Nikishin systems on star-like sets: algebraic properties and weak asymptotics of the associated multiple orthogonal polynomials

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Abstract. We investigate polynomials \( Q_n(z) \), \( n = 0, 1, \ldots \), that are multi-orthogonal with respect to a Nikishin system of \( p \geq 1 \) compactly supported measures over the star-like set of \( p + 1 \) rays \( S_+ := \{ z \in \mathbb{C} : z^{p+1} > 0 \} \). We prove that the Nikishin system is normal, that the polynomials satisfy a three-term recurrence relation of order \( p + 1 \) of the form \( zQ_n(z) = Q_{n+1}(z) + a_n Q_{n-p}(z) \) with \( a_n > 0 \) for all \( n \geq p \), and that the nonzero roots of \( Q_n \) are all simple and located in \( S_+ \). Under the assumption that the measures generating the Nikishin system are regular (in the sense of Stahl and Totik), we describe the asymptotic zero distribution and weak behaviour of the polynomials \( Q_n \) in terms of a vector equilibrium problem for logarithmic potentials. Under the same regularity assumptions, we prove a theorem on the convergence of the Hermite-Padé approximants to the Nikishin system of Cauchy transforms.

Bibliography: 16 titles.

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§ 1. Introduction

This work is motivated by the studies [2], [1] and [3] on sequences of polynomials \( \{Q_n\}_{n=0}^\infty \) satisfying a recurrence relation of the form

\[
zQ_n(z) = Q_{n+1}(z) + a_n Q_{n-p}(z), \quad a_n > 0, \quad n \geq p, \quad (1.1)
\]

where \( p \) is a fixed positive integer.

Some well-known families of polynomials satisfy this type of recurrence relation with the coefficients \( a_n \) all being equal to some constant \( a \). For instance, when \( p = 1 \) and \( a_n = 1 \) for all \( n \geq 1 \), the polynomials \( Q_n \) resulting from the pairs of initial conditions \( Q_0(z) = 2 \), \( Q_1(z) = z \) and \( Q_0(z) = 1 \), \( Q_1(z) = z \), are, respectively, the Chebyshev polynomials of the first and second kind for the interval \([-2, 2]\). As a way of generalizing the Chebyshev polynomials of the first kind, in (1.1) one can

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set \( a_n = 1/p, n \geq p \), and \( Q_0(z) = p + 1, Q_\ell = z^\ell, \ell = 1, \ldots, p \), which generates the sequence of Faber polynomials associated with a hypocycloid of \( p + 1 \) cusps. Many interesting properties of these Faber polynomials were established in \([8]\). For instance, their zeros are all located in the star-like set of \( p + 1 \) rays

\[
S_+ := \{ z \in \mathbb{C}: z^{p+1} > 0 \};
\]

more precisely, they are contained, interlace, and form a dense subset of \( \{ z \in S_+: |z| < (p + 1)/p \} \).

It was proved in \([2]\) that with the initial conditions

\[
Q_\ell(z) = z^\ell, \quad 0 \leq \ell \leq p,
\]

the polynomials generated by (1.1) are in fact multi-orthogonal (in the same non-Hermitian sense of Definition 2.3 below) with respect to a system of \( p \) complex measures \( \mu_1, \ldots, \mu_p \) supported on \( S_+ \). These measures can be viewed as spectral measures (see \([2]\) and \([1]\)) of the difference operator given, in the standard basis of the Hilbert space \( l^2(\mathbb{N}) \), by the infinite \((p + 2)\)-banded Hessenberg matrix

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \ldots & \ldots \\
0 & 0 & 1 & 0 & 0 & \ldots & \ldots \\
0 & 0 & 0 & 1 & 0 & \ldots & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
ap_p & 0 & 0 & 0 & 0 & \ldots & \ldots \\
0 & ap_p+1 & 0 & 0 & 0 & \ldots & \ldots \\
0 & 0 & ap_p+2 & 0 & 0 & \ldots & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\end{pmatrix}
\]

Their Cauchy transforms

\[
\int_{S_+} \frac{d\mu_j(t)}{z-t}, \quad 1 \leq j \leq p,
\]

are the resolvents or Weyl functions of the operator. We remark that the spectral measures are of the form \( d\mu_j(t) = t^{j-1} dt \nu_j(t^{-p+1}), j = 1, \ldots, p \), where \( \nu_j \) is a positive measure supported on \( S_+^{p+1} = \mathbb{R}_+ \). Hence \( \mu_1 \) is rotationally invariant, and the rest are rotationally invariant up to a monomial factor.

The \( l^1 \) perturbation of the constant coefficient case

\[
\sum_{n=p}^{\infty} |a_n - a| < \infty, \quad a > 0,
\]

was investigated in \([1]\). There, the strong asymptotics of the polynomials \( Q_n \) determined by (1.1), (1.2) and (1.4), as well as properties of the measures \( \mu_j \), were derived. For instance, it was proved that these spectral measures are absolutely continuous, and a formal connection of these measures with a Nikishin-type system was obtained. In \([3]\), this connection was explicitly established in the case of periodic recurrence coefficients (see §2.4 of \([3]\)), and many algebraic and asymptotic properties of the Riemann-Hilbert minors associated with the polynomials \( Q_n \) satisfying (1.1) and (1.2) were given.
Motivated by these results, in this paper we investigate polynomials $Q_n$ that are multi-orthogonal with respect to a Nikishin system of $p$ measures (defined in analogy to the classical sense) supported over the star-like set $S_+$. As we will see in §3 below, such $Q_n$’s happen to satisfy the recurrence relation (1.1). Our goal is to understand how the properties of the measures generating the Nikishin system affect the multi-orthogonal polynomials $Q_n$ and the recurrence coefficients $a_n$, in particular, what their asymptotic behaviour is as $n \to \infty$. Thus, in the context of inverse spectral problems, our investigation sheds some light on the properties of the operator (1.3).

Nikishin systems of functions (the Cauchy transforms of a Nikishin system of measures on intervals of the real line) were first introduced in [11] as the first wide class of functions possessing convergent Hermite-Padé approximants. While in his original paper [11] Nikishin proved this convergence only for a system of two measures and diagonal multi-indices, great progress has been made since then for any number of intervals and arbitrary multi-indices (see, for instance, [5]). Our work can also be viewed within the context of rational approximation as a generalization of Nikishin systems from real intervals to star-like sets.

The content of this paper is organized into five sections. Sections 2 and 3 are, for the most part, of an algebraic nature, and they have been structured linearly, so as to have any result needed for a given topic stated and proved beforehand. The Nikishin system and other related hierarchies of measures, together with the multi-orthogonal polynomials and their associated functions of the second kind, are introduced in §2. Among the many relations and properties proved in that section we have included the normality of the Nikishin system and the location of the zeros of the multi-orthogonal polynomials and of the functions of the second kind. In §3 we prove the recurrence relation (1.1) for the multi-orthogonal polynomials, including the (nontrivial) positivity of the recurrence coefficients. In §5 we describe the asymptotic zero distribution and weak behaviour of the polynomials $Q_n$ in terms of a vector equilibrium problem for logarithmic potentials, under the assumption that the measures generating the Nikishin system are regular in the sense of Stahl and Totik. A weak convergence theorem for the coefficients of the recurrence relation is also obtained. Finally, in §6, and under the same regularity assumptions, a theorem on the convergence of the Hermite-Padé approximants to the Nikishin system of Cauchy transforms is proved.

Many of the results in this paper were already obtained in [9] for Nikishin systems of $p = 2$ measures on a star-like set of three rays. The case when $p \geq 3$ is technically much more difficult, with many subtleties that do not appear when $p = 2$.

§2. Nikishin systems on stars

2.1. Definition and basic properties of the Nikishin system. Let $p \geq 1$ be an integer, and let

$$S_\pm := \{z \in \mathbb{C} : z^{p+1} \in \mathbb{R}_\pm\}, \quad \mathbb{R}_+ = [0, +\infty), \quad \mathbb{R}_- = (-\infty, 0].$$

Then

$$S_- = e^{\pi i/(p+1)} S_+.$$
We construct \( p \) finite stars contained in \( S_\pm \) as follows:

\[
\Gamma_j := \{ z \in \mathbb{C} : z^{p+1} \in [a_j, b_j] \}, \quad 0 \leq j \leq p - 1,
\]

where

\[
0 \leq a_j < b_j < \infty, \quad j \equiv 0 \mod 2,
\]

\[
-\infty < a_j < b_j \leq 0, \quad j \equiv 1 \mod 2,
\]

so that \( \Gamma_j \subset S_+ \) if \( j \) is even, and \( \Gamma_j \subset S_- \) if \( j \) is odd. We assume throughout that \( \Gamma_j \cap \Gamma_{j+1} = \emptyset \) for all \( 0 \leq j \leq p - 2 \), that is, no two consecutive stars meet at the origin.

We now define a Nikishin system on \((\Gamma_0, \ldots, \Gamma_{p-1})\). For each \( 0 \leq j \leq p - 1 \), let \( \sigma_j \) denote a positive, rotationally invariant (over the angle \( 2\pi/(p+1) \)) measure on \( \Gamma_j \), with infinitely many points in its support. These will be the measures generating the Nikishin system.

Let

\[
\tilde{\mu}(x) := \int \frac{d\mu(t)}{x - t}
\]

denote the Cauchy transform of a complex measure \( \mu \), and let \( \mu_1, \ldots, \mu_N \) be \( N \geq 1 \) measures such that \( \mu_j \) and \( \mu_{j+1} \) have disjoint supports for every \( 1 \leq j \leq N - 1 \). We define the measure \( \langle \mu_1, \ldots, \mu_N \rangle \) by the following recursive procedure. For \( N = 1 \), \( \langle \mu_1 \rangle := \mu_1 \), for \( N = 2 \),

\[
d\langle \mu_1, \mu_2 \rangle(x) := \tilde{\mu}_2(x) d\mu_1(x),
\]

and for \( N > 2 \),

\[
\langle \mu_1, \ldots, \mu_N \rangle := \langle \mu_1, \langle \mu_2, \ldots, \mu_N \rangle \rangle.
\]

We then define the Nikishin system \((s_0, \ldots, s_{p-1}) = \mathcal{N}(\sigma_0, \ldots, \sigma_{p-1})\) generated by the vector of \( p \) measures \((\sigma_0, \ldots, \sigma_{p-1})\) by setting

\[
s_j := \langle \sigma_0, \ldots, \sigma_j \rangle, \quad 0 \leq j \leq p - 1.
\]

Notice that these measures \( s_j \) are supported on the first star \( \Gamma_0 \).

It is convenient, however, to think of this Nikishin system as the first row of the following hierarchy of measures \( s_{k,j} \),

\[
s_{0,0} \quad s_{0,1} \quad s_{0,2} \ldots \quad s_{0,p-1} \\
s_{1,1} \quad s_{1,2} \ldots \quad s_{1,p-1} \\
s_{2,2} \ldots \quad s_{2,p-1} \\
\vdots \quad \vdots \\
s_{p-1,p-1}
\]

where

\[
s_{k,j} = \langle \sigma_k, \ldots, \sigma_j \rangle, \quad 0 \leq k \leq j \leq p - 1.
\]

More descriptively, the measures \( s_{k,j} \) are inductively defined by setting

\[
s_{k,k} := \sigma_k, \quad 0 \leq k \leq p - 1,
\]

\[
\begin{aligned}
ds_{k,j}(z) &= \int_{\Gamma_{k+1}} \frac{ds_{k+1,j}(t)}{z - t} d\sigma_k(z), \quad 0 \leq k < j \leq p - 1.
\end{aligned}
\]
Notice that then for each pair \( k \) and \( j \) with \( 0 \leq k \leq j \leq p - 1 \), \((s_{k,k}, \ldots, s_{k,j}) = \mathcal{N}((\sigma_k, \ldots, \sigma_j))\) is the Nikishin system generated by \((\sigma_k, \ldots, \sigma_j)\).

Throughout the paper we will use the notation
\[
\omega := e^{2\pi i/(p+1)}.
\]
The following proposition summarizes several basic properties that will be needed later.

**Proposition 2.1.** For every \( 0 \leq k \leq j \leq p - 1 \), the measure \( s_{k,j} \) satisfies the symmetry property
\[
ds_{k,j}(\omega z) = \omega^{k-j} ds_{k,j}(z). \tag{2.5}
\]
Also, for every integrable \( f \) on \( \Gamma_k \), we have
\[
\int_{\Gamma_k} f(\omega z) ds_{k,j}(z) = \omega^{j-k} \int_{\Gamma_k} f(z) ds_{k,j}(z) \tag{2.6}
\]
and
\[
\int_{\Gamma_k} \overline{f(z)} ds_{k,j}(z) = \int_{\Gamma_k} f(z) ds_{k,j}(z). \tag{2.7}
\]

**Proof.** Relation (2.5) holds trivially for \( k = p - 1 \), and for \( 0 \leq k < p - 1 \) it is proved by reverse induction on \( k \). Formula (2.6) follows immediately from (2.5). Relation (2.7) is also proved by reverse induction on \( k \), using (2.4) and the fact that, due to its rotational invariance, \( d\sigma_k(t) = d\sigma_k(\overline{t}) \).

For every \( 0 \leq j \leq p - 1 \), we shall denote the push-forward of \( \sigma_j \) under the map \( z \mapsto z^{p+1} \) by \( \sigma_j^* \), that is, \( \sigma_j^* \) is the measure on \([a_j, b_j]\) such that for every Borel set \( E \subset [a_j, b_j] \),
\[
\sigma_j^*(E) := \sigma_j(\{z: z^{p+1} \in E\}). \tag{2.8}
\]

We now construct, out of these \( \sigma_j^* \), a new hierarchy of measures \( \mu_{k,j} \), \( 0 \leq k \leq j \leq p - 1 \):
\[
\begin{align*}
\mu_{0,0} & \quad \mu_{0,1} & \quad \mu_{0,2} & \quad \cdots & \quad \mu_{0,p-1} \\
\mu_{1,1} & \quad \mu_{1,2} & \quad \cdots & \quad \mu_{1,p-1} \\
\mu_{2,2} & \quad \cdots & \quad \mu_{2,p-1} \\
\vdots & \quad \vdots & \quad \vdots & \quad \vdots \\
\mu_{p-1,p-1}
\end{align*}
\tag{2.9}
\]
where the measures \( \mu_{k,j} \) are inductively defined by setting
\[
\mu_{k,k} := \sigma_k^*, \quad 0 \leq k \leq p - 1,
\]
\[
d\mu_{k,j}(\tau) = \left(\tau \int_{a_k}^{b_i+1} \frac{d\mu_{k+1,j}(s)}{\tau - s} \right) d\sigma_k^*(\tau), \quad \tau \in [a_k, b_k], \quad 0 \leq k < j \leq p - 1. \tag{2.10}
\]

In the following result we describe the relationship between the measures \( \mu_{k,j} \) and \( s_{k,j} \).
Proposition 2.2. For every \(0 \leq k \leq j \leq p - 1\), we have
\[
\int_{\Gamma_k} \frac{ds_{k,j}(t)}{z - t} = z^{p + k - j} \int_{a_k}^{b_k} \frac{d\mu_{k,j}(\tau)}{z^{p+1} - \tau},
\]
that is,
\[
\hat{s}_{k,j}(z) = z^{p + k - j} \hat{\mu}_{k,j}(z^{p+1}). \tag{2.11}
\]
Hence, for every continuous function \(f\) on \([a_k, b_k]\),
\[
\int_{a_k}^{b_k} f(\tau) \, d\mu_{k,j}(\tau) = \int_{\Gamma_k} f(z^{p+1}) z^{j - k} \, ds_{k,j}(z). \tag{2.12}
\]

Proof. Using (2.5) we find that
\[
\int_{\Gamma_k} \frac{t^l \, ds_{k,j}(t)}{z - t^{p+1}} = 0 \quad \text{when} \quad l \not\equiv j - k \mod(p + 1).
\]
Hence, it follows that
\[
\int_{\Gamma_k} \frac{ds_{k,j}(t)}{z - t} = z^{p + k - j} \int_{\Gamma_k} t^{j - k} \, ds_{k,j}(t). \tag{2.13}
\]
This proves (2.11) for \(k = j\). If (2.11) holds for some value of \(k \in \{0, \ldots, j\}\), then using (2.13), (2.10) and (2.8) yields that (2.11) is also true for \(k - 1\). The proof of (2.12) also follows by backward induction on \(k\).

