A Note on the Relativistic Covariance of the B–Cyclic Relations

Valeri V. Dvoeglazov

Escuela de Física, Universidad Autónoma de Zacatecas

Antonio Dovalí Jaime s/n, Zacatecas 98068, ZAC., México

Internet address: VALERI@CANTERA.REDUAZ.MX

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Abstract

It is shown that the Evans-Vigier modified electrodynamics is compatible

with the Relativity Theory.

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Recently a new version of non-Maxwellian theories of electromagnetism has been proposed [1,2]. As a matter of fact, the Evans-Vigier $B^{(3)}$ theory includes a spin variable in the classical theory and presents itself straightforward development of the Belinfante, Ohanian and Kim ideas [3–5]. In the present note I restrict myself only one particular question of the relativistic covariance of this theory. I would not like to speak here about a numerous variety of other generalizations of the Maxwell’s theory referring a reader to the recent review [6]. All those theories are earlier given either strong critics (while not always perfectly reasonable) or ignorance and only in the nineties several new versions appeared at once, what ensures that the question would obtain serious, careful and justified consideration. The $B^{(3)}$ model is not an exception. A list of works criticizing this theory was presented in ref. [8] and that author wrote several critical comments too [7,8]. A serious objection to the Evans-Vigier theory which was presented by Comay is that he believes that the modified electrodynamics is not a relativistic covariant theory. Questions of Dr. Comay may arise in future analyses of the $B^{(3)}$ theory because he correctly indicated some notational misunderstandings in the Evans and Vigier works. Therefore, they are required detailed answers.

According to [3, Eq.(11.149)] the Lorentz transformation rules for electric and magnetic fields are the following:

\[
E' = \gamma(E + c\beta \times B) - \frac{\gamma^2}{\gamma + 1} \beta(\beta \cdot E), \tag{1a}
\]

\[
B' = \gamma(B - \beta \times E/c) - \frac{\gamma^2}{\gamma + 1} \beta(\beta \cdot B) \tag{1b}
\]

where $\beta = v/c$, $\beta = |\beta| = \tanh \phi$, $\gamma = \frac{1}{\sqrt{1-\beta^2}} = \cosh \phi$, with $\phi$ being the parameter of the
Lorentz boost. We shall further use the natural unit system \( c = \hbar = 1 \). After introducing the spin matrices \( (S_i)_{jk} = -i\epsilon_{ijk} \) and deriving relevant relations:

\[
(S \cdot \beta)_{jk} a_k \equiv i[\beta \times a]_j ,
\]

\[
\beta_j \beta_k \equiv [\beta^2 - (S \cdot \beta)^2]_{jk}
\]

one can rewrite Eqs. (1a,1b) to the form

\[
E'_i = \left( \gamma I + \frac{\gamma^2}{\gamma + 1} \right) [(S \cdot \beta)^2 - \beta^2]_{ij} E_j - i\gamma(S \cdot \beta)_{ij} B_j , \tag{2a}
\]

\[
B'_i = \left( \gamma I + \frac{\gamma^2}{\gamma + 1} \right) [(S \cdot \beta)^2 - \beta^2]_{ij} B_j + i\gamma(S \cdot \beta)_{ij} E_j . \tag{2b}
\]

First of all, one should mention that these equations are valid for electromagnetic fields of various polarization configurations. Next, Eqs. (2a,2b) preserve properties of the vectors \( B \) (axial) and \( E \) (polar) with respect to the space inversion operation. Furthermore, if we consider other field configurations like \( \phi_{L,R} = E \pm iB \) or \( B \mp iE \), the Helmoltz bivectors, which may already not have definite properties with respect to the space inversion operation (namely, they transform as \( \phi_R \leftrightarrow \pm \phi_L \)) we obtain

\[
(B' \pm iE')_i = \left( 1 \pm \gamma(S \cdot \beta) + \frac{\gamma^2}{\gamma + 1} (S \cdot \beta)^2 \right)_{ij} (B \pm iE)_j , \tag{3}
\]

