|\nabla r_\omega(\tau)\|_{L^2} \leq \|\nabla^2 r_\omega(\tau)\|_{L^2}^{\frac{1}{2}} \|r_\omega\|_{L^2}^{\frac{1}{2}}. \]

Thus we get,
\[ |I_2| \leq C_\omega |v| \frac{3}{2} e^{-\delta \sqrt{|v|} |t|} \|r_\omega(\tau)\|_E. \]
And this concludes the proof of the estimates (4.10) and (4.11).

Due to (4.10), (4.11) and the fact that $|v| > 1$ we have

$$\|J_1(r_\omega)\|_{H^2} \leq C_\omega \|v\|^2 e^{-\delta \sqrt{\omega}|v|t} \|r_\omega\|_E.$$  

Then

$$\|J_1(r_\omega)\|_E \leq \frac{C_\omega}{|v|} \|r_\omega\|_E.$$

(3) **Estimate for $J_2$.**

Recall that $A_2(r_\omega(t,x)) = -R(t,x) r_\omega^2(t,x) - 2R(t,x) |r_\omega(t,x)|^2$.

$$\|J_2(r_\omega(t))\|_{H^2} \leq C \int_t^{+\infty} \|A_2(\tau)\|_{H^2} d\tau.$$  

Using the fact that $H^2$ is an algebra we obtain

$$\|J_2(r_\omega(t))\|_{H^2} \leq C \int_t^{+\infty} |\tau| |r_\omega\|_{H^2}^2 d\tau \leq C \int_t^{+\infty} |v| |v|^6 e^{-2\delta \sqrt{\omega}|v| \tau} \|r_\omega\|_E^2$$

$$\leq C \|v\|^2 e^{-2\delta \sqrt{\omega}|v| t} \|r_\omega\|_{H^2}^2.$$  

Then

$$\|J_2(r_\omega)\|_E \leq C \|v\|^4 e^{-\delta \sqrt{\omega}|v| T_0} \|r_\omega\|_{H^2}^2.$$  

(4) **Estimate for $J_3$.**

We have $A_3(r_\omega(t,x)) = -|r_\omega(t,x)|^2 r_\omega(t,x)$.

$$\|J_3(r_\omega(t))\|_{H^2} \leq \int_t^{+\infty} \|A_3(\tau)\|_{H^2}^3 d\tau \leq \int_t^{+\infty} \|r_\omega(\tau)\|_{H^2}^3 d\tau$$

$$\leq \int_t^{+\infty} |v|^9 e^{-3\delta \sqrt{\omega}|v| \tau} \|r_\omega\|_{H^2}^3$$

$$\leq C \|v\|^8 e^{-3\delta \sqrt{\omega}|v| t} \|r_\omega\|_{H^2}^3.$$  

This implies that

$$\|J_3(r_\omega)\|_E \leq C \|v\|^5 e^{-2\delta \sqrt{\omega}|v| T_0} \|r_\omega\|_{H^2}^3.$$  

(5) **Stability of $\Phi$.**

Recall that $\Phi(r_\omega(t,x)) = -i J_0(t) - i \sum_{k=1}^3 J_k(r_\omega(t,x))$.

Using the fact that the velocity $v$ is large enough in each estimate (4.2), (4.3), (4.4) and (4.5), we get

$$\forall r_\omega \in B_E(0,1), \quad \|\Phi(r_\omega)\|_E \leq \|J_0\|_E + \sum_{k=1}^3 \|J_k(r_\omega)\|_E \leq 1.$$  

□
\begin{itemize}
  \item Step 2: Contraction mapping.
  Let \( f, g \in B_E(0, 1) \)
  \[
  \| \Phi(f(t)) - \Phi(g(t)) \|_{H^2} \leq \left\| \int_t^{+\infty} S(t - \tau) (A_1(f(\tau)) - A_1(g(\tau))) d\tau \right\|_{H^2} J_1(f) - J_1(g)
  \]
  \[
  + \left\| \int_t^{+\infty} S(t - \tau) (A_2(f(\tau)) - A_2(g(\tau))) d\tau \right\|_{H^2} J_2(f) - J_2(g)
  \]
  \[
  + \left\| \int_t^{+\infty} S(t - \tau) (A_3(f(\tau)) - A_3(g(\tau))) d\tau \right\|_{H^2} J_3(f) - J_3(g)
  \]

  \textbf{Lemma 4.4.} For all \( T_0 > 0, \omega > 0 \), there exists \( V_0 > 0 \) such that for \( |v| > V_0 \), for all \( f, g \in B_E \), we have
  \[
  d_E(\Phi(f) - \Phi(g)) \leq \frac{1}{2} d_E(f - g).
  \]

  \textit{Proof.} Due to Lemma 4.2 we have
  \[
  \|J_1(f) - J_1(g)\|_E \leq \frac{C_\omega}{|v|} \|f - g\|_E,
  \]
  Let \( V_0 > 0 \) large enough to be chosen below such that for \( |v| > V_0 \), we have
  \begin{equation}
  \frac{1}{|v|} \leq \frac{1}{8}.
  \end{equation}
  Then,
  \begin{equation}
  \|J_1(f) - J_1(g)\|_E \leq \frac{1}{8} \|f - g\|_E.
  \end{equation}
  Recall that \( A_2(h(t, x)) = -R(t, x)h^2(t, x) - 2R(t, x)h(t, x)^2 \).
  \[
  \|J_2(f(t)) - J_2(g(t))\|_{H^2} \leq C \int_t^{+\infty} \|R(\tau)\|_{H^2} \|f^2(t) - g^2(t)\|_{H^2} d\tau
  \]
  \[
  \leq C_\omega |v|^2 \int_t^{+\infty} |v|^6 e^{-2\delta \sqrt{\omega}|v|^6} d\tau (\|f\|_E + \|g\|_E) \|f - g\|_E
  \]
  \[
  \leq C_\omega |v|^7 e^{-\delta \sqrt{\omega}|v|^7} T_0 e^{-\delta \sqrt{\omega}|v|^7} (\|f\|_E + \|g\|_E) \|f - g\|_E.
  \]
  This implies that
  \[
  \|J_2(f(t)) - J_2(g(t))\|_E \leq C_\omega |v|^4 e^{-\delta \sqrt{\omega}|v|^4} T_0 (\|f\|_E + \|g\|_E) \|f - g\|_E.
  \]
  Since the velocity \( v \) is large enough we have
  \begin{equation}
  \forall f, g \in B_E(0, 1), \ C_\omega |v|^4 e^{-\delta \sqrt{\omega}|v|^4} T_0 (\|f\|_E + \|g\|_E) \leq \frac{1}{8},
  \end{equation}
  then
  \begin{equation}
  \|J_2(f) - J_3(g)\|_E \leq \frac{1}{8} \|f - g\|_E.
  \end{equation}
\end{itemize}
Recall that, $A_3(h(t, x)) = -h(t, x)|h(t, x)|^2$.

