Vibration analysis of structural elements using differential quadrature method

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Abstract The method of differential quadrature is employed to analyze the free vibration of a cracked cantilever beam resting on elastic foundation. The beam is made of a functionally graded material and rests on a Winkler–Pasternak foundation. The crack action is simulated by a line spring model. Also, the differential quadrature method with a geometric mapping are applied to study the free vibration of irregular plates. The obtained results agreed with the previous studies in the literature. Further, a parametric study is introduced to investigate the effects of geometric and elastic characteristics of the problem on the natural frequencies.

Introduction

In recent years, differential quadrature method (DQM) has become increasingly popular in the numerical solution of initial and boundary value problems. The advantages of this method lie in its easy use and flexibility with regard to arbitrary grid spacing. Also, DQM method can yield accurate results with relatively much fewer grid points compared with the previous numerical techniques such as the finite element and finite difference methods. The present work aims to realize the ability of DQM to solve two complicated problems. The first one concerns with the free vibration of elastically supported cracked beams and the second problem concerns with the free vibration of irregular plates.

In general, there are two approaches to analyze the free vibration of the cracked beams. The first one employs the variational principles through a continuous model, see for example, those in [1,2]. The second approach employs the line spring models (LSMs) to simulate existence of the cracks. Shen and Pierre [3] analyzed the free vibration of beams with pairs of symmetric open cracks. Yokoyama and Chen [4] examined the vibration characteristics of a Bernoulli–Euler beam with an edge crack. Qian et al. [5] explained the dynamic behavior and crack detection of a beam with a crack by using the finite element method. Gudmundson [6,7] discussed the dynamic model for beams with cross sectional crack and predicted the changes in resonance frequencies of a structure.
resulting from the crack. Rizos et al. [8] analyzed the cracked structures by measuring the modal characteristics. All of these works [3–8] concerned with the cracked beams made of an isotropic materials.

Also, there are several publications concerned with the vibration analysis of plates. Leissa [9] derived exact solutions for the free vibration problems of the rectangular plates. Xiang et al. [10] used Ritz method to analyze the vibration of rectangular Mindlin plates resting on elastic edge supports. More recently, DQM is extensively applied for solving vibration problems. Bert and Malik [11] introduced a review on the early stages of the method development and its applications. Also, they [12] made the first attempt to apply DQM for vibration analysis of irregular plates. Liew et al. [13] and Han and Liew [14] also used a similar approach to analyze irregular quadrilateral thick plates. Lam [15] introduced a mapping technique to apply the DQ method to conduction, torsion, and heat flow problems with arbitrary geometries.

Functionally graded materials (FGMs), a novel class of macroscopically nonhomogeneous composites with spatially continuous material properties, have attracted considerable research efforts over the past few years due to their increasing applications in many engineering fields. Numerous studies have been conducted on FGM beams and plates, dealing with a variety of subjects such as thermal elasticity [16,17], fracture mechanics [18,19], and vibration analysis [20–25].

The present work aims to extend the applications of DQM to solve two difficult problems. The first one concerns with the free vibration of an elastically supported cracked beam. The beam is made of a FGM and rests on a Winkler–Pasternak foundation. The line spring model is employed to simulate the crack actions. In the second problem, the DQM with a mapping technique are applied to analyze the free vibration of irregular plates. The obtained results are agreed with the previous similar ones. Further, a parametric study is introduced to explain the effects of elastic and geometric characteristics of the problem on the values of natural frequencies.

**Methodology**

*Free vibration analysis of cracked beams*

Consider an elastically cantilever beam of length $L$ and thickness $h$, containing an edge crack of depth $a$ located at a distance $L_1$ from the left end, as shown in Fig. 1. The beam is made of a FGM, such that shear modulus, Young’s modulus, and mass density of the beam vary in the thickness direction only as follows [28]:

$$
\mu(z) = \mu_0 e^{az}, \quad E(z) = E_0 e^{az}, \quad \rho(z) = \rho_0 e^{az},
$$

where $\mu_0$, $E_0$, and $\rho_0$ are shear modulus, Young’s modulus, and mass density at the mid-plane, $(z = 0)$, of the beam. $z$ is a constant characterizing the beam material grading, $z = \ln(E_z/E_0)/h$, where $E_1$ and $E_2$ are the values of Young’s modulus at the lower and upper beam surfaces, respectively.

