A note on the thermodynamic stability of a black ring at quantum scales

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Abstract: In this note, the thermodynamic properties of a thin black ring in AdS space-time is explored when the size of the ring is comparable to quantum scales. The angular momentum to mass ratio of this system has an upper limit, which is the cosmological radius of the black ring. It is found that the small black ring will be thermodynamically stable due to the effects introduced by thermal fluctuations. However we find that the black ring is less stable than thermal AdS. Thermodynamics analysis indicates that there is no critical point, but there is Hawking-Page transition to radiation, which is confirmed by the Helmholtz free energy analysis.

Keywords: Black Ring, Thermodynamics, Statistics, Stability.
1 Introduction

Four-dimensional black holes are natural solutions of Einstein’s theory of general relativity (GR). The simplest of these is the spherically symmetric solution known as the Schwarzschild metric \[1\], which describes a static black hole (BH) that has no electric charge and no angular momentum. The only parameter that distinguishes various Schwarzschild BHs is the mass. The linear stability of these BHs was investigated by Dotti \[2\]. Adding an electric charge yields the Reissner-Nordström BH, which is super-radiantly unstable against spherical perturbations of a charged scalar field \[3\]. Alternatively, adding rotation yields the Kerr metric \[4\], which is an axially-symmetric vacuum solution of Einstein equations describing a BH with mass and angular momentum. An interesting parameter here is the ratio \(b\) of angular momentum to squared mass, known as the Kerr parameter, whose absolute value is smaller than unity (\(|b| < 1\)). The most general vacuum solution of the Einstein-Maxwell equations in GR is called the Kerr-Newman metric. characterized by three parameters: mass, charge and angular momentum. This metric describes a charged and rotating BH, whose linear mode is stable within Einstein-Maxwell theory \[5\], though they have super-radiant instabilities \[6\]. If a cosmological constant \(\Lambda\) is included \[7\] then all of these BHs generalize to what are called anti-de Sitter (AdS) (\(\Lambda < 0\)) and de-Sitter (dS) BHs (\(\Lambda > 0\)).

These four-dimensional BHs can be extended to higher dimensional spacetimes \[8\]. However their stability properties can change. For example, it has been argued by Konoplya and Zhidenko \[9\] that the higher-dimensional Reissner-Nordström-de Sitter BHs are gravitationally unstable. Along with gravitational stability, thermodynamic stability of black objects is also an interesting topic of study \[10\].

One of the important black objects in higher dimensions is the black ring \[11\], which has horizon topology of \(S^1 \times S^{d-3}\) in \(d\)-dimensional space-time. Black rings are classified as either fat or thin, depending on their shape. Numerical methods have been used to show that a black ring in five dimensions is gravitationally unstable \[12\]. This instability may be just like those of black strings and p-branes in higher dimensions \[13\], and numerical evidence that the end state of these instabilities can violate the weak cosmic censorship conjecture has been provided \[14\]. Fat rings are unstable \[12\], which has been demonstrated using local Penrose inequalities.

There is very little information in literature about the stability of thin rings. A supersymmetric five-dimensional black ring having an event horizon of topology \(S^1 \times S^2\) has been constructed \[15\]. and a certain two-charge supersymmetric state of such configuration was studied from the statistical viewpoint \[16\]. The exact microscopic entropy of a black ring using the M-theory was subsequently obtained \[17\], and the mass-angular-momentum inequality for such black rings has been used to gain insight into the standard picture of gravitational collapse \[18\]. A string theoretic description...
of near extremal black rings was proposed by Larsen [19]. The thermodynamical properties of a dipole black ring was investigated by Astefanesei and Radu using the quasilocal formalism [20].

We are interested in this paper in investigating the thermodynamical stability of thin black rings in AdS space that incorporate expected quantum corrections to the entropy that are logarithmic in the horizon area [21]. Although a black ring in AdS space is gravitationally and thermodynamically unstable, in the presence of thermal fluctuations, the scenario we consider is expected to be slightly different at small scales: the thin black ring in AdS space may be stable due to thermal fluctuations at small scales due to the quantum effects.

We begin by briefly describing the black ring parameters and their relationship with classical angular momenta in Sec. 2. In Sec. 3, we explore the thermodynamic aspects of the system and end with a discussion and conclusions in Sec. 4.

