Complex systems: physics beyond physics

Yurij Holovatch\textsuperscript{1,2}, Ralph Kenna\textsuperscript{2,3}, Stefan Thurner\textsuperscript{4,5,6,7}

\textsuperscript{1} Institute for Condensed Matter Physics, National Academy of Sciences of Ukraine, UA–79011 Lviv, Ukraine
\textsuperscript{2} Doctoral College for the Statistical Physics of Complex Systems, Leipzig-Lorraine-Lviv-Coventry (\textsuperscript{4}L\textsuperscript{3}), Europe
\textsuperscript{3} Applied Mathematics Research Centre, Coventry University, Coventry CV1 5FB, United Kingdom
\textsuperscript{4} Section for Science of Complex Systems, Medical University of Vienna, A–1090 Vienna, Austria
\textsuperscript{5} Santa Fe Institute, Santa Fe, NM 87501, USA
\textsuperscript{6} IIASA, Schlossplatz 1, A–2361 Laxenburg, Austria
\textsuperscript{7} Complexity Science Hub, Josefstädterstraße 39, A–1080 Vienna, Austria

Abstract. Complex systems are characterized by specific time-dependent interactions among their many constituents. As a consequence they often manifest rich, non-trivial and unexpected behavior. Examples arise both in the physical and non-physical world. The study of complex systems forms a new interdisciplinary research area that cuts across physics, biology, ecology, economics, sociology, and the humanities. In this paper we review the essence of complex systems from a physicist’s point of view, and try to clarify what makes them conceptually different from systems that are traditionally studied in physics. Our goal is to demonstrate how the dynamics of such systems may be conceptualized in quantitative and predictive terms by extending notions from statistical physics and how they can often be captured in a framework of co-evolving multiplex network structures. We mention three areas of complex-systems science that are currently studied extensively, the science of cities, dynamics of societies, and the representation of texts as evolutionary objects. We discuss why these areas form complex systems in the above sense. We argue that there exists plenty of new land for physicists to explore and that methodical and conceptual progress is needed most.

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E-mail: hol@icmp.lviv.ua, r.kenna@coventry.ac.uk, stefan.thurner@meduniwien.ac.at
1. Introduction

As a natural science, physics is based on the language of mathematics, the interplay between theory and empirics, and the ultimate primacy of experiment. Its foundational purpose is the predictive understanding and modelling of natural phenomena. Its prime driver is curiosity. The curiosity of physicists is taking the discipline to a wider set of fields, tackling problems traditionally seen to belong to the remits of other sciences and academic domains. New types of problem areas are being explored. One of these, the nature of complex systems, is the subject of the following thoughts.

The notion of a complex system is gradually emerging to be one of the inherent concepts of modern science. On a wider scale, it also appears in sociological and cultural contexts. The continual expansion of the dominion where the notion of complexity is used, as well as the discovery or recognition of wider ranges of phenomena where it is applicable, leads to difficulties in establishing a strict definition. Sometimes the term is applied to any system consisting of many interconnected parts which, as a whole, possesses properties that are not trivial aggregates of the properties of its separate constituents. A reference point often cited in the context of collective effects of complex systems is the paper *More is different*, written by Philipp Anderson, a Nobel Laureate in physics. The science of complex systems addresses the ways in which the constituent parts give rise to the collective behaviour of a whole system. However, such an interpretation has limited usefulness for a physicist, as it encompasses too broad a set of circumstances.

Historically, another more useful step in defining complex systems appeared in physics some time ago: a system is complex if its behaviour crucially depends on its details. In this context, one means phenomena such as deterministic chaos, quantum entanglement, protein folding, spin glasses, etc. Collective complex behaviour can arise under the influence of frustration and structural disorder. As a consequence, an equilibrium state is hard to reach and responses to external perturbations are slow and very often random. Such very different phenomena are studied in different fields of physics such as dynamical systems, quantum mechanics and statistical physics. Their common feature is that infinitesimal changes in initial conditions (albeit of very different natures) lead to radically different scenarios in the time evolution of these systems.

There is a second component that is essential for the definition of complex systems. On the one hand, interactions between constituent parts lead to collective behaviour and define the macrostate but on the other hand, the interactions are modified in the course of the system’s evolution and are influenced by the macrostate. In other words, the macrostate and microstates dynamically update each other. Analysis of such effects has lead to the creation of methods and development of concepts that

\[\text{‡} \quad \text{Given the rapidly evolving nature of the subject of this paper, references to the literature are intended to connect to some important or interesting sources, rather than to deliver an exhaustive or complete list.}\]
were successfully applied to the description of formally similar phenomena occurring in chemical, biological, social and other systems which are formed by agents of non-physical origin [6,7].

Finally, the notion of complex systems needs the possibility that the interactions between constituents are time varying, and many different interaction types can be present at the same time. Interactions between elements can be very specific. These interactions change the states of the constituent elements. The essence of many complex systems is that the states of constituents and interactions co-evolve over time, meaning that interactions change states of constituents, and the states of the constituents change the networks of interactions between constituents [8]. If physics is the science of the four fundamental forces which hold matter together, the science of complex systems is its generalization to forces and matter of a broader concept. Forces can be anything that change states of constituents, matter is anything where a force can apply [9]. Examples include in condensed matter, ecology, biology, stock markets, economics, sociology and humanities.