It is assumed that the crack is always open and its surfaces are free of traction, such that the beam can be treated as a two sub-beams connected by an elastic rotational spring at the cracked section which has no mass and no length. The bending stiffness of the cracked section, $K_F$, is related to the flexibility $G$ by:

$$
K_F = \frac{1}{G}
$$

Referring to Broek’s approximation [26], one can find that flexibility is governed by:

$$
da G = \frac{2\pi(1 - \nu^2)K_F^2}{E(z)M_1^2},
$$

where $\nu$ is Poisson’s ratio. $M_1$ is bending moment at the cracked section. $K_F$ is mode I stress intensity factor, which can be obtained as a special case of the results introduced by Erdogan and Wu [27].

The governing equations, for the prescribed cracked FGM beam, can be written as [28]:

$$
A_i \frac{\partial^2 w_i}{\partial x^2} - B_{11} \frac{\partial^2 w_i}{\partial x^2} = 0 \quad (i = 1, 2),
$$

$$
\left( D_{11} - \frac{B_{11}^2}{A_{11}} \right) \frac{\partial^2 w_i}{\partial x^2} + K_{11} - k \frac{\partial w_i}{\partial x} + I_{11} \frac{\partial^2 w_i}{\partial x^2} = 0, \quad (i = 1, 2),
$$

where the subscript $i = 1$ stands for the left sub-beam in $(0 \leq x \leq L_1)$, while $i = 2$ holds true for the right sub-beam occupying $(L_1 \leq x \leq L)$, see Fig. 1.

$w_i$, $u_i$ are the components of displacement vector in the directions of $z$ and $x$, respectively. $t$ is time. $K$, $k$ are elastic and shear modules of the foundation reaction, respectively.

$$
(A_{11}, B_{111}, D_{11}) = \int_{-h/2}^{h/2} \frac{E(z)}{1 - \nu^2} (1, z, z^2) dz, \quad I_1 = \int_{-h/2}^{h/2} \rho(z) dz
$$

The normal force $N_i(X, t)$, bending moment $M_i(X, t)$, and transverse shear force $Q_i(X, t)$ are related to the displacement components as follows [28]:

$$
N_i(X, t) = A_{11} \frac{\partial u_i}{\partial X} - B_{11} \frac{\partial^2 w_i}{\partial X^2}, \quad (i = 1, 2),
$$

$$
M_i(X, t) = B_{11} \frac{\partial u_i}{\partial X} + D_{11} \frac{\partial^2 w_i}{\partial X^2}, \quad (i = 1, 2),
$$

$$
Q_i(X, t) = B_{11} \frac{\partial u_i}{\partial X^2} - D_{11} \frac{\partial^2 w_i}{\partial X^2}, \quad (i = 1, 2).
$$

Let $x$ be a normalized parameter defined as:

$$
x = \begin{cases} 
X/L_1 & i = 1 \\
(X - L_1)/L_2 & i = 2
\end{cases}, \quad L_2 = L - L_1.
$$

Also for free vibration analysis, let the prescribed field quantities can be expressed as:

$$
u_i(x, t) = U_i(x) \sin \omega t, \quad w_i(x, t) = W_i(x) \sin \omega t,
$$

$$
N_i(x, t) = N_i(x) \sin \omega t, \quad M_i(x, t) = M_i(x) \sin \omega t, \quad Q_i(x, t) = Q_i(x) \sin \omega t, \quad (i = 1, 2),
$$

where $\omega$ is the natural frequency of the cracked FG beam.
Then, by substituting from Eq. (10) into Eqs. (4), (5), and (7)-(9), the problem can be reduced to a quasi-static one as follows:

\[ A_{ij} \frac{d^{2}U_{j}}{dx^{2}} - B_{ij} \frac{dW_{j}}{dx} = 0, \quad (i = 1, 2), \quad (11) \]

\[ \left( D_{11} - A_{11} \right) \frac{1}{L_{1}} \frac{d^{3}W_{1}}{dx^{3}} + KW_{1} - k \frac{1}{L_{1}} \frac{d^{2}W_{1}}{dx^{2}} = \alpha^{2}I_{1}W_{1}, \quad (i = 1, 2), \quad (12) \]

