Remarks on the completeness of trajectories of accelerated particles in Riemannian manifolds and plane waves

A.M. Candela, A. Romero and M. Sánchez

Abstract

Recently, classical results on completeness of trajectories of Hamiltonian systems obtained at the beginning of the seventies, have been revisited, improved and applied to Lorentzian Geometry [5]. Our aim here is threefold: to give explicit proofs of some technicalities in the background of the specialists, to show that the introduced tools allow to obtain more results for the completeness of the trajectories, and to apply these results to the completeness of spacetimes that generalize classical plane and pp–waves.

MSC2010: 34A26, 34C40, 37C60, 35Q75, 83C35.

Key words and phrases: generalized plane wave, pp–wave, geodesic completeness, non-autonomous system, completeness of inextensible trajectories.

1 Introduction

In Classical Mechanics, one of the most venerable equations on a (connected) Riemannian manifold \((M_0, g_0)\) is:

\[
\frac{D\dot{\gamma}}{dt}(t) = -\nabla^{M_0}V(\gamma(t), t)
\]

\( (E_0) \)

where \(D/dt\) denotes the covariant derivative along \(\gamma\) induced by the Levi-Civita connection of \(g_0\) and \(\dot{\gamma}\) represents the velocity field along \(\gamma\), while \(V : M_0 \times \mathbb{R} \to \mathbb{R}\) is a smooth time–dependent potential. In fact, when

---

1Partially supported by Spanish Grants with FEDER funds MTM2010-18099 (MICYCINN) and P09-FQM-4496 (J. de Andalucía).

2Partially supported by M.I.U.R. (research funds ex 40% and 60%) and of the G.N.A.M.P.A. Research Project 2011 Analisi Geometrica sulle Varietà di Lorentz ed Applicazioni alla Relatività Generale.
(M₀, g₀) is ℜ³, this is just Newton’s second law for forces that come from an external time-dependent potential. A basic property that may have its solutions is completeness i.e. the extendability of their domain to all ℜ. At the beginning of the seventies, some authors studied systematically this property (see, e.g., [7, 11, 14] or also [1, Theorem 3.7.15]) but, essentially, they focus only in the autonomous case, that is, when V(x, t) ≡ V(x) (V is independent of time).

Very recently, the authors have considered the completeness of the trajectories not only for the general equation (E₀) but also for more general forces (see [5]). Concretely, −∇M₀V was generalized to an arbitrary time-dependent vector field X and forces linearly dependent with the velocity by means of an operator F, were also allowed. Nevertheless, it is specially interesting to understand and analyze accurately the differences between the autonomous and the non-autonomous case for a potential. Moreover, as pointed out in [4] (see also [6]), the completeness for (E₀) is equivalent to the completeness for the geodesics of a class of relativistic spacetimes that generalizes the classical plane and pp–waves. So, the aim of the present paper is, first, to analyze further the completeness in the non-autonomous case \( X = −∇M₀V \) (even admitting the linear dependence of the force with the operator F, see equation (E) below) and, then, to analyze the applications to generalized plane waves.

This paper is organized as follows. In Section 2, we recall the framework for the completeness of Riemannian trajectories (Subsection 2.1), and give a new theorem on completeness (Subsection 2.2). The proofs of two results are provided. The first one is a technical comparison lemma that is commonly taken into account in the results on completeness (Lemma 2.2). The second one is a theorem on completeness (Theorem 2.3), obtained by developing further the techniques in [5]. In Section 3 we introduce plane wave type spacetimes (Subsection 3.1) and explain the relation between the problem of completeness of trajectories and the geodesic completeness of generalized plane waves (Subsection 3.2). Moreover, we give further results on geodesic completeness (Corollaries 3.3, 3.4) as a consequence of the previous result of completeness of trajectories.