Here we mainly focus on complex systems comprising constituents or agents which arise beyond physics, and even beyond the physical. We intentionally omit examples from within traditional physics because these are adequately covered elsewhere [10,11]. Our principal goal is to demonstrate how the behaviours of systems of agents which originate beyond physics, or even those which have non-physical origin, may be described in terms of physics concepts (e.g., universality, scaling, phase transitions, entropy, random walks, percolation, diffusion, etc.) and how they can be quantitatively analysed using tools associated with modern statistical physics. Our paper is primarily addressed to those sceptical physicists who consider that dealing with systems of agents of non-physical origin can not be called physics. We believe that there is a physics beyond the study of planets, atoms, molecules and energy. We promote the opinion that it is not the object of study that matters, but rather the system of notions and methods that are used for its quantitative and predictive (possibly probabilistic) description [12].

The paper is structured as follows. In section 2 we discuss the notion of predictability and how its understanding has changed in the course of evolution and revolutions in physics. In section 3 we formulate what is a complex system and introduce some observables used for its description. In section 4 we discuss a dynamical-network formalism that currently serves as a universal language in complex-systems science. Finally, section 5 presents examples of complex systems outside physics: we discuss evolution of societies, cities, and culture itself. Conclusions and an outlook are given in section 6.

2. Predictability in physics

The mode of thinking that has lead to the emergence of a science of complex systems is closely connected to the way the notions of determinism and predictability evolved in physics. The concept of predictability evolved from strict determinism of trajectories to
a probabilistic description in statistical mechanics and quantum mechanics.

In the early nineteenth century by “predictability” of a many-particle system one meant that the coordinates of all particles and their states (momenta) should in principle be calculable as functions of time, no matter how large the number of particles is and provided that the initial conditions and equations of motion were known. This understanding is illustrated in the famous statement by Pierre Simon Laplace in *Essai philosophique sur les probabilités* (see Ref. [13], p.4 for an English translation):

“...We ought then to regard the present state of the universe as the effect of its anterior state and as the cause of the one which is to follow. Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situation of the beings who compose it an intelligence sufficiently vast to submit these data to analysis, it would embrace in the same formula the movements of the greatest bodies of the universe and those of the lightest atom; for it, nothing would be uncertain and the future, as the past, would be present to its eyes....”

Two major changes in understanding predictability of a physical system lead to (and were caused by) the appearance of statistical mechanics and quantum mechanics. In the framework of the former, microscopic states of many-particle systems are described by probabilities. There, predictions associated with a certain observable (including the state of a particle) no longer mean strict determinism of trajectories, but the determination of quantities such as mean values, variances, distribution functions, etc. The latter, quantum mechanics, does not even allow exact measurements of more than one state anymore. The notion of “predictions”, even for a single-particle state, have now become of a probabilistic nature: either they involve mean values of observables for a single particle from a system (ensemble) of interacting classical particles or they deal with mean values of observables for a unique quantum particle measured during a sequence of experiments.

Yet a new change in the notion of predictability (of classical systems) was caused by the understanding of deterministic chaos, where it became clear, that for certain classes of dynamical systems long-time predictions are impossible in practice: small changes in the initial conditions would lead to enormous changes in the outcome. Note that equations of motion of such systems do not contain stochastic terms: they are fully deterministic, however they are non-linear. For example in weather forecasting, running the same computer code on different machines can deliver strongly different outputs due to the accumulation of differing rounding errors during computations [14]. These effects are caused by the non-linearity of equations of motion and can be present even for a single classical object.

In complex systems we will keep the notion of probabilistic predictions. What is different to classical statistical mechanics and quantum mechanics is that the underlying systems not only might have non-linear interactions, but also interactions are time-varying. This may lead to much richer phase diagrams and macroscopic behaviours.
3. What is a complex system for a physicist?

3.1. Complex vs complicated

Most complicated systems in physics are not complex. Certainly particle physics is more or less complicated, however it shows no signs of complexity. In the standard theory of elementary particles interactions between particles always happen in the same way. However, it is true that most complex systems are complicated [15].

Let us consider a collection of classical nodes or entities (e.g., spins, agents) connected by links (interactions). Each node can be in one of a number of possible states described by a scalar, vector, matrix, or other mathematical object. For example, the state of the node can be the value of the local magnetic moment of a spin or the opinion of an agent (god exists or does not exist). The overall system is also characterized by a state, in a spin system that would be the overall magnetization, in the social network it could be the majority opinion. A single configuration of all nodes is a microstate, the state of the whole system is the macrostate. We first consider a ‘simple’ (not complex) system; e.g., a random network or a regular lattice. Denoting the state of a node by $\sigma_i$, one may describe its evolution by a system of equations,

$$\frac{d\sigma_i}{dt} = F(\{\sigma_i(t)\}, M), \quad i = 1, \ldots, N,$$

(1)

where $N$ is the number of agents, $M$ is the interaction matrix with elements $M^\alpha_{ij}$ that span all pairs of agents and the index $\alpha$ accounts for possible different types of interactions. The function $F$ can be stochastic or deterministic. In the latter case, solutions of Eq. (1) are uniquely determined by the interaction matrix and the initial conditions, $\{\sigma_1(0), \sigma_2(0), \ldots, \sigma_N(0)\}$. In this sense, the latter may be considered as a ‘boundary condition’ for the system that determines the differential equations (1). Of course, the equations might be (and usually are) complicated but, at least in principle, they are analytically tractable. Henceforth, we may call such systems ‘complicated’ but not complex. Needless to say, with an increase in number of agents $N$, methods of statistical physics will come into play, as briefly outlined in Section 2.

