Nilmanifolds with a calibrated $G_2$-structure

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Abstract

We introduce obstructions to the existence of a calibrated $G_2$-structure on a Lie algebra $g$ of dimension seven, not necessarily nilpotent. In particular, we prove that if there is a Lie algebra epimorphism from $g$ to a six-dimensional Lie algebra $h$ with kernel contained in the center of $g$, then $h$ has a symplectic form. As a consequence, we obtain a classification of the nilpotent Lie algebras that admit a calibrated $G_2$-structure.

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1 Introduction

A Riemannian 7-manifold with holonomy contained in $G_2$ can be characterized by the existence of an associated parallel 3-form. The first examples of complete metrics with holonomy $G_2$ were given by Bryant and Salamon in [4], and the first examples of compact manifolds with such a metric were constructed by Joyce in [16]. Explicit examples on solvable Lie groups were constructed in [3]; examples on nilpotent Lie groups can be obtained by taking a nilpotent six-dimensional group with a half-flat structure and solving Hitchin’s evolution equations (see [9, 15]). More generally, one can consider $G_2$-structures where the associated 3-form $\varphi$ is closed: then $\varphi$ defines a calibration ([14]), and the $G_2$-structure is said to be calibrated. An equivalent condition is that the intrinsic torsion lies in the component $\mathcal{X}_2 \cong g_2$ ([10]).

Compact calibrated $G_2$ manifolds have interesting curvature properties. It is well known that a $G_2$ holonomy manifold is Ricci-flat, or equivalently, both Einstein and scalar-flat. On a compact calibrated $G_2$ manifold, both the Einstein condition ([6]) and scalar-flatness ([8]) are equivalent to the holonomy being contained in $G_2$. In fact, [4] shows that the scalar curvature is always non-positive.

Constructing examples is not a straightforward task. For instance, [7] classifies calibrated $G_2$-manifolds on which a simple groups acts with cohomogeneity one, and no compact manifold occurs in this list. On the other hand, the second author exhibited the first example of a compact calibrated $G_2$-manifold that
does not have holonomy $G_2$ \[11\]. This example is given in terms of a nilpotent Lie algebra $\mathfrak{g}$ and an element of $\Lambda^3 \mathfrak{g}^*$ that corresponds to a closed left-invariant 3-form on the associated simply-connected Lie group. Since the structure constants are rational, there exists a uniform discrete subgroup \[18\]; the quotient, called a nilmanifold, has an induced calibrated $G_2$-structure.

In this paper we pursue this approach, and classify the nilpotent 7-dimensional Lie algebras that admit a calibrated $G_2$-structure. Since the structure constants turn out to be rational, each Lie algebra determines a nilmanifold. So, we obtain 12 compact calibrated $G_2$ nilmanifolds (see Theorem \[4\] Lemma \[5\] and Lemma \[6\]). Three of them are reducible: they are the product of a circle with a symplectic half-flat nilmanifold, the latter being classified in \[8\]. The remaining nine are new.

The proof is based on two necessary conditions that a Lie algebra must satisfy for a calibrated $G_2$-structure to exist (see Proposition \[1\] and Lemma \[3\]).

Our first obstruction is related to a construction of \[11\]. Suppose that $M$ is a 7-manifold with a calibrated $G_2$ form $\varphi$, and $X$ is a unit Killing field, i.e. $\mathcal{L}_X \varphi = 0$. Then if $\eta = X^\flat$, we can write

$$\varphi = \omega \wedge \eta + \psi^+,$$

where $\omega$, $\psi^+$ and $d\eta$ are basic forms with respect to the 1-dimensional foliation defined by $X$. Suppose in addition that $X$ is the fundamental vector field of a free $S^1$ action; then basic forms can be identified with forms on $M/S^1$. By

$$0 = \mathcal{L}_X \varphi = d(X \lrcorner \varphi) = d\omega,$$

$\omega$ is a symplectic form on $M/S^1$. Moreover

$$0 = \omega \wedge d\eta + d\psi^+$$

implies that $[d\eta]$ is in the kernel of the map

$$H^2(M/S^1) \to H^4(M/S^1), \quad [\beta] \to [\beta \wedge \omega].$$

If this map is an isomorphism, then the $S^1$-bundle is trivial: this puts topological restrictions on $M$, which translate to algebraic conditions in our setup. A similar method was used in \[8\].

In principle, these restrictions reduce our problem to the classification of symplectic nilpotent Lie algebras of dimension six for which the map $H^2 \to H^4$ is non-injective (see the remark before Lemma \[3\]). The complexity of the required calculations, however, motivate a different approach. In analogy with \[9\], we introduce a second obstruction, that requires computing the space of closed 3-forms. It consists in the observation that $(X \lrcorner \varphi)^3$ must be nonzero, whenever $X$ is a nonzero vector and $\varphi$ a 3-form defining a $G_2$-structure.

The final ingredient is Gong's classification of 7-dimensional indecomposable nilpotent Lie algebras \(12\). This list contains 140 Lie algebras and 9 one-parameter families; in addition, there are 35 decomposable nilpotent Lie algebras \(17, 19\). Calculations on a case-by-case basis show that our list of 12 examples is complete.
2 Calibrated $G_2$-structures and obstructions

In this section we show obstructions to the existence of a calibrated $G_2$ form on a Lie algebra (not necessarily nilpotent). First, we recall some definitions and results about $G_2$-structures.

Let us consider the space $O$ of the Cayley numbers, which is a non-associative algebra over $\mathbb{R}$ of dimension 8. Thus, we can identify $\mathbb{R}^7$ with the subspace of $O$ consisting of pure imaginary Cayley numbers. Then, the product on $O$ defines on $\mathbb{R}^7$ the 3-form given by

$$e^{127} + e^{347} + e^{567} + e^{135} - e^{236} - e^{146} - e^{245}$$

(1)

(see [10] and [13] for details), where $\{e^1, \ldots, e^7\}$ is the standard basis of $(\mathbb{R}^7)^*$. Here, $e^{127}$ stands for $e^1 \wedge e^2 \wedge e^7$, and so on. The group $G_2$ is the stabilizer of (1) under the standard action of $GL(7, \mathbb{R})$ on $\Lambda^3(\mathbb{R}^7)^*$. $G_2$ is one of the exceptional Lie groups, and it is a compact, connected, simply connected simple Lie subgroup of $SO(7)$ of dimension 14.

A $G_2$-structure on a 7-dimensional Riemannian manifold $(M, g)$ is a reduction of the structure group $O(7)$ of the frame bundle to $G_2$. Manifolds admitting a $G_2$-structure are called $G_2$ manifolds. The existence of a $G_2$-structure on $(M, g)$ is determined by a global 3-form $\varphi$ (the $G_2$ form) which can be locally written as (1) with respect to some (local) basis $\{e^1, \ldots, e^7\}$ of the (local) 1-forms on $M$. We say that the $G_2$ manifold $M$ has a calibrated $G_2$-structure if $\varphi$ is closed, i.e. $d\varphi = 0$, where $d$ denotes the Chevalley-Eilenberg differential on $\mathfrak{g}^*$. If $\Gamma$ is a discrete subgroup of $G$, a $G_2$-structure on $\mathfrak{g}$ induces a $G_2$-structure on the quotient $\Gamma \backslash G$. Moreover, in [18] it is proved that if $\mathfrak{g}$ is nilpotent with rational structure constants, then the associated simply connected Lie group $G$ admits a uniform discrete subgroup $\Gamma$. Therefore, a $G_2$-structure on $\mathfrak{g}$ determines a $G_2$-structure on the compact manifold $\Gamma \backslash G$, which is called a compact nilmanifold; and if $\mathfrak{g}$ has a calibrated $G_2$-structure, the $G_2$-structure on $\Gamma \backslash G$ is also calibrated.