2 Completeness of Riemannian trajectories

2.1 Framework

Let \((M₀, g₀)\) be a (connected) smooth n–dimensional Riemannian manifold and \(V : M₀ × ℜ → ℜ\) a given smooth function. Taking \(p \in M₀\) and \(v \in T_pM₀\),
there exists a unique inextensible smooth curve $\gamma : I \to M_0$, $0 \in I$, solution of $(E_0)$ which satisfies the initial conditions
\[
\gamma(0) = p, \quad \dot{\gamma}(0) = v. \quad (1)
\]

An inextensible solution of $(E_0)$ is complete if it is defined on the whole real line. Note that equation $(E_0)$ in the trivial case $V \equiv 0$ is the equation of the geodesics in $(M_0, g_0)$. Let us recall that a Riemannian manifold $(M_0, g_0)$ is geodesically complete if any of its inextensible geodesics is defined on $\mathbb{R}$ or, equivalently, the metric distance induced by $g_0$ is complete.

In [11, Theorem 2.1] Gordon proved the completeness of the trajectories of $(E_0)$ if the potential $V$ is time–independent, bounded from below and satisfying either $(M_0, g_0)$ is complete or $V$ is proper (i.e., $V^{-1}(K)$ is compact in $M_0$ for any compact $K \subset \mathbb{R}$). Other results in the autonomous case were given in [7, 14] and [1, Theorem 3.7.15].

Following [5], we generalize such results to the non–autonomous case by including also the action of a $(1,1)$ tensor field $F$ along the natural projection $\pi : M_0 \times \mathbb{R} \to M_0$, i.e., we consider the second order differential equation
\[
\frac{D\dot{\gamma}}{dt}(t) = F_{(\gamma(t), t)} \dot{\gamma}(t) - \nabla^{M_0} V(\gamma(t), t). \quad (E)
\]

Let us remark that the existence and uniqueness result of inextensible solutions of $(E)$, under the same initial conditions (1), remains now true, and, obviously, one has the notion of complete inextensible trajectory of $(E)$.

Now, let us introduce some terminology in order to express natural conditions on $F$ and $V$. Notice that, in general, $F$ is neither self-adjoint nor skew-adjoint with respect to $g_0$, and denote by $S$ the self-adjoint part of $F$. For each $t \in \mathbb{R}$, put
\[
\|S(t)\| := \max \{|S_{\sup}(t)|, |S_{\inf}(t)|\}
\]
where
\[
S_{\sup}(t) := \sup_{v \in T_{M_0}} \frac{g(v, S_{(p, t)} v)}{\|v\| = 1} \quad \text{and} \quad S_{\inf}(t) := \inf_{v \in T_{M_0}} \frac{g(v, S_{(p, t)} v)}{\|v\| = 1}.
\]

We say that $S$ is bounded (resp. upper bounded, lower bounded) along finite times when, for each $T > 0$, there exists a constant $N_T$ such that
\[
\|S(t)\| \leq N_T \quad (\text{resp.} \quad S_{\sup}(t) \leq N_T, -S_{\inf}(t) \leq N_T) \quad \text{for all} \quad t \in [-T, T]. \quad (2)
\]
Moreover, the potential $V$ is bounded from below along finite times if there exists a continuous function $\beta_0 : \mathbb{R} \to \mathbb{R}$ such that

$$V(p, t) \geq \beta_0(t) \quad \text{for all} \quad (p, t) \in M_0 \times \mathbb{R}. \quad (3)$$

In order to investigate the completeness of the inextensible solutions of equation \((E)\), let us recall that an integral curve $\rho$ of a vector field on a manifold, defined on some bounded interval $[a, b)$, $b < +\infty$, can be extended to $b$ (as an integral curve) if and only if there exists a sequence $\{t_n\}_n$, $t_n \to b^-$, such that $\{\rho(t_n)\}_n$ converges [13, Lemma 1.56]. The following technical result follows directly from this fact and [5, Lemma 3.1].

**Lemma 2.1.** Let $\gamma : [0, b) \to M_0$ be a solution of equation \((E)\) with $0 < b < +\infty$. The curve $\gamma$ can be extended to $b$ as a solution of \((E)\) if and only if there exists a sequence $\{t_n\}_n \subset [0, b)$ such that $t_n \to b^-$ and the sequence of velocities $\{\dot{\gamma}(t_n)\}_n$ is convergent in the tangent bundle $TM_0$.

