Anomaly Matching and a Hopf-Wess-Zumino Term in 6d, $\mathcal{N} = (2,0)$ Field Theories

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We point out that the low energy theory of 6d $\mathcal{N} = (2,0)$ field theories, when away from the origin of the moduli space of vacua, necessarily includes a new kind of Wess-Zumino term. The form of this term is related to the Hopf invariant associated with $\pi_7(S^4)$. The coefficient of the Wess-Zumino term is fixed by an anomaly matching relation for a global flavor symmetry. For example, in the context of a single M5 brane probe in the background of $N$ distant M5 branes, the probe must have the Hopf-WZ term with coefficient proportional to $N(N+1)$. Various related checks and observations are made. We also point out that there are skyrmionic strings, and propose that they are the $W$-boson strings.

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1. Introduction

Low energy effective field theories can have effects, generated by integrating out massive fields, which do not decouple even when the masses of these fields is taken to be infinite. The classic example is the Goldstone-Wilczek current [1], which is generated by integrating out fermions which get a mass via Yukawa coupling to a scalar which gets an expectation value. The GW current does not depend on the masses of the integrated out fermions (as long as they are non-zero), so it plays a role in the low energy theory even when the masses are taken to be infinite (via large Yukawas or scalar vevs). A related effect is the Wess-Zumino term associated with integrating out massive fields, which is often needed in the low energy theory on symmetry grounds. For example, a Wess-Zumino term, with a particular coefficient, can be needed to reproduce the contribution to the ’t Hooft anomalies of integrated out fermions, which get a mass via Yukawa couplings to a scalar which gets a vev. The size of the WZ term can not depend on any parameters (e.g. Yukawa or gauge couplings, vevs, etc.), or the RG scale: since its coefficient must be quantized [2], it can not be renormalized. See e.g. [1-3] and references cited therein.

This paper will be concerned with flavor anomaly matching and Wess-Zumino terms in the 6d $\mathcal{N} = (2,0)$ field theories. Much about these theories, including how to properly formulate them as field theories, remains mysterious. They have the exotic property of having, rather than ordinary gauge fields, interacting (somehow!, despite [3]) two-form gauge fields, with self-dual three-form field strengths. The existence of these field theories, as well as all of their known properties, has come from string theory, where they occur in various related contexts: the IR limit of the M5 or IIA NS-5 brane world-volume theory, IIB string theory on a ALE singularity [7], M theory on $AdS_7 \times S^4$ [8], etc.

The 6d $\mathcal{N} = (2,0)$ theories are interesting, and worthy of further study, both because of these connections to string theory and duality and, in their own right, as field theories. They are the maximally supersymmetric conformal field theories, in the highest possible dimension [3,10], and other interesting theories can be obtained by compactification and RG flow. For example, compactifying on a $T^2$ gives 4d $\mathcal{N} = 4$ theories and makes $SL(2,Z)$ electric-magnetic duality manifest, as the geometric symmetry of the complex structure of $T^2$. Instead compactifying on a $T^2$ with supersymmetry breaking boundary conditions leads to the theory known as MQCD, which is hoped to be in the same universality class, but more tractable than, ordinary, non-supersymmetric, pure glue, QCD [11].
The 6d $\mathcal{N} = (2,0)$ theories are chiral, with an $SO(5)_R$ flavor symmetry. Although the gauge fields are two-forms rather than one-forms, there is a correspondence with non-Abelian groups $G$. String theory indicates that $G$ can be an arbitrary ADE group: $SU(N)$, $SO(2N)$, or $E_{6,7,8}$. (And $G = U(1)$ for the free $\mathcal{N} = (2,0)$ tensor multiplet.) Upon compactification to lower dimensions, there is an ordinary gauge symmetry with gauge group $G$. In 6d, there is a moduli space of supersymmetric vacua $\mathcal{M} = (\mathbb{R}^5)^{r(G)}/W_G$, where $r(G) = \text{rank}(G)$ and $W_G$ is the Weyl group of $G$, with real scalar expectation value coordinates given by $\Phi_i^a$, where $a = 1 \ldots 5$ is an index in the 5 of $SO(5)_R$ and $i = 1 \ldots r(G)$.

The theory is interacting at the origin and, more generally, at the boundaries of $\mathcal{M}$, where $\mathcal{M}$ is singular. On the other hand, for the generic vacuum in the bulk of $\mathcal{M}$, the massless spectrum is that of $r(G)$ free, 6d, $\mathcal{N} = (2,0)$ tensor multiplets. Naively, any effects associated with degrees of freedom which were massless in the interacting theory, but become massive for the generic vacuum in the bulk of $\mathcal{M}$, would decouple at energy scales much less than their mass, which can be made arbitrarily large by going to large vevs in $\mathcal{M}$. One such degree of freedom are BPS strings, which couple to the $r$ two-form gauge fields with charges $\alpha^i$, $i = 1 \ldots r$. These charge vectors span the root lattice of $G$, and the string with charges $\alpha^i$ has tension $|\alpha^i \Phi_i|$ (here $|\Phi_i| \equiv \sqrt{\sum_a \Phi_i^a \Phi_i^a}$, the length of the $SO(5)$ vector), which can be made arbitrarily large by taking the scalar expectation values $\Phi_i^a$ to be huge. In the realization via M5 branes, with separations $\Phi_i^a$ in the 11d bulk, these strings come from M2 branes which stretch between the M5 branes. Upon $S^1$ compactification, the W-bosons of $G$ come from these strings wrapped on $S^1$. [4]

However, no matter how far the vacuum is from the origin of the moduli space, there are effects associated with the interacting theory at the origin which can not decouple from the low energy theory. The reason is that the interacting theory at the origin generally has a non-trivial 't Hooft anomaly associated with the global $SO(5)_R$ symmetry. This anomaly differs from that of the $r$ tensor multiplets comprising the massless spectrum away from the origin. We will argue that a Wess-Zumino term must be present in the low energy theory to account for what would otherwise be a deficit in the 't Hooft anomaly.

As mentioned above, everything which is presently known about the interacting, 6d, $\mathcal{N} = (2,0)$ field theory has been obtained from string theory. (A hope is that it will eventually be understood how to properly formulate these theories, and recover the properties predicted via string theory, directly in the context of some sort of quantum field theory.) In particular, the non-trivial $SO(5)_R$ 't Hooft anomaly mentioned above was found in [12] in the context of 11d $M$ theory, which gave the anomaly for the case $G = SU(N)$, realized
as $N$ parallel M5 branes. The interesting anomaly coefficient for the $G = SU(N)$ case was found [12] to be $c(SU(N)) = N^3 - N$. The generalization for the $G = SO(2N)$ and $G = E_{6,7,8}$ cases has not yet appeared in the literature.

In the next section, we discuss anomaly matching and the Hopf-Wess-Zumino term which it requires. In particular, the coefficient of this term is the difference between the anomaly $c(G)$ of the interacting theory at the origin and that of the low energy theory away from the origin. Nontrivial maps in $\pi_7(S^4)$ imply a non-trivial quantization condition on the WZ term (which is related to the Hopf invariant of the map) and, consequently, on the anomaly: $c(G) \in 6\mathbb{Z}$. Skyrmionic strings associated with $\pi_4(S^4)$ are also discussed, and it is proposed that they are the $W$-boson strings.

In sect. 3, we briefly discuss 4d $\mathcal{N} = 4$ theories, ’t Hooft anomaly matching, and the WZ term thus required in the low-energy theory when away from the origin. In this case, the WZ term can be derived by a standard [1,3] 1-loop calculation [13]. We also review some math facts concerning the Hopf invariant and map.

