Markovianity of the invariant distribution of probabilistic cellular automata on the line.

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Abstract

We revisit the problem of finding the conditions under which synchronous probabilistic cellular automata indexed by the line \( \mathbb{Z} \), or the periodic line \( \mathbb{Z}/n\mathbb{Z} \), depending on 2 neighbours, admit as invariant distribution the distribution of a Markov chain. A famous result in the literature asserts that under mild conditions, positive rate PCA admit an invariant Markovian measure defined with two Markov kernels \( (D,U) \) on a certain zigzag traversal on two consecutive lines of the space time diagram (corresponding to two successive times) if and only if \( (D,U,T) \) satisfies a certain algebraic system of equations, where \( T \) is the local transition matrix of the PCA. Here we go further and provide the condition on \( T \) only, and go a bit beyond the positive rate condition. These advances are valid for \( T \) inducing the quasi-reversibility of the PCA under its stationary distribution.

We show that the condition for Markovianity of an invariant measure for a PCA on the line or the zigzag (in the periodic case or not) can be turned into some questions of algebra. This approach allows us to compare the different structures with respect to this property.

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1 Introduction

Foreword. Take a colouring \( X := (x_i, i \in \mathbb{Z}) \in \{0,1,\ldots,\kappa\}^\mathbb{Z} \) of \( \mathbb{Z} \) with numbers taken in \( E_\kappa = \{0,1,\ldots,\kappa\} \) for some \( \kappa \geq 1 \), and let \( A \) be a probabilistic cellular automata (PCA) depending on a neighbourhood of size 2 with transition matrix \( T = [T_{(a,b),c}, (a,b) \in E_2^2, c \in E_\kappa] \). This PCA allows one to define a random process taking its values in \( \{0,1,\ldots,\kappa\}^\mathbb{Z} \), defined by taking \( X_0 = X \) and for \( t \geq 0 \), \( X_t = (x_i(t), i \in \mathbb{Z}) \) and where

\[
\mathbb{P}(x_i(t+1) = c \mid x_i(t) = a, x_{i+1}(t) = b) = T_{(a,b),c},
\]

where we assume that all transitions are space and time independent.

Here is a simple question: under which condition on \( T \) does it exist the distribution of a Markov process invariant by the PCA with transition matrix \( T \)? The answer is not that simple... and the complete answer is still not known even when \( \kappa = 2 \). What is known is that, when the Markov chain \( (X_t, t \geq 0) \) is quasi-reversible under its stationary distribution, then this stationary distribution is Gibbs...
on a graph built on two copies of \( \mathbb{Z} \) (which is also stated in the literature as the fact that under the invariant distribution \( \pi_{[1/2]}(i \mod 2, i \in \mathbb{Z}) \) is a Markov process).

In the paper we give a complete characterisation of the transition matrices \( T \) having this property (Theorem 2.6) and we provide a similar criterion for the case where the PCA is defined on \( \mathbb{Z}/n\mathbb{Z} \) instead (Theorem 2.11). The property "to have the distribution of a Markov process as invariant distribution" depends on the graph where the PCA is defined. In Section 3 we compare the conditions needed to have this property.

We start with formal definitions. **Cellular automata** (CA) are dynamical systems in which space and time are discrete. A CA is a 4-tuple \( A := (L, E_\kappa, N, f) \) where:

- \( L \) is the lattice, the set of cells. It will be \( \mathbb{Z} \) or \( \mathbb{Z}/n\mathbb{Z} \) in the paper,
- \( E_\kappa = \{0, 1, \ldots, \kappa\} \) for some \( \kappa \geq 1 \), is the set of states of the cells,
- \( N \) is the neighbourhood function: for \( x \in L \), \( N(x) \) is a finite sequence of elements of \( L \), the list of neighbours of \( x \); its cardinality is \( |N| \). Here,
  \[
  N(x) = (x, x + 1) \quad \text{when } L = \mathbb{Z} \quad \text{and} \quad N(x) = (x, x + 1 \mod n) \quad \text{when } L = \mathbb{Z}/n\mathbb{Z},
  \]
- \( f \) is the local rule. It is a function \( f : E_\kappa^{|N|} \rightarrow E_\kappa \).

The CA \( A = (L, E_\kappa, N, f) \) defines a global function \( F : E_\kappa^L \rightarrow E_\kappa^L \) on the set of configurations indexed by \( L \). For any \( S_0 = (S_0(x), x \in L) \), \( S_1 = (S_1(x), x \in L) := F(S_0) \) is defined by

\[
S_1(x) = f([S_0(y), y \in N(x)]), \quad x \in L.
\]

In words the states of all the cells are updated simultaneously. The state \( S_1(x) \) of \( x \) at time 1 depends only on the states \( S_0(x) \) and \( S_0(x + 1) \) of its neighbours at time 0. Starting from configuration \( \eta \in E_\kappa^L \) at time \( t_0 \), meaning \( S_{t_0} = \eta \), the sequence of configurations

\[
S := (S_t = (S(x,t), x \in L), t \geq t_0)
\]

where \( S_{t+1} := F(S_t) \) for \( t \geq t_0 \) forms what we call the space-time diagram of \( A \).

**Probabilistic cellular automata** (PCA) are generalisations of CA in which the states \( (S(x,t), x \in L, t \in T) \) are random variables (r.v.) defined on a common probability space \( (\Omega, \mathcal{A}, P) \), each of the r.v. \( S(x,t) \) taking a.s. its values in \( E_\kappa \). Seen as a random process, \( S \) is equipped with the \( \sigma \)-fields generated by the cylinders. In PCA the local deterministic function \( f \) is replaced by a transition matrix \( \text{Tr} \) which gives the distribution of the state of a cells at time \( t + 1 \) conditionally on those of its neighbours at time \( t \):

\[
\mathbb{P}(S(x, t + 1) = b \mid [S(y,t), y \in N(x)] = [a_1, \ldots, a_{|N|}]) = \text{Tr}(a_1, \ldots, a_{|N|}, b).
\]

Conditionally on \( S_t \), the states in \( (S(x, t + 1), x \in L) \) are independent.
The transition matrix (TM) is then an array of non negative numbers
\[ \text{Tr} = \left( \text{Tr}_{(a_1,\ldots,a_{|N|}),b} \right)_{(a_1,\ldots,a_{|N|}),b} \in E^{|N|}_\kappa \times E_\kappa, \]  
 satisfied \( \sum_{b \in E_\kappa} \text{Tr}_{(a_1,\ldots,a_{|N|}),b} = 1 \) for any \( (a_1,\ldots,a_{|N|}) \in E^{|N|}_\kappa \).

Formally a PCA is a 4-tuple \( (L, E_\kappa, N, \text{Tr}) \). Instead of considering \( A \) as a random function on the set of configurations \( E^{|N|}_\kappa \), \( A \) is considered as an operator on the set of probability distributions \( \mathcal{M}(E^{|N|}_\kappa) \) on the configuration space. If \( S_0 \) has distribution \( \mu_0 \) then the distribution of \( S_1 \) will be denoted by \( \text{Tr}(\mu_0) \): the meaning of this depends on the lattice \( L \), but this latter will be clear from the context. The process \( (S_t, t \in T) \) is defined as a time-indexed Markov chain, the evolutions being independent in time and in space, that is the distribution of \( S_{t+1} \) knowing \( \{S_{t'}, t' \leq t\} \) is the same as that knowing \( S_t \) only. Conditionally on \( S_t = \eta \) it is \( \text{Tr}(\delta_\eta) \), the Dirac measure at \( \eta \). A measure \( \mu \in \mathcal{M}(E^{|N|}_\kappa) \) is said to be invariant for the PCA \( A \) if \( \text{Tr} (\mu) = \mu \). We will simply say that \( \mu \) is invariant by \( \text{Tr} \) when no confusion on the lattice \( L \) exists.

The literature on CA, PCA, and asynchronous PCA is huge. We here concentrate on works related to PCA’s only, and refer to Kari [14] for a survey on CA (see also Ganguly et al. [11] and Bagnoli [2]), to Wolfram [26] for asynchronous PCA and to Liggett [17] for more general interacting particle systems. For various links with statistical mechanics, see Chopard et al. [6], Lebowitz et al. [16]. PCA are studied by different communities: in statistical mechanics and probability theory in relation with particle systems as Ising (Verhagen [25]), hard particles models (Dhar [9, 10]), Gibbs measures ([7, 8, 23, 18]), percolation theory, in combinatorics ([9, 10, 4, 15, 1, 21]) where they emerge in relation with directed animals, and in computer science around the problem of stability of computations in faulty CA (the set of CA form a Turing-complete model of computations), see e.g. Gács [13], Toom et al. [23]. In a very nice survey Mairesse & Marcovici [20] discuss these different aspects of PCA (see also the PhD thesis of Marcovici [22]).

**Notation .** The set of PCA on the lattice \( \mathbb{L} \) equal to \( \mathbb{Z} \) (or \( \mathbb{Z}/n\mathbb{Z} \)) and neighbourhood function \( N(x) = (x, x+1) \) (or \( N(x) = (x, x+1 \mod n) \)) with set of states \( E_\kappa \) will be denoted by PCA \( (\mathbb{L}, E_\kappa) \). This set is parametrised by the set of TM \( \{(\text{Tr}_{(a,b),c}, (a,b,c) \in E^3_\kappa)\} \). A TM \( \text{Tr} \) which satisfies \( \text{Tr}_{(a,b),c} > 0 \) for any \( a,b,c \in E_\kappa \) is called a positive rate TM, and a PCA \( A \) having this TM will also be called a positive rate PCA. The subset of PCA \( (\mathbb{L}, E_\kappa) \) of PCA with positive rate will be denoted by PCA \( (\mathbb{L}, E_\kappa)^+ \). In order to get more compact notation, on which the time evolution is more clearly represented, we will write \( T_{a,b,c}^\kappa \) instead of \( \text{Tr}_{(a,b),c} \).

Given a PCA \( A := (\mathbb{L}, E_\kappa, N, T) \) the first question arising is that of the existence, uniqueness and description of the invariant distribution(s) and sometimes the question of convergence to this latter. Important difficulties arise here and finally very few is known about these questions. In most cases, no description is known for the (set of) invariant distributions, and the question of ergodicity in general is not solved: the weak convergence of \( T^m(\mu) \) when \( m \to +\infty \) to a limit law independent from \( \mu \) is only partially known for some TM \( T \)’s even when \( \kappa = 1 \), as discussed in
Toom & al. [23, Part 2, Chapters 3–7], and Gács [13] for a negative answer in general. Besides the existence of a unique invariant measure does not imply ergodicity (see Chassaing & Mairesse [5]).

For $A$ in $\text{PCA}(\mathbb{Z}/n\mathbb{Z}, E^n)$ the situation is different since the state space is finite. When $A \in \text{PCA}(\mathbb{Z}/n\mathbb{Z}, E^n)^*$ the Markov chain $(S_t, t \geq 0)$ is aperiodic and irreducible and then owns a unique invariant distribution which can be computed explicitly for small $n$, since $\mu = \text{Tr}(\mu)$ is a linear system.

1.1 The structures

We present now the geometric structures that will play a special role in the paper. The $t$th (horizontal) line on the space-time diagram is

$$H_t := \{(x, t), x \in \mathbb{Z}\},$$

and we write $H_t(n) := \{(x, t), x \in \mathbb{Z}/n\mathbb{Z}\}$ for a line on the space-time diagram in the cyclic case. The $t$th horizontal zigzag on the space-time diagram is

$$\text{HZ}_t := \left\{\left\lfloor \frac{x}{2} \right\rfloor, t + \frac{1 + (-1)^{x+1}}{2}, x \in \mathbb{Z}\right\},$$

as represented on Figure 1. Define also $\text{HZ}_t(n)$ by taking $\left\lfloor \frac{x}{2} \right\rfloor \pmod{n}$ instead of $\left\lfloor \frac{x}{2} \right\rfloor$ in (5).

