Generating photons from vacuum with counter rotating wave interaction

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(Dated: December 27, 2018)

Recently, the studies on the light-matter interaction have been pushed into the ultrastrong coupling regime, which motivates the exploration of applications of the counter rotating wave (CRW) interaction. Even in the ultrastrong coupling regime, however, few photons can be generated from the vacuum by switching on the CRW interaction. Here we propose a scheme to enhance the photon generation from the vacuum by using a bang-bang (switching on/off) control of the CRW interaction. By developing a pruning greedy algorithm to search the optimal control sequence, we find that the maximum photon number obtained for a given time period in our scheme can be dramatically increased up to several orders than that from switching on the CRW interaction.

I. INTRODUCTION

Recently, the researches on the atom-photon interaction have been pushed into the ultrastrong coupling regime, where the coupling is so strong that the rotating wave approximation (RWA) is no longer valid. In this regime, the counter rotating wave (CRW) interaction cannot be ignored and it will lead to new phenomena, such as the inelastic scattering process in the single-photon scattering with an atom in a one dimensional supercavity or waveguide [2–6], the ground state with non-zero photons in the Rabi model [7, 8], and the multi-photon quantum Rabi oscillations [9]. Nowadays, many researchers have observed the CRW effect on diverse systems, such as circuit QEDs [5, 6, 10–12], the spiropyran molecules [13], and the trapped ion [14]. All these achievements show the growing concern from scientists to the investigation of the CRW interaction.

In general, we study the CRW interaction based on the Rabi model [7, 15] which describes the coupling between a single-mode cavity and a two-level atom. Its Hamiltonian can be expressed as (we set \( \hbar = 1 \))

\[
\hat{H} = \omega_c \hat{a}^\dagger \hat{a} + \frac{\omega_0}{2} \hat{\sigma}_z + g \hat{\sigma}_x (\hat{a}^\dagger + \hat{a}),
\]

(1)

where the operator \( \hat{a} \) \((\hat{a}^\dagger)\) is the photon creation (annihilation) operator for the cavity with its intrinsic frequency \( \omega_c \), \( \hat{\sigma}_- = |g\rangle \langle e| \) \((\hat{\sigma}_+ = \hat{\sigma}_-^\dagger)\) is the atomic lowering (raising) operator with \( |g\rangle \) and \( |e\rangle \) being the ground state and the excited state of the two-level atom respectively, the Pauli operators \( \hat{\sigma}_z = \hat{\sigma}_+ \hat{\sigma}_- - \hat{\sigma}_- \hat{\sigma}_+ \) and \( \hat{\sigma}_x = \hat{\sigma}_+ + \hat{\sigma}_- \), \( \omega_0 \) is the atomic energy splitting, and \( g \) is the coupling strength between atom and cavity. In the Hamiltonian (1), the interaction term can be divided into two parts:

\[
\hat{H}_{\text{int}} = \hat{H}_{\text{int}}^{\text{RW}} + \hat{H}_{\text{int}}^{\text{CRW}},
\]

(2)

where \( \hat{H}_{\text{int}}^{\text{RW}} = g (\hat{\sigma}_+ \hat{a}^\dagger + \hat{a}^\dagger \hat{\sigma}_- ) \) is the ‘rotating wave’ term, and

\[
\hat{H}_{\text{int}}^{\text{CRW}} = g (\hat{\sigma}_+ \hat{a}^\dagger + \hat{a} \hat{\sigma}_- )
\]

(3)

is the CRW term.

Within the RWA, we neglect the CRW term \( \hat{H}_{\text{int}}^{\text{CRW}} \) to get the well-known Jaynes–Cummings model [1] whose ground state is \( |0\rangle \otimes |g\rangle \) where \( |0\rangle \) represents the vacuum state of the cavity. When the CRW term is relevant, non-zero photons will be observed in the ground state of the Rabi model. In this sense, it is natural to treat the CRW term as a photon generator. However, even in the ultrastrong coupling regime, we can only generate few photons from the vacuum merely by the free time evolution of the Rabi model. Besides, it’s difficult to prepare an extremely large coupling strength [16, 17] in current experiments. Thus it is still a challenge to use the CRW interaction to generate more photons from the vacuum \( |0\rangle \otimes |g\rangle \).

Up to now, the qubit-resonator coupling strength can be tuned with time [18, 19]. In particular, we have designed a ‘symmetry protected charge qubit’ and tuned the coupling between the qubit and the superconducting resonator from extremely weak regime to the ultrastrong coupling one [19] with unchanged qubit frequency. These processes make it possible to control the coupling strength varying with time, which will be a key element in our following protocol to generate photons via the CRW interaction.

