Weak singularity dynamics in a nonlinear viscous medium

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Abstract

We consider a system of nonlinear equations which can be reduced to a degenerate parabolic equation. In the case $x \in \mathbb{R}^2$ we obtained necessary conditions for the existence of a weakly singular solution of heat wave type (codim sing supp = 1) and of vortex type (codim sing supp = 2). These conditions have the form of a sequence of differential equations and allow one to calculate the dynamics of the singularity support. In contrast to the methods used traditionally for degenerate parabolic equations, our approach is not based on comparison theorems.

Key words: degenerate parabolic equations, singularities, heat wave, vortex.

MSC: 35K65, 35D05.

1 Introduction

We consider the system of equations arising in problems of water purification:

$$\frac{\partial c}{\partial t} = D\Delta c - D\kappa \text{div} (c\text{div} (\text{div} u - 3\beta c)), \quad (1.1)$$

$$\Delta u = (3\omega - 1)\beta \nabla c, \quad (1.2)$$

which describes diffusion in media with nonlinear viscosity. Here $c \geq 0$ is a scalar function, $u$ is a vector, $x \in \Omega \subset \mathbb{R}^n$, $t > 0$, $D, \kappa, \beta > 0$ and $\omega > 1$ are several constants.

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We understand all derivatives in (1.1), (1.2) in the weak sense, consider the case \( n = 2 \), and obtain necessary conditions for the existence of a solution of Eqs. (1.1) and (1.2) with a weak singularity. These conditions, which allow us to calculate the dynamics of a singularity, are obtained by using the asymptotic expansion of the solution with respect to smoothness.

Prior to studying (1.1), (1.2), we consider the model example

\[
\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial x^2} u^2, \quad x \in \mathbb{R}^1.
\]  

One can readily see that a necessary and sufficient condition for the existence of a solution finite in \( x \) is satisfied for (1.3) (e.g., see [1, 24] and the references therein). On the other hand, from the viewpoint of distributions, only the elements of \( D' \) that admit multiplication can be singular solutions of a nonlinear equation. As is well known [18], in the one-dimensional case this restriction leads to an algebra with generators 1, \( \theta(x) \), \( \varepsilon \delta(x) \) and \( x^\lambda \pm \), \( \lambda > 0 \).

Here 1 stands for all smooth functions, \( \varepsilon \delta(0) = 1 \) and \( \varepsilon \delta(x) = 0 \) for \( x \neq 0 \),

\[
\theta(x) = \begin{cases} 
0, & x < 0 \\
1, & x > 0
\end{cases}, \quad x^\lambda_+ = \begin{cases} 
0, & x \leq 0 \\
x^\lambda, & x > 0
\end{cases}, \quad x^\lambda_- = \begin{cases} 
|x|^\lambda, & x \leq 0 \\
0, & x \geq 0
\end{cases},
\]

and \( \theta^2 = \theta, \ (x^\lambda_\pm)^2 = x^{2\lambda}_\pm \), and \( \theta(x)x^\lambda_+ = x^\lambda_+ \).

One can readily verify that both the Heaviside function and \( \varepsilon \delta \) cannot satisfy Eq. (1.3). Nevertheless, if we set \( u = (x - a(t))^\lambda_\pm \), then this singularity is admissible for \( \lambda = 1 \). Such solutions of degenerate parabolic equations are well known and are called the heat wave. In particular, the simplest exact solution of Eq. (1.3) is the one-parameter family (e.g., see [24]):

\[
u = (6t)^{-1}(x + \eta t^{1/3})^\lambda_+ (x - \eta t^{1/3})^-_+, \quad \eta = \text{const} > 0, \quad t > 0.
\]  

A natural multidimensional analog of the heat wave \( (x - a(t))^\lambda_\pm \) is the function \( S_\pm: S_+ = S \) for \( S \geq 0 \) and \( S_+ = 0 \) for \( S \leq 0 \), where \( S = S(x, t) \) is a nondegenerate smooth function. The fact that \( S \) is nondegenerate means that \( \nabla S|_{S=0} \neq 0 \) for any chosen \( t \), and thus the surface \( \Gamma_t = \{ x, S(x, t) = 0 \} \) is of codimension 1. Solutions of this type for a wide class of degenerate equations have been studied thoroughly (e.g., see [1, 7, 10, 24]). In particular, it is known that, starting from some time, the degeneration surface \( \Gamma_t \) becomes differentiable [1, 5, 13, 22].
Another possible generalization of \((x - a(t))_+\) to the multidimensional case is a singularity of the form \(|x - a(t)|, x, a \in \mathbb{R}^n\). Solutions of such form, whose singular support is of codimensions \(> 1\), are called vortex type solutions, since they are used for the mathematical description of a typhoon eye motion [4]. It is known that vortex type solutions exists for shallow water equations [4]. However, as far as we know, the existence problem for such solutions of parabolic equations has not been discussed.

In this paper we obtain necessary conditions under which system (1.1), (1.2) have singular solutions of both types mentioned above.

We study system (1.1), (1.2) by using the asymptotic expansion of the solution with respect to smoothness. For linear equations, the idea to expand the solution with respect to smoothness was proposed and developed by R. Courant, D. Ludwig, V. P. Maslov, V. M. Babich and others. A further development of this idea applied to quasilinear equations and special solutions lying in \(D'\) and admitting multiplication was performed by many authors [7, 8, 23] starting from Maslov’s paper [19]. A possibility to study a more general class of nonsmooth solutions of quasilinear equations [2, 3, 6, 9, 11, 12, 21] appeared after the construction of algebras of generalized functions, which include distributions from \(D'\) (V. K. Ivanov, G. Colombeau, Yu. V. Egorov, and others).