$$
\|J_3(f(t)) - J_3(g(t))\|_{H^2} \leq \int_{t}^{+\infty} \left\| f(\tau)f(\tau) - g(\tau)g(\tau) \right\|_{H^2} d\tau \\
\leq \int_{t}^{+\infty} \| f(\tau)(f(\tau) - g(\tau)) + g(\tau)(f(\tau) - g(\tau)) \|_{H^2} d\tau \\
\leq C \int_{t}^{+\infty} \| f(\tau) - g(\tau) \|_{H^2} \left( \| f(\tau) \|^2_{H^2} + \| g(\tau) \|^2_{H^2} \right) d\tau \\
\leq C \int_{t}^{+\infty} \| v \|^9 e^{-3\delta\sqrt{|v|}T}\left( \| f \|^2_{E} + \| g \|^2_{E} \right) \| f - g \|_{E} d\tau \\
\leq C_\omega \| v \|^8 e^{-2\delta\sqrt{|v|}T_{0} e^{-\delta\sqrt{|v|}T}} \left( \| f \|^2_{E} + \| g \|^2_{E} \right) \| f - g \|_{E}.
$$

Hence

$$
\|J_3(f(t)) - J_3(g(t))\|_{E} \leq C_\omega \| v \|^5 e^{-2\delta\sqrt{|v|}T_{0} e^{-\delta\sqrt{|v|}T}} \left( \| f \|^2_{E} + \| g \|^2_{E} \right) \| f - g \|_{E}.
$$

Due to the choice of the high velocity $v$ we have

$$
\forall f, g \in B_{E}(0, 1), \quad C_\omega \| v \|^5 e^{-2\delta\sqrt{|v|}T_{0} e^{-\delta\sqrt{|v|}T}} \left( \| f \|^2_{E} + \| g \|^2_{E} \right) \leq \frac{1}{8},
$$

and thus

$$
\|J_3(f) - J_3(g)\|_{E} \leq \frac{1}{8} \| f - g \|_{E}.
$$

The inequalities (4.12), (4.14), (4.16) specifies how large $V_{0}$ needs to be taken and from (4.13), (4.15) and (4.17) we have

$$
\forall f, g \in B_{E}(0, 1), \quad \| \Phi(f) - \Phi(g) \|_{E} \leq \frac{1}{2} \| f - g \|_{E}.
$$

Thus $\Phi$ is a contraction mapping for $v$ large enough.

\[\square\]

• Step 3: Conclusion.

Due to steps 1 and 2, $\Phi$ is a contraction mapping for high velocity on the complete metric space $(B_{E}, d_{E})$. By the fixed point Theorem there exists a unique solution,

$$r_{\omega}(t, x) = \Phi(r_{\omega}(t, x)) = -i J_0(t) - i \sum_{k=1}^{3} J_k(r_{\omega}(t, x)),
$$

such that

$$\|r_{\omega}(t)\|_{H^2} \leq C_\omega \| v \|^3 e^{-\delta\sqrt{|v|}t} \quad \forall t \in [T_{0}, +\infty),
$$

which concludes the proof of Theorem 1.4.

\[\square\]

**Appendix A. Proof of some Technical results**

*Proof of Lemma 2.4.* Recall that for all $f \in H^1\setminus\{\lambda Q; \; \lambda \in \mathbb{R}\}$ real valued, we have $\int (L^+ - f) \geq 0$. Denote $y_1 = \text{Re}(Y^+)$ and $y_2 = \text{Im}(Y^+)$. Since $y_2$ is not colinear to $Q$, we have

$$
(A.1) \quad - \text{Im} \int Y^+ Y^- = 2 \int y_1 y_2 = \frac{2}{\varepsilon_0} \int -(L^- y_2) y_2 \neq 0.
$$
Let \( h \in H^1 \) such that \( h = h_1 + i h_2 \), we can write \( h \) as,
\[
h = h^\perp + g,
\]
where,
\[
\begin{cases}
h^\perp \in G^\perp = \{ h \in H^1, (h, iQ) = (h, i\mathcal{V}^\pm) = (h, \partial_x Q) = 0, j = 1, 2, 3\}, \\
g \in \text{Span}\{i\mathcal{V}^+, i\mathcal{V}^-, (\partial_x Q)_{j=1,2,3}, iQ\}.
\end{cases}
\]
\( \Phi(h) \geq c \|h\|_{H^1}^2 \),
where, \( \Phi(h) = \frac{1}{2}(L^+ h_1, h_1) + \frac{1}{2}(L^- h_2, h_2) \). Next, we have
\[
\|h\|_{H^1}^2 = \|h^\perp + g\|_{H^1}^2 \leq c \|h^\perp\|_{H^1}^2 + c \|g\|_{H^1}^2 
\]
\[
\leq C \Phi(h) + C \left( \text{Im} \int \mathcal{V}^+ \bar{h} \right)^2 + C \left( \text{Im} \int \mathcal{V}^- \bar{h} \right)^2 + C \left( \int Q h_2 \right)^2 
\]
\[
+ C \sum_{j=1}^3 \left( \int \partial_x Q h_1 \right)^2.
\]
\[\square\]

**Proof of Lemma 2.10.** We will only prove the local existence statement. The construction of a maximal solution is standard and we omit it. Let us recall that the usual Strichartz estimates are also available outside a convex obstacle, see [20] and [17]:

**Theorem A.** Let \( d \geq 2 \), \( \Omega \subset IR^d \) be the exterior of a smooth compact strictly convex obstacle. Let \( q, \tilde{q} > 2 \) and \( 2 \leq r, \tilde{r} \leq \infty \) satisfy the scaling conditions: \( \frac{2}{q} + \frac{d}{r} = \frac{2}{\tilde{q}} + \frac{d}{\tilde{r}} \)
Then
\[
(A.3) \quad \left\| e^{it\Delta} u_0 \pm i \int_0^t e^{i(t-s)\Delta} F(s) \, ds \right\|_{L_t^q L_x^r} \leq C_s \left( \|u_0\|_{L^2(\Omega)} + \|F\|_{L_t^{\tilde{q}} L_x^{\tilde{r}}} \right).
\]

For the proof of the Lemma 2.10, we claim the following result .
As a consequence, fixing \( \forall \) second estimate. So, let us prove the second estimate. Note that

\[
\text{Proof.} \quad (A.6)
\]

\[
(A.4)
\]

\[
(A.5)
\]

\[
(A.6)
\]

if \( u \in L^a W^{s,p+1} \),

\[
\left\| u \right\|_{L^a W^{s,p+1}} \leq C \left\| u \right\|_{L^a L^{p+1}} \leq C \left\| u \right\|_{L^a L^{p+1}}
\]

Proof. Note that \( a > 2 \) since \( p < 5 \). For the first estimate it suffices to take \( v = 0 \) in the second estimate. So, let us prove the second estimate.

We use the following elementary inequality

\[
(A.7) \quad \forall (\xi, \zeta) \in \mathbb{C}^2, \quad \left| \xi \right|^{p-1} \xi - \left| \zeta \right|^{p-1} \zeta \leq C_p \left( \left| \xi \right|^{p-1} + \left| \zeta \right|^{p-1} \right) \left| \xi - \zeta \right|
\]

As a consequence, fixing \( t \), we deduce

\[
\left\| u \right\|_{L^a L^{p+1}} \leq C \left\| u \right\|_{L^a L^{p+1}} \leq C \left\| u \right\|_{L^a L^{p+1}} \leq C \left\| u \right\|_{L^a L^{p+1}} \leq C \left\| u \right\|_{L^a L^{p+1}}
\]

Then taking the \( L^a \)-norm in time, we obtain (A.5).