2.2. Multiple orthogonal polynomials and functions of the second kind.

Definition 2.3. Let \(\{Q_n(z)\}_{n=0}^{\infty}\) be the sequence of monic polynomials of lowest degree that satisfy the following non-Hermitian orthogonality conditions:
\[
\int_{\Gamma_0} Q_n(z) z^l \, ds_j(z) = 0, \quad l = 0, \ldots, \left\lfloor \frac{n - j - 1}{p} \right\rfloor, \quad 0 \leq j \leq p - 1. \tag{2.14}
\]
In what follows we will use the notation
\[
d_n := \deg(Q_n), \quad n \geq 0.
\]
Using (2.6) and (2.7), it is easy to see that the polynomials \(Q_n(z)\), \(Q_n(\omega z)\) and \(Q_n(\overline{z})\) satisfy the same orthogonality relations (2.14). Thus, by the uniqueness of \(Q_n\) we have
\[
Q_n(\omega z) = \omega^{d_n} Q_n(z) \quad \text{and} \quad Q_n(z) = \overline{Q_n(\overline{z})}, \quad n \geq 0. \tag{2.15}
\]
Let \(0 \leq \ell \leq p\) be such that \(d_n \equiv \ell \mod(p + 1)\), so that
\[
d_n = d(p + 1) + \ell, \quad d := \left\lfloor \frac{d_n}{p + 1} \right\rfloor.
\]
Then the first relation in (2.15) implies that
\[
Q_n(t) = t^\ell \mathcal{Q}_d(t^{p+1}) \tag{2.16}
\]
for some polynomial \(\mathcal{Q}_d\) of exact degree \(d\).
We note that in this paper we use the following standard notations for a real number $x$:

$$[x] = \sup \{ m \in \mathbb{Z} : m \leq x \}$$

and

$$[x] = \inf \{ m \in \mathbb{Z} : m \geq x \}.$$

The polynomials $Q_n$ are intrinsically related to the so-called functions of the second kind, which we define next.

**Definition 2.4.** Set $\Psi_{n,0} = Q_n$ and let

$$\Psi_{n,k}(z) = \int_{\Gamma_{k-1}} \frac{\Psi_{n,k-1}(t)}{z-t} d\sigma_{k-1}(t), \quad k = 1, \ldots, p.$$

Observe that $\Psi_{n,k}$ is analytic in $\mathbb{C} \setminus \Gamma_{k-1}$. Our next proposition shows that the function $\Psi_{n,k}$ satisfies multiple orthogonality conditions similar to those satisfied by $Q_n$ but with respect to the Nikishin system given by the $k$th row of the hierarchy (2.2). Note that the function $\Psi_{n,p}$ is excluded from this proposition.

**Proposition 2.5.** For each $k = 0, \ldots, p-1$ the function $\Psi_{n,k}$ satisfies the following orthogonality conditions

$$\int_{\Gamma_{k}} \Psi_{n,k}(z) z^l ds_{k,j}(z) = 0, \quad 0 \leq l \leq \left\lfloor \frac{n-j-1}{p} \right\rfloor, \quad k \leq j \leq p-1. \quad (2.17)$$

**Proof.** The result follows by induction on $k$, based on the equality

$$\int_{\Gamma_{k+1}} \Psi_{n,k+1}(z) z^l ds_{k+1,j}(z) = \int_{\Gamma_{k}} \Psi_{n,k}(t) p_l(t) d\sigma_k(t) - \int_{\Gamma_{k}} \Psi_{n,k}(t) t^l ds_{k,j}(t),$$

where $p_l$ denotes the polynomial $p_l(t) = \int_{\Gamma_{k+1}} \frac{z^l-t^l}{z-t} ds_{k+1,j}(z)$.

**Proposition 2.6.** The functions $\Psi_{n,k}$ satisfy the symmetry property

$$\Psi_{n,k}(\omega z) = \omega^{d_n-k} \Psi_{n,k}(z), \quad k = 0, \ldots, p, \quad n \geq 0, \quad (2.18)$$

where, as above, $d_n$ is the degree of $Q_n$.

**Proof.** The proof is again by induction on $k$. The case $k = 0$ is the symmetry property (2.15) for the polynomials $Q_n$ which we have already proved. Assuming that (2.18) holds for $k$ we have

$$\Psi_{n,k+1}(\omega z) = \int_{\Gamma_{k}} \frac{\Psi_{n,k}(t)}{\omega z - t} d\sigma_k(t) = \int_{\Gamma_{k}} \frac{\Psi_{n,k}(\omega t)}{\omega z - \omega t} d\sigma_k(t) = \omega^{d_n-k-1} \Psi_{n,k+1}(z).$$

We now seek to find an analogue of the polynomial $\mathcal{Q_d}$ in (2.16) for the functions $\Psi_{n,k}$. To accomplish this, we first need the following representation.
Proposition 2.7. Assume that \( d_n \equiv \ell \mod(p+1) \) with \( 0 \leq \ell \leq p \). Then, for each \( k = 1, \ldots, p \) we have

\[
\Psi_{n,k}(z) = z^{p-s} \int_{\Gamma_{k-1}} \frac{\Psi_{n,k-1}(t) t^s}{z^{p+1} - t^{p+1}} \, d\sigma_{k-1}(t),
\]

where \( s \) is the only integer in \( \{0, \ldots, p\} \) such that \( s \equiv k - 1 - \ell \mod(p+1) \), that is,

\[
s = \begin{cases} 
  k - 1 - \ell, & \ell < k, \\
  p + k - \ell, & k \leq \ell.
\end{cases}
\]  

Proof. Let \( 1 \leq k \leq p \). From Definition 2.4 we deduce that

\[
\Psi_{n,k}(z) = \sum_{l=0}^{p} z^{p-l} \int_{\Gamma_{k-1}} \frac{\Psi_{n,k-1}(t) t^l}{z^{p+1} - t^{p+1}} \, d\sigma_{k-1}(t).
\]

By (2.18), the integral of the \( l \)th term of this sum equals 0 if \( d_n - k + 1 + l \not\equiv 0 \mod(p+1) \), which proves (2.19).

Definition 2.8. Set \( \psi_{n,0} := \mathcal{D}_d \), and for \( 1 \leq k \leq p \), let \( \psi_{n,k} \) be the function, analytic in \( \mathbb{C} \setminus [a_{k-1}, b_{k-1}] \), defined by

\[
\psi_{n,k}(z) = \begin{cases} 
  z \int_{\Gamma_{k-1}} \frac{\Psi_{n,k-1}(t) t^{k-1-\ell}}{z - t^{p+1}} \, d\sigma_{k-1}(t), & \ell < k, \\
  \int_{\Gamma_{k-1}} \frac{\Psi_{n,k-1}(t) t^{p+k-\ell}}{z - t^{p+1}} \, d\sigma_{k-1}(t), & k \leq \ell.
\end{cases}
\]

This definition is what one naturally gets by substituting the expressions in (2.20) for \( s \) in (2.19), and doing this also immediately yields the following corollary.

Corollary 2.9. Suppose \( d_n \equiv \ell \mod(p+1) \) with \( 0 \leq \ell \leq p \), and define

\[
d\sigma_{n,k}(\tau) := \begin{cases} 
  d\sigma_{k}^*(\tau), & \ell \leq k, \\
  \tau \, d\sigma_{k}^*(\tau), & k < \ell.
\end{cases}
\]  

Then

\[
z^{k-\ell} \Psi_{n,k}(z) = \psi_{n,k}(z^{p+1}), \quad 0 \leq k \leq p,
\]

and for all \( 1 \leq k \leq p \),

\[
\psi_{n,k}(z) = \begin{cases} 
  z \int_{a_{k-1}}^{b_{k-1}} \frac{\psi_{n,k-1}(\tau)}{z - \tau} \, d\sigma_{n,k-1}(\tau), & \ell < k, \\
  \int_{a_{k-1}}^{b_{k-1}} \frac{\psi_{n,k-1}(\tau)}{z - \tau} \, d\sigma_{n,k-1}(\tau), & k \leq \ell.
\end{cases}
\]

We have seen that the functions \( \Psi_{n,k} \) satisfy orthogonality relations with respect to the hierarchy (2.2). We now show that their associated functions \( \psi_{n,k} \) do the same with respect to the hierarchy (2.9).
Proposition 2.10. Let \(0 \leq k \leq p - 1\) and assume that \(d_n \equiv \ell \mod(p + 1)\) with \(0 \leq \ell \leq p\). Then the function \(\psi_{n,k}\) satisfies the following orthogonality conditions:

\[
\int_{\alpha_k}^{b_k} \psi_{n,k}(\tau) \tau^s d\mu_{k,j}(\tau) = 0,
\]

\[
\left[\frac{\ell - j}{p + 1}\right] \leq s \leq \left[\frac{n + p\ell - 1 - j(p + 1)}{p(p + 1)}\right], \quad k \leq j \leq p - 1.
\] (2.24)

Proof. Let \(0 \leq k \leq j \leq p - 1\). From orthogonality conditions (2.17) and relations (2.22) and (2.12) we find that

\[
\int_{\alpha_k}^{b_k} \psi_{n,k}(\tau)\tau^{(\ell + l - j)/(p + 1)} d\mu_{k,j}(\tau) = 0,
\]

\[
0 \leq l \leq \left[\frac{n - j - 1}{p}\right], \quad \ell + l - j \equiv 0 \mod(p + 1).
\] (2.25)

If we take \(l\) satisfying \(\ell + l - j \equiv 0 \mod(p + 1)\) in (2.25), then we can write

\(l = j - \ell + s(p + 1)\) and we obtain the orthogonality conditions

\[
\int_{\alpha_k}^{b_k} \psi_{n,k}(\tau)\tau^s d\mu_{k,j}(\tau) = 0,
\]

\[
\left[\frac{\ell - j}{p + 1}\right] \leq s \leq \left[\frac{1}{p + 1}\left[\frac{n + p\ell - 1 - j(p + 1)}{p}\right]\right].
\] (2.26)

Since

\[
\left[\frac{x}{p + 1}\right] = \left[\frac{x}{p + 1}\right]
\]

for all \(x \in \mathbb{R}\), the range for \(s\) in (2.26) takes the form in (2.24).

2.3. Counting the number of orthogonality conditions.

Definition 2.11. Let \(n\) and \(\ell\) be nonnegative integers with \(0 \leq \ell \leq p\). For each \(0 \leq j \leq p - 1\) let \(M_j = M_j(n, \ell)\) be the number of integers \(s\) satisfying the inequalities

\[
\left[\frac{\ell - j}{p + 1}\right] \leq s \leq \left[\frac{n + p\ell - 1 - j(p + 1)}{p(p + 1)}\right].
\] (2.27)

For each \(0 \leq k \leq p - 1\), we define

\[
Z(n, k) = Z(n, \ell, k) := \sum_{j=k}^{p-1} M_j.
\] (2.28)

We also agree to set \(Z(n, p) := 0\).

Hereafter we shall always write \(Z(n, k)\) instead of \(Z(n, \ell, k)\) because in all future situations the number \(\ell\) will be dependent on \(n\). We easily deduce from (2.27) and (2.28) that

\[
Z(n, k) = \frac{n(p - k)}{p(p + 1)} + O(1), \quad n \to \infty.
\] (2.29)
It is also clear from the definition that for every \( n \) and \( \ell \),
\[
Z(n, k) \geq Z(n, k + 1), \quad 0 \leq k \leq p - 2,
\]
and
\[
Z(n, k) - Z(n, k + 1) = \# \left\{ s : \left\lfloor \frac{\ell - k}{p + 1} \right\rfloor \leq s \leq \left\lfloor \frac{n + p\ell - 1 - k(p + 1)}{p(p + 1)} \right\rfloor \right\}. \quad (2.30)
\]
Hence, choosing \( j = k \) in (2.24) and noticing that
\[
\left\lfloor \frac{\ell - j}{p + 1} \right\rfloor = \begin{cases} 
0 & \text{if } \ell \leq j, \\
1 & \text{if } j < \ell,
\end{cases} \quad (2.31)
\]
we arrive at the following corollary.

**Corollary 2.12.** Let \( 0 \leq k \leq p - 1 \) and assume that \( d_n \equiv \ell \mod (p + 1) \) with \( 0 \leq \ell \leq p \). Then the function \( \psi_{n,k} \) satisfies the orthogonality conditions
\[
\int_{a_k}^{b_k} \psi_{n,k}(\tau) \tau^s d\sigma_{n,k}(\tau) = 0, \quad 0 \leq s \leq Z(n, k) - Z(n, k + 1) - 1. \quad (2.32)
\]

Let us fix nonnegative integers \( n \) and \( \ell \), with \( \ell \) satisfying \( 0 \leq \ell \leq p \). We associate to \( n \) and \( \ell \) three numbers \( \alpha \), \( \beta \) and \( v \), taking \( \alpha \) and \( \beta \) to be the quotient and the remainder, respectively, on dividing \( n + p\ell - 1 \) by \( p(p + 1) \), and \( v \) to be the quotient on dividing \( \beta \) by \( p + 1 \). That is,
\[
n + p\ell - 1 = \alpha p(p + 1) + \beta, \quad 0 \leq \beta \leq p(p + 1) - 1, \\
\alpha = \left\lfloor \frac{n + p\ell - 1}{p(p + 1)} \right\rfloor \quad \text{and} \quad v = \left\lfloor \frac{\beta}{p + 1} \right\rfloor. \quad (2.33)
\]
Notice that
\[
0 \leq v \leq p - 1.
\]

**Lemma 2.13.** If \( \ell \leq v \), then inequality (2.27) is equivalent to
\[
1 \leq s \leq \alpha \quad \text{if} \quad 0 \leq j < \ell, \\
0 \leq s \leq \alpha \quad \text{if} \quad \ell \leq j \leq v, \\
0 \leq s \leq \alpha - 1 \quad \text{if} \quad v < j \leq p - 1, \quad (2.34)
\]
while if \( v < \ell \), then (2.27) is equivalent to
\[
1 \leq s \leq \alpha \quad \text{if} \quad 0 \leq j \leq v, \\
1 \leq s \leq \alpha - 1 \quad \text{if} \quad v < j < \ell, \\
0 \leq s \leq \alpha - 1 \quad \text{if} \quad \ell \leq j \leq p - 1. \quad (2.35)
\]
Moreover,

\[
Z(n, k) = \begin{cases} 
\left\lfloor \frac{n - \ell}{p + 1} \right\rfloor - k\alpha & \text{if } k \leq \ell, v, \\
\left\lfloor \frac{n - \ell}{p + 1} \right\rfloor - k\alpha + \ell - v - 1 & \text{if } \ell, v < k, \\
\left\lfloor \frac{n - \ell}{p + 1} \right\rfloor - k(\alpha + 1) + \ell & \text{if } 0 \leq \ell < k \leq v, \\
\left\lfloor \frac{n - \ell}{p + 1} \right\rfloor - k(\alpha - 1) - v - 1 & \text{if } 0 < v < k \leq \ell.
\end{cases} \tag{2.36}
\]

**Proof.** We begin by writing (2.27) in the form

\[
\left\lfloor \frac{\ell - j}{p + 1} \right\rfloor \leq s \leq \left\lfloor \alpha + \frac{\beta - j(p + 1)}{p(p + 1)} \right\rfloor. \tag{2.37}
\]

Now, since

\[
0 \leq v \leq p - 1 \quad \text{and} \quad v(p + 1) \leq \beta < (v + 1)(p + 1), \tag{2.38}
\]

we have

\[
-1 < -\frac{(p - 1)}{p} \leq \frac{\beta - j(p + 1)}{p(p + 1)} \leq \frac{p(p + 1) - 1}{p(p + 1)} < 1,
\]

and since \(\alpha\) is an integer, this implies that

\[
\left\lfloor \alpha + \frac{\beta - j(p + 1)}{p(p + 1)} \right\rfloor = \begin{cases} 
\alpha & \text{if } 0 \leq j \leq v, \\
\alpha - 1 & \text{if } v < j \leq p - 1.
\end{cases} \tag{2.39}
\]

Inequalities (2.34) and (2.35) follow from (2.31) and (2.39).

As for (2.36), we shall only prove it for the case \(k \leq \ell, v\), since the remaining cases listed in (2.36) are proved similarly. For every \(j \in \{k, \ldots, p - 1\}\) we can use (2.39) to express, in terms of \(\alpha\), the number \(M_j\) of integer values \(s\) that satisfy (2.37).

