MULTIPLICATION FORMULAS FOR THE ELLIPTIC GAMMA FUNCTION

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Abstract. The elliptic gamma function is a generalization of the Euler gamma function. Its trigonometric and rational degenerations are the Jackson q-gamma function and the Euler gamma function. We prove multiplication formulas for the elliptic gamma function, whose degenerations are the Gauss-Askey multiplication formula for the Euler and trigonometric gamma functions.

1. Introduction

Special functions defined by infinite products often have duplication formulas. Here are some examples.

\[
\sin (2\pi z) = 2 \sin (\pi z) \sin (\pi (z + \frac{1}{2})) ,
\]

\[
\Gamma(2z) \sqrt{\pi} = 2^{2z-1} \Gamma(z) \Gamma(z + \frac{1}{2}) ,
\]

\[
\Gamma_q(2z) \Gamma_q^2(\frac{1}{2}) = [2]_q^{2z-1} \Gamma_q(z) \Gamma_q^2(z + \frac{1}{2}) ,
\]

\[
\theta_0(2z, \tau) = \theta_0(z, \tau) \theta_0(z + \frac{1}{2}, \tau) \theta_0(z + \frac{\tau}{2}, \tau) \theta_0(z + \frac{1+\tau}{2}, \tau) .
\]

The function \( \Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt \) is the Euler gamma function. It satisfies the functional equation \( \Gamma(z+1) = z \Gamma(z) \). Formula (1) is Legendre’s duplication formula.

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The function $\Gamma_q(z)$ is Jackson’s $q$-gamma function. Set $x = e^{2\pi i z}$, $q = e^{2\pi i \tau}$, and denote

$$(x; q) = \prod_{j=0}^{\infty} (1 - x q^j).$$

Then

$$\Gamma_q(z) = \Gamma \text{trig}(z, \tau) = (1 - q) \frac{(q; q)}{(q^z; q)} .$$

The $q$-gamma function obeys the functional equation

$$\Gamma_q(z + 1) = [z]_q \Gamma_q(z),$$

where $[z]_q = \frac{1 - e^{2\pi i \tau z}}{1 - e^{2\pi i \tau}}$ is the trigonometric analog of the number $z$. The $q$-gamma function degenerates to Euler’s gamma function,

$$\lim_{\tau \to 0} \Gamma \text{trig}(z, \tau) = \Gamma(z).$$

Formula (2) is Askey’s duplication formula, see [A].

The function $\theta_0(z, \tau) = (x, q)(q/x, q)$ in (3) is one of Jacobi’s theta functions. Formula (3) see for instance in [Ra].

In this paper we give two duplication formulas for elliptic analogs of the gamma function,

$$\Gamma(2z, \tau, \sigma) = \Gamma(z, \tau, \sigma)\Gamma(z + \frac{\tau}{2}, \tau, \sigma)\Gamma(z + \frac{\sigma}{2}, \tau, \sigma)\Gamma(z + \frac{\tau + \sigma}{2}, \tau, \sigma)$$

$$\Gamma(z + \frac{1}{2}, \tau, \sigma)\Gamma(z + 1 + \frac{\tau}{2}, \tau, \sigma)\Gamma(z + 1 + \frac{\sigma}{2}, \tau, \sigma)\Gamma(z + 1 + \frac{\tau + \sigma}{2}, \tau, \sigma),$$

$$\Gamma(z, \tau, \sigma)\Gamma(z + \frac{1}{2}, 2\tau, \sigma) = \left(\frac{\theta_0(2\tau, \sigma)}{\theta_0(\tau, \sigma)}\right)^{2z-1} \Gamma(z, 2\tau, \sigma)\Gamma(z + \frac{1}{2}, 2\tau, \sigma),$$

see definitions below. The expression $\frac{\theta_0(z\tau, \sigma)}{\theta_0(\tau, \sigma)}$ is an elliptic analog of the number $z$. We have the trigonometric and rational limits of the theta function:

$$\frac{\theta_0(z\tau, \sigma)}{\theta_0(\tau, \sigma)} \xrightarrow{\sigma \to i\infty} \frac{1 - e^{2\pi i \tau z}}{1 - e^{2\pi i \tau}} \xrightarrow{\tau \to 0} z .$$

2. **Elliptic Gamma Function**

The *elliptic gamma function* is an elliptic generalization of the Euler gamma function. It is the meromorphic function of three complex variables $z, \tau, \sigma$, with $\text{Im} \, \tau, \text{Im} \, \sigma > 0$ defined by the convergent infinite product

$$\Gamma(z, \tau, \sigma) = \prod_{j,k=0}^{\infty} \frac{1 - e^{2\pi i ((j+1)\tau + (k+1)\sigma - z)}}{1 - e^{2\pi i (j\tau + k\sigma + z)}} .$$
It is the unique solution of a functional equation involving the Jacobi theta function $\theta_0$.