Next, we will prove the last estimate (A.6). For that, we have to use some fractional estimate for the non-linearity \( \left| u \right|^{p-1} u \). We refer to [20], for the following Proposition.

**Proposition B.** (Fractional chain rule)

Suppose \( G \in C^1(\mathbb{C}) \), \( s \in (0, 1] \), and \( 1 < p, p_1, p_2 < \infty \) are such that \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \) and \( 0 < s < \min \left( 1 + \frac{1}{p_1}, \frac{a}{p_2} \right) \). Then

\[
(A.8) \quad \left\| (-\Delta)^{\frac{s}{2}} G(f) \right\|_{L^p(\Omega)} \leq \left\| G'(f) \right\|_{L^{p_1}(\Omega)} \left\| (\Delta)^{\frac{s}{2}} f \right\|_{L^{p_2}(\Omega)},
\]

Uniformly for \( f \in C^\infty_c(\Omega) \).

**Remark A.2.** For the sake of simplicity, we will write the Dirichlet Laplacian as \( \Delta \) instead of \( \Delta_\Omega \).

By (A.8), we have

\[
(A.9) \quad \left\| (-\Delta)^{\frac{s}{2}} \left| u \right|^{p-1} u \right\|_{L^p(\Omega)} \leq C \left\| u \right\|_{L^a L^{p+1}} \left\| (\Delta)^{\frac{s}{2}} u \right\|_{L^{p+1}}
\]
Due to (A.4) and the above estimate (A.9), we have
\[
\left\| |u|^{p-1} u \right\|_{L^a W^{s,p+1}_x} = \left\| (1 - \Delta) \frac{\dot{u}}{u} |u|^{p-1} u \right\|_{L^a L^{p+1}_x} \\
\leq C \left\| |u|^{p-1} u \right\|_{L^a L^{p+1}_x} + C \left\| (-\Delta) \frac{\dot{u}}{u} |u|^{p-1} u \right\|_{L^a L^{p+1}_x} \\
\leq C \left\| |u|^{p-1} u \right\|_{L^\infty L^{p+1}_x} \left\| u \right\|_{L^a L^{p+1}_x} + \left\| |u|^{p-1} u \right\|_{L^\infty L^{p+1}_x} \left\| u \right\|_{L^a W^{s,p+1}_x} \\
\leq C \left\| |u|^{p-1} u \right\|_{L^\infty L^{p+1}_x} \left\| u \right\|_{L^a W^{s,p+1}_x}.
\]

\[\square\]

Fix \( M > 0 \) to be specified later. Let \( B \) be the ball of \( X = C([-T, T], L^2) \cap L^\infty_t H^s_x \cap L^a_t W^{s,p+1}_x \), with radius \( M > 0 \) and center 0, i.e. the set of functions \( u \in X \) such that
\[
\left\| u \right\|_{L^\infty_x H^s} \leq M \text{ and } \left\| u \right\|_{L^a W^{s,p+1}_x} \leq M.
\]

Denote
\[
d_B(u, v) = \left\| u - v \right\|_{L^\infty_x L^2} + \left\| u - v \right\|_{L^a_x L^{p+1}_x}
\]

**Lemma A.3.** \((B, d_B)\) is a complete metric space.

**Proof.** It is an immediate consequence of the easy fact that \( B \) is a closed subset of the following Banach space
\[
Y := C([-T, T], L^2) \cap L^a_t L^{p+1}_x.
\]

For \( v \in B \) we define \( \Phi(v)(t) := e^{it\Delta} u_0 + D(v)(t) \), where \( D(v) \) is the Duhamel term given by
\[
D(v)(t) := -i \int_0^t e^{i(t-s)\Delta} |v(s)|^{p-1} v(s) ds.
\]

\[\bullet\] **Step 1:** Stability of \( B \).

We will prove that: for \( v \in B \implies \Phi(v) \in B \), for a good choose of \( M \) and \( T \).

We have
\[
\left\| e^{it\Delta} u_0 \right\|_{L^\infty_x H^s} = \left\| u_0 \right\|_{H^s} \leq \frac{M}{2}.
\]

If the following conditions satisfied
\[
(A.10) \quad 2 \left\| u_0 \right\|_{H^s} \leq M,
\]
and we have
\[
\left\| e^{it\Delta} u_0 \right\|_{L^a W^{s,p+1}_x} = \left\| (1 - \Delta) \frac{\dot{u}}{u} e^{it\Delta} u_0 \right\|_{L^a L^{p+1}_x} = \left\| e^{it\Delta} (1 - \Delta) \frac{\dot{u}}{u} u_0 \right\|_{L^a L^{p+1}_x}.
\]

Using Strichartz estimate (recall that \((a, p+1)\) is an admissible pair) we obtain
\[
\left\| e^{it\Delta} u_0 \right\|_{L^a W^{s,p+1}_x} \leq C_s \left\| (1 - \Delta) \frac{\dot{u}}{u} u_0 \right\|_{L^2} = C_s \left\| u_0 \right\|_{H^s} \leq \frac{M}{2}.
\]

If \( M \) is chosen so that
\[
(A.11) \quad M \geq 2C_s \left\| u_0 \right\|_{H^s},
\]

So, we have proved that
\[
(A.12) \quad \max \left( \left\| e^{it\Delta} u_0 \right\|_{L^\infty_x H^s}, \left\| e^{it\Delta} u_0 \right\|_{L^a W^{s,p+1}_x} \right) \leq \frac{M}{2}, \text{ if (A.10), (A.11) are satisfied.}
\]
We next treat the Duhamel term.

\[ \|D(v)\|_{L^\infty H^s} = \left\| \int_0^t e^{i(t-\sigma)\Delta} |u(\sigma)|^{p-1} u(\sigma) d\sigma \right\|_{L^\infty H^s} = \left\| \int_0^t e^{i(t-\sigma)\Delta} \left(1 - \Delta\right)^{\frac{s}{2}} |u(\sigma)|^{p-1} u(\sigma) d\sigma \right\|_{L^\infty L^2}. \]

By Strichartz estimate, we have

\[ \|D(v)\|_{L^\infty H^s} \leq C_s \left\| (1 - \Delta)^{\frac{s}{2}} |u|^{p-1} u \right\|_{L^a L^{\frac{p+1}{p}}} \]

\[ \text{Since } s_p = \frac{3}{2} - \frac{3}{p+1} \text{ and } s_p \leq s < 1, \text{ we have } H^s(\Omega) \subset L^{p+1}(\Omega), \text{ see [3].} \]

Now, using Hölder inequality in the time variable and Claim A.1 we obtain

\[ \|D(v)\|_{L^\infty H^s} \leq C_s T^{1-\frac{2}{a}} \|u\|_{L^\infty L^{p+1}}^{p-1} \|u\|_{L^a W^{s, p+1}} \]

\[ \leq C T \|u\|_{L^\infty L^{p+1}}^{p-1} \|u\|_{L^a W^{s, p+1}} \]

\[ \leq C_2 T^{1-\frac{2}{a}} \|u\|_{L^\infty H^s} \|u\|_{L^a W^{s, p+1}}. \]

We can obtain the same thing for \( L^a W^{s, p+1} \)-norm of the Duhamel term.

