ON INVARIANCE OF DOMAINS WITH
SMOOTH BOUNDARIES WITH RESPECT TO
STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. We prove constructible sufficient conditions of lack of exit by solutions of stochastic differential Itô’s equations from domains with smooth boundaries.

Consider a stochastic differential equation for process $\xi(t) \in \mathbb{R}^n$.

$$d\xi(t) = a(t, \xi)dt + \sum_{k=1}^{n} b_k(t, \xi)dw_k, \quad \xi(0) = \xi_0 \quad (1)$$

here

$a(t, x) := (a_i(t, x), 1 \leq i \leq n), \quad b_k(t, x) := (b_{ki}(x), 1 \leq i \leq n), \quad x \in \mathbb{R}^n$.

It is assumed that there is such constant $L$ that for functions $b_{ij}(t, x), \ a_i(t, x)$ the following conditions take place

$$|a(s, x) - a(s, y)| + \sum_{k} |b_k(s, x) - b_k(s, y)| \leq L|x - y|.$$
\[ |a(s, x)|^2 + \sum_{k}^{n} |b_k(s, x)|^2 \leq L^2(1 + |x|^2). \quad (2) \]

for all \( x, y \in R^n \).

Here \( | \cdot | \) is the norm (length) of a vector.

It follows from \([7, \text{p.480}]\) that this is a sufficient condition for existence of a unique solution of (1).

Let there be given a measurable set \( K \subseteq R^n \). A set \( K \) is said to be an invariant set of equation (1) if under condition \( P(\xi_0 \in K) = 1 \) the following equalities hold:

\[ P(\xi(t) \in K) = 1 \quad \text{for all} \quad t \geq 0. \quad (3) \]

The property (3) of trajectories of solution of (1) sometimes is called viability. The necessary and sufficient conditions of viability were proved for the first time in \([1]\). Such conditions were proved for more general constructions of equations in \([2]\). The methods of investigations of these articles are different but the set \( K \) is the same: it is convex and closed.

This problem was reduced to viability of ordinary differential equations with the help of approximation theorems Ikeda- Nakao- Yamato for homogeneous stochastic differential equations in \([4]\).

The conditions of viability were formulated in terms of asymptotic behavior of distance to the considered closed set. The analogy conditions of viability were proved for inhomogeneous stochastic differential equations and relative closed sets in \([5]\). We observe that test of conditions in terms of distance to sets requires the additional investigations. They are checked for convex sets effectively. For example, it was done in \([5]\).

It was proved necessary and sufficient conditions or only sufficient conditions of viability (3) in \([3, 6]\) by probabilistic methods for the specific domains \( K \).

Our purpose is to obtain verifiable sufficient conditions of viability (3) for domains with smooth boundaries. Our method of investigation is different from other. It is based on the use of Ostrogradskii- Gauss theorem.

Consider a closed set \( K \) in \( R^n \) with boundary \( S( \text{or} \partial K) \).
Let $U(x, r)$ denote open ball with center in point $x$ and with radius $r$. The union of balls with centers in $K$ is called $\varepsilon$-neighborhood $K_\varepsilon$ of the set $K : K_\varepsilon = \bigcup_{x \in K} U(x, \varepsilon)$. We will denote by $S_\varepsilon$ the boundary of $K_\varepsilon$.

We introduce the following function

$$\omega_\varepsilon(x) = \begin{cases} c_\varepsilon e^{-\varepsilon^2 - |x|^2}, & \text{if } |x| \leq \varepsilon, \\ 0, & \text{if } |x| > \varepsilon. \end{cases}$$

The constant is chosen $c_\varepsilon$ such that the following equality holds

$$\int \omega_\varepsilon(x) dx = 1.$$ 

Thus

$$c_\varepsilon \varepsilon^n \int_{|\xi| < 1} e^{-\frac{1}{(1-|\xi|^2)} d\xi} = 1.$$ 

If $\chi(\cdot)$ be characteristic function of set $K_{2\varepsilon}$ then for any $\varepsilon > 0$ the function

$$\eta_\varepsilon(x) = \int \chi(z) \omega_\varepsilon(x - z) dz.$$ 

satisfies the following relations [8, p.89]:

$$0 \leq \eta_\varepsilon(x) \leq 1, \quad \eta_\varepsilon(x) = 1, \quad x \in K_\varepsilon,$$

$$\eta_\varepsilon(x) = 0, \quad x \notin K_{3\varepsilon}, \quad \eta_\varepsilon(x) \in C^\infty(R^n), \quad |\eta_\varepsilon^{(\alpha)}(x)| \leq L_\alpha \varepsilon^{-|\alpha|}. \quad (4)$$

The next statement follows from the axiom of continuity:

**Statement.** If $\zeta$ be random vector in space $R^n$, then the following representation takes place

$$E\eta_\varepsilon(\zeta) = P(\zeta \in K) + l_\varepsilon, \quad \text{where} \quad l_\varepsilon \geq 0, \quad l_\varepsilon \to 0, \quad \text{when} \ \varepsilon \to \infty. \quad (5)$$
Lemma 1. If $P(\xi(0) \in K) = 1$ and for some number $\epsilon_0 > 0$ and any numbers $0 < \epsilon \leq \epsilon_0$ the following inequality takes place

$$E\eta_\epsilon(\xi(t)) \geq E\eta_\epsilon(\xi(0)), \quad t \geq 0,$$

then the following equality is true

$$P(\xi(t) \in K) = 1, \quad t \geq 0.$$

Proof. Let the condition of Lemma be fulfilled but statement of Lemma don’t fulfill. If statement of Lemma don’t fulfill then there exists such $t_*$ that

$$P(\xi(t_*) \in K) < 1$$  \hspace{1cm} (6).$$

Futher according to the statement (5) and the condition of Lemma 1 we have the the following inequality in point $t_*$

$$P(\xi(t_*) \in K) + l_{3,\epsilon} \geq P(\xi(0) \in K) + l_{2,\epsilon}.$$

Letting $\epsilon \to 0$, we arrive at

$$P(\xi(t_*) \in K) \geq P(\xi(0) \in K) = 1.$$

The latter one contradicts to (6). This contradiction proves the Lemma 1.

We make the following assumption: the boundary of $\partial K_\epsilon$ belongs to class $C^l$, when the following condition of smoothness of boundary of $K_\epsilon$ holds for $\epsilon < \epsilon_0$ under small $\epsilon_0 > 0$. The intersection of boundary of set $K_\epsilon$ with ball $U(x, \epsilon)$, $x \in K_{2\epsilon}$:

$$\triangle_\epsilon(x) := \overline{U}(x, \epsilon) \cap \partial K_\epsilon$$

is surface whose equation in local coordinates $(y_1, \ldots, y_{n-1})$ with origin of coordinates in point $x_0 \in \triangle_\epsilon(x)$ has form $y_n = \varphi(y_1, \ldots, y_{n-1})$.

The function $\varphi$ belongs to class $C^l$ in region $\overline{D}_\epsilon$, which is projection of $\triangle_\epsilon(x)$ on the plane $y_n = 0$.

Let us denote by $\nu(z) = (\nu_i(z), \quad i = 1, 2, \ldots, n)$ the unit vector of external normal to boundary $S$ in point $z \in S$.

It is known, that if surface is given by relation $Q(y_1, \ldots, y_n) = 0$, here $Q(\cdot)$
be smooth function, then the unit vector of normal \( \vec{n} \) has the following form

\[
\vec{n} = \left( \frac{Q_{yi}}{\sqrt{\sum_k Q_{yk}^2}}, \ i = 1, n \right).
\]

Thus, if the \( \varphi \) is differentiable, then the next representation for \( \nu(z) \) takes place locally

\[
\nu(z) = \left( \frac{1}{\sqrt{1 + \sum_{i \leq n-1} \varphi_{yi}^2}}, \frac{-\varphi_{yk}}{\sqrt{1 + \sum_{i \leq n-1} \varphi_{yi}^2}}, \ k = 1, n-1 \right).
\]

Suppose now that the boundary \( K_{\epsilon}, \epsilon_0 \geq \epsilon \geq 0 \) under some \( \epsilon_0 > 0 \) belongs to class \( C^2 \).

**Theorem 1.** If the following conditions are fulfilled

1. The functions \( a(t, x), b_k(t, x), 1 \leq k \leq n \), in addition to properties (2), under fixed \( t \) belong according to classes \( C^2(R^n), C^3(R^n) \).
2. \( \sup_{s \geq 0} \sup_{z \in S_\epsilon} \left( \sum_i b_{ji}(s, z)\nu_i(z) \right) = o(\epsilon), \ \epsilon \to 0. \ 1 \leq j \leq n, \ s \geq 0; \)
3. \( \lim_{\epsilon \to 0} \sup_{s \geq 0} \sup_{z \in S_\epsilon} \left( \sum_i a_i(s, z)\nu_i(z) - \frac{1}{2} \sum_{i,j,k} \frac{\partial b_{ki}(s, z)}{\partial z_j} \nu_i(z) b_{kj}(s, z) \right) < 0. \)

then (3) takes place.