Furthermore, we need also the following result (compare with [1, Example 2.2.H]).

**Lemma 2.2 (Comparison Lemma).** Let $\varphi : [a, +\infty) \to \mathbb{R}$ be a continuous monotone increasing function such that

$$\varphi(s) > 0 \quad \text{for all} \quad s \geq a \quad \text{and} \quad \int_a^{+\infty} \frac{ds}{\varphi(s)} = +\infty. \quad (4)$$

If a $C^1$ function $v_0 = v_0(t)$ satisfies the equation

$$v_0'(t) = \varphi(v_0(t)) \quad \text{with} \quad v_0(0) \geq a, \quad (5)$$

and it is inextensible, then it is defined for all $t \geq 0$.

Furthermore, if $v : [0, b) \to \mathbb{R}$ is a continuous function such that

$$\begin{cases} a \leq v(t) \leq v(0) + \int_0^t \varphi(v(s))ds \quad \text{for all} \quad t \in [0, b), \\ v(0) \leq v_0(0), \end{cases} \quad (6)$$

then $v(t) \leq v_0(t)$ for all $t \in [0, b)$.

**Proof.** Even though this is a simple exercise, we prefer to give here a complete argument by completeness. If $v_0 = v_0(t)$ is a $C^1$ inextensible solution of \((5)\) in the interval $[0, \bar{b})$, then

$$v_0(t) \geq v_0(0) \geq a \quad \text{for all} \quad t \in [0, \bar{b}), \quad (7)$$
whence, for all $t \in [0, \bar{b})$, $\varphi(v_0(t))(> 0)$ is well defined and $v_0$ becomes strictly monotone increasing. Thus, dividing both the terms of (5) by $\varphi(v_0(t))$ and integrating in $[0, t]$, $0 < t < \bar{b}$, we have

$$\int_0^t \frac{v'_0(\tau)}{\varphi(v_0(\tau))} d\tau = t,$$

hence, $v_0 = v_0(t)$ is the inverse of

$$t(v_0) = \int_{v_0(0)}^{v_0} \frac{ds}{\varphi(s)},$$

with the maximum $\bar{b}$ equal to $\lim_{v_0 \to +\infty} t(v_0)$ in (8). From (4) it follows $\bar{b} = +\infty$.

Now, let $v = v(t)$, $t \in [0, b)$, be such to satisfy (6) and define

$$h(t) = v_0(0) + \int_0^t \varphi(v(s))ds.$$

Clearly, $h$ is a $C^1$ function such that

$$h(0) = v_0(0) \quad \text{and} \quad h'(t) = \varphi(v(t)) \quad \text{for all} \quad t \in [0, b).$$

Moreover, from (4) it follows

$$a \leq v(t) \leq h(t) \quad \text{for all} \quad t \in [0, b),$$

whence the monotonicity of $\varphi$ implies

$$h'(t) \leq \varphi(h(t)) \quad \text{for all} \quad t \in [0, b).$$

Thus, from (4), (5) and (10) we have

$$\frac{h'(t)}{\varphi(h(t))} \leq 1 = \frac{v'_0(t)}{\varphi(v_0(t))} \quad \text{for all} \quad t \in [0, b),$$

whence direct computations give

$$\int_{v_0(0)}^{h(t)} \frac{ds}{\varphi(s)} \leq \int_{v_0(0)}^{v_0(t)} \frac{ds}{\varphi(s)} \quad \text{for all} \quad t \in [0, b).$$

Now, assume that $\bar{t} \in (0, b)$ exists such that $h(\bar{t}) > v_0(\bar{t})$. Hence, (4) and (7) imply

$$\int_{v_0(0)}^{h(\bar{t})} \frac{ds}{\varphi(s)} > \int_{v_0(0)}^{v_0(\bar{t})} \frac{ds}{\varphi(s)}$$

in contradiction with (11). So, we have $h(t) \leq v_0(t)$ for all $t \in [0, b)$ and the proof follows from (9).
Our main result on the non–autonomous problem \((E)\)

Now, we are ready to state our main result on the completeness of inextensible trajectories of the non–autonomous problem \((E)\).