\[ \|D(v)\|_{L^a W^{s, p+1}} = \left\| \int_0^t e^{i(t-\sigma)\Delta} \left(1 - \Delta\right)^{\frac{s}{2}} |u(s)|^{p-1} u(s) ds \right\|_{L^a L^{p+1}} \]

\[ \leq C_s \left\| (1 - \Delta)^{\frac{s}{2}} |u|^{p-1} u \right\|_{L^a L^{p+1}} \]

\[ \leq C T \|u\|_{L^\infty L^{p+1}}^{p-1} \|u\|_{L^a W^{s, p+1}} \]

\[ \leq C T^{1-\frac{2}{a}} \|u\|_{L^\infty H^s} \|u\|_{L^a W^{s, p+1}}. \]

Finally, we have obtained

\[ \|D(v)\|_{L^\infty H^s} + \|D(v)\|_{L^a W^{s, p+1}} \leq \frac{M}{2}. \]

If the following condition are satisfied

\[ (A.13) \quad C_{1,2} T^{1-\frac{2}{a}} M^{p-1} \leq \frac{1}{2}. \]

- Step 2: Contraction property

Let \( u, v \in B \),

\[ \|\Phi(u) - \Phi(v)\|_{L^\infty L^2 \cap L^a L^{p+1}} = \|D(u) - D(v)\|_{L^\infty L^2 \cap L^a L^{p+1}} \]

\[ \leq C_s \left\| |u|^{p-1} u - |v|^{p-1} v \right\|_{L^a L^{p+1}} \]

\[ \leq C_s T^{1-\frac{2}{a}} \left\| |u|^{p-1} u - |v|^{p-1} v \right\|_{L^a L^{p+1}} \]

\[ \leq C T \|u - v\|_{L^a L^{p+1}} \left( \|u\|_{L^\infty L^{p+1}}^{p-1} + \|v\|_{L^\infty L^{p+1}}^{p-1} \right) \]

\[ \leq C_3 T^{1-\frac{2}{a}} M^{p-1} \|u - v\|_{L^a L^{p+1}}, \]
which yields,
\[ d_B(\Phi(u) - \Phi(v)) \leq C_3 T^{1 - \frac{2}{p}} M^{p-1} d_B(u, v). \]

And this prove that \( \Phi \) is a contraction if the following condition is satisfied
\[(A.14) \quad C_3 T^{1 - \frac{2}{p}} M^{p-1} < 1. \]

- Step 3: Fixed point
We have proved that \( \Phi \) is a contraction on the metric space \((B, d_B)\) if the conditions \((A.12), (A.13)\) and \((A.14)\) on \( M \) and \( T \) hold. However it is easy to find \( M \) and \( T \) satisfying these conditions, we can take \( M \) as
\[ M := \max(2, 2C_s) \| u_0 \|_{H^s}. \]
So that \((A.12)\) is satisfied, then \( T \leq \tau \), where
\[ \tau := \frac{1}{C \| u_0 \|_{H^s}^{(p-1)/2}} \]
and \( C := \max(C_{1,2}, C_3) \) is a large constant, so that \((A.14), (A.13)\) hold for \( T \leq \tau \). Due to the fixed point theorem there exist a unique solution \( u \).

**Remark A.4.** Note that we can choose \( T \), up to multiplicative constant, as an explicit negative power of \( \| u_0 \|_{H^s} \). Also, if \( T \) is chosen as above, then \( u \in L^p_t W^{s,p+1} \) and
\[ \| u \|_{L^p_t W^{s,p+1}} + \| u \|_{L^\infty_t H^s} \leq C \| u_0 \|_{H^s}, \]
for a constant \( C > 0 \) which is independent of \( T \).

It remains to check that \( u \in C([-T, T], H^s) \), which will be done in step 4 and in step 5 we prove also the uniqueness of \( u \) among the \( C([-T, T], H^s) \) solutions.

- Step 4: Continuity
\[ u(t) = e^{it\Delta} u_0 + i \int_0^t e^{i(t-s)\Delta} |u(s)|^{p-1} u(s) ds. \]

It is well known that the function: \( t \mapsto e^{it\Delta} u_0 \) is in \( C([-T, T], H^s) \).

Next, we recall that from Step 1 and Step 2 that
\[ t \mapsto |u(t)|^{p-1} u(t) \in L^p_t W^{s,p+1}. \]

By Strichartz inequality, we have that the Duhamel term \( D(u) \in C([-T, T], H^s) \). Thus, we get \( u = e^{it\Delta} u_0 + D(u) \in C([-T, T], H^s) \).

- Step 5: Uniqueness.
Let \( u \) and \( v \) be two solutions in \( C([-T, T], H^s(\Omega)) \) with the same initial data \( u_0 \). Then
\[ u(t) - v(t) = i \int_0^t e^{i(t-s)\Delta} \left(|u(s)|^{p-1} u(s) - |v(s)|^{p-1} v(s)\right) ds, \]
by Strichartz inequality, if \( \theta > 0 \)
\[
\| u - v \|_{L^p_t L^r_x} \leq C \left( \| u \|_{L^p_t L^r_x}^{p-1} \| u - v \|_{L^p_t L^r_x} \right)^{p-1}.
\]
\[
\leq C \left( \| u \|_{L^p_t L^r_x}^{p-1} \| v \|_{L^p_t L^r_x} \right)^{p-1}.
\]
Choosing \( \theta > 0 \) small enough, so that
\[
C \left( \| u \|_{L^p_t L^r_x}^{p-1} \| v \|_{L^p_t L^r_x} \right)^{p-1} \theta^{1-\frac{\alpha}{2}} < 1.
\]
We deduce that \( \| u - v \|_{L^p_t L^r_x} = 0 \), then \( u = v \) in \([-\theta, \theta] \). Iterating this argument, we obtain \( u = v \) in \([-T, T] \).

\[ \square \]

**Proof of Lemma 3.2.** Let \( \rho = u - R \) and let
\[
\Phi : L^2 \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^4
\]
\[
(\rho, y, \mu) \mapsto \left( \text{Re} \int (\rho + R - \tilde{R})\nabla \tilde{Q}_\omega \Psi e^{-i(\frac{1}{2}(x,v)+\theta)} e^{-i\mu} dx, \text{Im} \int (\rho + R - \tilde{R})\tilde{R} dx \right).
\]
Denote:
\[
\Phi_1(\rho, y, \mu) = \text{Re} \int (\rho + R - \tilde{R})\nabla \tilde{Q}_\omega \Psi e^{-i(\frac{1}{2}(x,v)+\theta)} e^{-i\mu} dx,
\]
\[
\Phi_2(\rho, y, \mu) = \text{Im} \int (\rho + R - \tilde{R})\tilde{R}.
\]

