A Formula for the Static Potential Energy in Quantum Gravity.

Giovanni Modanese

Center for Theoretical Physics
Laboratory for Nuclear Science
Department of Physics
Massachusetts Institute of Technology
Cambridge, Massachusetts, 02139, U.S.A.

Abstract

We give a general expression for the static potential energy of the gravitational interaction of two massive particles, in terms of an invariant vacuum expectation value of the quantized gravitational field. This formula holds for functional integral formulations of euclidean quantum gravity, regularized to avoid conformal instability. It could be regarded as the analogue of the Wilson loop for gauge theories and allows in principle, through numerical simulations or other approximation techniques, non perturbative evaluations of the potential or of the effective coupling constant. The geometrical meaning of this expression is quite simple, as it represents the “average proper-time delay”, respect to two neighboring lines, of a very long geodesic with unit timelike tangent vector.

*On leave from University of Pisa, Pisa, Italy. This work is supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under contract #DE-AC02-76ER03069.
1 Introduction.

The present paper is concerned with the problem of the energy of the gravitational field. This energy has been under investigation since the birth of General Relativity and some issues, like the determination of the total energy of a field configuration, have been settled in a rigorous way in the ADM formalism \[5\] or through Noether’s theorem \[4\]. Other points, however, like the possibility of “localizing” the gravitational energy, are still obscure. Before presenting our contribution, which concerns in fact the particular issue of the static potential energy, we shall briefly review a few general facts.

In principle, the energy of the gravitational field is physically as important as the energy of any other field. In fact, one of the basic principles of relativistic field theories is that any interaction between two particles is not instantaneous, but it is transmitted by a field which propagates with finite velocity. Suppose that the two particles exchange an amount of energy \(E\), as a result of their interaction. If the first particle loses the energy \(E\) at the time \(t\), and the second particle receives that energy at the time \(t + \Delta t\), it is usually assumed that in the interval \((t, t + \Delta t)\) the energy is “stored” in the field that carries the interaction.

In practice, however, the gravitational energy turns out to be much more “elusive” than other forms of energy, in the sense that it seems not possible to “localize” it.

For example, the electromagnetic field is known to possess the local energy density
\[
\tau^{00}(x) = \frac{1}{2} \left[ E^2(x) + B^2(x) \right],
\]
which is a component of the energy-impulse tensor
\[
\tau^{\mu\nu} = F^{\mu\alpha} F^{\nu}_\alpha - \frac{1}{4} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}.
\]
(See §\[2\] for our convention about the signature of the metric and others.) The tensor \(\tau^{\mu\nu}\) is locally conserved, that is
\[
\partial_{\mu}\tau^{\mu\nu} = 0.
\]
This assures that the spatial integral of \(\tau^{00}\) is conserved, provided \(E\) and \(B\) vanish sufficiently fast at infinity.

Eq. (3) has suggested a generalization of the tensor \(\tau^{\mu\nu}\) to the gravitational case, called the “Bel superenergy tensor” \[1\]. It has the form
\[
\mathcal{T}^{\mu\nu\rho\sigma} = R^{\mu\nu}_{\alpha\beta} R_{\alpha\beta}^{\rho\sigma} + R^{*\mu\nu}_{\alpha\beta} R^{*\rho\sigma}_{\alpha\beta},
\]
where \(R^{*\rho\sigma}_{\alpha\beta}\) is a suitably defined dual of the Riemann tensor.

The tensor \(\mathcal{T}^{\mu\nu\rho\sigma}\) has a positivity property, is covariantly conserved and allows to define a criterion for the presence of a gravitational wave in a point of empty spacetime. We remind that, due to the equivalence principle, the curvature tensor is the true physical field in General Relativity, since we cannot set it equal to zero at a given point by a transformation of the coordinates. So it could be fairly expected that a local energy density contains this tensor. However, the tensor \(\mathcal{T}^{\mu\nu\rho\sigma}\) defined in (4) is quadratic in the curvature, and this prevents us from relating it to the Hamiltonian of General Relativity.