i.e, it becomes obviously that they transform as the right- and the left- parts of the Weinberg’s \( 2(2S + 1) \)- component field function \([10]\).
of $B^{(1)} \times B^{(2)} = E^{(1)} \times E^{(2)} = iB^{(0)}B^{(3)*}$. In the first frame transversal modes of the electromagnetic field have the following explicit forms, ref. [11], $\phi = \omega t - k \cdot r$:

$$B^{(1)} = \frac{B^{(0)}}{\sqrt{2}} (i \mathbf{i} + j \mathbf{j} ) e^{i \phi}, \quad E^{(1)} = -i B^{(1)} = \frac{E^{(0)}}{\sqrt{2}} (i - j \mathbf{i} ) e^{i \phi}, \quad (4a)$$

$$B^{(2)} = \frac{B^{(0)}}{\sqrt{2}} (-i \mathbf{i} + j \mathbf{j} ) e^{-i \phi}, \quad E^{(2)} = +i B^{(2)} = \frac{E^{(0)}}{\sqrt{2}} (i + j \mathbf{i} ) e^{-i \phi}. \quad (4b)$$

We have implied that in a free-space circularly-polarized radiation $B^{(0)} = E^{(0)}$. The pure Lorentz transformations (without inversions) do not change the sign of the phase of the field functions, so we should consider separately properties of the set of $B^{(1)}$ and $E^{(1)}$, which can be regarded as the negative-energy solutions in QFT (cf. the Dirac case [12]), and another set of $B^{(2)}$ and $E^{(2)}$, as the positive-energy solutions. The opposite interpretation is also possible and, in fact, was used by E. Comay ($\phi \rightarrow -\phi$). But these issues are indicated not to be relevant to the present discussion. Thus, in this framework one can deduce from Eqs. (2a,2b)

$$B^{(1)}'_{i} = \left( 1 + \frac{\gamma^2}{\gamma + 1} (\mathbf{S} \cdot \mathbf{\beta})^2 \right)_{ij} B^{(1)}_{j} + i \gamma (\mathbf{S} \cdot \mathbf{\beta})_{ij} E^{(1)}_{j}, \quad (5a)$$

$$B^{(2)}'_{i} = \left( 1 + \frac{\gamma^2}{\gamma + 1} (\mathbf{S} \cdot \mathbf{\beta})^2 \right)_{ij} B^{(2)}_{j} + i \gamma (\mathbf{S} \cdot \mathbf{\beta})_{ij} E^{(2)}_{j}, \quad (5b)$$

$$E^{(1)}'_{i} = \left( 1 + \frac{\gamma^2}{\gamma + 1} (\mathbf{S} \cdot \mathbf{\beta})^2 \right)_{ij} E^{(1)}_{j} - i \gamma (\mathbf{S} \cdot \mathbf{\beta})_{ij} B^{(1)}_{j}, \quad (5c)$$

$$E^{(2)}'_{i} = \left( 1 + \frac{\gamma^2}{\gamma + 1} (\mathbf{S} \cdot \mathbf{\beta})^2 \right)_{ij} E^{(2)}_{j} - i \gamma (\mathbf{S} \cdot \mathbf{\beta})_{ij} B^{(2)}_{j}. \quad (5d)$$

Using relations between transversal modes of electric and magnetic field (4a,4b) one can formally write:

1One would wish to study properties of this physical system with respect to the space inversion
\[
B^{(1)}_i = \left(1 + \gamma (S \cdot \beta) + \frac{\gamma^2}{\gamma + 1} (S \cdot \beta)^2\right)_{ij} B_j^{(1)}, \quad (6a)
\]
\[
B^{(2)}_i = \left(1 - \gamma (S \cdot \beta) + \frac{\gamma^2}{\gamma + 1} (S \cdot \beta)^2\right)_{ij} B_j^{(2)}, \quad (6b)
\]
\[
E^{(1)}_i = \left(1 + \gamma (S \cdot \beta) + \frac{\gamma^2}{\gamma + 1} (S \cdot \beta)^2\right)_{ij} E_j^{(1)}, \quad (6c)
\]
\[
E^{(2)}_i = \left(1 - \gamma (S \cdot \beta) + \frac{\gamma^2}{\gamma + 1} (S \cdot \beta)^2\right)_{ij} E_j^{(2)}, \quad (6d)
\]

