Limit formulas for the normalized fundamental matrix of the north-west-corner truncation of Markov chains: Matrix-infinite-product-form solutions of block-Hessenberg Markov chains

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Abstract

This paper considers the normalized fundamental matrix of the north-west-corner (NW-corner) truncation of an ergodic infinitesimal generator (i.e., the infinitesimal generator of an ergodic continuous-time Markov chain). We first present several limit formulas associated with the normalized fundamental matrix. One of the limit formulas shows that, as the order (size) of the NW-corner truncation of the ergodic generator diverges to infinity, the corresponding normalized fundamental matrix converges to a stochastic matrix whose rows are all equal to the stationary distribution vector of the ergodic generator, though some technical conditions are required. Using such results, we derive the matrix-infinite-product-form (MIP-form) solutions of the stationary distribution vectors of upper and lower block-Hessenberg Markov chains. From the MIP-form solutions, we also develop numerically stable and easily implementable algorithms that generate the sequences of probability vectors convergent to the corresponding stationary distribution vectors of block-Hessenberg Markov chains under appropriate conditions.

Keywords: Fundamental matrix; Northwest-corner truncation (NW-corner truncation); Block-Hessenberg Markov chain; Level-dependent M/G/1-type Markov chain; Level-dependent GI/M/1-type Markov chain; Level-dependent quasi-birth-and-death process (LD-QBD); Matrix-infinite-product-form solution (MIP-form solution)

Mathematics Subject Classification: 60J22; 60K25

*This research was supported in part by JSPS KAKENHI Grant Numbers JP15K00034.
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1 Introduction

Let \( \{Z(t); t \geq 0\} \) denote an ergodic (i.e., irreducible and positive-recurrent) continuous-time Markov chain with state space \( \mathbb{Z}_+ := \{0, 1, 2, \ldots\} \). Let \( Q := (q(i, j))_{i,j \in \mathbb{Z}_+} \) denote the infinitesimal generator (or \( Q \)-matrix; see, e.g., [1, Section 2.1]) of the ergodic Markov chain \( \{Z(t)\} \), i.e., for all \( i \in \mathbb{Z}_+ \),

\[
\sum_{j \in \mathbb{Z}_+} q(i, j) = 0; \quad q(i, i) \in (-\infty, 0); \quad q(i, j) \geq 0, \ j \in \mathbb{Z}_+ \setminus \{i\}.
\]

We refer to \( Q \) as the \textit{ergodic (infinitesimal) generator}. We then define \( \pi := (\pi(i))_{i \in \mathbb{Z}_+} \) as an unique and positive stationary distribution vector of the ergodic generator \( Q \) (see, e.g., [1, Chapter 5, Theorems 4.4 and 4.5]), i.e., \( \pi \) is a positive vector such that \( \pi Q = 0 \) and \( \pi e = 1 \), where \( e \) denotes a column vector of ones whose order depends on the context.

For \( n \in \mathbb{Z}_+ \), let \( (n)Q := ((n)q(i, j))_{i,j \in \{0,1,\ldots,n\}} \) denote the \((n + 1) \times (n + 1)\) northwest-corner (NW-corner) truncation of the ergodic generator \( Q \), i.e., \( (n)q(i, j) = q(i, j) \) for all \( i, j \in \{0, 1, \ldots, n\} =: \mathbb{Z}_n \). It should be noted that \( (n)Q \) can be considered the \textit{transient} generator of an absorbing Markov chain with transient states \( \mathbb{Z}_n \) and absorbing states \( \mathbb{Z}_n := \mathbb{Z}_+ \setminus \mathbb{Z}_n \) (see, e.g., [4, Chapter 8, Section 6.2]). The absorbing Markov chain characterized by \( (n)Q \) eventually reaches the absorbing states from any transient state with probability one due to the ergodicity of the original Markov chain \( \{Z(t); t \geq 0\} \). Therefore, \( (-Q)^{-1} \geq O, \neq O \) exists (see, e.g., [15, Theorem 2.4.3]), where \( O \) denotes the zero matrix. The nonnegative matrix \( (-Q)^{-1} \) is called the \textit{fundamental matrix} of the transient generator \( (n)Q \), which is a continuous-time counterpart of the fundamental matrix defined for the discrete-time absorbing Markov chains (see [4, Chapter 4, Section 6] and [11, Chapter 5]).

We now define \( (n)F, n \in \mathbb{Z}_+ \), as

\[
(n)F = \text{diag}^{-1}\{(-Q)^{-1}e\}(-Q)^{-1}, \tag{1.1}
\]

where \( \text{diag}\{\cdot\} \) denotes the diagonal matrix whose \( i \)-th diagonal element is equal to the \( i \)-th element of the vector in the braces. It follows from (1.1) that \( (n)F \geq O \) and \( (n)Fe = e \), i.e., \( F \) is row stochastic (stochastic, for short, hereafter). We refer to \( (n)F \) as the \textit{normalized fundamental matrix} of \( (n)Q \).

The main purpose of this paper is twofold. The first purpose is to present several limit formulas associated with the normalized fundamental matrix \( (n)F \). For example, (for a precise statement, see Corollary 2.2 below),

\[
(n)F = \begin{pmatrix}
(n)\pi\{0\} \\
(n)\pi\{1\} \\
\vdots \\
(n)\pi\{n\}
\end{pmatrix} \to e\pi \quad \text{as} \ n \to \infty, \tag{1.2}
\]

where \( (n)\pi\{\nu\} \) denotes the stationary distribution vector of the \( \nu \)-th-column-augmented \((n + 1) \times (n + 1)\) NW-corner truncation (\( \nu \)-th-column-augmented truncation, for short) \( (n)Q_{\{\nu\}} \) of the ergodic generator \( Q \). The limit formula (1.2) is derived from the following formula (see Theorem 2.2
where \((n)\alpha\) is a \((1 \times (n + 1))\) probability vector. The second purpose is to derive, from the limit formula (1.3) and related ones, the matrix-infinite-product-form (MIP-form) solutions of the stationary distribution vectors of upper and lower block-Hessenberg Markov chains (i.e., level-dependent M/G/1-type and GI/M/1-type Markov chains). The MIP-form solutions require some technical conditions because of their underlying limit formulas including (1.3).

It should be noted that if \((n)\alpha = (0, \ldots, 0, 1, 0, \ldots, 0)\) then \((n)\pi = (n)\pi_{\nu}\). The probability vector \((n)\pi\) is called the linearly augmented truncation approximation to the stationary distribution vector \(\pi\) of the original Markov chain. For discrete-time ergodic Markov chains, Wolf [34] discussed the convergence of several augmented truncation approximations including the linearly augmented one (see also [6]). Wolf [34]'s results are directly applicable to uniformizable continuous-time Markov chains (see, e.g., [32, Section 4.5.2]). As for the continuous-time Markov chain, there are some studies on the convergence of augmented truncation approximations. Hart and Tweedie [8] proved that \(\lim_{n \to \infty} (n)\pi_{\nu} = \pi\) under the condition that \(Q\) is exponentially ergodic. Hart and Tweedie [8] also assumed that \(Q\) is (stochastically) monotone, under which they proved the convergence of any augmented truncation approximation. Masuyama [22] presented computable and convergent error bounds for the last-column-block-augmented truncation approximation, under the condition that \(Q\) is block monotone and exponentially ergodic. Without block monotonicity, Masuyama [23] derived such convergent error bounds under the condition that \(Q\) satisfies the \(f\)-modulated drift condition.

We now remark that the idea of the normalized fundamental matrix is inspired by the studies of Shin [29] and Takine [31]. Shin [29] presented an algorithm for computing the fundamental matrix of the transient generator of finite level-dependent quasi-birth-and-death processes (LD-QBDs) with absorbing states, based on matrix analytic methods [7, 15, 25]. The matrix analytic methods are the foundation of many iterative algorithms [2, 3, 5, 12, 26, 30] for computing the stationary distribution vectors of upper and lower block-Hessenberg Markov chains (including LD-QBDs). Such iterative algorithms usually require us to solve the system of linear equations for the boundary probabilities. Recently, for a special upper block-Hessenberg Markov chain, Takine [31] proposed an algorithm for computing the conditional stationary distribution vector of the levels below a given one, which does not require solving the system of linear equations for the boundary probabilities. Takine’s algorithm [31] is based on a limit formula for a submatrix of \((- (n)Q)^{-1}\), which is related to (1.2) but different from it (for details, see Section 3.1).

In fact, we prove some limit formulas for the normalized linear combination of the truncated rows of \((- (n)Q)^{-1}\). One of the limit formulas does not require any technical condition except the ergodicity of generator \(Q\) if \(Q\) is in the upper block-Hessenberg form. Therefore, for the general upper block-Hessenberg Markov chain, we can establish an algorithm for computing the conditional stationary distribution vector of an arbitrary finite substate space. To save space, we do not present a detailed discussion of this topic. Instead, for the (unconditioned) stationary distribution vector, we develop numerically stable and easily implementable algorithms for the block-Hessenberg Markov chains that have the MIP-form solutions. Thus, our algorithms are not neces-
sarily applicable to any block-Hessenberg Markov chain. However, our algorithms do not require us to determine the maximum number of blocks (or levels) involved in computing. The existing algorithms in [2, 3, 5, 12, 26, 30] require such input parameters, though it is, in general, difficult to determine the parameters appropriately. Our algorithms are free from this problem, which is the advantage over the existing ones.

The rest of this paper is divided into two sections. Section 2 derives the limit formulas associated with the normalized fundamental matrix by relating it to linearly augmented truncation approximation. Section 3 presents the MIP-form solutions for block-Hessenberg Markov chains and the algorithms for computing the solutions.

2 Normalized fundamental matrix of the NW-corner truncation

This section discusses the normalized fundamental matrix \( (n)F \) though its relation to the linearly augmented truncation of the ergodic generator \( Q \). Thus, we perform calculations involving vectors and matrices that originally have different orders. For simplicity of notation, we extend, according to the context, a finite matrix (possibly vector) to an infinite one by appending an infinite number of zeros to the original matrix in such a way that the existing elements remain in their original positions. For example, when we write \((n)Q - (n+1)Q - Q\), we set \((n)q(i, j) = 0\) for \(i, j \in \{ n + 1, n + 2, \ldots \}\) and \((n+1)q(i, j) = 0\) for \(i, j \in \{ n + 2, n + 3, \ldots \}\). In addition, we use the following notation: Suppose that \( H_n := (h_n(i, j))_{i,j \in \mathbb{Z}_n}, n \in \mathbb{Z}_+ \), is a matrix such that \( \lim_{n \to \infty} h_n(i, j) = h(i, j) \) for \(i, j \in \mathbb{Z}_+\). We then represent this as \( \lim_{n \to \infty} H_n = H := (h(i, j))_{i,j \in \mathbb{Z}_+} \).

2.1 Relation between the normalized fundamental matrix and linearly augmented truncation approximation

For any \(n \in \mathbb{Z}_+\), let \( (n)\overline{Q} := ((n)q(i, j))_{i,j \in \mathbb{Z}_n} \) denote a \(Q\)-matrix such that

\[
(n)\overline{Q} = (n)Q - (n)Qe_{(n)\alpha},
\]

(2.1)

where \((n)\alpha := ((n)\alpha(i))_{i \in \mathbb{Z}_n}\) is an arbitrary probability vector such that \(\sum_{i=0}^{n} (n)\alpha(i) = 1\). Note here that \((n)Qe \leq 0\) for all \(n \in \mathbb{Z}_+\) and \(\lim_{n \to \infty} (n)Qe = 0\). It thus follows from (2.1) that

\[
\lim_{n \to \infty} (n)\overline{Q} = Q,
\]

(2.2)

and all the nondiagonal elements of \((n)\overline{Q}\) are nonnegative and \((n)\overline{Q}e = 0\), which implies that \((n)\overline{Q}\) is conservative (see, e.g., [1, Section 1.2]). We refer to the \(Q\)-matrix \( (n)\overline{Q} \) as the linear-augmented truncation of \(Q\). We also refer to the probability vector \((n)\overline{\alpha}\) as the augmentation distribution of \((n)\overline{Q}\).

We now fix the augmentation distribution \((n)\alpha = (n)e_{\{\nu\}}^{\top}\), where \((n)e_{\{\nu\}}, \nu \in \mathbb{Z}_n\), denotes the \((n + 1) \times 1\) unit vector whose \(\nu\)-th element is equal to one. We then denote the resulting matrix by \((n)Q_{\{\nu\}} := ((n)q_{\{\nu\}}(i, j))_{i,j \in \mathbb{Z}_n}, \) i.e.,

\[
(n)Q_{\{\nu\}} = (n)Q - ((n)Qe)_{(n)e_{\{\nu\}}^{\top}}.
\]
We call \((n)Q_{\{\nu\}}\) the \(\nu\)-th-column-augmented \((n+1) \times (n+1)\) NW-corner truncation (\(\nu\)-th-column-augmented truncation, for short) of \(Q\). By definition, the \(\nu\)-th-column-augmented truncation \((n)Q_{\{\nu\}}\) is a special case of the linear-augmented truncation \((n)\bar{Q}\).

**Theorem 2.1** For each \(n \in \mathbb{Z}_+\), the \(Q\)-matrix \((n)\bar{Q}\) has a single closed communicating class and therefore has a unique stationary distribution vector \((n)\bar{\pi} := \{(n)\bar{\pi}(i)\}_{i \in \mathbb{Z}_+}\), which is given by

\[
(n)\bar{\pi} = \frac{(n)\alpha (-(n)Q)^{-1}\alpha e}{(n)\alpha (-(n)Q)^{-1}e}, \quad n \in \mathbb{Z}_+.
\] (2.3)

Furthermore,

\[
(n)\bar{\pi} = (n)\beta (n)F, \quad n \in \mathbb{Z}_+,
\] (2.4)

where \((n)F\) is given in (1.1), and \((n)\beta\) is a probability vector such that

\[
(n)\beta = \frac{(n)\alpha \text{diag}\{(-(n)Q)^{-1}e\}}{(n)\alpha (-(n)Q)^{-1}e}.
\] (2.5)

**Proof.** We first prove that \((n)\bar{Q}\) has a single closed communicating class. To this end, we consider a Markov chain with state space \(Z_n\) characterized by \((n)\bar{Q}\). Since \((n)\bar{Q}\) is finite and conservative, the state space \(Z_n\) of the Markov chain is decomposed into some closed communicating classes and transient states. If each communicating class shares at least one state with \(A_n := \{i \in Z_n : (n)\alpha(i) > 0\}\), then \((n)\bar{Q}\) has at least state connecting to all closed communicating classes and thus has a single communicating class. We now suppose that there exists a closed communicating class \(E \subseteq Z_n\) such that \(E \cap A_n = \emptyset\). It then follows from (2.1) and \(\sum_{j \in A_n} (n)\alpha(j) = 1\) that, for all \(i \in E\),

\[
0 = \sum_{j \in Z_n \setminus E} (n)\bar{\pi}(i, j)
= \sum_{j \in A_n \cap (Z_n \setminus E)} (n)\bar{\pi}(i, j) + \sum_{j \in Z_n \setminus (E \cup A_n)} q(i, j)
= \sum_{j \in A_n} \left\{ q(i, j) + (n)\alpha(j) \sum_{\ell \in Z_n \setminus Z_n} q(i, \ell) \right\} + \sum_{j \in Z_n \setminus (E \cup A_n)} q(i, j)
= \sum_{j \in Z_n \setminus Z_n} q(i, j) + \sum_{j \in Z_n \setminus (E \cup A_n)} q(i, j)
= \sum_{j \in Z_+ \setminus E} q(i, j),
\]

which implies that the original Markov chain \(\{Z(t)\}\) with generator \(Q\) cannot move from \(E\) to \(Z_+ \setminus E\). This contradicts the ergodicity of \(Q\), which implies that each closed communicating class shares at least one state with \(A_n\). Consequently, \((n)\bar{Q}\) has a single closed communicating class and has the unique stationary distribution vector (see, e.g., [1, Chapter 5, Theorems 4.4 and 4.5]).