The boundary conditions (at the clamped and free ends), can be written as:

\[ U_{i}(0) = W_{i}(0) = \left. \frac{dW_{i}}{dx} \right|_{x=0} = 0, \quad (13) \]

\[ N_{2}(1) = Q_{2}(1) = M_{2}(1) = 0, \quad (14) \]

The boundary conditions, (at the cracked section), can be described through a line spring model as follows [28]:

\[ U_{i}(1) = U_{2}(0), \quad W_{i}(1) = W_{2}(0), \quad (15) \]

\[ N_{1}(1) = N_{2}(0), \quad M_{1}(1) = M_{2}(0), \quad Q_{1}(1) = Q_{2}(0), \quad (16) \]

\[ \frac{1}{L_{1}} \frac{dW_{1}}{dx} - \frac{1}{L_{2}} \frac{dW_{2}}{dx} = \frac{1}{K_{f}}M_{1}(1), \quad (17) \]

**Differential quadrature solution**

The method of DQ requires to discretize the domain of the problem into N points. Then the derivatives at any point are approximated by a weighted linear summation of all the functional values along the discretized domain, as follows [29–32]:

\[ \frac{d^{m}f(x)}{dx^{m}} \approx \sum_{j=1}^{N} C_{ij}f(x_{j}), \quad (i = 1, N), \quad (m = 1, M), \quad (18) \]

where M is the order of the highest derivative appearing in the problem.

\[ f(x_{i}) \] are the values of the function at the sampling points \( x_{i} \), \( (i = 1, N); N > M, \) \( C_{ij} \), \( (i, j = 1, N) \), are the weighting coefficients relating the mth derivative to the functional values at \( x_{j} \). These coefficients can be determined by making use of Lagrange interpolation formula as follows [29,30]:

\[ f(x) = \sum_{j=1}^{N} \frac{L(x)}{(x-x_{j})L(x_{j})}f(x_{j}), \quad (19) \]

where \( L(x) = \prod_{j=1}^{N} (x-x_{j}), \) \( L_{j}(x) = \prod_{j=1}^{N} (x_{j}-x_{i}), \) \( (i = 1, N). \)

On substitution from Eq. (19) into (18), the weighting coefficients relating the 1st order derivative to the functional values at \( x_{j} \) can be obtained as [29,30]:

\[ C_{ij}^{(i)} = \begin{cases} \frac{L_{j}(x)}{L_{j}(x_{i})}, & \text{when } (i \neq j), \\ - \sum_{j=1}^{N} C_{ik} \quad & \text{when } (i = j), \end{cases}, \quad (i, j = 1, N). \quad (20) \]

Further, \( C_{ij}^{m} \) relating the higher order derivatives can be obtained as [29,30]:

\[ C_{ij}^{m} = \sum_{k=0}^{m} C_{k}^{m} C_{ij}^{m-k}, \quad (m = 1, M), \quad (i, j = 1, N) \quad (21) \]

The accuracy of DQ results, is affected by choosing of the number, \( N \), and type of the sampling points, \( x_{i} \). It is found that the optimal selection of the sampling points in vibration problems, is the normalized Gauss–Chebyshev–Lobatto points [29,30,33]:

\[ x_{i} = \frac{1}{2} \left[ 1 - \cos \left( \frac{i - 1}{N-1} \pi \right) \right], \quad (i = 1, N). \quad (22) \]

On suitable substitution from Eqs. (18)-(22) into (11)-(17), one can reduce the problem to the following system of linear algebraic equations:

\[ A_{11} \sum_{j=1}^{N} C_{ij}U_{j}(x_{i}) - B_{11} \sum_{j=1}^{N} C_{ij}W_{j}(x_{i}) = 0, \quad (i = 1, N), \quad (23) \]

\[ A_{11} \sum_{j=1}^{N} C_{ij}U_{j}(x_{i}) - B_{11} \sum_{j=1}^{N} C_{ij}W_{j}(x_{i}) = 0, \quad (i = 1, N), \quad (24) \]