In sect. 4, we review how the $N^3$ dependence of the entropy and Weyl anomaly, which is related by supersymmetry to the $SO(5)_R$ anomaly $c(SU(N))$, was originally found [14,15], via $M$ theory on $AdS_7 \times S^4$. We generalize this argument to $M$ theory on $AdS_7 \times X_4$ for general Einstein space $X_4$, finding the anomaly (in the large $N$ limit) $c(N; X_4) = N^3/\text{vol}(\hat{X}_4)^2$. This argument shows that the anomaly for the $\mathcal{N} = (2,0)$ theory associated with $G = SO(2N)$ is $c(SO(2N)) = 4N^3 + \text{terms lower order in large } N$.

In sect. 5, we discuss how the needed WZ term of the $\mathcal{N} = (2,0)$ theory indeed arises in the world-volume of a M5 brane, which probes $N$ distant M5 branes.

In sect. 6, we discuss anomaly matching and a Hopf-WZ term in the 2d $\mathcal{N} = (0,4)$ CFT which arises in the world-volume of strings in 5d. This occurs via M theory on a Calabi-Yau three-fold, with the 5d the uncompactified directions and the strings coming from M5 branes wrapped on a 4-cycle of the Calabi-Yau. As will be discussed, perhaps the story of this section is a fantasy, since there is no moduli space in 2d.

One might expect that, in the context of field theory, it would be possible to derive directly the Wess-Zumino term, by some analog of the 1-loop computation of [1,3] for integrating out some massive degrees of freedom. Turning around our anomaly matching discussion, this would give a derivation of the anomaly of the interacting theory at the origin; e.g. the result of [12] could be re-derived and checked directly in the context of field theory, without having to invoke M-theory. A hope is that these issues could lead to a better understanding of the interacting 6d $\mathcal{N} = (2,0)$ field theory. In sect. 7, we speculate on deriving the WZ term via integrating out tensionful strings and on a possible formula for the ’t Hooft anomaly for general $G = A, D, E$ type $\mathcal{N} = (2,0)$ theories: $c(G) = |G|C_2(G)$. 3
2. Six dimensional, $\mathcal{N} = (2, 0)$ effective field theory

The $\mathcal{N} = (2, 0)$ theory associated with arbitrary group $G$ is expected to have an anomaly of the following general form when coupled to a background $SO(5)_R$ gauge field 1-form $A$, and in a general gravitational background:

$$I_8(G) = r(G)I_8(1) + c(G)p_2(F)/24,$$  \hfill (2.1)

where $p_i$ are the Pontryagin classes for the background $SO(5)_R$ field strength $F$,

$$p_1(F) = \frac{1}{2}(\frac{i}{2\pi})^2 \text{tr} F^2, \quad p_2(F) = \frac{1}{8}(\frac{i}{2\pi})^4((\text{tr} F^2) \wedge (\text{tr} F^2) - 2\text{tr} F^4).$$  \hfill (2.2)

(Writing the Chern roots of $F/2\pi$ as $\lambda_1$ and $\lambda_2$, $p_1 = \lambda_1^2 + \lambda_2^2$ and $p_2 = \lambda_1^2 \lambda_2^2$.) $I_8$ is the anomaly polynomial 8-form, which gives the anomaly by the descent formalism: $I_8 = dI_7^{(0)}$, $\delta I_7^{(0)} = dI_6^{(1)}$, with $I_6^{(1)}$ the anomalous variation of the Lagrangian under a gauge variation $\delta$. $I_8(1)$ is the anomaly polynomial for a single, free, $\mathcal{N} = (2, 0)$ supermultiplet [16,17]:

$$I_8(1) = (p_2(F) - p_2(R) + \frac{1}{4}(p_1(R) - p_1(F))^2)/48.$$  \hfill (2.1)$^1$

The gravitational anomalies, associated with any non-trivial curvature $R$, appear only in $I_8(1)$. In (2.1), $r(G)$ is the rank of the group $G$ associated with the $\mathcal{N} = (2, 0)$ theory and the quantity $c(G)$, which we refer to as the ’t Hooft anomaly of the $SO(5)_R$ flavor symmetry, also depends on $G$. The anomaly (2.2) was found via $M$ theory in [12], for the case $G = SU(N)$, with the result that $c(SU(N)) = N^3 - N$. The analog for for other $G$ has not yet appeared in the literature.

The $SO(5)_R$ current is in the same supermultiplet as the stress-tensor, and thus the ’t Hooft anomaly $c(G)$ also enters in a term in the Weyl anomaly. The entropy of the $\mathcal{N} = (2, 0)$ theory at finite temperature is also proportional to $c(G)$. Indeed, the $N^3$ behavior of $c(G = SU(N))$ was first discovered in these two ways, in the context of $N$ M5 branes in 11d SUGRA [14,15]. Viewing $c(G)$ as a $c$-function, it should decrease in RG flows to the IR. E.g. compactifying the 6d theory and flowing in the IR to 4d $\mathcal{N} = 4$, this suggests that in all cases $c_{UV} = c(G) > c_{IR} = |G| \equiv \text{dim}(G)$.

The result (2.1) gives the anomaly at the origin, where the $SO(5)_R$ global symmetry is unbroken. Away from the origin, $SO(5)_R$ is spontaneously broken. Nevertheless, we argue that the ’t Hooft anomaly of (2.1) must be reproduced everywhere on the moduli space of vacua. The argument for ’t Hooft anomaly matching is same as the original argument of ’t Hooft in 4d [18]: we could imagine adding spectator fields, which remain

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$^1$ In the context of M5 branes, the role of these “spectators” is played by contributions from the 11d bulk: the anomaly inflow and the Chern-Simons term contributions of [13,12].
decoupled from the rest of the dynamics, to cancel the anomalies, allowing the global symmetry to be weakly gauged. The Ward identities of the symmetry must then always be satisfied. Thus, subtracting the constant contribution of the spectators, the anomalous Ward identities of the original theory must be independent of any deformations, including the scalar expectation values. Away from the origin, a Wess-Zumino term, with specific coefficient, is needed to ensure that this is the case.

For simplicity, we consider the case that the scalar vacuum expectation values are chosen to be $\Phi^a = \phi^a(T)_{ii}$, where $(T)_{ii}$ are the diagonal components of a generator of the Cartan of $G$ whose little group is $H \times U(1) \subset G$. The massless spectrum for $\langle \phi^a \rangle \neq 0$ is that of the $\mathcal{N} = (2, 0)$ CFT associated with $H$, along with a single additional $\mathcal{N} = (2, 0)$ multiplet associated with the $U(1)$. The $\phi^a$ are the scalars in the $\mathcal{N} = (2, 0)$ supermultiplet associated with this $U(1)$. Naively, for energy $E \ll \sqrt{|\phi|} \equiv (\phi^a \phi^a)^{1/4}$, the $H$ and $U(1)$ theories are decoupled and the $\mathcal{N} = (2, 0)$ multiplet associated with the $U(1)$ is free.