With this expression to hand, it is easy to see that if \(k \leq \ell, v\), then

\[
Z(n, k) = \sum_{j=k}^{p-1} M_j = \alpha p + v - \ell + 1 - k\alpha.
\]

Now, using the expression that defines \(\alpha\) in (2.33), we find

\[
\alpha p + v - \ell + 1 = \frac{n - \ell + (v + 1)(p + 1) - (\beta + 1)}{p + 1},
\]

which together with (2.38) yields

\[
\frac{n - \ell}{p + 1} \leq \alpha p + v - \ell + 1 \leq \frac{n - \ell}{p + 1} + \frac{p}{p + 1}.
\]

As \(\alpha p + v - \ell + 1\) is an integer, we deduce that it must be equal to \(\left\lceil (n - \ell)/(p + 1) \right\rceil\).
2.4. The AT-system property. The system of continuous functions \( u_1(x), \ldots, u_n(x) \) is said to be an algebraic Chebyshev system (AT-system) over the interval \([a, b]\) for the set of integers \((d_1, \ldots, d_n)\), \(d_j \geq 0\), if for any choice of polynomials \((P_1(x), \ldots, P_n(x)) \neq (0, 0, \ldots, 0)\), with \(\deg(P_j) \leq d_j - 1\), the polynomial combination

\[
P_1(x)u_1(x) + \cdots + P_n(x)u_n(x)
\]

has at most \(d_1 + \cdots + d_n - 1\) zeros on \([a, b]\). Here, and in what follows, a polynomial of degree \(-1\) is understood to be the constant zero function.

Since \(\mu_{k,k} = \sigma_k^*\) and \(d\mu_{k,j}(t) = t\tilde{\mu}_{k+1,j}(t)d\sigma_k^*(t)\) for \(k < j \leq p - 1\), the orthogonality conditions (2.24) can equivalently be written as \(\psi_{n,k}\) being orthogonal to polynomial linear combinations of functions of the form

\[
1, \ t\tilde{\mu}_{k+1,k+1}(t), \ \cdots, \ t\tilde{\mu}_{k+1,m}(t), \ \tilde{\mu}_{k+1,m+1}(t), \ \cdots, \ \tilde{\mu}_{k+1,p-1}(t),
\]

for some \(k \leq m \leq p - 1\). We now prove that any such collection of functions forms an AT-system over \([a_k, b_k]\).

**Proposition 2.14.** Let \(k\) and \(m\) be integers such that \(0 \leq k \leq m \leq p - 1\). For each \(j\) in the range \(k \leq j \leq p - 1\), let \(P_j\) be a polynomial of degree at most \(d_j - 1\), with \(d_j \geq 0\), and suppose that

\[
d_k \geq d_{k+1} \geq \cdots \geq d_m \geq d_{m+1} - 1 \geq d_{m+2} - 1 \geq \cdots \geq d_{p-1} - 1.
\]

If \((P_k, \ldots, P_{p-1}) \neq (0, 0, \ldots, 0)\), then

\[
H(z) = P_k(z) + \sum_{k+1 \leq j \leq m} P_j(z) z\tilde{\mu}_{k+1,j}(z) + \sum_{m < j \leq p-1} P_j(z) \tilde{\mu}_{k+1,j}(z) \tag{2.40}
\]

has at most \(D_H := \sum_{j=k}^{p-1} d_j - 1\) zeros in \([a_k, b_k]\).

**Proof.** The proof is by induction on \(k\). If \(k = p - 1\), the statement is trivially true, as in this case we simply have \(H(z) = P_{p-1}(z)\) and \(D_H = d_{p-1} - 1\). Assume that the statement of the proposition is also true for \(k + 1\), \(0 < k + 1 \leq p - 1\), but that for the value \(k\), there is a corresponding function \(H\) of the form (2.40) with at least \(D_H + 1\) zeros in \([a_k, b_k]\).

Then, for this \(H\) not all the polynomials \(P_j\) corresponding to \(k + 1 \leq j \leq p - 1\), can be zero simultaneously. Let \(T\) be a monic polynomial that vanishes at the zeros of \(H\) in \([a_k, b_k]\), and let \(\gamma\) be a positively oriented simple contour around the interval \([a_{k+1}, b_{k+1}]\) that leaves the zeros of \(T\) outside. Since \(H/T\) is analytic outside \([a_{k+1}, b_{k+1}]\) and

\[
\frac{H(z)}{T(z)} = O\left(\frac{1}{z^{D_H + 2 - d_k}}\right), \quad z \to \infty, \tag{2.41}
\]

we have

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{z^u H(z)}{T(z)} \, dz = 0, \quad u = 0, \ldots, D_H - d_k. \tag{2.42}
\]
By definition, \( \hat{\mu}_{k+1,j}(z) = \int_{a_{k+1}}^{b_{k+1}} \frac{d\mu_{k+1,j}(\tau)}{z-\tau} \), and so an application of Fubini’s theorem and Cauchy’s integral formula give
\[
\frac{1}{2\pi i} \int \frac{z^u P_j(z)}{T(z)} \hat{\mu}_{k+1,j}(z) \, dz = \int_{a_{k+1}}^{b_{k+1}} \frac{\tau^u P_j(\tau)}{T(\tau)} \, d\mu_{k+1,j}(\tau), \quad u \in \mathbb{N} \cup \{0\}. \quad (2.43)
\]
If we now replace \( H \) in (2.42) by the right-hand side of (2.40), from (2.43) and (2.10) we see that
\[
\int_{a_{k+1}}^{b_{k+1}} \tau^u G(\tau) \frac{\tau^{1-\delta_{mk}} d\sigma_{k+1}(\tau)}{T(\tau)} = 0, \quad u = 0, \ldots, D_H - d_k, \quad (2.44)
\]
where \( \delta_{mk} \) is the Kronecker delta and \( G \) is the function that, when \( m = k \), is given by
\[
G(z) = P_{k+1}(z) + \sum_{k+2 \leq j \leq p-1} P_j(z)z\hat{\mu}_{k+2,j}(z),
\]
while if \( k+1 \leq m \leq p-1 \),
\[
G(z) = P_{k+1}(z) + \sum_{k+2 \leq j \leq m} zP_j(z)\hat{\mu}_{k+2,j}(z) + \sum_{m < j \leq p-1} P_j(z)\hat{\mu}_{k+2,j}(z).
\]
By the induction hypothesis, the function \( G \) has at most \( D_G = D_H - d_k \) zeros in \([a_{k+1}, b_{k+1}]\) which, together with (2.44), implies that \( G \) must be identically zero, yielding a contradiction, since at least one of the \( P_j \) for \( k+1 \leq j \leq p-1 \) is not identically zero.

**Corollary 2.15.** Let \( k \) and \( m \) be integers such that \( 0 \leq k \leq m \leq p-1 \). Let \( \{d_j\}_{j=k}^{p-1} \) be a finite sequence of nonnegative integers such that
\[
d_k \geq d_{k+1} \geq \cdots \geq d_m \geq d_{m+1} - 1 \geq d_{m+2} - 1 \geq \cdots \geq d_{p-1} - 1.
\]
Suppose \( F \neq 0 \) is a real-valued analytic function on \([a_k, b_k]\), satisfying the orthogonality conditions
\[
\int_{a_k}^{b_k} F(\tau) \tau^{s+\delta} d\mu_{k,j}(\tau) = 0, \quad 0 \leq s \leq d_j - 1, \quad k \leq j \leq m, \quad (2.45)
\]
and
\[
\int_{a_k}^{b_k} F(\tau) \tau^{s} d\mu_{k,j}(\tau) = 0, \quad 0 \leq s \leq d_j - 1, \quad m < j \leq p-1, \quad (2.46)
\]
where \( \delta = 1 \) if \( m < p-1 \) and \( d_{m+1} = d_m + 1 \), and otherwise \( \delta \) can be taken to be either 1 or 0. Then, \( F \) has at least
\[
N := \sum_{j=k}^{p-1} d_j
\]
zeros of odd multiplicity in \((a_k, b_k)\).
Proof. Orthogonality conditions (2.45) and (2.46) imply that
\[
\int_{a_k}^{b_k} \tau^\delta F(\tau) H(\tau) d\sigma^*_k(z) = 0 
\] (2.47)
for every $H$ of the form
\[ H(z) = P_k(z) + \sum_{k+1 \leq j \leq m} P_j(z) z \tilde{\mu}_{k+1,j}(z) + \sum_{m < j \leq p-1} P_j(z) \tilde{\mu}_{k+1,j}(z), \]
where $P_j$ is a polynomial of degree at most $d_j - 1$ for each $k \leq j \leq p-1$, and if $\delta = 0$ then we take $m = p-1$. Applying Proposition 2.14 we see that any such function $H$ which is not identically zero has at most $N - 1$ zeros in $[a_k, b_k]$. Consequently, if $F$ has $D < N$ zeros with odd multiplicity in $(a_k, b_k)$, say $x_1, \ldots, x_D$, we can find $H$ with simple zeros at these $x_k$, and with $N - D - 1$ zeros (counting multiplicities) at the endpoints of the interval $[a_k, b_k]$. Since $H$ does not admit any more zeros on this closed interval, the integral (2.47) cannot be zero. Therefore, $D \geq N$.

2.5. The normality of the Nikishin system and the zeros of $Q_n$. The Nikishin system of measures $(s_0, \ldots, s_p)$ is said to be normal provided that the degree of the multi-orthogonal polynomial $Q_n$ is maximal, that is, $d_n = n$ for all $n \geq 0$. We will prove this normality in this section.

Proposition 2.16. Let $n$, $k$ and $\ell$ be nonnegative integers satisfying $0 \leq k \leq p-1$, and let $d_n \equiv \ell \mod(p+1)$, $0 \leq \ell \leq p$. Then, the function $\psi_{n,k}$ has at least $Z(n,k)$ zeros with odd multiplicity in the open interval $(a_k, b_k)$. In particular, if follows that
\[ d_n = n, \quad n \geq 0, \]
that is, the polynomial $Q_n$ has degree $n$, and the associated polynomial $\varphi_d$ has exactly
\[ d = \frac{n - \ell}{p+1} \]
zeros, which are all simple and located in $(a_0, b_0)$.

Proof. The total number of orthogonality conditions in (2.24) is given by $Z(n,k)$ as defined in (2.28). Using Lemma 2.13, these orthogonality conditions can be written more specifically, but since in (2.24) $j$ only varies from $k$ to $p-1$, we split the analysis into the following five cases:\ 1) $k \leq \ell \leq v \leq p-1$; 2) $k \leq v < \ell \leq p$; 3) $0 \leq \ell, v < k$; 4) $0 \leq \ell < k \leq v$; and 5) $v < k \leq \ell$. In each of these cases for each $j \in \{k, \ldots, p-1\}$ we use (2.34) and (2.35) to write relation (2.24) as an orthogonality relation for $\psi_{n,k}$ against $\tau^s$ or against $\tau^{s+1}$, according to whether the range of $s$ in (2.34) and (2.35) starts from $s = 0$ or $s = 1$, respectively. For instance, in the first case $k \leq \ell \leq v \leq p-1$, (2.24) takes the form
\[
\int_{a_k}^{b_k} \psi_{n,k}(\tau) \tau^{s+1} d\mu_{k,j}(\tau) = 0, \quad 0 \leq s \leq \alpha - 1, \quad k \leq j < \ell, 
\]
\[
\int_{a_k}^{b_k} \psi_{n,k}(\tau) \tau^s d\mu_{k,j}(\tau) = 0, \quad 0 \leq s \leq \alpha, \quad \ell \leq j \leq v, 
\]
\footnote{Recall that we defined $v$ in (2.33).}
and
\[
\int_{a_k}^{b_k} \psi_{n,k}(\tau) \tau^s d\mu_{k,j}(\tau) = 0, \quad 0 \leq s \leq \alpha - 1, \quad v < j \leq p - 1.
\]

Written in this way, in each of the five cases the orthogonality relations (2.24) adhere to the form of (2.45) and (2.46), and so Corollary 2.15 ensures that \( \psi_{n,k} \) has at least \( Z(n,k) \) zeros of odd multiplicity in \((a_k, b_k)\). In particular, for \( k = 0 \), \( \psi_{n,0} = \mathcal{D}_d \) has at least \( Z(n,0) = \lceil (n - \ell)/(p + 1) \rceil \) zeros of odd multiplicity in \((a_0, b_0)\). Hence
\[
\left\lfloor \frac{n - \ell}{p + 1} \right\rfloor \leq d = \frac{d_n - \ell}{p + 1} \leq \frac{n - \ell}{p + 1},
\]
finishing the proof of the proposition.

2.6. On the difference \( Z(n, j) - Z(n, j + 1) \). We now prove an auxiliary result that will be useful later.

Lemma 2.17. Let \( n \) be any nonnegative integer. Suppose that \( n \equiv \ell \mod(p + 1) \) and \( n \equiv r \mod p \), \( 0 \leq \ell \leq p \), \( 0 \leq r \leq p - 1 \), and let
\[
\lambda = \left\lfloor \frac{n}{p(p + 1)} \right\rfloor. \tag{2.48}
\]
Then, for every \( j = 0, \ldots, p - 1 \) we have
\[
Z(n, j) - Z(n, j + 1) = \begin{cases} 
\lambda, & j < \ell \leq r, \\
\lambda + 1, & \ell \leq j < r, \\
\lambda, & \ell \leq r \leq j, \\
\lambda + 1, & j < r \leq \ell, \\
\lambda, & r \leq j \leq \ell, \\
\lambda + 1, & r < \ell \leq j.
\end{cases} \tag{2.49}
\]

Proof. Write \( n = mp + r \) with \( m \geq 0 \), and
\[
m = \lambda(p + 1) + q, \quad 0 \leq q \leq p,
\]
so that
\[
n = \lambda p(p + 1) + pq + r, \quad \text{where} \ 0 \leq pq + r \leq p(p + 1) - 1, \tag{2.50}
\]
and \( \lambda \) is given by (2.48). In terms of these quantities, the remainder \( \ell \) on dividing \( n \) by \( p + 1 \) is given by
\[
\ell = \begin{cases} 
\frac{r - q}{p + 1 + r - q}, & 0 \leq \ell \leq r, \\
\frac{r - q}{p + 1 + r - q}, & r < \ell.
\end{cases} \tag{2.51}
\]
Combining (2.30), (2.50) and (2.51), for all \( 0 \leq j \leq p - 1 \) we obtain
\[
Z(n, j) - Z(n, j + 1) = \lambda + \left\lfloor \frac{(r - j)(p + 1) - 1}{p(p + 1)} \right\rfloor + \begin{cases} 
0, & \ell \leq r, \\
1, & r < \ell,
\end{cases} + \begin{cases} 
0, & j < \ell,
\end{cases}
\]
thus (2.49) follows.
2.7. Order of decay and the zeros of functions of the second kind.

**Proposition 2.18.** Let \( 1 \leq k \leq p \), and suppose that \( n \equiv \ell \mod (p+1) \). Then, as \( z \to \infty \),
\[
\psi_{n,k}(z) = O(z^{-N(n,k)}),
\]
where
\[
N(n,k) = \begin{cases} 
Z(n,k-1) - Z(n,k), & \ell < k, \\
Z(n,k-1) - Z(n,k) + 1, & k \leq \ell.
\end{cases}
\]
(Recall that \( Z(n,p) = 0 \).)

**Proof.** From (2.23) and (2.21) we see that if \( \ell < k \), the Laurent expansion of \( \psi_{n,k} \) at infinity has the form
\[
\psi_{n,k}(z) = \sum_{s=0}^{\infty} z^{-s} \int_{a_{k-1}}^{b_{k-1}} \psi_{n,k-1}(\tau) \tau^s d\sigma_{k-1}^*(\tau),
\]
while if \( k \leq \ell \), the expansion of \( \psi_{n,k} \) at infinity is as in (2.53) but with the series starting from \( s = 1 \) (instead of from \( s = 0 \)). Combining these expansions with (2.32) yields (2.52).

We are now in a position to prove the following result.

**Proposition 2.19.** For each \( n \geq 0 \) and \( k = 0, \ldots, p-1 \) the function \( \psi_{n,k} \) has exactly \( Z(n,k) \) zeros in \( \mathbb{C} \setminus ([a_{k-1}, b_{k-1}] \cup \{0\}) \); they are all simple and lie in the open interval \( (a_k, b_k) \). The function \( \psi_{n,p} \) has no zeros in \( \mathbb{C} \setminus ([a_{p-1}, b_{p-1}] \cup \{0\}) \).