Another problem for a local energy density resides in the fact that the ordinary conservation equation of a tensor (like (eq.\[3\])) is not generally covariant, and the introduction of the covariant derivative spoils in fact the conservation. All we can obtain in General Relativity is thus a conserved energy-impulse “pseudotensor” \[2\]. It leads to an energy which is the integral of some derivatives of the metric on a surface at spatial infinity, namely
\[
P^0 = -\frac{1}{16\pi G} \int \left( \frac{\partial h_{ij}(x)}{\partial x^4} - \frac{\partial h_{ij}(x)}{\partial x^j} \right) n^i \tau^j d\Omega,
\]
(5)
where it is assumed that the metric is quasi-minkowskian at infinity and is decomposed as \( g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \).

The most natural way to obtain the pseudotensor is to apply Noether’s theorem to the case of coordinates transformations, whose generators are the Lie derivatives \( \mathcal{L} \). The nice formalism of “improvement” of the Noether currents also works in this case \( \mathcal{L} \); we symmetrize the tensor obtained from translational invariance and we finally obtain for the energy density a quantity which amounts to a spatial divergence and reduces to (5) upon spatial integration. This result is typical of the application of Noether’s theorem to gauge symmetries, which usually produce “trivially conserved currents” of the form

\[
J_\mu = \partial_\nu f^{\mu\nu},
\]

where \( f^{\mu\nu} \) is an antisymmetric tensor.

For known asymptotically flat metrics like the Schwartzschild metric, (5) reproduces the total energy obtained from the ADM hamiltonian \( \mathcal{H} \). (See also ref. \( \mathcal{H} \) for a comparison of different “quasilocal energies” and references.)

The purpose of this paper is to relate the potential energy of the gravitational interaction of two particles to an invariant quantum-field average, which might play in gravity the same role the Wilson loop plays in the usual gauge theories.

In other words, this energy turns out to be related in a general fashion to the vacuum average of a simple and well-defined invariant functional of the field. While evaluation of this average for a weak field on a flat background yields the usual Newton potential energy, non-perturbative evaluations of the same average are likely to give rise to modifications in the coupling constant or in the dependence of the energy on the distance between the particles. We shall prove, however, that the energy is always negative.

Like in the case of the Wilson loop, our formula can be implemented quite naturally on a lattice version of the theory, in order to allow numerical computations.

The outline of the paper is the following. In § 2, mainly in order to fix our conventions, we compute the Newton potential starting from the graviton propagator. In § 3 we recall the connection between the ADM mass formula and the static gravitational potential. Through the ADM formula it is possible, in principle, to find the relativistic corrections to the Newton potential. In § 4 we give a formula which, treating two masses as external sources for a quantized gravitational field on a flat background, allows to write the potential energy of their interaction. This is done using a known technique of euclidean quantum field theory \( \mathcal{E} \). Such a formula would allow to compute the quantum corrections to the Newton potential. In § 5 the definition of the external source is generalized, avoiding use of the background metric and introducing the idea of nearby parallel lines in curved space. Finally, in § 6 we generalize the formula for the potential energy to the case of “strong” gravity and in § 7 we suggest some possible applications.

## 2 Newton potential. Conventions.

In this paper we work in \((3+1)\) dimensions and follow the conventions of Weinberg \( \mathcal{W} \). Our metric has signature \((-1, 1, 1, 1)\). The Einstein action is given by

\[
S_{\text{Einst.}} = -\frac{1}{16\pi G} \int d^4x \sqrt{|g(x)|} R(x)
\]
and the action of a material particle of mass \( m \) is

\[
S_{\text{Mat.}} = -m \int dp \sqrt{-g_{\mu\nu}(x(p)) \dot{x}^\mu(p) \dot{x}^\nu(p)},
\]

where \( x^\mu(p) \) is the trajectory of the particle and \( p \) is any parameter. The dots will always denote differentiation with respect to the parameter. Finally, the Einstein equations have the form

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}.
\]

Let us decompose the metric in the traditional way

\[
g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)
\]

and denote the linearized Einstein equations in the harmonic gauge as

\[
K_{\rho\sigma}^{\mu\nu} h_{\rho\sigma} = T_{\mu\nu}.
\]