However, the $U(1)$ multiplet of the theory on the Coulomb branch is actually never really free: it must always include a WZ interaction term. The WZ term is needed to compensate for what would otherwise be a difference in the ’t Hooft anomaly between the $G$ theory at the origin and the massless $H \times U(1)$ $\mathcal{N} = (2, 0)$ theory for $|\phi| \neq 0$:

$$I_8(G) - I_8(H \times U(1)) = \frac{1}{24} (c(G) - c(H)) p_2(F). \quad (2.3)$$

For $\langle \phi^a \rangle \neq 0$, the global $SO(5)_R$ symmetry is broken to $SO(4)_R$ and the configuration space, for fixed non-zero $|\phi|$, is $\mathcal{M}_c = SO(5)/SO(4) = S^4$, with coordinates $\widehat{\phi}^a = \phi^a/|\phi|$, $a = 1 \ldots 5$. The needed Wess-Zumino term is given by the following term in the action

$$S_{WZ} = \frac{1}{6} (c(G) - c(H)) \int_{\Sigma_7} \Omega_3(\widehat{\phi}, A) \wedge d\Omega_3(\widehat{\phi}, A) + \ldots, \quad (2.4)$$

\[2\] In 6d, conjugate group representations contribute to the anomaly polynomial $I_8$ with the same sign. Massless fermions of chiralities $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ under the $SO(4) \cong SU(2) \times SU(2)$ little group contribute with opposite signs.

\[3\] Although there is no potential which requires $\langle \phi^a \rangle \neq 0$, it is a modulus labeling superselection sectors, so we can always choose this to be the case by our choice of boundary conditions at infinity. The requirement that $|\phi|$ be fixed is not essential: we only need $\mathcal{M}_c \cong S^4$ topologically, and requiring $\langle \phi^a \rangle \neq 0$ is enough.
where . . . are terms related by supersymmetry. \( \Sigma \) is a 7 dimensional space, whose boundary is the 6d spacetime \( W_6 \) of the \( \mathcal{N} = (2, 0) \) field theory, \( \partial \Sigma_7 = W_6 \) (e.g. \( \Sigma_7 \) could be \( AdS_7 \)). \( \Omega_3(\hat{\phi}, A) \) is a 3-form which is defined as follows. Consider the 4-form

\[
\eta_4(\hat{\phi}, A) \equiv \frac{1}{2} e_4^\Sigma \equiv \frac{1}{64 \pi^2} \epsilon_{a_1 . . . a_5} \left[ (D_{i_1} \hat{\phi})^{a_1} (D_{i_2} \hat{\phi})^{a_2} (D_{i_3} \hat{\phi})^{a_3} (D_{i_4} \hat{\phi})^{a_4} - 2 F_{i_1 i_2}^{a_1 a_2} (D_{i_3} \hat{\phi})^{a_3} (D_{i_4} \hat{\phi})^{a_4} + F_{i_1 i_2}^{a_1 a_2} F_{i_3 i_4}^{a_3 a_4} \right] \hat{\phi}^a dx^1 \wedge . . . \wedge dx^4,
\]

with \((D_i \phi)^a \equiv \partial_i \phi^a - A_i^{ab} \phi^b\) the covariant derivative of \( \phi^a \), involving the background \( SO(5)_R \) gauge field \( A_i^{ab} = -A_i^{ba} \), with \( a, b \in 5 \) of \( SO(5)_R \) (\( F_{ij}^{ab} \) is its field strength). The \( x^i \) are the coordinates on \( \Sigma \). In (2.5), \( e_4^\Sigma \equiv \hat{\phi}^*(e_4) \) is the pullback to \( \Sigma_7 \), via \( \hat{\phi} : \Sigma_7 \rightarrow S^4 \), of the global, angular, Euler class 4-form \( e_4 \) which also entered in \([19]\). The \( \eta_4 \) in (2.3) is normalized so that \( \eta_4(\hat{\phi}, A = 0) = \hat{\phi}^*(\omega_4) \), the pullback of the \( S^4 \) unit volume form, \( \int_{S^4} \omega_4 = 1 \). The form (2.5) is closed and, because we take \( \Sigma_7 \) such that \( H^4(\Sigma_7) \) is trivial, it must be exact, \( \eta_4(\hat{\phi}, A) = d\Omega_3(\hat{\phi}, A) \). This defines the \( \Omega_3 \) appearing in (2.4).

The Wess-Zumino term (2.4) has the desired non-trivial gauge variation under \( SO(5)_R \) gauge transformations. To see this, we note that eqn. (2.7) in \([20]\) implies that

\[
d\Omega_3 \wedge d\Omega_3 = \frac{1}{4} e_4^\Sigma \wedge e_4^\Sigma = \frac{1}{4} p_2(F) + d\chi,
\]

where \( \chi \) is invariant under \( SO(5)_R \) gauge transformations. Writing the left hand side as \( d(\Omega_3 \wedge d\Omega_3) \) and \( p_2(F) = dp_2^{(0)}(A) \), the \( SO(5)_R \) gauge variation of (2.6) implies that

\[
\delta \int_{\Sigma_7} \Omega_3 \wedge d\Omega_3 = \frac{1}{4} \int_{\Sigma_7} \delta p_2^{(0)}(A) = \frac{1}{4} \int_{W_6} p_2^{(1)}(A),
\]

where \( p_2^{(1)}(A) \) is the anomaly 6-form found by descent, \( \delta p_2^{(0)} = dp_2^{(1)} \). Note that the \( \phi \) dependence in \( \Omega_3(\hat{\phi}, A) \) has dropped out in the gauge variation (2.7). Using (2.7), the \( SO(5)_R \) gauge variation of the WZ term (2.4) indeed compensates for the deficit (2.3).

As an example, consider \( G = SU(N + 1) \) and \( H = SU(N) \). Using the result of \([12]\) that \( c(G) = (N + 1)^3 - (N + 1) \) and \( c(H) = N^3 - N \), the Wess-Zumino term (2.4) is

\[
\frac{1}{2} N(N + 1) \int_{\Sigma_7} \Omega_3(\hat{\phi}, A) \wedge d\Omega_3(\hat{\phi}, A).
\]

This Wess-Zumino term must be present in the world-volume of a M5 brane, when in the background of \( N \) other M5 branes, and thus for a M5 brane in \( AdS_7 \times S^4 \).

\[\text{Note that to construct the WZ term requires extending } \hat{\phi} : W_6 \rightarrow S^4 \text{ to } \hat{\phi} : \Sigma_7 \rightarrow S^4, \text{ which can have an obstruction if the original } \hat{\phi} \text{ is in the non-trivial component of } \pi_6(S^4) = \mathbb{Z}_2.\]
By a general analysis\textsuperscript{5} \cite{21}, WZ terms are generally of the form $\int_{\Sigma_{d+1}} \hat{\phi}^*(\omega_{d+1})$, with $\omega_{d+1} \in H^{d+1}(\mathcal{M}_c, \mathbb{R})$. However, our WZ term (2.4) is not of this form, as $\Omega_3 \wedge d\Omega_3 \neq \hat{\phi}^*(\omega_7)$. Indeed, here $\mathcal{M}_c = SO(5)/SO(4) \cong S^4$, and obviously $H^7(S^4, \mathbb{R}) = 0$; nevertheless, even for $A = 0$, (2.4) is non-zero. An aspect of the present case, which sets it apart from the general analysis of \cite{21}, is mentioned at the end of this section.

The 7-form $\Omega_3 \wedge d\Omega_3$ in (2.4) is not exact, so the ambiguity in the choice of $\Sigma_7$ is non-trivial. The difference between choosing $\Sigma_7$ and $\Sigma'_7$, both with boundary $W_6$, is the integral over $\Sigma_7 - \Sigma'_7 \cong S^7$

$$\frac{1}{6}(c(G) - c(H)) \int_{S^7} \Omega_3(\hat{\phi}, A) \wedge d\Omega_3(\hat{\phi}, A). \tag{2.9}$$

This can be non-trivial. Indeed, e.g. for zero background $SO(5)_R$ field, $A = 0$, the integral in (2.9) gives the Hopf number of the map $\hat{\phi}^a : S^7 \to S^4$, which can be an arbitrary integer, corresponding to $\pi_7(S^4) = \mathbb{Z} + \mathbb{Z}_{12}$. The coefficient of (2.9) thus must be quantized in order for $e^{2\pi i S}$ to be well-defined and invariant under the choice of $\Sigma_7$:

$$\frac{1}{6}(c(G) - c(H)) \in \mathbb{Z}. \tag{2.10}$$

To have (2.10) hold for arbitrary ADE groups $G$ and subgroups $H$ requires

$$\frac{1}{6}c(G) \in \mathbb{Z} \tag{2.11}$$

for all ADE groups $G$. Happily, (2.11) is indeed satisfied by $c(G = SU(N)) = N^3 - N$.

