Many-Body Entanglement in Short-Range Interacting Fermi Gases for Metrology

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We explore many-body entanglement in spinful Fermi gases with short-range interactions, for metrology purposes. We characterize the emerging quantum phases via Density-Matrix Renormalization Group simulations and quantify their entanglement content for metrological usability via the Quantum Fisher Information (QFI). Our study establishes a method, promoting the QFI to be an order parameter. Short-range interactions reveal to build up metrologically promising entanglement in the XY-ferromagnetic and cluster ordering, the cluster physics being unexplored so far.

Strongly-correlated systems are progressively becoming a paradigm for precision metrology, attracting broad interest [1]. Quantum gases represent a powerful platform to develop quantum measurement devices [2, 3], bridging between engineering of quantum states of matter [4] and progress in atom interferometry [5, 6]. Atom interferometry has many sources of uncertainty, classifiable into device and statistics-driven causes [7]. Accurate experimental schemes have blossomed, providing significant reduction of the former, now comparable or even lower than statistical error [7–12]. Further precision improvements can be obtained by addressing the statistical uncertainty problem, in particular the quantum phase estimation [13, 14]. A conceptual tool to reduce statistical uncertainty may come from entanglement, specifically quantum squeezing [15, 16]. Squeezed states are entangled states whose quantum uncertainty in a selected observable can be made smaller than the Heisenberg lower bound, which remains preserved at expenses of larger uncertainty in a conjugate observable [17]. The concept can be applied to any quantum variables set subject to an uncertainty principle, e.g., real or pseudo-spins like motional or internal atomic states [15, 18, 19]. Atomic spin squeezing has been implemented in numerous experimental setups, using interactions either collision-driven or light-mediated in optical cavities [15, 20–23].

Entanglement is a necessary but not sufficient condition for squeezing, its metrological usefulness being quantified via the Quantum Fisher Information (QFI) from Cramér-Rao bound for statistical estimation of variances [1, 14, 26]. Long-range interactions are often considered useful to progressively build up entanglement [15, 27, 28], but many-body short-range interactions can drive long-range correlations. An interesting question thus arises, whether short-range interactions can provide phases with useful entanglement content for metrology.

In this Letter, we tackle the problem from a conceptual perspective and investigate many-body entanglement via a minimal model able to reproduce the essential desirable features of a strongly-correlated quantum fluid with short-range interactions and motional degrees of freedom [29]. To this aim, we consider a system of $N$ fermionic atoms in two spin states within the $tUJ$ model [24, 30], correlated via nearest-neighbor coupling $J$ and on-site $U$, and in the presence of tunneling processes $t$. We use Density-Matrix Renormalization Group (DMRG) simulations to characterize the system quantum phases and classify them by finding a quantitative correspondence between the QFI and the order parameters characterizing the quantum fluid, conveying two central messages. First, this idea acquires methodological significance, since QFI can be seen as an order param-

FIG. 1: (Color online) System concept. Top. The $tUJ$ Hamiltonian [1]: $t$ drives the hopping, $U$ the on-site interaction and $J$ the spin-exchange coupling. Bottom. Qualitative phase diagram at quarter filling in the $U/t$-$J/t$ parameter space, including the following phases: Luttinger-Liquid (LL), Superfluid (SF), Charge-Density-Wave-like (CDW), Spin-Density-Wave (SDW $^{x,y,z}$), XY Ferromagnetic (XY-FM), Clusters with internal XY-FM or antiferromagnetic (AFM) spin ordering, and hemmed clusters (HC) (see text for descriptions). Simulations have been performed along the solid lines. Thick solid straight lines: $|J/U| = 1$. Thick curves: guidelines delimiting cluster phases. Dot-dashed straight lines: studies from [24] (tilted) and [25] (horizontal) (see text). We explore the metrological usability of these phases, finding XY-FM and XY-FM cluster phases especially convenient (see text).
eter. Second, two particular ground states in a short-range interacting system result especially promising for metrological use, because of their QFI scaling with the number of atoms \(N\). These phases correspond to an XY-ferromagnet and a cluster ordering, the latter being here identified and quantitatively analyzed in the whole \(U-J\) phase diagram. Possible experimental realizations are discussed.