In this work, we aim to obtain more photons from vacuum by using a bang-bang (switching on/off) control [20–24] of the CRW interaction. As will be shown below, the maximum photon number obtained for a given time period in our scheme can be dramatically increased up to several orders than that from simply switching on the CRW interaction. In the process, we develop a pruning greedy algorithm (PGA) to search for the optimal control sequence and analyze the new phenomena we observed, such as the existence of the photon number steps, the variation of the maximum photon number with the

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atomic frequency, and the upper limit of the maximum photon number for the fixed switching on time of the CRW interaction.

The rest of this paper is organized as follows. In Sec. II we introduce our model and method. With the help of the PGA proposed in Appendix B, we obtain the optimal control sequence of the CRW interaction and observe several new phenomena in Sec. III. In Sec. IV we draw the conclusions. To test the accuracy of our results, we derive the non-linear dynamical equations which go beyond the mean field theory (MFT) in Appendix A. Furthermore, inspired by our work, we propose a different scheme which might be more easier to be achieved in experiments, and show our corresponding results in Appendix C.

II. MODEL AND METHOD

In the bang-bang protocol, when the CRW interaction is switched on, the time evolution of our system is controlled by the Hamiltonian of the Rabi model \( \hat{H} \); when the CRW interaction is switched off, the time evolution is controlled by the free Hamiltonian

\[
\hat{H}_0 = \omega_c \hat{a}^{\dagger} \hat{a} + \frac{\omega_a}{2} \hat{\sigma}_z. \tag{4}
\]

Assume that our system is prepared at the vacuum state \( |0\rangle \otimes |g\rangle \) initially. Note that the parity is conserved both in the Rabi model \( \hat{H} \) and the free Hamiltonian \( \hat{H}_0 \), where the parity is defined as the even-odd check of the total excitation \( \hat{N}_e = \hat{a}^{\dagger} \hat{a} + |e\rangle \langle e| \). Because the parity of our initial state is even, we can truncate the Hilbert space into the subspace with the even parity [5]. Since we only care about the photon number obtained rather than the atomic state, it is convenient to rewrite Eq. (4) as the effective Hamiltonian [25]

\[
\hat{H}_{\text{eff}} = \omega_c \hat{b}^{\dagger} \hat{b} - \frac{\omega_a}{2} (-1)^{\hat{b}^{\dagger} \hat{b}} + g (\hat{b}^{\dagger} + \hat{b}). \tag{5}
\]

Similarly, the effective Hamiltonian of Eq. (4) is

\[
\hat{H}_{\text{eff}}^0 = \omega_c \hat{b}^{\dagger} \hat{b} - \frac{\omega_a}{2} (-1)^{\hat{b}^{\dagger} \hat{b}}, \tag{6}
\]

whose eigenstate is \( |m\rangle \) which satisfies \( \hat{b}^{\dagger} \hat{b} |m\rangle = m |m\rangle \).

\[
\text{It’s worth pointing out that the average photon numbers obtained by the time evolution of } \hat{H} \text{ and } \hat{H}_{\text{eff}} \text{ are the same, i.e., } \langle \hat{a}^{\dagger} \hat{a} \rangle = \langle \hat{b}^{\dagger} \hat{b} \rangle.
\]

\[
\text{Apparently, the free time evolution under the free Hamiltonian } \hat{H}_{\text{eff}}^0 \text{ only changes the phase differences between components in a quantum state but leaves the average photon number unchanged. Thus the time evolution under the Hamiltonian } \hat{H}_{\text{eff}} \text{ determines the variation of photon number with time, which motivates us to examine the way of the dynamics under } \hat{H}_{\text{eff}} \text{ to change the average photon number.}
\]

Let’s consider the time evolution of our system under the effective Hamiltonian \( \hat{H}_{\text{eff}} \) with the initial state \( |0\rangle \). The average photon number \( N_{\text{ph}}^0 \) as a function of \( t \) in the free time evolution of the Rabi model. Here, we take \( \omega_c = \omega_a \), \( g/\omega_c = 0.1 \) and \( T = 15/\omega_c \).

FIG. 1. (Color online). The photon number \( N_{\text{ph}}^0 \) as a function of time \( t \) in the free time evolution of the Rabi model. Here, we take \( \omega_c = \omega_a \), \( g/\omega_c = 0.1 \) and \( T = 15/\omega_c \).
as a series of binary numbers, where the numbers of 1s and 0s are represented by \( n_a \) and \( n_0 \) respectively, and they satisfy \( n_a + n_0 = \frac{T}{\delta t} \). Because the control sequence space contains \( 2^{\frac{T}{\delta t}} \) possible sequences, it is very difficult for us to exhaustively search the optimal sequence for a large \( T \) and a small \( \delta t \).