For a complex (as opposed to a complicated) system, the states of the nodes again determine the state of the system but they also influence the interactions between the nodes. In other words, the microstates (nodes) and the nature of the interactions $M$ dynamically update each other. The evolution of such a system is therefore governed by a system of equations,

$$\frac{d\sigma_i}{dt} = F(\{\sigma_i(t)\}, M(t)), \quad i = 1, \ldots, N,$$

(2)

$$\frac{dM^\alpha_{ij}}{dt} = G(\{\sigma_i(t)\}, M(t)), \quad i < j = 1, \ldots, N.$$  

(3)

The principal difference between the systems of differential equations (1) and (2) is that in the former case the matrix $M$ does not change over time (the ‘boundary conditions’ of
the system defined as a set of equations are fixed) whereas in the latter case evolution of $M$ is governed by Eq. (3). Rephrasing the famous chicken-or-egg causality dilemma, one can ask: do the interactions define the states (as in the above example of a ‘complicated’ system) or, vice versa, do the states define the interactions? It is the second equation (3) that makes this system analytically intractable: given the system of differential equations with the boundary conditions that are not fixed, one is not able to get an analytic solution. The dynamical update of the two sets of equations in Eq. (2) and Eq. (3) is similar to the dynamics of how an algorithm works. It changes its internal states (here that would be the interactions) as it evolves. These processes no longer follow analytic dynamics (solutions to equations of motion) but algorithmic dynamics. Physics in its traditional scheme (from Newton to spin glasses) is a subset of these equations, namely Eqs. (2) alone. In this sense it becomes very intuitive in what sense complex systems generalize traditional physics. In some cases the analysis of complex systems can be (and is) treated by reducing Eqs. (2) and (3) to a system like Eq. (1). This can be achieved for example when the dynamical processes described by Eq. (2) and by Eq. (3) occur on different time scales.

For example, consider a social system, wherein agents (nodes) are individuals that interact by means of specific social interactions (links), like communication, trading, liking, etc. In real life the ‘states’ $\sigma_i$ in which an individuals $i$ can be, could include its wealth, education level, social influence, religion, etc. For the purpose of describing human behaviour during elections, it is often enough to consider them being in one of two possible states: ‘god exists’ and ‘does not exist’ \[16\]. Clearly, it is conceivable that my interactions ($M$) with others will determine or at least influence who I am going to vote for $\sigma_i$. At the same time, who I am (typically) voting for will have an influence on my future social environment ($M$). Similarly, when describing the spreading of infections in the framework of a SIR model, one considers individuals being in one of three possible states: susceptible to infection, infected, or recovered with immunity \[17\]. Depending on the states the interactions between individuals are different: if infected meets recovered the interaction is: pass infection, whereas if two infected meet, nothing happens, no interaction (and no state-change). It is not the number of possible states, nor those of interactions that makes social systems complex, but the co-evolution of states and interactions.

An example of co-evolution that happens at very distinct time scales is the transportation system of a city. Consider bus and metro stations as nodes $i$ and represent the public transportation network as a set of links between them ($M_{ij}$) \[18,19\]. The transportation network is changing over time as a function of preferences of individuals using the system, and new bus lines, stations, bridges are opened, etc. However, these infrastructural changes evolve much slower than the typical urban mobility processes.

Spin glasses are frequently cited as examples of physical complex systems \[2,10,11\]. Also protein folding has become an increasingly popular theme in statistical physics \[4\]. For a spin glass, the state of a single node is described by a (classical) spin variable, $\sigma_i$ which can be $\pm 1$ in the Ising case. The interactions $M_{ij}$ account for two main
ingredients that lead to the spin-glass state: frustration and randomness. Within the Edwards-Anderson model $M_{ij}$ are taken to be random variables with the distribution [20],

$$P(M_{ij}) = \sqrt{\frac{1}{2\pi M^2}} e^{-M_{ij}^2/2M^2},$$

which has zero mean and variance $M^2$. In this representation frustrations arise through competing ferro- and antiferromagnetic interactions (note that $M_{ij}$ can have different signs). This leads to typical features of the spin-glass state: the presence of many relevant but non-equivalent macrostates, slow reaction to external perturbations, etc. Although these features are inherent to complex systems, such a description misses an essential attribute: the dynamical co-evolutionary update between microstates and interactions (cf. Eqs. (2), (3)) is substituted by a randomness in the interactions $M_{ij}$. Again, the presence of different time scales allows one to treat certain types of disordered magnets reducing Eqs. (2) and (3) to a system of equations of the form (1). In particular, this concerns cases when magnetic degrees of freedom in a structurally-disordered magnet relax much faster than its structure.[8]

An example from biology that contains all attributes of the complex system behaviour is gene expression. There, the fact that a certain gene is expressed (meaning that the gene is involved in the synthesis of a functional gene product) depends on whether some other genes are expressed. As soon as gene products are present a series of specific non-linear interactions may take place that change the expression levels of yet other genes. In other words, the state of a gene (its expression) is dynamically updated with the interactions that regulate the states of other genes [21, 22].