- **Step 1:** Compute \( d_{(y, \mu)} \Phi_1 \). Let \( z \in \mathbb{R}^3 \), \( l \in \mathbb{R} \).
\[
(A.15) \quad (d_y \Phi_1(\rho, y, \mu), z)_j = \text{Re} \left( z_j \int \partial_{x_j} \tilde{Q}_\omega \partial_{x_j} \tilde{Q}_\omega \Psi^2 dx + \sum_{k \neq j} z_k \int \partial_{x_k} \tilde{Q}_\omega \partial_{x_j} \tilde{Q}_\omega \Psi^2 dx \right.
\]
\[
- \left. \sum_{k=1}^{3} \int (\rho + R - \tilde{R})\partial_{x_k} \partial_{x_j} \tilde{Q}_\omega \Psi e^{-i(\frac{1}{2}(x,v)+\theta)} e^{-i\mu} z_k dx \right).
\]
\[
(A.16) \quad (d_y \Phi_1(\rho, y, \mu), l)_j = \text{Re} \left( i \int l (\rho + R - \tilde{R})\partial_{x_j} \tilde{Q}_\omega \Psi e^{-i(\frac{1}{2}(x,v)+\theta)} e^{-i\mu} dx \right).
\]

**Claim A.5.**
\[
(A.17) \quad (d_y \Phi_1(\rho, y, \mu), z)_j = z_j \left\| \partial_{x_j} \tilde{Q}_\omega \Psi \right\|_{L^2}^2 + O \left( |z| \left( \| \rho \|_{L^2} + |y| \right) \right).
\]
\[
(A.18) \quad d_y \Phi_1(0, 0, 0) = \text{diag} \left( \left\| \partial_{x_j} \tilde{Q}_\omega \Psi \right\|_{L^2}^2 \right).
\]
\[
(A.19) \quad d_y \Phi_1(\rho, y, \mu) = O \left( \| \rho \|_{L^2} + |y| \right).
\]
\[
(A.20) \quad d_y \Phi_1(0, 0, 0) = 0.
\]

**Proof.** For the first estimate we have
\[
\left| \int \rho \partial_{x_k} \partial_{x_j} \tilde{Q}_\omega \Psi e^{-i(\frac{1}{2}(x,v)+\theta)} e^{-i\mu} dx \right| \leq C \| \rho \|_{L^2},
\]
and
\[
\int (R - \overline{R}) \partial_{x_k, x_j} Q_\omega \Psi dx = \int \int_0^1 \frac{d}{dt} R(x - ty) dt \partial_{x_k, x_j} Q_\omega \Psi dx
\]
\[
= \int \int_0^1 y \nabla R(x - ty) dt \partial_{x_k, x_j} Q_\omega \Psi dx,
\]
\[
\left| \int (R - \overline{R}) \partial_{x_k, x_j} Q_\omega \Psi dx \right| \leq C |y|.
\]
This implies that
\[
(A.21) \quad \text{Re} \left( \sum_{k=1}^3 \int z_k (\rho + R - \overline{R}) \partial_{x_k, x_j} Q_\omega \Psi e^{-i \left( \frac{1}{2} (x \cdot v) + \theta \right)} e^{-i \mu} dx \right) = O (|z| (\|\rho\|_{L^2} + |y|)).
\]

Since $Q_\omega$ is radial, we have
\[
\forall k \neq j, \quad \int \partial_{x_k} Q \partial_{x_j} Q dx = 0,
\]
which yields, for $k \neq j$
\[
\int \partial_{x_k} \overline{Q}_\omega \partial_{x_j} \overline{Q}_\omega \Psi^2 dx \leq \int \partial_{x_k} \overline{Q}_\omega \partial_{x_j} \overline{Q}_\omega dx = 0.
\]
Then
\[
(A.22) \quad \text{Re} \left( \sum_{k \neq j} z_k \int \partial_{x_k} \overline{Q}_\omega \partial_{x_j} \overline{Q}_\omega \Psi^2 dx \right) = 0.
\]

The estimate (A.17) it is a consequence of (A.21) and (A.22). Applying (A.17) at point $(0, 0, 0)$, we get (A.18).

Due to (A.16), we have
\[
(d_{\mu} \Phi_1 (\rho, y, \mu))_j = \text{Re} \left( i \int l (\rho + R - \overline{R}) \partial_{x_j} \overline{Q}_\omega \Psi e^{-i \left( \frac{1}{2} (x \cdot v) + \theta \right)} e^{-i \mu} dx \right).
\]
Then
\[
d_{\mu} \Phi_1 (\rho, y, \mu) = \text{Im} \int l (\rho + R - \overline{R}) \partial_{x_j} \overline{Q}_\omega \Psi e^{-i \left( \frac{1}{2} (x \cdot v) + \theta \right)} e^{-i \mu} dx.
\]

Similarly to the proof of the estimate (A.21), we have
\[
d_{\mu} \Phi_1 (\rho, y, \mu) = O (|l| (\|\rho\|_{L^2} + |y|)).
\]

Finally, due to the above equality it is easy to see that
\[
d_{\mu} \Phi_1 (0, 0, 0) = 0,
\]
which concludes the proof of the Claim A.5

\[\Box\]

- **Step 2:** Compute $d_{(y, \mu)} \Phi_2$.
  Recall that
  \[
  \Phi_2 (\rho, y, \mu) = \text{Im} \int (\rho + R - \overline{R}) \overline{R}.
  \]
\[
(A.23) \quad d_y \Phi_2(\rho, y, \mu).l = - \text{Im} \left( \sum_{j=1}^{3} \int l_j (\rho + R - \bar{R}) \partial_{x_j} \bar{Q}_{\omega} \Psi e^{-i(\frac{1}{2}(x.v) + \theta) e^{-i\mu}} \right).
\]

\[
(A.24) \quad d_\mu \Phi_2(\rho, y, \mu).q = - \int q \bar{Q}_{\omega}^2 \Psi^2 - \text{Re} \int q (\rho + R - \bar{R}) \bar{R}.
\]

**Claim A.6.** Let \(l \in \mathbb{R}^3, q \in \mathbb{R} \).
\[
(A.25) \quad d_y \Phi_2(\rho, y, \mu).l = O(|l| (\|\rho\|_{L^2} + |y|)).
\]
\[
(A.26) \quad d_y \Phi_2(0, 0, 0) = 0.
\]
\[
(A.27) \quad d_\mu \Phi_2(\rho, y, \mu).q = - \int q \bar{Q}_{\omega}^2 \Psi^2 + O(|q| (\|\rho\|_{L^2} + |y|)).
\]
\[
(A.28) \quad d_\mu \Phi_2(0, 0, 0) = - \left\| \bar{Q}_{\omega} \Psi^2 \right\|_{L^2}.
\]

**Proof.** Using the same argument as in the proof of Claim A.5, we obtain
\[
\text{Im} \left( \sum_{j=1}^{3} \int l_j (\rho + R - \bar{R}) \partial_{x_j} \bar{Q}_{\omega} \Psi e^{-i(\frac{1}{2}(x.v) + \theta) e^{-i\mu}} \right) = O(|l| (\|\rho\|_{L^2} + |y|)).
\]

Due to (A.23), we obtain the first estimate. Applying (A.25) at point \((0, 0, 0)\), we obtain
\[
d_y \Phi_2(0, 0, 0) = 0.
\]
Similarly to the proof of \(d_{y,\mu} \Phi_1\), we have
\[
\text{Re} \int q (\rho + R - \bar{R}) \bar{R} = O(|q| (\|\rho\|_{L^2} + |y|)).
\]
Using the above estimate and (A.24), we get
\[
d_\mu \Phi_2(\rho, y, \mu).q = - \int q \bar{Q}_{\omega}^2 \Psi^2 + O(|q| (\|\rho\|_{L^2} + |y|)).
\]