2 Black ring

It is known that $d$-dimensional AdS space-time admits a thin black ring, constructed from a thin black string of width $r_0$ transformed to a circle of radius $R$, with horizon topology of $S^1 \times S^{d-3}$ [22]. Beginning with global AdS spacetime

$$ds^2 = -f d\tau^2 + \frac{d\rho^2}{f} + \rho^2 (d\theta^2 + \sin^2 \theta d\Omega_{d-4}^2 + \cos^2 \theta d\psi^2),$$

(2.1)

where

$$f = 1 + \frac{\rho^2}{l^2},$$

(2.2)

and $d\Omega_{d-4}^2$ is the metric of a $(D - 4)$-dimensional unit sphere with volume,

$$\Omega_{d-4} = \frac{2\pi^{\frac{d-3}{2}}}{\Gamma\left(\frac{d-3}{2}\right)}.$$  

(2.3)

the black ring is located at $\rho = R$ on the $\theta = 0$ plane, so that $r_0 \ll R$ and $r_0 \ll l$, where $l$ is the cosmological radius [23]. At large distances the gravitational field created by the ring (in directions transverse to the ring) is the same as that of an equivalent circular distribution of energy-momentum centered at $\rho = R, \theta = 0$. Close to this it is possible to choose a set of adapted coordinates in AdS so that the radial coordinate $r$ measures the transverse distance away from the circle at $r = 0$ (the location of the ring), where surfaces of constant $r$ have ring-like topology. A complete description appears in ref. [23].

The mass of the ring in units of $G$ is

$$M = \frac{r_0^{d-4} \Omega_{d-3} R (d-2) \left(1 + \frac{R^2}{l^2}\right)^{\frac{3}{2}}}{8},$$

(2.4)

where $\Omega_{d-3}$ obtained via (2.3). Clearly $M$ is an increasing function of $R$, whereas it is a decreasing function of $d$, which we illustrate in Fig. 1. The other conserved quantity associated with the black ring is its angular momentum, given by [23]

$$J = \frac{r_0^{d-4} \Omega_{d-3} R^2 \sqrt{(1 + (d-2) \frac{R^2}{l^2})(d-3 + (d-2) \frac{R^2}{l^2})}}{8}.$$  

(2.5)

It is clear that the angular momentum is also an increasing function of $R$. The angular momentum per unit mass [4]

$$a \equiv \frac{J}{M} = \frac{R}{d-2} \sqrt{\frac{(1 + (d-2) \frac{R^2}{l^2})(d-3 + (d-2) \frac{R^2}{l^2})}{\left(1 + \frac{R^2}{l^2}\right)^3}} \leq l$$

(2.6)
Figure 1. Typical behavior of the mass as a function of $R/l$ for $r_0/l = 0.01$. (a) in terms of $R/l$ for $d = 5$; (b) in terms of $d$ for $R/l = 10$.

Figure 2. Typical behavior of the values of the angular momentum per unit mass in terms of $R/l$ for $d = 5$. becomes a $d$-independent constant at large values of $R$, whose upper bound is always given by $l$; $a \rightarrow l$ for $R/l \gg 1$. We illustrate this in Fig. 2 for $d = 5$; other dimensions also yield qualitatively similar results. Note from (2.6) that $a$ is independent of $r_0$. The angular velocity of the horizon

$$
\Omega = \sqrt{(1 + R^2 l^2)(1 + (d - 2)R^2 l^2)} \frac{R^2 (d - 3 + (d - 2)R^2 l^2)}{4} \right) \frac{1}{\sqrt{1 + (d - 2)R^2 l^2}}.
$$

(2.7)

is the thermodynamic conjugate of $J$, and is a decreasing function of $R$.

The moment of inertia of the thin black ring is

$$
I = \frac{\Omega_{d-3} R^4}{8} \frac{(d - 3 + (d - 2)R^2 l^2) \frac{2}{3}}{(1 + R^2 l^2) \sqrt{1 + (d - 2)R^2 l^2}}.
$$

(2.8)

obtained from the classical relation $J = I \Omega^2$. It is an increasing function of $R$ and a decreasing function of $d$. We depict its typical behavior in Fig. 3 (a) for $d = 6$, while in Fig. 3 (b) we plot $I_M \equiv \frac{I}{M}$ and see that it becomes constant at large values of $R$. We can rewrite (2.8) as

$$
I = \mathcal{M} R^2,
$$

(2.9)
in order to have a match with the classical relation by defining a new mass parameter

\[ M ≡ \frac{MR}{2} \]  (2.10)

in complete agreement with the mass given in Ref. [12], where

\[ R = \sqrt{\frac{2}{d-2} R \left( \frac{d-3 + (d-2)\frac{R^2}{l^2}}{1 + \frac{R^2}{l^2}} \right)^3} \]  (2.11)

is the reduced radius. For \( d = 4 \) and \( l → ∞ \) we see that \( R = R \).

