FOURTH MOMENT THEOREMS FOR COMPLEX GAUSSIAN APPROXIMATION

SIMON CAMPESE

Abstract. We prove a bound for the Wasserstein distance between vectors of smooth complex random variables and complex Gaussians in the framework of complex Markov diffusion generators. For the special case of chaotic eigenfunctions, this bound can be expressed in terms of certain fourth moments of the vector, yielding a quantitative Fourth Moment Theorem for complex Gaussian approximation on complex Markov diffusion chaos. This extends the results of [ACP14] and [CNPP15] for the real case. Our main ingredients are a complex version of the so called $\Gamma$-calculus and Stein’s method for the multivariate complex Gaussian distribution.

1. Introduction

Let $F_n$ be a sequence of random variables and $\gamma$ be a probability measure. A Fourth Moment Theorem for $F_n$ holds, if there exists a polynomial $P = P(m_1(F_n), \ldots, m_4(F_n))$ in the first four moments of $F_n$ such that $P(m_1(F_n), \ldots, m_4(F_n)) \to 0$ characterizes, or at least implies the convergence of the laws of $F_n$ towards $\gamma$. The first discovery of such a Fourth Moment Theorem dates back to 2005, when Nualart and Peccati, in their seminal paper [NP05], characterized convergence in distribution of a sequence $F_n$ of multiple Wiener-Itô integrals towards a Gaussian distribution by the convergence of the moment sequence $m_4(F_n) - 3m_2(F_n)^2$. The next developments were an extension to the multidimensional case by Peccati and Tudor in [PT05], the introduction of Malliavin calculus by Nualart and Ortiz-Latorre in [NOL08] and, starting in [NP09] by Nourdin and Peccati and then followed by numerous other contributions of the same two authors and their collaborators, the combination of Malliavin calculus and Stein’s method to obtain quantitative central limit theorems in strong distances. This Malliavin-Stein method has found widespread applications, for example in statistics, mathematical physics, stochastic geometry or free probability (see the webpage [Nou15] for an overview). For a textbook introduction to the method, we refer to [NP12].

Recently, initiated by Ledoux in [Led12] and then further refined by Azmoodeh, Campese and Poly in [ACP14], it was shown that Fourth Moment Theorems hold in the very general framework of Markov diffusion generators (see [BGL14] for an exhaustive treatment of this framework), in which the
aforementioned setting of multiple Wiener-Itô integrals appears as the special case of the Ornstein-Uhlenbeck generator. This abstract point of view not only allows for a drastically simplified proof of Nualart and Peccati’s classical Fourth Moment Theorem, but also provides new Fourth Moment Theorems for previously uncovered structures such as Laguerre and Jacobi chaos and new target laws (such as the Beta distribution). We refer to [ACP14] for details. Stein’s method continues to work in this framework as well and can be used to associate quantitative estimates to these results. The abstract framework also led to new results for the Ornstein-Uhlenbeck setting, such as an “even-moment theorem” (see [AMP13]) and advances regarding the Gaussian product conjecture (see [MNPP15]).

In this paper, we extend this abstract diffusion generator framework to the complex case, thus covering complex valued random variables. Our main result is a quantitative bound between the Wasserstein distance of a multivariate complex Gaussian random vector and vectors of square integrable complex random variables in the domain of a carré du champ operator associated to a diffusive Markov generator (all unexplained terminology will be introduced below). For the case of chaotic eigenfunctions, this bound can be expressed purely in terms of the second and fourth absolute moments of the vector and thus yields a quantitative complex Fourth Moment Theorem. To obtain these results, we extend Stein’s method to cover the complex Gaussian distribution and develop a complex version of the so called Γ-calculus, in which a central role will be played by the Wirtinger derivatives $\partial_z$ and $\partial_{\overline{z}}$.

Of course, from a purely algebraic point of view, our approach is equivalent to the multidimensional real case, which has been treated in [CNPP15], and indeed one could handle sequences of complex valued random variables by separating real and imaginary part and considering them as two-dimensional real vectors. For the complex Ornstein-Uhlenbeck generator, such an ad-hoc strategy has been followed in [CL14] and [Che14]. However, by staying completely inside the complex domain, our approach has the advantage of making available many powerful tools connected to concepts such as holomorphy from complex analysis which have no equivalents in the real case. Although not needed for an abstract derivation of our main results, these tools might be useful in the future, also when taking the reverse route and trying to prove results for the real case by translating them to the complex domain. Even for the special case of the complex Ornstein-Uhlenbeck generator, there is much structure present in the complex domain, such as a fine decomposition of the eigenspaces as shown in Example 3.3 or the unitary group from complex White noise analysis (see [Hid80, Ch. 7]), which has no real counterparts.

Complex random variables are encountered naturally in many applications and indeed this paper was motivated by the study of spin random fields arising in cosmology (see for example [BR14, GM10, Mal13, MP11]). A followup paper with an application of our results to this case is in preparation.

The rest of this paper is organized as follows. In Section 2, we introduce the notation used throughout the rest of this paper and provide some necessary background material. The abstract complex diffusion generator
framework is presented in Section 3, while our main results are stated and proved in Section 4.

2. Preliminaries

2.1. Wirtinger calculus. Let \( x, y \in \mathbb{R} \) be two real variables and define the complex variables \( z = x + iy \) and \( \overline{z} = x - iy \), where \( i^2 = -1 \). Then, every function \( \tilde{f} = f(x, y) : \mathbb{R}^2 \to \mathbb{C} \) can be considered as a function \( f = f(z) : \mathbb{C} \to \mathbb{C} \) via the identity

\[
\tilde{f}(x, y) = \tilde{f}\left(\frac{z + \overline{z}}{2}, \frac{z - \overline{z}}{2i}\right) = f(z).
\]

Conversely, every function \( f = f(z) : \mathbb{C} \to \mathbb{C} \) can be considered as a function \( \tilde{f} = \tilde{f}(x, y) : \mathbb{R}^2 \to \mathbb{C} \) by writing

\[
f(z) = f(x + iy) = \tilde{f}(x, y).
\]

With this notation, the Wirtinger derivatives are defined as

\[
\partial_z = \frac{1}{2}(\partial_x - i\partial_y) \quad \text{and} \quad \partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y),
\]

where, here and in the following, we use the shorthand \( \partial_a = \frac{\partial}{\partial_a} \) to denote derivatives with respect to the variables \( a_1, \ldots, a_d \) (the concrete interpretation as a partial or Wirtinger derivative will always be clear from the context). Starting from the Wirtinger derivatives \( \partial_z \) and \( \partial_{\overline{z}} \), one can get back the partial derivatives \( \partial_x \) and \( \partial_y \) by the identities

\[
\partial_x = \partial_z + \partial_{\overline{z}} \quad \text{and} \quad \partial_y = i(\partial_z - \partial_{\overline{z}}).
\]

It is straightforward to see that both Wirtinger derivatives are linear and merit their names by satisfying the derivation properties (product rules)

\[
\partial_z (fg) = (\partial_z f)g + f\partial_z g \quad \text{and} \quad \partial_{\overline{z}} (fg) = (\partial_{\overline{z}} f)g + f\partial_{\overline{z}} g.
\]

Moreover, the conjugation identities

\[
\overline{\partial_z f} = \partial_{\overline{z}} \overline{f} \quad \text{and} \quad \overline{\partial_{\overline{z}} f} = \partial_z \overline{f}
\]
hold. The chain rules take the form

\[
\partial_z (f \circ g) = ((\partial_z f) \circ g) \partial_z g + ((\partial_{\overline{z}} f) \circ g) \partial_{\overline{z}} g,
\]

\[
\partial_{\overline{z}} (f \circ g) = ((\partial_z f) \circ g) \partial_{\overline{z}} g + ((\partial_{\overline{z}} f) \circ g) \partial_z g.
\]

In particular, we see that \( \partial_z z = 1, \partial_z \overline{z} = 0, \partial_{\overline{z}} z = 1 \) and \( \partial_{\overline{z}} \overline{z} = 0 \), which, in view of the chain and product rule, allows to formally treat \( z \) and \( \overline{z} \) as if they were independent variables when differentiating. Heuristically, when applying the Wirtinger derivatives to a function \( f : \mathbb{C} \to \mathbb{C} \), one does not need to consider \( f(z) \) as a function \( \tilde{f}(x, y) \) and then compute the partial derivatives with respect to \( x \) and \( y \), but can instead directly apply the formal rules of differentiation to the complex variables \( z \) and \( \overline{z} \). For example, we have for \( p, q \neq -1 \) that \( \partial_z (z^{p+q}) = p z^{p-1} \overline{z} \) and \( \partial_{\overline{z}} (z^{p+q}) = q z^{p+1} \overline{z} \).