**Proof.** The proof is by induction on \( k \). It was already shown in Proposition 2.16 that the polynomial \( \psi_{n,0} = \mathcal{Q}_d \) has degree \( Z(n,0) = (n-\ell)/(p+1) \), all of its zeros are simple and lie in the interval \( (a_0, b_0) \).

Assume that the result holds for \( k-1, k \geq 1 \), but \( \psi_{n,k} \) has at least \( Z(n,k) + 1 \) zeros in \( \mathbb{C} \setminus ([a_{k-1}, b_{k-1}] \cup \{0\}) \), counting multiplicities. Let \( P_{n,k}(z) \) denote the monic polynomial whose zeros are the zeros of \( \psi_{n,k} \) in \( \mathbb{C} \setminus ([a_{k-1}, b_{k-1}] \cup \{0\}) \). Since \( \psi_{n,k}(\overline{z}) = \overline{\psi_{n,k}(z)} \), the complex zeros of \( \psi_{n,k} \), if any, must come in conjugate pairs, so \( P_{n,k} \) is a polynomial with real coefficients with \( \deg(P_{n,k}) \geq Z(n,k) + 1 \).

Now we set \( \mathcal{P}_{n,k} := zP_{n,k} \) if \( \ell < k \), and \( \mathcal{P}_{n,k} := P_{n,k} \) otherwise. Then it follows from (2.52) that
\[
\frac{\psi_{n,k}(z)}{\mathcal{P}_{n,k}(z)} = O\left(\frac{1}{zZ(n,k-1)+2}\right), \quad z \to \infty,
\]
and this function is analytic outside \( [a_{k-1}, b_{k-1}] \). Suppose that \( \gamma \) is a closed Jordan curve that surrounds \( [a_{k-1}, b_{k-1}] \) and that the zeros of \( P_{n,k} \) lie outside it. Then it follows from (2.54) and (2.23) that for \( j = 0, \ldots, Z(n,k-1) \) we have
\[
0 = \frac{1}{2\pi i} \int_{\gamma} z^j \frac{\psi_{n,k}(z)}{\mathcal{P}_{n,k}(z)} \, dz = \int_{a_{k-1}}^{b_{k-1}} \psi_{n,k-1}(\tau)\tau^j \frac{d\sigma_{n,k-1}(\tau)}{P_{n,k}(\tau)},
\]
where we have applied Cauchy’s theorem, Cauchy’s integral formula and Fubini’s theorem. The above orthogonality conditions for \( \psi_{n,k-1} \) with respect to the measure...
\( d\sigma_{n,k-1}(\tau)/P_{n,k}(\tau) \) imply that \( \psi_{n,k-1} \) has at least \( Z(n, k-1) + 1 \) zeros in \( (a_{k-1}, b_{k-1}) \), contrary to our initial hypothesis.

Thus, the function \( \psi_{n,k} \) has at most \( Z(n, k) \) zeros in \( \mathbb{C} \setminus ([a_{k-1}, b_{k-1}] \cup \{0\}) \). This together with Proposition 2.16 gives the result.

For the asymptotic analysis that will be performed later it is crucial to consider the polynomials whose zeros coincide with those of the functions \( \psi_{n,k} \). We introduce now notation for these polynomials.

**Definition 2.20.** For any integers \( n \geq 0 \) and \( k \) with \( 0 \leq k \leq p-1 \), let \( P_{n,k} \) denote the monic polynomial whose zeros are the zeros of \( \psi_{n,k} \) in \( (a_k, b_k) \). For convenience we also define the polynomials \( P_{n,-1} \equiv 1, P_{n,p} \equiv 1 \).

Hence by Proposition 2.19 we know that \( P_{n,k} \) has degree \( Z(n, k) \) and all its zeros are simple. Note that \( P_{n,0} = \psi_{n,0} \).

**Proposition 2.21.** Let \( 0 \leq k \leq p-1 \). Then the function \( \psi_{n,k} \) satisfies the following orthogonality conditions:

\[
\int_{a_k}^{b_k} \psi_{n,k}(\tau) \tau^s d\sigma_{n,k}(\tau) P_{n,k+1}(\tau) = 0, \quad s = 0, \ldots, Z(n, k) - 1. \tag{2.56}
\]

**Proof.** For \( 0 \leq k \leq p-2 \), these orthogonality conditions are just (2.55). For \( k = p-1 \), (2.56) follows from (2.24) and (2.36), since \([((\ell - (p-1))/(p+1)] = 0 \) if \( \ell \leq p-1 \) and \([((\ell - (p-1))/(p+1)] = 1 \) if \( \ell = p \).

**Corollary 2.22.** Let \( 0 \leq k \leq p-1 \), and let \( I \) be any connected component of \([a_k, b_k] \setminus \text{supp}(\sigma^*_k)\). Then the polynomial \( P_{n,k} \) has at most one zero in the closure of \( I \).

**Proof.** Suppose that \( P_{n,k} \) has two distinct zeros \( \tau_1 \) and \( \tau_2 \) in \( \overline{I} \) and assume that \( \ell \leq k \) (the case \( k < \ell \) follows along the same lines). Set \( L_{n,k}(\tau):= P_{n,k}(\tau)/((\tau-\tau_1)(\tau-\tau_2)) \). Then, according to (2.56) we have

\[
\int_{a_k}^{b_k} \psi_{n,k}(\tau) L_{n,k}(\tau) d\sigma^*_k(\tau) P_{n,k+1}(\tau) = 0, \tag{2.57}
\]

since \( L_{n,k} \) is a polynomial of degree \( Z(n, k) - 2 \). On the other hand, the function \( \psi_{n,k} L_{n,k} \) has constant sign and finitely many zeros on \( \text{supp}(\sigma^*_k) \), therefore its integral with respect to the measure \( d\sigma^*_k(\tau)/P_{n,k+1}(\tau) \) must be different from zero, contradicting (2.57).

2.8. The auxiliary functions \( H_{n,k} \). We now introduce certain functions that will play an important role in the analysis that will follow.

**Definition 2.23.** For integers \( n \geq 0 \) and \( 0 \leq k \leq p \), set

\[
H_{n,k}(z):= \frac{P_{n,k-1}(z) \psi_{n,k}(z)}{P_{n,k}(z)}. \tag{2.58}
\]
Note that $H_{n,0} \equiv 1$. Since the zeros of $P_{n,k}$ are zeros of $\psi_{n,k}$ lying outside $[a_{k-1}, b_{k-1}]$, we have

$$H_{n,k} \in \mathcal{H}(\mathbb{C} \setminus [a_{k-1}, b_{k-1}]), \quad 1 \leq k \leq p.$$

Putting together (2.21), (2.56) and (2.58), we readily obtain the following result.

**Proposition 2.24.** For any $k = 0, \ldots, p - 1$, the polynomial $P_{n,k}$ satisfies the following orthogonality conditions:

$$\int_{a_k}^{b_k} P_{n,k}(\tau) \frac{H_{n,k}(\tau)}{P_{n,k-1}(\tau) P_{n,k+1}(\tau)} d\sigma_{n,k}(\tau) = 0, \quad s = 0, \ldots, Z(n, k) - 1. \quad (2.59)$$

(Recall that $P_{n,-1}, P_{n,p} \equiv 1$.)

We prove now a formula analogous to (2.23) for the functions $H_{n,k}$.

**Proposition 2.25.** Let $1 \leq k \leq p$ and $n \equiv \ell \mod (p + 1)$, $0 \leq \ell \leq p$. Then

$$H_{n,k}(z) = \begin{cases} z \int_{a_k}^{b_k} \frac{P_{n,k-1}^2(\tau)}{z - \tau} \frac{H_{n,k-1}(\tau) d\sigma_{n,k-1}(\tau)}{P_{n,k-2}(\tau) P_{n,k}(\tau)}, & \ell < k, \\ \int_{a_k}^{b_k} \frac{P_{n,k-1}^2(\tau)}{z - \tau} \frac{H_{n,k-1}(\tau) d\sigma_{n,k-1}(\tau)}{P_{n,k-2}(\tau) P_{n,k}(\tau)}, & k \leq \ell. \end{cases} \quad (2.60)$$

**Proof.** We know by (2.59) that for any polynomial $Q$ with $\deg(Q) \leq Z(n, k - 1)$, $1 \leq k \leq p$, we have

$$\int_{a_k}^{b_k} \frac{Q(z) - Q(\tau)}{z - \tau} P_{n,k-1}(\tau) \frac{H_{n,k-1}(\tau) d\sigma_{n,k-1}(\tau)}{P_{n,k-2}(\tau) P_{n,k}(\tau)} = 0. \quad (2.61)$$

In particular, if we take $Q = P_{n,k-1}$ in (2.61), then we obtain

$$P_{n,k-1}(z) \int_{a_k}^{b_k} \frac{P_{n,k-1}(\tau)H_{n,k-1}(\tau) d\sigma_{n,k-1}(\tau)}{(z - \tau)P_{n,k-2}(\tau) P_{n,k}(\tau)} = \int_{a_k}^{b_k} \frac{P_{n,k-1}^2(\tau)H_{n,k-1}(\tau) d\sigma_{n,k-1}(\tau)}{(z - \tau)P_{n,k-2}(\tau) P_{n,k}(\tau)}. $$

Since $Z(n, k) \leq Z(n, k - 1)$, we can also apply (2.61) for $Q = P_{n,k}$, which together with this last equality yields that for $k = 1, \ldots, p$,

$$\frac{1}{P_{n,k}(z)} \int_{a_k}^{b_k} \frac{\psi_{n,k-1}(\tau)}{z - \tau} d\sigma_{n,k-1}(\tau) = \frac{1}{P_{n,k-1}(z)} \int_{a_k}^{b_k} \frac{P_{n,k-1}(\tau)H_{n,k-1}(\tau) d\sigma_{n,k-1}(\tau)}{(z - \tau)P_{n,k-2}(\tau) P_{n,k}(\tau)}, \quad (2.62)$$

and the result follows from (2.62), (2.23) and (2.58).

In what follows we shall use the notation $\text{sign}(f, I)$ to define the sign of the function $f$ on the interval $I$, and $\Delta_k$ shall denote the open interval $(a_k, b_k)$. 


Corollary 2.26. Let $1 \leq k \leq p - 1$ and $n \equiv \ell \mod(p + 1)$, $0 \leq \ell \leq p$. Then, with the convention that $Z(n, -1) = 0$, we have

\[
\sign(H_{n,k}, \Delta_k) = \begin{cases} 
(-1)^{(k+1)[Z(n,k-2)-Z(n,k)]} \sign(H_{n,k-1,1}, \Delta_{k-1}), & \ell < k, \\
(-1)^{1+(k+1)[Z(n,k-2)-Z(n,k)]} \sign(H_{n,k-1,1}, \Delta_{k-1}), & k \leq \ell.
\end{cases}
\]  

(2.63)

Proof. Suppose first that $k \leq \ell$. Then, by (2.60) and (2.21),

\[
H_{n,k}(z) = \int_{\Delta_{k-1}} \frac{P_{n,k-1}^2(\tau)}{z - \tau} \frac{H_{n,k-1}(\tau) d\sigma_{k-1}^*(\tau)}{P_{n,k-2}(\tau) P_{n,k}(\tau)},
\]

If $k$ is even, then $\Delta_{k-1}$ lies in $(-\infty, 0]$, while $\Delta_{k-2}$ and $\Delta_k$ lie in $[0, \infty)$. Since the monic polynomials $P_{n,k-2}$ and $P_{n,k}$ have their zeros in $\Delta_{k-2}$ and $\Delta_k$, respectively, and $\deg(P_{n,k}) = Z(n,k)$, the above equality gives

\[
\sign(H_{n,k}, \Delta_k) = (-1)^{Z(n,k-2)-Z(n,k)+1} \sign(H_{n,k-1,1}, \Delta_{k-1}).
\]

If $k$ is odd, then $\Delta_{k-1}$ lies in $[0, \infty)$ and $P_{n,k-2}$ and $P_{n,k}$ are both positive in $\Delta_{k-1}$, so that

\[
\sign(H_{n,k}, \Delta_k) = - \sign(H_{n,k-1,1}, \Delta_{k-1}).
\]

Suppose now that $\ell < k$. Then by (2.60) and (2.21),

\[
H_{n,k}(z) = z \int_{\Delta_{k-1}} \frac{P_{n,k-1}^2(\tau)}{z - \tau} \frac{H_{n,k-1}(\tau) d\sigma_{k-1}^*(\tau)}{P_{n,k-2}(\tau) P_{n,k}(\tau)},
\]

so that if $k$ is even,

\[
\sign(H_{n,k}, \Delta_k) = (-1)^{Z(n,k-2)-Z(n,k)} \sign(H_{n,k-1,1}, \Delta_{k-1}),
\]

while for $k$ odd,

\[
\sign(H_{n,k}, \Delta_k) = \sign(H_{n,k-1,1}, \Delta_{k-1}).
\]

§ 3. The recurrence relation and positivity of the recurrence coefficients

Proposition 3.1. The polynomials $Q_n$ satisfy the following three-term recurrence relation of order $p + 1$:

\[
zQ_n(z) = Q_{n+1}(z) + a_n Q_{n-p}(z), \quad n \geq p, \quad a_n \in \mathbb{R},
\]

(3.1)

where

\[
Q_\ell(z) = z^\ell, \quad \ell = 0, \ldots, p.
\]

(3.2)

Proof. Equation (3.2) is clear since we know that if $n = d(p + 1) + \ell$, $0 \leq \ell \leq p$, then $Q_n(z) = z^\ell Q_d(z^{p+1})$ for some monic polynomial $Q_d$ of degree $d$. Moreover, this also implies that for $n \geq p$, $zQ_n(z) - Q_{n+1}(z) = c_{n-p} z^{n-p} + \cdots$. Thus, we can write

\[
zQ_n = Q_{n+1} + \sum_{j=0}^{n-p} b_j Q_j
\]

(3.3)
for some real coefficients \( \{b_j\}_{j=0}^{n-p} \). The goal is to show that

\[
b_0 = b_1 = \cdots = b_{n-p-1} = 0. \tag{3.4}
\]

Assume that \( n = mp + k, \quad 0 \leq k \leq p - 1 \). If we integrate (3.3) term by term with respect to the first measure \( s_0 \) of the Nikishin system, we observe that the only nonvanishing integral is \( \int Q_0 \, ds_0 \), and consequently \( b_0 = 0 \). Integrating (3.3) successively with respect to \( s_j \) we obtain \( b_j = 0 \) for \( j = 1, \ldots, p - 1 \).

In general, we can prove inductively that for all \( l \) such that \( 0 \leq l \leq m - 2 \) we have

\[
b_{lp} = b_{lp+1} = \cdots = b_{lp+(p-1)} = 0. \tag{3.5}
\]

The case \( l = 0 \) was described above. Assume now that all coefficients \( b_s \) in (3.3) are zero for \( s < lp \). If we multiply (3.3) by \( z^l \) and integrate with respect to \( s_0 \), then the only nonvanishing integral in the resulting expression is \( \int z^l Q_{lp}(z) \, ds_0(z) \). Indeed, all other integrals vanish because of the orthogonality conditions, and

\[
\int z^l Q_{lp}(z) \, ds_0(z) = 0 \quad \text{would imply that} \quad Q_{lp+1} \quad \text{and} \quad Q_{lp} \quad \text{satisfy the same orthogonality conditions, violating the normality of the Nikishin system. So} \quad b_{lp} = 0. \quad \text{Integrating successively with respect to the rest of the measures} \quad s_j \quad \text{gives (3.5).}
\]

The remaining part of (3.4) is

\[
b_{(m-1)p} = b_{(m-1)p+1} = \cdots = b_{(m-1)p+k-1} = 0;
\]

which is proved multiplying by \( z^{m-1} \) and integrating with respect to \( s_0, \ldots, s_k-1 \).

We now show that the functions of the second kind satisfy a similar recurrence relation.