**Theorem.** [FV1] Suppose that $\tau, \sigma$ are complex numbers with positive imaginary part. Then $u(z) = \Gamma(z, \tau, \sigma)$ is the unique meromorphic solution of the difference equation

$$u(z + \sigma) = \theta_0(z, \tau) u(z)$$

such that:

(i) $u(z)$ obeys $u(z + 1) = u(z)$ and is holomorphic on the upper half plane $\text{Im} \, z > 0$,

(ii) $u((\tau + \sigma)/2) = 1$.

The elliptic gamma function first appeared in [R]. The modular properties of the elliptic gamma function and their relations to $SL(3, \mathbb{Z})$ are discussed in [FV1], appearances and applications of the elliptic gamma function can be found in [B, DP, JMN, JKKMW, FTV, FV2, FV3, FV4].

Let $\bar{\Gamma}$ be the function

$$\bar{\Gamma}(z, \tau, \sigma) = (q; q) \theta_0(\tau, \sigma)^{1-z} \Gamma(z\tau, \tau, \sigma), \quad q = e^{2\pi i\tau}, \quad r = e^{2\pi i\sigma}.$$

Then $u(z) = \bar{\Gamma}(z, \tau, \sigma)$ is a solution of the functional equation

$$u(z + 1) = \frac{\theta_0(\tau z, \sigma)}{\theta_0(\tau, \sigma)} u(z).$$

The normalization was chosen here so that $u(1) = 1$. As $\sigma \to i\infty$ we recover Jackson’s $q$-gamma function,

$$\Gamma_{\text{trig}}(z, \tau) = \lim_{\sigma \to i\infty} \bar{\Gamma}(z, \tau, \sigma).$$

3. Multiplication Formulas

3.1. The first multiplication formula.

**Theorem.** For any natural $n$ we have

$$\Gamma(nz, \tau, \sigma) = \prod_{k_1, k_2, k_3 = 0}^{n-1} \Gamma(z + \frac{k_1 + k_2\tau + k_3\sigma}{n}, \tau, \sigma).$$
Proof. Let \( w = e^{2\pi i/n} \). Then the right hand side of this formula is

\[
\prod_{k_1,k_2,k_3=0}^{n-1} \prod_{l,m=0}^{\infty} \frac{1 - w^{-k_1}q^{l+\frac{n-k_3}{n}}r^{m+\frac{n-k_3}{n}}x^{-1}}{1 - w^{k_1}q^{l+\frac{k_3}{n}}r^{m+\frac{k_3}{n}}x} = \prod_{l,m=0}^{\infty} \frac{1 - q^{l+1}r^{m+1}x^{-n}}{1 - q^{n}r^{n}x^{-n}} = \Gamma(nz, \tau, \sigma).
\]

\[\square\]

3.2. The second multiplication formula.

Theorem. For any natural \( n \) we have

\[
\tilde{\Gamma}(nz, \tau, \sigma) \tilde{\Gamma}(\frac{1}{n}, n\tau, \sigma) \tilde{\Gamma}(\frac{2}{n}, n\tau, \sigma) \ldots \tilde{\Gamma}(\frac{n-1}{n}, n\tau, \sigma) = \left(\frac{\theta_0(n\tau, \sigma)}{\theta_0(\tau, \sigma)}\right)^{n-1} \Gamma(z, n\tau, \sigma) \Gamma(z + \frac{1}{n}, n\tau, \sigma) \Gamma(z + \frac{2}{n}, n\tau, \sigma) \ldots \Gamma(z + \frac{n-1}{n}, n\tau, \sigma) .
\]

The theorem is an easy consequence of the following two lemmas.

Lemma. For any natural \( m \) and \( n \), we have

\[
\Gamma(z, \tau, \sigma) = \prod_{a=0}^{m-1} \prod_{b=0}^{n-1} \Gamma(z + a\tau + b\sigma, m\tau, n\sigma) .
\]

Lemma. For any natural \( n \), we have

\[
\Gamma(\tau, n\tau, \sigma) \Gamma(2\tau, n\tau, \sigma) \ldots \Gamma((n-1)\tau, n\tau, \sigma) = \frac{1}{(q, q^n)(q^2, q^n)\ldots(q^{n-1}, q^n)} .
\]

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