**Theorem 2.3.** Let \((M_0, g_0)\) be a complete Riemannian manifold, \(F\) a smooth time–dependent \((1, 1)\) tensor field with self–adjoint component \(S\) and \(V : M_0 \times \mathbb{R} \to \mathbb{R}\) a smooth potential. Assume that \(|S(t)|\) is bounded along finite times, \(V\) is bounded from below along finite times and there exists a continuous function \(\alpha_0 : \mathbb{R} \to \mathbb{R}\) such that

\[
\left| \frac{\partial V}{\partial t}(p, t) \right| \leq \alpha_0(t)(V(p, t) - \beta_0(t)) \quad \text{for all } (p, t) \in M_0 \times \mathbb{R}
\]

with \(\beta_0\) as in \((3)\).

Then, each inextensible solution of equation \((E)\) must be complete.

The proof of Theorem 2.3 is a direct consequence of the following more general result.

**Proposition 2.4.** Let \((M_0, g_0)\) be a complete Riemannian manifold, \(F\) a smooth time–dependent \((1, 1)\) tensor field with self–adjoint component \(S\) and \(V : M_0 \times \mathbb{R} \to \mathbb{R}\) a smooth potential bounded from below along finite times with \(\beta_0\) as in \((3)\).

If \(S_{\text{sup}}(t)\) is upper bounded along finite times and a continuous function \(\alpha_0 : \mathbb{R} \to \mathbb{R}\) exists such that

\[
\frac{\partial V}{\partial t}(p, t) \leq \alpha_0(t)(V(p, t) - \beta_0(t)) \quad \text{for all } (p, t) \in M_0 \times \mathbb{R},
\]

then each inextensible solution of equation \((E)\) must be forward complete.

Conversely, if \(S_{\text{inf}}(t)\) is lower bounded along finite times and a continuous function \(\alpha_0 : \mathbb{R} \to \mathbb{R}\) exists such that

\[
-\frac{\partial V}{\partial t}(p, t) \leq \alpha_0(t)(V(p, t) - \beta_0(t)) \quad \text{for all } (p, t) \in M_0 \times \mathbb{R},
\]

then each inextensible solution of equation \((E)\) must be backward complete.

**Proof.** Let \(\gamma\) be a non–constant forward inextensible solution of equation \((E)\) defined on the interval \([0, b) \subset \mathbb{R}\). Arguing by contradiction, assume that \(\gamma\) is not forward complete, i.e., \(b < +\infty\), so a real positive constant \(T > b\) can be fixed so that \((2)\) holds for \(S_{\text{sup}}(t)\), furthermore

\[
V(p, t) - B_T \geq 1 \quad \text{and} \quad \frac{\partial V}{\partial t}(p, t) \leq A_T(V(p, t) - B_T) \quad (12)
\]
Completeness of Trajectories and Plane Waves

for all \((p, t) \in M_0 \times [-T, T]\), with \(A_T \geq \max_0([-T, T])\) and \(B_T \leq \min_0([-T, T]) - 1\).

Now, for simplicity, denote

\[
u(t) = g(\dot{\gamma}(t), \dot{\gamma}(t)) \quad \text{and} \quad v(t) = \frac{1}{2} u(t) + V(\gamma(t), t) - B_T, \quad t \in [0, b).
\]

From (12) it follows

\[u(t) + 1 \leq 2v(t),\]

hence if \(v(t)\) is bounded in \([0, b)\) so is \(u(t)\), that is a constant \(k > 0\) exists such that

\[u(t) \leq k \quad \text{for all} \quad t \in [0, b). \quad (13)\]

Note that this inequality is enough for contradicting that \(b\) is finite. In fact, (13) implies that \(\dot{\gamma}([0, b])\) is bounded in \(TM_0\) and, being \((M_0, g_0)\) complete, Lemma 2.1 is applicable because of the completeness of \(M_0\). Hence, \(\gamma\) can be extended to \(b\) in contradiction with its maximality assumption.