In the sequel, we will work in general dimension \( d \geq 1 \), considering functions \( f : \mathbb{C}^d \to \mathbb{C} \) and \( \tilde{f} : \mathbb{R}^{2d} \to \mathbb{C} \) which continue to be related through (1) and (2), where now \( x = (x_1, \ldots, x_d) \), \( y = (x_1, \ldots, x_d) \) and \( z = (z_1, \ldots, z_d) \)
are vectors of variables. For these functions, we define the gradients $\nabla f$ and $\nabla f^*$ by
\[
\nabla f = \left( \partial_z f, \partial_{\bar{z}} f, \ldots, \partial_\bar{z}^d f \right)^T
\]
and
\[
\nabla f^* = \left( \partial_z f, \partial_{\bar{z}} f, \ldots, \partial_\bar{z}^d f \right)^T,
\]
and the complex Hessians $\nabla \nabla f$, $\nabla \nabla f^*$, $\nabla \nabla f^*$ and $\nabla \nabla f^*$ by
\[
\begin{align*}
\nabla \nabla f &= \left( \partial_{zz} f \right)_{1 \leq j, k \leq d}, \\
\nabla \nabla f^* &= \left( \partial_{z\bar{z}} f \right)_{1 \leq j, k \leq d}, \\
\nabla \nabla f^* &= \left( \partial_{\bar{z}\bar{z}} f \right)_{1 \leq j, k \leq d}, \\
\nabla \nabla f^* &= \left( \partial_{\bar{z}z} f \right)_{1 \leq j, k \leq d}.
\end{align*}
\]
With some abuse of notation, we say that $f$ is an element of $C^m(\mathbb{C}^d)$, if the associated function $\tilde{f}$ belongs to $C^m(\mathbb{R}^{2d})$. Similarly, a function $f$ has bounded Wirtinger derivatives up to some order $m$, if the associated function $\tilde{f}$ has bounded partial derivatives up to this order.

2.2. The complex normal distribution. Given a probability space $\mathcal{P} = (\Omega, \mathcal{F}, P)$ and two real-valued random variables $X$ and $Y$, the quantity $Z = X + iY$ is called a complex valued random variable. The characteristic function, law and, if it exists, density of $Z$ are defined as being the corresponding quantities of the two-dimensional real random vector $(X, Y)$.

With the notation of the previous subsection, we clearly have that if $f(z)$ is the density of $Z$, then $\tilde{f}(x, y)$ is the density of $(X, Y)$, and the analogous statement is true for the law. From the characteristic function $\tilde{\rho}(\xi, \nu)$ of $(X, Y)$, we readily calculate the characteristic function $\rho(\zeta) = \rho(\xi + iv)$ of $Z$ to be
\[
\tilde{\rho}(\zeta) = E \left[ e^{i\Re(\langle \zeta, Z \rangle_{\mathbb{C}^d})} \right],
\]
where, here and in the following, $E$ denotes mathematical expectation and $\Re(z)$ and $\Im(z)$ stand for the real and imaginary parts of a complex number $z$.

**Definition 2.1** (multivariate complex normal distribution). For $d \geq 1$, $\mu \in \mathbb{C}^d$ and a positive definite $d \times d$ Hermitian matrix $\Sigma$, a complex random vector $Z \in \mathbb{C}^d$ is said to follow a multivariate complex normal distribution with mean $\mu$ and covariance $\Sigma$, short: $Z \sim \mathcal{CN}_d(\mu, \Sigma)$, if it has the density function
\[
f(z) = \frac{1}{\pi^d |\det \Sigma|} \exp \left( - (z - \mu)^T \Sigma^{-1} (z - \mu) \right),
\]
where $A^T$ denotes the transpose of $A$.

**Remark 2.2.** Let $Z \sim \mathcal{CN}_d(\mu, \Sigma)$.

(i) Straightforward calculations show that
\[
E \left[ (Z - \mu)(Z_k - \mu)^T \right] = \Sigma
\]
and
\[
E \left[ (Z - \mu)(Z - \mu)^T \right] = 0.
\]
Furthermore, it can be shown that $Z$ is circularly symmetric: For any $\alpha \in \mathbb{R}$, the rotated vector $e^{i\alpha} Z$ has the same distribution as
Each circularly symmetric complex Gaussian vector can be obtained via a linear transformation of a standard complex Gaussian vector $Z \sim CN_d(0, \text{Id}_d)$ whose real and imaginary part are independent real-valued standard Gaussian vectors. Some authors drop the independence assumption and thus obtain more general complex Gaussian vectors for which the matrix on the left hand side of (4), sometimes called the relation matrix, is no longer zero. However, when we speak of a complex Gaussian vector, we always mean the circularly symmetric case of Definition 2.1.

(ii) The characteristic function $\rho$ of $Z$ is given by

$$\rho(\zeta) = \exp \left( i \Re \langle \mu, \zeta \rangle + \frac{1}{4} \langle \Sigma \zeta, \zeta \rangle \right),$$

which shows that $Z$ is determined by its moments, i.e. (assuming $\mu = 0$ for notational convenience) any $d$-dimensional complex random vector $W$ satisfying

$$E \left[ \prod_{j=1}^{d} (W_j - \mu_j)^{p_j} (\overline{W}_j - \overline{\mu}_j)^{q_j} \right] = E \left[ \prod_{j=1}^{d} (Z_j - \mu_j)^{p_j} (\overline{Z}_j - \overline{\mu}_j)^{q_j} \right]$$

for all $p_j, q_j \in \mathbb{N}_0$, $j = 1, \ldots, d$, has the same law as $Z$.

We will need the following complex version of the Gaussian integration by parts formula, which for convenience will be stated for the centered case.

**Lemma 2.3.** Let $Z \sim CN_d(0, \Sigma)$ and $\varphi : \mathbb{C}^d \to \mathbb{C}$ such that, for $1 \leq i \leq d$, the Wirtinger derivatives $\partial_{z_i} \varphi$ and $\overline{\partial}_{\bar{z}_i} \varphi$ exist and have at most polynomial growth in $z_i$ and $\overline{z}_i$ respectively. Then it holds that

$$E[Z_i \varphi(Z_1, \ldots, Z_d)] = \sum_{j=1}^{d} E[Z_j \bar{Z}_j] E[\partial_{z_j} \varphi(Z_1, \ldots, Z_d)]$$

and

$$E[\overline{Z}_i \varphi(Z_1, \ldots, Z_d)] = \sum_{j=1}^{d} E[Z_j \overline{Z}_j] E[\partial_{\bar{z}_j} \varphi(Z_1, \ldots, Z_d)]$$