A method based on asymptotic expansions of solutions with respect to smoothness is very fruitful, since in this method the maximum principle and the comparison theorems are not supposed to be valid. On the other hand, the following significant drawback is typical: this method implies a system of model equations (the so-called Hugoniot type conditions), which is an infinite nontriangular sequence in the case of nonlinear equations.

For example, by setting

\[
u = a_1(t)(x - \varphi)_+ + a_2(t)(x - \varphi)_+^2 + \ldots \quad (1.5)
\]

in a neighborhood of the singularity support \(x = \varphi(t)\) and substituting (1.5) into (1.3), we obtain the system

\[
\dot{\varphi} = -a_1, \quad \dot{a}_1 = 4a_1a_2, \quad \dot{a}_2 = 3(2a_2^2 + 3a_1a_3), \ldots \quad (1.6)
\]

In the special case \(a_1(t_0) = \eta/(3t_0^{2/3}), a_2(t_0) = -1/(6t_0), a_i(t_0) = 0, i \geq 3,\) for \(t \geq t_0 > 0\) it follows from (1.6) that

\[
\varphi = -\eta t^{1/3}, \quad a_1 = \eta/3t^{2/3}, \quad a_2 = -1/6t, \quad a_i = 0, \quad i \geq 3.
\]
Thus we arrive at the exact solution (1.4) considered near the left-hand front \( x = -\eta t^{1/3} \).

In the general case system (1.6) is an infinite sequence of coupled equations. Of course, infinite sequences cannot be used for practical computations, hence we need to close them. If \( \text{codim sing supp } u = 1 \), then this can be performed by passing to a free boundary problem. In this case conditions of the free boundary follow from a local description of the solution near the singularity support and from the first Hugoniot type condition. For example, it follows from this procedure that in the region \( Q_t = \Omega_t \times \{ t > t_0 \} \), \( \Omega_t = \{ x, \varphi_-(t) < x < \varphi_+(t) \} \) we need to study Eq. (1.3) with the conditions

\[
\left. u \right|_{x = \varphi_{\pm} \mp 0} = u \left|_{x = \varphi_{\pm} \mp 0} \right. = 0, \quad \dot{\varphi}_{\pm} = \frac{\partial u}{\partial \nu} \left|_{x = \varphi_{\pm} \mp 0} \right.
\]

(1.7)

and a natural initial condition. The first condition in (1.7), which follows from the structure of solution (1.5), implies that \( u \) and the flow \( uu_x \) are continuous on \( \partial \Omega_t \). The second condition in (1.7), which relates the boundary velocity to the limit value of the derivative of \( u \) along the outward normal \( \nu \), is exactly the first Hugoniot type condition in (1.6). In this statement of the problem we can ignore all other conditions in (1.6), since they are automatically satisfied due to the fact that the free boundary problem has a solution.

From the ideological viewpoint, this method of closing system (1.6) is similar to closing the Hugoniot type conditions for the shock wave by passing to the classical statement of the problem describing the shock wave motion \([8]\). However, here we obtain a problem that differs from the classical one-phase Stefan problem (see \([20]\) and \([14, 25]\) for related problems) by degeneration of the parabolic operator on the free boundary.

Another possible approach is to truncate the sequence of Hugoniot type conditions. By choosing a number \( N > 0 \) and setting all “extra” function equal to zero, we can calculate the dynamics of a singularity by using the first \( N \) Hugoniot type conditions. However, in this cases we need to justify this procedure. In general, this problem, which is similar to the well-known problem of truncating Bogolyubov–Born–Green–Kirkwood–Yvon chains, remains open. Nevertheless, the first result in justifying the truncation procedure has been obtained \([8]\), and numerical results show a good agreement between the exact solution and those obtained by using truncated sequences \([4, 23]\) for a reasonably large time interval.
In the case codim sing supp $u > 1$ the truncation procedure is apparently the only real method for calculating the singularity dynamics. Indeed, in this case, instead of the free boundary problem, we must consider a nonlinear equation in a domain with a remote set of codimension $> 1$. The position of this set, on which we pose additional conditions, varies in time and is to be determined. As far as we know, such problems have not been studied till recently.

Now let us consider system (1.1), (1.2) and write it in a simple form. To this end, we apply (in the $D'$-sense) the operator div to Eq. (1.2). We assume that $c$ has a singularity of an above-listed type and obtain the relation

$$\Delta (\text{div } u - \beta (3\omega - 1)c) = 0,$$

which holds for all $x$ including the singularity points. Hence

$$\text{div } u = \beta (3\omega - 1)c + w,$$

where $w$ is an arbitrary harmonic function. Now Eq. (1.1) acquires the form

$$\frac{\partial c}{\partial t} = D\{ \text{div } ((1 - \kappa \beta (3\omega - 4)c)\nabla c) - \kappa \langle \nabla c, \nabla w \rangle \}.$$  \hspace{1cm} (1.8)

We assume that $\omega > 4/3$, since only in this case Eq. (1.8) is degenerate and, correspondingly, can have singular solutions bounded in the $C$-norm.

By changing the variables, we rewrite (1.8) in the compact form

$$\frac{\partial c}{\partial t} + \langle \nabla w, \nabla c \rangle = \Delta (c - \mu c^2), \hspace{1cm} \mu = \text{const} > 0.$$  \hspace{1cm} (1.9)

Our goal is to derive necessary conditions for the existence of a solution with a special local structure. Therefore, we abstract our consideration from the initial and boundary (on the outer boundary $\partial \Omega$) conditions. However, it is clear that in this case the initial data cannot be arbitrary but must satisfy requirement that follow from the Hugoniot type conditions obtained below.