**Proposition 3.2.** Let \( a_n, \quad n \geq p, \) be the coefficients of the recurrence relation (3.1). For every \( n \geq p \) and \( 0 \leq k \leq p \), we have

\[
z \Psi_{n,k}(z) = \Psi_{n+1,k}(z) + a_n \Psi_{n-p,k}(z), \tag{3.6}
\]

and if \( n \equiv \ell \mod(p+1), \quad 0 \leq \ell \leq p - 1 \), then

\[
\psi_{n,k}(z) = \psi_{n+1,k}(z) + a_n \psi_{n-p,k}(z), \tag{3.7}
\]

while if \( n \equiv p \mod(p+1), \) then

\[
z \psi_{n,k}(z) = \psi_{n+1,k}(z) + a_n \psi_{n-p,k}(z). \tag{3.8}
\]

**Proof.** For \( k = 0 \), by definition, \( \Psi_{n,0} = Q_n \), and so (3.6) reduces to (3.1). Assume that (3.6) holds for some \( 0 \leq k \leq p - 1 \). Then, by the very definition of \( \Psi_{n,k} \), we have

\[
\int_{\Gamma_k} \frac{t \Psi_{n,k}(t) \, ds_k(t)}{z - t} = \int_{\Gamma_k} \frac{\Psi_{n+1,k}(t) \, ds_k(t)}{z - t} + a_n \int_{\Gamma_k} \frac{\Psi_{n-p,k}(t) \, ds_k(t)}{z - t} = \Psi_{n+1,k+1}(z) + a_n \Psi_{n-p,k+1}(z). \]
Now, from Proposition 2.5, we know that
\[ \int_{\Gamma_k} \Psi_{n,k}(z) z^l d\sigma_k(z) = 0, \quad 0 \leq l \leq \left\lfloor \frac{n - k - 1}{p} \right\rfloor, \]
and since \( n - k - 1 \geq p - 1 - k \geq 0 \), we have \( \int_{\Gamma_k} \Psi_{n,k}(z) d\sigma_k(z) = 0 \) and so
\[ \int_{\Gamma_k} \frac{t\Psi_{n,k}(t)}{z - t} d\sigma_k(t) = z \int_{\Gamma_k} \frac{\Psi_{n,k}(t)}{z - t} d\sigma_k(t) - \int_{\Gamma_k} \Psi_{n,k}(t) d\sigma_k(t) = z\Psi_{n,k+1}(z). \]

Now, using (2.22) in (3.6) we find that for \( n \geq p, 0 \leq k \leq p \),
\[ z^{\ell_n+1-k} \psi_{n,k}(z^{p+1}) = z^{\ell_n+1-k} \psi_{n+1,k}(z^{p+1}) + a_n z^{\ell_n-p-k} \psi_{n-p,k}(z^{p+1}), \]
where we are using the notation \( \ell_n \) to denote the remainder of \( n \) on dividing by \( p+1 \), \( 0 \leq \ell_n \leq p \).

The relations (3.7) and (3.8) then follow from the fact that \( \ell_{n+1} = \ell_{n-p} = \ell_n + 1 \) when \( \ell_n \leq p-1 \), while \( \ell_{n+1} = \ell_{n-p} = 0 \) when \( \ell_n = p \).

**Lemma 3.3.** Let \( n \geq p \) and suppose that \( n \equiv k \mod p \), \( 0 \leq k \leq p-1 \), and \( n \equiv \ell \mod (p+1) \), \( 0 \leq \ell \leq p \). Then
\[ \int_{a_k}^{b_k} \psi_{n+1,k}(\tau) \tau^{Z(n,k)-Z(n,k+1)} d\sigma_{n,k}(\tau) = 0, \quad \ell \leq p-1, \tag{3.9} \]
\[ \int_{a_k}^{b_k} \psi_{n+1,k}(\tau) \tau^{Z(n,k)-Z(n,k+1)-1} d\sigma_{n,k}(\tau) = 0, \quad \ell = p, \tag{3.10} \]
and
\[ \tau^{Z(n-p,k)-Z(n-p,k+1)} d\sigma_{n-p,k}(\tau) = \begin{cases} \tau^{Z(n,k)-Z(n,k+1)} d\sigma_{n,k}(\tau), & \ell \leq p-1, \\ \tau^{Z(n,k)-Z(n,k+1)-1} d\sigma_{n,k}(\tau), & \ell = p. \end{cases} \tag{3.11} \]

**Proof.** Obviously,
\[ n + 1 \equiv \begin{cases} \ell + 1 \mod (p+1), & \ell \leq p-1, \\ 0 \mod (p+1), & \ell = p. \end{cases} \tag{3.12} \]

With this in mind, we readily get from (2.21) that (3.9) and (3.10) are equivalent to
\[ \int_{a_k}^{b_k} \psi_{n+1,k}(\tau) \tau^{Z(n,k)-Z(n,k+1)} d\sigma_{n+1,k}(\tau) = 0, \quad \ell \neq k, \tag{3.13} \]
\[ \int_{a_k}^{b_k} \psi_{n+1,k}(\tau) \tau^{Z(n,k)-Z(n,k+1)-1} d\sigma_{n+1,k}(\tau) = 0, \quad \ell = k. \tag{3.14} \]

Suppose \( n \geq p, n \equiv k \mod p \), \( 0 \leq k \leq p-1 \). Then
\[ n + 1 \equiv \begin{cases} k + 1 \mod p, & k < p-1, \\ 0 \mod p, & k = p-1. \end{cases} \tag{3.15} \]
We then use (2.49) to analyze all possible cases arising from (3.12) and (3.15). Using the notation
\[ \lambda(n) = \left\lfloor \frac{n}{p(p+1)} \right\rfloor, \]
we find that
\[ Z(n, k) - Z(n, k + 1) = \lambda(n) \]
and
\[ Z(n + 1, k) - Z(n + 1, k + 1) = \begin{cases} 
\lambda(n + 1), & \ell = k, \\
\lambda(n + 1), & \ell = p, \; k = p - 1, \\
\lambda(n + 1) + 1 & \text{otherwise.}
\end{cases} \]
Now, \( \lambda(n + 1) = \lambda(n) \) if \( n + 1 \) is not a multiple of \( p(p+1) \), and \( \lambda(n + 1) = \lambda(n) + 1 \) otherwise. The latter case holds exactly when \( \ell = p \) and \( k = p - 1 \). We then conclude that in all instances the exponent of \( \tau \) in (3.13) and (3.14) equals \( Z(n + 1, k) - Z(n + 1, k + 1) - 1 \). Notice that this quantity is nonnegative since the smallest integer \( n \geq p \) satisfying that \( \ell = k \) (that is, having the same remainder when divided by \( p \) and by \( p+1 \)) is \( n = p(p+1) \). This together with (2.32) yields (3.9) and (3.10).

Now, both \( n \) and \( n - p \) leave the same remainder \( k \) when they are divided by \( p \). If \( \ell \leq p - 1 \), then \( n - p \equiv \ell + 1 \mod(p+1) \), and (2.49) yields
\[ Z(n - p, k) - Z(n - p, k + 1) = \begin{cases} 
Z(n, k) - Z(n, k + 1), & \ell \neq k, \\
Z(n, k) - Z(n, k + 1) - 1, & \ell = k,
\end{cases} \]
while from (2.21) we obtain
\[ d\sigma_{n-p,k}(\tau) = \begin{cases} 
d\sigma_{n,k}(\tau), & \ell \neq k, \\
\tau d\sigma_{n,k}(\tau), & \ell = k.
\end{cases} \]
So we see that (3.11) holds in the case \( \ell \leq p - 1 \). Similarly, if \( \ell = p \), then \( n - p \equiv 0 \mod(p+1) \), so that, again by (2.30) and (2.21), we have
\[ Z(n - p, k) - Z(n - p, k + 1) = Z(n, k) - Z(n, k + 1), \quad \ell = p, \]
and
\[ \tau d\sigma_{n-p,k}(\tau) = d\sigma_{n,k}(\tau), \quad \ell = p. \]

**Lemma 3.4.** Suppose that \( n \equiv \ell \mod(p+1) \), \( 0 \leq \ell \leq p \), and that \( n = mp + k \) with \( 0 \leq k \leq p - 1 \). With the notation \( \Delta_k := (a_k, b_k) \), we have
\[ \text{sign}(P_{n,k-1}P_{n,k+1}, \Delta_k) = \begin{cases} 
1 & \text{if } k \text{ is even}, \\
-1 & \text{if } k \text{ is odd and } k \neq \ell, \\
1 & \text{if } k \text{ is odd and } k = \ell.
\end{cases} \quad (3.16) \]
Also, for every \( j \) in the range \( 0 \leq j \leq k \), we have
\[ \text{sign}(H_{n,j}, \Delta_j) = \begin{cases} 
(-1)^j, & j \leq \ell, \\
1, & \ell < j.
\end{cases} \quad (3.17) \]
Proof. From (2.63), we know that for every $1 \leq j \leq p - 1$,

$$
\text{sign}(H_{n,j}, \Delta_j) = \begin{cases} 
(1)^{1+(j+1)[Z(n,j-2)-Z(n,j)]} \text{sign}(H_{n,j-1}, \Delta_{j-1}), & j \leq \ell, \\
(1)^{j+1}[Z(n,j-2)-Z(n,j)] \text{sign}(H_{n,j-1}, \Delta_{j-1}), & \ell < j,
\end{cases}
$$

(3.18)

and this allows us to compute the sign of $H_{n,j}$ recursively.

From (2.49), for all $j < k$ we obtain

$$
Z(n,j) - Z(n,j+1) = \begin{cases} 
\lambda + 1, & \ell \leq j \text{ or } k < \ell, \\
\lambda, & j < \ell \leq k,
\end{cases}
$$

while if $k \leq j$, then

$$
Z(n,j) - Z(n,j+1) = \begin{cases} 
\lambda, & \ell \leq k \text{ or } j < \ell, \\
\lambda + 1, & k < \ell \leq j,
\end{cases}
$$

Since

$$
Z(n,j-2) - Z(n,j) = Z(n,j-2) - Z(n,j-1) + Z(n,j-1) - Z(n,j),
$$

this implies that if $2 \leq j \leq k$, then

$$
Z(n,j-2) - Z(n,j) = \begin{cases} 
2(\lambda + 1), & \ell < j - 1 \text{ or } k < \ell, \\
2\lambda + 1, & \ell = j - 1, \\
2\lambda, & j - 1 < \ell \leq k,
\end{cases}
$$

(3.19)

and if $j = k + 1$, then

$$
Z(n,k-1) - Z(n,k+1) = \begin{cases} 
2\lambda + 1, & \ell \neq k, \\
2\lambda, & \ell = k.
\end{cases}
$$

(3.20)

That (3.16) holds for $k$ even is trivial, since in such a case $\Delta_k \subseteq (0, \infty)$ while the zeros of the monic polynomials $P_{n,k+1}$ all lie in $\Delta_{k+1} \subseteq (-\infty,0)$.

If $k \geq 1$ is odd, then $\Delta_k \subseteq (-\infty,0)$ and $\Delta_{k-1}, \Delta_{k+1} \subseteq (0,\infty)$, so that

$$
\text{sign}(P_{n,k-1}P_{n,k+1}, \Delta_k) = (-1)^{Z(n,k-1)-Z(n,k+1)},
$$

and (3.16) for $k$ odd follows from (3.20).

Now, directly from (3.18) we get

$$
\text{sign}(H_{n,1}, \Delta_1) = \begin{cases} 
\text{sign}(H_{n,0}, \Delta_0), & \ell = 0, \\
-\text{sign}(H_{n,0}, \Delta_0), & \ell \geq 1,
\end{cases}
$$

(3.21)

while from (3.19) and (3.18), for all $2 \leq j \leq k$ we obtain

$$
\text{sign}(H_{n,j}, \Delta_j) = \begin{cases} 
-\text{sign}(H_{n,j-1}, \Delta_{j-1}), & k < \ell, \\
\text{sign}(H_{n,j-1}, \Delta_{j-1}), & \ell < j - 1, \\
(-1)^{j+1}\text{sign}(H_{n,j-1}, \Delta_{j-1}), & \ell = j - 1, \\
-\text{sign}(H_{n,j-1}, \Delta_{j-1}), & j \leq \ell \leq k.
\end{cases}
$$

(3.22)
This implies that
\[ \text{sign}(H_{n,j}, \Delta_j) = -\text{sign}(H_{n,j-1}, \Delta_{j-1}), \quad 1 \leq j \leq \ell, \]
and iterating this relation we obtain
\[ \text{sign}(H_{n,j}, \Delta_j) = (-1)^j, \quad 0 \leq j \leq \ell \] (3.23)
(recall that \( H_{n,0} \equiv 1 \)).

We now get from (3.22) and (3.23) that if \( \ell < j \leq k \) and \( j > 2 \), then
\[ \text{sign}(H_{n,j}, \Delta_j) = \text{sign}(H_{n,\ell+1}, \Delta_{\ell+1}) = (-1)\ell \text{sign}(H_{n,\ell}, \Delta_{\ell}) = 1. \]

From (3.21) we see that this last relation also holds if \( j = 1 > \ell = 0 \), completing the proof of (3.17).

**Theorem 3.5.** The coefficients \( a_n \) of the recurrence relation (3.1) are all positive, that is, \( a_n > 0 \) for every \( n \geq p \).

**Proof.** It follows directly from (3.7), (3.8) and Lemma 3.3 that for all \( n \geq p \), with \( n \equiv k \mod p \), we have
\[ \int_{a_k}^{b_k} \tau Z(n,k) - Z(n,k+1) \psi_{n,k}(\tau) \, d\sigma_{n,k}(\tau) = a_n \int_{a_k}^{b_k} \tau Z(n-p,k) - Z(n-p,k+1) \psi_{n-p,k}(\tau) \, d\sigma_{n-p,k}(\tau). \] (3.24)

Since \( \deg(P_{n,k}) = Z(n,k) \), from (2.56) we obtain
\[ \int_{a_k}^{b_k} \tau Z(n,k) - Z(n,k+1) \psi_{n,k}(\tau) \, d\sigma_{n,k}(\tau) = \int_{a_k}^{b_k} \tau Z(n,k) - Z(n,k+1) P_{n,k+1}(\tau) \psi_{n,k}(\tau) \, d\sigma_{n,k}(\tau) \]
\[ = \int_{a_k}^{b_k} P_{n,k}^2(\tau) \frac{H_{n,k}(\tau)}{P_{n,k-1}(\tau)P_{n,k+1}(\tau)} \, d\sigma_{n,k}(\tau). \] (3.25)

It follows from Lemma 3.4 and (2.21) that if \( n = mp + k \), then
\[ \text{sign} \left( \int_{a_k}^{b_k} P_{n,k}^2(\tau) \frac{H_{n,k}(\tau)}{P_{n,k-1}(\tau)P_{n,k+1}(\tau)} \, d\sigma_{n,k}(\tau) \right) = (-1)^k, \]
and since \( n - p = (m - 1)p + k \), we conclude that the two integrals in (3.24) have the same sign, and thus \( a_n > 0 \).

**Corollary 3.6.** The nonzero roots of the polynomials \( Q_n \) and \( Q_{n+1} \) interlace on \( \Gamma_0 \) for every \( n \geq p + 1 \), that is, between two consecutive nonzero roots of \( Q_n \) there is exactly one nonzero root of \( Q_{n+1} \) and vice versa.

**Proof.** This interlacing property is a consequence of (3.1), (3.2) and the fact that the recurrence coefficients are positive, as shown in Theorem 2.2 in [14].

We remark that for each \( 0 \leq k \leq p - 1 \) the zeros of the polynomials \( P_{n,k} \) and \( P_{n+1,k} \) interlace on \( (a_k, b_k) \). This result was proved in [10], Theorem 3.1.
§ 4. Normalization

In this section we introduce a convenient normalization of the polynomials $P_{n,k}$ and the functions $H_{n,k}$.

It follows from the definition of the functions $H_{n,k}$ and the polynomials $P_{n,k}$ that the measures
\[
\frac{H_{n,k}(\tau) d\sigma_{n,k}(\tau)}{P_{n,k-1}(\tau) P_{n,k+1}(\tau)}, \quad 0 \leq k \leq p - 1,
\]
have constant sign on the interval $[a_k, b_k]$. We then denote by
\[
\left| H_{n,k}(\tau) \right| d\sigma_{n,k}(\tau) = \left| P_{n,k-1}(\tau) P_{n,k+1}(\tau) \right|
\]
the positive normalization of this measure and we have
\[
\int_{a_k}^{b_k} P_{n,k}(\tau) \ \tau^s \left| H_{n,k}(\tau) \right| d\sigma_{n,k}(\tau) = 0, \quad s = 0, \ldots, Z(n,k) - 1, \quad k = 0, \ldots, p - 1. \tag{4.1}
\]

Let
\[
K_{n,-1} := 1, \quad K_{n,p} := 1, \quad K_{n,k} := \left( \int_{a_k}^{b_k} P_{n,k}(\tau) \ \tau^s \left| H_{n,k}(\tau) \right| d\sigma_{n,k}(\tau) \right)^{-1/2}, \quad k = 0, \ldots, p - 1. \tag{4.2}
\]

We also define the constants
\[
\kappa_{n,k} := \frac{K_{n,k}}{K_{n,k-1}}, \quad k = 0, \ldots, p. \tag{4.4}
\]

**Definition 4.1.** For $k = 0, \ldots, p$, we define
\[
p_{n,k} := \kappa_{n,k} P_{n,k}, \quad h_{n,k} := K_{n,k}^2 H_{n,k}, \tag{4.5}
\]
where the constants $\kappa_{n,k}$ and $K_{n,k}$ are given in (4.4) and (4.2), (4.3), respectively.

