SUBGROUPS OF THE MAPPING CLASS GROUP
AND QUADRUPLE POINTS OF REGULAR HOMOTOPY

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Abstract. Let $F$ be a closed orientable surface. If $i, i' : F \to \mathbb{R}^3$ are two regularly homotopic generic immersions, then it has been shown in [N] that all generic regular homotopies between $i$ and $i'$ have the same number mod 2 of quadruple points. We denote this number by $Q(i, i') \in \mathbb{Z}/2$. We show that for any generic immersion $i : F \to \mathbb{R}^3$ and any diffeomorphism $h : F \to F$ such that $i$ and $i \circ h$ are regularly homotopic,

$$Q(i, i \circ h) = \left( \text{rank}(h_* - Id) + (n + 1)\epsilon(h) \right) \mod 2,$$

where $h_*$ is the map induced by $h$ on $H_1(F, \mathbb{Z}/2)$, $n$ is the genus of $F$ and $\epsilon(h)$ is 0 or 1 according to whether $h$ is orientation preserving or reversing, respectively.

1. Introduction

For $F$ a closed surface and $i, i' : F \to \mathbb{R}^3$ two regularly homotopic generic immersions, we are interested in the number mod 2 of quadruple points occurring in generic regular homotopies between $i$ and $i'$. It has been shown in [N] that this number is the same for all such regular homotopies, and so it is a function of $i$ and $i'$ which we denote $Q(i, i') \in \mathbb{Z}/2$. There then arises the problem of finding explicit formulae for $Q(i, i')$.

Assuming $F$ is orientable, we give an explicit formula for $Q(i, i \circ h)$, where $i : F \to \mathbb{R}^3$ is any generic immersion and $h : F \to F$ is any diffeomorphism such that $i$ and $i \circ h$ are regularly homotopic (Theorem [5.7]). For two special cases a formula for $Q(i, i \circ h)$ has already been known: The case where $F$ is a sphere has appeared in [MB] and [N], and the case where $F$ is a torus and $i$ is an embedding has appeared in [N].

Based on the Smale-Hirsch Theorem, Pinkall in [P] gave a useful tool for determining when two immersions are regularly homotopic, namely, any immersion $i : F \to \mathbb{R}^3$ induces a quadratic form $g^i : H_1(F, \mathbb{Z}/2) \to \mathbb{Z}/2$, and two immersions $i, i' : F \to \mathbb{R}^3$ are regularly homotopic iff $g^i = g^{i'}$. Let $\widehat{\mathcal{M}}$ denote the group of all diffeomorphisms $h : F \to F$ up to isotopy. Given $i : F \to \mathbb{R}^3$ we are interested in the group of all $h \in \widehat{\mathcal{M}}$ such that $Q(i, i \circ h)$
is defined, that is the group of all \( h \in \hat{M} \) such that \( i \) and \( i \circ h \) are regularly homotopic. It follows from the above criterion that this is precisely the group \( \hat{M}_{g'} \) of all \( h \in \hat{M} \) which preserve the quadratic form \( g^i \) on \( H_1(F, \mathbb{Z}/2) \). We are thus lead to study the groups \( \hat{M}_g \), starting with their index 2 subgroup \( M_g \) of orientation preserving maps.

The plan of the paper is as follows: In Section 2 we present the known results on quadratic forms which we will need. In Section 3 we show that the expression \( \text{rank}(T - \text{Id}) \mod 2 \) appearing in our proposed formula for \( Q(i, i \circ h) \) defines a homomorphism on appropriate subgroups of \( GL(H_1(F, \mathbb{Z}/2)) \) (Theorem 3.3). In Section 4 we show that (except for one special case) the group \( M_g \) is generated by Dehn twists and squares of Dehn twists (Theorem 4.9). Our main result is then proved in Section 5. We start with surfaces of genus 0 and 1 where we compute \( Q(i, i \circ h) \) by giving explicit regular homotopies for generators of \( \hat{M}_{g'} \). We then continue by induction on the genus, using the special nature of the generators of \( M_{g'} \) that we have found and of an additional generator for \( \hat{M}_{g'} \). Each generator respects a separation of \( F \) into surfaces of smaller genus, a fact which will enable us to construct regular homotopies for \( F \) by combining regular homotopies of surfaces of smaller genus.

2. Quadratic Forms over \( \mathbb{Z}/2 \)

In this section we summarize the definitions and known properties of quadratic forms over \( \mathbb{Z}/2 \) which will be needed in our work. Proofs to all facts stated in this section may be found in [C], except for those relating to the Arf invariant, which may be found in [L].

Let \( V \) be a finite dimensional vector space over \( \mathbb{Z}/2 \). A function \( g : V \rightarrow \mathbb{Z}/2 \) is called a quadratic form if \( g \) satisfies: \( g(x + y) = g(x) + g(y) + B(x, y) \) for all \( x, y \in V \), where \( B(x, y) \) is a bilinear form. The following properties follow: (a) \( g(0) = 0 \). (b) \( B(x, x) = 0 \) for all \( x \in V \). (c) \( B(x, y) = B(y, x) \) for all \( x, y \in V \). \( g \) is called non-degenerate if \( B \) is non-degenerate, i.e. for any \( 0 \neq x \in V \) there is \( y \in V \) with \( B(x, y) \neq 0 \).

**Proposition 2.1.** If \( g \) is non-degenerate then \( V \) is necessarily of even dimension and there exists a basis \( a_1, \ldots, a_n, b_1, \ldots, b_n \) for \( V \) such that \( B(a_i, a_j) = B(b_i, b_j) = 0 \) and \( B(a_i, b_j) = \delta_{ij} \) for all \( 1 \leq i, j \leq n \) and such that one of the following two possibilities holds:

1. \( g(a_i) = g(b_i) = 0 \) for \( i = 1 \ldots n \).
2. \( g(a_1) = g(b_1) = 1 \) and \( g(a_i) = g(b_i) = 0 \) for \( i = 2 \ldots n \).
$g$ is completely determined by the values $g(v_i)$ and $B(v_i, v_j)$ on a basis $v_1, \ldots, v_{2n}$ and so for given dimension $2n$ there are two isomorphism classes of non-degenerate quadratic forms, and they are in fact distinct. The invariant $\text{Arf}(g) \in \mathbb{Z}/2$ is then defined to be $0$ or $1$ according to whether $1$ or $2$ of Proposition 2.1 holds respectively. (In the more general setting of [C], this is equivalent to $g$ having index $n$ or $n-1$ respectively.) The Arf invariant is additive in the following sense:

**Proposition 2.2.** If $g_i : V_i \to \mathbb{Z}/2$, $i = 1, 2$, are non-degenerate quadratic forms, then $g_1 \oplus g_2 : V_1 \oplus V_2 \to \mathbb{Z}/2$ defined by $(g_1 \oplus g_2)(x_1, x_2) = g_1(x_1) + g_2(x_2)$ is a non-degenerate quadratic form with $\text{Arf}(g_1 \oplus g_2) = \text{Arf}(g_1) + \text{Arf}(g_2)$.

From now on we will always assume that our quadratic form $g$ is non-degenerate.

**Proposition 2.3.** If $a_1, \ldots, a_k \in V$ are independent and $B(a_i, a_j) = 0$ for all $1 \leq i, j \leq k$ then there are $b_1, \ldots, b_k \in V$ with $B(b_i, b_j) = 0$ and $B(a_i, b_j) = \delta_{ij}$ for all $1 \leq i, j \leq k$. ($a_1, \ldots, a_k, b_1, \ldots, b_k$ are then necessarily independent.)

A linear map $T : V \to V$ is called orthogonal with respect to $g$ if $g(T(x)) = g(x)$ for all $x \in V$. It then follows that $B(T(x), T(y)) = B(x, y)$ for all $x, y \in V$ and that $T$ is invertible. The group of all orthogonal maps of $V$ with respect to $g$ will be denoted $O(V, g)$.

**Definition 2.4.** Given $a \in V$, define $T_a : V \to V$ by $T_a(x) = x + B(x, a)a$.

**Proposition 2.5.** $T_a \in O(V, g)$ iff $g(a) = 1$ or $a = 0$.

**Theorem 2.6** (Cartan, Dieudonné). Except for the case when $\dim V = 4$ and $\text{Arf}(g) = 0$, $O(V, g)$ is generated by the elements $T_a$ with $g(a) = 1$.

Theorem 2.6 will also follow from Theorem 4.9 below. See Remark 4.10.