In order to show obstructions to the existence of a calibrated $G_2$ form on a Lie algebra $\mathfrak{g}$, let us consider first a direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$. If $\varphi$ is a $G_2$ form on $\mathfrak{g}$, and the decomposition is orthogonal with respect to the underlying metric, then

$$\varphi = \omega \wedge \eta + \psi^+,$$
where $\omega, \psi^+$ are forms on $\mathfrak{h}$ and $\eta$ generates the dual of the ideal $\mathbb{R}$. The pair $(\omega, \psi^+)$ defines an SU(3)-structure on $\mathfrak{h}$. The condition that $\varphi$ is closed is equivalent to both $\omega$ and $\psi^+$ being closed; this means that the SU(3)-structure is symplectic half-flat. There are exactly three nilpotent Lie algebras of dimension six that admit a symplectic half-flat structure, classified in [8]. So, if we focus our attention on decomposable nilpotent Lie algebras, there are at least three 7-dimensional Lie algebras with a calibrated G$_2$-structure; we will see that these are all.

More generally, every 7-dimensional nilpotent Lie algebra fibres over a nilpotent Lie algebra of dimension six. In fact if $\xi$ is in the center of $\mathfrak{g}$, then the quotient $\mathfrak{g}/\text{Span}\{\xi\}$ has a unique Lie algebra structure that makes the projection map $\mathfrak{g} \rightarrow \mathfrak{g}/\text{Span}\{\xi\}$ a Lie algebra morphism. Moreover, due to the nilpotency assumption every epimorphism $\mathfrak{g} \rightarrow \mathfrak{h}$, with $\mathfrak{h}$ of dimension six, is of this form. Using the pullback, we can identify forms on the quotient with basic forms on $\mathfrak{g}$; in this setting, $\alpha$ is basic if $\xi \lrcorner \alpha = 0$.

Given a G$_2$-structure on $\mathfrak{g}$ with associated 3-form $\varphi$ and a nonzero vector $\xi$ in the center, let $\eta = \xi \flat$; then we can write

$$\varphi = \omega \wedge \eta + \psi^+, \quad \xi \lrcorner \omega = 0 = \xi \lrcorner \psi^+, \quad \xi \lrcorner \eta = 0,$$

and up to a normalization coefficient the forms $(\omega, \psi^+)$ define an SU(3)-structure on the six-dimensional quotient (see also [1]). In analogy with the case of a circle bundle, we shall think of $\eta$ as a connection form, and $d\eta$ as the curvature.

**Proposition 1.** Let $\mathfrak{g}$ be a 7-dimensional Lie algebra with a calibrated G$_2$-structure and a non-trivial center. If $\pi: \mathfrak{g} \rightarrow \mathfrak{h}$ is a Lie algebra epimorphism with kernel contained in the center, and $\mathfrak{h}$ of dimension six, then $\mathfrak{h}$ admits a symplectic form $\omega$, and the curvature form is in the kernel of

$$H^2(\mathfrak{h}^*) \xrightarrow{\wedge \omega} H^4(\mathfrak{h}^*). \quad (2)$$

If the curvature form is exact on $\mathfrak{h}$, then $\mathfrak{g} \cong \mathfrak{h} \oplus \mathbb{R}$ as Lie algebras.

**Proof.** Write

$$\varphi = \pi^* \omega \wedge \eta + \pi^* \psi^+$$

where $(\omega, \psi^+)$ are forms on $\mathfrak{h}$. Since $d$ commutes with the pullback,

$$0 = d\varphi = d\pi^* \omega \wedge \eta + \pi^* \omega \wedge d\eta + \pi^* d\psi^+, \quad \pi^* d\omega, \; d\eta \text{ and } \pi^* d\psi^+ \text{ are basic.}$$

Thus $\omega$ is a symplectic form and $d\eta$ is in the kernel of (2).

Now suppose that $d\eta$ is exact on $\mathfrak{h}$. Then, the epimorphism $\pi: \mathfrak{g} \rightarrow \mathfrak{h}$ is trivial, that is $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$. More precisely, we can choose a different, closed connection form $\tilde{\eta}$, and $\mathfrak{g} = \ker \tilde{\eta} \oplus \ker \pi$ is a direct sum of Lie algebras; by construction, $\ker \tilde{\eta}$ is isomorphic to $\mathfrak{h}$.

\qed
Remark. In the previous Proposition, we must notice that when the curvature form is zero, \((\omega, \psi^+\)) is a symplectic half-flat structure on \(\mathfrak{h}\). Therefore, if \(\mathfrak{h}\) is nilpotent, by [8], \(\mathfrak{h}\) is one of
\[(0, 0, 0, 0, 0, 0), \quad (0, 0, 0, 0, 12, 13), \quad (0, 0, 0, 12, 13, 23).
\]
With notation from [19], \((0, 0, 0, 0, 12, 13)\) represents a Lie algebra with a fixed basis \(e^1, \ldots, e^6\) of \(\mathfrak{g}^*\), satisfying
\[de^1 = 0 = de^3 + de^4, \quad de^5 = e^{12}, \quad de^6 = e^{13}.
\]

Remark. Another obstruction to the existence of a calibrated \(G_2\)-structure on a nilpotent Lie algebra is given by the condition \(b_3 > 0\). Indeed, if \(\varphi\) is a closed \(G_2\) form on a nilpotent Lie algebra \(\mathfrak{g}\), and \(X\) is a nonzero vector in the center of \(\mathfrak{g}\), then \(\mathcal{L}_X \varphi = 0\), so \(X \cdot \varphi\) is closed. If \(\varphi\) were exact, say \(\varphi = d\beta\), then the 7-form
\[(X \cdot \varphi) \wedge (X \cdot \varphi) \wedge \varphi = d((X \cdot \varphi) \wedge (X \cdot \varphi) \wedge \beta)
\]
would also be exact, hence zero, which is absurd. On the other hand, \(b_3\) is always positive on a nilpotent Lie algebra of dimension seven.

Proposition 1 motivates the following definition. We say that a 6-dimensional Lie algebra \(\mathfrak{h}\) satisfies the 2-Lefschetz property if, for every symplectic structure on \(\mathfrak{h}\), the map (2) is an isomorphism. This condition holds trivially when \(\mathfrak{h}\) has no symplectic structure, namely when \(\mathfrak{h}\) is one of
\[(0, 0, 0, 12, 23, 14 + 35), \quad (0, 0, 0, 12, 23, 14 - 35), \quad (0, 0, 0, 12, 15 + 34),
\]
\[(0, 0, 0, 0, 12 + 34), \quad (0, 0, 12, 13, 14 + 23, 34 + 52), \quad (0, 0, 12, 13, 14, 34 + 52), \quad (0, 0, 0, 12, 14, 24).
\]

It is well known [2] that if \((\mathfrak{h}, \omega)\) is a 6-dimensional, symplectic nilpotent Lie algebra, the map
\[H^1(\mathfrak{h}^*) \overset{\wedge \omega^2}{\longrightarrow} H^5(\mathfrak{h}^*).
\]
is not surjective. However, in the next proposition, we prove that some of those Lie algebras satisfy the 2-Lefschetz property.