\[ \sum_{j=1}^{N} \left[ C_{ij}^{2} \frac{A_{11}}{L_{1}^{2}} \left( A_{11}D_{11} - B_{11}^{2} \right) C_{ij}^{2} \right] W_{j}(x_{i}) = i^{4} W_{j}(x_{i}), \quad (i = 1, N), \quad (25) \]

\[ \sum_{j=1}^{N} \left[ C_{ij}^{2} \frac{A_{11}}{L_{2}^{2}} \left( A_{11}D_{21} - B_{11}^{2} \right) C_{ij}^{2} \right] W_{j}(x_{i}) = i^{4} W_{j}(x_{i}), \quad (i = 1, N), \quad (26) \]

where

\[ x_{i}^{4} = \frac{A_{11}(I_{1} \alpha^{2} - K)}{A_{11}D_{11} - B_{11}^{2}} \quad (27) \]

The boundary conditions can be rewritten using the DQM as follows:

At the clamped end:

\[ U_{i}(x_{1}) = 0, \quad W_{i}(x_{1}) = 0, \quad (28) \]

\[ \sum_{j=1}^{N} C_{ij}W_{j}(x_{1}) = 0. \quad (29) \]

At the free end:

\[ A_{11} \sum_{j=1}^{N} C_{ij}U_{j}(x_{1}) - B_{11} \sum_{j=1}^{N} C_{ij}W_{j}(x_{1}) = 0. \quad (30) \]

\[ A_{11} \sum_{j=1}^{N} C_{ij}U_{j}(x_{1}) - B_{11} \sum_{j=1}^{N} C_{ij}W_{j}(x_{1}) = 0. \quad (31) \]

At the crack location:

\[ U_{i}(x_{N}) = U_{2}(0), \quad W_{i}(x_{N}) = W_{2}(0), \quad (32) \]

\[ A_{11} \sum_{j=1}^{N} C_{ij}U_{j}(x_{N}) - B_{11} \sum_{j=1}^{N} C_{ij}W_{j}(x_{N}) = 0. \quad (33) \]

\[ A_{11} \sum_{j=1}^{N} C_{ij}U_{j}(x_{N}) - B_{11} \sum_{j=1}^{N} C_{ij}W_{j}(x_{N}) = 0. \quad (34) \]

\[ A_{11} \sum_{j=1}^{N} C_{ij}U_{j}(x_{N}) - B_{11} \sum_{j=1}^{N} C_{ij}W_{j}(x_{N}) = 0. \quad (35) \]
Consider a curvilinear quadrilateral plate in Cartesian \( x-y \) plane, see Fig. 2a. A geometric mapping can be applied to transform this irregular plate into a rectangular one in \( \xi - \eta \) plane, as in Fig. 2. The following blending function may be applied to do for this mapping [12,34,35]:

\[
s = \frac{1}{2} \left[ (1-\eta)\tilde{s}_i(\xi) + (1+\xi)\tilde{s}_2(\eta) + (1+\eta)\tilde{s}_3(\xi) + (1-\xi)\tilde{s}_4(\eta) \right] - \frac{1}{4} \left[ (1-\xi)(1-\eta)s_1 + (1+\xi)(1-\eta)s_2 + (1+\xi)(1+\eta)s_3 + (1-\xi)(1+\eta)s_4 \right],
\]

where \( s = x, y, \tilde{s}_i(\xi), \tilde{s}_i(\eta), \tilde{y}_i(\xi), \tilde{y}_i(\eta), \ (i = 1, 4) \), are the parametric forms of the curvilinear boundaries, \( x_i, y_i, (i = 1, 4) \), are the Cartesian coordinates of the corner points of the physical domain, as shown in Fig. 2a.