Since $\text{HZ}_t$ is made by the two lines $H_t$ and $H_{t+1}$, a PCA $A = (\mathbb{Z}, E^n, N, T)$ on $\mathbb{Z}$ can be seen as acting on the configuration distributions on $\text{HZ}_t$. A transition from $\text{HZ}_t$ to $\text{HZ}_{t+1}$ amounts to a transition from $H_{t+1}$ to $H_{t+2}$, with the additional condition that the first line of $\text{HZ}_{t+1}$ coincides with the second line of $\text{HZ}_t$ (the transition probability is 0 if this is not the case) (see also the proof of Theorem 2.3 for more details).

1.2 A notion of Markovianity per structure

We first fix the matrix notation: $[A_{x,y}]_{a \leq x, y \leq b}$ designates the square matrix with size $(b - a + 1)^2$; the line index is $x$, the column one is $y$. The line vectors (resp. column vectors) will be written $[v_x]_{a \leq x \leq b}$ (resp. $[^t v_x]_{a \leq x \leq b}$).
We define here what we call Markov chains (MC) on $H$, $H(n)$, $HZ$ and $HZ(n)$. As usual a MC indexed by $H_t$ is a random process $(S(x,t), x \in Z)$ whose finite dimensional distribution are characterised by $(\rho, M)$, where $\rho$ is an initial probability distribution $\rho := [\rho_a]_{a \in E_\nu}$ and $M := [M_a, b]_{0 \leq a, b \leq \kappa}$ a Markov kernel as follows:

$$P(S(i, t) = a_i, n_1 \leq i \leq n_2) = \rho_{a_1} \prod_{i=n_1}^{n_2-1} M_{a_i, a_{i+1}},$$

where $\rho$ may depend on the index $n_1$. Observing what happens far away from the starting point, one sees that if a $(\rho, M)$-MC with kernel $M$ is invariant under a PCA $A$ with TM $T$ on the line, then there is a $(\rho', M)$-MC invariant under $T$ with $\rho' = \rho'M$ (in other words, one of the invariant distribution of the MC with kernel $M$ is invariant under $T$). In the sequel, when we simply talk about the $M$-MC for short, but we will specify the initial distribution when needed.

A process $S_t$ indexed by $HZ_t$ and taking its values in $A$ is said to be Markovian if there exists a probability measure $\rho := (\rho_x, x \in E_\nu)$ and two Markov kernels $D$ and $U$ such that, for any $n \geq 0$, any $a_i \in E_\nu$, $b_i \in E_\nu$,

$$P(S(i, t) = a_i, S(i, t + 1) = b_i, 0 \leq i \leq n) = \rho_{a_0} \prod_{i=0}^{n-1} D_{a_i, b_i} U_{b_i, a_{i+1}} D_{a_n, b_n}$$

(6)

in which case $\rho$ is said to be the initial distribution. Again we are interested in shift invariant processes. We then suppose that $\rho$ is an invariant measure for the Markov kernel $DU$ in the sequel, that is $\rho = \rho DU$. We will call such a process a $(\rho, D, U)$ HZMC (horizontal zigzag Markov chain), or for short a $(D, U)$ HZMC.

A process $S_t$ indexed by $H_t(n)$ and taking its values in $E_\nu$ is called a cyclic Markov chain (CMC) if there exists a Markov kernel $M$ such that for all $a = (a_0, \ldots, a_{n-1}) \in E_\nu^n$,

$$P(S(i, t) = a_i, i \in Z/nZ) = Z_n^{-1} \prod_{i=0}^{n-1} M_{a_i, a_{i+1} \bmod n}$$

(7)

where $Z_n = \text{Trace}(M^n)$. The terminology cyclic Markov chain is borrowed to Albenque [1]. It corresponds to Gibbs distributions (see e.g. Georgii [12, Theo. 3.5]). For two Markov kernels $D$ and $U$, a process $S$ indexed by $HZ_t(n)$ and taking its values in $E_\nu$ is said to be a $(D, U)$-cyclic Markov chain (HZCMC) if for any $a_i \in E_\nu$, $b_i \in E_\nu$,

$$P(S(i, t) = a_i, S(i, t + 1) = b_i, 0 \leq i \leq n - 1) = Z_n^{-1} \prod_{i=0}^{n-1} D_{a_i, b_i} U_{b_i, a_{i+1} \bmod n}$$

(8)

where $Z_n = \text{Trace}((DU)^n)$. Again, HZMC corresponds to some Gibbs measures on this space.

We will also consider product measures measure of the form $\mu(x_1, \ldots, x_k) = \prod_{i=1}^{n} \nu_{x_i}$. 


1.3 References and contributions of the present paper

In the paper our main contribution concerns the case $\kappa > 1$. Our approach, mainly algebraic, has for object to find the conditions on PCA, or rather on TM, to have as invariant measure a Markov chain. Above we have brought to the reader attention that (different) PCA with same TM $T$ may be defined on each of the structure $H$, $H(n)$, $HZ$ and $HZ(n)$. The transitions $T$ for which they admit a Markovian invariant distribution depends on the structure. A part of the paper is devoted to these comparisons, the conclusions being summed up in Figure 2, in Section 3.

The main contribution of the present paper concerns the full characterisation of the TM with positive rates and beyond for which there exists a Markovian invariant distribution on, on the one hand $HZ$ and on the other hand $HZ(n)$. One finds in the literature two main families of contributions in the same direction. We review them first before presenting our advances.

The first family of results we want to mention is the case $\kappa = 1$ for which much is known.

1.3.1 Case $\kappa = 1$. Known results

Here is to the best knowledge of the authors the exhaustive list of existing results concerning PCA having a Markov chain as invariant measure on $H$, $H(n)$, $HZ$ or $HZ(n)$ for $\kappa = 1$ and $N(x) = (x, x + 1)$.

On the line $H$: The first result we mention is due to Beliaev & al. [3] (see also Toom & al. [23, section 16]). A PCA $A = (Z, E_1, N, T) \in \text{PCA}(Z, E_1)^*$ (with positive rate) admits the distribution of a MC on $H$ as invariant measure if and only if (iff) any of the three following conditions hold.

(i) $T_{0,0} T_{1,1} T_{1,0} T_{0,1} = T_{1,1} T_{0,0} T_{0,1} T_{1,0}$,

(ii) $T_{0,0} + T_{1,1} = T_{0,1} + T_{1,0} = 1$,

(iii) $T_{0,1} T_{1,0} = T_{1,1} T_{0,0}$ or $T_{1,0} T_{0,1} = T_{1,1} T_{0,0}$.

In case (iii), the MC is in fact a product measure with marginal $(\rho_0, \rho_1)$, and

$$\rho_0 = \begin{cases} 
T_{0,0} T_{1,1} - T_{0,0} T_{1,0} & \text{if } T_{0,0} + T_{1,1} \neq T_{0,1} + T_{1,0}, \\
T_{0,0} + T_{1,1} & \text{if } T_{0,0} + T_{1,1} = T_{0,1} + T_{1,0}.
\end{cases}$$

(the same condition is given in Marcovici & Mairesse, Theorem 3.2 in [19]; see also this paper for more general condition, for different lattices, and for $\kappa > 1$).

In case (ii), the Markov kernel $M$ of the invariant distribution satisfies $M_{0,0} = M_{1,1}$ and

$$M_{0,0} = \frac{T_{0,0} T_{0,1}}{T_{0,1} T_{0,0} + (T_{0,0} T_{1,1} T_{0,1} T_{0,0})^{1/2}}.$$

In case (i), $M$ satisfies $T_{0,1} T_{1,0} M_{1,0} M_{0,1} = T_{0,0} T_{1,1} M_{0,0} M_{1,1}$ and $M_{0,0} T_{0,0} = M_{1,1} T_{1,1}$.

Without the positive rate condition the additional solutions are also solution to (i), (ii) or (iii),
or $T_{0,0} = 1$ or $T_{1,1} = 1$ as one can show using Proposition 2.13 since this finite system can be solved by computing a Gröbner basis, which can be done explicitly using a computer algebra system like sage or singular. Without the positive rate condition some pathological cases arise. Consider for example, the cases $(T_{1,0}, T_{0,1}) = (1, 1)$ (case (a)) or $(T_{0,1}, T_{1,0}) = (1, 1)$ (case (b)) or $(T_{0,0}, T_{1,1}) = (1, 1)$ (case (c)). In these cases some periodicity may occur if one starts from some special configurations. Let $C_i$ be the constant sequence (indexed by $\mathbb{Z}$) equals to $i$, and $C_{0,1}$ the sequence $(\frac{1+(-1)^{n+1}}{2}, n \in \mathbb{Z})$ and $C_{1,0}$, the sequence $(\frac{1+(-1)^{n}}{2}, n \in \mathbb{Z})$. It is easy to check that in case (a), $(\delta C_{01} + \delta C_{1,0})/2$ is an invariant measure, in case (b), $p\delta C_{01} + (1-p)\delta C_{1,0}$ is invariant for any $p \in [0, 1]$, in case (c), $(\delta C_{1} + \delta C_{0})/2$ is invariant. Each of these invariant measures are Markov ones with some ad hoc initial distribution. Case (a) is given in Chassaing & Mairesse [5] as an example of non ergodic PCA with a unique invariant measure (they add the conditions $T_{0,0} = T_{1,1} = 1/2$).

**On the periodic line $HZ(a)$:** It is Proposition 4.6 in Bousquet-Mélou [4]. Let $A$ be a PCA in $PCA(\mathbb{Z}/n\mathbb{Z}, E_1)$ such that $A$ has a unique invariant distribution. This invariant distribution is of that of a HZMC iff

$$T_{0,0}T_{1,1}T_{0,1}T_{1,0} = T_{1,1}T_{0,0}T_{0,1}T_{1,0}.$$  \hspace{1cm} (9)

The positive rate condition ensures the uniqueness of the invariant distribution. The general case (that can be solved using Theorem 2.11) contains always the TM $T$ solutions to (9) (for $T$ with positive rate or not), the case $T_{0,0} = 1$, the case $T_{1,1} = 1$ and some additional results that depend on the size of the cylinder.

**On the periodic line $HZ$:** Condition (9) is necessary and sufficient too (Theorem 2.6) for positive rate automata. This result is also a simple consequence of Toom & al. [23, Section 16].

The other family of results are also related to this last zigzag case but are much more general. The point is that, under mild condition, Gibbs measures defined on the configuration space corresponding to two successive configurations of the PCA are invariant by some PCA defined thanks to this given measure. This is the object of the following section, valid for $\kappa \geq 1$.

### 1.3.2 Markovianity on the horizontal zigzag. Known results

Assume that a PCA $A = (\mathbb{Z}, E_\kappa, N, T)$ seen as acting on HZ admits as invariant distribution a $(D, U)$ HZMC. Since $HZ_t$ is made of $H_t$ and $H_{t+1}$, the distribution of $S_{t+1}$ knowing $S_t$ that can be computed using (6), relates also directly ($D, U$) with $T$. From (6) we check that in the positive rate case

$$T_{a,b} = \frac{\rho_a D_{a,c}U_{c,b}}{\rho_a (DU)_{a,b}} = \frac{D_{a,c}U_{c,b}}{(DU)_{a,b}}; \hspace{1cm} (10)$$

where $\rho$ is the invariant distribution of the $DU$-MC (solution to $\rho = \rho DU$). Since the $(D, U)$ HZMC is invariant by $T$, and since the Markov kernel of $S_t$ and $S_{t+1}$ are respectively $DU$ and $UD$, 7
we must also have in the positive rate case,

\[ DU = UD. \]  

(11)

Indeed the distribution of \( S_t \) and \( S_{t+1} \) must be equal since they are both first line of some horizontal zigzags. From Lemma 16.2 in Toom \& al. [23], we can deduce easily the following proposition:

**Proposition 1.1.** Conditions (10) and (11) are necessary and sufficient for the HZMC \((D,U)\) to be invariant under \( T \) in the positive rate case.