**Proof.** Applying the Ito’s formula, we get the following equality

\[
E\eta_\epsilon(\xi(t)) - E\eta_\epsilon(\xi(0)) = E \int_0^t A\eta_\epsilon(\xi(s))ds.
\]

Here

\[
A := \sum_{i=1}^n a_i(s, x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n \sigma_{ij}(s, x) \frac{\partial^2}{\partial x_i \partial x_j}.
\]
The matrix \( \sigma(s, x) = (\sigma_{ij}(x), 1 \leq i, j \leq n) \), is defined in the following way:

\[
\sigma(s, x) = B^T(s, x)B(s, x), \quad B(s, x) := (b_{ki}(s, x), 1 \leq k, i \leq n).
\]

According to the Lemma 1 and definition of function \( \eta_\varepsilon(x) \) for proof of invariance of set \( K \) it suffices to prove the next inequality:

\[
A\eta_\varepsilon(x) \geq 0, \quad x \in (K_{3\varepsilon} \setminus K_\varepsilon), \quad s \geq 0.
\]

It is not difficult to check the following properties of function \( \omega_\varepsilon(x - z) \)

\[
-\frac{\partial}{\partial z_i}(\omega_\varepsilon(x - z)) = \frac{\partial}{\partial x_i}(\omega_\varepsilon(x - z)); \quad \frac{\partial^2}{\partial z_i \partial z_j}(\omega_\varepsilon(x - z)) = \frac{\partial^2}{\partial x_i \partial x_j}(\omega_\varepsilon(x - z));
\]

\[
\omega_\varepsilon(x - z)|_{|x-z|=\varepsilon} = \frac{\partial}{\partial x_i}(\omega_\varepsilon(x - z))|_{|x-z|=\varepsilon} = \frac{\partial^2}{\partial x_i \partial x_j}(\omega_\varepsilon(x - z))|_{|x-z|=\varepsilon} = 0.
\]

We define set

\[
K_\varepsilon(x) := \{z : |z - x| \leq \varepsilon \cap (K_{3\varepsilon} \setminus K_\varepsilon)\}.
\]

Further, applying the properties of function \( \omega_\varepsilon(x) \) and Taylor-series expansion of functions \( a_i(s, x), \sigma_{ij}(s, x) \) in point \( z \) we get

\[
A\eta_\varepsilon(x) = \left( \sum_i a_i(s, x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j} \sigma_{ij}(s, x) \frac{\partial^2}{\partial x_i \partial x_j} \right) \int_{R^n} \chi(z) \omega_\varepsilon(x - z) dz =
\]