A possible way to overcome the difficulty is to use the greedy algorithm, in which we take 0 or 1 at each time period depending on which one makes us to get more photons at the end of the time period. We will show that the above greedy method is an effective way to increase the average photon number.

In general, however, the greedy algorithm can not find out the optimal control sequence to obtain maximal average photon number for a given evolution time. To obtain the optimal control sequence, we develop the PGA to search for the optimal sequence by constructing a much smaller searching subspace by sorting and pruning. Compared to the exhaustive searching method, the PGA makes us to search out a possible optimal control sequence for a larger \( T \) and a smaller \( \delta t \). The details of the PGA are given in Appendix. B.

### III. NUMERICAL RESULTS

In this section, we’ll show the numerical results on the average photon numbers in the bang-bang control scheme from the greedy algorithm and the PGA algorithm, and analyze its underlying physics.

![FIG. 2.](image1.png) (Color online). The photon number \( N_{\text{ph}} \) as a function of \( t \). The red solid line represents the photon number \( N_{\text{ph}}^{\text{op}} \) obtained by the free time evolution of the Rabi model. The blue solid line represents the photon number \( N_{\text{ph}}^{\text{PGA}} \) obtained by the optimal quantum control of the CRW interaction. The purple dashed line stands for the photon number \( N_{\text{ph}}^{\text{gre}} \) obtained from the greedy algorithm. Here, we take \( \omega_c = \omega_a \), \( g/\omega_c = 0.1 \), \( T = 15/\omega_c \) and \( \delta t = 0.2/\omega_c \).

First let us demonstrate much more photons can be generated in a proper control sequence with a typical example shown in Fig. 2. In the figure, we plot two numerical results on the time variation of average photon number, \( N_{\text{ph}}^{\text{op}} \) and \( N_{\text{ph}}^{\text{gre}} \), whose control sequences are obtained from the greedy algorithm and the PGA algorithm respectively. Here the photon number plateaux correspond to the time periods when the CRW interaction is switched off. We find that the photon number \( N_{\text{ph}}^{\text{op}} \) is a little larger than the photon number \( N_{\text{ph}}^{\text{gre}} \). For comparison, we also plot the time variation of average photon number \( N_{\text{ph}} \) under the effective Hamiltonian all the time. It is worthy to point out that the photon number \( N_{\text{ph}}^{\text{op}} \) at time \( T = 15/\omega_c \) can be approximately \( 0.277/0.001 = 277 \) times larger than the maximum photon number in \( N_{\text{ph}}^{\text{op}} \).

The average photon numbers for different evolution time \( T \) are shown in Fig. 3 where the results from the PGA are denoted as a red solid line, and those from the greedy algorithm are denoted as a blue dashed line. Note that the average photon number from the PGA can be approximately regarded as the maximum photon number \( N_{\text{ph}} \) at time \( T \) we can get in a bang-bang control scheme. In the curve of \( N_{\text{ph}}^{\text{max}} \), we find that there also exist a series of photon number plateaux, which implies that we must switch off the CRW interactions during the time corresponding to the plateau to get more photons. With the increase of \( T \), the average photon number becomes larger and larger. We expect that there does not exist an upper bound for the average photon number we can get in the long time limit, which is due to the fact the CRW interaction can always excite larger multi-photon states during adjacent photon number plateaux.

![FIG. 3.](image2.png) (Color online). The maximum photon number \( N_{\text{ph}}^{\text{max}} \) obtained by optimal quantum control of the CRW interaction for different evolution time \( T \). The red solid line represents results obtained via the pruning greedy algorithm (PGA). The blue dashed line represents results obtained by the greedy algorithm. Here, we take \( \omega_c = \omega_a \), \( g/\omega_c = 0.1 \) and \( \delta t = 0.2/\omega_c \).

We also observe that the results from the greedy algorithm are only a little smaller than those from the PGA, which implies that the greedy algorithm is effective to solve our problem, although it can not yield an optimal solution in most cases. When carefully examining the results shown in Fig. 3, we find that the results from the
PGA and those from the greedy algorithm are almost the same in the time $T$ corresponding to the photon number plateaux with almost the same bang-bang control series, see Fig. 2 for an example. The main differences between the results from the PGA and the greedy algorithm appear at the time $T$ between the nearest photon number plateaux, see an example for the case of $T = 13.6/\omega_c$ shown in Fig. 3. In this case, we observe that, compared with the results from the greedy algorithm, the photon plateaux from the PGA have obvious left displacements in time such that we have more time in the last increasing regime, where the average photon number increases beyond that from the greedy algorithm.