Consider next the quantity \( b = a/M = J/M^2 \). This is an interesting quantity that varies as \( 1/\eta^d \) (in units of \( c = G = 1 \)) that (for sufficiently large \( R/l \)) is generally a decreasing function of \( R \) and vanishes at large values of \( R \). It has a maximum at

\[ \frac{R}{l} = \sqrt{\frac{6 - 3d + \sqrt{25d^2 - 196d + 388}}{8(d-2)}} \]  (2.12)

provided \( d ≥ 7 \); otherwise it is maximized at \( R = 0 \). We illustrate this behavior in Fig. 4 for \( d = 5, 10, 11 \). As \( d \) increases the maximum value of \( b \) is a monotonically increasing function of \( d \). The value of \( b \) in contrast to that of a 4D Kerr black hole where \( -1 < b < 1 \).

Using (2.10), we find that the maximum for \( \bar{a} ≡ \frac{a}{M} \) is at

\[ R(\bar{a}_{max}) = \sqrt{\frac{d^2 - 7d + 13}{d-2}} \]  (2.13)

Using it in \( \bar{a} \) we find,

\[ \bar{a}_{max} = \frac{2\sqrt{(1 + \sqrt{d^2 - 7d + 13}) (d-3 + \sqrt{d^2 - 7d + 13})}}{(d-2) \left(1 + \sqrt{\frac{d^2 - 7d + 13}{d-2}} \right)^{3/2}} \]  (2.14)

so that \( \bar{a}_{max} \) is a decreasing function of \( d \) (independent of \( l \)) and we find for \( 5 ≤ d ≤ 11 \) that \( 1.075 ≥ \bar{a}_{max} ≥ 1.028 \).
Figure 4. Typical behavior of the values of the angular momentum per mass squared in terms of $R/l$ for $r_0/l = 0.01$.

3 Thermodynamics

The thermodynamic quantities for a thin AdS black ring have been previously calculated with

$$T = \frac{(d - 4)^2}{4 \pi r_0} \sqrt{1 + \frac{R}{r_0}}$$

being the temperature, computed from the surface gravity at the horizon and

$$S_0 = \frac{\pi}{2} r_0^{d-3} \Omega_{d-3} R \sqrt{d - 3 + (d - 2) \frac{R^2}{r_0^2}}$$

being the entropy [23].

It is clear that $T$ is a decreasing function of $R/l$. As the black ring radiates its mass decreases and so it will reduce its size. However other effects such as thermal (or quantum) fluctuations can eventually become important. Such corrections to lowest order yield [21, 24],

$$S = S_0 - \frac{\alpha}{2} \ln (S_0),$$

for the entropy of black objects, where $\alpha$ is a dimensionless parameter that tracks such correction terms [25].

We depict the behavior of the logarithmically corrected entropy (3.3) in Fig. 5. We see that effects of thermal fluctuations are extremely significant at smaller values of $R$. In Fig. 5 (a), corresponding to $d = 5$, we see that there is a critical radius (denoted by $R_c$) where $S = S_0$, below which corrections due to thermal fluctuations dominate over the semiclassical value given in (3.2) — in Fig. 5 (a) the critical radius $R_c/l = 18$. It is clear that $S \to S_0$ at large $R/l$; indeed for $R \gtrsim R_c$ the entropy $S \approx S_0$.

The critical value $R_c$ is easily obtained from the vanishing of the logarithmic term in (3.3)

$$\frac{R_c}{l} = \sqrt{2} \sqrt{\frac{\sqrt{16 r_0^6 (d - 4) \left[ \Gamma \left( \frac{d}{2} \right) \right]^2 + \pi^d r_0^{2d} (d - 2)(d - 3)^2}{r_0^d \pi^{d/2} (d - 2)^{d/2}}} - \frac{d - 3}{d - 2}}$$

which is clearly a decreasing function of $r_0$. At sufficiently small $r_0$ this will violate the approximations used to obtain the thermodynamic parameters for the black ring.
Figure 5. Typical behavior of the entropy (3.2) (dashed blue curve) and logarithmic corrected entropy (solid red curve) in terms of $R/l$ for $r_0/l = 0.01$.

In Fig. 5 it is given by the intersection of the solid red (presence of thermal fluctuation) and the dashed blue (absence of thermal fluctuation) lines. The first law of thermodynamics using $S$ in (3.3) is satisfied exactly at $R = R_c$ since the entropy is given by its classical value $S_0$.

The specific heat is

$$C = T \left( \frac{dS}{dT} \right) = T \frac{dS}{dR} \left( \frac{dT}{dR} \right)^{-1}$$

using (3.3), where

$$C_0 = T \left( \frac{dS_0}{dT} \right)$$

is the semiclassical specific heat.

