On Successive Approximations
To The Choice Problem and Logic

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This paper studies the formation of logical operations from pre-logical processes. We are concerned with the reasons for certain mental processes taking form of logical reasoning and the underlying drives for consolidation of logical operations in human mind. Starting from Piaget's approach to Logic (Piaget, 1956) we discuss whether the evolutionary adaptation can be such a driving force and whether the limits of human mind can result in the standard system of logical operations. The paper demonstrates that the classical two-valued propositional logic can begin from a method of successive approximations applied to a decision-making problem within a framework of Subject-in-an-environment survival. The presented results shed a new light on the known model of human choice by Lefebvre (Lefebvre, 1991, 1995).

Key Words: Piaget's theory, successive approximation method, approximating forms, logic connectives, choice problem, Lefebvre's model.

1. INTRODUCTION

This paper studies the formation of logical operations from pre-logical processes. We are concerned with the reasons for certain mental processes taking form of logical reasoning and the underlying drives for consolidation of logical operations in human mind.

In order to enable a wider appreciation of the results, we opted to present our model on the basis of Piaget's approach to Logic (Piaget, 1956). Thus, even though this research was carried out independently, the reader will find numerous references to Piaget's concepts and definitions throughout the paper.

According to J.Piaget, logic is not an inherent form of thinking and "logical operations result from coordinations of the actions of combining, dissociating, ordering and setting up of correspondences, which then acquire the form of reversible systems" (Piaget, 1956, p. 13). In his works Piaget suggested a representation of logical operations in terms of certain more elementary ("pre-logical") operations of the kinds listed above.

On the other hand, logical operations are the only means to carry out logical inference. Therefore, it seems to be reasonable to assume that the consolidation of logical operations in a developing mind goes within a framework of some mental processes which gradually take on the form of logical reasoning. What could be the cause of such a transformation? How is it possible to describe in strict mathematical terms such a "pre-logical" process that comes to logical inference at the end?

In this paper, we attempt to demonstrate (i) that the evolution-caused adaptation necessity could play the role of the underlying driving force; (ii) that the processes of a successive approach to the adaptation purpose can be behind logical inference; and (iii) that then the limited nature of human mind leads to the ordinary systems of logical operations.
The adaptive character of human intelligence is commonly accepted. So it is natural to begin with the acceptance of an elementary (i.e., as commonly applicable as possible) scheme of adaptation problems and some fundamental solution processes for them. Such a scheme should be as elementary as to admit a formulation in terms of the pre-logical operations and relations Piaget listed.

This research project studies one such scheme based on a model of Subject's behavior choice (Bulitko, 2000). The model background lies with two preference relations on the same set of Subject's states. They are referred to as the internal (subjective) preference relation and the external (objective) preference relation.

The internal preference relation is defined by a partial order on a set of inner states of the Subject. The external preference relation on the state set is defined by a mapping from the state set to another partially ordered set. The mapping will be called evaluation mapping (evaluation function).

We consider the process of getting to a better state with respect to the external preference in a fashion that uses the external preference relation as little as possible. On the contrary, the Subject may use the internal preference without any restriction.

Several interpretations of this framework are possible. In our primary interpretation the Subject must pay for its access to the external preference whenever the access to the internal preference is free. For example, we can think of the true external preference as induced by a computationally expensive value function defined over states while the subjective internal preference relation is easy to compute. Thus, we strive to minimize the access to the former while allowing unlimited access to the latter.

We use a successive approximation method as the core of our solution approach. Humans are constantly involved in problem solving that requires planning and execution of action sequences. Therefore, it is reasonable to make the conjecture that an apparatus enabling multi-step planning is among humans' innate functions.

The successive approximation method proposed herein strongly connects with Piaget's actions of "combining and dissociating". In our model functions ⊞, ⊟ are the counterparts to these actions. Since these pre-logical actions are ambiguously described in (Piaget, 1956) we introduce our counterparts in an axiomatic fashion.

The successive approximation method decomposes the evaluation mapping into a superposition of functions ⊞, ⊟ as well as "easier" functions (the details will be given below). Such superpositions are called approximating forms.

We are now able to interpret Piaget's "actions of combining, dissociating, ordering and setting up of correspondences" with operations ⊞, ⊟, the two preferences, and the evaluation mapping correspondingly.

In this paper, several approximating forms are developed for different systems of axioms. Furthermore, two of them are relevant to the hypothetical origin of logic we are discussing in this research. These two are the cases in which the external and internal preferences take certain canonical forms. Indeed, every finite partially ordered set can be embedded into an appropriate Boolean cube. Thus, it is always reasonable to replace a given external preference with a two-element preference of the "admissible - inadmissible" or "good-bad" kind.

Certain standard systems of logical operations obey these axioms. Then the corresponding approximating forms translate into logical formulæ. Finally, the subject’s action sequences map to logical inferences. Thus, in our framework it is possible to explain the transition from pre-logical to logical form of mental processes via a canonical simplification of the internal and external preferences.

Developing a comprehensive theory of algorithm design on the basis of approximating forms is beyond the focus of this paper. Yet, we pose an example of such an algorithm design in the final section of the paper.

The approach proposed in this paper illuminates the well known model of human choice by Lefebvre (Lefebvre 1991,1995) from a different angle. In particular, the primary formula of that model follows from the our approach under certain natural assumptions.
2. BEHAVIOR CHOICE WITH TWO PREFERENCES

In this section we consider the Subject faced with a choice of a state among a set of states in the environment. Some of the states may be better than others and some are incomparable. Subject’s objective is to reach a satisfactory terminal state.

A fundamental feature of many real-world behavior problems is the difference between the evaluations of a state before and after the state is arrived at. We will now attempt to formalize this phenomenon by introducing two preference relations over the state space.

One of the two relations will specify the “internal” system of values based on the Subject’s internal representation (or model) of the world (including the Subject itself). We argue that the internal preference relation is intrinsic to the Subject’s mentality since the Subject perceives the world in terms of this relation. Thus, there are no restrictions on the usage of this internal relation.

The other, “external”, relation is based on effects of state choices. Thus, the external preference reflects the actual nature of the interaction between the environment and the Subject and, generally speaking, only a part of the external preference is available to the Subject. Naturally, the known portion of the external preference relation includes the information the Subject has uncovered so far in its exploration of the environment.

The external preference is objective and determines the Subject rewards/penalties and ultimately its survival. No specific limitations are imposed on the two preference relations making the framework quite general. It is natural to pose an extremum (as defined by the external preference) as a goal state for the Subject. Indeed, maximum and the greatest elements can be expressed in terms of preference relation by means of predicate logic. It is clear reaching an extremum is a simpler problem than reaching the optimal state.

Discrepancies between the internal and external preference relations may cause problems for Subject. Note that there is a cost associated with accessing the external preference relation. It is not only the cost of accessing the information but also the cost of changing the Subject’s behavior patterns. (Here we abstract from computing the actual cost values).

Thus informally, the problem studied herein is to find an extremum of the external preference relation under certain given restrictions on the information access to the external preference and an unlimited access to the internal preference relation.