We also note that there are topologically stable, solitonic “skyrmion” field configurations in the theory with non-zero $|\phi|$. In $d$ spacetime dimensions, a $p$-brane skyrmion is a field configuration $\hat{\phi}^a(X_t)$ which only depends on the $d - p - 1$ space coordinates of $X_t$, the space transverse to the $p$-brane worldvolume. In order for this to be a finite-energy configuration, $\hat{\phi}^a$ must approach a constant value when the coordinates of $X_t$ are taken to infinity. Such field configurations are thus topologically classified by $\pi_{d-p-1}(\mathcal{M}_c)$. In the present case, $\pi_4(S^4) = \mathbb{Z}$ means that there are non-trivial $p = 1$ branes in $d = 6$, i.e. there are skyrmionic strings. (There are also $\mathbb{Z}_2$ particles since $\pi_5(S^4) = \mathbb{Z}_2$.) The topological charge density for the skyrmionic strings is $\eta_4$, defined as in (2.5): the string number is $N_s = \int_{X_t} \eta_4$. The WZ term means that there is a Goldstone-Wilczek contribution \textsuperscript{4}{2} to the $SO(5)_R$ flavor current, which can give the skyrmions $SO(5)_R$ charges.

\textsuperscript{5} I thank E. D’Hoker for pointing this out to me and for related correspondences.
We propose that these skyrmionic strings are actually the “W-boson” BPS strings mentioned in the introduction. (Other works, including [22,23], have briefly considered solitonic strings in the M5 brane theory, but not specifically the $\pi_4(S^4)$ skyrmionic solitons.) In line with this proposal, the skyrmionic string density $\eta_4$ should act as electric and magnetic flux sources for the $H_3$ in the $U(1)$ $\mathcal{N} = (2,0)$ multiplet:

$$dH_3 = J_{mag} = \alpha_m \eta_4, \quad d \ast H_3 = J_{elec} = \alpha_e \eta_4$$

(2.12)

for some non-zero constants $\alpha_{m,e}$. The $H_3$ in (2.12) is not self-dual, rather it is related to a self-dual tensor $h_3$ by a non-linear transformation [24], so $\alpha_e$ and $\alpha_m$ need not be equal. The electric relation in (2.12) means there is an interaction

$$S_{sky} = \alpha_e \int_{W_6} B_2 \wedge \eta_4 = -\alpha_e \int_{W_6} dB_2 \wedge \Omega_3(\hat{\phi}, A).$$

(2.13)

Given that the skyrmionic string is charged under $H_3$ as outlined above, it follows from completely general considerations that the supersymmetry algebra has central term $Z = |Q\phi|$, and the tension of such a string satisfies $T \geq |Q\phi|$. Here $Q \sim N_s$, with $N_s$ the $\pi_4(S^4)$ topological string number, $N_s = \int_X \eta$. For each $N_s$ charge, there is a BPS field configuration $\hat{\phi}^a$ which minimizes the energy, satisfying $T = |Q\phi|$. It is these BPS skyrmionic string solitons which should be identified with the BPS W-boson strings.

In the particular context of $N$ M5 branes, corresponding to the $G = SU(N)$ $\mathcal{N} = (2,0)$ theories, the magnetic relation in (2.12) and the coupling (2.13) have already appeared in [22], though the interpretation of these relations in terms of the $\pi_4(S^4)$ skyrmionic strings was not discussed there. The argument of [22] for the magnetic relation in (2.12) involves accounting for the fact that M5-branes act as $G_4$ sources; this is also related to the analysis and results of [13,12], and is further discussed in sect. 5.

The argument of [22] for the coupling (2.13), which immediately generalizes to general $\mathcal{N} = (2,0)$ $G \to H \times U(1)$ Coulomb branch Higgsing, is as follows (see also [23,13]): consider $S^1$ reducing to 5d, where the theory is ordinary Yang-Mills and is IR free. The 5d $U(1)$ gauge field in the Higgsing $G \to H \times U(1)$ arises from $B_{\mu 6}$ in 6d. The coupling (2.13) then arises by a standard type of 1-loop calculation, much as in [13], with the $n_W = |G| - |H| - 1$ massive gauginos running in the loop. Taking care with the normalization, we get (2.13) with $\alpha_e = \frac{1}{2} n_W$. E.g. for $G = SU(N + 1)$ and $H = SU(N)$, the term (2.13) is generated with coefficient $\frac{1}{2} N$. The constant $\alpha_m$ in (2.12) is more difficult to determine.
We can also consider $S^1$ dimensional reduction of the relations (2.12), using $H_3 \rightarrow H_3 + F_2 \wedge dx_6$ and $\eta_4 \rightarrow \eta_4$, where now, using (2.3), $\eta_4$ has no $dx_6$ component because we take $\hat{\phi}$ to be independent of $x_6 \in S^1$ in dimensional reduction. Then (2.12) gives $dH_3 = \alpha_m \eta_4$, $dF_2 = 0$, $d \ast H_3 = 0$, and $d \ast F_2 = \alpha_e \eta_4$ (now $\ast$ acts in the uncompactified 5d). The $F_2$ equations show that there are no magnetic charges in 5d, and that the $\pi_4(M_c = S^4)$ skyrmions are electrically charged particles in 5d which, since $\alpha_e = \frac{1}{4} n_W$, can be identified with the $n_W$ electrically charge $W$-bosons. Naively, one might identify $H_3$ as $\ast F_2$, making the $H_3$ equations repeats of the $F_2$ equations and suggesting that $\alpha_m$ be identified with $\alpha_e$. However, as inherited from 6d where $H_3$ is not simply self-dual, the 5d $H_3$ and $\ast F_2$ are not simply equal. This again makes $\alpha_m$ more difficult to determine.

We emphasize that, unlike the term (2.13), it does not seem possible (at least in any obvious way) to get our 6d Hopf-Wess-Zumino term (2.4) by a direct calculation in the dimensionally reduced 5d gauge theory. Indeed, upon $S^1$ dimensional reduction, the term (2.4) actually vanishes (unless the Kaluza-Klein $S^1$ momentum modes are included) since, for $\hat{\phi}$ independent of $x_6$, (2.3) shows that $\eta_4$, and thus also $\Omega_3$, have no $dx_6$ component. It could have been anticipated that the 6d WZ term (2.4) would be difficult to obtain by dimensional reduction because (unlike (2.13)) its coefficient is not simply $n_W$; e.g. in (2.8) the coefficient is proportional to $N(N + 1)$ rather than just $n_W \sim N$.