In the present paper we restrict ourselves to a discussion of Eq. (1.9). However, the method proposed here for constructing singular solutions can be easily generalized to a wide class of degenerate parabolic equations and systems.
2 Heat wave dynamics

In $D'$ it readily follows from Eq. (1.9) that for all $t$ the wave front $\Gamma_t = \{x, S(x,t) = 0\}$ lies on the degeneration surface $\Sigma = \{(x,y), c = 1/2\mu\}$ corresponding to Eq. (1.9). This fact implies that the asymptotic expansion of the solution with respect to smoothness in the neighborhood of $\Gamma_t$ has the form

$$c = (2\mu)^{-1} + a_1 S_+ + a_2 S_+^2 + \ldots$$  \hspace{1cm} (2.1)

Here $S \in C^\infty$ is a function such that $\nabla S|_{\Gamma_t} \neq 0$, $a_1 < 0$, the coefficients $a_i = a_i(x,t)|_{\Gamma_t} \in C^\infty$ have the meaning of the normal derivatives of $c$ on $\Gamma_t$, and $\partial a_i/\partial \nu|_{\Gamma_t} = 0$, where $\nu = -\nabla S/|\nabla S|$ is the outward normal. Just as in the case of the model equation (1.3), the structure of the solution guarantees that the flow $(1 - 2\mu c)\partial c/\partial \nu$ is continuous on $\Gamma_t$. To find $S$ and $a_i$, we substitute (2.1) into (1.9) and expand $\nabla w$ in the Taylor series on $\Gamma_t$:

$$\nabla w = V_0 + V_1 S + V_2 S^2 + \ldots$$

Now by setting the coefficients of $\theta(S), S_+, S_+^2, \ldots$ equal to zero, we obtain the infinite nontriangular system

$$\{ S_t + \langle V_0, \nabla S \rangle + 2\mu|\nabla S|^2 a_1 \} \bigg|_{S=0} = 0,$$  \hspace{1cm} (2.2)

$$\{ a_{it} + \langle V_0, \nabla \psi \rangle a_{i\xi} + 2(i + 1)^2 \mu|\nabla S|^2 a_1 a_{i+1} + F_i \} \bigg|_{S=0} = 0, \quad i = 1, 2, \ldots$$

Here $\xi = \psi(x,t)$ is a parameter along $\Gamma_t$ that is determined in each chart as the solution of the equation $S = 0$, and $F_i$ are functions of $S, a_1, \ldots, a_i$ and their derivatives. In particular, we have

$$F_1 = a_1 \langle V_1, \nabla S \rangle + 2\mu a_1^2 \Delta S,$$

$$F_2 = 2a_2 \langle V_1, \nabla S \rangle + a_1 \langle V_1, \nabla \psi \rangle + a_1 \langle V_2, \nabla S \rangle + \mu\{6a_1 a_2 \Delta S + 12a_2^2 |\nabla S|^2 + |\nabla \psi|^2 (a_1^2)_{\xi \xi} + \Delta \psi (a_1^2)_{\xi} \}.$$  

Equations (2.2) are written in the form that is inconvenient for analysis and numerical calculations, since in (2.2) we need to calculate the trace on $\Gamma_t$. This can be avoided by taking into account the geometry of the front $\Gamma_t$. For example, if for $t \in [0, T)$ the front can be uniquely projected on the $x_2$-axis, then we can set $S = x_1 - \varphi(x_2,t)$, obtain $\nabla S = (1, -\varphi_{x_2})^T$, $\xi = x_2$, ...
\( a_i = a_i(x_2, t) \), and rewrite Eq. (2.2) as

\[
\begin{align*}
\varphi_t + w_{x_2} |_{x_1 = \varphi} \varphi_{x_2} &= 2\mu(1 + \varphi^2) a_1 + w_{x_1} |_{x_1 = \varphi}, \\
a_{1t} + w_{x_2} |_{x_1 = \varphi} a_{1x_2} + a_1 \langle V_1 |_{x_1 = \varphi}, \nabla S \rangle \\
&+ 2\mu(-a_1^2 \varphi_{x_2 x_2} - 2(a_1^2)_{x_2} \varphi_{x_2} + 4(1 + \varphi^2) a_1 a_2) = 0, \ldots,
\end{align*}
\]

If \( \Gamma_t \) is a closed smooth curve without self-intersections, then we can avoid a local description of the front by setting \( S = t - \Phi(x) \). Then \( a_i = a_i(x) \) and, instead of (2.2), we obtain

\[
1 - \langle \nabla \Phi, \nabla w \rangle |_{t = \Phi} + 2\mu a_1 |\nabla \Phi|^2 = 0,
\]

\[
-\langle \nabla a_1, \nabla w \rangle |_{t = \Phi} + a_1 \langle V_1, \nabla \Phi \rangle \\
+ 2\mu a_1 \{ 4\langle \nabla \Phi, \nabla a_1 \rangle + a_1 \Delta \Phi - 4a_2 |\nabla \Phi|^2 \} = 0, \ldots,
\]

where \( V_1 = (\nabla w_i) |_{t = \Phi} \).

By writing the Hugoniot conditions in the form (2.3) or (2.4), we arrive at more trivial problem of posing the initial conditions. Namely, for (2.3) we specify the values of \( \varphi_1, a_1, \ldots \) for \( t = 0 \), and for (2.4) we specify the values \( \Phi = 0 \) and \( a_1 = a_1^0, \ldots \) on the initial curve \( \Gamma_0 \) whose position is assumed to be known in advance.

Thus we arrive at the statement.

**Theorem 1** For the existence of a heat wave type solution of Eq. (1.9), it is necessary that the Hugoniot type conditions (2.2) be satisfied on the front \( \Gamma_t \).