If $W \subseteq V$ is a subspace, then the conjugate space $W^\perp$ of $W$ is defined by $W^\perp = \{x \in V : B(x, y) = 0 \text{ for all } y \in W\}$. If $a \in V$ we similarly define $a^\perp = \{x \in V : B(x, a) = 0\}$. Let $\text{Id}$ denote the identity map on $V$, let $\text{Im}(T)$ denote the image of $T$ and let $F(T) = \{x \in V : T(x) = x\}$.

**Proposition 2.7.** If $T \in O(V, g)$ then $\text{Im}(T - \text{Id}) = (F(T))^\perp$. 

3. A Homomorphism from $O(V,g)$ to $\mathbb{Z}/2$.

Let $a \in V$, then $F(T_a) = a^\perp$ and so if $a \neq 0$ $\dim F(T_a) = 2n - 1$, where $2n = \dim V$.

**Lemma 3.1.** Let $T \in O(V,g)$ and $a \in V$ with $g(a) = 1$.

1. If $F(T) \subseteq F(T_a)$ then $\dim F(T \circ T_a) = \dim F(T) + 1$.
2. If $F(T) \not\subseteq F(T_a)$ then $\dim F(T \circ T_a) = \dim F(T) - 1$.

**Proof.** We first note: (a) If $x \notin a^\perp$ then $B(x,a) = 1$ so $T_a - T(x+a) = T(x+a)$, and so $T \circ T_a = \perp$ (Proposition 2.2) implies that $F(T \circ T_a) \not\subseteq a^\perp$. Since $a^\perp = F(T_a)$ we conclude that $F(T \circ T_a) \not\subseteq F(T_a)$.

Clearly $F(T) \cap F(T_a) = F(T \circ T_a) \cap F(T_a)$ and let $k$ denote the dimension of this subspace. $F(T_a)$ is of codimension 1 in $V$ and so it follows: (1) If $F(T) \subseteq F(T_a)$ then $\dim F(T) = k$ and $F(T \circ T_a) \not\subseteq F(T_a)$ and so $\dim F(T \circ T_a) = k + 1$. (2) If $F(T) \not\subseteq F(T_a)$ then $\dim F(T) = k + 1$ and $F(T \circ T_a) \not\subseteq F(T_a)$ and so $\dim F(T \circ T_a) = k$.

We now define $\psi : O(V,g) \to \mathbb{Z}/2$ by:

$$\psi(T) = \text{rank}(T - \text{Id}) \mod 2.$$  

**Remark 3.2.** Since $F(T) = \ker(T - \text{Id})$ (or by Proposition 2.7) we may also write: $\psi(T) = \text{codim}F(T) \mod 2$, and since $V$ is of even dimension we also have: $\psi(T) = \dim F(T) \mod 2$.

**Theorem 3.3.** $\psi : O(V,g) \to \mathbb{Z}/2$ is a (non-trivial) homomorphism.

**Proof.** We will be using the equivalent definition $\psi(T) = \dim F(T) \mod 2$ of Remark 3.2. Assume first that $(V,g)$ is not of the special case excluded from Theorem 2.6, and so $O(V,g)$ is generated by the elements $T_a$ with $g(a) = 1$. If $T = T_{a_1} \circ \cdots \circ T_{a_k}$ $(g(a_i) = 1)$ then since $\psi(Id) = \dim V \mod 2 = 0$, induction on Lemma 3.1 implies $\psi(T) = k \mod 2$ which clearly implies that $\psi$ is a homomorphism.

We are left with the case $\dim V = 4$, $\text{Arf}(g) = 0$. By Proposition 2.2, $(V,g) \cong (V' \oplus V', g' \oplus g')$ where $\dim V' = 2$, $\text{Arf}(g') = 1$. We identify $V$ with $V' \oplus V'$ via such an isomorphism. The set of all elements in $V$ with $g = 1$ is $V_1 \cup V_2$ where $V_1 = \{(x,0) : 0 \neq x \in V'\}$ and
$V_2 = \{(0, x) : 0 \neq x \in V'\}$. If $a \in V_1$ and $b \in V_2$ then $B(a, b) = 0$, whereas if $a \neq b$ are in the same $V_k$ then $B(a, b) = 1$. It follows that any $T \in O(V, g)$ must either map each $V_k$ into itself or map $V_1$ into $V_2$ and $V_2$ into $V_1$. So $T$ is of the form $(x, y) \mapsto (T_1(x), T_2(y))$ or $(x, y) \mapsto (T_1(y), T_2(x))$ where $T_1, T_2 \in O(V', g') (= GL(V')).$ Such a map will be denoted by $(T_1, T_2)_0$ or $(T_1, T_2)_1$ respectively. If $T = (T_1, T_2)_0$ then $T(x, y) = (x, y)$ iff $T_1(x) = x$ and $T_2(y) = y$ and so $\mathbf{F}(T) = \mathbf{F}(T_1) \oplus \mathbf{F}(T_2)$ so $\dim \mathbf{F}(T) = \dim \mathbf{F}(T_1) + \dim \mathbf{F}(T_2)$ so $\psi(T) = \psi(T_1) + \psi(T_2)$. (The $\psi$ on the left is the function on $O(V, g)$ and the $\psi$ on the right is the function on $O(V', g')$.) If $T = (T_1, T_2)_1$ then $T(x, y) = (x, y)$ iff $T_1(y) = x$ and $T_2(x) = y$ so $\psi(T) = \psi(T_1 \circ T_2)$. Now, since $\dim V' = 2$, $V'$ belongs to the general case, and so we already know $\psi(T_1 \circ T_2) = \psi(T_1) + \psi(T_2)$. So we have shown for both $u = 0$ and $u = 1$ that $\psi((T_1, T_2)_u) = \psi(T_1) + \psi(T_2)$ . Now if $T = (T_1, T_2)_u$ and $S = (S_1, S_2)_{u'}$ then $T \circ S$ is of the form $(T_1 \circ S_1, T_2 \circ S_2)_{u''}$ or $(T_1 \circ S_2, T_2 \circ S_1)_{u''}$. In any case (again using the fact that $\psi$ on $V'$ is a homomorphism) we get that $\psi(T \circ S) = \psi(T_1) + \psi(T_2) + \psi(S_1) + \psi(S_2) = \psi(T) + \psi(S)$.

Finally, $\psi : O(V, g) \to \mathbb{Z}/2$ is not trivial since $\psi(T_a) = 1$ for any $a \in V$ with $g(a) = 1$.

\( \square \)

Remark 3.4. 1. For $A \in O_k(\mathbb{R})$ (the group of $k \times k$ orthogonal matrices over $\mathbb{R}$) $\text{codim} \mathbf{F}(A) = 0 \mod 2$ iff $\det A = 1$. And so by Remark 3.2, $\psi : O(V, g) \to \mathbb{Z}/2$ may be thought of as an analogue of the homomorphism $\det : O_k(\mathbb{R}) \to \{1, -1\}$ (det on $O(V, g)$ is of course trivial.)

2. Our expression for $\psi$ is meaningful on the whole of $GL(V)$, however $\psi$ is in general not a homomorphism on $GL(V)$ or even on its subgroup $Sp(V) \supseteq O(V, g)$ of maps preserving $B(x, y)$.

For $\dim V = 4, \text{Arf}(g) = 0$, we note that though the identification of $V$ with $V' \oplus V'$ in the proof of Theorem 3.3 is not unique, the (unordered) pair of sets $V_1, V_2$ is uniquely defined by its mentioned properties, namely, $V_1 \cup V_2 = \{v \in V : g(v) = 1\}, B(a, b) = 0$ for $a \in V_1, b \in V_2$, and $B(a, b) = 1$ for $a \neq b \in V_k$, $k = 1, 2$. It follows, as we have noticed, that any $T \in O(V, g)$ either preserves each $V_k$ (then $T = (T_1, T_2)_0$) or interchanges the $V_k$s (then $T = (T_1, T_2)_1$.)

Definition 3.5. Let $\dim V = 4, \text{Arf}(g) = 0$. $T \in O(V, g)$ will be called a $U$-map if $T$ interchanges $V_1$ and $V_2$. 

Lemma 3.6. Let $\dim V = 4$, $\text{Arf}(g) = 0$. If $T$ is a $U$-map such that $T^2 = \text{Id}$ then $\psi(T) = 0$.