Next, we confirm that the remaining statements are true. Pre-multiplying both sides of (2.1) by \((n)\bar{\pi}\) in (2.3) and using \((n)\alpha e = 1\) results in

\[
(n)\bar{\pi} (n)\bar{Q} = \frac{-(n)\alpha + (n)\alpha e (n)\alpha}{(n)\alpha (-(n)Q)^{-1}e} = 0.
\]
which shows that \((n)\overline{\pi}\) is the stationary distribution vector of \((n)\overline{Q}\). Furthermore, from (1.1), we have

\[ (-n)Q^{-1} = \text{diag}\{(-n)Q^{-1}e\}(n)F. \]

Substituting this into (2.3) and using \((n)Fe = e\), we obtain

\[
(n)\overline{\pi} = \frac{(n)\alpha \text{diag}\{(-n)Q^{-1}e\}(n)F}{(n)\alpha \text{diag}\{(-n)Q^{-1}e\}(n)Fe} = \frac{(n)\alpha \text{diag}\{(-n)Q^{-1}e\}(n)F}{(n)\alpha \text{diag}\{(-n)Q^{-1}e\}e(n)F} = \frac{(n)\alpha (-n)Q^{-1}e(n)F}{(n)\beta(n)F},
\]

where the last equality follows from (2.5). The proof is completed.

\[ \square \]

**Corollary 2.1** For any \(n \in \mathbb{Z}_+\) and \(\nu \in \mathbb{Z}_n\), let \((n)\pi_{\{\nu\}} := ((n)\overline{\pi}_{\{\nu\}}(i))_{i \in \mathbb{Z}_n}\) denote the stationary distribution vector of the \(\nu\)-th-column-augmented truncation \((n)Q_{\{\nu\}}\). We then have

\[
(n)\pi_{\{\nu\}} = (n)e_{\{\nu\}}^T(n)F = \frac{(n)e_{\{\nu\}}^T(-n)Q^{-1}}{(n)e_{\{\nu\}}(-n)Q^{-1}e}, \quad n \in \mathbb{Z}_+, \nu \in \mathbb{Z}_n, \tag{2.6}
\]

or equivalently,

\[
(n)F = \begin{pmatrix} (n)\pi_{\{0\}} \\ (n)\pi_{\{1\}} \\ \vdots \\ (n)\pi_{\{n\}} \end{pmatrix}, \quad n \in \mathbb{Z}_+.
\]

**Proof.** Fix \((n)\alpha = (n)e_{\{\nu\}}^T\). We then have \((n)\overline{\pi} = (n)\pi_{\{\nu\}}\). From (2.5), we also have

\[
(n)\beta = \frac{(n)e_{\{\nu\}}^T \text{diag}\{(-n)Q^{-1}e\}}{(n)e_{\{\nu\}}(-n)Q^{-1}e} = (n)e_{\{\nu\}}. \tag{2.7}
\]

Substituting this and (1.1) into (2.4) yields

\[
(n)\overline{\pi} = (n)e_{\{\nu\}}^T(n)F = (n)e_{\{\nu\}}^T \text{diag}^{-1}\{(-n)Q^{-1}e\}(-n)Q^{-1}. \tag{2.7}
\]

Note here that

\[
(n)e_{\{\nu\}}^T \text{diag}^{-1}\{(-n)Q^{-1}e\} = \frac{(n)e_{\{\nu\}}}{(n)e_{\{\nu\}}(-n)Q^{-1}e}. \tag{2.8}
\]

Combining (2.7) and (2.8) results in (2.6). \[\square\]
2.2 Limit formulas for the normalized fundamental matrix

For later use, we introduce the notation. Let \( \mathbb{N} = \{1, 2, 3, \ldots \} \). For any row vector \( \alpha := (\alpha(j)) \) and matrix \( A := (a(i, j)) \), let

\[
\| \alpha \| = \sum_j |\alpha(j)|, \quad \| A \| = \sup_i \sum_j |a(i, j)|,
\]

respectively. Furthermore, let \( \text{abs}\{ \cdot \} \) denote the element-wise absolute operator for vectors and matrices. Thus, \( \text{abs}\{ \alpha \} e = \| \alpha \| \). Finally, we define the empty sum as zero, e.g., \( \sum_{m=\ell}^k a_m = 0 \) if \( \ell > k \), where \( \{a_m; m = 0, \pm 1, \pm 2, \ldots \} \) are a sequence of numbers.

We now define \( \Delta := (\Delta(i, j))_{i,j \in \mathbb{Z}_+} \) as the diagonal matrix such that

\[
\Delta(i, i) = \sup_{j \in \mathbb{Z}_+} |q(j, j)|, \quad i \in \mathbb{Z}_+.
\]

We then assume the following.

**Assumption 2.1** There exist some \( b \in (0, \infty) \), column vector \( v := (v(i))_{i \in \mathbb{Z}_+} \geq 0 \) and finite set \( C \subset \mathbb{Z}_+ \) such that

\[
Qv \leq -\Delta e + b1_C,
\]

where, for any set \( S \subset \mathbb{Z}_+ \), \( 1_S := (1_S(i))_{i \in \mathbb{Z}_+} \) denotes a column vector whose \( i \)-th element \( 1_S(i) \) is given by

\[
1_S(i) = \begin{cases} 1, & i \in S, \\ 0, & i \in \mathbb{Z}_+ \setminus S. \end{cases}
\]

**Remark 2.1** Suppose that the generator \( Q \) is irreducible and regular (non-explosive). It then follows from [13, Theorem 1.1] that Assumption 2.1 holds if and only if \( Q \) is ergodic with an unique stationary distribution vector \( \pi \) such that

\[
\pi \Delta e < \infty.
\]

Under Assumption 2.1, we obtain a row-wise limit formula for the normalized fundamental matrix \( (n)F \).

**Lemma 2.1** If Assumption 2.1 holds, then

\[
\lim_{n \to \infty} \| (n)e_{(\nu)}^T (n)F - \pi \| = 0 \quad \text{for any fixed} \ \nu \in \mathbb{Z}_+. \tag{2.9}
\]

**Proof.** Let \( P := (p(i, j))_{i,j \in \mathbb{Z}_+} \) denote

\[
P = I + \Delta^{-1}Q. \tag{2.10}
\]

Since the generator \( Q \) is ergodic, \( P \) is an irreducible stochastic matrix. Let \( \varpi \) denote

\[
\varpi = \frac{\pi \Delta}{\pi \Delta e}, \tag{2.11}
\]
which is well-defined due to Assumption 2.1 (see Remark 2.1). From (2.10), (2.11) and \( \pi Q = 0 \),
we have \( \varpi P = \varpi \) and thus \( \varpi \) is an unique stationary distribution vector of \( P \).
From (2.11), we also have
\[
\pi = \frac{\varpi \Delta^{-1}}{\varpi \Delta^{-1} e}.
\]  
(2.12)

Let \( (n)P, n \in \mathbb{Z}_+ \), denote the \((n + 1) \times (n + 1)\) NW-corner truncation of \( P \), i.e.,
\[
(\nu)P = I + (\nu)\Delta^{-1}(\nu)Q,
\]  
(2.13)

where \( (\nu)\Delta \) denotes the \((n + 1) \times (n + 1)\) NW-corner truncation of \( \Delta \). We then define \( (\nu)P_{(\nu)} \),
\( n \in \mathbb{Z}_{\nu - 1} = \mathbb{Z}_+ \setminus \mathbb{Z}_{\nu - 1} \), as the \( \nu \)-th-column-augmented \((n + 1) \times (n + 1)\) NW-corner truncation
\( (\nu)P \) of \( P \), i.e.,
\[
(\nu)P_{(\nu)} = (\nu)P + (I - (\nu)P)e \cdot (\nu)\top e_{(\nu)}.
\]

We also define \( (\nu)\varpi_{(\nu)}, n \in \mathbb{Z}_{\nu - 1} \), as the stationary distribution vector of \( (\nu)P_{(\nu)} \). It is known [28, Lemma 7.2] that
\[
(\nu)\varpi_{(\nu)} = \frac{(\nu)\top e_{(\nu)}(I - (\nu)P)^{-1}}{(\nu)\top e_{(\nu)}(I - (\nu)P)^{-1} e}, \quad n \in \mathbb{Z}_{\nu - 1},
\]  
(2.14)

and [34, Theorem 5.1] (see also [6, Theorem 3.1]) that
\[
\lim_{n \to \infty} \| (\nu)\varpi_{(\nu)} - \varpi \| = 0 \quad \text{for any fixed } \nu \in \mathbb{Z}_+.
\]  
(2.15)

Substituting (2.13) into (2.14), we have
\[
(\nu)\varpi_{(\nu)} = \frac{(\nu)\top e_{(\nu)}(- (\nu)Q)^{-1}(\nu)\Delta}{(\nu)\top e_{(\nu)}(- (\nu)Q)^{-1}(\nu)\Delta e}, \quad n \in \mathbb{Z}_{\nu - 1}.
\]  
(2.16)

From (2.6) and (2.16), we obtain
\[
(\nu)\pi_{(\nu)} = \frac{(\nu)\varpi_{(\nu)}(\nu)\Delta^{-1}}{(\nu)\varpi_{(\nu)}(\nu)\Delta^{-1} e}, \quad n \in \mathbb{Z}_{\nu - 1}.
\]  
(2.17)

We now define \( d = \Delta^{-1}e \). It then follows from (2.6), (2.12) and (2.17) that
\[
(\nu)\top e_{(\nu)}(\nu)F - \pi
= (\nu)\pi_{(\nu)} - \pi
= (\nu)\varpi_{(\nu)}\Delta^{-1} - \frac{\varpi \Delta^{-1}}{\varpi d}
= \frac{1}{(\nu)\varpi_{(\nu)}d} \left[ (\nu)\varpi_{(\nu)} - \varpi \right] + \left( \frac{1}{(\nu)\varpi_{(\nu)}d} - \frac{1}{\varpi d} \right) \varpi \Delta^{-1}
= \frac{1}{(\nu)\varpi_{(\nu)}d} \left[ (\nu)\varpi_{(\nu)} - \varpi \right] + \varpi - (\nu)\varpi_{(\nu)} \frac{\varpi d}{\varpi d} \Delta^{-1},
\]  
(2.18)

where, as mentioned in the beginning of this section, the finite probability vectors \( (\nu)\pi_{(\nu)} \) and
\( (\nu)\varpi_{(\nu)} \) are extended to the infinite ones by appending zeros to these vectors. From (2.18), we
have
\[
\begin{align*}
\text{abs}\{ (n)e^T \varpi (n) F - \pi \} e \\
\quad \leq \frac{1}{(n) \varpi (n) d} \left[ \text{abs}\{ (n) \varpi (n) \varpi \} + \text{abs}\{ (n) \varpi (n) \varpi \} \right] d
\end{align*}
\]
\[= \frac{2 \cdot \text{abs}\{ (n) \varpi (n) \varpi \} e}{(n) \varpi (n) d}. \]

(2.19)

We discuss the convergence of the right hand side of (2.19). It follows from \( \inf_{i \in \mathbb{Z}_+} |\Delta(i, i)| = |q(0, 0)| \in (0, \infty) \) that
\[
d = \Delta^{-1} e \leq C e, \tag{2.20}
\]
where \( C := |q(0, 0)|^{-1} \in (0, \infty) \). It also follows from (2.20) that \( (n) \varpi (n) d \leq C(n) \varpi (n) e = C \) for \( n \in \mathbb{Z}_{\nu-1} \). Therefore, using (2.15) and the dominated convergence theorem, we obtain
\[
\lim_{n \to \infty} (n) \varpi (n) d = \varpi d \in (0, \infty) \quad \text{for any fixed} \ \nu \in \mathbb{Z}_+. \tag{2.21}
\]

In addition, using (2.15) and (2.20), we have
\[
\begin{align*}
\text{abs}\{ (n) \varpi (n) \varpi \} e d & \leq C \cdot \text{abs}\{ (n) \varpi (n) \varpi \} e \\
& = C \| (n) \varpi (n) \varpi \| \to 0 \quad \text{as} \ n \to \infty, \tag{2.22}
\end{align*}
\]
where the limit holds for any fixed \( \nu \in \mathbb{Z}_+ \). Applying (2.21) and (2.22) to (2.19) yields
\[
\lim_{n \to \infty} \text{abs}\{ (n)e^T (n) F - \pi \} e = 0 \quad \text{for any fixed} \ \nu \in \mathbb{Z}_+.
\]
which implies that (2.9) holds. \( \square \)

Lemma 2.1 does not necessarily implies that if Assumption 2.1 holds then, for any fixed \( m \in \mathbb{Z}_+ \),
\[
\lim_{n \to \infty} \| (n+m)e^T (n+m) F - \pi \| = 0,
\]
or equivalently, \( \lim_{n \to \infty} \| (n+m)\pi (n) - \pi \| = 0 \). In fact, although this is the discrete-time case, Gibson and Seneta [6] provided an example such that the last-column-augmented truncation approximation \( (n)\pi (n) \) does not converge to \( \pi \) as \( n \to \infty \) (see also [34]). Such an example implies that an additional assumption is required by the limit formulas: \( \lim_{n \to \infty} (n)\pi = \pi \) and \( \lim_{n \to \infty} (n) F = \pi e \). To discuss the convergence of \( \{ (n)\pi \} \) and \( \{ (n) F \} \), we introduce the following assumption, in addition to Assumption 2.1.