Notice that in the proposition the condition concerns the 3-tuple \((D,U,T)\) not only \( T \).

The bit more general Theorem 2.3 is stated and proved further in the paper. This theorem has a counterpart for positive rate PCA defined on more general lattices as \( \mathbb{Z}^d \), with more general neighbourhood, where what the authors consider are the cases where a Gibbs measure defined on a pair of two (time) consecutive configurations is invariant. The analogous of (10) in this setting connects the transition matrix with the potential of the Gibbs measure and \( DU = UD \) is replaced by the reversibility of the global Markov chain \( S_0, S_1, \ldots \) under an invariant distribution. Because of this representation only particular symmetric Gibbs measure appears. We send the interested reader in Vasilyev [24], Toom \& al. [23, section 18], Dai Pra \& al. [8], PhD thesis of Louis [18] (see also Marcovici [22, section 1.4]) for additional details. In this case only some symmetric Gibbs measures occur as invariant distributions, and an important result in [8] is that the reversibility yields the uniqueness of the invariant measure when the ambient space is \( \mathbb{Z} \). Hence, the results we have just discussed can be applied on the line, and yields that the cases where the reversibility occurs have a unique invariant measure, which is a Gibbs measure (that is a HZMC) on \( \mathbb{H} \).

Nevertheless in Remark 2.8 we will see that reversibility implies \( D = U \).

Our main Theorems 2.3 and 2.6 apply to this case, and in the more general case of quasi-reversibility (the case where under the invariant distribution, the passage from \( S_t \) to \( S_{t-1} \) is given by a PCA with some TM \( T' \)). These cases have been identified by [24] as corresponding to the cases where the invariant distribution is Gibbs on \( \mathbb{H} \). In the paper, we then provide a full characterisation of the TM \( T \) having this property (see the important Remark 2.8).

### 1.3.3 Content

Some elementary facts about Markov chain often used in the paper are recalled in Section 2.1. Section 2.2 contains Theorem 2.6 which gives the full characterisation of PCA with positive rate (and beyond) having a Markov distribution as invariant measure on \( \mathbb{H} \). It is one of the main contributions of the paper. This goes further than Proposition 1.1 (or Theorem 2.3) since the condition given in Theorem 2.6 is given in terms of the transition matrix only. This condition is reminiscent to the conditions obtained in mathematical physics to obtain an integrable system, conditions that are in general algebraic relations on the set of parameters. Theorem 2.9 extends the results of Theorem 2.6 to a class of PCA having some non positive rate TM.
Section 2.3 contains Theorem 2.11 which gives the full characterisation of PCA with positive rate (and beyond) having a Markov distribution as invariant measure on $HZ(n)$. It is much similar to Theorem 2.6.

The rest of Section 2 is devoted to the conditions on $T$ under which Markov distribution are invariant measure on $H$ and $H(n)$. Unfortunately the condition we found are stated under some (finite) system of equations relating the TM $T$ of a PCA and the kernels of the Markov distributions. Nevertheless this systematic approach sheds some lights on the structure of the difficulties: they are difficult problems of algebra! Indeed the case that can be treated completely, for example the case where the invariant distribution is a product measure and the TM $T$ symmetric (that is for any $a, b, c$, $T_{a,b} = T_{b,a}$) need some “not that obvious algebra”, not available in the general case. The present work leads to the idea that full characterisations of the TM $T$ having Markov distribution as invariant measure $H$ and $H(n)$ involve some combinatorics (of the set $\{(a, b, c) : T_{a,b} = 0\}$) together with some linear algebra considerations as those appearing in Proposition 2.13 and in its proof, and in Section 2.4.1.

In Section 3 we discuss the different conclusions we can draw from the Markovianity of an invariant distribution of a PCA with TM $T$ on one of the structure $H$, $H(n)$, $HZ$ and $HZ(n)$, on the other structures (which is summed up in Figure 2). Apart the fact that this property on $HZ$ implies that on $H$ (and $HZ(n)$ implies that on $H(n)$) all the other implications are false, to say the least, not total.

Last, Section 4 is devoted to the proof of Theorems 2.6, 2.9 and 2.11.

2 Algebraic criteria for Markovianity

2.1 Markov chains: classical facts and notation

We now recall two classical results of probability theory for sake of completeness.

Proposition 2.1. [Perron-Frobenius] Let $A = [a_{i,j}]_{1 \leq i,j \leq n}$ be an $n \times n$ matrix with positive entries and $\Lambda = \{\lambda_i, 1 \leq i \leq n\}$ be the multiset of its eigenvalues. Set $m = \max |\lambda_i| > 0$ the maximum of the modulus of the eigenvalues of $A$. The positive real number $m$ is a simple eigenvalue for $A$ called the Perron eigenvalue of $A$; all other eigenvalues $\lambda \in \Lambda \setminus \{m\}$ satisfy $|\lambda| < r$. The eigenspace associated to $m$ has dimension 1, and the associated left (resp. right) eigenvectors $L = [\ell_i]_{1 \leq i \leq n}$ (resp. $R = [r_i]_{1 \leq i \leq n}$) can be normalised such that its entries are positive. We have $\lim_{k \to \infty} A^k/m^k = RL$ for $(L, R)$ moreover normalised so that $LR = 1$. We will call Perron-LE (resp Perron-RE) these vectors $L$ and $R$. We will call them stochastic Perron-LE (or RE) when they are normalised to be probability distributions. We will denote by $ME(A)$ the maximum eigenvalue of the matrix $A$, and call it the Perron eigenvalue.

One can extend this theorem to matrices $A$ for which there exists $k \geq 1$ such that all coefficients of $A^k$ are positive. These matrices are called primitive in the literature.
Proposition 2.2. Let $P$ be a Markov Kernel with a unique invariant measure $\pi$; this invariant measure can be expressed in terms of the coefficients of $P$ as follows:

$$\pi_y = \frac{\det \left( (\text{Id}[\kappa] - P)\{y\} \right)}{\sum_x \det \left( (\text{Id}[\kappa] - P)\{x\} \right)},$$

where $P^{\{y\}}$ stands for $P$ where have been removed the $y$th column and line.

2.2 Markovianity of an invariant distribution on $\mathbb{H}Z$: complete solution

Here is a slight generalisation of Proposition 1.1. It gives a condition for a $(D,U)$-HZMC to be invariant by $T$ in terms of the 3-tuple $(D,U,T)$.

Theorem 2.3. Let $A := (\mathbb{Z}, E_\kappa, N, T)$ be a PCA seen as acting on $\mathcal{M}(E_\kappa^{\mathbb{H}Z})$ and $(D,U)\ a$ pair of Markov kernels such that for any $0 \leq a,b \leq \kappa$, $(DU)_{a,b} > 0$. The $(D,U)$ HZMC is invariant by $A$ iff the two following conditions are satisfied:

Cond 1 : there exists two Markov kernels $D$ and $U$ s.t.:
- if $T_{a,b} > 0$ then $T_{a,b} = D_{a,c}U_{c,b}/(DU)_{a,b}$,
- if $T_{a,b} = 0$ then $T_{a,b} = D_{a,c}U_{c,b} = 0$.

Cond 2 : $DU = UD$.

Lemma 16.2 in Toom & al. [23], asserts that if two Markov kernels $D$ and $U$ (with positive coefficients) satisfy $DU = UD$, then the $DU$-HMC is stable by the TM $T$ defined by $T_{a,b} = D_{a,c}U_{c,b}/(DU)_{a,b}$. These authors do not consider HZMC but only Markov chain on the line so they do not have the equivalence in their theorem.

Remark 2.4. If $T$ is a positive rate TM then if a HZMC with kernel $M = DU$ is invariant by $T$ then $M_{a,b} > 0$ for any $0 \leq a,b \leq \kappa$ since any finite configuration has a positive probability to occur at time 1 whatever is the configuration at time 0. If a product measure $\rho_\mathbb{Z}$ is invariant then $\rho_a > 0$ for any $0 \leq a \leq \kappa$.

Remark 2.5. Under Cond 1, if for some $a,b,c$ we have $T_{a,b} = 0$ then either all the $T_{a',c} = 0$ for $b' \in E_\kappa$ or all the $T_{a',b} = 0$ for $a' \in E_\kappa$.

Notice that the we do not assume the positive rate condition but something weaker $(DU)_{a,b} > 0$; under this condition, the $DU$-MC admits a unique invariant distribution.

Without the condition $(DU)_{a,b} > 0$, for any $a,b$, some problems arise. Assume a $(D,U)$ HZMC is invariant under $T$ but $(DU)_{a,b} = 0$. Under the invariant distribution, the event \{S(i,t) = a, S(i+1,t) = b\} (a.s.) does not occur, and then the transitions $(T_{a,b},c)$ do not matter. For this reason, they do not need to satisfy Cond 1. In other words the condition $(DU)_{a,b} > 0$ implies that each transition $T_{a,b}$ will play a role (for some $x$). Without this condition “pathological cases” for the
behaviour of PCA are possible as discussed in Section 1.3.1. For example if \( T_{\alpha} = 1 \) the constant process \( \alpha \) is invariant. Hence sufficient conditions for Markovianity can be expressed on only one single value \( T_{\alpha} \) and only few values of \( D \) and \( U \) (if \( T_{1,1} = 1, D_{1,1} = U_{1,1} = 1 \), the additional conditions \( DU^c \) and \( D_{a,c} U_{c,b} / (DU)_{a,b} = T_{a,b} \) for \((a,b,c) \neq (1,1,1)\) are not needed). It turns out that designing necessary and sufficient conditions for general PCA on \( E_\kappa \) seems to us a quite intricate problem, where the “reasons” why a PCA can admit a Markov invariant distribution on \( HZ \) increases over \( \kappa \), and depends somehow on the combinatorics of the periodicity of the PCA.

Proof of Theorem 2.3. Assume first that \( S_0 \) is a \((D,U)\)-HZMC invariant by \( A \). Using the argument developed in Section 1.3.2 we check that \( T_{\alpha} = D_{a,c} U_{c,b} (DU)_{a,b} \) (again when \((DU)_{a,b} > 0\), the invariant distribution of the \( DU \)-MC has full support) and that \( DU = UD \).