**Proof.** The proof is standard and straightforward but is included nevertheless to demonstrate the use of Wirtinger calculus. We will prove formula (5) and note that (6) then follows by conjugation. As $\Sigma$ is positive definite Hermitian, it admits a normal square root $A$ such that $\Sigma = A^* A$, where $A^*$ denotes the conjugate transpose of $A$. Clearly, both $A$ and $A^*$ are invertible and we have that $\Sigma^{-1} = (A^*)^{-1} A^{-1}$. The linear transformation

$$\xi = A^{-1} z$$

induces a linear transformation on $\mathbb{R}^{2d}$ whose constant volume element will be denoted by $v$. Writing $z = x + iy$ and $\xi = \xi + iv$ (with $z_i = x_i + iy_i$ and
\( \zeta_i = \xi_i + iv_i \), we get that

\[
E [Z_i \varphi(Z)] = \frac{1}{\pi^d |\det \Sigma|} \int_{\mathbb{R}^{2d}} z_i \varphi(z) e^{-z^* \Sigma^{-1} z} \, d(x, y)
\]

\[
= \frac{v}{\pi^d |\det \Sigma|} \sum_{j=1}^d A_{ij} \int_{\mathbb{R}^{2d}} \zeta_j \varphi(A\zeta) e^{-\zeta^* \zeta} \, d(\xi, v)
\]

\[
= \frac{-v}{\pi^d |\det \Sigma|} \sum_{j=1}^d A_{ij} \int_{\mathbb{R}^{2d}} \varphi(A\zeta) \left( \partial_{\zeta_j} e^{-\zeta^* \zeta} \right) \, d(\xi, v).
\]

By the product rule, it holds that

\[
\varphi(A\zeta) \left( \partial_{\zeta_j} e^{-\zeta^* \zeta} \right) = \partial_{\zeta_j} \left( \varphi(A\zeta)e^{-\zeta^* \zeta} \right) - \left( \partial_{\zeta_j} \varphi(A\zeta) \right) e^{-\zeta^* \zeta}.
\]

Now, by a Fubini argument,

\[
\int_{\mathbb{R}^{2d}} \partial_{\zeta_j} \left( \varphi(A\zeta)e^{-\zeta^* \zeta} \right) \, d(\xi, v) = 0.
\]

Furthermore, by the chain rule,

\[
\partial_{\zeta_j} \varphi(A\zeta) = \sum_{k=1}^d A_{k,j} (\partial_{\zeta_k} \varphi)(A\zeta).
\]

Therefore,

\[
E [Z_i \varphi(Z)] = \frac{v}{\pi^d |\det \Sigma|} \sum_{j,k=1}^d A_{ij} A_{k,j} \int_{\mathbb{R}^{2d}} (\partial_{\zeta_k} \varphi)(A\zeta)e^{-\zeta^* \zeta} \, d(\xi, v)
\]

\[
= \frac{1}{\pi^d |\det \Sigma|} \sum_{j,k=1}^d A_{ij} A_{k,j} \int_{\mathbb{R}^{2d}} \partial_{\zeta_k} \varphi(z)e^{-z^* z} \, d(x, y).
\]

Noting that \( \sum_{j=1}^d A_{i,j} A_{k,j} = \sum_{j=1}^d A_{i,j} A_{j,k}^* = \Sigma_{i,k} = E [Z_i Z_k] \) proves (5).

As an immediate consequence of Lemma 2.3, we see that for \( Z \sim \mathcal{CN}_d(0, \Sigma) \) and all multi-indices \( p = (p_1, \ldots, p_d) \in \mathbb{N}_0^d \) of order at least one it holds that

\[
E \left[ \prod_{j=1}^d Z_j^{p_j} \right] = E \left[ \prod_{j=1}^d \overline{Z}_j^{p_j} \right] = 0.
\]

Furthermore, for the case \( Z \sim \mathcal{CN}_1(0, \sigma^2) \), Lemma 2.3 yields the well-known moment-formula

\[
\begin{cases}
\quad p! \, \sigma^{2p} & \text{if } p = q \\
\quad 0 & \text{if } p \neq q,
\end{cases}
\]

valid for \( p, q \in \mathbb{N}_0 \) (with the usual convention that \( 0! = 1 \)).
2.3. Stein’s method for the complex normal distribution. For a quadratic matrix $A$, the Hilbert-Schmidt norm $\|A\|_{HS}$ is defined via the inner product $(A, B)_{HS} = \text{tr}(A^* B)$.

The next lemma is an adaptation of the Stein characterization for the multivariate real normal distribution (see [CM08, Lemma 2.1], [NPR10, Lemma 3.3] and also [RR09, Lemma 2.6]) to the complex case. Recall the definitions of complex gradients and Hessians given in Section 2.1.

**Lemma 2.4** (Stein’s Lemma for the complex Gaussian distribution). For $d \geq 1$, let $\Sigma$ be a positive definite, Hermitian matrix and $Z \sim CN_d(0, \Sigma)$.

i) A $d$-dimensional complex random vector $Y$ has the complex normal distribution $CN_d(0, \Sigma)$, if, and only if, the identity

$$E \left[ \langle \nabla \nabla f(Y), \Sigma \rangle_{HS} \right] + E \left[ \langle \nabla \nabla f(Y), \Sigma \rangle_{HS} \right] - E \left[ \langle \nabla f(Y), \Sigma \rangle_{HS} \right] = 0$$

holds for any $f \in C^2(\mathbb{C}^d)$ which satisfies

$$E \left[ \langle \nabla \nabla f(Y), \Sigma \rangle_{HS} \right] + E \left[ \langle \nabla \nabla f(Y), \Sigma \rangle_{HS} \right] + E \left[ \langle \nabla f(Y), \Sigma \rangle_{HS} \right] < \infty.$$

ii) Given $h \in C^2(\mathbb{C}^d)$ with bounded derivatives up to order two, the function

$$U_h(z) = \int_0^1 \frac{1}{2t} \left( E[h(Z_{z,t})] - E[h(Z)] \right) dt,$$

where $Z_{z,t} = \sqrt{t}z + \sqrt{1-t}Z$, is a solution to the complex Stein equation

$$(7) \quad \langle \nabla \nabla f(z), \Sigma \rangle_{HS} + \langle \nabla \nabla f(z), \Sigma \rangle_{HS} - \langle \nabla f(z), \Sigma \rangle_{HS} = h(z) - E[h(Z)].$$

**Proof.** The real counterpart of this lemma has been proven by direct calculations using Gaussian integration by parts in [CM08, Lemma 2.1], and based on the generator approach of [Bar90] in [NPR10, Lemma 3.3]. Both of these proofs can be straightforwardly adapted to the complex case (either using complex Gaussian by parts or the complex Ornstein-Uhlenbeck generator) which is why we omit the details. □

We also need the following technical lemma, which can be deduced by adapting the proof of inequality (3.4) in [NPR10].

**Lemma 2.5.** In the setting and with the notation of Lemma 2.4, let $c(\Sigma) = \|\Sigma^{-1}\|_{op} \|\Sigma\|_{op}^{1/2}$. Then, for any $\alpha \in [0, 1]$, the complex Hessians of the Stein solution (7) satisfy the bounds

$$\|\nabla \nabla U_h(z)\|_{HS} \leq c(\Sigma) \left( \alpha \max_{1 \leq k \leq d} \|\partial_z h\|_\infty + (1 - \alpha) \max_{1 \leq k \leq d} \|\partial_{z_k} h\|_\infty \right),$$

$$\|\nabla \nabla U_h(z)\|_{HS} \leq c(\Sigma) \left( \alpha \max_{1 \leq k \leq d} \|\partial_z h\|_\infty + (1 - \alpha) \max_{1 \leq k \leq d} \|\partial_{z_k} h\|_\infty \right),$$

$$\|\nabla \nabla U_h(z)\|_{HS} \leq c(\Sigma) \max_{1 \leq k \leq d} \|\partial_{z_k} h\|_\infty.$$
and
\[ \| \nabla \nabla U_h(z) \|_{HS} \leq c(\Sigma) \max_{1 \leq k \leq d} \| \partial_z h \|_\infty. \]