**Assumption 2.2** For \( n \in \mathbb{Z}_+ \),
\[
(\nu, \bar{Q} \nu) \leq -\Delta e + b' 1_C \quad \text{for some} \ b' \in (0, \infty), \tag{2.23}
\]
and
\[
\lim_{n \to \infty} (n) Q \nu = Q \nu, \tag{2.24}
\]
\[
\sup_{i \in \mathbb{Z}_+ \setminus C} \left\{ \frac{\sum_{j \in \mathbb{Z}_+ \setminus C} q(i, j) (v(j) - v(i))^+}{\Delta(i, i)} \right\} < \infty, \tag{2.25}
\]
where \( (x)^+ = \max(x, 0) \) for \( x \in (-\infty, \infty) \).
**Theorem 2.2** If Assumptions 2.1 and 2.2 hold, then

\[
\lim_{n \to \infty} \| (n)\pi - \pi \| = \lim_{n \to \infty} \| (n)\beta P - \pi \| = 0.
\]

**Proof.** According to (2.4), it suffices to prove that \( \lim_{n \to \infty} \| (n)\pi - \pi \| = 0 \). For \( n \in \mathbb{Z}_+ \), let \((n)\overline{\omega}\) and \((n)\overline{P}\) denote

\[
(n)\overline{\omega} = \frac{(n)\alpha (I - (n)P)^{-1} \Delta}{(n)\alpha (I - (n)P)^{-1} e}, \tag{2.26}
\]

\[
(n)\overline{P} = (n)P + (I - (n)P)e(n)\alpha, \tag{2.27}
\]

where \((n)P\) is given in (2.13), i.e., \((n)P\) is the \((n + 1) \times (n + 1)\) NW-corner of \( P \) in (2.10). It then follows that \((n)\overline{\omega}\) is a stationary distribution vector of \((n)\overline{P}\). Using (2.13), we rewrite (2.26) as

\[
(n)\overline{\omega} = \frac{(n)\alpha (- (n)Q)^{-1} (n)\Delta}{(n)\alpha (- (n)Q)^{-1} (n)\Delta e}. \tag{2.28}
\]

Using (2.1) and (2.13), we also rewrite (2.27) as

\[
(n)\overline{P} = I + (n)\Delta^{-1} (n)Q - (n)Qe(n)\alpha
= I + (n)\Delta^{-1} (n)\overline{Q}, \tag{2.29}
\]

Combining Assumptions 2.1 and 2.2 with (2.10) and (2.29), we obtain

\[
Pv \leq v - e + \frac{b}{q(0, 0)} 1_C
\]

\[
(n)\overline{P}v \leq v - e + \frac{b'}{q(0, 0)} 1_C,
\]

\[
\lim_{n \to \infty} (n)\overline{P}v = Pv,
\]

and

\[
\sup_{i \in \mathbb{Z}_+ \setminus C} \left\{ \sum_{j \in \mathbb{Z}_+ \setminus C} p(i, j)(v(j) - v(i))^+ \right\} < \infty. \tag{2.30}
\]

Therefore, Corollary 4.5 of [34] implies that \( \lim_{n \to \infty} \| (n)\overline{\omega} - \omega \| = 0 \). Furthermore, from (2.3) and (2.28), we have

\[
(n)\overline{\pi} = \frac{(n)\overline{\omega} (n)\Delta^{-1} \Delta e}{(n)\overline{\omega} (n)\Delta^{-1} e}, \quad n \in \mathbb{Z}_+.
\]

Using these results, and following the proof of Lemma 2.1, we can readily prove that \( \lim_{n \to \infty} \| (n)\overline{\pi} - \pi \| = 0 \). \(\square\)

**Remark 2.2** Suppose that Assumption 2.1 holds. If

\[
(n)\overline{Q}v \leq Qv, \quad n \in \mathbb{Z}_+,
\]

(2.31)
then the conditions (2.23) and (2.24) of Assumption 2.2 are satisfied. A sufficient condition for (2.31) is that \( \{v(i); i \in \mathbb{Z}_+\} \) is nondecreasing. The last condition (2.25) of Assumption 2.2 implies (2.30), which guarantees that a superharmonic vector \( v \geq 0 \) of \( \mathbf{P}_C := \text{diag}\{1_{\mathbb{Z}_+}\} \mathbf{P} \text{diag}\{1_{\mathbb{Z}_+}\} \) (i.e., \( \mathbf{P}_C v \leq v \)) is a potential of \( \mathbf{P}_C \), or equivalently,

\[
v = \sum_{m=0}^{\infty} (\mathbf{P}_C)^m (v - \mathbf{P}_C v).
\]

Although the last condition (2.25) is not easy to handle, this cannot be removed. For details, see [34].

The following corollary is an immediate consequence of Theorem 2.2 and Remark 2.2.

**Corollary 2.2** If Assumption 2.1, (2.25) and (2.31) hold, then

\[
\lim_{n \to \infty} (n)F = e\pi,
\]

where the convergence is in row-wise total variation.

In what follows, we present some limit formulas for the normalized linear combination of the truncated rows of the fundamental matrix \( -(n)Q^{-1} \). These formulas do not require Assumption 2.1 or 2.2. To prove the formulas, we need the following lemma.

**Lemma 2.2** For any fixed \( m \in \mathbb{Z}_+ \), the set \( \mathbb{Z}_m \) of states is included by the single closed communicating class of the \( Q \)-matrix \( (n)Q \) for all sufficiently large \( n \geq m \).

**Proof.** Let \( \mathbf{P}(t) := (p(t)(i, j))_{i, j \in \mathbb{Z}_+} \) denote the transition matrix function of the Markov chain \( \{Z(t)\} \) with ergodic generator \( Q \), i.e., \( P(Z(t) = j \mid Z(0) = i) \) for all \( i, j \in \mathbb{Z}_+ \). It then follows from [1, Chapter 2, Proposition 2.14] that, for \( i, j \in \mathbb{Z}_+ \) and \( t \geq 0 \),

\[
\begin{bmatrix}
\exp\{ (n)Qt \}
\end{bmatrix}_{i,j} \xrightarrow{\text{as } n \to \infty} \begin{bmatrix}
\mathbf{P}(t)
\end{bmatrix}_{i,j}
\]

where \([\cdot]_{i,j}\) denotes the \((i, j)\)-th element of the matrix between the square brackets. Thus, using the monotone convergence theorem, we have, for \( n \in \mathbb{Z}_+ \) and \( i, j \in \mathbb{Z}_n \),

\[
\left[\left( -\left(\begin{array}{c}
(n)Q^{-1}
\end{array}\right)\right)_{i,j}\right] = \left[\int_{0}^{\infty} \exp\{ (n)Qt \} dt\right]_{i,j} \xrightarrow{\text{as } n \to \infty} \left[\int_{0}^{\infty} \mathbf{P}(t) dt\right]_{i,j} = \infty.
\]

Recall here (see (2.1)) that \( (n)Q \) is a conservative \( Q \)-matrix obtained by linearly augmenting the NW-corner \( (n)Q \) of \( Q \). Therefore, we have

\[
\exp\{ (n)Qt \} \leq \exp\{ (n)Qt \}, \quad t > 0,
\]

which leads to

\[
-\left(\begin{array}{c}
(n)Q^{-1}
\end{array}\right) \leq \left[\int_{0}^{\infty} \exp\{ (n)Qt \} dt\right].
\]
It follows from (2.33) and (2.34) that the \((m + 1) \times (m + 1)\) NW-corner of \(\int_0^\infty \exp\{\frac{1}{m} Q t\} dt\) is positive (possibly positive infinite) for all sufficiently large \(n \geq m\) and thus \(\mathbb{Z}_m\) is a communicating class of \((n)\overline{Q}\). This fact and Theorem 2.1 imply that the statement of the present lemma is true.

For any finite \(B \subseteq \mathbb{Z}_+\) and \(n \in \mathbb{Z}_+ \setminus B\), let \((n)\mu_B\) denote

\[
(n)\mu_B = (n)\alpha (-(n)Q)^{-1} (n)E_B, \tag{2.35}
\]

where \((n)\alpha\) is a \(1 \times (n + 1)\) probability vector and \((n)E_B, B \subseteq \mathbb{Z}_n\), is a matrix that can be permuted such that

\[
(n)E_B = \begin{bmatrix} \mathbb{B} \\ \mathbb{Z}_n \setminus \mathbb{B} \begin{bmatrix} I \\ O \end{bmatrix} \end{bmatrix}.
\]

Let \(\pi^*_B, B \subseteq \mathbb{Z}_+\), denote

\[
\pi^*_B = \frac{\pi_B}{\pi_B e}, \tag{2.36}
\]

where \(\pi_B = (\pi(i))_{i \in B} > 0\). In this setting, we prove a limit formula for \((n)\mu_B\) by using Lemma 2.2.

**Theorem 2.3** For any fixed finite \(B \subseteq \mathbb{Z}_+\), let \(\mathbb{N}^+_B\) denote

\[
\mathbb{N}^+_B = \{n \in \mathbb{Z}_+: (n)\mu_B e > 0\},
\]

and suppose that \(\mathbb{N}^+_B\) has infinitely many elements. We then have

\[
\lim_{n \to \infty} \frac{(n)\mu_B}{(n)\mu_B e} = \pi^*_B. \tag{2.37}
\]

**Proof.** It is not assumed that the ergodic generator \(Q\) has a special structure. Thus, we fix \(m \in \mathbb{Z}_+\) arbitrarily and prove the statement of this theorem with \(B = \mathbb{Z}_m\), which does not lose generality.

Let \((n)\overline{\pi}_m^+ \in \mathbb{N}^+_Z, n \in \mathbb{N}^+_Z \setminus \mathbb{Z}_m\), denote

\[
(n)\overline{\pi}_m^+ = \frac{(n)\overline{\pi}_m}{(n)\overline{\pi}_m e}, \tag{2.38}
\]

where \((n)\overline{\pi}_m = (\pi(0), (n)\overline{\pi}(1), \ldots, (n)\overline{\pi}(m))\). It then follows from (2.3), (2.35) and (2.38) that

\[
(n)\overline{\pi}_m^* = \frac{(n)\overline{\pi}(n)E_{Z_m}}{(n)\overline{\pi}(n)E_{Z_m} e} = \frac{(n)\alpha (-(n)Q)^{-1} (n)E_{Z_m}}{(n)\alpha (-(n)Q)^{-1} (n)E_{Z_m} e} = \frac{(n)\mu_{Z_m}}{(n)\mu_{Z_m} e} \tag{2.39}
\]

As a result, it suffices to prove that

\[
\lim_{n \to \infty} \frac{(n)\overline{\pi}_m^+}{n \in \mathbb{N}^+_Z} = \overline{\pi}_m^*. \tag{2.40}
\]
We partition $\overline{Q}$ and $(n)\overline{Q}$ as

$$Q = \frac{Z_m}{Z_m} \begin{pmatrix} Z_m & Z_m \\ Q_{zm, zm} & Q_{zm, zm} \end{pmatrix} ,$$

$$\frac{\bar{Q}}{Z_m} = \frac{Z_m}{Z_m \ Z_m} \begin{pmatrix} Z_m & Z_m \\ (n)\bar{Q}_{zm, zm} & (n)\bar{Q}_{zm, zm} \end{pmatrix} ,$$

respectively. We then define $Q_{zm}^*$ and $(n)\overline{Q}_{zm}^*$ as

$$Q_{zm}^* = Q_{zm} + Q_{zm, zm} (-Q_{zm})^{-1} Q_{zm, zm} ,$$

$$\frac{(n)\overline{Q}_{zm}^*}{Z_m} = \frac{(n)\overline{Q}_{zm}}{Z_m} + (n)\overline{Q}_{zm, zm} (-1) (n)\overline{Q}_{zm, zm} ,$$

respectively. The $Q$-matrix $Q_{zm}^*$ (resp. $(n)\overline{Q}_{zm}^*$) is the generator of a censored Markov chain with state space $Z_m$, which emulates the behavior of the original Markov chain with state space $Z_+$ (resp. $Z_m$) and generator $Q$ (resp. $(n)\overline{Q}$) while the original chain is running in $Z_m$. It thus follows from (2.36) and the ergodicity of $Q$ that the generator $Q_{zm}^*$ is ergodic with stationary distribution vector $\pi_{zm}^*$. Similarly, it follows from (2.38) and Lemma 2.2 that, for all sufficiently large $n \in \mathbb{Z}_m$, the generator $(n)\overline{Q}_{zm}^*$ is ergodic with stationary distribution vector $(n)\overline{\pi}_{zm}^*$.

In fact, we can prove (see Appendix A) that

$$\lim_{n \to \infty} (n)\overline{Q}_{zm}^* = Q_{zm}^* .$$

We also have, from [9, Section 4.1, Eq. (9)],

$$(n)\overline{\pi}_{zm}^* - \pi_{zm}^* = (n)\overline{\pi}_{zm}^* (n)\overline{Q}_{zm}^* D_{zm}^* , \quad n \in \mathbb{Z}_m ,$$

where $D_{zm}^*$ is the deviation matrix of the transition matrix function with generator $Q_{zm}^*$, i.e.,

$$D_{zm}^* = \int_0^\infty (\exp\{Q_{zm}^* t\} - e^{\pi_{zm}^*}) \, dt .$$

Applying (2.45) to (2.46) results in (2.40). □

From Theorem 2.3, we have the following corollary.

**Corollary 2.3** Suppose that the conditions of Theorem 2.3 are satisfied. If

$$\sum_{i \in \mathbb{B}} (n)\alpha(i) > 0 \quad \text{for all sufficiently large } n \in \mathbb{Z}_+ ,$$

then (2.37) holds.

**Proof.** It follows from (2.33) that, for all sufficiently large $n \in \mathbb{Z}_+$,

$$[(- (n) D_{zm})^{-1}]_{i,j} > 0 , \quad i, j \in \mathbb{B} .$$

Using this and (2.47), we have

$$(n)\mu_B = (n)\alpha (- (n) Q)^{-1} (n)E_B > 0 \quad \text{for all sufficiently large } n \in \mathbb{Z}_+ ,$$

which implies that $\mathbb{N}^+_B$ has infinitely many elements. Therefore, (2.37) follows from Theorem 2.3.
3 Matrix-infinite-product-form solutions for block-Hessenberg Markov chains

In this section, we consider the case where the ergodic generator \( Q \) is in block-Hessenberg form. To this end, we rewrite \( Q \) as a block-structured generator.

Let \( m_\ell \)'s, \( \ell \in \mathbb{Z}_+ \), denote positive integers. Let \( n_{-1} = -1 \) and \( n_s = \sum_{\ell=0}^{s} m_\ell - 1 \) for \( s \in \mathbb{Z}_+ \). We then partition the state space \( \mathbb{Z} \) into the substate spaces \( \mathbb{L}_s \)'s, \( s \in \mathbb{Z}_+ \), where

\[
\mathbb{L}_s = \{ n_{s-1} + 1, n_{s-1} + 2, \ldots, n_s \}, \quad s \in \mathbb{Z}_+.
\]

The substate spaces \( \mathbb{L}_s \)'s, \( s \in \mathbb{Z}_+ \), are often referred to as *levels*, and \( \mathbb{L}_s \) is referred to as *level* \( s \). By definition, the cardinality of \( \mathbb{L}_s \) is equal to \( m_s \).

We now partition \( Q \) as follows:

\[
Q = \begin{pmatrix}
\mathbb{L}_0 & \mathbb{L}_1 & \mathbb{L}_2 & \mathbb{L}_3 & \cdots \\
\mathbb{L}_0 & Q_{0,0} & Q_{0,1} & Q_{0,2} & Q_{0,3} & \cdots \\
\mathbb{L}_1 & Q_{1,0} & Q_{1,1} & Q_{1,2} & Q_{1,3} & \cdots \\
\mathbb{L}_2 & Q_{2,0} & Q_{2,1} & Q_{2,2} & Q_{2,3} & \cdots \\
\mathbb{L}_3 & Q_{3,0} & Q_{3,1} & Q_{3,2} & Q_{3,3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\end{pmatrix}, \quad (3.1)
\]

where \( Q_{k,\ell} = (q(i, j))_{(i, j) \in \mathbb{L}_k \times \mathbb{L}_\ell} \) for \( k, \ell \in \mathbb{Z}_+ \). Similarly, we partition matrices (including vectors) associated with \( Q \). For example,

\[
(n_s)Q = \begin{pmatrix}
\mathbb{L}_0 & \mathbb{L}_1 & \cdots & \mathbb{L}_{s} \\
\mathbb{L}_0 & (n_s)X_{0,0} & (n_s)X_{0,1} & \cdots & (n_s)X_{0,s} \\
\mathbb{L}_1 & (n_s)X_{1,0} & (n_s)X_{1,1} & \cdots & (n_s)X_{1,s} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbb{L}_s & (n_s)X_{s,0} & (n_s)X_{s,1} & \cdots & (n_s)X_{s,s} \\
\end{pmatrix}, \quad s \in \mathbb{Z}_+, \quad (3.2)
\]

\[
(-n_s)Q^{-1} = \begin{pmatrix}
\mathbb{L}_0 & \mathbb{L}_1 & \cdots & \mathbb{L}_{s} \\
\mathbb{L}_0 & (n_s)X_{0,0} & (n_s)X_{0,1} & \cdots & (n_s)X_{0,s} \\
\mathbb{L}_1 & (n_s)X_{1,0} & (n_s)X_{1,1} & \cdots & (n_s)X_{1,s} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbb{L}_s & (n_s)X_{s,0} & (n_s)X_{s,1} & \cdots & (n_s)X_{s,s} \\
\end{pmatrix}, \quad s \in \mathbb{Z}_+, \quad (3.2)
\]
We assume that the generator \( Q \) is of GI/M/1 type (see, e.g., [7]), i.e., that is, \( Q \) is of GI/M/1 type in upper block-Hessenberg form. Utilizing the result in Section 3.1, Section 3.2 develops an algorithm for \( Q \) in upper block-Hessenberg form. Finally, Section 3.3 considers a special case where \( Q \) is of GI/M/1 type (see, e.g., [7]), i.e., \( Q \) is a block-Toeplitz-like generator in upper block-Hessenberg form.