Given the restrictions and costs associated with accessing the true (external) preference relation, the Subject strives to reach its goal (i.e., to locate an extremum) using the internal preference as much as possible. Naturally, in order for the internal preference to be beneficial to the subject, it needs to approximate the external preference. This interpretation of the choice problem corresponds to a certain conservatism on the side of the Subject when it is necessary to follow a certain external pressure. Indeed, even if the Subject is aware of its incomplete and/or incorrect representation it often might not be able to correct it. Therefore, it will need to refine/reconstruct its representation starting with whatever is available.

In order to address these problems, we consider a successive approximation principle that will guide our further investigation. Namely, in the following we will demonstrate that the problem is decidable by some version of successive approximation method. The underlying idea of the method is as follows. The Subject follows a certain part of its internal preference as long as the preference doesn’t deviate significantly from the external one. Then on the basis of accessible information on the external preference the Subject reverses the corresponding part of the internal reference and uses it to explore the environment further. The process then repeats.

Thus, the Subject needs a means and a scheme to select and manipulate corresponding parts of the preference relations. The following section is devoted to a theory of such schema. We believe the framework proposed below can be viewed as a possible formalization of operations and relations listed by Piaget.
3. AN EXPLICATION OF THE SUCCESSIVE APPROXIMATION METHOD

Let $S$ be a set of Subject states, $(M, \leq_M), (L, \leq_L)$ be partially ordered sets of internal and external estimates correspondingly. Let $\varphi : S \rightarrow M, \psi : S \rightarrow L$ be mappings that link corresponding estimates to states. In this way it is possible to set internal and external preferences on $S$. Generally speaking, these preferences are pre-order relations on the set (Birkhoff, 1967).

We simplify this description by introducing an order $\leq_S$ on $S$ through the mapping $\varphi$ and the poset $(M, \leq_M)$ as follows. Let us set $s \leq s' \iff \varphi(s) \leq_M \varphi(s') \vee s = s'$. This reformulation does not put any restrictions on $\leq_S$. Therefore, we can from now on consider the description $((M, \leq_M), (L, \leq_L), \psi : M \rightarrow L)$. Furthermore, $(M, \leq_M)$ plays the role of $(S, \leq_S)$ above and $\psi$ is called the evaluation mapping (evaluation function).

It is worth noting that generally in each instance of the choice problems the Subject gets the corresponding internal and external preferences and the evaluation function. These three objects can depend functionally on some parameters of the choice problem.

First, we consider the case of a single problem of choice. In a section related to Lefebvre’s model, we will consider a family of choice problems.

If the evaluation function $\psi$ is a monotonic mapping (i.e., the condition $\forall m_1, m_2 \in M)[m_1 \leq_M m_2 \Rightarrow \psi(m_1) \leq_L \psi(m_2)]$ is met) then both preference relations $(M, \leq_M)$ and $(L, \leq_L)$ are mutually compatible (concordant) and de facto the Subject may follow its internal preference to reach the target state (that is, a state with the maximum value).

Otherwise, it is natural to represent $\psi$ by a superposition of monotonic evaluation mappings from $(M, \leq_M)$ to $(L, \leq_L)$ and several connecting operations. We look for representations that can be used for successive approximations. In finding a representation of this kind that uses as few monotonic evaluation mappings as possible, we attempt to use the external preference relation as little as possible.

3.1. Axiom system $\mathcal{A}$

This section proposes collections of operations providing representations of the aforementioned kind for any given evaluation function $\psi$. These representations are called "approximating forms”.

Our first collection uses three operations: $\square : L \times L \rightarrow L, \boxplus : 2^L \rightarrow L, \odot : L \rightarrow L$. In our model, the first two represent Piaget’s operations of "dissociating" and "combining" respectively. The third operation $\odot$ represents the conception of the "null" element $o$ that we also encounter in (Piaget, 1956). Following his theory, Piaget developed a special algebra of numerous concrete operations.

We feel it is quite natural to define the sought model via an appropriate axiomatic system. We start with system $\mathcal{A}$:

- $A_1$: $\forall S \subseteq M)(\exists \bar{s} \subseteq S)(\forall s \in S)(\exists \bar{s} \in \bar{S})[\bar{s} \leq_M s] \& (\forall \bar{s}, \bar{s}' \in \bar{S})[\bar{s} \not\leq_M \bar{s}']$.
- $A_2$: $\forall L', L'' \subseteq L)(\forall x \in L')[(x \leq_L \boxplus(L')) \& (L' \subseteq L'' \Rightarrow \boxplus(L') \leq_L \boxplus(L''))]$.
- $A_3$: $\forall l, l' \in L)[\exists (l, \odot(l)) = l \& (l \leq_L l' \Rightarrow \odot(l) \leq_L \odot(l'))]$.
- $A_4$: $\forall l, l' \in L)[l \leq_L l' \Rightarrow (\exists l'' \in L)[\exists (l', l'') = l \& \odot(l') \leq_L l'']]$.

Axiom $A_1$ demands the internal preference to have no infinite decreasing chains. The axiom is trivially true for finite state set. It is clear that the restriction of finite state sets is not overly constraining in practice. It is worth noting that there is only one axiom relating to the internal preference.

Axiom $A_2$ describes "combining" $\boxplus$ whereas $A_3, A_4$ tie operation of "dissociating" $\square$ and operation $\odot$ of coming to a "null".

Operation $\boxplus$ combines element set $L' \subseteq L$ into a single element while respecting the monotonicity property. This property is one of the main properties of set-theoretical operation $\cup$. Thus, our definition preserves the primary property of the concept of combination as used by Piaget.
Axiom $A_4$ postulates the property of reversibility for "dissociating". Element $l''$ such that $\square(l,l'')=l'$ represents the "difference" between $l$ and $l'$ (again, this preserves the flavor of Piaget's definition).

Axiom $A_3$ fixes some sufficient properties of the concept of "null". Note that many "null" elements may exist (but not required to).

### 3.2. Approximating forms

For every function $\nu : M \rightarrow L$ we call set $n(\nu) = \{(m,m')|(m \leq_M m') \& (\nu(m) \not\leq_L \nu(m'))\}$ non-monotonicity domain of $\nu$. If $n(\nu) = \emptyset$ then $\nu$ is called monotonic function. Also for every poset $(R,\leq_R)$ the standard mappings $(\cdot^R, (\cdot)^R) : R \rightarrow 2^R$ are defined by $t^R = \{t' \in R | t' \leq_R t\}, t^R = \{t' \in R | t \leq_R t'\}$.

**Theorem 1.** Let for $(M, \leq_M), (L, \leq_L)$ all axioms of the system $A$ be satisfied and lengths of all increasing chains in $(M, \leq_M)$ do not exceed some integer $D$. Then for every function $\psi : M \rightarrow L$ there exists a representation $\psi = \sqcap(\varphi_1, \sqcap(\varphi_2, \sqcap(\varphi_3, \ldots)))$ such that all $\varphi_i, i = 1, 2, 3, \ldots$, are monotonic functions from $(M, \leq_M)$ to $(L, \leq_L)$.

Furthermore, the number of occurrences of operation $\sqcap$ in this representation does not exceed $D$.

**Proof.** Let us reduce the problem for a given function $\psi$ to the same problem for a simpler function $\varphi_1$ such that the following holds $\psi = \sqcap(\varphi_1, \psi_1)$ and $n(\psi_1) \subseteq n(\psi)$.