A truly complex physical “system” in the above sense is general relativity. There, the already mentioned chicken-and-egg dilemma is obviously present: the dynamics of the agents (their trajectories in space) unfolds as given by Eq. (2) for a fixed space $M$, however as a result of the dynamics, the space $M$ changes itself according to Eq. (3). We will consider three examples of this “algorithmic” type in detail in the following section, however outside traditional realm of physics. Before doing so we introduce the notions of power law statistics and measures of complexity.

3.2. Power laws

Complex systems are almost never characterized by Gaussian statistics. This is due to the fact that the central limit theorem often no longer holds for systems that are strongly interacting, path-dependent and non-ergodic, as those that are governed by our Eqs. (2)-(3) typically are. Instead, the statistics of complex systems are dominated by fat-tailed distributions and very often these are power laws. Depending on context and exponents, these are known under various names, such as student t-distributions,

§ This last statement concerns only so-called quenched magnets [23]. In annealed magnets both magnetic and non-magnetic degrees of freedom relax on the same time scale [24].
Zipf, Pareto, Mandelbrot, Cauchy, Lorenz, or Tsallis distributions, just to mention a few.

Often the distribution functions that are available are not probability distribution functions of measured variables (such as the velocity distribution of gas particles) but are frequency distributions. A famous example is the analysis of word frequencies \( f \) in texts. When all words of the text are ordered by descending frequency one finds \[ f(r) = Ar^\kappa. \] where \( A \) is a normalization constant and the power exponent \( \kappa \) was for a long time considered to be the same and universal for all languages, \( \kappa \simeq -1 \) and independent of factors such as the author, genre, time when the text was written, etc. This is no-longer believed, see [26] and the references therein.

Power laws that govern statistics of systems of many interacting agents were discovered in different disciplines at different times. Very often these laws hold names of their discoverers: in economics it is the distribution of wealth among individuals (V. Pareto, 1896); in demography – distribution of towns according to their size (F. Auerbach, 1913); in biology – distribution of sizes of biological genera according to the number of species they contain (J.C. Willis, G. Yule, 1922); in scientometrics – distribution of papers written by separate scientists (A.J. Lotka, 1926), of scientific journals according to the number of papers they contain (S.C. Bradford, 1934), citations (D. de S. Price, 1965). The list is terribly incomplete and goes on.

While in most cases where Gaussian or Boltzmann distributions occur, the Central Limit Theorem is at work in one way or the other, the origin of power law distributions is not so simple. The wide range of phenomena that exhibit fat tailed distributions suggests that the reasons for their appearance have to be quite general and should not depend on the individual peculiarities of their constituting parts or interactions. On the other hand, the fact that different phenomena are described by the same power law does not mean that the origin of these laws is the same. The origin of power laws can be explained by several very different mechanisms. In particular, the main routes to power laws are

- critical phenomena [28, 30],
- preferential processes [27, 31, 32],
- self-organized criticality [33],
- multiplicative processes with constraints [34, 35],
- optimisation [36],
- path dependent, non-ergodic processes that reduce their sample space as they unfold [37].

A typical example of the appearance of power laws in condensed matter physics is given by critical phenomena. As the critical point is approached, universal power

|| George Zipf was not the first who noticed the power-law decay in the function (5). Earlier observations were made by J. B. Estoup (in 1916) and by E. U. Condon (in 1928) [24, 35, 38].
laws govern divergencies in several thermodynamic observables (like compressibility of fluids or susceptibility of magnets) as well as structural (large distance asymptotics of the pair correlation function) and statistical (correlated cluster size distribution) properties \cite{28,30}. There, an increase of fluctuations in the fine-tuned vicinity of the critical point provides a mechanism for the emergence of power laws.

Preferential processes that are sometimes referred to as the “the rich get richer” phenomenon were suggested by Yule and later by Herbert Simon \cite{27}. If outcomes of processes occur proportional to the number of times they have occurred in the past, power laws in the occurrence frequencies appear. This mechanism has gained recent popularity in the form of the preferential attachment mechanism in the formation of scale-free networks \cite{32}. With preferential attachment, in the course of system evolution, new elements tend to create links with those, who already have more links.

Another class is self-organized criticality, where power laws appear due to non-linear interactions between system constituents \cite{33}. The system adjusts automatically to the critical point and no fine tuning is necessary. The classical example is the slope that self-organizes in a sandpile when sand is dropped grain by grain on a surface.

Multiplicative processes (products of sequences of random numbers) lead to lognormal distributions, that sometimes are hard to distinguish from true power laws in data. This is a trivial result of the Central Limit Theorem when applied to logarithmic variables. If multiplicative processes are subjected to simple constraints they can very easily show true power laws, as reviewed for example in \cite{34}.

An optimization scenario that leads to power laws was suggested by Benoit Mandelbrot, which is based on information theoretical arguments. In this scenario, power laws appear as a result of the optimization of the costs associated with the transmission of information.