\[
= -\int_{K_\varepsilon(x)} \sum_i \left( a_i(s, z) + \sum_k \frac{\partial a_i(s, z)}{\partial z_k} (x_k - z_k) + \frac{1}{2} \sum_{k,j} \frac{\partial^2 a_i(s, \theta(x, z))}{\partial z_k \partial z_j} (x_k - z_k)(x_j - z_j) \right) \frac{\partial}{\partial z_i} \omega_\varepsilon(x - z) dz +
\]
\[ + \int_{K_\epsilon(x)} \frac{1}{2} \sum_{i,j} \left( \sigma_{ij}(s,z) + \sum_k \frac{\partial \sigma_{ij}(s,z)}{\partial z_k} (x_k - z_k) \right) + \]
\[ + \frac{1}{2} \sum_{k,m} \frac{\partial^2 \sigma_{ij}(s,z)}{\partial z_k \partial z_m} (x_k - z_k)(x_m - z_m) + \]
\[ + \frac{1}{6} \sum_{k,m,l} \frac{\partial^3 \sigma_{ij}(s,\theta_1(x,z))}{\partial z_k \partial z_m \partial z_l} (x_k - z_k)(x_m - z_m)(x_l - z_l) \frac{\partial^2 \omega_\epsilon(x-z)}{\partial z_i \partial z_j} dz = \]
\[ = - \int_{K_\epsilon(x)} \sum_i \left\{ \frac{\partial}{\partial z_i} \left( a_i(s,z) \omega_\epsilon(x-z) \right) - \omega_\epsilon(x-z) \frac{\partial a_i(s,z)}{\partial z_i} \right\} \]
\[ + \sum_k \frac{\partial}{\partial z_i} \left( (x_k - z_k) \omega_\epsilon(x-z) \frac{\partial a_i(s,\theta(x,z))}{\partial z_k} \right) + \omega_\epsilon(x-z) \frac{a_i(s,z)}{\partial z_i} - \]
\[ - \sum_k (x_k - z_k) \omega_\epsilon(x-z) \frac{\partial^2 a_i(s,z)}{\partial z_i \partial z_k} + \]
\[ + \frac{1}{2} \sum_{k,j} \frac{\partial^2 a_i(s,\theta(x,z))}{\partial z_k \partial z_j} (x_k - z_k)(x_j - z_j) \frac{\partial \omega_\epsilon(x-z)}{\partial z_i} \} dz + \]
\[ + \frac{1}{2} \int_{K_\epsilon(x)} \sum_{i,j} \left\{ \frac{\partial}{\partial z_i} \left( \sigma_{ij}(s,z) \frac{\partial \omega_\epsilon}{\partial z_j} \right) - \right\} \]
\[ - \frac{\partial \sigma_{ij}(s,z)}{\partial z_i} \frac{\partial \omega_\epsilon(x-z)}{\partial z_j} + \sum_k \frac{\partial}{\partial z_i} \left( (x_k - z_k) \frac{\partial \sigma_{ij}(s,z)}{\partial z_k} \frac{\partial \omega_\epsilon(x-z)}{\partial z_j} \right) + \]
\[ + \frac{\partial \sigma_{ij}(s,z)}{\partial z_i} \frac{\partial \omega_\epsilon(x-z)}{\partial z_j} - \sum_k (x_k - z_k) \frac{\partial^2 \sigma_{ij}(s,z)}{\partial z_i \partial z_k} \frac{\partial \omega_\epsilon(x-z)}{\partial z_j} + \]
\[ + \frac{1}{2} \sum_{k,m} \frac{\partial}{\partial z_i} \left( (x_k - z_k)(x_m - z_m) \frac{\partial^2 \sigma_{ij}(s, z)}{\partial z_k \partial z_m} \frac{\partial \omega_\epsilon(x - z)}{\partial z_j} \right) + \]
\[ + \frac{1}{2} \sum_m (x_m - z_m) \frac{\partial^2 \sigma_{ij}(s, z)}{\partial z_i \partial z_m} \frac{\partial \omega_\epsilon(x - z)}{\partial z_j} + \]
\[ + \frac{1}{2} \sum_k (x_k - z_k) \frac{\partial^2 \sigma_{ij}(s, z)}{\partial z_k \partial z_i} \frac{\partial \omega_\epsilon(x - z)}{\partial z_j} - \]
\[ - \frac{1}{2} \sum_{k,m} (x_k - z_k)(x_m - z_m) \frac{\partial^3 \sigma_{ij}(s, z)}{\partial z_i \partial z_k \partial z_m} \frac{\partial \omega_\epsilon(x - z)}{\partial z_j} + \]
\[ + \sum_{k,m,l} \frac{\partial^3 \sigma_{ij}(s, \theta_1(x, z))}{\partial x_k \partial x_m \partial x_l} (x_k - z_k)(x_m - z_m)(x_l - z_l) \frac{\partial^2 \omega_\epsilon(x - z)}{\partial z_i \partial z_j} \right \} dz. \quad (7) \]

Here \( \theta(x, z) = z + \theta(x - z), \ \theta_1(x, z) = z + \theta_1(x - z); \ 0 \leq \theta_1, \theta \leq 1. \)

Applying Ostrogradskii - Gauss theorem we transform some integrals in right part of (7) to integrals on the surface \( \triangle_\epsilon(x) \). Further, we obtain the estimate of smallness for some surface integrals and the volume integrals.

We set

\[ f_\epsilon(i, x, z) = \frac{2\epsilon^2(x_i - z_i)}{(\epsilon^2 - |x - z|^2)^2}, \quad \text{now} \quad \frac{\partial \omega_\epsilon(x - z)}{\partial z_i} = f_\epsilon(i, x, z) \omega_\epsilon(x - z). \]

So

\[ \int_{K_\epsilon(x)} \sum_i \frac{\partial}{\partial z_i} \left( a_i(s, z) \omega_\epsilon(x - z) \right) dz = \int_{\Delta_\epsilon(x)} \sum_i a_i(s, z) \nu_i(z) \omega_\epsilon(x - z) d\beta_z. \quad (8) \]

We make use of Cuachy- unyakovskii inequality in (9) and later on.

\[ | \int_{K_\epsilon(x)} \sum_i \frac{\partial}{\partial z_i} \left( \sum_k (x_k - z_k) \omega_\epsilon(x - z) \frac{\partial a_i(s, z)}{\partial z_k} \right) | = \]
\[
\| \int_{\triangle_{\epsilon}(x)} \sum_{i} \nu_i(z) \sum_{k} (x_k - z_k) \omega_\epsilon(x - z) \frac{\partial a_i(s, z)}{\partial z_k} d\beta_z \| \leq \\
\leq \int_{\triangle_{\epsilon}(x)} \sqrt{\sum_{k} (x_k - z_k)^2} \sqrt{\sum_{k} \left( \sum_{i} \nu_i(z) \frac{\partial a_i(s, z)}{\partial z_k} \right)^2} \omega_\epsilon(x - z) d\beta_z \leq \epsilon c_1,
\]
\[c_1 < \infty. \quad (9)\]