### 3.1 Upper block-Hessenberg Markov chain

We assume that the generator \( Q \) is in upper block-Hessenberg form (i.e., is of level-dependent M/G/1-type):

\[
Q = \begin{pmatrix}
L_0 & L_1 & L_2 & L_3 & \\
L_0 & Q_{0,0} & Q_{0,1} & Q_{0,2} & Q_{0,3} & \\
L_1 & Q_{1,0} & Q_{1,1} & Q_{1,2} & Q_{1,3} & \\
L_2 & O & Q_{2,1} & Q_{2,2} & Q_{2,3} & \\
L_3 & O & O & Q_{3,2} & Q_{3,3} & \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

(3.6)

where \( Q_{k,\ell} = O \) for \( k \in \mathbb{Z}_+ \) and \( \ell = 0, 1, \ldots, \max(k-1,0) \).
**Remark 3.1** If $Q$ in (3.6) is block-tridiagonal, i.e., $Q_{k,\ell} = 0$ for all $k, \ell \in \mathbb{Z}_+$ such that $|k - \ell| \geq 2$, then $Q$ can be considered the generator of a level-dependent quasi-birth-and-death process (LD-QBD) (see [5, 27]).

It follows from (3.6) that

$$(n_\uparrow)Q = \begin{pmatrix}
Q_{0,0} & Q_{0,1} & Q_{0,2} & \cdots & Q_{0,s-2} & Q_{0,s-1} & Q_{0,s} \\
Q_{1,0} & Q_{1,1} & Q_{1,2} & \cdots & Q_{1,s-2} & Q_{1,s-1} & Q_{1,s} \\
O & Q_{2,1} & Q_{2,2} & \cdots & Q_{2,s-2} & Q_{2,s-1} & Q_{2,s} \\
& & & & \vdots & \vdots & \vdots \\
O & O & O & \cdots & Q_{n-1,s-2} & Q_{n-1,s-1} & Q_{n-1,s}
\end{pmatrix}, \quad s \in \mathbb{Z}_+. \tag{3.7}
$$

The NW-corner truncation $(n_\uparrow)Q$ of $Q$ is also in the upper block-Hessenberg form. Therefore, as we will see later, we can derive an efficient recursive formula for the last block

$$( (n_\uparrow)X_{s,0}, (n_\uparrow)X_{s,1}, \ldots, (n_\uparrow)X_{s,s})$$

of $(-(n_\uparrow)Q)^{-1}$ in (3.2). To derive this formula, we define $\{U^*_k; k \in \mathbb{Z}_+\}$ recursively as follows:

$$U^*_k = \begin{cases}
(-Q_{0,0})^{-1}, & k = 0, \\
(-Q_{k,k} - \sum_{\ell=0}^{k-1} U^*_{k,\ell} Q_{\ell,k})^{-1}, & k \in \mathbb{N},
\end{cases} \tag{3.8}
$$

where $U_{k,\ell}$'s, $k \in \mathbb{N}, \ell \in \mathbb{Z}_{k-1}$, are given by

$$U_{k,\ell} = (Q_{k,k-1} U^*_{k-1})(Q_{k-1,k-2} U^*_{k-2}) \cdots (Q_{\ell+1,\ell} U^*_\ell). \tag{3.9}
$$

Since the empty sum is defined as zero, Eq. (3.8) is expressed as the single equation (i.e., the equation for $k \in \mathbb{N}$ is extended to the one for $k \in \mathbb{Z}_+$). Note here that $U^*_k$ is nonsingular, which is proved in Appendix C.1.

The following lemma provides a matrix-product-form expression of the $(n_\uparrow)X_{s,\ell}$’s.

**Lemma 3.1** If the ergodic generator $Q$ is in upper block-Hessenberg form (3.6), then

$$(n_\uparrow)X_{s,\ell} = U^*_s U_{s,\ell}, \quad s \in \mathbb{Z}_+, \quad \ell \in \mathbb{Z}_u,
$$

where $U_{s,s} = I$ for $s \in \mathbb{Z}_+$.

**Remark 3.2** Shin [29] presented the similar expressions of all the blocks $(n_\uparrow)X_{k,\ell}$’s in a special case where $Q$ in (3.6) is reduced to be block tridiagonal (see Theorem 2.1 therein), i.e., to the generator of an LD-QBD (see Remark 3.1).

**Remark 3.3** It is stated in [31, Remark 2] that if the ergodic generator $Q$ is in upper block-Hessenberg form (3.6) then

$$\pi_\ell = \pi_k U_{k,\ell}, \quad k \in \mathbb{N}, \quad \ell \in \mathbb{Z}_{k-1}. \tag{3.10}
$$

For the reader’s convenience, we provide a complete proof of (3.10) in Appendix B.
Proof of Lemma 3.1

The inverse \((n_s)Q^{-1}\) of \((n_s)Q\) is the unique solution of \((n_s)Q^{-1}(n_s)Q = I\). Thus, the last block row \((n_s)X_{s,0} \cdot (n_s)X_{s,1} \cdot \ldots \cdot (n_s)X_{s,s}\) of \((−(n_s)Q)^{-1}\) is the unique solution of the following equations:

\[
O = (n_s)X_{s,0}Q_{0,0} + (n_s)X_{s,1}Q_{1,0}, \quad (3.11)
\]

\[
O = \sum_{\ell=0}^{k+1} (n_s)X_{s,\ell}Q_{\ell,k}, \quad k = 1, 2, \ldots, s - 1, \quad (3.12)
\]

\[
-I = \sum_{\ell=0}^{s} (n_s)X_{s,\ell}Q_{\ell,s}. \quad (3.13)
\]

Solving (3.11) with respect to \((n_s)X_{s,0}\) and applying (3.8) to the result, we have

\[
(n_s)X_{s,0} = (n_s)X_{s,1}Q_{1,0}U_0^* = (n_s)X_{s,1}U_{1,0}, \quad (3.14)
\]

where the second equality follows from (3.9).

We now suppose that, for some \(k \in \{1, 2, \ldots, s - 1\},\)

\[
(n_s)X_{s,\ell} = (n_s)X_{s,k}U_{k,\ell} \quad \text{for all } \ell \in \mathbb{Z}_{k-1},
\]

which holds at least for \(k = 1\) due to (3.14). Substituting (3.15) into (3.12) and using (3.8), we obtain

\[
O = \sum_{\ell=0}^{k-1} (n_s)X_{s,k}U_{k,\ell}Q_{\ell,k} + (n_s)X_{s,k}Q_{k,k} + (n_s)X_{s,k+1}Q_{k+1,k}
\]

\[
= (n_s)X_{s,k} \left( Q_{k,k} + \sum_{\ell=0}^{k-1} U_{k,\ell}Q_{\ell,k} \right) + (n_s)X_{s,k+1}Q_{k+1,k}
\]

\[
= (n_s)X_{s,k}(-U_k^*)^{-1} + (n_s)X_{s,k+1}Q_{k+1,k},
\]

which leads to

\[
(n_s)X_{s,k} = (n_s)X_{s,k+1}Q_{k+1,k}U_k^*. \quad (3.16)
\]

Using (3.16) and (3.9), we rewrite (3.15) as

\[
(n_s)X_{s,\ell} = (n_s)X_{s,k+1}Q_{k+1,k}U_k^*U_{k,\ell}
\]

\[
= (n_s)X_{s,k+1}U_{k+1,\ell} \quad \text{for all } \ell \in \mathbb{Z}_k.
\]

Therefore, by induction, we have

\[
(n_s)X_{s,\ell} = (n_s)X_{s,s}U_{s,\ell} \quad \text{for all } \ell \in \mathbb{Z}_{s-1}. \quad (3.17)
\]

It remains to prove that \((n_s)X_{s,s} = U_s^*\). Applying (3.17) to (3.13) and following the derivation of (3.16), we obtain

\[
-I = (n_s)X_{s,s}Q_{s,s} + \sum_{\ell=0}^{s-1} (n_s)X_{s,s}U_{s,\ell}Q_{\ell,s}
\]

\[
= (n_s)X_{s,s} \left( Q_{s,s} + \sum_{\ell=0}^{s-1} U_{s,\ell}Q_{\ell,s} \right) = (n_s)X_{s,s}(-U_s^*)^{-1};
\]
which results in \((n_s)X_{s,s} = U_s^*\).

It follows from (3.2) and (3.5) that

\[(n_s)\pi = \frac{(n_s)\alpha_s (X_{s,0}, (n_s)X_{s,1}, \ldots, (n_s)X_{s,s})}{(n_s)\alpha_s \sum_{\ell=0}^s X_{s,\ell}e}. \tag{3.18}\]

Applying Lemma 3.1 to (3.18), we have

\[(n_s)\pi_k = \frac{(n_s)\alpha_s U_{s,k}^* U_{s,k}}{(n_s)\alpha_s \sum_{\ell=0}^s U_{s,\ell}^* U_{s,\ell}e}, \quad s \in \mathbb{Z}_+, k \in \mathbb{Z}_s. \tag{3.19}\]

Combining (3.19) and Theorem 2.2 yields the following result.

**Theorem 3.1** Suppose that the ergodic generator \(Q\) is in upper block-Hessenberg form (3.6). If Assumptions 2.1, 2.2 and 3.1 hold, then

\[
\pi_k = \lim_{s \to \infty} \frac{(n_s)\alpha_s U_{s,k}^* U_{s,k}}{(n_s)\alpha_s \sum_{\ell=0}^s U_{s,\ell}^* U_{s,\ell}e}, \quad k \in \mathbb{Z}_+. \tag{3.20}
\]

We now define \(U_k, k \in \mathbb{Z}_+\), as

\[U_k = Q_{k+1,k} U_k^*, \quad k \in \mathbb{Z}_+.
\]

It then follows from (3.9) that

\[U_{s,\ell} = U_{s-1} U_{s-2} \cdots U_{\ell}, \quad \ell \in \mathbb{Z}_{s-1}.
\]

Substituting this into (3.20) yields

\[
\pi_k = \lim_{s \to \infty} \frac{(n_s)\alpha_s U_{s,k}^* U_{s-1} U_{s-2} \cdots U_k}{(n_s)\alpha_s \sum_{\ell=0}^s U_{s,\ell}^* U_{s-1} U_{s-2} \cdots U_{\ell}e}, \quad k \in \mathbb{Z}_+, \tag{3.21}
\]

which shows that \(\pi_k\) is expressed as an infinite product of matrices with a normalizing constant. Therefore, we refer to the expression (3.21), equivalently (3.20) as the *matrix-infinite-product-form (MIP-form)* solution of \(\pi\).

In what follows, we discuss the computation of the MIP-form solution (3.20) of \(\pi = (\pi_0, \pi_1, \ldots)\). From (3.19), we have

\[(n_s)\pi_k = \frac{(n_s)\alpha_s U_{s,k}^*}{(n_s)\alpha_s u_s^*}, \quad s \in \mathbb{Z}_+, k \in \mathbb{Z}_s, \tag{3.22}\]

where

\[U_{s,k}^* = U_s^* U_{s,k}, \quad s \in \mathbb{Z}_+, k \in \mathbb{Z}_s, \tag{3.23}\]

\[u_s^* = \sum_{\ell=0}^s U_{s,\ell}^* e = \sum_{\ell=0}^s U_{s,\ell}^* U_{s,\ell} e, \quad s \in \mathbb{Z}_+. \tag{3.24}\]
Combining (3.23) with (3.8) and (3.9), we have

\[
U^*_0,0 = U^*_0 = (-Q_{0,0})^{-1},
\]

\[
U^*_{s,k} = \begin{cases} U^*_s Q_{s,s-1} \cdot U^*_{s-1,k}, & s \in \mathbb{N}, \quad k \in \mathbb{Z}_{s-1}, \\ U^*_s, & s \in \mathbb{N}, \quad k = s. \end{cases}
\]  

(3.25)

(3.26)

From (3.24), (3.25) and (3.26), we also have

\[
u^*_0 = U^*_0 e = (-Q_{0,0})^{-1} e, \\
u^*_s = U^*_s (e + Q_{s,s-1} u^*_{s-1}), \quad s \in \mathbb{N}.
\]

(3.27)

Furthermore, applying (3.9) to (3.8) with \( k = s \in \mathbb{N} \), we obtain

\[
U^*_s = \left(-Q_{s,s} - Q_{s,s-1} \sum_{\ell=0}^{s-1} U^*_{s-1,\ell} Q_{\ell,s}\right)^{-1}
\]

\[
= \left(-Q_{s,s} - Q_{s,s-1} \sum_{\ell=0}^{s-1} U^*_{s-1,\ell} Q_{\ell,s}\right)^{-1}, \quad s \in \mathbb{N},
\]

(3.28)

where the second equality follows from (3.23). Recall here (see Theorem 2.2) that if Assumptions 2.1 and 2.2 are satisfied then

\[
\lim_{s \to \infty} \| (n_s) \pi - \pi \| = 0.
\]

As a result, under these assumptions, we can establish an algorithm for computing \( \pi \), which is described in Algorithm 1 below.

**Algorithm 1: Upper block-Hessenberg Markov chain**

**Input:** \( Q \) and \( \varepsilon \in (0, 1) \).

**Output:** \((n_s) \pi = ((n_s) \pi_0, (n_s) \pi_1, \ldots, (n_s) \pi_s)\), where \( s \in \mathbb{Z}_+ \) is fixed when the iteration stops.

(i) Set \( s = 0 \).

(ii) Compute \( U^*_0 = U^*_{0,0} = (-Q_{0,0})^{-1} \) and \( u^*_0 = U^*_0 e \).

(iii) Compute \((n_0) \pi_0 = (n_0) \alpha_0 U^*_0 / (n_0) \alpha_0 u^*_0\).

(iv) Iterate the following:

(a) Increment \( s \) by one.

(b) Compute \( U^*_s = U^*_{s,s} \) by (3.28).

(c) Compute \( \{U^*_{s,k}; k \in \mathbb{Z}_{s-1}\} \) by (3.26) and \( u^*_s \) by (3.27).

(d) Compute \( \{ (n_s) \pi_k; k \in \mathbb{Z}_s \} \) by (3.22).

(e) If \( \| (n_s) \pi - (n_{s-1}) \pi \| < \varepsilon \), then stop the iteration; otherwise return to step (a).

**Remark 3.4** If Assumptions 2.1 and 2.2 are satisfied, then Algorithm 1 stops after a finite number of iterations. Otherwise, it is possible that Algorithm 1 never stops.
Remark 3.5 If Algorithm 1 stops at \( s = N \), then an \( N + 1 \) number of the inverse matrices \( U_s^* \)'s are computed because one inverse matrix \( U_s^* \) is computed for each \( s \in \mathbb{Z}_+ \). The computation of such inverse matrices is the most time-consuming part of Algorithm 1. However, we can compute \( U_s^* \) by a stable and efficient procedure proposed by Le Boudec [16] (see Proposition C.1 in Appendix C.2).