The inverse of the kinetic operator \( K \) is the well-known Feynman-De Witt propagator

\[
K_{\mu\nu}^{-1}(x-y) = \langle h_{\mu\nu}(x)h_{\rho\sigma}(y) \rangle = -\frac{2G}{\pi} \frac{\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\rho\nu} - \eta_{\mu\nu}\eta_{\rho\sigma}}{(x-y)^2 - i\epsilon}.
\]

Let us now compute the Newton potential starting from the preceding equations. The field produced by a generic four-momentum source \( T_{\rho\sigma} \) in the linearized approximation is given by

\[
h_{\mu\nu}(x) = \int d^4y [K^{-1}]^{\rho\sigma}_{\mu\nu}(x-y) T_{\rho\sigma}(y);
\]

when the source is a particle of mass \( m \) at rest in the origin, the only non-vanishing component of \( T_{\rho\sigma} \) is

\[
T_{00}(y) = m \delta^3(y),
\]

so we have

\[
h_{00}(x^0, x) = \int d^4y [K^{-1}]_{00}^{00}(x-y) m \delta^3(y)
\]

\[
= m \int_{-\infty}^{+\infty} dy^0 \int d^3y \frac{-2G \delta^3(y)}{(x-y)^2 - (x^0 - y^0)^2 - i\epsilon}
\]

\[
= m \int_{-\infty}^{+\infty} dy^0 \frac{-2G}{\frac{x^2 - (x^0 - y^0)^2}{|x|} - i\epsilon} = \frac{2mG}{|x|}.
\]

This is the correct result, since in the Newtonian approximation we have

\[
g_{00} = -1 - 2V,
\]

where \( V \) is the Newton potential. We shall encounter the integral appearing in (15) also in §4.

3 Potential Energy Versus ADM Energy.

In classical General Relativity the total energy (mechanical + gravitational) of a physical system is given by the ADM mass formula \( M \), which has the remarkable property of involving only the gravitational field on a
surface at spatial infinity. We shall briefly recall here the connection between the ADM energy and the static gravitational potential.

A generic static metric $g_{\mu\nu}$ can be written at spatial infinity in the form

\[
g_{00} \simeq -1 + \frac{2M_1 G}{|x|} + O \left( \frac{1}{|x|^2} \right);
\]

\[
g_{0i} \simeq O \left( \frac{1}{|x|^2} \right);
\]

\[
g_{ij} \simeq \delta_{ij} + \frac{2M_2 G}{|x|^3} x_i x_j + O \left( \frac{1}{|x|^2} \right);
\]

(17) (18) (19)

Performing the integral (5) with $h_{ij}$ given by (19) one sees that $M_2$ is the ADM energy (“total mass”). On the other hand, $M_1$ is the mass observed by measuring the newtonian force at infinity. Substituting (17) - (19) into Einstein’s equations $R_{\mu\nu} = 0$, it is easy to see that $M_1 = M_2$. This is a quite natural result [5]. In other words it means that, according to special relativity conceptions, the source of the newtonian field is not only the mass, but also the energy density. For instance, in the gravitational collapse of a star a part of the gravitational energy is converted into kinetic energy and eventually this energy is employed to produce heavier elements from hydrogen or helium. If we disregard the radiation emitted into space, the newtonian field far away from the star remains unchanged during the whole process and the same holds for the ADM mass, which is a conserved quantity.

The gravitational potential energy can be found, by definition, assuming a static distribution of matter and computing the metric it generates at infinity. This has been done by Murchada and York [8] for a spherical matter distribution of uniform density, using conformal transformations and a special formulation of the initial-value equations of General Relativity. For a sphere of (small) density $\rho$ and unit radius they found the right newtonian gravitational binding energy, namely the ADM mass is given in this case by (reintroducing the radius $R$ and the velocity of light $c$)

\[
M_{\text{TOT}} = \frac{4}{3} \pi R^3 \rho - \frac{1}{c^2} \frac{16}{15} \pi^2 \rho^2 G R^5 + O(\rho^3).
\]

(20)

Remembering that $M_{\text{TOT}}$ also represents the effective source of the newtonian field, we see that the second term in (20) gives rise to a deviation from the famous law which states the independence of the potential on the radius of the source. Nevertheless, this effect is usually unobservable, due to the very small factor $c^{-2}$.