We still observe that \(B^{(2)}\) can be related with \(B^{(1)}\) by the unitary matrix:
\[
B^{(2)} = U B^{(1)} = e^{-2i\phi} \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} B^{(1)}. \quad (7)
\]

Since this unitary transformation results in the change of the basis of spin operators only, we deduce that the concepts of properties of some geometrical object with respect to Lorentz transformations and with respect to space-inversion transformations can be simultaneously well-defined concepts only after defining corresponding “bispinors” of the \((j, 0) \oplus (0, j)\) representations and keeping the same spin basis for both parts of the bispinor. See also [13] for the example in the \((1/2, 0) \oplus (0, 1/2)\) rep.

Operation. Since, explicit forms of transversal modes of electric field in the first frame are proportional (with imaginary coefficients) to the transversal modes of the magnetic field \((4a, 4b)\) some fraction of \(E^{(k)}\) or \(B^{(k)}\) can be formally substituted by the vector of other parity (like we are doing in the process of calculations). Furthermore, one can take any combinations of Eq. \((5a)\) and Eq. \((5c)\) multiplied by an arbitrary phase factor, or that of Eq. \((5b)\) and Eq. \((5d)\) multiplied by a phase factor. The parity properties of the field functions in the general case would be different in the left-hand side and in the right-hand side of resulting equations. Generally speaking, the notation \((6a-6d)\) is used in this paper only for simplification of calculations. In fact, one can also proceed further with the forms \((5a-5d)\).
We still advocate that a) the properties $E^{(k)'}$ to be proportional to $B^{(k)'}$ with imaginary coefficients are preserved and b) $B^{(1)}$ and $E^{(1)}$ in Eqs. (6a-6d) transform like $B + iE$ of the Cartesian basis, i.e., like the right part of the Weinberg field function and $B^{(2)}$ and $E^{(2)}$, like $B - iE$, i.e., like the left part of the Weinberg field function. Using the above rules to find the transformed 3-vector $B^{(3)'}$ is just an algebraic exercise. Here it is

$$B^{(1)'} \times B^{(2)'} = E^{(1)'} \times E^{(2)'} = i\gamma(B^{(0)})^2(1 - \beta \cdot \hat{k}) \left[ \hat{k} - \gamma\beta + \frac{\gamma^2(\beta \cdot \hat{k})\beta}{\gamma + 1} \right],$$

(8)

where $\hat{k}$ is the orth vector of the axis $OZ$. We know that the longitudinal mode in the Evans-Vigier theory is defined as $B^{(3)} = B^{(3)'} = B^{(0)}k$. Thus, considering that $B^{(0)}$ transforms as zero-component of the four-vector and $B^{(3)}$ as space components of the four-vector: [9, Eq.(11.19)]

$$B^{(0)'} = \gamma(B^{(0)} - \beta \cdot B^{(3)}),$$

(9a)

$$B^{(3)'} = B^{(3)} + \frac{\gamma - 1}{\beta^2(\beta \cdot B^{(3)})}\beta - \gamma\beta B^{(0)},$$

(9b)

we find from (8) that the relation between transversal and longitudinal modes preserves its form:

$$B^{(1)'} \times B^{(2)'} = iB^{(0)'}B^{(3)''}.$$

(10)

A reader interested in these matters can exercise to prove the covariance of other cyclic relations [11]. Next, when the boost is made in the $x$ direction we obtain

$$B^{(3)'} = ( -\gamma\beta B^{(0)}, 0, B^{(0)}) \quad \text{, in the coordinates of the old frame},$$

(11a)

$$B^{(3)'} = (0, 0, B^{(0)'}) \quad \text{, in the coordinates of the new frame}.$$
One can see that the transformations (9a, 9b) are the ones for a light-like 4-vector of the Minkowski space, formed by \((B^{(0)}, B^{(3)})\). They are similar (while not identical) to the transformation rules for the spin vector [3, Eq.(11.159)]. The difference with that consideration of a massive particle is caused by impossibility to find a rest system for the photon which is believed to move with the invariant velocity \(c\). Nevertheless, some relations between the concept of the Pauli-Lubanski vector of the antisymmetric tensor fields and the \(B^{(3)}\) concept have been found elsewhere [14,6,15].