**Proposition 2.** Among 6-dimensional nilpotent Lie algebras with a symplectic structure, those that satisfy the 2-Lefschetz property are
\[(0, 0, 0, 0, 0, 0); \quad (0, 0, 12, 13, 23, 14); \quad (0, 0, 12, 13, 23, 14 + 25).
\]

**Proof.** In the abelian case, the bilinear map
\[H^2 \otimes H^2 \rightarrow H^4
\]
induced by the wedge product is non-degenerate, in the sense that for every nonzero \(\beta \in H^2\), the induced linear map \(\cdot \wedge \beta: H^2 \rightarrow H^4\) is an isomorphism.
For the second Lie algebra, the cohomology class of a generic symplectic form is represented by
\[ \omega = \lambda_1 e^{16} + \lambda_2 (e^{15} + e^{24}) + \lambda_3 e^{25} + \lambda_4 (e^{34} - e^{26}); \]
non-degeneracy implies \( \lambda_4 \neq 0 \). The map \( H^2 \to H^4 \) of (2) is represented by the matrix
\[
\begin{pmatrix}
\lambda_3 & 2\lambda_4 & \lambda_1 & 2\lambda_2 \\
\lambda_4 & 0 & 0 & \lambda_1 \\
0 & 0 & 0 & -2\lambda_4 \\
0 & 0 & \lambda_4 & \lambda_3
\end{pmatrix}
\]
which is invertible by the assumption \( \lambda_4 \neq 0 \).

Similarly, for the last Lie algebra
\[ \omega = \lambda_1 e^{14} + \lambda_2 (e^{15} + e^{24}) - \lambda_3 (e^{26} - e^{34}) + \lambda_4 (e^{16} + e^{35}). \]
The map (2) is represented by
\[
\begin{pmatrix}
\lambda_3 & 2\lambda_4 & \lambda_1 & 2\lambda_2 \\
-\lambda_4 & 2\lambda_3 & 2\lambda_2 & -\lambda_1 \\
0 & 0 & -2\lambda_3 & 2\lambda_4 \\
0 & 0 & -\lambda_4 & -\lambda_3
\end{pmatrix}
\]
which is invertible unless \( \lambda_2^2 + \lambda_3^2 = 0 \), which makes \( \omega \) degenerate.

For all but three of the remaining Lie algebras, we observe that the bilinear map
\[ H^2 \otimes H^2 \to H^4 \]
is degenerate in the sense that, for every nonzero \( \beta \in H^2 \), the map
\[ \alpha \to \alpha \wedge \beta, \quad H^2 \to H^4 \]
is non-injective. The three exceptions are
\( (0, 0, 12, 13, 23, 14 - 25), \quad (0, 0, 0, 12, 13, 23), \quad (0, 0, 0, 0, 0, 12). \)

However, either Lie algebra has a symplectic form that makes the map (2) non-injective. In fact, on the Lie algebra \( \mathfrak{h} \) defined by the equations \( (0, 0, 12, 13, 23, 14 - 25) \), consider the symplectic form
\[ \omega = -e^{16} + e^{15} + e^{35} + e^{34} + e^{24} - e^{26}. \]
Then one can check that \( e^{14} + e^{25} + e^{15} + e^{24} \) defines a non-trivial class in \( H^2(\mathfrak{h}^*) \), but
\[ (e^{14} + e^{25} + e^{15} + e^{24}) \wedge \omega = 2e^{1245} = 2d(e^{146}). \]
Now, on the Lie algebra \( (0, 0, 0, 12, 13, 23) \) we consider the symplectic form \( \omega = e^{14} + e^{26} + e^{35} \). Then,
\[ (-e^{15} - e^{24} + e^{36}) \wedge \omega = d(e^{356}); \]
finally, on the Lie algebra \( (0, 0, 0, 0, 0, 12) \),
\[ (e^{16} + e^{25} + e^{34}) \wedge e^{13} = -d(e^{356}). \]
In principle, one could try to classify all pairs \((\mathfrak{h}, \omega)\), with \(\mathfrak{h}\) nilpotent of dimension six and \(\omega\) a symplectic form on \(\mathfrak{h}\), for which (2) is non-injective. This means that \(\omega \wedge \gamma = d\psi^+\), for some \(\psi^+ \in \Lambda^3\mathfrak{h}^*\) and some closed non-exact 2-form \(\gamma \in \Lambda^2\mathfrak{h}^*\). If in addition, \((\omega, \psi^+)^\perp\) are compatible in the sense that they define an SU(3)-structure, then declaring \(de^\gamma = \gamma\) one obtains a 7-dimensional Lie algebra \(\mathfrak{g}\) with a calibrated \(G_2\)-structure. By Proposition 1 all calibrated \(G_2\)-structures on indecomposable nilpotent Lie algebras are obtained in this way.

However, these calculations turn out to be difficult (although in one dimension less a similar approach was pursued successfully in [8]), and for this reason we shall use a different method (see Section 4), starting with Gong’s classification of 7-dimensional Lie algebras. In fact, given a Lie algebra, it is straightforward to compute the space of its closed 3-forms. In the spirit of [9], the existence of a calibrated \(G_2\)-structure puts restrictions on this space. Whilst straightforward, the following result turns out to give an effective obstruction.

**Lemma 3.** Let \(\mathfrak{g}\) be a 7-dimensional nilpotent Lie algebra. If there is a nonzero \(X\) in \(\mathfrak{g}\) such that \((X \cdot \phi)^3 = 0\) for every closed 3-form on \(\mathfrak{g}\), then \(\mathfrak{g}\) has no calibrated \(G_2\)-structure.

**Proof.** Obvious.

**Remark.** When \(\mathfrak{g}\) fibers over a non-symplectic Lie algebra \(\mathfrak{h}\), this obstruction is satisfied automatically. Indeed, suppose \(\pi: \mathfrak{g} \to \mathfrak{h}\) is a Lie algebra epimorphism; then any closed 3-form on \(\mathfrak{g}\) can be written as
\[
\pi^*\omega \wedge \eta + \pi^*\psi^+,
\]
as in the proof of Proposition 1. So \(\omega\) is a closed form on \(\mathfrak{h}\); if we assume \(\mathfrak{h}\) has no symplectic form, then \(\omega^3 = 0\). Then the condition of Lemma 3 is satisfied with \(X\) a generator of \(\ker \pi\).

## 3 Decomposable case

In this section we classify the decomposable nilpotent Lie algebras with a calibrated \(G_2\)-structure. Indeed, we prove:

**Theorem 4.** Among the 35 decomposable nilpotent Lie algebras of dimension 7, those that have a calibrated \(G_2\)-structure are
\[
(0, 0, 0, 0, 0, 0, 0), \quad (0, 0, 0, 0, 12, 13, 0), \quad (0, 0, 0, 12, 13, 23, 0).
\]

**Proof.** By the remark at the beginning of Section 2 we know that these three Lie algebras have a calibrated \(G_2\)-structure (see [11] where the second of these Lie algebras was considered). In fact, on the non-abelian Lie algebras \((0, 0, 0, 0, 12, 13)\) and \((0, 0, 0, 12, 13, 23)\) we can consider the symplectic half-flat structure \((\omega_1, \psi_1^+)\) and \((\omega_2, \psi_2^+)\), respectively, defined by
\[
\omega_1 = e^{14} + e^{26} + e^{35}, \quad \psi_1^+ = e^{123} + e^{156} + e^{245} - e^{346},
\]
and 
\[ \omega_2 = e^{16} + 2e^{25} + e^{34}, \quad \psi_2^+ = e^{123} + e^{145} + e^{246} - e^{356}. \]

Using Lemma 3 we can see that the decomposable Lie algebra 
\[ 0, 0, 0, 12, 34, 36 \]
has no calibrated $G_2$-structure. Indeed a basis of the space $Z^3$ of the closed 3-forms is given by
\[ e^{123}, e^{124}, e^{125}, e^{134}, e^{135}, e^{136}, e^{137}, e^{145}, e^{146}, e^{234}, e^{235}, e^{236}, \]
\[ e^{237}, e^{245}, e^{246}, -e^{126} + e^{345}, e^{346}, e^{347}, e^{127} + e^{356}, e^{367}, e^{467}. \]

Thus $e_7 \cdot Z^3$ is the span of $e^{13}, e^{23}, e^{12}, e^{14}, e^{25}, e^{34}, e^{36}, e^{46}$, which contains only degenerate forms.