It is also possible to find the following corrections to (20), proportional to $\rho^3$, $\rho^4$, ... They denote the existence of general-relativistic corrections to the potential energy $m_1 m_2 / r$. For instance, the term proportional to $\rho^3$ would contribute to $M_{\text{TOT}}$ a term of the form

\[
\Delta M \propto \frac{1}{c^7} \rho^3 G^2 R^7 + O(\rho^4).
\]

(21)

In the case of a source constituted by two pointlike bodies of masses $m_1$ and $m_2$, kept at rest at a fixed distance $r$, the method of solution mentioned above is not applicable. (An approximate solution is still possible [8].) From the preceding discussion we may infer that the ADM mass is given in this case by

\[
M_{\text{TOT}} = m_1 + m_2 - \frac{1}{c^2} \frac{G m_1 m_2}{r} + o \left( \frac{1}{c^2} \right).
\]

(22)

We are not able, however, to deduce the relativistic corrections to the two-body potential from (21), because the corresponding potential does not admit a continuum limit. For instance, if we try to integrate a potential of the form $G^2 m_1^{3/2} m_2^{3/2} / r^2$ to obtain the term proportional to $\rho^3$, we find that the binding energy of the sphere depends on the way it has actually been put together.
4 Quantum formula for the potential energy on a flat background.

The same result we found in the preceding Section using the classical equations of motion can be obtained in a completely different way. It is known that the ground state energy of a system described by an action $S_0[\phi] = \int d^4x L(\phi(x))$ in the presence of external sources $J(x)$ can in euclidean quantum field theory be expressed as

$$E = \lim_{T \to \infty} -\frac{1}{T} \log \frac{\int d[\phi] \exp \{ -\frac{1}{T} \int d^4x L(\phi(x)) + \int d^4x \phi(x)J(x) \}}{\int d[\phi] \exp \{ -\frac{1}{T} \int d^4x L(\phi(x)) \} },$$

(23)

where, outside the interval $(-\frac{1}{2}T, \frac{1}{2}T)$, the source has been switched off.

This formula has been proved exactly in perturbation theory \cite{11} for the case of a linear local coupling between the field and the external source, but it can be generalized if we assume that in any case the vacuum-to-vacuum transition amplitude is given by

$$\langle 0^+|0^- \rangle_J = \frac{\int d[\phi] \exp \{ -S_0[\phi] + S_{\text{Inert.}}[\phi, J] \}}{\int d[\phi] \exp \{ -S_0[\phi] \} }.$$

(24)

In fact, inserting a complete set of energy eigenstates we can write

$$\langle 0^+|0^- \rangle_J = \langle 0|e^{-HT}|0 \rangle = \sum_n \langle 0|e^{-HT}|n \rangle \langle n|0 \rangle = \sum_n |\langle 0|n \rangle|^2 e^{-E_n T}. $$

(25)

The smallest energy eigenvalue $E_n$ corresponds to the ground state, and in the limit $T \to \infty$ it dominates the sum. So taking the logarithm and dividing by $(-T)$ we obtain that energy. This is a well-known technique in QCD (see for instance \cite{12}).

In the case of a weak gravitational field quantized on a flat background, we may consider the source constituted by two masses $m_1$, $m_2$, placed at rest near the origin at a distance $L$ each from the other (see eq. (27)). The action of this system is

$$S = S_{\text{Einst.}} + S_{\text{Mat.,1}} + S_{\text{Mat.,2}} = \frac{1}{16\pi G} \int d^4x \sqrt{g(x)} R(x) - m_1 \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_1 \sqrt{-g_{\mu\nu}[x(t_1)]} \dot{x}^\mu(t_1) \dot{x}^\nu(t_1)$$

$$- m_2 \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_2 \sqrt{-g_{\mu\nu}[y(t_2)]} \dot{y}^\mu(t_2) \dot{y}^\nu(t_2),$$