Assume now that \textbf{Cond 1} and \textbf{Cond 2} hold for \( D \) and \( U \) (with \((DU)_{a,b} > 0\), for any \( a,b \)). Let us show that the \((D,U)\)-HZMC is invariant by \( A \). For this start from the \((D,U)\)-HZMC on \( HZ_0 \), meaning that for any \( a_i,b_i \in E_\kappa \),

\[
\mathbb{P}(S(i,0) = a_i, i = 0, \ldots, n + 1, S(i,1) = b_i, i = 0, \ldots, n) = \rho_{a_0} \prod_{i=0}^{n} D_{a_i,b_i} U_{b_i,a_{i+1}},
\]

and let us compute the induced distribution on \( HZ_1 \). Assume that the configuration on \( HZ_1 \) is obtained by a transition of the automata from \( HZ_0 \)

\[
\mathbb{P}
\left(S(i,1) = b_i, i = 0, \ldots, n, \quad S(i,2) = c_i, i = 0, \ldots, n - 1 \right)
\]

\[
= \sum_{(a_i,0 \leq i \leq n+1)} \rho_{a_0} \left( \prod_{i=0}^{n} D_{a_i,b_i} U_{b_i,a_{i+1}} \right) \left( \prod_{i=0}^{n-1} T_{b_i,b_{i+1}} \right)
\]

\[
= \left( \sum_{a_0} \rho_{a_0} D_{a_0,b_0} \right) \left( \prod_{i=0}^{n} D_{a_i,b_i} U_{b_i,a_{i+1}} \right) \left( \sum_{x} U_{b_n,x} \right)
\]

\[
\times \prod_{i=0}^{n-1} \left( \frac{D_{b_i,c_i} U_{c_i,b_{i+1}}}{(DU)_{b_i,b_{i+1}}} \right)
\]

The first parenthesis equals \( \rho_{b_0} \), the second \( \prod_{i=0}^{n-1} (UD)_{b_i,b_{i+1}} \), the third 1, and the denominator of the fourth vanishes when multiplied by the second since \( DU = UD \). This gives the desired result. 

We now define some quantities needed to state Theorem 2.6.

Let \( \nu := \nu[T] \) be the stochastic Perron-LE of the stochastic matrix

\[
Y := Y[T] = \begin{bmatrix} T_{i,j} \end{bmatrix}_{0 \leq i,j \leq \kappa}
\]

and \( \gamma := \gamma[T] \) be the stochastic Perron-LE of the matrix

\[
X := X[T] = \begin{bmatrix} T_{a,b} \nu_a \end{bmatrix}_{0 \leq d,a \leq \kappa}
\]
associated with $\lambda := \lambda[T] > 0$ the Perron-eigenvalue of $X$. Then $(\gamma_i, 0 \leq i \leq \kappa)$ is solution to:

$$\sum_d \frac{\gamma_d}{T_{a,d}^0} = \lambda \frac{\gamma_a}{T_{a,a}^0}.$$  \hspace{1cm} (12)

By Proposition 2.2, $\nu$ and $\gamma$ can be computed in terms of $T$ (but difficulties can of course arise for effective computation starting from that of $\lambda$). Define further for any $\eta = (\eta_a, 0 \leq a \leq \kappa) \in M^*(\kappa)$ (distribution on $E_\kappa$ with full support), the Markov kernels $D^\eta$ and $U^\eta$:

$$D^\eta_{a,c} = \frac{\sum \ell \eta_\ell T_{a,\ell}^0 T_{\ell,c}^0}{\sum b' \eta_{b'} T_{a,b'}^0 T_{b',c}^0}, \quad U^\eta_{c,b} = \frac{\sum b' \eta_{b'} T_{0,b'}^0 T_{0,c}^0}{\sum b' \eta_{b'} T_{0,b'}^0 T_{0,b'}^0}, \quad \text{for } 0 \leq a, b, c \leq \kappa. \hspace{1cm} (13)$$

The indices are chosen to make easier some computations in the paper. When no specified the sum are taken on $E_\kappa$.

**Theorem 2.6.** Let $A := (Z, E_\kappa, N, T) \in \text{PCA}(Z, E_\kappa)^*$ be a positive rate PCA seen as acting on $\mathcal{M}(E_\kappa^{HZ})$. $A$ admits a HZMC on $HZ$ iff $T$ satisfies the two following conditions

**Cond 3:** for any $0 \leq a, b, c \leq \kappa, T_{a,b}^c = T_{0,a}^0 T_{a,b}^c T_{0,b}^c$.

**Cond 4:** $D^\eta U^\eta = U^\eta D^\eta$, for $\eta = \gamma$.

In this case the $(D^\gamma, U^\gamma)$-HZMC is invariant under $A$ and the common invariant distribution for the MC with Markov kernels $D^\gamma, U^\gamma, D^\gamma U^\gamma$ or $U^\gamma D^\gamma$ is $\rho = [\gamma_i \mu_i]_{0 \leq i \leq \kappa}$ where $\mu = \{\mu_i\}_{0 \leq i \leq \kappa}$ is the Perron-RE of $X$ normalised so that $\rho$ is a probability distribution.

When $\kappa = 1$ (the two-colour case), when **Cond 3** holds, then so does **Cond 4**, and then the only condition is **Cond 3** (which is equivalent to (9)).

Even if **Cond 4** seems much similar to **Cond 2**, it is not ! In Theorem 2.3 the question is that of the existence of a pair $(D, U)$ satisfying a condition. In Theorem 2.6 the pair $(D, U)$ is known, it is $(D^\gamma, U^\gamma)$, and the remaining question is: does the equality $D^\gamma U^\gamma = U^\gamma D^\gamma$ holds or not?

The proof of this theorem is postponed in Section 4, as well as the fact that for $\kappa = 1$, **Cond 4** disappears whilst this fact is far to be clear at the first glance. An important ingredient in the proof is Lemma 4.1 which says that **Cond 1** and **Cond 3** are equivalent.

By (12), **Cond 4** can be rewritten

$$\sum_c \left( \frac{T_{c,c}^\nu \eta_c}{T_{0,c}^0 T_{a,c}^0} \right) \left( \sum_d \frac{\gamma_d}{T_{c,d}^0 T_{a,d}^0} \right) = \frac{T_{a,a} \nu_a}{\gamma_a} \frac{\gamma_b}{T_{a,b}^0}, \quad \text{for } 0 \leq a, b \leq \kappa. \hspace{1cm} (14)$$
Remark 2.7. Condition Cond 3 is bit asymmetric. In Lemma 4.1 we will show that this condition is equivalent in the positive rate case to the following condition:

Cond 5: for any $0 \leq a, a', b, b', c, c', \leq \kappa$, 

\[
T_{a',b'} T_{a,b} T_{a',b} = T_{a,b} T_{a',b'} T_{a',b},
\]

Remark 2.8. A consequence of Theorem 2.3 is that if $(D, U, T)$ is solution to Cond 1 and Cond 2 then ($U, D, T'$) with $T'_{a,b} = \frac{U_{a,c} D_{a,b}}{(UD)_{a,b}}$ is also solution to Cond 1 and Cond 2. Since $UD = DU$ the $M$-MC with kernel $M = DU = UD$ is invariant by $T$ and $T'$. This property is called quasi-reversibility in the literature (see Vasilyev [24, Theorem 3.1 and Corollaries 3.2 and 3.7]). As explained in Corollary 3.2 [24], reversibility, that is, when $T = T'$ is equivalent to the fact that (in our words)

\[
T_{x_1, x_2} = \frac{A[x_1, x_2, y]}{\sum_{y' \in E_{\kappa}} A[x_1, x_2, y']}
\]

where

\[
A[x_1, x_2, y] = \exp(-\phi(y) - \phi_D(x_1, y) - \phi_U(x_2, y))
\]

with $\phi : E_{\kappa} \to \mathbb{R}$, $\phi_D : E_{\kappa}^2 \to \mathbb{R}$, $\phi_U : E_{\kappa}^2 \to \mathbb{R}$ are some functions (that defines the potential of the Gibbs measure) and $\phi_D$ and $\phi_U$ satisfies

\[
\phi_D(x, y) = \phi_U(y, x), \text{ for any } (x, y) \in E_{\kappa}^2.
\]

To express this in terms of $(D, U)$ set

\[
D_{a,c} = \frac{\exp(-\frac{1}{2} (\phi(c) + \phi(a)) - \phi_D(a, c))}{\sum_{c'} \exp(-\frac{1}{2} (\phi(c') + \phi(a)) - \phi_D(a, c'))}
\]

\[
U_{c,b} = \frac{\exp(-\frac{1}{2} (\phi(c) + \phi(b)) - \phi_U(b, c))}{\sum_{b'} \exp(-\frac{1}{2} (\phi(c) + \phi(b'))) - \phi_U(b', c))}
\]

From here we get

\[
D_{a,c} U_{c,b} / (D U)_{a,b} = \frac{A[a, b, c]}{\sum_{c'} A[a, b, c']}
\]

and $D = U$. Hence the hypothesis of reversibility satisfies Cond 1 and Cond 2. According to [24, Theorem 3.1], the quasi reversible case corresponds to the cases where the invariant distribution of the PCA seen as acting on HZ has a Gibbs measure as invariant distribution (which by Georgii [12, Theo. 3.5] corresponds to MC). From what is said above, Theorem 2.3 applies then to the quasi-reversible case exactly. In the positive rate case, Cond 3 says that for any $a, b$, the map $c \mapsto T_{a,b} T_{0,0} = T_{a,b} T_{0,0}$ is a constant map. Under Cond 3 for any $a, b$, $\sum_c T_{a,b,c} = 1$ and then

\[
\sum_c \frac{T_{a,b,c}}{T_{0,0,c}} = \frac{T_{a,b}}{T_{0,0}}.
\]
We can then rewrite
\[ T_{a,b} = A[a,b,c] / \sum_{c'} A[a,b,c'] \]
with \( A[a,b,c] = T_{a,b} T_{0,c} / T_{0,0} \), which can be of course written under the form (15) with \( \phi(c) = \log(T_{0,c}) \), \( \phi_D(a,c) = -\log(T_{a,c}) \), \( \phi_U(b,c) = -\log(T_{0,b}) \). This time \( \phi_D \) and \( \phi_U \) do not satisfy necessarily (16) but satisfy the implicit conditions Cond 3 and Cond 4. The form we found for \( D \) and \( U \) are more complex (in terms of \( \phi, \phi_D, \phi_U \) than (17) and (18)).

Relaxation of the positive rate condition. We won’t consider all PCA that admit some Markov invariant distribution on \( \text{HZ} \) here, but only those for which the invariant distribution \((D,U)\) satisfies, for any \( a,b \in E_\kappa \), \( (DU)_{a,b} > 0 \), one of the hypothesis already discussed of Theorem 2.3. We will assume that exists for \( i = 0 \), for any \( a,b,c \), \( T_{a,b} > 0 \) and \( T_{1,c} > 0 \) (if it is true for another \( i \), then relabel the elements of \( E_\kappa \)).

Cond 6 : for any \( a,b,c \in E_\kappa \), \( T_{a,b} > 0, T_{0,c} > 0 \).

Under Cond 6, Cond 3 is well defined. Notice that for a fixed pair \( (a,b) \), not all the \( T_{a,b} \) can be 0; hence Cond 3 implies that \( T_{a,b} > 0 \) for any \( a,b \). It follows that \( U_{c,b}^\gamma \) and \( D_{a,c}^\gamma \) as defined in (13) are still well defined and \( (D^\gamma U^\gamma)_{a,b} > 0 \) for all \( a,b \). We have the following result

**Theorem 2.9.** Theorem 2.6 holds if instead of considering \( A := (\mathbb{Z}, E_\kappa, N,T) \) in \( \text{PCA}(\mathbb{Z}, E_\kappa)^* \), \( T \) satisfies Cond 6 instead with the slight modification that \( U_{c,b}^\gamma = 0 \) when \( \{T_{a,c}, a \in E_\kappa\} = \{0\} \).

The proof is postponed at the end of Section 4. It is similar to that of Theorem 2.6.