I would like to indicate origins of why Dr. Comay achieved the opposite incorrect result:

1) Obviously, one is not allowed to identify \(B_z\) and \(B^{(3)}\) (as the authors of previous papers did, see, e.g., the formula (6) in ref. [7]), the first one is an entry of the antisymmetric tensor and the second one is a 3-vector quantity, the entry of the 4-vector. They are different geometric objects. Of course, the Poynting vector must be perpendicular to the \(E\) and \(B\), the Cartesian 3-vectors, whose components are entries of the antisymmetric tensor field. The \(B^{(3)}\) vector is a vector of different nature, while, in its turn, it forms an “isotopic” vector with \(B^{(1)}\) and \(B^{(2)}\) in a circular complex basis and while its physical effect is similar to that of the Cartesian \(B\), namely, magnetization. 2) One is not allowed to forget about the fact that \(B^{(0)}\) is not a scalar quantity, it is a zero component of the 4-vector; so Comay’s “appropriate

\[\text{2Let me remind that the Cartesian basis is a pure real basis. Introducing complex vectors we, in fact, enlarge the space; a number of independent components may increase and, the bases, in general, are not equivalent mathematically.}\]
“units” would transform too from the first to the second frame.\footnote{3} As we have found the axial 3-vector $\mathbf{B}^{(3)}$ is always aligned with the $OZ$ axis in all frames like the Poynting vector (polar) is, provided that the ordinary electric and magnetic fields lie in the $XY$ plane in these frames. If this is not the case one can do that by rotation, using the unitary matrix. So while the questions raised in the paper by Comay \cite{7} may be useful for deeper understanding of the Evans-Vigier theory and the Relativity Theory but the conclusion is *unreasonable*. Briefly referring to the paper \cite{8a} one can apparently note that, in my opinion, the $\mathbf{B}^{(3)}$ field is a property of one photon and when considering the many-photon problem with various types of polarizations in a superposition the question whether the circulation of this vector would be different from zero (?) must be regarded more carefully; furthermore, the applicability of the dynamical equations to this vector, which Comay refers to, is not obvious for me.

The conclusion follows in a straightforward manner: the $\mathbf{B}^{(3)}$ Evans-Vigier modified electrodynamics is a relativistic covariant theory if one regards it mathematically correctly. This construct may be the simplest and most natural classical representation of a particle spin. The $\mathbf{B} –$ cyclic relations manifest relations between Lorentz group generators answering for the angular momentum \cite{1,2,4,11}. Moreover, as realized by E. Comay himself “the modified electrodynamics relates its longitudinal magnetic field $\mathbf{B}^{(3)}$ to the expectation value of the quantum mechanical intrinsic angular momentum operator”. Therefore, recent critics by Profs. L. D. Barron, A. D. Buckingham, E. Comay, D. Grimes, A. Lakhtakia of the Evans-Vigier \cite{3} theory have been rather unjustified.

\footnote{3}{Surprisingly, he noted this fact himself in the fourth Section but ignored it in the Section 3.}
Vigier $B^{(3)}$ theory appear to signify that, in fact, they doubt existence of the helicity variable for a photon (additional discrete phase-free variable according to Wigner) and, hence, all development of physics since its (helicity) discovery. Undoubtedly, such a viewpoint could lead to deep contradictions with experimental results (the spin-spin interaction, the inverse Faraday effect, the optical Cotton-Mouton effect, the Tam and Happer experiment (1977), etc., etc.). Whether it has sufficient reasons? On an equal footing claims of “it is unknowable” and/or “is not fundamental” seem to me to be based on the unknowable logic. As opposed to them with introduction of this variable in a classical manner nobody wants to doubt all theoretical results of QED and other gauge models. As a matter of fact, existence of the spin variable and of different polarization states are accounted in calculations of