(26)

where the trajectories $x^\mu(t_1)$ and $y^\mu(t_2)$ of the particles with respect to the background are simply given by

$$x^\mu(t_1) = \left(t_1, -\frac{L}{2}, 0, 0 \right); \quad y^\mu(t_2) = \left(t_2, \frac{L}{2}, 0, 0 \right).$$

(27)

So we have, denoting by $\hat{S}$ the euclidean action,

$$E = \lim_{T \to \infty} -\frac{1}{T} \times \log \frac{\int d[h] \exp \{ -\hat{S}_{\text{Einst.}} - m_1 \int dt_1 \sqrt{1 - h_{00}[x(t_1)]} - m_2 \int dt_2 \sqrt{1 - h_{00}[y(t_2)]} \}}{\int d[h] \exp \{ -\hat{S}_{\text{Einst.}} \} }.$$

(28)
Since it is known that the euclidean Einstein action is not bounded from below, due to the existence of “conformal singularities” (see [13]), we actually mean by $\hat{S}_{\text{Einst}}$ a regularized version of (7). Such a regularization can be achieved adding $R^2$-terms to the original Einstein lagrangian [14] or through a more recent technique, called “stochastic regularization” [13]. We shall return to this point in §7.

Returning to (28), by standard perturbation techniques it is straightforward to see that for weak fields and to lowest order in $G$ it reduces to

$$E = m_1 + m_2 + \lim_{T \to \infty} -\frac{1}{T} \log \left\{ 1 + \frac{m_1 m_2}{4} \int_{-T}^{T} dt_1 \int_{-T}^{T} dt_2 \langle h_{00}[x(t_1)]h_{00}[y(t_2)] \rangle \right\}$$

$$\simeq m_1 + m_2 + \lim_{T \to \infty} -\frac{1}{T} \frac{m_1 m_2}{4} \int_{-T}^{T} dt_1 \int_{-T}^{T} dt_2 K^{-1}_{0000}(\tau_1 - \tau_2, L, 0, 0)$$

$$= m_1 + m_2 - \frac{m_1 m_2 G}{L}.$$ \hspace{1cm} (29)

The next term in the perturbative series is the first quantum correction, proportional to $\hbar G^2$. We shall compute it exactly in a forthcoming paper.

Eq. (28), like the corresponding ones in QED or QCD, has the physically appealing feature of showing how the force between the sources ultimately arises from the exchange of massless bosons. However, let us make a closer comparison with electrodynamics. In that case the analogue of the functional integral which appears in the logarithm of (28) has the form [10]

$$\left\langle \exp \left\{ g \int_{-T}^{T} dt_1 A_0[x(t_1)] - g \int_{-T}^{T} dt_2 A_0[y(t_2)] \right\} \right\rangle .$$ \hspace{1cm} (30)

(The two charges have been chosen to be opposite: $q_1 = g, q_2 = -g$.) Reversing the direction of integration in the second integral and closing the contour at infinity, one is able to show that the quantity (30) coincides with the Wilson loop of a single charge $g$, thus giving a gauge invariant expression for the potential energy.

In gravity this is not possible: we may imagine that an expression like (31) could be obtained in the first-order formalism (with $A_0$ replaced by the tetrad $e^i_0$), but the masses necessarily have the same sign.

Luckily, our formula for the energy is invariant as it stands, as we shall see better in the next Sections, where we generalize it to the case where no background metric is fixed.

5 Parallel lines in curved space.

In order to generalize eq. (28) beyond the case of weak fluctuations of the gravitational field around a fixed flat background, we need a definition of the source that does not depend on such a background.

We assume that a functional integral for regularized euclidean gravity exists, denoted by

$$z = \int d[g] \exp \left\{ -\hat{S}_{\text{Einst}}[g] \right\} ,$$ \hspace{1cm} (31)

and we require that all the field configurations in this functional integral are asymptotically flat.