Remark 2.6.

i) The proofs of Lemmas 2.4 and 2.5 crucially depend on the characterizing property that any complex Gaussian vector \( Z \sim CN_d(0, \Sigma) \) can be obtained via a linear transformation of a standard complex Gaussian vector \( \tilde{Z} \sim CN_d(0, \text{Id}_d) \). An adaptation of these proofs to the larger class of not necessarily circularly symmetric complex Gaussian vectors hinted at in Remark 2.2 is thus not possible.

ii) As \( \Sigma \) and its inverse are positive definite and Hermitian, the operator norms of these two matrices coincide with their spectral radii. Thus, the constant \( c(\Sigma) = \| \Sigma^2 \|_{op} \| \Sigma \|_{op}^{1/2} \) appearing in the bounds for \( U_h \) in Lemma 2.5 coincides with \( \sqrt{\lambda_{\text{max}}/\lambda_{\text{min}}} \) where \( \lambda_{\text{max}} \) and \( \lambda_{\text{min}} \) denote the largest and smallest eigenvalues of \( \Sigma \), respectively. In particular, in the case \( d = 1 \), where \( \Sigma = \lambda > 0 \), this constant becomes \( 1/\sqrt{\lambda} \).

For the case \( d = 1 \), it is actually possible to derive a simpler characterizing differential equation.

**Lemma 2.7.** A complex valued random variable \( Z \) has the standard complex normal distribution if, and only if
\[(9)\quad \mathbb{E} [\partial_z f(Z)] - \mathbb{E} [\overline{Z} f(Z)] = 0\]
for all Wirtinger differentiable functions \( f : \mathbb{C} \rightarrow \mathbb{C} \) such that \( \partial_z f \) has at most polynomial growth.

**Proof.** Necessity of condition (9) is implied by Gaussian integration by parts (Lemma 2.3). Sufficiency follows by inserting the polynomials \( f(z) = \overline{z}^q \), which immediately yields the moment recursion for the standard complex Gaussian, and noting that the complex Gaussian distribution is determined by its moments. \( \Box \)

Remark 2.8. From identity (9), one is led to the “Stein equation”
\[(10)\quad \partial_z f(z) - \overline{z} f(z) = h(z) - \mathbb{E} [h(Z)],\]
where \( h : \mathbb{C} \rightarrow \mathbb{C} \) is some given function and \( Z \) has the standard complex Gaussian distribution. Note that if we formally replace the complex variable \( z \) with a real variable \( x \) and the Wirtinger derivative \( \partial_z \) with the partial derivative \( \partial_x \), we obtain the classical Stein equation of the one-dimensional Gaussian distribution. However, as one sees after writing \( z = x + iy \) and separating real and imaginary parts, this equation is not solvable in general. This was to be expected, as otherwise, using Stein’s method for the real case, one could obtain bounds in total variation and Kolmogorov distance for two-dimensional real Gaussian approximation, which is not possible using this approach (see for example [CM08, pp.263]. Lemma 2.7 can thus not be quantified.
3. Complex Markov diffusion generators

As in the real case, we start with a good measurable space $E$ in the sense of [BGL14, p.7] (for example, take $E$ to be a Polish space), equipped with a probability measure $\mu$. On $L^2(E, \mathbb{R}, \mu)$, let $L$ be a symmetric Markov diffusion generator $L$ with discrete spectrum $S = \{-\lambda_k\}$, where the eigenvalues $-\lambda_k$ are ordered by magnitude, i.e. $0 = \lambda_0 < \lambda_1 < \ldots$. In the language of functional analysis, $-L$ is a positive, self-adjoint linear operator vanishing on the constants. The associated bilinear carré du champ operator $\Gamma$, acting on a set $\mathcal{A}_0$ which we assume to be dense in $L^p(E, \mathbb{R}, \mu)$ for all $p \geq 1$, is defined in the usual way as

$$2\Gamma(U_1, U_2) = L(U_1 U_2) - U_1 L U_2 - U_2 L U_1$$

and for any smooth function $\varphi: \mathbb{R}^d \to \mathbb{R}$ the diffusion property

$$L \varphi(U) = L \varphi(U_1, \ldots, U_d)$$

$$= \sum_{k=1}^d \partial_{x_k} \varphi(U_1, \ldots, U_d) L U_k + \sum_{j,k=1}^d \partial_{x_j x_k} \varphi(U_1, \ldots, U_d) \Gamma(U_j, U_k)$$

holds. As is well known, this diffusion property is equivalent to the chain rule

$$\Gamma(\varphi(U_1, \ldots, U_d), V) = \sum_{j=1}^d \partial_j \varphi(U_1, \ldots, U_d) \Gamma(U_k, V).$$

Through straightforward complexification, we can extend $L$ and $\Gamma$ to act on the space $L^2(E, \mathbb{C}, \mu)$. Writing $F = U + iV$ for a generic element of $L^2(E, \mathbb{C}, \mu)$, this extension of $L$, which we denote for the moment by $\hat{L}$, is simply defined as

$$\hat{L} = \hat{L}(U + iV) = L U + iL V.$$

We immediately see that $-\hat{L}$ remains a positive, self-adjoint operator vanishing on constants and that its spectrum coincides with the one of $L$. Indeed, assuming that $U + iV$ is an eigenfunction of $\hat{L}$ with some eigenvalue $\lambda$, we have that

$$\lambda(U + iV) = \hat{L}(U + iV) = L U + iL V.$$

Comparing imaginary and real parts, this gives that $\lambda$ lies in the spectrum of $L$ and that both, $U$ and $V$ are eigenfunctions of $L$ with eigenvalue $\lambda$. Similarly, for $U_1, U_2, V_1, V_2 \in \mathcal{A}_0$, the extended carré du champ operator $\hat{\Gamma}$, defined on $\hat{\mathcal{A}}_0 = \mathcal{A}_0 \oplus i\mathcal{A}_0$, is given by

$$\hat{\Gamma}(U_1 + iV_1, U_2 + iV_2) = \Gamma(U_1, U_2) + \Gamma(V_1, V_2) + i(\Gamma(V_1, U_2) - \Gamma(U_1, V_2)).$$

It follows that $\hat{\Gamma}$ is sesquilinear, positive ($\hat{\Gamma}(F, F) \geq 0$) and Hermitian ($\hat{\Gamma}(F, G) = \overline{\hat{\Gamma}(G, F)}$). Furthermore, using identity (11), we get that

$$2\hat{\Gamma}(F, G) = \hat{L}(F\overline{G}) - F\hat{L}\overline{G} - \overline{G}\hat{L}F,$$

which yields the integration by parts formula

$$\int_E \hat{\Gamma}(F, G) d\mu = -\int_E F\hat{L}\overline{G} d\mu.$$
As $-\hat{L}$ is positive, it holds that $\hat{L}F = \hat{L} \overline{F}$. In particular, all eigenspaces of $\hat{L}$ are closed under complex conjugation and, if $\pi_j$ denotes the orthogonal projection onto $\ker(\hat{L} + \lambda_k I_d)$, we have for all $F \in L^2(E, \mathbb{C}, \mu)$ that $\pi_j(F) = \pi_j(\overline{F})$. Furthermore, by the defining equation (16) of $\Gamma$, we also see that $\Gamma(F, G) = \overline{\Gamma(F, \overline{G})}$. Using (14) and (15), it is straightforward to verify the diffusion property