First, we define $M_1 = \{x \in M | n(\psi) \cap (x^0 \times x^0) \neq \emptyset\}$, $M^1 = \overline{M_1}$:

$$\varphi_1(x) = \begin{cases} \sqcap(\psi(x^0)), & x \in M_1, \\ \psi(x), & x \in M^1. \end{cases}$$

Then we set $\psi_1(x)$ to any such $z \in L$ that $\sqcap(\varphi_1(x), z) = \psi(x) \& \sqcap(\varphi_1(x)) \leq_L z$ if $\varphi_1(x) \neq \psi(x)$. Otherwise, we set $\psi_1(x) = \sqcap(\psi(x))$.

Existence of element $z$ in the definition is guaranteed by axioms $A_3, A_4$. Now, the equality $\psi(x) = \sqcap(\varphi_1(x), \psi_1(x))$ holds due to the definitions of $\varphi_1, \psi_1$.

Let us prove that function $\varphi_1 : (M \leq_M) \rightarrow (L, \leq_L)$ is monotonic.

First, $\varphi_1 = \psi$ over $M^1$ and we may use the condition $x, y \in M^1 \& x \leq_M y \Rightarrow \psi(x) \leq_L \psi(y)$. Indeed, otherwise $\psi(x) \not\leq_L \psi(y), x \leq_M y, \psi(x) \neq \psi(y)$ and, therefore, $y \in M_1 \cap M^1$. However, $M^1 \cap M_1 = \emptyset$ which leads to a contradiction.

Second, $\varphi_1$ maps $(M_1, \leq_M)$ into $(L, \leq_L)$ monotonically in accordance with $A_2$.

Finally, let us consider the "mixed" case when $x \in M^1, y \in M_1$ and all elements of $M$ are comparable with respect to $\leq_M$. It is clear that $y \leq_M x$ is impossible since $z \in M_1 \Rightarrow Z \subseteq M_1$ immediately follows from the definition of $M_1$.

Thus, it remains to consider the possibility of $x \leq_M y$. In that case $\varphi_1(y) = \sqcap(\psi(y^0)) \geq_L \psi(x)$ in accordance to $A_2$. On the other hand, $\psi(x) = \varphi_1(x)$ on $M^1$ follows from the definition of $\varphi_1$. Hence, function $\varphi_1$ is monotonic.

We are now ready to prove the last assertion of the theorem. For that it is sufficient to demonstrate the inclusion $M^1 \cup M_1 \subseteq M^2$. Here $M^2, M_2$ are defined for $\psi_1$ in the same way as $M^1, M_1$ were defined for $\psi$ above. $M_1$ is the set of all minimal elements of set $M_2$, see $A_1$. Namely: $M^2 = \overline{M_2}$ and $M_2 = \{x \in M | n(\psi_1) \cap (x^0 \times x^0) \neq \emptyset\}$.

From here we have $M_2 \subseteq (M_1 \setminus \tilde{M}_1)$ and $n(\psi_1) \subseteq n(\psi_1) \setminus \tilde{M}_1 \times M_1$. Thus, sequence $M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots$ ends on a step with the number can not be greater than the highest of the lengths of the increasing chains in poset $(M, \leq_M)$. Indeed, since $\tilde{M}_2 \subseteq M_1 \setminus \tilde{M}_1$ then in accordance with $A_1$ for every element $y \in \tilde{M}_2$ there exists some $x \in \tilde{M}_1$ such that $x \leq_L y$. Therefore, one can choose an increasing chain of representatives of sets $M_1, M_2, M_3, \ldots$ which are mutually disjoint sets.

We will now prove that $M^1 \cup \tilde{M}_1 \subseteq M^2$. First, $\varphi_1(x) = \psi(x)$ holds for every $x \in M^1$. From here $\psi_1(x) = \sqcap(\psi(x))$. However, mapping $\psi_1$ is monotonic on $M^1$ due to $A_3$ and since $\psi$ is monotonic on $M^1$. So $(M^1 \times M^1) \cap n(\psi_1) = \emptyset$ and therefore $M^1 \subseteq M^2$. 

5
Further, let \( x, y \in M^1 \cup \tilde{M}_1 \) and \( x \leq_M y \). Then we can show that \( \psi_1(x) \leq_L \psi_1(y) \). Indeed, the case \( x, y \in M^1 \) was considered above. The case \( x, y \in \tilde{M}_1 \) is impossible since all elements of \( \tilde{M}_1 \) are incomparable by the definition. Above, we saw that \( x \in M_1 \) \& \( x \leq_M y \Rightarrow y \in M_1 \). Besides \( M^1 \cap M_1 = \emptyset \). Therefore, \( x \in M^1, y \in \tilde{M}_1 \) is the only case remaining to consider. By definition \( \psi_1(x) = \odot(\psi(x)) \) and relation \( \square(\psi_1(y), \psi_1(y)) = \psi(y) \) holds. Moreover, \( \psi(y) <_{L} \varphi(y) \). In accordance with \( A_4 \) we have \( \odot(\varphi_1(y)) \leq_L \psi_1(y) \). Hence, \( \psi_1(x) \leq_L \psi_1(y) \) takes place since \( \odot \) is a monotonic operation due to \( A_3 \) and \( \psi(z) \leq_L \varphi_1(z) \), \( z \in M \) in accordance to \( A_2 \) and the construction.

Let us denote by \( \mathcal{M} \) the class of all monotonic mappings from \( (M, \leq_M) \) to \( (L, \leq_L) \). Also let \( S(\varphi) = \{ x | \varphi(x) >_L \odot(x) \}, \varphi \in \mathcal{M} \).

**Corollary 1.** Under the conditions of theorem 1 for every \( \psi : M \rightarrow L \) there exists a substitution \( p : \{ z_1, \ldots, z_{D+1} \} \rightarrow \mathcal{M} \) such that \( S(p(z_{n+1})) \subseteq S(p(z_n)), n = 1, D, \) and

\[
\psi = \mathcal{S}b_{z_1, \ldots, z_{D+1}} \square(z_1, \square(z_2, \square(\ldots \square(z_D, z_{D+1}) \ldots))).
\]

**Proof.** Let us fix a formula \( \Phi(z_1, \ldots, z_{D+1}) = \square(z_1, \square(z_2, \square(\ldots \square(z_D, z_{D+1}) \ldots))) \) and consider substitutions of monotonic functions instead of variables \( z_1, \ldots, z_{D+1} \) when their results are determined.

According to theorem 1, for every \( \psi : M \rightarrow L \) there exists representation

\[
\psi = \square(\varphi_1, \square(\varphi_2, \square(\varphi_3, \ldots)))
\]

where all \( \varphi_i, i = 1, 2, 3 \ldots \) are monotonic mappings from \( (M, \leq_M) \) into \( (L, \leq_L) \). Condition \( S(\varphi_{n+1}) \subseteq S(\varphi_n) \) follows the construction of functions \( \varphi_1 \) made in the proof of theorem 1. The number of occurrences of operation \( \square \) in this representation does not exceed \( D \).