Let us suppose that a field configuration is given. We consider a geodesic line of length $T$, which starts at an arbitrary point in the “past” asymptotically flat region with unit timelike velocity.
To fix the ideas, this curve could be written in its first part as

\[
\xi^\mu(\tau) = \left(-\frac{1}{2}T + \tau, 0, 0, 0\right); \quad 0 \leq \tau \leq \bar{\tau},
\]

where \(\tau\) is the proper time measured along the curve and we have chosen the spatial coordinates of the starting point to be equal to \((0, 0, 0)\) (this is an irrelevant arbitrariness, since at the end we shall integrate over all the configurations of the field). As usual, \(T\) denotes a very long time interval. After a time \(\simeq \bar{\tau}\) the curve enters the region of spacetime where the gravitational field is non vanishing. It continues as a geodesic, which means that \(\xi^\mu(\tau)\) satisfies the equation

\[
\Gamma^\rho_{\mu\nu}[\xi(\tau)]\dot{\xi}^\mu(\tau)\dot{\xi}^\nu(\tau) + \ddot{\xi}^\rho(\tau) = 0,
\]

where \(\Gamma^\rho_{\mu\nu}\) is the Christoffel symbol of the metric. The curve terminates at \(\tau = \frac{1}{2}T\), again in the flat region.

Let us then take in the initial point \(\xi^\mu(0)\) a unit vector \(q^\mu(0)\), orthogonal to \(\dot{\xi}^\mu(0)\) (for instance, in our preceding example, \(q^\mu(0) = (0, 1, 0, 0)\)), and define a vector \(q^\mu(\tau)\) along the curve \(\xi^\mu(\tau)\) by parallel transport of \(q^\mu(0)\). We remind that \(\dot{\xi}^\mu(\tau)\), being the tangent vector of a geodesic, is parallel transported along the geodesic itself, and that the parallel transport preserves the norms and the scalar products. Then the following relations hold along the curve

\[
\begin{align*}
\dot{\xi}^\mu(\tau)q^\nu(\tau)g^\mu_{\nu}(\tau) &= -1; \quad (34)
q^\mu(\tau)q^\nu(\tau)g^\mu_{\nu}(\tau) &= 1; \quad (35)
\ddot{\xi}^\mu(\tau)q^\nu(\tau)g^\mu_{\nu}(\tau) &= 0. \quad (36)
\end{align*}
\]

Next we consider two masses \(m_1, m_2\), and a length \(L\) which we may regard as infinitesimal, compared to the scale \(T\). We assume that the two masses follow the trajectories \(x^\mu(\tau)\) and \(y^\mu(\tau)\), respectively, given by

\[
\begin{align*}
x^\mu(\tau) &= \xi^\mu(\tau) - L_1q^\mu(\tau); \quad (37)
y^\mu(\tau) &= \xi^\mu(\tau) + L_2q^\mu(\tau), \quad (38)
\end{align*}
\]

where \(L_1\) and \(L_2\) are two positive lengths such that

\[
L_1 + L_2 = L \quad \text{and} \quad -m_1L_1 + m_2L_2 = 0. \quad (39)
\]

The physical meaning of the preceding geometrical construction is apparent: it represents an observer which falls freely in the center of mass of the system composed by \(m_1\) and \(m_2\), while holding the two masses at rest at a distance \(L\) each from the other. This is a generalization of the source introduced in §4 that is naturally dictated by the equivalence principle.

We notice that if the two masses were allowed to fall freely in the field, they would not keep at a constant distance from each other. In fact, as it is well known from the so-called geodesic deviation equation, the distance between two neighboring geodesics varies according to the sign of the curvature in the region they are traversing.

We can reparameterize the two curves \(x^\mu(\tau)\) and \(y^\mu(\tau)\) introducing their proper times \(\tau_1\) and \(\tau_2\), respectively. The ratio between the proper time \(\tau_1\) and the proper time \(\tau\) is given by the equation

\[
d\tau_1 = \sqrt{-g^\mu_{\nu}[x(\tau_1)]\dot{x}^\mu(\tau_1)\dot{x}^\nu(\tau_1)}d\tau,
\]

where \(\tau_1 = \tau_1(\tau)\); using \((37), (38)\), we have

\[
\left(\frac{d\tau_1(\tau)}{d\tau}\right)^2 = 1 + L_1q^\alpha(\tau)\partial_\alpha g^\mu_{\nu}[\xi(\tau)]\dot{\xi}^\mu(\tau)\dot{\xi}^\nu(\tau) +
+L_1 g^\mu_{\nu}[\xi(\tau)]\left\{\ddot{\xi}^\mu(\tau)q^\nu(\tau) + \ddot{q}^\nu(\tau)\dot{\xi}^\mu(\tau)\right\} + O(L_1^2).
\]