As mentioned in Remark 3.1, the LD-QBD is a special case of upper block-Hessenberg Markov chains. Thus, Algorithm 1 is applicable to LD-QBDs and works more efficiently for them because (3.28) is reduced to

\[
U_s^* = (-Q_{s,s} - Q_{s,s-1}U_{s-1,s-1}Q_{s-1,s})^{-1}, \quad s \in \mathbb{N}.
\]

It is well-known (see [5, 27]) that if \( Q \) in (3.6) is reduced to be the generator of the LD-QBD then the stationary distribution vector \( \pi = (\pi_0, \pi_1, \ldots) \) is in the matrix product form

\[
\pi_k = \pi_0 R^{(1)} R^{(2)} \cdots R^{(k)}, \quad k \in \mathbb{N},
\]

where \( \pi_0 \) is the solution of

\[
\pi_0 (Q_{0,0} + R^{(1)} Q_{1,0}) = 0,
\]

\[
\pi_0 \left( e + \sum_{\ell=1}^{\infty} R^{(1)} R^{(2)} \cdots R^{(\ell)} e \right) = 1;
\]

and the matrices \( R^{(k)} \)'s are the minimal nonnegative solutions of

\[
Q_{k-1,k} + R^{(k)} Q_{k,k} + R^{(k)} R^{(k+1)} Q_{k+1,k} = O, \quad k \in \mathbb{N}.
\]

From (3.29), we obtain

\[
R^{(k)} = Q_{k-1,k} (-Q_{k,k} - R^{(k+1)} Q_{k+1,k})^{-1}, \quad k \in \mathbb{N}.
\]

By (3.30), we can compute the matrices \( R^{(N-1)}, R^{(N-2)}, \ldots, R^{(1)} \) given \( R^{(N)} \) for some \( N \in \mathbb{N} \). Thus, we refer to (3.30) as the backward recursion for \( \{ R^{(k)} ; k \in \mathbb{N} \} \).

Based on the above results, Phung-Duc et al. [26] developed a simple algorithm for LD-QBDs (a similar algorithm is discussed in [2]). According to the algorithm, we first choose a sufficiently large \( N \in \mathbb{N} \) such that \( \sum_{k=N+1}^{\infty} \pi_k e \) is expected to be negligible. Next, for a sufficiently large \( L \in \mathbb{N} \), we compute an approximation \( R_L^{(N)} \) to \( R^{(N)} \) by the backward recursion (3.30) with \( R^{(N+L)} = O \). It is shown (see [26, Proposition 2.4]) that \( \lim_{L \to \infty} R_L^{(N)} = R^{(N)} \). However, we have to determine \( L \) by trial and error (for the details, see [26, Algorithms 1 and 3]).

Bright and Taylor [5] developed an elaborate algorithm for LD-QBDs, which generates \( R_L^{(N)} \) with \( L = 2^{\ell+1} - 1 (\ell \in \mathbb{Z}_+) \) by using the logarithmic reduction approach for level-independent QBDs (see [14]). The computational complexity of their algorithm is of the same order as Phung-Duc et al.’s algorithm [26], though the former is more memory-consuming than the latter (for details, see [26, Section 3.2]).

We now suppose that \( L \) is given in advance. In this case, Phung-Duc et al.’s algorithm [26] computes \( L \) inverse matrices to obtain \( R_L^{(N)} \), and then computes \( N - 1 \) inverse matrices in generating \( R_L^{(k)} \)'s, \( k = N - 1, N - 2, \ldots, 1 \), by the backward recursion (3.30) with \( R^{(N)} = R_L^{(N)} \), in
order to obtain \((\pi_0, \pi_1, \ldots, \pi_N)\) (of course, approximately). Note that the computation of \(R_L^{(N)}\) corresponds to the situation that the iteration index \(s\) of Algorithm 1 reaches \(N + L - 1\) and thus \((n_{N+L-1})\), i.e., an approximation to \((\pi_0, \pi_1, \ldots, \pi_{N+L-1})\), is obtained as the result of computing \(N + L\) inverse matrices (see Remark 3.5). Consequently, the computational complexity of Algorithm 1 is of the same order as that of Phung-Duc et al.’s algorithm [26] even when the situation is best for the latter one, i.e., \(L\) is given in advance.

There are some studies on upper block-Hessenberg Markov chains. Shin and Pearce [30] established a method for computing the stationary distribution vector of the discrete-time upper block-Hessenberg Markov chain. Li et al. [17] and Klimenko and Dudin [12] proposed similar algorithms, respectively, for a BMAP/M/1 generalized processor-sharing queue and for an asymptotically level-independent M/G/1-type Markov chain. The key of their algorithms is to transform the original transition probability matrix (or the original generator in the continuous-time case) into a level-independent one (i.e., block-Toeplitz-like one) except for a finite number, say \(N\), of levels. These algorithms require us to compute, from scratch, Neuts’ \(G\)-matrix [25] and the stationary probabilities of the first \(N\) levels (i.e., \(\mathbb{I}_0, \mathbb{I}_1, \ldots, \mathbb{I}_{N-1}\)) every time \(N\) is incremented one by one. On the other hand, each iteration of Algorithm 1 inherits the results from the previous iteration.

In the rest of this subsection, we compare our result with one of the main results of Takine [31], which is closely related to the result in this subsection. To clarify the difference between our result and Takine’s one, we present the following corollary.

**Corollary 3.1** Suppose that the ergodic generator \(Q\) is in upper block-Hessenberg form (3.6). We then have

\[
\lim_{s \to \infty} \frac{e^{\top}U_s^{*}U_{s,k}e}{e^{\top}U_s^{*}U_{s,k}e} = \frac{\pi_k}{\nu_k e}, \quad k \in \mathbb{Z}_+.
\]  

(3.31)

Furthermore, if there exists \(k \in \mathbb{Z}_+\) such that

\[
U_s^{*}U_{s,k}e > 0 \quad \text{for all sufficiently large } s \in \mathbb{Z}_k,
\]  

(3.32)

then

\[
\lim_{s \to \infty} \text{diag}^{-1}\{U_s^{*}U_{s,k}e\}U_s^{*}U_{s,k} = \frac{e\pi_k}{\nu_k e}.
\]  

(3.33)

**Proof:** Let \((n_s)\alpha\) satisfy (3.3). It then follows from (2.35), (3.2) and Lemma 3.1 that, for all \(k \in \mathbb{Z}_+\) and \(s \in \mathbb{Z}_k\),

\[
(n_s)\mu_{s,k} = (0, \ldots, 0, (n_s)\alpha_s)(-\langle n_s \rangle Q)^{-1}\langle n_s \rangle E_{L,k}
\]

\[
= (n_s)\alpha_s\langle n_s \rangle X_{s,k}
\]

\[
= (n_s)\alpha_s(U_s^{*}U_{s,k}^{*}).
\]  

(3.34)

Note that, if (3.32) holds, the conditions of Theorem 2.3 are satisfied with \(n = n_s, \mathbb{B} = \mathbb{L}_k\) and \((n_s)\alpha = (n_s)\mu_{s,k}^{\top}, \nu = 1, 2, \ldots, m_s\). Therefore, the latter statement of the present corollary is true.

In what follows, we prove the former statement (3.31). Since \(Q\) is in upper block-Hessenberg form (3.6), the Markov chain \(\{Z(t)\}\) must go through \(\mathbb{L}_s\) in order to move from \(\bigcup_{k=s+1}^{\infty} \mathbb{L}_k\) to \(\bigcup_{k=0}^{s-1} \mathbb{L}_k\). Therefore, for each \(s \in \mathbb{N}\) and \(j \in \bigcup_{k=0}^{s-1} \mathbb{L}_k\), there exists at least one state \(i \in \mathbb{L}_s\) from
which the Markov chain \( \{Z(t)\} \) can reach state \( j \in \bigcup_{k=0}^{s-1} \mathbb{I}_k \) avoiding \( \bigcup_{k=s}^{\infty} \mathbb{I}_k \). This implies that, for each \( s \in \mathbb{N} \) and \( j \in \bigcup_{k=0}^{s-1} \mathbb{I}_k \),

\[
\left[ (-(n_s)Q)^{-1} \right]_{i,j} > 0 \quad \text{for some } i \in \mathbb{I}_s. \tag{3.35}
\]

We now fix \( (n_s)\alpha_s = e^\top/m_s \) in (3.3). It then follows from (3.34) and (3.35) that, for all \( k \in \mathbb{Z}_+ \) and \( s \in \mathbb{Z}_k \),

\[
(n_s)\mu_{s,k} = (0, \ldots, 0, e^\top/m_s)(-(n_s)Q)^{-1}(n_s)E_{L_k}
= (e^\top/m_s)U_s^*U_{s,k} > 0. \tag{3.36}
\]

From (3.36), Theorem 2.3 and \( \pi_{L_k} = \pi_k \), we obtain

\[
\lim_{s \to \infty} \frac{(0, \ldots, 0, e^\top)(-(n_s)Q)^{-1}(n_s)E_{L_k}}{(0, \ldots, 0, e^\top)(-(n_s)Q)^{-1}(n_s)E_{L_k}e} = \frac{\pi_k}{\pi_k e}, \quad k \in \mathbb{Z}_+,
\]

which shows that (3.31) holds.

The first limit formula (3.31) of Corollary 3.1 holds for the general ergodic generator \( Q \) in the upper block-Hessenberg form (3.6). On the other hand, the second one (3.33) requires the additional condition (3.32). A similar formula to (3.33) is presented by Takine [31] (see Theorem 3 therein):

\[
\lim_{s \to \infty} \text{diag}^{-1}\{U_{s,k}e\} \cdot U_{s,k} = \frac{e\pi_k}{\pi_k e}, \quad k \in \mathbb{Z}_+,
\]

under the assumption [31, Assumption 1] that, for all sufficiently large \( s \in \mathbb{N} \), the \( Q_{s,s-1} \)'s are nonsingular and the \( Q_{s,s} \)'s are of the same order.

The limit formula (3.37) does not always hold because it is possible that \( U_{s,k} \) has a row of zeros for infinitely many \( s \in \mathbb{N} \). Indeed, [31, Assumption 1] is a sufficient condition under which \( U_{s,k}e > 0 \) for all sufficiently large \( s \in \mathbb{N} \). Note here that [31, Assumption 1] implies (3.32). Note also that (3.37) can be derived from (3.33) (of course, if the former holds), as follows:

\[
\lim_{s \to \infty} \text{diag}^{-1}\{U_{s,k}e\} \cdot U_{s,k} = \lim_{s \to \infty} \text{diag}^{-1}\{U_{s,k}e\}(U_s^*)^{-1}\text{diag}\{U_s^*U_{s,k}e\}
\times \text{diag}^{-1}\{U_s^*U_{s,k}e\} \cdot U_s^*U_{s,k}
= \lim_{s \to \infty} \text{diag}^{-1}\{U_{s,k}e\} \cdot (U_s^*)^{-1}\text{diag}\{U_s^*U_{s,k}e\} \cdot \frac{e\pi_k}{\pi_k e}
= \lim_{s \to \infty} \text{diag}^{-1}\{U_{s,k}e\} \cdot U_{s,k}e \frac{\pi_k}{\pi_k e}
= \frac{e\pi_k}{\pi_k e},
\]

where the second last equality holds because

\[
(U_s^*)^{-1}\text{diag}\{U_s^*U_{s,k}e\}e = (U_s^*)^{-1}U_s^*U_{s,k}e = U_{s,k}e.
\]

Similarly, we can show that (3.37) leads to (3.31).
Based on (3.10) and (3.37), Takine [31] proposed an algorithm for computing \( \{ \pi_k; k \in \mathbb{Z}_N \} \) for a chosen sufficiently large \( N \) under the additional assumption mentioned above (Assumption 1 therein). The outline of the algorithm is as follows (for details, see [31, Section 3]): (i) Start with choosing \( N \in \mathbb{N} \) sufficiently large; (ii) compute

\[
x_{s,N} := m_s^{-1}e^\top \text{diag}^{-1}\{U_{s,N}e\} \cdot U_{s,N},
\]

for a sufficiently large \( s \in \mathbb{Z}_N \), where \( m_s \) is equal to the cardinality of \( L_s \); and (iii) compute

\[
x_{s,k} := x_{s,N}U_{N,k}, \quad k = N - 1, N - 2, \ldots, 0,
\]

and then

\[
x^{(N)}_{s,k} := \frac{x_{s,k}}{\sum_{\ell=0}^{N} x_{s,\ell}e} = \frac{x_{s,N}U_{N,k}}{\sum_{\ell=0}^{N} x_{s,N}U_{N,\ell}e}, \quad k \in \mathbb{Z}_N.
\]

This algorithm generates the probability vector \( x^{(N)}_{s,N} := (x^{(N)}_{s,0}, x^{(N)}_{s,1}, \ldots, x^{(N)}_{s,N}) \), which can be considered an approximation to \( \pi = (\pi_0, \pi_1, \ldots) \). Indeed, applying (3.37) to (3.38) yields

\[
\lim_{s \to \infty} x_{s,N} = \frac{\pi_N}{\pi_N e};
\]

and combining (3.40), (3.39) and (3.10) leads to

\[
\lim_{s \to \infty} x^{(N)}_{s,k} = \frac{\pi_N U_{N,k}}{\sum_{\ell=0}^{N} \pi_N U_{N,\ell}e} = \frac{\pi_k}{\sum_{\ell=0}^{N} \pi_\ell e}, \quad k \in \mathbb{Z}_N,
\]

which results in

\[
\lim_{N \to \infty} \lim_{s \to \infty} x^{(N)}_{s,k} = \lim_{N \to \infty} \frac{\pi_k}{\sum_{\ell=0}^{N} \pi_\ell e} = \pi_k, \quad k \in \mathbb{Z}_+.
\]

Recall that [31, Assumption 1] is required by the positivity of \( U_{s,N}e \) (i.e., \( U_{s,N}e > 0 \)) and thus by the definition (3.38) of \( x_{s,N} \). This additional assumption is not required by the definition of the following probability vector:

\[
\mu_{s,N} = \frac{e^\top U_s^e U_{s,N}e}{e^\top U_s^e U_{s,N}e}, \quad N \in \mathbb{Z}_+, \ s \in \mathbb{Z}_N.
\]

(3.41)

It follows from (3.41) and Corollary 3.1 that

\[
\lim_{s \to \infty} \mu_{s,N} = \frac{\pi_N}{\pi_N e}, \quad N \in \mathbb{Z}_+.
\]

Therefore, replacing \( x_{s,N} \) by \( \mu_{s,N} \), we can establish an alternative algorithm for computing an approximation to \( \pi \). Note here that Takine’s algorithm [31] and its alternative algorithm cannot start without input parameter \( N \). In fact, such a parameter \( N \) can be determined by using the \( f \)-modulated drift condition (for details, see [13] and [24, Section 14.2.1]).

**Condition 3.1 (\( f \)-modulated drift condition)** There exist some \( b \in (0, \infty) \), column vectors \( v := (v(i))_{i \in \mathbb{Z}_+} \geq 0 \) and \( f := (f(i))_{i \in \mathbb{Z}_+} \geq e \) and finite set \( \mathbb{C} \subset \mathbb{Z}_+ \) such that

\[
Qv \leq -f + b1_\mathbb{C}.
\]

(3.42)
**Remark 3.6** Assumption 2.1 and (2.23) are special cases of the $f$-modulated drift condition.