Since this is the only decomposable nilpotent Lie algebra of dimension seven which does not have the form $\mathfrak{h} \oplus \mathbb{R}$, it remains to prove that if $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$ has a calibrated $G_2$ form, then $\mathfrak{g}$ must be as in the statement.

Clearly, if $\mathfrak{h}$ is one of the eight Lie algebras defined by (2), Proposition 1 implies that the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$ has no calibrated $G_2$ form.

Also, one can check that none of the five Lie algebras defined by
\[
(0, 0, 12, 13, 23, 14, 0), \quad (0, 0, 12, 13, 23, 14 + 25, 0), \quad (0, 0, 12, 13, 23, 14 - 25, 0), \\
(0, 0, 0, 13 + 42, 14 + 23, 0), \quad (0, 0, 0, 0, 12, 14 + 23, 0),
\]
has a calibrated $G_2$ form because each of these is a bundle over a non-symplectic Lie algebra of dimension six. Explicitly, the base of the bundle and curvature form are given by
\[
\pi: (0, 0, 12, 13, 23, 14, 0) \rightarrow (0, 0, 12, 13, 23, 0), \quad d\eta = e^{14}, \\
\pi: (0, 0, 12, 13, 23, 14 + 25, 0) \rightarrow (0, 0, 12, 13, 23, 0), \quad d\eta = e^{14} + e^{25}, \\
\pi: (0, 0, 12, 13, 23, 14 - 25, 0) \rightarrow (0, 0, 12, 13, 23, 0), \quad d\eta = e^{14} - e^{25},
\]
\[
\pi: (0, 0, 0, 13 + 42, 14 + 23, 0) \rightarrow (0, 0, 0, 0, 13 + 42, 0), \quad d\eta = e^{14} + e^{23}, \\
\pi: (0, 0, 0, 0, 12, 14 + 23, 0) \rightarrow (0, 0, 0, 0, 14 + 23, 0), \quad d\eta = e^{12}.
\]

For each of the remaining 18 Lie algebras of the form $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$, listed in Table 1 along with a basis of the space of closed 3-forms, one can check that the hypothesis of Lemma 3 is satisfied with $X = e_6$.

\[\square\]

### 4 Indecomposable case

In this section we complete the classification of 7-dimensional nilpotent Lie algebras with a calibrated $G_2$-structure. We have seen that there are exactly three decomposable Lie algebras of this type. In order to discuss the indecomposable
Table 1: Closed 3-forms on decomposable Lie algebras

| (0, 0, 12, 13, 14 + 23, 24 + 15, 0) | e_{123}, e_{124}, e_{125}, e_{126}, e_{127}, e_{134}, e_{137}, e_{136} + e_{135}, e_{137}, e_{154}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_{147}, e_
Lie algebras, we refer to Gong’s classification in [12]. This list consists of 140 Lie algebras and 9 one-parameter families.

The one-parameter families are the following:

\[147E = (0, 0, 0, e^{12}, e^{23}, -e^{13}, \lambda e^{26} - e^{15} - (-1 + \lambda)e^{34}), \quad \lambda \neq 0, 1;\]
\[1357M = (0, 0, e^{12}, 0, e^{24} + e^{13}, e^{14}, -(1 + \lambda)e^{34} + e^{15} + e^{26}\lambda), \quad \lambda \neq 0;\]
\[1357N = (0, 0, e^{12}, 0, e^{13} + e^{24} + e^{14}, e^{46} + e^{34} + e^{15} + e^{23}\lambda);\]
\[1357S = (0, 0, e^{12}, 0, e^{13}, e^{24} + e^{23}, e^{25} + e^{34} + e^{16} + e^{15} + \lambda e^{26}), \quad \lambda \neq 1;\]
\[12457N = (0, 0, e^{12}, e^{13}, e^{23}, e^{24} + e^{15}, \lambda e^{25} + e^{26} + e^{34} - e^{35} + e^{16} + e^{14});\]
\[123457I = (0, 0, e^{12}, e^{13}, e^{14} + e^{23}, e^{15} + e^{24}, \lambda e^{25} - (1 + \lambda)e^{34} + e^{16});\]
\[147E1 = (0, 0, 0, e^{12}, e^{23}, -e^{13}, 2e^{26} - 2e^{34} - e^{16}\lambda + \lambda e^{25}), \quad \lambda > 1;\]
\[1357QRS1 = (0, 0, e^{12}, 0, e^{13} + e^{24}, e^{14} - e^{23}, e^{25}\lambda + e^{15} - e^{34}(-1 + \lambda)), \quad \lambda \neq 0;\]
\[12457N2 = (0, 0, e^{12}, e^{13}, e^{23}, -e^{14} - e^{25}, e^{15} - e^{35} + e^{16} + e^{24} + e^{25}\lambda), \quad \lambda \geq 0.\]

Recall that a 3-form of type $G_2$ has the form $\omega$ with respect to some coframe $e^1, \ldots, e^7$; such a coframe identifies the $G_2$-structure.

**Lemma 5.** Exactly three of the above Lie algebras admit a calibrated $G_2$-structure. Explicit examples are given in terms of a coframe by

\[1357N(\lambda = 1): \quad \sqrt{3}(2e^1 - e^7 - e^6 - e^5), \sqrt{3}(e^4 + e^3 - 2e^2 - e^6 + e^5), 2e^3 - e^6, 2e^5,\]
\[-e^3 + 3e^4 + e^5 - e^6, 2e^3 - e^5 - e^6 + 3e^7, -\sqrt{3}e^6;\]
\[1357S(\lambda = -3): \quad \sqrt{7}(2e^1 + e^2 - e^5 + e^6), 7e^2 + 3e^5 + 5e^6, \sqrt{7}(e^3 + 2e^4 - e^7),\]
\[3e^3 + \frac{7}{2}e^7, -\sqrt{70}e^6, \sqrt{10}(2e^5 + e^6), -2\sqrt{10}e^3.\]
\[147E1(\lambda = 2): \quad \sqrt{3}(2e^1 + e^5 - e^2 - e^6), 3e^2 - e^5 + e^6, e^3 + 2e^4, \sqrt{3}(e^3 + e^7),\]
\[\sqrt{5}(e^6 - e^5), \sqrt{6}(e^5 + e^6), 2\sqrt{2}(e^4 - e^3).\]

**Proof.** It is straightforward to verify that each coframe in the statement determines a calibrated $G_2$-structure on the corresponding Lie algebra. Conversely, for each Lie algebra $\mathfrak{g}$ the vector $e^7$ is in the center, and determines an epimorphism on a 6-dimensional Lie algebra $\mathfrak{h}$; we view $de^7$ as the curvature form on $\mathfrak{h}$, and apply Proposition I.