Once representation:

\[
\psi = \square(\varphi_1, \square(\varphi_2, \square(\varphi_3, \ldots)))
\]

with \( k < D \) is obtained for mapping \( \psi \), one can always continue the expression on the right side of the representation until \( k = D \). For that it is sufficient to set

\[
\varphi_i(x) = \odot(\varphi_{i-1}(x)), i = k + 2, D + 1.
\]

In accordance with axiom \( A_3 \) the obtained functions are monotonic and

\[
\psi = \square(\varphi_1, \square(\varphi_2, \square(\varphi_3, \ldots)))
\]

For any given \( D \) the corollary states existence of the universal formula

\[
\square(z_1, \square(z_2, \square(\ldots \square(z_D, z_{D+1}) \ldots)))
\]

which describes a structure of the representations. However, the cost of the universality lies with the fact that the length of the representation in theorem 1 can be essentially lower than the lengths of the representations suggested by the corollary.

In order to apply the theory developed in the last section, we need to specialize monotonic functions in approximating forms. Thus, we define the functions via the following auxiliary construction.

Let us denote by \( R^k \) the set of all minimal elements of \( (R, \leq_R) \). On the basis of axiom \( A_1 \) let us split set \( M \):

\[
M^1 = M^1; \\
M_{n+1} = (M \setminus \bigcup_{j \leq n} M_j)^+.
\]

Any two elements of \( M_j \) are incomparable in \( (M, \leq_M) \) for any \( j \).

We denote by \( \theta \)-function of rank \( i \) any monotonic mapping \( \theta : M \rightarrow L \) such that \( \theta(x) = \odot(x) \) for all \( x \in \bigcup_{j < i} M_j \) as well as \( \theta(x) \in \max(L, \leq_L) \) for all \( x \in \bigcup_{j > i} M_j \). The rank of a given function \( \theta \) is denoted as \( \rho(\theta) \). Let \( \Theta \) be the class of all \( \theta \)-functions.
THEOREM 2. Let conditions of theorem 1 be fulfilled, D be the exact upper bound of the lengths of the increasing chains in \((M, \leq_M)\) and \((L, \leq_L)\) contain its greatest element. Then for any function \(\psi : M \rightarrow L\) there exists a substitution \(p : \{z_1, \ldots, z_{D+1}\} \rightarrow \Theta\), such that \(\rho(p(z)) = i, i = 1, D+1\) and

\[
\psi = \text{Sh}_{p(z)}^{z_1} \cdots z_{D+1} (z_1, \Xi(z_2, \Xi(\cdots \Xi(z_D, z_{D+1}) \cdots))).
\]

Proof. We denote by \(\gamma\) the greatest element of \((L, \leq_L)\) and use induction on \(D\). In the case of \(D = 0\) the statement is obvious since there are no restrictions on \(\theta\)-functions. Therefore, \(\psi \in \Theta\).

Induction step: Let us define \(\theta_1\) of rank 1 in the following manner:

\[
\theta_1(x) = \begin{cases} 
\psi(x), & \text{if } x \in M_1, \\
\gamma, & \text{else.}
\end{cases}
\]

Then we may state \(\psi = \Xi(\theta_1, \psi_1)\) where for \(\psi_1\) we have \(\psi_1(x) = \otimes(x)\) if \(x \in M_1\) else \(\psi_1(x)\) satisfies \(\Xi(\gamma, \psi_1(x)) = \psi(x)\).

It remains to obtain the desirable representation of \(\psi_1\) on set \(M \setminus M_1\) with the partial order \(\leq_M\) induced by \(\leq_M\). Since the length of the longest increasing chain in \((M \setminus M_1, \leq_M)\) is \(D - 1\) we may use the induction supposition.

3.3. Axiom system \(\mathcal{B}\).

Let us define binary operations \(\sqcup, \sqsubset : L \times L \rightarrow L\) and a unary operation \(\ominus : L \rightarrow L\) in such a way that the system \(\mathcal{B} = \{\mathcal{B}_1, \ldots, \mathcal{B}_4\}\) of axioms takes place. Here \(\mathcal{B}_i\) coincides with \(\mathcal{A}_i\) for \(i = 1, 3, 4\). Also:

\(\mathcal{B}_2\): \((\forall x, y \in L)[x \leq_L \psi(x, y) \& y \leq_L \psi(x, y)]\).

THEOREM 3. Let for \((M, \leq_M), (L, \leq_L)\) all axioms of the system \(\mathcal{B}\) be satisfied, lengths of all increasing chains in \((M, \leq_M)\) do not exceed some integer \(D\), and every increasing chain in \((L, \leq_L)\) be a finite one. Then for every \(\psi : M \rightarrow L\) there exists a representation \(\psi = \Xi(\varphi_1, \Xi(\varphi_2, \Xi(\varphi_3, \cdots)))\) where \(\varphi_i, i = 1, 2, 3, \ldots\) are monotonic functions from \((M, \leq_M)\) to \((L, \leq_L)\).

The number of occurrences of the operation \(\sqcup\) in this representation does not exceed \(D\).

Proof. First, in the case when \((\forall x \in M)[|x|^\bar{x} < \infty]\) is true we can prove this theorem using theorem 1. For that we will only need to note that in this case it is possible to replace \(\Xi(\psi(x^\bar{x}))\) with any expression of the kind \(\psi(\psi(z_1), \psi(z_2), \psi(z_3))\). Here \(z_1, \ldots, z_n\) is an enumeration of the finite set \(x^\bar{x}\). Indeed, in the proof of theorem \(\Xi\) we used axiom \(\mathcal{A}_2\) only for subsets of \(L\) of the form \(\psi(x^\bar{x})\). Thus, it is sufficient to check that axiom \(\mathcal{A}_2\) is respected for sets of the kind \(\psi(x^\bar{x})\). This check is trivial on the basis of axiom \(\mathcal{B}_2\) for operation \(\sqcup\).

Otherwise, when there are infinite sets \(x^\bar{x}\) we can make use of the same scheme for the operation \(\sqcup\) basing on the condition of finiteness of increasing chains in \((L, \leq_L)\). For that let us enumerate elements \(z_1, z_2, \ldots, z_n, \ldots\) of set \(x^\bar{x}\) for a given \(x \in M\). Simultaneously we compute a series of expressions:

\[
\psi(\psi(z_1), \psi(z_2)), \\
\psi(\psi(z_1), \psi(z_2), \psi(z_3)), \\
\vdots \\
\psi(\psi(z_1), \psi(z_2), \psi(z_3), \psi(z_4)), \\
\vdots
\]

By axiom \(\mathcal{B}_2\) the values of these expressions are comparable and do not decrease in \((L, \leq_L)\). In view of the finiteness supposition for increasing chains in \((L, \leq_L)\) the sequence of computed values becomes stable from a certain element. We set \(\varphi_1(x)\) to this final value.

Thus, \(\varphi_1\) is a monotonic mapping and \(\psi(x) \leq_L \varphi_1(x), x \in M\). The last part of the proof is analogous to the corresponding part of theorem \(\Xi\).
3.4. Dual problem

Following the case of maximization investigated above, we now consider the minimization problem whose formulation can be obtained from the previous case simply via replacing the word "maximization" with the word "minimization".