\begin{equation}
\hat{L}\varphi(F) = \hat{L}\varphi(F_1, \ldots, F_d) = \sum_{j=1}^d \left( \partial_{z_j} \varphi(F) \hat{L}F_j + \partial_{\overline{z}_j} \varphi(F) \hat{L}\overline{F}_j \right) + \sum_{j,k=1}^d \left( \partial_{z_j z_k} \varphi(F) \hat{\Gamma}(F_j, \overline{F}_k) + \partial_{z_j \overline{z}_k} \varphi(F) \hat{\Gamma}(\overline{F}_j, F_k) \right) + \sum_{j,k=1}^d \left( \partial_{z_j \overline{z}_k} \varphi(F) \hat{\Gamma}(F_j, F_k) + \partial_{z_j z_k} \varphi(F) \hat{\Gamma}(\overline{F}_j, \overline{F}_k) \right),
\end{equation}

and the chain rule

\begin{equation}
\hat{\Gamma}(\varphi(F), G) = \hat{\Gamma}(\varphi(F_1, \ldots, F_d)) = \sum_{j=1}^d \left( \partial_{z_j} \varphi(F) \hat{\Gamma}(F_j, G) + \partial_{\overline{z}_j} \varphi(F) \hat{\Gamma}(\overline{F}_j, G) \right),
\end{equation}

both valid for smooth functions $\varphi: \mathbb{C}^d \to \mathbb{C}$ and $F = (F_1, \ldots, F_d) \in \hat{A}_0^d$. Here, $\partial_z$, $\partial_{\overline{z}}$, $\partial_{zz}$ etc. denote the (iterated) Wirtinger derivatives introduced in Section 2.1. To be more clear, the diffusion property (17) and the chain rule (18) can be translated into versions of their real counterparts (i.e. identities (12) and (13)), by simply writing $\varphi(z_1, \ldots, z_d) = u(x_1, \ldots, x_d, y_1, \ldots, y_d) + iv(x_1, \ldots, x_d, y_1, \ldots, y_d)$, where $z_j = x_j + iy_j$ and the functions $u$ and $v$ are real valued, decomposing the vector $F$ into real and imaginary parts and writing the Wirtinger derivatives in terms of derivatives with respect to the real variables $x_j$ and $y_j$. Because of this, we also see that, as in the real case, the diffusion property of $\hat{L}$ is equivalent to the chain rule of $\hat{\Gamma}$. Of course, using the fact that $\hat{\Gamma}$ is Hermitian, we can derive a chain rule for the second argument:

\begin{equation}
\hat{\Gamma}(F, \varphi(G)) = \hat{\Gamma}(F, \varphi(G_1, \ldots, G_d)) = \sum_{j=1}^d \left( \partial_{z_j} \varphi(G) \hat{\Gamma}(F, G_j) + \partial_{\overline{z}_j} \varphi(G) \hat{\Gamma}(F, \overline{G}_j) \right).
\end{equation}

We are now ready to define a complex Markov diffusion generator.

**Definition 3.1.** Given a good measurable space $(E, \mathcal{F})$, equipped with a probability measure $\mu$, a self-adjoint, linear operator $\hat{L}$ acting on $L^2(E, \mathbb{C}, \mu)$ is called a **complex symmetric Markov diffusion generator** with invariant measure $\mu$, if $-L$ is positive, $L1 = 0$ and the diffusion property (17) holds.
Note that the construction above also works in reverse: Given a complex symmetric Markov diffusion generator \( \hat{L} \), we obtain a corresponding generator on \( L^2(E, \mathbb{R}, \mu) \). From this abstract point of view, the real and complex approaches are thus completely equivalent. We state this as a short proposition.

**Proposition 3.2.** \( \hat{L} \) is a complex Markov diffusion generator, if, and only if, there exists a real Markov diffusion generator \( L \) with (the same) spectrum such that 
\[
\hat{L}F = \mathcal{L} \Re(F) + i \mathcal{L} \Im(F)
\]
for all \( F \in \text{dom} \hat{L} \).

Note that \( \hat{L} \) and \( \hat{\Gamma} \) coincide with \( L \) and \( \Gamma \) when restricted to real valued arguments. Because of this and Proposition 3.2 we will from now on no longer notationally distinguish \( L \) and \( \Gamma \) from their complexified versions \( \hat{L} \) and \( \hat{\Gamma} \) and instead denote both versions by \( L \) and \( \Gamma \), respectively.

As already mentioned in the introduction, it should be noted that although everything is equivalent from an abstract point of view, the complex case often introduces features absent from the real case when turning to special cases such as the complex Ornstein-Uhlenbeck generator (see the next example). Furthermore, in many applications it is often much more natural to use complex Gamma-calculus, taking the direct route through the complex domain instead of making a detour through \( \mathbb{R}^2 \).

**Example 3.3 (The complex Ornstein-Uhlenbeck generator).** Starting from a standard isonormal Gaussian process framework (see for example [Nua06, Ch. 1] or [NP12, Ch. 2]) for the real-valued, infinite-dimensional Ornstein-Uhlenbeck generator, we obtain the complex Ornstein-Uhlenbeck generator \( L_{OU} \) by carrying out the construction outlined above (i.e. via Proposition 3.2). Equivalently, one can directly start from an isonormal complex Gaussian process (see [CL14]). The generator \( L_{OU} \) can then be decomposed in the form 
\[
L_{OU} = -\delta D, \quad \text{where} \quad \delta \text{ and } D \text{ are the complex Malliavin divergence and Malliavin derivative operators (see [Jan97, Ch. 15] for more details on complex Malliavin calculus}).
\]
The carré du champ operator \( \Gamma_{OU} \) takes the form 
\[
\Gamma_{OU} = \langle DF, DG \rangle_H, \quad \text{where the inner product is now taken in a complex Hilbert space} \ H \text{ (coming from the underlying complex isonormal Gaussian process) and} \ D \text{ denotes the complex Malliavin derivative}. \]
The spectrum of \( L_{OU} \) is \(-N_0\) and by our findings above we know that all eigenfunctions \( F_\lambda \in \ker (L_{OU} + \lambda \text{Id}) \), \( \lambda \in N_0 \), are of the form \( F_\lambda = U_\lambda + iV_\lambda \), where \( U_\lambda \) and \( V_\lambda \) are eigenfunctions of the real-valued Ornstein-Uhlenbeck generator and thus multiple Wiener-Itô integrals. In contrast to the real case, however, one has a finer decomposition of the eigenspaces with a rich structure: For each \( \lambda \in N_0 \), it holds that 
\[
\ker (L_{OU} + \lambda \text{Id}) = \bigoplus_{p,q \in N_0, p+q = \lambda} \mathcal{H}_{p,q},
\]
where the sum on the right is orthogonal and the spaces \( \mathcal{H}_{p,q} \) consist of complex Wiener-Itô integrals of the form \( I_{p,q}(f) \) (see [Itô52]). Furthermore, one can show that \( \mathcal{H}_{p,q} = \mathcal{H}_{q,p} \) and that only the eigenspaces of even eigenvalues \( \lambda = 2p \) contain real-valued eigenfunctions, belonging to \( \mathcal{H}_{p,p} \). Let us briefly outline the construction of an orthonormal basis for \( \mathcal{H}_{p,q} \) (see again [Itô52].
For integers \( p, q \geq 0 \), the complex Hermite polynomials \( H_{p,q} \) are given by

\[
H_{p,q}(z) = (-1)^{p+q} e^{\frac{|z|^2}{2}} \frac{\partial^p}{\partial z^p} \frac{\partial^q}{\partial \bar{z}^q} e^{-|z|^2}
\]

where summation ends at the smaller of the two parameters \( p \) and \( q \).