Proof. $T = (T_1, T_1^{-1})_1$ and so by the proof of Theorem 3.3, $\psi(T) = \psi(T_1) + \psi(T_1^{-1}) = 0$. □

4. Generators for the Orthogonal Mapping Class Group

Let $F$ be a closed orientable surface. $H_1$ from now on will always denote $H_1(F, \mathbb{Z}/2)$ (considered as a vector space over $\mathbb{Z}/2$.) Let $g : H_1 \to \mathbb{Z}/2$ be a quadratic form whose associated bilinear form $B(x, y)$ is the algebraic intersection form $x \cdot y$ of $H_1$. (In particular, $g$ is non-degenerate.) Let $\mathcal{M}$ denote the mapping class group of $F$ i.e. the group of all orientation preserving diffeomorphisms $h : F \to F$ up to isotopy. For $h : F \to F$, let $h_*$ denote the map it induces on $H_1$. The orthogonal mapping class group of $F$ with respect to $g$ will be the subgroup $\mathcal{M}_g$ of $\mathcal{M}$ defined by $\mathcal{M}_g = \{h \in \mathcal{M} : h_* \in O(H_1, g)\}$.

A simple closed curve will be called a circle. If $c$ is a circle in $F$, the homology class of $c$ in $H_1$ will be denoted by $[c]$. Given a circle in $F$, a Dehn twist along $c$ will be denoted $T_c$.

Definition 4.1. A map $h : F \to F$ will be called good if it is of one of the following forms:

1. $h = (T_c)^2$ for some circle $c$.
2. $h = T_c$ for a circle $c$ with $g([c]) = 1$.
3. $h = T_c$ for a circle $c$ with $[c] = 0$.

A good map will be called of type 1, 2 or 3 accordingly.

The purpose of this section is to show that except for the special case when $\text{genus}(F) = 2$ and $\text{Arf}(g) = 0$, $\mathcal{M}_g$ is generated by the good maps. For the mentioned special case, we will show that one more generator is required.

Whenever we consider two circles in $F$, we will assume that they intersect transversally. $|c_1 \cap c_2|$ will then denote the number of intersection points between circles $c_1$ and $c_2$. (And so the algebraic intersection $[c_1] \cdot [c_2]$ in $H_1$ is the reduction mod 2 of $|c_1 \cap c_2|$.) Given two circles $a, b$ in $F$ with $|a \cap b| = 1$, there are two ways for joining them into one circle $c$ via surgery at their intersection point. $c$ will be called a merge of $a$ and $b$. If $a$ and $b$ are oriented, then
c will be called the positive or negative merge of a and b, according to whether the surgery is performed so that the orientations of a and b match or do not match, respectively.

**Lemma 4.2.** Let a, b be two oriented circles in $F$ with $|a \cap b| = 1$, let P be their intersection point and let c be their negative merge.

1. $T_c$ followed by an isotopy performed in a thin neighborhood of $a \cup b$, maps a orientation preservingly onto b.

2. If d is another circle passing P, and otherwise disjoint form a and b, and if a and b cross d at P in the same direction, then the above Dehn twist and isotopy may be performed while fixing d.

**Proof.** See Fig. 1.

![Figure 1](image1.png)

**Figure 1.**

![Figure 2](image2.png)

**Figure 2.**

**Lemma 4.3.** Let a be a circle in $F$ and b an arc connecting two points of a, and whose interior is disjoint from a. Assume that at the two endpoints of b, a passes b in the same direction (as in Fig. 2a.) Let $a', a''$ be the two parts of a into which it is separated by $\partial b$ and let $c = b \cup a'$. Then $(T_c)^2$ followed by an isotopy performed in a thin neighborhood of c, maps a onto the circle obtained by surgering a along the arc b as in Fig. 2d.

**Proof.** See Fig. 2.
The setting for the following lemma is that of sections 2, 3:

**Lemma 4.4.** Assume $(V, g)$ is not of the two special cases $\dim V = 2$, $\text{Arf}(g) = 0$ and $\dim V = 4$, $\text{Arf}(g) = 0$. Let $w_1, \ldots, w_k \in V$, $k \geq 0$, be independent vectors with $g(w_1) = 1$ and $B(w_i, w_j) = 0$ for all $1 \leq i, j \leq k$. Let $W = \langle w_1, \ldots, w_k \rangle$ (the subspace spanned by $w_1, \ldots, w_k$) and let $a_1, a_2 \in W^\perp - W$ be two vectors with $g(a_1) = g(a_2) = 1$ and $B(a_1, a_2) = 0$. Then there exists $c \in W^\perp$ with $g(c) = 1$ and $B(a_1, c) = B(a_2, c) = 1$.

**Proof.** Assume first $k > 0$. We first find $b \in W^\perp$ such that $B(a_1, b) = B(a_2, b) = 1$. Since $a_1, a_2 \notin W = (W^\perp)^\perp$ there are $b_1, b_2 \in W^\perp$ with $B(a_1, b_1) = B(a_2, b_2) = 1$. If also $B(a_1, b_2) = 1$ or $B(a_2, b_1) = 1$ then we have a $b$. Otherwise $b_1 + b_2$ is our $b$. If $g(b) = 1$ we are done with $c = b$, otherwise take $c = b + w_1$.

Now assume $k = 0$, so $W = \{0\}$ and $W^\perp = V$. If $a_1 = a_2 = a$, take $b \in V$ with $B(a, b) = 1$. If $g(b) = 1$ we are done with $c = b$, otherwise define $U = \langle a, b \rangle$. If $a_1 \neq a_2$, take $b_1, b_2 \in V$ with $B(b_1, b_2) = 0$ and $B(a_i, b_j) = \delta_{ij}$ (Proposition 2.3). If $g(b_1 + b_2) = 1$ we are done with $c = b_1 + b_2$, otherwise define $U = \langle a_1, a_2, b_1, b_2 \rangle$. In either case $B(x, y)$ is non-degenerate on $U$, and so it is non-degenerate on $U^\perp$. If $\dim V > \dim U$ it follows that $g$ cannot be identically 0 on $U^\perp$. Take any element $d \in U^\perp$ with $g(d) = 1$ then we are done with $c = b + d$ or $c = b_1 + b_2 + d$ respectively. So we are left with the case $\dim V = \dim U$. If $\dim V = 2$ then we have assumed $\text{Arf}(g) = 1$ and so we must have had $g(b) = 1$. If $\dim V = 4$ then again we have assumed $\text{Arf}(g) = 1$ and so $g(b_1) \neq g(b_2)$ (since $g(a_1) = g(a_2) = 1$) so again we must have had $g(b_1 + b_2) = 1$.

\[ \square \]

**Remark 4.5.** When $\dim V = 4$ then in the proof of Lemma 4.4 above, we haven’t used the additional assumption that $\text{Arf}(g) = 1$ in the following two cases: (1) When $k > 0$. (2) When $k = 0$ and $a_1 = a_2$ (since then $\dim V > \dim U$.)

**Lemma 4.6.** Assume $F, g$ are not of the two special cases genus($F$) = 1, $\text{Arf}(g) = 0$ and genus($F$) = 2, $\text{Arf}(g) = 0$. Let $w_1, \ldots, w_k$ be disjoint circles in $F$ with $g([w_1]) = 1$ and such that $[w_1], \ldots, [w_k]$ are independent in $H_1$ (which is equivalent to $\bigcup_i w_i$ not separating $F$.) Let $a_1, a_2$ be oriented circles in $F$ with $g([a_1]) = g([a_2]) = 1$ and such that $a_1, a_2$ are each disjoint from $\bigcup_i w_i$ and $[a_1], [a_2] \notin \langle [w_1], \ldots, [w_k] \rangle$. Then there is a sequence $h_1, \ldots, h_m$ of good maps
of type 1 and 2 which all fix $\bigcup_i w_i$ and such that $h_1 \circ \cdots \circ h_m$ (followed by an isotopy fixing $\bigcup_i w_i$) maps $a_1$ orientation preservingly onto $a_2$.

Proof. Assume first that $[a_1] \cdot [a_2] = 1$. If actually $|a_1 \cap a_2| = 1$ then we are done by Lemma 4.12 since the merge $c$ of $a_1$ and $a_2$ satisfies $g([c]) = g([a_1]) + g([a_2]) + [a_1] \cdot [a_2] = 1$ and so $T_c$ is a good map of type 2. So assume $|a_1 \cap a_2|$ is some odd number $> 1$. Then necessarily there exist two consecutive crossings along $a_2$, at which $a_1$ crosses $a_2$ in the same direction. Applying the map $(T_c)^2$ of Lemma 4.3 (a is here $a_1$ and $b$ is a portion of $a_2$) reduces $|a_1 \cap a_2|$ by precisely 2, and so again $[a_1] \cdot [a_2] = 1$ and so we may continue by induction.