There is an easy reduction of the minimization problem to the maximization. To do the reduction we first replace ≤ with ≥ in both preferences. Secondly, we replace functions \( \varphi_i \) with dual ones that remain to be non-decreasing monotonic and then increasing chains with decreasing ones, maximal elements with minimal ones, etc. This way we arrive at a new set of axioms for operations denoted by an asterisk. System \( \mathcal{A} \) is replaced with \( \mathcal{A}^* \):

\[
\mathcal{A}_1': (\forall S \subseteq M)(\exists \tilde{s} \subseteq S)[(\forall s \in S)(\exists \tilde{s} \in \tilde{S})[s \geq_M s] \& (\forall \tilde{s}, \tilde{s}' \in \tilde{S})[\tilde{s} \nless_M \tilde{s}']]].
\]

\[
\mathcal{A}_2': (\forall l', L'' \subseteq L)(\forall x \in L')[x \geq_L \psi^*(L'))\&(L' \subseteq L'' \Rightarrow \psi^*(L') \geq_L \psi^*(L'')].
\]

\[
\mathcal{A}_3': (\forall l, l' \in L)[l \geq_L l' \Rightarrow (\exists l'' \in L)[\psi^*(l'', l') = l \& \psi^*(l') \geq_L l'']].
\]

Then the following theorem, that is a dual for theorem 1, can be proven:

**Theorem 4.** Let for posets \((M, \leq_M), (L, \leq_L)\) all axioms of the system \( \mathcal{A}^* \) be respected and the lengths of all decreasing chains in \((M, \leq_M)\) do not exceed a certain integer \( D \). Then for every function \( \psi : M \to L \) there exists representation \( \psi = \psi^*(\varphi_1, \psi^*(\varphi_2, \psi^*(\varphi_3, \ldots))) \) where all \( \varphi_i, i = 1, 2, \ldots, \) are monotonic functions from \((M, \leq_M)\) to \((L, \leq_L)\).

The number of occurrences of the operation \( \psi^* \) in this representation does not exceed \( D \).

In the same manner we can formulate system \( \mathcal{B}^* \) of axioms:

\[
\mathcal{B}_1': (\forall S \subseteq M)(\exists \tilde{s} \subseteq S)[(\forall s \in S)(\exists \tilde{s} \in \tilde{S})[s \geq_M s] \& (\forall \tilde{s}, \tilde{s}' \in \tilde{S})[\tilde{s} \nless_M \tilde{s}']]].
\]

\[
\mathcal{B}_2': (\forall x, y \in L)[x, y \geq_L \psi^*(x, y)].
\]

\[
\mathcal{B}_3': (\forall l, l' \in L)[\psi^*(\psi^*(l), l) = l \& (l \geq_L l' \Rightarrow \psi^*(l) \geq_L \psi^*(l'))].
\]

\[
\mathcal{B}_4': (\forall l, l' \in L)[l \geq_L l' \Rightarrow (\exists l'' \in L)[\psi^*(l'', l') = l \& \psi^*(l') \geq_L l'']].
\]

Then a dual to theorem 3 holds:

**Theorem 5.** Let for posets \((M, \leq_M), (L, \leq_L)\) all axioms of the system \( \mathcal{B}^* \) be respected, the lengths of all decreasing chains in \((M, \leq_M)\) do not exceed some integer \( D \), and every decreasing chain in \((L, \leq_L)\) be a finite one. Then for every function \( \psi : M \to L \) there exists representation \( \psi = \psi^*(\varphi_1, \psi^*(\varphi_2, \psi^*(\varphi_3, \ldots))) \) where all \( \varphi_i, i = 1, 2, \ldots, \) are monotonic functions from \((M, \leq_M)\) to \((L, \leq_L)\).

The number of occurrences of the operation \( \psi^* \) in this representation does not exceed \( D \).

It is said that \((L, \leq_L)\) admits a dual isomorphism \( \eta \) if \( \eta \) is an one-to-one mapping of \( L \) onto \( L \) such that \( \forall l, l' \leq_L l' \iff \eta(l') \leq_L \eta(l) \) holds.

If the external preference \((L, \leq_L)\) admits a dual isomorphism \( \eta \) then the following identities hold:

\[
\chi^* = \eta^{-1} \circ \chi \circ \eta, \chi \in \{\oplus, \boxplus, \ominus\}
\]

where \( \circ \) denotes the composition of functions.

Thus generally speaking, we obtain new operation systems and new representations that we refer to as approximating forms.
4. A POSSIBLE ORIGIN OF LOGIC

In this section we include a complexity notion into our considerations. First of all, the Subject might reduce the external preferences to the simplest kind such as "acceptable-unacceptable" or "good-bad", etc. So in this case we can set $L = \{0, 1\}, \leq_L = \{(0, 0), (0, 1), (1, 1)\}$. 

Now it is natural to use the simplest collection of operations. As well known, poset $(B^n, \preceq)$ is a self-dual poset for any $n$. In particular, given aforementioned $(L, \leq_L), (n = 1)$, we have $\eta(0) = 1, \eta(1) = 0$ with identity $\eta = \eta^{-1}$. Theorems 3 and 3* offer two-argument operations $\sqcup$ and $\sqcap^*$ correspondingly (unlike many-place operations $\sqcup, \sqcup^*$ from theorems 1,1*) for this case. Both representations introduced in theorems 3, 3* holds and $\eta(l) = l$ is true.

**Lemma 1.** Let $L = \{0, 1\}$ and $\leq_L = \{(0, 0), (0, 1), (1, 1)\}$. Then:

1. Operation $\lambda x, y [\neg x \& y]$ as $\sqcup$, operation $0 : \{0, 1\} \to \{\varnothing\}$ as $\sqcup^*$, and operation $\lor$ as $\sqcup$ obey the axiom set $B$.

2. Operation $\lambda x, y [y \to x]$ as $\sqcap$, operation $1 : B^n \to \{\varnothing\}$ as $\sqcap^*$, and operation $\land$ as $\sqcap^*$ obey the axiom set $B^*$.

3. There exists only one boolean interpretation of the operations $\sqcup, \sqcap^*$.

The lemma can be proved via a routine check of the axiom systems.

Let us recall that $x \to^* y = \neg(y \to x) = \neg x \& y$ (see for example (Kleene, 1967)). Henceforth, we refer to the approximating forms constructed with operations $\to^*, \lor, 0$ or with $\to, \land, 1$ as boolean approximating forms.

The important question here is why natural human languages do not contain any connective that represents operation $\to^*$ (in the way like the connective "and" represents $\&$, for example). A possible answer is offered below.

Dual approximating forms of theorem $3^*$ begin with a given function $\psi$ and approximate it by means of successive simplifications: $\psi_i = \sqcup^*(\varphi_{i+1}, \psi_{i+1})$, where $\psi_0 = \psi$ and $i$ runs integers $0, 1, 2, \ldots$, while $\psi_i$ is not a monotonic function (i.e., not an "easy" one). Taking in account the meaning of $\sqcup^*(x, y)$ is $y \to x$ we get $\psi = \psi_1 \to \varphi_1 = (\psi_2 \to \varphi_2) \to \varphi_1, \ldots$.

Since $\varphi_1 \to \psi$ follows from $\psi = \psi_1 \to \varphi_1$, we can think that the transition from $\psi$ to $\varphi_1$ means the transition from the general notion $\psi$ to the specific notion $\varphi_1$. On the contrary, in the dual case we have only $\psi^* \to \varphi_1^*$. Taking into account the fact that a developing mind forms classes from specific examples we see a support to the claim that the first transition (from general to specific) is easier to implement.