An analogous relation holds for $\tau_2$. We agree to adjust the function $\tau_1(\tau)$ in such a way that $\tau_1(0) = 0$. Then we shall denote $\tau_1(-\frac{1}{T}) = -\frac{1}{T} T'_1$ and $\tau_1(\frac{1}{T}) = \frac{1}{T} T''_1$. For flat geometries we have $T'_1 = T''_1 = T$. Analogous relations hold for $\tau_2$.

We notice that eq. (11) takes a much simpler form in the coordinate system where $\xi^\mu(\tau)$ defines one axis of a normal coordinates system (if they can be globally defined). In that case, we have $\dot{\xi}^\mu(\tau) = (1, 0, 0, 0)$, $\ddot{\xi}^\mu(\tau) = 0$ and (11) reduces to

$$\left(\frac{d \tau_1}{d \tau}\right)^2 = 1 + L_1 \partial_t g_{00}(\tau, 0, 0) + O(L_1^2),$$

which obviously reminds us of the equations of § 4. However, we are interested only in coordinates-independent quantities in the following.

## 6 General formula for the potential energy.

According to the discussion of the preceding Section, in the absence of a background eq. (8) must be rewritten as

$$\mathcal{E} = \lim_{T \to \infty} -\frac{1}{T} \log \left\langle \exp \left\{ -m_1 \int_{-\frac{1}{T}}^{\frac{1}{T}} d \tau_1 \sqrt{-g_{\mu \nu} [x(\tau_1)]} \tilde{x}^\mu(\tau_1) \tilde{x}^\nu(\tau_1) \right\} \right\rangle \mathcal{S}_\text{Einst.}$$

where, for brevity, we have denoted by brackets the functional average weighted by the exponential of the (regularized) Einstein action.

Now we exploit the property, characteristic of timelike geodesics in a Lorentzian manifold, of having maximal length with respect to neighboring lines. This means that

$$\frac{1}{2} m_1 (T'_1 + T''_1) = T \{1 - \delta_1 (L_1, [g])\};$$

$$\frac{1}{2} m_2 (T'_2 + T''_2) = T \{1 - \delta_2 (L_2, [g])\},$$

where $\delta_{1,2}$ are small positive adimensional functionals of the geometry $g$, which also depend on $L_{1,2}$ and thus, – through eq. (38) – on $L$ and $m_1, m_2$. So we have

$$\mathcal{E} = \frac{1}{T} \log \left\langle e^{-T (m_1 + m_2)} \exp \{ T (m_1 \delta_1 (L_1, [g]) + m_2 \delta_2 (L_2, [g])) \} \right\rangle \mathcal{S}_\text{Einst.}$$

The quantity in the bracket represents the energy of the gravitational interaction. More exactly, it represents, apart from $m_1, m_2$, the ground state energy of the system constituted by the source described in § 3 coupled to a quantum gravitational field. By construction, $\mathcal{E}$ is invariant with respect to coordinate transformations.

We see from eq. (47) that the interaction energy is always negative. Apart from this, (47) alone does not give us any precise indication on the dependence of this energy on the masses, on $L$ and on $G$ (which is contained
in the action). This dependence is a nontrivial result of the dynamics, as we have seen already in the simple perturbative example of §4. We can apply to the functional integral appearing in (47) other approximation techniques, like the semiclassical approximation, or discretize it and use numerical methods.

In any case, the geometrical meaning of (47) is quite simple. In practice, this formula implies the following: (1) trace a geodesic with unit timelike tangent vector through one (asymptotically flat) field configuration; (2) measure the total “delay” of this geodesic with respect to two neighboring lines; (3) average on many configurations.