Assume now $[a_1] \cdot [a_2] = 0$. By Lemma 4.4 there is $x \in H_1$ with $g(x) = 1$, $x \cdot [a_1] = x \cdot [a_2] = 1$ and $x \cdot [w_i] = 0$ for all $i$. ($x \notin \langle [w_1], \ldots, [w_k] \rangle$ follows.) There exists a circle $c$ in $F$ with $[c] = x$ and such that $c$ is disjoint from each $w_i$. (Start with any embedded representative and surger it along the $w_i$s until it is disjoint from all of them. This is possible since the number of intersection points with each $w_i$ is even. Then connect the various components to each other by surgery. This is possible since $F - \bigcup_i w_i$ is connected and since there are no orientations to consider.) By the previous case we may now map $a_1$ onto $c$, and from there, orientation preservingly onto $a_2$. \hfill \Box

Remark 4.7. If $F, g$ are of the special case genus$(F) = 2$, Arf$(g) = 0$ then in Lemma 4.6 we further assume either that $[a_1] \cdot [a_2] = 1$ or that $[a_1] = [a_2]$ or that $k > 0$ then it follows from the proof of Lemma 4.6 and from Remark 4.3, that the conclusion of Lemma 4.6 still holds.

Definition 4.8. Let genus$(F) = 2$, Arf$(g) = 0$. A map $U \in M_g$ such that $U_*$ is a $U$-map on $H_1$ (Definition 3.3) will again be called a $U$-map. (Such maps clearly exist.)

Theorem 4.9. If $F, g$ are not of the special case genus$(F) = 2$, Arf$(g) = 0$ then $M_g$ is generated by the good maps. In the mentioned special case, $M_g$ is generated by the good maps and any one $U$-map.

Proof. We first assume we are not in the two special cases appearing in Lemma 4.6, in particular, we are not in the special case of this theorem. Let $n = \text{genus}(F)$ and let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be circles such that $|a_i \cap a_j| = |b_i \cap b_j| = 0$ ($i \neq j$), $|a_i \cap b_j| = \delta_{ij}$ and $g([a_i]) = 1$. (Start with such circles without the assumption on $g$. Then if for some $i$
both circles have $g = 0$, replace one of them with their merge. Then exchange $a_i$ with $b_i$ if necessary.)

Now let $h \in \mathcal{M}_g$. We will compose $h$ with good maps until (after isotopy) we arrive at the identity. We may first use Lemma 4.6 to bring the $a_i$s one by one back to place. Indeed, after $a_1, \ldots, a_k$ are in place, consider $a_1, \ldots, a_k, h(a_{k+1})$, $a_{k+1}$, as the $w_1, \ldots, w_k, a_1, a_2$ of Lemma 4.6 respectively, assigning orientations to $a_{k+1}$ and $h(a_{k+1})$ which correspond via $h$.

So assuming $h$ fixes all $a_i$s, we bring each $b_i$ back to place, and assume we have already done this for all $i < j$. Denote $a = a_j, b = b_j$ and let $P$ be the intersection point of $a$ and $b$. Since $a$ is fixed by $h$, $h(b)$ must also pass $P$, and since $h$ is orientation preserving, $b$ and $h(b)$ must cross $a$ at $P$ in the same direction, (where orientations on $b$ and $h(b)$ correspond via $h$.) Our permanent assumption that any circles we consider intersect transversally, may still be maintained for the intersection of $b$ and $h(b)$ at $P$. Assume first that $P$ is the only intersection point between $b$ and $h(b)$. Let $c$ be the negative merge of $b$ and $h(b)$. $g([h(b)]) = g([b])$ (since $h$ preserves $g$) and so $g([c]) = g([b]) + g([h(b)]) + [b] \cdot [h(b)] = 1$. By Lemma 1.2, $\mathcal{T}_c$ and an isotopy bring $h(b)$ orientation preservingly onto $b$, and the Dehn twist and isotopy may be performed while fixing $a$ and while fixing all other $a_i$s and all $b_i$s with $i < j$. (Our $h(b), b, c$ and $a$, correspond to $a, b, c$ and $d$ of Lemma 1.2 respectively.)

So assume now that there are additional intersection points between $h(b)$ and $b$ besides $P$. Choose a side of $a$ in $F$. Let $X$ be the first additional intersection point, when moving from $P$ along $b$ into the chosen side. Let $a'$ be a circle parallel and close to $a$ on the chosen side. $g([a']) = 1$. If $a'$ is close enough to $a$, then $a'$ intersects both $b$ and $h(b)$ each at a single point, and so $\mathcal{T}_{a'}$ applied to $h(b)$ will add precisely one intersection point between $h(b)$ and $b$. We denote this new intersection point by $P'$. If we choose the orientation of the Dehn twist $\mathcal{T}_{a'}$ correctly, then $h(b)$ will cross $b$ at $P'$ in the same direction that it crosses $b$ at $X$. Let $c$ be the circle which is the union of the subarcs of $b$ and $h(b)$ which are defined by $P'$ and $X$, and which do not contain $P$, and so $c$ is disjoint from $a$. By Lemma 1.3, $(\mathcal{T}_c)^2$ applied to $h(b)$ reduces the intersection between $h(b)$ and $b$ by precisely 2. And so we have first increased the intersection by 1 and then decreased it by 2, and so we may continue by induction.

We are now in the situation that all $a_i$s and $b_i$s are fixed, so we are left with performing a map fixing all $a_i$s and $b_i$s, which is equivalent to performing a boundary fixing map on a disc with $n$ holes. The group of all such maps is known to be generated by Dehn twists. Now,
any circle in the complement of the $a_i$s and $b_i$s is bounding in $F$, and so these Dehn twists are good maps of type 3. This completes the proof for the general case.

The case genus($F$) = 1, $\operatorname{Arf}(g) = 0$ does not rely on the above, and will not be used in the sequel. We differ it to the end of Section 5.1.

We are left with the case genus($F$) = 2, $\operatorname{Arf}(g) = 0$. We will show that by adding any $U$-map $U$, the above proof can be made to work. By Remark 4.7, the only problem we have is when moving the first circle $a_1$. Let $V_1, V_2$ be the pair of subsets of $H_1$ from the definition of $U$-map, and say $[a_1] \in V_1$. If $[h(a_1)] \in V_2$ then $[U \circ h(a_1)] \in V_1$, and so we may assume $[a_1]$ and $[h(a_1)]$ are both in $V_1$. But then either $[a_1] = [h(a_1)]$ or $[a_1] \cdot [h(a_1)] = 1$. By Remark 4.7 again, Lemma 4.6 applies, and so the above process works.

Remark 4.10. If $p : \mathcal{M} \to GL(H_1)$ denotes the map $h \mapsto h_*$ then $p(\mathcal{M})$ is known to be $Sp(H_1)$, the group of maps preserving the intersection form on $H_1$. Since $O(H_1, g) \subseteq Sp(H_1)$ and $\mathcal{M}_g = p^{-1}(O(H_1, g))$, $p|_{\mathcal{M}_g} : \mathcal{M}_g \to O(H_1, g)$ is onto. Since $B(x, y)$ is unique up to isomorphism for every given dimension $2n$, we see that our Theorem 4.9 implies Theorem 2.6 (the Cartan-Dieudonne Theorem for the field $\mathbb{Z}/2$.)

5. Quadruple Points of Regular Homotopies

Let $A$ be an annulus. There are two regular homotopy classes of immersions $i : A \to \mathbb{R}^3$. This follows from the Smale-Hirsch Theorem ([H]) and the fact that $\pi_1(SO_3) = \mathbb{Z}/2$. The first class is represented by an embedding whose image lies in a plane in $\mathbb{R}^3$, and the second class, by an embedding that differs from the former by one full twist. For an immersion $i : A \to \mathbb{R}^3$ we let $G(i) \in \mathbb{Z}/2$ be 0 or 1 according to whether $i$ belongs to the first or second class, respectively.

Definition 5.1. If $X \subseteq Y$ then $N(X, Y)$ will denote a regular neighborhood of $X$ in $Y$.

Let $F$ be a closed orientable surface, and $i : F \to \mathbb{R}^3$ an immersion, then $i$ determines a quadratic form $g^i : H_1 \to \mathbb{Z}/2$ whose associated bilinear form is the intersection form on $H_1$, as follows: Let $x \in H_1$, choose a circle $c$ in $F$ such that $[c] = x$ and define $g^i(x) = G(i|_{N(c, F)})$. One needs to verify that $g^i(x)$ is independent of the choice of $c$ and that $g^i(x + y) = g^i(x) + g^i(y) + x \cdot y$. This has been done in [P] in the more general setting of surfaces which are not necessarily orientable. (It is then necessary for the quadratic form to attain values in
\(\frac{1}{2}\mathbb{Z}/2 = \mathbb{Z}/4\) rather than \(\mathbb{Z}/2\), to accommodate for the half twists of Mobious bands, and the Arf invariant attains values in \(\mathbb{Z}/8\).) For immersions \(i, i' : F \to \mathbb{R}^3\), \(i \sim i'\) will denote that \(i\) and \(i'\) are regularly homotopic in \(\mathbb{R}^3\). The following has been shown in \(\text{[P]}\):

**Theorem 5.2.** Let \(i, i' : F \to \mathbb{R}^3\) be two immersions.

1. \(g^i = g^{i'}\) iff \(i \sim i'\).
2. \(\text{Arf}(g^i) = \text{Arf}(g^{i'})\) iff there exists a diffeomorphism \(h : F \to F\) such that \(i \sim i' \circ h\).
3. \(\text{Arf}(g^e) = 0\) for any embedding \(e : F \to \mathbb{R}^3\).