Now let \( \{e_j: j \geq 1\} \) be an orthonormal basis of the underlying complex Hilbert space \( H \) and \( \{Z(h): h \in H\} \) denote the complex isonormal Gaussian process. Furthermore, for \( n \in \mathbb{N}_0 \), denote by \( M_n \) the set of all multi-indices of order \( n \) (sequences with a finite number of positive non-zero entries which sum up to \( n \)) and, for \( (m_p,m_q) \in M_p \times M_q \), define

\[
\Phi_{m_p,m_q} = \prod_{j=1}^{\infty} H_{m_p(j),m_q(j)}(Z(e_j)).
\]

Then, the family \( \{ \Phi_{m_p,m_q}: (m_p,m_q) \in M_p \times M_q \} \) is an orthonormal basis of \( H_{p,q} \). In particular, as \( H_{p,0} = z^p \), we see that for any multi-index \( m \in M_p \), the monomial \( \prod_{j=1}^{\infty} Z(e_j)^{m(j)} \) is an element of \( H_{p,0} \) and remains an eigenfunction when taking powers. In other words, for these eigenfunctions the Wick product coincides with the ordinary product. In the real case, there exist no non-constant eigenfunctions with this property.

### 4. Fourth Moment Theorems for Complex Gaussian Approximation

Throughout the whole section, \( L \) denotes a complex symmetric Markov diffusion generator with invariant measure \( \mu \) and discrete spectrum \( S = \{-\lambda_k\} \), acting on \( L^2(E,\mathbb{C},\mu) \), where \( E \) is a good measurable space. The associated carré du champ operator, acting on \( \mathcal{A}_0 \) which we assume to be dense in \( L^p(E,\mathbb{C},\mu) \) for all \( p \geq 1 \), is denoted by \( \Gamma \).

We start by introducing the notion of chaos, which for the real case was given in [ACP14] and extended to the multidimensional case in [CNPP15]. These definitions can be generalized as follows to the complex setting.

**Definition 4.1.**

(i) Two eigenfunctions \( F_1 \in \ker (L+\lambda_{p_1} \text{Id}) \) and \( F_2 \in \ker (L+\lambda_{p_2} \text{Id}) \) are called **jointly chaotic**, if \( F_1 F_2, F_1 \overline{F_2} \) (and thus, as the eigenspaces are closed under conjugation, also \( \overline{F_1 F_2} \) and \( \overline{F_1} \overline{F_2} \)) have an expansion over the first \( p_1 + p_2 + 1 \) eigenspaces. In formulas, we require that

\[
F_1 F_2 \in \bigoplus_{k=0}^{p_1+p_2} \ker (L+\lambda_k \text{Id}) \quad \text{and} \quad F_1 \overline{F_2} \in \bigoplus_{k=0}^{p_1+p_2} \ker (L+\lambda_k \text{Id}).
\]

(ii) A single eigenfunction is called chaotic, if it is jointly chaotic with itself.

(iii) A vector \( F = (F_1, \ldots, F_d) \) of eigenfunctions \( F_j \in \ker (L+\lambda_{p_j} \text{Id}) \) is called **chaotic**, if any two of its components are jointly chaotic (in particular, each component is chaotic in the sense of part (iii)).
Note that indeed, by taking all involved eigenfunctions to be real valued, we obtain the corresponding notions of real Markov chaos (namely [ACP14, Def. 2.2] and [CNPP15, Def. 3.2]) as special cases of Definition 4.1. As in the real case, a crucial ingredient for our main results will be the following general principle. The proof for the real case (see [ACP14, Thm. 2.1]) can be straightforwardly generalized to the complex case and is therefore omitted.

**Theorem 4.2.** Let \( F \in \bigoplus_{k=0}^{p} \ker (L + \lambda_k \text{Id}) \). Then, for any \( \eta \geq \lambda_p \) it holds that

\[
\int_E F(L + \eta \text{Id})^2 F \, d\mu \leq \eta \int_E F(L + \eta \text{Id}) F \, d\mu \leq c \int_E F(L + \lambda_k \text{Id})^2 F \, d\mu,
\]

where \( 1/c \) is the minimum of the set \( \{ \eta - \lambda_k \mid 0 \leq k \leq p \} \setminus \{0\} \).

Again, we note that by specializing to real valued eigenfunctions, we obtain [ACP14, Thm. 2.1] as a special case. From Theorem 4.2, we immediately deduce the following corollary.

**Corollary 4.3.** For two jointly chaotic eigenfunctions \( F_1 \in \ker (L + \lambda_{p_1} \text{Id}) \) and \( F_2 \in \ker (L + \lambda_{p_2} \text{Id}) \) it holds that

\[
(19) \quad \int_E |\Gamma(F_1, F_2)|^2 \, d\mu \leq \frac{p_1 + p_2}{2} \int_E \overline{F_1} \Gamma(F_1, F_2) \, d\mu.
\]

**Proof.** By definition, \( 2\Gamma(F_1, F_2) = (L + (p_1 + p_2) \text{Id}) (F_1 \overline{F_2}) \) and thus, using the fact that \( L \) and the identity are both self-adjoint and then Theorem 4.2, it follows that

\[
\int_E |\Gamma(F_1, F_2)|^2 \, d\mu = \frac{1}{4} \int_E \overline{F_1} F_2 (L + (p_1 + p_2) \text{Id})^2 (F_1 \overline{F_2}) \, d\mu \\
\leq \frac{p_1 + p_2}{4} \int_E \overline{F_1} F_2 (L + (p_1 + p_2) \text{Id}) (F_1 \overline{F_2}) \, d\mu \\
= \frac{p_1 + p_2}{2} \int_E \overline{F_1} F_2 \Gamma(F_1, F_2) \, d\mu.
\]

\[\square\]

Before continuing, we need to introduce some notation. For \( d \geq 1 \), let \( F = (F_1, \ldots, F_d) \) and \( G = (G_1, \ldots, G_d) \) be two complex random vectors. The Wasserstein distance \( d_W(F, G) \) between \( F \) and \( G \) is then defined as

\[
(20) \quad d_W(F, G) = \sup_{h \in \mathcal{H}} |E[h(F)] - E[h(G)]|,
\]

where \( \mathcal{H} = \{ h: \mathbb{C}^d \to \mathbb{C} \mid \|h\|_{\text{Lip}} \leq 1 \} \) and \( \|h\|_{\text{Lip}} \) denotes the Lipschitz norm, defined as

\[
\|h\|_{\text{Lip}} = \sup_{w,z \in \mathbb{C}^d} \frac{|h(w) - h(z)|}{\|w - z\|_2} = \sup_{w,z \in \mathbb{C}^d} \frac{|h(w) - h(z)|}{\sqrt{\sum_{j=1}^{d} |w_j - z_j|^2}}.
\]

Furthermore, we will use the shorthand \( \Gamma(F, L^{-1} G) \) to denote the matrix \( (\Gamma(F_j, L^{-1} G_k))_{1 \leq j,k \leq d}^d \).

The following Theorem provides a quantitative bound on the Wasserstein distance between a complex random vector and a multivariate complex Gaussian.
**Theorem 4.4.** For $d \geq 1$, let $Z \sim C\mathcal{N}_d(0, \Sigma)$ and denote by $F = (F_1, \ldots, F_d)$ a complex random vector whose components are elements of $\mathcal{A}_0$. Then it holds that

$$d_W(F, Z) \leq 2 \|\Sigma^{-1}\|_{op} \|\nabla\|_{op}^{1/2} \left( \int_E \|\Gamma(F, -L^{-1} F)\|_{HS}^2 d\mu \right)^{1/4}.$$  

**Remark 4.5.** In the case of the real Ornstein-Uhlenbeck generator, a similar bound was obtained in [NPR10, Theorem 3.3] through the use of Malliavin calculus. Formulated in the language of Markov diffusion generators, this bound reads

$$d_W(U, W) \leq \|C^{-1}\|_{op} \|C\|_{op}^{1/2} \left( \int_E \|\Gamma(U, -L^{-1} U) - C\|_{HS}^2 d\mu \right)^{1/4},$$

where $U$ is a real-valued, smooth random vector and $W \sim \mathcal{N}_d(0, C)$ a $d$-dimensional centered real Gaussian vector. Compared to the bound (21), we see that in the complex setting a second $\Gamma$-term appears.