In order to prove that \(v(t)\) is bounded in \([0, b)\), taking any \(t \in [0, b)\) by using equation (E) and estimates (2) and (12) we have

\[
\frac{dv}{dt}(t) = g\left(D_{\gamma} \dot{\gamma}(t), \dot{\gamma}(t)\right) + g\left(\nabla_{M_0} V(\gamma(t), t), \dot{\gamma}(t)\right) + \frac{\partial V}{\partial t}(\gamma(t), t)
\]

\[= g(F_{\gamma(t), t}\dot{\gamma}(t), \dot{\gamma}(t)) + \frac{\partial V}{\partial t}(\gamma(t), t)
\]

\[= g(S_{\gamma(t), t}\dot{\gamma}(t), \dot{\gamma}(t)) + \frac{\partial V}{\partial t}(\gamma(t), t)
\]

\[\leq N_T u(t) + A_T (V(\gamma(t), t) - B_T).
\]

Whence, \(A_T^* \in \mathbb{R}\) exists such that

\[
\frac{dv}{dt}(t) \leq A_T^* v(t) \quad \text{for all} \quad t \in [0, b). \quad (14)
\]

On the other hand, if we consider the linear equation

\[
w'(t) = A_T^* w(t), \quad (15)
\]

let \(v_0 = v_0(t)\) be the unique (global) solution of (15) satisfying the initial condition \(v_0(0) = v(0)\), with \(v(0) \geq 1\) from (12). Thus, from (14) and Lemma 2.2 with \(\varphi(s) = A_T^* s\) and \(a = 1\), we have that \(v(t) \leq v_0(t)\) for all \(t \in [0, b)\), with \(v_0(t)\) bounded in \([0, b]\); whence, \(v(t)\) is bounded in \([0, b]\).

Conversely, let us assume that \(\gamma\) is not backward complete in \((-b, 0]\) with \(b < +\infty\), then we can consider \(T > b\) and \(\hat{\gamma}(t) := \gamma(-t)\) in \([0, b)\). From
the lower boundedness of $S_{\inf}(t)$ in $[-T, T]$ and the estimate on $-\frac{\partial V}{\partial t}$ along finite times, we have

$$\frac{dv}{dt}(-t) = -g(S_{\gamma(-t), -t} \dot{\gamma}(-t), \dot{\gamma}(-t)) - \frac{\partial V}{\partial t}(\gamma(-t), -t) \leq N_T u(-t) + A_T (V(\gamma(-t), -t) - B_T),$$

and we repeat the above argument for $\tilde{\gamma}(t)$.

Remark 2.5. Both in Theorem 2.3 and in Proposition 2.4 the assumption on the completeness of $(M_0, g_0)$ can be replaced by the condition “$V$ is proper”. In fact, in the above proof once we have proven that $v(t)$ is bounded in $[0, b)$, the properness of $V$ implies that $\dot{\gamma}([0, b))$ lies in a compact subset of $TM_0$, so $\gamma$ can be extended to $b$.

Remark 2.6. As commented in the Introduction, other completeness results on the inextensible trajectories of equation $(E)$ as well as their comparison with Theorem 2.3 can be found in [5].

3 Geodesic completeness of GPW

3.1 Plane waves and their generalizations

A parallely propagated wave spacetime, or a pp–wave in brief, is a relativistic spacetime $(\mathbb{R}^4, ds^2)$ where the Lorentzian metric $ds^2$ has the form

$$ds^2 = dx^2 + dy^2 + 2dudv + H(x, y, u)du^2,$$

being $(x, y, u, v)$ the natural coordinates of $\mathbb{R}^4$ and $H : \mathbb{R}^3 \to \mathbb{R}$ a non–zero smooth function. If the expression of $H$ is quadratic in $x, y$, i.e.,

$$H(x, y, u) = f_1(u)x^2 - f_2(u)y^2 + 2f(u)xy,$$  \hspace{1cm} (16)

for some smooth real functions $f_1$, $f_2$ and $f$, then the spacetime is called plane wave, and, in particular, an (exact plane fronted) gravitational wave if $f_1 \equiv f_2$ (for example, see [2]).