Let \(\widehat{\mathcal{M}}\) be the group of all diffeomorphisms of \(F\) (not necessarily orientation preserving) up to isotopy. Given a quadratic form \(g\) on \(H_1\) (whose associated bilinear form is the intersection form) let \(\widehat{\mathcal{M}}_g\) be the subgroup of \(\widehat{\mathcal{M}}\) defined by \(\widehat{\mathcal{M}}_g = \{h \in \widehat{\mathcal{M}} : h_* \in O(H_1, g)\}\). \((\mathcal{M}_g\) is then a subgroup of index 2 in \(\widehat{\mathcal{M}}_g\).) Now let \(i : F \to \mathbb{R}^3\) be an immersion and \(h : F \to F\) a diffeomorphism. By Theorem 5.2(1), \(i \sim i' \circ h\) iff \(g^i = g^{i' \circ h}\). It is easy to see that \(g^{i \circ h} = g^i \circ h_*\) and so we get:

**Proposition 5.3.** \(i \sim i' \circ h\) iff \(h \in \widehat{\mathcal{M}}_{g^i}\).

Let \(H_t : F \to \mathbb{R}^3\) be a generic regular homotopy. We denote by \(q(H_t) \in \mathbb{Z}/2\) the number mod 2 of quadruple points occurring in \(H_t\). The following has been shown in \(\text{[N]}\):

**Theorem 5.4.** Let \(F\) be any closed surface (not necessarily orientable or connected.) If \(H_t, G_t : F \to \mathbb{R}^3\) are two generic regular homotopies between the same two generic immersions, then \(q(H_t) = q(G_t)\).

**Definition 5.5.** Let \(i, i' : F \to \mathbb{R}^3\) be two regularly homotopic generic immersions. We define \(Q(i, i') \in \mathbb{Z}/2\) by \(Q(i, i') = q(H_t)\), where \(H_t\) is any generic regular homotopy between \(i\) and \(i'\). This is well defined by Theorem 5.4.

If \(H_t, G_t : F \to \mathbb{R}^3\) are two regular homotopies such that the final immersion of \(H_t\) is the initial immersion of \(G_t\), then \(H_t \ast G_t\) will denote the regular homotopy that performs \(H_t\) and then \(G_t\).

**Lemma 5.6.** Let \(i : F \to \mathbb{R}^3\) be a generic immersion. The map \(\widehat{\mathcal{M}}_{g^i} \to \mathbb{Z}/2\) given by \(h \mapsto Q(i, i \circ h)\) is a homomorphism.
Proof. Let \( h_1, h_2 \in \widehat{\mathcal{M}}_g \) and let \( H^k_t \) be a generic regular homotopy from \( i \) to \( i \circ h_k, k = 1, 2 \). Then \( H^1_t \ast (H^2_t \circ h_1) \) is a regular homotopy from \( i \) to \( i \circ h_2 \circ h_1 \) and \( q(H^1_t \ast (H^2_t \circ h_1)) = q(H^1_t) + q(H^2_t) \).

Recall that for \( T \in O(V, g) \) we have defined \( \psi(T) = \text{rank}(T - \text{Id}) \mod 2 \), and have shown that \( \psi : O(V, g) \rightarrow \mathbb{Z}/2 \) is a homomorphism (Theorem 3.3). For \( h \in \widehat{\mathcal{M}}_g \), let \( \epsilon(h) \in \mathbb{Z}/2 \) be 0 or 1 according to whether \( h \) is orientation preserving or reversing, respectively, and let \( n = \text{genus}(F) \). Since \( \epsilon : \widehat{\mathcal{M}}_g \rightarrow \mathbb{Z}/2 \) and \( h \mapsto h_* \) are also homomorphisms, the following \( \Psi : \widehat{\mathcal{M}}_g \rightarrow \mathbb{Z}/2 \) is a homomorphism:

\[
\Psi(h) = \psi(h_*) + (n + 1)\epsilon(h) = \left( \text{rank}(h_* - \text{Id}) + (n + 1)\epsilon(h) \right) \mod 2.
\]

Our purpose is to show:

**Theorem 5.7.** Let \( i : F \rightarrow \mathbb{R}^3 \) be a generic immersion. Then for any \( h \in \widehat{\mathcal{M}}_g \):

\[
Q(i, i \circ h) = \Psi(h).
\]

Let \( i : F \rightarrow \mathbb{R}^3 \) be an immersion and let \( c \) be a circle in \( F \) such that \( c \) is disjoint from the multiplicity set of \( i \). Adding a ring to \( i \) along \( c \) will mean to change \( i \) into a new immersion \( i' \) in the following way: Let \( U = N(i(c), \mathbb{R}^3) \), thin enough so that \( A = i^{-1}(U) \) is an annulus which is still disjoint from the multiplicity set. Let \( D \) and \( a \) be a disc and an arc. Let \( f_1 : a \rightarrow D \) be a proper embedding and let \( f_2 : a \rightarrow D \) be a proper immersion with one transverse intersection and such that \( f_1|_{N(\partial_a, a)} = f_2|_{N(\partial_a, a)} \). Parametrize \( U \) and \( A \) as \( D \times S^1 \) and \( a \times S^1 \) so that \( i|_A : A \rightarrow U \) will be given by \( f_1 \times \text{Id} : a \times S^1 \rightarrow D \times S^1 \). Now, the new immersion \( i' \) will be given by \( f_2 \times \text{Id} \) on \( A \), and will be the same as \( i \) outside \( A \). There are basically two ways for adding a ring to \( i \) along \( c \), depending on what side of \( A \) in \( \mathbb{R}^3 \) the ring will be facing (which in turn depends on our choice of \( f_2 : a \rightarrow D \).) If \( i \) is an embedding, then \( i(F) \) separates \( \mathbb{R}^3 \) into two pieces, one compact and one non-compact. They will be denoted \( C_i \) and \( N_i \) respectively. And so if \( i \) is an embedding then we have a natural way for distinguishing the two possibilities for adding a ring along a given circle \( c \), namely, the ring is facing either \( C_i \) or \( N_i \).

Note that \( f_1 \times \text{Id}, f_2 \times \text{Id} : A \rightarrow U \) are homotopic relative \( \partial A \) (but not regularly homotopic.) And so if \( N = N(i(F), \mathbb{R}^3) \) with \( N \supset U \), then \( i \) and \( i' \) are homotopic in \( N \) (but in general not regularly homotopic.)
We now present two moves on immersions, that have been introduced in [N]. Let $S_0$ be an annulus and $S_1$ be a disc. Move $A$ (resp. $B$) is a regular homotopy which is applied to a proper immersion of $S_0$ (resp. $S_1$) into a ball $E$ and which fixes $N(\partial S_k, S_k), k = 0, 1$. Move $A$ begins with the standard embedding of $S_0$ in $E$, and adds a ring and a Dehn twist along parallel essential circles in $S_0$. The ring may face either side of $S_0$ in $E$ and the Dehn twist may have either orientation. Fig. 3 shows one of the possibilities. The reverse move, going from right to left in Fig. 3, will be called an $A^{-1}$ move. Move $B$ is described in Fig. 4. It begins with a specific immersion of $S_1$, with two arcs of intersection, and replaces them with two other arcs of intersection. It is easy to see (and has been shown in [N]) that the initial and final immersions that we have presented for the $A$ and $B$ moves, are indeed regularly homotopic in $E$ (while keeping $N(\partial S_k, S_k)$ fixed.) Move $A$ (resp. $B$) will be applied to an immersion $i : F \to \mathbb{R}^3$ inside a ball $E$ in $\mathbb{R}^3$ for which $i^{-1}(E)$ is an annulus (resp. disc) and $i|_{i^{-1}(E)} : i^{-1}(E) \to E$ is as above. (The rest of $F$ will be kept fixed.) In particular, an $A$ move may be applied to a neighborhood of a circle $c$ in $F$ iff $c$ is disjoint from the multiplicity set of $i$ and there is an embedded disc $D$ in $\mathbb{R}^3$ such that $D \cap i(F) = i(c)$. The move will then
be performed in a thin $N(D, \mathbb{R}^3)$. If the circle along which we perform an $A$ move happens to bound a disc in $F$, then the Dehn twist that is produced is trivial, and may be cancelled by rotating this disc. The following has been shown in [N]:

**Proposition 5.8.** Let $S_0, S_1, E$ denote an annulus, disc and ball respectively.

1. For any generic regular homotopy $A_t : S_0 \to E$ that realizes an $A$ move, $q(A_t) = 1$.
2. For any generic regular homotopy $B_t : S_1 \to E$ that realizes the $B$ move, $q(B_t) = 1$.