Then
\[
d_\mu \Phi_2(0, 0, 0) = - \left\| \bar{Q}_{\omega} \Psi^2 \right\|_{L^2}.
\]
This concludes the proof of the Claim A.6 \(\square\)

**Step 3: Conclusion**
From Step 1 and Step 2 we get
\[
d_{(y,\mu)} \Phi(\rho, y, \mu) = \begin{pmatrix}
\left\| \partial_{x_1} \bar{Q}_{\omega} \Psi \right\|_{L^2}^2 & 0 & 0 & 0 \\
0 & \left\| \partial_{x_2} \bar{Q}_{\omega} \Psi \right\|_{L^2}^2 & 0 & 0 \\
0 & 0 & \left\| \partial_{x_3} \bar{Q}_{\omega} \Psi \right\|_{L^2}^2 & 0 \\
0 & 0 & 0 & - \left\| \bar{Q}_{\omega} \Psi \right\|_{L^2}^2
\end{pmatrix} + O(\|\rho\|_{L^2} + |y|).
\]
We can deduce that \(d_{(y,\mu)} \Phi(0, 0, 0)\) is invertible and we have \(\Phi(0, 0, 0) = 0\).
Then, by the Implicit function theorem, there exists $\varepsilon_0 > 0$, $\varepsilon_0 \leq \eta$ and a $C^1$-function
\[ g : B_{L^2}(0,\varepsilon) \rightarrow B_{L^1}(0,1) \]
\[ \rho \mapsto g(\rho) = ((y(\rho), \mu(\rho)) \]
such that $\Phi(\rho, y, \mu) = 0$ in $B_{L^2}(0,\varepsilon) \times g(B_{L^2}(0;\varepsilon))$ is equivalent to $(y, \mu) = g(\rho)$. Finally we set
\[ r := r(\rho) = \rho + R(\cdot - y(\rho))e^{i\mu(\rho)}. \]

Proof of Lemma 3.4.
\[ \sigma : \mathbb{R}^2 \rightarrow H_0^1 \quad \Gamma : B_{H_0^1}(\varepsilon) \rightarrow H_0^1 \times \mathbb{R}^3 \times \mathbb{R} \]
\[ \lambda := \lambda^\perp \mapsto i(\lambda^+ Y_+(T_n) + \lambda^- Y_-(T_n)), \quad \rho \mapsto (r, y, \mu). \]

Where, $(r, y, \mu)$ is the modulation of $u(T_n)$ around $R(T_n)$ and $B_{H_0^1}(\varepsilon)$ is a ball of radius $\varepsilon > 0$ which is defined in the proof of the Lemma 3.2.
\[ \Lambda : H_0^1 \times \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^2 \]
\[ (r, y, \mu) \mapsto \left( \alpha^+(T_n) = \text{Im} \int \tilde{Y}_-(T_n, x) \bar{\varphi}(T_n, x)dx, \alpha^-(T_n) = \text{Im} \int \tilde{Y}_+(T_n, x) \bar{\varphi}(T_n, x)dx \right). \]

We have, $\sigma(0) = 0$, $\Gamma(0) = (0, 0, 0)$, $\Lambda(0, 0, 0) = (0, 0)$. Denote: $\Theta = \Lambda \circ \Gamma \circ \sigma$.

Now let us prove that $\Theta$ is a diffeomorphism on a $\mathcal{V}_0$ a neighbourhood of $0 \in \mathbb{R}^2$ by computing $d\Theta = d\Lambda \circ d\Gamma \circ d\sigma$.

Firstly, we have that $d\sigma(\lambda) = \sigma$, for all $\lambda \in \mathbb{R}^2$. Secondly, let $l \in H_0^1, z \in \mathbb{R}^3, q \in \mathbb{R}$ such that
\[ d\Lambda(r, y, \mu).l, z, q = \left( \text{Im} \int \tilde{Y}_-(x) \bar{\varphi}(x)dx + \sum_{j=1}^{3} z_j \partial_{x_j} \tilde{Y}_-(x) \bar{\varphi}(x) +iq \tilde{Y}_-(x) \bar{\varphi}(x)dx, \right. \]
\[ \left. \text{Im} \int \tilde{Y}_+(x) \bar{\varphi}(x)dx + \sum_{j=1}^{3} z_j \partial_{x_j} \tilde{Y}_+(x) \bar{\varphi}(x) +iq \tilde{Y}_+(x) \bar{\varphi}(x)dx \right). \]

Finally, we have to compute $d\Gamma$. Let $\Phi$ and $g$ defined as in the proof of the Lemma 3.2 for $R(t_n)$. Then, we obtain
\[ \Gamma(\rho) = \left( \rho + R(T_n) - R(T_n, \cdot - y(\rho)), y(\rho), \mu(\rho) \right). \]
\[ (A.29) \]
\[ d\Gamma(\rho).l = \left( l + \nabla R(T_n, \cdot - y(\rho))e^{i\mu(\rho)}dy(\rho), l + iR(\cdot - y(\rho))e^{i\mu(\rho)}d\mu(\rho), l, dy(\rho), d\mu(\rho) \right). \]

we have
\[ \Phi(\rho, y(\rho), \mu(\rho)) = 0 \implies \begin{cases} \Phi_1(\rho, y(\rho), \mu(\rho)) = 0 \\ \Phi_2(\rho, y(\rho), \mu(\rho)) = 0 \end{cases} \implies \begin{cases} d_1 \Phi_1 + d_2 \Phi_1 dy(\rho) + d_3 \Phi_1 d\mu(\rho) = 0 \\ d_1 \Phi_2 + d_2 \Phi_2 dy(\rho) + d_3 \Phi_2 d\mu(\rho) = 0 \end{cases}. \]
(A.30) $$\begin{align*}
\Longrightarrow \begin{cases}
\quad dy(\rho) = (d_2\Phi_1)^{-1}[-(d_1\Phi_1) - (d_3\Phi_1) d\mu(\rho)] \\
\quad d\mu(\rho) = (d_3\Phi_2)^{-1} \left( (d_2\Phi_2) (d_2\Phi_1)^{-1} (d_1\Phi_1) - (d_3\Phi_2)^{-1} (d_1\Phi_2) - (d_3\Phi_1)^{-1} (d_1\Phi_1) \right) \\
\quad + (d_3\Phi_1)^{-1} (d_2\Phi_2)^{-1} (d_1\Phi_2).
\end{cases}
\end{align*}$$

Recall that

$$d\Theta(\lambda, \tilde{\lambda}) = d\Lambda(d\Gamma(\sigma(\lambda))). d\Gamma(\sigma(\lambda)) \cdot \sigma(\tilde{\lambda}),$$

by (A.29) we get

$$d\Gamma(\sigma(\lambda)) \cdot \sigma(\tilde{\lambda}) = \left( \sigma(\tilde{\lambda}) - y(\sigma(\lambda)) e^{i\mu(\sigma(\lambda))} dy(\sigma(\lambda)) \cdot \sigma(\tilde{\lambda}) \right) + i R(T_n, \cdot - y(\sigma(\lambda)) e^{i\mu(\sigma(\lambda))} d\mu(\sigma(\lambda)) \cdot \sigma(\tilde{\lambda}), dy(\sigma(\lambda)) \cdot \sigma(\tilde{\lambda}), d\mu(\sigma(\lambda)) \cdot \sigma(\tilde{\lambda}))$$

We claim the following estimate which will be proved at the end of this proof.