In the case of 1357M, the generic element of $H^2(\mathfrak{h}^*)$ is represented by

\[\omega = \lambda_6 e^{46} + \lambda_3 e^{23} + \lambda_1 e^{13} + \lambda_5 (e^{15} + e^{34}) + \lambda_2 e^{16} + \lambda_4 (e^{15} + e^{26}).\]

Assume $de^7 \wedge \omega$ is exact. Then $\lambda_3, \lambda_6$ are zero, $\lambda_4 = -\lambda_5 \lambda$ and

\[(\lambda - \lambda^2 - 1)\lambda_5 = 0.\]

Since $\lambda^2 - \lambda + 1$ has no real zeroes, $\lambda_4$ and $\lambda_5$ are zero as well, and therefore $\omega^3$ is zero. So there is no symplectic form in the cohomology class of $\omega$. By
Proposition 1 if a calibrated $G_2$-structure existed, then $\mathfrak{g}$ would have to be decomposable, which is absurd.

The other cases are similar.

We now turn to the rest of the list, where no parameters appear.

**Lemma 6.** In Gong’s list, only six Lie algebras with no parameters in their definition admit a calibrated $G_2$-structure, which can be expressed in terms of a coframe as follows:

\[
\begin{align*}
0, 0, 12, 0, 0, 13 + 24, 15 & \quad e^1, e^2, e^5, e^6, e^3, e^7, e^4 \\
0, 0, 12, 0, 0, 13, 14 + 25 & \quad e^1, e^3, e^5, e^7, e^2, e^6, e^4 \\
0, 0, 0, 12, 13, 14, 15 & \quad e^1, e^2, e^4, e^7, e^3, e^6, e^3 \\
0, 0, 0, 12, 13, 14 + 23, 15 & \quad e^2 + e^7, e^3 + e^6, e^7, e^6, e^5, e^4, e^1 \\
0, 0, 12, 13, 23, 15, 15 + 24, 16 + 34 & \quad e^2 + e^4, e^7, e^2, e^5, e^3, e^6, e^1 \\
0, 0, 12, 13, 23, 15 + 24, 16 + 25 + 34 & \quad \sqrt{3}(2e^2 + e^5 + e^7), 2e^4 - 3e^5 - e^7, \sqrt{3}(e^1 - e^3 + 2e^6), \\
& \quad e^1 + 3e^3, \sqrt{6}e^7, \sqrt{2}(2e^4 - e^7), 2\sqrt{2}e^1
\end{align*}
\]

**Theorem 7.** Up to isomorphism, there are exactly 12 nilpotent Lie algebras that admit a calibrated $G_2$-structure, namely those appearing in Theorem 4, Lemma 5 and Lemma 6.

**Proof.** We must show that the remaining Lie algebras in Gong’s list satisfy one of the two obstructions of Section 2; we do so in the Appendix, where we reproduce Gong’s list, and note which obstruction applies to each Lie algebra (as a preference, we try to use Proposition 1 rather than Lemma 3 whenever possible, because the former does not require computing the space of closed 3-forms).

**Appendix**

This appendix contains a list of all indecomposable nilpotent Lie algebras of dimension 7, taken from [12], except the 9 one-parameter families that we listed at the beginning of Section 4. Alongside each Lie algebra $\mathfrak{g}$, we give a chosen vector $\xi \in \mathfrak{g}$ which satisfies the conditions of Proposition 1 (when marked with a (P)) or Lemma 3 and the structure constants of the quotient $\mathfrak{g}/\text{Span}\{\xi\}$. The word “resists” marks instead the six Lie algebras that resist the obstructions. Below each Lie algebra, we give a basis of its space of closed 3-forms, except when Proposition 1 applies.
Table 2: Step 2 nilpotent Lie algebras of dimension 7

| 0, 0, 0, 0, 12, 23, 24 | $e_7$ | [0, 0, 0, 0, $e_{12}^\parallel, e_{23}^\parallel$] |
|------------------------|------|---------------------------------------------|
| $e_{123}, e_{124}, e_{125}, e_{126}, e_{127}, e_{134}, e_{135}, e_{136}, e_{145}, e_{137}, e_{146} + e_{127}, e_{147}$ |
| $e_{234}, e_{235}, e_{236}, e_{237}, e_{245}, e_{246}, e_{256}, e_{257}, e_{345} + e_{137}$ |

| 0, 0, 0, 0, 12, 23, 34 | $e_7$ | [0, 0, 0, 0, $e_{12}^\parallel, e_{24}^\parallel$] |
|------------------------|------|---------------------------------------------|
| $e_{123}, e_{124}, e_{125}, e_{126}, e_{134}, e_{135}, e_{136}, e_{145}, e_{137}, e_{146} - e_{127}, e_{147}$ |
| $e_{234}, e_{235}, e_{236}, e_{237}, e_{245}, e_{246}, e_{256}, -e_{127} + e_{345}, e_{346}, e_{347}, e_{367}$ |

| 0, 0, 0, 0, 12 + 34, 23, 24 | $e_5$ | [0, 0, 0, 0, $e_{12}^\parallel, e_{24}^\parallel$] |
|-----------------------------|------|---------------------------------------------|
| $e_{123}, e_{124}, e_{125}, e_{126}, e_{134}, e_{135}, e_{136}, e_{137} + e_{125}, e_{145}, -e_{125} + e_{146}$ |
| $e_{147}, e_{234}, e_{235}, e_{236}, e_{237}, e_{245}, e_{246}, e_{247}, e_{267}, -e_{125} + e_{345}, e_{346}, e_{347}$ |
| $e_{347}, -e_{256} + e_{367}, -e_{257} + e_{367}$ |

| 0, 0, 0, 0, 12 + 34, 13, 24 | $e_7$ | [0, 0, 0, 0, $e_{12}^\parallel, e_{24}^\parallel, e_{13}^\parallel$] |
|-----------------------------|------|---------------------------------------------|
| $e_{123}, e_{124}, e_{125}, e_{126}, e_{134}, e_{135}, e_{136}, e_{137} + e_{125}, e_{145}, e_{146}, e_{147}$ |
| $e_{234}, e_{235}, e_{236}, e_{237}, e_{245}, e_{246} + e_{125}, e_{247}, -e_{125} + e_{345}, e_{346}, e_{347}$ |

| 0, 0, 0, 0, 12, 14 + 35 | $e_6(P)$ | [0, 0, 0, 0, 0, $e_{14}^\parallel, e_{15}^\parallel$] |
|--------------------------|--------|---------------------------------------------|
| $e_{123}, e_{124}, e_{125}, e_{126}, e_{134}, e_{135}, e_{136}, e_{145}, e_{146}, e_{147}$ |
| $e_{234}, e_{235}, e_{236}, e_{237}, e_{245}, e_{246} + e_{125}, e_{247}, -e_{125} + e_{345}, e_{346}, e_{347}$ |

| 0, 0, 0, 0, 12 + 34, 15 + 23 | $e_7(P)$ | [0, 0, 0, 0, 0, $e_{12}^\parallel] |
|-----------------------------|------|---------------------------------------------|
| $e_{123}, e_{124}, e_{125}, e_{126}, e_{134}, e_{135}, e_{136}, e_{145}, e_{146}, e_{147}$ |
| $e_{234}, e_{235}, e_{236}, e_{237}, e_{245}, e_{246} + e_{125}, e_{247}, -e_{125} + e_{345}, e_{346}, e_{347}$ |