It is implied in [13, Theorem 1.1] that Condition 3.1 holds if and only if $Q$ is ergodic, provided $Q$ is irreducible. It also follows from (3.42) that $\pi f \leq b$ and thus

$$\pi(i) \leq \frac{b}{f(i)}, \quad i \in \mathbb{Z}_+. \quad (3.43)$$

We now define $N_\varepsilon, \varepsilon > 0$, as a positive integer such that

$$\sum_{k=N_\varepsilon+1}^{\infty} \sum_{i \in \mathbb{Z}_k} b \leq \frac{\varepsilon}{2}. \quad (3.44)$$

Combining (3.43) and (3.44) yields

$$\sum_{k=N_\varepsilon+1}^{\infty} \pi_k e = \sum_{k=N_\varepsilon+1}^{\infty} \sum_{i \in \mathbb{Z}_k} \pi(i) \leq \frac{\varepsilon}{2}. \quad (3.45)$$

We also define $\pi^{(N_\varepsilon)} := (\pi^{(N_\varepsilon)}_0, \pi^{(N_\varepsilon)}_1, \ldots, \pi^{(N_\varepsilon)}_{N_\varepsilon})$ as a probability vector such that

$$\pi^{(N_\varepsilon)}_k = \frac{\pi_k}{\sum_{\ell=0}^{N_\varepsilon} \pi_{\ell} e}, \quad k \in \mathbb{Z}_{N_\varepsilon}.$$ 

Using (3.45), we have

$$\text{abs}\{\pi - \pi^{(N_\varepsilon)}\} e = \sum_{k=0}^{N_\varepsilon} (\pi^{(N_\varepsilon)}_k - \pi_k) e + \sum_{k=N_\varepsilon+1}^{\infty} \pi_k e$$

$$= \sum_{k=0}^{N_\varepsilon} \left( \frac{\pi_k}{\sum_{\ell=0}^{N_\varepsilon} \pi_{\ell} e} - \pi_k \right) e + \sum_{k=N_\varepsilon+1}^{\infty} \pi_k e$$

$$= 1 - \sum_{k=0}^{N_\varepsilon} \pi_k e + \sum_{k=N_\varepsilon+1}^{\infty} \pi_k e$$

$$= 2 \sum_{k=N_\varepsilon+1}^{\infty} \pi_k e \leq \varepsilon, \quad (3.46)$$

which shows that $\pi$ is approximated by $\pi^{(N_\varepsilon)}$ within error $\varepsilon$ measured in terms of the total variation distance. Furthermore, from (3.46), we have

$$\text{abs}\{\pi - x^{(N_\varepsilon)}_s\} e \leq \text{abs}\{\pi - \pi^{(N_\varepsilon)}\} e + \text{abs}\{\pi^{(N_\varepsilon)} - x^{(N_\varepsilon)}_s\} e$$

$$\leq \varepsilon + \text{abs}\{\pi^{(N_\varepsilon)} - x^{(N_\varepsilon)}_s\} e.$$ 

Therefore, we can estimate the distance (error) of computable probability vector $x^{(N_\varepsilon)}_s$ from $\pi$ through the total variation distance $\text{abs}\{\pi^{(N_\varepsilon)} - x^{(N_\varepsilon)}_s\} e$, which depends on the difference $x_{s,N} - \pi_{N_\varepsilon}/\pi_{N_\varepsilon} e$ (see (3.39) and (3.40)). The estimation of this difference is discussed in [31, Section 3].

We now go back to Algorithm 1. Algorithm 1 is free from the problem of parameter $N$, though the convergence conditions are required (see Remark 3.4). Algorithm 1 has another advantage over Takine’s algorithm [31]. Recall here that Algorithm 1 generates a sequence of the
linear-augmented truncation approximations \((n_s)\pi\)'s. Therefore, we can obtain an upper bound for 
\(\text{abs}\{\pi - (n_s)\pi\}e\), following the studies [18, 19, 22, 20, 21, 23, 33] on the error estimation of the truncation approximation of Markov chains (the \(f\)-modulated drift condition plays an important role therein). Furthermore, using such an upper bound, we can establish sophisticated stopping criteria for Algorithm 1, which guarantee the accuracy of the resulting approximation to \(\pi\). The details of this topic are beyond the scope of this paper and thus are omitted here.

3.2 Lower block-Hessenberg Markov chain

We assume that the ergodic generator \(Q\) is in lower block-Hessenberg form (i.e., is of level-dependent GI/M/1-type):

\[
Q = \begin{pmatrix}
QL_0 & QL_1 & QL_2 & QL_3 & \cdots \\
QL_0 & QL_1 & QL_2 & O & \cdots \\
QL_1 & QL_2 & QL_2 & QL_3 & \cdots \\
QL_0 & QL_1 & QL_2 & QL_3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\tag{3.47}
\]

where \(Q_{k,\ell} = O\) for \(k \in \mathbb{Z}_+\) and \(\ell \in \mathbb{Z}_{k+1}\). We then have

\[
(n_s)Q = \begin{pmatrix}
Q_{0,0} & Q_{0,1} & O & \cdots & O & O & O \\
Q_{1,0} & Q_{1,1} & Q_{1,2} & \cdots & O & O & O \\
Q_{2,0} & Q_{2,1} & Q_{2,2} & \cdots & O & O & O \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
Q_{s-2,0} & Q_{s-2,1} & Q_{s-2,2} & \cdots & Q_{s-2,s-2} & Q_{s-2,s-1} & O \\
Q_{s-1,0} & Q_{s-1,1} & Q_{s-1,2} & \cdots & Q_{s-1,s-2} & Q_{s-1,s-1} & Q_{s-1,s} \\
Q_{s,0} & Q_{s,1} & Q_{s,2} & \cdots & Q_{s,s-2} & Q_{s,s-1} & Q_{s,s}
\end{pmatrix}.
\]

For \(s \in \mathbb{Z}_+\), we define \(\{(s)R^*_k; k \in \mathbb{Z}_s\}\) by the recursion: For \(k = s, s-1, \ldots, 0\),

\[
(s)R^*_k = \left(-Q_{k,k} - \sum_{\ell=k+1}^{s} (s)R_{k,\ell}Q_{\ell,k}\right)^{-1},
\tag{3.48}
\]

where \((s)R_{k,\ell}\)'s, \(k \in \mathbb{Z}_{s-1}, \ell = k+1, k+2, \ldots, s\), are given by

\[
(s)R_{k,\ell} = (Q_{k,k+1}(s)R^*_{k+1}) (Q_{k+1,k+2}(s)R^*_{k+2}) \cdots (Q_{\ell-1,\ell}(s)R^*_{\ell}).
\tag{3.49}
\]

For convenience, let \((s)R_{\ell,\ell} = I\) for \(\ell \in \mathbb{Z}_s\). We then have the following lemma, which is the counterpart of Lemma 3.1.

**Lemma 3.2** If the ergodic generator \(Q\) is in lower block-Hessenberg form (3.47), then

\[
(n_s)X_{0,\ell} = (s)R^*_{0}(s)R_{0,\ell}, \quad s \in \mathbb{Z}_+, \ell \in \mathbb{Z}_s.
\tag{3.50}
\]
We also partition \((n_s)\) \(Q\) by arranging the subsets \(\{I_k; k = 0, 1, \ldots, s\}\) of the state space \(Z_+\) in the descending order \(\{I_{s}, I_{s-1}, \ldots, I_{0}\}\). We denote the resulting matrix by \((n_s)\) \(\tilde{Q}\). Clearly, \((n_s)\) \(\tilde{Q}\) is in the same form as \((n_s)\) \(Q\) in (3.7), i.e., in the upper block-Hessenberg form:

\[
(n_s)\tilde{Q} = \begin{pmatrix}
Q_{s,s} & Q_{s,s-1} & Q_{s,s-2} & \cdots & Q_{s,2} & Q_{s,1} & Q_{s,0} \\
Q_{s-1,s} & Q_{s-1,s-1} & Q_{s-1,s-2} & \cdots & Q_{s-1,2} & Q_{s-1,1} & Q_{s-1,0} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
O & O & O & \cdots & Q_{1,2} & Q_{1,1} & Q_{1,0} \\
O & O & O & \cdots & O & Q_{0,1} & Q_{0,0}
\end{pmatrix}
\] (3.51)

We also partition \((- (n_s)\tilde{Q})^{-1}\) as

\[
(n_s)\tilde{Q}^{-1} = \begin{pmatrix}
\mathbb{I}_s & \mathbb{I}_{s-1} & \cdots & \mathbb{I}_0 \\
\mathbb{I}_{s-1} & \begin{pmatrix}
(n_s)Y_{0,0} & (n_s)Y_{0,1} & \cdots & (n_s)Y_{0,s}
\end{pmatrix} \\
\vdots & \vdots & \ddots & \vdots \\
\mathbb{I}_0 & \begin{pmatrix}
(n_s)Y_{s,0} & (n_s)Y_{s,1} & \cdots & (n_s)Y_{s,s}
\end{pmatrix}
\end{pmatrix}
\] (3.52)

It follows from (3.2), (3.51) and (3.52) that

\[
(n_s)X_{0,\ell} = (n_s)Y_{s,\ell-s}, \quad \ell \in Z_s.
\] (3.53)

For \(s \in Z_+\), we now define \(\{(s)\tilde{U}_k^*; k \in Z_s\}\) recursively as follows:

\[
(s)\tilde{U}_k^* = \left(-Q_{s-k,s-k} - \sum_{\ell=0}^{k-1} (s)\tilde{U}_{k,\ell}Q_{s-\ell,s-k}\right)^{-1}, \quad k \in Z_s,
\] (3.54)

where \(s)\tilde{U}_{k,\ell}\)'s, \(k = 1, 2, \ldots, s, \ell \in Z_{k-1}\), are given by

\[
(s)\tilde{U}_{k,\ell} = (Q_{s-k,s-k+1}(s)\tilde{U}_{k-1}^*) \\
\times (Q_{s-k+1,s-k+2}(s)\tilde{U}_{k-2}^*) \cdots (Q_{s-\ell-1,s-\ell}(s)\tilde{U}_{\ell}^*).
\] (3.55)

Note here that \((s)\tilde{U}_k^*\) and \(s)\tilde{U}_{k,\ell}\) are obtained by replacing, with \(Q_{s-k,s-\ell}, Q_{k,\ell}\) in (3.8) and (3.9), respectively. Thus, Lemma 3.1 implies that

\[
(n_s)Y_{s,\ell} = (s)\tilde{U}_s^*(s)\tilde{U}_{s,\ell}, \quad s \in Z_+, \ell \in Z_s,
\] (3.56)

where \((s)\tilde{U}_{\ell,\ell} = I\) for \(\ell \in Z_s\). Substituting (3.56) into (3.53), we have

\[
(n_s)X_{0,\ell} = (s)\tilde{U}_s^*(s)\tilde{U}_{s,\ell-s}, \quad s \in Z_+, \ell \in Z_s.
\] (3.57)

Note also that (3.54) yields

\[
(s)\tilde{U}_{s-k}^* = \left(-Q_{k,k} - \sum_{\ell=0}^{s-k-1} (s)\tilde{U}_{s-k,\ell}Q_{s-\ell,k}\right)^{-1}
\] (3.58)
Therefore it follows from (3.48), (3.49), (3.55) and (3.58) that

\[
(s)R_k^* = (s)U_{s-k}^*, \quad s \in \mathbb{Z}_+, \quad k \in \mathbb{Z}_s, \quad (3.59)
\]

\[
(s)R_{k,\ell} = (s)U_{s-k,s-\ell}, \quad s \in \mathbb{Z}_+, \quad k \in \mathbb{Z}_s, \quad \ell = k, k+1, \ldots, s. \quad (3.60)
\]

Combining (3.57), (3.59) and (3.60) leads to (3.50).

As with the upper block-Hessenberg case, we readily obtain the MIP-form solution of \(\pi = (\pi_0, \pi_1, \ldots)\) in the lower block-Hessenberg case. It follows from (3.2), Corollary 2.1 and Lemma 3.2 that

\[
(\pi_{\{0\}}, k) = \frac{e_F^T (s)R_0^* (s)R_{0,k}}{e_F^T \sum_{\ell=0}^s (s)R_{\ell}^* (s)R_{\ell}e}, \quad k \in \mathbb{Z}_s, \quad (3.61)
\]

where \(e_F\) denotes a finite column unit vector whose first element equals to 1. We now define \((s)R_k, k \in \mathbb{N}\), as

\[
(s)R_k = Q_{k-1,k}(s)R_k^*, \quad k \in \mathbb{N}. \quad (3.62)
\]

Substituting (3.62) into (3.49), we have, for \(s \in \mathbb{N}\) and \(k \in \mathbb{Z}_{s-1},

\[
(s)R_{k,\ell} = (s)R_{k+1}(s)R_{k+2} \cdots (s)R_{\ell}, \quad \ell = k+1, k+2, \ldots, s. \quad (3.63)
\]

Using (3.63), we rewrite (3.61) as

\[
(\pi_{\{0\}}, k) = \frac{e_F^T (s)R_0^* (s)R_1(s)R_2 \cdots (s)R_k}{e_F^T \sum_{\ell=0}^s (s)R_{\ell}^* (s)R_1(s)R_2 \cdots (s)R_{\ell}e}, \quad k \in \mathbb{Z}_+. \quad (3.64)
\]

Furthermore, Lemma 2.1 implies that if Assumption 2.1 holds then

\[
\lim_{s \to \infty} \| (\pi_{\{0\}} - \pi \| = 0.
\]

Combining this with (3.61) and (3.64) yields the MIP-form solution of \(\pi = (\pi_0, \pi_1, \ldots)\) in the lower block-Hessenberg case, which is summarized in the following theorem.

**Theorem 3.2** Suppose that the ergodic generator \(Q\) is in lower block-Hessenberg form (3.47). If Assumption 2.1 holds, then

\[
\pi_k = \lim_{s \to \infty} \frac{e_F^T (s)R_0^* (s)R_{0,k}}{e_F^T \sum_{\ell=0}^s (s)R_{\ell}^* (s)R_{\ell}e}, \quad k \in \mathbb{Z}_+, \quad (3.65)
\]

or equivalently,

\[
\pi_k = \lim_{s \to \infty} \frac{e_F^T (s)R_0^* (s)R_1(s)R_2 \cdots (s)R_k}{e_F^T \sum_{\ell=0}^s (s)R_{\ell}^* (s)R_1(s)R_2 \cdots (s)R_{\ell}e}, \quad k \in \mathbb{Z}_+. \quad (3.66)
\]

Using the MIP-form solution (3.65), we establish an algorithm that generates the sequence \(\{\pi_{\{0\}}; s \in \mathbb{Z}_+\}\) convergent to \(\pi\) in the lower block-Hessenberg case. The MIP-form solution (3.65) consists of \((s)R_0^*\) and \((s)R_{0,k} k = 1, 2, \ldots, s\). To obtain \((s)R_0^*\), we compute
we can compute \((s)R^*_s, (s)R^*_{s-1}, \ldots, (s)R^*_1\) by (3.48) and (3.49), where \((s)R^*_s = (-Q_{s,s})^{-1}\). Given the \((s)R^*_k\)'s, we can compute \(\{(s)R^*_{0,k}; k = 1, 2, \ldots, s\}\) by the recursion:

\[
(s)R^*_{0,k} = \begin{cases} 
I, & k = 0, \\
(s)R^*_{0,k-1}Q_{k-1,k}^* & k = 1, 2, \ldots, s,
\end{cases}
\]

which follows from (3.49). It should be noted that, for different values of \(s\), we have to independently compute the component matrices \((s)R^*_0\) and \((s)R^*_{0,k}\)'s of the MIP-form solution (3.65). This fact implies that the algorithm in the lower block-Hessenberg case (Algorithm 2 below) is less effective than Algorithm 1 in the upper block-Hessenberg case.

**Algorithm 2**: Lower block-Hessenberg Markov chain

**Input**: \(Q\) and \(\varepsilon \in (0, 1)\).

**Output**: \((n_s)\pi_{\{0\}} = ((n_s)\pi_{\{0\},0}; (n_s)\pi_{\{0\},1}, \ldots, (n_s)\pi_{\{0\},s})\), where \(s \in \mathbb{Z}_+\) is fixed when the iteration stops.

(i) Set \(s = 0\).

(ii) Compute \((0)R^*_0 = (-Q_{0,0})^{-1}\).