We denote by $\nu_{n,k}$ the measure on $[a_k, b_k]$ given by
\[
d\nu_{n,k}(\tau) := \frac{h_{n,k}(\tau) d\sigma_{n,k}(\tau)}{P_{n,k-1}(\tau) P_{n,k+1}(\tau)}, \quad k = 0, \ldots, p - 1.
\]

Again this measure has constant sign in $[a_k, b_k]$, and we denote its sign by $\varepsilon_{n,k}$ and its positive normalization by $|\nu_{n,k}|$; hence
\[
d|\nu_{n,k}|(\tau) = \frac{|h_{n,k}(\tau)| d|\sigma_{n,k}|(\tau)}{|P_{n,k-1}(\tau) P_{n,k+1}(\tau)|} = \varepsilon_{n,k} \frac{h_{n,k}(\tau) d|\sigma_{n,k}|(\tau)}{P_{n,k-1}(\tau) P_{n,k+1}(\tau)}. \tag{4.7}
\]
Proposition 4.2. For each \( k = 0, \ldots, p-1 \), the polynomial \( p_{n,k} \) defined in (4.5) satisfies the following:

\[
\int_{a_k}^{b_k} p_{n,k}(\tau) \tau^s \, d|\nu_{n,k}|(\tau) = 0, \quad s = 0, \ldots, Z(n,k) - 1,
\]

and

\[
\int_{a_k}^{b_k} p_{n,k}^2(\tau) \, d|\nu_{n,k}|(\tau) = 1,
\]

that is, \( p_{n,k} \) is the orthonormal polynomial of degree \( Z(n,k) \) with respect to the positive measure \( |\nu_{n,k}| \).

For each \( k = 1, \ldots, p \), the function \( h_{n,k} \) defined in (4.6) satisfies

\[
h_{n,k}(z) = \begin{cases} 
\varepsilon_{n,k-1} \int_{a_k}^{b_{k-1}} \frac{p_{n,k-1}(\tau)}{z - \tau} \, d|\nu_{n,k-1}|(\tau) & \text{if } k \leq \ell, \\
\varepsilon_{n,k-1} z \int_{a_k}^{b_{k-1}} \frac{p_{n,k-1}(\tau)}{z - \tau} \, d|\nu_{n,k-1}|(\tau) & \text{if } k > \ell.
\end{cases}
\]

Proof. The orthogonality conditions (4.8) are obvious in view of (4.1). The formulae (4.9) and (4.10) follow immediately from (4.5), (4.6), (4.4), (4.2), (4.3) and (2.60).

§ 5. Zero asymptotic distribution

5.1. Definitions and results. In this section we investigate the zero asymptotic distribution of the polynomials \( Q_n \). This distribution will be described in terms of a vector equilibrium problem for logarithmic potentials. Before describing this problem, we introduce some definitions and notation.

Let \( E_k, k = 0, \ldots, p-1 \), be a system of compact subsets of the real line satisfying

\[
E_k \cap E_{k+1} = \emptyset, \quad k = 0, \ldots, p-2.
\]

We assume that

\[
\text{cap}(E_k) > 0, \quad k = 0, \ldots, p-1,
\]

where \( \text{cap}(E) \) denotes the logarithmic capacity of the compact set \( E \). A vector measure

\[
\vec{\nu} = (\nu_0, \nu_1, \ldots, \nu_{p-1})
\]

is called admissible if

1) \( \nu_k \) is a positive Borel measure supported on \( E_k \) for all \( k = 0, \ldots, p-1 \);

2) \( \nu_k \) has total mass \( \|\nu_k\| = 1 - k/p \) for all \( k = 0, \ldots, p-1 \).

We denote by \( \mathcal{M} \) the class of all admissible vector measures.

Given a pair of compactly supported measures \( \nu_1 \) and \( \nu_2 \), let \( I(\nu_1) \) and \( I(\nu_1, \nu_2) \) denote the logarithmic energy of \( \nu_1 \) and the mutual logarithmic energy of \( \nu_1 \) and \( \nu_2 \), respectively, defined by

\[
I(\nu_1) = \iint \log \frac{1}{|x-y|} \, d\nu_1(x) \, d\nu_1(y) \quad \text{and} \quad I(\nu_1, \nu_2) = \iint \log \frac{1}{|x-y|} \, d\nu_1(x) \, d\nu_2(y).
\]
On the class of admissible vector measures $\vec{\nu} = (\nu_0, \ldots, \nu_{p-1})$ we consider the energy functional $J$ defined by

$$J(\vec{\nu}) := \sum_{k=0}^{p-1} I(\nu_k) - \sum_{k=0}^{p-2} I(\nu_k, \nu_{k+1}).$$  \hfill (5.3)

Observe that $J$ is well-defined and $J(\vec{\nu}) \in (-\infty, +\infty]$ for all $\vec{\nu} \in \mathcal{M}$. This type of energy interaction is typical in the study of Nikishin systems on the real line.

The vector equilibrium problem that is relevant in this work is the problem of finding an extremal vector measure $\vec{\mu} \in \mathcal{M}$ that satisfies

$$J(\vec{\mu}) = \inf_{\vec{\nu} \in \mathcal{M}} J(\vec{\nu}) < \infty. \hfill (5.4)$$

Such a measure exists and is unique; see [12] for a proof of this fact and several other important results on logarithmic vector equilibrium problems in the complex plane. The extremal measure $\vec{\mu}$ is the vector equilibrium measure.

The vector equilibrium measure can be characterized in terms of certain equilibrium conditions which we describe next. Given a vector measure $\vec{\nu} = (\nu_0, \ldots, \nu_{p-1}) \in \mathcal{M}$, we consider the combined potentials $W_k^{\vec{\nu}}$ defined by

$$W_k^{\vec{\nu}}(z) = U_{\nu_k}(z) - \frac{1}{2} U_{\nu_{k-1}}(z) - \frac{1}{2} U_{\nu_{k+1}}(z), \quad k = 0, \ldots, p - 1,$$  \hfill (5.5)

where $U^\nu$ denotes the logarithmic potential associated with $\nu$, that is,

$$U^\nu(z) = \int \log \frac{1}{|z - t|} \, d\nu(t),$$

and in (5.5) we take $U^{\nu_{-1}} \equiv 0$ and $U^{\nu_p} \equiv 0$. The following result is an adaptation of a well-known result in the theory of logarithmic vector equilibrium problems; see [12].

Lemma 5.1. The measure $\vec{\mu} = (\mu_0, \ldots, \mu_{p-1}) \in \mathcal{M}$ is the equilibrium measure satisfying (5.4) if and only if there exist finite constants $\{w_k\}_{k=0}^{p-1}$ such that for every $k = 0, \ldots, p - 1$ the following conditions hold:

$$W_k^{\vec{\mu}}(x) \leq w_k \quad \text{for all } x \in \text{supp}(\mu_k),$$  \hfill (5.6)

$$W_k^{\vec{\mu}}(x) \geq w_k \quad \text{for q.e. } x \in E_k.$$  \hfill (5.7)

Let $E \subset \mathbb{C}$ be a compact set, let $\{\nu_n\}_n$ be a sequence of finite positive measures supported on $E$, and let $\nu$ be another finite positive measure on $E$. We write

$$\nu_n \overset{*}{\rightharpoonup} \nu \quad \text{as } n \to \infty,$$

if for every $f \in C(E)$,

$$\lim_{n \to \infty} \int f \, d\nu_n = \int f \, d\nu,$$

if for every $f \in C(E)$,
that is, when the sequence of measures converges to \( \nu \) in the weak-star topology. Given a polynomial \( P \) of degree \( n \geq 1 \), we denote the associated normalized zero counting measure by

\[
\mu_P = \frac{1}{n} \sum_{P(z) = 0} \delta_z,
\]

where \( \delta_z \) is the Dirac mass at \( z \) (the zeros are repeated according to their multiplicity in the sum).

The weak asymptotic result that we present in this paper is obtained under mild assumptions on the measures \( \sigma \) generating the Nikishin system. One of these assumptions is the so-called regularity of the measures in the sense of Stahl and Totik. A measure \( \sigma \) is said to be in the class \( \text{Reg} \) if

\[
\lim_{n \to \infty} \frac{1}{n} \| \pi_n \|_{L^2(\sigma)} = \text{cap}(\text{supp}(\sigma)),
\]

where \( \pi_n \) denotes the \( n \)th monic orthogonal polynomial associated with \( \sigma \).

We refer the reader to [16] for a detailed analysis of the orthogonal polynomials associated with measures in the class \( \text{Reg} \). It is well known that the regularity assumption is indeed a mild condition. For instance, measures \( \sigma \) supported on a compact interval \( I \subset \mathbb{R} \) on which \( \sigma'(x) > 0 \) a.e. are regular.

Let \( E \subset \mathbb{C} \) be a compact set with \( \text{cap}(E) > 0 \) and let \( \varphi \) be a continuous function on \( E \). Recall that the equilibrium measure \( \overline{\mu} \) in the presence of the external field \( \varphi \) is the unique probability measure that minimizes the energy functional \( I(\mu) + 2 \int \varphi \, d\mu \) among all probability measures on \( E \); cf. [15]. The equilibrium measure \( \overline{\mu} \) satisfies

\[
U_{\overline{\mu}}(z) + \varphi(z) \begin{cases} 
\leq w & \text{for all } z \in \text{supp}(\overline{\mu}), \\
\geq w & \text{for q.e. } z \in E
\end{cases}
\]

for some constant \( w \) (called the equilibrium constant). These equilibrium conditions also characterize the equilibrium measure, and we emphasize that if \( E \) is regular with respect to the Dirichlet problem, then in (5.8) the first inequality can be replaced by an equality and the second inequality holds for all \( z \in E \).

We will need the following auxiliary result concerning the zero asymptotic distribution of a sequence of orthogonal polynomials with respect to varying measures.

**Lemma 5.2.** Let \( \sigma \in \text{Reg}, E = \text{supp}(\sigma) \subset \mathbb{R} \), where \( E \) is regular with respect to the Dirichlet problem. Let \( \{\phi_l\}, l \in \Lambda \subset \mathbb{Z}_+, \) be a sequence of positive continuous functions on \( E \) such that

\[
\lim_{l \in \Lambda} \frac{1}{2l} \log \frac{1}{|\phi_l(x)|} = \varphi(x) > -\infty
\]

uniformly on \( E \). Let \( q_l, l \in \Lambda, \) be a sequence of monic polynomials such that \( \deg q_l = l \) and

\[
\int x^k q_l(x) \phi_l(x) \, d\sigma(x) = 0, \quad k = 0, \ldots, l - 1.
\]
Then
\[ \mu_{q_i} \xrightarrow{\ast} \mu \]
and
\[ \lim_{l \in \Lambda} \left( \int |q_l(x)|^2 \phi_l(x) \, d\sigma(x) \right)^{1/2l} = e^{-w}, \]
where \( \mu \) and \( w \) are the equilibrium measure and equilibrium constant in the presence of the external field \( \varphi \) on \( E \).

The above result was proved in [4]. It is a generalization of a result due to Gonchar and Rakhmanov [6], obtained under the more restrictive assumption that \( \text{supp}(\sigma) \) is an interval on which \( \sigma' > 0 \) a.e.

In the following asymptotic results, the measures \( \mu_k \) are the components of the vector equilibrium measure \( \vec{\mu} = (\mu_0, \ldots, \mu_{p-1}) \) that minimizes the energy functional \( (5.3) \) on the space \( \mathcal{M} \) of all admissible vector measures supported on \( E_k = \text{supp}(\sigma_k^*) \), \( k = 0, \ldots, p-1 \), and the constants \( w_k \) are the equilibrium constants satisfying the variational conditions \( (5.6) \) and \( (5.7) \).

**Theorem 5.3.** Let \( (s_0, \ldots, s_{p-1}) = \mathcal{N}(\sigma_0, \ldots, \sigma_{p-1}) \) be the Nikishin system generated by the measures \( \sigma_0, \ldots, \sigma_{p-1} \). Assume that for each \( k = 0, \ldots, p-1 \) the measure \( \sigma_k \) satisfies \( \sigma_k \in \text{Reg} \) and \( \text{supp}(\sigma_k) \) is regular for the Dirichlet problem. Then, for each \( k = 0, \ldots, p-1 \), we have \( \text{supp}(\mu_k) = \text{supp}(\sigma_k^*) \), and the following three limit relations hold:

\[ \mu_{P_n,k} \xrightarrow{\ast} \frac{p}{p-k} \mu_k, \]

\[ \lim_{m \to \infty} \left( \prod_{j=1}^{m} a_{p_j+k} \right)^{1/m} = \exp \left( -\frac{2p}{p+1} \sum_{j=0}^{k} w_j \right) \]

and

\[ \lim_{n \to \infty} |\psi_{n,k}(z)|^{1/Z(n,0)} = \exp \left( -U^{\mu_k}(z) + U^{\mu_{k-1}}(z) - 2 \sum_{j=0}^{k-1} w_j \right) \]

uniformly on compact subsets of \( \mathbb{C} \setminus \{ [a_{k-1}, b_{k-1}] \cup [a_k, b_k] \cup \{0\} \} \). In \( (5.11) \) we take \( U^{\mu_{-1}}, U^{\mu_p} \equiv 0 \), and this formula is also valid for \( k = p \).

We now state the corresponding asymptotic results on the stars \( \Gamma_k \). For each \( k = 0, \ldots, p-1 \) let \( \tilde{\mu}_k \) be the unique rotationally symmetric measure supported on \( \Gamma_k \) such that for every Borel set \( E \subset [a_k, b_k] \),

\[ \tilde{\mu}_k(\{z: z^{p+1} \in E\}) = \mu_k(E). \]

Let \( \omega_{k,j}, j = 0, \ldots, p \), be the \( p+1 \) distinct roots of the equation \( z^{p+1} = (-1)^k \), numbered as usual in such a way that \( 0 \leq \arg \omega_{k,j} < \arg \omega_{k,j+1} < 2\pi \). Then we can write \( \Gamma_k = \bigcup_{j=0}^{p} \Gamma_{k,j} \), with

\[ \Gamma_{k,j} = \left\{ z: z^{p+1} \in [a_k, b_k], \frac{z}{\omega_{k,j}} \geq 0 \right\}. \]
Then, for every Borel set \( F \subset \Gamma_{k,j} \),
\[
\tilde{\mu}_k|_{\Gamma_{k,j}}(F) = \frac{1}{p+1} \mu_k(\{z^{p+1}: z \in F\}).
\]

**Corollary 5.4.** For the zero counting measures \( \mu_{Q_n} \) of the multi-orthogonal polynomials \( Q_n \), we have
\[
\mu_{Q_n} \xrightarrow{\ast} \mu_0 \quad (5.13)
\]
and for every \( k = 0, \ldots, p \),
\[
\lim_{n \to \infty} |\Psi_{n,k}(z)|^{1/n} = \exp\left(-U\tilde{\mu}_k(z) + U\tilde{\mu}_{k-1}(z) - \frac{2}{p+1} \sum_{j=0}^{k-1} w_j\right) \quad (5.14)
\]
uniformly on compact subsets of \( \mathbb{C} \setminus (\Gamma_k \cup \Gamma_{k-1} \cup \{0\}) \). In (5.14) we take \( U\tilde{\mu}^{-1} \), \( U\tilde{\mu}_p \equiv 0 \).

The proofs of these asymptotic results make use of the following auxiliary lemma.

**Lemma 5.5.** Let \( \sigma_j \) be a positive, rotationally symmetric measure on the star \( \Gamma_j = \{z \in \mathbb{C}: z^{p+1} \in [a_j, b_j]\} \), for some \( j = 0, \ldots, p-1 \), and suppose that \( \sigma_j \in \text{Reg} \). Then the measures \( d\sigma_j^*(\tau) \) and \( |\tau|d\sigma_j^*(\tau) \) on \([a_j, b_j] \), where \( \sigma_j^* \) is defined in (2.8), are also in the class \( \text{Reg} \).