| 0, 0, 0, 0, 0, 12 + 34 + 56 | $e_7$ | [0, 0, 0, 0, 0, 0] |
|-----------------------------|------|---------------------------------------------|
| $e_{123}, e_{124}, e_{125}, e_{126}, e_{134}, e_{135}, e_{136}, e_{145}, e_{146}, e_{147}$ |
| $e_{234}, e_{235}, e_{236}, e_{237}, e_{245}, e_{246}, e_{256}, e_{345}, e_{346}, e_{356}, e_{456}$ |

| 0, 0, 0, 0, 12 + 34, 13 + 24, 14 | $e_6$ | [0, 0, 0, 0, 0, $e_{12}^\parallel, e_{24}^\parallel, e_{13}^\parallel$] |
|-----------------------------|------|---------------------------------------------|
| $e_{123}, e_{124}, e_{125}, e_{126}, e_{134}, e_{135}, e_{136}, e_{145}, e_{146}, e_{147}$ |
| $e_{234}, e_{235}, e_{236}, e_{237}, e_{245}, e_{246}, -e_{125} + e_{246}, e_{247}, e_{125} + e_{345}, e_{346}, e_{347}$ |
| $e_{347}, e_{457} + e_{167}, e_{467} + e_{157}$ |

| 0, 0, 0, 0, 12 + 34, 13 + 24, 14 - 23 | $e_7$ | [0, 0, 0, 0, 0, $e_{12}^\parallel, e_{24}^\parallel, e_{13}^\parallel, e_{14}^\parallel$] |
|-----------------------------|------|---------------------------------------------|
| $e_{123}, e_{124}, e_{125}, e_{126}, e_{134}, e_{135}, e_{136}, e_{145}, e_{146}, e_{147}$ |
| $e_{234}, e_{235}, e_{236}, e_{237}, e_{245}, -e_{125} + e_{246}, e_{247}, e_{125} + e_{345}, e_{346}, e_{347}$ |
Table 3: Step 3 nilpotent Lie algebras of dimension 7