Since the pioneer papers dealing with gravitational waves [3, 8], these spacetimes have been widely studied by many authors (see [4] and references therein or the summary in [15]) not only for their geometric interest but above all for their physical interpretation. In fact, as explained in [12], a gravitational wave represents ripples in the shape of spacetime which propagate across spacetime, as water waves are small ripples in the shape of
the ocean’s surface propagating across the ocean. The source of a gravitational wave is the motion of massive particles; in order to be detectable, very massive objects under violent dynamics must be involved (binary stars, supernovas, gravitational collapses of stars...). With more generality, pp–waves may also taken into account the propagation of non–gravitational effects such as electromagnetism.

Here, we focus only on the property of geodesic completeness. In particular, we add further information to the study of the geometric properties for the family of generalized plane waves, already developed in [4, 5, 9, 10].

The key fact is that the geodesic completeness of a pp–wave reduces to the completeness of the inextensible trajectories that are solutions of the second order differential equation (E₀) when \((M₀, g₀) = \mathbb{R}²\). However, this last restriction is not important and, following [9], the classical notion of pp–wave can be generalized as follows:

**Definition 3.1.** A Lorentzian manifold \((M, g)\) is called **generalized plane wave**, briefly **GPW**, if there exists a connected \(n\)–dimensional Riemannian manifold \((M₀, g₀)\) such that \(M = M₀ × \mathbb{R}²\) and

\[
g = g₀ + 2dudv + \mathcal{H}(x, u)du², \tag{17}
\]

where \(x \in M₀\), the variables \((u, v)\) are the natural coordinates of \(\mathbb{R}²\) and the smooth function \(\mathcal{H} : M₀ × \mathbb{R} → \mathbb{R}\) is such that \(\mathcal{H} \not≡ 0\).

### 3.2 Application to geodesic completeness

In order to investigate the properties of geodesics in a GPW, it is enough studying the behavior of the Riemannian trajectories under a suitable potential \(V\). In particular, the problem of geodesic completeness is fully reduced to a purely Riemannian problem: the completeness of the inextensible trajectories of particles moving under the potential \(V(x, u) = -\frac{1}{4} \mathcal{H}(x, u)\) as the following result shows (see [4, Theorem 3.2] for more details).

**Theorem 3.2.** A GPW is geodesically complete if and only if \((M₀, g₀)\) is a complete Riemannian manifold and the inextensible trajectories of

\[
\frac{D\dot{\gamma}}{dt} = \frac{1}{2} \nabla^{M₀} \mathcal{H}(\gamma(t), t) \tag{E₀^*}
\]

are complete.

Now, we can use Theorem 2.3 to obtain the completeness of the inextensible trajectories of equation \((E₀^*)\). Then, the following result on the geodesic completeness on GPW can be stated:
Corollary 3.3. Let $M = M_0 \times \mathbb{R}^2$ be a GPW such that $(M_0, g_0)$ is a geodesically complete Riemannian manifold and $H : M_0 \times \mathbb{R} \to \mathbb{R}$ is a smooth function. If there exist two continuous functions $\alpha_0, \beta_0 : \mathbb{R} \to \mathbb{R}$ such that

$$H(x, u) \leq \beta_0(u) \quad \text{and} \quad \left| \frac{\partial H}{\partial u}(x, u) \right| \leq \alpha_0(u)(\beta_0(u) - H(x, u))$$

for all $(x, u) \in M_0 \times \mathbb{R}$, then $(M, g)$ is geodesically complete.