Recently it was understood that all history-dependent processes that become more constrained as they unfold (sample-space reducing processes) lead to power laws. It constitutes an independent class of processes that lead to the power law statistics \cite{26,37}.

All the above mechanisms are present in our framework of Eqs. (2) and (3) as limiting cases. It remains a challenge to see which of these processes is dominating for a given complex system at hand. Very often for a specific complex system it might be a mixture of these processes that lead to power laws.

### 3.3. Measures of complexity

As the notion of complexity entered the domain of the mathematical sciences, there were attempts to define quantitative “measures of complexity”. The current abundance of such measures is caused partly by the fact that they were introduced in different fields and that they quantify different aspects of complex systems. Many of these measures are similar and show strong overlaps, so that one can try to group these measures into different taxonomies. Currently, there exist different classifications of complexity measures, depending on which features are chosen to be essential for a given group. One
classification is suggested by Lloyd [39] where measures are grouped according to the
questions they are supposed to answer, namely

- (i) how hard is it to describe?
- (ii) how hard its is to create?
- (iii) what is its degree of organization?

The degree of difficulty in describing a complex system completely (i) is usually
quantified with measures such as information, entropy and algorithmic complexity that
is sometimes called Kolmogorov complexity [40, 41]. Measures to answer how hard it
is to create a complex system (ii) include computational complexity, logical [42] and
thermodynamic [43] depth, and cost. The answer to the degree of organization in a
system (iii) is quantified in measures like effective complexity [15], fractal dimension,
and stochastic complexity [44]. Other measures of complexity that are sometimes used
are classified as: (i) non-computable vs. computable and (ii) deterministic vs statistical;
see e.g. [45] for more details.

In our opinion several new measures are needed that capture the degree of
coevolutionariness. If it is low, traditional physics can be used. If there exists separation
of characteristic time scales at which the dynamics in Eqs. (2), (3) unfolds, i.e. time
scales that can be clearly separated, co-evolutionariness is low. If it becomes hard to
disentangle dynamics of $\sigma$ and $M$ in Eqs. (2), (3) co-evolutionariness is high. This is
the challenge and promises new land for physicists. In terms of conceptual difficulty it
is not more complicated that general relativity.

4. Dynamical multilayer networks

In mathematics a network is a graph comprising a set of vertices and a set of edges.
In physics one often uses the terms nodes and links instead [46–48]. In the science
of complex systems networks play a central role because they offer a way to describe
different types of interactions specifically between agents (not everyone interacts with
everyone else). Interactions that change over time, stochastic interactions, interactions
that occur on multiple levels and that are not embedded in Euclidean space can all be
described within a network formalism.

Depending on the type of interaction, a network can take the form of an undirected
or directed graph. In the former case, interactions are bi-directional, as is the case for
physical interactions, or for example include scientific-collaboration networks, where two
scientists are linked if they co-author a paper. Directed networks, on the other hand,
arise when relationships between nodes are not symmetric or mutual [49]. For example,
food webs that describe which agents (species) eat other agents (other species) are
directed, people eat chicken, but chicken don’t eat humans. The strength of interactions
can be indicated by the weight of a link in a graph. One considers unweighted and
weighted graphs. For example, in a transportation network, connectivity simply marks
the presence of a link (e.g., a road) and can be represented by an unweighted network.
Traffic capacities or loads may vary from road to road and links may be characterized by different weights [50].

To describe the situation where many different types of interactions are simultaneously present between constituents or nodes, one may use the notion of multilayer networks [51–55]. Sometimes a distinction is made between multilayer networks and multiplex networks or ‘networks of networks’. In so-called multiplex networks, the same set of nodes is connected by links of different types. In ‘network of networks’ nodes of different networks are connected by inter-links. For example, a connected set of energy-supply sources (e.g., electric power stations) and another connected set of computers that control them, together form an example of a network of networks. In this case, inter-links between the two sets determine not only the transmission of control signals but also the power supply [56, 57]. The relevance of the topology of the resulting network of networks becomes clear if one analyzes the reactions of the entire system to the random removal of some of its nodes. One finds that undesired collective cascading dynamics can lead to propagation of a failure and dysfunction of the network as a whole [56]. The cascading is due to feedback between both interacting networks; the communication network of computers controls the network of power stations, which in turn control energy supply to the computers. This is an example of a percolation phenomenon on interconnected networks with the uncommon features of an abrupt first order phase transition [61, 62].

Prior to the recent emergence of network science, a mathematical theory had been developed for the simplest of networked objects, so-called Erdős-Rényi, or random graphs [60]. These are characterised by a degree distribution which is Poisson, for large graphs. The degree \( k \) of a node is the number of links attached to it and the distribution of degrees across nodes is one of the most fundamental characteristics of the network structure. Poissonian degree distributions are not observed in many real-world networks. Instead typically fat tailed degree distributions are observed, sometimes they are real power laws,

\[
P(k) \sim k^{-\lambda} \quad k \gg 1,\]

where \( P(k) \) is the probability that a randomly picked node in the network has exactly degree \( k \). Typically \( k \sim 1 - 3 \). Networks that are characterized by a power-law decay of the node degree distribution are called scale-free; sometimes they are also called “complex networks”. This has of course nothing to do with the complexity (or simplicity) of the underlying system.