Note that \( a_1 |\nabla S| |_{\Gamma_t} = -\partial c / \partial \nu |_{\Gamma_t} \) and \( v_\nu = S_t / |\nabla S| \) is the velocity of the boundary \( \Gamma_t \) along its outward normal. Therefore, the first Hugoniot type condition can be written in the form

\[
v_\nu = 2\mu \frac{\partial c}{\partial \nu} |_{\Gamma_t} - \frac{\partial w}{\partial \nu} |_{\Gamma_t}.
\]

As was already noted in Introduction, condition (2.5), together with the condition \( c |_{\Gamma_t} = (2\mu)^{-1} \), allows us to reduce calculations of the heat wave to a one-phase free boundary problem. Obviously, the solvability of this problem implies that all Hugoniot type conditions are satisfied.
Remark If instead of (1.9) we consider the equation
\[ \frac{\partial c}{\partial t} + \langle V, \nabla c \rangle = \langle \nabla, k(c) \nabla \rangle, \]
where \( k = k_0 c^\gamma (1 + O(c)) \) as \( c \to 0 \), \( k_0 > 0 \), and \( \gamma > 0 \), then by analogy with (2.1) the asymptotic expansion of the solution with respect to smoothness has the form
\[ c = a_1 S_+^{\gamma-1} + a_2 S_+^{\gamma-1+1} + \ldots \]
In this case the first Hugoniot type condition is similar to (2.2), namely,
\[ \left\{ S_t + \langle V, \nabla S \rangle - \frac{k_0}{\gamma} |\nabla S|^2 a_1 \right\}_{S=0} = 0. \]

However, now the closure of the sequence of Hugoniot type conditions leads to a one-phase Stefan type problem, which consists of the original equation considered in the domain \( Q_t = \Omega_t \times \{ t > t_0 \} \), \( \Omega_t = \{ x, S(x,t) > 0 \} \) with the initial condition and the boundary conditions on the free boundary \( \Gamma_t = \partial \Omega_t \):
\[ c\big|_{\Gamma_t} = 0, \quad v = \langle V, \nu \rangle\big|_{\Gamma_t} - k_0 c^{\gamma-1} \frac{\partial c}{\partial \nu}\big|_{\Gamma_t}, \]
where \( f\big|_{\Gamma_t} \) is understood, just as in (2.2), as the limit obtained by passing to \( \Gamma_t \) along the inward normal. Obviously, we can rewrite the last term in the Stefan condition in the form \(-k(c)c^{-1} \partial c/\partial \nu\big|_{\Gamma_t}\).

3 Dynamics of a vortex type singularity

The solution of Eq. (1.9) with a vortex type singularity can be written in the form
\[ c = c^0(x,t) + \sqrt{S(x,t)} c^1(x,t), \]
where \( S, c^0, c^1 \) are some smooth function. We assume that for each fixed \( t \)
\[ S \geq 0, \quad \nabla S\big|_{S=0} = 0, \quad S''\big|_{S=0} > 0, \]
and moreover, \( T = \bigcup_t \Gamma_t \subset \mathbb{R}_+ \times \mathbb{R}^2 \) is a curve in the \( C^\infty \)-class. Here \( \Gamma_t = \{ x, S(x,t) = 0 \} \) is the singular support of the solution and \( S'' \) is the Hessian of \( S \). By \( a(t) \) we denote a vector-function, which describes the position of
a singularity at each time instant $t$, i.e., we have $\Gamma_t = \{x, x + a(t) = 0\}$.

We perform the change $x' = x + a(t)$. Then, omitting the prime on the new variable, instead of Eq. (1.9), we obtain the equation

$$\dot{c} + \langle \dot{a} + \nabla w, \nabla c \rangle = \Delta(c - \mu c^2).$$

(3.3)

Here and in the following, the dot indicates the derivative with respect to $t$.

Substituting (3.1) into (3.3) and grouping smooth and singular terms, we obtain the relation

$$D^0 + S^{-3/2} D^1 = 0,$$

(3.4)

where

$$D^0 = \dot{c}^0 + \langle V, \nabla c^0 \rangle - \Delta G^0, \quad V = \dot{a} + \nabla w,$$

$$D^1 = -\kappa G^1 + 4S\left(\frac{1}{2} \langle \nabla S, V \rangle c^1 - \langle \nabla S, \nabla G^1 \rangle \right) + 4S^2(-\Delta G^1 + \dot{c}^1 + \langle V, \nabla c^1 \rangle), \quad G^0 = c^0 - \mu(c^0 + Sc^2),$$

$$G^1 = (1 - 2\mu c^0)c^1, \quad \kappa = 2S\Delta S - |\nabla S|^2.$$

The smooth functions $D^i$ satisfy relation (3.4) at the point $x = 0$ only if for any $N \geq 0$ we have

$$\frac{\partial^k D^0}{\partial x^\alpha} \bigg|_{x=0} = 0, \quad |\alpha| = k, \quad k \leq N,$$

(3.6)

$$\frac{\partial^k D^1}{\partial x^\alpha} \bigg|_{x=0} = 0, \quad |\alpha| = k, \quad k \leq N,$$

(3.7)

where, as usual, $\alpha = (\alpha_1, \alpha_2)$ is a multiindex.

We write the functions $S$ and $c^i$, $i = 0, 1$, as the formal Taylor expansions

$$c^i = \sum_{k=0}^{\infty} p_k^i(x, t), \quad S = \sum_{k=2}^{\infty} S_k(x, t),$$

(3.8)

$$p_k^i = \sum_{|\alpha|=k} c^i_\alpha(t) x^\alpha, \quad S_k = \sum_{|\alpha|=k} s_\alpha(t) x^\alpha.$$

In view of conditions (3.2), we assume that $s_{11} = 0, s_{20} > 0$, and $s_{02} > 0$.