![FIG. 4.](image1.png)

**FIG. 4.** (Color online). The maximum photon number as a function of $t$ for $T = 13.6/\omega_c$. The red solid line represents the results $N_{ph}^{\text{opt}}$ obtained by the PGA. The blue dashed line stands for the results obtained by the greedy algorithm. Here, we take $\omega_c = \omega_0$, $g/\omega_c = 0.1$ and $\delta t = 0.2/\omega_c$.

Now we extend our studies to the nonresonant case with $\omega_0 \neq \omega_c$. The numerical results of the photon number $N_{ph}^{\text{opt}}$ as a function of $T$ and $\omega_0$ from the greedy algorithm are shown in Fig. 5. When $\omega_0$ is near $\omega_c$, we can observe similar phenomena as shown in Fig. 3. However, when we tune $\omega_0$ far away from $\omega_c$, different behaviors of the average photon number appear. For $\omega_0 \gg \omega_c$, few photons are generated at time $T$ from the greedy algorithm, since the coupling strength $g$ in this case is too small to provide enough energy for the quantum jump from $|0\rangle \otimes |g\rangle$ to higher energy levels. As $\omega_0$ becomes smaller from $\omega_c$, the visibility of the photon number plateaux increases, and the width of the photon number plateau gets wider which tends to a certain value.

![FIG. 5.](image2.png)

**FIG. 5.** (Color online). The photon number $N_{ph}^{\text{opt}}$ obtained by the greedy algorithm as a function of $T$ and $\omega_0$ for $g/\omega_c = 0.1$ and $\delta t = 0.2/\omega_c$.

In Fig. 4, the total evolution time of the CRW interaction is $n_0 \delta t = 2/\omega_c > t_0$ which implies that the photon number would be less than 0.01 for $n_0 = 0$. With the increasing of $n_0$, the average photon number in the PGA reaches a finite limit. This maximal photon number characterizes the capacity of generating photons for the CRW interaction in time $T$ with the aid of the free evolution under $H_0^{\text{eff}}$. Although a proper arrangement of switching off the CRW interaction can enhance the photon generation, the CRW interaction and its action time determine the maximal photons generated.

![FIG. 6.](image3.png)

**FIG. 6.** (Color online). The maximum photon number $N_{ph}^{n_0}$ for different $n_0$ with the fixed $n_g$ and $\delta t$. The red solid line represents the results obtained by the PGA, and the blue dashed line represents the results obtained by the greedy algorithm. Here, we take $\omega_c = \omega_0$, $g/\omega_c = 0.1$, $n_g = 10$ and $\delta t = 0.2/\omega_c$.

Since photons are generated via the CRW interaction, it is interesting to ask at most how many photons can be generated when the total time with the CRW interaction switched on is fixed. The numerical results of the maximal photon number $N_{ph}^{n_0}$ as a function of $n_0$ for $\delta t = 0.2/\omega_c$ and $n_g = 10$ are shown in Fig. 6, where the red solid line is from the PGA, and the blue dashed line is from the greedy algorithm. According to

In Fig. 6, we also plot the maximal photon number $N_{ph}^{n_0}$ as a function of $n_0$ by the greedy algorithm, where the maximal photon number obtained is obviously smaller than that from the PGA algorithm. For a small $n_0(< 6)$, we even find that the maximal photon number is lower than that of $n_0 = 0$. All these facts indicate that the
PGA algorithm is more suitable to search the optimal sequence in the case with fixed $n_g$ than the greedy algorithm.

IV. CONCLUSION AND DISCUSSION

Our above numerical results show that we can enhance the photon generation from the vacuum via a bang-bang control scheme. Here we emphasize that the vacuum is of the Hamiltonian $H_0$ but not of $H$, which reminds us of its similarity with the dynamical Casimir effect \cite{26,28}, where the photons are generated by variation of the vacuum via the motion of one cavity mirror. In this sense, our bang-bang protocol to generate photons from vacuum can be regarded as a generalized dynamical Casimir effect.

Based on the above explanation, we may ask whether some similar bang-bang control protocols to generate photons via the CRW interaction exist. We give a positive answer to this question by presenting an alternative bang-bang control protocol via the $\sigma_z$ gates, which is even easier to be implemented in experiments than the current protocol. The details of this alternative bang-bang protocol are given in Appendix C.