Scale-free networks are often characterised by very short average distances between randomly chosen pairs of nodes, a circumstance sometimes referred to as a small world. Networks with scale-free structures that underly complex systems may have a strong impact on the dynamics of the system and may effect percolation properties or self-organization. Note that complex networks by themselves are not necessarily “complex systems”.

\[\text{¶} \quad \text{Usually percolation phenomena have a continuous phase transition [59].}\]
In physics we frequently consider systems that are homogeneous and isotropic. If interactions take place on networks the resulting “physics” plays out in a very different environment. The fat-tail behaviour of degree distributions of interactions (1) reflects strong inhomogeneities. Nodes with a high degree (hubs) can play very different roles in a network than those with a low degree. Another important observable that is used to capture the inhomogeneities in interaction networks is the so-called clustering coefficient $c_i$ of node $i$, which is is defined as the proportion of neighbours of $i$ that are mutually linked. A power law decrease of the clustering $c_i$ as a function of the degree $k_i$, is a signal for a hierarchical organization in the network [63]. The degree of clustering has severe dynamical consequences for complex systems that are built on such networks.

Another useful quantity is the assortativity of the network. This is the extent to which similar nodes are mutually linked. It is measured for example by using Pearson’s correlation coefficient. The degree assortativity $r_k$ of a network measures the correlations between the degrees of those nodes which are connected by a link [64]. If $r_k > 0$ the network is called assortative and if $r_k < 0$ it is said to be disassortative. Assortativity measures the extent to which similar vertices associate with each other and, again, has impact on how interactions in a complex system manifest themselves and evolve over time.

5. Examples of physics outside physics and the physical

5.1. Societies as complex systems – a new generation of computational sociology

Societies are obvious complex systems in the sense of Eqs. (2) and (3). They are composed of many individuals and institutions $i$ that can be seen as nodes in a multiplex network. Nodes are described by state vectors that can represent wealth, education, social roles etc. The set of all states of a node can be collected into a state vector $\vec{\sigma}_i$. These state vectors are not static but change over time, depending on the interactions the nodes are involved in. In societies, these interactions $M_{ij}^\alpha$ that always happen between individuals (or institutions) $i$ and $j$, can be of very different types, $\alpha$. For example, interaction type $\alpha = 1$ could represent communications, $\alpha = 2$ trade, $\alpha = 3$ friendship, $\alpha = 4$ family relation, $\alpha = 5$ collaboration, etc. Interactions are not static but evolve over time and often explicitly depend on the states of the nodes. Often interactions are discrete and happen in interaction events.

The complexity now arises through the coevolution of interactions and state changes. If one considers the act of trading for example, typically a trade interaction is preceded by a sequence of communication events (deciding on a price for a particular good). Once the interactions of the trade happens the states of the involved parties change, the state of wealth in terms of cash of the buyer diminishes, while the state of owning a particular quantity of the traded good increases for the buyer. Trading interactions change states of cash positions, communication events change the state of informedness of communicators, hostile interactions may result in changes of the state
of health, etc. Note that interactions do not happen all the time and are often very specific, and depend on the states of the involved nodes. No trading event will happen if the state of wealth of a potential buyer is below the minimum price expected of the seller.

Interactions do not happen independently of each other; they mutually influence each other. The networks of one interaction type $\alpha$ influences the link formation and destruction in an other layer $\beta$. In this sense there exists a network-network interaction, a completely unexplored field that could open new and wide territory for physicists.

This notion of complexity in the dynamics of societies is of course useless in a scientific sense without the ability to measure and track state changes and interaction events. With the availability of new generation of data this seems indeed to be possible. For example, since several decades every trade in financial markets is recorded. Data on every single paid medical intervention between patients and doctors within a country is becoming available [65], and practically every social interaction and state change amongst human beings can be recorded in the framework of massive multiplayer online games. In such games where players, in the form of avatars, live an alternative life in a virtual universe. Life there is open ended and extremely high dimensional. Avatars pursue economic activities (such as being employed or being entrepreneurs), they buy goods and services from other players, and they form social relationships on various levels such as friendships, clubs, gangs, parties, kingdoms, countries, financial systems, etc. All interactions between all players and all their state changes in wealth, skills, education, leadership role, etc. can be monitored and recorded completely. In the case of the game Pardus.at these data have been used to study a society of about 500,000 individuals, as a complex systems in the form of Eqs. (2) and (3) [53]. The amazing situation arises that every component of these equations is recorded at all times. The state vectors $\mathbf{\sigma}_i(t)$ as well as all the interactions $M_{ij}^{\alpha}(t)$ are available, where $\alpha$ stands for the different interaction types such as friendships, trades, enmity relations, communications, hostilities, revenge, etc. In the particular case of the Pardus.at game, time $t$ ranges over more than ten years, with a time resolution of a second.