Here is a possible insight into the origin of the 6d WZ term (2.4): in direct analogy with the discussion in [2], the action of an electric current in a magnetic background is $\int_{\Sigma_7} H_3 \wedge J_{elec}$, again with $\partial \Sigma_7 = W_6$. Solving (2.12) for $H_3$ as

$$H_3 = dB_2 + \alpha_m \Omega_3(\hat{\phi}, A),$$

(2.14)

with $\eta_4 = d\Omega_3$, and plugging in $J_{elec}$ from (2.12), this gives the action

$$\int_{\Sigma_7} \alpha_e (dB_2 + \alpha_m \Omega_3) \wedge d\Omega_3.$$  

(2.15)

The first term in (2.13) gives the coupling (2.13) and the second term gives the WZ term (2.4), with the right coefficient (assuming that the WZ term indeed arises entirely from (2.13)) provided that $\frac{1}{6}(c(G) - c(H)) = \alpha_e \alpha_m = \frac{1}{4} n_W \alpha_m$. In this light, (2.10) is simply Dirac quantization. Note that, in $\Sigma_7$, $\eta_4$ becomes a density for skyrmionic membranes, whose ends on $\partial \Sigma_7 = W_6$ are the skyrmionic strings of $W_6$ discussed above. In the M5 brane realization, these are like skyrmionic M2 branes living, e.g. in $\Sigma_7 = AdS_7$. The WZ term is proportional to $\int_{\Sigma_7} \Omega_3 \wedge d\Omega_3$, which measures membrane winding number in $\Sigma_7$.  

9
Lastly, we tie up a loose end: our definition of $\Omega_3$, via $\eta_4 = d\Omega_3$, only defines $\Omega_3$ up to exact forms, $\Omega_3 \to \Omega_3 + d\Lambda_2$. Under such a change, (2.4) changes by
\[ S_{WZ} \to S_{WZ} - \frac{1}{6} (c(G) - c(H)) \int_{W_6} d\Lambda_2 \wedge \Omega_3; \] (2.16)
this freedom must somehow be fixed in order for the effective action to be well-defined.\(^6\) Noting that the physical quantity $H_3$ must also be well-defined, (2.14) shows that the change $\Omega_3 \to \Omega_3 + d\Lambda_2$ requires a compensating shift $B_2 \to B_2 - \alpha_m \Lambda_2$. (In the M5-brane realization, to be discussed in sect. 5, this is the freedom of $C_3$ gauge transformations.) If the WZ term arises from (2.15), it is unchanged by this combined shift of $\Omega_3$ and $B_2$, with the change in (2.13) under $B_2 \to B_2 - \alpha_m \Lambda_2$ cancelling (2.16). Alternatively, we can simply use (2.14) to define $\Omega_3$ on $W_6$ as $\alpha_m^{-1} H_3$. The remaining ambiguity on $\Sigma_7$ of taking $\Omega_3 \to \Omega_3 + d\Lambda_2$, with $d\Lambda_2|_{W_6} = 0$, is harmless in (2.16). In short, our WZ term (2.4) needs $B_2$ to be well-defined, an aspect which sets it apart from the general analysis of [21].

3. Miscellaneous Notes

3.1. The WZ Term of 4d $\mathcal{N} = 4$ Theories

A completely analogous relation between ’t Hooft anomaly matching and a WZ term holds in the 4d $\mathcal{N} = 4$ theory. The $\mathcal{N} = 4$ theory has a global $SU(4)_R \cong SO(6)_R$ flavor symmetry, with ’t Hooft anomaly $\text{tr}SU(4)_R^3 = |G|$, i.e. in a background $SU(4)_R$ gauge field $A_B$, with field strength $F_B$, there is an anomaly determined via descent from
\[ I_6(G) = \frac{|G|}{6} (\frac{i}{2\pi})^3 \text{tr} F_B^3, \] (3.1)
with $|G|$ the dimension of the gauge group $G$. This anomaly comes from the $|G|$ gauginos in the 4 of $SU(4)_R$ and is not renormalized.

Consider now moving away from the origin of the moduli space via $\Phi^a = \phi^a T$, with $T$ a generator of the Cartan of $G$, with little group $H \times U(1) \subset G$. Here $a \in 6$ of $SU(4)_R$ and taking $\langle \phi^a \rangle \neq 0$ breaks the gauge symmetry $G \to H \times U(1)$ and the flavor symmetry $SU(4) \to SO(5)$. For fixed $|\phi| \equiv \sqrt{\phi^a \phi^a}$, the configuration space is $\mathcal{M}_c = SU(4)/SO(5) \cong S^5$. The massless spectrum is that of the $\mathcal{N} = 4$ theory with decoupled

\(^6\) I thank E. Witten for correspondences, which stressed the need to fix this issue and suggested the following discussion.
groups $H \times U(1)$ and, for energies $E \ll |\phi|$, one might be tempted to forget about the effects of the $n_W = |G| - |H| - 1$, ultra-massive, $G/H \times U(1)$ gauge field multiplets. However, the $n_W$ gauginos in these multiplets contributed to the anomaly (3.1); without them there is a deficit in (3.1) of $I_0(G) - I_0(H) - I_0(U(1)) = \frac{1}{6} n_W (\frac{1}{2\pi})^3 \text{tr} F_B^3$. This deficit must be accounted for by a Wess-Zumino term in the low energy theory.

The WZ term thus required in the low-energy theory is

$$\frac{1}{2} n_W \Gamma[\hat{\phi}, A_B] + \text{superpartners}, \quad (3.2)$$

where $\Gamma[\hat{\phi}, A_B]$ is conventionally [2] written as $\Gamma[\hat{\phi}, A_B] = \Gamma[\hat{\phi}] + Z[\hat{\phi}, A_B]/48\pi^2$ with

$$\Gamma[\hat{\phi}] = \frac{1}{240\pi^2} \int_{\Sigma_5} \epsilon_{a_1...a_6} \partial_{i_1} \hat{\phi}^{a_1} \ldots \partial_{i_5} \hat{\phi}^{a_5} d\Sigma^{i_1} \wedge \ldots \wedge d\Sigma^{i_5}, \quad (3.3)$$

where $\partial_5 = W_4$. $\Gamma[\hat{\phi}, A] = \frac{1}{2} \int_{\Sigma_5} \hat{\phi}^a(e_5)$, with $e_5$ the $S^5$ global, angular, Euler class form in the appendix of [12], normalized so that $\frac{1}{2} \epsilon_5(A = 0) = \omega_5$, the unit $S^5$ volume form. (Since $\pi_4(S^5) = 0$, there is no obstruction to extending $\hat{\phi} : W_4 \to S^5$ to $\hat{\phi} : \Sigma_5 \to S^5$.)

Corresponding to (3.2), there is an induced Goldstone-Wilczek current

$$j^{\mu}_{a_1 a_2} = \frac{1}{2} n_W \frac{1}{24\pi^2} \epsilon^{\mu \nu \rho \sigma} \epsilon_{a_1 a_2 ... a_6} \partial_{\nu} \hat{\phi}^{a_3} \partial_{\rho} \hat{\phi}^{a_4} \partial_{\sigma} \hat{\phi}^{a_5} \hat{\phi}^{a_6}. \quad (3.4)$$

The same $\Gamma[\hat{\phi}, A_B]$ appeared in [1] in the context of $N = 1$ SUSY QCD with $N_f = N_c = 2$, as in both cases there is a $SU(4)$ flavor symmetry with order parameter in the 6 of constant magnitude. As shown in [1], the $SU(4)$ variation of $\Gamma[\hat{\phi}, A_B]$ contributes to the $SU(4)^3$ flavor ’t Hooft anomalies the same as with two fermions in the 4 of $SU(4)$. Thus, with the coefficient of the WZ term as in (3.2), it properly accounts for the contribution to the ’t Hooft anomaly of the $n_W$ gauginos, in the 4 of $SU(4)$, which got a mass via Yukawa couplings to $\phi$ in the Higgsing $G \to H \times U(1)$. The fact that integrating out the $n_W$ massive fermions actually does generate precisely the WZ term (3.2) follows from the standard 1-loop calculation of the type appearing in [13]. See, in particular, [13].