In what follows, we essentially use the fact that the vector-function $V$ is of a special structure. Namely, since $w$ is a harmonic function, we have

$$w = \sum_{k=0}^{\infty} W_k(x, t), \quad W_k = \sum_{|\alpha|=k} \omega_\alpha(t) x^\alpha,$$

(3.9)
where $W_k$ are harmonic polynomials. Therefore, in particular, $W_0 = W_0(t)$ and $W_1(x, t)$ are arbitrary functions, and

$$W_2 = \omega_{20}(x_1^2 - x_2^2) + \omega_{11}x_1x_2,$$

$$W_3 = \omega_{30}x_1(x_1^2 - 3x_2^2) + \omega_{03}x_2(x_2^2 - 3x_1^2),$$

where $\omega_\alpha = \omega_\alpha(t)$ are arbitrary smooth functions. Thus in the Taylor expansion

$$V = \sum_{k=0}^{\infty} V_k(x, t), \quad V_k = \sum_{|\alpha|=k} v_\alpha(t)x^\alpha,$$

we must take into account that $V_0 = \dot{a} + \nabla W_1$ and $V_i = \nabla W_{i+1}$, $i \geq 1$.

Substituting expansions (3.8)–(3.11) into (3.6) and (3.7) and setting the coefficients of $x^\alpha$ equal to zero, we obtain an infinite system of equations. One can readily see that this system is not triangular, i.e., the first $N$ equations contain $M > N$ unknowns. The inequality $M \geq N$ is necessary for the existence (at least, for small $t$) of a solution of these equations, but, in general, is not sufficient. The matter is that the compatibility conditions for these equations contain less than $M$ unknown functions. Since we do not know in advance whether the equations obtained are solvable, a detailed analysis of these equations is required.

To avoid too cumbersome formulas, we shall successively analyze the equations obtained for $k = 0, 1, \ldots$

We consider (3.7) for $k = 2$. Since

$$\kappa = \sum_{k=2}^{\infty} \kappa_k(x, t), \quad \kappa_2 = 2S_2\Delta S_2 - |\nabla S_2|^2 = 4s_{20}s_{02}x_2^2,$$

we have the relation

$$s_{20}s_{02}c_0^1(1 - 2\mu c_0^0) = 0.$$

In view of (3.2) and the natural assumptions that

$$c_0^1 \neq 0$$

(otherwise, the singularity of the solution (3.1) becomes weaker than $|x|$), we derive the relation

$$c_0^0 = 1/2\mu,$$
which readily implies that the condition \( \mu > 0 \) is necessary for the existence of a physically meaningful \( (c \geq 0) \) solution of the form (3.1).

Now we consider (3.7) for \( k = 3 \). Since \( G^1_0 = 0 \) in view of (3.14), we have the relation

\[
\kappa_2 G^1_1 + 4 S_2 (\nabla S_2, \nabla G^1_1 - \frac{1}{2} V_0 c^1_0) = 0. \tag{3.15}
\]

Here and in the following, relations of the form \( P_k(x,t) \equiv \sum |\alpha| = k P_\alpha(t) x^\alpha = 0 \) mean that the coefficients \( P_\alpha(t), |\alpha| = k \), are equal to zero. We also use expansions of the functions \( G^i \) that are similar to (3.8):

\[
G^i = \sum_{k=0}^{\infty} G^i_k(x,t), \quad G^i_k = \sum_{|\alpha| = k} g_\alpha(t) x^\alpha, \quad i = 1, 2. \tag{3.16}
\]

In particular, by taking into account (3.14), we have

\[
G^0_2 = -\mu (2 c^1_0 p^1_2 + p^0_1 + p^2_1), \quad G^1_0 = 0, \quad G^1_1 = -2 \mu c^1_0 p^1_1, \quad G^1_2 = -2 \mu (c^1_0 p^1_2 + p^1_1 p^1_1).
\tag{3.17}
\]

By substituting explicit formulas for \( \kappa_2, S_2, G^1_1 \) into (3.15) and matching the coefficients of like powers of \( x \), we obtain the four equations

\[
6 \mu c^0_\alpha + V_{0,\alpha} = 0, \quad 2 \mu (s^0_{02} / s^0_{20}) g^0_\alpha c^0_\alpha + V_{0,\alpha} + 4 \mu c^0_\alpha = 0, \tag{3.18}
\]

where \( \alpha \) attains the values \((1,0)\) and \((0,1)\), \( V_{0,\alpha} = V_{0,1} \) and \( q_\alpha = 1 \) for \( \alpha = (1,0) \), \( V_{0,\alpha} = V_{0,2} \) and \( q_\alpha = -1 \) for \( \alpha = (0,1) \), and \( V_{0,i} \) are components of the vector \( \mathbf{V}_0 = (V_{0,1}, V_{0,2}) \).

Obviously, Eqs. (3.18) are compatible only for \( s^0_{20} = s^0_{02} \). Without loss of generality, we obtain

\[
s^0_{20} = s^0_{02} = 1. \tag{3.19}
\]

The (3.18) implies the relation

\[
V_0 + 6 \mu \nabla p^0_1 = 0 \tag{3.20}
\]

between the velocity \( \dot{a} \) of the singularity and the coefficients in expansions (3.8) and (3.9).

Next, from (3.6) with \( k = 0 \) we obtain

\[
\dot{c}_0^0 + (V_0, \nabla p^0_1) = \Delta G^0_2.
\]
In view of (3.14), (3.19), and (3.20), this immediately implies
\[ c_0^{12} = \| \nabla p_1^0 \|^2. \] (3.21)

Now let us consider (3.7) for \( k = 4 \). In view of (3.19), we have
\[ x^2 R_2 + S_3 L_1 = 0 \] (3.22)

with homogeneous polynomials \( R_2 \) and \( L_1 \) of degree 2 and 1, respectively.