It is noted that Eq. (38) achieves exact geometric transformation for the boundaries of the curvilinear quadrilateral domain [12,35]. To apply the DQM, one must discretize the computational domain to a grid of dimensions \( N_\xi \times N_\eta \), where \( N_\xi \) and \( N_\eta \) are the number of sampling points in the \( \xi \) and \( \eta \) directions, respectively. The first order partial derivatives at a sampling point \((\xi_i, \eta_j)\), \( (i = 1, N_\xi, \text{and } j = 1, N_\eta) \), can be written as [29–32]:

\[
\frac{\partial f}{\partial x_{ij}} = \frac{1}{|J|} \left[ \left( \frac{\partial y}{\partial \eta} \right)_j \sum_{k=1}^{N_\xi} P_{ik} f_k - \left( \frac{\partial x}{\partial \eta} \right)_j \sum_{i=1}^{N_\eta} R_{ij} f_i \right],
\]

\[
\frac{\partial f}{\partial y_{ij}} = \frac{1}{|J|} \left[ - \left( \frac{\partial x}{\partial \eta} \right)_j \sum_{k=1}^{N_\xi} P_{ik} f_k + \left( \frac{\partial y}{\partial \eta} \right)_j \sum_{i=1}^{N_\eta} R_{ij} f_i \right],
\]

that can be rewritten as [12,35]:

\[
\frac{\partial f}{\partial x_{m}} = \sum_{n=1}^{N_\eta} G_{mn} f_n, \quad \frac{\partial f}{\partial y_{m}} = \sum_{n=1}^{N_\xi} H_{mn} f_n,
\]

where \( N_\xi = N_\xi \times N_\eta \), \( m, n = (i-1)N_\eta + j, \ (i = 1, N_\xi \text{ and } j = 1, N_\eta) \).
$G^{(1)}_{mn}$ and $H^{(1)}_{mn}$ are the weighting coefficients of the first order partial derivatives with respect to $x$ and $y$, respectively. Similarly, one can find

$$\frac{\partial f}{\partial x} = \sum_{n=1}^{N_{x}} G^{(r)}_{mn} f_{n}, \quad \frac{\partial f}{\partial y} = \sum_{n=1}^{N_{y}} H^{(r)}_{mn} f_{n}, \quad \frac{\partial^{r+s} f}{\partial x^{r} \partial y^{s}} = \sum_{n=1}^{N_{x}} V^{(r+s)}_{mn} f_{n},$$

in which the weighting coefficients can be obtained as [12,35]:

$$[G^{(r)}] = [G^{(1)}]^{r}, \quad r \geq 2,$$

$$[H^{(s)}] = [H^{(1)}]^{s}, \quad s \geq 2,$$

$$[V^{(r+s)}] = [G^{(r)}][H^{(s)}], \quad r, s \geq 1.$$

The equation of motion governing the free harmonic vibration of a thin isotropic plate in a dimensionless form, can be written as [33,35]:

Fig. 3  Variation of the fundamental frequency ratio with elastic and geometric characteristics of the problem.
where $X, Y = x/a, y/a$ are dimensionless Cartesian coordinates in the plane of the mid-surface of the plate. $W = W(X, Y)$ is the mode function of plate deflection. $\Omega = \omega a^2 \sqrt{\rho h/D}$ is the dimensionless frequency, where $\omega$ is a natural frequency, $a$ is a characteristic plate dimension, $\rho$ is the plate material density, $h$ is the plate thickness, and $D = E h^3/(1 - \nu^2)$, is the flexural rigidity of the plate, $E$ and $\nu$ being Young’s modulus and Poisson’s ratio of the plate material, respectively.

The boundary conditions, (at the clamped edges), can be written as:

$$W = 0, \quad \cos \theta \frac{\partial W}{\partial X} + \sin \theta \frac{\partial W}{\partial Y} = 0,$$

(45)

The boundary conditions, (at simply supported edges), can be written as:
\[ W = 0, \quad (\cos^2 \theta + \nu \sin^2 \theta) \frac{\partial^2 W}{\partial X^2} + \left( \sin^2 \theta + \nu \cos^2 \theta \right) \frac{\partial^2 W}{\partial Y^2} \]
\[ \times \frac{\partial^2 W}{\partial X \partial Y} + (1 - \nu) \sin 2\theta \frac{\partial^2 W}{\partial X \partial Y} = 0, \quad (46) \]

where \( \theta \) is the angle between normal to the plate boundary and the \( x \)-axis. Using the quadrature rules in Eq. (42), the equation of motion can be reduced to

\[ \sum_{n=1}^{N_{na}} \left[ G_{mn}^1 + 2V_{mn}^{22} + H_{mn}^1 \right] W_n = \Omega^2 W_m, \quad (47) \]