In addition, we neglect the dissipation effect in our bang-bang control of generating photons. In general, we expect the dissipation will decrease the average photon number in our scheme, which is essential for comparison with experimental realizations, and needs further investigations.

In summary, we propose a bang-bang control method to use the CRW interaction to enhance the photon generation. We find that the maximum photon number we obtained can be dramatically increased by using the greedy algorithm and the PGA. Compared with the greedy algorithm, the PGA algorithm can find better control sequence to obtain more photons although it often takes more time. We hope that our work will stimulate further studies on the practical applications of CRW interaction in the diverse fields of coherent quantum manipulation.

ACKNOWLEDGMENTS

This work is supported by NSF of China (Grant Nos. 11475254 and 11775300), NKBRSF of China (Grant No. 2014CB921202), the National Key Research and Development Program of China (2016YFA0300603).

Appendix A: Dynamics of the Rabi model in the Heisenberg picture

In this appendix, we develop an alternative solution for the dynamics of the Rabi model in the Heisenberg picture. Because the dynamics of the photon number $\hat{n}$ in the Rabi model is not be closed for a set of finite operators, we need to resort a truncation method based on the cumulant of operators \cite{29,30}.

As is well known, the expectation value $\langle \hat{A}\hat{B} \rangle$ can be approximated as

$$\langle \hat{A}\hat{B} \rangle \approx \langle \hat{A} \rangle \langle \hat{B} \rangle \tag{A1}$$

when we neglect the correlation between $\hat{A}$ and $\hat{B}$. In general, we expand the expression of the expectation value for product of $N$ operators based on cumulant $\hat{O}^{12\cdots N}$ as

$$\langle \hat{O}^{12\cdots N} \rangle \approx \sum_{\{S_i\}} (-1)^M (M-1)! \left( \prod_i \hat{O}^{S_i} \right), \tag{A2}$$

where $\hat{O}^{12\cdots N} = \prod_{i=1}^M \hat{O}^i$, and $M$ ($M \geq 2$) is the number of partition in the set $\{S_i\} = \{S_1, S_2, \ldots, S_M\}$ which satisfies $S_j \neq \emptyset$, $S_j \cap S_k = \emptyset$ and $\bigcup_{j=1}^M S_j = \{1, 2, \ldots, N\}$ for $\forall j, k \in \{1, 2, \ldots, M\}$.

With $N = 3$ and $N = 4$ in Eq. (A2), we’ll get

$$\langle ABC \rangle \approx \langle A \rangle \langle BC \rangle + \langle B \rangle \langle AC \rangle + \langle C \rangle \langle AB \rangle - 2 \langle A \rangle \langle B \rangle \langle C \rangle, \tag{A3}$$

and

$$\langle ABCD \rangle \approx 6 \langle A \rangle \langle B \rangle \langle C \rangle \langle D \rangle + \langle A \rangle \langle BCD \rangle + \langle B \rangle \langle ACD \rangle + \langle C \rangle \langle ABD \rangle + \langle AB \rangle \langle CD \rangle + \langle AC \rangle \langle BD \rangle$$

$$+ \langle AD \rangle \langle BC \rangle - 2 \langle A \rangle \langle CD \rangle - 2 \langle A \rangle \langle BD \rangle - 2 \langle A \rangle \langle BC \rangle - 2 \langle A \rangle \langle AD \rangle - 2 \langle B \rangle \langle AD \rangle - 2 \langle B \rangle \langle DC \rangle \tag{A4}$$

$$- 2 \langle C \rangle \langle BD \rangle.$$
Now we introduce the following dynamical operators:

\[
\hat{x} = \hat{b}^\dagger + \hat{b}, \quad \hat{p} = i(\hat{b}^\dagger - \hat{b}), \quad \hat{n} = \hat{b}^\dagger \hat{b}, \quad \hat{\gamma} = (-1)^\hat{n}, \quad \hat{\delta} = \frac{i}{2}(\hat{x}\hat{\gamma} - \hat{\gamma}\hat{x}), \quad \hat{\epsilon} = \frac{i}{2}(\hat{p}\hat{\gamma} - \hat{\gamma}\hat{p}),
\]

(A5a)

\[
\hat{a} = \hat{x}^2, \quad \hat{b} = \hat{p}^2, \quad \hat{\theta} = \frac{1}{2}(\hat{x}\hat{\theta} + \hat{\theta}\hat{x}), \quad \hat{\lambda} = \frac{1}{2}(\hat{\beta}\hat{\gamma} + \hat{\gamma}\hat{\beta}), \quad \hat{\mu} = \frac{1}{2}(\hat{\theta}\hat{\gamma} + \hat{\gamma}\hat{\theta}).
\]

(A5b)

In the Heisenberg picture, we have

\[
x(0) = p(0) = n(0) = \delta(0) = \epsilon(0) = \theta(0) = \lambda(0) = 0, \quad \gamma(0) = \alpha(0) = \beta(0) = \kappa(0) = \mu(0) = 1,
\]

(A6)

since the system is at the vacuum state initially.