The sign of the specific heat determines the thermodynamic stability of the system. The system is in a stable phase if $C \geq 0$ and phase transitions may occur at the asymptotic points. In Fig. 6 we plot the specific heat for $d = 5$ and $d = 6$ as a function of $R/l$; other dimensions yield similar results. Dash dotted green lines of Fig. 6 show the plot of the uncorrected specific heat. From the figure it is clear that the thin black ring in AdS space-time is completely unstable because of the negativity of its specific heat – we find that this occurs for all values of $r_0$. However the corrected specific heat, obtained using (3.3) in (3.6), is positive for sufficiently small $R/l$, and asymptotes to its classical value at large $R/l$. The thin black ring in AdS space-time may be therefore be thermodynamically

Figure 6. Typical behavior of the specific heat in terms of $R/l$ for $r_0/l = 0.01$. 

In Fig. 6 it is given by the intersection of the solid red (presence of thermal fluctuation) and the dashed blue (absence of thermal fluctuation) lines. The first law of thermodynamics using $S$ in (3.3) is satisfied exactly at $R = R_c$ since the entropy is given by its classical value $S_0$. The specific heat is

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stable once quantum corrections and/or thermal fluctuations are taken into account. Similar results hold for any $d$.

We also see that the behavior of specific heat indicates that there is no phase transition involved, which is confirmed by the analysis of Helmholtz free energy obtained by the relation,

$$ F = - \int S dT. \quad (3.7) $$

Numerically, we find that there is no extremum for the Helmholtz free energy, and hence there are no first or second order phase transitions. For $\alpha = 0$, the Helmholtz free energy is an increasing function of $R/l$ with positive values in all dimensions. However, in the presence of thermal fluctuations the value of the Helmholtz free energy may be negative (or positive) for smaller (or larger) values of $R/l$. These are illustrated by plots in Fig. 7 (a) and (b). However, both curves are nearly identical for large values of $R/l$, which confirms that the effect of the logarithmic corrections are significant only when $R/l$ is small. We can see from the Fig. 7 (b) that the black ring is stable for small $R/l$, while for larger $R/l$ there is a Hawking-Page transition to radiation (ie, the black ring is less stable than thermal AdS).

Consider next the Gibbs free energy. Here we need the thermodynamic volume of a thin AdS black ring, which is given by \[ 22 \]

$$ V = \frac{\pi \Omega_{d-3}}{d-1} R^d \sqrt{1 + \frac{R^2}{l^2}}. \quad (3.8) $$

Hence, Gibbs free energy is given by,

$$ G = F + PV, \quad (3.9) $$

where the thermodynamic pressure is

$$ P = - \left( \frac{\partial F}{\partial V} \right)_T. \quad (3.10) $$

In the Fig. 8 we can see typical behavior of pressure and the effects of quantum correction. We can see there are no important differences for the large $R/l$. An important effect of thermal fluctuation at small $R/l$ is increasing pressure.

In the Fig. 9 we draw Gibbs free energy in $d = 5$ (similar results hold in other dimensions) to see effect of thermal fluctuations. We see that the Gibbs free energy is negative for small $R/l$ in the presence of thermal fluctuations, possibly indicating thermodynamic stability in this regime. At large $R/l$ the free energy is positive, indicative of a Hawking-Page phase transition.
Discussions and Conclusion

A small thin AdS black ring may be thermodynamically stable once thermal (or quantum) fluctuations are taken into account. Such fluctuations correct the entropy of black objects with a logarithmic term \[ 21, 24, \] given in Eq. (3.3). These corrections become as large as the entropy at a critical value of the black ring radius, below which there are significant departures from standard semi-classical behavior. As \( R/l \rightarrow 0 \) we observe a spike in the modified entropy (given by red bold line in Fig. 5 (a)), with the trajectory eventually asymptotic to the \( R/l = 0 \) axis. Below the critical radius the modified entropy of the black ring is a decreasing function of \( R/l \). As the number of spacetime dimensions increases the critical radius becomes larger, but the qualitative features are the same. Under certain conditions the first law of thermodynamics can be satisfied \[ 26, 27, \]; in our case it holds exactly at \( R = R_c \). In general other thermodynamic quantities will be modified along with the entropy, in turn modifying the first law.

To probe the thermodynamic stability of the ring, we performed a specific heat analysis. We found that the modified specific heat is positive at sufficiently small \( R/l \), whereas for the uncorrected specific heat it is negative. Moreover, the trajectory for the modified specific heat spikes near \( R/l = 0 \) towards the positive end, before becoming asymptotic with the vertical axis (see Fig. 6). This clearly suggests stability of the ring at quantum scales. These results are corroborated by an
analysis of the thermodynamic potentials. The uncorrected Helmholtz free is always positive, but becomes negative values once fluctuations are taken into account at sufficiently small $R/l$. The transition point is indicative of a Hawking-Page phase transition.

The stability of black rings is rather delicate, and a more thorough stability analysis taking fluctuation effects more fully into account would be of interest.

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