One can readily see that the compatibility condition for the equations obtained from (3.22) is
\[ \Lambda \tilde{t}_1 = 0, \]
where \( \Lambda \) is the \( 2 \times 2 \) matrix with coefficients \( \Lambda_{11} = -\Lambda_{22} = s_{21} - s_{03}, \Lambda_{12} = \Lambda_{21} = s_{30} - s_{12} \) and \( \tilde{t}_1 = (l_{10}, l_{01}) \), where \( l_\alpha \) are coefficients in the polynomial \( L_1 \). The choice \( \tilde{t}_1 = 0 \) implies that the solution constructed is trivial. Therefore, \( \det \Lambda = 0 \), and hence, in turn,
\[ S_3 = x^2 Q_1(x, t), \] (3.23)
where \( Q_1 \) is a homogeneous polynomial of the first order.

By using (3.23) and the obvious identity
\[ \langle x, \nabla p_n \rangle = n p_n, \] (3.24)
we transform (3.22) as follows:
\[ 5 p_2^0 + Q_1 p_1^0 + \frac{2}{c_0} p_1^1 p_1^0 + \frac{1}{\mu} W_2 = \frac{1}{2} x^2 \left( \langle \nabla Q_1, \nabla p_1^0 \rangle + \frac{1}{\mu c_0^0} \theta_0 \right), \] (3.25)
where
\[ \theta_0 = -\dot{c}_0^1 + 6 \mu \langle \nabla p_1^0, \nabla p_1^1 \rangle + \Delta G_2^1. \]

Relation (3.25) implies three equations for the coefficients in expansions (3.8).

We combine these equations and, in particular, obtain the system
\[ A \nabla Q_1 = -\frac{2}{c_0} A \nabla p_1^1 - f, \] (3.26)
where \( A \) is the \( 2 \times 2 \) matrix with coefficients \( A_{11} = A_{22} = c_{01}^0, A_{12} = -A_{21} = c_{10}^0, f = (f_1, -f_2), f_1 = 5 c_{11}^0 + \omega_{11}/\mu, \) and \( f_2 = 5 (c_{20}^0 - c_{02}^0) + 2 \omega_{20}/\mu. \)
By using (3.21) and the assumption (3.13), we find from (3.26) the coefficients in the polynomial $Q_1$:

$$
\nabla Q_1 = \frac{2}{c_0} \nabla p_1 - \psi_0, \quad \psi_0 = A^{-1} f.
$$

(3.27)

Now the third equation obtained from (3.25) can be reduced to the form

$$
\dot{c}_0 + \frac{9}{2} \mu c_0 \Delta p_2^0 = 0.
$$

(3.28)

Next, we consider (3.6) for $k = 1$ and arrive at the relation

$$
p_1^0 - 2\mu \langle \nabla p_1^0, \nabla p_2^0 \rangle + \langle V_1, \nabla p_1^0 \rangle + 2\mu (p_1^0 \Delta p_2^0 - 4\psi_1) = 0,
$$

(3.29)

where $\psi_1$ is a homogeneous polynomial such that $\nabla \psi_1 = c_0^2 \psi_0$. One can readily see that the system of relations (3.21), (3.28), (3.29) is consistent only if the compatibility condition

$$
\langle \nabla p_1^0, B \nabla p_1^0 \rangle = 0
$$

(3.30)

is satisfied. Condition (3.30) can be readily obtained by differentiating (3.21) with respect to $t$ and calculating $\nabla \dot{p}_1^0$ with the help of (3.29). In (3.30)

$$
B = 2p_2'' + \frac{5}{2} I \Delta p_2^0 + 8f - \mu^{-1} W_2''.
$$

$I$ is the unit $2 \times 2$ matrix, the double prime indicates the Hessian of a function, and $f$ is the $2 \times 2$ matrix with coefficients $f_{11} = -f_{22}$ and $f_{12} = f_{21} = f_1$.

It follows from (3.30) that

$$
B \nabla p_1^0 = 7\sigma \nabla p_1^{0\perp},
$$

(3.31)

where $f^\perp = Tf$ is the vector orthogonal to $f$, $T$ is the rotation matrix, i.e., $T_{11} = T_{22} = 0$ and $T_{12} = -T_{21} = -1$, and $\sigma = \sigma(t)$ is an arbitrary scalar.

Let us transform (3.31). To this end, we note that $B$ can be rewritten in the form $B = 7(B^0 + \mu^{-1} W_2'')$, where $B^0$ is the matrix with coefficients $B_{11}^0 = 7c_{20} - 5c_{02}, B_{22}^0 = -5c_{20} + 7c_{02}, B_{12}^0 = B_{21}^0 = 6c_{11}$.

Moreover, the following relations hold: $W_2'' \nabla p_1^0 = D \bar{\omega}_2$ and $B^0 \nabla p_1^0 = B^1 \bar{C}_2 + c_{02} r$, where $\bar{\omega}_2 = (\omega_{20}, \omega_{11})^T$, $\bar{C}_2 = (c_{20}^0, c_{11}^0)^T$, and $r = (-5c_{10}^0, 7c_{01}^0)^T$. 