2.3 Markovianity of an invariant distribution on \( \text{HZ}(n) \): complete solution

In the cyclic zigzag, we have

**Theorem 2.10.** Let \( A := (\mathbb{Z}/n\mathbb{Z}, E_\kappa, N,T) \) be a PCA seen as acting on \( \mathcal{M}(E_\kappa^{\text{HZ}(n)}) \) for some \( n \geq 1 \) and \( (D,U) \) a pair of Markov kernels such that for any \( 0 \leq a,b \leq \kappa \), \( (DU)_{a,b} > 0 \). The \( (D,U)\)-HZCMC on \( \text{HZ}(n) \) is invariant by \( A \) iff Cond 1 holds and

Cond 7 :

\[ \text{Diagonal}((DU)^k) = \text{Diagonal}((UD)^k) \text{ for all } 1 \leq k \leq \min(\kappa + 1, n). \]

Notice that Cond 7 is equivalent to the fact that for all \( j \leq |E_\kappa| \), for all \( a_0, \ldots, a_{j-1} \in E_\kappa \),

\[ \prod_{i=0}^{j-1} (DU)_{a_i, a_{i+1} \mod j} = \prod_{i=0}^{j-1} (UD)_{a_i, a_{i+1} \mod j}. \]

It does not imply \( DU = UD \) (but the converse holds).
Proof. Suppose that the \((D,U)\)-HZCMC on \(\text{HZ}(n)\) is invariant by \(T\). The reason why **Cond 1** holds is almost the same as in Section 1.3.2:

\[
\mathbb{P}(S(0,1) = c | S(0,0) = a, S(1,0) = b) = \frac{D_{a,c}U_{c,b}((DU)^{n-1})_{k,a}}{(DU)_{a,b}((DU)^{n-1})_{k,a}} = \frac{D_{a,c}U_{c,b}}{(DU)_{a,b}}
\]

If \(S\) is a \((D,U)\)-HZCMC on \(\text{HZ}(n)\) then \(S|_{\text{HZ}(n)}\) and \(S|_{\text{HZ}(1)}\) are respectively \(DU\) and \(UD\) CMC on \(\mathbb{Z}/n\mathbb{Z}\). Moreover the distributions of \(S|_{\text{HZ}(n)}\) and \(S|_{\text{HZ}(1)}\) must be equal since they are respectively first line of \(\text{HZ}_0\) and \(\text{HZ}_1\). Now take a pattern \(w = (w_1, \ldots, w_k)\) in \(E_k^\ell\), for some \(\ell \leq |E_\kappa|\), and consider the word \(W\) obtained by \(j\) concatenations of \(w\). The probability that \(S|_{\text{HZ}(j\ell)}\) and \(S|_{\text{HZ}(j\ell)}\) take value \(W\), are both equal to

\[
\left(\prod_{i=0}^{j-1} (DU)_{x_i,x_{i+1} \mod \ell}\right)^j \frac{\text{Trace}((DU)^{n\ell})}{\text{Trace}((UD)^{nj})},
\]

where the denominator are equal. Therefore, we deduce **Cond 7**.

Assume that **Cond 1** and **Cond 7** hold true for \(D\) and \(U\) some Markov kernels. Assume that \(S\) is a \((D,U)\)-HZCMC on \(\text{HZ}_0\). Again \(S|_{\text{HZ}(n)}\) and \(S|_{\text{HZ}(1)}\) are respectively \(DU\) and \(UD\) CMC on \(\mathbb{Z}/n\mathbb{Z}\). By **Cond 1** one sees that \(S|_{\text{HZ}(1)}\) is obtained from \(S|_{\text{HZ}(n)}\) by the PCA \(A\). Let us see why **Cond 7** implies that \(S|_{\text{HZ}(n)}\) and \(S|_{\text{HZ}(1)}\) have the same distribution: we have to prove that any word \(W = (w_0, \ldots, w_{n-1})\) occurs equally likely for \(S|_{\text{HZ}(n)}\) or \(S|_{\text{HZ}(1)}\), when **Cond 7** says that it is the case only when \(n \leq |E_\kappa|\). We will establish that

\[
\prod_{i=0}^{n-1} (UD)_{w_i,w_{i+1} \mod n} = \prod_{i=0}^{n-1} (DU)_{w_i,w_{i+1} \mod n}.
\]

For any letter \(a \in E_\kappa\) which occurs at successive positions \(j_1^a, \ldots, j_k^a\) for some \(k_a\) in \(W\) let \(d_n(j_i^a, j_{i+1}^a)\) be the distance between these indexes in \(\mathbb{Z}/n\mathbb{Z}\) that is \(\min(j_{i+1}^a - j_i^a, n-j_i^a+j_{i+1}^a)\). Since \(|E_\kappa| < +\infty\) is bounded, there exists \(a\) and indexes \(j_i^a\) and \(j_{i+1}^a\) for which \(d_n(j_i^a, j_{i+1}^a) \leq |E_\kappa|\) (by the so called pigeonhole principle): to show that \(W\) occurs equally likely in \(S|_{\text{HZ}(1)}\) and in \(S|_{\text{HZ}(n)}\) it suffices to establish that \(W'\) obtained by removing the cyclic-pattern \(W' = w_{j_1^a+1}, \ldots, w_{j_k^a+1}\) from \(W\) occurs equally likely in \(S|_{\text{HZ}(n-j_{k+1}^a+j_i^a)}\) and \(S|_{\text{HZ}(n-j_{k+1}^a+j_i^a)}\) (since the contribution to the weight of the cyclic-pattern \(W'\) is \(\prod_{i=j_i^a}^{j_i^a+1} (DU)_{w_i',w_{i+1}'} = \prod_{i=0}^{n-1} (DU)_{w_i',w_{i+1}'}\) in both \(S|_{\text{HZ}(1)}\) and \(S|_{\text{HZ}(n)}\)). This ends the proof by induction. \(\Box\)

Recall the definitions of \(U^n, D^n, \gamma\) defined above Theorem 2.6.

**Theorem 2.11.** Let \(A := (\mathbb{Z}/n\mathbb{Z}, E_\kappa, N, T)\) be a positive rate PCA seen as acting on \(\mathcal{M}(E_\kappa^{\text{HZ}(n)})\).

\(A\) admits a HZCMC as invariant distribution on \(\text{HZ}(n)\) iff **Cond 3** holds and if

**Cond 8**:

\[
\text{Diagonal}((D^nU^n)^k) = \text{Diagonal}((U^nD^n)^k) \text{ for all } 1 \leq k \leq \kappa + 1 \text{ for } \eta = \gamma.
\]

(21)
In this case the \((D^\gamma, U^\gamma)\)-HZCMC is invariant under \(A\). When \(\kappa = 1\) (the two-colour case), when \textbf{Cond } 3 holds, then so does \textbf{Cond} 8, and then the only condition is \textbf{Cond} 3.

Again, one can state a version of this Theorem without the positive rate condition with \textbf{Cond} 6 instead (the analogous of Theorem 2.9 in the cyclic case). The proof in this case is the same as that of Theorem 2.9.

2.4 Markov invariant distribution on the line

In this section, we discuss some necessary and sufficient conditions on \((M, T)\) for the \(M\)-MC to be invariant under \(T\) on \(H\) and \(H(n)\).

2.4.1 Markovian invariant distribution on \(H\) or \(H(n)\)

Let \(T\) be a TM for a PCA \(A\) in \(\text{PCA}(\mathbb{L}, E_\kappa)\). Let \(M\) be a Markov kernel on \(E_\kappa\), and \(\rho = \left[\sqrt{\rho_i}\right]_{0 \leq i \leq \kappa}\) an element of \(M(E_\kappa)^*\). Consider the matrices \((Q^M_x, x \in E_\kappa)\) defined by

\[
Q^M_x = \left[\frac{\sqrt{\rho_i}}{\sqrt{\rho_j}} M_{i,j} T_{i,j}\right]_{0 \leq i,j \leq \kappa},
\]

and set \(\rho^{1/2} := \left[\sqrt{\rho_i}\right]_{0 \leq i \leq \kappa}\) (we should write \(Q^M_x(\rho, T)\) instead, but \(\rho\) and \(T\) will be implicit).

**Lemma 2.12.** Let \(T\) be a TM for a PCA \(A\) in \(\text{PCA}(\mathbb{L}, E_\kappa)\) (with positive rate or not).

(i) The \((\rho, M)\)-MC is invariant by \(T\) on \(H\) iff for any \(m > 0\), any \(x_1, \ldots, x_m \in E_\kappa\),

\[
\rho_{x_1} \prod_{j=1}^{m-1} M_{x_j, x_{j+1}} = \rho^{1/2} \left( \prod_{j=1}^{m} Q^M_{x_j} \right)^{1/2}. \tag{22}
\]

(ii) The \(M\)-CMC is invariant by \(T\) on \(H(n)\) iff for any \(x_1, \ldots, x_n \in E_\kappa\),

\[
\prod_{j=1}^{n} M_{x_j, x_{j+1 \mod n}} = \text{Trace} \left( \prod_{j=1}^{n} Q^M_{x_j} \right). \tag{23}
\]

**Proof.** Just expand the right hand side.

In the rest of this section, (i) and (ii) will always refer to the corresponding item in Lemma 2.12. We were not able to fully describe the set of solutions \((M, T)\) to (i) and (ii). Nevertheless, in the rest of this section we discuss various necessary and sufficient conditions on \((M, T)\). We hope that the following results will shed some light on the algebraical difficulties that arise here.

**Proposition 2.13.** [I.I. Piatetski-Shapiro] Lemma 2.12 still holds if in (i) the conditions (22) holds only for all \(m \leq \kappa + 2\).
Proof. We borrow the argument in Toom & al. [23, Theorem 16.3]. In words, assume we have (22) for \( m \leq \kappa + 2 \). This can be rewritten
\[
\left( \frac{\rho^{1/2}}{\rho_{x_1}} Q_{x_1}^M Q_{x_2}^M - \frac{M_{x_1,x_2}}{\rho_{x_2}} \rho^{1/2} Q_{x_2}^M \right) (Q_{x_3}^M \ldots Q_{x_m}^M) t \rho^{1/2} = 0.
\]
To get the result for all \( m \), the idea is to investigate the form of the vectors \( c \) for which
\[
e P(Q_0^M, \ldots, Q_\kappa^M) t \rho^{1/2} = 0,
\]
for all \( P(Q_0^M, \ldots, Q_\kappa^M) \) monomial in the \( Q_t^M \), that is an ordered product constituted with the matrices \( Q_0^M, \ldots, Q_\kappa^M \). Let then take some non zero vector \( c \) for which (24) holds for all \( P \) such that \( \text{deg}(P) \leq \kappa + 2 \). Consider now the vector space \( L_{1} = \text{Vect}(c) \) and for any \( m \geq 1 \),
\[
L_{m+1} = \text{Vect}(L_m, \{ xQ_g^M, x \in L_m, 0 \leq y \leq \kappa \}).
\]
The sequence \( L_m \) is strictly increasing till it becomes constant, because its dimension is bounded by that of the ambient space \( \kappa + 1 \). For this reason, it reaches its final size for some \( m \leq \kappa + 1 \).

To end the proof, take all \( c \) of the form
\[
\frac{\rho^{1/2}}{\rho_{x_1}} Q_{x_1}^M Q_{x_2}^M - \frac{M_{x_1,x_2}}{\rho_{x_2}} \rho^{1/2} Q_{x_2}^M.
\]

Since the asymptotics of \( \text{Trace}(A^n) \) or \( \rho^{1/2} A^n t \rho^{1/2} \) are driven by the largest eigenvalues of \( A \) (under mild conditions on \( (\rho^{1/2}, A) \)), we have the following statement which can be used as some necessary conditions on the system \( (M, T) \).