At the clamped edges, the boundary conditions reduces to

\[ W_m = 0, \quad \sum_{n=1}^{N_{na}} \left[ (\cos \theta_n)G_{mn}^1 + (\sin \theta_n)H_{mn}^1 \right] W_n = 0, \quad (48) \]
Table 2 Natural frequencies of a clamped rectangular plate \((N_x = N_y = 11)\).  

| \(\lambda = a/b\) | Mode | \(\omega_1\) | \(\omega_2\) | \(\omega_3\) | \(\omega_4\) | \(\omega_5\) |
|-----------------|------|------------|------------|------------|------------|------------|
|                 | Leissa [9] | Present | Leissa [9] | Present | Leissa [9] | Present | Leissa [9] | Present | Leissa [9] | Present |
| 2/5             | 23.648 | 23.645 | 27.817 | 27.815 | 35.446 | 35.399 | 46.702 | 47.500 | 61.554 | 63.067 |
| 2/3             | 27.010 | 27.007 | 41.716 | 41.706 | 66.143 | 66.108 | 66.552 | 66.354 | 79.850 | 79.831 |
| 1.0             | 35.992 | 35.987 | 73.413 | 73.383 | 73.413 | 73.383 | 108.270 | 108.248 | 131.640 | 131.706 |
| 3/2             | 60.772 | 60.764 | 93.860 | 93.840 | 148.820 | 148.743 | 149.740 | 149.296 | 179.660 | 179.619 |
| 5/2             | 147.800 | 147.784 | 173.850 | 173.847 | 221.540 | 221.245 | 291.890 | 293.885 | 384.710 | 389.163 |

Table 3 Natural frequencies of a simply supported rectangular plate \((N_x = N_y = 11)\).  

| \(\lambda = a/b\) | Mode | \(\omega_1\) | \(\omega_2\) | \(\omega_3\) | \(\omega_4\) | \(\omega_5\) |
|-----------------|------|------------|------------|------------|------------|------------|
|                 | Leissa [9] | Present | Leissa [9] | Present | Leissa [9] | Present | Leissa [9] | Present | Leissa [9] | Present |
| 2/5             | 11.4487 | 11.4487 | 16.1862 | 15.83103 | 24.0818 | 24.0813 | 35.1358 | 36.160 | 41.0576 | 41.0307 |
| 2/3             | 14.2561 | 14.2561 | 27.4156 | 26.7598 | 43.8649 | 43.8395 | 49.3480 | 49.1034 | 57.0244 | 56.0732 |
| 1.0             | 19.7361 | 19.7392 | 49.3480 | 49.325 | 78.9568 | 78.3735 | 98.6960 | 98.6713 | 128.3049 | 126.1394 |
| 3/2             | 32.0762 | 32.0762 | 61.6850 | 61.6674 | 98.6960 | 97.7441 | 111.0330 | 110.4827 | 128.3049 | 126.1394 |
| 5/2             | 71.5564 | 71.5547 | 101.1634 | 101.1529 | 150.5115 | 150.1146 | 229.5987 | 230.1461 | 256.6097 | 255.6151 |

At simply supported edges, the boundary conditions reduces to  
\[
W_m = 0, \quad \sum_{n=1}^{N_x} \left[ (\cos^2 \theta_n + v \sin^2 \theta_n)G_{nm}^2 + (\sin^2 \theta_n + v \cos^2 \theta_n) \right] \times H_{nm}^2 + \left[ (1 - v) \sin 2\theta_n \right] u_{nm}^1, W_m = 0,
\]

On suitable substitutions from Eqs. (48) and (49) into (47), the problem be reduced to eigenvalue one of dimensions \((N_x - 4)(N_y - 4)\). A Matlab program has been designed to solve the problem.