**Proof.** We begin by observing that, since \( \text{supp}(\sigma_j) = \{z: z^{p+1} \in \text{supp}(\sigma_j^*)\} \), we have
\[
[\text{cap}(\text{supp}(\sigma_j))]^{p+1} = \text{cap}(\text{supp}(\sigma_j^*))
\]
(see [13], Theorem 5.2.5)

Let \( \pi_n \) be the \( n \)th monic orthogonal polynomial associated with the measure \( \sigma_j \). Then \( \sigma_j \in \text{Reg} \) means that
\[
\lim_{n \to \infty} \|\pi_n\|_{L^2(\sigma_j)}^{1/n} = \text{cap}(\text{supp}(\sigma_j)). \quad (5.15)
\]
By the rotational symmetry of \( \sigma_j \), the monic polynomial \( \omega^{-n}\pi_n(\omega z) \) (where \( \omega = e^{2\pi i/(p+1)} \)) is also orthogonal with respect to \( \sigma_j \), and therefore, for every integer \( m \geq 0 \), \( \pi_{m(p+1)}(z) = L_m(z^{p+1}) \), where \( L_m \) is the \( n \)th monic orthogonal polynomial with respect to \( \sigma_j^* \). Moreover, \( \|\pi_{m(p+1)}\|_{L^2(\sigma_j)}^2 = ||L_m||_{L^2(\sigma_j^*)}^2 \), and so from (5.15) it follows that
\[
\lim_{m \to \infty} \|L_m\|_{L^2(\sigma_j^*)}^{1/m} = [\text{cap}(\text{supp}(\sigma_j))]^{p+1} = \text{cap}(\text{supp}(\sigma_j^*)).
\]
This proves that \( \sigma_j^* \in \text{Reg} \).

Now that we know that \( \sigma_j^* \) is regular, we want to conclude that the measure \( d\lambda(\tau) := |\tau|d\sigma_j^*(\tau) \) is also regular. Let \( \{l_n\} \) be the sequence of monic orthogonal polynomials associated with \( \lambda \), and let \( \{L_n\} \) be the corresponding sequence for \( \sigma_j^* \).

Without loss of generality we assume that \( 0 \leq a_j \leq b_j \). Then, by the extremality property of the monic orthogonal polynomials, we have
\[
b_j^{-1}\|L_{n+1}\|_{L^2(\sigma_j^*)}^2 \leq b_j^{-1}\int_{a_j}^{b_j} |l_n(\tau)|^2 \tau^2 d\sigma_j^*(\tau) \leq ||l_n||_{L^2(\lambda)}^2 \leq \int_{a_j}^{b_j} |L_n|^2 d\lambda \leq b_j\|L_n\|_{L^2(\sigma_j^*)}^2.
\]
Since supp($\sigma_j^*$) = supp($\lambda$), taking nth roots and letting $n \to \infty$ yields the desired result.

5.2. Proof of Theorem 5.3. Let $\Lambda \subset \mathbb{N}$ be a sequence of integers such that for every $k = 0, \ldots, p - 1$,

$$c_k \mu_{P_{n,k}} \xrightarrow{s_{n \in \Lambda}} \mu_k, \quad c_k := 1 - \frac{k}{p},$$

(5.16)

for some positive measures $\mu_k$ on $[a_k, b_k]$. Our goal is to show that the vector measure $\vec{\mu} = \mu_0, \ldots, \mu_{p-1}$ is the unique equilibrium measure satisfying (5.4). This implies that for each $k = 0, \ldots, p - 1$ the sequence of measures $(\mu_{P_{n,k}})_n$ has a unique limit point in the weak-star topology. By the compactness of the unit ball in the space of Borel positive measures with the weak-star topology, we obtain that the limits hold.

Note that $\mu_k$ has mass $c_k$. From (5.16) we obtain

$$\lim_{n \in \Lambda} \frac{c_k}{Z(n,k)} \log |P_{n,k}(z)| = -U^{\mu_k}(z), \quad k = 0, \ldots, p - 1,$$

(5.17)

uniformly on compact subsets of $\mathbb{C} \setminus [a_k, b_k]$. Note also that supp($\mu_k$) $\subset E_k :=$ supp($\sigma_k^*$) for every $k = 0, \ldots, p - 1$, which follows immediately from Corollary 2.22.

For each $k = 0, \ldots, p - 1$ we have the orthogonality conditions

$$\int_{a_k}^{b_k} P_{n,k}(\tau)^s \frac{|h_{n,k}(\tau)| d|\sigma_{n,k}|(\tau)}{P_{n,k-1}(\tau) P_{n,k+1}(\tau)} = 0, \quad s = 0, \ldots, Z(n,k) - 1,$$

(5.18)

where $h_{n,k}$ is defined in (4.6). Recall that $P_{n,-1}, P_{n,p} \equiv 1$. We consider the expression

$$\frac{1}{2Z(n,k)} \log \left( \frac{|P_{n,k-1}(\tau)||P_{n,k+1}(\tau)|}{|h_{n,k}(\tau)|} \right) = \frac{\log |P_{n,k-1}(\tau)| + \log |P_{n,k+1}(\tau)| - \log |h_{n,k}(\tau)|}{2Z(n,k)},$$

(5.19)

associated with the orthogonality measure in (5.18). Applying (5.17) and (2.29) we obtain

$$\lim_{n \in \Lambda} \frac{\log |P_{n,k-1}(\tau)|}{2Z(n,k)} = \lim_{n \in \Lambda} \frac{Z(n,k-1) \log |P_{n,k-1}(\tau)|}{Z(n,k) 2Z(n,k-1)} = -\frac{p}{2(p-k)} U^{\mu_k-1}(\tau),$$

(5.20)

and

$$\lim_{n \in \Lambda} \frac{\log |P_{n,k+1}(\tau)|}{2Z(n,k)} = \lim_{n \in \Lambda} \frac{Z(n,k+1) \log |P_{n,k+1}(\tau)|}{Z(n,k) 2Z(n,k+1)} = -\frac{p}{2(p-k)} U^{\mu_k+1}(\tau),$$

(5.21)

uniformly on $E_k = $ supp($\sigma_k^*$).

To analyze the expression $\log |h_{n,k}(\tau)|/Z(n,k)$, observe that as $p_{n,k-1}^2 |\nu_{n,k-1}|$ is a probability measure (cf. (4.9)), we can find constants $d_k > 0$ and $D_k > 0$, independent of $n$, such that

$$d_k \leq \int \frac{p_{n,k-1}^2(\tau)}{|z-\tau|} d|\nu_{n,k-1}|(\tau) \leq D_k, \quad z \in [a_k, b_k].$$

(5.22)
These bounds together with (4.10) yield

$$\lim_{n \in \Lambda} \frac{\log |h_{n,k}(\tau)|}{2Z(n,k)} = 0$$

(5.23)

uniformly on $E_k = \text{supp}(\sigma_k^*)$. Here we need an observation concerning the possibility that $0 \in E_k$. If this happens and $\ell(n) < k$, then $|h_{n,k}(z)|$ has a factor $|z|$ (see (4.10)) which may destroy (5.23) at $\tau = 0$. To avoid this, we could, if necessary, take the limit as $n \to \infty$ along a subsequence $\Lambda' \subset \Lambda$ such that $\ell(n) < k$ for all $n \in \Lambda'$ and incorporate the factor $|z|$ into the measure $|\sigma_{n,k}|$ in (5.18). So we may assume that (5.23) holds in any case. As a result of (5.20), (5.21) and (5.23) we have the convergence of (5.19) to

$$\lim_{n \in \Lambda} \frac{1}{2Z(n,k)} \log \left( \frac{|P_{n,k-1}(\tau)||P_{n,k+1}(\tau)|}{|h_{n,k}(\tau)|} \right) = -\frac{p}{2(p-k)}(U^{\mu_{k-1}}(\tau) + U^{\mu_{k+1}}(\tau))$$

uniformly on $E_k$.

Now we can apply Lemma 5.2 to the sequence $(P_{n,k})$, identifying $q_l$ with this sequence, $\phi_l$ with the weight $|h_{n,k}|/|P_{n,k-1}P_{n,k+1}|$, and $\phi$ with

$$\varphi(z) = -\frac{p}{2(p-k)}(U^{\mu_{k-1}}(z) + U^{\mu_{k+1}}(z)).$$

(5.24)

By Lemma 5.5, the measures $|\sigma_{n,k}|$ are in the class $\text{Reg}$, and the regularity of $\text{supp}(\sigma_k)$ with respect to the Dirichlet problem also implies that $\text{supp}(\sigma_{n,k})$ is regular in the same sense. Hence by Lemma 5.2 we get that for each $0 \leq k \leq p-1$,

$$\mu_{P_{n,k}} \overset{*}{\to}_{n \to \Lambda} \mu_k$$

(5.25)

and

$$\lim_{n \in \Lambda} \left( \int_{a_k}^{b_k} P_{n,k}^2(\tau) \frac{|h_{n,k}(\tau)| d|\sigma_{n,k}|(\tau)}{|P_{n,k-1}(\tau)| |P_{n,k+1}(\tau)|} \right)^{1/(2Z(n,k))} = e^{-\overline{w}_k},$$

(5.26)

where $\overline{\mu}_k$ and $\overline{w}_k$ are the equilibrium measure and equilibrium constant, respectively, in the presence of the external field (5.24) on $E_k$. Hence, from (5.16) and (5.25) it follows that for each $0 \leq k \leq p-1$ we have $\mu_k = c_k \overline{\mu}_k$ and consequently

$$U^{\mu_k}(x) = \frac{1}{2} U^{\mu_{k-1}}(x) - \frac{1}{2} U^{\mu_{k+1}}(x) \begin{cases} = c_k \overline{w}_k & \text{for } x \in \text{supp}(\mu_k), \\ \geq c_k \overline{w}_k & \text{for } x \in E_k \setminus \text{supp}(\mu_k); \end{cases}$$

(5.27)

cf. (5.8).

Finally, let $w_k := c_k \overline{w}_k$, $k = 0, \ldots, p-1$. Then (5.27) shows that the vector measure $\overline{\mu} = (\mu_0, \ldots, \mu_{p-1}) \in \mathcal{M}$ satisfies the variational conditions (5.6) and (5.7) for every $k = 0, \ldots, p-1$ (cf. (5.5)). Therefore, by Lemma 5.1, $\overline{\mu}$ is the unique equilibrium measure satisfying (5.4). This completes the proof of (5.9).

From (5.27) and (6.12) we deduce that $E_k \setminus \text{supp}(\mu_k) = \emptyset$ for all $k$, hence $\text{supp}(\mu_k) = \text{supp}(\sigma_k^*)$ for all $k$. 
We now prove (5.11). From (2.58) and (4.6) we see that
\[ |\psi_{n,k}(z)| = \frac{K_{n,k}^{-2} |h_{n,k}(z)||P_{n,k}(z)|}{|P_{n,k-1}(z)|}, \quad 1 \leq k \leq p. \quad (5.28) \]
Thus, we seek to establish the asymptotic behaviour of each of the factors on the right-hand side of (5.28).

We set
\[ f_{n,k}(z) := \begin{cases} h_{n,k}(z), & \ell(n) < k, \\ zh_{n,k}(z), & \ell(n) \geq k. \end{cases} \]
From (4.9) and (4.10), we see that all functions in the family \( \{f_{n,k}\}_{n} \) are analytic in \( U := \mathbb{C} \setminus ([a_{k-1}, b_{k-1}] \cup \{0\}) \), and for every closed subset \( E \) of \( U \), we have \( \sup_{n \geq 0} \max_{z \in E} |f_{n,k}(z)| < \infty \). By Montel’s theorem, \( \{f_{n,k}\}_{n} \) is a normal family in \( U \). Since no \( f_{n,k} \) vanishes in \( U \) and, in particular, \( |f_{n,k}(\infty)| = 1 \), Hurwitz’s theorem tells us that every normal limit point of \( \{f_{n,k}\}_{n} \) is zero-free in \( U \), which, in view of the normality of the family, implies that for every closed subset \( E \subset U \),
\[ \inf_{n \geq 0} \min_{z \in E} |f_{n,k}(z)| > 0. \]
Therefore, as \( \lim_{n \to \infty} Z(n,0) = \infty \), we have
\[ \lim_{n \to \infty} |h_{n,k}(z)|^{1/Z(n,0)} = 1 \quad (5.29) \]
locally uniformly on \( \mathbb{C} \setminus ([a_{k-1}, b_{k-1}] \cup \{0\}) \).

Applying (5.26) and equalities (4.3) and (4.6) we obtain
\[ \lim_{n \to \infty} \left[ \frac{K_{n,k}}{K_{n,k-1}} \right]^{1/Z(n,k)} = e^{w_k/c_k}, \quad k = 0, \ldots, p - 1. \]

Since \( K_{n,-1} = 1 \) and \( c_0 = 1 \), this yields
\[ \lim_{n \to \infty} K_{n,0}^{1/Z(n,0)} = e^{w_0}. \]

More generally,
\[ \lim_{n \to \infty} K_{n,k}^{1/Z(n,0)} = e^{\sum_{j=0}^{k} w_j}, \quad k = 0, \ldots, p - 1, \quad (5.30) \]
which easily follows by mathematical induction. Indeed,
\[ \lim_{n \to \infty} \frac{Z(n,k)}{Z(n,0)} = \lim_{n \to \infty} \prod_{j=1}^{k} \frac{Z(n,j)}{Z(n,j-1)} = \prod_{j=1}^{k} \frac{p-j}{p-(j-1)} = \frac{p-k}{p} = c_k, \quad 0 \leq k \leq p - 1, \]
so that
\[ K_{n,k}^{1/Z(n,0)} = \left( \frac{K_{n,k}}{K_{n,k-1}} \right)^{1/Z(n,k)} Z(n,k)/Z(n,0), \quad K_{n,k-1} \xrightarrow{n \to \infty} e^{\sum_{j=0}^{k} w_j}. \]

In virtue of (5.9) we have
\[ \lim_{n \to \infty} |P_{n,k}(z)|^{1/Z(n,0)} = e^{-U^{n,k}(z)}, \quad 0 \leq k \leq p - 1, \quad (5.31) \]
locally uniformly on $\mathbb{C} \setminus [a_k, b_k]$. This last equality proves (5.11) in the case $k = 0$. For $1 \leq k \leq p$, the corresponding result follows from (5.28)–(5.31).

Finally, we prove (5.10). Since the coefficients $a_n$ are positive, it follows from (3.24), (3.25) and (4.3) that for all $n \geq p, n \equiv k \mod p$,

$$a_n = \frac{K^2_{n-p,k}}{K^2_{n,k}},$$

so that

$$\prod_{j=1}^m a_{pj+k} = \frac{K^2_{k,k}}{K^2_{mp+k,k}}, \quad 0 \leq k \leq p - 1.$$

Then, using (5.30) and (2.29), we obtain

$$\lim_{m \to \infty} \left( \prod_{j=1}^m a_{pj+k} \right)^{1/m} = \lim_{m \to \infty} \left[ \left( \frac{K^2_{k,k}}{K^2_{mp+k,k}} \right)^{1/Z(mp+k,0)} Z(mp+k,0)/m \right]^{1/m} = \exp \left( -\frac{2p}{p+1} \sum_{j=0}^k w_j \right).$$

5.3. Proof of Corollary 5.4. For each $0 \leq j \leq p$, let $f_j: \Gamma_{0,j} \to [a_0, b_0]$ be the function given by $f_j(t) = t^{p+1}$, which is clearly a homeomorphism. Since $Q_n(z) = z^\ell P_{n,0}(z^{p+1})$, $Q_n$ has a zero at the origin of order $\ell$, and its remaining zeros are the elements of the set $\{f_j^{-1}(\tau): 0 \leq j \leq p, P_{n,0}(\tau) = 0\}$. Thus, for every continuous function $F$ on $\Gamma_0 \cup \{0\}$, we have

$$\int F \, d\mu_{Q_n} = \frac{\ell F(0)}{n} + \frac{n - \ell}{n(p+1)} \sum_{j=0}^p \frac{1}{(n - \ell)/(p + 1)} \sum_{P_{n,0}(\tau) = 0} F(f_j^{-1}(\tau)),$$

whence (5.13) easily follows.