\[
\| \int_{K_{\epsilon}(x)} \sum_{i} \sum_{k} (x_k - z_k) \omega_\epsilon(x - z) \frac{\partial^2 a_i(s, z)}{\partial z_i \partial z_k} dz \| \leq \\
\leq \int_{K_{\epsilon}(x)} \sqrt{\sum_{k} (x_k - z_k)^2} \sqrt{\sum_{k} \left( \sum_{i} \frac{\partial^2 a_i(s, z)}{\partial z_i \partial z_k} \right)^2} \omega_\epsilon(x - z) dz \leq \epsilon c_2,
\]
\[c_2 < \infty. \quad (10)\]

\[
\| \int_{K_{\epsilon}(x)} \sum_{k,j} \frac{\partial^2 a_i(\theta(x, z))}{\partial z_k \partial z_j} (x_k - z_k)(x_j - z_j) \frac{\partial \omega_\epsilon(x - z)}{\partial z_i} dz \| \leq \\
\leq \int_{K_{\epsilon}(x)} \sqrt{\sum_{j,k,l} \left( \frac{\partial^2 a_i(s, \theta(x, z))}{\partial z_k \partial z_j} \right)^2} \sum_{k} (x_k - z_k)^2 \times \\
\times \sqrt{\sum_{i} f_\epsilon^2(i, x, z) \omega_\epsilon(x - z) dz} \leq \epsilon c_3, \quad c_3 < \infty. \quad (11)\]

We exploit the condition (2) for estimate in (12)

\[
\left| \int_{K_{\epsilon}(x)} \sum_{i,j} \frac{\partial}{\partial z_i} \left( \sigma_{ij}(s, z) \frac{\partial \omega_\epsilon}{\partial z_j} \right) dz \right| = \frac{1}{2} \left| \int_{\triangle_{\epsilon}(x)} \sum_{i} \nu_i(z) \sigma_{ij}(s, z) \frac{\partial \omega_\epsilon}{\partial z_j} d\beta_z \right| =
\]
\begin{align*}
&= \frac{1}{2} \left| \int_{\Delta_\epsilon(x)} \sum_k \sum_i b_{ki}(s, z) \nu_i(z) \sum_j b_{kj}(s, z) \frac{\partial \omega_\epsilon}{\partial z_j} d\beta_z \right| \leq \\
&\leq \frac{1}{2} \left| \int_{\Delta_\epsilon(x)} \left| \sum_i b_{ki}(s, z) \nu_i(z) \right| \sqrt{\sum_j b_{kj}^2(s, z)} \sqrt{\sum_j f_j(\epsilon, x, z)} \times \\
&\times \omega_\epsilon(x - z) d\beta_z = o(1). \quad (12)
\end{align*}

We exploit the relation \( \sigma_{ij}(s, z) = \sigma_{ji}(s, z) \) for estimate in (13)

\begin{align*}
&= \frac{1}{2} \left| \int_{K_\epsilon(x)} \sum_{i,j} (\sum_k (x_k - z_k) \frac{\partial^2 \sigma_{ij}(s, z)}{\partial z_i \partial z_k} \frac{\partial \omega_\epsilon(x - z)}{\partial z_j} + \\
&+ \frac{1}{2} \sum_m (x_m - z_m) \frac{\partial^2 \sigma_{ij}(s, z)}{\partial z_i \partial z_m} \frac{\partial \omega_\epsilon(x - z)}{\partial z_j} + \\
&+ \frac{1}{2} \sum_k (x_k - z_k) \frac{\partial^2 \sigma_{ij}(s, z)}{\partial z_k \partial z_i} \frac{\partial \omega_\epsilon(x - z)}{\partial z_j}) dz \right| = \\
&= \frac{1}{2} \left| \int_{\Delta_\epsilon(x)} \sum_{i,j} \nu_j(z) (\sum_k (x_k - z_k) \frac{\partial^2 \sigma_{ij}(s, z)}{\partial z_i \partial z_k} + \frac{1}{2} \sum_m (x_m - z_m) \frac{\partial^2 \sigma_{ij}(s, z)}{\partial z_i \partial z_m} + \\
&+ \frac{1}{2} \sum_k (x_k - z_k) \frac{\partial^2 \sigma_{ij}(s, z)}{\partial z_k \partial z_i} \omega_\epsilon(x - z) d\beta_z + \\
&+ \frac{1}{2} \int_{K_\epsilon(x)} \sum_{i,j} (\frac{\partial^2 \sigma_{ij}(s, z)}{\partial z_i \partial z_j} - \frac{1}{2} \frac{\partial^2 \sigma_{ij}(s, z)}{\partial z_i \partial z_j} - \frac{1}{2} \frac{\partial^2 \sigma_{ij}(s, z)}{\partial z_j \partial z_i}) \omega_\epsilon(x - z) dz + \\
&+ \frac{1}{2} \int_{K_\epsilon(x)} \sum_{i,k,j} (x_k - z_k) \left( \frac{\partial^3 \sigma_{ij}(s, z)}{\partial z_j \partial z_i \partial z_k} + \frac{1}{2} \frac{\partial^3 \sigma_{ij}(s, z)}{\partial z_j \partial z_i \partial z_k} + \frac{1}{2} \frac{\partial^3 \sigma_{ij}(s, z)}{\partial z_j \partial z_k \partial z_i} \right) \omega_\epsilon(x - z) d\beta_z \leq \right|
\end{align*}
\[ \begin{align*}
&\leq \int_{\Delta_\epsilon(x)} \sqrt{\sum_j \nu_j^2(z)} \sqrt{\sum_k (x_k - z_k)^2} \sqrt{\sum_{i,j,k} \left( \frac{\partial^2 \sigma_{ij}(s,z)}{\partial z_i \partial z_k} \right)^2} \omega_\epsilon(x - z) dz + \\
&\int_{K_\epsilon(x)} \sqrt{\sum_k (x_k - z_k)^2} \sqrt{\sum_{i,j,k} \left( \frac{\partial^3 \sigma_{ij}(s,z)}{\partial z_j \partial z_i \partial z_k} \right)^2} \omega_\epsilon(x - z) dz \leq c_4 \epsilon, \quad c_4 < \infty.
\end{align*} \]