(iii) Compute \((n_0)\pi_{\{0\},0} = e_1^\top (0)R^*_0/(e_1^\top (0)R^*_0e)\).

(iv) Iterate the following:

(a) Increment \(s\) by one.

(b) For \(k = s, s-1, \ldots, 0\), compute \((s)R^*_k\) by (3.48) and (3.49).

(c) For \(k = 1, 2, \ldots, s\), compute \((n_s)\pi_{\{0\},k}\) by (3.66).

(d) Compute \(\{(n_s)\pi_{\{0\},k}; k \in \mathbb{Z}_s\}\) by (3.61).

(e) If \(\|(n_s)\pi_{\{0\}} - (n_{s-1})\pi_{\{0\}}\| < \varepsilon\), then stop the iteration; otherwise return to step (a).

**Remark 3.7** Algorithm 2 generates the sequence of the first-column-augmented truncation approximations \(\{(n_s)\pi_{\{0\}}; s \in \mathbb{Z}_+\}\). Therefore, Lemma 2.1 guarantees that Algorithm 2 stops after a finite number of iterations if Assumption 2.1 holds.

**Remark 3.8** Algorithm 2 increments the iteration index \(s\) one by one and thus generates the probability vector \((n_s)\pi_{\{0\}}\) of the smallest order that satisfies the stopping criterion \(\|(n_s)\pi_{\{0\}} - (n_{s-1})\pi_{\{0\}}\| < \varepsilon\). Unlike Algorithm 1, however, the iterations of Algorithm 2 are performed independently one another. More specifically, for each \(s \in \mathbb{Z}_+\), Algorithm 2 computes \(\{(s)R^*_k; k \in \mathbb{Z}_s\}\) from scratch and thus \(s + 1\) inverse matrices. Therefore, until the iteration index \(s\) reaches \(N \in \mathbb{N}\), Algorithm 2 computes \((N+1)(N+2)/2\) inverse matrices whereas Algorithm 1 computes \(N + 1\) inverse matrices (see Remark 3.5). To reduce the computational cost and accelerate the convergence of the resulting probability vectors, we can increment the iteration index \(s\) in such a way that \(s = s_0, s_1, \ldots\) where \(\{s_i; i \in \mathbb{Z}_+\}\) is an increasing and divergent sequence of nonnegative integers. A possible choice of \(\{s_i\}\) is that \(s_i = 2^i - 1\) for \(i \in \mathbb{Z}_+\). In this case, \(2^{i+1} - 1\) inverse matrices have been computed when the \(i\)-th iteration ends, i.e., when level \(2^i - 1\) is the maximum of levels involved in computing, which shows that the total number of inverse matrices computed increases linearly with the maximum level.
To the best of our knowledge, there are no previous studies on computing the stationary distribution vector of the lower block-Hessenberg Markov chain, except for Baumann and Sandmann’s work [3]. They proposed an algorithm for a special case of lower block-Hessenberg Markov chains, which is referred to as the level-dependent quasi-birth-and-death process (LD-QBD) with catastrophes therein. Their algorithm is very similar to the ones for ordinary LD-QBDs in [2, 26] and thus requires the maximum $N \in \mathbb{N}$ of levels involved in computing. When $N$ is given, Baumann and Sandmann’s algorithm [3] generates $N$ inverse matrices by the backward recursion (3.30) and computes the system of linear equations for $\pi_0$. Therefore, the computational complexity of their algorithm is of the same order as that of Algorithm 2 in the situation where the maximum level $N$ is determined by trial and error.

### 3.3 GI/M/1-type Markov chain

In this subsection, we consider the GI/M/1-type Markov chain. Since the GI/M/1-type Markov chain is a special case of lower block-Hessenberg Markov chains, the results presented in this subsection can be directly obtained from those in Section 3.2. However, as we will see later, we can establish an effective algorithm like Algorithm 1 for the upper block-Hessenberg Markov chain by using the special structure of the GI/M/1-type Markov chain. To achieve this, we utilize the results in Section 3.2 in an (apparently) indirect way.

We fix $s \in \mathbb{N}$ arbitrarily. We assume that the ergodic generator $Q$ in (3.47) is reduced to

$$Q = \begin{pmatrix} L_0 & B_0 & B_1 & O & O & \cdots \\ L_1 & B_0 & A_0 & A_1 & O & \cdots \\ L_2 & B_1 & A_0 & A_1 & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (3.67)$$

In this case, $(n_s) \bar{Q}$ in (3.51) is reduced to

$$(n_s) \bar{Q} = \begin{pmatrix} L_s & L_{s-1} & L_{s-2} & \cdots & L_2 & L_1 & L_0 \\ L_s & A_0 & A_1 & A_2 & \cdots & A_{s+1} & B_{s-1} \\ L_{s-1} & A_1 & A_0 & A_1 & \cdots & A_{s+2} & B_{s-2} \\ L_{s-2} & O & A_1 & A_0 & \cdots & A_{s+3} & B_{s-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (3.68)$$

Note here that $(n_s) \bar{Q}$ in (3.68) is equivalent to $(n_s) \bar{Q}$ in (3.51) with

$$Q_{k, \ell} = \begin{cases} A_{\ell-k}, & k, \ell \in \{1, 2, \ldots, s\}, \\ B_{\ell-k}, & k = 0 \text{ or } \ell = 0, \end{cases} \quad (3.69)$$
where \( \ell \leq k + 1 \). Substituting (3.69) into (3.54) yields, for \( s \in \mathbb{N} \),

\[
(s)\tilde{U}_s^* = \left( -B_0 - \sum_{\ell=0}^{s-1} (s)\tilde{U}_{s,\ell} B_{\ell-s} \right)^{-1},
\]

(3.70)

\[
(s)\tilde{U}_k^* = \left( -A_0 - \sum_{\ell=0}^{k-1} (s)\tilde{U}_{k,\ell} A_{k-\ell} \right)^{-1}, \quad k \in \mathbb{Z}_{s-1},
\]

(3.71)

Furthermore, substituting (3.69) into (3.55) yields the following: For \( s \in \mathbb{N} \),

\[
(s)\tilde{U}_{s,\ell} = (B_1(s)\tilde{U}_{s-1}^*)(s)\tilde{U}_{s-1,\ell}, \quad \ell \in \mathbb{Z}_{s-1},
\]

(3.72)

and, for \( k = 1, 2, \ldots, s - 1 \),

\[
(s)\tilde{U}_{k,\ell} = (A_1(s)\tilde{U}_{k-1}^*)(A_1(s)\tilde{U}_{k-2}^*) \cdots (A_1(s)\tilde{U}_{\ell}^*), \quad \ell \in \mathbb{Z}_{k-1}.
\]

(3.73)

Since \( \tilde{U}_0^* = (-A_0)^{-1} \), we can prove by induction that \((s)\tilde{U}_{k}^*\)'s in (3.71) and \((s)\tilde{U}_{k,\ell}^*\)'s in (3.73) are independent of \( s \). To utilize this fact, we introduce the notation:

\[
\tilde{U}_k^* = \left( -A_0 - \sum_{\ell=0}^{k-1} \tilde{U}_{k,\ell} A_{k-\ell} \right)^{-1}, \quad k \in \mathbb{Z}_{+},
\]

(3.74)

where \( \tilde{U}_{k,\ell}^* \)'s, \( k \in \mathbb{N}, \ell \in \mathbb{Z}_{k-1} \), are given by

\[
\tilde{U}_{k,\ell} = (A_1\tilde{U}_{k-1}^*)(A_1\tilde{U}_{k-2}^*) \cdots (A_1\tilde{U}_{\ell}^*).
\]

(3.75)

Using (3.74) and (3.75), we rewrite (3.70)–(3.72) as

\[
(s)\tilde{U}_s^* = \left( -B_0 - B_1 \sum_{\ell=0}^{s-1} \tilde{U}_{s,\ell} B_{\ell-s} \right)^{-1},
\]

(3.76)

\[
(s)\tilde{U}_k^* = \left( -A_0 - \sum_{\ell=0}^{k-1} \tilde{U}_{k,\ell} A_{k-\ell} \right)^{-1}, \quad k \in \mathbb{Z}_{s-1},
\]

\[
(s)\tilde{U}_{s,\ell} = B_1 \cdot \tilde{U}_{s-1}^* \tilde{U}_{s-1,\ell}, \quad \ell \in \mathbb{Z}_{s-1},
\]

(3.77)

where \( \tilde{U}_{\ell,\ell} = I \) for \( \ell \in \mathbb{Z}_{+} \). Note here that combining (3.61) with (3.59) and (3.60) leads to

\[
(n_s)\pi \{0\}, k = \frac{e_F^\top (n_s)\tilde{U}_s^* (n_s)\tilde{U}_{s-k}}{e_F^\top \sum_{\ell=0}^{s} (n_s)\tilde{U}_{s,\ell} e}, \quad k \in \mathbb{Z}_s.
\]

(3.78)

Substituting (3.77) into (3.78), we readily obtain, for \( s \in \mathbb{N} \),

\[
(n_s)\pi \{0\}, 0 = \frac{e_F^\top (n_s)\tilde{U}_s^*}{e_F^\top (n_s)\tilde{U}_s^* (e + B_1 \sum_{\ell=0}^{s-1} \tilde{U}_{s-1,\ell} e)},
\]

(3.79)

\[
(n_s)\pi \{0\}, k = \frac{e_F^\top (n_s)\tilde{U}_s^* B_1 \cdot \tilde{U}_{s-1}^* \tilde{U}_{s-1,\ell} e}{e_F^\top (n_s)\tilde{U}_s^* (e + B_1 \sum_{\ell=0}^{s-1} \tilde{U}_{s-1,\ell} e)}, \quad k = 1, 2, \ldots, s.
\]

(3.80)

In addition, Lemma 2.1 shows that \( \lim_{s \to \infty} \| (n_s)\pi - \pi \| = 0 \) and thus the following result holds.
**Theorem 3.3** If the ergodic generator $Q$ is given by (3.67), then

$$
\pi_0 = \lim_{s \to \infty} \frac{e_F^T(s) \tilde{U}_s^*}{e_F^T(s) \tilde{U}_s^* (e + B_1 \sum_{\ell=0}^{s-1} \tilde{U}_{s-1,\ell} e)},
$$

$$
\pi_k = \lim_{s \to \infty} \frac{e_F^T(s) \tilde{U}_s^* B_1 \cdot \tilde{U}_{s-1,\ell-k}}{e_F^T(s) \tilde{U}_s^* (e + B_1 \sum_{\ell=0}^{s-1} \tilde{U}_{s-1,\ell} e)}, \quad k \in \mathbb{N}.
$$

**Remark 3.9** The GI/M/1-type structure (3.67) of $Q$ implies that $\sup_{i \in \mathbb{Z}_+} |q(i, i)| < \infty$, which leads to $\sum_{i \in \mathbb{Z}_+} \pi(i) |q(i, i)| < \infty$, i.e., Assumption 2.1 holds (see Remark 2.1).

Using Theorem 3.3, we develop an algorithm for computing the stationary distribution vector of the GI/M/1-type Markov chain, which is performed in a similar way to Algorithm 1. For $k \in \mathbb{Z}_+$, let $\tilde{u}_k$ and $\tilde{U}_{k,\ell}$'s, $\ell \in \mathbb{Z}_k$, denote

$$
\tilde{u}_k = \sum_{\ell=0}^{k} \tilde{U}_{k,\ell} e, \quad k \in \mathbb{Z}_+,
$$

$$
\tilde{U}_{k,\ell} = \tilde{U}_k^{*} \tilde{U}_{k,\ell}, \quad k \in \mathbb{Z}_+, \ell \in \mathbb{Z}_k.
$$

Note that, since $\tilde{U}_k = I$ for $k \in \mathbb{Z}_+$, we have $\tilde{u}_0^* = \tilde{U}_0^* e = (-A_0)^{-1} e$ and $\tilde{U}_{k,k} = \tilde{U}_k^*$ for $k \in \mathbb{Z}_+$. Note also that (3.79) and (3.80) can be rewritten in terms of $\tilde{u}_k^*$, as follows:

$$
^{(n_s)} \pi \{0\}, 0 = \frac{e_F^T(s) \tilde{U}_s^*}{e_F^T(s) \tilde{U}_s^* (e + B_1 \tilde{u}_{s-1}^*)},
$$

$$
^{(n_s)} \pi \{0\}, k = \frac{e_F^T(s) \tilde{U}_s^* B_1 \tilde{U}_{s-1-k}}{e_F^T(s) \tilde{U}_s^* (e + B_1 \tilde{u}_{s-1}^*)}, \quad k = 1, 2, \ldots, s,
$$

where $^{(s)} \tilde{U}_s^*$ is given by (3.85) below (which follows from (3.76) and (3.82)):

$$
^{(s)} \tilde{U}_s^* = \left( -B_0 - B_1 \sum_{\ell=0}^{s-1} \tilde{U}_{s-1,\ell} B_{\ell-s} \right)^{-1}, \quad s \in \mathbb{N}.
$$

In what follows, we derive the recursion of $\{\tilde{u}_k^*\}$ and $\{\tilde{U}_{k,\ell}^*\}$. From (3.75) and (3.82), we obtain

$$
\tilde{U}_{k,\ell} = A_1 \tilde{U}_{k-1,\ell} = A_1 \tilde{U}_{k-1,\ell}, \quad k \in \mathbb{N}, \ell \in \mathbb{Z}_{k-1}.
$$

Applying (3.86) to (3.81), (3.82) and (3.74) yields

$$
\tilde{u}_k^* = \tilde{U}_k^* (e + A_1 \tilde{u}_k^*), \quad k \in \mathbb{N},
$$

$$
\tilde{U}_{k,\ell}^* = \tilde{U}_k^* A_1 \tilde{U}_{k-1,\ell}, \quad k \in \mathbb{N}, \ell \in \mathbb{Z}_{k-1},
$$

and

$$
\tilde{U}_k^* = \left( -A_0 - A_1 \sum_{\ell=0}^{k-1} \tilde{U}_{k-1,\ell} A_{\ell-k} \right)^{-1}, \quad k \in \mathbb{N},
$$

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**Limit Formulas for Normalized Fundamental Matrix**
respectively.

We are now ready to present the algorithm for the GI/M/1-type Markov chain, which is described in Algorithm 3 below.

Algorithm 3: GI/M/1-type Markov chain

**Input:** \( Q \) and \( \varepsilon \in (0, 1) \).

**Output:** \((n_s)\pi_0 = ((n_s)\pi_{0,0}, (n_s)\pi_{0,1}, \ldots, (n_s)\pi_{0,n_s})\), where \( s \in \mathbb{N} \) is fixed when the iteration stops.

(i) Set \( s = 0 \) and \((n_s)\pi_0 = 0\).

(ii) Compute \( \tilde{U}_0^* = \tilde{U}_{0,0} = (-A_0)^{-1}, \quad \tilde{u}_0^* = \tilde{U}_0^*e \).

(iii) Iterate the following:

(a) Increment \( s \) by one.

(b) Compute \((n_s)\tilde{U}_s^* \) by (3.85).

(c) Compute \{(n_s)\pi_{0,k}; k \in \mathbb{Z}_s\} by (3.83) and (3.84).

(d) If \( \| (n_s)\pi_0 - (n_{s-1})\pi_0 \| < \varepsilon \), then stop the iteration; otherwise go to step (iii.e).

(e) Compute \( \tilde{U}_s^* = \tilde{U}_{s,s}^* \), \( \tilde{u}_s^* \) and \{\( \tilde{U}_s^*; \ell \in \mathbb{Z}_{s-1}\} \) by (3.89), (3.87) and (3.88), where \( k = s \); and then return to step (iii.a).