7 Possible applications and concluding remarks.

The functional integral representation for the static gravitational potential given in (47) can serve as a basis for various approximations (weak-field, semiclassical) and for numerical simulations. The latter require a discretized version of the theory in which the geodesic lengths or the metric represent the fundamental variables. The most suitable technique under this respect is probably the “quantum Regge calculus” of Hamber [14]; another candidate, which might have the advantage of simpler algorithms, is the “stochastic stabilized gravity” of Greensite [13]. As we pointed out in the last Section, the geometrical meaning of our formula is quite simple. This should allow us to write a dedicated algorithm for its evaluation, in order to improve the efficiency of the method.

The most simple quantity to be “measured” in this way in a lattice version of gravity is the effective coupling constant $G$. In order to compute it through $E$, we just need to set $m_1 = m_2$, $L_1 = L_2$ and assign some fixed values to them, so that the algorithm can be even more simplified.

There have been some suggestions (see for ex. [15]) that the effective coupling constant in gravity is scale dependent and decreases at small distances. This “antiscreening” of the gravitational interaction seems to be quite natural, since the longer is the cloud of virtual particles, the stronger is the gravitational force. In a forthcoming paper we shall employ our formula to compute the correction of order $\hbar$ to the potential energy, and possibly to exhibit such an effect.

In conclusion, we remind that this paper is part of a program which aims to study physical observables in four-dimensional quantum gravity. Other observables considered were the vacuum correlations at geodesic distance [16] and the loops of the Christoffel connection [17].

8 Acknowledgments.

It is a pleasure to thank prof. Roman Jackiw for the kind hospitality at M.I.T. and for charming discussions about the problem of energy in General Relativity, as well as for helpful criticism and suggestions concerning this work. I also am grateful to prof. S. Deser for invaluable clarifications about the ADM energy. The author is supported by a fellowship of the Foundation “A. Della Riccia” of Florence, Italy.

References

[1] V.D. Zakharov, *Gravitational waves in Einstein’s theory*, J. Wiley, New York, 1973.
[2] S. Weinberg, *Gravitation and cosmology: principles and applications of the general theory of relativity*, J. Wiley, New York, 1972.

[3] R. Jackiw, “Field theoretical investigations in current algebra” and “Topological investigations of quantized field theories”, in *Current algebra and anomalies*, by S.B. Treiman, R. Jackiw, B. Zumino, E. Witten, Princeton University Press, Princeton, 1985.

[4] S. Bak, D. Cangemi and R. Jackiw, “Energy-momentum conservation in General Relativity”, report CTP # 2245, September 1993.

[5] R. Arnowitt, S. Deser and C. Misner, *Phys. Rev.* **116** (1959) 1322. B. De Witt, *Phys. Rev.* **160** (1967) 1113.

[6] L. Faddeev, *Sov. Phys. Usp.* **25** (1982) 130.

[7] S.A. Hayward, “Quasi-local gravitational energy”, report PRINT-93-0302 (Garching), March 1993.

[8] N.O. Murchada and J.W. York, *Phys. Rev.* **D 10** (1974) 2345;

[9] S. Deser, private communication, June 1993.

[10] W. Fischler, *Nucl. Phys. B* **129** (1977) 157.

[11] K. Symanzik, *Comm. Math. Phys.* **16** (1970) 48.

[12] M. Bander, *Phys. Rep.* **75 C** (1981) 205.

[13] J. Greensite, *Nucl. Phys. B* **361** (1991) 729; *Phys. Lett. B* **291** (1992) 405; *Nucl. Phys. B* **390** (1993) 439.

[14] H.W. Hamber, *Phys. Rev.* **D 45** (1992) 507; *Nucl. Phys. B* (Proc. Suppl.) **25 A** (1992) 150; *Nucl. Phys. B* **400** (1993) 347.

[15] A.M. Polyakov, “A few projects in string theory”, Princeton preprint PUPT-1394, April 1993.

[16] G. Modanese and M. Toller, *J. Math. Phys.* **31** (1990) 452; P. Menotti, G. Modanese and D. Seminara, *Ann. Phys. (N.Y.)* **224** (1993) 110; G. Modanese, *Phys. Lett. B* **288** (1992) 69.

[17] G. Modanese, *Phys. Rev.* **D 47** (1993) 502; “Wilson loops in four-dimensional quantum gravity”, report CTP # 2225, July 1993.