**The Fermionic \(tUJ\) model** - We consider an ensemble of fermions in two (real or pseudo)-spin states, moving in a one-dimensional (1D) geometry in the presence of a short-range interaction. We model the system as卡通ized in top Fig. 1 according to the \(tUJ\) Hamiltonian:

\[
H = \sum_i \left[ -t \left( c_{i\sigma}^\dagger c_{i+1\sigma} + \text{h.c.} \right) + U n_{i\uparrow} n_{i\downarrow} + J \left( s_{i\uparrow}^z s_{i+1\uparrow} + \text{h.c.} \right) \right].
\]

(1)

Here, \(c_{j,\sigma}^{(1)}\) are destruction (creation) operators for a fermion with spin \(\sigma\) on site \(j\), \(n_j = \sum_{\sigma} c_{j\sigma}^{(1)} c_{j\sigma}\) is the number operator, and \(s_j^{+(-)} = c_{j\uparrow(\downarrow)}^{(1)} c_{j\downarrow(\uparrow)}^{(1)}\) the spin raising (lowering) operators acting on site \(j\). In \(1\), the \(t\)-term mimics atomic motion via hopping. The \(U\) and \(J\) terms represent, respectively, the contact and nearest-neighbor pairs of a same two-body interaction, from now on in \(|t| = 1\) units.

**DMRG method** - In order to explore the quantum phases of the analytically unsolved \(tUJ\) model, we resort to a DMRG method \([31,32]\), described in detail in the Supplemental material \([33]\). We adapted the code provided by Rossini et al. \([34]\), with a finite size algorithm improving thermalization \([32]\). As we focus on the connection between the system quantum phases and their metrological usability, we only display results for \(\nu = 1/4\) filling, though results at \(\nu = 1/2\) are also discussed. The nature of the ground state has been explored by probing the different quantum correlation functions \(\langle O_i^\dagger O_j \rangle\) on the ground state, with \(O_k\) an operator acting on site \(k\). We have considered Spin Density Waves (SDW) correlations with \(O = s^{x,y,z}\). Charge Density Waves (CDW) with \(O = n\), and superfluid pairing (SF) with \(O = c c^\dagger\). Boundary effects, embodied by Friedel oscillations \([35]\), are ruled out by averaging the correlation functions over a number of sites corresponding to multiples of the Friedel wavelength.

**Quantum phases** - The bottom Fig. 1 displays the system quantum phases. We first discuss the phase diagram for \(-\infty < U < +\infty\) and \(|J|\)-values below the solid thickest curves. Large and negative \(U\) favor a SF phase in 1D sense \([36]\) with a large fraction of doubly-occupied sites. In fact, small \(J\) couplings are ineffective, as there are no opposite-spins to pair. Moving towards \(U \to 0\), onsite pairs progressively become disfavored, and hopping begins to dominate. As expected, this leads to CDW ordering for \(J > 0\) and SDW for \(J < 0\), both characterized by a typical \(2k_F\) oscillation in respective correlation functions. Overall, the behavior around the origin is consistent with a smooth merging into a Luttinger-Liquid (LL) description. Larger and positive \(U\) values drive instead a dominance of antiferromagnetic (AFM)-like ordering in the form of SDW-oscillating correlation functions for \(J > 0\). For \(J < 0\), these phases retain their AFM-like features, with spin-\(z\) correlations. We call this XY-Ferromagnetic (XY-FM) phase in 1D sense, the power-law decay being the longest range ordering possible \([36]\). In the FM phase, the \(x\)-, \(y\) expectation values on each lattice site are solid zeros. Positive correlations and zero expectation values suggest the many-body ground state to be an even superposition of two \(x\)-, \(y\)-spin aligned states oppositely directed. This is consistent with an expected degeneracy, as the \(J/U < 0\) coupling favors alignment in the \(x\), \(y\)-plane without specifying a direction. The SDW-oscillations can be understood noticing that \(+J \sum_i (s_i^+ s_{i+1}^- + \text{h.c.})\) can be cast as \(\sim s_i^+ s_{i+1}^- + s_i^z s_{i+1}^z\), so that spin-exchange coupling favors spin (anti-)alignment in the \(x, y\) plane.

We remark that a similar \(tUJ\) model has been investigated by Dziurzik, Japaridze et al. \([24,30]\) in the context of high-temperature superconductivity via bosonization and DMRG techniques, exploring the \(J, U\) space at different fillings. While we find good agreement on the phases nature and boundaries discussed so far (tilted dot-dashed lines in Fig. 1 \([24]\)), our analysis provides qualitative and quantitative evidence of a new phase. In this phase, particles clusterize, i.e. form regions with unit density surrounded by zero density. Inside the clusters, spins are strongly aligned (FM) or antialigned (AFM) in their \(x, y\)-components. In Fig. 1 these orderings are named XY-FM and XY-AFM cluster phases, emerging for \(J < 0\) and \(J > 0\), respectively above and below a \(U\)-dependent threshold \(J_c\).