Because the WZ 5-form term in $\Gamma[\hat{\phi}, A_B]$ is not exact, there is a quantization condition on its coefficient (3.2) in order to have $e^{2\pi i S}$ be invariant under $\Sigma_5 \to \Sigma'_5$ with $\partial \Sigma'_5 = \partial \Sigma_5 = W_4$. The difference involves the 5-form of (3.3) integrated over $\Sigma'_5 - \Sigma_5 \cong S^5$, which is an arbitrary integer associated with $\pi_5(M_c = S^5) = \mathbb{Z}$. The quantization condition is thus

$$\frac{1}{2} n_W \equiv \frac{1}{2} (|G| - |H| - 1) \in \mathbb{Z}. \quad (3.5)$$

Fortunately, this is indeed satisfied for arbitrary group $G$, with subgroup $H \times U(1)$ obtained via adjoint Higgsing. Since all $\pi_{3-p}(M_c = S^5) = 0$, now there are no $p$-brane skyrmions.

The 4d WZ term (3.2) is related to the dimensional reduction of (2.13), not the 6d WZ term (2.4). Again, the dimensional reduction of the 6d WZ term vanishes.
3.2. Some math notes on the Hopf invariant

We now summarize some facts which can be found e.g. in [26]. The Hopf invariant \( H(f) \) of a mapping \( f : S^{2n-1} \rightarrow S^n \) is an integer which can be defined as the winding coefficient of curves \( f^*(a) \) and \( f^*(b) \) in \( S^{2n-1} \) for distinct \( a \) and \( b \) in \( S^n \); \( f^*(a) \) is the \((n-1)\) dimensional curve in \( S^{2n-1} \) which is mapped by \( f \) to a point \( a \in S^n \). \( H(f) \) can be written as an integral over \( S^{2n-1} \) as follows: consider the pullback \( f^*(\omega_n) \), where \( \omega_n \) is the unit volume form of \( S^n \),
\[
\int_{S^n} \omega_n = 1.
\]
The form \( f^*(\omega_n) \) is closed and, as \( H^n(S^{2n-1}) \) is trivial, must be exact, \( f^*(\omega_n) = d\theta_{n-1} \). The Hopf invariant can be written as
\[
H(f) = \int_{S^{2n-1}} \theta_{n-1} \wedge d\theta_{n-1}.
\]
(3.6)
Clearly, \( H(f) = 0 \) for \( n \) odd. For \( n = 2k \) even, \( H(f) \in \mathbb{Z} \), taking all integer values for various maps \( f \). Thus \( \pi_{4k-1}(S^{2k}) \) is at least \( \mathbb{Z} \). E.g. \( \pi_3(S^2) = \mathbb{Z} \) and \( \pi_4(S^7) = \mathbb{Z} \oplus \mathbb{Z}_{12} \).

The basic map \( S^3 \rightarrow S^2 \) with Hopf number 1 is given by writing \( S^3 \) as \((z_1, z_2)\), with \( z_i \in \mathbb{C} \) and \( |z_1|^2 + |z_2|^2 = 1 \), and writing \( S^2 \) as \( CP^1 \), i.e. \([z_1, z_2]\) with \( z_i \in \mathbb{C}^* \) and \([z_1, z_2] \sim [\lambda z_1, \lambda z_2]\) for arbitrary \( \lambda \in \mathbb{C}^* \). The map is then simply \( f : (z_1, z_2) \rightarrow [z_1, z_2] \).

The map with Hopf number 1 for \( S^7 \rightarrow S^4 \) is exactly the same as that above, with the simple replacement that \( z_i \) and \( \lambda \) now take values in the quaternionic rather than the complex numbers.

4. Getting the \( N^3 \) via gravity and the \( G = SO(2N) \) case via an orbifold

We now review how \( c \approx N^3 \) appears via 11d sugra, generalizing to \( M \) theory on \( AdS_7 \times X_4 \), where \( X_4 \) is a general, compact, Einstein space. The anomaly coefficient \( c \) arises as the coefficient of a Chern-Simons term in \( AdS_7 \). This term is related by supersymmetry to the coefficient of the 7d Einstein-Hilbert action in \( AdS_7 \). It thus follows that
\[
c = \frac{L^5}{G_7},
\]
(4.1)
where \( G_7 \) is the 7d Newtons constant, and the powers of \( L \), which is the horizon size of \( AdS_7 \) (related to the size of the negative cosmological constant), are determined by dimensional analysis; for simplicity, we will everywhere drop universal constants (factors of 2 and \( \pi \)). The entropy [14] and Weyl anomaly [15] are also proportional to (4.1).
By the dimensional reduction from 11d SUGRA or M theory on compact space $X_4$, $G_7^{-1} = \text{vol}(X_4)/l_P^9$, with $l_P$ the 11d Planck length. We thus write (4.1) as

$$c = \text{vol}(\hat{X}_4) \frac{L^9}{l_P^9},$$  (4.2)

where $\text{vol}(\hat{X}_4)$ is the dimensionless volume of $X_4$ measured in units of $L$ (normalized so that $\text{vol}(\hat{X}_4) = 1$ for $X_4 = S^4$ of radius $L$). The $G_4$ flux quantization condition gives

$$\int_{X_4} G_4 = L^3 \text{vol}(\hat{X}_4) = Nl_P^3,$$  (4.3)

so (4.2) leads to the general result

$$c = \frac{N^3}{(\text{vol}(\hat{X}_4))^2} + \text{lower order in } N.$$  (4.4)

In particular, for orbifolds $X_4 = S^4/\Gamma$, (4.4) gives

$$c = N^3 |\Gamma|^2 + \text{lower order in } N,$$  (4.5)

in the normalization where $c = N^3$ (plus lower order) for the $\mathcal{N} = (2, 0)$ theory with $G = SU(N)$, corresponding to $X_4 = S^4$. This argument is analogous to that of [27] for IIB on $AdS_5 \times X_5$, which gave $c = N^2/\text{vol}(\hat{X}_5) = N^2|\Gamma|$.

In particular, the $\mathcal{N} = (2, 0)$ theory with group $G = SO(2N)$ arises from M theory on $AdS_7 \times RP^4$ and, writing $RP^4 = S^4/\Gamma$ with $\Gamma = \mathbb{Z}_2$, (4.3) implies that the anomaly is

$$c(G = SO(2N)) = 4N^3 + \text{lower order in } N.$$  (4.6)

5. The WZ term via the M5-brane worldvolume action

Branes in string or M theory always have some sort of “Wess-Zumino” terms, e.g. for $Dp$ branes it is usually written as [28]

$$S_{WZ} = \int_{W_{p+1}} C \wedge \text{tr} \exp(i(F - B)/2\pi) \wedge \sqrt{\frac{A(R_T)}{A(F_N)}},$$  (5.1)

and the presence of some similar terms for the M5 brane is well-known [29]. As written, these could not be exactly the Wess-Zumino terms of the type we have argued for, as they are written as local integrals over the world-volume $W$ and not over a higher dimensional
space $\Sigma$ with $\partial \Sigma = W$. Of course, they could be written as an integral over $\Sigma$ of an exact form, but our WZ term is the integral over $\Sigma$ of a form which is not exact. Nevertheless, we argue that writing the “Wess-Zumino” term of [29] as the integral over $\Sigma_7$ of a 7-form, which is naively exact, actually gives the Hopf-Wess-Zumino term which we want. The point is that the naively exact 7-form actually is not exact upon properly taking into account the fact that 5-branes act as a non-trivial source for $G_4$ in M theory.