These kind of massive data brings the science of complex systems to an entirely new quality level. By looking at how Eqs. (2) and (3) unfold over time it becomes possible to develop a quantitative feeling of how societies work. In particular, the co-evolution aspects of interactions and state changes can for the first time be visualized and in a next step be understood. It is needless to say that this offers an opportunity to understand human behaviour and social dynamics on a level of precision that was so far only reached in the natural sciences. In the particular case of the Pardus.at game first steps were taken in this direction in [53, 54, 66–74]. This situation tempts one to state that for the first time it becomes possible to turn sociology into a quantitative and predictive science, a statement, that most sociologists would – of course – object to. For physicists, however, there seems to be new land ahead that needs conceptual as well as methodological progress.
5.2. Cities as complex systems

Another archetypal example of a complex system arises when one attempts to apply quantitative analysis to understand inherent features of a city as a whole, its organization, development, impact and correlations between global processes that define city life. In fact, an analogy between behaviour of a living system and a city has far reaching consequences. One of the first arguments in favour of such analogy can be found in Aristotle ‘Politics’ [75].

In the context of what has been said above, interpretation of a city as a complex system involves considerations of nodes of various types (e.g. subway stations in the public transportation network, houses linked into the energy production and consumption network, gas storage facilities etc.), forming inter-connected and co-evolving networks (streets, information systems, traffic routes, distribution systems such as energy and waste infrastructures, and other infrastructure facilities). Much in cities has to do with distribution of people, goods and services, similar to the way in which nutrients are distributed through the body to reach every cell of an organism or, indeed, an ecosystem. The co-evolutionary aspect of city dynamics as described by Eqs. (2)–(3) is obvious. Success or failure of distribution events may change the states of the various nodes. In turn, changes of states in the nodes may lead to changes in the distribution networks. For example, chronic overloading of a subway station might eventually lead to the construction of a new subway line. On shorter timescales the information provided by a navigation device on traffic density in a part of a city might lead to changes of route planning for individual drivers. This may lead to congestion in other parts of the system: the traffic-flow network on a city’s street network is changing as a consequence of an information flow of current traffic. These examples, and many more, obviously follow a co-evolution dynamics as described in Eqs. (2)–(3).

If one considers the growth of a city as its expansion in geographical space, this poses a challenge to quantitative approaches. City growth has been shown to exhibit self-similar (or fractal) patterns, an observation that might imply a universality of processes that drive city agglomeration and clustering [76,78]. Several physical growth processes that are known to lead to such geometry (percolation or diffusion limited aggregation) have been exploited to explain such growth in cities [76,77,79]. Fat-tailed distributions that very often govern statistics of complex systems make ‘rare events’ to be essential for emerging system properties. In particular, inhomogeneities in structure lead to another typical feature of complex systems: their resilience to random failures and their vulnerability to targeted attacks. In the context of public transportation this issue was studied in various cities [19,80–82].

Obviously, urban growth is much more than expansion in space. Many observables that may be used to quantify various aspects of the evolution of a city are governed by power laws. Some of the observed exponents are listed in table 1. Power laws in various urban indicators as a function of city size (i.e. number of inhabitants) are ubiquitous [77,83,87], signalling that complex economic, demographic and social
processes take place in cities. An understanding of the origin of these power laws is only partially complete. Data about scaling of various city indicators are universal in the sense that similar exponents are observed for cities of very different historical, cultural and economical backgrounds. Another interesting observation that comes from table 1 is that the data seems to fall into two “universality classes” [83, 85]: (i) indicators that describe social interaction of some kind (upper part of the table) manifest super-linear scaling with exponents $\beta \sim 1.15 > 1$, and (ii) indicators that describe the infrastructure which typically scale sub-linearly, $\beta \sim 0.85 < 1$.

Returning to the analogy of cities and living organisms, it is worth noting that numerous physiological characteristics of different organisms scale with body mass as power laws with exponents that are multiples of 1/4. A theoretical explanation for this particular linear scaling laws has been provided [88, 89]. An essential ingredient of the theory is the evidence of hierarchical branching networks that terminate in size-invariant units. It is the structure of these networks that leads to the universal allometric scaling observed in nature. Similarly, scaling laws observed in various city indicators might turn out as manifestations of city network structures, and of universal patterns of human social interaction networks.

Big efforts are being currently undertaken to analyze networks and timeseries of city data. Examples include analyses of creative output of a city (measured e.g. in innovations found in a city [86]), of mobile phone records [87, 91] or taxi movements [92] as proxies for spatio-temporal distributions of people, etc. In fact, cities are data-producing entities generating data on the level of nodes or agents of different type and the level of links that are formed through their interactions. Co-evolutionary aspects of changes in states of agents and simultaneous changes in their interaction network structure (as described by Eqs. (2), (3)) are not yet systematically studied.

| $Y$                        | $\beta$ |
|---------------------------|---------|
| Number of new patents     | 1.27    |
| Number of inventors       | 1.25    |
| Private R&D employment    | 1.34    |
| “Supercreative” employment| 1.15    |
| Number of R&D establishments| 1.19  |
| R&D employment            | 1.26    |
| New AIDS cases            | 1.23    |
| Serious crimes            | 1.16    |
| Number of Gasoline stations| 0.77  |
| Gasoline sales            | 0.79    |
| Length of all electrical cables| 0.87 |
| Road surface of city      | 0.83    |

Table 1. Scaling exponents for several urban indicators $Y$ vs. city size $N$: $Y \sim N^\beta$, where $N$ is population size. See [85] for the data sources.
simple example are traffic guiding systems where traffic lights that govern the traffic, are themselves governed by local traffic loads [93].