(13)

\[ \begin{align*}
&\frac{1}{4} \left| \int_{K_\epsilon(x)} \sum_i \frac{\partial}{\partial z_i} \left( \sum_j \sum_{k,m} (x_k - z_k)(x_m - z_m) \frac{\partial^2 \sigma_{ij}(s,z)}{\partial z_k \partial z_m} \frac{\partial \omega_\epsilon(x - z)}{\partial z_j} \right) dz \right| = \\
&= \frac{1}{4} \int_{\Delta_\epsilon(x)} \sum_i \nu_i(z) \sum_j \sum_{k,m} (x_k - z_k)(x_m - z_m) \frac{\partial^2 \sigma_{ij}(s,z)}{\partial z_k \partial z_m} \frac{\partial \omega_\epsilon(x - z)}{\partial z_j} f_\epsilon(j, x, z) \times \\
&\times \omega_\epsilon(x - z) d\beta_z \leq \frac{1}{4} \int_{\Delta_\epsilon(x)} \sum_k (x_k - z_k)^2 \sqrt{\sum_j f_\epsilon^2(j, x, z)} \sqrt{\sum_i \nu_i^2(z)} \times \\
&\times \sqrt{\sum_{i,j,k,m} \left( \frac{\partial^2 \sigma_{ij}(s,z)}{\partial z_k \partial z_m} \right)^2} d\beta_z \leq \epsilon c_5, \quad c_5 < \infty.
\end{align*} \]

(14)

\[ \begin{align*}
&\frac{1}{4} \left| \int_{K_\epsilon(x)} \sum_{k,m} (x_k - z_k)(x_m - z_m) \frac{\partial^3 \sigma_{ij}(s,z)}{\partial z_i \partial z_k \partial z_m} \frac{\partial \omega_\epsilon(x - z)}{\partial z_j} dz \right| \leq \\
&\leq \frac{1}{4} \int_{K_\epsilon(x)} \sum_k (x_k - z_k)^2 \sqrt{\sum_j f_\epsilon^2(j, x, z)} \times \\
&\sum_i \sqrt{\sum_{j,k,m} \left( \frac{\partial^3 \sigma_{ij}(s,z)}{\partial z_i \partial z_k \partial z_m} \right)^2} \omega_\epsilon(x - z) dz \leq \epsilon c_6, \quad c_6 < \infty.
\end{align*} \]

(15)
We exploit the following relation

\[ \frac{\partial^2 \omega_{\epsilon}(x - z)}{\partial z_i \partial z_j} = \]

\[ = \left( f_{\epsilon}(i, x, z) f_{\epsilon}(j, x, z) - \frac{2 \epsilon^2 \delta_{ij}}{(\epsilon^2 - |x - z|^2)^2} - \frac{8 \epsilon^2 (x_i - z_i)(x_j - z_j)}{(\epsilon^2 - |x - z|^2)^3} \right) \omega_{\epsilon}(x - z), \]

here \( \delta_{ij} \) be Kronecker’s symbol,

for estimate in (16).