**Proof of Theorem 4.4.** Let $h \in C^2(\mathbb{C}^d)$ with bounded first and second derivatives and denote by $U_h$ the solution (7) to the Stein equation of Lemma 2.4. By the integration by parts formula and the chain rule for $\Gamma$, it holds that

$$\int_E \langle \nabla U_h(F), \mathcal{T}\rangle_{\mathbb{C}^d} d\mu = \sum_{k=1}^d \int_E \langle \partial z_k U_h(F) \rangle F_k d\mu$$

$$= \sum_{k=1}^d \int_E \langle \partial z_k U_h(F) \rangle L L^{-1} F_k d\mu$$

$$= \sum_{k=1}^d \int_E \Gamma(\partial z_k U_h(F), -L^{-1} F_k) d\mu$$

$$= \sum_{j,k=1}^d \int_E \left( \partial z_j z_k U_h(F) \Gamma(F_j, -L^{-1} F_k) + \partial z_j z_k U_h(F) \Gamma(F_k, -L^{-1} F_j) \right) d\mu$$

$$= \int_E \langle \nabla \nabla U_h(F), \Gamma(F, -L^{-1} F) \rangle_{HS} d\mu$$

$$+ \int_E \langle \nabla f(F) - \nabla f(F) \rangle_{HS} d\mu.$$
Plugging these two identities into the complex Stein equation yields
\[
\int_{E} h(F) \, d\mu - E[h(Z)] = \int_{E} \left< \nabla \nabla U_{h}(F), \Gamma(F, -L^{-1}F) \right>_{\text{HS}} d\mu \\
+ \int_{E} \left< \nabla \nabla U_{h}(F), \Gamma(F, -L^{-1}\bar{F}) \right>_{\text{HS}} d\mu \\
+ \int_{E} \left< \nabla \nabla U_{h}(F), \Gamma(F, -L^{-1}F) - \Sigma \right>_{\text{HS}} d\mu \\
+ \int_{E} \left< \nabla \nabla U_{h}(F), \Gamma(\bar{F}, -L^{-1}\bar{F} - \Sigma) \right>_{\text{HS}} d\mu
\]
so that
\[
\left| \int_{E} h(F) \, d\mu - E[h(Z)] \right| = \sqrt{|I_1 + I_2 + I_3 + I_4|^2} \\
\leq 2 \sqrt{|I_1|^2 + |I_2|^2 + |I_3|^2 + |I_4|^2}.
\]
(22)

Using Lemma 2.5, we obtain
\[
|I_1|^2 + |I_3|^2 \leq \int_{E} \left( \left| \left< \nabla \nabla U_{h}(F), \Gamma(F, -L^{-1}F) \right>_{\text{HS}} \right|^2 \\
+ \left| \left< \nabla \nabla U_{h}(F), \Gamma(F, -L^{-1}\bar{F}) \right>_{\text{HS}} \right|^2 \right) d\mu \\
\leq \|\Sigma^{-1}\|_{\text{op}}^2 \|\Sigma\|_{\text{op}} \|h\|_{\text{Lip}}^2 \int_{E} \|\Gamma(F, -L^{-1}F)\|_{\text{HS}}^2 d\mu
\]
and similarly
\[
|I_2|^2 + |I_4|^2 \leq \|\Sigma^{-1}\|_{\text{op}}^2 \|\Sigma\|_{\text{op}} \|h\|_{\text{Lip}}^2 \int_{E} \|\Gamma(F, -L^{-1}F) - \Sigma\|_{\text{HS}}^2 d\mu.
\]
Plugged back into (22), this gives
\[
\left| \int_{E} h(F) \, d\mu - E[h(Z)] \right| \\
\leq 2 \|\Sigma^{-1}\|_{\text{op}}^2 \|\Sigma\|_{\text{op}} \|h\|_{\text{Lip}} \left( \int_{E} \|\Gamma(F, -L^{-1}F)\|_{\text{HS}}^2 d\mu \\
+ \int_{E} \|\Gamma(F, -L^{-1}F) - \Sigma\|_{\text{HS}}^2 d\mu \right)^{1/2},
\]
where \( h \in C^2(\mathbb{C}^d, \mathbb{C}) \) with bounded first and second derivatives. The proof is finished by noting that any Lipschitz function \( g \) can be uniformly approximated by functions of this type (take for example \( g_\varepsilon(z) = E[g(z + \varepsilon Z)] \), where \( Z \sim C \mathcal{N}_d(0, \text{Id}_d) \); see [NPR10, proof of Lemma 3.3]).

\[\Box\]

In the framework of real Markov diffusion generators, one can obtain bounds for the stronger Kolmogorov and total variation distances when specializing to dimension one. As a complex random variable corresponds to a two-dimensional real random vector, this strengthening is of course no longer possible using this approach.

For chaotic complex random vectors, the integrals appearing in the bound (21) of Theorem 4.4 can be expressed purely in terms of moments as follows.
Theorem 4.6. For \( d \geq 1 \), let \( Z \sim CN_d(0, \Sigma) \), where \( \Sigma = (\sigma_{j,k})_{1 \leq j,k \leq d} \), and \( F = (F_1, \ldots, F_d) \) be a chaotic complex random vector. Then it holds that

\[
\tag{23}
d_W(F, Z) \leq \left\| \Sigma^{-1} \right\|_{\text{op}} \left\| \Sigma \right\|_{\text{op}}^{1/2} \sqrt{\Psi_1(F) + \Psi_2(F) + \Psi_3(F)},
\]

where

\[
\Psi_1(F) = \sum_{j,k=1}^{d} \left| \int_E F_j \overline{F}_k \, d\mu - \sigma_{j,k} \right|^2,
\]

\[
\Psi_2(F) = \sum_{j,k=1}^{d} \sqrt{\int_E |F_j|^4 \, d\mu} \left( \frac{1}{2} \int_E |F_k|^4 \, d\mu - \left( \int_E |F_k|^2 \, d\mu \right)^2 \right),
\]

and

\[
\Psi_3(F) = \sum_{j,k=1}^{d} \int_E |F_j F_k|^2 \, d\mu - \int_E |F_j|^2 \, d\mu \int_E |F_k|^2 \, d\mu - \left| \int_E F_j \overline{F}_k \, d\mu \right|^2.
\]

Proof. In view of Theorem 4.4, we have to show that

\[
\int_E \left( \left\| \Gamma(F, -L^{-1} F) \right\|_{\text{HS}}^2 + \left\| \Gamma(F, -L^{-1} F) - \Sigma \right\|_{\text{HS}}^2 \right) \, d\mu \leq \Psi_1(F) + \Psi_2(F) + \Psi_3(F).
\]

When expanding the two Hilbert-Schmidt norms, the integral on the left hand side becomes

\[
\tag{24}
\sum_{j,k=1}^{d} \int_E \left( \left| \Gamma(F_j, -L^{-1} F_k) \right|^2 + \left| \Gamma(F_j, -L^{-1} F_k) - \sigma_{j,k} \right|^2 \right) \, d\mu
\]