The science of cities in terms of dynamical and co-evolving complex systems as we define them here has only begun. It is largely scientific new territory. The availability of increasingly complete datasets will make it possible to make this science “experimental”, in the sense that it can be formulated in quantitative, predictive and eventually testable ways.

5.3. Complex systems in humanities

History itself may be considered to comprise a complex system. One can consider the nodes of such a system to be individuals that existed, still exist or even those that might have existed. They can again be characterized by states which can, for example, be social functions in a society, such as being a king, a revolutionary, a writer of political texts, a demagogue, a simple voter, a public-opinion former (e.g., newspaper journalist) or similar. They are linked to each other through social ties like family relations, marriage, friendship, business and so on, or even through conflict or competition. Forms of interactions may then be captured in a multiplex network and layers include kinship, political coalitions, financial dependencies, taxation, information flow networks through letters, emails or even through fighting.

Networks were of importance to thought and culture long before the invention of modern telecommunications systems and examples include in ancient Icelandic and Scandinavian literature, through Uppsala romanticism, the Flemish movement, and French modernists, to more modern Norwegian poetics [94]. Efforts have been made by medievalists to map communication networks over time in this context [95]. The notion of place has been explored in literature both as geographic and narrative phenomena [96]. Another perspective is to represent historical records in the form of books or annals into co-evolving structures.

In many texts such as novels, chronicles, sagas, etc, one can identify individuals together with their social ties, dependencies and influence with each other. That it is possible to extract meaningful social networks from such texts was demonstrated, for example, in the context of ancient myths in Refs. [97, 98]. There it was shown that epic narratives from the world of mythology have certain universal characteristics. The exploration of such characteristics was inspired by the notion of universality from the study of critical phenomena in statistical physics. For example, the networks tend to be highly clustered, structurally balanced, hierarchical small worlds with right skewed degree distributions and coherent community structures. These are properties very similar to the social networks that we find ourselves in today.

Does that mean that such ancient stories are based on reality? We don’t know for sure, however we can make sensible investigations by considering such systems as physicists. For example, it was found that although the famous Anglo-Saxon epic Beowulf has many properties of real social networks, it lacks one crucial one:
assortativity. This is the tendency for similar nodes to connect to each other. In our society, popular people tend to know each other and the few friends of less popular individuals tend also to have few acquaintances. So what would it take to make *Beowulf* assortative? In other words, how far is the narrative away from appearing realistic?

To investigate, a defining characteristic of complex systems was exploited. As discussed above, for such systems, changes of states of individual nodes may lead to macroscopic changes in the entire network. In Refs. [97], the eponymous protagonist Beowulf was removed from the social network of the narrative *Beowulf*, and the properties of the remainder were determined. The resulting network was (marginally) assortative, meaning that it had all the properties of a real social network. Intriguingly, although the narrative is embellished with obvious fiction, archaeological excavations in Denmark and Sweden support the historicity of some of the characters in *Beowulf*. The lead character Beowulf is mostly not believed to have existed, however [113, 114]. Thus it is entirely possible that the character Beowulf is himself fictional but that the backdrop to the narrative – the society within which the story unfolds – is realistic. It is interesting, and encouraging, that the network analysis which is based on considering the society as a complex system, delivers the same conclusion as coming from traditional humanities considerations [113, 114].

With the availability of electronic libraries, it is possible to perform similar explorations to practically any text that involves people or institutions. Not only can these texts then be studied in terms of their co-evolutionary dynamics of ‘states’ of individuals and their interactions, but also individuals that were identified in several distinct texts can be brought into relation through a multiplex network. A classic example is the Icelandic sagas. These contain an abundance of characters, some of whom appear in more than one narrative. The main characters in one often appear as minor ones in another. In text α the local social network of a given character represented by node $i$ is given by the matrix elements $M_{ij}^α$. We can then combine societies to obtain an enormous network and to consider its properties. By seeking to compare communities in the combined network with individual sagas, one can speculate as to whether one was a source for another [98]. Similarly, the consistency of social networks across texts could, for example, be useful to speculate to what extent a given text was embellished; the amount of inconsistencies across layers would offer a new approach to historical research.

There is no reason why such modes of thinking could not be extended to every character that ever appeared in a text. This would offer an entirely new way in understanding the written history about humankind.

6. Conclusions and outlook

If physics is the science of matter and its interactions, the science of complex systems is its natural extension where both matter and interactions are seen in a much broader context. Interactions can be anything that leads to a change of a state in a constituent in a complex system, and matter is anything that can have at least two states and
is able to interact. In this sense complex systems are a natural extension of physics. The framework that complex systems are co-evolving multiplex networks [8], where interactions between elements change states of these elements and where the collection of states in a system changes the interaction networks, is similar in spirit as the framework of general relativity, where dynamics needs a space that is itself changed as a function of the dynamics. Needless to say that the dynamics of such systems will be impossible to be solved analytically and without use of massive computational power.

The availability of more and more data in disciplines and fields beyond physics allows us to observe the states (and their changes) of elements and their interactions (and their changes) in great detail. With the electronic fingerprints we leave practically everywhere, we are getting toward a situation where we have – more and more often – even complete information on dynamical complex systems. For many systems we already have such information in the sense that literally all state changes and all interaction events are recorded. This situation will make it possible in the next years to transform the science of complex systems, from presently a collection of computational and network based methods, into a fully experimental science.

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