Since $\bar{\mu}_k$ is rotationally symmetric and $\mu_k$ is the push forward of $\bar{\mu}_k$ by the map $z \mapsto z^{p+1}$, we have

$$U^{\bar{\mu}_k}(z^{p+1}) = \sum_{j=0}^p \int_{\Gamma_k} \log \frac{1}{|z - t\omega^j|} \, d\bar{\mu}_k(t) = (p + 1)U^{\bar{\mu}_k}(z). \quad (5.32)$$

Then, (5.14) follows by combining (2.22), (2.29), (5.11) and (5.32).

§ 6. Hermite-Padé approximation

6.1. Definitions and results. In this section we study the Hermite-Padé approximation to the system of functions

$$\hat{s}_j(z) = \int_{\Gamma_0} \frac{ds_j(t)}{z - t}, \quad 0 \leq j \leq p - 1, \quad (6.1)$$

where $(s_0, \ldots, s_{p-1}) = \mathcal{N}(\sigma_0, \ldots, \sigma_{p-1})$ is the Nikishin system of measures defined in (2.1). For this, we closely follow the method employed by Gonchar, Rakhmanov
and Sorokin [7] in their study of Hermite-Padé approximants for generalized Nikishin systems on the real line.

The problem of Hermite-Padé approximation for the system of functions \((6.1)\) is the following. Given a multi-index \(\vec{n} = (n_0, n_1, \ldots, n_{p-1}) \in \mathbb{Z}_+^p\), we seek a nonzero polynomial \(Q_{\vec{n}}\) with \(\deg(Q_{\vec{n}}) \leq |\vec{n}| := n_0 + \cdots + n_{p-1}\) such that for every \(j = 0, \ldots, p-1\),

\[
Q_{\vec{n}}(z) \overset{\sim}{s}_j(z) - Q_{\vec{n},j}(z) = O\left(\frac{1}{z^{n_j+1}}\right), \quad z \to \infty,
\]

where \(Q_{\vec{n},j}\) is the polynomial part in the Laurent series expansion of \(Q_{\vec{n}} \overset{\sim}{s}_j\) at \(z = \infty\). It is easy to see that such a polynomial \(Q_{\vec{n}}\) exists, since the \(p\) conditions \((6.2)\) can be expressed equivalently as a homogeneous system of \(|\vec{n}| + 1\) linear equations with \(|\vec{n}| + 1\) unknowns (the coefficients of \(Q_{\vec{n}}\)), which always has a nontrivial solution. The vector of rational functions

\[
\begin{pmatrix}
Q_{\vec{n},0} \\
Q_{\vec{n},1} \\
\vdots \\
Q_{\vec{n},p-1} \\
Q_{\vec{n}}
\end{pmatrix},
\]

is called a Hermite-Padé approximant associated with \(\vec{n}\) for the system of functions \((6.1)\). If we integrate the expression \(z^l(Q_{\vec{n}}(z) \overset{\sim}{s}_j(z) - Q_{\vec{n},j}(z))\), \(l = 0, \ldots, n_j - 1\), along a closed contour that surrounds \(\Gamma_0\), it easily follows from \((6.2)\) and \((6.1)\) that the polynomial \(Q_{\vec{n}}\) satisfies the multi-orthogonality conditions

\[
\int_{\Gamma_0} Q_{\vec{n}}(z) z^l ds_j(z) = 0, \quad l = 0, \ldots, n_j - 1, \quad 0 \leq j \leq p - 1.
\]

In this paper we will only consider Hermite-Padé approximants associated with multi-indices \(\vec{n} = (n_0, \ldots, n_{p-1}) \in \mathbb{Z}_+^p\) that are defined by

\[
n_j = \left\lfloor \frac{n - j - 1}{p} \right\rfloor + 1, \quad j = 0, \ldots, p - 1,
\]

for a given integer \(n \geq 0\). These multi-indices can equivalently be described as those satisfying the conditions \(n_0 \geq n_1 \geq \cdots \geq n_{p-1}\) and \(n_{p-1} \geq n_0 - 1\), and are uniquely determined by their norm \(|\vec{n}|\), which equals \(n\) if \(\vec{n}\) is defined by \((6.3)\). Let \(I\) denote the sequence of such multi-indices.

Given \(\vec{n} \in I\) with \(|\vec{n}| = n\), we see that \(Q_{\vec{n}}\) satisfies \((2.14)\). Thus, if we assume, as we will, that \(Q_{\vec{n}}\) is monic, then from the normality of the Nikishin system we see that \(Q_{\vec{n}}\) is unique and is given by \(Q_{\vec{n}} = Q_n\). Moreover, since

\[
\int_{\Gamma_0} \frac{Q_n(t)}{z - t} ds_j(t) = O\left(\frac{1}{z^{n_j+1}}\right), \quad z \to \infty,
\]

from \((6.2)\) and the identity

\[
Q_n(z) \overset{\sim}{s}_j(z) - \int_{\Gamma_0} \frac{Q_n(z) - Q_n(t)}{z - t} ds_j(t) = \int_{\Gamma_0} \frac{Q_n(t)}{z - t} ds_j(t)
\]

it follows that the polynomials

\[
Q_{n,j}(z) := \int_{\Gamma_0} \frac{Q_n(z) - Q_n(t)}{z - t} ds_j(t), \quad j = 0, \ldots, p - 1,
\]

are the numerators of the Hermite-Padé approximant associated with \(\vec{n}\).
Definition 6.1. Given \( n > 0 \), for each \( j = 0, \ldots, p - 1 \) we define

\[
\Phi_{n,j+1}(z) = \int \frac{Q_n(t)}{z-t} ds_j(t)
\]

and

\[
\delta_{n,j}(z) = \frac{\Phi_{n,j+1}(z)}{Q_n(z)}.
\]

From (6.4)–(6.7) we deduce that

\[
\delta_{n,j} = \hat{s}_j - \frac{Q_{n,j}}{Q_n}, \quad j = 0, \ldots, p - 1,
\]

that is, \( \delta_{n,j} \) is the remainder in the approximation of \( \hat{s}_j \) by the \( j \)th component of the \( n \)th Hermite-Padé approximant.

Theorem 6.2. Under the same assumptions as Theorem 5.3, we have that for every \( j = 0, \ldots, p - 1 \),

\[
\lim_{n \to \infty} |\delta_{n,j}(z)|^{1/n} = \exp \left( -U_{\bar{\mu}_1}(z) + 2U_{\bar{\mu}_0}(z) - \frac{2}{p+1}w_0 \right)
\]

uniformly on compact subsets of \( \mathbb{C} \setminus (\bigcup_{i=0}^{j+1} \Gamma_i \cup \{0\}) \), where the measures \( \bar{\mu}_0 \) and \( \bar{\mu}_1 \) are defined in (5.12), and \( w_0 = \bar{w}_0 \) is the equilibrium constant in (5.27). In particular, for every \( j = 0, \ldots, p - 1 \),

\[
\lim_{n \to \infty} \frac{Q_{n,j}(z)}{Q_n(z)} = \hat{s}_j(z)
\]

uniformly on compact subsets of \( \mathbb{C} \setminus (\bigcup_{i=0}^{j+1} \Gamma_i \cup \{0\}) \).

Before we give the proof of Theorem 6.2, we make some remarks and prove an auxiliary result.

Note that \( \Phi_{n,1} = \Psi_{n,1} \). More generally, it can be shown (see, for example, [7], §5.3) that for every \( k = 1, \ldots, p \),

\[
\Phi_{n,k}(z) = \sum_{i=1}^{k} (-1)^{i-1} \hat{s}_{i,k-1}(z) \Psi_{n,i}(z), \quad z \in \mathbb{C} \setminus \bigcup_{i=0}^{k-1} \Gamma_i,
\]

where \( \hat{s}_{i,k-1}(z) \) denotes the Cauchy transform of the measure \( s_{i,k-1} = (\sigma_i, \ldots, \sigma_{k-1}) \) (cf. (2.3)), and we take \( \hat{s}_{k,k-1}(z) \equiv 1 \). Observe that (2.11) implies that the function \( \hat{s}_{i,k-1}(z) \) does not vanish on \( \mathbb{C} \setminus (\Gamma_i \cup \{0\}) \).

Lemma 6.3. Let \( (\mu_0, \ldots, \mu_{p-1}) \) be the vector equilibrium measure satisfying (5.9), and \( w_0, \ldots, w_{p-1} \) the associated equilibrium constants. Then for each \( k = 0, \ldots, p - 1 \) we have

\[
2U_{\mu_k}(z) - U_{\mu_{k-1}}(z) - U_{\mu_{k+1}}(z) - 2w_k < 0 \quad \text{for all } z \in \overline{\mathbb{C}} \setminus \text{supp}(\mu_k),
\]

where \( U_{\mu_{-1}} \equiv 0 \) and \( U_{\mu_p} \equiv 0 \).
Proof. According to (5.27), we have

$$2U^{\mu_k}(x) - U^{\mu_k-1}(x) - U^{\mu_k+1}(x) - 2w_k = 0, \quad x \in \text{supp}(\mu_k), \quad k = 0, \ldots, p - 1.$$ \hfill (6.13)

This implies that $U^{\mu_k}$ is continuous on $\text{supp}(\mu_k)$, and hence $U^{\mu_k}$ is continuous on $\mathbb{C}$ for all $k$. The measure $2\mu_k - \mu_{k-1} - \mu_{k+1}$ has total mass $1 + 1/p$ if $k = 0$ and has total mass 0 for all other values of $k$. Therefore the function $2U^{\mu_k}(z) - U^{\mu_k-1}(z) - U^{\mu_k+1}(z) - 2w_k$ is subharmonic on $\overline{\mathbb{C}} \setminus \text{supp}(\mu_k)$. By the maximum principle for subharmonic functions applied to this function, (6.13) implies (6.12).

6.2. Proof of Theorem 6.2. We already observed that $\Phi_{n,1} = \Psi_{n,1}$. In view of (5.13), (5.14) and (6.7) we obtain

$$\lim_{n \to \infty} |\delta_{n,0}(z)|^{1/n} = \lim_{n \to \infty} \frac{|\Psi_{n,1}(z)|^{1/n}}{|Q_n(z)|^{1/n}} = \exp\left( -U^{\tilde{\mu}_1}(z) + 2U^{\tilde{\mu}_0}(z) - \frac{2}{p + 1}w_0 \right)$$

uniformly on compact subsets of $\mathbb{C} \setminus (\Gamma_1 \cup \Gamma_0 \cup \{0\})$, which is (6.9) for $j = 0$.

Let $1 \leq j \leq p - 1$. By (6.11) we can write for $z \in \mathbb{C} \setminus (\bigcup_{l=0}^j \Gamma_l \cup \{0\})$,

$$\Phi_{n,j+1}(z) = \tilde{s}_{1,j}(z)\Psi_{n,1}(z) \left( 1 - \frac{\tilde{s}_{2,j}(z)\Psi_{n,2}(z)}{\tilde{s}_{1,j}(z)\Psi_{n,1}(z)} + \cdots + (-1)^j \frac{\Psi_{n,j+1}(z)}{\tilde{s}_{1,j}(z)\Psi_{n,1}(z)} \right). \hfill (6.14)$$

Now, applying (5.32) and (6.12), we see that for every $i = 1, \ldots, j$,

$$-U^{\tilde{\mu}_i}(z) + U^{\tilde{\mu}_{i-1}}(z) - \frac{2}{p + 1} \sum_{l=0}^{i-1} w_l > -U^{\tilde{\mu}_{i+1}}(z) + U^{\tilde{\mu}_i}(z) - \frac{2}{p + 1} \sum_{l=0}^i w_l$$

for all $z \in \mathbb{C} \setminus \text{supp}(\tilde{\mu}_i)$. This implies, in view of (5.14), that

$$\lim_{n \to \infty} \frac{\Psi_{n,i+1}(z)}{\Psi_{n,1}(z)} = 0, \quad 1 \leq i \leq j,$$

locally uniformly on $\mathbb{C} \setminus (\bigcup_{l=0}^{j+1} \Gamma_l \cup \{0\})$. This and (6.14) give

$$\lim_{n \to \infty} |\delta_{n,j}(z)|^{1/n} = \lim_{n \to \infty} \frac{|\Phi_{n,j+1}(z)|^{1/n}}{|Q_n(z)|^{1/n}} = \lim_{n \to \infty} \frac{|\Psi_{n,1}(z)|^{1/n}}{|Q_n(z)|^{1/n}} = \exp\left( -U^{\tilde{\mu}_1}(z) + 2U^{\tilde{\mu}_0}(z) - \frac{2}{p + 1}w_0 \right)$$

uniformly on compact subsets of $\mathbb{C} \setminus (\bigcup_{l=0}^{j+1} \Gamma_l \cup \{0\})$. From (6.13) for $k = 0$ and (5.32) we see that $-U^{\tilde{\mu}_1}(z) + 2U^{\tilde{\mu}_0}(z) - (2/(p + 1))w_0 < 0$ on $\mathbb{C} \setminus (\bigcup_{l=0}^{j+1} \Gamma_l \cup \{0\})$; hence (6.10) follows from (6.7) and (6.9).

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Bibliography

[1] A.I. Aptekarev, V.A. Kalyagin and E.B. Saff, “Higher-order three-term recurrences and asymptotics of multiple orthogonal polynomials”, Constr. Approx. 30:2 (2009), 175–223.
[2] A. I. Aptekarev, V. A. Kaliaguine (Kalyagin) and J. Van Iseghem, “The genetic sums’ representation for the moments of a system of Stieltjes functions and its applications”, Constr. Approx. 16:4 (2000), 487–524.

[3] S. Delvaux and A. López, “High-order three-term recursions, Riemann-Hilbert minors and Nikishin systems on star-like sets”, Constr. Approx. 37:3 (2013), 383–453.

[4] U. Fidalgo Prieto, A. López García, G. López Lagomasino and V. N. Sorokin, “Mixed type multiple orthogonal polynomials for two Nikishin systems”, Constr. Approx. 32:2 (2010), 255–306.

[5] U. Fidalgo Prieto and G. López Lagomasino, “Nikishin systems are perfect”, Constr. Approx. 34:3 (2011), 297–356.

[6] A. A. Gonchar and E. A. Rakhmanov, “Equilibrium measure and the distribution of zeros of extremal polynomials”, Mat. Sb. (N.S.) 125(167):1(9) (1984), 117–127; English transl. in Math. USSR-Sb. 53:1 (1986), 119–130.

[7] A. A. Gonchar, E. A. Rakhmanov and V. N. Sorokin, “Hermite-Padé approximants for systems of Markov-type functions”, Mat. Sb. 188:5 (1997), 33–58; English transl. in Sb. Math. 188:5 (1997), 671–696.

[8] M. X. He and E. B. Saff, “The zeros of Faber polynomials for an $m$-cusped hypocycloid”, J. Approx. Theory 78:3 (1994), 410–432.

[9] A. López García, “Asymptotics of multiple orthogonal polynomials for a system of two measures supported on a starlike set”, J. Approx. Theory 163:9 (2011), 1146–1184.

[10] A. López-García and G. López Lagomasino, “Nikishin systems on star-like sets: ratio asymptotics of the associated multiple orthogonal polynomials”, J. Approx. Theory 225 (2018), 1–40; arXiv:1612.01149.

[11] E. M. Nikišin (Nikishin), “On simultaneous Padé approximants”, Mat. Sb. (N.S.) 113(155):4(12) (1980), 499–519; English transl. in Math. USSR-Sb. 41:4 (1982), 409–425.

[12] E. M. Nikishin and V. N. Sorokin, Rational approximations and orthogonality, Nauka, Moscow 1988, 256 pp.; English transl. in Transl. Math. Monogr., vol. 92, Amer. Math. Soc., Providence, RI 1991, viii+221 pp.

[13] T. Ransford, Potential theory in the complex plane, London Math. Soc. Stud. Texts, vol. 28, Cambridge Univ. Press, Cambridge 1995, x+232 pp.

[14] N. Ben Romdhane, “On the zeros of $d$-symmetric $d$-orthogonal polynomials”, J. Math. Anal. Appl. 344:2 (2008), 888–897.

[15] E. B. Saff and V. Totik, Logarithmic potentials with external fields, Grundlehren Math. Wiss., vol. 316, Springer-Verlag, Berlin–Heidelberg 1997, xvi+505 pp.

[16] H. Stahl and V. Totik, General orthogonal polynomials, Encyclopedia Math. Appl., vol. 43, Cambridge Univ. Press, Cambridge 1992, xii+250 pp.

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