With Eqs. (A4) and (A5), we obtain

\[
\frac{dx}{dt} = \omega_c p + \omega_a \delta, \quad \frac{dp}{dt} = -\omega_c x + \omega_a \epsilon - 2g, \quad \frac{dn}{dt} = -gp, \quad \frac{d\gamma}{dt} = 2g\delta, \quad \frac{d\delta}{dt} = \omega_c \epsilon - \omega_a x - 2g\kappa, \quad \frac{d\epsilon}{dt} = -\omega_c \delta - \omega_a p - 2g\lambda, \quad \frac{d\theta}{dt} = 2\omega_c \theta, \quad \frac{d\beta}{dt} = -2\omega_c \theta - 4gp, \quad \frac{d\lambda}{dt} = -\omega_c \kappa + \omega_c \mu + 2g\alpha\epsilon - 4g^2x^2\epsilon - 8gx\rho\delta + 4g\theta\delta, \quad \frac{d\kappa}{dt} = 2\omega_c \lambda + 6g\alpha\delta - 12g^2x^2\delta, \quad \frac{d\mu}{dt} = -2\omega_c \lambda + 2g\beta\delta + 4g\theta\epsilon - 4g^2p^2\delta - 8gxp\epsilon.
\]

(A7a) \hspace{1cm} (A7b) \hspace{1cm} (A7c) \hspace{1cm} (A7d) \hspace{1cm} (A7e) \hspace{1cm} (A7f) \hspace{1cm} (A7g) \hspace{1cm} (A7h) \hspace{1cm} (A7i) \hspace{1cm} (A7j) \hspace{1cm} (A7k) \hspace{1cm} (A7l)

With the initial condition (A6) and the dynamical equations (A7), we get the photon number as a function of the evolution time with \( \omega_a = \omega_c \) and \( g/\omega_c = 0.1 \), which is shown as the blue dashed line in Fig. 7a. We find that our results agree well with those obtained from the exact diagonalization which is represented as the red solid line.

In addition, we also apply our method to obtain the results on the average photon number in the bang-bang protocol demonstrated in Fig. 7b, which agree well with those obtained via the exact diagonalization method. From a different angle, these facts show the validness of our numerical results in Sec. III.

Appendix B: The pruning greedy algorithm

In this appendix, we shall introduce the pruning greedy algorithm, which is used to search the optimal sequence in our problem.

For a fixed evolution time \( T \) and a small \( \delta t \), each control sequence is denoted as a sequence with \( T/\delta t \) binary numbers, and the size of the searching space is \( 2^{T/\delta t} \), which is impossible to use the exhaustive searching to get the optimal sequence.

The idea of the pruning greedy algorithm is to combine the exhaustive searching algorithm and the greedy algorithm. Here we introduce an integer \( N \) \((1 \leq N \leq T/\delta t)\) as the size of our truncated searching subspace which contains \( 2^N \) possible sequences.
We describe the procedure of the PGA as follows. When $T/\delta t \leq N$, we adopt the exhaustive searching algorithm to get the optimal sequence. When $T/\delta t > N$, we need to truncate the searching space to its subspace whose size is $2^N$, and then search the optimal sequence exhaustively. To truncate the searching space, we start at the time $t = N\delta t$, when the size of the searching space is $2^N$. When the time $t = (N + 1)\delta t$, the size of the searching space becomes $2^{N+1}$, and then we truncated it into its subspace whose size is $2^N$ by keeping the $2^N$ sequence with larger average photon numbers at the time. Repeating the truncating process until $t = T$, we will get the searching subspace at time $T$ with the size being $2^N$.

Notice that the PGA algorithm is the greedy algorithm for $N = 1$ and it becomes the exhaustive searching algorithm for $N = T/\delta t$. The convergence of the PGA algorithm is guaranteed by choosing a sufficiently large $N$.

In general, our problem is equivalent to searching the best path in a binary tree, where we truncate the subspace by pruning the tree step by step. Therefore, we name our algorithm ‘the pruning greedy algorithm’ and provide its pseudo-code in Algorithm. 1.