If $H_t : F \to \mathbb{R}^3$ is given by $H : F \times [0, 1] \to \mathbb{R}^3$ then we denote by $H_{-t}$ the regular homotopy given by $(x, t) \mapsto H(x, 1 - t)$. Clearly $H_{-t}$ is generic iff $H_t$ is generic, and $q(H_t) = q(H_{-t})$.

**Lemma 5.9.** If $i \sim i'$ then $Q(i, i \circ h) = Q(i', i' \circ h)$ for any $h \in \widehat{\mathcal{M}}_{g^i} = \widehat{\mathcal{M}}_{g^{i'}}$.

**Proof.** Let $J_t$ be a regular homotopy from $i$ to $i'$ and $H_t$ a regular homotopy from $i$ to $i \circ h$. Then $J_{-t} * H_t * (J_t \circ h)$ is a regular homotopy from $i'$ to $i' \circ h$ and $q(J_{-t} * H_t * (J_t \circ h)) = q(H_t)$. □

**Remark 5.10.** After we prove Theorem 5.7, we will know that the assumption $i \sim i'$ in Lemma 5.3 is actually unnecessary, as long as $h \in \widehat{\mathcal{M}}_{g^i} \cap \widehat{\mathcal{M}}_{g^{i'}}$. This is so since $\Psi(h)$ does not depend on $i$. (It is a function of $h$ only.)

We begin the proof of Theorem 5.7. First let $\operatorname{genus}(F) = 0$. By Lemma 5.9 (and since all immersions of $S^2$ in $\mathbb{R}^3$ are regularly homotopic) we may assume $i$ is an embedding onto the unit sphere. Let $r : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection with respect to the $xy$-plane, and $h \in \widehat{\mathcal{M}}$ be such that $i \circ h = r \circ i$, then $h$ is the unique non-trivial element of $\widehat{\mathcal{M}} = \widehat{\mathcal{M}}_{g^i}$, and $\Psi(h) = \epsilon(h) = 1$. The following is a regular homotopy from $i$ to $i \circ h$: Perform an $A$ move on the equator of the sphere, such that the ring formed will be facing $N_i$. Then exchange the northern and southern hemispheres, arriving at $i \circ h$. By Proposition 5.8 the $A$ move contributed $1 \mod 2$ quadruple points, then exchanging the hemispheres involved only double curves, and so $Q(i, i \circ h) = 1$. We proceed by induction on $\operatorname{genus}(F)$ beginning with $\operatorname{genus}(F) = 1$. For the starting point $\operatorname{genus}(F) = 1$, we will need to separate the cases $\operatorname{Arf}(g^i) = 0$ and $\operatorname{Arf}(g^i) = 1$.

5.1. **The case $\operatorname{genus}(F)=1$, $\operatorname{Arf}(g^i)=0$.** This has basically been done in [N]. We present it here with slight variation. Let $T$ denote the torus. We will say an embedding $e : T \to \mathbb{R}^3$ is *standard* if its image is the torus $\tilde{T} \subseteq \mathbb{R}^3$ obtained by rotating the circle $\{y = 0, (x-2)^2 + z^2 = 1\}$ around the $z$-axis. Let $\tilde{m}, \tilde{l}$ be the circles in $\tilde{T}$ given by $\tilde{m} = \{y = 0, (x-2)^2 + z^2 = 1\}$
and \( \tilde{l} = \{ z = 0, x^2 + y^2 = 1 \} \) and choose some orientations for \( \tilde{m} \) and \( \tilde{l} \). For a standard embedding \( e : T \to \mathbb{R}^3 \), let \( m_e \) and \( l_e \) denote the oriented circles in \( T \) such that \( e(m_e) = \tilde{m} \) and \( e(l_e) = \tilde{l} \).

Since \( \text{Arf}(g^i) = 0 \) then by Theorem 5.2(2,3) \( i \) is regularly homotopic to a standard embedding. (Take an arbitrary standard embedding \( \tilde{i}' \), then \( i \sim \tilde{i}' \circ h \) for some diffeomorphism \( h : T \to T \), but \( \tilde{i}' \circ h \) is again a standard embedding.) So by Lemma 5.9 we may assume \( i \) itself is a standard embedding. By viewing \( m = m_i, l = l_i \) as the basis for \( H_1(T, \mathbb{Z}) \) (note \( \mathbb{Z} \) coefficients) we identify \( \widehat{M} \) with \( \text{GL}_2(\mathbb{Z}) \). We will think of any \( h \in \widehat{M} \) both as a map from \( F \) to \( F \) and as a \( 2 \times 2 \) matrix. If \( h \in \widehat{M} \) then \( h_* : H_1 \to H_1 \) (now \( \mathbb{Z}/2 \) coefficients) presented with respect to the basis \([m],[l]\) is simply the \( \mathbb{Z}/2 \) reduction of the matrix \( h \). \( g^i([m]) = g^i([l]) = 0 \) and for \( x = [m] + [l] \), \( g^i(x) = 1 \). A matrix \( h \in \text{GL}_2(\mathbb{Z}) \) is in \( \widehat{M}_{g^i} \) if \( h_* \) preserves \( g^i \). This will happen iff \( h_*([m]) = [m] \) and \( h_*([l]) = [l] \), i.e. iff the \( \mathbb{Z}/2 \) reduction of \( h \) is either \( I = [1 \ 0 \ 0 \ 1] \) or \( J = [0 \ 1 \ 1 \ 0] \). By means of row and column operations one can show that this subgroup of \( \text{GL}_2(\mathbb{Z}) \) is generated by the following four elements: \( A_1 = (1/0 \ 2/1), A_2 = (1/0 \ 2/1), A_3 = (-1/0 \ 0/1), A_4 = (0/1 \ 0/1) \).

Since the \( \mathbb{Z}/2 \) reduction of \( A_1, A_2, A_3 \) is \( I \) and that of \( A_4 \) is \( J \), and since \( \psi(I) = 0, \psi(J) = 1 \) and \( n = 1 \), we have \( \Psi(A_1) = \Psi(A_2) = \Psi(A_3) = 0 \) and \( \Psi(A_4) = 1 \). We will now show that the values of \( Q(i, i \circ A_k), k = 1, \ldots, 4 \) are the same.

\( A_1 \) and \( A_2 \) are \((T_m)^2 \) and \((T_l)^2 \) respectively. Let \( D \) be a compressing disc for \( i(m) = \tilde{m} \) in \( \mathbb{R}^3 \) and \( B = N(D, \mathbb{R}^3) \) thin enough so that \( i(T) \cap B \) is a standard annulus in \( B \). In \( B \) we isotope \( i(T) \cap B \) to be a thin tube, then we perform the “belt trick” on this tube (Fig. 5) and then isotope \( i(T) \cap B \) back to place. The effect of this regular homotopy is precisely \((T_m)^2 \) and it involves only double curves, and so \( Q(i, i \circ A_1) = 0 \). In the same way \( Q(i, i \circ A_2) = 0 \).

Up to isotopy in \( T \), \( i \circ A_3 = r \circ i \) where \( r \) is the reflection of \( \mathbb{R}^3 \) with respect to the \( xy \)-plane. This may be achieved by a regular homotopy similar to the one we have used for the case.
genus\( (F) = 0 \), as follows: Perform an \( A \) move on each of the two circles \( \{ z = 0, x^2 + y^2 = 1 \} \) and \( \{ z = 0, x^2 + y^2 = 3 \} \), so that the ring formed by each of them is facing \( N_i \) and such that the two Dehn twists formed will have opposite orientations and so will cancel each other. (The \( A \) moves are performed inside thin neighborhoods of compressing discs for the two circles.) So we remain with just the two rings. We may now exchange the upper and lower halves of \( T \) until we arrive at \( i \circ A_3 \). The two \( A \) moves each contributed 1 mod 2 quadruple points and the final stage involved only double curves, and so all together \( Q(i, i \circ A_3) = 0 \).