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$D$ and $B^1$ are the matrices with coefficients $D_{11} = 2D_{22} = 2c_{10}^0$, $D_{21} = -2D_{12} = -2c_{01}^0$, $B_{11}^1/7 = B_{22}^1/6 = c_{10}^0$, and $B_{12}^1/6 = -B_{21}^1/5 = c_{01}^0$. Therefore, we can rewrite (3.31) in the form

$$D\omega_2 = \mu(\sigma^T - B^0)\nabla p_1^0 = \mu\{\sigma\nabla p_1^{01} - B^0\nabla p_2 - c_{02}^{01}\}. \quad (3.32)$$

Both matrices $D$ and $B^1$ are nondegenerate in view of the assumption (3.13) ($\det D = 2c_{10}^2$ and $\det B^1 = 6(5c_{01}^2 + 2c_{10}^2)$ def 6d). However, we must take into account that any condition imposed on $W_2$ implies restrictions on the boundary values of $u$. So, avoiding the appearance of such restrictions and inverting $B^1$, we obtain from (3.30) the algebraic relations

$$c_{20}^0 = \frac{1}{d}\{ -2\sigma c_{10}^0 c_{01} + c_{02}^0(5c_{10}^2 + 7c_{01}^2) - 2c_{12}^0 \omega_{20}/\mu \}, \quad (3.33)$$

$$c_{11}^0 = \frac{1}{6d}\{ -\sigma(5c_{01}^2 - 7c_{10}^2) - 24c_{02}c_{10}c_{01} + 4c_{10}c_{01}\omega_{20}/\mu \}
- (7c_{10}^2 + 5c_{01}^2)\omega_{11}/\mu \}.$$

Obviously, since the higher-order terms of the expansion contain differential equations for the coefficients of $p_2^0$, we must prove the validity of relations (3.33).

First, we note that (3.32) allows us to simplify Eq. (3.29). Namely, we invert the matrix $D$, calculate $V_1$ and $\psi_1$, and, according to (3.21), set

$$c_{10}^0 = c_{10}^1 \sin \varphi, \quad c_{01}^0 = c_{01}^1 \cos \varphi, \quad (3.34)$$

where $\varphi = \varphi(t)$ is a new unknown function. Then after some calculations we transform (3.29) to the final form

$$\dot{\varphi} + 7\mu \sigma = 0. \quad (3.35)$$

To prove that (3.33) is consistent, we consider (3.6) for $k = 2$, which leads to the equation

$$p_2^0 + \langle V_0, \nabla p_3^0 \rangle + \langle V_1, \nabla p_2^0 \rangle + \langle V_2, \nabla p_1^0 \rangle = \Delta G_4^0, \quad (3.36)$$

and (3.7) for $k = 5$, which leads to the relation

$$6\mu c_{01}^1 S_4 p_1^0 + x^2 \mathcal{R}_3 = 0, \quad (3.37)$$

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where $\mathcal{R}_3$ is a homogeneous polynomial of the third order. By analogy with (3.23), we can readily verify that (3.37) implies the relation

$$S_4 = x^2 Q_2(x, t),$$

(3.38)

where $Q_2$ is a homogeneous polynomial of the second order. Then (3.37) acquires the form

$$-4c_0^1p_1^0Q_2 + \mathcal{R}_3 + x^2 \{c_0^1(\langle \nabla Q_2, \nabla p_1^0 \rangle - p_1^0 \Delta Q_2) - \mu^{-1}p_1^0 + \mathcal{L}_1 \} = 0,$$

(3.39)

where

$$\mathcal{R}_3 = -2\{4p_1^0p_1^1 + 7c_0^1p_3^0 + 7p_1^1p_2^0 + Q_1(p_1^0(5p_1^1 + c_0^1Q_1) + 11c_0^1p_2^0)$$

$$+ (W_2(p_1^1 + 2Q_1c_0^1) + 3W_3c_0^1/2)/\mu \},$$

$$\mathcal{L}_1 = Q_1(c_0^1(\nabla Q_1, \nabla p_1^0) + 2\theta_0/\mu) + |\nabla Q_1|^2c_0^1p_1^0/2$$

$$+ \mu^{-1}(\langle \nabla Q_1, \nabla G_2^1 - (p_1^1V_0 + c_0^1V_1)/2 \rangle$$

$$+ \mu^{-1}(\Delta G_3^1 - \langle V_0, \nabla p_2^0 \rangle - \langle V_1, \nabla p_1^0 \rangle).$$

Relation (3.39) is equivalent to four scalar equations. The compatibility condition for these equations is

$$\mathcal{A}q_2 = -q_02\nabla p_1^{0\perp} - F/4c_0^1,$$

where $\mathcal{A}$ is the same matrix as in (3.26), $q_2 = (q_20, q_{21})^T$, $q_\alpha$, $|\alpha| = 2$, are the coefficients in the polynomial $Q_2$, and $F$ is the vector with components $F_1 = r_03 - r_21$ and $F_2 = r_30 - r_12$, where $r_\alpha$ are coefficients in $\mathcal{R}_3$. Hence we find

$$q_20 = q_02 + \langle F, \nabla p_1^{0\perp} \rangle/4c_0^{13}, \quad q_{21} = -\langle F, \nabla p_1^0 \rangle/4c_0^{13},$$

(3.40)

and see that $q_02$ is arbitrary.

Now it follows from (3.40) that the other two relations that follow from (3.39) are

$$\nabla p_1^1 = \mu F - 6\mu c_0^1q_02\nabla p_1^0.$$  

(3.41)

Here $\mathcal{F}$ is the vector with components

$$\mathcal{F}_1 = r_30 + l_1^{10} - \langle F, 4c_{10}^0\nabla p_1^{0\perp} + c_{01}^0\nabla p_1^0 \rangle/4c_0^{12},$$

$$\mathcal{F}_2 = r_03 + l_1^{01} - \langle F, 2c_{01}^0\nabla p_1^{0\perp} + c_{10}^0\nabla p_1^0 \rangle/4c_0^{12}.$$
where $l^1_0$ are coefficients in $L_1$.

In turn, relations (3.40) allow us to rewrite Eq. (3.36) in the form

$$p_2^0 + M + 16q_{02}c_0^{12}x^2 = 0,$$

(3.42)

where the coefficients in the polynomial $M$ are independent of $q_{02}$.