Similarly, for Dp-branes, the WZ term generally can not be written as the local term (5.1) on $W_{p+1}$. It must be written as $\int_{\Sigma_{p+2}} \Omega_{p+2}$, with $\Omega_{p+2}$ not exact, despite the fact that, naively, $\Omega_{p+2} = d\Omega_{p+1}$, with $\Omega_{p+1}$ the form in (5.1). E.g. for a D3 brane (5.1) contains $\int_{W_4} C_4$, which should really be written as $\int_{\Sigma_5} F_5$. Naively $F_5 = dC_4$ and there is no difference; however, in the presence of other D3 brane sources, $F_5$ is not exact. This is how (3.2) arises for a D3 brane probing other D3 branes.

The M5 brane world-volume theory depends on (with sign conventions of [17-12])

$$H_3 = dB_2 + C_3^W,$$  

(5.2)

with $B_2$ the two-form gauge field and $C_3^W$ the pull-back of the 11d $C_3$ field to the M5 brane world-volume $W_6$. This $H_3$ (5.2) is invariant under the gauge invariance $\delta C_3 = d\Lambda_2$, $\delta B_2 = -\Lambda_2^W$ and satisfies a generalized self-duality condition (it is only self-dual at linear order; there is a field transform to a 3-form $h$ which is exactly self-dual [24]).

We consider a probe brane in the background of the $N$ others; for large $N$, this should be equivalent to a M5 brane in $AdS_7 \times S^4$. Following [12], there is a $G_4$ background, with pullback $G_4^W = N\eta_4(\widehat{\phi}, A) +$ fluctuations in $C_3$, with $\eta_4$ the 4-form (2.3); thus

$$C_3^W = N\Omega_3(\widehat{\phi}, A) +$$ fluctuations,  

(5.3)

and (5.2) becomes (2.14) with $\alpha_m \approx N$. The fact that $dH_3 \approx N\eta_4$, which follows from (5.2) and (5.3), has already been suggested (with $A_{SO(5)} = 0$) in [22].

We re-write the WZ term of [29] as an integral over some $\Sigma_7$ with $\partial \Sigma_7 = W_6$:

$$\int_{\Sigma_7} (\star G_4 + \frac{1}{2} (dB_2 + C_3^\Sigma) \wedge G_4^\Sigma).$$  

(5.4)

Plugging in $C_3^\Sigma$ and $G_4^\Sigma = dC_3^\Sigma$, given by (5.3) extended to $\Sigma_7$, we get

$$\int_{\Sigma_7} (\star G_4 + \frac{1}{2} N (dB_2 + N\Omega_3) \wedge d\Omega_3).$$  

(5.5)
This indeed contains the Hopf-WZ term (2.8), with the correct leading order in large $N$ coefficient of $\frac{1}{2}N^2$. Indeed, ignoring the $*G_4$ term, (5.5) is of exactly the form (2.13) with $\alpha_e \approx \frac{1}{2} N$ and $\alpha_m \approx N$; so (5.5) also contains the coupling (2.13) needed for the $\pi_4(S^4)$ solitonic strings to couple electrically to the $B_2$ field as the $n_W = 2N$ “W-boson” strings. As discussed in sect. 2, $S^1$ dimensional reduction suggests that we get the term (2.13) with $\alpha_e = \frac{1}{4} n_W = \frac{1}{2} N$ (exact).

We should also get the term proportional to $N$ in (2.8). The term $*G_4$ in (5.4) will be order $N$, but $*G_4$ needs to be properly interpreted to see if it also contributes to the Hopf-Wess-Zumino term (naively it’s just a contribution to the $AdS_7$ vacuum energy). Perhaps a new term, similar to the $C_3 \wedge I_8^{inf}$ term of 11d SUGRA, is needed to get the order $N$ term in (2.8). If the WZ term indeed arises entirely as in (2.15), with coefficient $\alpha_e \alpha_m$, the order $N$ term in (2.8) should arise from correcting $\alpha_m \approx N$ to $\alpha_m = N + 1$.

6. Reduction on a Calabi-Yau 3-fold

Following the discussion in [19,12], we now consider $M$ theory on a Calabi-Yau 3-fold $X$, with M5 branes wrapping a four-cycle to yield strings. These strings live in the 5 uncompactified dimensions of $M$ theory on $X$ and their world-volume theory is a 2d $\mathcal{N} = (0,4)$ CFT, which has a $SO(3)_R$ global symmetry. The $SO(3)_R$ symmetry is that of the normal bundle of the three transverse directions of these strings in 5d. E.g. there are 2d world-volume scalars $\Phi^a$, $a = 1, 2, 3$, in the 3 of $SO(3)_R$, whose expectation values gives the positions of the strings in the 3 transverse directions.

Classically, we can consider the situation of separating one string from $N$ others in these three transverse directions. This would spontaneously break the $SO(3)_R$ global symmetry of the probe string world-volume theory to an $SO(2)_R$ subgroup. However, this can not really happen: there is no spontaneous symmetry breaking in 2d [30]. There is no moduli space, as the scalars $\Phi^a$ have a wavefunction which spreads over all values.

We now discuss how the story with anomalies and the WZ term would go if we ignore the fact that there is actually no moduli space in 2d. Perhaps this discussion is relevant in some sort of Born-Oppenheimer approximation, where the spreading of the wave-function is initially neglected or suppressed. Or perhaps this section is just a fairy tale.

The $SO(3)_R$ symmetry is an affine Lie algebra, with level $k$, which is the $SO(3)_R$ ’t Hooft anomaly in the 2d anomaly polynomial

$$I_4(G, X) = \frac{1}{4} k(G, X)p_1(F),$$  \hspace{1cm} (6.1)
with $F$ the $SO(3)$ background field and $p_1(F)$ as in \(2.2\). (We ignore gravitational contributions to $I_4$ since they do not require a WZ term.) We expect that $G$ can be $U(1)$ or an ADE group, corresponding to the ADE classification of $SU(2)_k$ modular invariant partition functions. For $G = U(N)$ \[12,31\]

$$k(U(N), X) = N^3D_0 + Nc_2 \cdot P_0/12,$$

(6.2)

where $D_0$ and $c_2 \cdot P_0$ are determined in terms of the geometry of the 3-fold $X$ and the 4-cycle of $X$ on which the $N$ M5 branes wrap.

In the fairy tale where we can consider fixed non-zero $|\phi| = \sqrt{\phi^a \phi^a}$, the classical configuration space is $\mathcal{M}_c = SO(3)/SO(2) = S^2$ and $G$ is broken to $H \times U(1)$. There must be a Wess-Zumino term on the probe string world-volume to compensate for the deficit $I_4(G, X) - I_4(H, X) - I_4(U(1), X)$.