**Algorithm 1:** The pseudo-code of the ‘pruning greedy algorithm’

**Input:** the evolution time $T$, the pulse duration $\delta t$, the truncated number $N$

**Output:** the maximum photon number $N_{\text{ph}}^{\text{op}}$ with its corresponding control sequence

1. **if** $T/\delta t \leq N$ **then**
   2. output the maximal photon number at time $T$ and its control sequence by exhaustive searching;
   3. **else**
   4. constructing the searching subspace with $2^N$ possible sequences at time $t_L = L\delta t$ with $L = N$;
   5. **while** $L < T/\delta t$ **do**
   6. add 0 or 1 as the next number for every binary sequence in the previous searching subspace;
   7. sorting these $2^{N+1}$ sequences in the decreasing order by the corresponding photon numbers;
   8. keep the first $2^N$ sequences as the new searching subspace;
   9. $L = L + 1$;
10. **end**
11. output the maximal photon number at time $T$ and its control sequence
12. **end**

Besides, our algorithm is used to investigate the optimal quantum control problem discretely, which indicates that the pulse duration $\delta t$ must be small enough to ensure the convergence of the numerical results to their continuous limits. In Fig. 4, we show the maximum photon number $N_{\text{ph}}^{\text{max}}$ obtained by PGA with $\delta t = 0.1/\omega_c$ and $\delta t = 0.2/\omega_c$ as a function of the evolution time $T$. We find that two curves obtained with different $\delta t$ agree well with each other which implies that $\delta t = 0.2/\omega_c$ in our manuscript is sufficient to guarantee the convergence of our numerical results.
FIG. 8. (Color online). The maximum photon number $N_{\text{ph}}^{\text{max}}$ as a function of the experimental time $T$ for the bang-bang protocol. The red solid line and blue dashed line represent the results obtained by the PGA with $\delta t = 0.1/\omega_c$ and $\delta t = 0.2/\omega_c$, respectively. Here, we take $\omega_a = \omega_c$ and $g/\omega_c = 0.1$.

Appendix C: Bang-bang control via $\sigma_z$ gates

In this appendix, we present an alternative bang-bang control to generating photons via the CRW interaction with the aid of the $\hat{\sigma}_z$ gates, which may be easier to be implemented in experiments than that via switching the coupling on/off.

The bang-bang protocol based on the $\hat{\sigma}_z$ gates is similar as that on switching on/off the CRW interaction. In the present protocol, rather than switch off the CRW interaction, we apply two $\hat{\sigma}_z$ gates at the start and the end of the time period respectively, which is equivalent the free Hamiltonian $\hat{H}_0$ is replaced by

$$H' = \hat{\sigma}_z \hat{H} \hat{\sigma}_z = \omega_c \hat{a}^{\dagger} \hat{a} + \frac{\omega_a}{2} \hat{\sigma}_z - g \hat{\sigma}_x (\hat{a}^{\dagger} + \hat{a}).$$

(C1)

FIG. 9. (Color online). (a) The maximum photon number $N_{\text{ph}}^{\text{max}}$ as a function of the experimental time $T$ for $\omega_a = \omega_c$. (b) The photon number $N_{\text{gr}}^{\text{ph}}$ obtained by the greedy algorithm as a function of $T$ and $\omega_a$. Here, we take $g/\omega_c = 0.1$ and $\delta t = 0.1/\omega_c$.

With the PGA discussed in Appendix B and the greedy algorithm, we get the maximum photon number $N_{\text{ph}}^{\text{max}}$ as a function of the evolution time $T$ shown in Fig. 9a. We find that the maximum photon number at time $T$ we obtained are much larger than that from the pure time evolution of the Rabi model. Besides, we find that the results obtained
by the greedy algorithm are close to those obtained with the PGA, which implies that the greedy method can also be helpful in the experiment. In addition, we calculate the maximum photon numbers $N^\text{gr}_{\text{ph}}$ for different $\omega_a$ with the greedy algorithm. As shown in Fig. 9b, we find that $N^\text{gr}_{\text{ph}}$ increases with the decrease of $\omega_a$ when $T$ is fixed, and $N^\text{gr}_{\text{ph}}$ tends to zero when $\omega_a \gg \omega_c$.