Surprisingly, the opposite claims (of the pure “longitudinal nature” of the massless antisymmetric tensor fields) by several authors are yet another unexplained statements. That was point out as long as 1939, by F. Belinfante in the comment to the paper by Durandin and Erschow [Phys. Z. Sowjet. 12 (1937) 466]: “Three directions of polarization are possible for a Proca quantum with given momentum and charge”. While the question for neutral particles (self/anti-self charge conjugate states) should be regarded properly in both the Majorana and the Dirac constructs, even in this case one can see from the first sight that those claims of the pure “longitudinal nature” contradict with a classical limit and with the Weinberg theorem $B - A = \lambda$, ref. [10b]. By the way, I do not understand reasons to call this field after the paper of M. Kalb and P. Ramond (1974). As a matter of fact, the antisymmetric tensor fields (and their “longitudity”) have earlier been investigated by many authors; first of all by E. Durandin and A. Erschow (1937), F. Belinfante (1939), V. Ogievetskiï and I. Polubarinov (1966), F. Chang and F. Gürsey (1969), Y. Takahashi and R. Palmer (1970), K. Hayashi (1973).
QED matrix elements. The proposed development of the Maxwell’s theory does not signify
the necessity of rejecting the results which have been obtained in regions where the old
models do work. Moreover, it was recently shown [3] that both transversal and longitudinal
classical modes of electromagnetism are naturally incorporated in the Weinberg quantum-
field formalism [10]. Thus, the aim of my work (and, I believe, Dr. Evans too) is to
systemize results on the basis of the Poincarè group symmetries, to simplify the theory, to
unify interactions and, perhaps, to predict yet unobserved phenomena. One should follow
the known advice of A. Einstein and W. Pauli to build a reliable theory on the basis of the
First Principles, namely, on the basis of relativistic covariance (irreducible representations
of the Poincaré group) and of causality. Some progress in this direction has already been
achieved while authors started from different viewpoints [6,16,17].

I understand that further discussions of the Evans-Vigier model will be desirable. First
of all, the questions arise, whether this theory implies a photon mass? namely, how does
this theory account these effects (mass appears to manifest itself here in somewhat different
form)? if so what is the massless limit of this theory? what are relations between the $E(2),
O(3)$ groups and the group of gauge transformations of the 4-potential electrodynamics [3]
and of other gauge models? can a massless field be particulate? and, finally, what is the
mass itself? It is also necessary to give relations of this construct with those presented by
L. Horwitz, M. Sachs, A. Staruszkiewicz, D. Ahluwalia and myself. This should be the aim
of the forthcoming papers.

I am grateful to Prof. M. Evans for many internet communications on the concept of

the $B^{(3)}$ field and estimate his efforts as considerable (while *not always* agree with him). I acknowledge the help of Prof. A. F. Pashkov, who informed me about the papers [3,4], and Prof. D. V. Ahluwalia for his kind comments.

*Note Added.* The main addition to the final version of the Comment by Prof. E. Comay is the claim that the object $(B^{(0)}, B^{(3)})$ does *not* form the (pseudo) 4-vector. I slightly touched the question of the parity properties in this Note. But, because after writing this my Comment I has become aware about several more papers of him, which are aimed at destruction of the Evans-Vigier modified electrodynamics, and, moreover, the questions of the properties of the spin-1 massive/massless fields with respect to the discrete symmetry operations deserve much attention, I think, it would be useful to discuss the matters of the parity covariance of the $B$ Cyclic Relations and other claims of Prof. E. Comay in detail in a separate paper. Here I want to indicate only that in the final version of his Comment (see this issue of FPL) Professor E. Comay has made conventions which are required the rigorous proof: namely, as a matter of fact he assumed that the complex-valued $B^{(1)}$ and $B^{(2)}$ are axial vectors in the sense that they transform under the space inversion operation as $B^{(1)} \rightarrow B^{(1)'}$, and $B^{(2)} \rightarrow B^{(2)'}$ (on the Comay’s opinion!). This wisdom (if take into account the Lorentz transformation rules of $B^{(1)}$ and $B^{(2)}$), in fact, puts shades on the rules $\phi_R \leftrightarrow \phi_L$ with respect to the space inversion operation [13] in the $2(2j + 1)-$ component theories.

Finally, I want apparently to note that this my paper is NOT a Reply to the Comay’s
Comment. In fact, it criticizes both the Evans’ work and the Comay’s work. So, it is the Comment and this fact was let to know to Prof. Comay before publication (with my permission and along with the draft of my Comment). I cannot figure out why does Prof. Comay insist that it is a Reply?
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