\[ \frac{1}{12} \left| \int_{K_{\epsilon}(x)} \sum_{i,j,k,m,l} \frac{\partial^3 \sigma_{ij}(s, \theta_1(x, z))}{\partial z_k \partial z_l \partial z_i} (x_k - z_k)(x_m - z_m)(x_l - z_l) \frac{\partial^2 \omega_{\epsilon}(x - z)}{\partial z_i \partial z_j} dz \right| \leq \]

\[ \leq \frac{1}{12} \int_{K_{\epsilon}(x)} \left( \sum_m (x_m - z_m)^2 \right)^{\frac{3}{2}} \sqrt{\sum_{i,j,k,m,l} \left( \frac{\partial^3 \sigma_{ij}(s, \theta_1(x, z))}{\partial z_k \partial z_m \partial z_l} \right)^2} \times \]

\[ \times \left( \sum_i f_{\epsilon}^2(i, x, z) + \frac{2 \epsilon^2 \sqrt{n}}{(\epsilon^2 - |x - z|^2)^2} + \frac{8 \epsilon^2 \sum_i (x_i - z_i)^2}{(\epsilon^2 - |x - z|^2)^3} \right) \omega_{\epsilon}(x - z) dz \leq \epsilon c_6, \]

\[ c_6 < \infty. \]  

(16)

Consider remaining summand in (7).

\[ \frac{1}{2} \int_{K_{\epsilon}(x)} \sum_k \frac{\partial}{\partial z_i} \left( (x_k - z_k) \frac{\partial \sigma_{ij}(s, z)}{\partial z_k} \frac{\partial \omega_{\epsilon}(x - z)}{\partial z_j} \right) = \]

\[ = \frac{1}{2} \int_{\Delta_{\epsilon}(x)} \sum_k \sum_j \sum_i \frac{\partial b_{pj}(z)}{\partial z_k} \nu_i(z) b_{pi}(s, z)(x_k - z_k) \frac{\partial}{\partial z_j} \omega_{\epsilon}(x - z) d\beta_z + \]
\[ + \frac{1}{2} \int_{\triangle_\epsilon(x)} \sum_k \sum_p \sum_j b_{pj}(s,z) \sum_i \nu_i(z) \frac{\partial b_{pi}(s,z)}{\partial z_k} (x_k - z_k) \frac{\partial}{\partial z_j} \omega_\epsilon(x - z) d\beta_z =: \]

\[ =: I_1 + I_2. \]

The first summand \( I_1 \) is estimated analogy to (11) with help condition 2 of theorem. Thus we have \( I_1 = o(1) \).

We observe that by construction the points of boundary \( \partial \triangle_\epsilon(x) \) of set \( \triangle_\epsilon(x) \) have the following properties:

\[ z \in \partial \triangle_\epsilon(x) \Rightarrow |x - z| = \epsilon \Rightarrow \omega_\epsilon(x - z) = 0 \]

Now we make use of local property of surface \( \partial K_\epsilon \) for more precise representation of summand \( I_2 \).

The variables \( z_i, i = 1, n \) in \( \triangle_\epsilon(x) \) have form \( z_i = y_i, i \leq n - 1, z_n = \varphi(y_1, \ldots, y_{n-1}) \).

Put \( \hat{y} = (y_1, \ldots, y_n) \), here \( y_n = \varphi(y_1, \ldots, y_{n-1}) \).

The domain \( \bar{D}_\epsilon(x) \) which corresponds to \( \triangle_\epsilon(x) \) has the following form

\[ \bar{D}_\epsilon(x) = \{(y_1, \ldots, y_{n-1}) : |x - \hat{y}| \leq \epsilon\}. \]  

(17)

The boundary \( \partial \bar{D}_\epsilon(x) \) is set of points for which in (17) the next equality is fulfilled. Let \( y' = (y_1, \ldots, y_{n-1}) \). Thus if \( y' \in \partial \bar{D}_\epsilon(x) \), then \( \omega_\epsilon(x - \hat{y}) = 0 \).

Set \( \omega_\epsilon(x - \hat{y}) = 0 \), under \( y' \notin \bar{D}_\epsilon(x) \).

Thus the function \( \omega_\epsilon(x - \hat{y}) \) is finite function in space \( R^{n-1} \) with support \( \bar{D}_\epsilon(x) \). The following formula of integration by parts is true for such functions [8, p.106].

\[ f \in C^1 \Rightarrow \int_{\bar{D}_\epsilon(x)} f \frac{\partial}{\partial y_i} \omega_\epsilon(x - \hat{y}) dy' = - \int_{\bar{D}_\epsilon(x)} \omega_\epsilon(x - \hat{y}) \frac{\partial}{\partial y_i} f dy' \quad i = 1, n - 1. \]

(18)

Applying (18) to integration in \( I_2 \), we get
2I_2 =

\[- \int_{\overline{D_\epsilon(x)}} \sum_j \sum_k \frac{\partial}{\partial y_j} \left( \sum_{p,i} b_{pj}(s, \hat{y}) \nu_i(\hat{y}) \frac{\partial}{\partial y_k} b_{pi}(s, \hat{y}) \right) (x_k - y_k) \omega_\epsilon(x - \hat{y}) dy' +

+ \int_{\overline{D_\epsilon(x)}} \sum_j \sum_k \sum_{p,i} b_{pj}(s, \hat{y}) \nu_i(\hat{y}) \frac{\partial}{\partial y_k} b_{pi}(s, \hat{y}) \frac{\partial}{\partial y_j} y_k \omega_\epsilon(x - \hat{y}) dy' =:

=: I_{21} + I_{22}.