We take the string world-volume to be $W_2 = \partial \Sigma_3$. Consider the two-form,

$$\eta_2(\hat{\phi}, A) \equiv \frac{1}{2} e_2^\Sigma = \frac{1}{8\pi} e_{abc}(D_i \hat{\phi}^a D_j \hat{\phi}^b - F_{ij}^{ab}) \hat{\phi}^c dx^i \wedge dx^j,$$

(6.3)

where the covariant derivatives include a background $SO(3)$ gauge field $A^{ab} = -A^{ba}$, with background field strength $F^{ab}$, and $\hat{\phi}^a = \phi^a/|\phi|$. $e_2^\Sigma$ is the pullback (via $\hat{\phi}$ : $\Sigma_3 \rightarrow S^2$) to $\Sigma_3$ of the global, angular, Euler-class form $e_2$ appearing in \[13\]. Because $H^2(\Sigma_3)$ is trivial, the form \(6.3\) must be exact, $\eta_2 = d\Omega_1(\hat{\phi}, A)$. The Hopf-Wess-Zumino term is

$$\Gamma = (k(G, X) - k(H, X) - k(U(1), X)) \int_{\Sigma_3} \Omega_1(\hat{\phi}, A) \wedge d\Omega_1(\hat{\phi}, A).$$

(6.4)

To see that \(6.4\) contributes to the anomaly matching, we note that \[20\]

$$d\Omega_1 \wedge d\Omega_1 \equiv \frac{1}{4} e_2^\Sigma(\hat{\phi}, A) \wedge e_2^\Sigma(\hat{\phi}, A) = \frac{1}{4} p_1(F) + d\chi,$$

(6.5)

where $p_1(F)$ is as in \(2.2\) and $\chi$ is invariant under $SO(3)$ gauge transformations. Writing $p_1 = dp_1^{(0)}$, with $\delta p_1^{(0)} = dp_1^{(1)}, \ (5.5)$ implies

$$\delta \int_{\Sigma_3} \Omega_1 \wedge d\Omega_1 = \frac{1}{4} \int_{\Sigma_3} \delta p_1^{(0)} = \frac{1}{4} \int_{W_2} p_1^{(1)},$$

(6.6)

so \(6.4\) compensates for the deficit in the anomaly \(6.1\) in the low energy theory.

The coefficient of the WZ term must be quantized and properly normalized in order for $e^{2\pi i \Gamma}$ to be invariant under changing $\Sigma \rightarrow \Sigma'$, with $\partial \Sigma = \partial \Sigma' = W_2$. The difference is \(5.4\) integrated over $S^3 \cong (\Sigma - \Sigma')$, which gives $(k(G, X) - k(H, X) - k(U(1), X))H[\hat{\phi}]$, where $H[\hat{\phi}]$ is the Hopf number of the map $\hat{\phi} : S^3 \rightarrow S^2$; $H[\hat{\phi}] \in \mathbb{Z}$, corresponding to $\pi_3(S^2) = \mathbb{Z}$. To have the functional integral be well defined under $\Sigma \rightarrow \Sigma'$ thus requires all $(k(G, X) - k(H, X) - k(U(1), X))$, and thus all $k(G, X)$, to always be an integer.

\[7\] There can be an obstruction to extending $\hat{\phi}$ from $W_2$ to $\Sigma_3$ if $\hat{\phi}$ is in the non-trivial component of $\pi_2(\mathcal{M}_c = S^2) = \mathbb{Z}$.
7. Speculations

Because the WZ term (2.4) of the $\mathcal{N} = (2, 0)$ theory is related to an anomaly, it is natural to expect that it can be found exactly by a 1-loop calculation, with some fields which become massive due to $\langle \phi \rangle \neq 0$ running in the loop. In the 4d $\mathcal{N} = 4$ theory, these were the $n_W = |G| - |H| - 1$ gauginos, which get a mass via Yukawa couplings to $\langle \phi \rangle \neq 0$. The analog in the 6d $\mathcal{N} = (2, 0)$ theory are the BPS strings, which get a tension $1/\alpha'(\phi) \sim \phi$. Perhaps, then, it is possible to derive the WZ term directly by a 1-loop string calculation, with these $n_W$ strings, coupling to $\phi$, running in the loop. This suggests a WZ term proportional to $n_W$, though we know from (2.8) that it can not be exactly just $n_W$. Indeed, following (2.15), we speculated that the WZ coefficient is

$$\frac{1}{6}(c(G) - c(H)) = \alpha_e \alpha_m = \frac{1}{4} n_W \alpha_m, \quad (7.1)$$

e.g. with $\alpha_m = N + 1$ for the case (2.8). So then the challenge is to get the factor of $\alpha_m$.

We have not yet demonstrated that such a derivation of the WZ term (2.4), via integrating out tensionful strings, is actually possible. One might object that the $\mathcal{N} = (2, 0)$ theory is really a field theory, and the strings are not fundamental but, rather, some kind of solitonic objects, e.g. the skyrmionic strings of sect. 2. Perhaps, then, these are not the correct degrees of freedom to be integrating out in deriving the WZ term. On the other hand, perhaps the distinction between fundamental vs composite degrees of freedom is irrelevant for deriving the WZ term, since it is related to 't Hooft anomalies. In any case, it is hoped that reproducing the answers for the WZ terms presented here could lead to a better understanding of the 6d $\mathcal{N} = (2, 0)$ field theories.

In analogy with ordinary QFT, one might suppose that the coefficient of the WZ term is some function of only those degrees of freedom which become massive when $G \to H \times U(1)$. E.g. we might try a function only of $n_W$ which, based on the (2.8) case, would then be the general guess $c(G) - c(H) = 3(n_W/2)(n_W/2 + 1)$. However, this guess does not work for the case $G = SO(2N)$ and $H = SU(N)$ in the large $N$ limit: using (4.0), we have $c(G) \approx 4N^3$ and $c(H) \approx N^3$, so we should be getting $c(G) - c(H) \approx 3N^3$; on the other hand, the guessed formula incorrectly gives $3(n_W/2)(n_W/2 - 1) \approx \frac{3}{4} N^4$ since $n_W = N(N - 1)$. Based on this failure, it seems that the coefficient of the WZ term must contain some explicit dependence on the massless, interacting, $H$ degrees of freedom. If (7.1) is correct, the explicit $H$ dependence is in $\alpha_m$. 

17
Our conjecture for $c(G)$ for general $G$, based on the $SU(N)$ case and (1.0), is

$$c(G) = |G|C_2(G),$$  (7.2)

where $C_2(G)$ is the dual Coxeter number, normalized to be $N$ for $SU(N)$. This gives $c(SU(N)) = N^3 - N$, $c(SO(2N)) = 2N(N - 1)(2N - 1)$, $c(E_6) = (78)(12) = 912$, $c(E_7) = (133)(18) = 2394$, and $c(E_8) = (248)(30) = 7440$. A check of (7.2) is that it is a multiple of 6, satisfying (2.11) and thus (2.10), for all $ADE$ groups $G$. It also satisfies the $c$-function condition $c(G) > |G|$ in all cases.

It would be interesting to derive the Hopf-Wess-Zumino term (2.4), and thus check (7.2), in the context of IIB string theory on a $C^2/\Gamma_G$ ALE space, where $\phi^a$ are the periods of the 3 Kahler forms and two $B$ fields on a blown-up two-cycle. Since (2.4) depends only on the angular $\hat{\phi}$, the size $|\phi|$ of the blown-up two cycle can be arbitrarily large. The $C_4 \wedge H_3 \wedge H_3^*$ interaction of the 10d IIB string looks promising for leading to WZ terms. The $\pi_4(S^4)$ skyrmionic strings should again be identified with the W-boson strings, which here arise [7] from D3 branes wrapped on the blown-up two-cycle.

Decomposing the adjoint of $G$ as $ad(G) \to W + ad(H)$ for some representations $W$ (which is the rep of the massive W-bosons, along with a singlet = $ad(U(1))$) of $G$, the conjectured formula (7.2) gives for the coefficient of the WZ term:

$$c(G) - c(H) = |H|C_2(W) + |W|(C_2(H) + C_2(W)).$$  (7.3)

Note that this expression depends explicitly on $H$, via $|H|$ and $C_2(H)$, and not only on the massive reps in $W$. This suggests that an eventual derivation of the WZ term must include effects which couple the massive degrees of freedom, which are integrated out, to the massless, interacting, $H$ degrees of freedom. Assuming (7.1), this could be just via $\alpha_m$, the magnetic charge of the skyrmionic strings in $W_6$ (or membranes in $\Sigma_7$), which would have explicit $H$ dependence as given by (7.3). It would be interesting to directly determine $\alpha_m$ and see if, and how, it is given as suggested by the above discussion.

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