**Claim A.7.** Let $\delta > 0$ such that

$$\begin{align*}
&dy(\sigma(\lambda)) \cdot \sigma(\tilde{\lambda}) = O \left( (e^{-\delta \sqrt{\omega}} |T_n + |\lambda||\tilde{\lambda}|) \right) \\
&d\mu(\sigma(\lambda)) \cdot \sigma(\tilde{\lambda}) = O \left( (e^{-\delta \sqrt{\omega}} |T_n + |\lambda||\tilde{\lambda}|) \right).
\end{align*}$$

By the claim above we have

$$d\Gamma(\sigma(\lambda)) \cdot \sigma(\tilde{\lambda}) = \left( \sigma(\tilde{\lambda}) , 0 , 0 \right) + O \left( (e^{-\delta \sqrt{\omega}} |T_n + |\lambda||\tilde{\lambda}|) \right).$$

Using the expression of $d\Lambda$, we get

$$d\Theta(\lambda, \tilde{\lambda}) = d\Lambda(\sigma(\lambda)) \cdot (\sigma(\tilde{\lambda}) , 0 , 0) + O \left( (e^{-\delta \sqrt{\omega}} |T_n + |\lambda||\tilde{\lambda}|) \right),$$

$$d\Theta(\lambda) = M + O \left( (e^{-\delta \sqrt{\omega}} |T_n + |\lambda|) \right).$$

Where $M$ is a matrix such that

$$M = \begin{pmatrix}
\text{Re} \int \tilde{Y}_-(T_n, x) Y_+(T_n, x) dx & \text{Re} \int \tilde{Y}_-(T_n, x) Y_-(T_n, x) dx \\
\text{Re} \int \tilde{Y}_+(T_n, x) Y_+(T_n, x) dx & \text{Re} \int \tilde{Y}_+(T_n, x) Y_-(T_n, x) dx
\end{pmatrix}$$

Since $Y_+$ and $Y_-$ are linearly independent, then the following matrix is invertible

$$A = \begin{pmatrix}
\text{Re} \int Y_-(T_n, x) Y_+(T_n, x) dx & \text{Re} \int Y_-(T_n, x) Y_-(T_n, x) dx \\
\text{Re} \int Y_+(T_n, x) Y_+(T_n, x) dx & \text{Re} \int Y_+(T_n, x) Y_-(T_n, x) dx
\end{pmatrix}$$

And we have

$$\left| \text{Re} \int \left( \tilde{Y}_-(T_n, x) - Y_-(T_n, x) \right) Y_+(T_n, x) dx \right| \leq C |y| \leq C |\lambda|.$$
Let $\eta > 0$ be such that $B_{\mathbb{R}^2}(\eta) \subset \mathcal{U}$. Then, for any $\alpha^+ \in B_{\mathbb{R}}(\eta)$, there exist a unique $\lambda = \lambda(\alpha^+) \in B_{\mathbb{R}^2}(\beta)$ such that
\[
\Theta(\lambda(\alpha^+)) = (\alpha^+, 0) \quad \text{and} \quad |\lambda(\alpha^+)| \leq C |\alpha^+|.
\]
And this concludes the proof of Lemma 3.4. $\square$

**Proof of Claim A.7.** From (A.30), we have
\[
dy = (d_2\Phi_1)^{-1} \left[ - (d_1\Phi_1) - (d_3\Phi_1) d\mu(\rho) \right],
\]
\[
d\mu(\rho) = (d_3\Phi_2)^{-1} (d_2\Phi_2) (d_2\Phi_1)^{-1} (d_1\Phi_1) - (d_3\Phi_2)^{-1} (d_1\Phi_2) - (d_3\Phi_1)^{-1} (d_1\Phi_1)
\]- (d_3\Phi_1)^{-1} (d_2\Phi_1) (d_2\Phi_2)^{-1} (d_1\Phi_2).
\]
Remark that it suffices to prove that
\[
d_1\Phi_1,\sigma(\tilde{\lambda}) = O \left( |\lambda \tilde{\lambda}| \right)
\]
\[
d_1\Phi_2,\sigma(\tilde{\lambda}) = O \left( (e^{-\delta \sqrt{\omega}} + |\lambda|) |\tilde{\lambda}| \right).
\]
Let $l \in H^1_0$, we have
\[
d_1\Phi_1(\rho, y, \mu). l = \text{Re} \int l(x) \nabla \tilde{Q}_\omega(T_n, x) \Psi(x) e^{-i\tilde{\omega}(T_n, x)} dx,
\]
\[
d_1\Phi_2(\rho, y, \mu). l = \text{Im} \int l(x) \bar{R}(T_n, x) dx.
\]
Recall that $\sigma(\tilde{\lambda}) = i \left( \tilde{\lambda}^+ Y_+(T_n, x) + \tilde{\lambda}^- Y_-(T_n, x) \right)$.
\[
d_1\Phi_1,\sigma(\tilde{\lambda}) = \text{Re} \int i \left( \tilde{\lambda}^+ Y_+ + \tilde{\lambda}^- Y_- \right) \nabla \tilde{Q}_\omega \Psi e^{-i\tilde{\omega}} dx
\]
\[
= \text{Im} \left[ e^{-i\mu \tilde{\lambda}} \left( \int_{Y^+} \nabla \tilde{Q}_\omega \Psi dx + e^{-i\mu \tilde{\lambda}} \int_{Y^-} \nabla \tilde{Q}_\omega \Psi dx \right) \right].
\]
\[
I_1 + I_2 = \int Y^+_\omega \nabla Q_\omega \Psi dx + \int Y^-_\omega \nabla Q_\omega \Psi dx + O(|y|).
\]
Since $Y^\pm_\omega$ and $Q_\omega$ are radial, we have
\[
\int Y^+_\omega \nabla Q_\omega \Psi dx \leq \int Y^+_\omega \nabla Q_\omega dx = 0,
\]
and using $|y| \leq |\lambda|$ we get
\[
d_1\Phi_1,\sigma(\tilde{\lambda}) = O \left( |\lambda \tilde{\lambda}| \right).
\]
Denote $y_1 = \text{Re} (Y^+_\omega) = \text{Re} (Y^-_\omega)$ and $y_2 = \text{Im} (Y^+_\omega) = -\text{Im} (Y^-_\omega)$.
Recall that $\mathcal{L}_\omega Y^\pm_\omega = \pm e_\omega Y^\pm_\omega$.
\[
d_1\Phi_2,\sigma(\tilde{\lambda}) = \text{Im} \int i \left( \tilde{\lambda}^+ Y_+ + \tilde{\lambda}^- Y_- \right) \bar{Q}_\omega \Psi e^{-i\tilde{\omega}} dx
\]
\[
= \text{Re} \left[ e^{-i\mu \tilde{\lambda}} \left( \int_{Y^+_\omega} \bar{Q} \Psi dx + e^{-i\mu \tilde{\lambda}} \int_{Y^-_\omega} \bar{Q} \Psi dx \right) \right].
\]
\[ J_1 + J_2 = \int (-L^{-}_\omega y_2 + i L^+_\omega y_1) \tilde{Q}_\omega \Psi \, dx + \int - (L^{-}_\omega y_2 + i L^+_\omega y_1) \tilde{Q}_\omega \Psi \, dx \]

\[ = -2i \int L^{-}_\omega y_2 (\tilde{Q}_\omega \Psi) \, dx. \]