We differentiate relation (3.32) with respect to $t$ and calculate the derivatives $\dot{p}_{01}^0$ and $\dot{p}_{02}^0$ according to (3.28), (3.34), (3.35), and (3.42). After elementary calculations, we obtain

$$(\dot{\sigma} - 7\mu\sigma\Delta p_2^0)\nabla p_{01}^0 + (32c_0^{12}q_{02} - 14\mu\sigma^2)\nabla p_1^0 = F^1.$$ (3.43)

Here $F^1 = \mu^{-1}\tilde{D}\omega^2 - \mathcal{M}\nabla p_0^0$ and $\mathcal{M}$ is the matrix with coefficients $\mathcal{M}_{11} = 7M_{20} - 5M_{02}$, $\mathcal{M}_{22} = 7M_{02} - 5M_{20}$, and $\mathcal{M}_{12} = \mathcal{M}_{21} = 6M_{11}$, where $M_\alpha$ are coefficients in the polynomial $M$.

Obviously, (3.43) implies the differential equation

$$\dot{\sigma} - 7\mu\sigma\Delta p_2^0 = \langle F^1, \nabla p_{01}^0 \rangle / c_0^{12}$$ (3.44)

and the relation

$$q_{02} = (7\mu\sigma^2 + \langle F^1, \nabla p_1^0 \rangle / 2c_0^{12}) / 16c_0^{12},$$ (3.45)

which guarantee that (3.33) and (3.36) are compatible.

In turn, relations (3.33) and (3.45) mean that (3.42) can readily be reduced to the scalar equation

$$\dot{c}_{02}^0 + 7\mu\sigma^2 + M_{02} + \langle F^1, \nabla p_1^0 \rangle / 2c_0^{12} = 0.$$ (3.46)

Let us summarize the preceding. Relations (3.14) and (3.20) mean that Eq. (1.9) degenerates on the singularity support (since we have $c = c_0^0 = 1/2\mu$ there). The construction of the first terms $p_1^0, p_2^0$ and $c_{01}^1, p_1^1$ in the asymptotic expansion of the solution with respect to smoothness (3.1), (3.8) is reduced to solving ordinary differential equations (3.28), (3.35), (3.41), (3.44), and (3.46) with regard to algebraic relations (3.21), (3.27), (3.33), (3.40), and (3.45).

We introduce the notation $y = (y_1, \ldots, y_4)$, where

$$y_1 = \sigma(t), \quad y_2 = c_{10}^1(t), \quad y_3 = c_{01}^1(t), \quad y_4 = c_{02}^0(t),$$
and \( z = (z_1, \ldots, z_7) \), where \( z_j \) is a function from \( c_\alpha^0(t), |\alpha| = 3 \), or from \( c_\beta^1(t), |\beta| = 2 \). Then we can rewrite the above equations as follows:

\[
\begin{align*}
\dot{a}_1 &= -6\mu c_0^1 \sin \varphi - \omega_{10}, \\
\dot{a}_2 &= -6\mu c_0^1 \cos \varphi - \omega_{01}, \\
\dot{\varphi} &= -7\mu y_1, \\
\dot{c}_0^1 &= \mu c_0^1 \frac{9}{d'} (y_1 \sin 2\varphi - 12y_4 + 2\omega_{20}/\mu), \\
\dot{y}_i &= P_i(y, \omega_{20}, \omega_{11}; \varphi, \mu) + c_0^1 L_i(\omega_{30}, \omega_{03}; \varphi, \mu) \\
&\quad + M_i(\omega_{20}, \dot{\omega}_{11}; \varphi, \mu) + c_0^1 N_i(z; \varphi, \mu), \quad i = 1, \ldots, 4.
\end{align*}
\]

Here \( d' = \cos 2\varphi - 6 \), \( P_i(\tau; \varphi, \mu) \) are homogeneous second-order polynomials in \( \tau \), \( L_i(\xi; \varphi, \mu) \), \( M_i(\xi; \varphi, \mu) \), and \( N_i(z; \varphi, \mu) \) are homogeneous first-order polynomials in \( \xi \) and \( z \). The coefficients of these polynomials are uniformly bounded smooth functions of \( \varphi \) and the parameter \( \mu \geq \text{const} > 0 \).

The right-hand sides in (3.47) can readily be calculated by using computers and “Mathematica” software. Nevertheless, the explicit formulas thus obtained are too cumbersome, and we do not write them here. We also note that system (3.47) is nonclosed, since it contains coefficients of the polynomials \( p_0^0 \) and \( p_1^1 \).

Finally, it was proved that the function \( S \) is of the form \( S = x^2(1 + Q_1 + Q_2 + \ldots) \), and thus implies that the assumptions (3.2) are satisfied.

We consider the higher-order terms in the expansion with respect to smoothness following the above scheme and arrive at the statement of the theorem.

**Theorem 2** For the existence of a vortex type solution of Eq. (1.9), it is necessary that the singularity support \( x = -a(t) \) be the curve of degeneration of Eq. (1.9), i.e., \( c|_{x=-a(t)} = 1/2\mu \), and that the limit values of the derivatives of \( c \) as \( x \to -a(t) \) satisfy Hugoniot type conditions the first of which have the form (3.47).

For the initial data, it should be noted that relations (3.21) and (3.33) impose restrictions on the initial values of \( c_\alpha^0, |\alpha| = 1, 2 \), while \( c_\alpha^1|_{t=0}, |\alpha| \leq 1 \), and \( \omega_\alpha(t), t \geq 0 \), can be chosen arbitrarily.

The plots show the trajectories (corresponding to different initial data) of the singularity support motion calculated without regard for the drift (for
These plots were calculated by using the truncated system (3.47), where all “extra” functions \( c_0^\alpha, |\alpha| = 3, \) and \( c_1^\beta, |\beta| = 2, \) were set to be equal to zero.

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