Until now we have not assumed any properties about preference $(M, \leq_M)$. The second step of the simplification process is an isotonic embedding a given finite internal preference $(M, \leq_M)$ into an appropriate Boolean cube $B^n$ where $n$ is the dimension of the cube. This step is always possible for finite preferences (Birkhoff, 1967). Therefore, let the given inner preference be $(B^n, \preceq)$. Then direct corollaries of the lemma 3 and theorems above are as follows:

**Corollary 2.** Every classical logic function can be represented by a boolean approximating form.

**Corollary 3.** Every $n$-argument logical (boolean) function $f$ can be represented by an implicative normal form of the kind $f = P_k \rightarrow P_{k-1} \rightarrow \cdots \rightarrow P_0$, where $k \leq n$, and $P_i, i = 0, k$, are monotonic boolean function.

Therefore, one can consider the classical two-valued propositional logic merely as an application of the above-mentioned principle of successive approximations to the problem of decision-making within the Subject-environment survival framework. Thus, this viewpoint suggests a way for the classical propositional logic to develop from the survival problem. It is also important that this hypothetical origin of logic appears quite natural.
5. APPLICATION TO ONE MODEL BY LEFEBVRE

Lefebvre proposed (Lefebvre, 1991, 1995) a model of Subject facing a choice among a set of alternatives. In the model the Subject is represented by function $X_1 = f(x_1, x_2, x_3)$ where $X_1, x_1, x_2, x_3$ run over $[0, 1]$. The value of $X_1$ is interpreted as "the readiness to choose a positive pole" (Lefebvre, 1991) with probability $x_3$, and the value of $x_3$ as the Subject’s plan or intention to choose a positive pole with probability $x_3$. Variables $x_1$ and $x_2$ represent the world influence on the subject.

Furthermore, function $f$ is required to obey the following axioms:

$L_1$: $(\forall x_3 \in [0, 1])[f(0, 0, x_3) = x_3]$ ("the axiom of free choice");

$L_2$: $(\forall x_3 \in [0, 1])[f(0, 1, x_3) = 0]$ ("the axiom of credulity");

$L_3$: $(\forall x_2, x_3 \in [0, 1])[f(1, x_2, x_3) = 1]$ ("the axiom of non-evil-inclinations");

$L_4$: $(\forall i, j, k)[\{i, j, k\} = \{1, 2, 3\} \Rightarrow (\forall x_j, x_k \in [0, 1])(\exists c, c' \in \mathbb{R})(\forall x_i \in [0, 1])[f(x_1, x_2, x_3) = cx_i + c']$ ("the postulate of simplicity").

Through this model Lefebvre gave explanations of several psychological experiments putting it in the spotlight (e.g., see bibliography in (Lefebvre, 1995)). It is, however, worthwhile to ponder if the model is mainly a compact representation (i.e., a "roll-up") of certain empirical data or whether it describes a fundamental structure governing human behavior.

In order to substantiate his model Lefebvre used various arguments including the well-known "antropic principle" (Lefebvre, 1995). In addition to our previous comments (Bulitko, 1997), in the following we present an alternative justification to Lefebvre’s model rooted in the theory of the approximating forms presented in the prior sections.

First, we show a reduction of the general case to the boolean case. Second, we demonstrate that the system of the first three axioms by Lefebvre can be replaced with a postulate of special poset $(M, \leq_M)$ and a special algorithm computing a decision (choice). Namely, the poset can be chosen in the form of a linear ordered three-element set. We furthermore suggest a natural interpretation of such poset $(M, \leq_M)$ and the algorithm.

5.1. Lefebvre’s ensembles

It is easy to check that in the boolean case $(X_1, x_1, x_2, x_3 \in \{0, 1\})$ the axioms $L_1 - L_3$ completely define $f$. Namely, $f(x_1, x_2, x_3) = (x_3 \rightarrow x_2) \rightarrow x_1$. The "postulate of simplicity" $L_4$ sets $f$ on the interior of the three-dimensional cube $[0, 1]^3$ in the real-valued case.

Let us consider a set $Q$ of Subjects $s_i$ each being described by a probabilistic collection $\alpha_i$ of values of the boolean variables $(n_1, n_2, n_3)$. Let us assume that the probability of encountering a Subject with a collection $\alpha$ of the variable values in $Q$ is equal to $p_\alpha$.

If behavior $z_i$ of each $s_i \in Q$ is described by the function $(n_3 \rightarrow n_2) \rightarrow n_1$ then we refer to $Q$ as Lefebvre’s ensemble (L-ensemble or simply ensemble) $(Q, P)$ with characteristic $P = (p_0, \ldots, p_7)$. We call elements of the L-ensemble L-Subjects. Here $p_k$ denotes $p_\alpha$ and $k$ is the decimal representation of the binary sequence $\alpha$.

Ensemble $(Q, P)$ averaging boolean variables $n_1, n_2, n_3, z_i$ yields real numbers $x_1, x_2, x_3, z \in [0, 1]$. Given the truth table of the boolean function $n_3 \rightarrow n_2 \rightarrow n_1$ elementary probabilistic considerations lead to the
following equalities:

\[ 1 = \sum_{k=0}^{n} p_k, \quad (1) \]
\[ x_1 = p_4 + p_5 + p_6 + p_7, \quad (2) \]
\[ x_2 = p_2 + p_3 + p_6 + p_7, \quad (3) \]
\[ x_3 = p_1 + p_3 + p_5 + p_7, \quad (4) \]
\[ z = p_1 + p_4 + p_5 + p_6 + p_7. \quad (5) \]

It is therefore reasonable to inquire which \( L \)-ensembles \((Q, P)\) values of \( x_1, x_2, x_3, z \) satisfy Lefebvre’s equation \( z = x_1 + (x_1 - x_2 + x_2 x_3)x_3 \).

The following examples show that, generally speaking, \( z \neq f (x_1, x_2, x_3) \). Indeed, let us set \( p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = p_7 = 0.1 \). Then \( x_1 = x_2 = x_3 = 0.4 \) and \( f (x_1, x_2, x_3) = 0.544 \). However, the ensemble average \( z \) equals 0.5. Interestingly enough, the difference can be quite substantial as the following example demonstrates. Namely, \( p_0 = p_1 = p_2 = p_4 = p_6 = p_7 = 0, p_3 = 5 = 0.5 \) correspond to \( x_1 = x_2 = 0.5, x_3 = 1 \). Then \( z = 0.5 \) but \( f(0.5, 0.5, 1) = 0.75 \). Thus, the error can reach at least 30%.

On the other hand, equality \( z = f(x_1, x_2, x_3) \) is met for all possible (i.e., obeying equations (1)-(4)) characteristics \( P \) when \((x_1, x_2, x_3) \in \{(x_1, x_2, x_3) | (x_1 = 0) \cup (x_1, x_2, x_3) | (x_1 = 1) \} \cup \{(x_1, x_2, x_3) | (x_1 = 0) \cup (x_1, x_2, x_3) | (x_1 = 1) \} \cup \{(x_1, x_2, x_3) | (x_1 = 0) \cup (x_1, x_2, x_3) | (x_1 = 1) \} \).

**Proposition 1.** For every collection \( x_1, x_2, x_3 \in [0, 1] \) there exists an \( L \)-ensemble \((Q, P(x_1, x_2, x_3)) \) with characteristic \( P(x_1, x_2, x_3) \) such that \( z = f(x_1, x_2, x_3) \).