Proposition 2.14. (a) Assume that \( (M, T) \) is solution to (i) with \( T \) a positive rate \( TM \), then for any \( \ell \geq 1 \), any \( x_1, \ldots, x_\ell \) we have
\[
\prod_{i=1}^\ell M_{x_i,x_{i+1} \mod \ell} = \text{ME}(Q_{x_1}^M \ldots Q_{x_\ell}^M).
\]
(b) Let \( \ell \geq 1 \) be fixed. Assume that \( (M, T) \) is solution to (ii) for at least \( \kappa + 1 \) (this is \( |E_n| \)) different positive integers \( n \) of the form \( n = k\ell \). In this case, for any \( x_1, \ldots, x_\ell \), \( \text{ME}(Q_{x_1}^M \ldots Q_{x_\ell}^M) = \prod_{i=1}^\ell M_{x_i,x_{i+1} \mod \ell} \). Moreover, all the matrices \( Q_{x_1}^M \ldots Q_{x_\ell}^M \) have rank 1.

Remark 2.15. In Proposition 2.14, we can replace the positive rate condition by a weaker one: we only need the primitivity of the matrices \( Q_{x_1}^M \ldots Q_{x_\ell}^M \) for any \( \ell \), \( x_1, \ldots, x_\ell \). But this condition is a bit difficult to handle since it does not follow the primitivity of the family of matrices \( Q_x^M \).

Proof. We give a proof in the case \( \ell = 1 \) and for case (i) and (ii) sake of simplicity, but exactly the same argument applies for larger \( \ell \) (by repeating the pattern \( (x_1, \ldots, x_\ell) \) instead of \( x \) alone). Following Remark 2.4, the positive rate condition on \( T \) implies that any Markovian invariant distribution with kernel \( M \) will be as stated in the remark, from which we deduce that all the matrices \( Q_x^M \) have positive coefficients.

(a) Let \( m \geq 1 \). Taking \( x_1 = \cdots = x_m = x \) in Lemma 2.12, we get
\[
\rho_x M_{x,x}^{m-1} = \rho^{1/2} (Q_x^M)^m t \rho^{1/2}.
\]
By Perron-Frobenius we obtain for \( R_{Q_x^M} \) and \( L_{Q_x^M} \) the Perron-RE and LE of \( Q_x^M \) normalised so that
\[
L_{Q_x^M} R_{Q_x^M} = 1, \quad \rho_x M_{x,x}^{m-1} \sim_{m \to \infty} \text{ME}(Q_x^M)^m \left( \rho^{1/2} R_{Q_x^M} L_{Q_x^M} t \rho^{1/2} \right).
\]
Hence, necessarily,
\[
\text{ME}(Q_x^M) = M_{x,x}.
\]
(b) Let $x$ be fixed. Assume that (ii) holds for $\kappa + 1$ different integers $n = n_i$ for $i = 0, \ldots, \kappa$. For all $n \in \{n_0, \ldots, n_\kappa\}$, $M^M_{x,x} = \text{Trace} \left( \left( Q^M_x \right)^n \right) = \sum \lambda^n_i$ where $(\lambda_i, 0 \leq i \leq \kappa)$ are the eigenvalues of $Q^M_x$ from what we deduce that all the eigenvalues of $Q^M_x$ equals 0, but 1 which is $M^M_{x,x}$. □

One can design various sufficient conditions for $T$ to satisfy (i), (ii). For example, for the case (i) following the proof of Proposition 2.13, it suffices that for any $x_1, x_2$

$$\frac{\rho^{1/2}}{\rho_{x_1}} Q^M_{x_1} Q^M_{x_2} = \frac{M_{x_1,x_2}}{\rho_{x_2}} \rho^{1/2} Q^M_{x_2}$$  \hspace{1cm} (25)$$

for the $M$-MC to be invariant under $T$. Notice that this does not imply that $\rho^{1/2}$ is a common eigenvector for all the matrices $Q^M_x$ (since the multiplication by a matrix $Q^M_{x_2}$ remains). But of course, one can design a sufficient condition with such a property.

### 2.4.2 I.i.d. case

If we are interested in product measures instead of Markov chains, the content of Section 2.4.1 still applies since product measure are Markov chains of the type $M_{a,b} = \rho_b$. In this case

$$Q^M_x = Q^p = \left[ \sqrt{\rho_j} T_{i,j} \sqrt{\rho_i} \right]_{0 \leq i,j \leq \kappa}$$

The iid case is also interesting, as has been shown by Mairesse & Marcovici [19]. We can design some additional sufficient conditions for the product measure $\rho^Z$ to be invariant under $T$:

- the following condition is sufficient:

\[
\begin{align*}
\rho^{1/2} \prod_{i=0}^{\kappa} (Q^p_i)^{n_i} t^{1/2} & = \prod_{i=0}^{\kappa} \rho_i^{n_i}, \text{ for any } n_i \geq 0, \\
Q^p_{W_1} (Q^p_x Q^p_y - Q^p_y Q^p_x) Q^p_{W_2} & = 0 \text{ for any words } W_1 \text{ and } W_2, \text{ and any } 0 \leq x, y \leq \kappa.
\end{align*}
\]

(26)

- Considering again (25): it suffices that $\frac{\rho^{1/2}}{\rho_{x_1}} Q^p_{x_1} Q^p_{x_2} = \rho^{1/2} Q^p_{x_2}$. It suffices that $\frac{\rho^{1/2}}{\rho_{x_1}} Q^p_{x_1} = \rho^{1/2}$ (see also Mairesse & Marcovici [19, Theorem 5.6] for the i.i.d. case).

Additional sufficient or necessary conditions can be imagined for both cases $H$ and $H(n)$, but necessary and sufficient conditions on $T$ seem out of reach for the moment.

Let us compare $H$ and $H(n)$ with respect to the existence of a product measure as the invariant distribution of some PCA.

**Proposition 2.16.** (a) If $\rho^{Z/Z_n}$ is invariant by $T$ on $H(n)$ then for any $m < n$, for any $x_1, \ldots, x_m \in E_\kappa$,

$$\prod_{j=1}^{m} \rho_{x_j} = \rho^{1/2} \left( \prod_{j=1}^{m} Q^p_{x_j} \right) t^{1/2}$$

(that is, starting from a $m+1$ tuple of i.i.d. $\rho$-distributed random variables, a transition by $T$ gives a $m$-tuple of i.i.d. $\rho$-distributed random variables).
If $\rho^Z_n$ is invariant by $T$ on $H(n)$ for infinitely many $n$ then $\rho^Z$ is invariant by $T$ on $H$.

**Proof.** First (b) is a consequence of (a). (a) can be proven by a computation but also by a picture! The fact that the $n$ first variables are arranged around the cylinder or along a line at time 0 does not matter when we look at the distribution of $m < n$ consecutive states at time 1 (but it matters for $m = n$).

We now discuss a subcase where the complete characterisation of TM $T$ having a product measure as invariant distribution on $H$ and $H(n)$ is possible.

### 2.4.3 Symmetric transition matrices and i.i.d. invariant measure

We say that a TM $T$ is symmetric if for any $a, b, c \leq \kappa$, $T_{a,b} = T_{b,a}$. Let $T$ be a symmetric transition matrix of a PCA $A$ in PCA$(L, E_\kappa)^*$ with positive rate and let $\rho \in M^*(E_\kappa)$ be a distribution on $E_\kappa$ with full support. A distribution $\mu$ in $M(E^Z_\kappa)$ is said to be symmetric if $\mu(x_1, \ldots, x_n) = \mu(x_n, \ldots, x_1)$ for any $n \geq 1, 0 \leq x_1, \ldots, x_n \leq \kappa$. We start by a simple proposition.

**Proposition 2.17.** Let $T$ be a symmetric TM. There exists a symmetric distribution $\mu$ in $M(E^Z_\kappa)$ invariant by $T$.

**Proof.** It suffices to notice that if $\mu_0$ is symmetric, then so does $\mu_1 = \mu_0 T$, and to apply a compacity argument.

When $T$ is a symmetric TM, for any $x$ the matrix $Q^x_\rho$ is symmetric and then Hermitian. We can then use some classical additional properties of this class of matrices to go further in our analysis.

**Lemma 2.18.**

(a) $r$ is a right eigenvector for an Hermitian matrix $A$ associated with the eigenvalue $\lambda$ (that is $rA = \lambda A$) iff $^t r$ is a right eigenvector of $A$ associated with $\lambda$ (that is $A^t r = \lambda^t r$).

(b) If $A$ and $B$ are two Hermitian matrices then $\text{ME}(A + B) \leq \text{ME}(A) + \text{ME}(B)$. The equality holds only if the (left, and then right by (a)) eigenspaces of the matrices $A$ and $B$ associated with the respective eigenvalues $\text{ME}(A)$ and $\text{ME}(B)$ are equal.

**Proposition 2.19.** Let $T$ be a symmetric TM with positive rate.

(a) that is $\rho^Z$ is invariant by $T$ on $H$ iff for all $x \in E_\kappa$, the Perron eigenvalue of $Q^x_\rho$ is $\rho_x$ and $\rho^{1/2}$ and $^t \rho^{1/2}$ are Perron-LE and RE of $Q^x_\rho$.

(b) if $\rho^Z(n)$ is invariant by $T$ on $H(n)$ for at least $\kappa + 1$ different positive integers $n$ iff for any $i, j, x \in E_\kappa$, $T_{i,j} = \rho_x$.

The positive rate condition allows one to use Perron-Frobenius theorem on the matrices $(Q^x_\rho, x)$ in the case where the Perron-eigenspaces have dimension 1. The proposition still holds if we replace the positive rate condition by a weaker one for example the primitivity of the matrices $(Q^x_\rho, x)$. 

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Proposition 2.20.

and ∑

Suppose (a) Assume ρ^2 is invariant by T on H. We apply Lemma 2.18: the matrices Q^ρ are Hermitian and ∑ x Q^ρ = \left[\sqrt{\rho_0} M_{i,j}\right]_{0 \leq i, j < \kappa}. Hence ME(∑ x Q^ρ) = ∑ x ME(Q^ρ) = 1. Therefore, all the matrices Q^ρ and ∑ x Q^ρ have same Perron-LE and RE that are ρ^{1/2} and tρ^{1/2}.

Proof. (b) Consider a TM T

H

This case, for any x, that ρ^{1/2} is a Perron-LE of Q^ρ associated to the Perron eigenvalue ρ_x. Then for any m for any x_1, \ldots, x_m \in E_\kappa,

\rho^{1/2} \left( \prod_{j=1}^{m} Q^ρ \right) tρ^{1/2} = \rho_{x_1} \rho^{1/2} \left( \prod_{j=2}^{m} Q^ρ \right) tρ^{1/2} = \cdots = \prod_{j=1}^{m} \rho_{x_j} \rho^{1/2} tρ^{1/2} = \prod_{j=1}^{m} \rho_{x_j}.

Proof of (b).

But, (Q^ρ)_{i,j} = tρ^{1/2} T_{i,j} \rho_j^{1/2}. Then, for all x, i, j, T_{i,j} = ρ_x. (The converse is trivial) □

A (ρ, M)-MC is called reversible, if for any a, b, ρ_a M_{a,b} = ρ_b M_{b,a}.

Proposition 2.20. The reversible (ρ, M)-MC is invariant by the symmetric TM T on H iff

\sum_j T_{i,j} M_{i,j} = ρ_x, \quad \text{for any } 0 \leq i, x \leq \kappa.

(27)

In this case, for any 0 \leq a, b \leq \kappa, M_{a,b} = ρ_b (the Markov chain is a product measure).