Now note that by integration by parts and Corollary 4.3,

\[
\int_E \left| \Gamma(F_j, -L^{-1} F_k) - \sigma_{j,k} \right|^2 \, d\mu = \int_E \left| \Gamma(F_j, -L^{-1} F_k) \right|^2 \, d\mu + |\sigma_{j,k}|^2
\]

\[
- 2 \text{Re} \left( \int_E \Gamma(F_j, -L^{-1} F_k) \, d\mu \sigma_{j,k} \right)
\]

\[
= \int_E \left| \Gamma(F_j, -L^{-1} F_k) \right|^2 \, d\mu + |\sigma_{j,k}|^2
\]

\[
- 2 \text{Re} \left( \int_E F_j \overline{F}_k \, d\mu \sigma_{j,k} \right)
\]

\[
= \int_E \left| \Gamma(F_j, -L^{-1} F_k) \right|^2 \, d\mu - \left| \int_E F_j \overline{F}_k \, d\mu \right|^2
\]

\[
+ \left| \int_E F_j \overline{F}_k \, d\mu - \sigma_{j,k} \right|^2
\]

\[
\tag{25}
\leq \int_E \overline{F}_j F_k \Gamma(F_j, -L^{-1} F_k) \, d\mu - \left| \int_E F_j \overline{F}_k \, d\mu \right|^2
\]

\[
+ \left| \int_E F_j \overline{F}_k \, d\mu - \sigma_{j,k} \right|^2.
\]
On the other hand, by Corollary 4.3, the chain rule and integration by parts,
\[
\int_E |\Gamma(F_j, -L^{-1} F_k)|^2 \, d\mu \leq \int_E F_j F_k \Gamma(F_j, -L^{-1} F_k) \, d\mu
\]
\[
= \int_E \Gamma(F_j F_k, -L^{-1} F_k) \, d\mu
\]
\[
- \int_E |F_j|^2 \Gamma(F_k, -L^{-1} F_k) \, d\mu
\]
\[
- \int_E \overline{F_j} F_k \Gamma(F_j, -L^{-1} F_k) \, d\mu
\]
(26)

Plugging (25) and (26) into (24) yields that
\[
\sum_{j,k=1}^d \int_E \left( |\Gamma(F_j, -L^{-1} F_k)|^2 + |\Gamma(F_j, -L^{-1} F_k) - \sigma_{jk}|^2 \right) \, d\mu
\]
\[
\leq \Psi_1 - \sum_{j,k=1}^d \int_E |F_j|^2 \left( \Gamma(F_k, -L^{-1} F_k) - \int_E |F_k|^2 \right) \, d\mu + \Psi_3.
\]

To see that the sum in the middle is bounded by \(\Psi_3(F)\), we apply Cauchy-Schwarz to each summand and then make use of the complex Gamma calculus once more to transform the remaining \(\Gamma\)-expression into a moment:
\[
\int_E \Gamma(F_k, -L^{-1} F_k)^2 \, d\mu \leq \int_E F_k \overline{F_k} \Gamma(F_k, -L^{-1} F_k) \, d\mu
\]
\[
= \frac{1}{2} \int_E \Gamma(F_k^2, -L^{-1} F_k) \, d\mu
\]
\[
- \int_E F_k^2 \Gamma(F_k, -L^{-1} F_k) \, d\mu
\]
\[
\leq \frac{1}{2} \int_E \Gamma(F_k^2, -L^{-1} F_k) \, d\mu
\]
\[
= \frac{1}{2} \int_E |F_k|^4 \, d\mu,
\]
where the last inequality follows from Corollary 4.3, which implies that \(\int_E F_k^2 \Gamma(F_k, -L^{-1} F_k) \, d\mu \geq 0\).

For eigenfunctions of the real Ornstein-Uhlenbeck generator, a bound of a similar type has been proven in [NN11, Theorem 1.5] using Malliavin calculus, with the notable difference that only non-mixed fourth moments appear. The same strategy could be followed to prove a refined version of the bound (23) for eigenfunctions of the complex Ornstein-Uhlenbeck generator (i.e. complex multiple Wiener-Itô integrals; see Example 3.3), exclusively involving the second moments \(\int_E F_j F_k \, d\mu\) and the non-mixed fourth moments \(\int_E |F_j|^4 \, d\mu\).

Applying the Gaussian integration by parts formula, one sees for \(Z \sim \mathcal{CN}_d(0, \Sigma)\) and \(j = 1, 2, 3\) that indeed \(\Psi_j(Z) = 0\). Therefore, we have the following corollary.
**Corollary 4.7.** For \( d \geq 1 \), let \( Z \sim CN_d(0, \Sigma) \) and \( F_n = (F_{1,n}, \ldots, F_{d,n}) \) be a sequence of centered chaotic complex random vectors. Then, \( (F_n) \) converges in distribution towards \( Z \), if, and only if,

\[
\int_E F_{j,n} \overline{F}_{k,n} \, d\mu \to \mathbb{E}[Z_j \overline{Z}_k]
\]

and

\[
\int_E |F_{j,n} F_{k,n}|^2 \, d\mu \to \mathbb{E}[|Z_j Z_k|^2]
\]

for \( 1 \leq j, k \leq d \).

**Remark 4.8.**

(i) For \( d = 1 \) and \( \Sigma = \sigma^2 > 0 \), Corollary 4.7 says that a sequence of centered chaotic eigenfunctions converges in distribution towards a one-dimensional centered complex Gaussian random variable with variance \( \sigma^2 \), if, and only if, its second and fourth absolute moments converge towards \( \sigma^2 \) and \( 2\sigma^4 \), respectively. This is the complex counterpart of the abstract Fourth Moment Theorem for Gaussian approximation ([ACP14, Corollary 3.3]). If, in addition, we take \( L \) to be the complex Ornstein-Uhlenbeck generator (see Example 3.3), we obtain [CL14, Theorem 1.1.1].

(ii) For \( d \geq 2 \), Corollary 4.7 is the complex counterpart of [CNPP15, Theorem 1.2].

If \( L \) is the real Ornstein-Uhlenbeck generator, the Peccati-Tudor Theorem ([PT05]) says that a centered sequence \( (F_n) \) of vectors of eigenfunctions of \( L \) (i.e. multiple integrals) converges jointly in distribution towards a centered Gaussian random vector with covariance \( \Sigma \), if, and only if, \( \text{Var}(F_n) \to 0 \) and each component sequence converges separately towards a (one-dimensional) Gaussian. This result has been generalized in [CNPP15, Proposition 3.5] to the abstract diffusion generator framework. A straightforward adaptation of the latter finding yields the following complex Peccati-Tudor Theorem:

**Proposition 4.9.** For \( d \geq 2 \), let \( Z \sim CN_d(0, \Sigma) \), where \( \Sigma \) is positive definite and Hermitian, and let \( F_n = (F_{1,n}, \ldots, F_{d,n}) \) be a centered chaotic vector whose covariance converges towards \( \Sigma \) as \( n \to \infty \). Furthermore, assume that

1. The underlying generator \( L \) is ergodic, in the sense that its kernel only consists of constants
2. If, for \( 1 \leq j < k \leq d \) and \( j \neq k \), the pair \( (F_{j,n}, F_{k,n})_n \) has a subsequence \( (F_{j,n_l}, F_{k,n_l})_l \) such that \( F_{j,n_l} \) and \( F_{k,n_l} \) are elements of the same eigenspace with eigenvalue \( \lambda_l \) for all \( l \), it holds that

\[
\int_E \pi_{2\lambda_l}(F_{j,n_l})^2 \pi_{2\lambda_l}(\overline{F}_{j,n_l})^2 \, d\mu - 2 \left( \int_E F_{j,n_l} \overline{F}_{j,n_l} \, d\mu \right)^2 \to 0,
\]

where \( \pi_\lambda \) denotes the orthogonal projection onto \( \ker (L + \lambda\text{Id}) \).

Then, the following two assertions are equivalent.

(i) \( F_n \overset{d}{\to} Z \).

(ii) For \( 1 \leq j \leq d \) it holds that \( F_{j,n} \overset{d}{\to} Z_j \).
In [ACP14], Fourth Moment Theorems for the Gamma and Beta distribution were derived. We would like to mention that our techniques could be readily applied to extend these results and cover target random variables whose real and imaginary parts are independent real Gamma or Beta random variables (in the Ornstein-Uhlenbeck case, this has been done for the Gamma case in [CL14] by treating real and imaginary part separately).

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E-mail address: campese@mat.uniroma2.it

Università degli Studi di Roma Tor Vergata, Dipartimento di Matematica, Via della Ricerca Scientifica 1, 00133 Roma, Italy