We emphasize that other results on autonomous and non-autonomous potentials can be translated into results of geodesic completeness of GPW. So, as a consequence of [5, Corollary 3.6] we have:

Corollary 3.4. A GPW with complete $(M_0, g_0)$ is geodesically complete if $\nabla^M H$ grows at most linearly in $M$ along finite times.

Remark 3.5. The particular case of this corollary for pp–waves (i.e. its application for $(M_0, g_0) = \mathbb{R}^2$) was discussed in [5], and it has a clear interpretation: not only classical plane waves are geodesically complete but also each pp–wave such that its coefficient $H$ behaves qualitatively as the one of a plane wave, are. This can be understood as a result of stability of the completeness of plane waves in the class of all pp–waves. So, Corollary 3.4 also ensures stability of completeness in the class of generalized plane waves.

Even though the physical interpretation of Corollary 3.3 is not so clear, it is logically independent of Corollary 3.4 (a discussion as the one below Proposition 3.7 in [5] also holds here). This shows that the application of the techniques are not exhausted and, under motivated assumptions, further results could be obtained.

References

[1] R. Abraham and J. Marsden, Foundations of Mechanics, 2nd ed. (6th printing), Addison–Wesley Publishing Co., Boston MA, 1987.

[2] J.K. Beem, P.E. Ehrlich and K.L. Easley, Global Lorentzian Geometry, 2nd Ed., Monogr. Textbooks Pure Appl. Math. 202, Marcel Dekker Inc., New York, 1996.

[3] H. Brinkmann, Einstein spaces which are conformally mapped on each other, Math. Ann. 94 (1925), 119–145.
[4] A. Candela, J.L. Flores and M. Sánchez, On general plane fronted waves. Geodesics, *Gen. Relativity Gravitation* **35** (2003), 631–649.

[5] A.M. Candela, A. Romero and M. Sánchez, Completeness of the trajectories of particles coupled to a general force field. [arXiv:1202.0523v1 [math.DS]].

[6] A. Candela and M. Sánchez, Geodesics in semi–Riemannian manifolds: geometric properties and variational tools, In: *Recent Developments in pseudo–Riemannian Geometry* (D.V. Alekseevsky & H. Baum eds.), Special Volume in the ESI–Series on Mathematics and Physics, EMS Publ. House, Zürich, 2008, 359–418.

[7] D.G. Ebin, Completeness of Hamiltonian vector fields, *Proc. Amer. Math. Soc.* **26** (1970), 632–634.

[8] A. Einstein and N. Rosen, On gravitational waves, *J. Franklin Inst.* **223** (1937), 43–54.

[9] J.L. Flores and M. Sánchez, Causality and conjugate points in general plane waves, *Classical Quantum Gravity* **20** (2003), 2275–2291.

[10] J.L. Flores and M. Sánchez, The causal boundary of wave–type spacetimes, *J. High Energy Phys.* **03** (2008), 036–079.

[11] W.B. Gordon, On the completeness of Hamiltonian vector fields, *Proc. Amer. Math. Soc.* **26** (1970), 329–331.

[12] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation*, W.H. Freeman & Co., San Francisco, 1973.

[13] B. O’Neill, *Semi–Riemannian Geometry with Applications to Relativity*, Pure Appl. Math. **103**, Academic Press Inc., New York, 1983.

[14] A. Weinstein and J. Marsden, A comparison theorem for Hamiltonian vector fields, *Proc. Amer. Math. Soc.* **26** (1970), 629–631.

[15] U. Yurtsever, Colliding almost–plane gravitational waves: colliding plane waves and general properties of almost–plane–wave spacetimes, *Phys. Rev. D* **37** (1988), 2803–2817.
Università degli Studi di Bari “Aldo Moro”,
Via E. Orabona 4, 70125 Bari, Italy
e-mail: candela@dm.uniba.it

Alfonso Romero† and Miguel Sánchez‡
Departamento de Geometría y Topología,
Facultad de Ciencias, Universidad de Granada,
Avenida Fuentenueva s/n,
18071 Granada, Spain
e-mail addresses: †aromero@ugr.es, ‡sanchezm@ugr.es