**Proof.** Let us consider three independent boolean random variables \( \zeta, \eta, \theta : \mathbb{N} \rightarrow \{0, 1\} \) with the mean values \( x_1, x_2, x_3 \) correspondingly. Then random variable \((\zeta, \eta, \theta) : \mathbb{N} \rightarrow \{0, 1\}^3 \) runs over the desired ensemble \((Q, P(x_1, x_2, x_3)) \). For the \( i \)-th component of the characteristic \( p_1(x_1, x_2, x_3) = \prod_{j=1,2,3} (1 - \sigma_j + (-1)^{1-\sigma_j} x_j) \) holds where \( \sigma_j \in \{0, 1\}, j = 1, 2, 3, \) and \( i = \sum_{j=1,2,3} \sigma_j 2^{3-j} \). A simple verification shows that the relations (1)-(4) are fulfilled and if \( z \) satisfies (5), then \( z = f(x_1, x_2, x_3) \).

We call the ensembles described in this proposition pure Lefebvre’s ensembles (PL-ensembles). Thus, a PL-ensemble is a collection of \( L \)-subjects with random parameters \((n_1, n_2, n_3) \) distributed independently in such a way that the probability \( P\{n_1 = 1\} \) equals the given number \( x_1, x_1 \in [0, 1], i = 1, 2, 3 \).

L-ensembles seems to be a more flexible means than Lefebvre’s real number function \( f \) for some aspects. For example, let us consider how ”golden section” for categorization of stimuli without measurable intensity can be explained in terms of Lefebvre’s theory (Lefebvre, 1995, p.51) and in terms of PL-ensembles.

In this case Lefebvre adds equation \( x_1 = x_2, x_1 = 1 - x_3 \) to his ”Realist’ condition” \( x_3 = f (x_1, x_2, x_3) \) (an justification is given in (Lefebvre, 1995, p.51)). In turn, that yields the equation \( x_3 = 2x_3 + 1 = 0 \) for the choice of \( x_3 \). One possible solution is the well known ”golden section” \( x_3 = \frac{\sqrt{5} - 1}{2} \).

Following the alternative approach, we construct the desired PL-ensemble by first postulating the boolean ”Realist’ condition” \( n_3 \rightarrow n_2 \rightarrow n_1 = n_3 \). Then considering the truth area \( R = \{000, 001, 010, 101, 111\} \) of the condition we form the ensemble by means of boolean random variables \( \zeta, \eta, \theta \) in the following fashion. The variables \( \zeta, \eta \) are independent with the mean value of \( 1 - x_3 \), and the value of the random variable \( \theta \) depending on the values of \( \zeta, \eta \) as illustrated in Table 5.1.

It is important that in the first line of the table value 1 is chosen with the probability of \( x_3 \). Thus if \( x_3 \) satisfies \( x_3 = 2x_3 + 1 = 0 \) then we obtain the desired PL-ensemble. Indeed, every element of the ensemble is a ”Realist” and the probability to encounter an \( L \)-Subject with parameters \((n_1, n_2, 1) \) is determined by solutions to the equation \( x_3 = 2x_3 + 1 = 0 \). Finally, we arrive at the ”golden section” choosing the corresponding solution exactly as it was done by Lefebvre.

We believe that the \( L \)-ensemble tool provides additional opportunities for Lefebvre’s theory and its applications. Indeed, the ensemble structure is a powerful parameter for modelling because it can vary even though the average values are fixed.
### TABLE 1
The solution list for boolean equation $n_3 \rightarrow n_2 \rightarrow n_1 = n_3$.

| ζ | η | θ |
|---|---|---|
| 0 | 0 | 0.1 |
| 0 | 1 | 0 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

**FIG. 1.** Subject structure in Lefebvre’s model.

#### 5.2. Application of approximating forms

Now we propose an alternative model of the binary choice for the same inputs and outputs. This model is constructed within the framework of the theory proposed in this paper.

First we need to represent every choice that a Subject makes in Lefebvre’s model as a solution of the corresponding extremalization problem of the aforementioned kind. Second we will provide an algorithm for extremalization that computes results in concordance with Lefebvre’s theory.

The problem to pose such a extremalization problem is not trivial. However, in our case it can be solved easily on the basis of the interpretation of $x_i, i = 1, 3$, given by Lefebvre. Indeed, on one hand $x_1, x_2, x_3$ are connected to the motivations: $x_1$ corresponds to the impulse (we continue to use Lefebvre’s terms) induced by the external world, $x_2$ corresponds to the impulse induced by Subject’s experience, and, finally, $x_3$ corresponds to Subject’s will. On the other hand, the values of these variables describe objectives of the impulses. Thus, in Lefebvre’s model (boolean value of variable $x_i$ equals to 1(0)) if and only if (motivation $x_i$ pushes the Subject to the positive(negative) pole).

In order to avoid the ambiguity we will denote the boolean value of variable $x_i$ in bold: $x_i$. There are just eight problems of choice in Lefebvre’s model as: $S = \{(x_1, x_2, x_3)|x_i \in \{0, 1\}, i = 1, 3\}$. For each of these problems function $f$ computes a chosen pole $z$ (Figure 1).

It is easy to check that $z \in \{x_1, x_2, x_3\}$. So in order to be accurate one needs to reconstruct impulse $z$ that stands behind $z$ and is implicit in Lefebvre’s model. Thus it is logical to think that Subject chooses one of the initial impulse set which is defined by a current choice problem. Then the Subject tries to implement the chosen impulse. Therefore within the framework of the two-preference scheme we ought to set $M = \{x_1, x_2, x_3\}$.

Then for a given boolean 3-tuple $(x_1, x_2, x_3)$ one needs to propose a routine computing $\leq_M, (L, \leq_L)$, evaluating mapping $\psi$, and an optimizing algorithm $\mathfrak{B}$ in such a way that for every input boolean 3-tuple $\mathfrak{B}$ computes an extremum $x_i$ obeying condition $x_i = f(x_1, x_2, x_3)$. We can do this so that $\psi$ depends on $(x_1, x_2, x_3)$ only.

Our further consideration is based mainly on theorem $\mathfrak{B}$ which postulates the existence of a universal...
representation of all mapping of kinds \( \psi : M \to L \) when preferences \((M, \leq_M), (L, \leq_L)\) are fixed. The representation operates with the set of \( \theta \)-functions.

First, using the universality it is possible to define any evaluating mapping \( \psi \) by means of an appropriate substitution of \( \theta \)-functions into the corresponding universal form. For that it is enough to link any \((x_i, x_i)\) with an appropriate \( \theta \)-function.

Second, one needs to use the entire set \( \Theta \). Taking into account that different 3-tuples define different choice problems we come to

\[
|\Theta| = |\{(x_i, x_i) | i = 1, 3, x_i \in \{0, 1\}\}| = 6.
\]

It is easy to see that this is possible only when \((M, \leq_M), (L, \leq_L)\) are linear orderings and \(|L| = 2\). So we may define the external preference by equalities:

\[
L = \{0, 1\}, \leq_L = \{(0, 0), (0, 1), (1, 1)\}.
\]

Further we choose the following linear order as the internal preference:

\[
\leq_M = \{(x_1, x_1), (x_2, x_2), (x_3, x_3), (x_1, x_2), (x_2, x_3), (x_1, x_3)\}.
\]

This is because in the considered case the internal (Subjective) preference might be based on a degree of dependence of the states on Subject’s will. The world pressure \(x_1\) depends on Subject to the least extent. On the contrary, the dependence of \(x_3\) on the Subject is maximum. So the degree of dependence of \(x_2\) on the Subject lies in between the those two. The model can be now finalized (Figure 2).