| Dimension | Lie Algebra | Injection | Description |
|-----------|-------------|-----------|-------------|
| 0, 0, 12, 0, 13, 24, 14 | \(e_5\) | \([0, 0, e^{12}, 0, e^{24}, e^{14}]\) | \(e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{146}, e^{147}, e^{157}, e^{234}, e^{235}, e^{236}, e^{237}, e^{136} - e^{136}, e^{124} - e^{24}, e^{247}, e^{248}, e^{246}, e^{145} - e^{156}, e^{154} + e^{267}, e^{347} + e^{167}, e^{467}\) |
| 0, 0, 12, 0, 13 + 24, 14 | \(e_7(P)\) | \([0, 0, e^{12}, 0, e^{13}, e^{23}]\) | \(e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{146}, e^{147}, e^{157}, e^{234}, e^{235}, e^{236}, e^{237} + e^{135}, -e^{135} + e^{245}, e^{246}, e^{247}, e^{248} - e^{345} + e^{167}, e^{346} + e^{267}, e^{347} + e^{157}\) |
| 0, 0, 12, 0, 13 + 24, 15 | \(e_6\) | \([0, 0, e^{12}, 0, e^{24} + e^{14}, e^{14}]\) | \(e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{146} + e^{135}, e^{147}, e^{234}, e^{235}, e^{236}, e^{237} + e^{135}, -e^{135} + e^{245}, e^{246}, e^{247}, e^{248} - e^{345} + e^{167}, e^{346} + e^{267}, e^{347} + e^{157}\) |
| 0, 0, 12, 0, 13 + 24, 15 | \(e_7\) | \([0, 0, e^{12}, 0, e^{14} + e^{14}]\) | \(e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{146} + e^{147}, e^{156}, e^{157}, e^{234}, e^{235}, e^{236}, e^{237} + e^{135}, -e^{135} + e^{245}, e^{246}, e^{247}, e^{248} - e^{345} + e^{167}, e^{346} + e^{267}, e^{347} + e^{157}\) |
| 0, 0, 12, 0, 13 + 24, 15 | \(e_7(P)\) | \([0, 0, e^{12}, 0, e^{14} + e^{14}]\) | \(e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{146} + e^{147}, e^{156}, e^{157}, e^{234}, e^{235}, e^{236}, e^{237} + e^{135}, -e^{135} + e^{245}, e^{246}, e^{247}, e^{248} - e^{345} + e^{167}, e^{346} + e^{267}, e^{347} + e^{157}\) |
| 0, 0, 12, 0, 13 + 45, 24 | \(e_5\) | \([0, 0, e^{12}, 0, e^{24}, e^{14}]\) | \(e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{146}, e^{147}, e^{157}, e^{234}, e^{235}, e^{236}, e^{237} + e^{135}, -e^{135} + e^{245}, e^{246}, e^{247}, e^{248} - e^{345} + e^{167}, e^{346} + e^{267}, e^{347} + e^{157}\) |
| 0, 0, 12, 0, 13 + 45, 15 + 24 | \(e_7(P)\) | \([0, 0, e^{12}, 0, e^{14} + e^{14}]\) | \(e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{146}, e^{147}, e^{156}, e^{157}, e^{234}, e^{235}, e^{236}, e^{237} + e^{135}, -e^{135} + e^{245}, e^{246}, e^{247}, e^{248} - e^{345} + e^{167}, e^{346} + e^{267}, e^{347} + e^{157}\) |
| 0, 0, 12, 0, 13 + 45, 15 + 24 | \(e_6\) | \([0, 0, e^{12}, 0, e^{14} + e^{13}]\) | \(e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{146}, e^{147}, e^{156}, e^{157}, e^{234}, e^{235}, e^{236}, e^{237} + e^{135}, -e^{135} + e^{245}, e^{246}, e^{247}, e^{248} - e^{345} + e^{167}, e^{346} + e^{267}, e^{347} + e^{157}\) |
| 0, 0, 12, 0, 13 + 45, 15 + 24 | \(e_7(P)\) | \([0, 0, e^{12}, 0, e^{14} + e^{14}]\) | \(e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{146}, e^{147}, e^{156}, e^{157}, e^{234}, e^{235}, e^{236}, e^{237} + e^{135}, -e^{135} + e^{245}, e^{246}, e^{247}, e^{248} - e^{345} + e^{167}, e^{346} + e^{267}, e^{347} + e^{157}\) |
| 0, 0, 12, 0, 13 + 45, 15 + 24 | \(e_6\) | \([0, 0, e^{12}, 0, e^{14} + e^{14}]\) | \(e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{146}, e^{147}, e^{156}, e^{157}, e^{234}, e^{235}, e^{236}, e^{237} + e^{135}, -e^{135} + e^{245}, e^{246}, e^{247}, e^{248} - e^{345} + e^{167}, e^{346} + e^{267}, e^{347} + e^{157}\) |
| 0, 0, 12, 0, 13 + 45, 15 + 24 | \(e_7(P)\) | \([0, 0, e^{12}, 0, e^{14} + e^{14}]\) | \(e^{123}, e^{124}, e^{125}, e^{126}, e^{127}, e^{134}, e^{135}, e^{137}, e^{145}, e^{146}, e^{147}, e^{156}, e^{157}, e^{234}, e^{235}, e^{236}, e^{237} + e^{135}, -e^{135} + e^{245}, e^{246}, e^{247}, e^{248} - e^{345} + e^{167}, e^{346} + e^{267}, e^{347} + e^{157}\) |
| Page | Description |
|------|-------------|
| 0.0, 0, 12, 13, 14, 15 | \( e_7 \) | \( [0, 0, 0, e_{12}, e_{13}, e_{14}] \) |
| 0.0, 0, 12, 13, 14, 35 | \( e_7 \) | \( [0, 0, 0, e_{12}, e_{13}, e_{14}] \) |
| 0.0, 0, 12, 13, 14 + 35, 15 | \( e_7(P) \) | \( [0, 0, 0, e_{12}, e_{13}, e_{14}] \) |
| 0.0, 0, 12, 13, 14 + 25 + 34 | \( e_7(P) \) | \( [0, 0, 0, e_{12}, e_{13}, e_{14}] \) |
| 0.0, 0, 12, 13, 14 + 24 + 35, 25 + 34 | \( e_7(P) \) | \( [0, 0, 0, e_{12}, e_{13}, e_{14}] \) |
| 0.0, 0, 12, 13, 14 + 15 + 24 + 35, 25 + 34 | \( e_7(P) \) | \( [0, 0, 0, e_{12}, e_{13}, e_{14}] \) |
| 0.0, 0, 12, 13, 14 + 23, 15 | \( e_7 \) | \( [0, 0, 0, e_{12}, e_{13}, e_{14}] \) |
| 0.0, 0, 12, 13, 15 + 24, 23 | \( e_7(P) \) | \( [0, 0, 0, e_{12}, e_{13}, e_{14}] \) |
| 0.0, 0, 12, 13, 14 + 23, 25 + 34 | \( e_7(P) \) | \( [0, 0, 0, e_{12}, e_{13}, e_{14}] \) |
| 0.0, 0, 12, 13, 14 + 15 + 23, 25 + 34 | \( e_7(P) \) | \( [0, 0, 0, e_{12}, e_{13}, e_{14}] \) |
| 0.0, 0, 12, 13, 14 + 24 + 35, 15 | \( e_7 \) | \( [0, 0, 0, e_{12}, e_{13}, e_{14}] \) |
| 0.0, 0, 12, 13, 14 + 23, 25 + 34 | \( e_7(P) \) | \( [0, 0, 0, e_{12}, e_{13}, e_{14}] \) |
| 0.0, 0, 12, 13, 14 + 15 + 23, 25 + 34 | \( e_7(P) \) | \( [0, 0, 0, e_{12}, e_{13}, e_{14}] \) |
| 0.0, 0, 12, 13, 14 + 23, 15 | \( e_7 \) | \( [0, 0, 0, e_{12}, e_{13}, e_{14}] \) |
| 0.0, 0, 12, 13, 15 + 24, 23 | \( e_7(P) \) | \( [0, 0, 0, e_{12}, e_{13}, e_{14}] \) |
| 0.0, 0, 12, 13, 14 + 23, 25 + 34 | \( e_7(P) \) | \( [0, 0, 0, e_{12}, e_{13}, e_{14}] \) |
| 0.0, 0, 12, 13, 14 + 15 + 23, 25 + 34 | \( e_7(P) \) | \( [0, 0, 0, e_{12}, e_{13}, e_{14}] \) |
| 0.0, 0, 12, 13, 0, 13 + 24 + 56 | \( e_7 \) | \( [0, 0, 0, e_{12}, e_{13}, e_{14}] \) |
| 0.0, 0, 12, 13, 0, 16 + 25 + 34 | \( e_7 \) | \( [0, 0, 0, e_{12}, e_{13}, e_{14}] \) |
| 0.0, 0, 12, 13, 0, 14 + 26 + 35 | \( e_7 \) | \( [0, 0, 0, e_{12}, e_{13}, e_{14}] \) |
| 0.0, 0, 12, 23, -13, 15 + 26 + 16 - 2 * 34 | \( e_7 \) | \( [0, 0, 0, e_{12}, e_{13}, e_{14}] \) |
| 0.0, 0, 12, 34, 15 + 36 | \( e_7 \) | \( [0, 0, 0, e_{12}, e_{13}, e_{14}] \) |
| Model | Score | Name | Text |
|-------|-------|------|------|
| 0, 0, 0, 0, 12, 34, 15 + 24 + 36 | e_7 | $e^{123}, e^{124}, e^{125}, e^{134}, e^{135}, e^{136}, e^{137}, e^{138}, e^{145}, e^{146}, e^{147}, e^{234}, e^{235}, e^{236}, e^{245}, e^{246}, e^{247}, e^{256}, e^{345}, e^{346}, e^{347}, e^{356}, e^{357}, e^{367}$ |
| 0, 0, 0, 0, 12, 14 + 23, 16 − 35 | e_7 | $e^{123}, e^{124}, e^{125}, e^{134}, e^{135}, e^{136}, e^{137}, e^{138}, e^{145}, e^{146}, e^{147}, e^{234}, e^{235}, e^{236}, e^{245}, e^{246}, e^{247}, e^{256}, e^{345}, e^{346}, e^{347}, e^{356}, e^{357}, e^{367}$ |
| 0, 0, 0, 0, 12, 14 + 23, 16 + 24 − 35 | e_7 | $e^{123}, e^{124}, e^{125}, e^{134}, e^{135}, e^{136}, e^{137}, e^{138}, e^{145}, e^{146}, e^{147}, e^{234}, e^{235}, e^{236}, e^{245}, e^{246}, e^{247}, e^{256}, e^{345}, e^{346}, e^{347}, e^{356}, e^{357}, e^{367}$ |
| 0, 0, 0, 0, 12, 14 + 23, 16 + 24 + 25 | e_7 | $e^{123}, e^{124}, e^{125}, e^{134}, e^{135}, e^{136}, e^{137}, e^{138}, e^{145}, e^{146}, e^{147}, e^{234}, e^{235}, e^{236}, e^{245}, e^{246}, e^{247}, e^{256}, e^{345}, e^{346}, e^{347}, e^{356}, e^{357}, e^{367}$ |
| 0, 0, 0, 0, 12, 13, 24 + 35 | e_6(P) | $0, 0, 0, 0, e^{12}, e^{13}, e^{14}, e^{24} + e^{25}$ |
| 0, 0, 0, 0, 12, 13, 24 − 35, 25 + 34 | e_7(P) | $0, 0, 0, 0, e^{12}, e^{13}, e^{14}, e^{25} − e^{26} + e^{24} + e^{25}$ |
| 0, 0, 0, 0, 12, 13, 24 + 35, 25 − 34 | e_7(P) | $0, 0, 0, 0, e^{12}, e^{13}, e^{14}, e^{25} − e^{26} + e^{25} − e^{24} + e^{26}$ |
| 0, 0, 0, 0, 12, 13, 0, 16 + 24 + 35 | e_7 | $0, 0, 0, 0, e^{12}, e^{13}, e^{14}, 0$ |
| 0, 0, 0, 0, 13 + 24, 14 − 23, 15 + 26 | e_7 | $0, 0, 0, 0, e^{12}, e^{14}, e^{24}, e^{14} − e^{24}$ |
| 0, 0, 0, 0, 13 + 24, 14 − 23, 15 + 26 | e_7 | $0, 0, 0, 0, e^{12}, e^{14}, e^{24}, e^{14} − e^{24}$ |