A Proof of (2.45)

To avoid repeating the same phrase, fix \( m \in \mathbb{Z}_+ \) arbitrarily, and let \( n \in \mathbb{Z}_+ \setminus \mathbb{Z}_m \). It follows from (2.2), (2.41) and (2.42) that

\[
\lim_{n \to \infty} (n)Q_{\mathbb{Z}_m} = Q_{\mathbb{Z}_m}, \quad \lim_{n \to \infty} (n)Q_{\mathbb{Z}_m, \mathbb{Z}_m} = Q_{\mathbb{Z}_m, \mathbb{Z}_m}, \\
\lim_{n \to \infty} (n)\overrightarrow{Q}_{\mathbb{Z}_m} = Q_{\mathbb{Z}_m}, \quad \lim_{n \to \infty} (n)\overrightarrow{Q}_{\mathbb{Z}_m, \mathbb{Z}_m} = Q_{\mathbb{Z}_m, \mathbb{Z}_m}.
\]

According to these limits together with (2.43) and (2.44), it suffices to show that

\[
\lim_{n \to \infty} (- (n)Q_{\mathbb{Z}_m})^{-1} = (- Q_{\mathbb{Z}_m})^{-1}.
\]

Note that \( \mathbb{Z}_n \setminus \mathbb{Z}_m \nearrow \mathbb{Z}_+ \setminus \mathbb{Z}_m \) as \( n \to \infty \) and that \((n)Q_{\mathbb{Z}_m}\) is a principal submatrix of the \( Q\)-matrix \( Q_{\mathbb{Z}_m} \). Thus, we have (see [1, Chapter 2, Proposition 2.14]), for \( i, j \in \mathbb{Z}_+ \setminus \mathbb{Z}_m \) and \( t > 0 \),

\[
\left[ \exp \left\{ (n)Q_{\mathbb{Z}_m} t \right\} \right]_{i,j} \nearrow \left[ \exp \left\{ Q_{\mathbb{Z}_m} t \right\} \right]_{i,j} \quad \text{as} \ n \to \infty.
\]  

(A.1)

From (2.1), we also have

\[
[n)Q]_{i,j} = \left[ (n)Q_{\mathbb{Z}_m} \right]_{i,j} + \sum_{\ell=n+1}^{\infty} q(i, \ell) (n)\alpha(j), \quad i, j \in \mathbb{Z}_n \setminus \mathbb{Z}_m,
\]

(A.2)

which leads to

\[
\left[ \exp \left\{ (n)Q_{\mathbb{Z}_m} t \right\} \right]_{i,j} \geq \left[ \exp \left\{ (n)Q_{\mathbb{Z}_m} t \right\} \right]_{i,j}, \quad i, j \in \mathbb{Z}_n \setminus \mathbb{Z}_m, \ t > 0.
\]  

(A.3)
In addition, for $i, j \in \mathbb{Z}_n \setminus \mathbb{Z}_m$ and $t > 0$,

\[
\left[ \exp \left\{ (n)Q_{zm} t \right\} \right]_{i,j} = \mathbb{P} \left( Z(t) = j, \sup_{u \in [0,t]} Z(u) \leq n, \inf_{u \in [0,t]} Z(u) \geq m + 1 \mid Z(0) = i \right). \tag{A.4}
\]

We now define \((n)\delta_{zm}^{(t)}(i), i \in \mathbb{Z}_n \setminus \mathbb{Z}_m, t > 0\), as

\[
(n)\delta_{zm}^{(t)}(i) = \sum_{j=m+1}^{\infty} \left[ \exp \left\{ Q_{zm} t \right\} \right]_{i,j} - \sum_{j=m+1}^{n} \left[ \exp \left\{ (n)Q_{zm} t \right\} \right]_{i,j}
\]

\[
= \mathbb{P} \left( \sup_{u \in [0,t]} Z(u) \geq n + 1, \inf_{u \in [0,t]} Z(u) \geq m + 1 \mid Z(0) = i \right). \tag{A.5}
\]

It then follows from (A.2), (A.4), (A.5) and \(\sum_{j=m+1}^{n} (n)\alpha(j) \leq 1\) that, for $i, j \in \mathbb{Z}_n \setminus \mathbb{Z}_m$ and $t > 0$,

\[
\left[ \exp \left\{ (n)Q_{zm} t \right\} \right]_{i,j} \leq \left[ \exp \left\{ (n)Q_{zm} t \right\} \right]_{i,j} + (n)\delta_{zm}^{(t)}(i).
\]

Combining this and (A.3) yields, for $i, j \in \mathbb{Z}_n \setminus \mathbb{Z}_m$ and $t > 0$,

\[
\left[ \exp \left\{ (n)Q_{zm} t \right\} \right]_{i,j} \leq \left[ \exp \left\{ (n)Q_{zm} t \right\} \right]_{i,j} \leq \left[ \exp \left\{ (n)Q_{zm} t \right\} \right]_{i,j} + (n)\delta_{zm}^{(t)}(i). \tag{A.6}
\]

It also follows from (A.1) and (A.5) that, for $i \in \mathbb{Z}_+ \quad$ and $t > 0$,

\[
(n)\delta_{zm}^{(t)}(i) \searrow 0 \quad \text{as} \quad n \to \infty. \tag{A.7}
\]

Using (A.7) and the monotone convergence theorem, we have

\[
\int_{0}^{\infty} (m+1)\delta_{zm}^{(t)}(i) dt - \lim_{n \to \infty} \int_{0}^{\infty} (n)\delta_{zm}^{(t)}(i) dt
\]

\[
= \lim_{n \to \infty} \int_{0}^{\infty} \left\{ (m+1)\delta_{zm}^{(t)}(i) - (n)\delta_{zm}^{(t)}(i) \right\} dt
\]

\[
= \int_{0}^{\infty} (m+1)\delta_{zm}^{(t)}(i) dt,
\]

which yields

\[
\lim_{n \to \infty} \int_{0}^{\infty} (n)\delta_{zm}^{(t)}(i) dt = 0, \quad i \in \mathbb{Z}_+, \ t > 0. \tag{A.8}
\]

Furthermore, using (A.1), (A.6), (A.8) and the monotone convergence theorem, we obtain

\[
\lim_{n \to \infty} \int_{0}^{\infty} \exp\{ (n)Q_{zm} t \} dt = \lim_{n \to \infty} \int_{0}^{\infty} \exp\{ (n)Q_{zm} t \} dt = \int_{0}^{\infty} \exp\{ Q_{zm} t \} dt,
\]

which leads to

\[
\lim_{n \to \infty} (- (n)Q_{zm}^{-1}) = \lim_{n \to \infty} (- (n)Q_{zm}^{-1}) = (-Q_{zm}^{-1}).
\]

The proof is completed.
B Proof of (3.10)

It follows from (3.6) and (3.7) that the first \(k\) blocks of \(\pi Q = 0\) are given by

\[
(\pi_0, \pi_1, \ldots, \pi_{k-1}) (n_{k-1}) Q + (\pi_k, \pi_{k+1}, \ldots) \begin{pmatrix}
O & \cdots & O & Q_{k,k-1} \\
O & \cdots & O & O \\
\vdots & \ddots & \vdots & \vdots \\
O & \cdots & O & O
\end{pmatrix} = 0,
\]

and thus

\[
(\pi_0, \pi_1, \ldots, \pi_{k-1}) = \begin{pmatrix}
O & \cdots & O & Q_{k,k-1} \\
O & \cdots & O & O \\
\vdots & \ddots & \vdots & \vdots \\
O & \cdots & O & O
\end{pmatrix} \left( - (n_{k-1}) Q \right)^{-1}
\]

where the last equality follows from (3.2). Therefore,

\[
\pi_\ell = \pi_k Q_{k,k-1} (n_{k-1}) X_{k-1,\ell}, \quad \ell \in \mathbb{Z}_{k-1}.
\]

Applying Lemma 3.1 and (3.9) to the above equality, we have

\[
\pi_\ell = \pi_k U^*_k U_{k-1,\ell} = \pi_k U_{k,\ell}, \quad \ell \in \mathbb{Z}_{k-1},
\]

which shows that (3.10) holds.

C Discussion on matrix \(U^*_k\)

C.1 Nonsingularity of \(U^*_k\)

Let \(T^*_k, k \in \mathbb{Z}_+\), denote

\[
T^*_k = Q_{k,k} + \sum_{\ell=0}^{k-1} U_{k,\ell} Q_{\ell,k}, \quad k \in \mathbb{Z}_+.
\]

Substituting (C.1) into (3.8) yields \(U^*_k = (-T^*_k)^{-1}\) for \(k \in \mathbb{Z}_+\). Furthermore, (C.1) shows \(T^*_0 = Q_{0,0}\) since the empty sum is defined as zero. Note here that \(Q_{0,0}\) is the \((0,0)\)-th block, i.e., the zero-th diagonal block of the partitioned ergodic generator \(Q\) in (3.1), which implies that \(T^*_0 = Q_{0,0}\) is nonsingular. In what follows, we prove by induction the nonsingularity of \(T^*_k\) for \(k \in \mathbb{N}\).
We suppose that there exists some $k \in \mathbb{N}$ such that, for all $m \in \mathbb{Z}_{k-1}$, $T_m^*$ is nonsingular and thus $U_m^* = (-T_m^*)^{-1}$ is well-defined. We then partition $(n_k)Q$ as

$$
(n_k)Q = \begin{pmatrix}
(n_{k-1})Q & Q_{0,k} \\
(n_{k-1})Q & Q_{1,k} \\
& \vdots \\
(n_{k-1})Q & Q_{k-1,k} \\
O & O & \cdots & O & Q_{k-1,k} \end{pmatrix}.
$$

(C.2)

Since the generator $Q$ is ergodic, its diagonal blocks $Q_{k,k}$ and $(n_{k-1})Q$ are nonsingular. Furthermore, $(n_k)X_{k,k}$, i.e., the $(k,k)$-th block of $(-Q_{n,k}^{-1})$, is given by (see, e.g., [10, Section 0.7.3])

$$
(n_k)X_{k,k} = \begin{pmatrix}
-Q_{k,k} - (O, \ldots, O, Q_{k-1,k})(-Q_{n,k}^{-1}) \\
& \vdots \\
& \vdots \\
& Q_{k-1,k}
\end{pmatrix}^{-1}.
$$

(C.3)

where the second equality follows from (3.2); more specifically, the fact that the last block row of $(-Q_{n,k}^{-1})$ is equal to

$$
(n_{k-1})X_{k-1,0}, (n_{k-1})X_{k-1,1}, \ldots, (n_{k-1})X_{k-1,k-1}.
$$

Applying Lemma 3.1 to (C.3) and using (3.9) and (C.1) yields

$$
(n_k)X_{k,k} = \left(-Q_{k,k} - Q_{k,k-1} \left[ \sum_{\ell=0}^{k-1} (n_{k-1})X_{k-1,\ell} Q_{\ell,k} \right] \right)^{-1}.
$$

(C.4)

As a result, we have proved by induction that $T_k^*$ is nonsingular for all $k \in \mathbb{Z}_+$.

### C.2 Computation of $U_k^*$

In this subsection, we discuss the computation of $U_k^* = (-T_k^*)^{-1}$. We begin with the following lemma.

**Lemma C.1** For $k \in \mathbb{Z}_+$, the matrix $T_k^*$ is a $Q$-matrix, i.e., all the nondiagonal elements of $T_k^*$ are nonnegative and $T_k^*e \leq 0$. 
Proof. From (C.3), (C.4) and (3.2), we have

\[ T_k^* = (- (n_k) X_{k,k})^{-1} \]

\[ = Q_{k,k} + (O, \ldots, O, Q_{k,k-1})(- (n_{k-1}) Q)^{-1} \begin{pmatrix} Q_{0,k} \\ Q_{1,k} \\ \vdots \\ Q_{k-1,k} \end{pmatrix} \]  \hfill (C.5)

where \( Q_{k,k} \) is a \( Q \)-matrix and the second term of (C.5) is nonnegative. Therefore, it suffices to show \( T_k^* e \leq 0 \). It follows from (C.2) and \( (n_k) Q e \leq 0 \) that

\[ (n_k-1) Q e + \begin{pmatrix} Q_{0,k} \\ Q_{1,k} \\ \vdots \\ Q_{k-1,k} \end{pmatrix} e \leq 0, \]

which leads to

\[ (- (n_{k-1}) Q)^{-1} \begin{pmatrix} Q_{0,k} \\ Q_{1,k} \\ \vdots \\ Q_{k-1,k} \end{pmatrix} e \leq e. \]

Using this inequality and (C.5), we obtain

\[ T_k^* e = Q_{k,k} e + (O, \ldots, O, Q_{k,k-1})(- (n_{k-1}) Q)^{-1} \begin{pmatrix} Q_{0,k} \\ Q_{1,k} \\ \vdots \\ Q_{k-1,k} \end{pmatrix} e \]

\[ \leq Q_{k,k} e + Q_{k,k-1} e \leq \sum_{\ell=0}^{\infty} Q_{k,\ell} e = 0, \]

where the last inequality follows from \( \sum_{\ell \in \mathbb{Z}_+} Q_{k,\ell} e = 0 \) and \( \sum_{\ell \in \mathbb{Z}_+ \setminus \{k\}} Q_{k,\ell} e \geq 0 \). The statement of the present lemma has been proved. \( \square \)

We now define \( P_k^*, k \in \mathbb{Z}_+ \) as

\[ P_k^* = I + T_k^*/\theta_k, \]  \hfill (C.6)

where \( \theta_k \) denotes the maximum of the absolute values of the diagonal elements of \( T_k^* \). It follows from Lemma C.1 and the nonsingularity of \( T_k^* \) that \( P_k^* \) is strictly substochastic, i.e., \( P_k^* \geq O \), \( P_k^* e \leq e, \neq e \) and \( \text{sp}(P_k^*) < 1 \), where \( \text{sp}(P_k^*) \) denotes the spectral radius of \( P_k^* \). Thus, from (C.6), we have

\[ (- T_k^*)^{-1} = \theta_k^{-1} (I - P_k^*)^{-1} = \theta_k^{-1} \sum_{m=0}^{\infty} (P_k^*)^m \geq O, \neq O. \]  \hfill (C.7)

According to (C.7), we can obtain \((- T_k^*)^{-1}\) approximately by computing \( P_k^*, (P_k^*)^2, \ldots, (P_k^*)^M \) for sufficiently large \( M \in \mathbb{N} \) and summing them up. However, Le Boudec [16] proposed a more efficient algorithm for computing \((- T_k^*)^{-1}\), which is based on the following proposition.
Proposition C.1 ([16, Proposition 1]) Let \( \{V_n; n \in \mathbb{Z}_+\} \) and \( \{W_n; n \in \mathbb{Z}_+\} \) denote sequences of matrices such that

\[
V_n = \begin{cases} 
P^*_k, & n = 0, \\ (V_{n-1})^2, & n \in \mathbb{N}, \end{cases} \tag{C.8}
\]

\[
W_n = \begin{cases} 
I, & n = 0, \\
(I + V_{n-1})W_{n-1}, & n \in \mathbb{N}. \end{cases} \tag{C.9}
\]

It then holds that

\[
\lim_{n \to \infty} W_n = (I - P^*_k)^{-1}.
\]

It follows from (C.8) and (C.9) that \( W_n = \sum_{m=0}^{2^n-1} (P^*_k)^m \). Therefore, Le Boudec’s algorithm [16] logarithmically reduces the number of iterations for computing \( \sum_{m=0}^{M} (P^*_k)^m \).

Acknowledgments

The author thanks Mr. Masatoshi Kimura and Dr. Tetsuya Takine for their invaluable comments on the convergence of the limit formula (2.32). The author also thanks Dr. Tetsuya Takine for sharing an early version of [31]. In addition, the author acknowledges stimulating discussions on Algorithm 2 with Kazuya Fukuoka.

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