Results and discussions

Cracked beam results

For practical purposes the values of the natural frequencies of the concerned cracked FG beam are divided by that of the un-cracked isotropic beam, such that one can define the fundamental frequency ratio as: \(\omega_{C1}/\omega_1\), where \(\omega_{C1}\) is the fundamental frequency of the cracked FG beam. \(\omega_1\) is that of the un-cracked isotropic beam. A parametric study is introduced to investigate the effects of crack location, crack depth, Young’s modulus gradation ratio, \((E_x/E_y)\), and the foundation moduli, \((K, k)\), on the values of the fundamental frequency ratio, \((\omega_{C1}/\omega_1)\) and the mode shapes. Eqs. (23)–(37) are solved with \(N = (6, 20)\). It is observed that the results for \(N = 10\) are the same as those corresponding to \(N = (11, 20)\). Therefore, the present results are implemented with \(N = 10\).

To examine the validity of the obtained results, the problem of a clamped – free cracked beam made of an isotropic material is considered. It corresponds to a special case of the present analysis when \(E_1 = E_2\) and \(K = k = 0\). This problem was previously solved by Yokoyama and Chen [4], Yang and Chen [28]. Table 1 shows the agreement between the present results and the previous ones in [4,28].

Further, Figs. 3–5 explain the effects of geometric and elastic characteristics of the problem on the values of fundamental frequency ratio and the mode shapes. Fig. 3 show that the values of natural frequencies increase with increasing both of foundation elastic modulus \((K)\) and the distance of crack site from the clamped end \((L_1/L)\). While, these values decrease with increasing of foundation shear modulus \((k)\), the crack depth \((a/b)\), and the gradation of Young’s modulus, \((E_x/E_y)\) across the beam depth as shown. Fig. 4a and b shows the first three mode shapes of the instantaneous lateral and longitudinal displacements, \((w(L_1, t), u(L_1, t))\) at the crack tip when \(L_1/L = 0.35\). At the crack location, Fig. 4c shows that the values of instantaneous lateral fundamental amplitude increase with decreasing of the crack depth. Fig. 5 report that existence of cracks affects on the local flexibility of the beam. Further, these figures show that the values of the lateral amplitude, (along the whole beam), increase with the decreasing both of foundation elastic modulus \((K)\) and the distance of crack site from the clamped end \((L_1/L)\). While, these values are increased with increasing of Young’s graded modulus, \((E_x/E_y)\) ratio.

Irregular plate results

To examine the validity of DQ results, the free vibration problem of rectangular plate is considered. Tables 2 and 3 show the first five natural frequencies \(\omega\) for different aspect ratios \(\lambda = a/b = 2/5, 2/3, 13/2, 5/2\), where \(a\) and \(b\) are lengths of the rectangular plate. Table 2 also, shows the results corresponding to all of the plate edges are clamped (CCCC), while Table 3 shows the results corresponding to all of the plate edges are simply supported (SSSS). For a numerical scheme with \(N_x = N_y = 11\), the DQ results agreed with the previous ones were obtained by Leissa [9].

Further, DQM with a geometric mapping are applied to solve a free vibration problem of an irregular parabolic plate, as in Fig. 6. Referring to Eq. (38), one can find a blending
function for such mapping as: \( (x, y) = \frac{1}{1+\eta} \sqrt{b^2 - (b^2 - c^2)} \)

Furthermore, Fig. 6 explain the effects of aspect ratio \( \lambda = a/b \), vortex angle \( \theta = 2 \tan^{-1}(2ac/(b^2 - c^2)) \), and the flexural rigidity of the plate \( D \) on values of the fundamental frequency ratio \( \omega_{1r}/\omega_1 \), where \( \omega_1 \) is the fundamental frequency of the irregular plate while the \( \omega_1 \) is that of the rectangular one with \( \lambda = 1 \). Fig. 6 show that (for such irregular parabolic plate), the values of \( \omega_{1r}/\omega_1 \) increase with increasing both of \( \lambda \) and \( D \), while the converse is true with the vortex angle \( \theta \).

Conclusion

The method of DQ is applied to analyze the free vibration of an elastically supported cracked beam. Also, the method of DQ with a geometric mapping are employed to solve the free vibration problem of an irregular plate. The new trends in this work are the method of solution (DQM), material of the beam (FGM), elastic foundation model (Winkler–Pasternak), and the irregular boundaries of the plate. So, this work can be considered as an extension for the applications of DQM. Further, the obtained results may be employed to detect, locate, and quantify the extent of the cracks or damages in FG beams.

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