Table 4: Step 4 nilpotent Lie algebras of dimension 7
Table 5: Step 5 nilpotent Lie algebras of dimension 7

| 0.0, 0.0 | 12.13, 14, 15, 23 | e_6(P) | [0.0, 0.0, e^{12}, e^{13}, e^{14}, e^{15}, e^{23}] |
| 0.0, 0.0 | 12.13, 14, 25−34, 23 | e_7(P) | [0.0, 0.0, e^{12}, e^{13}, e^{14}, e^{15}, e^{23}] |
| 0.0, 0.0 | 12.13, 14, 15+23, 25−34 | e_6(P) | [0.0, 0.0, e^{12}, e^{13}, e^{14}, e^{15}, e^{23}] |
| 0.0, 0.0 | 12.13, 14, 15+23, 25−34 | e_6(P) | [0.0, 0.0, e^{12}, e^{13}, e^{14}, e^{15}, e^{23}] |
| 0.0, 0.0 | 12.13, 14, 15+23, 25−34 | e_7(P) | [0.0, 0.0, e^{12}, e^{13}, e^{14}, e^{15}, e^{23}] |
| 0.0, 0.0 | 12.13, 14, 15+23, 25−34 | e_6(P) | [0.0, 0.0, e^{12}, e^{13}, e^{14}, e^{15}, e^{23}] |
| 0.0, 0.0 | 12.13, 14, 15+23, 25−34 | e_7(P) | [0.0, 0.0, e^{12}, e^{13}, e^{14}, e^{15}, e^{23}] |
| 0.0, 0.0 | 12.13, 14, 15+23, 25−34 | e_6(P) | [0.0, 0.0, e^{12}, e^{13}, e^{14}, e^{15}, e^{23}] |
| 0.0, 0.0 | 12.13, 14, 15+23, 25−34 | e_7(P) | [0.0, 0.0, e^{12}, e^{13}, e^{14}, e^{15}, e^{23}] |
| 0.0, 0.0 | 12.13, 14, 15+23, 25−34 | e_6(P) | [0.0, 0.0, e^{12}, e^{13}, e^{14}, e^{15}, e^{23}] |
| 0.0, 0.0 | 12.13, 14, 15+23, 25−34 | e_7(P) | [0.0, 0.0, e^{12}, e^{13}, e^{14}, e^{15}, e^{23}] |
| Equation | Description |
|----------|-------------|
| $0, 0, 12, 13, 14 + 23, 0, 16 + 25 - 34$ | $e_7$, $e_{123}, e_{124}, e_{125}, e_{126}, e_{131}, e_{132}, e_{136}, e_{145} + e_{127}, e_{146}, e_{147}, e_{157} + e_{158}$, $e_{236}, e_{237} + e_{156}, e_{137} + e_{245}, e_{246} + e_{156}, e_{147} + e_{145}, e_{145}, e_{146}, e_{147}, e_{156}, e_{157}, e_{158}$ |
| $0, 0, 12, 13, 14, 23, 15 + 26$ | $e_7(P)$, $0, 0, e_{12}, e_{13}, e_{14}, e_{23}$ |
| $0, 0, 12, 13, 14, 23, 16 + 24 + 25 - 34$ | $e_7(P)$, $0, 0, e_{12}, e_{13}, e_{14}, e_{23}$ |
| $0, 0, 12, 13, 14, 23, 15 + 26 - 34$ | $e_7(P)$, $0, 0, e_{12}, e_{13}, e_{14}, e_{23}$ |
| $0, 0, 12, 13, 0, 14 + 25, 16 + 35$ | $e_7$, $e_{123}, e_{124}, e_{125}, e_{126}, e_{134}, e_{145} + e_{127}, e_{136}, e_{145}, e_{146}, e_{157}$, $e_{236}, e_{237} - e_{245} - e_{246} - e_{146}$, $-e_{135}, e_{146}, e_{147}, -e_{235}, e_{237} - e_{257} + e_{156} - e_{147}, e_{157}$ |
| $0, 0, 12, 13, 0, 14 + 25, 16 + 25 + 35$ | $e_7$, $e_{123}, e_{124}, e_{125}, e_{126}, e_{134}, e_{145} + e_{127}, e_{136}, e_{145}, e_{146}, e_{157}$, $e_{237}, e_{235} - e_{245} + e_{246} + e_{146}$, $-e_{257} + e_{156} + e_{146}, e_{147}, -e_{235}, e_{237} - e_{257} + e_{156} - e_{147}$ |
| $0, 0, 12, 13, 0, 14 + 25, 26 - 34$ | $e_7$, $e_{123}, e_{124}, e_{125}, e_{126}, e_{134}, e_{145} + e_{127}, e_{136}, e_{145}, e_{146}, e_{147}$, $e_{235}, e_{236} + e_{137}, e_{146}, e_{147}, e_{157} + e_{158}, e_{237}$ |
| $0, 0, 12, 13, 0, 14 + 25, 15 + 26 - 34$ | $e_7$, $e_{123}, e_{124}, e_{125}, e_{126}, e_{134}, e_{145} + e_{127}, e_{136}, e_{145}, e_{146}, e_{157}$, $e_{235}, e_{236} + e_{137}, e_{146}, e_{147}, e_{157} + e_{158}, e_{237}$ |
| $0, 0, 12, 13, 0, 14 + 23 + 25, 16 + 24 + 35$ | $e_7$, $0, 0, e_{12}, e_{13}, e_{14}, e_{23} + e_{25} + e_{23}$ |
| $0, 0, 12, 13, 0, 14 + 23 + 25, 16 - 34$ | $e_7$, $0, 0, e_{12}, e_{13}, 0, e_{23} + e_{24} + e_{25}$ |
| $0, 0, 12, 13, 0, 14 + 23 + 25, 26 - 34$ | $e_7$, $e_{123}, e_{124}, e_{125}, e_{126}, e_{134}, e_{145} + e_{127}, e_{136}, e_{145}, e_{146}, e_{157}$, $e_{235}, e_{236} + e_{137}, e_{146}, e_{147}, e_{157} + e_{158}, e_{237}$ |
| $0, 0, 12, 13, 0, 14 + 23 + 25, 15 + 26 - 34$ | $e_7$, $0, 0, e_{12}, e_{13}, 0, e_{23} + e_{25} + e_{23}$ |
| $0, 0, 12, 13, 0, 14 + 23 + 25, 15 + 26 + 24 + 34$ | $e_7$, $0, 0, e_{12}, e_{13}, 0, e_{23} + e_{25} + e_{23}$ |
| $0, 0, 12, 13, 15 + 24, 16 + 34$ | $e_7$, $e_{123}, e_{124}, e_{125}, e_{126}, e_{134}, e_{135}, e_{145} + e_{127}, e_{136}, e_{145} + e_{146}$, $e_{147}, e_{234}, e_{235}, e_{236} + e_{137}, e_{146}, e_{147}, e_{148}, e_{156}, e_{157}, e_{158}$, $e_{235}, e_{236} + e_{137}, e_{146}, e_{147}, e_{148}, e_{156}, e_{157}, e_{158}$ |
| $0, 0, 12, 13, 15 + 24, 16 + 25 + 24$ | $e_7$, $e_{123}, e_{124}, e_{125}, e_{126}, e_{134}, e_{135}, e_{145} + e_{127}, e_{136}, e_{145} + e_{146}$, $e_{147}, e_{234}, e_{235}, e_{236} + e_{137}, e_{146}, e_{147}, e_{148}, e_{156}, e_{157}, e_{158}$ |
| $0, 0, 12, 13, 15 + 24, 16 + 25 + 34$ | $e_7$, $0, 0, e_{12} + e_{13}, e_{23} + e_{24} + e_{25}$ |
| $0, 0, 12, 13, 15 + 24, 16 + 25 + 34$ | $e_7$, $0, 0, e_{12} + e_{13}, e_{23} + e_{24} + e_{25}$ |
| $0, 0, 12, 13, 15 + 24, 16 + 25 + 34$ | $e_7$, $0, 0, e_{12} + e_{13}, e_{23} + e_{24} + e_{25}$ |
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