Proof. Suppose (M, T) is solution to (i). We apply Lemma 2.18: the matrices Q^M are Hermitian and ∑ x Q^M = \left[\frac{\sqrt{\kappa}}{\kappa} M_{i,j}\right]. Hence ME(∑ x Q^M) = ∑ x ME(Q^M) = 1. Therefore, all the matrices Q^M and ∑ x Q^M have same Perron-LE and RE that are ρ^{1/2} and tρ^{1/2}. The extraction of the i th coordinate of the relation Q^M tρ^{1/2} = ρ_x tρ^{1/2} gives (27). Conversely, assume that (27) holds; this says that the ith coordinate of Q^M tρ^{1/2} coincides with the ith coordinate ρ_x tρ^{1/2} for any x and i, from which we deduce Q^M tρ^{1/2} = ρ_x tρ^{1/2}, and then ρ^{1/2} Q^M x_1 \ldots Q^M x_t tρ^{1/2} = ρ_{x_1} \ldots ρ_{x_t} tρ^{1/2} = ρ_{x_1} \ldots ρ_{x_t} tρ^{1/2} = ρ_{x_1} \ldots ρ_{x_t}. This is a product measure. □

3 Markov invariant distribution on \text{H vs } \text{H(n) vs } \text{HZ vs } \text{HZ(n)}

Consider a TM T seen as acting on H, H(n), HZ and HZ(n). In this section we discuss the different conclusions we can draw from the Markovianity of the invariant distribution under T on one of these structures. Figure 2 gathers most of the results obtained in this section.

From H(n) to H. The following Proposition is already known (see Albenque [1] and a “formal” version is also used in Bousquet-Mélou [4]).
Proposition 3.1. If a PCA $A := (\mathbb{Z}/n\mathbb{Z}, E_\kappa, N, T)$ admits a CMC with an irreducible kernel $M$ on $H(n)$ as an invariant distribution for infinitely many $n$ then the PCA $A := (\mathbb{Z}, E_\kappa, N, T)$ admits the $M$-MC as an invariant distribution on $H$.

Notice that if $A := (\mathbb{Z}/n\mathbb{Z}, E_\kappa, N, T)$ with positive rate admits a $M$-CMC as invariant distribution then $M$ is irreducible.

Proof. Several proofs are possible. We adapt slightly the argument of Theorem 3 in [1]. The idea is to prove that the distribution of a $M$-CMC on $H(n)$ converges to a $M$-MC on the line (limit taken in the set $O$, the set of integers $n$ for which the $M$-MC is invariant by $T$ on $H(n)$). Proceed as follows. Choose some $k \geq 1$. For $n \geq k$ in $O$, the probability of any pattern $b_1, \ldots, b_k$ in $E_\kappa$ (in successive positions) is for this distribution

$$\left( \prod_{i=1}^{k-1} M_{b_i, b_{i+1}} \right) (M^{n-k})_{b_k, b_1} = \sum_{(a_1, \ldots, a_{k+1}) \in E^{k+1}_\kappa} \left( \prod_{i=1}^{k} M_{a_i, a_{i+1}} T_{a_i, a_{i+1}}^{a_i, a_{i+1}} \right) (M^{n-k-1})_{a_{k+1}, a_1}. \quad (28)$$

Since $M$ is an irreducible Markov kernel, by Perron-Frobenius theorem, $M^n \to M^\infty$ where $M^\infty$ is the matrix whose lines equal the stochastic LE $\rho$ of $M$. Therefore $(M^{n-k})_{b_k, b_1} \to \rho_{b_1}$ and the limit distribution for $H(n)$ exists and satisfies

$$\mathbb{P}(S_i = b_i, i = 1, \ldots, k) = \rho_{b_1} \prod_{i=1}^{k-1} M_{b_i, b_{i+1}}$$

and satisfies, taking the limit in (28),

$$\rho_{b_1} \prod_{i=1}^{k-1} M_{b_i, b_{i+1}} = \sum_{(a_1, \ldots, a_{k+1}) \in E^{k+1}_\kappa} \rho_{a_1} \left( \prod_{i=1}^{k} M_{a_i, a_{i+1}} T_{a_i, a_{i+1}}^{a_i, a_{i+1}} \right). \quad \square \quad (29)$$
From H to HZ.

**Proposition 3.2.** If a M-MC is an invariant distribution for a PCA $A := (Z, E, N, T)$ on the line, then seen as acting on the set of measures indexed by HZ, A admits a HZMC with memory 2 as invariant distribution.

**Proof.** Take $D_{a,c} = \sum_i M_{a,i}T_{a,i}$ and $U_{a,c,b} = \frac{M_{a,b}T_{a,b}}{\sum_i M_{a,i}T_{a,i}}$ or 0 if the denominator is 0 (in which case the numerator is 0 too). These kernels have to be understood as follows:

$$P(S(0, 1) = c | S(0, 0) = a) = D_{a,c}, \quad P(S(1, 0) = b | S(0, 0) = a, S(0, 1) = c) = U_{a,c,b},$$

and they satisfy

$$D_{a,c}U_{a,c,b} = M_{a,b}T_{a,b}. \quad (30)$$

Roughly the Markov 2 property along the zigzag is Markov 1 along a $D$ steps and Markov 2 along a $U$ step. Now if a M-MC is invariant on H, then for $\rho$ stochastic LE of $M$, we have by (30)

$$P(S(i, 0) = a_i, i = 0, \ldots, n + 1, S(i, 1) = b_i, i \in 0, \ldots, n) = \rho_{a_0} \prod_{i=0}^{n} M_{a_i,a_{i+1}}T_{a_i,b_{i+1}}$$

$$= \rho_{a_0} \prod_{i=0}^{n} D_{a_i,b_i}U_{a_i,b_i,a_{i+1}}$$

which is indeed the representation of a Markov 2 process with kernel $(D, U)$ on HZ. □

**Remark 3.3.** In the previous proof we saw that if $M$ is Markov on H, then it is Markov 2 on HZ with memory 1 on a down step, and 2 on a up step. What it is true too, is that to this kind of process one can associate a Markov 1 process with kernel $M'$ on H with values in $E^2_\kappa$ (as illustrated on Figure 3) by “putting together” the state $S_t(i)$ and $S_{t+1}(i)$. The associated PCA is

![Figure 3](image)

Figure 3: From PCA with Markov 2 invariant distribution to PCA with Markov 1.

$A' = (Z, E^2_\kappa, N, T')$ with $T'_{(a_1,b_1),(a_2,b_2)} = 1_{b_1 = a_3}T_{b_1,b_2}$ and the Markov kernel is

$$M'_{(a_1,b_1),(a_2,b_2)} = U_{a_1,b_1,a_2}D_{a_2,b_2}.$$
From HZ(n) to H(n) and from HZ to H. We have already said that the restrictions of a HZMC on HZt (resp. a HZCMC on HZt(n)) on the lines on Ht and Ht+1 (resp. Ht(n) and Ht+1(n)) were MC (resp. CMC). As a consequence, if a PCA \( A := (L, E, N, T) \) seen as acting on \( M(E_{\kappa}) \) (resp. \( M(E_{\kappa}(n)) \)) admits an HZMC (resp. HZCMC) invariant distribution, then seen as acting on \( M(E_{\kappa}) \) (resp. \( M(E_{\kappa}(n)) \)), it admits a MC (resp. CMC) as invariant distribution.

Remark 3.4. • The converse is not true. Indeed as explained in Section 1.3.1 if \( T_{1,0} = T_{0,1} \) or \( T_{1,1} = T_{0,0} \), a Markov chain is invariant on H but the stationary distribution on HZ is not a HZMC.

• [Any Markov measure on H is the invariant measure for a PCA] We recall in our setting, the following fact, appearing in Prop. 16.1 in [23]: Let \( M \) be the Markov kernel of a MC on the line. There exists a TM \( T \) that lets the \( M \)-MC invariant. Indeed it suffices to find two Markov kernels \( (D, U) \) such that \( DU = UD = M \) since the TM defined by \( T_{a,b} = \frac{D_{a,c} U_{c,b}}{(DU)_{a,b}} \) will do the job. Take \( D = (1 + \varepsilon)M(1 + \varepsilon M)^{-1} \) and \( U = (1 + \varepsilon M)/(1 + \varepsilon) \) for a small enough \( \varepsilon \).

From HZ to HZ(n).

Proposition 3.5. Let \( A := (Z, E, N, T) \) be a PCA. If the \( (D,U) \)-HZMC on HZ is invariant by \( A \) then the \( (D,U) \)-HZMC on HZ(n) is invariant by \( A \).

Proof. Just compare the hypothesis of Theorems 2.3 and 2.10. □

From H to H(n) In the case \( \kappa = 1 \), there exists some PCA that have a product measure invariant on H that are not Markov on H(n). To be invariant on H(n) for infinitely many \( n \) implies that the matrices \( (Q^x, x \in \{0,1\}) \) have rank 1 (Proposition 2.14 (b)). In Section 1.3.1 we have seen that when \( T_{1,0} = T_{0,1} \), a product measure was invariant on H. The computation in this case (in the positive rate case) gives \( \rho_0 = \left( \frac{1-T_{1,1}}{T_{1,0}+T_{0,1}} \right) \); and with this value one checks that neither \( Q^0 \) nor \( Q^1 \) have rank 1. This does not prove the non existence of a product measure depending on \( n \), invariant by the PCA acting on H(n).

4 Proofs of Theorems 2.6, 2.11 and 2.9

We prove Theorem 2.9 at the end of the section.

To prove Theorem 2.6 and 2.11 we will use the characterisation given by Theorem 2.3. First we will show that the three conditions Cond 3, Cond 5 and Cond 1 are equivalent (this is Lemma 4.1 below). We focus on the proof of Theorem 2.6 and will prove Theorem 2.11 incidentally.

Lemma 4.1. Let \( T \) be a positive rate TM. Then the three conditions Cond 1, Cond 5, and Cond 3 are equivalent.
Proof. Proof of Cond 1 $\Rightarrow$ Cond 5: substitute $T_{k,j}$ by their expression in terms of $(D,U)$ as specified in Cond 1, and check that both sides are equal. Proof of Cond 5 $\Rightarrow$ Cond 3: take $a' = b' = c' = 0$ in Cond 5. Proof of Cond 3 $\Rightarrow$ Cond 1. Suppose Cond 3 holds and let us find $D$ and $U$ such that

$$\frac{D_{a,b}U_{b,a'}}{(DU)_{a,a'}} = T_{b,b},$$

for any $a,b,a'$. (31)

It suffices to find $D$ and $U$ s.t.

$$D_{a,b}U_{b,a'} = \frac{T_{a,b}T_{0,a'}}{T_{0,b}} G[a,a']$$

(32)

for some numbers $(G[i,j], 0 \leq i, j \leq \kappa)$, since in this case $D_{a,b}U_{b,a'} = \frac{T_{a,b}T_{0,a'}}{T_{0,b}} G[a,a']$.

Now, a solution to (32) is given by

$$D_{a,b} = \frac{T_{a,b}}{T_{0,b}} A_{a}B_{b}, \quad U_{b,a'} = \frac{T_{a,b}}{T_{0,b}} C_{a'}B_{b}, \quad G[a,a'] = A_{a}C_{a'},$$

(33)

where $C = (C_{a}, 0 \leq a \leq \kappa)$ is any array of positive numbers, $B = (B_{a}, 0 \leq a \leq \kappa)$ is chosen such that $U$ is a Markov kernel, and then $A = (A_{a}, 0 \leq a \leq \kappa)$ such that $D$ is a Markov kernel.

We now characterise the set of solutions $(D,U)$ to Cond 1 when $T$ satisfies Cond 3.