To estimate of summand \(I_{21}\) with help Cauchy - Bunyakovskii’s inequality we will use the condition 1 and the supposition that surface belongs to class \(C^2\).

Later on it is convenient to omit the argument of functions.

\[|I_{21}| \leq \int_{\overline{D_\epsilon(x)}} \sum_j \sum_k \sum_{p,i} \left| \sum_k \frac{\partial}{\partial y_j} \left( b_{pj} \nu_i \frac{\partial}{\partial y_k} b_{pi} \right) (x_k - y_k) \right| \omega_\epsilon(x - y) dy' \leq \]

\[\leq \int_{\overline{D_\epsilon(x)}} \left( \sum_k (x_k - y_k)^2 \right)^{\frac{1}{2}} \sum_j \sum_{p,i} \left( \sum_k \left\{ \frac{\partial b_{pj}}{\partial y_j} \nu_i \frac{\partial b_{pi}}{\partial y_k} \right\}^2 + (b_{pj} \nu_i \frac{\partial^2 b_{pi}}{\partial y_j \partial y_k})^2 \right)^{\frac{1}{2}} \omega_\epsilon(x - \hat{y}) dy' \leq \epsilon c_{21}.

Here \(c_{21}\) is bounded constant.

Combining (8)-(16) and latter one gives the following representation

\[A \eta_\epsilon(x) = - \int_{\overline{D_\epsilon(x)}} \left\{ \sum_i a_i(s, \hat{y}) \nu_i(\hat{y}) - \right\} \]
\[-\frac{1}{2} \sum_{j,k,p,i} b_{jp}(s, \hat{y}) \nu_i(\hat{y}) \frac{\partial b_{pi}(s, \hat{y})}{\partial y_k} \frac{\partial}{\partial y_j} y_k \\right\} \omega_\varepsilon(x - \hat{y}) dy' + o(1). \tag{19}\]

Let \(G(s, y)\) denote the second summ in braces of right part of latter equality. It takes place the following relation for new variables

\[\frac{\partial}{\partial y_n} = \sum_{i=1}^{n-1} \varphi_{y_i} \frac{\partial}{\partial y_i}. \tag{20}\]

It is not hard to calculate the following equalities for partial derivatives in summands from \(G(s, y)\)

\[\frac{\partial b_{pi}}{\partial y_k} \frac{\partial}{\partial y_j} y_k = \begin{cases} \frac{\partial}{\partial y_k} b_{pi}, & \text{if } j = k < n \\ 0, & \text{if } j \neq k, j < n, k < n \\ \frac{\partial}{\partial y_n} b_{pn} \varphi_{x_j}, & \text{if } j < n, k = n \\ \frac{\partial b_{pi}}{\partial y_k} \varphi_{x_k}, & \text{if } k < n, j = n \\ \frac{\partial}{\partial y_n} b_{pi} \sum_{m=1}^{n-1} \varphi_{y_m}^2, & \text{if } j = k = n. \end{cases}\]

Now we will show that the function \(G(s, y)\) coincides with the following function from condition 3 of theorem completely

\[\sum_{i,j,k} \frac{\partial b_{ki}(s, z)}{\partial z_j} \nu_i(z) b_{kj}(s, z), \quad s \geq 0; \tag{21}\]

in case when differentiation is fulfilled in coordinates \(\hat{y}\).

Applying (20), we get the following equalities for differentiation in (21)

\[\frac{\partial b_{pi}}{\partial z_j} = \begin{cases} \frac{\partial}{\partial y_j} b_{pi} + \frac{\partial b_{pi}}{\partial z_n} \varphi_{x_j}, & \text{if } j < n \\ \sum_{k}^{n-1} \frac{\partial b_{pi}}{\partial y_k} \varphi_{x_k} + \frac{\partial}{\partial z_n} \sum_{l=1}^{n-1} \varphi_{x_l}^2, & \text{if } j = n. \end{cases}\]

It is clear that latter one defines the summands in (21) which is identical to the summands in the \(G(s, y)\).
Thus it follows from representation (19) that under conditions theorem there exists such $\epsilon^* > 0$ that the inequality $A_{\eta}(x) \geq 0$ is fulfilled for all $\epsilon \leq \epsilon^*$. Theorem is prooved.

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