For \( A_4 \), isotope \( T \) until it has the shape of a large sphere with a tiny handle at its north pole. Now exchange the northern and southern hemispheres. This will involve only double curves, and will result in a sphere having a tiny handle at the south pole and a ring along the equator. We cancel this ring with an \( A^{-1} \) move in a thin neighborhood of the plane of the equator, resulting in an embedding again. We may think of \( m \) and \( l \) as being contained in the tiny handle, and so tiny compressing discs for \( m \) and \( l \) may be pulled along with the regular homotopy. The compressing disc of \( m \) now lies in \( N_i \), and the compressing disc of \( l \) now lies in \( C_i \). It follows that the final embedding may be isotoped to a standard embedding i.e. to a map of the form \( i \circ h \), and this \( h \) is orientation reversing and exchanges \( m \) and \( l \). And so \( h \) must be either \( (0 1 \, 1 0) \) or \( (0 -1 \, -1 0) \). These two maps composed with \( i \) are isotopic in \( \mathbb{R}^3 \) (via a half revolution about the \( x \)-axis) and so we may assume \( h = (0 1 \, 1 0) = A_4 \). Our regular homotopy had a portion involving just double curves, then one \( A^{-1} \) move, and finally some isotopy, and so \( Q(i, i \circ A_4) = 1 \).

This completes the proof that \( Q(i, i \circ h) = \Psi(h) \) for every \( h \in \hat{M}_g \) for \( F \) a torus and \( \text{Arf}(g^i) = 0 \). We now give the promised completion of the proof of Theorem 4.9. We need to show that \( M_g \) is generated by good maps when \( \text{genus}(F) = 1 \), \( \text{Arf}(g) = 0 \). Choose two oriented circles \( a, b \) with \( g([a]) = g([b]) = 0 \) and \( |a \cap b| = 1 \) as a basis for \( H_1(T, \mathbb{Z}) \), thus identifying \( M \) with \( SL_2(\mathbb{Z}) \). Again we see \( h \in SL_2(\mathbb{Z}) \) is in \( M_g \) iff its \( \mathbb{Z}/2 \) reduction is either \( I \) or \( J \). By row and column operations, we then see that \( M_g \) is generated by \( A_1 = (1 0 \, 0 1) \), \( A_2 = (1 0 \, 0 1) \) and \( A' = (0 1 \, -1 0) \). Now \( A_1 \) and \( A_2 \) are \( (T_a)^2 \) and \( (T_b)^2 \). If \( c \) is the positive merge of \( a \) and \( b \) then \( g([c]) = 1 \) and \( T_c \) is \( (0 1 \, -1 2) \). Since \( (0 1 \, -1 2) = (0 1 \, 1 2) \) we see \( M_g \) is generated by \( (T_a)^2 \), \( (T_b)^2 \) and \( T_c \).

5.2. The case \( \text{genus}(F) = 1 \), \( \text{Arf}(g^i) = 1 \). By Theorem 5.2(1), \( i \) is regularly homotopic to an immersion which is obtained from a standard embedding \( e : T \to \mathbb{R}^3 \) by adding a ring along the circle \( c \) which is the positive merge of \( m_e \) and \( l_e \), and such that the ring is facing \( N_e \).
(One checks directly that such an immersion \(i'\) has \(\text{Arf}(g^{i'}) = 1\), but then \(g^i = g^{i'}\) since on \(V\) of dimension 2 there is only one \(g\) with \(\text{Arf}(g) = 1\.) By Lemma 5.9 we may assume \(i\) itself is this new immersion (Fig. b). Since all non-zero elements in \(H_1\) have \(g^i = 1\), it follows that \(O(H_1, g^i) = GL(H_1)\) and so \(\widehat{\mathcal{M}}_{g^i} = \widehat{\mathcal{M}}\). Since \(\mathbb{Z}/2\) is abelian, it is enough to verify \(Q(i, i \circ h) = \Psi(h)\) only on normal generators, and we claim that \(B_1 = (\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix})\) and \(B_2 = (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})\) are normal generators of \(\widehat{\mathcal{M}}\) where the identification with \(GL_2(\mathbb{Z})\) is via \(m_e, l_e\). (There are no \(m_i, l_i\) since such are defined only for standard embeddings.) Indeed, \(\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}\) which as we have noticed in the end of section 5.1. is a Dehn twist along \(c\) (the positive merge of \(m_e\) and \(l_e\).)

Now, any Dehn twist is a normal generator of \(\mathcal{M}\), and since \(B_1, B_2\) are orientation reversing, they normally generate the whole of \(\widehat{\mathcal{M}}\). As above, we see that \(\Psi(B_1) = 0\) and \(\Psi(B_2) = 1\).

The regular homotopy we construct for \(B_1\) is as follows: Let the ring become thicker, and at the same time let the “body” of the torus become thinner, until they are equal in width, (Fig. b→a.) Now using the intersection circle as an axis of rotation, we perform a half revolution of the torus around it, interchanging the two equal-width rings, (Fig. b→c.) We then return to our original position by reversing our first step. (Fig. c→d.) We thus arrive at an immersion of the form \(i \circ h\). The map \(h : T \to T\) may best be understood by looking
at the intermediate stage of the regular homotopy, Fig. 3b→c. We see here that \( h \) maps \( m_e \) to \(-m_e \) and \( c \) to itself. i.e. the column \((1,0)^t \) to \((-1,0)^t \) and \((1,1)^t \) to itself. And so indeed \( h = B_1 \). We had no quadruple points (actually no singular occurrences at all) and so \( Q(i, i \circ B_1) = 0 \).

For \( B_2 \), we first imitate the regular homotopy we have had for \( A_4 = B_2 \) of Section 5.1 i.e. we perform that regular homotopy on \( e \) and carry the ring along. If before exchanging the upper and lower hemispheres we make sure that the ring is situated at the tiny handle, then this exchange will have at most triple points, and the ring will not interfere with the \( A^{-1} \) move, and so at the end of this process we will have \( q = 1 \). The immersion \( j \) we arrive at, is the immersion obtained by adding a ring \( R \) to the embedding \( e \circ B_2 \) along the circle \( e \circ B_2(c) = e(c) \), that is the same circle along which \( R \) was originally situated, but now \( R \) is facing \( C_e \) instead of \( N_e \) and so \( j \) is not of the form \( i \circ h \). We fix this with a regular homotopy as follows: Perform an \( A \) move on a little disc bounding circle in \( T \) near \( R \), forming a ring \( R' \) facing \( N_e \). See Fig. 4a. (The dotted line in Fig. 7 is the intersection curve of \( R \). \( R \) itself is not seen since it is facing \( C_e \).) We then elongate \( R' \) along side \( R \), until it approaches itself from all the way around. (Fig. 4a→b→c.) We then perform a \( B \) move, turning \( R' \) into two rings which are parallel to \( R \), but facing \( N_e \). (Fig. 4c→d.) It is then easy to construct an
explicit regular homotopy so that $R$ and the new ring which is adjacent to it, will cancel each other, and with no quadruple points at all. (The idea is as follows: Let $f : a \to D$ be a proper immersion of an arc $a$ into a disc $D$ with two loops facing opposite sides, as in the front disc of Fig. 8a. There is a regular homotopy $f_t : a \to D$ fixing $N(\partial a, a)$, from $f$ to an embedding as in the front disc of Fig. 8b. If $f_t$ is generic then it has at most triple points. Now the regular homotopy $f_t \times Id : a \times S^1 \to D \times S^1$ is a regular homotopy which begins with a pair of rings facing opposite sides, and cancels them. Fig. 8 depicts a portion of $a \times S^1$ in the initial and final immersions. Indeed since $f_t$ has at most triple points, so will $f_t \times Id$, but $f_t \times Id$ is not generic. Nevertheless, any specific $f_t \times Id$ may serve as a guide for constructing an explicit generic regular homotopy with no quadruple points, and which begins and ends with the same immersions.)

Since $R$ and one of the new rings have disappeared, we are left with one ring facing $N_e$ and situated along a circle parallel to $e(c)$. By pushing it precisely to $e(c)$ we finally get an immersion of the form $i \circ h$. The regular homotopy which started with $j$ and replaced the ring $R$ with a ring facing $N_e$, took place inside some $N = N(e(F), \mathbb{R}^3)$, and so the homotopy class into $N$ is still that of $e \circ B_2$ and so (up to isotopy in $F$) the new immersion is indeed $i \circ B_2$. ($i : F \to N$ is a homotopy equivalence, and so two diffeomorphisms $h, h' : F \to F$ are isotopic in $F$ iff $i \circ h, i \circ h' : F \to N$ are homotopic in $N$.) Finally, our regular homotopy from $i$ to $j$ involved $1 \mod 2$ quadruple points, then from $j$ to $i \circ B_2$ we had one $A$ move, one $B$ move, and a regular homotopy with no quadruple points, and so all together indeed $Q(i, i \circ B_2) = 1$. 