Since \( L^{-}_\omega \) is self-adjoint operator.

\[ J_1 + J_2 = -2i \int L^{-}_\omega y_2 (\tilde{Q}_\omega \Psi) \, dx + O(|y|). \]

Using the fact that \( \partial_x \Psi \) has a compact support, \( L^{-}_\omega (Q_\omega) = 0 \) and \( |y| \leq |\lambda| \) we get

\[ d_1 \Phi_2, \sigma (\tilde{\lambda}) = O \left( (e^{-\delta \sqrt{|v|} T_n} + |\lambda|)|\tilde{\lambda}| \right). \]

This concludes the proof of the Claim A.7.

\[ \square \]

**Appendix B. Computation of some estimates**

**Proof of Claim 4.3.** Using (4.9) and the compact support of \( \nabla^k \Psi \), we obtain the first estimate. Let us prove the second inequality.

Notice that \( F : z \mapsto \nabla^2 F(z) = |z|^2 \) is differentiable on \( \mathbb{C} \) and \( \nabla \Psi \), we obtain the first estimate. Let us prove the second inequality.

Notice that \( F : z \mapsto |z|^2 \) is differentiable on \( \mathbb{C} \) and

\[ \frac{dF}{dz}(z) = 2|z|^2, \quad \frac{d^2 F}{dz^2}(z) = 2z, \quad \frac{dF}{dz}(z) = z, \quad \frac{d^2 F}{dz^2}(z) = 0. \]

Since \( x \mapsto H(t, x) \) is smooth.

Then we have,

\[ \nabla \left( \left| H(t, x) \right|^2 \right) = \nabla H(t, x) \nabla^2 F(H(t, x)) + \nabla \overline{H}(t, x) \nabla \bar{z} F(H(t, x)) \]

\[ \nabla^2 \left( \left| H(t, x) \right|^2 \right) = \nabla^2 H(t, x) \nabla^2 F(H(t, x)) + \nabla^2 \overline{H}(t, x) \nabla \bar{z} F(H(t, x)) + \nabla^2 \bar{z} F(H(t, x)) + \nabla \overline{z} F(H(t, x)) \]

where, \( \nabla f = (\partial_i f) \), \( i = 1, 2, 3 \).

Using again the fact that \( \nabla^k \Psi \) has a compact support and the exponential decay of \( Q_\omega \) to conclude the proof. \[ \square \]

**References**

[1] Abou Shakra, F. On 2D nonlinear Schrödinger equation on non-trapping exterior domains. *Rev. Mat. Iberoam.* 31, 2 (2015), 657–680.

[2] Berestycki, H., and Lions, P.-L. Nonlinear scalar field equations. II. Existence of infinitely many solutions. *Arch. Rational Mech. Anal.* 82, 4 (1983), 347–375.

[3] Burq, N., Gérard, P., and Tzvetkov, N. On nonlinear Schrödinger equations in exterior domains. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 21, 3 (2004), 295–318.

[4] Cazenave, T. *Semilinear Schrödinger equations*, vol. 10 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 2003.

[5] Cazenave, T., and Lions, P.-L. Orbital stability of standing waves for some nonlinear Schrödinger equations. *Comm. Math. Phys.* 85, 4 (1982), 549–561.
[6] Combet, V. Multi-existence of multi-solitons for the supercritical nonlinear Schrödinger equation in one dimension. *Discrete Contin. Dyn. Syst.* 34, 5 (2014), 1961–1993.

[7] Côte, R., Martel, Y., and Merle, F. Construction of multi-soliton solutions for the $L^2$-supercritical gKdV and NLS equations. *Rev. Mat. Iberoam.* 27, 1 (2011), 273–302.

[8] Duyckaerts, T., Holmer, J., and Roudenko, S. Scattering for the non-radial 3D cubic nonlinear Schrödinger equation. *Math. Res. Lett.* 15, 6 (2008), 1233–1250.

[9] Duyckaerts, T., and Merle, F. Dynamic of threshold solutions for energy-critical NLS. *Geom. Funct. Anal.* 18, 6 (2009), 1787–1840.

[10] Duyckaerts, T., and Roudenko, S. Threshold solutions for the focusing 3d cubic Schrödinger equation. *Rev. Mat. Iberoam.* 26, 1 (2010), 1–56.

[11] Fang, D., Xie, J., and Cazenave, T. Scattering for the focusing energy-subcritical nonlinear Schrödinger equation. *Sci. China Math.* 54, 10 (2011), 2037–2062.

[12] Grillakis, M. Analysis of the linearization around a critical point of an infinite-dimensional Hamiltonian system. *Comm. Pure Appl. Math.* 43, 3 (1990), 299–333.

[13] Grillakis, M. Analysis of the linearization around a critical point of an infinite-dimensional Hamiltonian system. *Comm. Pure Appl. Math.* 43, 3 (1990), 299–333.

[14] Grillakis, M., Shatah, J., and Strauss, W. Stability theory of solitary waves in the presence of symmetry. I. *J. Funct. Anal.* 74, 1 (1987), 160–197.

[15] Guevara, C. D. Global behavior of finite energy solutions to the $d$-dimensional focusing nonlinear Schrödinger equation. *Math. Res. Lett.* 15, 6 (2008), 1233–1250.

[16] Guevara, C. D. On the Schrödinger equation outside strictly convex obstacles. *Analysis & PDE* 3, 3 (2010), 261–293.

[17] Grillakis, M. Analysis of the linearization around a critical point of an infinite-dimensional Hamiltonian system. *Comm. Pure Appl. Math.* 43, 3 (1990), 299–333.

[18] Grillakis, M., Shatah, J., and Strauss, W. Stability theory of solitary waves in the presence of symmetry. I. *J. Funct. Anal.* 74, 1 (1987), 160–197.

[19] Guevara, C. D. Global behavior of finite energy solutions to the $d$-dimensional focusing nonlinear Schrödinger equation. *Math. Res. Lett.* 15, 6 (2008), 1233–1250.

[20] Guevara, C. D. On the Schrödinger equation outside strictly convex obstacles. *Analysis & PDE* 3, 3 (2010), 261–293.

[21] Grillakis, M. Analysis of the linearization around a critical point of an infinite-dimensional Hamiltonian system. *Comm. Pure Appl. Math.* 43, 3 (1990), 299–333.

[22] Grillakis, M., Shatah, J., and Strauss, W. Stability theory of solitary waves in the presence of symmetry. I. *J. Funct. Anal.* 74, 1 (1987), 160–197.

[23] Guevara, C. D. Global behavior of finite energy solutions to the $d$-dimensional focusing nonlinear Schrödinger equation. *Math. Res. Lett.* 15, 6 (2008), 1233–1250.
[33] Weinstein, M. I. Lyapunov stability of ground states of nonlinear dispersive evolution equations. *Comm. Pure Appl. Math.* 39, 1 (1986), 51–67.

[34] Yang, K. The focusing NLS on exterior domains in three dimensions. *Commun. Pure Appl. Anal.* 16, 6 (2017), 2269–2297.

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