We now investigate the nature of these phases, turning our attention to the density profiles displayed in Fig. 2 for the illustrative value \(J/U = -0.1\) \([33]\). While for \(U < 0\) and \(U \lesssim 3\) (top panel), the density profiles show the usual Friedel oscillations around average density, for values \(U \gtrsim 38\) we encounter the typical situation depicted in the lower panel. The system’s bulk ceases to be translationally invariant, and fermions form clusters of singly occupied sites. Simultaneously, very strong spin-\(x\) correlations arise among particles inside clusters \([34]\). A similar simulation for the Hubbard model with \(J = 0\) shows no trace of this phase (inset). We may thus infer that the cluster phase is driven by the dominance of the local nearest-neighbor spin-\(x\)-coupling, both FM and AFM, over the delocalizing hopping term. In order to assess the robustness of this phase, we performed a number of runs against variations of simulation parameters. While clusters positions and number are seen to change in sensible manner, their qualitative behavior persists as detailed in the SM \([33]\). In essence, with our DMRG algorithm, single
clusters more likely form at relatively small system sizes ($L \lesssim 40$), and moving clusters may merge when driven by larger numbers of finite-size algorithm iterations. We infer that the variability of the clusters positions can be due to the vanishing energetic cost of moving around one of them in the surrounding free space. This means that the ground state is strongly degenerate, preventing extraction of definite information on clusters positions. In fact, we found traces of this state in the exact diagonalization results of the $t-J$ model by Ogata et al. [25], occurring at a critical coupling $J_c$. This is computed by comparing the energies of a pair of spin-$x$ correlated atoms in a cluster and in a Luttinger liquid-like state of the infinite chain. Their argument leads to $J_c = 3.22$, quite close to the $J_c \approx 3.15$ value found in a DMRG study of the $t-J$ model by [37], where the phenomenon is called phase separation. From our density profiles, we infer that the cluster phase appears at $U_c \approx 38$ which - given $J/U = -0.1$ - corresponds to $J_c \approx -3.8$. We are thus led to infer that this phase transition is driven by the same mechanism, but occurring at a critical $J$-value modified by the presence of $U$. Actually, their no-double occupancy setting can be viewed as our $U \rightarrow +\infty$ limit. Indeed, the symmetric critical value $|J_c| \approx 3.8$ is met in our results at $U \rightarrow +\infty$ reported in Fig. 1. $J_c \approx 3.8$ being the boundary between AFM-like SDW and XY-AFM Cluster phases, while $J_c \approx -3.8$ between XY-FM and XY-FM Clusters. For large attractive $U$, the boundary is instead dictated by the lines $|J/U| \sim \pm 0.85$. As one would expect the lines to be $|J/U| = \pm 1$, the observed modified value could be due to super-exchange. In the $-1 < J/U < -0.85$ gap, we observe peculiar clusters characterized by double-occupancy at the density edges, which we have named hemmed clusters (HC) [33]. This is not the case in the symmetric region with $J/U > 0$.

Quantum Fisher Information (QFI)- Having characterized our quantum phases, we can now turn to measure their degree of many-body entanglement via Quantum Fisher Information and test the system’s metrological usability. The quantum Cramér-Rao lower bound [14] on an estimator variance is given by the inverse of the Quantum Fisher Information, defined as $F[\rho, J] = \langle (\partial L_\theta/\partial \theta)^2 \rangle$. Here $L_\theta$ is the logarithmic derivative operator defined by $(\partial \rho/\partial \theta) = (\rho L + L \rho)/2$. The QFI depends on both the system’s initial state and the transformation performed by the physical phenomenon to be measured, and considerably simplifies for a pure state undergoing a unitary transformation $\exp iD\hat{S}$, becoming $F[\psi, \hat{S}] = 4\langle \Delta S \hat{\Delta} S \rangle$. $S = a_0 S^a$ is here a linear combination of global (pseudo-)spin operators [14]. $F[\psi, \hat{S}]$ fixes a criterion for evaluating the metrological usability of a quantum state, here the ground state of the many-fermion system. It is known that for a N-body uncorrelated product state, $F \sim N$ corresponds to the shot-noise limit [14]. For good metrological usability then, the QFI needs to scale as $N^\gamma$ with $1 < \gamma < 2$ limited by the Heisenberg principle [13].