**Figure 8.**
5.3. **The general case.** Assume genus($F$) > 1. By Theorem 4.9, $\hat{M}_{g'}$ is generated by Dehn twists and squares of Dehn twists, and in the special case genus($F$) = 2, $\text{Arf}(g') = 0$ we also need a $U$-map. Other than the special $U$-map generator, which will be dealt with last, each generator $h$ fixes all but a regular neighborhood of a circle $a$ (and we may assume $a$ does not bound a disc in $F$.) If $a$ is non-separating then there is a circle $b$ in $F$ with $|a \cap b| = 1$. $N(a \cup b, F)$ is a punctured torus, and so $c = \partial N$ is a circle separating $F$ into two subsurfaces $F_1, F_2$ of smaller genus than $F$ and with $h(F_k) = F_k$. If $a$ is separating, then a nearby parallel circle $c$ will again separate $F$ in this way. Now $\hat{M}_{g'}$ needs one additional generator. Choosing any separating circle $c$ in $F$ there is clearly an orientation reversing $h : F \to F$ which preserves the two sides of $c$ and which induces the identity on $H_1$ and so $h \in \hat{M}_{g'}$. And so finally we have a set of generators for $\hat{M}_{g'}$, each of which (except for the $U$-map) preserves such a separation of $F$ by a circle $c$ into $F_1, F_2$ of smaller genus.

Let $A = N(c, F)$. Slightly diminishing $F_1, F_2$ to be the components of $F - \text{int} A$, we may still assume $h(F_k) = F_k, k = 1, 2$. Since $c$ is separating in $F$, $g'([c]) = 0$ and so $i|_A$ is regularly homotopic to a standard embedding of $A$, in the shape of a thin tube. By means of $|S|$ (namely the proof of Theorem 2.1) we may extend such a regular homotopy of $A$ to the whole of $F$. We now stretch this tube to be very long, at the same time pulling $F_1$ and $F_2$ rigidly away from each other until they are disjoint. See Fig. 9a. By taking a smaller $A$ if necessary, we may assume $i(A)$ is disjoint from $i(F - A)$, Fig. 9b. By Lemma 5.9, we
may assume that this is in fact our immersion \( i \). Let \( \bar{F}_1, \bar{F}_2 \) be the closed surfaces obtained by gluing a disc \( D_k \) to \( F_k \) and let \( h_k : \bar{F}_k \to \bar{F}_k \) be an extension of \( h|_{F_k} : F_k \to F_k \). If the tube \( i(A) \) is very thin, then there is also a naturally defined extension \( i_k : \bar{F}_k \to \mathbb{R}^3 \) of \( i|_{F_k} \). We may further assume that the thin ball \( B \) in \( \mathbb{R}^3 \) which is bounded by the sphere \( i_1(D_1) \cup i(A) \cup i_2(D_2) \), is disjoint from \( i(F - A) \).

Since \( h|_{F_k} \) preserves \( g^1|_{H_1(F_k, \mathbb{Z}/2)} \) then \( h_k \) preserves \( g^{i_k} \). It follows that there is a regular homotopy \( H^k \) between \( i_k \) and \( i_k \circ h_k \). We perform \( H^1_t \) and \( H^2_t \) inside disjoint balls, and we let the thin tube \( A \) be carried along. If we make sure no quadruple points occur in \( D_1 \) and \( D_2 \), and the thin tube \( A \) does not pass triple points, then the regular homotopy \( H_t \) induced on \( F \) in this way will have the sum of the number of quadruple points of \( H^1_t \) and \( H^2_t \).

Now if \( h \) is orientation preserving then so are \( h_k \), in particular \( h_k|_{D_k} \) is orientation preserving. So if we had carried the thin ball \( B \) along with the tube \( A \), then it would now approach the \( D_k \)'s from the same side it had for \( i \). And so we may continue the regular homotopy on the tube \( A \), still not passing through triple points, and cancelling all knotting by having the thin tube pass itself, until it is back to its original place, and this will not contribute any quadruple points. However, the new embedding of \( A \) may differ from \( i \circ h|_{A} \) by some number of Dehn twists as in Fig. 9c. We may resolve this by rigidly rotating say \( F_1 \) around the axis of the tube.

If on the other hand \( h \) is orientation reversing, then after applying \( H^1_t \) and \( H^2_t \) and carrying the tube along, the thin ball \( B \) will approach both \( D_k \)'s from the wrong side. And so after we cancel all knotting, the tube \( A \) will be as in Fig 9d. Fig. 10 presents a regular homotopy that resolves this, and has 1 mod 2 quadruple points. Fig. 10a depicts the relevant part of Fig. 9d, where the regular homotopy will take place. Fig. 10a→b→c is a regular homotopy with no singular occurrences, or alternatively may be thought of as an ambient isotopy of \( \mathbb{R}^3 \). It shows that we may view the immersion of \( A \) as a sphere with two rings facing outward, each of which has a tube attached to it. We now perform a \( B \) move which joins the two rings into one ring with two tubes attached to it, Fig. 10c→d. Again by ambient isotopy, the ring may be brought to the equator, Fig. 10d→e. Finally we exchange the northern and southern halves of the sphere, arriving at an embedding, Fig. 10e→f. This regular homotopy involved a \( B \) move and a portion involving only double curves, and so indeed it had 1 mod 2 quadruple points. We then continue to bring \( A \) back to place. As above, the new embedding
of $A$ may differ from $i \circ h|_A$ by some number of Dehn twists, and those may be cancelled by rigidly rotating $F_1$.

We have thus constructed a regular homotopy between $i$ and $i \circ h$ such that the number mod 2 of quadruple points, is the sum of the number occurring in the $\bar{F}_k$s in case $h$ is orientation preserving, and the sum plus 1, in case $h$ is orientation reversing. In other words

$$Q(i, i \circ h) = Q(i_1, i_1 \circ h_1) + Q(i_2, i_2 \circ h_2) + \epsilon(h).$$

By the induction hypothesis,

$$Q(i_k, i_k \circ h_k) = \Psi(h_k), \quad k = 1, 2.$$ Let $n_k = \text{genus}(F_k)$ and notice $n = n_1 + n_2$, $\epsilon(h_k) = \epsilon(h)$ and

$$\text{rank}(h_* - Id) = \text{rank}(h_{1*} - Id) + \text{rank}(h_{2*} - Id)$$

and so $\psi(h_*) = \psi(h_{1*}) + \psi(h_{2*})$. So finally:

$$Q(i, i \circ h) = \Psi(h_1) + \Psi(h_2) + \epsilon(h) = \psi(h_{1*}) + (n_1 + 1)\epsilon(h) + \psi(h_{2*}) + (n_2 + 1)\epsilon(h) + \epsilon(h) = \psi(h_*) + (n_1 + n_2 + 1)\epsilon(h) = \Psi(h).$$

We now deal with the special $U$-map generator which appears when genus$(F) = 2$, Arf$(g^i) = 0$. Since Arf$(g^i) = 0$, then by Theorem 5.2(2,3) and Lemma 5.9 as before, we may assume $i$ is an embedding whose image is two embedded tori connected to each other with a tube, and such that a half revolution around some line in $\mathbb{R}^3$ maps $i(F)$ onto itself and interchanges the two tori. Let $h : F \to F$ be the map such that $i \circ h$ is the final embedding of this half.
revolution, then \( h \) is orientation preserving and so \( h \in \mathcal{M}_{g} \). Take a circle \( c \) in one of the tori with \( g^{i}([c]) = 1 \), then \( h(c) \) lies in the other torus, and so \( [c] \neq h_{*}([c]) \) and \( [c] \cdot h_{*}([c]) = 0 \). It follows that \( [c] \) and \( h_{*}([c]) \) are not in the same \( V_{k} \) of the definition of \( U \)-map, so \( h_{*} \) must be a \( U \)-map, and so \( h \) is indeed a \( U \)-map on \( F \). Since \( h^{2} = Id \) then by Lemma 3.6, \( \psi(h_{*}) = 0 \) and so \( \Psi(h) = \psi(h_{*}) + \epsilon(h) = 0 \). Since there is a rigid rotation between \( i \) and \( i \circ h \), \( Q(i, i \circ h) = 0 \). This completes the proof of Theorem 5.7.

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