[1] E. T. Jaynes and F. W. Cummings, Proc. IEEE 51, 89 (1963).
[2] L. Zhou, Z. R. Gong, Y.-x. Liu, C. P. Sun, and F. Nori, Phys. Rev. Lett. 101, 100501 (2008).
[3] E. Sanchez-Burillo, D. Zueco, J. J. Garcia-Ripoll, and L. Martin-Moreno, Phys. Rev. Lett. 113, 263604 (2014).
[4] Q.-K. He, W. Zhu, Z. H. Wang, and D. L. Zhou, J. Phys. B: At. Mol. Opt. Phys. 50, 145002 (2017).
[5] P. Forn-Díaz, J. Garcia-Ripoll, B. Peropadre, J.-L. Or-giazzi, M. Yurtalan, R. Belyansky, C. Wilson, and A. Lupascu, Nat. Phys. 13, 39 (2017).
[6] F. Yoshihara, T. Fuse, S. Ashhab, K. Kakuyanagi, S. Saito, and K. Semba, Nat. Phys. 13, 44 (2017).
[7] I. I. Rabi, Phys. Rev. 49, 324 (1936).
[8] D. Braak, Phys. Rev. Lett. 107, 100401 (2011).
[9] L. Garziano, R. Stassi, V. Macrì, A. F. Kockum, S. Savasta, and F. Nori, Phys. Rev. A 92, 063830 (2015).
[10] T. Niemczyk, F. Deppe, H. Huebl, E. P. Menzel, F. Hocke, M. J. Schwarz, J. J. Garcia-Ripoll, D. Zueco, T. Hummer, E. Solano, A. Marx, and R. Gross, Nat Phys 6, 772 (2010).
[11] P. Forn-Díaz, J. Lisenfeld, D. Marcos, J. J. Garcia-Ripoll, E. Solano, C. J. P. M. Harmans, and J. E. Mooij, Phys. Rev. Lett. 105, 237001 (2010).
[12] A. Baust, E. Hoffmann, M. Haeberlein, M. J. Schwarz, P. Eder, J. Goetz, F. Wulfscher, E. Xie, L. Zhong, F. Quijandría, D. Zueco, J.-G. Ripoll, L. García-Alvarez, G. Romero, E. Solano, K. G. Fedorov, E. P. Menzel, F. Deppe, A. Marx, and R. Gross, Phys. Rev. B 93, 214501 (2016).
[13] T. Schwartz, J. A. Hutchison, C. Genet, and T. W. Ebbesen, Phys. Rev. Lett. 106, 196405 (2011).
[14] D. Lv, S. An, Z. Liu, J.-N. Zhang, J. S. Pedernales, L. Lamata, E. Solano, and K. Kim, Phys. Rev. X 8, 021027 (2018).
[15] I. I. Rabi, Phys. Rev. 51, 652 (1937).
[16] B. Peropadre, P. Forn-Díaz, E. Solano, and J. J. García-Ripoll, Phys. Rev. Lett. 105, 023601 (2010).
[17] Z.-L. Xiang, S. Ashhab, J. Q. You, and F. Nori, Rev. Mod. Phys. 85, 623 (2013).
[18] Y. Lu, S. Chakram, N. Leung, N. Earnest, R. K. Naik, Z. Huang, P. Groszkowski, E. Kapit, J. Koch, and D. I. Schuster, Phys. Rev. Lett. 119, 150502 (2017).
[19] Q. He and D. L. Zhou, arXiv: 1805.10794 (2018).
[20] Z. Artstein, Siam Review 22, 172 (1980).
[21] L. M. Sonneborn and F. S. Van Vleck, Journal of The Society for Industrial and Applied Mathematics, Series A: Control 2, 151 (1964).
[22] L. Viola and S. Lloyd, Phys. Rev. A 58, 2733 (1998).
[23] L. Viola, E. Knill, and S. Lloyd, Phys. Rev. Lett. 82, 2417 (1999).
[24] L. Viola, S. Lloyd, and E. Knill, Phys. Rev. Lett. 83, 4888 (1999).
[25] J. Casanova, G. Romero, I. Lizuain, J. J. García-Ripoll, and E. Solano, Phys. Rev. Lett. 105, 263603 (2010).
[26] H. B. G. Casimir, Proc. K. Ned. Akad. Wet. B 51, 793 (1948).
[27] G. Moore, J. Math. Phys. 11, 26792691 (1970).
[28] C. M. Wilson, G. Johansson, A. Pourkabirian, M. Simoen, J. R. Johansson, T. Duty, F. Nori, and P. Delsing, Nature 479, 376 (2011).
[29] D. L. Zhou, B. Zeng, Z. Xu, and L. You, Phys. Rev. A 74, 052110 (2006).
[30] A. Vardi and J. R. Anglin, Phys. Rev. Lett. 86, 568 (2001).