**Proposition 4.2.** Let $T$ with positive rate satisfying Cond 3. The set of pairs $(D,U)$ solutions to Cond 1 is the set of pairs $\{(D^\eta,U^\eta), \eta \in M^*(\kappa)\}$ (indexed by the set of distributions $\eta = (\eta_a, 0 \leq a \leq \kappa)$ with full support) as defined in (13).

Proof. Assume that Cond 3 holds. By Lemma 4.1, there exists $(D,U)$ satisfying Cond 1. Cond 1 is equivalent to the existence of two Markov kernels $D$ and $U$ such that for any $0 \leq a, b, c \leq \kappa$, $T_{a,b}(DU)_{a,b} = D_{a,c}U_{c,b}$. If all the $T_{a,b}$ are not 0, then for $(D,U)$ solution to Cond 1, $D_{a,b}$ and $U_{a,b}$ are not 0 for any $a,b$. Then Cond 1 implies

$$D_{a,c}U_{c,b} = \frac{D_{a,c}U_{0,b}T_{a,b}}{T_{a,b}}, \text{ and then } (DU)_{a,b} = \frac{D_{a,0}U_{0,b}}{T_{a,0}}.$$  

(34)

In the positive rate case, Cond 1 is equivalent to this pair of statements (both of them, not each of them). Observe that (34) implies (summing over $b$ at the left hand side),

$$D_{a,c} = D_{a,b} \sum_{b} \frac{U_{0,b}}{T_{a,b}} T_{a,c},$$

(35)
and then by (34) again (replacing $D_{a,c}$ by the right hand side of (35)) we get

$$U_{c,b} = \frac{U_{0,b} T_{a,b}}{\sum_{b'} U_{0,b'} T_{a,b'} c}.$$  \hfill (36)

We claim this last quantity is independent of $a$. To prove this it suffices to check that for any $a$,

$$\frac{U_{0,b}}{T_{a,b} \sum_{b'} U_{0,b'} T_{a,b'} c} T_{a,b} = \frac{U_{0,b}}{T_{0,b} \sum_{b'} U_{0,b'} T_{0,b'} c} T_{0,b}.$$  \hfill (37)

Now $U_{0,b} \neq 0$ (since $\text{Cond} 1$ holds, and $T$ has positive rate). Divide the left and right hand side by $U_{0,b} T_{0,b} T_{a,b}$, then take the inverse and the difference between the right and left hand side (after inversion), we see that this is equivalent to

$$\sum_{b'} U_{0,b'} \left( \frac{T_{a,b} T_{0,b} T_{0,b'}}{T_{a,b'} c} - \frac{T_{a,b} T_{0,b} T_{0,b'}}{T_{0,b'} c} \right) = 0$$  \hfill (38)

but since $\text{Cond} 5$ holds, this quantity is 0 for any $(U_{0,b'}, b' = 0, \ldots, \kappa)$ which proves the claim. Now

$$D_{a,*} = 1 = \sum_c \sum_{b'} D_{a,b} U_{0,b} T_{a,b} = \sum_{b'} D_{a,b} T_{a,b}$$  \hfill (39)

which implies

$$D_{a,0} = \left( \sum_{b} U_{0,b} / T_{a,b} \right)^{-1}.$$  \hfill (40)

We then see clearly that the distributions $\eta = (U_{0,b'}, b' = 0, \ldots, \kappa)$ can be used to parametrise the set of solutions. Plugging (40) in (35) and (36) we get the result.

We end now the proof of Theorem 2.6. A consequence of the previous considerations is that there exists $(D, U)$ satisfying $\text{Cond} 1$ and $\text{Cond} 2$ if there exists $\eta$ such that $D^\eta U^\eta = U^\eta D^\eta$, and of course, in this case $(D, U) = (D^\eta, U^\eta)$ satisfies $\text{Cond} 1$. Not much remains to be done: we need to determine the existence (or not) and the value of $\eta$ for which $D^\eta U^\eta = U^\eta D^\eta$ and if such a $\eta$ exists compute the invariant distribution of the MC with Markov kernel $U^\eta$ and $D^\eta$.

We claim now that if $D^\eta U^\eta = U^\eta D^\eta$, then $\eta = \gamma$ the stochastic Perron-LE of $X$. As a consequence such a $\eta$ is unique or does not exist. To show this claim proceed as follows. Assume that there exists $\eta$ such that $D^\eta U^\eta = U^\eta D^\eta$ (where $D^\eta U^\eta$ is given in (41), and $U^\eta D^\eta$ is computed as usual, starting from (13)). By (34) and (40) we have

$$(D^\eta U^\eta)_{a,b} = \sum_d \frac{\eta_b}{T_{a,d} c} T_{a,b} c.$$  \hfill (41)
Hence $D^n U^n = U^n D^n$ is equivalent to

$$\frac{1}{\sum_d \eta_d T_{a,d}} \eta_b T_{a,b} = \sum_c \frac{\eta_c T_{a,c}}{\sum_d \eta_d T_{a,d}} \sum_b \frac{\eta_b T_{b,c}}{\sum_d \eta_d T_{b,d}}$$

for any $a,b$.

Replace $T_{c,d}$ by $(T_{0,c},T_{0,d})$ and introduce

$$g_a = \left( \sum_d \frac{\eta_d}{T_{a,d}} \right)^{-1}, \quad f_a = \sum_{b,c} \frac{\eta_c}{T_{0,c,b}} T_{0,c,b}$$

(42) rewrites

$$g_a \eta_b T_{a,b} = \sum_c \frac{\eta_c T_{a,c}}{f_a} f_b T_{0,b,c} g_c$$

for any $a,b$.

(44) is equivalent to

$$g_a \eta_b T_{a,b} = \sum_c \frac{\eta_c T_{a,c}}{f_a} f_b T_{0,b,c} g_c$$

and, using Cond 3 again,

$$g_a \eta_b T_{a,b} = \sum_c \frac{\eta_c T_{a,c}}{f_a} f_b T_{0,b,c} g_c$$

for any $a,b$.

The question is still here to find/guess, for which TM $T$ there exists $\eta$ solving this system of equations (some $\eta'$s are also hidden in $g$ and $f$). In the sequel we establish that there exists at most one $\eta$ that solves the system: it is $\gamma$. For this we notice that for $a=b$ this system (45) simplifies: $(D^n U^n)_{a,a} = (U^n D^n)_{a,a}$ (for any $a$) is equivalent to

$$g_a \eta_a T_{a,a} = \sum_c \frac{\eta_c T_{a,c}}{f_a} f_b T_{0,b,c} g_c$$

for any $a$.

which is equivalent to the matrix equation:

$$\begin{bmatrix} g_a \eta_a T_{a,a} & \ldots & \eta_a T_{a,a} \end{bmatrix} = \lambda \nu,$$

where $\nu$ is the stochastic Perron-LE of $Y$ and $\lambda$ some free parameter. By (43), this rewrites

$$\sum_d \frac{\eta_d}{T_{a,d}} \eta_a T_{a,a} = \lambda \nu_a$$

and taking the inverse, we see that $\eta$ needs to be solution to

$$\sum_d \eta_d T_{a,d} \nu_a = \frac{1}{\lambda} \eta_a.$$

(47) The only possible $\eta$ is then $\gamma$ the unique Perron-LE of $X$ (which can be normalised to be stochastic), and $1/\lambda$ must be the Perron-eigenvalue of $X$ (which also provides the uniqueness of the eigenvector.
with positive coordinates). Hence \( D^\eta U^\gamma = U^\gamma D^\eta \) implies \( \eta = \gamma \). Nevertheless this does not imply \( D^\gamma U^\gamma = U^\gamma D^\gamma \) and then the condition \( D^\gamma U^\gamma = U^\gamma D^\gamma \) remains in Theorem 2.6 (what is true in all cases is \( (D^\gamma U^\gamma)_{a,a} = (U^\gamma D^\gamma)_{a,a} \) for any \( a \)).

However when \( \kappa = 1 \) this is sufficient since one can deduce the equality of two Markov kernel \( K \) and \( K' \) from \( K_{0,0} = K'_{0,0} \) and \( K_{1,1} = K'_{1,1} \) only. This is why in Theorem 2.6 a slight simplification occurs for the case \( \kappa = 1 \). When \( \kappa > 1 \) this is no more sufficient.

Remark 4.3. This ends the proof of Theorem 2.11 since we see that \( \text{Cond} 3 \) and \( \text{Diagonal}(DU) = \text{Diagonal}(UD) \) imply \( \eta = \gamma \) (the converse in Theorem 2.11 is easy). And the discussion just above the remark suffices to check the statement concerning the case \( \kappa = 1 \).

It remains to show that the stochastic Perron-RE \( \rho \) of \( X \) satisfies \( \rho D^\gamma U^\gamma = \rho \). Consider (41), where \( \eta \) is now replaced by \( \gamma \). By (47), we have

\[
(D^\gamma U^\gamma)_{a,b} = \lambda \frac{T_{a,a} \nu_a \gamma_b}{T_{a,b} \gamma_a}.
\]

The stochastic Perron-RE of \( D^\gamma U^\gamma \) is the unique vector \( \delta \) such that \( \delta D^\gamma U^\gamma = \delta \), that is

\[
\sum_a \lambda \frac{T_{a,a} \nu_a \gamma_b}{T_{a,b} \gamma_a} \delta_a = \delta_b.
\]

Taking \( \mu_i = \delta_i / \gamma_i \), (48) is equivalent to

\[
\sum_a \frac{T_{a,a} \nu_a}{T_{a,b}} \mu_a = \frac{1}{\lambda} \mu_b,
\]

which means that \( \mu \) is the Perron-RE of \( X \). This ends the proof of Theorem 2.3.

### 4.1 Proof of Theorem 2.9

We follow the arguments of the proof of Theorem 2.6 and adapt them slightly to the present case. The only difference is that \( \text{Cond} 6 \) replaces the positive rate condition. First (19) still holds, since

\[
1 = \sum_c T_{a,c} = \sum_c \frac{T_{0,a} T_{a,b} T_{a,c} T_{0,b} T_{0,c}}{T_{a,b} T_{a,c}}.
\]

Lemma 4.1 still holds if instead of the positive rate condition we take \( \text{Cond} 6 \) (Remark 2.5 is needed to see why \( \text{Cond} 3 \Rightarrow \text{Cond} 5 \), and the positivity of \( T_{a,b} \) to see that \( (DU)_{a,b} > 0 \) for all \( a,b \)). Also, we have \( D_{a,0} > 0, U_{0,b} > 0, D_{0,a} > 0 \) and \( U_{0,b} > 0 \) for any \( a,b \) by \( \text{Cond} 1 \). In (13), \( D^\eta_{a,c} \) and \( U^\eta_{c,b} \) are well defined under \( \text{Cond} 6 \) only. (34) still holds for the same reason, and again the two conditions (34) are equivalent to \( \text{Cond} 3 \) under \( \text{Cond} 6 \) only. (35) still holds, but there is a small problem for (36) since the division by \( D_{a,c} \) is not possible for all \( a \). The \( D_{a,c} \) (for fixed \( c \)) are not 0 for all \( a \) since \( D_{0,c} > 0 \). So (36) holds for the \( a \) such that \( D_{a,c} > 0 \). If all the
\( T_{a,b} = 0 \) then take \( U_{c,b}^0 = 0 \). In (37) again \( U_{0,b} \neq 0 \) and by \textbf{Cond} 3 if \( T_{a,b} > 0 \) then so do \( T_{0,b} = 0 \).

We can then pass from (37) to (38) (with the same restrictions on the considered \( a \)) and end the proof of the Proposition. The rest of the proof of Theorem 2.6 can be adapted with no additional problem. □

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