Every choice problem the Subject is faced with can be characterized by a certain boolean 3-tuple \((x_1, x_2, x_3)\) of values of variables \(x_1, x_2, x_3\). We associate a particular evaluating mapping (“pure evaluation”) \(\theta^M_i \in \Theta\) with pair \((x_i, x_i), i = 1, 2, 3, x_i \in \{0, 1\}\). Thus:

\[
\theta^M_i(x_k) =
\begin{cases} 
  1, & \text{if } i < k, \\
  x_i, & \text{if } i = k, \\
  0, & \text{if } k < i.
\end{cases}
\]

For a given external preference we can set \(\sqsupseteq = \rightarrow^*\) (\(\rightarrow^*\) is the boolean operation dual to implication, see lemma 1 above). Hence every evaluation mapping can be determined by the following formula:

\[
\psi_{x_1, x_2, x_3} = \theta^M_1 \rightarrow^* (\theta^M_2 \rightarrow^* \theta^M_3).
\]

Thus, one can consider evaluation mapping to be some sort of ”mixture” of pure evaluations. (It is worth noting a vague analogy with quantum mechanics here. Lefebvre discussed a relation of his model to the mechanics in (Lefebvre, 1991)).

Furthermore, \(\psi_{x_1, x_2, x_3} : M \to \{0, 1\}\) determines Subject’s choice of state \(x_i, i = 1, 2, 3\), for any given choice problem and thereby the pole \(\psi_{x_1, x_2, x_3}(x_i)\).
In order to find a maximum value of a mapping of kind $\psi' : (M', \leq_{M'}) \rightarrow (L', \leq_{L'})$ when $(M', \leq_{M'})$, $(L', \leq_{L'})$ are linear orders, one can use an easy algorithm based on the representation from theorem 2. We do not formulate the algorithm or prove its correctness here. Instead, we formulate a simplification of the algorithm $A$ for $\psi_{x_1, x_2, x_3}$ and $(M, \leq_M), (L, \leq_L)$ that we have set above:

1. Starting in the state $x_1$ in order $(M, \leq_M)$ proceed to the nearest maximum $x_{j1}$ of $\theta_{x_1}^{x_1}$.
2. Continue from the state to the nearest minimum $x_{j2}$ of function $\theta_{x_2}^{x_2}$ (due to its place in the approximative form for $\psi_{x_1, x_2, x_3}$).
3. Finally, starting from $x_{j2}$ proceed to the nearest maximum $z$ of $\theta_{x_3}^{x_3}$.

The algorithm computes element $z$ of $\arg\max \psi_{x_1, x_2, x_3}$. Having the solution we know the pole $z$ chosen by Subject for parameters $x_1, x_2, x_3$. The results are presented in Table 2.

| parameters | choice by $A$ | choice by $a$ |
|------------|---------------|---------------|
| $x_1$      | $x_2$         | $F$           | $z$ | $f$ |
| 0          | 0             | 0             | $x_2$ | 0 |
| 0          | 0             | 1             | $x_3$ | 1 |
| 0          | 1             | 0             | $x_1$ | 0 |
| 0          | 1             | 1             | $x_3$ | 1 |
| $x_1$      | 0             | 1             | $x_2$ | 1 |
| $x_1$      | 1             | 0             | $x_3$ | 1 |
| $x_1$      | 1             | 1             | $x_3$ | 1 |

TABLE 2
Results produced by algorithms $A$ and $a$.

It turns out that boolean value $f(x_1, x_2, x_3)$ computed with algorithm $a$ for all boolean 3-tuples $(x_1, x_2, x_3)$ coincides with the value given by formula $(x_3 \rightarrow x_2) \rightarrow x_1$. Thus, algorithm $a$ de facto optimizes the external preference in concordance with Lefebvre’s axioms. Indeed, if $x_1 = 1$ then nothing happens: the start state $x_1$ is the result of the choice. Hence, it is in accordance with axiom $L_3$. Otherwise, if $x_1 = x_2 = 0$ then $x_3$
or $x_2$ are chosen. In both of these cases the boolean value of chosen variable coincides with the $x_3$ (axiom $L_1$). Otherwise, $x_1 = 0$ & $x_2 = 1$ and the algorithm chooses $x_1$. This corresponds to axiom $L_2$. Thus, we are able to derive these axioms from the algorithm.

5.3. Discussion

As shown above, the formula of human behavior proposed by Lefebvre can be derived from our model given certain specific preferences and optimization algorithm $a$. Therefore, Lefebvre’s subjects appear distinguished merely by particular internal and external orders $(M, \leq_M), (L, \leq_L)$.

Instead of evaluation mapping (pure evaluation) one may use preference (pure preference) induced by it. In our model any initial Subject’s impulse $(x_1, x_2)$ is linked to the partial order induced by mapping $\theta_{x_i}$. This order contributes to the external preference induced by $\psi_{x_1, x_2, x_3}$.

It should be noted that the statements are worded using ‘extremes’ and not ‘maxima’ and ‘minima’. This is so because the Subject can use an approximation to the exact algorithm if the latter is overly complex for it. Often such an approximation is sufficient in practice.

One of the key strengths of our approach is the natural generalization of the model for more than three states. In particular, this is applicable when the Subject has two or more levels of reflections.

Then, one can see that at the level of intentions (unlike the level of their boolean values) there is a difference between the case of $x_1 = x_2 = x_3 = 0$ when the algorithm $a$ computes $x_2$ and the case of $x_1 = x_2 = 0, x_3 = 1$ when the algorithm computes $x_3$. Thus, it appears that we can’t exactly follow Lefebvre’s reasoning on the ”free will” when $x_1 = x_2 = 0$.

If we adopt behavior function $F$ instead of $f$ then we would lose the opportunity to explain the ”golden section” effect considered in the previous subsections. Therefore, in our model the inexact algorithm appears to be the real cause of the effect.

It may seem that $x_2 < x_1$ ought to hold since we interpret $x_2$ as the ”past experience” and $x_1$ as the ”current pressure of the environment”. However, one should keep in mind that we are currently dealing with an internal order on states in the process of decision-making. In that process ”past experience” $x_2$ serves the role of Subject’s ”current base” and it is $x_1$ that initiates decision-making. Variable $x_3$ is a means to produce a solution and as such is most likely related to the future.

6. CONCLUSIONS

As the paper demonstrates, the classical two-valued propositional logic can be viewed as a realization of the method of successive approximations for a decision-making problem within a framework of Subject-in-an-environment survival.

Consequently, the classical propositional logic can take its beginning from the survival problem. It is important that such hypothetical origin of the logic appears quite natural.

Furthermore, this approach can serve as a background for considering other families of mappings from one poset to another with a chosen notion of simplicity of mapping. These families can generate corresponding logics. So one may say that the psychological effects described via Lefebvre’s model considered above can be interpreted as a logic rooted in evaluation functions of the kind:

$$\psi : (\{1, 2, 3\}, \{1 < 2, 1 < 3, 2 < 3\}) \rightarrow (\{0, 1\}, \{0 < 1\})$$

implemented with a limited algorithm of extremum finding.

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