Results on QFI- We now quantify these expectations by computing the QFI across the phase diagram and comparing it with the quantum phases order parameters. In all computations we select the spin axis which offers the largest QFI value from the the angular momentum covariance matrix $\text{Cov}_{ab} = \sum_{i,j} \langle s_i^a s_j^b \rangle$ [15], always obtaining the $x$-axis as non-granted outcome. A simple reasoning would lead us to infer that the QFI on SDW
or SF states would return a tiny value as compared even to shot-noise QFI~ $N$. In fact, the oscillating spin-$x$ correlations between different sites would add up to zero in the $SDW$ and vanish for each doubly occupied site of the $SF$ state. This view corresponds to our numerical findings. The QFI results to be large only in the XY-FM and XY-FM cluster phases. For a quantitative comparison, we now define the corresponding order parameters. The XY-FM parameter is defined as the normalized area of the zero-component peak $CC_x(0)$ in the Fourier transform of the spatial spin-$x$ correlation function $C_x(i-j)$: in fact, $CC_x(0)$ is its normalized spatial integral. The clusters order parameter is defined as the normalized density variance $L^{-1}\sum_i (\Delta n^2)$. Since we are interested in systems where $J$ and $U$ are effectively caused by the same term, we run simulations at different $J/U$ while varying $U$ to cross all possible phases. The results for the QFI (red points and curve), XY-FM (green points and curve), and Cluster (blue points and curve) order parameters are collected within one single graph in Fig. 3, one central result of the present work. We see that the Quantum Fisher Information shows a steep change in correspondence of the quantum phase transition to spin-$x$ ordering, the QFI and $CC_x(0)$ curves getting quite closely along with varying $U$. In fact, one may use the QFI to infer the occurrence of two quantum phase transitions around $U \sim 4$ and $U \sim 38$. However, while in the former a quantitative comparison is possible, it has to be intended qualitative in the latter, because particles in separate clusters are mutually uncorrelated. A quantitative treatment for the cluster phase is later recovered in the QFI scaling analysis. While the present results refer to a fixed $J/U = -0.1$ value for illustrative purposes, they are generalized below to the whole phase diagram.

We finally study the dependence of QFI on $J/U$ and filling, and assess the degree of metrological usability from the QFI scaling with the particle number $N = 2\nu L$. We display in Fig. 4 the QFI density $QFI/N$ at two commensurate fillings, $1/4$ (red) and $1/2$ (blue). As anticipated, the QFI vanishes for $U < 0$ and $J/U = +0.8$, where XY-FM and cluster phases are absent. In addition, at $\nu = 1/2$ the QFI density is larger and, unlike $\nu = 1/4$, is smooth because the whole system is in the form of a single cluster. For both fillings, larger (negative) values of $J$ favor cluster formation and steeper QFI rise. We study the $N$-scaling with special care at $\nu = 1/4$, where several uncorrelated clusters may form at large $U > 0$. Thus, we keep relatively small system sizes ($L < 40$) to have one single cluster! For both fillings, we fit QFI dependence on $N$ with the function $QFI = kN^\gamma$, as illustrated in the inset for e.g. $U = 60$ and $J/U = -0.1$ at $\nu = 1/4$. The table reports $\gamma$ for $U = 11$, corresponding to the QFI maximum in the XY-FM phase, and $U = 60$, the large-$U$ limit for the Cluster phase at $\nu = 1/4$. We see that half-filling possesses a better scaling outside the cluster region. Inside it, the scalings at half and quarter filling are compatible within error.

Conclusions- Our study conveys two unforeseen messages. First, short-range interactions are able to build metrologically useful entanglement in a many-fermions system. This is demonstrated by a large degree of Quantum Fisher Information, accompanied by interesting scaling with the number of particles. The best performing phase is indeed the cluster one, driven by the $J$ coupling, which in our study models the short-range interactions. Second, our results imply that the QFI represent a powerful tool to characterize the phases of the quantum fluid, acting as an order parameter. Implementations in ultracold gases platforms may in principle include currently realized systems of dipolar fermions in optical lattices and the suitably engineered versions of the Fermi-Hubbard setup investigated in, in both cases after further reduction of dimensionality to 1D. Finally, a microscopic origin of this $IUJ$ model can be provided by a photon-mediated effective interaction among fermions in an optical cavity, leading to a spin-squeezing Hamiltonian. Multimode optical cavities may bring in the short-range environment, though a realistic probe requires detailed modeling to include unavoidable dissipation processes.

We have observed similar results for hard-core bosons in the phases of interest, with QFI scalings compatible...
with the fermionic case. From a preliminar analysis at incommensurate fillings, a non-trivial scaling behavior of the QFI emerges, possibly caused by frustration. This requires better suited investigation tools and more extended studies, which are referred to future work.

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